MSc Economics
Econometrics BSP816 Resit

Summer 2008
Two Hours
Answer THREE questions

1. An investigator wants to estimate the parameters of a set of models

$$y_i = X_i\beta_i + u_i \quad u_i \sim N\left(0, \sigma_i^2 I_T\right)$$

for $i = 1, 2, ..., N$ cross-sectional units. There are T observation on each entity, y_i is a $T \times 1$ vector of observations on entity i, ordered by time, X_i is a $T \times K$ matrix, β_i is a $K \times 1$ parameter vector and u_i is a $T \times 1$ vector of errors for entity i. Although the number of regressors is the same for each entity, the list of regressors may be differen from entity to entity. It is assumed that the error term for each entity is serially uncorrelated and homoscedastic but that the errors for entiity i may be contemporaneously correlated with the errors for entity j so that

$$E\left(u_i u_j\right) = \sigma_{ij} I_N$$

(a) Show that the model can be written in the form

$$y = X\beta + u \quad u \sim N\left(0, \Omega\right)$$

where X is a block diagonal matrix.
Answer: Stacking the models

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix} = \begin{bmatrix} X_1 & 0 & \dots & 0 \\ 0 & X_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & X_N \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_N \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix}$$

$$V\left(u\right) = \Omega = \begin{bmatrix} \sigma_{11} I & \sigma_{12} I & \dots & \sigma_{1N} I \\ \sigma_{21} I & \sigma_{22} I & \cdots & \sigma_{2N} I \\ \vdots & \vdots & \vdots & \vdots \\ \sigma_{N1} I & \sigma_{N2} I & \cdots & \sigma_{NN} I \end{bmatrix}$$

(b) Explain why the N parameter vectors, β_i, can be estimated by applying OLS to each model in turn but that this is, in general, inefficient.
Answer: The OLS estimator satisfies

$$(X'X)\,\hat{\beta} = X'y$$

where

$$\begin{aligned}
X'X &= \begin{bmatrix} X_1 & 0 & \dots & 0 \\ 0 & X_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & X_N \end{bmatrix}' \begin{bmatrix} X_1 & 0 & \dots & 0 \\ 0 & X_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & X_N \end{bmatrix} \\
&= \begin{bmatrix} X_1' & 0 & \dots & 0 \\ 0 & X_2' & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & X_N' \end{bmatrix} \begin{bmatrix} X_1 & 0 & \dots & 0 \\ 0 & X_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & X_N \end{bmatrix} \\
&= \begin{bmatrix} X_1'X_1 & 0 & \dots & 0 \\ 0 & X_2'X_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & X_N'X_N \end{bmatrix}
\end{aligned}$$

and

$$\begin{aligned}
X'y &= \begin{bmatrix} X_1' & 0 & \dots & 0 \\ 0 & X_2' & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & X_N' \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix} \\
&= \begin{bmatrix} X_1'y_1 \\ X_2'y_2 \\ \vdots \\ X_N'y_N \end{bmatrix}
\end{aligned}$$

Hence,

$$\begin{bmatrix} X_1'X_1 & 0 & \dots & 0 \\ 0 & X_2'X_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & X_N'X_N \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{bmatrix} = \begin{bmatrix} X_1'y_1 \\ X_2'y_2 \\ \vdots \\ X_N'y_N \end{bmatrix}$$

so

$$b_i = \left(X_i'X_i\right)^{-1} X_i'y$$

However, Ω is not in the form $\sigma^2 I$ and so the conditions for the Gauss-Markov theoren=m re not satisfied.

(c) Show that the efficient estimator takes the form

$$\hat{\beta} = \left(X'\Omega^{-1}X\right)^{-1} X'\Omega^{-1}y$$

Answer: This is the GLS estimator.

(d) Explain why $\hat{\beta}$ is, in general, not a feasible estimator and outline a feasible procedure for estimating β.
Answer: Ω unknown. Feasible procdure: estimate exah model by OLS, collect residuals and use these to estimate $\dot{\Omega}$ Aplly GLS using $\hat{\Omega}$.

