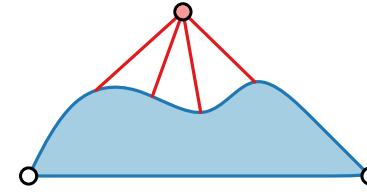
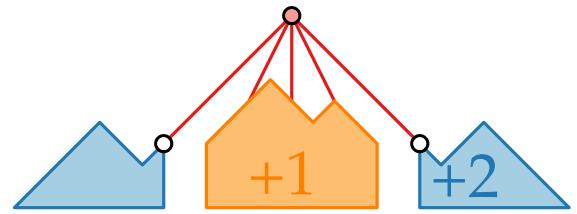


CS F402: Computational Geometry

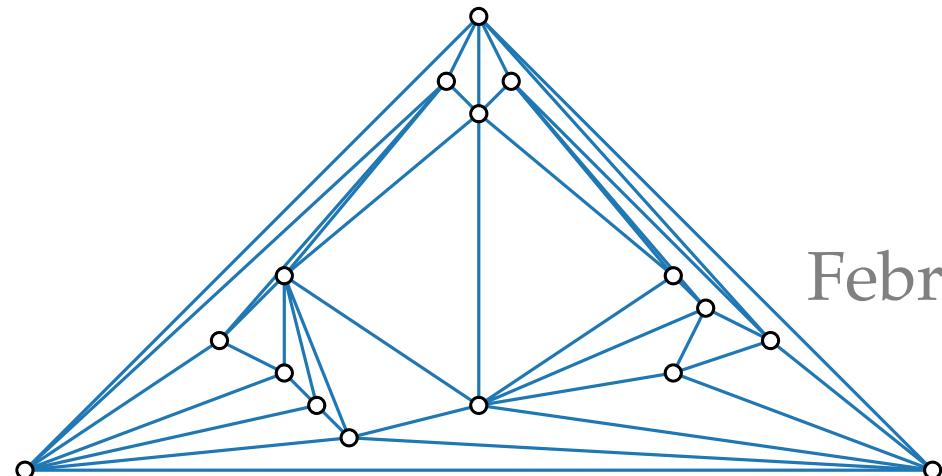


Lecture 11:

Straight-Line Drawings of Planar Graphs I: Canonical Ordering and Shift Method

Siddharth Gupta

February 21 + March 11+12+17, 2025



Motivation

Why planar and straight-line?

Motivation

Why planar and straight-line?

[Bennett, Ryall, Spaltzeholz and Gooch '07]

The Aesthetics of Graph Visualization

3.2. Edge Placement Heuristics

By far the most agreed-upon edge placement heuristic is to *minimize the number of edge crossings* in a graph [BMRW98, Har98, DH96, Pur02, TR05, TBB88]. The importance of avoiding edge crossings has also been extensively validated in terms of user preference and performance (see Section 4). Similarly, based on perceptual principles, it is beneficial to *minimize the number of edge bends* within a graph [Pur02, TR05, TBB88]. Edge bends make edges more difficult to follow because an edge with a sharp bend is more likely to be perceived as two separate objects. This leads to the heuristic of *keeping edge bends uniform* with respect to the bend's position on the edge and its angle [TR05]. If an edge must be bent to satisfy other aesthetic criteria, the angle of the bend should be as little as possible, and the bend placement should evenly divide the edge.

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Drawing conventions

- No crossings \Rightarrow planar
- No bends \Rightarrow straight-line

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Drawing aesthetics

- Area

Planar Graphs

Planar Graphs

Characterization

Planar Graphs

Characterization

Recognition

Planar Graphs

Characterization

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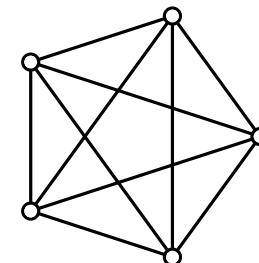
Drawing

Planar Graphs

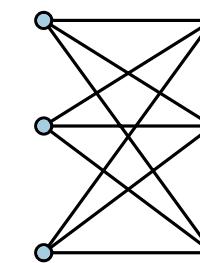
Theorem. [Kuratowski 1930]

G planar \Leftrightarrow
neither K_5 nor $K_{3,3}$ minor of G

K_5



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Characterization

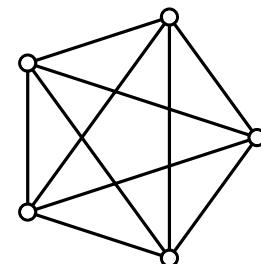
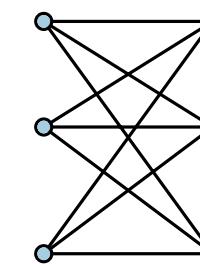
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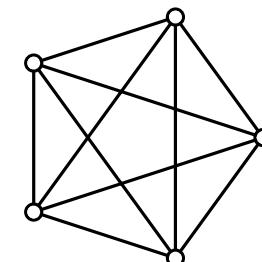
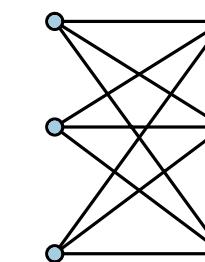
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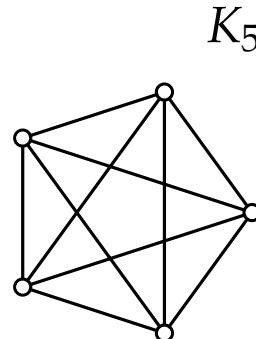
Also computes an embedding in $\mathcal{O}(n)$.

Drawing

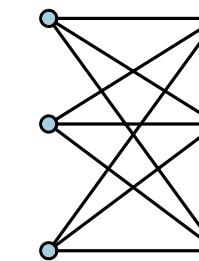
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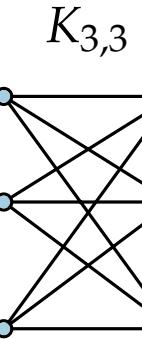
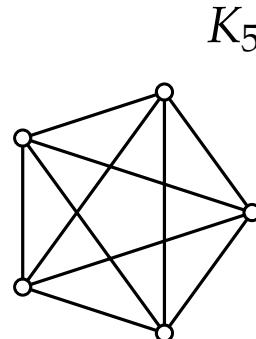
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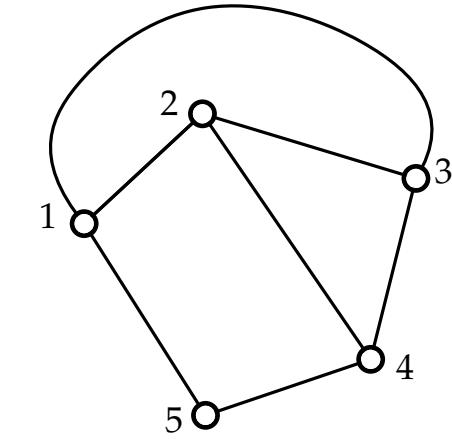
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Recognition

The algorithms implied by this theory produce drawings with area **not** bounded by any polynomial on n .

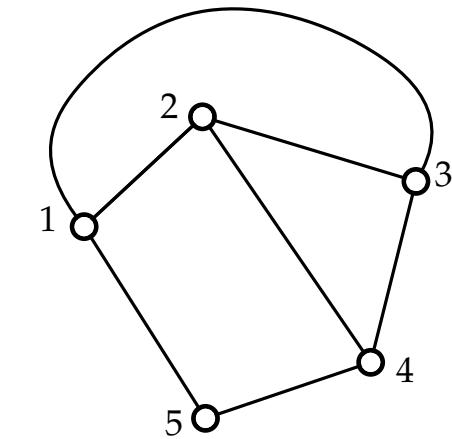
Drawing

Triangulations



Triangulations

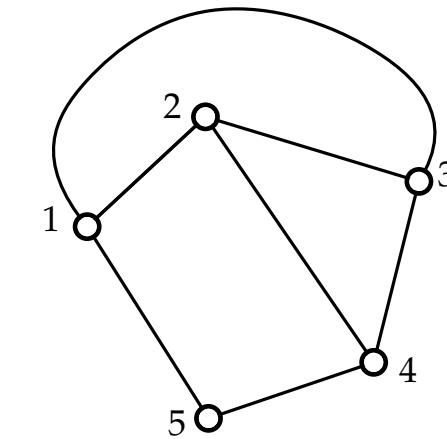
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Triangulations

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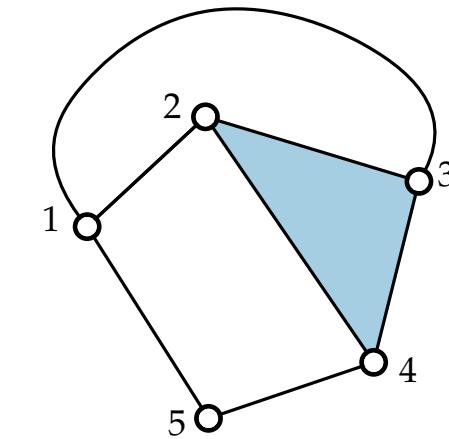
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Triangulations

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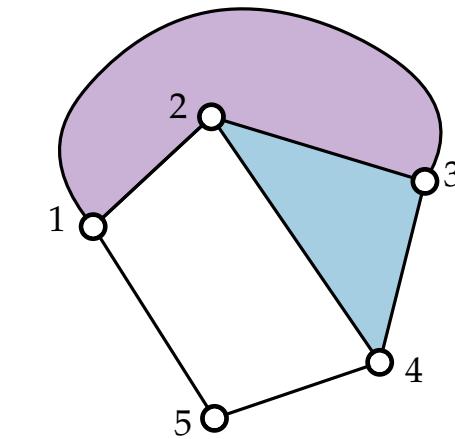
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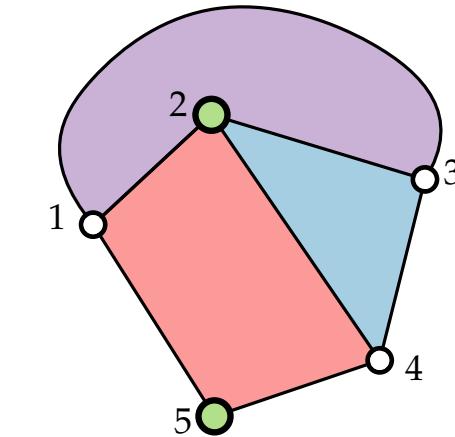
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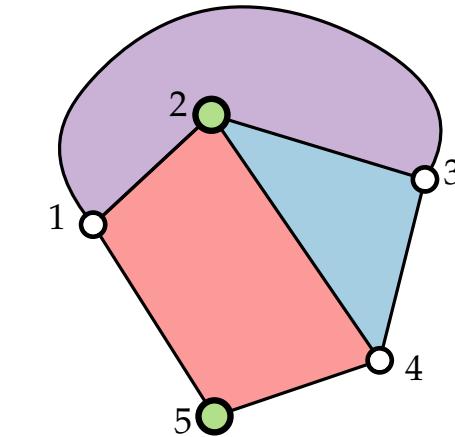
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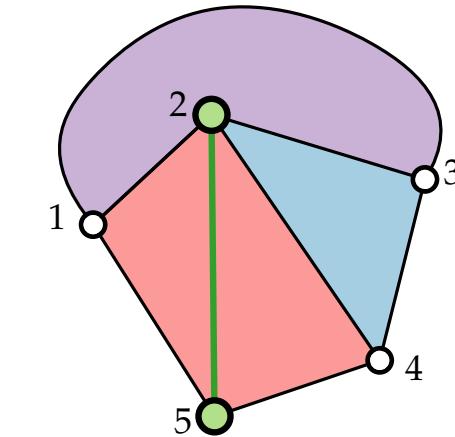
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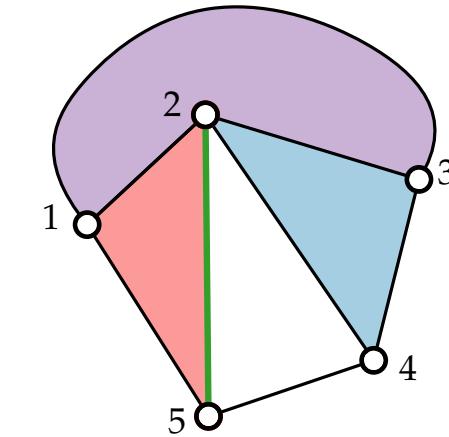
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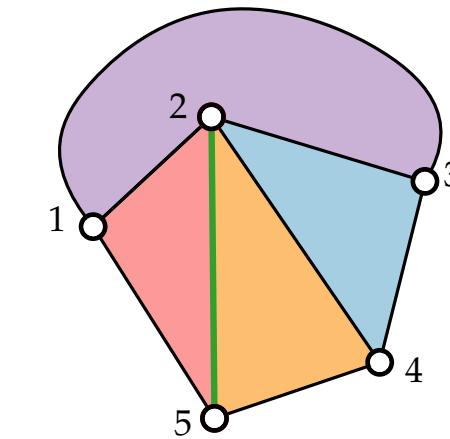
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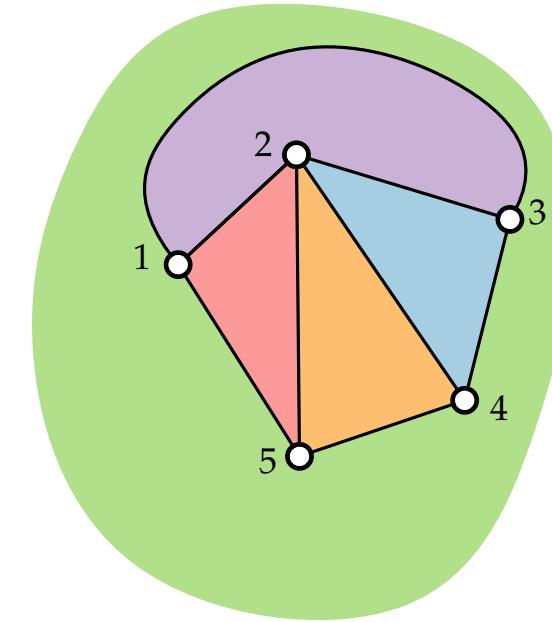
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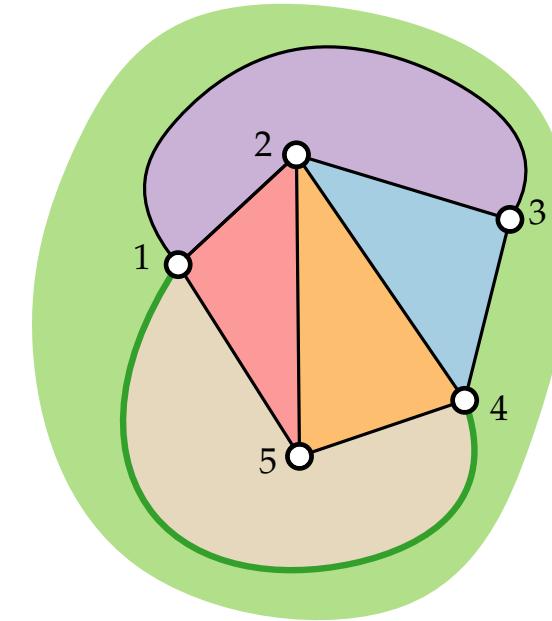
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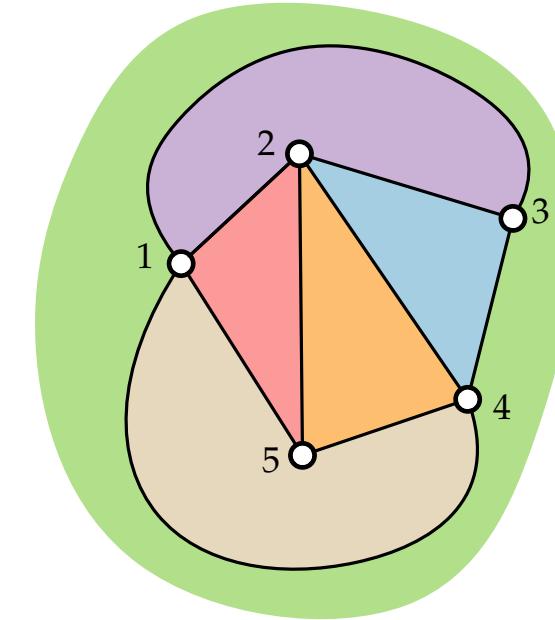
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Triangulations

