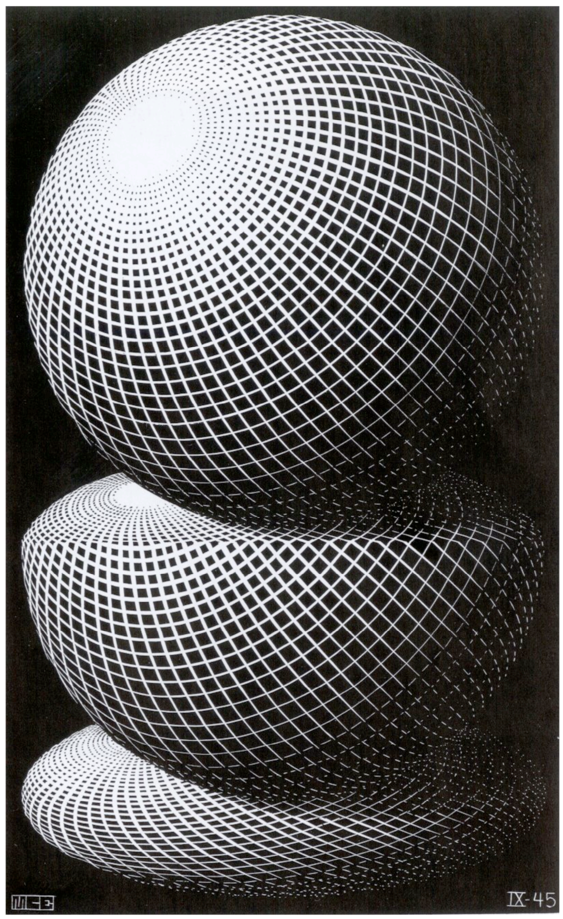
An Essay On Sensibility In Non-Euclidean Geometry



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Senior Essay

Class of 2007

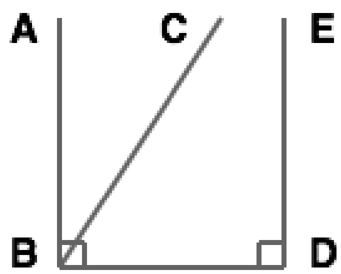
"Contrary to the opinion of Legendre, all other imperfections [of Euclidean Geometry]- for example, the definition of a straight line - show themselves foreign here and without any real influence."

-- Nikolai Lobechevsky, Introduction to Researches in The Theory of Parallels

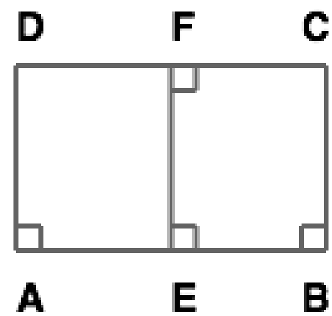
Preface: A Short Encounter With A Student

When the student initially encounters Lobechevsky, he is amazed and yet perplexed at this new world. The real amazement that the student feels can be seen by anyone - even to one who has never been initialized into the cult of non-Euclidean geometry.

For the most part this essay is my own inquiry into how to make sense of something so strange. I have set the student to be the central character. While the character does directly represent my own progression of thought on the matter, I believe and hope that the questions asked in the essay by the student are universal to every student who is determined to make sense of something so apparently absurd. To give an idea of this absurdity, I will present just three images that the student encounters at the beginning of such a study.

