An Equational Deductive System for Linear Temporal Logic (Draft)

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Abstract

This paper presents an equational deductive system for linear temporal logic. It differs from previous developments of such systems in several respects. First, it presents a numbered list of axioms and theorems to indicate which formulas are assumed, which formulas are derived, and for those that are derived, which previous formulas they depend on. Second, it gives a proof of every theorem. Third, the proofs are governed by an equational deductive system as opposed to the older Hilbert-style deductive systems. Fourth, it presents several new and interesting linear temporal theorems.

1 Introduction

Propositional calculus is a formal system of logic based on the unary operator negation $\neg$, the binary operators conjunction $\wedge$, disjunction $\vee$, implies $\Rightarrow$ (also written $\rightarrow$), and equivalence $\equiv$ (also written $\leftrightarrow$), variables (lowercase letters p, q, $\ldots$), and the constants $true$ and $false$. Hilbert-style logic systems, $\mathcal{H}$, are the deductive logic systems traditionally used in mathematics. In the late 1980's, Dijkstra and Scholten [**?**], and Feijen [**?**] developed a method of proving program correctness with a new logic based on an equational style. This equational deductive system, $\mathcal{E}$, has been the basis of textbooks by Kaldewaij [**?**], Cohen [**?**], and Gries and Schneider [**?**].

The equational deductive system of proofs is slow to be adopted by the computer science community. The problem is two-fold. First, a fair amount of technical detail must be mastered, and many computer science educators and practitioners do not have the requisite knowledge of formal logic systems, much less the equational deductive system. Scores of textbooks for discrete mathematics for computer science could be cited that give only a cursory introduction to formal logic. Most of these texts, such as the classic one by Rosen [**?**] are beginning to move to a more formal treatment of logic appropriate for computer science.

Second, even when formal logic is taught at a depth necessary to apply it to program proofs, the older Hilbert style still dominates. The previously-cited texts [**?**, **?**, **?**] are among the few that rely on the equational deductive system. More typical is Ben-Ari's book [**?**], which is based enitrely on natural deduction systems.

Linear temporal logic describes how the truth values of propositions change over time. It extends the propositional operators with the unary operators next $\bigcirc$, eventually $\Diamond$, and always $\Box$, and the binary operators until $\mathcal{U}$ and wait $\mathcal{W}$. Propositional calculus applies to program correctness with the formulation of the Hoare

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triple to establish invariants in programs that terminate. Temporal logic applies to program correctness with concurrent processing to establish safety and liveness properties in programs that possibly do not terminate.

As is the case for propositional and predicate calculus, most treatments of linear temporal logic use $\mathcal{H}$ instead of $\mathcal{E}$. Typical are Ben-Ari [**?**], Emerson [**?**], Kröger [**?**], Manna and Pnueli [**?**], and Schneider [**?**]. The only appearance of an equational proof of a temporal logic theorem appears to be a single example in [**?**], which otherwise uses a Hilbert-style system for temporal logic. The development of linear temporal logic in all these works is motivated by its use to prove correctness of concurrent programs. The presentation typically consists of lists of valid temporal formulas, with little emphasis of which formulas are required as axioms and which are theorems that can be proved using a deductive system.

This paper presents an equational deductive system for linear temporal logic. It differs from previous developments of such systems in several respects. First, it presents a numbered list of axioms and theorems to indicate which formulas are assumed, which formulas are derived, and for those that are derived, which previous formulas they depend on. Second, it gives a proof of every theorem. Third, the proofs are given in the equational style $\mathcal{E}$ instead of $\mathcal{H}$. Fourth, it presents several new and interesting linear temporal theorems.

Section 2 describes the deductive axioms and the proof rules for $\mathcal{E}$. It also defines the syntax and semantics of linear temporal logic. Section 3 presents the equational deductive system for linear temporal logic.

2 Background

The first section below summarizes the equational system $\mathcal{E}$ from [**?**]. The summary is minimal, and the remainder of the paper assumes familiarity with $\mathcal{E}$. The second section introduces temporal logic and assumes no prior familiarity with it. The paper can serve as an introduction to temporal logic for those familiar with $\mathcal{E}$.

2.1 Equational Deductive Systems

The definition of an expression has four parts:

- A constant or variable is an expression.
- If E is an expression, then (E) is an expression.
- If $\circ$ is a unary prefix operator and E is an expression, then $\circ E$ is an expression with operand E.
- If $\star$ is a binary infix operator and D and E are expressions, then $D \star E$ is an expression with operands D and E.

By convention, upper-case letters (*e.g.* $X, Y, \ldots$) represent expressions, and lower-case letters (*e.g.* $x, y, \ldots$) represent variables. In the propositional calculus, the constants are *true* and *false*.

Here is the table of precedences.

$[x := e]$ (textual substitution)	Highest precedence
$\neg \quad \bigcirc \quad \diamond \quad \square$	
$\mathcal{U} \quad \mathcal{W}$	
$=$ (conjunctional)	
$\vee \quad \wedge$	
$\Rightarrow \quad \Leftarrow$	
$\equiv$ (associative)	Lowest precedence

Textual substitution has the highest precedence. All the unary operators have the next highest precedence. They are necessarily right associative. For example, $\neg \bigcirc \neg p$ means $\neg(\bigcirc(\neg p))$. In this system, two binary operators that have the same precedence require parentheses to disambiguate. As in [**?**], conjunction $\wedge$ and disjunction $\vee$ have the same precedence so that $p \wedge q \vee r$ must be disambiguated as either $(p \wedge q) \vee r$ or $p \wedge (q \vee r)$. This contrasts with many systems in which conjunction has higher precedence than disjunction.

Also consistent with the equational system of [**?**] but different from most other deductive logic systems is the difference between operators equals $=$ and equivales $\equiv$. Equals applies to any mathematical type including, *e.g.*, boolean, natural number, and set. Equivales applies only to boolean, and is commonly denoted $\leftrightarrow$ in other systems. Another difference is that equals is conjunctive, while equivales is associative. For example, the expression $p = q = r$ means $(p = q) \wedge (q = r)$, while the expression $p \equiv q \equiv r$ can be taken as either $(p \equiv q) \equiv r$ or $p \equiv (q \equiv r)$. This property of equivales is the first axiom in the equational deductive system of [**?**].

The equational deductive system relies on the three deductive axioms for equality

Reflexivity: $x = x$

Symmetry: $(x = y) = (y = x)$

Transitivity: $\dfrac{X = Y, \quad Y = Z}{X = Z}$

and the proof rule

Leibniz: $\dfrac{X = Y}{E[z := X] = E[z := Y]}$

where the square bracket indicates textual substitution of expression X for variable z and substitution of expression Y for variable z. Roughly speaking, Leibniz allows for the substitution of equals for equals in a proof step. The general form of a proof step is

$$\begin{array}{ll} & E[z := X] \\ = & \quad \langle X = Y \rangle \\ & E[z := Y] \end{array}$$

where the expression enclosed in angle brackets $\langle\ \rangle$ called the "hint" is the justification for the step. An example of a proof step from the proof of theorem (5) below is

$$\begin{array}{ll} & \neg\bigcirc(\neg p \vee \neg q) \\ = & \quad \langle (4) \text{ with } p, q := \neg p, \neg q \rangle \\ & \neg(\bigcirc \neg p \vee \bigcirc \neg q) \end{array}$$

This proof step uses the previously proved theorem (4), distributivity of $\bigcirc$ over $\vee$, which is $\bigcirc(p \vee q) \equiv \bigcirc p \vee \bigcirc q$. The expressions in Leibniz for the step are

$$
\begin{array}{ll}
E: & \neg z \\
X: & \bigcirc(\neg p \vee \neg q) \\
Y: & \bigcirc \neg p \vee \bigcirc \neg q
\end{array}
$$

The textual substitutions are

$$
\begin{array}{ll}
E[z := X]: & \neg\bigcirc(\neg p \vee \neg q) \\
E[z := Y]: & \neg\bigcirc(\neg p \vee \neg q)
\end{array}
$$

And the justification in the hint $X = Y$ comes from the textual substitution of $\neg p$ for p and $\neg q$ for q in (4) as follows

$$(\bigcirc(p \vee q) \equiv \bigcirc p \vee \bigcirc q)[p, q := \neg p, \neg q]: \quad \bigcirc(\neg p \vee \neg q) \equiv \bigcirc \neg p \vee \bigcirc \neg q$$

Gries and Schneider [**?**] extend the proof format to incorporate implication using its transitive properties with itself and with equivales. An example is a proof of (27), $p \Rightarrow \Diamond p$.

$$
\begin{array}{ll}
 & \Diamond p \\
= & \langle \text{(26) Expansion of } \Diamond \rangle \\
 & p \vee \bigcirc \Diamond p \\
\Leftarrow & \langle \text{Weakening } p \Rightarrow p \vee q \text{ with } q := \bigcirc \Diamond p \rangle \\
 & p
\end{array}
$$

Because $\Diamond p$ equivales $p \vee \bigcirc \Diamond p$, and $p \vee \bigcirc \Diamond p$ follows from p, it follows by transitivity that $\Diamond p$ follows from p.

