Exam information

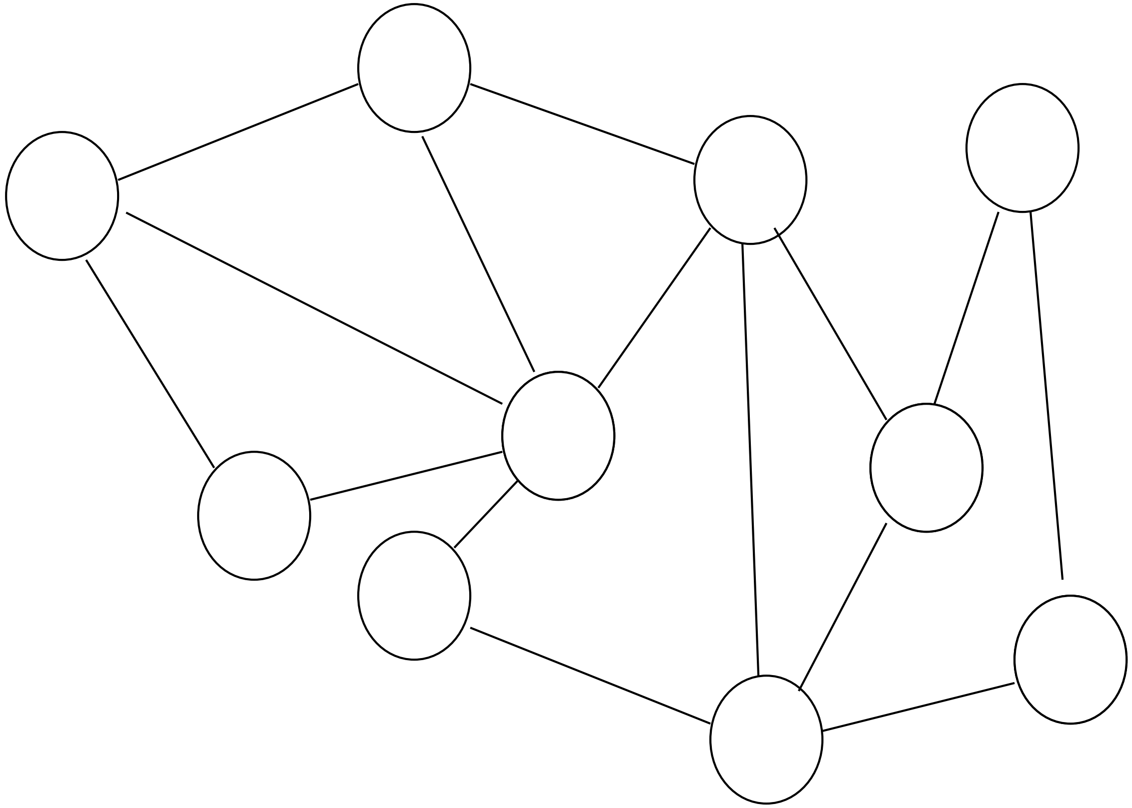
- The exam will be in <u>written</u> form
  - First exam: 31.07.25, 9:30-12:00 in A 151
  - Second exam: 1.10.25, 11:30-14:00 in A 151
- Content: everything discussed in the lecture and in the exercise classes, no extra material from the lecture notes

Spanning Tree Constructions

In the LOCAL and CONGEST models

Spanning Trees

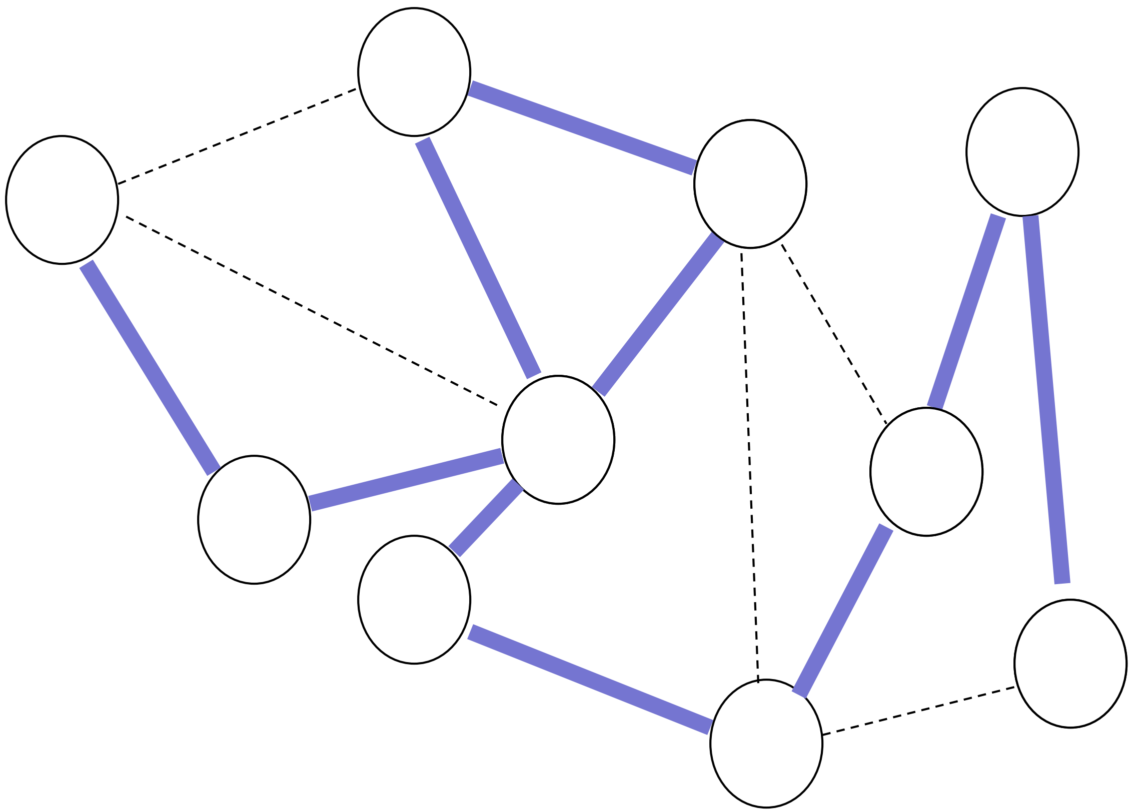
Attactive „infrastructure“: sparse subgraph („loop-free backbone“) connecting all nodes. E.g., cheap flooding.



Spanning Tree

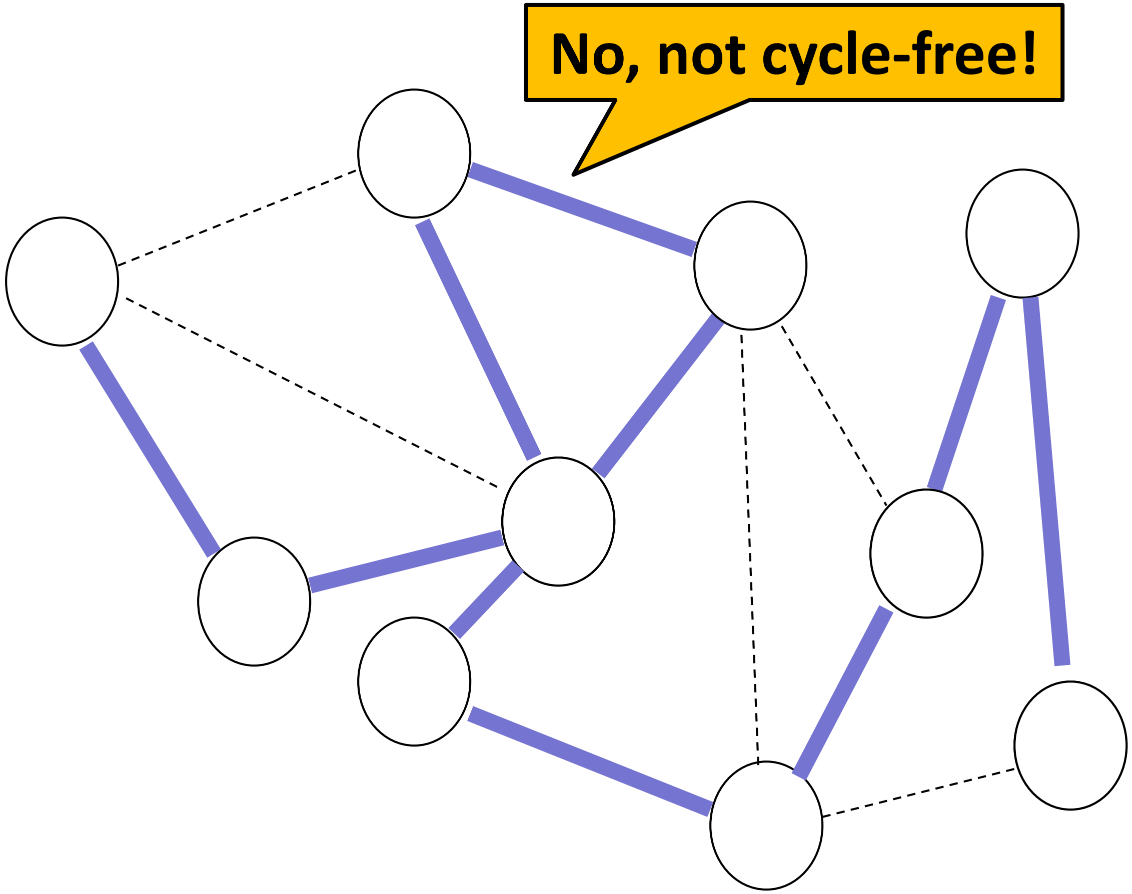
Cycle-free subgraph spanning all nodes.

Spanning Trees

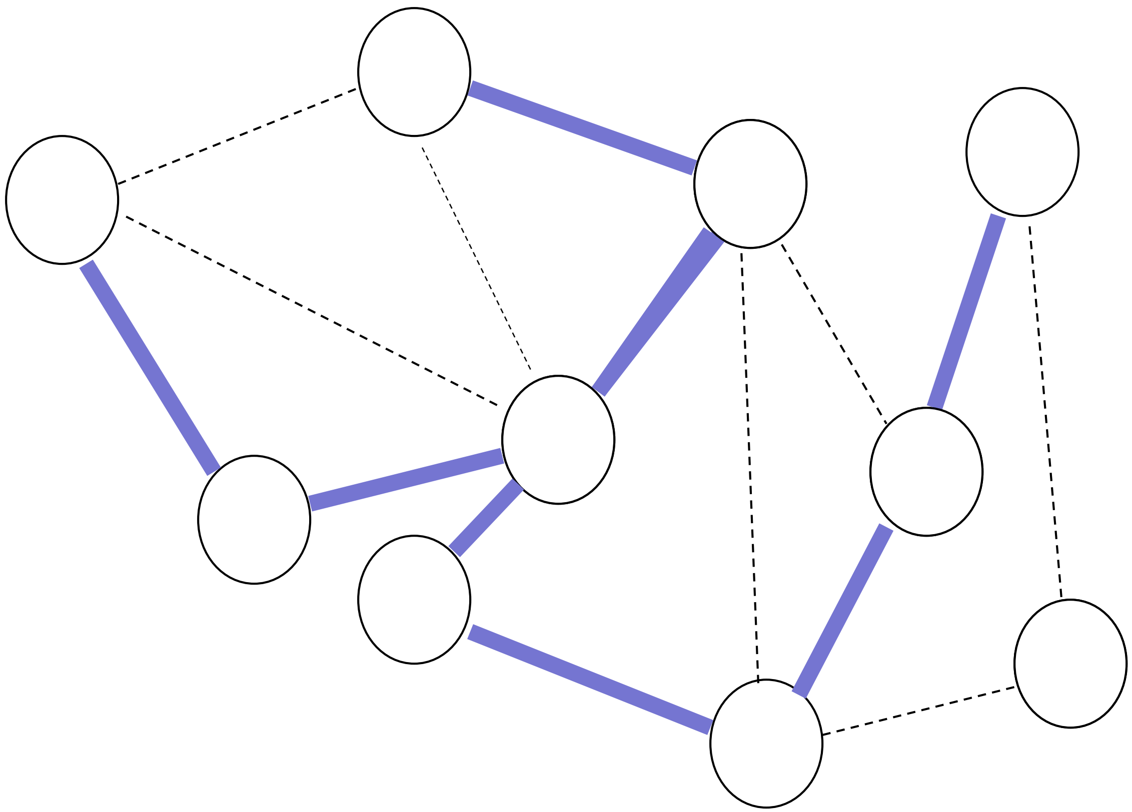


Is this a spanning tree?

Spanning Trees



Spanning Trees



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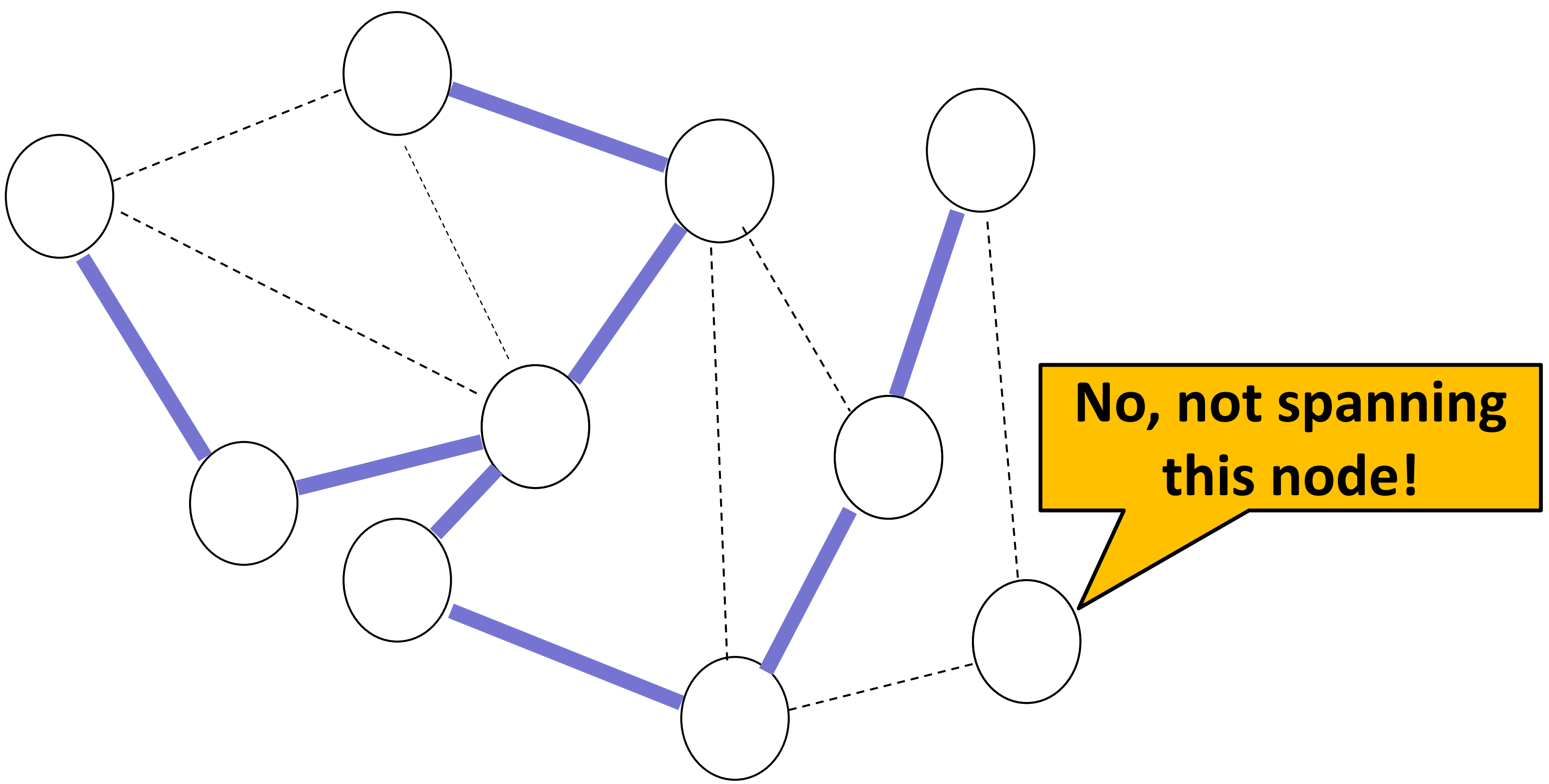
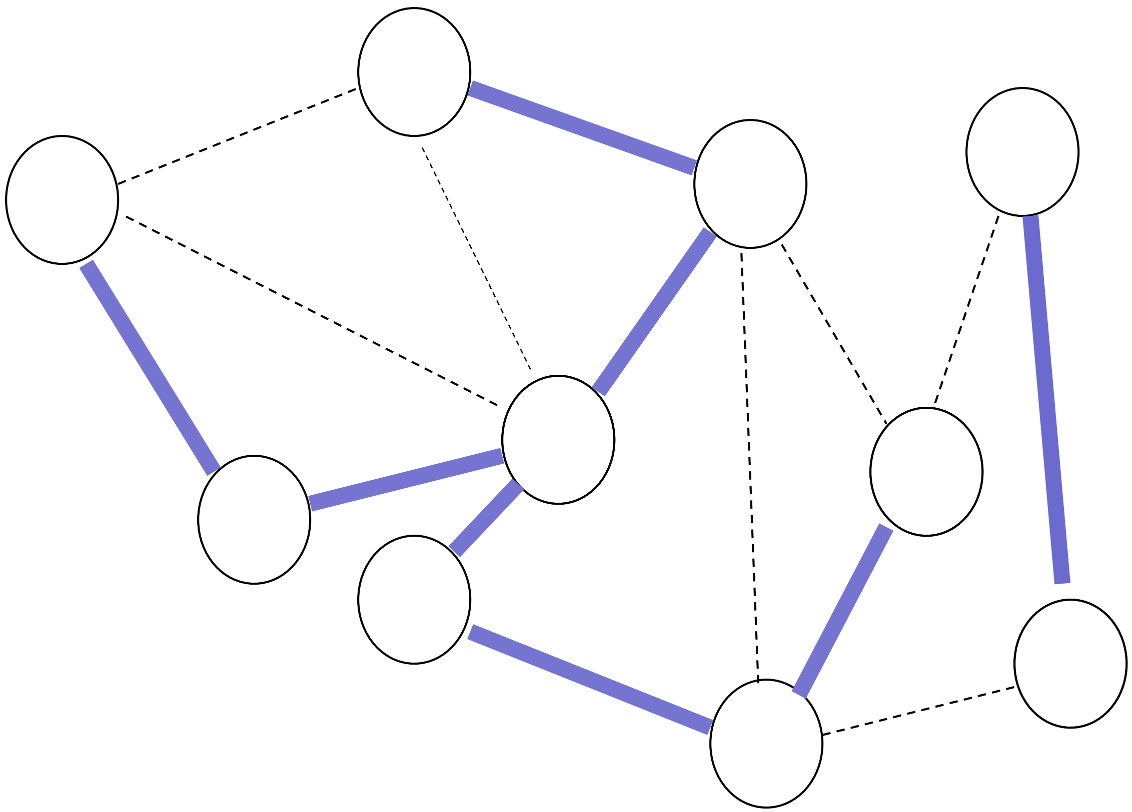


Spanning Trees



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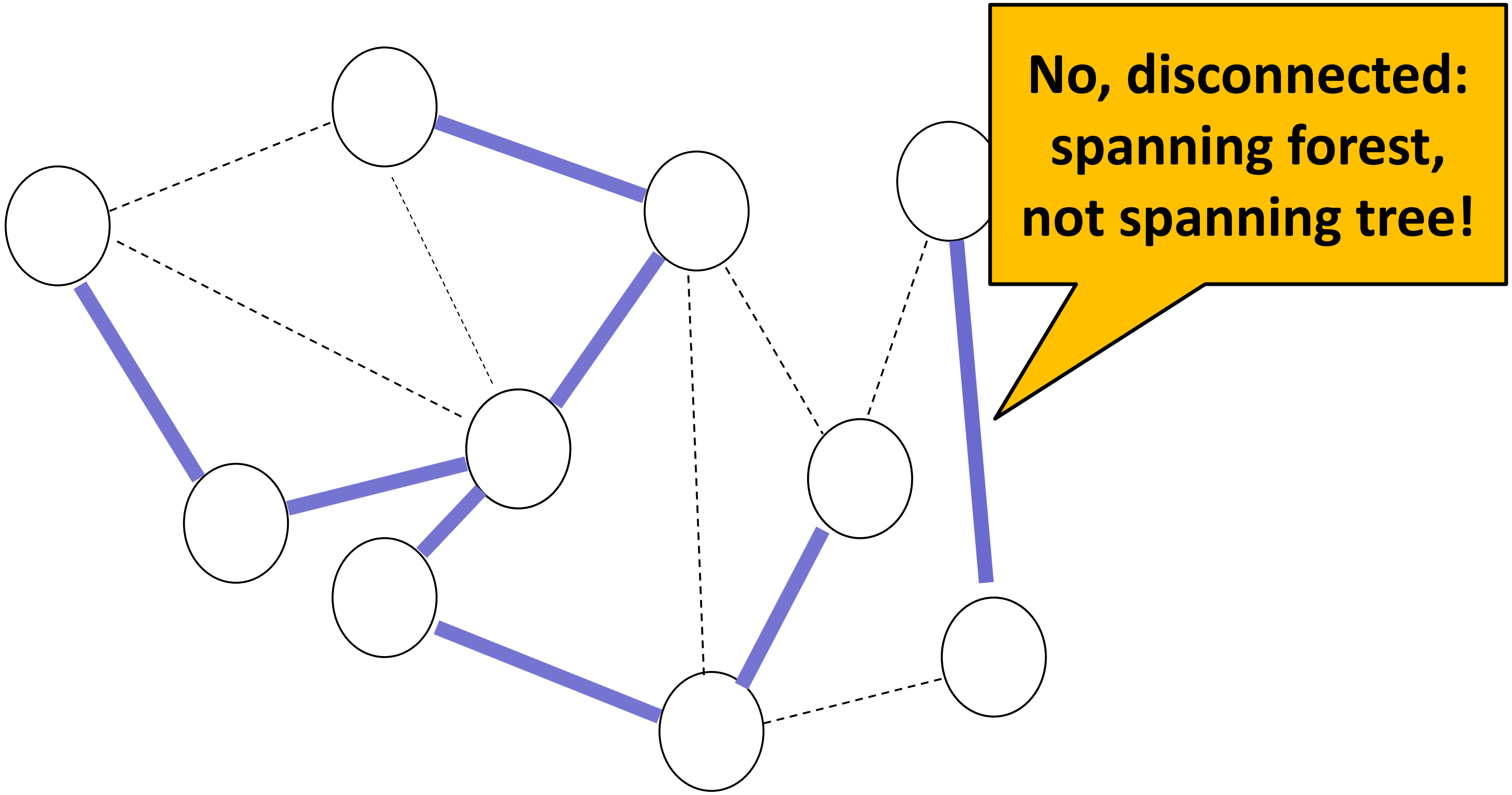
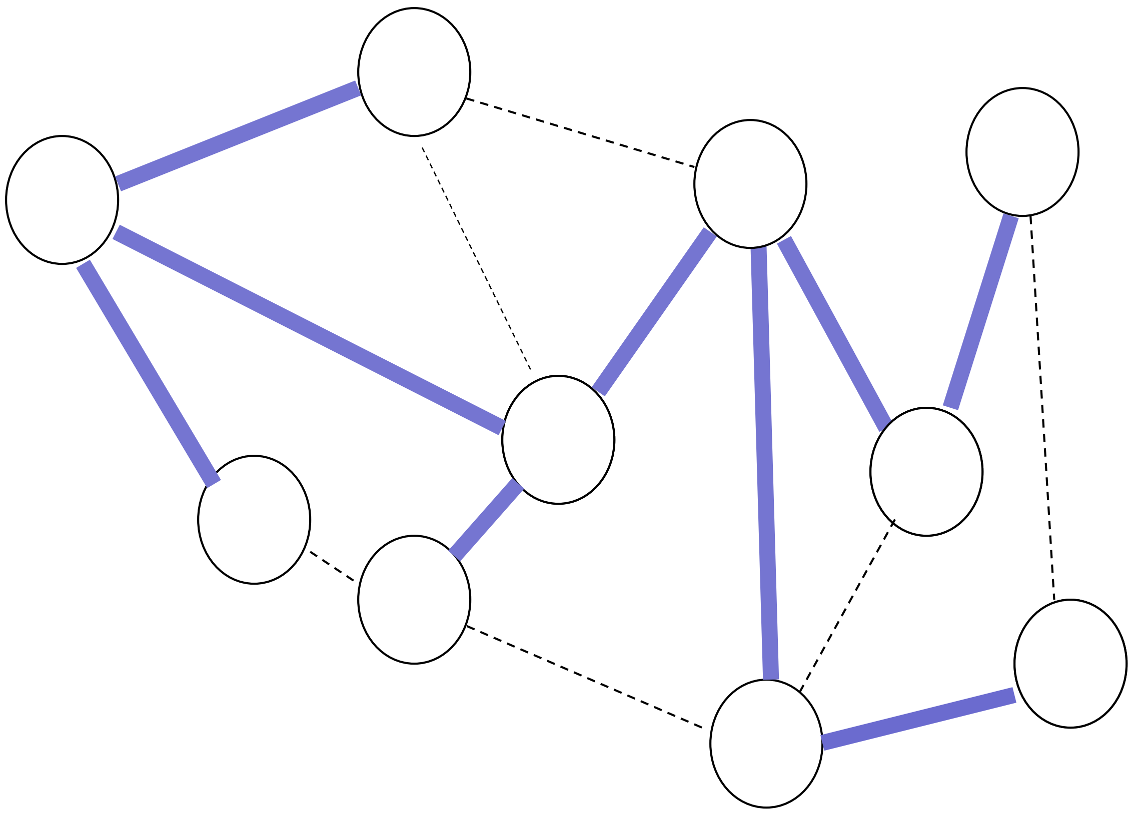


Spanning Trees



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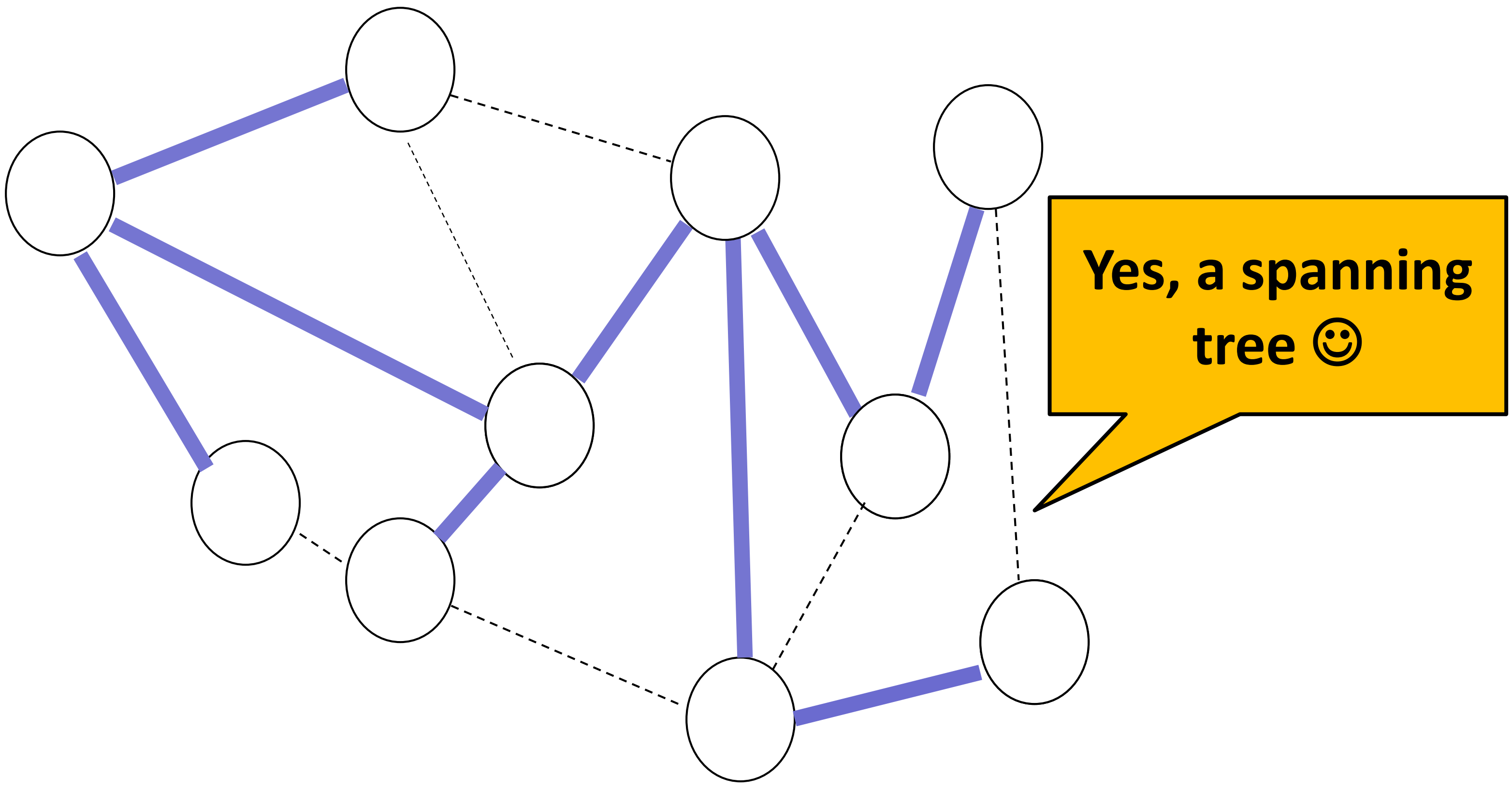
Spanning Trees



Applications

Efficient Broadcast and Aggregation

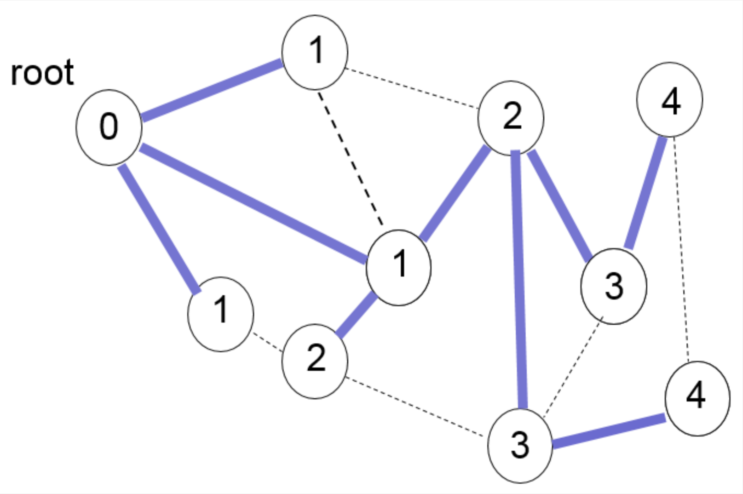
- ❑ Used in Ethernet network to avoid Layer-2 forwarding loops: Spanning Tree Protocol
- ❑ In ad-hoc networks: efficient backbone: broadcast and aggregate data using a linear number of transmissions

Algebraic Gossip

- ❑ Disseminating multiple messages in large communication network
- ❑ Random communication pattern with neighbors
- ❑ Gossip: based on local interactions

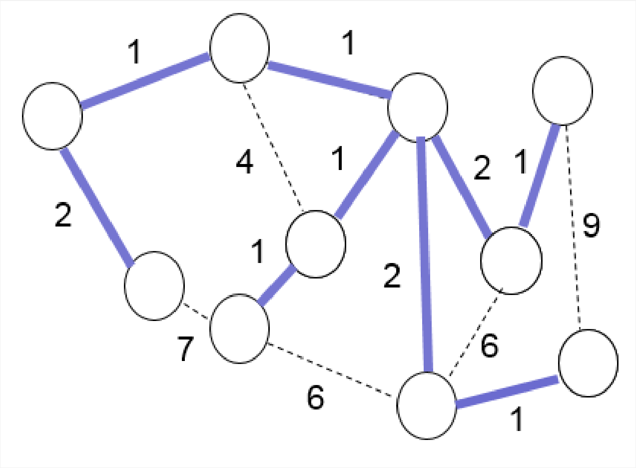
Types of Spanning Trees

BFS



- a.k.a. shortest distance spanning tree (may also be weighted)
- Spanning tree includes shortest paths from **a given root** to all nodes
- Interesting e.g. for fast broadcast

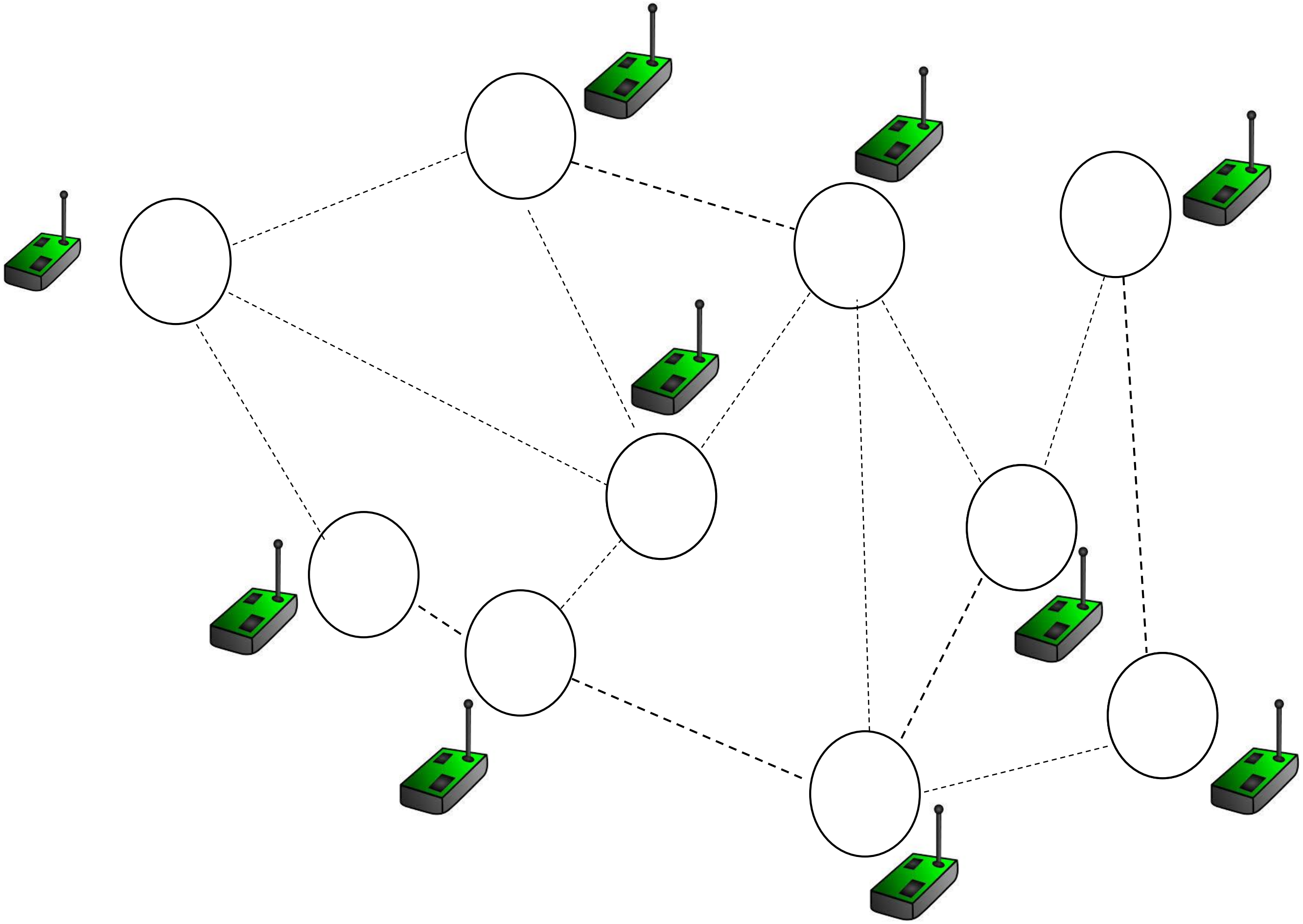
MST



- Minimum link cost spanning tree
- Interesting, e.g., for least routing cost or energy cost

How to compute a spanning tree in the LOCAL model?

A Fundamental Communication Primitive: ConvergeCast



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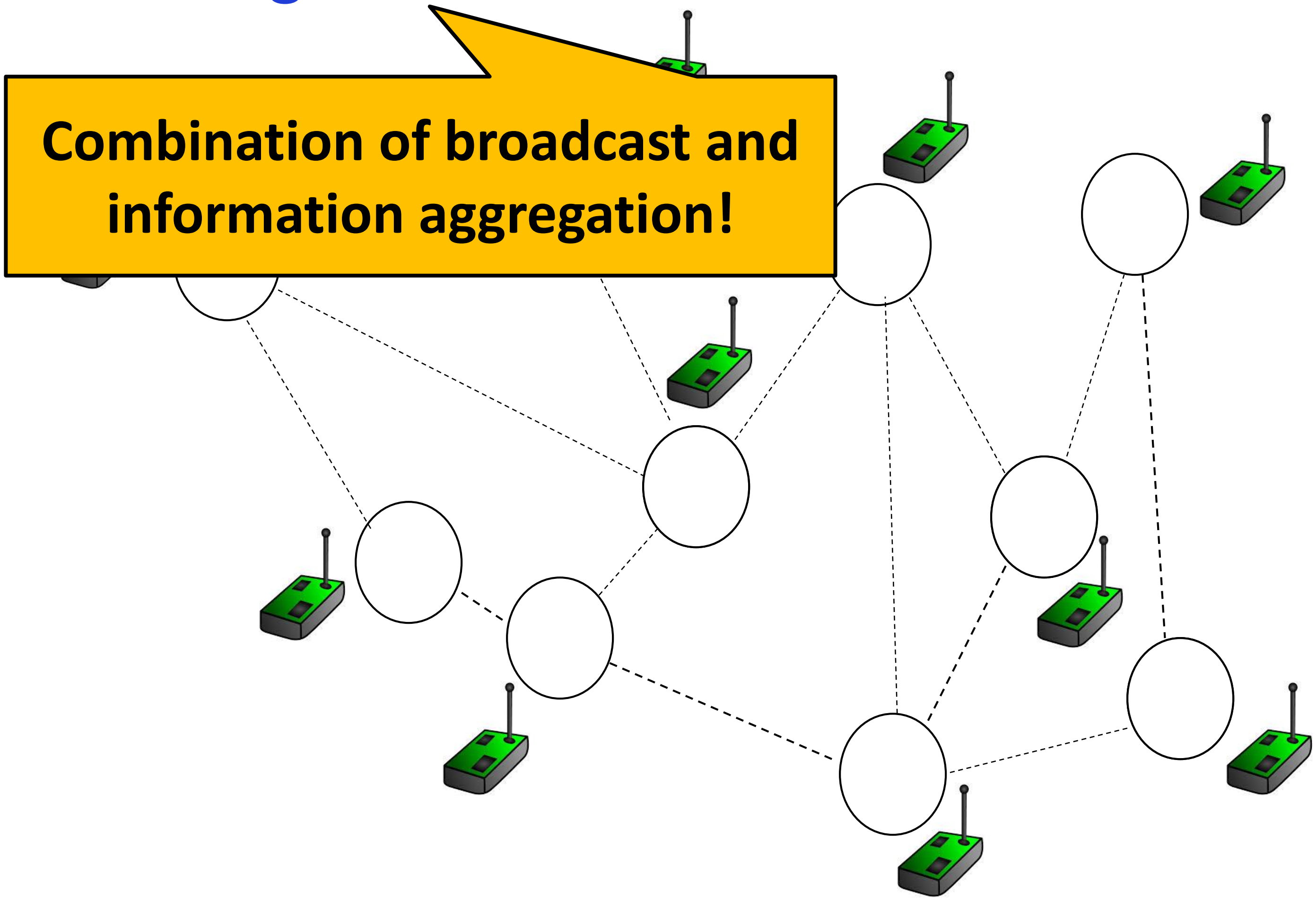

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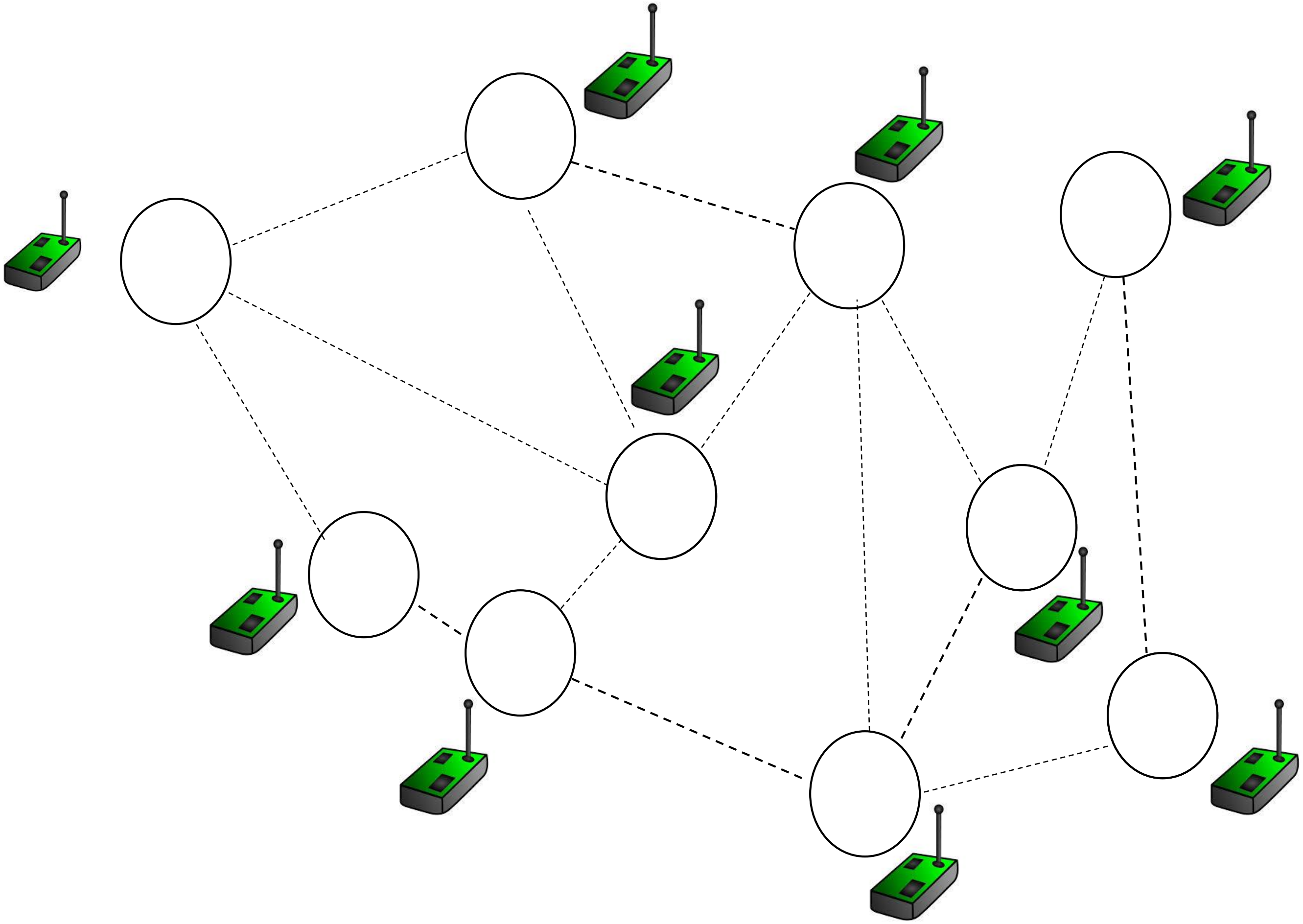

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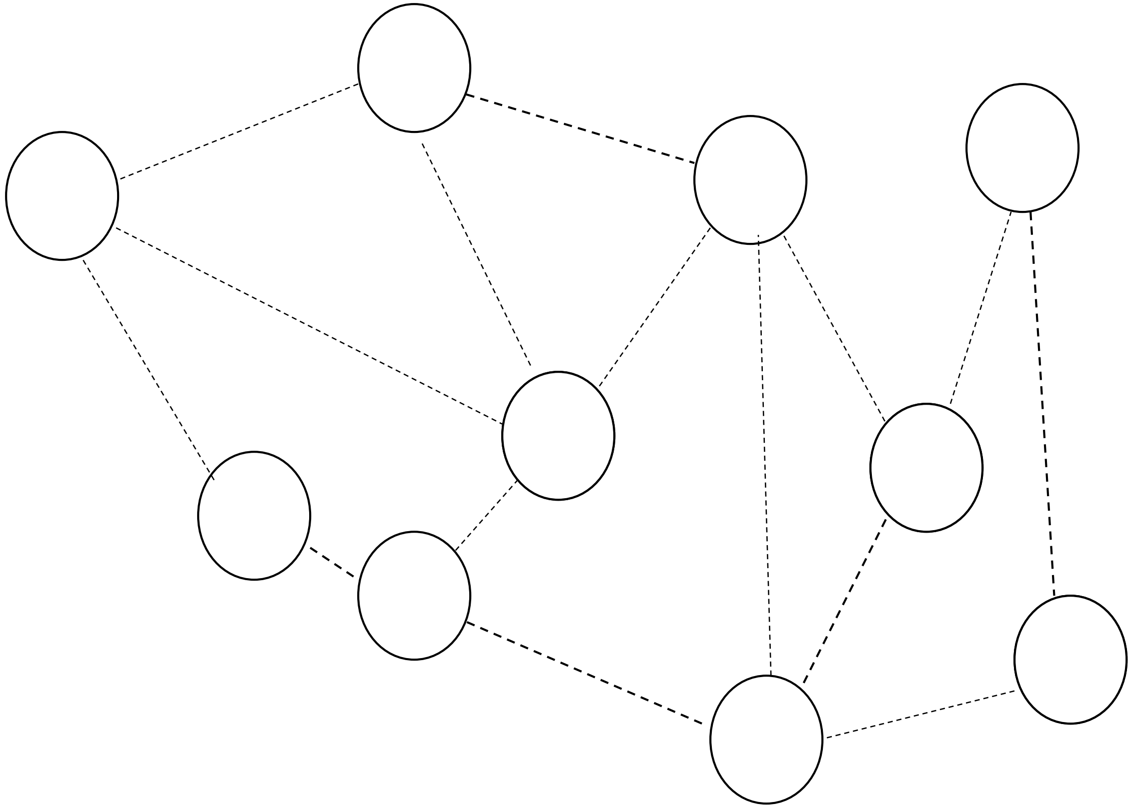

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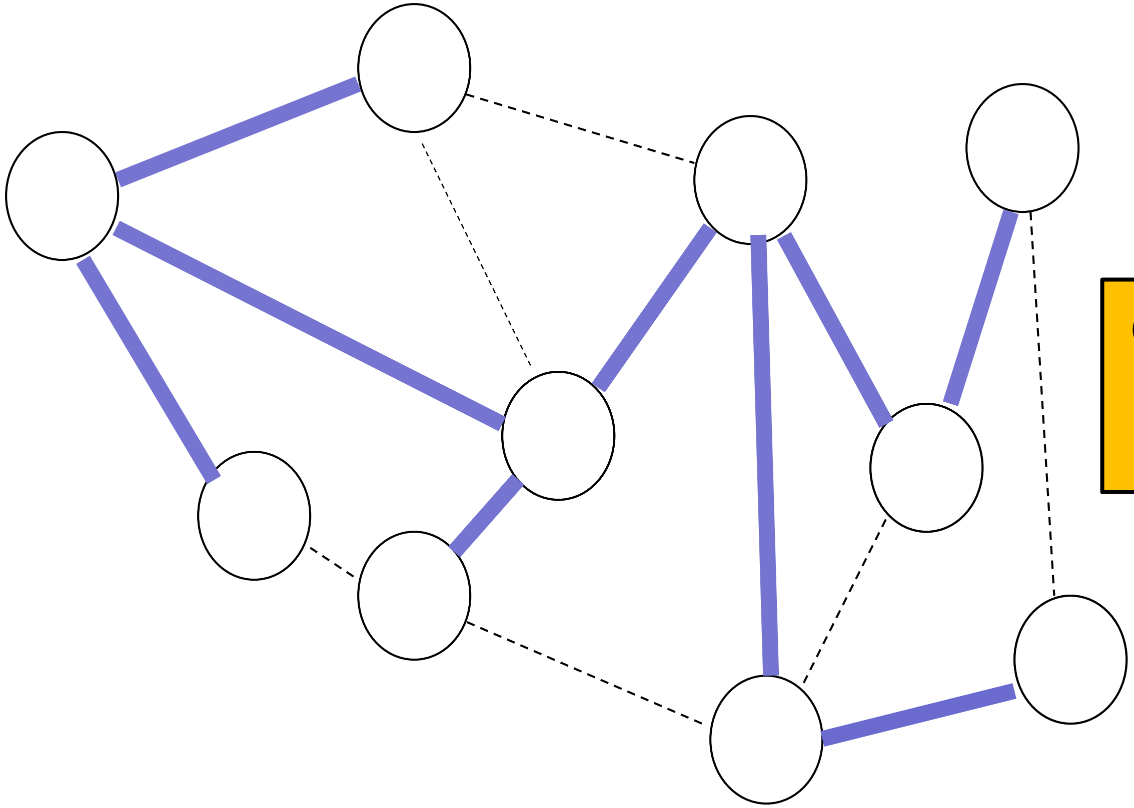
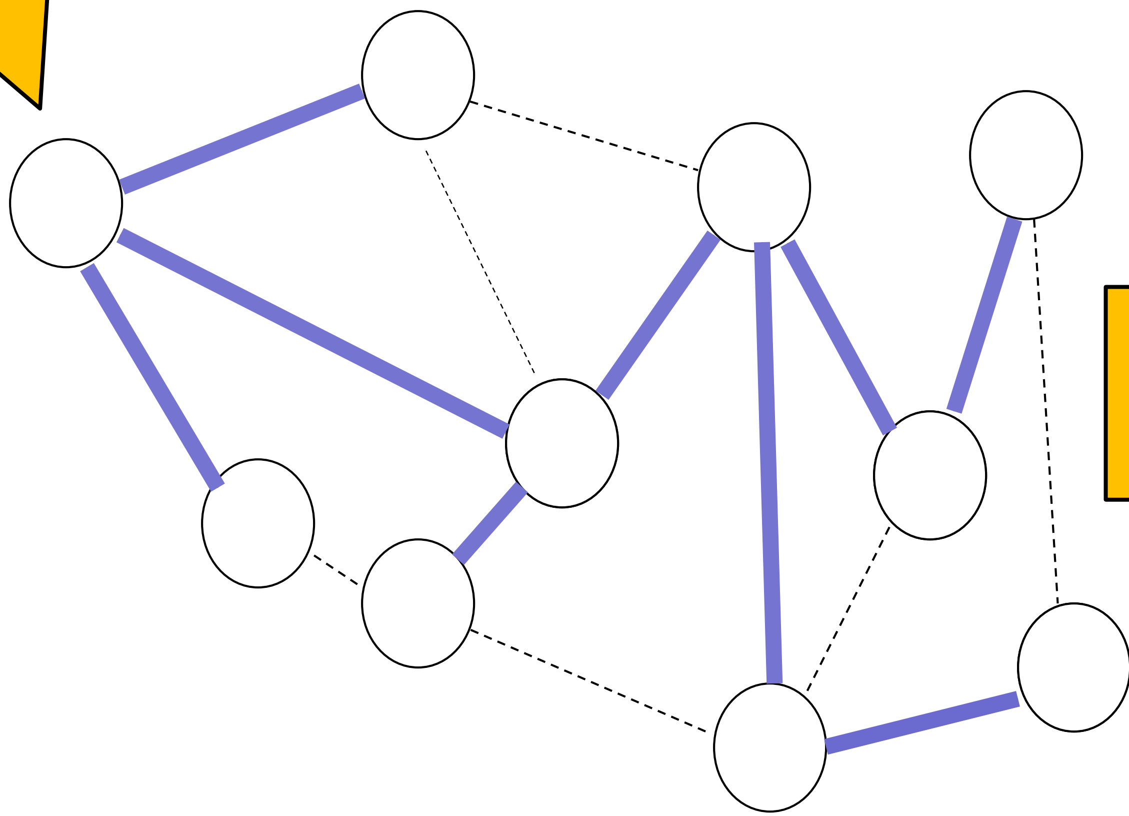


ConvergeCast: efficient if we already have a spanning tree!

