14th Week Thursday

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power series $\rightarrow \sum_{n=0}^{\infty} c_n (x-a)^n$

$x=a \rightarrow$ center of the series

$\rightarrow \underbrace{a-R < x < a+R}_{\text{interval of convergence}} \rightarrow R$: radius of convergence. ✓

2) $x=a$ conv. otherwise div. $R=0$

3) conv. everywhere $R=\infty$

$$\sum_{n=0}^{\infty} x^n = 1+x+x^2+\dots+x^n+\dots = \frac{1}{1-x}$$

$a=1$, $r=x$, geometric series converges $|x|<1 \rightarrow = \frac{a}{1-r}$

$|r|<1 \rightarrow$ conv.

$|x|<1$ → interval of conv.

Representations of functions as Power Series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad (|x|<1)$$

interval conv.

find a power series representation for $f(x)$.

find the radius of conv. (int. conv.).

$f(x) = \frac{1}{1-x}$

this power series represents $\frac{1}{1-x}$ in the interval $|x|<1$

We Need More functions to be represented with Power Series.

$$\underset{\text{function}}{\frac{1}{1-x}} = \underbrace{1+x+x^2+x^3+\dots+x^n+\dots}_{\text{infinite sum}} = \underbrace{\sum_{n=0}^{\infty} x^n}_{\text{power series}}$$

→ **Substitution** : (Yerine koyma)

$f(x) = \frac{1}{1-x}$ $\quad f(-x^2)$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x|<1$$

Ex/ Find the power series representation for $f(x) = \frac{1}{1+x^2}$.

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)}$$

just by substituting $-x^2$ in place of x :

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n$$

$a=1$ $r=-x^2$ geometric

$|-x^2|<1$

$\Rightarrow |x|<1$

! ex/ $\left(\frac{1}{1-x}\right)^2 = \left(\sum_{n=0}^{\infty} x^n\right)^2 = \left(1+x+x^2+x^3+\dots+x^n+\dots\right)^2$

$\rightarrow \ne \sum_{n=0}^{\infty} (x^n)^2 \ne (1+x+x^2+\dots+x^n+\dots)^2$

$1+x^2+x^4+x^6+\dots$

substitution is not possible, we'll try other methods.

$(a+b)^2 \ne a^2+b^2$

$(a+b)^3 \ne a^3+b^3$

! örn/ $f(x)=\frac{x^3}{1-x} \to ?$

$$\frac{1}{1-x}=\sum_{n=0}^{\infty} x^n = 1+x+x^2+\dots+x^n+\dots$$

$$\frac{x^3}{1-x} = x^3\cdot\sum_{n=0}^{\infty} x^n = x^3\left(1+x+x^2+\dots+x^n+\dots\right)$$

$x^3+x^4+x^5+\dots$

$$\frac{x^3}{1-x}=\sum_{n=0}^{\infty} x^{n+3}$$

ar^n $\quad |x|<1$

$a=x^3 \quad r=x$

x^3, x^n

! örn/ $f(x)=\frac{1}{x-x^2}=\frac{1}{x(1-x)}$

$$\frac{1}{1-x}=\sum_{n=0}^{\infty} x^n = 1+x+x^2+\dots$$

$$\frac{1}{x(1-x)}=\sum_{n=0}^{\infty}\frac{x^n}{x}=\sum_{n=0}^{\infty} x^{n-1}$$

x^n $\quad |x|<1$

$a=\frac{1}{x} \quad r=x \quad$ geo.

örn/ $f(x)=\frac{1}{2+x}$ Find a power series rep. for $f(x)$.

$$\frac{1}{1-x}=\sum_{n=0}^{\infty} x^n \quad |x|<1$$

$$\Rightarrow \frac{1}{2+x}=\frac{1}{2\left(1+\frac{x}{2}\right)}=\frac{1}{2}\cdot\frac{1}{1+\frac{x}{2}}=\frac{1}{2}\cdot\frac{1}{1-\left(-\frac{x}{2}\right)}$$

$\to$ substitude $-\frac{x}{2}$ in place of x and multiply each term with $\frac{1}{2}$

$$\frac{1}{2+x}=\sum_{n=0}^{\infty}\frac{1}{2}\cdot\left(-\frac{x}{2}\right)^n$$

$a=\frac{1}{2} \quad r=-\frac{x}{2} \quad$ geometric.

