Program Syntax; Operational Semantics

CS 536: Science of Programming, Fall 2021

Part I: Program Syntax

1. In our simple language, if $x < 0$ then $\{x := 0\}$ is (syntactically) equivalent to what other statement?
2. How are if B then $\{S_1\}$ else $\{S_2\}$ and B ? e_1 : e_2 different?

For Questions 3 – 8, translate the given C / C++ / Java program fragments into our simple programming language.

3. *++x; if (x < y) { x = y = y+1; }*
4. *y = z * ++x; z = z+x;*
5. *y = z * x++; z = z+x;*
6. *x = z = 0; while (x++ < n) z = z+x;*
7. *z = 1; for (x = n ; x >= 1 ; --x) z = z * x;*
8. *x = 0; while (x++ <= n) { y = (++x)*y;}*

Part II: Operational Semantics

9. Evaluate each of the following configurations to completion. If there are multiple steps, show each step individually.
   a. $\langle x := x+1, \{x = 5\}\rangle$
   b. $\langle y := 2*x, \{x = 6\}\rangle$
   c. $\langle x := x+1, \sigma\rangle$ (Your answer will be symbolic — you'll need to include $\sigma(x)$.)
   d. $\langle x := x+1;\ y := 2*x, \{x = 5\}\rangle$

10. Let $S \equiv$ if $x > 0$ then $\{x := x+1\}$ else $\{y := 2*x\}$.
    a. Let $\sigma(x) = 8$, evaluate $\langle S, \sigma\rangle$ to completion, showing the individual steps. Give the final state.
    b. Repeat, if $\sigma(x) = 0$.
    c. Repeat, if we don't know what $\sigma(x)$ is. (Your answer will be symbolic.)

11. Let $S \equiv$ if $x > 0$ then $\{x := x/z\}$ *else* $\{skip\}$. Evaluate S (starting) in σ, for each the σ below:

a. $\sigma = \{x = 8, z = 3\}$ (and don't forget, integer division truncates)
b. $\sigma = \{x = -2, z = 3\}$

12. Let $W \equiv$ while $x < 3$ { S } where $S \equiv x := x+1;\ y := y*x$.
    a. Show what evaluation of the body S in an arbitrary state τ does.
    b. Use your answer from part a to evaluate W in σ where $\sigma \vDash x = 4 \wedge y = 1$.
    c. Repeat part b where $\sigma \vDash x = 1 \wedge y = 1$.

CS 536: Solution to Practice 5 (Program Syntax; Operational Semantics)

Part I: Syntax

1. if $x < 0$ then $x := 0$ else skip fi
2. if B then S_1 else S_2 fi is a statement; its evaluation can change the state.
   B ? e_1 : e_2 is an expression; its evaluation produces a value.
3. $x{:=}x{+}1;$ if $x < y$ then $\{y{:=}y{+}1;\ x{:=}y\}$
4. $x{:=}x{+}1;\ y{:=}z{*}x;\ z{:=}z{+}x$
5. $y{:=}z{*}x;\ x{:=}x{+}1;\ z{:=}z{+}x$
6. $z{:=}0;\ x{:=}z;$ while $x < n$ $\{\ x{:=}x{+}1;\ z{:=}z{+}x\ \};\ x{:=}x{+}1$
7. $z{:=}1;\ x{:=}n;$ while $x >= 1$ $\{\ z{:=}z{*}x;\ x{:=}x{-}1\ \}$
8. In the solution below, the increment of x after the loop is for the $x{+}{+}$ of the test that breaks out of the loop. For the body of the loop, the first increment of x is for the $x{+}{+}$ in $x{+}{+} <= n$ after testing $x <= n$. The immediately following increment of x is for the ++x in $y = ({+}{+}x){*}y$ because the increment occurs before calculating $y = x{*}y$. You could certainly combine the two $x := x{+}1$ to just one $x := x{+}2$.

$x := 0;$ while $x <= n$ $\{\ x := x{+}1;\ x := x{+}1;\ y := x{*}y\ \};\ x := x{+}1$

Part II: Operational Semantics

9. (Calculate meanings of programs)

a. $\langle x := x{+}1, \{x = 5\}\rangle \rightarrow \langle E, \tau\rangle$ where $\tau = \{x = 5\}[x \mapsto \{x = 5\}(x{+}1)]$
   $= \{x = 5\}[x \mapsto 6] = \{x = 6\}.$

b. $\langle y := 2{*}x, \{x = 6\}\rangle \rightarrow \langle E, \tau\rangle$ where $\tau = \{x = 6\}[y \mapsto \{x = 6\}(2{*}x)]$
   $= \{x = 6\}[y \mapsto 12] = \{x = 6,\ y = 12\}$

c. $\langle x := x{+}1, \sigma\rangle \rightarrow \langle E, \sigma[x \mapsto \sigma(x{+}1)]\rangle = \langle E, \sigma[x \mapsto \sigma(x){+}1]\rangle$

d. $\langle x := x{+}1;\ y := 2{*}x, \{x = 5\}\rangle$
   $\rightarrow \langle y := 2{*}x, \{x = 5\}[x \mapsto \alpha]\rangle$ *where* $\alpha = \{x = 5\}(x{+}1) = 6$
   $= \langle y := 2{*}x, \{x = 5\}[x \mapsto 6]\rangle$
   $= \langle y := 2{*}x, \{x = 6\}\rangle$
   $\rightarrow \langle E, \{x = 6\}[y \mapsto \beta]\rangle$ *where* $\beta = \{x = 6\}(2{*}x) = 12$
   $= \langle E, \{x = 6,\ y = 12\}\rangle$