A Fundamental Communication Primitive:

„Want to know average temperature!“



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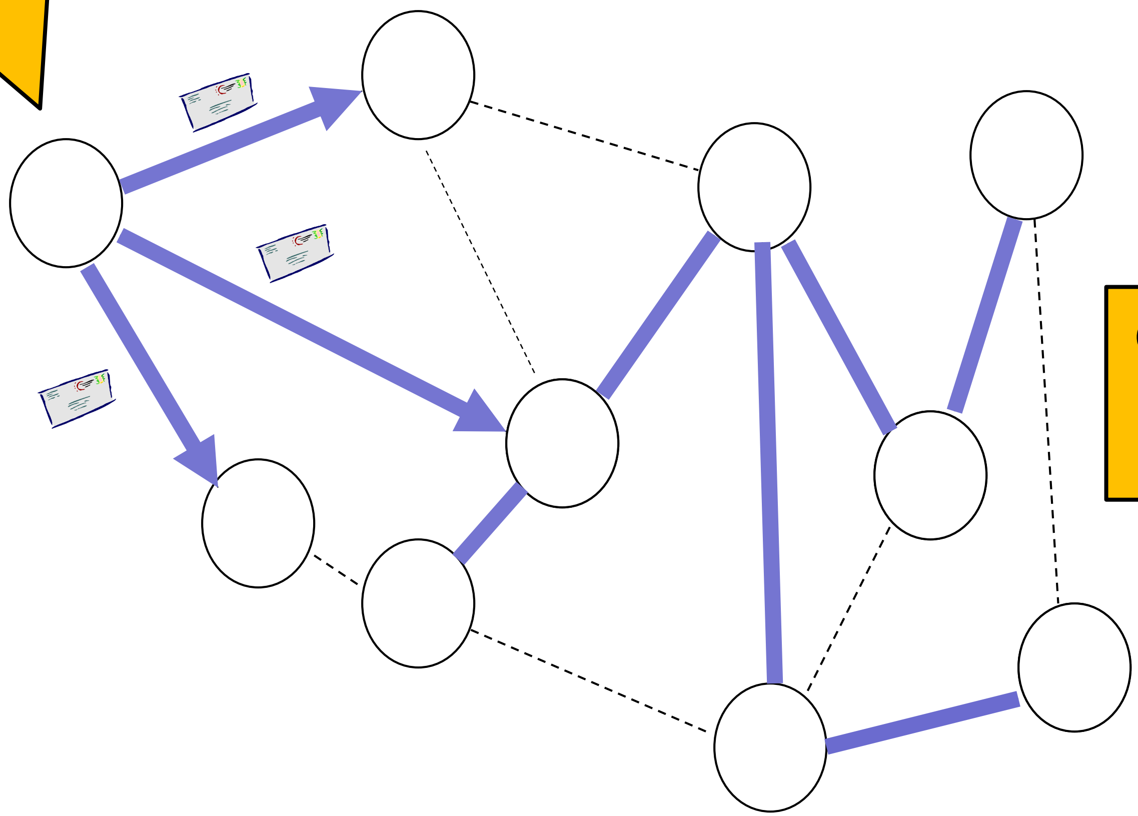


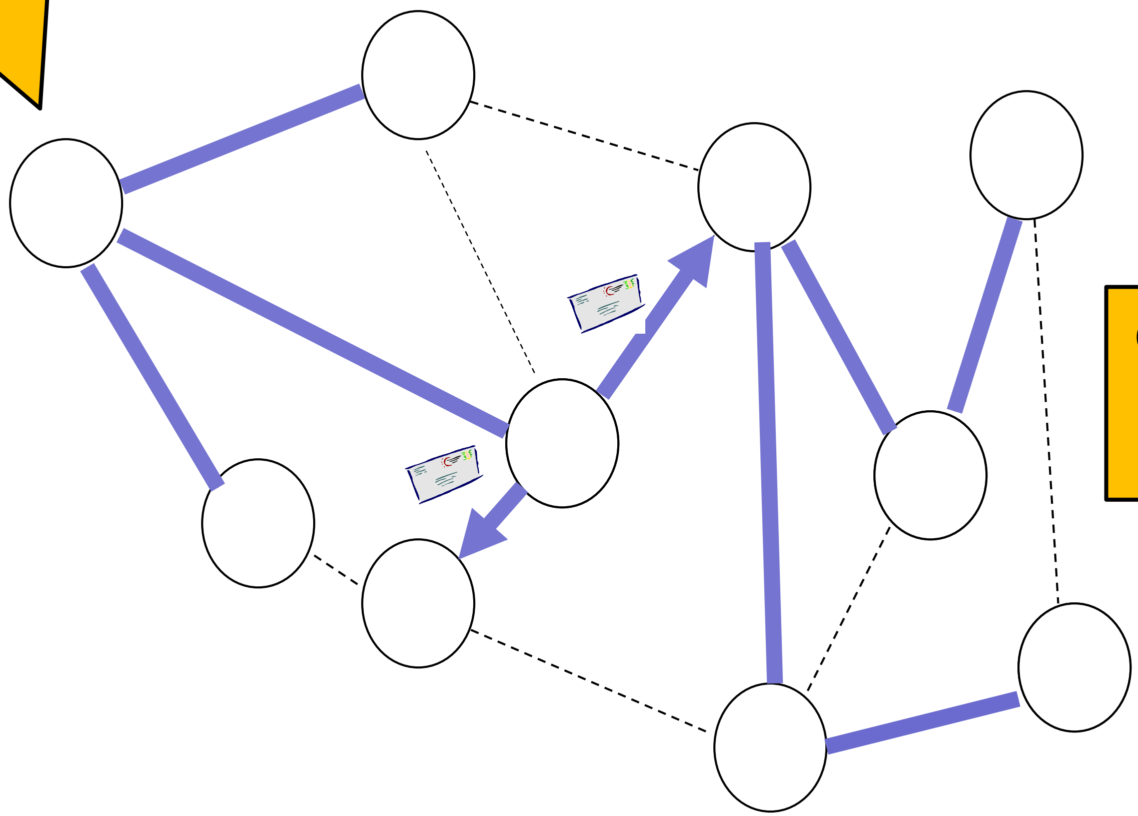
ConvergeCast: efficient if we already have a spanning tree!

Broadcast

Round 1

A Fundamental Communication Primitive:

„Want to know average temperature!“

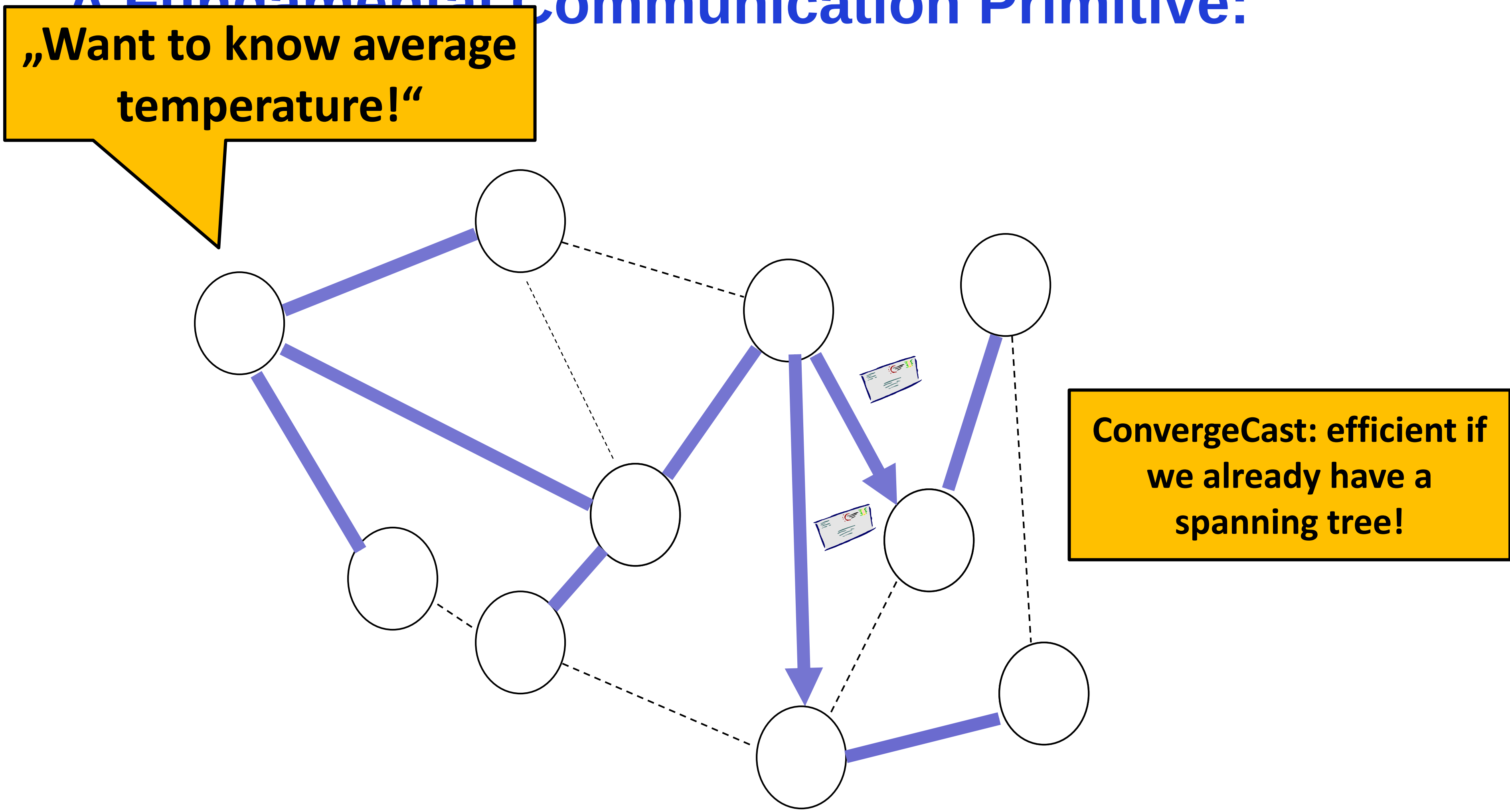


ConvergeCast: efficient if we already have a spanning tree!

Broadcast

Round 2

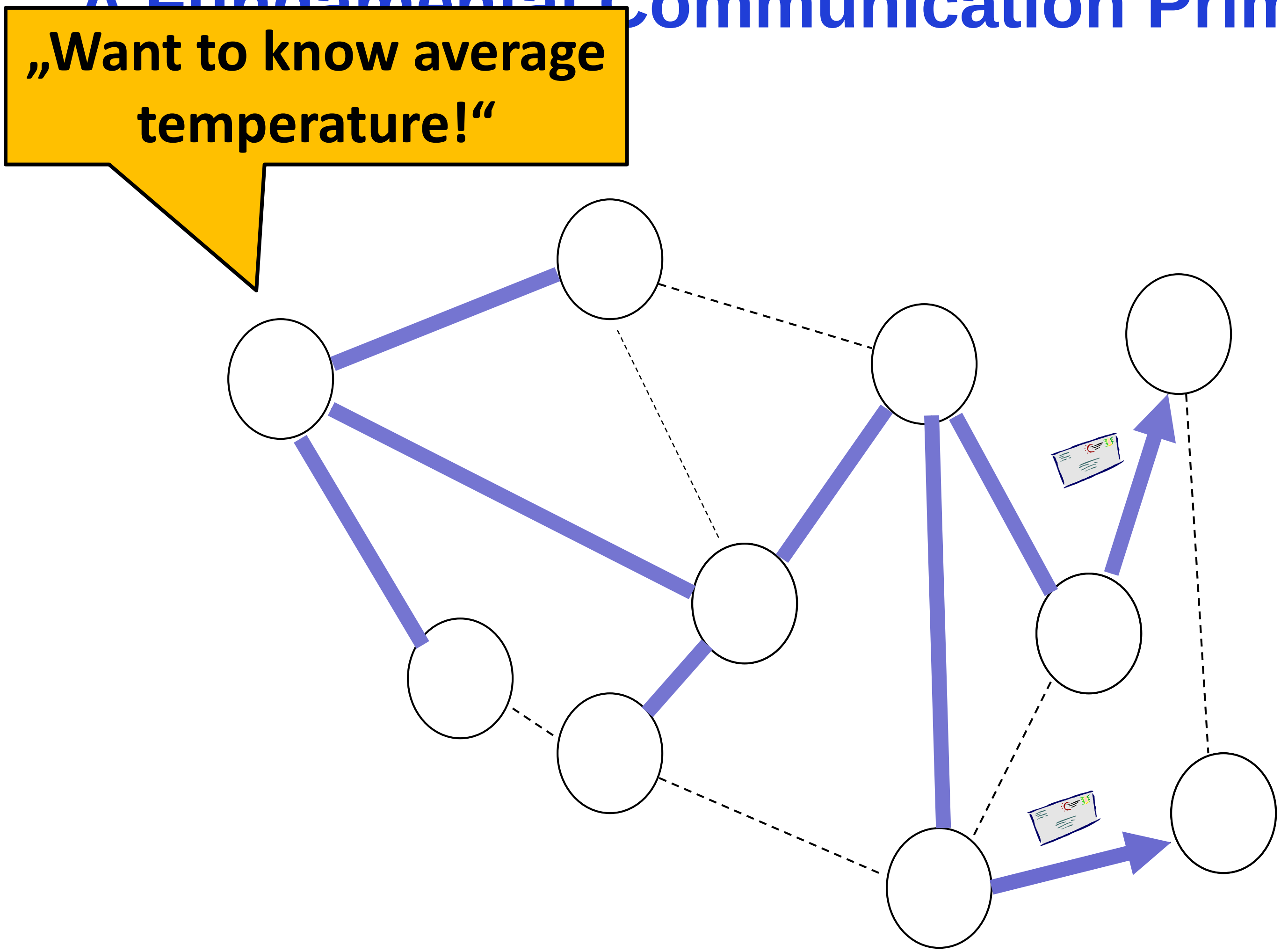
A Fundamental Communication Primitive:



Broadcast

Round 3

A Fundamental Communication Primitive:



Broadcast

Round 4

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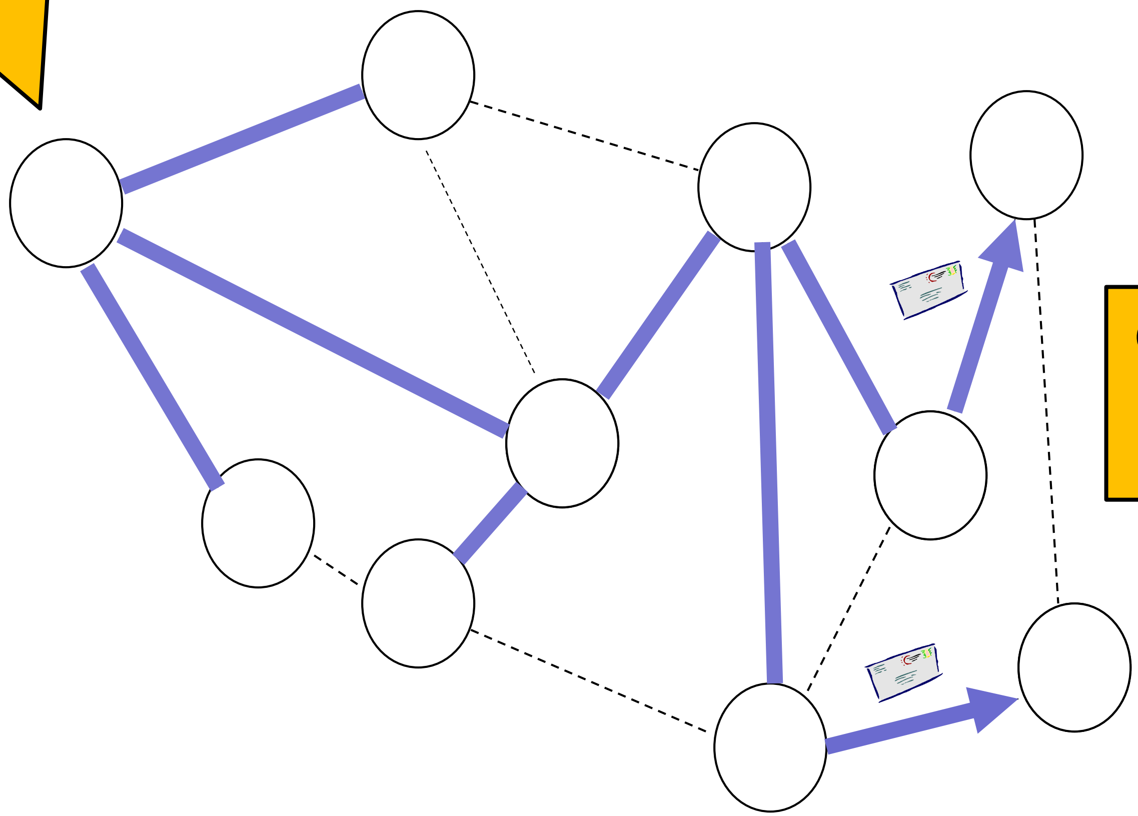
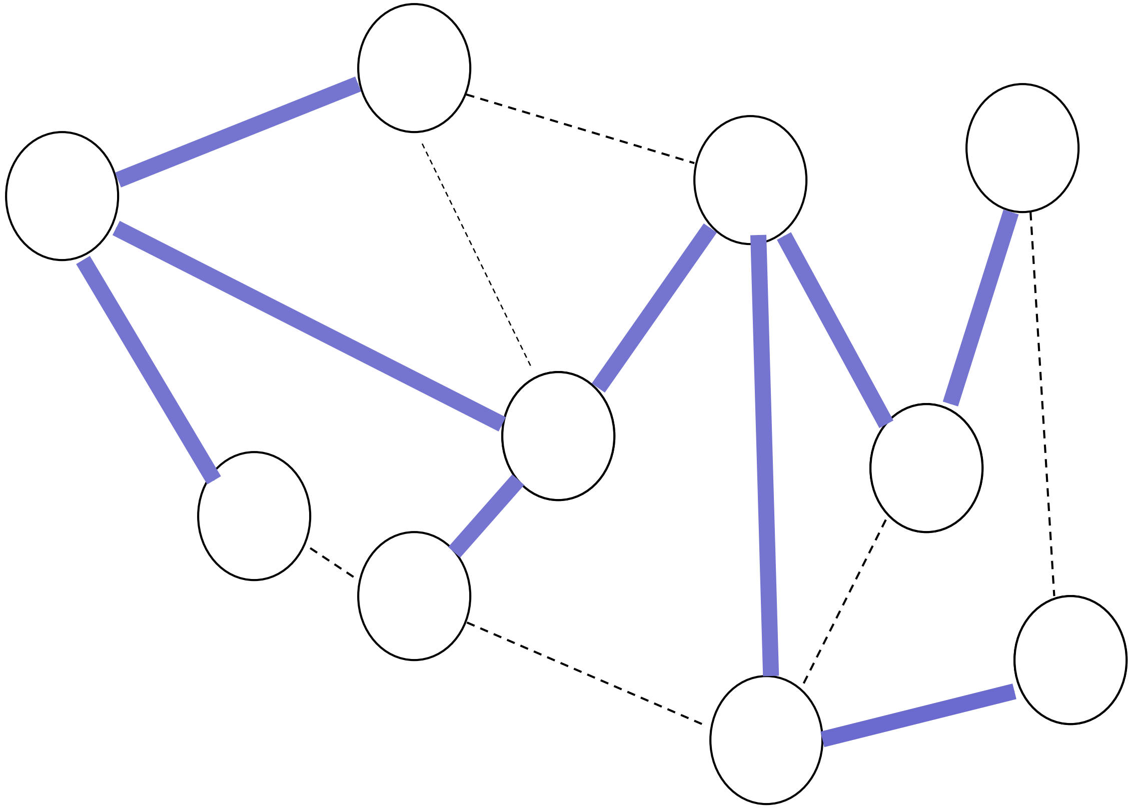


ConvergeCast: efficient if we already have a spanning tree!

Broadcast

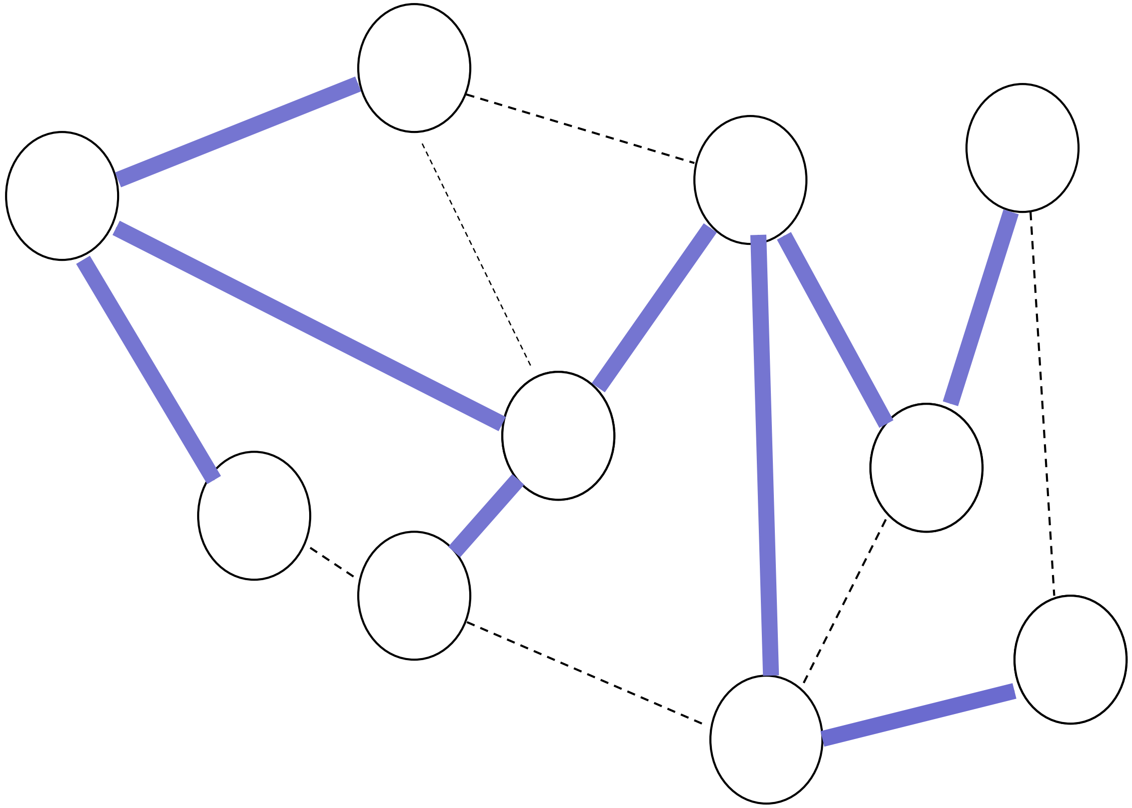
Round 4

A Fundamental Communication Primitive: ConvergeCast



Broadcast: O(n) messages, O(Tree-Depth) time!

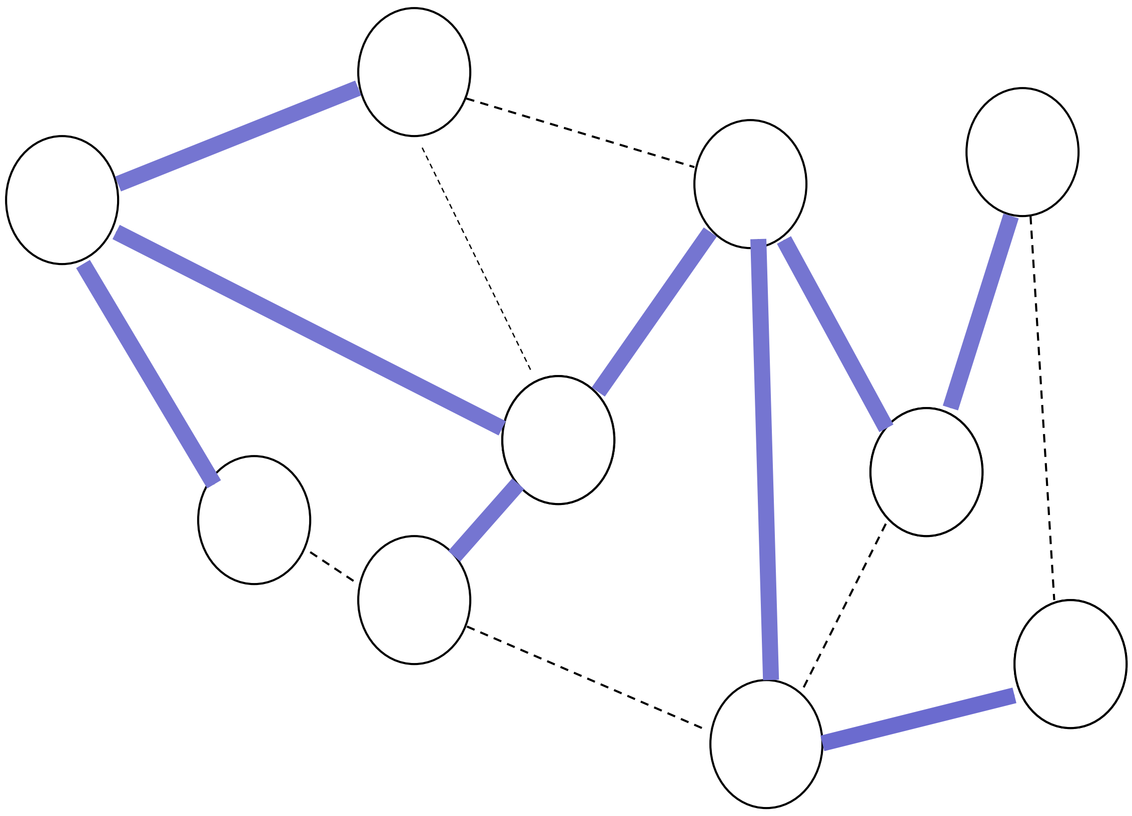
A Fundamental Communication Primitive: ConvergeCast



Broadcast: O(n) messages, O(Tree-Depth) time!

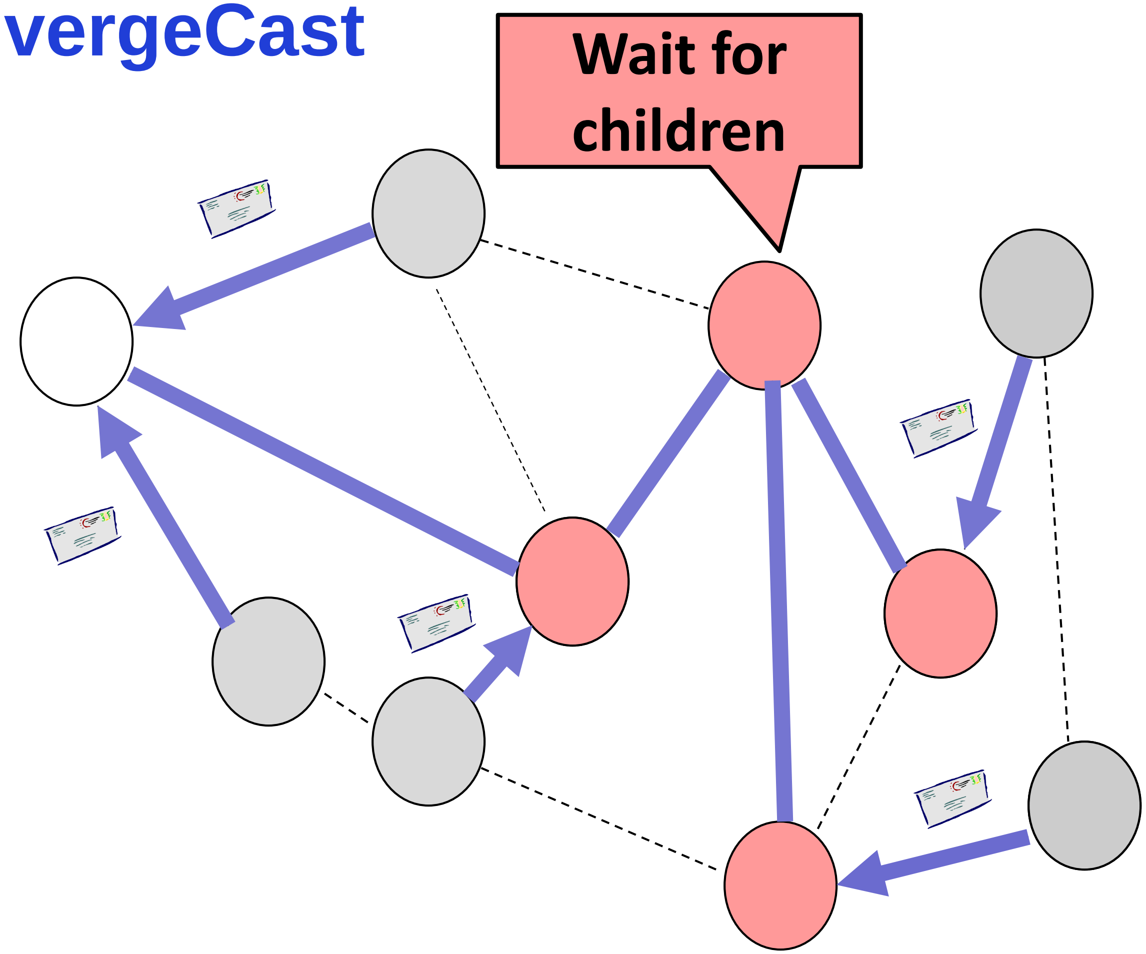
How to aggregate with O(n) messages?

A Fundamental Communication Primitive: ConvergeCast



Efficiently aggregate: from leaves toward root!

A Fundamental Communication Primitive: ConvergeCast

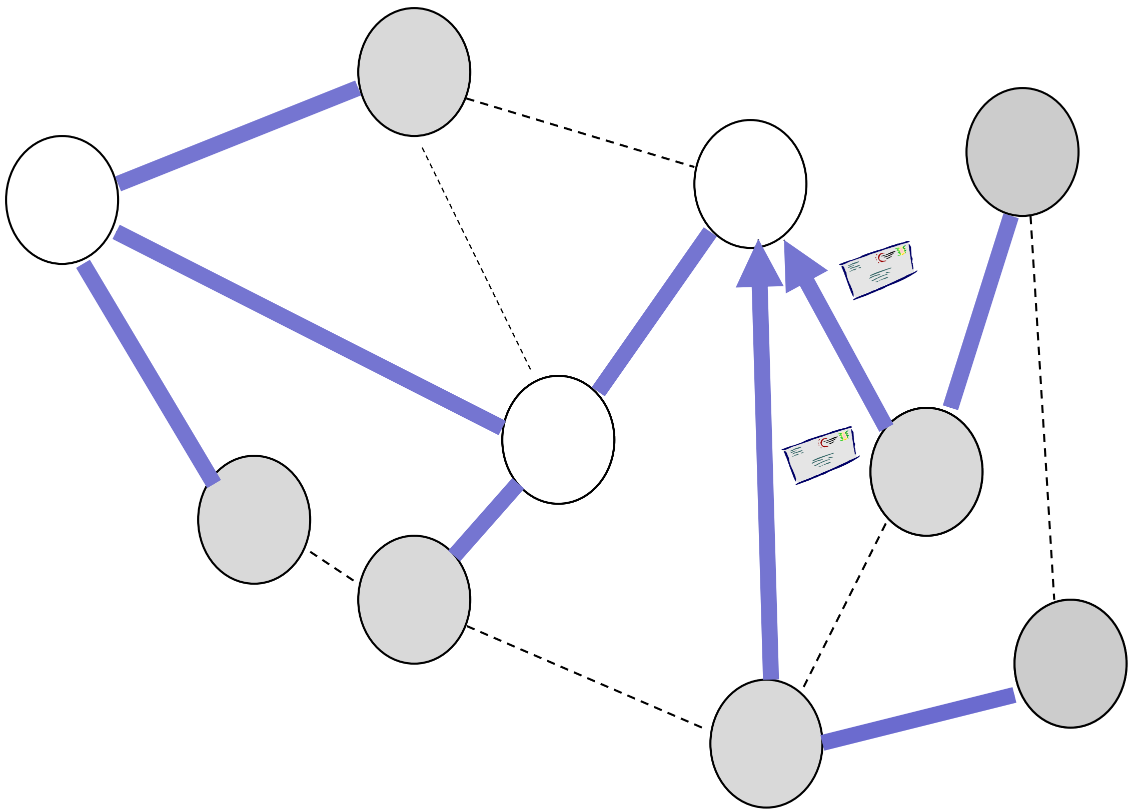


Efficiently aggregate: from leaves toward root!

Aggregate

Round 1

A Fundamental Communication Primitive: ConvergeCast

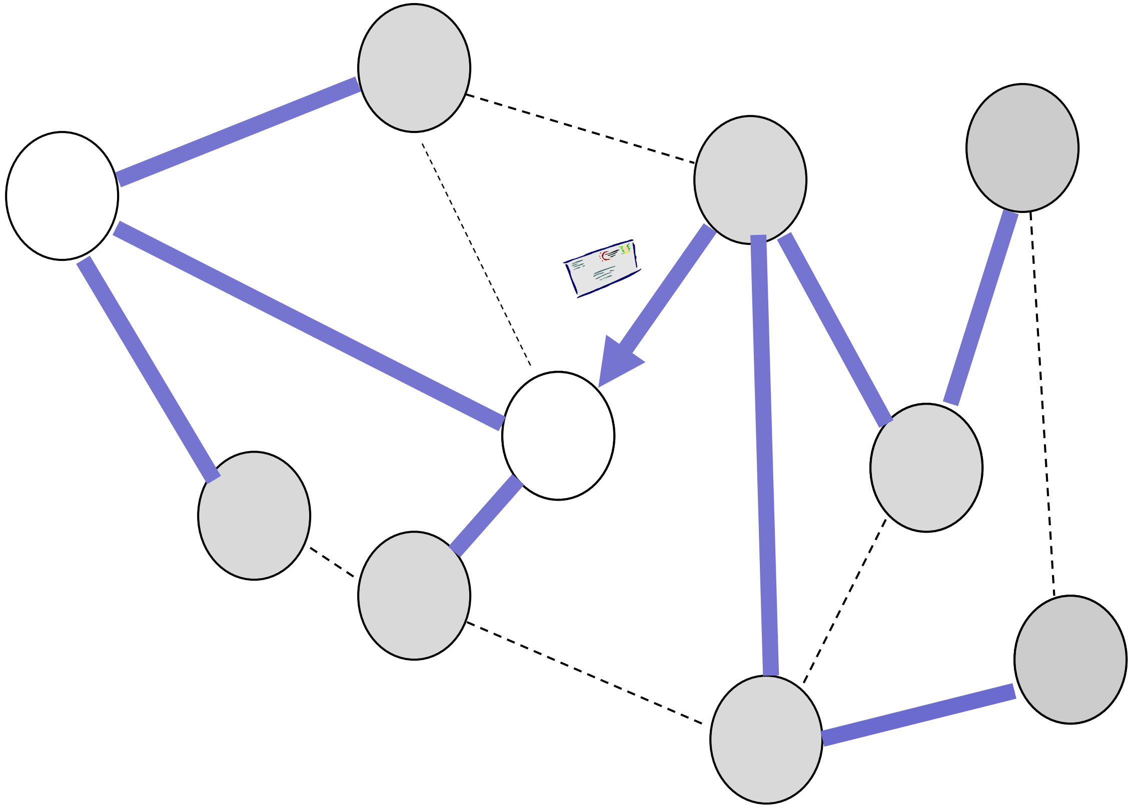


Efficiently aggregate: from leaves toward root!

Aggregate

Round 2

A Fundamental Communication Primitive: ConvergeCast

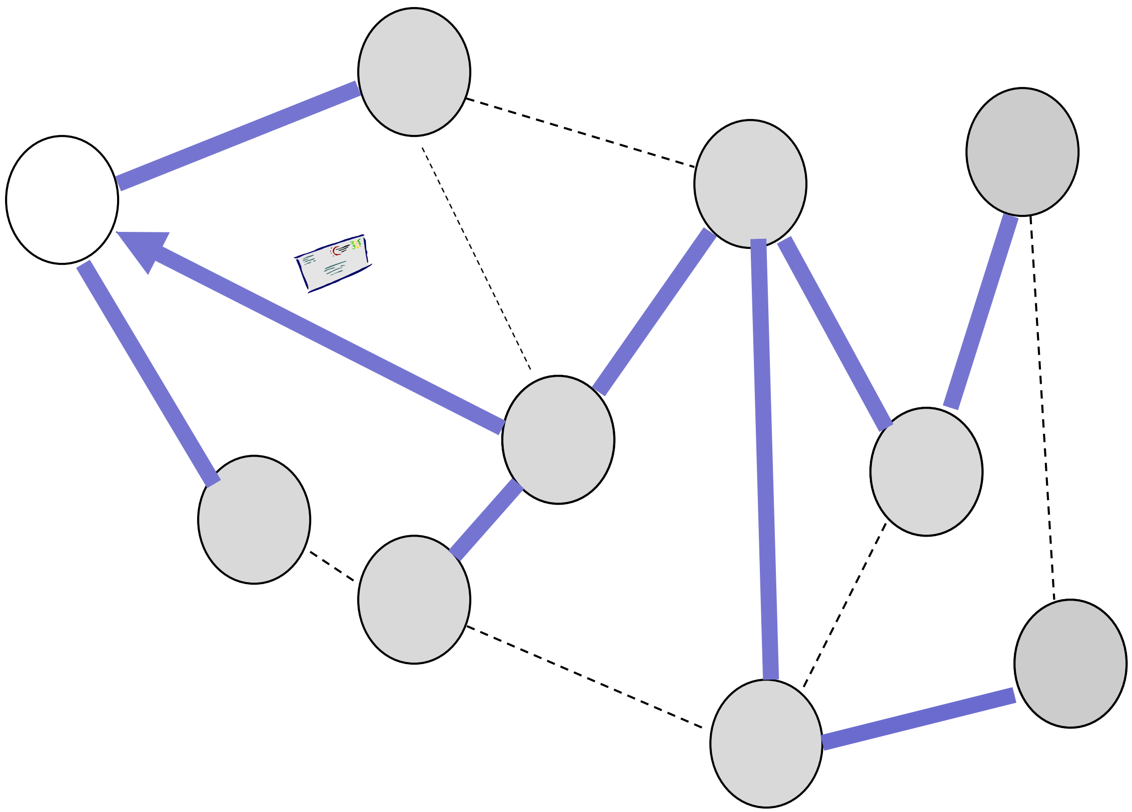


Efficiently aggregate: from leaves toward root!

Aggregate

Round 3

A Fundamental Communication Primitive: ConvergeCast



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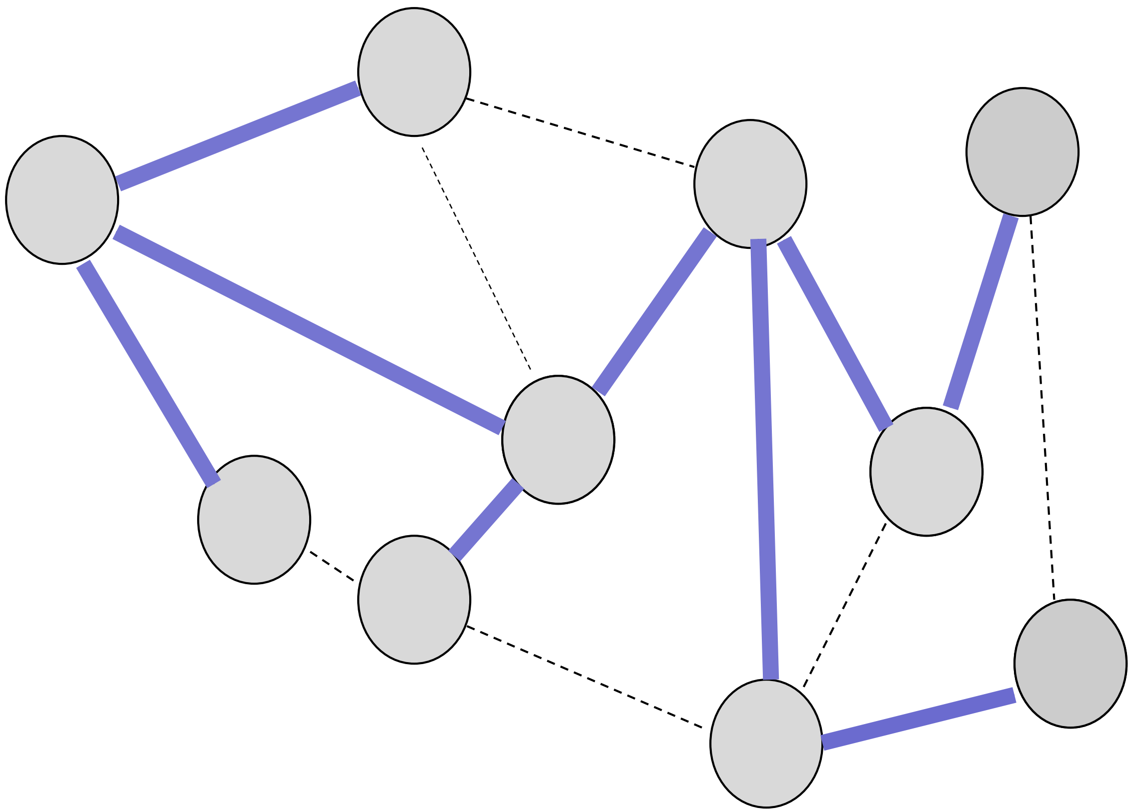
Aggregate

Round 4

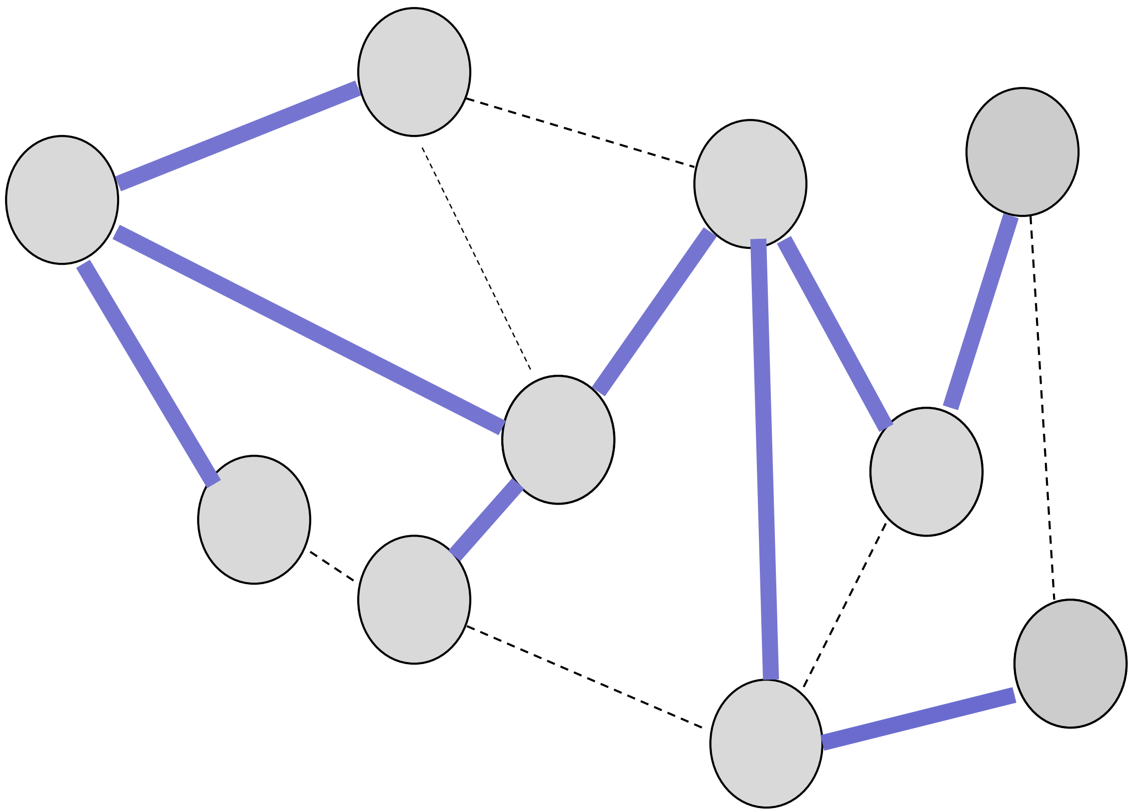
A Fundamental Communication Primitive: ConvergeCast



Efficiently aggregate: from leaves toward root!

ConvergeCast done!
O(n) messages and
O(Depth) time

A Fundamental Communication Primitive: ConvergeCast

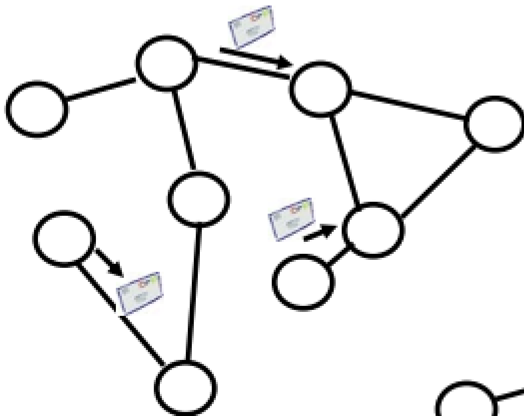


Optimality?
Lower bounds?

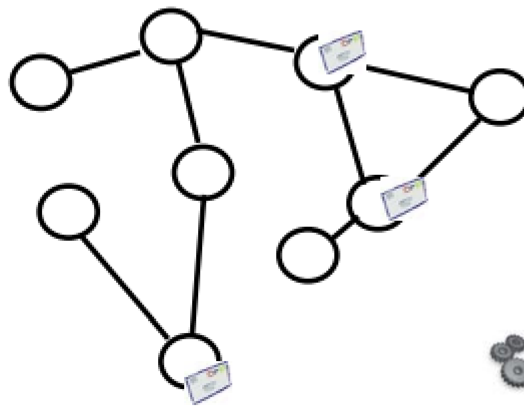
ConvergeCast done!
O(n) messages and
O(Depth) time

Recall: Local Algorithm

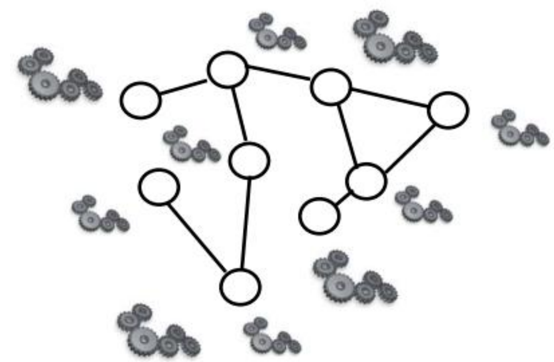
Send...



... receive...



... compute.



Let us introduce some definitions

Distance, Radius, Diameter

Distance between two nodes is # hops.
Radius of a node is max distance to any other node.
Radius of graph is *minimum* radius of any node.
Diameter of graph is *max* distance between any two nodes.

Relationship between R and D?

L definitions

In general: R ≤ D ≤ 2R.
max distance cannot be longer than going through this node.

Distance, Radius, Diameter

Distance between two nodes is # hops.
Radius of a node is max distance to any other node.
Radius of graph is *minimum* radius of any node.
Diameter of graph is *max* distance between any two nodes.

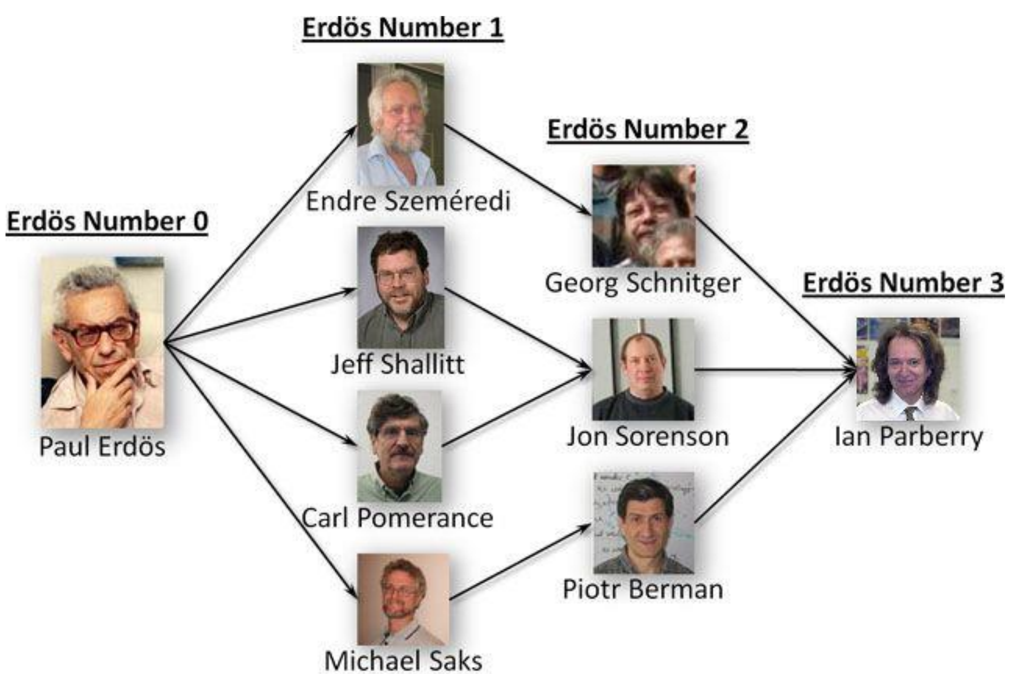
In the complete graph, for all nodes: R=D.

