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4. Consider the two polar curves

$$r = 1 - \cos\theta \text{ and } r = \sin\theta$$

(a) Identify the symmetries for each curve.

(b) Sketch the graph of the two curves on the same coordinates.

(c) Find all the intersection points.

(d) Find the slope of the tangent to the 1^{st} curve at $(1, \pi/2)$.

(e) Find the area shared by the two curves.

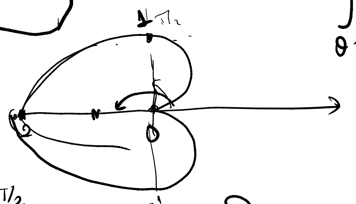
(f) Set up an integral that represents the arc length of the 1^{st} curve.

(g) Set up an integral (Do not evaluate) that represents the area of the surface generated by rotating the 1^{st} curve about the x-axis.

$$A_1 = \int_{\theta=0}^{\theta=\pi/2} \frac{1}{2}(1-\cos\theta)^2\, d\theta$$

$$A_2 = \int_{\theta=\pi/2}^{\theta=\pi} \frac{1}{2}(\sin\theta)^2\, d\theta$$

$$A_1 + A_2$$



$\cos\theta = -1$, $\theta = \pi$, $r = 2$

f) $$\int_{\theta=0}^{\theta=2\pi} \sqrt{(1-\cos\theta)^2 + \sin^2\theta}\, d\theta$$

$2\pi f(\theta)\sqrt{\ }$

$y = r\sin\theta = f(\theta)\sin\theta$
$x = r\cos\theta = f(\theta)\cos\theta$

$1-\cos\theta = \sin\theta$

$\sin\theta + \cos\theta = 1$ $\quad \theta = \pi/2 \quad \theta = 0$

Derivative

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta} \rightarrow \frac{dy}{dx} = \frac{\sin^2\theta + (1-\cos\theta)\cos\theta}{\cancel{\sin\theta\cos\theta} - (1-\cos\theta)\sin\theta}$$

$f(\theta) = 1-\cos\theta$

$$\theta = \pi/2 \rightarrow = \frac{1+0}{-(1-0)\cdot 1} = -1$$

6. Find the areas of the following regions:

(a) Inside the cardioid $r = a(1 + \sin\theta)$, $a > 0$,

(b) Shared by the circle $r = 2$ and the cardioid $r = 2(1 - \cos\theta)$,

(c) Outside the circle $r = 2$ and inside the cardioid $r = 2(1 - \cos\theta)$,

(d) Inside the circle $r = -2\cos\theta$ and outside the circle $r = 1$,

(e) Shared by the circles $r = 2\cos\theta$ and $r = 2\sin\theta$,

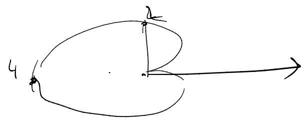
(f) Inside the circle $r = 4$ and above the line $r = 2\csc\theta$.

$$A = \int_{\theta=0}^{\theta=2\pi} \frac{1}{2}\left[a(1+\sin\theta)\right]^2 d\theta$$

$\frac{1}{\sin\theta}$

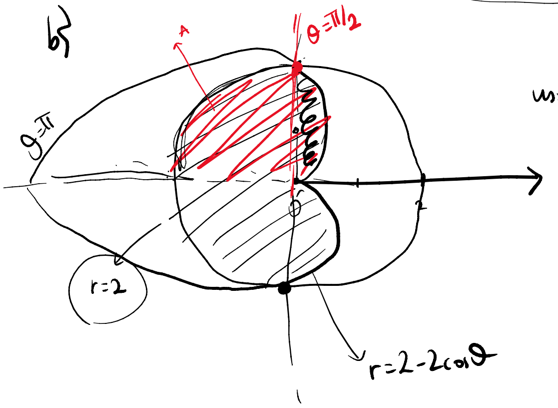
$r = 2\csc\theta$

$r = \frac{2}{\sin\theta} \Rightarrow r\sin\theta = 2$
y



$r = 2-2\cos\theta$

b)

