Measure Theory & Probability

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Basic Notions and Notation

Example 1.1.
Simplest σ-algebra:

- $\{\emptyset, \Omega\}$, ***contained in every*** σ-algebra on Ω,
- Family of all subsets of Ω, ***containing every*** σ-algebraon Ω.

Exercise 1.1.
Let $\mathcal{F}$ be a σ-algebra. Then $A_n \in \mathcal{F}$ for every integer $n \geqslant 1 \Rightarrow \bigcap_{n=1}^{\infty} A_n \in \mathcal{F}$.

Proposition 1.2.
Let P be a probability measure on σ-algebra $\mathcal{F}$. Then the following statements hold:

(i) $A, B \in \mathcal{F}$ s.t. $A \subseteq B \Rightarrow P(A) \leqslant P(B)$;

(ii) For ***increasing*** sequence $(A_n)_{n=1}^{\infty}$ we have
$$\lim_{n\to\infty} P(A_n) = P\left(\bigcup_{n=1}^{\infty} A_n\right);$$

(iii) For ***decreasing*** sequence $(A_n)_{n=1}^{\infty}$ we have
$$\lim_{n\to\infty} P(A_n) = P\left(\bigcap_{n=1}^{\infty} A_n\right).$$

Proposition 1.2 (General).
Let μ be a measure on σ-algebra $\mathcal{F}$. Then the following statements hold:

(i) $A, B \in \mathcal{F}$ s.t. $A \subseteq B \Rightarrow \mu(A) \leqslant \mu(B)$;

(ii) For ***increasing*** sequence $(A_n)_{n=1}^{\infty}$ we have
$$\lim_{n\to\infty} \mu(A_n) = \mu\left(\bigcup_{n=1}^{\infty} A_n\right);$$

(iii) For ***decreasing*** sequence $(A_n)_{n=1}^{\infty}$ we have
$$\lim_{n\to\infty} \mu(A_n) = \mu\left(\bigcap_{n=1}^{\infty} A_n\right).$$

Proposition (Bounding Intersections).
Let $A, B \in \mathcal{F}$. Then $\mu(A \cap B) \leqslant \mu(A)$.
Hint: σ-additivity and $A = (A \cap B) \cup (A \setminus B)$.

Proposition (Measure of Set Difference, I).
Let $A, B \in \mathcal{F}$, then $\mu(A \setminus B) = \mu(A) - \mu(A \cap B)$.

Proposition (Measure of Set Difference, II).
Let $A, B \in \mathcal{F}$ and $B \subseteq A$, then $\mu(A \setminus B) = \mu(A) - \mu(B)$.

Proposition (Complement of Limit Inferior/Superior).
Let $(A_n)_{n=1}^{\infty}$ be a sequence of sets in $\mathcal{F}$, then:

(i)
$$\left(\liminf_{n\to\infty} A_n\right)^C = \limsup_{n\to\infty} A_n^C$$

(ii)
$$\left(\limsup_{n\to\infty} A_n\right)^C = \liminf_{n\to\infty} A_n^C$$

Exercise Ws 2, 1 (Limit Inferior/Superior Properties).
Let $(A_n)_{n=1}^{\infty}$ be a sequence of sets in $\mathcal{F}$, then:

(i)
$$\liminf_{n\to\infty} A_n \coloneqq \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k$$
is the set of those ω that are ***in all but finitely many*** A_n, i.e. that uphold the property A_n captures for all except a finite amount of values of n.

(ii)
$$\limsup_{n\to\infty} A_n \coloneqq \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$$
is the set of those ω that are ***in infinitely many*** A_n, i.e. that uphold the property A_n captures for an infinite amount of values of n.

Proposition (Continuous Implies Borel-Measurability).
Let $f : \mathbb{R} \to \overline{\mathbb{R}}$ be a ***continuous*** function. Then f is Borel-measurable.

Proposition (Countable Sets).
Every countable subset of $\mathbb{R}$ is Borel-measurable.

Expectation Integrals

Proposition (Unknown).
Let $A, B \subseteq \Omega$. Then the following equalities hold:

- $\mathbf{1}_{A^C} = 1 - \mathbf{1}_A$,
- $\mathbf{1}_{A\cap B} = \mathbf{1}_A \mathbf{1}_B$.
- $\mathbf{1}_{A\cup B} = \mathbf{1}_A + \mathbf{1}_B - \mathbf{1}_{A\cap B}$.

Lemma 3.3.
Let X be a ***non-negative*** random variable. Then there exists a sequence of ***non-negative, simple*** random variables X_n converging to X for every $\omega \in \Omega$.
Hint: $h_n(x) = \min\{\lfloor 2^n x \rfloor / 2^n, n\}$ is non-negative, simple and increasing, approaching x. Consider $X_n \coloneqq h(X) \to X$.

Lemma (Simple Function Integral Properties).
Let $f, g : \Omega \to \overline{\mathbb{R}}$ be a ***non-negative***, simple functions and $a, b \geqslant 0$. Then the following holds:

- $\int_\Omega f \, d\mu \geqslant 0$,
- $\int_\Omega (af + bg) \, d\mu = a \int_\Omega f + b \int_\Omega g \, d\mu$.

Corollary (Positive Integral over Set).
Let $A \subseteq \Omega$ and $f : \Omega \to \overline{\mathbb{R}}$ a ***non-negative*** measurable function. Then $\int_A f \, d\mu \geqslant 0$.

