Measure Theory & Probability

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Basic Notions and Notation

Example 1.1.
Simplest σ-algebra:

- $\{\emptyset, \Omega\}$, ***contained in every*** σ-algebra on Ω,
- Family of all subsets of Ω, ***containing every*** σ-algebraon Ω.

Exercise 1.1.
Let $\mathcal{F}$ be a σ-algebra. Then $A_n \in \mathcal{F}$ for every integer $n \geqslant 1 \Rightarrow \bigcap_{n=1}^{\infty} A_n \in \mathcal{F}$.

Proposition (Unknown).
Let $A, B \in \mathcal{F}$. Then $\mu(A \cap B) \leqslant \mu(A)$.
Hint: σ-additivity and $A = (A \cap B) \cup (A \setminus B)$.

Expectation Integrals

Proposition (Unknown).
Let $A, B \subseteq \Omega$. Then the following equalities hold:

- $\mathbf{1}_{A^C} = 1 - \mathbf{1}_A$,
- $\mathbf{1}_{A \cap B} = \mathbf{1}_A \mathbf{1}_B$.
- $\mathbf{1}_{A \cup B} = \mathbf{1}_A + \mathbf{1}_B - \mathbf{1}_{A \cap B}$.

Lemma 3.3.
Let X be a ***non-negative*** random variable. Then there exists a sequence of ***non-negative, simple*** random variables X_n converging to X for every $\omega \in \Omega$.
Hint: $h_n(x) = \min\{\lfloor 2^n x \rfloor / 2^n, n\}$ is non-negative, simple and increasing, approaching x. Consider $X_n \coloneqq h(X) \to X$.

Lemma (Simple Function Integral Properties).
Let $f, g : \Omega \to \overline{\mathbb{R}}$ be a ***non-negative***, simple functions and $a, b \geqslant 0$. Then the following holds:

- $\int_\Omega f \, d\mu \geqslant 0$,
- $\int_\Omega (af + bg) \, d\mu = a \int_\Omega f + b \int_\Omega g \, d\mu$.

Corollary (Positive Integral over Set).
Let $A \subseteq \Omega$ and $f : \Omega \to \overline{\mathbb{R}}$ a ***non-negative*** measurable function. Then $\int_A f \, d\mu \geqslant 0$.

Lemma 3.3 (General).
Let $f : \Omega \to \overline{\mathbb{R}}$ be a ***non-negative***, measurable function. The there exists a sequence f_n of ***non-negative***, simple functions such that:

$$\lim_{n \to \infty} f_n = f$$

Hint: Use h_n from Lemma 3.3's hint.

Exercise 3.5.
Let $A \in \mathcal{F}$ s.t. $\mu(A) = 0$. Then for ***any*** measurable function $f : \Omega \to \overline{\mathbb{R}}$:

$$\int_A f \, d\mu = 0.$$

Exercise 3.6.
Let $f : \Omega \to \mathbb{R}$ be a measurable function, then:

(i) For any $c \in \mathbb{R}$ and $A \in \mathcal{F}$:
$$\int_A cf \, d\mu = c \int_A f \, d\mu,$$
provided the integral exists.

(ii) For any $A, B \in \mathcal{F}$, such that $A \cap B = \emptyset$:
$$\int_{A \cup B} f \, d\mu = \int_A f \, d\mu + \int_B f \, d\mu,$$
provided the left-hand or right-hand side is well-defined.

Theorem 3.8 (Monotone Convergence).
Let $(f_n)_{n=1}^{\infty}$ be increasing sequence of non-negative, measurable functions $f_n : \Omega \to \overline{\mathbb{R}}$, converging to some f. Then:

$$\int_\Omega \lim_{n \to \infty} f_n \, d\mu = \lim_{n \to \infty} \int_\Omega f_n \, d\mu$$

Proposition 3.18 (Markov-Chebyshev's Inequality).
Let X be a ***non-negative*** R.V., then

$$P(X \geqslant \lambda) \leqslant \lambda^{-\alpha} E(X^\alpha) \quad \forall \lambda > 0, \alpha > 0.$$

Remark 3.3.
Let $(\Omega, \mathcal{F}, \mu)$ be measure space, $f : \Omega \to \overline{\mathbb{R}}$ ***non-negative*** $\mathcal{F}$-measurable, then

$$\mu(f \geqslant \lambda) \leqslant \lambda^{-\alpha} \int_\Omega f^\alpha \, d\mu \quad \forall \lambda > 0, \alpha > 0.$$

