A Note on Dual Codes of Pseudo-cyclic Codes

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Contents

Objective

In this report, as a result of our periodic studies, we introduce a method to obtain a direct construction for the dual codes of pseudo-cyclic codes.

Introduction

- Pseudo-cyclic codes over finite fields were first introduced by (Peterson and Weldon, 1972).

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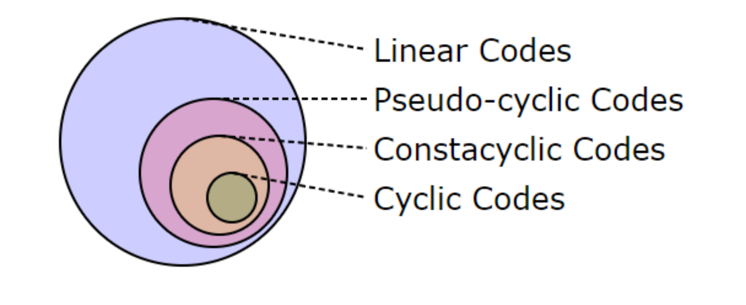
- Pseudo-cyclic codes over finite fields were first introduced by (Peterson and Weldon, 1972).
- Although every pseudo-cyclic code corresponds to a shortened cyclic code over finite fields, in terms of introducing a direct construction, pseudo-cyclic codes have attracted many researchers with their rich algebraic structure.

Generalization of Cyclic Codes



Pseudo-cyclic Shift

Let F_q be a finite field with q elements and let F_q^n be the $n-$dimensional vector space over F_q.
Let $c = (c_0, c_1, \ldots, c_{n-1})$ be any vector in F_q^n . We fix a shift vector $v = (v_0, v_1, \ldots, v_{n-1})$ and define the following linear transformation

$$\begin{aligned} \tau_v \colon \; F^n & \longrightarrow F^n \\ (c_0, c_1, \ldots, c_{n-1}) & \mapsto (v_0 c_{n-1}, c_0 + v_1 c_{n-1}, \ldots, c_{n-2} + v_{n-1} c_{n-1}) \end{aligned}$$

- It has the following representation matrix as $\tau_v(c) = c.T_v$ and T_v is exactly the companion matrix for $f(x) = x^n - v(x)$.

$$T_v = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & 0 \\ \vdots & 0 & \cdots & 0 & 1 \\ v_0 & v_1 & \cdots & v_{n-2} & v_{n-1} \end{bmatrix}_{nxn}$$

Example

Let $c = (c_0, c_1, c_2)$ be a vector in some vector space F_q^3. Let $v = (v_0, v_1, v_2)$ be the shift vector.
Thus we have

$$T_v = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ v_0 & v_1 & v_2 \end{bmatrix}$$

And the transformation τ_v moves $c = (c_0, c_1, c_2)$ to the vector $\tau_v(c) = (v_0c_2, c_0 + v_1c_2, c_1 + v_2c_2)$ as follows;

$$\begin{aligned} \tau_v(c) &= c \cdot T_v = \begin{bmatrix} c_0 & c_1 & c_2 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ v_0 & v_1 & v_2 \end{bmatrix} \\ &= \begin{bmatrix} v_0c_2 & c_0 + v_1c_2 & c_1 + v_2c_2 \end{bmatrix} \end{aligned}$$

- T_v is the companion matrix for $f(x) = x^3 - (v_0 + v_1x + v_2x^2)$.

Pseudo-cyclic Codes

Definition

A linear code C with length n over a finite field F_q is called $pseudo-cyclic$ with respect to the vector $v = (v_0, v_1, \ldots, v_{n-1})$, if whenever $c = (c_0, c_1, \ldots, c_{n-1})$ is in C, so is its $v-$pseudo-cyclic shift $(v_0 c_{n-1}, c_0 + v_1 c_{n-1}, \ldots, c_{n-2} + v_{n-1} c_{n-1})$.

- A pseudo-cyclic code with respect to v is invariant under τ_v.