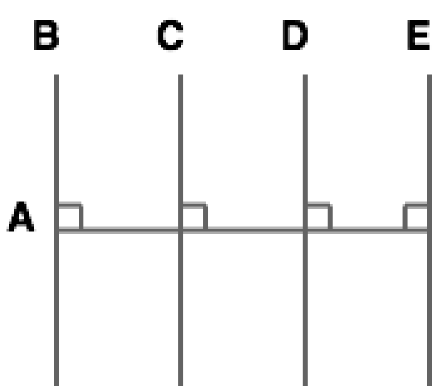


(1) Lobechevsky, Proposition 16



(2) The Saccheri Quadrilateral

1. *None of the straight lines between AB and CB will intersect ED*. CB is the first line in this series and is called “parallel.” The angle of parallelism, CBD, will vary inversely with the distance BD. As BD becomes longer, the angle CBD will lessen.
2. At the center of the figure, EF is a straight line that forms perpendiculars with both DC and AB. As the figure increases in size, the distance of a perpendicular to the base AB will increase in size. Hence DA is longer than FE. *All the lines in this figure are straight. The angles ADC and BCD are less than right*.



(3) Perpendicular Or "Ultra Parallel" Lines

3. All lines in the diagram are straight. A is a perpendicular to B, C, D, and E. It is the *only* perpendicular possible for these lines. The aforementioned lines (B, C, D, E) are called “ultra-parallel.” In this

example, the lines are closest where they are intersected by the perpendicular at A. *Everywhere else these four lines diverge* from each other.

As you see, the student finds himself in a predicament. He has just made his first request. Let's go directly him.

"Nikolai, Please extend those straight parallel lines in the first figure." But Lobachevsky is nowhere to be found. So the student extends the lines - only to find himself drawing one of them as curved. How can he reconcile the fact that he had no choice in drawing the line this way? He knows sight is no mathematical proof, but yet the student feels the contradiction. Contrary to Lobechevsky's claim, it appears that straightness has actually changed.

So the student labors in vain to find the most conclusive absurdity, a contradiction in terms (known in the mathematical world as a reductio ad absurdum) to disprove this nonsense once and for all. "Obviously the straight line is what has changed; the contradiction *must* be in it," the student thinks to himself. And how does he labor, generating theorem after theorem, only to lose hope time and again in finally finding the hidden trick up Lobachevsky's sleeve.

Finally the student is presented with a proof that if the Euclidean system is logically consistent, so must the non-Euclidean, and vice-versa. At this he is astonished. But he now knows better than to search for the answer in mathematics. He must go higher than mathematics; he must go to philosophy.

The First Alternative: A Logical Game

So far we have followed the student directly; now we shall use his thought as a somewhat arbitrary springboard to one particular philosopher, Immanuel Kant. The decision to investigate Kant's philosophy is only a somewhat arbitrary one, for Kant must be the philosopher who most appeals to the student. He has proved the apodictic (demonstrable) certainty of Euclidean Geometry. Although Kant's commentary on mathematics was finished prior to the development of this other geometry, we shall see that he yet has quite a bit to say on the student's dilemma.

We left the student at his discovery of the consistency of non-Euclidean geometry. His problem was that he did not know what could justify calling the parallel lines in the new system "straight." Clearly this was because he could not make the lines appear straight. In the example (the first one given which corresponds to Lobechevsky's Theorem 16), the student is sensible of a curved line. What is he to make of this? Either the term ‘straight’ is being misapplied or he is not truly seeing the straight lines.

Lobechevsky has said that straightness has not changed at all. The student therefore believes the images in the diagram do not represent the real straight lines; he now believes he has concepts but no (possible) sensible content to pin them on. At this time, the student turns to Immanuel Kant, as Kant spoke of concepts that did not have sensible content[1]. Two such examples given in The Critique of Pure Reason are spirit and God. The former necessarily leaves out anything sensible, (since a spirit must not have a body), while the later, Kant claims, is simply beyond us[2].

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[1] The Critique of Pure Reason, B74-82
[2] A96

Kant has also described a process of formal logic of which we are all well accustomed to. An A implies a B; a B implies a C, and so on. These formal systems abstract from all content and this is obvious because a 'B' does not stand for anything on the outset, but of course can stand for anything. The student now has one explanation for this other geometry, and quite a good one, because he believes this new “geometry” is entirely absent of any sensible content. Lobachevsky’s Geometry is nothing more than a logical game!

