CHAPTER

3

BASIC PRINCIPLES OF GROUND-WATER FLOW

Water flows in sediments or rocks through open spaces, which range from tiny imperfections along crystal boundaries in igneous rocks to huge caverns in limestone. This chapter introduces the basic concepts and principles of ground-water flow in sediments and rock.

3.1 POROSITY OF A SOIL OR ROCK

Total porosity of a rock or soil is defined as the ratio of the void volume to the total volume of material:

$$n_T = \frac{V_v}{V_T} = \frac{V_T - V_s}{V_T} \tag{3.1}$$

where n_T is the total porosity, V_v is the volume of voids, V_s is the volume of solids, and V_T is the total volume. In some cases, porosity is expressed as a percentage.

Primary porosity refers to the original interstices (or voids) created when some rock or soil was formed. These interstices include pores in soil or sedimentary rocks, and vesicles, lava tubes, and cooling fractures in basalt (Figure 3.1*a*, *b*; Heath 1988). *Secondary porosity* refers to joints, faults in igneous, metamorphic, and consolidated sedimentary rocks, and solution-enlarged openings in carbonate and other soluble rocks (e.g., Figure 3.1*c*–*f*; Heath, 1988). Porosity may also be defined in terms of grain density and bulk density.

$$n_T = 1 - \frac{\rho_b}{\rho_s} \tag{3.2}$$

where ρ_b is the *bulk density* (density of dry soil or rock sample) and ρ_s is the density of solids.

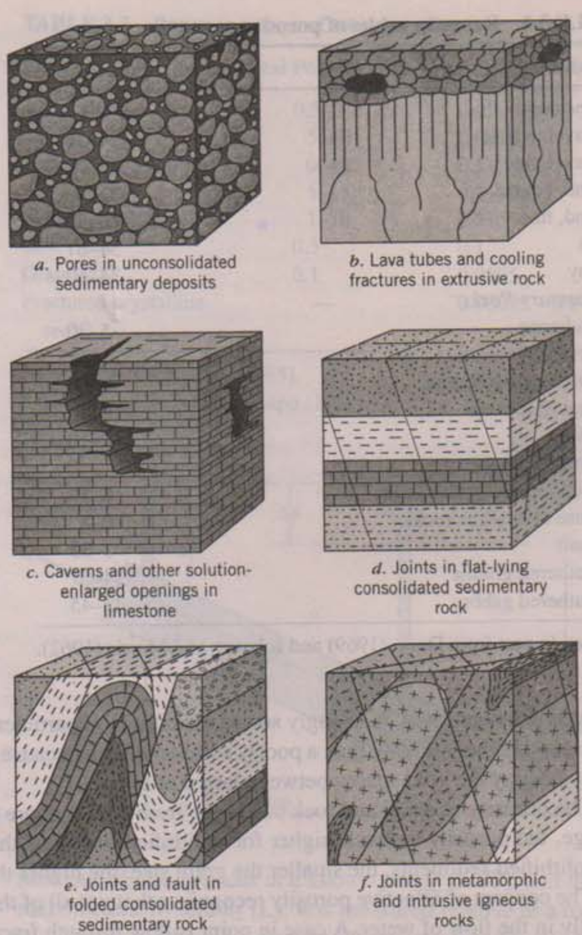


Figure 3.1 Types of openings in selected water-bearing rocks. Block (*a*) is a few millimeters to a few tens of centimeters wide depending on the medium. The remaining blocks are a few tens of meters wide. Openings in Panels (*a*) and (*b*) are primary; those in the others are secondary (from *Hydrogeology*, Heath, 1988). Reproduced with permission of the publisher, the Geological Society of America, Boulder, Colorado USA. Copyright ©1988 by the Geological Society of America, Inc.

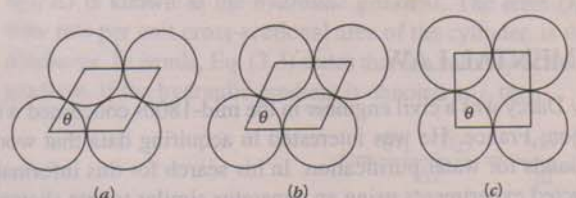


Figure 3.2 Sections of four contiguous spheres of equal size: (*a*) the most compact arrangement, lowest porosity; (*b*) less compact arrangement, higher porosity; (*c*) least compact arrangement, highest porosity (from Slichter, 1899).

The porosity of a soil or rock depends on the degree of compaction of grains, the shape of grains, and the particle-size distribution. If a material consists of spheres of equal size, the greater the compaction, the lower the porosity (Figure 3.2). The shape of the grains can cause the porosity to be larger or smaller than average, depending on how the grains are

TABLE 3.1 Range in values of porosity

Material	Porosity (%)
Sedimentary	
Gravel, coarse	24–36
Gravel, fine	25–38
Sand, coarse	31–46
Sand, fine	26–53
Silt	34–61
Clay	34–60
Sedimentary Rocks	
Sandstone	5–30
Siltstone	21–41
Limestone, dolomite	0–40
Karst limestone	0–40
Shale	0–10
Crystalline Rocks	
Fractured crystalline rocks	0–10
Dense crystalline rocks	0–5
Basalt	3–35
Weathered granite	34–57
Weathered gabbro	42–45

Source: In part from Davis (1969) and Johnson and Morris (1962).

arranged and connected. A strongly sorted medium (soil particles of relatively equal sizes) possesses a higher porosity than a poorly sorted medium because particles of a smaller size tend to occupy the void spaces between larger ones.

The porosity of a soil and rock can range from zero to more than 60% (Table 3.1). On average, the porosity is much higher for unlithified materials than for lithified materials. For unlithified sediments, the smaller the grain size, the higher the porosity.

The concept of effective porosity recognizes that not all of the pores participate meaningfully in the flow of water. A case in point is flow through fractured shale. Although the unfractured shale contains water-filled pores, almost all of the flow takes place through the fractures. In other words, the fracture system provides the effective pathway for flow through the rock. The *effective porosity* of a sediment or rock is the ratio of volume of the interconnected interstices to the total volume of the soil or rock. Effective porosity is more closely related to the flow of ground water in a medium than total porosity. In some cases, the effective porosity can differ significantly from the total porosity (Table 3.2).

▶ 3.2 DARCY'S EXPERIMENTAL LAW

Henry Darcy was a civil engineer in the mid-1800s concerned with the public water supply of Dijon, France. He was interested in acquiring data that would improve the design of filter sands for water purification. In his search for this information (Darcy, 1856), Darcy conducted experiments using an apparatus similar to that shown in Figure 3.3. His testing system consisted of a cylinder having a known cross-sectional area A (L^2), which was filled with various filter sands. Appropriate plumbing was provided to flow water through the column. The cylinder contained two manometers whose intakes were separated by a distance Δl (L). *Manometers* are nothing more than small open tubes that provide measurements of the energy available for flow at their open end in the medium. Water was flowed into

TABLE 3.2 Range in values of total porosity

Material	Total Porosity (%)	Effective Porosity (%)
Anhydrite[a]	0.5–5	0.05–0.5
Chalk[a]	5–40	0.05–2
Limestone, dolomite[a]	0–40	0.1–5
Sandstone[a]	5–15	0.5–10
Shale[a]	1–10	0.5–5
Salt[a]	0.5	0.1
Granite[b]	0.1	0.0005
Fractured crystalline rock[b]	—	0.00005 – 0.01

[a] Data from Croff et al. (1985).
[b] Data from Norton and Knapp (1977).

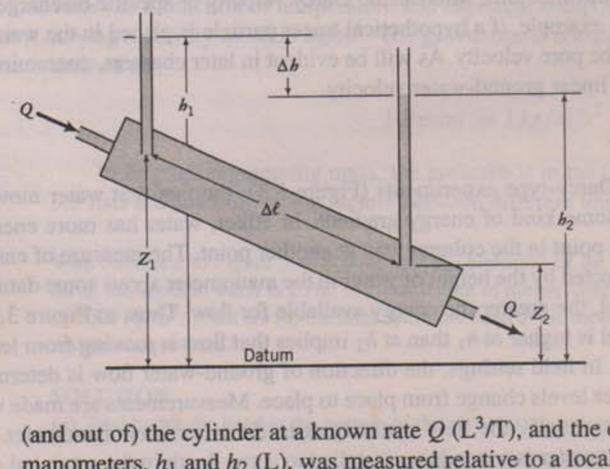


Figure 3.3 Laboratory apparatus to demonstrate Darcy's law (from Domenico and Schwartz, 1998. *Physical and chemical hydrogeology*). Copyright ©1998 by John Wiley & Sons, Inc. Reprinted by permission of John Wiley & Sons, Inc.

(and out of) the cylinder at a known rate Q (L^3/T), and the elevation of water levels in the manometers, h_1 and h_2 (L), was measured relative to a local datum.

Darcy conducted a variety of experiments in which the flow rate Q or the types of filter medium were changed. He derived the following relationship, now known as *Darcy's equation*

$$\frac{Q}{A} = K\frac{(h_1 - h_2)}{\Delta l} \tag{3.3}$$

where K is a constant of proportionality termed *hydraulic conductivity*. The term $(h_1 - h_2)/\Delta l$ is known as the *hydraulic gradient*. The term Q/A, representing the volumetric flow rate per unit cross-sectional area of the cylinder, is the *Darcy velocity* (q), or *specific discharge*. In words, Eq. (3.3) states that the velocity of flow is proportional to the hydraulic gradient. If the hydraulic gradient is denoted as i, then

$$i = \frac{(h_1 - h_2)}{\Delta l} = -\frac{dh}{dl} \tag{3.4}$$

and Darcy's equation is written as

$$q = K\,i \tag{3.5}$$

or

$$Q = K\,i\,A \tag{3.6}$$

Darcy's equation is valid for flow through most granular material as long as the flow is laminar. Under conditions of turbulent flow, the water particles take more tortuous paths. At the other extreme, in very-low-permeability materials, a minimum threshold gradient could be required before flow takes place (Bolt and Groenevelt, 1969).

Linear Ground-Water Velocity or Pore Velocity

Built into the Darcy velocity is an assumption that flow occurs over the entire surface area of the soil column. Because water only flows in the pore space, the actual flow velocity or pore velocity is greater than the Darcy velocity. The *pore velocity* (v), also termed the *linear velocity*, is defined as the volumetric flow rate per unit interconnected pore space.

$$v = \frac{q}{n_e} \tag{3.7}$$

where n_e is the effective porosity. The pore velocity is the true velocity of water flow in a porous medium, whereas the Darcy velocity or specific discharge is the apparent velocity. For example, if a hypothetical tracer particle is placed in the water, the particle will travel at the pore velocity. As will be evident in later chapters, contaminants are transported with the linear ground-water velocity.

Hydraulic Head

In Darcy-type experiments (Figure 3.3), the fact that water moves implies the existence of some kind of energy gradient. In effect, water has more energy available for flow at one point in the column than at another point. The measure of energy available for flow is reflected by the height of water in the manometer above some datum—the higher the water level, the greater the energy available for flow. Thus, in Figure 3.3, the fact that the water level is higher at h_1 than at h_2 implies that flow is moving from left to right in the column.

In field settings, the direction of ground-water flow is determined by looking at how water levels change from place to place. Measurements are made with a *piezometer*, which is essentially the field counterpart to the laboratory manometer. In its simplest form, a piezometer is a standpipe installed to some depth in the saturated system (Figure 3.4). The water elevation with sea level as the datum is a measure of the energy for flow at the intake or opening at the bottom of the piezometer.

This energy available for flow is given the name *hydraulic head*. It consists of three components, related to elevation, pressure, and velocity. The relationship among hydraulic head, elevation, pressure, and velocity is expressed by the Bernoulli equation.

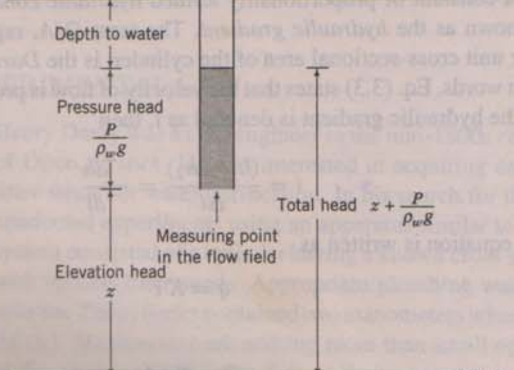


Figure 3.4 Diagram showing elevation, pressure, and total head for a point in the flow field (from Domenico and Schwartz, 1998. *Physical and chemical hydrogeology*). Copyright ©1998 by John Wiley & Sons, Inc. Reprinted by permission of John Wiley & Sons, Inc.

$$h = z + \frac{P}{\rho_w g} + \frac{v^2}{2g} \tag{3.8}$$

where h is the hydraulic head [L], z is the elevation [L], P is the pressure exerted by water column [M/LT^2], ρ_w is the fluid density [M/L], g is the gravitation acceleration [L/T], and v is the velocity [L/T]. In ground-water settings, the flow velocity is so low that the energy contained in velocity can be neglected when computing the total energy. Thus, the hydraulic head is written as

$$h = z + \frac{P}{\rho_w g} \tag{3.9}$$

This relationship is illustrated in Figure 3.4. The hydraulic head is the sum of elevation head and pressure head. In SI units, h is in meters (m), z is in meters (m) above the datum (usually sea level), P is in Pascal (Pa), ρ_w is in kg/m^3, and g is in m/s^2. The density ρ_w varies as a function of temperature and chemical composition, with fresh water at 15.5°C having a density of 1000 kg/m^3. The gravitational constant, g, is 9.81 m/s^2. The Pascal is defined as

$$1\ \text{Pascal} = 1\ kg/m/s^2 \tag{3.10}$$

In English engineering units, the pressure is in psi (pounds per square inches). Table 3.3 lists the unit conversion factors between different units of pressure.

► EXAMPLE 3.1

With reference to Figure 3.4, assume that the elevation of the ground surface is 1000 m above sea level, the depth to water is 25 m, the total length of the piezometer is 50 m, and the water has density of 1000 kg/m^3. What are (a) the total hydraulic head at the measurement point, (b) the pressure head, and (c) the pressure?

SOLUTION

(a) Total hydraulic head at the bottom of the piezometer

$$h = 1000 - 25 = 975\ \text{m}$$

(b) Pressure head

$$\frac{P}{\rho_w g} = h - z = 975 - 950 = 25\ \text{m}$$

(c) Pressure

$$P = \rho_w g(h - z) = (1000\ \text{kg/m}^3)(9.81\ \text{m/s}^2)(25\ \text{m}) = 2.45 \times 10^5\ \text{kg/m/s}^2 = 0.245\ \text{MPa}$$

TABLE 3.3 Unit conversion factors between different units of pressure

	Psi	Kg/cm^2	Pascal	Atmospheres	Inches of Hg	Millibar	Ft of H_2O
Psi	1	7.031×10^{-2}	6.895×10^3	6.807×10^{-2}	2.036	6.895×10^1	2.307
kg/cm^2	1.422×10^1	1	9.807×10^4	9.681×10^{-1}	2.896×10^1	9.807×10^2	3.281×10^1
Pascal	1.45×10^{-4}	1.02×10^{-5}	1	9.872×10^{-6}	2.953×10^{-4}	$1. \times 10^{-2}$	3.346×10^{-4}
Atmosphere	1.469×10^1	1.033	1.013×10^5	1	2.992×10^1	1.013×10^3	3.389×10^1
Inches of Hg	4.911×10^{-1}	3.453×10^{-2}	3.386×10^3	3.343×10^{-2}	1	3.386×10^1	1.133
Millibar	1.45×10^{-2}	1.02×10^{-3}	1×10^2	9.872×10^{-4}	2.953×10^{-2}	1	3.346×10^{-2}
ft of H_2O	4.335×10^{-1}	3.048×10^{-2}	2.989×10^3	2.951×10^{-2}	8.827×10^{-1}	2.989×10^1	1

Note: H_2O at 4°C and Hg at 0°C.

