10th Week Thursday

27 Nisan 2023 Perşembe 11:30

Ex $a_n = \sqrt[n]{n}$ Check convergence of the sequence $\{a_n\}$.

$$\lim_{n\to\infty} \underbrace{\sqrt[n]{n}}_{n^{1/n}} = \lim_{n\to\infty} n^{1/n} = y \qquad \ln \lim_{n\to\infty} n^{1/n} = \ln y$$

$$\lim_{n\to\infty} \ln n^{1/n} = \ln y$$

$$\lim_{n\to\infty} \frac{1}{n} \cdot \ln n = \ln y$$

$$\lim_{n\to\infty} \frac{\ln n}{n} \to \frac{\infty}{\infty} \xrightarrow{L'} = \lim_{n\to\infty} \frac{1/n}{1} = 0 \qquad 0 = \ln y \qquad y = e^0 = 1$$

$\Rightarrow \{a_n\}$ is convergent.

$$\boxed{\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e} \qquad \boxed{\lim_{n\to\infty} \left(1 + \frac{a}{n}\right)^n = e^a \quad a\in\mathbb{R}}$$

Ex $a_n = \left(\frac{n+5}{n}\right)^n$ Check convergence of the sequence $\{a_n\}$.

$$\lim_{n\to\infty} \left(1 + \frac{5}{n}\right)^n = e^5 \rightsquigarrow \{a_n\} \text{ is convergent.}$$

Ex $a_n = \left(\frac{n-2}{n}\right)^{n+3}$. Check convergence of the sequence $\{a_n\}$.

$$\lim_{n\to\infty} \underbrace{\left(1 + \frac{-2}{n}\right)^n}_{\text{by formula } = e^{-2}} \cdot \underbrace{\left(1 + \frac{-2}{n}\right)^3}_{1^3} = e^{-2} \cdot 1 = e^{-2}$$

$\rightsquigarrow \{a_n\}$ is convergent.

Ex $a_n = \left(\frac{n-5}{n}\right)^{2n}$ Check convergence of the sequence $\{a_n\}$.

$$\lim_{n\to\infty} \left[\left(1 - \frac{5}{n}\right)^n\right]^2 = \left(e^{-5}\right)^2 = e^{-10}$$

$\rightsquigarrow \{a_n\}$ is convergent.

$$\lim_{n\to\infty}\Big[\underbrace{\Big(1-\frac{5}{n}\Big)}_{\text{by formula } = e^{-5}}\Big] = (e^{-5}) = e^{-5} \quad \longrightarrow \{a_n\} \text{ is convergent.}$$

Ex

$$a_n = \left(\frac{n}{n+1}\right)^{n^2+1}$$

Check convergence of the sequence $\{a_n\}$.

kendimiz paydaya uydurmalıyız.

$$\lim_{n\to\infty}\left[\underbrace{\left(1+\frac{-1}{n+1}\right)^{n+1}}_{\text{by the formula } e^{-1}}\right]^{\frac{n^2+1}{n+1}} = \lim_{n\to\infty}\left(e^{-1}\right)^{\frac{n^2+1}{n+1}\to\infty}$$

$= n^2+1$ ✓

$n+1-1 = n$

$$= e^{-\infty} = \frac{1}{e^{\infty}} = 0$$

$$\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^{n} = e^a$$

$$\lim_{n\to\infty}\left(1+\frac{a}{n^2}\right)^{n^2} = e^a$$

$$\lim_{n\to\infty}\left(1+\frac{a}{n^2+3n+1}\right)^{n^2+3n+1} = e^a$$

$\Rightarrow \{a_n\}$ is convergent.

Ex

$$a_n = \left(\frac{2n+1}{2n-3}\right)^{n/3}$$

Check convergence of the sequence $\{a_n\}$.

$$\lim_{n\to\infty}\left[\underbrace{\left(1+\frac{4}{2n-3}\right)^{2n-3}}_{\text{by the formula } = e^4}\cdot\underbrace{\left(1+\frac{4}{2n-3}\right)^{3}}_{(1+0)^3 = 1}\right]^{1/6}$$

$\frac{2n}{6} = \frac{n}{3}$ ✓

$$\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^{n} = e^a$$

$$\lim_{n\to\infty}\left(1+\frac{a}{2n-3}\right)^{2n-3} = e^a$$

$$= \left[e^4\cdot 1\right]^{1/6} = e^{4/6} = e^{2/3} \quad \Rightarrow \{a_n\} \text{ is convergent}$$

Ex

$$a_n = \left(1-\frac{1}{4n^2}\right)^{n}$$

Check convergence of the sequence $\{a_n\}$.

$$\lim_{n\to\infty}\left[\underbrace{\left(1-\frac{1}{4n^2}\right)^{4n^2}}_{\text{by formula } e^{-1}}\right]^{1/4n}$$

$\frac{4n^2}{4n} = n$ ✓

$$= \lim_{n\to\infty}\left(e^{-1}\right)^{1/4n \to 0}$$

$$= e^0 = 1$$

$$\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^{n} = e^a$$

$$\lim_{n\to\infty}\left(1+\frac{a}{4n^2}\right)^{4n^2} = e^a$$

$\Rightarrow \{a_n\}$ is convergent.

