Department of Mathematics
School of Advanced Sciences
MAT 1011 – Calculus for Engineers (MATLAB)
Experiment 4–A
Double Integrals and change of order of integration

In this experiment, we consider a continuous function f such that $f(x,y) \geq 0$ for all (x,y) in a region R in the xy – plane, then the volume of the solid region that lies above R and below the graph of f is defined as the double integral $V = \iint_R f(x,y)dA$, where R is the region bounded by the curves $y = \phi_1(x)$ and $y = \phi_2(x)$ between $x = a$ and $x = b$.

In this case the inner integration is with respect to y and outer integration is with respect to x. Hence

$$V = \iint_R f(x,y)dA = \int_{x=a}^{b} [\int_{y=\phi_1(x)}^{\phi_2(x)} f(x,y)dy]dx$$

MATLAB Syntax

`int(int(f(x,y),y, phi1, phi2),x,a,b)` where y is the inner variable, x is the outer variable.

When R is a region bounded by the curves $x = \psi_1(y)$ and $x = \psi_2(y)$ between $y = c$ and $y = d$, i.e., the inner integration is with respect to x and outer integration is with respect to y. Then

$$V = \iint_R f(x,y)dA = \int_{y=c}^{d} [\int_{x=\psi_1(y)}^{\psi_2(y)} f(x,y)dx]dy$$

MATLAB Syntax

`int(int(f,x, psi1, psi2),y,c,d)` where x is the inner variable, y is the outer variable.

Supporting files required:

To visualize the surfaces two additional m-files viz., `viewSolid, viewSolidone` are required. These files are to be included in the current working directory before execution.

Syntax for double integral:

```
viewSolid(z,0+0*x+0*y,f,y,phi1,phi2,x,a,b)

viewSolidone (z,0+0*x+0*y,f,x,psi1,psi2,y,c,d)
```

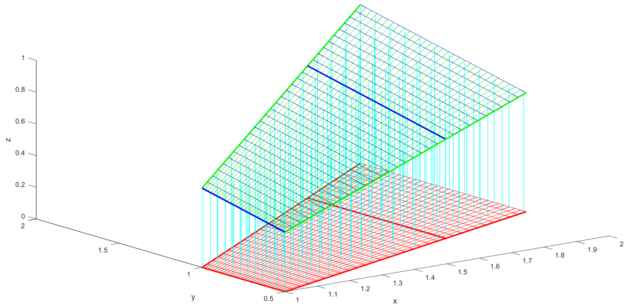
Example. 1

To find $\int_{1}^{2}\int_{x/2}^{x}\frac{x+y}{4}dydx$.

```
syms x y z
int(int((x+y)/4,y,x/2,x),x,1,2)
viewSolid(z,0+0*x+0*y,(x+y)/4,y,x/2,x,x,1,2)
```

Output

```
ans =
49/96
```



In this figure the required volume is above the plane z=0 (shown in red) and above the surface $z=\frac{x+y}{4}$ (shown in green) .

Example. 2

To find the volume of the prism whose base is the triangle in the $xy-$plane bounded by the $x-$axis and the lines $y=x$ and $x=1$ and whose top lies in the plane $z=f(x,y)=3-x-y$. The limits of integration here are $y=0$ to 1 while $x=y$ to 1. Hence

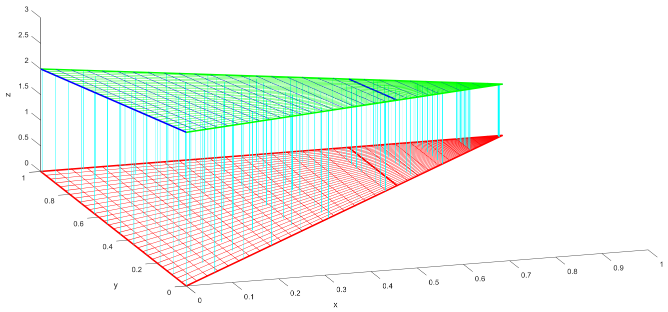
$$\iint_R (3-x-y)dA=\int_{0}^{1}\int_{y}^{1}(3-x-y)dxdy$$

MATLAB code:

```
syms x y z
int(int(3-x-y,x,y,1),y,0,1)
viewSolidone(z,0+0*x+0*y,3-x-y,x,y,1,y,0,1)
```

Output:

```
ans =
1
```



In this figure the triangular region on the *xy* plane is shown in red, while the plane surface *z*=3–*x*–*y* above the *xy* plane is shown in green .