Lemma 3.3 (General).
Let $f : \Omega \to \overline{\mathbb{R}}$ be a ***non-negative***, measurable function. The there exists a sequence f_n of ***non-negative***, simple functions such that:
$$\lim_{n\to\infty} f_n = f$$
Hint: Use h_n from Lemma 3.3's hint.

Exercise 3.5.
Let $A \in \mathcal{F}$ s.t. $\mu(A) = 0$. Then for ***any*** measurable function $f : \Omega \to \overline{\mathbb{R}}$:
$$\int_A f \, d\mu = 0.$$

Exercise 3.6.
Let $f : \Omega \to \mathbb{R}$ be a measurable function, then:

(i) For any $c \in \mathbb{R}$ and $A \in \mathcal{F}$:
$$\int_A cf \, d\mu = c \int_A f \, d\mu,$$
provided the integral exists.

(ii) For any $A, B \in \mathcal{F}$, such that $A \cap B = \emptyset$:
$$\int_{A\cup B} f \, d\mu = \int_A f \, d\mu + \int_B f \, d\mu,$$
provided the left-hand or right-hand side is well-defined.

Theorem 3.8 (Monotone Convergence).
Let $(f_n)_{n=1}^{\infty}$ be increasing sequence of non-negative, measurable functions $f_n : \Omega \to \overline{\mathbb{R}}$, converging to some f. Then:
$$\int_\Omega \lim_{n\to\infty} f_n \, d\mu = \lim_{n\to\infty} \int_\Omega f_n \, d\mu$$

Theorem 3.14 (Lebesgue Integral as Riemann Integral).
Let $f : \mathbb{R} \to \mathbb{R}$ be a Borel-function such that:

(i) the Riemann integral $\int_{-\infty}^{\infty} f(x) \, dx$ exists and

(ii) the Riemann integral $\int_{-\infty}^{\infty} |f(x)| \, dx < \infty$, i.e. is finite,

then the Lebesgue integral $\int_\mathbb{R} f(x)\lambda(dx)$ ***exists*** and
$$\int_\mathbb{R} f(x)\lambda(dx) = \int_{-\infty}^{\infty} f(x) \, dx,$$
i.e. the Lebesgue integral is equal to the Riemann integral.

Exercise 3.15.
Let ν be a measure that is absolutely continuous with respect to measure μ and density g, then $\mu(g < 0) = 0$. Moreover, ν is a probability measure $\Leftrightarrow g \geqslant 0$ μ-a.e. and $\int_\Omega g \, d\mu = 1$.

Proposition 3.16.
Let ν and μ be measures on σ-algebra $\mathcal{F}$ such that ν is absolutely continuous with respect to μ and density g. Then for every $\mathcal{F}$-measurable function f the following holds:
$$\int_\Omega f \, d\nu = \int_\Omega fg \, d\mu,$$
whenever one of the integrals exists.

Remark 3.3.
Let $(\Omega, \mathcal{F}, \mu)$ be measure space, $f : \Omega \to \overline{\mathbb{R}}$ ***non-negative*** $\mathcal{F}$-measurable, then
$$\mu(f \geqslant \lambda) \leqslant \lambda^{-\alpha} \int_\Omega f^\alpha \, d\mu \quad \forall \lambda > 0, \alpha > 0.$$

Lemma 3.10 (Fatou's Lemma).
Let $(f_n)_{n=1}^{\infty}$ be a sequence of ***non-negative***, measurable functions $f : \Omega \to \overline{\mathbb{R}}$, then
$$\int_\Omega \liminf_{n\to\infty} f_n \, d\mu \leqslant \liminf_{n\to\infty} \int_\Omega f_n \, d\mu.$$

Corollary 3.11 (Fatou's Lemma Extension).
Let $(f_n)_{n=1}^{\infty}$ be a sequence of measurable functions $f : \Omega \to \overline{\mathbb{R}}$. Then

(i) if there exists a $g \in L_1(\Omega, \mathcal{F}, \mu)$, i.e. $\int_\Omega |g| \, d\mu < \infty$ such that $g \leqslant f_n$ for all n, then:
$$\int_\Omega \liminf_{n\to\infty} f_n \, d\mu \leqslant \liminf_{n\to\infty} \int_\Omega f_n \, d\mu.$$

(ii) if there exists a $g \in L_1(\Omega, \mathcal{F}, \mu)$, i.e. $\int_\Omega |g| \, d\mu < \infty$ such that $g \geqslant f_n$, then:
$$\int_\Omega \limsup_{n\to\infty} f_n \, d\mu \geqslant \limsup_{n\to\infty} \int_\Omega f_n \, d\mu.$$

Theorem 3.12 (Lebegue's Theorem on Dominated Convergence).
Let $(f_n)_{n=1}^{\infty}$ be a sequence of Borel functions $f_n : \Omega \to \overline{\mathbb{R}}$ converging to some $f : \Omega \to \overline{\mathbb{R}}$. Assume there exists a (non-negative) Borel functions g such that $|f_n| \leqslant g$ for any $n \geqslant 1$ and $\int_\Omega g \, d\mu < \infty$. Then the following two statements hold:

(i)
$$\int_\Omega |f| \, d\mu < \infty,$$

(ii)
$$\int_\Omega f \, d\mu = \lim_{n\to\infty} \int_\Omega f \, d\mu.$$

Proposition (Restricted Expectation).

