A Computational Study of Muscle Biomechanics

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Abstract

"The relationship between the stress and strain that a particular material displays is known as that particular material's stress–strain curve." [1] The strain on fresh muscle increases as more stress is applied to it. A loading experiment can be performed to determine these values for any given muscle. These values can then be used and evaluated using the fundamental relationship for soft tissue. By using many computational methods, we can use this data to make analytic curves that closely resemble that of the original.

1. Introduction

In this paper, we are creating two lines of best fit, as well as several analytic curves based on data generated from a loading experiment. This data will then be analyzed using MATLAB. Finite difference methods were used to establish values from the initial data. With this, lines of regression were created that both had r^2, or coefficient of determinations, values which were close to 1, which means that they were accurate. Conversely, the first analytic curves found were not as accurate. After algebraic manipulations, however, these curves were changed and became much more accurate.

2. Numerical Analysis

The calculations in this project start off with the following equation:

$$\sigma = \frac{E_0}{a}\left(e^{a\varepsilon} - 1\right) \quad (1)$$

This equation is based off of soft tissue, and its behaviors in uniaxial tension while in the normal range of elongation. "σ is the stress (N/m2), ε is the strain (dimensionless), and E_0 and a are material constants that are determined experimentally." [2]

To determine the material constants, we need to take a derivative with respect to ε. Doing this gives the following equation, which establishes the fundamental relationship for soft tissue:

$$\frac{d\sigma}{d\varepsilon} = E_0 + a\sigma \quad (2)$$

Now, to determine the values for E_0 and a, we then plot **Equation (2)** plot as a line. We can then deduce from this that E_0 is the intercept, while a is the slope.

We then reference Table 1 for data points to use to solve these equations.

Stress σ, (10^2)	Strain ε, (10^{-3})
88	155
96	204
176	255
263	306
351	357
571	408
834	460
1230	510
1624	561
2107	612
2678	663
3380	714
4258	765

Table 1: Stress-strain data for fresh muscle.

To continue solving this equation, we need to use the bisection method. The bisection method is based on the Intermediate Value Theorem, which says "If f is continuous on [a, b] and $f(a)f(b) < 0$ then there exists at least one root r in (a, b) such that $f(r) = 0$."

The function *f* is **Equation (1)** set equal to zero. The points *a* and *b* are decided by the user, with *a* needing to be less than the root, and *b* needing to be greater than it. σ_{mean} and ε_{mean} are also required for using the bisection method, and were determined by simply taking the averages of the stress and strain columns.

We are solving **Equation (1)** for variable a, and user $\sigma = \sigma_{mean}$, $\varepsilon = \varepsilon_{mean}$, $E_0 = 1$.

An additional value, *c*, is then assigned and is equal to the middle point between *a* and *b*. We then make a comparison between the values of *a* and *c*. If the two values have the same sign, then *b* is changed to *c*, giving us an interval of [*a*,*c*]. If the two values have opposite signs, then *a* becomes *c*, giving us an interval of [*c*,*b*]. This process is repeated until *f(c)* is within the tolerance range set by the user at the beginning of the program. The value of *c* at this point is the root.

We then calculate the derivative dσ/dε using two finite difference schemes, forward and backward. "Finite-difference methods (FDM) are numerical methods for solving differential equations by approximating them with difference equations, in which finite differences approximate the derivatives. FDMs are thus discretization methods." [3]

A forward difference scheme uses the equation:

$$\frac{d\sigma}{d\varepsilon} = \frac{\sigma_{i+1} - \sigma_i}{\varepsilon_{i+1} - \varepsilon_i} \tag{3}$$

While a backward difference scheme uses the equation:

$$\frac{d\sigma}{d\varepsilon} = \frac{\sigma_i - \sigma_{i-1}}{\varepsilon_i - \varepsilon_{i-1}} \tag{4}$$

It is important to note that while the two equations look very similar, they will not provide the same results.

At this point, we have determined mostly linear points corresponding to dσ/dε. However, there will be points that are close to zero due to inability to read small values by the instrumentation, so these are discarded.

