Assignment 1

ELEC 442 - Introduction to Robotics

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1.

Given the homogenous transformation

$$\begin{bmatrix} \boldsymbol{y} \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} Q & \boldsymbol{d} \\ \mathbf{0}^\top & 1 \end{bmatrix}}_{T} \begin{bmatrix} \boldsymbol{x} \\ 1 \end{bmatrix}$$

where Q and $\boldsymbol{d}$ accounts for rotation and translation, respectively. We have that the inverse is on the form

$$T^{-1} = \begin{bmatrix} \tilde{Q} & \tilde{\boldsymbol{d}} \\ \mathbf{0}^\top & 1 \end{bmatrix}$$

where we know that $T^{-1}T$ is equal to the 4×4 identity matrix. This yields

$$\begin{aligned} T^{-1}T &= \begin{bmatrix} \tilde{Q} & \tilde{\boldsymbol{d}} \\ \mathbf{0}^\top & 1 \end{bmatrix} \begin{bmatrix} Q & \boldsymbol{d} \\ \mathbf{0}^\top & 1 \end{bmatrix} \\ &= \begin{bmatrix} \tilde{Q}Q & \tilde{Q}\boldsymbol{d} + \tilde{\boldsymbol{d}} \\ \mathbf{0}^\top & 1 \end{bmatrix} = \mathbf{I}_{4\times 4} \\ &\implies \begin{cases} \tilde{Q}Q & = \mathbf{I}_{3\times 3} \\ \tilde{Q}\boldsymbol{d} + \tilde{\boldsymbol{d}} & = \mathbf{0} \end{cases} \\ &\implies \begin{cases} \tilde{Q} & = Q^{-1} = Q^\top \\ \tilde{\boldsymbol{d}} & = -\tilde{Q}\boldsymbol{d} = -Q^\top \boldsymbol{d} \end{cases} \\ \implies T^{-1} &= \underline{\begin{bmatrix} Q^\top & -Q^\top \boldsymbol{d} \\ \mathbf{0}^\top & 1 \end{bmatrix}} \end{aligned} \tag{1}$$

For T^{-1} to exist obvously T must be invertible, and for this to be fulfilled we require full rank. In this case $\text{rank}(T) = 4$ since $\text{rank}(Q) = 3 \;\; \forall Q$ as Q is a rotation matrix, and the $T_{4,4} = 1 \;\; \forall T$. Thus $\forall \{Q, \boldsymbol{d}\}$ we have $\text{rank}(T) = 4$ and T^{-1} exists.

2.

Considering the homogenous transformation matrix

$$^0T_1 = \begin{bmatrix} Q & \boldsymbol{d} \\ \mathbf{0}^\top & 1 \end{bmatrix}$$

with

$$Q = \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{Q_1} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}}_{Q_2}$$

and

$$\boldsymbol{d} = \begin{bmatrix} -\frac{5}{\sqrt{2}} \\ \frac{5}{\sqrt{2}} \\ 4 \end{bmatrix} \text{ cm}$$

2a).

By obserwing the rotation matrices Q_1 and Q_2 we see that Q_1 is a simple rotation around the $\boldsymbol{k}$-axis. The angle of this rotation is given by $\theta = \arccos\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$. Q_2 is a simple rotation arond the $\boldsymbol{i}$-axis and the rotation angle is given by $\alpha = \arccos\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$. To determine d_1 and a_1 we recognize that a homogenous transformation matrix can be written ass the product of four transformation matrices; angle, offset, length and twist. This gives us

