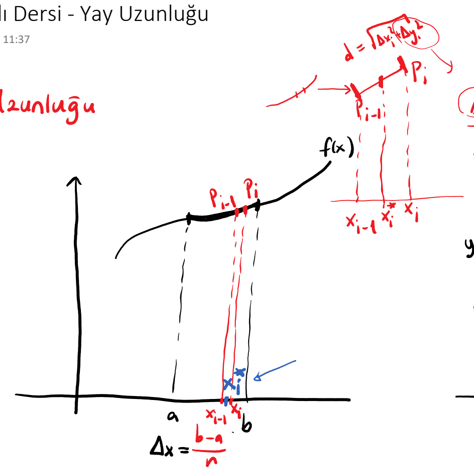
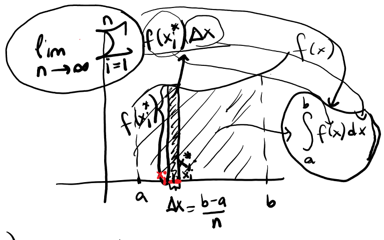
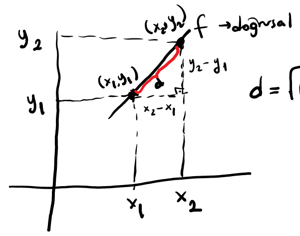
4. Hafta Salı Dersi - Yay Uzunluğu

16 Mart 2021 Salı 11:37

Yay Uzunluğu







$$d=\sqrt{\underbrace{(x_2-x_1)^2}_{\Delta x}+\underbrace{(y_2-y_1)^2}_{\Delta y}}$$

$$d=\sqrt{(\Delta x)^2+(\Delta y)^2}$$

x=a ile x=b arasındaki f eğrisinin yay uzunluğu

$$\rightarrow L=\lim_{n\to\infty}\sum_{i=1}^{n}\overbrace{|P_{i-1}P_i|}^{d}=\lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{(\Delta x_i)^2+(\Delta y_i)^2}$$

x_{i-1} x_i x_i^*

$$\left(f'(x_i^*).\Delta x_i\right)^2$$

ortalama değer teoremi ;

$$\frac{\overbrace{f(x_i)-f(x_{i-1})}^{\Delta y_i}}{\underbrace{x_i-x_{i-1}}_{\Delta x_i}}=f'(x_i^*)$$

$$\Rightarrow \Delta y_i = f'(x_i^*)\,\Delta x_i$$

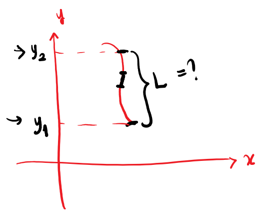
$$=\lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{(\Delta x_i)^2\left(1+f'(x_i^*)^2\right)}$$

$$=\lim_{n\to\infty}\sum\sqrt{1+f'(x_i^*)^2}\;\Delta x_i$$

$$L=\int_{x=a}^{x=b}\sqrt{1+f'(x)^2}\,dx \longrightarrow$$

y eksenine göre yay uzunluğu ;

$\rightarrow x=f(y)$



$$L=\int_{y_1}^{y_2}\sqrt{1+f'(y)^2}\,dy$$

Örn $y^2=x^3$ parabolünün $x=1$ ve $x=4$ noktaları arasındaki yay uzunluğunu bulunuz.

$f(x)=?$ $\qquad f(x)=\sqrt{x^3}$

$$f'(x)=\frac{3}{2}x^{1/2}=\frac{3\sqrt{x}}{2}$$

$$L=\int_1^4\sqrt{1+f'(x)^2}\,dx=\int_1^4\sqrt{1+\left(\frac{3\sqrt{x}}{2}\right)^2}\,dx$$

$$=\int_1^4\sqrt{1+\frac{9x}{4}}\,dx$$

$u=1+\frac{9x}{4}$

$du=\frac{9}{4}dx$

$x=4 \quad u=1+\frac{9x}{4}=10$

$x=1 \quad u=1+\frac{9x}{4}=\frac{13}{4}$

$$=\int\sqrt{u}\,\frac{4}{9}du=\frac{4}{9}.\frac{2}{3}u^{3/2}\Big]^{10}_{13/4}$$

$x=4 \quad u=\ldots$
$x=1 \quad u = 1+\frac{9x}{4} = \frac{13}{4}$

$$= \int_1^4 \sqrt{u}\ \frac{4}{9}\,du = \frac{4}{9}\cdot\frac{2}{3}\,u^{3/2}\Big]_{13/4}^{10}$$

veya

$$= \frac{8}{27}\left(1+\frac{9x}{4}\right)^{3/2}\Big]_1^4 = \frac{8}{27}\left[10^{3/2} - \left(\frac{13}{4}\right)^{3/2}\right]$$

Örn $f(x) = \frac{x^3}{12} + \frac{1}{x}$ eğrisinin $1 \leq x \leq 4$ aralığında uzunluğunu bulunuz.

$$f'(x) = \frac{3x^2}{12} - \frac{1}{x^2}$$

$$\frac{x^2}{4} - \frac{1}{x^2}$$

$$f'(x)^2 = \frac{x^4}{16} - \frac{1}{2} + \frac{1}{x^4}$$

$$L = \int_1^4 \sqrt{1+f'(x)^2}\,dx =$$

$$= \int_1^4 \sqrt{1+\left(\frac{x^4}{16} - \frac{1}{2} + \frac{1}{x^4}\right)}\,dx$$

$$\frac{x^4}{16} + \frac{1}{2} + \frac{1}{x^4} = \left(\frac{x^2}{4} + \frac{1}{x^2}\right)^2$$

$$= \int_1^4 \sqrt{\left(\frac{x^2}{4} + \frac{1}{x^2}\right)^2}\,dx = \int_1^4 \left(\frac{x^2}{4} + \frac{1}{x^2}\right)dx$$

$$= \frac{x^3}{12} + -\frac{1}{x}\Big]_1^4$$

$$= \left(\frac{4^3}{12} - \frac{1}{4}\right) - \left(\frac{1}{12} - 1\right)$$

$$= \frac{64-3-1+12}{12} = 6$$

Örn $y^2 = x$ parabolünün $x=0$ ve $x=1$ noktaları arasındaki uzunluğunu bulalım.

