

Assignment 1

ELEC 442 - Introduction to Robotics

Sondre Myrberg

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**THE UNIVERSITY
OF BRITISH COLUMBIA**

1.

Given the homogenous transformation

$$\begin{bmatrix} \mathbf{y} \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} Q & \mathbf{d} \\ \mathbf{0}^\top & 1 \end{bmatrix}}_T \begin{bmatrix} \mathbf{x} \\ 1 \end{bmatrix}$$

where Q and \mathbf{d} accounts for rotation and translation, respectively. We have that the inverse is on the form

$$T^{-1} = \begin{bmatrix} \tilde{Q} & \tilde{\mathbf{d}} \\ \mathbf{0}^\top & 1 \end{bmatrix}$$

where we know that $T^{-1}T$ is equal to the 4×4 identity matrix. This yields

$$\begin{aligned} T^{-1}T &= \begin{bmatrix} \tilde{Q} & \tilde{\mathbf{d}} \\ \mathbf{0}^\top & 1 \end{bmatrix} \begin{bmatrix} Q & \mathbf{d} \\ \mathbf{0}^\top & 1 \end{bmatrix} \\ &= \begin{bmatrix} \tilde{Q}Q & \tilde{Q}\mathbf{d} + \tilde{\mathbf{d}} \\ \mathbf{0}^\top & 1 \end{bmatrix} = \mathbf{I}_{4 \times 4} \\ &\Rightarrow \begin{cases} \tilde{Q}Q &= \mathbf{I}_{3 \times 3} \\ \tilde{Q}\mathbf{d} + \tilde{\mathbf{d}} &= \mathbf{0} \end{cases} \\ &\Rightarrow \begin{cases} \tilde{Q} &= Q^{-1} = Q^\top \\ \tilde{\mathbf{d}} &= -\tilde{Q}\mathbf{d} = -Q^\top \mathbf{d} \end{cases} \\ &\Rightarrow T^{-1} = \underline{\begin{bmatrix} Q^\top & -Q^\top \mathbf{d} \\ \mathbf{0}^\top & 1 \end{bmatrix}} \end{aligned} \tag{1}$$

For T^{-1} to exist obviously T must be invertible, and for this to be fulfilled we require full rank. In this case $\text{rank}(T) = 4$ since $\text{rank}(Q) = 3 \ \forall Q$ as Q is a rotation matrix, and the $T_{4,4} = 1 \ \forall T$. Thus $\forall \{Q, \mathbf{d}\}$ we have $\text{rank}(T) = 4$ and T^{-1} exists.

2.

Considering the homogenous transformation matrix

$${}^0T_1 = \begin{bmatrix} Q & \mathbf{d} \\ \mathbf{0}^\top & 1 \end{bmatrix}$$

with

$$Q = \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{Q_1} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}}_{Q_2}$$

and

$$\mathbf{d} = \begin{bmatrix} -\frac{5}{\sqrt{2}} \\ \frac{5}{\sqrt{2}} \\ 4 \end{bmatrix} \text{ cm}$$

2a).

By observing the rotation matrices Q_1 and Q_2 we see that Q_1 is a simple rotation around the \mathbf{k} -axis. The angle of this rotation is given by $\theta = \arccos\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$. Q_2 is a simple rotation around the \mathbf{i} -axis and the rotation angle is given by $\alpha = \arccos\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$. To determine d_1 and a_1 we recognize that a homogenous transformation matrix can be written as the product of four transformation matrices; angle, offset, length and twist. This gives us

