Propositional and Predicate Logic

CS 536: Science of Programming, Fall 2021

1. Fill in the missing rule names in the proof below of $\neg(p \leftrightarrow q) \Leftrightarrow (q \wedge \neg p) \vee (p \wedge \neg q)$, using the rules from the class notes. (See page 2.)

   $\neg(p \leftrightarrow q)$
   $\Leftrightarrow \neg((p \rightarrow q) \wedge (q \rightarrow p))$ by defn $\leftrightarrow$
   $\Leftrightarrow \neg(p \rightarrow q) \vee \neg(q \rightarrow p)$ __________
   $\Leftrightarrow ((p \wedge \neg q) \vee (q \wedge \neg p)$ __________
   $\Leftrightarrow (q \wedge \neg p) \vee (p \wedge \neg q)$ __________

2. Write a formal proof that shows that $(p \rightarrow p \vee q)$ (sometimes called the "$\vee$ introduction" rule) is a tautology: Prove $(p \rightarrow p \vee q) \Leftrightarrow T$'

   $p \rightarrow p \vee q$
   $\Leftrightarrow$ __________ by _________
   etc.

3. Some logical rules can be derived from others. Prove the rule of contraposition by proving $(p \rightarrow q) \rightarrow (\neg q \rightarrow \neg p) \Leftrightarrow T$, using only these rules: Definition of $\rightarrow$, double negation, commutativity of $\vee$, and excluded middle. (You may need to use a rule more than once.)

3. Some logical rules can be derived from others. Use (only) the rules given with each problem (not necessarily in that order). You may need to use a rule more than once.

   a. Prove the rule of contraposition, $(p \rightarrow q) \rightarrow (\neg q \rightarrow \neg p) \Leftrightarrow T$, using (only) Definition of $\rightarrow$, double negation, commutativity of $\vee$, and excluded middle.

   b. Prove the rule of transitive contradiction, $(p \rightarrow q) \wedge (q \rightarrow \neg p) \Leftrightarrow \neg p$, using defn $\rightarrow$, identity, contradiction, and distributivity.

   c. Prove the rule of left $\wedge$ elimination, $p \wedge q \rightarrow p \Leftrightarrow T$ using defn $\rightarrow$, distributivity, excluded middle and DeMorgan's law. (There's a similar rule of right $\rightarrow$ elimination, by the way.)

   d. Prove a rule that combines left and right $\vee$ introduction, $(p \rightarrow p \vee q) \wedge (q \rightarrow p \vee q) \Leftrightarrow T$, using excluded middle, identity, domination, and defn $\rightarrow$.

4. Let $q(x, y) \equiv x < y \rightarrow y < z \wedge f(x) = 2$. Expand $\neg q(x, y)$ to remove $\neg$ signs: Use the rules to find a predicate equivalent to $\neg(x < y \rightarrow y < z \wedge f(x) = 2)$ that doesn't use $\neg$. Hint: Use DeMorgan's laws to move the negation "inward" to smaller and smaller subexpressions. Show your reasoning as a formal proof. (Don't forget the rule names.)

6. In general, if $\forall x.\forall y.\ p(x, y)$ is valid, is $\forall y.\ \forall x. p(x, y)$ also valid? What about $\exists x. \exists y. p(x, y)$ and $\exists y. \exists x. p(x, y)$?

7. Using propositional and predicate proof rules, find a predicate equivalent to $\neg(\forall x. \exists y. p(x, y))$ that has no negation symbols (i.e., $\neg$), except possibly in front of $p(x, y)$. Write a formal proof that shows each step needed (don't forget the rule names!). Hint: Use DeMorgan's laws to move the negation inward.

8. Repeat the previous question on $\neg(\exists y . \forall x . p(x, y))$.
9. Write the definition of a predicate function *Repeats(b, m)* that is true exactly when the first *m* elements of *b* match the second *m* elements of *b*: I.e., *b[0] = b[m], b[1] = b[m+1], …, b[m-1] = b[2*m-1]*. (Alternatively, *b[0..m-1]* and *b[m..2*m-1]* are pointwise equal.) Example: If *b* is *[1, 3, 5, 1, 3, 5]*, then *Repeats(b, 3)* is true but *Repeats(b, 2)* is false.