We can also calculate pressure in Pa using Table 3.3.

$$P = 25 \text{ m of water} = 82 \text{ ft of water} = 82 \times 2989 \text{ Pa} = 0.245 \text{ MPa}$$

▶ 3.3 HYDRAULIC GRADIENT AND GROUND-WATER-FLOW DIRECTION

Darcy's experiments showed that for flow to occur there must be differences in hydraulic head creating a hydraulic gradient. The *hydraulic gradient* is formally defined as the change in hydraulic head in a given direction

$$i = -\frac{dh}{dl} = \frac{h_1 - h_2}{\Delta l} \tag{3.11}$$

where h_1 and h_2 is the hydraulic head at points 1 and 2, respectively, and Δl is the distance between points 1 and 2. The hydraulic gradient i in Eq. (3.11) is the hydraulic gradient from point 1 to 2.

In a field setting, it is possible to install a large number of piezometers in a unit and to contour the resulting hydraulic head values (Figure 3.5). Starting at one of the piezometers, for example, *a* in Figure 3.5, it is likely that the head will decrease in some directions and increase in others. The gradient is oriented in the direction of maximum head decrease. For the simple example, the maximum gradient is perpendicular to the lines of equal hydraulic head—called *equipotential lines*. The hydraulic gradient essentially defines the direction of ground-water flow from regions of high hydraulic head to regions of lower hydraulic head.

At a minimum, it takes three hydraulic-head measurements to determine the hydraulic gradient and the direction of ground-water flow. Let's demonstrate how this calculation is done. Consider three piezometers, located as shown in Figure 3.6 at the corners of a triangle. Here are the steps.

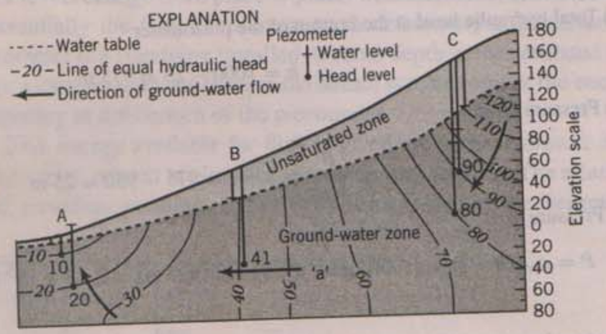


Figure 3.5 Example of the hydraulic-head distribution defined along a hypothetical cross section using piezometers and water-table observation wells (from Winter et al., 1998).

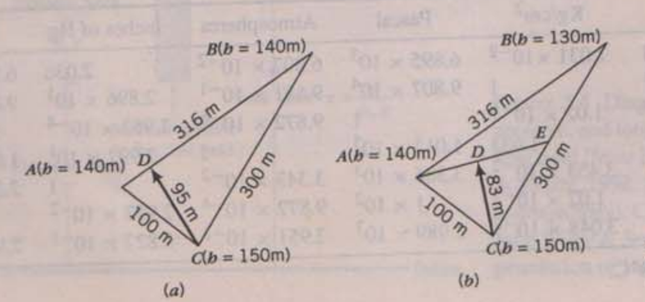


Figure 3.6 Determination of flow direction.

$$1\ \text{gpd/ft}^2 = 4.075 \times 10^{-2}\ \text{m/day} = 0.1337\ \text{ft/day} \tag{3.14}$$

EXAMPLE 3.3

Ground water flows through a buried-valley aquifer with a cross-sectional area of 1.0×10^6 ft^2 and a length of 2×10^4 ft. Hydraulic heads at the ground-water entry and exit points in the aquifer are 1000 and 960 ft, respectively. At the downstream end of the aquifer, ground water discharges into a stream at a rate of 1.0×10^5 ft^3/day. What is the hydraulic conductivity of the buried-valley aquifer in ft/day, m/day, and gpd/ft^2? If the effective porosity of the material is 0.3, what is the linear ground-water velocity?

SOLUTION

1. Calculation of the Darcy velocity:

$$q = \frac{Q}{A} = \frac{1.0 \times 10^5\ \text{ft}^3/\text{day}}{1.0 \times 10^6\ \text{ft}^2} = 0.1\ \text{ft/day}$$

2. Calculation of the hydraulic gradient:

$$-dh/dl = \frac{(1000\ \text{ft}) - (960\ \text{ft})}{2 \times 10^4\ \text{ft}} = 2.0 \times 10^{-3}$$

3. Calculation of the hydraulic conductivity:

$$K = -\frac{q}{dh/dl} = \frac{0.1\ \text{ft/day}}{2.0 \times 10^{-3}} = 50\ \text{ft/day} = 15.24\ \text{m/day} = 373\ \text{gpd/ft}^2$$

4. Calculation of the linear ground-water velocity:

$$v = \frac{q}{n_e} = \frac{0.1\ \text{ft/day}}{0.3} = 0.33\ \text{ft/day}$$

Many tens of thousands of hydraulic conductivity values have been measured for geological materials of all kinds. Subsequent chapters will describe hydraulic conductivity in a variety of geological materials and structural settings, and the variety of field and laboratory approaches designed to provide estimates.

In the absence of actual data, it is often necessary to estimate hydraulic-conductivity values from knowledge of the rock type. This method is only slightly better than an educated guess, but occasionally it is the only available approach. Hydraulic-conductivity values for common rocks and sediments are listed in Table 3.4.

Intrinsic Permeability

Experiments have shown that hydraulic conductivity depends on both properties of the porous medium and the fluid (for example, density and viscosity). For many ground-water studies, water is the fluid of interest, providing more or less constant values of density and viscosity (neglecting temperature dependencies). Thus, measurements of hydraulic conductivity are useful in comparing differences in hydraulic behavior of the actual materials. In looking more generally at systems where the fluids other than water are present (such as, air, oil, and gasoline), hydraulic conductivity becomes an awkward parameter because the density and viscosity of the fluid vary together with the medium properties.

A convenient alternative is to write Darcy's equation in a form where the properties of the medium and the fluid are represented explicitly

TABLE 3.4 Representative values of hydraulic conductivity for various rock types

Material	Hydraulic Conductivity (m/s)
Sedimentary	
Gravel	$3 \times 10^{-4} - 3 \times 10^{-2}$
Coarse sand	$9 \times 10^{-7} - 6 \times 10^{-3}$
Medium sand	$9 \times 10^{-7} - 5 \times 10^{-4}$
Fine sand	$2 \times 10^{-7} - 2 \times 10^{-4}$
Silt, loess	$1 \times 10^{-9} - 2 \times 10^{-5}$
Till	$1 \times 10^{-12} - 2 \times 10^{-6}$
Clay	$1 \times 10^{-11} - 4.7 \times 10^{-9}$
Unweathered marine clay	$8 \times 10^{-13} - 2 \times 10^{-9}$
Sedimentary Rocks	
Karst and reef limestone	$1 \times 10^{-6} - 2 \times 10^{-2}$
Limestone, dolomite	$1 \times 10^{-9} - 6 \times 10^{-6}$
Sandstone	$3 \times 10^{-10} - 6 \times 10^{-6}$
Siltstone	$1 \times 10^{-11} - 1.4 \times 10^{-8}$
Salt	$1 \times 10^{-12} - 1 \times 10^{-10}$
Anhydrite	$4 \times 10^{-13} - 2 \times 10^{-8}$
Shale	$1 \times 10^{-13} - 2 \times 10^{-9}$
Crystalline Rocks	
Permeable basalt	$4 \times 10^{-7} - 2 \times 10^{-2}$
Fractured igneous and metamorphic rock	$8 \times 10^{-9} - 3 \times 10^{-4}$
Weathered granite	$3.3 \times 10^{-6} - 5.2 \times 10^{-5}$
Weathered gabbro	$5.5 \times 10^{-7} - 3.8 \times 10^{-6}$
Basalt	$2 \times 10^{-11} - 4.2 \times 10^{-7}$
Unfractured igneous and metamorphic rocks	$3 \times 10^{-14} - 2 \times 10^{-10}$

Source: From Domenico and Schwartz, 1998. *Physical and chemical hydrogeology*. Copyright ©1998 by John Wiley & Sons, Inc. Reprinted by permission of John Wiley & Sons, Inc.

$$q = -\frac{k \rho_w g}{\mu} \frac{dh}{dl} \tag{3.15}$$

where q is the rate of flow per unit area, k is the intrinsic permeability, ρ_w is the density of water, g is the acceleration due to gravity, μ is the dynamic viscosity of water, and dh/dl is the unit change in hydraulic head per unit length of flow. The *intrinsic permeability* of a rock or soil is a measure of its ability to transmit fluid as the fluid moves through it. The permeability is independent of the fluid moving through the medium. If q is measured in m/sec, μ is in kg/(m.sec), ρ_w is in kg/m^3, g in m/sec^2, and dh/dl in m/m, the unit for k is m^2.

$$[k] = \frac{(\text{m/sec})(\text{kg/(m.sec)})}{(\text{kg/m}^3)(\text{m/sec}^2)(\text{m/m})} = m^2$$

Intrinsic permeability also has units like cm^2 or the darcy. Equations are available for conversion between the units.

CHAPTER

4

GEOLOGY AND GROUND WATER

Life as a hydrogeologist would be downright boring if field settings looked anything like the soil-filled pipes or simple media we have considered so far. The reality is that the complexities of the geology are magnificently manifested in the hydrogeological world that we are starting to explore. The geologic setting provides a context for hydrogeological investigations. This chapter will link elements of geology and hydrogeology.

The discussion begins with aquifers and confining beds, which are manifestations of the geological setting. Next, we examine how key hydrologic variables, like hydraulic conductivity and porosity, are influenced by the geology and geologic processes.

▸ 4.1 AQUIFERS AND CONFINING BEDS

From a resource perspective, the primary unit in ground-water investigations is the *aquifer*, a lithologic unit or combination of lithologic units capable of yielding water to pumped wells or springs (Domenico, 1972). An aquifer can be co-extensive with geologic formations, a group of formations, or part of a formation. It may cut across formations in a way that makes it independent of any geologic unit. Units of low permeability that bound an aquifer are called *confining beds*.

Linking the definition of an aquifer to features of water supply can create confusion. In areas with prolific aquifers, a low-permeability unit might be considered a confining bed. However, in ground-water-poor regions, the same deposit could be considered an aquifer. In actual field studies, this ambiguity in the definition of an aquifer turns out not to be much of a problem because hydraulic conductivity or porosity values explicitly define the hydraulic character of the unit.

Aquifers and confining beds come in flavors. The terms *water table* or *unconfined* are applied to aquifers where the water table forms the upper boundary (Figure 4.1). When shallow wells or piezometers are installed into such an aquifer, the water levels in these wells approximately define the position of the water table.

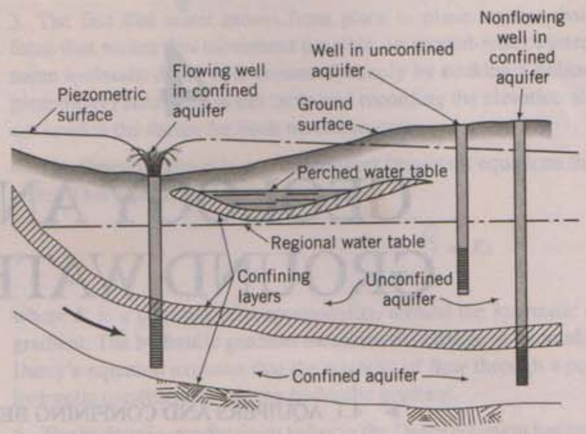


Figure 4.1 Conceptual model of aquifers developed in a field setting (modified from Bureau of Reclamation, 1995).

A *confined* (or *artesian*) *aquifer* has its upper and lower boundaries marked by confining beds (Figure 4.1). Stated another way, an aquifer is confined by overlying and underlying low-permeability beds. The water level of a well or piezometer installed in a confined aquifer occurs somewhere above its upper boundary. Occasionally, the water level of a well occurs above the ground surface. This condition can produce a flowing artesian well (Figure 4.1). As noted in Section 3.8, a contoured map of hydraulic heads for a large number of wells installed in the same aquifer is a potentiometric surface.

A *perched aquifer* is an unconfined aquifer that develops above the regional water table. In effect, there is an unsaturated zone below the low-hydraulic conductivity layer on which the perched zone develops.

Occasionally, the terms *aquifuge*, *aquitard*, and *aquiclude* are applied to various types of confining beds. The use of these terms has fallen out of favor, but sometimes readers might encounter them. An *aquifuge* is the ultimate low-hydraulic conductivity unit, which is a poor conductor of ground water and is essentially impermeable. An *aquitard* is a low-permeability unit that is capable of storing water and transmitting water between adjacent aquifers. This stored and transmitted water is available to wells being pumped in nearby aquifers. The term *aquiclude* is essentially a synonym for confining bed.

CASE STUDY 4-1

AQUIFERS OF LONG ISLAND, NEW YORK

Long Island, located on the East Coast of the United States (Figure 4.2), has important ground-water resources. A sequence of Cretaceous aquifers (Lloyd Aquifer, Magothy Aquifer) and confining beds (Raritan Confining unit) dip from north to south. Above the Cretaceous deposits is a series of marine clays and clayey sands (Gardiners Clay and Monmouth Greensand) that act as a confining unit above the Magothy Aquifer. The cross section (Figure 4.2) shows that the Lloyd Aquifer and some of the Magothy Aquifer can be classified as confined aquifers, given the presence of confining units above and below.

The uppermost aquifer consists of a thick sequence of outwash deposits, related to the most recent glaciation. These deposits consist mainly of stratified sand and gravel with little or no clay and silt. The water table is found at shallow depth in this unit, which makes the uppermost aquifer unconfined. Later in the book, we will revisit the unconfined glacial aquifer on Long Island because of its susceptibility to ground-water contamination.

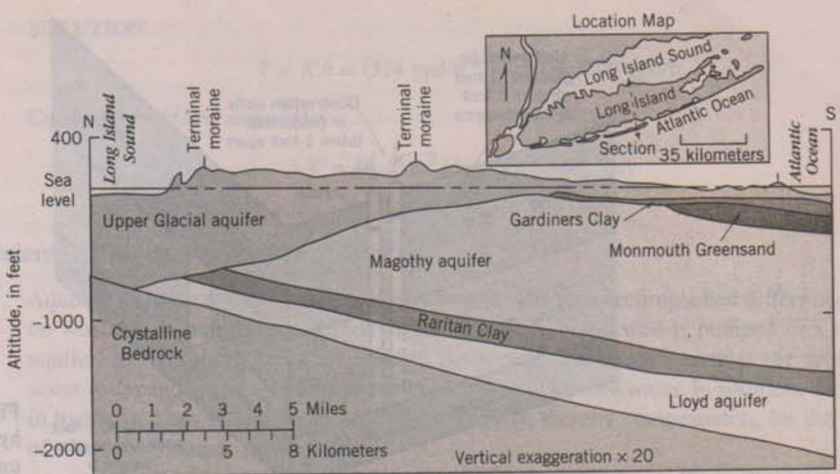


Figure 4.2 Generalized cross section showing the distribution of confined and unconfined aquifers, and confining beds on Long Island, New York (insert) (from Wexler, 1988).

▶ 4.2 TRANSMISSIVE AND STORAGE PROPERTIES OF AQUIFERS

Aquifers play a key role in supplying water to wells. When a pump is turned on in a well, the water level in the well casing (and the hydraulic head) is reduced, causing ground water to flow from the aquifer into the well. Of the water that is pumped from the well, much of it initially comes from "storage" in the aquifer. Thus, aquifers have at least two important characteristics—some ability to store ground water and to transmit this water to a nearby well. These properties depend to an important extent on the geologic setting.