$$\begin{aligned} {}^0T_1 &= \begin{bmatrix} Q & \mathbf{d} \\ \mathbf{0}^\top & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} \exp(\theta \mathbf{k} \times) & \mathbf{0} \\ \mathbf{0}^\top & 1 \end{bmatrix}}_{\text{angle}} \underbrace{\begin{bmatrix} \mathbf{I} & d\mathbf{k} \\ \mathbf{0}^\top & 1 \end{bmatrix}}_{\text{offset}} \underbrace{\begin{bmatrix} \mathbf{I} & a\mathbf{i} \\ \mathbf{0}^\top & 1 \end{bmatrix}}_{\text{length}} \underbrace{\begin{bmatrix} \exp(\alpha \mathbf{i} \times) & \mathbf{0} \\ \mathbf{0}^\top & 1 \end{bmatrix}}_{\text{twist}} \\ &= \begin{bmatrix} \exp(\theta \mathbf{k} \times + \alpha \mathbf{i} \times) & \exp(\theta \mathbf{k} \times)(a\mathbf{i} + d\mathbf{k}) \\ \mathbf{0}^\top & 1 \end{bmatrix} = \begin{bmatrix} Q & \mathbf{d} \\ \mathbf{0}^\top & 1 \end{bmatrix} \\ \Rightarrow \exp(\theta \mathbf{k} \times)(a\mathbf{i} + d\mathbf{k}) &= \begin{bmatrix} -\frac{5}{\sqrt{2}} \\ \frac{5}{\sqrt{2}} \\ 4 \end{bmatrix} \tag{2} \\ \Rightarrow \exp(\theta \mathbf{k} \times) \begin{bmatrix} a \\ 0 \\ d \end{bmatrix} &= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ 0 \\ d \end{bmatrix} = \begin{bmatrix} \frac{a}{\sqrt{2}} \\ -\frac{a}{\sqrt{2}} \\ d \end{bmatrix} = \begin{bmatrix} -\frac{5}{\sqrt{2}} \\ \frac{5}{\sqrt{2}} \\ 4 \end{bmatrix} \\ \Rightarrow \begin{cases} a = -5 \\ d = 4 \end{cases} \end{aligned}$$

And we have numeric values for all our four DH parameters.

2b).

For the point represented in coordinate system 1 by $\underline{\tilde{x}} = [1 \ 0 \ 0]^\top$ cm we get the representation in system 0 given by

$$\begin{aligned}
\begin{bmatrix} {}^0\mathbf{x} \\ 1 \end{bmatrix} &= {}^0T_1 \begin{bmatrix} {}^1\mathbf{x} \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{\sqrt{3}}{2\sqrt{2}} & -\frac{5}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{\sqrt{3}}{2\sqrt{2}} & \frac{5}{\sqrt{2}} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{4}{\sqrt{2}} \\ -\frac{4}{\sqrt{2}} \\ 4 \\ 1 \end{bmatrix} \\
\Rightarrow {}^0\tilde{\mathbf{x}} &= \begin{bmatrix} \frac{4}{\sqrt{2}} \\ -\frac{4}{\sqrt{2}} \\ 4 \end{bmatrix} \text{ cm}
\end{aligned}$$

2c).

For the opposite case, that we have a point represented in coordinate system 0 by $\underline{\tilde{x}} = [1 \ 0 \ 0]^\top$ cm we apply the inverse transformation matrix that is on the form we found in (1). This gives

$$\begin{aligned}
\begin{bmatrix} {}^1\mathbf{x} \\ 1 \end{bmatrix} &= {}^0T_1^{-1} \begin{bmatrix} {}^0\mathbf{x} \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 5 \\ -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{2} & -2\sqrt{3} \\ -\frac{\sqrt{3}}{2\sqrt{2}} & -\frac{\sqrt{3}}{2\sqrt{2}} & -\frac{1}{2} & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{10+\sqrt{2}}{2} \\ -\frac{8\sqrt{3}+\sqrt{2}}{4} \\ \frac{16-\sqrt{6}}{4} \\ 1 \end{bmatrix} \\
\Rightarrow {}^1\tilde{\mathbf{x}} &= \begin{bmatrix} \frac{10+\sqrt{2}}{2} \\ -\frac{8\sqrt{3}+\sqrt{2}}{4} \\ \frac{16-\sqrt{6}}{4} \end{bmatrix} \text{ cm}
\end{aligned}$$

2d).

The angular velocity vector represented in coordinate system 0 by $[1 \ 0 \ 0]^\top$ is given by

$$\begin{aligned}
 \begin{bmatrix} \omega_{1,1} \\ 0 \end{bmatrix} &= {}^0T_1^{-1} \begin{bmatrix} \omega_{1,0} \\ 0 \end{bmatrix} \\
 &= \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 5 \\ -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{2} & -2\sqrt{3} \\ -\frac{\sqrt{3}}{2\sqrt{2}} & -\frac{\sqrt{3}}{2\sqrt{2}} & -\frac{1}{2} & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\
 &= \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} \\ -\frac{\sqrt{3}}{2\sqrt{2}} \\ 0 \end{bmatrix} \\
 \Rightarrow \omega_{1,1} &= \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} \\ -\frac{\sqrt{3}}{2\sqrt{2}} \end{bmatrix}
 \end{aligned}$$

3.

