

# Measure Theory & Probability

Sebastian Müksch, v1, 2019/20

## Basic Notions and Notation

### Example 1.1.

Simplest  $\sigma$ -algebra:

- $\{\emptyset, \Omega\}$ , **contained in every**  $\sigma$ -algebra on  $\Omega$ ,
- Family of all subsets of  $\Omega$ , **containing every**  $\sigma$ -algebra on  $\Omega$ .

### Exercise 1.1.

Let  $\mathcal{F}$  be a  $\sigma$ -algebra. Then  $A_n \in \mathcal{F}$  for every integer  $n \geq 1 \Rightarrow \bigcap_{n=1}^{\infty} A_n \in \mathcal{F}$ .

### Proposition 1.2.

Let  $P$  be a probability measure on  $\sigma$ -algebra  $\mathcal{F}$ . Then the following statements hold:

- $A, B \in \mathcal{F}$  s.t.  $A \subseteq B \Rightarrow P(A) \leq P(B)$ ;
- For **increasing** sequence  $(A_n)_{n=1}^{\infty}$  we have

$$\lim_{n \rightarrow \infty} P(A_n) = P\left(\bigcup_{n=1}^{\infty} A_n\right);$$

- For **decreasing** sequence  $(A_n)_{n=1}^{\infty}$  we have

$$\lim_{n \rightarrow \infty} P(A_n) = P\left(\bigcap_{n=1}^{\infty} A_n\right).$$

### Proposition 1.2 (General).

Let  $\mu$  be a measure on  $\sigma$ -algebra  $\mathcal{F}$ . Then the following statements hold:

- $A, B \in \mathcal{F}$  s.t.  $A \subseteq B \Rightarrow \mu(A) \leq \mu(B)$ ;
- For **increasing** sequence  $(A_n)_{n=1}^{\infty}$  we have

$$\lim_{n \rightarrow \infty} \mu(A_n) = \mu\left(\bigcup_{n=1}^{\infty} A_n\right);$$

- For **decreasing** sequence  $(A_n)_{n=1}^{\infty}$  we have

$$\lim_{n \rightarrow \infty} \mu(A_n) = \mu\left(\bigcap_{n=1}^{\infty} A_n\right).$$

### Proposition (Bounding Intersections).

Let  $A, B \in \mathcal{F}$ . Then  $\mu(A \cap B) \leq \mu(A)$ .

*Hint:*  $\sigma$ -additivity and  $A = (A \cap B) \cup (A \setminus B)$ .

### Proposition (Measure of Set Difference, I).

Let  $A, B \in \mathcal{F}$ , then  $\mu(A \setminus B) = \mu(A) - \mu(A \cap B)$ .

### Proposition (Measure of Set Difference, II).

Let  $A, B \in \mathcal{F}$  and  $B \subseteq A$ , then

$$\mu(A \setminus B) = \mu(A) - \mu(B).$$

### Proposition (Complement of Limit Inferior/Superior).

Let  $(A_n)_{n=1}^{\infty}$  be a sequence of sets in  $\mathcal{F}$ , then:

- $$\left(\liminf_{n \rightarrow \infty} A_n\right)^C = \limsup_{n \rightarrow \infty} A_n^C$$
- $$\left(\limsup_{n \rightarrow \infty} A_n\right)^C = \liminf_{n \rightarrow \infty} A_n^C$$

### Exercise Ws 2, 1 (Limit Inferior/Superior Properties).

Let  $(A_n)_{n=1}^{\infty}$  be a sequence of sets in  $\mathcal{F}$ , then:

- $$\liminf_{n \rightarrow \infty} A_n := \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k$$

is the set of those  $\omega$  that are **in all but finitely many**  $A_n$ , i.e. that uphold the property  $A_n$  captures for all except a finite amount of values of  $n$ .

(ii)

$$\limsup_{n \rightarrow \infty} A_n := \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$$

is the set of those  $\omega$  that are **in infinitely many**  $A_n$ , i.e. that uphold the property  $A_n$  captures for an infinite amount of values of  $n$ .

### Proposition (Continuous Implies Borel-Measurability).

Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be a **continuous** function. Then  $f$  is Borel-measurable.

### Proposition (Countable Sets).

Every countable subset of  $\mathbb{R}$  is Borel-measurable.

## Expectation Integrals

### Proposition (Unknown).

Let  $A, B \subseteq \Omega$ . Then the following equalities hold:

- $\mathbf{1}_{A^C} = 1 - \mathbf{1}_A$ ,
- $\mathbf{1}_{A \cap B} = \mathbf{1}_A \mathbf{1}_B$ .
- $\mathbf{1}_{A \cup B} = \mathbf{1}_A + \mathbf{1}_B - \mathbf{1}_{A \cap B}$ .

### Lemma 3.3.

Let  $X$  be a **non-negative** random variable. Then there exists a sequence of **non-negative, simple** random variables  $X_n$  converging to  $X$  for every  $\omega \in \Omega$ .

*Hint:*  $h_n(x) = \min\{\lfloor 2^n x \rfloor / 2^n, n\}$  is non-negative, simple and increasing, approaching  $x$ . Consider  $X_n := h(X) \rightarrow X$ .

