Tutor: Mitja Krebs

Solution Sheet 1

Port Numbering

Exercise 1 (Port-Numbered Network Construction)**.** Construct a simple port-numbered network $N = (V, E, \{p_v\}_{v \in V})$ and its underlying graph $G = (V, E)$ that has *as few nodes as possible* and that satisfies the following properties:

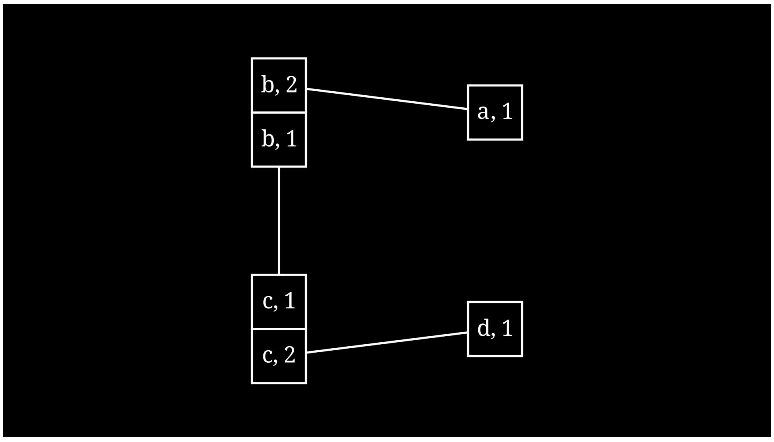
- Set E is nonempty.
- The set $M \subseteq E$ consisting of the edges $\{u, v\} \in E$ with $p_u(v) = 1$ and $p_v(u) = 2$, is a *perfect matching* in graph G, i.e., each vertex in V is incident to exactly one edge in M.

Please answer by listing all elements of sets V, E, and by listing all values of p.

Solution.
For example, you might specify a network with four nodes as follows:

$$
\begin{aligned}
V &= \{a, b, c, d\} \\
E &= \{\{a, b\}, \{b, c\}, \{c, d\}\} \\
p_a(b) &= 1 \\
p_b(a) &= 2 \\
p_b(c) &= 1 \\
p_c(b) &= 1 \\
p_c(d) &= 2 \\
p_d(c) &= 1
\end{aligned}
$$

which corresponds to the following port-numbered network:



End of Solution.

Exercise 2 (Vertex Cover and Matching Approximation)**.** Consider the following definition of an α-approximation:

> Let $\alpha \geq 1$, we say that an algorithm is a *α-approximation* for a given problem if that algorithm returns a solution that is always at most α times worse than the optimal solution.

Let M be a maximal matching. Let $C = \bigcup M$, i.e., C consists of all endpoints of matched edges. Show that:

1. C is a 2-approximation of the minimum vertex cover.
2. M is a 2-approximation of the maximum matching.

Solution.
In approximation algorithms, we typically have to prove two properties: *feasibility* (i.e., that the output is a valid solution) and the *approximation guarantee*.

1. To show that C is a valid vertex cover, assume for the sake of contradiction that there is an edge $\{u, v\}$ such that neither u nor v is included in C. This implies that M does not contain any edge incident to either u or v. Hence, we could add the edge $\{u, v\}$ to M, contradicting its maximality. Thus, C is indeed a vertex cover.

   To show the approximation ratio of 2, let C^* be a minimum vertex cover. Since M is a matching, its edges are disjoint, and therefore, any vertex cover must include at least one endpoint of each edge in M. It follows that:

   $$|C^*| \geq |M| = \frac{1}{2}|C|,$$

   which implies that $|C| \leq 2|C^*|$, i.e., C is a 2-approximation.

2. To show that M is a 2-approximation of a maximum matching, let M^* be a maximum matching. From the previous argument, we know that for each edge $\{u, v\} \in M^*$, at least one of u or v must be in C; otherwise, we could augment M, violating its maximality. Hence, each edge in M^* is incident to at least one vertex in C, which has size $2|M|$. This implies:

   $$|M^*| \leq |C| = 2|M|,$$

   and thus M is a 2-approximation.

