BST169 Autumn 2009
Misprints in the original paper have been corrected here.

1. Consider the model

$$y = X\beta + u \quad u \sim N\left(0, \sigma^2\Omega\right)$$

   (a) Write down the log-likelihood function based on T observations and show that the Maximum Likelihood estimator of β is

$$\beta_{ML} = \left(X'\Omega^{-1}X\right)^{-1} X'\Omega^{-1}y$$

Answer:

$$\begin{aligned} L &= -\frac{T}{2}\ln 2\pi - \frac{1}{2}\ln\left|\sigma^2\Omega\right| - \frac{1}{2\sigma^2}\left(y - X\beta\right)'\Omega^{-1}\left(y - X\beta\right) \\ &= -\frac{T}{2}\ln 2\pi - \frac{T}{2}\ln\sigma^2 - \frac{1}{2}\ln\left|\Omega\right| - \frac{1}{2\sigma^2}\left(y'\Omega^{-1}y - 2y'\Omega^{-1}X\beta + \beta'X'\Omega^{-1}X\beta\right) \end{aligned}$$

$$\frac{\partial L}{\partial \beta} = -\frac{1}{2\sigma^2}\left[-2X'\Omega^{-1}y - 2X'\Omega^{-1}X\beta\right] = \frac{1}{\sigma^2}\left[X'\Omega^{-1}y - X'\Omega^{-1}X\beta\right]$$

Hence,

$$\beta_{ML} = \left(X'\Omega^{-1}X\right)^{-1} X'\Omega^{-1}y$$

   (b) Show that this estimator is the best linear unbiased estimator of β
Answer:

$$\begin{aligned} V\left(\beta_{ML}\right) &= V\left\{\left[\left(X'\Omega^{-1}X\right)^{-1} X'\Omega^{-1}\right]u\right\} \\ &= \sigma^2\left(X'\Omega^{-1}X\right)^{-1} \end{aligned}$$

Let

$$b = Cy = \left[D + \left(X'\Omega^{-1}X\right)^{-1} X'\Omega^{-1}\right]y$$

be any other linear estimator.

$$\begin{aligned} E\left(b\right) &= E\left\{\left[D + \left(X'\Omega^{-1}X\right)^{-1} X'\Omega^{-1}\right]\left[X\beta + u\right]\right\} \\ &= DX\beta + \beta + \left[D + \left(X'\Omega^{-1}X\right)^{-1} X'\Omega^{-1}\right]u \end{aligned}$$

so require $DX = X'D' = 0$.

$$
\begin{aligned}
V(b) &= \sigma^2 C\Omega C' \\
&= \sigma^2 \left[D + \left(X'\Omega^{-1}X\right)^{-1} X'\Omega^{-1}\right] \Omega \left[D + \left(X'\Omega^{-1}X\right)^{-1} X'\Omega^{-1}\right]' \\
&= \sigma^2 \left\{D\Omega D' + \left(X'\Omega^{-1}X\right)^{-1} X'\Omega^{-1}\Omega D' + D\Omega\Omega^{-1}X\left(X'\Omega^{-1}X\right)^{-1} + \left(X'\Omega^{-1}X\right)^{-1} X'\Omega^{-1} \right. \\
&= \sigma^2 \left\{D\Omega D' + \left(X'\Omega^{-1}X\right)^{-1}\right\}
\end{aligned}
$$

which exceeds $V(\beta_{ML})$ by a *p.d.* matrix, $D\Omega D'$

(c) Show that the ML estimator can be obtained by regressing y_* on $X_*$where $y_* = P'y$ and $X_* = P'X$ for some matrix, P.
Answer:

$$
\begin{aligned}
\beta_{ML} &= \left(X'\Omega^{-1}X\right)^{-1} X'\Omega^{-1}y \\
&= \left(X'P'PX\right)^{-1} X'P'Py
\end{aligned}
$$

where $\Omega^{-1} = P'P$ since Ω is *p.d.* Hence, defining $X_* = PX, y_* = Py$

$$\beta_{ML} = \left(X_*'X_*\right)^{-1} X_*'y_*$$

(d) Comment on the issues that arise when P is the matrix:
$$
\begin{matrix}
\sqrt{1-\rho^2} & 0 & 0 & 0 & \cdots & 0 \\
-\rho & 1 & 0 & 0 & \cdots & 0 \\
0 & -\rho & 1 & 0 & \cdots & 0 \\
0 & 0 & -\rho & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & -\rho & 1
\end{matrix}
$$
Answer: This is an $AR(1)$ model. The first observation needs to be treated differently to give the error term the same variance.

