A Brief Review of Matrix Algebra

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This note based on Wooldridge (2008) summarizes some matrix and probability algebra , needed for the study of multiple linear regression models. It is not meant to be exhaustive. For the proofs and more details on a certain topic on linear algebra and probability, please refer to the relevant chapters in Greene (2011) and Meyer (2000).

1 Basic Definitions and Operations

Definition 1.1. A **matrix** is a rectangular array of numbers. More precisely, an $m \times n$ matrix has m rows and n columns. The positive integer m is called the row dimension, and n is called the column dimension. A typical element in a matrix A can be represented as a_{ij}, which means it is the ith row and jth column element, and A can also be written as $[a_{ij}]$. If $m = n$, the matrix is called a **square matrix**. A square matrix A is a **diagonal matrix** when all of its off-diagonal elements are zero, that is, $a_{ij} = 0$ for all $i \neq j$.

Definition 1.2. For a matrix, if $m = 1$ and $n > 1$, it is called a **row vector**; if $m > 1$ and $n = 1$, it is called a **column vector**; if $m = 1$ and $n = 1$, it is called a **scalar**. For our later discussion, we refer to a **vector** as column vector.

Definition 1.3. Matrix Addition: Two matrices A and B, each having dimension $m \times n$, can be added element by element: $A + B = [a_{ij} + b_{ij}]$. For example, $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{pmatrix}$.

Definition 1.4. Given a scalar λ, **scalar multiplication** is defined as $\lambda A = [\lambda a_{ij}]$. For example, $\lambda \times \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \lambda a_{11} & \lambda a_{12} \\ \lambda a_{21} & \lambda a_{22} \end{pmatrix}$.

Definition 1.5. Matrix and Matrix Multiplication: To multiply matrix A by matrix B to form the product AB, the column dimension of A must equal the row dimension of B. Therefore, matrix multiplication is defined as $\underset{m \times n}{A} \underset{n \times p}{B} = \underset{m \times p}{C} = [\sum_{k=1}^{n} a_{ik} b_{kj}] = [c_{ij}]$. For example, $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \times \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \end{pmatrix}$.

Note that $A + B = B + A$; $(A + B) + C = A + (B + C)$; $(AB)C = A(BC)$; $A(B + C) = AB + AC$; in general, $AB \neq BA$, even when both products are defined.

Definition 1.6. The **range of a matrix** $A_{m \times n}$, denoted as $range(A)$, is the set of all linear combinations of the columns of A, i.e. $\{y | y = Ax = \sum_{i}^{n} a_i x_i, x \in R^n\}$, where a_i is the ith column of A and x_i is the ith element of the vector x. It is also called the **column space** of A. If $n < m$, the column space of A is a subspace of R^m.

Definition 1.7. An **identity matrix**, denoted as I or sometimes I_n to emphasize its dimension, is a square matrix with ones on the diagonal line

and zeros elsewhere, i.e. $I_{ii} = 1$ and $I_{ij} = 0$ for $i \neq j$. For example
$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Identity matrix is an extension of the concept 1 in numbers. Note that $AI = IA = A$.

Definition 1.8. Inverse Matrix: For a square matrix A, if we can find another matrix B, such that $AB = BA = I$, then B is called the inverse of A and we can write it as $B = A^{-1}$. In this case, A is said to be **invertible** or **nonsingular**. Otherwise, it is said to be **noninvertible** or **singular**.

Note that $(A')^{-1} = (A^{-1})'$. If A and C are both invertible, then $(AC)^{-1} = C^{-1}A^{-1}$.

Definition 1.9. Transpose of a Matrix:Let $A = [a_{ij}]$ be an $m \times n$ matrix. The transpose of A, denoted A' or A^T(called A prime or A transpose), is the $n \times m$ matrix obtained by interchanging the rows and columns of A. We can write this as $A' = [a_{ji}]$. For example, $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}^T = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{pmatrix}$. A is called **symmertic** if $A' = A$.

Note that $(A+B)' = B' + A'$, $(AB)' = B'A'$. If x and y are both vectors, then $x'y = y'x$.

Definition 1.10. The **inner product of two vectors**, x and y, is $x'y$. If $x'y = 0$, we say x and y are **orthogonal** to each other.