2.2 Temporal Logic

The operators of propositional calculus, $\neg$, $=$, $\wedge$, $\vee$, $\Rightarrow$, $\Leftarrow$, and $\equiv$ are static. That is, they apply at a single point in time. Each operator has a truth table that dictates how to evaluate the truth value of an expression. A state is an assignment of a truth value to each variable in the expression. A given boolean expression may be false in all states, true in some states and false in others, or true in all states, in which case the expression is known as a theorem or validity or tautology.

The operators of temporal logic, $\bigcirc$, $\Diamond$, $\Box$, $\mathcal{U}$, and $\mathcal{W}$ are dynamic. That is, they do not apply at a single point in time, but apply over an infinite sequence of states. Each state corresponds to a discrete point in time that represents one point in the execution of a program, possibly having several threads running concurrently but whose instruction executions have been serialized. As one instruction in the program executes, the state changes, and hence the truth value of an expression may change as well.

A model σ is an infinite sequence of the form

$$\sigma : s_0, s_1, s_2, \ldots$$

where s_0 is the initial state and each state $s_i, 0 \leq i$ is the state at time i. For example, suppose x is an integer variable whose value varies at each step of the computation. Then x and the expression $x < 10$, known as a state expression, might evolve as follows.

σ	s_0	s_1	s_2	s_3	s_4	$\ldots$
x	8	9	10	11	12	$\ldots$
$x < 10$	T	T	F	F	F	$\ldots$

The bottom row shows the evaluation of the state expression for each state in the sequence. Temporal logic extends propositional logic by considering the evolution of expression evaluations in time. For example, if you assume that x in the above sequence keeps increasing by one you can assert informally in English, "For the sequence σ, eventually $x < 10$ will always be false."

The notation

$$(\sigma, j) \models p$$

means that the expression p holds at position j in a sequence σ. In the above example,

$$(\sigma, 1) \models x < 10$$

The symbol $\models$ means "satisfies", so the above expression is read as "State 1 of sequence σ satisfies $x < 10$". Or, using "holds", the same expression is read as, "$x < 10$ holds in state 1 of sequence σ". The following sections use $\models$ to formalize the interpretation of each temporal operator.

The *next* operator $\bigcirc$

The semantics of the unary prefix operator $\bigcirc$ is

$$(\sigma, j) \models \bigcirc p \quad \text{iff} \quad (\sigma, j+1) \models p$$

That is, $\bigcirc p$ holds at position j iff p holds at position $j + 1$.

For example, in the above sequence $\bigcirc x \geq 10$ holds at state s_1 because $x \geq 10$ holds at state s_2. In other words,

$$(\sigma, 1) \models \bigcirc x \geq 10 \quad \text{because} \quad (\sigma, 2) \models x \geq 10$$

The *until* operator $\mathcal{U}$

The semantics of the binary infix operator $\mathcal{U}$ is

$$(\sigma, j) \models p\,\mathcal{U}\,q \quad \text{iff} \quad (\exists k \mid k \geq j : (\sigma, k) \models q \land (\forall i \mid j \leq i < k : (\sigma, i) \models p))$$

If $p\,\mathcal{U}\,q$ holds at state s_j, then p holds at state s_j and continues to hold at every state after s_j until q holds at some future state. $p\,\mathcal{U}\,q$ guarantees that q will eventually hold at some future state, and that p will continue to hold until then. After the state in which q holds for the first time, there are no restrictions on either p or q.

For example, suppose x and y evolve in the computation as follows.

σ	s_0	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_8	$\ldots$
x	-1	0	1	2	3	4	5	6	7	8	$\ldots$
y	9	8	7	6	5	4	3	2	1	0	$\ldots$
$0 < x < y$	F	F	T	T	T	F	F	F	F	F	$\ldots$
$2 \leq y < 5$	F	F	F	F	F	T	T	T	F	F	$\ldots$
$(0 < x < y)\,\mathcal{U}\,(2 \leq y < 5)$	F	F	T	T	T	T	T	T	F	F	$\ldots$

The bottom row shows the evaluation of the expression $p\,\mathcal{U}\,q$ where $p \equiv 0 < x < y$ and $q \equiv 2 \leq y < 5$. In states s_0 and s_1, $p\,\mathcal{U}\,q$ is false because both p and q are false. Starting at state s_2, $p\,\mathcal{U}\,q$ is true because in that state p is true and will remain true until q eventually becomes true in state s_5.

From the semantics of $p\ \mathcal{U}\ q$, if q is true in any state, then $p\ \mathcal{U}\ q$ is true in that state regardless of p. For example, not only is $p\,\mathcal{U}\,q$ true in state s_5, before which p was true in several preceding states, it is also true in states s_6 and s_7, because in those states q is true. This behavior of $p\,\mathcal{U}\,q$ comes from the empty range and one-point rules [**?**] of the predicate calculus in the case that q holds in state s_j and $k = j$.

$$
\begin{array}{ll}
 & (\exists k \mid k \geq j : (\sigma, k) \models q \wedge (\forall i \mid j \leq i < k : (\sigma, i) \models p)) \\
= & \langle \text{Case } k = j \rangle \\
 & (\exists k \mid k = j : (\sigma, k) \models q \wedge (\forall i \mid j \leq i < j : (\sigma, i) \models p)) \\
= & \langle j \leq i < j \equiv false \rangle \\
 & (\exists k \mid k = j : (\sigma, k) \models q \wedge (\forall i \mid false : (\sigma, i) \models p)) \\
= & \langle \text{Empty range rule } (\star x \mid false : P) = u \text{ with } true \text{ the identity of } \wedge \rangle \\
 & (\exists k \mid k = j : (\sigma, k) \models q \wedge true) \\
= & \langle \text{Identity of } \wedge \text{ and one-point rule } (\star x \mid x = E : P) = P[x := E] \rangle \\
 & ((\sigma, k) \models q)[k := j] \\
= & \langle \text{Textual substitution} \rangle \\
 & (\sigma, j) \models q \\
= & \langle \text{Case } q \text{ holds in state } s_j \rangle \\
 & true
\end{array}
$$

This result is theorem (14) $p\ \mathcal{U}\ true \equiv true$ proved in the next section. $true$ is the right zero of the until operator.

The *eventually* operator $\Diamond$

The semantics of the unary prefix operator $\Diamond$ is

$$(\sigma, j) \models \Diamond p \quad \text{iff} \quad (\exists k \mid k \geq j : (\sigma, k) \models p)$$

So, $\Diamond p$ holds in state s_j if p holds in state s_j or in any other state s_k where $k \geq j$, that is, if p holds in the current state or in any other future state.

For example, suppose x evolves in the computation as follows.

σ	s_0	s_1	s_2	s_3	s_4	s_5	s_6	$\ldots$
x	1	2	3	4	5	6	7	$\ldots$
$3 \leq x < 6$	F	F	T	T	T	F	F	$\ldots$
$\Diamond(3 \leq x < 6)$	T	T	T	T	T	F	F	$\ldots$

The bottom row shows the evaluation of the expression $\Diamond p$ where $p \equiv 3 \leq x < 6$. In states s_0 and s_1, $\Diamond p$ is true because there is a state, either now or in the future, in which p will hold.

If $\Diamond p$ is ever false in any state s_i in a sequence σ, it must be false in all subsequent states s_j, $j \geq i$. If $\Diamond p$ is ever true in any state s_i in a sequence σ, it must be true in all preceding states s_j, $j \leq i$. For example, suppose p and q evolve in the computation as follows.