A MATLAB function named `DH_homog` is implemented with the code shown in Listing 1. This function has two return values; the homogenous transformation matrix and the rotation matrix.

```

1 function [T, C] = DH_homog(theta, d, a, alpha)
2     i=[1;0;0];
3     k=[0;0;1];
4     angle = [expm(theta*skew(k)) zeros(3,1); zeros(1,3) 1];
5     offset = [eye(3) d*k; zeros(1,3) 1];
6     length = [eye(3) a*i; zeros(1,3) 1];
7     twist = [expm(alpha*skew(i)) zeros(3,1); zeros(1,3) 1];
8     C = expm(theta*skew(k)) * expm(alpha*skew(i));
9     T = angle*offset*length*twist;
10 end

```

Listing 1: MATLAB code to generate homogenous transformation matrix based on the Denavit-Hartenberg convention

4.

A sketch of the “home” position can be found in Figure 1.

The Jacobian for a manipulator like this can generally be expressed as

$$\underline{J}_{i-1} = \begin{cases} \begin{bmatrix} \underline{k}_{i-1} \times (\underline{o}_n - \underline{o}_{i-1}) \\ \underline{k}_{i-1} \end{bmatrix} & \text{when joint } i \text{ is revolute} \\ \begin{bmatrix} \underline{k}_{i-1} \\ \mathbf{0} \end{bmatrix} & \text{when joint } i \text{ is prismatic} \end{cases}$$

as given in (10) in Salcudean’s notes. This gives us the Jacobian

$$\underline{J} = [\underline{J}_0 \quad \underline{J}_1 \quad \underline{J}_2] = \begin{bmatrix} \underline{k}_0 \times (\underline{o}_3 - \underline{o}_0) & \underline{k}_1 \times (\underline{o}_3 - \underline{o}_1) & \underline{k}_2 \\ \underline{k}_0 & \underline{k}_1 & 0 \end{bmatrix}$$

We know that singularities occur every time \underline{J} loses rank. This happens when $\underline{k}_0 \times (\underline{o}_3 - \underline{o}_0)$, $\underline{k}_1 \times (\underline{o}_3 - \underline{o}_1)$ and \underline{k}_2 are coplanar. By inspection we see that this happens when $\theta_2 = n\pi$, $\pi \in \mathbb{Z}$. At these instances θ_1 and θ_2 creates the same instantaneous movement. Singularities will also occur when $\theta_2 = \frac{\pi}{2} + n\pi$, $\pi \in \mathbb{Z}$. In this case θ_1 and d_3 will create the same instantaneous movement, hence singularities occur when $\theta_2 = n\frac{\pi}{2}$, $n \in \mathbb{Z}$.

5.

5a).

The different coordinate frames are sketched and the completed table is found in Figure 2. I see now that it should have been in degrees, but I have filled the table with the radian values.

5b).

We know that the relationship between base $\{\underline{o}_0, \underline{C}_0\}$ and the end effector $\{\underline{o}_6, \underline{C}_6\}$ is given by

$$\begin{aligned} \begin{bmatrix} \underline{C}_n & \underline{o}_n \\ \mathbf{0}^\top & 1 \end{bmatrix} &= \begin{bmatrix} \underline{C}_0 & \underline{o}_0 \\ \mathbf{0}^\top & 1 \end{bmatrix} {}^0T_1(q_1) {}^1T_2(q_2) \dots {}^{n-1}T_n(q_n) \\ \Rightarrow \begin{bmatrix} \underline{C}_6 & \underline{o}_6 \\ \mathbf{0}^\top & 1 \end{bmatrix} &= \begin{bmatrix} \underline{C}_0 & \underline{o}_0 \\ \mathbf{0}^\top & 1 \end{bmatrix} {}^0T_6 \end{aligned}$$

where 0T_6 is a series of transformations on the form as shown in the first line of (2), and we have the relationship between the base and the end effector. A chain of transformations

