Örnek 7. (Geometrik olmayan teleskopik seri)

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{n\to\infty} s_n$$

kısmi toplamı sadeleşebilen

ilk n terimin toplamını düşünelim;

lim s_n = ?

$$s_n = \sum_{i=1}^{n} \frac{1}{i(i+1)} = \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+1}\right)$$

$$= \underbrace{\left(1 - \cancel{\frac{1}{2}}\right)}_{i=1} + \underbrace{\left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right)}_{i=2} + \underbrace{\left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right)}_{i=3} + \cdots + \underbrace{\left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right)}_{i=n}$$

$$s_n = 1 - \frac{1}{n+1}$$

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(1 - \underbrace{\frac{1}{n+1}}_{0}\right) = 1 - 0 = 1 \checkmark$$

Örnek 8. (Harmonik Seri) → ıraksaktır.

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots = \lim_{n\to\infty} s_n$$

$\boxed{\text{Harmonik Seri} \quad \sum_{n=1}^{\infty} \frac{1}{n} \to \text{ıraksak}}$

Iraksak olduğunu gösterelim;

$s_2 = 1 + \frac{1}{2}$ → ilk iki terimin toplamı

$$s_4 = 1 + \frac{1}{2} + \underbrace{\left(\frac{1}{3} + \frac{1}{4}\right)} > 1 + \frac{1}{2} + \underbrace{\left(\frac{1}{4} + \frac{1}{4}\right)} = 1 + \frac{2}{2}$$

$$s_8 = 1 + \frac{1}{2} + \underbrace{\left(\frac{1}{3} + \frac{1}{4}\right)} + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)$$
$$> 1 + \frac{1}{2} + \underbrace{\left(\frac{1}{4} + \frac{1}{4}\right)}_{1/2} + \underbrace{\left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right)}_{1/2}$$
$$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1 + \frac{3}{2}$$

$$s_{16} = 1 + \frac{1}{2} + \left(\underbrace{\frac{1}{3} + \frac{1}{4}}_{2\text{ tane}}\right) + \left(\underbrace{\frac{1}{5} + \cdots + \frac{1}{8}}_{4\text{ tane}}\right) + \left(\underbrace{\frac{1}{9} + \cdots + \frac{1}{16}}_{8\text{ tane}}\right)$$
$$> 1 + \frac{1}{2} + \underbrace{\left(\frac{1}{4} + \frac{1}{4}\right)}_{2/4} + \underbrace{\left(\frac{1}{8} + \cdots + \frac{1}{8}\right)}_{4/8} + \underbrace{\left(\frac{1}{16} + \cdots + \frac{1}{16}\right)}_{8/16}$$
$$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1 + \frac{4}{2}$$

$s_4 > 1 + \frac{2}{2}$

$s_8 > 1 + \frac{3}{2}$

$s_{16} > 1 + \frac{4}{2}$

$s_{32} > 1 + \frac{5}{2}$

$s_{64} > 1 + \frac{6}{2}$

$s_{32} > 1 + \frac{5}{2}$, $s_{64} > 1 + \frac{6}{2}$

$$n\to\infty \quad \underbrace{s_{2^n}}_{\to\infty} > \underbrace{1 + \frac{n}{2}}_{\to\infty} \Rightarrow \lim_{n\to\infty} s_n = \text{ıraksaktır.}$$

TEOREM

$\underbrace{\sum_{n=1}^{\infty} a_n \text{ yakınsıyorsa}}_{p}$, $\underbrace{a_n \to 0}_{q}$ olur.

! Tersi doğru değil ~~($a_n \to 0 \Rightarrow \sum a_n$ yakınsar)~~

$p \Rightarrow q$ $\quad$ ~~$q \Rightarrow p$~~

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (\overset{a_1+a_2+\cdots+a_{n-1}+a_n}{s_n} - \overset{a_1+a_2+\cdots+a_{n-1}}{s_{n-1}}) = \underbrace{\lim_{n\to\infty} s_n}_{=s} - \underbrace{\lim_{n\to\infty} s_{n-1}}_{=s}$$
$$= s - s = 0$$

$n\to\infty$
$n-1\to\infty$

$p \Rightarrow q$ ✓

$\equiv q' \Rightarrow p'$ ✓

$= s - s = 0$ $\quad =s \quad =s$

Iraksaklık için n. Terim Testi

$\lim_{n\to\infty} a_n$ yoksa veya sıfırdan farklıysa, $\sum_{n=1}^{\infty} a_n$ ıraksar.

