Real Analysis

Sebastian Müksch, v2, 2018/19

Convergence

Remark [Wade 7.2].
Let $S \subseteq \mathbb{R}$, non-empty. A sequence of functions f_n ***converges pointwise*** if $\forall \varepsilon > 0, x \in S\, \exists N \in \mathbb{N}$ s.t.:

$$n \geqslant N \Rightarrow |f_n(x) - f(x)| < \varepsilon.$$

Theorem [Wade 7.9].
Let $S \subseteq \mathbb{R}$, non-empty, and suppose $f_n \to f$ ***uniformly*** on S as $n \to \infty$. Then ***each*** f_n continuous at $x_0 \in S \Rightarrow f$ continuous at $x_0 \in S$.

Theorem [Wade 7.10].
Suppose $f_n \to f$ ***uniformly*** on closed interval $[a, b]$. Then ***each*** f_n integrable on $[a, b] \Rightarrow f$ integrable on $[a, b]$ and

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b \left(\lim_{n\to\infty} f_n(x)\right) dx$$

Lemma [Wade 7.11] (Uniform Cauchy Criterion).
Let $S \subseteq \mathbb{R}$, non-empty, and $f_n : S \to \mathbb{R}$ a sequence of functions. Then f_n ***converges uniformly*** on $S \Leftrightarrow \forall \varepsilon > 0\, \exists N \in \mathbb{N}$ s.t.:

$$n, m \geqslant N \Rightarrow |f_n(x) - f_m(x)| < \varepsilon, \quad \forall x \in S.$$

Theorem [Wade 7.12].
Let (a, b) be a bounded interval and f_n converging at some $x_0 \in (a, b)$. Each f_n is differentiable on (a, b) and f_n' converges ***uniformly*** on $(a, b) \Rightarrow f_n$ converges uniformly on (a, b) and

$$\lim_{n\to\infty} f_n'(x) = \left(\lim_{n\to\infty} f_n(x)\right)'.$$

Exercise 7.1.3.
Let the sequence of $f_n : S \to \mathbb{R}$ be bounded and let $f_n \to f$ uniformly. Then f is bounded and moreover, sequence f_n is ***uniformly*** bounded.

Exercise 7.1.5.
Let $f_n \to f$ and $g_n \to g$ uniformly as $n \to \infty$ on $S \subseteq \mathbb{R}$. Then

a) $f_n + g_n \to f + g$, $\alpha f_n \to \alpha f$ ***uniformly*** on S as $n \to \infty$, for all $\alpha \in \mathbb{R}$;
b) $f_n g_n \to fg$ ***pointwise*** on S;
c) if f, g ***bounded***, then $f_n g_n \to fg$ ***uniformly*** on S;
d) if g unbounded, c) is false.

Exercise 7.1.9.
Let f, g be ***continuous*** on ***closed & bounded interval*** $[a, b]$ with $|g(x)| > 0$ for all $x \in [a, b]$. Let $f_n \to f$ and $g_n \to g$ ***uniformly*** on $[a, b]$. Then

a) $1/g_n$ is defined for large n and $f_n/g_n \to f/g$ ***uniformly*** on $[a, b]$;
b) a) is false if $[a, b]$ is replaced with (a, b).

Exercise 7.1.10.
Let $S \subseteq \mathbb{R}$, non-empty, f_n sequence of ***bounded*** functions on S s.t. $f_n \to f$ ***uniformly***. Then

$$\frac{f_1(x) + \ldots + f_n(x)}{n} \to f(x)$$

uniformly on S.

Theorem [Wade 7.14].
Let $S \subseteq \mathbb{R}$, non-empty, $f_n : S \to \mathbb{R}$.

i) Let each f_n is continuous at $x_0 \in E \Rightarrow$. Then $f = \sum_{n=1}^{\infty} f_n$ converging ***uniformly*** $\Rightarrow f$ ***continuous*** at x_0.
ii) Suppose $S = [a, b]$ and each f_n be integrable on $[a, b]$. Then $f = \sum_{n=1}^{\infty} f_n$ converging ***uniformly*** on $[a, b] \Rightarrow f$ ***integrable*** on $[a, b]$ and
$$\int_a^b \left(\sum_{n=1}^{\infty} f_n(x)\right) dx = \sum_{n=1}^{\infty} \int_a^b f_n(x)\,dx.$$
iii) Suppose S is ***bounded, open interval*** and each f_n differentiable on S. $\sum_{n=1}^{\infty} f_n$ convergent at some $x_0 \in S$ and $\sum_{n=1}^{\infty} f_n'$ ***uniformly*** convergent on $S \Rightarrow$ $f := \sum_{n=1}^{\infty} f_n$ ***uniformly*** convergent on S, f ***differentiable*** on S and
$$\left(\sum_{n=1}^{\infty} f_n(x)\right)' = \sum_{n=1}^{\infty} f_n'(x)$$
for $x \in S$.

Theorem [Wade 7.15] (Weierstrass M-Test).
Let $S \subseteq \mathbb{R}$, non-empty, and $f_n : S \to \mathbb{R}$. Suppose $M_n \geqslant 0$ satisfies $\sum_{n=1}^{\infty} M_n < \infty$. If $\forall n \in \mathbb{N}, x \in S:\ |f_n(x)| \leqslant M_n$, then $\sum_{n=1}^{\infty} f_n$ ***converges absolutely and uniformly*** on S

Workshop 2, Question 7.
Let $f_n : \mathbb{R} \to \mathbb{R}$ be a sequence of ***continuous*** functions converging ***uniformly*** to f. Let (x_n) be a sequence in $\mathbb{R}$ s.t. $x_n \to x \in \mathbb{R}$. Then $f_n(x_n) \to f(x)$.

Power Series

Theorem [Power Series, Thrm. 1].
Let R be radius of convergence of $\sum_{n=0}^{\infty} a_n(x - c)^n$.

(i) $|x - c| < R \Rightarrow$ series ***converges absolutely***;
(ii) $|x - c| > R \Rightarrow$ series ***diverges***.

Exercise (Radius of Convergence).

(i) If $\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$ exists, then it is radius of convergence;
(ii) If $\lim_{n\to\infty} |a_n|^{-\frac{1}{n}}$ exists, then it is radius of convergence.

Theorem [Power Series, Thrm. 2].
Let $R > 0$, then $\sum_{n=0}^{\infty} a_n(x - c)^n$ converges ***uniformly & absolutely*** on $|x - c| < R$ to a continuous function f, i.e.:

$$f(x) = \sum_{n=0}^{\infty} a_n(x - c)^n$$

defines a continuous function $f : (c - R, c + R) \to \mathbb{R}$.

Lemma [Power Series].
$\sum_{n=0}^{\infty} a_n(x - c)^n$ and $\sum_{n=0}^{\infty} na_n(x - c)^{n-1}$ have the same radius of convergence.