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- A pseudo-cyclic code with respect to v is invariant under τ_v.
- Any cyclic code is $pseudo-cyclic$ with respect to $v = (1, 0, \ldots, 0)$ and $v(x) = 1$.

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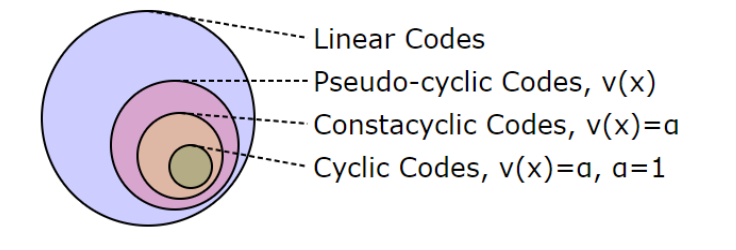
- A pseudo-cyclic code with respect to v is invariant under τ_v.
- Any cyclic code is $pseudo-cyclic$ with respect to $v = (1, 0, \ldots, 0)$ and $v(x) = 1$.
- Any constacyclic code with respect to α, is $pseudo-cyclic$ with respect to $v = (\alpha, 0, \ldots, 0)$ and $v(x) = \alpha$.

Polynomial Correspondence



- $C \lhd F_q[x]/(f(x))$, $f(x) = x^n - v(x)$

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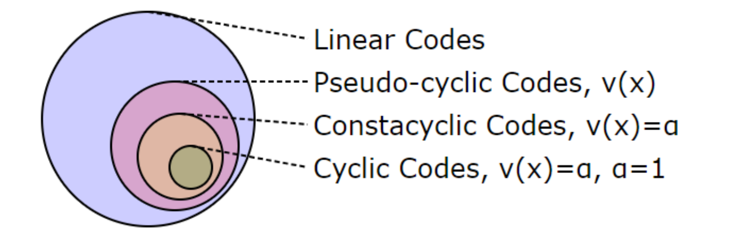

- $C \lhd F_q[x]/(f(x))$, $f(x) = x^n - v(x)$
- $C \lhd F_q[x]/(f(x))$, $f(x) = x^n - \alpha$, $\alpha \in F_q^*$

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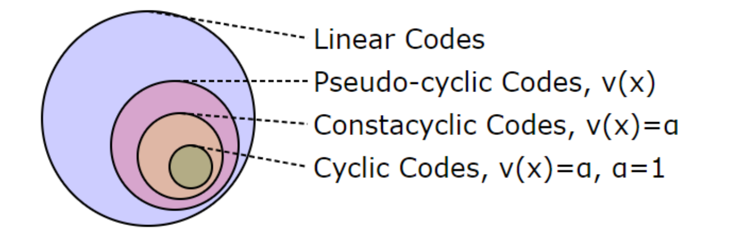

- $C \lhd F_q[x]/(f(x)),\ f(x) = x^n - v(x)$
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- $C \lhd F_q[x]/(f(x)),\ f(x) = x^n - 1$

- In terms of the usual correspondence to the polynomial ring $F_q[x]/(x^n - v(x))$, multiplying a polynomial by x corresponds to a pseudo-cyclic shift with respect to v, therefore a $pseudo-cyclic$ code over F_q^n corresponds to an ideal in $F_q[x]/(x^n - v(x))$.

Example

Consider $c(x) = c_0 + c_1x + c_2x^2$. Let $v(x) = v_0 + v_1x + v_2x^2$, so we are in $F_q[x]/(x^3 - v(x))$.
Multiplying $c(x)$ by x, we get;

$$\begin{aligned}
(c_0 + c_1x + c_2x^2).x &= c_0x + c_1x^2 + c_2x^3 \\
&= c_0x + c_1x^2 + c_2(v(x)) \\
&= c_0x + c_1x^2 + c_2(v_0 + v_1x + v_2x^2) \\
&= c_2v_0 + (c_0 + c_2v_1)x + (c_1 + c_2v_2)x^2
\end{aligned}$$