2. Consider the regression model

$$y = S\delta + X\beta + u \quad u \sim N\left(0, \sigma^2 I\right)$$

where X is a $T \times K$ matrix, β is a $K \times 1$ vector, S is a $T \times 4$ matrix of quarterly seasonal dummies of the form

$$S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

and δ is a 4×1 vector of seasonal dummy coefficients. For simplicity, assume that there are $T/4$ observatns on each season.

(a) An investigator estimates the model

$$y = S\delta + v$$

by OLS. Show that the fitted value of the dependent variable for each season is simply the average value of the dependent variable for that season.
Answer:

$$\hat{\delta} = \left(S'S\right)^{-1} S'y$$

where

$$S'S = \begin{bmatrix} T/4 & 0 & 0 & 0 \\ 0 & T/4 & 0 & 0 \\ 0 & 0 & T/4 & 0 \\ 0 & 0 & 0 & T/4 \end{bmatrix} \quad S'y = \begin{bmatrix} \sum_1 y \\ \sum_2 y \\ \sum_3 y \\ \sum_4 y \end{bmatrix}$$

so

$$\hat{\delta} = \begin{bmatrix} \bar{y}_1 \\ \bar{y}_2 \\ \bar{y}_3 \\ \bar{y}_4 \end{bmatrix}$$

and

$$\hat{y} = S\hat{\delta} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} \bar{y}_1 \\ \bar{y}_2 \\ \bar{y}_3 \\ \bar{y}_4 \end{bmatrix} = \begin{bmatrix} \bar{y}_1 \\ \bar{y}_2 \\ \bar{y}_3 \\ \bar{y}_4 \\ \bar{y}_1 \\ \bar{y}_2 \\ \bar{y}_3 \\ \bar{y}_4 \\ \vdots \end{bmatrix}$$

(b) Show that the this results in a biased estimator of δ unless $S'X = 0$.
Answer:

$$\begin{aligned} E\left(\hat{\delta}\right) &= \left(S'S\right)^{-1} S'\left(S\delta + X\beta + E\left(u\right)\right) \\ &= \delta + \left(S'S\right)^{-1} S'X\beta \end{aligned}$$

(c) Show that an unbiased estimator of δ can be obtained by regressing the residuals from the regression in (a) on the residuals obtained from regressing the K variables in X on the seasonal dummy variables in S.
Answer:

(d) What problem would arise if X contained a column of 1s?
Answer:

3. The model

$$y = X\beta + u \quad u \sim N\left(0, \sigma^2 I\right)$$

is to be estimated by OLS using T observations on the K variables in X. Let $\hat{\beta}$ be the OLS estimator of $\dot{\beta}$. The model is assumed to contain a constant so that we can write

$$X = [i, Z]$$

where i is the T-dimensional sum vector and Z is a $T \times (K-1)$ matrix.. As a result, the equation can be written in the partitioned form,

$$y = i\beta_1 + Z\beta_2 + u$$

Let b_1 and b_2 be the OLS estimators of β_1 and β_2, respectively. Finally, define the matrix M^0 by

$$M^0 = I_T - \frac{1}{T} ii'$$

where I_T is the identity matrix of order T.

(a) Show that M^0 is a symmetric, idempotent, matrix of rank $T-1$. (You are not required to prove any theorems which you may use regarding the rank of idempotent matrices.)
Answer: I_T is symmetric and ii' is a $T \times T$ matrix of 1s and so is symmetric. Hence,

$$\left(M^0\right)' = \left(I_T - \frac{1}{T}ii'\right)' = (I_T)' - \left(\frac{1}{T}ii'\right)' = I_T - \frac{1}{T}ii' = M^0$$

and

$$\begin{aligned} M^0 M^0 &= \left(I_T - \frac{1}{T}ii'\right)\left(I_T - \frac{1}{T}ii'\right)' \\ &= I_T - \frac{1}{T}ii' - \frac{1}{T}ii' + \frac{1}{T^2}ii'\left(ii'\right)' \\ &= I_T - \frac{1}{T}ii' - \frac{1}{T}ii' + \frac{1}{T^2}ii'\left(i'i\right) \\ &= I_T - \frac{2}{T}ii' + \frac{1}{T}ii' \\ &= I_T - \frac{1}{T}ii' \\ &= M^0 \end{aligned}$$

where the third equality uses the fact that $i'i = T$. Finally, since the rank of an idempotent matrix is equal to its trace, and since

$$\begin{aligned} tr\left(M^0\right) &= tr\left(I_T - \frac{1}{T}ii'\right) \\ &= tr\left(I_T\right) - \frac{1}{T}tr\left(ii'\right) \\ &= T - \frac{1}{T}T \\ &= T - 1 \end{aligned}$$

M^0 has rank $T-1$.

