11th Week Tuesday

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Infinite

SERIES

Sequences (Dizi)

$\{a_n\}$ a_n = general formula

Dizi (sequence) $\rightarrow$ $\{a_1, a_2, \ldots, a_n, \ldots\}$

Series (Seri)

Series $\rightarrow$ $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots + a_n + \ldots$

an infinite sum of the terms of an infinite sequence $\{a_n\}$

Ex/ $a_n = \frac{1}{2^n}$ $n \geq 1$

$\left\{ \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots \right\}$ $\rightarrow$ The sequence $\{a_n\}$

$\frac{1}{2} \downarrow a_1$, $\frac{1}{4} \downarrow a_2$, $\frac{1}{8} \downarrow a_3$, $\frac{1}{16} \downarrow a_4$

Ex/ $\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$ infinite sum.

$a_n = \frac{1}{2^n}$

Ex/ $\sum_{n=1}^{\infty} \frac{1}{2^n} = ?$

$\{a_n\} = \underset{a_1}{\frac{1}{2}}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$

Partial Sums

$s_1 = a_1 = \frac{1}{2} \longrightarrow \frac{1}{2}$

$s_2 = a_1 + a_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4} \longrightarrow \frac{3}{4}$

$s_3 = a_1 + a_2 + a_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8} \rightarrow \frac{7}{8}$

$1 - \frac{1}{2^n}$

$s_4 = a_1 + a_2 + a_3 + a_4 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16} \rightarrow \frac{15}{16}$

try to find a pattern

the sum of first n terms $\rightarrow$ $s_n = a_1 + a_2 + a_3 + \ldots + a_n = 1 - \frac{1}{2^n}$

$\lim_{n\to\infty} s_n = ?$

$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(1 - \frac{1}{2^n}\right) = 1$

$\frac{1}{2^n} \to 0$

D: $\sum_{n=1}^{\infty} a_n = \boxed{\lim_{n\to\infty} s_n}$ (s_n: the general term for the sequence of partial sums)

$\{a_n\}$ sequence

$\Rightarrow$ if this limit exists, we say the series $\sum_{n=1}^{\infty} a_n$ is convergent $= \lim_{n\to\infty} s_n$

$\Rightarrow$ if this limit DNE ($\pm\infty$), we say the serie $\sum_{n=1}^{\infty} a_n$ is divergent

Telescopic Series

Ex $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = ?$

$\frac{1}{n(n+1)} = \frac{1}{n} + \frac{-1}{n+1}$ $\quad \frac{\cancel{n}+1-\cancel{n}}{n(n+1)}$

$a_n = \frac{1}{n(n+1)}$ $a_1 = \frac{1}{1.2}$, $a_2 = \frac{1}{2.3}$, $a_3 = \frac{1}{3.4}$, $a_4 = \frac{1}{4.5}$ ----

$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5} + \ldots$ infinite

partial sums

$s_1 = \frac{1}{1.2}$

$$s_2 = \frac{1}{1.2} + \frac{1}{2.3}$$

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

$$s_3 = \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4}$$

$$\vdots$$

$$s_n = \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{5}}\right) + \cdots + \left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1}$$

$$\lim_{n\to\infty} s_n = ? \Rightarrow \lim_{n\to\infty} 1 - \underbrace{\frac{1}{n+1}}_{0} = 1 \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

and the series is convergent.

Ex

$$\sum_{n=1}^{\infty} \left(\cos\frac{1}{n^2} - \cos\frac{1}{(n+1)^2} \right) = ?$$

$$a_1 = \cos\frac{1}{1} - \cos\frac{1}{2^2}, \quad a_2 = \cos\frac{1}{2^2} - \cos\frac{1}{3^2}$$

partial sums

$$s_1 = a_1 = \cos 1 - \cos\frac{1}{4}$$

$$s_2 = a_1 + a_2 = \left(\cos 1 - \cos\frac{1}{4}\right) + \left(\cos\frac{1}{4} - \cos\frac{1}{9}\right)$$

$$s_3 = a_1 + a_2 + a_3 = \left(\cos 1 - \cos\frac{1}{4}\right) + \left(\cos\frac{1}{4} - \cos\frac{1}{9}\right) + \left(\cos\frac{1}{9} - \cos\frac{1}{16}\right)$$

$$\vdots$$

$$s_n = a_1 + a_2 + \dots + a_n = \left(\cos 1 - \cancel{\cos\frac{1}{4}}\right) + \left(\cancel{\cos\frac{1}{4}} - \cancel{\cos\frac{1}{9}}\right) + \left(\cancel{\cos\frac{1}{9}} - \cancel{\cos\frac{1}{16}}\right) + \dots + \left(\cancel{\cos\frac{1}{n^2}} - \cos\frac{1}{(n+1)^2}\right) = \cos 1 - \cos\frac{1}{(n+1)^2}$$

$$\lim_{n\to\infty} s_n = ? \rightarrow \lim_{n\to\infty} \left(\cos 1 - \cos\underbrace{\frac{1}{(n+1)^2}}_{0}\right) = \cos 1 - 1$$

$$\sum_{n=1}^{\infty} \cos\frac{1}{n^2} - \cos\frac{1}{(n+1)^2} = \cos 1 - 1$$

and this series is convergent.

