Integration of Rational Functions by Partial Fractions

basit kesir

$$\int \frac{p(x)}{q(x)}\,dx$$ polynomial, polynomial, no $\sqrt{x\ldots}$!

! $\int \frac{p(x)}{q(x)}\,dx$ $\quad deg(p(x)) < deg(q(x))$ (if otherwise, apply polynomial division)

$p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$, $\quad a_i \in \mathbb{R}$, powers $\in \mathbb{Z}^+$

If you already have a solution, you may not need IRFBPF.

Ex $\int \frac{5}{x-2}\,dx = 5\ln|x-2| + C$

Ex $\int \frac{4x}{3x^2-7}\,dx = \int \frac{6\,du}{4u} = \frac{6}{4}\ln|3x^2-7| + C$

$u = 3x^2-7$
$du = 6x\,dx$

Ex $\int \frac{3}{x^2+1}\,dx = 3\arctan(x) + C$

Ex $\int \frac{2x^3-4x^2+11x}{x^2-2x+4}\,dx = ?$

$\int 2x\,dx + \int \frac{3x}{x^2-2x+4}\,dx = x^2 + \ldots$ continue from here.

poly.

$$\begin{array}{r|l} 2x^3-4x^2+11x & x^2-2x+4 \\ -\;2x^3-4x^2+8x & 2x \\ \hline 3x & \end{array}$$

tam kısım

$\frac{8}{5} = 1\frac{3}{5}$

Ex $\int \frac{x-13}{x^2-2x-15}\,dx = \int \frac{x-13}{(x-5)(x+3)}\,dx = \int \frac{A}{x-5}\,dx + \int \frac{B}{x+3}\,dx$

factorize this!

distinct linear, distinct linear

constant, constant

$= \frac{A}{1}\ln|x-5| + \frac{B}{1}\ln|x+3| + C$, with $A=-1$, $B=2$

$\frac{x-13}{(x-5)(x+3)} = \frac{A}{x-5} + \frac{B}{x+3} \Rightarrow A=?\ B=?$

$(x+3)$, $(x-5)$

$\frac{x-13}{(x-5)(x+3)} = \frac{A(x+3)+B(x-5)}{(x-5)(x+3)} \Rightarrow x-13 = Ax+3A+Bx-5B$ → equating nominator polynomials!

-3 $\quad 1 = A+B$
$-13 = 3A-5B$
$-16 = -8B$

$B=2 \quad A=-1$

! We should factorize the denominator till the end.

! Every polynomial can be factored out into linear OR irreducible quadratic factors.

$mx+n$ ← linear

ax^2+bx+c ← indirgenemez quadratik ← irreducible quadratic

Ex ! (x^2-4) → not an irreducible quadratic. $(x-2)(x+2)$ → reducible, you can factorize it.

! x^2+1 → irreducible quadratic

! x^2+3x+1 → " "

! x^2-4x+4 → not an irreducible quadratic. $(x-2)(x-2)$

Ex Write out the partial fraction decomposition for the following:

$$\int \frac{p(x)}{(x+5)^2(x-3)(x^2-6x+9)(x+1)(x^2+1)(x^2+x+1)^3 \cdot x^2}\,dx = \int \frac{A}{x+1}\,dx + \int \frac{B}{x+5}\,dx + \int \frac{C}{(x+5)^2}\,dx + \int \frac{D}{x-3}\,dx + \int \frac{E}{(x-3)^2}\,dx + \int \frac{F}{(x-3)^3}\,dx$$

$$+ \int \frac{Gx+H}{(x^2+1)}\,dx + \int \frac{Ix+J}{(x^2+x+1)}\,dx + \int \frac{Kx+L}{(x^2+x+1)^2}\,dx + \int \frac{Mx+N}{(x^2+x+1)^3}\,dx$$

$$+ \int \frac{P}{x}\,dx + \int \frac{R}{x^2}\,dx$$

linear (repeated-2) $(x+5)^2$; linear; $(x-3)^2$ (repeated-3) $(x-3)^3$; linear (distinct); irr. quadratic (distinct); irr quadratic (repeated-3)

$x^2 = x \cdot x$ linear repeated-2

factors of denominator

IRFBPF

For each distinct linear

For each repeated linear

For each distinct irr. quadratic factor

For each repeated irr. quadratic factor $(ax^2+bx+c)^k$

For each distinct linear factor $(mx+n)$

$$\int \frac{A}{mx+n}\,dx \quad (A \to \text{constant})$$

For each repeated linear factor $(mx+n)^k$

$$\int \frac{A}{mx+n}\,dx + \int \frac{B}{(mx+n)^2}\,dx + \dots + \int \frac{M}{(mx+n)^k}\,dx$$

nominators $\Rightarrow$ constants $A, B, C, D, \dots$

For each distinct irr. quadratic factor (ax^2+bx+c)

