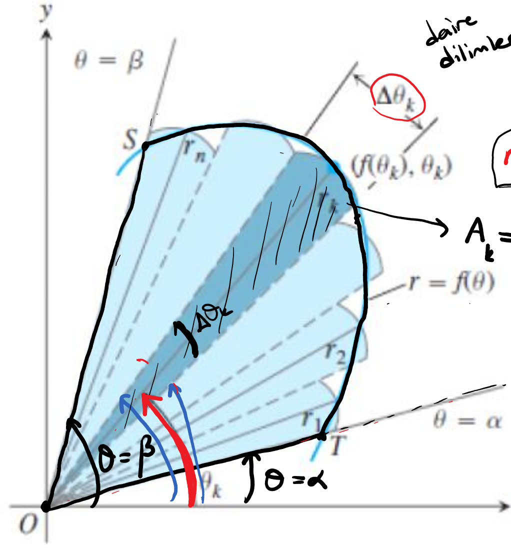
7. Hafta Perşembe Dersi - Kutupsal Eğrilerde Alan ve Uzunluk

06 Nisan 2021 Salı 11:28

$\int f(x)\,dx \qquad \int \sqrt{\Delta x^2 + \Delta y^2}$



daire dilimleri

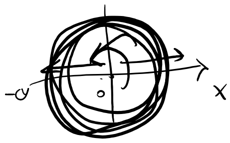
$\Delta\theta_k$

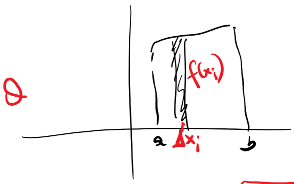
$\boxed{r_k = f(\theta_k)}$

$A_k = \cancel{\pi} r_k^2 \cdot \dfrac{\Delta\theta_k}{2\cancel{\pi}}$ → $d\theta$

$r_k^2 \to [f(\theta)]^2$

$$A = \int_\alpha^\beta \frac{1}{2} r^2 \, d\theta$$

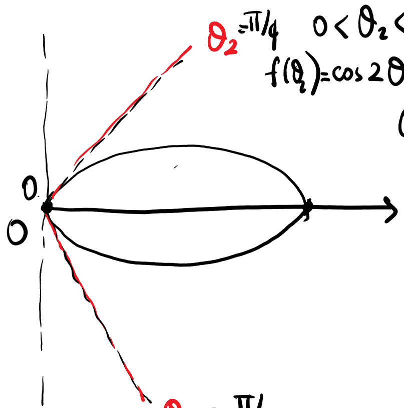




$f(x_i)\,\Delta x_i$

$$\boxed{A = \int_{\theta=\alpha}^{\theta=\beta} \frac{1}{2} f^2(\theta)\, d\theta}$$

Örn $f(\theta) = \cos 2\theta$ (4 yapraklı yoncasının) 1 yaprağının alanını bulalım.



$\theta_2 = \pi/4 \quad 0 < \theta_2 < \pi/2$

$f(\theta_2) = \cos 2\theta_2 = 0$

$\theta_2 = \pi/4$

$$A = \int_{\theta_1}^{\theta_2} \frac{1}{2} f(\theta)^2 \, d\theta$$

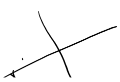
$\theta_1 = -\pi/4 \qquad 0 < \theta_1 < -\pi/2 \qquad \cos 2\theta_1 = 0 \qquad \theta_1 = -\pi/4$

$$A = \int_{\theta_1}^{\theta_2} \frac{1}{2} f(\theta)^2 \, d\theta = \int_{-\pi/4}^{\pi/4} \frac{1}{2} \cos^2 2\theta \, d\theta$$

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

$$= \int_{-\pi/4}^{\pi/4} \frac{1}{2}\left(\frac{1 + \cos 4\theta}{2}\right) d\theta$$

