Machine Learning

Lecture 12: Probability Review

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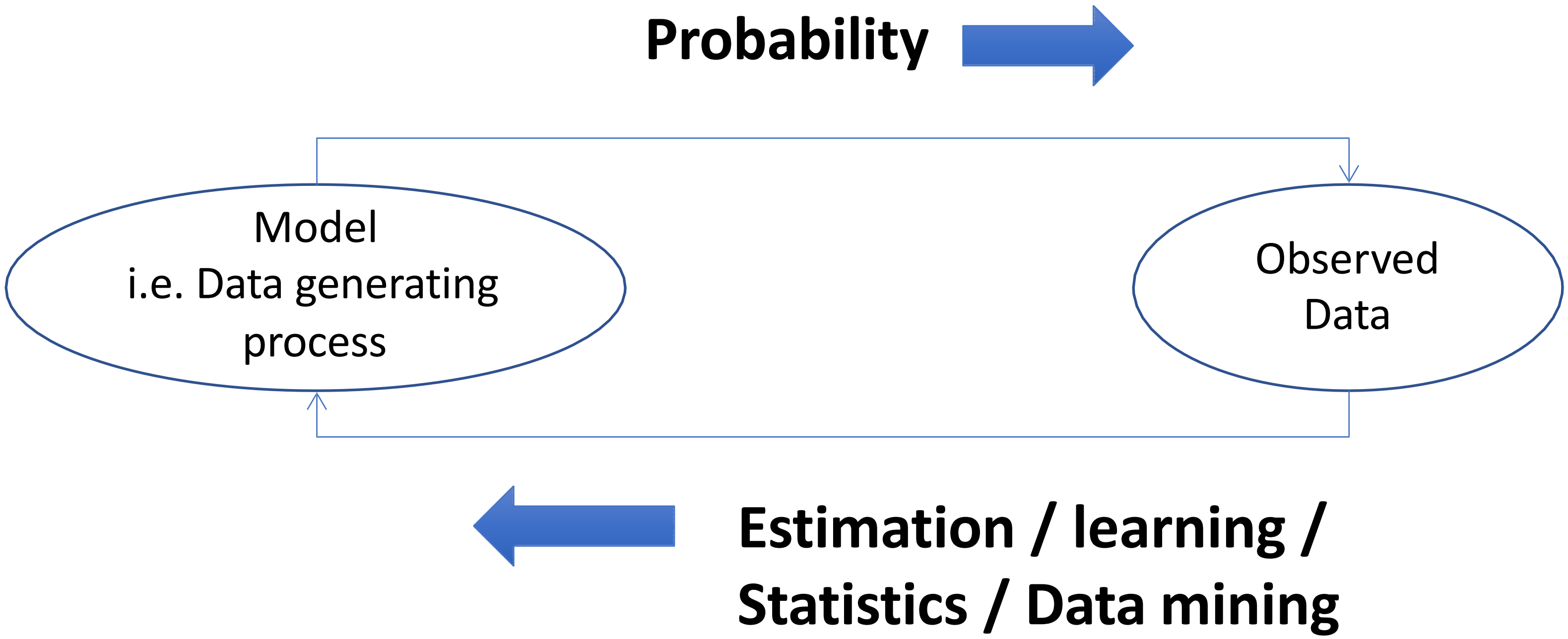
Today: Probability Review

- The big picture
- Events and Event spaces
- Random variables
- Joint probability, Marginalization, conditioning, chain rule, Bayes Rule, law of total probability, etc.
- Structural properties, e.g., Independence, conditional independence
- Maximum Likelihood Estimation

The Big Picture



Probability

- Counting
- Basics of probability
- Conditional probability
- Random variables
- Discrete and continuous distributions
- Expectation and variance
- Tail bounds and central limit theorem
- ……

Statistics

- Maximum likelihood estimation
- Bayesian estimation
- Hypothesis testing
- Linear regression
- ……

