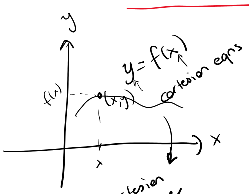
6th Week Tuesday

28 Mart 2023 Salı 13:49

Parametric Equations and Parametric Curves



derivatives ~ tangent lines
integral
arc length, surface area

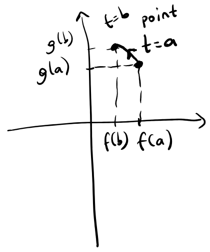
$$\boxed{\begin{array}{l} x = f(t) \\ y = g(t) \end{array}}$$

$a < t < c$

initial point — terminal point

t : parameter

t	x	y
$t=a$	$f(a)$	$g(a)$
$t=b$	$f(b)$	$g(b)$



! Finding cartesian equation $y=f(x)$ → by eliminating the parameter t

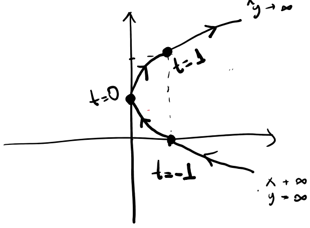
Ex Sketch and Find the cartesian equation corresponding to the curve parametrized as;

$$\boxed{\begin{array}{l} x = t^2 \\ y = t+1 \\ -\infty < t < \infty \end{array}}$$

$t \to -\infty$

$t \to \infty$

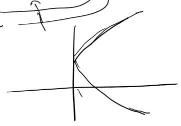
t	x	y
$t=-1$	$t^2=1$	$t+1=0$
$t=0$	$t^2=0$	$t+1=1$
$t=1$	$t^2=1$	$t+1=2$



$$\left\{\begin{array}{l} x = t^2 \\ y = t+1 \end{array}\right. \qquad \left.\begin{array}{l} \sqrt{x} = t \\ y-1 = t \end{array}\right\} \Rightarrow \boxed{\sqrt{x} = y-1}$$

cartesian eqn.

$y = \sqrt{x}+1$



Ex

$x = t^2 - t$

$y = t+1$

$-\infty < t < \infty$

$t = y-1$

$$\underbrace{t^2 - t + 1/4}_{(t-1/2)^2} - 1/4$$

$$t^2 - 2 \cdot t \cdot \frac{1}{2} + \frac{1}{4}$$

$$x = (t - 1/2)^2 - 1/4$$

$$\sqrt{x + 1/4} + 1/2 = t$$

$$\boxed{y - 1 = \sqrt{x + 1/4} + 1/2} \to \text{cartesian eqn.}$$

Ex

$$\boxed{\begin{array}{l} x = 2\cos t \\ y = 2\sin t \\ 0 \leq t \leq 2\pi \end{array}}$$

$$\frac{x}{2} = \cos t$$

$$\frac{y}{2} = \sin t$$

$$\sin^2 t + \cos^2 t = 1$$

$$\boxed{\frac{x^2}{4} + \frac{y^2}{4} = 1}$$

cartesian eqn.

Ex

$$\boxed{\begin{array}{l} x = 3\sin t \cos t \\ y = 4\sin 2t \end{array}}$$

$< t <$

$$\sin 2t = 2\sin t \cos t$$

$$\boxed{\frac{y}{4} = 2\,\frac{x}{3}}$$

$$\boxed{\begin{array}{l} x = a + r\cos t \\ y = b + r\sin t \\ 0 \leq t \leq 2\pi \end{array}} \to \frac{x-a}{r} = \cos t \qquad \frac{y-b}{r} = \sin t$$

