

Chapter 6 Polynomial Functions

6.0 Pre-assessment

Match each of the vocabulary terms on the left with the appropriate letter and definition on the right.

1. coefficient

2. like terms

3. root of an equation

4. x -intercept

5. maximum of a function
- A. the y -value of the highest point of the graph of the function

B. the horizontal number line that divides the coordinate plane

C. the numerical factor in a term

D. a value of the variable that makes the equation true

E. terms that contain the same variables raised to the same powers

F. the x -coordinate of a point where the graph intersects the x -axis.

Evaluate each expression.

6. 6^4

7. -5^4

8. $(-1)^5$

9. $\left(-\frac{2}{3}\right)^2$

Evaluate each expression for the given value of the variable.

10. $x^4 - 5x^2 - 6x - 8$ for $x = 3$

12. $2x^3 - x^2 - 8x + 4$ for $x = \frac{1}{2}$
11. $2x^3 - 3x^2 - 29x - 30$ for $x = -2$

13. $3x^4 + 5x^3 + 6x^2 + 4x - 1$ for $x = -1$

Multiply or divide.

14. $2x^3y \cdot 4x^2$

15. $-a^2b \cdot ab^4$

16. $\frac{-7t^4}{3t^2}$

17. $\frac{3p^3q^2r}{12pt^4}$

1. C 2. E 3. D 4. F 5. A 6. 1296 7. -1 9. 4/9
10. 10 11. 0 12. 0 13. -1 14. $8x^5y$ 15. $-a^3b^5$ 16. $-\frac{3}{2}t^2$ 17. $\frac{p^2q^2r}{4t^2}$

6.1 Polynomials

Objective: Identify and classify polynomials

Definition 6.1.1. A **monomial** is a number or a product of numbers and variables with whole number exponents. A **polynomial** is a monomial or a sum or difference of monomials. The **degree of a monomial** is the sum of the exponents of the variables.

Polynomials:	$3x^4$	$2z^{12} + 9z^3$	$\frac{1}{2}a^7$	$0.15x^{101}$	$3t^2 - t^3$
Not Polynomials:	3^x	$ 2b^3 - 6b $	$\frac{8}{5y^2}$	$\frac{1}{2}\sqrt{x}$	$m^{0.75} - m$

Example 1. Identify the degree of each monomial.

- (a) x^4 (c) $4a^2b$
 (b) 12 (d) x^3y^4z

Definition 6.1.2. The **degree of a polynomial** is given by the term with the greatest degree. A polynomial is in standard when its terms are written in descending order of degree. The **leading coefficient** the coefficient of the first term in standard form.

$$5x^3 + 8x^2 + 3x - 17$$

Definition 6.1.3. A polynomial with two terms is called a **binomial**, and a polynomial with three terms is called a **trinomial**.

Classifying Polynomials by Degree		
Name	Degree	Example
Constant	0	-9
Linear	1	$x - 4$
Quadratic	2	$x^2 + 3x - 1$
Cubic	3	$x^3 + 2x^2 + x + 1$
Quartic	4	$2x^4 + x^3 + 3x^2 + 4x - 1$
Quintic	5	$7x^5 + x^4 - x^3 + 3x^2 + 2x - 1$

Example 2. Rewrite each polynomial in standard form. Then identify the leading coefficient, degree, and number of terms. Name the polynomial.

- | | |
|--|---|
| <p>(a.) $2x + 4x^3 - 1$</p> <p>Standard Form:</p> <p>Leading Coefficient:</p> <p>Degree:</p> <p>Terms:</p> <p>Name:</p> | <p>(b.) $7x^3 - 11x + x^5 - 2$</p> <p>Standard Form:</p> <p>Leading Coefficient:</p> <p>Degree:</p> <p>Terms:</p> <p>Name:</p> |
|--|---|

Example 3. Add or subtract. Write your answer in standard form.

- (a.) $(3x^2 + 7 + x) + (14x^3 + 2 + x^2 - x)$ (b.) $(1 - x^2) - (3x^2 + 2x - 5)$

Polynomials (day 2)

Objective: Evaluate and Graph Polynomials

You Try It! 1. Add or subtract. Write your answer in standard form.

