The Action Principle

For the time being, we will consider a system with a single degree of freedom - with generalized coordinate q and corresponding generalized velocity $\dot{q}$.

Hamilton's principle of least action is analogous to Fermat's principle of least time in geometrical optics. It states that among all possible paths in configuration space that start at q_I at time t_I and end at q_F at time t_F, the actual path taken by a system is the one that minimizes (extremizes) the action S. That is, for the actual path taken by the system

$$\delta S = 0$$

for small variations of the path $q(t) \to q(t) + \delta q(t)$ (to the first order in δq), provided that the variation of the path $\delta q(t)$ satisfies

$$\delta q(t_I) = \delta q(t_F) = 0$$

The action is a *functional* [1] of the path $q(t)$, and is defined as

$$S = \int_{t_I}^{tt_F} L(q(t), \dot{q}(t), t)\, dt = S[q(t); t_I, t_F]$$

The function L of the generalized coordinate, generalized velocity and time is called the **Lagrangian**. We assume that it is a smooth function of its arguments.

Note:

- The arguments $q(t)$ and $\dot{q}(t)$ of the Lagrangian are *numbers* - the *values* of the *functions* $q(t)$ and $\dot{q}(t)$ at time t. This, I admit, is bad notation - but it is quite commonly used by physicists[2]. It is usually clear from the context whether one means the function or its value. Confusion between these two is often the cause of a lot of grief - so be warned!

- The value of q and that of $\dot{q}$ at a particular instant of time are **independent**. Knowing where a particle *is* gives you no clue about how fast it is moving - and *vice versa*. One can change q without changing $\dot{q}$

- The functions $q(t)$ and $\dot{q}(t)$ are *not independent!* If you know the function $q(t)$ you can simply obtain the generalized velocities $\dot{q}(t)$ by differentiating. In particular, if $q(t)$ is varied to $q(t) + \delta q(t)$, then $\dot{q}(t)$ changes to $\dot{q}(t) + \frac{d}{dt}\delta q(t)$ so that the change in the genralized velocity is

$$\delta \dot{q}(t) = \frac{d}{dt}\delta q(t)$$

[1] A functional is a map that takes in a function as argument - and yields a number

[2] no wonder that mathematicians hate us so much!

Variation of the action

When the path is varied : $q(t) \rightarrow q(t) + \delta q(t)$ (and consequently $\dot{q}(t) \rightarrow \dot{q}(t) + \frac{d}{dt}(\delta q(t))$), the action changes to

$$S + \delta S = \int_{t_I}^{t_F} L(q(t) + \delta q(t), \dot{q}(t) + \delta \dot{q}(t), t)\, dt$$

The variation in the action is

$$\begin{aligned}
\delta S &= \int_{t_I}^{t_F} L(q(t) + \delta q(t), \dot{q}(t) + \delta \dot{q}(t), t)\, dt - \int_{t_I}^{t_F} L(q(t), \dot{q}(t), t)\, dt \\
&= \int_{t_I}^{t_F} \left[L(q(t) + \delta q(t), \dot{q}(t) + \delta \dot{q}(t), t) - L(q(t), \dot{q}(t), t) \right] dt \\
&= \int_{t_I}^{t_F} \left[\frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} + \ldots \right] dt
\end{aligned}$$

The $\ldots$ in the last line above indicates higher order terms. Using the relation $\delta \dot{q}(t) = \frac{d}{dt} \delta q(t)$ in the above, we get

$$\begin{aligned}
\delta S &= \int_{t_I}^{t_F} \left[\frac{\partial L}{\partial q} \delta q(t) + \frac{\partial L}{\partial \dot{q}} \frac{d}{dt}(\delta q(t)) + \ldots \right] dt \\
&= \int_{t_I}^{t_F} \left[\frac{\partial L}{\partial q} \delta q(t) + \left\{ \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \delta q(t) \right) - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \delta q(t) \right\} + \ldots \right] dt \\
&= \int_{t_I}^{t_F} \left[\frac{\partial L}{\partial q} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \right] \delta q(t)\, dt + + \int_{t_I}^{t_F} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \delta q(t) \right) dt + \ldots \\
&= \int_{t_I}^{t_F} \left[\frac{\partial L}{\partial q} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \right] \delta q(t)\, dt + \left. \frac{\partial L}{\partial \dot{q}} \delta q(t) \right|_{t_I}^{t_F} + \ldots
\end{aligned}$$

