Rationalization

$\int \dfrac{\text{(drawing)}}{x^{7/2}} dx \longrightarrow$ not a rational func.

Evaluate $\int \dfrac{\sqrt{x+4}}{x} dx.$ $\longrightarrow$ Rationalize!

$u = \sqrt{x+4} \longrightarrow \boxed{u^2 = x+4}$ ← u^2-4

the center of the substitution

$2u\,du = dx$

$= \int \dfrac{u}{u^2-4} \cdot 2u\,du = \int \dfrac{2u^2}{u^2-4} du \longrightarrow$ IRFBPF ! first apply poly. division

$\begin{array}{r} 2u^2 \\ -2u^2-8 \\ \hline 8 \end{array} \Big| \dfrac{u^2-4}{2} \longrightarrow 2 + \dfrac{8}{u^2-4}$

$= \int 2\,du + \int \dfrac{8}{u^2-4} du$

$\underbrace{\qquad}_{2u} \qquad \underbrace{\qquad}_{IRFBPF}$

$\dfrac{8}{(u-2)(u+2)} = \dfrac{A}{u-2} + \dfrac{B}{u+2}$

$8 = Au + 2A + Bu - 2B$

$A+B = 0 \qquad 2A-2B = 8 \qquad \begin{array}{l} A=2 \\ B=-2 \end{array}$

$\begin{array}{l} A+B=0 \\ A-B=4 \end{array}$

$\int \dfrac{2}{u-2} du + \int \dfrac{-2}{u+2} du = 2\ln|u-2| - 2\ln|u+2|$

$2\ln\left| \dfrac{u-2}{u+2} \right|$

$= 2\sqrt{x+4} + 2\ln\left| \dfrac{\sqrt{x+4}-2}{\sqrt{x+4}+2} \right| + C$

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41. $\int \dfrac{dx}{x^2 + x\sqrt{x}}$

$u = \sqrt{x} \longrightarrow \boxed{u^2 = x}$ $2u\,du = dx$

center of the subst.

$u^4 = x^2 \qquad u^3 = x\sqrt{x}$

$\int \dfrac{2u\,du}{u^4 + u^3}$ ✓ rational function $\longrightarrow$ IRFBPF $\longrightarrow \int \dfrac{2}{u^3+u^2} du$ rationalizing

$\int \dfrac{2}{u^2(u+1)} du = \int \dfrac{A}{u} + \dfrac{B}{u^2} + \dfrac{C}{u+1}\, du$

$u(u+1) \quad u+1 \quad u^2$

$2 = Au^2 + Au + Bu + B + Cu^2$

$A+C = 0 \quad \Rightarrow C = 2$

$A+B = 0 \quad \Rightarrow A = -2$

$B = 2$

$= \int \dfrac{-2}{u} du + \int \dfrac{2}{u^2} du + \int \dfrac{2}{u+1} du$

$\ln|abs(u)|$

$= -2\ln|u| - \dfrac{2}{u} + 2\ln|u+1| + C$

$= -2\ln|\sqrt{x}| - \dfrac{2}{\sqrt{x}} + 2\ln|\sqrt{x}+1| + C$

$$= -2 \ln|\sqrt{x}| - \frac{2}{\sqrt{x}} + 2\ln|\sqrt{x}+1| + C$$

Strategies

think simple first!

1) anti-derivative (after a simple u-substitution)

2) u-substitution $\quad f, f' \quad \int \sin(\ln(x)) \frac{dx}{x} \quad$ u=lnx

$\int \sin u \, du$
$-\cos(\ln x)+C$

$\int \frac{1}{1+4x^2} dx \quad$ u=2x
$\frac{1}{2} \arctan(2x) + C$
$2 \cdot \frac{1}{1+4x^2}$

3) multiplication of functions
integration by parts → $\int u\, dv = uv - \int v\, du$ ✓

4) trig integral $\quad sin, cos, tan, sec \quad$ (u-substitution)

$\int sec^2(x) \ln(\tan(x)) \, dx$
u=tan(x)
$\int \ln u \, du$ → int by parts.

$\int \frac{\cos(x) \, dx}{1+\sin^2(x)} = \frac{du}{1+u^2}$
u=sin(x)
$\arctan(\sin(x))+C$

$\int \sin(8x)\cos(6x)\,dx$ →✗
$\sin a \cdot \cos b = \frac{1}{2}[\sin(a+b)+\sin(a-b)]$

$\int \frac{\cos^2(x)}{1+\cos 2x} \cdot \frac{}{2}$

5) $\int \frac{\frown}{\frown} dx$ → rational → IRFBPF

$\sqrt{x^2}^{3/2}$ → 1) trig subst. ✓ ← $a^2-b^2x^2$, $a^2+b^2x^2$, $b^2x^2-a^2$
   bx=a sinθ   bx=a tanθ   bx=a secθ

2) rationalization → IRB

$\int \sqrt{} \, dx$ → 1) trig. subst.

