

Measure Theory & Probability

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Basic Notions and Notation

Example 1.1.

Simplest σ -algebra:

- $\{\emptyset, \Omega\}$, **contained in every** σ -algebra on Ω ,
- Family of all subsets of Ω , **containing every** σ -algebra on Ω .

Exercise 1.1.

Let \mathcal{F} be a σ -algebra. Then $A_n \in \mathcal{F}$ for every integer $n \geq 1 \Rightarrow \bigcap_{n=1}^{\infty} A_n \in \mathcal{F}$.

Proposition 1.2.

Let P be a probability measure on σ -algebra \mathcal{F} . Then the following statements hold:

- $A, B \in \mathcal{F}$ s.t. $A \subseteq B \Rightarrow P(A) \leq P(B)$;
- For **increasing** sequence $(A_n)_{n=1}^{\infty}$ we have

$$\lim_{n \rightarrow \infty} P(A_n) = P\left(\bigcup_{n=1}^{\infty} A_n\right);$$

- For **decreasing** sequence $(A_n)_{n=1}^{\infty}$ we have

$$\lim_{n \rightarrow \infty} P(A_n) = P\left(\bigcap_{n=1}^{\infty} A_n\right).$$

Proposition 1.2 (General).

Let μ be a measure on σ -algebra \mathcal{F} . Then the following statements hold:

- $A, B \in \mathcal{F}$ s.t. $A \subseteq B \Rightarrow \mu(A) \leq \mu(B)$;
- For **increasing** sequence $(A_n)_{n=1}^{\infty}$ we have

$$\lim_{n \rightarrow \infty} \mu(A_n) = \mu\left(\bigcup_{n=1}^{\infty} A_n\right);$$

- For **decreasing** sequence $(A_n)_{n=1}^{\infty}$ we have

$$\lim_{n \rightarrow \infty} \mu(A_n) = \mu\left(\bigcap_{n=1}^{\infty} A_n\right).$$

Proposition (Bounding Intersections).

Let $A, B \in \mathcal{F}$. Then $\mu(A \cap B) \leq \mu(A)$.

Hint: σ -additivity and $A = (A \cap B) \cup (A \setminus B)$.

Proposition (Measure of Set Difference, I).

Let $A, B \in \mathcal{F}$, then $\mu(A \setminus B) = \mu(A) - \mu(A \cap B)$.

Proposition (Measure of Set Difference, II).

Let $A, B \in \mathcal{F}$ and $B \subseteq A$, then

$$\mu(A \setminus B) = \mu(A) - \mu(B).$$

Proposition (Complement of Limit Inferior/Superior).

Let $(A_n)_{n=1}^{\infty}$ be a sequence of sets in \mathcal{F} , then:

- $$\left(\liminf_{n \rightarrow \infty} A_n\right)^C = \limsup_{n \rightarrow \infty} A_n^C$$
- $$\left(\limsup_{n \rightarrow \infty} A_n\right)^C = \liminf_{n \rightarrow \infty} A_n^C$$

Exercise Ws 2, 1 (Limit Inferior/Superior Properties).

Let $(A_n)_{n=1}^{\infty}$ be a sequence of sets in \mathcal{F} , then:

- $$\liminf_{n \rightarrow \infty} A_n := \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k$$

is the set of those ω that are **in all but finitely many** A_n , i.e. that uphold the property A_n captures for all except a finite amount of values of n .

(ii)

$$\limsup_{n \rightarrow \infty} A_n := \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$$

is the set of those ω that are **in infinitely many** A_n , i.e. that uphold the property A_n captures for an infinite amount of values of n .

Proposition (Continuous Implies Borel-Measurability).

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a **continuous** function. Then f is Borel-measurable.

Proposition (Countable Sets).

Every countable subset of \mathbb{R} is Borel-measurable.

Expectation Integrals

Proposition (Unknown).

Let $A, B \subseteq \Omega$. Then the following equalities hold:

- $\mathbf{1}_{A^C} = 1 - \mathbf{1}_A$,
- $\mathbf{1}_{A \cap B} = \mathbf{1}_A \mathbf{1}_B$.
- $\mathbf{1}_{A \cup B} = \mathbf{1}_A + \mathbf{1}_B - \mathbf{1}_{A \cap B}$.

Lemma 3.3.

Let X be a **non-negative** random variable. Then there exists a sequence of **non-negative, simple** random variables X_n converging to X for every $\omega \in \Omega$.

Hint: $h_n(x) = \min\{\lfloor 2^n x \rfloor / 2^n, n\}$ is non-negative, simple and increasing, approaching x . Consider $X_n := h(X) \rightarrow X$.

Lemma (Simple Function Integral Properties).

Let $f, g : \Omega \rightarrow \mathbb{R}$ be a **non-negative**, simple functions and $a, b \geq 0$. Then the following holds:

- $\int_{\Omega} f d\mu \geq 0$,
- $\int_{\Omega} (af + bg) d\mu = a \int_{\Omega} f d\mu + b \int_{\Omega} g d\mu$.

Corollary (Positive Integral over Set).