Probability as frequency

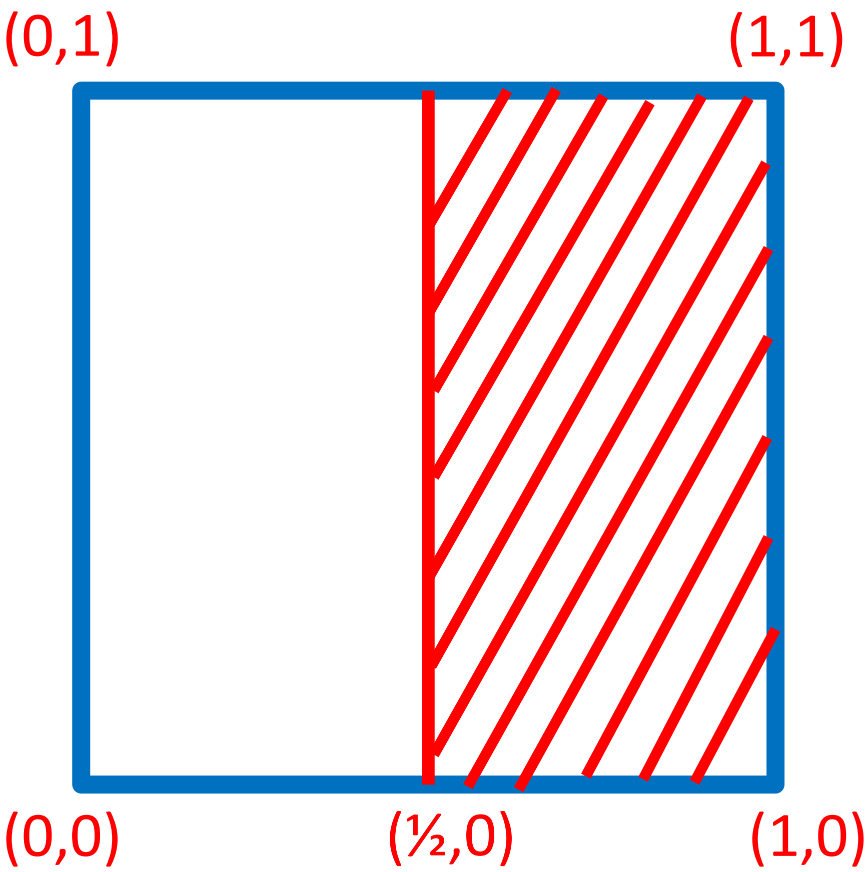
- Consider the following questions:
  - What is the probability that when I flip a coin it is "heads"? ➔ 50%

  - What is the probability of Zijin Mountains to have a mudslide in the near future? ➔ could not count

Message: The frequentist view is very useful, but it seems that we can also use domain knowledge to come up with probabilities.

Probability as a measure of uncertainty

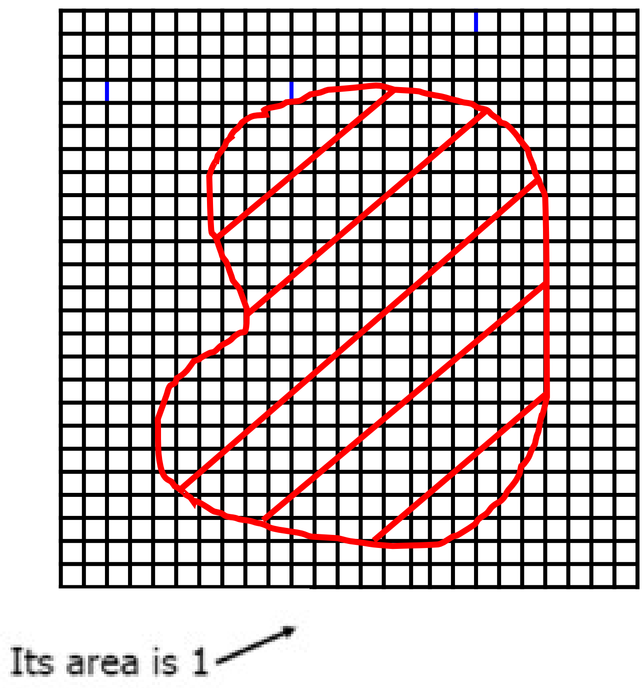
- Imagine we are throwing darts at a wall size 1x1 and that all darts are guaranteed to fall within this 1x1 wall.
- What is the probability that a dart will hit the shaded area?



Probability as a measure of uncertainty

- Probability is a measure of certainty of an event taking place.

- i.e. in the example, we were measuring the chances of hitting the shaded area.



$$prob = \frac{\#RedBoxes}{\#Boxes}$$

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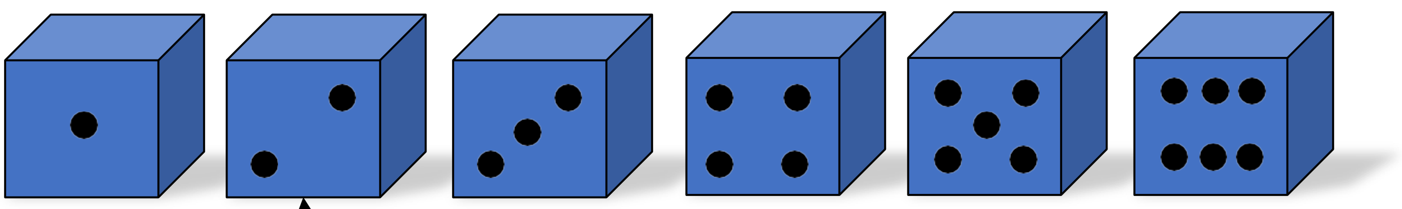
Probability

Probability is the formal study of the laws of chance.
Probability allows us to manage uncertainty.