$\int \frac{1}{1+e^x} dx \quad \int 1 - \frac{e^x}{1+e^x} dx = x - \ln|e^x+1| + C$
u=e^x+1
du=e^x dx
$\frac{du}{u}$

1–82 Evaluate the integral.

1. $\int \cos x \,(1+\sin^2 x)\, dx$
2. $\int_0^1 (3x+1)^{\sqrt{2}}\, dx$
3. $\int \frac{\sin x + \sec x}{\tan x} dx$ ✓
4. $\int \frac{\sin^3 x}{\cos x} dx$
5. $\int \frac{t}{t^4+2} dt$
6. $\int_0^1 \frac{x}{(2x+1)^3} dx$
7. $\int_{-1}^1 \frac{e^{\arctan y}}{1+y^2} dy$
8. $\int t \sin t \cos t \, dt$
9. $\int_1^e r^4 \ln r \, dr$
10. $\int_0^4 \frac{x-1}{x^2-4x-5} dx$
11. $\int \frac{x-1}{x^2-4x+5} dx$
12. $\int \frac{x}{x^4+x^2+1} dx$
13. $\int \sin^5 t \cos^4 t \, dt$
14. $\int \frac{x^3}{\sqrt{1+x^2}} dx$
15. $\int \frac{dx}{(1-x^2)^{3/2}}$
16. $\int_0^{\sqrt{7}/2} \frac{x^3}{\sqrt{1-x^2}} dx$
17. $\int_0^\pi t \cos^2 t \, dt$
18. $\int_1^4 \frac{e^{\sqrt{t}}}{\sqrt{t}} dt$
19. $\int e^{x+e^x} dx$
20. $\int e^x \, dx$
21. $\int \arctan \sqrt{x} \, dx$
22. $\int \frac{\ln x}{x\sqrt{1+(\ln x)^2}} dx$
23. $\int_0^1 (1+\sqrt{x})^8 dx$
24. $\int_0^4 \frac{6z+5}{2z+5} dz$

27. $\int \frac{dx}{1+e^x}$
28. $\int \sin\sqrt{at}\, dt$
29. $\int \ln(x+\sqrt{x^2-1})\, dx$
30. $\int_{-1}^2 |e^x - 1|\, dx$
31. $\int \sqrt{\frac{1+x}{1-x}} dx$
32. $\int \frac{\sqrt{2x-1}}{2x+3} dx$
33. $\int \sqrt{3-2x-x^2}\, dx$
34. $\int_{\pi/4}^{\pi/2} \frac{1+4\cot x}{4-\cot x} dx$
35. $\int \cos 2x \cos 6x \, dx$
36. $\int_{-\pi/4}^{\pi/4} \frac{x^2 \tan x}{1+\cos^4 x} dx$
37. $\int_0^{\pi/4} \tan^3\theta \, \sec^2\theta \, d\theta$
38. $\int_{\pi/6}^{\pi/3} \frac{\sin\theta \cot\theta}{\sec\theta} d\theta$
39. $\int \frac{\sec\theta \tan\theta}{\sec^2\theta - \sec\theta} d\theta$
40. $\int \frac{1}{\sqrt{4y^2-4y-3}} dy$
41. $\int \theta \tan^2\theta \, d\theta$
42. $\int \frac{\tan^{-1} x}{x^2} dx$
43. $\int \frac{\sqrt{x}}{1+x^3} dx$
44. $\int \sqrt{1+e^x}\, dx$
45. $\int x^5 e^{-x^2} dx$
46. $\int \frac{(x-1)e^x}{x^2} dx$
47. $\int x^3(x-1)^{-4} dx$
48. $\int_0^1 x\sqrt{2-\sqrt{1-x^2}}\, dx$
49. $\int \frac{1}{x\sqrt{4x+1}} dx$
50. $\int \frac{1}{x^2\sqrt{4x+1}} dx$

$-\int \frac{1-u^2}{u} du$

$\int \frac{t}{t^4+2} dt \quad$ u=t² du=2t dt

$\int \frac{1}{2} \frac{du}{(u^2+2)} = \frac{1}{\sqrt{2}} \arctan\left(\frac{t^2}{\sqrt{2}}\right) + C$

