15th Week Thursday

1 Haziran 2023 Perşembe 11:32

9:30-

Forms 10 soru final →

1.soru: forms dosya →
2.soru: forms →
10.soru: forms

Ex $\sum_{n=0}^{\infty} \frac{(-1)^n (x-2)^n}{2^{2n+1}}$ $4^n \cdot 2$ $\sum_{n=0}^{\infty} ar^n$ $r = -\frac{x-2}{4}$ $a = \frac{1}{2}$ → geometrik seri $|r|<1$

$\left|\frac{x-2}{4}\right| < 1$

$|x-2|<4$

$-2<x<6$

$a=2$ $R=4$

Ex $\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{3^{2n-1}} = ?$ $9^n \cdot \frac{1}{3}$ $\sum_{n=0}^{\infty} ar^n$ $r = \frac{-2}{9}$ $|r|<1$ $\frac{2}{9}<1$ → yakınsak

$a=3$

geom → $= \frac{a}{1-r} = \frac{3}{1-(-\frac{2}{9})}$

Ex $\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n \cdot 3^{2n-1}}$ geometrik değil.

geometrikte $a =$ $r =$ n'li ifade olamaz

oran testi

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\cancel{(-1)^{n+1}}\, 2^{n+1}}{(n+1)\, 3^{2(n+1)-1}} \cdot \frac{n \cdot 3^{2n-1}}{\cancel{(-1)^n}\, 2^n} \right| = \frac{2}{9} < 1$

seri yakınsaktır.

Ex $\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{\sqrt{n}}$ A.S.T. ile yakınsak

Ex $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{\ln n}}$ sürekli ✓ pozitif ✓ azalan ✓

A.S.T. $u_n = \frac{\ln(n)}{\sqrt{n}}$ pozitif ✓

azalan; $\frac{d}{dn} u_n = \frac{1/n \cdot \sqrt{n} - \ln n / 2\sqrt{n}}{n} \to \frac{2}{2\sqrt{n}} - \frac{\ln n}{2\sqrt{n}}$

$\frac{2-\ln(n)}{n\, 2\sqrt{n}}$ $\to$ azalan ✓ sonlu terimden sonra

$u_n \to 0$ $\lim_{n\to\infty} \frac{\ln n}{\sqrt{n}} = \lim_{n\to\infty} \frac{1/n}{1/2\sqrt{n}} = \frac{2\sqrt{n}}{n}$ $\frac{1}{\sqrt{n}} \to 0$ ✓

$\int \frac{1}{\sqrt{n}}$ $u^{-1/2} \to \frac{u^{1/2}}{1/2}$

$\int_1^{\infty} \frac{1}{x\sqrt{\ln x}} = \lim_{t\to\infty} 2\sqrt{\ln x}\,\Big]_{x=1}^{x=t}$ $u = \ln x$

$= \lim_{t\to\infty} 2\sqrt{\ln t} - 2\sqrt{\ln 1}$

$= \infty$

⇒ seri de ıraksaktır ⇒ ıraksaktır

Ex $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{n\sqrt{\ln n}}$ int. uygulanmaz A.S.T

$u_n = \frac{1}{n\sqrt{\ln n}}$ pozitif ✓ azalan ✓ $u_n \to 0$ ✓ ⇒ A.S.T. yakınsaktır.

converges conditionally.

$f(x)$ → power series representation

Famous well-known Maclaurin Series

interval of conv.

$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$ geometrik $a=1$ $r=x$ $|x|<1$ $R=1$

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ $R=\infty$

$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$ $R=\infty$

$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$ $R=\infty$

$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$ $R=1$

$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ $R=1$

$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$ $R=1$

→ substitution $\frac{1}{1-x}$ $\frac{1}{1+\pi x}$ $x \to -\pi x$

$\frac{1}{2+x}$ → $\frac{1}{2(1+\frac{x}{2})}$ → $x \to -\frac{x}{2}$ $\frac{1}{2}$

Differentiation / Integration $\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$ $|x|<1$

$\frac{1}{\ldots} = \sum \frac{1}{\ldots}\left(-\frac{x}{\ldots}\right)^n$ $\frac{1}{\ldots} = \sum^{\infty} (-x)^n = \sum^{\infty} (-1)^n x^n$ integ. $\arctan(x) = \sum^{\infty} (-1)^n \frac{x^{2n+1}}{\ldots}$ $|x|<1$

$$\frac{1}{2+x} = \sum_{n=0}^{\infty} \frac{1}{2}\left(-\frac{x}{2}\right)^n \qquad |x|<2$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n \qquad |x|<1$$

integral

$$\text{inty} \quad \arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \qquad |x|<1$$

$$\frac{1}{1-x} \xrightarrow{x \to -x^2} \frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

Taylor / Maclaurin

a merkezli $\qquad a=0$

$f(x) = \sim \qquad f(a) = \sim$

$f'(x) = \sim \qquad f'(a) = \sim$

$f''(x) = \sim \qquad f''(a) = \sim$

$\vdots$

$f^{(n)}(a) = n$ cinsinden

$$\boxed{c_n = \frac{f^{(n)}(a)}{n!}} \rightarrow \sum_{n=0}^{\infty} c_n (x-a)^n$$

$$f(x) = e^x \rightarrow \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad a=0$$

Ex/ $f(x) = x^2 . e^{x^2} \rightarrow e^x \xrightarrow[x^2]{x \text{ yerine}} e^{x^2} \xrightarrow{} x^2 e^{x^2}$

tüm terimleri x^2 ile çarp

$$e^{x^2} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} \qquad x^2 e^{x^2} = \sum_{n=0}^{\infty} \frac{x^{2n+2}}{n!}$$

4. Find the Taylor series for f centered at 4 if

$$f^{(n)}(4) = \frac{(-1)^n \cancel{n!}}{3^n(n+1)} \qquad a=4$$

$$c_n = \frac{f^{(n)}(4)}{\cancel{n!}} \qquad \sum_{n=0}^{\infty} c_n (x-4)^n$$

What is the radius of convergence of the Taylor series?

radius of convergence = 3

Interval of convergence $\Rightarrow 1 < x \leq 7 \qquad (1,7]$

absolute convergence? $\Rightarrow 1 < x < 7$

Conditional convergence $\Rightarrow x=7$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{3^n(n+1)} (x-4)^n$$

geometrik değil.

→ oran testi

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\cancel{(-1)^{n+1}} \, \cancel{(x-4)^{n+1}}}{\cancel{3^{n+1}} \, (n+1+1)} \cdot \frac{\cancel{3^n} (n+1)}{\cancel{(-1)^n} \, \cancel{(x-4)^n}} \right| = \lim_{n\to\infty} \frac{n+1}{3(n+2)} |x-4| = \frac{|x-4|}{3} < 1$$

ise converges

$$|x-4| < 3 \qquad 1 < x < 7$$

center $a=4$

$x=1$: ? divergent $\qquad x=7$: ? convergent

$x=1$:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{3^n(n+1)} (1-4)^n = \sum_{n=0}^{\infty} \frac{1}{n+1} \qquad b_n = \frac{1}{n} \rightarrow \text{harmonic divergent.}$$

~~D.C.T.~~

$$\text{L.C.T.}: \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{n+1} \cdot \frac{n}{1} = 1 > 0$$

series diverges

$x=7$:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{3^n(n+1)} (7-4)^n = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n+1}$$

$u_n = \frac{1}{n+1}$ → positive ✓, decreasing ✓, $u_n \to 0$ ✓ ⇒ A.S.T series converges.

abs. value serisi diverges → conditionally convergent.

Ex/ Find the Maclaurin series for $f(x) = \sin(x)$. $\qquad$ $a=0$'daki Taylor Serisi

$$c_n = \frac{f^{(n)}(0)}{n!}$$

$f(x) = \sin(x) \qquad f(0) = \sin 0 = 0 \qquad n=0 \qquad c_0 = \frac{0}{0!}$

$f'(x) = \cos(x) \qquad f'(0) = \cos 0 = 1 \qquad n=1 \qquad c_1 = \frac{1}{1!}$

$f(x) = \sin(x)$ $\quad f(0) = \sin 0 = 0$ $\quad n=0$ $\quad c_0 = \frac{}{0!}$ $\quad c_n = \frac{}{n!}$

$f'(x) = \cos(x)$ $\quad f'(0) = \cos 0 = 1$ $\quad n=1$ $\quad c_1 = \frac{1}{1!}$

$f''(x) = -\sin(x)$ $\quad f''(0) = -\sin 0 = 0$ $\quad n=2$ $\quad c_2 = \frac{0}{2!}$ $\quad \sum_{n=0}^{\infty} c_n x^n$

$f'''(x) = -\cos(x)$ $\quad f'''(0) = -\cos 0 = -1$ $\quad n=3$ $\quad c_3 = \frac{-1}{3!}$

$f^{(4)}(0)$

$$c_n = \begin{cases} 0 & n \text{ even} \\ 1 \\ -1 \end{cases} \Big\} n \text{ odd}$$

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots + c_n x^n + \dots$$

$$= \cancel{0} + x + \cancel{0x^2} + \frac{-1}{3!} x^3 + \cancel{0x^4} + \frac{1}{5!} x^5 + \cancel{0x^6} - \frac{1}{7!} x^7 + \cancel{0x^8} \dots$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = \sin(x)$$

Ratio test

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{2(n+1)+1}}{(2(n+1)+1)!} \cdot \frac{(2n+1)!}{x^{2n+1}} \right|$$

$2n+3$, x^2, $2n+2+1$

$(2n+3)(2n+2)(\cancel{2n+1})!$

$$= \lim_{n\to\infty} \frac{1}{(2n+3)(2n+2)} |x^2| = 0 < 1 \quad \text{converges for all } x.$$

$\to 0$

$R = \infty$

$f(x) = \cos(x)$ $\quad$ term by term differentiation for $f(x) = \sin(x)$:

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{(\cancel{2n+1})\, x^{2n}}{(\cancel{2n+1})!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \qquad R = \infty$$

Ex

$$\int \frac{\cos(x) - 1}{x}\, dx = ?$$

$$= \int \frac{\cos(x)}{x}\, dx - \int \frac{1}{x}\, dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \cdot \frac{x^{2n}}{2n} - \ln|x| + C$$

$\ln|x|$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \qquad \frac{\cos(x)}{x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n-1}}{(2n)!}$$

integrate

Ex Find Maclaurin Series for $f(x) = (1+x)^k$ $\quad k \in \mathbb{R}$

$a = 0$

$f(x) = (1+x)^k$ $\quad f(0) = 1$ $\quad n=0$ $\quad c_n = \frac{f^{(n)}(0)}{n!}$

$f'(x) = k(1+x)^{k-1}$ $\quad f'(0) = k.1$ $\quad n=1$

$f''(x) = k(k-1)(1+x)^{k-2}$ $\quad f''(0) = k.(k-1).1$ $\quad n=2$ $\quad \sum_{n=0}^{\infty} c_n x^n$

$f'''(x) = k(k-1)(k-2)(1+x)^{k-3}$ $\quad f'''(0) = k.(k-1)(k-2)$ $\quad n=3$

$n=4$

$$f^{(n)}(0) = k(k-1)(k-2)\dots(k-(n-1)) \qquad n=n$$

$$c_n = \frac{k(k-1)(k-2)\dots(k-(n-1))}{n!} \qquad \boxed{\sum_{n=0}^{\infty} \frac{k(k-1)(k-2)\dots(k-(n-1))}{n!} x^n = (1+x)^k}$$

$$c_n = \frac{k(k-1)(k-2)\cdots(k-(n-1))}{n!}$$

$$\boxed{\sum_{n=0} \frac{k(k-1)(k-2)\cdots(k-(n-1))}{n!} x^n = (1+x)}$$

Prev. $k=2$

$(1+x)^2 = 1+2x+x^2$

$k=3$ $(1+x)^3 = 1+3x+3x^2+x^3$

$\Bigg\}$ $k>0$ $\Rightarrow$ finite binomial expansion

Ex $k=3$: $(1+x)^3 = \sum_{n=0}^{\infty} \frac{k(k-1)\cdots(k-(n-1))}{n!} x^n$ $\binom{k}{n}$

$$= 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \frac{k(k-1)(k-2)\cancel{(k-3)}}{4!}x^4 + \cancel{\frac{k(k-1)(k-2)(k-3)(k-4)}{5!}x^5} + \cdots$$

$k=3$ 0, 0, 0, $+0+$

$$= 1 + 3x + 3x^2 + x^3 + 0 + 0 + \cdots + 0 \cdots$$

$\Rightarrow$ $\underline{\underline{k<0}}$ $\rightarrow$ $(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n$ $\rightarrow$ $\frac{k!}{(k-n)!\,n!} = \frac{k(k-1)\cdots(k-(n-1))\cancel{(k-n)!}}{\cancel{(k-n)!}\,n!}$

$\frac{1}{(1+x)^2} = (1+x)^{-2}$ $k=-2$

$\frac{1}{\sqrt{1+x}} = (1+x)^{-1/2}$ $k=-1/2$

Ex $\frac{1}{\sqrt{1+4x}}$ $k=-\frac{1}{2}$ $\rightarrow \sum_{n=0}^{\infty} \binom{-1/2}{n}(4x)^n$

x yerine 4x

interval of conv.

$$\sum_{n=0}^{\infty} \frac{k(k-1)(k-2)\cdots(k-(n-1))}{n!} x^n \rightarrow \text{ratio test}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\cancel{k(k-1)(k-2)}\cdots(k-(n+1-1))\,\cancel{x^{n+1}}}{\cancel{(n+1)!}\,(n+1)} \cdot \frac{\cancel{n!}}{\cancel{k(k-1)(k-2)\cdots(k-(n-1))}\,\cancel{x^n}}\right|$$

$k-(n-1)$, $k-n$, x

$$= \lim_{n\to\infty}\left|\frac{(k-n)}{(n+1)}x\right| = \underline{\underline{|x| < 1}} \text{ ise converges}$$

-1

interval of conv. $|x|<1$

$\underline{-1<x<1}$