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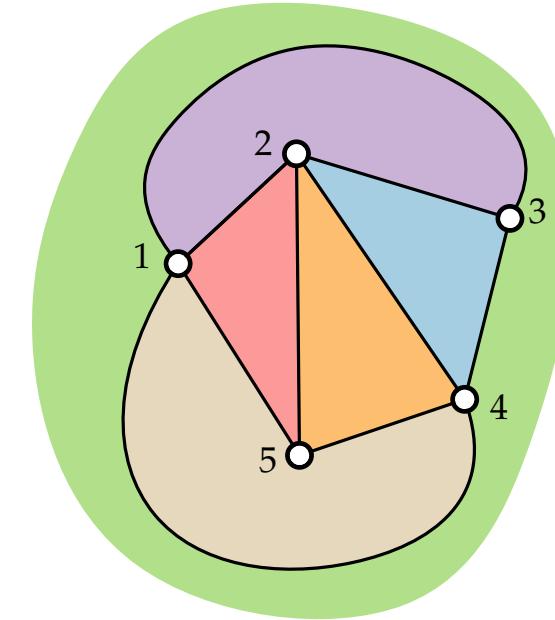


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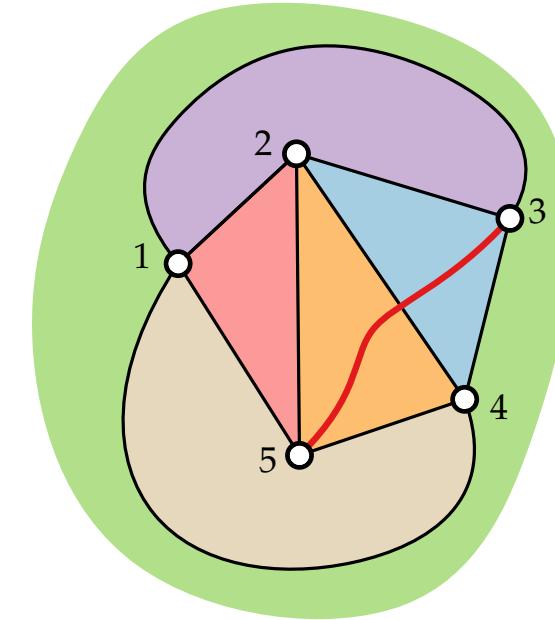


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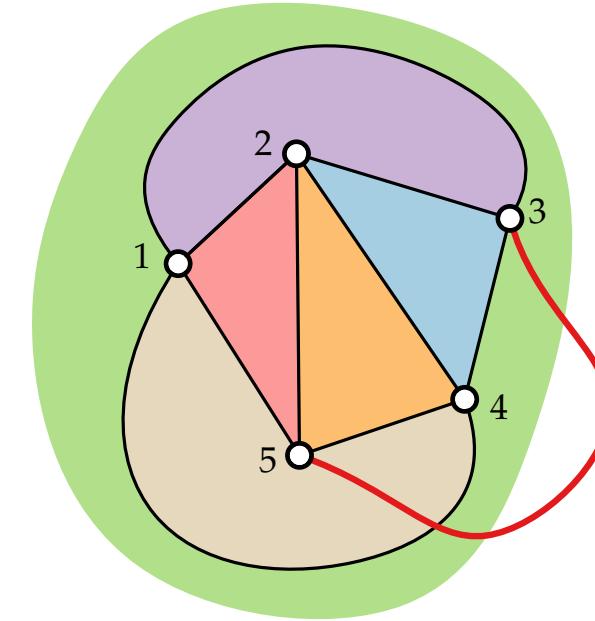


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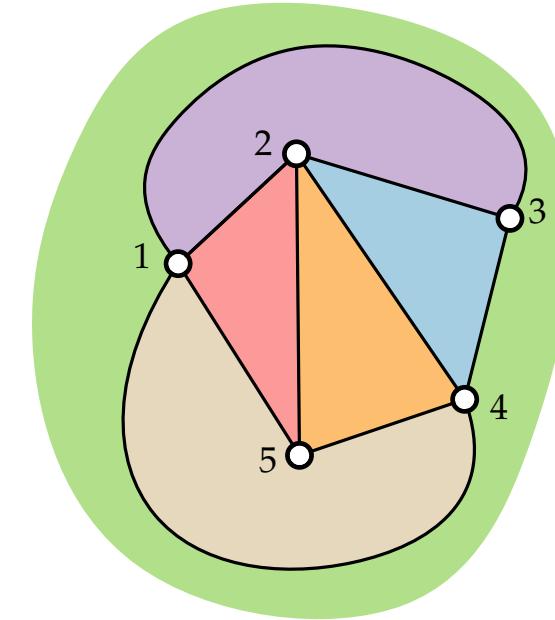


