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OPERATIONS RESEARCH (OR)

1. Introduction to OR
2. Linear programming models
3. Simplex Models
4. Transportation Models
5. Assignment Models
6. Network Analysis

Duality of LP models
Sensitivity analysis

INTRODUCTION - Ass 1

- Defn: Scientific mthd that involves use of mathematical techniques to solve real life problems to give an optimal result (soln).
- How it can be applied in real life ↳ help make optimal decisions.
- Features of OR
- Why you learn :- Applications : Finance, logistics
- Depends on other disciplines: technology, economics, medicine, etc. Areas of study.
- Phases of OR :- Judgemental, Action, Reactive (procedures)
- Various Models in OR :
  ↳ Allocation ↳ Simulation ↳ Network ↳ Sequential ↳ Queuing.
- Adv. of OR ---- limitations.

→ Transportation, Decision Making, Assignment

standard max problem : ≤
standard min problem : ≥

2. LINEAR PROGRAMMING MODELS (LP)

→ Can be defined mathematically as ~~an~~ ~~optimal~~ optimize (z)

optimize (z) = $C_1x_1 + C_2x_2 + C_3x_3 + \dots + C_nx_n$ subject to:

subject to (st) ... $a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \leq = \geq b_1$

$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \leq = \geq b_2$

$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \leq = \geq b_m$

and $x_1, x_2 \dots x_n \geq 0$ constraints ... $x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0$ (or $x \geq 0$)

Which can also be presented as follows:

Optimize (z): $\sum_{i=1}^{n} C_i x_i$

s.t: $\sum \sum a_{ij} x_i \leq = \geq b_j$ $x_i \geq 0$ $i = 1, 2, 3 \dots n$, $j = 1, 2, 3 \dots m$

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matrix → array of rows & columns.

(table form).

Similarly, in matrix form :

Resource	x_1	x_2	x_3	.....	x_n	Solution.
1	a_1	a_2	a_3	....	a_{1n}	b_1
2	a_{21}	a_{22}	a_{23}	---	a_{2n}	b_2
⋮	⋮	⋮	⋮			⋮
m	a_{m1}	a_{m2}	a_{m3}	....	a_{mn}	b_m
Z	c_1	c_2	c_3	....	c_n	

The above Lp model consists of 3 main parts → Definition.

i.e : i) Objective function

→ function which describes the output of the decision maker.

↳ function to be maximized or minimized.

ii) Constraints - Mathematical equations which describes the availability of resources that must be satisfied in order for the objective function to be achieved. • mathematical

iii) Non-negativity

- In this case, we assume that the decision variables take positive values.

Formulation of LP Models ✓

→ To Formulate an LP model, there are some (assumptions) that must be taken into consideration.

Assumptions

1. The decision maker is very knowledgeable abt a given problem (Certainty)
2. Linearity : The decision variables are linearly related. (when one decreases the other increases)
3. Non-negativity : The decision variables take positive values
4. Objectivity : You know what you want to achieve.

→ After considering the assumptions, the next step is to formulate an LP model:

Procedure

1. Study the problem and identify the decision variables.
2. Construct the objective function
3. Construct the constraints
4. Add the non-negativity part.

Example 1

Suppose a manifacturing company manufactures two products. It fetches a profit of 6/= per unit for product A and 5/= per unit for product B. Given that for product A to be manufactured, the company requires 2 labourers & 2 man hours per unit of A manufactured. Similarly, for product B to be manufuctured, the company requires 3 labourers & 4 man hours per unit of B manufactured.

If the availability of laboures and man hours for A & B to be manufactured is 30 labourers and 40 manhours, construct an LP model to maximize the profit for the 2 products.

A → 6 B → 5

	lab	man	profit
A	2	2	6
B	3	4	5
P.	30	40	

① decision var: → A - x_1 B → x_2

② Objective function max Z = $6x_1 + 5x_2$

③ Constraints: $2x_1 + 3x_2 \le 30$

$2x_1 + 4x_2 \le 40$

④ Non-negativity: $x_1, x_2 \ge 0$

⇒ Max $Z = 6x_1 + 5x_2$

s.t: $2x_1 + 3x_2 \le 30$

$2x_1 + 4x_2 \le 40$

$x_1, x_2 \ge 0$

Let products: A and B

① decision var: A & B

let product A = x_1 & B = x_2

② Objective function:

Profit = $6x_1 + 5x_2$

max $Z = 6x_1 + 5x_2$

③ Constraints:

lab: $2x_1 + 3x_2 \le 30$

manh: $2x_1 + 4x_2 \le 40$

④ Non-negativity:

$x_1 \ge 0 \quad x_2 \ge 0$

Summarize model:

max $Z = 6x_1 + 5x_2$

s.t: $2x_1 + 3x_2 \le 30$

$2x_1 + 4x_2 \le 40$

$x_1, x_2 \ge 0$

Example 2:

Jkuat produces & sells two diff. products under brand names: Black and White. Profits per unit for these products are ~~50~~ 50 & ~~40~~ 40 ksh respectively. Both products employ è same manufacturing process which has a total capacity of 50,000 man hours. As per the estimate of the JKUAT marketing .., there is demand for a max of 8000 units of Black & 10,000 units of White subject to 'overall demand' the products can be sold in any possible combination. If it takes 3 hours to produce 1 unit of black and 2 hours for white. Formulate the model for the linear programming. -> Maximization.

(4) Table ??

	Black	White	
man hrs	500	10000	50,000
Profit	50	40	

① Decision variables -> things you can achieve/(optimize) manipulate to key variable (goal).

let Black = x_1, White = x_2

~~Max z~~

② Objective function

Z_{max}

Max $Z = 50x_1 + 40x_2$

③ Constraints

$3x_1 + 2x_2 \leq 50,000$

$x_1 \leq 8000$

$x_2 \leq 10,000$

④ Non-negativity

$x_1, x_2 \geq 0$

Summary: Model

max $Z = 50x_1 + 40x_2$

s.t: $3x_1 + 2x_2 \leq 50,000$

$x_1 \leq 8000$

$x_2 \leq 10,000$

$x_1, x_2 \geq 0$

		manhrs	Profit
8000	Black	3	50
10000	White	2	40
		50000	

max $Z = 50x_1 + 40x_2$

s.t: $3x_1 + 2x_2 \leq 50000$

$x_1 \leq 8000$

$x_2 \leq 10000$

$x_1, x_2 \geq 0$.

Solving L.P Models/Problems

-> After formulating an LP model, the next step is to solve it.

-> Inorder to obtain the soln which satisfy the objective function, there are several techniques that can be used to solve an LP model.

Eg: Graphical method and Simplex method.

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1. Graphical Method

· Two decision variables.

→ In this method, we can obtain soln to LP problems involving two decision variables

· formulate model
· Plot points : lines
· Identify soln area
· Optimal : corner point
· Isoprofit

Procedure:

1. Formulate the LP model.
2. Plot the points & draw the lines accordingly.
3. Identify the sln area/ soln region
 - To identify the optimal soln based on graphical method, two techniques can be applied

(i) Corner-point Method / Extreme point Method

- The steps in order to use mthd
  - Identify each extreme point/ corner point
  - Calculate profit/cost corresponding to each of ē points identified
  - The point that gives the maximum value of the objective function becomes the solution of ē maximization problem.
  - The point that gives the minimum value of the objective function becomes the sln of ē minimization problem.

(ii) Isoprofit / Iso cost Method

- Method which shows any set of combination of points that produces profit and at ē same time minimizes cost

Procedure

+ Draw iso-profit lines within constraints of ē problem. Is a straight line connecting all points with same profit.
+ Move iso-profit line parallel to the objective function
+ Identify the feasible extreme point at which iso-profit is largest or minimal. This is the optimal soln.

Terms in graphical Method

value of decision variables which satisfy ē constraints of ē LP model

→ Solution: The value of decision variables which satisfy ē constraints of a given LP model

→ Basic Solutions: The variables which have 0 values are non-basic variables and the remaining variables which have non-zero values are called basic variables.

non-zero values

* For a set of, P, simultaneous equations in Q known. ($P > Q$), a solution obtained by setting $(P-Q)$ of variables is equals to 0 and solving the remaining P equations in P unknowns is known as basic solution.