10. Let $S \equiv$ if $x > 0$ then $x := x{+}1$ else $y := 2{*}x$ fi.

a. If $\sigma(x) = 8$, then $\sigma(x > 0) = T$,
   so $\langle S, \sigma\rangle \rightarrow \langle x := x{+}1, \sigma\rangle \rightarrow \langle E, \sigma[x \mapsto \sigma(x{+}1)]\rangle = \langle E, \sigma[x \mapsto 9]\rangle.$

b. If $\sigma(x) = 0$, then $\sigma(x > 0) = F$,
so $\langle S, \sigma\rangle \rightarrow \langle y := 2*x, \sigma\rangle \rightarrow \langle E, \sigma[y \mapsto \sigma(2*x)]\rangle = \langle E, \sigma[y \mapsto 0]\rangle$

c. If $\sigma(x) > 0$ then $\langle S, \sigma\rangle \rightarrow \langle x := x+1, \sigma\rangle = \langle E, \sigma[x \mapsto \sigma(x)+1]\rangle$.
If $\sigma(x) \le 0$ then $\langle S, \sigma\rangle \rightarrow \langle y := 2*x, \sigma\rangle = \langle E, \sigma[y \mapsto 2 \times \sigma(x)]\rangle$.

11. Let $S \equiv$ if $x > 0$ then $x := x/z$ fi $\equiv$ if $x > 0$ then $x := x/z$ else skip fi

a. If $\sigma = \{x = 8, z = 3\}$, then $\sigma(x > 0) = T$ so $\langle S, \sigma\rangle \rightarrow \langle x := x/z, \sigma\rangle \rightarrow \langle E, \sigma[x \mapsto \alpha]\rangle$ where $\alpha = \sigma(x/z) = \sigma[x \mapsto 8/3] = \sigma[x \mapsto 2]$, since integer division truncates.

b. If $\sigma = \{x = \text{-}2, z = 3\}$ then $\sigma(x > 0) = F$, so $\langle S, \sigma\rangle \rightarrow \langle \text{skip}, \sigma\rangle \rightarrow \langle E, \sigma\rangle$.

12. Let $W \equiv$ while $x < 3$ do S od where $S \equiv x := x+1;\ y := y*x$.

a. For arbitrary τ, $\langle S,\tau\rangle \rightarrow \langle x := x+1;\ y := y*x, \tau\rangle \rightarrow \langle y := y*x, \tau[x \mapsto \tau(x)+1]\rangle$
$\rightarrow \langle E, \tau[x \mapsto \tau(x)+1][y \mapsto \alpha]\rangle$ where $\alpha = \tau[x \mapsto \tau(x)+1](y*x) = \tau(y) \times (\tau(x)+1)$.

b. If $\sigma \vDash x = 4 \wedge y = 1$, then $\sigma(x < 3) = F$ so $\langle W, \sigma\rangle \rightarrow \langle E, \sigma\rangle$.

c. If $\sigma \vDash x = 1 \wedge y = 1$, then $\sigma(x < 3) = T$ so we have at least one iteration to do.
Let $\sigma_0 = \sigma$, let $\sigma_1 = \sigma_0(y) \times (\sigma_0(x)+1)$, and let $\sigma_2 = \sigma_1(y) \times (\sigma_1(x)+1)$. Then

$$\sigma_0 = \sigma[x \mapsto 1][y \mapsto 1]$$
$$\sigma_1 = \sigma_0[x \mapsto \sigma_0(x)+1][y \mapsto \sigma_0(y) \times (\sigma_0(x)+1)] = \sigma[x \mapsto 2][y \mapsto 2]$$
$$\sigma_2 = \sigma_1[x \mapsto 2+1][y \mapsto 2 \times (2+1)] = \sigma[x \mapsto 3][y \mapsto 6]$$

Since σ_0 and $\sigma_1 \vDash x < 3$ but $\sigma_2 \vDash x \ge 3$, we have
$\langle W, \sigma\rangle \rightarrow \langle S; W, \sigma_0\rangle \rightarrow^* \langle W, \sigma_1\rangle = \langle S; W, \sigma_1\rangle \rightarrow^* \langle W, \sigma_2\rangle \rightarrow \langle E, \sigma_2\rangle$, so σ_2 is the final state.