But what is he to make of the consistency of this geometry? An empty formal system, as described, does not necessarily lead us to a contradiction. In fact, in such a system, a contradiction is the only thing we are really concerned about. The only means we have of judging such a system is the law of non-contradiction.

But what if the student is wrong on all of this? What if the objects in the system are real? The student, after all, is sensible of these lines, but cannot see all of them as straight. Perhaps the term straight is being misapplied and something absurd is being laid down as a postulate.

One may think that a system with absurd premises will lead to a contradiction; this is not necessarily the case in the strictest sense of 'contradiction'. What one finds instead is an escalating degree of absurdity in the deductions of the system. The ancients clearly understood this idea, and hence had the joke that this sort of reasoning would lead one man to milking a male goat while the other attempted to catch the milk with a sieve[3].

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[3] B830

These explanations of non-Euclidean geometry may seem valid, but they also raise some red flags. Kant described the process of the mathematician and the philosopher, and according to our explanation, this geometry would be more a work of philosophy than mathematics. The process of generating these theorems does not abide by the typical rules we see in a formal logical system[4]. So we must ask: "If these theorems were not created through the intuition, did they then really arise through analysis? Could all of these constructions of lines and figures be some sort of empty symbol manipulation?" Has this system really been so artificially created?

The Second Alternative: A Non-Empirical Intuition

Without discarding the previous questions (ones which we must certainly revisit), we see another alternative on the horizon. Could it be possible that we have a priori intuitions of these objects with no corresponding sensible component?

In one sense, we must observe that the only objects we really know the validity of are the objects presented to us through sensibility. Yet this does not completely limit us to only that which is directly seen. We can still have a priori mathematics[5]. Nor does this rule out knowledge of

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[4] In a typical formal system we have rules such as "a B implies a C." Here, instead, we have *constructions* of what appear to be typical geometrical objects, such as figures, straight lines, etc.

[5] "Take for instance, the concepts of mathematics, considering them first of all in their pure intuitions. Space has three dimensions; between two points there can only be one straight line, etc. Although all these principles, and the representation of the object with which this science occupies itself, are generated in the mind completely a priori, they would mean nothing were we not always able to present their meaning in appearances, that is, in empirical objects. We therefore demand that a bare concept be made sensible, that is, that an object corresponding to it be present in intuition. Otherwise, the concept would, as we say, be without sense, that is, without meaning" B299

things of which we are not directly sensible. Hence, our dull senses do not directly perceive magnetism, but only the movement of the iron filings[6].

The only two things we can become conscious of are our thoughts, and the objects of which we are thinking about. The objects of mathematics are always generated formally, brought consciously into concepts, and later (or simultaneously) represented empirically. So if we are inclined to speak of formal intuitions that can only be formal, that is, never have the possibility of being represented sensibly, we must be speaking of concepts without (sensible) objects. This was exactly the first thing we considered.

The Third Alternative: Contradictory Synthesis

In the desire to explain mathematics, Kant has claimed that the intuition is a synthetic faculty. How a mathematician comes up with a proof is a very mysterious thing, one that I do not think even Kant fully understands. The mathematician has a triangle and wants to prove some relation of the interior angles to the triangle[7]. Synthetically, he extends the base, and then constructs a line parallel to the opposite side. It is only then that the proof can proceed analytically (where one relies on things previously proven). The mathematician then concludes that the angular sum of the triangle is equal to two right angles[8]. Kant's claim is that if one were to give a philosopher the triangle, he would analyze the concepts involved. The philosopher would make the concepts "three", "triangle", "straight", etc., crystal clear, but the philosopher would never get

[6] B273
[7] B740-749
[8] The example given is the proof of Euclid 1.32

to the answer that the mathematician has, and this is because mathematics engages in creation, or synthesis.