We then created a line of regression for both of the finite difference schemes by using the Least Squares Method. This method is based on the Weierstrass' Extreme Value Theorem: if *f* is continuous on the interval [a,b], then *f* has a maximum and minimum on [a,b]. The points at which *f* has extreme values are critical points.

As with any mathematical method, it is susceptible to error. To minimize this error, we use the following function:

$$S_r = \sum_{i=1}^{n} (y_i measured - y_i calculated)^2$$

$$= \sum_{i=1}^{n} (y_i - a_0 - a_1 x_i)^2$$

Where x_i and y_i are the data points.

The line of best fit is given by when the derivative of S_r with respect to a_0 and a_1 is equal to zero.

$$\frac{dS_r}{da_0} = -2 \sum_{i=1}^{n} (y_i - a_0 - a_1 x_i) = 0$$

$$\frac{dS_r}{da_1} = -2 \sum_{i=1}^{n} [(y_i - a_0 - a_1 x_i) x_i] = 0$$

We then solve the above equations for a_0 and a_1 and get the following:

$$a_1 = \frac{n \sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n \sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2}$$

$$a_0 = y_{mean} - a_1 x_{mean}$$

Where

$$y_{mean} = \frac{\sum_{i=1}^{n} y_i}{n}$$

and

$$x_{mean} = \frac{\sum_{i=1}^{n} x_i}{n}$$

This will give us our line of regression in the form of:

$$y = a_1 x_i + a_0$$

It is important to note that a_1 is equal to E_0 and a_0 is equal to α.

To ensure that our equation is accurate, we will find the coefficient of determination, r^2, using the formula:

$$\frac{\sum_{i=1}^{n} \left(y_i - \frac{\sum_{i=1}^{n} y_i}{n}\right)^2 - \sum_{i=1}^{n} (y_i - a_0 - a_1 x_i)^2}{\sum_{i=1}^{n} \left(y_i - \frac{\sum_{i=1}^{n} y_i}{n}\right)^2}$$

The main difference between using these calculations for both the forward and backward scheme is that their values for *i* and *n* will differ. For the forward scheme, *i* will be equal to 5, while *n* will be equal to 12. This is due to the elimination of erroneous data and being able to go up to one less than the maximum number of values. The backward scheme will have and *i* = 6 and an *n* =13 for similar reasons.

From this, we take the values just found for E_0 and α and substitute them in **Equation (1)**. This gives us an analytic curve, which we can compare to our plain data in Table 1 to test its accuracy. However, this frequently does not work well as E_0 can be difficult to evaluate using this technique. The fix this problem, we don't use E_0 and instead substitute σ_{mean} and ε_{mean} in **Equation (1)**. From there, we solve for E_0/α. This value of E_0/α is then substituted into **Equation (1)** giving us:

$$\sigma = \frac{\sigma_{mean}}{e^{\alpha\varepsilon_{mean}} - 1}(e^{\alpha\varepsilon} - 1)\ (5)$$

With this, we can once again plot our new curves and compare it to the raw data to test its accuracy.

4. Results

To being solving these equations, we need to find the mean values of both the stress and strain. These were found to be 208525 and $9.58 * 10^5$, respectively. Then, by using $\sigma = \sigma_{mean}$, $\varepsilon = \varepsilon_{mean}$, and $E_0 = 1$ for **Equation (1)**, we can get:

$$0 = \frac{1}{\alpha}\left(e^{\alpha(9.58*10^5)} - 1\right) - 208525$$

From this, we applied the bisection method. To get a value within the tolerance range of 0.000001 it took 23 iterations, and was calculated to be 33.375.

The forward and backward difference schemes gave us a multitude of values, which can be seen on **Figure 1**.

