Machine Learning **(Due: 21th March 12:00)**

Submission Assignment #1 Solution

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Problem 1: Basic Vector Operations (points)

(1) $||\mathbf{a}||_2 = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$
$||\mathbf{b}||_2 = \sqrt{8^2 + 1^2 + 2^2} = \sqrt{69}$

(2) $||\mathbf{a} - \mathbf{b}||_2 = \sqrt{9^2 + 1^2 + 1^2} = \sqrt{83}$

(3) $\because \mathbf{a}^{\mathrm{T}}\mathbf{b} = -8 + 2 + 6 = 0 \therefore$ $\mathbf{a}$ *and* $\mathbf{b}$ are orthogonal

Problem 2: Basic Matrix Operations (points)

(1) $(\mathbf{A}, \mathbf{E}) = \begin{pmatrix} 1 & -3 & 3 & 1 & 0 & 0 \\ 3 & -5 & 3 & 0 & 1 & 0 \\ 6 & -6 & 4 & 0 & 0 & 1 \end{pmatrix}$ Then through row transformation, we can get: $\begin{pmatrix} 1 & 0 & 0 & -\frac{1}{8} & -\frac{3}{8} & \frac{3}{8} \\ 0 & 1 & 0 & \frac{3}{8} & -\frac{7}{8} & \frac{3}{8} \\ 0 & 0 & 1 & \frac{3}{4} & -\frac{3}{4} & \frac{1}{4} \end{pmatrix}$
so $\mathbf{A}^{-1} = \begin{pmatrix} -\frac{1}{8} & -\frac{3}{8} & \frac{3}{8} \\ \frac{3}{8} & -\frac{7}{8} & \frac{3}{8} \\ \frac{3}{4} & -\frac{3}{4} & \frac{1}{4} \end{pmatrix}$
It's easy to get $|\mathbf{A}| = 16$

(2) $\because |\mathbf{A}| \neq 0, \therefore r(A) = 3$

(3) $tr(\mathbf{A}) = 1 + (-5) + 4 = 0$

(4) $\mathbf{A} + \mathbf{A}^{\mathrm{T}} = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix} + \begin{pmatrix} 1 & 3 & 6 \\ -3 & -5 & -6 \\ 3 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 9 \\ 0 & -10 & -3 \\ 9 & -3 & 8 \end{pmatrix}$

(5) Matrix $\mathbf{A}$ is not orthogonalbecause $(1, -3, 3)^T \cdot (3, -5, 3) \neq 0$

(6) $|\lambda\mathbf{E} - \mathbf{A}| = \begin{vmatrix} \lambda - 1 & 3 & -3 \\ -3 & \lambda + 5 & -3 \\ -6 & 6 & \lambda - 4 \end{vmatrix}$ After simplification, we can find $(\lambda - 4)(\lambda + 2)^2 = 0$. So we can find the three eigenvalues is: λ_1 = 4, $\lambda_2 = \lambda_3$ = -2.
When λ_1 = 4, there is one eigen vectors: $\mathbf{v_1} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{pmatrix}$. When $\lambda_2 = \lambda_3$ = -2, we can get two eigen vectors:
$\mathbf{v_2} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{v_3} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

(7) $\mathbf{P} = (\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}) = \begin{pmatrix} \frac{1}{2} & 1 & -1 \\ \frac{1}{2} & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$, *then we can calculate* $\mathbf{P}^{-1} = \begin{pmatrix} 1 & -1 & 1 \\ -\frac{1}{2} & \frac{3}{2} & -\frac{1}{2} \\ -1 & 1 & 0 \end{pmatrix}$, $\mathbf{\Lambda} = diag(4, -2, -2)$
so $\mathbf{A} = \mathbf{P}^{-1}\Lambda\mathbf{P}$

(8) $||\mathbf{A}||_F = \sqrt{1 + 9 + 9 + 9 + 25 + 9 + 36 + 36 + 16} = 5\sqrt{6}$

$||\mathbf{A}||_{l_{21}} = \sqrt{1+9+9} + \sqrt{9+25+9} + \sqrt{36+36+16} = \sqrt{19} + \sqrt{43} + 2\sqrt{22}$

