# 10th Week

25 Nisan 2023 Salı 13:30

## Sequences

Series

Dizi

Infinite Sequences of Real Numbers

Ex

$$\left\{ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots \right\}$$

$n=1$ $n=2$ $n=3$ $\longrightarrow \infty$

$a_1 = \frac{1}{2}$ $a_2 = \frac{2}{3}$ $a_3 = \frac{3}{4}$

$\frac{n}{n+1}$

general term, general formula

$$a_n = \frac{n}{n+1}, \quad n \geq 1$$

not a function of n every time.

$$\left\{ \frac{n}{n+1} \right\}_{n=1}^{\infty}$$

Ex

$$\left\{ -\frac{2}{3}, \frac{3}{9}, -\frac{4}{27}, \frac{5}{81}, \ldots \right\}$$

$n=1$ $n=2$ $n=3$ $n=4$

Find a general formula for the given sequence.

$$a_n = \frac{(-1)^n \cdot (n+1)}{3^n} \quad n \geq 1$$

OR

$$\left\{ -\frac{2}{3}, \frac{3}{9}, -\frac{4}{27}, \frac{5}{81}, \ldots \right\}$$

$n=0$ $n=1$ $n=2$ $n=3$

$$a_n = \frac{(-1)^{n+1}(n+2)}{3^{n+1}}, \quad n \geq 0$$

$n \geq 2$

Ex

Find a formula for the general term $a_n$ of the sequence

$$\left\{ \frac{3}{5}, -\frac{4}{25}, \frac{5}{125}, -\frac{6}{625}, \frac{7}{3125}, \ldots \right\}$$

$n=1$ $n=2$ $n=3$ $n=4$ $n=5$

$n \geq 1$

$$a_n = \frac{(-1)^{n+1}(n+2)}{5^n}, \quad n \geq 1$$

## Limit of a Sequence

$$\lim_{n \to \infty} a_n = ?$$

Mat-I Limits at infinity

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$a_n =$

! If $\lim_{n\to\infty} a_n$ exists $\Rightarrow$ $\{a_n\}$ is **convergent**.

! If $\lim_{n\to\infty} a_n$ = DNE, $\pm\infty$ $\Rightarrow$ $\{a_n\}$ is **divergent**.

$$a_n = f(n)$$

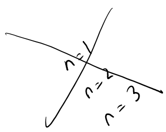
$\Rightarrow \lim_{n\to\infty} \dfrac{p(n)}{q(n)}$

$\deg(p) > \deg(q) \Rightarrow \lim = \pm\infty$ ✓

$\deg(p) < \deg(q) \Rightarrow \lim = 0$

$\deg(p) = \deg(q) \Rightarrow$ başkatsayıların oranı

ex/ $a_n = \dfrac{3n^2 - 5n + 7}{-6n^2 + 4}$ $\quad n \geq 1$ $\quad$ Is $\{a_n\}$ convergent?




$\lim_{n\to\infty} a_n = -\dfrac{3}{6} = -\dfrac{1}{2} \Rightarrow \{a_n\}$ is convergent.

$\Rightarrow a_n = \dfrac{\sim}{\sim} \rightarrow$ not rational

Ex/ $a_n = \dfrac{n^2+1}{\sqrt{4n^4 - 3n^3 + n - 1}}$ $\quad$ Is $\{a_n\}$ convergent?

$n \to \infty$: $\dfrac{1}{n} \to 0$

$$\lim_{n\to\infty} \frac{n^2\left(1 + \frac{1}{n^2}\right)}{\sqrt{n^4\left(4 - \frac{3}{n} + \frac{1}{n^3} - \frac{1}{n^4}\right)}} = \lim_{n\to\infty} \frac{n^2}{\sqrt{n^4 \cdot 4}} = \frac{1}{2}$$

$\frac{1}{n^2} \to 0$, $\frac{3}{n} \to 0$, $\frac{1}{n^3} \to 0$, $\frac{1}{n^4} \to 0$

$n \geq 1$

$2n^2$

$\Rightarrow \{a_n\}$ is convergent.