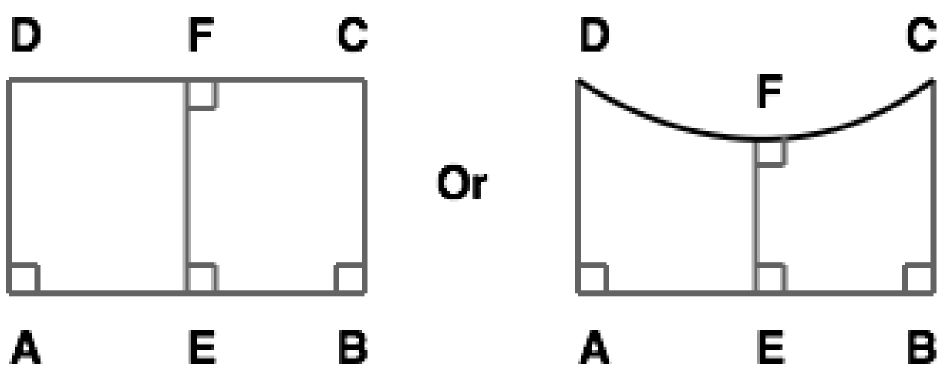
The obvious distinction that one can draw between analysis and synthesis is that where the first can be reduced down to a method (even a complete table of logical functions, as Kant has done), the later can never be. The first must proceed with non-contradiction as its criterion, but the later has no criterion at all (save only a vaguely defined formal intuition of space). Hence, here is the third alternative: Faced with Lobachevsky's geometry, Kant could claim that the intuition does produce contradictory opposites, which, when arising into concepts, baffles the understanding.

What is one to make of the idea just presented? The intuition has no capacity for judgment, so the phrase "contradictory intuitions" alone, without concepts, is altogether meaningless. The truth of this statement is seen when one understands that all the intuition presents us with are images. So what does one mean when speaking of contradictory intuitions? These a priori intuitions must be the creators of contradictory concepts. What would the understanding do with such concepts? Since non-contradiction is the absolute, negative principle of all judgments, one must conclude that the understanding would reject these two differing objects as both invalid, non-real entities. If this were the case, we would reject all forms of pure geometry (both the Euclidean and non-Euclidean) as some sort of fiction of the mind.

A Disconnect From The Sensible

Although Kant's *Critique* has a massive scope, one can view its main purpose as the limitations of reason's employment. How far shall reason go? Kant's claim is that it should be rooted in the sensible[9]. Looking back on this fact may give us some headway.

What the student felt to be contradictory in Lobechevsky's Proposition 16 can most clearly be seen in the Saccheri quadrilateral. In Proposition 16, the student knows of no way to draw the straight line except by curving it. But the Saccheri quadrilateral is different; the student is presented with an option immediately. The line at the top should be straight, but the angles should be less than right. If he draws the top line as straight, he can no longer draw the angles properly. Likewise if he draws the angles as they should be drawn (less than right), he can no longer draw a straight line.



(2) The Saccheri Quadrilateral

[9] And by this I mean that reason should be rooted in the objects of *possible* sensation.