Let X be a random variable and $A \in \mathcal{F}$, then:

$$E(X\mathbf{1}_A) = \int_A X\, dP.$$

Theorem 3.17 (Integration Over The Sample Space).

Let $f : \mathbb{R} \to \mathbb{R}$ be a Borel function and X a ***finite*** random variable, then:

$$Ef(X) = \int_{\mathbb{R}} f Q_X(dx).$$

Proposition 3.18 (Markov-Chebyshev's Inequality).

Let X be a ***non-negative*** R.V., then

$$P(X \geqslant \lambda) \leqslant \lambda^{-\alpha} E(X^\alpha) \quad \forall \lambda > 0, \alpha > 0.$$

Hint: $E(X^\alpha) \geqslant E(\mathbf{1}_{X \geqslant \lambda} X^\alpha) \geqslant E(\mathbf{1}_{X \geqslant \lambda} \lambda^\alpha) = \lambda^\alpha E(\mathbf{1}_{X \geqslant \lambda}) = \lambda^\alpha P(X \geqslant \lambda)$.

Proposition 3.18 (Markov-Chebyshev's Inequality (General)).

Let $f : \Omega \to \overline{\mathbb{R}}$ be a ***non-negative***, measurable function, then

$$\mu(f \geqslant \lambda) \leqslant \lambda^{-\alpha} \int_\Omega f^\alpha\, d\mu \quad \forall \lambda > 0, \alpha > 0.$$

L_p Spaces

Theorem (Hölder's Inequality).

Let $f, g : \Omega \to \overline{\mathbb{R}}$ be measurable functions, then

$$\int_\Omega |fg|\, d\mu \leqslant \|f\|_p \|g\|_q \quad \text{for } p \geqslant 1,$$

where

$$q := \begin{cases} \frac{p}{p-1} & p > 1, \\ \infty & p = 1 \end{cases}.$$

Theorem (Hölder's Inequality for Expectations).

Let X, Y be random variables, then

$$E|XY| \leqslant (E|X|^p)^{\frac{1}{p}} (E|Y|^q)^{\frac{1}{q}}$$

where

$$q := \begin{cases} \frac{p}{p-1} & p > 1, \\ \infty & p = 1 \end{cases}.$$

Proposition (Finite Second Momenta Implication).

Let X, Y be random variables with finite second momenta. Then $E|XY| < \infty$.

Hint: Use Hölder's Inequality with $p = 2$ on $E|XY| = \int_\Omega |XY|\, dP$.

Lemma 4.4 (Borel-Cantelli Lemma).

Let $(A)_{n=1}^\infty$ be a sequence of sets $A_n \in \mathcal{F}$ such that $\sum_{n=1}^\infty \mu(A_n) < \infty$, i.e. the series of measures of A_n converges. Then for:

$$A := \limsup_{n \to \infty} A_n := \bigcap_{n=1}^\infty \bigcup_{k=n}^\infty A_k,$$

we have $\mu(A) = 0$.

Hint: Define $B_n := \bigcup_{k=n}^\infty A_k$, then $(B_n)_{n=1}^\infty$ is decreasing and so $\bigcap_{n=1}^\infty B_n = \lim_{n\to\infty} B_n$ and realize that $\sum_{n=1}^\infty \mu(A_n) < \infty \Rightarrow$ tail sums $\sum_{k=n}^\infty \mu(A_k) \to 0$ as $n \to \infty$.

Convergence of Measurable Functions

Exercise 5.1.

Let $(f_n)_{n=1}^\infty$ be a sequence of $\mathcal{F}$-measurable functions $f_n : \Omega \to \mathbb{R}$. Then the set A of those $\omega \in \Omega$ such that $\lim_{n\to\infty} f_n(\omega)$ converges to some (finite) number belongs to $\mathcal{F}$.

Exercise 5.2 (Almost Finite, Converging Sequence is Bounded).

Assume that $\mu(\Omega) < \infty$. Let $(f_n)_{n=1}^\infty$ be ***μ-a.e. finite***, converging in measure to μ to some $f : \Omega \to \mathbb{R}$. Then the sequence of f_n is ***bounded in measure μ, uniformly in*** n, i.e.:

$$\lim_{K\to\infty} \sup_{n \geqslant 1} \mu(|f_n| \geqslant K) = 0.$$

Hint: f_n ***μ-a.e. finite and*** $\mu(\Omega) < \infty \Rightarrow f_n$ bounded in measure (not necessarily uniformly), so

$$\lim_{K\to\infty} \sup_{n \geqslant 1} \mu(|f_n| \geqslant K) =$$

$$\lim_{K\to\infty} \limsup_{n\to\infty} \mu(|f_n| \geqslant K).$$

Then use observation of splitting measures of inequalities.

Exercise 5.3 (Product of Bounded & Zero Convergent is Zero Convergent).

Let $(f_n)_{n=1}^\infty$ and $(g_n)_{n=1}^\infty$ be sequences of μ-a.e. finite measurable functions such that the f_n are bounded in measure μ, uniformly in n and $g_n \to 0$ in measure μ, as $n \to \infty$. Then $f_n g_n \to 0$ in measure μ, as $n \to \infty$.

Exercise Ws 3, 1.