Ex

$$\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right) = ?$$

$$a_1 = \ln\left(\frac{1}{2}\right), \quad a_2 = \ln\left(\frac{2}{3}\right), \quad a_3 = \ln\left(\frac{3}{4}\right) \dots$$

partial sums

$$s_1 = a_1 = \ln\left(\frac{1}{2}\right)$$

$$s_2 = a_1 + a_2 = \ln\left(\frac{1}{2}\right) + \ln\left(\frac{2}{3}\right)$$

$$s_3 = a_1 + a_2 + a_3 = \ln\left(\frac{1}{2}\right) + \ln\left(\frac{2}{3}\right) + \ln\left(\frac{3}{4}\right)$$

$$\vdots$$

$$s_n = a_1 + a_2 + a_3 + \dots + a_n = (\ln 1 - \cancel{\ln 2}) + (\cancel{\ln 2} - \cancel{\ln 3}) + (\cancel{\ln 3} - \cancel{\ln 4}) + \dots + (\cancel{\ln n} - \ln(n+1))$$

$$s_n = \underbrace{\ln 1}_{0} - \ln(n+1)$$

$$\lim_{n\to\infty} s_n = ? \Rightarrow \lim_{n\to\infty} -\underbrace{\ln(n+1)}_{\infty} = -\infty \rightarrow \text{The series is divergent.}$$

Ex

$$\sum_{n=2}^{\infty} \frac{1}{n^3 - n} = ?$$

$$a_2 + a_3 + \dots$$

$$\frac{1}{n^3 - n} = \frac{1}{n(n-1)(n+1)} = \frac{-1}{\underset{(n^2-1)}{n}} + \frac{1/2}{\underset{(n^2+n)}{n-1}} + \frac{1/2}{\underset{(n^2-n)}{n+1}}$$

$$A(n^2-1) + B(n^2+n) + C(n^2-n) = 1$$

$$A + B + C = 0 \qquad 2B = 1 \Rightarrow B = 1/2 = C$$

$$B - C = 0 \Rightarrow B = C$$

Partial Sums

Partial Sums

$A + B + C = 0 \quad 2B = 1 \Rightarrow B = 1/2 = C$
$B - C = 0 \Rightarrow B = C$
$-A = 1 \Rightarrow A = -1$

$s_1 = a_2 = -\frac{1}{2} + \frac{1}{2.1} + \frac{1}{2.3}$

$a_n = -\frac{1}{n} + \frac{1}{2(n-1)} + \frac{1}{2(n+1)}$

$s_2 = a_2 + a_3 = \left(-\frac{1}{2} + \frac{1}{2.1} + \frac{1}{2.3}\right) + \left(-\frac{1}{3} + \frac{1}{2.2} + \frac{1}{2.4}\right)$

$= a_2 + a_3 + a_4 = \left(-\frac{1}{2} + \frac{1}{2.1} + \cancel{\frac{1}{2.3}}\right) + \left(\cancel{-\frac{1}{3}} + \frac{1}{2.2} + \frac{1}{2.4}\right) + \left(-\frac{1}{4} + \cancel{\frac{1}{2.3}} + \frac{1}{2.5}\right)$

$\frac{-1}{2.3}$

$= a_2 + a_3 + a_4 + a_5 = \left(\left(-\frac{1}{2} + \frac{1}{2.1}\right) + \frac{1}{2.3}\right) + \left(+\frac{1}{2.2} - \frac{1}{3} + \frac{1}{2.4}\right) + \left(-\frac{1}{4} + \cancel{\frac{1}{2.3}} + \frac{1}{2.5}\right) + \left(-\frac{1}{5} + \cancel{\frac{1}{2.4}} + \frac{1}{2.6}\right)$

$= a_2 + a_3 + \dots + a_n = \text{''} + \dots + \left(-\frac{1}{n-2} + \frac{1}{2(n-3)} + \frac{1}{2(n-1)}\right) + \left(-\frac{1}{n-1} + \cancel{\frac{1}{2.(n-2)}} + \frac{1}{2(n)}\right) + \left(-\frac{1}{n} + \frac{1}{2(n-1)} + \frac{1}{2(n+1)}\right)$

$a_{n-2} \quad -\frac{1}{2(n-1)} \quad a_{n-1} \quad a_n$

$s_n = -\cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} + \frac{1}{4} + \frac{1}{2n} - \frac{1}{n} + \frac{1}{2(n+1)}$

$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{1}{4} - \frac{1}{2n} + \frac{1}{2(n+1)}$

$\frac{-(n+1)+n}{2n(n+1)} \quad \frac{-1 \to 0}{2n(n+1) \to \infty}$

$= \lim_{n\to\infty} \frac{1}{4} - 0 = \frac{1}{4}$

$\sum_{n=2}^{\infty} \frac{1}{n^3 - n} = \frac{1}{4}$ (series is convergent)

Geometric Series

geometric sequence $\rightarrow \{a, ar, ar^2, ar^3, \dots, ar^n, \dots\}$

$a \in \mathbb{R} \quad \frac{ar}{a} = r \quad \frac{ar^2}{ar} = r \quad \frac{ar^3}{ar^2} = r \longrightarrow$ Common Ratio $= r = \frac{a_{n+1}}{a_n}$

$\sum_{n=0}^{\infty} ar^n = \sum_{n=1} ar^{n-1}$!