→ Feasible solution : Soln which → satisfies all the constraints of linear programming equation
soln which satisfies all ē constraints of an LP eqn.

→ Optimal ~~Basic~~ feasible soln : Soln which optimizes the objective function

→ Basic " " : Is a feasible soln which is also a basic solution.

→ Degenerate soln : A basic soln is said to be degenerate if one or more basic variables become zero

→ Infeasible solution : soln which does not satisfy all the constraints of linear programming equations.

Example

Solve the LP problem given below using graphical method:

$$\max z = 5x_1 + 3x_2$$

$$\text{s.t: } 3x_1 + 5x_2 \leq 15$$

$$5x_1 + 2x_2 \leq 10$$

$$x_1, x_2 \geq 0$$

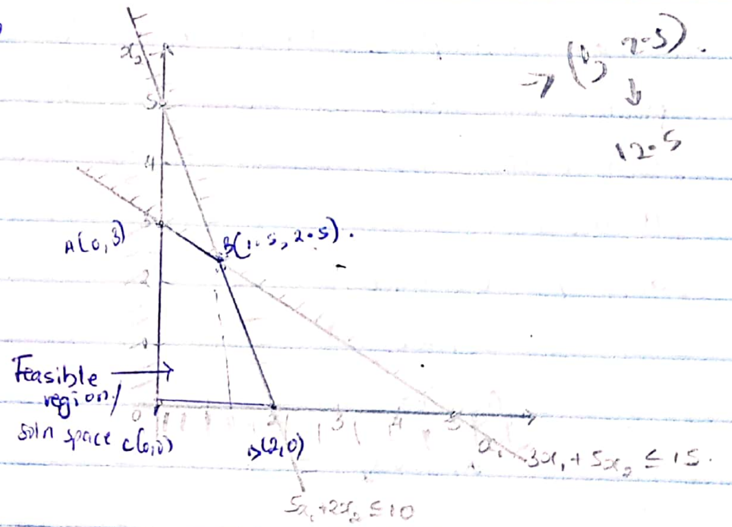
Soln:

$3x_1 + 5x_2 \leq 15$

x_1	0	5
x_2	3	0

$5x_1 + 2x_2 \leq 10$

x_1	0	2
x_2	5	0



→ (1.5, 2.5)
↓
12.5

C z = 0

A z = 9

D z = 10

B z = 15

B gives the optimum solution this means $x_1 = 1.5$ & $x_2 = 2.5$

→ Graphical methods are not suitable techniques for solving problems involving more than two decision variables.

Un-bounded solution
→ No clear corner point

→ In a situation where the values of e objective function & decision variables increases infinitely without violating feasibility conditions, the soln is said to be unbounded solution.

Example: DO

max $z = 6x_1 + x_2$

s.t: $2x_1 + x_2 \geq 3$

x_1	0	1.5
x_2	3	0

$x_2 - x_1 \geq 0$

x_1	0	0
x_2	0	0

$x_1, x_2 \geq 0$

Check if e soln is unbounded, i.e: No clear corner point.

Alternative solution (multiple optimal soln)
→ A.k.a Multiple optimal soln. Is a situation where the linear programming soln has more than 1 optimal solution. All the optimal solutions known as alternative solutions.

Example: DO

max $z = 4x_1 + 4x_2$

s.t: $x_1 + 2x_2 \leq 10$

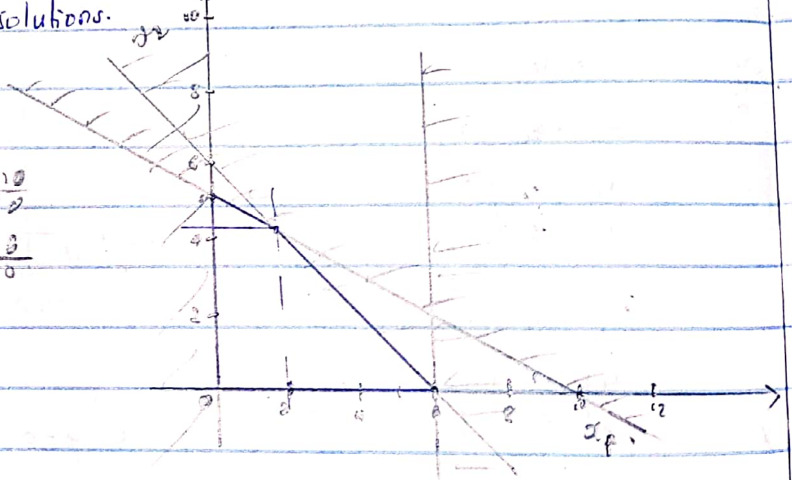
x_1	0	10
x_2	5	0

$6x_1 + 6x_2 \leq 36$

x_1	0	6
x_2	6	0

$x_1 \leq 6$ $x_1 \leq 6$

$x_2 \geq 0$ $x_2 \geq 0$



Check if it has alternate solutions.

Infeasible solution
→ No variable satisfies all e constraints.

→ Involves problems where no variable satisfy all the constraints, i.e: there is no unique feasible soln.

DO

$\max(z) = 5x_1 + 3x_2$

s.t: $4x_1 + 2x_2 \leq 8$

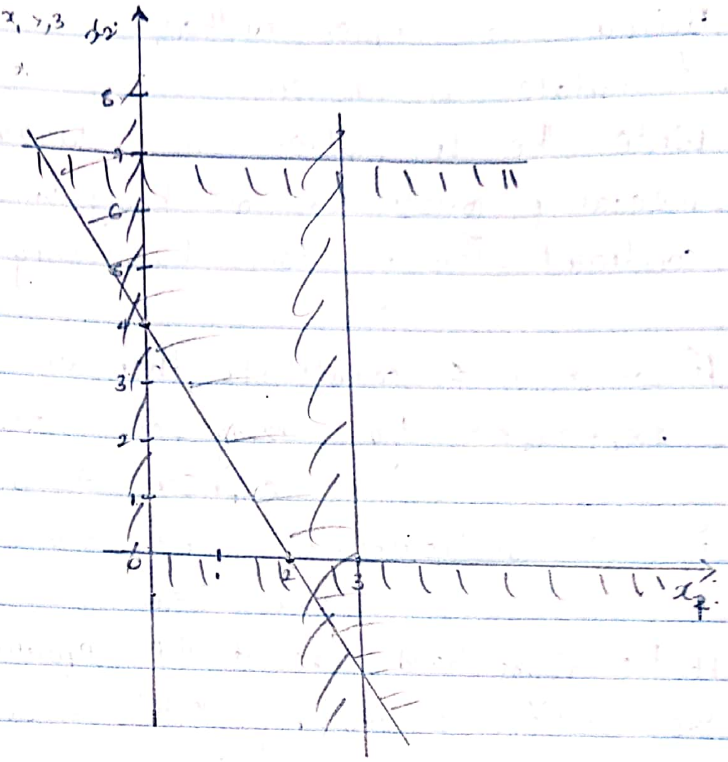
x_1	0	2
x_2	4	0

$x_1 \geq 3$

$x_2 \geq 7$

$x_1, x_2 \geq 0$

Check if it does not have feasible soln.



SIMPLEX METHOD

* Is another method of solving an LP problem, especially those problems where we have more than two decision variables

Procedure for simplex method:

1. formulate LP problem
2. Write the LP problem in standard form. ; In this case, the inequality constraints are being transformed to be equality constraints. This is done by adding slack/surplus & artificial variables

$\leq \rightarrow$ slack variable

If we have '$\leq$' constraints, then we add slack variable, eg:

$\geq \rightarrow$ surplus var

$2x_1 + 5x_2 \leq 5$ by adding slack variables: $\rightarrow s \geq 0$ or above 0.

$$2x_1 + 5x_2 + \underset{\text{slack (+S)}}{S} = 5$$

For '$\geq$' constraints. eg: $x_1 + 3x_2 \geq 6$, we add surplus:

$$x_1 + 3x_2 - \underset{\text{surplus (-S)}}{S} = 6$$

why ?? 6// * If the right-hand side of the equation is a negative value, then we have to multiply both sides of the eqn by -1 eg:

$$3x_1 + 2x_2 \geq -6 \quad -1$$

$$-3x_1 - 2x_2 \leq 6$$

$\rightarrow$ given mixture of $\leq$, $\geq$ add surplus & artificial.