$$\begin{aligned}
{}^0T_1 &= \begin{bmatrix} Q & \boldsymbol{d} \\ 0^\top & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} \exp(\theta\boldsymbol{k}\times) & \boldsymbol{0} \\ \boldsymbol{0}^\top & 1 \end{bmatrix}}_{\text{angle}} \underbrace{\begin{bmatrix} \boldsymbol{I} & d\boldsymbol{k} \\ \boldsymbol{0}^\top & 1 \end{bmatrix}}_{\text{offset}} \underbrace{\begin{bmatrix} \boldsymbol{I} & a\boldsymbol{i} \\ \boldsymbol{0}^\top & 1 \end{bmatrix}}_{\text{length}} \underbrace{\begin{bmatrix} \exp(\alpha\boldsymbol{i}\times) & \boldsymbol{0} \\ \boldsymbol{0}^\top & 1 \end{bmatrix}}_{\text{twist}} \\
&= \begin{bmatrix} \exp(\theta\boldsymbol{k}\times + \alpha\boldsymbol{i}\times) & \exp(\theta\boldsymbol{k}\times)(a\boldsymbol{i} + d\boldsymbol{k}) \\ \boldsymbol{0}^\top & 1 \end{bmatrix} = \begin{bmatrix} Q & \boldsymbol{d} \\ \boldsymbol{0}^\top & 1 \end{bmatrix} \\
&\implies \exp(\theta\boldsymbol{k}\times)(a\boldsymbol{i} + d\boldsymbol{k}) = \begin{bmatrix} -\frac{5}{\sqrt{2}} \\ \frac{5}{\sqrt{2}} \\ 4 \end{bmatrix} \qquad (2) \\
&\implies \exp(\theta\boldsymbol{k}\times) \begin{bmatrix} a \\ 0 \\ d \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ 0 \\ d \end{bmatrix} = \begin{bmatrix} \frac{a}{\sqrt{2}} \\ -\frac{a}{\sqrt{2}} \\ d \end{bmatrix} = \begin{bmatrix} -\frac{5}{\sqrt{2}} \\ \frac{5}{\sqrt{2}} \\ 4 \end{bmatrix} \\
&\implies \begin{cases} a = -5 \\ d = 4 \end{cases}
\end{aligned}$$

And we have numeric values for all our four DH parameters.

2b).

For the point represented in coordinate system 1 by $\underset{\sim}{\boldsymbol{x}} = \begin{bmatrix}1 & 0 & 0\end{bmatrix}^\top$ cm we get the representation in system 0 given by

$$\begin{aligned}
\begin{bmatrix} {}^0\boldsymbol{x} \\ 1 \end{bmatrix} &= {}^0T_1 \begin{bmatrix} {}^1\boldsymbol{x} \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{\sqrt{3}}{2\sqrt{2}} & -\frac{5}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{\sqrt{3}}{2\sqrt{2}} & \frac{5}{\sqrt{2}} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{4}{\sqrt{2}} \\ -\frac{4}{\sqrt{2}} \\ 4 \\ 1 \end{bmatrix} \\
\implies {}^0\underset{\sim}{\boldsymbol{x}} &= \begin{bmatrix} \frac{4}{\sqrt{2}} \\ -\frac{4}{\sqrt{2}} \\ 4 \end{bmatrix} \text{cm}
\end{aligned}$$

2c).

For the opposite case, that we have a point represented in cooridinate system 0 by $\underset{\sim}{\boldsymbol{x}} = \begin{bmatrix}1 & 0 & 0\end{bmatrix}^\top$ cm we apply the inverse transformation matrix that is on the form we found in (1). This gives

$$\begin{aligned}
\begin{bmatrix} {}^1\boldsymbol{x} \\ 1 \end{bmatrix} &= {}^0T_1^{-1} \begin{bmatrix} {}^0\boldsymbol{x} \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 5 \\ -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{2} & -2\sqrt{3} \\ -\frac{\sqrt{3}}{2\sqrt{2}} & -\frac{\sqrt{3}}{2\sqrt{2}} & -\frac{1}{2} & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{10+\sqrt{2}}{2} \\ -\frac{8\sqrt{3}+\sqrt{2}}{4} \\ \frac{16-\sqrt{6}}{4} \\ 1 \end{bmatrix} \\
\implies {}^1\underset{\sim}{\boldsymbol{x}} &= \begin{bmatrix} \frac{10+\sqrt{2}}{2} \\ -\frac{8\sqrt{3}+\sqrt{2}}{4} \\ \frac{16-\sqrt{6}}{4} \end{bmatrix} \text{cm}
\end{aligned}$$

2d).