Transmissivity

The term *transmissivity* describes the ease with which water can move through an aquifer. More explicitly, it is the rate at which water of prevailing kinematic viscosity is transmitted through a unit width of the aquifer under a unit hydraulic gradient (Figure 4.3). The concept of transmissivity is similar to hydraulic conductivity. The main difference is that transmissivity is a measurement that applies across the vertical thickness of an aquifer. If the thickness of the aquifer is b, the transmissivity (T) is

$$T = bK \tag{4.1}$$

where K is the hydraulic conductivity of the aquifer. Transmissivity has units of $[L^2/T]$ (for example, ft^2/day, m^2/day). In English engineering units, transmissivity has units of gpd/ft. The following set of factors provides a basis for unit conversions

$$1\ m^2/day = 10.76\ ft^2/day = 80.52\ gpd/ft \tag{4.2}$$

$$1\ ft^2/day = 0.0929\ m^2/day = 7.48\ gpd/ft \tag{4.3}$$

$$1\ gpd/ft = 0.01242\ m^2/day = 0.1337\ ft^2/day \tag{4.4}$$

Not surprisingly, one can develop a form of the Darcy equation that applies to an aquifer. This equation, which applies to a homogeneous confined aquifer (Figure 4.4),

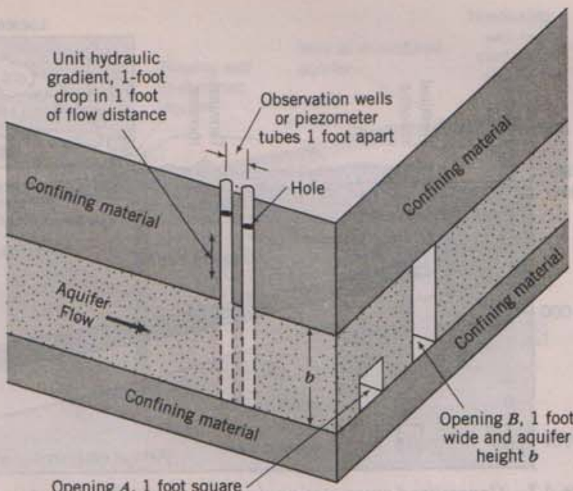


Figure 4.3 Diagram illustrating hydraulic conductivity and transmissivity in an aquifer (from Ferris et al., 1962).

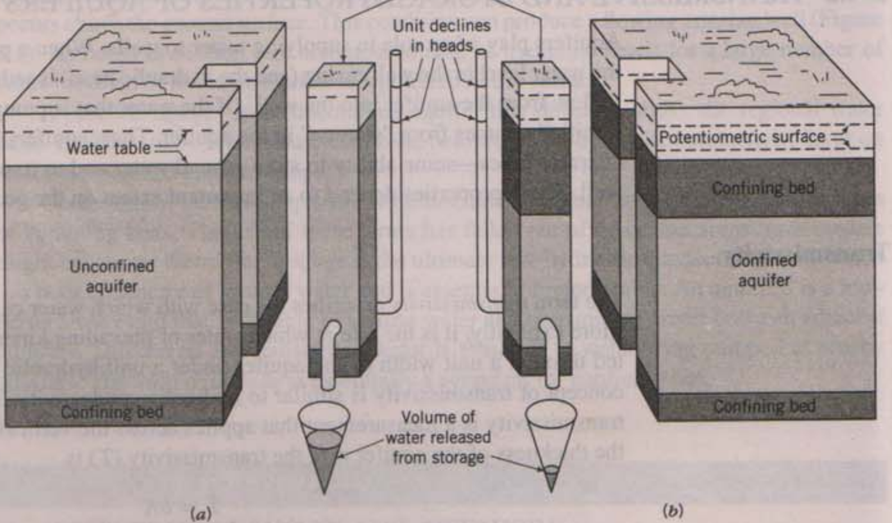


Figure 4.4 Diagrams illustrating the concept of storativity in (*a*) an unconfined aquifer and (*b*) a confined aquifer (from Heath, 1982).

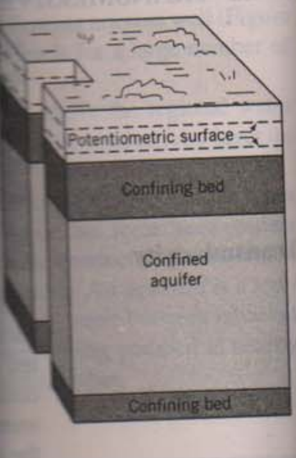
is written as

$$Q = -WT\frac{dh}{dl} \tag{4.5}$$

where W is the width of the aquifer [L], T is the transmissivity of the aquifer [L^2/T], and Q is discharge rate [L^3/T].

► **EXAMPLE 4.1** The hydraulic conductivity of a confined aquifer with a thickness of 10 ft is 374 gpd/ft^2. Calculate the transmissivity of the aquifer in gpd/ft, ft^2/day, and m^2/day.

Figure 4.3 Diagram illustrating hydraulic conductivity and transmissivity in an aquifer (from Ferris et al., 1962).



...) an unconfined aquifer and (b) a

(4.5)

...sivity of the aquifer [L^2/T], and

...ss of 10 ft is 374 gpd/ft^2. Calculate

SOLUTION

$$T = K\,b = (374\ \text{gpd/ft}^2)(10\ \text{ft}) = 3740\ \text{gpd/ft}$$

Conversion to ft^2/day and m^2/day

$$T = 46.45\ \text{m}^2/\text{day} = 500.0\ \text{ft}^2/\text{day}$$

Storativity (or Coefficient of Storage) and Specific Storage

Aquifers have the ability to store water. How this storage is accomplished differs depending on whether the aquifer is confined or unconfined. When a well is pumped in a confined aquifer, the declining hydraulic head in the vicinity of the well enables the pressurized water to expand slightly, adding a small volume of additional water. In addition, the decline in hydraulic head lets the aquifer collapse slightly, thereby compensating for the volume of water that flows to the well.

In an unconfined aquifer, the main source of water is the drainage of water from pores as the water table declines in response to pumping. For a comparable unit decline in hydraulic head, an unconfined aquifer releases much more water from storage than a confined aquifer (Figure 4.4).

The *storativity* of an aquifer is defined as the volume of water that an aquifer releases from or takes into storage per unit surface area of the aquifer per unit change in head (Figure 4.4).

$$S = \frac{\text{volume of water}}{\text{(unit area)(unit head change)}} = \frac{\text{m}^3}{\text{m}^3} \tag{4.6}$$

where S is storativity (dimensionless). In a confined aquifer, values of storativity range from 10^{-3} to 10^{-5}. A related measure of the water stored in an aquifer is specific storage. *Specific storage* is defined as the volume of water that an aquifer releases from or takes into storage per unit surface area of the aquifer per unit aquifer thickness per unit change in head.

$$S_s = \frac{\text{volume for water}}{\text{(unit area)(unit aquifer thickness)(unit head change)}} = \frac{1}{m} \tag{4.7}$$

where S_s is the specific storage of an aquifer [1/L]. Equation (4.7) indicates that the unit of specific storage in the SI unit system is 1/m. Specific storage is related to storativity by

$$S = S_s b \tag{4.8}$$

where b is the thickness of an aquifer.

Storage in Confined Aquifers

For a confined aquifer, the mathematical definition of specific storage reflects the storage coming from compression of the granular matrix and the expansion of water. The specific storage in a confined aquifer (see Domenico and Schwartz, 1998) is

$$S_s = \rho_w\, g(\beta_p + n\,\beta_w) \tag{4.9}$$

where ρ_w is the density of water [ML^{-3}], g is gravitational constant (9.81 m/s^2) [LT^{-2}], n is the porosity of the aquifer, β_p is the vertical compressibility of rock matrix, and β_w

is the compressibility of water. The unit for compressibility is the inverse of pressure. The compressibility of ground water (β_w) is 4.8×10^{-10} m^2/N (or 2.3×10^{-8} ft^2/lb) at 25°C. Table 4.1 lists the compressibility of some common geological materials. To use the compressibility, it is important to review the units of force.

$$1 \text{ Newton(N)} = 1 \text{ kg} \cdot \text{m/s}^2 \tag{4.10}$$

$$1 \text{ kg of force} = 9.80665 \text{ N} = 2.02046 \text{ lb of force} \tag{4.11}$$

Equation (4.9) can be used to estimate the range of specific storage and storativity of some aquifer types.

▶ EXAMPLE 4.2 A confined aquifer is composed of dense, sandy gravel with a thickness of 100 m and a porosity of 20%. Estimate the likely range for specific storage and storativity. For a total head drop of 100 m in an area of 1×10^9 m^2, how much water is released from the storage?

SOLUTION From Table 4.1, the compressibility of dense, sandy gravel ranges from 5.2×10^{-9} to 1.0×10^{-8} m^2/N. To calculate the specific storage, it is computationally convenient to replace unit N with kg · m/s^2.

The specific storage due to the compressibility of water is

$$S_s^w = \rho_w g n \beta_w = (1000 \text{ kg/m}^3)(9.81 \text{ m/s}^2)(0.2)\left(4.8 \times 10^{-10} \frac{\text{m}^2}{\text{kg} \cdot \text{m/s}^2}\right)$$

$$= 9.4 \times 10^{-7} \frac{1}{\text{m}}$$

The specific storage due to the compressibility of granular matrix is

$$S_s^M = \rho_w g \beta_p = (1000 \text{ kg/m}^3)(9.81 \text{ m/s}^2)\left[(0.52 \sim 1.0) \times 10^{-8} \frac{\text{m}^2}{\text{kg} \cdot \text{m/s}^2}\right]$$

$$= (5.1 \sim 9.81) \times 10^{-5} \frac{1}{\text{m}}$$

The specific storage for the aquifer combines these

$$S_s = \rho_w g(\beta_p + n\beta_w) = 9.4 \times 10^{-7} + (5.1 \sim 9.81) \times 10^{-5} = (5.2 \sim 9.9) \times 10^{-5} \frac{1}{\text{m}}$$

TABLE 4.1 Vertical compressibility

Material	Coefficient of Vertical Compressibility		
	ft^2/lb	m^2/N	bars^{-1}
Plastic clay	1×10^{-4}–1.25×10^{-5}	$2 \times 10^{-6} - 2.6 \times 10^{-7}$	$2.12 \times 10^{-1} - 2.65 \times 10^{-}$[illegible]
Stiff clay	1.25×10^{-5}–6.25×10^{-6}	$2.6 \times 10^{-7} - 1.3 \times 10^{-7}$	$2.65 \times 10^{-2} - 1.29 \times 10^{-}$[illegible]
Medium-hard clay	6.25×10^{-6}–3.3×10^{-6}	$1.3 \times 10^{-7} - 6.9 \times 10^{-8}$	$1.29 \times 10^{-2} - 7.05 \times 10^{-}$[illegible]
Loose sand	5×10^{-6}–2.5×10^{-6}	$1 \times 10^{-7} - 5.2 \times 10^{-8}$	$1.06 \times 10^{-2} - 5.3 \times 10^{-3}$
Dense sand	1×10^{-6}–6.25×10^{-7}	$2 \times 10^{-8} - 1.3 \times 10^{-8}$	$2.12 \times 10^{-3} - 1.32 \times 10^{-}$[illegible]
Dense, sandy gravel	5×10^{-7}–2.5×10^{-7}	$1 \times 10^{-8} - 5.2 \times 10^{-9}$	$1.06 \times 10^{-3} - 5.3 \times 10^{-4}$
Rock, fissured	3.3×10^{-7}–1.6×10^{-8}	$6.9 \times 10^{-10} - 3.3 \times 10^{-10}$	$7.05 \times 10^{-4} - 3.24 \times 10^{-}$[illegible]
Rock, sound	less than 1.6×10^{-8}	less than 3.3×10^{-10}	less than 3.24×10^{-5}
Water at 25° C	2.3×10^{-8}	4.8×10^{-10}	5×10^{-5}

Source: Modified from Domenico and Mifflin (1965), *Water resources res.* 4. p. 563–576. Copyright by America[n] Geophysical Union.

The storativity due to the compressibility of water is

$$S^W = b\,S_s^W = (100\,\mathrm{m})\left(9.4\times 10^{-7}\frac{1}{\mathrm{m}}\right) = 9.4\times 10^{-5}$$

The storativity due to the compressibility of the matrix is

$$S^M = b\,S_s^W = (100\,\mathrm{m})(5.1\sim 9.81)\times 10^{-5}\frac{1}{\mathrm{m}} = (5.1\sim 9.81)\times 10^{-3}$$

The overall storativity of the aquifer is

$$S = S^M + S^W = 9.4\times 10^{-5} + (5.1\sim 9.8)\times 10^{-3} = (5.2\sim 9.9)\times 10^{-3}$$

The volume of water, which is withdrawn from the storage due to a drop in hydraulic head of 100 m in an area of $10^9\ \mathrm{m}^2$, is

$$V = S\,\Delta h\,A = (5.2\sim 9.9)\times 10^{-3}(100\,\mathrm{m})(1\times 10^{9}\,\mathrm{m}^2) = (5.2\sim 9.9)\times 10^{8}\,\mathrm{m}^3$$

In this example, most of the water comes from the compression of the matrix. In some areas of California and Texas, overpumping of ground water leads to land subsidence.

Storage in Unconfined Aquifers

In an unconfined aquifer, the ground-water response to pumping is different from a that in a confined aquifer. At an early time, when there is no significant change of water level, water comes from expansion of the water and compression of the grains. Later on, water comes mainly from the gravity drainage of pores in the aquifer through which the water table is falling. The storativity of an unconfined aquifer is expressed as

$$S = S_y + b\,S_s \tag{4.12}$$

where S_y is the specific yield of the aquifer. The specific yield ranges from 0.1 to 0.3, while the product of aquifer thickness and specific storage is in the range of 10^{-3} to 10^{-5}. Thus, specific yield is the storage term for an unconfined aquifer.

In some cases, an aquifer may be confined at an early stage of pumping, only to become unconfined at a late time. Water levels that initially started out above the aquifer end up falling below the top of the aquifer as it dewaters. As the aquifer changes from a confined to an unconfined aquifer, storativity values change accordingly.

Specific Yield and Specific Retention

Specific yield is the water released from a water-bearing material by gravity drainage. The specific yield is expressed as the ratio of the volume of water yielded from soil or rock by gravity drainage, after being saturated, to the total volume of the soil or rock (Meinzer, 1923).

$$S_y = \frac{V_d}{V_T} \tag{4.13}$$

where S_y is the specific yield and V_d is the volume of water that drains from a total volume of V_T.

Not all of the water initially present in the rock or sediment is released from storage. The term *specific retention* describes the water that is retained as a film on the surface of grains or held in small openings by molecular attraction. The specific retention is expressed as the ratio of volume of water that is retained, after being saturated, to the total volume of the soil or rock (Meinzer, 1923).

$$S_r = \frac{V_r}{V_T} \tag{4.14}$$

where S_r is the specific retention and V_r is the volume water retained against gravity. The porosity defined in Section 3.1 is related to specific yield and specific retention by

$$n = S_y + S_r \tag{4.15}$$

That is, the sum of specific yield and specific retention equals porosity. The specific retention increases with decrease of grain size and pore size of a soil or rock (Table 4.2).

► EXAMPLE 4.3

After a soil sample is drained by gravity, the weight of the soil sample is 85 g. After the sample is oven-dried, the sample weighs 80 g. The bulk density of the wet soil is 1.65 g/cm^3, and the density of water is 1 g/cm^3. Calculate the specific yield, specific retention, and porosity of the sample. Assume water that was drained by gravity is 20 g.

