11th Week Thursday

4 Mayıs 2023 Perşembe 11:29

Ex/ $2.3\overline{17}$ → Express this number as a rational number using geometric series.

$$2.3\overline{17} = 2.3171717\ldots = 2.3 + 0.017 + 0.00017 + 0.0000017 + \ldots$$

17.10^{-3}, 17.10^{-5}, 17.10^{-7}

$\frac{\cancel{17}.10^{-5}}{\cancel{17}.10^{-3}} = 10^{-2}$ $\frac{17.10^{-7}}{17.10^{-5}} = 10^{-2}$

Common Ratio $r = 10^{-2} < 1$

⇒ series converges.

$$= \frac{a}{1-r} = \frac{17.10^{-3}}{1-10^{-2}} = \frac{0.017}{99} \times 100 = \frac{17}{990}$$

$$= 2.3 + \frac{17}{990} = \frac{23 \times 99 + 17}{990}$$

Ex/ $3 - 4 + \frac{16}{3} - \frac{64}{9} + \ldots = ?$ $\sum_{n=1}^{\infty} ar^{n-1}$

$\frac{-4}{3}$ $\frac{-4}{3}$ $\frac{16}{3.4}$ $\frac{-64}{9} \cdot \frac{\cancel{3}}{\cancel{16}} = \frac{-4}{3}$

$r = -\frac{4}{3}$ → common ratio.

$a = 3$ $|r| > 1$ ⇒ series diverges.

Ex/ $10 - 2 + 0.4 - 0.08 + \ldots = ?$ $\sum_{n=1}^{\infty} ar^{n-1}$

$\frac{-2}{10}$ $\frac{0.4}{-2} = -0.2$ $\frac{-0.08}{0.4} = -0.2$

$r = -0.2$ → common ratio $|r| < 1$

$a = 10$

series converges ⇒ $= \frac{a}{1-r} = \frac{10}{1 + \frac{2}{10}} = \frac{100}{12}$

• For convergent series: $\sum_{n=1}^{\infty} a_n = A$ (convergent) $\sum_{n=1}^{\infty} b_n = B$ (convergent)

$$\sum_{n=1}^{\infty} a_n \pm b_n = A \pm B \qquad \sum_{n=1}^{\infty} k.a_n = k.A \quad k \in \mathbb{R}$$

If one of the series diverges the result also diverges.

Ex $\sum_{n=1}^{\infty}\left(\frac{3}{n(n+1)}+\frac{1}{2^{2n-2}}\right)=?$

$$\sum_{n=1}^{\infty}\frac{3}{n(n+1)}+\sum_{n=1}^{\infty}\frac{1}{2^{2n+4}}=3+\frac{1}{48}$$

telescopic $=3$ Conv.

geometric $\sum_{n=1}^{\infty} ar^{n-1}$

$$\sum_{n=1}^{\infty}\frac{1}{2^{2n+4}}=\sum_{n=1}^{\infty}\frac{1}{4^{n+2}}=\sum_{n=1}^{\infty}\frac{1}{4^3}\left(\frac{1}{4}\right)^{n-1}$$

$a=\frac{1}{64}$ $r=\frac{1}{4}$

$r=\frac{1}{4}$ $|r|<1 \Rightarrow$ geometric series converges.

$$=\frac{a}{1-r}=\frac{1}{64}\cdot\frac{4}{3}=\frac{1}{48}$$

$$\sum_{n=1}^{\infty}\frac{3}{n(n+1)}=$$

partial fraction $\to$ $a_n=\frac{3}{n(n+1)}=\frac{3}{n+1\,n}-\frac{3}{}$

$s_1=a_1$

$s_2=a_1+a_2$

$\vdots$

$s_n=a_1+a_2+\dots+a_n=\left(\frac{3}{1}-\cancel{\frac{3}{2}}\right)+\left(\cancel{\frac{3}{2}}-\cancel{\frac{3}{3}}\right)+\left(\cancel{\frac{3}{3}}-\cancel{\frac{3}{4}}\right)+\cancel{\dots}+\left(\cancel{\frac{3}{n}}-\frac{3}{n+1}\right)=3-\frac{3}{n+1}$

∞ $\lim_{n\to\infty} s_n=?$ $\to$ $\lim_{n\to\infty} 3-\frac{3}{n+1}\to 0 =3$

Harmonic Series

: $\sum_{n=1}^{\infty}\frac{1}{n}\to$ diverges.