Theorem [Power Series, Thrm. 3].
Suppose $\sum_{n=0}^{\infty} a_n(x - c)^n$ has radius of convergence R. Then

$$f(x) = \sum_{n=0}^{\infty} a_n(x - c)^n$$

is ***infinitely differentiable*** on $|x - c| < R$ and for such x:

$$f'(x) = \sum_{n=0}^{\infty} na_n(x - c)^{n-1}$$

and the series converges ***uniformly & absolutely*** on $[c - r, c + r]$ for any $r < R$. Additionally

$$a_n = \frac{f^{(n)}(c)}{n!}.$$

Remark [Power Series].
Analytic functions are ***infinitely differentiable*** on $\{x \in \mathbb{R} : |x - c| < r\}$ and the coefficients of the power series are ***uniquely*** determined by $a_n = f^{(n)}(c)/n!$.

Exercise 7.2.2.
The geometric series

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1 - x}$$

converges ***uniformly*** on any $[a, b] \subset (-1, 1)$.

Exercise 7.3.3.
Let $\sum_{k=0} \infty a_k x^k$ have radius of convergence R. Then

a) $\sum_{k=0} \infty a_k x^{2k}$ has radius of convergence $\sqrt{R}$
b) $\sum_{k=0} \infty a_k^2 x^k$ has radius of convergence R^2

Exercise 7.3.4.
Let $|a_k| \leqslant |b_k|$ for ***large*** k and $\sum_{k=0} \infty b_k x^k$ converges on ***open*** interval I. Then $\sum_{k=0} \infty a_k x^k$ converges on I.
Hint: Supremum Definition.

Exercise 7.3.5.
Let (a_k) be ***bounded*** sequence of real numbers. Then $\sum_{k=0} \infty a_k x^k$ has ***positive*** radius of convergence.

Riemann Integration

Workshop 3, Question 5.
Let $I \subseteq \mathbb{R}$ be an open interval, $f : I \to \mathbb{R}$ ***differentiable*** with f' ***bounded*** on I. Then f is ***uniformly*** continuous.

Workshop 3, Question 7.
Let $I \subseteq \mathbb{R}$ be an open interval and let $f : I \to \mathbb{R}$ continuous. Then f ***uniformly*** continuous $\Leftrightarrow$ whenever sequences (s_n), (t_n) in I are s.t. $|s_n - t_n| \to 0$, then $|f(s_n) - f(t_n)| \to 0$.

Workshop 3, Question 8.
Let $f : [a, b] \to \mathbb{R}$ continuous. Then f is ***uniformly*** continuous.

Exercise (Step Function Vector Space).
The class of step functions is a vector space. Moreover, if ϕ and ψ are step functions, then $\max\{\phi, \psi\}$, $\min\{\phi, \psi\}$, $|\phi|$ and $\phi\psi$ are also step functions.

Exercise (Characterising Step Functions).
Function ϕ is a ***step function*** $\Leftrightarrow \phi$ is of form:

$$\phi(x) = \sum_{j=1}^{n} c_j \chi_{I_j}(x)$$

where each I_j is a ***bounded interval***.

Lemma (Set Independence).
Let ϕ be a step function. Then $\int \phi$ is ***independent*** of the particular set $\{x_0, x_1, \ldots, x_n\}$ with respect to which ϕ is a step function.

Proposition [Integration, Prop. 1].
Let ϕ, ψ be step functions, $\alpha, \beta \in \mathbb{R}$. Then

$$\int (\alpha\phi + \beta\psi) = \alpha \int \phi + \beta \int \psi.$$

Exercise (Integral Ordering).
Let ϕ, ψ be step functions. Then $\phi \leqslant \psi \Rightarrow \int \phi \leqslant \int \psi$.

Theorem [Integration, Thrm. 1].
Let $f : \mathbb{R} \to \mathbb{R}$. Then f ***Riemann-integrable*** $\Leftrightarrow$

$$\sup\left\{\int \phi : \phi \text{ step function}, \phi \leqslant f\right\} =$$
$$\inf\left\{\int \psi : \psi \text{ step function}, \psi \geqslant f\right\}.$$

Theorem [Integration, Thrm. 2].
Let $f : \mathbb{R} \to \mathbb{R}$. Then f is ***Riemann-integrable*** $\Leftrightarrow$ there exist sequences of step functions ϕ_n and ψ_n s.t. $\forall n \in \mathbb{N} : \phi_n \leqslant f \leqslant \phi_n$ and
$$\int \psi_n - \int \phi_n \to 0.$$
If ϕ_n and ψ_n are any sequences of step functions satisfying the above, then
$$\int \phi_n \to \int f \quad \text{and} \int \psi_n \to \int f$$
as $n \to \infty$.

Exercise (Sum of Powers Estimate).
Let $n \in \mathbb{N}$, then for any integer $m \geqslant 1$:
$$\frac{n^{m+1}}{m+1} \leqslant \sum_{j=1}^{n} j^m \leqslant \frac{(n+1)^{m+1}}{m+1}$$

Lemma [Integration, Lem. 1].
Let $f : \mathbb{R} \to \mathbb{R}$ be ***bounded*** with ***bounded support*** $[a, b]$. Then the following is equivalent:

(i) f is ***Riemann-integrable***;
(ii) $\forall \varepsilon > 0\ \exists\, a = x_0 < \ldots < x_n = b$ s.t. if
$$M_j = \sup_{x \in I_j} f(x), \quad m_j = \inf_{x \in I_j} f(x)$$
where $I_j = [x_{j-1}, x_j]$, then
$$\sum_{j=1}^{n} (M_j - m_j)(x_j - x_{j-1}) < \varepsilon;$$
(iii) $\forall \varepsilon > 0\ \exists\, a = x_0 < \ldots < x_n = b$ s.t., with $I_j = (x_{j-1}, x_j)$ for $j \geqslant 1$:
$$\sum_{j=1}^{n} \sup_{x,y \in I_j} |f(x) - f(y)||I_j| < \varepsilon.$$

Theorem [Integration, Thrm. 3].
Let f, g be ***Riemann-integrable***, $\alpha, \beta \in \mathbb{R}$. Then

(a) $\alpha f + \beta g$ is ***Riemann-integrable*** and
$$\int (\alpha f + \beta g) = \alpha \int f + \beta \int g;$$
(b) $f \geqslant 0 \Rightarrow \int f \geqslant 0$ and $f \geqslant g \Rightarrow \int f \geqslant \int g$;
(c) $|f|$ is ***Riemann-integrable*** and
$$\left|\int f\right| \leqslant \int |f|;$$
(d) $\max\{f, g\}$ and $\min\{f, g\}$ are ***Riemann-integrable***;
(e) fg is ***Riemann-integrable***

Theorem [Integration, Thrm. 4].
Let $g : [a, b] \to \mathbb{R}$ be ***continuous***, $f(x) = g(x)$ if $x \in [a, b]$, $f(x) = 0$ if $x \notin [a, b]$. Then f is ***Riemann-integrable***.

