Multivariable Calculus
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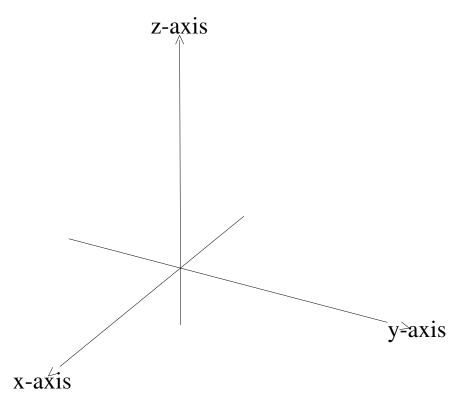
Chapter 11 Three-Dimensional Space; Vectors

11.1 Rectangular Coordinates in 3-Space; Spheres; and Cylindrical Surfaces

Definition 11.1.1. For this class we will call 3-dimensional space **3-space**, two-dimensional space **2-space**, and a one-dimensional space **1-space**.

Three space can placed in one-to-one correspondence with triples of real numbers by using three mutually perpendicular coordinate lines called the **x-axis**, **y-axis**, and **z-axis** positioned so that their origins.



Three space is divided into eight parts, called octants. The set of points with all positive coordinates form the **first octant**. The other seven octants have no standard numbering. Consider the following facts about three-dimensional rectangular coordinate systems:

Region	Description
xy-plane	Consists of all points of the form $(x, y, 0)$
xz-plane	Consists of all points of the form $(x, 0, z)$
yz-plane	Consists of all points of the form $(0, y, z)$
x-axis	Consists of all points of the form $(x, 0, 0)$
y-axis	Consists of all points of the form $(0, y, 0)$
z-axis	Consists of all points of the form $(0, 0, z)$

Definition 11.1.2. Recall that in 2-space the distance between two points (x_1, y_1) and (x_2, y_2) is given by the Pythagorean Theorem:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \tag{11.1}$$

The distance formula in 3-space is found by a similar method. The distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2) in 3-space is:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Definition 11.1.3. Recall that in 2-space the formula circle comes directly from equation (11.1) in Definition 11.1.2. A circle of radius r with center at (x_0, y_0) is:

$$(x - x_0)^2 + (y - y_0)^2 = r^2$$

It follows that **the standard equation of the sphere** in 3-space that has center (x_0, y_0, z_0) and radius r is:

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$$

Some examples of the standard equation of a sphere are given in the table below:

Equation	Graph
$(x-2)^2 + (y-3)^2 + (z+4)^2 = 9$	Sphere with center $(2, 3, -4)$ and radius 3
$(x+1)^2 + y^2 + (z-2)^2 = 5$	Sphere with center $(-1, 0, 2)$ and radius $\sqrt{5}$
$x^2 + y^2 + z^2 = 1$	Sphere with center $(0, 0, 0)$ and radius 1

Theorem 11.1.1. An equation of the form

$$x^2 + y^2 + z^2 + Gx + Hy + Iz + J = 0$$

represents a sphere, a point, or has no graph.

Equations in two variables are commonly graphed in 2-space, but they can also be graphed in 3-space. Consider below the graph of $y = x^2$ in both the xy-plane and the xyz-coordinate system.

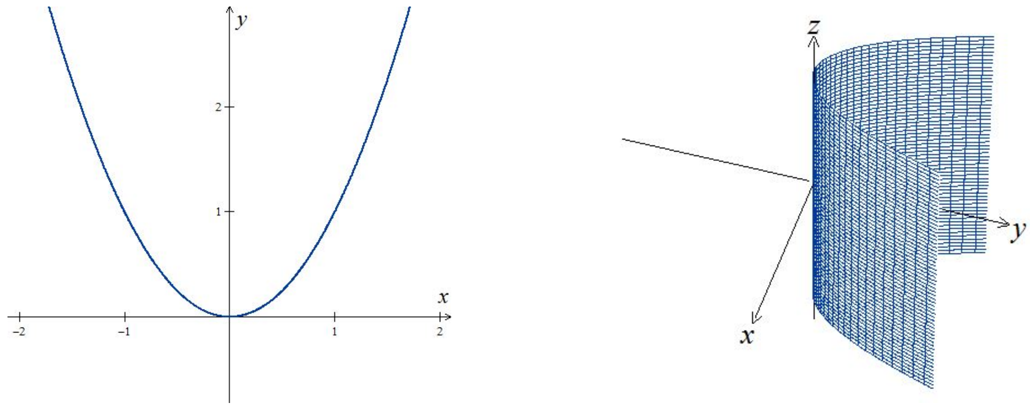


Figure 11.1: $y = x^2$ in xy-coordinate system and the xyz-coordinate system

Definition 11.1.4. The process of generating a surface by translating a plane curve parallel to some line is called **extrusion**, and surfaces that are generated by extrusion are called **cylindrical surfaces**. A good example of a cylindrical surface is given in the surface below of a right cylinder which is generated by extruding the unit circle in the direction of the y-axis.

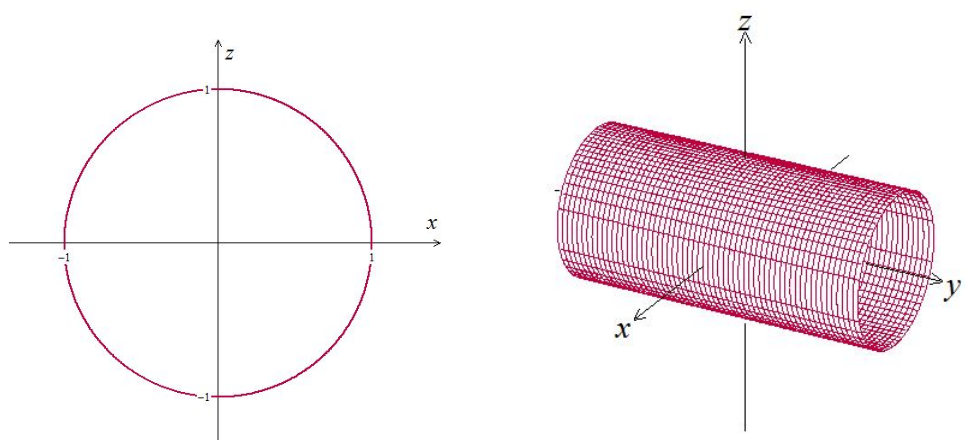


Figure 11.2: Extrusion of the unit circle, $x^2 + z^2 = 1$, in the direction of the y-axis

Theorem 11.1.2. An equation that contains only two of the variables x,y, and z represents a cylindrical surface in an xyz-coordinate system. The surface can be obtained by graphing the equation in the coordinate plane of the two variables that appear in the equation and then translating that graph parallel to the axis of the missing variable.

Homework Assignment
Section 11.1 : (Page 771: 3, 10, 13, 15, 23, 27, 31, 33, 35, 37, 39, 47)

Introductory Questions:

At most how many regions does one plane divide 3-space? __________.

At most how many regions does two planes divide 3-space? __________.

At most how many regions does three planes divide 3-space? __________.

Introduction to Set Notation:

$\mathbb{N} =$

$\mathbb{Z} =$

$\mathbb{Q} =$

$\mathbb{R} =$

$\mathbb{I} =$

Some Key Formulas:

Example 1. Find the distance between the points $(2, 3, -1)$ and $(4, -1, 3)$.

Example 2. Find the radius of the sphere $x^2 + y^2 + z^2 - 2x - 4y + 8z + 17 = 0$.

Example 3. Stetch the surface in 3-space, $2x + 3z = 6$.

Example 4. Sketch the graph of $z = \cos y$ in 3-space.

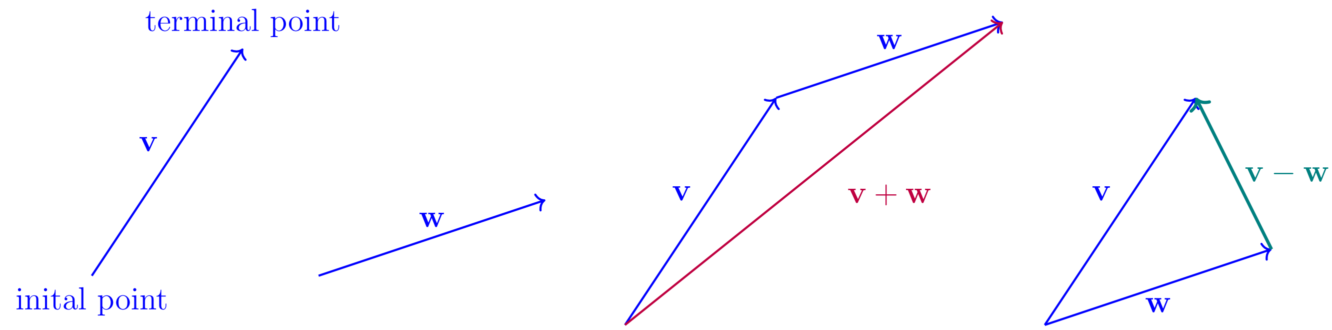
Example 5. Write and inequality to describe the given region:

(a) The half-space consisting of all points to the left of the xz-plane.

(b) The solid upper hemisphere of the sphere of radius 2 centered at the origin.

11.2 Vectors

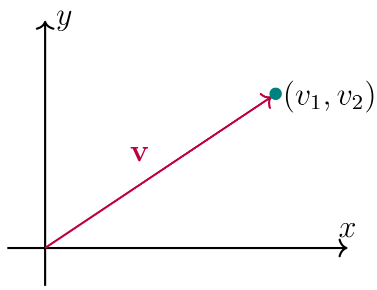
Definition 11.2.1. If $\mathbf{v}$ and $\mathbf{w}$ are vectors, then the **sum $\mathbf{v}+\mathbf{w}$** is the vector from the initial point of $\mathbf{v}$ to the terminal point of $\mathbf{w}$ when the vectors are positioned so that the initial point of $\mathbf{w}$ is at the terminal point of $\mathbf{v}$. The **difference $\mathbf{v}-\mathbf{w}$** forms the vector between their terminal points.

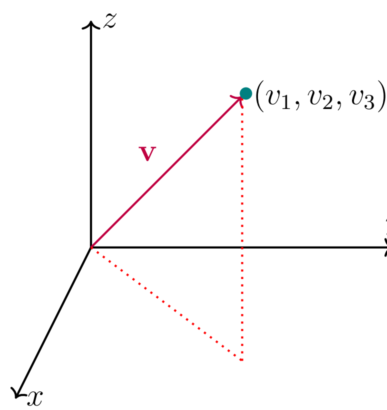


Definition 11.2.2. If $\mathbf{v}$ is a nonzero vector and k is a nonzero real number (a scalar), then the **scalar multiple** $k\mathbf{v}$ is defined to be the vector whose length is $|k|$ times the length of $\mathbf{v}$ and whose direction is the same as that of $\mathbf{v}$ if $k>0$ and opposite to that of $\mathbf{v}$ if $k<0$. We define $k\mathbf{v}=\mathbf{0}$ if $k=0$ or $\mathbf{v}=\mathbf{0}$.

Definition 11.2.3. If a vector $\mathbf{v}$ is positioned with its initial point at the origin of a rectangular coordinate system, then its terminal point will have coordinates of the form (v_1,v_2) or (v_1,v_2,v_3), depending on whether the vector is in 2-space or 3-space. We call these coordinates the **components** of $\mathbf{v}$, and we write $\mathbf{v}$ in *component form* using the **bracket notation**,

$$\mathbf{v}=\langle v_1,v_2\rangle \quad\text{or}\quad \mathbf{v}=\langle v_1,v_2,v_3\rangle.$$





Theorem 11.2.1. Two vectors are equivalent if and only if their corresponding components are equal.

Theorem 11.2.2. If $\mathbf{v}=\langle v_1,v_2\rangle$ and $\mathbf{w}=\langle w_1,w_2\rangle$ are vectors in 2-space and k is any scalar, then

$$\begin{aligned}\mathbf{v}+\mathbf{w} &= \langle v_1+w_1, v_2+w_2\rangle\\ \mathbf{v}-\mathbf{w} &= \langle v_1-w_1, v_2-w_2\rangle\\ k\mathbf{v} &= \langle kv_1, kv_2\rangle.\end{aligned}$$

Similarly, if $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ and $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$ are vectors in 3-space an k is any scalar, then

$$\begin{aligned} \mathbf{v} + \mathbf{w} &= \langle v_1 + w_1, v_2 + w_2, v_3 + w_3 \rangle \\ \mathbf{v} - \mathbf{w} &= \langle v_1 - w_1, v_2 - w_2, v_3 - w_3 \rangle \\ k\mathbf{v} &= \langle kv_1, kv_2, kv_3 \rangle. \end{aligned}$$

Theorem 11.2.3. If $\overrightarrow{P_1P_2}$ is a vector in 2-space with initial point $P_1(x_1, y_1)$ and terminal point $P_2(x_2, y_2)$, then

$$\overrightarrow{P_1P_2} = \langle x_2 - x_1, y_2 - y_1 \rangle$$

Similarly, if $\overrightarrow{P_1P_2}$ is a vector in 3-space with initial point $P_1(x_1, y_1, z_1)$ and terminal point $P_2(x_2, y_2, z_2)$, then

$$\overrightarrow{P_1P_2} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle.$$

Theorem 11.2.4. For any vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ and any scalars k and l, the following relationships hold:

(a) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

(b) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

(c) $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$

(d) $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$

(e) $k(l\mathbf{u}) = (kl)\mathbf{u}$

(f) $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$

(g) $(k + l)\mathbf{u} = k\mathbf{u} + l\mathbf{u}$

(h) $1\mathbf{u} = \mathbf{u}$

Definition 11.2.4. The distance between the initial point, at the origin, and terminal points of a vector $\mathbf{v}$ is called the **length**, the **norm**, or the **magnitude** of $\mathbf{v}$ and is denoted by $\|\mathbf{v}\|$. Since the components, v_1 and v_2, of the vector $\mathbf{v} = \langle v_1, v_2 \rangle$ are orthogonal (perpendicular) we can solve for the norm in 2-space using the Pythagorean Theorem,

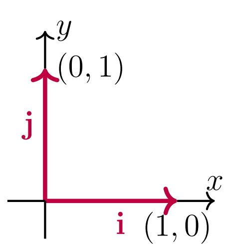
$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2}$$

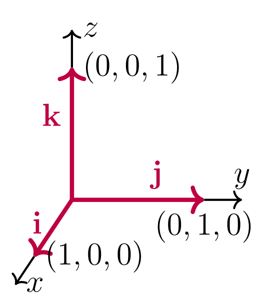
and the norm of a vector $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ in 3-space is given by

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}.$$

Definition 11.2.5. A vector of length 1 is called a **unit vector**. In an xy-coordinate system the unit vectors along the x- and y- axes are denoted by $\mathbf{i}$ and $\mathbf{j}$, respectively; and in an xyz-coordinate system the unit vectors along the x-, y-, and z-axes are denoted by $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, respectively. Thus,

$$\begin{array}{llll} \mathbf{i} = \langle 1, 0 \rangle, & \mathbf{j} = \langle 0, 1 \rangle, & & \text{in 2-space} \\ \mathbf{i} = \langle 1, 0, 0 \rangle, & \mathbf{j} = \langle 0, 1, 0 \rangle, & \mathbf{k} = \langle 0, 0, 1 \rangle, & \text{in 3-space} \end{array}$$





Every vector in 2-space is expressible uniquely in terms of **i** and **j**, and every vector in 3-space is expressible uniquely in terms of **i**, **j**, and **k** as follows:

$$\mathbf{v} = \langle v_1, v_2 \rangle = \langle v_1, 0 \rangle + \langle 0, v_2 \rangle = v_1\langle 1, 0 \rangle + v_2\langle 0, 1 \rangle = v_1\mathbf{i} + v_2\mathbf{j}$$

$$\mathbf{v} = \langle v_1, v_2, v_3 \rangle = v_1\langle 1, 0, 0 \rangle + v_2\langle 0, 1, 0 \rangle + v_3\langle 0, 0, 1 \rangle = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$$

Definition 11.2.6. The process of multiplying a vector **v** by the reciprocal of its length to obtain a unit vector with the same direction is called **normalizing v**.

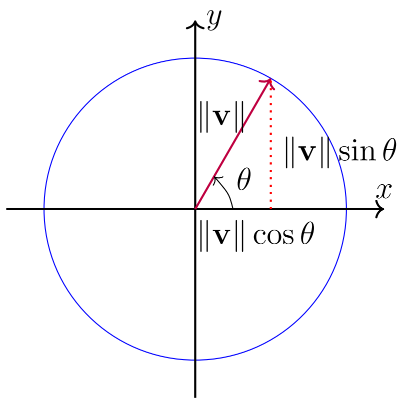
$$\mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$$

A nonzero vector **v** situated with its initial point at the origin with θ the angle from the positive x-axis to the radial line through **v** has a trigonometric form of

$$\mathbf{v} = \|\mathbf{v}\|\langle \cos\theta, \sin\theta \rangle \quad \text{or} \quad \mathbf{v} = \|\mathbf{v}\|\cos\theta\,\mathbf{i} + \|\mathbf{v}\|\sin\theta\,\mathbf{j}$$

In the case that **v** is a unit vector **u** this simplifies to

$$\mathbf{u} = \langle \cos\theta, \sin\theta \rangle \quad \text{or} \quad \mathbf{u} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$$



Homework Assignment
Section 11.2 : (Page 782: 1-45 odds)

Example 1. If $\mathbf{v} = \langle -3, 0, 2\rangle$ and $\mathbf{w} = \langle 3, 7, -1\rangle$ find

$\mathbf{v} + \mathbf{w} =$

$\mathbf{v} =$

$-\mathbf{w} =$

$\mathbf{w} - 2\mathbf{v} =$

Example 2. Find the vector ordered pairs for following points:

In 2-space the vector from $P_1(1, 3)$ to $P_2(4, -2)$ is

In 3-space $A(0, -2, 5)$ to $B(3, 4, -1)$ is

Prove parts (a), (b), and (c) of **Theorem 11.2.4**.

Proof:

Example 3. Find the norms (magnitude) of the following vectors if $\mathbf{v} = \langle -2, 3 \rangle$, $10\mathbf{v} = \langle -20, 30 \rangle$, and $\mathbf{w} = \langle 2, 3, 6 \rangle$.

Example 4. Write the following vectors in $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ notation:

$\langle 2, -3, 4 \rangle =$

$(3\mathbf{i} + 2\mathbf{j}) + (4\mathbf{i} + \mathbf{j}) =$

$\|v_1\mathbf{i} + v_2\mathbf{j}\| =$

Example 5. Find the unit vector that has the same direction as $\mathbf{v} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$

11.3 Dot Product; Projections

Definition 11.3.1. If $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ are vectors in 2-space, then the **dot product** of $\mathbf{u}$ and $\mathbf{v}$ is written as $\mathbf{u} \cdot \mathbf{v}$ and is defined as

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2$$

Similarly, if $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ are vectors in 3-space, then their dot product is defined as

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

Theorem 11.3.1. if $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in 2- or 3-space and k is a scalar, then:

(a.) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

(b.) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$

(c.) $k(\mathbf{u} \cdot \mathbf{v}) = (k\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (k\mathbf{v})$

(d.) $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2 \qquad \Longrightarrow \qquad \|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$

(e.) $\mathbf{0} \cdot \mathbf{v} = 0$

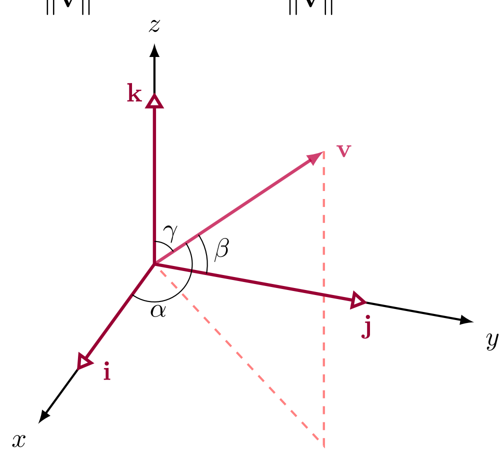
Theorem 11.3.2. If $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors in 2-space or 3-space, and if θ is the angle between them, then

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$$

Proof:

Theorem 11.3.3. The direction cosines of a nonzero vector $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ are

$$\cos\alpha = \frac{v_1}{\|\mathbf{v}\|}, \qquad \cos\beta = \frac{v_2}{\|\mathbf{v}\|}, \qquad \cos\gamma = \frac{v_3}{\|\mathbf{v}\|}$$



The directional cosines follow the following identity:

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

Verification:

In many physical applications it is useful to decompose a force (vector) into two orthogonal component directions, this leads us to the next definition.

Definition 11.3.2. Let $\mathbf{e}_1$ and $\mathbf{e}_2$ be perpendicular unit vectors and $\mathbf{v}$ any vector. Then the **vector components** of $\mathbf{v}$ along $\mathbf{e}_1$ and $\mathbf{e}_2$ are $(\mathbf{v}\cdot\mathbf{e}_1)\mathbf{e}_1$ and $(\mathbf{v}\cdot\mathbf{e}_2)\mathbf{e}_2$ respectively. We call $\mathbf{v}\cdot\mathbf{e}_1$ and $\mathbf{v}\cdot\mathbf{e}_2$ the **scalar components** for $\mathbf{v}$ along $\mathbf{e}_1$ and $\mathbf{e}_2$, respectively.

The scalar components can be written in trigonometric form as

$$\mathbf{v}\cdot\mathbf{e}_1 = \|\mathbf{v}\|\cos\theta \qquad \text{and} \qquad \mathbf{v}\cdot\mathbf{e}_2 = \|\mathbf{v}\|\sin\theta,$$

where θ is the angle between $\mathbf{v}$ and $\mathbf{e}_1$. Also, the vector components of $\mathbf{v}$ can be expressed as

$$(\mathbf{v}\cdot\mathbf{e}_1)\mathbf{e}_1 = (\|\mathbf{v}\|\cos\theta)\mathbf{e}_1 \qquad \text{and} \qquad (\mathbf{v}\cdot\mathbf{e}_2)\mathbf{e}_2 = (\|\mathbf{v}\|\sin\theta)\mathbf{e}_2.$$

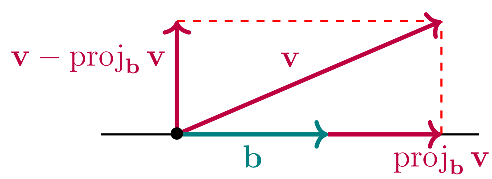
Therefore we can express $\mathbf{v}$ as

$$\mathbf{v} = (\|\mathbf{v}\|\cos\theta)\mathbf{e}_1 + (\|\mathbf{v}\|\sin\theta)\mathbf{e}_2.$$

Definition 11.3.3. Let $\mathbf{v}$ and $\mathbf{b}$ be nonzero vectors, then we define the **orthogonal projection of v onto b** to be

$$\operatorname{proj}_{\mathbf{b}}\mathbf{v} = \frac{\mathbf{v}\cdot\mathbf{b}}{\|\mathbf{b}\|^2}\mathbf{b}$$

Geometrically, if $\mathbf{b}$ and $\mathbf{v}$ have a common initial point, then $\operatorname{proj}_{\mathbf{b}}\mathbf{v}$ is the vector determined when a perpendicular is dropped from the terminal point of $\mathbf{v}$ to the line through $\mathbf{b}$ as illustrated below.



Homework Assignment
Section 11.3: (Page 792: 3, 5, 7, 11, 15, 23, 27, 31, 33, 37, 39)

Example 1. Calculate the following dot products

$$\langle 3,5\rangle \cdot \langle -1,2\rangle =$$

$$\langle 2,3\rangle \cdot \langle -3,2\rangle =$$

$$\langle 1,-3,4\rangle \cdot \langle 1,5,2\rangle =$$

Prove **Theorem 11.3.2**.

Example 2. Find the angle between $\mathbf{u} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ and

(a) $\mathbf{v} = -3\mathbf{i} + 6\mathbf{j} + 2\mathbf{k}$

(b) $\mathbf{w} = 2\mathbf{i} + 7\mathbf{j} + 6\mathbf{k}$

(c) $\mathbf{z} = -3\mathbf{i} + 6\mathbf{j} - 6\mathbf{k}$

It is helpful to remember that when:

$\mathbf{u} \cdot \mathbf{v} > 0$, θ is __________. $\mathbf{u} \cdot \mathbf{v} < 0$, θ is __________. $\mathbf{u} \cdot \mathbf{v} = 0$, θ is __________.

Example 3. Find the directional cosines of $\mathbf{v} = 2\mathbf{i} - 4\mathbf{j} + 4\mathbf{k}$, then approximate the angle to the nearest degree.

Example 4. Find the angle between the diagonal of the cube and one of its edges. What is your hypothesis?

Read vector and scalar components **Definition 11.3.2**.

Example 5. Let $\mathbf{v} = \langle 2, 3 \rangle$, $\mathbf{e}_1 = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$ and $\mathbf{e}_2 = \left\langle \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$. Find the scalar and vector components of $\mathbf{v}$ along $\mathbf{e}_1$ and $\mathbf{e}_2$.

Read vector and scalar components **Definition 11.3.3**.

Example 6. Find the orthogonal projection of $\mathbf{v} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ on $\mathbf{b} = 2\mathbf{i} + 2\mathbf{j}$, then find the vector component of $\mathbf{v}$ orthogonal to $\mathbf{b}$.

11.4 Cross Product

Definition 11.4.1. A **determinant** is a function that assigns numerical values to square arrays (*Matrices*) of numbers. For example, if a_1, a_2, b_1, and b_2 are real numbers then we define the $\mathbf{2} \times \mathbf{2}$ **determinant** by

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

and a $\mathbf{3} \times \mathbf{3}$ **determinant** is defined in terms of 2×2 determinants by

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

These definitions will be studied in depth in the next course, **linear algebra**.

Theorem 11.4.1.

(a.) If two rows in the array of a determinant are the same, then the value of the determinant is 0.

(b.) Interchanging two rows in the array of a determinant multiplies its value by -1.

Definition 11.4.2. If $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ are vectors in 3-space then the **cross product** $\mathbf{u} \times \mathbf{v}$ is the vector defined by

$$\mathbf{u} \times \mathbf{v} = \mathbf{i} \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

or, equivalently,

$$\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2)\mathbf{i} - (u_1 v_3 - u_3 v_1)\mathbf{j} + (u_1 v_2 - u_2 v_1)\mathbf{k}$$

You may have observed in the last definition that the cross product is the result of the determinant of the following 3×3 array:

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

However, this is just a helpful way to remember the cross product, and is not a true determinant since the entries in the first row are vectors instead of real numbers.

Theorem 11.4.2. If **u**, **v** and **w** are any vectors in 3-space and k is any scalar, then:

(a.) $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$

(b.) $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$

(c.) $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \times \mathbf{w}) + (\mathbf{v} \times \mathbf{w})$

(d.) $k(\mathbf{u} \times \mathbf{v}) = (k\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (k\mathbf{v})$

(e.) $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$

(f.) $\mathbf{u} \times \mathbf{u} = \mathbf{0}$

The previous theorem gives us the following unique and helpful cross product of the orthogonal unit vectors:

$$\begin{array}{lll} \mathbf{i} \times \mathbf{j} = \mathbf{k} & \mathbf{j} \times \mathbf{k} = \mathbf{i} & \mathbf{k} \times \mathbf{i} = \mathbf{j} \\ \mathbf{j} \times \mathbf{i} = -\mathbf{k} & \mathbf{k} \times \mathbf{j} = -\mathbf{i} & \mathbf{i} \times \mathbf{k} = -\mathbf{j} \end{array}$$

Theorem 11.4.3. if **u** and **v** are vectors in 3-space, then:

(a.) $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0$ ($\mathbf{u} \times \mathbf{v}$ is orthogonal to **u**)

(b.) $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 0$ ($\mathbf{u} \times \mathbf{v}$ is orthogonal to **v**)

Theorem 11.4.4. Let **u** and **v** be nonzero vectors in 3-space, and let θ be the angle between these vectors when they are positioned so their initial points coincide.

(a.) $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\| \sin\theta$

(b.) The area A of the parallelogram that has **u** and **v** as adjacent sides is

$$A = \|\mathbf{u} \times \mathbf{v}\|$$

(c.) $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ if and only if **u** and **v** are parallel vectors, that is, if and only if they are multiples of one another.

Definition 11.4.3. If $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$, $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, and $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$ are vectors in 3-space, then the number

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$$

is called the **scalar triple product** of $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$. It is computed using

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

Theorem 11.4.5. Let $\mathbf{u}, \mathbf{v}$, and **w** be nonzero vectors in3-space.

(a.) The volume V of the parallelepiped that has **u**, **v**, and **w** as adjacent edges is

$$V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$$

(b.) $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = 0$ if and only if **u**, **v**, and **w** lie in the same plane.

Homework Assignment
Section 11.4: (Page 803: 1, 3, 7, 11, 17, 19, 21, 27, 29, 30, 37)

Example 1. Let $\mathbf{u} = \langle 1, 2, -2 \rangle$ and $\mathbf{v} = \langle 3, 0, 1 \rangle$. Find
(a) $\mathbf{u} \times \mathbf{v}$ (b) $\mathbf{v} \times \mathbf{u}$

Example 2. Show that $\mathbf{u} \times \mathbf{u} = 0$ for any vector in 3-space.