SOLUTION The total volume of the sample is

$$V_T = \frac{(85\,\text{g} + 20\,\text{g})}{(1.65\,\text{g/cm}^3)} = 63.6\,\text{g/cm}^3$$

The volume of water retained in the sample after it was drained by gravity is

$$V_r = \frac{(85 - 80)\text{g}}{(1\,\text{g/cm}^3)} = 5\,\text{cm}^3$$

The volume of water that was drained by gravity is

$$V_d = \frac{(20\text{g})}{(1\,\text{g/cm}^3)} = 20\,\text{cm}^3$$

The specific retention is

$$S_r = \frac{V_r}{V_T} = \frac{(5\,\text{cm}^3)}{(63.636\,\text{cm}^3)} = 0.079 = 7.9\%$$

TABLE 4.2 Selected values of porosity, specific yield, and specific retention (values in percent by volume)

Material	Porosity	Specific Yield	Specific Retention
Soil	55	40	15
Clay	50	2	48
Sand	25	22	3
Gravel	20	19	1
Limestone	20	18	2
Sandstone (semiconsolidated)	11	6	5
Granite	0.1	0.09	0.01
Basalt (young)	11	8	3

Source: From Heath (1989).

while the specific yield is

$$S_y = \frac{V_d}{V_T} = \frac{(20\,\text{cm}^3)}{(63.636\,\text{cm}^3)} = 0.314 = 31.4\%$$

The porosity of the sample is now calculated as

$$n = S_y + S_r = 7.9\% + 31.4\% = 39.3\%$$

4.3 GEOLOGY AND HYDRAULIC PROPERTIES

The hydraulic properties of geological materials are closely related to sediment or rock type, and the historical fingerprint is impressed on the materials by geological events. For example, the hydraulic conductivity of sedimentary rocks depends mainly on the porosity of the original sediments. Subsequent erosion, tectonic, and other geological activities often cause fractures that can increase their hydraulic conductivity. This section discusses how geology determines the development of key aquifers. Following the lead of the U.S Geological Survey (Figure 4.5), we will examine aquifers related to five types of materials: unconsolidated sediments, semi-unconsolidated sediments, carbonate rocks, sandstone rocks, and volcanic and other crystalline rocks. To learn more about aquifers in the United States, visit http://capp.water.usgs.gov/gwa, which is the site for the Ground Water Atlas of the United States.

Aquifers in Unconsolidated Sediments

Various types of aquifers are composed of unconsolidated sediments. Examples include blanket sand and gravel aquifers, basin-fill aquifers, and glacial-deposit aquifers. *Basin-fill aquifers* consist of sand and gravel deposits that partly fill depressions that were formed by faulting or erosion, or both. These aquifers are also commonly called valley-fill aquifers because the basins that they occupy are topographic valleys. *Blanket sand* and *gravel aquifers* consist mostly of medium to coarse sand and gravel. These aquifers mostly contain water under unconfined, or water-table, conditions, but locally, confined conditions exist where the aquifers contain beds of low-permeability silt, clay, or marl. The majority of blanket sand and gravel aquifers form from alluvial deposits. Occasionally, some of these aquifers, such as the High Plains aquifer, contain windblown sand, whereas others, such as the surficial aquifer system of the southeastern United States, form as a complex assemblage of alluvium, beach deposits, and shallow marine sands.

Glacial-deposit aquifers are sediments that were deposited during cycles of continental glaciation across the northern United States and Canada. Glacial sediments incorporated in the ice were redistributed as ice-contact or meltwater deposits, or both, during retreats. Ice-laid or meltwater deposits are often called *glacial drift*. *Till* is an unsorted and unstratified material that ranges in size from boulders to clay, which is deposited directly by the ice. *Outwash*, which is mostly stratified sand and gravel, and glacial-lake deposits consisting mostly of stratified clay, silt, and fine sand are related to meltwater processes. In the Midwest, outwash is commonly deposited along river valleys cut into bedrock. *Ice-contact deposits* consisting of local bodies of sand and gravel were deposited at the face of the ice sheet or in cracks in the ice.

The hydraulic conductivity of unconsolidated sediments depends on the grain size, mineral composition, and sorting (Table 4.3). For example, the hydraulic conductivity of

CHAPTER

5

THEORY OF GROUND-WATER FLOW

In this chapter, we describe the theory of flow in saturated, ground-water systems and develop basic equations of ground-water flow. These equations are fundamental to the quantitative treatment of flow and provide the basis for calculating hydraulic heads, given an idealization of some hydrologic system, boundary, and initial conditions. This chapter also provides simple approaches for solving such equations. For example, flownet theory provides a straightforward graphical way to determine a hydraulic-head distribution and the resulting pattern of flow, especially for problems that have a simple geometry. This chapter also shows how analytical solutions are developed and used for solving problems of steady-state flow.

DIFFERENTIAL EQUATIONS OF GROUND-WATER FLOW IN SATURATED ZONES

What sets hydrogeology apart from many of the other geosciences is an emphasis on treating problems mathematically. For example, one might be interested in calculating how much water levels will fall in the vicinity of a well after 10 years of pumping, or how contaminant concentrations change after five years of aquifer remediation. These mathematical approaches also help us interpret measurements made in the field (for example, aquifer tests and slug tests).

Basically, the mathematical approach involves representing a ground-water process by an equation and solving that equation. Let's illustrate this idea with the simple ground-water flow problem shown in Figure 5.1*a*. For this two-dimensional section, assume we know the pattern of layering, the hydraulic conductivity of the various units, and the configuration of the water table. Given this information, can one calculate what the pattern of flow would look like? Developing this problem from a mathematical viewpoint requires (1) finding and using the appropriate equation to describe the flow of ground water, (2) establishing a domain or region where the equation is to be solved, and (3) defining flow conditions along

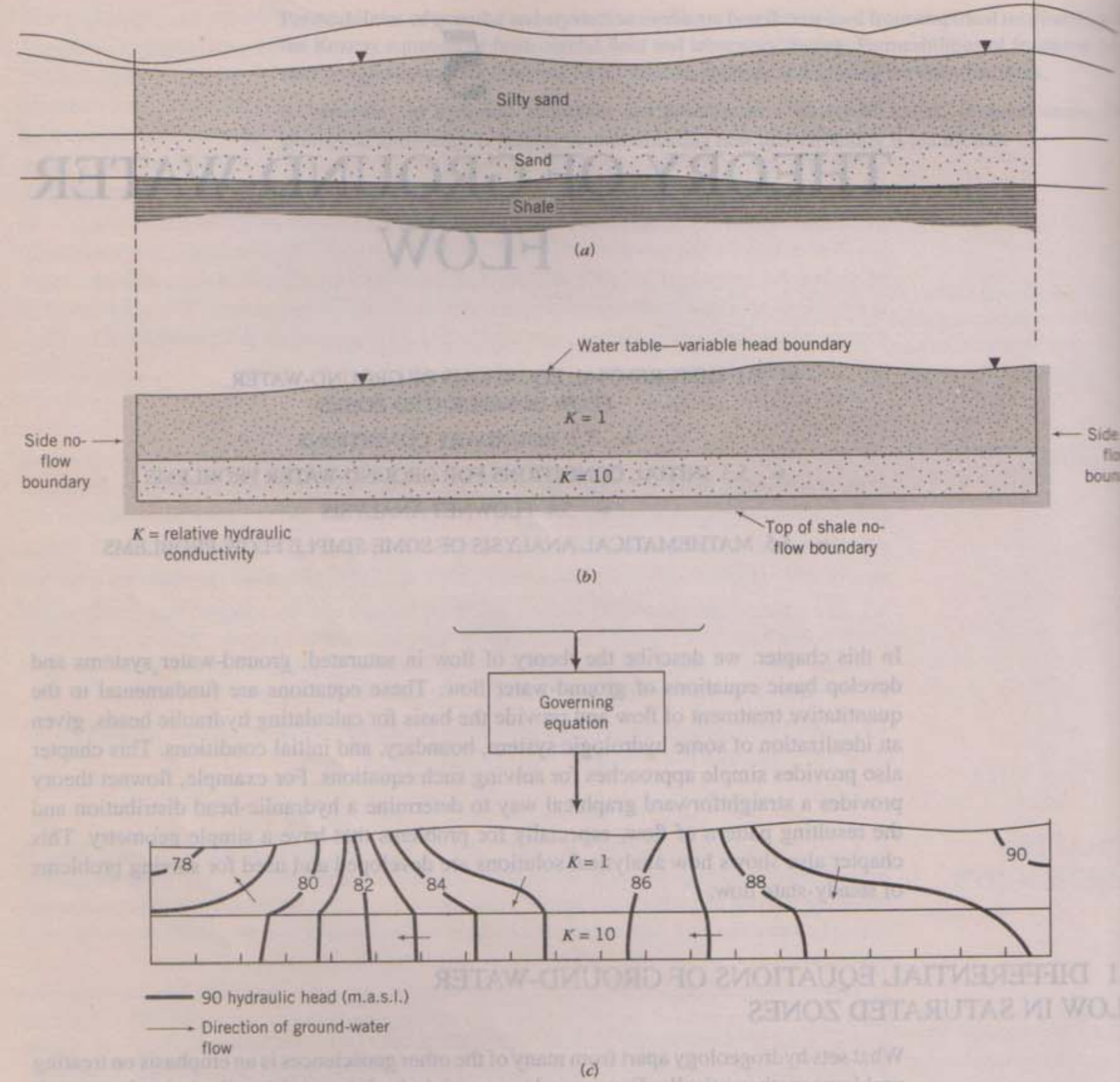
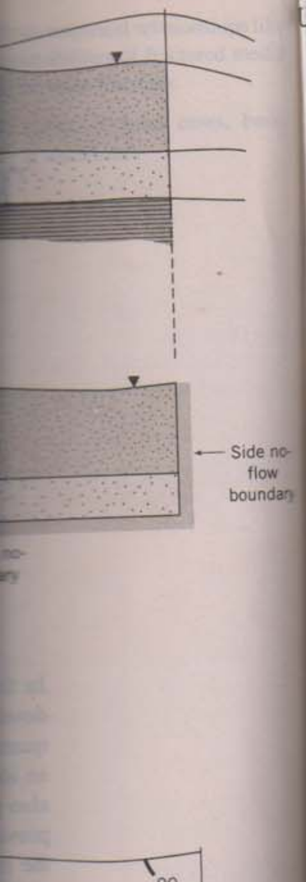


Figure 5.1 A geological problem (*a*) is conceptualized as a formal mathematical problem (*b*), which provides the basis for calculating the hydraulic head distribution (*c*) (from Domenico and Schwartz, 1998. *Physical and chemical hydrogeology*). Copyright ©1998 by John Wiley & Sons, Inc. Reprinted by permission of John Wiley & Sons, Inc.

the boundaries (the so-called boundary conditions)(Figure 5.1*b*). With this information we can calculate the hydraulic head at a large number of specified locations (x and z) within the domain. In principle, this step is like taking readings from a large number of hypothetical piezometers. Contouring the hydraulic-head distribution provides the equipotential distribution, from which we can deduce the patterns of flow (Figure 5.1*c*). This simple example helps to highlight some of the new knowledge that is required for the quantitative treatment of ground-water flow.



Useful Knowledge about Differential Equations

Many students of ground-water hydrology have difficulty in dealing with the quantitative aspects of this subject. At first glance, a differential equation describing ground-water flow immediately implies a need to understand advanced mathematical concepts:

$$\frac{\partial}{\partial x}\left(T_x\frac{\partial h}{\partial x}\right)+\frac{\partial}{\partial y}\left(T_y\frac{\partial h}{\partial y}\right)+\frac{\partial}{\partial z}\left(T_z\frac{\partial h}{\partial z}\right)=S\frac{\partial h}{\partial t} \tag{5.1}$$

Fortunately, at this introductory level, one doesn't need to do much with this equation except to recognize it. The basics of hydrogeology are set up to provide simplified approaches for dealing with these equations.

Let's consider this idea that these equations contain recognizable features. Even as toddlers we all could recognize things. Shown a picture of a rhinoceros, we could say—yes, that is a rhino because of the four stubby legs and two horns on an ugly-looking head. In looking at this picture, one wouldn't become consumed with trying to figure out why the legs were stubby and why there were two horns. Let's consider Eq. (5.1). What are its distinguishing features? Well, first, like all equations, it has some unknown that we are trying to evaluate. The unknown is hidden in the derivative terms, for example:

$$\frac{\partial h}{\partial x} \text{ or } \frac{\partial h}{\partial y} \text{ or } \frac{\partial h}{\partial z} \tag{5.2}$$

where h is the unknown. In other words, a solution to Eq. (5.1) will be of the form

$$h = \ldots\text{. stuff} \tag{5.3}$$

where the "stuff" on the right-hand side are terms that are simple functions of the time and space variables (t, x, y, z) and various parameters like T and S. For a solution to exist, all the terms on the right-hand side need to be known. Fortunately, for most of our applications, the "stuff" is algebraic in form and easy to evaluate.

Here are some simple steps in examining a differential equation. First look at the equation and determine the unknown. This step is straightforward—an equation containing

h : (hydraulic head) makes the equation a ground-water flow equation,

C : (concentration) makes the equation a mass transport equation, and

T : (temperature) makes the equation an energy-transport equation.

Thus, Eq. (5.1) with h as the unknown is a ground-water-flow equation. We know that it is used to apply to aquifers because it also contains the expected hydraulic parameters (T and S). Similarly, a mass transport equation will contain parameters (for example, D, a dispersion coefficient) related to processes involved with mass transport.

Equation (5.1) has other distinguishing features as well. For example, you can look at the space variables to determine the dimensionality of the problem. The dimensionality of a problem describes in how many directions the unknown (hydraulic head) is changing. For example, in Eq. (5.1), there are three space variables, x, y, and z. Three space variables make this equation a three-dimensional equation. Later in this chapter, you'll encounter one-dimensional flow equations that imply that values of hydraulic head change in one direction but not the other two.

Next, you need to decide whether hydraulic head changes with time. To figure this out, look and see whether there is a time variable (t) in the equation. In Eq. (5.1), t is there. With hydraulic head changing with time, the equation of flow is transient. If there is no t term, the equation describes a steady-state problem, where hydraulic head doesn't change with time. A partial differential equation is a concise way to represent hydrogeological processes. By looking at the unknown parameters, dimensionality, and transient nature, you will understand something about the problem that the equation is trying to portray. Such equations are still difficult to solve, but in looking at a partial differential equation, you'll discover plenty of useful information. The following example illustrates how to use these ideas.

▸ EXAMPLE 5.1

Shown here are two different differential equations that can be applied to ground-water problems. Look at each equation and determine what kind of problem it applies to, what dimensionality is involved, and whether the equation is a transient or steady-state form.

$$\frac{\partial^2 h}{\partial x^2} = 0$$

The unknown is h; therefore, it is a ground-water-flow equation. Only one space dimension (x) is included; therefore, it is one-dimensional. There is no time term; therefore, the equation is a steady-state form. There are no parameters (like K); therefore, you can conclude that the hydraulic-head distribution in this case doesn't depend on the parameter values.

$$D\frac{\partial^2 C}{\partial x^2} - v_x\frac{\partial C}{\partial x} = \frac{\partial C}{\partial t}$$

The unknown in this equation is C; therefore, it is a mass transport equation. With x as the only space dimension and t present, it is a one-dimensional transient equation.

More about Dimensionality

This chapter has made the point that the solution to ground-water-flow equations describes how hydraulic head varies within some domain or region of interest. Yet, it is not exactly clear how the dimensionality of the equation matches the dimensionality of the flow region. For example, is it necessary to use a three-dimensional form of a flow equation to calculate hydraulic head values in a three-dimensional domain?

