# Electromagnetic waves in matter



For linear and homogeneous medium (no free charge or current) the Maxwell's relations are

$$\vec{\nabla}\cdot\vec{E} = 0 \qquad \vec{\nabla}\times\vec{E} = -\frac{\partial}{\partial t}\vec{B}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{B} = \mu\epsilon\frac{\partial \vec{E}}{\partial t}$$

$\mu \rightarrow$ permeability

$\epsilon \rightarrow$ permittivity

In vacuum

$\epsilon \rightarrow \epsilon_0 = 8.85 \times 10^{-12}\ C^2 N^{-1} m^{-2}$

$\mu \rightarrow \mu_0 = 4\pi \times 10^{-7}\ C^{-2}\ N\ T^2$

$\epsilon = \epsilon_r \epsilon_0$, where $\epsilon_r \rightarrow$ dielectric constant

$\mu \sim \mu_0$, for most linear, homogeneous media

Linear: $\vec{P}$ = Polarization (induced) $= \epsilon_0 \chi_e \vec{E} \longleftarrow$ linear on $\vec{E}$

$\vec{M}$ = Magnetization (induced) $= \mu_0 \chi_m \vec{H} \longleftarrow$ linear on $\vec{H}$

Homogeneous: $\epsilon$ and $\mu$ do not depend on $\vec{r}$.

$\mu_0(\vec{H}+\vec{M}) = \vec{B}$

$\Rightarrow \mu_0(1+\chi_m)\vec{H} = \vec{B}$

$\Rightarrow \vec{H} = \frac{1}{\mu}\vec{B}$

with $\mu = \mu_0(1+\chi_m)$

To construct the wave equation we use,

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = \vec{\nabla}\times\left(-\frac{\partial\vec{B}}{\partial t}\right) = -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{B})$$

$$\Rightarrow \vec{\nabla}(\vec{\nabla}\cdot\vec{E}) - \nabla^2\vec{E} = -\frac{\partial}{\partial t}\left(\mu\epsilon\frac{\partial\vec{E}}{\partial t}\right)$$

As, $\vec{\nabla}\cdot\vec{E} = 0$, we have,

$$\boxed{\nabla^2\vec{E} = \mu\epsilon\frac{\partial^2\vec{E}}{\partial t^2}} \Rightarrow \boxed{c^2 = \frac{1}{\mu\epsilon}}$$

Also,

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{B}) = \vec{\nabla}\times\left(\mu\epsilon\frac{\partial\vec{E}}{\partial t}\right) = \mu\epsilon\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{E})$$

$$\Rightarrow \vec{\nabla}(\vec{\nabla}\cdot\vec{B}) - \nabla^2\vec{B} = \mu\epsilon\frac{\partial}{\partial t}\left(-\frac{\partial\vec{B}}{\partial t}\right)$$

$$\Rightarrow \boxed{\nabla^2\vec{B} = \mu\epsilon\frac{\partial^2\vec{B}}{\partial t^2}} \Rightarrow \boxed{c^2 = \frac{1}{\mu\epsilon}}$$

$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = ?$

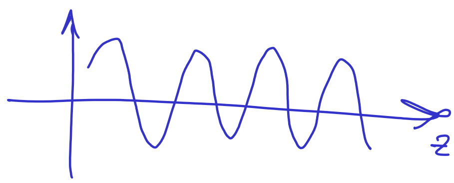
$\vec{\nabla} \times \vec{A} \equiv \epsilon_{ijk}\, \partial_j A_k$ using Levi-Civita symbols

$$
\begin{aligned}
\Rightarrow \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) &= \epsilon_{ijk}\, \partial_j\, \epsilon_{klm}\, \partial_l A_m \\
&= \epsilon_{ijk}\, \epsilon_{klm}\, \partial_j \partial_l A_m \\
&= \epsilon_{kij}\, \epsilon_{klm}\, \partial_j \partial_l A_m \\
&= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\, \partial_j \partial_l A_m \\
&= \partial_j \partial_i A_j - \partial_j^2 A_i \\
&= \partial_i\, \partial_j A_j - \partial_j^2 A_i \\
&\equiv \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}
\end{aligned}
$$

---

Consider monochromatic plane wave solutions.

