13th Week Thursday

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5. $\sum_{n=1}^{\infty} \frac{n^2 2^{n-1}}{(-5)^n} = \sum_{n=1}^{\infty} (-1)^n \frac{n^2 2^{n-1}}{5^n}$ → $u_n = \frac{n^2 . 2^{n-1}}{5^n}$ → positive ✓

$u_n = \frac{n^2 . 2^n}{2 . 5^n}$

→ A.S.T doesnot work!

decreasing?

$\frac{a_{n+1}}{a_n} = \frac{(n+1)^2 \cancel{2^{n+1}}\,2}{\cancel{2}.\cancel{5^{n+1}}\,5} \cdot \frac{\cancel{2}.\cancel{5^n}}{n^2 . \cancel{2^n}} = \frac{(n+1)^2}{5 . n^2}$

$= \frac{1}{5}\left(\frac{n+1}{n}\right)^2 > 1$ not decreasing!

Abs. value series

$\sum_{n=1}^{\infty} \frac{n^2 2^{n-1}}{5^n}$ → $\frac{n^2 2^n}{2 \; 5^n} > \frac{2^n}{2 . 5^n}$ D.C.T. does not work

$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \left(\frac{2}{5}\right)^n$ $a = \frac{2}{5}$ $r = \frac{2}{5} < 1$ conv. geo. series

L.C.T. $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2 . \cancel{2^n}}{2 . \cancel{5^n}} \cdot \frac{\cancel{5^n}}{\cancel{2^n}} = \infty$ L.C.T. does not work.

Ratio Test

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^2 \; \cancel{2^{n+1-1}}\,2}{\cancel{5^{n+1}}\,5} \cdot \frac{\cancel{5^n}}{n^2 \; \cancel{2^{n-1}}} = \lim_{n\to\infty} \underbrace{\frac{(n+1)^2}{n^2}}_{1} \cdot \frac{2}{5} = \frac{2}{5} < 1$

⇒ By ratio test, this series converges

Since the abs. value series is convergent

⇒ $\sum (-1)^n \frac{2^{n-1} . n^2}{5^n}$ is also convergent. } ⇒ absolutely convergent.

Ratio Test & Root Test

Ratio Test

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L \begin{cases} L < 1 & \Rightarrow \sum a_n \text{ is convergent} \\ L > 1 \text{ or } \infty & \Rightarrow \sum a_n \text{ is divergent} \\ L = 1 & \text{Test does not work!} \end{cases}$$

$\sum a_n$ factorials ✓ $2^n, 3^n, 5^n$ ✓

Root Test

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L \begin{cases} L < 1 & \Rightarrow \sum a_n \text{ is convergent} \\ L > 1 \text{ or } \infty & \Rightarrow \sum a_n \text{ is divergent} \\ L = 1 & \text{Test does not work!} \end{cases}$$

$\sum a_n$ n^n $(\ldots)^n$

Ex/ $\sum_{n=1}^{\infty} \frac{(2n)!}{n! . n!}$ → ratio test

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(\overbrace{2(n+1)}^{2n+2})!}{(n+1)! (n+1)!} \cdot \frac{n! \; n!}{(2n)!}$

$= \lim_{n\to\infty} \frac{(2n+2)(2n+1)\cancel{(2n)!}}{(n+1)\cancel{n!} \; (n+1) . \cancel{n!}} \cdot \frac{\cancel{n!} \; \cancel{n!}}{\cancel{(2n)!}} = \lim_{n\to\infty} \frac{\overbrace{(2n+2)(2n+1)}^{4n^2}}{\underbrace{(n+1)(n+1)}_{n^2}} = 4 > 1$

$\Rightarrow$ Series diverges by the ratio test.

EX $\sum_{n=1}^{\infty} \frac{(-1)^n n^3}{2^n}$

$u_n = \frac{n^3}{2^n}$ positive ✓ decreasin ✓

$f'(n) = \frac{3n^2 \cdot 2^n - n^3 \cdot 2^n \ln 2}{2^n \cdot 2^n} = \frac{3n^2 - n^3 \ln 2}{2^n} < 0$

$\Rightarrow$ decreasing.

