2nd Week Thursday

02 Mart 2023 Perşembe 11:35

Trigonometric Integrals

$\int \csc^m(x) \cot^n(x)\, dx$

m,n → integers

$$\int \sin^m(x) \cos^n(x)\, dx = ?$$

u-substitution: choose the one with even power to be u

if m is even: $\boxed{u=\sin(x)}$ or if n is even: $\boxed{u=\cos(x)}$

$$\boxed{\sin^2(x)+\cos^2(x)=1}$$

✓ if both are odd ⇒ any of them may be u.

✓ if both are even ⇒ use half-angle formulas $\cos^2(x)=\frac{1+\cos(2x)}{2}$, $\sin^2(x)=\frac{1-\cos(2x)}{2}$

$$\int \sec^m(x) \tan^n(x)\, dx = ?$$

if m is even ⇒ $\boxed{u=\tan(x)}$ ✓ $\boxed{u=\tan(x),\ du=\sec^2(x)dx}$ ✓

if n is odd ⇒ $\boxed{u=\sec(x)}$ ✓ $\boxed{u=\sec(x),\ du=\sec(x)\tan(x)dx}$ ✓

$$\boxed{1+\tan^2(x)=\sec^2(x)} \qquad \boxed{\tan^2(x)=\sec^2(x)-1}$$

$\cot^2(x)=\csc^2(x)-1$

$1+\cot^2(x)=\csc^2(x)$

otherwise, we should try integration by parts

$\int u\, dv \to$ $u=$, $du=$; $\int dv=$, v

$=uv-\int v du$ ✓

$$\boxed{\begin{aligned}\int \tan(x)dx &= \ln|\sec(x)|+C\\ \int \sec(x)dx &= \ln|\sec(x)+\tan(x)|+C\end{aligned}}$$

Ex

$$\int \underbrace{\sec^3(x)\, dx}_{u\, dv} = uv-\int v du = \sec(x)\tan(x) - \int \tan(x)\sec(x)\tan(x)\,dx$$

$u=\sec(x)$ $\qquad \int dv=\int \sec^2(x)\,dx$

$du=\sec(x)\tan(x)dx$ $\qquad v=\tan(x)$

$$\int \sec^3(x)\,dx = \int \underbrace{\sec^2(x)}_{1+\tan^2(x)}\sec(x)\,dx = \int\left(1+\tan^2(x)\right)\sec(x)\,dx = \int \sec(x)dx + \int \tan^2(x)\sec(x)\,dx$$

$\int \sec(x)dx \to \ln|\sec(x)+\tan(x)|$

$$\int \sec^3(x)\,dx = \ln|\sec(x)+\tan(x)| + \cancel{\int \tan^2(x)\sec(x)\,dx}$$
$$\int \sec^3(x)\,dx = \sec(x)\tan(x) - \cancel{\int \tan^2(x)\sec(x)dx}$$

$$\int \sec^3(x)dx = \frac{1}{2}\left[\ln|\sec(x)+\tan(x)| + \sec(x)\tan(x)\right]+C$$

$$\int \tan^2(x)\sec(x)\,dx = \frac{1}{2}\left[\sec(x)\tan(x) - \ln|\sec(x)+\tan(x)|\right]+C$$

→ ! Trigonometric Substitution

In Any place of the integral

1) $\int \sqrt{a^2-x^2}\,dx \to \boxed{x=a\sin\theta}$ $\in \mathbb{R}$

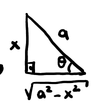
$\sqrt{a^2-x^2} = \sqrt{a^2-a^2\sin^2\theta} = \sqrt{a^2(1-\sin^2\theta)}$
$= \sqrt{a^2\cos^2\theta}$
$= a\cos\theta$

Convert everything in terms of θ

$x=a\sin\theta \to dx = a\cos\theta\, d\theta$

After solving the integral for θ, go back to $x \to x=a\sin\theta \Rightarrow \sin\theta=\frac{x}{a}$

use any trig value for θ from here



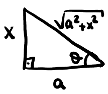
$\theta=\arcsin\left(\frac{x}{a}\right)$

2) $\int \sqrt{a^2+x^2}\,dx \to \boxed{x=a\tan\theta}$

$\sqrt{a^2+x^2} = \sqrt{a^2+a^2\tan^2\theta} = \sqrt{a^2(\underbrace{1+\tan^2\theta}_{\sec^2\theta})}$
$= \sqrt{a^2\sec^2\theta}$
$= \boxed{a\sec\theta}$

$x=a\tan\theta \to dx = a\sec^2\theta\, d\theta$

$\tan\theta=\frac{x}{a} \to \theta=\arctan\left(\frac{x}{a}\right)$



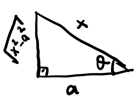
all trig values of θ are available in terms of x.

