

Measure Theory & Probability

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Basic Notions and Notation

Example 1.1.

Simplest σ -algebra:

- $\{\emptyset, \Omega\}$, *contained in every* σ -algebra on Ω ,
- Family of all subsets of Ω , *containing every* σ -algebra on Ω .

Exercise 1.1.

Let \mathcal{F} be a σ -algebra. Then $A_n \in \mathcal{F}$ for every integer $n \geq 1 \Rightarrow \bigcap_{n=1}^{\infty} A_n \in \mathcal{F}$.

Proposition 1.2.

Let P be a probability measure on σ -algebra \mathcal{F} . Then the following statements hold:

- $A, B \in \mathcal{F}$ s.t. $A \subseteq B \Rightarrow P(A) \leq P(B)$;
- For *increasing* sequence $(A_n)_{n=1}^{\infty}$ we have

$$\lim_{n \rightarrow \infty} P(A_n) = P\left(\bigcup_{n=1}^{\infty} A_n\right);$$

- For *decreasing* sequence $(A_n)_{n=1}^{\infty}$ we have

$$\lim_{n \rightarrow \infty} P(A_n) = P\left(\bigcap_{n=1}^{\infty} A_n\right).$$

Proposition 1.2 (General).

Let μ be a measure on σ -algebra \mathcal{F} . Then the following statements hold:

- $A, B \in \mathcal{F}$ s.t. $A \subseteq B \Rightarrow \mu(A) \leq \mu(B)$;
- For *increasing* sequence $(A_n)_{n=1}^{\infty}$ we have

$$\lim_{n \rightarrow \infty} \mu(A_n) = \mu\left(\bigcup_{n=1}^{\infty} A_n\right);$$

- For *decreasing* sequence $(A_n)_{n=1}^{\infty}$ we have

$$\lim_{n \rightarrow \infty} \mu(A_n) = \mu\left(\bigcap_{n=1}^{\infty} A_n\right).$$

Proposition (Unknown).

Let $A, B \in \mathcal{F}$. Then $\mu(A \cap B) \leq \mu(A)$.

Hint: σ -additivity and $A = (A \cap B) \cup (A \setminus B)$.

Proposition (Measure of Set Difference, I).

Let $A, B \in \mathcal{F}$, then $\mu(A \setminus B) = \mu(A) - \mu(A \cap B)$.

Proposition (Measure of Set Difference, II).

Let $A, B \in \mathcal{F}$ and $B \subseteq A$, then

$$\mu(A \setminus B) = \mu(A) - \mu(B).$$

Expectation Integrals

Proposition (Unknown).

Let $A, B \subseteq \Omega$. Then the following equalities hold:

- $\mathbf{1}_{A^c} = 1 - \mathbf{1}_A$,
- $\mathbf{1}_{A \cap B} = \mathbf{1}_A \mathbf{1}_B$.
- $\mathbf{1}_{A \cup B} = \mathbf{1}_A + \mathbf{1}_B - \mathbf{1}_{A \cap B}$.

Lemma 3.3.

Let X be a *non-negative* random variable. Then there exists a sequence of *non-negative, simple* random variables X_n converging to X for every $\omega \in \Omega$.

Hint: $h_n(x) = \min\{\lfloor 2^n x \rfloor / 2^n, n\}$ is non-negative, simple and increasing, approaching x . Consider $X_n := h(X) \rightarrow X$.

Lemma (Simple Function Integral Properties).

Let $f, g : \Omega \rightarrow \mathbb{R}$ be a *non-negative*, simple functions and $a, b \geq 0$. Then the following holds:

- $\int_{\Omega} f d\mu \geq 0$,
- $\int_{\Omega} (af + bg) d\mu = a \int_{\Omega} f d\mu + b \int_{\Omega} g d\mu$.

Corollary (Positive Integral over Set).

Let $A \subseteq \Omega$ and $f : \Omega \rightarrow \mathbb{R}$ a *non-negative* measurable function. Then $\int_A f d\mu \geq 0$.

Lemma 3.3 (General).

Let $f : \Omega \rightarrow \mathbb{R}$ be a *non-negative*, measurable function. Then there exists a sequence f_n of *non-negative*, simple functions such that:

$$\lim_{n \rightarrow \infty} f_n = f$$

Hint: Use h_n from Lemma 3.3's hint.

Exercise 3.5.

Let $A \in \mathcal{F}$ s.t. $\mu(A) = 0$. Then for *any* measurable function $f : \Omega \rightarrow \mathbb{R}$:

$$\int_A f d\mu = 0.$$

Exercise 3.6.

