14. Hafta Perşembe Dersi

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$\lim_{n\to\infty} \sqrt[n]{n} = 1$

1.) a) $a_n = \sqrt[n]{2n}$ $\qquad \lim_{n\to\infty} \underbrace{2^{1/n}}_{1} \underbrace{\sqrt[n]{n}}_{1} = 1 \Rightarrow$ yakınsak.

b) $a_n = \sin\left(\frac{n\pi}{2}\right)$ $\to$ n çift $\Rightarrow n=2k$ $\quad \sin(k\pi)=0$

$\to$ n tek $\Rightarrow n=2k+1 \to \sin\left(k\frac{\pi}{2}\right) \to (-1)^k$

$$\lim_{n\to\infty} a_n = \begin{cases} 0 : & n \text{ çift ise} \\ \mp 1 : & n \text{ tek ise.} \end{cases} \Rightarrow \text{Dizi ıraksaktır.}$$

c) $a_n = \dfrac{1+\sin^2(2n)}{n^2+\cos^2(n)} \to \lim_{n\to\infty} \dfrac{1+\sin^2(2n)}{n^2+\cos^2(n)} = \lim_{n\to\infty} \dfrac{\frac{1}{n^2} + \left(\frac{\sin(2n)}{n}\right)^2}{1+\left(\frac{\cos(n)}{n}\right)^2} = \dfrac{0}{1} = 0$

$n\to\infty \quad \frac{1}{n}\to 0$

$\Rightarrow$ Dizi yakınsaktır.

d) $a_n = \dfrac{n^2+2}{n^2+5} + \dfrac{2n+3}{n+4}$ $\qquad n\to\infty$

$a_n \to 1+2=3$ $\quad \Rightarrow$ Dizi yakınsaktır.

e) $a_n = \left(1-\dfrac{1}{4n^2}\right)^n$ $\qquad \lim_{n\to\infty}\left(\underbrace{\left(1-\frac{1}{2n}\right)^{2n}}_{e^{-1}}\underbrace{\left(1+\frac{1}{2n}\right)^{2n}}_{e^{1}}\right)^{1/2} = 1^{1/2} = 1$

$\left(1+\frac{x}{n}\right)^n \to e^x$ $\qquad \Rightarrow$ Dizi yakınsaktır.

f) $a_1 = 1 \qquad a_{n+1} = \frac{1}{3}(a_n+4)$ $\to$ rekürsif dizi

$a_{n+1} \;?\; a_n$

$a_1 = 1$

$a_2 = \frac{1}{3}(a_1+4) = \frac{5}{3}$ $\qquad \to a_{n+1} - a_n = \frac{1}{3}(a_n+4) - a_n$

$= -\frac{2}{3}a_n + \frac{4}{3}$

$a_3 = \frac{1}{3}\left(\frac{5}{3}+4\right) =$

$= \frac{2}{3}\underbrace{(2-a_n)}_{>0} > 0$

$a_n < 2$

$\downarrow$ üstten 2 ile sınırlıdır.

$\Rightarrow a_{n+1} > a_n \to$ Dizi artan bir dizidir.

$\Rightarrow$ Yakınsaktır.

4) a) $a_n = \dfrac{3n}{2n+1}$ $\qquad \lim_{n\to\infty} \dfrac{3n}{2n+1} = \dfrac{3}{2} \to$ üst sınır.

$n = 1, 2, 3, \ldots$

$(2n+2)(2n+1) - 3n(2n+3)$

$n = 1, 2, 3, \ldots$

$$a_{n+1} - a_n = \frac{3(n+1)}{\underbrace{2(n+1)+1}_{(2n+1)}} - \frac{3n}{\underbrace{2n+1}_{(2n+3)}} = \frac{(3n+3)(2n+1) - 3n(2n+3)}{(2n+1)(2n+3)}$$

$$= \frac{\cancel{6n^2} + \cancel{6n} + \cancel{3n} + 3 - \cancel{6n^2} - \cancel{9n}}{(2n+1)(2n+3)} = \frac{3}{\underbrace{(2n+1)(2n+3)}_{+}} > 0 \quad \text{Her } n \text{ için}$$

$\Rightarrow a_{n+1} > a_n \Rightarrow$ Dizi artan dizidir.

b) $a_n = \frac{2^n}{n!} > 0$

$n=0 \Rightarrow a_0 = 1$
$n=1 \Rightarrow a_1 = 2$
$n=2 \Rightarrow a_2 = \frac{4}{2} = 2$

$$\frac{a_{n+1}}{a_n} = \frac{2^{\cancel{n+1}}}{(n+1)\cancel{!}} \cdot \frac{\cancel{n!}}{\cancel{2^n}} = \frac{2}{n+1} < \frac{2}{n} < \frac{n}{n} = 1$$

$\forall n > 2$ için $\frac{a_{n+1}}{a_n} < 1$

$\Rightarrow a_{n+1} < a_n$

0 ile alttan sınırlıdır.

