# Linear Algebra
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Text: Lay, David C. *Linear Algebra and its Applications*, 3rd Ed. Pearson Education, Inc.

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# Chapter 1

# Linear Equations in Linear Algebra

## 1.1 Systems of Linear Equations

**Definition 1.1.1.** A **linear equation** in the variables $x_1, x_2, \ldots, x_n$ is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where $b$ and the **coefficients** $a_1, a_2, \ldots, a_n$ are real or complex numbers.

**Definition 1.1.2.** A **system of linear equations** (or linear system) is a collection of one or more linear equations involving the same variables say, $x_1, x_2, \ldots, x_n$.

**Definition 1.1.3.** A **solution** to a system is a list $(s_1, s_2, \ldots s_n)$ of numbers that makes eqach equation a true statement when the values $s_1, s_2, \ldots s_n$ are substituted for $x_1, x_2, \ldots, x_n$ respectively.

The set of all possible solutions is called the **solution set** of the linear system. Two linear systems are called equivalent if they have the same solution set.

**Note 1.1.1.** A system of linear equations has either

1. no solutions, or
2. exactly one solution, or
3. infinitely many solutions.

**Definition 1.1.4.** A system of linear equations is said to be **consistent** if it has either one solution or infinitely many solutions; a system is **inconsistent** if it has no solution.

**Definition 1.1.5.** The essential information of a linear system can be recored compactly in a rectangular array called a **matrix**. Each linear equation is placed in a row of the array and the respective coefficients of the variables of the system are placed in the columns, this is called the **coefficent matrix**.

| System of Linear Equations | Coefficient Matrix | Augmented Matrix |
|---|---|---|
| $\left\{\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \end{array}\right.$ | $\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix}$ | $\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$ |

**Definition 1.1.6.** The **size** of a matrix tells how many rows and columns it has. The augmented matrix on the previous page has 3 rows and 4 columns and is called a $3 \times 4$ matrix, read as a "three by four matrix". If a matrix has $m$ rows and $n$ columns, where $m$ and $n$ are positive integers, then we say it is an $\mathbf{m} \times \mathbf{n}$ **matrix** with $m \cdot n$ entries.

**Note 1.1.2** (Elementary Row Operations)**.**

1. (Interchange) Interchange two rows.
   notation: $R_i \leftrightarrow R_j$
2. (Scaling) Multiply all entries in a row by a nonzero constant, $k$.
   notation: $kR_i$
3. (Replacement) Replace one row by the sum of itself and a multiple of another row.
   notation: $R_i + kR_j \rightarrow R_i$

Row replacement is a common paraphrase for adding one row to a multiple of another row.

**Definition 1.1.7.** We say two matrices are **row equivalent** if there is a sequence of elementary row operations that transforms one matrix into the other.

**Homework Assignment for Section 1.1: (Page 11) 3, 5, 7, 13, 17, 19**

**Example 1.** *Solve the system below using elementary row operations.*

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ \phantom{x_1 -{}} 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}$$

**Example 2.** *Determine if the following system is consistent.*

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ \phantom{x_1 -{}} 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}$$

**Example 3.** *Determine if the following system is consistent.*

$$\begin{cases} x_2 - 4x_3 = 8 \\ 2x_1 - 3x_2 + 2x_3 = 1 \\ 5x_1 - 8x_2 + 7x_3 = 1 \end{cases}$$

## 1.2 Row Reductions and Echelon Forms

**Definition 1.2.1.** A rectangular matrix is in **echelon form** (or row echelon form) if it has the following three properties:

1. All nonzero rows are above any rows of all zeros.
2. Each leading entry (leftmost nonzero entry) of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zeros.

If a matrix in echelon form satisfies the following additional conditions, then it is in **reduced echelon form** (or **reduced row echelon form**):

4. The leading entry in each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column.

Echelon Form

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5/2 \end{bmatrix}$$

Reduced Row Echelon Form

$$\begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

**Definition 1.2.2.** A **pivot position** in a matrix $A$ is a location of an element in $A$ that corresponds to a leading 1 in the reduced row echelon form of $A$. A **pivot column** is a column of $A$ that contains a pivot position.

**Theorem 1.2.1.** Each matrix is row equivalent to one and only one reduced echelon matrix.

**Note 1.2.1.** (Gaussian Elimination) To solve a system of linear equations

1. Write an Augmented matrix of system.
2. Use elementary row operations to reduce matrix to row echelon.
3. Use **back-substitution** to solve equivalent system corresponding to the row reduced matrix.

Back-substitution is the iterative process of replacing known variables into equations to solve for unknown variables.

**Definition 1.2.3.** The variables $x_1$ and $x_2$ corresponding to the pivot columns in the augmented matrix below are called **basic variables**. The other variable, $x_3$, is called a **free variable**.

$$\begin{bmatrix} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{cases} x_1 \phantom{{}+x_2} - 5x_3 = 1 \\ \phantom{x_1 +{}} x_2 + x_3 = 4 \\ \phantom{x_1 + x_2 + {}} 0 = 0 \end{cases}$$

By saying $x_3$ is "free", we mean that we are free to choose any value for $x_3$. We can express the system of equations in a **general solution** where basic variables are expressed in terms of free variables

$$\begin{cases} x_1 = 1 + 5x_3 \\ x_2 = 4 - x_3 \\ x_3 \text{ is free} \end{cases} \qquad \longleftarrow \text{General Solution}$$

**Homework Assignment for Section 1.2: (Page 25) 3, 7, 13, 19, 21**

**Example 1.** *Row reduce the matrix A below to echelon form, and locate the pivot columns of A*

$$A = \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

**Example 2.** *Apply elementary row operations to transform the following matrix first into echelon form and then into reduced echelon form.*

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

**Example 3.** *Find the general solution of the linear system whose augmented matrix has been reduced to*

$$\begin{bmatrix} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$

**Example 4.** *Determine the existence and uniqueness of the solution(s) to the system*

$$\left\{ \begin{array}{rcrcrcrcrcr} & & 3x_2 & - & 6x_3 & + & 6x_4 & + & 4x_5 & = & -5 \\ 3x_1 & - & 7x_2 & + & 8x_3 & - & 5x_4 & + & 8x_5 & = & 9 \\ 3x_1 & - & 9x_2 & + & 12x_3 & - & 9x_4 & + & 6x_5 & = & 15 \end{array} \right.$$

## 1.3 Vector Equations

**Definition 1.3.1.** A matrix with only one column is called a **column vector**, or simply a **vector**. A column vector with only two rows with real entries can be thought of a vector in $\mathbb{R}^2$.

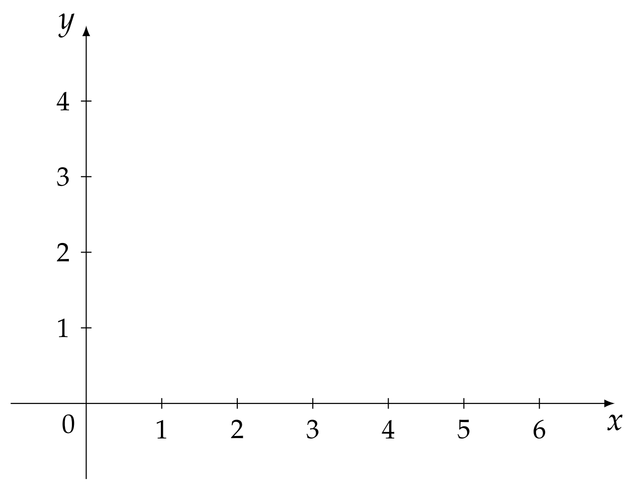
$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \text{ where } u_1, u_2 \in \mathbb{R}, \qquad \text{for example: } \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

We say that two vectors are **equal** if each component is equal. Given a vector $\mathbf{u}$ and a real number $c$, the **scalar multiple** of $\mathbf{u}$ by $c$ is the vector $c\mathbf{u}$ obtained by multiplying each entry in $\mathbf{u}$ by $c$.

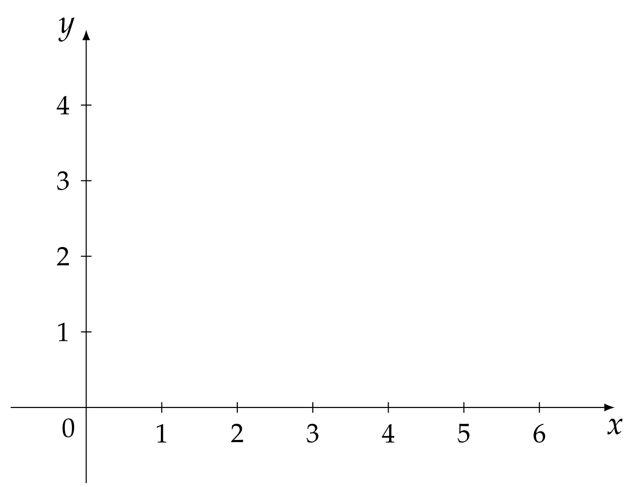
$$\text{if } \mathbf{u} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} \text{ and } c = 5, \qquad \text{then } c\mathbf{u} = 5\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 15 \\ -5 \end{bmatrix}$$

Ther real number $c$ is called a **scalar**.

**Rule 1.3.1** (The Head-to-Tail Rule for Addition)**.** Given vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$, translate $\mathbf{v}$ so that its tail coincides with the head of $\mathbf{u}$. The **sum** $\mathbf{u} + \mathbf{v}$ is the vector from the tail of $\mathbf{u}$ to the head of $\mathbf{v}$.




**Rule 1.3.2** (The Parallelogram Rule for Addition)**.** If $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$ are represented as points in the plane, then $\mathbf{u} + \mathbf{v}$ corresponds to the fourth vertex of the parallelogram whose other vertices are $\mathbf{u}, \mathbf{0}$, and $\mathbf{v}$. The vector $\mathbf{u} + \mathbf{v}$ is the diagonal of this parallelogram.



**Note 1.3.1.** Vectors in $\mathbb{R}^3$ are $3 \times 1$ column matrices with three entries. They are represented geometrically by points in a three-dimensional coordinate space, with arrows from the origin.

In general, if $n \in \mathbb{Z}^+$ (positive integers) then $\mathbb{R}^n$ denotes the collection of all the lists (or *ordered n-tuples*) of $n$ real numbers, usually written as $n \times 1$ column matrices, such as

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

The vector with zero in every position is called the zero vector denoted $\mathbf{0}$ and has zero magnitude and no direction.

**Property 1.3.1** (Algebraic Properties of $\mathbb{R}^n$)**.** For all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and all scalars $c$ and $d$:

(i) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

(ii) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

(iii) $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$

(iv) $\mathbf{u} + (-\mathbf{u}) = -\mathbf{u} + \mathbf{u} = \mathbf{0}$,
where $-\mathbf{u}$ denotes $(-1)\mathbf{u}$

(v) $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$

(vi) $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$

(vii) $c(d\mathbf{u}) = (cd)\mathbf{u}$

(viii) $1\mathbf{u} = \mathbf{u}$

**Definition 1.3.2.** Given vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_p \in \mathbb{R}^n$ and given scalars $c_1, c_2, \ldots c_p$, and the vector $\mathbf{y}$ defined by

$$\mathbf{y} = c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p$$

is called a **linear combination** of $\mathbf{v}_1, \ldots, \mathbf{v}_p$ with **weights** $c_1, \ldots, c_p$.

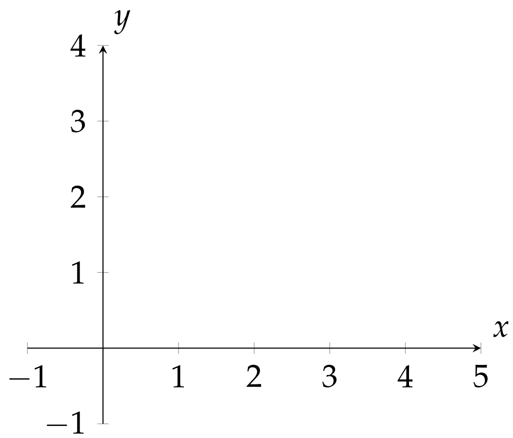
**Definition 1.3.3.** If $\mathbf{v}_1 \ldots, \mathbf{v}_p \in \mathbb{R}^n$ then the set of all linear combinations of $\mathbf{v}_1 \ldots, \mathbf{v}_p$ is denoted by Span$\{\mathbf{v}_1 \ldots, \mathbf{v}_p\}$ and is called the **subset of $\mathbb{R}^n$ spanned** ( or **generated) by** $\mathbf{v}_1 \ldots, \mathbf{v}_p$. That is, Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is the collection of all vectors that can be written in the form

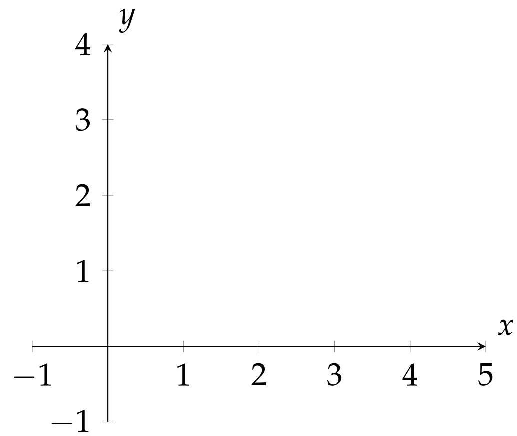
$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p$$

with scalars $c_1, \ldots, c_p$.

**Homework Assignment**
**Section 1.2**: (Page 25) 19, 21
**Section 1.3**: (Page 37) 3, 11, 13, 17, 19, 25

**Example 1.** *Graph column vectors* $\mathbf{u} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ *and* $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ *and* $\mathbf{u} + \mathbf{v}$.




(Head to Tail Rule)

(Parallelogram Rule)

**Example 2.** *Given* $\mathbf{u} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ *and* $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, *find* $-3\mathbf{u}$, $2\mathbf{v}$, *and* $2\mathbf{v} - 3\mathbf{u}$.

**Example 3.** *Is* $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$ *a linear combination of* $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ *and* $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

**Example 4.** *Let* $\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$, *and* $\mathbf{b} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$. *Determine whether* $\mathbf{b}$ *is a linear combination of* $\mathbf{a}_1$ *and* $\mathbf{a}_2$. *That is, find weights* $c_1$ *and* $c_2$ *such that* $c_1\mathbf{a}_1 + c_2\mathbf{a}_2 = \mathbf{b}$

**Example 5.** *Let* $\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 5 \\ -13 \\ -3 \end{bmatrix}$, *and* $\mathbf{b} = \begin{bmatrix} -3 \\ 8 \\ 1 \end{bmatrix}$. *Then* $\mathrm{Span}\{\mathbf{a}_1, \mathbf{a}_2\}$ *is a plane in* $\mathbb{R}^3$. *Is* $\mathbf{b}$ *in the plane?*

## 1.4 Matrix Equations

**Definition 1.4.1.** If $A$ is an $m \times n$ matrix, with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$, and if $\mathbf{x} \in \mathbb{R}^n$ then the **product of $A$ and $\mathbf{x}$**, denoted $A\mathbf{x}$, is the **linear combination of the columns of $A$ using the corresponding entries in $\mathbf{x}$ as weights**; that is ,

$$A\mathbf{x} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a_n} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n$$

An equation with the form $A\mathbf{x} = \mathbf{b}$ is called a **matrix equation**.

**Theorem 1.4.1.** If $A$ is an $m \times n$ matrix, with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$, and if $\mathbf{b} \in \mathbb{R}^m$, the matrix equation

$$A\mathbf{x} = \mathbf{b}$$

has the same solution set as the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

which, in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n & \mathbf{b} \end{bmatrix}$$

**Theorem 1.4.2.** The equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is a linear combination of the columns of $A$.

**Theorem 1.4.3.** Let $A$ be an $m \times n$ matrix. Then the following statements are logically equivalent. That is, for a particular $A$, either they are all true statements or they are all false.

a. For each $\mathbf{b} \in \mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.

b. Each $\mathbf{b} \in \mathbb{R}^m$ is a linear combination of the columns of $A$.

c. The columns of $A$ span $\mathbb{R}^m$.

d. $A$ has a pivot element in every row.

*(Caution: this theorem applies to the coefficient matrix, not the augmented matrix. It is possible for an augmented matrix to have a pivot element in every position and be inconsistent)*

**Theorem 1.4.4.** If $A$ is an $m \times n$ matrix, $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, and $c$ is a scalar, then:

a. $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$;

b. $A(c\mathbf{u}) = c(A\mathbf{u})$

**Homework Assignment**
**Section 1.4**: (Page 47) 3, 13, 15, 17, 21, 37, 39

**Example 1.** *Find the product of the given matrix and vector.*

(a) $\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix} =$

(b) $\begin{bmatrix} 2 & -3 \\ 8 & 0 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 7 \end{bmatrix} =$

**Example 2.** *Write the linear combination* $3\mathbf{v}_1 - 5\mathbf{v}_2 + 7\mathbf{v}_3$ *as a matrix times a vector.*

**Example 3.** *Let* $A = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix}$ *and* $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$. *Is the equation* $A\mathbf{x} = \mathbf{b}$ *consistent for all possible* $b_1, b_2, b_3$*? (In other words, do the columns of* $A$ *span* $\mathbb{R}^3$*?)*

**Example 4.** *Compute the product*

(a) $\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix} =$

(b) $\begin{bmatrix} 2 & -3 \\ 8 & 0 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 7 \end{bmatrix} =$

(c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r \\ s \\ t \end{bmatrix} =$

**Example 5.** *Let* $A = \begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$, *and* $\mathbf{v} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$. *Verify* ***Theorem 1.4.4*** *(a) by computing* $A(\mathbf{u} + \mathbf{v})$ *and* $A\mathbf{u} + A\mathbf{v}$.

## 1.5 Solution Sets of Linear Systems

**Definition 1.5.1.** A system of linear equations is said to be **homogeneous** if it can be written in the form $A\mathbf{x} = \mathbf{0}$, where $A$ is an $m \times n$ matrix and $\mathbf{0}$ is the zero vector in $\mathbb{R}^n$. A system of this form always has a solution, namely $\mathbf{x} = \mathbf{0}$. This zero solution is usually called the **trivial solution**.

**Theorem 1.5.1.** The homogeneous equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution if and only if the equation has at least one free variable.

**Note 1.5.1.** If the equation $A\mathbf{x} = \mathbf{0}$ has only one free variable, the solution set is a line through the origin. If there are two free variables the solution set is a plane through the origin.

**Definition 1.5.2.** The nontrivial solution of a homogeneous equation can be expressed in a **parametric vector equation** form where

$$\mathbf{x} = s\mathbf{u} + t\mathbf{v} \ \ (s, t \in \mathbb{R})$$

this form of the solution is called **parametric vector form**.

**Theorem 1.5.2.** Suppose the equation $A\mathbf{x} = \mathbf{b}$ is consistent for some given $\mathbf{b}$, and let $\mathbf{p}$ be a solution. Then the solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$ where $\mathbf{v}_h$ is any solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

**Note 1.5.2.** *(Writing A Solution Set of a Consistent System in Parametric Form)*

1. Row reduce the augmented matrix to reduced row echelon form.
2. Express each basic variable in terms of any free variable appearing in an equation.
3. Write a typical solution $\mathbf{x}$ as a vector whose entries depend on the free variables, if any.
4. Decompose $\mathbf{x}$ into a linear combination of vectors (with numeric entries) using the free variables as parameters (parametric vector form).

**Homework Assignment**
**Section 1.5**: (Page 55) 1, 2, 5, 7, 11, 19

**Example 1.** *Determine if the following homogeneous system has a nontrivial solution. Then describe the solution set*

$$\begin{cases} 3x_1 & + & 5x_2 & - & 4x_3 & = & 0 \\ -3x_1 & - & 2x_2 & + & 4x_3 & = & 0 \\ 6x_1 & + & x_2 & - & 8x_3 & = & 0 \end{cases}$$

**Example 2.** *Describe all the solutions of the homogeneous "system"*

$$3x_1 - 5x_2 - 7x_3 = 0$$

**Example 3.** *Describe the solutions of* $A\mathbf{x} = \mathbf{b}$, *where* $A = \begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix}$ *and* $\mathbf{b} = \begin{bmatrix} 7 \\ -1 \\ -4 \end{bmatrix}$.

**Example 4.** *Describe the solution to the following system in parametric form, and provide a geometric description.*

$$\left\{ \begin{array}{rcrcrcr} x_1 & + & 3x_2 & - & 5x_3 & = & 4 \\ x_1 & + & 4x_2 & - & 8x_3 & = & 7 \\ -3x_1 & - & 7x_2 & + & 9x_3 & = & -6 \end{array} \right.$$

## 1.6 Applications of Linear Systems

The goal of this section is to give realistic applications of systems with infinitely many solutions. We will first look at an example of a homogeneous system in economics.

**Example 1** (From text)**.** Suppose an economy consists of the Coal, Electric (power), and Steel sectors, and the output of each sector is distributed among the various sectors as in the table below, where the entries in a column represent the fractional parts of a sector's total output.

**Table 1** A Simple Economy

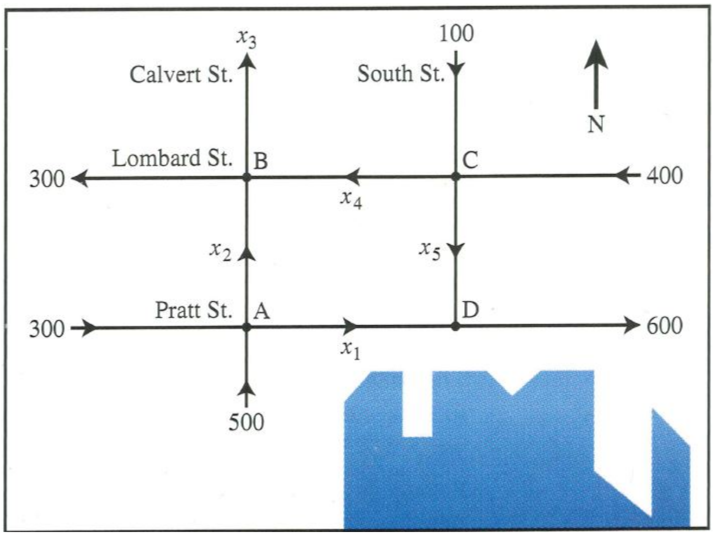
| Distribution of Output from: | | | |
|---|---|---|---|
| Coal | Electric | Steel | Purchased by: |
| .0 | .4 | .6 | Coal |
| .6 | .1 | .2 | Electric |
| .4 | .5 | .2 | Steel |

**Example 2.** Use a system of equations to balance the chemical equation below, in other words find whole numbers $x_1, \ldots, x_4$ so that the total number of each element is the same on both sides of the chemical reaction.

$$(x_1)C_3H_8 + (x_2)O_2 \longrightarrow (x_3)CO_2 + (x_4)H_2O$$

$$\text{Hint:} \quad C_3H_8 : \begin{bmatrix} 3 \\ 8 \\ 0 \end{bmatrix}, O_2 : \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, CO_2 : \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, H_2O : \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \begin{array}{l} \leftarrow \text{Carbon} \\ \leftarrow \text{Hydrogen} \\ \leftarrow \text{Oxygen} \end{array}$$

**Example 3.** **(a.)** The network in the picture below shows the traffic flow (in vehicles per hour) over several one-way streets in downtown Baltimore during a typical early afternoon. Determine the general flow pattern for the network. In other words, write equations that describe the flow, and then find the general solution of the system.




| Intersection | Flow in | | Flow out |
|---|---|---|---|
| A | $300 + 500$ | $=$ | $x_1 + x_2$ |
| B | $x_2 + x_4$ | $=$ | $300 + x_3$ |
| C | $100 + 400$ | $=$ | $x_4 + x_5$ |
| D | $x_1 + x_5$ | $=$ | $600$ |

**(b.)** Determine the possible range of the values of $x_1$ and $x_2$. What does this imply about $x_1$ and $x_2$?

**Homework Assignment**
**Section 1.6**: (Page 63) 1, 3, 11, 14

## 1.7 Linear Independence

**Definition 1.7.1.** An indexed set of vectors $\{v_1, \ldots, v_p\}$ in $\mathbb{R}^n$ is said to be **linearly independent** if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution. The set $\{v_1, \ldots, v_p\}$ is said to be **linearly dependent** if there exists weights $c_1, \ldots, c_p$, not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0} \tag{1.1}$$

Equation (1.1) is called a **linear dependence relation** among $\mathbf{v}_1, \ldots, \mathbf{v}_p$ when the weights are not all zero.

