# *Syntactic Substitution*

## *CS 536: Science of Programming, Fall 2021*

1. Calculate $[b+c/c][i+1/i](\ x+i*b+c = 0\ )$.
2. Let $p$ be $\exists x. x < y \wedge x^2 \geq y+k$
   a. What is $[5/x]p$?
   b. What is $[5/y]p$?
   c. What is $[5/z]p$?
   d. What is $[y*2/y]p$?
   e. What is $[y*k/y]p$?
   f. What is $[(x + y) \div 2/y]p$?
3. Give an example where $[e/v][e'/w](v * w)$ and $[e'/w][e/v](v * w)$ are
   a. Syntactically equal $(\equiv)$
   b. Syntactically unequal $(\not\equiv)$.
4. In the predicate $(\exists x\ .\ x < y \wedge x^2 \geq y+k)$, $x$ is bound, but in $(x < y \wedge x^2 \geq y+k)$, $x$ is free — is this a contradiction?
6. Let $p \equiv (\forall x. \exists y. R(x, y, z)) \wedge (\exists z. R(x, y, z))$ where $R$ is a boolean function over three arguments.
   a. What is $[17/w]p$ ?
   b. What is $[17/x]p$ ?
   c. What is $[y*2/y]p$ ?
   d. What is $[y*2/z]p$ ?
   e. What is $[a+b/z][a*z/y]p$ ?

 

## *Solution to Practice 12 (Syntactic Substitution)*

1. $[b+c/c][i+1/i](x+i*b+c = 0) \equiv [b+c/c]( x + (i+1)*b + c = 0 )$
   $\equiv x+(i+1)*b+(b+c) = 0$
2. Let $p \equiv \exists x . x < y \wedge x^2 \geq y+k$
   - 2a. $[5/x]p \equiv p$ unchanged
   - 2b. $[5/y]p \equiv [5/y](\exists x . x < y \wedge x^2 \geq y+k) \equiv \exists x . x < 5 \wedge x^2 \geq 5+k$
   - 2c. $[5/z]p \equiv p$ unchanged because $z$ doesn't occur in $p$
   - 2d. $[y*2/y]p \equiv [y*2/y](\exists x . x < y \wedge x^2 \geq y+k) \equiv \exists x . x < y*2 \wedge x^2 \geq y*2+k$
   - 2e. $[y*k/y]p \equiv [y*k/y](\exists x . x < y \wedge x^2 \geq y+k) \equiv \exists x . x < y*k \wedge x^2 \geq y*k+k$
   - 2f. $[(x + y) \div 2/y]p \equiv [(x + y) \div 2/y](\exists x . x < y \wedge x^2 \geq y+k)$

     $\equiv [(x + y) \div 2/y]\exists v.[v/x](x < y \wedge x^2 \geq y+k)$ (note renaming of $x$ to $v$)

     $\equiv [(x + y) \div 2/y]\ \exists v .(v < y \wedge v^2 \geq y+k)$

     $\equiv \exists v . v < (x + y) \div 2 \wedge v^2 \geq (x + y) \div 2 + k$
3. (Cases where $[e'/w][e/v](v * w)$ and $[e/v][e'/w](v * w)$ are $\equiv$ and $\not\equiv$.)
   - 3a. One case is when $v$ doesn't occur in $e'$ and $w$ doesn't occur in $e$.

     Example: $[a*w/w][v*2/v](v * w) \equiv [a*w/w](v*2 * w)$

     $\equiv v*2 *(a*w) \equiv [v*2/v](v *(a*w))$

     $\equiv [v*2/v][a*w/w](v * w)$
   - 3b. One case is when $w$ appears in $e$ and $v$ appears in $e'$, at least, for certain $e$ and $e'$.

   Example: $[a*v/w][w\text{-}3/v](v * w) \equiv [a*v/w]((w\text{-}3) * w) \equiv (w\text{-}3) * (a * v)$

   but $[w\text{-}3/v][a*v/w](v * w) \equiv [w\text{-}3/v](v *(a*v)) \equiv (w\text{-}3) *(a* (w\text{-}3))$
4. No, this is exactly what a quantifier does: It captures the x's that are free in its body and makes them bound with respect to any context that includes the quantified predicate.
6. Substitutions with $p \equiv (\forall x.\exists y.R(x, y, z)) \wedge \exists z.R(x, y, z)$:
   - 6a. $[17/w]p \equiv p$ (because $w$ doesn't occur in $p$)
   - 6b. $[17/x]p \equiv (\forall x.\exists y.R(x, y, z)) \wedge \exists z.R(17, y, z) )$
   - 6c. $[y*2/y]p \equiv (\forall x.\exists y.R(x, y, z)) \wedge \exists z.R(x, y*2, z) )$
   - 6d. $[y*2/z]p \equiv (\forall x.\exists v.[y*2/z][v/y]R(x, y, z)) \wedge \exists z.R(x, y, z) )$ (using $v$ as a fresh variable)

     $\equiv (\forall x.\exists v.R(x, v, y*2)) \wedge \exists z.R(x, y, z) )$
   - 6e. $[a+b/z][a*z/y]p$

     $\equiv [a+b/z](\forall x.\exists y.R(x, y, z)) \wedge \exists v.[a*z/y][v/z]R(x, y, z))$ (using $v$ as a fresh variable)

     $\equiv [a+b/z]( (\forall x.\exists y.R(x, y, z)) \wedge \exists v.[a*z/y]R(x, y, v) )$ (only the first $y$ is quantified)

     $\equiv [a+b/z]((\forall x.\exists y.R(x, y, z)) \wedge \exists v.R(x, a*z, v))$

     $\equiv ((\forall x.\exists y.R(x, y, a+b)) \wedge \exists v.R(x, a*(a+b), v))$ (parens around $a+b$ are required)

     (No renaming necessary because we have no quantification of $a$ or $b$.)