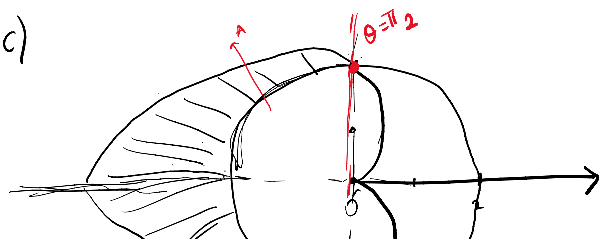


using symmetry,

$$A = \int_{0}^{\pi/2} \frac{1}{2}(2-2\cos\theta)^2\, d\theta + \int_{\theta=\pi/2}^{\theta=\pi} \frac{1}{2}(2)^2\, d\theta$$

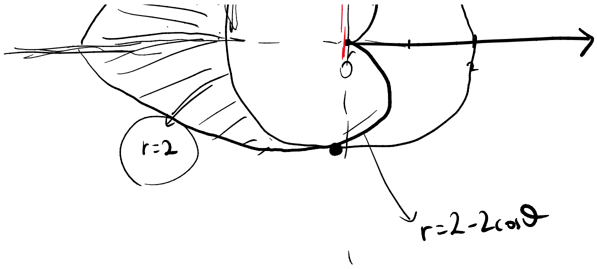
solution $= 2A$

c)



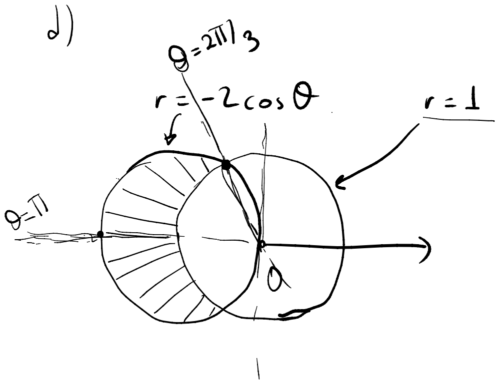
without using symmetry.

$$\int^{\theta=\pi/2} \frac{1}{2}\left((2-2\cos\theta)^2 - (2)^2\right) d\theta$$



$$\int_{\theta=\pi/2} \frac{1}{2}\left((2-2\cos\theta)^2 - (2)^2\right) d\theta$$

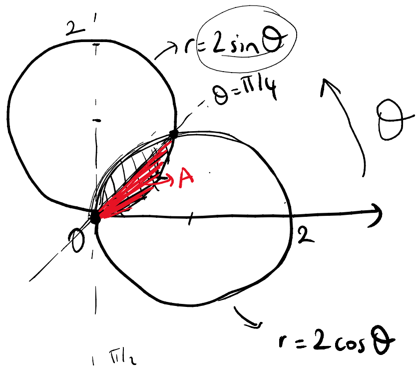
d)



$-2\cos\theta = 1 \Rightarrow \cos\theta = -\frac{1}{2}$ $\quad \theta = \frac{2\pi}{3}$

using symmetry:

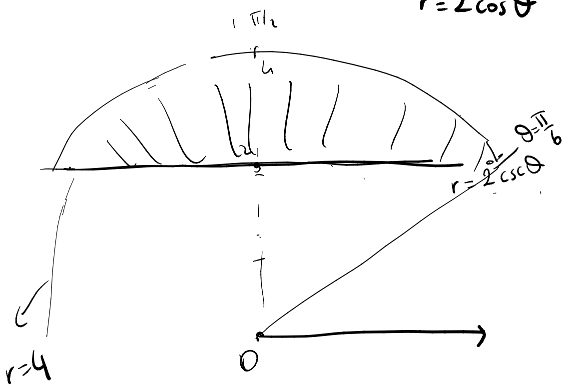
$$A = 2 \cdot \int_{\theta=2\pi/3}^{\theta=\pi} \frac{1}{2}\left[(-2\cos\theta)^2 - (1)^2\right] d\theta$$



$2\sin\theta = 2\cos\theta$

$\theta = \pi/4$

$$2 \cdot \underbrace{\int_{\theta=0}^{\theta=\pi/4} \frac{1}{2}(2\sin\theta)^2 \, d\theta}_{A}$$



$4 = 2\csc\theta$

$2 = \frac{1}{\sin\theta}$ $\quad \sin\theta = \frac{1}{2}$

$$A = 2\int_{\theta=\pi/6}^{\theta=\pi/2} \frac{1}{2}\left(4^2 - (2\csc\theta)^2\right) d\theta$$

Sequences $\quad a_n = \ldots$ monotone inc/dec. ?

$f'(n)<0$ dec. / $f'(n)>0$ inc.

