14th Week Tuesday

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Power Series

x

Ex
$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots + x^n + \dots$$

Ex $\sum_{n=0}^{\infty} x^n$ For which values of x, is the given series convergent?

$\sum_{n=0}^{\infty} 3^n$ ✗, ... $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$ ✓, $\sum_{n=0}^{\infty} \left(-\frac{5}{2}\right)^n$ ✗, ..., $\sum_{n=0}^{\infty} \left(\frac{7}{3}\right)^n$ ✗ $\quad \sum_{n=1}^{\infty} \left(\frac{7}{3}\right)^{n-1}$

geometric

$\sum_{n=0}^{\infty} ar^n$, $\sum_{n=1}^{\infty} ar^{n-1}$ a= r=

$|r|<1 \Rightarrow$ series converges $= \frac{a}{1-r}$

diverges otherwise

$\frac{1}{1-x}$

$\sum_{n=0}^{\infty} x^n \rightarrow$ geometric series where $a=1$ $r=x$ $\Rightarrow$ converges when $|x|<1$

$-1<x<1$

In general,

a power series : $\boxed{\sum_{n=0}^{\infty} c_n (x-a)^n}$

$$= c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \dots + c_n(x-a)^n + \dots$$

Ex $\sum_{n=0}^{\infty} \left(\frac{-1}{2}\right)^n (x-2)^n$ for which values of x, is the series convergent?

$\sum_{n=0}^{\infty} ar^n \rightarrow \sum_{n=0}^{\infty} \left(\frac{x-2}{-2}\right)^n$ $\quad a=1$, $r=\frac{x-2}{-2}$

geometric series : $(|r|<1 \Rightarrow \text{conv.})$

$\Rightarrow \left|\frac{x-2}{2}\right| < 1 \Rightarrow$ series converges

$|x-2|<2$

$x-2<2$, $-x+2<2$ $\Big\}$ $0<x<4$

interval of convergence.

THEOREM :

A power series $\sum^{\infty}$

A power serie[s]

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$ centered at $x=a$ may either

(*) have an interval of convergence such as $|x-a| < R$ → radius of convergence.
a → center of the power series.
$a-R < x < a+R$ → interval of convergence

OR

(*) be convergent at the center $\boxed{x=a}$, divergent elsewhere. $(R=0)$

OR

(*) be absolutely convergent ($\Rightarrow$ convergent) for all x. $(R=\infty)$

EX $$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$$ Find the interval of convergence. (x'in hangi değerleri için seri yakınsak olur?)

By Ratio Test; $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \dots$
- $<1 \Rightarrow$ series converges → interval of convergence
- $>1, \infty \Rightarrow$ series diverges
- $=1$ test does not work. ← endpoints → we'll investigate each one separately.

ratio test: $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\cancel{(x-3)^{n+1}}^{(x-3)}}{n+1}\cdot\frac{n}{\cancel{(x-3)^n}}\right| = \underbrace{\lim_{n\to\infty}\frac{n}{n+1}}_{1}|x-3| = |x-3|$$

ratio test says that $|x-3| < 1 \Rightarrow$ series converges.
$2 < x < 4 \Rightarrow$ " "
? ?

If $\boxed{|x-3| = 1} \Rightarrow$ test does not work
$x-3=1$ or $x-3=-1$
$x=4$ or $x=2$

If $x=2$: $$\sum_{n=1}^{\infty}\frac{(x-3)^n}{n} = \sum_{n=1}^{\infty}\frac{(2-3)^n}{n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n} \rightarrow$$ alternating harmonic series → converges.
$u_n = \frac{1}{n}$ pozitif ✓ dec. ✓ $u_n \to 0$ ✓

If $x=4$: $$\sum_{n=1}^{\infty}\frac{(x-3)^n}{n} = \sum_{n=1}^{\infty}\frac{(4-3)^n}{n} = \sum_{n=1}^{\infty}\frac{1}{n} \rightarrow$$ harmonic series $\Rightarrow$ diverges.

Therefore the interval of convergence is → $2 \le x < 4$ $[2,4)$

$a=3$ $R=1$ → radius of convergence

center of the power series.

The series diverges where $x \in (-\infty, 2) \cup [4, \infty)$

Ex $\displaystyle\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$ find the interval of convergence?

Ratio test $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{(-3)^{n+1} x^{n+1}}{\sqrt{n+1+1}} \cdot \frac{\sqrt{n+1}}{(-3)^n \cdot x^n} \right| = |-3x| \lim_{n\to\infty} \underbrace{\frac{\sqrt{n+1}}{\sqrt{n+2}}}_{1} = |-3x|$

$|-3x| < 1 \Rightarrow$ series converges by the ratio test

$\Rightarrow -\frac{1}{3} < x < \frac{1}{3}$ → convergence

$|-3x| = 1 \Rightarrow$ test does not work!