**Note 1.7.1.** A set containing only one vector, $\mathbf{v}$, is linearly independent if and only if $\mathbf{v} \neq \mathbf{0}$.

*Proof.*

**Theorem 1.7.1.** The columns of a matrix $A$ are linearly independent if and only if the equation $A\mathbf{x} = \mathbf{0}$ has *only* the trivial solution.

**Theorem 1.7.2.** A set of two vectors $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent if at least one of the vectors is a multiple of the other. The set is linearly independent if and only if neither of the vectors is a multiple of the other.

**Theorem 1.7.3.** If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$

*Proof.*

**Theorem 1.7.4** (Characterization of Linearly Dependent Sets)**.** An indexed set $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in $S$ is a linear combination of the others. In fact, if $S$ is linearly dependent and $\mathbf{v}_1 \neq \mathbf{0}$, then some $\mathbf{v}_j$ (with $j > 1$) is a linear combination of the preceding vectors $\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}$.

*Proof.*

**Theorem 1.7.5.** If a set $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ contains the zero vector, then the set is linearly dependent.

*Proof.*

**Homework Assignment**
**Section 1.7**: (Page 71) 1, 7, 9, 15, 17, 36

**Example 1.** *Let* $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$. *Determine if the set* $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ *is linearly independent. If possible, find a dependence relation among* $\mathbf{v}_1$, $\mathbf{v}_2$, *and* $\mathbf{v}_3$.

**Example 2.** *Determine if the columns of the matrix* $A = \begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 5 & 8 & 0 \end{bmatrix}$ *are linearly independent.*

**Example 3.** *Determine if the following sets of vectors are linearly independent.*

$$(a)\{\mathbf{v}_1, \mathbf{v}_2\}, \mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 6 \\ 2 \end{bmatrix} \qquad (b)\{\mathbf{v}_1, \mathbf{v}_2\}, \mathbf{v}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$$

**Example 4.** *Let* $\mathbf{u} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ *and* $\mathbf{v} = \begin{bmatrix} 1 \\ 6 \\ 0 \end{bmatrix}$. *Describe the* $\mathrm{Span}\{\mathbf{u}, \mathbf{v}\}$, *explain why a vector* $\mathbf{w}$ *is in* $\mathrm{Span}\{\mathbf{u}, \mathbf{v}\}$ *if and only if* $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ *is linearly dependent.*

**Example 5.** *Determine by inspection if the given set of vectors is linearly independent.*

$$(a) \begin{bmatrix} 1 \\ 7 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 9 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 8 \end{bmatrix} \qquad (b) \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 8 \end{bmatrix} \qquad (c) \begin{bmatrix} -2 \\ 4 \\ 6 \\ 10 \end{bmatrix}, \begin{bmatrix} 3 \\ -6 \\ -9 \\ 15 \end{bmatrix}$$

## 1.8 Introduction to Linear Transformations

**Definition 1.8.1.** A **transformation** (or **function** or **mapping**) $T$ from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector $\mathbf{x}$ in $\mathbb{R}^n$ a vector $T(\mathbf{x})$ in $\mathbb{R}^m$. The set $\mathbb{R}^n$ is called the **domain** of $T$, and $\mathbb{R}^m$ is called the **codomain** of $T$. The notation

$$T : \mathbb{R}^n \to \mathbb{R}^m$$

indicates that the domain of $T$ is $\mathbb{R}^n$ and the codomain is $\mathbb{R}^m$. For $\mathbf{x}$ in $\mathbb{R}^n$, the vector $T(x)$ in $\mathbb{R}^m$ is called the **image** of $\mathbf{x}$ (under the action of $T$). The set of all images $T(\mathbf{x})$ is called the **range** of $T$.

For the remainder of this section we will compute $T(\mathbf{x})$ as $A\mathbf{x}$, where $x \in \mathbb{R}^n$ and $A$ is an $m \times n$ matrix. Sometimes we will denote this **matrix transformation** as $\mathbf{x} \mapsto A\mathbf{x}$.

**Definition 1.8.2.** For a matrix $A = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$ where $b$ is a scalar, the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(\mathbf{x}) = A\mathbf{x}$ is called a **shear transformation**.

**Definition 1.8.3.** A transformation (or mapping) $T$ is **linear** if:

(i) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}$, $\mathbf{v}$ in the domain of $T$;

(ii) $T(c\mathbf{u}) = cT(\mathbf{u})$ for all $\mathbf{u}$ and all scalars $c$.

**Theorem 1.8.1.** If $T$ is a linear transformation, then

$$T(\mathbf{0}) = \mathbf{0}$$

and

$$T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$$

for all vectors $\mathbf{u}$, $\mathbf{v}$ in the domain of $T$ and all scalars $c$, $d$. This property can be generalized to what is known as the **superposition principle**.

$$T(c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p) = c_1 T(\mathbf{v_1}) + \cdots + c_p T(\mathbf{v_p})$$

**Homework Assignment**
**Section 1.8**: (Page 79) 1, 5, 7, 9, 11, 13, 19, 32, 36

**Example 1.** *Let* $A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$, *and define a transformation* $T : \mathbb{R}^2 \to \mathbb{R}^3$ *by* $T(\mathbf{x}) = A\mathbf{x}$, *so that*

$$T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - 3x_2 \\ 3x_1 + 5x_2 \\ -x_1 + 7x_2 \end{bmatrix}$$

(a) Find $T(\mathbf{u})$ "The image of $\mathbf{u}$ under the transformation $T$".

(b) Find $\mathbf{x} \in \mathbb{R}^2$ whose image under $T$ is $\mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$.

(c) Is there more than one $\mathbf{x}$ whose image under $T$ is $\mathbf{b}$?

(d) Determine if $\mathbf{c} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$ is in the range of the transformation $T$.

**Example 2.** *If* $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, *then the transformation* $\mathbf{x} \to A\mathbf{x}$ *projects points in* $\mathbb{R}^3$ *to the* $x_1x_2$*-plane.*

**Example 3.** *Let* $A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$*. The transformation* $T : \mathbb{R}^2 \to \mathbb{R}^2$ *defined by* $T(\mathbf{x}) = A\mathbf{x}$ *is a shear transformation. Transform the square with vertices* $\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix}$.

**Example 4.** *Given a scalar* $r$*, define* $T : \mathbb{R}^2 \to \mathbb{R}^2$ *by* $T(\mathbf{x}) = r\mathbf{x}$*. T is called a* <u>*contraction*</u> *when* $0 \leq r \leq 1$ *and a* <u>*dilation*</u> *when* $r > 1$*, Let* $r = 3$*, and show that T is a linear transformation.*

**Example 5.** *Define a linear transformation* $T : \mathbb{R}^2 \to \mathbb{R}^2$ *by*

$$T(\mathbf{x}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}$$

*Find the image under T of* $\mathbf{u} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$*, and* $\mathbf{u} + \mathbf{v} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$.

# 1.9 The Matrix of a Linear Transformation

**Theorem 1.9.1.** Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there exists a unique matrix $A$ such that

$$T(\mathbf{x}) = A\mathbf{x} \text{ for all } \mathbf{x} \in \mathbb{R}^n$$

In fact, $A$ is the $m \times n$ matrix whose $j$th column is the vector $T(\mathbf{e}_j)$, where $\mathbf{e}_j$ is the $j$th column of the identity matrix in $\mathbb{R}^n$:

$$A = [T(\mathbf{e}_1) \ \cdots \ T(\mathbf{e}_n)]$$

The matrix $A$ is called the **standard matrix for the linear transformation** $T$.

*Proof.*

**Definition 1.9.1.** A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be **onto** $\mathbb{R}^m$ if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of *at least one* $\mathbf{x}$ in $\mathbb{R}^n$.

**Definition 1.9.2.** A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be **one-to-one** if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of *at most one* $\mathbf{x}$ in $\mathbb{R}^n$.

**Note 1.9.1.** A one-to-one transformation is sometimes called *injective* or an *injection*. A common, equivalent statement for a one-to-one transformation $T$ is:

$$T : \mathbb{R}^n \to \mathbb{R}^m \text{ is one-to-one if for all } a, b \text{ in } \mathbb{R}^n \text{ such that } T(a) = T(b), \text{ then } a = b.$$

$$\text{Stated even more simply: } \forall\ a, b \in \mathbb{R}^n,\ T(a) = T(b) \Rightarrow a = b.$$

**Theorem 1.9.2.** Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then $T$ is one-to-one if and only if the equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.

*Proof.*

**Theorem 1.9.3.** Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and let $A$ be the standard matrix for $T$. Then:

a. $T$ maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of $A$ span $\mathbb{R}^m$;

b. $T$ is one-to-one if and only if the columns of $A$ are linearly independent.

**Homework Assignment**

**Section 1.9**: (Page 90) 1, 2, 5, 15, 17, 21, 25

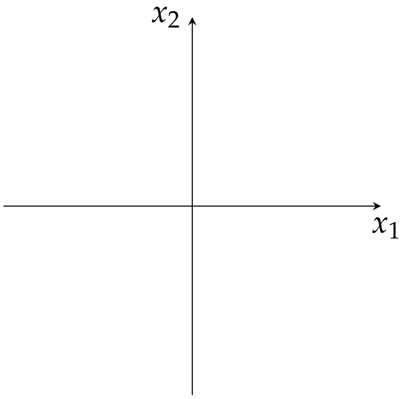
**Example 1.** *The columns of* $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ *are* $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ *and* $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. *Suppose* $T$ *is a linear transformation from* $\mathbb{R}^2$ *to* $\mathbb{R}^3$ *such that*

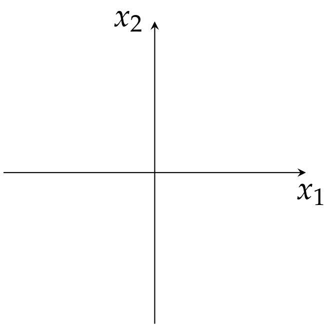
$$T(\mathbf{e}_1) = \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix} \qquad\qquad T(\mathbf{e}_2) = \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix}$$

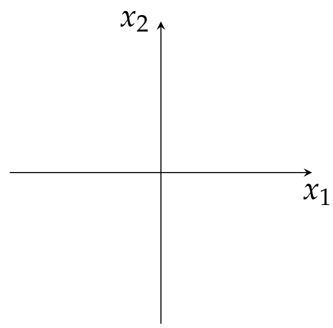
*with no additional information, find the formula for an arbitrary* $\mathbf{x} \in \mathbb{R}^2$.

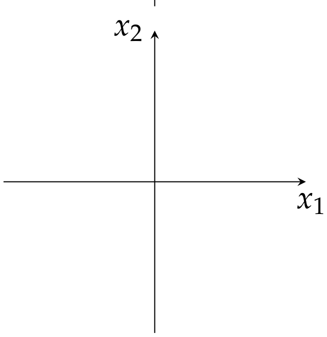
**Example 2.** *Find the standard matrix* $A$ *for the dilation transformation* $T(\mathbf{x}) = 3\mathbf{x}$ *for* $\mathbf{x} \in \mathbb{R}^2$.

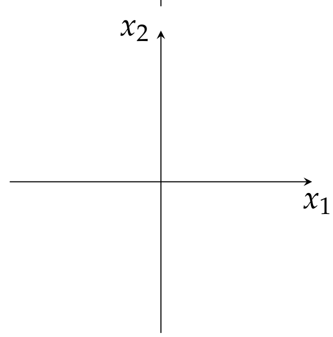
**Example 3.** *Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation that rotates each point in $\mathbb{R}^2$ about the origin, counterclockwise, through a positive angle $\phi$. Find the transformation matrix if we know that $T$ is linear.*




**Example 4.** *Let* $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 + x_2 \\ 5x_1 + 7x_2 \\ x_1 + 3x_2 \end{bmatrix}$. *Show that $T$ is a one-to-one linear transformation. Does $T$ map $\mathbb{R}^2 \to \mathbb{R}^3$? Is it onto?*

## 1.10 Linear Models in Business, Science, and Engineering

**Example 1.** If possible, find some combination of nonfat milk, soy flour, and whey to provide the exact amounts of protein, carbohydrate, and fat supplied by the diet in one day (see the table below)

**Nutrients Supplied in Various Ingredients**

| | Amounts (g) Supplied per 100 g of Ingredient | | | |
|---|---|---|---|---|
| Nutrient | Nonfat Milk | Soy Flour | Whey | Amounts (g) Supplied by the Cambridge Diet in One Day |
| Protein | 36 | 51 | 13 | 33 |
| Carbohydrate | 52 | 34 | 74 | 45 |
| Fat | 0 | 7 | 1.1 | 3 |

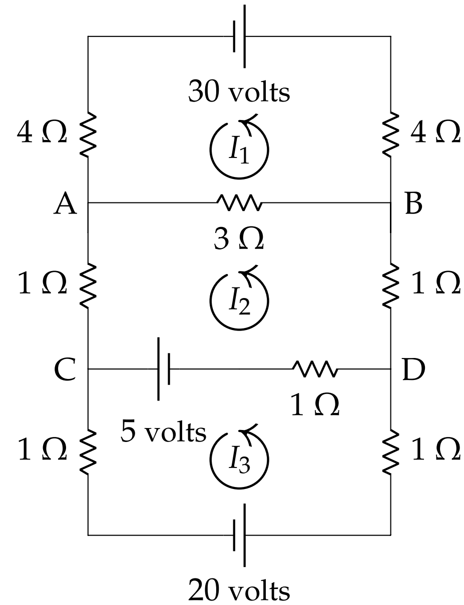
**Law 1. Ohm's Law**
A voltage source forces a current of electrons to flow through a network. When the current passes through a resistor some of the voltage is "used up". Ohm's Law gives the specific drop in voltage as:

$$V = IR$$

where the voltage $V$ is measured in *volts*, the resistance is measured in *ohms* (denoted by $\Omega$), and the current flow $I$ in *amperes* (*amps* for short).

**Law 2. Kirchhoff's Voltage Law** The Algebraic sum of the $IR$ voltage drops in one direction around a loop equals the algebraic sum of the voltage sources in the same direction around the loop.

**Example 2.** Determine the loop currents in the network in the figure below.




**Definition 1.10.1.** If there is a matrix $A$ such that $\mathbf{x}_1 = A\mathbf{x}_0$, $\mathbf{x}_2 = A\mathbf{x}_1$, and in general,

$$\mathbf{x}_{k+1} = A\mathbf{x}_k \quad \text{for } k = 0, 1, 2, \ldots \tag{1.2}$$

then (1.2) is called a **linear difference equation** (or **recurrence relation**). In a linear difference equation one can compute $\mathbf{x}_1$, $\mathbf{x}_2$ and on, provided that $\mathbf{x}_0$ is known.

**Example 3.** Compute the population of the region described in the table below for the years 2001 and 2002, given that the population in 2000 was 600,000 in the city and 400,000 in the suburbs.

**Annual Percent Migration between City and Suburbs**

| From: City | From: Suburbs | To: |
|---|---|---|
| 0.95 | 0.03 | City |
| 0.05 | 0.97 | Suburbs |

**Homework Assignment**
**Section 1.10**: (Page 100) 1, 7, 9

## 1.11 Chapter 1 Review

$3 \times 5$ note card and calculator permitted on the exam.

Content: Sections 1.1-1.10

- **Systems of Linear Equations**: 3 type of solutions to linear equations, augmented matrix, elementary row operations, row equivalent
- **Row Reductions and Echelon Forms**: REF and RREF, pivot elements and positions, basic and free variables, general solution, consistent vs. inconsistent
- **Vector Equations**: Column vectors, scalars, parallelogram rule, algebraic properties of $\mathbb{R}^n$, Span$\{S\}$ for set of vectors $S$.
- **Matrix Equations**: Linear Combinations, matrix equations, Theorem **1.4.3**
- **Solution Sets of Linear Equations**: Homogeneous system of equations, trivial solution, Theorem **1.5.1**, parametric vector form
- **Applications of Linear Systems**: Example 2 and example 3
- **Linear Independence**: Linearly independence and dependence, dependence relation, prove Theorem **1.7.5**
- **Introduction to Linear Transformations**: Transformations, domain, codomain, image, matrix transformation, shear transformation, linear transformation, prove a transformation is linear
- **The Matrix of a Linear Transformation**: Standard matrix, onto, one-to-one, Theorem **1.9.3**
- **Linear Models**: Ohm's law as it relates to current in a circuit. Set up a linear system based on population data.

**True/False** Questions:

1. _____. Every matrix is row equivalent to a unique matrix in echelon form.
2. _____. If matrices $A$ and $B$ are row equivalent , then they have the same reduced row echelon form.
3. _____. In some cases, it is possible for five vectors to span $\mathbb{R}^6$.
4. _____. The equation $A\mathbf{x} = \mathbf{0}$ has the trivial solution if and only if there are no free variables.
5. _____. Any system of $n$ linear equations in $n$ variables has at most $n$ solutions.

(*more true/false: p.27 # 21, 22 p.38 # 23, 24 p.48 #23, 23 p.55 # 23, 24 p.71 #21, 22 p.80 #21, 22*)

**Extended Response** Questions:

6. Solve the following system

$$\begin{cases} x_1 - 3x_2 + 4x_3 = -4 \\ 3x_1 - 7x_2 + 7x_3 = -8 \\ -4x_1 + 6x_2 - x_3 = 7 \end{cases}$$

7. Suppose the system below is consistent for all possible values of $f$ and $g$. What can you say about the coefficents $c$ and $d$? Justify your answer

$$\begin{cases} x_1 + 3x_2 = f \\ cx_1 + dx_2 = g \end{cases}$$

8. Find the general solution of the following system whose augmented matrix is given below

$$\begin{bmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ -1 & 7 & -4 & 2 & 7 \end{bmatrix}$$

9. Choose $h$ and $k$ such that the system has (a) no solution, (b) a unique solution, (c) many solutions.

$$\begin{cases} x_1 + 3x_2 = 2 \\ 3x_1 + hx_2 = k \end{cases}$$

10. Determine if $\mathbf{b}$ is a linear combination of the vectors formed by the columns of the matrix $A$.

$$A = \begin{bmatrix} 1 & -2 & -6 \\ 0 & 3 & 7 \\ 1 & -2 & 5 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 11 \\ -5 \\ 9 \end{bmatrix}$$

11. Give a geometric description of Span$\{\mathbf{v}_1, \mathbf{v}_2\}$ for the vectors
$$\mathbf{v}_1 = \begin{bmatrix} 8 \\ 2 \\ -6 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 12 \\ 3 \\ -9 \end{bmatrix}$$

12. Use the definition of $A\mathbf{x}$ to write the vector equation as a matrix equation.
$$x_1 \begin{bmatrix} 4 \\ -1 \\ 7 \\ -4 \end{bmatrix} + x_2 \begin{bmatrix} -5 \\ 3 \\ -5 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 7 \\ -8 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ -8 \\ 0 \\ -7 \end{bmatrix}$$

13. Refer to the matrix $A$ below. Can every vector of $\mathbb{R}^4$ be written as a linear combination of the columns of the matrix $A$? Do the columns of $A$ span $\mathbb{R}^4$?
$$A = \begin{bmatrix} 1 & 3 & 0 & 3 \\ -1 & -1 & -1 & 1 \\ 0 & -4 & 2 & -8 \\ 2 & 0 & 3 & -1 \end{bmatrix}$$

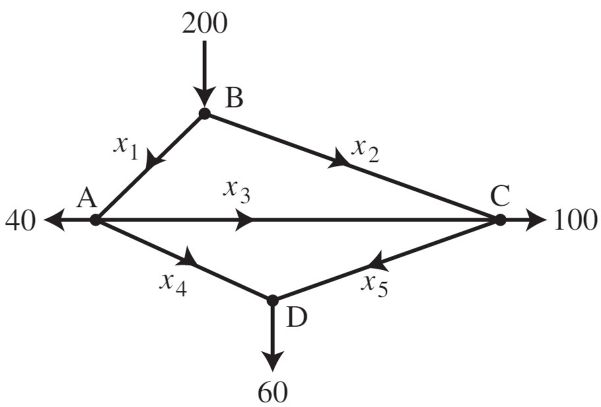
14. Describe all of the solutions of $A\mathbf{x} = \mathbf{0}$ in parametric vector form, where $A$ is row equivalent to the given matrix.
$$\begin{bmatrix} 3 & -9 & 6 \\ -1 & 3 & -2 \end{bmatrix}$$

15. Describe and compare the solution sets of $x_1 - 3x_2 + 5x_3 = 0$ and $x_1 - 3x_2 + 5x_3 = 4$.

16. Boron sulfide reacts violently with water to form boric acid and hydrogen sulfide gas (smell of rotten eggs). The unbalanced equation is

$$B_2S_3 + H_2O \longrightarrow H_3BO_3 + H_2S$$

17. (a) Find the general traffic pattern in the freeway interchange shown in the figure (flow rates given in cars per minute). (b) Describe the general traffic pattern when the road whose flow is $x_4$ is closed. (c) When $x_4 = 0$, what is the minimum value of $x_1$?




18. (a) For what value(s) of $h$ is $\mathbf{v}_3$ in Span$\{\mathbf{v}_1, \mathbf{v}_2\}$, and (b) for what value(s) of $h$ is $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ linearly *dependent*? Justify each answer.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -5 \\ -3 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -2 \\ 10 \\ 6 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 2 \\ -9 \\ h \end{bmatrix}$$

19. Determine by inspection (no calculations) whether the vectors are linearly *independent*. Justify each answer.

(a) $\begin{bmatrix} 1 \\ -5 \end{bmatrix}, \begin{bmatrix} 2 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 7 \end{bmatrix}$ (b) $\begin{bmatrix} 4 \\ -2 \\ 6 \end{bmatrix}, \begin{bmatrix} 6 \\ -3 \\ 9 \end{bmatrix}$ (c) $\begin{bmatrix} 1 \\ 4 \\ -7 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

20. With $T$ defined by $T(\mathbf{x}) = A\mathbf{x}$, find a vector $\mathbf{x}$ whose image under $T$ is $\mathbf{b}$, and determine whether $\mathbf{x}$ is unique.