On the line, for broder nodes: 2R=D.

Relevance: Radius

People enjoy identifying nodes of small radius in a graph!

E.g., Erdös number, Kevin Bacon number, joint Erdös-Bacon number, etc.

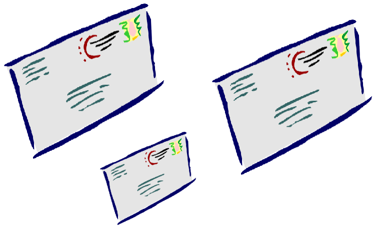


Kevin Bacon Number	# of People
0	1
1	3211
2	376831
3	1359872
4	347806
5	29593
6	3496
7	515
8	102
9	8
10	1

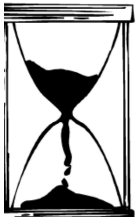
Total number of linkable actors: 2121436
Weighted total of linkable actors: 6401157
Average Kevin Bacon number: 3.017

Lower Bounds for Broadcast

Message complexity?

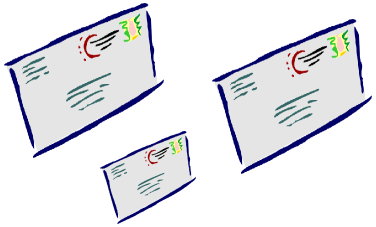


Time complexity?



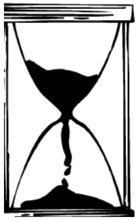
Lower Bounds for Broadcast

Message complexity?



Each node must receive message: so at least n-1.

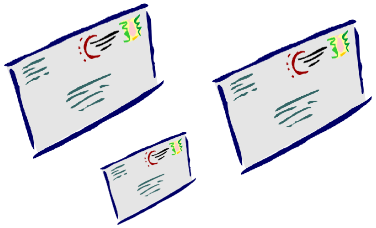
Time complexity?



The radius of the source: each node needs to receive message.

Lower Bounds for Broadcast

Message complexity?



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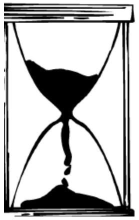


The radius of the source: each node needs to receive message.

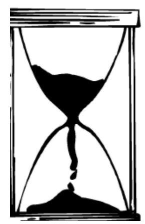
How to achieve this?

Lower Bounds for Broadcast

Message complexity?

Each node must receive message: so at least n-1.

Time complexity?

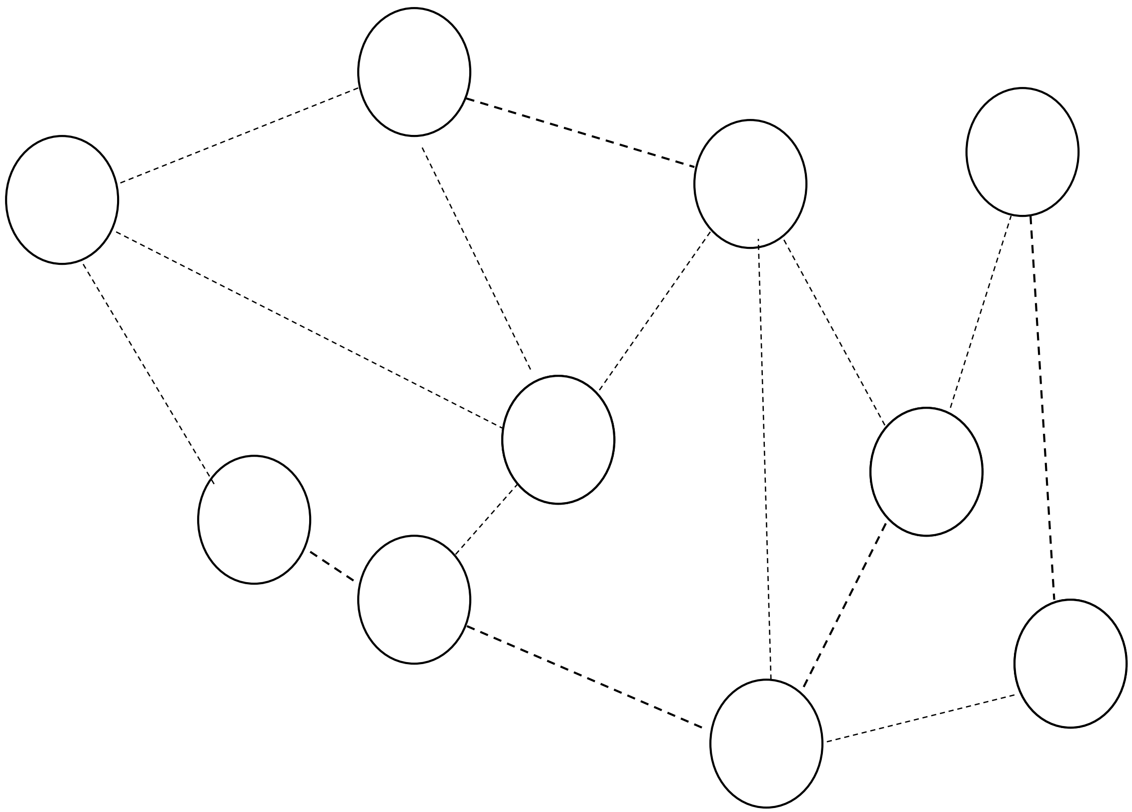


The radius of the source: each node needs to receive message.

How to achieve this?

Compute a breadth first spanning tree! ☺ But how?

Idea: Compute BFS using Flooding!



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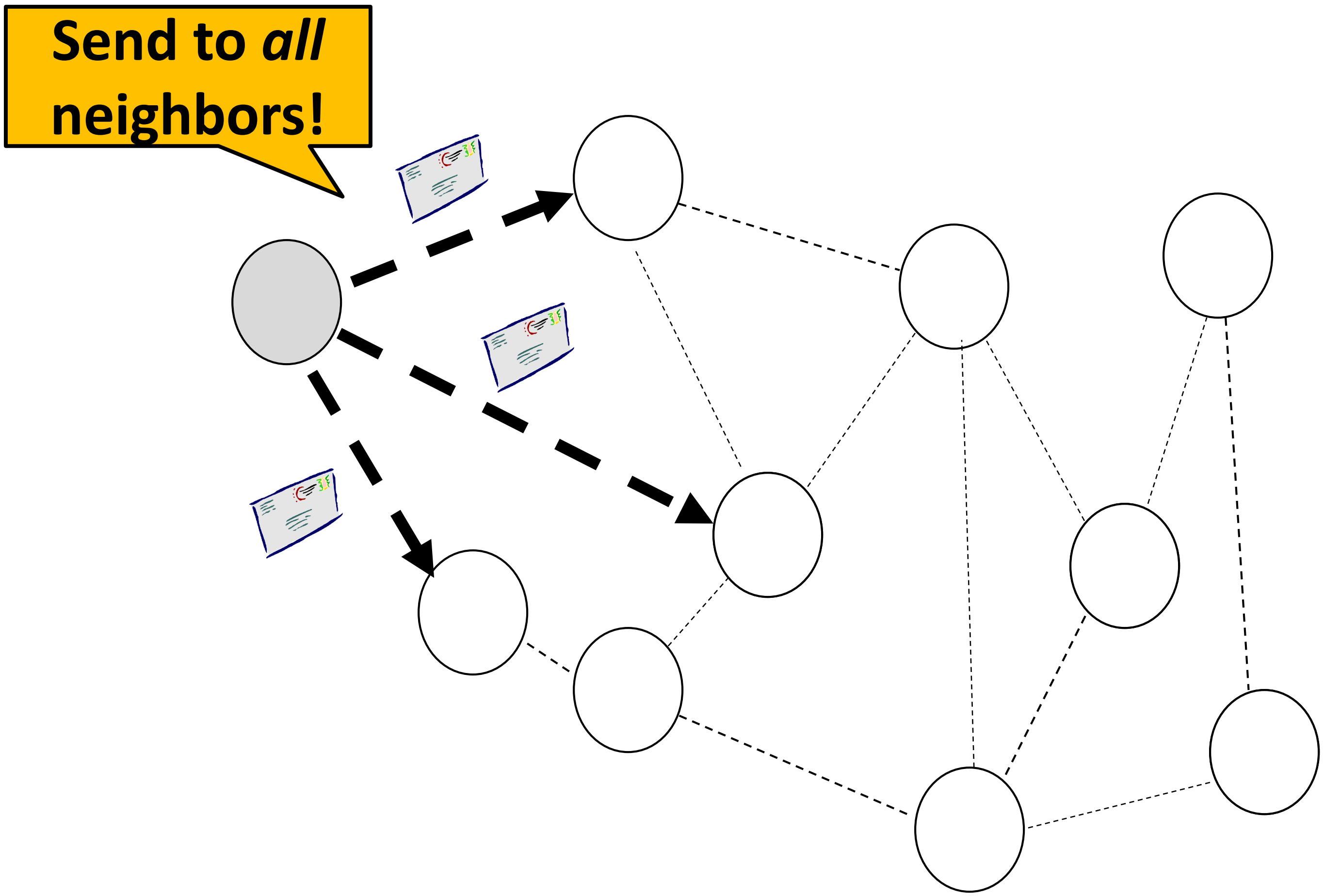
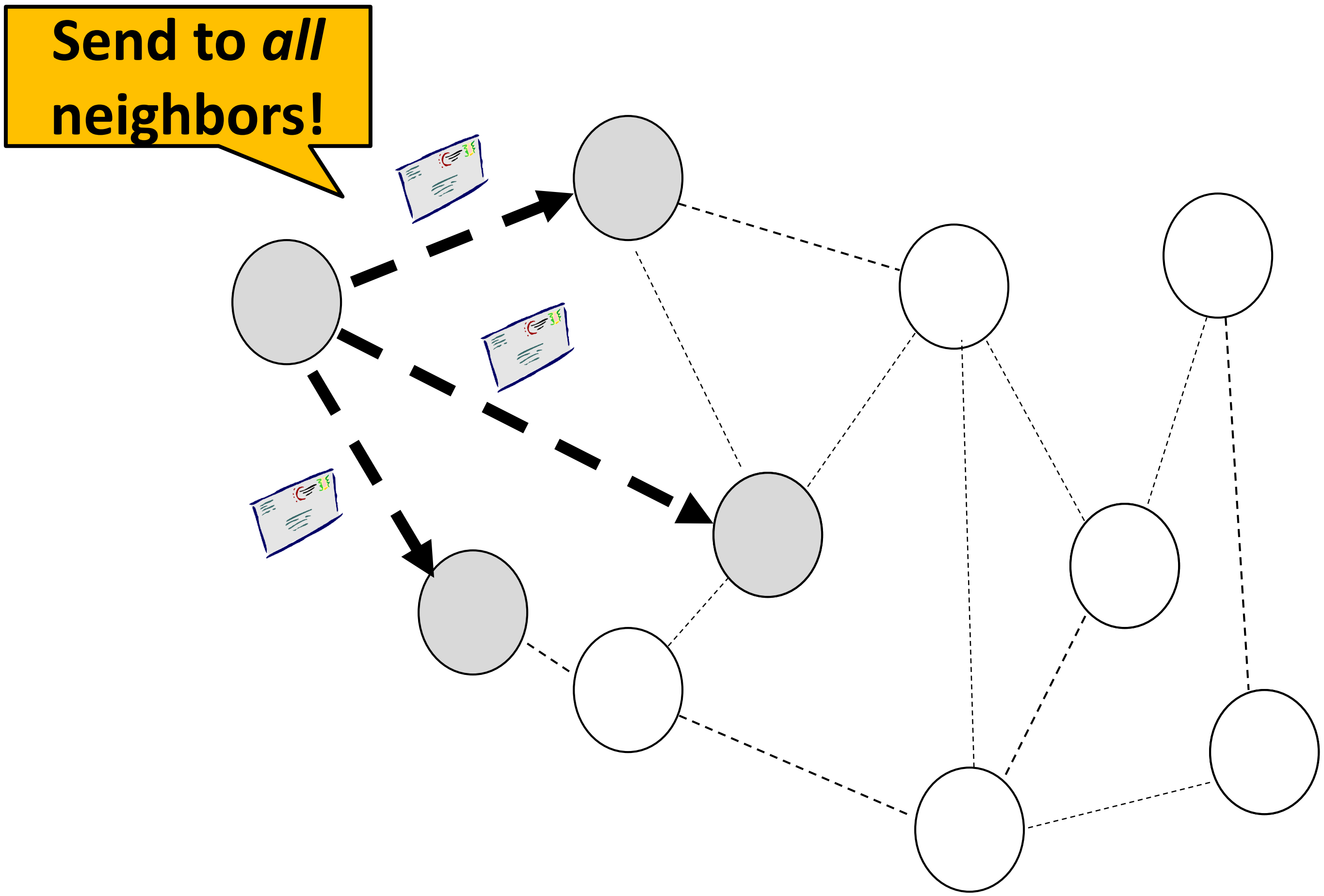


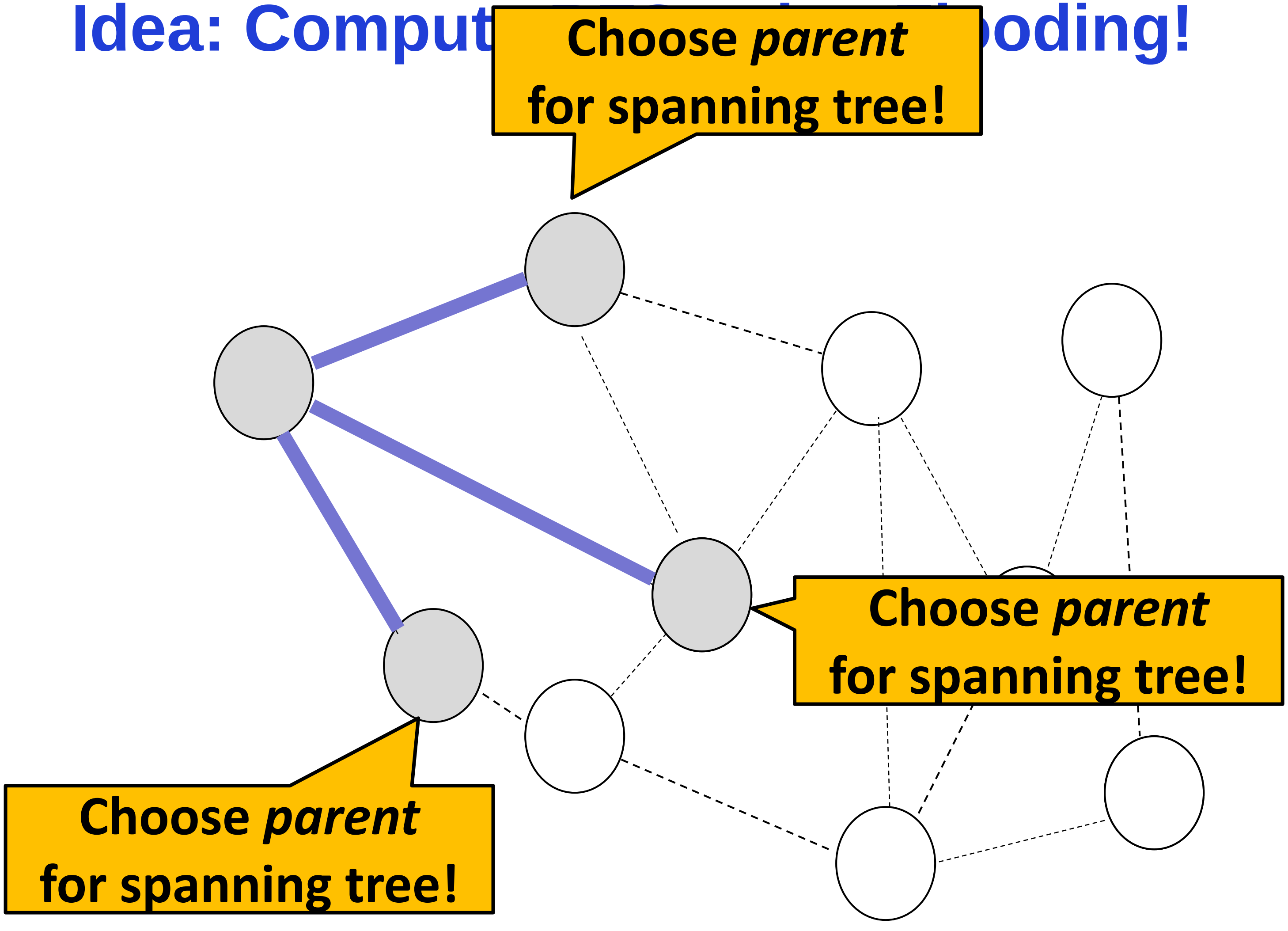
Round 1

Idea: Compute BFS using Flooding!



Round 1

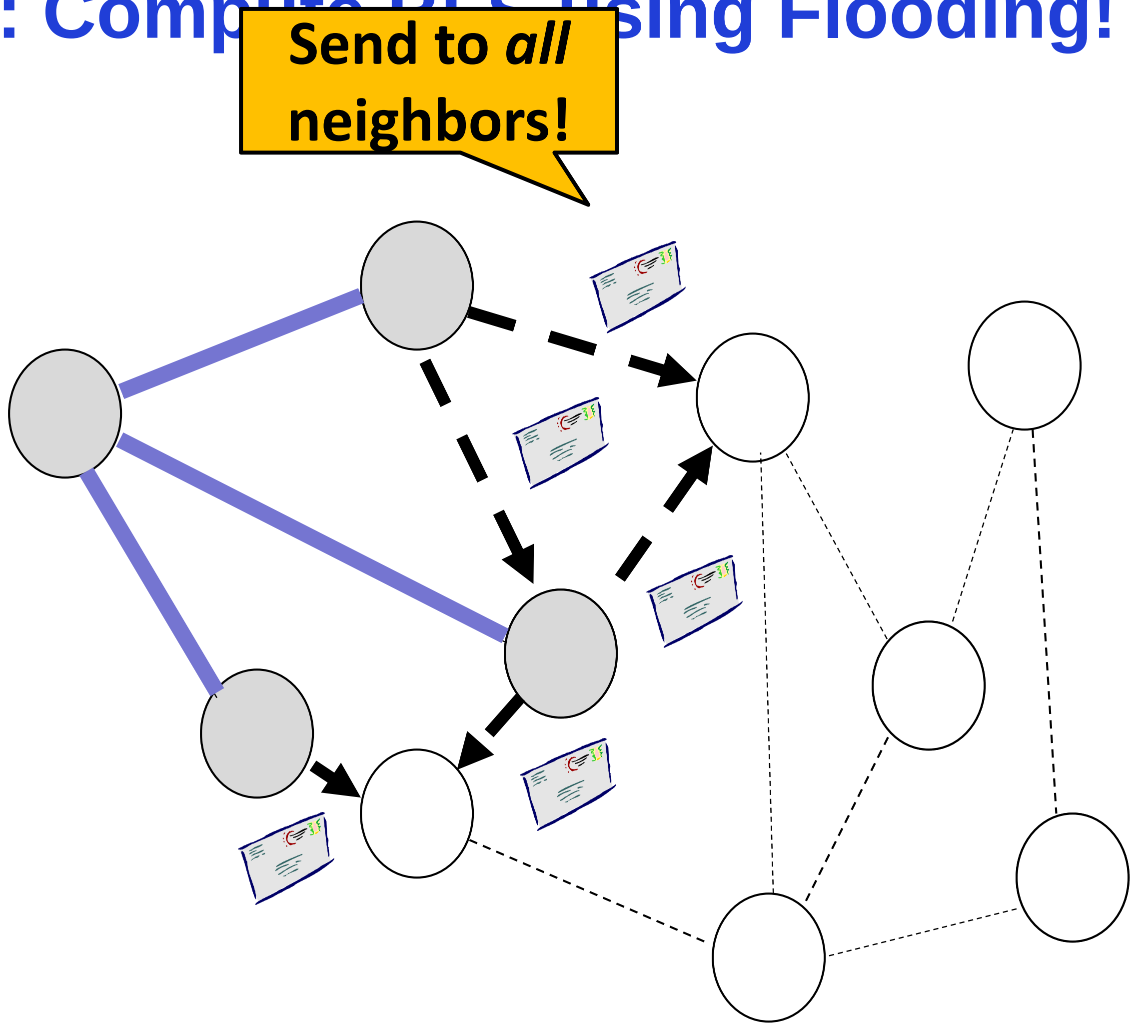
Idea: Comput… ooding!



Round 1

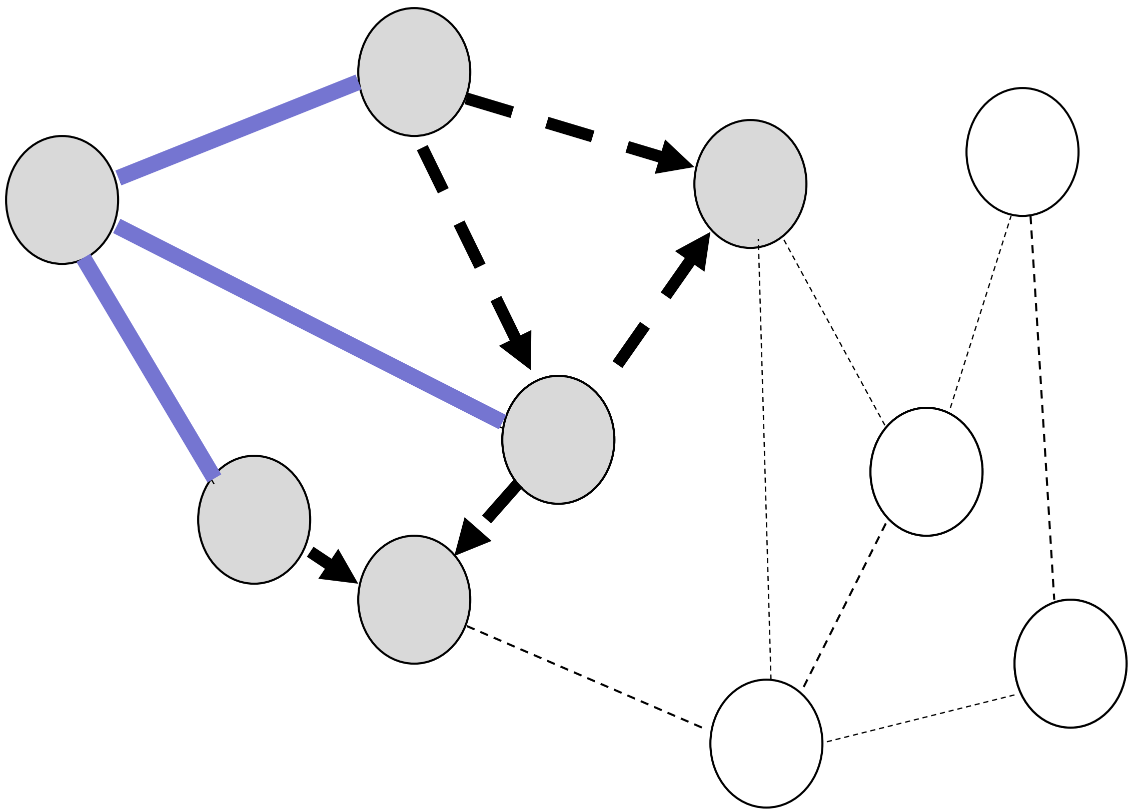
Invariant: parent has shorter distance to root: loop-free!

Idea: Compute BFS using Flooding!



Round 2

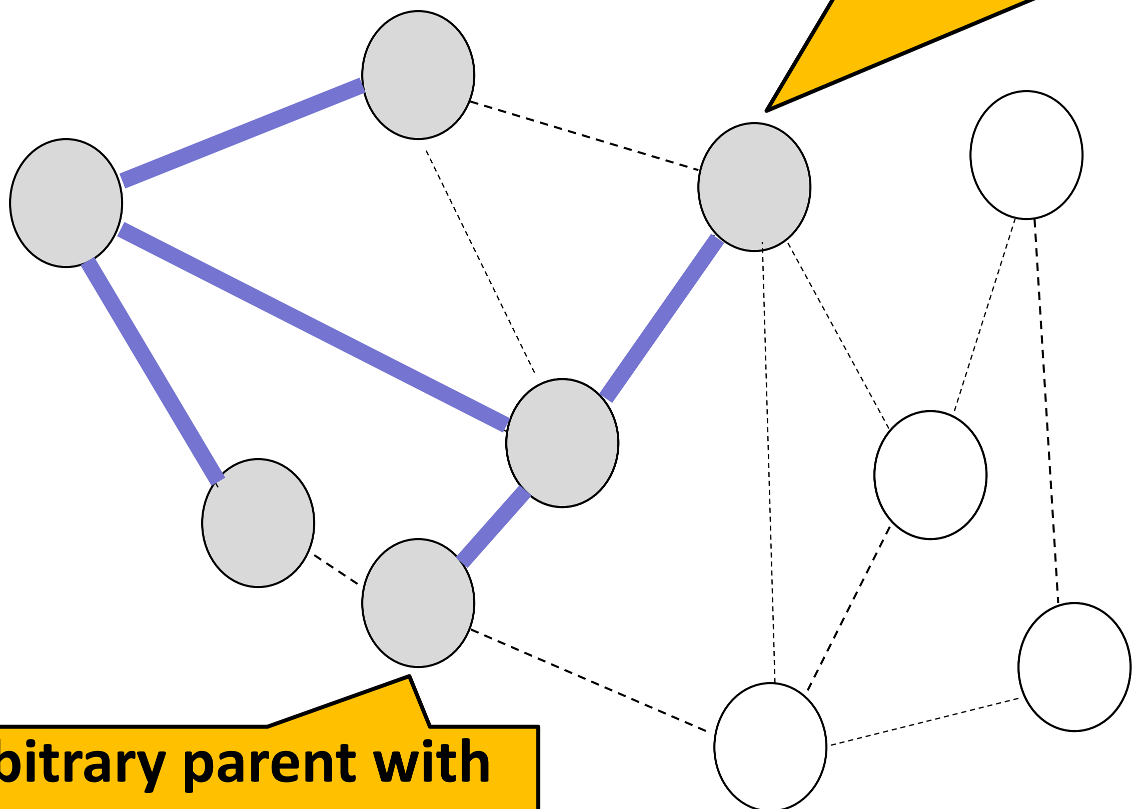
Idea: Compute BFS using Flooding!



Round 2

Idea: Compute BFS using Flooding!

Choose a parent: if multiple arrive at same time, take *arbitrary*!

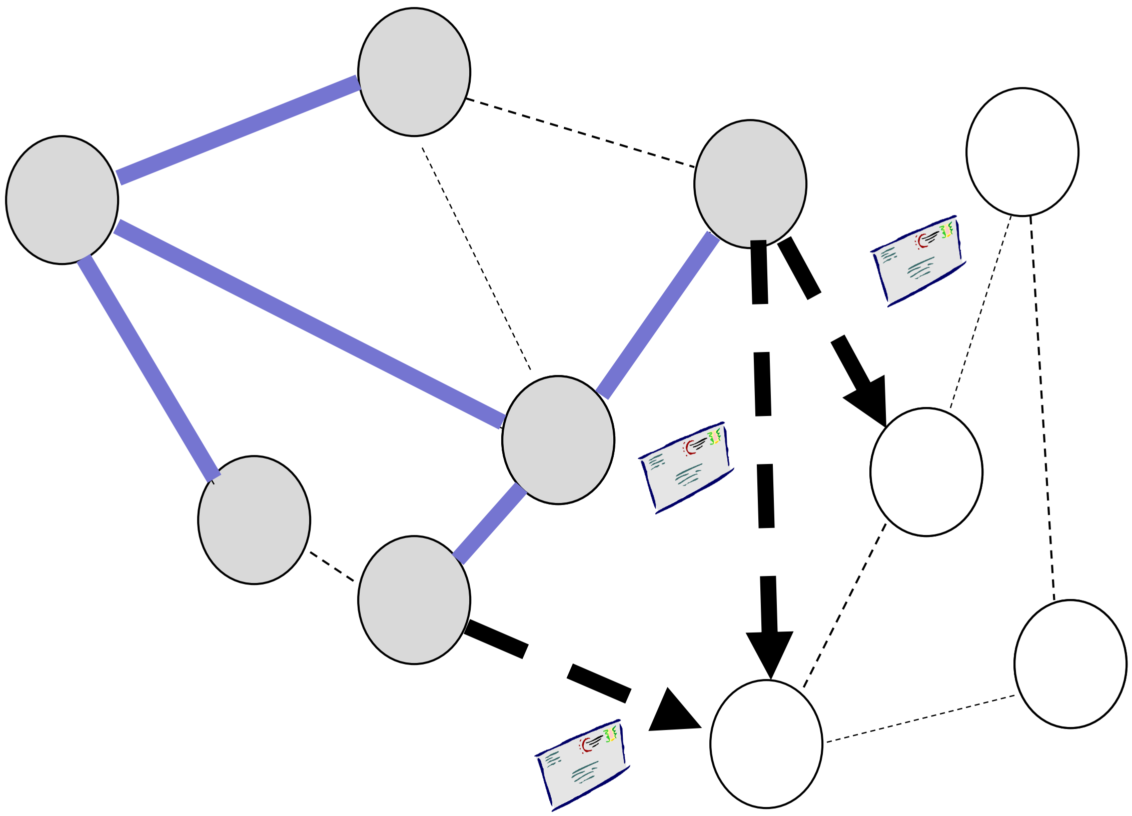


arbitrary parent with shorter distance!

Round 2

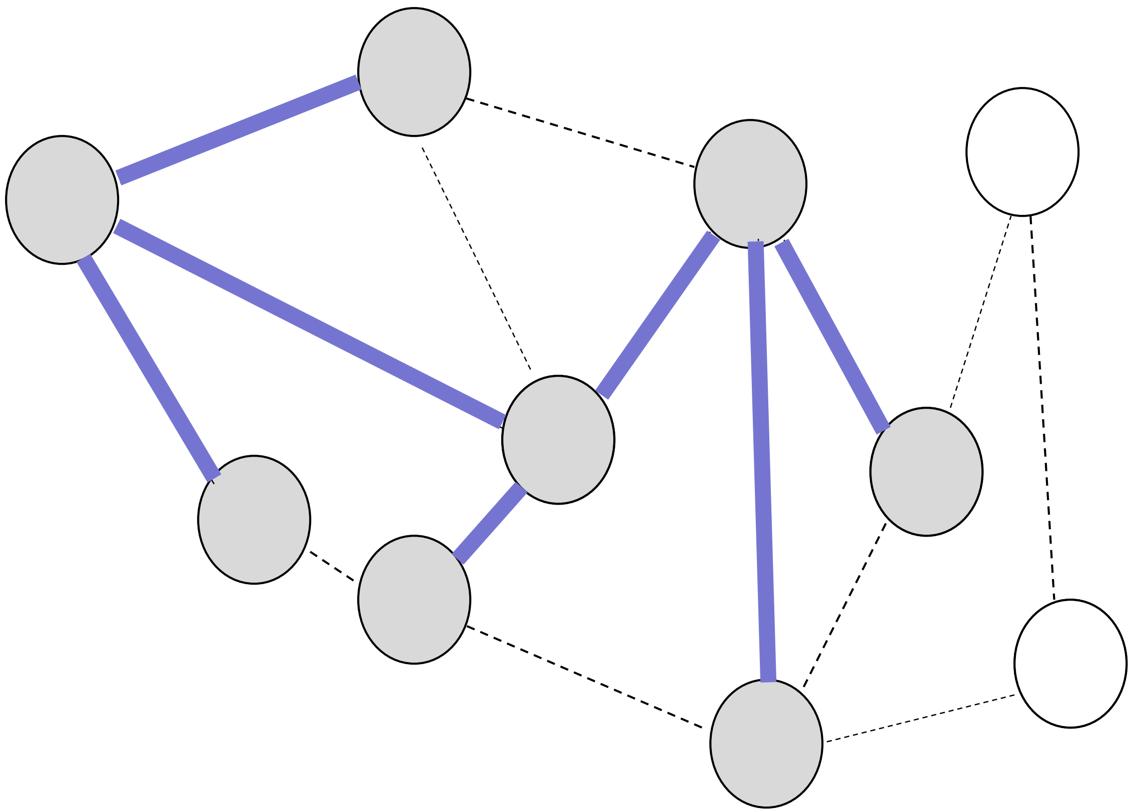
Invariant: parent has shorter distance to root: loop-free!

Idea: Compute BFS using Flooding!



Round 3

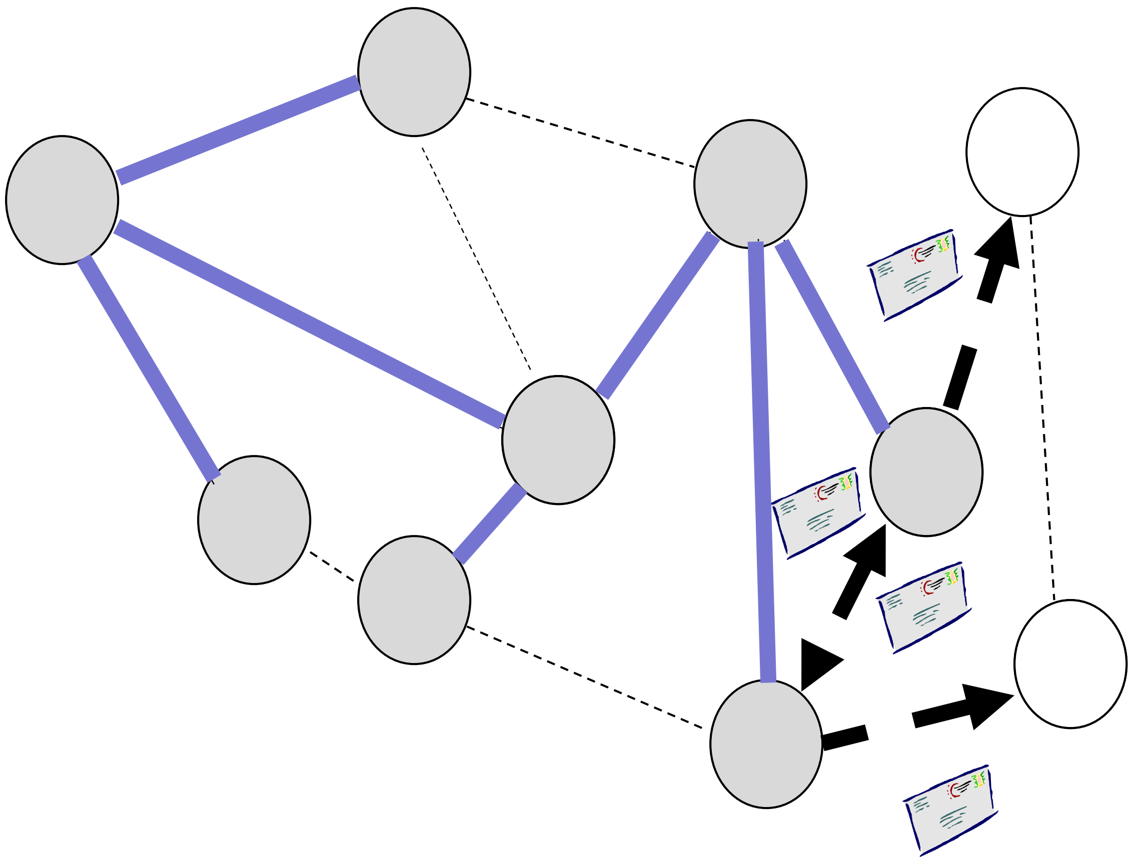
Idea: Compute BFS using Flooding!



Round 3

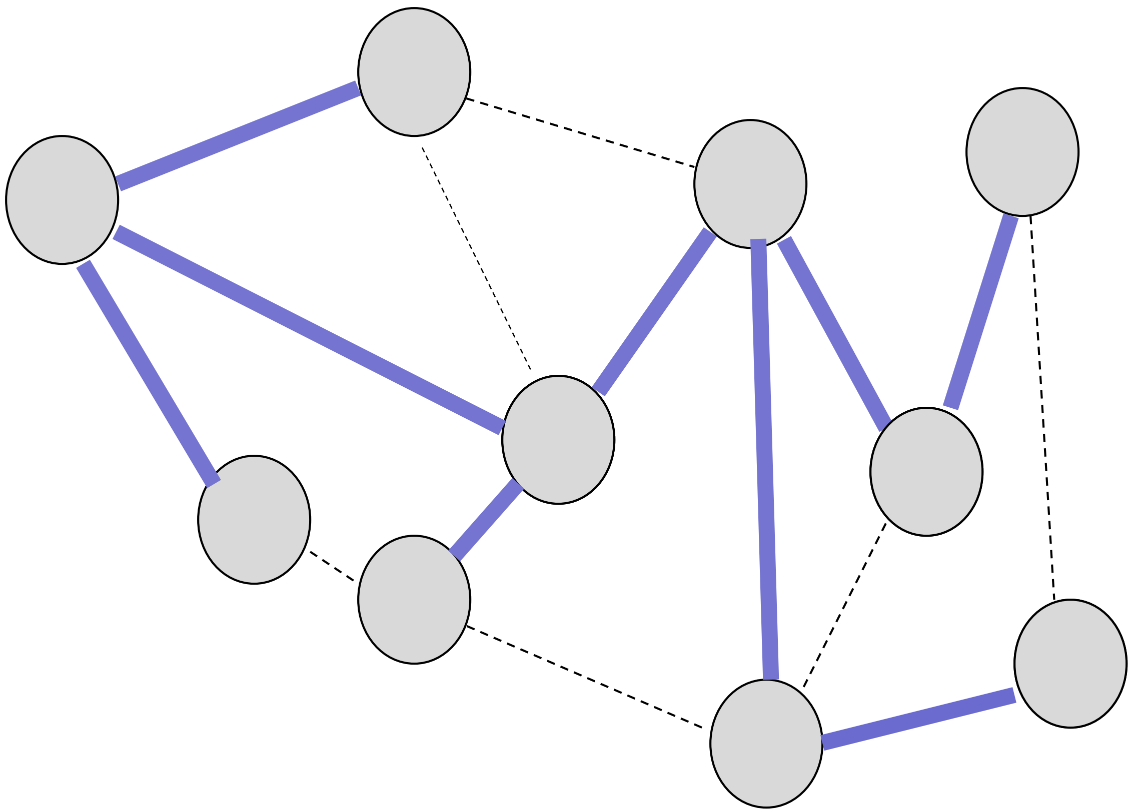
Invariant: parent has shorter distance to root: loop-free!

Idea: Compute BFS using Flooding!



Round 4

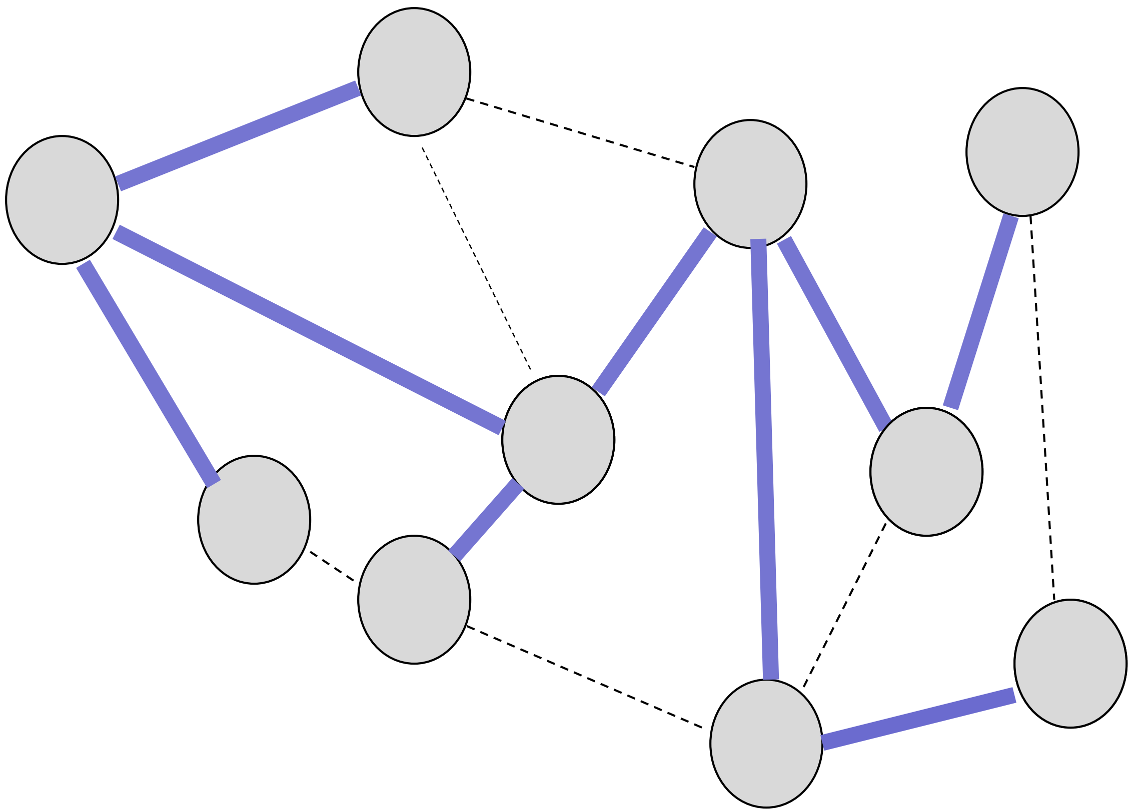
Idea: Compute BFS using Flooding!



BFS!

Invariant: parent has shorter distance to root: loop-free!

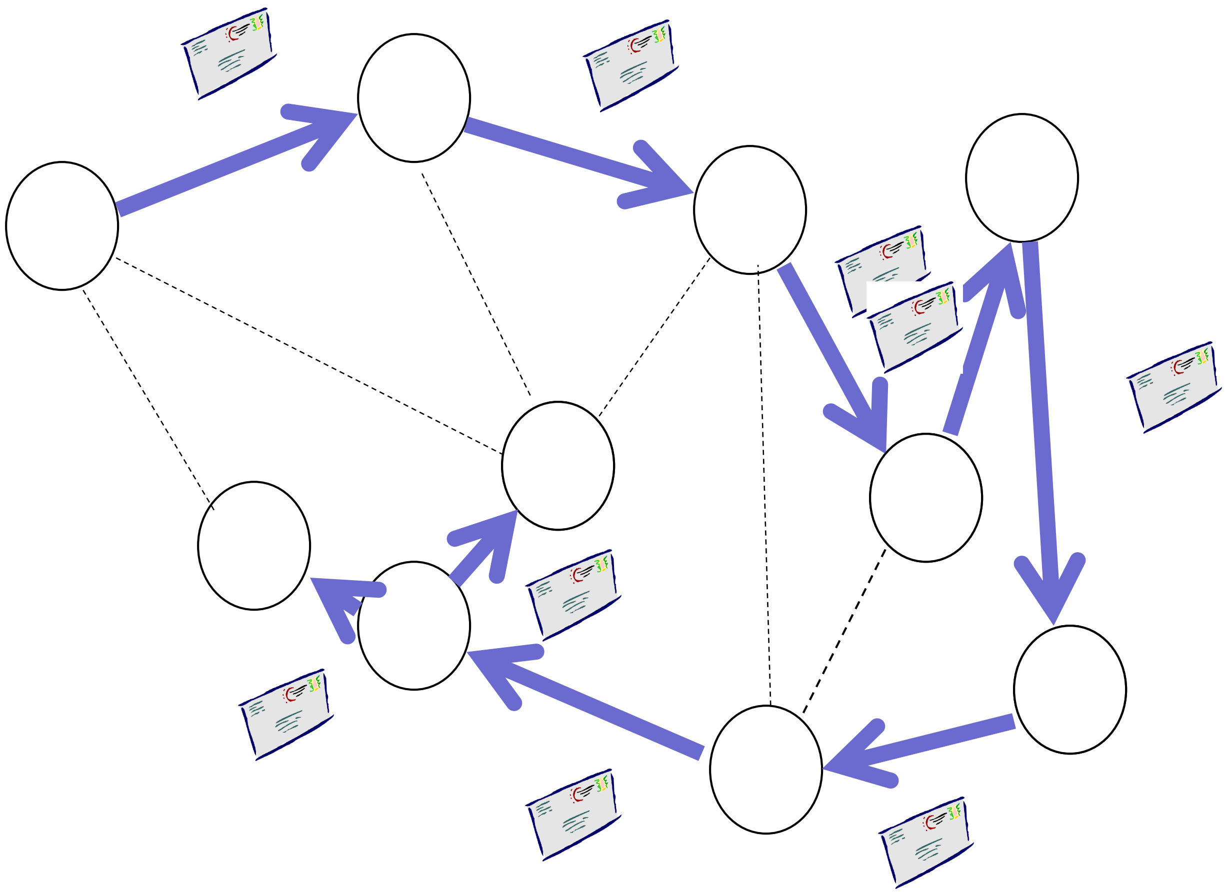
Idea: Compute BFS using Flooding!



BFS!

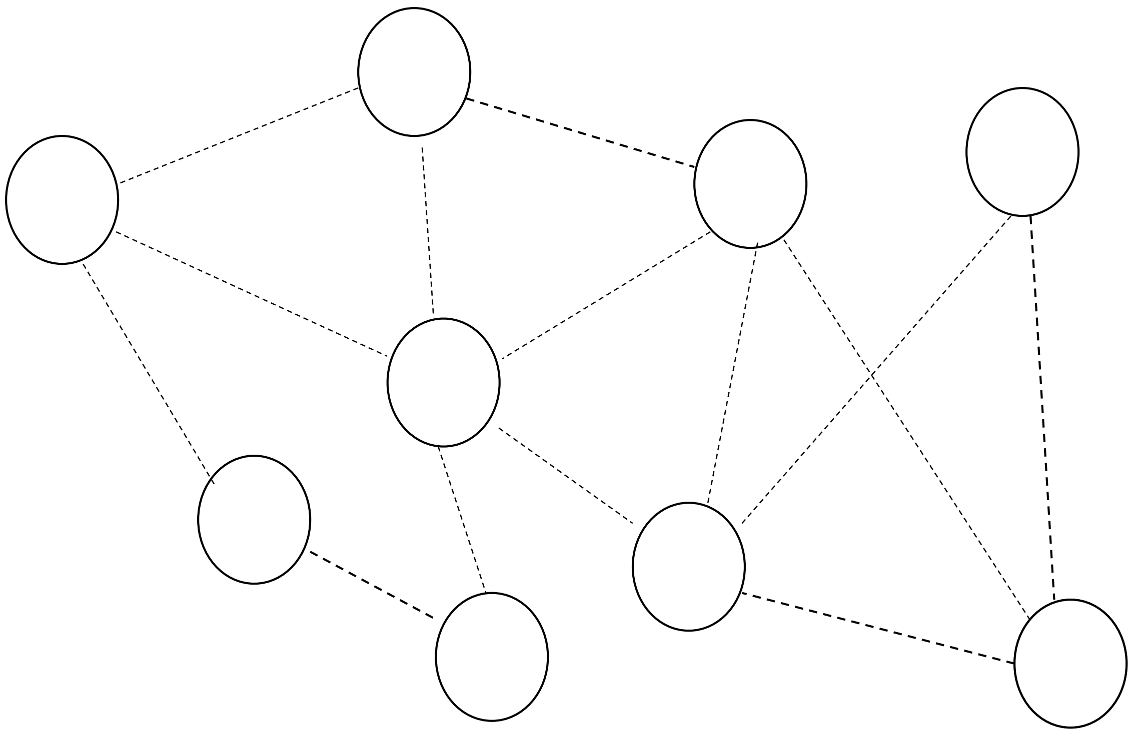
Careful: in *asynchronous* environment, should not make first successful sender my parent!

Bad example



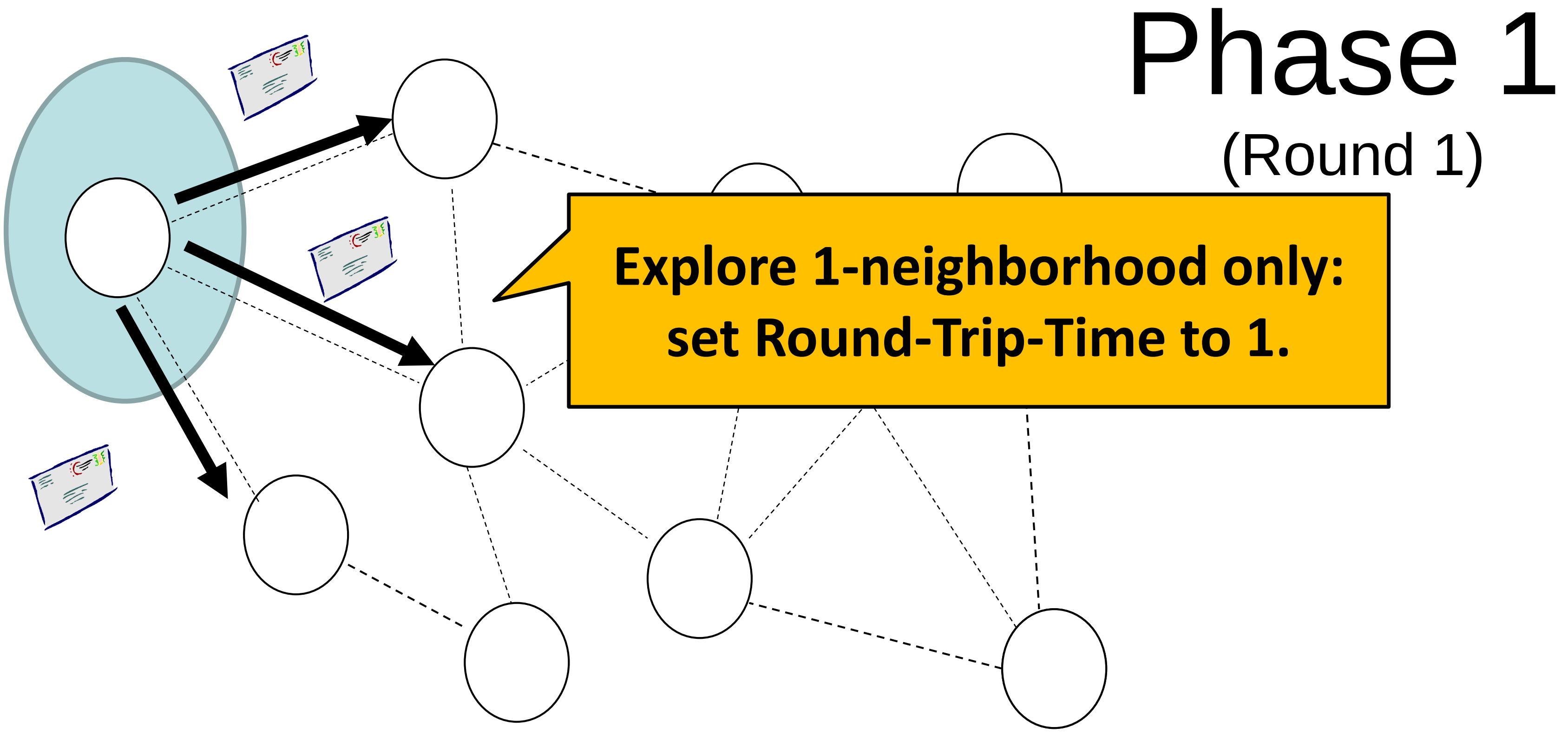
Careful: in *asynchronous* environment, should not make first successful sender my parent!