(b) Solve the normal equations of the model to show that the solution for b_2 is

$$b_2 = \left(Z'M^0Z\right)^{-1} Z'M^0y$$

Interpret this result.
Answer: The normal equations are

$$X'y = X'Xb$$

so we have

$$\begin{bmatrix} i' \\ Z' \end{bmatrix} y = \begin{bmatrix} i' \\ Z' \end{bmatrix} [i, Z] \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

or

$$\begin{aligned} i'y &= ii'b_1 + i'Zb_2 \\ Z'y &= Z'ib_1 + Z'Zb_2 \end{aligned}$$

where b_1 is a scalar. Solving for b_1 from the first equation and substituting into the second

$$Z'y = Z'i\left[\frac{i'y - i'Zb_2}{T}\right] + Z'Zb_2$$

Rearranging

$$Z'y - Z'i\frac{i'y}{T} = Z'Zb_2 - Z'i\frac{i'Z}{T}b_2$$

or

$$Z'\left[I_T - \frac{ii'}{T}\right]y = Z'\left[I_T - \frac{ii'}{T}\right]Zb_2$$

or

$$Z'M^0y = Z'M^0Zb_2$$

Since $Z'M^0Z = (M^0Z)'\left(M^0Z\right)$ is a positive definite matrix, it is non-singular and so we have

$$b_2 = \left(Z'M^0Z\right)^{-1}Z'M^0y$$

This means that β_2 can be estimated by first removing the means from y and the variables in Z and then running a regression on the mean-adjusted variables.

(c) Determine the variance-covariance matrix of b_2.
Answer:

$$\begin{aligned} b_2 &= \left(Z'M^0Z\right)^{-1}Z'M^0\left(X\beta + u\right) \\ &= \left(Z'M^0Z\right)^{-1}Z'M^0\left([i,Z]\left[\begin{array}{c}\beta_1 \\ \beta_2\end{array}\right] + u\right) \\ &= \left(\left(Z'M^0Z\right)^{-1}Z'M^0i\beta_1 + \left(Z'M^0Z\right)^{-1}Z'M^0Z\beta_2 + \left(Z'M^0Z\right)^{-1}Z'M^0u\right) \\ &= \beta_2 + \left(Z'M^0Z\right)^{-1}Z'M^0u \end{aligned}$$

since

$$M^0i = I_Ti - \frac{1}{T}i\left(i'i\right) = i - \frac{1}{T}iT = 0$$

Hence,

$$\begin{aligned} V\left(b_2\right) &= E\left(b_2 - \beta_2\right)\left(b_2 - \beta_2\right)' \\ &= E\left[\left(Z'M^0Z\right)^{-1}Z'M^0uu'M^0Z\left(Z'M^0Z\right)^{-1}\right] \\ &= \sigma^2\left(Z'M^0Z\right)^{-1} \end{aligned}$$

(d) Compare the variance-covariance matrix of b_2 with the variance-covariance matrix of the last $K-1$ elements of $\hat{\beta}$.
Answer: The variance-covariance matrix of $\hat{\beta}$ is

$$\begin{aligned}\sigma^2\left(X'X\right)^{-1} &= \sigma^2\left(\begin{bmatrix} i' \\ Z' \end{bmatrix}[i,Z]\right)^{-1} \\ &= \sigma^2\begin{bmatrix} i'i & i'Z \\ Z'i & Z'Z \end{bmatrix}^{-1}\end{aligned}$$

The inverse can be determined from

$$\begin{bmatrix} i'i & i'Z \\ Z'i & Z'Z \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

We want the $(K-1)\times(K-1)$ matrix, D, in the lower right of this inverse. We have

$$\begin{aligned}(i'i)\,A + (i'Z)\,C &= I \\ (i'i)\,B + (i'Z)\,D &= 0 \\ (Z'i)\,A + (Z'Z)\,C &= 0 \\ (Z'i)\,B + (Z'Z)\,D &= I\end{aligned}$$

From the second and fourth,

$$(Z'i)\left(-\frac{i'Z}{T}D\right) + (Z'Z)\,D = I$$

or

$$Z'\left[I_T - \frac{ii'}{T}\right]ZD = I$$

so

$$D = \left(Z'M^0Z\right)^{-1}$$

and the variance-covariance matrix of the last $K-1$ elements of $\hat{\beta}$ is

$$\sigma^2\left(Z'M^0Z\right)^{-1}$$

so the two are the same.

