

# Measure Theory & Probability

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## Basic Notions and Notation

### Example 1.1.

Simplest  $\sigma$ -algebra:

- $\{\emptyset, \Omega\}$ , *contained in every*  $\sigma$ -algebra on  $\Omega$ ,
- Family of all subsets of  $\Omega$ , *containing every*  $\sigma$ -algebra on  $\Omega$ .

### Exercise 1.1.

Let  $\mathcal{F}$  be a  $\sigma$ -algebra. Then  $A_n \in \mathcal{F}$  for every integer  $n \geq 1 \Rightarrow \bigcap_{n=1}^{\infty} A_n \in \mathcal{F}$ .

### Proposition (Unknown).

Let  $A, B \in \mathcal{F}$ . Then  $\mu(A \cap B) \leq \mu(A)$ .

*Hint:*  $\sigma$ -additivity and  $A = (A \cap B) \cup (A \setminus B)$ .

### Proposition (Measure of Set Difference, I).

Let  $A, B \in \mathcal{F}$ , then  $\mu(A \setminus B) = \mu(A) - \mu(A \cap B)$ .

### Proposition (Measure of Set Difference, II).

Let  $A, B \in \mathcal{F}$  and  $B \subseteq A$ , then

$$\mu(A \setminus B) = \mu(A) - \mu(B).$$

## Expectation Integrals

### Proposition (Unknown).

Let  $A, B \subseteq \Omega$ . Then the following equalities hold:

- $\mathbf{1}_{A^c} = 1 - \mathbf{1}_A$ ,
- $\mathbf{1}_{A \cap B} = \mathbf{1}_A \mathbf{1}_B$ .
- $\mathbf{1}_{A \cup B} = \mathbf{1}_A + \mathbf{1}_B - \mathbf{1}_{A \cap B}$ .

### Lemma 3.3.

Let  $X$  be a *non-negative* random variable. Then there exists a sequence of *non-negative, simple* random variables  $X_n$  converging to  $X$  for every  $\omega \in \Omega$ .

*Hint:*  $h_n(x) = \min\{[2^n x]/2^n, n\}$  is non-negative, simple and increasing, approaching  $x$ . Consider  $X_n := h(X) \rightarrow X$ .

### Lemma (Simple Function Integral Properties).

Let  $f, g : \Omega \rightarrow \overline{\mathbb{R}}$  be a *non-negative*, simple functions and  $a, b \geq 0$ . Then the following holds:

- $\int_{\Omega} f \, d\mu \geq 0$ ,
- $\int_{\Omega} (af + bg) \, d\mu = a \int_{\Omega} f \, d\mu + b \int_{\Omega} g \, d\mu$ .

### Corollary (Positive Integral over Set).

Let  $A \subseteq \Omega$  and  $f : \Omega \rightarrow \overline{\mathbb{R}}$  a *non-negative* measurable function. Then  $\int_A f \, d\mu \geq 0$ .

### Lemma 3.3 (General).

Let  $f : \Omega \rightarrow \overline{\mathbb{R}}$  be a *non-negative*, measurable function. There exists a sequence  $f_n$  of *non-negative*, simple functions such that:

$$\lim_{n \rightarrow \infty} f_n = f$$

*Hint:* Use  $h_n$  from Lemma 3.3's hint.

### Exercise 3.5.

Let  $A \in \mathcal{F}$  s.t.  $\mu(A) = 0$ . Then for *any* measurable function  $f : \Omega \rightarrow \overline{\mathbb{R}}$ :

$$\int_A f \, d\mu = 0.$$

### Exercise 3.6.

Let  $f : \Omega \rightarrow \overline{\mathbb{R}}$  be a measurable function, then:

- (i) For any  $c \in \mathbb{R}$  and  $A \in \mathcal{F}$ :

$$\int_A cf \, d\mu = c \int_A f \, d\mu,$$

provided the integral exists.

- (ii) For any  $A, B \in \mathcal{F}$ , such that  $A \cap B = \emptyset$ :

$$\int_{A \cup B} f \, d\mu = \int_A f \, d\mu + \int_B f \, d\mu,$$

provided the left-hand or right-hand side is well-defined.