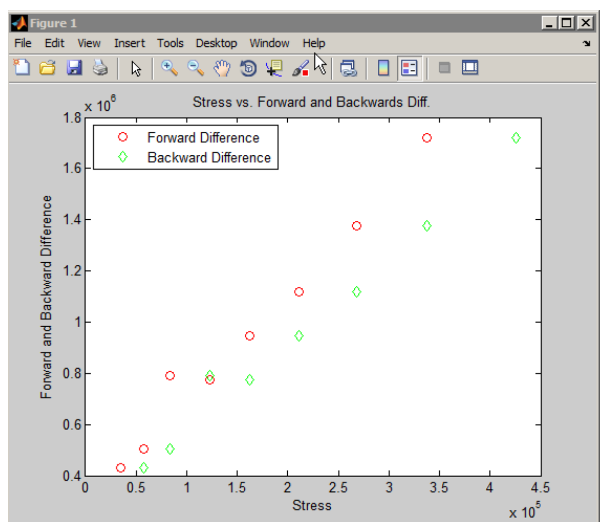

Figure 1. Stress vs. the values from the Forward and Backward Difference schemes.

These points represent the data points created by running the scheme with the values from **Table 1**. However, the first 4 data points for each are excluded from the graph, as they were erroneous.

Next, the linear regression analysis gives us the values for E_0 and a. These were found after using the Least Squares Method, which came out in the form of **Equation (2)**. The forward difference scheme produced an equation of:

$y = 4.048683x + 311775.559797.$

This means that

$E_0 = 311775.559797$

$a = 4.048683$

with an r^2 of 0.978746.

This is visually represented by **Figure 2**.

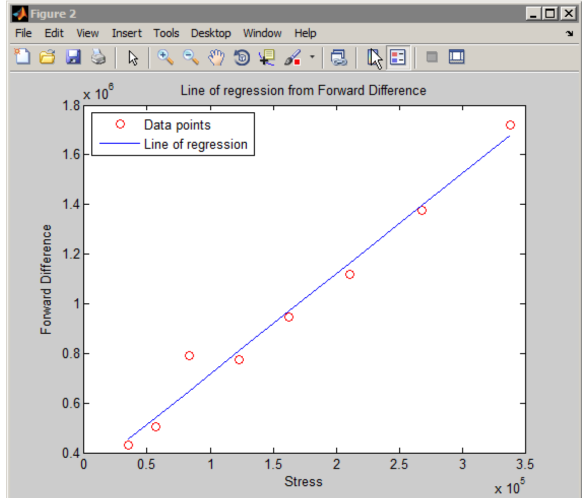

Figure 2. A line of regression comparing the stress to the data produced from the forward difference scheme.

The backwards difference scheme produced an equation of

$y = 3.365029x + 256606.982827$

This means

$E_0 = 256606.982827$

$a = 3.365029$

with an r^2 of 0.984847

This is visually represented by **Figure 3**.

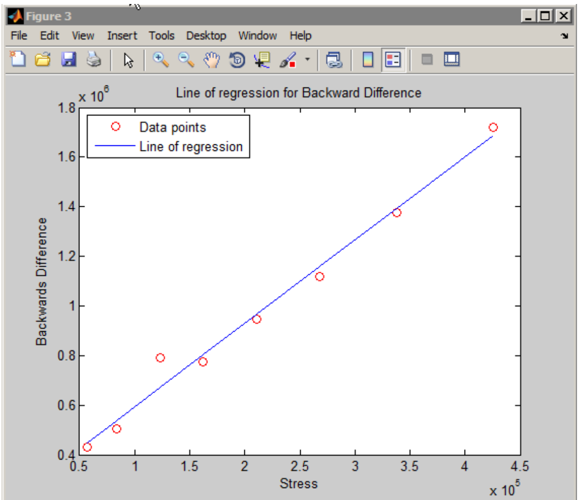

Figure 3. A line of regression comparing the stress to the data produced from the backwards difference scheme.

Next, we can plug in the values for E_0 and a into **Equation (1)**. This gives us two separate equations, one for each of the difference schemes. They are:

Forward

$$\sigma = \frac{311775.559797}{4.048683}(e^{4.048683\varepsilon} - 1)$$

Backwards

$$\sigma = \frac{256606.982827}{3.365029}(e^{3.365029\varepsilon} - 1)$$

These curves can then be compared to a curve of the actual data for accuracy. The forward curve can be seen in **Figure 4**, while the backward curve can be seen in **Figure 5**.