(a) $(-36x^2 + 6x - 11) + (6x^2 + 16x^3 - 5)$

(b) $(5x^3 + 12 + 6x^2) + (15x^2 + 3x - 2)$

Example 4. Cardiac output is the amount of blood pumped through the heart. The output is measured by a technique called dye dilution. A doctor injects dye into a vein near the heart and measures the amount of dye in the arteries over time.

The cardiac output of a particular patient can be approximated by the function

$$f(t) = 0.0056t^3 - 0.22t^2 + 2.33t,$$

where $f(t)$ represents the concentration of dye (in milligrams per liter).

(a) Evaluate $f(t)$ for $t = 0$ and $t = 3$.

(b) Describe what the values of the function in part (a) represent.

Example 5. Graph each polynomial on a graphing calculator. Describe the graph, and identify the number of real zeros.

(a) $f(x) = x^3 - x$

(b) $f(x) = -3x^3 + 2x + 1$

(c) $h(x) = x^4 - 8x^2 + 1$

(d) $k(x) = x^4 + x^3 - x^2 + 2x - 3$

6.2 Multiplying Polynomials

Objective: To Multiply Polynomials and Binomial Expansion

Example 1. Find each product.

(a) $3x^2(x^3 + 4)$

(b) $ab(a^3 + 3ab^2 - b^3)$

Example 2. Find each product.

(a) $(x - 2)(1 + 3x - x^2)$

(b) $(x^2 + 3x - 5)(x^2 - x + 1)$

Binomial Expansion

Example 3. Find the product.

$(x + y)^3$

Binomial Expansion	Pascal's Triangle (Coefficients)
$(a + b)^0 =$ 1	1
$(a + b)^1 =$ 1 $a +$ 1 b	1 1
$(a + b)^2 =$ 1 $a^2 +$ 2 $ab +$ 1 b^2	1 2 1
$(a + b)^3 =$ 1 $a^3 +$ 3 $a^2b +$ 3 $ab^2 +$ 1 b^3	1 3 3 1
$(a + b)^4 =$ 1 $a^4 +$ 4 $a^3b +$ 6 $a^2b^2 +$ 4 $ab^3 +$ 1 b^4	1 4 6 4 1
$(a + b)^5 =$ 1 $a^5 +$ 5 $a^4b +$ 10 $a^3b^2 +$ 10 $a^2b^3 +$ 5 $ab^4 +$ 1 b^5	1 5 10 10 5 1

Example 4. Expand each expression using Pascal's triangle.

(a) $(y - 3)^4$

(b) $(4z + 5)^3$

6.3 Dividing Polynomials

Objective: Use long and synthetic division to divide polynomials.

Example 1. Divide using arithmetic long division.

(a) $12 \overline{)277}$

You Try It! 2. Divide.

(b) $8 \overline{)347}$

Example 2. Divide using long division.

(a) $(4x^2 + 3x^3 + 10) \div (x - 2)$

(b) $(15x^2 + 8x - 12) \div (3x + 1)$

Example 3. Divide using synthetic division.

(a) $(4x^2 - 12x + 9) \div \left(x + \frac{1}{2}\right)$

(b) $(6x^2 - 5x - 6) \div (x + 3)$

Example 4. Use synthetic substitution to evaluate the polynomial for the given value.

(a) $P(x) = x^3 - 4x^2 + 3x - 5$ for $x = 4$

(b) $P(x) = 4x^4 + 2x^3 + 3x + 5$ for $x = -\frac{1}{2}$

6.3 (day 2)

Objective: Use long and synthetic division to divide polynomials.

You Try It! 3. Divide using long division.

(a) $(2x^2 + 7x + 7) \div (x + 2)$

(b) $(x^2 + 5x - 28) \div (x - 3)$

Example 5. Divide using synthetic division.

(a) $(x^2 - 3x - 18) \div (x - 6)$

(b) $(x^4 - 7x^3 + 9x^2 - 22x + 25) \div (x + 3)$

Remainder Theorem	
Theorem	Example
If the polynomial function $P(x)$ is divided by $x - \mathbf{a}$, then the remainder r is $P(\mathbf{a})$.	Divide $x^3 - 4x^2 + 5x + 1$ by $x - 3$ <div><div><div>3</div><div>1</div><div>-4</div><div>5</div><div>1</div></div><div><div>↓</div><div>3</div><div>-3</div><div>6</div></div><div><div>1</div><div>-1</div><div>2</div><div>7</div></div></div> <div>$P(\mathbf{3}) = \mathbf{7}$</div>

Example 6. Use synthetic substitution to evaluate the polynomial for the given value.