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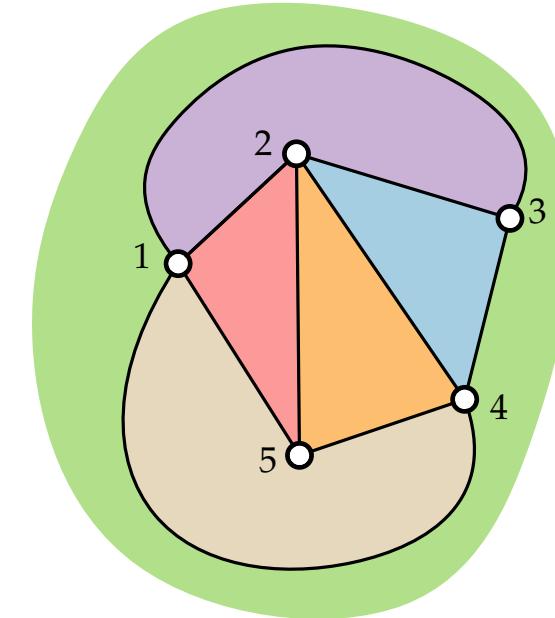
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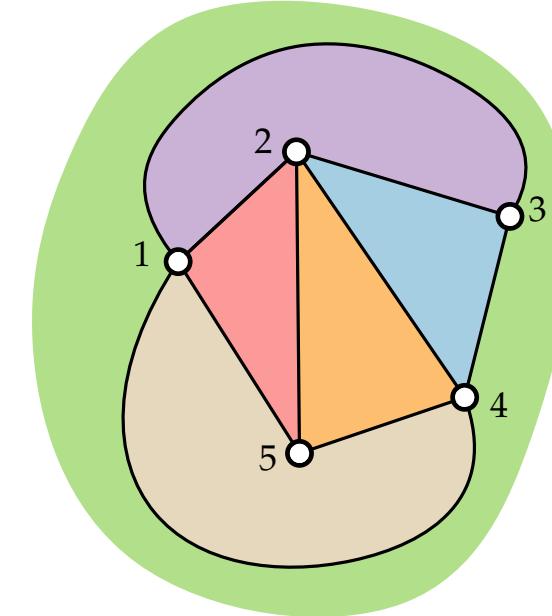
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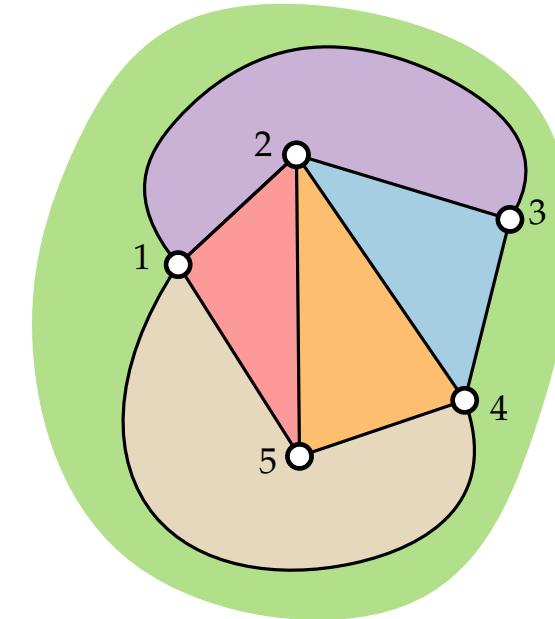
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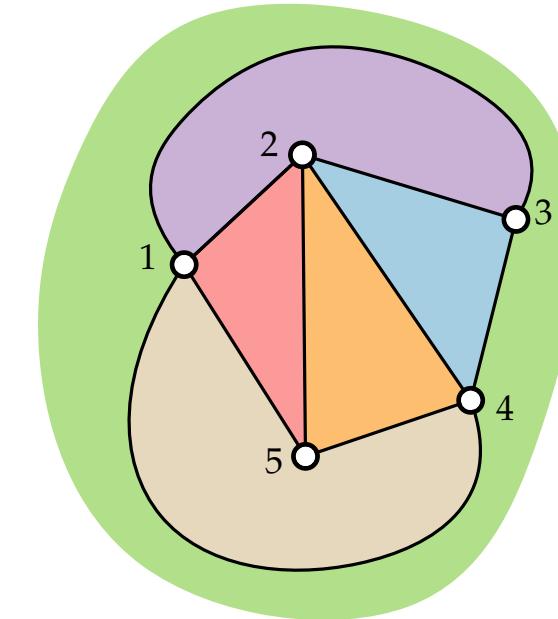
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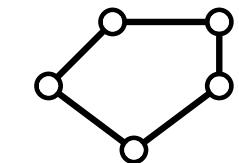
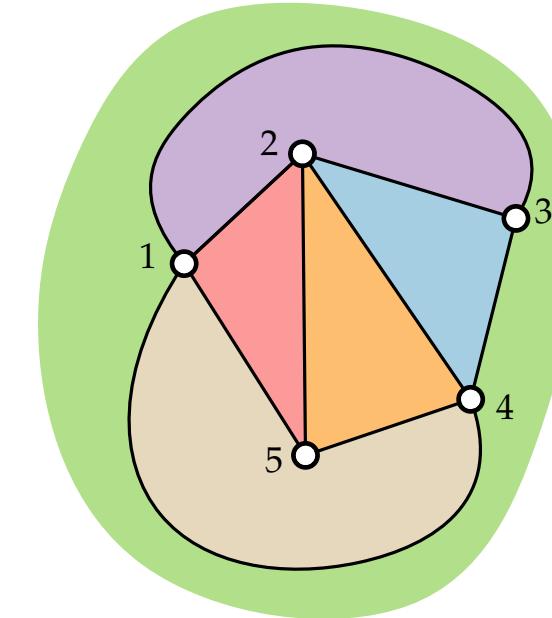
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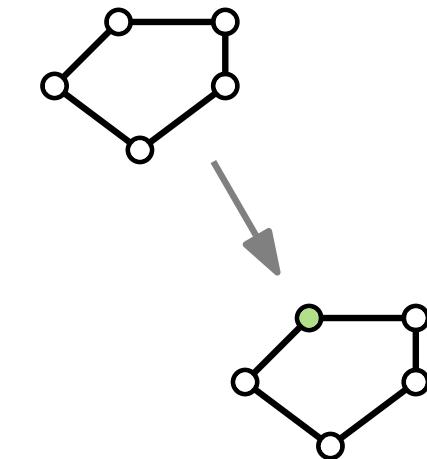
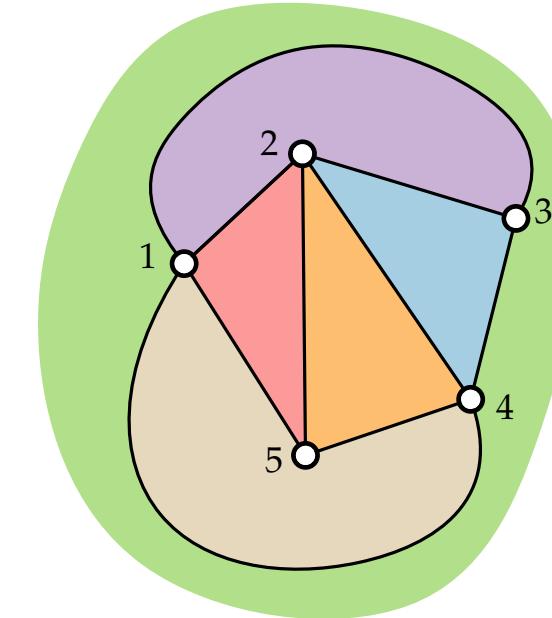
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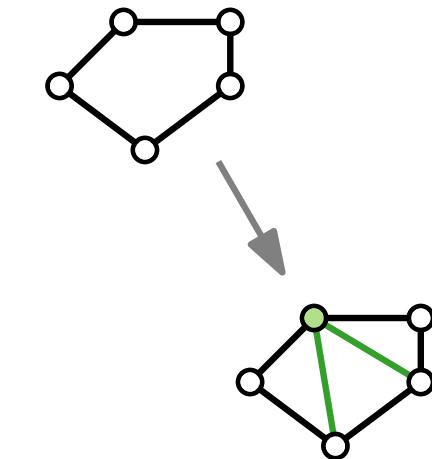
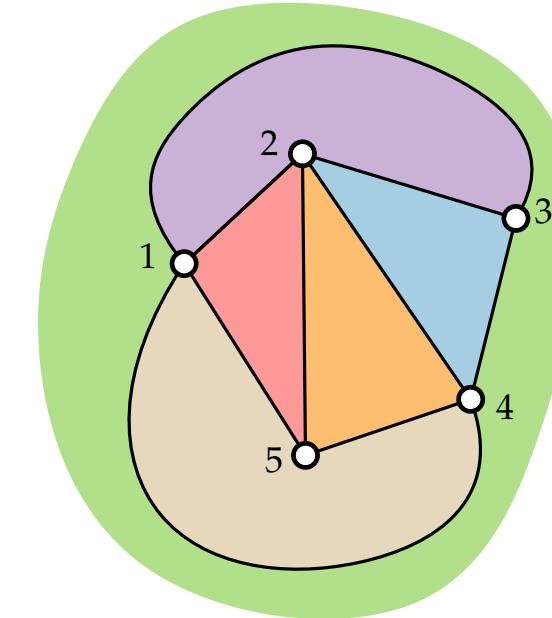
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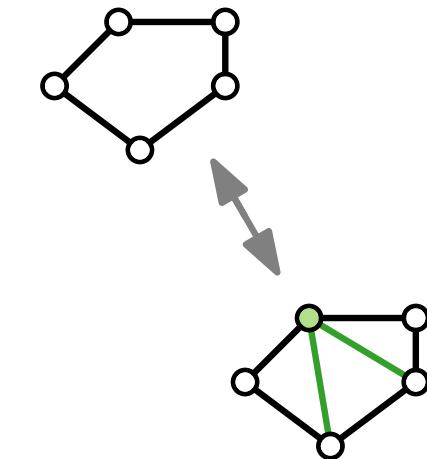
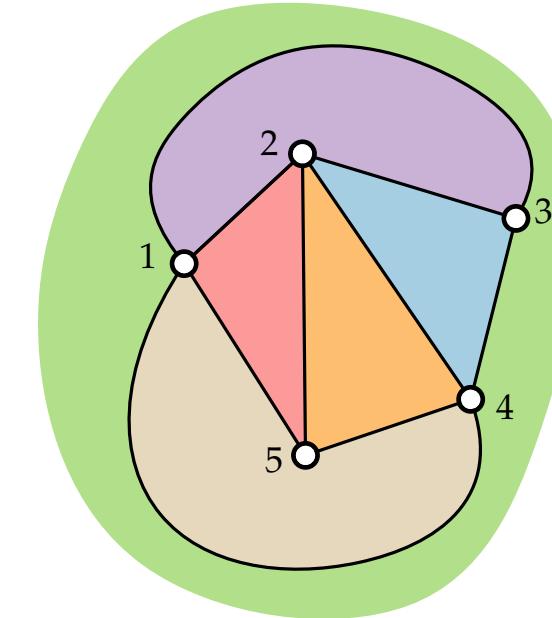
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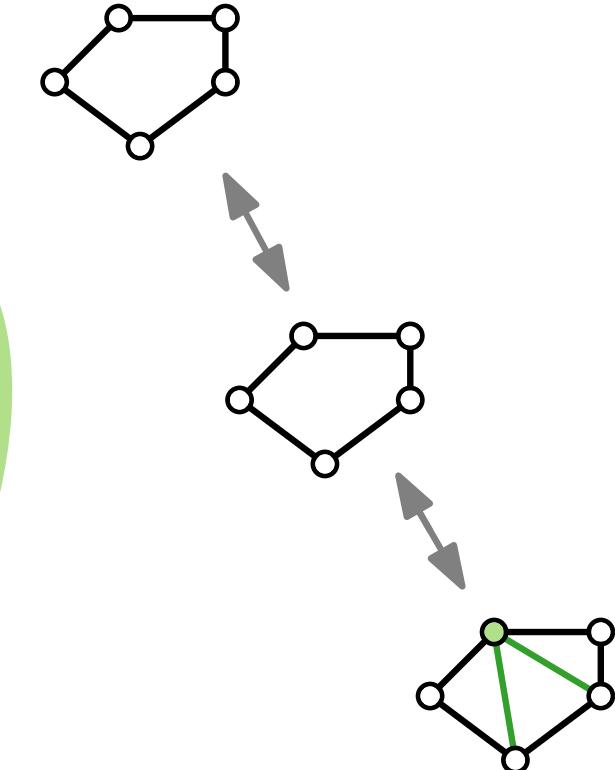
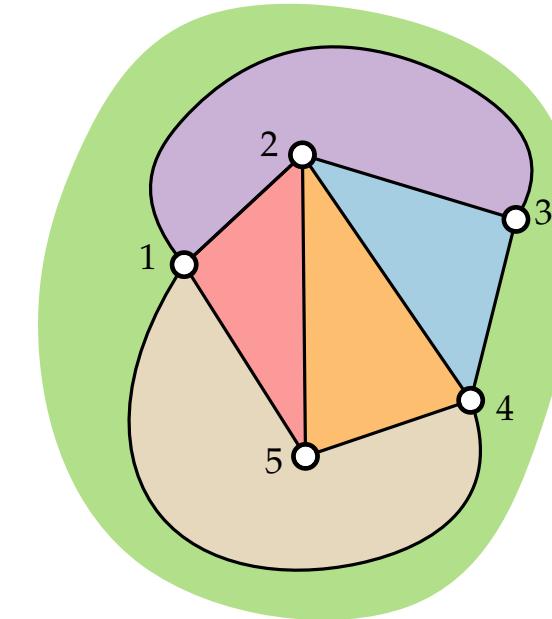
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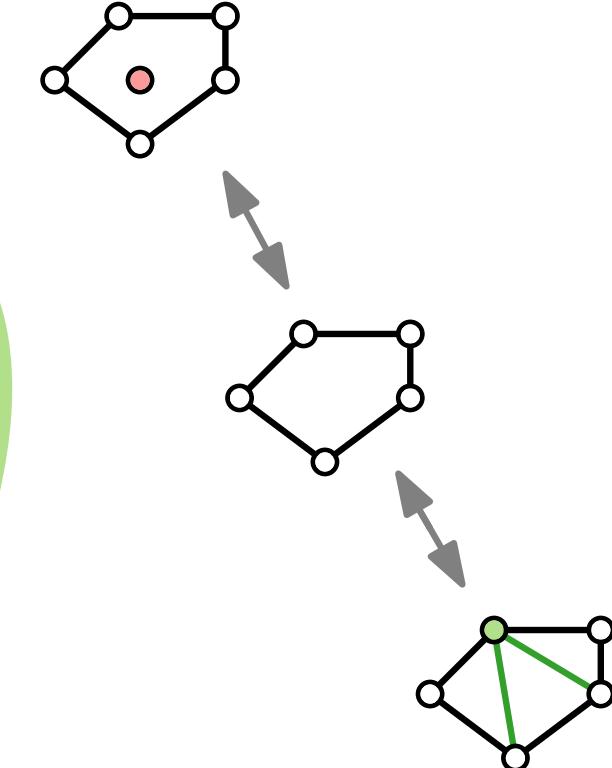
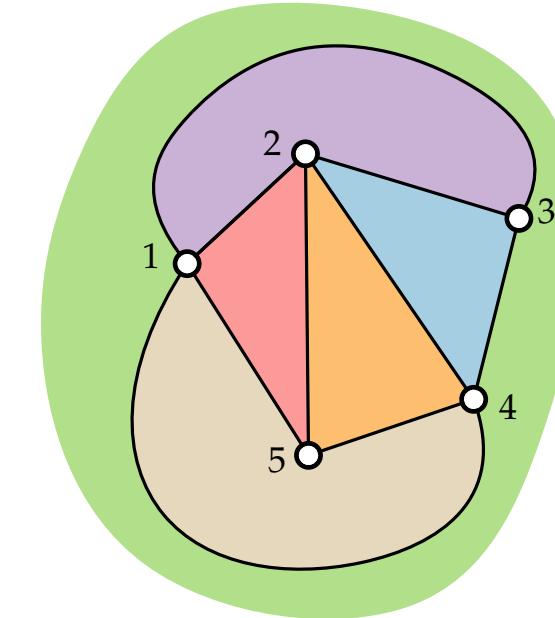
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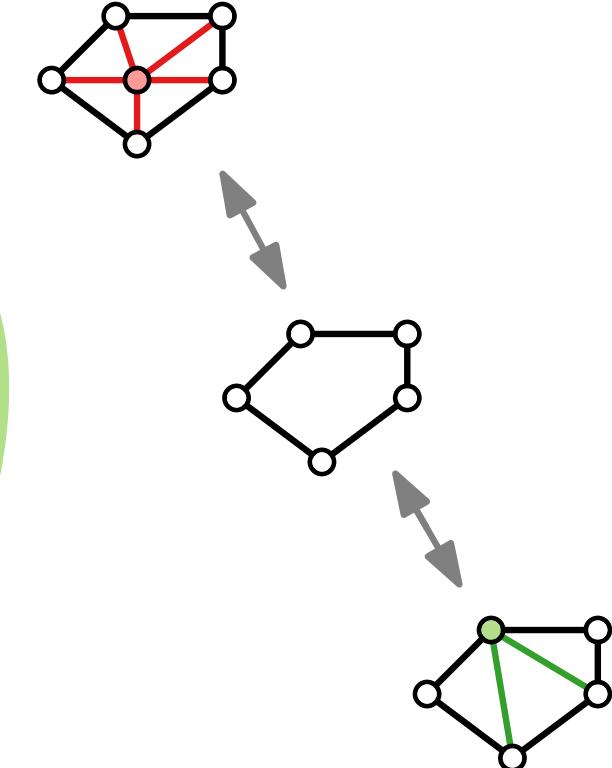
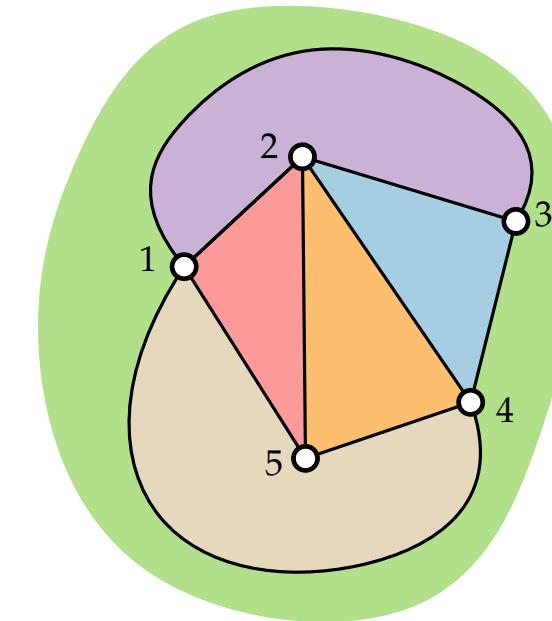
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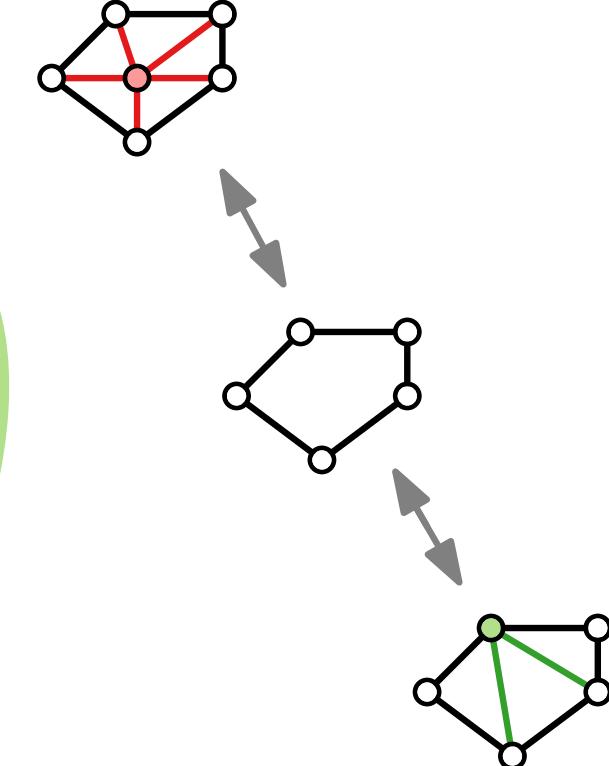
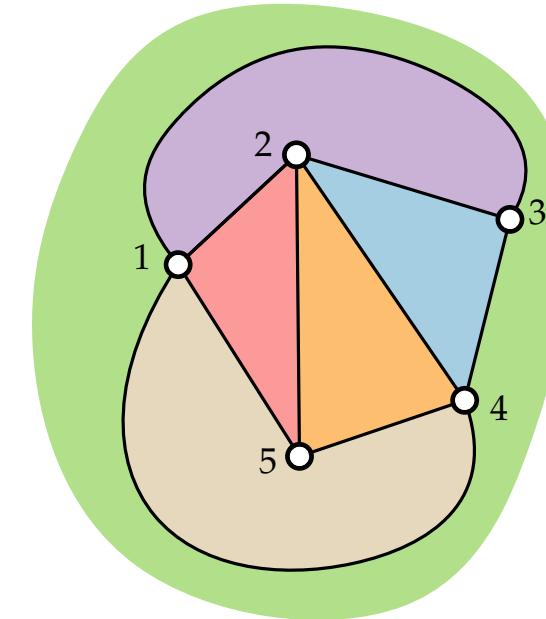
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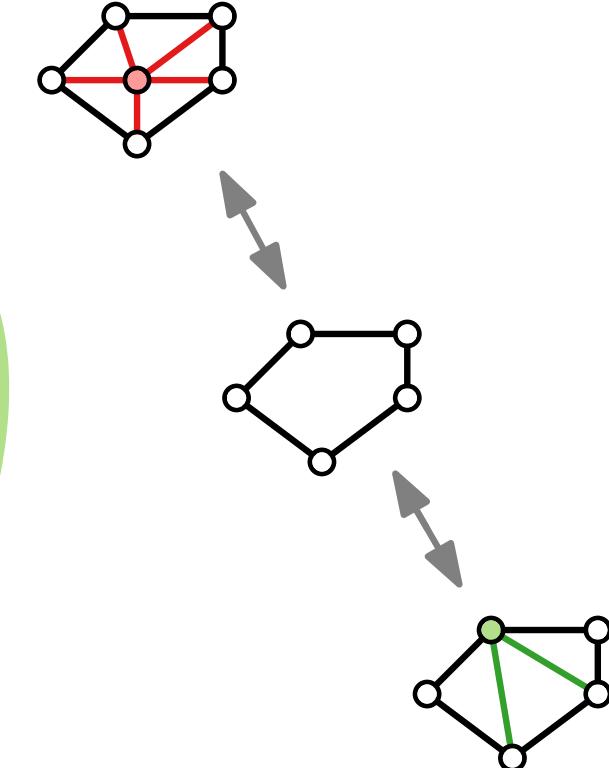
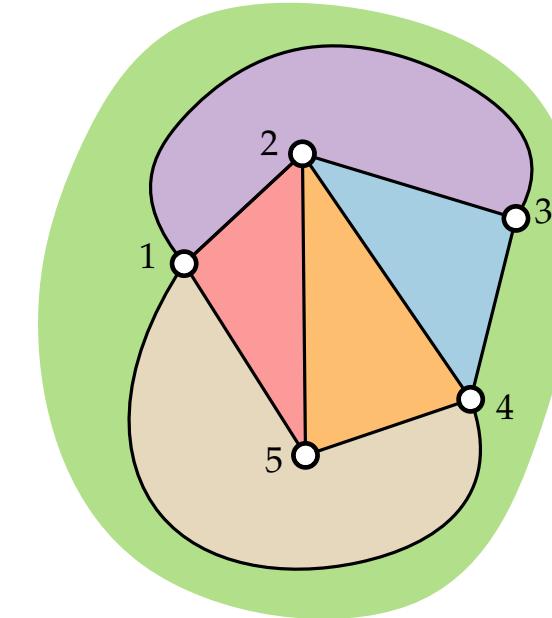
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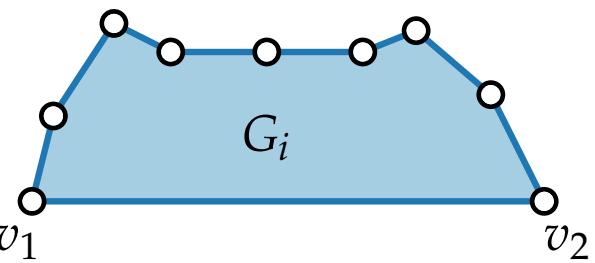
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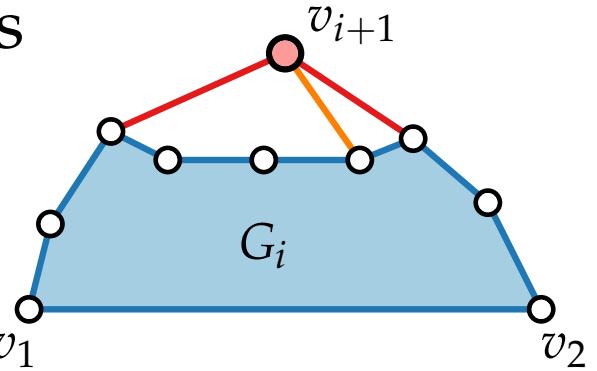
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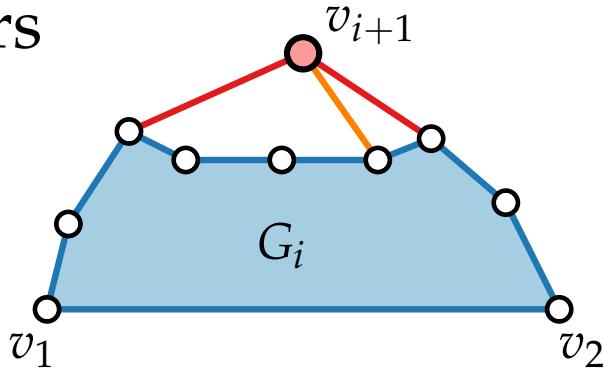
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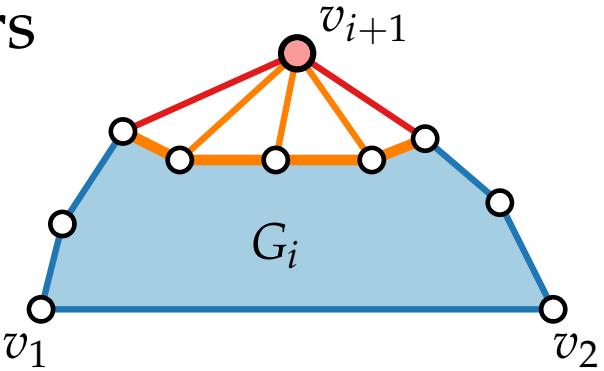
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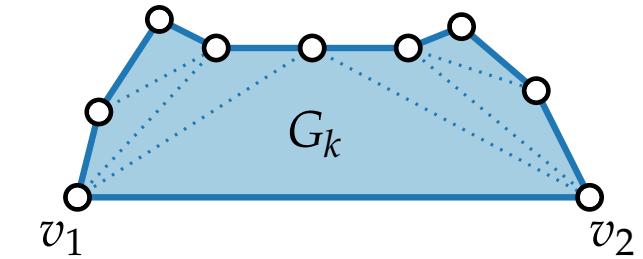
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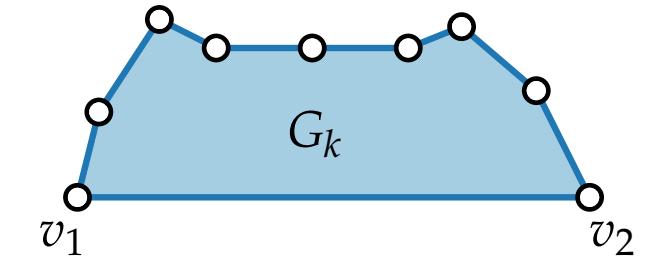
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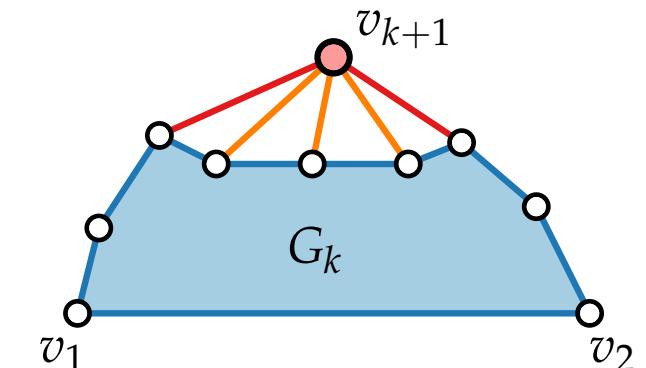
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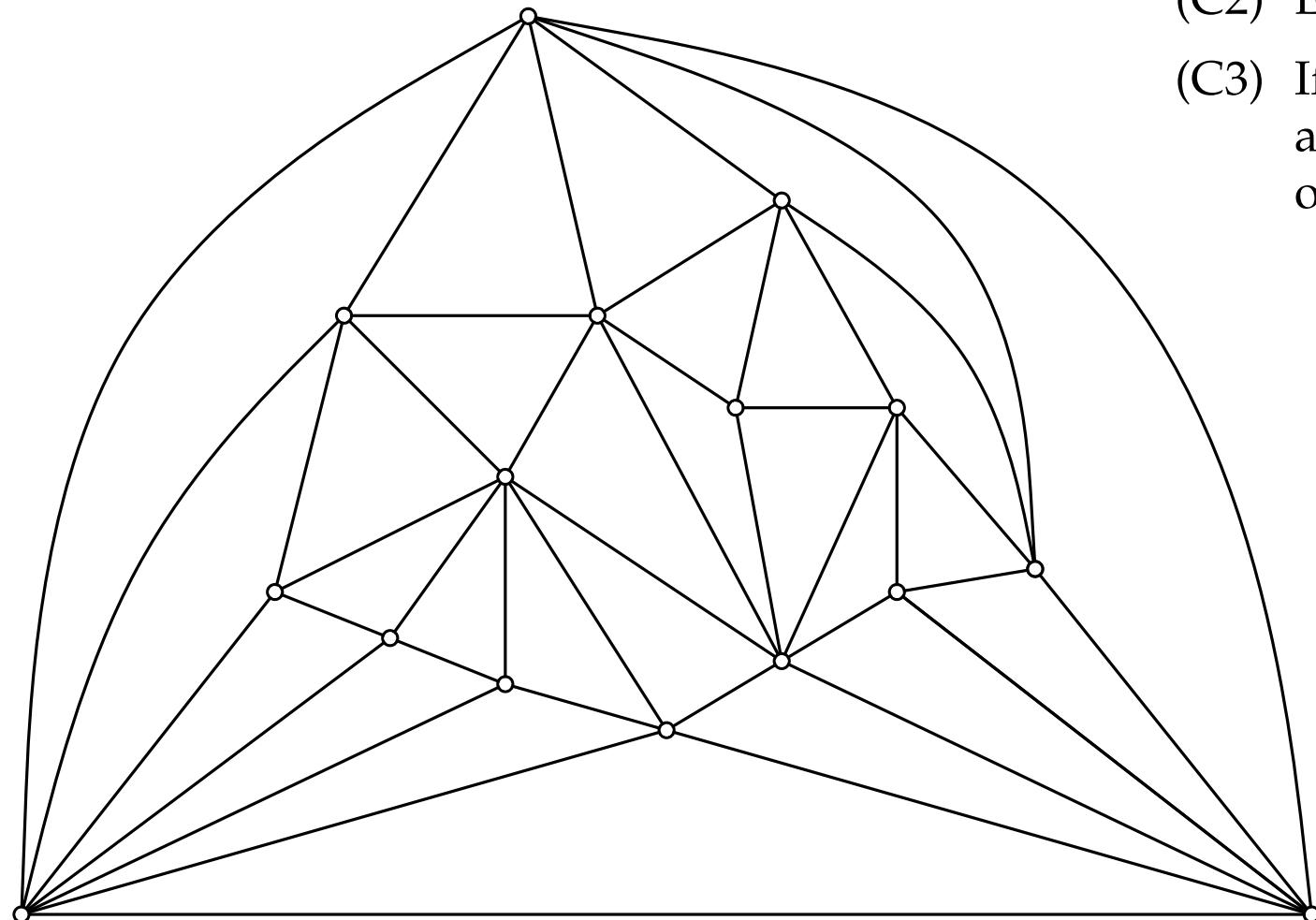
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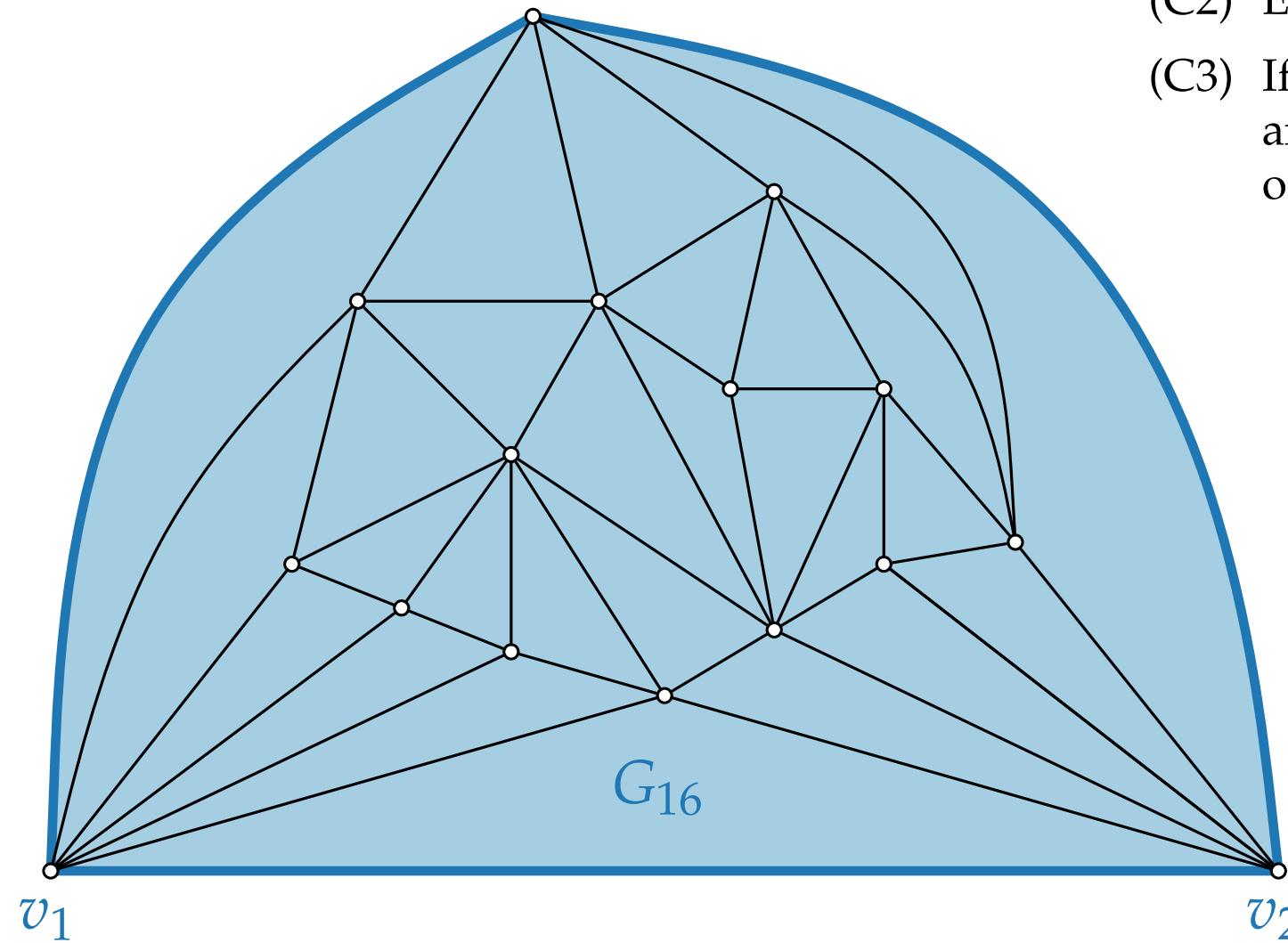


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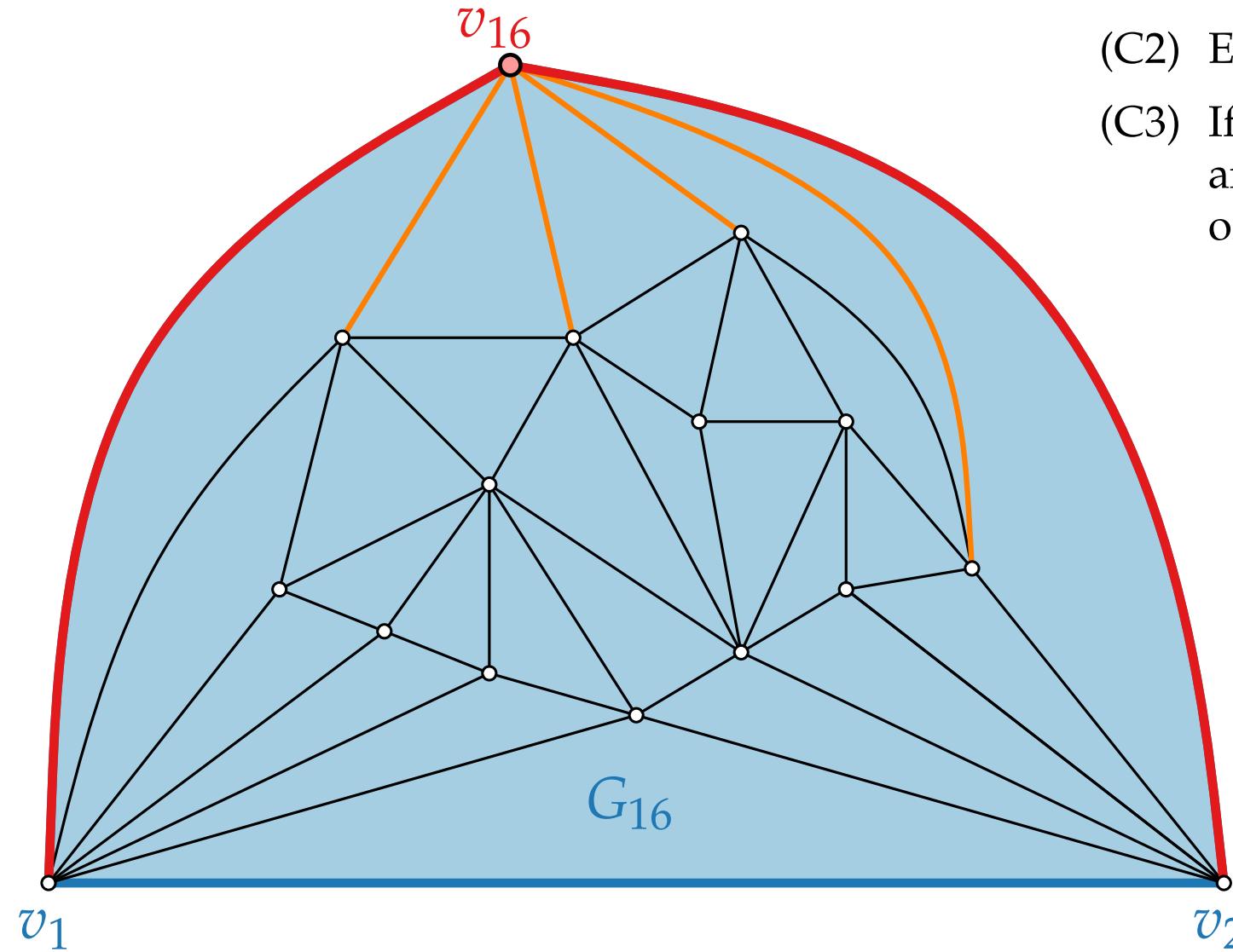
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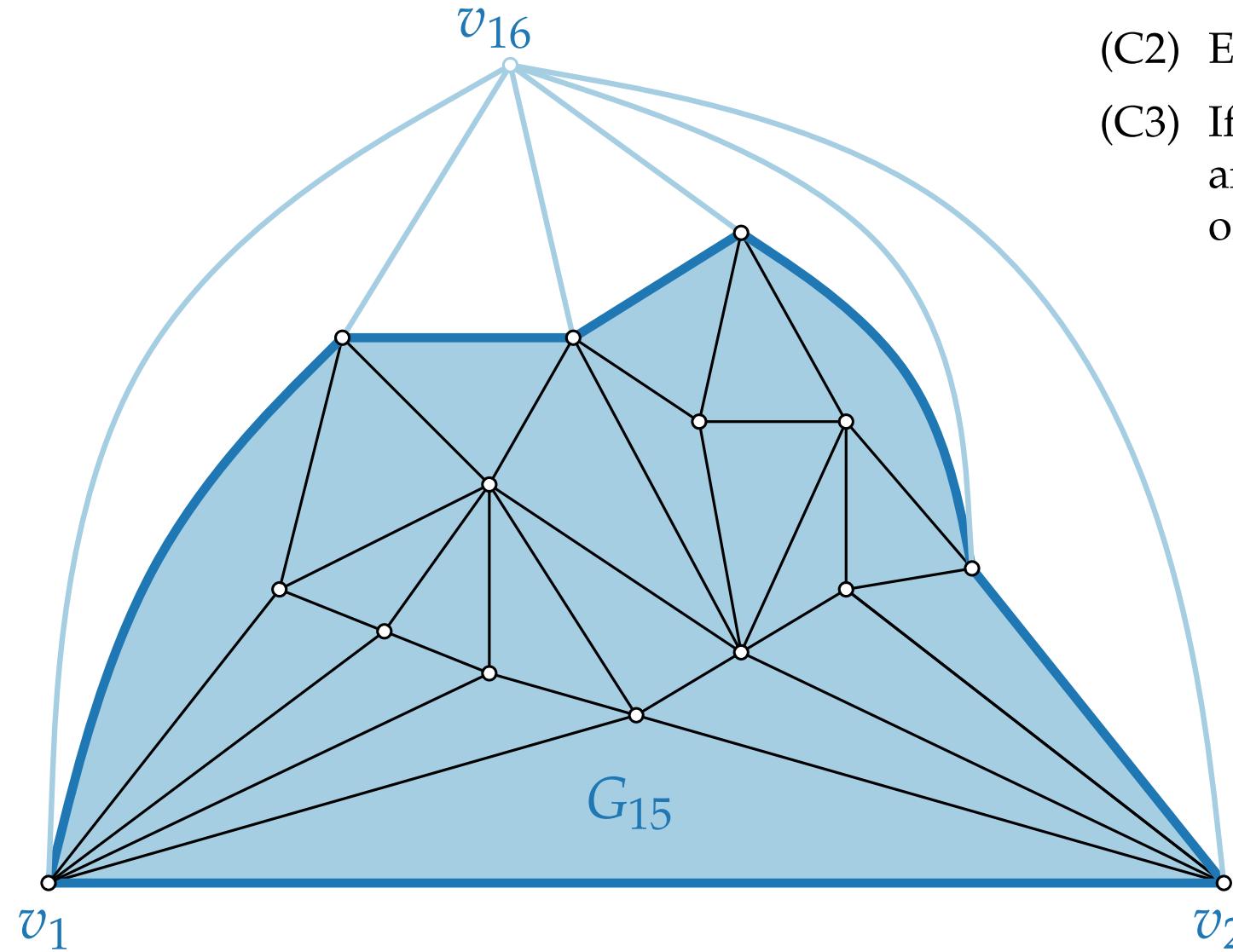
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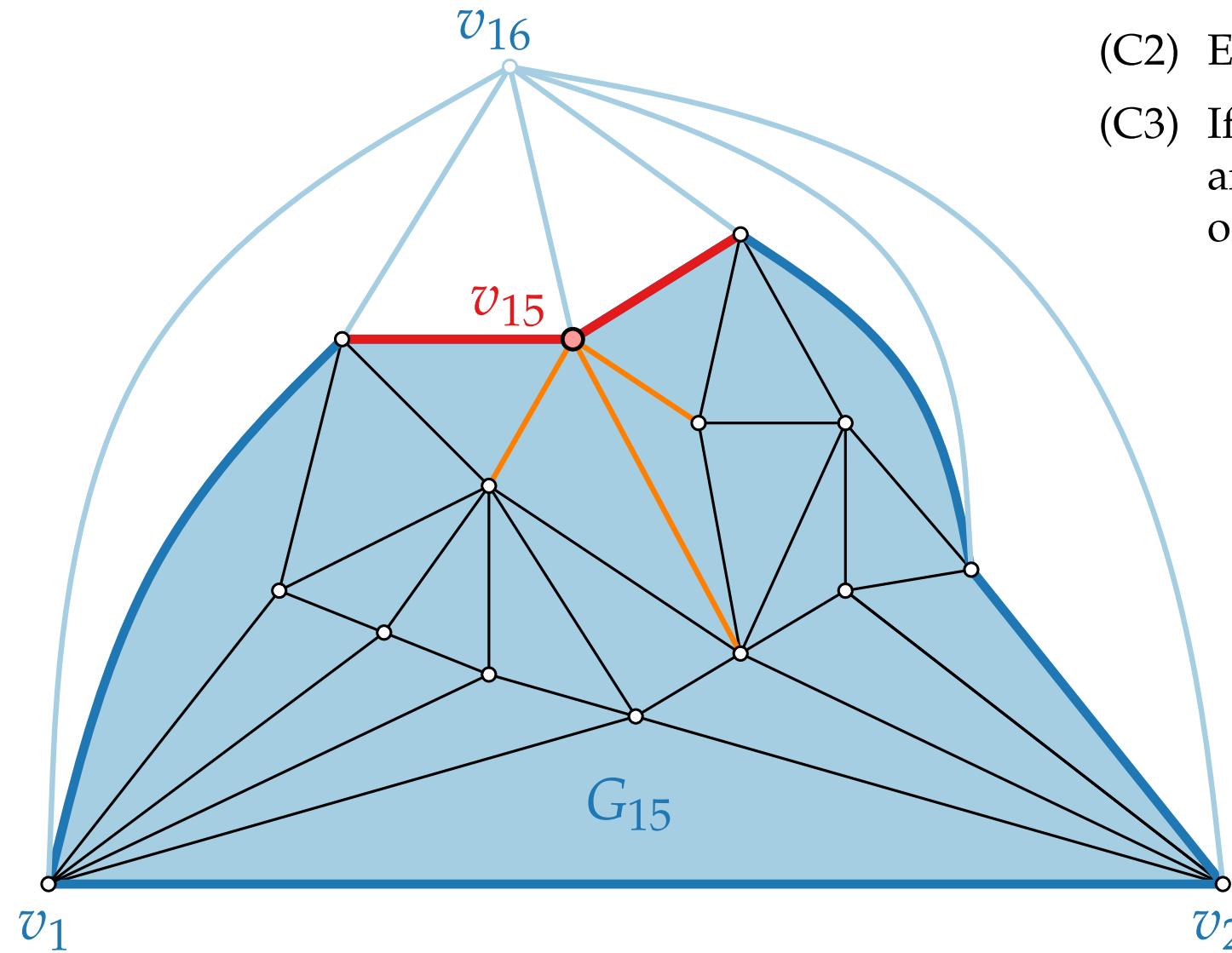
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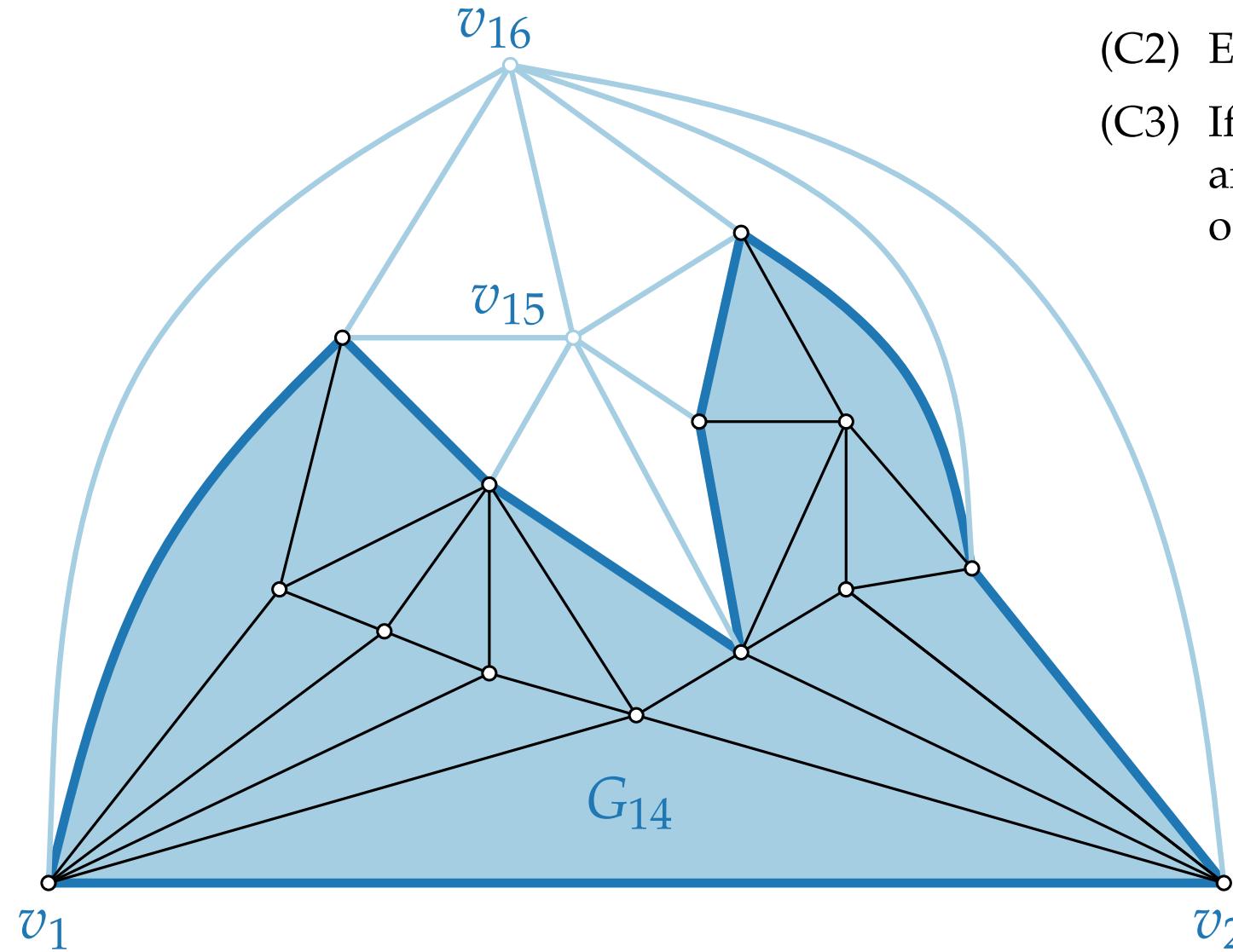
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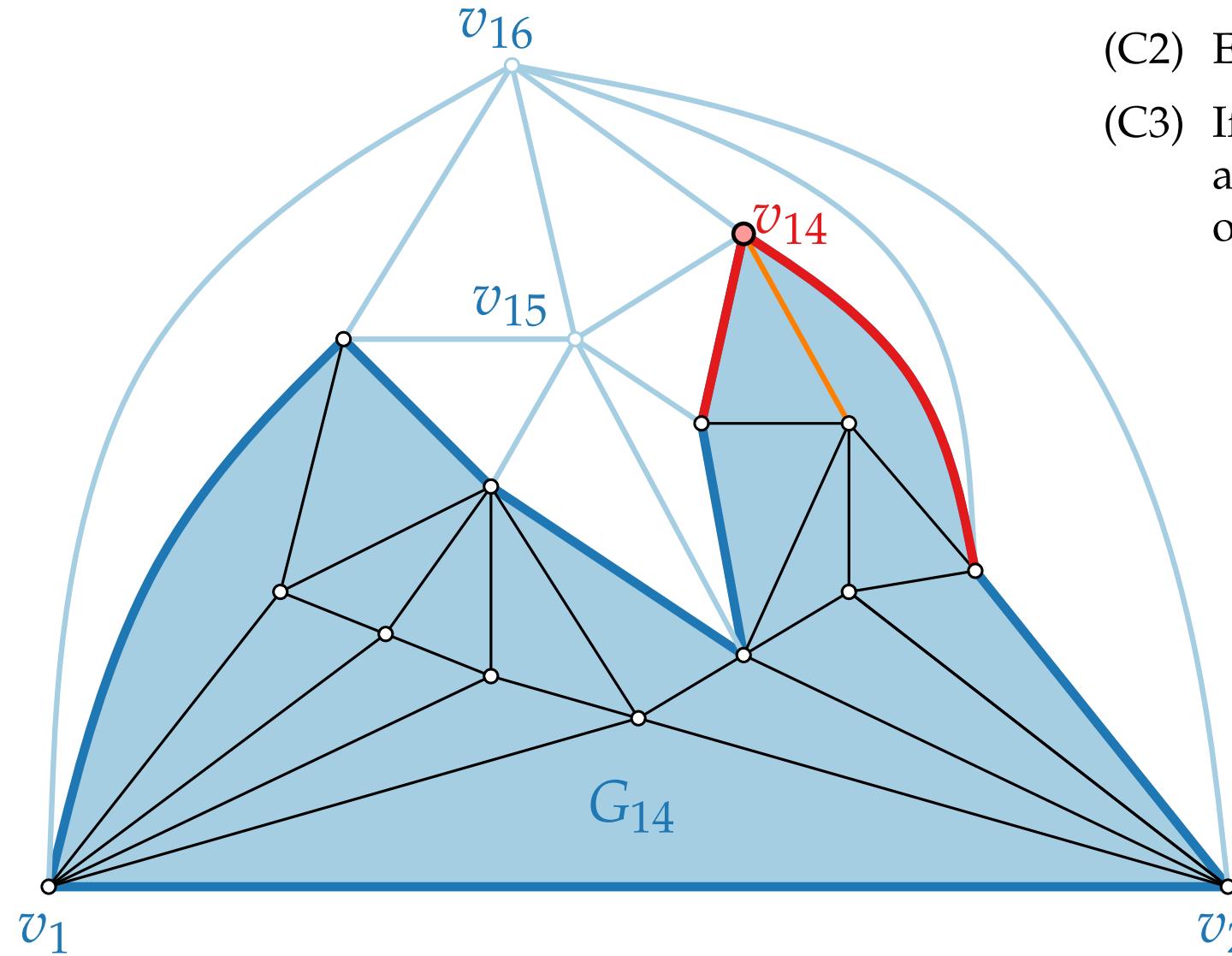
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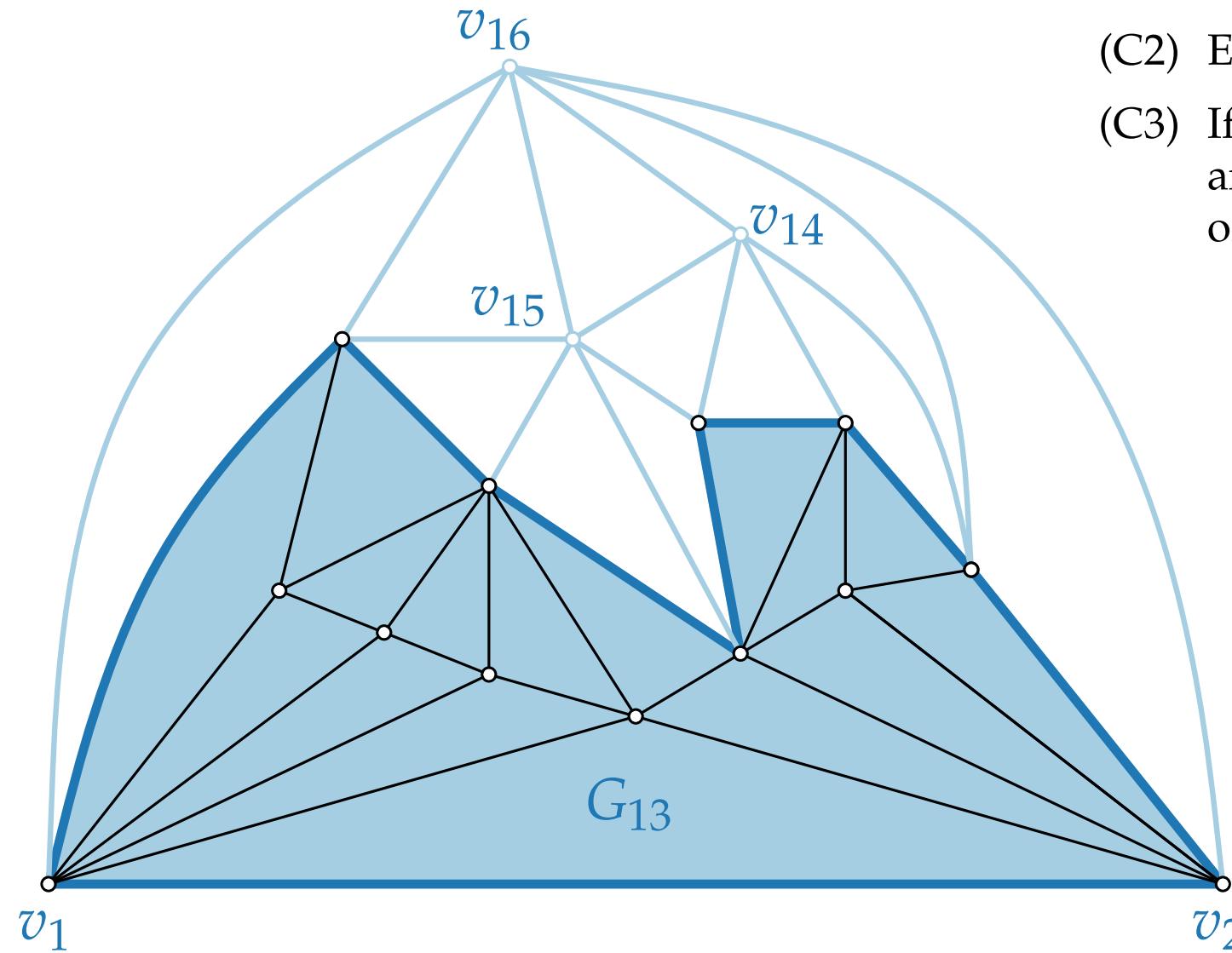
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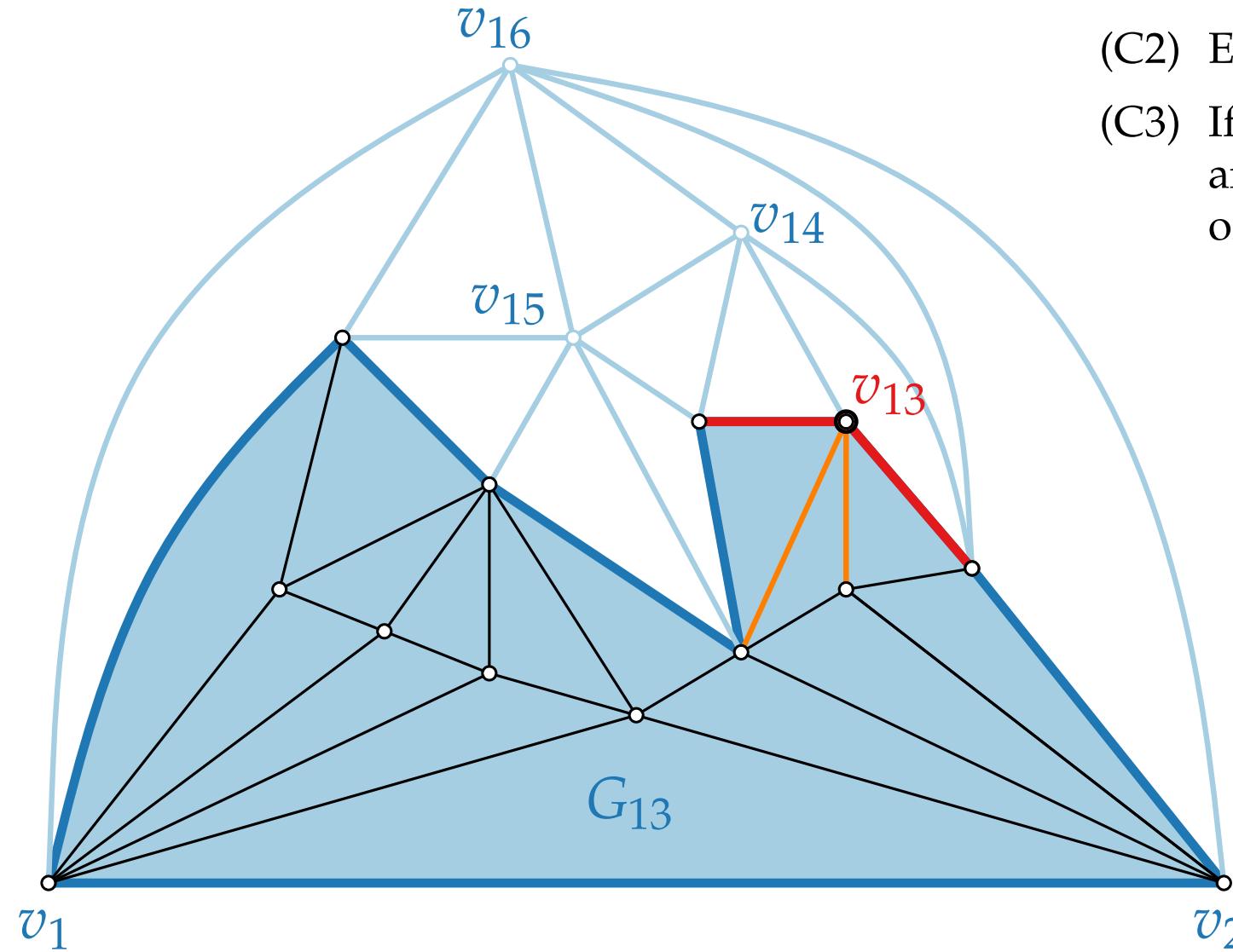
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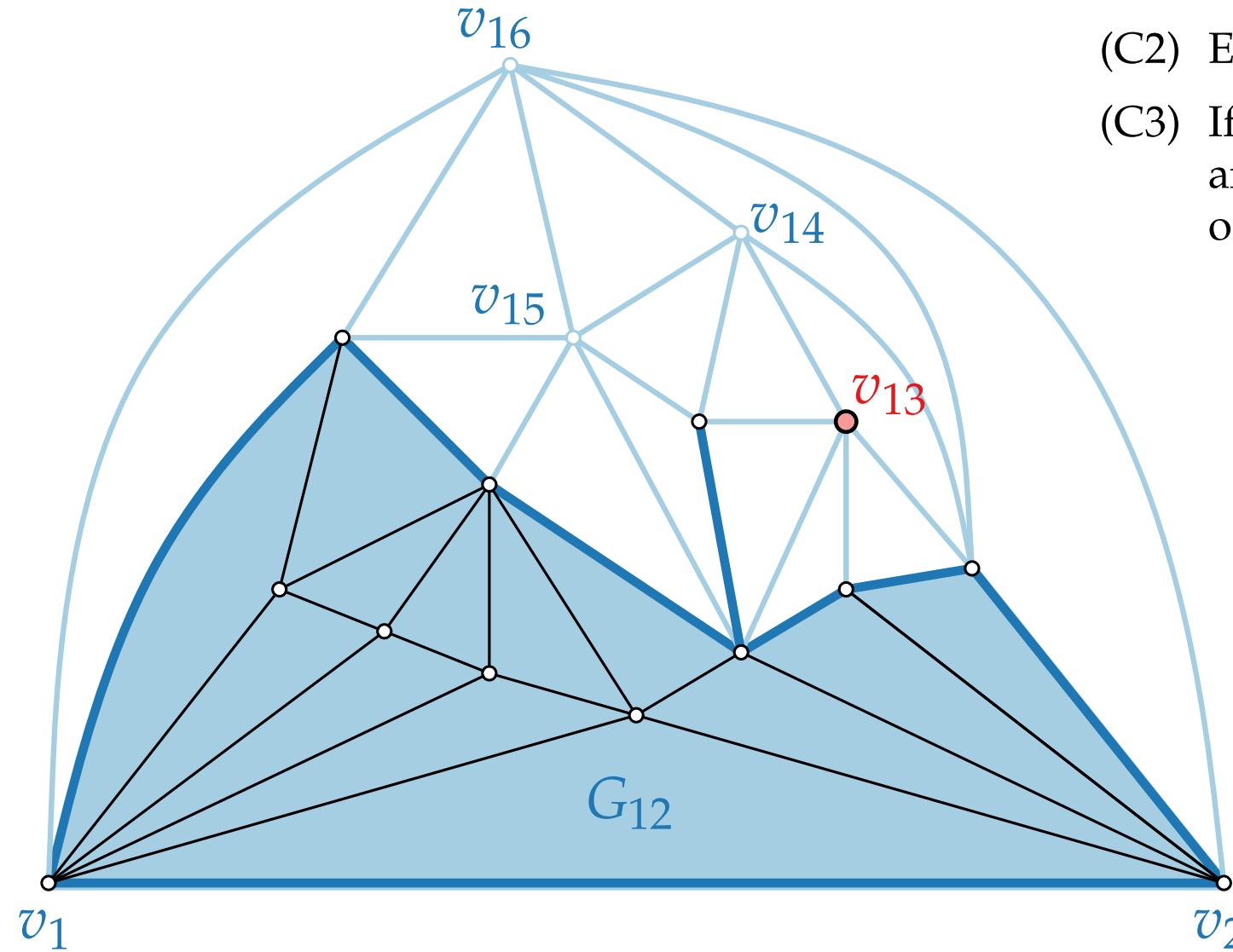
