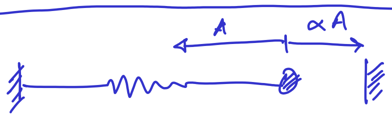
PH2101 Mid-Sem solutions



If there is no wall the amplitude of the S.H.M. is A and the time period is T.

As, $x(0) = -A$ and $\dot{x}(0) = 0$, hence

$$x(t) = -A\cos\omega t$$

(a) If it is at eq. position at t_1

then $\omega t_1 = \pi/2 \Rightarrow t_1 = \frac{\pi}{2\omega} = \frac{T}{4}$

If it is at αA at time t_2

then $-A\cos\omega t_2 = \alpha A \Rightarrow t_2 = \frac{1}{\omega}\cos^{-1}(-\alpha) = \frac{T}{2\pi}\cos^{-1}(-\alpha)$

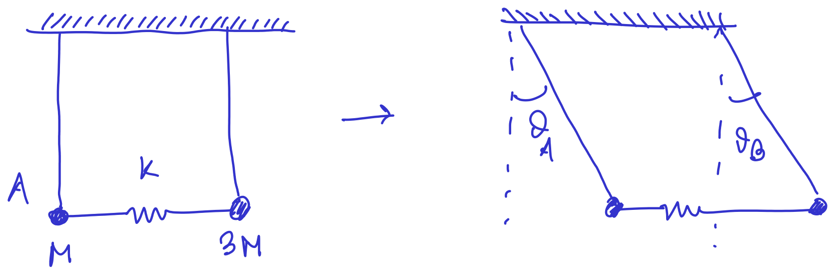
$\Rightarrow \Delta t = t_2 - t_1 = \frac{T}{2\pi}\cos^{-1}(-\alpha) - \frac{T}{4}$

(b) Time period $= 2t_2 = \frac{T}{\pi}\cos^{-1}(-\alpha)$

(c) For $\alpha = \sqrt{3}/2$, time period $= \frac{T}{\pi}\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

$= \frac{T}{\pi}\left(\frac{\pi}{2} + \frac{\pi}{3}\right) = \frac{5T}{6}$

2.



Equation of motion:

$$\underbrace{Ml^2}_{\text{Moment of inertia}}\underbrace{\ddot{\theta}_A}_{\text{ang. acc.}} = \underbrace{-Mgl\sin\theta_A}_{\text{torque from gravity}} - \underbrace{kl(\sin\theta_A - \sin\theta_B)\,l\cos\theta_A}_{\text{torque from spring.}}$$

Using small angle approximation

$$\ddot{\theta}_A = -\frac{g}{l}\theta_A - \frac{K}{M}(\theta_A - \theta_B)$$

or $$\ddot{\theta}_A = -\left(\frac{g}{l} + \frac{K}{M}\right)\theta_A + \frac{K}{M}\theta_B \quad \ldots\ (1)$$

and similarly, $$\ddot{\theta}_B = -\left(\frac{g}{l} + \frac{K}{3M}\right)\theta_B + \frac{K}{3M}\theta_A \quad \ldots\ (2)$$

(b) By taking ① + ② we have

$$\ddot{\theta}_A - \ddot{\theta}_B = -\left(\frac{g}{l} + \frac{K}{M}\right)\theta_A + \frac{K}{M}\theta_B + \left(\frac{g}{l} + \frac{K}{3M}\right)\theta_B - \frac{K}{3M}\theta_A$$

$$= -\left(\frac{g}{l} + \frac{4K}{3M}\right)(\theta_A - \theta_B) \quad \ldots ③$$

Now by taking ① − 3 × ② we get,

$$\ddot{\theta}_A + 3\ddot{\theta}_B = -\left(\frac{g}{l} + \frac{K}{M}\right)\theta_A + \frac{K}{M}\theta_B - 3\left(\frac{g}{l} + \frac{K}{3M}\right)\theta_B + \frac{K}{M}\theta_A$$

$$= -\frac{g}{l}(\theta_A + 3\theta_B) \quad \ldots ④$$

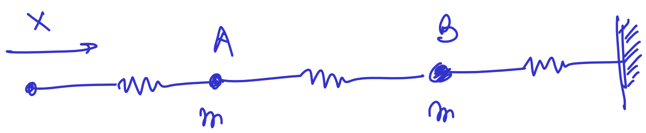
⇒ Normal mode frequencies are

$\sqrt{g/l}$ and $\sqrt{g/l + 4K/3M}$.