$\dfrac{1}{4} + \dfrac{\cos 4\theta}{4}$



$\to \theta = \pi/4$

$$\frac{1}{4} + \frac{\cos 4\theta}{4}$$

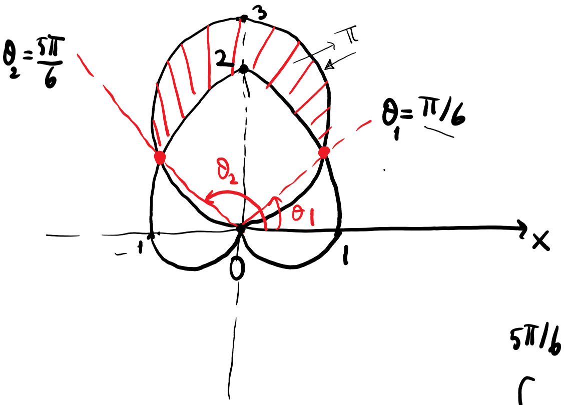
$$= \frac{\theta}{4} + \frac{1}{4}\,\frac{\sin 4\theta}{4} \Bigg]_{\theta=-\pi/4}^{\theta=\pi/4}$$

$$\frac{\sin(4.\pi/4)}{16}$$

$$\frac{\sin(4.-\pi/4)}{16}$$

$$= \left(\frac{\pi}{16} + 0\right) - \left(\frac{-\pi}{16} + 0\right) = \frac{2\pi}{16} = \frac{\pi}{8}$$

Örn $r = 3\sin\theta$ çemberinin içinde $1+\sin\theta$ kardiyoidinin dışında kalan alanı bulalım.



$$\boxed{3\sin\theta = 1+\sin\theta}$$

$$2\sin\theta = 1$$

$$\sin\theta = \frac{1}{2} \qquad 30°$$

$$\theta_1 = \frac{\pi}{6}$$

$$\theta_2 = \frac{5\pi}{6}$$

$$\frac{1}{2}\int f(\theta)^2\, d\theta$$

$$A = \frac{1}{2}\int_{\pi/6}^{5\pi/6} \left[(3\sin\theta)^2 - (1+\sin\theta)^2\right] d\theta$$

$$\sin^2 x = \frac{1-\cos 2x}{2}$$

$$\rightarrow = \frac{1}{2}\int \left(9\sin^2\theta - 1 - 2\sin\theta - \sin^2\theta\right) d\theta$$

$$= \frac{1}{2}\int \left(8\sin^2\theta - 1 - 2\sin\theta\right) d\theta$$

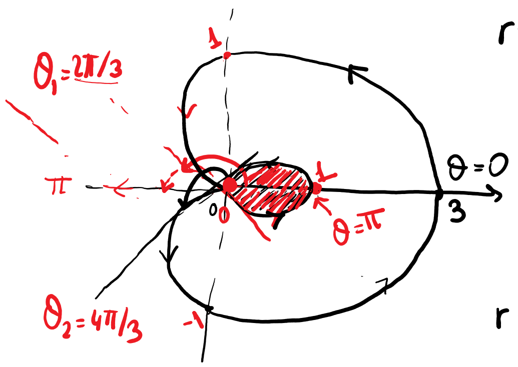
$$= \frac{1}{2}\int \left(8\left(\frac{1-\cos 2\theta}{2}\right) - 1 - 2\sin\theta\right) d\theta$$

$$\rightarrow = \frac{1}{2}\left(4\theta - 2\sin 2\theta - \theta + 2\cos\theta\right)\Bigg]_{\theta=\pi/6}^{\theta=5\pi/6}$$

$$= \frac{1}{2}\left(\underbrace{\cancel{3}.\left(\frac{4\pi}{\cancel{6}_2}\right)}_{2\pi} - \cancel{2}.\frac{\sqrt{3}}{2} + \cancel{2}.\frac{\sqrt{3}}{2}\right) = \frac{2\pi}{2} = \pi$$

8 Nisan Perşembe

Örn $r = 1 + 2\cos\theta$ limaçonunun içteki küçük döngüsünün alanını bulalım.