Let $A \subseteq \Omega$ and $f : \Omega \rightarrow \mathbb{R}$ a **non-negative** measurable function. Then $\int_A f d\mu \geq 0$.

Lemma 3.3 (General).

Let $f : \Omega \rightarrow \mathbb{R}$ be a **non-negative**, measurable function. Then there exists a sequence f_n of **non-negative**, simple functions such that:

$$\lim_{n \rightarrow \infty} f_n = f$$

Hint: Use h_n from Lemma 3.3's hint.

Exercise 3.5.

Let $A \in \mathcal{F}$ s.t. $\mu(A) = 0$. Then for **any** measurable function $f : \Omega \rightarrow \mathbb{R}$:

$$\int_A f d\mu = 0.$$

Exercise 3.6.

Let $f : \Omega \rightarrow \mathbb{R}$ be a measurable function, then:

- For any $c \in \mathbb{R}$ and $A \in \mathcal{F}$:

$$\int_A cf d\mu = c \int_A f d\mu,$$

provided the integral exists.

- For any $A, B \in \mathcal{F}$, such that $A \cap B = \emptyset$:

$$\int_{A \cup B} f d\mu = \int_A f d\mu + \int_B f d\mu,$$

provided the left-hand or right-hand side is well-defined.

Theorem 3.8 (Monotone Convergence).

Let $(f_n)_{n=1}^{\infty}$ be increasing sequence of non-negative, measurable functions $f_n : \Omega \rightarrow \mathbb{R}$, converging to some f . Then:

$$\int_{\Omega} \lim_{n \rightarrow \infty} f_n d\mu = \lim_{n \rightarrow \infty} \int_{\Omega} f_n d\mu$$

Theorem 3.14 (Lebesgue Integral as Riemann Integral).

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a Borel-function such that:

- the Riemann integral $\int_{-\infty}^{\infty} f(x) dx$ exists and
- the Riemann integral $\int_{-\infty}^{\infty} |f(x)| dx < \infty$, i.e. is finite,

then the Lebesgue integral $\int_{\mathbb{R}} f(x) \lambda(dx)$ **exists** and

$$\int_{\mathbb{R}} f(x) \lambda(dx) = \int_{-\infty}^{\infty} f(x) dx,$$

i.e. the Lebesgue integral is equal to the Riemann integral.

Exercise 3.15.

Let ν be a measure that is absolutely continuous with respect to measure μ and density g , then $\mu(g < 0) = 0$. Moreover, ν is a probability measure $\Leftrightarrow g \geq 0$ μ -a.e. and $\int_{\Omega} g d\mu = 1$.

Proposition 3.16.

Let ν and μ be measures on σ -algebra \mathcal{F} such that ν is absolutely continuous with respect to μ and density g . Then for every \mathcal{F} -measurable function f the following holds:

$$\int_{\Omega} f d\nu = \int_{\Omega} fg d\mu,$$

whenever one of the integrals exists.

Remark 3.3.

Let $(\Omega, \mathcal{F}, \mu)$ be measure space, $f : \Omega \rightarrow \mathbb{R}$ **non-negative** \mathcal{F} -measurable, then

$$\mu(f \geq \lambda) \leq \lambda^{-\alpha} \int_{\Omega} f^{\alpha} d\mu \quad \forall \lambda > 0, \alpha > 0.$$

Lemma 3.10 (Fatou's Lemma).

Let $(f_n)_{n=1}^{\infty}$ be a sequence of **non-negative**, measurable functions $f : \Omega \rightarrow \mathbb{R}$, then

$$\int_{\Omega} \liminf_{n \rightarrow \infty} f_n d\mu \leq \liminf_{n \rightarrow \infty} \int_{\Omega} f_n d\mu.$$

Corollary 3.11 (Fatou's Lemma Extension).

Let $(f_n)_{n=1}^{\infty}$ be a sequence of measurable functions $f : \Omega \rightarrow \mathbb{R}$. Then

- if there exists a $g \in L_1(\Omega, \mathcal{F}, \mu)$, i.e. $\int_{\Omega} |g| d\mu < \infty$ such that $g \leq f_n$ for all n , then:

$$\int_{\Omega} \liminf_{n \rightarrow \infty} f_n d\mu \leq \liminf_{n \rightarrow \infty} \int_{\Omega} f_n d\mu.$$

- if there exists a $g \in L_1(\Omega, \mathcal{F}, \mu)$, i.e. $\int_{\Omega} |g| d\mu < \infty$ such that $g \geq f_n$, then:

$$\int_{\Omega} \limsup_{n \rightarrow \infty} f_n d\mu \geq \limsup_{n \rightarrow \infty} \int_{\Omega} f_n d\mu.$$

Hint: TODO

Theorem 3.12 (Lebesgue's Theorem on Dominated Convergence).