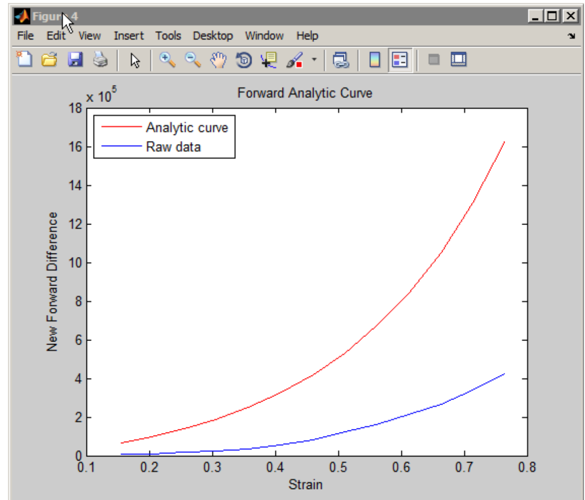

Figure 4. A graph of the strain on the muscle compared to the stress found from the Forward Difference scheme.

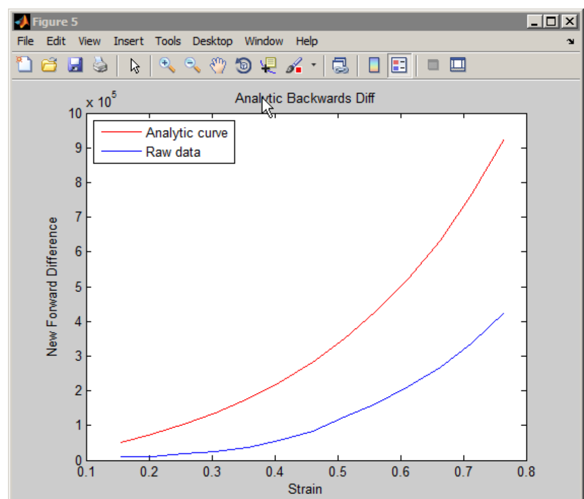

Figure 5. A graph of the strain on the muscle compared to the stress found from the Forward Difference scheme.

These curves are clearly not very accurate. To help improve on this, we discard E_0. Instead, we substitute σ_{mean} and ε_{mean} in **Equation (1)** and solve the equation for E_0/a. For both schemes, we get the following:

Forward

$$\sigma = \frac{208525}{e^{(4.048683)(9.58*10^5)} - 1}\left(e^{(4.048683)(9.58*10^5)} - 1\right)$$

Backwards

$$\sigma = \frac{208525}{e^{(3.365029)(9.58*10^5)} - 1}\left(e^{(3.365029)(9.58*10^5)} - 1\right)$$

These equations are once again compared to the data from **Table 1** for accuracy. They can be seen in **Figure 6** and **Figure 7**, respectively.

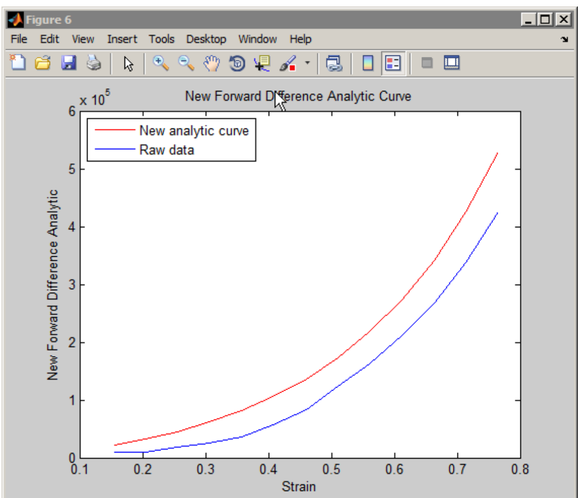

Figure 6. Strain vs. the new analytic curve solved using a forward difference scheme