Let $\mu - \lim f_n = f$, then there exists a subsequence $(f_{n_k})_{k=1}^\infty$ such that $(n_k)_{k=1}^\infty$ is increasing and $f_{n_k} \to f$ (μ-a.e.).

Hint: Borel-Cantelli with $A_k = \{|f_{n_k} - f| \geqslant 1/k\}$ s.t. $\mu(A_k) \leqslant 1/k^2$.

Theorem 5.4 (Measure Convergence Has Almost Everywhere Converging Subsequence).

Let $(f_n)_{n=1}^\infty$ be a sequence of functions converging in measure μ to some μ-a.e. finite function f. Then there exists a (strictly) increasing sequence $(n_k)_{k=1}^\infty$ of positive integers such that $\lim_{k\to\infty} f_{n_k} = f$ μ-almost everywhere.

Exercise 5.5.

Convergence in measure μ does not imple convergence μ-almost everywhere.

Hint: $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$ with $f_n = \mathbf{1}_{[k/2^m, (k+1)/2^m]}$ where $k = 0, 1, \ldots, 2^m - 1$ and $m = 0, 1, \ldots$ such that $n = 2^m + k$.

Exercise Ws 3, 2 (Convergence Implication).

Let $\mu(\Omega) < \infty$. Then $\lim_{n\to\infty} f_n = f$ (μ-a.e.) $\Rightarrow \mu - \lim_{n\to\infty} f_n = f$.

Exercise Ws 3, 3 (Relaxed Domnitated Convergence).

Lebegue's Theorem on Dominated convergence holds under the following, relaxed conditions:

(i) $\lim_{n\to\infty} f_n = f$ ***μ-a.e.***, $|f_n| \leqslant g|$ μ-a.e. and $g \in L_1(\Omega, \mathcal{F}, \mu)$, i.e. $\int_\Omega |g|\, d\mu < \infty$; and

(ii) $\mu - \lim_{n\to\infty} f_n = f$, $|f_n| \leqslant g|$ μ-a.e. and $g \in L_1(\Omega, \mathcal{F}, \mu)$, i.e. $\int_\Omega |g|\, d\mu < \infty$.

Independence of Events and Random Variables

Theorem 6.3 (Monotone Class Theorem).

Let Π be a π-system contained in a λ-system Λ. Then $\sigma(\Pi)$ is contained in Λ.

Proposition 6.4 (Extending π-System Independence).

Let C_1 and C_2 be two ***independent*** π-systems, i.e.

$$P(A \cap B) = P(A)P(B) \quad \forall A \in C_1, B \in C_2,$$

then the σ-algebras $\sigma(C_1)$ and $\sigma(C_2)$ are also independent.

Theorem 6.7 (Fubini-Tonelli Theorem).

Let $(\Omega_i, \mathcal{F}_i, \mu_i)$, for $i = 1, 2$, be measure spaces and $(\Omega, \mathcal{F}, \mu)$ be the product measure space of the two, i.e. $\Omega = \Omega_1 \times \Omega_2$, $\mathcal{F} = \mathcal{F}_i \otimes \mathcal{F}_2$ and $\mu = \mu_1 \otimes \mu_2$. Let $f : \Omega \to \overline{\mathbb{R}}$ be a ***non-negative*** $\mathcal{F}$-measurable function. If μ_i, for $i = 1, 2$, are ***finite measures*** on Ω_i, for $i = 1, 2$, respectively, then the following iterated integrals are well-defined and:

$$\int_{\Omega_1 \times \Omega_2} f\, d\mu_1 \otimes \mu_2 = \int_{\Omega_1} \int_{\Omega_2} f\, d\mu_2 d\mu_1 =$$

$$= \int_{\Omega_2} \int_{\Omega_1} f\, d\mu_1 d\mu_2.$$

Furthermore, this statement holds for $\mathcal{F}$-measurable functions if:

$$\int_{\Omega_1 \times \Omega_2} |f|\, d\mu_1 \otimes \mu_2 < \infty.$$

Lemma 6.9 (Borel-Cantelli (Full)).

Let $(A_n)_{n=1}^\infty$ be a sequence of sets and set

$$A := \limsup_{n \to \infty} A_n := \bigcap_{n=1}^\infty \bigcup_{k=n}^\infty A_k,$$

then the following statements holds:

(i) If $\sum_{n=1}^\infty \mu(A_n) < \infty$, then $\mu(A) = 0$.

(ii) If all A_n are ***jointly independent*** and $\sum_{n=1}^\infty P(A_n) = \infty$, then $P(A) = 1$.

Hint: (i) provided in general case. (ii) Prove $P((\limsup_{n\to\infty} A_n)^C) = 1$, define $B_n = \bigcap_{k=n}^\infty A_k^C$ and show that for a given $P(B_n) = P(\lim_{m\to\infty} \bigcap_{k=n}^m A_k) = 0$ using independence and observation that $1 - P(A) \leqslant e^{-P(A)}$. Finally, use ***sub-$\sigma$***-additivity for $P(\bigcup_{n=1}^\infty B_n)$. ***Do not*** attempt to argue through increasing sequences.

Exercise (Pulling Sum Through Variance).

Let $(X_i)_{i=1}^\infty$ be a sequence of ***pairwise independent*** random variables. Assume that $EX_i^2 < \infty$ for $i = 1, 2, \ldots, n$, then

$$\operatorname{Var}\left(\sum_{i=1}^n X_i\right) = \sum_{i=1}^n \operatorname{Var}(X_i).$$

Conditional Expectation

Exercise 8.1.

