12th Week Tuesday

9 Mayıs 2023 Salı 13:30

$\sum_{n=1}^{\infty} a_n$ conv. / div ?

Direct Comparison Test (D.C.T)

$\sum_{n=1}^{\infty} a_n \to$??? (conv/div)

$\sum_{n=1}^{\infty} b_n$ con/div ✓

D.C.T

By D.C.T.

$a_n < b_n$ and $\sum_{n=1}^{\infty} b_n$ conv. $\Rightarrow \sum_{n=1}^{\infty} a_n$ converges.

conv. using

$a_n > b_n$ and $\sum_{n=1}^{\infty} b_n$ divergent. $\Rightarrow \sum_{n=1}^{\infty} a_n$ diverges.

div.

! $\sum_{n=1}^{\infty} a_n$?

$a_n > b_n$ and $\sum b_n$ conv. $\to$ D.C.T doesn't work!

??? conv

$a_n < b_n$ and $\sum b_n$ div. $\to$ D.C.T doesn't work!

??? div.

geometric

$\sum_{n=1}^{\infty} ar^{n-1}$ $|r|<1$ conv, $|r|\geq 1$ div.

EX $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$ conv. / div.?

$\frac{1}{2^n+1} < \frac{1}{2^n}$! ✓ b_n

$\sum_{n=1}^{\infty} \frac{1}{2^n} \to r=\frac{1}{2} < 1$ conv. geometric series.

$a=1/2$

$\frac{1}{2}\cdot\left(\frac{1}{2}\right)^{n-1}$ (a, r)

!

$\Rightarrow$ By D.C.T, $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$ also converges.

Ex $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$

$\frac{1}{2^n-1} > \frac{1}{2^n}$ $r=\frac{1}{2}<1$ $\Rightarrow$ D.C.T does not work.

??? conv geometric series

EX $\sum_{n=1}^{\infty} \frac{n}{n^3+2n^2+3}$ conv. / div. ?

$\frac{n}{n^3+2n^2+3} < \frac{n}{n^3} = \frac{1}{n^2}$ b_n

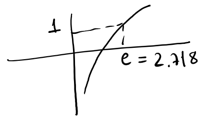
$\sum_{n=1}^{\infty} \frac{1}{n^2} \to p=2>1$ convergent p-series.

$\Rightarrow$ By D.C.T. $\sum \frac{n}{\ }$ also converges.

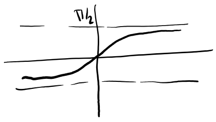
$n=1 \quad n^3+2n^2+3$

$\sin(n), \cos(n) < 1 \qquad 0 < \sin^2(n), \cos^2(n) < 1 \qquad \sin^2(n), \cos^2(n) < 1$

$\ln(n) > 1 \quad (n \geq 3)$



$\arctan(n) < \pi/2$



Ör. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n+\sqrt{n}}$ conv. / div ?

$b_n \to \frac{1}{3^n}$

$b_n \to \frac{1}{\sqrt{n}}$

$\frac{1}{3^n+\sqrt{n}} < \frac{1}{3^n} \to$ $r=\frac{1}{3}<1$ convergent geometric series.

$\Rightarrow$ By D.C.T. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n+\sqrt{n}}$ also converges.

(if we use $b_n = \frac{1}{\sqrt{n}}$ for comparison ;

$\frac{1}{3^n+\sqrt{n}} < \frac{1}{\sqrt{n}} \to \displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ $p=\frac{1}{2}<1$ divergent p-series.

XX div.

D.C.T does not work.)

Ex. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos^2 n}{3^n+\sqrt{n}}$ con./div.

$3^n+\sqrt{n} \to$ positive

$\cos^2 n < 1$

$\frac{\cos^2 n}{3^n+\sqrt{n}} < \frac{1}{3^n+\sqrt{n}} < \frac{1}{3^n}$

$r=\frac{1}{3}<1$ convergent geometric series

Converges by D.C.T.

Ex. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$ con./div ?

$\ln(n) > 1 \qquad n \geq 3$

$\frac{\ln(n)}{n} > \frac{1}{n}$ harmonic series $\to$ divergent

diverges by D.C.T.

