A Lower Bound for the Young Integral in the Ultrametric Setting

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Abstract

Path integrals of the form $\int Y\, dX$ play a fundamental role in stochastic and rough path analysis, where obtaining lower bounds for their magnitude is often difficult. In this note, we study the same problem over a non-Archimedean (ultrametric) field. Owing to the strong triangle inequality, additive cancellation is absent and each sum is dominated by its largest term. Exploiting this property, we prove that the Young integral admits an *exact equality* between its maximal oscillation and the supremum of the local product increments $(Y_u - Y_s)(X_t - X_u)$. As a consequence, the p-variation of the integral is bounded below by this supremum, providing a sharp analogue of classical upper bounds in the ultrametric setting.

Introduction

Path integrals of the form

$$\int_s^t Y_u dX_u$$

appears in many applications across science. In the past century, these integrals have been a central part of Stochastic Analysis where X and Y would normally be stochastic processes, or more generally, rough paths. While there are now established techniques to obtain upper bounds for such integrals, it is generally a difficult problem to obtain lower bounds for the same integral. For example, in the study of Norris' Lemma in rough path theory, see [GW], or in Theorem 5.6 of [CHLT15], the authors obtained a lower bound for

$$\sup_t \left| \int_0^t Y_u dX_u \right|$$

in terms of infinity norm of Y as well as quantities depending on X. Lower bounds for such integrals are also important in studying the inversion problem for path signature [LX]. In this paper, we show that it is possible to obtain an exact equality for a similar integral if X and Y takes value in an ultrametric field.

More precisely, let $(\mathbb{K}, |\cdot|)$ be a complete non-Archimedean (ultrametric) field. An *ultrametric* field is a field $(\mathbb{K}, |\cdot|)$ equipped with an absolute value satisfying, for all

$x, y \in \mathbb{K}$,

$$|x| = 0 \iff x = 0, \qquad |xy| = |x|\,|y|, \qquad |x + y| \leq \max(|x|, |y|).$$

The last condition, known as the *strong triangle inequality*, is what distinguishes ultrametric spaces from the familiar Archimedean case. The canonical example is the field of p-adic numbers $\mathbb{Q}_p$, obtained by completing $\mathbb{Q}$ with respect to the p-adic norm

$$|x|_p = p^{-\mathrm{ord}_p(x)},$$

where $\mathrm{ord}_p(x)$ denotes the highest power of p dividing x.

In the ultrametric setting, however, the strong triangle inequality gives

$$|x| \neq |y| \Longrightarrow |x + y| = \max(|x|, |y|),$$

[BGR]. This property implies that sums are often governed by a single "dominant" term. In this note, we show that for the Young integral in such a setting, the integral's oscillation is bounded below by the magnitude of its largest "local product" increment.

To define the Young integral, we say a map $X : [0, T] \to \mathbb{K}$ has finite p-variation if

$$\|X\|_{p,[0,T]} := \Big(\sup_P \sum_i |X_{t_i} - X_{t_{i-1}}|^p\Big)^{1/p} < \infty,$$

where the supremum runs over all partitions P. If X has finite p-variation and Y has finite q-variation with $\frac{1}{p} + \frac{1}{q} > 1$ and X and Y are both continuous, the *Young integral*

$$\int Y\,dX = \lim_{|P|\to 0} \sum_{i=0}^{r-1} Y_{t_i}(X_{t_{i+1}} - X_{t_i})$$

is well-defined and satisfies a natural continuity estimate. [LCL] Since Ultrametric is a metric, the same estimate holds under ultrametric.

Main result

Theorem 1 (Dominant increment equality)**.** *Let $X, Y : [0, T] \to \mathbb{K}$ be continuous paths having finite p, q variation with $\frac{1}{p} + \frac{1}{q} > 1$, and define*

$$F(s, u, t) := (Y_u - Y_s)(X_t - X_u), \qquad M := \sup_{0 \leq s < u < t \leq T} |F(s, u, t)|.$$

We assume throughout that $M > 0$. Then the Young integral satisfies

$$\sup_{0 \leq s < t \leq T} \left| \int_s^t (Y - Y_s)\,dX \right| = M.$$

Proof. We first establish the inequality "$\leq M$" and then show equality is attained.

(1) Partition removal identity. For a partition $P = (p_0, \dots, p_n)$ of $[s,t]$, define

$$S(P; Y, dX) := \sum_{i=1}^{n} Y_{p_{i-1}}(X_{p_i} - X_{p_{i-1}}).$$

Removing an interior point p_i changes only two terms, and

$$S(P; Y, dX) - S(P \setminus \{p_i\}; Y, dX) = (Y_{p_i} - Y_{p_{i-1}})(X_{p_{i+1}} - X_{p_i}) = F(p_{i-1}, p_i, p_{i+1}).$$

Thus removing all interior points iteratively yields

$$S(P; Y, dX) - S(\{s,t\}; Y, dX) = \sum_{j=1}^{n-1} F(s_j, u_j, t_j),$$

where (s_j, u_j, t_j) are the triples corresponding to each removal step.

(2) Ultrametric bound. The ultrametric inequality gives

$$|S(P; Y, dX) - S(\{s,t\}; Y, dX)| \leq \max_j |F(s_j, u_j, t_j)| \leq M.$$

Passing to the limit along partitions with mesh $\to 0$,

$$\Big| \int_s^t Y\, dX - Y_s(X_t - X_s) \Big| = \Big| \int_s^t (Y - Y_s)\, dX \Big| \leq M.$$

Hence $\sup_{s<t} |\int_s^t (Y - Y_s)\, dX| \leq M$.

(3) Existence of an isolated maximizer. Because $|F|$ is continuous on the compact simplex

$$\Delta := \{(s,u,t) \,:\, 0 \leq s < u < t \leq T\},$$

it attains its maximum M. Define the maximizer set

$$\Phi_{[0,T]} := \{(s,u,t) \in \Delta : |F(s,u,t)| = M\}.$$

This set is nonempty and compact. Consider the continuous map $f : \Phi_{[0,T]} \to \mathbb{R}$ given by $f(s,u,t) = t - s$. Since $\Phi_{[0,T]}$ is compact, f attains a minimum at some triple (s^*, u^*, t^*). By minimality, if $[s', t'] \subsetneq [s^*, t^*]$ with $s' < u^* < t'$, then $|F(s', u^*, t')| < M$.

(4) Dominant last removal. For any partition P of $[s^*, t^*]$ containing $\{s^*, u^*, t^*\}$, choose an order of removing points so that u^* is last. Every earlier removal yields $|F| < M$, while the last removal gives $|F(s^*, u^*, t^*)| = M$. By the ultrametric inequality, a single strictly dominant term determines the sum's magnitude:

$$\Big| S(P; Y, dX) - S(\{s^*, t^*\}; Y, dX) \Big| = M.$$

Letting the mesh $\to 0$,

$$\Big|\int_{s^*}^{t^*}(Y - Y_{s^*})\,dX\Big| = M.$$

(5) Conclusion. Thus the supremum over all intervals equals M, completing the proof. $\square$

Corollary 1 (Lower bound for the Young integral)**.** *Define* $I(t) := \int_0^t (Y_r - Y_0)\,dX_r$. *Then, assuming as before,*

$$\|I\|_{p,[0,T]} \geq M.$$

Proof. From the main theorem there exist $s^* < t^*$ such that

$$\Big|\int_{s^*}^{t^*}(Y - Y_{s^*})\,dX\Big| = M.$$

Then we have,

$$\int_{s^*}^{t^*}(Y - Y_0)\,dX + \int_{s^*}^{t^*}(Y_0 - Y_{s^*})\,dX = \int_{s^*}^{t^*}(Y - Y_0)\,dX - F(0, s^*, t^*),$$

so that

$$M = \Big|\int_{s^*}^{t^*}(Y - Y_0)\,dX - F(0, s^*, t^*)\Big| \leq \max\Big(\Big|\int_{s^*}^{t^*}(Y - Y_0)\,dX\Big|, |F(0, s^*, t^*)|\Big).$$

Case 1: If $|F(0, s^*, t^*)| < M$, then necessarily $\big|\int_{s^*}^{t^*}(Y - Y_0)\,dX\big| \geq M$.
Case 2: If $|F(0, s^*, t^*)| = M$, let

$$\Psi := \{(s,t) \in [0,T]^2 : 0 \leq s < t \leq T,\ |F(0,s,t)| = M\}.$$

By continuity and compactness, Ψ is nonempty and compact. Let $g(s,t) = t$; then g attains a minimum at some $(s_0, t_0) \in \Psi$. The same dominance argument used in the main theorem, now applied on $[0, t_0]$ with the last removal at s_0, yields

$$\Big|\int_0^{t_0}(Y - Y_0)\,dX\Big| = M.$$

Thus in either case,

$$\sup_{0 \leq a < b \leq T}\Big|\int_a^b (Y - Y_0)\,dX\Big| \geq M.$$

Since $\|I\|_{p,[0,T]} \geq \sup_{a<b}\big|\int_a^b (Y - Y_0)\,dX\big|$, the desired bound follows. $\square$

References

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