### Theorem 3.8 (Monotone Convergence).

Let  $(f_n)_{n=1}^{\infty}$  be increasing sequence of non-negative, measurable functions  $f_n : \Omega \rightarrow \overline{\mathbb{R}}$ , converging to some  $f$ . Then:

$$\int_{\Omega} \lim_{n \rightarrow \infty} f_n \, d\mu = \lim_{n \rightarrow \infty} \int_{\Omega} f_n \, d\mu$$

### Exercise 3.15.

Let  $\nu$  be a measure that is absolutely continuous with respect to measure  $\mu$  and density  $g$ , then  $\mu(g < 0) = 0$ . Moreover,  $\nu$  is a probability measure  $\Leftrightarrow g \geq 0$   $\mu$ -a.e. and  $\int_{\Omega} g \, d\mu = 1$ .

### Proposition 3.16.

Let  $\nu$  and  $\mu$  be measures on  $\sigma$ -algebra  $\mathcal{F}$  such that  $\nu$  is absolutely continuous with respect to  $\mu$  and density  $g$ . Then for every  $\mathcal{F}$ -measurable function  $f$  the following holds:

$$\int_{\Omega} f \, d\nu = \int_{\Omega} fg \, d\mu,$$

whenever one of the integrals exists.

### Proposition 3.18 (Markov-Chebyshev's Inequality).

Let  $X$  be a *non-negative* R.V., then

$$P(X \geq \lambda) \leq \lambda^{-\alpha} E(X^{\alpha}) \quad \forall \lambda > 0, \alpha > 0.$$

### Remark 3.3.

Let  $(\Omega, \mathcal{F}, \mu)$  be measure space,  $f : \Omega \rightarrow \overline{\mathbb{R}}$  *non-negative*  $\mathcal{F}$ -measurable, then

$$\mu(f \geq \lambda) \leq \lambda^{-\alpha} \int_{\Omega} f^{\alpha} \, d\mu \quad \forall \lambda > 0, \alpha > 0.$$

### Proposition (Restricted Expectation).

Let  $X$  be a random variable and  $A \in \mathcal{F}$ , then:

$$E(X \mathbf{1}_A) = \int_A X \, dP.$$

### Lemma 4.4 (Borel-Cantelli Lemma).

Let  $(A_n)_{n=1}^{\infty}$  be a sequence of sets  $A_n \in \mathcal{F}$  such that  $\sum_{n=1}^{\infty} \mu(A_n) < \infty$ , i.e. the series of measures of  $A_n$  converges. Then for:

$$A := \limsup_{n \rightarrow \infty} A_n := \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k,$$

we have  $\mu(A) = 0$ .

*Hint:* Define  $B_n := \bigcup_{k=n}^{\infty} A_k$ , then  $(B_n)_{n=1}^{\infty}$  is decreasing and so  $\bigcap_{n=1}^{\infty} B_n = \lim_{n \rightarrow \infty} B_n$  and realize that  $\sum_{n=1}^{\infty} \mu(A_n) < \infty \Rightarrow$  tail sums  $\sum_{k=n}^{\infty} \mu(A_k) \rightarrow 0$  as  $n \rightarrow \infty$ .

## Definitions

In the following,  $\Omega$  is a set,  $\mathcal{F}$  a  $\sigma$ -algebra on  $\Omega$ . If used, then  $\mu$  is a measure. Otherwise, the measure is the probability measure  $P$ .

### Definition 1.1.

Let  $\mathcal{F}$  be a family of subsets of set  $\Omega$ .  $\mathcal{F}$  is called a  *$\sigma$ -algebra* if:

- **Closed Under Complement:**  $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$ ,
- **Closed Under Arbitrary Union:**  $A_n \in \mathcal{F}$  for integer  $n \geq 1 \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$ ,
- **Contains Entire Set:**  $\Omega \in \mathcal{F}$

**Definition 1.2.** Let  $\mathcal{C}$  be a family of subsets of  $\Omega$ . There exists a  $\sigma$ -algebra which contains  $\mathcal{C}$  *and* which is contained in every  $\sigma$ -algebra that contains  $\mathcal{C}$  (take intersection of all  $\sigma$ -algebras). Such  $\sigma$ -algebra is *unique* and called *smallest  $\sigma$ -algebra containing  $\mathcal{C}$*  or  *$\sigma$ -algebra generated by  $\mathcal{C}$* , denoted by  $\sigma(\mathcal{C})$ . Simplest example, let  $A \subseteq \Omega$ :

$$\sigma(A) = \{\emptyset, A, A^c, \Omega\}.$$

### Definition 2.1.1.

Let  $A \subseteq \Omega$  and  $\mathbf{1}_A$  be defined as follows:

$$\mathbf{1}_A(\omega) = \begin{cases} 1, & \omega \in A \\ 0, & \omega \notin A \end{cases}.$$

Then  $\mathbf{1}_A$  is a R.V. and called the *indicator (function) of (events) A*.

