12th Week Thursday

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Alternating Series Test (A.S.T.)

$\sum_{n=1}^{\infty} a_n \to (-1)^n$ $-,+,-,+,-$

$- + - + -$

$\sum_{n=1}^{\infty} \underbrace{(-1)^n u_n}_{a_n} \to$ $u_n \to$ positive ✓, $u_n \to$ decreasing ✓, $\boxed{\lim_{n\to\infty} u_n = 0}$ ✓ $\Rightarrow \sum_{n=1}^{\infty} (-1)^n u_n$ this alternating series converges.

$n \to \infty$

(Test for divergence: $\lim_{n\to\infty} a_n \neq 0 \Rightarrow \sum_{n=1}^{\infty} a_n$ diverges)

$\to$! $\lim_{n\to\infty} (-1)^n u_n \to$ DNE, $\pm\infty$ $\Big\} \Rightarrow \sum_{n=1}^{\infty} (-1)^n u_n$ diverges.

$+,1,-1$ $\to \neq 0$

$\lim_{n\to\infty} \cancel{(-1)^n} u_n = 0$, $\to 0$ $\Rightarrow$ yakınsak veya ıraksak olabilir. ?

Ör/ $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n} \to$ alternating harmonic series

$\left(\sum_{n=1}^{\infty} \frac{1}{n}\right.$ harmonic series $\left.\to \text{divergent}\right)$

$\to$ pozitif ✓, $\boxed{\text{decreasing}}$ ✓, $\lim_{n\to\infty} \frac{1}{n} = 0$ ✓ $\Big\} \Rightarrow$ A.S.T ile $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ converges.

$\to f'(n) < 0$?

$\to \frac{a_{n+1}}{a_n} < 1$?

$\to a_{n+1} - a_n$?

Ör/ $\sum_{n=1}^{\infty} (-1)^n \underbrace{\left(\frac{3}{2}\right)^n}_{\to\infty}$ $\Rightarrow$ ıraksaklık testiyle de ıraksaktır.

$\Rightarrow \left(-\frac{3}{2}\right)^n$ $a = -3/2$, $r = -3/2$, geometrik seri; $|r| > 1 \Rightarrow$ ıraksaktır.

Ör/ $\sum_{n=1}^{\infty} (-1)^n \underbrace{\left(\frac{1}{3}\right)^n}_{\to 0}$ $\Rightarrow \left(-\frac{1}{3}\right)^n$ $a = -1/3$, $r = -1/3$, geometrik seri; $|r| < 1 \Rightarrow$ yakınsaktır.

$\Rightarrow$ A.S.T. ile. $u_n = \left(\frac{1}{3}\right)^n$ pozitif ✓, azalan ✓ $\frac{1}{3^n}$, $\lim_{n\to\infty} \frac{1}{3^n} = 0$ ✓ $\Big\} \Rightarrow$ A.S.T ile yakınsaktır.

Ör/ $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n+1}$ $u_n = \frac{n}{n+1}$ pozitif ✓, azalan ? ✗

türev $= \frac{1.(n+1) - n.1}{(n+1)^2} = \frac{1}{(n+1)^2} > 0$

A.S.T. çalışmaz.

$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$

$\lim (-1)^n n$ ... DNE $\to$ ıraksaklık testiyle ıraksaktır.

$n\to\infty$ $\quad n\to\infty$ $\quad n+1$

$\lim_{n\to\infty} (-1)^n \frac{n}{n+1} \to DNE \neq 0 \Rightarrow$ ıraksaklık testiyle ıraksaktır.

$+1,-1$

Örn

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1} \qquad u_n = \frac{n^2}{n^3+1}$$

pozitif ✓

azalan ✓

türev $= \frac{2n(n^3+1) - n^2 \cdot 3n^2}{(n^3+1)^2} = \frac{2n - n^4}{(n^3+1)^2} < 0$

$2n^4 + 2n - 3n^4$

$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n^2}{n^3+1} = 0$ ✓

$\Rightarrow$ A.S.T ile yakınsaktır.

2–20 Test the series for convergence or divergence.