$$\left(\frac{x-a}{r}\right)^2 + \left(\frac{y-b}{r}\right)^2 = 1$$

a circle centered at (a,b) with radius r ← $\boxed{(x-a)^2 + (y-b)^2 = r^2}$

Ex

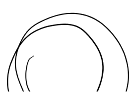
$x = \tan t$

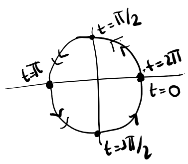
$y = \sec^2 t$

$-\pi/2 < t < \pi/2$

$$1 + \tan^2 t = \sec^2 t$$

$$\boxed{1 + x^2 = y}$$





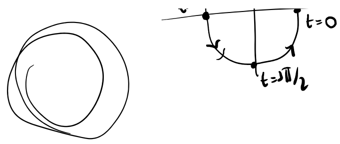
$x = \cos t$

$y = \sin t$

$0 \leq t \leq 4\pi$

$-\pi/2 < t < \pi/2$

$\boxed{1 + x^2 = y}$



$0 \leq t \leq 4\pi$

Ex

$x = e^{2t}$
$y = t^2 + 1$
$-\infty < t < \infty$

$2t = \ln x$
$\sqrt{y-1} = t$

$$\boxed{\frac{\ln x}{2} = \sqrt{y-1}}$$

Derivative for Parametric Curves

tangent line, vertical/horizontal tangents, concavity

$x = f(t)$
$y = g(t)$
$t \in I$

$$\boxed{\frac{dy}{dx} = \frac{dy/dt}{dx/dt}}$$

1st Derivative

$$\boxed{\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt}}$$

2nd Derivative

$$\frac{\frac{d}{dt}(\ )}{dx/dt}$$

3rd Deriv

n^{th} derivative $\rightarrow \dfrac{\frac{d}{dt}\left((n-1)^{th}\right)}{dx/dt}$

$\frac{d}{dx} \rightarrow$ derivative wrt x

y', $f'(x)$, $\frac{d}{dx}y$

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{(dx)^2}$$

$\rightarrow$ 2nd derivative

y'', $f''(x)$

Ex

$x = \sec t$
$y = \tan t$
$-\pi/2 < t < \pi/2$

Find the equation of the tangent line at the point $(\sqrt{2}, 1)$ for the curve parametrized on the left.

slope, point

We should find the 1st derivative at $(\sqrt{2}, 1)$ $\rightarrow$ For which $t = ?$ $x = \sqrt{2}$ $y = 1$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\sec^2 t}{\sec t \tan t} = \frac{\sec t}{\tan t} \rightarrow \text{1st Derivative}$$

$\sec t = \sqrt{2}$
$\tan t = 1$
$-\pi/2 < t < \pi/2$
$\Rightarrow t = \pi/4$

$t = \pi/4$

slope of the tangent line at $t = \pi/4$ $\Rightarrow$ $\dfrac{\sec \pi/4}{\tan \pi/4} = \dfrac{\sqrt{2}}{1} = \sqrt{2}$ ✓

$$\boxed{\sqrt{2} = \frac{y-1}{x-\sqrt{2}}} \rightarrow \text{tangent line eqn at } (\sqrt{2}, 1)$$

!

Horizontal Tangent $\rightarrow$ 1st Derivative $= 0$ $\Leftrightarrow$ $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$, $dy/dt = 0$, $dx/dt \neq 0$

Vertical Tangent $\rightarrow$ 1st Derivative $\rightarrow$ undefined $\Leftrightarrow$ $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$, $dy/dt \neq 0$, $dx/dt = 0$

EXAMPLE 1 A curve C is defined by the parametric equations $x = t^2$, $y = t^3 - 3t$.

(a) Show that C has two tangents at the point $(3, 0)$ and find their equations.

(b) Find the points on C where the tangent is horizontal or vertical.

(c) Determine where the curve is concave upward or downward. $\to$ 2nd derivative

(d) Sketch the curve.

$$\boxed{\begin{array}{l} x = t^2 \\ y = t^3 - 3t \\ -\infty < t < \infty \end{array}}$$

a) $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{3t^2 - 3}{2t}$ $\to$ 1st Derivative

For which t, $x = 3$ $y = 0$? point $(3,0)$

$\left.\begin{array}{l} t^2 = 3 \\ t^3 - 3t = 0 \end{array}\right\}$ ✓ $\Rightarrow t = \pm\sqrt{3}$

$t(t^2-3) = 0$

For $t = \sqrt{3}$, the slope $= \dfrac{3.(\sqrt{3})^2 - 3}{2\sqrt{3}} = \sqrt{3}$ $\to$ 1st tangent line ; $\boxed{\sqrt{3} = \dfrac{y-0}{x-3}}$

For $t = -\sqrt{3}$, the slope $= \dfrac{3(-\sqrt{3})^2 - 3}{2.(-\sqrt{3})} = -\sqrt{3}$ $\to$ 2nd tangent line : $\boxed{-\sqrt{3} = \dfrac{y-0}{x-3}}$

b) $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{3t^2 - 3}{2t}$ $\to$ 1st Derivative

$x = t^2$
$y = t^3 - 3t$

Horizontal tangent (1st Derivative $= 0$) $\to$ $\dfrac{dy}{dt} = 0$ $\dfrac{dx}{dt} \neq 0$