In general, there is no requirement that the dimensionality of the equation be the same as the dimensionality of the region. A one-dimensional equation could be applied to a three-dimensional domain. The number of directions in which the ground water actually moves determines the dimensionality of the equation. For any problem, it is the combination of the region shape, along with the boundary conditions and/or heterogeneity, which determines how the ground water is likely to move and the dimensionality of the equation.

Let's consider some simple examples. Figure 5.2a sets up a problem that we will be coming back to later in this chapter. Two parallel rivers fully penetrate a confined aquifer that receives no recharge. The river stages are assumed to be constant but different from each other. This difference in stage sets up a flow through the aquifer from one river to the other. As the arrows in Figure 5.2b imply, flow through the ground-water system is in only one direction. Thus, this problem can be represented mathematically using a one-dimensional flow equation, even though the aquifer itself has three dimensions.

With a few changes, it is not hard to make the flow more complicated and to require that the dimensionality of the governing equation be increased. For example, if the rivers didn't fully penetrate the aquifer, flows out of and into the rivers would have components in both the x- and z-directions requiring a two-dimensional flow equation. Adding a high-permeability lens in the middle of the aquifer would produce flow in all three coordinate directions, requiring a three-dimensional flow equation.

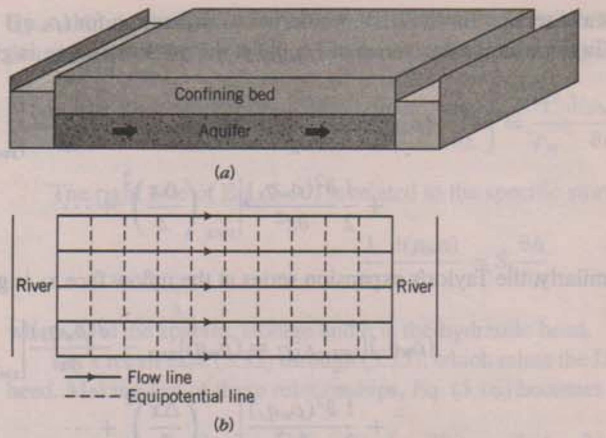


Figure 5.2 Panel (*a*) depicts ground-water flow in an aquifer due to inflow and outflow from rivers. Panel (*b*) suggests that although the system is three dimensional, hydraulic head varies in only one direction. Thus, flow is described by a one-dimensional flow equation.

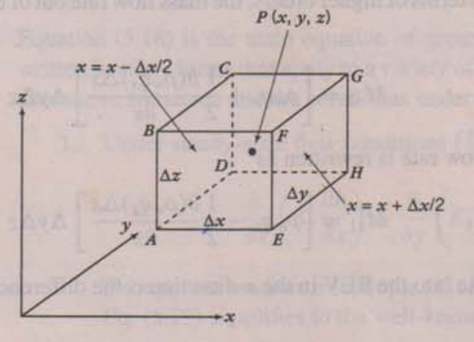


Figure 5.3 Conservation principles applied in relation to this representative elementary volume provide the basis for the development of ground-water flow equations.

Deriving Ground-Water-Flow Equations

The differential equations for ground-water flow are developed from principles of mass conservation in a representative elementary volume (Figure 5.3). In words, such a conservation statement can be written as

$$\text{mass inflow rate} - \text{mass outflow rate} = \text{change of mass storage with time} \tag{5.4}$$

A representative elementary volume (REV) is defined as a volume exhibiting the average properties of the porous media around a point $P(x, y, z)$, which is the center of the volume. For reference purposes, let's label the six faces of the REV as x_1, x_2, y_1, y_2, z_1, and z_2. Now, assume that the water mass fluxes through the six faces are M_{x1}, M_{x2}, M_{y1}, M_{y2}, M_{z1}, and M_{z2}, respectively. Furthermore, ρ_w is the fluid density [ML^{-3}] and n is the porosity in the REV. Equation (5.4) may be rewritten in mathematical terms as

$$M_{x1} - M_{x2} + M_{y1} - M_{y2} + M_{z1} - M_{z2} = \frac{\partial}{\partial t}(n\rho_w \Delta x \Delta y \Delta z) \tag{5.5}$$

The mass flux of water in the direction i is expressed as

$$M_i = \rho_w q_i \Delta S_i$$

where q_i is the i (= x, y, and z) component of the Darcy velocity vector, and ΔS_i is the area perpendicular to the flow direction i. To obtain the mass inflow and outflow rates in the x-direction for the REV, we derive an expression for $(\rho_w q_x)$ across the faces x_1 and x_2. A Taylor's expansion series of $(\rho_w q_x)$ at the outflow face x_2 is given by Thomas (1972) as

$$(\rho_w q_x)|_{x=x+\Delta x/2} = (\rho_w q)|_{x=x} + \left.\frac{\partial(\rho_w q_x)}{\partial x}\right|_{x=x}\left(\frac{\Delta x}{2}\right) + \frac{1}{2}\left.\frac{\partial^2(\rho_w q_x)}{\partial x^2}\right|_{x=x}\left(\frac{\Delta x}{2}\right)^2 + \cdots \quad (5.6)$$

Similarly, the Taylor's expansion series at the inflow face x_1 is given by

$$(\rho_w q_x)|_{x=x-\Delta x/2} = (\rho_w q)|_{x=x} - \left.\frac{\partial(\rho_w q_x)}{\partial x}\right|_{x=x}\left(\frac{\Delta x}{2}\right) + \frac{1}{2}\left.\frac{\partial^2(\rho_w q_x)}{\partial x^2}\right|_{x=x}\left(\frac{\Delta x}{2}\right)^2 + \cdots \quad (5.7)$$

By neglecting the terms of higher orders, the mass flow rate out of the REV in the x-direction is expressed as

$$M_{x2} = \left[\rho_w q_x + \frac{1}{2}\frac{\partial(\rho_w q_x)\Delta x}{\partial x}\right]\Delta y\Delta z \quad (5.8)$$

and the mass inflow rate is rewritten as

$$M_{x1} = \left[\rho_w q_x - \frac{1}{2}\frac{\partial(\rho_w q_x)\Delta x}{\partial x}\right]\Delta y\Delta z \quad (5.9)$$

The net inflow rate into the REV in the x-direction is the difference between the inflow and the outflow rates.

$$M_{x1} - M_{x2} = -\frac{\partial(\rho_w q_x)\Delta x\Delta y\Delta_z}{\partial x} \quad (5.10)$$

Similarly, the net flow rates into the REV in the y- and z-directions are

$$M_{y1} - M_{y2} = -\frac{\partial(\rho_w q_y)\Delta x\Delta y\Delta_z}{\partial y} \quad (5.11)$$

$$M_{z1} - M_{z2} = -\frac{\partial(\rho_w q_z)\Delta x\Delta y\Delta_z}{\partial z} \quad (5.12)$$

The sum of water inflow rate minus the sum of water outflow rate for the REV is

$$M_{x1} - M_{x2} + M_{y1} - M_{y2} + M_{z1} - M_{z2} = -\left[\frac{\partial(\rho_w q_x)}{\partial x} + \frac{\partial(\rho_w q_y)}{\partial y} + \frac{\partial(\rho_w q_z)}{\partial z}\right]\Delta x\Delta y\Delta z \quad (5.13)$$

The change in ground-water storage within the REV is

$$\text{change of water storage per unit time} = \frac{\partial(\rho_w n)}{\partial t}\Delta x\Delta y\Delta z \quad (5.14)$$

According to Eq. (5.4), the net rate of water inflow is equal to the change in storage. Collecting Eqs. (5.13) and (5.14) and dividing both sides by $\Delta x\Delta y\Delta z$ gives

$$-\left[\frac{\partial(\rho_w q_x)}{\partial x} + \frac{\partial(\rho_w q_y)}{\partial y} + \frac{\partial(\rho_w q_z)}{\partial z}\right] = \frac{\partial(\rho_w n)}{\partial t} \quad (5.15)$$

By making a further assumption that the density of the fluid does not vary spatially, the density term on the left-hand side can be taken out as a constant so that Eq. (5.15) becomes

$$-\left[\frac{\partial q_x}{\partial x}+\frac{\partial q_y}{\partial y}+\frac{\partial q_z}{\partial z}\right]=\frac{1}{\rho_w}\frac{\partial(\rho_w n)}{\partial t} \tag{5.16}$$

The right side of Eq. (5.16) is related to the specific storage of an aquifer by

$$\frac{1}{\rho_w}\frac{\partial(\rho_w n)}{\partial t}=S_s\frac{\partial h}{\partial t} \tag{5.17}$$

where S_s is the specific storage and h is the hydraulic head.

Let's recall Eqs. (3.33) through (3.35), which relate the Darcy velocity to the hydraulic head. Making use of these relationships, Eq. (5.16) becomes

$$\frac{\partial}{\partial x}\left(K_x\frac{\partial h}{\partial x}\right)+\frac{\partial}{\partial y}\left(K_y\frac{\partial h}{\partial y}\right)+\frac{\partial}{\partial z}\left(K_z\frac{\partial h}{\partial z}\right)=S_s\frac{\partial h}{\partial t} \tag{5.18}$$

Equation (5.18) is the main equation of ground-water flow in saturated media. It can be written in many forms that apply to a variety of different conditions. Here are some of these alternative equations and the conditions under which they apply.

1. Under steady-state flow conditions ($\frac{\partial h}{\partial t}=0$) Eq. (5.18) simplifies to

$$\frac{\partial}{\partial x}\left(K_x\frac{\partial h}{\partial x}\right)+\frac{\partial}{\partial y}\left(K_y\frac{\partial h}{\partial y}\right)+\frac{\partial}{\partial z}\left(K_z\frac{\partial h}{\partial z}\right)=0 \tag{5.19}$$

   If the porous medium is isotropic (K_x, K_y, K_z) and homogeneous ($K_{x,y,z}=$ constant), Eq. (5.19) simplifies to the well-known Laplace equation

$$\frac{\partial^2 h}{\partial x^2}+\frac{\partial^2 h}{\partial y^2}+\frac{\partial^2 h}{\partial z^2}=0 \tag{5.20}$$

2. With the same assumptions about isotropicity and homogeneity, Eq. (5.18) can be rewritten as

$$\frac{\partial^2 h}{\partial x^2}+\frac{\partial^2 h}{\partial y^2}+\frac{\partial^2 h}{\partial z^2}=\frac{S}{K_s}\frac{\partial h}{\partial t} \tag{5.21}$$

3. By dividing both sides of Eq. (5.18) by S_s, the equation is transformed into

$$\frac{\partial}{\partial x}\left(\frac{K_x}{S_s}\frac{\partial h}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{K_y}{S_s}\frac{\partial h}{\partial y}\right)+\frac{\partial}{\partial z}\left(\frac{K_z}{S_s}\frac{\partial h}{\partial z}\right)=\frac{\partial h}{\partial t} \tag{5.22}$$

   where K_x/S_s, K_y/S_s, and K_z/S_s are called hydraulic diffusivities in the x-, y-, and z-directions, respectively. A constant specific storage is assumed in Eq. (5.22). Writing the equation in this form shows that the ground-water-flow equation is a form of diffusion equation in which the hydraulic diffusivities and hydraulic gradients in the x-, y-, and z-direction are the determining factors.

4. Multiplying both sides of Eq. (5.18) by the aquifer thickness (b) gives

$$\frac{\partial}{\partial x}\left(T_x\frac{\partial h}{\partial x}\right)+\frac{\partial}{\partial y}\left(T_y\frac{\partial h}{\partial y}\right)+\frac{\partial}{\partial z}\left(T_z\frac{\partial h}{\partial z}\right)=S\frac{\partial h}{\partial t} \tag{5.23}$$

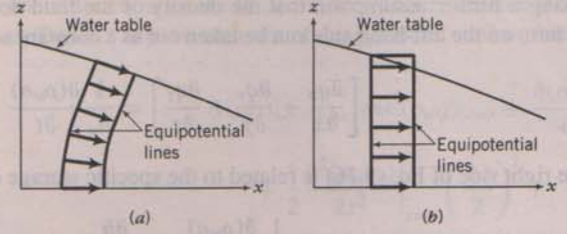


Figure 5.4 This figure illustrates how Dupuit's assumption simplifies flow in an unconfined aquifer. Panel (*a*) depicts the actual flow direction. Panel (*b*) depicts the flow with Dupuit's assumption.

where T is transmissivity and S is storativity.

$$T_x = K_x b, \quad T_y = K_y b, \quad T_z = K_z b, \quad S = S_s b \tag{5.24}$$

This form of the ground-water-flow equation is solved in numerical models like MODFLOW to predict hydraulic-head changes due to pumping in complex systems of aquifers and confining beds.

5. If there is no change of hydraulic head in the vertical direction, Eq. (5.23) is simplified to a two-dimensional ground-water-flow equation of the following form:

$$\frac{\partial}{\partial x}\left(T_x \frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(T_y \frac{\partial h}{\partial y}\right) = S\frac{\partial h}{\partial t} \tag{5.25}$$

6. The next equation relies on simplifications stemming from *Dupuit's assumption* where the direction of ground-water flow is assumed to be horizontal because the vertical hydraulic gradient is small and negligible (Figure 5.4). The differential equation for ground-water flow in an unconfined aquifer is

$$\frac{\partial}{\partial x}\left(K(x,y)h\frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(K(x,y)h\frac{\partial h}{\partial y}\right) = S_y\frac{\partial h}{\partial t} \tag{5.26}$$

where h is the hydraulic head, $K(x,y)$ is the average hydraulic conductivity, and S_y is the specific yield of the water-table aquifer. Equation (5.26) is also known as Boussinesq's equation. The equation only applies to ground-water regions where the vertical hydraulic gradient is very small. For general problems of flow in an unconfined aquifer, Eq. (5.18) should be used.

7. If a ground-water source exists, Eq. (5.18) is written as

$$\frac{\partial}{\partial x}\left(K_x\frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_y\frac{\partial h}{\partial y}\right) + \frac{\partial}{\partial z}\left(K_z\frac{\partial h}{\partial z}\right) + Q(x,y,z,t) = S_s\frac{\partial h}{\partial t} \tag{5.27}$$

where Q is the volumetric source rate per unit volume $[L^3T^{-1}\ L^{-3}]$. Equation (5.27) may be expressed in words as

$$\text{Inflow rate} - \text{outflow rate} + \text{source rate} = \text{change of storage} \tag{5.28}$$

5.2 BOUNDARY CONDITIONS

In order to solve a ground-water-flow equation, it is necessary to specify boundary conditions. When you define a simulation domain (for example, Figure 5.1*b*), you are selecting a small piece of some larger hydrologic system for detailed analysis. Unfortunately, the

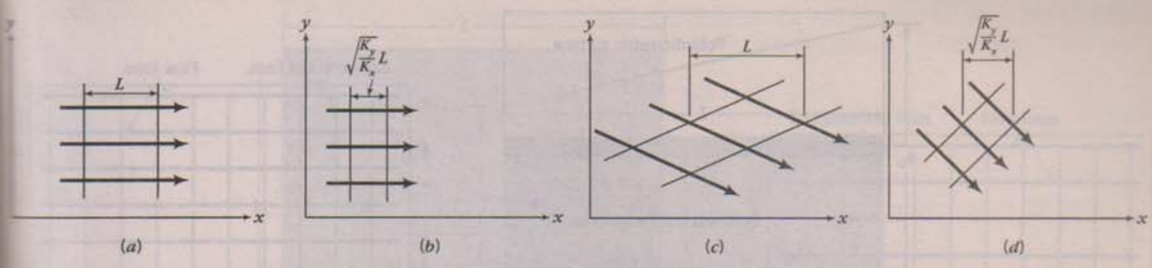


Figure 5.11 Flownet in anisotropic media. In Panel (a), the direction of flow is parallel to the principal direction (x) of hydraulic conductivity: the horizontal dimension of the porous medium cross section is changed by a ratio of $\sqrt{K_y/K_x}$. In Panel (b), a transformed cross section with square nets is formed when the same ratio is applied to horizontal dimension. In Panel (c), the direction of flow is at some angle to the principal direction (x) of hydraulic conductivity: the angles formed by the streamlines and the equipotential lines are no longer 90 degrees. In Panel (d), a transformed cross section with square nets can be formed by applying the ratio of $\sqrt{K_y/K_x}$ to the horizontal dimension of the flownet. Note: arrows point to the direction of flow.