Let $f : \Omega \rightarrow \mathbb{R}$ be a measurable function, then:

- For any $c \in \mathbb{R}$ and $A \in \mathcal{F}$:

$$\int_A cf d\mu = c \int_A f d\mu,$$

provided the integral exists.

- For any $A, B \in \mathcal{F}$, such that $A \cap B = \emptyset$:

$$\int_{A \cup B} f d\mu = \int_A f d\mu + \int_B f d\mu,$$

provided the left-hand or right-hand side is well-defined.

Theorem 3.8 (Monotone Convergence).

Let $(f_n)_{n=1}^{\infty}$ be increasing sequence of non-negative, measurable functions $f_n : \Omega \rightarrow \mathbb{R}$, converging to some f . Then:

$$\int_{\Omega} \lim_{n \rightarrow \infty} f_n d\mu = \lim_{n \rightarrow \infty} \int_{\Omega} f_n d\mu$$

Exercise 3.15.

Let ν be a measure that is absolutely continuous with respect to measure μ and density g , then $\mu(g < 0) = 0$. Moreover, ν is a probability measure $\Leftrightarrow g \geq 0$ μ -a.e. and $\int_{\Omega} g d\mu = 1$.

Proposition 3.16.

Let ν and μ be measures on σ -algebra \mathcal{F} such that ν is absolutely continuous with respect to μ and density g . Then for every \mathcal{F} -measurable function f the following holds:

$$\int_{\Omega} f d\nu = \int_{\Omega} fg d\mu,$$

whenever one of the integrals exists.

Proposition 3.18 (Markov-Chebyshev's Inequality).

Let X be a *non-negative* R.V., then

$$P(X \geq \lambda) \leq \lambda^{-\alpha} E(X^{\alpha}) \quad \forall \lambda > 0, \alpha > 0.$$

Remark 3.3.

Let $(\Omega, \mathcal{F}, \mu)$ be measure space, $f : \Omega \rightarrow \mathbb{R}$ *non-negative* \mathcal{F} -measurable, then

$$\mu(f \geq \lambda) \leq \lambda^{-\alpha} \int_{\Omega} f^{\alpha} d\mu \quad \forall \lambda > 0, \alpha > 0.$$

Lemma 3.10 (Fatou's Lemma).

Let $(f_n)_{n=1}^{\infty}$ be a sequence of *non-negative*, measurable functions $f : \Omega \rightarrow \mathbb{R}$, then

$$\int_{\Omega} \liminf_{n \rightarrow \infty} f_n d\mu \leq \liminf_{n \rightarrow \infty} \int_{\Omega} f_n d\mu.$$

Corollary 3.11 (Fatou's Lemma Extension).

Let $(f_n)_{n=1}^{\infty}$ be a sequence of measurable functions $f : \Omega \rightarrow \mathbb{R}$. Then

- if there exists a $g \in L_1(\Omega, \mathcal{F}, \mu)$, i.e. $\int_{\Omega} |g| d\mu < \infty$ such that $g \leq f_n$ for all n , then:

$$\int_{\Omega} \liminf_{n \rightarrow \infty} f_n d\mu \leq \liminf_{n \rightarrow \infty} \int_{\Omega} f_n d\mu.$$

- if there exists a $g \in L_1(\Omega, \mathcal{F}, \mu)$, i.e. $\int_{\Omega} |g| d\mu < \infty$ such that $g \geq f_n$, then:

$$\int_{\Omega} \limsup_{n \rightarrow \infty} f_n d\mu \geq \limsup_{n \rightarrow \infty} \int_{\Omega} f_n d\mu.$$

Hint: TODO

Theorem 3.12 (Lebesgue's Theorem on Dominated Convergence).

Let $(f_n)_{n=1}^{\infty}$ be a sequence of Borel functions $f_n : \Omega \rightarrow \mathbb{R}$ converging to some $f : \Omega \rightarrow \mathbb{R}$. Assume there exists a (non-negative) Borel functions g such that $|f_n| \leq g$ for any $n \geq 1$ and $\int_{\Omega} g d\mu < \infty$. Then the following two statements hold:

- $\int_{\Omega} |f| d\mu < \infty$,

- $\int_{\Omega} f d\mu = \lim_{n \rightarrow \infty} \int_{\Omega} f_n d\mu$.

Hint: TODO

Proposition (Restricted Expectation).

Let X be a random variable and $A \in \mathcal{F}$, then:

$$E(X \mathbf{1}_A) = \int_A X dP.$$

Lemma 4.4 (Borel-Cantelli Lemma).