$\left.\begin{array}{l} 3t^2 - 3 = 0 \\ 2t \neq 0 \end{array}\right\}$ $t^2 = 1$ $\Rightarrow$ $t = 1$ and $t = -1$

$(1,-2)$ $(1,2)$

Vertical tangent (1st Derivative undef.) $\to$ $\dfrac{dy}{dt} \neq 0$ $\dfrac{dx}{dt} = 0$

$\left.\begin{array}{l} 3t^2 - 3 \neq 0 \\ 2t = 0 \end{array}\right\}$ $\Rightarrow$ $t = 0$ $\to (0,0)$

c) $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{3t^2 - 3}{2t}$ $\to$ 1st Derivative

$$\frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt} = \frac{\frac{d}{dt}\left(\frac{3t^2-3}{2t}\right)}{2t} = \frac{\frac{3}{2}\left(1 + \frac{1}{t^2}\right)}{2t} \to \text{2nd Derivative}$$

$$\frac{3t^2-3}{2t} = \frac{3}{2}\left(\frac{t^2-1}{t}\right) = \frac{3}{2}\left(t - \frac{1}{t}\right)$$

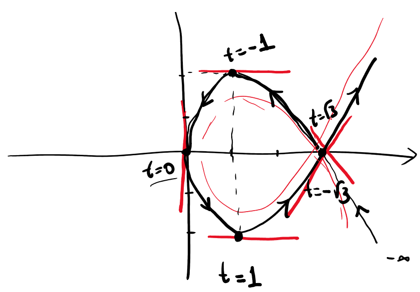
1 $\quad$ $-1/t^2$

✓ conc. up $\leftarrow$ $t > 0 \Rightarrow$ 2nd Deriv. > 0

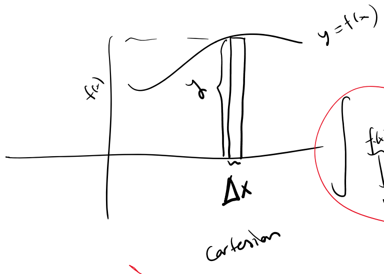
conc. down $\leftarrow$ $t < 0 \Rightarrow$ 2nd Deriv. < 0

d) $$\boxed{\begin{array}{l} x = t^2 \\ y = t^3 - 3t \\ -\infty < t < \infty \end{array}}$$

t	x	y
$-\infty$	$\to \infty$	$\to -\infty$
$-\sqrt{3}$	3	0
-1	1	2
0	0	0
1	1	-2
$\sqrt{3}$	3	0
∞	$\to \infty$	$\to \infty$



Areas under Parametric Curves (integral)



$\int f(x)\, dx$ — $f(x) \to y$, $dx \to$ derivative of x

$x = f(t)$

$y = g(t)$

$$\boxed{\int_{t=\dots}^{t=\dots} \underbrace{g(t)}_{y} \cdot \underbrace{f'(t)\,dt}_{dx}}$$

EXAMPLE 3 Find the area under one arch of the cycloid $0 \leqslant \theta \leqslant 2\pi$

$$x = r(\theta - \sin\theta) \qquad y = r(1 - \cos\theta)$$

$r\theta - r\sin\theta$

$$\int_{t=0}^{t=2\pi} \underbrace{r(1-\cos\theta)}_{y}\,(r - r\cos\theta)\, d\theta = r^2\int_{t=0}^{t=2\pi} (1-\cos\theta)^2\, d\theta$$

$$\frac{dx}{d\theta} = (r - r\cos\theta)$$

$1 - 2\cos\theta + \cos^2\theta$, $\quad \cos^2\theta = \frac{1+\cos 2\theta}{2}$

$$= r^2\left(\theta - 2\sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4}\right)\Big]_{t=0}^{t=2\pi}$$

Ex

$x = 1 - 2t$

$y = t/2 - 1$

$-2 \leqslant t \leqslant 2$

$\frac{dx}{dt} = -2 \qquad dx = -2\,dt$

The area under this parametric curve

$$\int_{t=-2}^{t=2} \underbrace{\left(\frac{t}{2} - 1\right)}_{y} \underbrace{-2\,dx}_{dx} = -\frac{t^2}{2} + 2t \Big]_{t=-2}^{t=2}$$

$-2 - 4 \qquad -2 + 4$

$$= 2 - (-6) = 8$$

Arc length

Surface Area