### Lemma (Simple Function Integral Properties).

Let  $f, g: \Omega \rightarrow \mathbb{R}$  be a **non-negative**, simple functions and  $a, b \geq 0$ . Then the following holds:

- $\int_{\Omega} f d\mu \geq 0$ ,
- $\int_{\Omega} (af + bg) d\mu = a \int_{\Omega} f d\mu + b \int_{\Omega} g d\mu$ .

### Corollary (Positive Integral over Set).

Let  $A \subseteq \Omega$  and  $f: \Omega \rightarrow \mathbb{R}$  a **non-negative** measurable function. Then  $\int_A f d\mu \geq 0$ .

### Lemma 3.3 (General).

Let  $f: \Omega \rightarrow \mathbb{R}$  be a **non-negative**, measurable function. Then there exists a sequence  $f_n$  of **non-negative**, simple functions such that:

$$\lim_{n \rightarrow \infty} f_n = f$$

*Hint:* Use  $h_n$  from Lemma 3.3's hint.

### Exercise 3.5.

Let  $A \in \mathcal{F}$  s.t.  $\mu(A) = 0$ . Then for **any** measurable function  $f: \Omega \rightarrow \mathbb{R}$ :

$$\int_A f d\mu = 0.$$

### Exercise 3.6.

Let  $f: \Omega \rightarrow \mathbb{R}$  be a measurable function, then:

- For any  $c \in \mathbb{R}$  and  $A \in \mathcal{F}$ :

$$\int_A cf d\mu = c \int_A f d\mu,$$

provided the integral exists.

- For any  $A, B \in \mathcal{F}$ , such that  $A \cap B = \emptyset$ :

$$\int_{A \cup B} f d\mu = \int_A f d\mu + \int_B f d\mu,$$

provided the left-hand or right-hand side is well-defined.

### Theorem 3.8 (Monotone Convergence).

Let  $(f_n)_{n=1}^{\infty}$  be increasing sequence of non-negative, measurable functions  $f_n: \Omega \rightarrow \mathbb{R}$ , converging to some  $f$ . Then:

$$\int_{\Omega} \lim_{n \rightarrow \infty} f_n d\mu = \lim_{n \rightarrow \infty} \int_{\Omega} f_n d\mu$$

### Exercise 3.15.

Let  $\nu$  be a measure that is absolutely continuous with respect to measure  $\mu$  and density  $g$ , then  $\mu(g < 0) = 0$ . Moreover,  $\nu$  is a probability measure  $\Leftrightarrow g \geq 0$   $\mu$ -a.e. and  $\int_{\Omega} g d\mu = 1$ .

### Proposition 3.16.

Let  $\nu$  and  $\mu$  be measures on  $\sigma$ -algebra  $\mathcal{F}$  such that  $\nu$  is absolutely continuous with respect to  $\mu$  and density  $g$ . Then for every  $\mathcal{F}$ -measurable function  $f$  the following holds:

$$\int_{\Omega} f d\nu = \int_{\Omega} fg d\mu,$$

whenever one of the integrals exists.

### Remark 3.3.

Let  $(\Omega, \mathcal{F}, \mu)$  be measure space,  $f: \Omega \rightarrow \mathbb{R}$  **non-negative**  $\mathcal{F}$ -measurable, then

$$\mu(f \geq \lambda) \leq \lambda^{-\alpha} \int_{\Omega} f^{\alpha} d\mu \quad \forall \lambda > 0, \alpha > 0.$$

### Lemma 3.10 (Fatou's Lemma).

Let  $(f_n)_{n=1}^{\infty}$  be a sequence of **non-negative**, measurable functions  $f: \Omega \rightarrow \mathbb{R}$ , then

$$\int_{\Omega} \liminf_{n \rightarrow \infty} f_n d\mu \leq \liminf_{n \rightarrow \infty} \int_{\Omega} f_n d\mu.$$

### Corollary 3.11 (Fatou's Lemma Extension).

Let  $(f_n)_{n=1}^{\infty}$  be a sequence of measurable functions  $f: \Omega \rightarrow \mathbb{R}$ . Then

- if there exists a  $g \in L_1(\Omega, \mathcal{F}, \mu)$ , i.e.  $\int_{\Omega} |g| d\mu < \infty$  such that  $g \leq f_n$  for all  $n$ , then:

$$\int_{\Omega} \liminf_{n \rightarrow \infty} f_n d\mu \leq \liminf_{n \rightarrow \infty} \int_{\Omega} f_n d\mu.$$

- if there exists a  $g \in L_1(\Omega, \mathcal{F}, \mu)$ , i.e.  $\int_{\Omega} |g| d\mu < \infty$  such that  $g \geq f_n$ , then:

$$\int_{\Omega} \limsup_{n \rightarrow \infty} f_n d\mu \geq \limsup_{n \rightarrow \infty} \int_{\Omega} f_n d\mu.$$

*Hint:* TODO

### Theorem 3.12 (Lebesgue's Theorem on Dominated Convergence).

Let  $(f_n)_{n=1}^{\infty}$  be a sequence of Borel functions  $f_n: \Omega \rightarrow \mathbb{R}$  converging to some  $f: \Omega \rightarrow \mathbb{R}$ .

