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HW4

1) 1D cylindrical

Central difference from class

$$\frac{U_{i+1}-2U_i+U_{i-1}}{h^2}=\frac{\partial \varphi^2}{\partial x^2}$$, translate to radial

$$\cancel{\frac{\partial^2 \varphi}{\partial r^2}=\frac{\partial^2\varphi}{\partial r}+\frac{\partial\varphi}{\partial r}}$$ $$\frac{\partial^2\varphi}{\partial x^2}=0=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\varphi}{\partial r}\right)$$

Wolfram alpha

$$\frac{\partial}{\partial r}\left(r\frac{\partial\varphi}{\partial r}\right)$$, Taylor series expansion $\Rightarrow r\frac{\partial\varphi^2}{\partial r^2}+\frac{\partial\varphi}{\partial r}$

$$\frac{1}{r}\left(r\frac{\partial\varphi}{\partial r^2}+\frac{\partial\varphi}{\partial r}\right)=0$$ $\frac{\partial\varphi}{\partial x}=\frac{U_{i+1}-U_{i-1}}{2h}$ central difference

$$\frac{\partial^2\varphi}{\partial r^2}+\frac{1}{r}\frac{\partial\varphi}{\partial r}=0$$

$$\boxed{\frac{U_{i+1}-2U_i+U_{i-1}}{h^2}+\frac{U_{i+1}-U_{i-1}}{2rh}=0}$$

2) $U_2=1$ $U_0=0$ $r=1.5$ $h=0.5$

$$\frac{1-2U_i+0}{(0.5)^2}+\frac{1-0}{2(1.5)(0.5)}=0$$

$$1-2U_i+\frac{0.5}{3}=0 \quad \frac{7}{6}=2U_i \quad U_i=\frac{7}{12}$$

$\phi(r=1.5)=7/12$

3) $U_3=2$ $U_0=0$ $U_1=4/3$ $U_2=5/3$ $h=1/3$

$$\frac{U_2 - 2U_1 + 0}{(1/3)^2} + \frac{U_2 - 0}{2(4/3)(1/3)} = 0$$

$$U_2 - 2U_1 + \frac{1/3\, U_2}{8/3} = 0$$

$$9/8\, U_2 = 2U_1 \qquad U_1 = 9/16\, U_2$$

$$\frac{1 - 2U_2 + U_1}{(1/3)^2} + \frac{1 - U_1}{2(5/3)(1/3)} = 0$$

$$1 - 2U_2 + 9/16\, U_2 + \frac{1/3(1 - 9/16\, U_2)}{10/3} = 0$$

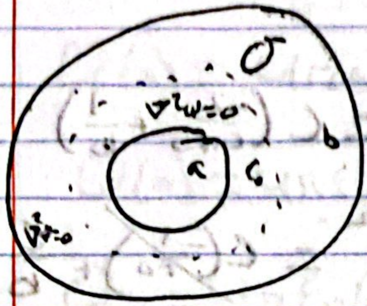
$$1 - 23/16\, U_2 + \frac{1 - 9/16\, U_2}{10} = 0$$

$$10 - 230/16\, U_2 + 1 - 9/16\, U_2$$

$$11 = 239/16\, U_2 \qquad 176/239 = U_2$$

$$U_1 = (9/16)\left(\frac{176}{239}\right) = \frac{1584}{3824}$$

4.2)



$r_0 = \frac{a+b}{2}$ $\qquad \sigma_0 = Q/A$

$4\pi r^2 \sigma = Q$

$\cancel{\psi_1 = Ar + B}$

$\cancel{\psi_2 = Cr + D}$

Sphere $\nabla^2 \psi = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\psi}{dr}\right) = 0$

$r^2 \frac{d\psi}{dr} = A \qquad \frac{d\psi}{dr} = \frac{A}{r^2} \qquad \psi_1 = \frac{-A}{r} + B$

$\psi_1(a) = 0 \qquad \psi_2 = \frac{-C}{r} + D$

$0 = -\frac{A}{a} + B \qquad B = \frac{A}{a}$

$\psi_2(b) = 0 \qquad 0 = -\frac{C}{b} + D \qquad D = \frac{C}{b}$

$\psi_1(r_0) = \psi_2(r_0) \qquad r_0 = \frac{a+b}{2}$

$A\left(\frac{-1}{r} + \frac{1}{a}\right) = C\left(\frac{-1}{r} + \frac{1}{b}\right)$

