CSE 5095: Types and Programming Languages

Rule Induction and Syntax

Stefan Muller

August 28, 2025

1 Rule Induction

Below are the proofs done in class, written out formally as you would write them on a homework.

First, here are the rules for our "is XML" judgment.

$$\frac{}{\text{“” is XML}}\ (\text{X-1}) \qquad \frac{}{\texttt{<T>}\ \textsf{is XML}}\ (\text{X-2}) \qquad \frac{X\ \textsf{is XML}}{\texttt{<T>}X\texttt{</T>}\ \textsf{is XML}}\ (\text{X-3}) \qquad \frac{X\ \textsf{is XML} \qquad Y\ \textsf{is XML}}{X\ Y\ \textsf{is XML}}\ (\text{X-4})$$

Let $\mathit{OpenAngle}(X)$ be the number of open angle brackets in a string X and $\mathit{CloseAngle}(X)$ be the number of close angle brackets in X.

Theorem 1. *If X is XML, then $\mathit{OpenAngle}(X) = \mathit{CloseAngle}(X)$.*

Proof. By induction on the derivation of X is XML.

- Case X-1. Then $X =$ “” and $\mathit{OpenAngle}(X) = \mathit{CloseAngle}(X) = 0$.
- Case X-2. Then $X = \texttt{<T>}$ and $\mathit{OpenAngle}(X) = \mathit{CloseAngle}(X) = 1$.
- Case X-3. Then $X = \texttt{<T>}Y\texttt{</T>}$ and Y is XML. We have $\mathit{OpenAngle}(X) = 2 + \mathit{OpenAngle}(Y)$ and $\mathit{CloseAngle}(X) = 2 + \mathit{CloseAngle}(Y)$. By induction, $\mathit{OpenAngle}(Y) = \mathit{CloseAngle}(Y)$, so $2 + \mathit{OpenAngle}(Y) = 2 + \mathit{CloseAngle}(Y)$.
- Case X-4. Then $X = Y\ Z$ and Y is XML and Z is XML. We have $\mathit{OpenAngle}(X) = \mathit{OpenAngle}(Y) + \mathit{OpenAngle}(Z)$ and $\mathit{CloseAngle}(X) = \mathit{CloseAngle}(Y) + \mathit{CloseAngle}(Z)$. By induction, $\mathit{OpenAngle}(Y) = \mathit{CloseAngle}(Y)$ and $\mathit{OpenAngle}(Z) = \mathit{CloseAngle}(Z)$, so $\mathit{OpenAngle}(X) = \mathit{CloseAngle}(X)$.

□

Now here are the rules for natural numbers (both constructing a natural number and the greater-than judgment)

$$\frac{}{0\ \textsf{is a natural number}}\ (\textsc{zero}) \qquad \frac{n\ \textsf{is a natural number}}{n+1\ \textsf{is a natural number}}\ (\textsc{succ}) \qquad \frac{n\ \textsf{is a natural number}}{n \geq n}\ (\textsc{ge-nat})$$

$$\frac{m\ \textsf{is a natural number} \qquad m \geq n}{m+1 \geq n}\ (\textsc{ge-succ})$$

Theorem 2. *If n is a natural number then $n \geq 0$.*

Proof. By induction on the derivation of n is a natural number.

- Rule ZERO. Then $n = 0$. By GE-NAT, we have $n \geq 0$.
- Rule SUCC. Then $n = m + 1$ and m is a natural number. By induction, $m \geq 0$. By GE-SUCC, we have $m + 1 \geq 0$.

□

2 E Language

2.1 Syntax

We will be working with a small language called E consisting of integer and string expressions. The grammar below is in BNF (Backus-Naur Form). We use $e ::= A \mid B \mid ...$ to mean that an expression (with metavariable e) can look like form A or B, and so on.

$$
\begin{array}{rcll}
e & ::= & \overline{n} & \text{Numbers} \\
  & \mid & \text{“}s\text{”} & \text{Strings} \\
  & \mid & e + e & \text{Addition} \\
  & \mid & e \mathbin{\hat{}} e & \text{Concatenation} \\
  & \mid & |e| & \text{String Length}
\end{array}
$$