Example 3

Evaluate the integral $\int_0^2\int_{x^2}^{2x}(4x+2)dydx$ by changing the order of integration.

As per the given limits of integration $x=0$ to 2 while $y=x^2$ to $2x$.

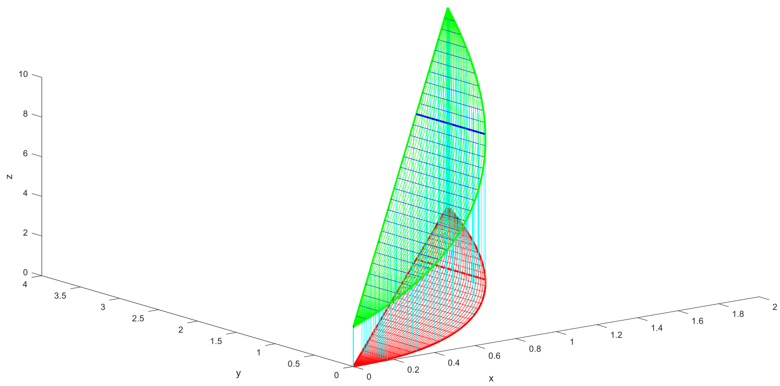
MATLAB Code:

```
syms x y z
int(int((4*x+2),y,x^2,2*x),x,0,2)
viewSolid(z,0+0*x+0*y, 4*x+2,y,x^2,2*x,x,0,2)
```

Output

```
ans =

8
```

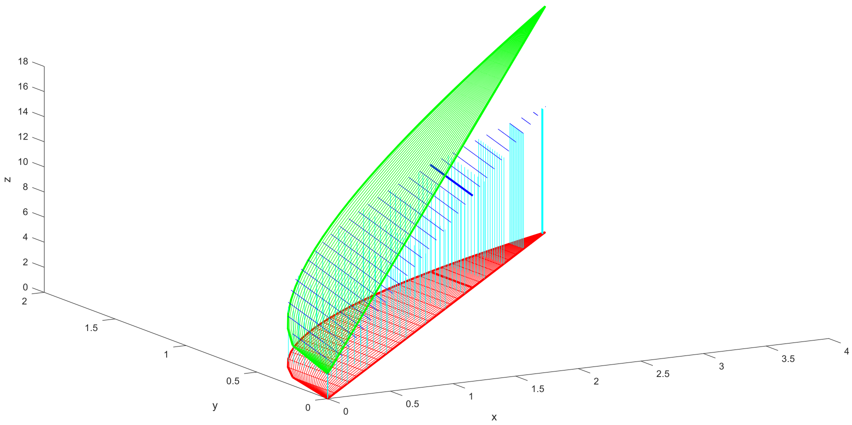


By changing the order of integration, the limits are $y=0$ to 4 while $x=\frac{y}{2}$ to $\sqrt{y}$.

```
int(int(4*x+2,x,y/2,sqrt(y)),y,0,4)

ans =

8
```



Example 4:

Consider the following mathematical problem

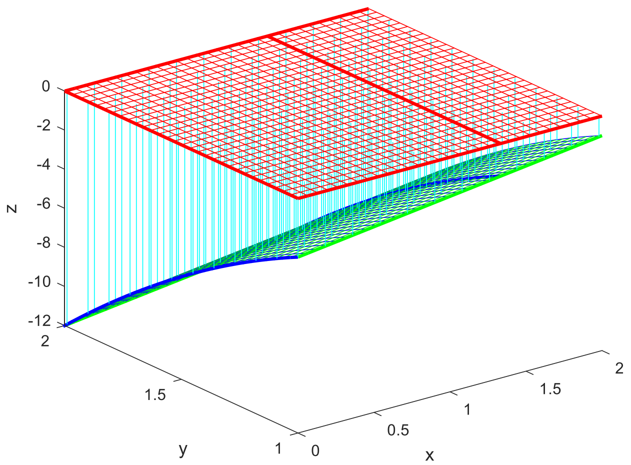
$$\text{Evaluate } \iint_R (x-3y^2)\,dA \text{ where } R=\{(x,y)\,|\,0\le x\le 2,\,1\le y\le 2\}$$

