15th Week Tuesday

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Power series

$\sum_{n=0}^{\infty} c_n (x-a)^n$ (center: a)

(i) converges $a-R < x < a+R$. $R \to$ radius of convergence

(ii) converges at $x=a$, diverges otherwise $R=0$

(iii) converges everywhere $R=\infty$

Representing functions as power series:

$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad |x|<1$ ($\frac{a}{1-r}$, $a=1$, $r=x$)

✓ substitution $x \to cx^d$

$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad |x|<1$

$\frac{1}{2+x} = \frac{1}{2(1+\frac{x}{2})}$, $x \to -\frac{x}{2}$, $\frac{1}{2}\cdot$ $\Rightarrow \sum_{n=0}^{\infty} \frac{1}{2}\cdot\left(-\frac{x}{2}\right)^n$

$a=\frac{1}{2}$, $r=-\frac{x}{2}$

$|\frac{x}{2}|<1 \to |x|<2$

$\frac{1}{1+x} \to$ $x \to -x$, $\frac{1}{1-x}$ $= \sum_{n=0}^{\infty} (-1)^n x^n \to |x|<1$

✓ Differentiation, Integration ($\frac{d}{dx}$, $\int dx$)

$\ln(1+x) = \int \frac{1}{1+x}\,dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} \quad |x|<1$

Taylor & MacLaurin Series

MacLaurin Series → Taylor series centered at $a=0$.

$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \to$ Taylor Series of $f(x)$ at a (centered at a, about $x=a$)

$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \dots \qquad f(a) = c_0 \qquad n=0$

$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \dots \qquad f'(a) = c_1 \qquad n=1$

$f''(x) = 2c_2 + 3.2.c_3(x-a) + 4.3.c_4(x-a)^2 + \dots \qquad f''(a) = 2c_2 \qquad n=2$

$f'''(x) = 3.2c_3 + 4.3.2c_4(x-a) + \dots \qquad f'''(a) = 3.2.c_3 \qquad n=3$

$f^{(4)}(x) = 4.3.2.c_4 + \dots \qquad f^{(4)}(a) = 4.3.2.c_4 \qquad n=4$

$\vdots$

$f^{(n)}(a) = n!\,c_n$ ← n^{th} derivative

$c_n = \frac{f^{(n)}(a)}{n!}$

$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \qquad c_n = \frac{f^{(n)}(a)}{n!}$$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \rightarrow$$ Taylor series of $f(x)$ centered at $x=a$.

Ex Find the Taylor series of $f(x) = \frac{1}{x}$ centered at $a=2$.

$f(x) = \frac{1}{x} \longrightarrow f(2) = \frac{1}{2} \longrightarrow n=0$

$f'(x) = -\frac{1}{x^2} = -x^{-2} \qquad f'(2) = -\frac{1}{4} \longrightarrow n=1$

$f''(x) = (-2)(-1)x^{-3} \qquad f''(2) = (-2)(-1) \cdot 2^{-3} \rightarrow n=2$

$f'''(x) = (-3)(-2)(-1)x^{-4} \qquad f'''(2) = (-3)(-2)(-1)2^{-4} \rightarrow n=3$

$\vdots \qquad \vdots$

$$\boxed{f^{(n)}(2) = (-1)^n . n! . 2^{-(n+1)}} \rightarrow c_n = \frac{(-1)^n . \cancel{n!} . 2^{-(n+1)}}{\cancel{n!}}$$

$$c_n = \frac{\boxed{f^{(n)}(2)}}{n!}$$

$$f(x) = \sum_{n=0}^{\infty} c_n (x-2)^n$$

$$\Rightarrow f(x) = \sum_{n=0}^{\infty} c_n (x-2)^n = \sum_{n=0}^{\infty} (-1)^n 2^{-(n+1)} (x-2)^n \rightarrow$$ Taylor series of $f(x)$ centered at 2.

$$\frac{1}{x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} (x-2)^n$$

$a = \frac{1}{2} \qquad r = \frac{(-1)(x-2)}{2}$

geometri

$\left|\frac{x-2}{2}\right| < 1$

$|x-2| < 2$

$0 < x < 4$

$a=2 \qquad R=2$

Ex Find Mac Laurin series for $f(x) = e^x$. (Find the interval of conv., radius of conv.)