$a_{n+1} - a_n < 0$ dec. / $a_{n+1} - a_n > 0$ inc.

$a_{n+1}/a_n < 1$ dec / $a_{n+1}/a_n > 1$ inc.

(n+1)

inc/dec. $a_{n+1} - a_n < 0$ $a_{n+1}/a_n > 1$ +

$a_{n+1}/a_n < 1$ dec /

$$\lim_{n\to\infty}\left(1+\frac{1}{n+1}\right)^{n+1} = e$$

boundedness ? ✓

⇓

conv./div. ? $\lim_{n\to\infty} a_n = L \to$ conv. $\left.\begin{matrix}\to\infty \\ \text{DNE}\end{matrix}\right\}\to$ div. ✓

$$a_n = \left(\frac{3n+1}{3n-1}\right)^n$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left[\underbrace{\left(1+\frac{2}{3n-1}\right)^{3n-1}}_{e^2}\cdot\underbrace{\left(1+\frac{2}{3n-1}\right)^{1}}_{1}\right]^{1/3} = e^{2/3}$$

(g*) $a_1 = 1, a_{n+1} = \frac{1}{3}(a_n+4)$. Show that $\{a_n\}$ is increasing and bounded above by 2. Find the limit of the sequence.

$\to a_1 = 1$ $\qquad a_{n+1} = \frac{1}{3}(a_n+4)$ $\qquad a_n = \ldots$

$\to a_2 = \frac{5}{3}$ $\to 6.\ 2.3$ $\qquad \frac{1}{3}\left(\frac{5}{3}+4\right)$

$\to a_3 = \frac{17}{9}$ $\to 18\ \ 2.9$ $\qquad \frac{1}{3}\left(\frac{17}{9}+4\right)$

$\to a_4 = \frac{53}{27}$ $\to 54\ \ 2.27$

$$a_n = \frac{2.3^{n-1}-1}{3^{n-1}} = 2 - \frac{1}{3^{n-1}}$$

$$a_n = 2 - \frac{1}{3^{n-1}}$$

$0 < a_n < 2$ → an upper bound.

a lower bound. bounded above

$\Rightarrow \{a_n\}$ is bounded.

increasing?

$$a_{n+1} - a_n = \left(\frac{1}{3}\right)(a_n+4) - a_n = \frac{a_n+4}{3} - a_n = \frac{a_n+4-3a_n}{3} = \frac{-2a_n+4}{3} > 0 \text{ since } a_n < 2.$$

$\Rightarrow \{a_n\}$ is increasing.

By mon. conv. thm $\Rightarrow \{a_n\}$ is conv.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} 2 - \underbrace{\frac{1}{3^{n-1}}}_{\to 0} = 2$$

(h) $a_n = \frac{3^n}{n^3}$

$$\frac{\overbrace{3.\ 3.\ 3.\ldots 3}^{n \text{ tane}}}{n.n.n}$$

$$\lim_{n\to\infty}\frac{3^n}{n^3} \xrightarrow[\frac{\infty}{\infty}]{L'} \lim_{n\to\infty}\frac{3^n.\ln 3}{3n^2} \xrightarrow{L'} \lim_{n\to\infty}\frac{3^n \ln 3.\ln 3}{6n}$$

$$\xrightarrow{L'} \lim_{n\to\infty}\frac{3^n \ln 3 \ln 3 \ln 3}{6} = \infty$$

$\{a_n\}$ is divergent

$n \to \infty$

$\{a_n\}$ is divergent

(a) $a_n = \frac{3n}{2n+1}, n = 1, 2, 3, \ldots$ ⇒ increasing, bounded. ⇒

(b) $a_n = \frac{2^n}{n!}, n = 0, 1, 2, \ldots$

$$f'(n) = \frac{3.(2n+1) - 3n.2}{(2n+1)^2} = \frac{\cancel{6n}+3-\cancel{6n}}{(2n+1)^2} > 0$$

0 → lower bound

$\lim_{n\to\infty} \frac{3n}{2n+1} = \frac{3}{2}$ → upper bound

$$\frac{a_{n+1}}{a_n} = \frac{\cancel{2^{n+1}}}{(n+1)!} \cdot \frac{n!}{\cancel{2^n}} = \frac{2}{n+1} < 1 \Rightarrow \{a_n\} \text{ is decreasing.}$$

lower bound.

$$0 < \frac{2.2.2\ldots 2}{1.2.3\ldots n} < \frac{2.2}{1.2} = 2 \rightarrow \text{upper bound.}$$

⇒ bounded.