Ex $\sum_{n=1}^{\infty}\frac{1}{2n}\rightsquigarrow$ diverges.

Test for Divergence: (n^{th} Term Test)

$p\Rightarrow q$ ✓

$q'\Rightarrow p'$ ✓

$q\Rightarrow p$???

Let $\sum_{n=1}^{\infty} a_n=$ converges $\to$ biber

Biber se $\Rightarrow$ acıdır. ✓

Acı değilse $\Rightarrow$ Biber değildir ✓

acı $\Rightarrow$ biber ?

olabilir de olmayabilir de.

$$\Rightarrow \sum_{n=1}^{\infty} a_n=\lim_{n\to\infty} s_n=L$$

$$\Rightarrow \lim_{n\to\infty} a_n=\lim_{n\to\infty}(s_n-s_{n-1})=\underbrace{\lim_{n\to\infty} s_n}_{L}-\underbrace{\lim_{n\to\infty} s_{n-1}}_{L}=L-L=0$$

s_1

s_2

s_{n-1}

s_n lim

$$\left.\begin{aligned}s_n&=a_1+a_2+\dots+a_n\\ s_{n-1}&=a_1+a_2+\dots+a_{n-1}\end{aligned}\right)\ s_n-s_{n-1}=a_n$$

$\Rightarrow \lim_{n\to\infty} a_n=0$ ← acı

n^{th} term test

Test for Divergence: $\boxed{\lim_{n\to\infty} a_n\neq 0 \Rightarrow \sum_{n=1}^{\infty} a_n \text{ diverges}}$ ✓

If $\lim\limits_{n\to\infty} a_n = 0 \Rightarrow \sum\limits_{n=1}^{\infty} a_n$ may be divergent or convergent

Ex/ $\sum\limits_{n=1}^{\infty} 3^{n^2+n+1}$ (a_n) $\quad \lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} 3^{n^2+n+1} = \infty \neq 0 \Rightarrow$ $\boxed{\text{By test for div.}}$ $\Rightarrow$ series diverges.

Ex/ $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ → harmonic series $\quad \lim\limits_{n\to\infty} \frac{1}{n} = 0$ but we know that $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ → diverges

Ex/ $\sum\limits_{n=1}^{\infty} \frac{1}{3^n}$ $\quad \lim\limits_{n\to\infty} \frac{1}{3^n} = 0$ ($3^n \to \infty$) but we know $\sum\limits_{n=1}^{\infty} \frac{1}{3^n}$ → geometric series $r = \frac{1}{3}$ $|r|<1$ → converges.

Ex/ $\sum\limits_{n=1}^{\infty} n^3$ $\quad \lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} n^3 = \infty \neq 0 \Rightarrow$ Series diverges by test for div.

Ex/ $\sum\limits_{n=1}^{\infty} e^{n^2+n}$ $\quad \lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} e^{n^2+n} \neq 0 \Rightarrow$ series diverges by test for div.

Ex/ $\sum (-1)^n \ldots$ → alternating series we'll come back here

Ex/ $\sum\limits_{n=1}^{\infty} \frac{n^2+n}{n-1}$? $\quad \lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{n^2+n}{n-1} = \infty \neq 0 \Rightarrow$ series diverges by test for div.

Ex/ $\sum\limits_{n=1}^{\infty} \frac{p(n)}{q(n)}$ ($p(n)$ → pol., $q(n)$ → pol.) $\quad \deg(p(n)) \geq \deg(q(n)) \Rightarrow$ series diverges by test for div.