Let $(A_n)_{n=1}^{\infty}$ be a sequence of sets $A_n \in \mathcal{F}$ such that $\sum_{n=1}^{\infty} \mu(A_n) < \infty$, i.e. the series of measures of A_n converges. Then for:

$$A := \limsup_{n \rightarrow \infty} A_n := \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k,$$

we have $\mu(A) = 0$.

Hint: Define $B_n := \bigcup_{k=n}^{\infty} A_k$, then $(B_n)_{n=1}^{\infty}$ is decreasing and so $\bigcap_{n=1}^{\infty} B_n = \lim_{n \rightarrow \infty} B_n$ and realize that $\sum_{n=1}^{\infty} \mu(A_n) < \infty \Rightarrow$ tail sums $\sum_{k=n}^{\infty} \mu(A_k) \rightarrow 0$ as $n \rightarrow \infty$.

Convergence of Measurable Functions

Exercise Ws 3, 1.

Let $\mu - \lim f_n = f$, then there exists a subsequence $(f_{n_k})_{k=1}^{\infty}$ such that $(n_k)_{k=1}^{\infty}$ is increasing and $f_{n_k} \rightarrow f$ (μ -a.e.).

Hint: Borel-Cantelli with

$$A_k = \{|f_{n_k} - f| \geq 1/k\} \text{ s.t. } \mu(A_k) \leq 1/k^2.$$

Exercise Ws 3, 2 (Convergence Implication).

Let $\mu(\Omega) < \infty$. Then $\lim_{n \rightarrow \infty} f_n = f$ (μ -a.e.) $\Rightarrow \mu - \lim_{n \rightarrow \infty} f_n = f$.

Exercise Ws 3, 3 (Relaxed Dominated Convergence).

Lebesgue's Theorem on Dominated convergence holds under the following, relaxed conditions:

- $\lim_{n \rightarrow \infty} f_n = f$ μ -a.e., $|f_n| \leq g$ μ -a.e. and $g \in L_1(\Omega, \mathcal{F}, \mu)$, i.e. $\int_{\Omega} |g| d\mu < \infty$;
- $\mu - \lim_{n \rightarrow \infty} f_n = f$, $|f_n| \leq g$ μ -a.e. and $g \in L_1(\Omega, \mathcal{F}, \mu)$, i.e. $\int_{\Omega} |g| d\mu < \infty$.

Hint: TODO

Definitions

Basic Notions and Notation

In the following, Ω is a set, \mathcal{F} a σ -algebra on Ω . If used, then μ is a measure. Otherwise, the measure is the probability measure P .

Definition 1.1.

Let \mathcal{F} be a family of subsets of set Ω . \mathcal{F} is called a σ -**algebra** if:

- **Closed Under Complement:**
 $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$,
- **Closed Under Arbitrary Union:**
 $A_n \in \mathcal{F}$ for integer $n \geq 1$
 $\Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$,
- **Contains Entire Set:** $\Omega \in \mathcal{F}$

Definition 1.2. Let \mathcal{C} be a family of subsets of Ω . There exists a σ -algebra which contains \mathcal{C} **and** which is contained in every σ -algebra that contains \mathcal{C} (take intersection of all σ -algebras. Such σ -algebra is **unique** and called **smallest σ -algebra containing \mathcal{C}** or **σ -algebra generated by \mathcal{C}** , denoted by $\sigma(\mathcal{C})$. Simplest example, let $A \subseteq \Omega$:

$$\sigma(A) = \{\emptyset, A, A^c, \Omega\}.$$

Definition (Finite Measure Space).

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. If $\mu(\Omega) < \infty$, then we call the measure space **finite**.

Random Variables

Definition 2.1.1.

Let $A \subseteq \Omega$ and $\mathbf{1}_A$ be defined as follows:

$$\mathbf{1}_A(\omega) = \begin{cases} 1, & \omega \in A \\ 0, & \omega \notin A \end{cases}.$$

Then $\mathbf{1}_A$ is a R.V. and called the **indicator (function) of (events) A**.

Expection Integrals

Definition (Indicator Integral).

Let $A \subseteq \Omega$, then:

$$\int_{\Omega} \mathbf{1}_A d\mu = \mu(A).$$

Definition (Simple Function).

Let $f : \Omega \rightarrow \mathbb{R}$ be a **simple function**, then f takes finitely many values. Formally, if I is a finite index set, $(A_i)_{i \in I}$ a family of **disjoint** subsets of Ω and $(c_i)_{i \in I}$ a family of real numbers, then:

$$f(\omega) = \sum_{i \in I} c_i \mathbf{1}_{A_i}(\omega).$$

Definition (Lebesgue Integral for Expection).

