PH2102 Midsem Solution

Soln 1)a) The length of the compartment according to Alice $= 16 \text{ m} \times \sqrt{1-0.6^2} = 16 \times 0.8$ m.

The velocity of the ball with respect to Alice $= \frac{0.8+0.6}{1+0.8\times 0.6}c = \frac{1.4}{1.48}c = \frac{35}{37}c$.

The rate of separation between the ball and the front wall of the compartment $= \frac{35}{37}c - \frac{3}{5}c = \frac{35\times 5 - 3\times 37}{37\times 5}c = \frac{64}{37\times 5}c$.

The time interval between $\mathcal{E}_2$ and $\mathcal{E}_1$ according to Alice is $\frac{16\times 0.8 \text{ m}}{\frac{64}{37\times 5}c} = 37\frac{\text{m}}{c}$

(*This is the kind of cancellation that you can get only if you use fractions!*)

b) After the collision, the velocity of the ball with respect to Bob is $-0.8\,c$.

The velocity of the ball with respect to Alice $= \frac{-0.8+0.6}{1-0.8\times 0.6}c = -\frac{0.2}{0.52}c = -\frac{5}{13}c$.

The rate of separation between the ball and Bob$= \frac{5}{13}c + \frac{3}{5}c = \frac{5\times 5+3\times 13}{13\times 5}c = \frac{64}{13\times 5}c$.

The time interval between $\mathcal{E}_3$ and $\mathcal{E}_2$ according to Alice is $\frac{16\times 0.8 \text{ m}}{\frac{64}{13\times 5}c} = 13\frac{\text{m}}{c}$

c) Bob is present at both $\mathcal{E}_3$ and $\mathcal{E}_1$. So he measures the proper time interval between these two events.

d) The time interval between $\mathcal{E}_3$ and $\mathcal{E}_1$:

- according to Alice $= 37\frac{\text{m}}{c} + 13\frac{\text{m}}{c} = 50\frac{\text{m}}{c}$
- according to Bob $= 2 \times \frac{16 \text{ m}}{0.8c} = 40\frac{\text{m}}{c}$

The ratio is 1.25 - exactly the value of γ for $\beta = 0.6$

Q 2) a) *Standard derivation.*

b) Given $U^\mu = \gamma_u\,(c, \vec{u})$. Then$A^\mu = \frac{dU^\mu}{d\tau} = \frac{dt}{d\tau}\frac{dU^\mu}{dt} = \gamma_u\left[\gamma_u\,(0,\vec{a}) + \frac{d\gamma_u}{dt}\,(c,\vec{u})\right]$.

Now $\gamma_u^{-2} = 1 - \frac{u^2}{c^2} \implies -2\gamma_u^{-3}\frac{d\gamma_u}{dt} = -\frac{2}{c^2}\vec{u}\cdot\vec{a} \implies \frac{d\gamma_u}{dt} = \gamma_u^3\frac{\vec{u}\cdot\vec{a}}{c^2}$. Thus

$$A^\mu = \gamma_u\left[\gamma_u\,(0,\vec{a}) + \gamma_u^3\frac{\vec{u}\cdot\vec{a}}{c^2}\,(c,\vec{u})\right] = \gamma_u^2\left(\gamma_u^2\frac{\vec{u}\cdot\vec{a}}{c}, \vec{a} + \gamma_u^2\frac{\vec{u}\cdot\vec{a}}{c^2}\vec{u}\right)$$

To decide whether this four vector is time-like, light-like or space-like, we need to evaluate its invariant 4-norm. A direct calculation of this norm (actually norm-squared, but we usually call it just the norm) would be

$$\begin{aligned}
\left(A^0\right)^2 - \vec{A}^2 &= \gamma_u^4 \left[\left(\gamma_u^2 \frac{\vec{u}\cdot\vec{a}}{c} \right)^2 - \left(\vec{a} + \gamma_u^2 \frac{\vec{u}\cdot\vec{a}}{c^2} \vec{u} \right)^2 \right] \\
&= \gamma_u^4 \left[\left(\gamma_u^2 \frac{\vec{u}\cdot\vec{a}}{c} \right)^2 - a^2 - 2\gamma_u^2 \frac{\vec{u}\cdot\vec{a}}{c^2} \vec{u}\cdot\vec{a} - \left(\gamma_u^2 \frac{\vec{u}\cdot\vec{a}}{c^2} \vec{u} \right)^2 \right] \\
&= -\gamma_u^4 \left[a^2 - \left(\gamma_u^2 \frac{\vec{u}\cdot\vec{a}}{c} \right)^2 \left(1 - \frac{u^2}{c^2} \right) + 2\gamma_u^2 \frac{\vec{u}\cdot\vec{a}}{c^2} \vec{u}\cdot\vec{a} \right] \\
&= -\gamma_u^4 \left[a^2 - \gamma_u^2 \left(\frac{\vec{u}\cdot\vec{a}}{c} \right)^2 + 2\gamma_u^2 \frac{\vec{u}\cdot\vec{a}}{c^2} \vec{u}\cdot\vec{a} \right] \\
&= -\gamma_u^4 \left[a^2 + \gamma_u^2 \left(\frac{\vec{u}\cdot\vec{a}}{c} \right)^2 \right] < 0
\end{aligned}$$