Let $(f_n)_{n=1}^{\infty}$ be a sequence of Borel functions $f_n : \Omega \rightarrow \mathbb{R}$ converging to some $f : \Omega \rightarrow \mathbb{R}$. Assume there exists a (non-negative) Borel functions g such that $|f_n| \leq g$ for any $n \geq 1$ and $\int_{\Omega} g d\mu < \infty$. Then the following two statements hold:

- $$\int_{\Omega} |f| d\mu < \infty,$$

- $$\int_{\Omega} f d\mu = \lim_{n \rightarrow \infty} \int_{\Omega} f_n d\mu.$$

Hint: TODO

Proposition (Restricted Expectation).

Let X be a random variable and $A \in \mathcal{F}$, then:

$$E(X\mathbf{1}_A) = \int_A X dP.$$

Proposition 3.18 (Markov-Chebyshev's Inequality).

Let X be a **non-negative** R.V., then

$$P(X \geq \lambda) \leq \lambda^{-\alpha} E(X^\alpha) \quad \forall \lambda > 0, \alpha > 0.$$

Hint: $E(X^\alpha) \geq E(\mathbf{1}_{X \geq \lambda} X^\alpha) \geq E(\mathbf{1}_{X \geq \lambda} \lambda^\alpha) = \lambda^\alpha E(\mathbf{1}_{X \geq \lambda}) = \lambda^\alpha P(X \geq \lambda)$.

Proposition 3.18 (Markov-Chebyshev's Inequality (General)).

Let $f : \Omega \rightarrow \mathbb{R}$ be a **non-negative**, measurable function, then

$$\mu(f \geq \lambda) \leq \lambda^{-\alpha} \int_\Omega f^\alpha d\mu \quad \forall \lambda > 0, \alpha > 0.$$

L_p Spaces

Theorem (Hölder's Inequality).

Let $f, g : \Omega \rightarrow \mathbb{R}$ be measurable functions, then

$$\int_\Omega |fg| d\mu \leq \|f\|_p \|g\|_q \quad \text{for } p \geq 1,$$

where

$$q := \begin{cases} \frac{p}{p-1} & p > 1, \\ \infty & p = 1 \end{cases}.$$

Hint: TODO

Theorem (Hölder's Inequality for Expectations).

Let X, Y be random variables, then

$$E|XY| \leq (E|X|^p)^{\frac{1}{p}} (E|Y|^q)^{\frac{1}{q}}$$

where

$$q := \begin{cases} \frac{p}{p-1} & p > 1, \\ \infty & p = 1 \end{cases}.$$

Proposition (Finite Second Momenta Implication).

Let X, Y be random variables with finite second momenta. Then $E|XY| < \infty$.

Hint: Use Hölder's Inequality with $p = 2$ on $E|XY| = \int_\Omega |XY| dP$.

Lemma 4.4 (Borel-Cantelli Lemma).

Let $(A_n)_{n=1}^\infty$ be a sequence of sets $A_n \in \mathcal{F}$ such that $\sum_{n=1}^\infty \mu(A_n) < \infty$, i.e. the series of measures of A_n converges. Then for:

$$A := \limsup_{n \rightarrow \infty} A_n := \bigcap_{n=1}^\infty \bigcup_{k=n}^\infty A_k,$$

we have $\mu(A) = 0$.

Hint: Define $B_n := \bigcup_{k=n}^\infty A_k$, then $(B_n)_{n=1}^\infty$ is decreasing and so $\bigcap_{n=1}^\infty B_n = \lim_{n \rightarrow \infty} B_n$ and realize that $\sum_{n=1}^\infty \mu(A_n) < \infty \Rightarrow$ tail sums $\sum_{k=n}^\infty \mu(A_k) \rightarrow 0$ as $n \rightarrow \infty$.

Convergence of Measurable Functions

Exercise 5.1.

Let $(f_n)_{n=1}^\infty$ be a sequence of \mathcal{F} -measurable functions $f_n : \Omega \rightarrow \mathbb{R}$. Then the set A of those $\omega \in \Omega$ such that $\lim_{n \rightarrow \infty} f_n(\omega)$ converges to some (finite) number belongs to \mathcal{F} .

Exercise 5.2 (Almost Finite, Converging Sequence is Bounded).

Assume that $\mu(\Omega) < \infty$. Let $(f_n)_{n=1}^\infty$ be **μ -a.e. finite**, converging in measure to μ to some

$f : \Omega \rightarrow \mathbb{R}$. Then the sequence of f_n is **bounded in measure** μ , **uniformly in** n , i.e.:

$$\lim_{K \rightarrow \infty} \sup_{n \geq 1} \mu(|f_n| \geq K) = 0.$$

Hint: f_n **μ -a.e. finite** and $\mu(\Omega) < \infty \Rightarrow f_n$ bounded in measure (not necessarily uniformly), so

$$\lim_{K \rightarrow \infty} \sup_{n \geq 1} \mu(|f_n| \geq K) =$$

$$\lim_{K \rightarrow \infty} \limsup_{n \rightarrow \infty} \mu(|f_n| \geq K).$$

Then use observation of splitting measures of inequalities.

Exercise 5.3 (Product of Bounded & Zero Convergent is Zero Convergent).