(9) $\mathbf{A}^T\mathbf{A} = \begin{pmatrix} 46 & -54 & 36 \\ -54 & 70 & -48 \\ 36 & -48 & 34 \end{pmatrix}$, and we can calculate eigenvalues of $\mathbf{A}^T\mathbf{A}$: $|\lambda\mathbf{E}{-}\mathbf{A}| = \begin{vmatrix} \lambda-46 & 54 & -36 \\ 54 & \lambda-70 & 48 \\ -36 & 48 & \lambda-34 \end{vmatrix} =$ $(\lambda-4)(\lambda-(73+9\sqrt{65}))(\lambda-(73-9\sqrt{65}))$ so we can find out: $\lambda_1 = 4, \lambda_2 = 73+9\sqrt{65}, \lambda 3 = 73-9\sqrt{65}$, so $||\mathbf{A}||_* = \sqrt{\lambda_1} + \sqrt{\lambda_2} + \sqrt{\lambda_3} = 2 + \sqrt{73+9\sqrt{65}} + \sqrt{73-9\sqrt{65}}$, $||\mathbf{A}||_2 = \sqrt{max\{\lambda_1, \lambda_2, \lambda_3\}} = \sqrt{73-9\sqrt{65}}$

Problem 3: Linear Equations (points)

(1) Through the equations, we can get the augmented matrix: $\begin{pmatrix} 2 & 2 & 3 & 1 \\ 1 & -1 & 0 & -1 \\ -1 & 2 & 1 & 2 \end{pmatrix}$

Through the row transformation of the matrix, we can find: $\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$, so we can get the solution of the equations is: $\begin{cases} x_1 = -1 \\ x_2 = 0 \\ x_3 = 1 \end{cases}$

(2) $\begin{pmatrix} 2 & 2 & 3 \\ 1 & -1 & 0 \\ -1 & 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

(3) Because the linear equations have unique solutions $r(A) = 3$

(4) $\begin{pmatrix} 2 & 2 & 3 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ -1 & 2 & 1 & 0 & 0 & 1 \end{pmatrix}$

Then we can through row transformation to find out the inverse of A:

$\mathbf{A}^{-1} = \begin{pmatrix} 1 & -4 & -3 \\ 1 & -5 & -3 \\ -1 & 6 & 4 \end{pmatrix}$

$|\mathbf{A}| = \begin{vmatrix} 2 & 2 & 3 \\ 1 & -1 & 0 \\ -1 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 4 & 3 \end{vmatrix} = -1$

(5) $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b} = \begin{pmatrix} 1 & -4 & -3 \\ 1 & -5 & -3 \\ -1 & 6 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

(6) $< \mathbf{x}, \mathbf{b} >= \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}^T \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = -1+0+2 = 1$

$\mathbf{x} \otimes \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 0 & 1 \\ 1 & -1 & 2 \end{vmatrix} = (1, 3, 1)$

(7) $||\mathbf{b}||_{l_1} = |1| + |-1| + |2| = 4$ $||\mathbf{b}||_{l_2} = \sqrt{1^2 + (-1)^2 + 2^2} = \sqrt{6}$ $||\mathbf{b}||_{l_\infty} = max\{|1|, |-1|, |2|\} = 2$

(8) $\mathbf{y}^T\mathbf{A}\mathbf{y} = (y_1, y_2, y_3) \begin{pmatrix} 2 & 2 & 3 \\ 1 & -1 & 0 \\ -1 & 2 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = 2y_1^2 - y_2^2 + y_3^2 + 3y_1y_2 + 2y_1y_3 + 2y_2y_3$

$\nabla_y \mathbf{y}^T\mathbf{A}\mathbf{y} = (4y_1 + 3y_2 + 2y_3, 3y_1 - 2y_2 + 2y_3, 2y_1 + 2y_2 + 2y_3)$

(9) $\begin{pmatrix} 2 & 2 & 3 \\ 1 & -1 & 0 \\ -1 & 2 & 1 \\ -1 & 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

(10) $\because r(\mathbf{A}_1) \leq 3, and\ r(\mathbf{A}_1) \geq r(\mathbf{A}),\ \therefore r(\mathbf{A}_1) = 3$

(11) $(\mathbf{A_1}, \mathbf{b}) = \begin{pmatrix} 2 & 2 & 3 & 1 \\ 1 & -1 & 0 & -1 \\ -1 & 2 & 1 & 2 \\ -1 & 2 & 1 & 2 \end{pmatrix} \Rightarrow \begin{pmatrix} 2 & 2 & 3 & 1 \\ 1 & -1 & 0 & -1 \\ -1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow\ r(\mathbf{A_1}, \mathbf{b}) = r(\mathbf{A}, \mathbf{b}),\ \because \mathbf{Ax} = \mathbf{x}\ have\ the\ only\ one\ solution,$
$r(\mathbf{A}) = 3. \therefore r(\mathbf{A_1}, \mathbf{b}) = 3 = r(\mathbf{A_1})$