$n \geq 0 \qquad n \geq 1$

$\sum_{n=3} ar^{n-3}$

Geometric Series with common ratio $= r$ ($a \in \mathbb{R}$)

$\sum_{n=1}^{\infty} ar^{n-1} = \underset{a_1}{a} + \underset{a_2}{ar} + \underset{a_3}{ar^2} + \underset{a_4}{ar^3} + \dots + ar^{n-1} + ar^n + \dots$ infinite

partial sums

$s_n = \dots ?$

$s_1 = a_1 = a$

$s_2 = a_1 + a_2 = a + ar$

$s_3 = a_1 + a_2 + a_3 = a + ar + ar^2$

$\vdots$

$s_n = a_1 + a_2 + a_3 + \dots + a_n = \underbrace{a + ar + ar^2 + \dots + ar^{n-1}}_{s_n}$

$\vdots$

$\infty \quad \lim_{n\to\infty} s_n = ?$

$s_n = a + ar + ar^2 + \dots + ar^{n-1}$

$r.s_n = (a + ar + ar^2 + \dots + ar^{n-1}).r$

$r.s_n = ar + ar^2 + ar^3 + \dots + ar^n$

$s_n - rs_n = s_n - (ar + ar^2 + ar^3 + \dots + ar^n)$

$(1-r)\,s_n = (a + \cancel{ar} + \cancel{ar^2} + \dots + \cancel{ar^{n-1}}) - (\cancel{ar} + \cancel{ar^2} + \cancel{ar^3} + \dots + ar^n)$

$(1-r)\,s_n = a - ar^n$

$$s_n = \frac{a(1-r^n)}{1-r}$$

$$\lim_{n\to\infty} \frac{a(1-r^n)}{1-r} \quad \begin{matrix} r^n \to 0 \text{ if } |r|<1 & \to \frac{a}{1-r} \\ \infty \text{ if } |r|>1 & \end{matrix}$$

$r=1 \Rightarrow \{a, ar, ar^2, \dots, ar^{n-1}, \dots\} = \{a, a, a, \dots, a, \dots\}$

$s_1 = a$

$s_2 = a + a$

$s_3 = a + a + a$

$\vdots$

$s_n = n.a \qquad \lim_{n\to\infty} n.a = \infty \longrightarrow \sum_{n=1}^{\infty} a$ divergent

$$\boxed{\sum_{n=1}^{\infty} ar^{n-1} = \begin{cases} \text{convergent and } = \frac{a}{1-r}, & \text{if } |r|<1 \\ \text{divergent}, & \text{if } |r|>1 \end{cases}}$$

$a=?,\ r=?$

$\Rightarrow$ Geometric Series

$\sum_{n=1}^{\infty} 3.5^{n-1} \quad a=3 \quad r=5$

Ex $\quad \sum_{n=1}^{\infty} 3\,5^{n-1} \quad a=3 \quad r=5 \qquad |r|>1 \Rightarrow$ divergent

$\sum_{n=1}^{\infty} ar^{n-1}$

Ex $\quad \sum_{n=1}^{\infty} 2.\left(\frac{1}{5}\right)^{n-1}$ a geometric series where $a=2$ $r=\frac{1}{5}$ $\quad \sum_{n=1}^{\infty} ar^{n-1}$

$|r|<1 \Rightarrow$ conv. $\qquad \frac{a}{1-r} = \frac{2}{1-\frac{1}{5}} = \frac{10}{4}$

$$\sum_{n=1}^{\infty} 2\left(\frac{1}{5}\right)^{n-1} = \frac{10}{4}$$

Ex $\quad \sum_{n=1}^{\infty} 2^{2n}.3^{1-n} = ?$

$\sum_{n=1}^{\infty} ar^{n-1}$

$$\sum_{n=1}^{\infty} 4^n.\left(\frac{1}{3}\right)^{n-1} = \sum_{n=1}^{\infty} 4.4^{n-1}\left(\frac{1}{3}\right)^{n-1}$$

$$= \sum_{n=1}^{\infty} 4\left(\frac{4}{3}\right)^{n-1}$$

A geometric series where $a=4$, $r=\frac{4}{3}$ $\qquad |r|>1 \Rightarrow$ the series is divergent