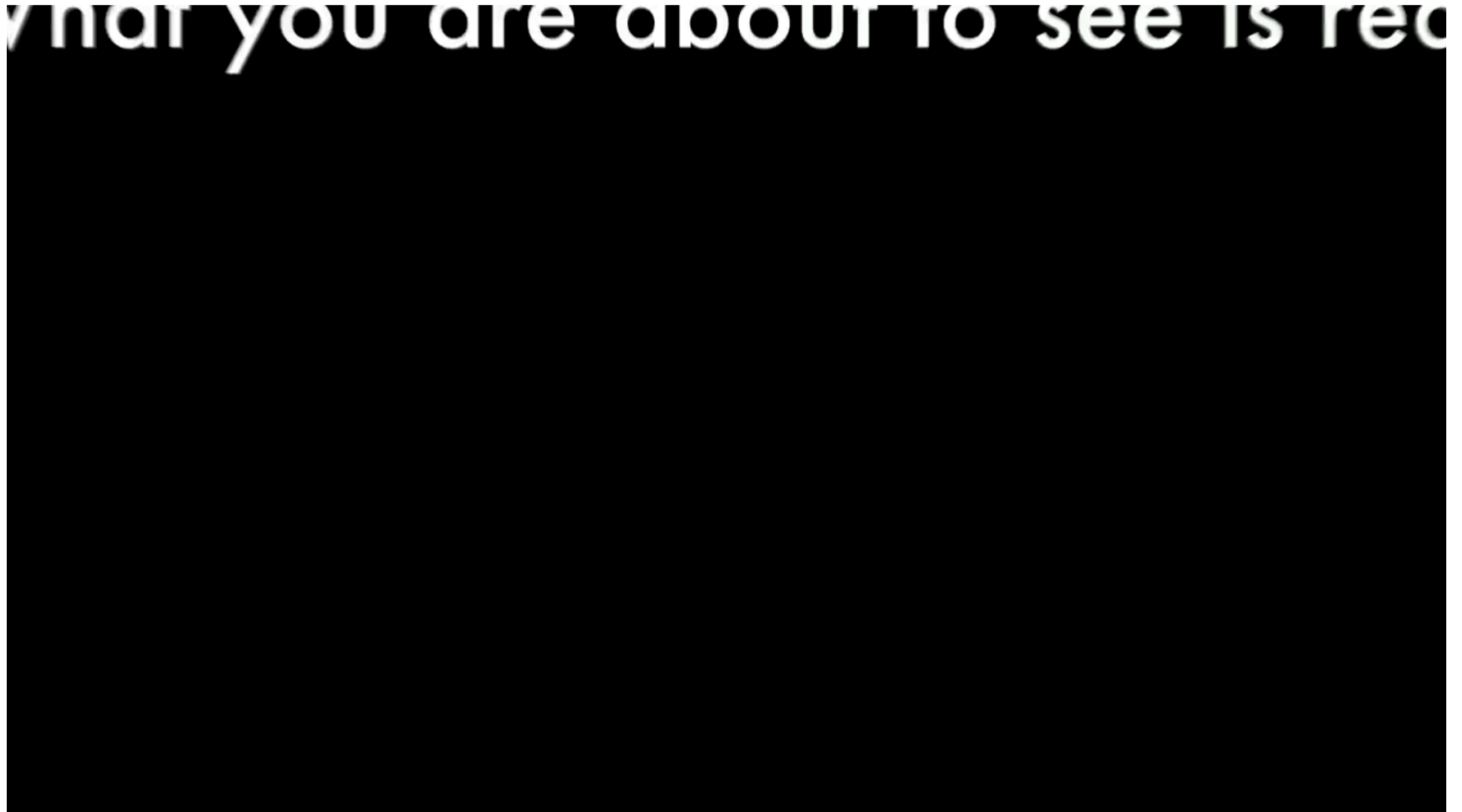
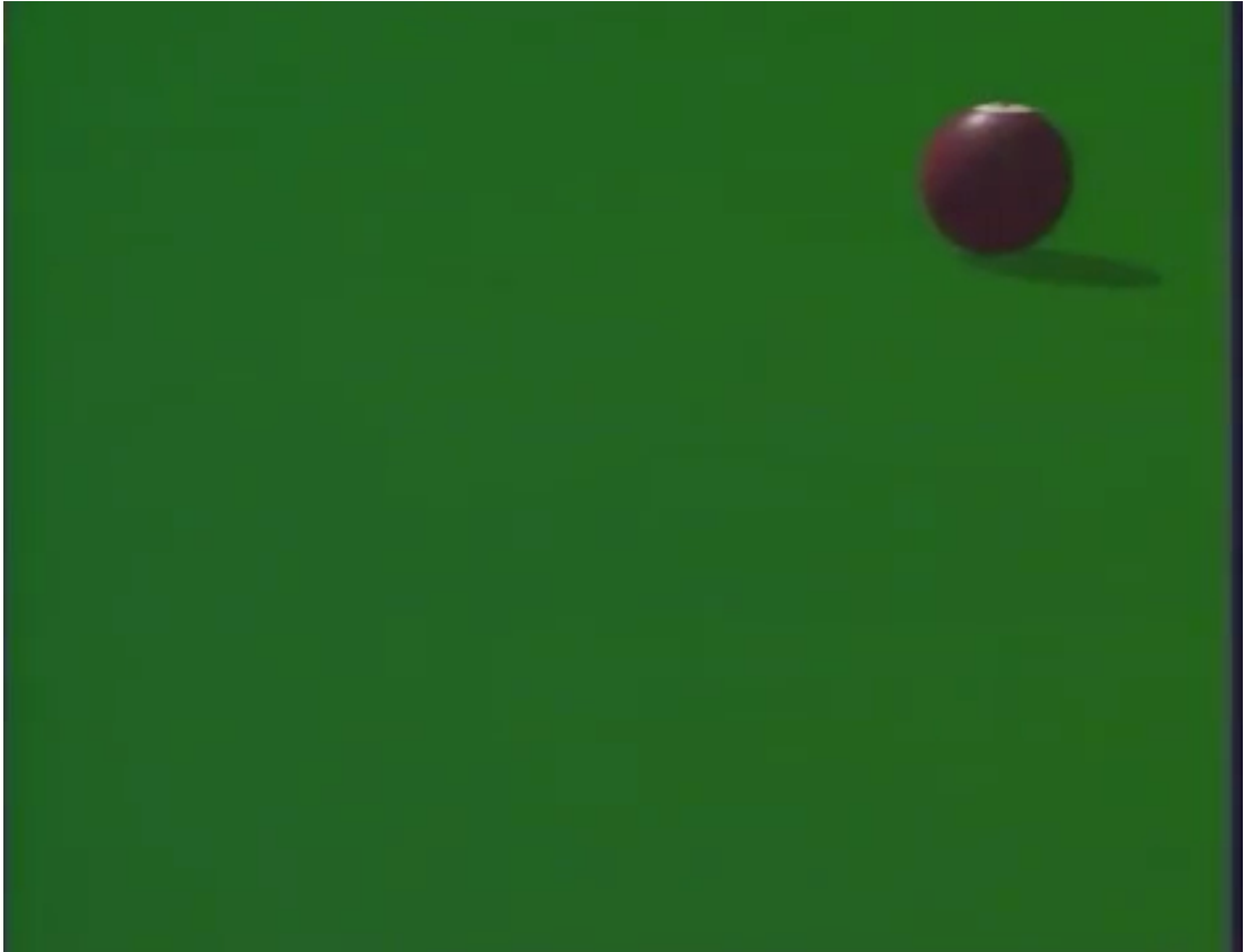


