4th week Tuesday

14 Mart 2023 Salı 13:39

Improper Integrals

$-\infty \quad \infty$

$\int_{-\infty}^{\infty}$

1^{st} Type ✓

Infinite Intervals

$\int_a^{\infty}$ or $\int_{-\infty}^{a}$ or $\int_{-\infty}^{\infty}$

2^{nd} Type !

Infinite Discontinuity

$\int_a^b f(x)dx \quad [a,b] \quad c \in (a,b)$

$\int_a^b f(x)dx$ or $\int_a^b f(x)dx$ or $\int_a^b f(x)dx$

$c \in (a,b)$ inf. disc.

sonuç = $\pm\infty$ → divergent! →→ infinity

sonuç = sayı → convergent! →→ finite

$0 \leftarrow \int_{-5}^{7} \frac{1}{x}\,dx$ → 2. tip

1^{st} Type (Infinite Intervals)

! limits at infinity from Mat-1

L'Hospital ✓

Indeterminacies ✓

$\frac{a}{0} = \infty$ / $\frac{a}{\infty} = 0$ / ...

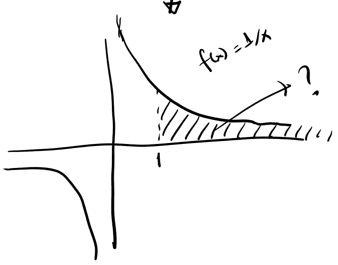
* $$\int_a^{\infty} f(x)\,dx = \lim_{t\to\infty}\left(\int_a^t f(x)\,dx\right)$$

∞ ← sorunlu

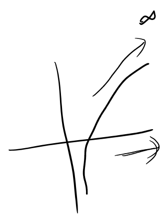
$F(t) - F(a)$ → t cinsinden

Ex/ $$\int_1^{\infty} \frac{1}{x}\,dx = \lim_{t\to\infty}\left(\int_1^t \frac{1}{x}\,dx\right) = \lim_{t\to\infty}\left(\ln|x|\Big]_{x=1}^{x=t}\right)$$

∞ → sorunlu



$$= \lim_{t\to\infty}\left(\underbrace{\ln|t|}_{\infty} - \underbrace{\ln|1|}_{0}\right)$$



$= \infty$ → divergent! (ıraksak)

{ sayı → convergent! (yakınsak) }

* $$\int_{-\infty}^{a} f(x)\,dx = \lim_{t\to-\infty}\left(\int_t^a f(x)\,dx\right)$$

$F(a) - F(t)$

Ex/ $$\int_{-\infty}^{0} xe^x\,dx = \lim_{t\to-\infty}\left(\int_t^0 xe^x\,dx\right)$$

poly. exp.

$u = x \quad dv = e^x dx \quad \{ \; uv - \int v\,du = xe^x - \int e^x dx = xe^x - e^x$

$$\int_{-\infty} xe^x\,dx = \lim_{t\to-\infty}\left(\int_t \ldots\right)$$

poly exp.

$u = x \quad dv = e^x dx$

$du = 1\,dx \quad v = e^x$

$$\Big\}\ uv - \int v\,du = xe^x - \int \underbrace{e^x dx}_{e^x} = xe^x - e^x$$

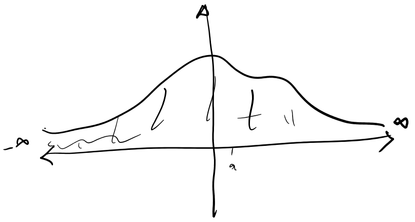
$$= \lim_{t\to-\infty}\left(xe^x - e^x \Big]_{x=t}^{x=0} \right) = \lim_{t\to-\infty}\left(\underbrace{(0.e^0 - e^0)}_{x=0} - \underbrace{(te^t - e^t)}_{x=t} \right)$$

şu andan itibaren mat-1'le bağlayalım.

$$= \lim_{t\to-\infty}\left(-1 - te^t + e^t \right) = -1 - 0 + 0 = -1 \quad \rightarrow \text{convergent!}$$

$e^{-\infty} = \frac{1}{e^{\infty}} = \frac{1}{\infty} = 0$

$$\lim_{t\to-\infty} t.e^t = \lim_{t\to-\infty} \frac{t}{e^{-t}} \ \left(\infty/\infty\right) \xrightarrow{L'} = \lim_{t\to-\infty} \frac{1}{-1.e^{-t}} = \frac{1}{\infty} = 0$$



$$* \quad \int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx$$

if any of the summands is divergent ⇒ divergent!

only if both of the summands are convergent ⇒ convergent!