$\Rightarrow$ Dizi azalandır

5) a) $\sum_{n=2}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+2}} \right) = ?$ $\quad \lim_{n\to\infty} s_n$

$$s_n = a_2 + a_3 + \ldots + a_n + a_{n+1} = \underbrace{\left(\frac{1}{\sqrt{2}} - \cancel{\frac{1}{\sqrt{4}}}\right)}_{n=2} + \underbrace{\left(\frac{1}{\sqrt{3}} - \cancel{\frac{1}{\sqrt{5}}}\right)}_{n=3} + \left(\cancel{\frac{1}{\sqrt{4}}} - \cancel{\frac{1}{\sqrt{6}}}\right)$$

$$\ldots + \underbrace{\left(\cancel{\frac{1}{\sqrt{n}}} - \frac{1}{\sqrt{n+2}}\right)}_{a_n} + \underbrace{\left(\cancel{\frac{1}{\sqrt{n+1}}} - \frac{1}{\sqrt{n+3}}\right)}_{a_{n+1}}$$

$$s_n = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \underbrace{\frac{1}{\sqrt{n+2}}}_{\to 0} - \underbrace{\frac{1}{\sqrt{n+3}}}_{\to 0} \qquad \lim_{n\to\infty} s_n = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}}$$

$$\sum^{\infty} \left(\overbrace{1}^{a_n} \quad \overbrace{2^{2n+1}}^{b_n} \right)$$

$$\sum_{n=1}^{\infty}\left(\overbrace{\frac{1}{n(n+1)}}^{a_n}+\frac{2^{2n+1}}{5^n}\right)$$

teleskopik geometrik

$$\frac{1}{n(n+1)}=\frac{1}{n}+\frac{-1}{(n+1)}$$

$$s_n=\underbrace{\left(1-\cancel{\tfrac{1}{2}}\right)}_{a_1}+\underbrace{\left(\cancel{\tfrac{1}{2}}-\cancel{\tfrac{1}{3}}\right)}_{a_2}+\frac{1}{3}-\frac{1}{4}$$

$$+\left(\cancel{\tfrac{1}{n}}-\frac{1}{n+1}\right)$$

$$s_n=1-\frac{1}{n+1}\qquad \lim_{n\to\infty}s_n=1$$

$$\sum_{n=1}^{\infty}a_n=1$$

$$\sum_{n=1}^{\infty}b_n=8$$

$$+\overline{\quad 9 \quad}$$

ar^{n-1}

$$\frac{2\cdot 4^{n-1}\cdot 4}{5\cdot 5^{n-1}}\ \checkmark$$

$a=\frac{8}{5}$ $r=\frac{4}{5}\checkmark$ $|r|<1$

$$\frac{a}{1-r}=\frac{8/5}{1-4/5}=8$$

$1.\underline{2345}\,\underline{345}\,\underline{345}\ldots$

$= 1.2 + 0.0345 + 0.0000345 + 0.0000000345 + \ldots$

$$=1.2+\underbrace{345.10^{-4}+345.10^{-7}+345.10^{-10}+\ldots}_{ar^{n-1}}=1.2+\frac{345}{9990}$$

$$\frac{10^{-7}}{10^{-4}}=10^{-7+4}=10^{-3}$$

$1+r+r^2$

$a=345\quad r=10^{-3}\longrightarrow 345(1+10^{-3}+10^{-6}+\ldots)\rightarrow\frac{a}{1-r}=\frac{345}{1-\frac{1}{10^3}}$

$$\frac{a}{1-r}=\frac{345.10^3}{999}$$

1000

$$\frac{999}{10^3}$$

$345(10^{-4}+10^{-7}+10^{-10}+\ldots)$

$34.5(10^{-3}+10^{-6}+10^{-9}+\ldots)=\frac{345}{9990}$ $\frac{345.10^3-999.345}{999}=\frac{345}{999}$

$$345(10^{-3}+10^{-6}+10^{-9}+\ldots)=\frac{345}{999}$$

8. For each of the following series determine whether it converges or diverges.

(a) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n+2}{n}\right)^{-n}$ → kök testi.

(e) $\displaystyle\sum_{n=1}^{\infty}\frac{1+n+n^2}{\sqrt{1-n^2+n^6}}$ ⇒ ıraksaktır $\frac{1}{n}$ → ıraksak

$\lim\limits_{n\to\infty}\dfrac{1+n+n^2}{\sqrt{1-n^2+n^6}}\cdot\dfrac{n}{1}$

$\lim\limits_{n\to\infty}\dfrac{1/n^3+\frac{1}{n}+1}{\sqrt{\frac{1}{n^6}-\frac{n^2}{n^6}+1}} = \dfrac{1}{1}=1$ aynı davranış

?(b) $\displaystyle\sum_{n=2}^{\infty}\frac{\ln^3 n}{n^2}$. → karşılaştırma

(f) $\displaystyle\sum_{n=2}^{\infty}(-1)^n\frac{n^5}{e^n}$.

(c) $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^2}$. → integral testi

(g) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{n^n}{n!}$.