$A\left(-\frac{2}{a+b} + \frac{1}{a}\right) = C\left(\frac{-2}{a+b} + \frac{1}{b}\right)$

$E = -\nabla\psi \qquad E \cdot n = \sigma_0/\varepsilon_0$ $\qquad$ 1 | 2, E, $n=-1$ | $n=1$

$-E = \sigma_0/\varepsilon_0 \qquad -\frac{\partial \psi_1}{\partial r} = \sigma_0/\varepsilon_0 \qquad \frac{\partial \psi_1}{\partial r} = \sigma_0/\varepsilon_0$

$E = \sigma_0/\varepsilon_0 \qquad -\frac{\partial \psi_2}{\partial r} = \sigma_0/\varepsilon_0 \qquad E_r(r_0^+) - E_r(r_0^-) = \sigma_0/\varepsilon_0$

$-\frac{\partial \psi_2}{\partial r} + \frac{\partial \psi_1}{\partial r} = \sigma_0/\varepsilon_0$

$\frac{\partial \psi_1}{\partial r} = \frac{A}{r^2} \qquad -\frac{C}{r^2} + \frac{A}{r^2} = \sigma_0/\varepsilon_0$

$A - C = \sigma_0 r^2/\varepsilon$

$$A = C + \sigma_0 r_0^2/\varepsilon_0$$

$$(C + \sigma_0 r^2/\varepsilon_0)\left(\frac{-2}{a+b} + \frac{1}{a}\right) = C\left(\frac{-2}{a+b} + \frac{1}{b}\right)$$

$$C\left(\cancel{\frac{-1}{r_0}}\right) + \frac{C}{a} - \frac{\sigma_0 r^2}{\varepsilon r_0} + \frac{\sigma_0 r^2}{a\varepsilon_0} = C\left(\cancel{\frac{-1}{r_0}}\right) + \frac{C}{b}$$

$$C\left(\frac{1}{a} - \frac{1}{b}\right) = \frac{\sigma_0 r}{\varepsilon_0} - \frac{\sigma_0 r^2}{a\varepsilon_0}$$

$$C = \frac{\sigma_0 r}{\varepsilon_0}\left(1 - \frac{r}{a}\right)\Big/\left(\frac{1}{a} - \frac{1}{b}\right)$$

$$A = \frac{\sigma_0 r}{\varepsilon_0}\left(\frac{(1 - r_0/a)}{\frac{1}{a} - \frac{1}{b}} + r_0\right)$$

$$B = \frac{\sigma_0 r_0}{a\varepsilon_0}\left(\frac{(1 - r_0/a)}{\frac{1}{a} - \frac{1}{b}} + r_0\right)$$

$$D = \frac{\sigma_0 r_0}{b\varepsilon_0}\left(\frac{1 - \frac{r_0}{a}}{\frac{1}{a} - \frac{1}{b}}\right)$$

$a \le r$ $$\Psi_1 = \frac{\sigma_0 r_0}{\varepsilon_0}\left(\left(\frac{1 - r_0/a}{\frac{1}{a} - \frac{1}{b}}\right) + r_0\right)\left(\frac{-1}{r} + \frac{1}{a}\right)$$

$r \le b$ $$\Psi_2 = \frac{\sigma_0 r_0}{\varepsilon_0}\left(\frac{1 - \frac{r_0}{a}}{\frac{1}{a} - \frac{1}{b}}\right)\left(\frac{1}{r} + \frac{1}{b}\right)$$

~~$Q_{inner} = \sigma 4\pi r^2$ $Q_{outer} = -\sigma 4\pi c^2$~~

$$Q(a) = A\,E(a)\,\epsilon = -\frac{A}{r^2} + B$$

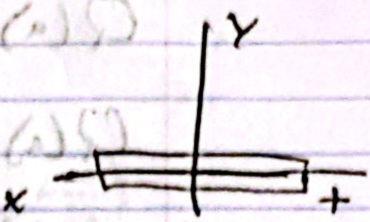
$$Q(a) = 4\pi a^2\left(-\frac{A}{a^2} + B\right)$$

$$Q(a) = 4\pi\epsilon_0(-A + a^2 B)$$ plug in A ; B solved above

$$Q_b = 4\pi\epsilon_0(-C + b^2 D)$$ " C ; D

3)