Since $\delta q(t_I) = \delta q(t_F) = 0$, the integrated term vanishes and we have, to first order in δq,

$$\delta S = \int_{t_I}^{t_F} \left[\frac{\partial L}{\partial q} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \right] \delta q(t)\, dt = \int_{t_I}^{t_F} E(q, \dot{q}, \ddot{q})\, \delta q(t)\, dt$$

where

$$\begin{aligned}
E(q, \dot{q}, \ddot{q}) &= \frac{\partial L}{\partial q} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \\
&= \frac{\partial L}{\partial q} - \frac{\partial^2 L}{\partial \dot{q} \partial q} \dot{q} - \frac{\partial^2 L}{\partial \dot{q}^2} \ddot{q}
\end{aligned}$$

The actual path

According to the principle of least action, the actual path followed by the system is given by

$$\delta S = 0$$

and thus

$$\int_{t_I}^{t_F} E\left(q,\dot{q},\ddot{q}\right)\delta q\left(t\right)dt=0$$

Why should δS vanish (to the first order in δq) for S to be minimum?

Note that if a path $q\left(t\right)$ yields a minimum for S, any small deviation should make S increase. So, shouldn't δS be positive? No! Note that $\delta S=\int_{t_I}^{t_F} E\left(q,\dot{q},\ddot{q}\right)\delta q\left(t\right)dt+\ldots$ is linear in δq. This means that even though the path $q+\delta q$ increases S $\left(\delta S>0\right)$, another one $q-\delta q$ decreases S $\left(\delta S<0\right)$. Thus a path cannot be one of minimum action if $\delta S>0$ to first order in δq. It is easy to see that $\delta S<0$ would not work either. Thus, for a minimum, we must have $\delta S=0$ (to first order in δq).

$\delta S=0$ does not have to mean that S is a minimum, however. It can also be a maximum - or just flex at the actual path. It is the higher order terms in δS that decide whether the actual path is a minimum or a maximum. Thus, the principle of least action should be better named the principle of extremal action.

We claim that this implies that

$$E=0$$

Proof: We prove this by contradiction. Assume that $E\neq 0$ for some $t\in\left(t_I,t_F\right)$. Then it is either +ve or -ve at t. Let's assume that it is +ve. Then, since L is a smooth function, so is E . Then there must be an interval $\left(a,b\right)\subsetneq\left(t_I,t_F\right)$ containing t where $E>0$. Now, since $\delta q\left(t\right)$ is arbitrary other than the restriction that it vanishes at t_I and t_F , we can choose

$$\delta q\left(t\right)\begin{cases}>0 & a<t<b\\ =0 & \text{otherwise}\end{cases}$$

For this δq we will have

$$\delta S=\int_{t_I}^{t_F} E\left(q,\dot{q},\ddot{q}\right)\delta q\left(t\right)dt=\int_a^b \underbrace{E\left(q,\dot{q},\ddot{q}\right)\delta q\left(t\right)}_{>0}dt>0$$

Which is a contradiction. Similarly $E<0$ will also lead to a contradiction.

Thus the actual path obeys

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{q}}\right)-\frac{\partial L}{\partial q}=0}$$

Systems with multiple degrees of freedom

In a system with n degrees of freedom, we can often[3] use n independent generalized coordinates $q_1, \ldots, q_n$. We will continue to write $L = L(q, \dot{q}, t)$ but now this is shorthand for $L(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t)$. Then a similar calculation will lead to

$$\delta S = \sum_{i=1}^{n} \int_{t_I}^{t_F} E_i(q, \dot{q}, \ddot{q})\, \delta q_i(t)\, dt$$

where

$$E_i(q, \dot{q}, \ddot{q}) = \frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)$$