Theorem [Integration, Thrm. 5].
Let $g : [a, b] \to \mathbb{R}$ be ***Riemann-integrable***. For $x \in [a, b]$ let
$$G(x) = \int_a^x g.$$
Then g ***continuous*** at some $x \in [a, b] \Rightarrow G$ ***differentiable*** at x and $G'(x) = g(x)$.

Theorem [Integration, Thrm. 6].
Let $f : [a, b] \to \mathbb{R}$ s.t. f has ***continuous*** derivative f' on $[a, b]$. Then
$$\int_a^b f' = f(b) - f(a).$$

Exercise (Integral Test).
Let (a_n) be a ***non-negative*** sequence of numbers and $f : [1, \infty) \to (0, \infty)$ s.t.

(i) $\int_1^n f \leqslant K$ for some K and all n and
(ii) $a_n \leqslant f(x)$ for $n \leqslant x < n + 1$.

Then $sum_n a_n$ converges to a real number which is at most K.

Exercise (p-Series Test).
For $p > 1$, $\sum 1/n^p$ converges.

Workshop 5, Question 1.
Let $f : \mathbb{R} \to \mathbb{R}$ be ***Riemann-integrable***. Then f is ***bounded*** with ***bounded support***.

Workshop 5, Question 7.
Let $g : [a, b] \to \mathbb{R}$, $a < b$, be ***continuous*** and ***non-negative***. Then $\int_a^b g = \Rightarrow g = 0$ on $[a, b]$.

Exercise 5.2.0 (b).
Let f be ***Riemann-integrable***, P ***any polynomial***, then $P \circ f$ is ***Riemann-integrable***.
Hint: f R-integrable $\Rightarrow f^n$ is R-integrable by Thrm. 3 linearity.

Exercise 5.2.6.
(a) Let $g_n \geqslant 0$ sequence of ***Riemann-integrable*** functions on $[a, b]$ s.t.
$$\lim_{n \to \infty} \int_a^b g_n = 0$$
Then f ***Riemann-integrable*** on $[a, b] \Rightarrow$
$$\lim_{n \to \infty} \int_a^b f g_n = 0$$
Hint: f is bounded $\Rightarrow f g_n$ is bounded & Squeeze Thrm.

Metric Spaces

Example [Wade 10.2].
Every Euclidean space $\mathbb{R}^n$ is a metric space with the ***usual metric*** $\rho(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\|$.

Definition [Wade 10.3].
$\mathbb{R}$ is a metric space with the ***discrete metric***:
$$\sigma(x, y) = \begin{cases} 0 & x = y, \\ 1 & x \neq y \end{cases}$$

Example [Wade 10.4].
Let (X, ρ) be a metric space and $E \subseteq X$. Then E is a metric space with metric ρ, called a ***subspace*** of X.

Exercise 10.4.10a.
$E \subset X$ ***compact*** $\Rightarrow E$ ***sequentially compact***.
Hint: Arbitrary $x \in$,
$S = \{n \in \mathbb{N} : x_n \in B_{r(x)}(x)\}$ must be finite for (x_n) not to have convergent subsequence. E has open cover $\{B_r(x_i) : 1 \leqslant i \leqslant k\} \Rightarrow \exists\, i$ s.t. $B_r(x_i)$ infinite $\Rightarrow$ contradicts S finite.

Example [Wade 10.6].
Let $\mathcal{C}[a, b]$ be the set of ***continuous*** functions $f : [a, b] \to \mathbb{R}$ and
$$\|f\| \coloneqq \sup_{x \in [a,b]} |f(x)|$$
Then $\rho(f, g) \coloneqq \|f - g\|$ is a metric on $\mathcal{C}[a, b]$.
N.B.: Convergence in this metric spaces means ***uniform*** convergence.

Remark [Wade 10.9].
Every open ball is ***open***, every closed ball is ***closed***.

Remark [Wade 10.10].
Let $a \in X$. Then $X \setminus \{a\}$ is ***open*** and $\{a\}$ is ***closed***.

Remark [Wade 10.11].
Let (X, ρ) be an ***arbitrary*** metric space. Then $\emptyset$ and X are ***both open & closed***.

Example [Wade 10.12].
Every subset of ***discrete*** space $\mathbb{R}$ is ***both open & closed***.

Theorem [Wade 10.14].
Let X be a metric space.

i) A sequence in X can have ***at most one*** limit.
ii) If $\{x_n\}$ in X converges to a and $\{x_{n_k}\}$ is ***any subsequence*** of $\{x_n\}$, then $\{x_{n_k}\}$ converges to a as well.
iii) $\{x_n\}$ in X is ***convergent*** $\Rightarrow \{x_n\}$ is ***bounded***
iv) $\{x_n\}$ in X is ***convergent*** $\Rightarrow \{x_n\}$ is ***Cauchy***

Remark [Wade 10.15].
Let $\{x_n\}$ in X. Then $x_n \to a$ as $n \to \infty \Leftrightarrow$ for ***every open set*** V s.t. $a \in V\ \exists N \in \mathbb{N}$ s.t. $n \geqslant N \Rightarrow x_n \in V$.

Theorem [Wade 10.16].
Let $E \subseteq X$. Then E is ***closed*** $\Leftrightarrow$ the limit of ***every convergent*** sequence $\{x_k\}$ in E ***lies in*** E, i.e.:
$$\lim_{k \to \infty} x_k \in E$$

Remark [Wade 10.17].
The discrete space contains ***bounded*** sequences with have ***no convergent subsequences***, e.g. $\{k\}$ with $k \in \mathbb{N}$.

Remark [Wade 10.18].
The metric space $\mathbb{Q}$ with usual metric contains ***Cauchy sequences*** which do ***not converge***, e.g. $\{q_k\}$ in $\mathbb{Q}$ s.t. $q_k \to \sqrt{2}$.

Exercise 10.1.4.
In ***discrete*** metric space, $x_n \to a$ as $n \to \infty \Leftrightarrow x_n = a$ for n large.

Exercise 10.1.5.
Let x_n, y_n sequences in (X, ρ) converge to same limit $a \in X$. Then $\rho(x_n, y_n) \to 0$ as $n \to \infty$. The ***converse*** is ***false***, e.g. $x_n = y_n = n$.

Exercise 10.1.6.
Let (x_n) be ***Cauchy*** in X. Then (x_n) ***converges*** $\Leftrightarrow (x_n)$ has a ***convergent subsequence***.

Remark [Wade 10.20].
If X is a ***complete*** metric space, then

1) ***every Cauchy*** sequence in X ***converges***;
2) the limit of ***every Cauchy*** sequence in X ***stays in*** X.

Theorem [Wade 10.21].
Let X be a ***complete*** metric space and $E \subseteq X$. Then E is ***complete*** $\Leftrightarrow E$ is ***closed***.