Read **Theorem 11.4.2** and **Theorem 11.4.3**

Example 3. Find a vector that is orthogonal to both of the vectors $\mathbf{u} = \langle 2, -1, 3 \rangle$ and $\mathbf{v} = \langle -7, 2, -1 \rangle$.

Read **Theorem 11.4.4**.

Example 4. *Find the are of the triangle that is determined by the points* $P_1(2, 2, 0)$*,* $P_2(-1, 0, 2)$*, and* $P_3(0, 4, 3)$*.*

Read **Definition 11.4.3** and **Theorem 11.4.5**.

Example 5. *Calculate the scalar triple product* $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ *of the vectors below. What is the volume of the parallepiped descriped by these vectors?*

$$\mathbf{u} = 3\mathbf{i} - 2\mathbf{j} - 5\mathbf{k}, \qquad \mathbf{v} = \mathbf{i} + 4\mathbf{j} - 4\mathbf{k} \qquad \mathbf{w} = 3\mathbf{j} + 2\mathbf{k}$$

11.5 Parametric Equations of Line

Theorem 11.5.1.

(a.) The line in 2-space that passes through the point $P_0(x_0, y_0)$ and is parallel to the nonzero vector $\mathbf{v} = \langle a, b\rangle = a\mathbf{i} + b\mathbf{j}$ has parametric equations

$$x = x_0 + at, \quad y = y_0 + bt$$

(b.) The line in 3-space that passes through the point $P_0(x_0, y_0, z_0)$ and is parallel to the nonzero vector $\mathbf{v} = \langle a, b, c\rangle = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ has parametric equations

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct$$

Definition 11.5.1. Two lines in 3-space that are not parallel and do not intersect are called **skew lines**.

Definition 11.5.2. Vector notation can be used to express the parametric equations of a line more compactly,

$$\langle x, y\rangle = \langle x_0 + at, y_0 + bt\rangle$$
$$\langle x, y, z\rangle = \langle x_0 + at, y_0 + bt, z_0 + ct\rangle$$

or, equivalently, as

$$\langle x, y\rangle = \langle x_0, y_0\rangle + t\langle a, b\rangle$$
$$\langle x, y, z\rangle = \langle x_0, y_0, z_0\rangle + t\langle a, b, c\rangle$$

For the equations in two space we define the vectors $\mathbf{r}, \mathbf{r}_0$, and $\mathbf{v}$ as

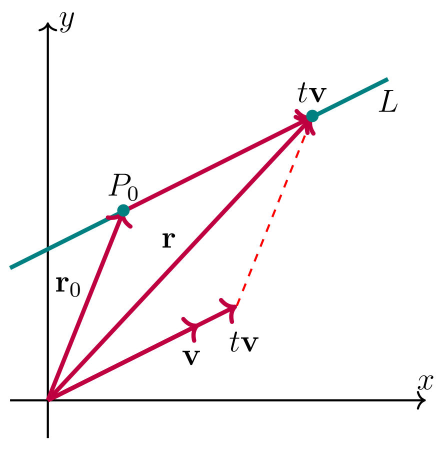
$$\mathbf{r} = \langle x, y\rangle, \qquad \mathbf{r}_0 = \langle x_0, y_0\rangle, \qquad \mathbf{v} = \langle a, b\rangle$$

and for the equations in 3-space we define them as

$$\mathbf{r} = \langle x, y, z\rangle, \qquad \mathbf{r}_0 = \langle x_0, y_0, z_0\rangle, \qquad \mathbf{v} = \langle a, b, c\rangle$$

which gives us the **vector equation of a line**

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$$



Homework Assignment
Section 11.5 : (Page 810: 1, 3, 5, 7, 9, 17, 21, 25, 29, 33, 37, 39, 41, 43, 49, 57)

Read **Theorem 11.5.1**.

Example 1. Find the parametric equations of the line

(a) Passing through $(4, 2)$ and parallel to $\mathbf{v} = \langle -1, 5 \rangle$.

(b) Passing through $(1, 2, -3)$ and parallel to $\mathbf{v} = \langle 4, 5, -7 \rangle$

(c) Passing through the origin in 3-space parallel to $\mathbf{v} = \langle 1, 1, 1 \rangle$

Example 2. Find parametric equations of the line passing through the points $P_1(2, 4, -1)$ and $P_2(5, 0, 7)$. Where does the line intersect the yz-plane?

Example 3. Let L_1 and L_2 be the lines

$$L_1: \; x = 1 + 4t, \; y = 5 - 4t, \; z = -1 + 5t$$

$$L_2: \; x = 2 + 8t, \; y = 4 - 3t, \; z = 5 + t$$

(a) Are the lines parallel? (b) Do the lines intersect?

Read **Definition 11.5.1**.

Example 4. Find parametric equations describing the line segment joining the points $P_1(2, 4, -1)$ and $P_2(5, 0, 7)$.

Read **Definition 11.5.2**.

Example 5. The equation $\langle x, y, z\rangle = \langle -1, 0, 2\rangle + t\langle 1, 5, -4\rangle$ is of the form $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$. Graph the line in 3-space and write parametric equations describing the line.

Example 6. Find the vector eqaution of the line passing through the points $P_1(3, -1, -2)$ and $P_2(6, 2, 1)$.

11.6 Planes in 3-Space

Definition 11.6.1. A vector perpendicular to a plane is called a **normal** to the plane.

Definition 11.6.2. Suppose a plane passes through a point $P_0(x_0, y_0, z_0)$ and is perpendicular to the vector $\mathbf{n} = \langle a, b, c\rangle$, then the **point-normal form** of the equation of the plane is

$$\langle a, b, c\rangle \cdot \langle x - x_0, y - y_0, z - z_0\rangle = 0$$

or when simplified,

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

Theorem 11.6.1. If a, b, c, and d are constants and a, b, and c are not all zero, then the graph of the equation

$$ax + by + cz + d = 0 \tag{11.2}$$

is a plane that has the vector $\mathbf{n} = \langle a, b, c\rangle$ as a normal. Equation (11.2) is called **general form** of the equation of a plane.

Note 1. Let θ be the angle between two intersecting planes with normals $\mathbf{n}_1$ and $\mathbf{n}_2$, then

$$\cos\theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\|\|\mathbf{n}_2\|}.$$

Theorem 11.6.2. The distance D between a point $P_0(x_0, y_0, z_0)$ and the plane $ax+by+cz+d = 0$ is

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$

Homework Assignment
Section 11.6: (Page 819: 1, 3, 9, 11, 13, 17, 19, 25, 29, 33, 37, 45, 49)

Example 1. Find an equation of the plane passing through the point $(3, -1, 7)$ and perpendicular to the vector $\mathbf{n} = \langle 4, 2, -5 \rangle$.

Read **Theorem 11.6.1**.

Example 2. Determine whether the planes are parallel $3x - 4y + 5z = 0$ and $-6x + 8y - 10z - 4 = 0$.

Example 3. Find an equation of the plane through the points $P_1(1, 2, -1)$, $P_2(2, 3, 1)$, and $P_3(3, -1, 2)$.

Example 4. Determine whether the line

$$x = 3 + 8t, \qquad y = 4 + 5t, \qquad z = -3 - t$$

is parallel to the plane $x - 3y + 5z = 12$.

Example 5. Find the intersection point of the line and the plane from **Example 4**.

Example 6. Find the acute angle of intersection between the two planes

$$2x - 4y + 4z = 6 \qquad \text{and} \qquad 6x + 2y - 3z = 4.$$

11.7 Quadratic Surfaces

Definition 11.7.1. The equation

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0$$

is called a **second degree equation in x, y, and z**. The graphs of such equations are called **quadratic surfaces** or sometimes just **quadratics**.

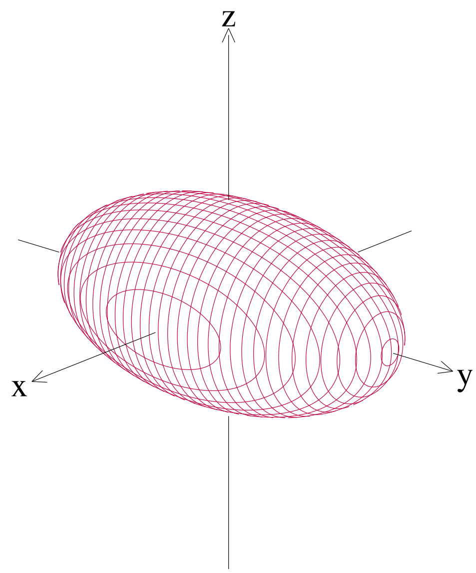


Figure 11.3: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$

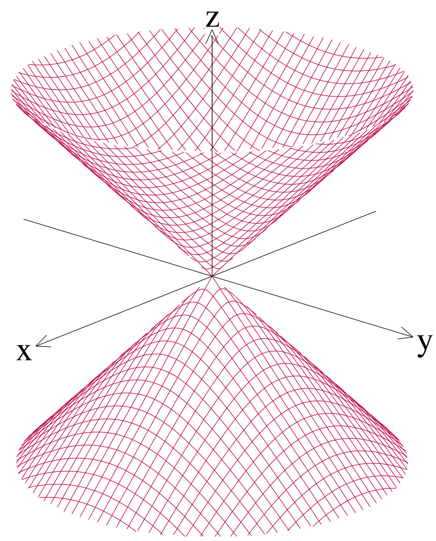


Figure 11.4: Elliptic Cone: $z^2 = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$

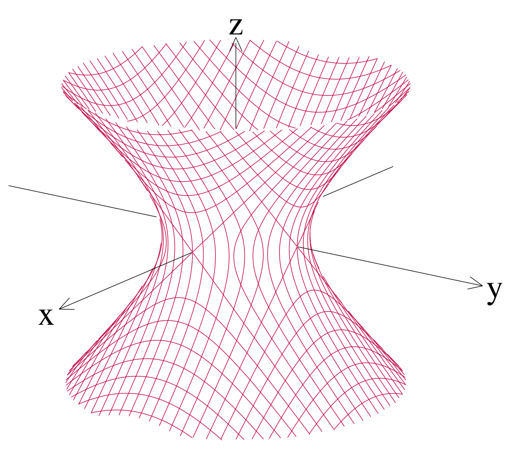


Figure 11.5: Hyperboloid of One Sheet:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

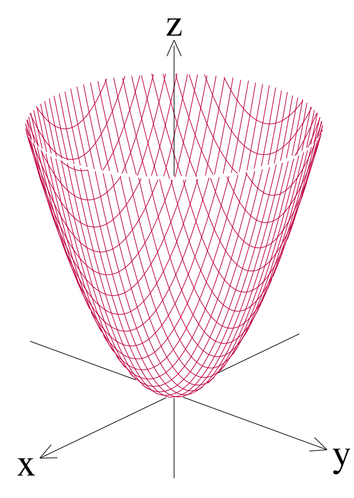


Figure 11.6: Elliptic Paraboloid:

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

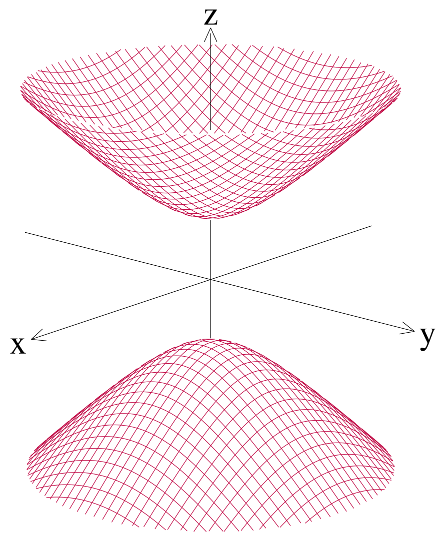


Figure 11.7: Hyperboloid of Two Sheets:
$$\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

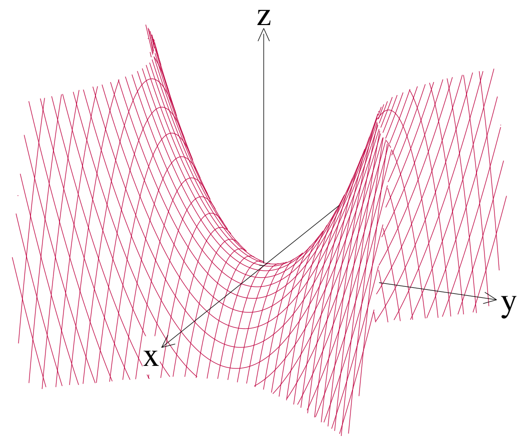
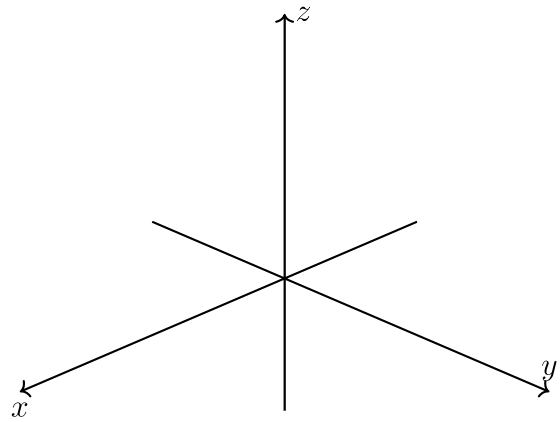


Figure 11.8: Hyperbolic Paraboloid:
$$z = \frac{y^2}{b^2} - \frac{x^2}{a^2}$$

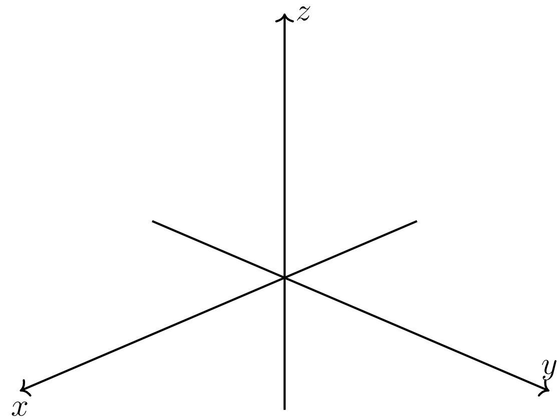
Definition 11.7.2. the curve that results when a surface is cut by a plane is called the **trace** of the surface in the plane.

Homework Assignment
Section 11.7 : (Page 830: 1, 5, 7, 15, 19, 23, 33, 41, 43, 45)

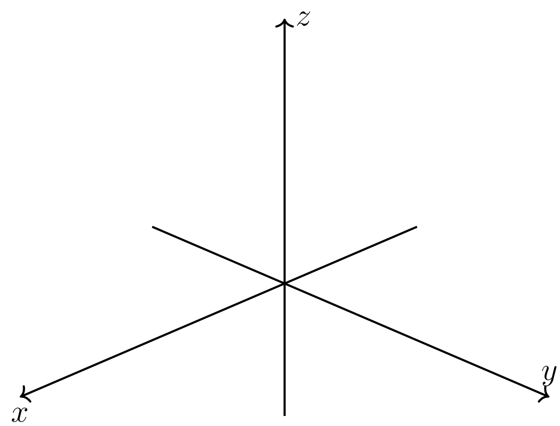
Example 1. Sketch the ellipsoid $\dfrac{x^2}{4} + \dfrac{y^2}{16} + \dfrac{z^2}{9} = 1$.



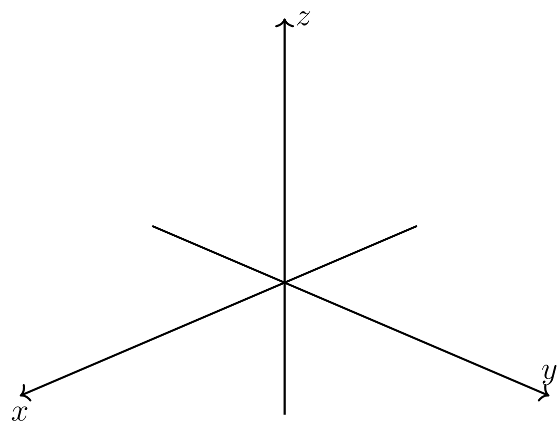
Example 2. Sketch the graph of the hyperboloid of one sheet, $x^2 + y^2 - \dfrac{z^2}{4} = 1$.



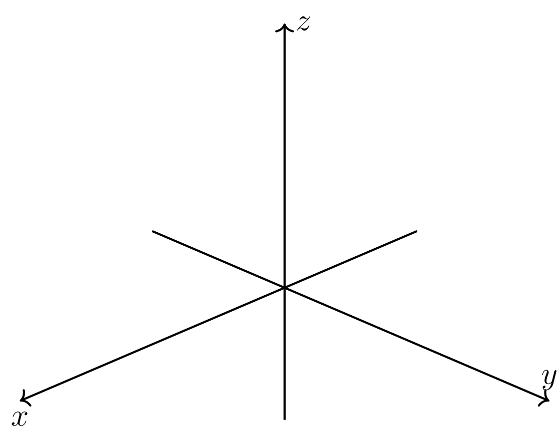
Example 3. Sketch the graph of the hyperboloid of two sheets, $z^2 - x^2 - \dfrac{y^2}{4} = 1$.



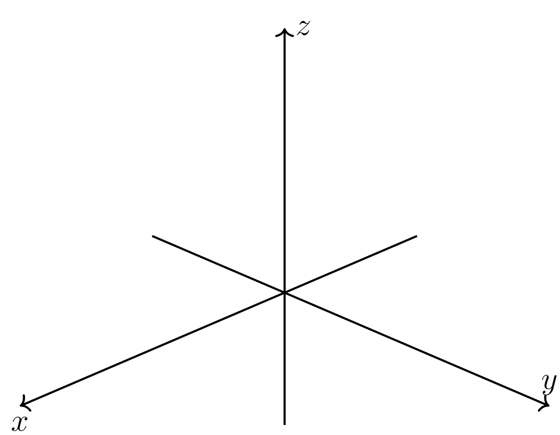
Example 4. Sketch the graph of the elliptic cone, $z^2 = x^2 + \dfrac{y^2}{4}$.



Example 5. Sketch the elliptic paraboloid $z = \dfrac{x^2}{4} + \dfrac{y^2}{9}$.

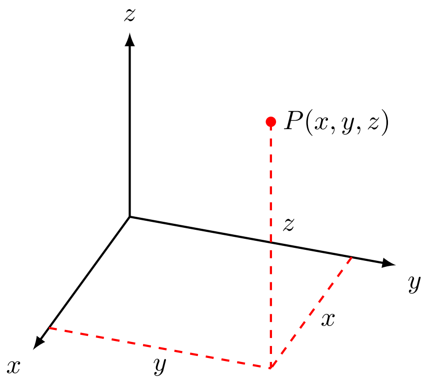


Example 6. Sketch the graph of the hyperbolic paraboloid $z = \dfrac{y^2}{4} - \dfrac{x^2}{9}$.

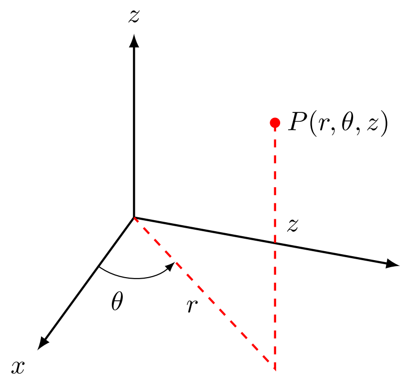


11.8 Cylindrical and Spherical Coordinates

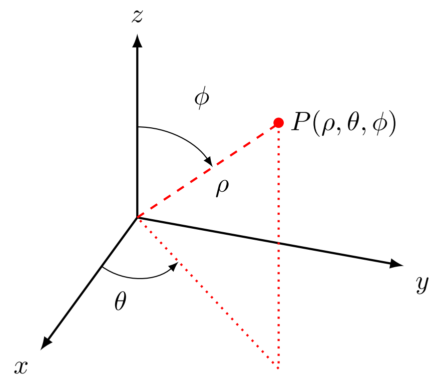
In three space three coordinates are required to find the location of a point. We have already used **rectangular coordinates** (x, y, z), but there are also, **cylindrical coordinates** (r, θ, z), and **spherical coordinates** (ρ, θ, ϕ). See the images below for restrictions on the cylindrical and spherical coordinates.



Rectangular Coordinates
(x, y, z)



Cylindrical Coordinates
(r, θ, z)
$(r \geq 0, 0 \leq \theta < 2\pi, z)$



Sphereical Coordinates
(ρ, θ, ϕ)
$(\rho \geq 0, 0 \leq \theta < 2\pi, 0 \leq \phi \leq \pi)$

Here are some useful conversion formulas for the coordinate systems:

Conversion		**Formulas**
Cylindrical to rectangular	$(r, \theta, z) \to (x, y, z)$	$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$
Rectangular to cylindrical	$(x, y, z) \to (r, \theta, z)$	$r = \sqrt{x^2 + y^2}, \quad \tan\theta = y/x, \quad z = z$
Spherical to cylindrical	$(\rho, \theta, \phi) \to (r, \theta, z)$	$r = \rho\sin\phi, \quad \theta = \theta, \quad z = \rho\cos\phi$
Cylindrical to spherical	$(r, \theta, z) \to (\rho, \theta, \phi)$	$\rho = \sqrt{r^2 + z^2}, \quad \theta = \theta, \quad \tan\phi = r/z$
Spherical to rectangular	$(\rho, \theta, \phi) \to (x, y, z)$	$x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta, \quad z = \rho\cos\phi$
Rectangular to spherical	$(x, y, z) \to (\rho, \theta, \phi)$	$\rho = \sqrt{x^2 + y^2 + z^2}, \quad \tan\theta = y/x,$ $\cos\phi = z/\sqrt{x^2 + y^2 + z^2}$

Here are some of the quadratic surfaces from the previous section represented in different coordinate systems.

	Cone	Cylinder	Sphere	Paraboloid	Hyperboloid
Rectangular	$z = \sqrt{x^2 + y^2}$	$x^2 + y^2 = 1$	$x^2 + y^2 + z^2 = 1$	$z = x^2 + y^2$	$x^2 + y^2 - z^2 = 1$
Cylindrical	$z = r$	$r = 1$	$z^2 = 1 - r^2$	$z = r^2$	$z^2 = r^2 - 1$
Spherical	$\phi = \pi/4$	$\rho = \csc\phi$	$\rho = 1$	$\rho = \cos\phi\csc^2\phi$	$\rho^2 = -\sec 2\phi$

Homework Assignment
Section 11.8 : (Page 836: 1, 3, 5, 7, 9, 11, 19, 23, 25, 29, 33, 37, 43, 47, 53)

Read full handout above.

Note 2. Memorizing formulas is a chore, I suggest memorizing the image below and you can find any conversion formula that you need.

Example 1. (a) Find the rectangular coordinates for the cylindrical point $(r, \theta, z) = \left(5, \frac{\pi}{3}, -7\right)$.

(b) Find the rectangular coordinates for the given spherical coordinate $(\rho, \theta, \phi) = \left(4, \frac{5\pi}{6}, \frac{3\pi}{4}\right)$.

Example 2. Find spherical and cylindrical coordinates given the rectangular coordinates.

$$(x, y, z) = (4, -4, 4\sqrt{6})$$

Example 3. Find the equation of the paraboloid $z = x^2 + y^2$ in both cylindrical and spherical coordinates.

Example 4. Write the equation of the given plane in cylindrical coordinates and in spherical coordinates.

$$x + 2y + 3z = 6$$

Here are some examples for **you** to try to check your understanding.

1. Find the rectangular coordinates given the cylindrical coordinates.
   $(\sqrt{2}, \frac{3\pi}{4}, 2)$

2. Find the cylindrical coordinates given the rectangular coordinates.
   $(2, 2, 2)$

3. Identify the surface and write the equation in cylindrical coordinates.
   $2x^2 + 2y^2 - z^2 = 4$

4. Find the rectangular coordinates given the spherical coordinates.
   $(6, \frac{\pi}{3}, \frac{\pi}{6})$

5. Find the spherical coordinates given the rectangular coordinates.
   $(\sqrt{3}, -1, 2\sqrt{3})$

6. Identify the surface and write the equation in spherical coordinates.
   $z = x^2 - y^2$

Solutions: 1. $(-1, 1, 2)$ 2. $(2\sqrt{2}, \frac{\pi}{4})$ 3. Hyperboloid of one sheet, $2r^2 - z^2 = 4$
4. $(\frac{3}{2}, \frac{3\sqrt{3}}{2}, 3\sqrt{3})$ 5. $(4, \frac{11\pi}{6}, \frac{\pi}{6})$ 6. Hyperbolic paraboloid, $\cos\phi = \rho\sin^2\phi\cos 2\theta$

Chapter 11 Three-Dimensional Space and Vectors Review

Content Overview

11.1: Rectangular Coordinates in 3-space; Spheres and Cylindrical Surfaces

- Distance formula
- Equation for a sphere
- Extrusions or cylindrical surfaces

11.2 Vectors

- Vector algebra properties (Theorem 11.2.4)
- Magnitude of a vector
- Unit Vectors
- Normalization and trigonometric form

11.3 Dot Product and Projections

- Properties of the dot product
- Angle between two vectors
- Directional cosines and projection
- Vector and scalar components in the direction of a perpendicular unit vectors

11.4 Cross Product

- How to find the cross product, its properties, and meaning
- Magnitude of the cross product
- Scalar triple product and its meaning (Theorem 11.4.5)

11.5 Parametric Equations of Lines

- Parametric and vector equations for a line
- Skew lines

11.6 Planes in 3-Space

- Normal vectors
- Point normal form of a plane
- General form of a plane
- Distance between a point and a plane, angle between two planes

11.7 Quadratic Surfaces

- Six surface types discussed in class and their respective equations
- Sketching surfaces by isolating one variable (trace)
- Trace of a surface (Definition 11.7.2)

11.8 Cylindrical and Spherical Surfaces

- Conversions between Rectangular, Cylindrical and Spherical Coordinates
- Convert quadratic surfaces and planes into different coordinate systems
- Restrictions on r, ϕ, and θ

Review Problems for Chapter 11

1. The vector with the same direction and one-half magnitude of $4\mathbf{i} - 6\mathbf{j}$ is ________________.

2. $\left(4, \frac{\pi}{2}, 3\right)$ is a point in cylindrical coordinates. Express this point in spherical coordinates.

3. Find a set of parametric equations for the line through the points $(-2, 0, 3)$ and $(4, 3, 3)$.

4. Lines L_1 and L_2 are parallel. Find the distance between the L_1 and L_2.

$$\begin{array}{llll} L_1: & x = 2t, & y = 3 + 4t, & z = 2 - 6t \\ L_2: & x = 1 + 3t, & y = 6t, & z = -9t \end{array}$$

5. Describe completely and draw the following surfaces:

   A. $r = 4\cos\theta$

   B. $\phi = \dfrac{\pi}{6}$

   C. $\rho\cos\phi = 4$

   D. $\rho = 16$

6. Classify and sketch the surface given by $2x^2 - 3y^2 + z^2 = 12$.

7. Let L_1 have parametric equations $x = 2 + 3t$, $y = 3 - t$, $z = 2t$, and let the line L_2 have parametric equations $x = 4 - s$, $y = 1 + s$, $z = 2 + s$. Determine if L_1 and L_2 intersect, find the point of intersection if it exists.

8. Determine the distance from the point $(1, 3, -2)$ to the plane $2x + y - z = 1$.

9. Consider the triangle ABC where A is the point $(2, 5, -3)$, B is the point $(0, 1, 2)$ and C is the point $(4, 1, 0)$. Find the cosine of angle BAC.

10. Find the volume of the parallelpiped with vertices $A(4, 1, 0)$, $B(6, -2, 1)$, $C(5, 2, 1)$, and $D(5, 2, -1)$.

11. Let $\mathbf{v} = 2\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$, and let $\mathbf{w}$ be a vector of magnitude 2 at an angle of $30°$ to $\mathbf{v}$. Find $\mathbf{v} \cdot \mathbf{w}$, $\|\mathbf{v} \times \mathbf{w}\|$, and find the component of $\mathbf{w}$ which is parallel to $\mathbf{v}$.

12. Consider the points $P(1, 2, 1)$, $Q(3, 4, -2)$, and $R(-1, 2, 2)$.

    A. Find the equation of the plane which contains P, Q, and R.

    B. Find the angle between $\overrightarrow{PQ}$ and $\overrightarrow{PR}$.

13. Describe the following surfaces:

    A. $z^2 - 2z - y = 0$

    B. $x^2 + 2y^2 + 2x - z = 0$

    C. $x^2 - y^2 + z^2 = 10$

    D. $x^2 = y^2 + z^2$

14. Describe the following surfaces:

    A. $r = 8$

    B. $z = -5$

    C. $\theta = 2$

    D. $z = 3r$

    E. $z = r^2$

    F. $\rho = 5$

    G. $\phi = 7$

Chapter 12 Vector-Valued Functions

12.1 Introduction to Vector-Valued Functions

Definition 12.1.1. A **plane curve** is defined as the set of ordered pairs $\big(f(t), g(t)\big)$ together with their defining parametric equations $x = f(t)$ and $y = g(t)$ where f and g are continuous functions of t on some interval $\mathscr{I}$. The **orientation** of the curve is the direction of increasing parameter.