Now, since the q_is are independent, we can vary any one, say q_1, without varying the rest[4]. Then $\delta S = \int_{t_I}^{t_F} E_1(q, \dot{q}, \ddot{q})\, \delta q_1(t)\, dt$ and the argument that we had used earlier shows that $E_1 = 0$. Similarly we can show that $E_2, \ldots, E_n$ all vanish. So, we get n Euler-Lagrange equations of motion:

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0}$$

These are equivalent to

$$\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}\ddot{q}_j = \frac{\partial L}{\partial q_i} - \frac{\partial^2 L}{\partial \dot{q}_i \partial q_j}\dot{q}_j$$

These are n 2nd order differential equations - one each for each degree of freedom. Note that these can be solved for the n accelerations (and thus for the motion) if (and only if) the matrix M defined by $M_{ij} = \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}$ is invertible (what happens if this were not is a fascinating, and important, story - but that is beyond the scope of these lectures).

In summary, given the Lagrangian, we can (usually) find n 2nd order equations of motion that describe the motion of the system.

What is the Lagrangian?

The Lagrangian describes the system. While there can be a wide variety of forms that the Lagrangian can take, the most common form that is useful particle mechanics with conservative forces is

$$L = T - V$$

where T and $V$are the kinetic and potential energies of the system, respectively. With this form, the Euler-Lagrange EoMs reproduce the equations given by Newton's laws.

Within this common scenario, we most often find that

[3] This may not always be possible - and then the analysis becomes more complicated

[4] This is exactly where the argument will go wrong if the coordinates were *not* independent!

- The potential energy depends on the generalized coordinates alone (independent of the generalized velocities)
- The kinetic energy depends homogeneously quadratically on the generalized velocities

Examples:

Particle moving in a straight line

Lagrangian

$$L = \frac{1}{2}m\dot{x}^2 - V(x)$$

Here $\frac{\partial L}{\partial \dot{x}} = m\dot{x}$ and $\frac{\partial L}{\partial x} = -\frac{dV}{dx}$, so that the EoM is

$$\frac{d}{dt}(m\dot{x}) = -\frac{dV}{dx} \quad \Longrightarrow \quad m\ddot{x} = -\frac{dV}{dx}$$

These is, of course, the standard equation for 1D motion.

Particle moving in 3D

$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - V(x, y, z)$$

- $\frac{\partial L}{\partial \dot{x}} = m\dot{x}$, $\frac{\partial L}{\partial x} = -\frac{\partial V}{\partial x}$ so that $m\ddot{x} = -\frac{\partial V}{\partial x}$
- $\frac{\partial L}{\partial \dot{y}} = m\dot{y}$, $\frac{\partial L}{\partial y} = -\frac{\partial V}{\partial y}$ so that $m\ddot{y} = -\frac{\partial V}{\partial y}$
- $\frac{\partial L}{\partial \dot{z}} = m\dot{z}$, $\frac{\partial L}{\partial z} = -\frac{\partial V}{\partial z}$ so that $m\ddot{z} = -\frac{\partial V}{\partial z}$

These are obviously the same as

$$m\frac{d^2\vec{r}}{dt^2} = -\nabla V$$

Note : the three derivatives of L with respect to the velocities are the corresponding components of momentum. By extension, the derivative $\frac{\partial L}{\partial \dot{q}_i}$ is called the i-th **generalized momentum** p_i. So the Euler-Lagrange EoMs can also be written as

$$\frac{dp_i}{dt} = \frac{\partial L}{\partial q_i}$$

Particle moving in 2D (polar coordinates)

Given that $x = r\cos\theta$ and $y = r\sin\theta$, we have

$$\dot{x} = \dot{r}\cos\theta - r\sin\theta\dot{\theta}$$
$$\dot{y} = \dot{r}\sin\theta + r\cos\theta\dot{\theta}$$

so that

$$v^2 = \dot{x}^2 + \dot{y}^2 = \left(\dot{r}\cos\theta - r\sin\theta\dot{\theta}\right)^2 + \left(\dot{r}\sin\theta + r\cos\theta\dot{\theta}\right)^2 = \dot{r}^2 + r^2\dot{\theta}^2$$

This can also be derived using geometry (or, equivalently, the fact that the radial and cross-radial components of the velocity are $\dot{r}$ and $r\dot{\theta}$, respectively).