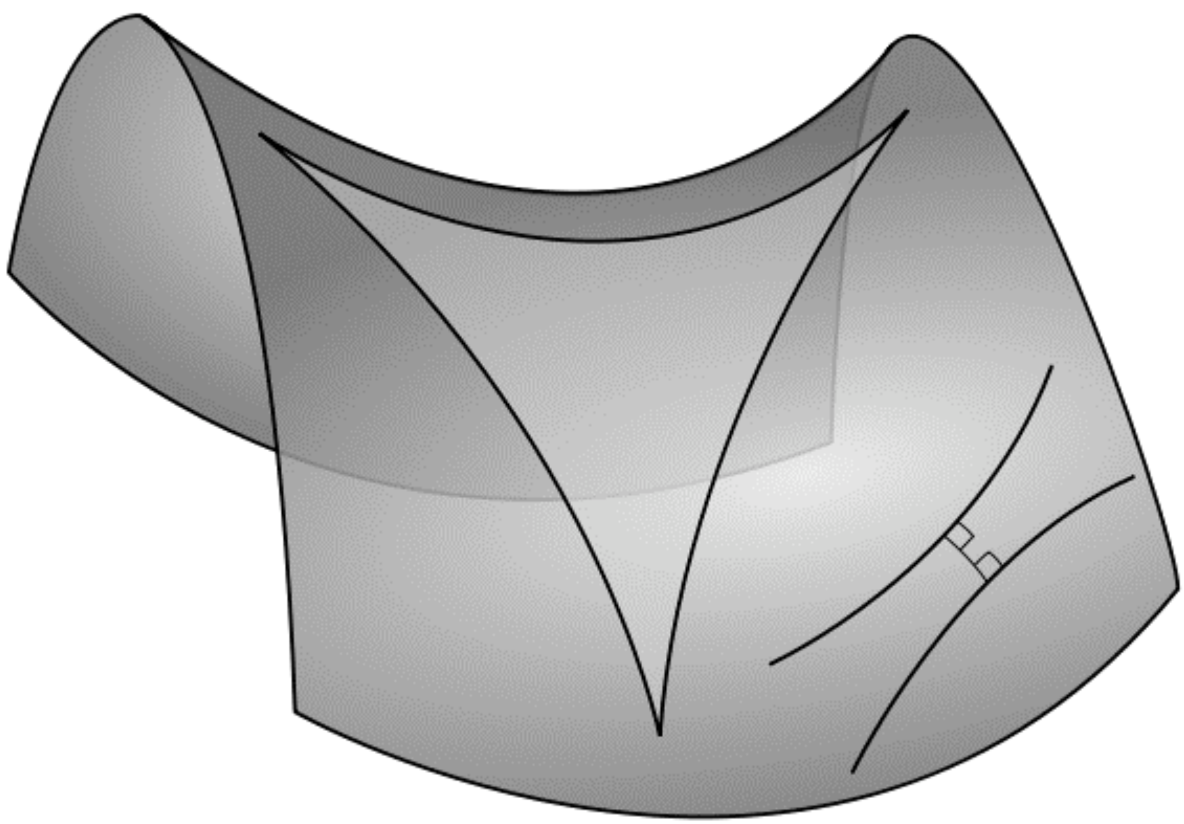
But the student comes to our rescue. "That is only appearance...That is only what I feel!" By now the student has totally rejected any connection between the pure objects of mathematics and the appearances that represent them. Pure mathematics has become truly pure - not only because it is an a priori science, but one which is completely divorced from the empirical.

<u>The Student's Eureka Moment, or Synthesis in Action</u>

But the student still feels the connection between the images and the pure objects, and so is left unsatisfied. The student asks himself, “Should the a priori laws of mathematics have absolutely no relation to the applied mathematics that takes place in every day life?” Why is it that those triangles I measure out in the world get so darn close to two right angles? Why is it that this measuring cup always fills that one, even though the person who made this one, has never seen that one,” and so on. The student wishes only that mathematics were not so rigorous. He would like to take the angle at the top of Saccheri quadrilateral, pick it, and place it on top of the right angle. He wished that this could in fact prove Lobechevsky to be absurd.

Faced with this quandary, the student finally realizes the meaning of synthesis. Many mathematicians, including both Euclid and Lobachevsky, have used this sort of procedure in the past. The student's act of picking up the angle and putting it on top of the other is no different from Euclid's synthetic act in Bk. 1, Prop. 4, where he lays one triangle on top of another. What the student felt initially was in fact valid. The Lobechevskian space cannot be represented in the Euclidean world (that is, it cannot be represented validly in flat space, or empirically on a flat board or a piece of paper). It was the student's a priori intuition, his *sense* of the very structure of space, that had generated the Fifth Postulate, and no concept whatsoever could contradict that.