Distributed BFS: Dijkstra Favor



Idea: overcome asynchronous problem by proceeding in phases!

Distributed BFS: Dijkstra Favor



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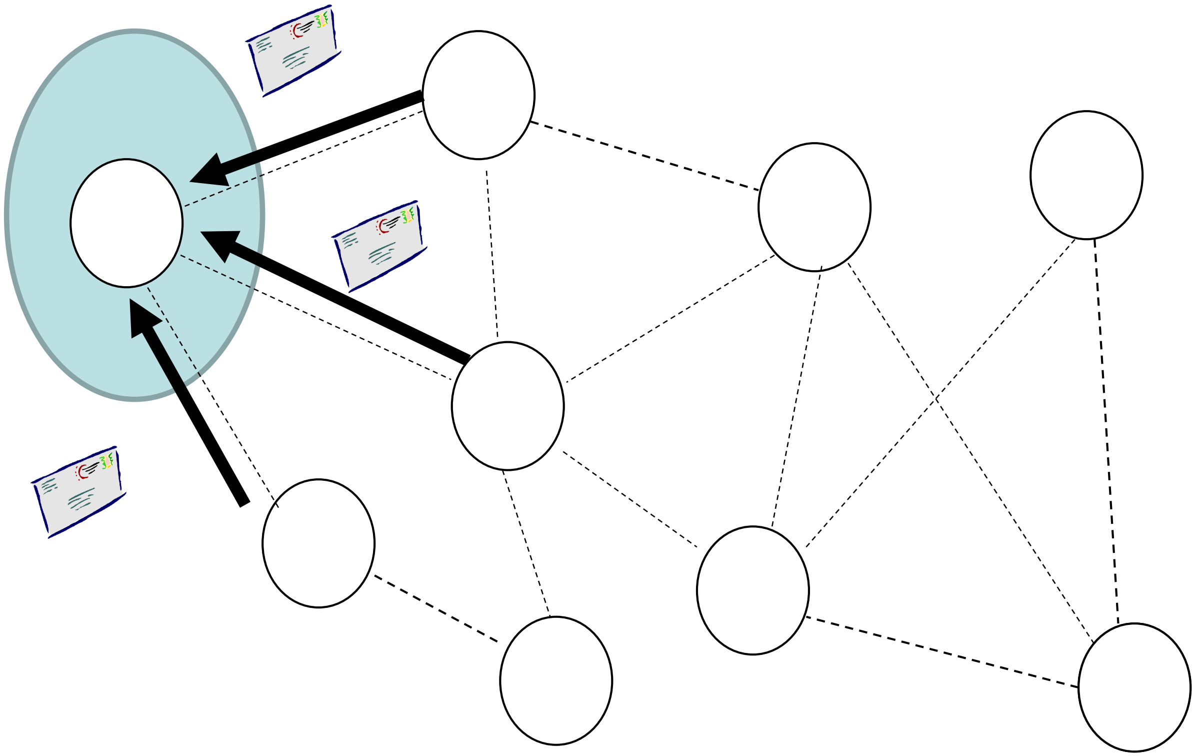
Distributed BFS: Dijkstra Favor



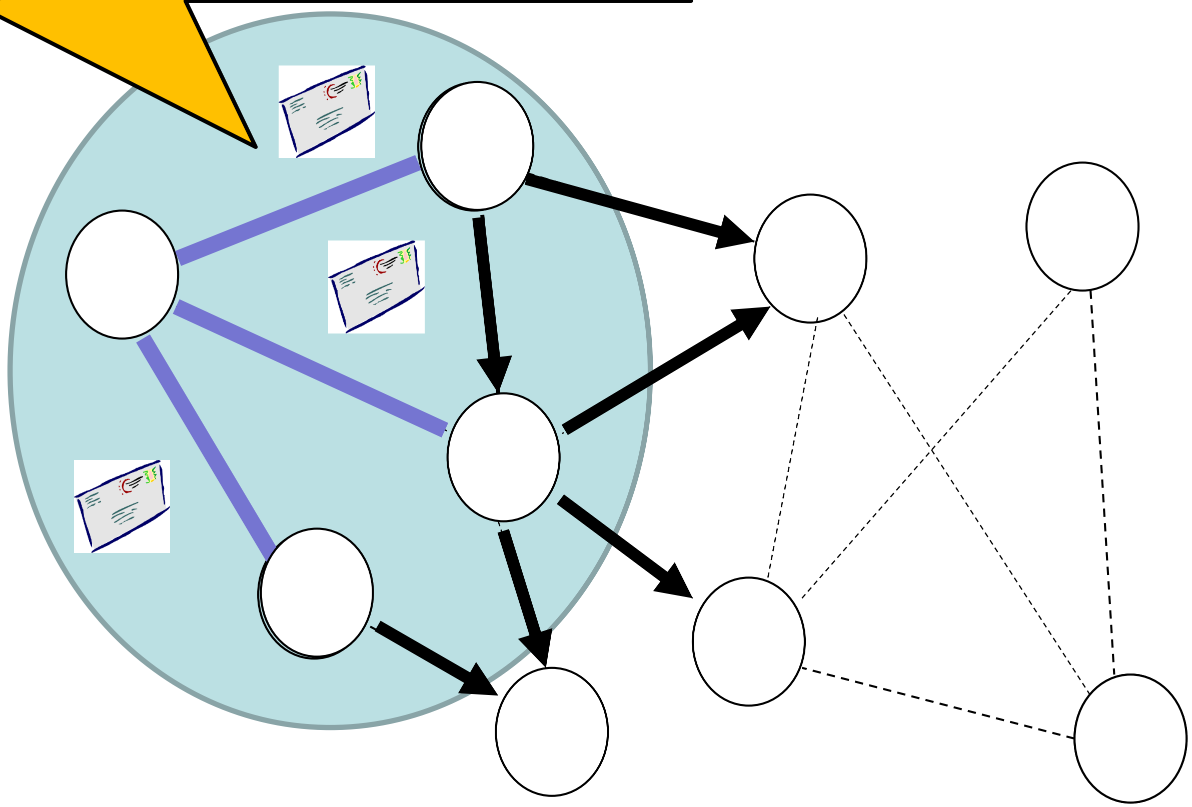
Phase 1

(Round 2)

Idea: overcome asynchronous problem by proceeding in phases!

Distributed BFS: Dijkstra Favor

Start Phase 2! (Propagate along existing spanning tree!)

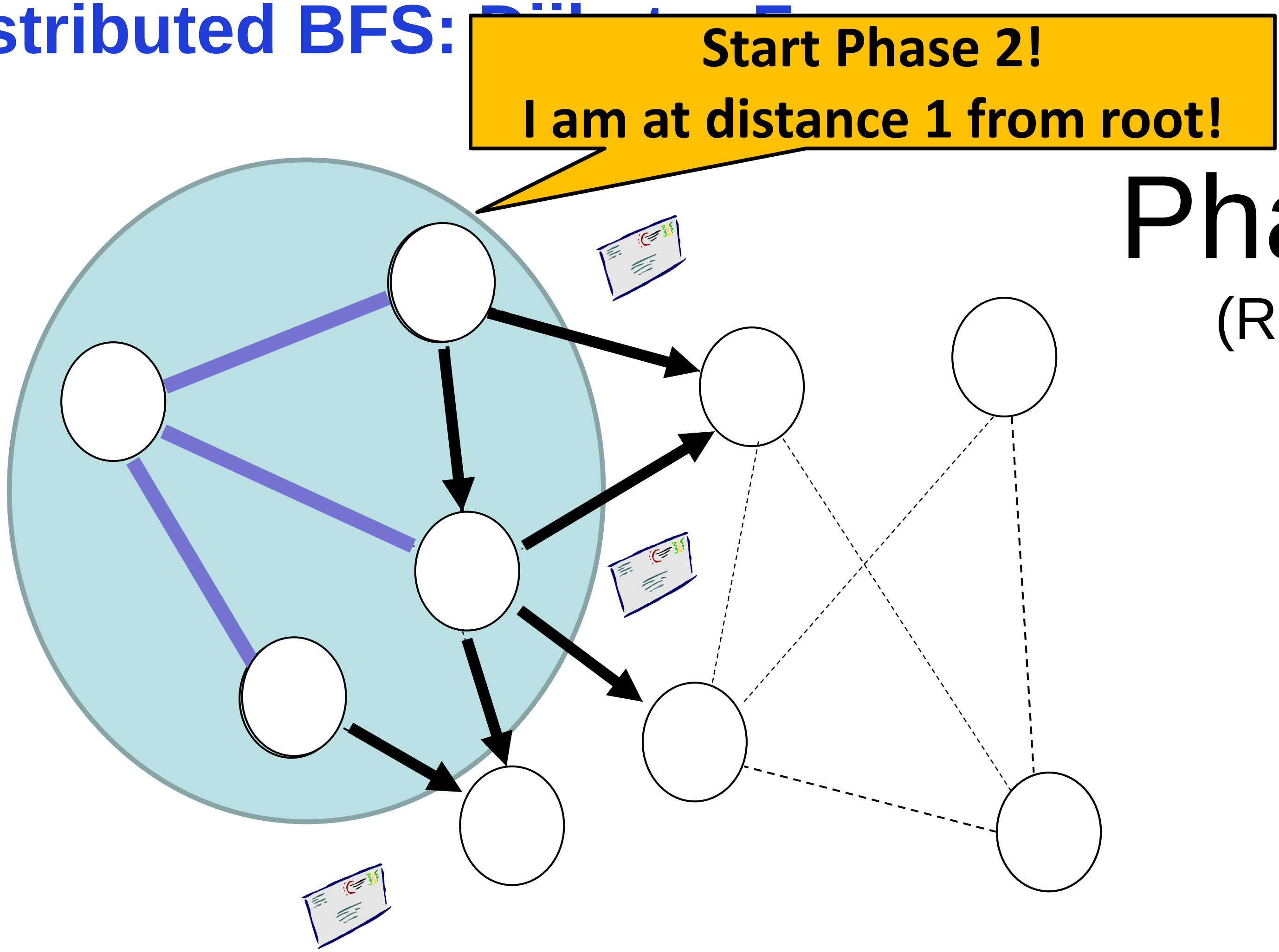


Phase 2

(Round 1)

Idea: overcome asynchronous problem by proceeding in phases!

Distributed BFS: Dijkstra F...

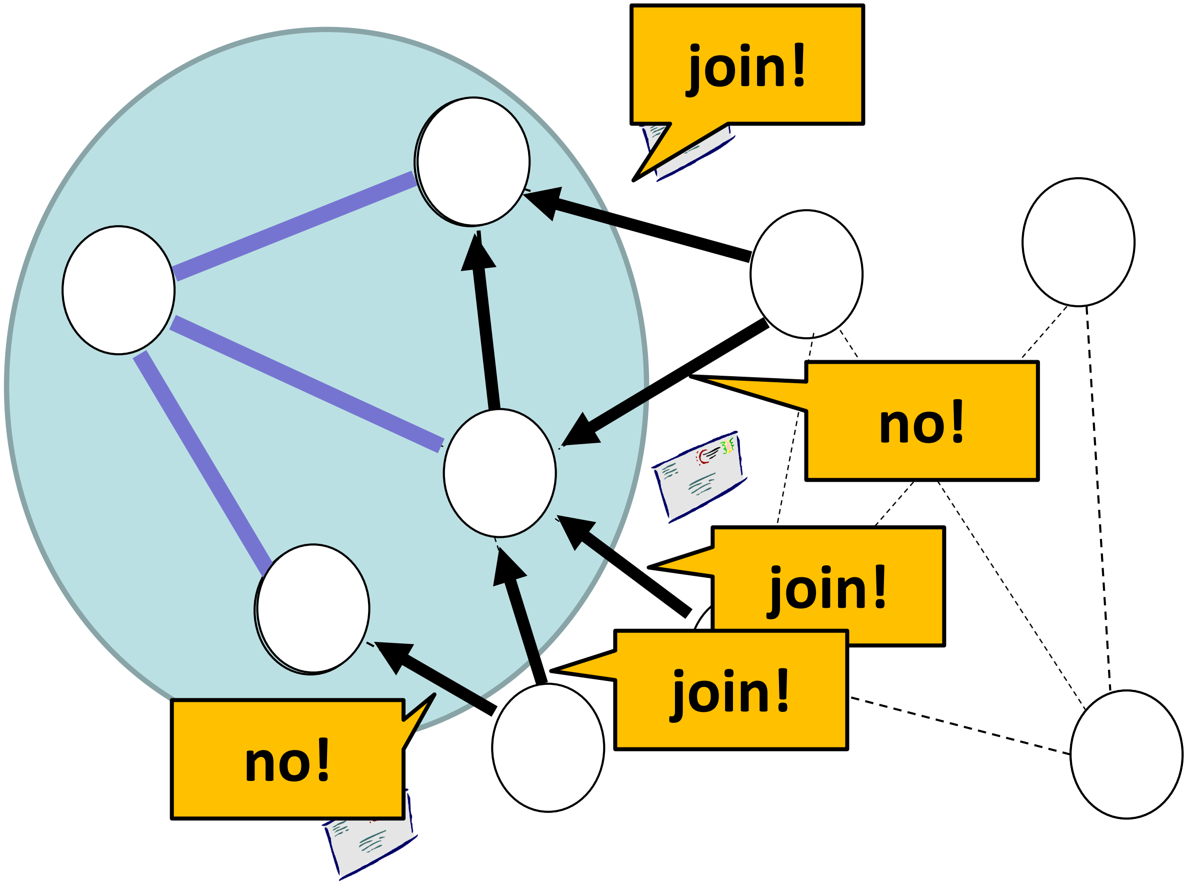


Phase 2

(Round 2)

Idea: overcome asynchronous problem by proceeding in phases!

Distributed BFS: Dijkstra Favor



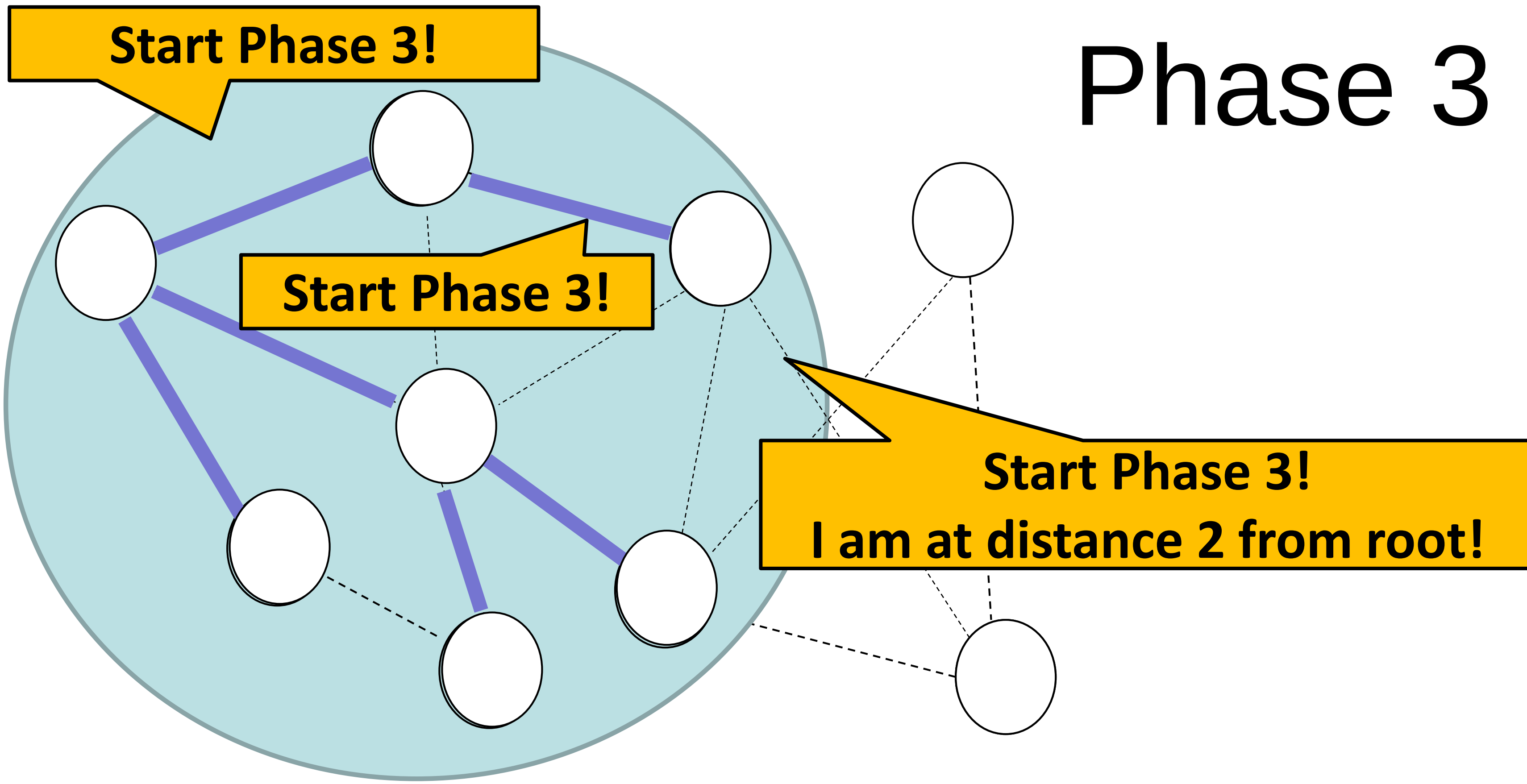
Phase 2

(Round 3)

Idea: overcome asynchronous problem by proceeding in phases!

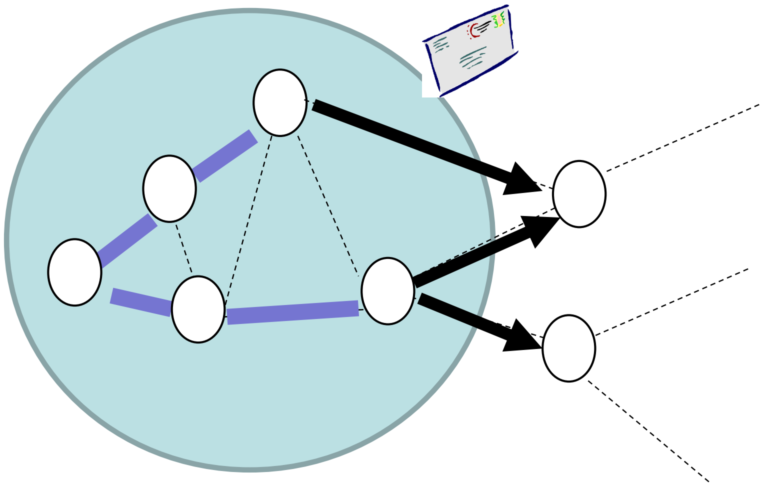
Choose parent with smaller distance!

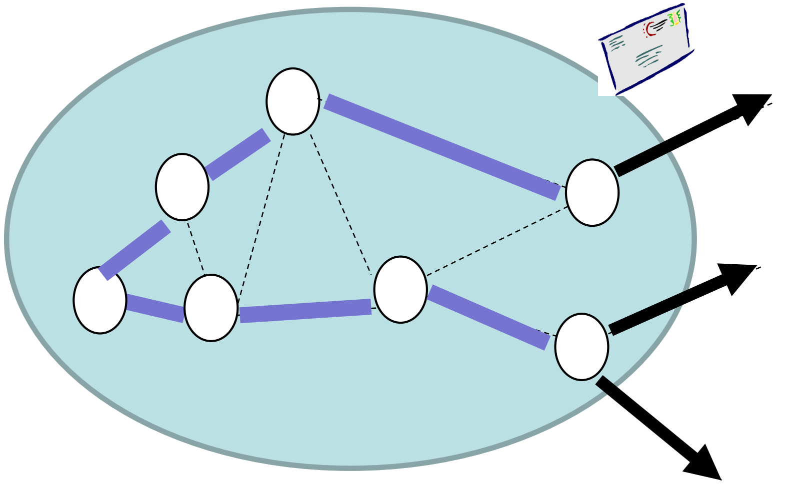
Distributed BFS: Dijkstra Favor



Idea: overcome asynchronous problem by proceeding in phases!

General Scheme

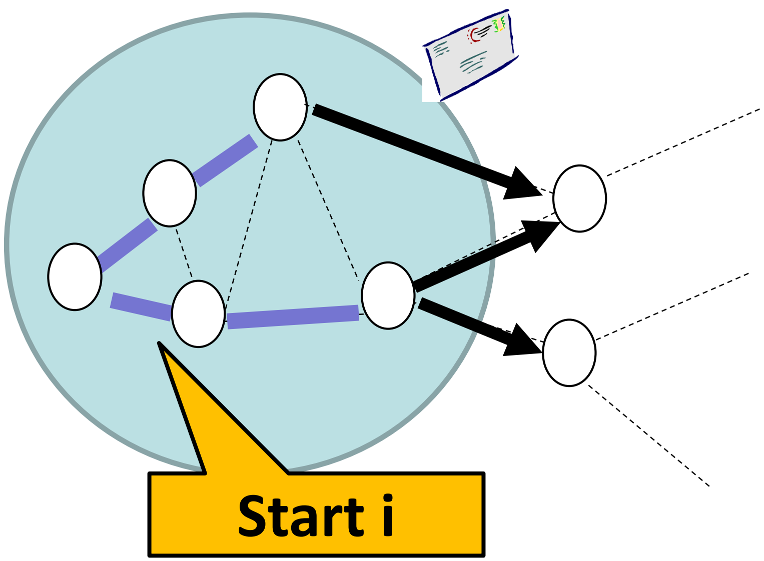


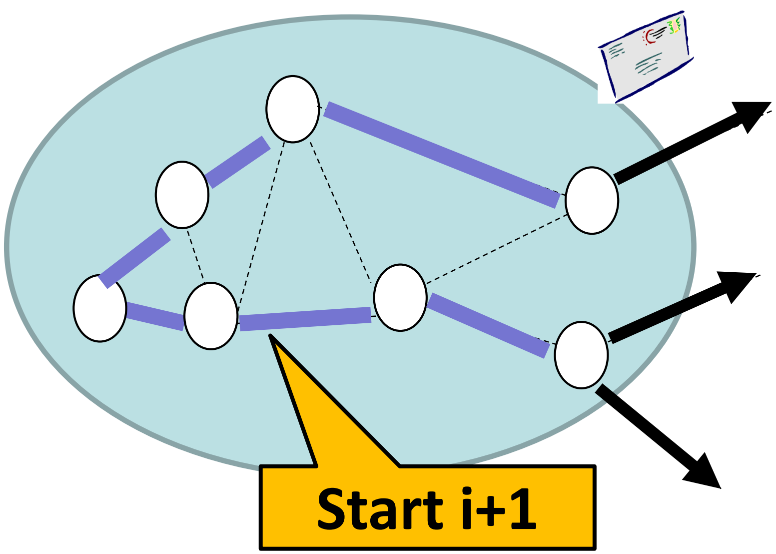


Phase i

Phase i+1

General Scheme

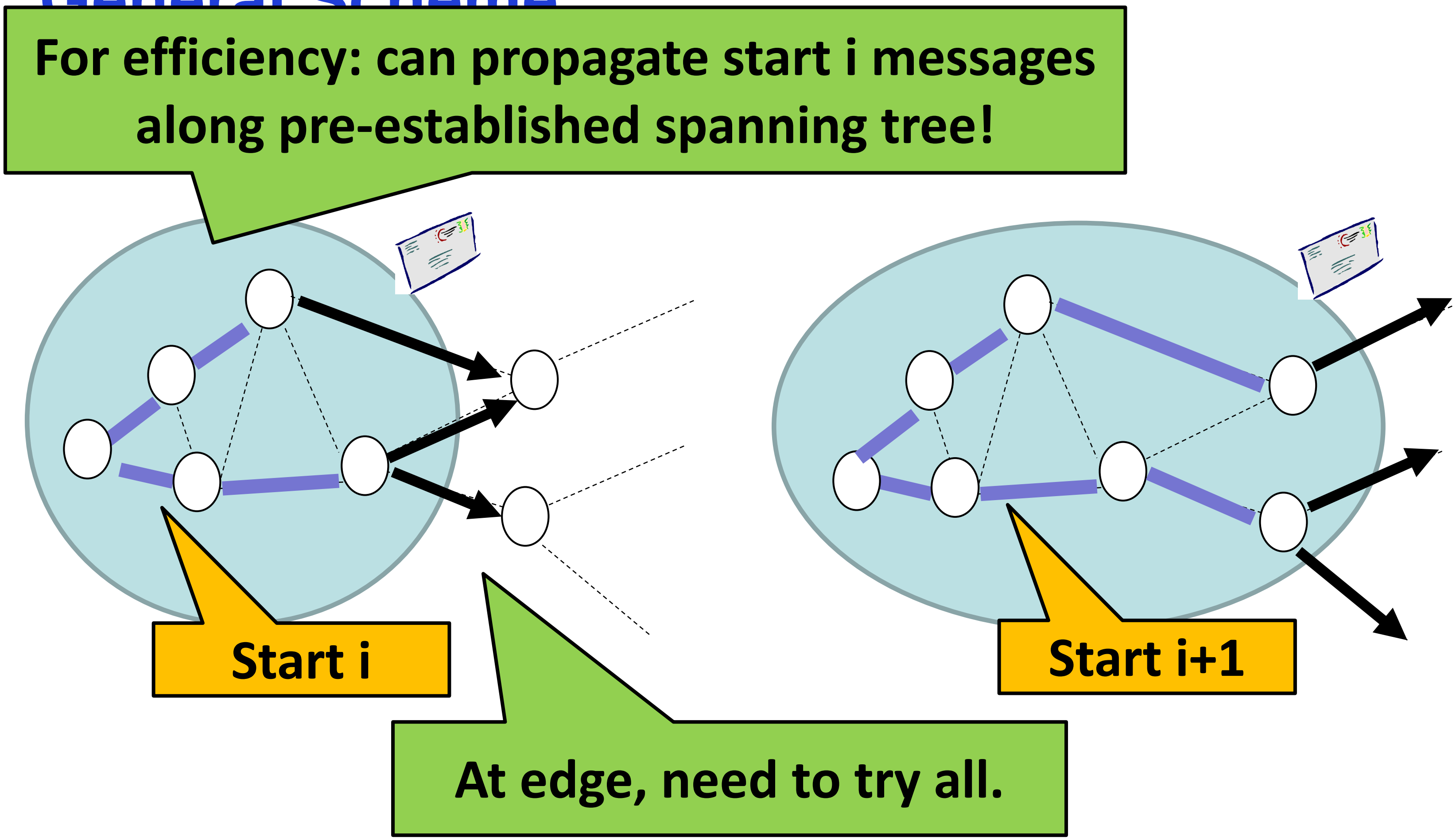




Phase i

Phase i+1

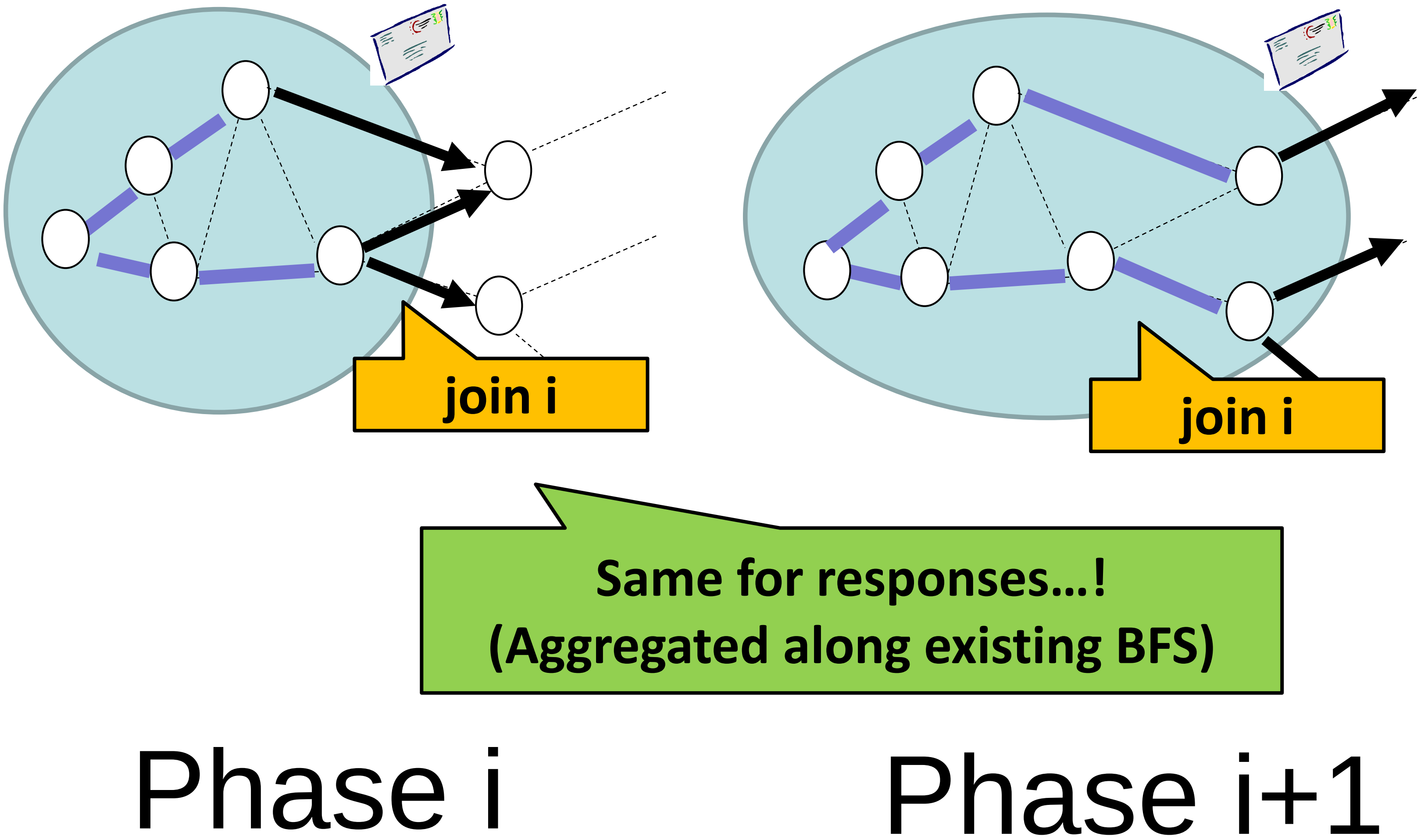
General Scheme



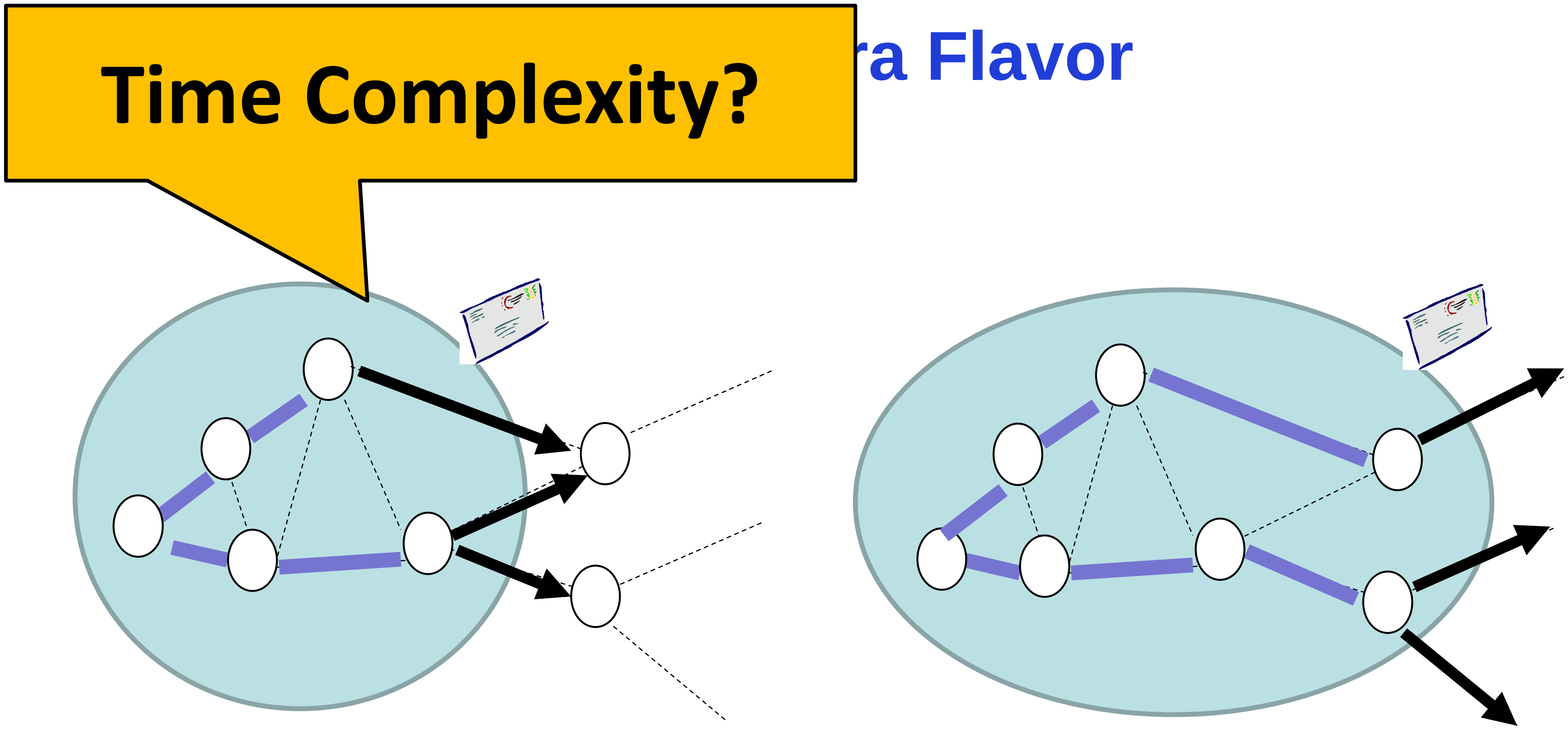
Phase i

Phase i+1

Distributed BFS: Dijkstra Flavor



ra Flavor

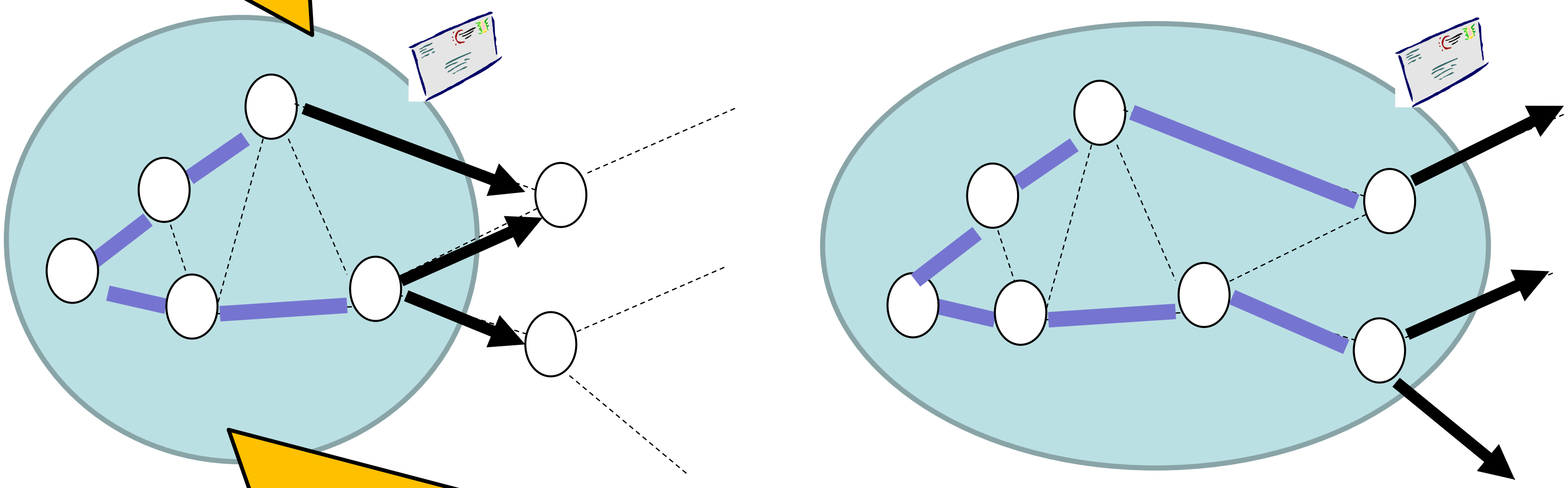


Phase i

Phase i+1

ra Flavor

Time Complexity?



O(R) phases, take time O(R): O(R^2)

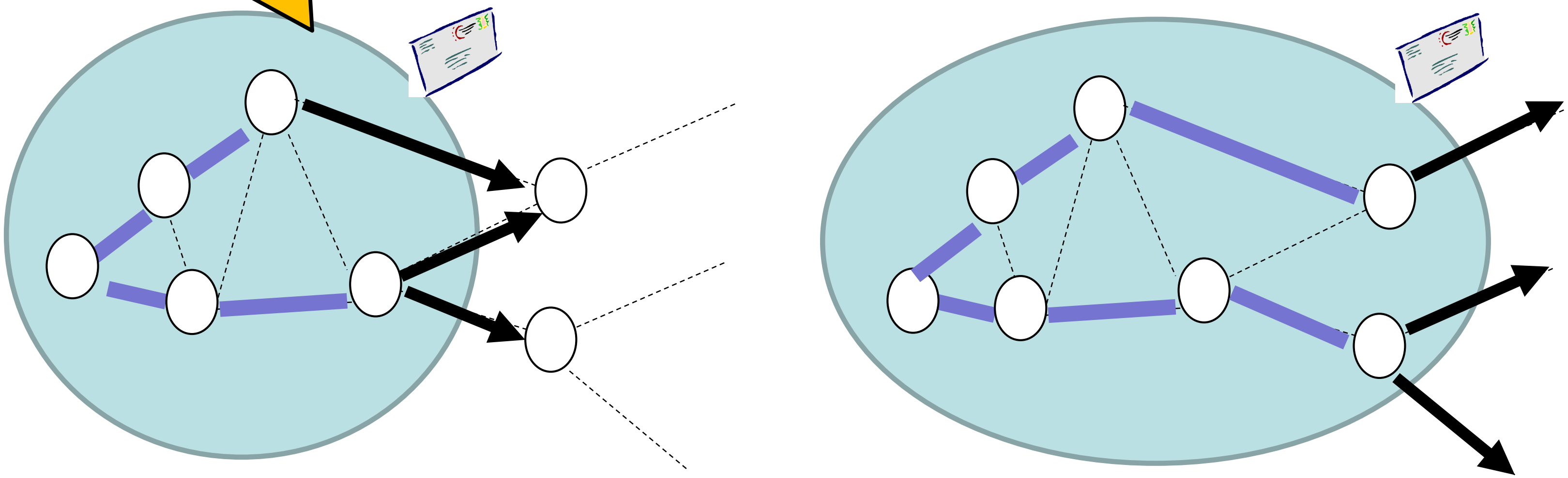
Where R is the radius from the root.

Phase i

Phase i+1

ra Flavor

Message Complexity?



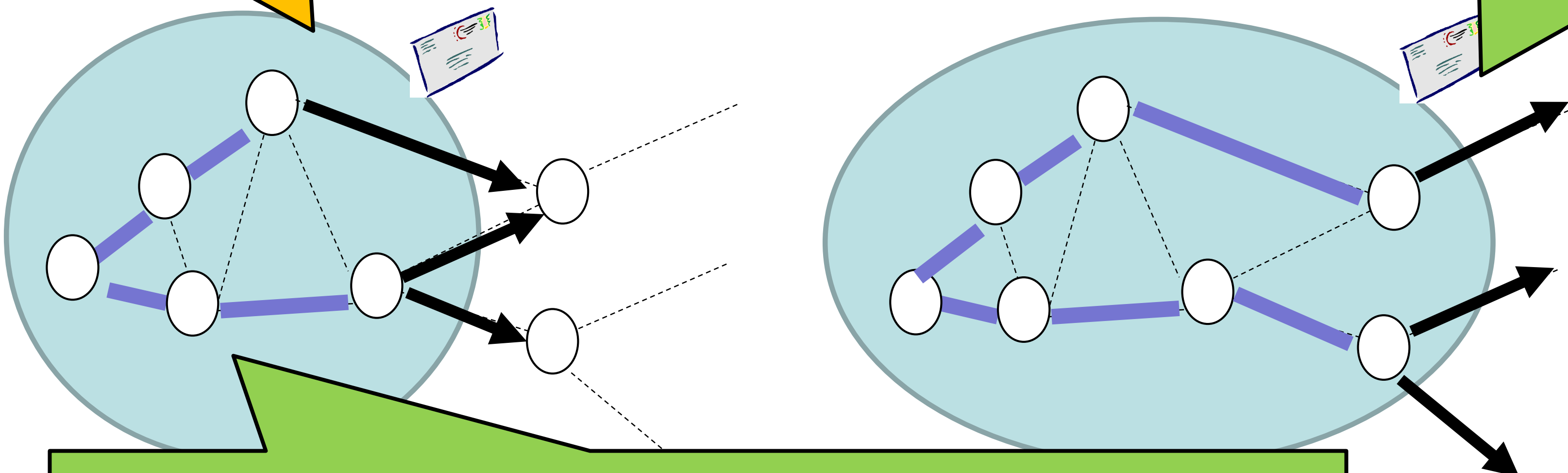
Phase i

Phase i+1

ra Flavo

Message Complexity?

Plus: test each edge once: join, ACK/NAK at edge: total O(m).



„start“ and „join“ propagation inside spanning tree: O(n) per phase: O(nD) in total.

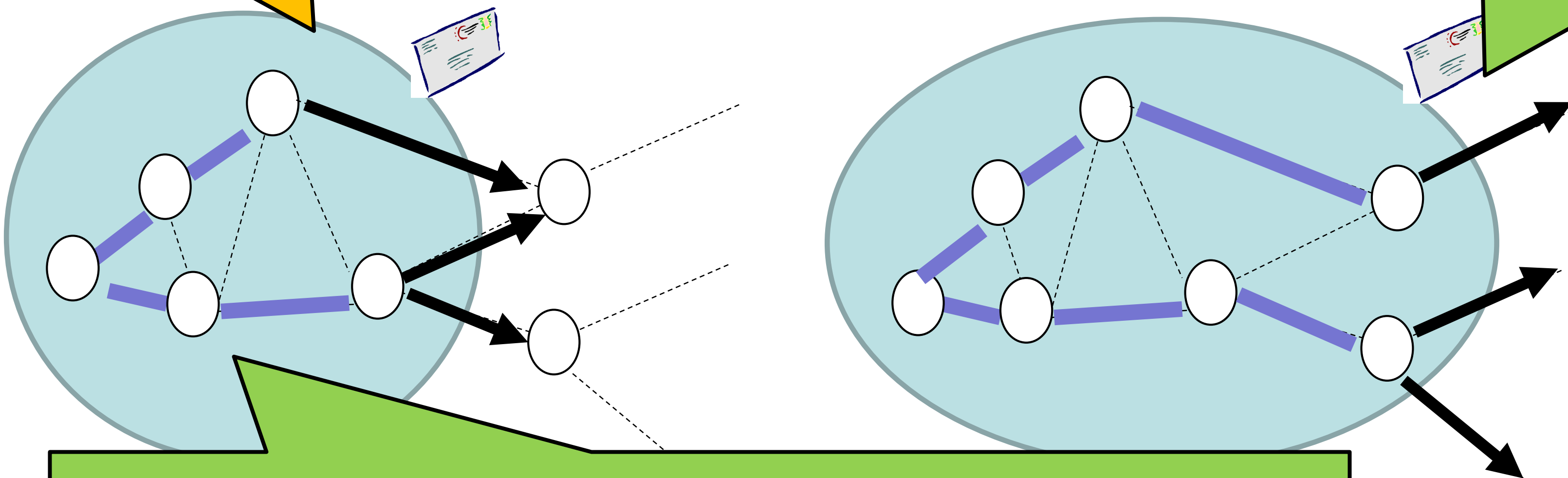
Phase i

Phase i+1

ra Flavo

Message Complexity?

Plus: test each edge once: join, ACK/NAK at edge: total O(m).



„start“ and „join“ propagation inside spanning tree: O(n) per phase: O(nD) in total.

Pha

O(nR+m)

e i+1

Distributed BFS: Dijkstra Flavor

Dijkstra: find next closest node („on border“) to the root

Dijkstra Style

Divide execution into *phases*. In phase p, nodes with distance p to the root are detected. Let T_p be the tree of phase p. T_1 is the root plus all direct neighbors.

Repeat (until no new nodes discovered):

1. Root starts phase p by broadcasting „**start p**“ within T_p
2. A leaf u of T_p (= node discovered only in last phase) sends „**join p+1**“ to all quiet neighbors v (u has not talked to v yet)
3. Node v hearing „join“ for first time sends back „**ACK**“: it becomes leave of tree T_{p+1}; otherwise v replied „**NACK**“ (needed since async!)
4. The leaves of T_p collect all answers and start Echo Algorithm to the root
5. Root initates next phase

Distributed BFS: Bellman-Ford Flavor

Distributed BFS: Bellman-Fo

Idea: Don‘t go through these time-consuming phases but blast out messages but with distance!

Distributed BFS: Bellman-Fo

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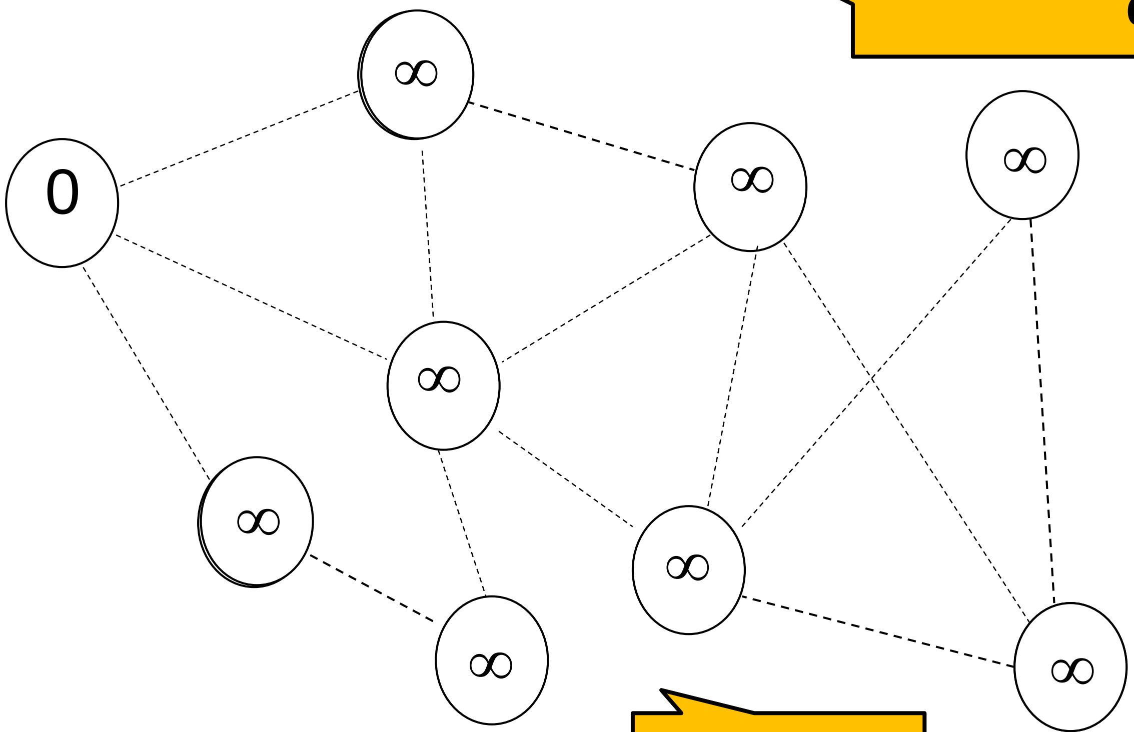


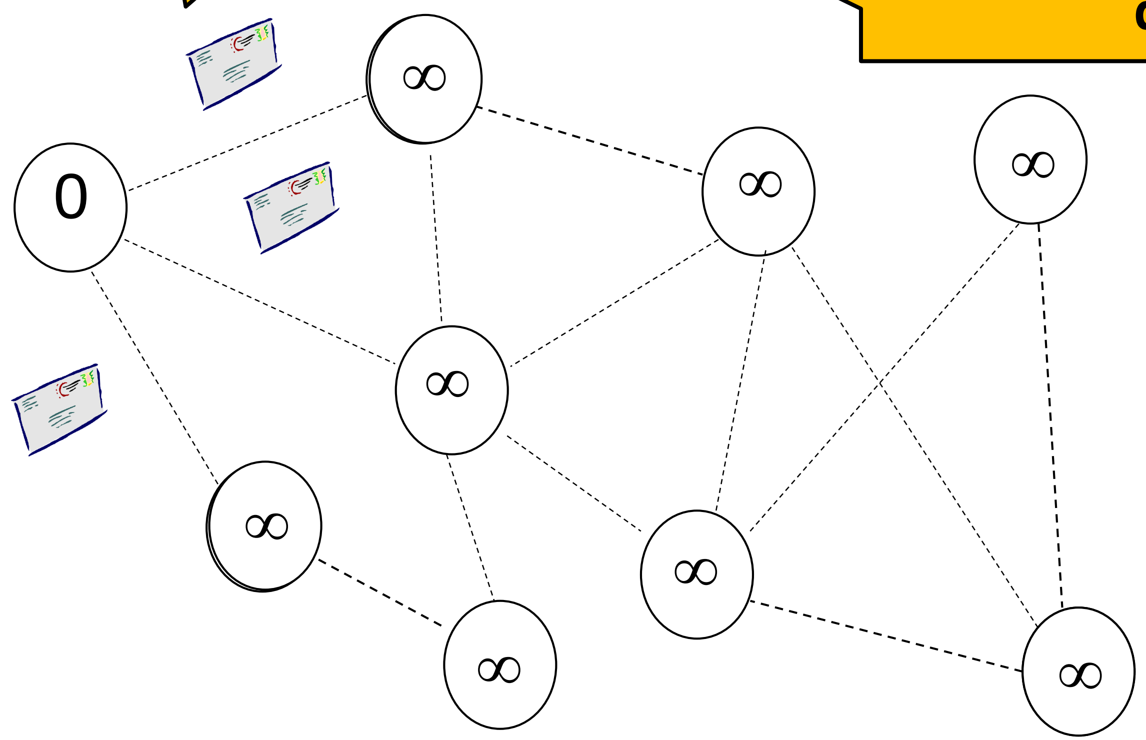
init to ∞

Initialize: root distance 0, other nodes ∞

Di FS: Bellman-Fo

distance 1

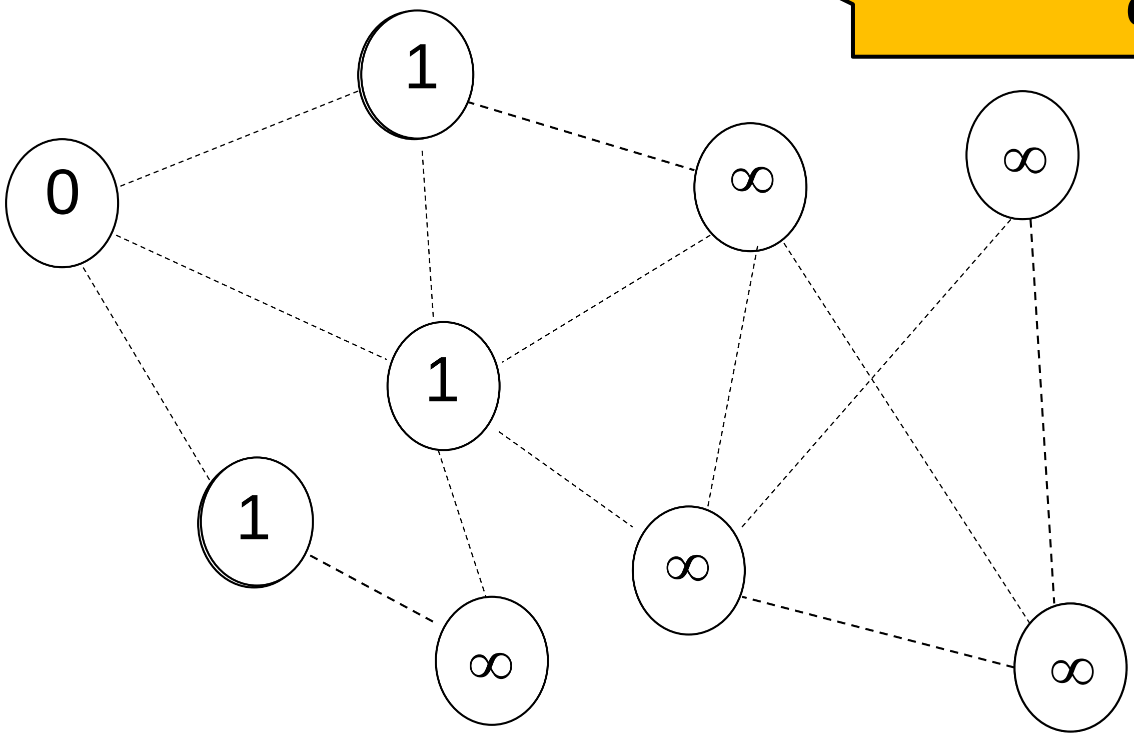
Idea: Don‘t go through these time-consuming phases but blast out messages but with distance!



Start: root sends distance 1 packet to neighbors

Distributed BFS: Bellman-Fo

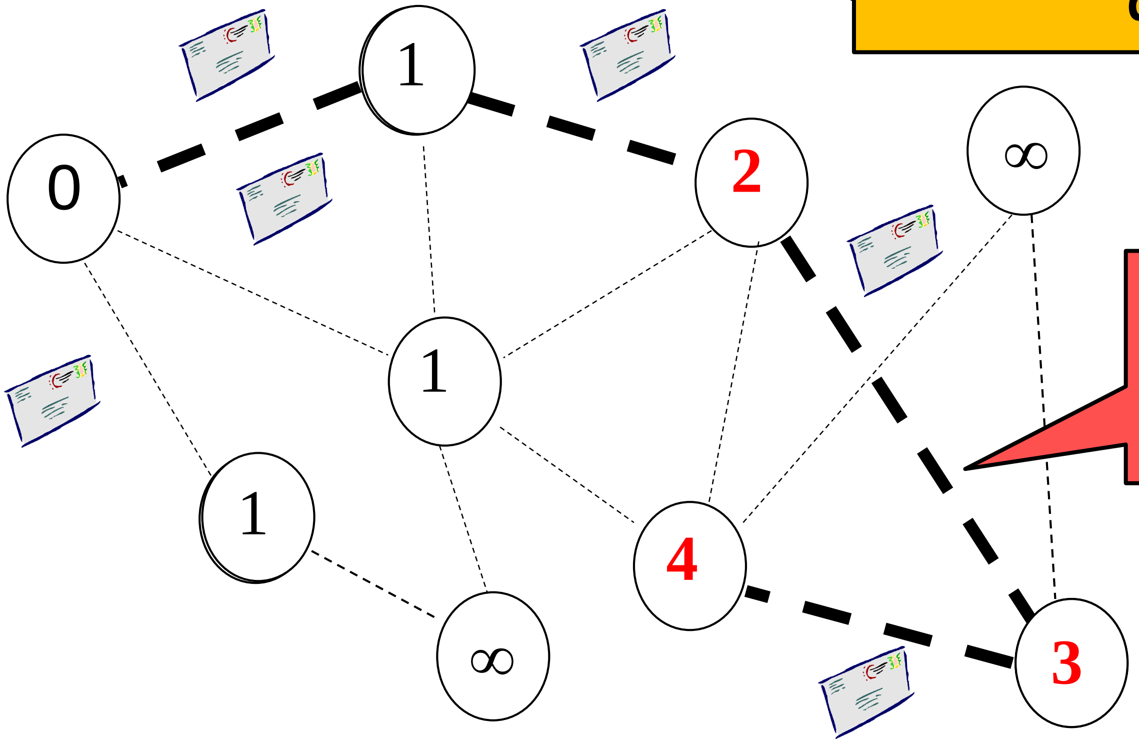
Idea: Don‘t go through these time-consuming phases but blast out messages but with distance!



Repeat: whenever receive new packet: check whether new minimal distance (if so change parent), and propagate!

Distributed BFS: Bellman-Fo

Idea: Don't go through these time-consuming phases but blast out messages but with distance!

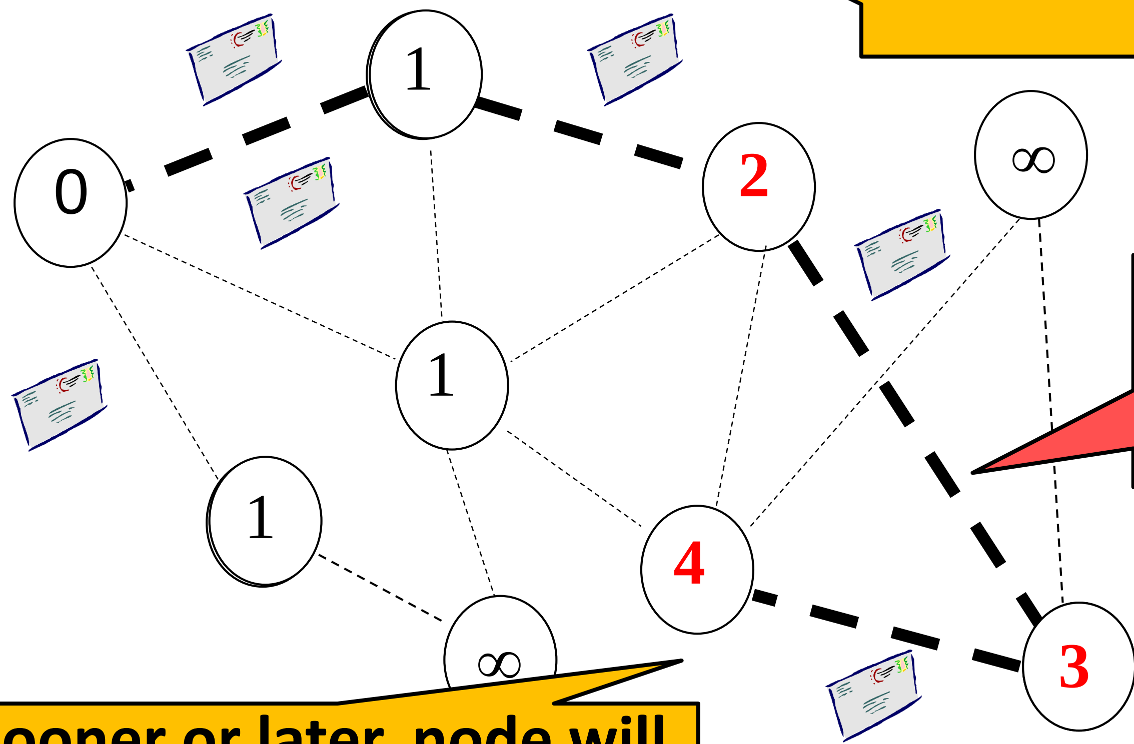


Packet may race through network: asynchronous!

Repeat: whenever receive new packet: check whether new minimal distance (if so change parent), and propagate!

Distributed BFS: Bellman-Fo

Idea: Don‘t go through these time-consuming phases but blast out messages but with distance!



Packet may race through network: asynchronous!

But sooner or later, node will learn shorter distance!

Repeat: whenever receive new packet: check whether new minimal distance (if so change parent), and propagate!

Distributed BFS: Bellman-Ford Flavor

Bellman-Ford: compute shortest distances by flooding an all paths; best predecessor = parent in tree

Bellman-Ford Style

Each node u stores d_u, the distance from u to the root.

Initially, d_{root}=0 and all other distances are ∞. Root starts algo by sending „1“ to all neighbors.

1. If a node u receives message „y“ with $y<d_u$

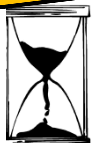
   $d_u := y$

   send „y+1“ to all other neighbors

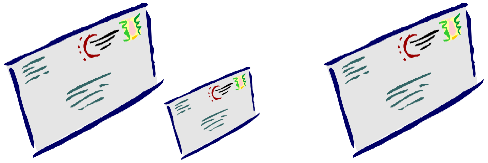
Analysis

How is this defined?! Assuming a unit upper bound on per link delay!

Time Complexity?



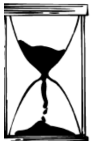
Message Complexity?



Analysis

Worst propagation time is simply the diameter.

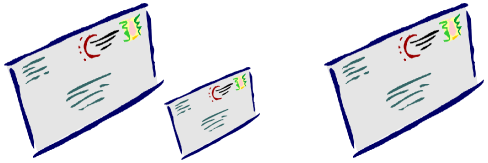
Time Complexity?



O(D) where D is diameter of graph. ☺

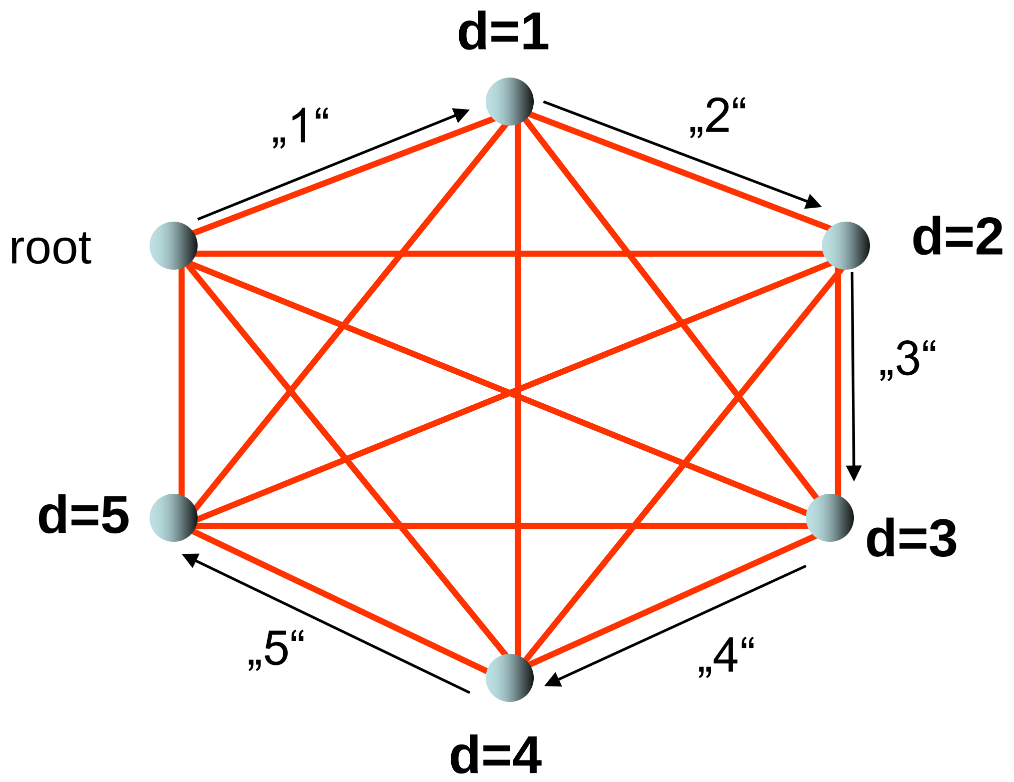
By induction: By time d, node at distance d got „d“.
Clearly true for d=0 and d=1.
A node at distance d has neighbor at distance d-1 that got „d-1“ on time by induction hypothesis. It will send „d“ in next time slot...

Message Complexity?

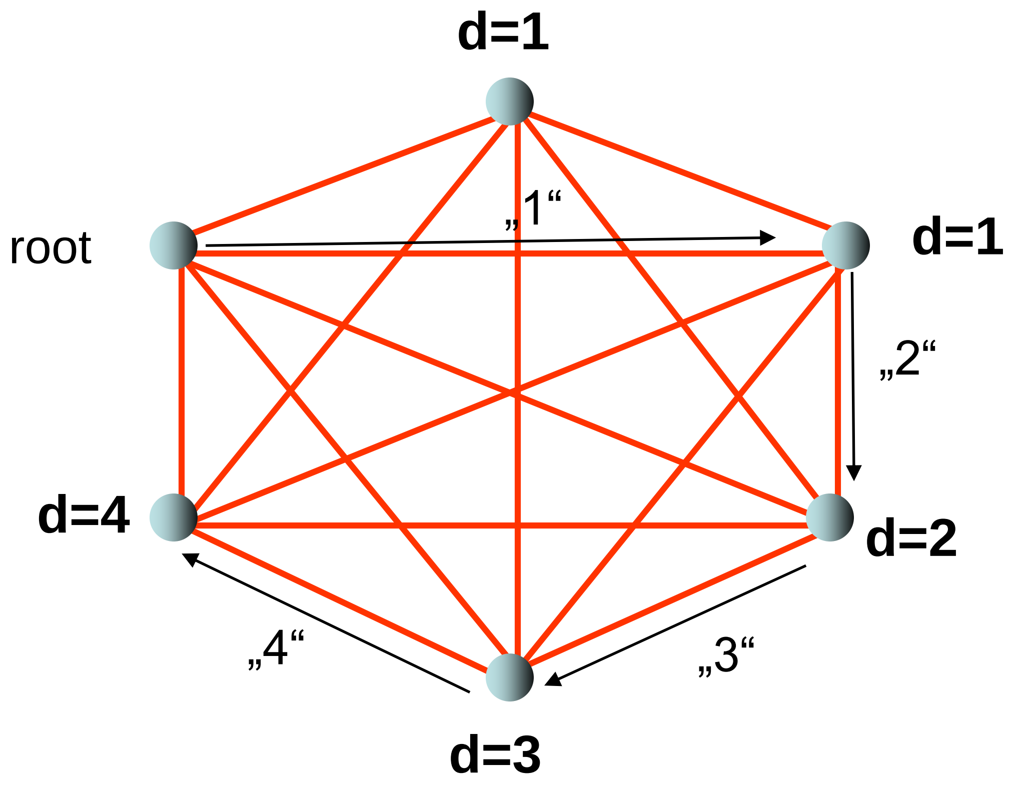


O(mn) where m is number of edges, n is number of nodes. ☹

Bellman-Ford with Many Messages

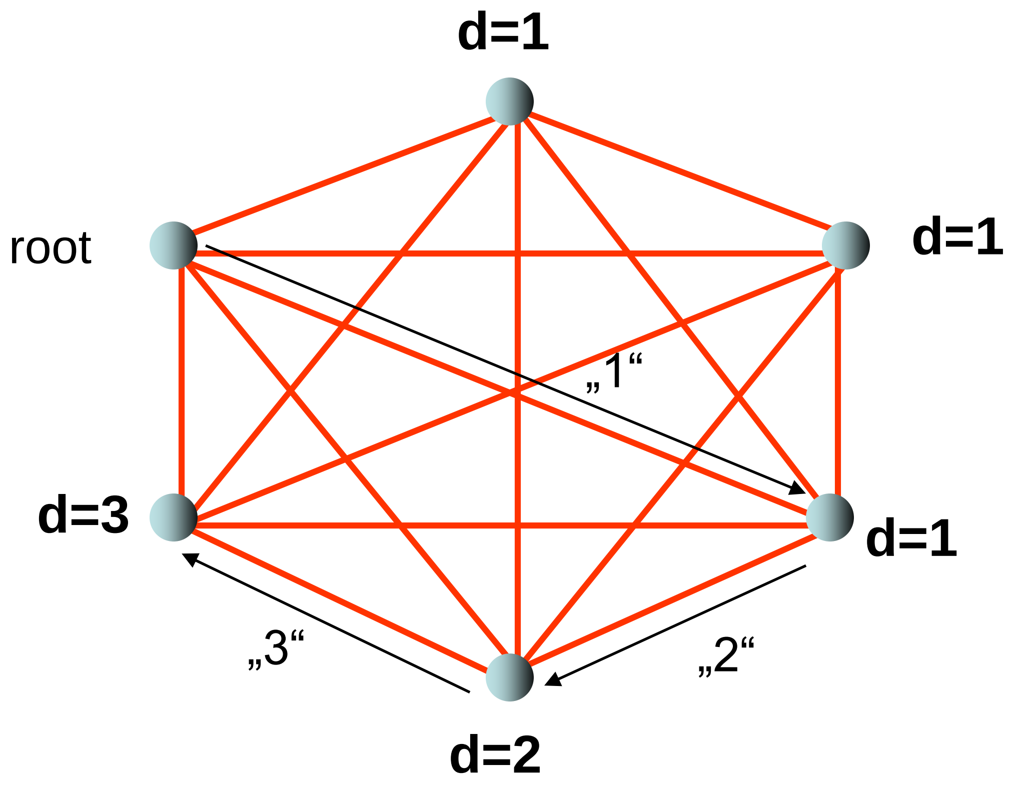


Bellman-Ford with Many Messages



Everyone has a new best distance and informs neighbors!

Bellman-Ford with Many Messages



Everyone has a new best distance and informs neighbors!

Discussion

Which algorithm is better?

Dijkstra has better message complexity, Bellman-Ford better time complexity.

Can we do better?

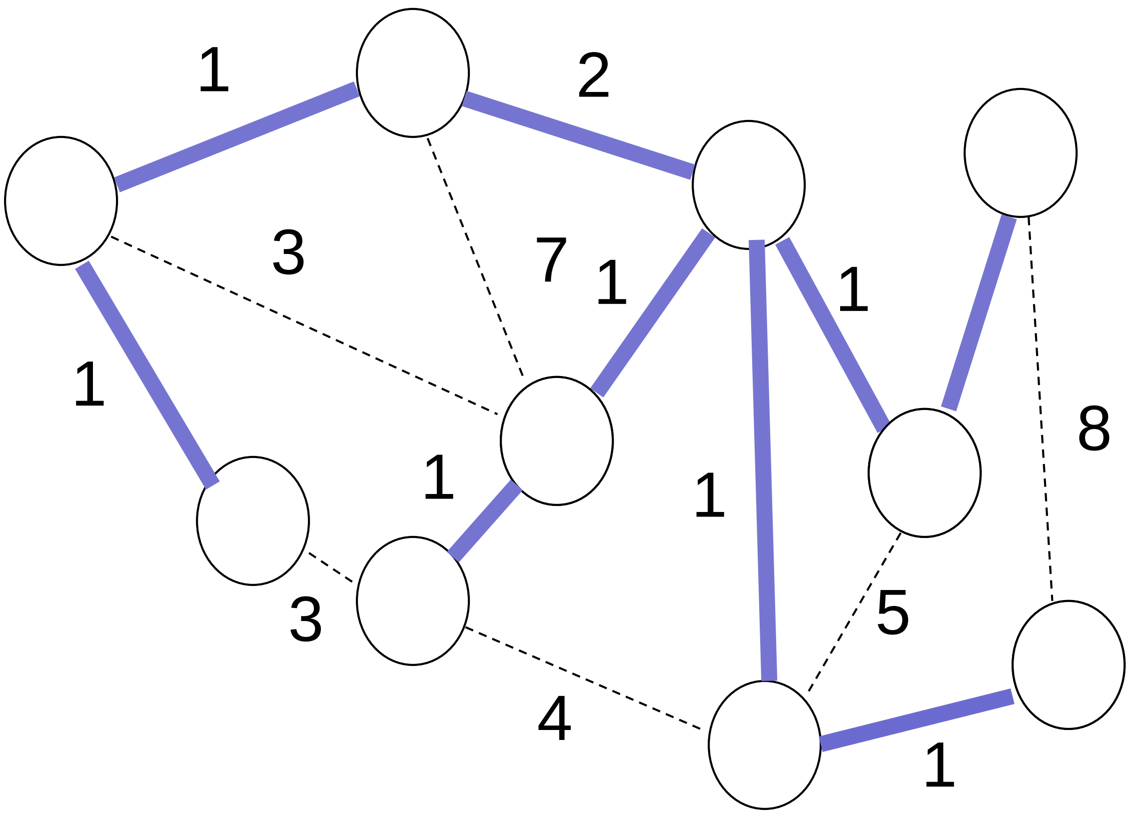
Yes, but not in this course... ☺

Remark: Asynchronous algorithms can be made synchronous... (e.g., by central controller or better: local synchronizers)

How to compute an MST?

MST

Tree with edges of minimal total weight.



Idea: Exploit Basic Fact of MST: Blue Edges

Blue Edge

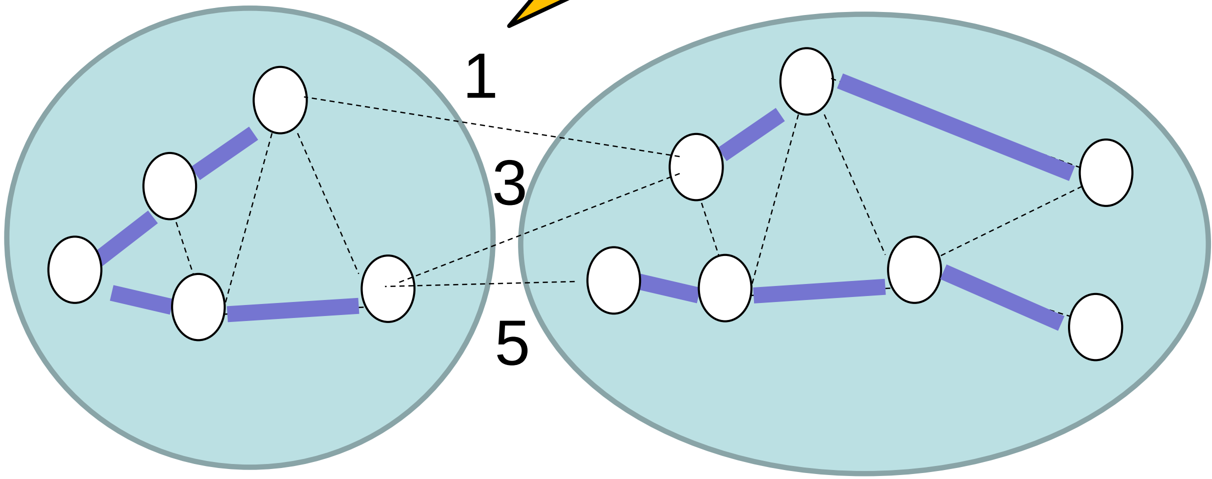
Let T be an MST and T‘ a subgraph of T.
Edge e=(u,v) is *outgoing edge* if u ϵ T‘ and v ∉ T‘.
The outgoing edge of minimal weight is called *blue edge*.

Lemma

If T is the MST and T‘ a subgraph of T, then the blue edge of T‘ is also part of T.

It holds: the lightest edge across a cut must be part of the MST!

By contradiction: otherwise get a cheaper MST by swapping the two cut edges!



Gallager-Humblet-Spira

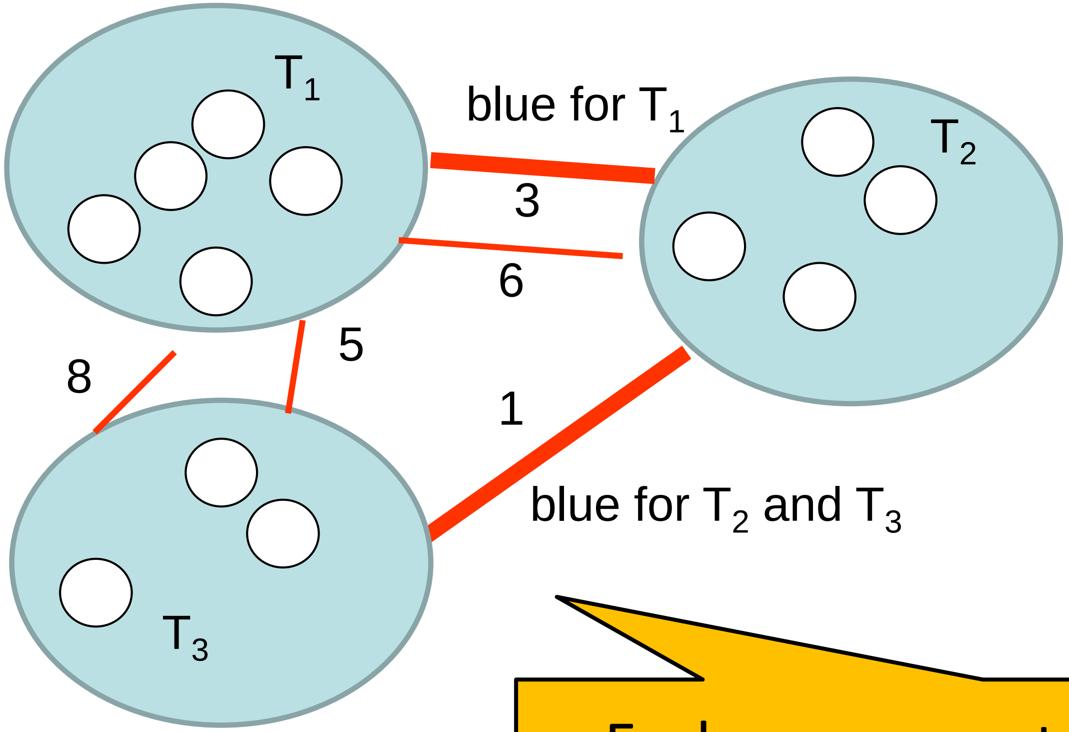
Gallager-Humblet-Sp

Basic idea: Grow components in parallel and merge them at the blue edge! Using Covergecast.

Gallager-Humblet-Sp

Basic idea: Grow components in parallel and merge them at the blue edge! Using Covergecast.

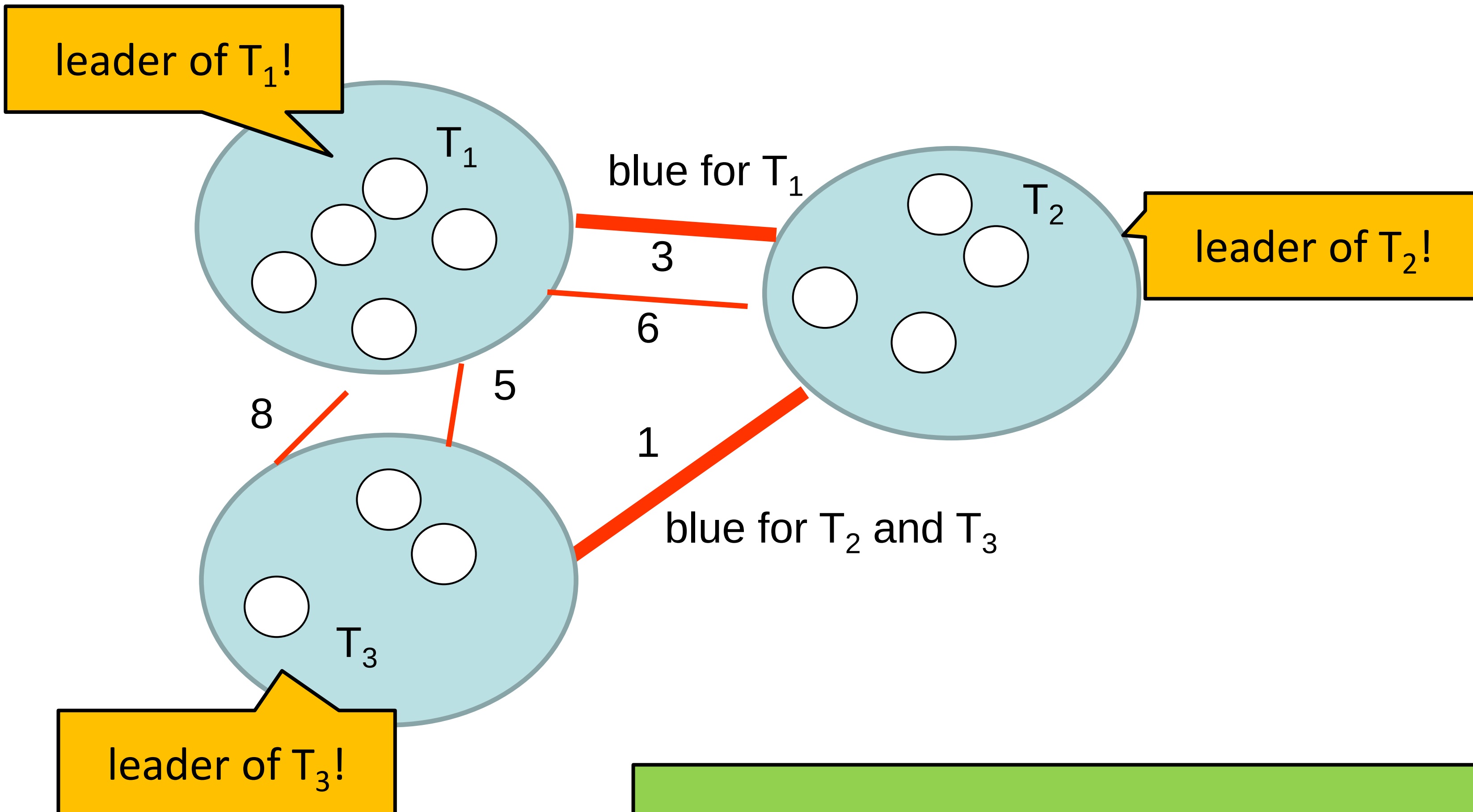
Assume some components have already emerged:



Each component has only one blue edge (cheapest outgoing): loops impossible, can take them in parallel!

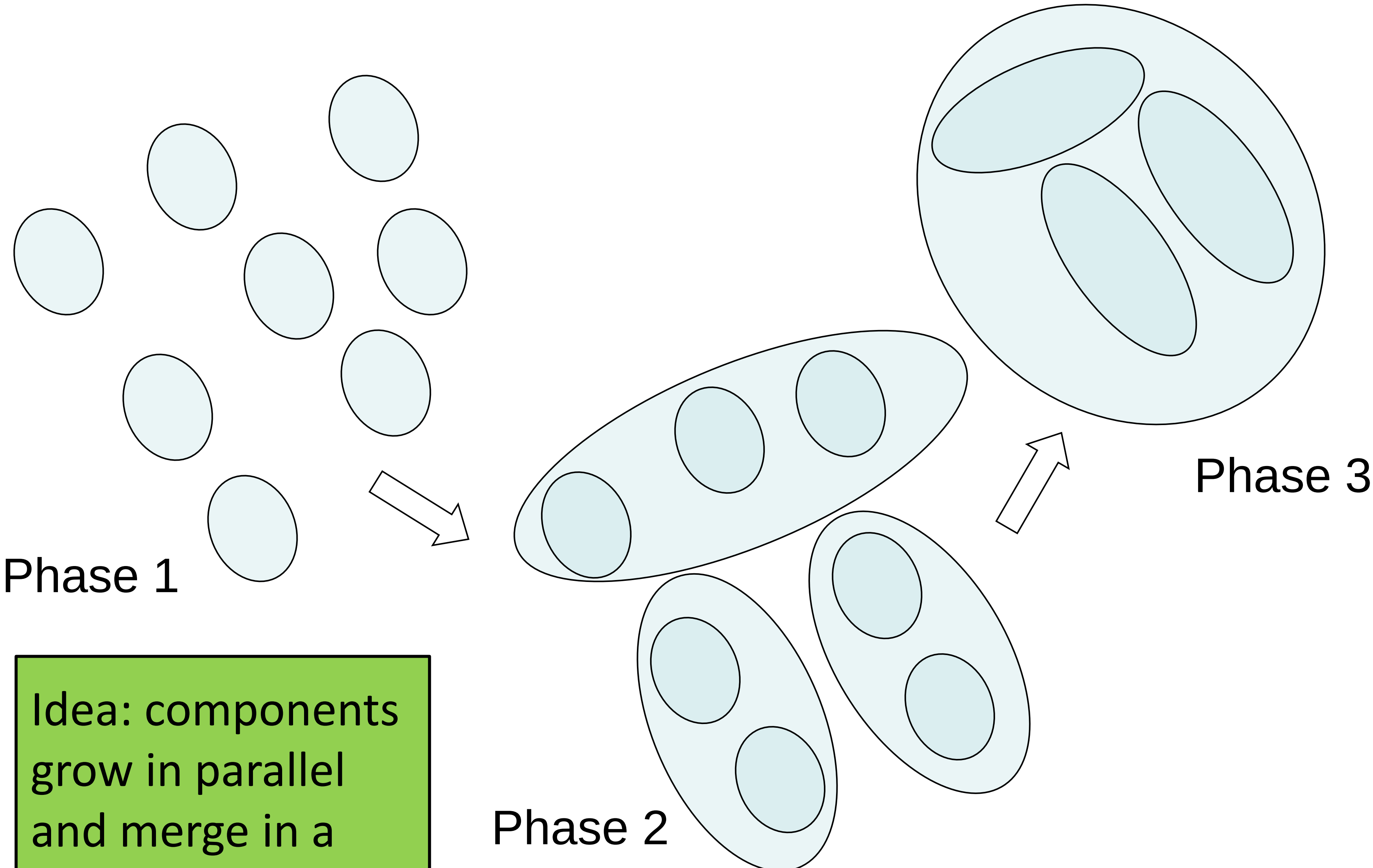
Gallager-Humblet-S

Basic idea: Grow components in parallel and merge them at the blue edge! Using Covergecast.



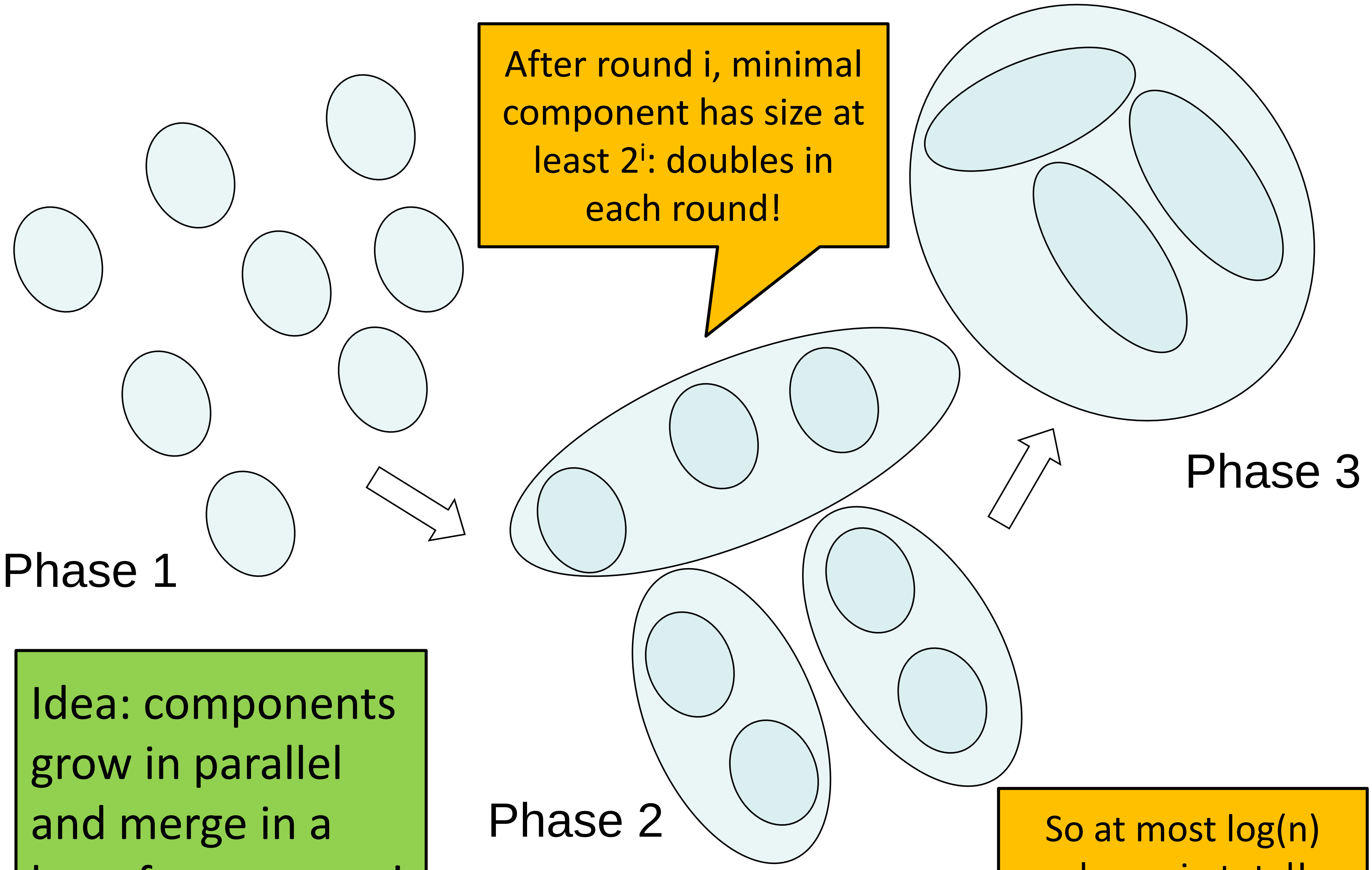
Idea: a leader in each cluster responsible of finding blue edge with convergecast!

Gallager-Humblet-Spira



Idea: components grow in parallel and merge in a loop-free manner!

Gallager-Humblet-Spira

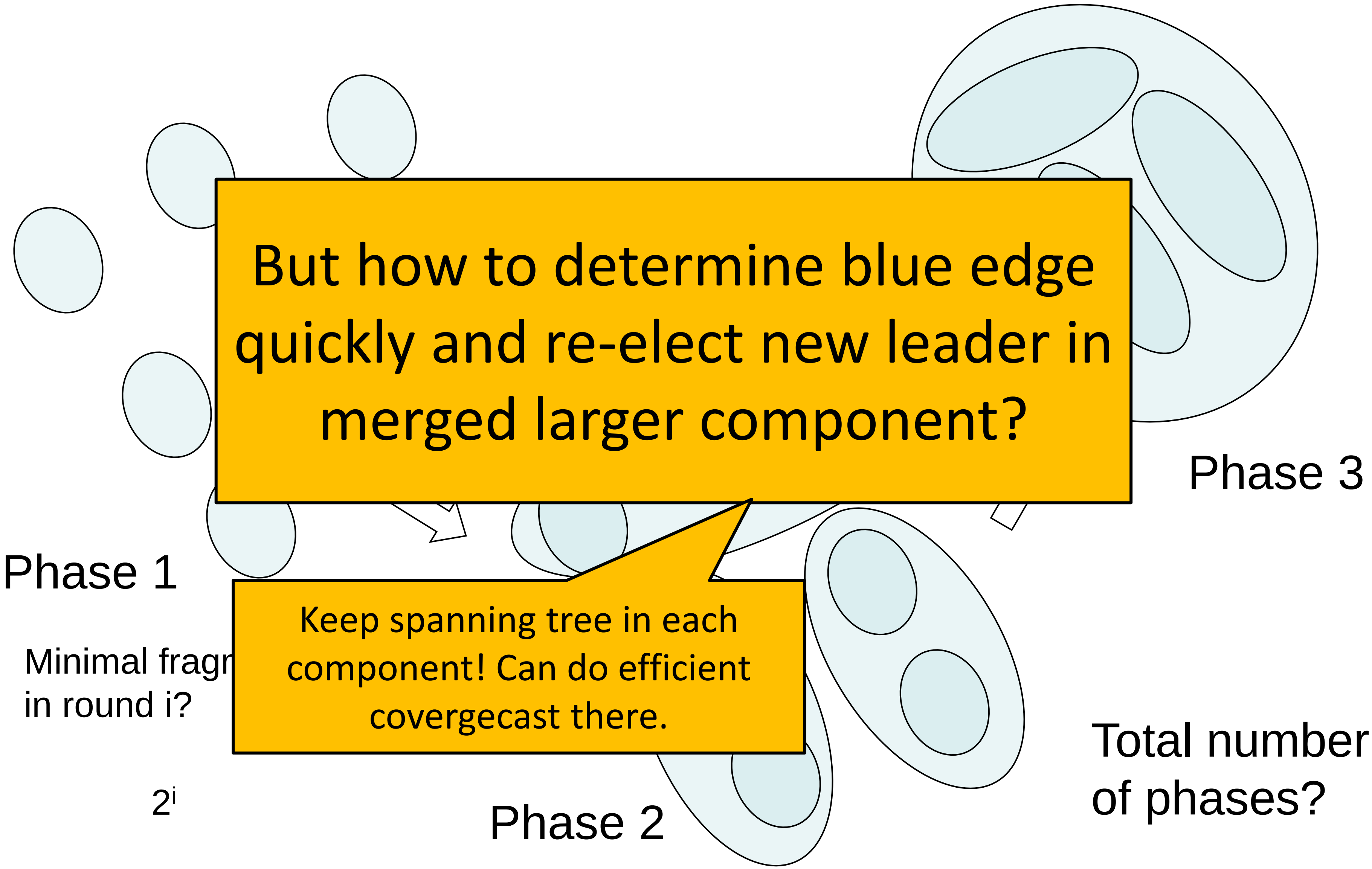


After round i, minimal component has size at least 2^i: doubles in each round!

Idea: components grow in parallel and merge in a loop-free manner!

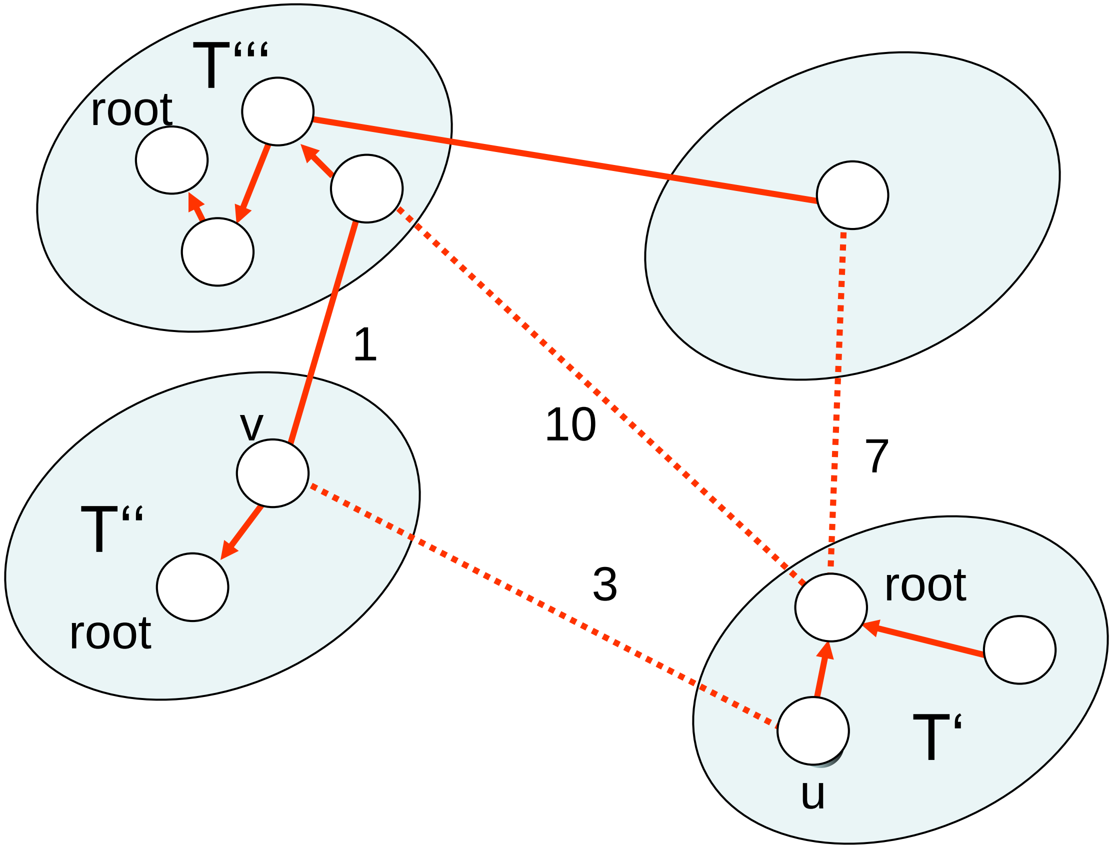
So at most log(n) phases in total!

Gallager-Humblet-Spira



Example: Agree on a New Root

How to merge T‘ and T‘‘ across (u,v)?



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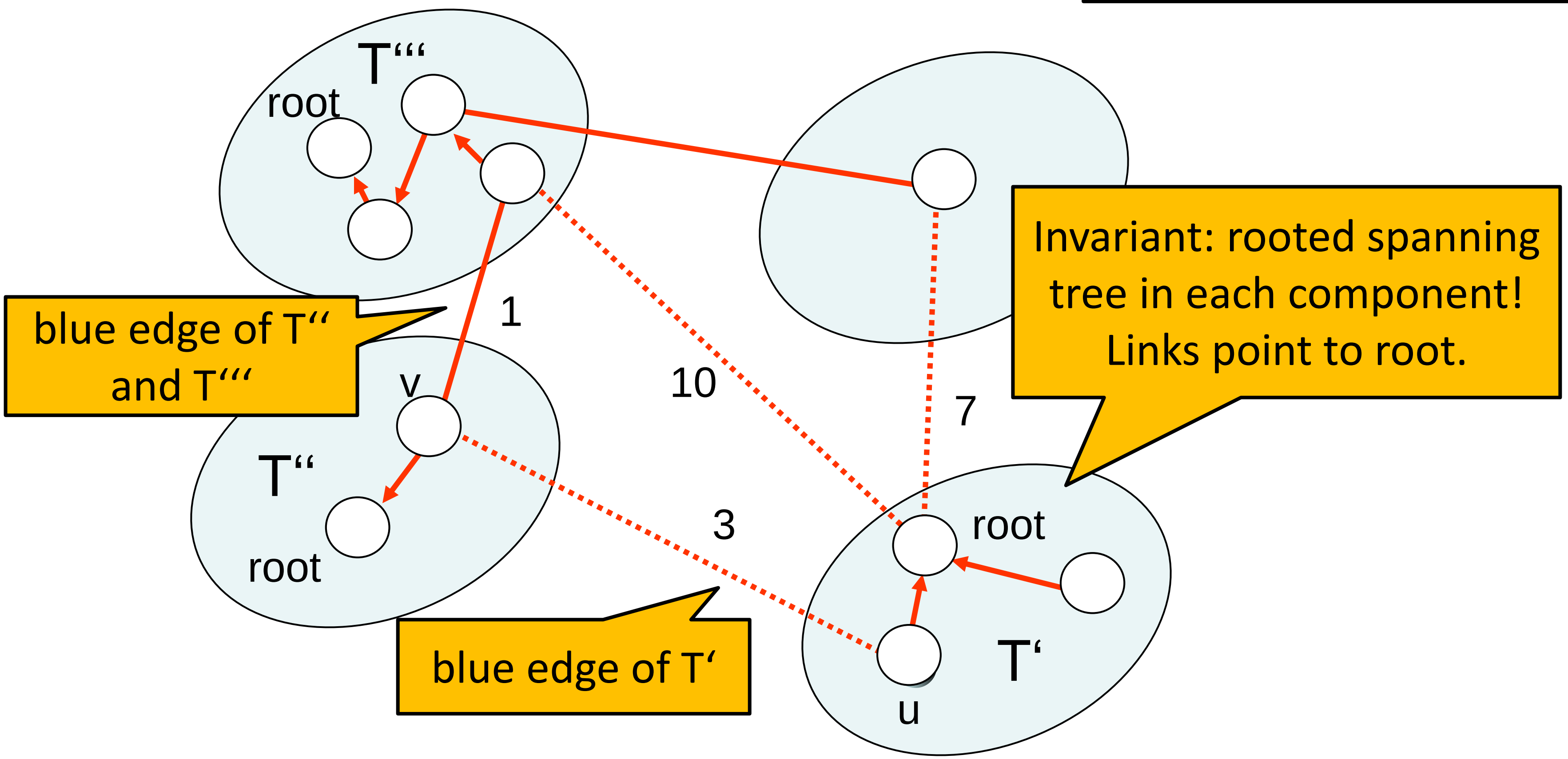

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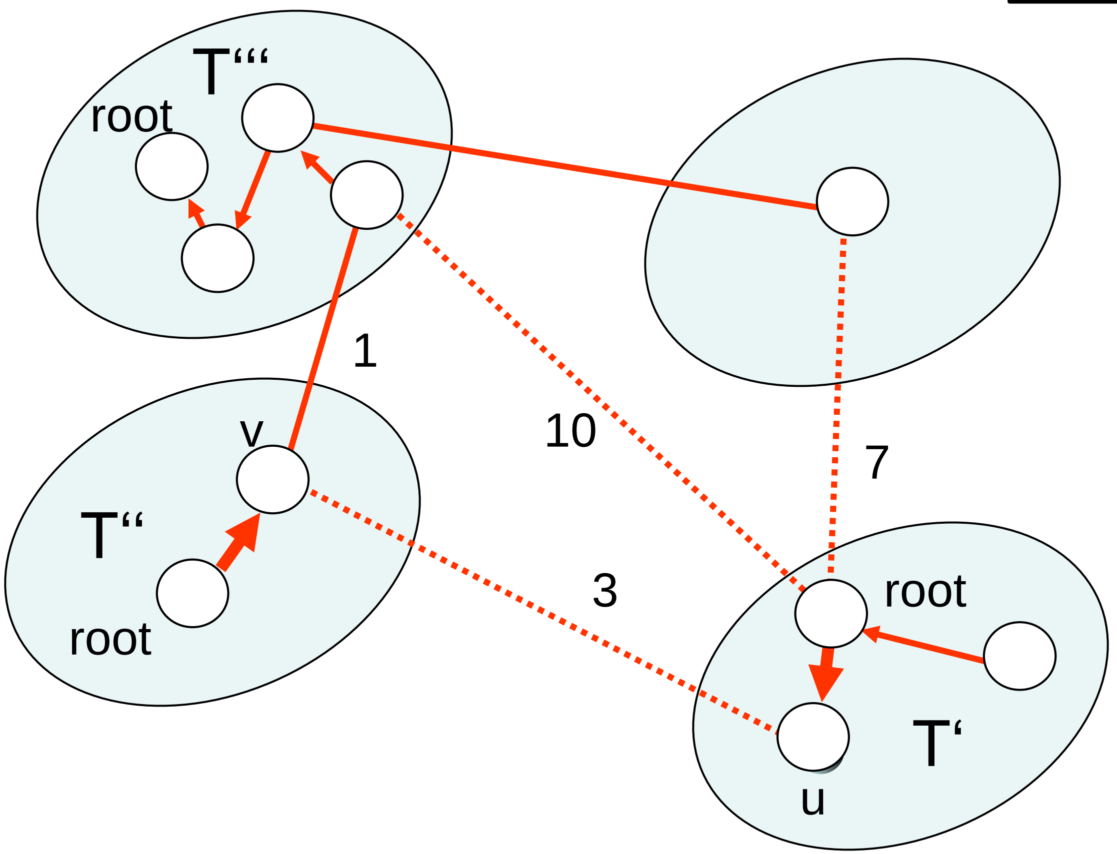

Step 1: invert path from root to u and v.

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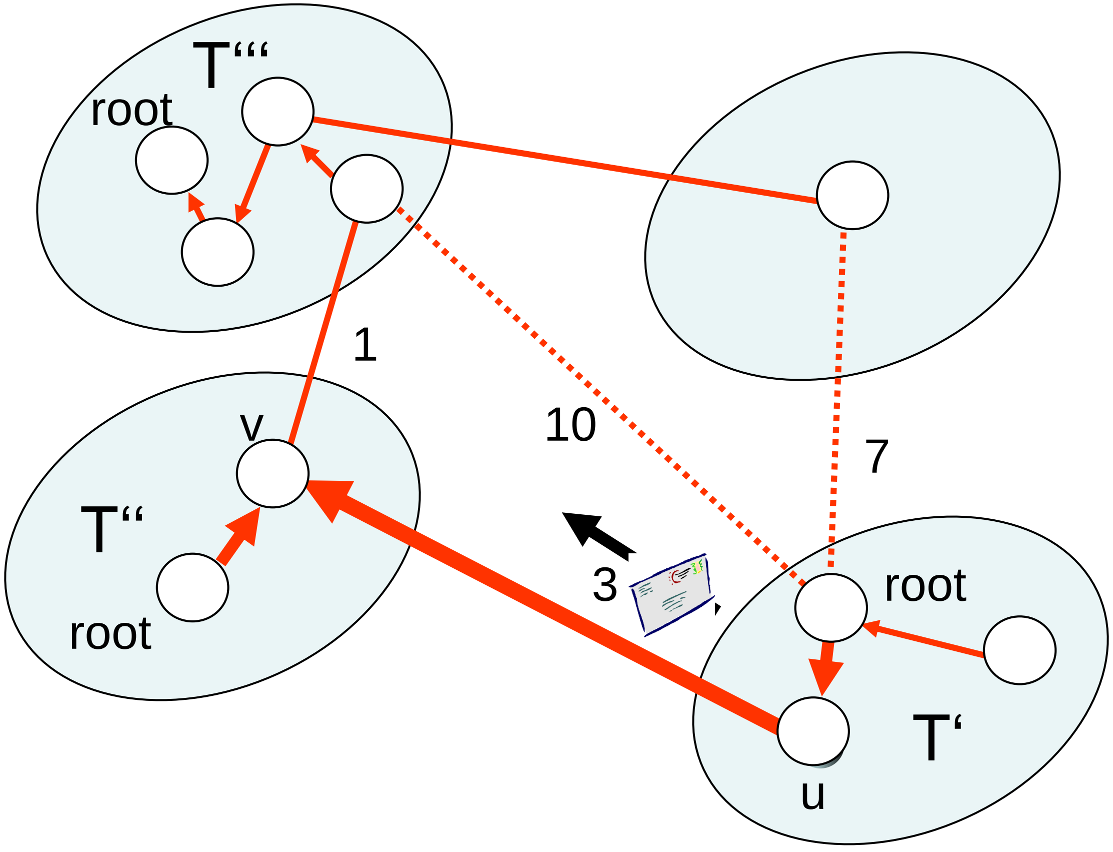

Step 1: invert path from root to u and v.

Step 2: send merge request across blue edge (u,v). Here only blue edge for T‘ so one message!

Step 3: v becomes new root overall!

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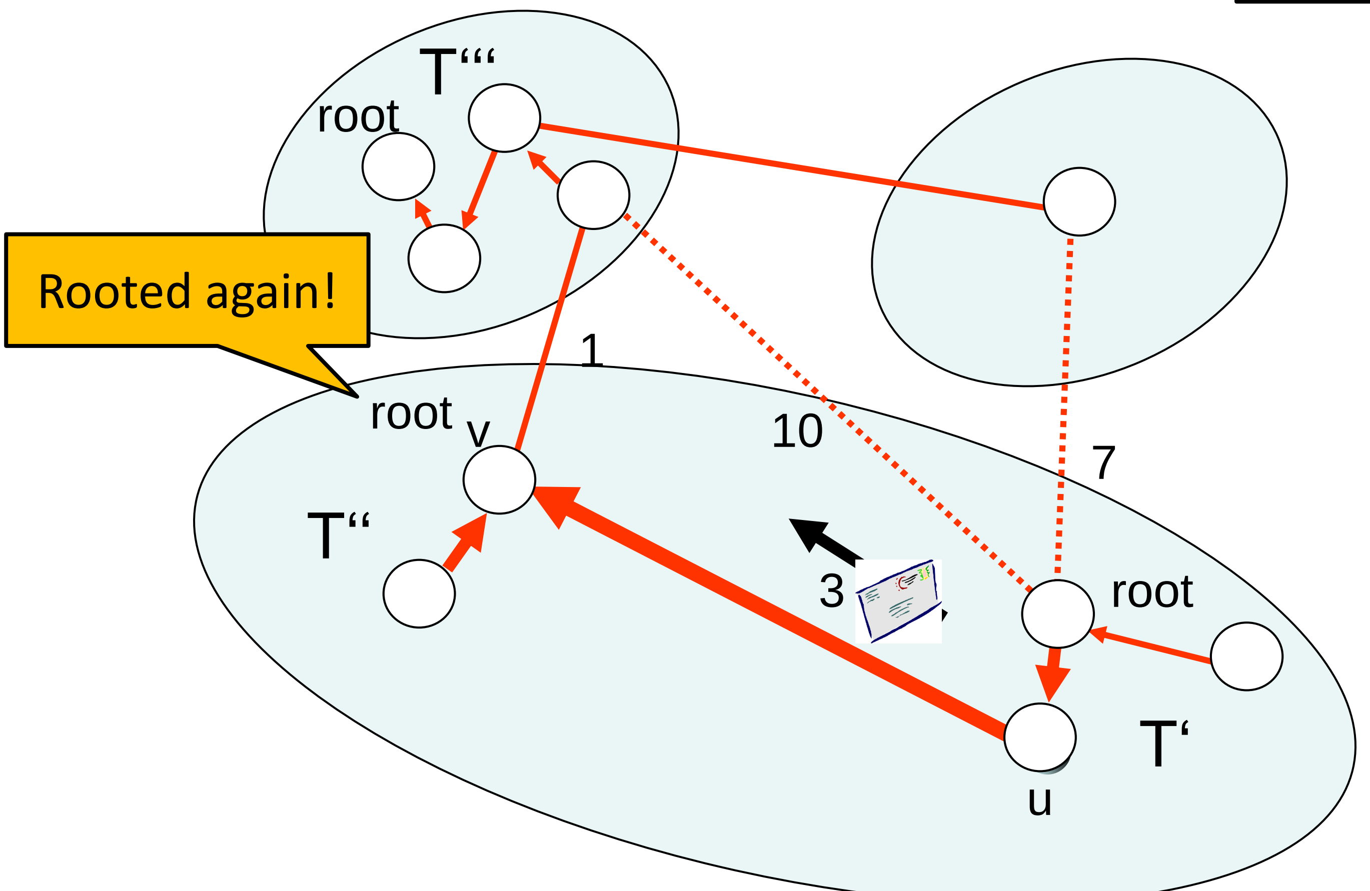

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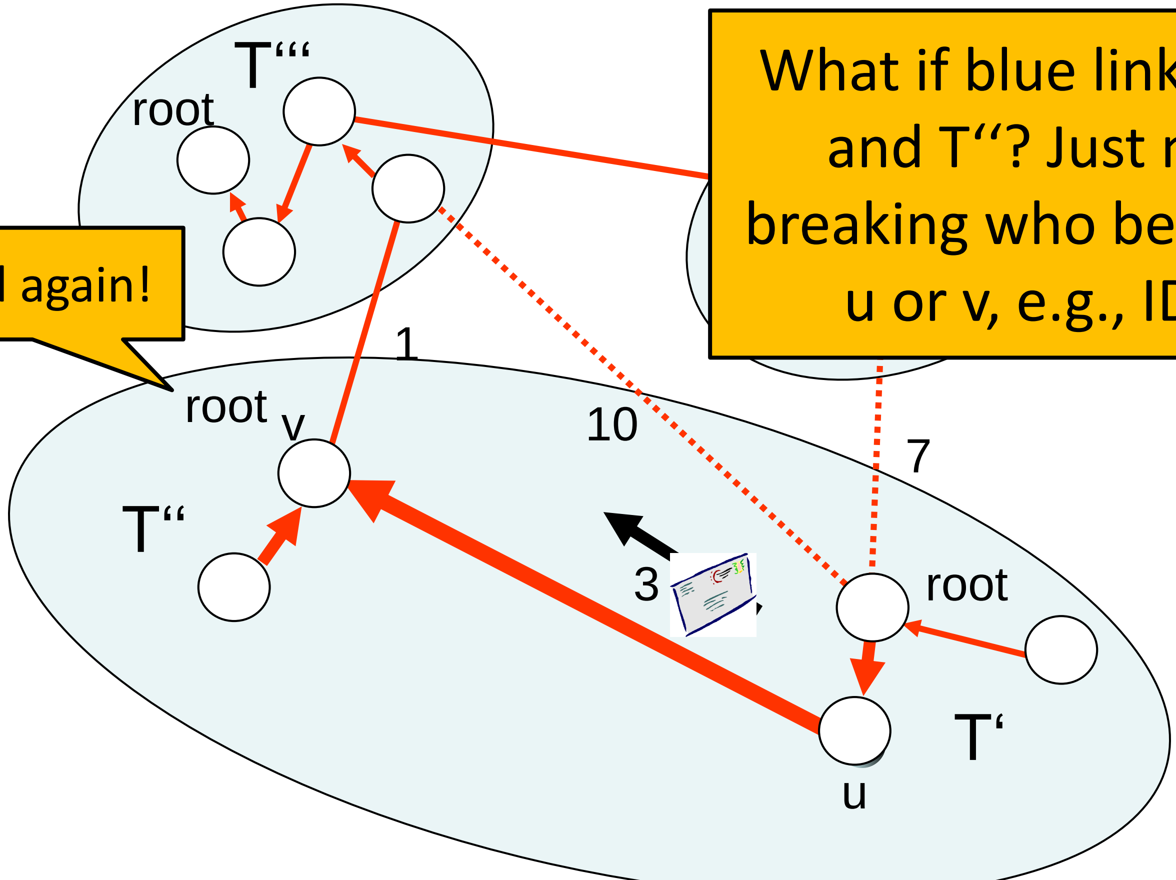
Example: Agree on a New Root

How to merge T‘ and T‘‘ across (u,v)?

What if blue link for both T‘ and T‘‘? Just make tie-breaking who becomes root, u or v, e.g., ID based!

Rooted again!



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Step 2: send merge request across blue edge (u,v). Here only blue edge for T‘ so one message!

Step 3: v becomes new root overall!

Distributed Kruskal

Idea: Grow components by learning blue edge!
But do many fragments in parallel!

Gallager-Humblet-Spira

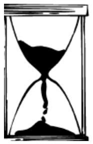
Initially, each node is root of its own fragment.

Repeat (until all nodes in same fragment)

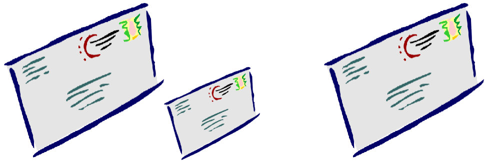
1. nodes learn fragment IDs of neighbors
2. root of fragment finds blue edge (u,v) by convergecast
3. root sends message to u (inverting parent-child)
4. if v also sent a merge request over (u,v), u or v becomes new root depending on smaller ID (make trees directed)
5. new root informs fragment about new root (convergecast on „MST“ of fragment): new fragment ID

Analysis

Time Complexity?



Message Complexity?

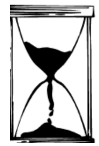


Each phase mainly consists of two convergecasts, so O(D) time and O(n) messages per phase?

Analysis

Log n phases with O(n) time convergecast: spanning tree is not BFS!

Time Complexity?

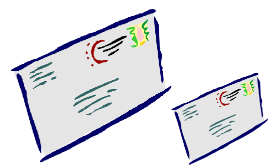


The size of the smallest fragment at least doubles in each phase, so it's logarithmic. But converge cast may take n hops

O(n log n) where n is graph size.

Log n phases but in each phase need to learn leader ID of neighboring fragments, for *all* neighbors!

Message Complexity?



O(m log n) where m is number of edges: at most O(1) messages on each edge in a phase.

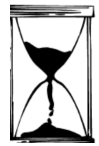
Really needed? Each phase mainly consists of two convergecasts, so O(n) time and O(n) messages. In order to learn fragment IDs of neighbors, O(m) messages are needed (again and again: ID changes in each phase).

Yes, we can do better. ☺

Analysis

Log n phases with O(n) time convergecast: spanning tree is not BFS!

Time Complexity?



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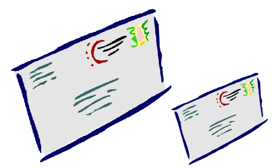
Message Complexity?



O(m log n) where m is number of edges: at most O(1) messages on each edge in a phase.

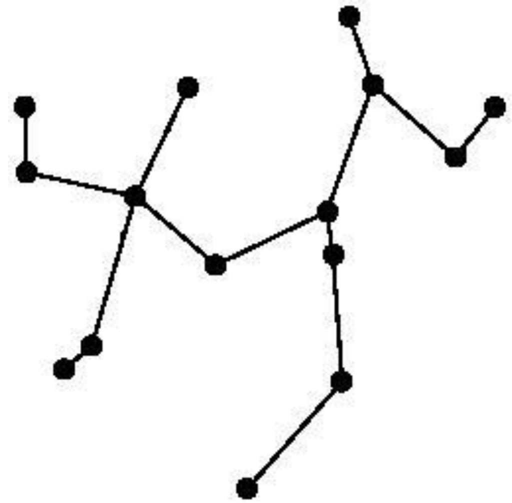
Really needed? Each phase mainly consists of two convergecasts, so O(n) time ... (m) mess...

Note: this algorithm can solve leader election! Leader = last surviving root!

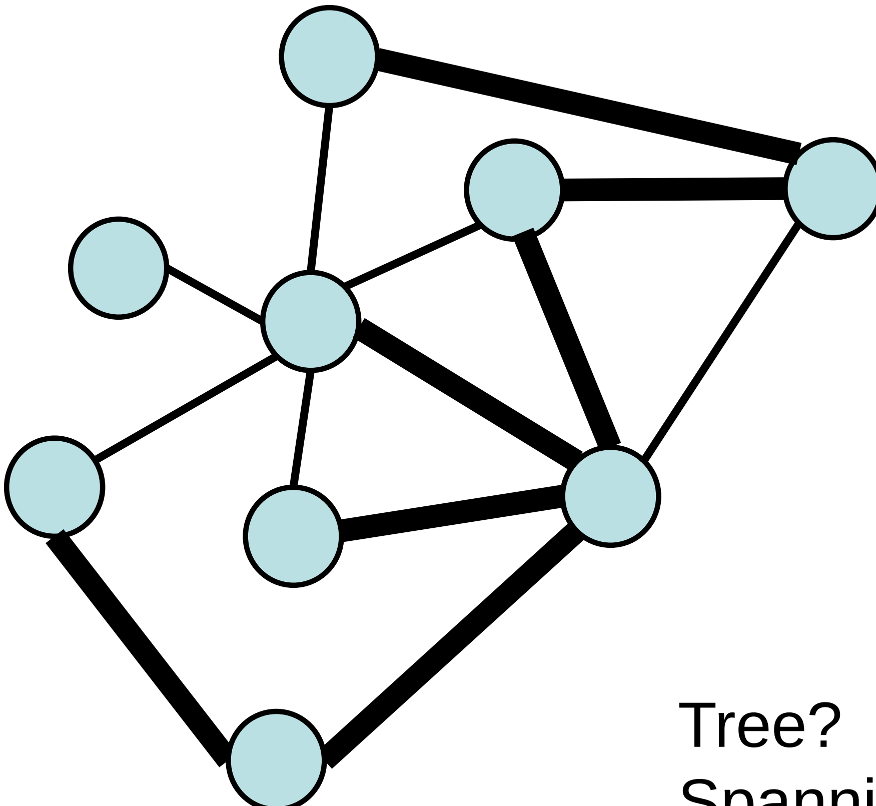
Next: Spanning Trees in the CONGEST Model

Repetition: MST

- Tree: connected graph without cycles
- Spanning subgraph: A subgraph that spans all vertices of a graph
- Minimum spanning tree (MST): Spanning tree with *the least total weight* among all the spanning trees of a weighted and connected graph



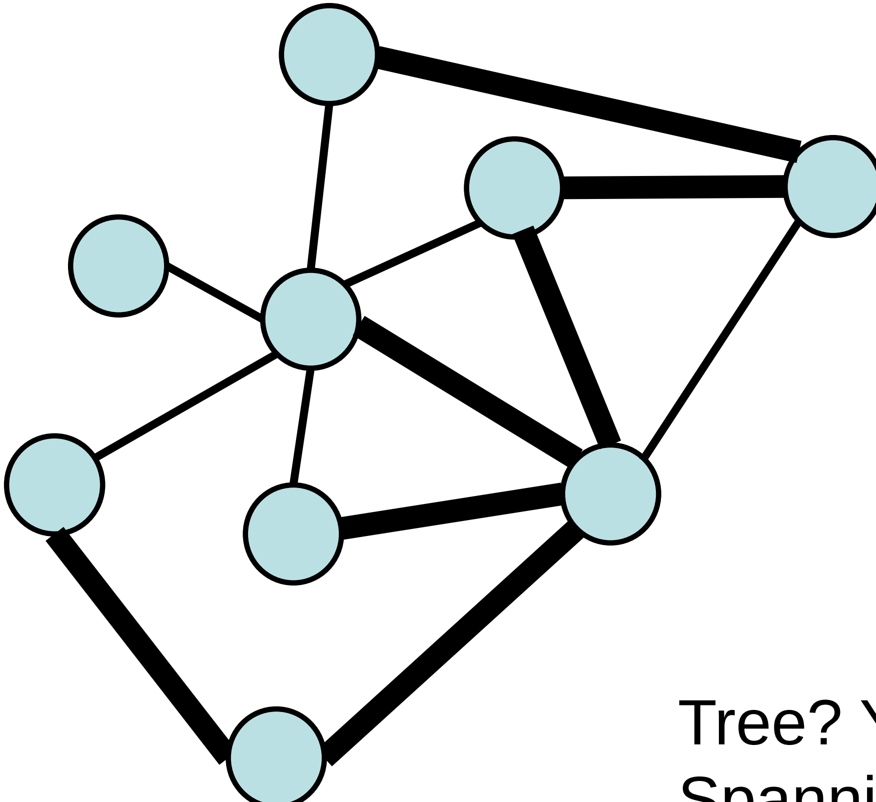
Definitions



Tree?

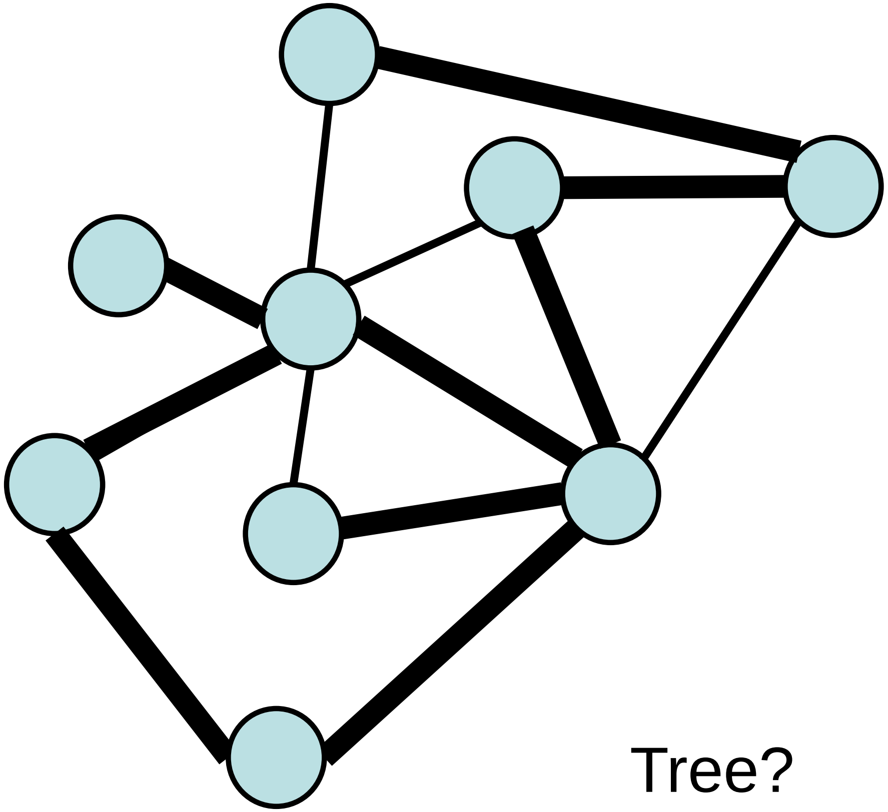
Spanning tree?

Definitions



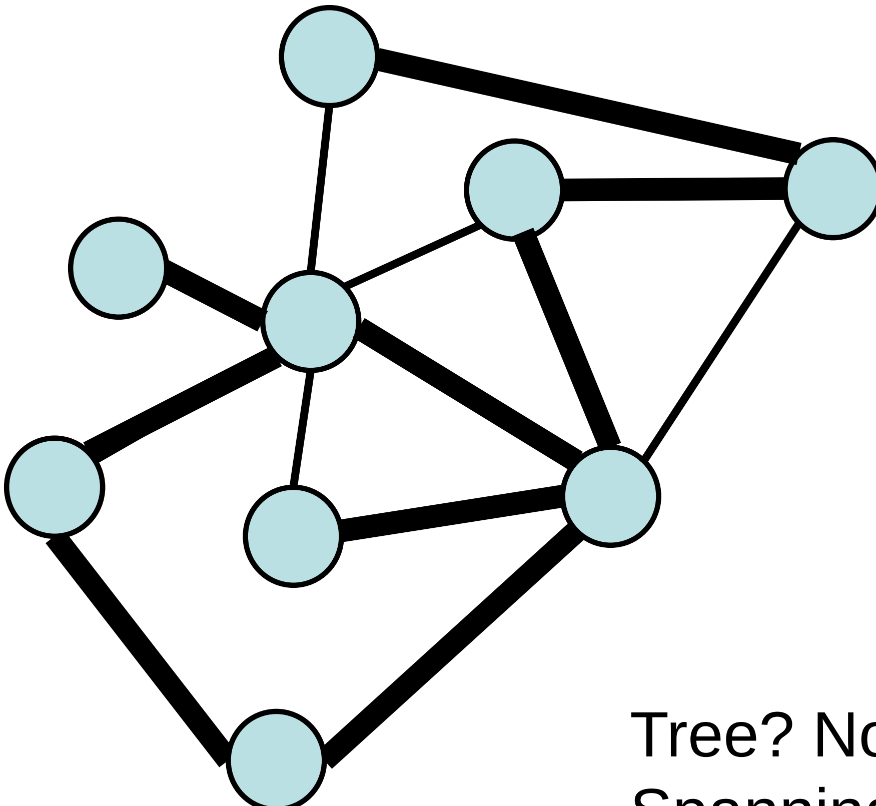
Tree? Yes
Spanning tree? No

Definitions



Tree?
Spanning tree?

Definitions

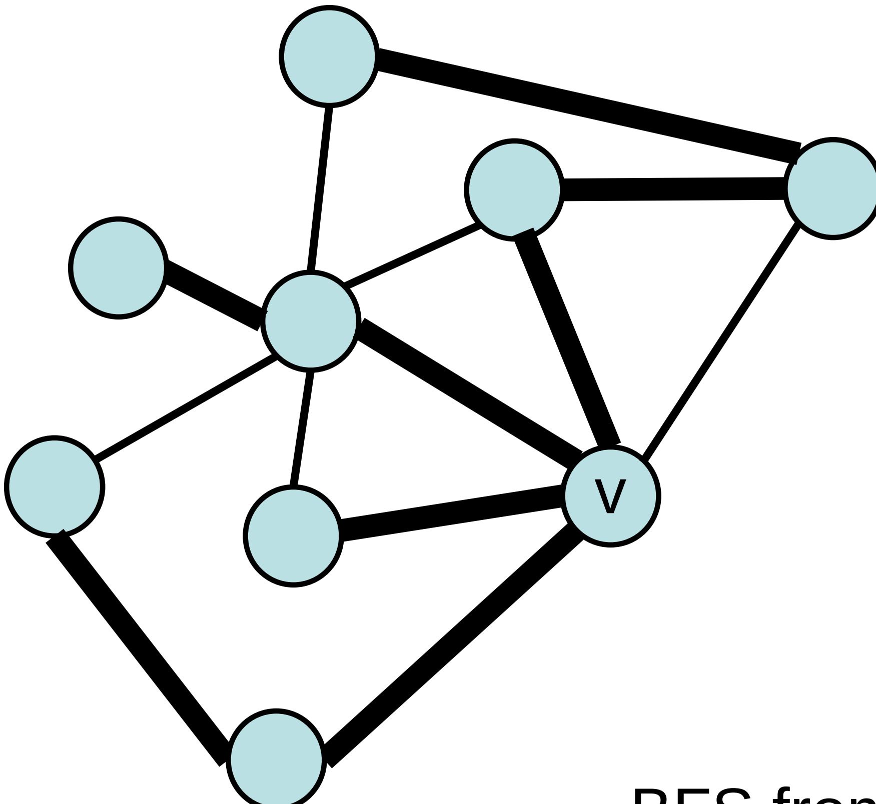


Tree? No

Spanning tree?

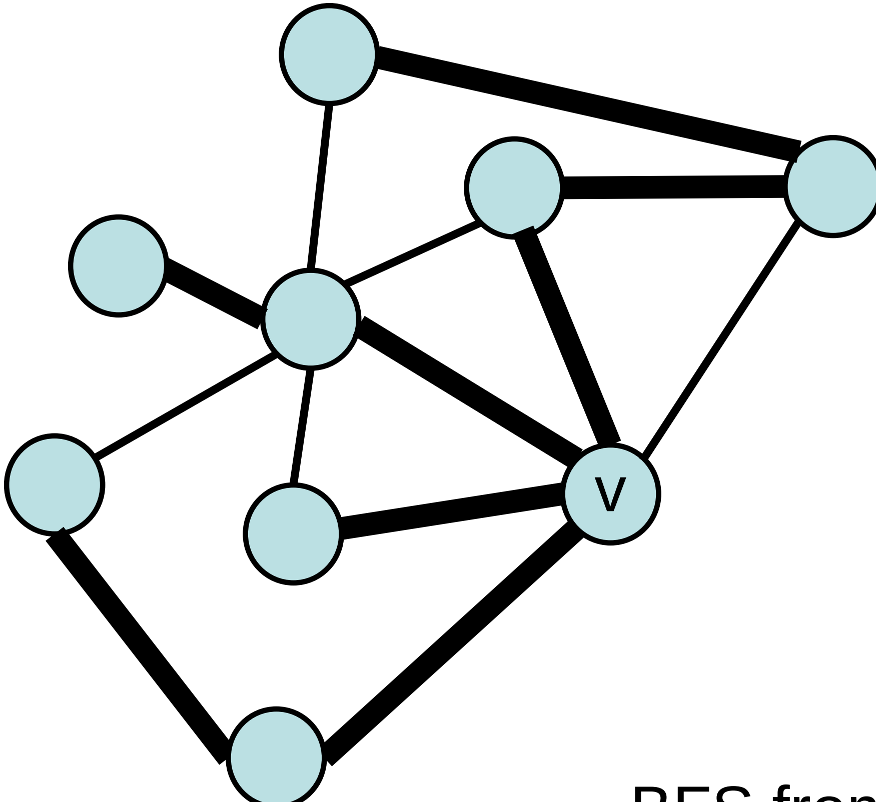
No, but spanning subgraph!

Definitions



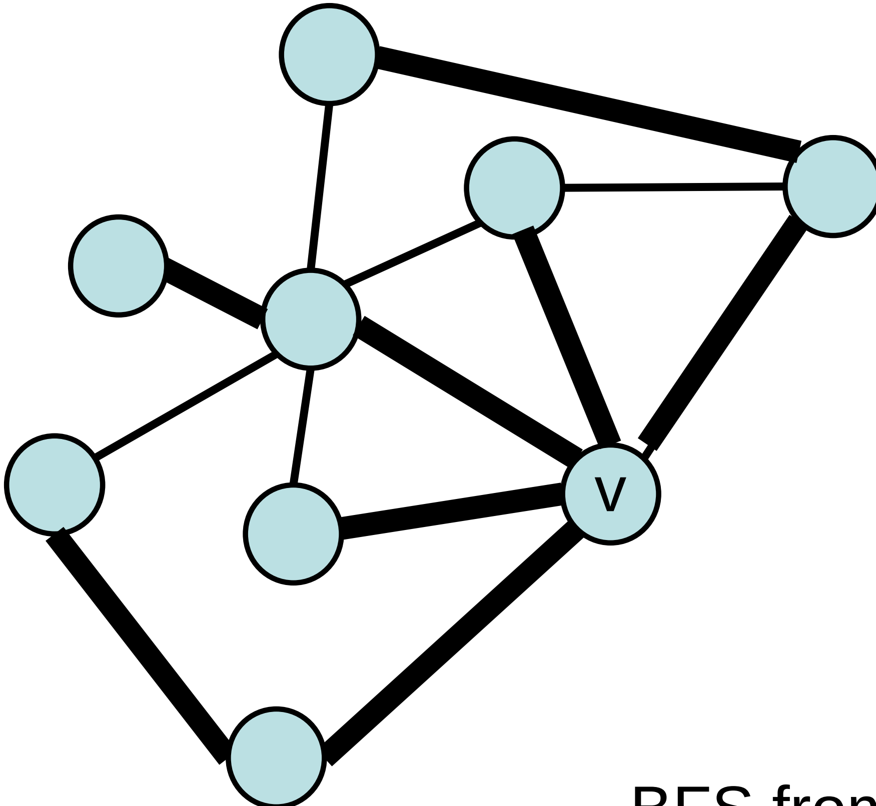
BFS from v?

Definitions



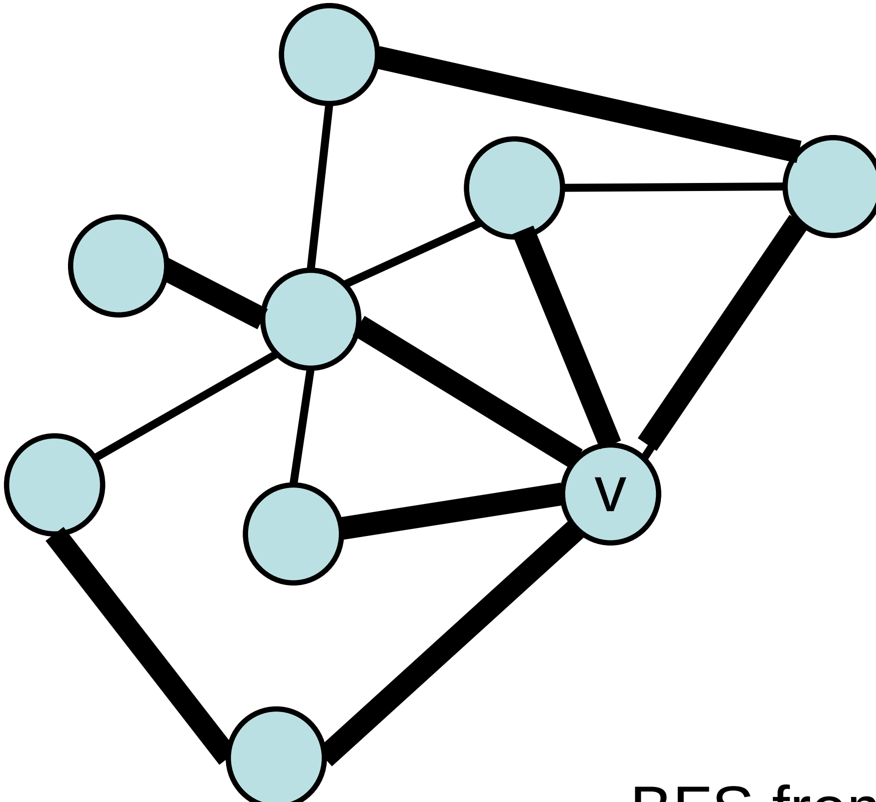
BFS from v? No.

Definitions



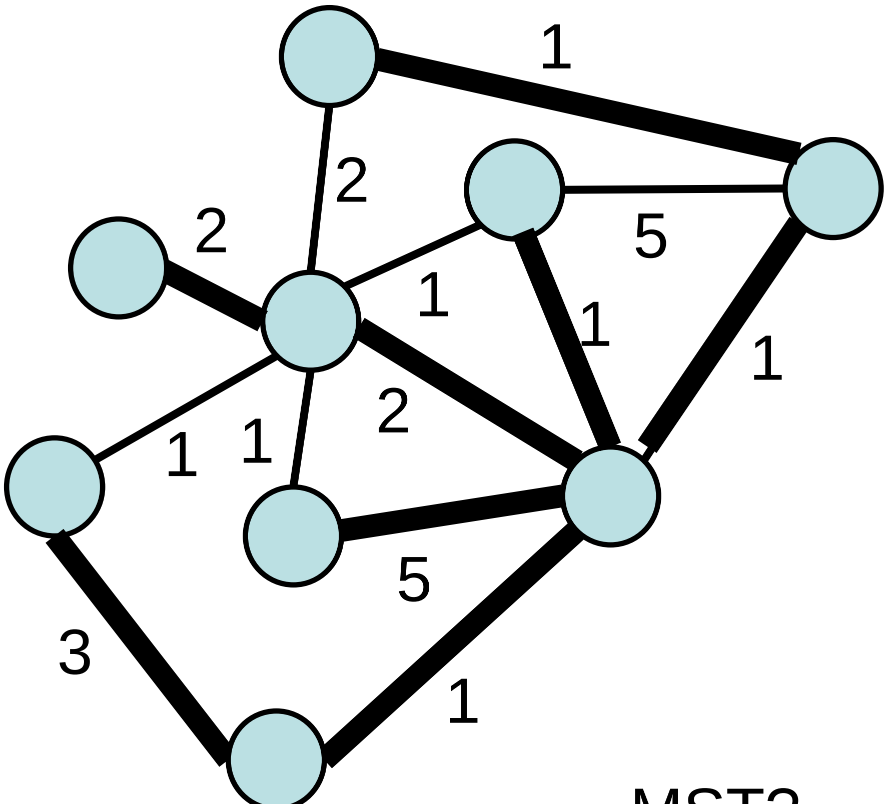
BFS from v?

Definitions



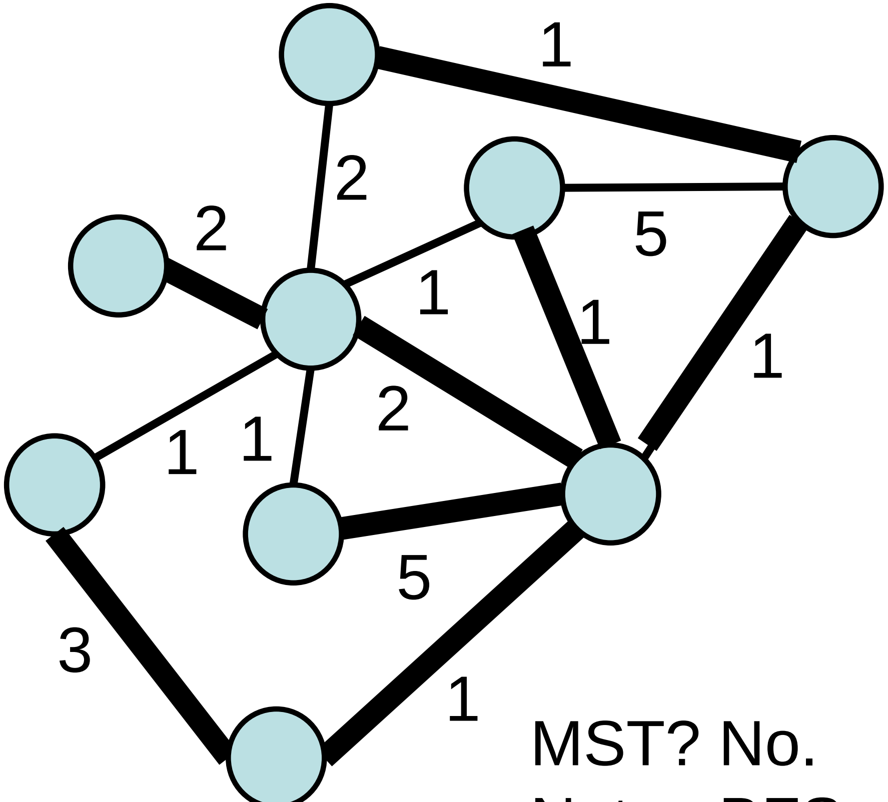
BFS from v? Yes.

Definitions



MST?

Definitions



MST? No.

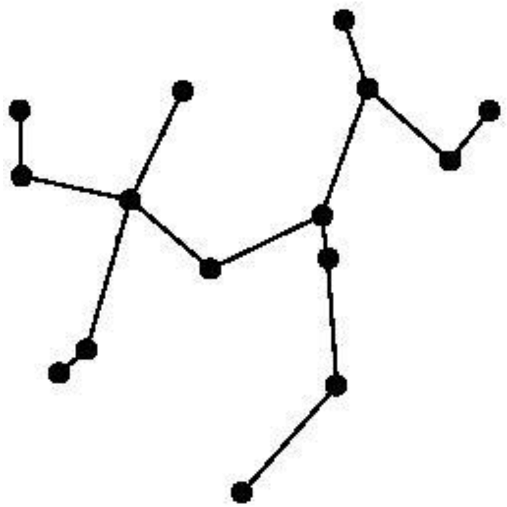
Note: BFS can also be defined wrt weights (shortest path spanning tree)!

System Model

- The network is a clique $G=(V,E,w)$ where $w(e)$ denotes the weight of edge $e \in E$ and $|V|=n$
- Edge weights unique (not critical, can break symmetries by Ids), can be represented with $O(\log n)$ bits
- Each node has a distinct ID of $O(\log n)$ bits
- Each node knows all the edges it is incident to and their weights
- Each node knows about all the other nodes
- The synchronous communication model is used
- Results so far?

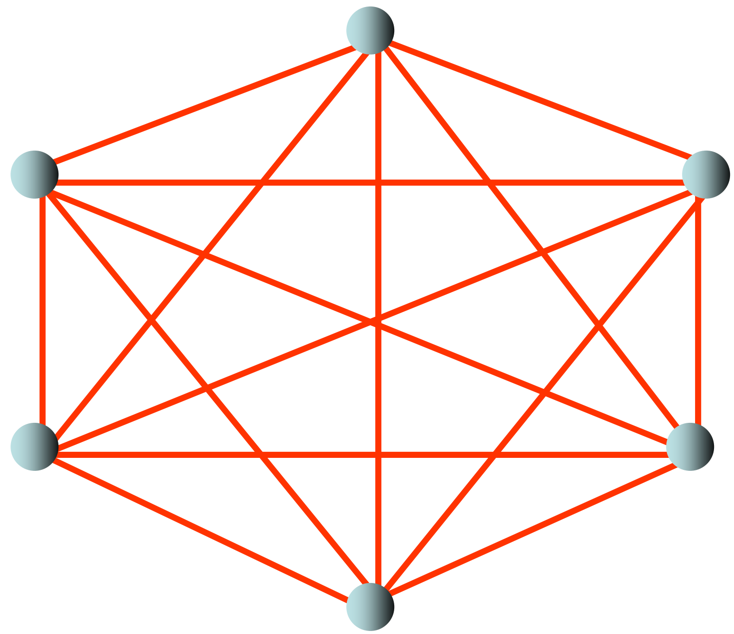
Definitions

- Many algorithms run in phases: in each phase, a subset of the |V|-1 MST edges are chosen. The MST «grows over time»!
- We say that nodes that are directly or indirectly connected by the edges chosen so far belong to the same fragment (or «cluster»)
- Blue edge : lightest edge out of a fragment
- Minimum weight outgoing edge (MWOE) is the edge with the lowest weight incident to *a given node* and leading to *another fragment*. This edge is a blue edge candidate!



MST on Clique

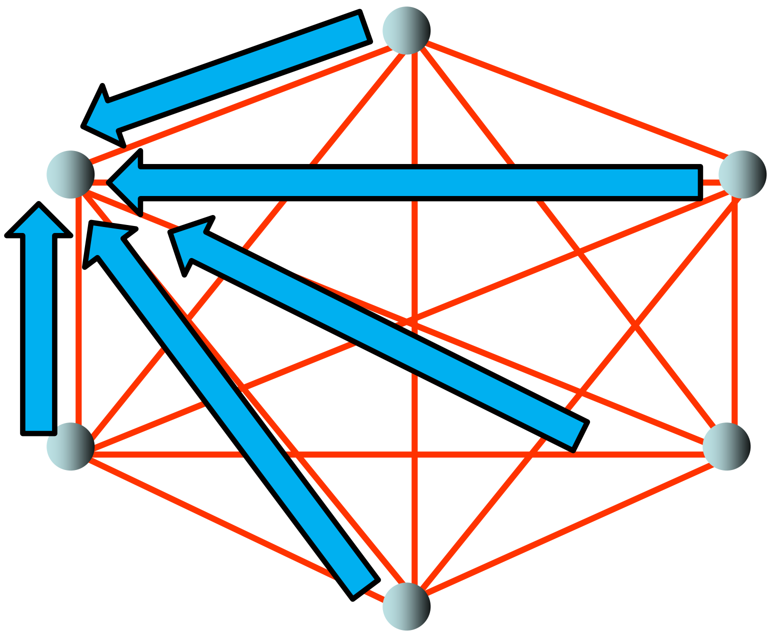
Can we do better on special networks?
E.g., the clique?



MST on Clique

Yes we can:

1. Send all weights to «leader» (or even to all other nodes)
2. Compute solution locally (Prim/Kruskal)
3. Broadcast result



Complexity? Time O(1), Messages O(n), so optimal! ☺

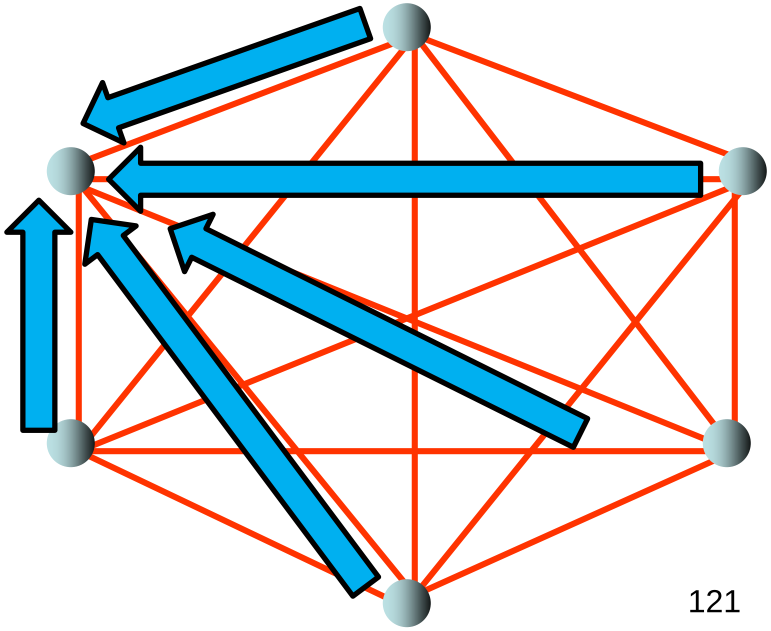
Bounded Message Size: CONGEST Model

Not very scalable! Messages are huge: contain n-1 weights

Message Size

Message size limited to O(log n) bits.
(Assume all variables in network all
of size O(log n), e.g., weights, identifiers, ...)

Simple algorithm required
messages of size O(n log n)
bits! So n rounds to transmit
over a single link.



Note: locality is no longer
the problem, but the
communication!

Bounded Message Size: CONGEST Model

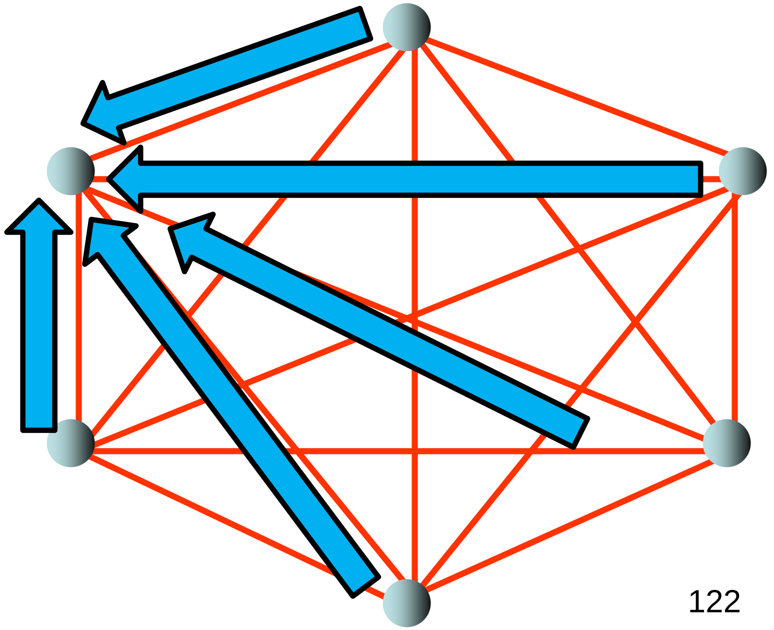
Not very scalable! Messages are huge: contain n-1 weights

Message Size

Message size limited to O(log n) bits.
(Assume all variables in network all
of size O(log n), e.g., weights, identifiers, ...)

So how to do it in time *less than O(n)* with bounded message size? What about GHS algo?

Guess possible performance!



A Simple Algorithm

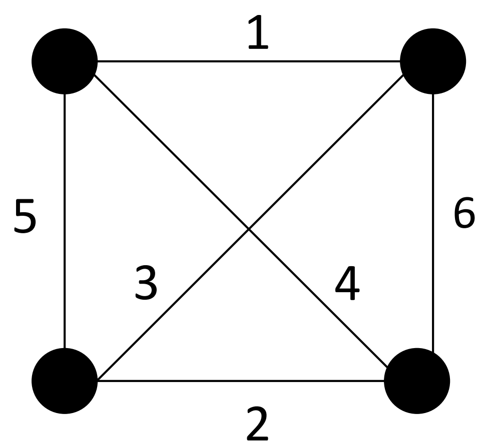
Phase k: Code for node v in fragment F

Input: Set of chosen edges that build node fragments

1. Compute the MWOE of v («blue edge candidate»)
2. Send the MWOE to all nodes in the same fragment
3. Receive messages from the other nodes
4. If own MWOE is the lightest in fragment, then broadcast it to all other nodes in the clique and add this edge. So all other nodes always know which fragments there are / merge currently!
5. Receive other broadcast messages and add those edges as well

A Simple Algorithm

Example: How will it proceed?