$\lim\limits_{n\to\infty} \frac{p(n)}{q(n)} \neq 0$

Ex/ $\sum\limits_{n=1}^{\infty} \frac{-n^2+3n-1}{5n^2+1}$ $\quad \lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{-n^2+3n-1}{5n^2+1} = -\frac{1}{5} \neq 0 \Rightarrow$ series diverges by test for div.

Integral Test

∞ main = $[1,\infty)$ ... ∞ evaluate the

$$\sum_{n=1}^{\infty} a_n \qquad \begin{matrix} a_n = f(n) & \text{Domain} = [1,\infty) \\ \text{continuous} ✓ \\ \text{positive} ✓ \\ \text{decreasing} ✓ \end{matrix} \Bigg\} \Rightarrow \int_1^{\infty} f(x)\,dx \to \text{evaluate the improper integral}$$

if this integral diverges $\Rightarrow \sum_{n=1}^{\infty} a_n$ also diverges.

if this intergal converges $\Rightarrow \sum_{n=1}^{\infty} a_n$ also converges.

($\int_1^{\infty} f(n)\,dn = m$ has nothing to do with the value of the series.)

Ex $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ $\qquad f(n) = \dfrac{1}{n^2}$ $\quad$ continuous ✓ $[1,\infty)$, positive ✓, decreasing ✓ $\quad a_{n+1} = \dfrac{1}{(n+1)^2} < a_n = \dfrac{1}{n^2}$

p-integral

$$\int_1^{\infty} \frac{1}{x^2}\,dx \to \text{converges.} \qquad p = 2 > 1$$

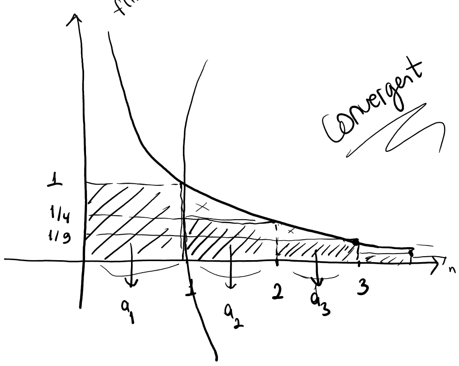
By the integral test $\Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ also converges.

$$\boxed{\text{p-integral ! } \int_1^{\infty} \frac{1}{x^p}\,dx = \begin{cases} \text{divergent}, & \text{if } p < 1 \\ \text{divergent}, & \text{if } p = 1 \\ \text{convergent}, & \text{if } p > 1 \end{cases}}$$

$\dfrac{n^{-1}}{-1} \quad -\dfrac{1}{n} \qquad n^{-2}$

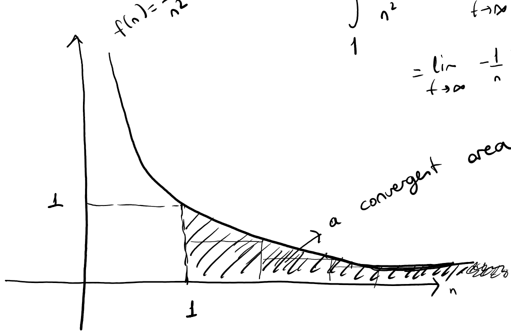
$$\int_1^{\infty} \frac{1}{n^2}\,dn = \lim_{t\to\infty} \int_1^{t} \frac{1}{n^2}\,dn = \lim_{t\to\infty} -\frac{1}{n}\Big]_{n=1}^{n=t} = \lim_{t\to\infty} \left(-\frac{1}{t}\right) + \frac{1}{1} = 1$$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ $\quad [1,\infty)$



convergent

$<$



a convergent area.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \underbrace{\frac{1}{1}}_{a_1} + \underbrace{\frac{1}{4}}_{a_2} + \underbrace{\frac{1}{9}}_{a_3} + \cdots = \text{sum of these rectangles} \Rightarrow \text{convergent}$$

p-series

$$\boxed{! \sum_{n=1}^{\infty} \frac{1}{n^p} = \begin{cases} \text{divergent}, & \text{if } p < 1 \\ \text{divergent}, & \text{if } p = 1 \\ \text{convergent}, & \text{if } p > 1 \end{cases}} \to \sum_{n=1}^{\infty} \frac{1}{n} \to \text{harmonic series} \quad \text{diverges}$$

