Assignment 1

ELEC 442 - Introduction to Robotics

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1.

Given the homogenous transformation

$$\begin{bmatrix} \boldsymbol{y} \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} Q & \boldsymbol{d} \\ \mathbf{0}^\top & 1 \end{bmatrix}}_{T} \begin{bmatrix} \boldsymbol{x} \\ 1 \end{bmatrix}$$

where Q and $\boldsymbol{d}$ accounts for rotation and translation, respectively. We have that the inverse is on the form

$$T^{-1} = \begin{bmatrix} \tilde{Q} & \tilde{\boldsymbol{d}} \\ \mathbf{0}^\top & 1 \end{bmatrix}$$

where we know that $T^{-1}T$ is equal to the 4×4 identity matrix. This yields

$$\begin{aligned} T^{-1}T &= \begin{bmatrix} \tilde{Q} & \tilde{\boldsymbol{d}} \\ \mathbf{0}^\top & 1 \end{bmatrix} \begin{bmatrix} Q & \boldsymbol{d} \\ \mathbf{0}^\top & 1 \end{bmatrix} \\ &= \begin{bmatrix} \tilde{Q}Q & \tilde{Q}\boldsymbol{d} + \tilde{\boldsymbol{d}} \\ \mathbf{0}^\top & 1 \end{bmatrix} = \mathbf{I}_{4\times4} \\ &\implies \begin{cases} \tilde{Q}Q & = \mathbf{I}_{3\times3} \\ \tilde{Q}\boldsymbol{d} + \tilde{\boldsymbol{d}} & = \mathbf{0} \end{cases} \\ &\implies \begin{cases} \tilde{Q} & = Q^{-1} = Q^\top \\ \tilde{\boldsymbol{d}} & = -\tilde{Q}\boldsymbol{d} = -Q^\top \boldsymbol{d} \end{cases} \\ \implies T^{-1} &= \underline{\begin{bmatrix} Q^\top & -Q^\top \boldsymbol{d} \\ \mathbf{0}^\top & 1 \end{bmatrix}} \end{aligned} \tag{1}$$

For T^{-1} to exist obvously T must be invertible, and for this to be fulfilled we require full rank. In this case $\text{rank}(T) = 4$ since $\text{rank}(Q) = 3\ \ \forall Q$ as Q is a rotation matrix, and the $T_{4,4} = 1\ \ \forall T$. Thus $\forall \{Q, \boldsymbol{d}\}$ we have $\text{rank}(T) = 4$ and T^{-1} exists.

2.

Considering the homogenous transformation matrix

$${}^0T_1 = \begin{bmatrix} Q & \boldsymbol{d} \\ \mathbf{0}^\top & 1 \end{bmatrix}$$

with

$$Q = \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{Q_1} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}}_{Q_2}$$

and

$$\boldsymbol{d} = \begin{bmatrix} -\frac{5}{\sqrt{2}} \\ \frac{5}{\sqrt{2}} \\ 4 \end{bmatrix} \text{ cm}$$

2a).

By obserwing the rotation matrices Q_1 and Q_2 we see that Q_1 is a simple rotation around the $\boldsymbol{k}$-axis. The angle of this rotation is given by $\theta = \arccos\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$. Q_2 is a simple rotation arond the $\boldsymbol{i}$-axis and the rotation angle is given by $\alpha = \arccos\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$. To determine d_1 and a_1 we recognize that a homogenous transformation matrix can be written ass the product of four transformation matrices; angle, offset, length and twist. This gives us

$$\begin{aligned}
{}^0T_1 = \begin{bmatrix} Q & \boldsymbol{d} \\ 0^\top & 1 \end{bmatrix} &= \underbrace{\begin{bmatrix} \exp(\theta\boldsymbol{k}\times) & \boldsymbol{0} \\ \boldsymbol{0}^\top & 1 \end{bmatrix}}_{\text{angle}} \underbrace{\begin{bmatrix} \boldsymbol{I} & d\boldsymbol{k} \\ \boldsymbol{0}^\top & 1 \end{bmatrix}}_{\text{offset}} \underbrace{\begin{bmatrix} \boldsymbol{I} & a\boldsymbol{i} \\ \boldsymbol{0}^\top & 1 \end{bmatrix}}_{\text{length}} \underbrace{\begin{bmatrix} \exp(\alpha\boldsymbol{i}\times) & \boldsymbol{0} \\ \boldsymbol{0}^\top & 1 \end{bmatrix}}_{\text{twist}} \\
&= \begin{bmatrix} \exp(\theta\boldsymbol{k}\times + \alpha\boldsymbol{i}\times) & \exp(\theta\boldsymbol{k}\times)(a\boldsymbol{i} + d\boldsymbol{k}) \\ \boldsymbol{0}^\top & 1 \end{bmatrix} = \begin{bmatrix} Q & \boldsymbol{d} \\ \boldsymbol{0}^\top & 1 \end{bmatrix} \\
\implies \exp(\theta\boldsymbol{k}\times)(a\boldsymbol{i} + d\boldsymbol{k}) &= \begin{bmatrix} -\frac{5}{\sqrt{2}} \\ \frac{5}{\sqrt{2}} \\ 4 \end{bmatrix} \\
\implies \exp(\theta\boldsymbol{k}\times) \begin{bmatrix} a \\ 0 \\ d \end{bmatrix} &= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ 0 \\ d \end{bmatrix} = \begin{bmatrix} \frac{a}{\sqrt{2}} \\ -\frac{a}{\sqrt{2}} \\ d \end{bmatrix} = \begin{bmatrix} -\frac{5}{\sqrt{2}} \\ \frac{5}{\sqrt{2}} \\ 4 \end{bmatrix} \\
\implies &\begin{cases} a = -5 \\ d = 4 \end{cases}
\end{aligned} \tag{2}$$