σ	s_0	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	$\ldots$
p	F	F	T	F	F	T	F	F	F	F	$\ldots$
q	F	F	T	T	F	F	T	T	F	F	$\ldots$
$\Diamond p$	T	T	T	T	T	T	F	F	F	F	$\ldots$
$\Diamond q$	T	T	T	T	T	T	T	T	T	T	$\ldots$

The bottom two rows show the evaluation of the expressions $\Diamond\, p$ and $\Diamond\, q$ assuming that p remains false indefinitely and q continues to switch between true and false indefinitely.

The eventually operator is a special case of the until operator. Namely, $true\ \mathcal{U}\ q$ is equivalent to $\Diamond\, q$ as follows.

$$
\begin{array}{ll}
 & (\sigma, j) \models true\, \mathcal{U}\ q \\
= & \langle \text{Semantics of } p\, \mathcal{U}\ q \text{ with } p := true \rangle \\
 & (\exists k \mid k \geq j : (\sigma, k) \models q \wedge (\forall i \mid j \leq i < k : (\sigma, i) \models true)) \\
= & \langle true \text{ holds in all states} \rangle \\
 & (\exists k \mid k \geq j : (\sigma, k) \models q \wedge (\forall i \mid j \leq i < k : true)) \\
= & \langle \text{Theorem (9.8) from [?]}, (\forall x \mid R : true) \equiv true \rangle \\
 & (\exists k \mid k \geq j : (\sigma, k) \models q \wedge true) \\
= & \langle \text{Identity of } \wedge \rangle \\
 & (\exists k \mid k \geq j : (\sigma, k) \models q) \\
= & \langle \text{Semantics of } \Diamond\, q \rangle \\
 & (\sigma, j) \models \Diamond\, q
\end{array}
$$

This relationship is the basis of the definition of $\Diamond\, p$ in equation (22) $\Diamond\, p \equiv true\ \mathcal{U}\ p$ assumed in the next section.

The *always* operator $\Box$

The semantics of the unary prefix operator $\Box$ is

$$(\sigma, j) \models \Box\, p \quad \text{iff} \quad (\forall k \mid k \geq j : (\sigma, k) \models p)$$

So, $\Box\, p$ holds in state s_j if p holds in state s_j and in all other states s_k where $k \geq j$, that is, if p holds in the current state and in all other future states. For example, suppose x evolves in the computation as follows.

σ	s_0	s_1	s_2	s_3	s_4	s_5	s_6	$\dots$
x	1	2	3	4	5	6	7	$\dots$
$x \geq 4$	F	F	F	T	T	T	T	$\dots$
$\Box\,(x \geq 4)$	F	F	F	T	T	T	T	$\dots$

The bottom row shows the evaluation of the expression $\Box\, p$ where $p \equiv x \geq 4$. In states s_0, s_1, and s_2, $\Box\, p$ is false because p does not hold in those states. In states s_3, s_4, s_5, s_6, and subsequent states, $\Box\, p$ is true because p holds in in those states and in all future states as well.

If $\Box\, p$ is ever true in any state s_i in a sequence σ, it must be true in all subsequent states s_j, $j \geq i$. If $\Box\, p$ is ever false in any state s_i in a sequence σ, it must be false in all preceding states s_j, $j \leq i$. For example, suppose p and q evolve in the computation as follows.

σ	s_0	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	$\dots$
p	T	T	F	T	T	F	T	T	T	T	$\dots$
q	T	T	F	F	T	T	F	F	T	T	$\dots$
$\Box\, p$	F	F	F	F	F	F	T	T	T	T	$\dots$
$\Box\, q$	F	F	F	F	F	F	F	F	F	F	$\dots$

The bottom two rows show the evaluation of the expressions $\Box\, p$ and $\Box\, q$ assuming that p remains true indefinitely and q continues to switch between true and false indefinitely.

$\Diamond\, p$ is an existential operator, while $\Box\, p$ is a universal operator. They are related through the generalized De Morgan theorem [**?**] $\neg(\exists x \mid R : \neg P) \equiv (\forall x \mid R : P)$ as follows.

$$
\begin{array}{ll}
 & (\sigma, j) \models \Box\, p \\
= & \langle \text{Semantics of } \Box\, p \rangle \\
 & (\forall k \mid k \geq j : (\sigma, k) \models p) \\
= & \langle \text{Generalized De Morgan } \neg(\exists x \mid R : \neg P) \equiv (\forall x \mid R : P) \rangle \\
 & \neg(\exists k \mid k \geq j : \neg((\sigma, k) \models p)) \\
= & \langle p \text{ does not hold in a state iff } \neg p \text{ holds in that state} \rangle \\
 & \neg(\exists k \mid k \geq j : (\sigma, k) \models \neg p) \\
= & \langle \text{Semantics of } \Diamond\, q \rangle \\
 & \neg((\sigma, j) \models \Diamond\, \neg q) \\
= & \langle p \text{ does not hold in a state iff } \neg p \text{ holds in that state} \rangle \\
 & (\sigma, j) \models \neg \Diamond\, \neg q
\end{array}
$$

This relationship is the basis of the definition of $\Box\, p$ in equation (33) $\Box\, p \;\equiv\; \neg \Diamond\, \neg p$ assumed in the next section.

The above demonstration that $(\sigma, j) \models \Box\, p \;\equiv\; (\sigma, j) \models \neg \Diamond\, \neg q$ depends on the rule, "p does not hold in a state iff $\neg p$ holds in that state", written formally as

$$\neg((\sigma, j) \models p) \quad \text{iff} \quad (\sigma, j) \models \neg p$$

The corresponding rules for the binary operators are

$$
\begin{array}{l}
((\sigma, j) \models p) \;\wedge\; ((\sigma, j) \models q) \quad \text{iff} \quad (\sigma, j) \models p \wedge q \\
((\sigma, j) \models p) \;\vee\; ((\sigma, j) \models q) \quad \text{iff} \quad (\sigma, j) \models p \vee q \\
((\sigma, j) \models p) \;\Rightarrow\; ((\sigma, j) \models q) \quad \text{iff} \quad (\sigma, j) \models p \Rightarrow q \\
((\sigma, j) \models p) \;\equiv\; ((\sigma, j) \models q) \quad \text{iff} \quad (\sigma, j) \models p \equiv q
\end{array}
$$

The *wait* operator $\mathcal{W}$

The semantics of the binary infix operator $\mathcal{W}$ in terms of $\mathcal{U}$ and $\Box$ is

$$(\sigma, j) \models p\, \mathcal{W}\, q \quad \text{iff} \quad (\sigma, j) \models p\, \mathcal{U}\, q \;\vee\; (\sigma, j) \models \Box\, p$$

The wait operator $\mathcal{W}$ is weaker than the until operator $\mathcal{U}$, because $p\ \mathcal{W}\ q$ does not require q to ever be true, while $p\, \mathcal{U}\ q$ does. For example, suppose p and q evolve in the computation as follows.

σ	s_0	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	s_{10}	$\dots$
p	F	F	T	T	F	F	F	F	T	T	T	$\dots$
q	F	F	F	F	T	T	F	F	F	F	F	$\dots$
$\Box\, p$	F	F	F	F	F	F	F	F	T	T	T	$\dots$
$p\, \mathcal{U}\, q$	F	F	T	T	T	T	F	F	F	F	F	$\dots$
$p\, \mathcal{W}\, q$	F	F	T	T	T	T	F	F	T	T	T	$\dots$

The bottom two rows show the evaluation of the expressions $p\, \mathcal{U}\ q$ and $p\ \mathcal{W}\ q$ assuming that p remains true indefinitely and q remains false indefinitely. From s_0 to s_7, $p\, \mathcal{U}\ q$ and $p\ \mathcal{W}\ q$ hold in the same states. From s_8 on, however, $p\, \mathcal{U}\ q$ does not hold because q never holds thereafter, while $p\ \mathcal{W}\ q$ does hold because p always holds thereafter.