MATLAB Code:

```
clc
clear all
syms x y z
viewSolid(z, 0+0*x+0*y, x-3*y^2,y,1+0*x, 2+0*x,x,0,2)
int(int(x-3*y^2,y,1,2),x,0,2)
```

Output:

```
>> ans
     -12
```



In this figure the required volume is below the plane $z = 0$ (shown in red) and above the surface $z = x - 3y^2$ (shown in green). The reason why the answer is negative is that the surface $z = x - 3y^2$ is below $z = 0$ for the given domain of integration.

Example 5:

Consider the next mathematical problem

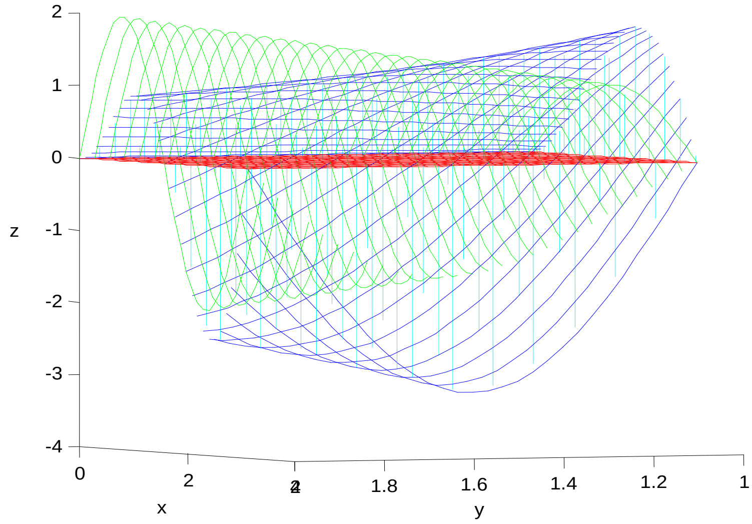
Evaluate $\iint_R y\sin(xy)\, dA$ where $R = [1,2]\times[0,\pi]$

MATLAB Code:

```
clc
clear all
syms x y z
viewSolidone(z, 0+0*x+0*y, y*sin(x*y),x,1+0*y, 2+0*y,y,0,pi)
int(int(y*sin(x*y),x,1,2),y,0,pi)
```

Output:

```
>>  ans
    0
```



For a function $f(x, y)$ that takes on both positive and negative values $\iint_R f(x, y)dA$ is a difference of volumes V_1-V_2, V_1 is the volume above R and below the graph of f and V_2 is the volume below R and above the graph. The integral in this example is 0 means V_1=V_2

Example 6:

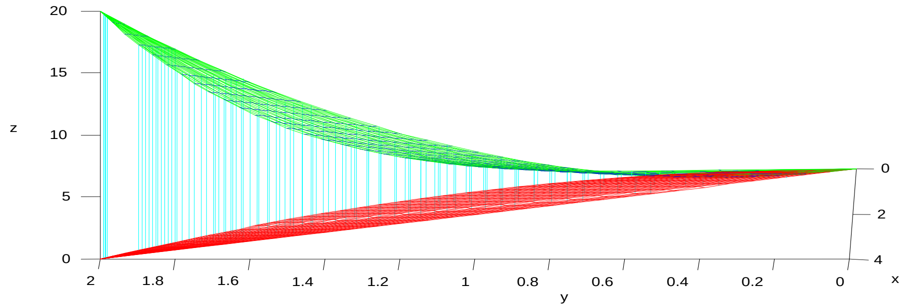
Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region D in the xy-plane bounded by the lines $y = 2x$ and the parabola $y = x^2$

MATLAB Code:

```
clc
clear all
syms x y z
viewSolidone(z, 0+0*x+0*y, x^2+y^2,x,y/2,sqrt(y),y,0,4)
int(int(x^2+y^2,x,y/2,sqrt(y)),y,0,4)
```