2. Suppose that observations on y_t are generated by the model

$$y = X_1\beta_1 + X_2\beta_2 + u \quad u \sim N\left(0, \sigma^2 I\right)$$

where X_1 is $Tx(K-q)$,and X_2 is Txq.

(a) Derive the normal equations for the fitted regression

$$y = X_1\hat{\beta}_1 + X_2\hat{\beta}_2 + e$$

and show that they imply $X_1'e = X_2'e = 0$ and ,$M_1e = M_2e = e$ where e is the vector of residuals, and $M_i = I - X_i\left(X_i'X_i\right)X_i',\ i = 1, 2$.
Answer: Minimise

$$
\begin{aligned}
S &= \left(y - X_1\beta_1 - X_2\beta_2\right)'\left(y - X_1\beta_1 - X_2\beta_2\right) \\
&= y'y - y'X_1\beta_1 - y'X_2\beta_2 - \beta_1'X_1'y + \beta_1'X_1'X_1\beta_1 + \beta_1'X_1'X_2\beta_2 - \beta_2'X_2'y + \beta_2'X_2'X_1\beta_1 + \beta_2'X_2' \\
&= y'y - 2y'X_1\beta_1 - 2y'X_2\beta_2 + \beta_1'X_1'X_1\beta_1 + 2\beta_2'X_2'X_1\beta_1 + \beta_2'X_2'X_2\beta_2
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial S}{\partial \beta_1} &= -2X_1'y + 2X_1'X_1\beta_1 + 2X_1'X_2\beta_2 = 0 \\
\frac{\partial S}{\partial \beta_2} &= -2X_{2'}y + 2X_2'X_1\beta_1 + 2X_2'X_2\beta_2 = 0
\end{aligned}
$$

implies

$$
\begin{aligned}
X_1'y &= X_1'X_1\beta_1 + X_1'X_2\beta_2 \\
X_{2'}y &= X_2'X_1\beta_1 + X_2'X_2\beta_2
\end{aligned}
$$

Hence

$$
\begin{aligned}
X_1'e &= X_1'\left(y - X_1\beta_1 - X_2\beta_2\right) = 0 \\
X_2'e &= X_2'\left(y - X_1\beta_1 - X_2\beta_2\right) = 0
\end{aligned}
$$

$$
\begin{aligned}
M_1e &= \left(I - X_1\left(X_1'X_1\right)X_1'\right)e = e \\
M_2e &= \left(I - X_2\left(X_2'X_2\right)X_2'\right)e = e
\end{aligned}
$$

(b) Under the null hypothesis $H_0 : \beta_2 = 0, y = X_1\beta_1 + u \quad u \sim N\left(0, \sigma^2 I\right)$. Let $e_R = M_1y$ be the residual vector for the restricted model. Show that $e_R'e_R = \hat{\beta}'_2\left(X_2'M_1X_2\right)\hat{\beta}_2 + e'e$. Interpret this result.
Answer:

$$
M_1y = M_1\left(X_1\hat{\beta}_1 + X_2\hat{\beta}_2 + e\right) = M_1X_2\hat{\beta}_2 + e
$$

$$
\begin{aligned}
e_R'e_R &= y'M_1y \\
&= \left(M_1X_2\hat{\beta}_2 + e\right)'\left(M_1X_2\hat{\beta}_2 + e\right) \\
&= \hat{\beta}'_2X_2'M_1X_2\hat{\beta}_2 + \hat{\beta}'_2X_2'M_1e + e'M_1X_2\hat{\beta}_2 + e'e \\
&= \hat{\beta}'_2X_2'M_1X_2\hat{\beta}_2 + \hat{\beta}'_2X_2'e + e'X_2\hat{\beta}_2 + e'e \\
&= \hat{\beta}'_2\left(X_2'M_1X_2\right)\hat{\beta}_2 + e'e.
\end{aligned}
$$

(c) Show that, if the null hypothesis is correct, $\hat{\beta}_2$ has an expected value of zero and a variance of $\sigma^2\left(X_2'M_1X_2\right)^{-1}$. Hence, show that

$$
\frac{\hat{\beta}'_2\left(X_2'M_1X_2\right)\hat{\beta}_2}{\sigma^2} \sim \chi^2_q
$$