2. $\frac{2}{3} - \frac{2}{5} + \frac{2}{7} - \frac{2}{9} + \frac{2}{11} - \cdots$

$n=1 \quad n=2 \quad n=3 \quad n=4$

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{2n+1} \to u_n$$

$u_n = \frac{2}{2n+1}$ $\to\infty$

pozitif ✓

azalan ✓

$\lim_{n\to\infty} \frac{2}{2n+1} = 0$ ✓

} A.S.T yak.

3. $-\frac{2}{5} + \frac{4}{6} - \frac{6}{7} + \frac{8}{8} - \frac{10}{9} + \cdots$

4. $\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} - \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{6}} - \cdots \to \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\sqrt{n+1}}$

$n=1$

$u_n = \frac{1}{\sqrt{n+1}}$

pozitif ✓

azalan ✓

$\lim_{n\to\infty} \frac{1}{\sqrt{n+1}} = 0$ ✓

türev $\to \frac{0 - 1 \cdot 1/2\sqrt{n+1}}{n+1} = \frac{-1}{(n+1)2(\sqrt{n+1})} < 0$

A.S.T ile yakınsaktır.

16. $\sum_{n=1}^{\infty} \frac{n\cos n\pi}{2^n} \Leftrightarrow \sum_{n=1}^{\infty} (-1)^n \frac{n}{2^n}$

$n=1$

$$\frac{1 \cdot \cos\pi}{2^1} + \frac{2 \cdot \cos 2\pi}{2^2} + \frac{3 \cos 3\pi}{2^3} + \frac{4 \cos 4\pi}{2^4} + \cdots$$

$-1 \quad 1 \quad -1 \quad +1$

$u_n = \frac{n}{2^n}$

pozitif ✓

azalan ? ✓

$\lim_{n\to\infty} u_n = ?\ 0$ ✓

türev $\to \frac{1 \cdot 2^n - n \cdot 2^n \cdot \ln 2}{2^{2n}} = \frac{2^n (1 - n\ln 2)}{2^{2n}} < 0$ ✓

$\to \frac{a_{n+1}}{a_n} = \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} = \frac{n+1}{2n} < 1$ ✓

$\to a_{n+1} - a_n = \frac{n+1}{2^{n+1}} - \frac{n}{2^n} = \frac{n+1-2n}{2^{n+1}} = \frac{1-n}{2^{n+1}} < 0$

$\lim_{n\to\infty} \frac{n}{2^n} \overset{L'}{=} \lim_{n\to\infty} \frac{1}{2^n \cdot \ln 2} = \frac{1}{\infty} = 0$

$\Rightarrow$ A.S.T ile seri yakınsaktır.

Örn $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^n}{n!}$ $\qquad u_n = \frac{n^n}{n!}$

pozitif ✓

azalan ?

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}}{\cancel{(n+1)!}_{\,n+1}} \cdot \frac{\cancel{n!}}{n^n} = \frac{(n+1)^n}{n^n} = \left(\frac{n+1}{n}\right)^n = \left(1+\frac{1}{n}\right)^n > 1$$

A.S.T çalışmaz.

$$\lim_{n\to\infty} \frac{n^n}{n!} = \lim_{n\to\infty} \frac{n.n.\dots.n}{n.(n-1)\dots 2.1} = \infty \neq 0$$

$$\sum_{n=1}^{\infty} (-1)^n \frac{n^n}{n!} \qquad \lim_{n\to\infty} (-1)^n \underbrace{\frac{n^n}{n!}}_{\neq 0} \neq 0 \Rightarrow$$ ıraksaklık testiyle ıraksaktır.

Örn

20. $\displaystyle\sum_{n=1}^{\infty} (-1)^n(\sqrt{n+1} - \sqrt{n})$ $\qquad u_n = \sqrt{n+1} - \sqrt{n}$

pozitif ✓

azalan ✓

$\displaystyle\lim_{n\to\infty} u_n \overset{?}{=} 0$ ✓

A.S.T. ile yakınsaktır.

$$\text{türev} = \frac{1}{2\sqrt{n+1}}_{(\sqrt{n})} - \frac{1}{2\sqrt{n}}_{(\sqrt{n+1})} = \frac{\sqrt{n} - \sqrt{n+1}}{2\sqrt{n}\sqrt{n+1}} < 0$$

$$\lim_{n\to\infty} \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{(\sqrt{n+1} + \sqrt{n})} = \lim_{n\to\infty} \frac{\cancel{n}+1-\cancel{n}}{(\sqrt{n+1} + \sqrt{n})} = \frac{1}{\infty} = 0$$

Alterne p-serileri : $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^p}$

! $\rightarrow p = 1 \rightarrow \displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ K.Y alterne har. seri yakınsaktır.