And we have numeric values for all our four DH parameters.

2b).

For the point represented in coordinate system 1 by $\underset{\sim}{\boldsymbol{x}} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^\top$ cm we get the representation in system 0 given by

$$
\begin{aligned}
\begin{bmatrix} {}^0\boldsymbol{x} \\ 1 \end{bmatrix} &= {}^0T_1 \begin{bmatrix} {}^1\boldsymbol{x} \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{\sqrt{3}}{2\sqrt{2}} & -\frac{5}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{\sqrt{3}}{2\sqrt{2}} & \frac{5}{\sqrt{2}} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{4}{\sqrt{2}} \\ -\frac{4}{\sqrt{2}} \\ 4 \\ 1 \end{bmatrix} \\
\implies {}^0\underset{\sim}{\boldsymbol{x}} &= \begin{bmatrix} \frac{4}{\sqrt{2}} \\ -\frac{4}{\sqrt{2}} \\ 4 \end{bmatrix} \text{cm}
\end{aligned}
$$

2c).

For the opposite case, that we have a point represented in cooridinate system 0 by $\underset{\sim}{\boldsymbol{x}} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^\top$ cm we apply the inverse transformation matrix that is on the form we found in (1). This gives

$$
\begin{aligned}
\begin{bmatrix} {}^1\boldsymbol{x} \\ 1 \end{bmatrix} &= {}^0T_1^{-1} \begin{bmatrix} {}^0\boldsymbol{x} \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 5 \\ -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{2} & -2\sqrt{3} \\ -\frac{\sqrt{3}}{2\sqrt{2}} & -\frac{\sqrt{3}}{2\sqrt{2}} & -\frac{1}{2} & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{10+\sqrt{2}}{2} \\ -\frac{8\sqrt{3}+\sqrt{2}}{4} \\ \frac{16-\sqrt{6}}{4} \\ 1 \end{bmatrix} \\
\implies {}^1\underset{\sim}{\boldsymbol{x}} &= \begin{bmatrix} \frac{10+\sqrt{2}}{2} \\ -\frac{8\sqrt{3}+\sqrt{2}}{4} \\ \frac{16-\sqrt{6}}{4} \end{bmatrix} \text{cm}
\end{aligned}
$$

2d).

The angular velocity vector represented in coordinate system 0 by $\begin{bmatrix}1 & 0 & 0\end{bmatrix}^\top$ is given by

$$
\begin{aligned}
\begin{bmatrix}\boldsymbol{\omega}_{1,1}\\ 0\end{bmatrix} &= {}^0T_1^{-1}\begin{bmatrix}\boldsymbol{\omega}_{1,0}\\ 0\end{bmatrix}\\
&= \begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 5\\ -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{2} & -2\sqrt{3}\\ -\frac{\sqrt{3}}{2\sqrt{2}} & -\frac{\sqrt{3}}{2\sqrt{2}} & -\frac{1}{2} & 4\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\\
&= \begin{bmatrix}\frac{1}{\sqrt{2}}\\ -\frac{1}{2\sqrt{2}}\\ -\frac{\sqrt{3}}{2\sqrt{2}}\\ 0\end{bmatrix}\\
\implies \boldsymbol{\omega}_{1,1} &= \begin{bmatrix}\frac{1}{\sqrt{2}}\\ -\frac{1}{2\sqrt{2}}\\ -\frac{\sqrt{3}}{2\sqrt{2}}\end{bmatrix}
\end{aligned}
$$

3.