⇒ convergent

Series $\sum_{n=1}^{\infty} a_n$

conv. use Değer bulunabilen

Telescopic → partial fractions ⇒ s_n $\quad \lim_{n\to\infty} s_n = ?$

factorize: a rational func. of n.

Geometric $\sum_{n=1}^{\infty} ar^{n-1}$ $\quad a = \ldots$ $\quad r = \ldots$

$|r| < 1 \Rightarrow$ conv. otherwise divergent

$$\sum_{n=1}^{\infty} a_n = \frac{a}{1-r}$$

12. Find the sum of the following series, if converges:

(a) $\sum_{n=2}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+2}}\right)$ → teleskopik

(c) $\sum_{n=2}^{\infty} \left(\frac{1}{2^n} + \frac{(-1)^n}{5^n}\right)$ $\quad \sum_{n=2}^{\infty} ar^{n-2}$ → geometric

geo geo

(b) $\sum_{n=1}^{\infty} \left(\frac{1}{n(n+2)} + \frac{2^{2n+1}}{5^n}\right)$ $\quad \sum ar^{n-1}$

teles

~~(d) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+2)}$~~

$$\sum_{n=2}^{\infty} \frac{1}{2^n} = \sum_{n=2}^{\infty} \frac{1}{2^2} \cdot \frac{1}{2^{n-2}}$$

$a = 1/4$

$r = 1/2 < 1$

conv.

$= \frac{a}{1-r}$ ✓

$$\sum_{n=2} \frac{(-1)^n}{5^n} = \sum_{n=2}^{\infty} \frac{(-1)^2}{5^2} \cdot \left(-\frac{1}{5}\right)^{n-2}$$

$a = 1/25 \quad r = -\frac{1}{5}$ conv. ✓

$\sum_{n=1} \left(\frac{}{n+2} \quad \frac{}{n+4}\right)$

16. Which of the following series converge absolutely, converge conditionally or diverge?

(a) $\sum_{n=1}^{\infty}(-1)^n \frac{1}{\sqrt{n}}$

~~(c) $\sum_{n=1}^{\infty}(-1)^n \frac{\sin n}{n^2}$~~

(b) $\sum_{n=1}^{\infty}(-1)^{n+1} \frac{n}{n^3+1}$

(d) $\sum_{n=1}^{\infty}(-1)^{n+1} \frac{1}{n \ln n}$

A.S.T. $\quad u_n \to$ positif ✓, decreasing ✓, $\lim_{n\to\infty} u_n = 0$ ✓ $\Rightarrow \sum_{n=1}^{\infty}(-1)^n u_n \Rightarrow$ conv.

a) $\sum_{n=1}^{\infty}(-1)^n \underbrace{\frac{1}{\sqrt{n}}}_{u_n}$ $\quad u_n \to$ positif ✓, decreasing ✓, $\lim_{n\to\infty}\frac{1}{\sqrt{n}} = 0$ ✓ $\Rightarrow$ series converges. $\to$ Conditionally convergent.

absolute value of this series : $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}} \to$ $p = 1/2 < 1$, p-series $\Rightarrow$ div.

b) $\sum_{n=1}^{\infty}(-1)^{n+1} \underbrace{\frac{n}{n^3+1}}_{u_n}$ $\quad u_n \to$ positif ✓, decreasing ✓, $\lim_{n\to\infty}\frac{n}{n^3+1} = 0$ ✓ $\Rightarrow$ series converges ✓ $\Rightarrow$ absolutely Convergent.

absolute value of this series : $\sum_{n=1}^{\infty}\frac{n}{n^3+1} \to$ Conv.

$\frac{n}{n^3+1} < \frac{n}{n^3} = \frac{1}{n^2} \to p = 2 > 1$ p-series Convergent.

By D.C.T. $\to$ converges

d) $\sum_{n=1}^{\infty}(-1)^{n+1} \underbrace{\frac{1}{n \ln n}}_{u_n}$: positif ✓, decreasing ✓, $\lim_{n\to\infty}\frac{1}{n \ln n} = 0$ ✓ $\Rightarrow$ Converges by A.S.T. $\Rightarrow$ Converges Conditionally.