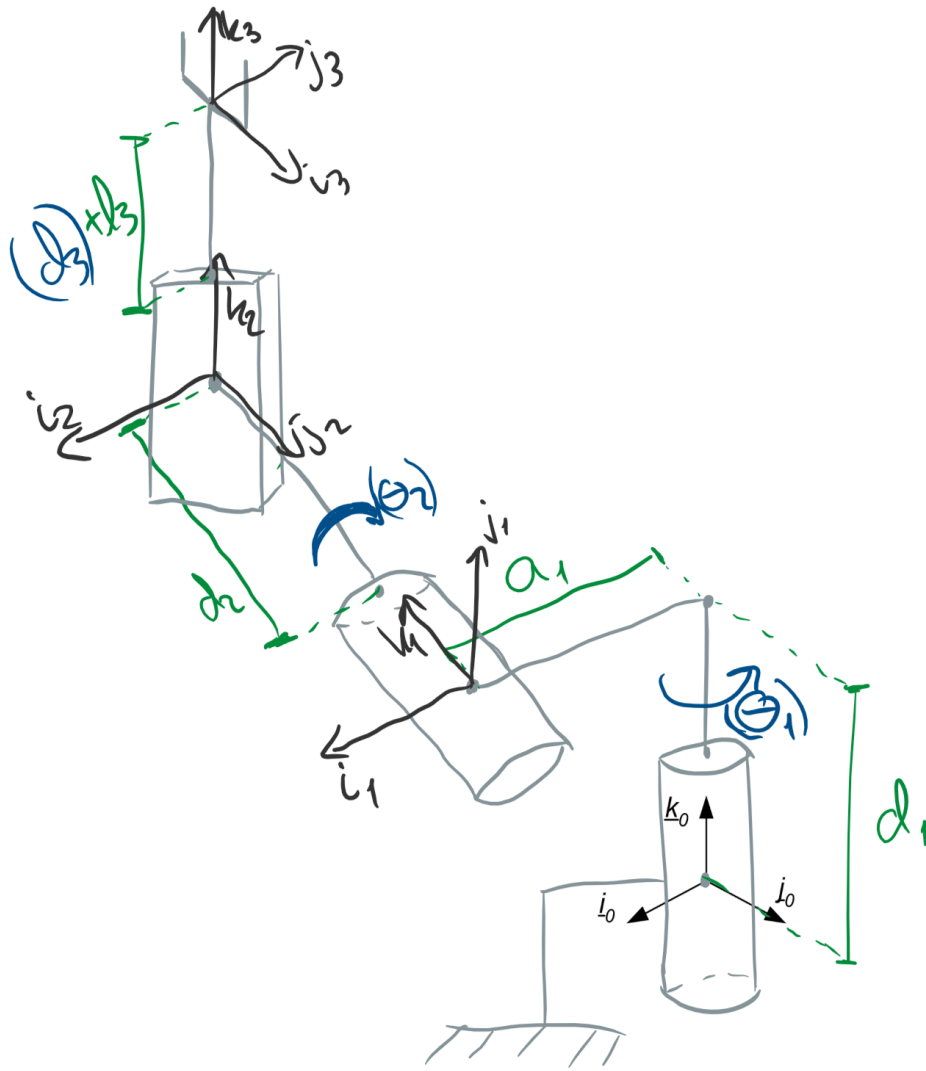


Figure 1: Sketch of “home” position based on the DH-table provided

that give the relationship between base $\{\underline{o}_0, \underline{C}_0\}$ and the end effector $\{\underline{o}_6, \underline{C}_6\}$ on the

