2. Hafta Salı Dersi

02 Mart 2021 Salı 11:57

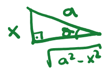
Trigonometrik Dönüşümler

1.) $x = a\sin\theta$ Dönüşümü:

$\int \sqrt{a^2 - x^2}\, dx$

$\boxed{x = a\sin\theta}$ → x'e geri dönerken $\theta = \arcsin\left(\frac{x}{a}\right)$

$dx = a\cos\theta\, d\theta$



$\sqrt{a^2 - a^2\sin^2\theta} = \sqrt{a^2(1-\sin^2\theta)} = \sqrt{a^2\cos^2\theta} = a|\cos\theta|$

$\int a|\cos\theta|\, a\cos\theta\, d\theta = \ldots$

$\sqrt{a^2-x^2}$ $a=3$

Örn $\int \frac{\sqrt{9-x^2}}{x^2}\, dx = ?$

$\boxed{x = 3\sin\theta}$

$dx = 3\cos\theta\, d\theta$

(t) ✓ $\int \frac{x\, dx}{\sqrt{9-x^2}}$ $-\frac{du}{2}$

$\int \frac{-du}{2\sqrt{u}} =$ $u = 9 - x^2$ $du = -2x\, dx$

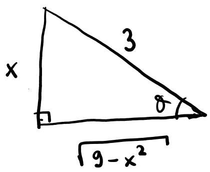
$\sqrt{9-x^2} = \sqrt{9 - 9\sin^2\theta} = 3|\cos\theta|$

$\cot^2\theta = \csc^2\theta - 1$

$\int \frac{\overbrace{\cancel{3}\cos\theta}^{\sqrt{9-x^2}}}{\underbrace{\cancel{9}\sin^2\theta}_{x^2}} \underbrace{\cancel{3}\cos\theta\, d\theta}_{dx} = \int \cot^2\theta\, d\theta = \int (\csc^2\theta - 1)\, d\theta$

$= \int \csc^2\theta\, d\theta - \int 1\, d\theta$

$x = 3\sin\theta$



$\cot\theta = \frac{\sqrt{9-x^2}}{x}$

$\theta = \arcsin\left(\frac{x}{3}\right)$

$= -\cot\theta - \theta + c$

$= -\frac{\sqrt{9-x^2}}{x} - \arcsin\left(\frac{x}{3}\right) + c$

$\sqrt{a^2-x^2}$ $x = a\sin\theta$ $a=3$

$\boxed{5x = 3\sin\theta}$ $25x^2 = 9\sin^2\theta$

$5\, dx = 3\cos\theta\, d\theta$

Örn $\int \frac{1}{\sqrt{9-25x^2}}\, dx = ?$

$\sqrt{9 - 9\sin^2\theta} = \sqrt{9\underbrace{(1-\sin^2\theta)}_{\cos^2\theta}} = 3|\cos\theta|$

$\frac{5x}{3} = \sin\theta \Rightarrow \theta = \arcsin\left(\frac{5x}{3}\right)$

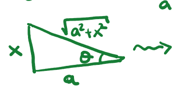
$= \int \frac{1}{\cancel{3}\cancel{\cos\theta}} \cdot \frac{\cancel{3}}{5}\cancel{\cos\theta}\, d\theta = \int \frac{1}{5}\, d\theta = \frac{\theta}{5} + c \rightarrow \frac{1}{5}\arcsin\left(\frac{5x}{3}\right) + c$

2) $x = a\tan\theta$ Dönüşümü:

$\int \sqrt{a^2 + x^2}\, dx$

$\boxed{x = a\tan\theta}$ → x'e geri dönüş $\theta = \arctan\left(\frac{x}{a}\right)$

$dx = a\sec^2\theta\, d\theta$



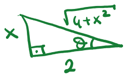
$\sqrt{a^2+x^2} = \sqrt{a^2 + a^2\tan^2\theta} = \sqrt{a^2\underbrace{(1+\tan^2\theta)}_{\sec^2\theta}} = a|\sec\theta|$

$\sqrt{a^2+x^2}$ $a=2$

$\sqrt{a^2+x^2} = \sqrt{a^2 + a^2\tan^2\theta} = \sqrt{a^2(\underbrace{1+\tan^2\theta}_{\sec^2\theta})} = a|\sec\theta|$