$A E^2(y) = \frac{Q_{enc}}{\varepsilon_0}$ $\qquad E = \frac{kQ}{r}$



$\oint \vec{E} \cdot d\vec{a} = Q_{enc}/\varepsilon_0$

$\int_{-t/2}^{t/2} EA = \rho_0 t A/\varepsilon_0$ $\qquad \frac{Q}{V} = \rho_0 \qquad Q = \rho_0 V$

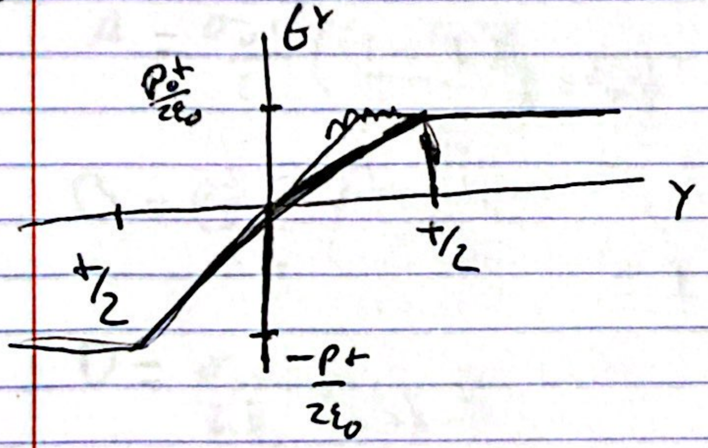
$E = \rho_0 t / 2\varepsilon_0$ outside $\qquad V = tA$

$Q = \rho_0 t A$

inside $Q_{enc} = \rho_0 A y$

$E = \rho_0 y / 2\varepsilon_0$

②



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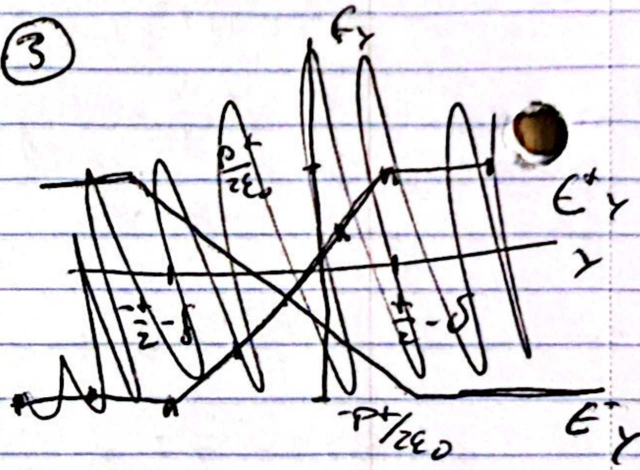
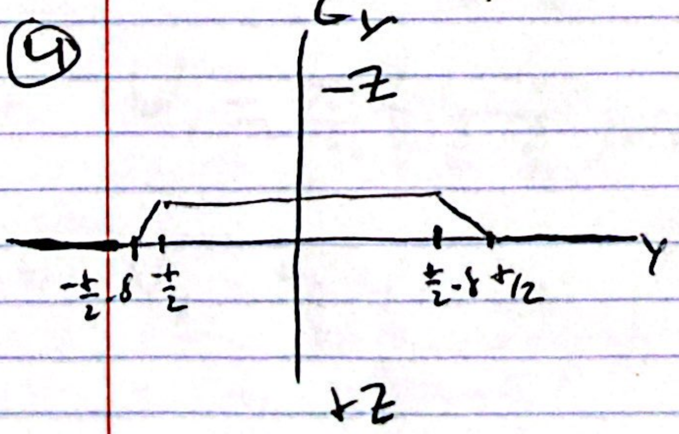
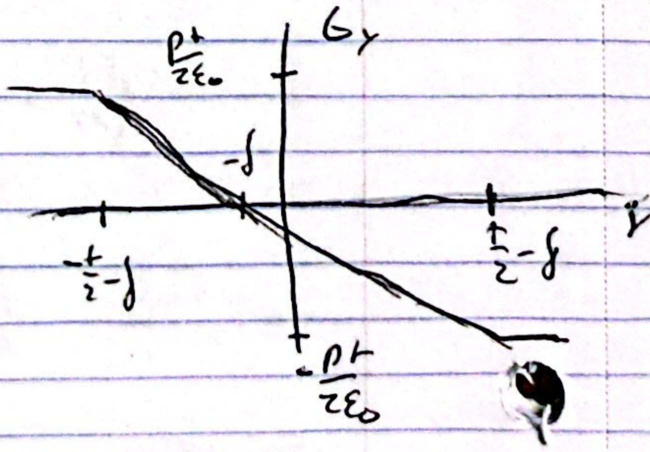