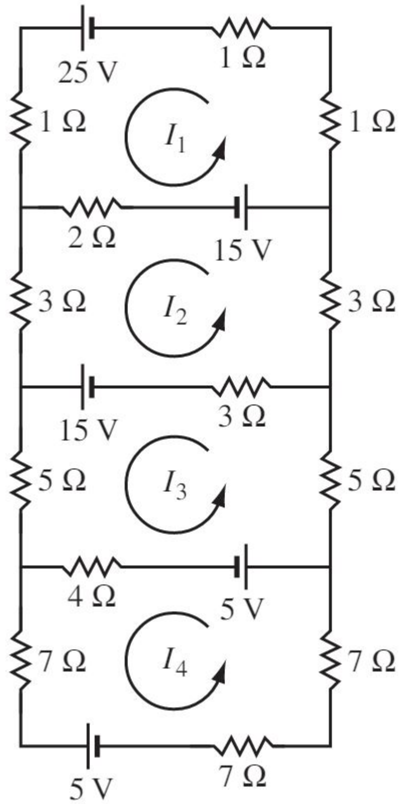
$$A = \begin{bmatrix} 1 & 0 & -2 \\ -2 & 1 & 6 \\ 3 & -2 & -5 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -1 \\ 7 \\ -3 \end{bmatrix}$$

21. Let $\mathbf{e_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\mathbf{y}_1 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$, and $\mathbf{y}_2 = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$, and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation that maps $\mathbf{e}_1$ into $\mathbf{y}_1$ and maps $\mathbf{e}_2$ into $\mathbf{y}_2$. Find the images of $\begin{bmatrix} 5 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

22. Assume that $T$ is a linear transformation. $T : \mathbb{R}^2 \to \mathbb{R}^2$ first reflects points through the vertical $x_2$-axis and then rotates points $\pi/2$ radians. Find the standard matrix of $T$.

23. Show that $T$ is a linear transformation by finding a matrix that implements the mapping. $T(x_1, x_2) = (2x_2 - 3x_1, x_1 - 4x_2, 0, x_2)$

24. Using a calulator, Determine the loop currents in the circuit shown below.




25. In a certain region, about 7% of a city's population moves to the surrounding suburbs each year, and about 3% of the suburban population moves into the city. In, 2000, there were 800,000 residents in the city and 500,000 in the suburbs. Set up a difference equation that describes this situation, where $\mathbf{x}_0$ is the initial population in 2000. Then estimate the population in the city and in the suburbs two years later, in 2002.

# Chapter 2 Matrix Algebra

## 2.1 Matrix Operations

**Definition 2.1.1.**

(i.) The **diagonal entries** in an $n \times m$ matrix $A = [a_{ij}]$ are $a_{11}, a_{22}, a_{33}, \ldots$ and they form the main diagonal of $A$.

(ii.) An $m \times n$ matrix whose entries are all zero is called a **zero matrix** and is written as 0.

(iii.) A **diagonal matrix** is a square matrix whose nondiagonal entries are zero.

(iv.) We say two matrices are **equal** if they have the same size, number of rows and columns, and each of their corresponding columns are equal.

(v.) The **sum** of two $m \times n$ matrices, $A + B$, is the $m \times n$ matrix whose columns are the sum of the corresponding columns of $A$ and $B$.

(vi.) If $r$ is a scalar and $A$ is a matrix then the **scalar multiple** $rA$ is the matrix whose columns are $r$ times the corresponding columns in $A$.

Matrix Notation

$$\begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$$

Zero Matrix

$$\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

Diagonal Matrix

$$\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & a_{nn} \end{bmatrix}_{n \times n}$$

**Theorem 2.1.1.** Let $A$, $B$, and $C$ be matrices of the same size, let $r$ and $s$ be scalars.

a. $A + B = B + A$

b. $(A + B) + C = A + (B + C)$

c. $A + 0 = A$

d. $r(A + B) = rA + rB$

e. $(r + s)A = rA + sA$

f. $r(sA) = (rs)A$

**Definition 2.1.2 (Matrix Multiplication).** If $A$ is an $m \times n$ matrix, and if $B$ is an $n \times p$ matrix with columns $\mathbf{b}_1, \ldots, \mathbf{b}_p$, then the product $AB$ is the $m \times p$ matrix whose columns are $A\mathbf{b}_1, \ldots, A\mathbf{b}_p$. That is,

$$AB = A\begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_p \end{bmatrix} = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_p \end{bmatrix}$$

In order for the product $AB$ to be defined $B$ must have the same number of rows as $A$ has columns. The resulting matrix will have the same number of rows as $A$ and the same number of columns as $B$.

**Rule 2.1.1** (Row-Column Rule for Computing $AB$)**.** If the product $AB$ is defined, then the entry in row $i$ and column $j$ of $AB$ is the sum of the products of corresponding entries from row $i$ of $A$ and column $j$ of $B$. If $(AB)_{ij}$ denotes the $(i,j)$-entry in $AB$, and if $A$ is an $m \times n$ matix, then

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

**Theorem 2.1.2.** Let $A$ be an $m \times n$ matrix, and let $B$ and $C$ have sizes for which the indicated sums and products are defined.

a. $A(BC) = (AB)C$ (associative law of multiplication)

b. $A(B + C) = AB + AC$ (left distributive law)

c. $(B + C)A = BA + CA$ (right distributive law)

d. $r(AB) = (rA)B = A(rB)$

e. $I_m A = A = AI_n$ (identity for matrix multiplication)

**Rule 2.1.2.** Some things to look out for while using matrix algebra:

1. In general, $AB \neq BA$. Not all matrices commute.
2. The cancellation laws do *not* hold for matrix multiplication. That is, if $AB = AC$, then it is *not* true in general that $B = C$.
3. If a product $AB$ is the zero matrix, you *cannot* conclude in general that either $A = 0$ or $B = 0$.

**Definition 2.1.3.** Given an $m \times n$ matrix $A$, the transpose of $A$ is the $n \times m$ matrix, denoted by $A^T$, whose columns are formed by the corresponding rows of $A$.

**Theorem 2.1.3.** Let $A$ and $B$ denote matrices whose sizes are appropriate for the following sums and products.

a. $(A^T)^T = A$

b. $(A + B)^T = A^T + B^T$

c. For any scalar $r$, $(rA)^T = rA^T$

d. $(AB)^T = B^T A^T$

**Homework Assignment**
**Section 2.1**: (Page 116) 1, 3, 5, 7, 9, 11, 20, 27

**Example 1.** *Let*

$$A = \begin{bmatrix} 3 & 8 & 0 \\ 2 & -3 & 5 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 3 & 1 \\ -1 & 7 & 2 \end{bmatrix}, \qquad C = \begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix}$$

*then calculuate $A + B$, $A + C$, $B + C$.*

**Example 2.** *Using $A$ and $B$ from* ***Example 1*** *find $2B$ and $A - 2B$.*

**Example 3.** *Compute $AB$, where $A = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix}$.*

**Matrix Multiplication:** $B$ is the matrix that transforms $\mathbf{x} \mapsto B\mathbf{x}$, if this new vector is in turn multiplied by a matrix $A$ the resulting vector is $A(B\mathbf{x})$

Picture Representation:

**Example 4.** *If $A$ is $3 \times 5$ and $B$ is a $5 \times 2$, what are the sizes of $AB$ and $BA$, if they are defined?*

**Example 5.** *Use the row-column rule to compute $AB$ from* ***Example 3****,*

$$A = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix}.$$

**Example 6.** *Find the entries of the second row of $AB$, where*

$$\begin{bmatrix} 2 & -5 & 0 \\ -1 & 3 & -4 \\ 6 & -8 & -7 \\ -3 & 0 & 9 \end{bmatrix} \text{ and } \begin{bmatrix} 4 & -6 \\ 7 & 1 \\ 3 & 2 \end{bmatrix}.$$

**Example 7.** *Let $A = \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix}$. Show that these matrices do not commute. That is, show that $AB \neq BA$.*

**Example 8.** *Let*

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad B = \begin{bmatrix} -5 & 2 \\ 1 & -3 \\ 0 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -3 & 5 & -2 & 7 \end{bmatrix}.$$

*Find $A^T, B^T, C^T$.*

## 2.2 Inverse of a Matrix

**Definition 2.2.1.** An $n \times n$ matrix $A$ is said to be **invertible** if there is an $n \times n$ matrix $C$ such that

$$CA = I \quad \text{and} \quad AC = I$$

where $I = i_n$, then $n \times n$ identity matrix.
In this case, $C$ is an **inverse** of $A$. In fact, $C$ is uniquely determined by $A$, because if there were another inverse $B$ we would have

$$B = BI = B(AC) = (BA)C = IC = C.$$

A matrix that is *not* invertible is sometimes called a **singular matrix**, an invertible matrix is called a **nonsingular matrix**.

**Theorem 2.2.1.** Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then $A$ is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $ad - bc = 0$, then $A$ is not invertible.

*Proof.*

**Definition 2.2.2.** The quantity $ad - bc$ is called the **determinant** of $A$, and we write

$$\det A = ad - bc$$

**Theorem 2.2.2.** If $A$ is an invertible $n \times n$ matrix, then for each $\mathbf{b} \in \mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

*Proof.*

**Theorem 2.2.3.**

a. If $A$ is an invertible matrix, then $A^{-1}$ is invertible and
$$(A^{-1})^{-1} = A$$

b. If $A$ and $B$ are $n \times n$ invertible matrices, then so is $AB$, and the inverse of $AB$ is the product of the inverses of $A$ and $B$ in the reverse order. That is,
$$(AB)^{-1} = B^{-1}A^{-1}$$
In general, the product of $n \times n$ invertible matrices is invertible, and the inverse is the product of their inverses in the reverse order.

c. If $A$ is an invertible matrix, then so is $A^T$, and the inverse of $A^T$ is the transpose of $A^{-1}$. That is,
$$(A^T)^{-1} = (A^{-1})^T$$

*Proof.*

**Definition 2.2.3.** An **elementary matrix** is one that is obtained by performing a single elementary row operation on an identity matrix. Each elementary matrix $E$ is invertible. The inverse of $E$ is the elementary matrix of the same type that transforms $E$ back into $I$.

**Theorem 2.2.4.** An $n \times n$ matrix $A$ is invertible if and only if $A$ is row equivalent to $I_n$, and in this case, any sequence of elementary row operations that reduces $A$ to $I_n$ also transforms $I_n$ into $A^{-1}$.

**Rule 2.2.1** (Alogorithim for Finding $A^{-1}$)**.** Row reduce the augmented matrix $\begin{bmatrix} A & I \end{bmatrix}$. If $A$ is row equivalent to $I$, then $\begin{bmatrix} A & I \end{bmatrix}$ is row equivalent to $\begin{bmatrix} I & A^{-1} \end{bmatrix}$. Otherwise, $A$ does not have an inverse.

**Homework Assignment**
**Section 2.2**: (Page 126) 1, 5, 7, 9, 11, 20, 27

**Example 1.** *If* $A = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}$ *and* $C = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$, *find* $AC$ *and* $CA$. *What do you observe?*

**Example 2.** *Find the inverse of* $A = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$ *if it exists.*

**Example 3.** *Use the inverse of the matrix* $A$ *from **Example 2** to solve the following system,*

$$\begin{cases} 3x_1 + 4x_2 = 3 \\ 5x_1 + 6x_2 = 7 \end{cases}$$

**Example 4.** *Let* $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$, *and* $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$. *Compute* $E_1A$, $E_2A$, $E_3A$, *describe how the products can be obtained by elementary row operations.*

**Example 5.** *Find the inverse of* $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$.

**Example 6.** *Find the inverse of the matrix* $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$ *if it exists.*

## 2.3 Characterization of Invertible Matrices

**Theorem 2.3.1** (The Invertible Matrix Theorem)**.** Let $A$ be a square $n \times n$ matrix. Then the following statements are equivalent.

a. $A$ is an invertible matrix.
b. $A$ is row equivalent to the $n \times n$ identity matrix.
c. $A$ has $n$ pivot positions.
d. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
e. The columns of $A$ form a linearly independent set.
f. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.
g. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.
h. The columns of $A$ span $\mathbb{R}^n$.
i. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.
j. There is an $n \times n$ matrix $C$ such that $CA = I$.
k. There is an $n \times n$ matrix $D$ such that $AD = I$.
l. $A^T$ is an invertible matrix.

*Proof.*

**Rule 2.3.1.** Let $A$ and $B$ be square matrices. If $AB = I$, then $A$ and $B$ are both invertible, with $B = A^{-1}$ and $A = B^{-1}$.

**Example 1.** Use the Invertible Matrix Theorem to decide if $A$ is invertible:

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{bmatrix}$$

**Definition 2.3.1.** A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is said to be **invertible** if there exists a function $S : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$S(T(\mathbf{x})) = \mathbf{x} \text{ for all } \mathbf{x} \text{ in } \mathbb{R}^n$$

$$T(S(\mathbf{x})) = \mathbf{x} \text{ for all } \mathbf{x} \text{ in } \mathbb{R}^n$$

We call $S$ the inverse of $T$ and write $T^{-1}$.

**Theorem 2.3.2.** Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and let $A$ be the standard matrix for $T$. Then $T$ is invertible if and only if $A$ is an invertible matrix.

*Proof.*

**Example 2.** What can you say about a one-to-one, linear transformation $T$ from $\mathbb{R}^n$ into $\mathbb{R}^n$.

**Homework Assignment**
**Section 2.3**: (Page 132) 1, 5, 11, 13, 15, 16, 17, 24

## 2.4 Partitioned Matrices

**Example 1.** The matrix

$$A = \left[\begin{array}{rrr|rr|r} 3 & 0 & -1 & 5 & 9 & -2 \\ -5 & 2 & 4 & 0 & -3 & 1 \\ \hline -8 & -6 & 3 & 1 & 7 & -4 \end{array}\right]$$

can also be written as the $2 \times 3$ **partitioned** (or **block**) **matrix**

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix}$$

whose entries are the *blocks* (or *submatrices*)

$$A_{11} = \begin{bmatrix} 3 & 0 & -1 \\ -5 & 2 & 4 \end{bmatrix}, \qquad A_{12} = \begin{bmatrix} 5 & 9 \\ 0 & -3 \end{bmatrix}, \qquad A_{13} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$A_{21} = \begin{bmatrix} -8 & -6 & 3 \end{bmatrix}, \qquad A_{22} = \begin{bmatrix} 1 & 7 \end{bmatrix}, \qquad A_{23} = \begin{bmatrix} -4 \end{bmatrix}$$

**Example 2.** Let

$$A = \left[\begin{array}{rrr|rr} 2 & -3 & 1 & 0 & -4 \\ 1 & 5 & -2 & 3 & -1 \\ \hline 0 & -4 & -2 & 7 & -1 \end{array}\right] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \qquad B = \left[\begin{array}{rr} 6 & 4 \\ -2 & 1 \\ -3 & 7 \\ \hline -1 & 3 \\ 5 & 2 \end{array}\right] = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

We say that the partitions of $A$ and $B$ are **conformable** for **block multiplication**, provided that the column partition of $A$ matches the row partition of $B$. Show that:

$$AB = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} A_{11}B_1 + A_{12}B_2 \\ A_{21}B_1 + A_{22}B_2 \end{bmatrix}$$

**Example 3.** Let $A = \begin{bmatrix} -3 & 1 & 2 \\ 1 & -4 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$. Verify that

$$AB = \mathrm{col}_1(A)\mathrm{row}_1(B) + \mathrm{col}_2(A)\mathrm{row}_2(B) + \mathrm{col}_3(A)\mathrm{row}_3(B)$$

**Theorem 2.4.1** (Column-Row Expansion of AB)**.** If $A$ is $m \times n$ and $B$ is $n \times p$, then

$$\begin{aligned} AB &= \begin{bmatrix} \mathrm{col}_1(A) & \mathrm{col}_2(A) & \cdots & \mathrm{col}_n(A) \end{bmatrix} \begin{bmatrix} \mathrm{row}_1(B) \\ \mathrm{row}_2(B) \\ \vdots \\ \mathrm{row}_n(B) \end{bmatrix} \\ &= \mathrm{col}_1(A)\mathrm{row}_1(B) + \mathrm{col}_2(A)\mathrm{row}_2(B) + \cdots + \mathrm{col}_n(A)\mathrm{row}_n(B) \end{aligned} \tag{2.3}$$

*Proof.*

**Example 4.** A matrix of the form

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

is said to be **block upper triangular**. Assume the $A_{11}$ is $p \times p$, $A_{22}$ is $q \times q$, and $A$ is invertible. Find a formula for $A^{-1}$.

**Homework Assignment**
**Section 2.4**: (Page 139) 1, 3, 5, 10, 11, 21

## 2.5 Matrix Factorizations

**Definition 2.5.1.** If $A$ is an $m \times n$ matrix that can be row reduced to echelon form, *without row interchanges*, then $A$ can be written in the form $A = LU$, where $L$ is an $m \times m$ lower triangular matrix with 1's on the diagonal and $U$ is an $m \times n$ echelon form of $A$, is called an **LU factorization** of $A$. The matrix $L$ is invertible and is called a *unit* lower triangular matrix.

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ \# & 1 & 0 & 0 \\ \# & \# & 1 & 0 \\ \# & \# & \# & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} \# & \# & \# & \# & \# \\ 0 & \# & \# & \# & \# \\ 0 & 0 & 0 & \# & \# \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{U}$$

An $LU$ factorization is useful for solving the equation $A\mathbf{x} = \mathbf{b}$. When $A = LU$ we can write $A\mathbf{x} = \mathbf{b}$ as $L(U\mathbf{x}) = \mathbf{b}$. By replacing $U\mathbf{x}$ with $\mathbf{y}$ we can find $\mathbf{x}$ by solving the pair of equations

$$\begin{aligned} L\mathbf{y} &= \mathbf{b} \\ U\mathbf{x} &= \mathbf{y} \end{aligned}$$

First solve $L\mathbf{y} = \mathbf{b}$ for $\mathbf{y}$, and then solve $U\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$.

**Rule 2.5.1** (Algorithm for an $LU$ Factorization)**.**

1. Reduce $A$ to an echelon form $U$ by a sequence of row replacement operations, if possible.

2. Place entries in $L$ such that the same sequence of row operations reduces $L$ to $I$.

**Homework Assignment**
**Section 2.5**: (Page 149) 1, 5, 7, 11, 19

**Example 1.** *The LU factorization of A is shown.*

$$A = \begin{bmatrix} 3 & -7 & -2 & 2 \\ -3 & 5 & 1 & 0 \\ 6 & -4 & 0 & -5 \\ -9 & 5 & -5 & 12 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -5 & 1 & 0 \\ -3 & 8 & 3 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}}_{U}$$

*Use this LU factorization of A to solve* $A\mathbf{x} = \mathbf{b}$ *for* $\mathbf{b} = \begin{bmatrix} -9 \\ 5 \\ 7 \\ 11 \end{bmatrix}$.

**Example 2.** *Find the LU factorization of*

$$A = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix}.$$

**Example 3.** *Find the LU factorization of*

$$A = \begin{bmatrix} 2 & -4 & -2 & 3 \\ 6 & -9 & -5 & 8 \\ 2 & -7 & -3 & 9 \\ 4 & -2 & -2 & -1 \\ -6 & 3 & 3 & 4 \end{bmatrix}.$$

**Key Concepts:**

- $L$ is a unit lower triangular, square matrix.
- $U$ is the same size as $A$, row echelon form of $A$
- No row interchanges are allowed when finding the LU factorization.
- Row operation used to reduce $A$ to $U$ form the matrix $L$.

## 2.7 Applications to Computer Graphics

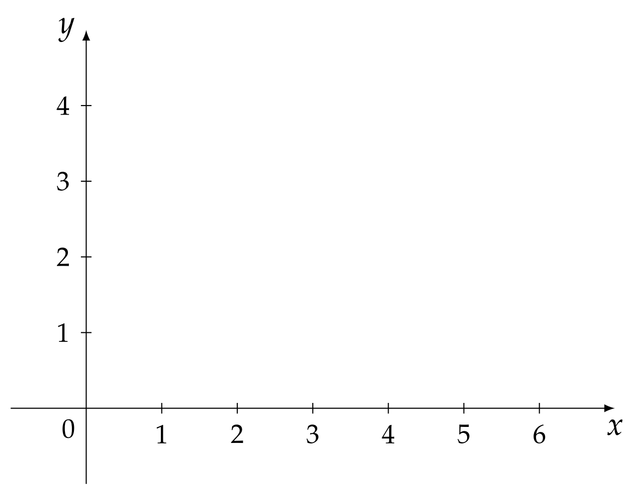
**Example 1.** The capital letter N is determined by eight points, or vertices. The coordinates of the points can be stored in a data matrix, $D$.

$$\begin{array}{rc} & \text{Vertex:} \\ & \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{array}{r}\text{x-coordinate}\\ \text{y-coordinate}\end{array} & \begin{bmatrix} 0 & .5 & .5 & 6 & 6 & 5.5 & 5.5 & 0 \\ 0 & 0 & 6.42 & 0 & 8 & 8 & 1.58 & 8 \end{bmatrix} \end{array} = D$$

**Example 2.** Given $A = \begin{bmatrix} 1 & .25 \\ 0 & 1 \end{bmatrix}$, describe the effect of the shear transformation $\mathbf{x} \mapsto A\mathbf{x}$ on the letter N from Example 1. This produces a wide italic *N*, to compensate for this use the scaling transformation $S = \begin{bmatrix} .75 & 0 \\ 0 & 1 \end{bmatrix}$ to shrink the width. What is the composition of these two transformations?

**Example 3.** A translation of the form $(x, y) \mapsto (x + h, y + k)$ is written in homogeneous coordinates as $(x, y, 1) \mapsto (x + h, y + k, 1)$. This transformation can be computed via matrix multiplication:

$$\begin{bmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x+h \\ y+k \\ 1 \end{bmatrix}$$



**Definition 2.7.1.** Each point $(x, y)$ in $\mathbb{R}^2$ can be identified with the point $(x, y, 1)$ on the plane in $\mathbb{R}^3$ that lies one unit above the $xy$-plane. We say that $(x, y)$ has **homogeneous coordinates** $(x, y, 1)$.

**Example 4.** Any linear transformation on $\mathbb{R}^2$ is represented with respect to homogeneous coordinates by a partitioned matrix of the form $\begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix}$, where $A$ is a $2 \times 2$ matrix. Typical examples are

$$\underset{\substack{\text{Counterclockwise} \\ \text{rotation about the} \\ \text{origin, angle } \phi}}{\begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}}, \qquad \underset{\substack{\text{Reflection} \\ \text{through } y = x}}{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}, \qquad \underset{\substack{\text{Scale } x \text{ by } s \\ \text{and } y \text{ by } t}}{\begin{bmatrix} s & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

**Example 5.** Find the $3 \times 3$ matrix that corresponds to the composite transformation of a scaling by .3, a counterclockwise rotation of 90°, and finally a transformation that adds $(-.5, 2)$ to each point of a figure.