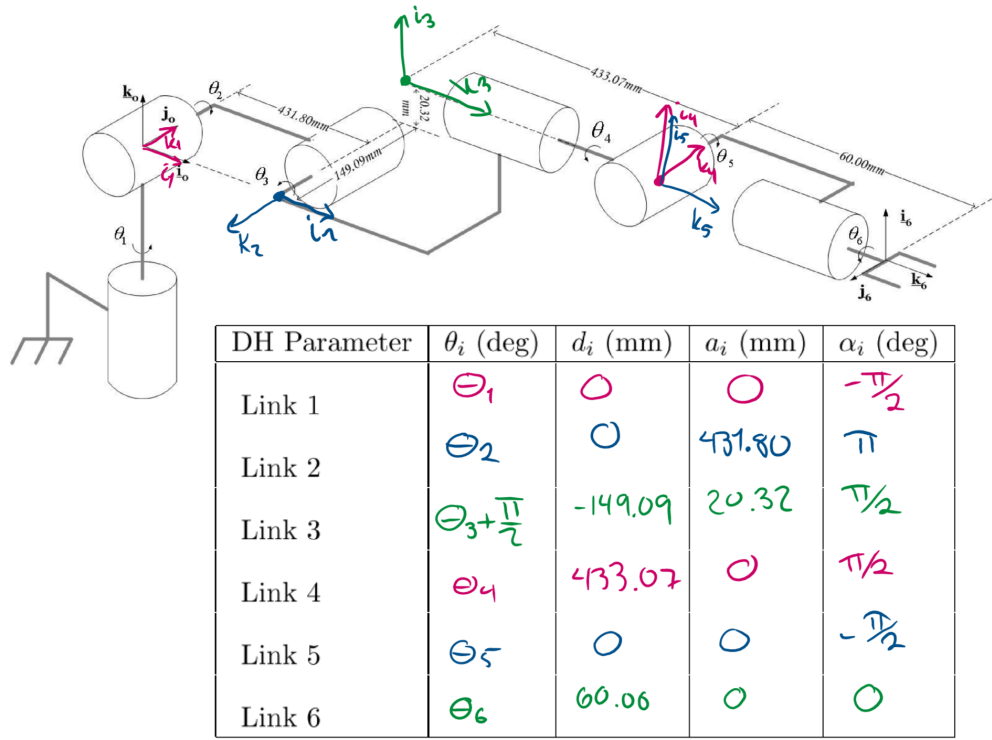


Figure 2: Sketch of the coordinate frames according to the DH-convention

form presented by Salcudean's example 2.5 is

$$\begin{aligned}
 \underline{C}_1 &= \underline{C}_0 \exp(\theta_1 \mathbf{k} \times) \exp(-\frac{\pi}{2} \mathbf{i} \times) & \underline{o}_1 &= \underline{o}_0 \\
 \underline{C}_2 &= \underline{C}_1 \exp(\theta_2 \mathbf{k} \times) \exp(\pi \mathbf{i} \times) & \underline{o}_2 &= \underline{o}_1 + \underline{C}_1 \exp(\theta_2 \mathbf{k} \times) (431.80 \mathbf{i}) \text{ mm} \\
 \underline{C}_3 &= \underline{C}_2 \exp(\theta_3 \mathbf{k} \times) \exp(\frac{\pi}{2} \mathbf{k} \times) \exp(\frac{\pi}{2} \mathbf{i} \times) & \underline{o}_3 &= \underline{o}_2 + \underline{C}_2 \exp(\theta_3 \mathbf{k} \times) (-149.09 \mathbf{k} + 20.32 \mathbf{j}) \text{ mm} \\
 \underline{C}_4 &= \underline{C}_3 \exp(\theta_4 \mathbf{k} \times) \exp(\frac{\pi}{2} \mathbf{i} \times) & \underline{o}_4 &= \underline{o}_3 + \underline{C}_3 \exp(\theta_4 \mathbf{k} \times) (433.07 \mathbf{k}) \text{ mm} \\
 \underline{C}_5 &= \underline{C}_4 \exp(\theta_5 \mathbf{k} \times) \exp(-\frac{\pi}{2} \mathbf{i} \times) & \underline{o}_5 &= \underline{o}_4 \\
 \underline{C}_6 &= \underline{C}_5 \exp(\theta_6 \mathbf{k} \times) & \underline{o}_6 &= \underline{o}_5 + \underline{C}_5 \exp(\theta_6 \mathbf{k} \times) (60.00 \mathbf{k}) \text{ mm}
 \end{aligned}
 \tag{3}$$

5c).

Do exercise 5ce

5d).