Ex. $\displaystyle\sum^{\infty} 1+\cos^2(n)$

EX $\sum_{n=1}^{\infty} \frac{1+\cos^2(n)}{\sqrt{n}}$ conv./div. !

$0 < \cos^2(n)$

$1 < 1+\cos^2(n)$

Reminder:

p-series

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ $p \leq 1$ divergent, $p > 1$ conv.

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ $\quad \frac{1}{\sqrt{n}} < \frac{1+\cos^2(n)}{\sqrt{n}} \Rightarrow$ By. D.C.T. $\sum_{n=1}^{\infty} \frac{1+\cos^2(n)}{\sqrt{n}}$ also diverges.

$p=\frac{1}{2}<1$ divergent p-series.

EX $\sum_{n=1}^{\infty} \frac{1}{n!}$ conv./div.?

$$\frac{1}{n!} = \frac{1}{\underbrace{n(n-1)\ldots 3.2.1}_{n \text{ tane}}} < \frac{1}{\underbrace{2.2.2\ldots.2}_{n \text{ tane}}} = \frac{1}{2^n}$$

✓

$\sum_{n=1}^{\infty} \frac{1}{2^n}$ $r=\frac{1}{2}<1$ Convergent geometric series

$\Rightarrow$ By D.C.T, $\sum_{n=1}^{\infty} \frac{1}{n!}$ also converges.

EX $\sum_{n=1}^{\infty} \frac{n \cdot \sin^2(n)}{1+n^3}$

$\sin^2(n) < 1$

$n. \sin^2(n) < n$

$$\frac{n. \sin^2(n)}{1+n^3} < \frac{n}{1+n^3} < \frac{n}{n^3} = \frac{1}{n^2} \rightarrow$$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ $p=2>1$ convergent p-series

By D.C.T, $\sum_{n=1}^{\infty} \frac{n \sin^2(n)}{1+n^3}$ also converges.

EX $\sum_{n=1}^{\infty} \frac{n-1}{n^2 \cdot \sqrt{n}}$ conv./div. ?

$$\frac{n-1}{n^2 \sqrt{n}} < \frac{n}{n^2\sqrt{n}} = \frac{1}{n\sqrt{n}}$$

$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ $p=\frac{3}{2}>1$ convergent p-series

By D.C.T. $\sum_{n=1}^{\infty} \frac{n-1}{n^2\sqrt{n}}$ also converges.

EX $\sum^{\infty} \frac{\arctan(n)}{\ldots}$ conv./div. ?

$\arctan(n) < \pi/2$

Ex $\sum_{n=1}^{\infty} \frac{\arctan(n)}{n^{1.2}}$ conv./div. ?

$\arctan(n) < \pi/2$

$\frac{\arctan(n)}{n^{1.2}} < \frac{\pi/2}{n^{1.2}} \to$ conv.

By D.C.T., $\sum_{n=1}^{\infty} \frac{\arctan(n)}{n^{1.2}}$ also converges.

$\sum_{n=1}^{\infty} \frac{1}{n^{1.2}}$ $p = 1.2 > 1$ convergent p-series

Limit Comparison Test

$\sum_{n=1}^{\infty} a_n$ conv/div. ???

b_n

L.C.T: $\lim_{n\to\infty} \frac{a_n}{b_n} = \begin{cases} > 0, & \sum_{n=1}^{\infty} a_n \text{ has the same con/div. status with } \sum_{n=1}^{\infty} b_n \\ = 0 \text{ and } \sum_{n=1}^{\infty} b_n \text{ conv.} & \Rightarrow \sum a_n \text{ conv.} \\ = \infty \text{ and } \sum_{n=1}^{\infty} b_n \text{ divergent} & \Rightarrow \sum a_n \text{ div.} \end{cases}$

Ex $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ conv./div?

$b_n = \frac{1}{2^n} \to$ conv. geometric series $r = \frac{1}{2} < 1$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{2^n-1}}{\frac{1}{2^n}}$

$= \lim_{n\to\infty} \frac{1}{2^n-1} \cdot \frac{2^n}{1} = \lim_{n\to\infty} \frac{2^n}{2^n-1}$

$= \lim_{n\to\infty} 1 + \frac{1}{2^n-1} = 1 + 0 = 1 > 0$

(with $\frac{1}{2^n-1} \to 0$)

By L.C.T. $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{2^n-1}$ also converges.