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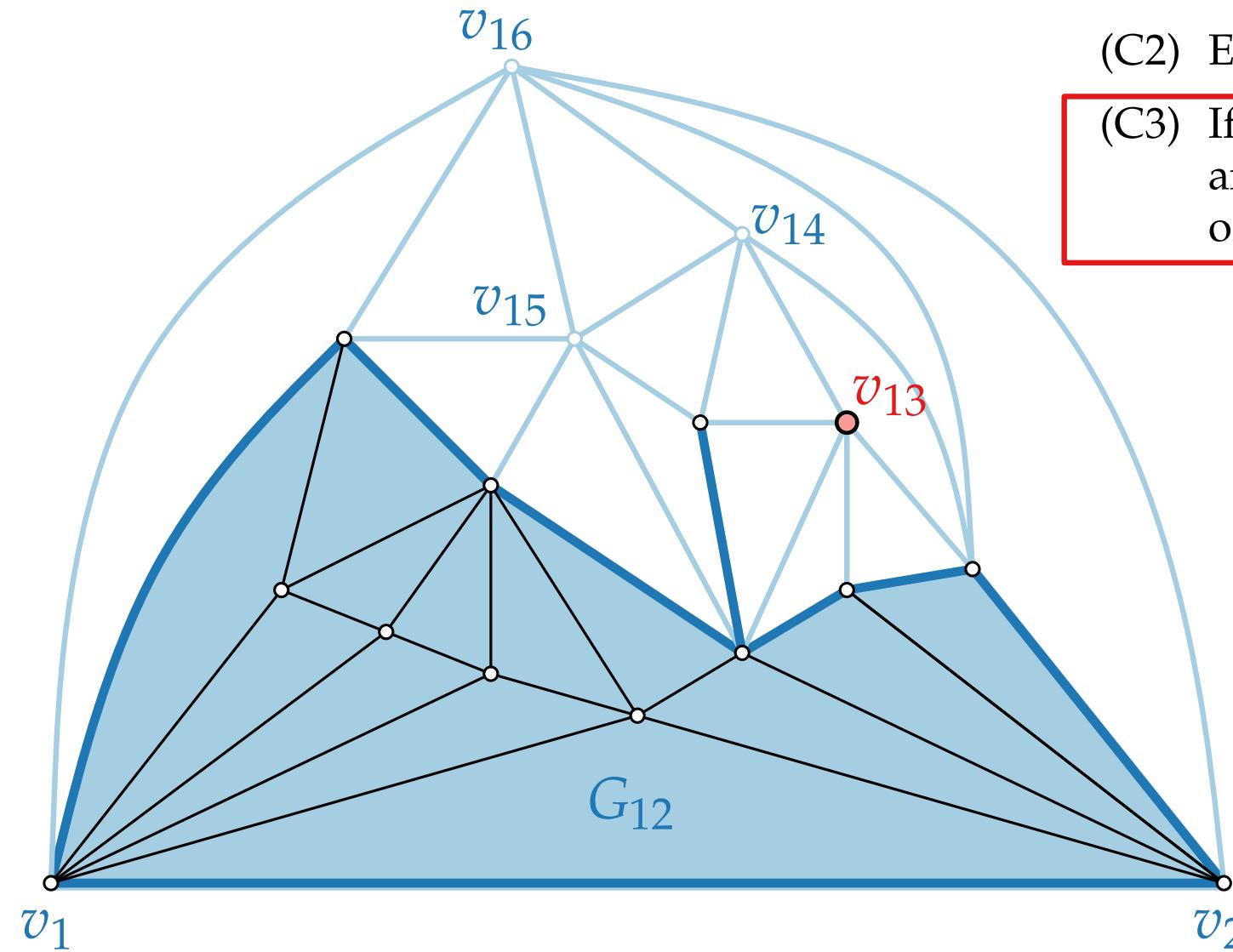
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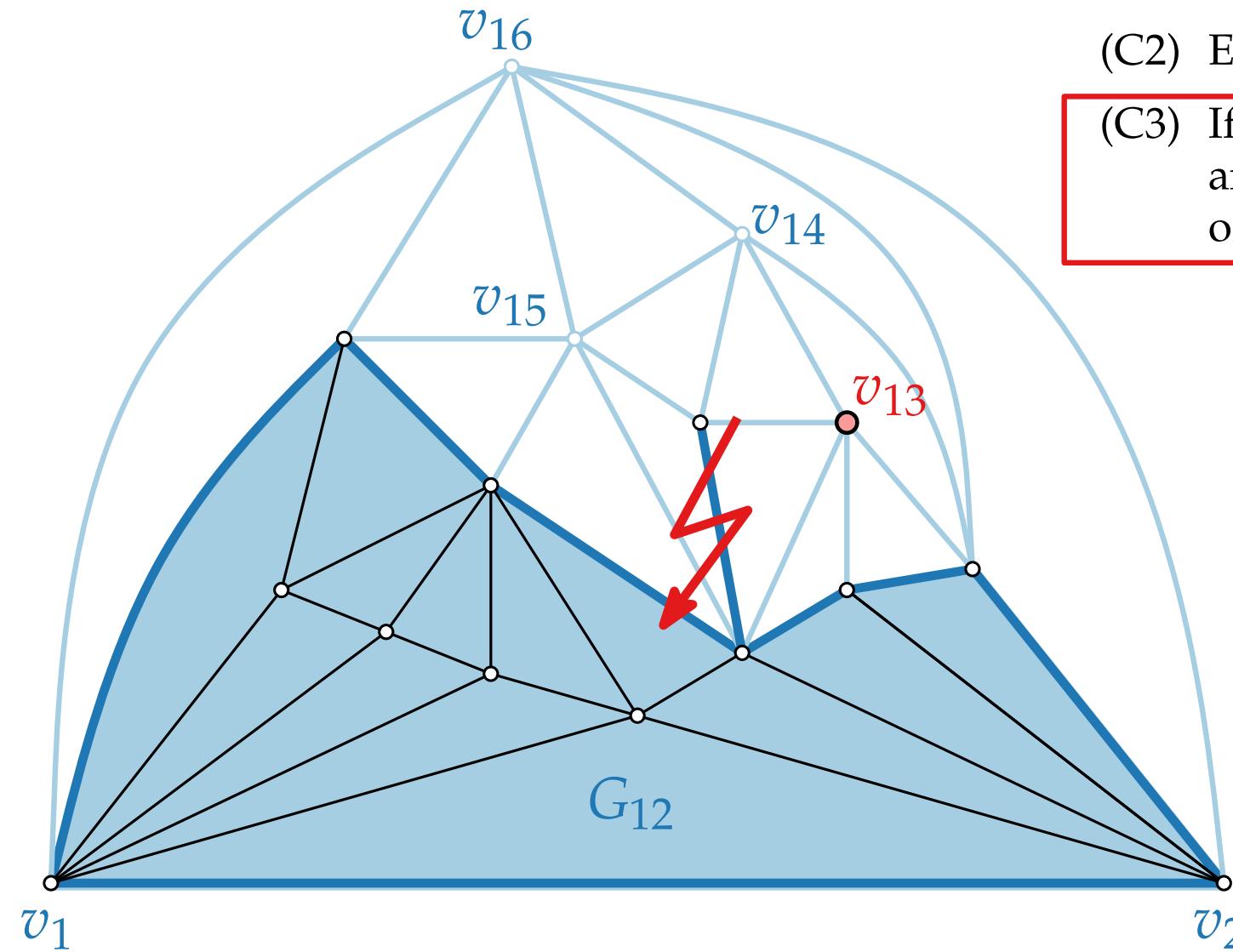
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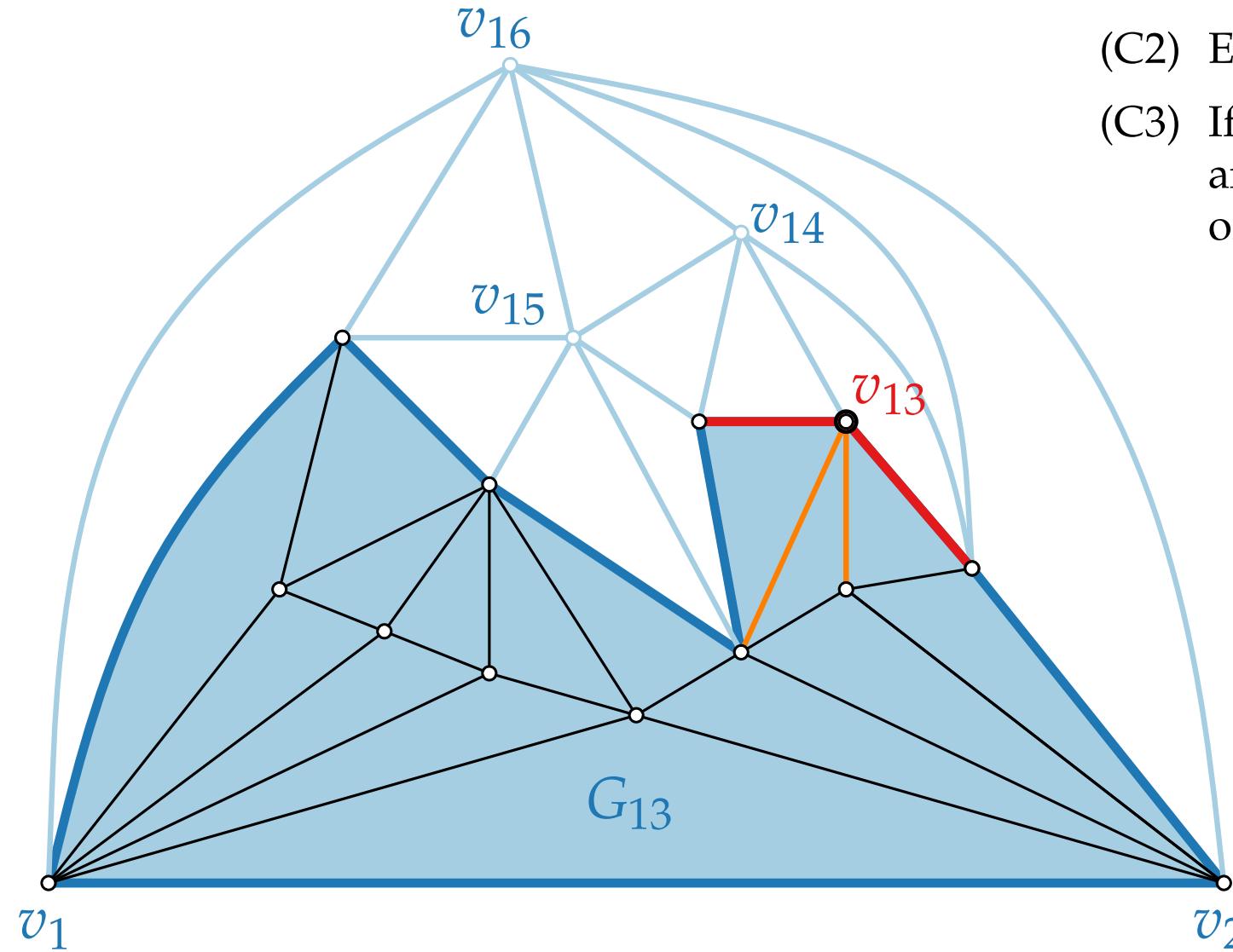
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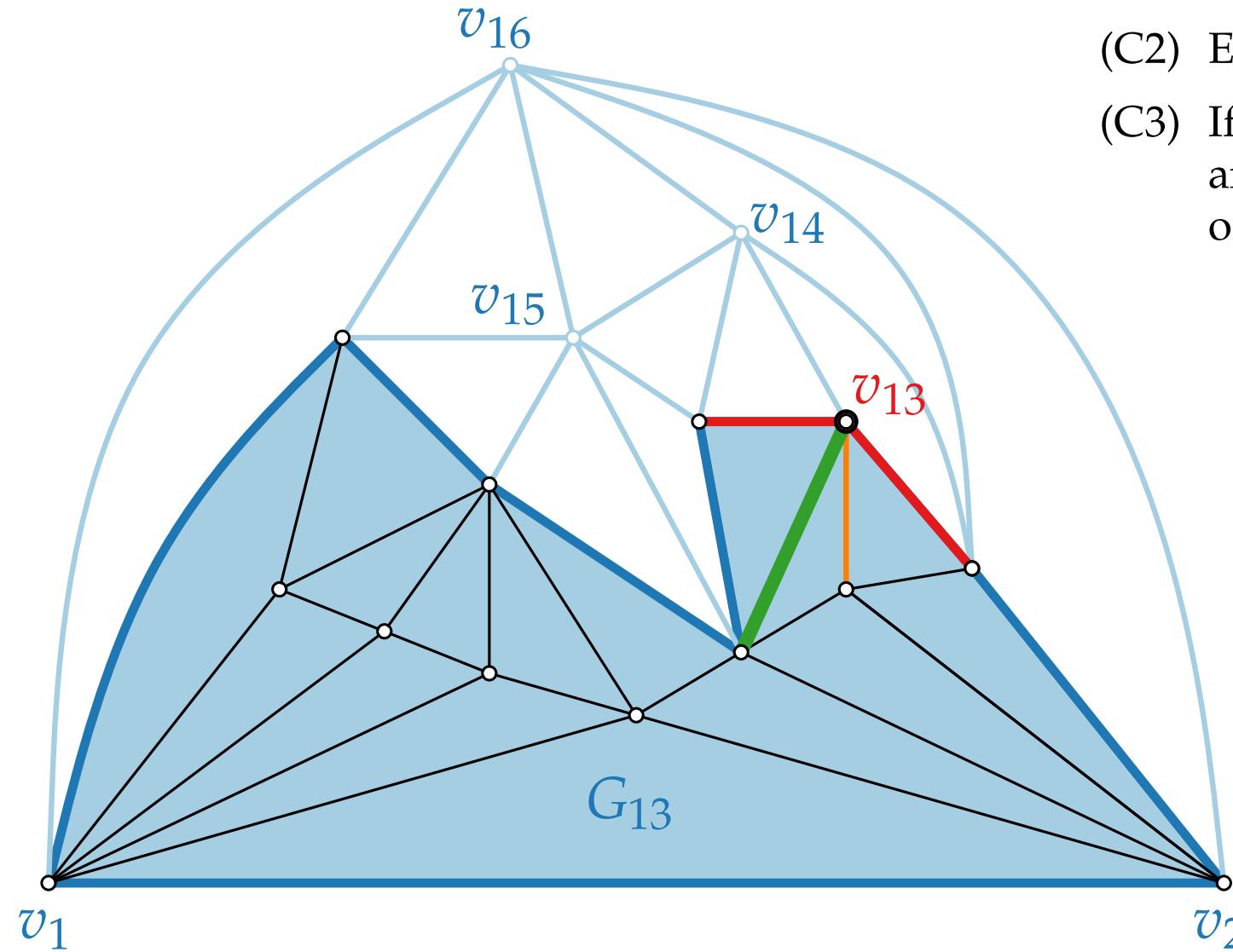