(a) $P(x) = x^3 + 3x^2 + 4$ for $x = -3$

(b) $P(x) = 5x^2 + 9x + 3$ for $x = \frac{1}{5}$

6.2 & 6.3 Review**Objective:** Multiply and Divide Polynomials

Find each product.

1. $3x^2(2x^2 + 9x - 6)$

2. $(2x + 5y)(3x^2 - 4xy + 2y^2)$

Expand each expression. (Use Pascal's triangle)

3. $(x - 3y)^3$

4. $(x - 2)^5$

Divide.

5. $7 \overline{)647}$

6. $9 \overline{)3452}$

Use long division to divide the polynomials. Write as Quotient + Remainder/Divisor.

7. $(2x^2 + 3x - 20) \div (x - 2)$

8. $(x^4 + 6x^3 + 6x^2) \div (x + 5)$

Use synthetic division to divide the polynomials. Write as Quotient + Remainder/Divisor.

9. $x^4 - 3x^3 - 7x - 14) \div (x - 4)$

10. $(x^2 + 9x + 6) \div (x + 8)$

Use synthetic substitution (The Remainder Theorem) to evaluate the polynomial for the given value.

11. $P(x) = 4x^3 - 5x^2 - x + 2$ for $x = -1$

12. $P(x) = 25x^2 - 16$ for $x = \frac{4}{5}$

13. $P(x) = 4x^3 - 5x^2 - x + 2$ for $x = -1$

14. $P(x) = 25x^2 - 16$ for $x = \frac{4}{5}$

6.4 Factoring Polynomials

Objective: Use the Factor Theorem to determine factors of a polynomial.

Factor Theorem	
Theorem	Example
For any polynomial $P(x)$, $(x - a)$ is a factor of $P(x)$ if and only if $P(a) = 0$.	Because $P(1) = 1^2 - 1 = 0$, $(x - 1)$ is a factor of $P(x) = x^2 - 1$.

Example 1. Determine whether the given binomial is a factor of the polynomial $P(x)$.

- (a) $(x - 3)$; $P(x) = x^2 + 2x - 3$
- (b) $(x + 4)$; $P(x) = 2x^4 + 8x^3 + 2x + 8$

Example 2. Factor by grouping.

- (a) $x^3 + 3x^2 - 4x - 12$
- (b) $x^3 - 2x^2 - 9x + 18$

You Try It! 4. Factor by grouping

- (a) $2x^3 + x^2 + 8x + 4$
- (b) $8y^3 - 4y^2 - 50y + 25$

6.4 (day 2) Factoring

Objective: Factor the sum and difference of two cubes.

Factoring The Sum and Difference of Two Cubes	
Method	Algebra
Sum of two cubes	$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$
Difference of two cubes	$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

S . O . A . P

same
opposite
always
positive

$(a \pm b)^3 = (a \pm b)(a^2 \mp ab + b^2)$

Example 3. Factor each expression using sum or difference of cubes.

- (a) $5x^4 + 40x$
- (b) $8y^3 - 27$

You Try It! 5. Factor each expression using sum or difference of cubes.

- (a) $8 + z^6$
- (b) $2x^5 - 16x^2$

6.4 Review of Factoring

Objective: Factor using the sum and difference of two cubes, difference of square, grouping, and GCF.

Factor using the greatest common factor (GCF).

1. $2x^5 - 6x^3$

3. $14x^3 - 49x^2 - 28x$

2. $5x^3 - 10x$

4. $27x^5 - 18x^4 + 9x^3$

Factor using difference of squares.

5. $q^2 - r^2$

8. $x^4 - y^4$

6. $25a^2 - 64b^2$

9. $a^6 - b^6$

7. $81x^2 - 100y^2$

10. $4x^4 - 9y^6$

Factor using sum and difference of cubes.

5. $x^3 - y^3$

8. $64x^3 + 125y^3$

6. $r^3 + s^3$

9. $a^6 - b^6$

7. $8a^3 - 27b^3$

10. $x^6 + y^6$

Factor using grouping.

5. $6x^3 + 2x^2 + 9x + 3$

7. $4x^3 + 8x^2 - 9x - 18$

6. $7x^3 - 35x^2 + 8x - 40$

8. $16x^3 - 64x^2 - 25x + 100$

6.5 Finding Real Roots of Polynomial Equations

Objective: Identify the multiplicity of roots, Use the Rational Root Theorem to solve polynomial equations.