Definition 12.1.2. The previous definition can be extended to 3-space and higher dimensions. A **space curve** is the set of all ordered triples $\big(f(t), g(t), h(t)\big)$ together with their defining parametric equations $x = f(t)$, $y = g(t)$, and $z = h(t)$ where f, g, and h are continuous functions of t on some interval $\mathscr{I}$.

Definition 12.1.3. If we view each point in of a parametric curve as the terminal point for a vector $\mathbf{r}$ whose initial point is at the origin,

$$\mathbf{r} = \langle x, y, z \rangle = \langle f(t), g(t), h(t) \rangle$$

then we obtain $\mathbf{r}$ as a function of the parameter t, $\mathbf{r}(t)$. We say $\mathbf{r}(t)$ defines a **vector-valued function of a real variable**, or more simply, a **vector-valued function**. The functions $f(t)$, $g(t)$, and $h(t)$ are called the **component functions** or the **components** of $\mathbf{r}(t)$.

Definition 12.1.4. The **two-point vector form of a line** that passes through the terminal points of $\mathbf{r}_0$ and $\mathbf{r}_1$ is

$$\mathbf{r} = \mathbf{r_0} + t(\mathbf{r}_1 - \mathbf{r}_0) \qquad \text{or} \qquad \mathbf{r} = (1-t)\mathbf{r}_0 + t\mathbf{r}_1.$$

Homework Assignment
Section 12.1 : (Page 845: 3, 5, 7, 9, 13, 19, 21, 27, 37, 41, 43, 47)

Read **Definitions 12.1.1, 12.1.2**.

Example 1. Describe the parametric curve described by the equations

$$x = a\cos t, \quad y = a\sin t, \quad z = ct$$

where a and c are positive constants.

Read **Definition 12.1.3**.

Example 2. Find the component functions of

$$\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$$

and the domain of $\mathbf{r}(t)$.

Example 3. Find the natural domain of $\mathbf{r}(t) = \langle \ln|t-1|, e^t, \sqrt{t} \rangle$.

Example 4. Describe the graph of the vector valued function $\mathbf{r}(t) = \langle t, \sin t, \cos t \rangle$.

Read **Definition 12.1.4**.

12.2 Calculus of Vector-Valued Functions

Definition 12.2.1. Let $\mathbf{r}(t)$ be a vector-valued function that is defined for all t in some open interval containing the number a, except that $\mathbf{r}(t)$ need not be defined at a.
We will write

$$\lim_{t\to a} \mathbf{r}(t) = \mathbf{L}$$

if and only if

$$\lim_{t\to a} \|\mathbf{r}(t) - \mathbf{L}\| = 0$$

Theorem 12.2.1.

(a) If $\mathbf{r}(t) = \langle x(t), y(t)\rangle$, then

$$\lim_{t\to a} \mathbf{r}(t) = \left\langle \lim_{t\to a} x(t), \lim_{t\to a} y(t) \right\rangle$$

(b) If $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$, then

$$\lim_{t\to a} \mathbf{r}(t) = \left\langle \lim_{t\to a} x(t), \lim_{t\to a} y(t), \lim_{t\to a} z(t) \right\rangle$$

provided the limits of the component functions exist. Conversely, the limits of the component functions exist provided $\mathbf{r}(t)$ *approaches a limiting vector as* t *approaches* a.

Definition 12.2.2. If $\mathbf{r}(t)$ is a vector-valued function, we define the **derivative of r with respect to t** to be the vector-valued function $\mathbf{r}'$ given by

$$\mathbf{r}'(t) = \lim_{h\to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$

The domain of $\mathbf{r}'$ consists of all values of t in the domain of $\mathbf{r}(t)$ for which the limit exists. We say $\mathbf{r}$ is **differentiable** at t if the limit exists.

Note 3. Suppose that $\mathscr{C}$ is the graph of a vector-valued function $\mathbf{r}(t)$ in 2 or 3-space and that $\mathbf{r}'(t)$ exists and is nonzero for a given value of t. If the vector $\mathbf{r}'(t)$ is positioned with its initial point at the terminal point of the radius vector $\mathbf{r}(t)$, then $\mathbf{r}'(t)$ is tangent to $\mathscr{C}$ and points in the direction of increasing parameter.

Theorem 12.2.2. If $\mathbf{r}(t)$ is a vector-valued function, then $\mathbf{r}$ is differentiable at t if and only if each of its component functions is differentiable at t, in which case the component functions of $\mathbf{r}'(t)$ are the derivatives of the corresponding component functions of $\mathbf{r}(t)$.

Definition 12.2.3. Let P be a point on the graph of a vector-valued function $\mathbf{r}(t)$, and let $\mathbf{r}(t_0)$ be the radius vector from the origin to the point P. If $\mathbf{r}'(t_0)$ exists and $\mathbf{r}'(t_0) \neq \mathbf{0}$ then we call $\mathbf{r}'(t_0)$ a **tangent vector** to the graph of $\mathbf{r}(t)$ at $\mathbf{r}(t_0)$, and we call the line through P that is parallel to the tangent vector the **tangent line** to the graph of $\mathbf{r}(t)$ at $\mathbf{r}(t_0)$ and it is

$$\mathbf{T} = \mathbf{r}(t_0) + t\mathbf{r}'(t_0)$$

Theorem 12.2.3 (Rules of Differentiation)**.** Let $\mathbf{r}(t)$, $\mathbf{r}_1(t)$, $\mathbf{r}_2(t)$ be differentiable vector-valued functions that are all in 2-space or all in 3-space, and let $f(t)$ be a differentiable real-valued function, k a scalar, and $\mathbf{c}$ a constant vector (that is, a vector that does not depend on t). The the following rules of differentiation hold:

(a) $\dfrac{d}{dt}[\mathbf{c}] = \mathbf{0}$

(b) $\dfrac{d}{dt}[k\mathbf{r}(t)] = k\dfrac{d}{dt}[\mathbf{r}(t)]$

(c) $\dfrac{d}{dt}[\mathbf{r}_1(t) + \mathbf{r}_2(t)] = \dfrac{d}{dt}[\mathbf{r}_1(t)] + \dfrac{d}{dt}[\mathbf{r}_2(t)]$

(d) $\dfrac{d}{dt}[\mathbf{r}_1(t) - \mathbf{r}_2(t)] = \dfrac{d}{dt}[\mathbf{r}_1(t)] - \dfrac{d}{dt}[\mathbf{r}_2(t)]$

(e) $\dfrac{d}{dt}[f(t)\mathbf{r}(t)] = f(t)\dfrac{d}{dt}[\mathbf{r}(t)] + \dfrac{d}{dt}[f(t)]\mathbf{r}(t)$

(f) $\dfrac{d}{dt}[\mathbf{r}_1(t) \cdot \mathbf{r}_2(t)] = \mathbf{r}_1(t) \cdot \dfrac{d\mathbf{r}_2}{dt} + \dfrac{d\mathbf{r}_1}{dt} \cdot \mathbf{r}_2(t)$

(g) $\dfrac{d}{dt}[\mathbf{r}_1(t) \times \mathbf{r}_2(t)] = \mathbf{r}_1(t) \times \dfrac{d\mathbf{r}_2}{dt} + \dfrac{d\mathbf{r}_1}{dt} \times \mathbf{r}_2(t)$

(h) $\dfrac{d}{dt}[\mathbf{r}(f(t))] = f'(t)\mathbf{r}'(f(t))$

Theorem 12.2.4. If $\mathbf{r}(t)$ is a differentiable vector-valued function in 2-space or in 3-space and $\|\mathbf{r}(t)\|$ is constant for all t, then

$$\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$$

that is, $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ are orthogonal vectors for all t.

Definition 12.2.4. If $\mathbf{r}(t)$ is a vector-valued function that is continuous on the interval $a \leq t \leq b$, then we define the **definite integral** of $\mathbf{r}(t)$ over the interval as the limit of Riemann sums,

$$\int_a^b \mathbf{r}(t)\,dt = \lim_{\max \Delta t_k \to 0} \sum_{k=1}^{n} \mathbf{r}(t_k^*)\Delta t_k$$

and in general

$$\int_a^b \mathbf{r}(t)\,dt = \left\langle \int_a^b x(t)\,dt, \int_a^b y(t)\,dt \right\rangle \qquad \text{in 2-space}$$

$$\int_a^b \mathbf{r}(t)\,dt = \left\langle \int_a^b x(t)\,dt, \int_a^b y(t)\,dt, \int_a^b z(t)\,dt \right\rangle \qquad \text{in 3-space}$$

Theorem 12.2.5. Let $\mathbf{r}(t)$, $\mathbf{r}_1(t)$, and $\mathbf{r}_2(t)$ be vector-valued functions in 2-space or 3-space that are continuous on the interval $a \leq t \leq b$, and let k be a scalar. Then the following rules of integration hold:

(a) $\displaystyle\int_a^b k\mathbf{r}(t)\,dt = k\int_a^b \mathbf{r}(t)\,dt$

(b) $\displaystyle\int_a^b [\mathbf{r}_1(t) + \mathbf{r}_2(t)]\,dt = \int_a^b \mathbf{r}_1(t)\,dt + \int_a^b \mathbf{r}_2(t)\,dt$

(c) $\displaystyle\int_a^b [\mathbf{r}_1(t) - \mathbf{r}_2(t)]\,dt = \int_a^b \mathbf{r}_1(t)\,dt - \int_a^b \mathbf{r}_2(t)\,dt$

Definition 12.2.5. An **antiderivative** for a vector-valued function $\mathbf{r}(t)$ is a vector-valued function $\mathbf{R}(t)$ such that

$$\mathbf{R}'(t) = \mathbf{r}(t)$$

or using integral notation

$$\int \mathbf{r}(t)\,dt = \mathbf{R}(t) + \mathbf{C}$$

where $\mathbf{C}$ represents an arbitrary constant vector.

Note 4. Most of the familiar integration properties have vector-function analogues. For example,

$$\frac{d}{dt}\left[\int \mathbf{r}(t)\,dt\right] = \mathbf{r}(t)$$

$$\int \mathbf{r}'(t)\,dt = \mathbf{r}(t) + \mathbf{C}$$

More over, if $\mathbf{R}(t)$ is an antiderivative of $\mathbf{r}(t)$ on an interval containing $t = a$ and $t = b$, then we have the following form of the Fundamental Theorem of Calculus:

$$\int_a^b \mathbf{r}(t)\,dt = \mathbf{R}(t)\Big|_a^b = \mathbf{R}(b) - \mathbf{R}(a)$$

Homework Assignment
Section 12.2 : (Page 856: 7, 11, 15, 19, 23, 27, 29, 32, 39, 45, 53)

Read **Definition 12.2.1** and **Theorem 12.2.1**. Discuss the limit of a vector valued function. (Revisit the formal definition of the limit of a function).

Example 1. Find $\lim_{t\to 0} \mathbf{r}(t)$ for $\mathbf{r}(t) = \langle t^2, e^t, -2\cos \pi t\rangle$.

Example 2. Let $\mathbf{r}(t) = t^2\mathbf{i} + e^t\mathbf{j} - (2\cos \pi t)\mathbf{k}$, find $\mathbf{r}'(t)$.

Read **Definition 12.2.2** and **Theorem 12.2.2**.

Example 3. Find parametric equations of the tangent line to the circular helix

$$x = \cos t, \quad y = \sin t, \quad z = t$$

where $t = \pi$.

Read **Definition 12.2.3** and **Theorem 12.2.3**.

Example 4. The two vector functions $\mathbf{r}_1(t) = \langle \tan^{-1} t, \sin t, t^2 \rangle$ and $\mathbf{r}_2(t) = \langle t^2 - t, 2t - 2, \ln t \rangle$ intersect at $(0, 0, 0)$. Find the angle measure between their tanget lines at this point.

Read and *prove* **Theorem 12.2.4**.

Read **Definition 12.2.4** and **Theorem 12.2.5** .

Example 5. Let $\mathbf{r}(t) = \langle t^2, e^t, -2\cos \pi t \rangle$. Find

$$\int_0^1 \mathbf{r}(t)\, dt =$$

Read **Definition 12.2.5**.

Example 6. Find $\displaystyle\int \mathbf{r}(t)\,dt$ for $\mathbf{r}(t) = \left\langle 2t, 3t^2, \dfrac{1}{t-2} \right\rangle$.

$$\int \mathbf{r}(t)\,dt =$$

Example 7. Find $\mathbf{r}(t)$ given $\mathbf{r}'(t) = \langle 3, 2t \rangle$ and $\mathbf{r}(1) = \langle 2, 5 \rangle$.

12.3 Change of Parameter; Arc Length

Definition 12.3.1. A curve represented by $\mathbf{r}(t)$ is **smoothly parameterized** by $\mathbf{r}(t)$, or is a **smooth function** of t if $\mathbf{r}'(t)$ is continuous and $\mathbf{r}'(t) \neq \mathbf{0}$ for any defined value of t.

Theorem 12.3.1. Recall from single variable calculus that the arc length L of a parametric curve

$$x = x(t), \qquad y = y(t), \qquad (a \leq t \leq b)$$
$$\mathbf{r}(t) = \langle x(t), y(t) \rangle, \quad \textit{as a vector-valued function}$$

is given by the following formula

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_a^b \|\mathbf{r}'(t)\|\, dt$$

The analogous formula for a parametric curve in 3-space is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt = \int_a^b \|\mathbf{r}'(t)\|\, dt$$

Arc Length as a Parameter

It is often very useful to **parameterize a curve with respect to arc length** since length is dependent on shape not a particular coordinate system. Here are three steps to help understand arc length as a parameter for a curve $\mathscr{C}$:

1. Select an arbitrary point, P, on $\mathscr{C}$ to serve as a **reference point** or **starting point**.
2. Starting from P, choose a direction along $\mathscr{C}$ to be the positive direction and the other, negative.
3. If Q is a point on $\mathscr{C}$, let s be the signed arc length along $\mathscr{C}$ from P to Q, where s is positive if Q is in the positive direction from the reference point and s is negative if Q is in the negative direction from the reference point.

Picture:

Theorem 12.3.2. Let $\mathbf{r}(t)$ be a vector-valued function in 2-space or 3-space that is differentiable with respect to t. If $t = g(\tau)$ is a change of parameter in which g is differentiable with respect to τ, then $\mathbf{r}(g(\tau))$ is differentiable with respect to τ and

$$\frac{d\mathbf{r}}{d\tau} = \frac{d\mathbf{r}}{dt}\frac{dt}{d\tau}$$

Theorem 12.3.3. Let $\mathscr{C}$ be the graph of a smooth vector-valued function $\mathbf{r}(t)$ in 2-space or 3-space, and let $\mathbf{r}(t_0)$ be any point on $\mathscr{C}$. Then the following formula defines a positive change in parameter from t to s, where s is an **arc length parameter** having $\mathbf{r}(t_0)$ as its reference point.

$$s = \int_{t_0}^{t} \left\| \frac{d\mathbf{r}}{du} \right\| du$$

The **arc length function** can be expressed in component form:

$$s(t) = \int_{t_0}^{t} \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2}\, du$$

$$s(t) = \int_{t_0}^{t} \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2}\, du$$

Where s(t) is the arc length of $\mathscr{C}$ between $\mathbf{r}(t_0)$ and $\mathbf{r}(t)$.

Note: The u's in the above equations are for integration purposes only and do not appear in the parameter conversion.

Homework Assignment

Section 12.3 : (Page 866: 3, 5, 9, 13, 19, 21, 25, 29, 35, 39, 43)

Read **Definition 12.3.1**.

Example 1. Determine whether the following vector-valued functions are smooth

(a) $\mathbf{r}(t) = \langle a \sin t, a \cos t, ct \rangle$ where $a > 0$ and $c > 0$.

(b) $\mathbf{r}(t) = \langle t^2, t^3 \rangle$.

Read **Theorem 12.3.1**.

Example 2. Find the arc length of the circular helix

$$x = 2\cos t, \quad y = 2\sin t, \quad z = t$$

from $(2, 0, 0)$ to $(2, 0, 2\pi)$.

Read *Arc Length Parameter*.

Example 3. Find the arc length parameterization of the circle $x^2 + y^2 = a^2$ with a clockwise orientation and $(a, 0)$ as the reference point.

Example 4. Find a change of parameter $t = g(\tau)$ of the circle

$$\mathbf{r}(t) = \langle \cos t, \sin t \rangle \qquad (0 \leq t \leq 2\pi)$$

so that

(a) the circle is traced counter clockwise as τ increases over the interval $[0, 1]$,

(b) the circle is traced clockwise as τ increases over the interval $[0, 1]$.

Example 5. Reparameterize the helix $\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$ with respect to arc length measured from $(1, 0, 0)$ in the direction of increasing parameter.

12.4 Unit Tangent, Normal, and Binormal Vectors

Definition 12.4.1. For the graph $\mathscr{C}$ of a smooth vector-valued function $\mathbf{r}(t)$ in 2-space or 3-space $\mathbf{r}'(t)$ is nonzero and tangent to $\mathscr{C}$ and points in the direction of increasing parameter. By normalizing $\mathbf{r}'(t)$ we obtain

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$$

that is tangent to $\mathscr{C}$. We call $\mathbf{T}(t)$ the **unit tangent vector** to $\mathscr{C}$ at t.

Definition 12.4.2. For the graph $\mathscr{C}$ of a smooth vector-valued function $\mathbf{r}(t)$ in 2-space or 3-space $\mathbf{T}(t)$ has constant norm 1. By **Theorem 12.2.4** $\mathbf{T}'(t)$ is normal to $\mathbf{T}(t)$ for all t. If $\mathbf{T}'(t) \neq \mathbf{0}$ by normalizing

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$$

that is normal to $\mathscr{C}$ and points in the same direction as $\mathbf{T}'(t)$. We call $\mathbf{N}(t)$ the **principal unit normal vector** to $\mathscr{C}$ at t.

<u>Computing **T** and **N** for Curves Parameterized by Arc Length</u>

In the case where $\mathbf{r}$ is parameterized by arc length, the procedure for computing the unit tangent vector $\mathbf{T}$ and the unit normal $\mathbf{N}$ are much simpler than the general case.

$$\mathbf{T}(s) = \mathbf{r}'(s)$$

and consequently the unit normal vector simplifies to

$$\mathbf{N}(s) = \frac{\mathbf{r}''(s)}{\|\mathbf{r}''(s)\|}.$$

Definition 12.4.3. If $\mathscr{C}$ is the graph of a vector-valued function $\mathbf{r}(t)$ in 3-space, then we define the **binormal vector** to $\mathscr{C}$ at t to be

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$$

It follows that $\mathbf{B}(t)$ is orthogonal to both $\mathbf{T}(t)$ and $\mathbf{N}(t)$ and is oriented relative to $\mathbf{T}(t)$ and $\mathbf{N}(t)$ by the right hand rule. Moreover, $\mathbf{T}(t) \times \mathbf{N}(t)$ is a unit vector since

$$\|\mathbf{T}(t) \times \mathbf{N}(t)\| = \|\mathbf{T}(t)\|\|\mathbf{N}(t)\| \sin(\pi/2) = 1$$

Definition 12.4.4. The three vectors $\mathbf{T}(t)$, $\mathbf{N}(t)$, and $\mathbf{B}(t)$ determine three mutually perpendicular planes that intersect in one point on the curve. The **TB**-plane called the **rectifying plane**, the **TN**-plane called the **osculating plane**, and the **NB**-plane called the **normal plane**.

Note 5. The three vectors $\mathbf{T}(t)$, $\mathbf{N}(t)$, and $\mathbf{B}(t)$ are related by the following:

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t), \qquad \mathbf{N}(t) = \mathbf{B}(t) \times \mathbf{T}(t), \qquad \mathbf{T}(t) = \mathbf{N}(t) \times \mathbf{B}(t).$$

Note 6. The binormal vector, $\mathbf{B}(t)$, is expressed in terms of $\mathbf{T}(t)$ and $\mathbf{N}(t)$. $\mathbf{B}(t)$ can alternatively be expressed in terms of $\mathbf{r}(t)$ as

$$\mathbf{B}(t) = \frac{\mathbf{r}'(t) \times \mathbf{r}''(t)}{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}$$

and in the case that $\mathbf{r}$ is parameterized by arc length $\mathbf{B}(t)$ becomes

$$\mathbf{B}(t) = \frac{\mathbf{r}'(s) \times \mathbf{r}''(s)}{\|\mathbf{r}''(s)\|}.$$

Homework Assignment
Section 12.4 : (Page 872: 1, 3, 5, 9, 11, 13, 15, 17, 19)

Read **Definition 12.4.1**.

Example 1. Find the unit tangent vector to the graph $\mathbf{r}(t) = \langle t^2, t^3 \rangle$ at the point $t = 2$.

Read **Definition 12.4.2**.

Example 2. Find $\mathbf{T}(t)$ and $\mathbf{N}(t)$ for the circular helix

$$x = a\cos t, \quad y = a\sin t, \quad z = ct \qquad \text{where } a > 0.$$

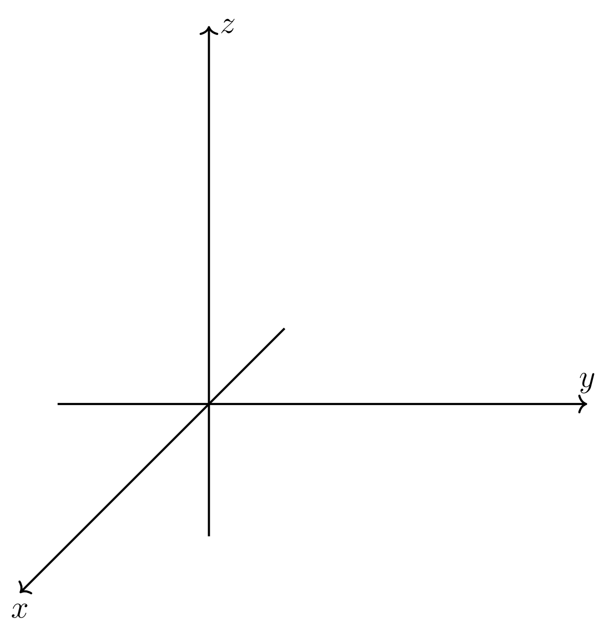
Read *Computing* **T** *and* **N** *for Curves Parameterized by Arc Length.*

Example 3. The circle of radius a with counter clockwise orientation and centered at the origin can be represented by

$$\mathbf{r}(t) = a\cos t\mathbf{i} + a\sin t\mathbf{j} \qquad (0 \leq t \leq 2\pi).$$

Parameterize the circle by arc length and find $\mathbf{T}(t)$ and $\mathbf{N}(t)$.

Read **Definition 12.4.3** and **Note 5**. Sketch the **TNB** frame for a curve in 3 space.



Read **Note 6**

Example 4. Find the unit binormal to the helix in **Example 2**

(a) using $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$,

(b) using $\mathbf{B}(t) = \dfrac{\mathbf{r}'(t) \times \mathbf{r}''(t)}{||\mathbf{r}'(t) \times \mathbf{r}''(t)||}$.

12.5 Curvature

Definition 12.5.1. If $\mathscr{C}$ is a smooth curve in 2-space or 3-space that is parameterized by arc length, then the **curvature** of $\mathscr{C}$, denoted $\kappa = \kappa(s)$ (κ = Greek "kappa"), is defined by

$$\kappa(s) = \left\| \frac{d\mathbf{T}}{ds} \right\| = \|\mathbf{r}''(s)\|$$

Notice that $\kappa(s)$ is a real valued function.

Theorem 12.5.1. If $\mathbf{r}(t)$ is a smooth vector-valued function in 2-space or 3-space, then for each each value of t at which $\mathbf{T}'(t)$ and $\mathbf{r}''(t)$ exist, the curvature κ can be expressed as

(a) $\kappa(t) = \dfrac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|}$

(b) $\kappa(t) = \dfrac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}$

Note 7. The equation (a) is useful when $\mathbf{T}(t)$ is known or easy to find. (b) is most often easier to use since it involves only derivatives of $\mathbf{r}(t)$.

Note 8. For the special case of a plane curve with equation y=f(x), we choose x as the parameter and write $\mathbf{r}(x) = x\mathbf{i} + f(x)\mathbf{j}$. As an exercise prove that the curvature κ is

$$\kappa(x) = \frac{|f''(x)|}{\left[1 + \left(f'(x)\right)^2\right]^{3/2}}.$$

Definition 12.5.2. If a curve $\mathscr{C}$ in 2-space or 3-space has nonzero curvature κ at a point P, then the circle of radius $\rho = 1/\kappa$ sharing a common tangent with $\mathscr{C}$ at P, and centered on the concave side of the curve at P, is called the **osculating circle** or **circle of curvature** at P. The radius ρ of the osculating circle at P is called the **radius of curvature** at P, and the center of the circle is called the **center of curvature** at P.

Note 9 (Formula Summary)**.**

- $\mathbf{T}(s) = \mathbf{r}'(s)$
- $\mathbf{N}(s) = \dfrac{1}{\kappa(s)}\dfrac{d\mathbf{T}}{ds} = \dfrac{\mathbf{r}''(s)}{\|\mathbf{r}''(s)\|} = \dfrac{\mathbf{r}''(s)}{\kappa(s)}$
- $\mathbf{B}(s) = \dfrac{\mathbf{r}'(s) \times \mathbf{r}''(s)}{\|\mathbf{r}''(s)\|} = \dfrac{\mathbf{r}'(s) \times \mathbf{r}''(s)}{\kappa(s)}$

- $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$
- $\mathbf{N}(t) = \mathbf{B}(t) \times \mathbf{T}(t)$
- $\mathbf{B}(t) = \dfrac{\mathbf{r}'(t) \times \mathbf{r}''(t)}{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}$

Homework Assignment
Section 12.5 : (Page 879: 1-9 odd, 13, 17, 19, 21, 27, 33, 47, 55, 57, 59, 63)

Read **Definition 12.5.1**.

Example 1. A circle of radius a centered at the origin can be parameterized be arc length as

$$\mathbf{r}(s) = a\cos\left(\frac{s}{a}\right)\mathbf{i} + a\sin\left(\frac{s}{a}\right)\mathbf{j} \quad (0 \le s \le 2\pi a).$$

Find the curvature, κ, of the circle.

Example 2. Let $\mathbf{r}(s) = \mathbf{r}_0 + s\mathbf{u}$ where $\mathbf{u}$ is a unit vector and $\mathbf{r}_0$ is an initial point for the line $\mathbf{r}(s)$. Find the curvature of $\mathbf{r}(s)$.

Read **Theorem 12.5.1**.

Example 3. Find $\kappa(t)$ for the circular helix

$$x = a\cos t, \quad y = a\sin t, \quad z = ct \quad \text{where } a > 0.$$

Example 4. The graph of the vector equation

$$\mathbf{r}(t) = 2\cos t\mathbf{i} + 3\sin t\mathbf{j} \quad (0 \le t \le 2\pi).$$

Find $\kappa(t)$ of $\mathbf{r}(t)$. What is the curvature at the vertices of the ellipse?

Finish Reading definitions, theorems, and notes for **Section 12.5**.

Example 5. Find the curvature of $f(x) = x^2$ at $(0, 0)$. How does it differ from $g(x) = x^4$? What are the radii of curvature at these points?

12.6 Motion Along a Curve

Definition 12.6.1. If $\mathbf{r}(t)$ is the position function of a particle moving along a curve in 2-space or 3-space, then the **instantaneous velocity, instantaneous acceleration, and instantaneous speed** of the particle at time t are defined by

$$\text{velocity } = \mathbf{v}(t) = \frac{d\mathbf{r}}{dt}$$

$$\text{acceleration } = \mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2}$$

$$\text{speed } = \|\mathbf{v}(t)\| = \frac{ds}{dt}$$

	2-space	3-space
Position	$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$	$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$
Velocity	$\mathbf{v}(t) = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j}$	$\mathbf{v}(t) = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}$
Acceleration	$\mathbf{a} = \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j}$	$\mathbf{a} = \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j} + \frac{d^2z}{dt^2}\mathbf{k}$
Speed	$\|\mathbf{v}(t)\| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$	$\|\mathbf{v}(t)\| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$

Definition 12.6.2. If $\mathbf{r}(t)$ is the position function of a particle moving along a curve in 2-space or 3-space, then the **displacement** of the particle over the time interval $t_1 \leq t \leq t_2$ is commonly denoted by $\Delta\mathbf{r}$ and is defined as

$$\Delta\mathbf{r} = \mathbf{r}(t_2) - \mathbf{r}(t_1)$$

In terms of the velocity function we can obtain displacement by

$$\Delta\mathbf{r} = \int_{t_1}^{t_2} \mathbf{v}(t)\,dt = \int_{t_1}^{t_2} \frac{d\mathbf{r}}{dt}\,dt = \mathbf{r}(t_2) - \mathbf{r}(t_1) \qquad \boxed{\text{displacement}}$$

The total distance traveled is merely the arc length of the position curve

$$s = \int_{t_1}^{t_2} \left\|\frac{d\mathbf{r}}{dt}\right\| d = \int_{t_1}^{t_2} \|\mathbf{v}(t)\|\,dt \qquad \boxed{\text{distance traveled}}$$

Theorem 12.6.1. If a particle moves along a smooth curve $\mathscr{C}$ in 2-space or 3-space, then at each point on the curve velocity and acceleration vectors can be written as

$$\mathbf{v} = \frac{ds}{dt}\mathbf{T} \qquad \mathbf{a} = \frac{d^2s}{dt^2}\mathbf{T} + \kappa\left(\frac{ds}{dt}\right)^2\mathbf{N}$$

where s is an arc length parameter for the curve, and $\mathbf{T}$, $\mathbf{N}$, and κ denote the unit tangent, unit normal vectors, and curvature at the point.