The angular velocity vector represented in coordinate system 0 by $\begin{bmatrix}1 & 0 & 0\end{bmatrix}^\top$ is given by

$$
\begin{aligned}
\begin{bmatrix}\boldsymbol{\omega}_{1,1}\\ 0\end{bmatrix} &= {}^0T_1^{-1}\begin{bmatrix}\boldsymbol{\omega}_{1,0}\\ 0\end{bmatrix}\\
&= \begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 5\\ -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{2} & -2\sqrt{3}\\ -\frac{\sqrt{3}}{2\sqrt{2}} & -\frac{\sqrt{3}}{2\sqrt{2}} & -\frac{1}{2} & 4\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\\
&= \begin{bmatrix}\frac{1}{\sqrt{2}}\\ -\frac{1}{2\sqrt{2}}\\ -\frac{\sqrt{3}}{2\sqrt{2}}\\ 0\end{bmatrix}\\
\implies \boldsymbol{\omega}_{1,1} &= \begin{bmatrix}\frac{1}{\sqrt{2}}\\ -\frac{1}{2\sqrt{2}}\\ -\frac{\sqrt{3}}{2\sqrt{2}}\end{bmatrix}
\end{aligned}
$$

3.

A MATLAB fuction named `DH_homog` is implemented with the code shown in Listing 1. This function has two return values; the homogenous transformation matrix and the rotation matrix.

```
function [T, C] = DH_homog(theta, d, a, alpha)
    i=[1;0;0];
    k=[0;0;1];
    angle = [expm(theta*skew(k)) zeros(3,1); zeros(1,3) 1];
    offset = [eye(3) d*k; zeros(1,3) 1];
    length = [eye(3) a*i; zeros(1,3) 1];
    twist = [expm(alpha*skew(i)) zeros(3,1); zeros(1,3) 1];
    C = expm(theta*skew(k)) * expm(alpha*skew(i));
    T = angle*offset*length*twist;
end
```

Listing 1: MATLAB code to generate homogenous transformation matrix based on the Denavit-Hartenberg convention

4.

A sketch of the “home” position can be found in Figure 1.