3) $\int \sqrt{x^2-a^2}\,dx \to \boxed{x=a\sec\theta}$

$\sqrt{x^2-a^2} = \sqrt{a^2\sec^2\theta-a^2}$
$= \sqrt{a^2(\underbrace{\sec^2\theta-1}_{\tan^2\theta})}$
$= \boxed{a\tan\theta}$

$x=a\sec\theta \Rightarrow dx = a\sec\theta\tan\theta\, d\theta$

$\sec\theta=\frac{x}{a}$ $\qquad \theta=\operatorname{arcsec}\left(\frac{x}{a}\right)$



values of θ are available in terms of x.

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\left(\frac{x}{a}\right)+C \quad \checkmark$$

~~a = x sin θ~~

Ex $\rightarrow \int \frac{1}{\sqrt{9-x^2}}\,dx = ?$ $\quad \checkmark\ \boxed{x = 3\sin\theta} \rightarrow dx = 3\cos\theta\, d\theta$

$x = a\sin\theta$

$x = 3\sin\theta \Rightarrow \frac{x}{3} = \sin\theta$

$\Rightarrow \theta = \arcsin\left(\frac{x}{3}\right)$

$\sqrt{9-x^2} = \ldots = \sqrt{9\cos^2\theta} = 3\cos\theta$!

$9\sin^2\theta$ $\quad \underbrace{1-\sin^2\theta}_{\cos^2\theta}$

$$= \int \frac{1}{\cancel{3\cos\theta}}\cancel{3\cos\theta}\,d\theta = \int 1.\,d\theta = \theta + C = \arcsin\left(\frac{x}{3}\right) + C$$

Ex $\int \frac{1}{\sqrt{4-25x^2}}\,dx$ $\quad \boxed{5x = 2\sin\theta} \longrightarrow 5\,dx = 2\cos\theta\, d\theta \rightarrow dx = \frac{2}{5}\cos\theta\, d\theta$

$\frac{5x}{2} = \sin\theta$

$\sqrt{4-25x^2} = \sqrt{4-4\sin^2\theta} = 2\cos\theta$

$\sqrt{a^2-x^2}$ $\quad x = a\sin\theta$

$$\int \frac{1}{\cancel{2\cos\theta}}\,\frac{\cancel{2}}{5}\cancel{\cos\theta}\,d\theta = \int \frac{1}{5}\,d\theta = \frac{\theta}{5} + C = \frac{1}{5}\arcsin\left(\frac{5x}{2}\right) + C$$

Ex $\int \frac{\sqrt{16-x^2}}{x^2}\,dx = ?$ $\quad \rightarrow \boxed{x = 4\sin\theta}$

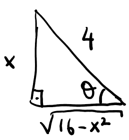
$\sqrt{16-x^2} \rightarrow 4\cos\theta$, $\quad dx \rightarrow 4\cos\theta\, d\theta$, $\quad x^2 \rightarrow 16\sin^2\theta$

$\sqrt{16-x^2} = \ldots = 4\cos\theta$

$$= \int \frac{\cancel{4}\cos\theta}{\cancel{16}\sin^2\theta}\,\cancel{4}\cos\theta\, d\theta = \int \cot^2\theta\, d\theta = \int (\csc^2\theta - 1)\,d\theta = \underbrace{\int \csc^2\theta\, d\theta}_{-\cot\theta} - \int 1\, d\theta$$

$1+\cot^2\theta = \csc^2\theta$

$= -\cot\theta - \theta + C$



$\sin\theta = \frac{x}{4} \rightarrow \theta = \arcsin\left(\frac{x}{4}\right)$

$\cot\theta = \frac{\sqrt{16-x^2}}{x}$

$$= -\frac{\sqrt{16-x^2}}{x} - \arcsin\left(\frac{x}{4}\right) + C$$

Ex $\int \frac{1}{\ldots}\,dx = ?$ $\quad \boxed{x = 5\tan\theta}$

Ex

$\boxed{x = 5\tan\theta}$

$$\int \frac{1}{x^2\sqrt{x^2+25}}\, dx = ? \qquad dx \to 5\sec^2\theta\, d\theta, \quad x^2 \to 25\tan^2\theta, \quad \sqrt{x^2+25} \to 5\sec\theta$$

$$= \int \frac{1}{25\tan^2\theta\, \cancel{5\sec\theta}}\, \cancel{5}\sec^{\cancel{2}}\theta\, d\theta = \int \frac{\sec\theta}{25\tan^2\theta}\, d\theta = -\frac{1}{25}\csc\theta + C$$

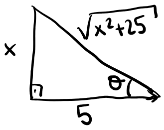
$$\int \frac{1}{\cancel{\cos\theta}} \cdot \frac{\cos^{\cancel{2}}\theta}{\sin^2\theta}\, d\theta = \int \frac{\cos\theta\, d\theta}{\sin^2\theta} = \int \frac{1}{u^2}\, du = -\frac{1}{\sin\theta}$$