$\left|-\frac{x}{2}\right|<1 \Rightarrow |x|<2$

$$\Rightarrow \boxed{\frac{1}{2+x}=\sum_{n=0}^{\infty}(-1)^n\frac{x^n}{2^{n+1}}} \quad \underline{|x|<2}$$

$\frac{1}{1-x}=\sum_{n=0}^{\infty} x^n$ $\quad$ ~~$\frac{1}{2+x}=\frac{1}{1-\underbrace{(-x-1)}_{x+2}}$~~ $\quad$ x yerine $(-x-1)$

~~$\frac{1}{2+x}=\sum_{n=0}^{\infty}(-x-1)^n=\sum_{n=0}^{\infty}(-1)^n(x+1)^n$~~ binom

~~$a=1 \quad r=-x-1$~~

$x\pm a \quad x\pm b$

Differentiation & Integration (term-by-term) $\to$ keeps the radius of convergence the same.

$\underline{|x|<1}$ $\quad \frac{1}{1-x}=\sum_{n=0}^{\infty} x^n$

! If $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ in an interval of conv. $|x-a| < R$

$\Rightarrow f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$ has the same interval of conv. $|x-a| < R$

If $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ in an interval of conv. $|x-a| < R$

$\Rightarrow \int f(x)dx = c_0 x + \sum_{n=1}^{\infty} \frac{c_n}{n+1} (x-a)^{n+1} \longrightarrow$ $+C$ $c_0(x-a)$

$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2$

$f'(x) = c_0 x + c_1 \frac{(x-a)^2}{2} +$

Örn $\frac{1}{(1-x)^2} \to ?$

$f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = \cancel{1} + x + x^2 + \quad |x| < 1$

$0 + 1 + 2x$

$f'(x) = (-1) \cdot -\frac{1}{(1-x)^2} = \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1} = 1 + 2 \cdot x^{2-1} + 3x^{3-1} + \dots$

$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n \cdot x^{n-1} \quad |x| < 1$

Ex

$f(x) = \frac{1}{(2+x)^3} \longrightarrow$ Find a power series rep. for $f(x)$.

$f(x) = \frac{1}{2+x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{2^{n+1}}$, $|x| < 2 \quad \longrightarrow \quad = \frac{1}{2} - \frac{x}{2^2} + \frac{x^2}{2^3} - \frac{x^3}{2^4} + \dots$

term by term differentiation

$f'(x) = -\frac{1}{(2+x)^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{2^{n+1}} n \cdot x^{n-1} \quad = \cancel{0} - \frac{1}{4} + \frac{2x}{8} - \frac{3x^2}{16} + \dots$

term by term diff.

$f''(x) = -(-2) \cdot \frac{1}{(2+x)^3} = \frac{2}{(2+x)^3} = \sum_{n=2}^{\infty} \frac{(-1)^n \cdot n}{2^{n+1}} \cdot (n-1) \cdot x^{n-2} \quad = \cancel{\ } \ \cancel{\ } + \frac{2}{8} +$

$\frac{1}{x}$ $-\frac{1}{x}$ x^{-2} $\frac{1}{x^2}$ x^{-3}

$\frac{1}{x^2}$

$-2 \cdot x^{-3}$

$f''(x) = -(-2) \cdot \frac{1}{(2+x)^3} = \left(\frac{2}{(2+x)^3}\right) = \left(\sum_{n=2} \frac{(-1)^n \cdot n(n-1) \cdot x}{2^{n+1}}\right) = / \quad / \quad + \frac{2}{8} +$

$\rightarrow \frac{1}{(2+x)^3} = \frac{1}{2} \quad // \quad = \sum_{n=2}^{\infty} \frac{(-1)^n \cdot n(n-1) \cdot x^{n-2}}{2^{n+2}}$

$|x| < 2$

$\frac{2}{16} + \cdots$ ✓

Ex/ $\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n} \qquad |x| < 1$

$\int \frac{1}{1+x^2} = \arctan(x) + C$

$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots$

$\underbrace{\int \frac{1}{1+x^2} dx}_{\arctan(x)+C} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} \cdots$

$x=0 \Rightarrow \underbrace{\arctan(0)}_{0} + C = 0 - 0 + 0 - 0 = 0$

$\Rightarrow C = 0$

$\Rightarrow \boxed{\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \qquad |x| < 1}$

Ex/ $f(x) = \frac{1}{1+x} \rightarrow \qquad \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad \frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n \qquad |x| < 1$

Ex/ $f(x) = \ln(1+x) \rightarrow$ Find a power series rep. for $f(x)$.