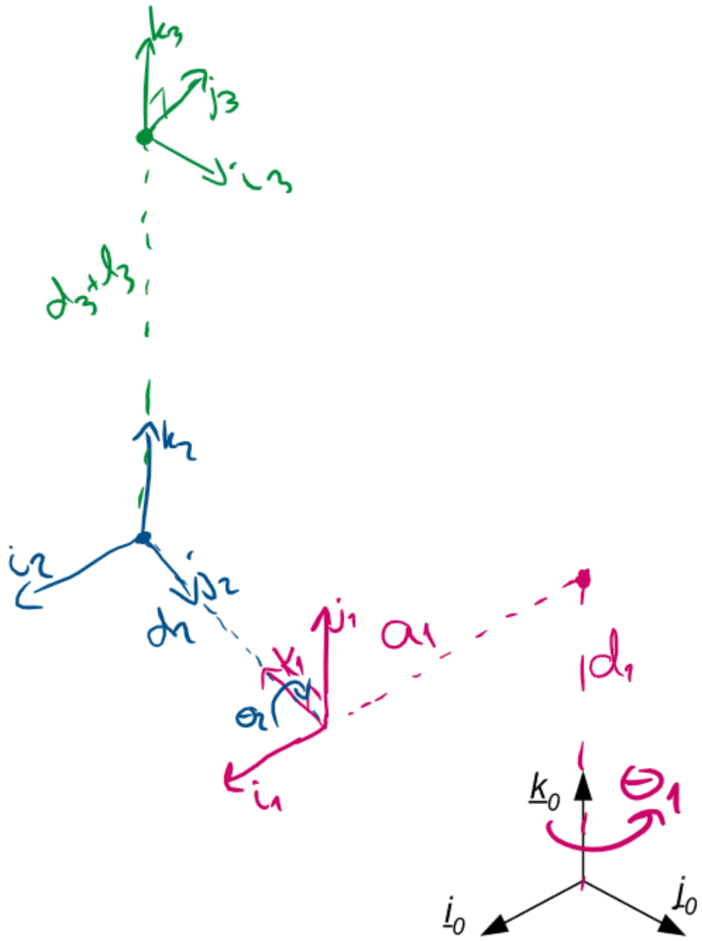


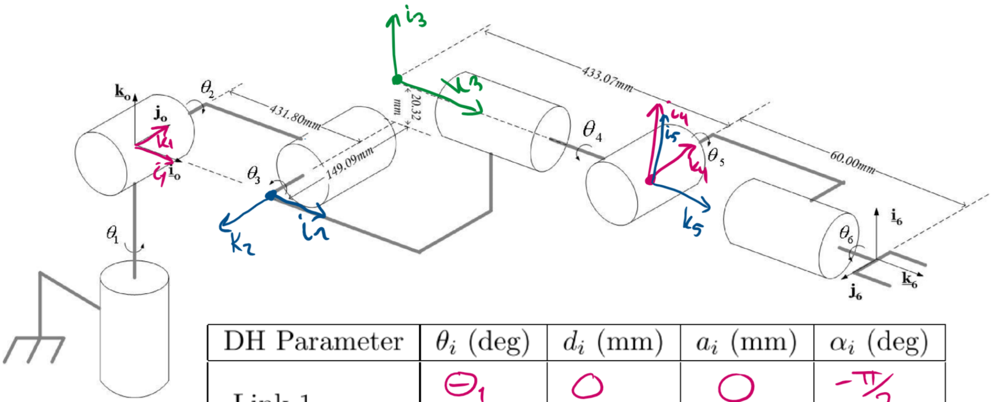
Figure 1: Sketch of “home” position based on the DH-table provided

Find the Jacobian and how to do this

5.

5a).

The different coordinate frames are sketched and the completed table is found in Figure 2. I see now that it should have been in degrees, but I have filled the table with the radian values.



DH Parameter	θ_i (deg)	d_i (mm)	a_i (mm)	α_i (deg)
Link 1	θ_1	0	0	$-\pi/2$
Link 2	θ_2	0	431.80	π
Link 3	$\theta_3 + \frac{\pi}{2}$	-149.09	20.32	$\pi/2$
Link 4	θ_4	433.07	0	$\pi/2$
Link 5	θ_5	0	0	$-\pi/2$
Link 6	θ_6	60.06	0	0

Figure 2: Sketch of the coordinate frames according to the DH-convention

5b).

We know that the relationship between base $\{\underset{\sim}{\boldsymbol{o}}_0, \underline{C}_0\}$ and the end effector $\{\underset{\sim}{\boldsymbol{o}}_6, \underline{C}_6\}$ is given by

$$\begin{bmatrix} \underline{C}_n & \underset{\sim}{\boldsymbol{o}}_n \\ \mathbf{0}^\top & 1 \end{bmatrix} = \begin{bmatrix} \underline{C}_0 & \underset{\sim}{\boldsymbol{o}}_0 \\ \mathbf{0}^\top & 1 \end{bmatrix} {}^0T_1(q_1)\, {}^1T_2(q_2) \ldots {}^{n-1}T_n(q_n)$$

$$\implies \begin{bmatrix} \underline{C}_6 & \underset{\sim}{\boldsymbol{o}}_6 \\ \mathbf{0}^\top & 1 \end{bmatrix} = \begin{bmatrix} \underline{C}_0 & \underset{\sim}{\boldsymbol{o}}_0 \\ \mathbf{0}^\top & 1 \end{bmatrix} {}^0T_6$$

where ${}^0T_6$ is a series of transformations on the form as shown in the first line of (2), and we have the relationship between he base and the end effector. A chain of transformations that give the relationship between base $\{\underset{\sim}{\boldsymbol{o}}_0, \underline{C}_0\}$ and the end effector $\{\underset{\sim}{\boldsymbol{o}}_6, \underline{C}_6\}$ on the form presented by Salcudean's example 2.5 is