Assume there exists a (non-negative) Borel functions  $g$  such that  $|f_n| \leq g$  for any  $n \geq 1$  and  $\int_{\Omega} g d\mu < \infty$ . Then the following two statements hold:

- $$\int_{\Omega} |f| d\mu < \infty,$$

- $$\int_{\Omega} f d\mu = \lim_{n \rightarrow \infty} \int_{\Omega} f_n d\mu.$$

*Hint:* TODO

### Proposition (Restricted Expectation).

Let  $X$  be a random variable and  $A \in \mathcal{F}$ , then:

$$E(X \mathbf{1}_A) = \int_A X dP.$$

### Proposition 3.18 (Markov-Chebyshev's Inequality).

Let  $X$  be a **non-negative** R.V., then

$$P(X \geq \lambda) \leq \lambda^{-\alpha} E(X^{\alpha}) \quad \forall \lambda > 0, \alpha > 0.$$

*Hint:*  $E(X^\alpha) \geq E(\mathbf{1}_{X \geq \lambda} X^\alpha) \geq E(\mathbf{1}_{X \geq \lambda} \lambda^\alpha) = \lambda^\alpha E(\mathbf{1}_{X \geq \lambda}) = \lambda^\alpha P(X \geq \lambda)$ .

**Proposition 3.18** (Markov-Chebyshev's Inequality (General)).

Let  $f : \Omega \rightarrow \mathbb{R}$  be a **non-negative**, measurable function, then

$$\mu(f \geq \lambda) \leq \lambda^{-\alpha} \int_{\Omega} f^\alpha d\mu \quad \forall \lambda > 0, \alpha > 0.$$

## $L_p$ Spaces

**Theorem** (Hölder's Inequality).

Let  $f, g : \Omega \rightarrow \mathbb{R}$  be measurable functions, then

$$\int_{\Omega} |fg| d\mu \leq \|f\|_p \|g\|_q \quad \text{for } p \geq 1,$$

where

$$q := \begin{cases} \frac{p}{p-1} & p > 1, \\ \infty & p = 1 \end{cases}.$$

*Hint:* TODO

**Theorem** (Hölder's Inequality for Expectations).

Let  $X, Y$  be random variables, then

$$E|XY| \leq (E|X|^p)^{\frac{1}{p}} (E|Y|^q)^{\frac{1}{q}}$$

where

$$q := \begin{cases} \frac{p}{p-1} & p > 1, \\ \infty & p = 1 \end{cases}.$$

**Proposition** (Finite Second Momenta Implication).

Let  $X, Y$  be random variables with finite second momenta. Then  $E|XY| < \infty$ .

*Hint:* Use Hölder's Inequality with  $p = 2$  on  $E|XY| = \int_{\Omega} |XY| dP$ .

**Lemma 4.4** (Borel-Cantelli Lemma).

Let  $(A_n)_{n=1}^{\infty}$  be a sequence of sets  $A_n \in \mathcal{F}$  such that  $\sum_{n=1}^{\infty} \mu(A_n) < \infty$ , i.e. the series of measures of  $A_n$  converges. Then for:

$$A := \limsup_{n \rightarrow \infty} A_n := \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k,$$

we have  $\mu(A) = 0$ .

*Hint:* Define  $B_n := \bigcup_{k=n}^{\infty} A_k$ , then  $(B_n)_{n=1}^{\infty}$  is decreasing and so  $\bigcap_{n=1}^{\infty} B_n = \lim_{n \rightarrow \infty} B_n$  and realize that  $\sum_{n=1}^{\infty} \mu(A_n) < \infty \Rightarrow$  tail sums  $\sum_{k=n}^{\infty} \mu(A_k) \rightarrow 0$  as  $n \rightarrow \infty$ .

## Convergence of Measurable Functions

**Exercise 5.1.**

Let  $(f_n)_{n=1}^{\infty}$  be a sequence of  $\mathcal{F}$ -measurable functions  $f_n : \Omega \rightarrow \mathbb{R}$ . Then the set  $A$  of those  $\omega \in \Omega$  such that  $\lim_{n \rightarrow \infty} f_n(\omega)$  converges to some (finite) number belongs to  $\mathcal{F}$ .

**Exercise 5.2** (Almost Finite, Converging Sequence is Bounded).

Assume that  $\mu(\Omega) < \infty$ . Let  $(f_n)_{n=1}^{\infty}$  be  **$\mu$ -a.e. finite**, converging in measure to  $\mu$  to some  $f : \Omega \rightarrow \mathbb{R}$ . Then the sequence of  $f_n$  is **bounded in measure**  $\mu$ , **uniformly in**  $n$ , i.e.:

$$\lim_{K \rightarrow \infty} \sup_{n \geq 1} \mu(|f_n| \geq K) = 0.$$

*Hint:*  $f_n$   **$\mu$ -a.e. finite and**  $\mu(\Omega) < \infty \Rightarrow f_n$  bounded in measure (not necessarily uniformly), so

$$\lim_{K \rightarrow \infty} \sup_{n \geq 1} \mu(|f_n| \geq K) =$$

$$\lim_{K \rightarrow \infty} \limsup_{n \rightarrow \infty} \mu(|f_n| \geq K).$$

Then use observation of splitting measures of inequalities.