Definition 1.11. The **euclidean norm of a vector** x is defined to be $||x|| = \sqrt{x'x}$. The **distance** between two vectors x and y is $||x - y||$.

Definition 1.12. Let $\{a_1, a_2, \ldots, a_r\}$ be a set of $n \times 1$ vectors. These are **linearly independent** vectors if and only if

$$\alpha_1 a_1 + \alpha_2 a_2 + \cdots + \alpha_r a_r = 0 \tag{1}$$

implies that $\alpha_1 = \alpha_2 = \cdots = \alpha_r = 0$. If (1) holds for a set of scalars that are not all zero, then $\{a_1, a_2, \ldots, a_r\}$ is **linearly dependent**.

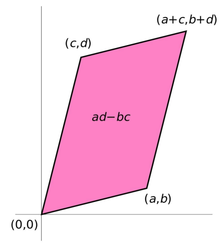
Definition 1.13. Let A be an $n \times m$ matrix. The **rank of a matrix** A, denoted $rank(A)$, is the maximum number of linearly independent columns of A. If A is $n \times m$ and $rank(A) = m$, then A has **full column rank**.

Note that $rank(A') = rank(A)$; if A is $n \times k$, then $rank(A) \leq min(n, k)$; if A is $k \times k$ and $rank(A) = k$, then A is nonsingular.

Definition 1.14. The **trace of a square matrix** $A_{n \times n}$, denoted $tr(A) = \sum_{i=1}^{n} a_{ii}$, is the sum of its diagonal elements.

Note that $tr(A') = tr(A)$, $tr(AB) = tr(BA)$ and $tr(A + B) = tr(A) + tr(B)$.

Definition 1.15. The **determinant** is a value associated with a square matrix, (denoted as $det(\cdot)$ or $|\cdot|$) which measures the volume spanned by the vectors in the matrix. For example, $\begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - bc$.



We usually use computer program to calculate matrix determinant. Note that the determinant of a triangular matrix is the product of all of its diagonal elements. A singular square matrix has 0 determinant while the determinant of a nonsingular matrix is nonzero.

Definition 1.16. A symmetric matrix A is said to be **positive definite (p.d.)** if $x'Ax > 0$ for all $n \times 1$ vectors x except $x = 0$. A symmetric matrix A is **positive semi-definite (p.s.d.)** if $x'Ax \geq 0$ for all $n \times 1$ vectors.

If a matrix is positive definite or positive semi-definite, it is automatically assumed to be symmetric. A positive definite matrix has diagonal elements that are strictly positive, while a p.s.d. matrix has nonnegative diagonal elements. If A is p.d., then A^{-1} exists and is p.d.. If X is $n \times k$, then $X'X$ and XX' are p.s.d. If X is $n \times k$ and $rank(X) = k$, then $X'X$ is p.d. (and therefore nonsingular).

Definition 1.17. A is said to be an **idempotent matrix** if and only if $AA = A$. If A also happens to be a symmetric matrix, then A is called an **orthogonal projection matrix**.

If A is idempotent, then $rank(A) = tr(A)$ and A is positive semi-definite. If $rank(X_{n\times k}) = k$, then $P_X = X(X'X)^{-1}X'$ is the orthogonal projection matrix, which projects any vector in R^n orthogonally onto the column space of X, i.e. $P_X y \in range(X)$ for any $y \in R^n$ and the distance between y and $P_X y$ is the smallest among the vectors in the column space of X. The matrix $M_X = I - P_X$ is also an orthogonal projection matrix, which projects any vector in R^n onto the space orthogonal to the column space of X, i.e. $(M_x y)'X = 0$. (*Q: What is the rank of P_x and M_x respectively?*)

2 Some Vector Calculus

For a given $n \times 1$ vector a, consider the linear function defined by $f(x) = a'x$, for all $n \times 1$ vectors x. The derivative of f with respect to x is the $1 \times n$ vector of partial derivatives[1], which is simply $\frac{\partial f(x)}{\partial x'} = a'$. For a given matrix A and the vector function $h(x) = Ax$, we have $\frac{\partial h(x)}{\partial x'} = A$. For a square matrix B, define the quadratic form $g(x) = x'Bx$. Then $\frac{\partial g(x)}{\partial x'} = x'(B + B')$.