$\lim_{n\to\infty} u_n = 0$.

$\lim_{n\to\infty} \frac{n^3}{2^n} \overset{L'}{\Rightarrow} \lim_{n\to\infty} \frac{3n^2}{2^n \ln 2} \overset{L'}{\Rightarrow} \lim_{n\to\infty} \frac{6n}{2^n \ln 2 \ln 2} \overset{L'}{\Rightarrow} \lim_{n\to\infty} \frac{6}{2^n \ln 2 \ln 2 \ln 2} = 0$ ✓

By A.S.T serie converges.

Ratio test

$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\cancel{(-1)^{n+1}} (n+1)^3}{\cancel{2^{n+1}}\, 2} \cdot \frac{\cancel{2^n}}{\cancel{(-1)^n}\, n^3} \right| = \lim_{n\to\infty} \frac{(n+1)^3}{2 \cdot n^3} = \frac{1}{2} < 1$

$\Rightarrow$ absolutely converges by ratio test.

$\Rightarrow$ " " "

Root test

$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\frac{n^3}{2^n}}$

$= \lim_{n\to\infty} \frac{(\sqrt[n]{n})^3}{2} = \frac{1}{2} < 1$ $\Rightarrow$ absolutely converges by root test.

$\Rightarrow$ " " "

EX $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ conv. / div. ?

Ratio test

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} = \lim_{n\to\infty} \frac{(n+1)^{n+\cancel{1}}}{\cancel{(n+1)}\cancel{n!}} \frac{\cancel{n!}}{n^n} = \lim_{n\to\infty} \frac{(n+1)^n}{n^n} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n$

$= \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e > 1$ $\Rightarrow$ series converges by ratio test.

EX $\sum_{n=1}^{\infty} \left(\frac{1}{1+n}\right)^n$?

Root test

$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{1}{1+n}\right)^n} = \lim_{n\to\infty} \frac{1}{1+n} = 0 < 1$

converges by the ratio test.

EX $\sum_{n=1}^{\infty} \left(\frac{n-3}{n+1}\right)^n$?

Try Root Test

$\lim \sqrt[n]{a_n} = \lim \sqrt[n]{\left(\frac{n-3}{n+1}\right)^n} = \lim_{n\to\infty} \frac{n-3}{n+1} = 1$:(Root test does not

Try Root Test

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{n-3}{n+1}\right)^n} = \lim_{n\to\infty} \frac{n-3}{n+1} = 1 \quad :($$

Root test does not work.

Test for Divergence?

$$\left(\lim_{n\to\infty} a_n \neq 0 \Rightarrow \sum a_n \text{ diverges} \right)$$

$$\lim_{n\to\infty} \left(\frac{n-3}{n+1}\right)^n = \lim_{n\to\infty} \underbrace{\left(1+\frac{-4}{n+1}\right)^{n+1}}_{e^{-4}} \cdot \underbrace{\left(1+\frac{-4}{n+1}\right)^{-1}}_{1} = e^{-4} \neq 0 \Rightarrow$$

$\sum a_n$ diverges by test for divergence

Ex $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n-1}\right)^n$

Root test : $\displaystyle\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{2n+3}{3n-1}\right)^n} = \frac{2}{3} < 1 \Rightarrow \sum a_n$ converges by the root test.

2–30 Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

2. $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{n^2}$ → diverges

3. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{5^n}$ → abs. conv.

4. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2+4}$ → conditionally conv.

2.) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{2^n}{n^2}$ → $u_n = \dfrac{2^n}{n^2}$

positive ✓

decreasing ? ✗

$$f'(n) = \frac{2^n \cdot \ln 2 \cdot n^2 - 2^n \cdot 2n}{n^2 \cdot n^2} = \frac{2^n (n \ln 2 - 2)}{n^3} > 0$$

⇒ A.S.T does not work.

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{2^n}{n^2} \overset{L'}{\to} \lim_{n\to\infty} \frac{2^n \cdot \ln 2}{2n} \overset{L'}{\to} \lim_{n\to\infty} \frac{2^n \ln 2 \ln 2}{2} = \infty$$

By test for divergence

$$\lim_{n\to\infty} (-1)^n \frac{2^n}{n^2} \neq 0 \Rightarrow \sum a_n \text{ diverges.}$$

or ratio test?