Figure 5.10 Flownets illustrating how flow lines refract in moving across hydraulic conductivity boundaries in systems of aquifers and confining beds (from Heath, 1983).

simple initial and boundary conditions. Computers are now used to help evaluate analytical solutions. However, they still can be treated with a calculator and various tables of functions.

Numerical approaches developed in concert with modern digital computers require the computational capabilities of these machines. These techniques are tremendously powerful and can be applied to evaluate the most complicated, real-world problems. They can handle variability in hydraulic properties, large numbers of wells, and complicated boundary conditions, which might include variable recharge/evaporation and ground-water/surface-water interactions.

In this section, we will introduce some simple applications of analytic approaches for solving steady-state flow problems, which are applied to flow in aquifers and flow to wells. Chapters 9 to 11 will address the application of more complicated transient analytic solutions to problems of well hydraulics. Numerical approaches to solving problems involving complex aquifer systems will be treated in Chapter 15.

Ground-Water Flow in a Confined Aquifer

We illustrate the mathematical analysis with a simple problem of steady-state ground-water flow in a confined aquifer. The flow is one dimensional, produced by imposing different constant heads on the opposite sides of a rectangular flow domain (Figure 5.12a). Assuming that the aquifer is homogeneous and that the system is at steady state, one can describe flow by

$$\frac{\partial^2 h}{\partial x^2} = 0 \tag{5.45}$$

with these boundary conditions:

$$h|_{x=0} = h_0 \tag{5.46}$$

and

$$h|_{x=l} = h_L \tag{5.47}$$

The resulting analytical solution provides the hydraulic head as a function of the x-position in the aquifer

$$h = h_0 + (h_L - h_0)\frac{x}{L} \tag{5.48}$$

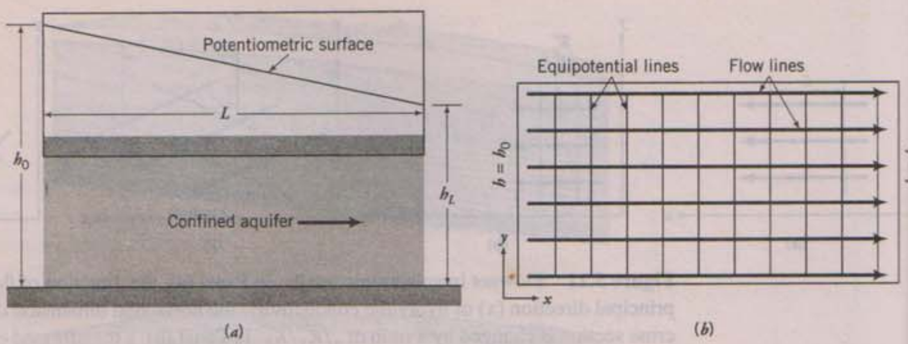


Figure 5.12 Unidirectional flow in a confined aquifer. Panel (*a*) is a cross section showing the region of interest, the aquifer, and the potentiometric surface. Panel (*b*) is a plan view showing the flownet.

Because the flow is one-dimensional and the medium is homogeneous, calculation of th Darcy flow velocity is straightforward:

$$q_x = K\frac{h_0 - h_L}{L} \qquad (5.4$$

Given the solution in Eq. (5.48), one can construct equipotential lines and sketch flow lines as shown in Figure 5.12*b*.

▶ EXAMPLE 5.5 Two rivers 1000 m apart penetrate a confined aquifer 20 m thick. The hydraulic conductivity is 2 m/day. The stages of the two rivers are 500 m and 495 m above sea level, respectively. What is th Darcy flow velocity for ground water in the aquifer? If the reaches of the rivers are 600 m long an there are no pumped wells between them, what is the volume of ground-water outflow/inflow per ye from the rivers? If a well were installed at a point located exactly between the rivers, what would the hydraulic head before any pumping?

SOLUTION The calculation of inflow/outflow from the rivers starts with the Darcy velocity

$$q = (20 \text{ m/day})\frac{(500 \text{ m}) - (495 \text{ m})}{1000 \text{ m}} = 0.1 \text{ m/day}$$

Given q, the inflow/outflow from the rivers is determined by multiplying q by the area of inflow/outfl areas, or

$$Q = (600 \text{ m})(20 \text{ m})(0.1 \text{ m/day})\left(365\frac{\text{days}}{\text{year}}\right) = 4.38 \times 10^5 \text{ m}^3/\text{year}$$

The hydraulic head in the aquifer midway between the two rivers is given from the analytic solut (Eq. 5.48):

$$h = 500 \text{ m} - \frac{(500 \text{ m}) - (495 \text{ m})}{(1000 \text{ m})}(500 \text{ m}) = 497.5 \text{ m}$$

Ground-Water Flow in an Unconfined Aquifer

This example is similar to the previous one except that now the aquifer is unconfin The equation for steady-state, ground-water flow in a homogeneous, unconfined aqui (Figure 5.13*a*) is

$$\frac{\partial}{\partial x}\left(h\frac{\partial h}{\partial x}\right) = 0 \qquad (5.5$$

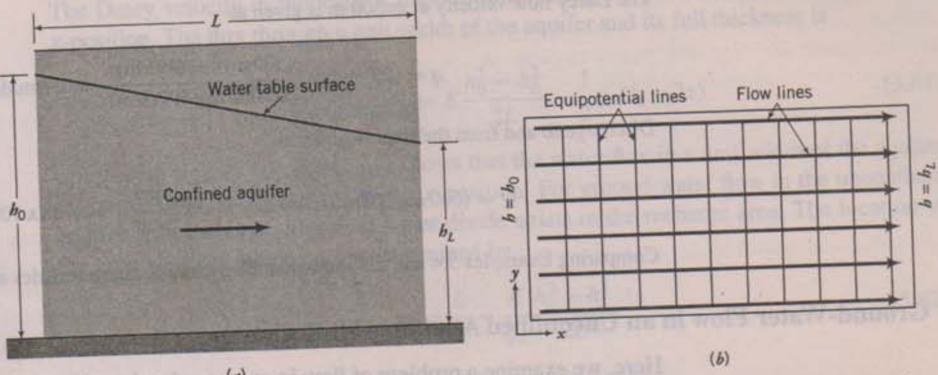


Figure 5.13 Unidirectional flow in an unconfined aquifer. Panel (*a*) is a cross section showing the region of interest and the water-table surface. Panel (*b*) is a plan view showing the flownet.

The boundary conditions are the same as before with

$$h|_{x=0} = h_0 \tag{5.51}$$

and

$$h|_{x=l} = h_L \tag{5.52}$$

The solution with Dupuit's assumption and these boundary conditions is

$$h = \sqrt{h_0^2 + (h_L^2 - h_0^2)\frac{x}{L}} \tag{5.53}$$

The Darcy flow velocity is

$$q = K\frac{h_0^2 - h_L^2}{2Lh} \tag{5.54}$$

The Darcy velocity will be different for different x locations because in a confined aquifer the thickness of the aquifer changes as a function of the hydraulic head. However, the flow rate through a unit width of the aquifer will be the same.

$$Q_{\text{unit width}} = K\frac{h_0^2 - h_L^2}{2L} \tag{5.55}$$

The solution provides the flow lines and equipotential lines shown in Figure 5.13*b*.

EXAMPLE 5.6 Return to Example 5.5, but now assume that the aquifer is unconfined. Calculate the hydraulic head in the aquifer at the midpoint, the Darcy flow velocity at $x = 500$ m, and the inflow/outflow from the rivers.

SOLUTION The hydraulic head at the midpoint (x=500 m) is calculated by appropriate substitution into the analytic solution to give

$$h = \sqrt{(500\text{ m})^2 + [(495\text{ m})^2 - (500\text{ m})^2]\frac{(500\text{ m})}{(1000\text{ m})}} = 497.5\text{ m}$$

CHAPTER

9

RESPONSE OF CONFINED AQUIFERS TO PUMPING

One of the important jobs for ground-water hydrologists is to find and to develop water supplies. High-capacity wells installed in productive aquifers are capable of providing thousands of gallons of water per minute. Research in the 1940s and 1950s produced quantitative analytical tools to predict how pumping would impact hydraulic head in the aquifers and to interpret the results of hydraulic tests. From the mid-1960s through the early 1980s, powerful numerical codes like MODFLOW became available to help analyze much more complicated systems.

There are fundamentally two types of problems related to an aquifer's responses to pumping. The so-called forward problem is concerned with predicting what the hydraulic-head distribution will be in an aquifer at times in the future, given boundary conditions, initial conditions, and information about transmissivity, storativity, and pumping rate. As we saw in Chapter 3, one can calculate this hydraulic-head distribution by solving a ground-water flow equation. Forward modeling is essential to designing well systems, analyzing whether drawdowns caused by a well are impacting other wells, and designing dewatering systems.

The inverse problem, as applied to well problems, involves using measurements of hydraulic head in an aquifer as a function of time to calculate values of transmissivity, storativity, specific yield, and so on. In other words, the mathematical theory provides the basis for interpreting the results of an aquifer test. Here and in following chapters, you will learn how to make drawdown predictions for various types of aquifers and how to interpret the results of aquifer tests.

▶ 9.1 AQUIFERS AND AQUIFER TESTS

The study of well hydraulics is complicated. For every situation where the
the pattern of layering of aquifers, or the length of the screen in relation to the
ness changes, we need a different analytical solution. If our goal was to treat
situation that one might conceivably encounter in the field, we could end up
or 50 different solutions. Fortunately, we have set a less ambitious goal of
a few of the most common situations. This chapter focuses on a confined
homogeneous, isotropic, and infinite in extent.

We have used the term *aquifer test* without discussing it in detail. An
involves pumping a well for the purpose of determining aquifer parameters
A test typically involves a pumping well and one or more observation wells.
well has a relatively large- diameter casing and is screened across all or part of
(Figure 9.1). A large-diameter casing is necessary because a pump and piping
to be installed down in the well. *Observation wells* are located at varying
the pumping well. They commonly are smaller in diameter and again are
all or part of the aquifer. Before an aquifer test is begun, water levels in all
are measured to provide the *prepumping or static water levels* (h_0). In other
measurements provide hydraulic heads in the wells at time zero. A test starts
pumping of water from the well. The *pumping rate* (Q) is the volume of water
from a well per unit time [L^3/T].

As the aquifer is pumped, water levels are measured periodically in the
observation wells. Water levels are measured frequently at first because at early
change rapidly. The term *pumping water level* (h) is used to describe the water
well during a test.

By convention, one works with the change in water levels through the test
water levels. The term *drawdown* ($s = h_0 - h$) is the difference between the static
and the pumping water level (Figure 9.1). Once the impact of pumping becomes
a well, drawdowns usually increase with time. The zone around the well in which
a measurable water-level change is called the cone of depression. The *cone of*
is a water level low in water table or potentiometric surface, which has the
inverted cone, centered on the pumped well. Away from the cone of depression,
caused by pumping is undetectable. The *radius of influence* (R) is the distance
pumped well to the edge of the cone of depression. Under steady-state conditions,
discharged by a well is assumed to be coming from sources beyond the radius of
Under transient-flow conditions, the water discharged by a well is assumed to be

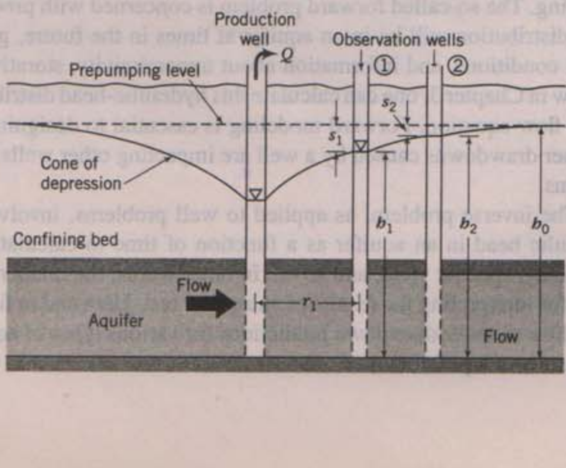


Figure 9.1 A confined
from which ground water
being withdrawn at a
rate Q. The cone of
spreads away from the
produces drawdowns s_1

from the aquifer storage within the radius of influence and sources beyond the radius of influence.

Units

The analytical solutions presented in this and the following chapters are all developed in terms of consistent units. Thus, it does not matter what units of length (for example, feet or meters) or time (seconds, day) you use, as long as all the units are consistent. For example, if meters and days are selected as the consistent units, discharge would have units of m^3/day, distances would be in meters, transmissivities in m^2/day, and so on. Readers will often find it necessary to convert units before using any of the equations in the following chapters. Conversion equations for transmissivity and hydraulic conductivity were introduced in Chapters 3 and 4. The following equations will help provide pumping rates as a consistent unit from the English unit.

$$1\ \text{gpm} = 192.5\ \text{ft}^3/\text{day} = 5.45\ \text{m}^3/\text{day} = 6.3 \times 10^{-5}\ \text{m}^3/\text{sec} \tag{9.1}$$

$$1\ \text{ft}^3/\text{day} = 5.19 \times 10^{-3}\ \text{gpm} = 2.832 \times 10^{-2}\ \text{m}^3/\text{day} = 3.28 \times 10^{-7}\ \text{m}^3/\text{sec} \tag{9.2}$$

$$1\ \text{m}^3/\text{day} = 35.31\ \text{ft}^3/\text{day} = 0.1835\ \text{gpm} = 1.1574 \times 10^{-5}\ \text{m}^3/\text{sec} \tag{9.3}$$

$$1\ \text{m}^3/\text{sec} = 3.051 \times 10^{6}\ \text{ft}^3/\text{day} = 1.58 \times 10^{4}\ \text{gpm} = 8.64 \times 10^{4}\ \text{m}^3/\text{day} \tag{9.4}$$

9.2 THIEM'S METHOD FOR STEADY-STATE FLOW IN A CONFINED AQUIFER

Historically, one of the first quantitative approaches for looking at flow in a confined aquifer was that of Thiem (1906). This theory applies to a homogeneous and isotropic aquifer that is infinite in extent. The analysis also assumes that there has been sufficient pumping to allow the ground-water system to achieve steady state. In other words, water levels in the wells do not change with time (Figure 9.2*a*). The map view in Figure 9.2*b* shows that the flow in this case is radial, toward the well, with hydraulic heads increasing away from the pumping well.

The hydraulic head in the aquifer can be determined as a solution to a ground-water flow equation, like those we presented in Chapter 3. However, in this case, it is beneficial to use radial coordinates, where distances (r) are measured from the well to some point of interest. A solution to the flow equation with appropriate boundary conditions is

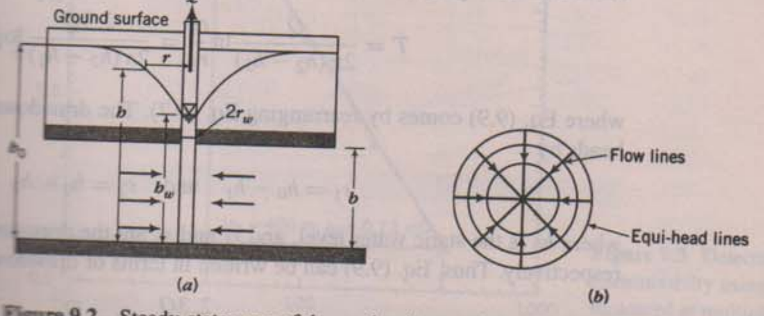


Figure 9.2 Steady-state cone of depression in a confined aquifer, (a) sectional view, (b) map view.