CS 536: Solution to Practice 2 (Propositional and Predicate Logic)

1. DeMorgan's law; Negation of → twice; commutativity of ∨.

2. $p \to p \vee q$
   $\Leftrightarrow \neg p \vee (p \vee q)$ Defn →
   $\Leftrightarrow (\neg p \vee p) \vee q$ Associativity of ∨
   $\Leftrightarrow T \vee q$ Excluded middle
   $\Leftrightarrow T$ Domination

3a. (contraposition)

$(p \to q) \to (\neg q \to \neg p)$
$\Leftrightarrow (\neg p \vee q) \to (\neg q \to \neg p)$ Defn →
$\Leftrightarrow (\neg p \vee q) \to (\neg\neg q \vee \neg p)$ Defn →
$\Leftrightarrow (\neg p \vee q) \to (q \vee \neg p)$ ¬ ¬
$\Leftrightarrow (\neg p \vee q) \to (\neg p \vee q)$ Comm. of ∨
$\Leftrightarrow \neg(\neg p \vee q) \vee (\neg\neg q \vee \neg p)$ Defn →
$\Leftrightarrow T$ Excluded middle (on $(\neg p \vee$
$q)$)

3b. (transitive contradiction)

$(p \to q) \wedge (q \to \neg p)$
$\Leftrightarrow (\neg p \vee q) \wedge (q \to \neg p)$ defn →
$\Leftrightarrow (\neg p \vee q) \wedge (\neg q \vee \neg p)$ defn →
$\Leftrightarrow \neg p \vee (q \wedge \neg q)$ distributivity
$\Leftrightarrow \neg p \vee F$ contradiction
$\Leftrightarrow \neg p$ identity

3c. (left ∧ elimination)

$p \wedge q \to p$
$\Leftrightarrow \neg(p \wedge q) \vee p$ defn →
$\Leftrightarrow (\neg p \vee \neg q) \vee p$ DeMorgan
$\Leftrightarrow (p \vee \neg p) \vee (\neg q \vee p)$ distributivity
$\Leftrightarrow T \vee (q \vee \neg q)$ excluded middle
$\Leftrightarrow T$ domination

3d. (left and right ∨ introduction)

$(p \to p \vee q) \wedge (q \to p \vee q)$
$\Leftrightarrow (\neg p \vee p \vee q) \wedge (\neg q \vee p \vee q)$ defn → (twice)
$\Leftrightarrow (T \vee q) \wedge (T \vee p)$ excluded middle (twice)
$\Leftrightarrow T \wedge T$ domination (twice)
$\Leftrightarrow T$ identity

4. If $q(x, y) \equiv x < y \to y < z \wedge f(x) = 2$, then

$\neg q(x, y)$
$\Leftrightarrow \neg(x < y \to y < z \wedge f(x) = 2)$ Defn of q
$\Leftrightarrow x < y \wedge \neg(y < z \wedge f(x) = 2)$ Negation of →

$\Leftrightarrow x<y \wedge (\neg(y < z) \vee \neg(f(x) = 2))$ DeMorgan's Law
$\Leftrightarrow x<y \wedge (y \geq z \vee f(x) \neq 2)$ Negation of comparison, 3 times

6. ($Q\,\mathrm{x}.Q\,y$ versus $Q\,y.Q\,\mathrm{x}$)
   a. Yes: $(\forall x.\forall y.\ p(x, y))$ is valid if and only if $(\forall y.\forall x.\,p(x, y))$ is valid
   b. Yes: $(\exists x.\exists y.\ p(x, y))$ is valid if and only if $(\exists y.\exists x.\,p(x, y))$ is valid

7. $\neg(\forall x.\exists y.\,p(x, y))$
   $\Leftrightarrow \exists x\,.\,\neg\exists y\,.\,p(x, y)$ DeMorgan's Law $\neg\forall$
   $\Leftrightarrow \exists x\,.\,\forall y\,.\,\neg\ p(x, y)$ DeMorgan's Law $\neg\exists$

8. $\neg(\exists y.\forall x.\,p(x, y))$
   $\Leftrightarrow \forall y.\neg(\forall x.\,p(x, y))$ DeMorgan's Law $\neg\exists$
   $\Leftrightarrow \forall y.\exists x.\neg p(x, y)$ DeMorgan's Law $\neg\forall$

9. First, here's a solution that doesn't check for *m* being too large:

   $Repeats(b, m) \equiv \forall j.0 \leq j < m \rightarrow b[j] = b[m+j]$

   You can also use a bounded quantifier: $Repeats(b, m) \equiv \forall 0 \leq j < m.\ b[j] = b[m+j]$.

   If we want to check for m being too large, then assuming ∧ is short-circuiting (like && in C etc.), we can write

   $Repeats(b, m) \equiv 0 \leq 2*m < size(b) \wedge \forall j.(0 \leq j < m \rightarrow b[j] = b[m+j])$.

