

Assignment 3

ELEC 442 - Introduction to Robotics

Sondre Myrberg (81113433) Ola Helbaek (68776772)

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**THE UNIVERSITY
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1 Computation of representation

As we have the relationship between the initial frame \underline{C}_i and the final frame \underline{C}_f given as

$$\begin{aligned}\underline{C}_f &= \underline{R}\underline{C}_i \\ &= e^{\theta \underline{s} \times} \underline{C}_i\end{aligned}$$

we could rearrange this as

$$\begin{aligned}\underline{C}_f &= \underline{C}_i R \\ &= \underline{C}_i e^{\theta \underline{s} \times}\end{aligned}$$

where

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

as given in the text. This gives us the relationship

$$e^{\theta \underline{s} \times} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

which we want to solve for \underline{s} . Firstly use that

$$\begin{aligned}e^{\theta \underline{s} \times} &= \sum_{n=0}^{\infty} \frac{(\theta \underline{s} \times)^n}{n!} \\ &= \mathbf{I} + (\theta \underline{s} \times) + \frac{1}{2!}(\theta \underline{s} \times)^2 + \dots\end{aligned}$$

and with the property that $(\underline{s} \times)^3 = -(\underline{s} \times)$ for a skew-symmetric matrix we get

$$\begin{aligned}e^{\theta \underline{s} \times} &= \mathbf{I} + \underbrace{\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots \right)}_{=\sin \theta} (\underline{s} \times) + \underbrace{\left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} - \dots \right)}_{=1 - \cos \theta} (\underline{s} \times)^2 \\ &= \mathbf{I} + \sin \theta (\underline{s} \times) + (1 - \cos \theta) (\underline{s} \times)^2 = R\end{aligned}$$

With

$$\underline{s} \times = \begin{bmatrix} 0 & -s_3 & s_2 \\ s_3 & 0 & -s_1 \\ -s_2 & s_1 & 0 \end{bmatrix}$$

and

$$(\underline{s} \times)^2 = \begin{bmatrix} -s_2^2 - s_3^2 & s_1 s_2 & s_1 s_3 \\ s_1 s_2 & -s_1^2 - s_3^2 & s_2 s_3 \\ s_1 s_3 & s_2 s_3 & -s_1^2 - s_2^2 \end{bmatrix}$$

we get

$$\begin{aligned}
\text{Tr}(R) &= \text{Tr}(\mathbf{I}) + \text{Tr}(\sin \theta (\mathbf{s} \times)) + \text{Tr}((1 - \cos \theta)(\mathbf{s} \times)^2) \\
\Rightarrow r_{11} + r_{22} + r_{33} &= 3 + (1 - \cos \theta) \underbrace{(-2s_1^2 - 2s_2^2 - 2s_3^2)}_{=-2} \\
\Rightarrow \cos \theta &= \frac{r_{11} + r_{22} + r_{33} - 1}{2}
\end{aligned}$$

and

$$\begin{aligned}
\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \sin \theta \begin{bmatrix} 0 & -s_3 & s_2 \\ s_3 & 0 & -s_1 \\ -s_2 & s_1 & 0 \end{bmatrix} \\
&+ (1 - \cos \theta) \begin{bmatrix} -s_2^2 - s_3^2 & s_1 s_2 & s_1 s_3 \\ s_1 s_2 & -s_1^2 - s_3^2 & s_2 s_3 \\ s_1 s_3 & s_2 s_3 & -s_1^2 - s_2^2 \end{bmatrix} \\
\Rightarrow \begin{cases} r_{21} &= \sin \theta s_3 + (1 - \cos \theta) s_1 s_2 \\ r_{12} &= -\sin \theta s_3 + (1 - \cos \theta) s_1 s_2 \\ r_{13} &= \sin \theta s_2 + (1 - \cos \theta) s_1 s_3 \\ r_{31} &= -\sin \theta s_2 + (1 - \cos \theta) s_1 s_3 \\ r_{32} &= \sin \theta s_1 + (1 - \cos \theta) s_2 s_3 \\ r_{23} &= -\sin \theta s_1 + (1 - \cos \theta) s_2 s_3 \end{cases}
\end{aligned}$$

By doing subtractions we get

$$2 \sin \theta \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$

and connect this to three equations on the form

$$\begin{aligned}
\tan \theta &= \frac{\sin \theta}{\cos \theta} = \frac{\frac{r_{32} - r_{23}}{2s_1}}{\frac{r_{11} + r_{22} + r_{33} - 1}{2}} = \frac{r_{32} - r_{23}}{s_1(r_{11} + r_{22} + r_{33} - 1)} \\
&= \frac{\frac{r_{13} - r_{31}}{2s_2}}{\frac{r_{11} + r_{22} + r_{33} - 1}{2}} = \frac{r_{13} - r_{31}}{s_2(r_{11} + r_{22} + r_{33} - 1)} \\
&= \frac{\frac{r_{21} - r_{12}}{2s_3}}{\frac{r_{11} + r_{22} + r_{33} - 1}{2}} = \frac{r_{21} - r_{12}}{s_3(r_{11} + r_{22} + r_{33} - 1)} \\
\Rightarrow \mathbf{s} &= \frac{1}{\tan \theta (r_{11} + r_{22} + r_{33} - 1)} \begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}
\end{aligned}$$