Example 1. Solve each polynomial equation by factoring. Check your answer using **Desmos**.

(a) $3x^5 + 18x^4 + 27x^3 = 0$

(b) $x^4 - 13x^2 = -36$

You Try It! 6. Solve each polynomial equation by factoring.

(a) $2x^6 - 10x^5 - 12x^4 = 0$

(b) $x^3 - 2x^2 - 25x = -50$

Definition 6.5.1. The **multiplicity** of root r is the number of times that $x - r$ is a factor of $P(x)$. Even multiplicity means the graph “touches” the x -axis at the root but does not cross. Odd multiplicity means the graph crosses the x -axis at the root.

Example 2. Identify the roots of each equation. State the multiplicity of each root.

(a) $x^3 - 9x^2 + 27x - 27 = 0$

(b) $-2x^3 - 12x^2 + 30x + 200 = 0$

6.5 Finding Real Roots of Polynomial Equations (day 2)

Theorem 6.5.1. (Rational Root Theorem) If the polynomial $P(x)$ has integer coefficients, then every rational root of the polynomial equation $P(x) = 0$ can be written in the form $\frac{p}{q}$, where p is a factor of the constant term of $P(x)$ and q is a factor of the leading coefficient of $P(x)$.

Example 3. Owen, a popcorn producer, is designing a new box for popcorn distribution. The marketing department has required a box with the width 2 inches less than the length and with the height 5 inches greater than the length. The volume of each box must be 24 cubic inches. What is the length of the box?

Example 4. Identify all of the real roots of:

(a) $4x^4 - 21x^3 + 18x^2 + 19x - 6 = 0$

(b) $2x^3 - 3x^2 - 10x - 4 = 0$

6.6 Fundamental Theorem of Algebra

Objective: Use the Fundamental Theorem of Algebra and corollary to write a polynomial equation given roots.

The following statements are equivalent:
A real number r is a root of the polynomial equation $P(x) = 0$.
$P(r) = 0$
r is an x -intercept of the graph of $P(x)$.
$x - r$ is a factor of $P(x)$.
When you divide the rule for $P(x)$ by $x - r$, the remainder is 0.
r is a zero of $P(x)$

Example 1. Write the simplest polynomial function with the given zeros.

(a) $-3, \frac{1}{2}$, and 1

(b) $-2, 2$, and 4

Theorem 6.6.1. (The Fundamental Theorem of Algebra) Every polynomial function of degree $n \geq 1$ has at least one zero, where a zero may be a complex number.

Theorem 6.6.2. (FTA Corollary) Every polynomial function of degree $n \geq 1$ has exactly n zeros, including multiplicities.

Example 2. Solve each polynomial by finding all roots.

(a) $x^4 + x^3 + 2x^2 + 4x - 8 = 0$

(b) $x^4 + 4x^3 - x^2 + 16x - 20 = 0$

6.6 Fundamental Theorem of Algebra (day 2)

Objective: Use the Fundamental Theorem of Algebra and corollary to write a polynomial equation given roots.

Theorem 6.6.3. If $\mathbf{a} + \mathbf{bi}$ is a root of a polynomial equation with real-number coefficients, then $\mathbf{a} - \mathbf{bi}$ is also a root.

Example 3. Write the simplest polynomial function with the given zeros.