Remark (Cluster Point in Subspace).
Let $E \subseteq X$ be a ***subspace*** of X. The $a \in E$ is a ***cluster point*** in $E \Leftrightarrow \forall \delta > 0$, the ***relative ball*** $B_\delta(a) \cap E$ contains ***infinitely*** many points.

Theorem [Wade 10.26].
Let $a \in X$ be a ***cluster point*** and $f, g : X \setminus \{a\} \to Y$.

i) $\forall x \in X \setminus \{a\} : f(x) = g(x)$ and $f(x)$ has a limit as $x \to a \Rightarrow g(x)$ has a limit as $x \to a$ and
$$\lim_{x\to a} g(x) = \lim_{x\to a} f(x).$$
ii) ***Sequential Characterization of Limits:***
$$L := \lim_{x\to a} f(x)$$
exists $\Leftrightarrow f(x_n) \to L$ as $n \to \infty$ for ***every*** sequence $\{x_n\}$ in $X \setminus \{a\}$ s.t. $x_n \to a$ as $n \to \infty$.

iii) Let $Y = \mathbb{R}^n$. $f(x)$ and $g(x)$ have a limit as $x \to a \Rightarrow (f+g)(x)$, $(fg)(x)$, $(\alpha f)(x)$ and if $Y = \mathbb{R}$ and limit of $g(x) \neq 0$ also $(f/g)(x)$ have limits. In this case, the usual algebra of limits applies.

iv) ***Squeeze Theorem:*** Let $Y = \mathbb{R}$. Let $h : X \setminus \{a\} \to \mathbb{R}$ s.t. $\forall x \in X \setminus \{a\}$: $g(x) \leqslant h(x) \leqslant f(x)$ and
$$\lim_{x\to a} g(x) = \lim_{x\to a} f(x) = L$$
$\Rightarrow$ limit of h as $x \to a$ exists and
$$\lim_{x\to a} h(x) = L.$$
v) ***Comparison Theorem:*** Let $Y = \mathbb{R}$. $\forall x \in X \setminus \{a\} : f(x) \leqslant g(x)$ and f, g have a limit as $x \to a$, then
$$\lim_{x\to a} f(x) \leqslant \lim_{x\to a} g(x).$$

Theorem [Wade 10.28].
Let $E \subseteq X$, non-empty, and $f, g : E \to Y$.

i) f ***continuous at*** $a \in E \Leftrightarrow f(x_n) \to f(a)$ as $n \to \infty$ for ***every*** sequence $\{x_n\}$ in E s.t. $x_n \to a$.

ii) Let $Y = \mathbb{R}^n$. f, g ***continuous at*** $a \in E \Rightarrow$ $f+g$, fg, αf, for $\alpha \in \mathbb{R}$ are ***continuous at*** $a \in E$. Also, if $Y = \mathbb{R}$ and $g(a) \neq 0$, then f/g ***continuous at*** $a \in E$.

Theorem [Wade 10.29].
Let X, Y, Z be metric spaces and $a \in X$ a ***cluster point***. Let $f : X \to Y$, $g : f(X) \to Z$. $f(x) \to L$ as $x \to a$ and g ***continuous at*** $L \Rightarrow$
$$\lim_{x\to a} (g \circ f)(x) = g\left(\lim_{x\to a} f(x)\right).$$

Exercise 10.2.2.
Let (X, d) be a metric space.

a) $a \in X$ ***isolated*** $\Leftrightarrow a$ ***not cluster point*** in X.
b) Discrete metric space has ***no cluster points***.

Hint: a) ($\Leftarrow$) not cluster $\Rightarrow B_r(a)$ finitely many elements, take ρ minimum of distance of those to a, then $X \cap B_\rho(a) = \{a\}$.

Exercise 10.2.3.
Let $E \subseteq X$. Then a is a ***cluster point*** $\Leftrightarrow$ there ***exists*** sequence (x_n) in $E \setminus \{a\}$ s.t. $x_n \to a$ as $n \to n$.
Hint: ($\Rightarrow$) $x_n \in E \cap B_{\frac{1}{n}}(a)$, ($\Leftarrow$) $E \cap B_r(a)$ infinite as $a \neq x_n$.

Exercise 10.2.4.

a) Let $E \subseteq X$, non-empty. Then a is a ***cluster point*** for of $E \Leftrightarrow \forall r > 0$: $(E \cap B_r(a)) \setminus \{a\} \neq \emptyset$.
b) Every ***bound infinite subset*** of $\mathbb{R}$ has ***at least one*** cluster point.

Hint: a) ($\Leftarrow$) $x_n \in (E \cap B_{\frac{1}{n}}(a)) \setminus \{a\}$ and Ex. 10.2.3. b) (x_n) sequence in E and Bolzano-Weierstrass.

Workshop 7, Question 5.
Metrics d, ρ ***strongly equivalent*** $\Rightarrow d$, ρ ***equivalent***.

Workshop 7, Question 7.
Let d, ρ be metrics on X. Then d, ρ ***equivalent*** $\Leftrightarrow$ ***every*** subset of X ***open*** with respect to d is ***also open*** with respect to ρ ***and vice-versa***.

Workshop 8, Question 11.
X ***compact*** $\Rightarrow \forall r > 0$, X can be covered by ***finitely*** many open balls of radius r.
Hint: Consider open cover of open balls of radius r.

Workshop 8, Question 12.
Let X be ***compact***. Then X is ***complete***. Additionally, X ***compact*** $\Leftrightarrow X$ is ***complete and*** can be covered by finitely many open balls of radius r for any $r > 0$.
Hint: X compact $\Rightarrow$ sequentially compact, so (x_n) Cauchy sequence has convergent subsequence (x_n) converges.

Workshop 8, Question 13.
X ***compact*** $\Leftrightarrow X$ ***sequentially compact***.
Hint: Take (x_n) Cauchy, has convergent subsequence by assumption $\Rightarrow$ converges $\Rightarrow X$ complete. Only need show that $\exists$ cover with finite number open balls. Assume none exists for $r > 0$. Pick $x_1 \in X$. Pick $x_2 \in X$ s.t. $d(x_1, x_2) > r$, repeat to get (x_n) s.t. $d(x_m, x_n) > r\ \forall m, n \Rightarrow$ not convergent $\Rightarrow$ contradiction.

Topology

Theorem [Wade 10.31].
Let X be a metric space.

i) The ***union*** of ***any collection*** of ***open*** sets in X is ***open***;
ii) The ***intersection*** of a ***finite collection*** of ***open*** sets in X is ***open***;
iii) The ***intersection*** of ***any collection*** of ***closed*** sets in X is ***closed***;
iv) The ***union*** of a ***finite collection*** of ***closed*** sets in X is ***closed***;
v) Let $V \subseteq X$ be ***open***, $E \subseteq X$ be ***closed***. Then $V \setminus E$ is ***open***, $E \setminus V$ is ***closed***.