The sample space is the set of all outcomes. For example, for a die we have 6 outcomes:

$$O_{die} = \{1,2,3,4,5,6\}$$

O:



Elementary Event "Throw 2"

The elements of O are called *elementary events.*

Probability

- Probability allows us to measure many events.
- The events are subsets of the sample space O. For example, for a die we may consider the following events: e.g.,

GREATER = {5, 6}

EVEN = {2, 4, 6}

- Assign probabilities to these events: e.g.,

P(EVEN) = 1/2

Sample space and Events

- O: Sample Space,
  - result of an experiment / set of all outcomes
  - If you toss a coin twice O = {HH,HT,TH,TT}

- Event: a subset of O
  - First toss is head = {HH,HT}
- S: Event Space, a set of events:
  - Contains the empty event and O

Axioms for Probability

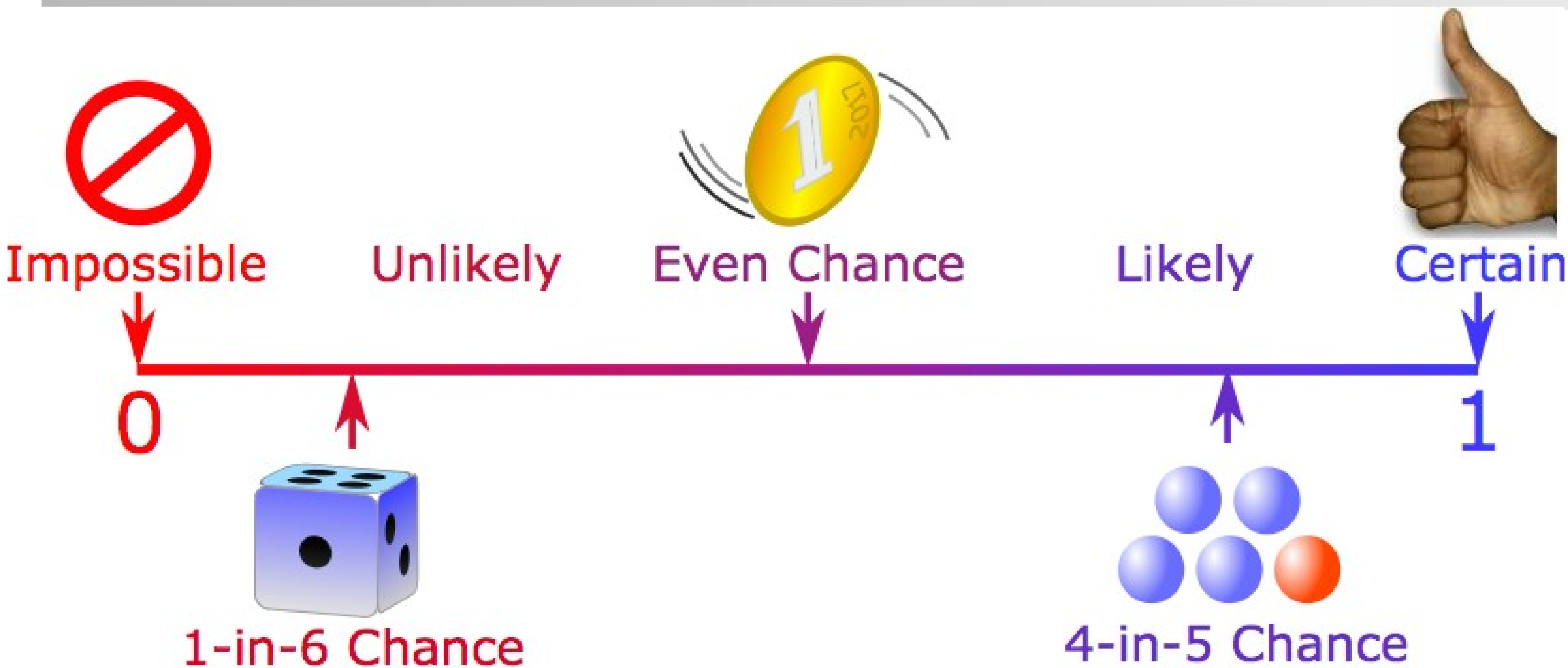
Sample space

Event space

- Defined over (Ω, S) s.t.
  - $1 >= P(\alpha) >= 0$ for all α in S
  - $P(\Omega) = 1$