So A^μ is space-like.

This result can be obtained much more simply by looking at A^μ from a frame in which the particle is, instantaneously, at rest. In this frame, we have $\vec{u} = 0$ and thus $\gamma_u = 1$. If the acceleration of the particle with respect to this frame is $\vec{a}_0$ - then in this frame A^μ is given by $(0, \vec{a}_0)$. This is obviously space-like!

Q 3) a) The invariance of $\left(x^0\right)^2 - \vec{x}^2 = \mathbf{x}^T \eta \mathbf{x}$ under $\mathbf{x} \mapsto \mathbf{x}' = L\mathbf{x}$ leads to the result that

$$\forall \mathbf{x}, \qquad \mathbf{x}^T L^T \eta L \mathbf{x} = \mathbf{x}^T \eta \mathbf{x} \qquad \Longrightarrow \qquad \mathbf{x}^T \left(L^T \eta L - \eta \right) \mathbf{x} = 0$$

Now if $\mathbf{x}^T A \mathbf{x} = 0$ for all $\mathbf{x}$, the square matrix A must be completely antisymmetric. Since $L^T \eta L - \eta$ is symmetric, it must be 0.

b) In terms of components, the equation $L^T \eta L = \eta$ becomes

$$\eta_{\mu\nu} = \sum_{\rho,\sigma} \left(L^T\right)_{\mu\rho} \eta_{\rho\sigma} \left(L\right)_{\sigma\nu} = \sum_{\rho,\sigma} \left(L\right)_{\rho\mu} \eta_{\rho\sigma} \left(L\right)_{\sigma\nu} = \eta_{\rho\sigma} \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu$$

c) Taking the determinant of both sides of $L^T \eta L = \eta$ we get

$$\det L^T \det \eta \det L = \det \eta$$

Using $\det L^T = \det L$ and $\det \eta \neq 0$ we get

$$\left(\det L\right)^2 = 1 \qquad \Longrightarrow \qquad \det L = \pm 1$$

d) Taking the 00-th element of $\eta_{\mu\nu} = \eta_{\rho\sigma} \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu$ we get and

$$1 = \eta_{00} = \eta_{\rho\sigma} \Lambda^\rho{}_0 \Lambda^\sigma{}_0 = \left(\Lambda^0{}_0\right)^2 - \left(\Lambda^1{}_0\right)^2 - \left(\Lambda^2{}_0\right)^2 - \left(\Lambda^3{}_0\right)^2$$

and thus

$$\left(\Lambda^0{}_0\right)^2 = 1 + \left(\Lambda^1{}_0\right)^2 + \left(\Lambda^2{}_0\right)^2 + \left(\Lambda^3{}_0\right)^2 \geq 1$$

e) $L^T\eta L = \eta$ can be rewritten in the form $L^{-1} = \eta^{-1}L^T\eta$. For the given L

$$
\begin{aligned}
L^{-1} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}^{-1} \frac{1}{64}\begin{pmatrix} 125 & -48 & -60 & 75 \\ -75 & 80 & 36 & -45 \\ -60 & 0 & 80 & -36 \\ 48 & 0 & 0 & 80 \end{pmatrix}^T \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \\
&= \frac{1}{64}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 125 & -75 & -60 & 48 \\ -48 & 80 & 0 & 0 \\ -60 & 36 & 80 & 0 \\ 75 & -45 & -36 & 80 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \\
&= \frac{1}{64}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 125 & 75 & 60 & -48 \\ -48 & -80 & 0 & 0 \\ -60 & -36 & -80 & 0 \\ 75 & 45 & 36 & 80 \end{pmatrix} \\
&= \frac{1}{64}\begin{pmatrix} 125 & 75 & 60 & -48 \\ 48 & 80 & 0 & 0 \\ 60 & 36 & 80 & 0 \\ -75 & -45 & -36 & 80 \end{pmatrix}
\end{aligned}
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