$$
\begin{aligned}
&\underline{C}_1 = \underline{C}_0 \exp\left(\theta_1 \boldsymbol{k}\times\right) \exp\left(-\tfrac{\pi}{2}\boldsymbol{i}\times\right) && \underset{\sim}{\boldsymbol{o}}_1 = \underset{\sim}{\boldsymbol{o}}_0 \\
&\underline{C}_2 = \underline{C}_1 \exp\left(\theta_2 \boldsymbol{k}\times\right) \exp\left(\pi\boldsymbol{i}\times\right) && \underset{\sim}{\boldsymbol{o}}_2 = \underset{\sim}{\boldsymbol{o}}_1 + \underline{C}_1 \exp\left(\theta_2 \boldsymbol{k}\times\right)(431.80\boldsymbol{i})\,\mathrm{mm} \\
&\underline{C}_3 = \underline{C}_2 \exp\left(\theta_3 \boldsymbol{k}\times\right) \exp\left(\tfrac{\pi}{2}\boldsymbol{k}\times\right) \exp\left(\tfrac{\pi}{2}\boldsymbol{i}\times\right) && \underset{\sim}{\boldsymbol{o}}_3 = \underset{\sim}{\boldsymbol{o}}_2 + \underline{C}_2 \exp\left(\theta_3 \boldsymbol{k}\times\right)(-149.09\boldsymbol{k} + 20.32\boldsymbol{j})\,\mathrm{mm} \\
&\underline{C}_4 = \underline{C}_3 \exp\left(\theta_4 \boldsymbol{k}\times\right) \exp\left(\tfrac{\pi}{2}\boldsymbol{i}\times\right) && \underset{\sim}{\boldsymbol{o}}_4 = \underset{\sim}{\boldsymbol{o}}_3 + \underline{C}_3 \exp\left(\theta_4 \boldsymbol{k}\times\right)(433.07\boldsymbol{k})\,\mathrm{mm} \\
&\underline{C}_5 = \underline{C}_4 \exp\left(\theta_5 \boldsymbol{k}\times\right) \exp\left(-\tfrac{\pi}{2}\boldsymbol{i}\times\right) && \underset{\sim}{\boldsymbol{o}}_5 = \underset{\sim}{\boldsymbol{o}}_4 \\
&\underline{C}_6 = \underline{C}_5 \exp\left(\theta_6 \boldsymbol{k}\times\right) && \underset{\sim}{\boldsymbol{o}}_6 = \underset{\sim}{\boldsymbol{o}}_5 + \underline{C}_5 \exp\left(\theta_6 \boldsymbol{k}\times\right)(60.00\boldsymbol{k})\,\mathrm{mm}
\end{aligned}
\tag{3}
$$

5c).

Do exercise 5ce

5d).

The MATLAB code used to in subsection 5d) is listed in Listing 2. The user is prompted for six joint avariables, and the total homogenous transformation matrix is calculated aswell as the link origins is plotted by using the `plot3` command. As Figure 3 shows there are 5 different link origins, as expected by (3) since origin 0 and 1, and 4 and 5 is the same point. There must however be a slight mistake somwhere in my code, as origin 2 gets moved $\sim$ 2500mm insted of 431.8mm, but I can't find it. The other origins does however seem to fit well with what's expected.

```
%% 5d
i = [1;0;0];
j = [0;1;0];
k = [0;0;1];

inputangle = ['1 ';'2 ';'3 ';'4 ';'5 ';'6 '];
theta = [0 0 0 0 0 0];
for i = 1:6
    theta(i) = degtorad(input(inputangle(i)));
end