Let $\mathcal{G} := \{\emptyset, \Omega\}$, i.e. the trivial σ-algebra. Then if random variable Y is $\mathcal{G}$-measurable, then Y is constant.

Lemma 8.2.

Let Z be a $\mathcal{G}$-measurable random variable such that:

$$\int_A Z\, dP \geqslant 0 \iff E(\mathbf{1}_A Z) \geqslant 0,$$

for any $A \in \mathcal{G}$, then $Z \geqslant 0$ (a.s.).

Theorem 8.6 (Properties of Conditional Expectations).

Let X be a random variable and $\mathcal{G} \subset \mathcal{F}$ be a σ-algebra. Then the following properties hold (under the given conditions):

(i) "Adding/Dropping Conditional Expectation":
$$EX = E(E(X|\mathcal{G}));$$
(ii) "Tower Rule": Let $\mathcal{H} \subset \mathcal{F}$ be a σ-algebra, such that $\mathcal{H}$ ***contains*** $\mathcal{G}$, then:
$$E(E(X|\mathcal{H})|\mathcal{G}) = E(X|\mathcal{G});$$
(iii) "Pulling/Pushing Random Variables Through": Let Y be a random variable, such that Y is $\mathcal{G}$-measurable ***and*** $E|XY| < \infty$, then:
$$E(XY|\mathcal{G}) = YE(X|\mathcal{G});$$
(iv) "Independence of Conditional": Let X and $\mathcal{G}$ be independent, i.e. $\sigma(X)$ and $\mathcal{G}$ are independent, then:
$$E(X|\mathcal{G}) = EX.$$

Definitions

Basic Notions and Notation

In the following, Ω is a set, $\mathcal{F}$ a σ-algebra on Ω. If used, then μ is a measure. Otherwise, the measure is the probability measure P.

Definition 1.1.
Let $\mathcal{F}$ be a family of subsets of set Ω. $\mathcal{F}$ is called a ***σ-algebra*** if:

- ***Closed Under Complement***: $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$,
- ***Closed Under Arbitrary Union***: $A_n \in \mathcal{F}$ for integer $n \geqslant 1$ $\Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$,
- ***Contains Entire Set***: $\Omega \in \mathcal{F}$

Definition 1.2. Let $\mathcal{C}$ be a family of subsets of Ω. There exists a σ-algebra which contains $\mathcal{C}$ ***and*** which is contained in every σ-algebra that contains $\mathcal{C}$ (take intersection of all σ-algebras. Such σ-algebra is ***unique*** and called ***smallest σ-algebra containing*** $\mathcal{C}$ or ***σ-algebra generated by*** $\mathcal{C}$, denoted by $\sigma(\mathcal{C})$. Simplest example, let $A \subseteq \Omega$:
$$\sigma(A) = \{\emptyset, A, A^c, \Omega\}.$$

Definition (Finite Measure Space).
Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. If $\mu(\Omega) < \infty$, then we call the measure space ***finite***.

Random Variables

Definition 2.1.1.
Let $A \subseteq \Omega$ and $\mathbf{1}_A$ be defined as follows:
$$\mathbf{1}_A(\omega) = \begin{cases} 1, & \omega \in A \\ 0, & \omega \notin A \end{cases}.$$
Then $\mathbf{1}_A$ is a R.V. and called the ***indicator (function) of (events) A***.

Definition 2.3 (Distribution Function).
Let X be a random variable. Then the function
$$\begin{aligned} F_X(x) &= P(X \leqslant x) = \\ &= P(X \in (-\infty, x]) = Q_X((-\infty, x]), \end{aligned}$$
for $x \in \mathbb{R}$ is called the ***distribution function*** of X.

Expectation Integrals

Definition (Indicator Integral).
Let $A \subseteq \Omega$, then:
$$\int_\Omega \mathbf{1}_A \, d\mu = \mu(A).$$

Definition (Simple Function).
Let $f : \Omega \to \mathbb{R}$ be a ***simple function***, then f takes finitely many values. Formally, if I is a finite index set, $(A_i)_{i \in I}$ a famility of ***disjoint*** subsets of Ω and $(c_i)_{i\in I}$ a family of real numbers, then:
$$f(\omega) = \sum_{i \in I} c_i \mathbf{1}_{A_i}(\omega).$$

Definition (Lebesgue Integral for Expectation).
Let X be a random variable. Then we write:
$$EX = \int_\Omega X \, dP.$$

Definition (Non-negative, Measurable Lebesgue Integral).
Let $f : \Omega \to \overline{\mathbb{R}}$ be a ***non-negative***, measurable function and $(f_n)_{n=1}^{\infty}$ a sequence of ***non-negative, simple*** functions sucht that $\lim_{n\to\infty} f_n = f$. Then
$$\int_\Omega f \, d\mu = \lim_{n\to\infty} f_n \, d\mu.$$

Definition (Lebesgue Integral).
Let $f : \Omega \to \overline{\mathbb{R}}$ be a measurable function. The ***Lebesgue Integral*** of f is defined as:
$$\int_\Omega f \, d\mu = \int_\Omega f^+ \, d\mu - \int_\Omega f^- \, d\mu,$$
where $f^+ = \max\{f, 0\}$ and $f^- = \max\{-f, 0\}$, if at least one of the integrals on the right-hand side is finite. If both are infinite, then we say that the Lebesgue Integral of f does not exist.