▶ 9.3 THEIS SOLUTION FOR TRANSIENT FLOW IN A FULLY PENETRATING, CONFINED AQUIFER

Theis theory lets us evaluate the behavior of a well pumping in a confined aquifer under transient (that is, nonsteady-state) conditions. The flow equation describing hydraulic head in a confined aquifer (Figure 9.4) can be written in polar coordinates as

$$\frac{\partial^2 h}{\partial r^2} + \frac{1}{r}\frac{\partial h}{\partial r} = \frac{S}{T}\frac{\partial h}{\partial t}$$

where h is the hydraulic head, r is the radial distance from a pumped well to an observation well, t is time since the pumping started, S is the storativity of the aquifer, and T is the transmissivity of the aquifer. The following initial and two boundary conditions apply to this problem

$$h(r, 0) = h_0$$

$$h(\infty, t) = h_0$$

$$\lim_{r \to 0}\left(r\frac{\partial h}{\partial r}\right) = \frac{Q}{2\pi T}$$

In words, the first condition means that at time zero and any distance r from the well, the head is equal to the initial head, h_0. The second condition means that at an infinite distance (one boundary) for all time the hydraulic head is fixed at h_0. The third condition prescribes a constant withdrawal rate at the pumping well (another boundary).

The solution of Eq. (9.12) was first derived by Theis (1935) and is expressed as

$$h_0 - h = s = \frac{Q}{4\pi T} W(u)$$

where Q is the pumping rate and T is the transmissivity of the aquifer. The well function $W(u)$ and the dimensionless variable u are expressed as:

$$W(u) = \int_u^\infty \frac{e^{-y}}{y} dy = -0.577216 - \ln(u) + u - \frac{u^2}{2!2} + \frac{u^3}{3!3} - \frac{u^4}{4!4} + \cdots$$

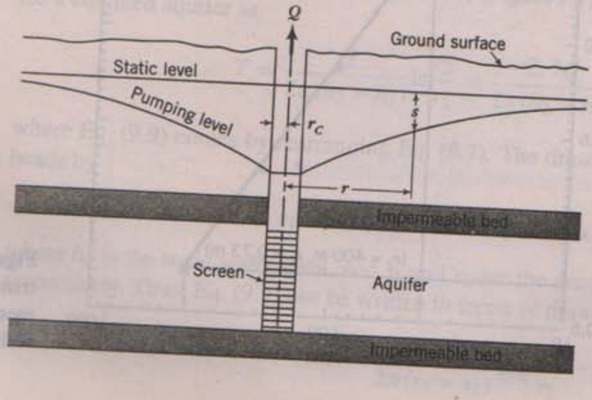


Figure 9.4 Illustration of a nonleaky, confined aquifer being pumped by a fully penetrating well (from Reed, 1980).

and

$$u = \frac{r^2 S}{4 T t} \tag{9.15}$$

Even though $W(u)$ is a complicated function, it can be evaluated by using well function tables (Table 9.2) or by approximating computer programs. The Theis solution is based on the following assumptions:

1. The pumping well is fully penetrating, with a constant discharge rate, infinitesimal diameter, and negligible storage.
2. The aquifer is confined, infinite in extent, homogeneous, and isotropic.
3. All water pumped by the well comes from the storage and is discharged instantaneously with the decline in head.

9.4 PREDICTION OF DRAWDOWN AND PUMPING RATE USING THE THEIS SOLUTION

The drawdown in an observation well at some future time can be calculated directly using Eq. (9.13) for known hydraulic parameters. For other problems, we might need to know what pumping rate provides a specified drawdown at a fixed place and time in the future. This calculation requires transforming the Theis equation into the following form.

$$Q = \frac{4\pi T s}{W(u)} \tag{9.16}$$

EXAMPLE 9.3 The transmissivity and storativity of a confined aquifer are 1000 m^2/day and 0.0001, respectively. An observation well is located 500 m away from a pumping well. For a pumping period of 220 min, calculate (a) the drawdown at the observation well if the discharge rate is 1000 m^3/day; (b) the pumping rate required to provide a drawdown of 1 m at that well after 220 minutes.

TABLE 9.2 Values of well function $W(u)$

u	1.0	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0
×1	0.219	0.049	0.013	0.0038	0.0011	0.00036	0.00012	0.000038	0.000012
$\times 10^{-1}$	1.82	1.22	0.91	0.70	0.56	0.45	0.37	0.31	0.26
$\times 10^{-2}$	4.04	3.35	2.96	2.68	2.47	2.30	2.15	2.03	1.92
$\times 10^{-3}$	6.33	5.64	5.23	4.95	4.73	4.54	4.39	4.26	4.14
$\times 10^{-4}$	8.63	7.94	7.53	7.25	7.02	6.84	6.69	6.55	6.44
$\times 10^{-5}$	10.94	10.24	9.84	9.55	9.33	9.14	8.99	8.86	8.74
$\times 10^{-6}$	13.24	12.55	12.14	11.85	11.63	11.45	11.29	11.16	11.04
$\times 10^{-7}$	15.54	14.85	14.44	14.15	13.93	13.75	13.60	13.46	13.34
$\times 10^{-8}$	17.84	17.15	16.74	16.46	16.23	16.05	15.90	15.76	15.65
$\times 10^{-9}$	20.15	19.45	19.05	18.76	18.54	18.35	18.20	18.07	17.95
$\times 10^{-10}$	22.45	21.76	21.35	21.06	20.84	20.66	20.50	20.37	20.25
$\times 10^{-11}$	24.75	24.06	23.65	23.36	23.14	22.96	22.81	22.67	22.55
$\times 10^{-12}$	27.05	26.36	25.96	25.67	25.44	25.26	25.11	24.97	24.86
$\times 10^{-13}$	29.36	28.66	28.26	27.97	27.75	27.56	27.41	27.28	27.16
$\times 10^{-14}$	31.66	30.97	30.56	30.27	30.05	29.87	29.71	29.58	29.46
$\times 10^{-15}$	33.96	33.27	32.86	32.58	32.35	32.17	32.02	31.88	31.76

Source: From Wenzel (1942).

SOLUTION Drawdown can be calculated using the Theis equation. All of the paramet... RHS of Eq. (9.13) are known except for $W(u)$. To evaluate $W(u)$, we first calculate u as

$$u = \frac{r^2 S}{4Tt} = \frac{500 \times 500\,\mathrm{m}^2 \times 0.0001}{4 \times 1000\dfrac{\mathrm{m}^2}{\mathrm{day}} \times \dfrac{1}{1440}\dfrac{\mathrm{day}}{\mathrm{min}} \times 220\,\mathrm{min}} = 0.041$$

The well function $W(u)$ at $u = 0.041$ is

$$W(0.041) = 2.66$$

For a pumping rate of 1000 m^3/day, the drawdown is calculated as

$$s = \frac{Q}{4\pi T} W(u) = \frac{1000\,\mathrm{m}^3/\mathrm{day}}{4 \times 3.14 \times 1000\,\mathrm{m}^2/\mathrm{day}}(2.66) = 0.21\,\mathrm{m}$$

For a drawdown of 1 m, the pumping rate is calculated as

$$Q = \frac{4\pi T s}{W(u)} = \frac{4 \times 3.14 \times 1000\,\mathrm{m}^2/\mathrm{day} \times 1\mathrm{m}}{(2.66)} = 4.72 \times 10^3\,\mathrm{m}^3/\mathrm{day}$$

▶ 9.5 THEIS TYPE-CURVE METHOD

Another important use of the Theis solution is in determinating the transmissivi... storativity from data collected from an aquifer test. A variety of aquifer testing appr... are available. We begin here with the so-called type-curve matching technique, wh... widely used. The test data are a series of drawdown values in an observation well, ... matched with a time since pumping began. The approach involves plotting the field d... one graph, which is overlain on a type curve plotted at the same scale. Here are the det...

1. Create the type curve by plotting the well function $W(u)$ versus $1/u$ on lo... graph paper (Figure 9.5). Usually, you can buy a copy of this curve.
2. Define a match point on the type curve. This match point only serves as a refe... and can be located anywhere on the graph. However, the math works ou... you choose a match point with "simple" coordinates like $W(u) = 1$ and $1/u$ = ... 10^2, or 10.

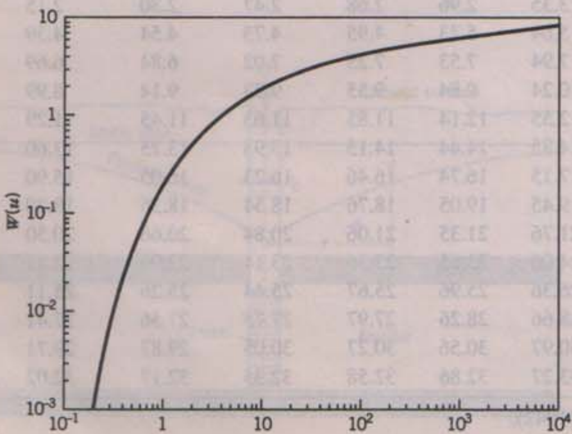


Figure 9.5 A plot of the well function $W(u)$ versus $1/u$ on log-log paper.

3. Prepare a transparent overlay with drawdown (s) plotted versus time (t) on log-log graph paper. This graph paper must be the same as the type curve. This step is where you use the set of field data from the observation well.
4. Superimpose the transparent graph of the field data on the type curve. Adjust the field curve until the collection of field points appears to fall along the type curve underneath. You must keep the axes of the two graphs parallel to each other.
5. Mark the point on the field curve that exactly corresponds with the match point on the type curve underneath. Now you will have points marked on both graphs with coordinates $W(u)$, $1/u$, and $s(t)$, t. These pairs of values will be substituted in step 5.
6. Calculate T and S using the following equations.

$$T = \frac{Q}{4\pi s} W(u) \tag{9.17}$$

and

$$S = \frac{4\,T\,t\,u}{r^2} \tag{9.18}$$

In the case of Eq. (9.17), Q is known from the pump-test data, $W(u)$ is the coordinate value of the match point on the type curve, and s is the coordinate value for the match point on the curve of the field data. In the case of Eq. (9.18), T is known from the previous calculation, r is known from the setup of the aquifer test, and $1/u$ and t are the match-point coordinates obtained via curve matching. Here is an example that illustrates these steps.

EXAMPLE 9.4 In a test of a confined aquifer, the pumping rate was 500 m^3/day. Drawdown/time data were collected at an observation well 300 m away (Table 9.3). Use the type-curve method to determine the hydraulic conductivity and storativity of the aquifer.

SOLUTION Figure 9.6 is the plot of drawdown versus time. Superimposing the field curve on to the type curve, as shown in Figure 9.7, gives the match-point coordinates $1/u = 10$, $W(u) = 1.0$, $t = 22$ min, and $s = 0.78$ m. Thus,

$$T = \frac{Q}{4\pi s} W(u) = \frac{(500\,\mathrm{m}^3/\mathrm{day})(1)}{(4\pi)(0.78\,\mathrm{m})} = 51\,\mathrm{m}^2/\mathrm{day}$$

and

$$S = \frac{4Ttu}{r^2} = \frac{4(51\,\mathrm{m}^2/\mathrm{day})\left(22\,\mathrm{min}\dfrac{1\,\mathrm{day}}{1440\,\mathrm{min}}\right)(0.1)}{(300\,\mathrm{m})^2} = 3.46\times10^{-6}$$

9.6 COOPER–JACOB STRAIGHT-LINE METHOD

Using the Theis solution (Eq. 9.13) is complicated by the fact that it contains the function $W(u)$, which is an exponential integral. The value of $W(u)$ is the sum of an infinite series (see Eq. 9.14), which we normally evaluate using the table of well functions. However, when the values of u are small, less than 0.01, the higher order terms of the infinite series are negligible and can be ignored (Cooper and Jacob, 1946; Jacob, 1940). With the Cooper–Jacob's assumption, drawdown is calculated as

$$s(t) = \frac{Q}{4\pi T}(-0.577216 - \ln(u)) = \frac{2.3Q}{4\pi T}\log\frac{2.25T\,t}{r^2S} \tag{9.19}$$

TABLE 9.3 Drawdowns measured at an observation well 300 m away

Time (min)	S(m)
1.00	0.03
1.27	0.05
1.61	0.09
2.04	0.15
2.59	0.22
3.29	0.31
4.18	0.41
5.30	0.53
6.72	0.66
8.53	0.80
10.83	0.95
13.74	1.11
17.43	1.27
22.12	1.44
28.07	1.61
35.62	1.79
45.20	1.97
57.36	2.15
72.79	2.33
92.37	2.52
117.21	2.70
148.74	2.89
188.74	3.07
239.50	3.26
303.92	3.45
385.66	3.64
489.39	3.83
621.02	4.02
788.05	4.21
1000.0	4.39

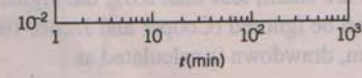


Figure 9.6 A plot of measured drawdown vers
on log-log paper.

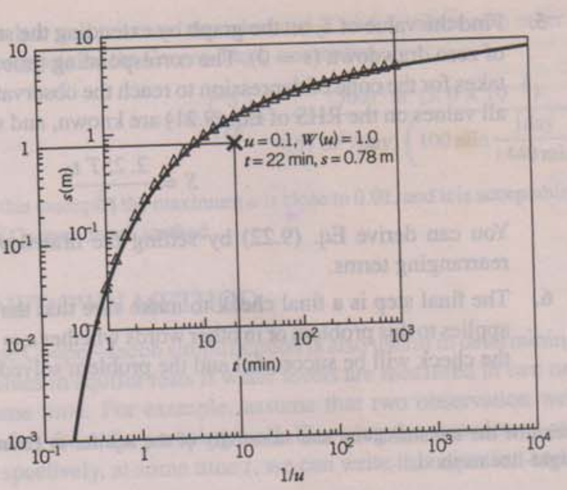


Figure 9.7 Illustration of the method of superposition used to determine transmissivity and storativity.

The result of this simplification is that integral expression in the Theis solution is replaced by a much simpler function. It can be evaluated using a calculator without the need for the table of well functions.

Another benefit of the Cooper and Jacob (1946) modification of the Theis equation is that it leads to a simple, graphical approach to evaluating aquifer test data. If drawdown/time data are plotted as drawdown versus the logarithm of time, the Cooper–Jacob theory predicts that the data will fall along a straight line. By extracting two numbers from the graph, we can solve equations to determine transmissivity and storativity. Here is a summary of the steps involved, followed by an example.

1. Plot drawdown versus time on semi-log graph paper, with time on the x-axis as a logarithmic scale and drawdown on the y-axis as an arithmetic scale. Often, zero drawdown is at the top of the y-axis.
2. Fit a straight line through the data points. If there is difficulty, use the later-time points.
3. Select two points (t_1, s_1 and t_2, s_2) on the line. The equation that is needed can be derived from Eq. (9.19) by writing one equation in terms of s_2 and one equation in terms of s_1 and subtracting them from each other. After some manipulation, the resulting equation is

$$\Delta s = s_2 - s_1 = \frac{2.3Q}{4\pi T} \log \frac{t_2}{t_1} \tag{9.20}$$

4. Choose t_1 and t_2 one log cycle apart, for example $t_1 = 10$ minutes and $t_2 = 100$ minutes, to give Δs or drawdown per log cycle. This choice simplifies the math. For example, with $t_1 = 10$ minutes and $t_2 = 100$ minutes, $\log(t_2/t_1) = \log(100/10) = 1$. The log term in Eq. (9.20) becomes one, and the equation simplifies to

$$T = \frac{2.3Q}{4\pi \Delta s} \tag{9.21}$$

where Δs is the drawdown per log cycle. All the terms on the RHS of Eq. (9.21) are known, so it is a simple matter to calculate transmissivity.