**Exercise 5.3** (Product of Bounded & Zero Convergent is Zero Convergent).

Let  $(f_n)_{n=1}^{\infty}$  and  $(g_n)_{n=1}^{\infty}$  be sequences of  $\mu$ -a.e. finite measurable functions such that the  $f_n$  are bounded in measure  $\mu$ , uniformly in  $n$  and  $g_n \rightarrow 0$  in measure  $\mu$ , as  $n \rightarrow \infty$ . Then  $f_n g_n \rightarrow 0$  in measure  $\mu$ , as  $n \rightarrow \infty$ .

**Exercise Ws 3, 1.**

Let  $\mu - \lim f_n = f$ , then there exists a subsequence  $(f_{n_k})_{k=1}^{\infty}$  such that  $(n_k)_{k=1}^{\infty}$  is increasing and  $f_{n_k} \rightarrow f$  ( $\mu$ -a.e.).

*Hint:* Borel-Cantelli with

$$A_k = \{|f_{n_k} - f| \geq 1/k\} \text{ s.t. } \mu(A_k) \leq 1/k^2.$$

**Theorem 5.4** (Measure Convergence Has Almost Everywhere Converging Subsequence).

Let  $(f_n)_{n=1}^{\infty}$  be a sequence of functions converging in measure  $\mu$  to some  $\mu$ -a.e. finite function  $f$ . Then there exists a (strictly) increasing sequence  $(n_k)_{k=1}^{\infty}$  of positive integers such that  $\lim_{k \rightarrow \infty} f_{n_k} = f$   $\mu$ -almost everywhere.

**Exercise 5.5.**

Convergence in measure  $\mu$  does not imply convergence  $\mu$ -almost everywhere.

*Hint:*  $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$  with  $f_n = \mathbf{1}_{[k/2^m, (k+1)/2^m]}$  where  $k = 0, 1, \dots, 2^m - 1$  and  $m = 0, 1, \dots$  such that  $n = 2^m + k$ .

**Exercise Ws 3, 2** (Convergence Implication).

Let  $\mu(\Omega) < \infty$ . Then  $\lim_{n \rightarrow \infty} f_n = f$  ( $\mu$ -a.e.)  $\Rightarrow \mu - \lim_{n \rightarrow \infty} f_n = f$ .

**Exercise Ws 3, 3** (Relaxed Domnitated Convergence).

Lebesgue's Theorem on Dominated convergence holds under the following, relaxed conditions:

- (i)  $\lim_{n \rightarrow \infty} f_n = f$   $\mu$ -a.e.,  $|f_n| \leq g$   $\mu$ -a.e. and  $g \in L_1(\Omega, \mathcal{F}, \mu)$ , i.e.  $\int_{\Omega} |g| d\mu < \infty$ ; and
- (ii)  $\mu - \lim_{n \rightarrow \infty} f_n = f$ ,  $|f_n| \leq g$   $\mu$ -a.e. and  $g \in L_1(\Omega, \mathcal{F}, \mu)$ , i.e.  $\int_{\Omega} |g| d\mu < \infty$ .

*Hint:* TODO

## Independence of Events and Random Variables

**Theorem 6.3** (Monotone Class Theorem).

Let  $\Pi$  be a  $\pi$ -system contained in a  $\lambda$ -system  $\Lambda$ . Then  $\sigma(\Pi)$  is contained in  $\Lambda$ .

*Hint:* TODO

**Proposition 6.4** (Extending  $\pi$ -System Independence).

Let  $C_1$  and  $C_2$  be two **independent**  $\pi$ -systems, i.e.

$$P(A \cap B) = P(A)P(B) \quad \forall A \in C_1, B \in C_2,$$

then the  $\sigma$ -algebras  $\sigma(C_1)$  and  $\sigma(C_2)$  are also independent.

*Hint:* TODO

**Theorem 6.7** (Fubini-Tonelli Theorem).

Let  $(\Omega_i, \mathcal{F}_i, \mu_i)$ , for  $i = 1, 2$ , be measure spaces and  $(\Omega, \mathcal{F}, \mu)$  be the product measure space of the two, i.e.  $\Omega = \Omega_1 \times \Omega_2$ ,  $\mathcal{F} = \mathcal{F}_1 \otimes \mathcal{F}_2$  and  $\mu = \mu_1 \otimes \mu_2$ . Let  $f : \Omega \rightarrow \mathbb{R}$  be a **non-negative**  $\mathcal{F}$ -measurable function. If  $\mu_i$ , for  $i = 1, 2$ , are **finite measures** on  $\Omega_i$ , for  $i = 1, 2$ , respectively, then the following iterated integrals are well-defined and:

$$\begin{aligned} \int_{\Omega_1 \times \Omega_2} f d\mu_1 \otimes \mu_2 &= \int_{\Omega_1} \int_{\Omega_2} f d\mu_2 d\mu_1 = \\ &= \int_{\Omega_2} \int_{\Omega_1} f d\mu_1 d\mu_2. \end{aligned}$$