3 The Equational Temporal System

This section presents an axiomatic deductive system of temporal logic whose theorems are proved with the equational logic $\mathcal{E}$ of [**?**]. Theorems cited in a proof hint take two forms. A numbered reference enclosed in parentheses *witout* a decimal point is a reference to an axiom or a previously-proved theorem in this paper. A numbered reference enclosed in parentheses *with* a decimal point is a reference to an axiom or a theorem from the propositional calculus in [**?**].

3.1 Next

The next operator $\bigcirc$ is defined by the following two axioms.

(1) **Axiom, Self-dual:** $\bigcirc \neg p \equiv \neg \bigcirc p$
(2) **Axiom, Distributivity of $\bigcirc$ over $\Rightarrow$:** $\bigcirc (p \Rightarrow q) \equiv \bigcirc p \Rightarrow \bigcirc q$

Linearity follows from self-dual and distributivity of $\bigcirc$ over $\Rightarrow$.

(3) **Linearity:** $\bigcirc p \equiv \neg \bigcirc \neg p$

Proof:

$$
\begin{array}{ll}
 & \bigcirc p \equiv \neg \bigcirc \neg p \\
= & \langle (3.11) \text{ with } p, q := \bigcirc \neg p, \bigcirc p \rangle \\
 & \neg \bigcirc p \equiv \bigcirc \neg p
\end{array}
$$

which is (1), Self-dual. ∎

Here are proofs that $\bigcirc$ distributes over $\vee$, $\wedge$, and $\equiv$.

(4) **Distributivity of $\bigcirc$ over $\vee$:** $\bigcirc (p \vee q) \equiv \bigcirc p \vee \bigcirc q$

Proof:

$$
\begin{array}{ll}
 & \bigcirc (p \vee q) \\
= & \langle (3.59) \text{ Implication} \rangle \\
 & \bigcirc (\neg p \Rightarrow q) \\
= & \langle (2) \text{ Distributivity of } \bigcirc \text{ over } \Rightarrow \rangle \\
 & \bigcirc \neg p \Rightarrow \bigcirc q \\
= & \langle (3.59) \text{ Implication} \rangle \\
 & \neg \bigcirc \neg p \vee \bigcirc q \\
= & \langle (3) \text{ Linearity} \rangle \\
 & \bigcirc p \vee \bigcirc q
\end{array}
$$

∎

(5) **Distributivity of $\bigcirc$ over $\wedge$:** $\bigcirc (p \wedge q) \equiv \bigcirc p \wedge \bigcirc q$

Proof:

$$
\begin{array}{ll}
 & \bigcirc (p \wedge q) \\
= & \langle\text{(3.12) Double Negation, twice}\rangle \\
 & \bigcirc (\neg\neg p \wedge \neg\neg q) \\
= & \langle\text{(3.47b) De Morgan}\rangle \\
 & \bigcirc \neg(\neg p \vee \neg q) \\
= & \langle\text{(1) with } p := (\neg p \vee \neg q)\rangle \\
 & \neg\bigcirc (\neg p \vee \neg q) \\
= & \langle\text{(4) with } p, q := \neg p, \neg q\rangle \\
 & \neg(\bigcirc \neg p \vee \bigcirc \neg q) \\
= & \langle\text{(1) twice}\rangle \\
 & \neg(\neg\bigcirc p \vee \neg\bigcirc q) \\
= & \langle\text{(3.47a) De Morgan}\rangle \\
 & \neg\neg(\bigcirc p \wedge \bigcirc q) \\
= & \langle\text{(3.12) Double Negation}\rangle \\
 & \bigcirc p \wedge \bigcirc q
\end{array}
$$

∎

(6) **Distributivity of $\bigcirc$ over $\equiv$:** $\bigcirc (p \equiv q) \equiv \bigcirc p \equiv \bigcirc q$

Proof:

$$
\begin{array}{ll}
 & \bigcirc (p \equiv q) \\
= & \langle\text{(3.80) Mutual Implication}\rangle \\
 & \bigcirc ((p \Rightarrow q) \wedge (q \Rightarrow p)) \\
= & \langle\text{(5) Distributivity of } \bigcirc \text{ over } \wedge\rangle \\
 & \bigcirc (p \Rightarrow q) \wedge \bigcirc (p \Rightarrow q) \\
= & \langle\text{(2) Distributivity of } \bigcirc \text{ over } \Rightarrow\rangle \\
 & (\bigcirc p \Rightarrow \bigcirc q) \wedge (\bigcirc q \Rightarrow \bigcirc p) \\
= & \langle\text{(3.80) Mutual Implication}\rangle \\
 & \bigcirc p \equiv \bigcirc q
\end{array}
$$

∎

Now, $true$ holds in the next state, and $false$ does not hold in the next state.

(7) **Truth:** $\bigcirc true \equiv true$

Proof:

$$
\begin{array}{ll}
 & \bigcirc true \\
= & \langle\text{(3.28) Excluded middle}\rangle \\
 & \bigcirc (p \vee \neg p) \\
= & \langle\text{(4) Distributivty of } \bigcirc \text{ over } \vee\rangle \\
 & \bigcirc p \vee \bigcirc \neg p
\end{array}
$$

$$
\begin{array}{ll}
= & \langle \text{(1) Self-dual} \rangle \\
& \bigcirc\, p \vee \neg \bigcirc\, p \\
= & \langle \text{(3.28) Excluded middle} \rangle \\
& true
\end{array}
$$

■

(8) **Falsehood:** $\bigcirc\, false \equiv false$

Proof:

$$
\begin{array}{ll}
& \bigcirc\, false \equiv false \\
= & \langle \text{(3.8) Definition of } false \rangle \\
& \bigcirc\, \neg true \equiv \neg true \\
= & \langle \text{(3.11) with } p, q := true, \bigcirc\, \neg true \rangle \\
& \neg \bigcirc\, \neg true \equiv true \\
= & \langle \text{(3) Linearity} \rangle \\
& \bigcirc\, true \equiv true
\end{array}
$$

which is (7). ■

3.2 Until

This system defines the until operator $\mathcal{U}$ with the following four axioms. The associativity of $\mathcal{U}$ does not seem to appear in the temporal logic literature. It is used here to prove theorems (20) and (21).

(9) **Axiom, Associativity of $\mathcal{U}$:** $p\,\mathcal{U}\,(q\,\mathcal{U}\,r) \;\equiv\; (p\,\mathcal{U}\,q)\,\mathcal{U}\,r$

(10) **Axiom, Distributivity of $\bigcirc$ over $\mathcal{U}$:** $\bigcirc\,(p\,\mathcal{U}\,q) \equiv \bigcirc\,p\,\mathcal{U}\,\bigcirc\,q$

(11) **Axiom, Expansion of $\mathcal{U}$:** $p\,\mathcal{U}\,q \equiv q \vee (p \wedge \bigcirc\,(p\,\mathcal{U}\,q))$

(12) **Axiom:** $p\,\mathcal{U}\,false \equiv false$

(13) **Idempotency of $\mathcal{U}$:** $p\,\mathcal{U}\,p \equiv p$

Proof:

$$
\begin{array}{ll}
& p\,\mathcal{U}\,p \\
= & \langle \text{(11) Expansion of } \mathcal{U} \rangle \\
& p \vee (p \wedge \bigcirc\,(p\,\mathcal{U}\,p)) \\
= & \langle \text{(3.43b) Absorption} \rangle \\
& p
\end{array}
$$

∎

(14) **Right zero of $\mathcal{U}$:** $p\,\mathcal{U}\,true \equiv true$

Proof:

$$
\begin{array}{ll}
 & p\,\mathcal{U}\,true \\
= & \langle \text{(11) Expansion of } \mathcal{U} \rangle \\
 & true \vee (p \wedge \bigcirc (p\,\mathcal{U}\,true)) \\
= & \langle \text{(3.29) Zero of } \vee \rangle \\
 & true
\end{array}
$$

∎

(15) $p\,\mathcal{U}\,q \Rightarrow p \vee q$

Proof:

$$
\begin{array}{ll}
 & p\,\mathcal{U}\,q \\
= & \langle \text{(11) Expansion of } \mathcal{U} \rangle \\
 & q \vee (p \wedge \bigcirc (p\,\mathcal{U}\,q)) \\
\Rightarrow & \langle \text{(3.76d) with } p, q, r := q, p, \bigcirc (p\,\mathcal{U}\,q) \rangle \\
 & p \vee q
\end{array}
$$

∎

The following four axioms describe the relations of conjunction and disjuction

(16) **Axiom:** $(p\,\mathcal{U}\,r) \vee (q\,\mathcal{U}\,r) \Rightarrow (p \vee q)\,\mathcal{U}\,r$
(17) **Axiom:** $p\,\mathcal{U}\,(q \wedge r) \Rightarrow (p\,\mathcal{U}\,q) \wedge (p\,\mathcal{U}\,r)$
(18) **Axiom:** $(p \wedge q)\,\mathcal{U}\,r \equiv (p\,\mathcal{U}\,r) \wedge (q\,\mathcal{U}\,r)$
(19) **Axiom:** $p\,\mathcal{U}\,(q \vee r) \equiv (p\,\mathcal{U}\,q) \vee (p\,\mathcal{U}\,r)$

(20) $p\,\mathcal{U}\,(p\,\mathcal{U}\,q) \equiv p\,\mathcal{U}\,q$

(21) $(p\,\mathcal{U}\,q)\,\mathcal{U}\,q \equiv p\,\mathcal{U}\,q$

3.3 Eventually

Eventually $\Diamond$ is a special case of $\mathcal{U}$ when the left hand side is true.