Answer: Using Frisch-Waugh

$$
\begin{aligned}
\hat{\beta}_2 &= \left(X_2'M_1X_2\right)^{-1}X_2'M_1y \\
&= \left(X_2'M_1X_2\right)^{-1}X_2'M_1\left(X_1\beta_1 + u\right) \\
&= \left(X_2'M_1X_2\right)^{-1}X_2'M_1u
\end{aligned}
$$

since $M_1 X_1 = 0$.

$$
\begin{aligned}
E(\beta_2 | X_1, X_2) &= 0 \\
V(\beta_2 | X_1, X_2) &= E\left[\left(\left(X_2' M_1 X_2\right)^{-1} X_2' M_1 u\right)\left(\left(X_2' M_1 X_2\right)^{-1} X_2' M_1 u\right)'\right] \\
&= E\left[\left(X_2' M_1 X_2\right)^{-1} X_2' M_1 u u' M_1 X_2 \left(X_2' M_1 X_2\right)^{-1}\right] \\
&= \sigma^2 \left(X_2' M_1 X_2\right)^{-1}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\hat{\beta}'_2 \left(X_2' M_1 X_2\right) \hat{\beta}_2}{\sigma^2} &= \frac{u' M_1 X_2 \left(X_2' M_1 X_2\right)^{-1} \left(X_2' M_1 X_2\right) \left(X_2' M_1 X_2\right)^{-1} X_2' M_1 u}{\sigma^2} \\
&= \frac{u' \left[M_1 X_2 \left(X_2' M_1 X_2\right)^{-1} X_2' M_1\right] u}{\sigma^2}
\end{aligned}
$$

The matrix in brackets is a real,symmetric, idempotent matrix of order t and rank equal to the number of columns in X_2, q. Hence,

$$
\frac{\hat{\beta}'_2 \left(X_2' M_1 X_2\right) \hat{\beta}_2}{\sigma^2} \sim \chi_q^2
$$

(d) Using the above results, explain why the null hypothesis, $H_0 : \beta_2 = 0$ can be tested using the test statistic

$$
F = \frac{\left(e_R' e_R - e' e\right)/q}{e' e / (T - K)} \sim F(q, T - K)
$$

Answer: Since

$$
e_R' e_R = \hat{\beta}'_2 \left(X_2' M_1 X_2\right) \hat{\beta}_2 + e' e.
$$

we have

$$
\frac{\hat{\beta}'_2 \left(X_2' M_1 X_2\right) \hat{\beta}_2}{\sigma^2} = \frac{\left(X_2' M_1 X_2\right) \left(X_2' M_1 X_2\right)^{-1} X_2' M_1 u}{\sigma^2} = \frac{\left(e_R' e_R - e' e\right)}{\sigma^2}
$$

so

$$
\frac{\hat{\beta}'_2 \left(\sigma^{-2} X_2' M_1 X_2\right) \hat{\beta}_2 / q}{\sigma^{-2} e' e / (T - K)} = \frac{\left(e_R' e_R - e' e\right)/q}{e' e / (T - K)}
$$

is the ratio of a χ_q^2 / q to a $\chi_{T-K}^2 / (T - K)$. These are independent since,

$$
\frac{\hat{\beta}'_2 \left(\sigma^{-2} X_2' M_1 X_2\right) \hat{\beta}_2 / q}{\sigma^{-2} e' e / (T - K)} = \frac{u' \left[M_1 X_2 \left(X_2' M_1 X_2\right)^{-1} X_2' M_1\right] u}{u' M u}
$$

and

$$
M_1 M = \left(I - X_1' \left(X_1' X_1\right) X_1\right) M = M
$$

so

$$M_1X_2\left(X_2'M_1X_2\right)^{-1}X_2'M_1M = M_1X_2\left(X_2'M_1X_2\right)^{-1}X_2'M = 0$$

Hence

$$F = \frac{\left(e_R'e_R - e'e\right)/q}{e'e/\left(T-K\right)} \sim F\left(q, T-K\right)$$

(e) Comment on the validity of this test statistic when the errors are not normally distributed?
Answer: Large sample validity.

3. Consider the model

$$y = X\beta + u$$

where y is a $T \times 1$ vector, X is a $T \times K$ matrix, β is a $K \times 1$ vector. It is believed that the coefficient vector for the first T_1 observations is different from that of the last $T_2 = T - T_1$ observations so $y_1 = X_1\beta_1 + u_1$ and $y_2 = X_2\beta_2 + u_2$. The unrestricted model can then be written

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Let $e_R = M_Xy$ be the residual vector from the restricted model and let $e_U = My$ be the residual vector for the unrestricted model.