Furthermore, this statement holds for  $\mathcal{F}$ -measurable functions if:

$$\int_{\Omega_1 \times \Omega_2} |f| d\mu_1 \otimes \mu_2 < \infty.$$

**Lemma 6.9** (Borel-Cantelli (Full)).

Let  $(A_n)_{n=1}^{\infty}$  be a sequence of sets and set

$$A := \limsup_{n \rightarrow \infty} A_n := \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k,$$

then the following statements holds:

- (i) If  $\sum_{n=1}^{\infty} \mu(A_n) < \infty$ , then  $\mu(A) = 0$ .
- (ii) If all  $A_n$  are **jointly independent** and  $\sum_{n=1}^{\infty} P(A_n) = \infty$ , then  $P(A) = 1$ .

*Hint:* (i) provided in general case. (ii) Prove  $P((\limsup_{n \rightarrow \infty} A_n)^C) = 1$ , define  $B_n = \bigcap_{k=n}^{\infty} A_k^C$  and show that for a given  $P(B_n) = P(\lim_{m \rightarrow \infty} \bigcap_{k=n}^m A_k) = 0$  using independence and observation that  $1 - P(A) \leq e^{-P(A)}$ . Finally, use **sub- $\sigma$ -additivity** for  $P(\bigcup_{n=1}^{\infty} B_n)$ . **Do not** attempt to argue through increasing sequences.

## Definitions

### Basic Notions and Notation

In the following,  $\Omega$  is a set,  $\mathcal{F}$  a  $\sigma$ -algebra on  $\Omega$ . If used, then  $\mu$  is a measure. Otherwise, the measure is the probability measure  $P$ .

**Definition 1.1.**

Let  $\mathcal{F}$  be a family of subsets of set  $\Omega$ .  $\mathcal{F}$  is called a  **$\sigma$ -algebra** if:

- **Closed Under Complement:**  
 $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$ ,
- **Closed Under Arbitrary Union:**  
 $A_n \in \mathcal{F}$  for integer  $n \geq 1$   
 $\Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$ ,
- **Contains Entire Set:**  $\Omega \in \mathcal{F}$

**Definition 1.2.** Let  $\mathcal{C}$  be a family of subsets of  $\Omega$ . There exists a  $\sigma$ -algebra which contains  $\mathcal{C}$  **and** which is contained in every  $\sigma$ -algebra that contains  $\mathcal{C}$  (take intersection of all  $\sigma$ -algebras). Such  $\sigma$ -algebra is **unique** and called **smallest  $\sigma$ -algebra containing  $\mathcal{C}$**  or  **$\sigma$ -algebra generated by  $\mathcal{C}$** , denoted by  $\sigma(\mathcal{C})$ . Simplest example, let  $A \subseteq \Omega$ :

$$\sigma(A) = \{\emptyset, A, A^c, \Omega\}.$$

**Definition** (Finite Measure Space).

Let  $(\Omega, \mathcal{F}, \mu)$  be a measure space. If  $\mu(\Omega) < \infty$ , then we call the measure space **finite**.

## Random Variables

**Definition 2.1.1.**

Let  $A \subseteq \Omega$  and  $\mathbf{1}_A$  be defined as follows:

$$\mathbf{1}_A(\omega) = \begin{cases} 1, & \omega \in A \\ 0, & \omega \notin A \end{cases}.$$

Then  $\mathbf{1}_A$  is a R.V. and called the **indicator (function) of (events)  $A$** .

**Definition 2.3** (Distribution Function).

Let  $X$  be a random variable. Then the function

$$\begin{aligned} F_X(x) &= P(X \leq x) = \\ &= P(X \in (-\infty, x]) = Q_X((-\infty, x]), \end{aligned}$$

for  $x \in \mathbb{R}$  is called the **distribution function** of  $X$ .

## Expectation Integrals

**Definition** (Indicator Integral).

Let  $A \subseteq \Omega$ , then:

$$\int_{\Omega} \mathbf{1}_A d\mu = \mu(A).$$

**Definition** (Simple Function).

Let  $f : \Omega \rightarrow \mathbb{R}$  be a **simple function**, then  $f$  takes finitely many values. Formally, if  $I$  is a finite index set,  $(A_i)_{i \in I}$  a family of **disjoint** subsets of  $\Omega$  and  $(c_i)_{i \in I}$  a family of real numbers, then:

$$f(\omega) = \sum_{i \in I} c_i \mathbf{1}_{A_i}(\omega).$$

**Definition** (Lebesgue Integral for Expectation).

Let  $X$  be a random variable. Then we write:

$$EX = \int_{\Omega} X dP.$$

**Definition** (Non-negative, Measurable Lebesgue Integral).

Let  $f : \Omega \rightarrow \mathbb{R}$  be a **non-negative**, measurable function and  $(f_n)_{n=1}^{\infty}$  a sequence of **non-negative**, **simple** functions such that  $\lim_{n \rightarrow \infty} f_n = f$ . Then

$$\int_{\Omega} f d\mu = \lim_{n \rightarrow \infty} \int_{\Omega} f_n d\mu.$$

**Definition** (Lebesgue Integral).

Let  $f : \Omega \rightarrow \mathbb{R}$  be a measurable function. The **Lebesgue Integral** of  $f$  is defined as:

$$\int_{\Omega} f d\mu = \int_{\Omega} f^+ d\mu - \int_{\Omega} f^- d\mu,$$

where  $f^+ = \max\{f, 0\}$  and  $f^- = \max\{-f, 0\}$ , if at least one of the integrals on the right-hand side is finite. If both are infinite, then we say that the Lebesgue Integral of  $f$  does not exist.

**Definition** (Restricted Integration).

Let  $A \in \mathcal{F}$  and  $f : \Omega \rightarrow \mathbb{R}$  is a measurable function, then we define:

$$\int_A f d\mu = \int_{\Omega} \mathbf{1}_A f d\mu,$$

when the integral of  $\mathbf{1}_A f$  w.r.t  $\mu$  exists.

**Definition 3.7** (Absolute Continuity).

Let  $\mu$  and  $\nu$  be measures on  $\sigma$ -algebra  $\mathcal{F}$  such that for some  $\mathcal{F}$ -measurable  $g : \Omega \rightarrow \mathbb{R}$ :

$$\nu(A) = \int_{\Omega} \mathbf{1}_A g d\mu = \int_A g \mu(dx),$$

for all  $A \in \mathcal{F}$ . Then  $\nu$  is called **absolutely continuous** with respect to  $\mu$  and  $g$  is called the **density** or **Radon-Nikodym derivative** (Notation:  $g = \frac{d\nu}{d\mu}$ ).

## Convergence of Measurable Functions

**Definition** ( $\mu$ -Almost Everywhere Finite).

Let  $f : \Omega \rightarrow \mathbb{R}$  be  $\mathcal{F}$ -measurable, then  $f$  is said to be  **$\mu$ -almost everywhere** ( $\mu$ -a.e.) finite if  $\mu(|f| = \infty) = 0$ .

**Definition** (Almost Surely Finite).

Let  $f : \Omega \rightarrow \mathbb{R}$  be  $\mathcal{F}$ -measurable, then  $f$  is said to be **almost surely** (a.s.) finite if  $P(|f| = \infty) = 0 \Leftrightarrow P(|f| < \infty) = 1$ .

**Definition 5.1** ( $\mu$ -Almost Everywhere Convergence).

Let  $(f_n)_{n=1}^{\infty}$  be  $\mathcal{F}$ -measurable functions. The  $f_n$  are said to **converge  $\mu$ -almost everywhere** to a  $\mu$ -a.e. **finite**  $f : \Omega \rightarrow \mathbb{R}$  as  $n \rightarrow \infty$  if there exists an  $A \in \mathcal{F}$  s.t.  $\mu(A) = 0$  and

$$\lim_{n \rightarrow \infty} f_n(\omega) = f(\omega) \in \mathbb{R}, \quad \forall \omega \in A^C.$$

**Notation:**  $\lim_{n \rightarrow \infty} f_n = f$  ( $\mu$ -a.e.) or  $f_n \rightarrow f$  ( $\mu$ -a.e.).

**Definition 5.1** (Almost Sure Convergence).

Let  $(f_n)_{n=1}^{\infty}$  be  $\mathcal{F}$ -measurable functions. The  $f_n$  are said to **converge almost surely** to a **a.s. finite**  $f : \Omega \rightarrow \mathbb{R}$  as  $n \rightarrow \infty$  if there exists an  $A \in \mathcal{F}$  s.t.  $P(A) = 0$  and

$$\lim_{n \rightarrow \infty} f_n(\omega) = f(\omega) \in \mathbb{R}, \quad \forall \omega \in A^C.$$

**Notation:**  $\lim_{n \rightarrow \infty} f_n = f$  (a.s.) or  $f_n \rightarrow f$  (a.s.).

**Definition 5.2** (Convergence in Measure).

Let  $(f_n)_{n=1}^{\infty}$  be  $\mathcal{F}$ -measurable functions. The  $f_n$  are said to **converge in measure**  $\mu$  to a  $\mu$ -a.e. **finite**  $f : \Omega \rightarrow \mathbb{R}$  as  $n \rightarrow \infty$  if

$$\lim_{n \rightarrow \infty} \mu(|f_n - f| \geq \varepsilon) = 0, \quad \forall \varepsilon > 0.$$

**Notation:**  $\mu - \lim_{n \rightarrow \infty} f_n = f$ .

**Definition 5.2** (Convergence in Probability).

Let  $(f_n)_{n=1}^{\infty}$  be  $\mathcal{F}$ -measurable functions. The  $f_n$  are said to **converge in probability** to a **a.s. finite**  $f : \Omega \rightarrow \mathbb{R}$  as  $n \rightarrow \infty$  if

$$\lim_{n \rightarrow \infty} P(|f_n - f| \geq \varepsilon) = 0, \quad \forall \varepsilon > 0.$$

**Definition** (Bounded in Measure).

Let  $(f_n)_{n=1}^{\infty}$  be a sequence of measurable functions, then it is **bounded in measure**  $\mu$  if

$$\lim_{K \rightarrow \infty} \mu(|f_n| \geq K) = 0,$$

for any  $n \geq 1$ .

