

Department of Mathematics
School of Advanced Sciences
MAT 1011 – Calculus for Engineers (MATLAB)
Experiment 3–B
Maxima and Minima of functions of two variables

Aim:

To find Maximum and Minimum values (Extreme values) of a function $f(x, y)$ using MATLAB.

Mathematical form:

Let $z = f(x, y)$ be the given function. Critical points are points in the xy – plane where the tangent plane is horizontal. The tangent plane is horizontal, if its normal vector points in the z – direction. Hence, critical points are solutions of the equations: $f_x(x, y) = 0$ and $f_y(a, b) = 0$.

Procedure for finding the maximum or minimum values of $f(x, y)$:

- (1) For the given function $f(x, y)$ find $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ and equate it to zero and solve them to find the roots $(x_1, y_1), (x_2, y_2), \dots$. These points may be maximum or minimum points.
- (2) Find the values $r = \frac{\partial^2 f}{\partial x^2}, s = \frac{\partial^2 f}{\partial x \partial y}, t = \frac{\partial^2 f}{\partial y^2}$ at these points.
- (3) (a) If $rt - s^2 > 0$ and $r < 0$ at a certain point, then the function is maximum at that point.
 (b) If $rt - s^2 > 0$ and $r > 0$ at a certain point, then the function is minimum at that point.
 (c) If $rt - s^2 < 0$ for a certain point, then the function is neither maximum nor minimum at that point. This point is known as saddle point.
 (d) If $rt - s^2 = 0$ at a certain point, then nothing can be said whether the function is maximum or minimum at that point. In this case further investigation are required.

MATLAB Syntax used:

diff	diff(expr) differentiates a symbolic expression expr with respect to its free variable as determined by symvar.
solve	Symbolic solution of algebraic equations, The input to solve can be either symbolic expressions or strings
size	Dimensions of data and model objects and to access a specific size output.
imag	Imaginary part of complex number, Y=imag(Z) returns the imaginary part of the elements of array Z.
figure	Create figure graphics object, Figure objects are the individual windows on the screen in which the MATLAB software displays graphical output
double	Convert to double precision, double(x) returns the double-precision value for X. If X is already a double-precision array, double has no effect.
sprintf	Format data into string. It applies the <i>format</i> to all elements of array A and any additional array arguments in column order, and returns the results to string str.
solve	Symbolic solution of algebraic equations, The input to solve can be either symbolic expressions or strings.
ezsurf	Easy-to-use 3-D colored surface plotter, ezsurf(fun) creates a graph of fun(x, y) using the surf function. fun is plotted over the default domain: $-2\pi < x < 2\pi, -2\pi < y < 2\pi$.
plot3	The plot3 function displays a three-dimensional plot of a set of data points.

MATLAB code:

```
clc
clear all
syms x y
f= input('Enter the function f(x,y):');
p= diff(f,x); q=diff(f,y);
[ax,ay]=solve(p,q);
ax=double(ax);ay=double(ay);
r= diff(p,x); s=diff(p,y); t =diff(q,y);D=r*t-s^2;
figure
ezsurf(f);
legstr={'Function Plot'};% for Legend
for i=1:size(ax)
T1=subs(D,{x,y},{ax(i),ay(i)});
T2=subs(r,{x,y},{ax(i),ay(i)});
T3=subs(f,{x,y},{ax(i),ay(i)});
if (double(T1) == 0)
sprintf('At (%f,%f) further investigation is required', ax(i),ay(i))
legstr=[legstr,{'Case of Further investigation'}];
mkr='ko';
elseif (double(T1) < 0)
sprintf('The point (%f,%f) is a saddle point', ax(i),ay(i))
legstr=[legstr,{'Saddle Point'}]; % updating Legend
mkr='bv'; % marker
else
if (double(T2) < 0)
sprintf('The maximum value of the function is f(%f,%f)=%f',
ax(i),ay(i),double(T3))
legstr=[legstr,{'Maximum value of the function'}];% updating Legend
mkr='g+';% marker
else
sprintf('The minimum value of the function is f(%f,%f)=%f',
ax(i),ay(i),double(T3))
legstr=[legstr,{'Minimum value of the function'}];% updating Legend
mkr='r*'; % marker
end
end
hold on
plot3(ax(i),ay(i),T3,mkr,'Linewidth',4);
end
legend(legstr,'Location','Best');
```

