Decay

A particle of rest mass M decays into two identical particles of rest mass m each.

- Firstly - decay is possible only if $M > 2m$ - so let's assume that this holds.
- Energy conservation gives us (in the rest frame of the mother particle)

  $$Mc^2 = E_1 + E_2$$

  and momentum conservation gives

  $$0 = \vec{p}_1 + \vec{p}_2$$

  Since $\vec{p}_1 = -\vec{p}_2$ and the two decay particles have identical mass m, we have

  $$E_1 = E_2 = \frac{1}{2}Mc^2$$

  So, the speed of either daughter particle is given by

  $$\frac{1}{2}Mc^2 = mc^2\gamma(u) \implies 1 - \frac{u^2}{c^2} = \gamma(u)^{-2} = \frac{4m^2}{M^2} \implies u = c\sqrt{1 - \frac{4m^2}{M^2}}$$

- In the rest frame of the mother particle, the two daughter particles have velocities $\pm c\sqrt{1 - \frac{4m^2}{M^2}}$. The speed of one with respect to the other is

  $$u_{\text{rel}} = \frac{u - (-u)}{1 - \frac{u(-u)}{c^2}} = \frac{2u}{1 + \frac{u^2}{c^2}} = c\frac{\sqrt{1 - \frac{4m^2}{M^2}}}{1 - \frac{2m^2}{M^2}}$$

- You can calculate the total energy, kinetic energy and magnitude of the momentum of one particle in the rest frame of the other directly using this speed. Alternatively - using the conservation of the 4-momentum, we get

  $$P_M = P_{1m} + P_{2m} \implies P_M^2 = P_{1m}^2 + P_{2m}^2 + 2P_{1m} \cdot P_{2m}$$

  Since in the rest frame of one decay particle, the invariant scalar product $P_{1m} \cdot P_{2m} = mE$ where E is the energy of the other particle we get

  $$M^2c^2 = 2mc^2 + 2mE$$

  and we can directly calculate the rest.

Force

- The force and acceleration are parallel under two situations - either when the velocity is parallel to the acceleration, or when it is perpendicular to it.

- When a force F acts through a distance d, we have $Fd = mc^2 (\gamma(u) - 1)$ (the kinetic energy)
- Either calculate $\vec{F} \cdot \vec{u}$ or $\frac{d}{dt}\left(mc^2 (\gamma(u) - 1)\right) = mc^2 \frac{d\gamma(u)}{dt}$ to get the rate of increase in kinetic energy.

Lagrangian

#1

$$L = \frac{1}{2} m\dot{x}^2 + 2Ax^3 t + 3Ax^2 \dot{x} t^2$$

This give $p = \frac{\partial L}{\partial \dot{x}} = m\dot{x} + 3Ax^2 t^2$, and $\frac{\partial L}{\partial x} = 6Ax^2 t + 6Ax\dot{x}t^2$. Thus, the EoM is

$$\frac{d}{dt}\left(m\dot{x} + 3Ax^2 t^2\right) = 6Ax^2 t + 6Ax\dot{x}t^2$$

$\implies$

$$m\ddot{x} + 6Ax^2 t + 6Ax\dot{x}t^2 = 6Ax^2 t + 6Ax\dot{x}t^2$$

$\implies$

$$m\ddot{x} = 0$$

#2

$$L = \frac{1}{2} m\dot{x}^2 + A\omega x^2 \cos(\omega t) + 2Ax\dot{x}\sin(\omega t) - \frac{1}{2} m\omega^2 x^2$$

$\frac{\partial L}{\partial \dot{x}} = m\dot{x} + 2Ax\sin(\omega t)$, $\frac{\partial L}{\partial x} = 2A\omega x\cos(\omega t) + 2A\dot{x}\sin(\omega t) - m\omega^2 x$, so that the EoM is

$$\frac{d}{dt}\left(m\dot{x} + 2Ax\sin(\omega t)\right) = 2A\omega x\cos(\omega t) + 2A\dot{x}\sin(\omega t) - m\omega^2 x$$

$\implies$

$$m\ddot{x} + 2A\omega x\cos(\omega t) + 2A\dot{x}\sin(\omega t) = 2A\omega x\cos(\omega t) + 2A\dot{x}\sin(\omega t) - m\omega^2 x$$

$\implies$

$$m\ddot{x} = -m\omega^2 x$$

#3

$$L = \frac{1}{2} m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + Ar^2\cos(\theta)\,\dot{\theta} + 2Ar\sin(\theta)\,\dot{r}$$

$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} + 2Ar\sin(\theta)$, $\frac{\partial L}{\partial r} = mr\dot{\theta}^2 + 2Ar\cos(\theta)\,\dot{\theta} + 2A\sin(\theta)\,\dot{r}$. Thus

$$\frac{d}{dt}\left(m\dot{r} + 2Ar\sin(\theta)\right) = mr\dot{\theta}^2 + 2Ar\cos(\theta)\,\dot{\theta} + 2A\sin(\theta)\,\dot{r}$$

$$m\ddot{r} + 2Ar\cos(\theta)\,\dot{\theta} + 2A\sin(\theta)\,\dot{r} = mr\dot{\theta}^2 + 2Ar\cos(\theta)\,\dot{\theta} + 2A\sin(\theta)\,\dot{r}$$

$\Longrightarrow$

$$\ddot{r} - r\dot{\theta}^2 = 0$$

Similarly - the θ equation gives

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0$$

All of the three Lagrangians above lead to simple equations of motion - despite having a complicated appearance. This is because $L' = L + \frac{dF}{dt}$ gives the same EoMs as L, if F is an arbitrary function of the qs and t.
The action calculated from L' differs from that from L by $F(q_f, t_f) - F(q_i, t_i)$ - which is independent of the path - so the same path extermizes the action for both of them.
It is also straightforward to show directly that the Euler-Lagrange EoMs for the two are the same by using the properties of derivatives (Do it!).

#4

$$L = \frac{1}{2}m\dot{x}^2 + A\omega x^2\cos(\omega t) + Ax\dot{x}\sin(\omega t) - \frac{1}{2}m\omega^2 x^2$$

This one is different, since $A\omega x^2\cos(\omega t) + Ax\dot{x}\sin(\omega t)$ is not $\frac{dF}{dt}$ for any $F(x,t)$.
$\frac{\partial L}{\partial \dot{x}} = m\dot{x} + Ax\sin(\omega t)$, $\frac{\partial L}{\partial x} = 2A\omega x\cos(\omega t) + A\dot{x}\sin(\omega t) - m\omega^2 x$, so that the EoM is

$$\frac{d}{dt}\left(m\dot{x} + Ax\sin(\omega t)\right) = 2A\omega x\cos(\omega t) + A\dot{x}\sin(\omega t) - m\omega^2 x$$

$\Longrightarrow$

$$m\ddot{x} + A\omega x\cos(\omega t) + A\dot{x}\sin(\omega t) = 2A\omega x\cos(\omega t) + A\dot{x}\sin(\omega t) - m\omega^2 x$$

$\Longrightarrow$

$$m\ddot{x} = A\omega x\cos(\omega t) - m\omega^2 x$$