Numerical Relativity 2022-2023

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Homework 2
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1 Advection Equation [max 2 pages]

Given the advection equation in 1D $\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0$ build a numerical code to solve it on a grid with extent $x \in [0, 10]$ and with initial conditions given by

$$u(x, t = 0) = \exp[-(x - x_0)^2] \, , \tag{1}$$

with $x_0 = 5$. Solve the equation using the following schemes:

1. FTCS
2. Lax-Friedrichs
3. Leapfrog
4. Lax-Wendroff

Use Courant factor $c_f = 0.5$ and compare the results obtained with the different methods, paying attention to their stability and dissipation properties. Plot $u(x, t)$ at different times (including $t = 0$ and $t = 20$) and the evolution of the L2-norm of $u(x, t)$. Use at least $J = 101$ points in the x direction, so that the spacing Δx is at least $0.1 = 10/(J - 1)$, and terminate your simulation at $t = 20$. Use periodic boundary conditions. Modify the number of points and/or the Courant factor c_f to check how your results change.

2 Step Function [max 2 pages]

Solve the advection equation, but using as initial data a step function instead of a Gaussian profile: $u(x, t = 0) = 1$ for $x \in [4, 6]$ and $u(x, t = 0) = 0$ in the rest of the domain. Compare the results obtained when using the Lax-Friedrichs and the Lax-Wendroff schemes. Use $c_f = 0.5$, $J = 101$, and terminate the evolution at $t = 20$. Plot $u(x, t)$ at different times and the evolution of the L2-norm of $u(x, t)$. **Optional:** check what happens when changing the number of points and/or the Courant factor.

3 Burgers' Equation [max 2 pages]

Given the Burgers' equation in 1D $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$ build a numerical code to solve it on a grid with extent $x \in [0, 10]$ and with initial conditions given by

$$u(x, t = 0) = 10\, \mathrm{e}^{-(x-x_0)^2} \,, \tag{2}$$

with $x_0 = 5$. Compute the solution using both the flux-conservative and the non flux-conservative versions of the upwind scheme. Use Courant factor $c_f = 0.5$, a grid with at least $J = 101$ points with periodic boundary conditions, and terminate the evolution at $t = 0.5$. Compare the solutions computed with the two different methods by plotting $u(x, t)$ at different times (including $t = 0.5$). What happens when you increase the resolution?

Figures do not count toward the maximum number of page limit. Use an A4 page format and a font size of at least 11.

Note: in order to get admitted to the oral exam you are requested to submit the answers to all these questions as a single pdf document via email at least two weeks before the oral exam. Include the source codes used to solve the exercises at the end of the pdf document.