(22) **Definition of $\Diamond$:** $\Diamond p \equiv true\, \mathcal{U}\, p$

$p\, \mathcal{U}\, q$ guarantees that q will eventually be true as follows.

(23) **Eventuality:** $p\, \mathcal{U}\, q \Rightarrow \Diamond q$

Proof:

$$
\begin{array}{ll}
 & p\, \mathcal{U}\, q \\
\Rightarrow & \langle \text{(3.76a) Weakening} \rangle \\
 & (p\, \mathcal{U}\, q) \vee (true\, \mathcal{U}\, q) \\
\Rightarrow & \langle \text{(16)} \rangle \\
 & (p \vee true)\, \mathcal{U}\, q \\
= & \langle \text{(3.29) Zero of } \vee \rangle \\
 & true\, \mathcal{U}\, q \\
= & \langle \text{(22) Definition of } \Diamond \rangle \\
 & \Diamond q
\end{array}
$$

∎

(24) **Truth:** $\Diamond true \equiv true$

Proof:

$$
\begin{array}{ll}
 & \Diamond true \\
= & \langle \text{(22) Definition of } \Diamond \rangle \\
 & true\, \mathcal{U}\, true \\
= & \langle \text{(13) Idempotency of } \mathcal{U} \rangle \\
 & true
\end{array}
$$

∎

(25) **Falsehood:** $\Diamond false \equiv false$

Proof:

$$
\begin{array}{ll}
 & \Diamond false \\
= & \langle \text{(22) Definition of } \Diamond \rangle \\
 & true\, \mathcal{U}\, false \\
= & \langle \text{(12)} \rangle \\
 & false
\end{array}
$$

∎

(26) **Expansion of $\Diamond$:** $\Diamond p \equiv p \vee \bigcirc \Diamond p$

Proof:

$$
\begin{array}{ll}
 & \Diamond p \\
= & \langle \text{(22) Definition of } \Diamond \rangle \\
 & true\, \mathcal{U}\, p \\
= & \langle \text{(11) Expansion of } \mathcal{U} \rangle \\
 & p \vee (true \wedge \bigcirc (true\, \mathcal{U}\, p)) \\
= & \langle \text{(22) Definition of } \Diamond \rangle \\
 & p \vee (true \wedge \bigcirc \Diamond p) \\
= & \langle \text{(3.39) Identity of } \wedge \rangle \\
 & p \vee \bigcirc \Diamond p
\end{array}
$$

∎

(27) **Weakening of $\Diamond$:** $p \Rightarrow \Diamond p$

Proof:

$$
\begin{array}{ll}
 & \Diamond p \\
= & \langle \text{(26) Expansion of } \Diamond \rangle \\
 & p \vee \bigcirc \Diamond p \\
\Leftarrow & \langle \text{(3.76a) Weakening} \rangle \\
 & p
\end{array}
$$

∎

(28) **Weakening of $\Diamond$:** $\bigcirc p \Rightarrow \Diamond p$

Proof:

$$
\begin{array}{ll}
 & \Diamond p \\
= & \langle \text{(22) Definition of } \Diamond \rangle \\
 & true\, \mathcal{U}\, p \\
= & \langle \text{(11) Expansion of } \mathcal{U} \rangle \\
 & p \vee (true \wedge \bigcirc (true\, \mathcal{U}\, p)) \\
= & \langle \text{(3.39) Identity of } \wedge \rangle \\
 & p \vee \bigcirc (true\, \mathcal{U}\, p) \\
= & \langle \text{(10) Distributivity of } \bigcirc \text{ over } \mathcal{U} \rangle
\end{array}
$$

$$
\begin{array}{ll}
 & p \vee (\bigcirc\, true\, \mathcal{U}\, \bigcirc\, p) \\
= & \langle \text{(11) Expansion of } \mathcal{U}\, \rangle \\
 & p \vee \bigcirc\, p \vee (\bigcirc\, true \wedge \bigcirc\, (\bigcirc\, true\, \mathcal{U}\, \bigcirc\, p)) \\
\Leftarrow & \langle \text{(3.76a) Weakening} \rangle \\
 & \bigcirc\, p
\end{array}
$$

∎

(29) **Absorption of $\Diamond$:** $\Diamond \Diamond p \equiv \Diamond p$

Proof:

$$
\begin{array}{ll}
 & \Diamond \Diamond p \\
= & \langle \text{(22) Definition of } \Diamond \text{, with } p := \Diamond p \rangle \\
 & true\, \mathcal{U}\, \Diamond p \\
= & \langle \text{(22) Definition of } \Diamond \rangle \\
 & true\, \mathcal{U}\, (true\, \mathcal{U}\, p) \\
= & \langle \text{(20) with } p, q := true, p \rangle \\
 & true\, \mathcal{U}\, p \\
= & \langle \text{(22) Definition of } \Diamond \rangle \\
 & \Diamond p
\end{array}
$$

∎

(30) $\bigcirc \Diamond p \equiv \Diamond \bigcirc p$

Proof:

$$
\begin{array}{ll}
 & \bigcirc \Diamond p \\
= & \langle \text{(22) Definition of } \Diamond \rangle \\
 & \bigcirc\, (true\, \mathcal{U}\, p) \\
= & \langle \text{(10) Distributivity of } \bigcirc \text{ over } \mathcal{U}\, \rangle \\
 & \bigcirc\, true\, \mathcal{U}\, \bigcirc\, p \\
= & \langle \text{(7)} \rangle \\
 & true\, \mathcal{U}\, \bigcirc\, p \\
= & \langle \text{(22) Definition of } \Diamond \rangle \\
 & \Diamond \bigcirc p
\end{array}
$$

∎

(31) **Distributivity of $\Diamond$ over $\vee$:** $\Diamond (p \vee q) \equiv \Diamond p \vee \Diamond q$

Proof:

$$\begin{array}{ll} & \Diamond\,(p \vee q) \\ = & \langle (22) \text{ Definition of } \Diamond \rangle \\ & true\,\mathcal{U}\,(p \vee q) \\ = & \langle (19) \rangle \\ & (true\,\mathcal{U}\,p) \vee (true\,\mathcal{U}\,q) \\ = & \langle (22) \text{ Definition of } \Diamond \text{ twice} \rangle \\ & \Diamond\,p \vee \Diamond\,q \end{array}$$

∎

(32) **Distributivity of $\Diamond$ over $\wedge$:** $\Diamond\,(p \wedge q) \Rightarrow \Diamond\,p \wedge \Diamond\,q$

Proof:

$$\begin{array}{ll} & \Diamond\,(p \wedge q) \\ = & \langle (22) \text{ Definition of } \Diamond \rangle \\ & true\,\mathcal{U}\,(p \wedge q) \\ \Rightarrow & \langle (17) \rangle \\ & (true\,\mathcal{U}\,p) \wedge (true\,\mathcal{U}\,q) \\ = & \langle (22) \text{ Definition of } \Diamond \text{ twice} \rangle \\ & \Diamond\,p \wedge \Diamond\,q \end{array}$$