$x=0 \rightarrow y=0$, $\quad x=1 \rightarrow y=1$

$f(y) = y^2 \qquad f'(y) = 2y$

$a^2+b^2y^2 \rightarrow by = a\tan\theta$

$$L = \int_0^1 \sqrt{1+f'(y)^2}\,dy = \int_0^1 \sqrt{1+4y^2}\,dy$$

$1+\tan^2\theta = \sec^2\theta$

$2y = 1\tan\theta$

$2\,dy = \sec^2\theta\,d\theta$

$$= \int_\alpha^\beta \frac{1}{2}\sec\theta\ \sec^2\theta\,d\theta = \int^\beta \frac{1}{2}\sec^3\theta\,d\theta$$

$u = \sec\theta \qquad dv = \sec^2\theta\,d\theta$

$$2y = 1\tan\theta \qquad = \int_{\alpha} \frac{1}{2}\sec\theta \, \sec^2\theta \, d\theta = \int_{\alpha} \frac{1}{2}\sec^3\theta \, d\theta$$

$$2\,dy = \sec^2\theta \, d\theta$$

$u = \sec\theta \quad dv = \sec^2\theta\, d\theta$

$du = \sec\theta\tan\theta\, d\theta \quad v = \tan\theta$

$y = 0 \quad \tan\theta = 0 \quad \theta = \arctan 0 = 0$

$y = 1 \quad \tan\theta = 2 \quad \theta = \arctan 2$

$1 + \tan^2\theta = \sec^2\theta$

$$\underbrace{\int \sec^3\theta\, d\theta}_{I} = \sec\theta\tan\theta - \int \tan^2\theta \sec\theta\, d\theta = \sec\theta\tan\theta - \underbrace{\int \sec^3\theta\, d\theta}_{I} + \int \sec\theta\, d\theta \ \checkmark$$

$$\tan^2\theta = (\sec^2\theta - 1), \qquad (\sec^3\theta - \sec\theta)\,d\theta$$

$$2I = \sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|$$

$$I = \left(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right)/2$$

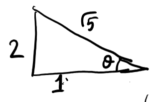
$$L = \frac{1}{4}\left(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right)\Big]_0^{\arctan(2)}$$

$y = 0 \rightarrow \theta = 0$

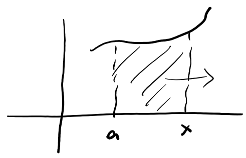
$y = 1 \rightarrow \theta = \arctan 2$

$$= \frac{1}{4}\left(2\sqrt{5} + \ln|\sqrt{5} + 2|\right)$$

$2y = \tan\theta$



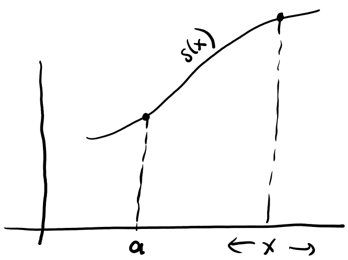
$\sec(\arctan(2))$



$$\int_a^x f(t)\,dt = g(x)$$

$$\boxed{g'(x) = f(x)}$$

Yay Uzunluğu Fonksiyonu



$$s(x) = \int_a^{x} \sqrt{1 + f'(t)^2}\, dt$$

$$\boxed{\frac{d}{dx} s(x) = \sqrt{1 + f'(x)^2}}$$

yay uzunluğu fonksiyonu

Örn/ $\boxed{y = x^2 - \frac{1}{8}\ln x}$ eğrisinin $(1,1)$ noktası başlangıç noktası seçilerek ifade edilecek yay uzunluğu fonksiyonunu bulunuz

$$s(x) = \int_1^x \sqrt{1 + f'(t)^2}\, dt$$

$$f(t) = t^2 - \frac{1}{8}\ln t$$

$$f'(t) = 2t - \frac{1}{8t}$$

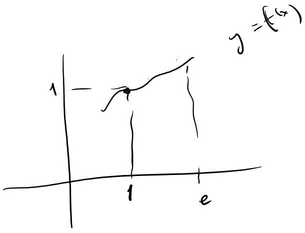
$$f'(t)^2 = 4t^2 - \cancel{2t}\cdot\frac{1}{\cancel{8t}_2} + \frac{1}{64t^2}$$

$$s(x) = \int_1^x \sqrt{1 + \underbrace{\left(4t^2 - \frac{1}{2} + \frac{1}{64t^2}\right)}}\, dt =$$

$$4t^2 + \frac{1}{2} + \frac{1}{64t^2} = \left(2t + \frac{1}{8t}\right)^2$$

$$= \int_1^x \sqrt{\left(2t + \frac{1}{8t}\right)^2}\, dt = \int_1^x \left(2t + \frac{1}{8t}\right) dt$$

$$= \left(t^2 + \frac{1}{8}\ln t\right)\Big]_1^x$$



$$s(x) = x^2 + \frac{1}{8}\ln x - 1$$