Let $(f_n)_{n=1}^\infty$ and $(g_n)_{n=1}^\infty$ be sequences of μ -a.e. finite measurable functions such that the f_n are bounded in measure μ , uniformly in n and $g_n \rightarrow 0$ in measure μ , as $n \rightarrow \infty$. Then $f_n g_n \rightarrow 0$ in measure μ , as $n \rightarrow \infty$.

Exercise Ws 3, 1.

Let $\mu - \lim f_n = f$, then there exists a subsequence $(f_{n_k})_{k=1}^\infty$ such that $(n_k)_{k=1}^\infty$ is increasing and $f_{n_k} \rightarrow f$ (μ -a.e.).

Hint: Borel-Cantelli with

$A_k = \{|f_{n_k} - f| \geq 1/k\}$ s.t. $\mu(A_k) \leq 1/k^2$.

Theorem 5.4 (Measure Convergence Has Almost Everywhere Converging Subsequence).

Let $(f_n)_{n=1}^\infty$ be a sequence of functions converging in measure μ to some μ -a.e. finite function f . Then there exists a (strictly) increasing sequence $(n_k)_{k=1}^\infty$ of positive integers such that $\lim_{k \rightarrow \infty} f_{n_k} = f$ μ -almost everywhere.

Exercise 5.5.

Convergence in measure μ does not imply convergence μ -almost everywhere.

Hint: $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$ with $f_n = \mathbf{1}_{[k/2^m, (k+1)/2^m]}$ where $k = 0, 1, \dots, 2^m - 1$ and $m = 0, 1, \dots$ such that $n = 2^m + k$.

Exercise Ws 3, 2 (Convergence Implication).

Let $\mu(\Omega) < \infty$. Then $\lim_{n \rightarrow \infty} f_n = f$ (μ -a.e.) $\Rightarrow \mu - \lim_{n \rightarrow \infty} f_n = f$.

Exercise Ws 3, 3 (Relaxed Dominated Convergence).

Lebesgue's Theorem on Dominated convergence holds under the following, relaxed conditions:

- (i) $\lim_{n \rightarrow \infty} f_n = f$ **μ -a.e.**, $|f_n| \leq |g|$ μ -a.e. and $g \in L_1(\Omega, \mathcal{F}, \mu)$, i.e. $\int_\Omega |g| d\mu < \infty$; and
- (ii) $\mu - \lim_{n \rightarrow \infty} f_n = f$, $|f_n| \leq |g|$ μ -a.e. and $g \in L_1(\Omega, \mathcal{F}, \mu)$, i.e. $\int_\Omega |g| d\mu < \infty$.

Hint: TODO

Independence of Events and Random Variables

Theorem 6.3 (Monotone Class Theorem).

Let Π be a π -system contained in a λ -system Λ . Then $\sigma(\Pi)$ is contained in Λ .

Hint: TODO

Proposition 6.4 (Extending π -System Independence).

Let C_1 and C_2 be two **independent** π -systems, i.e.

$$P(A \cap B) = P(A)P(B) \quad \forall A \in C_1, B \in C_2,$$

then the σ -algebras $\sigma(C_1)$ and $\sigma(C_2)$ are also independent.

Hint: TODO

Theorem 6.7 (Fubini-Tonelli Theorem).

Let $(\Omega_i, \mathcal{F}_i, \mu_i)$, for $i = 1, 2$, be measure spaces and $(\Omega, \mathcal{F}, \mu)$ be the product measure space of the two, i.e. $\Omega = \Omega_1 \times \Omega_2$, $\mathcal{F} = \mathcal{F}_1 \otimes \mathcal{F}_2$ and $\mu = \mu_1 \otimes \mu_2$. Let $f : \Omega \rightarrow \mathbb{R}$ be a **non-negative** \mathcal{F} -measurable function. If μ_i , for $i = 1, 2$, are **finite measures** on Ω_i , for $i = 1, 2$, respectively, then the following iterated integrals are well-defined and:

$$\begin{aligned} \int_{\Omega_1 \times \Omega_2} f d\mu_1 \otimes \mu_2 &= \int_{\Omega_1} \int_{\Omega_2} f d\mu_2 d\mu_1 = \\ &= \int_{\Omega_2} \int_{\Omega_1} f d\mu_1 d\mu_2. \end{aligned}$$

Furthermore, this statement holds for \mathcal{F} -measurable functions if:

$$\int_{\Omega_1 \times \Omega_2} |f| d\mu_1 \otimes \mu_2 < \infty.$$

Lemma 6.9 (Borel-Cantelli (Full)).

Let $(A_n)_{n=1}^\infty$ be a sequence of sets and set

$$A := \limsup_{n \rightarrow \infty} A_n := \bigcap_{n=1}^\infty \bigcup_{k=n}^\infty A_k,$$

then the following statements holds:

- (i) If $\sum_{n=1}^\infty \mu(A_n) < \infty$, then $\mu(A) = 0$.
- (ii) If all A_n are **jointly independent** and $\sum_{n=1}^\infty P(A_n) = \infty$, then $P(A) = 1$.

