6th Week Thursday

30 Mart 2023 Perşembe 11:49

Ex Find the cartesian eqn of the parametric curve

$x = \tan t + \sec t$

$y = \tan t - \sec t$

$1 + \tan^2 t = \sec^2 t \rightarrow 1 = \sec^2 t - \tan^2 t$

$xy = (\tan t + \sec t)(\tan t - \sec t) = \tan^2 t - \sec^2 t = -1$ $\qquad xy = -1$ $\qquad \boxed{y = -1/x}$

Ex $x = \cos t$

$y = \cos 2t$

$\cos^2 t = \dfrac{1 + \cos 2t}{2}$

$x^2 = \dfrac{1+y}{2}$ $\qquad \boxed{y = 2x^2 - 1}$

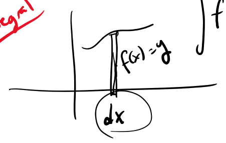
Calculus with Parametric Curves

$y = f(t)$, $x = g(t)$

Derivative

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \qquad \frac{d^2y}{(dx)^2} = \frac{d/dt\left(\frac{dy}{dx}\right)}{dx/dt}$$

Integral



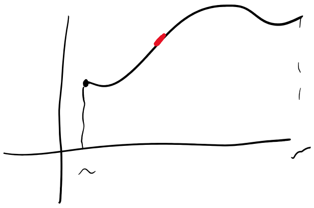
$\int f(x)\, dx$, $f(x) \to y$

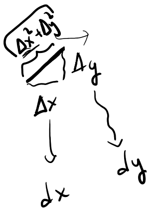
$$\int_{t=a}^{t=b} f(t)\, g'(t)\, dt \qquad y = f(t), \; x = g(t)$$

$x = g(t)$

$dx = g'(t)\, dt$

(Parametric) Arc Length





$y = f(t)$

$x = g(t)$

$dx = g'(t)\, dt$

$dy = f'(t)\, dt$

$$\boxed{\int_{t=a}^{t=b} \sqrt{f'(t)^2 + g'(t)^2}\; dt}$$

Ex $x = r\cos t$

$y = r\sin t$

$0 \leqslant t \leqslant 2\pi$

Find the arc length of the parametric curve.

$$\int_{t=0}^{t=2\pi} \sqrt{\underbrace{r^2\sin^2 t + r^2\cos^2 t}_{r^2(\underbrace{\sin^2 t + \cos^2 t}_{1})}}\; dt = \int_{\cancel{t=0}}^{\cancel{t=2\pi}} \underset{\text{sabit}}{r}\, \underset{\text{değişken}}{dt} = rt\Big]_{t=0}^{t=2\pi}$$

$= r \cdot 2\pi - r \cdot 0$

$= 2\pi r$

$dx = -r\sin t\, dt$

$dy = r\cos t\, dt$

$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$

$x^2 + y^2 = r^2$

(a,b) merkezli

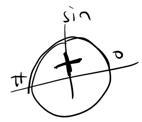
$(x-a)^2 + (y-b)^2 = r^2$

$x^2+y^2=r^2$

$(x-a)^2+(y-b)^2=r^2$

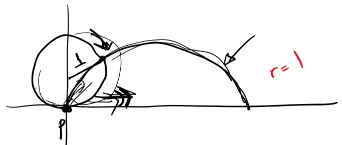
$x = a + r\cos t \qquad dx = -r\sin t\, dt$

$y = b + r\sin t \qquad dy = r\cos t\, dt$



$0 < t < 2\pi$

$0 < t/2 < \pi$



Ex

$x = rt - r\sin t$

$y = (1+\cos t)r$

$0 \le t \le 2\pi$

$$\int_{t=0}^{t=2\pi} \sqrt{(1-\cos t)^2+\sin^2 t}\; dt = \int_{t=0}^{t=2\pi} \sqrt{4\sin^2(t/2)}\; dt$$

$1 - 2\cos t + \underbrace{\cos^2 t + \sin^2 t}_{1} = 2 - 2\cos t = 2(1-\cos t)$

$2\sin^2(t/2)$

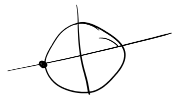
$dx = (1 - r\cos t)\,dt$

$dy = -r\sin t\, dt$

$\cos^2(t/2) = \dfrac{1+\cos t}{2}$

$\sin^2(t/2) = \dfrac{1-\cos t}{2}$

$2\sin^2(t/2)$



$$= \int_{t=0}^{t=2\pi} 2\sin(t/2)\, dt = 2\, \frac{-\cos(t/2)}{1/2}\Bigg]_{t=0}^{t=2\pi} = -4\underbrace{\cos\pi}_{-1} - (-4\cos 0)$$