∎

3.4 Always

(33) **Definition of $\Box$:** $\Box\,p \equiv \neg\Diamond\,\neg p$

(34) **Dual of $\Box$:** $\neg\Box\,p \equiv \Diamond\,\neg p$

Proof:

$$\begin{array}{ll} & \neg\Box\,p \equiv \Diamond\,\neg p \\ = & \langle (3.11) \text{ with } p, q := \Box\,p, \Diamond\,\neg p \rangle \\ & \Box\,p \equiv \neg\Diamond\,\neg p \end{array}$$

which is (33). ∎

(35) **Dual of $\Diamond$:** $\neg\Diamond\,p \equiv \Box\,\neg p$

Proof:

$$
\begin{array}{ll}
& \Box\,\neg p \\
= & \langle (33)\text{ Definition of } \Box\,\rangle \\
& \neg\Diamond\,\neg\neg p \\
= & \langle (3.12)\text{ Double Negation}\rangle \\
& \neg\Diamond\,p
\end{array}
$$

∎

(36) $\Diamond\,p \equiv \neg\Box\,\neg p$

Proof:

$$
\begin{array}{ll}
& \neg\Box\,\neg p \\
= & \langle (33)\text{ Definition of } \Box\,\rangle \\
& \neg\neg\Diamond\,\neg\neg p \\
= & \langle (3.12)\text{ Double Negation, twice}\rangle \\
& \Diamond\,p
\end{array}
$$

∎

(37) **Truth:** $\Box\,true \equiv true$

Proof:

$$
\begin{array}{ll}
& \Box\,true \\
= & \langle (33)\text{ Definition of } \Box\,\rangle \\
& \neg\Diamond\,\neg true \\
= & \langle (3.8)\text{ Definition of } false\rangle \\
& \neg\Diamond\,false \\
= & \langle (25)\text{ Falsehood}\rangle \\
& \neg false \\
= & \langle (3.13)\text{ Negation of } false\rangle \\
& true
\end{array}
$$

∎

(38) **Falsehood:** $\Box\,false \equiv false$

Proof:

$$
\begin{array}{ll}
& \Box\,false \equiv false \\
= & \langle (3.8)\text{ Definition of } false\text{, twice}\rangle
\end{array}
$$

$$\begin{array}{ll} & \Box\neg true \equiv \neg true \\ = & \langle(3.11)\rangle \\ & \neg\Box\neg\, true \equiv true \\ = & \langle(36)\rangle \\ & \Diamond\, true \equiv true \end{array}$$

which is (24) Truth. ∎

(39) **Expansion of $\Box$:** $\Box\, p \equiv p \wedge \bigcirc\Box\, p$

Proof:

$$\begin{array}{ll} & \Box\, p \\ = & \langle\text{(33) Definition of } \Box\,\rangle \\ & \neg\Diamond\neg p \\ = & \langle\text{(22) Definition of } \Diamond \text{ with } p := \neg p\rangle \\ & \neg(true\,\mathcal{U}\,\neg p) \\ = & \langle\text{(11) Expansion of } \mathcal{U}\,\rangle \\ & \neg(\neg p \vee (true \wedge \bigcirc(true\,\mathcal{U}\,\neg p))) \\ = & \langle\text{(3.39) Identity of } \wedge\rangle \\ & \neg(\neg p \vee \bigcirc(true\,\mathcal{U}\,\neg p)) \\ = & \langle\text{(3.47b) De Morgan's Law}\rangle \\ & \neg\neg p \wedge \neg\bigcirc(true\,\mathcal{U}\,\neg p) \\ = & \langle\text{(3.12) Double Negation}\rangle \\ & p \wedge \neg\bigcirc(true\,\mathcal{U}\,\neg p) \\ = & \langle\text{(22) Defintion of } \Diamond\,\rangle \\ & p \wedge \neg\bigcirc\Diamond\neg p \\ = & \langle\text{(34) Dual of } \Box\,\rangle \\ & p \wedge \neg\bigcirc\neg\Box\, p \\ = & \langle\text{(3) Linearity}\rangle \\ & p \wedge \bigcirc\Box\, p \end{array}$$

∎

(40) **Absorption of $\Box$:** $\Box\Box\, p \equiv \Box\, p$

Proof:

$$\begin{array}{ll} & \Box\Box\, p \equiv \Box\, p \\ = & \langle\text{(33) Definition of } \Box \text{ with } p := \Box\, p\rangle \\ & \neg\Diamond\neg\Box\, p \equiv \Box\, p \\ = & \langle\text{(3.11) with } p, q := \Diamond\neg\Box\, p, \Box\, p\rangle \\ & \Diamond\neg\Box\, p \equiv \neg\Box\, p \end{array}$$

$$
\begin{array}{ll}
= & \langle (34)\ \text{Dual of } \Box\,,\ \text{twice}\rangle \\
& \Diamond\,\Diamond\,\neg p \equiv \Diamond\,\neg p \\
= & \langle (29)\ \text{Absorption of } \Diamond\,\rangle \\
& \Diamond\,\neg p \equiv \Diamond\,\neg p
\end{array}
$$

which is (3.5) with $p := \Diamond\,\neg p$. ∎

(41) $\bigcirc\,\Box\,p \equiv \Box\,\bigcirc\,p$

Proof:

$$
\begin{array}{ll}
& \bigcirc\,\Box\,p \\
= & \langle (33)\ \text{Definition of } \Box\,\rangle \\
& \bigcirc\,\neg\Diamond\,\neg p \\
= & \langle (1)\ \text{Self-dual}\rangle \\
& \neg\bigcirc\,\Diamond\,\neg p \\
= & \langle (30)\ \text{with } p := \neg p\rangle \\
& \neg\Diamond\,\bigcirc\,\neg p \\
= & \langle (1)\ \text{Self-dual}\rangle \\
& \neg\Diamond\,\neg\bigcirc\,p \\
= & \langle (33)\ \text{Definition of } \Box\,\rangle \\
& \Box\,\bigcirc\,p
\end{array}
$$

∎

(42) **Strengthening of $\Box$:** $\Box\,p \Rightarrow p$

Proof:

$$
\begin{array}{ll}
& \Box\,p \\
= & \langle (33)\ \text{Definition of } \Box\,\rangle \\
& \neg\Diamond\,\neg p \\
= & \langle (26)\ \text{Expansion of } \Diamond\,\rangle \\
& \neg(\neg p \vee \bigcirc\,\Diamond\,\neg p) \\
= & \langle (3.47b)\ \text{De Morgan's Law}\rangle \\
& \neg\neg p \wedge \neg\bigcirc\,\Diamond\,\neg p \\
= & \langle (3.12)\ \text{Double Negation}\rangle \\
& p \wedge \neg\bigcirc\,\Diamond\,\neg p \\
\Rightarrow & \langle (3.76b)\ \text{Strengthening}\rangle \\
& p
\end{array}
$$

$\blacksquare$

(43) **Strengthening of $\Box$:** $\Box\, p \Rightarrow \Diamond\, p$

Proof:

$$
\begin{array}{ll}
 & \Box\, p \\
\Rightarrow & \langle \text{(42) Strengthening of } \Box \rangle \\
 & p \\
\Rightarrow & \langle \text{(27) Weakening of } \Diamond \rangle \\
 & \Diamond\, p
\end{array}
$$

$\blacksquare$

(44) **Strengthening of $\Box$:** $\Box\, p \Rightarrow \bigcirc\, p$

Proof:

$$
\begin{array}{ll}
 & \Box\, p \\
= & \langle \text{(39) Expansion of } \Box \rangle \\
 & p \wedge \bigcirc \Box\, p \\
= & \langle \text{(41)} \rangle \\
 & p \wedge \Box \bigcirc p \\
= & \langle \text{(39) Expansion of } \Box \text{ with } p := \bigcirc\, p \rangle \\
 & p \wedge \bigcirc\, p \wedge \bigcirc \Box \bigcirc p \\
\Rightarrow & \langle \text{(3.76b) Strengthening} \rangle \\
 & \bigcirc\, p
\end{array}
$$

$\blacksquare$

(45) **Strengthening of $\Box$:** $\Box\, p \Rightarrow \bigcirc \Box\, p$

Proof:

$$
\begin{array}{ll}
 & \Box\, p \\
= & \langle \text{(39) Expansion of } \Box \rangle \\
 & p \wedge \bigcirc \Box\, p \\
\Rightarrow & \langle \text{(3.76b) Strengthening} \rangle \\
 & \bigcirc \Box\, p
\end{array}
$$