Note 10. The coefficents of $\mathbf{T}$ and $\mathbf{N}$ in the above equation for acceleration, $\mathbf{a}$, are commonly denoted by

$$a_T = \frac{d^2s}{dt^2} \qquad\qquad a_N = \kappa\left(\frac{ds}{dt}\right)^2$$

in which case we can represent acceleration as

$$\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$$

If both $\mathbf{a}$ and a_T are known we can express a_N as

$$a_N = \sqrt{\|a\|^2 - a_T^2}$$

Theorem 12.6.2. If a particle moves along a smooth curve $\mathscr{C}$ in 2-space or 3-space, then at each point on the curve the velocity $\mathbf{v}$ and the acceleration $\mathbf{a}$ are related to a_T, a_N, and κ by the formulas

$$a_T = \frac{\mathbf{v}\cdot\mathbf{a}}{\|\mathbf{v}\|} \qquad\qquad a_N = \frac{\|\mathbf{v}\times\mathbf{a}\|}{\|\mathbf{v}\|} \qquad\qquad \kappa = \frac{\|\mathbf{v}\times\mathbf{a}\|}{\|\mathbf{v}\|^3}$$

Definition 12.6.3 (Projectile Motion in Vector Form)**.** Suppose a projectile is launched with initial velocity vector $\mathbf{v}_0$, initial height s_0, subject to the force of gravity, g. The vector equation that describes the position of the projectile can be written as

$$\mathbf{r}(t) = (-\tfrac{1}{2}gt^2 + s_0)\mathbf{j} + t\mathbf{v}_0 \tag{12.3}$$

Definition 12.6.4 (Parametric Equations for Projectile Motion)**.** Suppose a projectile is launched with speed v_0 and angle α with the horizon (positive x-axis) then the horizontal and vertical components of the velocity are given by

$$v_x = v_0\cos\alpha \qquad\qquad v_y = v_0\sin\alpha$$

or in vector form

$$\mathbf{v}_0 = v_x\,\mathbf{i} + v_y\,\mathbf{j} = v_0\cos\alpha\,\mathbf{i} + v_0\sin\alpha\,\mathbf{j}$$

By substituting this into equation **12.3** we have

$$\mathbf{r}(t) = (v_0\cos\alpha)t\mathbf{i} + \big(s_0 + (v_0\sin\alpha)t - \tfrac{1}{2}gt^2\big)\mathbf{j}$$

Homework Assignment
Section 12.6 : (Page 891: 1, 5, 9, 11, 15, 19, 23, 27, 31, 35, 37, 39, 41, 43, 45, 52, 54, 61, 67)

Read **Definition 12.6.1** and read the table.

Example 1. A prarticle moves along a circular path in such a way that its x and y coordinates at time t are

$$x = 2\cos t \qquad y = 2\sin t$$

(a) Find velocity and speed of the particle at any time t.

(b) Sketch the path of the particle (show position and velocity vectors at $t = \frac{\pi}{4}$)

(c) Show that at each instant the acceleration vector is perpendicular to velocity vector.

Example 2. A particle moves through space so that $\mathbf{v}(t) = \mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k}$. Find the position $\mathbf{r}(t)$ of the particle given the initial position at $t = 2$ is $(-1, 2, 4)$.

Read **Definition 12.6.2**.

Example 3. A particle moves along a circular helix in 3-space so that its position vector is $\mathbf{r}(t) = (4\cos \pi t)\mathbf{i} + (4\sin \pi t)\mathbf{j} + t\mathbf{k}$. Find total distance traveled and displacement on the interval $1 \le t \le 5$.

Read **Theorem 12.6.1 and 12.6.2**.

Example 4. The position of a particle is $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$

(a) Find $a_{\mathbf{N}}$ and $a_{\mathbf{T}}$ at t.

(b) Find $a_{\mathbf{N}}$ and $a_{\mathbf{T}}$ at $t = 1$.

Example 4. continued...

(c) Find the tangential and normal vector components of acceleration at $t = 1$.

(d) Find the curvature of $\mathbf{r}(t)$ at $t = 1$.

Multivariable Calculus Chapter 12 Vector Valued Fucntions

Content Overview

12.1 Intro to Vector Valued Functions

- Real valued verses vector valued functions
- Component functions
- Direction and orientation of a graph

12.2 Calculus of Vector Valued Functions

- Limits of vector valued functions
- Derivatives of vector valued functions
- Derivatives of dot and cross products
- Tangent line to a curve
- Integrals of vector valued functions

12.3 Change of Parameter; Arc Length

- Smoothly parameterized
- Arc length, arc length parameterization

12.4 Unit Tangent, Normal, and Binormal Vectors

- Find unit tangent, normal, and binormal functions given a curve
- Rectifying, Osculating, and normal planes

12.5 Curvature

- Formulas for curvature: $\kappa = \frac{\|\mathbf{r}'(t)\times\mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}$
- Curvature relating to $\mathbf{N}(t)$ and $\mathbf{B}(t)$
- Radius of Curvature

12.6 Motion Along a Curve

- Position, velocity, speed, and acceleration
- Normal and tangential components of acceleration
- Total distance and displacement

Supplementary Textbook Problems:

page 902 # 1-5, 8, 11, 13, 17, 20, 21, 22, 25;

1. What is the natural domain of $\mathbf{r}(t) = \left\langle 2t, -\frac{1}{t}, \sqrt{t+t} \right\rangle$

2. For $\mathbf{r}(t) = \langle \sin t, t^2 + 4, e^t \rangle$, find $\lim_{t \to 0} \mathbf{r}(t)$

3. Find the unit tangent vector to the curve $\mathbf{r}(t) = \langle t^2, t^3, t^4 \rangle$ at $t = 1$.

4. Find $\mathbf{r}(t)$ if $\mathbf{r}'(t) = \langle e^{2t}, -1, t^2 \rangle$ and $\mathbf{r}(0) = \mathbf{i} - \mathbf{j}$.

5. a) Find the arc length function for the curve $\mathscr{C}$ given by $\mathbf{r}(t) = \langle 1 - t^2, t^2, \sqrt{2}t^2 \rangle$ with intial point $t = 1$. Find arc length from $t = 1$ to $t = 4$.

   b) Reparametrize the curve $\mathscr{C}$ using the arc length function with $t = 0$ as a reference point.

6. Find the curvature at $t = 1$ for the curve $\mathbf{r}(t) = \left\langle t^2, \frac{2}{3}t^3, 2t \right\rangle$.

7. Find the curvature of $y = x^2$ and $x = \sqrt{6}$.

8. For $\mathbf{r}(t) = 2t\mathbf{i} - \frac{t^2}{2}\mathbf{j} + t\mathbf{k}$ find

   a) the unit tangent $\mathbf{T}$ at $t = 2$.

   b) the unit normal $\mathbf{N}$ at $t = 2$.

   c) the binormal $\mathbf{B}$ at $t = 2$

9. Find the velocity, speed, and acceleration of a particle whose position at time t is $\mathbf{r}(t) = \langle 3t^2, t^3 + 1, e^{-t} \rangle$

10. Find the velocity and position at time t of a particle whose initial position is $\mathbf{i} + \mathbf{j}$, initial velocity is $\mathbf{j} + \mathbf{k}$, and acceleration is $\mathbf{a}(t) = 12t\mathbf{i} + 2\mathbf{j}$.

11. Find the scalar tangential and normal components of acceleration for $\mathbf{r}(t) = \langle t^2, t^4, t^3 \rangle$.

12. For the given instantaneous velocity and acceleration find a_T, a_N, $\mathbf{T}$, and $\mathbf{N}$:
$\mathbf{v} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$, and $\mathbf{a} = \mathbf{i} + 2\mathbf{k}$

Solution in text: Section 12.6 problem # 38

Chapter 13 Partial Derivatives

13.1 Functions of Two or More Variables

Definition 13.1.1. A **function** f **of two variables**, x and y, is a rule that assigns a unique real number $f(x,y)$ to each point (x,y) in some set D in the xy-plane.

Definition 13.1.2. A **function f of three variables**, x, y, and z, is a rule that assigns a unique real number $f(x,y,z)$ to each point (x,y,z) in some set D in three-dimensional space.

Definition 13.1.3. A contour map of a surface is constructed by passing planes of constant elevation through the surface, projecting the resulting contours onto a flat surface, labeling the contours with their elevation.

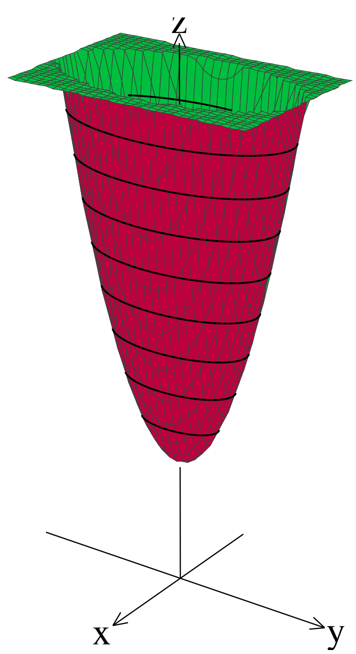


Figure 13.1: Surface with level curves shown

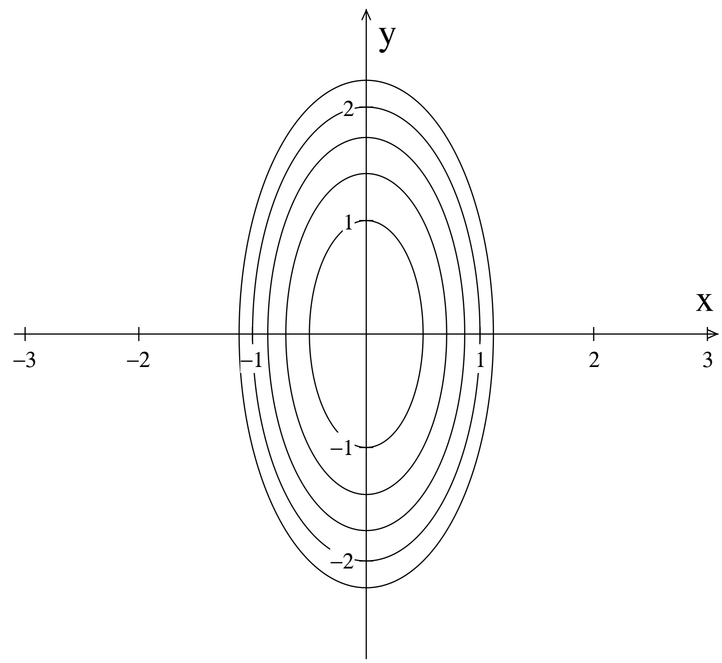


Figure 13.2: Contour Plot: k=1,2,3,4,5

Sometimes a contour plot is called a **contour map** and the curves of intersection are called the **level curves of height k**.

Homework Assignment
Section 13.1 : (Page 914: 1, 2, 10, 11, 15, 16, 17, 23, 25, 27, 35, 43, 47, 49, 50, 51, 57, 65)

Read **Definition 13.1.1** and **13.1.2**.

Example 1. Let $f(x, y) = \sqrt{y+1} + \ln(x^2 - y)$. Find $f(e, 0)$ and sketch the domain of f.

Example 2. Let $f(x, y, z) = \sqrt{1 - x^2 - y^2 - z^2}$ find $f(0, \frac{1}{2}, -\frac{1}{2})$ and the natural domain of f.

Example 3. Describe the graph of the function xyz-coordinate system

(a) $f(x, y) = 1 - x - \frac{1}{2}y$

(b) $g(x, y) = \sqrt{1 - x^2 - y^2}$

Read **Definition 13.1.3** and discuss level sets (level curves and surfaces) and contour plots.

Example 4. Describe the level curves for

(a) $f(x, y) = \sqrt{x^2 + y^2}$

(b) $g(x, y) = -2x + y$

Example 5. Write the equation for the level surface at the point $(1, 2, 3)$ of $f(x, y, z) = z^2y + xy^2$.

13.2 Limits and Continuity

Definition 13.2.1. *[Open and Closed Sets]* Let $\mathscr{C}$ be a circle in 2-space that is centered at (x_0, y_0) and has positive radius δ. The set of points enclosed by the circle, but not on the circle, is called the **open disk** of radius δ centered at (x_0, y_0), and the set of points enclosed by the circle together with those one that lie on the circle is called the **closed disk** of radius δ centered at (x_0, y_0).

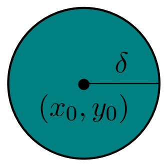


Figure 13.3: Closed Disk

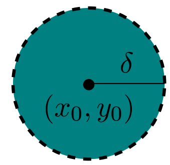


Figure 13.4: Open Disk

If S is a sphere in 3-space that is centered at (x_0, y_0, z_0) and has positive radius δ, then the set of points that are enclosed by the sphere, but do not lie on the sphere, is called the **open ball** of radius δ centered at (x_0, y_0, z_0), and the set of points that are enclosed by the sphere together with the points that lie on the sphere is called the **closed ball** of radius δ centered at (x_0, y_0, z_0).

Theorem 13.2.1. Let f be a function of two variables, and assume that f is defined at all points of some open disk centered at (x_0, y_0), except possibly at (x_0, y_0). We will write

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$$

if given any number $\epsilon > 0$, we can find a number $\delta > 0$ such that $f(x, y)$ satisfies

$$|f(x, y) - L| < \epsilon$$

whenever the distance between (x, y) and (x_0, y_0) satisfies

$$0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$$

Theorem 13.2.2.

(a) If $f(x, y) \to L$ as $(x, y) \to (x_0, y_0)$, then $f(x, y) \to L$ as $(x, y) \to (x_0, y_0)$ along any smooth curve.

(b) If the limit of $f(x, y)$ fails to exist as $(x, y) \to (x_0, y_0)$ along some smooth curve, or if $f(x, y)$ has different limits as $(x, y) \to (x_0, y_0)$ along two different smooth curves, then the limit of $f(x, y)$ does not exist as $(x, y) \to (x_0, y_0)$.

Definition 13.2.2. A function $f(x, y)$ is said to be **continuous at** $(\mathbf{x_0}, \mathbf{y_0})$ if $f(x_0, y_0)$ is defined and if

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$$

Also, if f is continuous at every point in an open set D, then we say that f is **continuous on D**, and if f is continuous at every point on the xy-plane we say that f is **continuous everywhere**.

Theorem 13.2.3.

(a) If $g(x)$ is continuous at x_0 and $h(y)$ is continuous at y_0, then $f(x,y) = g(x)h(y)$ is continuous at (x_0, y_0).

(b) If $h(x,y)$ is continuous at (x_0, y_0) and $g(u)$ is continuous at $u = h(x_0, y_0)$, then the composition $f(x,y) = g(h(x,y))$ is continuous at (x_0, y_0).

(c) If $f(x,y)$ is continuous at (x_0, y_0), and if $x(t)$ and $y(t)$ are continuous at t_0 with $x(t_0) = x_0$ and $y(t_0) = y_0$, then the composition $f(x(t), y(t))$ is continuous at t_0.

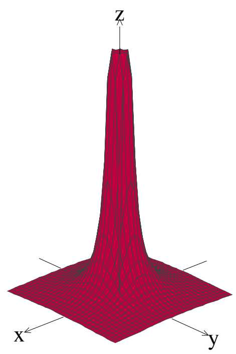


Figure 13.5: Infinte at the origin

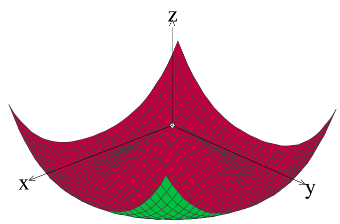


Figure 13.6: Hole at the origin

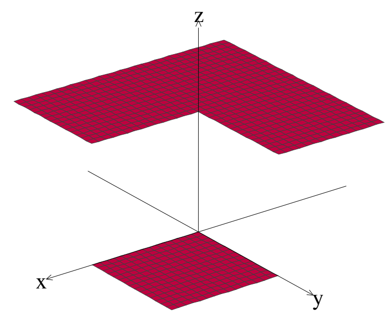


Figure 13.7: Vertical jump at the origin

Definition 13.2.3. Let f be a function of three variables, and assume that f is defined at all points within some open ball centered at (x_0, y_0, z_0), except possibly at (x_0, y_0, z_0). We will write

$$\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x,y,z) = L$$

if given any number $\epsilon > 0$, we can find a number $\delta > 0$ such that $f(x,y)$ satisfies

$$|f(x,y,z) - L| < \epsilon$$

whenever the distance between (x,y,z) and (x_0, y_0, z_0) satisfies

$$0 < \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2} < \delta$$

Homework Assignment
Section 13.2 : (Page 925: 1, 3, 7, 8, 9, 11, 13, 15, 16, 17, 19, 24, 33, 34, 41, 43, 50)

Read **Definition 13.2.1** and **Theorem 13.2.1**. Discuss the definition of a multivariable limit.

Example 1. Find the following limit

$$\lim_{(x,y)\to(1,4)} 5x^3y^2 - 9 =$$

Read **Theorem 13.2.2**

Example 2. Find the limit of $f(x,y) = \dfrac{-xy}{x^2+y^2}$ at $(0,0)$ along the:

(a) x-axis,

(b) y-axis,

(c) line $y = x$.

Example 3. Determine whether the limit exists. If so, find the value.

(a) $$\lim_{(x,y)\to(0,0)} \frac{x^3 y}{2x^6 + y^2} =$$

(b) $$\lim_{(x,y)\to(0,0)} \frac{x^4 - 25y^2}{x^2 + 5y} =$$

(c) $$\lim_{(x,y,z)\to(0,0,0)} \frac{\sin(x^2 + y^2 + z^2)}{\sqrt{x^2 + y^2 + z^2}} =$$

(d) $$\lim_{(x,y,z)\to(0,0,0)} \frac{e^{\sqrt{x^2+y^2+z^2}}}{\sqrt{x^2 + y^2 + z^2}} =$$

(e) $$\lim_{(x,y)\to(0,0)} \frac{x^2 y^2}{\sqrt{x^2 + y^2}} =$$

(f) $$\lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{\sqrt{x^2 + y^2 + z^2}} =$$

(g) $$\lim_{(x,y,z)\to(0,0,0)} \frac{yz}{\sqrt{x^2 + 4y^2 + 9z^2}} =$$

13.3 Partial Derivatives

Definition 13.3.1. The **partial derivative of f with respect to x at** (x_0, y_0) is the derivative at x_o of the function that results when $y = y_0$ is held fixed and x is allowed to vary. This partial derivative is denoted by $f_x(x_0, y_0)$ and is given by

$$f_x(x_0, y_0) = \frac{d}{dx}[f(x, y_0)]\bigg|_{x=x_0} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}.$$

The **partial derivative of f with respect to y at** (x_0, y_0) is the derivative at y_o of the function that results when $x = x_0$ is held fixed and y is allowed to vary. This partial derivative is denoted by $f_y(x_0, y_0)$ and is given by

$$f_y(x_0, y_0) = \frac{d}{dy}[f(x_0, y)]\bigg|_{y=y_0} = \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y}.$$

We will call $f_x(x_0, y_0)$ the slope of the surface in the x-direction at (x_0, y_0).
We will call $f_y(x_0, y_0)$ the slope of the surface in the y-direction at (x_0, y_0).

Definition 13.3.2. The **partial derivative of a function f with respect to** x in general is,

$$f_x(x, y) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}.$$

The **partial derivative of a function f with respect to** y in general is,

$$f_y(x, y) = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}.$$

Note 11. If $z = f(x, y)$ is a function with partial derivatives f_x and f_y then the partials are also denoted by the symbols,

$$\frac{\partial f}{\partial x}, \quad \frac{\partial z}{\partial x} \text{ and } \frac{\partial f}{\partial y}, \quad \frac{\partial z}{\partial y}.$$

Some notation for the partials of $z = f(x, y)$ at a point (x_0, y_0) are

$$\frac{\partial f}{\partial x}\bigg|_{x=x_0, y=y_0}, \quad \frac{\partial z}{\partial x}\bigg|_{(x_0,y_0)}, \quad \frac{\partial f}{\partial x}\bigg|_{(x_0,y_0)}, \quad \frac{\partial f}{\partial x}(x_0, y_0), \quad \frac{\partial z}{\partial x}(x_0, y_0).$$

Note 12. Higher-Order Partial Derivatives: There are four possible second order partial derivatives of a function $f(x, y)$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xx}, \quad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = f_{yy}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{xy}, \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{yx}.$$

Homework Assignment
Section 13.3 : (Page 936: 1, 3, 8, 12, 13, 15, 16, 25, 29, 31, 38, 40, 43, 47, 51, 55, 59, 63, 65)

Read **Definition 13.3.1** and **Definition 13.3.2**

Example 1. Find $f_x(1,3)$ and $f_y(1,3)$ for the function $f(x,y) = 2x^3y^2 + 2y + 4x$.

Example 2. Find $f_x(x,y)$ and $f_y(x,y)$ for $f(x,y)$ from **Example 1**. Use the partial derivatives to find $f_x(1,3)$ and $f_y(1,3)$.

Read **Note 11** and **12** .

Example 3. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ if $z = x^4 \sin(xy^3)$.

Example 4. Let $f(x, y) = x^2y + 5y^3$. Find the slope of the surface $z = f(x, y)$ in the x and y-direction at the point $(1, -2)$.

Example 5. Find the slope of the sphere $x^2 + y^2 + z^2 = 1$ in the x and y-direction at the point $(\frac{2}{3}, \frac{1}{3}, \frac{2}{3})$.

Example 6. Find all the second order partial derivatives of $f(x, y) = x^2y^3 + x^4y$.

Theorem 13.3.1. Let f be a function of two variables. If f_{xy} and fyx are continuous on some open disk, then $f_{xy} = f_{yx}$ on that disk.

Extra Practice with Partials

Show all work on another sheet of paper.

Find each the partial derivatives of the following functions:

1. $f(x,y) = 3x - 2y^4$
2. $z = xe^{3y}$
3. $f(x,y,z) = xy^2z^3 + 3yz$
4. $u = xe^{-t}\sin\theta$
5. $u = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$
6. $f(x,y) = x^y$

Use implicit differentiation to find $\partial z/\partial x$ and $\partial z/\partial y$:

7. $xy + yz = xz$
8. $x^2 + y^2 - z^2 = 2x(y+z)$
9. $1 = x^2 + y^2 - z^2$
10. $xyz = cos(x+y+z)$

Find all second partial derivatives of the following functions:

11. $f(x,y) = x^4 - 3x^2y^3$
12. $f(x,y) = \ln(3x + 5y)$

Solutions to Extra Practice:

1. $f(x,y) = 3x - 2y^4$
   $f_x(x,y) = 3$
   $f_y(x,y) = -8y^3$

2. $z = xe^{3y}$
   $\partial z/\partial x = e^{3y}$
   $\partial z/\partial y = 3xe^{3y}$

3. $f(x,y,z) = xy^2z^3 + 3yz$
   $f_x(x,y,z) = y^2z^3$
   $f_y(x,y,z) = 2xyz^3 + 3z$
   $f_z(x,y,z) = 3xy^2z^2 + 3y$

4. $u = xe^{-t}\sin\theta$
   $\partial u/\partial x = e^{-t}\sin\theta$
   $\partial u/\partial t = -xe^{-t}\sin\theta$
   $\partial u/\partial\theta = xe^{-t}\cos\theta$

5. $u = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$

   $$\frac{\partial u}{\partial x_i} = \frac{x_i}{\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}}$$

6. $f(x,y) = x^y$
   $f_x(x,y) = yx^{y-1}$
   $f_y(x,y) = x^y \ln x$

---

7. $xy + yz = xz$

   $$\frac{\partial z}{\partial x} = \frac{z-y}{y-x}$$

   $$\frac{\partial z}{\partial y} = \frac{x+z}{x-y}$$

8. $x^2 + y^2 - z^2 = 2x(y+z)$

   $$\frac{\partial z}{\partial x} = \frac{x-y-z}{x+z}$$

   $$\frac{\partial z}{\partial y} = \frac{y-x}{x+z}$$

9. $1 = x^2 + y^2 - z^2$

   $$\frac{\partial z}{\partial x} = \frac{x}{z}$$

   $$\frac{\partial z}{\partial y} = \frac{y}{z}$$

10. $xyz = cos(x+y+z)$

    $$\frac{\partial z}{\partial x} = -\frac{yz + \sin(x+y+z)}{xy + \sin(x+y+z)}$$

    $$\frac{\partial z}{\partial y} = -\frac{xz + \sin(x+y+z)}{xy + \sin(x+y+z)}$$

---

11. $f(x,y) = x^4 - 3x^2y^3$
    $f_{xx} = 12x^2 - 6y^3$

    $f_{xy} = -18xy^2$

    $f_{yx} = -18xy^2$

    $f_{yy} = -18x^2y$

12. $f(x,y) = \ln(3x+5y)$

    $$f_{xx} = -\frac{9}{(3x+5y)^2}$$

    $$f_{xy} = -\frac{15}{(3x+5y)^2}$$

    $$f_{yx} = -\frac{15}{(3x+5y)^2}$$

    $$f_{yy} = -\frac{25}{(3x+5y)^2}$$

13.4 Differentiability, Differentials, and Local Linearity

Definition 13.4.1. For a function $f(x, y)$, the symbol Δf, called the **increment** of f, denotes the change in the value of $f(x, y)$ that results when (x, y) varies from some initial position (x_0, y_0) to some new position $(x_0 + \Delta x, y_0 + \Delta y)$; thus

$$\Delta f = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0).$$

Furthermore, if both $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist then we can make the approximation

$$\Delta f \approx f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y.$$

Definition 13.4.2. A function f of two variables is said to be **differentiable at a point** (x_0, y_0) provided $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ both exist and

$$\lim_{(\Delta x, \Delta y)\to(0,0)} \frac{\Delta f - f_x(x_0, y_0)\Delta x - f_y(x_0, y_0)\Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = 0$$

Definition 13.4.3. If a function f of two variables is differentiable at each point of a region R in the xy-plane, then we say that f is **differentiable on R**; and if f is differentiable at every point in the xy-plane, then we say that f is **differentiable everywhere**

Theorem 13.4.1. If a function is differentiable at a point, then it is continuous at that point. (proof page 943)

Theorem 13.4.2. If all first order partial derivatives of f exist and are continuous at a point, then f is differentiable at that point.

Definition 13.4.4. If $z = f(x, y)$ is differentiable at a point (x_0, y_0), we let

$$dz = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy$$

denote a new function with dependent variable dz and independent variables dx and dy. We refer to this function as the **total differential of** z.

Definition 13.4.5. When $f(x, y)$ is differentiable at (x_0, y_0) we get

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

and refer to $L(x, y)$ as the **local linear approximation to** f **at** (x_0, y_0)

Homework Assignment
Section 13.4 : (Page 947: 1, 2, 3, 5, 6, 7, 8, 11, 12, 13, 17, 18, 21, 22, 25, 26, 31, 33, 35, 39, 41, 42, 43, 45, 46, 47, 49, 53, 54, 57)

Discuss the definition of differentials.

Read **Definition 13.4.1, 13.4.3, 13.4.2**, and **13.4.4**. Also, read **Theorem 13.4.1** and **13.4.2**.

Example 1. Find the total differential of $z = f(x, y) = e^{xy}$.

Example 2. Find the total differential, dz, of $z = f(x, y) = \sin(x^2y^3)$.

Example 3. Use the total differential to approximate the change in the values of f from P to Q where

$$f(x,y) = \frac{x+y}{xy} \qquad P(-1,-2) \qquad Q(-1.02,-2.04)$$

Read **Definition 13.4.5**.

Example 4. Find the local linear approximation L to $f(x,y) = x\sin y$ at $P(1,\pi)$. Use L to approximate f at $Q(1.01, 3.15)$.

Example 5. Find the local linear approximation L to $f(x,y) = \ln xy$ at $P(1,2)$. Use L to approximate f at $Q(1.01, 2.02)$.

13.5 The Chain Rule

Theorem 13.5.1. If $x = x(t)$ and $y = y(t)$ are differentiable at t, and if $z = f(x, y)$ is differentiable at the point $(x, y) = (x(t), y(t))$, then $z = f(x(t), y(t))$ is differentiable at t and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

where the ordinary derivatives are evaluated at t and the partial derivatives are evaluated at (x, y). Similarly, if $w = f(x(t), y(t), z(t))$ is differentiable at t then

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

where the ordinary derivatives are evaluated at t and the partial derivatives are evaluated at (x, y, z).