It is only now clear that when speaking of straightness, the student was partially correct and partially incorrect in regards to the quote given initially by Lobachevsky. The straight line has retained its properties. For instance, it is still the (shortest) distance between two points, lies evenly upon itself, two straight lines cannot enclose an area, etc. But the straight line has changed. It is now a different *sensible* object because it lies in the Lobachevskian plane, and this plane does not *appear* flat.



Self-Evidence of The Euclidean Postulates

There is no doubt that the postulates of Euclid are more obvious than many of the other theorems. The ancient Greeks prided themselves on believing what came first was also most obvious. This is wonderful for conversing with another and for appealing to the widest audience.

Obviously Euclid has achieved immense success at this. But there is no necessary, apparent reason why what comes first in a logical system is most obvious or self-evident. Another way to phrase such a truth is to ask if there are any proofs in Euclid's Geometry that are just as obvious as any of the postulates.

The postulates are in fact less obvious than the other theorems because they are the assumptions which one does not ask to be proven. One takes such things for granted when approaching the science. What is even scarier, though, is that few people are *even conscious* that they are assuming them.

The Meaning of The Postulates

For years I labored to figure out what the Fourth postulate was all about, and came up empty handed. Why was it there? What is its raison d'etre? It seemed so obvious, in fact, that I always believed it belonged in the Common Notions. It sounds like a self-evident truth. A given angle is always equal to itself.

What are the meanings of the Fourth and Fifth Postulates of Euclid? In retrospect it becomes very clear. The Fifth lays down the flatness of space, while the Fourth claims that the curvature is constant[10]. If all mathematicians understood what the Fourth Postulate was stating, they would have never ventured to prove the Fifth. They would have been attacking the Fourth from the

[10] The constant curvature in the Euclidean case is zero, or no curvature at all. The Fourth Postulate only lays down that the curvature is a constant. The Fifth then clarifies this, stating space has no curvature at all.

very beginning. If one cannot prove the consistency of curvature in space, how is one to prove that space is everywhere flat?

But of course the reason that mathematicians endeavored to prove the Fifth and not the Fourth was that it just *looked* so easy. The Fifth Postulate is wordy and sounds like a typical theorem. Yet when one sees all of the logical variations on it, the postulate begins to show its true colors. A triangle's angles are always equal to two right angles, a circle must always lie on a plane, a triangle can always be constructed greater than any given magnitude; these three and several others are equivalent. When one understands all of these as equivalent, one realizes how central flatness is to our every day (geometrical and non-geometrical) conception of space. Such a thing no longer seems as though it should be very easy, or even possible, to prove.

The lesson to be learned from the many postulates as one (or equivalent), and from our student (and Saccheri) is that one must first synthesize the nature of space before one can partake in geometry. Once the space has been synthesized (something we probably all did as very young children), all of the equivalent postulates follow.

Kant says that he sees the postulates of mathematics as valid not because they are postulates (that they come first), but because they are exhibited intuitively. We should remember what else he has claimed was exhibited intuitively - the Euclidean proposition that a triangle has two right angles (and in fact the rest of Euclidean Geometry). It is quite clear that Kant has relegated the role of the postulate (as logically primary) to a rather insignificant one. Now the postulates are nothing more than the things that come first in an argument.

Kant understood that one must first synthetically determine the nature of space (i.e. its curvature) before one could speak of it geometrically. This explains why, for the longest time, no one would even dream of proving the contradiction of the fifth. No one had yet synthesized such a space in which the plane was curved. Historically, proofs of the Fifth Postulate ended in the reduction form. It was only after carrying out the comprehensive, but unsatisfying, reductio of Saccheri, that geometry could progress to its opposite. The resistance to non-Euclidean geometry in part came about because of Leibniz's belief (and one probably held throughout much of the history of mathematics) that geometry was in fact analytic. If geometry were analytic one would be able to reduce the principles of this science down to a table of all possible principles (much like the logical functions of the understanding).