Ex $\sum_{n=1}^{\infty} \frac{1+4^n}{1+3^n}$ conv. / div. D.C.T. does not work.

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1+4^n}{1+3^n} \cdot \frac{3^n}{4^n} = \lim_{n\to\infty} \frac{1+4^n}{4^n} \cdot \frac{3^n}{1+3^n}$

$n\to\infty \quad b_n \qquad n\to\infty \quad 1+3^n \qquad 4^{\cdot\cdot} \qquad n\to\infty \quad 4$

$\boxed{b_n = \left(\frac{4}{3}\right)^n = \frac{4^n}{3^n} \to r=\frac{4}{3} \to \text{divergent geometric series.}}$

$$= \lim_{n\to\infty} \underbrace{\left(1+\frac{1}{4^n}\right)}_{1+0} \cdot \underbrace{\left(1-\frac{1}{1+5^n}\right)}_{1-0} = 1.1 = 1 > 0$$

By L.C.T., $\sum_{n=1}^{\infty} \frac{1+4^n}{1+3^n}$ also diverges.

Ex/ $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^3-3n^2+5n-7} \to a_n$

$b_n = \frac{\sqrt{n}}{n^3} = \frac{1}{n^{3-\frac{1}{2}=\frac{5}{2}}}$ $\quad p=\frac{5}{2}>1$ convergent p-series.

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\cancel{\sqrt{n}}}{n^3-3n^2+5n-7}\cdot\frac{n^3}{\cancel{\sqrt{n}}} = 1 > 0$$

$\Rightarrow$ By L.C.T, $\sum_{n=1}^{\infty} a_n$ also converges.

Ex/ $\sum_{n=1}^{\infty} \frac{2n^2+3n-5}{9n^7-5n^3+6n^2-4}$ conv./div. ?

$b_n = \frac{2n^2}{9n^7} \leftarrow \frac{1}{n^5}$ $\quad p=5>1$ convergent p-series

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{2n^2+3n-5}{9n^7-5n^3+6n^2-4}\cdot\frac{9n^7}{2n^2} = 1 > 0$$

Σa_n convergent by L.C.T.

Ex/ $\sum_{n=1}^{\infty} \frac{n^3-5n+1}{3n^4+2n-7}$ conv./div. ?

$b_n = \frac{n^3}{3n^4} \to$ divergent

$\frac{n^3}{n^4} = \frac{1}{n} \to$ harmonic series divergent.

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{n^3-5n+1}{3n^4+2n-7}\cdot\frac{3n^4}{n^3} = 1 > 0$$

Σa_n is divergent by L.C.T.

Ex/ $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^3}\right)$ conv. / div. ?

$b_n = \frac{1}{n^3} \to p=3>1$ convergent p-series

$\left(\frac{1}{\cdot}\right) \to 0$

$\sum_{n=1}$ n^3 | n conv. p-series

L.C.T. $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n^3}\right)}{\left(\frac{1}{n^3}\right)} = 1 > 0$

By. LCT, $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^3}\right)$ also converges.

$\frac{1}{e^n} \to r = \frac{1}{e} < 1$ Convergent geometric series

$\frac{1}{\sqrt{n}} \to p = \frac{1}{2} < 1$ divergent p-series

Ex $\sum_{n=1}^{\infty} \frac{1}{e^n \cdot \sqrt{n}}$ $b_n = \frac{1}{\sqrt{n}} \to$ div.

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{e^n \cancel{\sqrt{n}}} \cdot \frac{\cancel{\sqrt{n}}}{1} = 0$ LCT does not work

b_n $\to$ conv. olmalı

$b_n = \frac{1}{e^n} \to$ conv.

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{\cancel{e^n}\sqrt{n}} \cdot \frac{\cancel{e^n}}{1} = 0$

LCT ✓ $\Rightarrow$ By LCT $\sum_{n=1}^{\infty} \frac{1}{e^n\sqrt{n}}$ also converges.