$$\int \frac{Ax+B}{ax^2+bx+c}\,dx \quad (Ax+B \to \text{linear})$$

For each repeated irr. quadratic factor $(ax^2+bx+c)^k$

$$\int \frac{Ax+B}{ax^2+bx+c}\,dx + \int \frac{Cx+D}{(ax^2+bx+c)^2}\,dx + \int \frac{Ex+F}{(ax^2+bx+c)^3}\,dx + \dots + \int \frac{Yx+Z}{(ax^2+bx+c)^k}\,dx$$

nominator $\Rightarrow$ linear polynomials $Ax+B,\ Cx+D, \dots$

Ex $\displaystyle\int \frac{4x}{(x-1)^2(x+1)}\,dx =$

$$\frac{4x}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$$

linear (repeated-2) linear (distinct)

$$\to \int \frac{4x}{(x-1)^2(x+1)}\,dx = \int \frac{1}{(x-1)}\,dx + \int \frac{2}{(x-1)^2}\,dx + \int \frac{-1}{(x+1)}\,dx = 1.\ln|x-1| + \frac{-2}{(x-1)} - 1\ln|x+1| + C$$

Multiplying by the least common denominator, $(x-1)^2(x+1)$, we get

8 $\quad 4x = A\underbrace{(x-1)(x+1)}_{(x^2-1)} + B(x+1) + C(x-1)^2 = Ax^2 - A + Bx + B + Cx^2 - 2Cx + C$

$\quad = (A+C)x^2 + (B-2C)x + (-A+B+C)$

$A + C = 0 \Rightarrow \boxed{A = 1}$

$B - 2C = 4 \Rightarrow \boxed{C = -1}$

$-A + B + C = 0$

$$\int \frac{4x}{(x-1)^2(x+1)} = \int \frac{A}{x-1} + \int \frac{B}{(x-1)^2} + \int \frac{C}{x+1}$$

$(x-1)(x+1) \quad (x+1) \quad (x-1)^2$

$$A\ln|x-1| - \frac{B}{x-1} + C\ln|x+1|$$

$B + 2C = 0$

$+\ B - 2C = 4$

$2B = 4 \quad \boxed{B = 2}$

$$B\int \frac{1}{(x-1)^2}\,dx \quad \begin{matrix} u = x-1 \\ du = dx \end{matrix} \quad \to \int \frac{1}{u^2}\,du = -\frac{1}{u} + C$$

28. $\displaystyle\int \frac{x^2 - 2x - 1}{(x-1)^2(x^2+1)}\,dx = \int \frac{A^{\to 1}}{(x-1)}\,dx + \int \frac{B^{\to -1}}{(x-1)^2}\,dx + \int \frac{C^{\to -1}x + D^{\to 1}}{x^2+1}\,dx$

linear (repeated-2) irr. quadratic (distinct)

pay $\to$ $x^2 - 2x - 1 = A\underbrace{(x-1)(x^2+1)}_{(x^3-x^2+x-1)} + B(x^2+1) + (Cx+D)\underbrace{(x-1)^2}_{(x^2-2x+1)}$

$\boxed{x^2 - 2x - 1} = Ax^3 - Ax^2 + Ax - A + Bx^2 + B + Cx^3 - 2Cx^2 + Cx + Dx^2 - 2Dx + D$

✓ $A + C = 0 \to x^3 \quad C = -A$

$-A + B - 2C + D = 1 \to x^2$

$A + C - 2D = -2 \to x$

$-A + B + D = -1 \to$ sabit terim

$-A + B + 2A + 1 = 1 \Rightarrow A + B = 0$

$0 - 2D = -2 \Rightarrow \boxed{D = 1}$

$-A + B + 1 = -1$

$B - A = -2$

$2B = -2$

$\boxed{B = -1} \quad \boxed{A = 1}$

$\boxed{C = -1}$

$$\int \frac{A^{\to 1}}{(x-1)}\,dx + \int \frac{B^{\to -1}}{(x-1)^2}\,dx + \int \frac{C^{\to -1}x + D^{\to 1}}{x^2+1}\,dx = \ln|x-1| + \frac{1}{(x-1)} + \arctan(x) - \frac{1}{2}\ln|x^2+1| + C$$

✓

$$\int \frac{-1}{(x-1)^2}\,dx \quad \begin{matrix} u = x-1 \\ du = dx \end{matrix} \quad \int -\frac{1}{u^2}\,du = \frac{1}{u} + C$$

$$\int \frac{-x+1}{x^2+1}\,dx = \underbrace{\int \frac{1}{x^2+1}\,dx}_{\arctan(x)} - \int \frac{x}{x^2+1}\,dx \quad \begin{matrix} u = x^2+1 \\ du = 2x\,dx \end{matrix} \quad \int \frac{1}{2}\frac{du}{u}$$