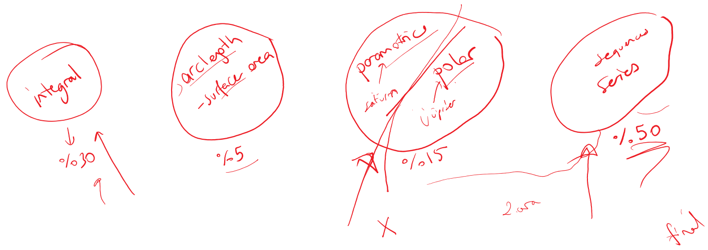
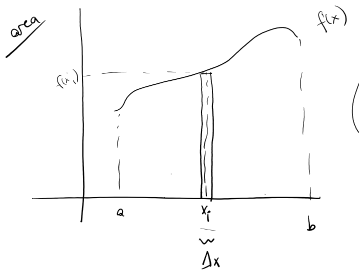
5th Week Tuesday

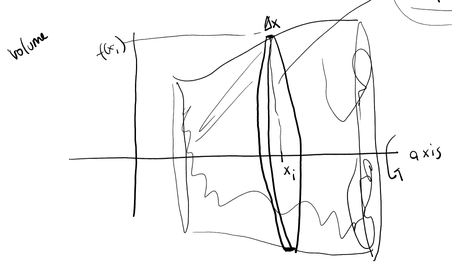
21 Mart 2023 Salı 13:29



Arc Length

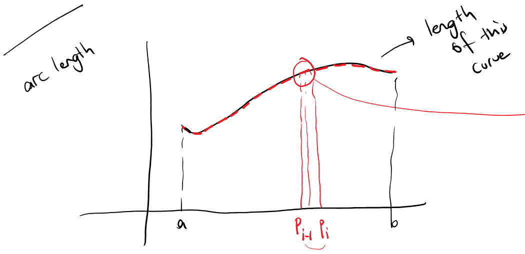


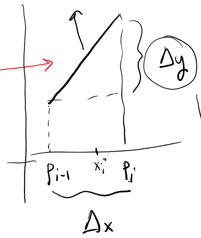
$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \int_{x=a}^{x=b} f(x_i)\,dx$$



$$\pi f(x_i)^2 \Delta x$$

$$\lim_{n\to\infty}\sum_{i=1}^{n} \pi f(x_i)^2 \Delta x = \int_{y=a}^{x=b} \pi f(x_i)^2\,dx$$





$$\ell = \sqrt{\Delta x^2 + \Delta y^2}$$

Mean Value Theorem

$$\frac{\Delta y}{\Delta x} = f'(x_i^*)$$

$$\Delta y = f'(x_i^*)\Delta x$$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{\Delta x_i^2 + \Delta y_i^2}$$

$c \in (a,b)$

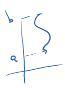
$$\frac{f(b)-f(a)}{b-a} = f'(c) \qquad \frac{\Delta y_i}{\Delta x_i}$$

$$\lim_{n\to\infty}\sum_{n=1}^{\infty}\sqrt{\Delta x_i^2 + f'(x_i^*)^2 \Delta x_i^2} = \lim_{n\to\infty}\sum_{n=1}^{\infty}\sqrt{\Delta x_i^2\left(1+f'(x_i^*)^2\right)}$$

$$= \lim_{n\to\infty}\sum_{n=1}^{\infty}\sqrt{1+f'(x_i^*)^2}\,\Delta x = \int_{x=a}^{x=b}\sqrt{1+f'(x)^2}\,dx$$

arc length of the curve $y=f(x)$ at $a \le x \le b$ $\Rightarrow$ $\boxed{L = \int_{x=a}^{x=b}\sqrt{1+f'(x)^2}\,dx}$

f(x) → türev → karesini → $\sqrt{1+f'(x)^2}$ ← karekök



$\boxed{x=f(y)}$ $L = \int_{y=a}^{y=b}\sqrt{1+f'(y)^2}\,dy$

Ex Find the arc length of the curve $y^2 = x^3$ between $\underline{x=1}$ and $\underline{x=4}$.

