3. Hafta Perşembe Dersi

11 Mart 2021 Perşembe 11:42

Örn (p - integrali)

$$\int_1^{\infty} \frac{1}{x^p}\,dx$$ integrali, p'nin hangi değerleri için yakınsaktır?

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{t\to\infty}\left(\int_1^{t} \frac{1}{x^p}\,dx\right) \qquad x^{-p} \to \frac{x^{-p+1}}{-p+1}$$

$$\frac{1}{x}$$

$$= \lim_{t\to\infty}\left(\left.\frac{x^{-p+1}}{-p+1}\right]_1^t\right) = \lim_{t\to\infty}\left(\frac{t^{-p+1}}{-p+1} - \frac{1^{-p+1}}{-p+1}\right) = \infty$$

$p<1 \Rightarrow -p+1>0$, $\quad -p+1 \neq 0$

$$p=1 \longrightarrow \int_1^{\infty} \frac{1}{x}\,dx = \infty \text{ ıraksak.}$$

$$p<1 \Rightarrow \lim_{t\to\infty} t^{(-p+1)>0} = \infty$$

$$p>1 \Rightarrow -p+1<0 \Rightarrow \lim_{t\to\infty}\left(\frac{t^{-p+1}}{-p+1} - \frac{1}{-p+1}\right) = \frac{1}{p-1}$$

$$\frac{x^{-p+1}}{-p+1}$$

!

$$\boxed{\int_1^{\infty} \frac{1}{x^p}\,dx = \begin{cases} \infty \text{ ıraksaktır}, & p<1 \\ \infty \text{ ıraksaktır}, & p=1 \\ \text{yakınsaktır}, = \dfrac{1}{p-1}, & p>1 \end{cases}}$$

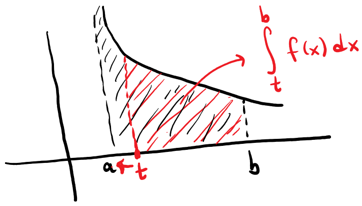
Örn

$$\int_1^{\infty} \frac{1}{x^5}\,dx = \frac{1}{4} \text{ yakınsak}, \qquad \int_1^{\infty} x^3\,dx = \infty \text{ ıraksaktır.}$$

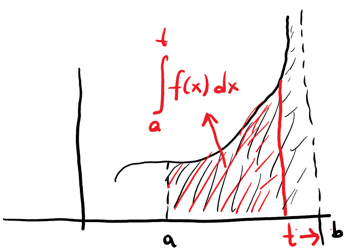
$$\int_1^{\infty} \frac{1}{\sqrt{x}}\,dx = \infty \text{ ıraksaktır.}$$

2. Tip Sonsuz Süreksizlik → (Dikey asimptot var)

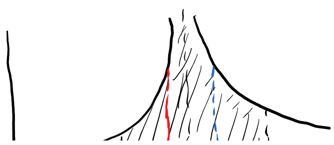
$$\boxed{\int_a^b f(x)\,dx = ?}$$



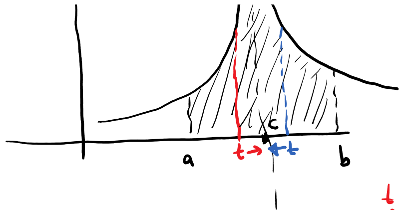
$$\lim_{t\to a^+}\left(\int_t^b f(x)\,dx\right)$$



$$\lim_{t\to b^-}\left(\int_a^t f(x)\,dx\right)$$



$$= \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$



$$= \int_a f(x)\,dx + \int_c f(x)\,dx$$

$$\lim_{t\to c^-}\left(\int_a^t f(x)\,dx\right) + \lim_{t\to c^+}\left(\int_t^b f(x)\,dx\right)$$

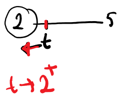
yakınsak + yakınsak = yakınsak ✓

herhangi biri ıraksak ise sonuç = ıraksak.

Örn

$$\int_{\boxed{2}}^{5} \frac{1}{\sqrt{x-2}}\,dx = ?$$

x=2'de sonsuz süreksizlik var.



$$= \lim_{t\to 2^+}\left(\int_t^5 \frac{1}{\sqrt{x-2}}\,dx\right) = ?$$

$u = x-2$ $\quad \frac{du}{u^{1/2}}$ $\quad u^{-1/2}$ $\quad \frac{u^{1/2}}{1/2} = 2\sqrt{u}$

$$= \lim_{t\to 2^+}\left(2\sqrt{x-2}\,\Big]_t^5\right)$$

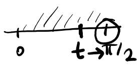
$$= \lim_{t\to 2^+}\left(2\sqrt{3} - 2\underbrace{\sqrt{t-2}}_{0}\right) = 2\sqrt{3}$$

yakınsaktır.