4. Consider the model

$$y = X\beta_1 + Z\beta_2 + u \quad u \sim N\left(0,\sigma^2 I\right)$$

Let b_1^U and b_2^U be the (unrestricted) OLS estimators of β_1 and β_2. Let b_1^R be the restricted estimator of β_1 obtained by regressing y on the columns of X.

(a) Show that the relationship between the restricted and unrestricted estimator of β_1 can be written

$$b_1^R = b_1^U + Q b_2^U$$

for some matrix, Q.

Answer: From the normal equaitons

$$X'y = X'Xb_1 + X'Zb_2$$

Multiplying through by $(X'X)^{-1}$

$$(X'X)^{-1} X'y = (X'X)^{-1} X'Xb_1 + (X'X)^{-1} X'Zb_2$$

or

$$b_1^R = b_1^U + Q b_2^U$$

where

$$Q = (X'X)^{-1} X'Z$$

(b) Show that the variance, $V\left(b_1^R\right)$, of the restricted estimator is no larger than the variance, $V\left(b_1^U\right)$ of the unrestricted estimator in the sense that

$$V\left(b_1^U\right) - V\left(b_1^R\right)$$

is a positive semidefinite matrix. (Hint: It is simpler to compare the precisions.)

Answer: We have,

$$V\left(b_1^R\right) = \sigma^2 (X'X)^{-1}$$

while, by the Frisch-Waugh theorem

$$V\left(b_1^U\right) = \sigma^2 (X'M_Z X)^{-1}$$

so, comparing precisions,

$$\begin{aligned} V\left(b_1^R\right)^{-1} - V\left(b_1^U\right)^{-1} &= \sigma^2 \left[(X'X) - (X'M_Z X)\right] \\ &= \sigma^2 X' (I - M_Z) X \\ &= \sigma^2 X' P_Z X \end{aligned}$$

The matrix $X'P_Z X$ is positive semidefinite since

$$a'X'P_Z X a = (P_Z X a)' P_Z X a \geq 0$$

(c) Under what circumstances are the two estimators and their variances identical?

Answer: The variances are the same when $P_Z X$, i.e. when the X variables are uncorrelated with the Z variables.

(d) Show that the relationship between the residual vectors, $e^R$and e^U, can be written

$$e^R = M_X Z b_2^U + e^U$$

Use this result to compare the residual sums of squares of the two regressions.

Answer: We have

$$e^R = y - X b_1^R$$

and

$$e^U = y - X b_1^U - Z b_2^U$$

so

$$\begin{aligned} e^R - e^U &= X\left(b_1^U - b_1^R\right) + Z b_2^U \\ &= \left(Z - XQ\right) b_2^U \\ &= \left(Z - X\left(X'X\right)^{-1} X'Z\right) b_2^U \\ &= M_X Z b_2^U \end{aligned}$$

To compare the sums of squares, we have

$$\begin{aligned} \left(e^R\right)' e^R &= \left(M_X Z b_2^U + e^U\right)' \left(M_X Z b_2^U + e^U\right) \\ &= \left(b_2^U\right)' Z' M_X Z b_2^U + \left(b_2^U\right)' Z' M_X e^U + \left(e^U\right)' M_X Z b_2^U + \left(e^U\right)' e^U \\ &= \left(b_2^U\right)' Z' M_X Z b_2^U + \left(e^U\right)' e^U \end{aligned}$$

since

$$Z' M_X e^U = Z' e^U = 0$$

5. Consider the Classical Linear Model

$$y = X\beta + u$$

where y is a $T \times 1$ vector, β is a $K \times 1$ vector, X is a non-stochastic $T \times K$ matrix with values of unity in the first column and u is a $T \times 1$ vector with zero mean and covariance matrix $\sigma^2 I$.