$q' \Rightarrow p'$

Örnekler

(a) $\sum_{n=1}^{\infty} n^2$ ıraksar, çünkü $n^2 \to \infty$

(b) $\sum_{n=1}^{\infty} \frac{n+1}{n}$ ıraksar, çünkü $\frac{n+1}{n} \to 1$

(c) $\sum_{n=1}^{\infty} (-1)^{n+1}$ ıraksar, çünkü $\lim_{n\to\infty}(-1)^{n+1}$ yoktur

(d) $\sum_{n=1}^{\infty} \frac{-n}{2n+5}$ ıraksar, çünkü $\lim_{n\to\infty}\frac{-n}{2n+5} = -\frac{1}{2} \neq 0$ dır.

TEOREM

$\sum a_n = A$ ve $\sum b_n = B$ yakınsak serilerse, aşağıdaki kurallar geçerlidir.

1. *Toplam Kuralı:* $\sum(a_n + b_n) = \sum a_n + \sum b_n = A + B$ ✓
2. *Fark Kuralı:* $\sum(a_n - b_n) = \sum a_n - \sum b_n = A - B$ ✓
3. *Sabitle Çarpım Kuralı:* $\sum ka_n = k\sum a_n = kA$ (Herhangi bir k) ✓

* Iraksak serinin sıfırdan farklı sabit bir katı yine ıraksaktır.

* $\sum a_n$ yakınsak $\sum b_n$ ıraksak olsun. Hem $\sum(a_n+b_n)$ hem de $\sum(a_n-b_n)$ ıraksaktır.

* $\sum a_n$ ve $\sum b_n$ ıraksak ise bakılmalıdır.

Ör/ $\sum 1$, $\quad \sum -1 \quad S_n = -n$ $\qquad \sum 1+(-1) = 0 \to$ yakınsak.

$= 1+1+1+\dots \quad S_n = n$ $\qquad = (-1)+(-1)+(-1)$ $\qquad 0+0+0\dots$

$\to$ ıraksak $\qquad \to$ ıraksak

Ör/

$$\sum_{n=1}^{\infty}\left(\underbrace{\frac{3}{n(n+1)}}_{\text{teleskopik}} + \underbrace{\frac{1}{2^n}}_{\text{geometrik}}\right) = 3\underbrace{\sum_{n=1}^{\infty}\frac{1}{n(n+1)}}_{3} + \underbrace{\sum_{n=1}^{\infty}\frac{1}{2^n}}_{1} = 3+1 = 4$$

yak ✓ $\quad$ yak ✓

$$3\underbrace{\sum_{n=1}^{\infty}\frac{1}{n(n+1)}}_{1}$$

$$\sum_{n=1}^{\infty} ar^{n-1} = \sum \frac{1}{2}\cdot\frac{1}{2^{n-1}} = \frac{a}{1-r} = \frac{1/2}{1-1/2} = 1$$

$a = 1/2 \quad r = 1/2$

$|r| < 1$ ✓

Ör/ $5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \dots$ serisi için $a=?$ $r=?$

seri yakınsak ise değeri nedir?

$$\sum_{n=1}^{\infty} ar^{n-1}$$

$5\left(1 - \frac{2}{3} + \frac{4}{9} - \frac{8}{27} + \dots\right)$

$a + ar + ar^2 + \dots$

$a(1+r+r^2+\dots)$

$a=5 \quad r = -\frac{2}{3}$ $\qquad \sum_{n=1}^{\infty} 5\cdot\left(-\frac{2}{3}\right)^{n-1}$ $\qquad |r|<1$ ✓ yakınsaktır ✓