The abs. value series $= \sum_{n=1}^{\infty}\frac{1}{n \ln n}$ continuous ✓ conv / div ? ✓

by integral test;

$$\int^{\infty} \frac{1}{x \ln x}\,dx = \lim_{t\to\infty} \ln(\ln|x|)\Big]_{x=2}^{x=t} = \lim_{t\to\infty} \ln|\ln t| - \ln|\ln 2| = \infty$$

$$\int_2^\infty \frac{1}{x \ln x}\,dx = \lim_{t\to\infty} \ln(\ln|x|) \Big]_{x=2}^{t\to\infty} \quad \infty = \infty$$

$u = \ln x \quad du$

$$\int \frac{1}{u}\,du = \ln|u| = \ln|\ln x|$$

test for divergence

14. For each of the following series det

(a) $\sum_{n=1}^{\infty}\left(\frac{n+2}{n}\right)^{-n}$ a_n

✓ (b) $\sum_{n=1}^{\infty}\frac{\sqrt[n]{n}}{n^2}$

(d) $\cancel{\sum_{n=2}^{\infty}\frac{\ln^3 n}{n^2}}$ → integral test

→ (c) $\sum_{n=1}^{\infty}\frac{n+5}{n^3-2n+3}$ → $\frac{1}{n^2}$ (LCT)

(e) $\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^2}$

→ (f) $\sum_{n=1}^{\infty}\frac{3+\cos n}{3^n}$

(g) $\sum_{n=1}^{\infty}\frac{1+n+n^2}{\sqrt{1-n^2+n^6}}$ → $\frac{1}{n}$ (LCT)

→ (k) $\sum_{n=1}^{\infty} n\sin\frac{1}{n}$

(h) $\sum_{n=2}^{\infty}(-1)^n\frac{n^5}{e^n}$ → A.S.T.

(1) $\sum_{n=1}^{\infty}\frac{e^n}{1+e^{2n}}$

(i) $\sum_{n=1}^{\infty}(-1)^n\frac{n^n}{n!}$ →

(m) $\sum_{n=1}^{\infty}\frac{8\arctan n}{n^2+1} < \frac{8\pi/2}{n^2+1} < \frac{4}{n^2}$ yeh.

(j) $\sum_{n=1}^{\infty}(-1)^n\frac{\ln n}{n^2+1}$ →

(n) $\sum_{n=3}^{\infty}\frac{1}{\ln(\ln n)}$ →

a) $$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\Bigg[\underbrace{\left(1+\frac{2}{n}\right)^n}_{e^2}\Bigg]^{-1} = e^{-2} \neq 0 \Rightarrow$$

By test for divergence (n^{th} term test) the series diverges

b) $$\sum_{n=1}^{\infty}\frac{\sqrt[n]{n}}{n^2} < \frac{1}{n^2}$$ by D.C.T, series converges.

$\sum \frac{1}{n^2}$ $p = 2 > 1$ conv. p-series.

By L.C.T,

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\sqrt[n]{n}}{\cancel{n^2}}\cdot\frac{\cancel{n^2}}{1} = 1 > 0$$

$\Rightarrow \sum a_n$ converges by LCT.

$$\frac{\ln^3(n)}{n^2} \qquad \frac{1}{n^2} \to \text{conv.}$$

$$\cancel{\frac{\ln^3(n)}{n^2}\cdot\frac{n^2}{1}} = \infty$$

$$\frac{\ln^3(n)}{n^2}$$ int. test

D.C.T ✗ L.C.T ✗

$$f'(n) = \frac{3\ln^2(n)\cdot 1/n\cdot n^2 - \ln^3(n)\cdot 2n}{n^4}$$

$$= \frac{3n\ln^2(n) - 2n\ln^3(n)}{n^4} = \frac{3\ln^2(n) - 2\ln^3(n)}{n^3} < 0$$

$3\ln^2(n) < 2\ln^3(n)$

$3 < 2\ln(n)$

$\frac{3}{2} < \ln(n)$

$$\cancel{\int^\infty \frac{\ln^3(x)}{x^2}\,dx}$$

$$\int \frac{\ln^3(x)}{x^2}\,dx$$

$\frac{3}{2} < \ln(n)$

/

$u = \ln^3(x)$ $\qquad dv = \frac{1}{x^2}\,dx$

$du = \frac{3\ln^2(x)}{x}\,dx$ $\qquad v = -\frac{1}{x}$

$-\frac{3\ln^2(x)}{x^2}\,dx$