**Homework Assignment**
**Section 2.7**: (Page 165) 2, 3, 5, 7, 11, 16

# Chapter 3 Determinants

## 3.1 Introduction to Determinants

The determinant is a scalar value that relates to the invertibility of a matrix. We want to consider the conditions for when a $3 \times 3$ matrix can be row reduced to echelon form. Assume $a_{11} \neq 0$.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \to \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{11}a_{21} & a_{11}a_{22} & a_{11}a_{23} \\ a_{11}a_{31} & a_{11}a_{32} & a_{11}a_{33} \end{bmatrix} \to \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{23} - a_{13}a_{21} \\ 0 & a_{11}a_{32} - a_{12}a_{31} & a_{11}a_{33} - a_{13}a_{31} \end{bmatrix}$$

and finally, assuming $a_{11}a_{22} - a_{12}a_{21} \neq 0$, by eliminating the last element in the second column we have...

$$A \to \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{23} - a_{13}a_{21} \\ 0 & 0 & a_{11}\Delta \end{bmatrix}$$

where

$$\begin{aligned} \Delta &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31} \\ &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) + a_{12}(a_{23}a_{31} - a_{21}a_{33}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \\ &= a_{11} \cdot \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} + a_{12} \cdot \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \cdot \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \\ &= a_{11} \cdot \det A_{11} - a_{12} \cdot \det A_{12} + a_{13} \cdot \det A_{13}. \end{aligned}$$

Where $A_{11}, A_{12}, A_{13}$ are obtained from $A$ by deleting the first row and one of the three columns. For $A$ to be invertible $\Delta$ must be nonzero. We call $\Delta$ the **determinant** of a $3 \times 3$ matrix $A$.
We can now *recursively* define the determinant for a general $n \times n$ matrix.

**Definition 3.1.1.** For $n \geq 2$, the **determinant** of an $n \times n$ matrix $A = [a_{ij}]$ is the sum of $n$ terms of the form $\pm a_{1j} \det A_{1j}$, with plus and minus signs alternating, where the entries $a_{11}, a_{12}, \ldots, a_{1n}$ are from the first row of $A$. In symbols,

$$\begin{aligned} \det A &= a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{1+n} a_{1n} \det A_{1n} \\ &= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j}. \end{aligned}$$

It is sometimes useful to look at $\det A$ in a different way using the $(\mathbf{i}, \mathbf{j})$**-cofactor**. Given $A = [a_{ij}]$ the $(\mathbf{i}, \mathbf{j})$**-cofactor** is the number $C_{ij}$ given by

$$C_{ij} = (-1)^{i+j} \det A_{ij}.$$

Then the formula for the determinant of $A$ becomes,

$$\det A = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}.$$

This formula is called a **cofactor expansion across the first row** of $A$.

**Theorem 3.1.1.** The determinant of an $n \times n$ matrix $A$ can be computed by a cofactor expansion across any row or down any column. The expansion across the $i$th row is

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in},$$

the cofactor expansion expansion down the $j$th column is

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}.$$

**Theorem 3.1.2.** If $A$ is a triangular matrix, then $\det A$ is the product of the entries on the main diagonal of $A$.

**Homework Assignment**
**Section 3.1**: (Page 190) 1, 5, 11, 20, 27, 29

**Example 1.** *Compute the determinant of* $A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$

**Example 2.** *Use the cofactor expansion across the third row to compute* $\det A$*, where* $A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$

$$\det A = \begin{vmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{vmatrix} =$$

**Example 3.** *Compute the determinant of* $A$.

$$\det A = \begin{vmatrix} 3 & -7 & 8 & 9 & -6 \\ 0 & 2 & -5 & 7 & 3 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 2 & 4 & -1 \\ 0 & 0 & 0 & -2 & 0 \end{vmatrix} =$$

**Example 4.** *Compute* $\det A = \begin{vmatrix} 3 & 5 & -8 & 4 \\ 0 & -2 & 3 & -7 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 2 \end{vmatrix} =$

**Try these on your own:**

**Exercise 1.** $\begin{vmatrix} 6 & 3 & 2 & 4 & 0 \\ 9 & 0 & -4 & 1 & 0 \\ 8 & -5 & 6 & 7 & 1 \\ 3 & 0 & 0 & 0 & 0 \\ 4 & 2 & 3 & 2 & 0 \end{vmatrix} =$

**Exercise 2.** $\begin{vmatrix} 3 & 0 & 4 \\ 2 & 3 & 2 \\ 0 & 5 & -1 \end{vmatrix} =$

**Exercise 3.** $\begin{vmatrix} 5 & -2 & 1 \\ 0 & 2 & -3 \\ 0 & 0 & -1 \end{vmatrix} =$

Solutions: **#1.** 9, **#2.** 1, **#3.** $-10$

## 3.2 Properties of Determinants

**Theorem 3.2.1.** Let $A$ be a square matrix.

**a.** If a multiple of one row of $A$ is added to another row to produce a matrix $B$, then $\det B = \det A$.

**b.** If two rows of $A$ are interchanged to produce $B$, then $\det B = -\det A$.

**c.** If one row of $A$ is multiplied by $k$ to produce $B$ then $\det B = k \cdot \det A$.

**Note 3.2.1.** Suppose a square matrix $A$ is reduced to row echelon form $U$ by row replacements and $r$ row interchanges. Then **Theorem 3.2.1** says that

$$\det A = \begin{cases} (-1)^r \cdot \begin{pmatrix} \text{product of} \\ \text{pivots in } U \end{pmatrix} & \text{when } A \text{ is invertible} \\ 0 & \text{when } A \text{ is not invertible} \end{cases}$$

**Theorem 3.2.2.** A square matrix $A$ is invertible if and only if $\det A \neq 0$.

**Theorem 3.2.3.** If $A$ is an $n \times n$ matrices, then $\det A^T = \det A$.

**Theorem 3.2.4.** If $A$ and $B$ are $n \times n$ matrices, then $\det AB = (\det A)(\det B)$.

**Note 3.2.2.** A common misconception about the determinant is that it distributes across sums, when in fact $\det(A + B) \neq \det A + \det B$

**Homework Assignment**
**Section 3.2**: (Page 5, 11, 19, 23, 25)

**Example 1.** *Compute* $\det A = \begin{vmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{vmatrix} =$

**Example 2.** *Compute* $\det A = \begin{vmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{vmatrix} =$

**Example 3.** *Discuss the difference between the following two equalities:*

$$\begin{bmatrix} 2 & 4 & 6 \\ 4 & 6 & 8 \\ 6 & 8 & 10 \end{bmatrix} = 2 \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix} \qquad \begin{vmatrix} 2 & 4 & 6 \\ 4 & 6 & 8 \\ 6 & 8 & 10 \end{vmatrix} = 2^3 \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}$$

Read **Note 3.2.1**

**Example 4.** *Compute* $\det A = \begin{vmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ -6 & 7 & -7 & 4 \\ -5 & -8 & 0 & 9 \end{vmatrix} =$

**Example 5.** *Compute* $\det A = \begin{vmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ -2 & -5 & 4 & -2 \end{vmatrix} =$

Read **Theorem 3.2.3** and **Theorem 3.2.4**

**Example 6.** *Verify* ***Theorem 3.2.4*** *for* $A = \begin{bmatrix} 6 & 1 \\ 3 & 2 \end{bmatrix}$ *and* $B = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$

## 3.3 Cramer's Rule, Volume, and Linear Transformations

**Note 3.3.1.** For an $n \times n$ matrix $A$ and any $\mathbf{b}$ in $\mathbb{R}^n$, let $A_i(\mathbf{b})$ be the matrix obtained from $A$ by replacing column $i$ by the vector $\mathbf{b}$.

$$A_i(\mathbf{b}) = \begin{bmatrix} \mathbf{a}_1 & \cdots & \underset{\substack{\uparrow \\ \text{column } i}}{\mathbf{b}} & \cdots & \mathbf{a}_n \end{bmatrix}$$

**Theorem 3.3.1** (Cramer's Rule)**.** Let $A$ be an invertible $n \times n$ matrix. For any $\mathbf{b}$ in $\mathbb{R}^n$, the unique solution $\mathbf{x}$ of $A\mathbf{x} = \mathbf{b}$ has entries given by

$$x_i = \frac{\det A_i(\mathbf{b})}{\det A}, \qquad i = 1, 2, \ldots, n$$

Cramer's rule leads to a generalized formula for the inverse of $A$. The $j$th column of the inverse of $A$ satisfies this equation

$$A\mathbf{x} = \mathbf{e}_j$$

where $\mathbf{e}_j$ is the $j$th column of the identity matrix, and the $i$th entry of $\mathbf{x}$ is the $(i, j)$-entry of $A^{-1}$. So, Theorem 3.3.1 tells us,

$$(i, j)\text{-entry of } A^{-1} = x_i = \frac{\det A_i(\mathbf{e}_j)}{\det A} \tag{3.4}$$

Since the only column of $A_i(\mathbf{b})$ that is changed from $A$ is the $i$th column, the cofactor expansion about that column will remain unchanged.

$$\det A_i(\mathbf{e}_j) = (-1)^{i+j} \det A_{ji} = C_{ji} \tag{3.5}$$

Now, by replacing equation (3.5) into equation (3.4) we have the following:

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

**Definition 3.3.1.** The matrix of cofactors in the equation above is called the **adjugate** (or the **classical adjoint**) of $A$, denoted by adj $A$

**Theorem 3.3.2** (An Inverse Formula)**.** Let $A$ be an invertible $n \times n$ matrix. Then

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A$$

**Theorem 3.3.3.** If $A$ is a $2 \times 2$ matrix, the area of the parallelogram determined by the columns of $A$ is $|\det A|$. If $A$ is a $3 \times 3$ matrix, the volume of the parallelepiped determined by the columns of $A$ is $|\det A|$.

**Note 3.3.2.** Let $\mathbf{a}_1$ and $\mathbf{a}_2$ be nonzero vectors. Then for any scalar $c$, the area of the parallelogram determined by $\mathbf{a}_1$ and $\mathbf{a}_2$ equals the area of the parallelogram determined by $\mathbf{a}_1$ and $\mathbf{a_2} + c\mathbf{a}_1$

**Theorem 3.3.4.** Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation determined by a $2 \times 2$ matrix $A$. If $S$ is a parallelogram in $\mathbb{R}^2$, then

$$\{\text{area of } T(S)\} = |\det A| \cdot \{\text{area of } S\}$$

If $T$ is determined by a $3 \times 3$ matrix $A$, and if $S$ is a parallelepiped in $\mathbb{R}^3$, then

$$\{\text{volume of } T(S)\} = |\det A| \cdot \{\text{volume of } S\}$$

**Note 3.3.3.** The conclusions of **Theorem 3.3.4** hold whenever $S$ is a region in $\mathbb{R}^2$ with finite area or a region in $\mathbb{R}^3$ with finite volume.

**Homework Assignment**
**Section 3.3**: (Page 209) 1, 5, 9, 11, 21, 23, 27, 29

**Example 1.** *Use Cramer's Rule to solve the system*

$$\begin{cases} 3x_1 - 2x_2 = 6 \\ -5x_1 + 4x_2 = 8 \end{cases}$$

**Example 2.** *Determine the values of s for which the system has a unique solution, use Cramer's Rule to describe the solution set.*

$$\begin{cases} 3sx_1 - 2x_2 = 4 \\ -6x_1 + sx_2 = 1 \end{cases}$$

**Example 3.** *Find the inverse of the matrix using **Theorem 3.3.2**.* $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 4 & -2 \end{bmatrix}$

**Example 4.** *Calculate the area of the parallelogram determined by the points* $(-2,-2)$, $(0,3)$, $(4,-1)$, *and* $(6,4)$.

**Example 5.** *Let* $a$ *amd* $b$ *be positive numbers. Find the area of the region* $E$ *bounded by the ellipse whose equation is*

$$\frac{(x_1)^2}{a^2} + \frac{(x_2)^2}{b^2} = 1.$$

# Chapter 2 and 3 Review

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$3 \times 5$ note card permitted on the exam.

Content: Sections 2.1, 2.2, 2.3, 2.4, 2.5, 2.7, 3.1, 3.2, 3.3

- **Matrix Operations**: When sum and product of matrices are defined, diagonal matrix, transpose,
  elementary matrices
- **Inverse of a Matrix**: Singular vs. nonsingular, determinant, inverse algorithm
- **Characterization of Invertible Matrices**: IMT, and the inverse of a transformation
- **Block Matrices**: How to write them, Column-Row Expansion of $AB$
- **LU Factorization** : Algorithm and how to use the factorization to solve $A\mathbf{x} = \mathbf{b}$
- **Applications in Computer Graphics**: Homogeneous coordinates, translations, rotations, reflections, scaling
- **Intro to Determinants**: Determinant of an $n \times n$, cofactor expansion
- **Propertires of the Determinant**: How row operations change the determinant
- **Cramer's Rule, Volume, and Linear Trans.**: Cramer's Rule, Adjugate, parallelogram area, parallelepiped volume using the determinant.

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**True/False** Questions:

1. _____. If $A$ and $B$ are $m \times n$ matrices, then both $AB^T$ and $A^TB$ are defined.
2. _____. If $BC = BD$ then $C = D$.
3. _____. If $AC = 0$, then either $A = 0$ or $C = 0$.
4. _____.The transpose of an elementary matrix is an elementary matrix.
5. _____. If $AB = BA$ and if $A$ is invertible, then $A^{-1}B = BA^{-1}$.

**Extended Response** Questions:

6. Suppose $CA = I_n$ (the $n \times n$ identity matrix). Show that the equation $A\mathbf{x} = \mathbf{b}$ has only the trivial solution. Explain why $A$ cannot have more rows than columns.

**Extended Response** Questions continued:

7. Let $A = \begin{bmatrix} 3 & -6 \\ -1 & 2 \end{bmatrix}$. Construct a $2 \times 2$ matrix $B$ such that $AB$ is the zero matrix. Use two different nonzero columns for $B$.

8. Show that $AI_n = A$ when $A$ is an $m \times n$ matrix. *Hint: use the "column" definition of the product $AI_n$.*

9. If $A = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}$ and $AB = \begin{bmatrix} -1 & 2 & -1 \\ 6 & -9 & 3 \end{bmatrix}$, determine the first and second column of $B$.

10. Find the inverse of the matices given below, if they exist.

$$A = \begin{bmatrix} 5 & 10 \\ 4 & 7 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & -2 & 1 \\ 4 & -7 & 3 \\ -2 & 6 & -4 \end{bmatrix}$$

**Extended Response** Questions continued:

10. Suppose $T$ and $U$ are linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^n$ such that $T(U(\mathbf{x})) = \mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$. Is it true that $U(T(\mathbf{x})) = \mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$? Why or why not?

11. Let $A = \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}$, where $B$ and $C$ are square. Show that $A$ is invertible *if and only if* both $B$ and $C$ are invertible.

12. Find an $LU$ factorization of $A = \begin{bmatrix} 2 & -4 & -2 & 3 \\ 6 & -9 & -5 & 8 \\ 2 & -7 & -3 & 9 \\ 4 & -2 & -2 & -1 \\ -6 & 3 & 3 & 4 \end{bmatrix}$.

**Extended Response** Questions continued:

13. Solve the equation $A\mathbf{x} = \mathbf{b}$ by using the $LU$ factorization given for $A$.

$$A = \begin{bmatrix} 4 & 3 & -5 \\ -4 & -5 & 7 \\ 8 & 6 & -8 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2 \\ -4 \\ 6 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 3 & -5 \\ 0 & -2 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

14. Find the $3 \times 3$ matrix that produces a translation by $(3, 1)$, and then a rotation of $45^\circ$ about the origin.

15. Compute the determinant using the cofactor expansion about the first row. To check your work complete the cofactor expansion down the second column.

$$\begin{vmatrix} 0 & 5 & 1 \\ 4 & -3 & 0 \\ 2 & 4 & 1 \end{vmatrix}$$

16. Find the determinant by row reducing to echlon form.

$$\begin{vmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ -1 & 2 & 8 & 5 \\ 3 & -1 & -2 & 3 \end{vmatrix}$$

17. Use Cramer's, **Theorem 3.3.1**, rule to compute the solutions of the system

$$\begin{aligned} 3x_1 - 2x_2 &= 7 \\ -5x_1 + 6x_2 &= -5 \end{aligned}$$

# 4 Vector Spaces

## 4.1 Vector Spaces and Subspaces

**Definition 4.1.1.** A **vector space** is a nonempty set $V$ of objects, called *vectors*, on which are defined two binary operations, called *addition* and *multiplication by scalars*, subject to the ten axioms listed below. The axioms must hold for all vectors $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ in $V$ and for all real (and complex) scalars $c$ and $d$.

1. The sum of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u}+\mathbf{v}$, is in $V$. (*Closure of Vector Addition*)
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$. (*Commutativity of Addition*)
3. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$. (*Associativity of Addition*)
4. There is a **zero** vector $\mathbf{0}$ in $V$ such that $\mathbf{u}+\mathbf{0}=\mathbf{u}$. (*Additive Identity*)
5. For each $\mathbf{u}$ in $V$, there is a vector $-\mathbf{u}$ in $V$ such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$. (*Additive Inverse*)
6. The scalar multipule of $\mathbf{u}$ by $c$, denoted by $c\mathbf{u}$, is in $V$. (*Closure Under Scalar Multiplication*)
7. $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$. (*Scalar Distribution*)
8. $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$. (*Vector Distribution Over Scalar Addition*)
9. $c(d\mathbf{u})=(cd)\mathbf{u}$. (*Associativity of Scalar Multiplication*)
10. $1\mathbf{u}=\mathbf{u}$. (*Multiplicative Identity*)

**Note 4.1.1.** For each $\mathbf{u}$ in $V$ and scalar $c$,

$$\begin{aligned} 0\mathbf{u} &= \mathbf{0}, \\ c\mathbf{0} &= \mathbf{0}, \\ -\mathbf{u} &= (-1)\mathbf{u}. \end{aligned}$$

**Definition 4.1.2.** A **subspace** of a vector space $V$ is a subset $H$ of $V$ that has three properties:

a. The zero vector of $V$ is in $H$. (*$H$ is nonempty*)

b. $H$ is closed under vector addition.

c. $H$ is closed under multiplication by scalars.

**Theorem 4.1.1.** If $\mathbf{v}_1, \dots, \mathbf{v}_p$ are in a vector space $V$, then $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is a subspace of $V$.

**Homework Assignment**
**Section 4.1**: (Page 223 1, 3, 5, 8, 11, 13, 17, 33)

Read **Definition 4.1.1** and **Note 4.1.1**

- $V = \mathbb{R}^n$, $n \geq 1$ is the best example of a vector space.
- Let $V$ be the set of all arrows in three space, and regarded as equal if they have the same length and direction. $V$ is a vector space using the parallelogram rule for addition.
- Let $V = \mathbb{S}$ be the space of all doubly infinite sequences of numbers.

$$\{y_k\} \in \mathbb{S} \text{ and } \{y_k\} = \{\ldots, y_{-2}, y_{-1}, y_0, y_1, y_2, \ldots\}$$

  If $\{z_k\}$ is another such sequence we can define an addition

$$\{y_k\} + \{z_k\} = \{y_k + z_k\},$$

  the sum of each element. Under this addition $\mathbb{S}$ is a vector space. (also scalar multiplication)

**Example 1.** *For $n \geq 0$, the set $\mathbb{P}_n$ of polynomials of degree at most $n$ consists of all polynomials of the form*

$$p(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n$$

*The degree of $p$ is the highest power of $t$ whose coefficent is nonzero. If $a_0 = a_1 = \cdots = a_n = 0$ then $p$ is called the zero polynomial. If $p, q \in \mathbb{P}_n$ with $p(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n$ and $q(t) = b_0 + b_1 t + b_2 t^2 + \cdots + b_n t^n$.*

**Example 2.** *Let $V$ be the set of all real valued functions on a set $D$, of real numbers or some interval of $\mathbb{R}$.*

**Example 3.** *The set containing only the zero vector from a vector space $V$ is a subspace of $V$.*

*proof:*

**Example 4.** $\mathbb{P}_n$, *the set of all polynomial functions of at most degree n with real coefficents is a subspace of the vector space of all real valued functions.*

**Example 5.** *Is* $\mathbb{R}^2$ *a subspace of* $\mathbb{R}^3$*? Is the set* $H = \left\{ \begin{bmatrix} s \\ t \\ 0 \end{bmatrix} : s, t \in \mathbb{R} \right\}$ *a subspace of* $\mathbb{R}^3$*?*

**Example 6.** *Is a plane in* $\mathbb{R}^3$ *a subspace of* $\mathbb{R}^3$*? Is a line in* $\mathbb{R}^2$ *a subspace of* $\mathbb{R}^2$*?*

**Example 7.** *Given* $\mathbf{v}_1$ *and* $\mathbf{v}_2$ *in* $V$*, let* $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$*. Show that* $H$ *is a subspace of* $V$*.*

**Example 8.** *Let* $H$ *be the set of all vectors of the form* $\begin{bmatrix} a - 3b \\ b - a \\ a \\ b \end{bmatrix}$*, where* $a, b \in \mathbb{R}$*. Show that* $H$ *is a subspace of* $\mathbb{R}^4$*.*

**Example 9.** *For what values of* $h$ *will* $\mathbf{y}$ *be in the subspace of* $\mathbb{R}^3$ *that is spanned by* $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$*.*

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{y} = \begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix}$$

## 4.2 Null Spaces, Column Spaces, and Linear Transformations

**Definition 4.2.1.** The **null space** of an $m \times n$ matrix $A$, written as Nul $A$, is the set of all solutions to the homogeneous equation $A\mathbf{x} = \mathbf{0}$. In set notation

$$\text{Nul}\, A = \{\mathbf{x} : \mathbf{x} \in \mathbb{R}^n \text{ such that } A\mathbf{x} = \mathbf{0}\}$$

**Theorem 4.2.1.** The null space of an $m \times n$ matrix $A$ is a subspace of $\mathbb{R}^n$. Equivalently, the set of all solutions to a system $A\mathbf{x} = \mathbf{0}$ of $m$ homogeneous linear equations in $n$ unknowns is a subspace of $\mathbb{R}^n$.

*Proof.*

**Definition 4.2.2.** The **column space** of an $m \times n$ matrix $A$, written as Col $A$, is the set of all linear combinations of the columns of $A$. If $A = \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{bmatrix}$, then

$$\text{Col}\, A = \text{Span}\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$$

**Theorem 4.2.2.** The column space of an $m \times n$ matrix $A$ is a subspace of $\mathbb{R}^m$.

**Note 4.2.1.** The column space of an $m \times n$ matrix $A$ is all of $\mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$ in $\mathbb{R}^m$.

**Definition 4.2.3.** A **linear transformation** $T$ from a vector space $W$ is a rule that assigns to each vector $\mathbf{x}$ in $V$ a unique vector $T(\mathbf{x})$ in $W$, such that

(i) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v}$ in $V$, and

(ii) $T(c\mathbf{u}) = cT(\mathbf{u})$ for all $\mathbf{u}$ in $V$ and all scalars $c$.

The **kernel** (or **null space**) of such a $T$ is the set of all $\mathbf{u}$ in $V$ such that $T(\mathbf{u}) = \mathbf{0} \in W$.
The **range** of $T$ is the set of all vectors in $W$ of the form $T(\mathbf{x})$ for some $\mathbf{x}$ in $V$.

**Homework Assignment**
**Section 4.2**: (Page 234 1, 5, 9, 13, 15, 17, 19, 23, 25, 31, 34, 37)

**Example 1.** *Let* $A = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix}$*, and let* $\mathbf{u} = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$*. Determine if* $\mathbf{u}$ *belongs to the null space of* $A$*.*

**Example 2.** *Let* $H$ *be the set of all vectors in* $\mathbb{R}^4$ *whose coordinates satisfy the equation* $a - 2b + 5c = d$ *and* $c - a = b$*. Show that* $H$ *is a subspace of* $\mathbb{R}^4$*.*

**Example 3.** *Find a spanning set for the null space of the matrix* $A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$

**Example 4.** *Find a matrix* $A$ *such that* $W = \text{Col}\, A$ *where* $W = \left\{ \begin{bmatrix} 6a - b \\ a + b \\ -7a \end{bmatrix} \middle| a, b \in \mathbb{R} \right\}$.