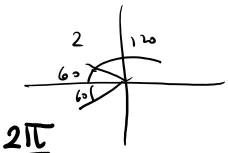


$r = 1 + 2\cos\theta$ $\quad \theta = 0$ $\quad (r=-1)\cos\pi$

$$A = 2\int_{\theta_1 = 2\pi/3}^{\theta_2 = 4\pi/3} \frac{1}{2} f(\theta)^2 \, d\theta$$

$$r = 1 + 2\cos\theta = 0$$

$$\cos\theta = -\frac{1}{2} \begin{cases} \text{2. bölgedeki açı} = \dfrac{2\pi}{3} \\ \text{3. bölgedeki açı} = 4\pi/3 \end{cases}$$



$$A = \int_{2\pi/3}^{4\pi/3} \frac{1}{2} (1 + 2\cos\theta)^2 \, d\theta$$

$$\cos^2\theta = \frac{1 + \cos 2\theta}{2}$$

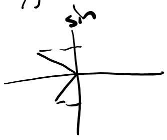
$$A = \frac{1}{2} \int_{\theta_1}^{\theta_2} \left(1 + 4\cos\theta + 4\cos^2\theta \right) d\theta$$

$$4\left(\frac{1}{2} + \frac{\cos 2\theta}{2}\right) \qquad 2\cos 2\theta \qquad \sin 2\theta$$

$$= \left(\frac{3\theta}{2} + 2\sin\theta + \frac{\sin 2\theta}{2} \right) \Bigg]_{2\pi/3}^{4\pi/3}$$

$2 \cdot \frac{4\pi}{3} = \frac{6\pi}{3} \Rightarrow \frac{2\pi}{3}$ $\quad 2 \cdot \frac{4\pi}{3}$ $\quad 2\pi/3$ $\quad 8\pi/3$

cos $\rightarrow -1/2$



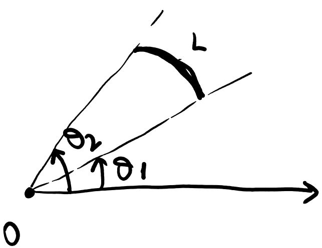
$\sin 2\pi/3 = \frac{\sqrt{3}}{2}$

$\sin 4\pi/3 = -\frac{\sqrt{3}}{2}$

$$= \left(2\pi - \sqrt{3} + \frac{\sqrt{3}}{4} \right) - \left(\pi + \sqrt{3} - \frac{\sqrt{3}}{4} \right)$$

$$= \pi - 2\sqrt{3} + \frac{\sqrt{3}}{2} = \pi - \frac{3\sqrt{3}}{2}$$

Kutupsal Koordinatlarda Yay Uzunluğu



Kutupsal Koordinatlarda;

$$x = r\cos\theta = \boxed{f(\theta)\cos\theta}$$
$$y = r\sin\theta = \boxed{f(\theta)\sin\theta}$$

$$\sqrt{\Delta x^2 + \Delta y^2} \longrightarrow L = \int_{\theta_1}^{\theta_2} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} \, d\theta$$

$$\frac{dx}{d\theta} = \frac{d}{d\theta}\left(f(\theta)\cos(\theta)\right) = f'(\theta)\cos\theta - f(\theta)\sin\theta$$

$$\frac{dy}{d\theta} = \frac{d}{d\theta}\left(f(\theta)\sin(\theta)\right) = f'(\theta)\sin\theta + f(\theta)\cos\theta$$

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = \left(f'(\theta)\cos\theta - f(\theta)\sin\theta\right)^2 + \left(f'(\theta)\sin\theta + f(\theta)\cos\theta\right)^2$$

$$= f'(\theta)^2\cos^2\theta - \cancel{2f(\theta)f'(\theta)\sin\theta\cos\theta} + f(\theta)^2\sin^2\theta$$

$$+ f'(\theta)^2\sin^2\theta + \cancel{2f(\theta)f'(\theta)\sin\theta\cos\theta} + f(\theta)^2\cos^2\theta$$

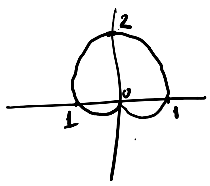
$$= f'(\theta)^2 \underbrace{(\sin^2\theta + \cos^2\theta)}_{1} + f(\theta)^2 \underbrace{(\sin^2\theta + \cos^2\theta)}_{1}$$

$$= f'(\theta)^2 + f(\theta)^2$$

$$\boxed{L = \int_{\theta_1}^{\theta_2} \sqrt{f'(\theta)^2 + f(\theta)^2}\, d\theta}$$

Örn $r = 1 + \sin\theta$ kardiyoidinin uzunluğunu bulalım.