Definition (Restricted Integration).
Let $A \in \mathcal{F}$ and $f : \Omega \to \overline{\mathbb{R}}$ is a measurable function, then we define:
$$\int_A f \, d\mu = \int_\Omega \mathbf{1}_A f \, d\mu,$$
when the integral of $\mathbf{1}_A f$ w.r.t μ exists.

Definition 3.7 (Absolute Continuity).
Let μ and ν be measures on σ-algebra $\mathcal{F}$ such that for some $\mathcal{F}$-measureable $g : \Omega \to \mathbb{R}$:
$$\nu(A) = \int_\Omega \mathbf{1}_A g \, d\mu = \int_A g\mu(dx),$$
for all $A \in \mathcal{F}$. Then ν is called ***absolutely continuous*** with respect to μ and g is called the ***density*** or ***Radon-Nikodym derivative*** (Notation: $g = \frac{d\nu}{d\mu}$).

Convergence of Measurable Functions

Definition (μ-Almost Everywhere Finite).
Let $f : \Omega \to \overline{\mathbb{R}}$ be $\mathcal{F}$-measurable, then f is said to be ***μ-almost everywhere*** (μ-a.e.) finite if $\mu(|f| = \infty) = 0$.

Definition (Almost Surely Finite).
Let $f : \Omega \to \overline{\mathbb{R}}$ be $\mathcal{F}$-measurable, then f is said to be ***almost surely*** (a.s.) finite if $P(|f| = \infty) = 0 \Leftrightarrow P(|f| < \infty) = 1$.

Definition 5.1 (μ-Almost Everywhere Convergence).
Let $(f_n)_{n=1}^{\infty}$ be $\mathcal{F}$-measurable functions. The f_n are said to ***converge μ-almost everywhere*** to a ***μ-a.e. finite*** $f : \Omega \to \overline{\mathbb{R}}$ as $n \to \infty$ if there exists an $A \in \mathcal{F}$ s.t. $\mu(A) = 0$ and
$$\lim_{n\to\infty} f_n(\omega) = f(\omega) \in \mathbb{R}, \quad \forall \omega \in A^C.$$
Notation: $\lim_{n\to\infty} f_n = f$ (μ-a.e.) or $f_n \to f$ (μ-a.e.).

Definition 5.1 (Almost Sure Convergence).
Let $(f_n)_{n=1}^{\infty}$ be $\mathcal{F}$-measurable functions. The f_n are said to ***converge almost surely*** to a ***a.s. finite*** $f : \Omega \to \overline{\mathbb{R}}$ as $n \to \infty$ if there exists an $A \in \mathcal{F}$ s.t. $P(A) = 0$ and
$$\lim_{n\to\infty} f_n(\omega) = f(\omega) \in \mathbb{R}, \quad \forall \omega \in A^C.$$
Notation: $\lim_{n\to\infty} f_n = f$ (a.s.) or $f_n \to f$ (a.s.).

Definition 5.2 (Convergence in Measure).
Let $(f_n)_{n=1}^{\infty}$ be $\mathcal{F}$-measurable functions. The f_n are said to ***converge in measure*** μ to a ***μ-a.e. finite*** $f : \Omega \to \overline{\mathbb{R}}$ as $n \to \infty$ if
$$\lim_{n\to\infty} \mu(|f_n - f| \geqslant \varepsilon) = 0, \quad \forall \varepsilon > 0.$$
Notation: $\mu - \lim_{n\to\infty} f_n = f$.

Definition 5.2 (Convergence in Probability).
Let $(f_n)_{n=1}^{\infty}$ be $\mathcal{F}$-measurable functions. The f_n are said to ***converge in probability*** to a ***a.s. finite*** $f : \Omega \to \overline{\mathbb{R}}$ as $n \to \infty$ if
$$\lim_{n\to\infty} P(|f_n - f| \geqslant \varepsilon) = 0, \quad \forall \varepsilon > 0.$$

Definition (Bounded in Measure).
Let $(f_n)_{n=1}^{\infty}$ be a sequence of measurable functions, then it is ***bounded in measure*** μ if
$$\lim_{K\to\infty} \mu(|f_n| \geqslant K) = 0,$$
for any $n \geqslant 1$.

Definition (Bounded Uniformly in Measure).
Let $(f_n)_{n=1}^{\infty}$ be a sequence of measurable functions, then it is ***bounded in measure μ, uniformly in*** n if
$$\lim_{K\to\infty} \sup_{n\geqslant 1} \mu(|f_n| \geqslant K) = 0.$$

Definition (Finite Second Moment).
Let X be a random variable. Then X has ***finite second moment*** if $EX^2 < \infty$.

Independence of Events and Random Variables

Definition 6.5 (λ-system).
Let Λ be a family o subsets of Ω. Then Λ is a λ-system, if it satisfies all of the following properties:

(i) (Contains whole set) $\Omega \in \Lambda$;
(ii) (Closed under Subset Set Subtraction) if $A, B \in \Lambda$, such that $B \subset A$, then $A \setminus B \in \Lambda$;
(iii) (Closed under Disjoint Union) if $(A_n)_{n=1}^{\infty}$ is a ***pairwise disjoint*** sequence, i.e. $A_i \cap A_j = \emptyset$ for $i \neq j$, of subsets, such that $A_i \in \Lambda$ for $i = 1, 2, \ldots$, then $\bigcup_{n=1}^{\infty} \in \Lambda$.