A MATLAB fuction named `DH_homog` is implemented with the code shown in Listing 1.

```matlab
function T = DH_homog(theta, d, a, alpha)
    i=[1;0;0];
    k=[0;0;1];
    angle = [expm(theta*skew(k)) zeros(3,1); zeros(1,3) 1];
    offset = [eye(3) d*k; zeros(1,3) 1];
    length = [eye(3) a*i; zeros(1,3) 1];
    twist = [expm(alpha*skew(i)) zeros(3,1); zeros(1,3) 1];
    T = angle*offset*length*twist;
end
```

Listing 1: MATLAB code to generate homogenous transformation matrix based on the Denavit-Hartenberg convention

4.

A sketch of the “home” position can be found in Figure 1.

Find the Jacobian and how to do this

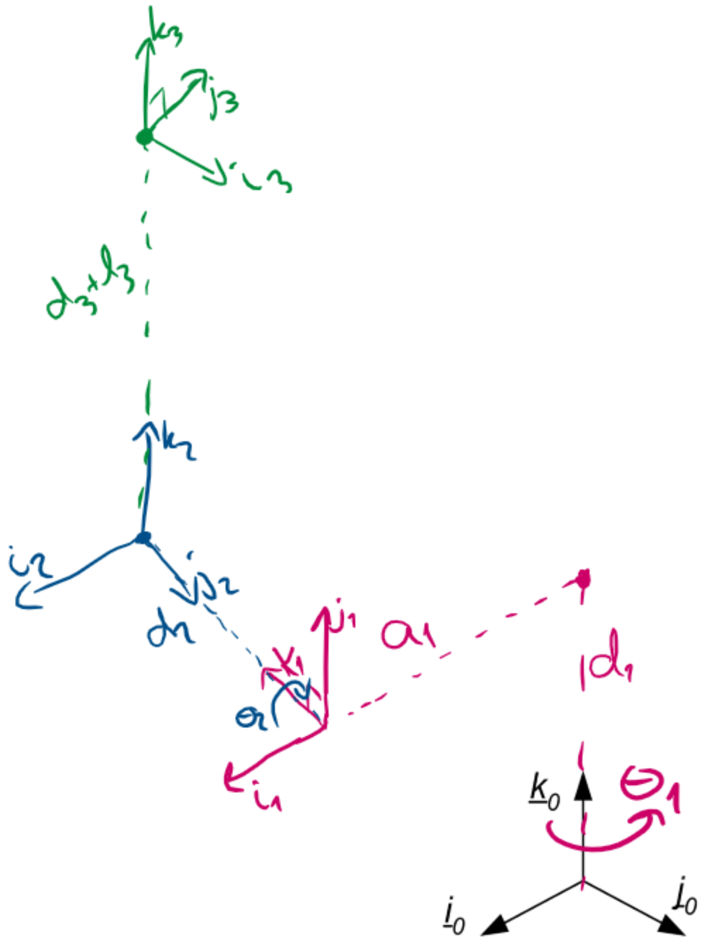
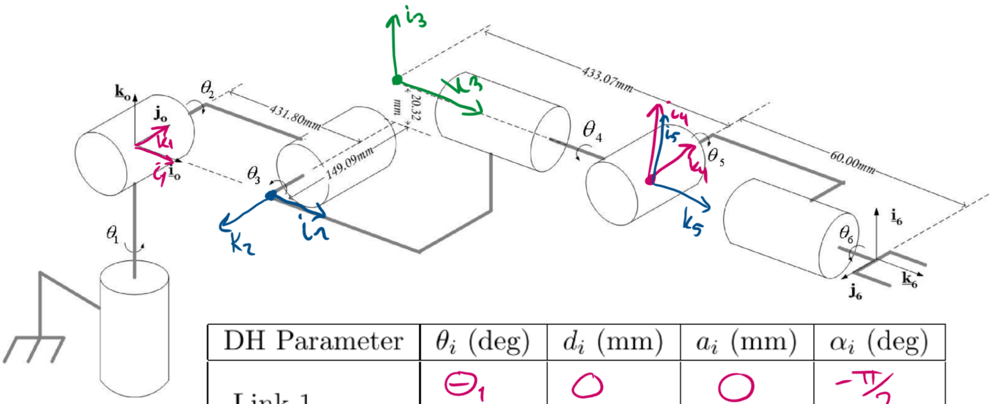


Figure 1: Sketch of “home” position based on the DH-table provided

5.

5a).

The different coordinate frames are sketched and the completed table is found in Figure 2. I see now that it should have been in degrees, but I have filled the table with the radian values.



