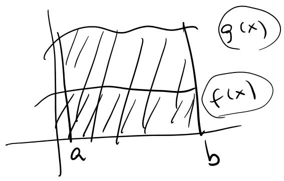
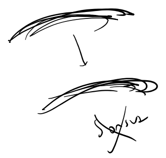
4th Week Thursday

16 Mart 2023 Perşembe 11:28

Comparison Test for Improper Integrals → convergent (sınırlı, yakınsak) / divergent? (sınırsız, sonsuz, ıraksak)

$$\int_a^b f(x)\,dx < \int_a^b g(x)\,dx \Leftarrow f(x) < g(x)$$





Ex/ $\displaystyle\int_1^\infty \frac{\sin(x)}{x^4+x^2+1}\,dx$ convergent/ divergent?

$-1 < \sin(x) < 1$

$$\frac{1}{x^4} > \frac{1}{x^4+x^2+1} > \frac{\sin(x)}{x^4+x^2+1}$$

$$\int_1^\infty \frac{1}{x^4}\,dx \to \text{convergent} \quad > \quad \int_1^\infty \frac{\sin(x)}{x^4+x^2+1}\,dx$$

sonsuz olamaz.

⇒ Convergent

p-integral !

$$\int_1^\infty \frac{1}{x^p}\,dx = \begin{cases} \text{divergent}, & \text{if } p<1 \\ \text{divergent}, & \text{if } p=1 \\ \text{convergent}, & \text{if } p>1 \quad \left(=\frac{1}{p-1}\right) \end{cases}$$

$\displaystyle\int_a^b g(x)\,dx$??

! If $f(x) > g(x)$ and $\displaystyle\int_a^b f(x)$ is convergent ⇒ $\displaystyle\int_a^b g(x)\,dx$ is convergent.

Ör/ $\displaystyle\int_1^\infty \frac{\sin(x)}{x}\,dx$?

$-1 < \sin(x) < 1$

$$\frac{1}{x} > \frac{\sin(x)}{x}$$

divergent > divergent/ convergent/ ?!

$\displaystyle\int_1^\infty \frac{1}{x}\,dx \to$ divergent.

⇒ Comparison test doesn't work here

$\displaystyle\int_a^b g(x)\,dx$??

! If $f(x) < g(x)$ and $\displaystyle\int_a^b f(x)$ is divergent ⇒ $\displaystyle\int_a^b g(x)\,dx$ is divergent.

convergent < divergent ??! Converge

2nd type improper

$$\int_0^1 \frac{1}{x^p}\,dx = \begin{cases} \text{divergent}, & \text{if } p \geq 1 \\ \text{Convergent}, & \text{if } p<1 \end{cases}$$

1st type improper, p-integral !

$$\int_1^\infty \frac{1}{x^p}\,dx = \begin{cases} \text{divergent}, & \text{if } p<1 \\ \text{divergent}, & \text{if } p=1 \\ \text{Convergent}, & \text{if } p>1 \quad \left(=\frac{1}{p-1}\right) \end{cases}$$

sğ 1 1

$= \frac{1}{p-1}$

Ex

$$\int_0^1 \frac{1}{x^5}\,dx = \lim_{t\to 0^+}\left(\int_t^1 \frac{1}{x^5}\,dx\right) = \lim_{t\to 0^+}\left(-\frac{1}{4x^4}\Big]_t^1\right) = \lim_{t\to 0^+}\left(-\frac{1}{4} + \frac{1}{4t^4}\right) = \infty \quad \text{divergent}$$

x^{-5}, $\frac{x^{-4}}{-4}$

$$\int_0^1 \frac{1}{x}\,dx = \lim_{t\to 0^+}\left(\ln(x)\Big]_t^1\right) = \lim_{t\to 0^+}\left(\underbrace{\ln 1}_{0} - \underbrace{\ln t}_{-\infty}\right) = \infty \quad \text{divergent}$$

$$\int_0^1 \frac{1}{x^{-3}}\,dx = \lim_{t\to 0^+}\left(\frac{x^4}{4}\Big]_t^1\right) = \lim_{t\to 0^+}\left(\frac{1}{4} - \frac{t^4}{4}\right) = \frac{1}{4} \quad \text{convergent.}$$