$x = \frac{1}{3}$ or $x = -\frac{1}{3}$

$x = \frac{1}{3}$: $\displaystyle\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{(-3)^n (1/3)^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$ → alternating series

$u_n = \frac{1}{\sqrt{n+1}}$ positive ✓, dec. ✓, $\lim_{n\to\infty} u_n = 0$ ✓ } conv. by A.S.T.

$x = -1/3$: $\displaystyle\sum_{n=0}^{\infty} \frac{(-3)^n (-1/3)^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}}$ ~~D.C.T~~ $b_n = \frac{1}{\sqrt{n}}$ → $p = 1/2 < 1$ divergent p-series

L.C.T. $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{1} = 1 > 0 \Rightarrow$ series diverges

Interval of convergence : $-\frac{1}{3} < x \leq \frac{1}{3}$ $(-1/3, 1/3]$ $R = 1/3$

Ex $\displaystyle\sum_{n=0}^{\infty} n!\, x^n$ Find the interval of convergence ... radius of conv.

Ratio test $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{(n+1)!\, x^{n+1}}{n!\, x^n} \right| = \lim_{n\to\infty} \underbrace{(n+1)}_{\infty} |x|$

If $x \neq 0$ $= \infty \Rightarrow$ Series diverges by ratio test

If $x=0 \Rightarrow$ series converges.

$R=0$

EX $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ Find the interval of convergence and radius of conv.

Ratio test $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\cancel{x^{n+1}}^{x}}{\cancel{(n+1)!}_{n+1}}\cdot\frac{\cancel{n!}}{\cancel{x^n}}\right| = \lim_{n\to\infty}|x|\underbrace{\frac{1}{n+1}}_{\to 0} < 1 \Rightarrow$ by ratio test

For all x, the series converges.

$R=\infty$

$(x-a)^n$ $a=-2$

EX $\displaystyle\sum_{n=0}^{\infty}\frac{n(x+2)^n}{3^{n+1}}$ find the interval convergence . . .

Ratio test $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)\cancel{(x+2)^{n+1}}^{(x+2)}}{\cancel{3^{n+2}}_{3}}\cdot\frac{\cancel{3^{n+1}}}{n\,\cancel{(x+2)^n}}\right| = \lim_{n\to\infty}\underbrace{\left(\frac{n+1}{n}\right)}_{\to 1}\left|\frac{x+2}{3}\right| = \left|\frac{x+2}{3}\right|$

If $\left|\frac{x+2}{3}\right|<1 \Rightarrow$ series converges by the ratio test. If $\left|\frac{x+2}{3}\right|=1 \Rightarrow$ test does not work

$\rightarrow -5 < x < 1$ //

$\frac{x+2}{3}=1 \Rightarrow x=1$

$\frac{x+2}{3}=-1 \Rightarrow x=-5$

For $x=-5$: $\displaystyle\sum_{n=0}^{\infty}\frac{n(-5+2)^n}{3^{n+1}} = \sum_{n=0}^{\infty}\frac{n(-3)^n}{3^{n+1}} = \sum_{n=0}^{\infty}\frac{n(-1)^n}{3} \rightarrow$ alternating $u_n=\frac{n}{3}$ positive ✓ dec ✗

$\displaystyle\lim_{n\to\infty}u_n=\lim_{n\to\infty}\frac{n}{3}=\infty \neq 0$

by test for divergence, series diverges.

For $x=1$: $\displaystyle\sum_{n=0}^{\infty}\frac{n(1+2)^n}{3^{n+1}} = \sum_{n=0}^{\infty}\frac{n}{3}$ Since $\displaystyle\lim_{n\to\infty}\frac{n}{3}=\infty\neq 0$, series diverges by test for divergence.

interval of conv. $= (-5,1)$ $R=3$ ← radius of conv. center $=-2$

EX $\displaystyle\sum_{n=1}^{\infty}\frac{(x-1)^n}{\sqrt[4]{n^3+1}}$ find the intervals where the series is

- absolutely convergent
- conditionally convergent
- divergent

find interval of convergence =

radius of conv. =

Ratio test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\cancel{(x-1)^{n+1}}^{(x-1)}}{\sqrt[4]{(n+1)^3+1}}\cdot\frac{\sqrt[4]{n^3+1}}{\cancel{(x-1)^n}}\right| = |x-1|\lim_{n\to\infty}\sqrt[4]{\underbrace{\frac{n^3+1}{(n+1)^3+1}}_{\to 1}} = |x-1|$$

If $|x-1|<1$ series converges by ratio test. If $|x-1|=1$, test does not work.