$0 \leq \theta \leq 2\pi$



$f(\theta) = 1 + \sin\theta$

$f'(\theta) = \cos\theta$

$$L = \int_0^{2\pi} \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta$$

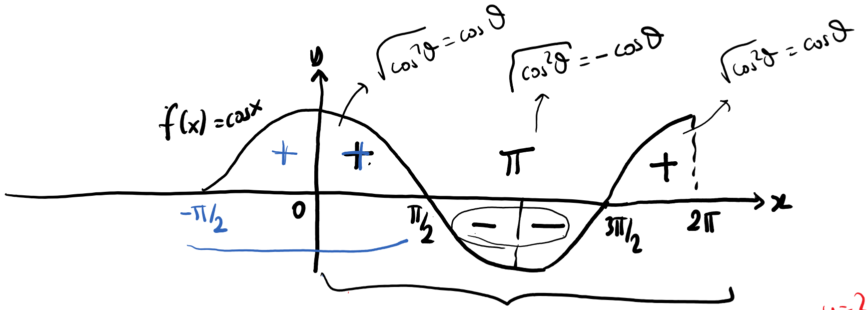
$$= \int_0^{2\pi} \sqrt{1 + 2\sin\theta + \sin^2\theta + \cos^2\theta}\, d\theta$$

$$= \int_0 \sqrt{1 + 2\sin\theta + \underbrace{\sin^2\theta + \cos^2\theta}_{1}} \, d\theta$$

$$= \int_0^{2\pi} \sqrt{2 + 2\sin\theta} \, d\theta$$

$$\sqrt{4 - 4\sin^2\theta} = 2\sqrt{1 - \sin^2\theta} = 2\sqrt{\cos^2\theta}$$

$$= \int_0^{2\pi} \frac{\sqrt{2 + 2\sin\theta}\,\sqrt{2 - 2\sin\theta}}{\sqrt{2 - 2\sin\theta}} \, d\theta = \int_0^{2\pi} \frac{2\,|\cos\theta|}{\sqrt{2 - 2\sin\theta}} \, d\theta$$



$$\underset{u=2}{\overset{u=0}{\int_0^{\pi/2}}} \frac{2\cos\theta}{\sqrt{2 - 2\sin\theta}} \, d\theta + \underset{u=0}{\overset{u=4}{\int_{\pi/2}^{3\pi/2}}} \frac{-2\cos\theta}{\sqrt{2 - 2\sin\theta}} \, d\theta + \underset{u=4}{\overset{u=2}{\int_{3\pi/2}^{2\pi}}} \frac{2\cos\theta}{\sqrt{2 - 2\sin\theta}} \, d\theta$$

$$\boxed{\begin{aligned} u &= 2 - 2\sin\theta \\ du &= -2\cos\theta \, d\theta \end{aligned}}$$

$$\int \frac{-du}{\sqrt{u}} \qquad \int \frac{du}{\sqrt{u}} \qquad \int -\frac{du}{\sqrt{u}}$$

$$= -2\sqrt{u}\,\Big]_2^0 \qquad = 2\sqrt{u}\,\Big]_0^4 \qquad = -2\sqrt{u}\,\Big]_4^2$$

$$= 0 - (-2\sqrt{2}) \; + \; 2\sqrt{4} - 0 \; + \; -2\sqrt{2} - (-4)$$

$$= \cancel{2\sqrt{2}} + 4 + \cancel{-2\sqrt{2}} + 4$$

$$= 8$$

$$2 \cdot \int_{-\pi/2} \rightarrow -2\sqrt{u}\,\Big]_{4}^{0} = -\cancel{2 \cdot 0} - (-2 \cdot \sqrt{4}) = 4$$

$u = 2 - 2\sin\theta$ $_{-\pi/2}$

$$2 \cdot 4 = 8$$