EX

$a_n = 3^n \Rightarrow \lim_{n\to\infty} 3^n = \infty \Rightarrow \{a_n\}$ is divergent.

$a_n = 1^n \quad \lim_{n\to\infty} a_n = 1 \Rightarrow \{a_n\}$ is convergent.

$a_n = (-3)^n \Rightarrow a_n = (-1)^n\, 3^n \qquad \lim_{n\to\infty} |a_n| = \lim_{n\to\infty} 3^n = \infty$ Theorem does not work.

$|x|<1$

$a_n = \left(\frac{1}{4}\right)^n \Rightarrow \lim_{n\to\infty} \frac{1}{4^n} = 0 \Rightarrow \{a_n\}$ is convergent

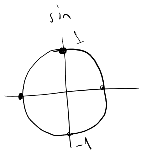
$a_n = \left(-\frac{1}{4}\right)^n \Rightarrow \lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{1}{4^n} = 0 \underset{\text{Theorem works.}}{\Rightarrow} \lim_{n\to\infty} a_n = 0$ ✓

$\Rightarrow \{a_n\}$ is convergent.

$a_n = x^n \Rightarrow$
- $\{a_n\}$ is convergent if $|x|<1$, $x=1$
- $\{a_n\}$ is divergent if $x>1$, $x=-1$
- $\{a_n\}$ " $x<-1$ $\rightsquigarrow$ we will not prove this now.

EX

$a_n = \sin\left(\frac{n\pi}{2}\right)$ Check convergence of the sequence $\{a_n\}$.



$\sin\left(\frac{n\pi}{2}\right) = 0$; if n is even.

$1, -1, 1, -1,$, if n is odd.

$a_n =$ changes always over $0, -1, 1$

$\Rightarrow \lim_{n\to\infty} a_n$ does NOT exist.

$\Rightarrow \{a_n\}$ is divergent.

EX

$a_n = \dfrac{1+\sin^2(2n)}{n^2+\cos^2(n)}$ Check convergence of the sequence $\{a_n\}$. $\lim_{n\to 0} \frac{\sin n}{n} = 1$

$$\lim_{n\to\infty} \frac{\dfrac{1+\sin^2(2n)}{n^2}}{\dfrac{n^2+\cos^2(n)}{n^2}} = \lim_{n\to\infty} \frac{\frac{1}{n^2} + \left(\frac{\sin(2n)}{n}\right)^2}{1 + \left(\frac{\cos(n)}{n}\right)^2}$$

$$= \frac{0+0}{1+0} = \frac{0}{1} = 0$$

Sandwich

$\lim_{n\to\infty} \frac{\sin 2n}{n} = 0$

n positive

$-\frac{1}{n} < \frac{\sin 2n}{n} < \frac{1}{n}$

$\lim$ 0 $\quad \lim = 0 \quad \lim$ 0

$\Rightarrow \{a_n\}$ is convergent.

$$\lim_{n\to\infty} \frac{\cos n}{n} = 0$$

n positive

$$\frac{-1}{n} < \frac{\cos n}{n} < \frac{1}{n}$$

$$\downarrow \quad \downarrow \quad \downarrow$$
$$0 \quad 0 \quad 0$$

Ex $a_n = \dfrac{n^4}{4n^2+1} \cdot \sin\left(\dfrac{1}{n^2}\right)$ Check convergence of the sequence $\{a_n\}$.

$$\lim_{n\to\infty} \frac{\cancel{n^4}^{\,n^2}}{4n^2+1} \cdot \underbrace{\frac{\sin\left(1/n^2\right)}{1/n^2}}_{\to 1} \cdot 1/\cancel{n^2}$$

$$\lim_{n\to 0} \frac{\sin(n)}{n} = 1$$

$$\lim \frac{\sin \square}{\square} = 1 \qquad (\square \to 0)$$

$$= \lim_{n\to\infty} \frac{n^2}{4n^2+1} = \frac{1}{4} \quad \Rightarrow \{a_n\} \text{ is convergent.}$$

Ex $a_n = \dfrac{2n^3}{3n+5} \cdot \tan\left(\dfrac{1}{n}\right)$ Check convergence of the sequence $\{a_n\}$.

$$\lim_{n\to\infty} \frac{2n^3}{3n+5} \cdot \underbrace{\frac{\tan\left(1/n\right)}{1/n}}_{\to 1} \cdot 1/n = \lim_{n\to\infty} \frac{\cancel{2n^3}^{\,n^2}}{3n+5} \cdot \frac{1}{\cancel{n}} = \infty$$

$$\lim \frac{\tan \square}{\square} = 1 \qquad (\square \to 0)$$

$\Rightarrow \{a\}$ is divergent.

