Stephanie Wu

Rate Parameter Calculation
October 9, 2018

1 $T|Z = 0 \sim Exp(\lambda_t)$

$$
\begin{aligned}
\frac{n_p}{N_p} &= P(T \leqslant C, T \leqslant \tau_{\max}|Z=0) \\
&= P(T \leqslant \min(C, \tau_{\max})|Z=0) \\
&= \int_0^\infty P(T \leqslant \min(c, \tau_{max})|Z=0)P(C=c)dc \\
&= \int_0^{\tau_{\max}} P(T \leqslant c|Z=0)f_c(c)dc + \int_{\tau_{\max}}^\infty P(T \leqslant \tau_{max}|Z=0)f_c(c)dc \\
&= 1) + 2) \\
1) &= \int_0^{\tau_{\max}} \big(1 - \exp(-\lambda_t * c)\big)\big(\lambda_c \exp(-\lambda_c * c)\big)dc \\
&= \int_0^{\tau_{\max}} \Big(\lambda_c \exp(-\lambda_c * c) - \lambda_c \exp(-c(\lambda_t + \lambda_c))\Big)dc \\
&= \left[\frac{\lambda_c}{-\lambda_c}\exp(-\lambda_c * c)\right]_0^{\tau_{\max}} - \left[\frac{\lambda_c}{-(\lambda_t + \lambda_c)}\exp(-(\lambda_t + \lambda_c) * c)\right]_0^{\tau_{\max}} \\
&= -\exp(-\lambda_c * \tau_{\max}) + \exp(0) + \Big(\frac{\lambda_c}{\lambda_t + \lambda_c}\Big)\Big(\exp(-(\lambda_t + \lambda_c) * \tau_{\max}) - \exp(0)\Big) \\
&= 1 - \exp(-\lambda_c * \tau_{\max}) - \frac{\lambda_c}{\lambda_t + \lambda_c} + \Big(\frac{\lambda_c}{\lambda_t + \lambda_c}\Big)\Big(\exp(-(\lambda_t + \lambda_c) * \tau_{\max})\Big) \\
2) &= P(T \leqslant \tau_{max}|Z=0)\int_{\tau_{\max}}^\infty f_c(c)dc \\
&= \Big(1 - \exp(-\lambda_t * \tau_{\max})\Big)\Big(\exp(-\lambda_c * \tau_{\max})\Big) \\
&= \exp(-\lambda_c * \tau_{\max}) - \exp\big(-(\lambda_t + \lambda_c) * \tau_{\max}\big) \\
\therefore \frac{n_p}{N_p} &= 1) + 2) \\
&= 1 - \exp(-\lambda_c * \tau_{\max}) - \frac{\lambda_c}{\lambda_t + \lambda_c} + \Big(\frac{\lambda_c}{\lambda_t + \lambda_c}\Big)\Big(\exp(-(\lambda_t + \lambda_c) * \tau_{\max})\Big) \\
&\quad + \exp(-\lambda_c * \tau_{\max}) - \exp\big(-(\lambda_t + \lambda_c) * \tau_{\max}\big) \\
&= 1 - \frac{\lambda_c}{\lambda_t + \lambda_c} + \exp\big(-(\lambda_t + \lambda_c) * \tau_{\max}\big)\big(\frac{\lambda_c}{\lambda_t + \lambda_c} - 1\big) \\
&= \frac{\lambda_t}{\lambda_t + \lambda_c} + \exp\big(-(\lambda_t + \lambda_c) * \tau_{\max}\big)\Big(\frac{-\lambda_t}{\lambda_t + \lambda_c}\Big) \\
&= \frac{\lambda_t}{\lambda_t + \lambda_c}\Big(1 - \exp\big(-(\lambda_t + \lambda_c) * \tau_{\max}\big)\Big)
\end{aligned}
$$

