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Solution Sheet 4

Lower Bounds

Exercise 1. Let $G = (V, E)$ be a cycle graph with n nodes and n edges, we assume that n is divisible by 3 for simplicity. We first prove that a 2-tuple dominating set of G of minimum size must contain at least $\frac{2}{3}n$ nodes. To prove this, let X be a 2-tuple dominating set and define $D(v) = |\text{ball}_G(v, 1) \cap X|$ for every node v; $D(v)$ is the number of nodes that belong to X among v and its two neighbors. Summing the $D(v)$ for all v counts each node that is in X three times, hence it holds:

$$\sum_{v \in V} D(v) = 3 \cdot |X|$$

Since X is a 2-tuple dominating set, it holds that

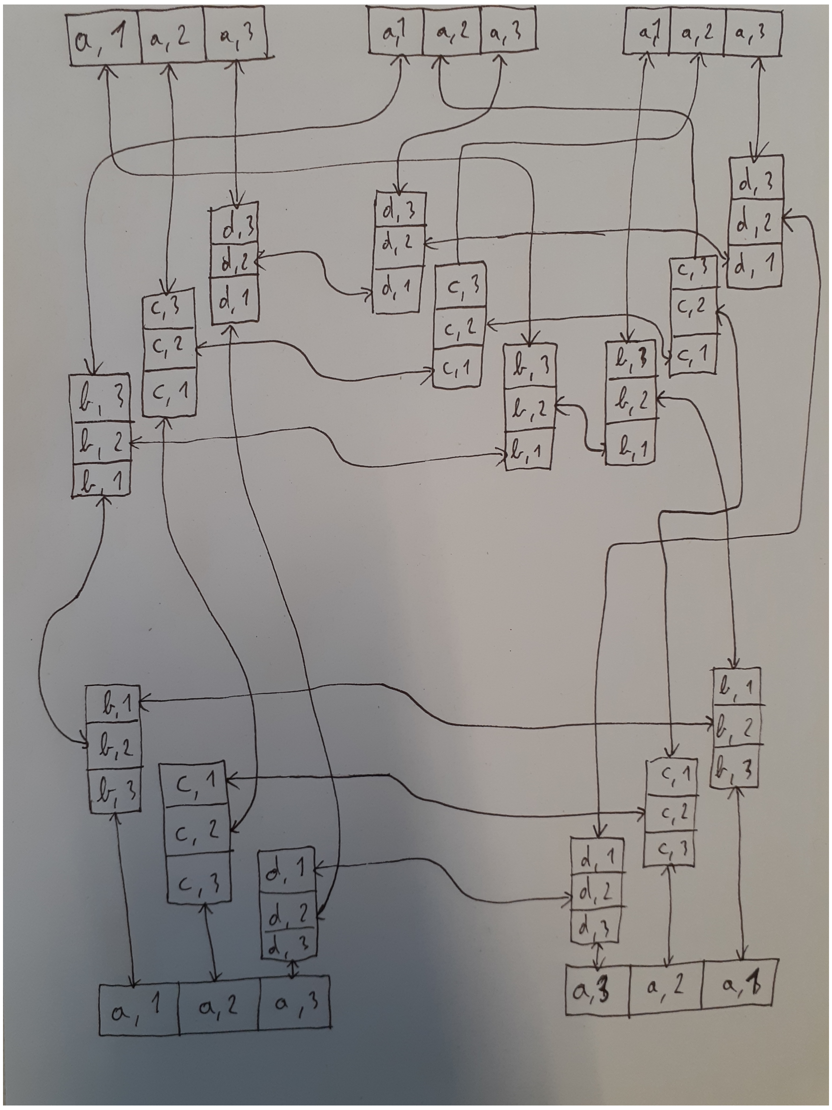
$$\forall v \in V,\ |D(v)| \geq 2$$

Combining the two previous lines gives:

$$3 \cdot |X| = \sum_{v \in V} D(v) \geq 2 \cdot n$$

which implies that $|X| \geq \frac{2}{3}n$ and completes the proof.
As seen during the lecture, any cycle is homogeneous; one can see this by alternating the port numbers 1 and 2 all around the cycle. As a result, any algorithm in the port-numbering model will have the same execution on all nodes and thus have the same output. So there are only two possible outputs: either all nodes are in the dominating set or none of them is. Out of those two, the only correct ouput is to assignall nodes to the dominating set which gives a solution of size n. We previously saw that any valid 2-tuple dominating set must contain at least $\frac{2}{3}n$ nodes, so our port-numbering algorithm that assigns all nodes to the dominating set is a $3/2 = 1.5$-approximation algorithm.