(a) By any appropriate means, show that the restricted estimator of β is $\hat{\beta} = \left(X_1'X_1 + X_2'X_2\right)^{-1}\left(X_1'y_1 + X_2'y_2\right)$ and that the estimators of β_1 and β_2 can be obtained by running separate regressions for the two subsamples.
Answer: In the restricted model

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}\beta + \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Hence

$$\begin{aligned} \hat{\beta} &= \left\{\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}'\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}\right\}^{-1}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}'\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ &= \left(X_1'X_1 + X_2'X_2\right)^{-1}\left(X_1'y_1 + X_2'y_2\right) \end{aligned}$$

The unrestricted estimators are

$$\begin{aligned} \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix} &= \left\{\begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}'\begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}\right\}^{-1}\begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}'\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ &= \left\{\begin{bmatrix} X_1'X_1 & 0 \\ 0 & X_2'X_2 \end{bmatrix}\right\}^{-1}\begin{bmatrix} X_1'y_1 \\ X_2'y_2 \end{bmatrix} \\ &= \begin{bmatrix} \left(X_1'X_1\right)^{-1}X_1'y_1 \\ \left(X_2'X_2\right)^{-1}X_2'y_2 \end{bmatrix} \end{aligned}$$

(b) Writing the model in the form $y = W\beta_1 + Z\beta_2 + u$ show that $M = I - P_W - P_Z$. Hence, or otherwise, show that $e_U'e_U = e_1'e_1 + e_2'e_2$ where e_1 and e_2 are the residual vectors from the subsample regressions.
Answer:

$$y = W\beta_1 + Z\beta_2 + u$$

where

$$W = \begin{bmatrix} X_1 \\ 0 \end{bmatrix}, Z = \begin{bmatrix} 0 \\ X_2 \end{bmatrix}, W'Z = Z'W = 0$$

$$\begin{aligned}
M &= I - \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix} \left\{ \begin{bmatrix} X_1' & 0 \\ 0 & X_2' \end{bmatrix} \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix} \right\}^{-1} \begin{bmatrix} X_1' & 0 \\ 0 & X_2' \end{bmatrix} \\
&= I - [W, Z] \left\{ \begin{bmatrix} W' \\ Z' \end{bmatrix} [W, Z] \right\}^{-1} [W, Z]' \\
&= I - [W, Z] \begin{bmatrix} (W'W)^{-1} & 0 \\ 0 & (Z'Z)^{-1} \end{bmatrix} \begin{matrix} W' \\ Z' \end{matrix} \\
&= I - [W, Z] \begin{bmatrix} (W'W)^{-1} W' \\ (Z'Z)^{-1} Z' \end{bmatrix} \\
&= I - W (W'W)^{-1} W' - Z (Z'Z)^{-1} Z' \\
&= I - P_W - P_Z
\end{aligned}$$

$$\begin{aligned}
P_W &= \begin{bmatrix} X_1 \\ 0 \end{bmatrix} \left\{ \begin{bmatrix} X_1' \\ 0 \end{bmatrix} \begin{bmatrix} X_1 & 0 \end{bmatrix} \right\}^{-1} \begin{bmatrix} X_1' \\ 0 \end{bmatrix} = \begin{bmatrix} (X_1'X_1)^{-1} X_1' & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} P_1 & 0 \\ 0 & 0 \end{bmatrix} \\
P_Z &= \begin{bmatrix} 0 & 0 \\ 0 & P_2 \end{bmatrix}
\end{aligned}$$

Hence,

$$\begin{aligned}
e_U'e_U &= y'My \\
& \quad y' (I - P_W - P_Z) y \\
&= y'y - y'P_W y - y'P_Z y \\
&= y_1'y_1 + y_2'y_2 - y_1'P_1 y_1 - y_2'P_2 y_2 \\
&= e_1'e_1 + e_2'e_2
\end{aligned}$$

(c) Show that the unrestricted model can be written $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} X_1 & 0 \\ X_2 & I \end{bmatrix} \begin{bmatrix} \beta \\ \gamma \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$. Interpret the coefficient vector, γ.