Remark 10.32.
The ***intersection*** of ***any collection*** of ***open*** sets is ***not*** necessarily ***open***, e.g.
$$\bigcap_{k\in\mathbb{N}} \left(-\frac{1}{k}, \frac{1}{k}\right) = \{0\}.$$
The ***union*** of ***any collection*** of ***closed*** sets is ***not*** necessarily ***closed***, e.g.
$$\bigcup_{k\in\mathbb{N}} \left[\frac{1}{k+1}, \frac{k}{k+1}\right] = (0, 1).$$

Theorem [Wade 10.34].
Let $E \subseteq X$. Then

i) $E^o \subseteq E \subseteq \overline{E}$;
ii) V ***open*** and $V \subseteq E \Rightarrow V \subseteq E^o$.
iii) C ***closed*** and $C \supseteq E \Rightarrow C \supseteq E$.

Theorem [Wade 10.39].
Let $E \subseteq X$. Then $\partial E = \overline{E} \setminus E^o$.

Theorem [Wade 10.40].
Let $A, B \subseteq X$. Then

i) $(A \cup B)^o \supseteq A^o \cup B^o$, $(A \cap B)^o = A^o \cap B^o$;
ii) $\overline{A \cup B} = \overline{A} \cup \overline{B}$, $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$;
iii) $\partial(A \cup B) \subseteq \partial A \cup \partial B$, $\partial(A \cap B) \subseteq (A \cap \partial B) \cup (B \cap \partial A) \cup (\partial A \cap \partial B)$.

Exercise 10.3.4.
Let $A \subseteq B \subseteq X$. Then $\overline{A} \subseteq \overline{B}$ & $A^o \subseteq B^o$.

Remark [Wade 10.43].
The empty set and ***all finite*** subsets of a metric space are ***compact***.

Remark 10.44.
Every ***compact*** set is ***closed***.
Hint: Assume H compact & not closed $\Rightarrow \exists$ sequence with limit x not in H. $y \in H$ and $r(y) := \rho(x, y)/2$, $x \neq H \Rightarrow r(y) > 0$. Open cover of $B_{r(y)}(y)$ w/ finite subcover $\{B_{r(y_j)}(y_j)\}$. $r = \min\{r(y_j)\}$. $x_k \to x \Rightarrow x_k \in B_r(x)$ for k large. $x_k \in B_r(x) \cap H \Rightarrow x_k \in B_{r(y_j)}(y_j)$ for some j. Then with $r_j \geqslant \rho(x_k, y_j) \geqslant \rho(x, y_j) - \rho(x_k, x) = 2r_j - \rho(x_k, x) > 2r_j - r \geqslant 2r_j - r_j \Rightarrow$ contradiction.

Remark [Wade 10.46].
Every ***closed subset*** of a ***compact*** set is ***compact***.
Hint: $E \subseteq H$ closed w/ H compact s.t. $\mathcal{V}$ is open cover of E. $E^c = X \setminus E$ open $\Rightarrow \mathcal{V} \cup E^c$ cover H. H compact $\Rightarrow$ finite subcover $\mathcal{V}_0$ and $H \subseteq E^c \cup \mathcal{V}_0$, but $E \cap E^c = \emptyset \Rightarrow \mathcal{V}_0$ finite subcover of E.

Theorem [Wade 10.46].
Let $H \subseteq X$, X being a metric space. H ***compact*** $\Rightarrow H$ ***closed & bounded***.

Remark 10.47.
Given an ***arbitrary*** metric space, ***closed & bounded*** $\not\Rightarrow$ ***compact*** in general.

Exercise 10.4.2.
Let $A, B \subseteq X$ be ***compact***. Then $A \cup B$ and $A \cap B$ are ***compact***.
Hint: Combine subcovers for $A \cup B$; note $A \cap B \subset A$ closed & Thrm. 10.46.

Exercise 10.4.3.
Let $E \subseteq \mathbb{R}$ be ***compact*** and non-empty. Then $\sup E$ and $\inf E$ belong to E.
Hint: Existence by boundedness. Approximation Property gives $\sup E \leqslant x_n \leqslant \sup E + 1/n$ and Squeeze Theorem.

Exercise 10.4.8.
(a) ***Cantor Intersection Theorem:*** Let $H_{k+1} \subseteq H_k$ be ***nested*** sequence of ***compact, non-empty*** sets in metric space X. Then $\bigcap_{k=1}^\infty H_k \neq \emptyset$.
Hint: Assume $\bigcap_{k=1}^\infty H_k = \emptyset$. $\{H_k^c\}$ open cover of $H_1 \Rightarrow$ finite subcover H_{k_i}, $1 \leqslant i \leqslant n$. H_k nested $\Rightarrow H_k^c$ nested $\Rightarrow s = \max\{k_i\}$ then $H_1 \subset H_s^c \Rightarrow \emptyset = H_s \cap H_1 = H_s$, contradiction.

Remark [Wade 10.55].
Let $E \subseteq X$. If $\exists A, B \subseteq X$, both ***open*** s.t.
$$E \subseteq A \cup B, \quad A \cap B = \emptyset$$
$$A \cap E \neq \emptyset, \quad B \cap E \neq \emptyset$$
i.e. A, B ***separate*** E, then E is ***not connected***.

Theorem [Wade 10.56].
$E \subseteq \mathbb{R}$ is ***connected*** $\Leftrightarrow E$ is an ***interval***.

Remark (Preimage of Open Balls).
Let X, Y be metric spaces and $f : X \to Y$. Then f is ***continuous*** $\Leftrightarrow$
$$B_\delta(a) \subseteq f^{-1}(B_\varepsilon(f(a))).$$

Theorem [Wade 10.58].
Let $f : X \to Y$. Then f ***continuous*** $\Leftrightarrow f^{-1}(V)$ is ***open*** in X for ***every open*** V in Y.
Hint: ($\Rightarrow$) $f^{-1}(V)$ non-empty, let $a \in f^{-1}(V)$, i.e. $f(a) \in V \Rightarrow$ choose ε s.t. $B_\varepsilon(f(a)) \subseteq V$. f continuous $\Rightarrow$ choose δ s.t.

$B_\delta(a) \subseteq f^{-1}(B_\varepsilon(f(a)))$. ($\Leftarrow$) $\varepsilon > 0$, $a \in X$. $V = B_\varepsilon(f(a))$ open and by assumption $f^{-1}(V)$ open. $a \in f^{-1}(V) \Rightarrow \exists\, \delta > 0$ s.t. $B_\delta(a) \subseteq f^{-1}(V) \Rightarrow f$ continuous.

Corollary [Wade 10.59].
Let $E \subseteq X$ and $f : E \to Y$. Then f ***continuous*** on $E \Leftrightarrow f^{-1}(V) \cap E$ is ***relatively open*** in E for ***every open*** V in Y.

Remark (Continuous Inverse Invariance).
Open & Closed sets are invariant under inverse images by ***continuous*** functions.