$0<x<2 \rightarrow$ absolutely conv. $\Rightarrow$ conv.

$x-1=1 \Rightarrow x=2$
$x-1=-1 \Rightarrow x=0$ $\Big\} \Rightarrow$

For $x=0$: $\displaystyle\sum_{n=1}^{\infty}\frac{(0-1)^n}{\sqrt[4]{n^3+1}} = \sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt[4]{n^3+1}} \rightarrow$ alternating; $u_n=\dfrac{1}{\sqrt[4]{n^3+1}}$ positive ✓ decreasing ✓ $\lim_{n\to\infty}u_n=0$ ✓ $\Big\} \Rightarrow$ conv. by A.S.T.

what about absolute? absolute value series $= \displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt[4]{n^3+1}} \rightsquigarrow \frac{1}{\sqrt[4]{n^3+1}}$ ~~D.C.T~~ $\dfrac{1}{\sqrt[4]{n^3}}$ (b_n) $\rightarrow p=\frac{3}{4}<1$ divergent p-series

L.C.T: $\displaystyle\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{1}{\sqrt[4]{n^3+1}}\cdot\frac{\sqrt[4]{n^3}}{1} = 1 > 0$ $\Rightarrow$ diverges by LCT

At $x=0$; the series converges conditionally.

For $x=2$: $\displaystyle\sum_{n=1}^{\infty}\frac{(2-1)^n}{\sqrt[4]{n^3+1}} = \sum_{n=1}^{\infty}\frac{1}{\sqrt[4]{n^3+1}} \rightarrow$ diverges (found above)

The series converges absolutely for $x\in$ $0<x<2$

" " " conditionally for $x=0$

" " converges for $x\in$ $0\le x<2$ $\rightarrow$ interval of conv. $=[0,2)$

" " diverges for $x\in(-\infty,0)\cup[2,\infty)$

centered at $a=1$

Radius of conv. $=1$

Ex $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2n}\left(\frac{3x+2}{-5}\right)^n$ . . . . . .

Ratio test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{1}{\cancel{2}(n+1)}\frac{\cancel{(3x+2)^{n+1}}^{(3x+2)}}{\cancel{(-5)^{n+1}}_{(-5)}}\cdot\frac{\cancel{2}n\,\cancel{(-5)^n}}{\cancel{(3x+2)^n}}\right| = \lim_{n\to\infty}\underbrace{\frac{n}{n+1}}_{\to 1}\left|\frac{3x+2}{-5}\right| = \left|\frac{3x+2}{-5}\right|$$

If $\left|\frac{3x+2}{-5}\right| < 1$, series converges by ratio test. If $\left|\frac{3x+2}{-5}\right| = 1$ test does not work

$\Rightarrow -7/3 < x < 1$

$a = -\frac{2}{3}$

$R = \frac{5}{3}$ $\quad 1 + \frac{7}{3} \quad \frac{10}{3} \quad \frac{5}{3}$

$3x+2 = 5 \Rightarrow x = 1$

$3x+2 = -5 \Rightarrow x = -7/3$

For $x=1$:

$$\sum_{n=1}^{\infty} \frac{1}{2n}\left(\frac{3.1+2}{-5}\right)^n = \sum (-1)^n \frac{1}{2n} \rightarrow \text{alternating}$$

$u_n = \frac{1}{2n}$ pos. ✓, dec ✓, $u_n \to 0$ ✓ $\Big\}$ conv. by A.S.T. $\Big\}$ cond. $\Rightarrow$ conv.

Abs. conv.? $\sum \frac{1}{2n} \rightarrow$ harmonic $\rightarrow$ divergent

For $x = -7/3$:

$$\sum_{n=1}^{\infty} \frac{1}{2n}\left(\frac{3\overset{-7/3}{x}+2}{-5}\right)^n = \sum_{n=1}^{\infty} \frac{1}{2n} \rightarrow \text{harmonic} \rightarrow \text{divergent.}$$

Series converges absolutely for $-7/3 < x < 1$
" " conditionally for $x = 1$ $\Big\} \Rightarrow$ Interval of conv. $-7/3 < x \leq 1$

diverges $(-\infty, -7/3] \cup (1, \infty)$ $\qquad$ radius $= \frac{5}{3}$ $\qquad$ center $= -\frac{2}{3}$