Brief Solutions

1. (a) From Frisch-Waugh Theorem, $\hat{\beta}_2 = \frac{x_2' M_1 y}{x_2' M_1 x_2}$ and $Var(\hat{\beta}_2) = \frac{\sigma^2}{x_2' M_1 x_2}$.

   (b) Note that $\hat{\beta}_2 = \beta + \frac{x_2' M_1 \epsilon}{x_2' M_1 x_2}$. If $\epsilon \sim N(0, \sigma^2 I)$, then $\hat{\beta}_2 \sim N\left(\beta, \frac{\sigma^2}{x_2' M_1 x_2}\right)$. Hence one can obtain the result in the question.

   (c) The expression will follow student t distribution. See Theorem 4.3 in Lecture 2.

   (d) The t statistic will be valid (its distribution will be close to normal) given the sample size is large and the conditions in Theorem 8.1 and 8.2 in Lecture 3 are satisfied.

2. See Page 3 in Lecture 6 for the first question. See Eq.(22) and Theorem 4.3 in Lecture 6 with $s(\hat{\theta})$ replaced by $R\hat{\theta}$ and $\frac{\partial s(\hat{\theta})}{\partial \theta}$ by R.

3. (a) $\hat{a_1}^{OLS}$ is consistent in [3.1] while not consistent in [3.2] since x_{1i} is not correlated with y_i while x_{2i} is correlated with y_i.

   (b) The asymptotic bias in [3.2] is $a_2 \frac{Cov(x_{2i}, y_i)}{Var(x_{2i})} = \frac{a_2^2 \sigma_y^2}{a_1^2 \sigma_x^2 + a_2^2 \sigma_y^2 + \sigma_u^2}$.

   (c) $Cov(x_{1i}, y_i | x_{2i}) = Cov(x_{1i}, y_i) - \frac{Cov(x_{1i}, x_{2i}) Cov(y_i, x_{2i})}{Var(x_{2i})} = -\frac{a_2 \sigma_y^2 a_1 \sigma_x^2}{a_1^2 \sigma_x^2 + a_2^2 \sigma_y^2 + \sigma_u^2}$.

   (d) The p-limit is $\frac{Cov(x_{1i}, y_i | x_{2i})}{Var(x_{1i} | x_{2i})} = -\frac{a_2 \sigma_y^2 a_1}{a_2^2 \sigma_y^2 + \sigma_u^2}$, which is not 0 if $a_1, a_2 \neq 0$. This implies we have to be careful about the causality relationship when running a regression. If one swaps the position of the dependent and the independent variable, one may find some spurious relationship.

4. (a) See Theorem 4.3 in Lecture 3.

   (b) See Theorem 5.7 in Lecture 3.

5. (a) If *spread* $= 0$, there is no favourite team. One can expect β_0 to be 0.5 and α_0 to be 0.

   (b) One should use the heteroskedasticity consistent standard error to obtain the t statistic $\frac{0.5769 - 0.5}{0.052} \approx 1.48$, which is not significant. Hence the hypothesis should not be rejected.

   (c) The probability for the favourite team to win when *spread* $= 5$ is $\Phi(0.452) > \Phi(0) = 0.5$.

(d) One can calculate the likelihiood ratio test statistic, $2(263.5622-262.6418) = 1.8408$, which is smaller than the 5% critical value (7.84) from a chi-squared distribution with degrees of freedom equal to 3.

6. (a) Define $X = [\iota, x]$. Using the information from $W'W$, one can find the sample size is 40 and the OLS estimates: $(\hat{\beta}_0, \hat{\beta}_1)' = (X'X)^{-1}X'y = (1.5, -0.5)'$.

   (b) Define $Z = [\iota, z]$, the IV estimates are $\hat{\beta}_{IV} = (Z'X)^{-1}Z'y = (0.8, 0.2)'$. Since the OLS estimate for β_1 is less than the IV estimate, one could expect the bias is negative and the correlation between x and ϵ is negative.

   (c) One can estimate the covariance matrix by $\widehat{\sigma^2}(Z'X)^{-1}Z'Z(X'Z)^{-1}$, where $\widehat{\sigma^2} = \frac{(y-X\hat{\beta}_{IV})'(y-X\hat{\beta}_{IV})}{40-2} \approx 0.21$. The corresponding variance estimate of $\hat{\beta}_1^{IV}$ is 0.0416 and hence the t statistic is $\frac{0.2}{\sqrt{0.0416}} = 0.98$, which is not significant at 5% level.