Example:

1. $\text{Max } (Z) = 2x_1 + 5x_2 + 6x_1$

s.t:

5 $\quad 3x_1 + 2x_2 \leq 6$	$3x_1 + 2x_2 + S_1 = 6$ -slack.
5 $\quad 3x_1 + 4x_2 \geq -6$	$-3x_1 - 4x_2 + S_2 = 6$ -surplus
5 $\quad 2x_1 + 6x_2 \geq 7$	$2x_1 + 6x_2 - S_3 + A_1 = 7$
$x_1, x_2 \geq 0$	

2. In standard form:

$$\text{Max } (Z) = 2x_1 + 5x_2 + 6x_1 + 0S_1 + 0S_2 + 0S_3$$

s.t:

$$3x_1 + 2x_2 + S_1 = 6$$

$$(-3x_1 - 4x_2 \leq 6)$$

$$-3x_1 - 4x_2 + S_2 = 6$$

$$2x_1 + 6x_2 - S_3 + A_1 = 7$$

* additional

③ Construct the initial simplex table ;

Basic variable	x_1 x_2 x_3 .... x_n	s_1 s_2 .... s_n	Sol	ratio co-efficient
s_1	a_{11} a_{12} a_{13} .... a_{1n}	1 0 0 ... 0	b_1	0
s_2	a_{21} a_{22} a_{23} .... a_{2n}	0 1 0 ... 0	b_2	0
s_3	a_{31} a_{32} a_{33} .... a_{3n}	0 0 1 ... 0	b_3	0
⋮	⋮	⋮	⋮	⋮
s_n	a_{mn} a_{mn} a_{mn} ... a_{mn}	0 0 0 1	b_n	0
c_i	c_1 c_2 c_3 ... c_n	0 0 0 .. 0		
z_j	z_1 z_2 z_3 .... z_n	z_1 z_2 z_3		
$c_i - z_j$	$c_1 - z_1$ $c_2 - z_2$ $c_3 - z_3$	$-z_1$ $-z_2$ $-z_3$		

④ Identify the leaving variable and the entering variable. and this can be done by establishing the pivot column and row.

*To obtain the pivot column: we examine the values of $c_i - z_j$

i) If $c_i - z_j$ values are equal to 0, or all of them are negative, then the current simplex table has an optimal soln.

ii) If any of the $c_i - z_j$ values is positive, then the current soln isn't optimal and the column corresponding to any of the most positive value of $c_i - z_j$ becomes the pivot column.

- The elements that corresponds to pivot column becomes ē pivot elements

- The variable that corresponds to ē pivot column is the entering variable

- To obtain the leaving variable, we first get the pivot row.

To get the pivot row:

a) Obtain the ratio column given by dividing soln column with the corresponding pivot column elements:

entering variable: pivot column
leaving variable: pivot row

Pivot row = smallest +ve ratio.

Ratio = $\dfrac{\text{Solution element}}{\text{Pivot column element}}$

b) Examine the ratios and to the row which corresponds to the smallest positive ratio becomes the pivot row.

NOTE:

Negative ratios and infinite ratios are always ignorored.
- The variable that corresponds to the pivot row is the leaving variable.

⑤ Calculate the pivot row.

Divide the current pivot row by corresponding pivot element.

New pivot row = $\dfrac{\text{current pivot row}}{\text{pivot element}}$

⑥ Calculate the remaining rows using Gau's Jordan Operation:

i.e;

New row = Current row - pivot element (New pivot row)

⑦ Construct new simplex table and repeat the procedure from step 4 until an optimal solution is obtained.

Surplus

Slack

Example:

Solve the following LP using simplex method.

$$\text{Max } (Z) = 12x_1 + 15x_2 + 14x_3$$

$$\text{St}: \quad -x_1 + x_2 \leq 0$$

$$-x_1 + 2x_3 \leq 0$$

$$x_1 + x_2 + x_3 \leq 100$$

$$x_1, x_2, x_3 \geq 0$$

Soln:

- In standard form:

Max (z) $= 12x_1 + 15x_2 + 14x_3 + 0s_1 + 0s_2 + 0s_3$

s.t : $-x_1 + x_2 + 1S_1 = 0$

$-x_1 + 2x_3 + S_2 = 0$

$x_1 + x_2 + x_3 + S_3 = 100$

$x_1, x_2, x_3, S_1, S_2, S_3 \geq 0$

- Construct simplex table:

Co-efficient of ⓘ basic
· calculate ⓙ

Basic	x_1	x_2	x_3	S_1	S_2	S_3	Soln	C of B	Ration.
Pivot row → S_1	-1	1	0	1	0	0	0	0	0/1 = 0 → Piv
S_2	-1	0	2	0	1	0	0	0	0/0 = ∞ → ignore
S_3	1	1	1	0	0	1	100	0	100/1 = 100
C_i	12	15	14	0	0	0			
$\alpha \cdot CB$ ← Z_j	0	0	0	0	0	0			
$C_i - Z_j$	12	15	14	0	0	0			

↑pivot column

Ration $= \frac{\text{Sol}}{\text{Pivot column}}$

(1 1 1 0 0 1 1 0 0) + 1(+1, -1, -1, 0, -1

S_1 is the leaving variable
x_2 - entering variable.

- Calculate the pivot row:

New pivot row $= \frac{\text{current pivot row}}{\text{pivot element}}$

New x_2 row $= (-1 1 0 1 0 0 0) \div 1$
$= -1 1 0 1 0 0 0$

(1-1)(-1)
1-1(-1) = 1+1 = 2v
1-1(1) OV
1-1(0) 1V
1-1(1)

⊕ New S_2 row = Current S_2 row − pivot element (new x_2 row)
$= (-1 0 2 0 1 0 0) - 0(-1 1 0 1 0 0 0)$ = 10

-1 - 0 (-1)

?? why multiply ?✓

$-1-0(-1), 0-0(1), 2-0(0), 0-0(1), 1-0(0), 0-0(0), 0-0(0)$
✓ -1 0 2 0 1 0 0

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New S_3 row = Current S_3 row - pivot element (New x_2 row).

$(1\ 1\ 1\ 0\ 0\ 1\ 100) - 1(\cancel{-1\ 1\ 0\ 1\ 0\ 0\ 0})$ $\quad 2\ 0\ 1\ -1\ 0\ 1 \mid 100$

$(1-1)(-1),\ 1-1(1),\ 1-1(0),\ 0-1(1),\ 0-1(0),\ 1-1(0),\ 100-1(0).$

- Construct a new table

Basic	x_1	x_2	x_3	s_1	s_2	s_3	Soln	C_B	Ratio
x_2	-1	1	0	1	0	0	0	15	0/-1 -
S_2	-1	0	2	0	1	0	0	0	0/-1 -
S_3	2	0	1	-1	0	1	100	0	$\frac{100}{2} = 50$ - Pivot row
C_i	12	15	14	0	0	0			
Z_j	-15	15	0	15	0	0			
$C_i - Z_j$	27	0	14	-15	0	0			

↑ Pivot column

co-efficient

$Z_j = (\Sigma\, C_B \cdot S_i)$

x_1 - entering variable

S_3 - leaving variable

New x_2 row $= (-1\ 1\ 0\ 1\ 0\ 0\ 0) - -1(1\ 0\ \tfrac{1}{2}\ -\tfrac{1}{2}\ 0\ \tfrac{1}{2}\ 50)$

Calculate the pivot row:

$$\text{New pivot row} = \frac{\text{Current pivot row}}{\text{pivot element}}$$

New x_1 row $= (2\ 0\ 1\ -1\ 0\ 1,\ 100) \div 2$

x_1 - $1\ 0\ \tfrac{1}{2}\ -\tfrac{1}{2}\ 0\ \tfrac{1}{2}\quad 50$

New S_2 row = Current S_2 row - pivot element (New x_1 row).