The MATLAB code used to in subsection 5d) is listed in Listing 2. The user is prompted for six joint avariables, and the total homogenous transformation matrix is calculated aswell as the link origins is plotted by using the `plot3` command. As Figure 3 shows there are 5 different link origins, as expected by (3) since origin 0 and 1, and 4 and 5 is the same point. There must however be a slight mistake somewhere in my code, as origin 2 gets moved $\sim 2500\text{mm}$ insted of 431.8mm , but I can't find it. The other origins does however seem to fit well with what's expected.

```
1 %% 5d
2 i = [1;0;0];
3 j = [0;1;0];
4 k = [0;0;1];
5
6 inputangle = ['1 ','2 ','3 ','4 ','5 ','6 '];
7 theta = [0 0 0 0 0 0];
8 for i = 1:6
9     theta(i) = degtorad(input(inputangle(i)));
10 end
11
12
13 [T1,C01] = DH_homog(theta(1), 0, 0, -pi/2);
14 [T2,C12] = DH_homog(theta(2), 0, 431.8, pi);
15 [T3,C23] = DH_homog(theta(3) + pi/2, -149.09, 20.32, pi/2);
16 [T4,C34] = DH_homog(theta(4), 433.07, 0, pi/2);
17 [T5,C45] = DH_homog(theta(5), 0, 0, -pi/2);
18 [T6,C56] = DH_homog(theta(6), 60, 0, 0);
19 T= T1*T2*T3*T4*T5*T6;
20
21 C0 = eye(3);
22 C1 = C0*C01;
23 C2 = C1*C12;
24 C3 = C2*C23;
25 C4 = C3*C34;
26 C5 = C4*C45;
27
28 o0 = [0;0;0];
29 o1 = o0;
30 o2 = o1 + C1*expm(theta(2)*skew(k))*431.8*i;
31 o3 = o2 + C2*expm(theta(3)*skew(k))*(-149.09*k + 20.32*j);
32 o4 = o3 + C3*expm(theta(4)*skew(k))*433.07*k;
33 o5 = o4;
```

```

34 o6 = o5 + C5*expm(theta(6)*skew(k))*60*k;
35
36
37 x = [o0(1) o1(1) o2(1) o3(1) o4(1) o5(1) o6(1)];
38 y = [o0(2) o1(2) o2(2) o3(2) o4(2) o5(2) o6(2)];
39 z = [o0(3) o1(3) o2(3) o3(3) o4(3) o5(3) o6(3)];
40
41 figure
42     hold on; view(3); grid on;
43     plot3(x,y,z, '*');
44     xlabel('$x$-coordinate', 'interpreter', 'latex'); xlim
45         ([-500 3500]);
46     ylabel('$y$-coordinate', 'interpreter', 'latex');
47     zlabel('$z$-coordinate', 'interpreter', 'latex'); zlim
48         ([-5 25]);

```

Listing 2: MATLAB code used to generate homogenous transformation matrix for angles chosen by the user

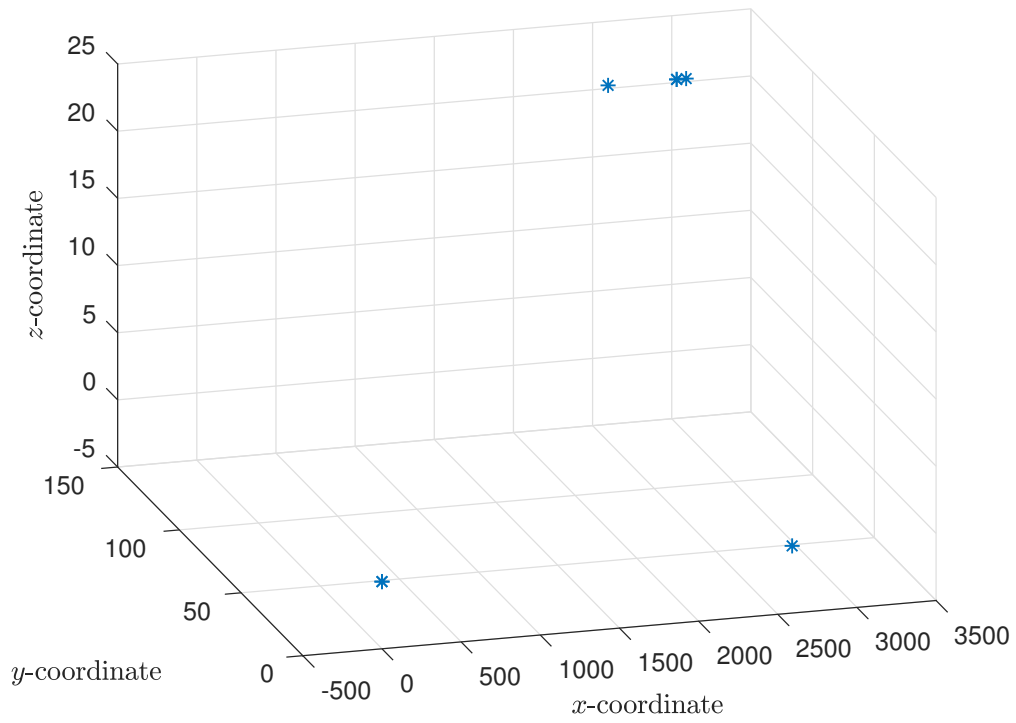


Figure 3: Location of the link origins with $\theta_i = 0 \ \forall i$