So the Lagrangian is given by

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - V(r,\theta)$$

Then

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}$$
$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

and the EoMs are

$$\frac{dp_r}{dt} = m\ddot{r} = mr\dot{\theta}^2 - \frac{\partial V}{\partial r}$$

and

$$\frac{dp_\theta}{dt} = mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} = -\frac{\partial V}{\partial \theta}$$

Note : the acceleration components in polar coordinates are $a_r = \ddot{r} - r\dot{\theta}^2$ and $a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$ - and thus the above equations are equivalent to

$$\vec{F} = m\vec{a}$$

since in polar coordinates $\nabla = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta}$.

The simple pendulum

The pendulum can be described by two coordinates (x, y) - but these are not independent. There is a constraint : $x^2 + y^2 = l^2$. Thus it is a system with a single degree of freedom. Fortunately, it is very easy to find a single generalized coordinate that describes its configuration - the angle θ.

- The kinetic energy is $T = \frac{1}{2}ml^2\dot{\theta}^2$.
- The potential energy is $V = -mgl\cos\theta$

Thus the Lagrangian is

$$L = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta$$

Thus the generalized momentum is $p_\theta = ml^2\dot{\theta}$. The EoM is

$$ml^2\ddot{\theta} = -mgl\sin\theta \quad \implies \quad \ddot{\theta} = -\frac{g}{l}\sin\theta$$

This is the familiar equation for the simple pendulum.

Note that for small deviations from equilibrium (small oscillations) this equation reduces to the harmonic equation

$$\ddot{\theta} = -\frac{g}{l}\theta$$

This shows that for small oscillations, the time period of the pendulum is $T = 2\pi\sqrt{\frac{l}{g}}$

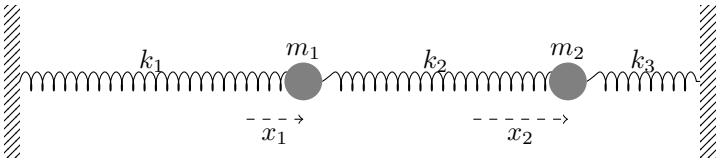
Also note that we could arrive at the equation for small oscillations directly by starting out with a Lagrangian that is correct to quadratic order in coordinates and velocities and writing down the EoMs from there (since we want the EoMs, which involve derivatives with respect to q and $\dot{q}$ to be correct to the first order, we need to keep terms up to the second order in the Lagrangian).

In this case, the Lagrangian correct to the second order is

$$L = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\left(1 - \frac{\theta^2}{2} + \ldots\right) \approx \frac{1}{2}ml^2\dot{\theta}^2 - \frac{1}{2}mgl\theta^2$$

In the last step above, we have dropped terms of higher order from L and have also omitted a constant - which has no bearing on the EoM. It is easy to see that the final form will directly lead to the linearized EoM.

Coupled oscillators



The kinetic energy of the system shown is obviously

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2$$

while the potential energy is

$$V = \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2x_2^2 + \frac{1}{2}k_3\left(x_2 - x_1\right)^2$$

and hence

$$L = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 - \frac{1}{2}k_1x_1^2 - \frac{1}{2}k_2x_2^2 - \frac{1}{2}k_3\left(x_2 - x_1\right)^2$$

Now,

$$p_1 = m_1\dot{x}_1, \qquad p_2 = m_2\dot{x}_2$$

and

$$\frac{\partial L}{\partial x_1} = -k_1x_1 + k_3\left(x_2 - x_1\right)$$
$$\frac{\partial L}{\partial x_2} = -k_2x_2 - k_3\left(x_2 - x_1\right)$$