### Definition (Indicator Integral).

Let  $A \subseteq \Omega$ , then:

$$\int_{\Omega} \mathbf{1}_A \, d\mu = \mu(A).$$

### Definition (Simple Function).

Let  $f : \Omega \rightarrow \mathbb{R}$  be a *simple function*, then  $f$  takes finitely many values. Formally, if  $I$  is a finite index set,  $(A_i)_{i \in I}$  a family of *disjoint* subsets of  $\Omega$  and  $(c_i)_{i \in I}$  a family of real numbers, then:

$$f(\omega) = \sum_{i \in I} c_i \mathbf{1}_{A_i}(\omega).$$

### Definition (Lebesgue Integral for Expectation).

Let  $X$  be a random variable. Then we write:

$$EX = \int_{\Omega} X \, dP.$$

### Definition (Non-negative, Measurable Lebesgue Integral).

Let  $f : \Omega \rightarrow \overline{\mathbb{R}}$  be a *non-negative*, measurable function and  $(f_n)_{n=1}^{\infty}$  a sequence of *non-negative, simple* functions such that  $\lim_{n \rightarrow \infty} f_n = f$ . Then

$$\int_{\Omega} f \, d\mu = \lim_{n \rightarrow \infty} \int_{\Omega} f_n \, d\mu.$$

### Definition (Lebesgue Integral).

Let  $f : \Omega \rightarrow \overline{\mathbb{R}}$  be a measurable function. The *Lebesgue Integral* of  $f$  is defined as:

$$\int_{\Omega} f \, d\mu = \int_{\Omega} f^+ \, d\mu - \int_{\Omega} f^- \, d\mu,$$

where  $f^+ = \max\{f, 0\}$  and  $f^- = \max\{-f, 0\}$ , if at least one of the integrals on the right-hand side is finite. If both are infinite, then we say that the Lebesgue Integral of  $f$  does not exist.

### Definition (Unknown).

Let  $A \in \mathcal{F}$  and  $f : \Omega \rightarrow \overline{\mathbb{R}}$  is a measurable function, then we define:

$$\int_A f \, d\mu = \int_{\Omega} \mathbf{1}_A f \, d\mu,$$

when the integral of  $\mathbf{1}_A f$  w.r.t  $\mu$  exists.

### Definition 3.7 (Absolute Continuity).

Let  $\mu$  and  $\nu$  be measures on  $\sigma$ -algebra  $\mathcal{F}$  such that for some  $\mathcal{F}$ -measurable  $g : \Omega \rightarrow \mathbb{R}$ :

$$\nu(A) = \int_{\Omega} \mathbf{1}_A g \, d\mu = \int_A g \, d\mu,$$

for all  $A \in \mathcal{F}$ . Then  $\nu$  is called *absolutely continuous* with respect to  $\mu$  and  $g$  is called the *density* or *Radon-Nikodym derivative* (Notation:  $g = \frac{d\nu}{d\mu}$ ).

### Definition ( $\mu$ -Almost Everywhere Finite).

Let  $f : \Omega \rightarrow \overline{\mathbb{R}}$  be  $\mathcal{F}$ -measurable, then  $f$  is said to be  $\mu$ -*almost everywhere* ( $\mu$ -a.e.) finite if  $\mu(|f| = \infty) = 0$ .

**Definition (Unknown).**

Let  $X$  be a random variable. Then  $X$  has *finite second moment* if  $EX^2 < \infty$ .

Tips and Tricks

**Trick** (Function As Integral).

Let  $f : \Omega \rightarrow \overline{\mathbb{R}}$  be a *non-negative* measurable function, then observe that

$$f(\omega) = \int_0^{f(\omega)} dx = \int_0^\infty \mathbf{1}_{x \leq f(\omega)} dx$$