Ex $a_n = \sqrt{n^2+n} - n$ Check convergence of the sequence $\{a_n\}$.

$$\lim_{n\to\infty} \frac{\left(\sqrt{n^2+n} - n\right)\left(\sqrt{n^2+n} + n\right)}{\left(\sqrt{n^2+n} + n\right)} = \lim_{n\to\infty} \frac{\cancel{n^2}+n - \cancel{n^2}}{\left(\sqrt{n^2+n} + n\right)} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+n} + n}$$

$$= \lim_{n\to\infty} \frac{n}{\sqrt{n^2\left(1+\frac{1}{n}\right)} + n} = \lim_{n\to\infty} \frac{\cancel{n}}{\cancel{n}\left(\sqrt{1+\frac{1}{n}} + 1\right)} = \frac{1}{2}$$

$\frac{1}{n} \to 0$, $1+0=1$

$\Rightarrow \{a_n\}$ is convergent.

Ex $a_n = n \cdot \left(2^{1/n} - 1\right)$ Check convergence of the sequence $\{a_n\}$.

$$\lim_{n\to\infty} \underbrace{n}_{\infty} \cdot \left(2^{1/n \to 0} - 1\right) \to 0 \cdot \infty \text{ indeterminacy.}$$

$$2^0 - 1 = 1 - 1 = 0$$

$$= \lim_{n\to\infty} \frac{2^{1/n} - 1}{1/n} \quad \frac{0}{0} \xrightarrow{L'} \lim_{n\to\infty} \frac{\cancel{-1/n^2} \cdot 2^{1/n} \cdot \ln 2}{\cancel{-1/n^2}} = \lim_{n\to\infty} 2^{1/n \to 0} \cdot \ln 2 = 1 \cdot \ln 2$$

üssünün türevi

$\Rightarrow \{a_n\}$ is convergent.

Ex $a_n = \dfrac{2^n}{(n+1)!}$ Check convergence of the sequence $\{a_n\}$.

- bounded ? lower bound + upper bound. ✓
- monotone ? inc. ? or dec.? ✓

$\Rightarrow$ Convergent

$n = 1, 2, \ldots \to \infty$

$\cancel{\lim_{n\to\infty} \dfrac{2^n}{(n+1)!}}$

Bounded ? $0 < a_n = \dfrac{2^n}{(n+1)!}$ (lower bound.)

$$\frac{2^n}{(n+1)!} < \overset{?}{M}$$

$$\frac{\overbrace{2 \cdot 2 \cdot 2 \cdots 2 \cdot 1}^{n}}{1 \cdot 2 \cdot 3 \cdot 4 \cdots n \cdot (n+1)} \quad 2 \cdot \frac{1}{1} \frac{1}{1} \frac{1}{1} \frac{1}{1} < 2 \text{ (upper bound.)}$$

$\{a_n\}$ is bounded.

Monotone inc./dec ?

$$\frac{a_{n+1}}{a_n} \quad >1 \;\; <1 \;\; \overset{?}{\lessgtr} 1 \qquad a_n = \frac{2^n}{(n+1)!}$$

$$a_{n+1} = \frac{2^{n+1}}{(n+1+1)!} \qquad \frac{a_{n+1}}{a_n} = \frac{\dfrac{2^{n+1}}{(n+2)!}}{\dfrac{2^n}{(n+1)!}} = \frac{\cancel{2^n} \cdot 2}{(n+2)\cancel{(n+1)!}} \cdot \frac{\cancel{(n+1)!}}{\cancel{2^n}} = \frac{2}{n+2} < 1$$

$n = 1, 2, \ldots \infty$

$\Rightarrow a_{n+1} < a_n \Rightarrow \{a_n\}$ is monotone decreasing.

$\Rightarrow$ By monotone-convergence thm. $\rightarrow \{a_n\}$ is convergent.

Bounded + Monotone inc/dec. $\Rightarrow$ Conv.

Ex $\quad a_n = \dfrac{3n}{2n+1}$

a) Is $\{a_n\}$ bounded ?

b) Is $\{a_n\}$ monotone inc / dec ?

c) Is $\{a_n\}$ convergent ? ✓

a) $\underbrace{0}_{\text{lower bound.}} < a_n \qquad \dfrac{3n}{2n+1} < \dfrac{3\cancel{n}}{2\cancel{n}} = \underbrace{\dfrac{3}{2}}_{\text{upper bound.}} \quad \Rightarrow \{a_n\}$ is bounded.

b) $a_{n+1} - a_n = \dfrac{3(n+1)}{2(n+1)+1} - \dfrac{3n}{2n+1} = \dfrac{3(n+1)}{\underset{(2n+1)}{2n+3}} - \dfrac{3n}{\underset{(2n+3)}{2n+1}} = \dfrac{3(n+1)(2n+1) - 3n(2n+3)}{(2n+3)(2n+1)} > 0$

$(3n+3)(2n+1) - 6n^2 - 9n$

$\cancel{6n^2} + \cancel{6n} + \cancel{3n} + 3 - \cancel{6n^2} - \cancel{9n}$

$= \dfrac{3}{(2n+3)(2n+1)} > 0$

$\Rightarrow a_{n+1} > a_n \qquad \Rightarrow \{a_n\}$ is monotone increasing.

$(a + b) \Rightarrow$ c) $\{a_n\}$ is convergent. $\square$.