∎

(46) $\Box \neg p \Rightarrow \neg \Box p$

Proof:

$$
\begin{array}{ll}
& \Box \neg p \\
\Rightarrow & \langle \text{(43) Strengthening of } \Box \rangle \\
& \Diamond \neg p \\
= & \langle \text{(34) Dual of } \Box \rangle \\
& \neg \Box p
\end{array}
$$

∎

(47) **Excluded Middle:** $\Diamond p \vee \Box \neg p$

Proof:

$$
\begin{array}{ll}
& \Diamond p \vee \Box \neg p \\
= & \langle \text{(35) Dual of } \Diamond \rangle \\
& \Diamond p \vee \neg \Diamond p
\end{array}
$$

which is (3.28) Excluded middle, with $p := \Diamond p$. ∎

(48) **Distributivity of $\Box$ over $\wedge$:** $\Box (p \wedge q) \equiv \Box p \wedge \Box q$

Proof:

$$
\begin{array}{ll}
& \Box (p \wedge q) \\
= & \langle \text{(33) Definition of } \Box \rangle \\
& \neg \Diamond \neg (p \wedge q) \\
= & \langle \text{(3.47a) De Morgan} \rangle \\
& \neg \Diamond (\neg p \vee \neg q) \\
= & \langle \text{(31) Distributivity of } \Diamond \text{ over } \vee \rangle \\
& \neg (\Diamond \neg p \vee \Diamond \neg q) \\
= & \langle \text{(3.47b) De Morgan} \rangle \\
& \neg \Diamond \neg p \wedge \neg \Diamond \neg q \\
= & \langle \text{(33) Definition of } \Box \text{, twice} \rangle \\
& \Box p \wedge \Box q
\end{array}
$$

∎

(49) **Distributivity of $\Box$ over $\vee$:** $(\Box p \vee \Box q) \Rightarrow \Box (p \vee q)$

Proof:

$$
\begin{array}{ll}
& \Box\, p \vee \Box\, q \Rightarrow \Box\,(p \vee q) \\
= & \langle \text{(3.60) Implication} \rangle \\
& (\Box\, p \vee \Box\, q) \wedge \Box\,(p \vee q) \equiv \Box\, p \vee \Box\, q \\
= & \langle \text{(3.46) Distributivity of } \wedge \text{ over } \vee \rangle \\
& (\Box\,(p \vee q) \wedge \Box\, p) \vee (\Box\,(p \vee q) \wedge \Box\, q) \equiv \Box\, p \vee \Box\, q \\
= & \langle \text{(48) Distributivity of } \Box \text{ over } \wedge \rangle \\
& \Box\,(p \wedge (p \vee q)) \vee \Box\,(q \wedge (p \vee q)) \equiv \Box\, p \vee \Box\, q \\
= & \langle \text{(3.43) Absorption, twice} \rangle \\
& \Box\, p \vee \Box\, q \equiv \Box\, p \vee \Box\, q
\end{array}
$$

which is (3.5) Reflexivity of $\equiv$. ∎

(50) **Distributivity of $\Box$ over $\equiv$:** $\Box\,(p \equiv q) \Rightarrow (\Box\, p \equiv \Box\, q)$

Proof:

$$
\begin{array}{ll}
& \Box\,(p \equiv q) \Rightarrow (\Box\, p \equiv \Box\, q) \\
= & \langle \text{(3.62)} \rangle \\
& \Box\,(p \equiv q) \wedge \Box\, p \equiv \Box\,(p \equiv q) \wedge \Box\, q \\
= & \langle \text{(48) Distributivity of } \Box \text{ over } \wedge\text{, twice} \rangle \\
& \Box\,((p \equiv q) \wedge p) \equiv \Box\,((p \equiv q) \wedge q) \\
= & \langle \text{(3.50), twice} \rangle \\
& \Box\,(p \wedge q) \equiv \Box\,(p \wedge q)
\end{array}
$$

which is (3.5) Reflexivity of $\equiv$. ∎

(51) **Distributivity of $\Box$ over $\Rightarrow$:** $\Box\,(p \Rightarrow q) \Rightarrow (\Box\, p \Rightarrow \Box\, q)$

Proof:

$$
\begin{array}{ll}
& \Box\,(p \Rightarrow q) \\
= & \langle \text{(3.60) Implication} \rangle \\
& \Box\,(p \wedge q \equiv p) \\
\Rightarrow & \langle \text{(50) Distributivity of } \Box \text{ over } \equiv \rangle \\
& \Box\,(p \wedge q) \equiv \Box\, p \\
= & \langle \text{(48) Distributivity of } \Box \text{ over } \wedge \rangle \\
& \Box\, p \wedge \Box\, q \equiv \Box\, p \\
= & \langle \text{(3.60) Implication} \rangle \\
& \Box\, p \Rightarrow \Box\, q
\end{array}
$$

∎

(52) **Distributivity of $\Box\Diamond$ over $\wedge$:** $\Box\Diamond(p \wedge q) \Rightarrow \Box\Diamond p \wedge \Box\Diamond q$

Proof:

$$\begin{array}{cl}
& \Box\Diamond(p \wedge q) \Rightarrow \Box\Diamond p \wedge \Box\Diamond q \\
= & \langle \text{(3.60) Implication} \rangle \\
& \Box\Diamond(p \wedge q) \wedge \Box\Diamond p \wedge \Box\Diamond q \equiv \Box\Diamond(p \wedge q) \\
= & \langle \text{(48) Distributivity of } \Box \text{ over } \wedge \rangle \\
& \Box(\Diamond(p \wedge q) \wedge \Diamond p \wedge \Diamond q) \equiv \Box\Diamond(p \wedge q) \\
= & \langle \text{Lemma: } \Diamond(p \wedge q) \wedge \Diamond p \wedge \Diamond q \equiv \Diamond(p \wedge q) \rangle \\
& \Box\Diamond(p \wedge q) \equiv \Box\Diamond(p \wedge q)
\end{array}$$

which is (3.5) Reflexivity of $\equiv$. ∎

Proof of Lemma:

$$\begin{array}{cl}
& \Diamond(p \wedge q) \wedge \Diamond p \wedge \Diamond q \equiv \Diamond(p \wedge q) \\
= & \langle \text{(3.60) Implication} \rangle \\
& \Diamond(p \wedge q) \Rightarrow \Diamond p \wedge \Diamond q
\end{array}$$

which is (32) Distributivity of $\Diamond$ over $\wedge$. ∎

(53) **Distributivity of $\Diamond\Box$ over $\vee$:** $\Diamond\Box p \vee \Diamond\Box q \Rightarrow \Diamond\Box(p \vee q)$

Proof:

$$\begin{array}{cl}
& \Diamond\Box p \vee \Diamond\Box q \Rightarrow \Diamond\Box(p \vee q) \\
= & \langle \text{(3.57) Definition of Implication} \rangle \\
& \Diamond\Box p \vee \Diamond\Box q \vee \Diamond\Box(p \vee q) \equiv \Diamond\Box(p \vee q) \\
= & \langle \text{(31) Distributivity of } \Diamond \text{ over } \vee \rangle \\
& \Diamond(\Box p \vee \Box q \vee \Box(p \vee q)) \equiv \Diamond\Box(p \vee q) \\
= & \langle \text{Lemma: } \Box p \vee \Box q \vee \Box(p \vee q) \equiv \Box(p \vee q) \rangle \\
& \Diamond\Box(p \vee q) \equiv \Diamond\Box(p \vee q)
\end{array}$$

which is (3.5) Reflexivity of $\equiv$. ∎

Proof of Lemma:

$$\begin{array}{cl}
& \Box p \vee \Box q \vee \Box(p \vee q) \equiv \Box(p \vee q) \\
= & \langle \text{(3.57) Definition of Implication} \rangle \\
& \Box p \vee \Box q \Rightarrow \Box(p \vee q)
\end{array}$$

which is (49) Distributivity of $\Box$ over $\vee$. ∎

(54) **Distributivity of $\Box\Diamond$ over $\vee$:** $\Box\Diamond(p \vee q) \equiv \Box\Diamond p \vee \Box\Diamond q$

