LS2103: Preparatory Problem Set (2024)

You should carefully read and understand each question before answering.

Some Useful Values:

Boltzmann constant, k_B = 1.3806 x 10^{-23} Joule K^{-1}
Gas constant, R = 8.3144 Joule K^{-1} mol^{-1}.
charge of electron, e = 1.6 x 10^{-19} Coulombs
density of pure water at 25 °C = 997 kg m^{-3}
absolute viscosity of pure water at 25 °C = 8.90 × 10^{-4} SI units.
absolute viscosity of pure water at 37 °C = 7.0 × 10^{-4} SI units.
absolute viscosity of 20% sucrose solution at 25 °C = 1.7 × 10^{-3} SI units
absolute viscosity of 40% sucrose solution at 25 °C = 5.3 × 10^{-3} SI units.
kinematic viscosity of ethanol at 25 °C = 1.52 x 10^{-2} SI units

A useful hint: Viscosity and diffusion originate from the same physical effect.

Q1) i) The magnitude of the force (f) between two point-charges q_1 and q_2 in vacuum, separated by a distance r, is given by the formula,

$$f = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}$$

where the constant ε_0 is the permittivity of free space. Find the dimensions of ε_0.

ii) The permittivity, ε_0, can also be expressed as 'capacitance per unit length'. What, then, are the dimensions of the capacitance?

iii) The stored electrostatic energy (E) depends on the capacitance (C), the stored charge (q), and a dimensionless constant (K). Use dimensional analysis to obtain the relationship of E with C, q and K.

Q2) The *Maxwell-Boltzmann distribution* describes the probability distribution, $P(v)$, of the velocities (v) of the molecules (of mass 'm') of an ideal gas at the temperature T and confined in a volume V as:

$$P(v) = 4\pi\left(\frac{m}{2\pi k_B T}\right)^{\frac{3}{2}} v^2 e^{-\left(\frac{mv^2}{2k_B T}\right)}$$

i) Determine how the velocity v_m, corresponding to maximum of the distribution, depends on the temperature.

ii) Suppose the system is now bombarded with some very fast moving particles, each moving with a velocity $v_{fast} >> v_m$. The distribution right after the bombardment is $P^{/}(v)$, and it becomes $P^{//}(v)$ after the system achieves equilibrium. Draw schematic diagrams comparing $P(v)$, $P^{/}(v)$ and $P^{//}(v)$. Comment on the velocities corresponding to the peak distributions in each case, and thus on the trends of their average temperatures.

Q3) Axons are long segments of neurons that transport neurotransmitters in vertebrates. Consider a neurotransmitter molecule that is continually synthesized at one end of an axon, thus maintaining a constant concentration 'c_0' at that end. The neurotransmitter, upon exiting the axon at the other terminal, is immediately used up, so that the concentration at the other end is 0. See Figure 1.

The axon is 1 metre long and is assumed to be fully stretched. The neurotransmitter molecule has a radius of 1nm.

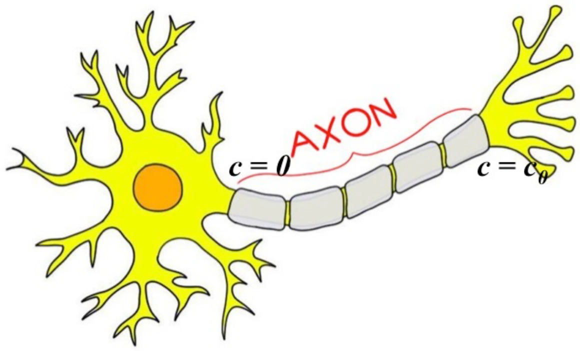

Figure 1

i) Using the Einstein's relationship and the Stokes' relationship, find the diffusion coefficient '*D*' of the neurotransmitter at the physiological temperature of 37 °C. Assume that the medium is fully aqueous. Use appropriate SI units.

ii) What is the expected time needed by the neurotransmitter to cover the length of the axon? You can assume the diffusive motion in the axon is one-dimensional.

iii) Write down the general form of the 1-dimensional Fick's law. Hence, express the flux of the neurotransmitter through the axon in terms of c_0 and the other parameters.

iv) What are the dimensions of the flux?

Q4) i) The one-dimensional *Diffusion equation* relating the rate of change of concentration (c) and its spatial dependence is,

$$\frac{dc}{dt} = D\frac{d^2c}{dx^2}$$

Combine this equation and the Fick's law to relate the rate of change of concentration with the spatial change (ie. gradient) in flux.

ii) Consider a concentration profile with spatial dependence as shown in Figure 2. Describe how the rate of change of concentration at points 'A', 'B' and 'C' will depend on time.

