SIE 606 Advanced Quality Engineering
Homework 1

Due: Feb 15 (9:15AM), 2018

1. A morning newspaper lists the following used-car prices for a foreign compact with age x_1 measured in years and selling price x_2 measured in thousands of dollars:

x_1	3	5	5	7	7	7	8	9	10	11
x_2	2.30	1.90	1.00	0.70	0.30	1.00	1.05	0.45	0.70	0.30

   (a). Construct a scatter plot of the data and marginal dot diagrams.
   (b). Infer the sign of the sample covariance s_{12} from the scatter plot.
   (c). Compute the sample means $\bar{x}_1$ and $\bar{x}_2$ and the sample variances s_{11} and s_{22}. Compute the sample covariance s_{12} and the sample correlation coefficient r_{12}. Interpret these quantities.
   (d). Calculate the sample mean array $\bar{\mathbf{x}}$, the sample variance-covariance array $\mathbf{S}_n$, and the sample correlation array $\mathbf{R}$.

2. Let

$$\mathbf{A} = \begin{bmatrix} 9 & -2 \\ -2 & 6 \end{bmatrix}$$

   (a). Is **A** symmetric?
   (b). Show that **A** is positive definite.

3. Verify the following properties of the transpose when

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 1 & 4 & 2 \\ 5 & 0 & 3 \end{bmatrix}, \text{ and } \mathbf{C} = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$$

   (a). $(\mathbf{A}^{\mathrm{T}})^{\mathrm{T}} = \mathbf{A}$
   (b). $(\mathbf{C}^{\mathrm{T}})^{-1} = (\mathbf{C}^{-1})^{\mathrm{T}}$
   (c). $(\mathbf{AB})^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$
   (d). For general $\mathbf{A}_{m\times k}$ and $\mathbf{B}_{k\times l}$, $(\mathbf{AB})' = \mathbf{B}'\mathbf{A}'$

4. Check that

$$\mathbf{Q} = \begin{bmatrix} \frac{5}{13} & \frac{12}{13} \\ -\frac{12}{13} & \frac{5}{13} \end{bmatrix}$$

   is an orthogonal matrix.

5. Using the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & -2 \\ 2 & 2 \end{bmatrix}$$

(a). Calculate $\mathbf{A}^{\mathrm{T}}\mathbf{A}$ and obtain its eigenvalues and eigenvectors.

(b). Calculate $\mathbf{A}\mathbf{A}^{\mathrm{T}}$ and obtain its eigenvalues and eigenvectors. Check that the nonzero eigenvalues are the same as those in part (a).

6. You are given the random vector $\mathbf{X}^{\mathrm{T}} = [X_1, X_2, X_3, X_4]$ with mean vector $\boldsymbol{\mu}_{\mathbf{X}}^{\mathrm{T}} = [4,3,2,1]$ and variance-covariance matrix

$$\boldsymbol{\Sigma}_{\mathbf{X}} = \begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 1 & 1 & 0 \\ 2 & 1 & 9 & -2 \\ 2 & 0 & -2 & 4 \end{bmatrix}$$

Partition X as

$$\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \\ \hdashline X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} \mathbf{X}^{(1)} \\ \hdashline \mathbf{X}^{(2)} \end{bmatrix}$$

Let

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}$$

and consider the linear combinations $\mathbf{AX}^{(1)}$ and $\mathbf{BX}^{(2)}$. Find

(a). $E\left(\mathbf{X}^{(1)}\right)$

(b). $E\left(\mathbf{AX}^{(1)}\right)$

(c). $Cov\left(\mathbf{X}^{(1)}\right)$

(d). $Cov\left(\mathbf{AX}^{(1)}\right)$

(e). $E\left(\mathbf{X}^{(2)}\right)$

(f). $E\left(\mathbf{BX}^{(2)}\right)$

(g). $Cov\left(\mathbf{X}^{(2)}\right)$

(h). $Cov\left(\mathbf{BX}^{(2)}\right)$

(i). $Cov\left(\mathbf{X}^{(1)}, \mathbf{X}^{(2)}\right)$

(j). $Cov\left(\mathbf{AX}^{(1)}, \mathbf{BX}^{(2)}\right)$

7. Let $\mathbf{X}$ be distributed as $N_3(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where $\boldsymbol{\mu}^{\mathrm{T}} = [1, -1, 2]$ and

$$\boldsymbol{\Sigma} = \begin{bmatrix} 4 & 0 & -1 \\ 0 & 5 & 0 \\ -1 & 0 & 2 \end{bmatrix}$$

Which of the following random variables are independent? Explain.

(a). X_1 and X_2

(b). X_1 and X_3

(c). X_2 and X_3

(d). (X_1, X_3) and X_2

(e). X_1 and $X_1 + 3X_2 - 2X_3$