Definition (π-system).
Let Π be a family of subsets of Ω. Then Π is a π-system, if it is closed under finite intersections, i.e. $A, B \in \Pi \Rightarrow A \cap B \in \Pi$.

Definition Ws 5, 1 (σ-Finite Measure).
Let μ be a measure, then μ is called ***σ-finite*** if there exists an increasing sequence $(\Omega_n)_{n=1}^{\infty}$ in $\mathcal{F}$, such that $\mu(\Omega_n) < \infty$ for all $n \geqslant 1$ and $\bigcap_{n=1}^{\infty} \Omega_n = \Omega$.

Conditional Expectation

Definition 8.1 (Sub-σ-Algebra Measurable). Let Y be a random variable and $\mathcal{G} \subset \mathcal{F}$ be a σ-algebra. Then Y is $\mathcal{G}$-measurable if $Y^{-1}(F) \in \mathcal{G}$ for any $F \in \mathcal{B}(\mathbb{R})$.

Definition 8.2 (Conditional Expectation). Let X, Y be random variables such that $E|X| < \infty$ and $\mathcal{G} \subset \mathcal{F}$ be a σ-algebra. Let Y satisfy the following properties:

(i) Y is $\mathcal{G}$-measurable and

(ii) for any $A \in \mathcal{G}$:

$$\int_A Y\, dP = \int_A X\, dP \iff E(\mathbf{1}_A Y) = E(\mathbf{1}_A X),$$

then Y is called the ***conditional expectation*** with respect of $\mathcal{G}$ of X and we write $Y = E(X|\mathcal{G})$.

Useful Observations

Observation (Bounding Measures).
The following inequalities to bound measures are ***always*** applicable, for ***any*** sets $A, B, C \in \mathcal{F}$:

1. "Dropping a set in an intersection gives an upper bound" ⇔ "Relaxing constraints":
$$\mu(A \cap B) \leqslant \mu(A).$$
2. "Dropping a set in a union gives an lower bound":
$$\mu(A \cup B) \geqslant \mu(A).$$
3. "Adding a set in a union gives an upper bound" ⇔ "Adding constraints":
$$\mu(A \cup B) \leqslant \mu(A \cup B \cup C).$$
4. "Intersections are less than a set and a set is less than a union":
$$\mu(A \cap B) \leqslant \mu(A) \leqslant \mu(A \cup B).$$

Observation (Adding Ω by Intersection).
If you would like to introduce a property to an existing set A to make it easier to work with, for instance easier to bound, you can add an intersection with Ω:

$$\mu(A) = \mu(\Omega \cap A).$$

Then Ω can be split into the set B that represents the property and B^C that does not have the property, where $\Omega = B \cup B^C$. Then:

$$\mu(A) = \mu(\Omega \cap A) = \mu((B \cup B^C) \cap A) =$$
$$\mu((B \cup B^C) \cap A) = \mu((B \cap A) \cup (B^C \cap A)).$$

Using σ-additivity, we get:

$$\mu(A) = \mu(B \cap A) + \mu(B^C \cap A).$$

Then by the observation on bounding measures, this can be made into an inequality:

$$\mu(A) = \mu(B \cap A) + \mu(B^C \cap A)$$
$$\leqslant \mu(B \cap A) + \mu(B^C).$$

Observation (Increasing Sequence of Sets).
For an ***increasing*** sequence of sets $(A_n)_{n=1}^{\infty}$ we can define:

$$\lim_{n\to\infty} A_n := \bigcup_{n=1}^{\infty} A_n$$

Observation (Decreasing Sequence of Sets).
For an ***decreasing*** sequence of sets $(A_n)_{n=1}^{\infty}$ we can define:

$$\lim_{n\to\infty} A_n := \bigcap_{n=1}^{\infty} A_n$$

Observation (μ-Almost Everywhere Finite, I).
If $f : \Omega \to \mathbb{R}$ is μ-a. e. finite, then note that if $A_n := \{|f| \geqslant n\}$, then $(A_n)_{n=1}^{\infty}$ is a decreasing sequence and so:

$$\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \mu\left(\lim_{n\to\infty} A_n\right) = \mu(|f| = \infty)$$
$$= 0.$$

Observation (μ-Almost Everywhere Finite, II).
If $f : \Omega \to \mathbb{R}$ is μ-a. e. finite, then observe

$$\mu(|f| = \infty) = \lim_{R\to\infty} \mu(|f| \geqslant R) = 0.$$

Observation (Almost Surely Finite, II).
If $f : \Omega \to \mathbb{R}$ is a.s. finite, then observe

$$P(|f| = \infty) = \lim_{R\to\infty} P(|f| \geqslant R) = 0.$$
$$\iff P(|f| < \infty) = \lim_{R\to\infty} P(|f| < R) = 1.$$

Observation (Almost Surely Finite).
If $f : \Omega \to \mathbb{R}$ is a. s. finite, then note that if $A_n := \{|f| \geqslant n\}$, then $(A_n)_{n=1}^{\infty}$ is a decreasing sequence and so:

$$P\left(\bigcap_{n=1}^{\infty} A_n\right) = P\left(\lim_{n\to\infty} A_n\right) = P(|f| = \infty)$$
$$= 0.$$

Observation (μ-Almost Everywhere Convergence I).
If $f_n \to f$ μ-a.e., then $\mu(f_n \not\to f) = 0$.