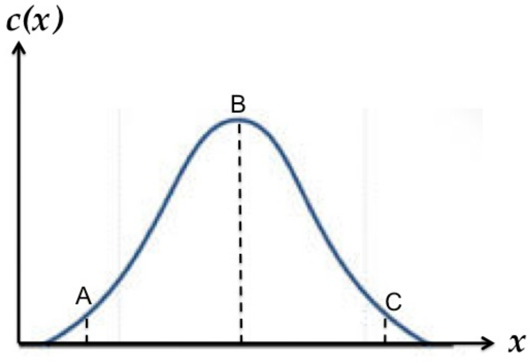

Figure 2

Q5) i) What does the Reynold's number (*Re*) of a fluid describe? Provide the expression of *Re* in terms of the i) absolute viscosity and the ii) kinematic viscosity. Clearly specify every term used, express its dimensions, and provide its SI units.

ii) What are laminar and turbulent flows? How can the *Re* be used to distinguish these flows?

iii) a) Consider an insect measuring 1.5 cm swimming at a speed of 5 times it own length per second in ethanol, at a temperature of 25 °C. b) Consider a unicellular organism moving at 100 μm s^{-1} in water at 25 °C. It has a protruding flagellum of length 2.0 μm, which can be assumed to move at the same speed.

Calculate the *Re* of the insect, and of the flagellum. In relative terms, which situation is more laminar (or turbulent)?

iv) Consider a dense and viscous fluid with a low *Re* in a cylindrical container of length 8 cm. A small blob of the coloured fluid is injected at a depth of 4 cm. Upon stirring it in *one direction*, the blob gets uniformly mixed. Immediately after this, the mixture is stirred in exactly the *opposite direction*, and the blob is found to re-appear.

a) "This *re-organization* of the blob is a clear violation of the 2nd law of thermodynamics." Do you agree with this statement? Provide your reasons.

b) What would you expect if the stirring in the opposite direction had been done after several hours of the first stirring? Why?

c) What would you expect if the fluid mentioned above was replaced by water, and the blob with a dilute ink solution? Comment specifically on what would happen after immediate stirring in the opposite direction.

Q6) The *Sackur-Tetrode* equation for the entropy 'S' of 'N' particles of an ideal gas occupying a volume 'V' and having a total energy 'E' can be expressed as:

$$S = Nk_B\left[\ln\left(\frac{V}{N}\left(\frac{4\pi mE}{3Nh^2}\right)^{3/2}\right)+\frac{5}{2}\right]$$

i) Based on this formula, what should you expect the units of entropy to be?

ii) Use dimensional analysis to establish the dimensions, and hence the SI units of the Planck's constant ('h').

iii) Express the entropy of an ideal gas in terms of its absolute temperature 'T'.

iv) Consider a protein molecule having 'N' atoms, whose volume in any given state can be obtained from its radius of gyration (R_g) as,

$$V = KR_g^{\ 3}$$

where 'K' is a dimensionless constant.

The protein expands from an *initial state* at a temperature T_1 and $R_g = R_1$, to a *final state* at temperature T_2 (= 1.5 T_1) and $R_g = R_2$ (= 2 R_1). Find the entropy difference (ΔS) between the final and the initial state under the ideal gas approximation. Express ΔS in terms of (Nk_B).

Q7) i) Consider a system with the discrete energy states $\{E_1, E_2, ..., E_i, ... ,E_{M-1}, E_M\}$, at an absolute temperature 'T'. Describe the *Boltzmann distribution* for this system in terms of the probability of occupation of any state.

ii) Suppose a physical parameter 'A' takes the discrete values $\{A_1, A_2, ..., A_i, ... ,A_{M-1}, A_M\}$ at the states above. Express the average (or measurable) value of this parameter.

iii) Consider a simpler system with just two energy states, E_1 and E_2. Derive the probabilities of occupation of either state as a function of the temperature and the energy difference between the states. Make sure the relevant terms are correctly defined.

Q9) Consider a protein molecule in solution with '*M*' number of possible energy states.
i) At a specified set of thermodynamic conditions, write down the Boltzmann formula for the probability (P_i) of occupation of the energy state E_i. You may take the proportionality constant, if any, as '*A*'. Specify all other constants and variables used.
ii) The level of disorder, *I*, in the system is specified by the Shannon's formula:

$$I = -B \sum_{i=0}^{i=M} P_i \ln (P_i)$$

'*B*' is a constant. Use the results of i) above to determine the level of disorder for the protein.
iii) For any system, what is the *minimum* value of the disorder? When will it be achieved?

Q10) The image below (Murata et al., Nature 2000, v407, p599) shows the cross section of the cellular 'water channel' known as Aquaporin.

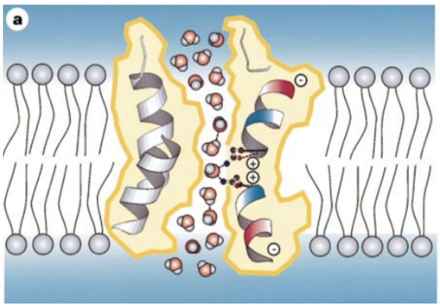

Figure 3

Though the passage of water is mediated by the internal structure and inward facing amino acids, for simplicity, one may assume that the process is diffusive. Hence, estimate the diffusion coefficient if experiments estimate that:

- The mean squared displacement of each water molecule is roughly the square of the length of the channel (20 Å)
- On average, aquaporins pass 10^9 water molecules per second.