(55) **Distributivity of $\Diamond\Box$ over $\wedge$:** $\Diamond\Box(p \wedge q) \equiv (\Diamond\Box p \wedge \Diamond\Box q)$

(56) $\Diamond\Box p \Rightarrow \Box\Diamond p$

(57) **Absorption of $\Diamond$ into $\Box$:** $\Diamond\Box\Diamond p \equiv \Box\Diamond p$

(58) **Absorption of $\Box$ into $\Diamond$:** $\Box\Diamond\Box p \equiv \Diamond\Box p$

(59) **Induction:** $\Box(p \Rightarrow \bigcirc p) \Rightarrow (p \Rightarrow \Box p)$

(60) **Monotonicity of $\bigcirc$:** $\Box(p \Rightarrow q) \Rightarrow (\bigcirc p \Rightarrow \bigcirc q)$

Proof:

$$
\begin{array}{ll}
 & \Box(p \Rightarrow q) \\
\Rightarrow & \langle \text{(44) Strengthening} \rangle \\
 & \bigcirc(p \Rightarrow q) \\
= & \langle \text{(2) Distributivity of } \bigcirc \text{ over } \Rightarrow \rangle \\
 & \bigcirc p \Rightarrow \bigcirc q
\end{array}
$$

∎

(61) **Monotonicity of $\Diamond$:** $\Box(p \Rightarrow q) \Rightarrow (\Diamond p \Rightarrow \Diamond q)$

Proof:

$$
\begin{array}{ll}
 & \Box(p \Rightarrow q) \Rightarrow (\Diamond p \Rightarrow \Diamond q) \\
= & \langle \text{(3.59) Implication, three times} \rangle \\
 & \neg\Box(\neg p \vee q) \vee \neg\Diamond p \vee \Diamond q \\
= & \langle \text{(34) Dual of } \Box \rangle \\
 & \Diamond\neg(\neg p \vee q) \vee \neg\Diamond p \vee \Diamond q \\
= & \langle \text{(3.47b) De Morgan, and Double negation} \rangle \\
 & \Diamond(p \wedge \neg q) \vee \neg\Diamond p \vee \Diamond q
\end{array}
$$

$= \quad \langle$(31) Distributivity of $\Diamond$ over $\vee\rangle$
$\Diamond\,((p \wedge \neg q) \vee q) \vee \neg\Diamond\, p$
$= \quad \langle$(3.44b) Absorption$\rangle$
$\Diamond\,(p \vee q) \vee \neg\Diamond\, p$
$= \quad \langle$(31) Distributivity of $\Diamond$ over $\vee\rangle$
$\Diamond\, p \vee \Diamond\, q \vee \neg\Diamond\, p$
$= \quad \langle$(3.28) Excluded Middle, with $p := \Diamond\, p\rangle$
$\Diamond\, q \vee true$
$= \quad \langle$(3.29) Zero of $\vee\rangle$
$true$

∎

(62) $\Diamond\,(p \Rightarrow q) \equiv (\Box\, p \Rightarrow \Diamond\, q)$

Proof:

$\Diamond\,(p \Rightarrow q)$
$= \quad \langle$(3.59) Implication$\rangle$
$\Diamond\,(\neg p \vee q)$
$= \quad \langle$(31) Distributivity of $\Diamond$ over $\vee\rangle$
$\Diamond\,\neg p \vee \Diamond\, q$
$= \quad \langle$(34) Dual of $\Box\rangle$
$\neg\Box\, p \vee \Diamond\, q$
$= \quad \langle$(3.59) Implication$\rangle$
$\Box\, p \Rightarrow \Diamond\, q$

∎

(63) $\Box\, p \wedge \Diamond\, q \Rightarrow \Diamond\,(p \wedge q)$

Proof:

$\Box\, p \wedge \Diamond\, q \Rightarrow \Diamond\,(p \wedge q)$
$= \quad \langle$(3.59) Implication$\rangle$
$\neg(\Box\, p \wedge \Diamond\, q) \vee \Diamond\,(p \wedge q)$
$= \quad \langle$(3.47a) De Morgan$\rangle$
$\neg\Box\, p \vee \neg\Diamond\, q \vee \Diamond\,(p \wedge q)$
$= \quad \langle$(34) Dual of $\Box\rangle$
$\Diamond\,\neg p \vee \neg\Diamond\, q \vee \Diamond\,(p \wedge q)$
$= \quad \langle$(31) Distributivity of $\Diamond$ over $\vee\rangle$
$\Diamond\,(\neg p \vee (p \wedge q)) \vee \neg\Diamond\, q$
$= \quad \langle$(3.44b) Absorption$\rangle$

$$
\begin{aligned}
& \Diamond\,(\neg p \vee q) \vee \neg\Diamond\, q \\
= \quad & \langle (31) \text{ Distributivity of } \Diamond \text{ over } \vee \rangle \\
& \Diamond\,\neg p \vee \Diamond\, q \vee \neg\Diamond\, q \\
= \quad & \langle (3.28) \text{ Excluded middle} \rangle \\
& \Diamond\,\neg p \vee true \\
= \quad & \langle (3.29) \text{ Zero of } \vee \rangle \\
& true
\end{aligned}
$$

∎

3.5 Wait

(64) **Definition of $\mathcal{W}$:** $p\,\mathcal{W}\,q \equiv (p\,\mathcal{U}\,q) \vee \Box\, p$

(65) $p\,\mathcal{W}\,q \equiv q \vee (p \wedge \bigcirc\,(p\,\mathcal{W}\,q))$

Proof:

$$
\begin{aligned}
& q \vee (p \wedge \bigcirc\,(p\,\mathcal{W}\,q)) \\
= \quad & \langle (64) \text{ Definition of } \mathcal{W} \rangle \\
& q \vee (p \wedge \bigcirc\,((p\,\mathcal{U}\,q) \vee \Box\, p)) \\
= \quad & \langle (4) \text{ Distributivity of } \bigcirc \text{ over } \vee \rangle \\
& q \vee (p \wedge (\bigcirc\,(p\,\mathcal{U}\,q) \vee \bigcirc\,\Box\, p)) \\
= \quad & \langle (3.46) \text{ Distributivity of } \wedge \text{ over } \vee \rangle \\
& q \vee (p \wedge \bigcirc\,(p\,\mathcal{U}\,q)) \vee (p \wedge \bigcirc\,\Box\, p) \\
= \quad & \langle (39) \text{ Expansion of } \Box \rangle \\
& q \vee (p \wedge \bigcirc\,(p\,\mathcal{U}\,q)) \vee \Box\, p \\
= \quad & \langle (11) \text{ Expansion of } \mathcal{U} \rangle \\
& (p\,\mathcal{U}\,q) \vee \Box\, p \\
= \quad & \langle (64) \text{ Definition of } \mathcal{W} \rangle \\
& p\,\mathcal{W}\,q
\end{aligned}
$$

∎

(66) $\Box\, p \Rightarrow p\,\mathcal{W}\,q$

Proof:

$$
\begin{aligned}
& \Box\, p \\
= \quad & \langle (3.76a) \text{ Weakening} \rangle \\
& \Box\, p \vee (p\,\mathcal{U}\,q) \\
= \quad & \langle (64) \text{ Definition of } \mathcal{W} \rangle \\
& p\,\mathcal{W}\,q
\end{aligned}
$$

∎

(67) $\Box p \equiv p \,\mathcal{W}\, false$

Proof:

$$\begin{array}{ll} & p \,\mathcal{W}\, false \\ = & \langle \text{(64) Definition of } \mathcal{W} \rangle \\ & (p \,\mathcal{U}\, false) \vee \Box p \\ = & \langle \text{(12)} \rangle \\ & false \vee \Box p \\ = & \langle \text{(3.30) Identity of } \vee \rangle \\ & \Box p \end{array}$$

∎

4 Conclusion

This paper presents an axiomatic deductive system of temporal logic whose theorems are proved with the equational logic $\mathcal{E}$ of [**?**]. It takes unary operator next $\bigcirc$ and binary operator until $\mathcal{U}$ as primitives and defines eventually $\Diamond$, always $\Box$, and wait $\mathcal{W}$ in terms of them.