$\rightarrow p > 1$ $\quad \displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^p}$ M.Y. $\quad u_n =$ pozitif ✓ azalan ✓ $\displaystyle\lim_{n\to\infty} \frac{1}{n^p} = 0$ ✓ A.S.T ile yakınsaktır.

! $\rightarrow 0 < p < 1$ $\quad \displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^p}$ K.Y (basit kesir) $\quad u_n =$ pozitif ✓ azalan ✓ $\displaystyle\lim_{n\to\infty} \frac{1}{n^p} = 0$ ✓ A.S.T ile yakınsaktır.

$\rightarrow p = 0$ $\quad \displaystyle\sum_{n=1}^{\infty} (-1)^n . 1$ $\quad \displaystyle\lim_{n\to\infty} (-1)^n \neq 0 \Rightarrow$ ıraksaklık testiyle ıraksaktır.

$\rightarrow p < 0$ $\quad \displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^p}$ (n^2, n^{-2}) $\quad u_n =$ pozitif ✓ azalan ✗ $\quad$ AST çalışmaz $\quad \displaystyle\lim_{n\to\infty} u_n = \infty$

$\displaystyle\lim_{n\to\infty} (-1)^n \frac{1}{n^p} \neq 0 \Rightarrow$ ıraksaklık testiyle ıraksaktır.

Bunların mutlak değer serileri p-serisi $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ = $p > 1$ yakınsak, $p \leq 1$ ıraksak

Absolute Convergence ✓ / Conditional Convergence ✓

$\sum_{n=1}^{\infty} a_n$ Teorem $\underbrace{\sum_{n=1}^{\infty} |a_n|}_{\text{absolute value}}$ convergent $\Rightarrow \sum_{n=1}^{\infty} a_n$ is convergent.

$\rightarrow$! $\sum_{n=1}^{\infty} |a_n|$ divergent but $\sum_{n=1}^{\infty} a_n$ convergent.

örn/ $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}} \rightarrow u_n = \frac{1}{\sqrt{n}}$ positive ✓ decreasing ✓ $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$ ✓

by A.S.T, converges.

conditionally convergent.

$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \rightarrow$ divergent, p-series $p = \frac{1}{2} < 1$

series where $\sum_{n=1}^{\infty} |a_n|$ is convergent are called "**Absolutely Convergent**" $\rightarrow \sum a_n$

series where $\sum_{n=1}^{\infty} |a_n|$ is divergent but $\sum a_n$ itself is convergent are called "Conditionally Convergent"

C.C.	A.C		Alternating p-series		Bunların Mutlak Değer serileri = Normal p-serisi
✓		$p=1$	$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n} \rightarrow$ conv.		$\sum_{n=1}^{\infty} \frac{1}{n} \rightarrow$ harmonik ıraksak div.
	✓	$p>1$	$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2} \rightarrow$ conv. by A.S.T.		$\sum_{n=1}^{\infty} \frac{1}{n^2} \rightarrow$ $p=2>1$ p-serisi yakınsak. conv.
✓		$0<p<1$	$\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}} \rightarrow$ conv. by A.S.T. (pozi, dec, $u_n \to 0$)		$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \rightarrow$ $p<1$ p-serisi ıraksaktır div.

örn/ $\sum_{n=1}^{\infty} \frac{\cos(n)}{n^2} \rightarrow$ alterne seri değil. $-1 < \cos(n) < 1$

mutlak değer serisi $\rightarrow \sum_{n=1}^{\infty} \frac{|\cos(n)|}{n^2}$ $0 < |\cos(n)| < 1$

$\frac{|\cos(n)|}{n^2} < \frac{1}{n^2}$ yakınsak p-serisi

D.K.T ile yakınsak.

$\Rightarrow$ Mutlak değeri yakınsak.

$\Rightarrow$ kendisi de yakınsaktır.