Ex $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ conv. / div ? $\qquad f(n) = \dfrac{1}{n^2+1}$ $\quad$ continuous ✓ positive ✓ decreasing ? $\quad a_{n+1} = \dfrac{1}{(n+1)^2+1} < a_n = \dfrac{1}{n^2+1}$ ✓

we may use integral test

$$\int_1^\infty \frac{1}{x^2+1}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^2+1}\,dx = \lim_{t\to\infty} \arctan(x)\Big]_{x=1}^{x=t}$$

$$= \lim_{t\to\infty} \arctan(t) - \arctan(1) = \frac{\pi}{4}$$

$\pi/2$ $\pi/4$

$\Rightarrow$ converges.

By the integral test, $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ also converges.

Ex

$$\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$$ conv / div ?

continuous ✓
positive ✓
decreasing ✓

$\ln(n)$, 1, $e = 2.717...$

$f'(n) < 0$?

$$f'(n) = \frac{1/n \cdot n - \ln(n) \cdot 1}{n^2} = \frac{1 - \ln(n)}{n^2} < 0$$

$n \geq 3 \Rightarrow \ln(n) > 1$ ✓

$$\int_1^\infty \frac{\ln(x)}{x}\,dx = \lim_{t\to\infty}\left(\frac{(\ln t)^2}{2} - \frac{\ln 1}{2}\right) = \infty \Rightarrow \text{diverges.}$$

$u = \ln(x)$
$du = \frac{1}{x}dx$

By the integral test, $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$ also diverges.

Series-Wrapped

4 Mayıs 2023 Perşembe
09:27

$a_n \to$ sequence $\sum_{n=1}^{\infty} a_n \to$ series $\Rightarrow$ $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n$ → kısmi toplamlar dizi (partial sums) —if exists→ seri convergent $= L$; —if ∞, DNE→ divergent

✓* **Geometric Series:** $\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \dots \Rightarrow$ conv., if $|r| < 1 \rightarrow \sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$
div., if $|r| \geq 1$

$\sum_{n=0}^{\infty} ar^n$ ✓ $\sum_{n=2}^{\infty} ar^{n-2}$ ✓ $\sum_{n=b} ar^{n-b}$ ✓ $a = \dots$, $r = \dots$

✓* **Telescopic Series:** $s_1, s_2, \dots s_n \dots \rightarrow s_n$ includes terms that can be cancelled. (if like $a_n = \frac{1}{n(n+1)} \to$ factorize and use partial fractions)

✓* **Harmonic Seri:** $\sum_{n=1}^{\infty} \frac{1}{n} \rightarrow$ diverges.

✓* **Test for Divergence (n^{th} term test):** $\lim_{n\to\infty} a_n \neq 0 \Rightarrow \sum_{n=1}^{\infty} a_n$ diverges.

✓* **Integral Test:** $a_n = f(n)$: on $[1,\infty)$ continuous ✓ positive ✓ decreasing ✓ (check!) if → the improper integral $\int_1^\infty f(x)dx$ → converges $\Rightarrow \sum_{n=1}^{\infty} a_n$ also converges; → diverges $\Rightarrow \sum_{n=1}^{\infty} a_n$ also diverges.

* finitely many terms are not important.

* **p-series:** $\sum_{n=1}^{\infty} \frac{1}{n^p} \rightarrow$ diverges $(p \leq 1)$; converges $(p > 1)$