$\vec{E} = \vec{E}_0\, e^{i(kz - \omega t)} = \vec{E}_0\, f(z,t)$




and

$\vec{B} = \vec{B}_0\, e^{i(kz - \omega t)} = \vec{B}_0\, f(z,t)$

[Note: no phase lag between $\vec{E}$ and $\vec{B}$, as dictated by Faraday's law]

Now, $\vec{\nabla} \cdot \vec{E} = 0 \Rightarrow E_{0z} = 0$

and $\vec{\nabla} \cdot \vec{B} = 0 \Rightarrow B_{0z} = 0$ $\Big\}$ The waves are transverse!

Also, $\vec{\nabla} \times \vec{E}$

$$
= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ E_{0x} f(z,t) & E_{0y} f(z,t) & 0 \end{vmatrix}
$$

$$
= \hat{i}\left(-ik\, E_{0y}\, f(z,t)\right) + \hat{j}\left(+ik\, E_{0x}\, f(z,t)\right) = -\frac{\partial \vec{B}}{\partial t}
$$

$$
= \hat{i}\left(+i\omega\, B_{0x}\, f(z,t)\right) + \hat{j}\left(+i\omega\, B_{0y}\, f(z,t)\right)
$$

$$
\left.\begin{aligned} \Rightarrow\ -k E_{0y} &= \omega B_{0x} \\ \text{and}\quad k E_{0x} &= \omega B_{0y} \end{aligned}\right\} \Rightarrow \vec{B} = \frac{k}{\omega}(\hat{k} \times \vec{E}_0)
$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & 1 \\ E_{0x} & E_{0y} & 0 \end{vmatrix} = -E_{0y}\,\hat{i} + E_{0x}\,\hat{j}$$

So, we have,

$$\vec{B}_0 = \frac{1}{c}\,\hat{k} \times \vec{E}_0 \Rightarrow B_0 = \frac{k}{\omega} E_0 = \frac{1}{c} E_0$$

$$\Rightarrow \vec{B} = \frac{1}{c}\big(\underset{\substack{\uparrow \\ \text{direction of propagation}}}{\hat{k}} \times \vec{E}\big)$$

We write $\vec{B}$ in terms of $\vec{E}$ and help reduce the complexity of the situation.

## Energy density and flux (intensity)

$$u = \frac{1}{2}\left(\epsilon E^2 + \frac{1}{\mu} B^2\right)$$

$$\begin{aligned}
B^2 &= \frac{1}{c^2}(\hat{k} \times \vec{E})\cdot(\hat{k} \times \vec{E}) \\
&= \frac{1}{c^2}\,\vec{E}\cdot\left[(\hat{k} \times \vec{E}) \times \hat{k}\right] \\
&= \frac{1}{c^2}\,\vec{E}\cdot\left[\hat{k} \times (\vec{E} \times \hat{k})\right] \\
&= \frac{1}{c^2}\,\vec{E}\cdot\left\{(\hat{k}\cdot\hat{k})\vec{E} - \underbrace{\cancel{(\hat{k}\cdot\vec{E})}}_{=0}\hat{k}\right\} \\
&= \frac{1}{c^2} E^2
\end{aligned}$$

Use,

$$\vec{A}\cdot(\vec{B} \times \vec{C}) = \vec{C}\cdot(\vec{A} \times \vec{B}) = \vec{B}\cdot(\vec{C} \times \vec{A})$$

$$\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C} = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$$

$$\Rightarrow u = \frac{1}{2}\left(\epsilon E^2 + \frac{1}{\mu}\frac{E^2}{c^2}\right) = \frac{1}{2}\left(\epsilon E^2 + \frac{1}{\mu}\cdot\epsilon\mu E^2\right) = \epsilon E^2$$

$E$ executes sinusoidal oscillation in time.

$$\Rightarrow \langle u \rangle = \langle \epsilon E^2 \rangle = \frac{1}{2}\epsilon E_0^2$$

Now, Poynting vector is

$$\begin{aligned}
\vec{S} &= \frac{1}{\mu}\vec{E} \times \vec{B} = \frac{1}{\mu}\vec{E} \times \frac{1}{v}(\hat{k} \times \vec{E}) \\
&= \frac{1}{\mu v}\left\{\hat{k}\,\vec{E}\cdot\vec{E} - \vec{E}(\vec{E}\cdot\hat{k})\right\} \\
&= \epsilon v E^2 \hat{k}
\end{aligned} \qquad \mu = \frac{1}{\epsilon v^2}$$

Energy flux is

$$\langle \vec{S} \rangle = v\epsilon\langle E^2 \rangle \hat{k} = \frac{1}{2}\epsilon v E_0^2\,\hat{k} = \underset{\substack{\uparrow \\ \text{intensity}}}{I}\,\hat{k}$$