So this gives us the pseudo-cyclic shift,

$$(c_0, c_1, c_2) \to (c_2v_0, c_0 + c_2v_1, c_1 + c_2v_2)$$

Different Definitions

- Pseudo-cyclic shift, Pseudo-cyclic codes
- Polycyclic shift, Polycyclic codes
- $p(x)-$circulants, Generalized cyclic codes
- $v-$vector cyclic shift, $v-$vector based codes
- $\theta-$polycyclic shift, module $\theta-$codes

Formulation of the Problem

- Pseudo-cyclic codes are fully characterized over finite fields and finite chain rings (Lopez et al., 2013; Bedir and Siap, 2015; Alahamdi et al., 2016)

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- Pseudo-cyclic codes are fully characterized over finite fields and finite chain rings (Lopez et al., 2013; Bedir and Siap, 2015; Alahamdi et al., 2016)
- They have been constructed as module $\theta-$codes over skew polynomial rings (Boucher and Ulmer, 2011).
- However, the problem of finding a concrete generator for the dual code was open both over the commutative and noncommutative cases.

For the commutative case, the following theorem gives an indirect method for generating the dual code;

Theorem (Bedir and Siap, 2015)

If $g(x) = f(x)/h(x)$ is the generating polynomial of a pseudo-cyclic code C, then $G = g(T_v)$ is a generator matrix for C and $H = h^R((T_v^{-1})^{tr})$ is a parity check matrix for C, where $h^R(x)$ is the reciprocal polynomial of $h(x)$ $(h^R(x) = h(1/x)x^{\deg(h(x))})$.

Sequential Codes

- The dual code of a polycyclic code is a type of "sequential code" (Lopez et al., 2009).

Definition

A linear code C with length n over a finite field F is called *sequential* with respect to the the vector $\omega = (\omega_0, \omega_1, \ldots, \omega_{n-1})$, if there is a function $\varphi_\omega : F^n \longrightarrow F$ such that whenever $c = (c_0, c_1, \ldots, c_{n-1})$ is in C, so is $(\varphi_\omega(c_0, c_1, \ldots, c_{n-1}), c_0, c_1, \ldots, c_{n-2})$.

The Dual Code of a Pseudo-cyclic Code

- Let C be a pseudo-cyclic code with respect to $v = (v_0, v_1, \ldots, v_{n-1})$, with generating polynomial $g(x) = g_0 + g_1 x + \cdots + g_{n-1} x^{n-1}$, and let $h(x) = (x^n - v(x))/g(x)$.

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- Set $\omega = (v_0^{-1}, -v_{n-1}/v_0, -v_{n-2}/v_0, \ldots, -v_1/v_0)$. And consider the following transformation;

$$\begin{aligned}\rho_\omega : \quad & F^n \longrightarrow F^n \\ & (c_0, c_1, \ldots, c_{n-1}) \longmapsto (\omega_{n-1}c_0 + \omega_{n-2}c_1 + \cdots + \omega_0 c_{n-1}, \\ & \qquad c_0, c_1, .., c_{n-2})\end{aligned}$$

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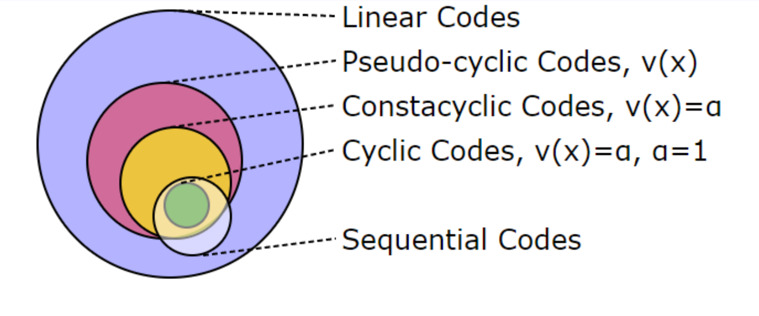
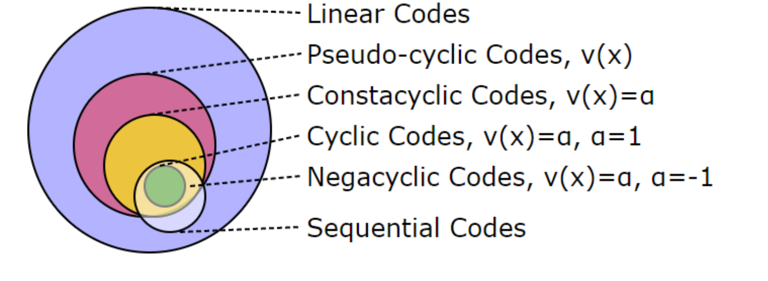
- Set $\omega = (v_0^{-1}, -v_{n-1}/v_0, -v_{n-2}/v_0, \ldots, -v_1/v_0)$. And consider the following transformation;