# Modern Physics Lecture #34

## Chapter 2: Special Relativity



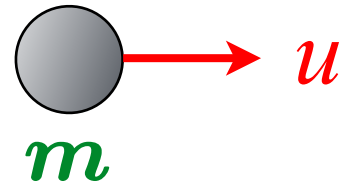
A collision of equal-mass billiard balls at slow velocities



A collision of equal-mass billiard balls at high velocities



# Velocity and Momentum



[Newtonian] velocity

$$u = dx/dt$$

between 0 and  $c$

[Newtonian] momentum

$$p = (m)(u)$$

between 0 and  $mc$

Proper velocity

$$\eta = dx/d\tau = \gamma_u u$$

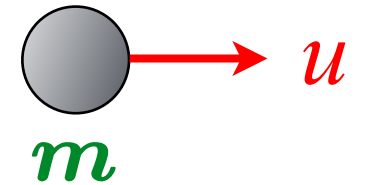
between 0 and  $\infty$

Proper momentum

$$p = (m)(\eta) = \gamma_u mu$$

between 0 and  $\infty$

# Total Energy, Rest Energy and Kinetic Energy



From Lecture 33 slide 19,  $E^2 = (pc)^2 + (mc^2)^2$

so the total energy is

$$\begin{aligned} E &= \sqrt{(pc)^2 + (mc^2)^2} \\ &= \sqrt{[(\gamma_u m u) c]^2 + (mc^2)^2} \\ &= \sqrt{[\gamma_u^2 u^2 + c^2] m^2 c^2} \\ &= \sqrt{[\gamma_u^2 c^2] m^2 c^2} \\ &= \gamma_u m c^2 \end{aligned}$$

while the rest energy is  $E_0 = mc^2$

so the kinetic energy is  $K \equiv E - E_0 = (\gamma_u - 1) mc^2$

# Energy at high velocities

$$W = \int F \, dx$$



**Invariant:** same value in all inertial frames

**Conserved:** same value as time progresses

Quantity	Conserved?	Invariant?
Mass		
Energy		
Momentum		
Electric charge		
Velocity		

**Invariant:** same value in all inertial frames

**Conserved:** same value as time progresses

Quantity	Conserved?	Invariant?
Mass	NO	YES
Energy	YES	NO
Momentum	YES	NO
Electric charge	YES	YES
Velocity	NO	NO



# Electric charge is a *Lorentz invariant*.

How do we know this?