$\sqrt{a^2+x^2}$ $a=2$

Örn $\displaystyle\int \frac{1}{x^2\sqrt{x^2+4}}\,dx = ?$

$\boxed{x = 2\tan\theta} \longrightarrow \frac{x}{2} = \tan\theta$  $\longrightarrow \sin\theta = \frac{x}{\sqrt{4+x^2}}$

$dx = 2\sec^2\theta\, d\theta$

$\sqrt{x^2+4} = 2\sec\theta$

$$= \int \frac{1}{\underbrace{4\tan^2\theta}_{x^2} \cdot \underbrace{\cancel{2}\sec\theta}_{\sqrt{x^2+4}}} \cdot \overbrace{\cancel{2}\sec^2\theta\, d\theta}^{dx} = \int \frac{1}{4} \cdot \frac{\sec\theta}{\tan^2\theta}\, d\theta = \int \frac{1}{4}\frac{\cos\theta}{\sin^2\theta}\,d\theta = \frac{1}{4}\int \frac{1}{u^2}\,du = -\frac{1}{4u} + c$$

$\frac{1}{\cos\theta} \cdot \frac{\cos^2\theta}{\sin^2\theta} = \frac{\cos\theta}{\sin^2\theta}$ $u = \sin\theta$, $du = \cos\theta\, d\theta$ $= -\frac{1}{4\sin\theta} + c$

$= -\frac{1}{4}\frac{\sqrt{4+x^2}}{x} + c$ //

Örn $\displaystyle\int_0^{3\sqrt{3}/2} \frac{x^3}{(4x^2+9)^{3/2}}\,dx.$ $x^3 \to \left(\frac{3\tan\theta}{2}\right)^3$

$\sqrt{4x^2+9}$ ③

$\rightarrow \boxed{2x = 3\tan\theta}$ $x = \frac{3\sqrt{3}}{2}$ $\theta = ?$

$2\,dx = 3\sec^2\theta\, d\theta$ $\frac{2x}{3} = \tan\theta$

$\sqrt{4x^2+9} = \sqrt{9\tan^2\theta + 9} = 3\sec\theta$ $\frac{\cancel{2}}{\cancel{3}} \cdot \frac{\cancel{3}\sqrt{3}}{\cancel{2}} = \tan\theta$

$$= \int \frac{\overbrace{\frac{\cancel{27}}{8}\tan^3\theta}^{x^3}}{\underbrace{\cancel{27}\sec^{\cancel{3}}\theta}_{\sqrt{4x^2+9}^{\,3}}} \overbrace{\frac{3}{2}\cancel{\sec^2\theta}\,d\theta}^{dx} = \int \frac{3}{16} \cdot \frac{\tan^3\theta}{\sec\theta}\,d\theta$$

$= \frac{3}{16}\int \frac{\sin^3\theta}{\cos^3\theta} \cdot \frac{\cos\theta}{1}\,d\theta = \frac{3}{16}\int \frac{\sin^3\theta}{\cos^2\theta}\,d\theta$ $\boxed{u = \cos\theta}$ $du = -\sin\theta\, d\theta$

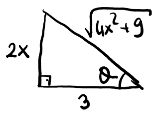
$= \frac{3}{16}\int \frac{u^2-1}{u^2}\,du = \frac{3}{16}\int \left(1 - \frac{1}{u^2}\right)du$

$= \frac{3}{16}u + \frac{3}{16u}$

$= \frac{3}{16}\cos\theta + \frac{3}{16}\sec\theta = \frac{3}{16} \cdot \frac{3}{\sqrt{4x^2+9}} + \frac{\cancel{3}}{16}\frac{\sqrt{4x^2+9}}{\cancel{3}}\Big]_0^{3\sqrt{3}/2} = \frac{3.\cancel{3}}{16.\cancel{6}\,2} + \frac{6}{\cancel{16}} - \left(\frac{3}{\cancel{16}} + \frac{3}{16}\right)$