$\int \frac{1}{x^2+2} dx = \frac{1}{\sqrt{2}} \arctan\left(\frac{x}{\sqrt{2}}\right)$
u = x²+2

arctan(x) $\frac{1}{x^2+2}$

23. $\int_0^1 (1+\sqrt{x})^8 dx$ **24.** $\int_0^1 \frac{6z+5}{2z+1} dz$ **49.** $\int \frac{1}{x\sqrt{4x+1}} dx$ **50.** $\int \frac{1}{x^2\sqrt{4x+1}} dx$

$\frac{1+\cos 2t}{2}$

$u=(1+$

$u^2 + \sqrt{x} = d^+$

$2udu = d^+$

$2u\,du\,(1+u)$

arctan(u)

$\frac{1}{x^2+2}$

$\frac{1}{x^2+2}$

15    $\int \frac{dx}{\sqrt{(1-x^2)^3}}$ $\xrightarrow{}$ $x = \sin\theta$

$dx = \cos\theta\, d\theta.$

$\int \frac{1}{\cos^3\theta} \cos\theta\, d\theta = \int \frac{1}{\cos^2\theta} d\theta = \int \sec^2\theta\, d\theta = \tan\theta + C$

$\int \cos(2t)\, dt = \frac{\sin 2t}{2} + C$

$= \frac{x}{\sqrt{1-x^2}} + C$

triangle with x, angle θ, hypotenuse 1, opposite side $\sqrt{1-x^2}$

19) $\int e^{x+e^x} dx$

$= \int e^x \cdot e^{e^x} \, dx$ $u = e^x$ $\rightarrow$ $\int e^u du = e^u = e^{e^x} + C$

du

21) $\int \arctan(\sqrt{x})\, dx$ $u = \arctan(\sqrt{x})$ $dv = 1\, dx$ $\rightarrow$ int. by parts.

$du = \frac{1}{2\sqrt{x}} \cdot \frac{1}{1+x} dx$ $v = x$

$uv - \int v\, du = x\cdot\arctan\sqrt{x} - \int \left(x \cdot \frac{1}{2\sqrt{x}} \cdot \frac{1}{1+x} dx \right) \rightarrow$

rationalize.

$\frac{\sqrt{x}}{2(1+x)} dx$ $u = \sqrt{x}$ $u^2 = x$ $2u\,du = dx$

$\left(\int (1) - \frac{1}{1+u^2} = u - \arctan(u) + C. \qquad \frac{u \cdot 2u\, du}{2(1+u^2)} \quad \frac{u^2}{1+u^2} \right)$

$= x \cdot \arctan\sqrt{x} - \left[\sqrt{x} - \arctan(\sqrt{x}) \right] + C$

12) $\int \frac{x}{x^4 + x^2 + 1} dx = \frac{1}{2} \int \frac{du}{u^2 + u + 1} = \frac{1}{\sqrt{3}} \cdot \arctan\left(\frac{x^2 + \frac{1}{2}}{\frac{\sqrt{3}}{2}} \right) + C$

$u = x^2$ $du = 2x\, dx$

$\frac{1}{x^2 + a^2} \rightarrow \frac{1}{a}\left(\arctan\left(\frac{x}{a}\right)\right)$

$\int \frac{1}{x^2 + x + 1} dx = \int \frac{1}{x^2 + x + \frac{1}{4} + \frac{3}{4}} dx = \int \frac{1}{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}} dx = \frac{2}{\sqrt{3}} \arctan\left(\frac{x + \frac{1}{2}}{\sqrt{3}/2} \right)$

$\underbrace{\qquad}_{\left(x+\frac{1}{2}\right)^2}$ a^2

$a = \frac{\sqrt{3}}{2}$

22. $\displaystyle\int \frac{\ln x}{x\sqrt{1+(\ln x)^2}}dx$

$u = \ln x$
$du = \frac{1}{x}dx$

$\displaystyle\int \frac{u\ du}{\sqrt{1+u^2}} = \sqrt{1+u^2}+C = \sqrt{1+\ln x^2}+C$

$\displaystyle\int \frac{x}{\sqrt{1+x^2}}dx$

$x = \tan\theta$

$\displaystyle\int \frac{\tan\theta\ \sec^2\theta\ d\theta}{\sec\theta} = \int \tan\theta\ \sec\theta\ d\theta = \sec\theta + C$

$= \sqrt{1+x^2}+C$