(a) $1 + i$, $\sqrt{2}$, and 3

(b) $2i$, $1 + \sqrt{2}$ and 3

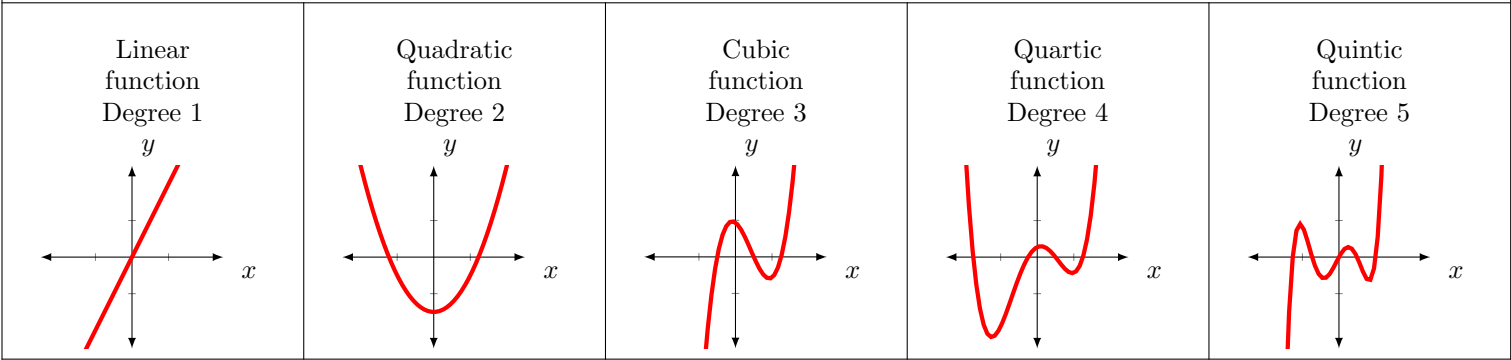
Example 4. An engineering class is designed model rockets for competition. The body of the rocket must be cylindrical with a cone-shaped top. The cylindrical portion must be 60 cm tall, and the height of the cone must be twice the radius. The volume of the payload region must be $558\pi \text{ cm}^3$ in order to hold the cargo. Find the radius of the rocket.

Example 5. A grain silo is in the shape of a cylinder with a hemisphere on top. The cylinder is 20 ft tall. The volume of the silo is 2106π cubic feet. Find the radius of the silo.

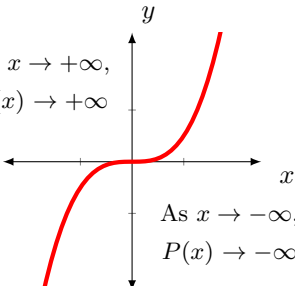
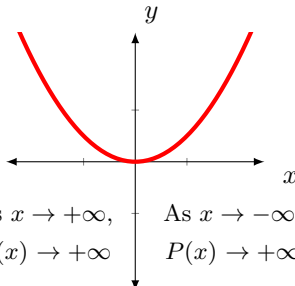
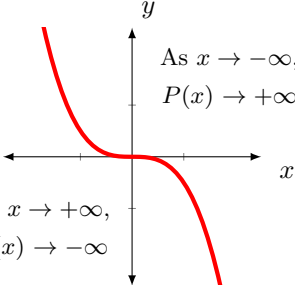
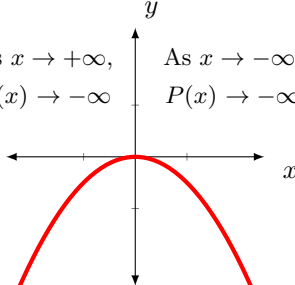
6.7 Investigating Graphs of Polynomial Functions

Objective: Use properties of end behavior to analyze, describe, and graph polynomial functions.

Graphs of Polynomial Functions



Definition 6.7.1. End behavior is a description of the values of a function (y -values) as x approaches positive infinity ($x \rightarrow +\infty$) or negative infinity ($x \rightarrow -\infty$).

Polynomial End Behavior		
P(x) has..	Odd Degree	Even Degree
Leading Coefficient $a > 0$	 <p>As $x \rightarrow +\infty$, $P(x) \rightarrow +\infty$</p> <p>As $x \rightarrow -\infty$, $P(x) \rightarrow -\infty$</p>	 <p>As $x \rightarrow +\infty$, $P(x) \rightarrow +\infty$</p> <p>As $x \rightarrow -\infty$, $P(x) \rightarrow +\infty$</p>
Leading Coefficient $a < 0$	 <p>As $x \rightarrow -\infty$, $P(x) \rightarrow +\infty$</p> <p>As $x \rightarrow +\infty$, $P(x) \rightarrow -\infty$</p>	 <p>As $x \rightarrow +\infty$, $P(x) \rightarrow -\infty$</p> <p>As $x \rightarrow -\infty$, $P(x) \rightarrow -\infty$</p>

Example 1. Identify the leading coefficient, degree and end behavior.