**Definition** (Bounded Uniformly in Measure).

Let  $(f_n)_{n=1}^{\infty}$  be a sequence of measurable functions, then it is **bounded in measure**  $\mu$ , **uniformly in  $n$**  if

$$\lim_{K \rightarrow \infty} \sup_{n \geq 1} \mu(|f_n| \geq K) = 0.$$

**Definition** (Finite Second Moment).

Let  $X$  be a random variable. Then  $X$  has **finite second moment** if  $EX^2 < \infty$ .

## Independence of Events and Random Variables

**Definition 6.5** ( $\lambda$ -system).

Let  $\Lambda$  be a family of subsets of  $\Omega$ . Then  $\Lambda$  is a  $\lambda$ -system, if it satisfies all of the following properties:

- (i) (Contains whole set)  $\Omega \in \Lambda$ ;
- (ii) (Closed under Subset Set Subtraction) if  $A, B \in \Lambda$ , such that  $B \subset A$ , then  $A \setminus B \in \Lambda$ ;
- (iii) (Closed under Disjoint Union) if  $(A_n)_{n=1}^{\infty}$  is a **pairwise disjoint** sequence, i.e.  $A_i \cap A_j = \emptyset$  for  $i \neq j$ , of subsets, such that  $A_i \in \Lambda$  for  $i = 1, 2, \dots$ , then  $\bigcup_{n=1}^{\infty} A_n \in \Lambda$ .

**Definition 6.6** ( $\pi$ -system).

Let  $\Pi$  be a family of subsets of  $\Omega$ . Then  $\Pi$  is a  $\pi$ -system, if it is closed under finite intersections, i.e.  $A, B \in \Pi \Rightarrow A \cap B \in \Pi$ .

## Conditional Expectation

**Definition** (Sub- $\sigma$ -Algebra Measurable).

Let  $Y$  be a random variable TODO FINISH

## Useful Observations

**Observation** (Bounding Measures).

The following inequalities to bound measures are **always** applicable, for **any** sets  $A, B, C \in \mathcal{F}$ :

1. “Dropping a set in an intersection gives an upper bound”  $\Leftrightarrow$  “Relaxing constraints”:

$$\mu(A \cap B) \leq \mu(A).$$

2. “Dropping a set in a union gives a lower bound”:

$$\mu(A \cup B) \geq \mu(A).$$

3. “Adding a set in a union gives an upper bound”  $\Leftrightarrow$  “Adding constraints”:

$$\mu(A \cup B) \leq \mu(A \cup B \cup C).$$

4. “Intersections are less than a set and a set is less than a union”:

$$\mu(A \cap B) \leq \mu(A) \leq \mu(A \cup B).$$

**Observation** (Adding  $\Omega$  by Intersection).

If you would like to introduce a property to an existing set  $A$  to make it easier to work with, for instance easier to bound, you can add an intersection with  $\Omega$ :

$$\mu(A) = \mu(\Omega \cap A).$$

Then  $\Omega$  can be split into the set  $B$  that represents the property and  $B^C$  that does not have the property, where  $\Omega = B \cup B^C$ . Then:

$$\begin{aligned} \mu(A) &= \mu(\Omega \cap A) = \mu((B \cup B^C) \cap A) = \\ &= \mu((B \cap A) \cup (B^C \cap A)). \end{aligned}$$

Using  $\sigma$ -additivity, we get:

$$\mu(A) = \mu(B \cap A) + \mu(B^C \cap A).$$

Then by the observation on bounding measures, this can be made into an inequality:

$$\begin{aligned} \mu(A) &= \mu(B \cap A) + \mu(B^C \cap A) \\ &\leq \mu(B \cap A) + \mu(B^C). \end{aligned}$$

**Observation** (Increasing Sequence of Sets).

For an **increasing** sequence of sets  $(A_n)_{n=1}^{\infty}$  we can define:

$$\lim_{n \rightarrow \infty} A_n := \bigcup_{n=1}^{\infty} A_n$$

**Observation** (Decreasing Sequence of Sets).

For an **decreasing** sequence of sets  $(A_n)_{n=1}^{\infty}$  we can define:

$$\lim_{n \rightarrow \infty} A_n := \bigcap_{n=1}^{\infty} A_n$$

**Observation** ( $\mu$ -Almost Everywhere Finite, I).

If  $f : \Omega \rightarrow \mathbb{R}$  is  $\mu$ -a. e. finite, then note that if  $A_n := \{|f| \geq n\}$ , then  $(A_n)_{n=1}^{\infty}$  is a decreasing sequence and so:

$$\begin{aligned} \mu\left(\bigcap_{n=1}^{\infty} A_n\right) &= \mu\left(\lim_{n \rightarrow \infty} A_n\right) = \mu(|f| = \infty) \\ &= 0. \end{aligned}$$