Exercise 2. Here is a solution port numbering network:



Exercise 3. Here are a few examples of problems that cannot be solved within the portnumber model if the underlying graph is homogeneous: counting the number of nodes of the graph, find a minimal dominating set. With the proper port-numbering, a port-numbering network must output the same result on all nodes.

Exercise 4. The graph G is reproduced in Figure 1.

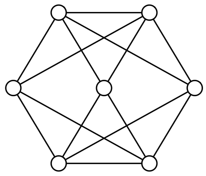
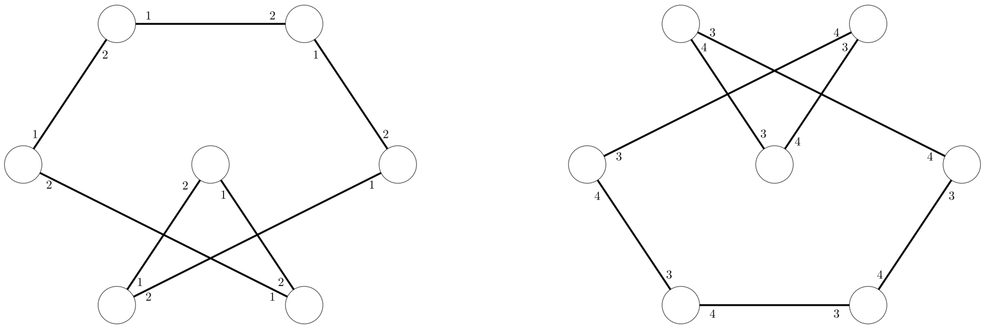


Figure 1: Graph G

The key to spot the homogeneity of the graph is to find two edge-disjoint cycles that cover the whole graph. Such cycles are shown in Figures 2a and 2b.



(a) A cycle with port numbers 1 and 2

(b) A cycle with port numbers 3 and 4

Figure 2: The two cycles that partition G

We gather the two port-numbered cycles in Figure 3 to prove that G is homogeneous.

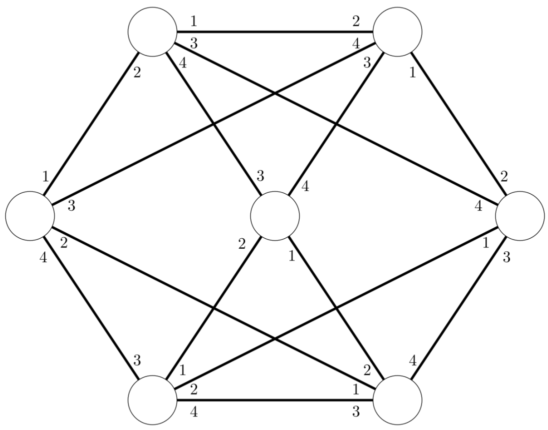


Figure 3: The graph G is homogeneous

Exercise 5. A full solution to this exercise contains non-trivial graph-theoretical parts that are not relevant for this course. I will thus skip those here and go to the essential. In this exercise, we assume that the following results are known: (1) it is possible to partition any $2k$-regular graph into k sets of cycles that cover all the nodes (the previous exercise is an example), and (2) any clique with $2k$ nodes has a 1-factor decomposition.

- Using the assumption (1), we Call $C_0, C_1 \dots C_{k-1}$ the sets of cycles that cover all the nodes of G. Note that C_i can be made of mutliple, disjoint cycles, but it must cover all the nodes of G. For each i, we assign the port numbers of the edges in C_i with alternating $2i$ and $2i+1$ along each cycle of C_i. The resulting port-numbering network can be mapped through a covering map to a port-numbering network with only one node and $2k$ ports and k self-loops from port $2i$ to port $2i+1$ for each i.

- This is already answered by assumption (2).

- If the clique has an odd number of nodes, say $2k+1$ then we can directly apply point one of this exercise, as then the graph is $2k$-regular. Otherwise if the clique has an even number of nodes, say $2k$, then we can apply point two of this exercise; Let M_1, $M_2 \dots M_{2k-1}$ be the matchings that partition the clique, we attribute the port numbers i to bot end-ports of the edges in M_i. The resulting port-numbering network can be mapped through a covering map to a port-numbering network with only one node, $2k$ ports and one self-loop form each port to itself.