End of Solution.

Exercise 3 (Vertex Cover 4-Approximation)**.** In this exercise, you will use the maximal matching algorithm for 2-colored graphs (Algorithm 1.13 from the lecture notes) to construct a 4-approximation for the minimum vertex cover in an arbitrary graph. We begin by transforming the given port-numbered network into a 2-colored network using a method called *bipartite duplication*:

> Let $G = (V, E, \{p_v\}_{v \in V})$ be a port-numbered network. A *bipartite duplication* of G is a port-numbered network $G' = (V', E', \{p'_v\}_{v \in V'})$ constructed as follows.
>
> 1. We double the number of nodes—for each node $v \in V$ we have two nodes v_1 and v_2 in V':
>
> $$V' = \{\, v_1, v_2 : v \in V \,\},$$
> $$E' = \big\{ \{v_i, w_j\} : \{v, w\} \in E, i \neq j \big\}$$

2. Then we define the port numbers. If $p_u(v) = i$ and $p_v(u) = j$, we set

$$p_{u_1}(v_2) = i \ \text{ and } \ p_{v_2}(u_1) = j$$
$$p_{u_2}(v_1) = i \ \text{ and } \ p_{v_1}(u_2) = j.$$

Now consider Algorithm 1.

Algorithm 1 Algorithm computing a 3-approximation of the minimum vertex cover.

1: Construct the bipartite duplication of the network, call it N'.
2: Simulate the algorithm for computing a maximal matching in a 2-colored graph (Algorithm 1.13 in the lecture notes) in N'. Each node v waits until both of its copies, v_1 and v_2, have stopped.
3: Node v says it is part of the vertex cover if at least one of its copies v_1 or v_2 becomes matched.

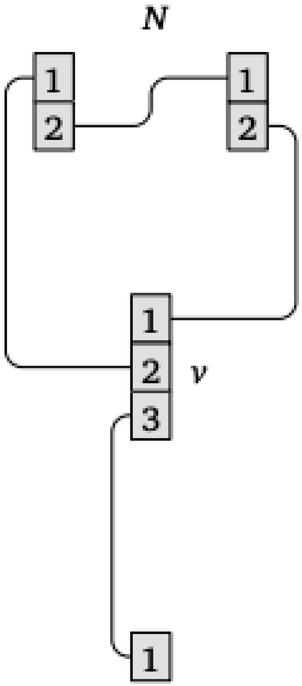


Figure 1: The port-numbered network for Exercise 3.

Tasks:

1. Run Algorithm 1 on the port-numbered network shown in Figure 1.

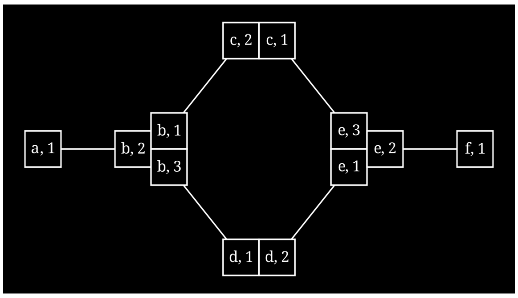
2. Prove that Algorithm 1 computes a vertex cover of graph G.

3. Show that Algorithm 1 returns a 4-approximation of the minimum vertex cover in $O(\Delta)$ rounds (where Δ is the maximum degree) in the port-numbering model.

Solution.
See the lecture notes.
End of Solution.

Exercise 4 (Tightness of Vertex Cover 3-Approximation)**.** The algorithm described in the previous exercise in fact achieves a 3-approximation for the minimum vertex cover (see Corollary 1.25 in the lecture notes). Is this approximation factor tight? That is, is it possible to construct a port-numbered graph G such that the algorithm outputs a vertex cover that is exactly 3 times as large as the minimum vertex cover of G?

Solution.
The following instance demonstrates that the approximation factor achieved by the algorithm cannot be better than 3: The minimum vertex cover is $\{b, e\}$, which has size 2, while the algorithm returns a cover consisting of all six nodes.



End of Solution.