**Observation** ( $\mu$ -Almost Everywhere Finite, II).

If  $f : \Omega \rightarrow \mathbb{R}$  is  $\mu$ -a. e. finite, then observe

$$\mu(|f| = \infty) = \lim_{R \rightarrow \infty} \mu(|f| \geq R) = 0.$$

**Observation** (Almost Surely Finite, II).

If  $f : \Omega \rightarrow \mathbb{R}$  is a.s. finite, then observe

$$P(|f| = \infty) = \lim_{R \rightarrow \infty} P(|f| \geq R) = 0.$$

$$\iff P(|f| < \infty) = \lim_{R \rightarrow \infty} P(|f| < R) = 1.$$

**Observation** (Almost Surely Finite).

If  $f : \Omega \rightarrow \mathbb{R}$  is a. s. finite, then note that if  $A_n := \{|f| \geq n\}$ , then  $(A_n)_{n=1}^\infty$  is a decreasing sequence and so:

$$P\left(\bigcap_{n=1}^\infty A_n\right) = P\left(\lim_{n \rightarrow \infty} A_n\right) = P(|f| = \infty) = 0.$$

**Observation** ( $\mu$ -Almost Everywhere Convergence I).

If  $f_n \rightarrow f$   $\mu$ -a.e., then  $\mu(f_n \not\rightarrow f) = 0$ .

**Observation** ( $\mu$ -Almost Everywhere Convergence II).

If  $A \in \mathcal{F}$  is a set such that  $\mu(A) = 0$  and

$$\lim_{n \rightarrow \infty} |f_n(\omega) - f(\omega)| = 0 \quad \forall \omega \in A^C,$$

then  $f_n \rightarrow f$   $\mu$ -almost everywhere.

**Observation** (Almost Sure Convergence).

If  $f_n \rightarrow f$  a.s., then  $P(f_n \not\rightarrow f) = 0$  or equivalently  $P(f_n \rightarrow f) = 1$ .

**Observation** (Splitting Measures of Inequalities).

Let  $f, g$  be measurable functions and  $a \in \mathbb{R}$ , then observe that:

$$\mu(|f| \geq a) \leq \mu\left(|f - g| \geq \frac{a}{2}\right) + \mu\left(|g| \geq \frac{a}{2}\right)$$

**Observation** (Using Borel-Cantelli).

If you can define sets  $(A_k)_{k=1}^\infty$  such that  $\mu(A_k) \leq 1/k^2$ , then you can use Borel-Cantelli as:

$$\sum_{k=1}^\infty \mu(A_k) \leq \sum_{k=1}^\infty \frac{1}{k^2} < \infty.$$

In fact, the choice of  $1/k^2$  is more or less arbitrary. This technique would work with any  $r_k$  s.t.  $\sum_{k=1}^\infty r_k < \infty$  and  $\mu(A_k) \leq r_k$ . Caution:  $r_k = 1/k$  does **not** work.

**Observation** (Function As Integral).

Let  $f : \Omega \rightarrow \overline{\mathbb{R}}$  be a **non-negative** measurable function, then observe that

$$f(\omega) = \int_0^{f(\omega)} dx = \int_0^\infty \mathbf{1}_{x \leq f(\omega)} dx$$

**Observation** (Bounding Complement Probabilities).

Note that  $1 - x \leq e^{-x}$ . Therefore, we can bound probabilities of a product of complement events, for instance:

$$\prod_{n=1}^\infty P(A_n^C) = \prod_{n=1}^\infty [1 - P(A_n)] \leq \prod_{n=1}^\infty e^{-P(A_n)} = e^{\sum_{n=1}^\infty -P(A_n)}$$

**Observation** (Interchanging Expectation & Infinite Sum).

Observe that if  $f$  is **non-negative**, then:

$$\begin{aligned} E\left(\sum_{n=1}^\infty f(X_n)\right) &= E\left(\lim_{N \rightarrow \infty} \sum_{n=1}^N f(X_n)\right) = \\ &= \lim_{N \rightarrow \infty} \sum_{n=1}^N E f(X_n) = \sum_{n=1}^\infty E f(X_n), \end{aligned}$$

where pulling the expectation through the sum can be done due to the Monotone Convergence Theorem, as  $\sum_{n=1}^N f(X_n)$  is an increasing sequence of **non-negative** random variables.

**Observation** (Markov-Chebyshev's Inequality & Norm).

The following is the general Markov-Chebyshev Inequality rewritten using the norm instead of an integral. Let  $f : \Omega \rightarrow \mathbb{R}$  be a **non-negative**, measurable function in  $L_\alpha(\Omega, \mathcal{F}, \mu)$ , then

$$\mu(f \geq \lambda) \leq \lambda^{-\alpha} \|f\|_\alpha^\alpha d\mu \quad \forall \lambda > 0, \alpha > 0.$$