Exercise 10.5.5.
Let $E \subseteq X$ and $E \subseteq A \subseteq \overline{E}$ and E ***connected***. Then A is ***connected***.
Hint: Assume A disconnected then Remark 10.55 for A. $U \cap E \neq \emptyset$ by contradiction $\Rightarrow$ $\exists\, x \in U$ s.t. $x \in A \setminus E$. $A \subset \overline{E} \Rightarrow x$ cluster point of $E \Rightarrow \exists\, r > 0$ s.t. $B_r(x) \subset U$ with infinitely many points from E so $E \cap U \neq \emptyset$. Similarly $E \cap V \neq \emptyset \Rightarrow$ contradicts E connected.

Exercise 10.5.11.
Let $\{E_\alpha\}_{\alpha \in A}$ collection of ***connected*** sets s.t. $\bigcap_{\alpha \in A} E_\alpha \neq \emptyset$. Then $\bigcup_{\alpha \in A} E_\alpha$ is ***connected***.
Hint: Contradiction and Remark 10.55.

Theorem [Wade 10.61].
$H \subseteq X$ ***compact*** and $f : H \to Y$ ***continuous*** $\Rightarrow$ $f(H)$ ***compact*** in Y.

Theorem [Wade 10.62].
$E \subseteq X$ ***connected*** and $f : E \to Y$ ***continuous*** $\Rightarrow f(E)$ ***connected*** in Y.

Theorem [Wade 10.63] (Extreme Value Theorem).
Let $H \subseteq X$, ***non-empty & compact*** and $f : H \to \mathbb{R}$ ***continuous***. Then

$$M \coloneqq \sup\{f(x) : x \in H\},$$
$$m \coloneqq \inf\{f(x) : x \in H\}$$

are ***finite real*** numbers and $\exists\, x_M, x_m \in H$ s.t. $M = f(x_M)$ and $m = f(x_m)$.

Theorem [Wade 10.64].
Let $H \subseteq X$ be ***compact*** and $f : H \to Y$ ***injective (1-1) & continuous***. Then f^{-1} is ***continuous*** on $f(H)$.

Workshop 11, Question 2-5.
Every ***open, connected*** set in $\mathbb{R}^n$ is ***path-connected***.
Hint: U set of $x, y \in E$ s.t. path exists, V s.t. does not. Show $E \subset U \cup V$, $U \cap V = \emptyset$, $U \cap E \neq \emptyset$. U is path-connected. Show U, V are open, $y \in U$ and as E open $B_r(y) \subseteq E$, let $z \in B_r(y)$ then x, z path-connected as x, y are. Similar reasoning for V open.

Exercise 10.6.5 (Intermediate Value Theorem).
Let $E \subseteq X$ be ***connected***, $f : E \to \mathbb{R}$ ***continuous*** and $a, b \in E$ with $f(a) < f(b)$. Then $\forall y$ s.t. $f(a) < y < f(b)$ $\exists x \in E$ s.t. $f(x) = y$
Hint: E connected, f continuous $\Rightarrow f(E)$ connected and as subset of $\mathbb{R}$ is interval, so $[f(a), f(b)] \subset f(E)$. So $f(a) < y < f(b) \Rightarrow$ $y \in f(E)$.

Exercise 10.6.9.
Let X be ***connected***. Then $f : X \to \mathbb{R}$ ***non-constant, continuous*** $\Rightarrow X$ ***uncountably*** many points.
Hint: Connected subsets in $\mathbb{R}$ are intervals (a, b) and $g : (a, b) \to X$ is injective, so $g((a, b)) \subset X$ same size as (a, b).

Contraction Mappings

Exercise [Contraction Mapping].
Let f be a ***contraction***. Then f is ***continuous***.

Theorem (Banach's Contraction Mapping Theorem).
Let (X, d) be a ***complete*** metric space, $f : X \to X$ a ***contraction***. Then there ***exists unique*** $x \in X$ s.t. $f(x) = x$.
N.B.: It is important that $f(X) \subseteq X$.
Hint: Pick $x_0 \in X$ and $f(x_n) = x_{n+1}$ as contraction $\Rightarrow d(x_n, x_{n+1}) \leqslant \alpha^n d(x_0, x_1)$. Use triangle inequality & finite geometric series to show $d(x_m, x_n) \leqslant \dfrac{\alpha^n}{1-\alpha} d(x_0, x_1) \Rightarrow (x_n)$ Cauchy, as X complete $\Rightarrow (x_n)$ converges to $x \in X$. f continuous $\Rightarrow f(x) = f(\lim x_n) = \lim f(x_n) = \lim x_{n+1} = x$. Uniqueness: $x, y \in X$, $f(x) = x$ & $f(y) = y \Rightarrow d(x, y) = d(f(x), f(y)) \leqslant \alpha d(x, y) \Rightarrow d(x, y) = 0$.

Exercise [Contraction Mapping].
Let (X, d) be a ***complete*** metric space and $f : X \to X$ s.t. $f^{(n)} = f \circ f \circ \ldots \circ f$ a ***contraction***. Then f has a ***unique fixed point***.
N.B.: f itself may not be a contraction.

Workshop 10, Question 8.
Let (X, d) be ***compact*** and $f : X \to X$ s.t. $d(f(x), f(y)) \leqslant d(x, y)$ for all $x \neq y \in X$. Then f has a ***unique*** fixed point.
Hint: $\phi(x) = d(x, f(x))$, continuous, so image is closed & bounded subset of $\mathbb{R}$ as X compact. f without fixed point $\Rightarrow \phi > 0$ and $\inf \phi = k > 0$ and $\exists x \in X$ s.t. $d(x, f(x)) = k$. $d(f(x), f(f(x))) < d(x, f(x)) = k$, contradicts k infimum.

Miscellaneous

Remark (Geometric Sum).

$$\sum_{k=0}^{n} r^k = \frac{1 - r^{n+1}}{1 - r}$$

Remark Product to Sum (
). $fg = \frac{1}{4}((f+g)^2 - (f-g)^2)$
N.B.: Used in proof of Cauchy-Schwarz for functions.

Definitions

Convergence

Definition [Wade 7.1].
Let $S \subseteq \mathbb{R}$, non-empty. A sequence of functions $f_n : S \to \mathbb{R}$ ***converges pointwise*** on $S \Leftrightarrow$ $f(x) = \lim_{n\to\infty} f_n(x)$ exists for each $x \in S$.
N.B.: N may depend on x.

Definition [Wade 7.7].
Let $S \subseteq \mathbb{R}$, non-empty. A sequence of functions $f_n : S \to \mathbb{R}$ ***converges uniformly*** on S to function $f \Leftrightarrow \forall \varepsilon > 0\, \exists N \in \mathbb{N}$ s.t.:

$$n \geqslant N \Rightarrow |f_n(x) - f(x)| < \varepsilon, \quad \forall x \in S.$$

N.B.: N independent of x.