$: (-1\ 0\ 2\ 0\ 1\ 0\ 0) - -1(1\ 0\ \tfrac{1}{2}\ -\tfrac{1}{2}\ 0\ \tfrac{1}{2}\ 50).$

- Construct a new table:

Basic	x_1	x_2	x_3	S_1	S_2	S_3	Soln	C_B	Ratio
x_1	1	0	1/2	-1/2	0	1/2	50	12	$\frac{50}{0.5} = 100$
x_2	0	1	1/2	1/2	0	1/2	50	15	$\frac{50}{0.5} = 100$
S_2	0	0	5/2	-1/2	1	1/2	50	0	$\frac{50}{2.5} = 50$
C_i	12	15	14	0	0	0			
Z_j	12	15	13.5	1.5	0	13.5			
$C_i - Z_j$	0	0	0.5	-1.5	0	-13.5			

P.C

S_2 - leaving variable
x_3 - entering variable

- Calculate pivot row:

New pivot row = $\dfrac{\text{Current pivot row}}{\text{Pivot element}}$

New x_3 row = $(0 \ \ 0 \ \ 5/2 \ \ -\frac{1}{2} \ \ 1 \ \ \frac{1}{2} \ \ 50) \div 5/2$

$0 \ \ 0 \ \ 1 \ \ -0.2 \ \ 0.4 \ \ 0.2 \ \ 20$

New x_2 row = Current x_2 row - pivot element (New x_1 row).

$(0 \ \ 1 \ \ \frac{1}{2} \ \ \frac{1}{2} \ \ 0 \ \ \frac{1}{2} \ \ 50) - \frac{1}{2}(0 \ \ 0 \ \ 1 \ \ -0.2 \ \ 0.4 \ \ 0.2 \ \ 20)$

$0 \ \ 1 \ \ 0 \ \ 3/5 \ \ -2/10 \ \ 2/5 \ \ 40$

New x_1 row = Current x_1 row - pivot element (New x_1 row)

$(1 \ \ 0 \ \ \frac{1}{2} \ \ \frac{-1}{2} \ \ 0 \ \ \frac{1}{2} \ \ 50) - \frac{1}{2}(0 \ \ 0 \ \ 1 \ \ -0.2 \ \ 0.4 \ \ 0.2 \ \ 20)$

$1 \ \ 0 \ \ 0 \ \ 0.4 \ \ 0.2 \ \ 0.4 \ \ 40$

Basic	x_1	x_2	x_3	S_1	S_2	S_3	Soln	C_B	Ratio
x_3	0	0	1	-0.2	0.4	0.2	20	14	
x_1	1	0	0	0.4	0.2	0.4	40	12	
x_2	0	1	0	0.6	-0.2	0.4	40	15	
C_i	12	15	14	0	0	0			
Z_j	12	15	14	1.4	5	13.6			
$C_i - Z_j$	0	0	0						

$x_1 = 40$ $\qquad$ $x_3 = 20$

$x_2 = 40$

Exercise:

Big M Method

→ This is another technique which involves solving LP problems with a mixture of inequalities, i.e: $\leq, \geq, =$.

→ In this case, a penalty is introduced for maximization and minimization problems, where a large capital $+M$ (penalty) is introduced for a minimization problem and a large capital $-M$ (penalty) introduced for a maximization problem.

Procedure:

1. Convert the inequality constraints to equality constraints by adding slack, surplus and artificial variables.
2. Calculate $C_j - Z_j$ of the last row
3. Examine the row that contains $C_j - Z_j$.
   i) If all the calculated values are ≥ 0, the solution is optimal.
   ii) If any of $C_j - Z_j < 0$, then the current solution is not optimal, and therefore, select the most negative $C_j - Z_j$ value and to the corresponding column becomes the pivot column. From pivot column we can obtain the pivot row, just like in ē simplex method.
4. If in the pivot column, all the values are negative, then the problem is said to be unbounded, i.e: has no solution.
5. Repeat the procedure, as in the simplex method until an optimal solution is reached.

Example:

Solve using big M method:

$\min(z) = 5x_1 + 3x_2$

s.t: $2x_1 + 4x_2 \le 12$

$2x_1 + 2x_2 = 10$

$5x_1 + 2x_2 \ge 10$

$x_1, x_2 \ge 0$

Solution:

① Write eqn in standard form:

$\le$ +S (slack), $\ge$ -S and A (surplus), = A → artificial variable

Min $z = 5x_1 + 3x_2 + 0S_1 + 0S_2 + A_1M + A_2M$ → because is minimization, add +M: co-efficient of Artificial variable.

s.t: $2x_1 + 4x_2 + S_1 = 12$

$2x_1 + 2x_2 + A_1 = 10$

$5x_1 + 2x_2 - S_2 + A_2 = 10$

$x_1, x_2, S_1, S_2, A_1, A_2 \ge 0$

* Penalize the objective function

$: 5x_1 + 3x_2 + 0S_1 + 0S_2 + A_1M + A_2M$

→ 20M ?? from where

* ② Construct Initial simplex table:

· S_2 - not Added bcz A_2 stands for the S_2 too - only 3 rows

Basic	x_1	x_2	S_1	S_2	A_1	A_2	Soln	C_B	Ratio → soln/P.C
S_1	2	4	1	0	0	0	12	0	6 (12/6)
~~S_2~~									
A_1	2	2	0	0	1	0	10	M	5 (10/2)
→ A_2	5	2	0	-1	0	1	10	M	2 (10/5)
C_j	5	3	0	0	M	M			
Z_j	7M	4M	0	-M	M	M	20M		
$C_j - Z_j$	5-7M	3-4M	0	M	0	0			

least positive → Pivot row

look at co-efficient of M. → 7 larger than 4

Pivot column

x_1 - entering variable

A_2 - leaving variable.

• New pivot row : Current row ÷ pivot element

$(5\ 2\ 0\ -1\ 0\ 10) \div 5$

New x_1 row $= 1\ \ \frac{2}{5}\ \ 0\ \ -\frac{1}{5}\ \ 0\ \ 2$

• If artificial variable is leaving, the column disappears.

• New S_1 row : Current S_1 row − new x_1 row

$(2\ 4\ 1\ 0\ 0\ 12) - 2(1\ \ \frac{2}{5}\ \ 0\ \ -\frac{1}{5}\ \ 0\ \ 2)$

$2-2(1)\quad 4-2(\frac{2}{5})\quad 1-2(0)\quad 0-2(-\frac{1}{5})\quad 0-2(0)\quad 12-2(2)$

$=(0\ \ \frac{16}{5}\ \ 1\ \ \frac{2}{5}\ \ 0\ \ 8)$

• New A_1 row :

$(2\ 2\ 0\ 0\ 1\ 10) - 2(\frac{1}{5}\ \ \frac{2}{5}\ \ 0\ \ -\frac{1}{5}\ \ 0\ \ 2)$

$(0\ \ \frac{6}{5}\ \ 0\ \ \frac{2}{5}\ \ 1\ \ 6)$

• ~~New A_2~~ New table:

Basic	x_1	x_2	S_1	S_2	A_1	Soln	C_B	Ratio
x_1	1	$\frac{2}{5}$	0	$-\frac{1}{3}$	0	2	5	$2 \div \frac{2}{5}$ 5
Pivot row → S_1	0	$\frac{16}{5}$	1	$\frac{2}{3}$	0	8	0	$8 \div \frac{16}{5}$ $\frac{5}{2}$
A_1	0	$\frac{6}{5}$	0	$\frac{2}{5}$	1	6	M	$6 \div \frac{6}{5}$ 5
C_j	5	3	0	0	M			
Z_j	5	$2+\frac{6}{5}M$	0	$-1+\frac{2}{5}M$	M	$10+6M$		
$C_j - Z_j$	0	$1-\frac{6M}{5}$	0	$1-\frac{2}{5}M$	0			

↑ Pivot column (x_2)

x_2 - Entering variable.