Örn

$$\int_0^{\boxed{\pi/2}} \sec x\,dx = \lim_{t\to \frac{\pi}{2}^-}\left(\int_0^t \sec x\,dx\right)$$

$\frac{1}{\cos x}$



$$= \lim_{t\to \frac{\pi}{2}^-}\left(\ln|\sec x + \tan x|\,\Big]_0^t\right)$$

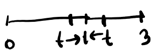
$$= \lim_{t\to \frac{\pi}{2}^-}\left(\ln|\underbrace{\overset{\infty}{\sec t} + \overset{\infty}{\tan t}}_{\frac{1}{\cos t}\ \infty}| - \ln|\underbrace{\sec 0}_{1} + \underbrace{\tan 0}_{0}|\right) = \infty$$

$\ln 1 = 0$

ıraksaktır. ✓

Örn

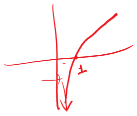
$$\int_0^3 \frac{dx}{x-1} = \int_0^{1} \frac{dx}{x-1} + \int_{1}^3 \frac{dx}{x-1}$$



$$= \lim_{t\to 1^-}\left(\int^t \frac{dx}{x-1}\right) + \lim_{t\to 1^+}\left(\int_t^3 \frac{dx}{x-1}\right)$$

$$= \lim_{t\to 1^-}\left(\int_0^t \frac{dx}{x-1}\right) + \lim_{t\to 1^+}\left(\int_t^{3} \frac{dx}{x-1}\right)$$

$$= \lim_{t\to 1^-}\Big(\underbrace{\ln|t-1|}_{-\infty} - \underbrace{\ln|0-1|}_{0}\Big) + \lim_{t\to 1^+}\Big(\underbrace{\ln|3-1|}_{\ln 2} - \underbrace{\ln|t-1|}_{\infty}\Big) \rightarrow \text{ıraksaktır.}$$



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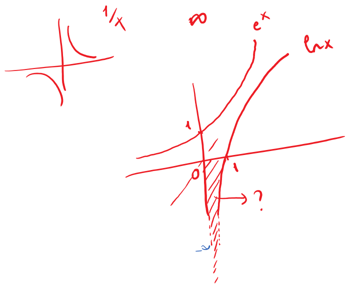


$x\ln x - x$

ÖRN/

$$\int_{\boxed{0}}^{1} \ln x\,dx = \lim_{t\to 0^+}\left(\int_t^1 \ln x\,dx\right)$$

0+t 1

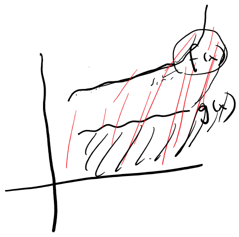
$$= \lim_{t\to 0^+}\left(\Big(x\ln x - x\Big)\Big]_t^1\right)$$

$$= \lim_{t\to 0^+}\Big[\big(\underbrace{1.\underset{0}{\ln 1} - 1}_{-1}\big) - \big(\underbrace{\overset{0\,\cdot\,\infty}{t\ln t}}_{0} - \underset{0}{t}\big)\Big] = -1 - \cancel{0} + \cancel{0} = -1 \rightarrow \text{yakınsaktır}$$

$$\lim_{t\to 0^+} t\ln t = \lim_{t\to 0^+} \frac{\ln t}{1/t} \xrightarrow{L'} \lim_{t\to 0^+} \frac{1/t}{\underbrace{-1/t^2}_{-t}} = 0$$

Genelleştirilmiş İntegraller İçin Karşılaştırma Testi

f, g sürekli fonk. olm. üzere, $f(x) \geq g(x) \geq 0$



* Eğer $\int_a^\infty f(x)\,dx$ yakınsak ise, $\int_a^\infty g(x)\,dx$ de yakınsaktır.

$\int_a^\infty f(x)\,dx$ ıraksak ise, bir şey söyleyemeyiz.

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! Büyük olan yakınsaksa, küçük olan da yakınsaktır.
Küçük olan ıraksaksa, büyük olan da ıraksaktır.

$$\int^\infty \frac{1}{x^p}\,dx = \begin{cases} p<1 & \text{ıraksak} \\ p=1 & \text{ıraksak} \\ p>1 & \text{yakınsak} \end{cases}$$

karşılaştırma için

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \begin{cases} p<1 & \text{ıraksak} \\ p=1 & \text{ıraksak} \\ p>1 & \text{yakınsak} \end{cases}$$

$$\int_0^{1} \frac{1}{x^p}\,dx = \begin{cases} p \geqslant 1 & \text{ıraksak} \\ p<1 & \text{yakınsak} \end{cases}$$

$$\int_0^{\infty} e^{ax}\,dx = \begin{cases} a<0 & \text{yakınsak} \\ a \geqslant 0 & \text{ıraksak} \end{cases}$$

Karşılaştırma için en sık kullanılan integraller.

Örn

$$\int_{\pi}^{\infty} \frac{1+\sin x}{x^2}\,dx$$ integralinin yakınsaklık durumunu inceleyiniz.

$$-1 \leqslant \sin x \leqslant 1$$

$$0 \leqslant 1+\sin x \leqslant 2$$

$$0 \leqslant \frac{1+\sin x}{x^2} \leqslant \frac{2}{x^2}$$ yakınsak.

Büyük olan yakınsak. $\Rightarrow \int_{\pi}^{\infty} \frac{1+\sin x}{x^2}\,dx$ yakınsaktır.

$$\int_{\pi}^{\infty} \frac{2}{x^2}\,dx = \lim_{t\to\infty}\left(-\frac{2}{x}\Big]_{\pi}^{t}\right) = \lim_{t\to\infty}\left(\underbrace{-\frac{2}{t}}_{\to 0} + \frac{2}{\pi}\right) = \frac{2}{\pi} \to \text{yakınsaktır.}$$