Theorem 13.5.2. If $x = x(u, v)$ and $y = y(u, v)$ have first-order partial derivatives at the point (u, v), and if $z = f(x, y)$ is differentiable at the point $(x, y) = (x(u, v), y(u, v))$, then $z = f(x(u, v), y(u, v))$ has first order partial derivatives at the point (u, v) given by

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \quad \text{and} \quad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}.$$

If $x = x(u, v)$, $y = y(u, v)$, and $z(u, v)$ have first-order partial derivatives at the point (u, v), and if $w = f(x, y, z)$ is differentiable at the point $(x, y, z) = (x(u, v), y(u, v), z(u, v))$, then $w = f(x(u, v), y(u, v), z(u, v))$ has first order partial derivatives at the point (u, v) given by

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u} \quad \text{and} \quad \frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}.$$

Theorem 13.5.3. If the equation $f(x, y) = c$ defines y implicitly as a differentiable function of x, and if $\partial f/\partial y \neq 0$, then

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}.$$

Theorem 13.5.4. If the equation $f(x, y, z) = c$ defines z implicitly as a differentiable function of x and y, and if $\partial f/\partial z \neq 0$, then

$$\frac{\partial z}{\partial x} = -\frac{\partial f/\partial x}{\partial f/\partial z} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{\partial f/\partial y}{\partial f/\partial z}.$$

Homework Assignment
Section 13.5 : (Page 956: 1, 2, 3, 7, 9, 11, 13, 15, 16, 17, 19, 24, 25, 31, 34, 36)

Read **Theorem 13.5.1**

Example 1. Suppose $z = x^2y$, $x = t^2$ and $y = t^3$. Use the chain rule to find dz/dt. Check your answer by expressing z in terms of t directly and then differentiating.

Example 2. If $f(x, y) = \dfrac{\sin(x^2y)}{x}$. Find $\dfrac{df}{dt}$ if $x(t) = t^2 + 1$ and $y(t) = 5 - t$.

Read **Theorem 13.5.2**

Example 3. Find $\dfrac{\partial z}{\partial u}$ and $\dfrac{\partial z}{\partial v}$ if $z = e^{xy}$, $x = 2u + v$ and $y = \dfrac{u}{v}$.

Example 4. Given that $x^3 + y^2x - 3 = 0$ find $\frac{dy}{dx}$ using partial derivatives. Check your answer using implicit differentiation.

Example 5. Consider the sphere $x^2 + y^2 + z^2 = 1$. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at the point $\left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$.

Example 6. Let $w = x^2 + y^2 + z^2$ where $z = x^2 + y^2$.

(a) Find $\frac{\partial w}{\partial x}$ by eliminating z, using $z = x^2 + y^2$.

(b) Find $\frac{\partial w}{\partial x}$ by using the chain rule.

(c) Find $\frac{\partial w}{\partial x}$ by thinking of x, y, and z as independent.

(d) Find $\frac{\partial w}{\partial x}$ by thinking of x and z as independent variables and replacing $y^2 = z - x^2$.

13.6 Directional Derivatives and Gradients

Definition 13.6.1. If $f(x, y)$ is a function of x and y, and if $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ is a unit vector, then the **directional derivative of f in the direction of $\mathbf{u}$** at (x_0, y_0) is denoted by $D_{\mathbf{u}}f(x_0, y_0)$ and is defined by

$$D_{\mathbf{u}}f(x_0, y_0) = \frac{d}{ds}[f(x_0 + su_1, y_0 + su_2)]_{s=0}$$

provided this derivative exists.

Definition 13.6.2. If $f(x, y, z)$ is a function of x, y and z, and if $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ is a unit vector, then the **directional derivative of f in the direction of $\mathbf{u}$** at (x_0, y_0, z_0) is denoted by $D_{\mathbf{u}}f(x_0, y_0, z_0)$ and is defined by

$$D_{\mathbf{u}}f(x_0, y_0, z_0) = \frac{d}{ds}[f(x_0 + su_1, y_0 + su_2, z_0 + su_3)]_{s=0}$$

provided this derivative exists.

Theorem 13.6.1.

(a) If $f(x, y)$ is differentiable at (x_0, y_0), and if $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ is a unit vector, then the directional derivative $D_{\mathbf{u}}f(x_0, y_0)$ exists and is given by

$$D_{\mathbf{u}}f(x_0, y_0) = f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2.$$

(b) If $f(x, y, z)$ is differentiable at (x_0, y_0, z_0), and if $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ is a unit vector, then the directional derivative $D_{\mathbf{u}}f(x_0, y_0, z_0)$ exists and is given by

$$D_{\mathbf{u}}f(x_0, y_0, z_0) = f_x(x_0, y_0, z_0)u_1 + f_y(x_0, y_0, z_0)u_2 + f_z(x_0, y_0, z_0)u_3.$$

Definition 13.6.3.

(a) If f is a function of x and y, then the **gradient of f** is defined by

$$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}.$$

(b) If f is a function of x, y, and z, then the **gradient of f** is defined by

$$\nabla f(x, y, z) = f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k}.$$

Note 13. Now we can write

$$D_{\mathbf{u}}f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \mathbf{u} \quad \text{and} \quad D_{\mathbf{u}}f(x_0, y_0, z_0) = \nabla f(x_0, y_0, z_0) \cdot \mathbf{u}.$$

Note 14. At (x, y), the surface $z = f(x, y)$ has its maximum slope in the direction of the gradient, and the maximum slope is $\|\nabla f(x, y)\|$.

Note 15. At (x, y), the surface $z = f(x, y)$ has its minimum slope in the direction opposite of the gradient, and the minimum slope is $-\|\nabla f(x, y)\|$.

Homework Assignment
Section 13.6 : (Page 968: 1, 2, 3, 5, 7, 9, 11, 17, 19, 20, 29, 30, 31, 32, 33, 38, 39, 43, 45, 47, 50, 53, 58, 59, 71, 73, 81, 82)

Read **Definition 13.6.1**, **13.6.2**

Example 1. If $f(x, y) = xy$ then find $D_u f(1, 2)$ if $u = \frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}$.

Read **Theorem 13.6.1**

Example 2. Find the directional derivative of $f(x, y) = e^{xy}$ at $(-2, 0)$ in the direction of the unit vector that makes an angle of $\frac{\pi}{3}$ with the positive x-axis.

Example 3. Find the directional derivative of $f(x, y, z) = x^2y - yx^3 + z$ at the point $(1, -2, 0)$ in the direction of the vector $\mathbf{a} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$.

Read **Definition 13.6.3** and **Note 13** and revisit **Example 3**

Example 4. If $f(x, y) = x^2e^y$ find the maximal value of a directional derivative at $(-2, 0)$, find the unit vector in that direction.

Example 5. A contour plot for the function $f(x, y) = (x-3)^2+2(y-2)^2$ is given for $k = 0, 1, \ldots, 8$. Sketch the direction of the gradient vectors at the points $P(1, 1)$, $R(2, 3)$, and $S(3, 4)$.

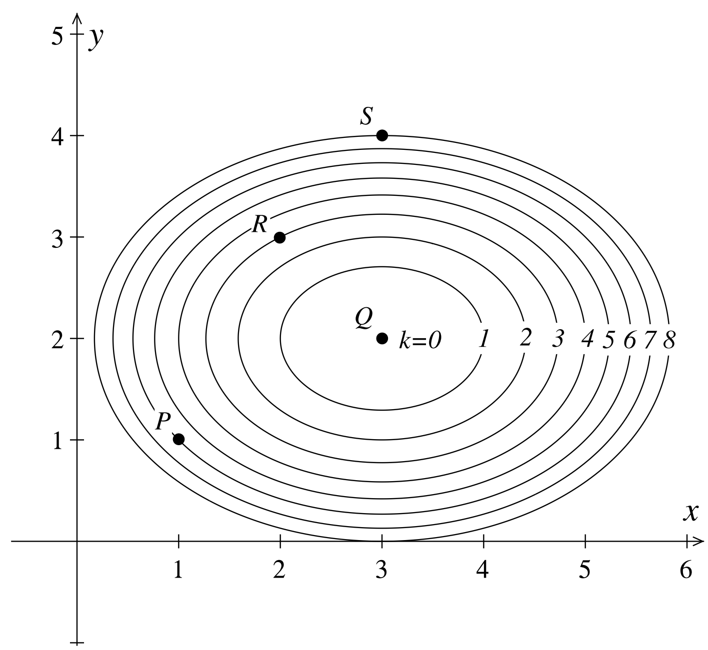


Figure 13.8: level curves for $f(x, y) = (x-3)^2+2(y-2)^2$

Example 6. Find the trajectory of a particle if it moves continuously in the direction of steepest decent starting at point P. Sketch the path on Figure 13.6.

13.7 Tangent Planes and Normal Vectors

Definition 13.7.1. Assume the $F(x, y, z)$ has continuous first-order partial derivatives and that $P_0(x_0, y_0, z_0)$ is a point on the level surface $S : F(x, y, z) = c$. If $\nabla f(x_0, y_0, z_0) \neq \mathbf{0}$, then $\mathbf{n} = \nabla F(x_0, y_0, z_0)$ is a **normal vector** to S at P_0 and the **tangent plane** to S at P_0 is the plane with the equation

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0.$$

Theorem 13.7.1. If $f(x, y)$ is differentiable at the point (x_0, y_0), then the tangent plane to the surface $z = f(x, y)$ at the point $P_0(x_0, y_0, f(x_0, y_0))$ [or (x_0, y_0)] is the plane

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

Note 16. The intersection of two surfaces $F(x, y, z) = 0$ and $G(x, y, z) = 0$ will be a curve in 3-space. If (x_0, y_0, z_0) is a point on the curve, then $\nabla F(x_0, y_0, z_0)$ will be normal to the surface $F(x, y, z) = 0$ at (x_0, y_0, z_0) and $\nabla G(x_0, y_0, z_0)$ will be normal to the surface $G(x, y, z) = 0$ at (x_0, y_0, z_0). If the curve of intersection can be smoothly parameterized, then the tangent vector $\mathbf{T}$ at (x_0, y_0, z_0) will be orthogonal to both $\nabla F(x_0, y_0, z_0)$ and $\nabla G(x_0, y_0, z_0)$. Consequently, if

$$\nabla F(x_0, y_0, z_0) \times \nabla G(x_0, y_0, z_0) \neq \mathbf{0}$$

this cross product will be parallel to $\mathbf{T}$ and hence tangent to the curve of intersection.

Homework Assignment
Section 13.7 : (Page 975: 1, 2, 3, 5, 7, 9, 11, 13, 14, 15, 24, 25, 27, 28, 29, 33)

Read **Definition 13.7.1**.

Example 1. Consider the ellipsiod $x^2 + 4y^2 + z^2 = 18$

(a) Find the tangent plane at $(1, 2, 1)$.

(b) Find the parametric equations of the normal line to the surface at $(1, 2, 1)$.

(c) Find the acute angle of intersection between the tangent plane at $(1, 2, 1)$ and the xy-plane.

Example 2. Find the eqution for the tangent plane and parametric equations for the normal line to the surface $x^2y - 4z^2 = -7$ at the point $(-3, 1, -2)$

Example 3. Show that the surfaces

$$z = \sqrt{x^2 + y^2} \text{ and } z = \frac{1}{10}(x^2 + y^2) + \frac{5}{2}$$

intersect at the point $(3, 4)$ and have a common tangent plane at that point.

Read **Theorem 13.7.1** and **Note 16**

Example 4. Find the parametric equation of the tangent line to the curve of intersection of the paraboloid $z = x^2 + y^2$ and the ellipsoid $3x^2 + 2y^2 + z^2 = 9$ at the point $(1, 1, 2)$.

13.8 Maxima and Minima of Functions of Two Variables

Definition 13.8.1. A function f of two variables is said to have a **relative maximum** at a point (x_0, y_0) if there is a disk centered at (x_0, y_0) such that $f(x_0, y_0) \geq f(x, y)$ for all points (x, y) that lie inside the disk, and f is said to have an **absolute maximum** at (x_0, y_0) if $f(x_0, y_0) \geq f(x, y)$ for all points (x, y) in the domain of f.

Definition 13.8.2. A function f of two variables is said to have a **relative minimum** at a point (x_0, y_0) if there is a disk centered at (x_0, y_0) such that $f(x_0, y_0) \leq f(x, y)$ for all points (x, y) that lie inside the disk, and f is said to have an **absolute minimum** at (x_0, y_0) if $f(x_0, y_0) \leq f(x, y)$ for all points (x, y) in the domain of f.

Theorem 13.8.1. *(Extreme-Value Theorem)* If $f(x, y)$ is continuous on a closed and bounded set R, then f has both an absolute maximum and an absolute minimum on R.

Theorem 13.8.2. If f has a relative extremum at a point (x_0, y_0), and if the first-order partial derivatives of f exist at this point, then

$$f_x(x_0, y_0) = 0 \quad \text{and} \quad f_y(x_0, y_0) = 0.$$

Definition 13.8.3. A point (x_0, y_0) in the domain of a function $f(x, y)$ is called a **critical point** of the function if $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$ or if one or both partial derivatives do not exist at (x_0, y_0).

Theorem 13.8.3. *(The Second Partials Test)* Let f be a function of two variables with continuous second order partial derivatives in some disk centered at a critical point (x_0, y_0), and let

$$D = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - f_{xy}^2(x_0, y_0)$$

(a) If $D > 0$ and $f_{xx}(x_0, y_0) > 0$, then f has a relative minimum at (x_0, y_0).

(b) If $D > 0$ and $f_{xx}(x_0, y_0) < 0$, then f has a relative maximum at (x_0, y_0).

(c) If $D < 0$, then f has a saddle point at (x_0, y_0).

(d) If $D = 0$, then no conclusion can be drawn.

Theorem 13.8.4. If a function f of two variables has an absolute extremum (either an absolute maximum or an absolute minimum) at an interior point of its domain, then this extremum occurs at a critical point.

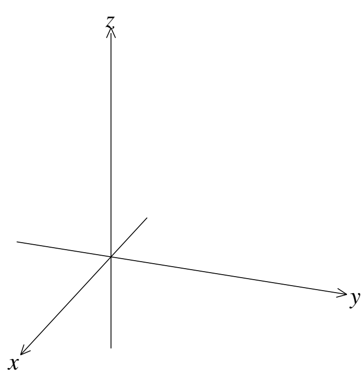
Homework Assignment
Section 13.8 : (Page 985: 1, 5, 6, 7, 9, 12, 13, 27, 29, 31, 33, 35, 37, 39, 43)

Read **Definitions 13.8.1, 13.8.2**

Note 17. A **bounded** set in the xy-plane is a set of points that can be contained by some rectangle. If the set of points cannot be contained by any rectangle it is said to be **unbounded**. A **bounded** set in 3-space is a set that can be contained by some rectangular prism.

Read **Theorem 13.8.1**, **13.8.2**, and **13.8.3**

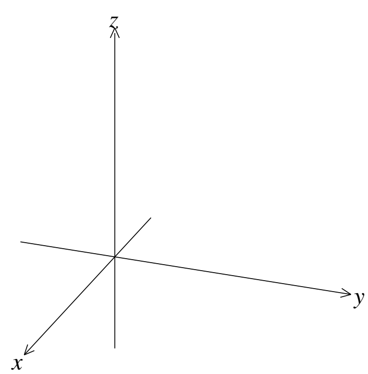
$$f(x,y) = x^2 + y^2$$



$$f_x(x,y) =$$

$$f_y(x,y) =$$

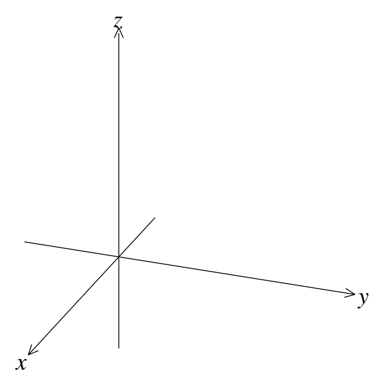
$$f(x,y) = 1 - x^2 - y^2$$



$$f_x(x,y) =$$

$$f_y(x,y) =$$

$$f(x,y) = \sqrt{x^2 + y^2}$$



$$f_x(x,y) =$$

$$f_y(x,y) =$$

Example 1. Locate all relative extrema and saddle points of

$$f(x, y) = 3x^2 - 2xy + y^2 - 8y$$

Note 18. Three steps to finding Absolute Extrema of a function f on a bounded set R:

1) Find the critical points of f that lie in the interior of R.
2) Find all boundry points at which extrema can occur.
3) Evaluate $f(x, y)$ at the points obtained in the previous steps, the largest of these values is the absolute max, and the smallest is the absolute minimum.

Example 2. Find all absolute extremum of $0 = (x - 2)^2 + (y - 2)^2 + z$ on the domain $D = \{(x, y)|0 \leq x \leq 3, 0 \leq y \leq 3\}$.

Example 3. Find the absolute maximum and minimum values of

$$f(x, y) = 3xy - 6x - 3y + 7$$

on the closed triangular region R with vertices $(0, 0), (3, 0), (0, 5)$.

Example 4. Determine the dimensions of a rectangular box, open at the top, having volume of $32\,\mathrm{ft}^3$, and requiring the least amount of material for its construction.

13.9 Lagrange Multipliers

Theorem 13.9.1. ***(The Constrained-Extremum Principle for Two Variables and One Constraint)*** Let f and g be functions of two variables with continuous first partial derivatives on some open set containing the constraint curve $g(x, y) = 0$, and assume the $\nabla g \neq \mathbf{0}$ at any point on this curve. If f has a constrained relative extremum, then this extremum occurs at a point (x_0, y_0) on which the gradient vectors $\nabla f(x_0, y_0)$ and $\nabla g(x_0, y_0)$ are parallel; that is, there is some number λ such that

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0).$$

This scalar λ is called a **Lagrange multiplier.**

Note 19. If c is a constant, then the functions $g(x, y)$ and $g(x, y) - c$ have the same gradient since the constant c drops out when we differentiate. Consequently, it is not essential to rewrite a constraint of the form $g(x, y) = c$ as $g(x, y) - c = 0$ in order to apply **Theorem 13.9.1**.

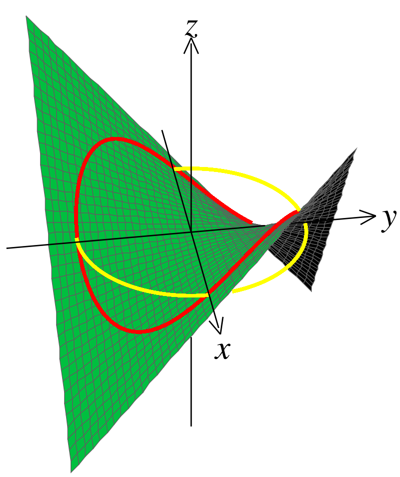
Theorem 13.9.2. ***(The Constrained-Extremum Principle for Three Variables and One Constraint)*** Let f and g be functions of three variables with continuous first partial derivatives on some open set containing the constraint surface $g(x, y, z) = 0$, and assume the $\nabla g \neq \mathbf{0}$ at any point on this surface. If f has a constrained relative extremum, then this extremum occurs at a point (x_0, y_0, z_0) on which the gradient vectors $\nabla f(x_0, y_0, z_0)$ and $\nabla g(x_0, y_0, z_0)$ are parallel; that is, there is some number λ such that

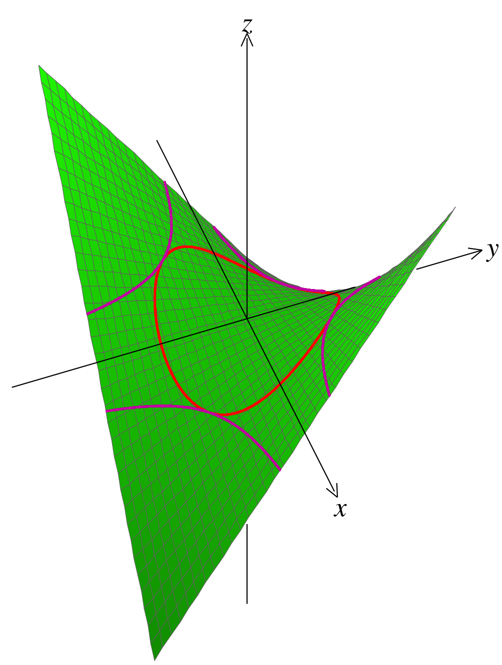
$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0).$$

Homework Assignment
Section 13.9 : (Page 996: 1, 3, 7, 9, 11, 19, 23, 28, 34)

Recall from an example in the previous section, we wanted to minimize $S(x, y, z) = xy + 2xz + 2yz$ with the constraint of $xyz = 32$ or $xyz - 32 = 0$. Read **Theorem 13.9.1, 13.9.2** and discuss the constrained extremum principle.

Example 1. At what point or points on the circle $x^2 + y^2 = 1$ does $f(x, y) = xy$ have an absolute extremum, and what are the extremum?





Example 2. Find the points on the sphere $x^2 + y^2 + z^2 = 36$ that are closest and farthest from the point $(1, 2, 2)$

Multivariable Calculus Chapter 13 Partial Derivatives-Review

Content Overview

13.1 Functions of Several Variables
- Domain and Range of a function of two or more variables
- Level Curves and Level Surfaces

13.2 Limits and Continuity
- Limits
- Continuity
- Two-Path Rule (Limit DNE if it differs along two paths)
- Limits using Polar Coordinates

13.3 Partial Derivatives
- Partial Derivatives
- Higher Order Partial Derivatives
- Notation for Partial Derivatives

13.4 Differentiability, Differentials, and Local Linearity
- Linearization, the function $L(x, y)$
- Linear Approximation or the Tangent Plane Approximation
- Differentiable Functions
- Increment of z, Δz
- Differentials, dx, dy, Total Differential, dz

13.5 Chain Rule
- Chain Rule (For Functions of Two or More Variables)
- Implicit Differentiation (Long way from Calculus I or using Partial Derivatives)

13.6 Directional Derivatives and Gradients
- Directional Derivative
- Gradient Vector (Maximal Directional Derivative, Normal to Level Curves and Level Surfaces, Direction of Steepest Slope)

13.7 Tangent Planes and Normal Vectors
- Normal Line to Level Surfaces
- Tangent Planes to Level Surfaces
- Tangent Lines to Curves of Intersection

13.8 Maxima and Minima of Functions of Two Variables
- Relative and Absolute Extrema
- Critical Points, (Also Called a Stationary Point), Saddle Point
- Closed, Bounded Set
- Extreme Value Theorem for Functions of Two Variables
- Second Derivative Test (D-test)

13.9 Lagrange Multipliers
- Lagrange Multipliers
- Constraints

Review Problems for Chapter 13

1. Sketch the domain of $f(x, y) = \ln(xy - 1)$ using solid lines for portions of the boundary included in the domain and dashed lines for portions not included.

2. Evaluate the limit $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}$

3. Evaluate the limit $\displaystyle\lim_{(x,y,z)\to(3,0,1)} e^{-xy}\sin(\frac{\pi z}{2})$

4. Find the partial derivatives as indicated for $f(x, y, z) = x^2y^4z^3 - 2x^3y^5 + 3x^2z^4 - 5y^2z^2$.

   (a) f_x, f_y, f_z

   (b) f_{xy}, f_{xz}, f_{xx}, f_{yy}, f_{zz}

   (c) f_{xyz}, f_{xzy}, f_{xxx}, f_{yyy}, f_{zzz}

5. Find $f_x(1,-1)$ for $f(x,y) = \frac{1}{2}\sin(x+y)$.

6. Find $f_x(0,2)$ for $f(x,y) = \arcsin\left(\dfrac{x}{y}\right)$

7. Find $\dfrac{\partial w}{\partial t}$ when $s=0$ and $t=1$ for $w = xy + yz + zx$, and $x = st$, $y = e^{st}$, $z = t^2$.

8. Find the directional derivative of $f(x,y) = x^2\sin(2y) + y^2 - 2y + 5$ at $(1, \frac{\pi}{2})$ in the direction of $\vec{v} = \langle 3, 4\rangle$.

9. Find the rate of change of $f(x, y) = xe^y$ at $P(2, 0)$ in the direction of the $\overrightarrow{PQ}$ for $Q(\frac{1}{2}, 2)$.

10. Find an equation of the tangent plane to the hyperboloid given by $z^2 - 2x^2 - 2y^2 = 12$ at the point $(1, -1, 4)$ and find the a parametric equation for the normal line.

11. Find $\dfrac{dy}{dx}$ using implicit differentiation for $x^2y^5 - 4x^3 = 6 - 2x^7y^2$.

12. What is the greatest area that a rectangle can have if the length of its diagonal is 2?

13. Find the points on the sphere $x^2 + y^2 + z^2 = 1$ where the tangent plane is parallel to the plane $2x + y - 3z = 2$

14. Use Lagrange multipliers to find the maximum and minimum values of $f(x, y) = 3x - 2y$ subject to the constraint $x^2 + 2y^2 = 44$.

Chapter 14 Multiple Integrals

14.1 Double Integrals Over Rectangular Regions

Recall: In single-variable calculus the *definite integral* of a function $f(x)$ over and interval $[a, b]$ was defined to be

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

where $\Delta x = (b-a)/n$, and for each i we have, $x_{i-1} \le x_i^* \le x_i$, where $x_i = a + i\Delta x$.

Now, let $f(x, y)$ be a function of two variables. We consider the problem computing the volume of the solid in three space bound by the surface $z = f(x, y)$ and the planes $x = a$, $x = b$, $y = c$, $y = d$, and $z = 0$ where a, b, c, and d are real valued constants. As before we divide the interval $[a, b]$ into n subintervals of width $\Delta x = (b-a)/n$, and similarly we divide the interval $[c, d]$ into m subintervals of width $\Delta y = (d-c)/m$. For ease of notation we will define $x_i = a + i\Delta x$ and $y_j = c + j\Delta y$.

We can now approximate the volume V of this solid by the sum of the volumes mn boxes. The base of each box will have dimensions Δx by Δy, and the height is given by $f(x_i^*, y_j^*)$, where, for each i and j we have, $x_{i-1} \le x_i^* \le x_i$ and $y_{j-1} \le y_j^* \le y_j$. That is to say,

$$V \approx \sum_{i=1}^{n}\sum_{j=1}^{m} f(x_i^*, y_j^*)\,\Delta y\,\Delta x.$$

Intuitively the volume of the solid is given exactly by letting the number of subintervals, n and m, tend to infinity. The result gives us our first definition for this section.

Definition 14.1.1. The ***double integral*** of the function $f(x, y)$ over the region R defined as $R = \{(x, y)\,|\, a \le x \le b,\ c \le y \le d\} = [a, b] \times [c, d]$ is

$$V = \iint\limits_R f(x, y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{m} f(x_i^*, y_j^*)\,\Delta y\,\Delta x,$$

where the dA corresponds to $\Delta A = \Delta x \Delta y$.

Theorem 14.1.1. ***(Fubini's Theorem)*** If $f(x, y)$ is continuous on the rectangular region $R = [a, b] \times [c, d]$, then

$$\iint\limits_R f(x, y)\,dA = \int_a^b \int_c^d f(x, y)\,dy\,dx = \int_c^d \int_a^b f(x, y)\,dx\,dy$$

Homework Assignment
Section 14.1 : page 1007 1, 3, 5, 9, 13, 15, 17, 19, 27, 29, 30, 31, 35, 37, 39

Well known Summation Rules

1. $\sum_{i=1}^{n} c =$

2. $\sum_{i=1}^{n} i =$

3. $\sum_{i=1}^{n} i^2 =$

4. $\sum_{i=1}^{n} i^3 =$

5. $\sum_{i=1}^{n} kf(i) =$

6. $\sum_{i=1}^{n} \big(g(i) + h(i)\big) =$

Example 1. Let $R = [0,1] \times [0,2]$ and $f(x,y) = x^2y + xy^3$, use Fubini's Theorem to evaluate.
$\iint\limits_R f(x,y)\,dA =$

Example 2. Find the volume of the region bound by the planes $x = 1$, $x = 4$, $y = 0$, $y = 2$, $z = 0$, and $x + y + z = 8$.

Example 3. Evaluate $\displaystyle\iint_R y\sin(xy)\,dA$ on $R = [1,2] \times [0,\pi]$. Integrate first with respect to x.

Example 3. continued... Integrate first with respect to y.

Sometimes it is better to integrate with respect to a specific variable first to simplify the second integral.

Here are some double integrals for you to practice on your own.

1. $\displaystyle\iint_R e^x \cos y \, dA$ on $R = [0, 1] \times [0, \frac{\pi}{2}]$
2. $\displaystyle\iint_R e^{x+y} \, dA$ on $R = [0, \ln 3] \times [0, \ln 2]$
3. $\displaystyle\iint_R \cos(x + y) \, dA$ on $R = [0, \frac{\pi}{4}] \times [0, \frac{\pi}{4}]$

Solutions: 1. $e - 1$ 2. 2 3. $\sqrt{2} - 1$

14.2 Double Integrals Over Nonrectangular Regions

When finding double integrals over nonrectangular regions there are two types of regions we encounter,

Definition 14.2.1.

(a) A **type I region** is bounded on the left and right by vertial lines $x = a$ and $x = b$ and is bounded below and above by continuous curves $y = g_1(x)$ and $y = g_2(x)$, where $g_1(x) \le g_2(x)$ for $a \le x \le b$.

(b) A **type II region** is bounded below and above by horizontal lines $y = c$ and $y = d$ and is bounded on the left and right by continuous curves $x = h_1(y)$ and $x = h_2(y)$ satisfying $h_1(x) \le h_2(y)$ for $c \le y \le d$.

Theorem 14.2.1.