Since the intuition is in fact synthetic, one can reject a postulate and immediately conceive of a world, albeit a strange one, for which the mathematics applies. This does not mean that any postulate can arbitrarily be laid down. Geometry must be confined to three dimensions. Kant goes further than this, though, and I believe he is somewhat over hasty in his support for continuity. One can conceive of the world of Zeno; But Kant is consistent in this regard: Truth for him is the agreement between thought and its object. One must have the ability to be sensible of a thing that can no longer be broken down. So far, we have never been sensible of the atom, in its original meaning[11], and our a priori intuition tells us that we never will be.

[11] That is, of a material object which can no longer be broken down

Possibility of Different Curvatures of Space

Gauss realizes that the example Kant has given (that the sum of the angles in a triangle = two right angles) is the fifth postulate in one of its many guises. The fact that one cannot prove the Fifth Postulate with out first assuming it (via the intuition) suggests to the student why Saccheri's proof was rejected. But this fact presents us with another question. Which geometry is the true one? Or closer to Gauss's words: "What is the constant (K) which determines the real curvature of space?"

Gauss knows that there is no a priori proof for the validity of one geometry over another. And because of this fact, he argues with Kant's claim that space is only formal. He seems to approach this problem in two different ways.

A Posteriori Determination of K and Objections

At the end of The Researches on Theory of Parallels, Lobechevsky calls out to measure the heavens. He reports that he has found inaccuracies in very large terrestrial triangles (up to one second of a degree off from 180 degrees). Hence, here we have the first way Gauss approaches the determination of K. Simply measure it empirically!

But how is this to be done? First, why should Gauss believe that as triangles become larger our values should become more accurate? One would assume this to be the case from common experience and it would be in accord with the theory. The current method would be to measure the stars. How are we to do so? Shall we use light? Are we assured that light does not bend? If light is not suitable, what material is? How are we to determine when we have the right material?

Lobechevsky's claim on the size of terrestrial triangles should also raise some eyebrows. He was only one sixtieth of a degree off from 180 degrees. Could this not be from human error? Even if the angular sum were further exaggerated, at what point shall we conclude it to no longer come about from the inaccuracies in our measuring devices and the error of our senses? How far can our sensation be applied?

The second major objection goes as follows: By measuring anything sensible we must predetermine what we consider straight to begin with. The problem is the same that Saccheri had in the proof of Euclidean geometry. One must first impose the system, and once done, the answer is predetermined. Effectively by trying to measure the curvature of space, one must first synthesize the plane (and hence what one considers straight for a comparison to curvature).

Space as Matter, not Form

The other objection that Gauss has to Kant's Formal Space is that of direction. In the Prolegomena to Any Future Metaphysics, Kant poses us with a "paradox." If one is to think of a hand and its mirror image, our understanding can think of no difference between the two. How are we to understand the differences? Only through the intuition. The intuition is what tells us that the image is in fact not the same hand, for the glove of the one would not fit on the other. Countless variations could be given on this example, but the basic principle is the same: one distinguishes direction based on the intuition and not through the understanding.

Gauss takes this example as an obvious proof that space is matter, since one must first fix a material object as 'up' or 'down', 'forward', or 'backward', to determine what would be 'right' or 'left'.

But isn't it quite a task to speak of space without speaking of matter! Gauss is correct; these relations only acquire meaning once one of the directions is fixed to a material object. But the very fact that we can fix 'up' in any direction we choose is a good indication that these directional concepts are formal, and not material, however dependent on each other they may be[12].

K as Neither Empirical Nor A Priori

Maybe Gauss is never expecting to actually measure K empirically. Here we have the second way Gauss approaches this constant. The constant is a transcendental object, a thing-in-itself, which we can never have access to. Gauss even speaks of the possibility of knowing the true curvature of space only in his next life.