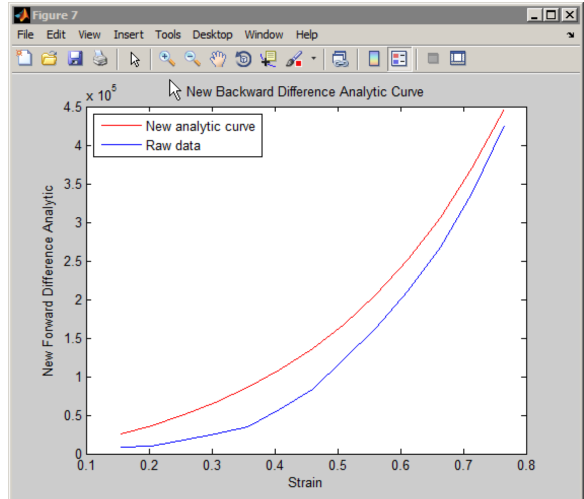

Figure 7. Strain vs. the new analytic curve solved using a backwards difference scheme

These new curve fittings are much more accurate than their respective originals. By looking at the graphs, one can notice how much closer the new analytic curves are compared to the old.

displacement matrix. Computations are done, and the real displacement is found. These values are then presented to the user.

Finally, the inverse of matrix A is found.

After working out the math on paper and comparing, we have determined that our methods are accurate.

5. Conclusion

In this paper, we used many computational methods in order to closely evaluate the relationship between stress and strain on fresh muscle. This information was compiled from a loading experiment and was analyzed using MATLAB. Finite difference schemes were used in order to produce values that were then used to create lines of regression. These lines were used to extract values, which were then used to create curves. These curves were compared to the original data, refined, and then compared once again. After being refined, it was found that the new data was much closer. There biggest errors during analysis occurred during the creation of the lines of regression, as the two difference schemes had different boundaries. Overall, the analysis produced results that are close to that with which they were aiming for.

References

1.) "Stress-strain curve." *Wikipedia*. Wikimedia Foundation, 7 Mar. 2015. Web. 28 Apr. 2015.

2.) Drapaca, Corina. *ESC261M Final Project 2015.* The Pennsylvania State University, 30 Mar. 2015. Web. 28 Apr. 2015.

3.) "Finite difference methods." *Wikipedia*. Wikimedia Foundation, 28 Apr. 2015. Web. 28 Apr. 2015.