Let X be a random variable. Then we write:

$$EX = \int_{\Omega} X dP.$$

Definition (Non-negative, Measurable Lebesgue Integral).

Let $f : \Omega \rightarrow \mathbb{R}$ be a **non-negative**, measurable function and $(f_n)_{n=1}^{\infty}$ a sequence of **non-negative, simple** functions such that $\lim_{n \rightarrow \infty} f_n = f$. Then

$$\int_{\Omega} f d\mu = \lim_{n \rightarrow \infty} \int_{\Omega} f_n d\mu.$$

Definition (Lebesgue Integral).

Let $f : \Omega \rightarrow \mathbb{R}$ be a measurable function. The **Lebesgue Integral** of f is defined as:

$$\int_{\Omega} f d\mu = \int_{\Omega} f^+ d\mu - \int_{\Omega} f^- d\mu,$$

where $f^+ = \max\{f, 0\}$ and $f^- = \max\{-f, 0\}$, if at least one of the integrals on the right-hand side is finite. If both are infinite, then we say that the Lebesgue Integral of f does not exist.

Definition (Restricted Integration).

Let $A \in \mathcal{F}$ and $f : \Omega \rightarrow \mathbb{R}$ is a measurable function, then we define:

$$\int_A f d\mu = \int_{\Omega} \mathbf{1}_A f d\mu,$$

when the integral of $\mathbf{1}_A f$ w.r.t μ exists.

Definition 3.7 (Absolute Continuity).

Let μ and ν be measures on σ -algebra \mathcal{F} such that for some \mathcal{F} -measurable $g : \Omega \rightarrow \mathbb{R}$:

$$\nu(A) = \int_{\Omega} \mathbf{1}_A g d\mu = \int_A g \mu(dx),$$

for all $A \in \mathcal{F}$. Then ν is called **absolutely continuous** with respect to μ and g is called the **density** or **Radon-Nikodym derivative** (Notation: $g = \frac{d\nu}{d\mu}$).

Convergence of Measurable Functions

Definition (μ -Almost Everywhere Finite).

Let $f : \Omega \rightarrow \mathbb{R}$ be \mathcal{F} -measurable, then f is said to be **μ -almost everywhere** (μ -a.e.) finite if $\mu(|f| = \infty) = 0$.

Definition (Almost Surely Finite).

Let $f : \Omega \rightarrow \mathbb{R}$ be \mathcal{F} -measurable, then f is said to be **almost surely** (a.s.) finite if $P(|f| = \infty) = 0 \Leftrightarrow P(|f| < \infty) = 1$.

Definition 5.1 (μ -Almost Everywhere Convergence).

Let $(f_n)_{n=1}^{\infty}$ be \mathcal{F} -measurable functions. The f_n are said to **converge μ -almost everywhere** to a **μ -a.e. finite** $f : \Omega \rightarrow \mathbb{R}$ as $n \rightarrow \infty$ if there exists an $A \in \mathcal{F}$ s.t. $\mu(A) = 0$ and

$$\lim_{n \rightarrow \infty} f_n(\omega) = f(\omega) \in \mathbb{R}, \quad \forall \omega \in A^C.$$

Notation: $\lim_{n \rightarrow \infty} f_n = f$ (μ -a.e.) or $f_n \rightarrow f$ (μ -a.e.).

Definition 5.1 (Almost Sure Convergence).

Let $(f_n)_{n=1}^{\infty}$ be \mathcal{F} -measurable functions. The f_n are said to **converge almost surely** to a **a.s. finite** $f : \Omega \rightarrow \mathbb{R}$ as $n \rightarrow \infty$ if there exists an $A \in \mathcal{F}$ s.t. $P(A) = 0$ and

$$\lim_{n \rightarrow \infty} f_n(\omega) = f(\omega) \in \mathbb{R}, \quad \forall \omega \in A^C.$$

Notation: $\lim_{n \rightarrow \infty} f_n = f$ (a.s.) or $f_n \rightarrow f$ (a.s.).

Definition 5.2 (Convergence in Measure).

Let $(f_n)_{n=1}^{\infty}$ be \mathcal{F} -measurable functions. The f_n are said to **converge in measure** μ to a **μ -a.e. finite** $f : \Omega \rightarrow \mathbb{R}$ as $n \rightarrow \infty$ if

$$\lim_{n \rightarrow \infty} \mu(|f_n - f| \geq \varepsilon) = 0, \quad \forall \varepsilon > 0.$$

Notation: $\mu - \lim_{n \rightarrow \infty} f_n = f$.