(a) If R is a type I region on which $f(x, y)$ is continuous, then

$$\iint\limits_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx.$$

(b) If R is a type II region on which $f(x, y)$ is continuous, then

$$\iint\limits_R f(x,y)\,dA = \int_c^d \int_{h_1(x)}^{h_2(x)} f(x,y)\,dx\,dy.$$

Determining Limits of Integration: Type I Region

Step 1. Move a vertical line across the region. There should be two points of intersection. The lower and upper points of intersection are the lower and upper y-limits of integration, $g_1(x)$ and $g_2(x)$ respectively.

Step 2. The farthest left and right points that the vertical line can be moved are the lower and upper x-limits of integration, a and b respectively.

Determining Limits of Integration: Type II Region

Step 1. Move a horizontal line across the region. There should be two points of intersection. The left and right points of intersection are the lower and upper x-limits of integration, $h_1(y)$ and $h_2(y)$ respectively.

Step 2. The lowest and highest points that the horizontal line can be moved are the lower and upper y-limits of integration, c and d respectively.

Homework Assignment
Section 14.2 page 1015: 1, 3, 5, 7-12, 15, 19, 23, 25, 31, 41, 42, 43, 47, 48, 51, 52, 53, 55, 56, 57

Example 1. Compute the volume of the solid under the plane $x + y + z = 8$ bounded by the surfaces $y = x$ and $y = x^2$.

Example 2. Compute the volume of a solid in the first octant bounded by the planes $z = 10+x+y$, $z = 2 - x - y$, and $x = 0$, as well as the surfaces $y = \sin x$ and $y = \cos x$ between $x = \pi/4$.

Example 3. Find the volume of the tetrahedron bound by the coordinate planes and the plane $z = 4 - 4x - 2y$.

Example 4. Evaluate the double integral for the region $R = \left\{(x, y) | 0 \leq x \leq 1, \sqrt{x} \leq y \leq 1\right\}$.

$$\iint\limits_R e^{y^3}\, dA =$$

14.3 Double Integrals in Polar Coordinates

Definition 14.3.1. A **simple polar region** in a polar coordinate system is a region that is enclosed between two rays, $\theta = \alpha$ and $\theta = \beta$, and two continuous polar curves, $r = r_1(\theta)$ and $r = r_2(\theta)$, where the equations of the rays and polar curves satisfy the following conditions:

$$(i)\ \alpha \le \beta \qquad (ii)\ \beta - \alpha \le 2\pi \qquad (iii)\ 0 \le r_1(\theta) \le r_2(\theta)$$

To calculate the net volume under the surface $f(r,\theta)$ over the region

$$R = \{(r,\theta) | a \le r \le b, \alpha \le \theta \le \beta\},$$

called a **polar rectangle**, we divide the interval $[a,b]$ into n subintervals with width $\Delta r = (a-b)/n$ and the interval $[\alpha,\beta]$ into m subintervals with width $\Delta\theta = (\beta-\alpha)/m$. The "center" of each polar subrectangle

$$R_{ij} = \{(r,\theta) | r_{i-1} \le r \le r_i, \theta_{j-1} \le \theta \le \theta_j\}$$

has polar coordinates (r_i^*, θ_j^*) where $r_i^* = \frac{1}{2}(r_{i-1}+r_i)$ and $\theta_j^* = \frac{1}{2}(\theta_{j-1}+\theta_j)$. The area of each polar subrectangle is given by

$$\begin{aligned}
\Delta A_{ij} &= \frac{1}{2}r_i^2(\theta_j - \theta_{j-1}) - \frac{1}{2}r_{i-1}^2(\theta_j - \theta_{j-1}) \\
&= \frac{1}{2}(r_i^2 - r_{i-1}^2)(\theta_j - \theta_{j-1}) \\
&= \frac{1}{2}(r_i + r_{i-1})(r_i - r_{i-1})\Delta\theta_j \\
&= r_i^*\Delta r_i \Delta\theta_j.
\end{aligned}$$

Definition 14.3.2. If $f(r,\theta)$ is continuous on R and has both positive and negative values, then

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(r_i^*,\theta_j^*)r_i^*\Delta r\Delta\theta$$

represents the approximation of the net signed volume between the region R and the surface $z = f(r,\theta)$ and is called the **polar Riemann sums**.

Definition 14.3.3. The limit of the polar Riemann sums is denoted by

$$\iint_R f(r,\theta)\,dA = \lim_{n\to\infty}\lim_{m\to\infty}\sum_{i=1}^{n}\sum_{j=1}^{m} f(r_i^*,\theta_j^*)r_i^*\Delta r\Delta\theta$$

which is called the **polar double integral** of $f(r,\theta)$ over R.

Theorem 14.3.1. If R is a simple polar region whose boundaries are the rays $\theta = \alpha$ and $\theta = \beta$ and the curves $r = r_1(\theta)$ and $r = r_2(\theta)$ and if $f(r,\theta)$ is continuous on R, then

$$\iint_R f(r,\theta)\,dA = \int_\alpha^\beta \int_{r_1(\theta)}^{r_2(\theta)} f(r,\theta)r\,dr\,d\theta$$

Homework Assignment
Section 14.3 page 1024: 1, 5, 7, 11, 13, 15, 16, 17, 19, 20, 23, 24, 25, 27, 28, 29, 33

Example 1. Evaluate $\iint\limits_R (3x + 4y^2)\,dA$, where R is the region in the upper half-plane bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

Example 2. Find the volume of the solid bounded by the plane $z = 0$ and the paraboloid $z = 1 - x^2 - y^2$.

Example 3. Find the formula for the volume of a sphere $x^2 + y^2 + z^2 = R_0^2$. Hint: Find the volume of the upper hemisphere and then double the volume.

Example 4. Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$, above the xy-plane, and inside the cylinder $x^2 + y^2 = 2x$

Example 5. Evaluate $\displaystyle\iint_D x + y\,dA$ where $D = \big\{(x, y)|1 \leq x^2 + y^2 \leq 4, x \leq 0\big\}$.

14.4 Surface Area; Parametric Surfaces

Theorem 14.4.1. Consider $z = f(x, y)$ defined over the region R in the xy-plane. Assume f has continuous first partial derivatives all all the points in the interior of R. The surface area is given by

$$S = \iint\limits_R \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA$$

Definition 14.4.1. The function $\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$ is a **vector-valued function of two variables**. The graph corresponds to the parametric equations

$$x = x(u, v), \qquad y = y(u, v), \qquad z = z(u, v).$$

As with vector functions of one variable, we say that $\mathbf{r}(u, v)$ is **continuous** if each component is continous.

Definition 14.4.2. Partial derivatives of vector-valued functions of two variables are obtained by taking partial derivatives of the components. For example, for

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$$

then

$$\frac{\partial \mathbf{r}}{\partial u} = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k}$$
$$\frac{\partial \mathbf{r}}{\partial v} = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}$$

Theorem 14.4.2. The surface generated by revolving the plane curve $y = f(x)$ about the x-axis can be represented parametrically as

$$x = u, \qquad y = f(u)\cos(v), \qquad z = f(u)\sin(v)$$

Definition 14.4.3. If a parametric surface σ is the graph of $\mathbf{r} = \mathbf{r}(u, v)$, and if $\dfrac{\partial \mathbf{r}}{\partial u} \times \dfrac{\partial \mathbf{r}}{\partial v} \neq \mathbf{0}$ at a point on the surface, then the **principal unit normal vector** to the surface at the point is denoted by $\mathbf{n}$ or $\mathbf{n}(u, v)$ and is defined to as

$$\mathbf{n} = \frac{\dfrac{\partial \mathbf{r}}{\partial u} \times \dfrac{\partial \mathbf{r}}{\partial v}}{\left\| \dfrac{\partial \mathbf{r}}{\partial u} \times \dfrac{\partial \mathbf{r}}{\partial v} \right\|}.$$

Homework Assignment
Section 14.4 page 1036: 1, 3, 6, 8, 11, 13, 17, 23, 25, 33, 35, 36, 41, 51, 52, 54, 55, 56, 57

Example 1. Find the surface area of the portion of the surface $z = \sqrt{4 - x^2}$ that lies above the rectangle R in the xy-plane, $R = [0, 1] \times [0, 4]$.

Example 2. Find the surface area of the paraboloid $z = x^2 + y^2$ below the plane $z = 1$.

Example 3. Find a way to represent the paraboloid $z = 4 - x^2 - y^2$ parametrically.

Example 4. Find the parametric equations representing the right cylinder $x^2 + z^2 = 9$.

Example 5. Find parametric equations for the surface generated by revolving the curve $y = f(x) = \dfrac{1}{x}$ about the x-axis.

Example 6. Find parametric equations for the surface generated by revolving $f(z) = y = e^{z^2+2z}$ about the z-axis.

Example 7. The paraboloid from example 3 with parameterization

$$x = u, \qquad y = v, \qquad z = 4 - u^2 - v^2$$

Find the vector-valued function that moves over this surface as well as the principal unit normal. Use the principal unit normal to find the tangent plane at the point (1,-1).

14.5 Triple Integrals

Theorem 14.5.1 (Fubini's Theorem). Let G be the rectangular box defined by the inequalities

$$a \leq x \leq b, \qquad c \leq y \leq d, \qquad k \leq z \leq l$$

If f is continuous on the region G, then

$$\iiint\limits_G f(x,y,z)\,dV = \int_a^b \int_c^d \int_k^l f(x,y,z)\,dz\,dy\,dx$$

Moreover, the iterated integral on the right can be replaced with any of the five other iterated integrals that result by altering the order of the integration.

Properties of Triple Integrals:

Triple integrals inherit their properties from double integrals and single variable integrals,

$$\iiint\limits_G cf(x,y,z)\,dV = c\iiint\limits_G f(x,y,z)\,dV \qquad (c \text{ a constant})$$

$$\iiint\limits_G [f(x,y,z) \pm g(x,y,z)]\,dV = \iiint\limits_G f(x,y,z)\,dV \pm \iiint\limits_G g(x,y,z)\,dV$$

Definition 14.5.1. Assume that the solid G is bounded above by a surface $z = g_2(x,y)$ and below by a surface $z = g_1(x,y)$ and that the projection of the solid on the xy-plane is a type I or type II region R. In addition, we will assume that $g_1(x,y)$ and $g_2(x,y)$ are continuous on R and that $g_1(x,y) \leq g_2(x,y)$ on R. We call a solid of this type a **simple xy-solid**. Similarly, we can define **simple xz solids** and **simple yz-solids**.

Theorem 14.5.2. Let G be a simple xy-solid with upper surface $z = g_2(x,y)$ and lower surface $z = g_1(x,y)$, and let R be the projection of G on the xy-plane. If $f(x,y,z)$ is continuous on G then

$$\iiint\limits_G f(x,y,z)\,dV = \iint\limits_R \left[\int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\,dz\right] dA$$

Note 20. One of the physical interpretations of triple integrals is volume,
this occurs when $f(x,y,z) = 1$,

$$\text{volume of } G = \iiint\limits_G dV$$

Homework Assignment
Section 14.5 page 1045: 1, 3, 5, 6, 7, 9, 10, 11, 15, 16, 17, 19, 21, 25, 33, 36

Example 1. Evaluate the triple integral over the rectangular box $D = \{[-1,2] \times [0,3] \times [0,2]\}$.

$$\iiint\limits_D 12xy^2z^3\,dV =$$

Example 2. Let D be the wedge in the first octant that is cut from the cylindrical solid $y^2+z^2 \leq 1$ by the planes $y = x$ and $x = 0$. Evaluate

$$\iiint\limits_D z\,dV =$$

Example 3. Use a triple integral to find the volume of the solid within the cylinder $x^2 + y^2 = 9$ and between the planes $z = 1$ and $x + z = 5$.

Example 4. Find the volume of the solid enclosed between the paraboloids $z = 5x^2 + 5y^2$ and $z = 6 - 7x^2 - y^2$.

14.6 Triple Integrals in Cylindrical and Spherical Coordinates

Triple Integrals in Cylindrical Coordinates:
A **cylindrical wedge** is the solid enclosed by the following surfaces

two cylinders:	$r = r_1, \quad r = r_2 \quad (r_1 < r_2)$
two vertical half planes:	$\theta = \theta_1, \quad \theta = \theta_2 \quad (\theta_1 < \theta_2)$
two horizontal planes:	$z = z_1, \quad z = z_2 \quad (z_1 < z_2)$

Sometimes a triple integral that is difficult to integrate in rectangular coordinates can be evaluated more easily by making the substitution $x = r\cos\theta$, $y = r\sin\theta$, $z = z$ to convert it to an **integral in cylindrical coordinates**. The integral becomes

$$\iiint_G f(x,y,z)\,dV = \iiint_{\substack{\text{appropriate}\\ \text{limits}}} f(r\cos\theta, r\sin\theta, z)r\,dz\,dr\,d\theta$$

Note 21. In the change of variables the differential dV becomes $r\,dz\,dr\,d\theta$

Triple Integrals in Spherical Coordinates:
The simplest regions in spherical coordinates are **spherical wedges** defined by

two spheres:	$\rho = \rho_1, \quad \rho = \rho_2 \quad (\rho_1 < \rho_2)$
two half planes:	$\theta = \theta_1, \quad \theta = \theta_2 \quad (\theta_1 < \theta_2)$
sections of two circular cones:	$\phi = \phi_1, \quad \phi = \phi_2 \quad (\phi_1 < \phi_2)$

$$\iiint_G f(\rho,\theta,\phi)\,dV = \iiint_{\substack{\text{appropriate}\\ \text{limits}}} f(\rho,\theta,\phi)\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

Note 22. In the change of variables the differential dV becomes $\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$

To convert a rectangular coordinate integral into a spherical integral use the relation below,

$$\iiint_G f(x,y,z)\,dV = \iiint_{\substack{\text{appropriate}\\ \text{limits}}} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

Homework Assignment
Section 14.6 page 1056: 1, 3, 5, 7, 9, 10, 11, 13, 17, 18, 19, 23, 27, 28, 29

Example 1. Use triple integration in cylindrical coordinates to find the volume of the solid G that is bound above by the hemisphere $z = \sqrt{25 - x^2 - y^2}$, below by the xy-plane $z = 0$, laterally bound by the cylinder $x^2 + y^2 = 9$

Example 2. Use cylindrical coordinates to evaluate

$$\int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{0}^{9-x^2-y^2} x^2 \, dz \, dy \, dx$$

Example 3. Use spherical coordinates to find the volume of the solid G bounded above by the sphere $x^2 + y^2 + z^2 = 16$ and below by the cone $z = \sqrt{x^2 + y^2}$.

Example 4. Use spherical coordinates to evaluate

$$V = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{0}^{\sqrt{4-x^2-y^2}} z^2 \sqrt{x^2 + y^2 + z^2}\, dz\, dy\, dx$$

14.7 Change of Variables in Multiple Integrals; Jacobian

Transformations of the Plane:
Recall the following parametric equations from earlier sections:

$$\begin{array}{lll} x = x(t), \quad y = y(t) & \longrightarrow & \text{(A curve in the plane)} \\ x = x(t), \quad y = y(t), \quad z = z(t) & \longrightarrow & \text{(A curve in three space)} \\ x = x(u,v), \quad y = y(u,v), \quad z = z(u,v) & \longrightarrow & \text{(A surface in three space)} \end{array}$$

We will now consider parametric equations of the form,

$$x = x(u,v), \quad y = y(u,v)$$

Parametric equations of this type associate points in the xy-plane with points in the uv-plane. These parametric equations can be related as a function

$$T(u,v) = (x(u,v), y(u,v)) \qquad \text{or} \qquad \mathbf{r} = \mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j}.$$

T is called a **transformation** from the uv-plane to the xy-plane. **Constant u-curves** and **constant v-curves** are the curves in the xy-plane resulting from constant u and v.

Jacobians in Two Variables

Definition 14.7.1. If T is a transformation from the uv-plane to the xy-plane defined by the equations $x = x(u,v)$ and $y = y(u,v)$, then the **Jacobian of** T is denoted by $J(u,v)$ or by $\partial(x,y)/\partial(u,v)$ and is defined by

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}$$

Definition 14.7.2 (Change of variables formula for double integrals)**.** If the transformation $x = x(u,v)$, $y = y(u,v)$ maps the region S in the uv-plane into the region R in the xy-plane, and if the Jacobian $\partial(x,y)/\partial(u,v)$ is nonzero and does not change sign on S, then it follows that

$$\iint_R f(x,y)\, dA_{xy} = \iint_S f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dA_{uv}$$

Note 23. The change in the area of the regions is given by $\Delta A \approx \left| \dfrac{\partial(x,y)}{\partial(u,v)} \right| \Delta u \Delta v$.

Jacobians in Three Variables

Definition 14.7.3. If T is a transformation from the uvw-space to the xyz-space defined by the equations $x = x(u, v, w)$, $y = y(u, v, w)$, and $z = (u, v, w)$, then the **Jacobian of** T is denoted by $J(u, v, w)$ or by $\partial(x, y, z)/\partial(u, v, w)$ and is defined by

$$J(u,v,w) = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

Definition 14.7.4 (Change of variables formula for triple integrals)**.** If the transformation $x = x(u, v, w)$, $y = y(u, v, w)$, $z = z(u, v, w)$, maps the solid S in uvw-space into the solid R in xy-space, and if the Jacobian $\partial(x, y, z)/\partial(u, v, w)$ is nonzero and does not change sign on S, then it follows that

$$\iiint\limits_R f(x,y,z)\,dA_{xyz} = \iiint\limits_S f(x(u,v,w), y(u,v,w), z(u,v,w)) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dA_{uvw}$$

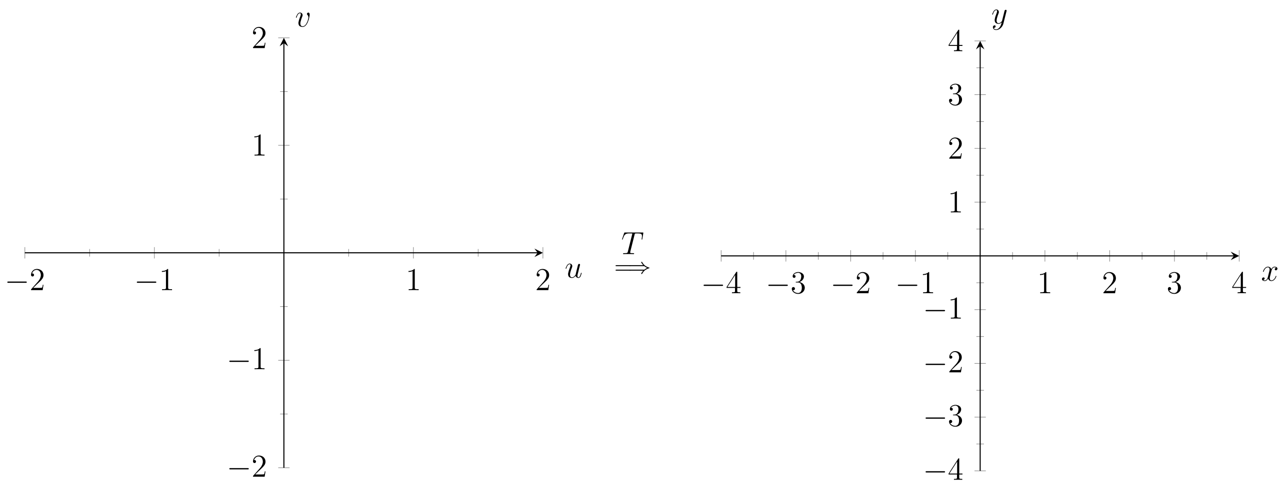
Note 24. The change in the volume of the solids is given by $\Delta V \approx \left|\dfrac{\partial(x,y,z)}{\partial(u,v,w)}\right| \Delta u \Delta v \Delta w$.

Homework Assignment
Section 14.7 page 1068: 1, 7, 11, 17, 18, 19, 20, 21, 23, 25, 26, 33, 34, 35, 37, 39, 41, 42

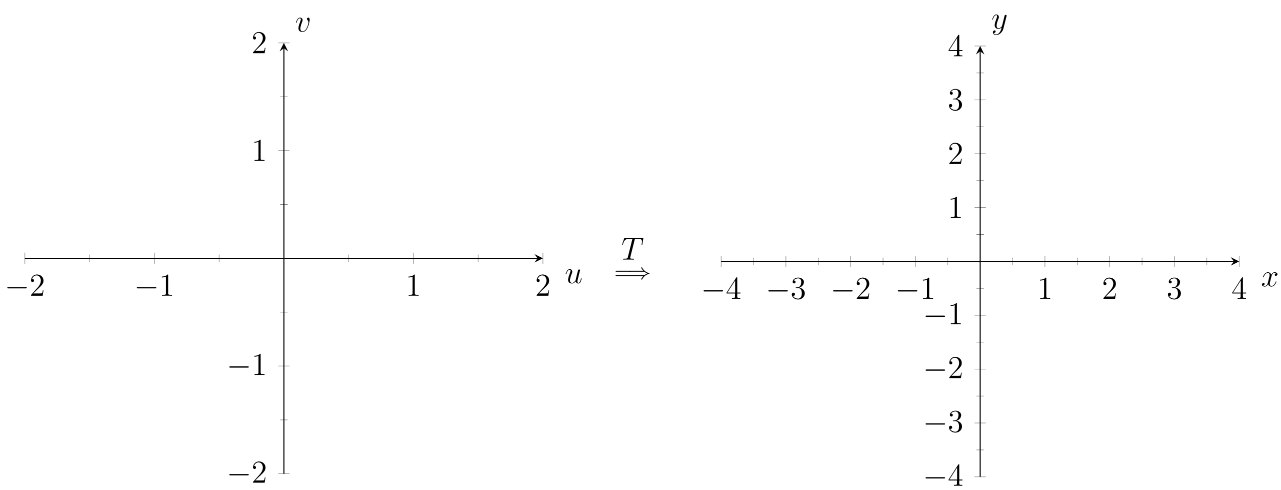
Example 1. Let T be the transformation from the uv-plane to the xy-plane defined by the equations $x = 1/4(u + v)$ $y = 1/2(u - v)$

(a) Find $T(1, 3)$

(b) Sketch the constant u-curves $u = -2, -1, 0, 1, 2$
Sketch the constant v-curves $v = -2, -1, 0, 1, 2$



(c) Sketch the image under T of the square region in the uv-plane bounded by $u = -2$, $u = 2$, $v = -2$, $v = 2$



Example 2. Evaluate

$$\iint\limits_R \frac{x-y}{x+y}\,dA$$

where R is the region enclosed by $x-y=0$, $x-y=1$, $x+y=1$, and $x+y=3$.

Note 25. *For polar coordinates the change of variables is*

$$x = r\cos\theta \qquad\qquad y = r\sin\theta$$

$$J(r,\theta) = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$$

Therefore,

$$\iint\limits_R f(x,y)\,dA = \iint\limits_S f(r\cos\theta, r\sin\theta) r\,dr\,d\theta$$

Example 3. Find the volume of the region G enclosed by the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Example 4. Evaluate

$$\iint_R e^{xy}\, dA$$

where R is the region enclosed by $y = \frac{1}{2}x$, $y = x$ and the hyperbolas $y = \frac{1}{x}$ and $y = \frac{2}{x}$

14.8 Centers of Gravity Using Multiple Integrals

Definition 14.8.1. An idealized flat object that is thin enough to be viewed as a two-dimensional plane region is called a **lamina**. A lamina is called **homogeneous** if its composition is uniform throughout and **inhomogeneous** otherwise. **Density** of a homogeneous lamina is defined as its mass per unit area, the density δ in terms of its mass M and area A is

$$\delta = M/A.$$

For an inhomogeneous lamina the composition may vary from point to point, and we require the **density function**, $\delta(x, y)$, defined to be

$$\delta(x, y) = \lim_{\Delta A \to 0} \frac{\Delta M}{\Delta A}.$$

Theorem 14.8.1 (Mass of a Lamina)**.** If a lamina with a continuous density function $\delta(x, y)$ occupies a region R in the xy-plane, then its total mass M is given by

$$M = \iint_R \delta(x, y)\, dA$$

Definition 14.8.2. The **center of gravity** of a lamina occupying a region R in the xy-plane is the point $(\bar{x}, \bar{y})$ such that the effect of gravity on the lamina is equivalent to that of a single force acting at $(\bar{x}, \bar{y})$.

Theorem 14.8.2 (Center of Gravity $(\bar{x}, \bar{y})$ of a Lamina)**.**

$$\bar{x} = \frac{M_y}{M} = \frac{\displaystyle\iint_R x\delta(x, y)\, dA}{\displaystyle\iint_R \delta(x, y)\, dA}, \qquad \bar{y} = \frac{M_x}{M} = \frac{\displaystyle\iint_R y\delta(x, y)\, dA}{\displaystyle\iint_R \delta(x, y)\, dA}$$

where M is the mass of the lamina and M_x is the **first moment of the lamina about the x-axis** and M_y is the **first moment of the lamina about the y-axis**.

Definition 14.8.3. The center of gravity of a homogeneous lamina is called the **centroid of the lamina** or sometimes the **centroid of the region** R. Because the density is constant the centroid is a geometric property of the region R.

Theorem 14.8.3 (Centroid of a Region R)**.**

$$\bar{x} = \frac{\displaystyle\iint_R x\, dA}{\displaystyle\iint_R dA} = \frac{1}{\text{area of } R}\iint_R x\, dA, \qquad \bar{y} = \frac{\displaystyle\iint_R y\, dA}{\displaystyle\iint_R dA} = \frac{1}{\text{area of } R}\iint_R y\, dA$$

Definition 14.8.4. For a three-dimensional solid G, the formulas for the center of gravity and centroid are similar to those for laminas. If G is homogeneous, then its **density** is defined to be its mass per unit volume. Thus, if the mass of the solid is M and the volume V, then its density δ is given by

$$\delta = \frac{M}{V}$$

If G is inhomogeneous and is in an xyz-coordinate system, then its density at a general point (x, y, z) is specified by a **density function** $\delta(x, y, z)$ whose value at a point can be viewed as a limit:

$$\lim_{\Delta V \to 0} \frac{\Delta M}{\Delta V}$$

Theorem 14.8.4. The mass of a solid G with a continuous density function is $\delta(x, y, z)$ is

$$M = \text{mass of } G = \iiint\limits_G \delta(x, y, z)\, dV$$

The formulas for center of gravity and centroid are as follows

Center of Gravity $(\bar{\mathbf{x}}, \bar{\mathbf{y}}, \bar{\mathbf{z}})$ of a Soild G

$$\bar{x} = \frac{1}{M} \iiint\limits_G x\delta(x, y, z)\, dV$$

$$\bar{y} = \frac{1}{M} \iiint\limits_G y\delta(x, y, z)\, dV$$

$$\bar{z} = \frac{1}{M} \iiint\limits_G z\delta(x, y, z)\, dV$$

Centroid $(\bar{\mathbf{x}}, \bar{\mathbf{y}}, \bar{\mathbf{z}})$ of a Soild G

$$\bar{x} = \frac{1}{V} \iiint\limits_G x\, dV$$

$$\bar{y} = \frac{1}{V} \iiint\limits_G y\, dV$$

$$\bar{z} = \frac{1}{V} \iiint\limits_G z\, dV$$

Homework Assignment
Section 14.8 page 1077: 1, 3, 5, 7, 13, 14, 19, 23, 25, 27, 29, 30, 33, 35, 37, 41

Example 1. A triagular lamina with vertices at $(0,0)$, $(0,1)$, and $(1,0)$ has density function $\delta(x,y) = xy$. Find the total mass.

Example 2. Find the center of gravity for the triangle in **Example 1**.

Example 3. Find the centroid of the semicircular region $y = \sqrt{a^2 - x^2}$.

Example 4. Find the mass and the center of gravity of a cylindrical solid of height h and radius a, assuming that the density at each point is proportional to the distance between the point and the base of the cylinder.

Multivariable Calculus Chapter 14 Multiple Integrals-Review

Name: ______________________

1. Reverse the order of integration of $\int_0^9 \int_0^{\sqrt{x}} f(x,y)\,dy\,dx$.

2. Consider the lamina with density $\delta(x,y)$ bounded by the closed region R. Determine the integral for the moment about the x-axis.

3. Evaluate the integral $\iint\limits_R (x^2+4y^2)\,dA$ where R is the region bounded by $y=2x$ and $y=x^2$.

4. Calculate the volume under the plane $z=2x+y$ and over the rectangle $R=\{(x,y)|2 \le x \le 4, 1 \le y \le 2\}$.

5. Write the given double integral in polar coordinates and evaluate. $\displaystyle\int_0^2 \int_x^{\sqrt{8-x^2}} \sqrt{x^2+y^2}\, dy\, dx.$

6. Sketch the region R and evaluate the double integral $\displaystyle\iint_R f(x,y)\, dA = \int_0^1 \int_{\sqrt{x}}^1 x^2 y\, dy\, dx$

7. Find the volume of the solid region R bounded above by the paraboloid $z = 1 - x^2 - y^2$ and below the plane $z = 1 - y$.

8. Find the surface area for the portion of $z = xy$ that is inside the cylinder $x^2 + y^2 = 1$.

9. Find the limits of integration for calculating the volume of the solid, S, enclosed by $y = x^2$, $z = 0$ and $y + z = 2$ if $\displaystyle\iiint\limits_S dz\,dy\,dx$.