(c) For $\sqrt{g/l}$, we must have $\theta_A - \theta_B = 0 \Rightarrow 1:1$

For $\sqrt{g/l + 4K/3M}$, we must have $\theta_A + 3\theta_B = 0 \Rightarrow 1:-3$

3.



The equations of motion are

A: $m\ddot{x}_1 = -k(x_1 - X) - k(x_1 - x_2)$

$$\Rightarrow \ddot{x}_1 = -\frac{2K}{m}x_1 + \frac{K}{m}x_2 + \frac{K}{m}X_0\cos\omega t \quad \ldots ①$$

B: $m\ddot{x}_2 = -k x_2 - k(x_2 - x_1)$

$$\Rightarrow \ddot{x}_2 = -\frac{2K}{m}x_2 + \frac{K}{m}x_1 \quad \ldots ②$$

(b) By taking ① + ② and ① − ② after setting $X_0 = 0$, we get,

$$\frac{d^2}{dt^2}(x_1 + x_2) = -\frac{K}{m}(x_1 + x_2)$$

and

$$\frac{d^2}{dt^2}(x_1 - x_2) = -\frac{3K}{m}(x_1 - x_2)$$

$\Rightarrow$ The normal mode frequencies are $\sqrt{K/m}$ and $\sqrt{3K/m}$.

(e) At the lowest normal mode both A and B have in phase displacement and the ratio is 1:1.

4. (a) $m\dot{v} = -m\gamma v \Rightarrow \dot{v} = -\gamma v$

$\Rightarrow v = v_0 e^{-\gamma t} \Rightarrow \dot{x} = v_0 e^{-\gamma t}$

$\Rightarrow x = C - \frac{v_0}{\gamma} e^{-\gamma t}$

(b) $\ddot{x} + \gamma \dot{x} = \frac{F_0}{m} \cos\omega t$

$\Rightarrow A = \frac{F_0}{m} \cdot \frac{1}{\omega\sqrt{\gamma^2+\omega^2}}$

and $\delta = \tan^{-1}\left(-\frac{\gamma}{\omega}\right)$

(c) $x(t) = C - \frac{v_0}{\gamma} e^{-\gamma t} + \frac{F_0}{m\omega} \frac{1}{\sqrt{\gamma^2+\omega^2}} \cos(\omega t - \delta)$

$x(0) = C - \frac{v_0}{\gamma} + \frac{F_0}{m\omega} \frac{1}{\sqrt{\gamma^2+\omega^2}} \cos\delta = 0$

$\Rightarrow C = \frac{v_0}{\gamma} - \frac{F_0}{m} \cdot \frac{1}{\omega^2+\gamma^2}$ as $\cos\delta = \frac{\omega}{\sqrt{\gamma^2+\omega^2}}$

$\dot{x}(0) = v_0 - \frac{F_0}{m} \frac{1}{\sqrt{\gamma^2+\omega^2}} \cdot \sin(-\delta)$

$= v_0 - \frac{F_0}{m} \cdot \frac{\gamma}{(\gamma^2+\omega^2)} = 0 \Rightarrow v_0 = \frac{F_0}{m} \cdot \frac{\gamma}{(\gamma^2+\omega^2)}$

$\Rightarrow C = \frac{1}{\cancel{\gamma}} \frac{F_0}{m} \cdot \frac{\cancel{\gamma}}{\gamma^2+\omega^2} - \frac{F_0}{m} \frac{1}{\gamma^2+\omega^2} = 0$

and $v_0 = \frac{F_0}{m} \frac{\gamma}{\gamma^2+\omega^2}$