Appendix

MATLAB:

```
clear all;

clc;


stress=fopen('stress.dat','r');

strain=fopen('strain.dat','r');

% getting values from files

x=csvread('stress.dat');

y=csvread('strain.dat');


stressmean = mean(x);

strainmean = mean(y);


%Bisection Method

maxiter=100;

a = 30;

b = 40;

delta = input('Please enter root tolerance.\n');

eps = input('Please enter residual tolerance.\n');

delta = 0.000001; eps = 0.000001;

E0=1;
```

```
%equation with left end point plugged in

u = (E0 / a)*(exp(a*strainmean) - 1) - stressmean;

%function value at left point

e = b - a; % interval length

for k = 1:maxiter

e = e*0.5;

% shrink interval by half

c = a + e;

%update middle point

w = (E0 / c)*(exp(c*strainmean) - 1) - stressmean;

%function value at middle point

   if (abs(e) < delta || abs(w) < eps)

   break;

   end

   if ((u > 0) && (w < 0)) || ((u < 0) && (w > 0))

   b = c;

   else

a = c; u = w;

   end

   fprintf('Nr. of iterations it took the bisection method to converge to root: %f \n',k);

   fprintf('Approximated root found by the bisection method is: %f \n',c);

end
```

```
n=12;

%Forward and Backward difference schemes

for i=1:n

    FDiff(i) = (x(i+1) - x(i)) / (y(i+1) - y(i));

end


for i=n+1:-1:2

    BDiff(i) = (x(i) - x(i-1)) / (y(i) - y(i-1));

end


%Plot Forward Difference scheme

plot(x(5:12),FDiff(5:12),'ro');

hold on

%Plot Backward Difference scheme

plot(x(6:13),BDiff(6:13),'gd');

hold off

title('Stress vs. Forward and Backwards Diff.');

xlabel('Stress')

ylabel('Forward and Backward Difference')

legend('Forward Difference','Backward Difference','Location','NorthWest');


% Least Square Method with Forward Diff.

Ex = 0; Exy = 0; st=0; Ey = 0; Ex2=0; sr=0;
```

```
for i = 5:12
    Ex = Ex + x(i);
            Ey = Ey + FDiff(i);
            Exy = Exy + (x(i) * FDiff(i));
            Ex2 = Ex2 + x(i)^2;
  end

            xm = Ex/8;
            ym = Ey/8;
            a1 = (8*Exy - Ex*Ey) / (8*Ex2 - Ex^2);
            a0 = ym - (a1*xm);

      for i = 5:12
            st = st + (FDiff(i) - ym)^2;
            sr = sr + (FDiff(i) - a0 - a1*x(i))^2;
   end

r2 = (st - sr) / st;
% Outputting solution to system
fprintf('The line of regression is \n y= % fx + % f \n',a1,a0);
fprintf('r^2 = % f \n',r2);
```

```
%Plot Line of regression for Forward Difference scheme

figure, plot(x(5:12),FDiff(5:12), 'ro', x(5:12),a0+a1*x(5:12), 'b-');

title('Line of regression from Forward Difference');

xlabel('Stress')

ylabel('Forward Difference')

legend('Data points','Line of regression','Location','NorthWest');


% Least Square Method with Backward Diff.

Ex = 0; Exy = 0; st=0; Ey = 0; Ex2=0; sr=0;


for i = 6:13

    Ex = Ex + x(i);

            Ey = Ey + BDiff(i);

            Exy = Exy + (x(i) * BDiff(i));

            Ex2 = Ex2 + x(i)^2;

  end


            xm = Ex/8;

            ym = Ey/8;

            a1B = (8*Exy - Ex*Ey) / (8*Ex2 - Ex^2);

            a0B = ym - (a1B*xm);


      for i = 6:13
```

```
            st = st + (BDiff(i) - ym)^2;

            sr = sr + (BDiff(i) - a0B - a1B*x(i))^2;

    end


r2 = (st - sr) / st;

% Outputting solution to system

fprintf('The line of regression is \n y= % fx + % f \n',a1B,a0B);

fprintf('r^2 = % f \n',r2);


%Plot Line of regression for Backward Difference scheme

figure, plot(x(6:13),BDiff(6:13), 'ro', x(6:13),a0B+a1B*x(6:13), 'b-');

title('Line of regression for Backward Difference');

xlabel('Stress')

ylabel('Backwards Difference')

legend('Data points','Line of regression','Location','NorthWest');


FNew = (a0 / a1)* (exp(a1*y)-1);

BNew = (a0B / a1B)* (exp(a1B*y)-1);


%Plotting analytic curves

figure, plot(y,FNew, 'r', y,x, 'b');

title('Forward Analytic Curve');

xlabel('Strain')
```

```
ylabel('New Forward Difference')

legend('Analytic curve','Raw data','Location','NorthWest');

figure, plot(y,BNew, 'r', y,x, 'b');

title('Analytic Backwards Diff');

xlabel('Strain')

ylabel('New Forward Difference')

legend('Analytic curve','Raw data','Location','NorthWest');


%Plot new analytic curves

EA=(stressmean/(exp(a1*strainmean)-1))*(exp(a1*y)-1);

figure, plot(y,EA, 'r', y,x, 'b');

title('New Forward Difference Analytic Curve');

xlabel('Strain')

ylabel('New Forward Difference Analytic')

legend('New analytic curve','Raw data','Location','NorthWest');

EB=(stressmean/(exp(a1B*strainmean)-1))*(exp(a1B*y)-1);

figure, plot(y,EB, 'r', y,x, 'b');

title('New Backward Difference Analytic Curve');

xlabel('Strain')

ylabel('New Forward Difference Analytic')

legend('New analytic curve','Raw data','Location','NorthWest');
```