Example 1. Obtain the maximum and minimum values of $f(x, y) = 2(x^2 - y^2) - x^4 + y^4$

Solution: $f(x, y) = 2(x^2 - y^2) - x^4 + y^4$, $\frac{\partial f}{\partial x} = 4x - 4x^3$, $\frac{\partial f}{\partial y} = 4y^3 - 4y$, $r = \frac{\partial^2 f}{\partial x^2} = 4 - 12x^2$,

$$s = \frac{\partial^2 f}{\partial x \partial y} = 0, \quad t = \frac{\partial^2 f}{\partial y^2} = 12y^2 - 4, \quad \frac{\partial f}{\partial x} = 0 \Rightarrow 4x - 4x^3 = 0, \quad \frac{\partial f}{\partial y} = 0 \Rightarrow 4y^3 - 4y = 0$$

S.No.	Critical Points	r	$D=rt-s^2$	Remarks
1	(0,0)	4	$-16 < 0$	Saddle Point
2	(0,1)	4	32	Minimum
3	(0, -1)	4	32	Minimum
4	(1,0)	-8	32	Maximum
5	(1,1)	-8	$-64 < 0$	Saddle Point
6	(1, -1)	-8	$-64 < 0$	Saddle Point
7	(-1,0)	-8	32	Maximum
8	(-1,1)	-8	$-64 < 0$	Saddle Point
9	(-1, -1)	-8	$-64 < 0$	Saddle Point

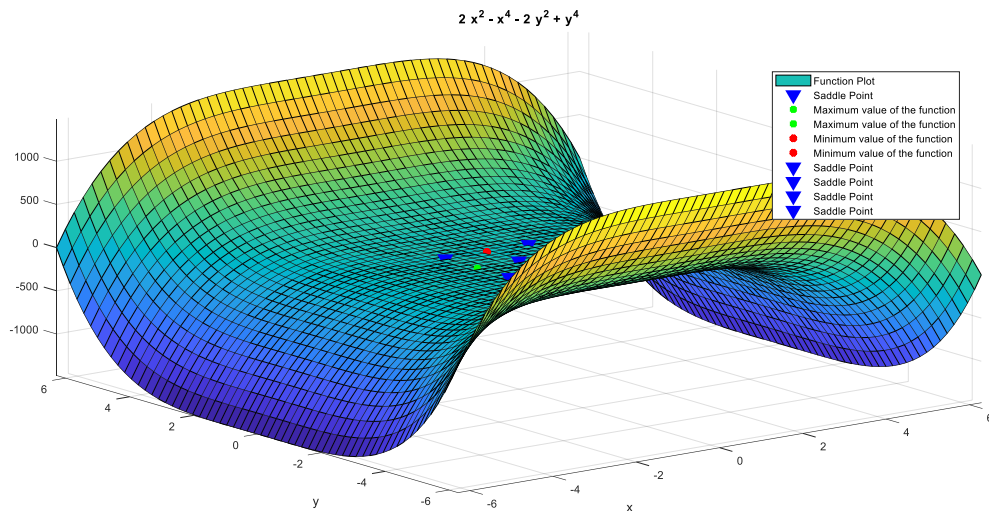
The Minimum value of $f(x,y)$ is -1 at $(0,1)$ & $(0, -1)$ and the Maximum value for $f(x,y)$ is $+1$ at $(1,0)$ & $(-1,0)$

Input:

Enter the function $f(x,y): 2*(x^2-y^2)-x^4+y^4$

Out Put:

```
ans =
'The point (0.000000,0.000000) is a saddle point'
ans =
'The maximum value of the function is f(-1.000000,0.000000)=1.000000'
ans =
'The maximum value of the function is f(1.000000,0.000000)=1.000000'
ans =
'The minimum value of the function is f(0.000000,-1.000000)=-1.000000'
ans =
'The minimum value of the function is f(0.000000,1.000000)=-1.000000'
ans =
'The point (-1.000000,-1.000000) is a saddle point'
ans =
'The point (1.000000,-1.000000) is a saddle point'
ans =
'The point (-1.000000,1.000000) is a saddle point'
ans =
'The point (1.000000,1.000000) is a saddle point'
```



Example. 2 Four small towns in a rural area wish to pool their resources to build a television station. If the towns are located at the points $(-5, 0)$, $(1, 7)$, $(9, 0)$ and $(0, -8)$ on a rectangular map grid, where units are in miles, at what point $S(x, y)$ should the station be located to minimize the sum of the distances from the towns?

Solution: Let $S(x, y)$ be the location where the television station is to be set up.

The location of the towns are $A(-5, 0)$, $B(1, 7)$, $C(9, 0)$ and $D(0, -8)$.

The point $S(x, y)$ where the sum of the distances from the above points is to be minimized is the same point that minimizes the sum of the squares of the distances; namely,

$$S(x, y) = [(x+5)^2 + y^2] + [(x-1)^2 + (y-7)^2] + [(x-9)^2 + y^2] + [x^2 + (y+8)^2]$$

$$S_x = 2(x+5) + 2(x-1) + 2(x-9) + 2x.$$

$$S_y = 2y + 2(y-7) + 2y + 2(y+8).$$

$$\text{Then } S_x = 0 \Rightarrow x = 5/4 \text{ and } S_y = 0 \Rightarrow y = -1/4.$$

$$r = S_{xx} = 8 > 0, \quad s = S_{xy} = 0 \text{ and } t = S_{yy} = 8 > 0.$$

$$\text{Hence } rt - s^2 > 0, \quad r > 0 \text{ at } (5/4, -1/4).$$

Therefore, the television station can be set up at the location $S(5/4, -1/4)$ on the rectangular map grid such that the distance from S to each of the towns is a minimum.

Input

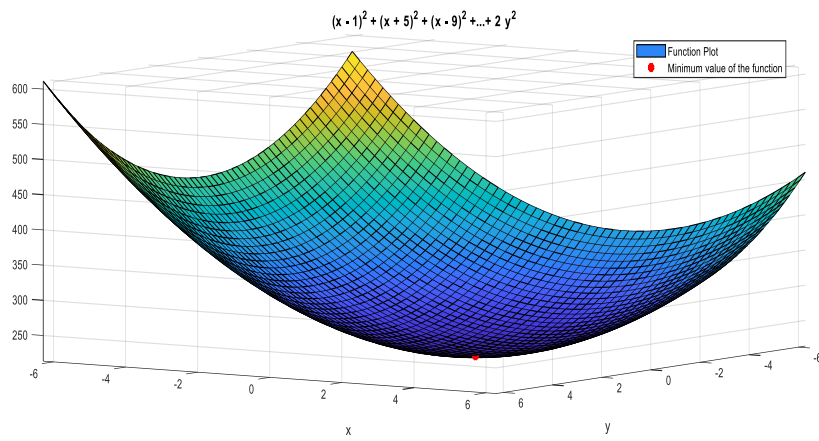
Enter the function $f(x, y)$:

```
(x+5)^2+y^2+(x-1)^2+(y-7)^2+(x-9)^2+y^2+x^2+(y+8)^2
```

Output

ans =

```
'The minimum value of the function is f(1.250000,-0.250000)=
213.500000'
```



Exercise

Find the maxima and minima for the following functions

1. $f(x, y) = x^4 + y^4 - x^2 - y^2 + 1$.
2. $f(x, y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$.