$= 4 + 4 = 8$

5. Determine the arc length of the path $x(t) = e^t + e^{-t}$, $y(t) = 5 - 2t$, $0 \le t \le 4$.

$$\int_{t=0}^{t=4} \sqrt{(e^t - e^{-t})^2 + 4}\; dt = \int_{t=0}^{t=4} \left(e^t + \frac{1}{e^t}\right) dt = \left(e^t - e^{-t}\right)\Big]_{t=0}^{t=4} = e^4 - e^{-4} - (e^0 - e^{-0})$$

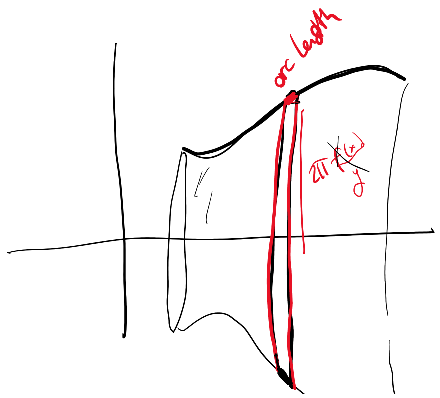
$$\underbrace{e^{2t} - 2e^t\cdot\frac{1}{e^t} + \frac{1}{e^{2t}}}_{\left(e^t - \frac{1}{e^t}\right)^2} + 4 = \underbrace{e^{2t} + 2 + \frac{1}{e^{2t}}}_{\left(e^t + \frac{1}{e^t}\right)^2}$$

$x = e^t + e^{-t} \qquad y = 5 - 2t$

$dx = e^t - e^{-t}\; dt \qquad dy = -2\,dt$

$2e^t \frac{1}{e^t}$

Surface Area

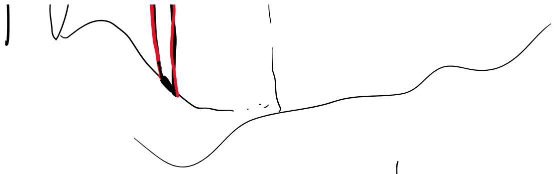


$\int 2\pi y$ arclength

$y = h(t)$

$x = g(t)$

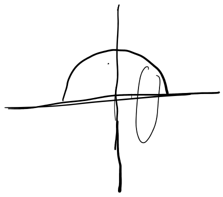
$$\boxed{\int_{t=a}^{t=b} 2\pi\, h(t) \sqrt{h'(t)^2 + g'(t)^2}\; dt}$$



$$t=a$$

$4\pi r^2$

EX



$x = r\cos t$
$y = r\sin t$
$0 < t < \pi$

surface area = ?

$$\int_{t=0}^{t=\pi} 2\pi\, r\sin t \underbrace{\sqrt{r^2\sin^2 t + r^2\cos^2 t}}_{r}\, dt = \int_{t=0}^{t=\pi} 2\pi r^2 \sin t\, dt = 2\pi r^2 \int_{t=0}^{t=\pi} \sin t\, dt$$

$dx = -r\sin t\, dt$

$dy = r\cos t\, dt$

$$= 2\pi r^2 \left(-\cos t \Big|_{t=0}^{t=\pi} \right) = 2\pi r^2 . 2 = 4\pi r^2$$

$$\to -\cos\pi - (-\cos 0) \quad 1+1$$

61–63 Find the exact area of the surface obtained by rotating the given curve about the x-axis.

61. $x = t^3, \quad y = t^2, \quad 0 \leqslant t \leqslant 1$

62. $x = 3t - t^3, \quad y = 3t^2, \quad 0 \leqslant t \leqslant 1$

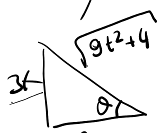
63. $x = a\cos^3\theta, \quad y = a\sin^3\theta, \quad 0 \leqslant \theta \leqslant \pi/2$

61) $x = t^3$
$y = t^2$

$dx = 3t^2\, dt$
$dy = 2t\, dt$

$$\int_{t=0}^{t=1} 2\pi t^2 \sqrt{9t^4 + 4t^2}\, dt = \frac{2\pi . 8}{27} \left(\left(\frac{\sqrt{9t^2+4}}{3t} \right)^5 \Big/ 5 - \ldots \right)\Big|_{t=0}^{t=1}$$

$$t^3 \sqrt{\underbrace{9t^2+4}_{4(\tan^2\theta+1)}}\, dt$$



$3t = 2\tan\theta$
$3dt = 2\sec^2\theta\, d\theta$

$$\frac{8}{27} \tan^3\theta\, 2\sec\theta\, 2\sec^2\theta\, d\theta$$

$$\tan^3\theta \sec^3\theta\, d\theta = (\sec^2\theta - 1)\sec^2\theta \sec\theta\tan\theta\, d\theta$$

$\tan^2\theta = \sec^2\theta - 1$

$u = \sec\theta$
$du = \sec\theta\tan\theta\, d\theta$

$(u^2-1)u^2\, du$
$u^4 - u^2$

$$\frac{u^5}{5} - \frac{u^3}{3} \qquad \frac{\sec^5\theta}{5} - \frac{\sec^3\theta}{3}$$