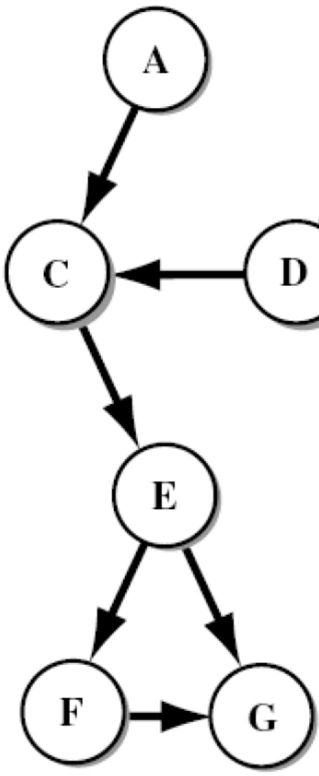
Chapter 5 Bayesian Belief Network

Directed Acyclic Graph (DAG) 有向无环图

$$G = (V, E)$$

- V: a set of nodes in graph G
- E: a set of directed edges in G
- Basic assumption: **no directed loop in G**

An illustrative example



$$V = \{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{E}, \mathbf{F}, \mathbf{G}\}(|V| = 7)$$
$$E = \{(\mathbf{A}, \mathbf{C}), (\mathbf{B}, \mathbf{D}), (\mathbf{D}, \mathbf{C}), (\mathbf{C}, \mathbf{E}), (\mathbf{E}, \mathbf{F}), (\mathbf{E}, \mathbf{G}), (\mathbf{F}, \mathbf{G})\}(|E| = 7)$$

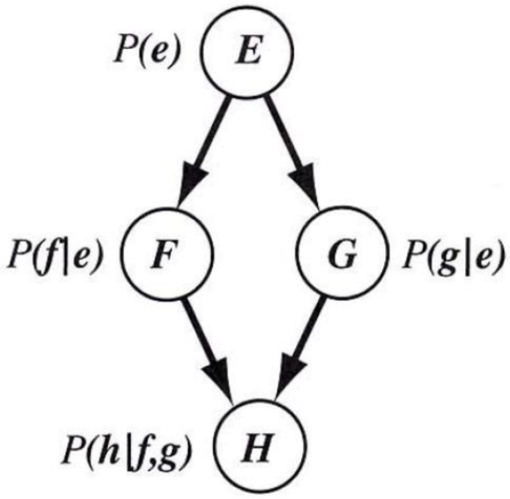
Goal

- Model the **joint distribution** of a set of random variables w.r.t. the network's DAG structure
- The joint distribution can **be factorized into the product of the conditional probability** of each random variable given its parent variables

Notations

- Node: $\mathbf{A}, \mathbf{B}, \ldots$
- Random Variable: $\mathbf{a}, \mathbf{b}, \ldots$ Values of Random Variable: $\{a_1, a_2, \ldots\}, \ldots$
- Parent variables: $\mathcal{G}(\mathbf{a}), \mathcal{G}(\mathbf{b}), \ldots$
  - e.g. $\mathcal{G}(\mathbf{c}) = \{\mathbf{a}, \mathbf{d}\}, \mathcal{G}(\mathbf{f}) = \{\mathbf{e}\}$