10. Find the moment about the z-axis of the cylinder $x^2 + y^2 = 4$ between $z = 0$ and $z = 6$ uniform density.

11. Set up and evaluate the iterated integral in polar coordinates to find the surface area of that portion of the sphere $x^2 + y^2 + z^2 = 4$ that is above the xy-plane and within the cylinder $x^2 + y^2 = 1$.

12. Use spherical coordinates to find the mass of the sphere $x^2 + y^2 + z^2 = 9$ if its density at a point is proportional to its distance from the origin.

13. Sketch the solid whose volume is given by the triple integral $\int_0^3 \int_0^{1-\frac{y}{3}} \int_0^3 dx\, dz\, dy$ and rewrite the integral in the order $dz\, dy\, dx$.

14. Sketch the solid whose volume is given by $\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^2 \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$ and evaluate the integral.

15. Convert the integral $\int_0^{2\pi} \int_0^2 \int_0^{\sqrt{4-r^2}} r^2\, dz\, dr\, d\theta$ to rectangular coordinates and sketch the solid.

Chapter 15 Topics in Vector Calculus

15.1 Vector Fields

Definition 15.1.1 (Vector Fields)**.** A **vector field** in a plane is a function that associates with each point P in the plane a unique vector $\mathbf{F}(P)$ parallel to the plane. Similarly, a vector field in 3-space is a function that associates with each point P in 3-space a unique vector $\mathbf{F}(P)$ in 3-space. $\mathbf{F}(P)$ can be expressed as

$$\mathbf{F}(x, y, z) = f(x, y, z)\mathbf{i} + g(x, y, z)\mathbf{j} + h(x, y, z)\mathbf{k}.$$

Definition 15.1.2 (Inverse-Square Field)**.** If $\mathbf{r}$ is a radius vector in 2-space or 3-space, and if c is a constant, then a vector field of the form

$$\mathbf{F}(\mathbf{r}) = \frac{c}{||\mathbf{r}||^3}\mathbf{r}$$

is called an **inverse-square field**. In the two-dimensional case,

$$\mathbf{F}(x, y) = \frac{c}{(x^2 + y^2)^{3/2}}(x\mathbf{i} + y\mathbf{j})$$

and three dimensional case,

$$\mathbf{F}(x, y) = \frac{c}{(x^2 + y^2 + z^2)^{3/2}}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}).$$

Definition 15.1.3. An important class of vector fields arises from the process of finding gradients. Recall that if ϕ is a function of three variables, then the gradient of ϕ is defined to as

$$\nabla\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}$$

This formula defines a vector field in 3-space called the **gradient field**.

Definition 15.1.4 (Potential Function)**.** A vector field $\mathbf{F}$ in 2-space or 3-space is said to be **conservative** in a region if it is the gradient field for some function ϕ in that region, that is, if

$$\mathbf{F} = \nabla\phi$$

The function ϕ is called a **potential function** for $\mathbf{F}$ in the region.

Definition 15.1.5. If $\mathbf{F}(x,y,z) = f(x,y,z)\mathbf{i} + g(x,y,z)\mathbf{j} + h(x,y,z)\mathbf{k}$, then we define the **divergence of F**, written div $\mathbf{F}$, to be the function given by

$$\text{div } \mathbf{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

Definition 15.1.6. If $\mathbf{F}(x,y,z) = f(x,y,z)\mathbf{i} + g(x,y,z)\mathbf{j} + h(x,y,z)\mathbf{k}$, then we define the **curl of F**, written curl $\mathbf{F}$, to be the function given by

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ f & g & h \end{vmatrix} = \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\mathbf{i} + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right)\mathbf{j} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\mathbf{k}$$

[1]

Definition 15.1.7 (Del Operator)**.** So far, the symbol ∇ that appears in the gradient expression $\nabla\phi$ has not been given a meaning of its own. However, it is often convenient to view ∇ as an operator

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$$

which when applied to $\phi(x,y,z)$ produces the gradient

$$\nabla\phi = \frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k}$$

we call ∇ the **del operator**.

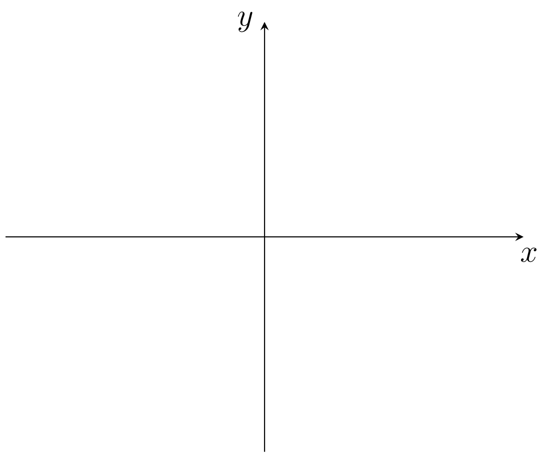
Definition 15.1.8 (Laplacian Operator)**.** The operator that results by taking the dot product of the del operator with itself is denoted by ∇^2 and is called the **Laplacian operator**. This operator has the form

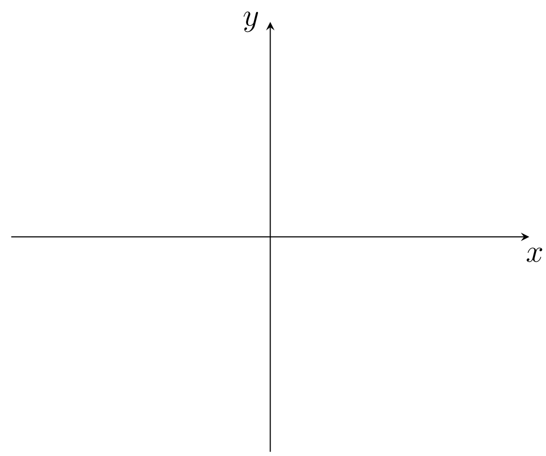
$$\nabla^2 = \frac{\partial^2}{\partial x^2}\mathbf{i} + \frac{\partial^2}{\partial y^2}\mathbf{j} + \frac{\partial^2}{\partial z^2}\mathbf{k}$$

Section 15.1 page 1093: 1, 3, 7, 8, 15, 17, 20, 23, 25, 27, 37, 38, 41, 43

[1] This is only a mnemonic to help remember the formula for the curl, it's not the determinant.

Example 1. *Two of the most famous pair of formulas in physics create inverse square fields.*



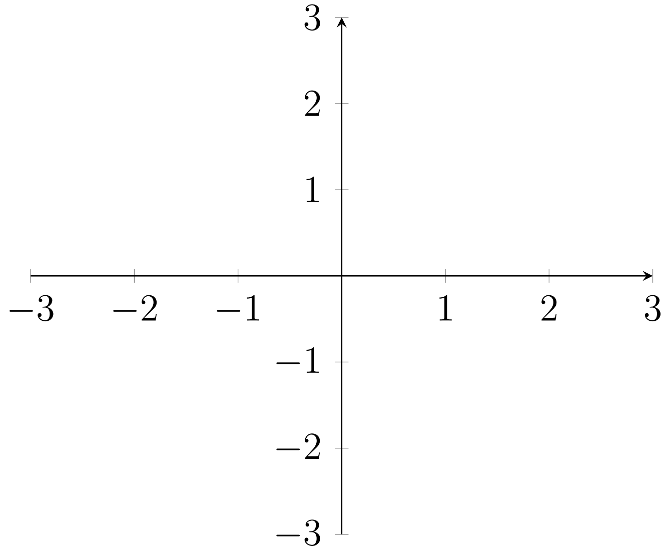


Example 2. *Inverse square fields are conservative in any region that does not contain the origin. For example, in the two dimensional case the function*

$$\phi(x, y) = -\frac{c}{\sqrt{x^2 + y^2}}$$

is a potiental function for the inverse square field in any region not containing the origin.

Example 3. *Sketch the gradient field of $\phi(x, y) = x + y$.*



Sketch the level curves of ϕ.

Can you find another potiental function for $\vec{\mathbf{F}} = \nabla\phi$?

Example 4. *Find the divergence and the curl of the vector field*

$$\vec{\mathbf{F}}(x, y, z) = x^2 y\mathbf{i} + 2y^3 z\mathbf{j} + 3z\mathbf{k}.$$

15.2 Line Integrals

To motivate the idea of line integrals we consider a thin wire whose linear density function $f(x,y)$ (mass per unit length) is known. Assume we are given parametric equations for a smooth plane curve C that models the wire,

$$x = x(t) \qquad y = y(t) \qquad a \leq t \leq b$$

or equivalently we can write the vector equation $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$. Since we have assumed that C is a smooth curve $\mathbf{r}'(t) \neq \mathbf{0}$. If we divide the parameter interval $[a,b]$ into n subintervals $[t_{i-1}, t]$ of equal width and we let $x_i = x(t_i)$ and $y_i = y(t_i)$, then the corresponding points $P_i(x_i, y_i)$ divide C into n subarcs with widths $\Delta s_1, \Delta s_2 \ldots, \Delta s_n$. Choose any point $P_i^*(x_i^*, y_i^*)$ in the i^{th} subarc. To find an approximation to the total density of the wire we can evaluate the following sum

$$\sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i.$$

Take the limit of this sum of partitions to make the following definition.

Definition 15.2.1 (Line Integral)**.** If C is a smooth curve in 2-space or 3-space, then the **the line integral of f with respect to s along C** is

$$\int_C f(x,y)\,ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i \qquad \boxed{\text{2-space}}$$

or

$$\int_C f(x,y,z)\,ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\Delta s_i \qquad \boxed{\text{3-space}}$$

if this limit exists.

Recall from Section 12.3 that the length of C is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

Similarly we arrive at the following form of the line integral

$$\int_C f(x,y)\,ds = \int_a^b f(x(t), y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$\int_C f(x,y,z)\,ds = \int_a^b f(x(t), y(t), z(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$

Note 26. Suppose C is a **piecewise-smooth curve**; that is, C is a union of a finite number of smooth curves C_1, C_2, …, C_n, where the initial point of C_{i+1} is the terminal point of C_i. Then we define the integral of f along C as the sum of the integrals of f along each of the smooth pieces of C:

$$\int_C f(x,y)\,ds = \int_{C_1} f(x,y)\,ds + \int_{C_2} f(x,y)\,ds + \cdots + \int_{C_n} f(x,y)\,ds$$

Note 27 (Mass of a Wire)**.** The mass M of a wire described by path C and linear density function $\delta(x, y)$ is

$$M = \int_C \delta(x, y)\, ds.$$

Definition 15.2.2 (Line Integrals with Respect to x and y)**.** Two other line integrals are obtained by replacing Δs_i by either $\Delta x = x_i - x_{i-1}$ or $\Delta y = y_i - y_{i-1}$ in Definition 15.2.1 or more precisely

$$\int_C f(x, y)\, dx = \int_a^b f(x(t), y(t))x'(t)\, dt$$

$$\int_C f(x, y)\, dy = \int_a^b f(x(t), y(t))y'(t)\, dt$$

Suppose C is a smooth oriented curve, we will let $-C$ denote the oriented curve consisting of the same points as C but with the opposite direction. We then have the following properties

$$\int_{-C} f(x, y)\, dx = -\int_C f(x, y)\, dx \qquad \text{and} \qquad \int_{-C} g(x, y)\, dy = -\int_C g(x, y)\, dy$$

while

$$\int_{-C} f(x, y)\, ds = \int_C f(x, y)\, ds$$

Frequently, the line integrals with respect to x and y occur in combination, in which case we write

$$\int_C f(x, y)\, dx + g(x, y)\, dy = \int_C f(x, y)\, dx + \int_C g(x, y)\, dy$$

Note 28 (Segment Parameterizations)**.** Sometimes the most difficult part of a problem is choosing a parametrization for a line segment. So, it is useful to remember the vector representation of the line segment that starts at $\mathbf{r_0}$ and ends at $\mathbf{r_1}$

$$\mathbf{r}(t) = (1 - t)\mathbf{r_0} + t\mathbf{r_1} \qquad 0 \le t \le 1$$

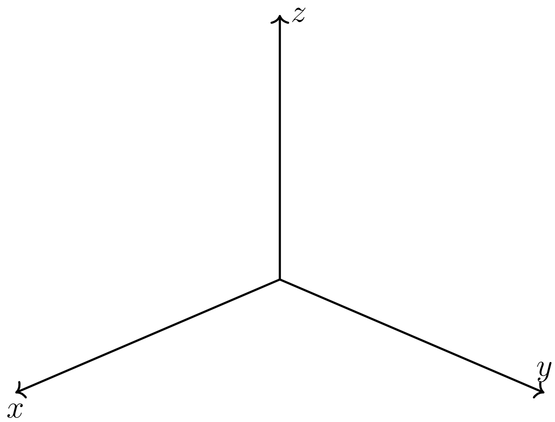
Definition 15.2.3 (Line Integrals in Vector Fields)**.** Let $\mathbf{F}$ be a continuous vector field defined on a smooth curve C given by a vector function $\mathbf{r}(t)$, $a \le t \le b$. Then the **line integral of F along C** is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$$

Definition 15.2.4 (Work Performed By Force Fields)**.** Suppose that under the influence of a continous force field $\mathbf{F}$ a particle moves along a smooth curve C and that C is oriented in the direction of the motion of the particle. The the **work performed by the force field** on the particle is

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

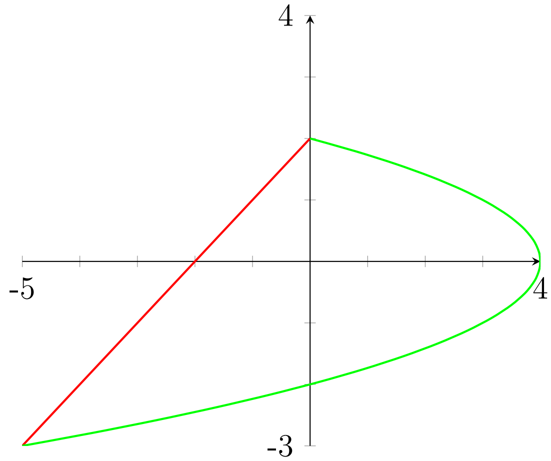
Section 15.2 page 1108: 1, 3, 5, 8, 13, 19, 21, 23, 25, 27, 33, 37, 39, 49, 51



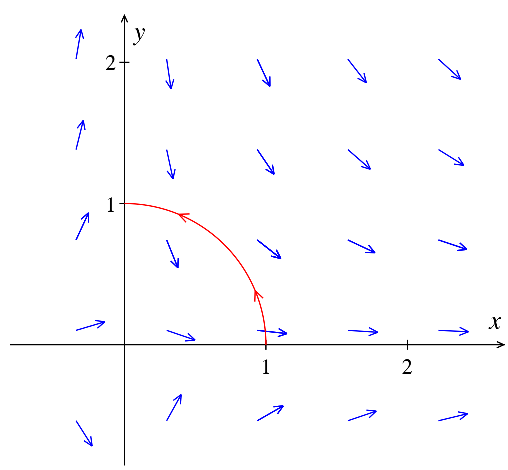
Example 1. *Evaluate* $\int_C (2 + x^2 y)\, ds$, *where* C *is the upper half of the unit circle,* $x^2 + y^2 = 1$.

Example 2. *If you have a semicircular wire* $y = \sqrt{25 - x^2}$ *with linear density function* $\delta(x, y) = 15 - y$, *what is the mass of the wire?*

Example 3. *Evaluate* $\displaystyle\int_C y^2\,dx + x\,dy$ *where* ***(a)*** $C = C_1$ *is the line segment from* $(-5,-3)$ *to* $(0,2)$ *and* ***(b)*** $C = C_2$ *is the arc of the parabola* $x = 4 - y^2$ *from* $(-5,-3)$ *to* $(0,2)$.



Example 4. *Find the work done by the force field* $\mathbf{F}(x,y) = x^2\mathbf{i} - xy\mathbf{j}$ *in moving a particle along the quarter circle* $\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j}$, $0 \le t \le \pi/2$.



Section 15.2: Line Integrals (Practice)

1. C is the line segment from $(1, 3)$ to $(5, -2)$, compute $\displaystyle\int_C x - y\, ds$. Solution: $\dfrac{5}{2}\sqrt{41}$

2. C is the line segment from (3,4,0) to (1,4,2), compute $\displaystyle\int_C z + y^2\, ds$. Solution: $17\sqrt{8}$

3. Evaluate the line inetgral $\displaystyle\int_C (xy+z^3)\, ds$, where C is the portion of the helix $\mathbf{r}(t) = \langle \cos t, \sin t, t\rangle$ from $t = 0$ to $t = \pi$ Solution: $\dfrac{\pi^4\sqrt{2}}{4}$

15.3 Independence of Path; Conservative Vector Fields

Definition 15.3.1. If $\mathbf{F}$ is a force field in 2-space or 3-space, then the *work* performed by the field on a particle moving along a parametric curve C from an initial point P to a final point Q is given by the integral

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C f(x,y)\,dx + g(x,y)\,dy,$$

where $\mathbf{F}(x,y) = f(x,y)\mathbf{i} + g(x,y)\mathbf{j}$ and $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$. We call an integral of this type a **work integral**.

Note 29. The parametric curve C in a work integral is called the **path of integration**. We will see through the next theorem if a force field $\mathbf{F}$ is conservative (i.e., is the gradient of some potential function ϕ), then the work that the field performs on a particle that moves from P to Q does not depend on the particular path C.

Theorem 15.3.1 (The Fundamental Theorem of Line Integrals)**.** Suppose that

$$\mathbf{F}(x,y) = f(x,y)\mathbf{i} + g(x,y)\mathbf{j}$$

is a conservative vector field in some open region D containing the points (x_0, y_0) and (x_1, y_1) and that $f(x,y)$ and $g(x,y)$ are continuous in this region. If

$$\mathbf{F}(x,y) = \nabla\phi(x,y)$$

and if C is any piecewise smooth parametric curve that starts at (x_0, y_0) , ends at (x_1, y_1), and lies in the region D, then

$$\int_C \mathbf{F}(x,y) \cdot d\mathbf{r} = \phi(x_1, y_1) - \phi(x_0, y_0)$$

or, equivalently,

$$\int_C \nabla\phi \cdot d\mathbf{r} = \phi(x_1, y_1) - \phi(x_0, y_0)$$

In plain terms this means the value of a line integral of a conservative vector field along a piecewise smooth path is ***independent of the path****.*

Note 30. For line integrals of conservative vector fields is is common to express the work integral as

$$\int_{(x_0,y_0)}^{(x_1,y_1)} \mathbf{F} \cdot d\mathbf{r} = \int_{(x_0,y_0)}^{(x_1,y_1)} \nabla\phi \cdot d\mathbf{r} = \phi(x_1, y_1) - \phi(x_0, y_0)$$

Definition 15.3.2 (Closed)**.** A parametric curve that is represented by the vector valued function $\mathbf{r}(t)$ for $a \le t \le b$ is said to be **closed** if the initial point $\mathbf{r}(a)$ and the terminal point $\mathbf{r}(b)$ coincide; that is, $\mathbf{r}(a) = \mathbf{r}(b)$.

Definition 15.3.3 (Connected)**.** A domain D is said to be **connected** if any two points in D can be joined by some piecewise smooth curve that lies entirely in D.

Theorem 15.3.2. If $f(x, y)$ and $g(x, y)$ are continuous on some open connected region D, then the following statements are equivalent (all true or all false):

(a) $\mathbf{F}(x, y) = f(x, y)\mathbf{i} + g(x, y)\mathbf{j}$ is a conservative vector field on the region D.

(b) $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every piecewise smooth closed curve C in D.

(c) $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of the path from any point P in D to any point Q in D for every piecewise smooth curve C in D.

Definition 15.3.4 (Simple)**.** A curve is called **simple** if it does not intersect itself between its endpoints. A region is said to be **simply connected** if no simple closed curve in D encloses points that are not in D. A set with one or more holes is said to be **multiply connected**.

Theorem 15.3.3 (Conservative Field Test)**.** If $f(x, y)$ and $g(x, y)$ are continuous and have continuous first partial derivatives on some open region D, and if the vector field $\mathbf{F}(x, y) = f(x, y)\mathbf{i} + g(x, y)\mathbf{j}$ is conservative on D, then

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$$

at each point in D. Conversely, if D is simply connected and $\partial f/\partial y = \partial g/\partial x$ holds at each point in D, then $\mathbf{F}(x, y)$ is conservative.

Section 15.3: page 1120: 1, 3, 5, 7, 13, 17, 23, 27, 29

Example 1. *The force field* $\mathbf{F}(x, y) = y\mathbf{i} + x\mathbf{j}$ *is conservative since the gradiendt of* $\phi(x, y) = xy$ *(verify). Thus, work performed by the field on a particle that moves from* $(0, 0)$ *to* $(1, 1)$ *should be independent of the chosen path. Confirm using the work integral.*

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{f}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$$

(a) $y = x$ (b) $y = x^2$ (c) $y = x^3$

Example 2. *Confirm that the force field* $\mathbf{F}(x, y) = y\mathbf{i} + x\mathbf{j}$ *in* **Example 1** *is conservative and use* ***Theorem 15.3.1*** *to evaluate.*

Example 3. *Use* ***Theorem 15.3.2*** *to determine whether the vector field*

$$\mathbf{F}(x, y) = (y + x)\mathbf{i} + (y - x)\mathbf{j}$$

is conservative on some open set.

Example 4. *Let* $\mathbf{F}(x, y) = 2xy^3\mathbf{i} + (1 + 3x^2y^2)\mathbf{j}$.

(a) Show that $\mathbf{F}$ is a conservative vector field on the entire xy-plane.

(b) Find ϕ by first integrating $\dfrac{\partial \phi}{\partial x}$ and then $\dfrac{\partial \phi}{\partial y}$.

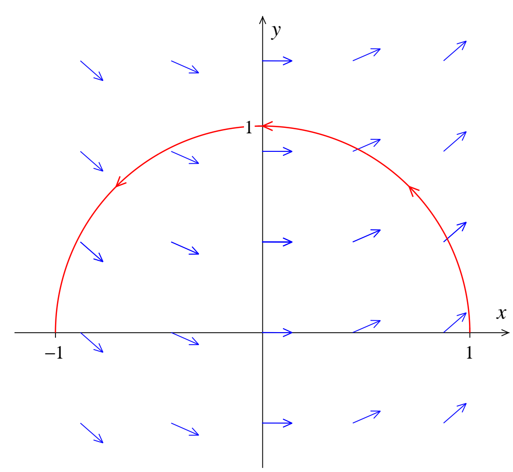
Example 5. *Use the potiential function obtained in* **Example 4** *to evaluate*

$$\int_{(1,4)}^{(3,1)} 2xy^3\,dx + (1+3x^2y^2)\,dy$$

Example 6. *Let* $\mathbf{F}(x,y) = e^y\mathbf{i} + xe^y\mathbf{j}$ *denote a force field in the xy-plane.*

(a) Verify that $\mathbf{F}$ *is conservative on the entire xy-plane.*

(b) Find work done by $\mathbf{F}$ *on the particle that moves along the semicircular path of radius 1 in the upper half plane.*



15.4 Green's Theorem

Theorem 15.4.1 (Green's Theorem)**.** Let R be a simply connected plane region whose boundary is a simple, closed, piecewise smooth curve C oriented counterclockwise. If $f(x, y)$ and $g(x, y)$ are continuous and have continuous first partial derivatives on some open set containing R, then

$$\int_C f(x,y)\,dx + g(x,y)\,dy = \iint\limits_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA$$

Note 31. It is common to denote a line integral around a simple closed curve by an integral sign with a superimposed circle over the integral. For example, Green's Theorem would be written in this way:

$$\oint_C f(x,y)\,dx + g(x,y)\,dy = \iint\limits_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA.$$

Sometimes a direction arrow is added to the circle to indicate whether the integration is clockwise or counterclockwise. Hence, if we wanted to emphasize the counterclockwise direction required by Green's Theorem we could write:

$$\ointctrclockwise_C f(x,y)\,dx + g(x,y)\,dy = \iint\limits_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA.$$

Note 32. Green's Theorem gives some useful formulas for the area A of the region R that satisfies the conditions of the theorem. The two formulas can be found using the following:

$$A = \iint\limits_R dA = \ointctrclockwise_C x\,dy \qquad \text{and} \qquad A = \iint\limits_R dA = \ointctrclockwise_C (-y)\,dx$$

Or equivalently,

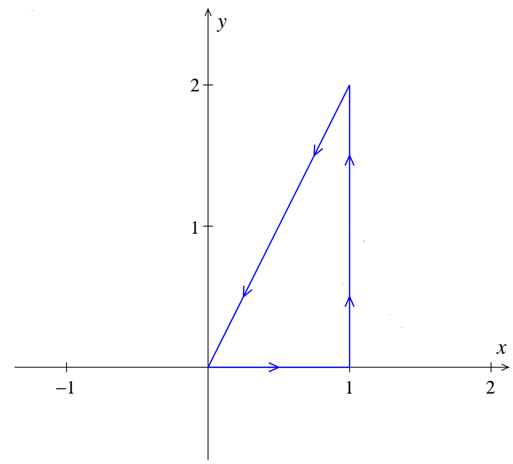
$$A = \frac{1}{2}\ointctrclockwise_C -y\,dx + x\,dy.$$

Note 33 (See image in handwritten notes)**.** Suppose we have a multiply connected region R such that the C_1 and C_2 define the boundary of R where C_2 lies in the interior of C_1. We can extend Green's Theorem to multiply connected regions by the following formula:

$$\iint\limits_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA = \ointctrclockwise_{C_1} f(x,y)\,dx + g(x,y)\,dy + \ointclockwise_{C_2} f(x,y)\,dx + g(x,y)\,dy$$

Section 15.4: page 1127: 1, 3, 7, 9, 11, 13, 21, 25, 27

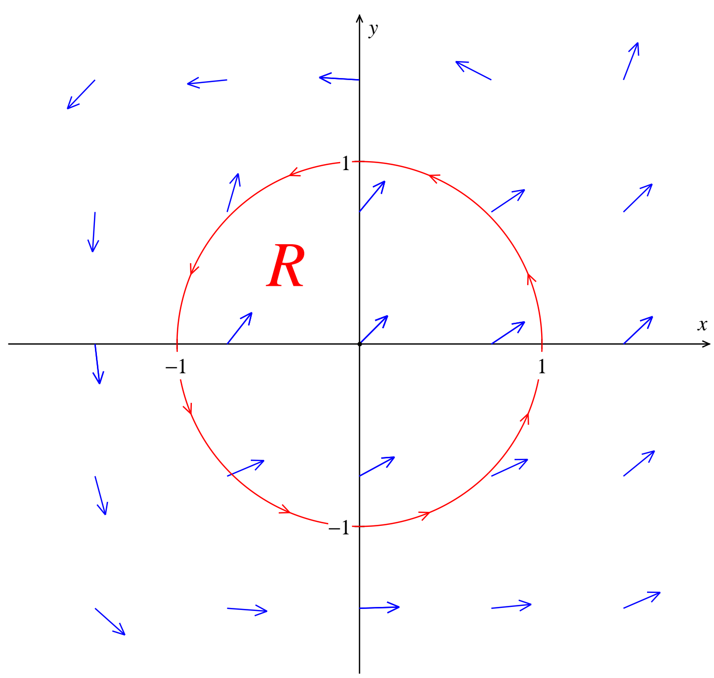
Example 1. *Use Green's Theorem to evaluate* $\int_C x^2y\,dx + x\,dy$ *along the trangular path, C, shown below.*



Example 2. *Find the work done by the force field*

$$\mathbf{F}(x, y) = (e^x - y^3)\mathbf{i} + (\cos y + x^3)\mathbf{j}$$

on a particle that travels around the unit circle in the counterclockwise direction.



Example 3. *Use a line integral to find the area enclosed by the ellipse*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Example 4. *Use Green's Theorem to evaluate* $\oint_C \mathbf{F} \cdot \mathbf{dr}$ *where* $\mathbf{F}(x, y) = \langle y^2 \cos x, x^2 + 2y \sin x\rangle$ *and* C *is the triangle from* $(0, 0)$ *to* $(2, 6)$ *to* $(2, 0)$ *to* $(0, 0)$.

Remark on **Note *33*** :

Example 5. *Evaluate the integral* $\oint_C \frac{-y\,dx + x\,dy}{x^2 + y^2}$ *if* C *is a piecewise, smooth, simple, closed curve oriented counterclockwise such that*

(a) C doesn't enclose the origin.

(b) C encloses the origin.

15.5 Surface Integrals

Similar to a line integral we can consider the mass of a curved lamina or a **curved lamina**. A curved lamina is an idealized object that is thin enough to be viewed as a bent surface in 3-space. Given a point (x, y, z) on a surface parametric σ, we will let $f(x, y, z)$ denote the corresponding value of the density function.