S_1 - leaving variable

• New pivot row : $(0\ \ \frac{16}{5}\ \ 1\ \ \frac{2}{5}\ \ 0\ \ 8) \div \frac{5}{2}$

New x_2 row : $(0\ \ 1\ \ \frac{5}{16}\ \ \frac{1}{8}\ \ 0\ \ \frac{5}{2})$

• New x_1 row : $(1\ \ \frac{2}{5}\ \ 0\ \ -\frac{1}{5}\ \ 0\ \ 2) - \frac{2}{5}(0\ \ 1\ \ \frac{5}{16}\ \ \frac{1}{8}\ \ 0\ \ \frac{5}{2})$

$1\ \ 0\ \ -\frac{1}{8}\ \ -\frac{1}{4}\ \ 0\ \ 1$

• New A_1 row : $(0\ \ \frac{6}{5}\ \ 0\ \ \frac{2}{5}\ \ 1\ \ 6) - \frac{6}{5}(0\ \ 1\ \ \frac{5}{16}\ \ \frac{1}{8}\ \ 0\ \ \frac{5}{2})$

$0\ \ 0\ \ -\frac{3}{8}\ \ \frac{1}{4}\ \ 1\ \ 3$

$\frac{16}{15} \times \frac{3}{5}$ 4 - 1

$\frac{3}{8} + \frac{5}{8}$ — $\frac{13}{8}$

Basic	x_1	x_2	S_1	S_2	A_1	Soln	C_B	Ratio
x_2	0	1	$\frac{5}{16}$	$\frac{1}{8}$	0	$\frac{5}{2}$	3	20
A_1	0	0	$-\frac{3}{8}$	$\frac{1}{4}$	1	3	M	12 ⟵ Pivot row
x_1	1	0	$-\frac{1}{8}$	$-\frac{1}{4}$	0	1	5	-4
C_j	5	3	0	0	M			
Z_j	5	3	$\frac{5}{16}-\frac{3}{8}M$	$\frac{1}{4}M-\frac{7}{8}$	M			
$C_j - Z_j$	0	0	$\frac{3}{8}M+\frac{5}{16}$	$\frac{7}{8}-\frac{1}{4}M$	0			

↑ Pivot column

$\frac{15}{16} - \frac{3}{8}m - \frac{5}{8}$

$\frac{15}{16} - \frac{5}{8}$

$\frac{15-10}{16}$

5

Leaving: A_1

Entering: S_2

$\frac{5}{16} - \frac{1}{8}\left(-\frac{3}{2}\right)$

New pivot row: $(0 \quad 0 \quad -\frac{3}{8} \quad \frac{1}{4} \quad 3) \div \frac{1}{4}$

New S_2 row $= (0 \quad 0 \quad -\frac{3}{2} \quad 1 \quad 12)$

New x_2 row: $(0 \quad 1 \quad \frac{5}{16} \quad \frac{1}{8} \quad \frac{5}{2}) - \frac{1}{8}(0 \quad 0 \quad -\frac{3}{2} \quad 1 \quad 12)$

$= (0 \quad 1 \quad \frac{1}{2} \quad 0 \quad 1)$

New x_1 row: $(1 \quad 0 \quad -\frac{1}{8} \quad -\frac{1}{4} \quad 1) + \frac{1}{4}(0 \quad 0 \quad -\frac{3}{2} \quad 1 \quad 12)$

$(1 \quad 0 \quad -\frac{1}{2} \quad 0 \quad 4)$

$\frac{5}{2} - \frac{3}{2}$

New table:

Basic	x_1	x_2	s_1	s_2	Soln	C_B	Ratio
x_2	0	1	$\frac{1}{2}$	0	1	3	
S_2	0	0	$-\frac{3}{2}$	1	12	0	
x_1	1	0	$-\frac{1}{2}$	0	4	5	
C_j	5	3	0	0			
Z_j	5	3	-1	0	23		
$C_j - Z_j$	0	0	1	0			

$\frac{5-3}{2}$ $\frac{2}{2}$

> It is an optimal soln because all values are 0 and above.

$x_1 = 4$

$x_2 = 1$

~~Exercise:~~

Note

→ For maximization problem, we examine the most positive value of $c_j - z_j$ and ~~all~~ if all are negative, an optimal soln has been found.

→ For minimization problem, we require the most negative valuse for $c_j - z_j$.

- **Exercise**

$$\text{Max } (z) = 6x_1 + 4x_2$$

s.t: $2x_1 + 3x_2 \leq 30$

$3x_1 + 2x_2 \leq 24$

$x_1 + x_2 = 3$

$x_1, x_2 \geq 0$

$$\text{Max } Z = 6x_1 + 4x_2 + 0S_1 + 0S_2 - A_1 M$$

s.t: $2x_1 + 3x_2 + S_1 = 30$

$3x_1 + 2x_2 + S_2 = 24$

$x_1 + x_2 + A_1 = 3$

$x_1, x_2, S_1, S_2, A_1 \geq 0$

	x_1	x_2	S_1	S_2	A_1	Soln	C_B	Ratio
S_1	2	3	1	0	0	30	0	
S_2	3	2	0	1	0	24	0	
A_1	1	1	0	0	1	3	-M	
c_j	6	4	0	0	-M			
$c_j - z_j$								

found.

Transportation problem

- Transportation can be described as the movement of items from one point, i.e source point / supply point / origin, to a given destination, i.e: demand point
- The objective in this case is to minimize the cost of transportation of items from ~~the begin~~ one point to another
- In transportation problem, the value of the items from the supply point is always denoted by a_i, while the number of items demanded at the destination point is always denoted by b_i
- Usually, we assume that the transportation problem is balanced, i.e: the number of items supplied is equals to the number of items demanded.

~~Procedure for~~

- Mathematically, transportation problem can be defined as follows:

$$\text{Min } (c) = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$$

$$\text{s.t: } \sum_{i=1}^{n} x_{ij} = a_i$$

$$\text{Also: } \quad \sum_{j=1}^{n} x_{ij} = b_i \qquad x_{ij} \geq 0$$

- where a_i is the qty of products available at source point i, and b_j is quantity of products needed at destination j
- C_{ij} is the cost of transporting one unit of a product from source point i to destination j.
- x_{ij} is the quality of the product transported from origin i to destination j

Procedure for solving transportation problem

1. Formulate the given problem in matrix form (table form)

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2. Find the initial feasible solution for the transportation problem. Can be done using the following methods:

i. North West Corner Method (NWCM)

ii. Least Cost Method (LCM)

iii. Vogel's Approximation Method (VAM)

3. Test the optimality of the soln obtained using the Modified Distribution Method (MoDI).

4. Update the soln accordingly and repeat step 3 until the most feasible solution is obtained.

→ The general format for transportation problem table is as follows:

Source \ Destination	D_1	D_2	...	D_n	
S_1	C_{11}	C_{12}	...	C_{1n}	a_1
S_2	C_{21}	C_{22}	...	C_{2n}	a_2
⋮	⋮	⋮		⋮	⋮
S_m	C_{m1}	C_{m2}	---	C_{mn}	a_m
Demand	b_1	b_2	...	b_n	

North West Corner Method

→ This method is used to find the initial feasible solution. It is the simplest method where the allocation of the available resource begins with the North West Corner cell of the table.

~~Pro~~

Procedure

1. Select the upper-Left hand corner cell of the given transportation table and allocate as much as possible the available resource to this cell

- This is done by considering the capacity of the first row and the first column, such that either of them is exhausted, i.e: x_{11} should be the minimum of $(a_i\ b_j)$ $x_{11} = \min(a_i\ b_j)$

2. i) If b_j is $> a_i$ move vertically down words to the second row and make the second allocation, i.e: ~~x_{21}~~
$x_{21} = \min(a_2, b_1 - a_1)$

ii) If $b_j < a_i$ then move horizontally to the next adjacent cell which occurs in column two., i.e: $x_{12} = \min(a_1 - b_1, b_2)$

iii) If $b_j = a_i$, move diagonally to the next cell in row 2 column 2 and make an allocation to this cell.