$a=5 \quad r=-\frac{2}{3}$

$$\sum_{n=1}^{\infty} 5\cdot\left(-\frac{2}{3}\right)^{n-1} \qquad |r|<1 \checkmark \qquad a(1+r+r^2+\dots)$$

yakınsaktır ✓

$$= \frac{a}{1-r} = \frac{5}{1+2/3} = \frac{5}{5/3} = 3$$

örn

$$10 - 2 + 0.4 - 0.08 + \dots\dots \qquad \sum_{n=1}^{\infty} a r^{n-1}$$

$2^2 \; 10^{-1} \qquad 2^3 \; 10^{-2}$

$$10\left(1 - 2.10^{-1} + 4.10^{-2} - 2^3.10^{-3} \dots\right)$$

$$10\left(1 - \frac{2}{10} + \frac{4}{10^2} - \frac{8}{10^3} \dots\dots\right)$$

$a=10 \quad r=-2/10 \quad |r|<1$

$\Rightarrow$ yakınsak

$$\sum_{n=1}^{\infty} 10.\left(-\frac{2}{10}\right)^{n-1} = \frac{a}{1-r} = \frac{10}{1+\frac{2}{10}} = \frac{100}{12}$$

örn

$$2,3\overline{17} = 2,3171717\dots$$

$$= 2,3 + 0,017 + 0,00017 + \dots$$

$$= 2,3 + \left(\frac{17}{10^3} + \frac{17}{10^5} + \frac{17}{10^7} + \frac{17}{10^9} + \dots\right) \qquad a(1+r+r^2+r^3+\dots$$

$$= 2,3 + \underbrace{\frac{17}{10^3}\left(1 + \frac{1}{10^2} + \frac{1}{10^4} + \frac{1}{10^6} + \dots\right)}$$

$a = \frac{17}{10^3} \quad r = \frac{1}{10^2} \quad |r|<1 \checkmark \qquad \frac{a}{1-r} = \frac{17/10^3}{1-1/10^2} = \frac{17 \cdot 10^2}{10^3 \cdot 99} = \frac{17}{990}$

$$= \frac{23}{10}_{(99)} + \frac{17}{990} = \frac{2294}{990}$$

örn

$$\sum_{n=1}^{\infty} \ln\frac{n}{n+1}$$ serisinin yakınsaklığı nedir, yakınsak ise değeri nedir?

$$S_1 = \ln\frac{1}{2}$$

$$S_2 = \ln\frac{1}{2} + \ln\frac{2}{3}$$

$$S_3 = \ln\frac{1}{2} + \ln\frac{2}{3} + \ln\frac{3}{4} = (\ln 1 - \ln 2) + (\ln 2 - \ln 3) + (\ln 3 - \ln 4)$$

$\vdots$

$$S_n = \ln\frac{1}{2} + \ln\frac{2}{3} + \dots\dots \ln\frac{n}{n+1} = (\cancel{\ln 1}^{0} - \cancel{\ln 2}) + (\cancel{\ln 2} - \cancel{\ln 3}) + \dots + (\cancel{\ln n} - \ln(n+1))$$

$$= -\ln(n+1)$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} -\ln(n+1) = -\infty \to \text{ıraksaktır.}$$

46. $\sum_{n=1}^{\infty}\left(\cos\frac{1}{n^2}-\cos\frac{1}{(n+1)^2}\right)$

$s_1 = \cos 1 - \cos\frac{1}{4}$

$s_2 = s_1 + \cos\frac{1}{4} - \cos\frac{1}{9}$

$s_3 = s_1 + s_2 + \cos\frac{1}{9} - \cos\frac{1}{16}$

$s_n = \left(\cos 1 - \cancel{\cos\frac{1}{4}}\right) + \left(\cancel{\cos\frac{1}{4}} - \cancel{\cos\frac{1}{9}}\right) + \left(\cancel{\cos\frac{1}{9}} - \cancel{\cos\frac{1}{16}}\right) + \dots + \left(\cancel{\cos\frac{1}{n}} - \cos\frac{1}{(n+1)^2}\right)$

$s_n = \cos 1 - \cos\frac{1}{(n+1)^2}$

$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \cos 1 - \cos\frac{1}{(n+1)^2} = \cos 1 - 1$