Mac Laurin → Taylor series at $a=0$

$$c_n = \frac{f^{(n)}(0)}{n!}$$

$f(x) = e^x \qquad f(0) = e^0 = 1 \qquad n=0$

$c_n = \frac{\ldots}{n!}$

$f(x) = e^x \qquad f(0) = e^0 = 1 \qquad n=0$

$f'(x) = e^x \qquad f'(0) = e^0 = 1 \qquad n=1$

$\sum_{n=0}^{\infty} c_n (x-0)^n$

$f''(x) = e^x \qquad f''(0) = e^0 = 1 \qquad n=2$

$\vdots$

$f^{(n)}(0) = 1 \Rightarrow c_n = \frac{1}{n!} \Rightarrow \sum_{n=0}^{\infty} \frac{1}{n!} \cdot x^n$

$$\boxed{f(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}}$$

ratio test $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\cancel{x^{n+1}}^{x}}{(n+1)\cancel{!}} \cdot \frac{\cancel{n!}}{\cancel{x^n}} \right| = \lim_{n\to\infty} \frac{1}{n+1} |x| = 0 < 1$ converges for every x.

$\frac{1}{n+1} \to 0$

$\boxed{R = \infty}$

Ex/ Find $\int e^{x^2} dx$ using Mac Laurin Series.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \xrightarrow[x \to x^2]{\text{substitution}} e^{x^2} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!}$$

$$\int e^{x^2} dx = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{x^{2n+1}}{2n+1}$$

Ex/ Find Mac Laurin Series for $f(x) = 2^x$.

$a=0$

$f(x) = 2^x \qquad f(0) = 1 \qquad n=0$

$f'(x) = 2^x . \ln 2 \qquad f(0) = \ln 2 \qquad n=1$

$f''(x) = 2^x . (\ln 2)^2 \qquad f''(0) = (\ln 2)^2 \qquad n=2$

$f^{(n)}(0) = (\ln 2)^n$

$$c_n = \frac{f^{(n)}(0)}{n!} = \frac{(\ln 2)^n}{n!}$$

$c_n(x-\ldots)$

$$\sum_{n=0}^{\infty} \frac{(\ln 2)^n}{n!} x^n$$

$$f^{(n)}(0) = (\ln 2)^n \qquad \sum_{n=0} \frac{(\ldots)}{n!} x$$

Taylor Approximation

Taylor Polinomial : $P_n(x) = \underline{\quad n=0 \quad} + \underline{\quad n=1 \quad} + \ldots + \underline{\quad n=n \quad}$. $\leftarrow$ n^{th} degree Taylor Polinomial

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \longrightarrow P_n(x) = \frac{f^{(0)}(a)}{0!} + \frac{f^{(1)}(a)}{1!}(x-a) + \ldots + \frac{f^{(n)}(a)(x-a)^n}{n!}.$$

$$f(x) \approx P_n(x) \qquad f(x) = P_n(x) + \underbrace{R_n(x)}_{\text{error}} \qquad R_n(x) = f(x) - P_n(x)$$

Ex: Given $f(x) = (1+2x)^{1/3}$

a) Find the first 4 terms of the Taylor expansion of $f(x)$ centered at $x=0$

b) Find an approximation for $\sqrt[3]{2}$ using $P_2(x)$ (2^{nd} degree Taylor polynomial)

a) $f(x) = (1+2x)^{1/3}$ $\qquad f(0) = 1$ $\qquad n=0$ ✓ $\qquad c_n = \dfrac{f^{(n)}(0)}{n!}$

$f'(x) = \frac{1}{3} \cdot 2 \cdot (1+2x)^{-2/3}$ $\qquad f'(0) = \frac{1}{3} \cdot 2$ $\qquad n=1$ ✓ $\qquad \sum_{n=0}^{\infty} c_n x^n$

$f''(x) = -\frac{2}{3} \cdot \frac{1}{3} \cdot 2^2 \cdot (1+2x)^{-5/3}$ $\qquad f''(0) = -\frac{2}{3} \cdot \frac{1}{3} \cdot 2^2$ $\qquad n=2$ ✓

$f'''(x) = -\frac{5}{3} \cdot -\frac{2}{3} \cdot \frac{1}{3} \cdot 2^3 (1+2x)^{-8/3}$ $\qquad f'''(0) = -\frac{5}{3} \cdot -\frac{2}{3} \cdot \frac{1}{3} \cdot 2^3$ $\qquad n=3$ ✓