Phase k: Code for node v in cluster F

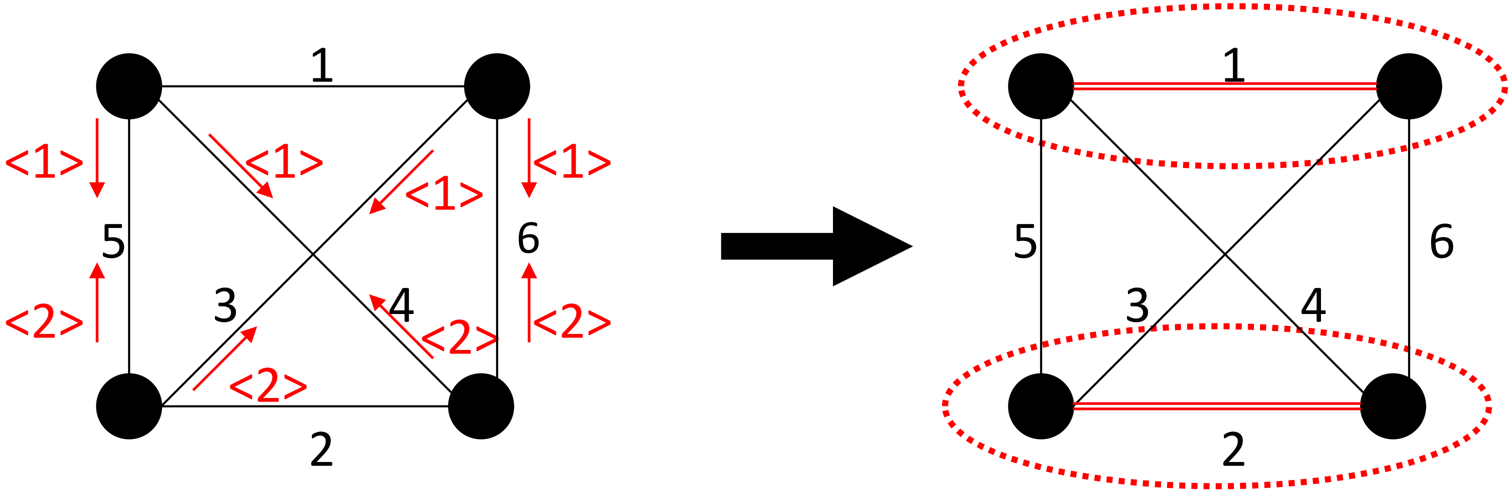
Input: Set of chosen edges that build node clusters

1. Compute the MWOE («blue edge candidate»)
2. Send the MWOE to all nodes in the same cluster
3. Receive messages from the other nodes
4. If own MWOE is the lightest, then broadcast it to all other nodes in the clique and add this edge. So all other nodes always know which clusters there are / merge currently!
5. Receive other broadcast messages and add those edges as well

A Simple Algorithm

Example:

Round 1:



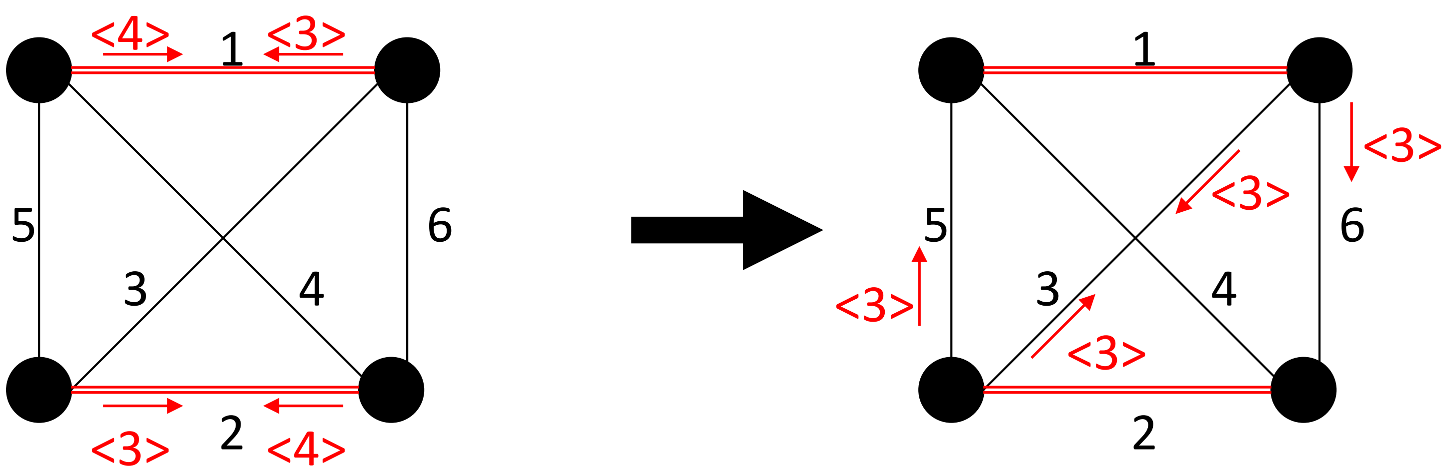
Single node component: Broadcast the lightest edge to all fragments = neighbors. Here, the lightest incident edge of each individual node is a blue edge!

Add edges and update fragment

A Simple Algorithm

Example:

Round 2:



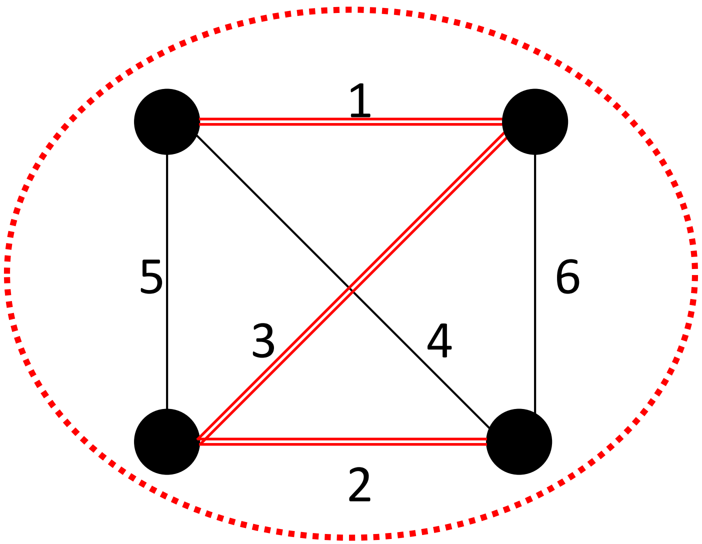
Send MWOE edge to all nodes in the same fragment

Broadcast the lightest edge to all other nodes in remaining fragments.

A Simple Algorithm

Example:

Round 2:



Add edges and update fragments

Why correct?

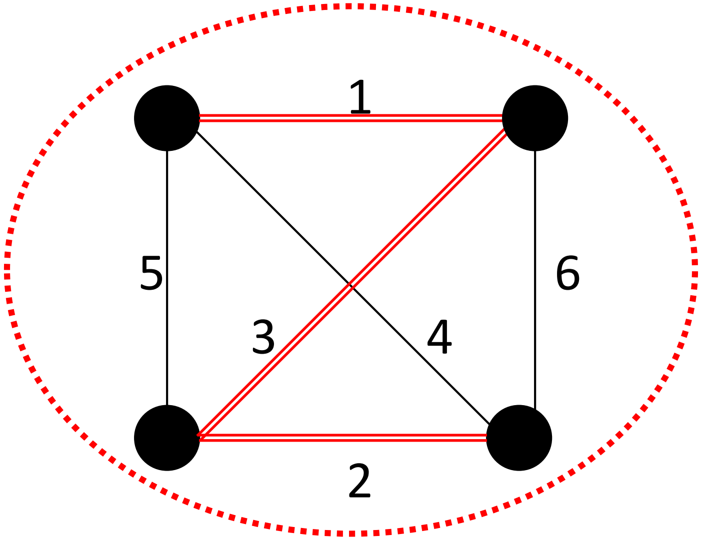
-

What is runtime?

A Simple Algorithm

Example:

Round 2:



Add edges and update fragments

Why correct?

- By blue edge rule, MST will emerge
- No message too large: one weight per edge and time (recall: it is a *clique*)

What is runtime?

Since the minimum fragment size **doubles** in each round, the algorithm computes the MST in O(log n) rounds!

Can we do better?

Note: we did not exploit fact that we can send different messages to different neighbors! But could be exploited to speed up!

Fast MST Algorithm: General Idea

- To reduce the number of rounds, fragments have to **grow faster**!
- In our simple algorithm, we used the MWOE / blue edge of each fragment to merge clusters.
- With this approach, the minimum fragment size **doubled** in each phase.
- Idea how to speed it up?
- What if we could use the k lightest outgoing edges of each fragment, where k is the fragment size?

Impact of Fragment Growth

"What if we could use the k lightest outgoing edges of each fragment, where k is the fragment size?"

- Cluster of size k merges with k clusters of size k, so next fragment of quadratic size k*k (in contrast to 2*k so far)
- So:
  - $1=2^0$, $2=2^1$, $4=2^2$, $16=2^4$, $256=2^8$ ....
- In general:
  - After i steps, 2^{2^i} in contrast to 2^i so far.
- How fast is this?
  - $n = 2^{2^i} \Leftrightarrow \log n = 2^i \Leftrightarrow \log \log n = i$

Impact of Fragment Growth

Let β_k denote the minimum fragment size after phase k, then it holds for our simple algorithm that

$$\beta_{k+1} \geq 2 \cdot \beta_k$$

and $\quad \beta_0 := 1$

thus $\quad \beta_k \geq 2^k \rightarrow k \in O(\log n)$

Impact of Fragment Growth

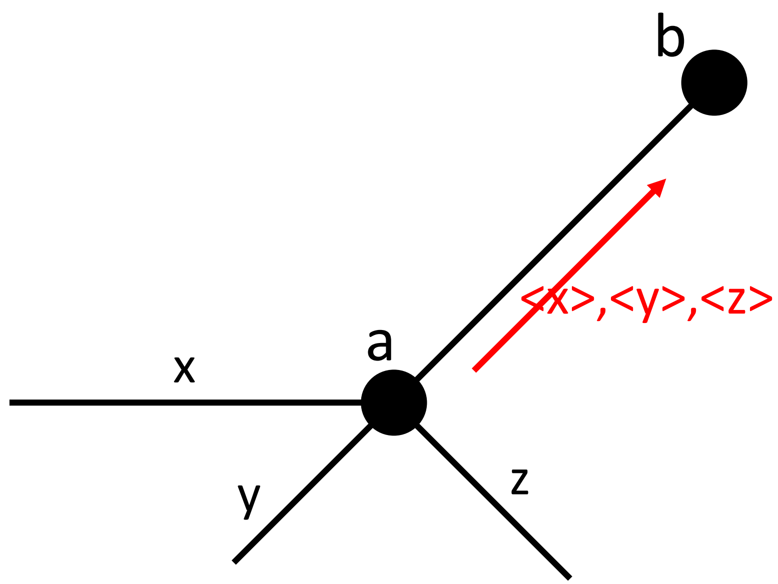
- We will derive an algorithm for which it holds that:

$$\beta_{k+1} \geq \beta_k \cdot (\beta_k + 1)$$

- Thus the fragment sizes grow quadratically as opposed to merely double
- In order to achieve such a rate, information has to be spread faster!
- We will use a simple trick for that...

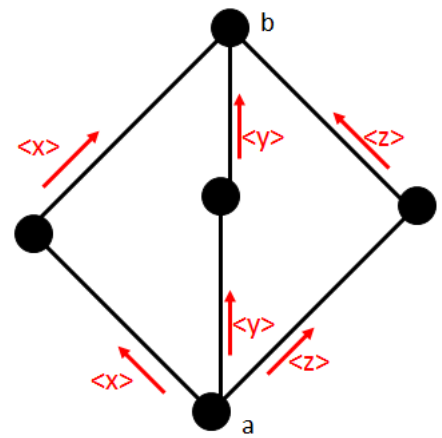
A Trick to Avoid Link Congestion

- Link capacity bounded: we cannot send much information over a single link...



A Trick to Avoid Link Congestion

- However, much information can be sent from different nodes to a particular node v_0!



We will use this trick twice in our algorithm!

- A node can simply send parts of the information that it wants to transmit to a specific node to some other nodes. These nodes can send all parts to the specific node in one step!

- This can be used to **share workload**!

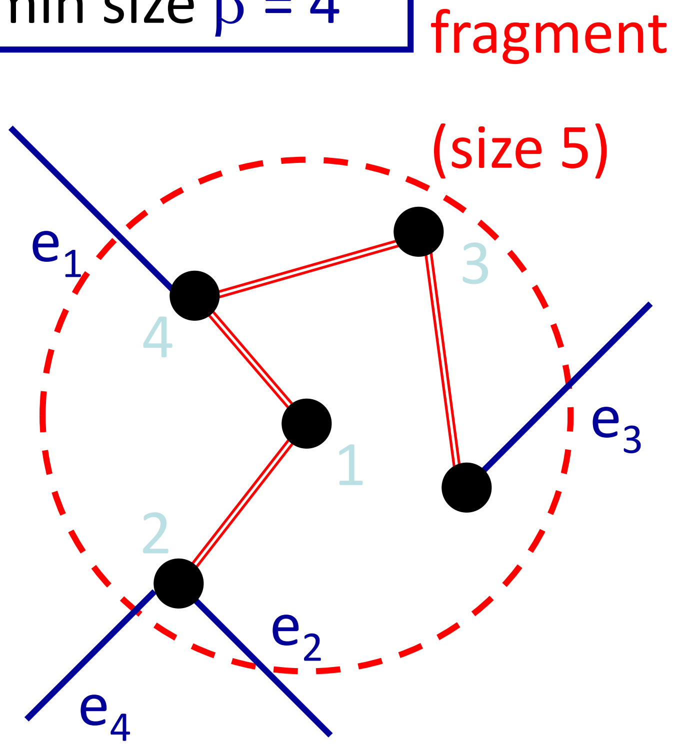
Fast MST Algorithm: General Idea

- Our new algorithm will execute the following steps in each phase.
- Let β be the minimal fragment size (the actual size can be larger!)

1. Each fragment computes the β lightest edges $e_1,\ldots,e_\beta$ to other distinct fragments

2. Assign at most one of those lightest edges to the members of the fragment («responsible» for this edge)!

min size $\beta = 4$

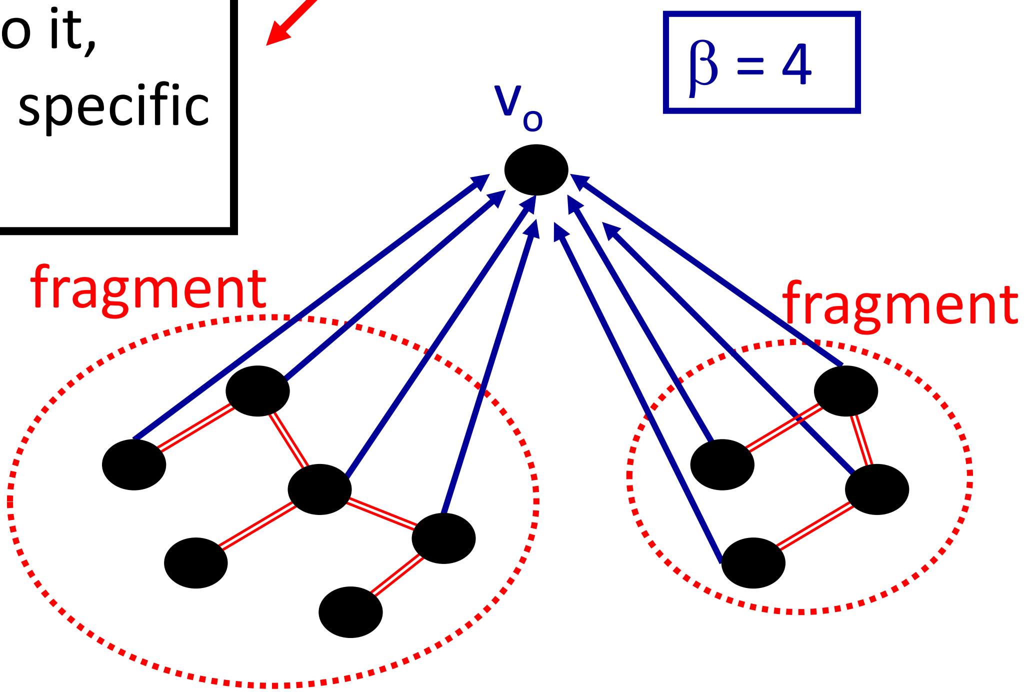


Fast MST Algorithm: General Idea

3. Each node with an edge <v,u,w({v,u})> assigned to it, sends <v,u,w({v,u})> to a specific node v_0 (global!)

Step 2 and 3 together is exactly our trick!

4. Node v_0 computes the lightest edges that can be *safely added* to the spanning tree (no cycle)

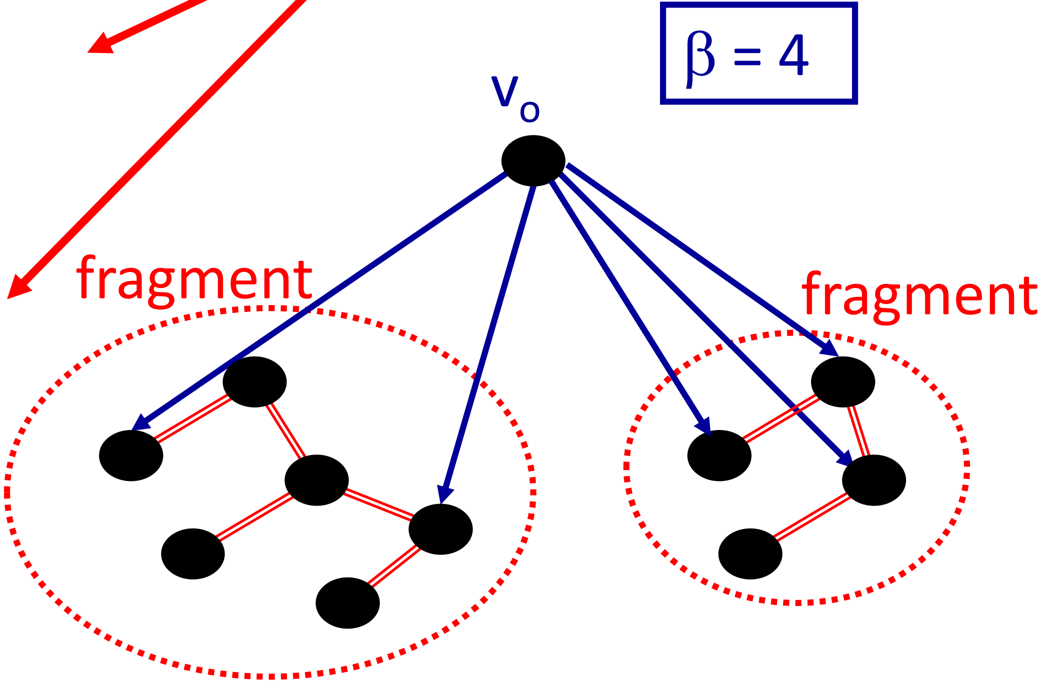


Fast MST Algorithm: General Idea

Our trick again!

5. Node v_0 sends a message to a node, if its assigned edge is added to the spanning tree

6. Each node, that received a message, broadcasts it to all other nodes (→ All nodes have to know about all added edges and new fragments!)



Fast MST Algorithm: General Idea

- This way, more edges can be added in one phase!
- However, how does it really work?
- There are a few obvious problems...

Fast MST Algorithm: General Idea

First problem:

• How to compute the β lightest outgoing edges of a specific fragment?

→ Not so difficult: procedure Cheap_Out (idea?)

Fast MST Algorithm: General Idea

Second problem:

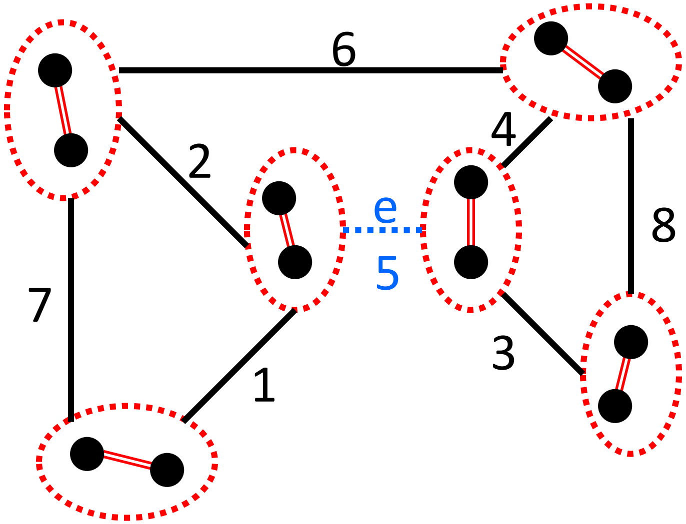
- How can the designated node v_0 know which edges can be added **safely**?
- Let's illustrate this problem with an example graph!

Fragment Cycle Detection

- In our example:

  $|V| = n = 12$

  $\beta = 2$ (minimum fragment size)

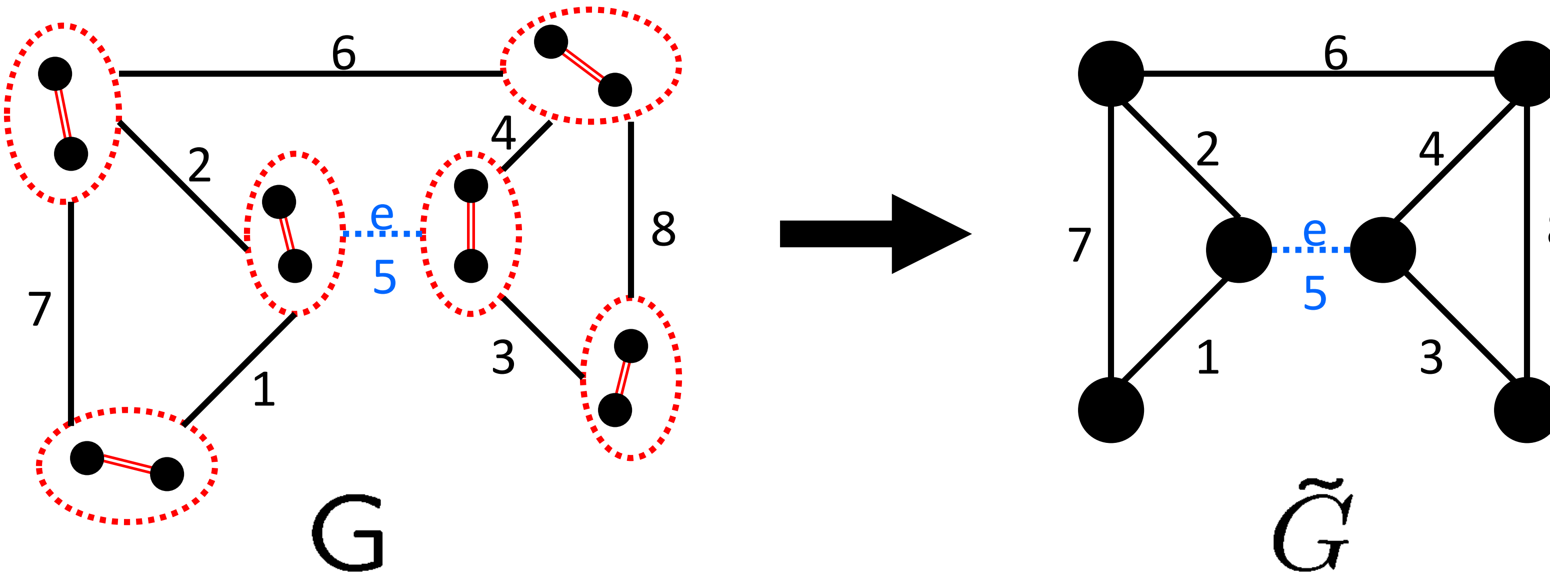


- Which are the lightest $\beta = 2$ outgoing edges of the fragments?
- All edges except for edge e! So this is the picture of the designated node v_0 after receiving the $\beta = 2$ lightest outgoing edges of each fragment
- v_0 does not know about the edge e! It is the 3rd lightest edge of both adjacent nodes!

Fragment Cycle Detection

v_0 can construct a logical graph:

its nodes are the fragments and its edges are the $\beta = 2$ lightest outgoing edges



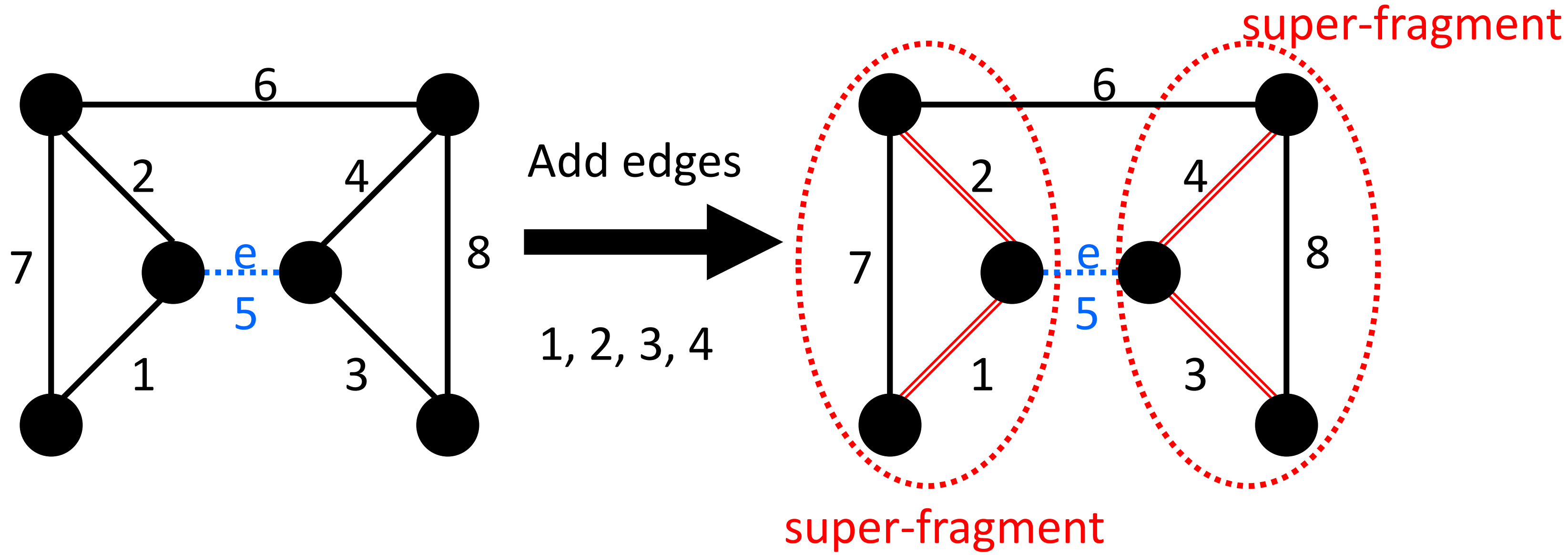
Fragment Cycle Detection

Based on the knowledge of the $\beta = 2$ lightest outgoing edges, v_0 can locally merge nodes of the logical graph into fragments.

Can for sure take the MWOE / blue edges of each component!

So build super-fragments by connecting any pair with the lightest connecting edge: edges with weights 1, 2, 3 and 4 are safe.

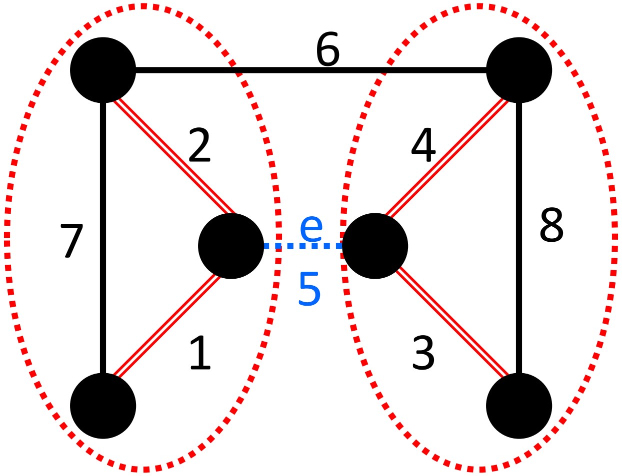
But can I continue adding edges in ascending order of weight, as long as no loop occurs?



Fragment Cycle Detection

Can I add edge with weight 6? Loop-free in logical graph!

No! The edge is not part of the MST! Not a blue edge between the components!



The problem is that in **both** (super)fragments at least one of the nodes has already used up all of its β outgoing edges.
The (β+1)th outgoing edge might be lighter than other edges, but does not appear in logical graph!

So, when is it safe to add an edge?

Fast MST Algorithm: The Algorithm Step by Step

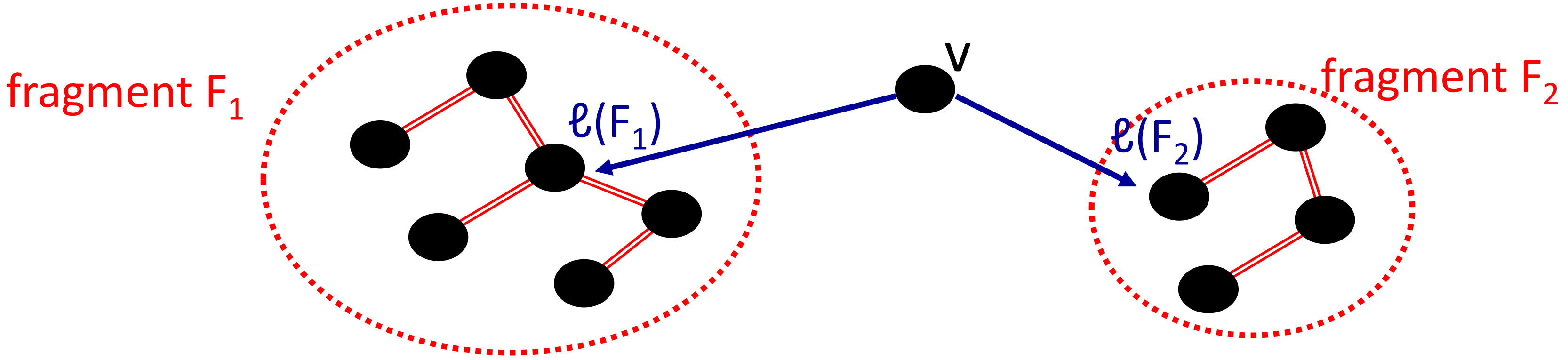
- Let's put everything together and solve the open problems!
- Initially, each node is itself a fragment of size 1 and no edges are selected.
- The algorithm consists of 6 steps. Each step can be performed in constant time (communication round).
- All 6 steps together build one phase of the algorithm, thus the time complexity of one phase is $O(1)$.
- A specific node in each fragment F, e.g. the node with the smallest ID, is considered the leader $\ell(F)$ of the fragment F (since our algorithm ensures that nodes know fragment, they also know leader).

Fast MST Algorithm: The Algorithm Step by Step

Step 1

a) Each node v computes the minimum-weight edge e(v,F) that connects v to any node of some other fragment F, for all other fragments F

b) Each node v sends e(v,F) to the leader ℓ(F) for all fragments other than the own fragment



Fast MST Algorithm: The Algorithm Step by Step

Procedure Cheap_Out

Code for the leader of fragment F

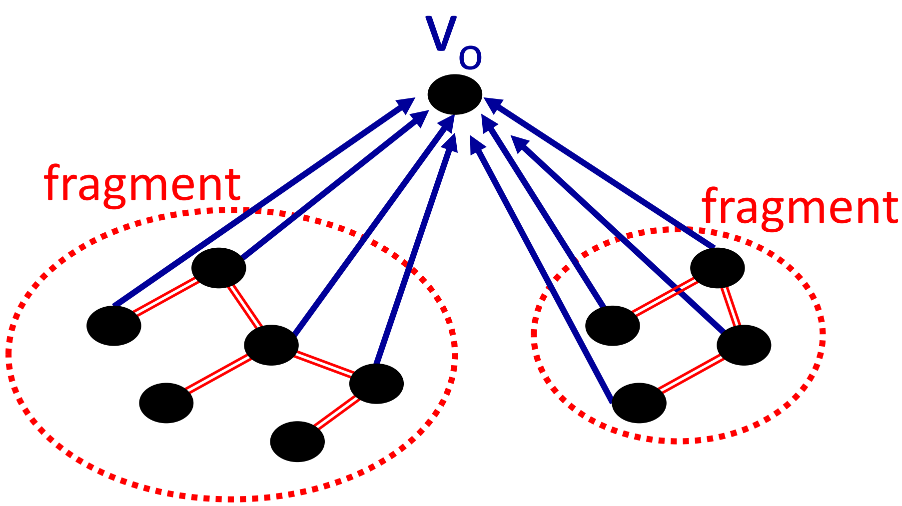
Input: Lightest edge e(F,F') for every other fragment F'

1. Sort the input edges in increasing order of weight
2. Define β = min{|F|, <# of fragments>}
   (Size of own fragment as long as many other fragments. Why not more? Cannot communicate so much info!)
3. Choose the first β nodes of the sorted list
4. Appoint the node with the i-th largest ID as the guardian of the i-th edge, for i = 1,...,β
5. Send a message about the edge to the node it is appointed to

Fast MST Algorithm: The Algorithm Step by Step

Step 3

All nodes that are guardians for a specific edge send a message to the designated node v_0, e.g. the node with the smallest ID in the whole graph



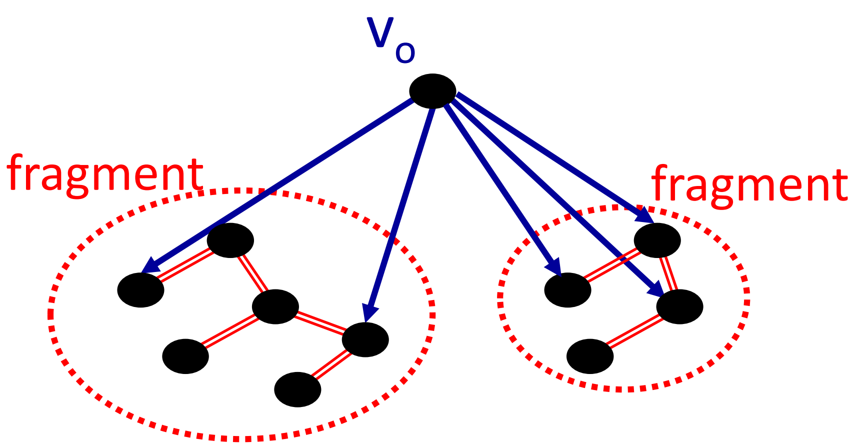
v_0 knows the β lightest outgoing edges of each fragment!

Fast MST Algorithm: The Algorithm Step by Step

Step 4

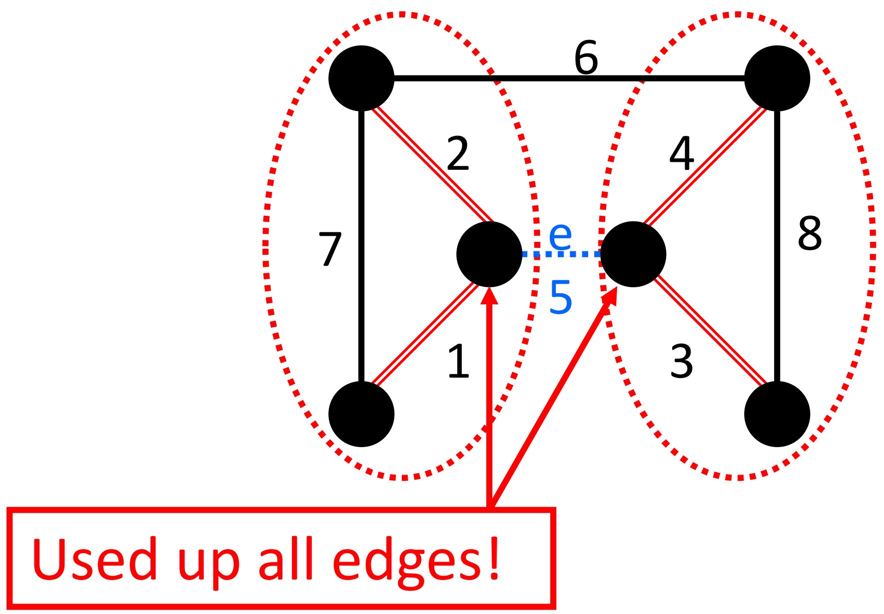
a) v_0 locally performs procedure Const_Frags → Computes the edges to be added

b) For all added edges, v_0 sends a message to g(e)



Fast MST Algorithm: The Algorithm Step by Step

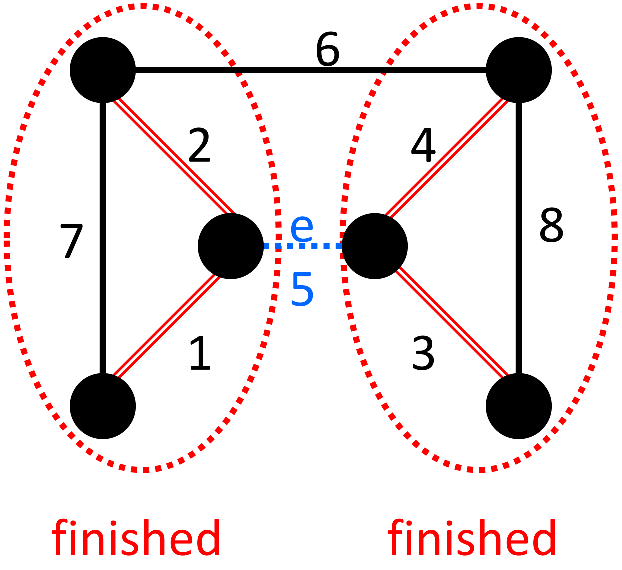
- How does Const_Frags work?
- We have seen: problem occurs when all β outgoing edges of a fragment are used up!
- More precisely, a problem occurs **only if** there is at least one "full" fragment **in each** of the two super-fragments which are supposed to be merged! (Otherwise we would see the edge.)



Fast MST Algorithm: The Algorithm Step by Step

- How does Const_Frags work?
- We call a super-cluster containing a cluster that used up all of its β edges finished ("not safe").

If an edge is the lightest outgoing edge of one super-fragment that is **not finished**, then it is still safe to add it, no matter if the other super-fragment is finished, since we are sure that there is no better edge to connect the unfinished super-fragment to other fragments.



Fast MST Algorithm: The Algorithm Step by Step

Procedure Const_Frags

Code for the designated node v_0

Input: the β lightest outgoing edges of each fragment

1. Construct the logical graph
2. Sort the input edges in increasing order of weight
3. Go through the list, starting with the lightest edge:

    If the edge can be added without creating a cycle then

        add it

    else

        drop it

Fast MST Algorithm: The Algorithm Step by Step

Procedure Const_Frags

If two (super-)fragments are merged, then the new super-fragment is declared finished if

- the edge is the heaviest edge of a fragment in any of the two super-fragment or
- any of the two super-fragments is already finished.

If the edge is dropped (→ both clusters already belong to the same super-fragment), then the super-fragment is declared finished if

- the edge is the heaviest edge of any of the two fragments

Note: If a super-fragment is declared finished then it will remain finished until the end of the phase.

Final Step

All edges between finished super-clusters are **deleted** (before looking at the next lightest edge)

Fast MST Algorithm: The Algorithm Step by Step

Step 5

All nodes that received a message from v_0 broadcast their edge to all other nodes

Step 6

Each node adds all edges and computes the new fragments (complete view!).

If the number of clusters is greater than 1, then the next phase starts.

Fast MST Algorithm: The Algorithm Step by Step

The entire algorithm for node v in fragment F

1. Compute the minimum-weight edge e(v,F') that connects v to fragment F' and send it to ℓ(F') for all fragments F' ≠ F

2. if v = ℓ(F): Compute lightest edge between F and every other fragment. Perform Cheap_Out

3. if v = g(e) for some edge e: Send <e> to v_0

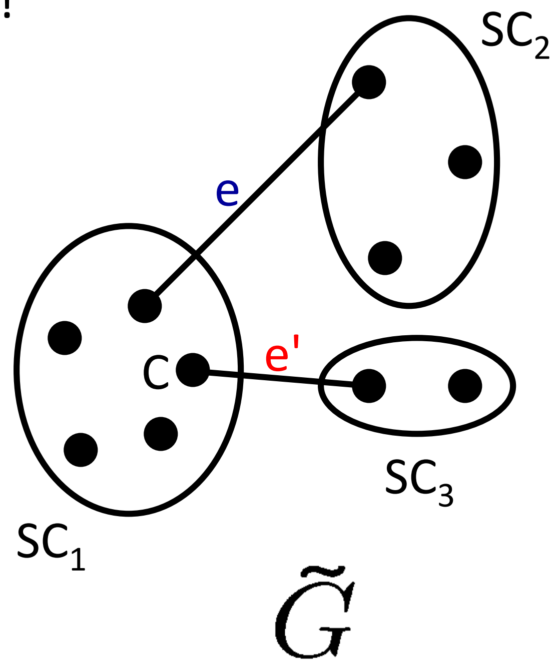
4. if v = v_0: Perform Const_Frags. Send message to g(e) for each added edge e

5. if v received a message from v_0: Broadcast it

6. Add all received edges and compute the new fragments

Analysis: Correctness

- Assume edge e is used to merge super-fragment SC_1 and SC_2. W.l.o.g., assume that SC_1 is not finished and that e is one of the β lightest outgoing edges of its fragment.
- We will show now that e is the MWOE of SC_1!
- Assume that there is a lighter outgoing edge e' (w(e') < w(e)), incident to a fragment C that connects super-fragment SC_1 to super-fragment SC_3.



Analysis: Correctness

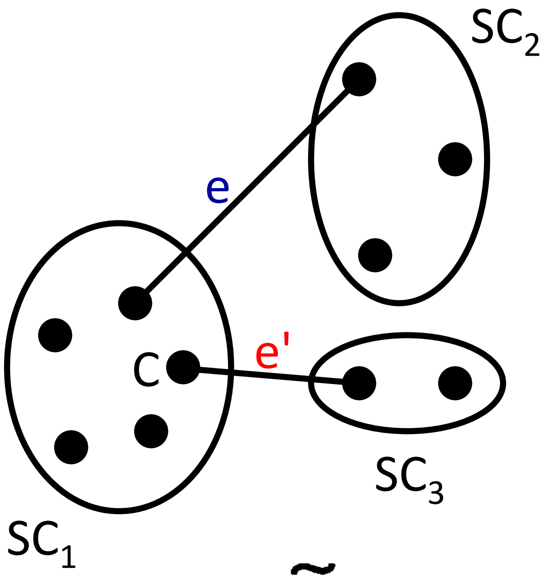
- It suffices to show that whenever an edge is added, it is part of the MST → We only have to analyze Const_Frags!

- Proof [Sketch]: We only have to show that we always add the lightest outgoing edge of each super-fragment. This is always the right choice!

Analysis: Correctness

- Case 1: e' is among the β lightest outgoing edges of its fragment C.

→ Since w(e') < w(e), e' must have been considered before e, thus either SC_1 and SC_3 have been merged before or e' was dropped because $SC_1 = SC_3$. Either way, e' cannot be an outgoing edge when the algorithm adds e.
→ Contradiction!



$\tilde{G}$

● is a fragment

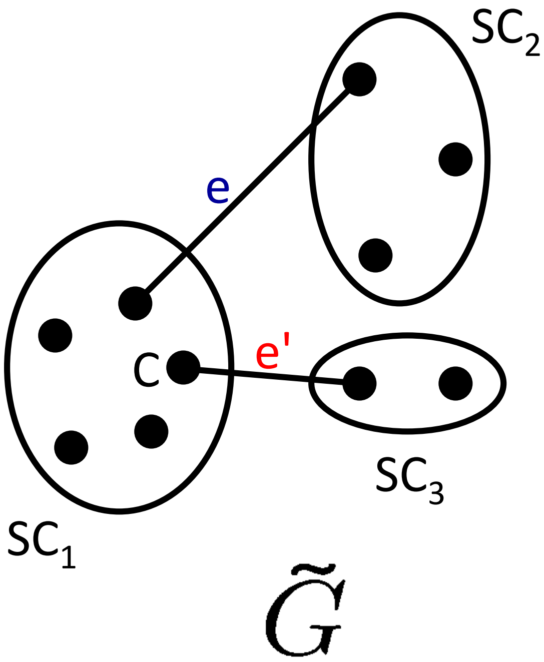
Analysis: Correctness

- Case 2: e' is **not** among the β lightest outgoing edges of its fragment C.

- Case 2.1: There is an edge e'' among the β lightest outgoing edges from fragment C leading to the same fragment C'.

It follows that w(e'') < w(e'). Since $SC_1 \neq SC_3$, e'' has not been considered yet, thus w(e) < w(e'').
Hence we have that w(e) < w(e').

→ Contradiction!

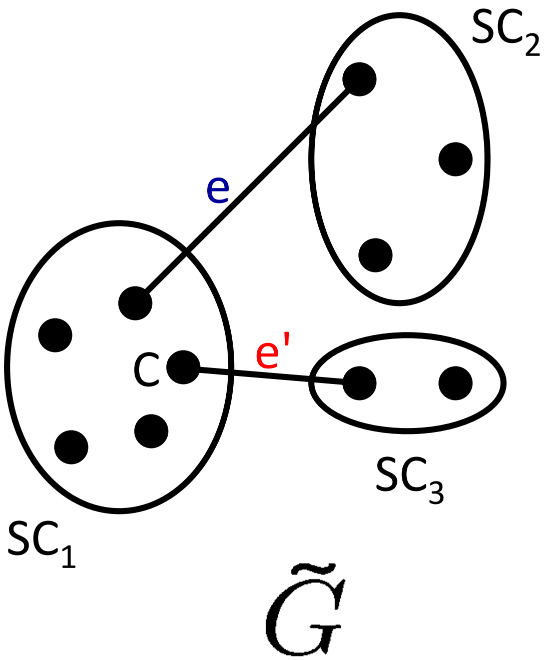


Analysis: Correctness

- Case 2: e' is **not** among the β lightest outgoing edges of its fragment C.
- Case 2.2: None of the β lightest outgoing edges of C lead to C'.

Thus, all β outgoing edges have lower weights than e', also the heaviest of these edges e'', i.e., w(e'') < w(e') < w(e). Hence, edge e'' must have been inspected already. Since it is the heaviest (last) edge of some fragment, SC_1 must now be finished.

→ Contradiction!



■

Analysis: Time Complexity

- Each phase requires O(1) rounds, but how many phases are required until termination?

- Reminder: β_k denotes the minimum fragment size in phase k.

Lemma 2: It holds that

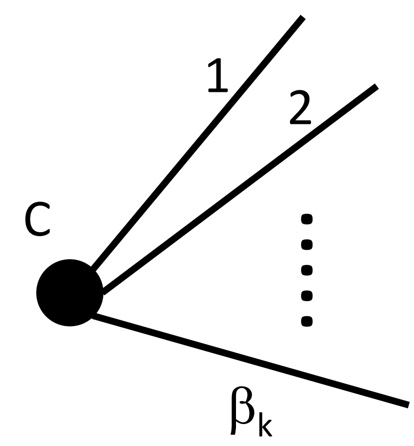
$$\beta_{k+1} \geq \beta_k(\beta_k+1).$$

Analysis: Time Complexity

- Proof [Sketch]:

We prove a stronger claim: Whenever a super-fragment is declared finished in phase k+1, it contains at least β_k+1 fragments.

- Each fragment has (at least) β_k outgoing edges in phase k+1, since β_k is the minimum fragment size after phase k.

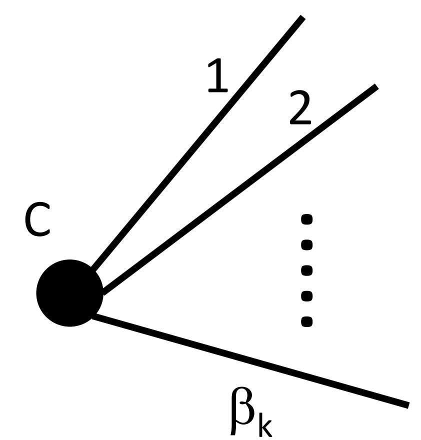


Analysis: Time Complexity

- Case 1: The super-fragment is declared finished after one of its fragment has used up all of its β_k outgoing edges. Let C be this fragment.

- Let's call those edges 1, 2, ..., β_k leading to the fragments $C_1, C_2, \ldots, C_\beta$.

- If the inspection of an edge **does not** result in a merge, then the clusters already belong to the same super-fragment! If there is a merge, then they belong to the same super-fragment afterwards.

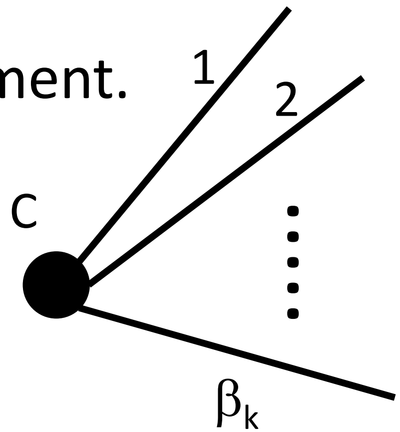


Analysis: Time Complexity

• Thus, at the end, the super-fragment contains at least $C, C_1, C_2, \ldots, C_\beta$!
→ The super-fragment contains at least β_k+1 fragments.

• Case 2: The super-fragment is declared finished after merging with an already finished super-fragment.

• Using an inductive argument, the finished super-fragment must already contain at least β_k+1 clusters, since one of its clusters has used up all of its β_k edges.



Analysis: Time Complexity

Theorem 1: The time complexity is O(log log n) rounds.

- Proof: According to Lemma 2, it holds that $\beta_{k+1} \geq \beta_k(\beta_k+1)$. Furthermore, we have that $\beta_0 := 1$. Hence it follows that

$$\beta_k \geq 2^{2^{k-1}}$$

for every $k \geq 1$. Since $\beta_k \leq n$, it follows that $k \leq \log(\log n)+1$.
Since each phase requires O(1) rounds, the time complexity is O(log log n).