  - If A, B are disjoint, then
    - $P(A \cup B) = P(A) + P(B)$

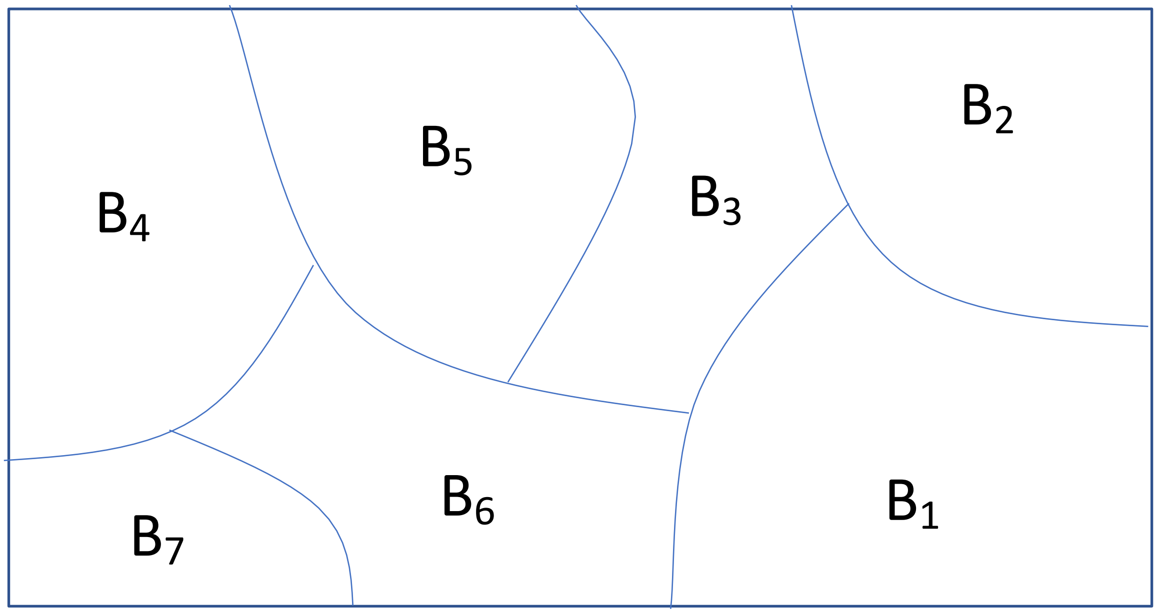
Axioms for Probability



Probability is always between 0 and 1

Axioms for Probability

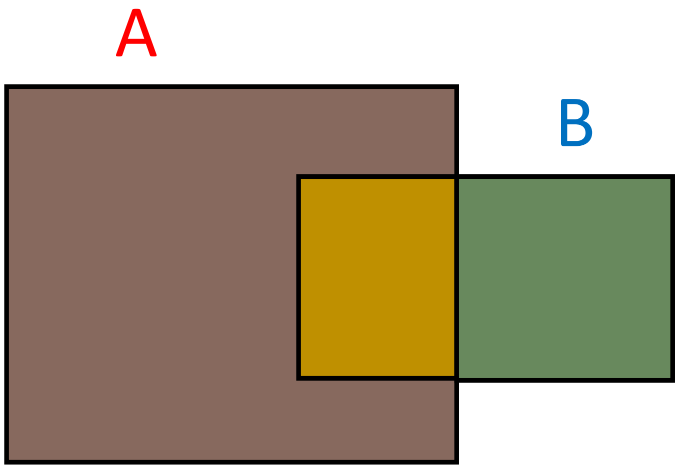
- $P(O) = \sum P(B_i) = 1$



OR & AND operation for Probability

- We can deduce other axioms from the above ones
  - Ex: P(A U B) for non-disjoint events

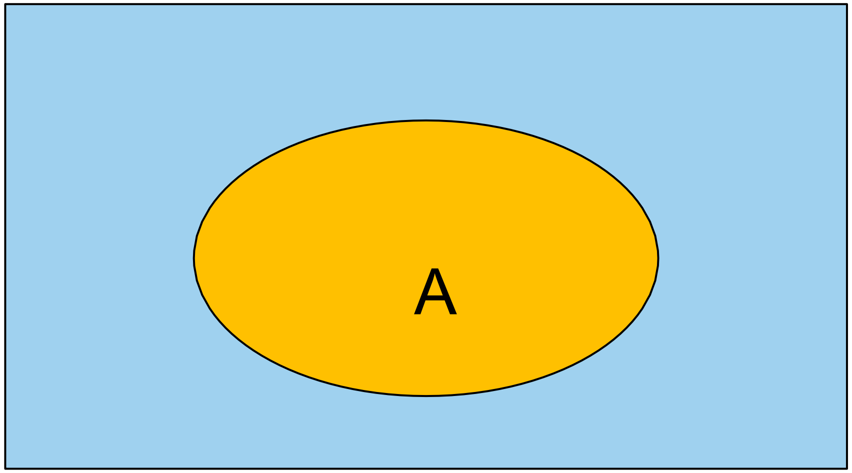
$$P(A \; or \; B) = P(A) + P(B) - P(A \; and \; B)$$



NOT operation for Probability

- $0 \leq P(A) \leq 1$
- $P(A\ or\ B) = P(A) + P(B) - P(A\ and\ B)$
- From these we can prove:

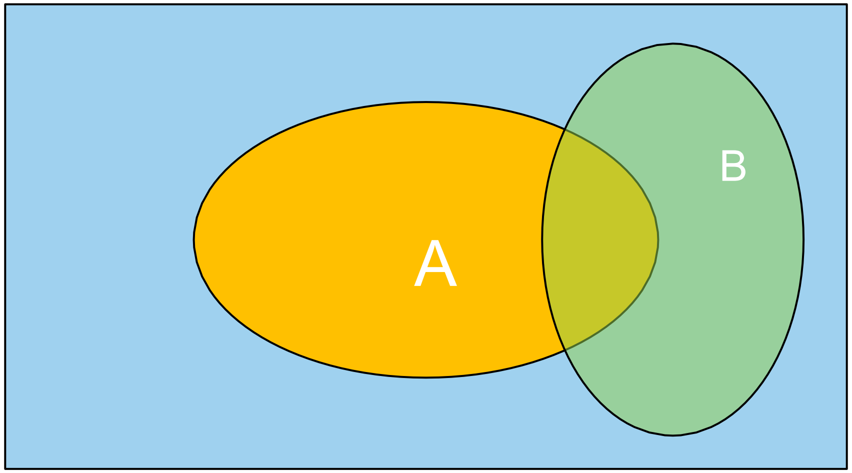
$$P(not\ A) = P(\sim A) = 1 - P(A)$$



Law of Total Probability

- $0 \le P(A) \le 1$
- $P(A\ or\ B) = P(A) + P(B) - P(A\ and\ B)$
- From these we can prove:

$$P(A) = P(A \wedge B) + P(A \wedge \sim B)$$



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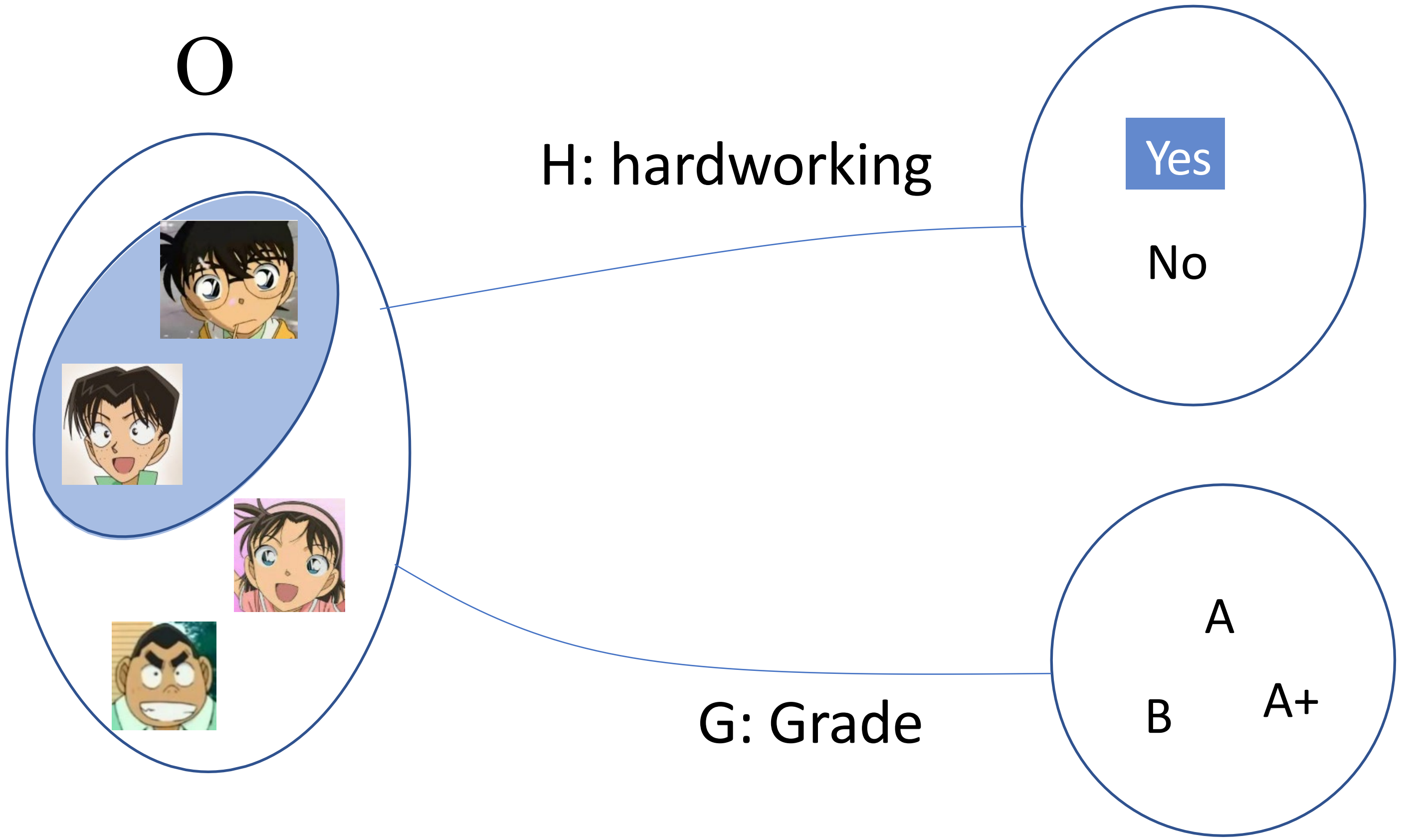
From Events to Random Variable (RV)

- Concise way of specifying attributes of outcomes
- Modeling students (Grade and Intelligence):
  - Ω = all possible students (sample space)
  - What are events (subset of sample space)
    - Grade_A = all students with grade A
    - Grade_B = all students with grade B
    - HardWorking_Yes = … who works hard
  - Very cumbersome

  - Need "functions" that maps from Ω to an attribute space T.
  - P(H = YES) = P({student ϵ Ω : H(student) = YES})

Random Variables (RV)



P(H = Yes) = P({all students who is working hard on the course})

- "functions" that maps from O to an attribute space T.