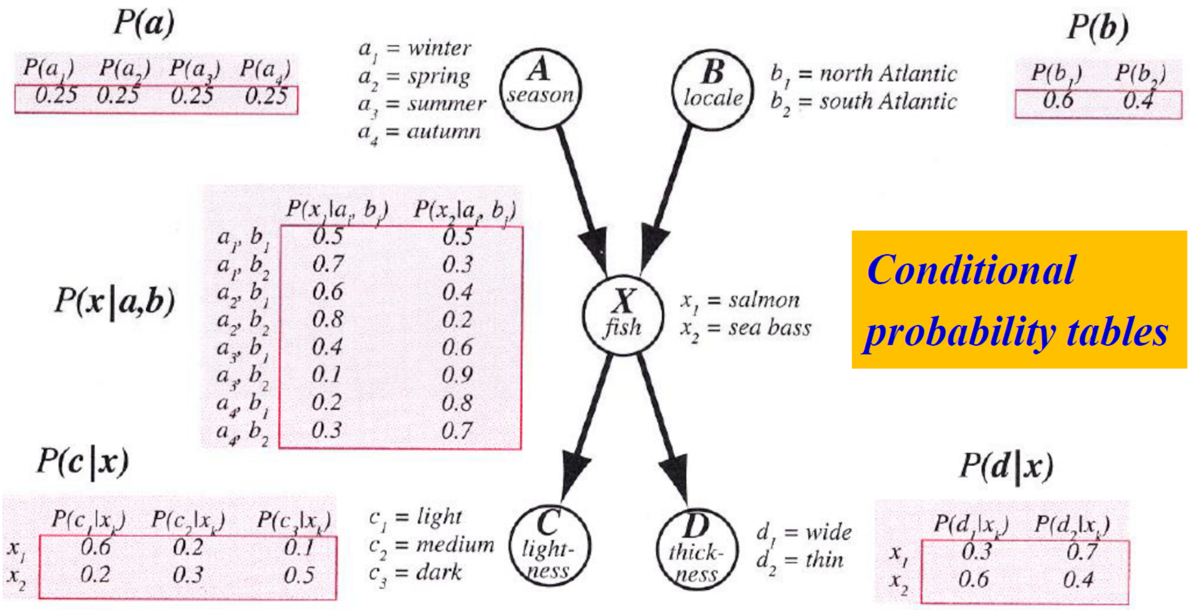
Example



$$
\begin{aligned}
P(\mathbf{e}, \mathbf{f}, \mathbf{g}, \mathbf{h}) & =P(\mathbf{e} \mid \mathcal{G}(\mathbf{e})) \cdot P(\mathbf{f} \mid \mathcal{G}(\mathbf{f})) \cdot P(\mathbf{g} \mid \mathcal{G}(\mathbf{g})) \cdot P(\mathbf{h} \mid \mathcal{G}(\mathbf{h})) \\
& =P(\mathbf{e}) \cdot P(\mathbf{f} \mid \mathbf{e}) \cdot P(\mathbf{g} \mid \mathbf{e}) \cdot P(\mathbf{h} \mid \mathbf{f}, \mathbf{g})
\end{aligned}
$$

$$
\begin{aligned}
P(\mathbf{f}, \mathbf{g}, \mathbf{h}) & =\sum_{\mathbf{e}} P(\mathbf{e}, \mathbf{f}, \mathbf{g}, \mathbf{h}) \\
& =P(\mathbf{h} \mid \mathbf{f}, \mathbf{g}) \sum_{\mathbf{e}} P(\mathbf{e}) \cdot P(\mathbf{f} \mid \mathbf{e}) \cdot P(\mathbf{g} \mid \mathbf{e})
\end{aligned}
$$

$$
\begin{aligned}
P(\mathbf{h}) & =\sum_{\mathbf{e}, \mathbf{f}, \mathbf{g}} P(\mathbf{e}, \mathbf{f}, \mathbf{g}, \mathbf{h}) \\
& =\sum_{\mathbf{e}} P(\mathbf{e}) \sum_{\mathbf{f}, \mathbf{g}} P(\mathbf{f} \mid \mathbf{e}) \cdot P(\mathbf{g} \mid \mathbf{e}) \cdot P(\mathbf{h} \mid \mathbf{f}, \mathbf{g})
\end{aligned}
$$



Q1: What is the probability that the fish was caught in the **summer** in the **north Atlantic** and is **sea bass** that is **dark** and **thin**?

$$
\begin{aligned}
P\left(a_3, b_1, x_2, c_3, d_2\right) & =P\left(a_3\right) \cdot P\left(b_1\right) \cdot P\left(x_2 \mid a_3, b_1\right) \cdot P\left(c_3 \mid x_2\right) \cdot P\left(d_2 \mid x_2\right) \\
& =0.25 \times 0.6 \times 0.6 \times 0.5 \times 0.4 \\
& =0.018
\end{aligned}
$$

Q2:Suppose we know a fish is **light** and caught in the **south Atlantic**, how shall we classify the fish?

Solution:

to compare $P\left(\mathbf{x}=x_1 \mid b_2, c_1\right)$ with $P\left(\mathbf{x}=x_2 \mid b_2, c_1\right)$

$$
\begin{aligned}
& P\left(x_1 \mid b_2, c_1\right) \\
& =P\left(x_1, b_2, c_1\right) / P\left(b_2, c_1\right) \\
& =\alpha \cdot \textstyle\sum_{\mathbf{a}, \mathbf{d}} P\left(\mathbf{a}, x_1, b_2, c_1, \mathbf{d}\right) \\
& =\alpha \sum_{\mathbf{a}, \mathbf{d}} P(\mathbf{a}) P\left(b_2\right) P\left(x_1 \mid \mathbf{a}, b_2\right) P\left(c_1 \mid x_1\right) P\left(\mathbf{d} \mid x_1\right) \\
& =\alpha \cdot P\left(b_2\right) P\left(c_1 \mid x_1\right)\left[\sum_{\mathbf{a}} P(\mathbf{a}) P\left(x_1 \mid \mathbf{a}, b_2\right)\right]\left[\sum_{\mathrm{d}} P\left(\mathbf{d} \mid x_1\right)\right] \\
& =\alpha \cdot(0.4)(0.6)[(0.25)(0.7)+(0.25)(0.8)+(0.25)(0.1)+(0.25)(0.3)](1.0) \\
& =\alpha \cdot 0.114
\end{aligned}
$$

Similarly, we can have $P\left(x_2 \mid b_2, c_1\right)=\alpha \cdot 0.066$