Hint: (i) provided in general case. (ii) Prove $P((\limsup_{n \rightarrow \infty} A_n)^C) = 1$, define $B_n = \bigcap_{k=n}^\infty A_k^C$ and show that for a given $P(B_n) = P(\lim_{m \rightarrow \infty} \bigcap_{k=n}^m A_k) = 0$ using independence and observation that $1 - P(A) \leq e^{-P(A)}$. Finally, use **sub**- σ -additivity for $P(\bigcup_{n=1}^\infty B_n)$. **Do not** attempt to argue through increasing sequences.

Exercise (Pulling Sum Through Variance).

Let $(X_i)_{i=1}^\infty$ be a sequence of **pairwise independent** random variables. Assume that $EX_i^2 < \infty$ for $i = 1, 2, \dots, n$, then

$$\text{Var} \left(\sum_{i=1}^n X_i \right) = \sum_{i=1}^n \text{Var}(X_i).$$

Conditional Expectation

Exercise 8.1.

Let $\mathcal{G} := \{\emptyset, \Omega\}$, i.e. the trivial σ -algebra. Then if random variable Y is \mathcal{G} -measurable, then Y is constant.

Lemma 8.2.

Let Z be a \mathcal{G} -measurable random variable such that:

$$\int_A Z dP \geq 0 \iff E(\mathbf{1}_A Z) \geq 0,$$

for any $A \in \mathcal{G}$, then $Z \geq 0$ (a.s.).

Definitions

Basic Notions and Notation

In the following, Ω is a set, \mathcal{F} a σ -algebra on Ω . If used, then μ is a measure. Otherwise, the measure is the probability measure P .

Definition 1.1.

Let \mathcal{F} be a family of subsets of set Ω . \mathcal{F} is called a σ -algebra if:

- **Closed Under Complement:**
 $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$,
- **Closed Under Arbitrary Union:**
 $A_n \in \mathcal{F}$ for integer $n \geq 1$
 $\Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$,
- **Contains Entire Set:** $\Omega \in \mathcal{F}$

Definition 1.2. Let \mathcal{C} be a family of subsets of Ω . There exists a σ -algebra which contains \mathcal{C} and which is contained in every σ -algebra that contains \mathcal{C} (take intersection of all σ -algebras). Such σ -algebra is **unique** and called **smallest σ -algebra containing \mathcal{C}** or **σ -algebra generated by \mathcal{C}** , denoted by $\sigma(\mathcal{C})$. Simplest example, let $A \subseteq \Omega$:

$$\sigma(A) = \{\emptyset, A, A^c, \Omega\}.$$

Definition (Finite Measure Space).

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. If $\mu(\Omega) < \infty$, then we call the measure space **finite**.

Random Variables

Definition 2.1.1.

Let $A \subseteq \Omega$ and $\mathbf{1}_A$ be defined as follows:

$$\mathbf{1}_A(\omega) = \begin{cases} 1, & \omega \in A \\ 0, & \omega \notin A \end{cases}.$$

Then $\mathbf{1}_A$ is a R.V. and called the **indicator (function) of (events) A** .

Definition 2.3 (Distribution Function).

Let X be a random variable. Then the function

$$F_X(x) = P(X \leq x) = P(X \in (-\infty, x]) = Q_X((-\infty, x]),$$

for $x \in \mathbb{R}$ is called the **distribution function** of X .

Expectation Integrals

Definition (Indicator Integral).

Let $A \subseteq \Omega$, then:

$$\int_{\Omega} \mathbf{1}_A d\mu = \mu(A).$$

Definition (Simple Function).

Let $f: \Omega \rightarrow \mathbb{R}$ be a **simple function**, then f takes finitely many values. Formally, if I is a finite index set, $(A_i)_{i \in I}$ a family of **disjoint** subsets of Ω and $(c_i)_{i \in I}$ a family of real numbers, then:

$$f(\omega) = \sum_{i \in I} c_i \mathbf{1}_{A_i}(\omega).$$

Definition (Lebesgue Integral for Expectation).

Let X be a random variable. Then we write:

$$EX = \int_{\Omega} X dP.$$

Definition (Non-negative, Measurable Lebesgue Integral).

Let $f: \Omega \rightarrow \mathbb{R}$ be a **non-negative**, measurable function and $(f_n)_{n=1}^{\infty}$ a sequence of **non-negative, simple** functions such that $\lim_{n \rightarrow \infty} f_n = f$. Then

$$\int_{\Omega} f d\mu = \lim_{n \rightarrow \infty} \int_{\Omega} f_n d\mu.$$

Definition (Lebesgue Integral).

Let $f: \Omega \rightarrow \mathbb{R}$ be a measurable function. The **Lebesgue Integral** of f is defined as:

$$\int_{\Omega} f d\mu = \int_{\Omega} f^+ d\mu - \int_{\Omega} f^- d\mu,$$

where $f^+ = \max\{f, 0\}$ and $f^- = \max\{-f, 0\}$, if at least one of the integrals on the right-hand side is finite. If both are infinite, then we say that the Lebesgue Integral of f does not exist.

Definition (Restricted Integration).