Ex

$$\int_1^\infty \frac{x}{x^6+2x^4+1}\,dx \qquad \text{convergent / divergent ?}$$

Comparison test ✓

$$\frac{x}{x^6+2x^4+1} < \frac{x}{x^6} = \frac{1}{x^5}$$

certainly positive

$$\int_1^\infty \frac{1}{x^5}\,dx \to \text{p-integral convergent}$$

Convergent ✓

Convergent

$$\int_0^\infty e^{ax}\,dx = \begin{cases} \text{divergent, if } a \geq 0 \\ \text{convergent, if } a < 0 \end{cases}$$

$$\int_0^\infty e^{2x}\,dx = \lim_{t\to\infty}\left(\int_0^t e^{2x}\,dx\right) = \lim_{t\to\infty}\left(\frac{e^{2t}}{2} - \frac{e^0}{2}\right) = \infty \quad \text{divergent}$$

$$\int_0^\infty e^{-x}\,dx = \lim_{t\to\infty}\left(\int_0^t e^{-x}\,dx\right) = \lim_{t\to\infty}\left(-e^{-t} + e^0\right) = 1 \quad \text{convergent}$$

$-\frac{1}{e^t} \to 0$

Ex $\int_0^{\infty} e^{-x^2}\,dx$

$\int e^{-x^2}\,dx$??

$e^{-x} > e^{-x^2}$ convergent ✓

$\Uparrow$ comparison test

$x > x^2$ $\quad$ $0-1$

$x < x^2$ $\quad$ $1-\infty$

$-x > -x^2$

$\int_1^{\infty} e^{-x}\,dx \Rightarrow$ convergent

$\underbrace{\int_0^{1} e^{-x^2}\,dx}_{\text{definite integral, convergent}} + \int_1^{\infty} e^{-x^2}\,dx$

Ex $\int_1^{\infty} \frac{1+e^{-x}}{x}\,dx = \underbrace{\int_1^{\infty} \frac{1}{x}\,dx}_{\text{p-integral, divergent}} + \underbrace{\int_1^{\infty} \frac{e^{-x}}{x}\,dx}_{\text{whatever}} \to$ divergent

2nd way

$\frac{1+e^{-x}}{x} > \frac{1}{x}$ divergent

divergent ✓ $\xleftarrow{\text{Comparison}}$

2. Which of the following integrals are improper? Why?

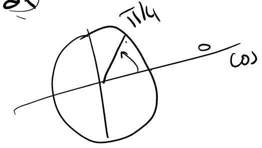
(a) $\int_0^{\pi/4} \tan x\,dx$ $\to$ definite integral

(b) $\int_0^{\pi} \tan x\,dx$ $\to$ 2nd type improper integral

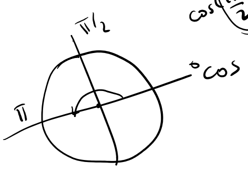
$\int_0^{\pi/4} \frac{\sin(x)}{\cos(x)}\,dx$

$-\ln(x)$

$\ln \sec(x)$



$\cos\left(\frac{\pi}{2}\right)=0$



$\int_0^{\pi/2} \tan(x)\,dx + \int_{\pi/2}^{\pi} \tan(x)\,dx$

26. $\int_0^{\infty} \frac{x \arctan x}{(1+x^2)^2}\,dx$

Ex

$$\int_1^\infty \frac{x}{(1+x^2)^2}\,dx = \lim_{t\to\infty}\left(\int_1^t \frac{x}{(1+x^2)^2}\,dx\right) = \lim_{t\to\infty}\left(-\frac{1}{2(1+x^2)}\Big]_1^t\right) = \lim_{t\to\infty}\left(-\frac{1}{2(t^2+1)}+\frac{1}{4}\right) = \frac{1}{4}$$

