Untyped λ Calculus

"Terms" $M ::= x \mid \lambda x.M \mid M\,M$

one arg. function   application

Semantics

Usually defined in terms of equivalence

$M \equiv M$

$$\frac{}{M \equiv M}\ (\text{Reflexivity}) \qquad \frac{M \equiv M'}{M' \equiv M}\ (\text{Symmetry}) \qquad \frac{M_1 \equiv M_2 \quad M_2 \equiv M_3}{M_1 \equiv M_3}\ (\text{Transi}$$

$$\frac{M \equiv M'}{\lambda x.M \equiv \lambda x.M'}\ (\text{Cong }\lambda) \qquad \frac{M_1 \equiv M_1' \quad M_2 \equiv M_2'}{M_1 M_2 \equiv M_1' M_2'}\ (\text{Cong App})$$

$$\frac{y \notin FV(M)}{\lambda x.M \equiv \lambda y.[y/x]M}\ (\alpha) \qquad \frac{}{(\lambda x.M)N \equiv [N/x]M}\ (\beta) \qquad \frac{x \notin FV(M)}{\lambda x.M\,x \equiv M}\ (\eta)$$

Can we define it as reduction $M \mapsto M'$? Yes. Many diff. ways

"Full β reduction" - β reduce (apply $(\lambda x.M)N \mapsto [N/x]M$) anywhere until you can't.

Convention: body of λ extends until a close-paren.

$(\lambda x.x)\ ((\lambda x.x)\ (\lambda z.\ (\lambda x.x)\ z))$

$\lambda x.x = id$ (Identity func.)

$id\ \ (id\ \ (\lambda z.\ id\ z))$

$\mapsto\ id\ (\lambda z.\ id\ z)$ $\qquad$ $id\ (id\ (\lambda z.z))$

$\mapsto\ (\lambda z.\ id\ z)$ $\qquad$ $(id\ (\lambda z.z)$

$\mapsto\ \lambda z.z$ $\qquad$ $\lambda z.z$

β normal form - No more β reductions possible!

Can also define call-by-value, call-by-name semantics
(or impose orderings on full β reduction

CBV - Evaluate arg. before substituting

$$\begin{aligned} & id\ (id\ (\lambda z.\ id\ z)) \\ \to\ & id\ (\lambda z.\ id\ z) \\ \mapsto\ & \lambda z.\ id\ z \end{aligned}$$

CBN - Evaluate function and then substitute

$$\begin{aligned} & id\ (id\ (\lambda z.\ id\ z)) \\ \mapsto\ & id\ (\lambda z.\ id\ z) \\ \mapsto\ & \lambda z.\ id\ z \end{aligned}$$

CBV, CBN may not reach full (β-)normal form!

Normal form may not exist

$$\begin{aligned} & (\lambda x.\ x\ x)\ (\lambda x.\ x\ x) \\ \mapsto\ & (\lambda x.\ x x)\ (\lambda x.\ x x) \\ \mapsto\ & \ldots \end{aligned}$$

CBN can reach a normal form when CBV doesn't.

$$\begin{aligned} & (\lambda x.\ (\lambda z.\ z))\ ((\lambda x.\ x x)\ (\lambda x.\ x x)) \quad \mapsto_{CBN} \lambda z.\ z \\ \mapsto_{CBV}\ & (\lambda x.\ (\lambda z.\ z))\ ((\lambda x.\ x x)\ (\lambda x.\ x x)) \\ \mapsto_{CBV}\ & \ldots \end{aligned}$$

"Parallell reduction" - allows the most choice $M \Rrightarrow M$

$$\frac{}{M \Rrightarrow M}\ (\text{Refl}) \qquad \frac{M \Rrightarrow M' \quad N \Rrightarrow N'}{M\ N \Rrightarrow M'\ N'}\ (\text{App}) \qquad \frac{M \Rrightarrow M' \quad N \Rrightarrow N'}{(\lambda x.\ M)\ N \Rrightarrow [N'/x]M'}\ (\text{App}\beta)$$

$$\frac{M \Rrightarrow M'}{\lambda x.\ M \Rrightarrow \lambda x.\ M'}\ (\text{Abs})$$

Lemma (Diamond Property) If $M \Mapsto N_1$ and $M \Mapsto N_2$, then $\exists M'$ s.t. $N_1 \Mapsto M'$ and $N_2 \Mapsto M'$

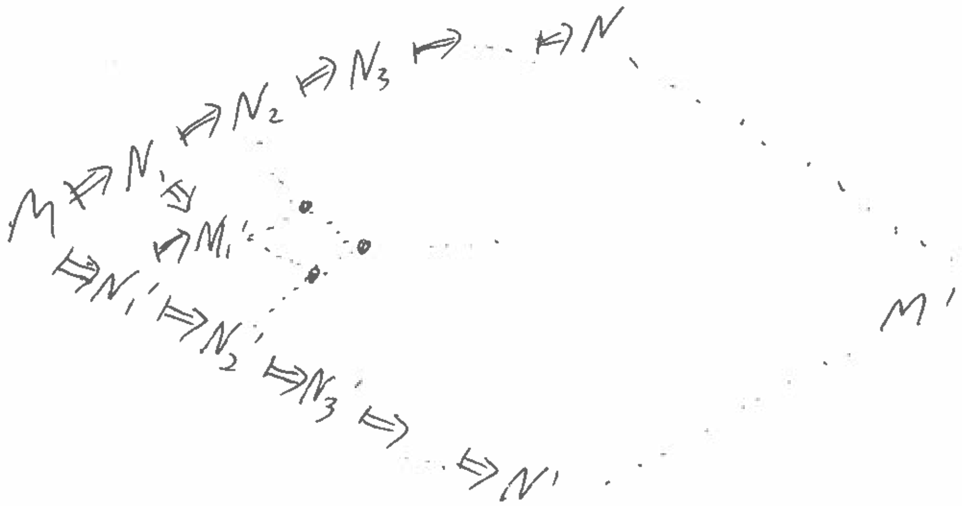
$$\begin{tikzcd} M \arrow[r, Mapsto] \arrow[dr, Mapsto] & N_1 \arrow[r, Mapsto] & M' \\ & N_2 \arrow[ur, Mapsto] & \end{tikzcd}$$

Pf. By induction on the derivation of $M \Mapsto N_1$ and $M \Mapsto N_2$.

Thm (Church-Rosser / Confluence)

If $M \Mapsto^* N_1$ and $M \Mapsto^* N_2$, then $\exists M'$ s.t. $N_1 \Mapsto^* M'$ and $N_2 \Mapsto^* M'$

Pf Sketch.



A language has the "Church-Rosser Property" if different evaluati[on] strategies/orders lead to the same result (normal form)