(a) Prove the Gauss-Markov theorem that the OLS estimator is the best linear unbiased estimator of β.

Answer: Consider any other linear estimator

$$\begin{aligned} b &= Cy \\ &= CX\beta + Cu \end{aligned}$$

where

$$C = \left(X'X\right)^{-1} X' + D$$

If b is to be unbiased, we require

$$CX = I$$

so

$$DX = 0$$

We then have,

$$\begin{aligned} COV\,(b|X) &= E\left[(b-\beta)(b-\beta)'\,|X\right] \\ &= E\left[(Cu)(Cu)'\,|X\right] \\ &= \sigma^2 CC' \\ &= \sigma^2\left[(X'X)^{-1} + DD'\right] \end{aligned}$$

Hence

$$Var\,(b|X) - Var\left(\hat{\beta}|X\right) = DD'$$

The matrix $DD' = (D'D)'$ is non-negative definite (c/f $X'X$), Thus $Var\,(b|X)$ exceeds $V\left(\hat{\beta}|X\right)$ by a non-negative definite matrix.

(b) Let $e = y - X\hat{\beta}$ be the vector of residuals from the regression. Suppose e is regressed on X. Show that the estimated coefficients from this regression are all zero.
Answer: Let $\hat{\delta}$ be the coefficient vector. Then

$$\begin{aligned} \hat{\delta} &= (X'X)^{-1} X'e \\ &= (X'X)^{-1} X'(y - X'\hat{\beta}) \\ &= (X'X)^{-1} X'y - (X'X)^{-1} X'X'\hat{\beta} \\ &= \hat{\beta} - \hat{\beta} \\ &= 0 \end{aligned}$$

(c) Let Z be a $T \times L$ matrix of additional variables and suppose that an investigator estimates

$$y = X\beta + Z\gamma + u$$

Show that the least squares estimator, $\hat{\beta}$, is unbiased but inefficient.
Answer: By the Frisch-Waugh Theorem

$$\begin{aligned} \hat{\beta} &= (X'M_Z X)^{-1} X'M_Z y \\ &= (X'M_Z X)^{-1} X'M_Z\,(X\beta + Z\gamma + u) \\ &= \beta + 0 + (X'M_Z X)^{-1} X'M_Z u \end{aligned}$$

Hence

$$E\left(\hat{\beta}\right) = \beta$$

$$COV\left(\hat{\beta}\right) = \sigma^2 \left(X' M_Z X\right)^{-1}$$

However, the efficient estimator is

$$\hat{\beta} = \left(X'X\right)^{-1} X'y$$

with variance

$$COV\left(\hat{\beta}\right) = \sigma^2 \left(X'X\right)^{-1}$$

(d) Examine the conditions under which the OLS estimator is consistent.
Answer:

6. Consider the linear regression model

$$y = X\beta + u \quad u \sim N\left(0, \sigma^2 I\right)$$

where β is a $K \times 1$ vector and X is a $T \times K$ matrix. We wish to test the linear restriction

$$R\beta = r$$

where R is a $q \times K$ matrix.

(a) Set up the constrained minimisation problem and derive the first order conditions.

(b) Show that these imply

$$\tilde{\beta} = \hat{\beta} - \left(X'X\right)^{-1} R' \left[R\left(X'X\right)^{-1} R'\right]^{-1} \left(R\hat{\beta} - r\right)$$

where $\hat{\beta}$ is the unrestricted estimator of β, and, hence, that

$$R\hat{\beta} - r \backsim N\left(0, \sigma^2 R\left(X'X\right)^{-1} R'\right)$$

(c) Show that

$$\frac{\left(R\hat{\beta} - r\right)' \left[s^2 R\left(X'X\right)^{-1} R'\right]^{-1} \left(R\hat{\beta} - r\right)}{q} \sim F\left(q, T-K\right)$$

(You do not need to give a formal derivation.)