**Example 5.** *Let* $A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$

(a) *If the column space of* $A$ *is a subspace of* $\mathbb{R}^k$*, what is* $k$*?*

(b) *If the null space of* $A$ *is a subspace of* $\mathbb{R}^n$*, what is* $n$*?*

**Example 6.** *With $A$ as in **Example 5** find a nonzero vector in* Col $A$ *and a nonzero vector in* Nul $A$.

$$A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$$

**Example 7.** *With $A$ as in **Example 5**, let* $\mathbf{u} = \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix}$ *and* $\mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}$.

*(a) Determine if* $\mathbf{u}$ *is in* Nul $A$*. Could* $\mathbf{u} \in$ Col $A$*?*

*(b) Determine if* $\mathbf{v}$ *is in* Col $A$*. Could* $\mathbf{v} \in$ Nul $A$*?*

**Example 8.** *Let $V$ be the space of all real valued functions $f$ on $[a, b]$ that are continuously differentiable on $[a, b]$. Let $W$ be the space of all continuous functions. Define $D : f \mapsto f'$ in which $D : V \to W$, called the derivative transformation. Is $D$ linear?*

**Example 9.** *Define a linear transformation $T : \mathbb{P}_2 \to \mathbb{R}^2$ by $T(\mathbf{p}) = \begin{bmatrix} \mathbf{p}(0) \\ \mathbf{p}(0) \end{bmatrix}$. Find polynomials $\mathbf{p}_1$ and $\mathbf{p}_2$ in $\mathbb{P}_2$ that span the kernal of $T$, and describe the range of $T$.*

## 4.3 Linear Independent Set; Bases

**Definition 4.3.1.** An indexed set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ in $V$ is said to be **linearly independent** if the vector equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0} \tag{4.6}$$

has *only* the trivial solution, $c_1 = 0, \ldots, c_n = 0$.
The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is said to be **linearly dependent** if equation (4.6) has a nontrivial solution. Equation (4.6) is called a **linear dependence relation** among $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

**Theorem 4.3.1.** An indexed set $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ of two or more vectors, with $\mathbf{v}_1 \neq \mathbf{0}$, is linearly dependent if and only if some $\mathbf{v}_j$ (with $j > 1$) is a linear combination of the preceding vectors $\mathbf{v}_1, \ldots, \mathbf{v}_j$ .

**Definition 4.3.2.** Let $H$ be a subspace of a vector space $V$. An indexed set of vectors $\mathscr{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ in $V$ is a **basis** for $H$ if

(i) $\mathscr{B}$ is a linearly independent set, and

(ii) the subspace spanned by $\mathscr{B}$ coincides with $H$; that is,

$$H = \text{Span}\{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$$

**Theorem 4.3.2** (The Spanning Set Theorem)**.** Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ be a set in $V$, and let $H = \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ .

a. If one of the vectors in $S$, $\mathbf{v}_k$ is a linear combination of the remaining vectors in $S$, then the set formed from $S$ by removing $\mathbf{v}_k$ still spans $H$.

b. If $H \neq \{\mathbf{0}\}$, some subset of $S$ is a basis for $H$.

**Note 4.3.1.** Elementary row operations on a matrix do not affect the linear dependence relations among the columns of the matrix.

**Theorem 4.3.3.** The pivot columns of a matrix $A$ form a basis for Col $A$.

**Homework Assignment**
**Section 4.3**: (Page 243 1-13 odd, 17, 19, 21, 33, 36)

**Example 1.** *Let* $\mathbf{p}_1(t) = 1$, *and* $\mathbf{p}_2(t) = t$, *and* $\mathbf{p}_3(t) = 4 - t$. *Is the set* $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ *linearly independent?*

**Example 2.** *Is the set* $\{\sin t, \cos t\}$ *a linearly independent set in* $C[0,1]$, *the space of all continous functions on* $[0,1]$*?*

**Example 3.** *Let* $A$ *be an invertible matrix* $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}$. *Do the columns of* $A$ *form a basis for* $\mathbb{R}^n$*?*

**Example 4.** *Let* $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 0 \\ -6 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -4 \\ 1 \\ 7 \end{bmatrix}$, *and* $\mathbf{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}$. *Determine if* $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ *is a basis for* $\mathbb{R}^3$.

**Example 5.** *Let* $S = \{1, t, t^2, \ldots, t^n\}$*. Verify that* $S$ *is a basis for* $\mathbb{P}_n$*.*
*This is called the* <u>*standard basis*</u> *for* $\mathbb{P}_n$*.*

**Example 6.** *Let* $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 6 \\ 16 \\ -5 \end{bmatrix}$*, and* $H = \operatorname{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$*. Find a basis for* $H$*.*

**Example 7.** *Find a basis for* $\operatorname{Col} B$*, where* $B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_5 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

**Example 8.** *It can be shown that the matrix* $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_5 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix}$ *is row equivalent to the matrix B is **Example 7**. Find a basis for* Col $A$.

**Example 9.** *Consider the following three sets of* $\mathbb{R}^3$. *Which form a basis for* $\mathbb{R}^3$? *Which* Span $\mathbb{R}^3$? *Which are linearly independent?*

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \right\}, \qquad \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right\}, \qquad \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \right\}$$

## 4.4 Coordinate Systems

**Theorem 4.4.1** (The Uniqueness Representation Theorem)**.** Let $\mathscr{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ be a basis for a vector space $V$. The for each $\mathbf{x}$ in $V$, there exists a unique set of scalars $c_1, \dots, c_n$ such that

$$\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$$

**Definition 4.4.1.** Suppose $\mathscr{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ is a basis for $V$ and $\mathbf{x}$ is in $V$, The **coordinates of x relative to the basis** $\mathscr{B}$ (or the $\mathscr{B}$**-coordinates of x**) are the weights $c_1, \dots, c_n$ such that $\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$. We write,

$$[\mathbf{x}]_{\mathscr{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}.$$

for the $\mathscr{B}$**-coordinates of x**. The mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathscr{B}}$ is the **coordinate mapping determined by B**.

**Note 4.4.1.** The vector equation $\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$ for a basis $\mathscr{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ is equivalent to

$$\mathbf{x} = P_{\mathscr{B}}[\mathbf{x}]_{\mathscr{B}}, \qquad \text{where } P_{\mathscr{B}} = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n \end{bmatrix}$$

$P_{\mathscr{B}}$ is called the **change-of-coordinates matrix** from $\mathscr{B}$ to the standard basis in $\mathbb{R}^n$.

**Theorem 4.4.2.** Let $\mathscr{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ be a basis for a vector space $V$. Then the coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathscr{B}}$ is a one-to-one linear transformation from $V$ onto $\mathbb{R}^n$.

*Proof.*

**Definition 4.4.2.** In general, a one-to-one linear transformation from a vector space $V$ onto a vector space $W$ is called an **isomorphism** from $V$ onto $W$. **Theorem 4.4.2** is an example of an isomorphism from $V$ onto $\mathbb{R}^n$.

**Homework Assignment**
**Section 4.4**: (Page 253: 3, 7, 9, 11, 13, 17, 21, 27, 29, 31, 33 )

**Example 1.** *Consider a basis* $\mathscr{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ *for* $\mathbb{R}^2$, *where* $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ *and* $\mathbf{b}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. *Suppose an* $\mathbf{x} \in \mathbb{R}^2$ *has the coordinate vector* $[\mathbf{x}]_{\mathscr{B}} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$. *Find* $\mathbf{x}$.

**Example 2.** *The entries in the vector* $\mathbf{x} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$ *are the coordinates of* $\mathbf{x}$ *relative to the standard basis* $\mathscr{E} = \{\mathbf{e}_1, \mathbf{e}_2\}$, *since...*

**Example 3.** *Let* $\mathbf{b}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$, *and* $\mathscr{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$. *Find the coordinate vector* $[\mathbf{x}]_{\mathscr{B}}$. *That is the vector* $\mathbf{x}$ *relative to* $\mathscr{B}$.

**Example 4.** *Use coordinate vectors to verify that the polynomials* $1 + 2t^2$, $4 + t + 5t^2$, *and* $3 + 2t$ *are linearly dependent in* $\mathbb{P}_2$.

**Example 5.** *Let* $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, *and* $\mathbf{x} = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}$, *and* $\mathscr{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$. *Then* $\mathscr{B}$ *is a basis for* $H = \operatorname{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$. *Determine if* $\mathbf{x} \in H$, *and if it is find the coordinates of* $\mathbf{x}$ *relative to* $\mathscr{B}$.

## 4.5 Dimension of a Vector Space

**Theorem 4.5.1.** If a vector space $V$ has a basis $\mathscr{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, then any set in $V$ containing more than $n$ vectors must be linearly dependent.

**Theorem 4.5.2.** If a vector space $V$ has a basis of $n$ vectors, then every basis of $V$ must consist of exactly $n$ vectors.

**Definition 4.5.1.** If $V$ is spanned by a finite set, then $V$ is said to be **finite-dimensional**, and the **dimension** of $V$, written as $\dim V$, is the number of vectors in a basis for $V$. The dimension of the zero vector space $\{\mathbf{0}\}$ is defined to be zero. If $V$ is not spanned by a finite number set, then $V$ is said to be **infinite-dimensional**.

**Theorem 4.5.3.** Let $H$ be a subspace of a finite-dimensional vector space $V$. Any linearly independent set in $H$ can be expanded, if necessary, to be a basis for $H$. Also, $H$ is finite-dimensional and

$$\dim H \leq \dim V$$

**Theorem 4.5.4.** Let $V$ be a $p$-dimensional vector space, $p \geq 1$. Any linearly independent set of exactly $p$ elements in $V$ is automatically a basis for $V$. Any set of exactly $p$ elements that spans $V$ is automatically a basis for $V$.

**Note 4.5.1.** The dimension of Nul $A$ is the number of free variables in the equation $A\mathbf{x} = \mathbf{0}$, and the dimension of Col $A$ is the number of pivot columns in $A$.

**Homework Assignment**
**Section 4.5**: (Page 260 1, 5, 7, 11, 13, 15, 17, 21 )

**Example 1.** *Find the dimension of the following vector spaces*

$\mathbb{R}^n$

$\mathbb{P}_n$

**Example 2.** *Let* $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$, *where* $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}$, *and* $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. *Then* $H$ *is a plane in* $\mathbb{R}^3$.

*Find* $\dim H$.

**Example 3.** *Find the dimension of the subspace*

$$H = \left\{ \begin{bmatrix} a - 3b + 6c \\ 5a + 4d \\ b - 2c - d \\ 5d \end{bmatrix} \middle| a, b, c, d \in \mathbb{R} \right\}$$

**Example 4.** *Find the dimensions of the null space and the column space of*

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

# 4.6 Rank

**Definition 4.6.1.** The **rank** of $A$ is the dimension of the column space of $A$.

**Definition 4.6.2.** the set of all linear combinations of the row vectors is called the **row space** of $A$ and is denoted by Row $A$.

**Theorem 4.6.1** (The Rank Theorem)**.** The dimensions of the column space and the row space of an $m \times n$ matrix $A$ are equal. This common dimension, the rank of $A$, also equals the number of pivot positions in $A$ and satisfies the equation

$$\operatorname{rank} A + \dim \operatorname{Nul} A = n$$

**Equivalent Description of Rank**

The rank of an $m \times n$ matrix $A$ may be described in several ways:

- the dimension of the column space of $A$, $\dim \operatorname{Col} A$ (definition),
- the number of pivot positions in $A$,
- the maximum number of linearly independent columns in $A$,
- the dimension of the row space of $A$,
- the maximum number of linearly independent rows in $A$,
- the maximum number of columns in an invertible submatrix of $A$.

**Effects of Elementary Row Operations**

- Row operations *do not* affect linear dependence relation among columns.
- Row operations *usually change* the column space.
- Row operations *never* change the row space.
- Row operations *never* change the null space.

**Theorem 4.6.2** (Invertible Matrix Theorem (continued))**.** Let $A$ be an $n \times n$ matrix. Then the following statements are each equivalent to the statement that $A$ is an invertible matrix.

m. The columns of $A$ form a basis of $\mathbb{R}^n$.

n. $\operatorname{Col} A = \mathbb{R}^n$

o. $\dim \operatorname{Col} A = n$

p. $\operatorname{rank} A = n$

q. $\operatorname{Nul} A = \{\mathbf{0}\}$

r. $\operatorname{nullity} A = \dim \operatorname{Nul} A = 0$

**Homework Assignment**
**Section 4.6**: (Page 269 3, 17, 27, 35, 5 – 16 *in class* )

**Example 1.** *Let* $A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$ *give a spanning set for* Row $A$

**Example 2.** *Find bases for the row space, the column space, and the null space of the matrix*

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$$

**Example 3.** *(a) If $A$ is a $7 \times 9$ matrix with a two-dimensional null space, what is the rank of $A$?*

*(b) Could a $6 \times 9$ have a two-dimensional null space?*

**Example 4.** *Let* $A = \begin{bmatrix} 3 & 0 & -1 \\ 3 & 0 & -1 \\ 4 & 0 & 5 \end{bmatrix}$*. Find and consider the subspaces determined by $A$ (i.e.* Nul $A$, Col $A$, Row $A$ *and* Row $A^T$*) Describe these subspaces inside the vector space $\mathbb{R}^3$.*

**Example 5.** *A scientist found two solutions to a homogeneous system of 40 equations in 42 variables. The two solutions are not multiples, and all other solutions can be constructed by adding together multiples of these two solutions. Can the scientist be certian that an associated nonhomogeneous system has a solution?*

## 4.7 Change of Basis

**Theorem 4.7.1.** Let $\mathscr{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ and $\mathscr{C} = \{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$ be a bases of a vector space $V$. Then there is a unique $n \times n$ matrix $\underset{\mathscr{C}\leftarrow\mathscr{B}}{P}$ such that

$$[\mathbf{x}]_{\mathscr{C}} = \underset{\mathscr{C}\leftarrow\mathscr{B}}{P}[\mathbf{x}]_{\mathscr{B}}$$

The columns of $\underset{\mathscr{C}\leftarrow\mathscr{B}}{P}$ are the $\mathscr{C}$-coordinate vectors of the vectors in the basis $\mathscr{B}$. That is,

$$\underset{\mathscr{C}\leftarrow\mathscr{B}}{P} = \begin{bmatrix} [\mathbf{b}_1]_{\mathscr{C}} & [\mathbf{b}_2]_{\mathscr{C}} & \cdots & [\mathbf{b}_n]_{\mathscr{C}} \end{bmatrix}$$

The matrix $\underset{\mathscr{C}\leftarrow\mathscr{B}}{P}$ is called the **change-of-coordinates matrix from $\mathscr{B}$ to $\mathscr{C}$**.

**Homework Assignment**
**Section 4.7**: (Page 276 1, 3, 5, 9, 13, 17, 18)

**Example 1.** *Consider two bases* $\mathscr{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ *and* $\mathscr{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ *for a vector space* $V$*, such that*

$$\mathbf{b}_1 = 4\mathbf{c}_1 + \mathbf{c}_2 \text{ and } \mathbf{b}_2 = -\mathbf{c}_1 + \mathbf{c}_2$$

*Suppose* $\mathbf{x} = 3\mathbf{b}_1 + \mathbf{b}_2$. *Find* $[\mathbf{x}]_{\mathscr{C}}$.

**Example 2.** *Let* $\mathbf{b}_1 = \begin{bmatrix} -9 \\ 1 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$, $\mathbf{c}_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$, $\mathbf{c}_2 = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$, *and consider the bases for* $\mathbb{R}^2$ *given by* $\mathscr{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ *and* $\mathscr{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$. *Find the change-of-coordinates matrix from* $\mathscr{B}$ *to* $\mathscr{C}$.

**Example 3.** *Let* $\mathbf{b}_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$, $\mathbf{c}_1 = \begin{bmatrix} -7 \\ 9 \end{bmatrix}$, $\mathbf{c}_2 = \begin{bmatrix} -5 \\ 7 \end{bmatrix}$, *and consider the bases for* $\mathbb{R}^2$ *given by* $\mathscr{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ *and* $\mathscr{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$.

(a) *Find* $\underset{\mathscr{C} \leftarrow \mathscr{B}}{P}$ *and* $\underset{\mathscr{B} \leftarrow \mathscr{C}}{P}$.

(b) *If* $\mathbf{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ *find* $[\mathbf{x}]_{\mathscr{B}}$ *and* $[\mathbf{x}]_{\mathscr{C}}$.

## 4.9 Applications to Markov Chains

**Definition 4.9.1.** A vector with nonnegative entries that add up to 1 is called a **probability vector**.

**Definition 4.9.2.** A **stochastic matrix** is a square matrix whose columns are probability vectors.

**Definition 4.9.3.** A **Markov chain** is a sequence of probability vectors $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots$, together with a stochastic matrix $P$, such that

$$\mathbf{x}_1 = P\mathbf{x}_0, \quad \mathbf{x}_2 = P\mathbf{x}_1, \quad \mathbf{x}_3 = P\mathbf{x}_2, \ldots$$

Thus the Markov chain is described by the first order difference equation

$$\mathbf{x}_{k+1} = P\mathbf{x}_k \quad \text{for } k = 0, 1, 2, \ldots$$

In this recursive equations $\mathbf{x}_k$ is often called the **state vector**.

**Definition 4.9.4.** If $P$ is a stochastic matrix, then a **steady-state vector** (or **equilibrium vector**) for $P$ is a probability vector $\mathbf{q}$ such that

$$P\mathbf{q} = \mathbf{q}$$

It can be shown that every stochastic matrix has a steady-state vector.

**Definition 4.9.5.** A stochastic matrix is **regular** if some matrix power $P^k$ contains only strictly positive entries.

**Definition 4.9.6.** A sequence of vectors $\{\mathbf{x}_k : k = 1, 2, \ldots\}$ **converges** to a vector $\mathbf{q}$ as $k \to \infty$ if for any $\epsilon > 0$ there is a positive $M$ such that if $k > M$ then $|\mathbf{x}_k - \mathbf{q}| < \epsilon$. Where $||$ is the vector magnitude.

**Theorem 4.9.1.** If $P$ is an $n \times n$ regular stochastic matrix, then $P$ has a unique steady-state vector $\mathbf{q}$. Further, if $\mathbf{x}_0$ is any initial state and $\mathbf{x}_{k+1} = P\mathbf{x}_k$ for $k = 0, 1, 2, \ldots$, then the Markov chain $\{\mathbf{x}_k\}$ converges to $\mathbf{q}$ as $k \to \infty$.

**Homework Assignment**
**Section 4.9**: (Page 296 1,3, 5, 9, 11, 13, 15)

**Example 1.** *The annual migration between two metropolitan region is governed by the matrix Suppose*

**Annual Percent Migration between City and Suburbs**

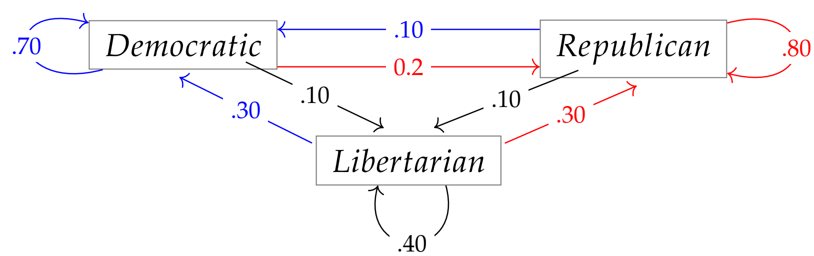
| From: City | From: Suburbs | To: |
|---|---|---|
| 0.95 | 0.03 | City |
| 0.05 | 0.97 | Suburbs |

*that in year 2000 the population of the region is 600,000 in the city and 400,000 in the suburbs. What is the population distribution of the city and suburbs in the years 2001, 2002?*

**Example 2.** *Suppose voting results were put into a vector of* $\mathbb{R}^3$ *so that* $\begin{bmatrix} \textit{\% vote Democratic} \\ \textit{\% vote Republican} \\ \textit{\% vote Libertarian} \end{bmatrix}$. *Suppose the trend of transitioning from one party to another were observed on a yearly basis to formulate the following stocastic matrix*

From:

$$\begin{array}{cc} & \begin{array}{ccc} \text{D} & \text{R} & \text{L} \end{array} \quad \text{To:} \\ P = & \begin{bmatrix} .70 & .10 & .30 \\ .20 & .80 & .30 \\ .10 & .10 & .40 \end{bmatrix} \begin{array}{c} D \\ R \\ L \end{array} \end{array}$$




*Determine the likely outcome of the elections for the next two years if the previous election outcome was* $\mathbf{x} = \begin{bmatrix} .55 \\ .40 \\ .05 \end{bmatrix}$.

**Example 3.** *Let* $P = \begin{bmatrix} .5 & .2 & .3 \\ .3 & .8 & .3 \\ .2 & 0 & .4 \end{bmatrix}$ *and* $x_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. *Consider a system whose state is described by the Markov Chain* $\mathbf{x}_{k+1} = P\mathbf{x}_k$ *for* $k = 0, 1, 2, \ldots$. *What happens to the state vectors as time passes? Calculate* $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{15}$.

**Example 4.** *The probability vector* $\mathbf{q} = \begin{bmatrix} .375 \\ .625 \end{bmatrix}$ *is a steady state vector for the population migration matrix M in* ***Example 1***.

**Example 5.** *Let* $P = \begin{bmatrix} .6 & .3 \\ .4 & .7 \end{bmatrix}$. *Find a steady-state vector for P.*

**Example 6.** *Find the steady state vector for* ***Example 2***. *What percentage of voters are likely to vote republican many years from now?*

**Example 7.** *Determine if the following two stocastic matrices are regular. If not, explain why.*

(a) $P = \begin{bmatrix} 1 & 0.4 \\ 0 & 0.6 \end{bmatrix}$

(b) $P = \begin{bmatrix} 0.2 & 1 \\ 0.8 & 0 \end{bmatrix}$

# Chapter 4 Review

$3 \times 5$ note card permitted on the exam.

Content: Sections 4.1, 4.2, 4.3, 4.4, 4.5, 4.6, 4.7, 4.9

- **Vector Spaces and Subspaces**: 10 axioms of a vector space, 3 requirements for subspace
- **Null and Column Spaces, and Linear Trans.**: Their definitions and how to find them
- **Linearly Independent Sets and Bases**: Dependence relation, Two things required to be a basis
- **Coordinate Systems**: $\mathscr{B}$-coordinates of $\mathbf{x}$, isomorphisms
- **Dimension of a Vector Space**: Definition of dimension, dimension of Nul $A$ and Col $A$
- **Rank**: Know equivalent descriptions of the rank, and the Rank Theorem
- **Change of Basis**: How to find the change-of-coordinate matrix from $\mathscr{B}$ to $\mathscr{C}$
- **Markov Chains**: Probability vectors, stochastic matrices, and steady-state vectors and how to find them

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**True/False** Questions:

1. ______. Row operations on a matrix change the null space.
2. ______. If $A$ is $m \times n$ and rank $A = m$, then the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one. (What if rank $A = n$ ?)
3. ______. If $A$ is $m \times n$ and the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is onto, then rank $A = m$.
4. ______. The rank of a matrix equals the number of nonzero rows.
5. ______. If $H$ is a subspace of $\mathbb{R}^3$, then there is a $3 \times 3$ matrix $A$ such that $H = \text{Col}\, A$.

