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Exercise Sheet 4

Lower Bounds - Covering Maps

We use the following definition in the exercises. A graph G is *homogeneous* if there are port-numbered networks N and N' and a covering map ϕ from N to N' such that N is simple, the underlying graph of N is G, and N' has only one node.

Exercise 1 (Quiz)**.** Let $G = (V, E)$ be a graph. A set $X \subseteq V$ is a *k-tuple dominating set* if for every $v \in V$ we have $|\operatorname{ball}_G(v, 1) \cap X| \geq k$. Consider the problem of finding a minimum 2-tuple dominating set in *cycles*. What is the best (i.e. smallest) approximation ratio we can achieve in the PN model?

Exercise 2 (finding port numbers)**.** Consider the graph G and network N' illustrated in Figure 1. Find a simple port-numbered network N such that N has G as the underlying graph and there is a covering map from N to N'.

Exercise 3 (homogeneity)**.** Assume that G is homogeneous and it contains a node of degree at least two. Give several examples of graph problems that cannot be solved with any deterministic PN-algorithm in any family of graphs that contains G.

Exercise 4 (regular and homogeneous)**.** Show that the following graphs are homogeneous:

- graph G illustrated in Figure 2,
- graph G illustrated in Figure 1.

Exercise 5 (complete graphs)**.** Recall that we say that a graph $G = (V, E)$ is *complete* if for all nodes $u, v \in V$, $u \neq v$, there is an edge $\{u, v\} \in E$. Show that

- any $2k$-regular graph is homogeneous,
- any complete graph with $2k$ nodes has a 1-factorization,
- any complete graph is homogeneous.

Exercise 6 (dominating sets)**.** Let $\Delta \in \{2, 3, \dots\}$, let $\epsilon > 0$, and let $\mathcal{F}$ consist of all graphs of maximum degree at most Δ. Show that it is possible to find a $(\Delta + 1)$-approximation of a minimum dominating set in constant time in family $\mathcal{F}$ with a deterministic PN-algorithm. Show that it is not possible to find a $(\Delta + 1 - \epsilon)$-approximation with a deterministic PN-algorithm.

Exercise 7 (3-regular and not homogeneous)**.** Consider the graph G illustrated in Figure 3.

- Show that G is not homogeneous.
- Present a deterministic PN-algorithm A with the following property: if N is a simple port-numbered network that has G as the underlying graph, and we execute A on N, then A stops and produces an output where at least one node outputs 0 and at least one node outputs 1.
- Find a simple port-numbered network N that has G as the underlying graph, a port-numbered network N', and a covering map ϕ from N to N' such that N' has the smallest possible number of nodes.

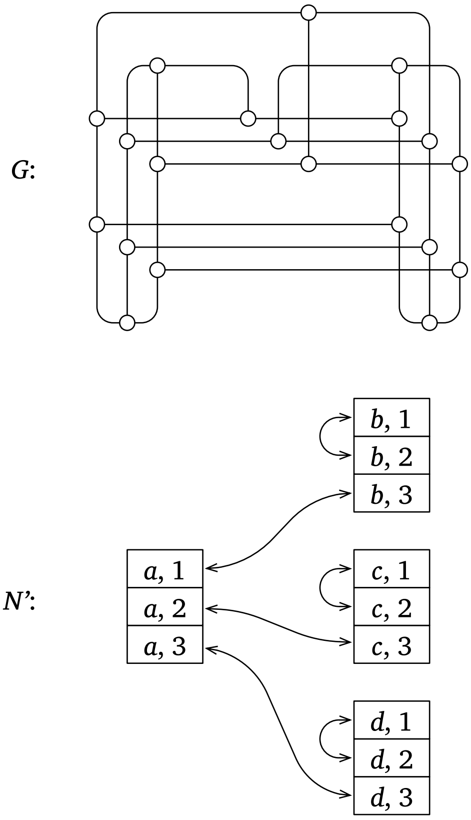

Figure 1: Graph G and network N' for Exercises 2 and 4b.

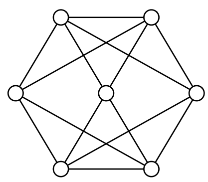

Figure 2: Graph G for Exercise 4a.

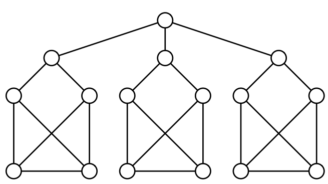

Figure 3: Graph G for Exercise 7.

Exercise 8 (covers and connectivity)**.** Assume that $N = (V, P, p)$ and $N' = (V', P', p')$ are simple port-numbered networks such that there is a covering map ϕ from N to N'. Let G be the underlying graph of network N, and let G' be the underlying graph of network N'.

- Is it possible that G is connected and G' is not connected?
- Is it possible that G is not connected and G' is connected?

Exercise 9 (k-fold covers)**.** Let $N = (V, P, p)$ and $N' = (V', P', p')$ be simple port-numbered networks such that the underlying graphs of N and N' are connected, and assume that $\phi\colon V \to V'$ is a covering map from N to N'. Prove that there exists a positive integer k such that the following holds: $|V| = k|V'|$ and for each node $v' \in V'$ we have $|\phi^{-1}(v')| = k$. Show that the claim does not necessarily hold if the underlying graphs are not connected.