M + N ✓

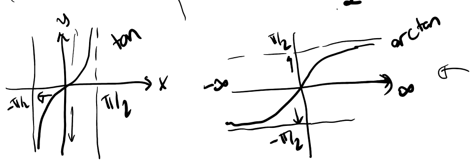
Ex

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

convergent. ✓

$$\lim_{t\to-\infty} \int_t^0 \frac{1}{1+x^2}\,dx \qquad\qquad \lim_{t\to\infty} \int_0^t \frac{1}{1+x^2}\,dx$$

$$= \lim_{t\to-\infty}\Big[\underbrace{\arctan(0)}_{0} - \underbrace{\arctan(t)}_{-\frac{\pi}{2}} \Big] = +\frac{\pi}{2} \qquad\qquad \lim_{t\to\infty}\Big[\underbrace{\arctan(t)}_{\frac{\pi}{2}} - \underbrace{\arctan(0)}_{0} \Big] = \frac{\pi}{2}$$



!* Ex (p-integral)

$$\int^{\infty} \frac{1}{x^p}\,dx \Rightarrow \lim \left(\int^{t} \frac{1}{x^p}\,dx \right) = \lim \left(\int^{t} x^{-p}\,dx \right) = \lim \left(\frac{x^{-p+1}}{-p+1} \Big]^{x=t} \right)$$

• Ex/ (p integral)

$$\int_1^{\infty} \frac{1}{x^p}\,dx \Rightarrow \lim_{t\to\infty}\left(\int_1^t \frac{1}{x^p}\,dx\right) = \lim_{t\to\infty}\left(\int_1^t x^{-p}\,dx\right) = \lim_{t\to\infty}\left(\frac{x^{-p+1}}{-p+1}\Big]_{x=1}\right)$$

$p \neq 1$

$$\int_0^{\infty} = \int_0^1 + \int_1^{\infty}$$

2. tip   1. tip

$$= \lim_{t\to\infty}\left(\frac{t^{-p+1}}{-p+1} - \frac{1^{-p+1}}{-p+1}\right)$$

$p>1$

0

$$t^{-\infty} = \frac{1}{t^{\infty}} \quad \frac{1}{\infty} = 0 + \frac{1}{p-1}$$

$= \frac{1}{p-1}$ conv.

Ex/ $\displaystyle\int_1^{\infty} \frac{1}{x^1}\,dx = \lim_{t\to\infty}(\ln t - \ln 1) = \infty \to$ divergent

$p=1$

$$\frac{x^{-5+1}}{-5+1} = -\frac{1}{4x^4}$$

Ex/ $$\int_1^{\infty} \frac{1}{x^5}\,dx = \int_1^{\infty} x^{-5}\,dx = \lim_{t\to\infty}\left(\int_1^t x^{-5}\,dx\right) = \lim_{t\to\infty}\left(-\frac{1}{4x^4}\Big]_{x=1}^{x=t}\right) = \lim_{t\to\infty}\left(\underbrace{\frac{-1}{4t^4}}_{0} - \frac{-1}{4}\right) = \frac{1}{4} \Rightarrow \text{convergent}$$

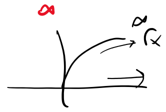
x^7 x^{-7} $-\frac{1}{6x^6}$ $-\frac{-1}{6}$ $\frac{1}{6}$

$p>1 \Rightarrow$ . . . . conv.

conv $\to \frac{1}{p-1}$

Ex/ $$\int_1^{\infty} \frac{1}{\sqrt{x}}\,dx = \int_1^{\infty} \frac{1}{\sqrt{x}}\,dx = \lim_{t\to\infty}\left(2\sqrt{x}\Big]_{x=1}^{x=t}\right) = \lim_{t\to\infty}(\underbrace{2\sqrt{t}}_{\infty} - 2) = \infty \to \text{divergent.}$$

$p=\frac{1}{2}$ $\qquad \frac{1}{2\sqrt{x}}$ $\qquad 2\sqrt{x}$



Ex/ $$\int_1^{\infty} \frac{1}{x^{-3}}\,dx = \int_1^{\infty} x^3\,dx = \lim_{t\to\infty}\left(\frac{x^4}{4}\Big]_{x=1}^{x=t}\right) = \lim_{t\to\infty}\left(\underbrace{\frac{t^4}{4}}_{\infty} - \frac{1}{4}\right) = \infty \to \text{divergent.}$$

$p=-3$

$p<1 \Rightarrow$ . . . . $\infty \to$ divergent

p-integral !