3. Repeat step. 1 and step 2 while moving vertically or horizontally until the South East Corner Cell of the transportation cell has been allocated.

Example:
Consider the transportation table given below and obtain the Initial feasible solution ~~g~~ using NWCM

Total supply = total demand
, balanced transportation problem

	Nairobi	Eldoret	Kisumu	Supply	
1	13 200	21	15	200	
2	20 50	5 100	10	150	100
3	2	14 110	6 310	420	-310
Demand	250	210	310	770	

Soln:

	Nairobi	Eldoret	Kisumu	Supply a_i	$\min(a_1, b_1)$
1	13 200	21	15	200	
2	20 50	5 100	10	150 · 100	
3	2	14 110	6 310	420 310	
Demand b_j	250	210	310		

Soln

$b_j > a_i$

• If supply met: move horizontally

$:(b_1 - a_1)$ $250 - 200 = 50$

move vertically downwards

Row 2: $\min(b_1 - a_1, a_2)$

$210 - 100 : 110$

Initial Cost of transportation:

$(13 \times 200) + (20 \times 50) + (5 \times 100) + (14 \times 110) + (6 \times 310)$

$\Rightarrow 7500$

Least Cost Method

→ Based on the cell with the smallest cost as the starting point in allocation process.

Procedure

1. Select cell with the lowest cost in the transportation table and to this cell, allocate as much as possible the available resource.
2. Eliminate row or column which have been satisfied.
3. To the remaining rows and columns, identify the cell with least cost value and to this cell allocate as much as possible the available resource.
4. Repeat the procedure in steps 1 and 2 until only one row or one column is remaining.

Note

If row or column have been satisfied simultaneously, then elliminate any of them at a time.

Example.

Apply Least Cost method technique to the above table: in obtaining the initial soln:

	Nairobi	Eldoret	kisumu	Supply
1	13	21 60	15 140	200
2	20	5 150	10	150
3	2 250	14	6 170	420 170
Demand	250	210 60	310 140	

Initial Cost

$: (21 \times 60) + (15 \times 140) + (5 \times 150) + (2 \times 250) + (6 \times 170)$

$\Rightarrow 5630$

Vogel's Approximation Method

→ Also known as penalty Method.

Procedure

1. Calculate the opportunity cost for each row and column. This is done by substracting the lowest cost cell from the second lowest cost cell for every column and row respectively.

2. Select row or column with the largest opportunity cost and to this row/column identify the least cost cell and to this cell allocate as much as possible, the available resources.

3. To the row/column which has been satisfied, eliminate and to the remaining rows and columns calculate new opportunity cost and repeat steps 1 and 2 until only one row/column is remaining.

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Note:

For a time when row/column has been satisfied, simultaneously select either row or column to be eliminated:

Example

Use VAM to obtain the initial feasible solution:

	D_1	D_2	D_3	Supply.	P_1	P_2	P_3
S_1	13	21 60	15 140	200	15-13 =2	6	6
→ S_2	20	5 150	10	150	10-5 =5	5	-
→ S_3	2 250	14	6 170	420 170	6-2 =4	8	8
demand.	250	210 60	310				
P_1	13-2	9	4				
P_2	11	9	4				

	D_1	D_2	D_3
P_2 →	-	7	9

① Opportunity (penalty) cost for each row:
second lowest cost cell - lowest cost cell

② Identify row/column with largest penalty

P_2 P_2

6

$$Cost = (21\times 60) + (15\times 140) + (5\times 150) + (2\times 250) + (6\times 170)$$

=>

Note

→ VAM is the most preferred method in finding the initial soln of transportation method because the solution obtained is near or an optimal solution to the transportation problem.

Exercise

Use the transportation table given below and find the initial soln using:

i) North West Corner Method.

ii) Least Cost Method

iii) Vogel's Approximation Method.

	D_1	D_2	D_3	D_4	Supply:
1	2	3	11	7	6
2	1	00	6	1	1
3	5	8	15	9	10
demand	7	5	3	2	

i) North West Corner Method:

	D_1	D_2	D_3	D_4	Supply
1	2 6	3	11	7	6
2	1 1	00	6	1	1
3	5	8 5	15 3	9 2	10
Demand	7	5	3	2	

cost :

$(2\times6)+(1\times1)+(8\times5)+(15\times3)+(9\times2)$

$\Rightarrow 116$

ii) Least Cost Method

	D_1	D_2	D_3	D_4	Supply
1	2 6	3	11	7	6
2	1	00 1	6	1	1
3	5 1	8 4	15 3	9 2	10
Demand	7	5	3	2	

Cost: $(2\times6)+(0\times1)+(5\times1)+(8\times4)+(15\times3)+(9\times2)$

$\Rightarrow 112$

iii) Vogel's Approximation

	D_1	D_2	D_3	D_4	Supply	P_1	P_2	P_3
1	2 (1)	3 (5)	11	7	6	1	1	5
2	1	0	6	1 (1)	1	1	-	-
3	5 (6)	8	15 (3)	9 (1)	10	3	3	4
Demand	7	5	3	2				
P_1	1	3	5	6				
P_2	3	5	4	2				
P_3	3	↑	4	2				

Cost: $(2\times1)+(3\times5)+(1\times1)+(5\times6)+(15\times3)+(9\times1)$
$\Rightarrow 102$

Modified Distribution Method (MoDi) / U-V Method.

→ After finding the initial soln for the transportation model, the next step is to test the optimality of the initial solution. This can be done using MoDi

→ To apply MoDi; We have to calculate every unoccupied cell in terms of an opportunity cost so as to reduce the total cost. Then the cell which has the largest -ve value of the opportunity cost is selected and is exchanged by an already occupied cell in a unique loop (closed path.) whose allocation will become zero at ē first move as more units are allocated to the newly selected unoccupied cell.

→ This process occurs repeatedly until there is no negative opportunity cost which indicates the sign of an optimal soln.

Procedure

1. Obtain the initial soln of the transportation problem using NWCM, LCM or VAM.

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45
45
93
·9
102

2. Count the number of occupied cells, if they are less than $(M+n-1)$, then there is degeneracy, i.e: the soln does not exist.

- In this case, we will then introduce a small +ve allocation of $\epsilon = 0$, to make the number of occupied cells equal to $(m+n-1)$

M - number of rows n - number of columns.

3. Solve the eqn: $C_{ij} = U_i + V_j$, for each occupied cell in the table, starting with any of $U_i = 0$ or $V_j = 0$ and solve progressively, the other values of U_i and V_j in the table.

4. Calculate the opportunity cost given by $d_{ij} = C_{ij} - (U_i + V_j)$ for each unoccupied cell.

5. Identify the cell with the most -ve value of d_{ij}. If none of the cells i) has a -ve value of d_{ij}, then the current soln in the table is optimal. $(d_{ij} \geq 0)$.

ii) two or more cells have $d_{ij} < 0$, then select the cell with ē largest negative value.

6. To the cell identified in step 5, allocate unknown quantity θ. Identify a closed loop that starts and ends at this cell, while connecting occupied cells. Add and subtract interchangeably, θ to and from the cells in the closed loop.

7. Assign the maximum value to θ in such a way that the value of any one of the basic cells (occupied) becomes 0 while the other remains +ve or 0

8. Repeat the procedure from step 3 until an optimal soln is reached.

u_i & v_j :- Dummy costs/ variables.