2 $C|Z = 0 \sim Exp(\lambda_c)$

$$
\begin{aligned}
0.1 &= P(C \leqslant T, C \leqslant \tau_{\max}|Z=0) \\
&= P(C \leqslant \min(T, \tau_{\max})|Z=0) \\
&= \int_0^\infty P(C \leqslant \min(c, \tau_{max})|Z=0) P(T=t)dt \\
&= \int_0^{\tau_{\max}} P(C \leqslant t|Z=0)P(T=t)dt + P(C \leqslant \tau_{max}|Z=0)\int_{\tau_{\max}}^\infty P(T=t)dt \\
&= 1) + 2) \\
1) &= \int_0^{\tau_{\max}} \big(1-\exp(-\lambda_c * t)\big)\big(\lambda_t \exp(-\lambda_t * t)\big)dt \\
&= \int_0^{\tau_{\max}} \Big(\lambda_t \exp(-\lambda_t * t) - \lambda_t \exp(-(\lambda_t+\lambda_c) * t)\Big)dt \\
&= \left[\frac{\lambda_t}{-\lambda_t}\exp(-\lambda_t * t)\right]_0^{\tau_{\max}} - \left[\frac{\lambda_t}{-(\lambda_t+\lambda_c)}\exp(-(\lambda_t+\lambda_c) * t)\right]_0^{\tau_{\max}} \\
&= -\exp(-\lambda_t * \tau_{\max}) + 1 + \Big(\frac{\lambda_t}{\lambda_t+\lambda_c}\Big)\Big(\exp(-(\lambda_t+\lambda_c) * \tau_{\max}) - \frac{\lambda_t}{\lambda_t+\lambda_c} \\
2) &= \Big(1-\exp(-\lambda_c * \tau_{\max})\Big)\Big(\exp(-\lambda_t * \tau_{\max})\Big) \\
&= \exp(-\lambda_t * \tau_{\max}) - \exp\big(-(\lambda_t+\lambda_c) * \tau_{\max}\big) \\
\therefore 0.1 &= 1) + 2) \\
&= -\exp(-\lambda_t * \tau_{\max}) + 1 + \Big(\frac{\lambda_t}{\lambda_t+\lambda_c}\Big)\Big(\exp(-(\lambda_t+\lambda_c) * \tau_{\max}) - \frac{\lambda_t}{\lambda_t+\lambda_c} \\
&\quad + \exp(-\lambda_t * \tau_{\max}) - \exp\big(-(\lambda_t+\lambda_c) * \tau_{\max}\big) \\
&= 1 - \frac{\lambda_t}{\lambda_t+\lambda_c} + \exp\big(-(\lambda_t+\lambda_c) * \tau_{\max}\big)\big(\frac{\lambda_t}{\lambda_t+\lambda_c} - 1\big) \\
&= \frac{\lambda_c}{\lambda_t+\lambda_c} + \exp\big(-(\lambda_t+\lambda_c) * \tau_{\max}\big)\Big(\frac{-\lambda_c}{\lambda_t+\lambda_c}\Big) \\
&= \frac{\lambda_c}{\lambda_t+\lambda_c}\Big(1 - \exp\big(-(\lambda_t+\lambda_c) * \tau_{\max}\big)\Big)
\end{aligned}
$$

3 System of equations

$$\frac{n_p}{N_p} = \frac{\lambda_t}{\lambda_t + \lambda_c}\Big(1 - \exp\big(-(\lambda_t + \lambda_c) * \tau_{\max}\big)\Big)$$

$$\iff \frac{n_p}{N_p} * \frac{\lambda_t + \lambda_c}{\lambda_t} = \Big(1 - \exp\big(-(\lambda_t + \lambda_c) * \tau_{\max}\big)\Big)$$

Plug into: $$0.1 = \frac{\lambda_c}{\lambda_t + \lambda_c}\Big(1 - \exp\big(-(\lambda_t + \lambda_c) * \tau_{\max}\big)\Big)$$

$$\iff 0.1 = \frac{\lambda_c}{\lambda_t + \lambda_c} * \frac{n_p}{N_p} * \frac{\lambda_t + \lambda_c}{\lambda_t}$$

$$\iff 0.1 * \frac{N_p}{n_p} = \frac{\lambda_c}{\lambda_t}$$

$$\iff 0.1 * \frac{N_p}{n_p} * \lambda_t = \lambda_c$$

Plugging λ_c in, we have: $$0.1 = \frac{0.1 * \frac{N_p}{n_p} * \lambda_t}{\lambda_t + 0.1 * \frac{N_p}{n_p} * \lambda_t} * \Big(1 - \exp\big(-(\lambda_t + 0.1 * \frac{N_p}{n_p} * \lambda_t) * \tau_{\max}\big)\Big)$$

$$\iff 0.1 = \frac{0.1 * \frac{N_p}{n_p}}{1 + 0.1 * \frac{N_p}{n_p}} * \Big(1 - \exp\big(-\lambda_t * \tau_{\max}(1 + 0.1 * \frac{N_p}{n_p})\big)\Big)$$

$$\iff \frac{0.1(1 + 0.1 * \frac{N_p}{n_p})}{0.1 * \frac{N_p}{n_p}} = 1 - \exp\big(-\lambda_t * \tau_{\max}(1 + 0.1 * \frac{N_p}{n_p})\big)$$

$$\iff 1 - \big(1 + 0.1 * \frac{N_p}{n_p})(\frac{n_p}{N_p})\big) = \exp\big(-\lambda_t * \tau_{\max}(1 + 0.1 * \frac{N_p}{n_p})\big)$$

$$\iff log[1 - \big(1 + 0.1 * \frac{N_p}{n_p})(\frac{n_p}{N_p})\big)] = -\lambda_t * \tau_{\max}(1 + 0.1 * \frac{N_p}{n_p})$$

$$\iff \frac{log[1 - \big(1 + 0.1 * \frac{N_p}{n_p})(\frac{n_p}{N_p})\big)]}{-\tau_{\max}(1 + 0.1 * \frac{N_p}{n_p})} = \lambda_t$$

Thus, $\lambda_c = 0.1 * \dfrac{N_p}{n_p} * \lambda_t$ and

$$\lambda_t = \frac{log[1 - \big(1 + 0.1 * \frac{N_p}{n_p})(\frac{n_p}{N_p})\big)]}{-\tau_{\max}(1 + 0.1 * \frac{N_p}{n_p})}$$