DH Parameter	θ_i (deg)	d_i (mm)	a_i (mm)	α_i (deg)
Link 1	θ_1	0	0	$-\pi/2$
Link 2	θ_2	0	431.80	π
Link 3	$\theta_3+\frac{\pi}{2}$	-149.09	20.32	$\pi/2$
Link 4	θ_4	433.07	0	$\pi/2$
Link 5	θ_5	0	0	$-\pi/2$
Link 6	θ_6	60.06	0	0

Figure 2: Sketch of the coordinate frames according to the DH-convention

5b).

We know that the relationship between base $\{\underset{\sim}{\boldsymbol{o}}_0, \underline{C}_0\}$ and the end effector $\{\underset{\sim}{\boldsymbol{o}}_6, \underline{C}_6\}$ is given by

$$\begin{bmatrix} \underline{C}_n & \underset{\sim}{\boldsymbol{o}}_n \\ \mathbf{0}^\top & 1 \end{bmatrix} = \begin{bmatrix} \underline{C}_0 & \underset{\sim}{\boldsymbol{o}}_0 \\ \mathbf{0}^\top & 1 \end{bmatrix} {}^0T_1(q_1)\,{}^1T_2(q_2)\ldots{}^{n-1}T_n(q_n)$$

$$\implies \begin{bmatrix} \underline{C}_6 & \underset{\sim}{\boldsymbol{o}}_6 \\ \mathbf{0}^\top & 1 \end{bmatrix} = \begin{bmatrix} \underline{C}_0 & \underset{\sim}{\boldsymbol{o}}_0 \\ \mathbf{0}^\top & 1 \end{bmatrix} {}^0T_6$$

where ${}^0T_6$ is a series of transformations on the form as shown in the first line of (2), and we have the relationship between he base and the end effector. A chain of transformations that give the relationship between base $\{\underset{\sim}{\boldsymbol{o}}_0, \underline{C}_0\}$ and the end effector $\{\underset{\sim}{\boldsymbol{o}}_6, \underline{C}_6\}$ in this exact case on the form presented by Salcudean's example 2.5 is

$$\underline{C}_1 = \underline{C}_0 \exp\left(\theta_1 \boldsymbol{k}\times\right) \exp\left(-\tfrac{\pi}{2}\boldsymbol{i}\times\right) \qquad \underset{\sim}{\boldsymbol{o}}_1 = \underset{\sim}{\boldsymbol{o}}_0$$

$$\underline{C}_2 = \underline{C}_1 \exp\left(\theta_2 \boldsymbol{k}\times\right) \exp\left(\pi\boldsymbol{i}\times\right) \qquad \underset{\sim}{\boldsymbol{o}}_2 = \underset{\sim}{\boldsymbol{o}}_1 + \underline{C}_1 \exp\left(\theta_2 \boldsymbol{k}\times\right)\left(431.80\boldsymbol{i}\right)\mathrm{mm}$$

$$\underline{C}_3 = \underline{C}_2 \exp\left(\theta_3 \boldsymbol{k}\times\right) \exp\left(\tfrac{\pi}{2}\boldsymbol{k}\times\right) \exp\left(\tfrac{\pi}{2}\boldsymbol{i}\times\right) \qquad \underset{\sim}{\boldsymbol{o}}_3 = \underset{\sim}{\boldsymbol{o}}_2 + \underline{C}_2 \exp\left(\theta_3 \boldsymbol{k}\times\right)\left(-149.09\boldsymbol{k} + 20.32\boldsymbol{j}\right)\mathrm{mm}$$

$$\underline{C}_4 = \underline{C}_3 \exp\left(\theta_4 \boldsymbol{k}\times\right) \exp\left(\tfrac{\pi}{2}\boldsymbol{i}\times\right) \qquad \underset{\sim}{\boldsymbol{o}}_4 = \underset{\sim}{\boldsymbol{o}}_3 + \underline{C}_3 \exp\left(\theta_4 \boldsymbol{k}\times\right)\left(433.07\boldsymbol{k}\right)\mathrm{mm}$$

$$\underline{C}_5 = \underline{C}_4 \exp\left(\theta_5 \boldsymbol{k}\times\right) \exp\left(-\tfrac{\pi}{2}\boldsymbol{i}\times\right) \qquad \underset{\sim}{\boldsymbol{o}}_5 = \underset{\sim}{\boldsymbol{o}}_4$$

$$\underline{C}_6 = \underline{C}_5 \exp\left(\theta_6 \boldsymbol{k}\times\right) \qquad \underset{\sim}{\boldsymbol{o}}_6 = \underset{\sim}{\boldsymbol{o}}_5 + \underline{C}_5 \exp\left(\theta_6 \boldsymbol{k}\times\right)\left(60.00\boldsymbol{k}\right)\mathrm{mm}$$

5c).

Do exercise 5cde