- ✓ 2 protons and 2 electrons in hydrogen molecule are neutral overall to a very high precision. The protons are  $\sim 10^{-10}$  m apart ( $\sim 0.1$  nanometer) and moving at a non-relativistic speed of  $\sim 0.001c$ .
- ✓ 2 protons and 2 electrons in helium atom are neutral overall to a very high precision. The protons are  $\sim 10^{-15}$  m apart ( $\sim 1$  femtometer) and moving at a much faster speed of  $\sim 0.2c$ .
- ✓ The cancellation of proton and electron charge is unaffected by the vast range in proton speeds.

– from Purcell's *Electricity and Magnetism*, 2e

If you observe a baseball at rest, what is its four-momentum?

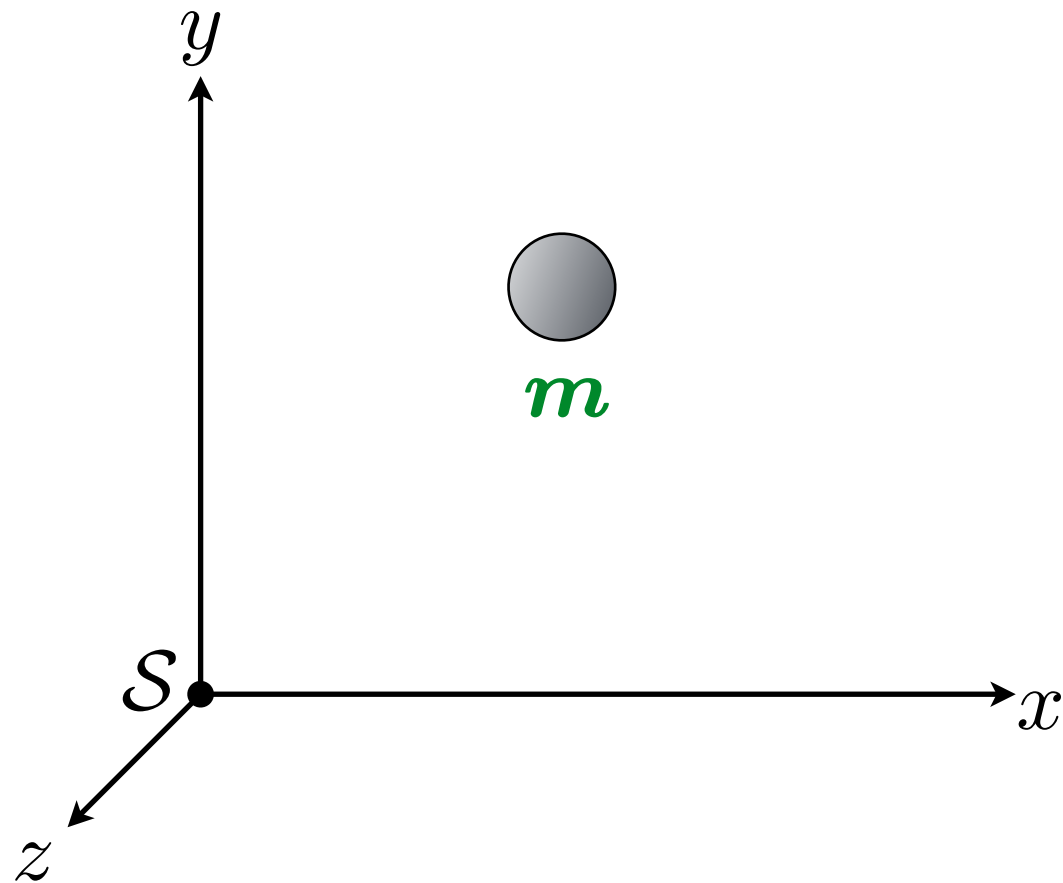
A) zero

B)  $(-mc, 0, 0, 0)$

C)  $(mc, mc, mc, mc)$

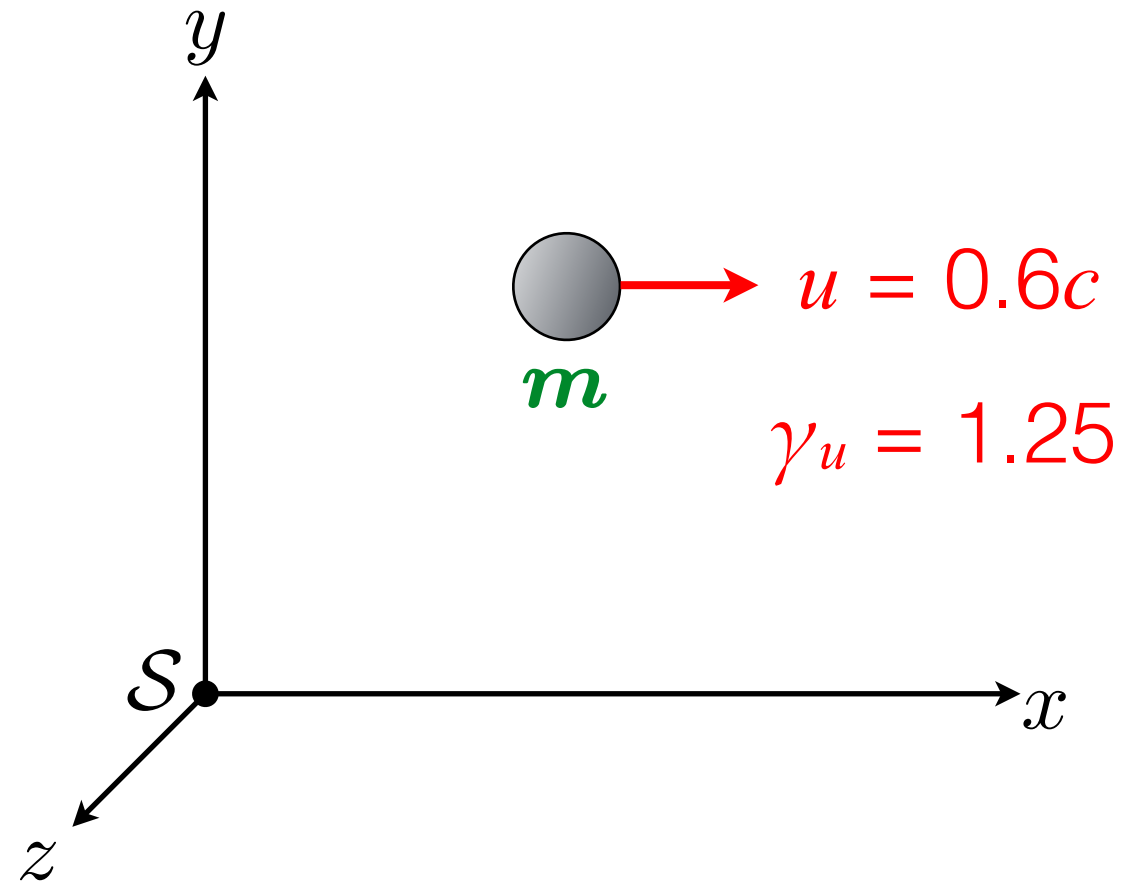
✓ D)  $(mc, 0, 0, 0)$

E)  $(0, 0, 0, 0)$



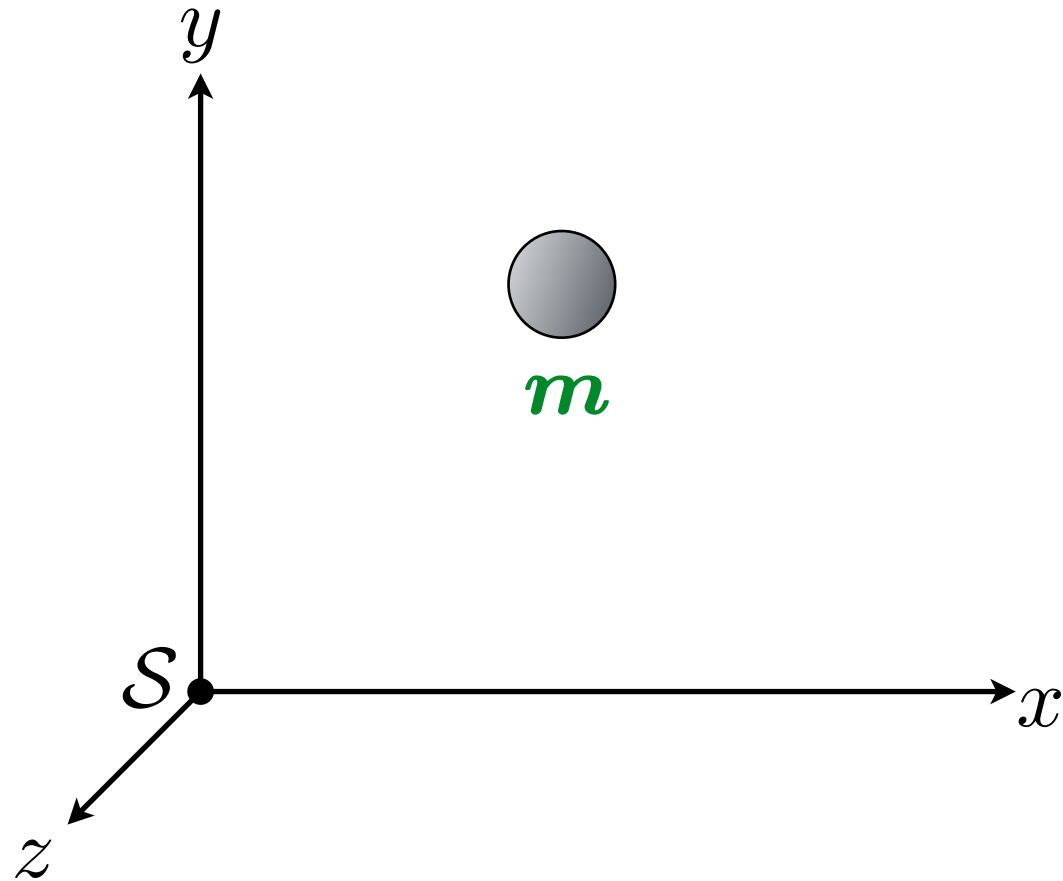
If you observe a baseball moving along the  $x$  axis at 0.6 times the speed of light, what is its four-momentum?