$$\begin{array}{lll} \rho_\omega : & F^n & \rightarrow F^n \\ & (c_0, c_1, \ldots, c_{n-1}) & \mapsto (\omega_{n-1} c_0 + \omega_{n-2} c_1 + \cdots + \omega_0 c_{n-1}, \\ & & c_0, c_1, .., c_{n-2}) \end{array}$$

- The matrix represenation for ρ_ω is exactly $(T_v^{-1})^{tr}$, and note that v_0 should be invertible in any case.

- The dual code of a pseudo-cyclic code with respect to $v = (v_0, v_1, \ldots, v_{n-1})$ is therefore a sequential code with respect to $\omega = (v_0^{-1}, -v_{n-1}/v_0, \ldots, -v_1/v_0)$, where $\varphi_\omega(c_0, c_1, \ldots, c_{n-1}) = \omega_{n-1}c_0 + \omega_{n-2}c_1 + \cdots + \omega_0 c_{n-1}$.


Linear Codes
Pseudo-cyclic Codes, v(x)
Constacyclic Codes, v(x)=a
Cyclic Codes, v(x)=a, a=1
Sequential Codes


Linear Codes
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Sequential Codes

Pseudo-cyclic Codes vs Sequential Codes

- Pseudo-cyclic codes have an ideal structure, and over the corresponding polynomial ring, we are able to find a generating polynomial; which gives a generating vector and provides constructing a vector-circulant generating matrix.

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- Pseudo-cyclic codes have an ideal structure, and over the corresponding polynomial ring, we are able to find a generating polynomial; which gives a generating vector and provides constructing a vector-circulant generating matrix.
- However, sequential codes do not have an ideal structure. The transformation does not correspond to multiplication by x in the polynomial correspondence.
- So, how can we obtain a generating polynomial/ generating vector a for sequential codes (as the dual code of pseudo-cyclic codes), so that we obtain a direct construction as follows;

$$\begin{aligned} H &= \begin{bmatrix} \cdots & a & \cdots \\ \cdots & \rho_{\omega}(a) & \cdots \\ \cdots & \rho_{\omega}^{2}(a) & \cdots \\ & \vdots & \\ \cdots & \rho_{\omega}^{n-1}(a) & \cdots \end{bmatrix}_{nxn} . \\ a &= ??? \end{aligned}$$

Shortening Method

Let C' be an $[n, k, d]$-linear code over F_q. For a fixed $1 \leq i \leq n$, form the subset A of C' consisting of the codewords with the i^{th} position equal to 0. Delete the i^{th} position from all the words in A to form a code C. Then C is an $[n-1, k, d]$-linear code over F_q with $k-1 \leq k \leq k$, $d \geq d$. (Ling and Xing, 2004).

- A pseudo-cyclic code with generating polynomial $g(x) = g_0 + g_1 x + \cdots + g_{n-1} x^{n-1}$ can be obtained by shortening a cyclic code C' generated by $g(x)$.

Puncturing Method

Let D' be an $[n+r, k, d+r]$-linear code over F_q. Choose a set B of codewords in D' with distance $d+r$. Choose r non-zero coordinates, and delete these coordinates from all the codewords of D'. Then the new code D, is an $[n, k, d]$-linear code over F_q (Ling and Xing, 2004).