(d) Explain why this result implies that the set of linear restrictions can be tested using the statistic

$$\frac{\left(RSS^R - RSS^U\right)/q}{RSS^U/\left(T-K\right)}$$

where RSS^U is the sum of squared residuals from the unrestricted regression and RSS^R is the sum of squared residuals from the unrestricted regression

7. Carefully distinguish between Likelihood ratio, (LR), Lagrange Multiplier, (LM), and Wald, (W), test procedures giving examples of situations where each is appropriate.

8. Suppose the true model is

$$y_t = \beta x_t + \varepsilon_t$$

where x_t is a stochastic variable such that $E(x_t e_t) = 0$ and

$$\varepsilon_t \sim IN\left(0, \sigma^2\right)$$

but an investigator decides to estimate the model

$$x_t = \gamma y_t + u_t$$

using OLS.

(a) Show that

$$u_t \sim IN\left(0, \frac{1}{\beta^2}\sigma^2\right)$$

What is the covariance between y_t and u_t?
Answer: From the true model

$$\begin{aligned} x_t &= \gamma y_t + u_t \\ &= \frac{1}{\beta} y_t - \frac{1}{\beta}\varepsilon_t \end{aligned}$$

so

$$u_t = -\frac{1}{\beta}\varepsilon_t$$

Hence,

$$\begin{aligned} E(u_t) &= -\frac{1}{\beta}E(\varepsilon_t) = 0 \\ V(u_t) &= E\left(-\frac{1}{\beta}\varepsilon_t\right)^2 = \frac{1}{\beta^2\sigma^2} \end{aligned}$$

$$\begin{aligned} Corr(y_t, u_t) &= E(y_t - Ey_t)(u_t - Eu_t) \\ &= E(\beta x_t + \varepsilon_t)\left(-\frac{1}{\beta}\varepsilon_t\right) \\ &= -E\left(x_t\varepsilon_t + \frac{1}{\beta}\varepsilon_t^2\right) \\ &= -\frac{1}{\beta}\sigma^2 \end{aligned}$$

(b) The investigator decides to estimate β using $1/\hat{\gamma}$. Show that $1/\hat{\gamma} = \hat{\beta}$ only when the regression has an R^2 of 1.

Answer:

$$\hat{\gamma} = \frac{\sum xy}{\sum y^2}, \qquad \hat{\beta} = \frac{\sum xy}{\sum x^2}$$

while

$$\begin{aligned}
R^2 &= \frac{\hat{\gamma}^2 \sum y^2}{\sum x^2} = \left(\frac{\sum xy}{\sum y^2}\right)^2 \frac{\sum y^2}{\sum x^2} \\
&= \frac{\left(\sum xy\right)^2}{\sum x^2 \sum y^2} \\
&= \hat{\beta}\hat{\gamma}
\end{aligned}$$

(c) Is $1/\hat{\gamma}$ an unbiased estimator of β? Is it a consistent estimator of β?

Answer:

$$\begin{aligned}
\frac{1}{\hat{\gamma}} &= \frac{\sum \left(\beta x_t + \varepsilon_t\right)^2}{\sum x \left(\beta x_t + \varepsilon_t\right)} \\
&\therefore \;\; E\left(\frac{1}{\hat{\gamma}}\right) = E\left[\frac{\sum \left(\beta x_t + \varepsilon_t\right)^2}{\sum x \left(\beta x_t + \varepsilon_t\right)}\right] \\
&\neq \frac{E\left[\sum \left(\beta x_t + \varepsilon_t\right)^2\right]}{E\left[\sum x \left(\beta x_t + \varepsilon_t\right)\right]} \\
&= \beta
\end{aligned}$$

so cannot show unbiasedness. However,

$$\begin{aligned}
plim\left(\frac{1}{\hat{\gamma}}\right) &= plim \frac{\sum \left(\beta x_t + \varepsilon_t\right)^2}{\sum x \left(\beta x_t + \varepsilon_t\right)} \\
&= \\
&= \frac{\beta^2 \sum x^2 + \sigma^2}{\beta \sum x^2} \\
&= \beta + \frac{\sigma^2/T}{\beta \sum x^2/T}
\end{aligned}$$

so the estimator is consistent given that $\sum x^2/T$ is finite.

(d) Briefly comment on the nature of the issues raised by the investigator's procedure.

Answer: Estimating the regression the 'wrong way around' has induced correlation between the regressor and the error term.