Definition 5.2 (Convergence in Probability).

Let $(f_n)_{n=1}^{\infty}$ be \mathcal{F} -measurable functions. The f_n are said to **converge in probability** to a **a.s. finite** $f : \Omega \rightarrow \mathbb{R}$ as $n \rightarrow \infty$ if

$$\lim_{n \rightarrow \infty} P(|f_n - f| \geq \varepsilon) = 0, \quad \forall \varepsilon > 0.$$

Definition (Unknown).

Let X be a random variable. Then X has **finite second moment** if $EX^2 < \infty$.

Useful Observations

Observation (Increasing Sequence of Sets).

For an **increasing** sequence of sets $(A_n)_{n=1}^{\infty}$ we can define:

$$\lim_{n \rightarrow \infty} A_n := \bigcup_{n=1}^{\infty} A_n$$

Observation (Decreasing Sequence of Sets).

For an **decreasing** sequence of sets $(A_n)_{n=1}^{\infty}$ we can define:

$$\lim_{n \rightarrow \infty} A_n := \bigcap_{n=1}^{\infty} A_n$$

Observation (μ -Almost Everywhere Finite, I).

If $f : \Omega \rightarrow \mathbb{R}$ is μ -a. e. finite, then note that if $A_n := \{|f| \geq n\}$, then $(A_n)_{n=1}^{\infty}$ is a decreasing sequence and so:

$$\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \mu\left(\lim_{n \rightarrow \infty} A_n\right) = \mu(|f| = \infty) = 0.$$

Observation (μ -Almost Everywhere Finite, II).

If $f : \Omega \rightarrow \mathbb{R}$ is μ -a. e. finite, then observe

$$\mu(|f| = \infty) = \lim_{R \rightarrow \infty} \mu(|f| \geq R) = 0.$$

Observation (Almost Surely Finite, II).

If $f : \Omega \rightarrow \mathbb{R}$ is a.s. finite, then observe

$$\begin{aligned} P(|f| = \infty) &= \lim_{R \rightarrow \infty} P(|f| \geq R) = 0. \\ \Leftrightarrow P(|f| < \infty) &= \lim_{R \rightarrow \infty} P(|f| < R) = 1. \end{aligned}$$

Observation (Almost Surely Finite).

If $f : \Omega \rightarrow \mathbb{R}$ is a. s. finite, then note that if $A_n := \{|f| \geq n\}$, then $(A_n)_{n=1}^{\infty}$ is a decreasing sequence and so:

$$P\left(\bigcap_{n=1}^{\infty} A_n\right) = P\left(\lim_{n \rightarrow \infty} A_n\right) = P(|f| = \infty) = 0.$$

Observation (μ -Almost Everywhere Convergence I).

If $f_n \rightarrow f$ μ -a.e., then $\mu(f_n \not\rightarrow f) = 0$.

Observation (μ -Almost Everywhere Convergence II).

If $A \in \mathcal{F}$ is a set such that $\mu(A) = 0$ and

$$\lim_{n \rightarrow \infty} |f_n(\omega) - f(\omega)| = 0 \quad \forall \omega \in A^C,$$

then $f_n \rightarrow f$ μ -almost everywhere.

Observation (Almost Sure Convergence).

If $f_n \rightarrow f$ a.s., then $P(f_n \not\rightarrow f) = 0$ or equivalently $P(f_n \rightarrow f) = 1$.

Observation (Splitting Measures of Inequalities).

Let f, g be measurable functions and $a \in \mathbb{R}$, then observe that:

$$\mu(|f| \geq a) \leq \mu\left(|f - g| \geq \frac{a}{2}\right) + \mu\left(|g| \geq \frac{a}{2}\right)$$

Observation (Using Borel-Cantelli).

If you can define sets $(A_k)_{k=1}^{\infty}$ such that $\mu(A_k) \leq 1/k^2$, then you can use Borel-Cantelli as:

$$\sum_{k=1}^{\infty} \mu(A_k) \leq \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty.$$

In fact, the choice of $1/k^2$ is more or less arbitrary. This technique would work with any r_k s.t. $\sum_{k=1}^{\infty} r_k < \infty$ and $\mu(A_k) \leq r_k$. Caution: $r_k = 1/k$ does **not** work.

Observation (Function As Integral).

Let $f : \Omega \rightarrow \mathbb{R}$ be a **non-negative** measurable function, then observe that

$$f(\omega) = \int_0^{f(\omega)} dx = \int_0^{\infty} \mathbf{1}_{x \leq f(\omega)} dx$$