- The dual code of a pseudo-cyclic code with generating polynomial $g(x) = g_0 + g_1 x + \cdots + g_{n-1} x^{n-1}$ can be obtained by puncturing the dual code of a cyclic code C' generated by $g(x)$.

From Shortening and Puncturing to Pseudo-cyclic Codes and Their Duals

- Following the intiutions we get from the above correspondences, we derived a formula to obtain a generating vector for the dual codes of pseudo-cyclic codes.

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- Following the intiutions we get from the above correspondences, we derived a formula to obtain a generating vector for the dual codes of pseudo-cyclic codes.
- We used the cyclic code with the smallest length N for which $f(x)$ divides $x^N - 1$.

Theorem

Let $h(x)g(x) = f(x)$, $\deg(g(x)) = n - k$, $p(x) = \frac{x^N - 1}{f(x)} = \sum_{i=0}^{N-n} p_i x^i$ *and let* C *be the pseudo-cyclic code generated by* $g(x)$*. Then the dual code* D *is generated by the vector* $a = (a_0, a_1, \ldots, a_{n-1})$ *and its* $n - k - 1$ *sequential shifts, where*

$$a_0 = p_0 h_0,\ a_i = \sum_{j=0}^{i-1} p_{N-n-j} h_{n-i+j}, 1 \leq i \leq n-1$$

Example

Let F be the finite field with 4 elements;
$F_4 = \{0, 1, \alpha, \alpha^2 = \alpha + 1\}$. Let
$g(x) = \alpha^2 + \alpha x^2 + x^3, h(x) = 1 + \alpha x + x^2$ and
$f(x) = g(x)h(x) = x^5 + \alpha x^3 + x^2 + x + \alpha^2$.
Let T_v be the companion matrix of $f(x)$.
Consider the pseudo-cyclic code C generated by $g(x)$ over F_4.
We obtain the generating matrix of C as follows;

$$\begin{aligned} G &= \begin{bmatrix} \cdots & g & \cdots \\ \cdots & g \cdot T_v & \cdots \end{bmatrix}_{2x5} \\ &= \begin{bmatrix} \alpha^2 & 0 & \alpha & 1 & 0 \\ 0 & \alpha^2 & 0 & \alpha & 1 \end{bmatrix}_{2x5} \end{aligned}$$

Example (contd.)

We have $f(x)|x^{15}-1$ and we set $N=15.$
In this case we have

$$\begin{aligned} p(x) &= \frac{x^N-1}{f(x)} \\ &= \alpha+\alpha^2x+\alpha x^2+\alpha^2x^3+\alpha^2x^5+\alpha x^6+x^7+\alpha x^8+x^{10} \end{aligned}$$

Using the above formula

$$a_0 = p_0h_0 \text{ and } a_i = \sum_{j=0}^{i-1} p_{N-n-j}h_{n-i+j},$$

we get

$$a = (a,0,0,1,a)$$

Example (contd.)

So the dual code can be generated as follows;

$$\begin{aligned} H &= \begin{bmatrix} \cdots & a & \cdots \\ \cdots & a\cdot(T_v^{-1})^{tr} & \cdots \\ \cdots & a\cdot((T_v^{-1})^{tr})^2 & \cdots \end{bmatrix}_{3x5} \\ &= \begin{bmatrix} \alpha & 0 & 0 & 1 & \alpha \\ 0 & \alpha & 0 & 0 & 1 \\ 1 & 0 & \alpha & 0 & 0 \end{bmatrix}_{3x5} \end{aligned}$$

Conclusion and Future Work

- We have derived a formula to obtain the generators of the dual codes of pseudo-cyclic codes and we gave an example over a commutative structure.

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- We further improved our result for the non-commutative case over skew polynomial rings as a near future work.
- We will apply these results to skew quasi-cyclic codes and skew multi-twisted codes.

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