Discrete Random Variables

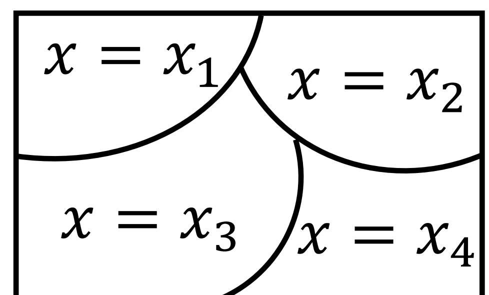
- Random variables (RVs) which may take on only a countable number of distinct values

- X is a RV with arity k if it can take on exactly one value out of $\{x_1, \ldots, x_k\}$

Probability of Discrete RV

- Probability mass function (pmf): $P(X = x_i)$
- Easy facts about pmf
  - $\Sigma_i P(X = x_i) = 1$
  - $P(X = x_i \cap X = x_j) = 0$ if $i \neq j$
  - $P(X = x_i \cup X = x_j) = P(X = x_i) + P(X = x_j)$ if $i \neq j$
  - $P(X = x_1 \cup X = x_2 \cup \ldots \cup X = x_k) = 1$

$$\sum_{i=1}^{4} P(X = x_i) = 1$$



e.g. Coin Flips

- You flip a coin
  - Head with probability *p*
  - Binary random variable
  - Bernoulli trial with success probability *p*

$$Binary = \{H, T\}$$

- You flip *a* coin for *k* times
  - How many heads would you expect
  - Number of heads X is a discrete random variable
  - Binomial distribution with parameters *k* and *p*

$$Integer = \{1,2,\ldots,k\}$$

Discrete Random Variables

- Random variables (RVs) which may take on only a countable number of distinct values
  - E.g. the total number of heads X you get if you flip 100 coins
- X is a RV with arity *k* if it can take on exactly one value out of
  - E.g. the possible values that X can take on are 0, 1, 2, …, 100

$$\{x_1, \ldots, x_k\}$$

e.g., two common distributions

- Uniform $X \sim U[1, \dots, N]$
  - X takes values 1, 2, …, *N* $P(X = i) = \frac{1}{N}$
  - E.g. picking balls of different colors from a box

- Binomial
  - X takes values 0, 1, …, *k* $X \sim B(k, p)$
  - E.g. coin flips k times $P(X = i) = \binom{k}{i} p^i (1 - p)^{k-i}$

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If hard to directly estimate from data, most likely we can estimate

- Joint probability
  - Use Chain Rule

$$P(A, B) = P(B)P(A|B)$$

- Marginal probability
  - Use the total law of probability

$$P(B) = P(B, A) + P(B, \sim A)$$
$$= P(B, A \cup \sim A)$$

- Conditional probability
  - Use the Bayes Rule

$$P(B|A) = \frac{P(A, B)}{P(A)} = \frac{P(A|B)P(B)}{P(A)}$$

(1) To calculate Joint Probability: Use Chain Rule

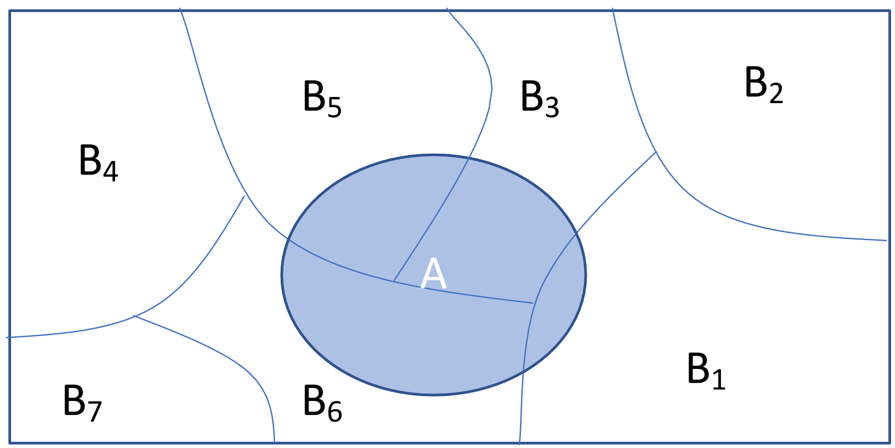
- Two ways to use chain rules on joint probability

joint conditional marginal

$$P(A,B) = P(B|A)P(A)$$

$$P(A,B) = P(A|B)P(B)$$

(2) To calculate Marginal Probability: Use Rule of total probability (e.g. event version)



$$p(A)=\sum P(B_i)P(A|B_i)$$

(2) To calculate Marginal Probability: Use Rule of total probability (e.g. RV version)

- Given two discrete RVs X and Y, which take values in:

$$\{x_1,\ldots,x_k\} \qquad \{y_1,\ldots,y_m\}$$

$$\mathrm{P}(\mathrm{X}=x_i)=\sum_j \mathrm{P}(\mathrm{X}=x_i \cap \mathrm{Y}=y_j)$$

$$=\sum_j \mathrm{P}(\mathrm{X}=x_i \mid \mathrm{Y}=y_j)\mathrm{P}(\mathrm{Y}=y_j)$$