Ex/ $a_n = \dfrac{n}{\sqrt{10+n}}$

$\sqrt{n}$

$\sqrt{10+n}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{\sqrt{n\left(\frac{10}{n}+1\right)}} = \lim_{n\to\infty} \frac{\cancel{n}^{\sqrt{n}}}{\cancel{\sqrt{n}}} = \lim_{n\to\infty} \sqrt{n} = \infty$$

$\frac{10}{n} \to 0$

$\Rightarrow \{a_n\}$ is divergent.

Ex/ $a_n = \frac{\ln n}{n}$ $\quad n \geq 1$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\ln n}{n} \quad \frac{\infty}{\infty} \xrightarrow{\text{L'Hospital}} \lim_{n\to\infty} \frac{1/n}{1} = 0$$

$\{a_n\}$ is convergent.

What If $a_n = f(n) \rightarrow$ not a function!

! Ex/ $a_n = (-1)^n$ $\quad \{-1, 1, -1, 1, -1, 1, -1, \ldots\}$

$n \geq 1$ $\quad \lim_{n\to\infty} a_n = \text{DNE} \Rightarrow \{a_n\}$ is divergent.

Teorem

$$\lim_{n\to\infty} |a_n| = 0 \Rightarrow \lim_{n\to\infty} a_n = 0 \quad \{\text{convergent}\}$$

$\Leftarrow$

Ex/ $a_n = \frac{(-1)^n}{n}$ $\quad$ convergent/? div.

$n \geq 1$

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{1}{n} = 0$$

$|a_n| = \frac{1}{n}$

$\Rightarrow \lim_{n\to\infty} a_n = 0 \Rightarrow \{a_n\}$ is convergent.

Ex/ $a_n = \frac{n!}{n^n}$ $\quad$ Is $\{a_n\}$ convergent?

n tane

$n^n$

$n \geq 1$

$$\lim_{n\to\infty} \frac{n!}{n^n}$$

$$\frac{n!}{n^n} = \frac{\overbrace{1 \cdot 2 \cdot 3 \cdot \ldots \cdot n-1 \cdot n}^{n \text{ tane}}}{n \cdot n \cdot n \cdot \ldots \cdot n \cdot n}$$

$$< \frac{1 \cdot \overbrace{\cancel{n} \cdot \cancel{n} \cdot \cancel{n} \cdot \cancel{n} \cdots \cancel{n}}^{n-1}}{n \cdot \cancel{n} \cdot \cancel{n} \cdot \cancel{n} \cdot \cancel{n} \cdots \cancel{n}} = \frac{1}{n}$$

Sandwich thm

$$b_n < a_n < c_n$$

$a_n = 0$

$b_n \to \lim = L$, $c_n \to \lim = L$, $a_n \to \lim = L$

$$b_n = 0 < a_n = \frac{n!}{n^n} < c_n = \frac{1}{n}$$

$b_n$: $\lim = 0$, $c_n$: $\lim = 0$

by sand. thr.

$$\lim_{n\to\infty} a_n = 0 \quad \Rightarrow \{a_n\} \text{ is convergent.}$$

## ! Monotonic Sequences $(n \geq 1)$

$n \to \infty$

If for all $n \geq 1$, $a_{n+1} > a_n \Rightarrow \{a_n\}$ is monotone increasing.

If for all $n \geq 1$, $a_{n+1} < a_n \Rightarrow \{a_n\}$ is monotone decreasing.

$\Rightarrow$ If $\{a_n\}$ is mon. inc. or mon. dec $\Rightarrow \{a_n\}$ is monotone.

örn

$$a_n = \frac{n}{n+1} \Rightarrow a_{n+1} = \frac{n+1}{(n+1)+1} = \frac{n+1}{n+2}$$

$n \geq 1$

$a_{n+1} \underset{?}{>} a_n \quad \forall n \geq 1 \Rightarrow$ mon. inc. ✓

$a_{n+1} < a_n \quad \forall n \geq 1 \Rightarrow$ mon. dec.

$$a_n = \frac{n}{n+1} \qquad a_{n+1} = \frac{n+1}{n+2}$$

$$a_n = \frac{n(n+2)}{(n+1)(n+2)} < a_{n+1} = \frac{(n+1)(n+1)}{(n+1)(n+2)}$$

$n^2+2n \qquad n^2+2n+1$

! $a_n = f(n)$ → a function of n.