Definition (Ex. 7.1.3).
Let $f_n : S \to \mathbb{R}$ be a sequence of functions. If $\exists M > 0\, \forall x \in S, n \in \mathbb{N}$ s.t. $|f_n(x)| \leqslant M$, then the sequence of functions is ***uniformly bounded***.

Definition [Wade 7.13].
Let $S \subseteq \mathbb{R}$, $f_k : S \to \mathbb{R}$ and $s_n(x) \coloneqq \sum_{k=1}^{n} f_k(x)$, for $x \in S$, $n \in \mathbb{N}$.

i) $\sum_{k=1}^{\infty} f_k$ converges ***pointwise*** on $S \Leftrightarrow$ sequence $s_n(x)$ converges pointwise on S;
ii) $\sum_{k=1}^{\infty} f_k$ converges ***uniformly*** on $S \Leftrightarrow$ sequence $s_n(x)$ converges uniformly on S;
iii) $\sum_{k=1}^{\infty} f_k$ converges ***absolutely (pointwise)*** on $S \Leftrightarrow$ sequence $\sum_{k=1}^{\infty} |f_k|$ converges for each $x \in S$.

Power Series

Definition (Power Series).
Let (a_n) be sequence of real numbers, $c \in \mathbb{R}$. A ***power series*** is a series of the form:

$$\sum_{n=0}^{\infty} a_n (x - c)^n$$

where a_n are the ***coefficients***, c is the ***centre***.

Definition (Radius of Convergence).
The ***radius of convergence*** R of power series $\sum_{n=0}^{\infty} a_n (x-c)^n$ is

$$R = \sup\{r \geqslant 0 : (a_n r^n) \text{ is bounded}\}$$

unless $(a_n r^n)$ is bounded for all $r \geqslant 0$, then $R = \infty$. I.e. R is ***unique*** number s.t. for $r < R$, $(a_n r^n)$ is bound, for $r > R$, $(a_n r^n)$ is unbound.

Definition (Analytic Function).
A function f is ***analytic*** on $S = \{x \in \mathbb{R} : |x - c| < r\}$ if there is a power series centred at c that converges to f on S.

Riemann Integration

Definition (Uniform Continuity).
Let $I \subseteq \mathbb{R}$ be an interval, $f : I \to \mathbb{R}$. We say f is ***uniformly continuous*** on I if $\forall \varepsilon > 0\, \exists\, \delta > 0$ s.t. for $x, y \in I$:

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$$

Definition (Characteristic Function).
Let $E \subseteq \mathbb{R}$, then $\chi_E : \mathbb{R} \to \mathbb{R}$ is the ***characteristic function*** if $\chi_E(x) = 1$ if $x \in E$, $\chi_E(x) = 0$ if $x \notin E$.

Definition (Area Under the Curve).
Let $I \subset \mathbb{R}$ be a ***bounded interval***. Then

$$\int \chi_I = \text{length}(I).$$

Definition [Integration, Def. 1].
We say $\phi : \mathbb{R} \to \mathbb{R}$ is a ***step function*** if there exist real numbers $x_0 < x_1 < \ldots < x_n$, for some $n \in \mathbb{N}$, s.t.

(i) $\phi(x) = 0$ for $x < x_0$ and $x > x_n$;
(ii) ϕ constant on (x_{j-1}, x_j), $1 \leqslant j \leqslant n$.

Definition (Bounded Support).
A function f has ***bounded support*** if $f(x) = 0$ for $x \notin [c, d]$, where $[c, d]$ is some bounded interval.

Definition [Integration, Def. 2].
Let ϕ be a step function with respect to $\{x_0, x_1, \ldots, x_n\}$, where $\phi(x) = c_j$ for $x \in (x_{j-1}, x_j)$, then

$$\int \phi \coloneqq \sum_{j=1}^{n} c_j (x_j - x_{j-1}).$$

Definition [Integration, Def. 3].
Let $f : \mathbb{R} \to \mathbb{R}$. Then f is ***Riemann-integrable*** if $\forall \varepsilon > 0\, \exists\, \phi, \psi$ step functions s.t. $\phi \leqslant f \leqslant \psi$ and

$$\int \psi - \int \phi < \varepsilon.$$

Definition [Integration, Def. 4].
If f is ***Riemann-integrable***, then we define:

$$\int f := \sup\left\{\int \phi : \phi \text{ step function}, \phi \leqslant f\right\} = \inf\left\{\int \psi : \psi \text{ step function}, \psi \geqslant f\right\}.$$

Definition (Definite Integral).
Let $f : I \to \mathbb{R}$, where I is ***bounded interval*** open/closed at end points $a \leqslant b$. Let $\tilde{f}(x) = f(x)$ for $x \in I$ and $f(x) = 0$ for $x \notin I$. $\tilde{f}$ ***Riemann-integrable*** $\Rightarrow f$ ***Riemann-integrable on*** I and

$$\int_I f = \int_a^b f = \int_a^b f(x)\,dx := \int \tilde{f}$$

is the ***definite integral of*** f ***on*** I.

Definition (Improper Integral).
Let $f : \mathbb{R} \to \mathbb{R}$ be ***possibly unbounded***, let

$$f_n(x) = \operatorname{mid}\{-n, f(x), n\}\chi_{[-n,n]}(x)$$

and

$$F_n(x) = \min\{|f(x)|, n\}\chi_{[-n,n]}(x)$$

If $\sup_n \int F_n < \infty$, then the ***improper integral*** of f over interval I is

$$\int_I f := \lim_{n\to\infty} \int_I f_n.$$

Metric Spaces

Definition [Wade 10.1].
A ***metric space*** is a set X together with a function $\rho : X \times X \to \mathbb{R}$ (the ***metric*** of X) which satisfies the following properties for $x, y, z \in X$:

(i) ***Positive definite:*** $\rho(x, y) \geqslant 0$ with $\rho(x, y) = 0 \Leftrightarrow x = y$;
(ii) ***Symmetric:*** $\rho(x, y) = \rho(y, x)$;
(iii) ***Triangle Inequality:*** $\rho(x, y) \leqslant \rho(x, z) + \rho(z, y)$

N.B.: $\rho(x, y)$ is finite valued by definition.

Definition [Wade 10.7].
Let $a \in X$ and $r > 0$. The ***open ball*** (in X) with ***centre*** a and ***radius*** r is the set

$$B_r(a) := \{x \in X : \rho(x, a) < r\}$$

and the ***closed ball*** (in X) with ***centre*** a and ***radius*** r is the set

$$\{x \in X : \rho(x, a) \leqslant r\}$$

Definition [Wade 10.8].

i) A set $V \subseteq X$ is ***open*** $\Leftrightarrow \forall x \in V\, \exists\, \varepsilon > 0$ s.t. open ball $B_\varepsilon(x) \subseteq V$.
ii) A set $E \subseteq X$ is ***closed*** $\Leftrightarrow$ complement $E^c := X \setminus E$ is ***open***.