Proposition (Restricted Expectation).
Let X be a random variable and $A \in \mathcal{F}$, then:

$$E(X \mathbf{1}_A) = \int_A X \, dP.$$

Lemma 4.4 (Borel-Cantelli Lemma).
Let $(A)_{n=1}^{\infty}$ be a sequence of sets $A_n \in \mathcal{F}$ such that $\sum_{n=1}^{\infty} \mu(A_n) < \infty$, i.e. the series of measures of A_n converges. Then for:

$$A \coloneqq \limsup_{n \to \infty} A_n \coloneqq \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k,$$

we have $\mu(A) = 0$.
Hint: Define $B_n \coloneqq \bigcup_{k=n}^{\infty} A_k$, then $(B_n)_{n=1}^{\infty}$ is decreasing and so $\bigcap_{n=1}^{\infty} B_n = \lim_{n \to \infty} B_n$ and realize that $\sum_{n=1}^{\infty} \mu(A_n) < \infty \Rightarrow$ tail sums $\sum_{k=n}^{\infty} \mu(A_k) \to 0$ as $n \to \infty$.

Definitions

In the following, Ω is a set, $\mathcal{F}$ a σ-algebra on Ω. If used, then μ is a measure. Otherwise, the measure is the probability measure P.

Definition 1.1.
Let $\mathcal{F}$ be a family of subsets of set Ω. $\mathcal{F}$ is called a ***σ-algebra*** if:

- ***Closed Under Complement***: $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$,
- ***Closed Under Arbitrary Union***: $A_n \in \mathcal{F}$ for integer $n \geqslant 1$ $\Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$,
- ***Contains Entire Set***: $\Omega \in \mathcal{F}$

Definition 1.2. Let $\mathcal{C}$ be a family of subsets of Ω. There exists a σ-algebra which contains $\mathcal{C}$ ***and*** which is contained in every σ-algebra that contains $\mathcal{C}$ (take intersection of all σ-algebras. Such σ-algebra is ***unique*** and called ***smallest σ-algebra containing*** $\mathcal{C}$ or ***σ-algebra generated by*** $\mathcal{C}$, denoted by $\sigma(\mathcal{C})$. Simplest example, let $A \subseteq \Omega$:

$$\sigma(A) = \{\emptyset, A, A^c, \Omega\}.$$

Definition 2.1.1.
Let $A \subseteq \Omega$ and $\mathbf{1}_A$ be defined as follows:

$$\mathbf{1}_A(\omega) = \begin{cases} 1, & \omega \in A \\ 0, & \omega \notin A \end{cases}.$$

Then $\mathbf{1}_A$ is a R.V. and called the ***indicator (function) of (events) A***.

Definition (Indicator Integral).
Let $A \subseteq \Omega$, then:

$$\int_\Omega \mathbf{1}_A \, d\mu = \mu(A).$$

Definition (Simple Function).
Let $f : \Omega \to \mathbb{R}$ be a ***simple function***, then f takes finitely many values. Formally, if I is a finite index set, $(A_i)_{i \in I}$ a family of ***disjoint*** subsets of Ω and $(c_i)_{i \in I}$ a family of real numbers, then:

$$f(\omega) = \sum_{i \in I} c_i \mathbf{1}_{A_i}(\omega).$$

Definition (Lebesgue Integral for Expectation).
Let X be a random variable. Then we write:

$$EX = \int_\Omega X \, dP.$$

Definition (Non-negative, Measurable Lebesgue Integral).
Let $f : \Omega \to \overline{\mathbb{R}}$ be a ***non-negative***, measurable function and $(f_n)_{n=1}^{\infty}$ a sequence of ***non-negative, simple*** functions sucht that $\lim_{n \to \infty} f_n = f$. Then

$$\int_\Omega f \, d\mu = \lim_{n \to \infty} f_n \, d\mu.$$

Definition (Lebesgue Integral).
Let $f : \Omega \to \overline{\mathbb{R}}$ be a measurable function. The ***Lebesgue Integral*** of f is defined as:

$$\int_\Omega f \, d\mu = \int_\Omega f^+ \, d\mu - \int_\Omega f^- \, d\mu,$$

where $f^+ = \max\{f, 0\}$ and $f^- = \max\{-f, 0\}$, if at least one of the integrals on the right-hand side is finite. If both are infinite, then we say that the Lebesgue Integral of f does not exist.

Definition (Unknown).
Let $A \in \mathcal{F}$ and $f : \Omega \to \overline{\mathbb{R}}$ is a measurable function, then we define:

$$\int_A f \, d\mu = \int_\Omega \mathbf{1}_A f \, d\mu,$$

when the integral of $\mathbf{1}_A f$ w.r.t μ exists.

Definition (Unknown).
Let X be a random variable. Then X has ***finite second moment*** if $EX^2 < \infty$.