$4.$ $\sqrt{4x^2+9} \to 6$

$= \frac{3}{32}$ //

$2x = 3\tan\theta$ 

~~Expression~~ İfade	~~Substitution~~ Dönüşüm	~~Identity~~ Özdeşlik
$\sqrt{a^2-x^2}$	$x = a\sin\theta,\quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$	$1-\sin^2\theta = \cos^2\theta$
$\sqrt{a^2+x^2}$	$x = a\tan\theta,\quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1+\tan^2\theta = \sec^2\theta$
$\sqrt{x^2-a^2}$	$x = a\sec\theta,\quad 0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

3) $x = a\sec\theta$ Dönüşümü:

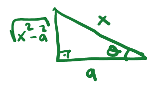
$\displaystyle\int \sqrt{x^2-a^2}\,dx$ $\boxed{x = a\sec\theta}$ $\longrightarrow$ x'e geri dönüş: $\operatorname{arcsec}\left(\frac{x}{a}\right) = \theta$

$\rightarrow dx = a\sec\theta\tan\theta\, d\theta$

3) $x = a\sec\theta$ Dönüşümü :

$$\int \sqrt{x^2-a^2}\,dx \qquad \boxed{x = a\sec\theta} \longrightarrow \text{x'e geri dönüş} \quad \operatorname{arcsec}\left(\frac{x}{a}\right) = \theta$$

$$\to dx = a\sec\theta\tan\theta\, d\theta$$



$$\sec\theta = \frac{x}{a}, \quad \frac{1}{\cos\theta} = \frac{x}{a}, \quad \cos\theta = \frac{a}{x}$$

$$\sqrt{x^2-a^2} = \sqrt{a^2\sec^2\theta - a^2} = \sqrt{a^2(\underbrace{\sec^2\theta-1}_{\tan^2\theta})} = a|\tan\theta|$$

Örn

$$\int \frac{dx}{x\sqrt{x^2-4}} = ? \qquad \boxed{x = 2\sec\theta}$$

$$dx = 2\sec\theta\tan\theta\, d\theta$$

$$\sqrt{x^2-4} = 2\tan\theta \qquad \theta = \operatorname{arcsec}\left(\frac{x}{2}\right)$$

$$= \int \frac{\overbrace{\cancel{2}\cancel{\sec\theta}\cancel{\tan\theta}\,d\theta}^{dx}}{\underbrace{2\cancel{\sec\theta}}_{x}\ \underbrace{\cancel{2}\cancel{\tan\theta}}_{\sqrt{x^2-4}}} = \int \frac{1}{2}\,d\theta = \frac{1}{2}\theta + c$$

$$= \frac{1}{2}\operatorname{arcsec}\left(\frac{x}{2}\right) + c$$

$\sqrt{a^2-x^2}$ $\quad \sqrt{a^2+x^2}$ $\quad \sqrt{x^2-a^2}$

Örn

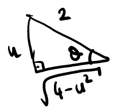
$$\int \frac{x}{\underbrace{\sqrt{3-2x-x^2}}}\,dx \longrightarrow \text{Tam kareye çevir.}$$

$$3 - \underline{2x - x^2} = -(\underbrace{x^2 + 2x + 1}_{x+1}) + 4 = -\underbrace{(x+1)}_{u}{}^2 + 4$$

$$\boxed{u = x+1}, \quad du = dx, \quad x = u-1$$

$$\int \frac{\overbrace{u-1}^{x}}{\underbrace{\sqrt{4-u^2}}_{\sqrt{-(x+1)^2+4}}}\,\overbrace{du}^{dx} \qquad u = 2\sin\theta, \quad du = 2\cos\theta\, d\theta \longrightarrow \sin\theta = \frac{u}{2}$$

$$\sqrt{4-u^2} = 2\cos\theta$$



$$\to \cos\theta = \frac{\sqrt{4-u^2}}{2}$$

$$= \int \frac{\overbrace{2\sin\theta - 1}^{u-1}}{\underbrace{\cancel{2\cos\theta}}_{\sqrt{4-u^2}}}\ \overbrace{\cancel{2\cos\theta}\ d\theta}^{du} = \int (2\sin\theta - 1)\,d\theta = -2\cos\theta - \theta + c$$

$$= -\cancel{2}\cdot\frac{\sqrt{4-u^2}}{\cancel{2}} - \arcsin(u) + c$$

$$= -\sqrt{4-(x+1)^2} - \arcsin(x+1) + c$$