**Extended Response** Questions:

6. Find a basis for the set of all vectors of the form

$$\begin{bmatrix} a - 2b + 5c \\ 2a + 5b - 8c \\ -a - 4b + 7c \\ 3a + b + c \end{bmatrix}.$$

7. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.
   a. What is the dimension of the range of $T$ if $T$ is a one-to-one mapping? Explain.
   b. What is the dimension of the kernel of $T$ if $T$ maps $\mathbb{R}^n$ onto $\mathbb{R}^m$?

**Extended Response** Questions continued:

8. Let $A$ be an $m \times n$ matrix, and let $B$ be an $n \times p$ matrix such that $AB = 0$. Show that $\operatorname{rank} A + \operatorname{rank} B \leq n$. [*Hint: One of the four subspaces* $\operatorname{Nul} A, \operatorname{Nul} B, \operatorname{Col} A$, *and* $\operatorname{Col} B$ *is contained in one of the other three subspaces*.]

9. Let $W$ be the set of vectors of the form $\begin{bmatrix} 3a+b \\ 4 \\ a-5b \end{bmatrix}$ where $a, b$, and $C$ represent arbitrary real numbers. Find a set $S$ that spans $W$ or give an example to show why $W$ is not a vector space.

10. Let $W$ be the set of vectors of the form $\begin{bmatrix} -a+1 \\ a-6b \\ 2b+a \end{bmatrix}$ where $a, b$, and $C$ represent arbitrary real numbers. Find a set $S$ that spans $W$ or give an example to show why $W$ is not a vector space.

11. Given subspaces $H$ and $K$ of a vector space $V$, the **sum** of $H$ and $K$, written as $H+K$, is the set of all vectors in $V$ that can be written as the sum of two vectors, one in $H$ and the other in $K$; that is

$$H + K = \{\mathbf{w} \mid \mathbf{w} = \mathbf{h} + \mathbf{k} \text{ for some } \mathbf{h} \in H \text{ and some } \mathbf{k} \in K\}$$

   a. Show that $H+K$ is a subspace of $V$.
   b. Show that $H$ is a subspace of $H+K$ and $K$ is a subspace of $H+K$.

12. Find an explicit description of $\operatorname{Nul} A$, by listing vectors that span the null space.

$$A = \begin{bmatrix} 1 & -2 & 0 & 4 & 0 \\ 0 & 0 & 1 & -9 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

<u>**Extended Response** Questions continued:</u>

13. Let $T : V \to W$ be a linear transformation from a vector space $V$ into a vector space $W$. Prove that the range of $T$ is a subspace of $W$. [*Hint: Typical elements of the range have the form* $T(\mathbf{x})$ *and* $T(\mathbf{w})$ *in* $W$ *for some* $\mathbf{x}$ *and* $\mathbf{w}$ *in* $V$. *In set notation* Range $T = \{T(\mathbf{x}) \mid \mathbf{x} \in V\}$ ]

14. Assume that $A$ is row equivalent to $B$. Find rank $A$ and bases for Nul $A$, Col $A$, Row $A$.

$$A = \begin{bmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & -3 & 1 \\ -3 & 8 & 2 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 6 & 5 \\ 0 & 2 & 5 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

15. Find a basis for the space spanned by the given vectors $\mathbf{v}_1, \ldots, \mathbf{v}_5$.

$$\begin{bmatrix} 1 \\ 0 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} -3 \\ -4 \\ 1 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ -8 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -6 \\ 9 \end{bmatrix}$$

16. Let $\mathscr{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathscr{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ be bases for $\mathbb{R}^2$.

Let $\mathbf{x} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$, $\mathbf{b}_1 = \begin{bmatrix} -1 \\ 8 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$, $\mathbf{c}_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$, and $\mathbf{c}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

a. Find $[\mathbf{x}]_{\mathscr{B}}$ and $[\mathbf{x}]_{\mathscr{C}}$.

b. Find the change-of-coordinate matrix from $\mathscr{B}$ to $\mathscr{C}$ and then from $\mathscr{C}$ to $\mathscr{B}$.

17. Find the steady-state vector for,

$$\begin{bmatrix} .7 & .1 & .1 \\ .2 & .8 & .2 \\ .1 & .1 & .7 \end{bmatrix}$$

# Chapter 5

# Eigenvalues and Eigenvectors

## 5.1 Eigenvalues and Eigenvectors

**Definition 5.1.1.** An **eigenvector** of an $n \times n$ matrix $A$ is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar $\lambda$. A scalar $\lambda$ is called an **eigenvalue** of $A$ if there is a nontrivial solution $\mathbf{x}$ of $A\mathbf{x}$ of $A\mathbf{x} = \lambda\mathbf{x}$; such an $\mathbf{x}$ is called an *eigenvector corresponding to* $\lambda$.

**Definition 5.1.2.** Let $A$ be an $n \times n$ matrix. $\lambda$ is an eigenvalue of $A$ if and only if

$$(A - \lambda I)\mathbf{x} = 0 \tag{5.7}$$

has a nontrivial solution. The set of all solutions, $\mathbf{x}$, to equation 5.7 is the null space of the matrix $A - \lambda I$. This set is a subspace of $\mathbb{R}^n$ called the **eigenspace** of $A$ corresponding to $\lambda$.

**Theorem 5.1.1.** The eigenvalues of a triangular matrix are the entries on its main diagonal.

**Theorem 5.1.2.** If $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ of an $n \times n$ matrix $A$, then the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is linearly independent.

**Definition 5.1.3.** If $A$ is an $n \times n$ matrix, then a *recursive* description of a sequence $\{\mathbf{x}_k\}$ in $\mathbb{R}^n$ is

$$\mathbf{x}_{k+1} = A\mathbf{x}_k \quad (k = 0, 1, 2, \ldots). \tag{5.8}$$

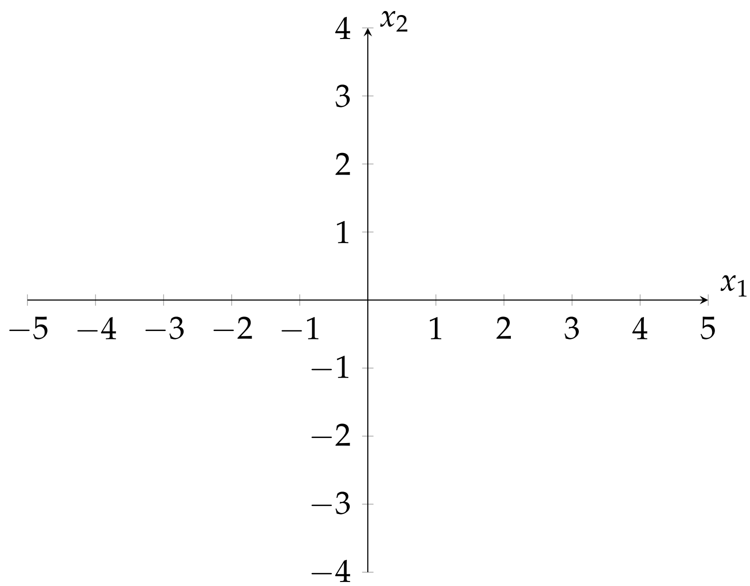
A **solution** of 5.8 is an explicit description of $\{\mathbf{x}_k\}$ whose formula for each $\mathbf{x}_k$ does not depend directly on $A$ or the preceding terms other than the initial term $\mathbf{x}_0$.

Solutions of 5.8 can be built using an eigenvector $\mathbf{x}_0$ and its corresponding eigenvalue $\lambda$ by using the relationship $\mathbf{x}_k = \lambda^k \mathbf{x}_0$ for $k = 0, 1, 2, \ldots$. Therefore,

$$\mathbf{x}_{k+1} = A\mathbf{x}_k = A(\lambda^k \mathbf{x_0}) = \lambda^k (A\mathbf{x}_0) = \lambda^k (\lambda \mathbf{x}_0) = \lambda^{k+1}\mathbf{x}_0.$$

**Homework Assignment**
**Section 5.1**: (Page 308 5, 7, 13, 17, 19, 21, 23)

**Example 1.** *Let* $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, *and* $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. *Graph the images of* $\mathbf{u}$ *and* $\mathbf{v}$ *under multiplication by* $A$. *What do you observe?*




**Example 2.** *Let* $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$, *and* $\mathbf{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$. *Are* $\mathbf{u}$ *and* $\mathbf{v}$ *eigenvectors of* $A$? *If so, what are their eigenvalues?*

**Example 3.** *Show that 7 is an eigenvalue of matrix* $A$ *from* ***Example 2*** *, and then find its corresponding eigenvector.*

**Example 4.** *Let* $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$. *An eigenvalue of* $A$ *is* 2. *Find a basis for the corresponding eigenspace.*

**Example 5.** *Let* $A = \begin{bmatrix} 3 & 6 & -8 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{bmatrix}$ *and* $B = \begin{bmatrix} 4 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & 3 & 4 \end{bmatrix}$. *Find the eigenvalues for* $A$ *and* $B$.

**Example 6.** *Something about discovery of eigenvalues... (Matrix must be singular...implies... det (A-lambda I)=0*

## 5.2 The Characteristic Equation

Recall the following definition of a determinant from Section 3.2.

**Definition 5.2.1.** Suppose a square matrix $A$ is reduced to row echelon form $U$ by row replacements and $r$ row interchanges. Then **Theorem 3.2.1** says that

$$\det A = \begin{cases} (-1)^r \cdot \begin{pmatrix} \text{product of} \\ \text{pivots in } U \end{pmatrix} & \text{when } A \text{ is invertible} \\ 0 & \text{when } A \text{ is not invertible} \end{cases}$$

**Theorem 5.2.1** (The Invertible Matrix Theorem (continued...))**.** Let $A$ be an $n \times n$ matrix. Then $A$ is invertible if and only if:

s. The number 0 is *not* an eigenvalue of $A$.

t. The determinate of $A$ is *not* zero.

This next theorem combines the Theorems of Sections 3.1 and 3.2 for reference in Chapter 5

**Theorem 5.2.2.** Let $A$ and $B$ be $n \times n$ matrices.

a. $A$ is invertible if and only if $\det A \neq 0$.

b. $\det AB = (\det A)(\det B)$.

c. $\det A^T = \det A$.

d. If $A$ is triangular, then $\det A$ is the product of the entries on the main diagonal of $A$.

e. A row replacement operation on $A$ does not change the determinant. A row interchange changes the sign of the determinant. A row scaling also scales the determinant by the same scale factor.

**Definition 5.2.2.** The scalar equation $\det(A - \lambda I) = 0$ is called the **characteristic equation** of $A$.

**Note 5.2.1.** A scalar $\lambda$ is an eigenvalue of an $n \times n$ matrix $A$ if and only if $\lambda$ satisfies the characteristic equation

$$\det(A - \lambda I) = 0$$

**Definition 5.2.3.** If $A$ is a $n \times n$ matrix, then $\det(A - \lambda I)$ is a polynomial of degree $n$ called the **characteristic polynomial** of $A$.

**Definition 5.2.4.** The **algebraic multiplicity** of an eigenvalue $\lambda$ is the multiplicity as a root of the characteristic equation.

**Definition 5.2.5.** If $A$ and $B$ are $n \times n$ matrices, then $A$ **is similar** $B$ if there is an invertible matrix $P$ such that $P^{-1}AP = B$, or equivalently, $A = PBP^{-1}$. We can write $Q = P^{-1}$ and then $Q^{-1}BQ = A$. This means that $B$ is also similar to $A$, therefore we may simplify this statement to: $A$ and $B$ **are similar**.

**Note 5.2.2.** Changing $A$ into $P^{-1}AP$ is called a **similarity transformation**.

**Theorem 5.2.3.** If $n \times n$ matrices $A$ and $B$ are similar, then they have the same characteristic polynomial and hence the same eigenvalues (with the same multiplicities).

*Proof.*

**Note 5.2.3.** Similarity is not the same as row equivalence. (If $A$ is row equivalent to $B$, then $B = EA$ for some invertible matrix $E$.) Row operations on a matrix usually change its eigenvalues.

**Homework Assignment**
**Section 5.2**: (Page 317: 18, 21, 25a, 27a )

**Example 1.** *Find the eigenvalues of* $A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$.

**Example 2.** *Compute* det $A$ *for* $A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$.

**Example 3.** *Find the characteristic equation of* $A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

**Example 4.** *The characteristic polynomial of a* $6 \times 6$ *matrix is* $\lambda^6 - 4\lambda^5 - 12\lambda^4$. *Find the eigenvalues and their algebraic multiplicities.*

## 5.3 Diagonalization

**Definition 5.3.1.** A square matrix $A$ is **diagonalizable** if $A$ is similar to a diagonal matrix, that is, if $A = PDP^{-1}$ for some invertible matrix $P$ and diagonal matrix $D$.

**Theorem 5.3.1** (The Diagonalization Theorem)**.** An $n \times n$ matrix $A$ is diagonalizable if and only if $A$ has $n$ linearly independent eigenvectors.
In fact, $A = PDP^{-1}$, with $D$ a diagonal matrix, if and only if the columns of $P$ are $n$ linearly independent eigenvectors of $A$. In this case, the diagonal entries of $D$ are the eigenvalues of $A$ that correspond, respectively, to the eigenvectors in $P$.
Stated again another way, $A$ is diagonalizable if and only if there are enough eigenvectors to form a basis for $\mathbb{R}^n$.

**Definition 5.3.2.** If an $n \times n$ matrix $A$ has eigenvectors that form a basis for $\mathbb{R}^n$, then $A$ is diagonalizable by **Theorem 5.3.1**. We call such a basis an **eigenvector basis**.

### Steps to Diagonalize a Matrix

**Step 1.** Find the eigenvalues of $A$ using the characteristic equation.

**Step 2.** Find $n$ linearly independent eigenvectors of $A$.

**Step 3.** Construct $P$ from vectors in **Step 2.**

**Step 4.** Construct $D$ from the corresponding eigenvalues.

**Theorem 5.3.2.** An $n \times n$ matrix with $n$ distinct eigenvalues is diagonalizable.

**Theorem 5.3.3.** Let $A$ be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \ldots, \lambda_p$.

a. For $1 \leq k \leq p$, the dimension of the eigenspace for $\lambda_k$ is less than or equal to the multiplicity of the eigenvector $\lambda_k$.

b. The matrix $A$ is diagonalizable if and only if the sum of the dimensions of the distinct eigenspaces equals $n$, and this happens if and only if the dimension of the eigenspace for each $\lambda_k$ equals the multiplicity of $\lambda_k$.

c. If $A$ is diagonalizable and $\mathscr{B}_k$ is a basis for the eigenspace corresponding to $\lambda_k$ for each $k$, then the total collection of vectors in the sets $\mathscr{B}_1, \ldots, \mathscr{B}_p$ forms an eigenvector basis for $\mathbb{R}^n$.

**Homework Assignment**
**Section 5.3**: (Page 325 1, 3, 5, 9, 11, 15, 21)

**Example 1.** *If* $D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$ *find* $D^2$, $D^3$, *and* $D^k$.

**Example 2.** *Let* $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$. *Find a formula for* $A^k$, *given that* $A = PDP^{-1}$, *when* $P = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$ *and* $D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$.

**Example 3.** *Diagonalize the following matrix, if possible.*

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

*Find an invertible matrix P and diagonal matrix D such that* $A = PDP^{-1}$.

**Example 4.** *Diagonalize the following matrix if possible,*

$$A = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}.$$

**Example 5.** *Determine if the following matrix is diagonalizable*

$$A = \begin{bmatrix} 5 & -8 & 1 \\ 0 & 0 & 7 \\ 0 & 0 & -2 \end{bmatrix}.$$

**Example 6.** *Determine if the following matrix is diagonalizable*

$$A = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 1 & 4 & -3 & 0 \\ -1 & -2 & 0 & -3 \end{bmatrix}.$$

## 5.4 Eigenvectors and Linear Transformations

**Definition 5.4.1.** Let $V$ be an $n$ dimensional vector space, $W$ an $m$-dimensional vector space, and $T$ ant linear transformation from $V$ to $W$. Associate bases $\mathscr{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ and $\mathscr{C} = \{\mathbf{c}_1, \ldots, \mathbf{c}_m\}$ to vector spaces $V$ and $W$, respectively. For $[\mathbf{x}]_{\mathscr{B}} \in \mathbb{R}^n$ and $[T(\mathbf{x})]_{\mathscr{C}} \in \mathbb{R}^m$ we have

$$[T(\mathbf{x})]_{\mathscr{C}} = M[\mathbf{x}]_{\mathscr{B}}$$

where

$$M = \begin{bmatrix} [T(\mathbf{b}_1)]_{\mathscr{C}} & [T(\mathbf{b}_2)]_{\mathscr{C}} & \cdots & [T(\mathbf{b}_n)]_{\mathscr{C}} \end{bmatrix}.$$

The matrix $M$ is a matrix representation of $T$, called the **matrix for $T$ relative to the bases $\mathscr{B}$ and $\mathscr{C}$**.

$$\begin{tikzcd}
\mathbf{x} \arrow[r, "T"] \arrow[d] & T(\mathbf{x}) \arrow[d] \\
{[\mathbf{x}]_{\mathscr{B}}} \arrow[r, "\text{Multiplication}", "\text{by } M"'] & {[T(\mathbf{x})]_{\mathscr{C}}}
\end{tikzcd}$$

**Note 5.4.1.** In the common case where $W$ is the same space as $V$ and the basis $\mathscr{C}$ is the same as $\mathscr{B}$, the matrix $M$ is called the **matrix for $T$ relative to $\mathscr{B}$** or the **$\mathscr{B}$-matrix for $T$**, and is denoted by $[T]_{\mathscr{B}}$. The $\mathscr{B}$-matrix for $T : V \to V$ satisfies

$$[T(\mathbf{x})]_{\mathscr{B}} = [T]_{\mathscr{B}}[\mathbf{x}]_{\mathscr{B}}, \text{ for all } \mathbf{x} \text{ in } V$$

$$\begin{tikzcd}
\mathbf{x} \arrow[r, "T"] \arrow[d] & T(\mathbf{x}) \arrow[d] \\
{[\mathbf{x}]_{\mathscr{B}}} \arrow[r, "\text{Multiplication}", "\text{by } {[T]_{\mathscr{B}}}"'] & {[T(\mathbf{x})]_{\mathscr{B}}}
\end{tikzcd}$$

**Theorem 5.4.1** (Diagonal Matrix Representation)**.** Suppose $A = PDP^{-1}$, where $D$ is a diagonal $n \times n$ matrix. If $\mathscr{B}$ is the basis for $\mathbb{R}^n$ formed by the columns for $P$, then $D$ is the $\mathscr{B}$-matrix for the transformation $\mathbf{x} \mapsto A\mathbf{x}$.

*Proof.*

**Homework Assignment**

**Section 5.4**: (Page 333 1, 3, 5, 11, 15, 17, 20, 31)

**Example 1.** *Suppose* $\mathscr{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ *is a basis for* $V$ *and* $\mathscr{C} = \{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$ *is a basis for* $W$. *Let* $T : V \to W$ *be a linear transformation with the property that*

$$T(\mathbf{b}_1) = 3\mathbf{c}_1 - 2\mathbf{c}_2 + 5\mathbf{c}_3 \quad \textit{and} \quad T(\mathbf{b}_2) = 4\mathbf{c}_1 + 7\mathbf{c}_2 - \mathbf{c}_3$$

*Find the matrix* $M$ *for* $T$ *relative to* $\mathscr{B}$ *and* $\mathscr{C}$.

**Example 2.** *The mapping* $T : \mathbb{P}_2 \to \mathbb{P}_2$ *defined by*

$$T(a_0 + a_1 t + a_2 t^2) = a_1 + 2a_2 t$$

*Is a linear transformation (differential operator)*

(a) *Find the* $\mathscr{B}$*-matrix for* $T$, *when* $\mathscr{B}$ *is* $\{1, t, t^2\}$.

(b) *Verify that* $[T(\mathbf{p}(t))]_{\mathscr{B}} = [T]_{\mathscr{B}}[\mathbf{p}(t)]_{\mathscr{B}}$

**Example 3.** *Define* $T : \mathbb{R}^2 \to \mathbb{R}^2$ *by* $T(\mathbf{x} = A\mathbf{x}$, *where* $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$. *Find a basis for* $\mathbb{R}^2$ *with the property that the* $\mathscr{B}$*-matrix for* $T$ *is a diagonal matrix.*

## 5.5 Complex Eigenvalues

**Definition 5.5.1.** A complex scalar $\lambda$ satisfies $\det(A\lambda I) = 0$ if and only if there is a nonzero vector in $\mathbb{C}^n$ such that $A\mathbf{x} = \lambda\mathbf{x}$. We call $\lambda$ a (**complex**) **eigenvalue** and $mathbf{x}$ a (**complex**) **eigenvector** corresponding to $\lambda$.

**Example 1.** Let $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, what is the transformation $\mathbf{x} \mapsto A\mathbf{x}$?

(a) Find the eigenvalues of $A$.

(b) Allow the transformation to act on $\mathbb{C}$ so that $\mathbb{C}^2 \to \mathbb{C}^2$. Consider vectors $\begin{bmatrix} 1 \\ -i \end{bmatrix}$ and $\begin{bmatrix} 1 \\ i \end{bmatrix}$

**Example 2.** Let $A = \begin{bmatrix} 0.5 & -0.6 \\ 0.75 & 1.1 \end{bmatrix}$. Find the eigenvalues of $A$, and find a basis for each eigenspace.

**Definition 5.5.2.** The complex conjugate of a complex vector $\mathbf{x}$ in $\mathbb{C}^n$ is the vector $\overline{\mathbf{x}}$ in $\mathbb{C}^n$ whose entries are the complex conjugates of the enteries of $\mathbf{x}$. The **real** and **imaginary parts** of a complex vector $\mathbf{x}$ are the vectors Re $\mathbf{x}$ and Im $\mathbf{x}$ formed from the real and imaginary parts of the entries of $\mathbf{x}$.

**Note 5.5.1.** Let $A$ be an $n \times n$ matrix whose entries are real. Then $\overline{A\mathbf{x}} = \overline{A}\overline{\mathbf{x}} = A\overline{\mathbf{x}}$. If $\lambda$ is an eigenvalue of $A$ and $\mathbf{x}$ is a corresponding eigenvector in $\mathbb{C}^n$, then

$$A\overline{\mathbf{x}} = \overline{A\mathbf{x}} = \overline{\lambda\mathbf{x}} = \overline{\lambda}\overline{\mathbf{x}}$$

**Example 3.** The eigenvalues of the real matrix in Example 2 are complex conjugates.

**Note 5.5.2.** The solutions of a matrix equation $\mathbf{x}' = A\mathbf{x}$ are

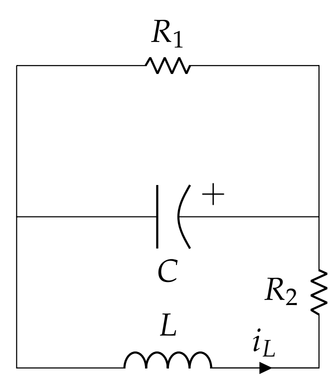
$$\mathbf{x}_1(t) = \mathbf{v}e^{\lambda t} \quad \text{and} \quad \mathbf{x}_2(t) = \overline{\mathbf{v}}e^{\overline{\lambda}t}$$

where $\mathbf{v}$ is an eigenvector and $\lambda$ is its corresponding eigenvalue.