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \begin{cases} \text{divergent}, & \text{if } p<1 \\ \text{divergent}, & \text{if } p=1 \\ \text{convergent}, & \text{if } p>1 \\ = \frac{1}{p-1} & \end{cases}$$

$$\int_1^{\infty} \frac{1}{x^5}\,dx \qquad x^{-5} \qquad \frac{x^{-4}}{-4} \qquad \frac{1}{-4x^4} \xrightarrow{t\to\infty} 0 \quad \checkmark \quad \to \text{conv.}$$

! 2^{nd} Type – (Infinite Discontinuity)

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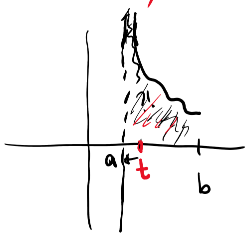
* $\displaystyle\int_a^b f(x)\,dx$ — undefined at $x=a$, vertical asymptot; a → sıkıntılı



$$= \lim_{t\to a^+}\left(\int_t^b f(x)\,dx\right)$$

* $\displaystyle\int_a^b f(x)\,dx$ — undefined at $x=b$ → vertical asimtot; b → sıkıntılı

$$= \lim_{t\to b^-}\left(\int_a^t f(x)\,dx\right)$$

$F(t)-F(a)$

* $\displaystyle\int_a^b f(x)\,dx$ — undefined at $c\in(a,b)$, $x=c$

$$= \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

$$\lim_{t\to c^-}\left(\int_a^t f(x)\,dx\right) + \lim_{t\to c^+}\left(\int_t^b f(x)\,dx\right)$$

if any of the summands is divergent ⇒ divergent!

only if both of the summands are convergent ⇒ convergent!

M + N ✓

Ex/

$$\int_3^5 \frac{1}{\sqrt{x-3}}\,dx = \lim_{t\to 3^+}\left(\int_t^5 \frac{1}{\sqrt{x-3}}\,dx\right)$$

undefined at $x=3$

$u=x-3$, $du=dx$ $\quad \displaystyle\int \frac{1}{\sqrt{u}}\,du \to 2\sqrt{u}+C$

$$= \lim_{t\to 3^+}\left(2\sqrt{x-3}\,\Big]_{x=t}^{x=5}\right)$$

$$= \lim_{t\to 3^+}\left(2\sqrt{2} - 2\sqrt{t-3}\right) = 2\sqrt{2}-0 = 2\sqrt{2}$$

convergent ✓

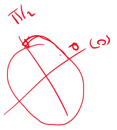
Ex/

$$\int^{\pi/2} \sec(x)\,dx = \lim_{t\to \frac{\pi}{2}^-}\left(\int^t \sec(x)\,dx\right)$$

$\pi/2$ → sıkıntılı

$\int \sec(x)\,dx = \ln|\sec(x)+\tan(x)| + C$

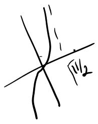
$$\int_0^{\pi/2} \sec(x)\,dx = \lim_{t\to\frac{\pi}{2}^-}\left(\int_0^{t} \underline{\sec(x)\,dx}\right) \qquad \int \sec(x)\,dx = \ln\left|\sec(x)+\tan(x)\right| + C$$



$\frac{1}{\cos(x)}$

$$= \lim_{t\to\frac{\pi}{2}^-}\left(\ln\left|\sec(x)+\tan(x)\right|\Big]_{x=0}^{x=t}\right)$$

$$= \lim_{t\to\frac{\pi}{2}^-}\left(\ln\left|\underbrace{\sec t}_{\frac{1}{\cos\frac{\pi}{2}}} + \underbrace{\tan t}_{\downarrow}\right| - \underbrace{\ln\left|\underbrace{\sec 0}_{1} + \underbrace{\tan 0}_{0}\right|}_{\ln|1| = 0}\right) = \infty \to \underline{\text{divergent}}$$



$$\underbrace{\infty + \infty}_{\infty}$$

EX

$$\int_0^3 \frac{1}{\underline{x-1}}\,dx = \underbrace{\int_0^{1} \frac{1}{x-1}\,dx}_{\swarrow} + \underbrace{\int_1^3 \frac{1}{x-1}\,dx}_{\searrow} \qquad \int \frac{1}{x-1}\,dx = \ln|x-1| + C$$

undefined at $x=1$

$\cancel{\ln 2 - \ln 1}$

$$\lim_{t\to 1^-}\left(\ln\underbrace{|t-1|}_{-\infty} - \underbrace{\ln|0-1|}_{0}\right) + \lim_{t\to 1^+}\left(\underbrace{\ln|3-1|}_{\ln 2} - \underbrace{\ln|t-1|}_{+\infty}\right) \to \underline{\text{divergent}}$$

$-\infty$