Analysis: Time Complexity

Theorem 2: The message complexity is $O(n^2 \log n)$.

Number of **bits**!

- The proof is simple: Count the messages exchanged in Steps 1, 3, 4, and 5. We will not do this here.

- Adler et al. showed that the minimum number of bits required to solve the MST problem in this model is $\Omega(n^2 \log n)$. Thus, this algorithm is asymptotically optimal!

Overview

I. Introduction

II. Previous Results

III. Fast MST Algorithm

IV. Analysis

V. Summary

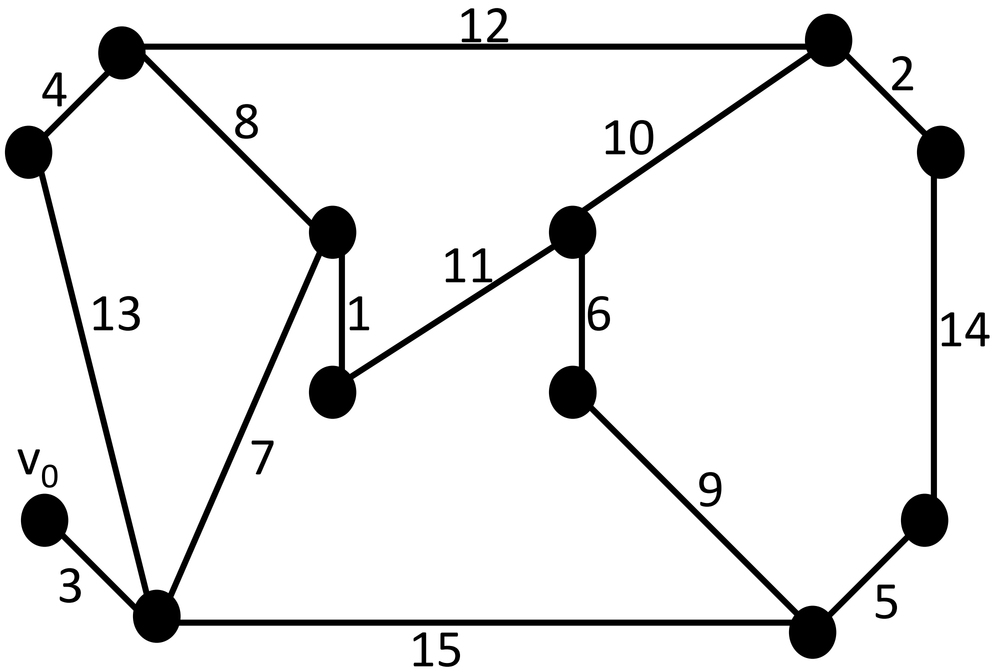
- Results & Conclusions

VI. Extensive Example

Summary: Conclusions

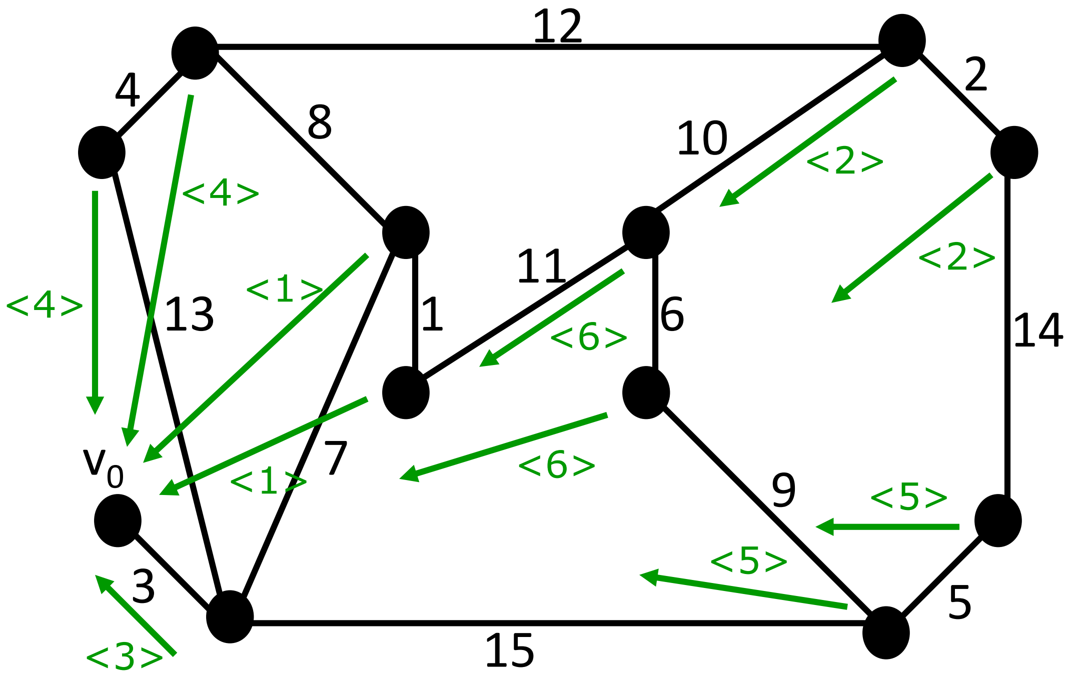
"An obvious question we leave open is whether the algorithm can be improved, or whether there is an inherent lower bound of $\Omega(\log \log n)$ on the number of communication rounds required to construct an MST in this model."

Extensive Example: The Graph



All other edges are heavier!!!

Extensive Example: Phase 1

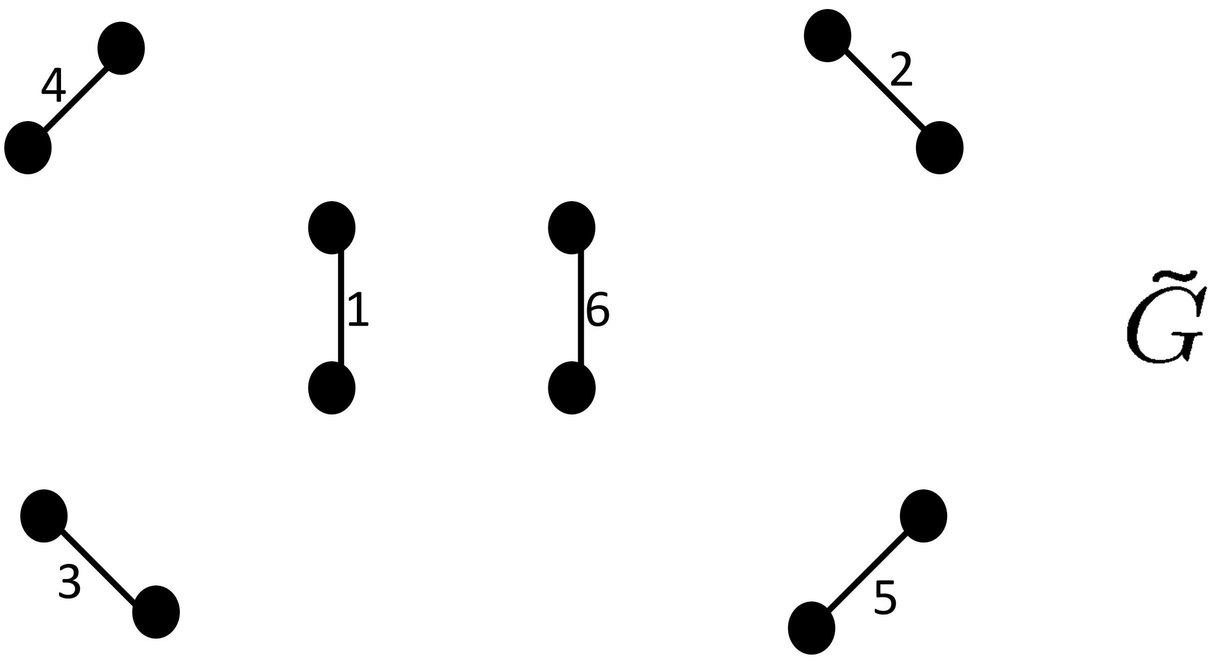


1. Not necessary
2. Not necessary
3. Send MWOE to v_0
4. Const_Frags!

Extensive Example: Phase 1

Const_Frags

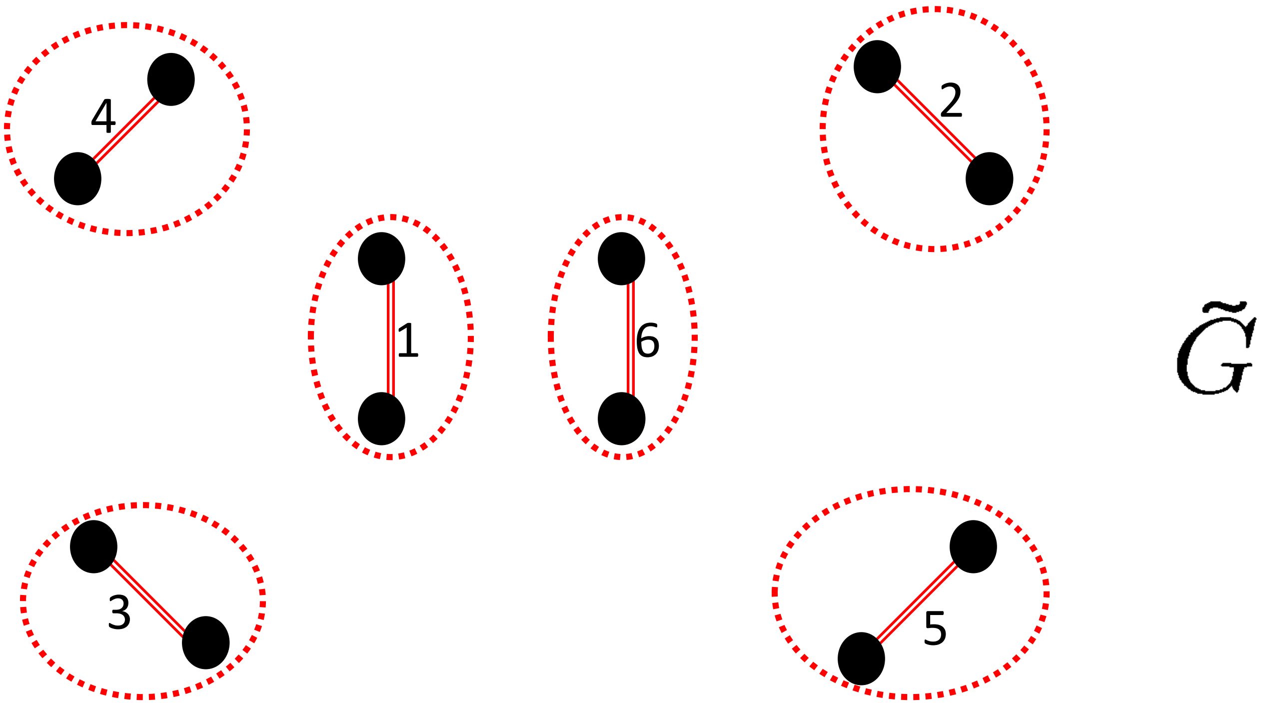
1. Construct logical graph



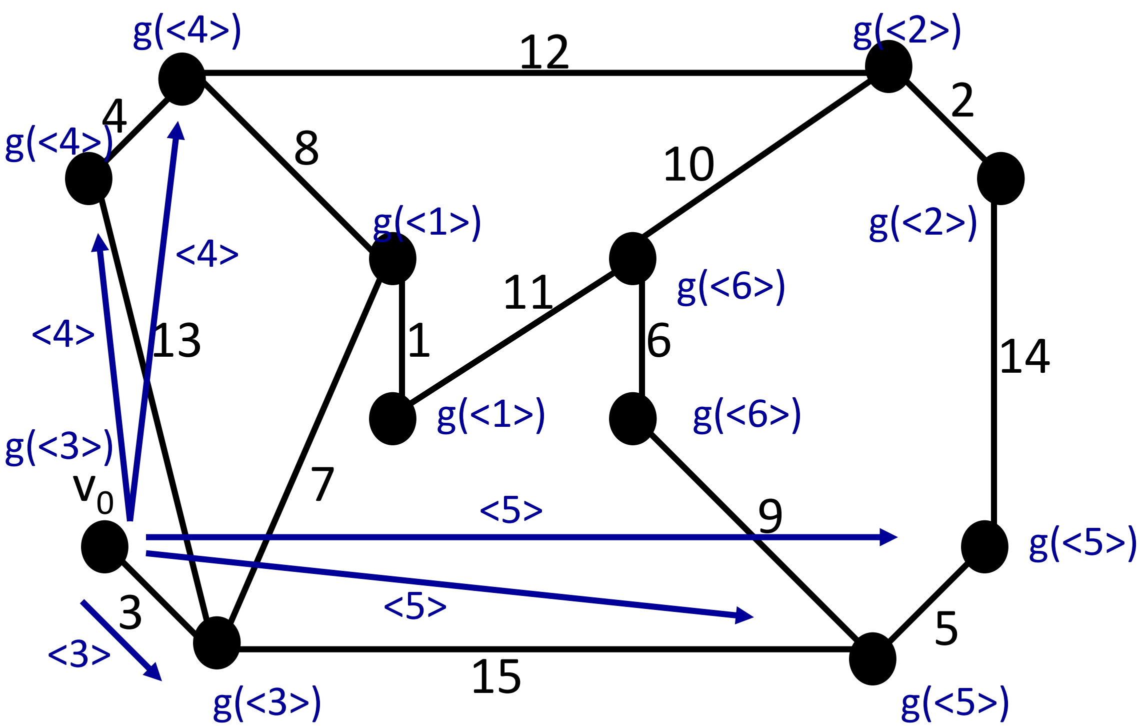
Extensive Example: Phase 1

Const_Frags

2. Add edges



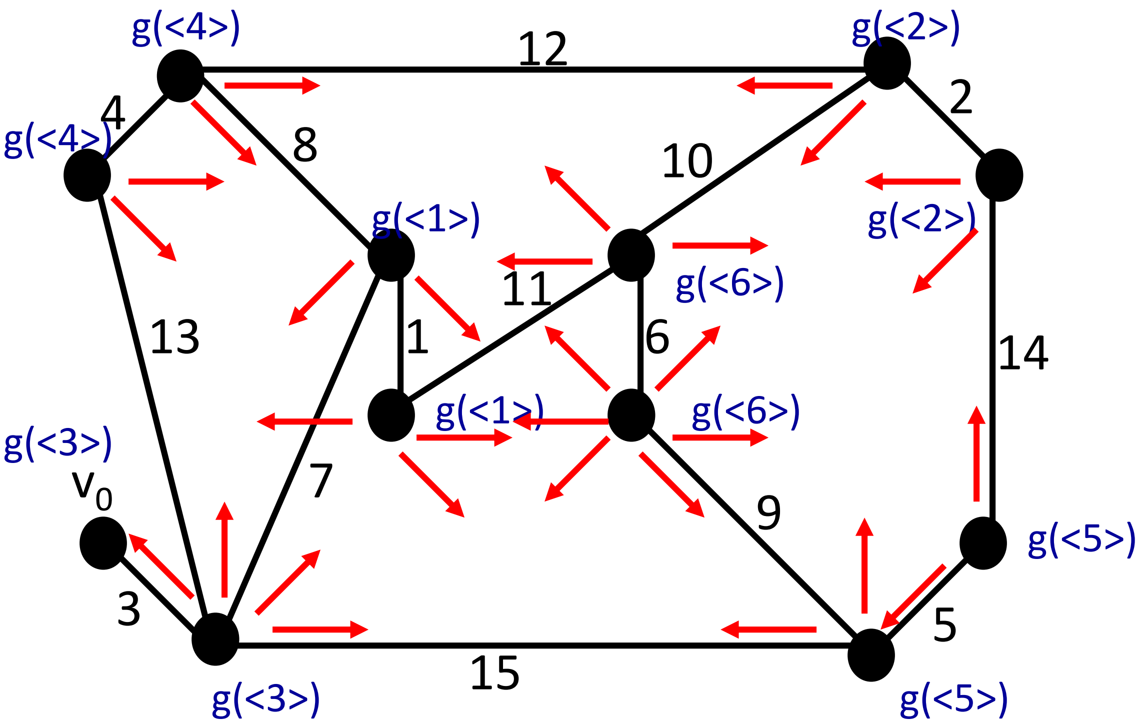
Extensive Example: Phase 1



1. Not necessary
2. Not necessary
3. Send MWOE to v_0
4. Const_Frags!
5. Send e to g(e)

Only some messages are displayed

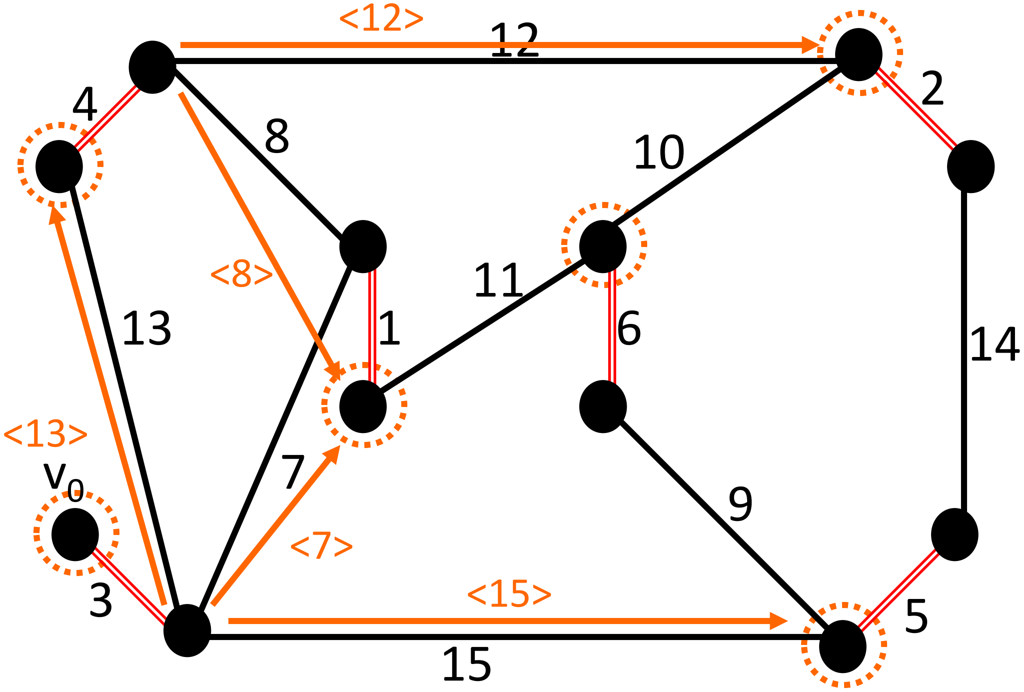
Extensive Example: Phase 1



1. Not necessary
2. Not necessary
3. Send MWOE to v_0
4. Const_Frags!
5. Send e to g(e)
6. Broadcast e and update the fragments

Only some messages are displayed

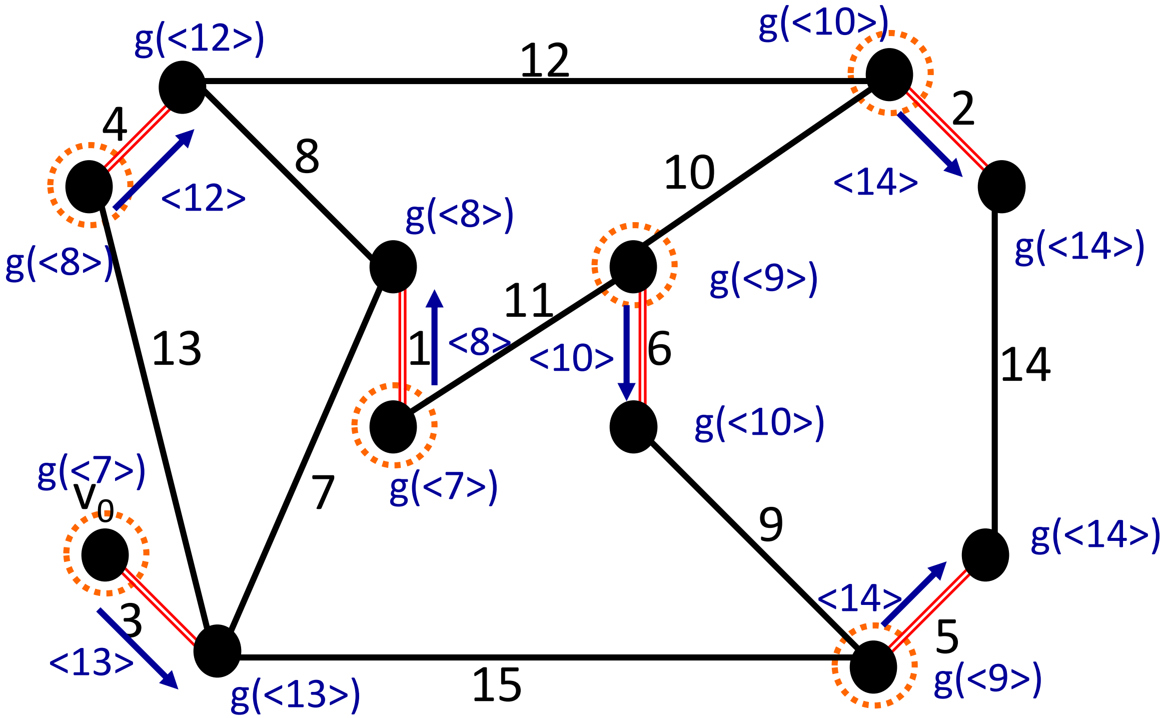
Extensive Example: Phase 2



1. Compute e(v,F‘) and send it to ℓ(F‘)

Only some messages are displayed

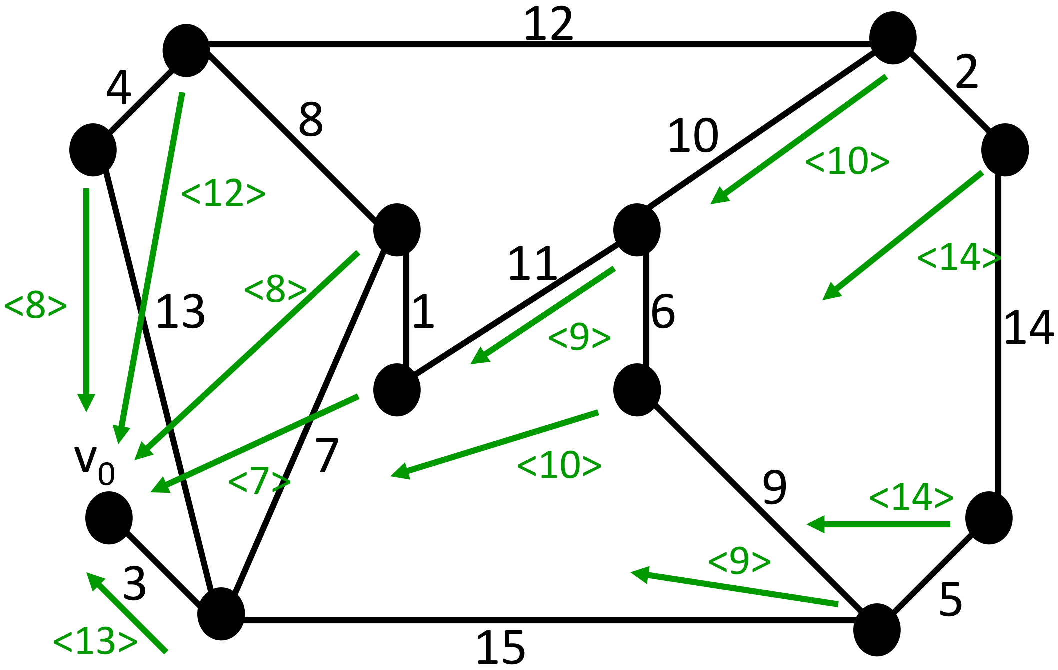
Extensive Example: Phase 2



1. Compute e(v,F') and send it to ℓ(F')
2. Select β = 2 lightest outgoing edges and appoint guardians

Only some messages are displayed

Extensive Example: Phase 2

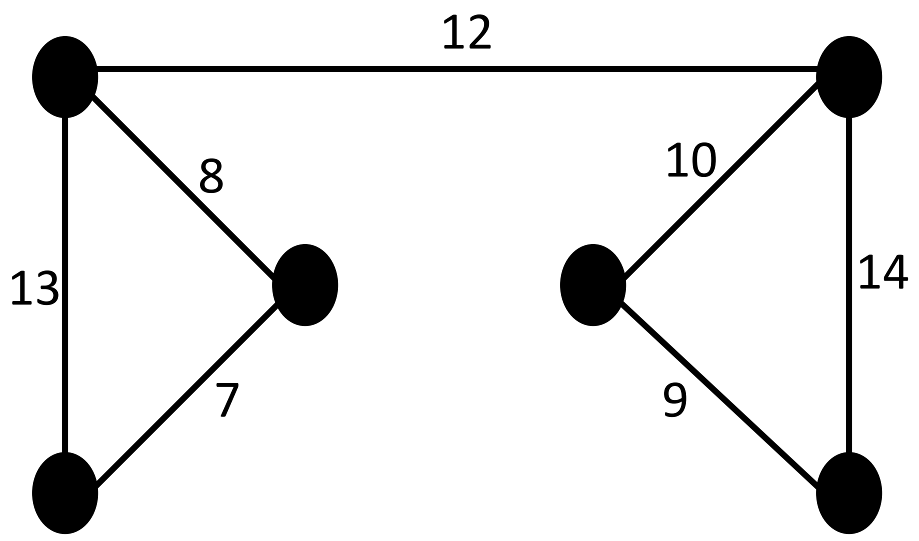


1. Compute e(v,F‘) and send it to ℓ(F‘)
2. Select β = 2 lightest outgoing edges and appoint guardians
3. Send appointed edge to v_0
4. Const_Frags!

Extensive Example: Phase 2

Const_Frags

1. Construct logical graph



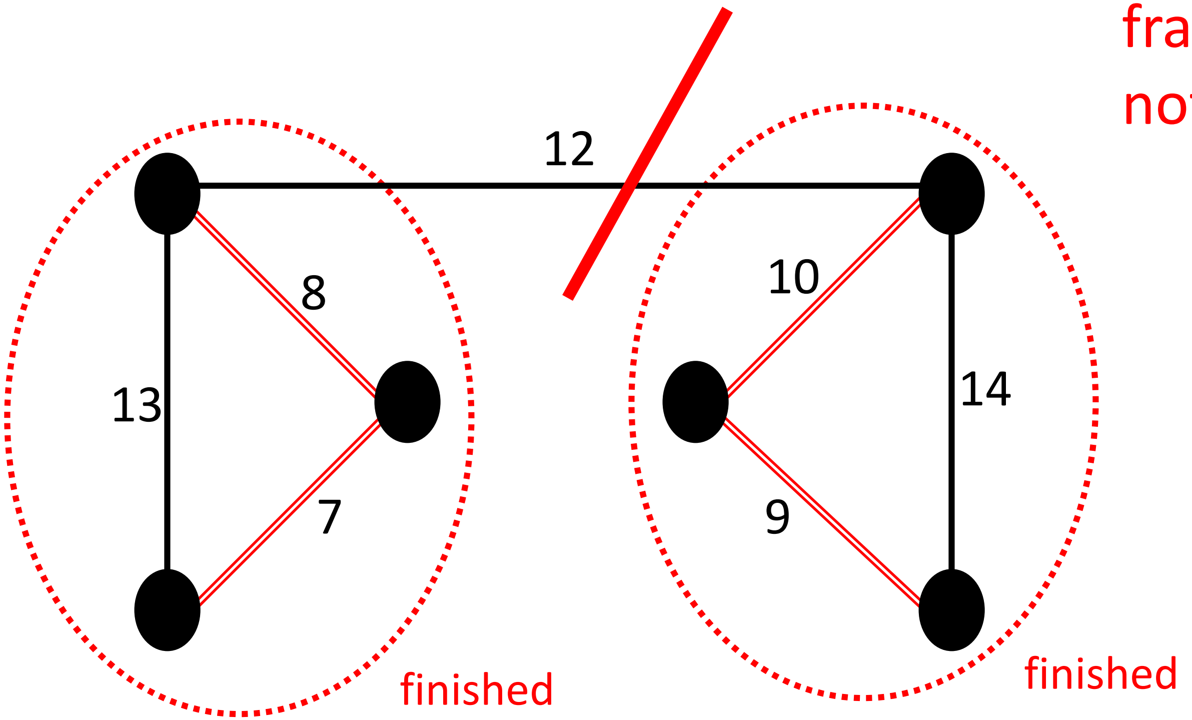
$\tilde{G}$

Extensive Example: Phase 2

Const_Frags

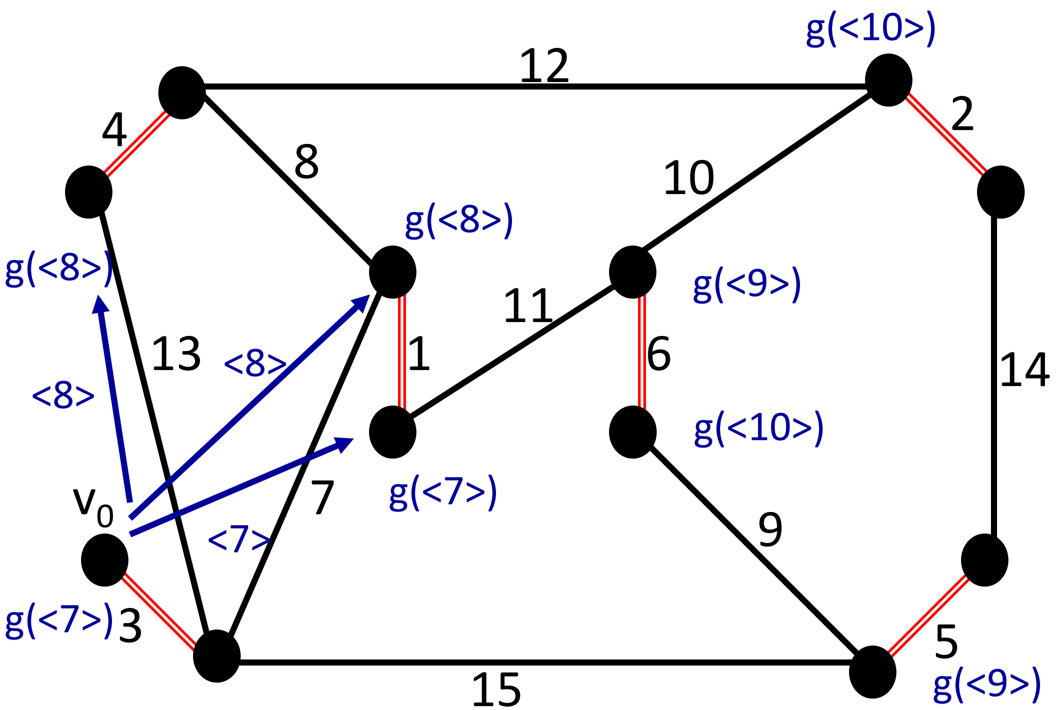
2. Add edges

Edges between finished super-fragments must not be added!



$\tilde{G}$

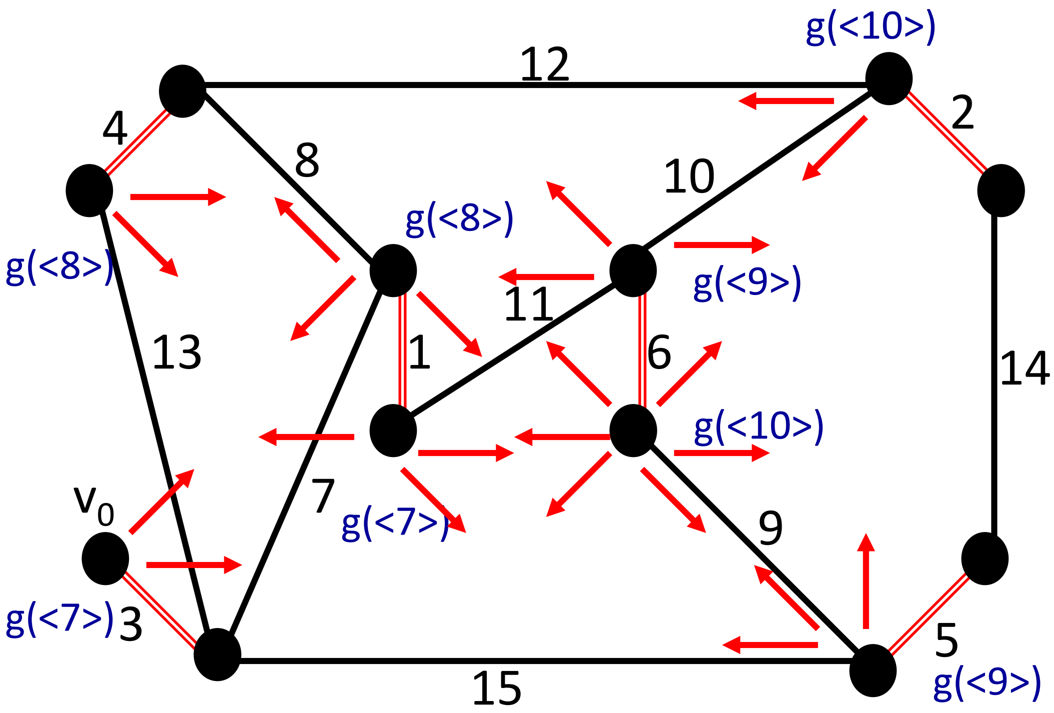
Extensive Example: Phase 2



Only some messages are displayed

1. Compute e(v,F') and send it to ℓ(F')
2. Select β = 2 lightest outgoing edges and appoint guardians
3. Send appointed edge to v_0
4. Const_Frags!
5. Send e to g(e)

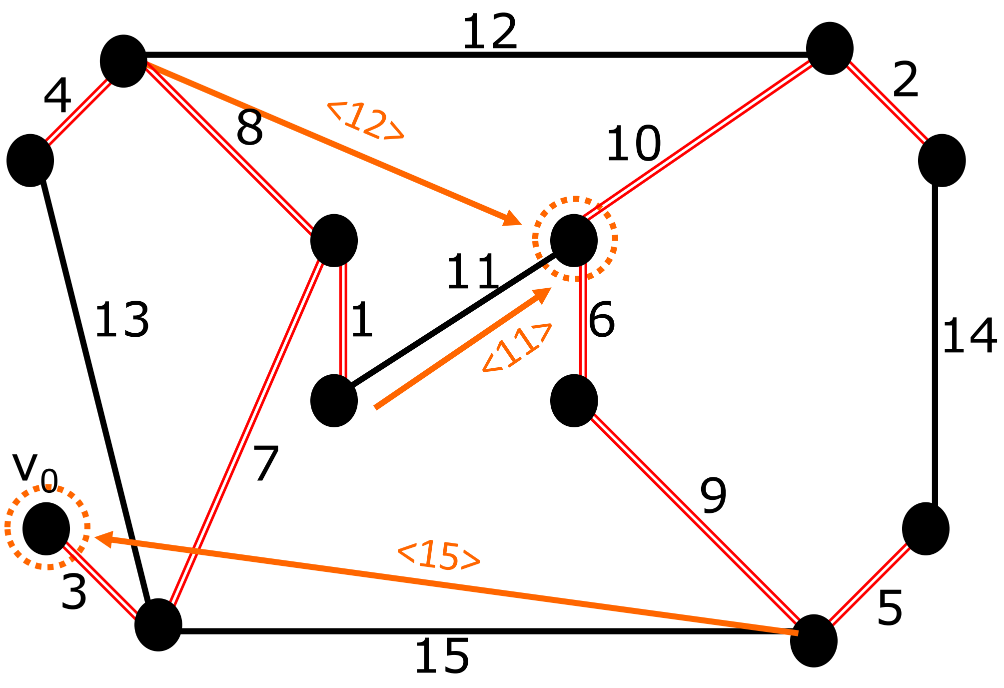
Extensive Example: Phase 2



Only some messages are displayed

1. Compute e(v,F‘) and send it to ℓ(F‘)
2. Select β = 2 lightest outgoing edges and appoint guardians
3. Send appointed edge to v_0
4. Const_Frags!
5. Send e to g(e)
6. Broadcast e and update the clusters

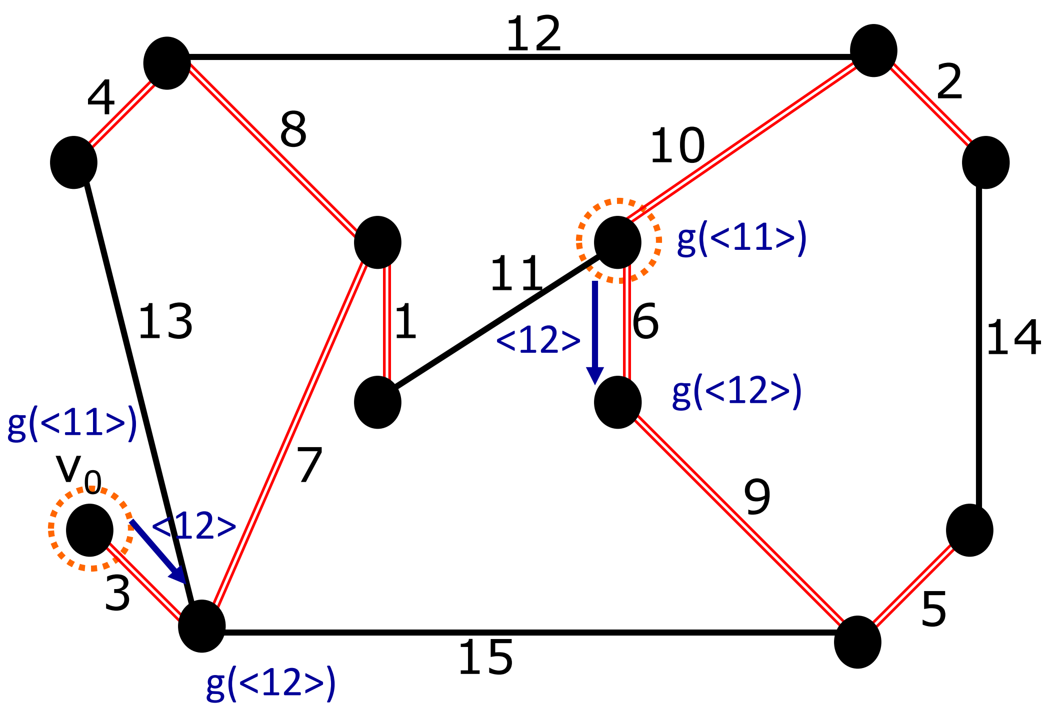
Extensive Example: Phase 3



1. Compute e(v,F‘) and send it to ℓ(F‘)

Only some messages are displayed

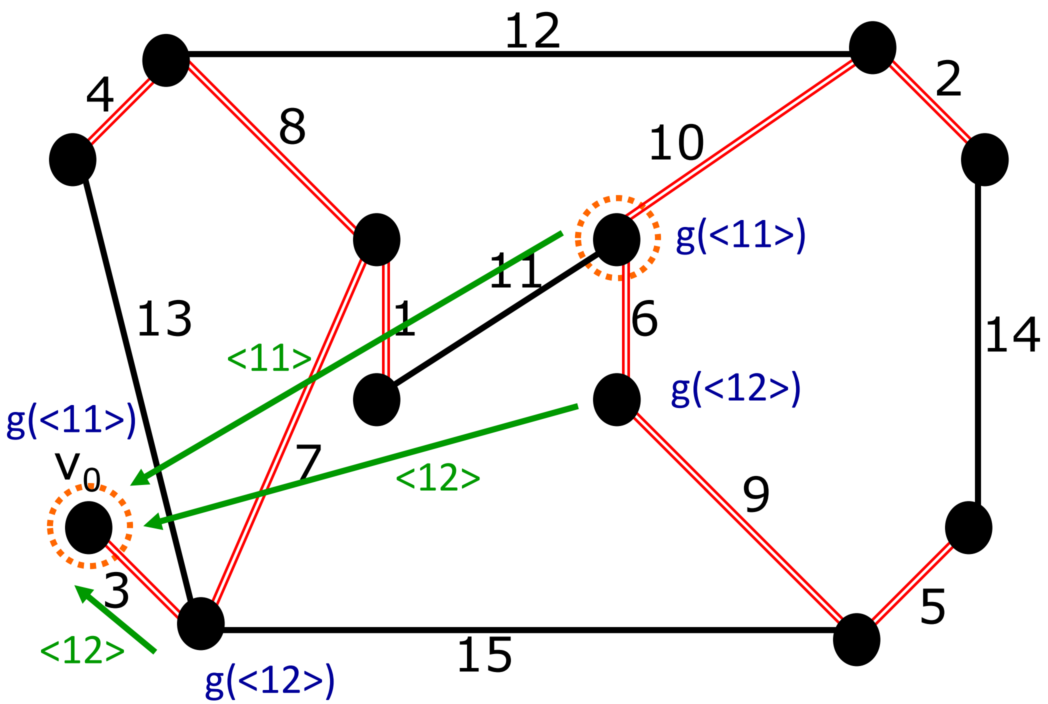
Extensive Example: Phase 3



Only some messages are displayed

1. Compute e(v,F‘) and send it to ℓ(F‘)
2. Select β = 2 lightest outgoing edges and appoint guardians

Extensive Example: Phase 3



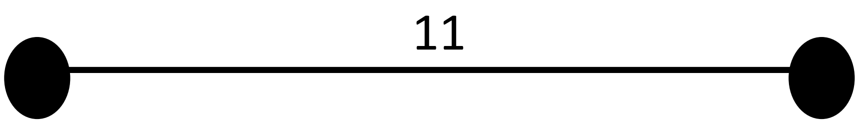
Only some messages are displayed

1. Compute e(v,F‘) and send it to ℓ(F‘)
2. Select β = 2 lightest outgoing edges and appoint guardians
3. Send appointed edge to v_0
4. Const_Frags!

Extensive Example: Phase 3

Const_Frags

1. Construct logical graph



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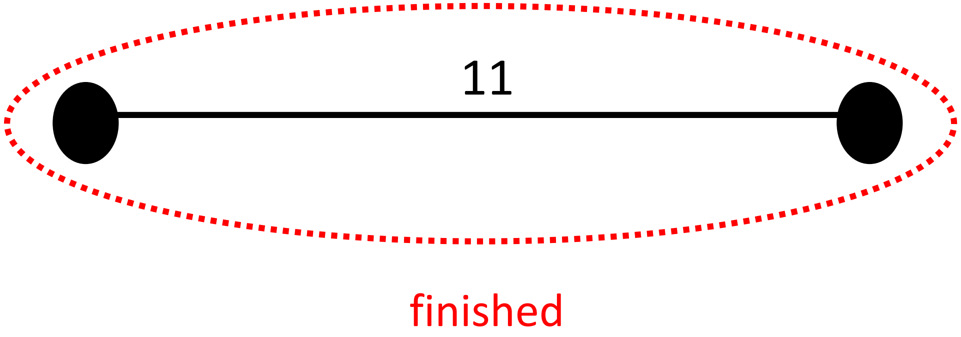
Extensive Example: Phase 3

Const_Frags

2. Add edges



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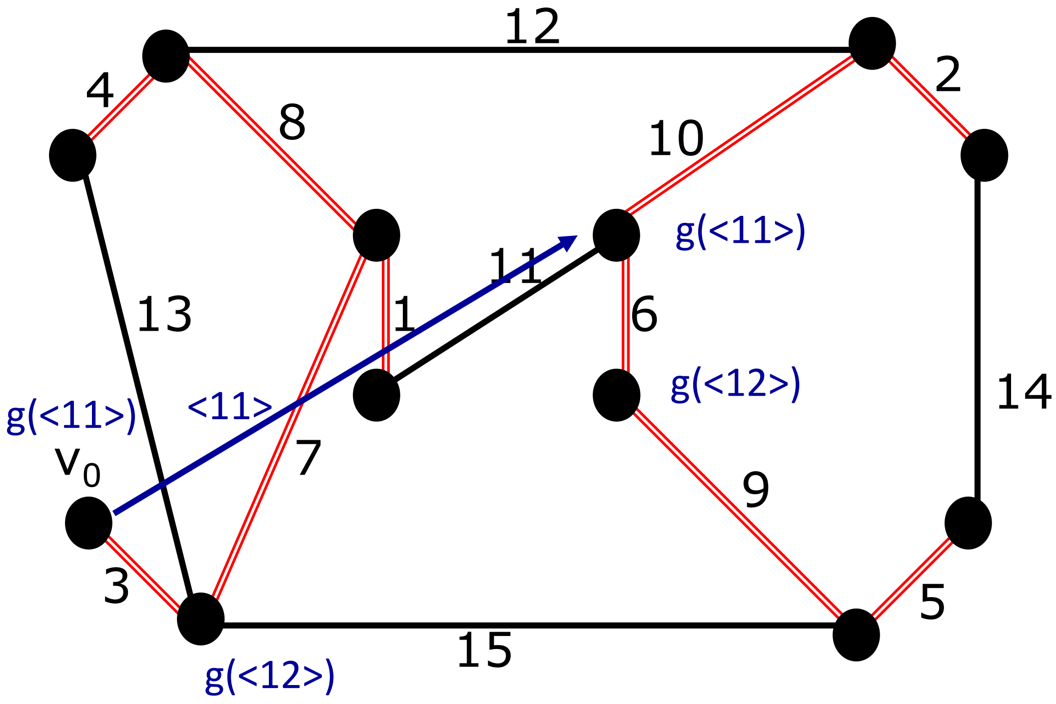
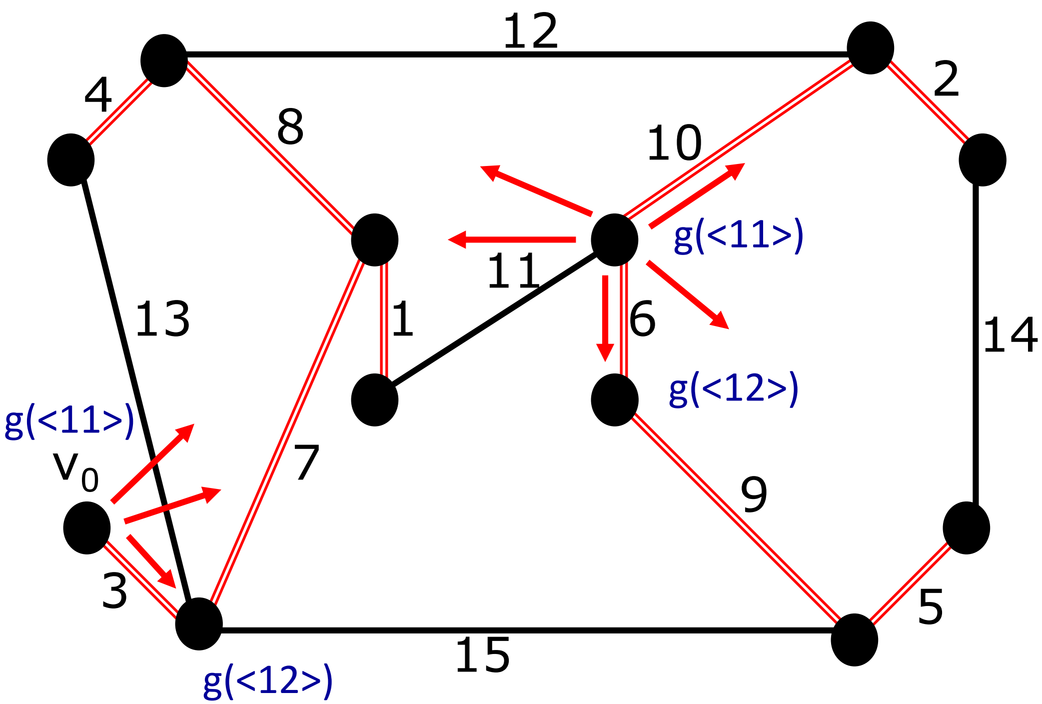
Extensive Example: Phase 3



1. Compute e(v,F') and send it to ℓ(F')
2. Select β = 2 lightest outgoing edges and appoint guardians
3. Send appointed edge to v_0
4. Const_Frags!
5. Send e to g(e)

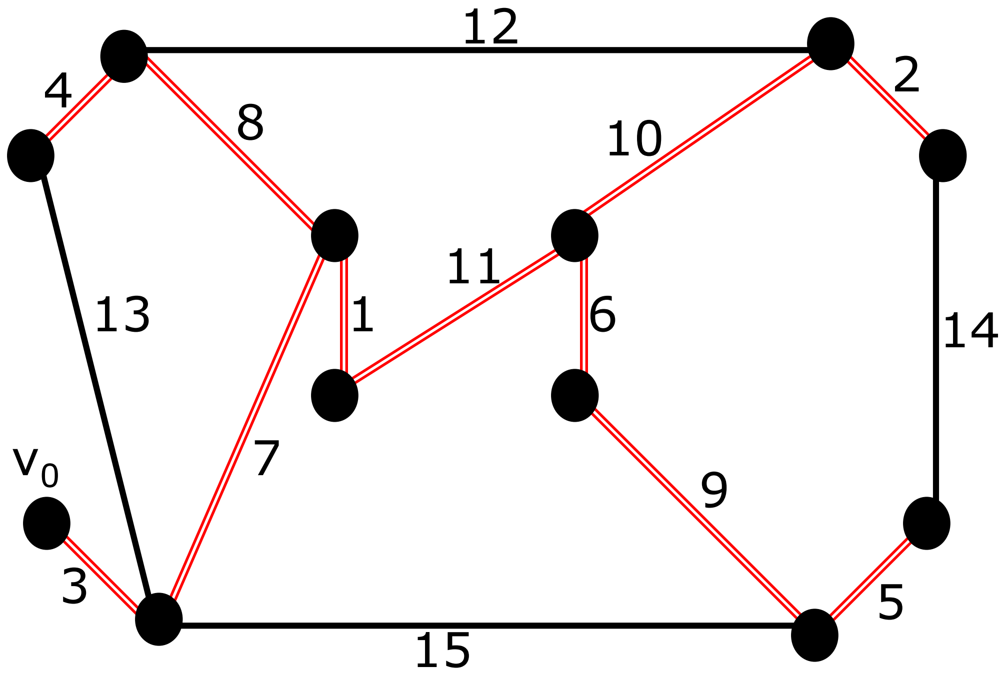
Extensive Example: Phase 3



Only some messages are displayed

1. Compute e(v,F') and send it to ℓ(F')
2. Select β = 2 lightest outgoing edges and appoint guardians
3. Send appointed edge to v_0
4. Const_Frags!
5. Send e to g(e)
6. Broadcast e and update the fragments

Extensive Example: Phase 3



Done!

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