- A)  $(mc, 0.6mc, 0, 0)$
- B)  $(1.25mc, 0.6mc, 0, 0)$
- ✓ C)  $(1.25mc, 0.75mc, 0, 0)$
- D)  $(0.8mc, 0.48mc, 0, 0)$
- E)  $(0.8mc, 0.6mc, 0, 0)$



What is the “momentum invariant” for a baseball at rest?

- A) zero
- B)  $m^2c^2$
- ✓ C)  $-m^2c^2$
- D)  $2m^2c^2$
- E)  $-2m^2c^2$



$$\mathbf{P} = -\frac{E^2}{c^2} + p_x^2 + p_y^2 + p_z^2$$

What is the “momentum invariant” for a baseball moving at a speed of 0.6 times the speed of light?

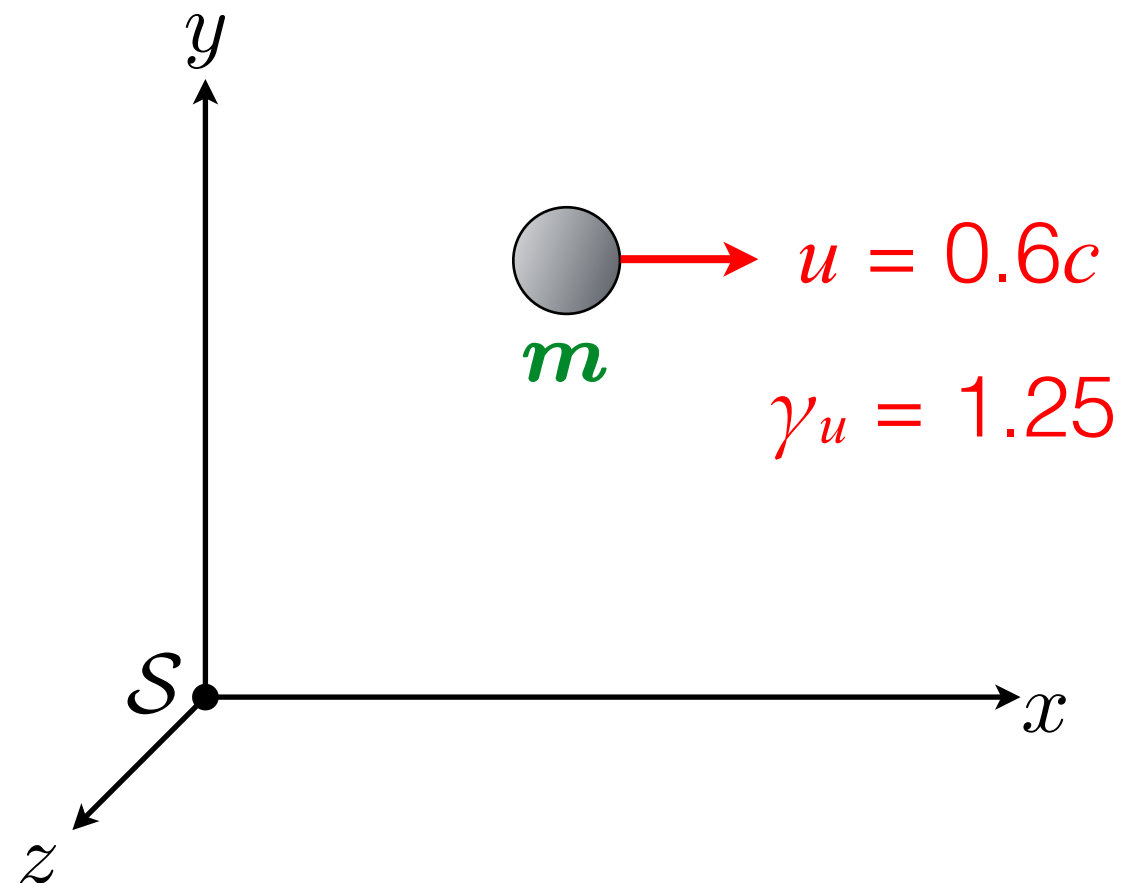
✓ A)  $-m^2c^2$

B)  $-1.5625m^2c^2$

C)  $-0.36m^2c^2$

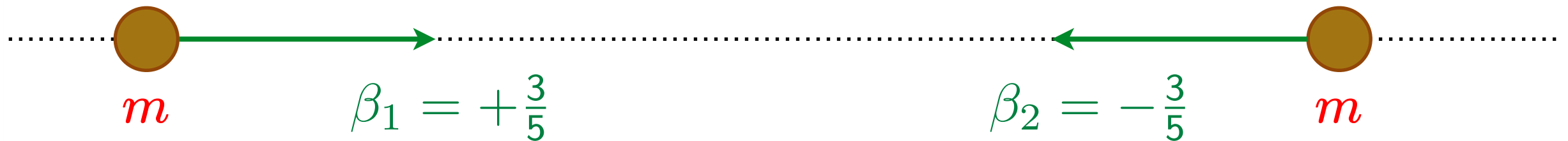
D)  $-0.64m^2c^2$

E)  $-0.5625m^2c^2$

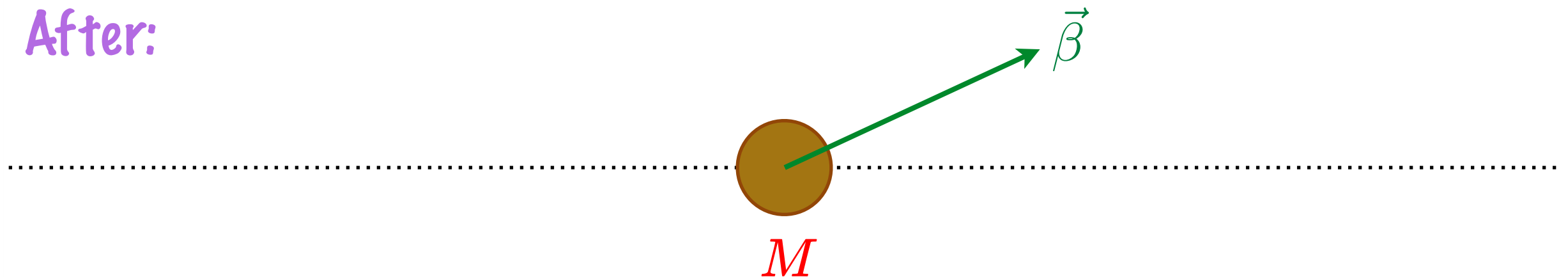


# Relativistic 4-momentum conservation in a collision of billiard balls

Before:



After:

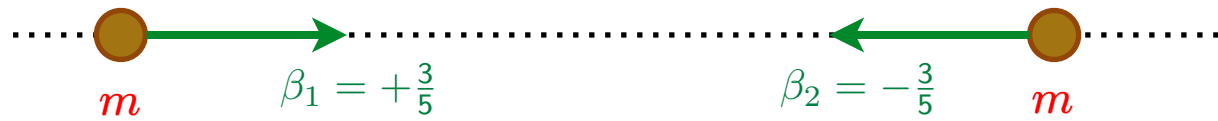


What are  $\vec{\beta}$  and  $M$ , given  $m$ ?

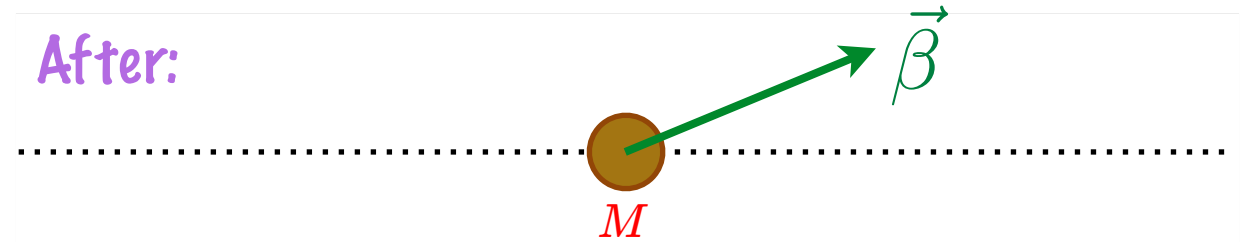
$c = 1$

# Relativistic 4-momentum conservation in a collision of billiard balls

Before:



After:



**Before:**  $p_1^\mu = \frac{5}{4}m(1, +\frac{3}{5}, 0, 0)$

$p_2^\mu = \frac{5}{4}m(1, -\frac{3}{5}, 0, 0)$

$$\gamma_1 = \gamma_2 = \frac{1}{\sqrt{1 - (\frac{3}{5})^2}} = \frac{5}{4}$$

**After:**  $p^\mu = \gamma M(1, \beta_x, \beta_y, \beta_z)$

*not the same as  $\gamma_1$   
– to be determined*

What are  $\vec{\beta}$  and  $M$ , given  $m$ ?

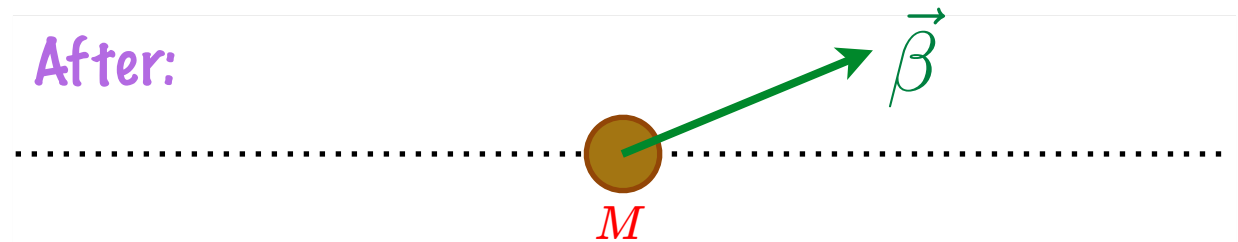
Use conservation of 4-momentum!