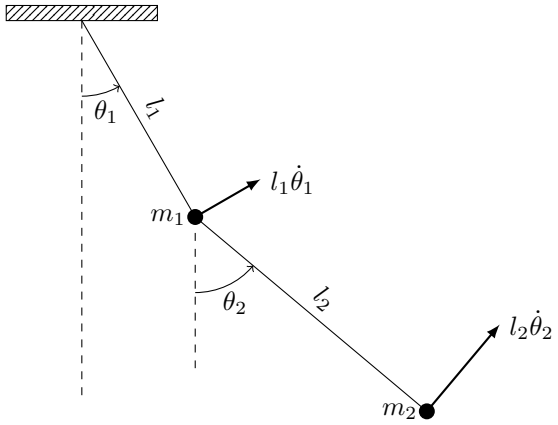
so that

$$m_1\ddot{x}_1 = -\left(k_1 + k_3\right)x_1 + k_3x_2$$
$$m_2\ddot{x}_2 = +k_3x_1 - \left(k_2 + k_3\right)x_2$$

These are the same equations that you can figure out using Newton's laws - but here the derivation is simplified considerably because you begin with scalars - thus not having to worry about the direction of the forces.

Planar double pendulum

This is a system with two degrees of freedom, and the two angles θ_1 and θ_2 indicated can be used as generalized coordinates. We can figure out the kinetic energy by resorting to Cartesian coordinates



$$x_1 = l_1\sin\theta_1$$
$$y_1 = l_1\cos\theta_1$$
$$x_2 = l_1\sin\theta_1 + l_2\sin\theta_2$$
$$y_2 = l_1\cos\theta_1 + l_2\cos\theta_2$$

This immediately shows that

$$v_1^2 = \dot{x}_1^2 + \dot{y}_1^2 = l_1^2\dot{\theta}_1^2$$
$$v_2^2 = \dot{x}_2^2 + \dot{y}_1^2 = l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1l_2\dot{\theta}_1\dot{\theta}_2\cos\left(\theta_2 - \theta_1\right)$$

The first of these is simply the relationship between angular and linear speeds for circular motion. The second follows from the fact that the speed of the second bob with respect to the first is $l_2\dot{\theta}_2$, so that its velocity vector with respect to the inertial observer fixed to the support is the sum of this velocity with that of the first bob (the angle between these two vectors is $\theta_2 - \theta_1$). In this case,

$$T = \frac{1}{2}m_1l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2\left(l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1l_2\dot{\theta}_1\dot{\theta}_2\cos\left(\theta_2 - \theta_1\right)\right)$$
$$V = -m_1gl_1\cos\theta_1 - m_2g\left(l_1\cos\theta_1 + l_2\cos\theta_2\right)$$

We can write down the Lagrangian and determine the two Euler-Lagrange EoMs. **Challenge :** derive the same using Newton's laws only!

Small oscillations in the double pendulum

If we are only interested in small oscillations where θ_1 and θ_2 are small, we can write down the quadratic order Lagrangian

$$L = \frac{1}{2}m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2}m_2 \left(l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \right) + \frac{1}{2} m_1 g l_1 \theta_1^2 + \frac{1}{2} m_2 g \left(l_1 \theta_1^2 + l_2 \theta_2^2 \right)$$

Then

$$\begin{aligned} p_1 &= \left(m_1 + m_2 \right) l_1^2 \dot{\theta}_1 + m_2 l_1 l_2 \dot{\theta}_2 \\ p_2 &= m_2 l_1 l_2 \dot{\theta}_1 + m_2 l_2^2 \dot{\theta}_2 \end{aligned}$$

So that the EoMs for small oscillations are

$$\begin{aligned} \left(m_1 + m_2 \right) l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2 &= \left(m_1 + m_2 \right) g l_1 \theta_1 \\ m_2 l_1 l_2 \ddot{\theta}_1 + m_2 l_2^2 \ddot{\theta}_2 &= m_2 g l_2 \theta_2 \end{aligned}$$

Cyclic coordinates and conserved momenta

If a particular generalized coordinate q_i is missing from the Lagrangian, then $\frac{\partial L}{\partial q_i} = 0$. In this case, changing $q_i(t)$ without changing any of the other coordinates leaves the action unchanged. This is a sign of a symmetry - a transformation that you can carry out on a system which does not change it. Now, for such a coordinate, the Euler Lagrange EoM implies that $\dot{p}_i = 0$. Thus p_i is conserved by the dynamics - it does not change in value as the system evolves in time.