Output:

```
>>  ans
   216/35
```



Exercise:

1. Find the volume of the solid S that is bounded by the elliptic paraboloid $x^2 + 2y^2 + z = 16$, the planes $x = 2$ *and* $y = 2$, and the three coordinate planes.
2. Evaluate $\iint_R \sin x \cos y \, dA$ where $R = [0, \pi/2] \times [0, \pi/2]$

Triple Integrals

Triple integrals enable us to solve more general problems such as to calculate the volumes of three – dimensional shapes, the masses and moments of solids of varying density, and the average value of a function over a three –dimensional region.

The triple integral of $f(x,y,z)$ over the region D is given by

$$\iiint_D f(x,y,z)dV$$

where the region D is bounded by the surfaces $x=a$, $x=b$, $y=\psi_1(x)$ to $y=\psi_2(x)$, $z=\phi_1(x,y)$ to $z=\phi_2(x,y)$.

Hence

$$\iiint_D f(x,y,z)dV=\int_a^b\int_{\psi_1(x)}^{\psi_2(x)}\int_{\phi_1(x,y)}^{\phi_2(x,y)} f(x,y,z)dzdydx.$$

Similarly when the region D is bounded by the surfaces $y=c$, $y=d$, $x=\psi_1(y)$ to $x=\psi_2(y)$, $z=\phi_1(x,y)$ to $z=\phi_2(x,y)$.

Hence

$$\iiint_D f(x,y,z)dV=\int_a^b\int_{\psi_1(y)}^{\psi_2(y)}\int_{\phi_1(x,y)}^{\phi_2(x,y)} f(x,y,z)dzdxdy.$$

Volume using Triple Integral

The volume of a closed, bounded region D in space is given by

$$V=\iiint_D dV$$

Syntax for evaluation of triple integral:

```
int(int(int(f,z,za,zb),y,ya,yb),x,xa,xb)
```

or

```
I=int(int(int(f,z,za,zb),x,xa,xb),y,ya,yb)
```

Syntax for visualization of region bounded by the limits of integration:

```
viewSolid(z,za,zb,y,ya,yb,x,xa,xb)
```

```
viewSolidone(z,za,zb,xa,xb,y,ya,yb)
```

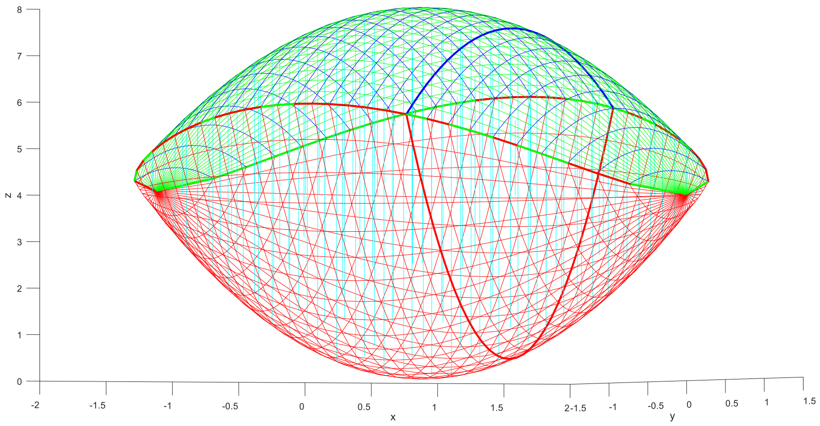
Example 1.

Find the volume of the region D enclosed by the surfaces $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$.

```
clear all
clc
syms x y z
xa=-2;
xb=2;
ya=-sqrt(2-x^2/2);
yb=sqrt(2-x^2/2);
za=x^2+3*y^2;
zb=8-x^2-y^2;
I=int(int(int(1+0*z,z,za,zb),y,ya,yb),x,xa,xb)
viewSolid(z,za,zb,y,ya,yb,x,xa,xb)
```

Output

```
I =
8*pi*2^(1/2)
```



Example 2.