Let $A \in \mathcal{F}$ and $f: \Omega \rightarrow \mathbb{R}$ is a measurable function, then we define:

$$\int_A f d\mu = \int_{\Omega} \mathbf{1}_A f d\mu,$$

when the integral of $\mathbf{1}_A f$ w.r.t μ exists.

Definition 3.7 (Absolute Continuity).

Let μ and ν be measures on σ -algebra \mathcal{F} such that for some \mathcal{F} -measurable $g: \Omega \rightarrow \mathbb{R}$:

$$\nu(A) = \int_{\Omega} \mathbf{1}_A g d\mu = \int_A g \mu(dx),$$

for all $A \in \mathcal{F}$. Then ν is called **absolutely continuous** with respect to μ and g is called the **density** or **Radon-Nikodym derivative** (Notation: $g = \frac{d\nu}{d\mu}$).

Convergence of Measurable Functions

Definition (μ -Almost Everywhere Finite).

Let $f: \Omega \rightarrow \mathbb{R}$ be \mathcal{F} -measurable, then f is said to be **μ -almost everywhere** (μ -a.e.) finite if $\mu(|f| = \infty) = 0$.

Definition (Almost Surely Finite).

Let $f: \Omega \rightarrow \mathbb{R}$ be \mathcal{F} -measurable, then f is said to be **almost surely** (a.s.) finite if $P(|f| = \infty) = 0 \Leftrightarrow P(|f| < \infty) = 1$.

Definition 5.1 (μ -Almost Everywhere Convergence).

Let $(f_n)_{n=1}^{\infty}$ be \mathcal{F} -measurable functions. The f_n are said to **converge μ -almost everywhere** to a μ -a.e. **finite** $f: \Omega \rightarrow \mathbb{R}$ as $n \rightarrow \infty$ if there exists an $A \in \mathcal{F}$ s.t. $\mu(A) = 0$ and

$$\lim_{n \rightarrow \infty} f_n(\omega) = f(\omega) \in \mathbb{R}, \quad \forall \omega \in A^c.$$

Notation: $\lim_{n \rightarrow \infty} f_n = f$ (μ -a.e.) or $f_n \rightarrow f$ (μ -a.e.).

Definition 5.1 (Almost Sure Convergence).

Let $(f_n)_{n=1}^{\infty}$ be \mathcal{F} -measurable functions. The f_n are said to **converge almost surely** to a **a.s. finite** $f: \Omega \rightarrow \mathbb{R}$ as $n \rightarrow \infty$ if there exists an $A \in \mathcal{F}$ s.t. $P(A) = 0$ and

$$\lim_{n \rightarrow \infty} f_n(\omega) = f(\omega) \in \mathbb{R}, \quad \forall \omega \in A^c.$$

Notation: $\lim_{n \rightarrow \infty} f_n = f$ (a.s.) or $f_n \rightarrow f$ (a.s.).

Definition 5.2 (Convergence in Measure).

Let $(f_n)_{n=1}^{\infty}$ be \mathcal{F} -measurable functions. The f_n are said to **converge in measure** μ to a μ -a.e. **finite** $f: \Omega \rightarrow \mathbb{R}$ as $n \rightarrow \infty$ if

$$\lim_{n \rightarrow \infty} \mu(|f_n - f| \geq \varepsilon) = 0, \quad \forall \varepsilon > 0.$$

Notation: $\mu - \lim_{n \rightarrow \infty} f_n = f$.

Definition 5.2 (Convergence in Probability).

Let $(f_n)_{n=1}^{\infty}$ be \mathcal{F} -measurable functions. The f_n are said to **converge in probability** to a **a.s. finite** $f: \Omega \rightarrow \mathbb{R}$ as $n \rightarrow \infty$ if

$$\lim_{n \rightarrow \infty} P(|f_n - f| \geq \varepsilon) = 0, \quad \forall \varepsilon > 0.$$

Definition (Bounded in Measure).

Let $(f_n)_{n=1}^{\infty}$ be a sequence of measurable functions, then it is **bounded in measure** μ if

$$\lim_{K \rightarrow \infty} \mu(|f_n| \geq K) = 0,$$

for any $n \geq 1$.

Definition (Bounded Uniformly in Measure).

Let $(f_n)_{n=1}^{\infty}$ be a sequence of measurable functions, then it is **bounded in measure** μ , **uniformly** in n if

$$\lim_{K \rightarrow \infty} \sup_{n \geq 1} \mu(|f_n| \geq K) = 0.$$

Definition (Finite Second Moment).

Let X be a random variable. Then X has **finite second moment** if $EX^2 < \infty$.

Independence of Events and Random Variables

Definition 6.5 (λ -system).

Let Λ be a family of subsets of Ω . Then Λ is a λ -system, if it satisfies all of the following properties:

- (i) (Contains whole set) $\Omega \in \Lambda$;
- (ii) (Closed under Subset Set Subtraction) if $A, B \in \Lambda$, such that $B \subset A$, then $A \setminus B \in \Lambda$;
- (iii) (Closed under Disjoint Union) if $(A_n)_{n=1}^{\infty}$ is a **pairwise disjoint** sequence, i.e. $A_i \cap A_j = \emptyset$ for $i \neq j$, of subsets, such that $A_i \in \Lambda$ for $i = 1, 2, \dots$, then $\bigcup_{n=1}^{\infty} A_n \in \Lambda$.