```

```
[T1,C01] = DH_homog(theta(1), 0, 0, -pi/2);
[T2,C12] = DH_homog(theta(2), 0, 431.8, pi);
[T3,C23] = DH_homog(theta(3) + pi/2, -149.09, 20.32, pi/2);
[T4,C34] = DH_homog(theta(4), 433.07, 0, pi/2);
[T5,C45] = DH_homog(theta(5), 0, 0, -pi/2);
[T6,C56] = DH_homog(theta(6), 60, 0, 0);
T= T1*T2*T3*T4*T5*T6;

C0 = eye(3);
C1 = C0*C01;
C2 = C1*C12;
C3 = C2*C23;
C4 = C3*C34;
C5 = C4*C45;

o0 = [0;0;0];
o1 = o0;
o2 = o1 + C1*expm(theta(2)*skew(k))*431.8*i;
o3 = o2 + C2*expm(theta(3)*skew(k))*(-149.09*k + 20.32*j);
o4 = o3 + C3*expm(theta(4)*skew(k))*433.07*k;
o5 = o4;
o6 = o5 + C5*expm(theta(6)*skew(k))*60*k;


x = [o0(1) o1(1) o2(1) o3(1) o4(1) o5(1) o6(1)];
y = [o0(2) o1(2) o2(2) o3(2) o4(2) o5(2) o6(2)];
z = [o0(3) o1(3) o2(3) o3(3) o4(3) o5(3) o6(3)];

figure
    hold on; view(3); grid on;
    plot3(x,y,z, '*');
    xlabel('$x$-coordinate', 'interpreter', 'latex');xlim
       ([-500 3500]);
    ylabel('$y$-coordinate', 'interpreter', 'latex');
    zlabel('$z$-coordinate', 'interpreter', 'latex');zlim
       ([-5 25]);
```

Listing 2: MATLAB code used to generate homogenous transformation matrix for angles chosen by the user

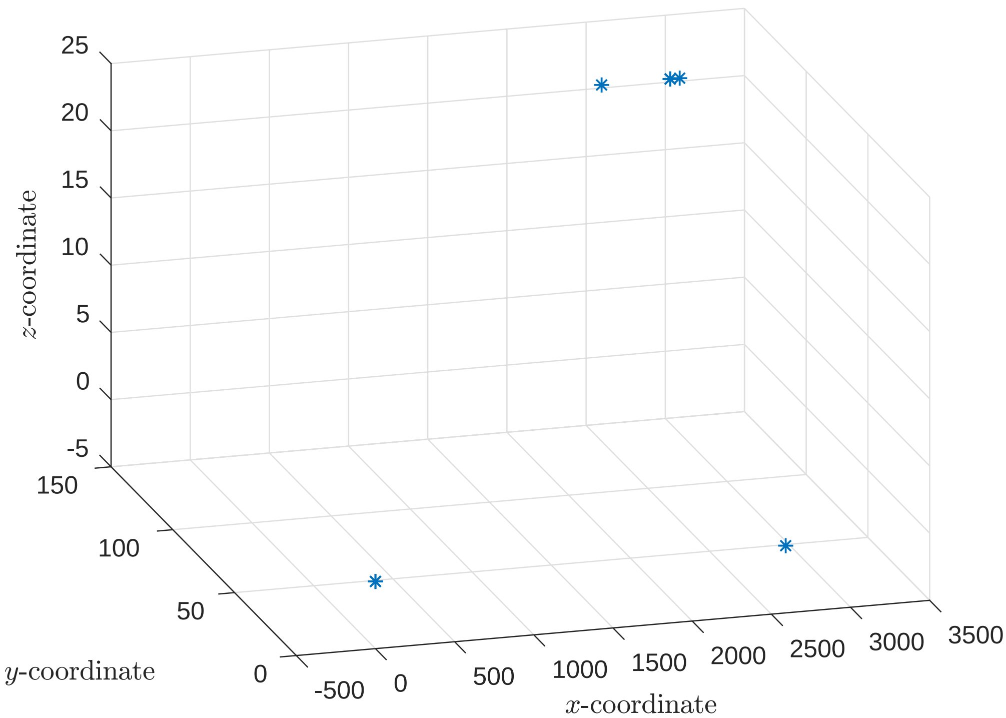


Figure 3: Location of the link origins with $\theta_i = 0\ \forall i$