Alterne

(d) $\displaystyle\sum_{n=1}^{\infty}\frac{3+\cos n}{3^n}$. → Karşılaştırma

(h) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{\ln n}{n^2+1}$.

a) $\lim\limits_{n\to\infty}\left|\sqrt[n]{a_n}\right| = \lim\limits_{n\to\infty}\sqrt[n]{\left(\frac{n+2}{n}\right)^{-n}} = \lim\limits_{n\to\infty}\dfrac{n}{n+2}=1$ kök testi çalışmaz

Iraksaklık n. terim testi $\quad a_n=\dfrac{1}{\left(\frac{n+2}{n}\right)^n}\qquad \lim\limits_{n\to\infty}a_n=\lim\limits_{n\to\infty}\dfrac{1}{\underbrace{\left(1+\frac{2}{n}\right)^n}_{e^2}}=e^{-2}\neq 0$ ⇒ seri ıraksaktır.

$\left(1+\frac{x}{n}\right)^n \xrightarrow[n\to\infty]{} e^x$

b) $\displaystyle\sum_{n=2}^{n}\frac{\ln^3 n}{n^2} > \frac{1}{n^2}$ p-serisi yakınsak

(direk) karşılaştırma çalışmaz

limit karşılaştırma deneyelim;

$\lim\limits_{n\to\infty}\dfrac{a_n}{b_n}=\lim\limits_{n\to\infty}\dfrac{\ln^3 n}{\cancel{n^2}}\cdot\dfrac{\cancel{n^2}}{1}=\infty$

$b_n=\dfrac{1}{n^2}$

c) $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^2}$ → yakınsaktır.

$\displaystyle\int\frac{1}{x(\ln x)^2}dx=\int\frac{du}{u^2}=-\frac{1}{u}=-\frac{1}{\ln x}$

$u=\ln x$

pozitif ✓
sürekli ✓
azalan ✓

$\displaystyle\int_2^{\infty}\frac{1}{x(\ln x)^2}dx=\lim_{t\to\infty}\int_2^{t}\frac{1}{x(\ln x)^2}dx=\lim_{t\to\infty}\left(-\frac{1}{\ln x}\right)_{x=2}^{x=t}$

$=\lim\limits_{t\to\infty}\left(-\underbrace{\frac{1}{\ln t}}_{0}+\frac{1}{\ln 2}\right)=\dfrac{1}{\ln 2}$ → yakınsak

d) $\displaystyle\sum_{n=1}^{\infty}\frac{3+\cos n}{3^n} \overset{\text{karşılaştırma}}{<} \frac{3+1}{3^n}=\frac{4}{3^n}$ geometrik

$-1<\cos n<1$

⇐ büyük olan yakınsak.

yakınsaktır.

$\displaystyle\sum\frac{4}{3^n}\qquad ar^{n-1}\qquad a=\frac{4}{3}\quad r=\frac{1}{3}$

$\dfrac{4}{3.3^{n-1}}$

$|r|<1$ yakınsak

yakınsak

f) $\sum_{n=2}^{\infty} (-1)^n \frac{n^5}{e^n}$ u_n

pozitif ✓

$(u_n)' = \frac{5n^4 e^n - n^5 e^n}{e^{2n}} = \frac{e^n n^4 (5-n)}{e^{2n}} < 0 \quad \forall n > 5 \Rightarrow$

azalan ✓

$\lim_{n\to\infty} \frac{n^5}{e^n} \quad \frac{5n^4}{e^n} \quad \frac{20n^3}{e^n} \quad \frac{60n^2}{e^n} \quad \frac{120n}{e^n} \quad \frac{120}{e^n} = 0$ ✓

$\Rightarrow$ Alterne seri testi gereği yakınsaktır.

g) $\sum_{n=1}^{\infty} (-1)^n \frac{n^n}{n!}$ $\not\to 0$ ıraksaktır.

$u_n \to$ pozitif

azalan?

limit?

$|a_n| \to \frac{n^n}{n!}$ Oran testi $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^{n+1}}{(n+1)\cancel{!}} \cdot \frac{\cancel{n!}}{n^n} = \lim_{n\to\infty} \underbrace{\left(\frac{n+1}{n}\right)^n}_{\left(1+\frac{1}{n}\right)^n \to e} \frac{(n+1)}{\infty} = \infty$

$\sum_{n\to\infty}^{\infty} |a_n|$ ıraksaktır.

10) $\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{(n^2+1)}$

$u_n \to$ pozitif

azalan mı? ✓

$u_n' = \frac{\frac{1}{n}(n^2+1) - \ln n\,(2n)}{(n^2+1)^2 > 0} = \frac{\overbrace{n^2+1 - 2n^2 \ln n}^{\frac{n^2+1}{n} - \ln n \cdot 2n}}{\underbrace{n(n^2+1)^2}_{>0}} < 0$

$\underbrace{n^2}_{>0}\overbrace{(1-2\ln n)}^{<0}$

$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{\ln n}{n^2+1} \quad \frac{1/n}{2n} = \frac{1}{2n^2} \to 0$ ✓

$\Rightarrow$ alterne seri testi gereği yakınsak