Answer: Define $\delta = \beta_2 - \beta_1$

$$\begin{aligned}
\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} &= \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\
&= \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_1 + \delta \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\
&= \begin{bmatrix} X_1\beta_1 \\ X_2\beta_1 + X_2\delta \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\
&= \begin{bmatrix} X_1 & 0 \\ X_2 & I \end{bmatrix} \begin{bmatrix} \beta \\ \gamma \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}
\end{aligned}$$

where $\beta = \beta_1$ and $\gamma = X_2\delta$ is a dummy variable.

(d) Describe a procedure for testing the hypothesis that $\beta_1 = \beta_2$ based on $\hat{\gamma}$.
Answer: If $\beta_1 = \beta_2$, $\delta = 0$ implying $\gamma = 0$. Hence. joint exclusion test on the γ parameters.

4. Consider the market model

$$\begin{aligned}
q_t^d &= \alpha p_t + \beta y_t + u_t^d \\
q_t^s &= \gamma p_t + u_t^s \\
q_t^d &= q_t^s
\end{aligned}$$

where q is quantity, p is price and y is income. All variables are in logarithms. The error terms in the demand and supply equations are uncorrelated and y is exogenous. We wish to estimate the supply parameter, γ.

(a) Assess the identifiability of the equations of the model.
Answer: We have

$$\begin{aligned}
q_t - \alpha p_t - \beta y_t &= u_t^d \\
q_t - \gamma p_t &= u_t^s
\end{aligned}$$

or

$$\begin{bmatrix} 1 & -\alpha \\ 1 & -\gamma \end{bmatrix} \begin{bmatrix} q_t \\ p_t \end{bmatrix} + \begin{bmatrix} -\beta \\ 0 \end{bmatrix} y_t = u_t$$

The Order Condition for the parameters of the g^{th}equation to be identified is that

$$(K - K_g) + (G - G_g) \geq G - 1$$

Here $G = 2$, $K = 1$ and $G = 1 = 1$. For the demand equation

$$(K - K_g) + (G - G_g) = 0$$

andf for the supply equation

$$(K - K_g) + (G - G_g) = 1$$

Hence the supply equaiotn is exactly identified by the Order Condition. The Rank Condition is that the matrix formed by removing from $[B', \Gamma']$ the g^{th} row and all columns with non-zero coefficients in row g has at least one subdeterminant of rank $G-1$. Only the supply equation is at issue.

$$[B', \Gamma'] = \begin{bmatrix} 1 & 1 & -\beta \\ -\alpha & \gamma & 0 \end{bmatrix}$$

so the relevant determinant is β so the supply equaiton is identified if $\beta \neq 0$.

(b) Show that the OLS estimator of γ is biased and inconsistent.
Answer:

(c) Describe a suitable estimator of γ and discuss its properties.

(d) How, if at all, you would carry out the estimation if the supply equation took the form $q_t^s = \gamma p_t + \delta w_t + u_t^s$ where w is an exogenous variable measuring costs of production.
Answer:

5. Discuss TWO of the following

   (a) Specification Tests
   (b) Prediction
   (c) Instrumental Variables Estimation
   (d) Large Sample Properties of Estimators

6. Consider the Classical Linear Model

$$y = X\beta + u$$

where y is a $T \times 1$ vector, β is a $K \times 1$ vector, X is a non-stochastic $T \times K$ matrix with values of unity in the first column and u is a $T \times 1$ vector with zero mean and covariance matrix $\sigma^2 I$. Let $\hat{\beta}$ be the OLS estimator of β and let e be the vector of OLS residuals

(a) Show that $COV\left(\hat{\beta}, e\right) = 0$
Answer:

$$\begin{aligned} COV\left(\hat{\beta}, e\right) &= COV\left((X'X)^{-1} X'y, My\right) \\ &= E\left[(X'X)^{-1} X'u\,(Mu)'\right] \\ &= \sigma^2 E\left[(X'X)^{-1} X'M\right] \\ &= 0 \end{aligned}$$

since $MX = 0$.