P(A) = P(A^B)+P(A^~B)

(3) To calculate Conditional Probability: Use Bayes Rule (e.g. RV version)

$$\mathrm{P}\left(\mathrm{X}=x \middle| \mathrm{Y}=y\right)=\frac{\mathrm{P}\left(\mathrm{X}=x \cap \mathrm{Y}=y\right)}{\mathrm{P}\left(\mathrm{Y}=y\right)}$$

One Example: Joint

Assume we have a dark box with 3 red balls and 1 blue ball. That is, we have the set{r, r, r, b}. What is the probability of drawing 2 red balls in the first 2 tries?

$$\boldsymbol{P(B_1 = r, B_2 = r) =}$$

One Example: Marginal

What is the probability that the 2^{nd} ball drawn from the set{r, r, r, b} will be red?

$$\boldsymbol{P(B_2 = r) =}$$

One Example: Conditional

What is the probability that the 2nd ball will be red if the 1st ball drawn from the set{r, r, r, b} is red?

$$\boldsymbol{P(B_2 = r \mid B_1 = r) =}$$

Use both Bayes Rule and Marginal

- X and Y are discrete RVs…

$$P\left(X = x_i \middle| Y = y_j\right) = \frac{P\left(X = x_i \cap Y = y_j\right)}{P(Y = y_i)}$$

$$\{x_1, \dots, x_k\}$$

$$\Downarrow$$

$$P\left(X = x_i \middle| Y = y_j\right) = \frac{P\left(Y = y_j | X = x_i\right) P\left(X = x_i\right)}{\sum_k P\left(Y = y_j | X = x_k\right) P\left(X = x_k\right)}$$

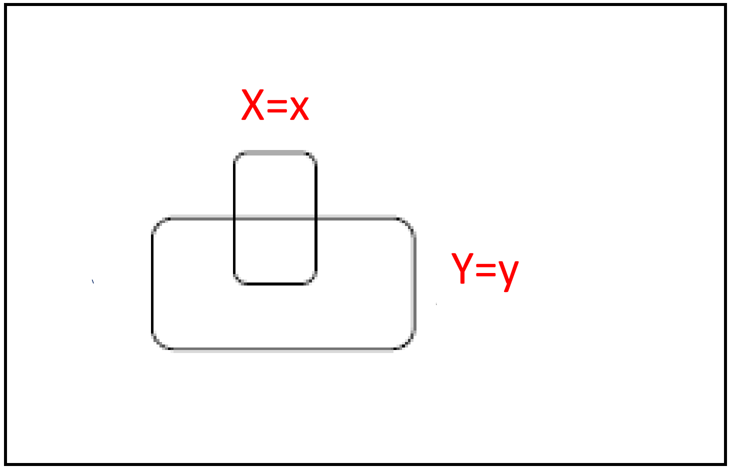
Simplify Notation: Conditional Probability

events

$$P(X = x|Y = y) = \frac{P(X = x \cap Y = y)}{P(Y = y)}$$

- But we will always write it this way:

$$P(x|y) = \frac{P(x, y)}{P(y)}$$



- P(X=x true) -> P(X=x) -> P(x)

Simplify Notation: An Example of estimating conditional

- We know that P(rain) = 0.5
  - If we also know that the grass is wet, then how this affects our belief about whether it rains or not?

$$P(rain|wet) = \frac{P(rain)P(wet|rain)}{P(wet)}$$

Simplify Notation: An Example of estimating conditional

- We know that P(rain) = 0.5
  - If we also know that the grass is wet, then how this affects our belief about whether it rains or not?

$$P(rain|wet) = \frac{P(rain)P(wet|rain)}{P(wet)}$$

$$P(x|y) = \frac{P(x)P(y|x)}{P(y)}$$

Simplify Notation: Conditional

- Bayes Rule

$$P(x|y) = \frac{P(x)P(y|x)}{P(y)}$$

- You can condition on more variables

$$P(x|y,z) = \frac{P(x|z)P(y|x,z)}{P(y|z)}$$

Simplify Notation: Marginal

- We know P(X, Y), what is P(Y=y) or P(X=x)?
- We can use the law of total probability

$$P(x) = \sum_{y} P(x, y)$$

$$= \sum_{y} P(y) P(x|y)$$

$$P(x) = \sum_{y,z} P(x, y, z)$$

$$= \sum_{z,y} P(y, z) P(x|y, z)$$

Simplify Notation: An Example

- We know that P(rain) = 0.5
- If we also know that the grass is wet, then how this affects our belief about whether it rains or not?

$$P(\underbrace{rain}_{\substack{\text{weather}\\ \{\text{rain, sunny}\}}}|\underbrace{wet}_{\substack{\text{grass}\\ \{\text{wet, dry}\}}}) = \frac{\overbrace{P(rain)}^{0.5}\overbrace{P(wet|rain)}^{1}}{\underbrace{P(wet)}_{\substack{P(rain)P(rain|wet)\\ + P(sunny)P(sunny|wet)}}}$$

