# Assignment 3

**ELEC 442 - Introduction to Robotics**

Sondre Myrberg (81113433) Ola Helbaek (68776772)

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OF BRITISH COLUMBIA

# 1 Computation of representation

As we have the relationship between the initial frame $\underline{C}_i$ and the final frame $\underline{C}_f$ given as

$$\begin{aligned} \underline{C}_f &= \underline{\underline{R}}\underline{C}_i \\ &= \mathrm{e}^{\theta \underline{s} \times} \underline{C}_i \end{aligned}$$

we could rearrange this as

$$\begin{aligned} \underline{C}_f &= \underline{C}_i R \\ &= \underline{C}_i \mathrm{e}^{\theta \boldsymbol{s} \times} \end{aligned}$$

where

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

as given in the text. This gives us the relationship

$$\mathrm{e}^{\theta \boldsymbol{s} \times} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

which we want to solve for $\boldsymbol{s}$. Firstly use that

$$\begin{aligned} \mathrm{e}^{\theta \boldsymbol{s} \times} &= \sum_{n=0}^{\infty} \frac{(\theta \boldsymbol{s} \times)^n}{n!} \\ &= \boldsymbol{I} + (\theta \boldsymbol{s} \times) + \frac{1}{2!} (\theta \boldsymbol{s} \times)^2 + \dots \end{aligned}$$

and with the property that $(\boldsymbol{s} \times)^3 = -(\boldsymbol{s} \times)$ for a skew-symmetric matrix we get

$$\begin{aligned} \mathrm{e}^{\theta \boldsymbol{s} \times} &= \boldsymbol{I} + \underbrace{\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots\right)}_{= \sin\theta} (\boldsymbol{s} \times) + \underbrace{\left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} - \dots\right)}_{= 1 - \cos\theta} (\boldsymbol{s} \times)^2 \\ &= \boldsymbol{I} + \sin\theta (\boldsymbol{s} \times) + (1 - \cos\theta)(\boldsymbol{s} \times)^2 = R \end{aligned}$$

With

$$\boldsymbol{s} \times = \begin{bmatrix} 0 & -s_3 & s_2 \\ s_3 & 0 & -s_1 \\ -s_2 & s_1 & 0 \end{bmatrix}$$

and

$$(\boldsymbol{s} \times)^2 = \begin{bmatrix} -{s_2}^2 - {s_3}^2 & s_1 \, s_2 & s_1 \, s_3 \\ s_1 \, s_2 & -{s_1}^2 - {s_3}^2 & s_2 \, s_3 \\ s_1 \, s_3 & s_2 \, s_3 & -{s_1}^2 - {s_2}^2 \end{bmatrix}$$

we get

$$
\begin{aligned}
\operatorname{Tr}(R) &= \operatorname{Tr}(\boldsymbol{I}) + \cancel{\operatorname{Tr}(\sin\theta(\boldsymbol{s}\times))} + \operatorname{Tr}\big((1-\cos\theta)(\boldsymbol{s}\times)^2\big) \\
\implies r_{11} + r_{22} + r_{33} &= 3 + (1-\cos\theta)\underbrace{(-2s_1^2 - 2s_2^2 - 2s_3^2)}_{=-2} \\
\implies \cos\theta &= \frac{r_{11}+r_{22}+r_{33}-1}{2}
\end{aligned}
$$

and

$$
\begin{aligned}
\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \sin\theta \begin{bmatrix} 0 & -s_3 & s_2 \\ s_3 & 0 & -s_1 \\ -s_2 & s_1 & 0 \end{bmatrix} \\
&+ (1-\cos\theta) \begin{bmatrix} -{s_2}^2 - {s_3}^2 & s_1\, s_2 & s_1\, s_3 \\ s_1\, s_2 & -{s_1}^2 - {s_3}^2 & s_2\, s_3 \\ s_1\, s_3 & s_2\, s_3 & -{s_1}^2 - {s_2}^2 \end{bmatrix} \\
\implies &\begin{cases} r_{21} &= \sin\theta s_3 + (1-\cos\theta)s_1 s_2 \\ r_{12} &= -\sin\theta s_3 + (1-\cos\theta)s_1 s_2 \\ r_{13} &= \sin\theta s_2 + (1-\cos\theta)s_1 s_3 \\ r_{31} &= -\sin\theta s_2 + (1-\cos\theta)s_1 s_3 \\ r_{32} &= \sin\theta s_1 + (1-\cos\theta)s_2 s_3 \\ r_{23} &= -\sin\theta s_1 + (1-\cos\theta)s_2 s_3 \end{cases}
\end{aligned}
$$

By doing subtractions we get

$$
2\sin\theta \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}
$$

and connect this to three equations on the form

$$
\begin{aligned}
\tan\theta = \frac{\sin\theta}{\cos\theta} &= \frac{\frac{r_{32}-r_{23}}{2s_1}}{\frac{r_{11}+r_{22}+r_{33}-1}{2}} = \frac{r_{32}-r_{23}}{s_1(r_{11}+r_{22}+r_{33}-1)} \\
&= \frac{\frac{r_{13}-r_{31}}{2s_2}}{\frac{r_{11}+r_{22}+r_{33}-1}{2}} = \frac{r_{13}-r_{31}}{s_2(r_{11}+r_{22}+r_{33}-1)} \\
&= \frac{\frac{r_{21}-r_{12}}{2s_3}}{\frac{r_{11}+r_{22}+r_{33}-1}{2}} = \frac{r_{21}-r_{12}}{s_3(r_{11}+r_{22}+r_{33}-1)} \\
\implies \boldsymbol{s} &= \underline{\frac{1}{\tan\theta(r_{11}+r_{22}+r_{33}-1)} \begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}}
\end{aligned}
$$

In the special case of $\sin\theta = 0$ we know that $\theta = n\pi$. For $n$ even we get $\mathrm{e}^{\theta\boldsymbol{s}\times}v = v$, and for $n$ odd we get that $\mathrm{e}^{\theta\boldsymbol{s}\times}v = -v$. This applies to all $\boldsymbol{s}$ and all $\theta = n\pi$ where $n \in \mathbb{Z}$.