**Example 4.** The circuit below can be described by the equation

$$\begin{bmatrix} i_L' \\ v_C' \end{bmatrix} = \begin{bmatrix} -R_2/L & -1/L \\ 1/C & -1/(R_1C) \end{bmatrix} \begin{bmatrix} i_L \\ v_C \end{bmatrix}$$

where $i_L$ is the current passing through the inductor $L$ and $v_C$ is the voltage drop across the capacitor $C$. Suppose $R_1$ is 5 ohms, $R_2$ is 0.8 ohms, $C$ is 0.1 farad, and $L$ is 0.4 henry. Find the formulas for $i_L$ and $v_C$, if the intial current through the inductor is 3 amps and the intial voltage is 3 volts.



# Chapter 6 Orthogonality and Least Squares

## 6.1 Inner Product, Length, and Orthogonailty

**Definition 6.1.1.** If $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$, then think of $\mathbf{u}$ and $\mathbf{v}$ as $n \times 1$ matrices. The transpose $\mathbf{u}^T$ is a $1 \times n$ matrix, and the matrix product $\mathbf{u}^T\mathbf{v}$ is a $1 \times 1$ matrix, which we write as a single real number (a scalar) without brackets. The number $\mathbf{u}^T\mathbf{v}$ is called the **inner product** of $\mathbf{u}$ and $\mathbf{v}$, and often it is written as $\mathbf{u} \cdot \mathbf{v}$. The inner product is sometimes called the **dot product**.

If $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ then the inner product of $\mathbf{u}$ and $\mathbf{v}$ is

$$\langle \mathbf{u}, \mathbf{v} \rangle = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1v_1 + u_2v_2 + \cdots + u_nv_n.$$

**Theorem 6.1.1.** Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in $\mathbb{R}^n$, and let $c$ be a scalar. Then

a. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

b. $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{v}$

c. $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (c\mathbf{v})$

d. $\mathbf{u} \cdot \mathbf{u} \geq 0$, and $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$

**Definition 6.1.2.** The **length** (or **norm**) of $\mathbf{v}$ is the nonnegative scalar $\|v\|$ defined by

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}, \text{ and } \|\mathbf{v}\|^2 = \mathbf{v} \cdot \mathbf{v}$$

**Definition 6.1.3.** A vector whose length is 1 is called a **unit vector**. If we *divide* a nonzero vector $\mathbf{v}$ by its length — that is, multiply by $1/\|\mathbf{v}\|$ — we obtain a unit vector $\mathbf{u}$ because the length of $\mathbf{u}$ is $(1/\|\mathbf{v}\|)\|\mathbf{v}\|$. The process of creating $\mathbf{u}$ from $\mathbf{v}$ is sometimes called **normalizing v**, and we say that $\mathbf{u}$ is *in the same direction* as $\mathbf{v}$.

**Definition 6.1.4.** For $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, the **distance between u and v**, written as $\text{dist}(\mathbf{u}, \mathbf{v})$, is the length of the vector $\mathbf{u} - \mathbf{v}$. That is,

$$\text{dist}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$$

**Definition 6.1.5.** Two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ are **orthogonal** if $\mathbf{u} \cdot \mathbf{v} = 0$.

**Theorem 6.1.2.** Two vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if $\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$.

**Definition 6.1.6.** If a vector $\mathbf{z}$ is orthogonal to every vector in a subspace $W$ of $\mathbb{R}^n$, then $\mathbf{z}$ is said to be **orthogonal to** $W$. The set of all vectors $\mathbf{z}$ that are orthogonal to $W$ is called the **orthogonal complement** of $W$ and is denoted by $W^\perp$ (read as "$W$ perpendicular" or "$W$ perp").

**Note 6.1.1.** A vector $\mathbf{x}$ is in $W^\perp$ if and only if $\mathbf{x}$ is orthogonal to every vector in a set that spans $W$.

**Note 6.1.2.** $W^\perp$ is a subspace of $\mathbb{R}^n$.

**Theorem 6.1.3.** Let $A$ be an $m \times n$ matrix. The orthogonal complement of the row space of $A$ is the nullspace of $A$, and the orthogonal complement of the column space of $A$ is the null space of $A^T$:

$$(\text{Row}\, A)^\perp = \text{Nul}\, A \quad \text{and} \quad (\text{Col}\, A)^\perp = \text{Nul}\, A^T$$

**Note 6.1.3.** If $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$, then

$$\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|\cos\theta$$

where $\theta$ is the angle between $\mathbf{u}$ and $\mathbf{v}$.

**Homework Assignment**

**Section 6.1**: (Page 382 1, 5, 7, 13, 15, 17, 19, 23, 34 )

**Example 1.** *Compute* $\mathbf{u} \cdot \mathbf{v}$ *and* $\mathbf{v} \cdot \mathbf{u}$ *when* $\mathbf{u} = \begin{bmatrix} 2 \\ -5 \\ -1 \end{bmatrix}$ *and* $\mathbf{v} = \begin{bmatrix} 3 \\ 2 \\ -3 \end{bmatrix}$.

**Example 2.** *Let* $\mathbf{v} = \begin{bmatrix} 1 \\ -2 \\ 2 \\ 0 \end{bmatrix}$. *Find a unit vector in the same direction as* $\mathbf{v}$

**Example 3.** *Let* $W$ *be the subspace of* $\mathbb{R}^2$ *spanned by* $\mathbf{x} = \begin{bmatrix} 2/3 \\ 1 \end{bmatrix}$. *Find a unit vector* $\mathbf{z}$ *that forms a basis for* $W$.

**Example 4.** *Compute the distance between the vectors* $\mathbf{u} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$ *and* $\mathbf{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

**Example 5.** *If* $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ *and* $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ *find* $\text{dist}(\mathbf{u}, \mathbf{v})$.

**Example 6.** *Suppose* $\text{dist}(\mathbf{u}, \mathbf{v}) = \text{dist}(\mathbf{u}, -\mathbf{v})$. *What can you say about the vectors* $\mathbf{u}$ *and* $\mathbf{v}$*?*

**Example 7.** *Let $W$ be a plane through the origin in $\mathbb{R}^3$, and let $L$ be the line through the origin and perpendicular to $W$. If $\mathbf{z}$ and $\mathbf{w}$ are nonzero, $\mathbf{z}$ is on $L$, and $\mathbf{w}$ is in $W$, then $\mathbf{z}$ is perpendicular to $\mathbf{w}$, that is $\mathbf{z} \cdot \mathbf{w} = 0$.*

**Example 8.** Proof *of* ***Theorem 6.1.3***
Let $A$ be an $m \times n$ matrix. The orthogonal complement of the row space of $A$ is the nullspace of $A$, and the orthogonal complement of the column space of $A$ is the null space of $A^T$:

$$(\text{Row}\, A)^{\perp} = \text{Nul}\, A \quad \text{and} \quad (\text{Col}\, A)^{\perp} = \text{Nul}\, A^T$$

*Proof*

## 6.2 Orthogonal Sets

**Definition 6.2.1.** A set of vectors $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ in $\mathbb{R}^n$ is said to be an **orthogonal set** if each pair of distinct vectors from the set is orthogonal, that is, if $\mathbf{u}_i \cdot \mathbf{u}_j = 0$ whenever $i \neq j$.

**Theorem 6.2.1.** If $S = \{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is an orthogonal set of nonzero vectors in $\mathbb{R}^n$, then $S$ is linearly independent and hence is a basis for the subspace spanned by $S$.

*Proof.*

**Definition 6.2.2.** An **orthogonal basis** for a subspace $W$ of $\mathbb{R}^n$ is a basis for $W$ that is also an orthogonal set.

**Theorem 6.2.2.** Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ be an orthogonal basis for a subspace $W$ of $\mathbb{R}^n$. For each $\mathbf{y}$ in $W$, the weights in the linear combination

$$\mathbf{y} = c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p$$

are given by

$$c_j = \frac{\mathbf{y} \cdot \mathbf{u}_j}{\mathbf{u}_j \cdot \mathbf{u}_j} \qquad (j = 1, \ldots, p)$$

**Definition 6.2.3.** Given a nonzero vector $\mathbf{u}$ in $\mathbb{R}^n$ a vector $\mathbf{y}$ in $\mathbb{R}^n$ can be decomposed into the sum of two vectors, one a multiple of $\mathbf{u}$ and the other orthogonal to $\mathbf{u}$. So that,

$$\mathbf{y} = \widehat{\mathbf{y}} + \mathbf{z}$$

where $\widehat{\mathbf{y}} = \alpha\mathbf{u}$ for some scalar $\alpha$ and $\mathbf{z}$ is some vector orthogonal to $\mathbf{u}$. Then $\mathbf{z} = \mathbf{y} - \alpha\mathbf{u}$, and

$$0 = \mathbf{z} \cdot \mathbf{u} = (\mathbf{y} - \alpha\mathbf{u}) \cdot \mathbf{u} = \mathbf{y} \cdot \mathbf{u} - (\alpha\mathbf{u}) \cdot \mathbf{u} = \mathbf{y} \cdot \mathbf{u} - \alpha(\mathbf{u} \cdot \mathbf{u})$$

Hence $\alpha = \dfrac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}$ and $\widehat{\mathbf{y}}$ is therefore $\widehat{\mathbf{y}} = \dfrac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}\mathbf{u}$.

The vector $\widehat{\mathbf{y}}$ is called the **orthogonal projection of y onto u**, and the vector $\mathbf{z}$ is called the **complement of y orthogonal to u**.

**Definition 6.2.4.** The orthogonal projection of $\mathbf{y}$ onto $\mathbf{u}$ is exactly the same as the orthogonal projection of $\mathbf{y}$ onto $c\mathbf{u}$ for some nonzero scalar $c$. Hence the projection is determined by the *subspace* $L$ spanned by $\mathbf{u}$. Sometimes $\widehat{\mathbf{y}}$ is denoted by $\text{proj}_L\,\mathbf{y}$ and is called the **orthogonal projection of y onto** $L$. That is,

$$\widehat{\mathbf{y}} = \text{proj}_L\,\mathbf{y} = \frac{\mathbf{y}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}$$

**Definition 6.2.5.** A set $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is an **orthonormal set** if it is an orthogonal set of unit vectors. If $W$ is the subspace spanned by such a set, then $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is an **orthonormal basis** for $W$.

**Theorem 6.2.3.** An $m \times n$ matrix $U$ has orthonormal columns if and only if $U^T U = I$.

**Theorem 6.2.4.** Let $U$ be an $m \times n$ matrix with orthonormal columns, and let $\mathbf{x}$ and $\mathbf{y}$ be in $\mathbb{R}^n$. Then

a. $\|U\mathbf{x}\| = \|\mathbf{x}\|$

b. $(U\mathbf{x}) \cdot (U\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$

c. $(U\mathbf{x}) \cdot (U\mathbf{y}) = 0$ if and only if $\mathbf{x} \cdot \mathbf{y} = 0$.

**Definition 6.2.6.** An **orthogonal matrix** is a square matrix $U$ such that $U^{-1} = U^T$.

**Homework Assignment**

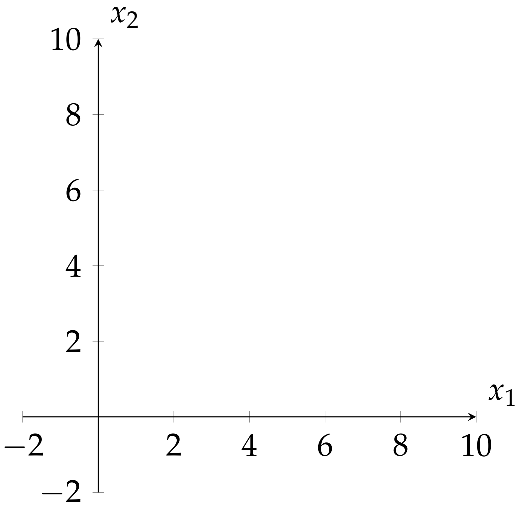
**Section 6.2**: (Page 392: 5, 9, 11 )

**Example 1.** *Show that* $\{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}\}$ *is an orthogonal set, where* $\mathbf{u_1} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{u_2} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$,
$\mathbf{u_3} = \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix}$.

**Example 2.** *The set* $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ *from* ***Example 1*** *is an orthogonal basis for* $\mathbb{R}^3$*. Express the vector* $\mathbf{y} - \begin{bmatrix} 6 \\ 1 \\ -8 \end{bmatrix}$ *as a linear combination of the basis vectors in* $S$*.*

**Example 3.** *Find the orthogonal projection of* $\mathbf{y}$ *onto* $\mathbf{u}$ *where* $\mathbf{y} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$ *and* $\mathbf{u} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$*. Write* $\mathbf{y}$ *as the sum of two orthogonal vectors, one in* $\text{Span}\{\mathbf{u}\}$ *and one orthogonal.*

**Example 4.** *Find the closest point of* $L = \text{Span}\{\mathbf{u}\}$ *to* $\mathbf{y}$.




**Example 5.** *Show that* $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ *is an orthonormal basis of* $\mathbb{R}^3$*, where*
$\mathbf{v}_1 = \begin{bmatrix} 3/\sqrt{11} \\ 1/\sqrt{11} \\ 1/\sqrt{11} \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -1/\sqrt{66} \\ -4/\sqrt{66} \\ 7/\sqrt{66} \end{bmatrix}$

**Example 6.** *Let* $U = \begin{bmatrix} 1/\sqrt{2} & 2/3 \\ 1/\sqrt{2} & -2/3 \\ 0 & 1/3 \end{bmatrix}$ *and* $\mathbf{x} = \begin{bmatrix} \sqrt{2} \\ 3 \end{bmatrix}$*. Notice that U has orthonormal columns and*

$$U^T U = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 2/3 & -2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 2/3 \\ 1/\sqrt{2} & -2/3 \\ 0 & 1/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

*Verify that* $||U\mathbf{x}|| = ||x||$

## 6.3 Orthogonal Projections

**Theorem 6.3.1** (The Orthogonal Decomposition Theorem)**.**
Let $W$ be a subspace of $\mathbb{R}^n$. Then each $\mathbf{y}$ in $\mathbb{R}^n$ can be written uniquely in the form

$$\mathbf{y} = \widehat{\mathbf{y}} + \mathbf{z}$$

where $\widehat{\mathbf{y}}$ in in $W$ and $\mathbf{z}$ is in $W^{\perp}$. In fact, if $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is any orthogonal basis of $W$, then

$$\widehat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 + \cdots + \frac{\mathbf{y} \cdot \mathbf{u}_p}{\mathbf{u}_p \cdot \mathbf{u}_p}\mathbf{u}_p$$

and $\mathbf{z} = \mathbf{y} - \widehat{\mathbf{y}}$.

**Theorem 6.3.2** (The Best Approximation Theorem)**.**
Let $W$ be a subspace of $\mathbb{R}^n$, $\mathbf{y}$ and vector in $\mathbb{R}^n$, and $\widehat{\mathbf{y}}$ the orthogonal projection of $\mathbf{y}$ onto $W$. Then $\widehat{\mathbf{y}}$ is the closest point in $W$ to $\mathbf{y}$, in the sense that

$$\|\mathbf{y} - \widehat{\mathbf{y}}\| < \|\mathbf{y} - \mathbf{v}\|$$

for all $\mathbf{v}$ in $W$ distinct from $\widehat{\mathbf{y}}$. Sometimes $\widehat{\mathbf{y}}$ is called **the best approximation to $\mathbf{y}$ by the elements of** $W$.

**Theorem 6.3.3.** If $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is an orthonormal basis for a subspace $W$ of $\mathbb{R}^n$, then

$$\operatorname{proj}_W \mathbf{y} = (\mathbf{y} \cdot \mathbf{u}_1)\mathbf{u}_1 + (\mathbf{y} \cdot \mathbf{u}_2)\mathbf{u}_2 + \cdots + (\mathbf{y} \cdot \mathbf{u}_p)\mathbf{u}_p$$

If $U = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_p \end{bmatrix}$, then

$$\operatorname{proj}_W \mathbf{y} = UU^T\mathbf{y} \text{ for all } \mathbf{y} \text{ in } \mathbb{R}^n$$

**Homework Assignment**
**Section 6.3**: (Page 400: 1, 3, 7, 9, 11, 13, 15, 17, 19, 21 )

**Example 1.** *Let* $\{\mathbf{u}_1, \ldots, \mathbf{u}_5\}$ *be an orthogonal basis for* $\mathbb{R}^5$ *and let* $y = c_1\mathbf{u}_1 + \cdots + c_5\mathbf{u}_5$. *Condsider the subspace spanned by* $\mathbf{u}_1$ *and* $\mathbf{u}_2$, $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$, *and write* $\mathbf{y}$ *as the sum of a vector* $\mathbf{z}_1$ *in* $W$ *and a vector* $\mathbf{z}_2$ *in* $W^{\perp}$

**Example 2.** *Let* $\mathbf{u}_1 = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. *Observe that* $\{\mathbf{u}_1, \mathbf{u}_2\}$ *is an orthogonal basis for* $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$. *Write* $\mathbf{y}$ *as the sum of a vector in* $W$ *and a vector in* $W^{\perp}$.

**Example 3.** *If* $\mathbf{u}_1 = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ *find the closest point of* $W$ *to* $\mathbf{y}$

**Example 4.** *Find the distance from* $\mathbf{y}$ *to* $W = \mathrm{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$ *where* $\mathbf{y} = \begin{bmatrix} -1 \\ -5 \\ 10 \end{bmatrix}$, $\mathbf{u}_1 = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$. *The distance between a vector and a subspace is defined to be the distance from* $\mathbf{y}$ *to the nearest point in* $W$.

**Example 5.** *Let* $\mathbf{y} = \begin{bmatrix} 7 \\ 9 \end{bmatrix}$, $\mathbf{u}_1 = \begin{bmatrix} 1/\sqrt{10} \\ -3/\sqrt{10} \end{bmatrix}$, *and* $W = \mathrm{Span}\{\mathbf{u}_1\}$.

(a) *Let* $U$ *be the* $2 \times 1$ *matrix whose only column is* $u_1$. *Compute* $U^T U$ *and* $UU^T$.

(b) *Compute* $\mathrm{proj}_W \mathbf{y}$ *and* $(UU^T)\mathbf{y}$

## 6.4 The Gram-Schmidt Process

**Theorem 6.4.1** (Gram-Schmidt Process)**.** The Gram-Schmidt Process is an algorithm for producing an orthonormal basis for any nonzero subspace of $\mathbb{R}^n$. Given a basis $\{\mathbf{x}_1, \ldots, \mathbf{x}_p\}$ for a subspace $W$ of $\mathbb{R}^n$, define

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{x}_1 \\
\mathbf{v}_2 &= \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 \\
\mathbf{v}_3 &= \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 \\
&\vdots \\
\mathbf{v}_p &= \mathbf{x}_p - \frac{\mathbf{x}_p \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_p \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 - \cdots - \frac{\mathbf{x}_p \cdot \mathbf{v}_{p-1}}{\mathbf{v}_{p-1} \cdot \mathbf{v}_{p-1}}\mathbf{v}_{p-1}
\end{aligned}$$

Then $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is an orthogonal basis for $W$. In addition

$$\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} = \text{Span}\{\mathbf{x}_1, \ldots, \mathbf{x}_k\} \qquad \text{for } 1 \leq k \leq p.$$

Note that this process only gives an orthogonal basis, one still needs to normalize each of the vectors in the orthogonal basis to achieve an orthonormal basis.

**Theorem 6.4.2** (The $QR$ Factorization)**.** If $A$ is an $m \times n$ matrix with linearly independent columns, then $A$ can be factored as $A = QR$, where $Q$ is an $m \times n$ matrix whose columns form an orthonormal basis for Col $A$ and $R$ is an $n \times n$ upper triangular invertible matrix with positive entries on its diagonal.

**Homework Assignment**
**Section 6.4**: (Page 407: 1, 5, 11, 13, 15, 17 )

**Example 1.** *Let* $W = \text{Span}\{\mathbf{x}_1, \mathbf{x}_2\}$ *where* $\mathbf{x}_1 = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$ *and* $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$*. Construct an orthogonal basis* $\{\mathbf{v}_1, \mathbf{v}_2\}$ *for* $W$*.*

**Example 2.** *Let* $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$*,* $\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$*,* $\mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$*. Then* $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ *is clearly independent an thus a basis for the subspace.* $W = \text{Span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ *in* $\mathbb{R}^4$*. Construct an orthogonal basis for* $W$*.*

**Example 3.** *In **Example 1** we constructed the orthogonal basis* $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$ *and* $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$. *Find an orthonormal basis from this orthogonal basis.*

**Example 4.** *Find the QR factorization of* $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

## 6.5 Least-Squares Problems

When $A\mathbf{x} = \mathbf{b}$ doesn't have a solution then we can find an $\mathbf{x}$ so that $||A\mathbf{x} - \mathbf{b}||$ is minimum. This is called a general "least-squares" problem since $||A\mathbf{x} - \mathbf{b}||$ is the square root of a sum of squares.

**Definition 6.5.1.** *If $A$ is $m \times n$ and $\mathbf{b}$ is in $\mathbb{R}^m$, a **least-squares solution** of $A\mathbf{x} = \mathbf{b}$ is an $\hat{\mathbf{x}}$ in $\mathbb{R}^n$ such that*

$$||\mathbf{b} - A\hat{\mathbf{x}}|| \leq ||\mathbf{b} - A\mathbf{x}||$$

*for all $\mathbf{x}$ in $\mathbb{R}^n$.*

For a system $A\mathbf{x} = \mathbf{b}$ that is inconsistent. The "best" approximation $A\hat{\mathbf{x}}$ that is in Col $A$ is $\hat{\mathbf{b}} = \text{proj}_{\text{Col } A}\, \mathbf{b}$.
*Picture:*

*Derivation and Motivation for The Normal Equations*:

**Definition 6.5.2.** *The matrix equation $A^T A\mathbf{x} = A^T\mathbf{b}$ represents a system of equations called the **normal equations** for $A\mathbf{x} = \mathbf{b}$. A solution to the normal equations is often denoted $\hat{\mathbf{x}}$.*

**Theorem 6.5.1.** *The set of least-square solutions of $A\mathbf{x} = \mathbf{b}$ coincides with the nonempty set of solutions of the normal equations $A^T A\mathbf{x} = A^T\mathbf{b}$.*

With each least-squares solution there is an associated **least-squares error** $||\mathbf{b} - A\hat{\mathbf{x}}||$.

**Theorem 6.5.2.** *The matrix $A^T A$ is invertible if and only if the columns of $A$ are linearly independent. In this case, the equation $A\mathbf{x} = \mathbf{b}$ has only one least-squares solution $\hat{\mathbf{x}}$, and it is given by*

$$\hat{\mathbf{x}} = (A^T A)^{-1} A^T\mathbf{b}$$

**Theorem 6.5.3.** *Given an $m \times n$ matrix $A$ with linearly independent columns, let $A = QR$ be a $QR$ factorization of $A$ as in **Theorem 6.4.2**. Then, for each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a unique least-squares solution, given by*

$$\hat{\mathbf{x}} = R^{-1} Q^T\mathbf{b}$$

**Homework Assignment**
**Section 6.5**: (Page 416: 3, 5, 7, 9 ,15 )

**Example 1.** *Find a least-squares solution of the inconsistent system* $A\mathbf{x} = \mathbf{b}$ *for*

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}$$

**Example 2.** *Find the least squares solution for* $A\mathbf{x} = \mathbf{b}$ *where* $A = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix}$ *and* $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ -3 \\ 4 \\ 1 \end{bmatrix}$.

