2nd Week Wednesday

01 Mart 2023 Çarşamba 14:23

Row Echelon Form (REF)

Reduced Row Echelon Form (RREF)

Gauss Method

system of lin. eqns. → Augmented matrix → REF (at least) / → RREF

Gauss-Jordan

REF

- ✓ * Number of "leading zeroes" (up to the first nonzero number) should get greater downside.
- ✓ * The (leftmost) leading nonzero element in each row, should be 1. "leading 1"
- ✓ * If there is an all-zero row, it should be at the bottom.

row, row, row

for RREF: +1

- * leading 1's of each row, should be the only nonzero element in their columns.

1. Which of the matrices that follow are in row echelon form? Which are in reduced row echelon form?

(a) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ 1✓ 2✓ 3✓ 4✗ REF✓ RREF✗

(b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ REF✗ 1✗ 3✗

(c) $\begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ 1✓ 2✓ 3✓ 4✓ REF✓ RREF✓

(d) $\begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ 1✓ 2✓ 3✓ 4✓

(e) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$ 1✓ 2✗ REF✗

(f) $\begin{bmatrix} 1 & 4 & 6 \\ 0 & 0 & 1 \\ 0 & 1 & 3 \end{bmatrix}$ 1✗

(g) $\begin{bmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 2 & 4 \\ 0 & 0 & 1 & 3 & 6 \end{bmatrix}$ 1✓ 2✓ 3✓ 4✓ REF✓ RREF✓

(h) $\begin{bmatrix} 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ 1✓ 2✓ 3✓ 4✗ REF✓ RREF✗

// leading zeroes = 2

0 0 1 2 0 3 4

- ✓ * Number of "leading zeroes" (up to the first nonzero number) should get greater downside.
- * The (leftmost) leading nonzero element in each row, should be 1. "leading 1" !
- * If there is an all-zero row, it should be at the bottom. (maybe more than 1)

✓ ✓ or RREF: +1

- * leading 1's of each row, should be the only nonzero element in their columns.

augmented

coeffs. results

2. The augmented matrices that follow are in row echelon form. For each case, indicate whether the corresponding linear system is consistent. If the system has a unique solution, find it.

already in REF

the system has a solution

- 1) unique solution
- 2) infinitely many solution

} → system is consistent

- 3) no solution. → the system is inconsistent

(a) $\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{array}\right]$

(b) $\left[\begin{array}{cc|c} 1 & 3 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right]$

(c) $\left[\begin{array}{ccc|c} 1 & -2 & 4 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$

(d) $\left[\begin{array}{ccc|c} 1 & -2 & 2 & -2 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right]$ unique soln✓

(e) $\left[\begin{array}{ccc|c} 1 & 3 & 2 & -2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{array}\right]$ $0=1$! no solution!

! In order to give a direct decision you should at least transform the system into REF !

a) $\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{array}\right]$ → $1x_1 + 2x_2 = 4$, $0x_1 + 1x_2 = 3$, $0x_1 + 0x_2 = 1$ → $0 = 1$! impossible!

! First, look at the last row of the REF. (not all zero) ⇒ NO solution!

a') $\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ → $x_1 + 2x_2 = 4$, $0x_1 + 1.x_2 = 0$, $0x_1 + 0x_2 = 0$ ✓

$x_1 = 4$, $x_2 = 0$ → $\{(4,0)\}$ → the solution set. unique sln. case✓

(b) $\begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \left[\begin{array}{cc|c} 1 & 3 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right] \end{array}$ → unique soln✓

→ $x_1 + 3x_2 = 1$, $x_2 = -1$, $0x_1 + 0x_2 = 0$

$x_1 + 3.-1 = 1$ ⇒ $x_1 = 4$

Solution set $= \{(4,-1)\}$ → unique solution.