Example

Confirm whether soln obtained from NWCM is optimal

$C_{13} = u_1 + v_3$

	D_1	D_2	D_3	D_4	Supply	u_i
1	2 6	3	11	7	6	u_1
2	1 1	0	(6) +	1	1	u_2
3	5	8 5 +	15 3 -	9 2	10	u_3
Demand v_j	7 v_1	5 v_2	3 v_3	2 v_4		

m+n-1

- For occupied cells :   • Set any u_i / v_j to 0

$C_{ij} = u_i + v_j$ (cost)   ∴ Let $u_1 = 0$

step 3:

$C_{11} = u_1 + v_1 \rightarrow 2$ hence $u_1 = 0$, $v_1 = 2$

$C_{21} = u_2 + v_1 \rightarrow 1$ $u_2 = -1$ $v_1 = 2$

$C_{32} = u_3 + v_2 \rightarrow 8$ let $u_3 = 0$ $v_2 = 8$

$C_{33} = u_3 + v_3 \rightarrow 15$ $u_3 = 0$ $v_3 = 15$

$C_{34} = u_3 + v_4 \rightarrow 9$ $u_3 = 0$ $v_4 = 9$

step 4: For unoccupied cells :-

$$d_{ij} = C_{ij} - (u_i + v_j)$$

$d_{12} = 3 - (0+8) = -5$

$d_{13} = 11 - (0+15) = -4$

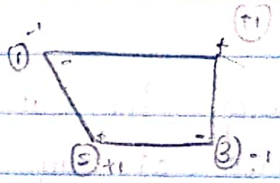
$d_{14} = 7 - (0+9) = -2$

$d_{22} = 0 - (-1+8) = -7$

$d_{23} = 6 - (-1+15) = -8$ → mark with +ve value → most -ve value.

$d_{24} = 1 - (-1+9) = -7$

$d_{31} = 5 - (0+2) = 3$



$\theta = 1$

	D_1	D_2	D_3	D_4	Supply
1	2	3	11	7	6
2	1 0	0	(6) + 1	1	1
3	5	8 + 6 5	15 2 -	9	10
Demand	7	5	3	2	

Repeat step 3 untill all $d_{ij} = 0$

Example 2:

Apply VAM to obtain ē initial soln.

	D_1	D_2	D_3	D_4	Supply	P_1	P_2	P_3	P_4
1	21 5	32	32	12 2	~~7~~ 2	9	9	20	20
2	72	32	42 7	62 2.	9	10	20	20	20
3	42	10 8	72	22 10	~~18~~ 10	12	20	50	-
Demand	5	8	7	14 ~~2~~ ~~4~~					
P_1	21	22	10	10					
P_2	21	↑	10	10					
P_3	-	-	10	10					
P_4	-	-	10	50					

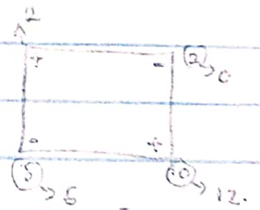
Cost => 547

Determine Whether it is Optimal: (MoDi

	D_1	D_2	D_3	D_4	Supply	
1	21 5	32	32	12 2	7	u_1
2	72	32 +	42 7	62 − 2	9	u_2
3	42	10 8 −	72	22 + 10	18	u_3
Demand	5	8	7	14		
	V_1	V_2	V_3	V_4		

→ use corners.
→ start from corner with min allocation

$\theta = 2.$



For occupied cells: $C_{ij} = U_i + V_j$

$C_{11} = 21 = U_1 + V_1$ let $u_1 = 0$ $V_1 = 21$

$12 = U_1 + V_4$ $V_4 = 12$

$42 = U_2 + V_3$ $U_2 = 50$ $V_3 = -12$

$62 = U_2 + V_4$ $U_2 = 50$ $V_4 = 12.$

$10 = U_3 + V_2$ $U_3 = 10$ $V_2 = 0$

$22 = U_3 + V_4$ $U_3 = 10$ $V_4 = 12.$

d_{ij} for un-occupied cells: $= C_{ij} - (u_i + v_j).$

$d_{12} : 32 - (0 + 0) = 32$

$d_{13} = 32 - (0 + 12) = 44$

$d_{21} = 72 - (50 + 21) \Rightarrow 1.$

$d_{22} = 32 - (50 + 0) = -18$

$d_{31} = 42 - (10 + 21) = 11$

$d_{32} = 72 - (10 - 12) = 74$

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	D_1	D_2	D_3	D_4	Supply	
1	21 (5)	32	32	12 2		u_1 - 21
2	72	32 2	42 7	62 0		u_2 - 53
3	42	10 6	72.	22 12		u_3 - 31
Demand.						
	$v_1 = 0$	$v_2 = -21$	$v_3 = -11$	$v_4 = -9$		

Cost = 811

Occupied:

$21 = u_1 + v_1$ let $v_1 = 0$ $u_1 = 21$

$12 = u_1 + v_4$ $v_4 = -9$

$32 = u_2 + v_2$ $u_2 = 53$

$42 = u_2 + v_3$ $v_3 = -11$

$10 = u_3 + v_2$ $v_2 = -21$

$22 = u_3 + v_4$ $u_3 = 31$

Un-occupied: $d_{ij} = C_{ij} - (u_i + v_j)$

$d_{12} = 32 - (21 - 21) \Rightarrow 32.$

$d_{13} = 32 - (21 - 11) \Rightarrow 22$

$d_{21} = 72 - (53 - 0) \Rightarrow 19$

$d_{24} = 62 - (53 - 9) \Rightarrow 18$

$d_{31} = 42 - (31 + 0) \Rightarrow 9$

$d_{33} = 72 - (31 - 11) \Rightarrow 52$

$\downarrow$

All of them are positive $\therefore$ Optimal.

Assignment Models

→ Another kind of transportation problem. Is basically a degenerated form of the transportation problem. The objective is to assign a number of resources to an equal number of activities, either to minimize the total cost or to maximize the total profit.

→ The units available and units demanded at each destination shld be equals to one i.e: there should be exactly one occupied cell in each row and each column of the table.

n = no. of occupied cell in place of $2n-1$ (the required one).

→ The mathematical formulation of assignment model is based on the matrix table shown below;

		Activity					
		A_1	A_2	A_3	....	A_n	Availability
Resource	1	a_{11}	a_{12}	a_{13}	....	a_{1n}	1
	2	a_{21}	a_{22}	a_{23}	....	a_{2n}	1
	⋮	⋮	⋮	⋮		⋮	⋮
	R_n	a_{n1}	a_{n2}	a_{n3}	....	a_{nn}	1
Required		1	1	1	....	1	

Mathematically:

$$Z = \sum_{i=1} \sum_{j=1} C_{ij} X_{ij}$$

s.t: $\sum_{i=1}^{n} X_{ij} = 1$ where $i = 1, 2 \dots n$ & $j = 1, 2 \dots m$.

$\sum_{j=1}^{n} X_{ij} = 1$

$X_{ij} = 0$ or 1.

Techniques:

1. Enumeration method.
2. Hungarian method.
3. Algebraic method

Hungarian Method

Procedure:

1. Determine the cost table of the given problem:
   i) If the no. of sources = no. of destination go to step 3
   ii) If the no. of sources ≠ no. of destinations go to step 2.

2. Add a dummy source or destination for making ē cost table a square matrix. The entries of a dummy source/destination are always zero.

3. Select the smallest element in each row of ē given cost matrix, and then subtract it from each element of that row.

4. In the reduced matrix of step 3, select the smallest element of each column and subtract it from each element of that column. Each row and column will have atleast 1 zero.

5. In the modified matrix of step 4, search an optimal assignment as follows: i) Examine the rows until a row with a single zero is found.
   ii) Highlight the zero with this mark: □ and X all other zeros in its column. Continue the process for all rows.
   iii) Repeat the above process for each column.
   iii) If a row/colum has ≥1 zeros, any of the zeros could be selected or highlighted and the remaining will be crossed out.
   iv) Repeat (i) to (iii) until chain of assignment (□) or X ends.

6. An optimal soln is considered to be reached if ē no. of assignment □ are equals to the order of the cost matrix, otherwise if the no. of □ are less, go to step 7

7. Draw min. number of horizontal and vertical lines to cover all the zeros. Steps:-

i) Mark (✓) on the rows that do not have any assigned zero.
ii) " " all columns that have zeros in ē marked rows
iii) " " on the rows that have assigned zeros in ē marked column
iv) Repeat (ii) & (iii) until ē chain of marking is completed.
v) Draw straight lines through all unmarked rows and marked columns.

8. Generate new matrix as follows:

i. Select smallest matrix not covered by any lines
ii. Subtract this element from all uncovered and add ē same to all elements of ē intersection of the two lines.
iii. Repeat the process from step 6 until an optimum soln is reached.