Divide σ into small patched $\sigma_1, \sigma_2, \ldots, \sigma_n$ with areas $\Delta S_1, \Delta S_2, \ldots, \Delta S_n$. Let (x_k^*, y_k^*, z_k^*) be a sample point in the kth patch with mass $\Delta M_k \approx f(x_k^*, y_k^*, z_k^*)\Delta S_k$. Therefore the mass of the entire lamina can be approximated by

$$M = \sum_{k=1}^{n} \Delta M_k \approx \sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)\Delta S_k$$

Definition 15.5.1. If σ is a smooth parametric surface, then the **surface integral** of $f(x, y, z)$ over σ is

$$\iint_{\sigma} f(x, y, z)\, dS = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)\Delta S_k$$

provided this limit exists and does not depend on the way the subdivisions of σ are made or how the sample points (x_k^*, y_k^*, z_k^*) are chosen

Theorem 15.5.1. Let σ be a smooth parametric surface whose vector equation is

$$\mathbf{r} = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$$

where (u, v) varies over a region R in the uv-plane. If $f(x, y, z)$ is continuous on σ, then

$$\iint_{\sigma} f(x, y, z)\, dS = \iint_{R} f(x(u, v), y(u, v), z(u, v)) \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| dA$$

Note 34. When σ is of the form $z = g(x, y)$, take $x(u, v) = u$ and $y(u, v) = v$ as the parameters then

$$\mathbf{r} = u\mathbf{i} + v\mathbf{j} + g(u, v)\mathbf{k}$$

gives

$$\left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| = \sqrt{\left(\frac{\partial g}{\partial u}\right)^2 + \left(\frac{\partial g}{\partial v}\right)^2 + 1}$$

Theorem 15.5.2.

(a). Let σ be a surface with equation $z = g(x, y)$ and let R be its projection on the xy-plane. If g has continuous first partial derivatives on R and $f(x, y, z)$ is continuous on σ, then

$$\iint_{\sigma} f(x,y,z)\,dS = \iint_{R} f(x,y,g(x,y))\sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}\,dA$$

(b). Let σ be a surface with equation $y = g(x, z)$ and let R be its projection on the xz-plane. If g has continuous first partial derivatives on R and $f(x, y, z)$ is continuous on σ, then

$$\iint_{\sigma} f(x,y,z)\,dS = \iint_{R} f(x,g(x,z),z)\sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial z}\right)^2 + 1}\,dA$$

(c). Let σ be a surface with equation $x = g(y, z)$ and let R be its projection on the yz-plane. If g has continuous first partial derivatives on R and $f(x, y, z)$ is continuous on σ, then

$$\iint_{\sigma} f(x,y,z)\,dS = \iint_{R} f(g(y,z),y,z)\sqrt{\left(\frac{\partial g}{\partial y}\right)^2 + \left(\frac{\partial g}{\partial z}\right)^2 + 1}\,dA$$

Note 35. Surface integrals have the same physical consequence as line integrals. Consider $\rho(x, y, z)$ to be the density function of a thin sheet, σ. Then the mass of σ is

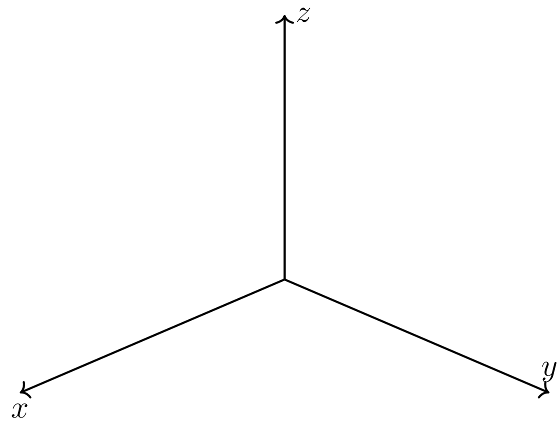
$$M = \iint_{\sigma} \rho(x,y,z)\,dS$$

Moreover, the center of mass of σ is $(\overline{x}, \overline{y}, \overline{z})$ where

$$\overline{x} = \frac{1}{M}\iint_{\sigma} x\rho(x,y,z)\,dS \qquad \overline{y} = \frac{1}{M}\iint_{\sigma} y\rho(x,y,z)\,dS \qquad \overline{z} = \frac{1}{M}\iint_{\sigma} z\rho(x,y,z)\,dS$$

and for the centriod $M = (\text{Area of } \sigma)$ and $\rho(x, y, z) = 1$.

Section 15.5: page 1127: 1, 3, 5, 13, 14, 15, 16, 17, 19, 23, 27, 34



Example 1. *Compute the surface integral* $\iint_\sigma x^2\, dS$*, where* σ *is the unit sphere.*

Example 2. *Evaluate the surface integral* $\iint_\sigma xz\,dS$*, where* σ *is the part of the plane* $x+y+z=1$ *in the first octant.*

Example 3. *Evaluate the surface integral* $\iint_\sigma y^2z^2\,dS$*, where* σ *is part of the cone* $z=\sqrt{x^2+y^2}$ *that lies between* $z=1$ *and* $z=2$*.*

Example 4. *Find* $\displaystyle\iint_{\sigma} f(x, y, z)\, dS$ *for* $f(x, y, z) = x + y$ *and* σ *is the portion of the plane* $z = 6 - 2x - 3y$ *in the first octant.*

15.6 Application of Surface Integrals; Flux

In section we will discuss applications of surface integrals to vector fields associated with fluid flow and electrostatic forces.

$$\text{Recall: for parametric surface } \mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle, \quad \mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{||\mathbf{r}_u \times \mathbf{r}_v||}$$

Definition 15.6.1. A two sided surface is said to be **orientable**, and a one sided surface is said to be **nonorientable**. The above formula defines a **positive orientation** for the parameterized surface. The orientation defined by $-\mathbf{n}$ is called the **negative orientation**.

Definition 15.6.2. (For the purpose of our course) **Flux** is the volume that passes through a surface in one given unit of time.

picture:

Definition 15.6.3. The **flux of F across** σ can be expressed as the surface integral

$$\Phi = \iint_\sigma \mathbf{F}(x,y,z) \cdot d\mathbf{S} = \iint_\sigma \mathbf{F}(x,y,z) \cdot \mathbf{n}(x,y,z).$$

Theorem 15.6.1. Let σ be a smooth parametric surface represented by the vector equation $\mathbf{r} = \mathbf{r}(u,v)$ in which (u,v) varies over a region R in the uv-plane. If the component functions of the vector field $\mathbf{F}$ are continuous on σ, and if $\mathbf{n}$ determines the positive orientation of σ, then

$$\Phi = \iint_\sigma \mathbf{F} \cdot \mathbf{n}\, dS = \iint_R \mathbf{F} \cdot \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right) dA$$

where it is understood that the integrand on the right hand side is expressed in terms of u and v.

Some useful versions of the flux integral are:

$$\begin{aligned}
\iint_\sigma \mathbf{F} \cdot \mathbf{n}\, dS &= \iint_R \mathbf{F} \cdot \left(-\frac{\partial z}{\partial x}\mathbf{i} - \frac{\partial z}{\partial y}\mathbf{j} + \mathbf{k}\right) dA, && \sigma \text{ of the form } z = g(x,y) \text{ oriented up} \\
\iint_\sigma \mathbf{F} \cdot \mathbf{n}\, dS &= \iint_R \mathbf{F} \cdot \left(\frac{\partial z}{\partial x}\mathbf{i} + \frac{\partial z}{\partial y}\mathbf{j} - \mathbf{k}\right) dA, && \sigma \text{ of the form } z = g(x,y) \text{ oriented down}
\end{aligned}$$

Section 15.6: page 1146: 9-15 (odd), 23, 24, 27

Example 1. Consider surface, σ, of the cylinder $x^2 + z^2 = 16$ where $0 \leq y \leq 5$.

(a) Sketch σ.

(b) Find a parameterization.

(c) Find $\mathbf{n}$ sketch a few normal vectors to show the positive orientation of σ.

(d) Find an oppositely oriented parameterization.

Example 2. Find the flux of $\mathbf{F}(x, y, z) = 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$ through the disk of radius 5 in the plane $z = 1$ oriented upward.

Example 3. Find the flux of $\mathbf{F}(x, y, z) = (y+4)\mathbf{j}$ through a square of side length 2 in the xz-plane oriented in the negative y-direction.

Example 4. Find the flux of the vector field $\mathbf{F}(x, y, z) = z\mathbf{k}$ across the outward oriented sphere $x^2 + y^2 + z^2 = 9$.

Example 5. Evaluate the flux integral $\displaystyle\iint_\sigma \mathbf{F} \cdot d\mathbf{S}$ for the vector field $\mathbf{F}(x, y, z) = y\mathbf{i} + (z - y)\mathbf{j} + x\mathbf{k}$ where σ is the surface of the tetrahedron with verticies (0,0,0), (1,0,0), (0,1,0), and (0,0,1) with outward orientation.

15.7 The Divergence Theorem

In this section we are concerned with surfaces that are boundaries of finite solids. Such surfaces are said to be **closed**. A closed surface may not be a smooth surface, but most applications in this sections the surfaces will be at least **piecewise smooth**. We will discuss surfaces that can have an **inward** (toward the interior) and **outward** (away from the interior) orientation. The following theorem will provide us with a physical interpretation of divergence in the context fluid flow.

Theorem 15.7.1 (Gauss's Theorem)**.** Let G be a solid whose surface σ is oriented outward. If

$$\mathbf{F}(x, y, z) = f(x, y, z)\mathbf{i} + g(x, y, z)\mathbf{j} + h(x, y, z)\mathbf{k}$$

where f, g, and h have continuous first partial derivatives on some open set containing G, and if $\mathbf{n}$ is the outward normal on σ, then

$$\iint_\sigma \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_G \operatorname{div} \mathbf{F}\, dV$$

The Divergence Theorem applied to inverse-square fields produces a result called **Gauss's Law for Inverse-Square Fields**.

Theorem 15.7.2. If

$$\mathbf{F}(\mathbf{r}) = \frac{c}{||\mathbf{r}||^3}\mathbf{r}$$

is an inverse-square field in 3-space, and if σ is a closed orientable surface that surrounds the origin, then the outward flux of $\mathbf{F}$ across σ is

$$\Phi = \iint_\sigma \mathbf{F} \cdot \mathbf{n}\, dS = 4\pi c$$

Section 15.7: page 1157: 1, 3, 5, 9, 11, 13

Example 1. Find the outward flux for the closed sphere surface in **Example 4** of the previous section using the Divergence Theorem.

Example 2. Find the outward flux for the closed tetrahedron surface in **Example 5** of the previous section using the Divergence Theorem. The tetrahedron with verticies (0,0,0), (1,0,0), (0,1,0), and (0,0,1).

Example 3. Use the Divergence Theorem to find the outward flux of the vector field

$$\mathbf{F}(x, y, z) = 2x\mathbf{i} + 3y\mathbf{j} + z^2\mathbf{k}$$

across the unit cube in the first octant with one corner on the origin.

Example 4. Use the Divergence Theorem to find the outward flux of the vector field

$$\mathbf{F}(x, y, z) = x^3\mathbf{i} + y^3\mathbf{j} + z^2\mathbf{k}$$

across the surface of the region that is enclosed by the circular cylinder $x^2 + y^2 = 9$ and the planes $z = 0$ and $z = 2$.

Section 15.7: The Divergence Theorem (Practice)

Use the Divergence Theorem to evaluate the outward flux.

1. $\mathbf{F}(x, y, z) = 5\mathbf{j} + 7\mathbf{k}$; σ is the spherical surface $x^2 + y^2 + z^2 = 1$. Solution: 0

2. $\mathbf{F}(x, y, z) = xy\mathbf{i} + yz\mathbf{j} + xz\mathbf{k}$; σ is the surface of the cube bounded by the planes $x = 0$, $x = 2$, $y = 0$, $y = 2$, $z = 0$, $z = 2$. Solution: 24

3. $\mathbf{F}(x, y, z) = z^3\mathbf{i} - x^3\mathbf{j} + y^3\mathbf{k}$; σ is the surface $x^2 + y^2 + z^2 = a^2$ Solution: 0

15.8 Stokes' Theorem

Theorem 15.8.1 (Stokes' Theorem)**.** Let σ be a piecwise smooth oriented surface that is bounded by a simple closed, piecwise smooth curve C with positive oritenation. If the components of the vector field

$$\mathbf{F}(x, y, z) = f(x, y, z)\mathbf{i} + g(x, y, z)\mathbf{j} + h(x, y, z)\mathbf{k}$$

are continuous and have continuous first partial derivatives on some open set containing σ, and if $\mathbf{T}$ is the unit tangent vector to C, then

$$\oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \iint_\sigma (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS$$

The work performed by a force field on a particle that traverses a simple, closed, piecewise smooth curve C in the positive direction can be obtained by integrating the normal compoment of the curl over <u>an</u> oriented surface σ bounded by C.

It's common to express Stokes' Theorem this way:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_\sigma (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS$$

Section 15.8: page 1164: 1, 3, 5, 9, 13

Example 1. Let σ be the triangular surface of the plane $2x + 2y + z = 6$ bounded in the first octant oriented with normal away from the origin and C be the oriented boundary of σ. Find the work performed by the field $\mathbf{F}(x, y, z) = -y^2\mathbf{i} + z^2\mathbf{j} + x\mathbf{k}$.

Example 2. Find the work performed by the force field

$$\mathbf{F}(x, y, z) = x^2\mathbf{i} + 4xy^3\mathbf{y} + y^2x\mathbf{k}$$

on a particle that traverses the rectangle from $(0, 0, 0)$ to $(0, 3, 3)$ to $(1, 3, 3)$ to $(1, 0, 0)$.

Example 3. Verfiy Stokes' Theorem for the vector field $\mathbf{F}(x, y, z) = 2z\mathbf{i} + 3x\mathbf{j} + 5y\mathbf{k}$, taking σ to be the portion of the paraboloid $z = 4 - x^2 - y^2$ for which $z \geq 0$ with upward orientation, and C to be the positively oriented circle $x^2 + y^2 = 4$ that forms the boundary of σ in the xy-plane.

Content Overview: Sections 15.1-15.8

15.1: Vector Fields

- Inverse Square Field
- Conservative Fields and Potiential Functions
- Divergence and Curl
- Del ∇ and Laplacian Operator ∇^2

15.2: Line Integrals

- Line integral along a curve (Parameterize Curves)
- Mass of a wire
- Work performed by a force field

15.3: Independence of Path; Conservative Vector Fields

- Fundamental Theorem of Line Integrals
- Independence of path
- Closed, Connected, Simple, Simply Connected, Multiply Connected

15.4 Green's Theorem

- Green's Theorem
- Integral notation for closed curves
- Extension of Green's Theorem to multiply connected regions

15.5 Surface Integrals

- **Theorem 15.5.1** Surface Integrals to Double Integrals
- Parameterizing surfaces
- Integrating over $xy-$, $xz-$, $yz-$ solids (**Theorem 15.5.2**)

15.6 Applications of Surface Integrals; Flux

- Orientable Surfaces verses Nonorientable Surfaces
- Φ the flux integral

15.7 The Divergence Theorem

- Inward verses outward orientation
- Using the Divergence Theorem to find flux

15.8 Stokes' Theorem

- Positive Orientation of boundary C
- Stokes' Theorem (use to find work)

Review Problems for Chapter 15

1. Let $\mathbf{F}(x,y,z) = (x^3 \ln z)\mathbf{i} + (xe^{-y})\mathbf{j} - (y^2+2z)\mathbf{k}$. Calculate the divergence of $\mathbf{F}$, div $\mathbf{F}$, at the point $(2, \ln 2, 1)$.

2. Let $\mathbf{F}(x,y,z) = \cos x\mathbf{i} + \sin y\mathbf{j} + e^{xy}\mathbf{k}$. Calculate curl $\mathbf{F}$ at the point (1,1,1).

3. Let $\mathbf{F}(x,y,z) = (2xy+z^2)\mathbf{i} + x^2\mathbf{j} + (2xz + \pi\cos\pi z)\mathbf{k}$. Find the potential function of $\mathbf{F}$.

4. Let C be the line segment from (0,0,0) to (1,3,-2). Find $\displaystyle\int_C x + y^2 - 2z\, ds$.

5. Find $\nabla\left(\dfrac{x+y}{x-y}\right)$ (solution should be a vector valued function).

6. Evaluate $\displaystyle\int_C x - y\,ds$; $C:\ x^2+y^2=1$.

7. Evaluate $\displaystyle\int_C x\,dx + z\,dy - 2y^2\,dz$; $C:\ x=\cos t, y=\sin t, z=t,\ (0\le t\le 2\pi)$.

8. A particle moves upward along the circular helix parameterized by $\mathbf{r}(t)=\cos t\mathbf{i}+\sin t\mathbf{j}+t\mathbf{k}$ for $0\le t\le 2\pi$ under a force given by $\mathbf{F}(x,y,z)=-zy\mathbf{i}+zx\mathbf{j}+xy\mathbf{k}$. Find the work done on the particle by the force field.

9. Let $\mathbf{F}(x,y,z) = y\mathbf{i} + x\mathbf{j} + z^2\mathbf{k}$, and evaluate $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r}$ for the curve $\mathbf{r}(t) = t\mathbf{i} + \cos t\mathbf{j} + \sin t\mathbf{k}$, for $0 \le t \le 2\pi$.

10. State Green's, Divergence, and Stokes' Theorems including the required hypotheses.

11. Use Green's Theorem to Evaluate the line integral $\displaystyle\int_C x - y^3\,dx + x^3\,dy$ where C is the circle $x^2 + y^2 = 4$.

12. Use Green's Theorem to evaluate $\displaystyle\oint_C e^x + y^2\,dx + e^y + x^2\,dy$, where C is the boundry of the region between $y = x^2$ and $y = x$.

13. Set up, but do not evaluate, two different iterated integrals equal to the given intgral.
$\iint_\sigma x^2y\,dS$, where σ is the portion of the cylinder $y^2+z^2=a^2$ in the first octant between the planes $x=0$, $x=9$, $z=y$, and $z=\sqrt{3}y$.

14. Evaluate $\iint_\sigma xyz\,dS$, where σ is the cone with parametric equations $x=u\cos v$, $y=u\sin v$, $z=u$, $0\le u\le 1$, $0\le v\le \pi/2$.

15. Find flux for $\mathbf{F}(x,y,z)=x\mathbf{i}-y\mathbf{j}+z^3\mathbf{k}$, σ is the cone $z=\sqrt{x^2+y^2}$ between the planes $z=1$ and $z=3$ with downward orientation.

16. Find flux for $\mathbf{F}(x,y,z)=yz\mathbf{i}+xz\mathbf{j}+xy\mathbf{k}$, σ is the surface $z=x\sin y$, $0\le x\le 2$, $0\le y\le\pi$ with upward orientation.

17. Using The Divergence Theorem find the flux for $\mathbf{F}(x, y, z) = z\mathbf{i} + y\mathbf{j} + x\mathbf{k}$, σ is the tetrahedron enclosed by the coordinate planes and the plane

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

where a,b, and c are positive numbers.

18. Use the divergence theorem to calculate the flux of $\mathbf{F} = xe^{y}\mathbf{i} + (z - e^{y})\mathbf{j} - xy\mathbf{k}$ over S the ellipsoid $x^2 + 2y^2 + 3z^2 = 4$.

19. Let $\mathbf{F} = z\tan^{-1}(y^2)\mathbf{i} + z^3\ln(x^2+1)\mathbf{j} + z\mathbf{k}$. Find the flux of $\mathbf{F}$ across the part of the paraboloid $x^2 + y^2 + z = 2$ that lies above the plane $z = 1$ and is oriented upward. (Hint: you must subtract the circular base of the paraboloid after using the divergence theorem)

20. Use Stokes' Theorem to find the work performed by the field $\mathbf{F}$ over the curve C if $\mathbf{F} = xy\mathbf{i} + yz\mathbf{j} + xz\mathbf{k}$, C is the boundary of the part of the paraboloid $z = 1 - x^2 - y^2$ in the first octant, C oriented clockwise as viewed from above.

Content Overview

Chapter 11

- **Rectangular Coordinates, Planes, Spherical and Cylindrical Surfaces**
- **Vectors-** Magnitude, Addition, Subtraction, Scalar Multiplication, and Unit Vectors
- **The Dot Product-** Properties, Direction Cosines, Direction Angles, Orthogonality
- **The Cross Product-** Properties, Parallelogram Area, Parallelepiped Volume
- **Parametric Equations of Lines and Planes-** Direction numbers, Normal Vector, Vector equation of a plane, Scalar equation of a plane, Parallel Lines, Distance from a point to a plane.
- **Cylinders and Quadratic Surfaces-** Ellipsoids, Hyperboloids, Paraboloids, and their Equations
- **Cylindrical and Spherical Coordinates-**Conversion of rectangular to each and vise versa

Chapter 12

- **Vector Functions-**Plane curve, and space curves, limits of vector functions
- **Derivatives and Integrals of Vector Functions-**Unit tangent, Differentiation rules
- **Arc Length and Change of Parameter-** Parameterizations and definition of arc length
- **Unit Tangent, Normal, and Binormal Vectors**
- **Curvature-**The different ways to express curvature, Circle of Curvature
- **Motion Along a Curve-**Velocity vector, speed, acceleration, new expressions for κ

Chapter 13

- **Functions of Two or more Variables-** Level curves and surfaces
- **Limits and Continuity-**Limit definition, limits along curves, definition of continuous
- **Partial Derivatives-**Definition of partial derivative, higher order partial derivatives, notation
- **Differentials-**Local linearity, total differential, local linear approximation to f and (x_0, y_0).
- **The Chain Rule-**Implicit differentiation, chain rule for partial derivatives
- **Directional Derivatives and Gradients-** Properties of the gradient
- **Normal Lines and Tangent Planes-**Equation of tangent plane
- **Maxima and Minima** Absolute, relative max and min (extremum), critical point, saddle point, Second Partials Test
- **Lagrange Multipliers-**Extremum problems with constraints, parallel to level curves

Chapter 14

- **Double Integrals Over Rectangular Regions**
- **Double Integrals Over Nonrectangular Regions**-Type I and Type II regions
- **Double Integrals in Polar Coordinates**-Simple polar regions, areas of polar regions
- **Surface Area, Parametric Surfaces**-Surface area integral, tangent planes
- **Triple Integrals**-Volume, simple solids
- **Triple Integrals in Cylindrical and Spherical Coordinates**
- **Change of Variables and Jacobians**-transformation of a plane
- **Centers of Gravity using Multiple Integrals**-mass, center of gravity, centroid

Chapter 15

- **Vector Fields**-Curl, divergence, and gradient field of ϕ, square inverse field
- **Line Integrals**-Parameterizing curves, Line integrals with respect to arc length, work as a line integral
- **Independence of Path; Conservative Vector Fields**-Fundamental Theorem of Line Integrals, simple, simply connected, multiply connected region, conservative field test
- **Green's Theorem**-Hypotheses, Conclusion, and notation.
- **Surface Integrals- Theorems 15.5.1-2** ,Parameterizing surfaces
- **Applications of Surface Integrals; Flux**- Orientable vs nonorientable surfaces, Flux
- **The Divergence Theorem**- Inward verses outward orientation, Hypotheses of the Theorem
- **Stokes' Theorem**- Positive orientation of boundary C, Hypotheses of the Theorem

Fundamental Theorem for Line Integrals:

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)), \quad \leftarrow \text{ we used } \phi \text{ for } f \text{ in class}$$

Green's Theorem:

$$\iint_D \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA = \int_C f(x,y)\,dx + g(x,y)\,dy$$

The Divergence Theorem:

$$\iiint_G \operatorname{div} \mathbf{F}\,dV = \iint_\sigma \mathbf{F} \cdot d\mathbf{S}$$

Stokes' Theorem:

$$\iint_\sigma \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r}$$

1. Describe the surface with equation $x^2 + y^2 + z^2 + 10x + 4y + 2z - 19 = 0$.

2. Find the equation of the plane that passes through the points $(2, 1, -4)$, $(-3, 1, 3)$, and $(-2, 7, 0)$.

3. Write the equation of a plane that contains the y-axis and the point $(1, 0, 2)$.

4. True/False:_____ The norm of the sum of two vectors is equal to the sum of the norms of the two vectors

5. True/False:_____ The dot product of two unit vectors is 1.

6. True/False:_____ The cross product of two unit vectors is a unit vector.

7. Find the arc length: $x = t^2$, $y = 2t^2 - 1$, $1 < t < 4$.

8. Let $\mathbf{u} = 3\mathbf{i} - \mathbf{j} - 2\mathbf{k}$, $\mathbf{v} = -2\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$. Find $\text{proj}_{\mathbf{v}}\mathbf{u}$.

9. Find the velocity vector for an object having $\mathbf{a}(t) = e^t\mathbf{j} - 32\mathbf{k}$, if $\mathbf{v}(0) = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$.

10. Find the tangential component of acceleration for $\mathbf{r}(t) = t\mathbf{i} + \dfrac{\sqrt{6}}{2}t^2\mathbf{j} + t^3\mathbf{k}$.

11. Graph the surface $x^2 + \dfrac{y^2}{4} - z = 0$.

12. Let $\mathbf{u} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\mathbf{v} = 3\mathbf{i} - \mathbf{j} + 4\mathbf{k}$. Find the angle between $\mathbf{u}$ and $\mathbf{v}$.

13. Find the curvature, κ, for the function $\mathbf{r}(t) = e^t \cos(t)\mathbf{i} + e^t \sin(t)\mathbf{j} + t\mathbf{k}$.

14. Evaluate the following limit, $\lim_{t\to 1} \left[\frac{\sin(t-1)}{t-1}\mathbf{i} + \frac{t+3}{t-2}\mathbf{j} + \cos(\pi t)\mathbf{k} \right]$.

15. Find the equation in rectangular coordinates for the spherical coordinate equation $\rho = 9 \csc\phi \csc\theta$ and identify the surface.

16. The hyperbolic paraboloid is given by $z = y^2 - x^2$. Sketch the contour map for the surface using $z = -4, -1, 0, 1, 4$.

17. Show that $f(x, y) = \ln(x^2 + y^2)$ satisfies Laplace's equation $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = 0$.

18. Evaluate the limit, $\displaystyle\lim_{(x,y)\to(0,0)} \left(\frac{x^2 - y^2}{x^2 + y^2}\right)^2$.

19. For $f(x, y, z) = \sqrt{x}e^y \arctan z$, find the gradient at $(4, 0, 1)$.

20. Find the directional derivative of $f(x, y) = 3x^2y$ at the point $(1, 2)$ in the direction of $\mathbf{v} = 3\mathbf{i} + 4\mathbf{j}$.

21. Use the chain rule to find $\dfrac{\partial z}{\partial u}$ if $z = \cos x \sin y$, $x = u - v$, and $y = u^2 + v^2$. Write your answer as a function of u and v.

22. Let $f(x,y) = \dfrac{x}{x^2+y^2}$. Use a total differential to approximate the change in $f(x,y)$ as (x,y) varies from the point $(1,2)$ to the point $(0.98, 2.01)$.

23. Find an equation for the tangent plane to the surface given by $f(x,y) = x^2y$ at the point $(2,-1,\underline{\quad})$.

24. Find the saddle point for $f(x,y) = x^2 - y^2 - 2x - 6y - 3$. Justify your answer.

25. Use Lagrange multipliers to find the maximum and minimum values of f subject to the given constraints: $f(x,y) = x - 3y - 1$; with $x^2 + 3y^2 = 16$.

26. Use Lagrange multipliers to find 3 positive numbers whose sum is 33 and whose product is maximized (What does your intuition tell you?).

27. Integrate $\displaystyle\iint_R xy\, dA$ where R is the region bounded by $y = \sqrt{x}$, $y = \dfrac{1}{2}x$, $x = 2$ and $x = 4$.

28. Evaluate $\displaystyle\int_0^1 \int_{\sqrt{y}}^1 \sin(\pi x^3)\, dx\, dy$ by reversing the order of integration.

29. Evaluate $\displaystyle\int_0^1 \int_0^{\sqrt{1-x^2}} e^{-(x^2+y^2)}\, dy\, dx$ using polar coordinates.

30. Find the volume of the solid in the first octant bounded by the graphs of $z = 1 - y^2$, $y = 2x$ and $x = 3$.

31. A thin lamina has the shape of a triangle with vertices at $(0,0)$, $(2,0)$ and $(2,4)$. The density function associated with the lamina has the equation $\delta(x,y) = 4x + 2y + 2$. Find the center of mass of the lamina.

32. Find the surface area of that portion of the sphere $x^2 + y^2 + z^2 = 4$ that is above the xy-plane within the cylinder $x^2 + y^2 = 1$.

33. Compute curl $\mathbf{F}$ if $\mathbf{F}(x,y,z) = xy\mathbf{i} + (y^2 - 2z)\mathbf{j} + \sin(yz)\mathbf{k}$.

34. Evaluate the line integral $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}$ is the vector field, $\mathbf{F} = x^2\mathbf{i} + xy\mathbf{j} + xz\mathbf{k}$ and C is the line segment from $(0,0,0)$ to $(1,3,-2)$.

35. Evaluate the line integral $\int_C xy\,dx + z\,dy - xz\,dz$ where C is the line segment from $P(1,1,3)$ to $Q(3,-1,4)$.

36. Show that $\mathbf{F} = (z^2 + 2xy)\mathbf{i} + x^2\mathbf{j} + (2xz + 3z^2)\mathbf{k}$ is conservative.

37. Let C be a curve from the point $(0,0,1)$ to the point $(2,-3,1)$ in the shape of an electric eel. Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$.

38. Find the flux of the vector field $\mathbf{F}(x,y,z) = 2yz\mathbf{i} - 2xz\mathbf{j} + 3z^2\mathbf{k}$ upward through the hemisphere of radius 3 with $z \geq 0$ and centered at the origin.