Definition (π -system).

Let Π be a family of subsets of Ω . Then Π is a π -system, if it is closed under finite intersections, i.e. $A, B \in \Pi \Rightarrow A \cap B \in \Pi$.

Definition Ws 5, 1 (σ -Finite Measure).

Let μ be a measure, then μ is called **σ -finite** if there exists an increasing sequence $(\Omega_n)_{n=1}^{\infty}$ in \mathcal{F} , such that $\mu(\Omega_n) < \infty$ for all $n \geq 1$ and $\bigcap_{n=1}^{\infty} \Omega_n = \Omega$.

Conditional Expectation

Definition 8.1 (Sub- σ -Algebra Measurable).

Let Y be a random variable and $\mathcal{G} \subset \mathcal{F}$ be a σ -algebra. Then Y is \mathcal{G} -measurable if $Y^{-1}(F) \in \mathcal{G}$ for any $F \in \mathcal{B}(\mathbb{R})$.

Definition 8.2 (Conditional Expectation).

Let X, Y be random variables such that $E|X| < \infty$ and $\mathcal{G} \subset \mathcal{F}$ be a σ -algebra. Let Y satisfy the following properties:

- (i) Y is \mathcal{G} -measurable and
- (ii) for any $A \in \mathcal{G}$:

$$\int_A Y dP = \int_A X dP \iff E(\mathbf{1}_A Y) = E(\mathbf{1}_A X),$$

then Y is called the **conditional expectation** with respect of \mathcal{G} of X and we write $Y = E(X|\mathcal{G})$.

Useful Observations

Observation (Bounding Measures).

The following inequalities to bound measures are *always* applicable, for *any* sets $A, B, C \in \mathcal{F}$:

1. “Dropping a set in an intersection gives an upper bound” \Leftrightarrow “Relaxing constraints”:

$$\mu(A \cap B) \leq \mu(A).$$

2. “Dropping a set in a union gives an lower bound”:

$$\mu(A \cup B) \geq \mu(A).$$

3. “Adding a set in a union gives an upper bound” \Leftrightarrow “Adding constraints”:

$$\mu(A \cup B) \leq \mu(A \cup B \cup C).$$

4. “Intersections are less than a set and a set is less than a union”:

$$\mu(A \cap B) \leq \mu(A) \leq \mu(A \cup B).$$

Observation (Adding Ω by Intersection).

If you would like to introduce a property to an existing set A to make it easier to work with, for instance easier to bound, you can add an intersection with Ω :

$$\mu(A) = \mu(\Omega \cap A).$$

Then Ω can be split into the set B that represents the property and B^C that does not have the property, where $\Omega = B \cup B^C$. Then:

$$\mu(A) = \mu(\Omega \cap A) = \mu((B \cup B^C) \cap A) =$$

$$\mu((B \cup B^C) \cap A) = \mu((B \cap A) \cup (B^C \cap A)).$$

Using σ -additivity, we get:

$$\mu(A) = \mu(B \cap A) + \mu(B^C \cap A).$$

Then by the observation on bounding measures, this can be made into an inequality:

$$\begin{aligned} \mu(A) &= \mu(B \cap A) + \mu(B^C \cap A) \\ &\leq \mu(B \cap A) + \mu(B^C). \end{aligned}$$

Observation (Increasing Sequence of Sets).

For an *increasing* sequence of sets $(A_n)_{n=1}^\infty$ we can define:

$$\lim_{n \rightarrow \infty} A_n := \bigcup_{n=1}^\infty A_n$$

Observation (Decreasing Sequence of Sets).

For an *decreasing* sequence of sets $(A_n)_{n=1}^\infty$ we can define:

$$\lim_{n \rightarrow \infty} A_n := \bigcap_{n=1}^\infty A_n$$

Observation (μ -Almost Everywhere Finite, I).

If $f : \Omega \rightarrow \mathbb{R}$ is μ -a. e. finite, then note that if $A_n := \{|f| \geq n\}$, then $(A_n)_{n=1}^\infty$ is a decreasing sequence and so:

$$\begin{aligned} \mu\left(\bigcap_{n=1}^\infty A_n\right) &= \mu\left(\lim_{n \rightarrow \infty} A_n\right) = \mu(|f| = \infty) \\ &= 0. \end{aligned}$$

Observation (μ -Almost Everywhere Finite, II).

If $f : \Omega \rightarrow \mathbb{R}$ is μ -a. e. finite, then observe

$$\mu(|f| = \infty) = \lim_{R \rightarrow \infty} \mu(|f| \geq R) = 0.$$

Observation (Almost Surely Finite, II).

If $f : \Omega \rightarrow \mathbb{R}$ is a.s. finite, then observe

$$\begin{aligned} P(|f| = \infty) &= \lim_{R \rightarrow \infty} P(|f| \geq R) = 0. \\ \Leftrightarrow P(|f| < \infty) &= \lim_{R \rightarrow \infty} P(|f| < R) = 1. \end{aligned}$$

Observation (Almost Surely Finite).