$u = \sin\theta$

$du = \cos\theta\, d\theta$

$u^{-2} \to \frac{u^{-1}}{-1} = -\frac{1}{u}$

$x = 5\tan\theta$



$$\csc\theta = \frac{\sqrt{x^2+25}}{x}$$

$$= -\frac{1}{25}\, \frac{\sqrt{x^2+25}}{x} + C$$

Evaluate $\int \frac{x}{\sqrt{3-2x-x^2}}\, dx. = \int \frac{x}{\sqrt{4-(x+1)^2}}$ $\quad x \to 2\sin\theta - 1, \quad dx \to 2\cos\theta\, d\theta = ?, \quad \sqrt{4-(x+1)^2} \to 2\cos\theta$

$3 - (x^2+2x+1) + 1$

$4 - x^2 - 2x - 1 = 3 - x^2 - 2x$

$\boxed{x+1 = 2\sin\theta}$

$dx = 2\cos\theta\, d\theta$

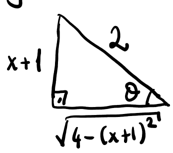
$$\sqrt{4-(x+1)^2} = \sqrt{4 - 4\sin^2\theta} = \sqrt{4(1-\sin^2\theta)} = 2\cos\theta$$

$$\int \frac{2\sin\theta - 1}{\cancel{2\cos\theta}}\, \cancel{2\cos\theta}\, d\theta = \int (2\sin\theta - 1)\, d\theta = -2\cos\theta - \theta + C$$

$x+1 = 2\sin\theta$

$\sin\theta = \frac{x+1}{2}$

$\theta = \arcsin\left(\frac{x+1}{2}\right)$



$$= -2\, \frac{\sqrt{4-(x+1)^2}}{2} - \arcsin\left(\frac{x+1}{2}\right) + C$$

23. $\int \sqrt{5+4x-x^2}\, dx = \int \sqrt{9-(x^2-4x+4)}\, dx = \int \sqrt{9-(x-2)^2}\, dx$

$-(x^2 - 4x + 4)$

$-x^2 + 4x - 4 + 9$

$9 - x^2 + 4x - 4$

$5 - x^2 + 4x$ ✓

$\boxed{x-2 = 3\sin\theta}$

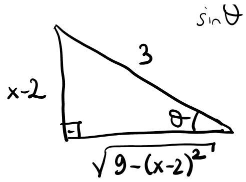
$$= \int 3\cos\theta \cdot 3\cos\theta\, d\theta = \int 9\cos^2\theta\, d\theta = \int 9\left(\frac{1+\cos 2\theta}{2}\right) d\theta$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$\sin\theta = \frac{x-2}{3}$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$\sin\theta = \frac{\ }{3}$



$$= \frac{9}{2}\theta + \frac{9}{2}\,\frac{\sin 2\theta}{2} + C$$

$$= \frac{9}{2}\theta + \frac{9}{2}\,\frac{\cancel{2}\sin\theta\cos\theta}{\cancel{2}} + C = \frac{9}{2}\arcsin\left(\frac{x-2}{3}\right) + \frac{9}{2}\cdot\frac{x-2}{3}\cdot\frac{\sqrt{9-(x-2)^2}}{3} + C$$

25. $\displaystyle\int \frac{x}{\sqrt{x^2+x+1}}\,dx = \int \frac{x}{\sqrt{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}}\,dx$

$$x+\frac{1}{2} = \frac{\sqrt{3}}{2}\tan\theta$$

$$dx = \frac{\sqrt{3}}{2}\sec^2\theta\, d\theta.$$

$$\left(x+\frac{1}{2}\right)^2+\frac{3}{4} = \left(x^2 + \cancel{2}\cdot x\cdot\frac{1}{\cancel{2}} + \frac{1}{4}\right) + \frac{3}{4}$$

$$\int \frac{\frac{\sqrt{3}}{2}\tan\theta - \frac{1}{2}}{\cancel{\frac{\sqrt{3}}{2}}\cancel{\sec\theta}}\,\cancel{\frac{\sqrt{3}}{2}}\,\cancel{\sec^2}\theta\, d\theta = \int\left(\frac{\sqrt{3}}{2}\tan\theta - \frac{1}{2}\right)\sec\theta\, d\theta$$

$$= \int \frac{\sqrt{3}}{2}\tan\theta\sec\theta\, d\theta - \int \frac{1}{2}\sec\theta\, d\theta.$$

$$= \frac{\sqrt{3}}{2}\sec\theta - \frac{1}{2}\ln\left|\sec\theta+\tan\theta\right| + C$$

x x