(c) $\begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & -2 & 4 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$ → $x_1 - 2x_2 + 4x_3 = 1$, $x_3 = 3$

$1 - 12 + 2r$

$x_2 = r \in \mathbb{R}$

$x_1 = -11 + 2r$

Solution set $= \{(-11+2r, r, 3) : r \in \mathbb{R}\}$ infinitely many solutions✓

no leading 1 in the corresponding column for x_2 → x_2: free variable. → at least 1 free variable → infinitely many solutions.

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$$\begin{array}{c|c} & \begin{matrix} x_1 & x_2 & x_3 & \end{matrix} \\ \textbf{(d)} & \left[\begin{array}{ccc|c} 1 & -2 & 2 & -2 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right] \end{array}$$

← unique soln

$\longrightarrow$

$x_1 - 2x_2 + 2x_3 = -2 \Rightarrow x_1 - 10 + 4 = -2 \quad x_1 = 4$

$x_2 - x_3 = 3 \rightarrow x_2 = 5$

$\boxed{x_3 = 2}$

Soln set $= \{(4,5,2)\}$ → unique soln.

Ex REF ✓

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array}$$
$$\left[\begin{array}{ccccc|c} 1 & 0 & 2 & 3 & 1 & 6 \\ 0 & 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & -3 \end{array}\right]$$

x_2, x_3 → free variables

$\boxed{x_2 = r \in \mathbb{R}}$ $\boxed{x_3 = s \in \mathbb{R}}$

$x_1 + 2x_3 + 3\overset{5}{x_4} + \overset{-3}{x_5} = 6$ $\quad 9-15-2s \Rightarrow \boxed{x_1 = -6-2s}$

$\boxed{x_4 = 5}$

$\boxed{x_5 = -3}$

Solution set $= \{(-6-2s, r, s, 5, -3) : r,s \in \mathbb{R}\}$

infinitely many solutions ✓

on the REF!

Ex

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array}$$
$$\left[\begin{array}{ccccc|c} 1 & & & & & \# \\ 0 & 0 & 1 & & & \# \\ 0 & 0 & & 1 & & \# \\ 0 & 0 & & & 1 & \# \\ & & & & 1 & \# \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

n = # of variables

on REF you should at least have n ~~not-all-zero~~ rows, having leading 1's.

1) unique soln.

2) inf. many soln.

3) no soln. → $\boxed{0 = \text{nonzero}}$

on REF at least one variable should not have a leading 1 on its corresponding column. (⇒ at least one free variable)

$$\begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array}$$

(j) $\quad x_1 + 2x_2 - 3x_3 + x_4 = 1$
$\quad -x_1 - x_2 + 4x_3 - x_4 = 6$
$\quad -2x_1 - 4x_2 + 7x_3 - x_4 = 1$

$\longrightarrow$ augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 2 & -3 & 1 & 1 \\ -1 & -1 & 4 & -1 & 6 \\ -2 & -4 & 7 & -1 & 1 \end{array}\right]$$

not in REF

⇝ ⇝ ⇝ REF

will have at most 3 not-all-zero rows. without $0=1$ case

not enough for 4 variables.

→ at least 1 free variable.

the only direct decision that we can give here; ⇒

! → the system can not have a unique soln.

Ex

$x_1 + x_2 + x_3 + 2x_4 = 5$

$-x_1 - x_2 - x_3 - 2x_4 = -5$

$2x_1 + 2x_2 + 2x_3 + 4x_4 = 10$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 2 & 5 \\ -1 & -1 & -1 & -2 & -5 \\ 2 & 2 & 2 & 4 & 10 \end{array}\right] \xrightarrow[-2r_1 + r_3 \to r_3]{r_1 + r_2 \to r_2} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

some row operations

REF

$x_2 = r \in \mathbb{R}$
$x_3 = s \in \mathbb{R}$
$x_4 = t \in \mathbb{R}$
free variables

$x_1 = 5 - r - s - 2t$

Solution set $= \{(5-r-s-2t, r, s, t) : r,s,t \in \mathbb{R}\}$

inf. many. soln.