1^{st} term	2^{nd} term	3^{rd} term	4^{th} term
$n=0$	$n=1$	$n=2$:	$n=3$:
$c_0 x^0 = \frac{1}{0!} x^0 = 1$	$c_1 x = \frac{2}{3 \cdot 1!} x$	$c_2 x^2 = -\frac{2}{3} \cdot \frac{1}{3} \cdot 2^2 \cdot \frac{1}{2!} \cdot x^2$	$c_3 x^3 = -\frac{5}{3} \cdot -\frac{2}{3} \cdot \frac{1}{3} \cdot 2^3 \cdot \frac{1}{3!} x^3$
1	$\frac{2x}{3}$	$-\frac{4}{9}x^2$	$\frac{40}{81}x^3$

b) $P_2(x) = \overset{n=0}{1} + \overset{n=1}{\frac{2x}{3}} + \overset{n=2}{\frac{-4x^2}{9}} \rightarrow$ 2nd degree Taylor polynomial

$f(x) \approx P_2(x)$

$\sqrt[3]{1+2x}$

$\sqrt[3]{2} \approx P_2\left(\frac{1}{2}\right) = 1 + \frac{2}{3}\cdot\left(\frac{1}{2}\right) - \frac{4}{9}\cdot\left(\frac{1}{2}\right)^2 = \frac{11}{9} = 1{,}22222$

$1+2x=2 \Rightarrow x=\frac{1}{2}$

$1+\frac{1}{3}-\frac{1}{9}$

$\sqrt[3]{2} = P_2\left(\frac{1}{2}\right) + \underbrace{R_2\left(\frac{1}{2}\right)}_{\text{error}}$

$R_2\left(\frac{1}{2}\right) = \sqrt[3]{2} - \frac{11}{9}$

Ex a) Find the Taylor Series for $f(x) = \frac{2+2x}{x}$ centered at $\underline{a=1}$.

b) Find $P_3(x)$.

a) $f(x) = \frac{2}{x_{=1}} + 2 \longrightarrow f(1) = 4$ $\quad n=0$

$(-1)^n . n! . 2$

$f'(x) = -\frac{2}{x^2} = -2x^{-2} \longrightarrow f'(1) = -2$ $\quad n=1$

$(-1).n!.2$

$f''(x) = (-2)(-2)x^{-3} \longrightarrow f''(1) = (-2)(-2)$ $\quad n=2$

$f'''(x) = (-3)(-2)(-2)x^{-4} \rightarrow f'''(1) = \underbrace{(-3)(-2)(-2)}_{(-1)^3 . 3! . 2}$ $\quad n=3$

$f^{(4)}(x) = (-4)(-3)(-2)(-2)x^{-5} \rightarrow f^{(4)}(1) = \underbrace{(-4)(-3)(-2)(-2)}_{(-1)^4 . 4! . 2}$

$\vdots$

$f^{(n)}(1) = (-1)^n . n! . 2$

$c_n = \frac{f^{(n)}(1)}{n!}$

$\sum_{n=0}^{\infty} c_n (x-1)^n$

$c_0 = \frac{f(1)}{1!} = 4$

$\sum_{n=0}^{\infty} \frac{(-1)^n . \cancel{n!} . 2}{\cancel{n!}} . (x-1)^n$

$$f(x) = \frac{2+2x}{x} = 4 + \sum_{n=1}^{\infty} (-1)^n . 2 . (x-1)^n$$

$c_0 + c_1(x-1) + c_2(x-2)^2 + \dots$

b) $P_3(x) = \underset{n=0}{4} + \underset{n=1}{-2(x-1)} + \underset{n=2}{2(x-1)^2} + \underset{n=3}{-2(x-1)^3}$ $\rightarrow$ 3rd ~~degree~~ taylor poly. (Centered at $x=1$).

Ex Find the value of 9^{th} derivative of $\boxed{f(x) = \arctan\left(\frac{x^3}{3}\right)}$ at $x=0$.

$f^{(9)}(0) = ?$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \rightarrow \text{mac laurin series}$$

$$\frac{f^{(9)}(0)\, x^9}{9!}$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$x \to -x^2$

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-x^2)^n$$

integrate

$$\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

$x \to \frac{x^3}{3}$

$$\arctan\left(\frac{x^3}{3}\right) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^3/3)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+3}}{(2n+1).\, 3^{2n+1}}$$

$6n+3=9$

$n=1$

$$\frac{f^{(9)}(0)\, x^9}{9!} = \frac{(-1)^1 x^9}{3.\, 3^3}$$

$$f^{(9)}(0) = -\frac{9!}{3^4}$$