For example, for a particle in a central potential the Lagrangian, using polar coordinates in 2D, is

$$L = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 \right) - V(r)$$

Here $r$is present in the Lagrangian, but θ is not. As a result $p_\theta = m r^2 \dot{\theta}$ is conserved.

In this case, the generalized coordinate that is not present happens to be an angle - something that keeps on returning to the same value periodically. This is why such coordinates are called **cyclic coordinates**. The result that we have stressed in this section is that the generalized momentum corresponding to a cyclic coordinate is conserved.

Time translation invariance - conservation

Consider the quantity called the **Jacobi function**, defined by

$$h\left(q,\dot{q},t\right)\equiv\sum_{i=1}^{n}p_i\dot{q}_i-L$$

The rate of change of this function with time is given by

$$\frac{dh}{dt}=\sum_{i=1}^{n}\left(p_i\ddot{q}_i+\dot{p}_i\dot{q}_i\right)-\left(\sum_{i=1}^{n}\left(\frac{\partial L}{\partial q_i}\dot{q}_i+\frac{\partial L}{\partial\dot{q}_i}\ddot{q}_i\right)+\frac{\partial L}{\partial t}\right)$$

Now $p_i=\frac{\partial L}{\partial\dot{q}_i}$ (by definition) and $\dot{p}_i=\frac{\partial L}{\partial q_i}$(by the Euler-Lagrange EoM) - and so almost all the terms on the right hand side cancel. This leads to

$$\frac{dh}{dt}=-\frac{\partial L}{\partial t}$$

This implies that if a Lagrangian is explicitly independent of time (*i.e.* depends on time only through the time dependence of q and $\dot{q}$), then the Jacobi function is conserved.

Homogeneous functions and Euler's formula

A function $f$of multiple variables is called **homogeneous of degree** n if it obeys

$$f\left(\lambda x_1,\ldots,\lambda x_2\right)=\lambda^n f\left(x_1,\ldots,x_n\right)$$

where λ is any parameter. For example

- $h_0\left(x,y\right)=a\left(\frac{x}{y}\right)^3+b\left(\frac{x}{y}\right)^2+c\left(\frac{x}{y}\right)+d$, where a,b,c,d are constants is homogeneous of degree 0
- $h_1\left(x,y,z\right)=\frac{x^2y+xyz+xz^2+z^3}{xy-yz+zx}$ is homogeneous of degree 1
- $h_2\left(x,y\right)=ax^2+by^2+cxy$ is homogeneous of degree 2

etc.
Differentiating both sides of the defining equation for homogeneous functions with repect to the parameter λ gives us

$$x_1\frac{\partial f}{\partial\left(\lambda x_1\right)}+\ldots+x_n\frac{\partial f}{\partial\left(\lambda x_n\right)}\bigg|_{\left(\lambda x_1,\ldots,\lambda x_n\right)}=n\lambda^{n-1}f\left(x_1,\ldots,x_n\right)$$

Finally, putting $\lambda=1$ above leads to Euler's formula for homogeneous functions:

$$x_1\frac{\partial f}{\partial x_1}+\ldots+x_n\frac{\partial f}{\partial x_n}=nf$$

Interpreting the Jacobi function

As we said earlier, the most common situation in mechanics is that $L = T - V$ where V is independent of the $\dot{q}s$ and where T is a purely homogeneous function of degree 2 in the $\dot{q}$s. This means that

$$p_1\dot{q}_1 + \ldots + p_n\dot{q}_n = \frac{\partial L}{\partial \dot{q}_1}\dot{q}_1 + \ldots + \frac{\partial L}{\partial \dot{q}_n}\dot{q}_n = \frac{\partial T}{\partial \dot{q}_1}\dot{q}_1 + \ldots + \frac{\partial T}{\partial \dot{q}_n}\dot{q}_n = 2T$$

In this scenario

$$h = \sum_{i=1} p_i\dot{q}_i - L = 2T - (T - V) = T + V$$

So that in this case, the Jacobi function is equal to the total energy.

Thus, the fact that a Lagrangian explicitly independent of t leads to a conserved h can be interpreted as *"time translation invariance implies conservation of energy"!*