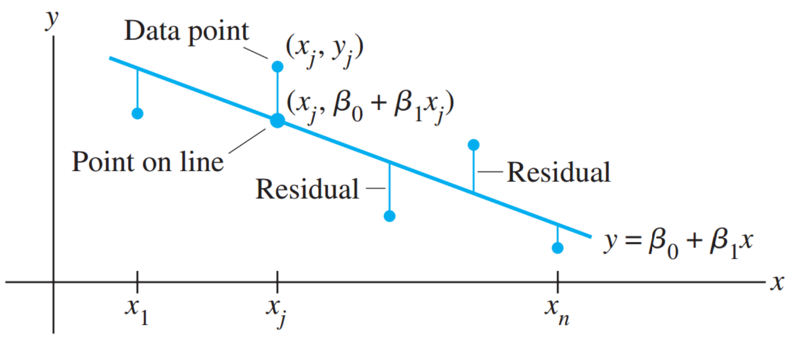
**Example 3.** *Given* $A$ *and* $\mathbf{b}$ *from* ***Example 1****, determine the least-squares error of the solution of* $A\mathbf{x} = \mathbf{b}$.

**Example 4.** *Find the least-squares solution of* $A\mathbf{x} = \mathbf{b}$ *for* $A = \begin{bmatrix} 1 & -6 \\ 1 & -2 \\ 1 & 1 \\ 1 & 7 \end{bmatrix}$ *and* $\mathbf{b} = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 6 \end{bmatrix}$.

**Example 5.** *Find the least-squares solution of* $A\mathbf{x} = \mathbf{b}$ *for* $A = \begin{bmatrix} 1 & 3 & 5 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 3 & 3 \end{bmatrix}$ *and* $\begin{bmatrix} 3 \\ 5 \\ 7 \\ -3 \end{bmatrix}$. *Use only the QR factorization of* $A$.

## 6.6 Applications to Linear Models

Suppose $\beta_0$ and $\beta_1$ are fixed, and consider the line $y = \beta_0 + \beta_1 x$ as shown below. Corresponding to each data point $(x_j, y_j$ there is a point $(x_j, \beta_0 + \beta_1 x_j)$ on the line. We call $y_j$ the *observed* value of $y$ and $\beta_0 + \beta_1 x_j$ the *predicted* $y$-value.




**Definition 6.6.1.** *The difference between an observed y-value and a predicted y-value is called a* ***residual****.*

**Definition 6.6.2.** *The* ***Least-Square Line*** *is the line $y = \beta_0 + \beta_1 x$ that minimizes the sum of the squares of the residuals. This is also called the* ***line of regression of*** *$y$* ***on*** *$x$.*

These data points would satisfy the following equations:

| Predicted $y$-value | | Observed $y$-value |
|---|---|---|
| $\beta_0 + \beta_1 x_1$ | $=$ | $y_1$ |
| $\beta_0 + \beta_1 x_2$ | $=$ | $y_2$ |
| $\vdots$ | | $\vdots$ |
| $\beta_0 + \beta_1 x_n$ | $=$ | $y_n$ |

We can express this system as:

$$X\boldsymbol{\beta} = \mathbf{y}, \qquad \text{where } X = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}, \quad \boldsymbol{\beta} = \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

Computing the least-squares solution to the above is equivalent to finding the $\boldsymbol{\beta}$ that determines the least-squares line.

**The General Linear Model**

**Definition 6.6.3.** *In some applications it is necessary to fit data points with something other than a straight line, the form of the linear model is still $X\boldsymbol{\beta} = \mathbf{y}$. Statisticians usually introduce a* ***residual vector*** $\boldsymbol{\epsilon}$*, defined by $\boldsymbol{\epsilon} = \mathbf{y} - X\boldsymbol{\beta}$, and write*

$$\mathbf{y} = X\boldsymbol{\beta} + \boldsymbol{\epsilon}$$

*Any equation of this form is refered to as a* ***linear model****.*

Once $X$ and $\mathbf{y}$ are determined, the goal is to minimize the length of $\boldsymbol{\epsilon}$, which amounts to finding the least-squares solution of $X\boldsymbol{\beta} = \mathbf{y}$. In each case, the least squares solution $\hat{\boldsymbol{\beta}}$ is a solution of the normal equations

$$X^T X\boldsymbol{\beta} = X^T\mathbf{y}.$$

**Homework Assignment**

**Section 6.5**: (Page 416: 1, 3, 7, 9)

**Example 1.** *Find the equation $y = \beta_0 + \beta_1 x$ of the least-square line that best fits the data points* $(2, 1)$, $(5, 2)$, $(7, 3)$, $(8, 3)$.

**Example 2.** *Suppose data points* $(x_1, y_1)$, ..., $(x_n, y_n)$ *appear to lie along some sort of parabola instead of a straight line. Suppose we wish to approximate the data by an equation of the form*

$$y = \beta_0 + \beta_1 x + \beta_2 x^2$$

*Describe the linear model that produces a "least-squares fit" of the data.*

**Example 3.** *Find the equation* $y = \beta_0 + \beta_1 x + \beta_2 x^2$ *of the least-squares curve for the data points* $(1, 3.7)$, $(2, 3.1)$, $(3, 3.3)$, $(4, 5.3)$, $(5, 6.6)$, $(6, 10.4)$.

**Example 4.** *Find the least-squares curve of the form* $y = \beta_1 x + \beta_2 x^2 + \beta_3 x^3$ *(constants not included) to fit the data* $(4, 1.58)$, $(6, 2.08)$, $(8, 2.5)$, $(10, 2.8)$, $(12, 3.1)$, $(14, 3.4)$, $(16, 3.8)$, $(18, 4.32)$.

# Chapter 5 & 6 Review

$3 \times 5$ note card permitted on the exam.

Contents: Sections 5.1, 5.2, 5.3, 5.4, 6.1, 6.2, 6.3, 6.4, 6.5, 6.6

- **Eigenvalues and Eigenvectors**: eigenspace and solutions to a recursive formula
- **Characteristic Equation and Polynomial**: Algebraic multiplicity, similar matrices
- **Diagonalization**: Steps to diagonalize, Theorem 5.3.3, eigenvector basis
- **Eigenvectors and Linear Transformations**: $\mathscr{B}$-matrix for $T$
- **Inner product, length, and orthogonality**: Normalize, distance, orthogonality
- **Orthogonal Sets**: Orthogonal basis, orthogonal projection, orthonormal basis
- **Orthogonal Projections**: Best approximation to $\mathbf{y}$, $\widehat{\mathbf{y}}$
- **The Gram-Schmidt Process**: Finding an orthonormal basis, and $QR$ Factorization
- **Least-Square Problems** Normal Equations, and least-squares error
- **Application to Linear Models** Best Fit Curves using least-square solutions

Review Questions

**Chapter 5**

- **Section 5.1** : 6, 8, 16, 22 (a), (c)
- **Section 5.2** : 7, 11, 14, 15
- **Section 5.3**: 7, 11, 16
- **Section 5.4** : 12
- **Chapter 5 Supplementary Excercises**: 1a, 1b, 1d, 1e, 1g, 1h, 2, 6a

**Chapter 6**

- **Section 6.1**: 2, 11, 14, 20, 24
- **Section 6.2**: 1, 4, 10, 15
- **Section 6.3**: 5, 12, 21
- **Section 6.4**: 3, 12, 16, (19, 20 optional)
- **Section 6.5**: 4, 16
- **Section 6.6**: 2, 8(b)
- **Chapter 6 Supplementary Excercises**: 1a, 1b, 1e, 1f, 1i, 1n, 1o, (13 optional)

**True/False** Questions:

1. ______. If $A\mathbf{x} = \lambda\mathbf{x}$ for some $\lambda$, then $\mathbf{x}$ is an eigenvector of $A$.

2. ______. A steady-state vector for a stochastic matrix is actually an eigenvector.

3. ______. If $A$ is invertible and 1 is an eigenvalue of $A$, then 1 is also an eigenvalue of $A^{-1}$.

4. ______. For any scalar $c$, $||c\mathbf{v}|| = c||\mathbf{v}||$.

5. ______. $\left\{ \begin{bmatrix} 2 \\ -5 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 6 \end{bmatrix} \right\}$ is an orthogonal set.

6. ______. The best approximation to $\mathbf{y}$ by the elements of a subspace $W$ is given by the vector $\mathbf{y} - \text{proj}_W\,\mathbf{y}$.

**Extended Response** Questions:

7. Is $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ an eignvector of $\begin{bmatrix} 3 & 6 & 7 \\ 3 & 3 & 7 \\ 5 & 6 & 5 \end{bmatrix}$? If so, find the eigenvalue.

8. Is $\lambda = 4$ an eigenvalue of $\begin{bmatrix} 1 & 2 & 2 \\ 3 & -2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$? If so, find the corresponding eigenvector.

9. Find a basis for the eigenspace corresponding to $\lambda = 4$ and
$$A = \begin{bmatrix} 3 & 0 & 2 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}.$$

10. Find the characteristic polynomial and the eigenvalues of the matrix $\begin{bmatrix} 5 & 3 \\ -4 & 4 \end{bmatrix}$.

**Extended Response** Questions continued:

11. Find the characteristic polynomial of each matrix, using either a cofactor expansion or the special formula for $3 \times 3$ determinants described in Section 3.1.

(a) $\begin{bmatrix} 4 & 0 & 0 \\ 5 & 3 & 2 \\ -2 & 0 & 2 \end{bmatrix}$

(b) $\begin{bmatrix} 5 & -2 & 3 \\ 0 & 1 & 0 \\ 6 & 7 & -2 \end{bmatrix}$

12. For the following matrix list the eigenvalues, repeated according to their multiplicities.
$\begin{bmatrix} 4 & -7 & 0 & 2 \\ 0 & 3 & -4 & 6 \\ 0 & 0 & 3 & -8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

13. Diagonalize the following matrices, if possible,

(a) $\begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}$

(b) $\begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}, \lambda = 1, 2, 3$

**Extended Response** Questions continued:

14. Find the $\mathscr{B}$-matrix for the transformation $\mathbf{x} \mapsto A\mathbf{x}$, when $\mathscr{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$
$A = \begin{bmatrix} -1 & 4 \\ -2 & 3 \end{bmatrix}, \quad \mathbf{b}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$

15. Show that if $\mathbf{x}$ is an eigenvector of the matrix product $AB$ and $B\mathbf{x} \neq \mathbf{0}$, then $B\mathbf{x}$ is an eigenvector of $BA$.

16. Suppose $A = PDP^{-1}$ where $D$ is a diagonal matrix. Let $B = 5I - 3A + A^2$. Show that $B$ is diagonalizable by finding a suitable factorization of $B$.

17. Find the distance between $\mathbf{u} = \begin{bmatrix} 0 \\ -5 \\ 2 \end{bmatrix}$ and $\mathbf{z} = \begin{bmatrix} -4 \\ -1 \\ 8 \end{bmatrix}$.

18. Show that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal basis for $\mathbb{R}^3$, then express $\mathbf{x}$ as a linear combination of the $\mathbf{u}$'s.
$\mathbf{u}_1 = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}.$

19. Verify that $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthogonal set, and then find the orthogonal projection of $\mathbf{y}$ onto $\text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$.
$\mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} -1 \\ 2 \\ 6 \end{bmatrix}.$

20. The given set is a basis for a subspace $W$ of $\mathbb{R}^3$. Use the Gram-Schmidt process to produce an orthogonal basis for $W$.
$\begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}.$

21. Find an orthogonal basis for the column space of the following matrix. Use the basis for form the $QR$ factorization for $A$.
$A = \begin{bmatrix} 1 & 3 & 5 \\ -1 & -3 & 1 \\ 0 & 2 & 3 \\ 1 & 5 & 2 \\ 1 & 5 & 8 \end{bmatrix}$

22. Find a least-squares solution of $A\mathbf{x} = \mathbf{b}$ using the normal equations.
$A = \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$.

23. Use the factorization $A = QR$ to find the least-squares solution of $A\mathbf{x} = \mathbf{b}$.
$$A = \begin{bmatrix} 1 & -1 \\ 1 & 4 \\ 1 & -1 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \\ 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -1 \\ 6 \\ 5 \\ 7 \end{bmatrix}. \qquad (\textit{Hint}: \ R\mathbf{x} = Q^T\mathbf{b})$$

24. Find the equation $y = \beta_0 + \beta_1 x$ of the least-squares line that best fits the given the data points:
$(1,0)$, $(2,1)$, $(4,2)$, $(5,3)$,

# Final Exam Review

$8\frac{1}{2} \times 11$ notes sheet permitted on the exam.

Content Overview: Chapters: **1:1-10**; **2:1-5,7**; **3:1-3**; **4:1-7,9**; **5:1-4**; **6:1-3**

**Chapter 1**

- **Section: 1.1 Systems of Linear Equations** : Solution, consistent, inconsistent, row operations
- **Section: 1.2 Row Reductions and Echelon Forms**: REF, RREF, pivots, row equivalent, basic and free variables, general solutions
- **Section: 1.3 Vector Equations** : Vector, scalar, linear combination and spanning
- **Section: 1.4 Matrix Equations** : Matrix equation, Theorem 1.4.3
- **Section: 1.5 Solution Sets of Linear Systems**: homogeneous, parametric vector form
- **Section: 1.6 Applications of Linear Systems**: balance chemical equations and flow diagrams
- **Section: 1.7 Linear Independence**: Linearly independent vectors, dependence relation
- **Section: 1.8 Intro to Linear Transformations**: Domain, codomain, range, linear transformation
- **Section: 1.9 Matrix of a Linear Transformation**: Standard matrix, one-to-one, onto, 1.9.3
- **Section: 1.10 Linear Models**: Ohm's and Kirchhoff's Laws, linear difference equation

**Chapter 2**

- **Section 2.1 Matrix Operations**: Sum and product, diagonal, transpose, elementary matrices
- **Section 2.2 Inverse of a Matrix**: Singular vs. nonsingular, determinant, inverse algorithm
- **Section 2.3 Characterization of Invertible Matrices**: IMT, inverse of a transformation
- **Section 2.4 Block Matrices**: How to write them, Column-Row Expansion of $AB$
- **Section 2.5 LU Factorization**: Algorithm and how to use the factorization to solve $A\mathbf{x} = \mathbf{b}$
- **Section 2.7 Applications in Computer Graphics**: Homogeneous coordinates, translations, rotations, reflections, scaling

**Chapter 3**

- **Section 3.1 Intro to Determinants**: Determinant of an $n \times n$, cofactor expansion
- **Section 3.2 Propertires of the Determinant**: How row operations change the determinant
- **Section 3.3 Cramer's Rule, Volume, and Linear Trans.**: Cramer's Rule, Adjugate, parallelogram area, parallelepiped volume

**Chapter 4**

- **Section 4.1 Vector Spaces and Subspaces**: 10 axioms, 3 requirements for subspace
- **Section 4.2 Null and Column Spaces, and Linear Trans.**: Definitions and how to find them
- **Section 4.3 Linearly Independent Sets and Bases**: Dependence relation, basis
- **Section 4.4 Coordinate Systems**: $\mathscr{B}$-coordinates of $\mathbf{x}$, isomorphisms
- **Section 4.5 Dimension of a Vector Space**: Dimension, dimension of Nul $A$ and Col $A$
- **Section 4.6 Rank**: Know equivalent descriptions of the rank, and the Rank Theorem
- **Section 4.7 Change of Basis**: How to find the change-of-coordinate matrix from $\mathscr{B}$ to $\mathscr{C}$
- **Section 4.9 Markov Chains** : Probability vectors, stochastic matrices, and steady-state vectors

**Chapter 5**

- **Section 5.1 Eigenvalues and Eigenvectors**: eigenspace and solutions to a recursive formula
- **Section 5.2 Characteristic Equation and Polynomial**: Algebraic multiplicity, similar matrices
- **Section 5.3 Diagonalization**: Steps to diagonalize, Theorem 5.3.3, eigenvector basis
- **Section 5.4 Eigenvectors and Linear Transformations**: $\mathscr{B}$-matrix for $T$

**Chapter 6**

- **Section 6.1 Inner product, length, and orthogonality**: Normalize, distance, orthogonality
- **Section 6.2 Orthogonal Sets**: Orthogonal basis, orthogonal projection, orthonormal basis
- **Section 6.3 Orthogonal Projections**: Best approximation to $\mathbf{y}$, $\widehat{\mathbf{y}}$
- **Section 6.4 The Gram-Schmidt Process**: Finding an orthonormal basis, and $QR$ Factorization
- **Section 6.5 Least-Square Problems**: Normal Equations, and least-squares error
- **Section 6.6 Application to Linear Models**: Best Fit Curves using least-square solutions

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**True/False** Questions:

1. ______. Any system of $n$ linear equations in $n$ variables has at most $n$ solutions.
2. ______. The equation $A\mathbf{x} = \mathbf{0}$ has the trivial solution if and only if there are no free variables.
3. ______. If $A$ is a $3 \times 3$ matrix and the equation $A\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ has a unique solution, then $A$ is invertible.
4. ______. If $A$ and $B$ are $n \times n$, then $(A + B)(A - B) = A^2 - B^2$.
5. ______. If rows of a $3 \times 3$ matrix $A$ are the same, then $\det A = 0$.
6. ______. If $A$ is invertible, then $(\det A)(\det A^{-1}) = 1$.
7. ______. The rank of a matrix equals the number of nonzero rows.
8. ______. The nonzero rows of a matrix $A$ form a basis for Row $A$.
9. ______. Eigenvalues must be nonzero scalars.
10. ______. If $A$ and $B$ are invertible $n \times n$ matrices, then $AB$ is similar to $BA$.
11. ______. If two vectors are orthogonal then they are linearly independent.
12. ______. If $W$ is a subspace, then $\| \operatorname{proj}_W \mathbf{v} \|^2 + \| \mathbf{v} - \operatorname{proj}_W \mathbf{v} \|^2 = \| \mathbf{v} \|^2$.

**Extended Response** Questions:

13. Boron Sulfide reacts violently with water to form boric acid and hydrogen sulfide gas (the smell of rotten eggs). The unbalanced chemical equation is

$$\mathrm{B_2S_3 + H_2O \rightarrow H_3BO_3 + H_2S}.$$

For each compound, construct a vector from $\mathbb{R}^4$ that lists the numbers of atoms of boron, sulfur, hydrogen, and oxygen and balance the equation.

14. Let $\mathbf{u} = \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -6 \\ 1 \\ 7 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 0 \\ -5 \\ 2 \end{bmatrix}$, $\mathbf{z} = \begin{bmatrix} 3 \\ 7 \\ -5 \end{bmatrix}$. Is $\{\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z}\}$ linearly dependent? Is it wise to check if, say, $\mathbf{w}$ is a linear combination of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{z}$ to determine this?

15. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the transformation that projects each vector $\mathbf{x} = (x_1, x_2, x_3)$ onto the plane $x_2 = 0$, so $T(\mathbf{x}) = (x_1, 0, x_3)$. Show that $T$ is a linear transformation.

16. Find the matrix $C$ whose inverse is $C^{-1} = \begin{bmatrix} 7 & 6 \\ 5 & 4 \end{bmatrix}$

17. Let $A = \begin{bmatrix} 1 & 3 & 8 \\ 2 & 4 & 11 \\ 1 & 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 5 \\ 1 & 5 \\ 3 & 4 \end{bmatrix}$. Complute $A^{-1}B$ without computing $A^{-1}$. [*Hint*: $A^{-1}B$ is the solution of the equation $AX = B$.]

18. Find the $LU$ factorization of the matrix. $A = \begin{bmatrix} 1 & 4 & -1 & 5 \\ 3 & 7 & -2 & 9 \\ -2 & -3 & 1 & -4 \\ -1 & 6 & -1 & 7 \end{bmatrix}$.

19. Use row operations to find the determinant, $\det A = \begin{vmatrix} 12 & 13 & 14 \\ 15 & 16 & 17 \\ 18 & 19 & 20 \end{vmatrix}$.

20. Compute the adjugate of the given matrix, and then use **Theorem 3.3.2** to compute the inverse. $A = \begin{bmatrix} 3 & 5 & 4 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix}$.

21. Use Cramer's Rule (**Theorem 3.3.1**) to compute the solution of the following system,

$$\begin{cases} 2x_1 + x_2 = 7 \\ -3x_1 + x_3 = -8 \\ x_2 + 2x_3 = -3 \end{cases}$$

22. Show that if $B$ is $n \times p$, then rank $AB \leq$ rank $A$. [*Hint:* Explain why every vector in the column space of $AB$ is in the column space of $A$.]

23. Find the steady-state vector of $\begin{bmatrix} .7 & .2 & .2 \\ 0 & .2 & .4 \\ .3 & .6 & .4 \end{bmatrix}$.

24. Let $\mathscr{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathscr{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ be bases for a vector space $V$, and suppose $\mathbf{b}_1 = -\mathbf{c}_1 + 4\mathbf{c}_2$ and $\mathbf{b}_2 = 5\mathbf{c}_1 - 3\mathbf{c}_2$.

    a. Find the change-of-coordinate matrix from $\mathscr{B}$ to $\mathscr{C}$.

    b. Find $[\mathbf{x}]_{\mathscr{C}}$ for $\mathbf{x} = 5\mathbf{b}_1 + 3\mathbf{b}_2$.

25. Suppose $\mathbf{x}$ is an eigenvector of $A$ corresponding to an eigenvalue of $\lambda$.

    a. Show that $\mathbf{x}$ is an eigenvector of $5I - A$. What is the corresponding eigenvalue?

    b. Show that $\mathbf{x}$ is an eigenvector of $5I - 3A + A^2$. What is the corresponding eigenvalue?

26. Diagonalize the following matrix, if possible. $A = \begin{bmatrix} 5 & -3 & 0 & 9 \\ 0 & 3 & 1 & -2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$.

27. Find the characteristic polynomial and the eigenvalues of $A = \begin{bmatrix} -1 & 0 & 1 \\ -3 & 4 & 1 \\ 0 & 0 & 2 \end{bmatrix}$.

28. Compute $\left(\frac{\mathbf{x}\cdot\mathbf{w}}{\mathbf{x}\cdot\mathbf{x}}\right)\mathbf{x}$ for $\mathbf{x} = \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix}$.

29. Let $\mathbf{y} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$. Write $\mathbf{y}$ as the sum of a vector in Span$\{\mathbf{u}\}$ and one orthogonal to $\mathbf{u}$.

30. Let $\mathbf{y} = \begin{bmatrix} 4 \\ 8 \\ 1 \end{bmatrix}$, $\mathbf{u}_1 = \begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -2/3 \\ 2/3 \\ 1/3 \end{bmatrix}$, and $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$.

   a. Let $U = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix}$. Compute $U^TU$ and $UU^T$.

   b. Compute $\text{proj}_W\, \mathbf{y}$ and $(UU^T)\mathbf{y}$. [*Hint*: Consider **Theorem 6.3.3**]