(a) $P(x) = -4x^3 - 3x^2 + 5x + 6$

Degree:

Leading Coefficient

End Behavior: As $x \rightarrow +\infty$

As $x \rightarrow -\infty$

(b) $R(x) = x^6 - 7x^5 + x^3 - 2$

Degree:

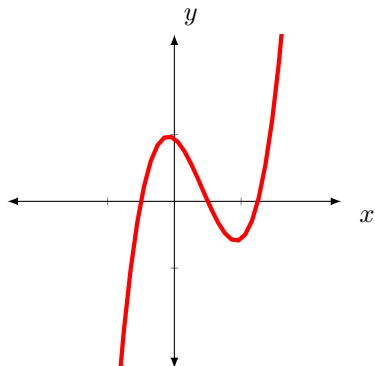
Leading Coefficient

End Behavior: As $x \rightarrow +\infty$

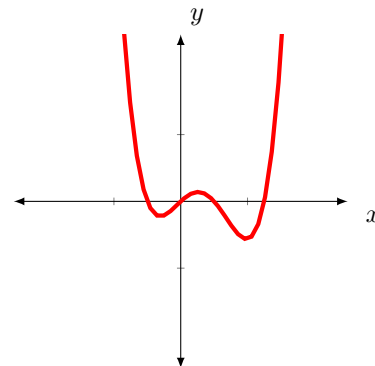
As $x \rightarrow -\infty$

Example 2. Identify whether the function graphed has an odd or even degree and a positive or negative leading coefficient.

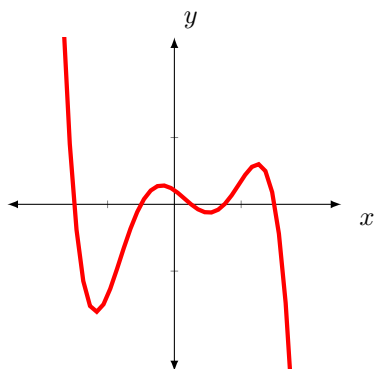
(a)

Degree:Leading Coefficient:

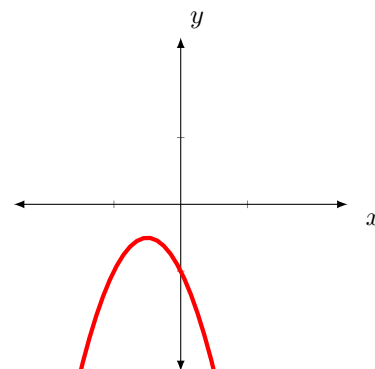
(b)

Degree:Leading Coefficient:

(c)

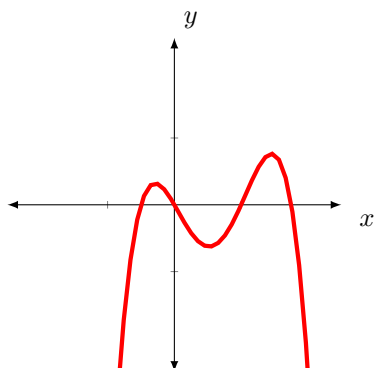
Degree:Leading Coefficient:

(d)

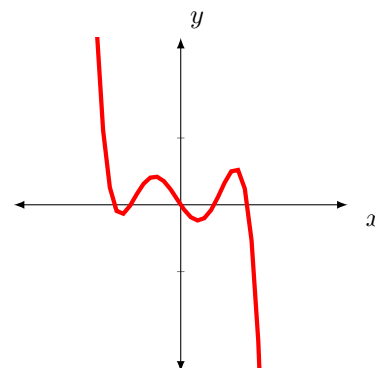
Degree:Leading Coefficient:

You Try It! 7. Identify whether the function graphed has an odd or even degree and a positive or negative leading coefficient.

(a)

Degree:Leading Coefficient:

(b)

Degree:Leading Coefficient:

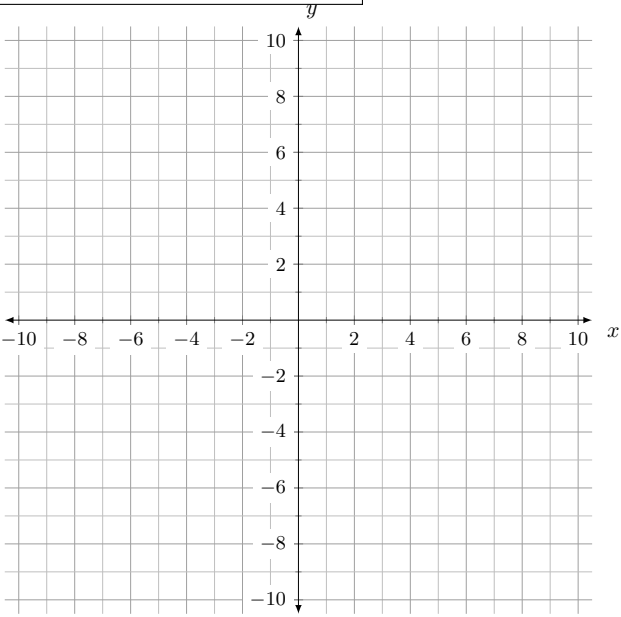
6.7 Investigating Graphs of Polynomial Functions (day 2)

Objective: Identify and use maxima and minima of polynomial functions to solve problems.

Steps for Graphing a Polynomial Function
1. Find all zeros (x-intercepts) and y-intercepts of the function.
2. Plot the x - and y -intercept.
3. Make a table for several x -values that lie between the real zeros.
4. Plot the points from your table.
5. Determine end behavior.
6. Sketch the graph.

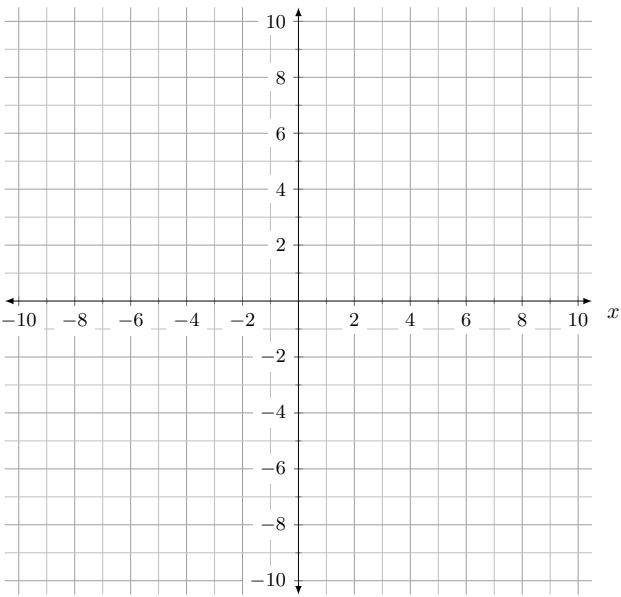
Example 3. Graph the function.

$f(x) = x^3 + 3x^2 - 6x - 8$



Example 4. Graph the function.

$f(x) = -x^3 + 2x^2 + 5x - 6$



Definition 6.7.2. For a function $f(x)$, $f(a)$ is a **local maximum** if there is an interval around a such that $f(x) < f(a)$ for every x -value in the interval except a .

Definition 6.7.3. For a function $f(x)$, $f(a)$ is a **local minimum** if there is an interval around a such that $f(x) > f(a)$ for every x -value in the interval except a .

Determine Maxima and Minima with a calculator

Step 1: Type equations into $\boxed{Y=}$ and press $\boxed{\text{GRAPH}}$.

The graph appears to have one local maximum or minimum.

Step 2: Find the maximum.

Press $\boxed{2\text{nd}}$ $\boxed{\text{TRACE}}$ to access the **CALC** menu. Choose **4: maximum**. Arrow to the left of maximum and press $\boxed{\text{ENTER}}$, next arrow to the right of the maximum and press $\boxed{\text{ENTER}}$.

Step 3: Find the minimum.

Press $\boxed{2\text{nd}}$ $\boxed{\text{TRACE}}$ to access the **CALC** menu. Choose **3: minimum**. Arrow to the left of minimum and press $\boxed{\text{ENTER}}$, next arrow to the right of the minimum and press $\boxed{\text{ENTER}}$.

Example 5. Graph the function on a calculator, and estimate the local **maxima** and **minima**.

(a) $g(x) = 2x^3 - 12x + 6$

(b) $f(x) = x^3 - 2x - 3$

Example 6. A welder plans to construct an open box from an 18.5 ft by 24.5 ft sheet of metal by cutting squares from the corners and folding up the sides. Find the maximum volume of the box and the corresponding dimensions.

6.8

Objective:

Definition 6.8.1. Definition

Example 1. Example

6.9

Objective:

Definition 6.9.1. Definition

Example 1. Example