Observation (μ-Almost Everywhere Convergence II).
If $A \in \mathcal{F}$ is a set such that $\mu(A) = 0$ and

$$\lim_{n\to\infty} |f_n(\omega) - f(\omega)| = 0 \quad \forall \omega \in A^C,$$

then $f_n \to f$ μ-almost everywhere.

Observation (Almost Sure Convergence).
If $f_n \to f$ a.s., then $P(f_n \not\to f) = 0$ or equivalently $P(f_n \to f) = 1$.

Observation (Splitting Measures of Inequalities).
Let f, g be measurable functions and $a \in \mathbb{R}$, then observe that:

$$\mu(|f| \geqslant a) \leqslant \mu\left(|f - g| \geqslant \frac{a}{2}\right) + \mu\left(|g| \geqslant \frac{a}{2}\right)$$

Observation (Using Borel-Cantelli).
If you can define sets $(A_k)_{k=1}^{\infty}$ such that $\mu(A_k) \leqslant 1/k^2$, then you can use Borel-Cantelli as:

$$\sum_{k=1}^{\infty} \mu(A_k) \leqslant \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty.$$

In fact, the choice of $1/k^2$ is more or less arbitrary. This technique would work with any r_k s.t. $\sum_{k=1}^{\infty} r_k < \infty$ and $\mu(A_k) \leqslant r_k$. Caution: $r_k = 1/k$ does ***not*** work.

Observation (Function As Integral).
Let $f : \Omega \to \overline{\mathbb{R}}$ be a ***non-negative*** measurable function, the obvserve that

$$f(\omega) = \int_0^{f(\omega)} dx = \int_0^{\infty} \mathbf{1}_{x \leqslant f(\omega)}\, dx$$

Observation (Bounding Complement Probabilities).
Note that $1 - x \leqslant e^{-x}$. Therefore, we can bound probabilities of a product of complement events, for instance:

$$\prod_{n=1}^{\infty} P(A_n^C) = \prod_{n=1}^{\infty} [1 - P(A_n)] \leqslant$$
$$\prod_{n=1}^{\infty} e^{-P(A_n)} = e^{\sum_{n=1}^{\infty} -P(A_n)}$$

Observation (Interchanging Expectation & Infinite Sum).
Observe that if f is ***non-negative***, then:

$$E\left(\sum_{n=1}^{\infty} f(X_n)\right) = E\left(\lim_{N\to\infty} \sum_{n=1}^{N} f(X_n)\right) =$$
$$= \lim_{N\to\infty} \sum_{n=1}^{N} Ef(X_n) = \sum_{n=1}^{\infty} Ef(X_n),$$

where pulling the expectation through the sum can be done due to the Monotone Convergence Theorem, as $\sum_{n=1}^{N} f(X_n)$ is an increasing sequence of ***non-negative*** random variables.

Observation (Markov-Chebyshev's Inequality & Norm).
The following is the general Markov-Chebyshev Inequality rewritten using the norm instead of an integral. Let $f : \Omega \to \overline{\mathbb{R}}$ be a ***non-negative***, measurable function in $L_\alpha(\Omega, \mathcal{F}, \mu)$, then

$$\mu(f \geqslant \lambda) \leqslant \lambda^{-\alpha} \|f\|_\alpha^\alpha\, d\mu \quad \forall \lambda > 0, \alpha > 0.$$

Observation (Distribution Function as Expectation).
Let X be a random variable and F_X its distribution function. Then:

$$F_X(a) = P(X \leqslant a) = \int_\Omega \mathbf{1}_{X \leqslant a}\, dP = E\mathbf{1}_{X \leqslant a}.$$

Observation (Distribution Function as Expectation, II).
Let X be a random variable and F_X its distribution function. Then:

$$F_X(x + a) - F_X(x) = E\mathbf{1}_{x < X \leqslant x+a}.$$

Observation (Tightening/Relaxing Expectations).
Let X be a random variable and $\lambda \in \mathbb{R}$. Then the following holds:

$$EX \geqslant E(\mathbf{1}_{X \geqslant \lambda} X) \geqslant E(\mathbf{1}_{X \geqslant \lambda} \lambda).$$

Left-to-right can be thought of as "tightening" the constraints and thus (potentially) decreasing the area that is integrated over, right-to-left as "loosening" and thus (potentially) increasing the area that is integrated over.

Observation (Identical Distribution Giving Equal Probability).
Let $(X_n)_{n=1}^{\infty}$ be a sequence of ***independent, identically distributed*** random variables. Let A_n be an event depending on X_n, for instance $A_n := \{X_n \geqslant K\}$ for some $K \in \mathbb{R}$, then all $P(A_n)$ are equal due to X_n being identically distributed, i.e.

$$P(A_n) = p \quad \text{for } n \geqslant 1, p \in [0, 1].$$

Observation (Identical Distribution & Infinite Sum).
Let $(X_n)_{n=1}^{\infty}$ be a sequence of ***independent, identically distributed*** random variables. Let A_n be an event depending on X_n, for instance $A_n := \{X_n \geqslant K\}$ for some $K \in \mathbb{R}$, then

$$\sum_{n=1}^{\infty} P(A_n) < \infty \Rightarrow P(A_n) = 0 \text{ for } n \geqslant 1.$$