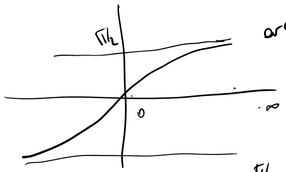
$u = 1+x^2$ $\quad \frac{du}{2u^2}$ $\quad \frac{1}{2}u^{-2}$ $\quad -\frac{1}{2}u^{-1}$ $\quad -\frac{1}{2u}$

$t^2+1 \to \infty$, $\; -\frac{1}{2(t^2+1)} \to 0$ $\quad$ convergent

$$\frac{x}{1+2x^2+x^4} < \frac{x}{x^4} = \frac{1}{x^3}$$

convergent by comp. test. $\Leftarrow$ $\quad \int_1^\infty \frac{1}{x^3} \to$ convergent

26. $\displaystyle\int_0^\infty \frac{x\arctan x}{(1+x^2)^2}\,dx$



$0 < x < \infty$ $\qquad$ $\arctan(x) < \pi/2$

$$\frac{\pi}{2}\cdot\frac{1}{x^3} = \frac{x\cdot\pi/2}{x^4} > \frac{x\cdot\pi/2}{(1+x^2)^2} > \int^\infty \frac{x\arctan(x)}{(1+x^2)^2}$$

$(1+x^2)^2 = x^4 + (2x^2+1)$

$\frac{\pi}{2}\int_1^\infty \frac{1}{x^3}\,dx \to$ Convergent

$\Rightarrow$ Convergent by comp. test.

$$\int_0^1 \frac{x\arctan x}{(1+x^2)^2}\,dx + \int_1^\infty \frac{x\arctan x}{(1+x^2)^2}\,dx$$

definite integral convergent $\qquad$ Convergent $\qquad \to$ Convergent

54. $\displaystyle\int_0^\pi \frac{\sin^2 x}{\sqrt{x}}\,dx$ $\qquad$ convergent / divergent ?

$-1 < \sin(x) < 1$

$0 < \sin^2(x) < 1$

$$\frac{\sin^2 x}{\sqrt{x}} < \frac{1}{\sqrt{x}} \qquad \Rightarrow \int_0^\pi \frac{\sin^2(x)}{\sqrt{x}}\,dx \to \text{convergent by comp. test.}$$

convergent

2^{nd} type

$$\int_0^\pi \frac{1}{\sqrt{x}}\,dx = \lim_{t\to 0^+}\left(\int_t^\pi \frac{1}{\sqrt{x}}\,dx\right) = \lim_{t\to 0^+}\left(2\sqrt{\pi} - 2\sqrt{t}\right) = 2\sqrt{\pi} \to \text{convergent}$$

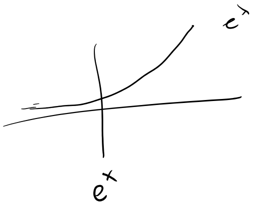
51. $\displaystyle\int_1^\infty \frac{x+1}{\sqrt{x^4-x}}\,dx \to \int_1^\infty \frac{x+1}{\sqrt{x^4-x}}\,dx$ $\qquad$ divergent ✓

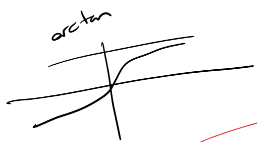
$$\frac{x+1}{\sqrt{x^4-x}} > \frac{x}{\sqrt{x^4-x}} > \frac{x}{\sqrt{x^4}} = \frac{1}{x} \Rightarrow \boxed{\frac{x+1}{\sqrt{x^4-x}} > \frac{1}{x}}$$

$$\int_1^\infty \frac{1}{x}\,dx = \text{divergent}$$

divergent by comp test. $\Leftarrow$

52. $\displaystyle\int_0^\infty \frac{\arctan x}{2+e^x}\,dx$





$x > 0$

$\arctan(x) < \pi/2$

$$\boxed{\frac{\arctan x}{2+e^x}} < \frac{\pi/2}{2+e^x} < \frac{\pi/2}{e^x}$$

$$\int_0^\infty \frac{\arctan x}{2+e^x}\,dx \Longleftarrow \frac{\pi}{2}\int_0^\infty e^{-x}\,dx$$

convergent by comp. test.

convergent