Same peaks shifted left by δ

④





4.4)

$P =$ dipole moment per unit volume

$\sigma_b = P_n \qquad \rho_b = -\nabla \cdot P$

$P = P_0 \hat{s} \qquad \hat{s} = \hat{n}$ (normal to cylinder)

$\sigma_b = P_0$

Question if we need to know this for exam

Cylindrical divergence $\rho_b = -\left(\frac{1}{s}\frac{\partial}{\partial s}(sP_s) + \frac{1}{s}\frac{\partial P_\phi}{\partial \phi} + \frac{\partial P_z}{\partial z}\right)$

we assume symmetry around

$\rho_b = -\frac{1}{s}\frac{\partial}{\partial s}(sP_s) \qquad$ z-axis $\frac{\partial P_\phi}{\partial \phi} = 0$

and

$\rho_b = -\frac{1}{s}P_s \qquad$ z is long $\therefore \frac{\partial P_z}{\partial z} = 0$

2) $V(r) = k\oint_S \frac{\sigma_b}{s}da + k\int \frac{\rho_b}{s}d\tau \qquad$ 4.18

Length $= L$

~~$V(r) = k\left(\frac{\sigma_b}{s}(\ldots) + \int \frac{\rho_b}{s} dz\, d\phi\, ds\right)$~~

~~$= k\left(\frac{P_0}{s}2\pi a L \iiint \frac{-P_0}{s^2}dz\right)$~~

~~$= \int_0^{2\pi} \frac{-P_0 L}{s^2}d\phi$~~

~~$\int_0^a \frac{-P_0 2\pi L}{s^2}ds$~~

From

$V(r) = k\left(\oint_S \frac{1}{s}P\,da + \int_V \frac{1}{s}(\nabla\cdot P)d\tau\right) \qquad$ 4.10

$d\tau = s\,ds\,dz\,d\phi$

$V(r) = k\left(\int_0^{2\pi}\int_0^L \frac{\sigma_b a}{s}dz\,d\phi + \int_0^{2\pi}\int_0^L\int_0^a \frac{\rho_b}{s} s\,ds\,dz\,d\phi\right)$

$k\left(2\pi L a\frac{\sigma_b}{s} + 2\pi L a \rho_b\right) \qquad \sigma_b = P_0$

$\rho_b = \frac{-P_0}{s}$

$V(r) = k\left(2\pi L a \frac{\rho_0}{s} - 2\pi L a \frac{\rho_0}{s}\right)$

$V(r) = 0$ ✓ makes sense for a dipole

3) find $E_0(s)$

$$\oint_S \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\varepsilon_0}$$

Outside of sphere $E_0(s) = 0$ $s > a$

dipole $\Rightarrow Q_{enc} = 0$

$Q_{enc} = \int_V \rho_0 \, d\tau$ $d\tau = s\,ds\,dz\,d\phi$

$= \int_0^{2\pi} \int_0^L \int_0^s \rho_0 \, s\,ds\,dz\,d\phi$ $\rho_0 = -\frac{\rho_0}{s}$

$= -2\pi L s \rho_0$

Simplify Gauss $EA = \frac{Q_{enc}}{\varepsilon_0}$

$E(2\pi L s) = \frac{-2\pi L s \rho_0}{\varepsilon_0}$ $E_0(s) = -\frac{\rho_0}{\varepsilon_0}$