3 Moments of Random Vectors

Definition 3.1. If y is an $n \times 1$ random vector, the **expectation** of y, denoted $E(y)$, is the vector of expected values: $E(y) = [E(y_1), E(y_2), \ldots, E(y_n)]'$. If Z is an $n \times m$ random matrix, $E(Z)$ is the $n \times m$ matrix of expected values:$E(Z) = [E(z_{ij})]$.

If A is an $m \times n$ matrix and b is an $n \times 1$ vector, where both are nonrandom, then $E(Ay + b) = AE(y) + b$. If A is $p \times n$ and B is $m \times k$, where both are nonrandom, then $E(AZB) = AE(Z)B$ and $E(tr(Z)) = tr(E(Z))$.

[1]To see why it is defined in this way, please refer to Magnus and Neudecker (2007).

Definition 3.2. If y is an $n \times 1$ random vector, its variance-covariance matrix, denoted $Var(y)$, is defined as $E\left((y - E(y))(y - E(y))'\right)$, i.e. $\begin{pmatrix} \sigma_1^2 & \sigma_{12} & \dots & \sigma_{1n} \\ \sigma_{21} & \sigma_2^2 & \dots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \dots & \sigma_n^2 \end{pmatrix}$ where $\sigma_j^2 = Var(y_j)$ and $\sigma_{ij} = Cov(y_i, y_j)$.

In other words, the variance-covariance matrix has the variances of each element of y down its diagonal, with covariance terms in the off diagonals. Because $Cov(y_i, y_j) = Cov(y_j, y_i)$, it immediately follows that a variance-covariance matrix is symmetric.

If a is an $n \times 1$ nonrandom vector, then $Var(a'y) = a'[Var(y)]a \geq 0$. If $Var(a'y) > 0$ for all $a \neq 0$, $Var(y)$ is positive definite. If the elements of y are uncorrelated, $Var(y)$ is a diagonal matrix. If, in addition, $Var(y_j) = \sigma^2$ for $j = 1, 2, \dots, n$, then $Var(y) = \sigma^2 I_n$. If A is an $m \times n$ nonrandom matrix and b is an $n \times 1$ nonrandom vector, then $Var(Ay + b) = A[Var(y)]A'$.

4 Distributions of Random Vectors

Definition 4.1. If y is an $n \times 1$ **multivariate normal random vector** with mean μ and variance-covariance matrix Σ, we write $y \sim Normal(\mu, \Sigma)$.

Ay and By are independent if and only if $A\Sigma B' = 0$.(Why?) If $y \sim Normal(0, \sigma^2 I_n)$, A is a $k \times n$ nonrandom matrix, and B is an $n \times n$ symmetric, idempotent matrix, then Ay and $y'By$ are independent if and only if $AB = 0$; if A and B are nonrandom symmetric, idempotent matrices, then $y'Ay$ and $y'By$ are independent if and only if $AB = 0$.

If $u \sim Normal(0, I_n)$, then $u'u \sim \chi^2(n)$ (**chi-squared distribution with degrees of freedom** n); if A is an $n \times n$ symmetric, idempotent matrix with $rank(A) = q$, then $u'Au \sim \chi^2(q)$.

If $u \sim Normal(0, I_n)$, c is an $n \times 1$ nonrandom vector, A is a nonrandom $n \times n$ symmetric, idempotent matrix with rank q, and $Ac = 0$, then $\frac{c'u/\sqrt{(c'c)}}{\sqrt{u'Au/q}} \sim t_q$ (**student's t distribution with degrees of freedom** q).

If $u \sim Normal(0, I_n)$ and A and B are $n \times n$ nonrandom symmetric, idempotent matrices with $rank(A) = k_1$, $rank(B) = k_2$, and $AB = 0$, then $\frac{u'Au/k1}{u'Bu/k2} \sim F_{k_1,k_2}$ (**F distribution with degrees of freedom** k_1 **and** k_2).

References

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[3] C. Meyer, *Matrix Analysis and Applied Linear Algebra*, Society for Industrial and Applied Mathematics, 2000.

[4] J.M. Wooldridge, *Introductory Econometrics (a modern approach)*, 4th ed., South Western College, 2008.