14. Hafta Salı Dersi

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Örn/ Binom Serisi ($f(x)=(1+x)^k$, $k\in\mathbb{R}$ 'nin MacLaurin Serisi)

$\underline{f(x)=(1+x)^k}$ fonksiyonunun $(k\in\mathbb{R})$ $\underset{a=0}{\underline{\text{MacLaurin}}}$ Serisini bulunuz.

Taylor: $f(x)=\sum_{n=0}^{\infty} c_n (x-a)^n \longrightarrow c_n=\dfrac{f^{(n)}(a)}{n!}$

MacLaurin: $f(x)=\sum_{n=0}^{\infty} c_n x^n \longrightarrow c_n=\dfrac{f^{(n)}(0)}{n!}$

$x=0$

$f(x)=(\underbrace{1+x}_{})^k$ $\qquad f(0)=1$

$f'(x)=k(\underbrace{1+x}_{1})^{k-1}$ $\qquad f'(0)=k.1$

$f''(x)=k(k-1)(\underbrace{1+x}_{1})^{k-2}$ $\qquad f''(0)=k(k-1)$

$f'''(x)=k(k-1)(k-2)(\underbrace{1+x}_{1})^{k-3}$ $\qquad f'''(0)=k(k-1)(k-2)$

$\vdots$

n: $k-(n-1)=k-n+1$

$f^{(n)}(x)=k(k-1)(k-2)\ldots\ldots(\underbrace{k-n+1}_{k-(n-1)})(1+x)^{k-n} \xrightarrow{x=0} 1$ $\qquad f^{(n)}(0)=k(k-1)(k-2)\ldots(k-n+1)$

$\vdots$

$c_n=\dfrac{f^{(n)}(0)}{n!}$

$$\sum_{n=0}^{\infty} c_n x^n=\sum_{n=0}^{\infty}\frac{k(k-1)(k-2)\ldots(k-n+1)}{n!}x^n \longrightarrow \text{Binom Serisi}$$

$(n+1)$. terim

$k-n$

$$\binom{k}{n}=\frac{k!}{(k-n)!\,n!}=\frac{k(k-1)\ldots(k-n+1)\cancel{(k-n)!}}{\cancel{(k-n)!}\,n!}=c_n$$

Eğer k pozitif ise $\Rightarrow$ Seri sonlu olur.

Eğer k negatif ise; (Oran testi)

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{\cancel{k(k-1)(k-2)\ldots(k-n+1)}(k-n)\;x^{n+1}}{(n+1)\cancel{!}}\cdot\frac{\cancel{n!}}{\cancel{k(k-1)\ldots(k-n+1)}\,x^n}\right|$$

$$=\lim_{n\to\infty}\frac{|(k-n)\,x|}{(n+1)}=|x|\underbrace{\lim_{n\to\infty}\frac{|k-n|}{n+1}}_{1}=|x|$$

$\boxed{|x|<1 \Rightarrow \text{seri yakınsaktır}}$

$|x| > 1 \Rightarrow$ seri ıraksaktır.

$$\boxed{k \in \mathbb{R}, \quad |x| < 1 \text{ ise;} \qquad (1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n}$$

$(1+x)^2 = (1 + 2x + x^2) + \cdots$ $n > 2$ $n \to \infty$

$n > k \Rightarrow \binom{k}{n} \to k(k-1)(k-2) \cdots \cancel{(k-k)} \cdots (k-(n-1))$ $\to 0$

Not: Uç noktalar $|x| = 1$ ise ne olur? $\to$ k değerine bağlıdır.

Eğer $-1 < k \leq 0 \Rightarrow x = 1$'de yakınsaktır.

$k \geq 0 \Rightarrow x = \pm 1$'de yakınsaktır.

Örn/ $f(x) = \dfrac{1}{\sqrt{4-x}}$'nin MacLaurin serisini ve yakınsaklık aralığını bulunuz.

$(1+x)^k$ $\quad (4-x)^{-1/2}$

$$f(x) = \frac{1}{\sqrt{4-x}} = \frac{1}{\sqrt{4\left(1-\frac{x}{4}\right)}} = \frac{1}{2} \cdot \frac{1}{\sqrt{1-\frac{x}{4}}} = \frac{1}{2} \cdot \underbrace{\left(1 - \frac{x}{4}\right)}^{-1/2}$$

$x \to -\dfrac{x}{4}$

$k \to -1/2$

$$\sum_{n=0}^{\infty} \binom{k}{n} x^n \xrightarrow[x \to -\frac{x}{4}]{k = -1/2} \frac{1}{\sqrt{4-x}} = \frac{1}{2} \sum_{n=0}^{\infty} \binom{-1/2}{n} \left(-\frac{x}{4}\right)^n$$

$|x| < 1 \longrightarrow \left|-\dfrac{x}{4}\right| < 1$

$\longrightarrow |x| < 4$ yeni yakınsaklık aralığı

$c_0 = 1 \quad c_1 = \dfrac{k}{1!} \quad c_2 = \dfrac{k(k-1)}{2!}$

$c_3 = \dfrac{k(k-1)(k-2)}{3!}$

$$\frac{1}{\sqrt{4-x}} = \frac{1}{2}\left[1 + \left(-\frac{1}{2}\right)\left(-\frac{x}{4}\right) + \frac{-\frac{1}{2} \cdot -\frac{3}{2}}{2!}\left(-\frac{x}{4}\right)^2 + \frac{-\frac{1}{2} \cdot \frac{-3}{2} \cdot \frac{-5}{2}}{3!}\left(-\frac{x}{4}\right)^3 + \cdots\right]$$

$|x| < 4$ iken bu eşitlik geçerlidir.

Elementer Olmayan İntegraller

Örn/

$$\int e^{-x^2}\,dx = ?$$

$r=\infty$ $\quad f(x) = e^{x} = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad x \to -x^2 \quad :$

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!} = 1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \dots$$

$$\int e^{-x^2}\,dx = \int\left(1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \dots\right)dx$$

$$= C + \underbrace{x}_{n=0} - \underbrace{\frac{x^3}{3.1!}}_{n=1} + \underbrace{\frac{x^5}{5.2!}}_{n=2} - \underbrace{\frac{x^7}{7.3!}}_{n=3} + \underbrace{\frac{x^9}{9.4!}}_{n=4} - \dots + (-1)^n \frac{x^{2n+1}}{(2n+1).n!} + \dots$$

$$\int e^{-x^2}\,dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1).n!} + C$$

Örn/

$$\int \sin(x^2)\,dx = ?$$

$R=\infty$ $\quad f(x) = \sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \to$ Maclaurin serisi

$x \to x^2$

$$\sin(x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n+1}}{(2n+1)!} = \underbrace{x^2}_{n=0} - \frac{\overset{x^6}{(x^2)^3}}{3!} + \frac{(x^2)^5}{5!} - \frac{\overset{14}{(x^2)^7}}{7!} + \dots$$

$2(2n+1)+1$

$4n+3$

$$\int \sin(x^2)\,dx = C + \frac{x^3}{3} - \frac{x^7}{7.3!} + \frac{x^{11}}{11.5!} - \frac{x^{15}}{15.7!} + \quad (-1)^n \frac{x^{4n+3}}{(4n+3)(2n+1)!}$$

$$\int \sin(x^2)\,dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+3}}{(4n+3)(2n+1)!} + C$$