If $f : \Omega \rightarrow \mathbb{R}$ is a. s. finite, then note that if $A_n := \{|f| \geq n\}$, then $(A_n)_{n=1}^\infty$ is a decreasing sequence and so:

$$\begin{aligned} P\left(\bigcap_{n=1}^\infty A_n\right) &= P\left(\lim_{n \rightarrow \infty} A_n\right) = P(|f| = \infty) \\ &= 0. \end{aligned}$$

Observation (μ -Almost Everywhere Convergence I).

If $f_n \rightarrow f$ μ -a.e., then $\mu(f_n \not\rightarrow f) = 0$.

Observation (μ -Almost Everywhere Convergence II).

If $A \in \mathcal{F}$ is a set such that $\mu(A) = 0$ and

$$\lim_{n \rightarrow \infty} |f_n(\omega) - f(\omega)| = 0 \quad \forall \omega \in A^C,$$

then $f_n \rightarrow f$ μ -almost everywhere.

Observation (Almost Sure Convergence).

If $f_n \rightarrow f$ a.s., then $P(f_n \not\rightarrow f) = 0$ or equivalently $P(f_n \rightarrow f) = 1$.

Observation (Splitting Measures of Inequalities).

Let f, g be measurable functions and $a \in \mathbb{R}$, then observe that:

$$\mu(|f| \geq a) \leq \mu\left(|f - g| \geq \frac{a}{2}\right) + \mu\left(|g| \geq \frac{a}{2}\right)$$

Observation (Using Borel-Cantelli).

If you can define sets $(A_k)_{k=1}^\infty$ such that $\mu(A_k) \leq 1/k^2$, then you can use Borel-Cantelli as:

$$\sum_{k=1}^\infty \mu(A_k) \leq \sum_{k=1}^\infty \frac{1}{k^2} < \infty.$$

In fact, the choice of $1/k^2$ is more or less arbitrary. This technique would work with any r_k s.t. $\sum_{k=1}^\infty r_k < \infty$ and $\mu(A_k) \leq r_k$. Caution: $r_k = 1/k$ does *not* work.

Observation (Function As Integral).

Let $f : \Omega \rightarrow \mathbb{R}$ be a *non-negative* measurable function, then observe that

$$f(\omega) = \int_0^{f(\omega)} dx = \int_0^\infty \mathbf{1}_{x \leq f(\omega)} dx$$

Observation (Bounding Complement Probabilities).

Note that $1 - x \leq e^{-x}$. Therefore, we can bound probabilities of a product of complement events, for instance:

$$\begin{aligned} \prod_{n=1}^\infty P(A_n^C) &= \prod_{n=1}^\infty [1 - P(A_n)] \leq \\ &\prod_{n=1}^\infty e^{-P(A_n)} = e^{\sum_{n=1}^\infty -P(A_n)} \end{aligned}$$

Observation (Interchanging Expectation & Infinite Sum).

Observe that if f is *non-negative*, then:

$$\begin{aligned} E\left(\sum_{n=1}^\infty f(X_n)\right) &= E\left(\lim_{N \rightarrow \infty} \sum_{n=1}^N f(X_n)\right) = \\ &= \lim_{N \rightarrow \infty} \sum_{n=1}^N E f(X_n) = \sum_{n=1}^\infty E f(X_n), \end{aligned}$$

where pulling the expectation through the sum can be done due to the Monotone Convergence Theorem, as $\sum_{n=1}^N f(X_n)$ is an increasing sequence of *non-negative* random variables.

Observation (Markov-Chebyshev's Inequality & Norm).

The following is the general Markov-Chebyshev Inequality rewritten using the norm instead of an integral. Let $f : \Omega \rightarrow \mathbb{R}$ be a *non-negative*, measurable function in $L_\alpha(\Omega, \mathcal{F}, \mu)$, then

$$\mu(f \geq \lambda) \leq \lambda^{-\alpha} \|f\|_\alpha^\alpha \quad \forall \lambda > 0, \alpha > 0.$$

Observation (Distribution Function as Expectation).

Let X be a random variable and F_X its distribution function. Then:

$$F_X(a) = P(X \leq a) = \int_\Omega \mathbf{1}_{X \leq a} dP = E \mathbf{1}_{X \leq a}.$$

Observation (Distribution Function as Expectation, II).

Let X be a random variable and F_X its distribution function. Then:

$$F_X(x+a) - F_X(x) = E \mathbf{1}_{x < X \leq x+a}.$$

Observation (Tightening/Relaxing Expectations).

Let X be a random variable and $\lambda \in \mathbb{R}$. Then the following holds:

$$EX \geq E(\mathbf{1}_{X \geq \lambda} X) \geq E(\mathbf{1}_{X \geq \lambda} \lambda).$$

Left-to-right can be thought of as “tightening” the constraints and thus (potentially) decreasing the area that is integrated over, right-to-left as “loosening” and thus (potentially) increasing the area that is integrated over.