$f'(n) > 0 \Rightarrow \{a_n\}$ is mon inc.

$f'(n) < 0 \Rightarrow \{a_n\}$ is mon. dec.

! $a_{n+1} - a_n$ ⇝ n cinsinden

$> 0 \Rightarrow \{a_n\}$ is mon. inc.

$< 0 \Rightarrow \{a_n\}$ is mon. dec.

! $\dfrac{a_{n+1}}{a_n}$ ⇝ n cinsinden

$> 1 \Rightarrow \{a_n\}$ is mon. inc.

$< 1 \Rightarrow \{a_n\}$ is mon. dec.

Ex $a_n = \dfrac{1}{n+1} + \dfrac{1}{n+2} + \dfrac{1}{n+3} + \dots + \dfrac{1}{2n}$ , $n \geq 1$ Is the sequence $\{a_n\}$ monotone inc/dec?

~~$n=1$ $a_1 = \frac{1}{2}$~~

~~$n=2$ $a_2 = \frac{1}{3} + \frac{1}{4}$~~

~~$n=3$ $a_3 = \frac{1}{4} + \frac{1}{5} + \frac{1}{6}$~~

$$a_{n+1} = \cancel{\frac{1}{\underbrace{n+1+1}_{n+2}}} + \cancel{\frac{1}{\underbrace{n+1+2}_{n+3}}} + \cancel{\frac{1}{\underbrace{n+1+3}_{n+4}}} + \cancel{\dots} + \frac{1}{2n} + \frac{1}{2n+1} + \frac{1}{\underbrace{2(n+1)}_{2n+2}}$$

$$a_n = \frac{1}{n+1} + \cancel{\frac{1}{n+2}} + \cancel{\frac{1}{n+3}} + \cancel{\dots} + \cancel{\frac{1}{2n}}$$

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$$a_{n+1} - a_n = \frac{1}{2n+1} + \underbrace{\frac{1}{2n+2} - \frac{1}{n+1}}_{\frac{1}{2(n+1)} - \frac{2}{2(n+1)} = -\frac{1}{2(n+1)}} = \frac{1}{2n+1} - \frac{1}{\underbrace{2(n+1)}_{2n+2}} > 0$$

$\Rightarrow \{a_n\}$ is convergent

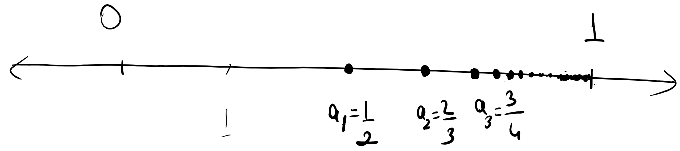
# Boundedness (Sınırlılık)

$\forall n, \quad a_n < M$ If $M \in \mathbb{R}$ exists, $\{a_n\}$ is bounded from above.

$\forall n, \quad a_n > m$ If $m \in \mathbb{R}$ exists, $\{a_n\}$ is bounded from below.

If $\{a_n\}$ is bounded both from above & below $\Rightarrow$ we say $\{a_n\}$ is bounded.

Ex/ $a_n = \dfrac{n}{n+1}$




$n=1$
$n=2$
$n=3$

$\underbrace{0 < a_n}_{\text{lower bound}}$ $\quad$ $\underbrace{a_n < 1}_{\text{upper bound}}$

$\Rightarrow \{a_n\}$ is bounded.

? Are all bounded sequences convergent?

Ex/ $a_n = (-1)^n \qquad \{-1, 1, -1, 1, \ldots\}$

$\lim_{n\to\infty} a_n = \text{DNE}$ → divergent.

$\underbrace{-2 < a_n}_{\text{lower bound}} \qquad a_n < 2 \rightsquigarrow$ upper bound.

$\Rightarrow \{a_n\}$ is bounded.

? Are all monotonic sequences convergent?

Ex/ $a_n = n \longrightarrow$ monotone increasing.

but $\lim_{n\to\infty} a_n = \infty \Rightarrow$ divergent!

! All monotone & bounded sequences are convergent.

$a_{159} = ?$

$a_n = \dots$

Recursive seq.

$a_1 = 1$

$a_2 = 2$

fibonacci

$a_n = a_{n-1} + a_{n-2}, \quad n \geq 2$