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{2^{n+1}}{(n+1)^2} \cdot \frac{n^2}{2^n} = 2 > 1 \Rightarrow \sum a_n \text{ diverges by the ratio test.}$$

Ex $\displaystyle\sum_{n=1}^{\infty} \frac{n}{5^n}$

ratio test ⇒ $\displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n+1}{5^{n+1}} \cdot \frac{5^n}{n} = \frac{1}{5} < 1 \Rightarrow \sum a_n$ converges by the ratio test.

4. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2+4}$ → $u_n = \frac{n}{n^2+4}$

positive ✓
decreasing ✓
$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n}{n^2+4} = 0$ ✓

⇒ converges by the A.S.T.

Abs. value : $\sum_{n=1}^{\infty} \frac{n}{n^2+4}$

$\frac{n}{n^2+4} < \frac{n}{n^2} = \frac{1}{n}$???

$\sum_{n=1}^{\infty} \frac{1}{n}$ Σb_n harmonic series diverges.

D.C.T. does NOT work.

L.C.T. : $\lim_{n\to\infty} \frac{|a_n|}{b_n} = \lim_{n\to\infty} \frac{n}{n^2+4} \cdot n = 1 > 0$

>0 Σb_n
$L=0$ Σb_n conv ⇒ Σa_n
$L=\infty$ b_n

same with Σb_n ⇒ $\Sigma |a_n|$ diverges.

12. $\sum_{n=1}^{\infty} \frac{\sin 4n}{4^n}$

~~+, -, +, -~~ not alternating
but it has some negative terms.

$0 < \left| \frac{\sin 4n}{4^n} \right| < \frac{1}{4^n}$

$\sum_{n=1}^{\infty} \frac{1}{4^n}$

$a = \frac{1}{4}$ $r = \frac{1}{4} < 1$ ⇒ conv. geometric series

By D.C.T, ⇐
this series also converges.

$\Sigma |a_n|$ conv. ⇒ Σa_n conv.

Σa_n is absolutely convergent.

19. $\sum_{n=1}^{\infty} \frac{\cos(n\pi/3)}{n!}$

not alternating
not allways positive.

$0 < \left| \frac{\cos(n\pi/3)}{n!} \right| < \frac{1}{n!} < \frac{1}{2^n}$ → $\sum_{n=1}^{\infty} \frac{1}{2^n}$

$n(n-1)\dots 2.1$ $\quad 2.2.2.2.2.$

$a = \frac{1}{2}$ $r = \frac{1}{2} < 1$ conv. geometric series.

Converges by D.C.T.

$\Sigma |a_n|$ conv. ⇒ Σa_n conv.

Σa_n is absolutely convergent.

15. $\sum_{n=1}^{\infty} \frac{(-1)^n \arctan n}{n^2}$ abs. conv. ✓

16. $\sum_{n=1}^{\infty} \frac{3 - \cos n}{n^{2/3} - 2}$

$\sum_{n=1}^{\infty} |a_n| = \sum \frac{\arctan n}{n^2}$

$\arctan n < \frac{\pi}{2}$

$-1 < \cos n < 1$

$\frac{3 - \cos n}{n^{2/3} - 2} > \frac{3 - 1}{n^{2/3} - 2} = \frac{2}{n^{2/3} - 2} > \frac{2}{n^{2/3}}$

$$\sum_{n=1} |a_n| = \sum \left(\frac{\arctan n}{n^2}\right)$$

$$\underbrace{\frac{\arctan n}{n^2}}_{\text{conv. by D.C.T.}} < \frac{\pi/2}{n^2}$$

$p=2>1$
conv. p-series

$\sum |a_n|$ conv. $\Rightarrow$ $\sum a_n$ conv.

$$\underbrace{\frac{3-\cos n}{n^{2/3}-2}}_{} > \underbrace{\frac{3-1}{n^{2/3}-2}}_{} = \frac{2}{n^{2/3}-2} > \underbrace{\frac{2}{n^{2/3}}}_{}$$

$p=2/3<1$
divergent

diverges
by D.C.T.