Find the volume of the region cut from the cylinder $x^2 + y^2 = 4$ by the plane $z = 0$ and the plane $x + z = 3$.

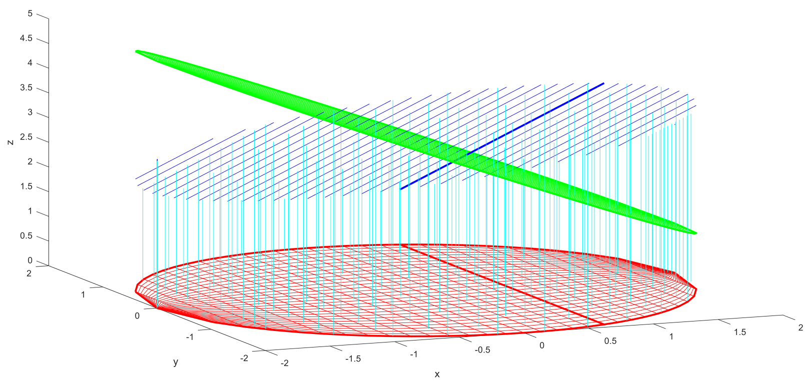
The limits of integration are $z = 0$ to $3 - x$, $x = -\sqrt{4-y}$ to $\sqrt{4-y}$, $y = -2$ to 2.

```
clear all
clc
syms x y z
ya=-2;
yb=2;
xa=-sqrt(4-y^2);
```

```
xb=sqrt(4-y^2);
za=0+0*x+0*y;
zb=3-x-0*y;
I=int(int(int(1+0*z,z,za,zb),x,xa,xb),y,ya,yb)
viewSolidone(z,za,zb,x,xa,xb,y,ya,yb)
```

Output

```
I =
12*pi
```



Example 3.

Find the volume of the region in the first octant bounded by the coordinate planes, the plane $y=1-x$, and the surface $z=\cos(\pi x/2)$, $0\le x\le 1$.

The limits of integration are $z=0$ to $\cos(\pi x/2)$, $y=0$ to $1-x$, $x=0$ to 1.

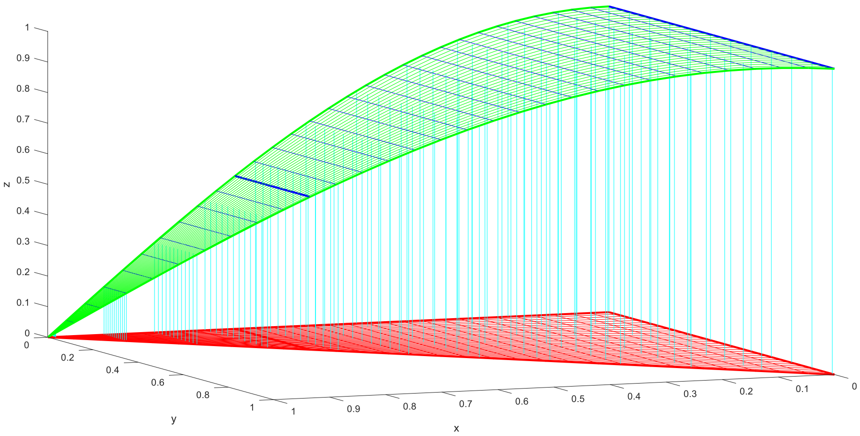
MATLAB code.

```
clear all
clc
syms x y z real
xa=0;
xb=1;
ya=0+0*x;
yb=1-x;
za=0*x+0*y;
zb=cos(pi*x/2)+0*y;
I=int(int(int(1+0*z,z,za,zb),y,ya,yb),x,xa,xb)
viewSolid(z,za,zb,y,ya,yb,x,xa,xb)
```

Output.

```
I =

4/pi^2
```



Exercise.

1. Find the volume of the region bounded between the planes $x+y+2z=2$ and $2x+2y+z=4$ in the first octant.
2. Find the volume of the region cut from the solid elliptical cylinder $x^2+4y^2 \le 4$ by the $xy-$ plane and the plane $z=x+2$.
3. The finite region bounded by the planes $z=x$, $x+z=8$, $z=y$, $y=8$ and $z=0$.