Satisfaction w/ Quantifiers

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$\forall x \in \mathbb{Z}. x \neq 0 \to x \leq x^2$ Is it a tautology?

Remember: $\models P$ if $\sigma \models P$ for all well-formed, proper σ
What does that mean here?
What states are proper?
Before: Needs to assign values to all variables in P
Do we need to know the value of x to eval. $\forall x \in \mathbb{Z}. x \neq 0 \to x \leq x^2$? No.
Now: Needs to assign values to all free vars in P.

Free variable: Not bound by a quantifier

$(\forall x \in \mathbb{Z}. \underbrace{x \neq 0 \to \exists y. y^2 < x}) \wedge (F \to T)$
x bound, y bound

$x \neq 0 \to \exists y. y^2 < x$ y bound, x free.

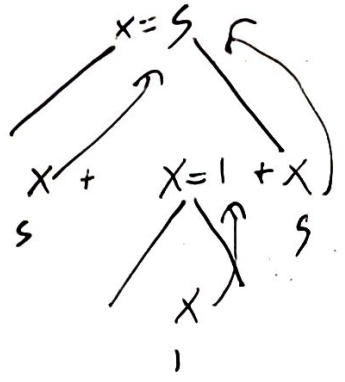
Back to $\forall x \in \mathbb{Z}. x \neq 0 \to x \leq x^2$
Or any $\forall x \in \mathbb{Z}. P$
When does $\sigma \models \forall x. P$? When P is true for all values of x... and σ

State updates
Need to add x to σ
$\sigma[x \mapsto 5]$ = σ with the added binding $x = 5$

$\sigma \models \forall x \in \mathbb{Z}. P$ when $\sigma[x \mapsto n] \models P$ for all $n \in \mathbb{Z}$.

If x is already in σ, replace it.
Not a real update, just shadowing
(should really call it extend rather than update)

let $x = 5$ in $x + (\text{let } x = 1 \text{ in } x) + x$



$\{x = 5, y = 6\}[x \mapsto 7] \models x \geq y$ b.c. $x=5$ is shadowed

Can always avoid confusion by renaming the different x's
"α-conversion"

let $x_1 = 5$ in $x_1 + (\text{let } x_2 = 1 \text{ in } x_2) + x_1$

Multiple updates go left to right

$\sigma[x \mapsto 5][y \mapsto 6] \not\models x \geq y$
$\sigma[x \mapsto 5][y \mapsto 6][x \mapsto 7] \models x \geq y$

Back to $\forall x \in \mathbb{Z}.\ x \neq 0 \to x \leq x^2$
Tautology iff $\sigma[x \mapsto n] \models x \neq 0 \to x \leq x^2$ for all $n \in \mathbb{Z}$
(regardless of $\sigma(x)$; doesn't matter bc. it's shadowed)

Similar: $\forall x \in S.\ P$ for any set S.

$\sigma \models \exists x \in S, P$ iff $\sigma[x \mapsto v] \models P$ for some $v \in S$

$\{\} \models \exists x \in \mathbb{Z}. x \neq 0 \wedge x \geq x^2$?
Yes. $\{\}[x \mapsto 1] \models x \neq 0 \wedge x \geq x^2$
Is it a tautology? Yes. State is irrelevant.

$x > 0 \rightarrow \exists y. y^2 < x$
Proper state needs to have value for x. (y is bound; doesn't matter)

$\sigma \models x > 0 \rightarrow \exists y. y^2 < x$
if for all states σ where $\sigma(x) > 0$, $\sigma[y \mapsto n] \models y^2 < x$ for some n.
True: $n = 0$
$\Rightarrow$ Tautology

If P has no free variables, can it be a contingency?
No. No free vars $\Rightarrow$ state doesn't matter $\Rightarrow$ true or false $\forall \sigma$

$\sigma \models \forall x. x > y \rightarrow \exists z. z \geq x + y^2$ σ must have y
Tautology: true for all y
Contingency: some
Contradiction: no

For every $n_1 \in \mathbb{Z}$, if $n_1 > \sigma(y)$ then $\sigma[x \rightarrow n_1][z \mapsto n_2] \models z \geq 2x + y^2$ for some z
Tautology: No matter what $\sigma(y)$, n_1, are, we can always pick a bigger n_2.