$y = x^{3/2}$

$\int_{x=1}^{x=4} \sqrt{1+f'(x)^2}\,dx$ ... $\int_{x=1}^{x=4}\sqrt{\ldots 9x \ldots}\,dx$

$y = x^{3/2}$

$f(x) = x^{3/2}$

$$L = \int_{x=1}^{x=4} \sqrt{1+f'(x)^2}\, dx = \boxed{\int_{x=1}^{x=4} \sqrt{1+\frac{9x}{4}}\, dx}$$

$f'(x) = \frac{3}{2}\cdot x^{1/2} = \frac{3\sqrt{x}}{2}$

$f'(x)^2 = \frac{9x}{4}$

$u = 1 + \frac{9x}{4}$

$du = \frac{9}{4}dx$

$\int \frac{4}{9}\sqrt{u}\, du$ $\quad u^{1/2} \quad \frac{u^{3/2}}{3/2} \quad \frac{2}{3}$

$$= \frac{4}{9}\cdot\frac{2}{3}\sqrt{\left(1+\frac{9x}{4}\right)^3}\Bigg]_{x=1}^{x=4}$$

$$= \frac{8}{27}\left[10^{3/2} - \left(\frac{13}{4}\right)^{3/2}\right]$$

Ex Find the arclength of the curve $\boxed{f(x) = \frac{x^3}{12} + \frac{1}{x}}$ between $1 \le x \le 4$

$$L = \int_{x=1}^{x=4} \sqrt{1+f'(x)^2}\, dx$$

$f'(x) = \frac{3x^2}{12} - \frac{1}{x^2} = \frac{x^2}{4} - \frac{1}{x^2}$

$(a-b)^2 = a^2 - 2ab + b^2$
$(a+b)^2 = a^2 + 2ab + b^2$

$$[f'(x)]^2 = \left(\frac{x^2}{4} - \frac{1}{x^2}\right)^2 = \frac{x^4}{16} - 2\cdot\frac{\cancel{x^2}}{4}\cdot\frac{1}{\cancel{x^2}} + \frac{1}{x^4} = \frac{x^4}{16} - \frac{1}{2} + \frac{1}{x^4}$$

$$\int_{x=1}^{x=4} \sqrt{1+f'(x)^2}\, dx = \int_{x=1}^{x=4} \sqrt{1 + \frac{x^4}{16} - \frac{1}{2} + \frac{1}{x^4}}\, dx = \int_{x=1}^{x=4} \sqrt{\frac{x^4}{16} + \frac{1}{2} + \frac{1}{x^4}}\, dx = \int_{x=1}^{x=4} \sqrt{\left(\frac{x^2}{4} + \frac{1}{x^2}\right)^2}\, dx$$

$$= \int_{x=1}^{x=4} \left(\frac{x^2}{4} + \frac{1}{x^2}\right) dx$$

V EXAMPLE 2 Find the length of the arc of the parabola $y^2 = x$ from $(0, 0)$ to $(1, 1)$.

$y = \sqrt{x}$ $\quad \frac{1}{2\sqrt{x}}$

$x = y^2$

$f(y) = y^2$

$f'(y) = 2y$

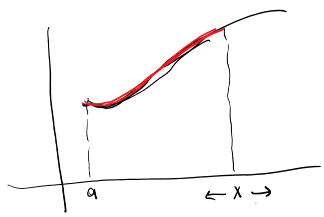
$f'(y)^2 = 4y^2$

$$\int_{y=0}^{y=1} \sqrt{1+f'(y)^2}\, dy = \int_{y=0}^{y=1} \sqrt{1+4y^2}\, dy$$

$2y = \tan\theta \cdot \quad 2dy = \sec^2\theta\, d\theta$

$\sec\theta \sec^2\theta\, d\theta \rightarrow \int \sec^3\theta\, d\theta$

Arc Length Function



$$\boxed{s(x) = \int_{a}^{x} \sqrt{1+f'(t)^2}\, dt}$$

x cinsinden fonksiyon

Ex **V EXAMPLE 4** Find the arc length function for the curve $y = x^2 - \frac{1}{8}\ln x$ taking $P_0(1, 1)$ as the starting point.

$f(t) = t^2 - \frac{1}{8}\ln t$

$f'(t) = 2t - \frac{1}{8t}$

$f'(t)^2 = \left(2t - \frac{1}{8t}\right)^2$

$$s(x) = \int^{x} \sqrt{1+f'(t)^2}\, dt = \int^{x} \left(2t + \frac{1}{8t}\right) dt$$

$$8t$$

$$s(x) = \int_1 \sqrt{1 + f'(t)^2}\, dt = \int_1 \left(2t + \frac{1}{8t}\right) dt \rightarrow t^2 + \frac{1}{8} \ln t$$

$$f'(t)^2 = \left(2t - \frac{1}{8t}\right)^2$$

$$= 4t^2 - 2.2t \frac{1}{8t} + \frac{1}{64t^2}$$

$$1/2$$

$$s(x) = x^2 + \frac{1}{8} \ln(x) - (1 + 0)$$

$$1 + f'(t)^2 = 1 + 4t^2 - \frac{1}{2} + \frac{1}{64t^2} = 4t^2 + \frac{1}{2} + \frac{1}{64t^2} = \left(2t + \frac{1}{8t}\right)^2$$