(b) Suppose e is regressed on X. Show that the estimated coefficients from this regression are all zero.
Answer: The coefficient vector is

$$(X'X)^{-1} X'e = (X'X)^{-1} X' \left(y - X\hat{\beta}\right) = \hat{\beta} - \hat{\beta} = 0$$

(c) Let A be a non-singular $K \times K$ matrix and define $Z = XA$. Suppose that y is regressed on Z to estimate the parameter vector, γ. What is the relationship between the estimated parameter vectors, $\hat{\beta}$ and $\hat{\gamma}$ What is the relationship between the two sets of fitted values and residuals?
Answer:

$$\begin{aligned}
\hat{\gamma} &= (Z'Z)^{-1} Z'y \\
&= (A'X'XA)^{-1} A'X'y \\
&= A^{-1} (X'X)^{-1} (A')^{-1} A'X'y \\
&= A^{-1}\hat{\beta}
\end{aligned}$$

Resids are

$$y - Z\hat{\gamma} = y - ZA^{-1}\hat{\beta} = y - XAA^{-1}\hat{\beta} = y - X\hat{\beta}$$

(d) Let Z be a $T \times L$ matrix of additional variables and suppose that an investigator estimates $y = X\beta + Z\gamma + u$. Show that the least squares estimator, $\hat{\beta}$, is unbiased but inefficient
Answer: By the Frisch-Waugh Theorem

$$\begin{aligned}
\hat{\beta} &= (X'M_Z X)^{-1} X'M_Z y \\
&= \beta + (X'M_Z X)^{-1} X'M_Z u
\end{aligned}$$

Hence,

$$E\left(\hat{\beta}\right) = \beta + (X'M_Z X)^{-1} X'M_Z E(u) = \beta$$

and

$$COV\left(\hat{\beta}\right) = \sigma^2 (X'M_Z X)^{-1}$$

whereas the variance of the 'correct' estinator is

$$COV(b) = \sigma^2 (X'X)^{-1}$$

Comparing precisions

$$\frac{X'X}{\sigma^2} - \frac{X'M_Z X}{\sigma^2} = \frac{X'P_Z X}{\sigma^2} \geq 0$$

since

$$X'P_Z X = (P_Z X)' P_Z X$$

is positive semidefinite.

7. Using appropriate illustrations, examine the ways in which 'auxiliary regressions' can be used to test hypotheses in regression models.

8. The tables below gives Eviews output for estimating the cost function for a sample of 158 electricity-generating firms. The model is

$$\log(C) = \beta_0+\beta_1 \log(Q)+\beta_3 \log(Q)^2+\beta_4 \log(p_k)+\beta_5 \log(p_l)+\beta_6 \log(p_f)+u$$

Where C is total cost, Q is output, and the other variables are the prices of capital services, labour services and fuel, respectively.

(a) Economic Theory implies that the coefficients on the price variables sum to 1. Test this hypothesis using a Wald test and a Likelihood Ratio test.
Answer: Wald Test:

$$\frac{\left(RSS^R - RSS^U\right)/q}{RSS^U/(T-K)} = \frac{(2.904896 - 2.904618)}{2.904896/152} = 0.0145$$

Clearly, cannot reject. There's a 90% prob value. (0.9042)
LR Test

$$LR = -2\,(91.50731 - 91.51486) = 0.0151$$

which has a prob value of 0.9022

(b) Specify a version of the model which would allow you test the above null hypothesis using a t-test.
Answer:

$$\log(C) = \beta_0+\beta_1 \log(Q)+\beta_3 \log(Q)^2+\beta_4 [\log(p_k/p_f)]+\beta_5 [\log(p_l/p_f)]+\gamma \log(p_f)+u$$

where

$$\gamma = \beta_4 + \beta_5 + \beta_6$$

which is 1 under the null. Hence

$$t_{T-K} = \frac{\hat{\gamma} - 1}{est\ s.e.(\hat{\gamma})}$$

Alternatively, rewriting as

$$\log(C/p_f) = \beta_0+\beta_1 \log(Q)+\beta_3 \log(Q)^2+\beta_4 [\log(p_k/p_f)]+\beta_5 [\log(p_l/p_f)]+\gamma \log(p_f)+u$$

and reported t ratio will do the job.

(c) The last 35 firms have a different structure to the first 123. Test the hypothesis that the coefficients for these two subsets of firms are different
Answer:

$$\begin{aligned} RSS^U &= RSS_1 + RSS_2 \\ &= 2.441520 + 0.309006 \end{aligned}$$

$$F\left(K,T-2K\right)=\frac{\left(RSS^{R}-RSS^{U}\right)/K}{RSS^{U}/\left(T-2K\right)}=\frac{\left(2.904618-2.441520-0.309006\right)/6}{\left(2.441520+0.309006\right)/146}=1.3632$$

probval $= 0.2332$ so do not reject same coefficients.

(d) .Comment on the results presented in these regressions.
Answer: