

# Chapter 6 Polynomial Functions

## 6.0 Pre-assessment

Match each of the vocabulary terms on the left with the appropriate letter and definition on the right.

1. coefficient

2. like terms

3. root of an equation

4.  $x$ -intercept

5. maximum of a function
- A. the  $y$ -value of the highest point of the graph of the function

B. the horizontal number line that divides the coordinate plane

C. the numerical factor in a term

D. a value of the variable that makes the equation true

E. terms that contain the same variables raised to the same powers

F. the  $x$ -coordinate of a point where the graph intersects the  $x$ -axis.

Evaluate each expression.

6.  $6^4$

7.  $-5^4$

8.  $(-1)^5$

9.  $\left(-\frac{2}{3}\right)^2$

Evaluate each expression for the given value of the variable.

10.  $x^4 - 5x^2 - 6x - 8$  for  $x = 3$

12.  $2x^3 - x^2 - 8x + 4$  for  $x = \frac{1}{2}$
11.  $2x^3 - 3x^2 - 29x - 30$  for  $x = -2$

13.  $3x^4 + 5x^3 + 6x^2 + 4x - 1$  for  $x = -1$

Multiply or divide.

14.  $2x^3y \cdot 4x^2$

15.  $-a^2b \cdot ab^4$

16.  $\frac{-7t^4}{3t^2}$

17.  $\frac{3p^3q^2r}{12pt^4}$

1. C 2. E 3. D 4. F 5. A 6. 1296 7. -1 9. 4/9  
10. 10 11. 0 12. 0 13. -1 14.  $8x_5^8y$  15.  $-a_3b_5^8$  16.  $-\frac{3}{2}t^2$  17.  $\frac{p^2q^2r}{4t^2}$

## 6.1 Polynomials

**Objective:** Identify and classify polynomials

**Definition 6.1.1.** A **monomial** is a number or a product of numbers and variables with whole number exponents. A **polynomial** is a monomial or a sum or difference of monomials. The **degree of a monomial** is the sum of the exponents of the variables.

<b>Polynomials:</b>	$3x^4$	$2z^{12} + 9z^3$	$\frac{1}{2}a^7$	$0.15x^{101}$	$3t^2 - t^3$
<b>Not Polynomials:</b>	$3^x$	$ 2b^3 - 6b $	$\frac{8}{5y^2}$	$\frac{1}{2}\sqrt{x}$	$m^{0.75} - m$

**Example 1.** Identify the degree of each monomial.

- (a)  $x^4$  (c)  $4a^2b$   
 (b) 12 (d)  $x^3y^4z$

**Definition 6.1.2.** The **degree of a polynomial** is given by the term with the greatest degree. A polynomial is in standard when its terms are written in descending order of degree. The **leading coefficient** the coefficient of the first term in standard form.

$$5x^3 + 8x^2 + 3x - 17$$

**Definition 6.1.3.** A polynomial with two terms is called a **binomial**, and a polynomial with three terms is called a **trinomial**.

Classifying Polynomials by Degree		
Name	Degree	Example
Constant	0	$-9$
Linear	1	$x - 4$
Quadratic	2	$x^2 + 3x - 1$
Cubic	3	$x^3 + 2x^2 + x + 1$
Quartic	4	$2x^4 + x^3 + 3x^2 + 4x - 1$
Quintic	5	$7x^5 + x^4 - x^3 + 3x^2 + 2x - 1$

**Example 2.** Rewrite each polynomial in standard form. Then identify the leading coefficient, degree, and number of terms. Name the polynomial.

- |                      |                             |
|----------------------|-----------------------------|
| (a.) $2x + 4x^3 - 1$ | (b.) $7x^3 - 11x + x^5 - 2$ |
| Standard Form:       | Standard Form:              |
| Leading Coefficient: | Leading Coefficient:        |
| Degree:              | Degree:                     |
| Terms:               | Terms:                      |
| Name:                | Name:                       |

**Example 3.** Add or subtract. Write your answer in standard form.

- (a.)  $(3x^2 + 7 + x) + (14x^3 + 2 + x^2 - x)$  (b.)  $(1 - x^2) - (3x^2 + 2x - 5)$

## Polynomials (day 2)

**Objective:** Evaluate and Graph Polynomials

**You Try It! 1.** Add or subtract. Write your answer in standard form.

(a)  $(-36x^2 + 6x - 11) + (6x^2 + 16x^3 - 5)$

(b)  $(5x^3 + 12 + 6x^2) + (15x^2 + 3x - 2)$

**Example 4.** Cardiac output is the amount of blood pumped through the heart. The output is measured by a technique called dye dilution. A doctor injects dye into a vein near the heart and measures the amount of dye in the arteries over time.

The cardiac output of a particular patient can be approximated by the function

$$f(t) = 0.0056t^3 - 0.22t^2 + 2.33t,$$

where  $f(t)$  represents the concentration of dye (in milligrams per liter).

(a) Evaluate  $f(t)$  for  $t = 0$  and  $t = 3$ .

(b) Describe what the values of the function in part (a) represent.

**Example 5.** Graph each polynomial on a graphing calculator. Describe the graph, and identify the number of real zeros.

(a)  $f(x) = x^3 - x$

(b)  $f(x) = -3x^3 + 2x + 1$

(c)  $h(x) = x^4 - 8x^2 + 1$

(d)  $k(x) = x^4 + x^3 - x^2 + 2x - 3$

## 6.2 Multiplying Polynomials

**Objective:** To Multiply Polynomials and Binomial Expansion

**Example 1.** Find each product.

(a)  $3x^2(x^3 + 4)$

(b)  $ab(a^3 + 3ab^2 - b^3)$

**Example 2.** Find each product.

(a)  $(x - 2)(1 + 3x - x^2)$

(b)  $(x^2 + 3x - 5)(x^2 - x + 1)$

### Binomial Expansion

**Example 3.** Find the product.

$(x + y)^3$

Binomial Expansion	Pascal's Triangle (Coefficients)
$(a + b)^0 =$ <span style="color: red;">1</span>	<span style="color: red;">1</span>
$(a + b)^1 =$ <span style="color: red;">1</span> $a +$ <span style="color: red;">1</span> $b$	<span style="color: red;">1</span> <span style="color: red;">1</span>
$(a + b)^2 =$ <span style="color: red;">1</span> $a^2 +$ <span style="color: blue;">2</span> $ab +$ <span style="color: red;">1</span> $b^2$	<span style="color: red;">1</span> <span style="color: blue;">2</span> <span style="color: red;">1</span>
$(a + b)^3 =$ <span style="color: red;">1</span> $a^3 +$ <span style="color: red;">3</span> $a^2b +$ <span style="color: blue;">3</span> $ab^2 +$ <span style="color: red;">1</span> $b^3$	<span style="color: red;">1</span> <span style="color: red;">3</span> <span style="color: blue;">3</span> <span style="color: red;">1</span>
$(a + b)^4 =$ <span style="color: red;">1</span> $a^4 +$ <span style="color: blue;">4</span> $a^3b +$ <span style="color: blue;">6</span> $a^2b^2 +$ <span style="color: blue;">4</span> $ab^3 +$ <span style="color: red;">1</span> $b^4$	<span style="color: red;">1</span> <span style="color: blue;">4</span> <span style="color: blue;">6</span> <span style="color: blue;">4</span> <span style="color: red;">1</span>
$(a + b)^5 =$ <span style="color: red;">1</span> $a^5 +$ <span style="color: blue;">5</span> $a^4b +$ <span style="color: blue;">10</span> $a^3b^2 +$ <span style="color: blue;">10</span> $a^2b^3 +$ <span style="color: blue;">5</span> $ab^4 +$ <span style="color: red;">1</span> $b^5$	<span style="color: red;">1</span> <span style="color: blue;">5</span> <span style="color: blue;">10</span> <span style="color: blue;">10</span> <span style="color: blue;">5</span> <span style="color: red;">1</span>

**Example 4.** Expand each expression using Pascal's triangle.

(a)  $(y - 3)^4$

(b)  $(4z + 5)^3$

## 6.3 Dividing Polynomials

**Objective:** Use long and synthetic division to divide polynomials.

**Example 1.** Divide using arithmetic long division.

(a)  $12 \overline{)277}$

**You Try It! 2.** Divide.

(b)  $8 \overline{)347}$

**Example 2.** Divide using long division.

(a)  $(4x^2 + 3x^3 + 10) \div (x - 2)$

(b)  $(15x^2 + 8x - 12) \div (3x + 1)$

**Example 3.** Divide using synthetic division.

(a)  $(4x^2 - 12x + 9) \div \left(x + \frac{1}{2}\right)$

(b)  $(6x^2 - 5x - 6) \div (x + 3)$

**Example 4.** Use synthetic substitution to evaluate the polynomial for the given value.

(a)  $P(x) = x^3 - 4x^2 + 3x - 5$  for  $x = 4$

(b)  $P(x) = 4x^4 + 2x^3 + 3x + 5$  for  $x = -\frac{1}{2}$

6.3 (day 2)

**Objective:** Use long and synthetic division to divide polynomials.

**You Try It! 3.** Divide using long division.

(a)  $(2x^2 + 7x + 7) \div (x + 2)$

(b)  $(x^2 + 5x - 28) \div (x - 3)$

**Example 5.** Divide using synthetic division.

(a)  $(x^2 - 3x - 18) \div (x - 6)$

(b)  $(x^4 - 7x^3 + 9x^2 - 22x + 25) \div (x + 3)$

Remainder Theorem	
Theorem	Example
If the polynomial function $P(x)$ is divided by $x - \mathbf{a}$ ,  then the remainder $r$ is $P(\mathbf{a})$ .	Divide $x^3 - 4x^2 + 5x + 1$ by $x - 3$ <div><div><div>3</div><div>1</div><div>-4</div><div>5</div><div>1</div></div><div><div>↓</div><div>3</div><div>-3</div><div>6</div></div><div><div>1</div><div>-1</div><div>2</div><div>7</div></div></div> <div><math>P(\mathbf{3}) = \mathbf{7}</math></div>

**Example 6.** Use synthetic substitution to evaluate the polynomial for the given value.

(a)  $P(x) = x^3 + 3x^2 + 4$  for  $x = -3$

(b)  $P(x) = 5x^2 + 9x + 3$  for  $x = \frac{1}{5}$

**6.2 & 6.3 Review****Objective:** Multiply and Divide Polynomials

Find each product.

1.  $3x^2(2x^2 + 9x - 6)$

2.  $(2x + 5y)(3x^2 - 4xy + 2y^2)$

Expand each expression. (Use Pascal's triangle)

3.  $(x - 3y)^3$

4.  $(x - 2)^5$

Divide.

5.  $7 \overline{)647}$

6.  $9 \overline{)3452}$

Use long division to divide the polynomials. Write as Quotient + Remainder/Divisor.

7.  $(2x^2 + 3x - 20) \div (x - 2)$

8.  $(x^4 + 6x^3 + 6x^2) \div (x + 5)$



Use synthetic division to divide the polynomials. Write as Quotient + Remainder/Divisor.

9.  $x^4 - 3x^3 - 7x - 14) \div (x - 4)$

10.  $(x^2 + 9x + 6) \div (x + 8)$

Use synthetic substitution (The Remainder Theorem) to evaluate the polynomial for the given value.

11.  $P(x) = 4x^3 - 5x^2 - x + 2$  for  $x = -1$

12.  $P(x) = 25x^2 - 16$  for  $x = \frac{4}{5}$

13.  $P(x) = 4x^3 - 5x^2 - x + 2$  for  $x = -1$

14.  $P(x) = 25x^2 - 16$  for  $x = \frac{4}{5}$

## 6.4 Factoring Polynomials

**Objective:** Use the Factor Theorem to determine factors of a polynomial.

Factor Theorem	
Theorem	Example
For any polynomial $P(x)$ , $(x - a)$ is a factor of $P(x)$ if and only if $P(a) = 0$ .	Because $P(1) = 1^2 - 1 = 0$ , $(x - 1)$ is a factor of $P(x) = x^2 - 1$ .

**Example 1.** Determine whether the given binomial is a factor of the polynomial  $P(x)$ .

(a)  $(x - 3)$ ;  $P(x) = x^2 + 2x - 3$

(b)  $(x + 4)$ ;  $P(x) = 2x^4 + 8x^3 + 2x + 8$

**Example 2.** Factor by grouping.

(a)  $x^3 + 3x^2 - 4x - 12$

(b)  $x^3 - 2x^2 - 9x + 18$

**You Try It! 4.** Factor by grouping

(a)  $2x^3 + x^2 + 8x + 4$

(b)  $8y^3 - 4y^2 - 50y + 25$

6.4 (day 2) Factoring

Objective: Factor the sum and difference of two cubes.

Factoring The Sum and Difference of Two Cubes	
Method	Algebra
Sum of two cubes	$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$
Difference of two cubes	$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

S . O . A . P

same  
opposite  
always  
positive

$(a \pm b)^3 = (a \pm b)(a^2 \mp ab + b^2)$

Example 3. Factor each expression using sum or difference of cubes.

- (a)  $5x^4 + 40x$
- (b)  $8y^3 - 27$

You Try It! 5. Factor each expression using sum or difference of cubes.

- (a)  $8 + z^6$
- (b)  $2x^5 - 16x^2$

## 6.4 Review of Factoring

**Objective:** Factor using the sum and difference of two cubes, difference of square, grouping, and GCF.

Factor using the greatest common factor (GCF).

1.  $2x^5 - 6x^3$

3.  $14x^3 - 49x^2 - 28x$

2.  $5x^3 - 10x$

4.  $27x^5 - 18x^4 + 9x^3$

Factor using difference of squares.

5.  $q^2 - r^2$

8.  $x^4 - y^4$

6.  $25a^2 - 64b^2$

9.  $a^6 - b^6$

7.  $81x^2 - 100y^2$

10.  $4x^4 - 9y^6$

Factor using sum and difference of cubes.

5.  $x^3 - y^3$

8.  $64x^3 + 125y^3$

6.  $r^3 + s^3$

9.  $a^6 - b^6$

7.  $8a^3 - 27b^3$

10.  $x^6 + y^6$

Factor using grouping.

5.  $6x^3 + 2x^2 + 9x + 3$

7.  $4x^3 + 8x^2 - 9x - 18$

6.  $7x^3 - 35x^2 + 8x - 40$

8.  $16x^3 - 64x^2 - 25x + 100$

## 6.5 Finding Real Roots of Polynomial Equations

**Objective:** Identify the multiplicity of roots, Use the Rational Root Theorem to solve polynomial equations.

**Example 1.** Solve each polynomial equation by factoring. Check your answer using **Desmos**.

(a)  $3x^5 + 18x^4 + 27x^3 = 0$

(b)  $x^4 - 13x^2 = -36$

**You Try It! 6.** Solve each polynomial equation by factoring.

(a)  $2x^6 - 10x^5 - 12x^4 = 0$

(b)  $x^3 - 2x^2 - 25x = -50$

**Definition 6.5.1.** The **multiplicity** of root  $r$  is the number of times that  $x - r$  is a factor of  $P(x)$ . Even multiplicity means the graph “touches” the  $x$ -axis at the root but does not cross. Odd multiplicity means the graph crosses the  $x$ -axis at the root.

**Example 2.** Identify the roots of each equation. State the multiplicity of each root.

(a)  $x^3 - 9x^2 + 27x - 27 = 0$

(b)  $-2x^3 - 12x^2 + 30x + 200 = 0$

## 6.5 Finding Real Roots of Polynomial Equations (day 2)

**Theorem 6.5.1.** (Rational Root Theorem) If the polynomial  $P(x)$  has integer coefficients, then every rational root of the polynomial equation  $P(x) = 0$  can be written in the form  $\frac{p}{q}$ , where  $p$  is a factor of the constant term of  $P(x)$  and  $q$  is a factor of the leading coefficient of  $P(x)$ .

**Example 3.** Owen, a popcorn producer, is designing a new box for popcorn distribution. The marketing department has required a box with the width 2 inches less than the length and with the height 5 inches greater than the length. The volume of each box must be 24 cubic inches. What is the length of the box?

**Example 4.** Identify all of the real roots of:

(a)  $4x^4 - 21x^3 + 18x^2 + 19x - 6 = 0$

(b)  $2x^3 - 3x^2 - 10x - 4 = 0$

## 6.6 Fundamental Theorem of Algebra

**Objective:** Use the Fundamental Theorem of Algebra and corollary to write a polynomial equation given roots.

<b>The following statements are equivalent:</b>
A real number $r$ is a root of the polynomial equation $P(x) = 0$ .
$P(r) = 0$
$r$ is an $x$ -intercept of the graph of $P(x)$ .
$x - r$ is a factor of $P(x)$ .
When you divide the rule for $P(x)$ by $x - r$ , the remainder is 0.
$r$ is a zero of $P(x)$

**Example 1.** Write the simplest polynomial function with the given zeros.

(a)  $-3, \frac{1}{2}$ , and 1

(b)  $-2, 2$ , and 4

**Theorem 6.6.1.** (The Fundamental Theorem of Algebra) Every polynomial function of degree  $n \geq 1$  has at least one zero, where a zero may be a complex number.

**Theorem 6.6.2.** (FTA Corollary) Every polynomial function of degree  $n \geq 1$  has exactly  $n$  zeros, including multiplicities.

**Example 2.** Solve each polynomial by finding all roots.

(a)  $x^4 + x^3 + 2x^2 + 4x - 8 = 0$

(b)  $x^4 + 4x^3 - x^2 + 16x - 20 = 0$



## 6.6 Fundamental Theorem of Algebra (day 2)

**Objective:** Use the Fundamental Theorem of Algebra and corollary to write a polynomial equation given roots.

**Theorem 6.6.3.** If  $\mathbf{a} + \mathbf{bi}$  is a root of a polynomial equation with real-number coefficients, then  $\mathbf{a} - \mathbf{bi}$  is also a root.

**Example 3.** Write the simplest polynomial function with the given zeros.

(a)  $1 + i$ ,  $\sqrt{2}$ , and 3

(b)  $2i$ ,  $1 + \sqrt{2}$  and 3

**Example 4.** An engineering class is designed model rockets for competition. The body of the rocket must be cylindrical with a cone-shaped top. The cylindrical portion must be 60 cm tall, and the height of the cone must be twice the radius. The volume of the payload region must be  $558\pi \text{ cm}^3$  in order to hold the cargo. Find the radius of the rocket.

**Example 5.** A grain silo is in the shape of a cylinder with a hemisphere on top. The cylinder is 20 ft tall. The volume of the silo is  $2106\pi$  cubic feet. Find the radius of the silo.

6.7

Objective:

Definition 6.7.1. Definition

Example 1. Example

6.8

Objective:

Definition 6.8.1. Definition

Example 1. Example

6.9

Objective:

Definition 6.9.1. Definition

Example 1. Example