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Independent RVs

- Definition: X and Y are independent ***iff***

$$P\left(X = x \cap Y = y\right) = P\left(X = x\right)P\left(Y = y\right)$$

More on Independence

$$P(X = x \cap Y = y) = P(X = x)P(Y = y)$$

$$P(X = x | Y = y) = P(X = x)$$

$$P(Y = y | X = x) = P(Y = y)$$

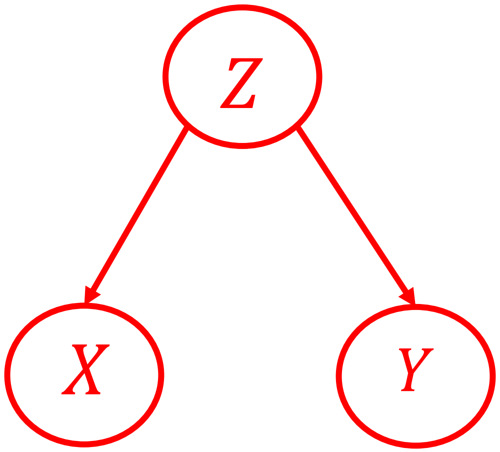
- E.g. no matter how many heads you get, your friend will not be affected, and vice versa

More on Independence

- X is independent of Y means that knowing Y does not change our belief about X.
- The following forms are equivalent:
  - $P(X=x, Y=y) = P(X=x)\, P(Y=y)$
  - $P(X=x \mid Y=y) = P(X=x)$

  - The above should hold for all x_i, y_j
  - It is symmetric and written as $X \perp Y$

Conditionally Independent RVs



- Intuition: X and Y are conditionally independent given Z means that once Z is known, the value of X does not add any additional information about Y
- Definition: X and Y are conditionally independent given Z *iff*

$$\mathrm{P}\left(\mathrm{X}=x \cap \mathrm{Y}=y \middle| \mathrm{Z}=z\right)=\mathrm{P}\left(\mathrm{X}=x \middle| \mathrm{Z}=z\right)\mathrm{P}\left(\mathrm{Y}=y \middle| \mathrm{Z}=z\right)$$

If holding for all x_i, y_j, z_k $\quad X \perp Y | Z$

More on Conditional Independence

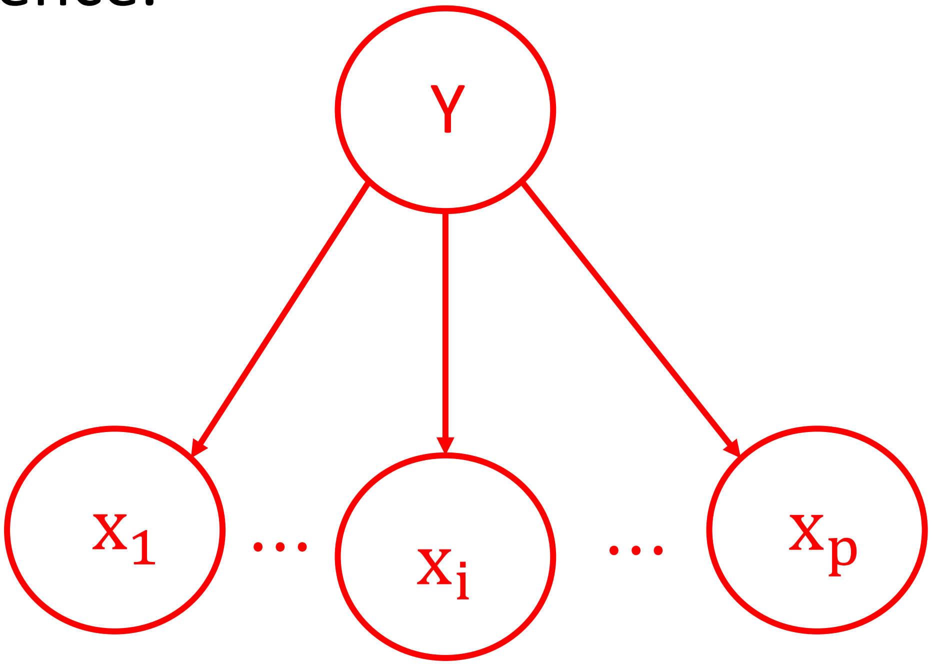
$$P\left(X = x \cap Y = y \middle| Z = z\right) = P\left(X = x \middle| Z = z\right) P\left(Y = y \middle| Z = z\right)$$

$$P\left(X = x \middle| Y = y, Z = z\right) = P\left(X = x \middle| Z = z\right)$$

$$P\left(Y = y \middle| X = x, Z = z\right) = P\left(Y = y \middle| Z = z\right)$$

Independence and Conditional Independence

- Independence does not imply conditional independence.
- Conditional independence does not imply independence.



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References

- https://qiyanjun.github.io/2019f-UVA-CS6316-MachineLearning/
- Prof. Andrew Moore’s review tutorial
- Prof. Nando de Freitas’s review slides
- Prof. Carlos Guestrin recitation slides

Thanks for listening