5. Find the value of t_0 on the graph by extending the straight line to intersect ... of zero drawdown ($s = 0$). The corresponding time (t_0) is in effect the time ... takes for the cone of depression to reach the observation well. With this value ... all values on the RHS of Eq. (9.21) are known, and storativity can be calcul...

$$S = \frac{2.25T\,t_0}{r^2}$$

You can derive Eq. (9.22) by setting the drawdown in Eq. (9.19) to zero ... rearranging terms.

6. The final step is a final check to make sure that the Cooper–Jacob simplific... applies to this problem or in other words whether $u = r^2S/(4Tt) < 0.01$. Hope... the check will be successful and the problem solved.

► EXAMPLE 9.5 Determine the transmissivity and storativity of the aquifer in Example 9.4 using the Cooper–... straight-line method.

SOLUTION Figure 9.8 shows a plot of drawdown versus log(t) for the data set. A line is ... late-time section of the curve. On the figure, we find drawdowns corresponding to two times, ... are a factor of 10 different from each other. For example, with $t_1 = 100$ min, $s_1 = 2.58$ m, and ... $t_2 = 1000$ min, $s_2 = 4.39$ m. Thus, the drawdown per log cycle is $4.39 - 2.58$ or 1.81 m

$$T = \frac{2.3Q}{4\pi\,\Delta s} = \frac{(2.3)(500\ \mathrm{m^3/day})}{4\pi(1.81\ \mathrm{m})} = 51\ \mathrm{m^2/day}$$

The next step is to substitute the value of t_0, 3.4 min (determined from the graph) in Eq. (9.22), ... with the other known parameters, and to calculate S

$$S = \frac{2.25T\,t_0}{r^2} = \frac{(2.25)(51\ \mathrm{m^2/day})\left(3.4\ \mathrm{min}\,\dfrac{1\ \mathrm{day}}{1440\ \mathrm{min}}\right)}{(300\ \mathrm{m})^2} = 3.0 \times 10^{-6}$$

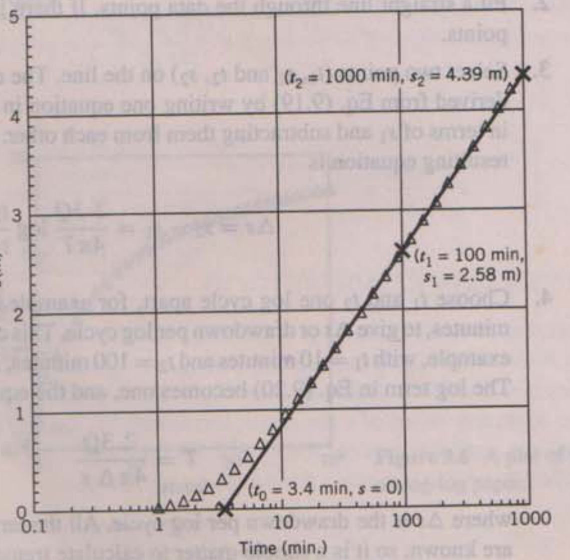


Figure 9.8 Illustration of ... the Cooper–Jacob straight-li... method is used with observ... well data.

The last step is to use the calculated T and S values with other parameters to see whether u is appropriate for the Cooper–Jacob assumption

$$u = \frac{r^2 S}{4Tt} = \frac{(300)^2\,\mathrm{m}^2(3.0 \times 10^{-6})}{4(51\,\mathrm{m}^2/\mathrm{day})\left(100\,\mathrm{min}\,\dfrac{1\mathrm{day}}{1440\,\mathrm{min}}\right)} = 0.02$$

In this example, the maximum u is close to 0.01, and it is acceptable to calculate T and S values using the Cooper–Jacob method.

9.7 DISTANCE-DRAWDOWN METHOD

The Cooper–Jacob simplification is also useful in determining transmissivity and storativity values in aquifer tests if water levels are measured in two or more observation wells at the same time. For example, assume that two observation wells are located at distances r_1 and r_2 from the pumping well. Knowing the drawdowns at these two wells are s_1 and s_2, respectively, at some time t, we can write this equation based on Eq. (9.19):

$$s_1 = \frac{2.3Q}{4\pi T}\log\frac{2.25T\,t}{r_1^2 S} \tag{9.23}$$

and

$$s_2 = \frac{2.3Q}{4\pi T}\log\frac{2.25T\,t}{r_2^2 S} \tag{9.24}$$

The combination of Eqs. (9.23) and (9.24) yields

$$s_1 - s_2 = \frac{2.3Q}{2\pi T}\log\frac{r_2}{r_1} \tag{9.25}$$

The procedures for determining the transmissivity and storativity for an aquifer using the distance-drawdown method are as follows:

1. For a selected time, plot the drawdown and distance information for the observation wells on semi-log graph paper. Distance is plotted as a logarithmic scale on the x-axis, and drawdown is plotted on a linear scale on the y-axis.
2. Fit a straight line through the data points.
3. Select two points (r_1, r_2) on the line, one log cycle apart, and determine the drawdown Δs. The transmissivity is calculated by

$$T = \frac{2.3Q}{2\pi(\Delta s)} \tag{9.26}$$

4. Extend the straight line to $s = 0$ and determine the distance r_0. The storativity is calculated by

$$S = \frac{2.25T\,t}{r_0^2} \tag{9.27}$$

EXAMPLE 9.6 A confined aquifer is pumped at 220 gpm. At time = 220 min, drawdowns were recorded in nine observation wells (Table 9.4). Calculate the transmissivity and storativity of the aquifer.

SOLUTION Drawdown versus distance is plotted in Figure 9.9. The pumping rate in gpm is first converted to ft^3/day using Eq. (9.1).

TABLE 9.4 Values of drawdown versus distance measured at 220 minutes

r(ft)	s(ft)
10	35.20
50	24.35
100	19.68
150	16.96
200	15.03
250	13.54
300	12.32
400	10.42
500	8.97

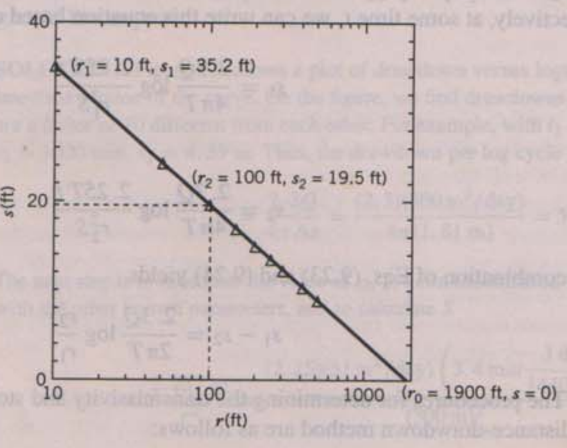


Figure 9.9 Drawdown data for nine observation wells at different radial distances from a pumping well are fit with a straight line to provide Δs and r_0.

$$Q = (220\,\text{gpm})\left(192.5\frac{\text{ft}^3/\text{d}}{\text{gpm}}\right) = 42350\,\text{ft}^3/\text{d}$$

Selecting r_1 as 10 ft and $r_2 = 100$ ft, the drawdown per log cycle is $s_1 - s_2$, or $35.2 - 19.5 = 15.7$ ft. The value of $r_0 = 1900$ ft. Thus,

$$T = \frac{2.3Q}{2\pi(\Delta s)} = \frac{2.3(42350\,\text{ft}^3/\text{d})}{2\pi(15.7)\text{ft}} = 987\,\text{ft}^2/\text{d}$$

and

$$S = \frac{2.25(987\,\text{ft}^2/\text{d})(220\,\text{min})\left(\frac{1}{1440}\frac{\text{d}}{\text{min}}\right)}{(1900\,\text{ft})^2} = 9.4 \times 10^{-5}$$

▶ 9.8 ESTIMATING T AND S USING RECOVERY DATA (THEIS, 1935)

If pumping of a well is halted, theory predicts that the water level in the aquifer will return to its prepumping level, h_0. Water-level data obtained during this recovery phase of a test provides a basis for determining the transmissivity of the aquifer. The residual drawdown during the recovery period for a confined aquifer is expressed as

$$s' = \frac{Q}{4\pi T}\left[\int_u^\infty \frac{e^{-u}}{u}du - \int_{u'}^\infty \frac{e^{-u'}}{u'}du'\right] \tag{9.28}$$

where

$$u = \frac{r^2 S}{4\,T\,t} \tag{9.29}$$

and

$$u' = \frac{r^2 S}{4\,T\,t'} \tag{9.30}$$

where t is the time since pumping starts, t' is time since pumping stops, and s' is the residual drawdown. Because recovery measurements are made in the pumped well or in a nearby observation well, the radius to the measurement point, r, is typically small. A small r usually leads to a small value u', which enables us to take advantage of the Cooper-Jacob simplification. Under the Cooper–Jacob assumption, Eq. (9.28) reduces to

$$s' = \frac{2.3Q}{4\pi T}\log\left(\frac{t}{t'}\right) \tag{9.31}$$

and transmissivity can be determined as

$$T = \frac{2.3Q}{4\pi s'}\log\left(\frac{t}{t'}\right) \tag{9.32}$$

The procedure for determining transmissivity using recovery data is as follows:

1. Plot the residual drawdown (s') on an arithmetic scale versus the time ratio (t/t') on a logarithmic scale.
2. Choose two points on the graph. Again it helps to select the two points one log cycle apart.

The transmissivity is obtained by

$$T = \frac{2.3Q}{4\pi\,\Delta s'} \tag{9.33}$$

where $\Delta s'$ is the change in residual drawdown over one log cycle (t/t').

Storativity can be calculated if the recovery data are collected in an observation well rather than the pumping well. The drawdown (s_P) when the pump is turned off at time (t_P) is expressed as

$$s_P = \frac{2.3Q}{4\pi T}\log\frac{2.25\,T\,t_P}{r^2 S} \tag{9.34}$$

Once T is known, the storativity is obtained by

$$S = \frac{2.25\,T\,t_P}{r^2}10^{-\frac{4\pi T s_P}{2.3Q}} \tag{9.35}$$

EXAMPLE 9.7

In an aquifer test reported by USBR in 1995, drawdowns are recorded in the pumped well (Table 9.5) and an observation well (Table 9.6). In both tables, the first column is the time since the pumping started, the second column is the drawdown during the pumping period, the third column is the time since pumping stopped, the fourth column is the time ratio, and the last column is the residual drawdown. A constant pumping rate of 162.9 ft^3/min was maintained during the pumping part of the test.

TABLE 9.5 Aquifer test information from a pumped well

t(min)	s(ft)	t'(min)	t/t'	s'(ft)
3	10.2	3	267.67	−20
8	10.6	8	101.00	−5
13	10.8	13	62.54	−0.5
20	11.3	20	41.00	1.5
80	11.6	80	11.00	1
140	11.8	140	6.71	0.8
195	11.8	195	5.10	0.69
255	11.8	255	4.14	0.59
315	12	315	3.54	0.51
375	12.2	375	3.13	0.49
435	12.2	435	2.84	0.46
495	12.2	495	2.62	0.38
560	12.2	560	2.43	0.34
616	12.3	616	2.30	0.33
668	12.4	668	2.12	0.33
737	12.5	727	2.10	0.22
800	12.5	800	2.00	0.22

Source: Modified from USBR (1995).

TABLE 9.6 Aquifer test information at an observation well

t(min)	s(ft)	t'(min)	t/t'	s'(ft)
5	0.08	5	161.00	1.78
10	0.22	10	81.00	1.64
15	0.33	15	54.33	1.53
20	0.41	20	41.00	1.45
25	0.5	25	33.00	1.37
30	0.55	30	27.67	1.32
40	0.66	40	21.00	1.22
50	0.73	50	17.00	1.15
60	0.8	60	14.33	1.09
70	0.86	70	12.43	1.03
80	0.92	80	11.00	0.97
90	0.96	90	9.89	0.94
100	1	100	9.00	0.9
110	1.04	110	8.27	0.87
120	1.07	120	7.67	0.85
180	1.24	180	5.44	0.7
240	1.35	240	4.33	0.61
300	1.45	300	3.67	0.54
360	1.52	360	3.22	0.49
420	1.59	420	2.90	0.46
480	1.65	480	2.67	0.4
540	1.71	540	2.48	0.36
600	1.73	600	2.33	0.36
660	1.77	660	2.21	0.34
720	1.81	720	2.11	0.31
800	1.86	800	2.00	0.29

Source: Modified from USBR (1995).

The observation well is 100 ft away from the pumped well. Calculate the hydraulic parameters using the recovery data.

SOLUTION Drawdown data for the pumped well and the observation well are plotted in Figures 9.10 and 9.11, respectively. A straight line approximates the drawdown versus time curve in each figure. We have derived $\Delta s' = 0.91$ ft and 0.87 ft for the pumped and observation wells, respectively. Therefore, the transmissivity determined from the recovery data in the pumped well is

$$T = \frac{2.3Q}{4\pi\,\Delta s'} = \frac{2.3(162.9\,\text{ft}^3/\text{min})}{4\pi(0.91\,\text{ft})} = 32.8\,\text{ft}^2/\text{min} = 4.7\times 10^4\,\text{ft}^2/\text{day}$$

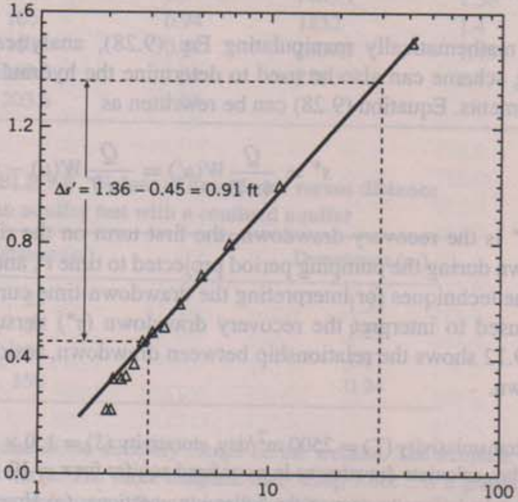


Figure 9.10 This plot illustrates the straight-line methods for determining transmissivity using residual drawdown data from the pumped well.

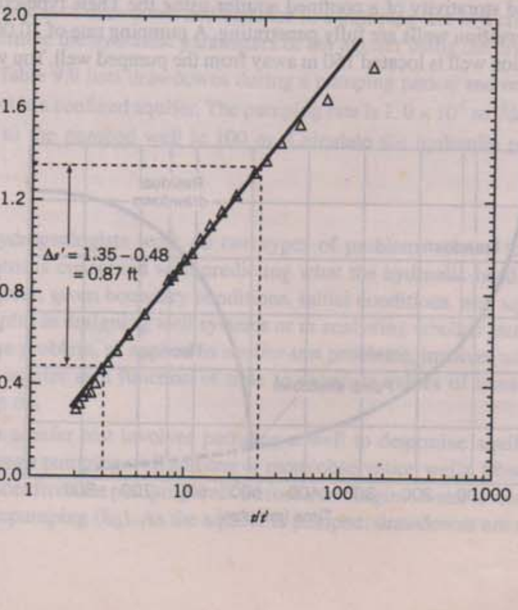


Figure 9.11 This plot illustrates the straight-line methods for determining transmissivity using residual drawdown data from the observation well.