$f(x) = \frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n \qquad |x| < 1$

$1 - x + x^2 - x^3$

int.

$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$

$\underbrace{\int \frac{1}{1+x} dx}_{\ln|1+x| + C} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}, \quad |x| < 1$

$x = 0 \Rightarrow \underbrace{\ln(1)}_{0} + C = 0 - 0 + 0 + \cdots \qquad C = 0$

$\boxed{\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}, \quad |x| < 1}$

Ex/ $\int \frac{1}{1+x^7}\,dx$ → express the integral as power series.

$$\frac{1}{1+x^7} = \frac{1}{1-(-x^7)} = \sum_{n=0}^{\infty} (-x^7)^n = \sum_{n=0}^{\infty} (-1)^n x^{7n} = 1 - x^7 + x^{14} - x^{21} + \dots$$

$|x^7|<1$

$|x|<1$

$\sum x^n = \frac{1}{1-x}$ $\qquad x \to -x^7$

$$\int \frac{1}{1+x^7}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{7n+1}}{7n+1} = x - \frac{x^8}{8} + \frac{x^{15}}{15} - \frac{x^{22}}{22} + \dots$$

$|x|<1$

15. $f(x) = \ln(5-x)$ **16.** $f(x) = x^2\tan^{-1}(x^3)$

17. $f(x) = \frac{x}{(1+4x)^2}$ **18.** $f(x) = \left(\frac{x}{2-x}\right)^3$ → $\frac{x^3}{(2-x)^3}$

19. $f(x) = \frac{1+x}{(1-x)^2}$ **20.** $f(x) = \frac{x^2+x}{(1-x)^3}$

$\arctan(x) = \sum_{n=0} \frac{(-1)^n x^{2n+1}}{2n+1}$ $\quad |x|<1$

16.) $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^3)^{2n+1}}{2n+1} \cdot x^2$

$\frac{1}{2-x} \xrightarrow{\text{türev}} \frac{1}{(2-x)^2} \xrightarrow{\text{türev}} \frac{-2}{(2-x)^3} \rightsquigarrow \left(-\frac{1}{2}\right)\cdot \rightsquigarrow \frac{1}{(2-x)^3} \longrightarrow x^3\cdot \rightsquigarrow \frac{x^3}{(2-x)^3}$

15) $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ $\qquad \frac{1}{5-x} = \frac{1}{5(1-\frac{x}{5})} = \sum_{n=0}^{\infty} \frac{1}{5}\left(\frac{x}{5}\right)^n$ $\qquad a=\frac{1}{5}$, $r=\frac{x}{5}$ $\qquad |\frac{x}{5}|<1$ $\quad |x|<5$

$\frac{1}{(1+x)^2}$

$\ln(5-x) \rightsquigarrow -1\cdot\frac{1}{5-x}$

$$\ln(5-x) = (-1)\cdot\int \frac{1}{5-x}\,dx = (-1)\cdot\sum_{n=0}^{\infty} \frac{1}{5^{n+1}}\,\frac{x^{n+1}}{n+1}, \quad |x|<5$$

17) $\frac{x}{(1+4x)^2}$ $\qquad \frac{1}{1+4x}$ $\qquad \frac{1}{1-x} = \sum x^n$ $\qquad$ x yerine $(-4x)$ $\qquad \frac{1}{1+4x} = \sum_{n=0}^{\infty} (-4x)^n$ $\qquad |4x|<1$ $\quad |x|<\frac{1}{4}$

$\frac{1}{(1+4x)^2}$

türev

$$-\frac{4}{(1+4x)^2} = \sum_{n=1}^{\infty} (-4)^n\, n\, x^{n-1}$$

$$\frac{1}{(1+4x)^2} = \sum_{n=1}^{\infty} (-4)^{n-1}\, n\, x^{n-1}$$

$$\frac{x}{(1+4x)^2} = \sum_{n=1}^{\infty} (-4)^{n-1}\, n\, x^{n}$$

$|x|<\frac{1}{4}$