# Relativistic 4-momentum conservation in a collision of billiard balls

Before:



After:



**Before:**  $p_1^\mu = \frac{5}{4}m(1, +\frac{3}{5}, 0, 0)$

$p_2^\mu = \frac{5}{4}m(1, -\frac{3}{5}, 0, 0)$

**After:**  $p^\mu = \gamma M(1, \beta_x, \beta_y, \beta_z)$

①

②

③

④

$$\gamma M = \frac{5}{4}m + \frac{5}{4}m = \frac{5}{2}m$$

$$\beta_x = +\frac{3}{5} - \frac{3}{5} = 0$$

$$\beta_y = 0 + 0 = 0$$

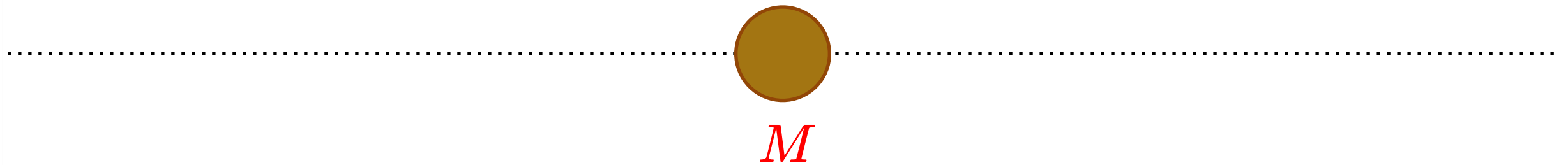
$$\beta_z = 0 + 0 = 0$$

Then  $\beta = \sqrt{\beta_x^2 + \beta_y^2 + \beta_z^2} = 0 \Rightarrow \gamma = 1 \Rightarrow M = \frac{5}{2}m$

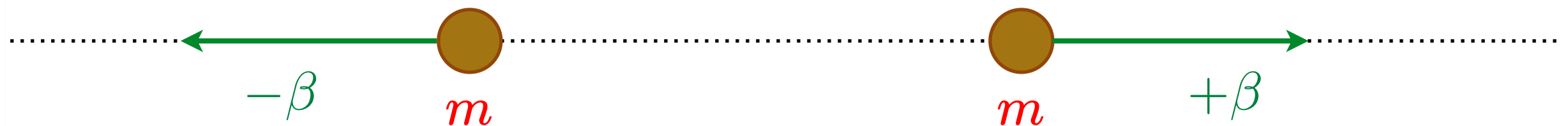


# Relativistic 4-momentum conservation in a disintegration into two equal-mass daughters

Before:



After:



What is  $\beta$ , given  $M$  and  $m$ ?

$$c = 1$$

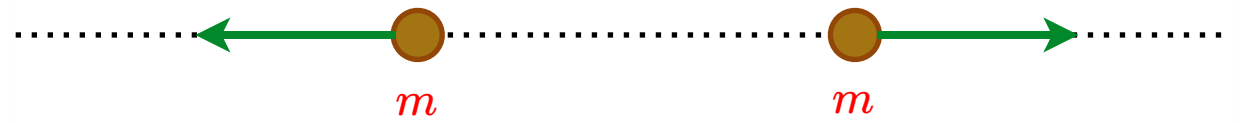
# Relativistic 4-momentum conservation in a disintegration into two equal-mass daughters

Before:



**Before:**  $p^\mu = \textcolor{red}{M}(1, 0, 0, 0)$

After:



**After:**  $p_1^\mu = \textcolor{green}{\gamma}\textcolor{red}{m}(1, -\textcolor{green}{\beta}, 0, 0)$

$$p_2^\mu = \textcolor{green}{\gamma}\textcolor{red}{m}(1, +\textcolor{green}{\beta}, 0, 0)$$

What is  $\beta$ , given  $\textcolor{red}{M}$  and  $\textcolor{red}{m}$ ?

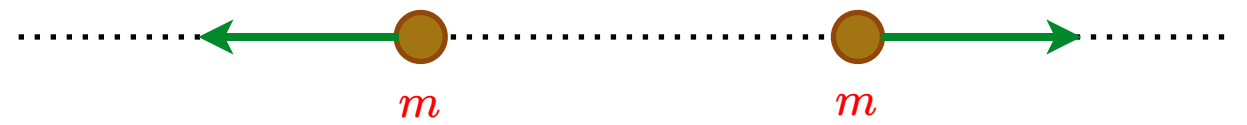
Use conservation of 4-momentum!

# Relativistic 4-momentum conservation in a disintegration into two equal-mass daughters

Before:



After:



**Before:**  $p^\mu = M(1, 0, 0, 0)$

**After:**  $p_1^\mu = \gamma m(1, -\beta, 0, 0)$

$p_2^\mu = \gamma m(1, +\beta, 0, 0)$

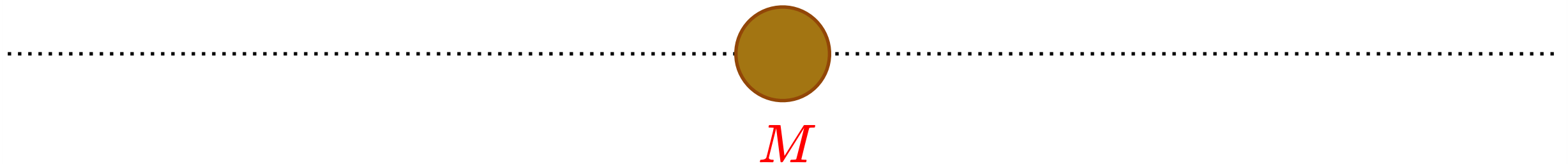
①

$$\boxed{M = \gamma m + \gamma m} \Rightarrow \gamma = \frac{M}{2m} = \frac{1}{\sqrt{1 - \beta^2}} \Rightarrow \beta = \sqrt{1 - \left(\frac{2m}{M}\right)^2}$$

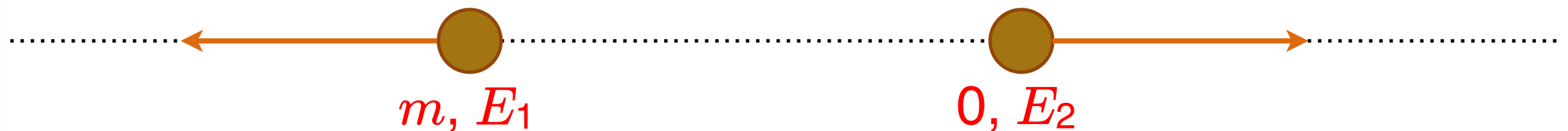
E.g.,  $\Upsilon(4S) \rightarrow B^0 + \bar{B}^0$  :  $\beta = \sqrt{1 - \left(\frac{2 \times 5279.6}{10579.4}\right)^2} = 0.0618$

# Relativistic 4-momentum conservation in a disintegration into two unequal-mass daughters

Before:



After:



What are  $E_1$  and  $E_2$ , given  $M$  and  $m$ ?

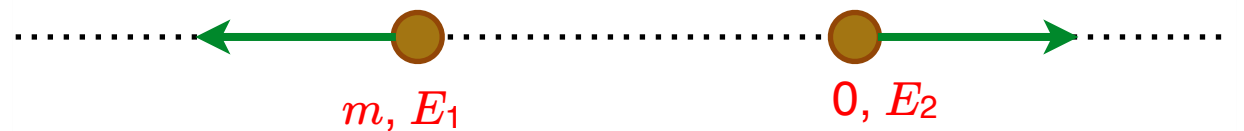
$c = 1$

# Relativistic 4-momentum conservation in a disintegration into two unequal-mass daughters

Before:



After:



**Before:**  $p^\mu = M(1, 0, 0, 0)$

**After:**  $p_1^\mu = (E_1, -\sqrt{E_1^2 - m^2}, 0, 0)$

$p_2^\mu = (E_2, +E_2, 0, 0)$

using  $E^2 = p^2 + m^2$  for  $p_1$   
and  $E = p$  for  $p_2$  ( $m = 0$ )

What are  $E_1$  and  $E_2$ , given  $M$  and  $m$ ?

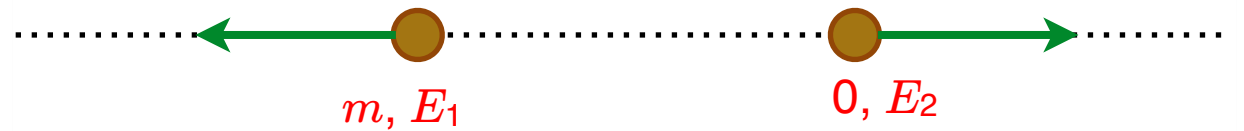
Use conservation of 4-momentum!

# Relativistic 4-momentum conservation in a disintegration into two unequal-mass daughters

Before:



After:



**Before:**  $p^\mu = M(1, 0, 0, 0)$

**After:**  $p_1^\mu = (E_1, -\sqrt{E_1^2 - m^2}, 0, 0)$

$$p_2^\mu = (E_2, +E_2, 0, 0)$$

①

$$M = E_1 + E_2$$

②

$$0 = -\sqrt{E_1^2 - m^2} + E_2$$

$$\Rightarrow E_1 = \frac{M^2 + m^2}{2M}$$

$$E_2 = \frac{M^2 - m^2}{2M}$$

# Relativistic 4-momentum conservation in a disintegration into two unequal-mass daughters

Before:



After:



**Example:**

$$\pi^+ \rightarrow \mu^+ + \bar{\nu}_\mu$$

Energy and mass  
measured in MeV.

$$E_1 = \frac{M^2 + m^2}{2M} = \frac{139.57^2 + 105.66^2}{2 \cdot 139.57} = 109.78$$

$$E_2 = \frac{M^2 - m^2}{2M} = \frac{139.57^2 - 105.66^2}{2 \cdot 139.57} = 29.8$$

Then, muon kinetic energy is

$$K_1 = E_1 - m = 109.78 - 105.66 = 4.12$$

and muon's relativistic factors are

$$\gamma = \frac{E_1}{m} = \frac{109.78}{105.66} = 1.04 \quad \beta = 0.27$$