What is a transcendental object? It is nothing more than an object outside of us, one that is "out there." A building, a triangle, God – these are all examples of transcendental objects[13]. But the

[12] : "Now that which, as representation, can be antecedent to any and every act of thinking anything, is intuition; and if it contains nothing but relations, it is the form of intuition." B68

[13] Since we are only presented with the noumenal things like buildings and triangles as representations, such distinctions no longer have any meaning when speaking of the noumenal. What do we mean when we say a building? A building is a thing constructed by the human, for human dwelling, or storage, etc. Notice that such definitions are always phenomenal. When we point and say, "This is a building," we are referencing only our perceptions, and likewise for the triangle.

representations of these things in our mind (via concepts, with or without corresponding intuitions) are *not* transcendental objects.

Unfortunately, as humans, we have access to only that which comes from our intuition. This fundamentally limits what we can speak of. We no longer have access to the real things in the world except as they are represented to us, sensibly.

This talk of phenomenal objects representing the actual, noumenal objects is very unsatisfying for mathematicians because it only guarantees that one *must think* of an object in a certain way, and not that these objects *actually exist* in this way. In a word, our a priori intuition are no longer apodictically certain. I believe that after the début of the First Edition of his *Critique*, Kant realized this problem with a priori intuitions as representations of transcendental objects. One can see it in the “middle course” he considers in The Transcendental Deduction (B167-168):

"A middle course may be proposed between the two above mentioned, namely, that the categories are neither self-thought first principles a priori of our knowledge nor derived from experience, but subjective dispositions of though, implanted in us from the first moment of our existence, and so ordered by our Creator that their employment is in complete harmony with the laws of nature in accordance with which experience proceeds..."

"There is this decisive objection against the suggested middle course, that the necessity of the categories, which belongs to their very conception, would then have to be sacrificed. The concept of cause, for instance...would be false if it rested only on an arbitrary subjective necessity, implanted in us, of connecting certain empirical representations according to the rule of causal relation. I would not then be able to say that the effect is connected with the cause in the object, that is to say, necessarily, but only that I am so constituted that I cannot think this representation otherwise than as thus connected. This is exactly what the skeptic most desires."

This is not a shift in philosophy for Kant, but a clarification. Notice what he has left out of the Second Edition:

“All our representations are, it is true, referred by the understanding to some object; and since appearances are nothing but representations, the understanding refers them to a *something*, as the object of sensible intuition. But this something, thus conceived, is only the transcendental object; and by that is meant a something=X, of which we know, and with the present constitution of our understanding can know, nothing whatsoever.” (A250)

In the Second Edition, Kant has replaced this with:

“At the very outset, however, we come upon an ambiguity which may occasion serious misapprehension. The understanding, when it entitles an object in a [certain] relation mere phenomenon, at the same time forms, *apart from that relation* [my emphasis], a representation of an *object in itself* and so comes to represent itself as also being able to form concepts of such objects.” (B306-307)

The shift or clarification here is that where before the empirical intuition was the representation of the transcendental object, now the representation is separated from the intuition. The representation exists alongside the empirical, but is generated by, and for the sake of, the understanding.

Where does this leave us with Gauss? Gauss has seen that K cannot be determined empirically nor given a priori. His only option left is to turn to the object in-itself. How can K be an object in itself? Since we can know absolutely nothing about an object except as given to us by intuition, all objects, when considered apart from our mode of intuiting them, become similar; the thing-in-itself must be thought in general. What does the phrase "[an object] thought in general" mean? Only that the object exists, outside and apart from us. Nothing more can be said about it apart from how *we* must see the object. The contradiction present (what Gauss calls "repugnant to his intellect") is that we know something more about this object than simply its bare existence. We know that it determines the curvature of space. In fact, we even become sensible of it every time we experience the validity of Euclidean or non-Euclidean geometry. The constant is in fact shown to be formal *because* it can be experienced in all of its manifest forms.

What Kant's development in phenomenology has given us is that the true objects "out there" are *our* creations, by virtue of the understanding. The true objects are not what enable the faculty of the intuition to begin with. The intuition, as synthetic, has the ability to synthesize space as it pleases. Space is not something that exists outside of us, but is in fact *created by us*. To ask which space is more primary is to ask which space one has synthesized at any given moment.