Normalization of STLC

Goal: Prove that if $\bullet \vdash e : \tau$ then there exists v such that v val and $e \mapsto^* v$.

Obvious but wrong idea: Prove by induction on the derivation of $\bullet \vdash e : \tau$.

~~Type~~ Unit-I Then $e = ()$ and $e \mapsto^* ()$ and $()$ val ✓

$\to$E Then $e = e_1\ e_2$ and $\bullet \vdash e_1 : \tau' \to \tau$ and $\bullet \vdash e_2 : \tau$.

By induction, $e_1 \mapsto^* v_1$ and $e_2 \mapsto^* v_2$.

So (need some lemmas for this) $e \mapsto^* v_1\ v_2 = (\lambda x.\ e_0)\ v_2 \mapsto [v_2/x]\,e_0$

It's true that $\bullet \vdash [v_2/x]\,e_0 : \tau$ but we don't have an IH that says $[v_2/x]\,e_0 \mapsto^* v$!

Instead, define a set R_τ of "halting" terms of type τ

$R_{()}(e)$ iff $e \mapsto^* ()$

$R_{\tau_1 \to \tau_2}(e)$ iff $e \mapsto^* v$ where v val and $\underbrace{\text{if } R_{\tau_1}(e') \text{ we have } R_{\tau_2}(e\ e')}_{\text{Needed for strong enough IH.}}$

New goal: If $\bullet \vdash e : \tau$ then $R_\tau(e)$

Lemma 1: If $R_\tau(e)$ then e halts   "e halts" $\triangleq \exists v$ val s.t. $e \mapsto^* v$

Pf: Clear from the definition.

Lemma 2: If $\bullet \vdash e : \tau$ and $e \mapsto e'$ then $R_\tau(e)$ iff $R_\tau(e')$

Pf: By induction on the structure of τ.

- $\tau = \text{unit}$. Then we need that e halts iff e' halts. But this is clear. ✓
- $\tau = \tau_1 \to \tau_2$.

  $R_\tau(e) \Rightarrow R_\tau(e')$. As above, e' halts because e does.

  We have: if $R_{\tau_1}(\hat{e})$ then $R_{\tau_2}(e\,\hat{e})$

  Need: if $R_{\tau_1}(\hat{e})$ then $R_{\tau_2}(e'\hat{e})$. Suppose $\bullet \vdash \hat{e} : \tau_1$ and $R_{\tau_1}(\hat{e})$.

  By StepSearchApp left, $e\,\hat{e} \mapsto e'\hat{e}$. By IH, $R_{\tau_2}(e'\hat{e})$.

  $R_\tau(e') \Rightarrow R_\tau(e)$. Similar. $\square$

To show goal, we need something stronger to reason about the lambda case.

Lemma 3: If $\underbrace{x_1 : \tau_1, \ldots, x_n : \tau_n}_{\Gamma} \vdash e : \tau$ and $\bullet \vdash v_i : \tau_i \; \forall i \in [1,n]$ and $R_{\tau_i}(v_i)$, then $R_\tau([v_1/x_1] \cdots [v_n/x_n] e)$.

Proof: By induction on the typing derivation.

Unit-I. Then $[v_1/x_1] \cdots [v_n/x_n] e = e = ()$. Clearly, $R_{unit}(())$.

Var. Then $e = x_i$ for some $i \in [1,n]$ and $\tau = \tau_i$.

$[v_1/x_1] \cdots [v_n/x_n]\, e = v_i$. By assumption, $R_{\tau_i}(v_i)$.

$\to$E Then $e = e_1\, e_2$ and $\Gamma \vdash e_1 : \tau' \to \tau$ and $\Gamma \vdash e_2 : \tau_1$

By induction, $R_{\tau' \to \tau}([\Gamma] e_1)$ and $R_{\tau'}([\Gamma] e_2)$.

$[\Gamma] = [v_1/x_1] \cdots [v_n/x_n]$

We have $[\Gamma](e_1\, e_2) = [\Gamma] e_1\; [\Gamma] e_2$

By definition of $R_{\tau' \to \tau}$, we have $R_\tau([\Gamma] e_1\; [\Gamma] e_2)$.

$\to I$ Then $e = \lambda x{:}\tau'.\, e_0$ and $\Gamma, x{:}\tau' \vdash e_0 : \tau''$ and $\tau = \tau' \to \tau''$.

Need: $[\gamma]e$ halts and if $R_{\tau'}(e_1)$ then $R_{\tau''}([\gamma]e\ e_1)$

Suppose $R_{\tau'}(e_1)$. Then (by Lemma 1) $\exists v$ s.t. $e_1 \mapsto^* v$.

By repeated apps of Lemma 2, $R_{\tau'}(v)$.

By IH, $R_{\tau''}([\gamma][v/x]e_0)$.

And we have $[\gamma]e\ v = (\lambda x{:}\tau'.[\gamma]e_0)v \mapsto [\gamma][v/x]e_0$.

By Lemma 2, $R_{\tau''}([\gamma]e\ v)$.

Also: $[\gamma]e\ e_1 \mapsto^* [\gamma]e\ v$. So by Lemma 2, $R_{\tau''}([\gamma]e\ e_1)$ ✓ □

Corollary. If $\cdot \vdash e:\tau$, then e halts.

Pf. We have $R_\tau(e)$ by Lemma 3, so e halts by Lemma 1. □