Example:
A department has 4 sub-ordinates & 4 tasks to be performed. Determine how ē task should be allocated so as to minimize total man hours.

	Task			
labour	A	B	C	D
1	20	15	40	21
2	28	30	21	28
3	19	16	20	26
4	13	28	17	12

Least element: subtract on each row.

• Row reduced matrix:

$$\begin{bmatrix} 5 & 0 & 25 & 6 \\ 7 & 9 & 0 & 7 \\ 3 & 0 & 4 & 10 \\ 1 & 16 & 5 & 0 \end{bmatrix}$$

Use to get column reduced matrix

- Column Reduced Matrix:

$$\begin{bmatrix} 4 & 0 & 25 & 6 \\ 6 & 9 & 0 & 7 \\ 2 & 0 & 4 & 10 \\ 0 & 16 & 5 & 0 \end{bmatrix}$$

- Make assignment:-

→ Examine rows

3<4

* Cannot repeat column/row where assignment has been made.

$$\begin{bmatrix} 4 & \boxed{0} & 25 & 6 \\ 6 & 9 & \boxed{0} & 7 \\ 2 & \cancel{0} & 4 & 10 \\ \boxed{0} & 16 & 5 & \cancel{0} \end{bmatrix} \begin{matrix} \checkmark \\ \\ \checkmark \\ \\ \end{matrix}$$

$\checkmark$ (under column 2)

· optimal if □ = order of matrix (4×4).

Hence go to next step:

no assignment (row).

→ also tick row corresponding to ☒ assignment

$$\begin{bmatrix} 4 & \boxed{0} & 25 & 6 \\ 6 & 9 & \boxed{0} & 7 \\ 2 & \cancel{0} & 4 & 10 \\ \boxed{0} & 16 & 5 & \cancel{0} \end{bmatrix} \begin{matrix} \checkmark \\ \text{— On un-ticked rows} \\ \checkmark \\ \\ \end{matrix}$$

on ticked columns.

- For the uncovered values:- subtract least element → Covered element do not change. → At intersection: add the least element

$$\begin{array}{c|cccc} & A & B & C & D \\ \hline 1. & 2 & \boxed{0} & 23 & 4 \\ 2. & 6 & 11 & \boxed{0} & 7 \\ 3. & \boxed{0} & \cancel{0} & 2 & 8 \\ 4. & \cancel{0} & 18 & 5 & \boxed{0} \end{array}$$

· Repeat from step 6: Assignment

□ = Order, i.e: 4 = 4

worker	task	time(min)
1	B	15
2	C	21
3	A	19
4	D	12

Min time:

15+21+19+12

=> 67 min.

Example 2:

Consider assignment table given below and determine how ē jobs will be assigned to various machines so that ē total cost is minimized.

	M_1	M_2	M_3	M_4
J_1	5	7	11	6
J_2	8	5	9	6
J_3	4	7	10	7
J_4	10	4	8	3

- Reduced row matrix

$$\begin{bmatrix} 0 & 2 & 6 & 1 \\ 3 & 0 & 4 & 1 \\ 0 & 3 & 6 & 3 \\ 7 & 1 & 5 & 0 \end{bmatrix}$$

- Column reduced:

$$\begin{bmatrix} 0 & 2 & 2 & 1 \\ 3 & 0 & 0 & 1 \\ 0 & 3 & 2 & 3 \\ 7 & 1 & 1 & 0 \end{bmatrix}$$

- Assigment

$$\begin{bmatrix} \boxed{0} & 2 & 2 & 1 \\ 3 & \boxed{0} & \cancel{0} & 1 \\ \cancel{0} & 3 & 2 & 3 \\ 7 & 1 & 1 & \boxed{0} \end{bmatrix}$$

3 < 4

Examine:

$$\begin{bmatrix} \boxed{0} & 2 & 2 & 1 \\ 3 & \boxed{0} & \cancel{0} & 1 \\ \cancel{0} & 3 & 2 & 3 \\ 7 & 1 & 1 & \boxed{0} \end{bmatrix}$$

All zeroes should be covered by lines

- New matrix

$$\begin{bmatrix} \cancel{0} & 1 & 1 & \cancel{0} \\ 4 & \boxed{0} & \cancel{0} & 1 \\ \boxed{0} & 2 & 1 & 2 \\ 8 & 1 & 1 & \boxed{0} \end{bmatrix} \qquad \begin{bmatrix} \cancel{0} & 1 & 1 & \cancel{0} \\ 4 & \boxed{0} & \cancel{0} & 1 \\ \boxed{0} & 2 & 1 & 2 \\ 8 & 1 & 1 & \boxed{0} \end{bmatrix}$$

$3 < 4$

- New matrix

$$\begin{bmatrix} \boxed{0} & \cancel{0} & \cancel{0} & \cancel{0} \\ 5 & \boxed{0} & \cancel{0} & 2 \\ \cancel{0} & 1 & \boxed{0} & 2 \\ 8 & \cancel{0} & \cancel{0} & \boxed{0} \end{bmatrix}$$

$4 = 4$

Optimal soln.

$J_1 \quad M_1 \rightarrow 5$

$J_2 \quad M_2 \rightarrow 5$

$J_3 \quad M_3 \rightarrow 10$

$J_4 \quad M_4 \rightarrow 3$

$\Rightarrow 23$

→ The above examples illustrated, demonstrate an assignment problem which involves minimization case

→ For maximization problem, the technique changes slightly and one has to convert maximization problem into minimization problem.

→ This could be achieved by subtracting all ē elements from the highest element in ē matrix:

Example:

Consider the below assignment table aimed at maximizing profit by 4 sales men. Find the assignment of how salesmen will be distributed.

Salesmen	Region R_1	R_2	R_3	R_4
A	18	12	16	13
B	16	13	17	17
C	17	17	15	14
D	15	14	16	17

~~make min~~

- Make min: subtract all from largest

$$\begin{bmatrix} 0 & 6 & 2 & 5 \\ 2 & 5 & 1 & 1 \\ 1 & 1 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{bmatrix}$$

Scanned with CamScanner

- Row reduced:

$$\begin{bmatrix} 0 & 6 & 2 & 5 \\ 1 & 4 & 0 & 0 \\ 0 & 0 & 2 & 3 \\ 2 & 3 & 1 & 0 \end{bmatrix}$$

- Column reduced:

$$\begin{bmatrix} 0 & 6 & 2 & 5 \\ 1 & 4 & 0 & 0 \\ 0 & 0 & 2 & 3 \\ 2 & 3 & 1 & 0 \end{bmatrix}$$

- Assignment

$$\begin{bmatrix} \boxed{0} & 6 & 2 & 5 \\ 1 & 4 & \boxed{0} & 0 \\ 0 & \boxed{0} & 2 & 3 \\ 2 & 3 & 1 & \boxed{0} \end{bmatrix}$$

4 = 4 : Optimal.

A - R_1 - 18.
B - R_3 - 17
C - R_2 - 17 ⇒ 69
D - R_4 - 17

- **Duality of LP models**

Eg:

Primal form

$\text{Max } Z = 2x_1 + 4x_2$

$\text{s.t: } 3x_1 + 4x_2 \le 4$

$2x_1 + x_2 \le 5$

$x_1, x_2 \geq 0$

	x_1	x_2	Soln
	3	4	4
	2	1	5
Z	2	4	0

① Into matrix form.

$$\rightarrow \left[\begin{array}{cc|c} 3 & 4 & 4 \\ 2 & 1 & 5 \\ \hline 2 & 4 & 0 \end{array}\right]$$

- Transpose : A^T
  interchange row & column

$$A^T = \left[\begin{array}{cc|c} 3 & 2 & 0 \\ 4 & 1 & 4 \\ \hline 4 & 5 & 0 \end{array}\right]$$

New LP model: $\text{Min } Z = 4y_1 + 5y_2$

$\text{s.t: } 3y_1 + 2y_2 \geq 2$

$4y_1 + y_2 \geq 4$