Definition [Wade 10.13].
Let $\{x_n\}$ be a sequence in X.

i) $\{x_n\}$ ***converges*** (in X) if $\exists\, a \in X$ (the ***limit*** of x_n) s.t. $\forall \varepsilon > 0\, \exists N \in \mathbb{N}$ s.t.:
$$n \geqslant N \Rightarrow \rho(x_n, a) < \varepsilon.$$
ii) $\{x_n\}$ is ***Cauchy*** if $\forall \varepsilon > 0\, \exists N \in \mathbb{N}$ s.t.:
$$n, m \geqslant N \Rightarrow \rho(x_n, x_m) < \varepsilon.$$
iii) $\{x_n\}$ is ***bounded*** if $\exists M > 0, b \in X$ s.t.
$$\rho(x_n, b) \leqslant M, \quad \forall n \in \mathbb{N}.$$

Definition [Wade 10.19].
A metric space X is ***complete*** $\Leftrightarrow$ ***every Cauchy*** sequence $\{x_n\}$ in X ***converges*** to some point ***in*** X.

Definition [Wade 10.22].
A point $a \in X$ is a ***cluster point*** $\Leftrightarrow \forall \delta > 0$, $B_\delta(a)$ contains ***infinitely*** many points.

Definition (Relative Ball).
Let $E \subseteq X$ be a ***subspace*** of X. An ***open ball*** in E centred at a is defined as

$$B_r^E(a) := \{x \in E : \rho(x, a) < r\}$$

and as metric on X and E are the same, is of the form

$$B_r^E(a) = B_r(a) \cap E$$

where $B_r(a)$ is an open ball in X. $B_r^E(a)$ is called ***relative ball*** (in E). The case with closed balls is analogous.

Definition [Wade 10.25].
Let $a \in X$ be a ***cluster point*** and $f : X \setminus \{a\} \to Y$. Then $f(x) \to L$ as $x \to a \Leftrightarrow \forall \varepsilon > 0\, \exists\, \delta > 0$ s.t.:

$$0 < \rho(x, a) < \delta \Rightarrow \tau(f(x), L) < \varepsilon.$$

Definition [Wade 10.27].
Let $E \subseteq X$, non-empty, and $f : E \to Y$.

i) f is ***continuous at point*** $a \in E \Leftrightarrow \forall \varepsilon > 0\, \exists\, \delta > 0$ s.t.
$$\rho(x, a) < \delta \text{ and } x \in E \Rightarrow \tau(f(x), f(a)) < \varepsilon.$$
ii) f is ***continuous on*** $E \Leftrightarrow f$ ***continuous*** for ***every*** $x \in E$.
N.B.: This is valid whether a is cluster point or not.

Definition (Isolated Points).
Let (X, d) be a metric space, $a \in X$. Then a is ***isolated*** if $\exists\, r > 0$ s.t. $B_r(a) = \{a\}$.

Definition (Strong Equivalence).
Two metrics d and ρ on X are ***strongly equivalent*** if $\exists\, A, B$ s.t.

$$d(x, y) \leqslant A\rho(x, y)$$
$$\rho(x, y) \leqslant Bd(x, y), \quad \forall x, y \in X.$$

Definition (Equivalence).
Two metrics d and ρ on X are ***equivalent*** if $\forall x \in X, \varepsilon > 0\, \exists\, \delta > 0$ s.t.

$$d(x, y) < \delta \Rightarrow \rho(x, y) < \varepsilon \text{ and}$$
$$\rho(x, y) < \delta \Rightarrow d(x, y) < \varepsilon$$

Topology

Definition [Wade 10.33].
Let X be a metric space and $E \subseteq X$.

i) The ***interior*** of E is the set
$$E^o := \bigcup\{V : V \subseteq E \text{ and } V \textbf{ open} \text{ in } X\}.$$
ii) The ***closure*** of E is the set
$$\overline{E} := \bigcap\{B : B \supseteq E \text{ and } B \textbf{ closed} \text{ in } X\}.$$

Definition [Wade 10.37].
Let $E \subset X$. The ***boundary*** of E is the set

$$\partial E := \{x \in X : \forall r > 0, B_r(x) \cap E \neq \emptyset \text{ and } B_r(x) \cap E^c \neq \emptyset\}.$$

Definition [Wade 10.41].
Let $\mathcal{V} = \{V_\alpha\}_{\alpha\in A}$ be a ***collection of subsets*** of metric space X and let $E \subseteq X$.

i) $\mathcal{V}$ ***covers*** E ($\mathcal{V}$ is a ***covering*** of E) $\Leftrightarrow$
$$E \subseteq \bigcup_{\alpha\in A} V_\alpha.$$
ii) $\mathcal{V}$ is an ***open covering*** of $E \Leftrightarrow \mathcal{V}$ ***covers*** E and each V_α is ***open***.
iii) Let $\mathcal{V}$ be a ***covering*** of E. $\mathcal{V}$ has a ***finite/countable subcovering*** $\Leftrightarrow$ there is a ***finite/countable*** subset $A_0 \subseteq A$ s.t. $\{V_\alpha\}_{\alpha\in A_0}$ ***covers*** E.

Definition [Wade 10.42].
Let $H \subseteq X$ with X being a metric space. H is ***compact*** $\Leftrightarrow$ ***every open covering*** of H has ***finite subcover***.

Definition 10.4.10a.
$E \subseteq X$ is ***sequentially compact*** $\Leftrightarrow$ every sequence (x_n) in E has a ***convergent subsequence*** with limit in E.

Definition [Wade 10.53].
Let X be a metric space.

i) A pair of ***non-empty open*** sets U, V in X ***separates*** $X \Leftrightarrow X = U \cup V$ and $U \cap V = \emptyset$.
ii) X is ***connected*** $\Leftrightarrow X$ ***cannot*** be ***separated*** by ***any pair*** of open sets U, V.

Definition [Wade 10.54].
Let X be a metric space and $E \subseteq X$.

i) $U \subseteq E$ is ***relatively open*** in $E \Leftrightarrow \exists V \subseteq X$, s.t. V ***open*** and $U = E \cap V$.
ii) $A \subseteq E$ is ***relatively closed*** in $E \Leftrightarrow \exists C \subseteq X$, s.t. C ***closed*** and $A = E \cap C$.

Contraction Mappings

Definition (Contraction).
Let (X, d) be a metric space. A function $f : X \to X$ is a ***contraction*** if $\exists\, \alpha$ with $0 < \alpha < 1$ s.t.:

$$d(f(x), f(y)) \leqslant \alpha d(x, y), \quad \forall x, y \in X.$$

Constant α is called the ***contraction constant*** of f.

Definition (Fixed Point).
Let $f : X \to X$. If $x \in X$ is s.t. $f(x) = x$, then x is a ***fixed point*** of f.