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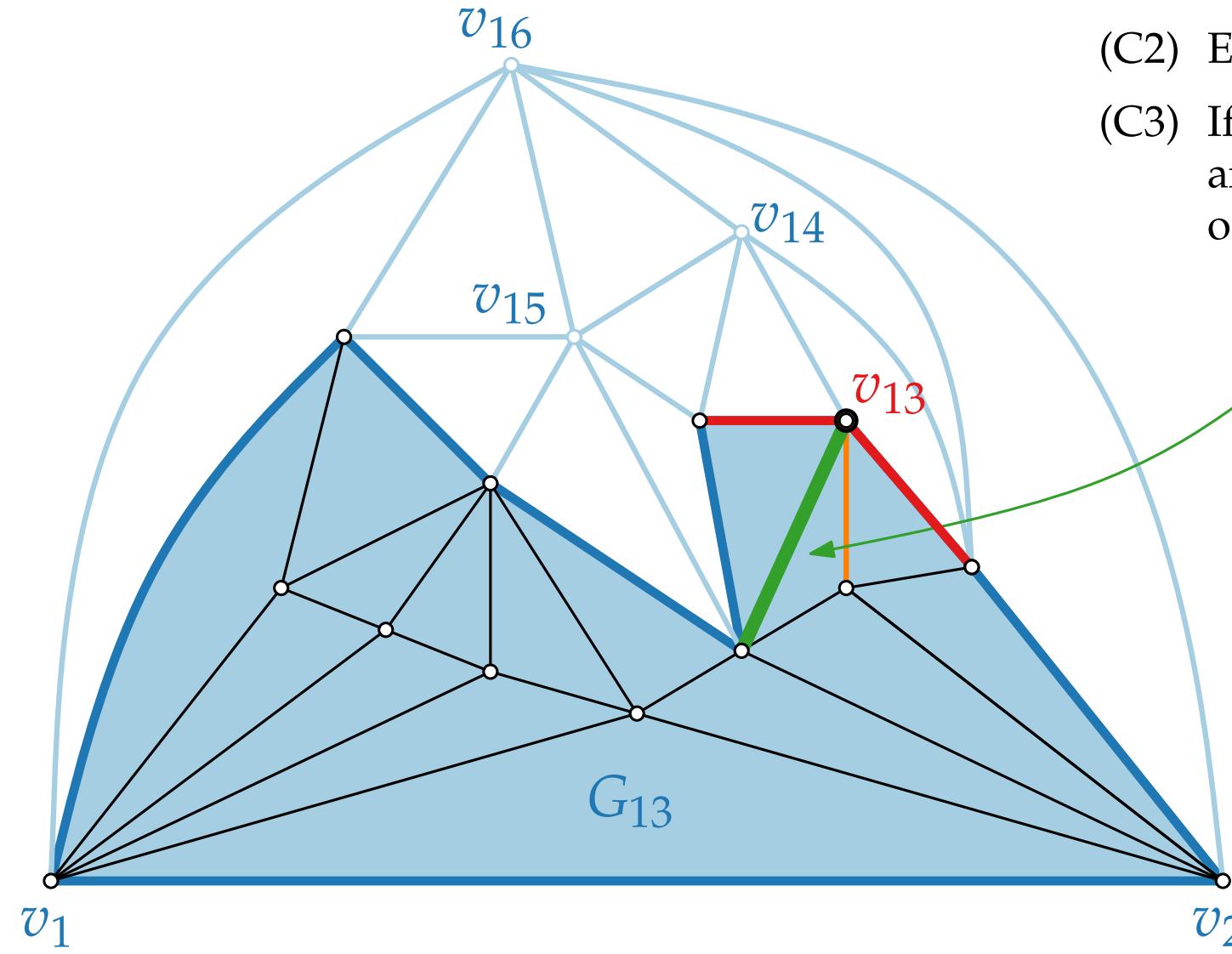
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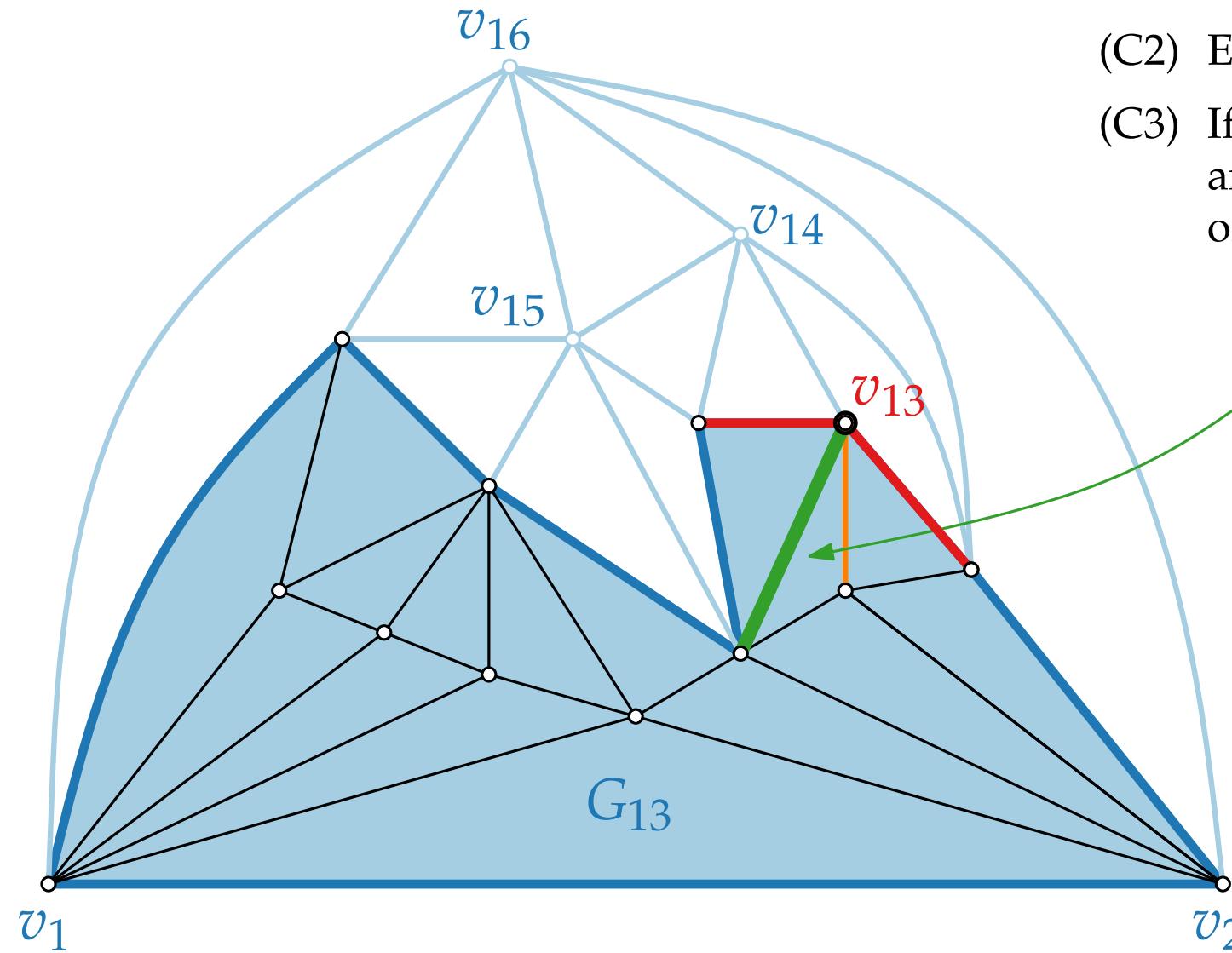
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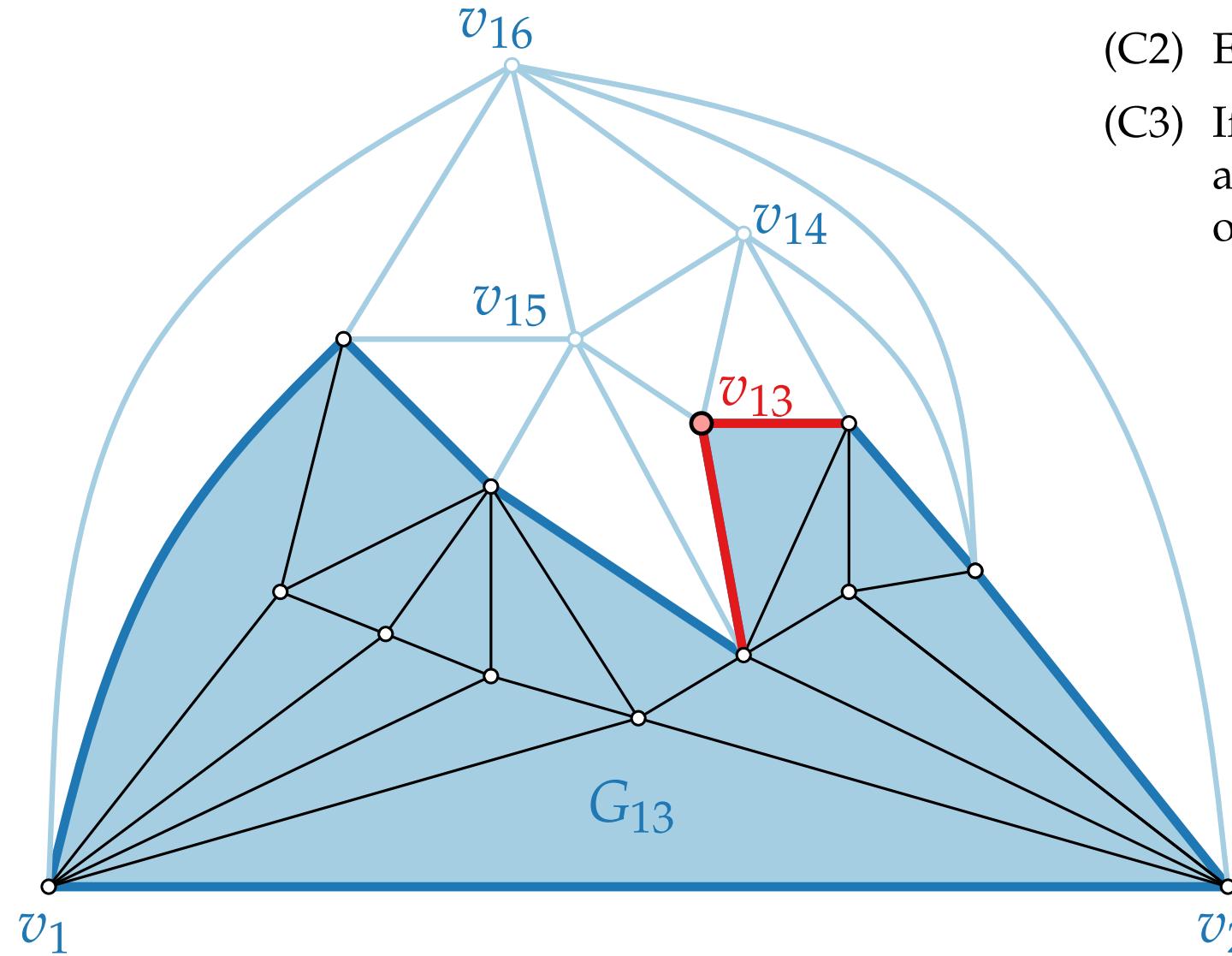
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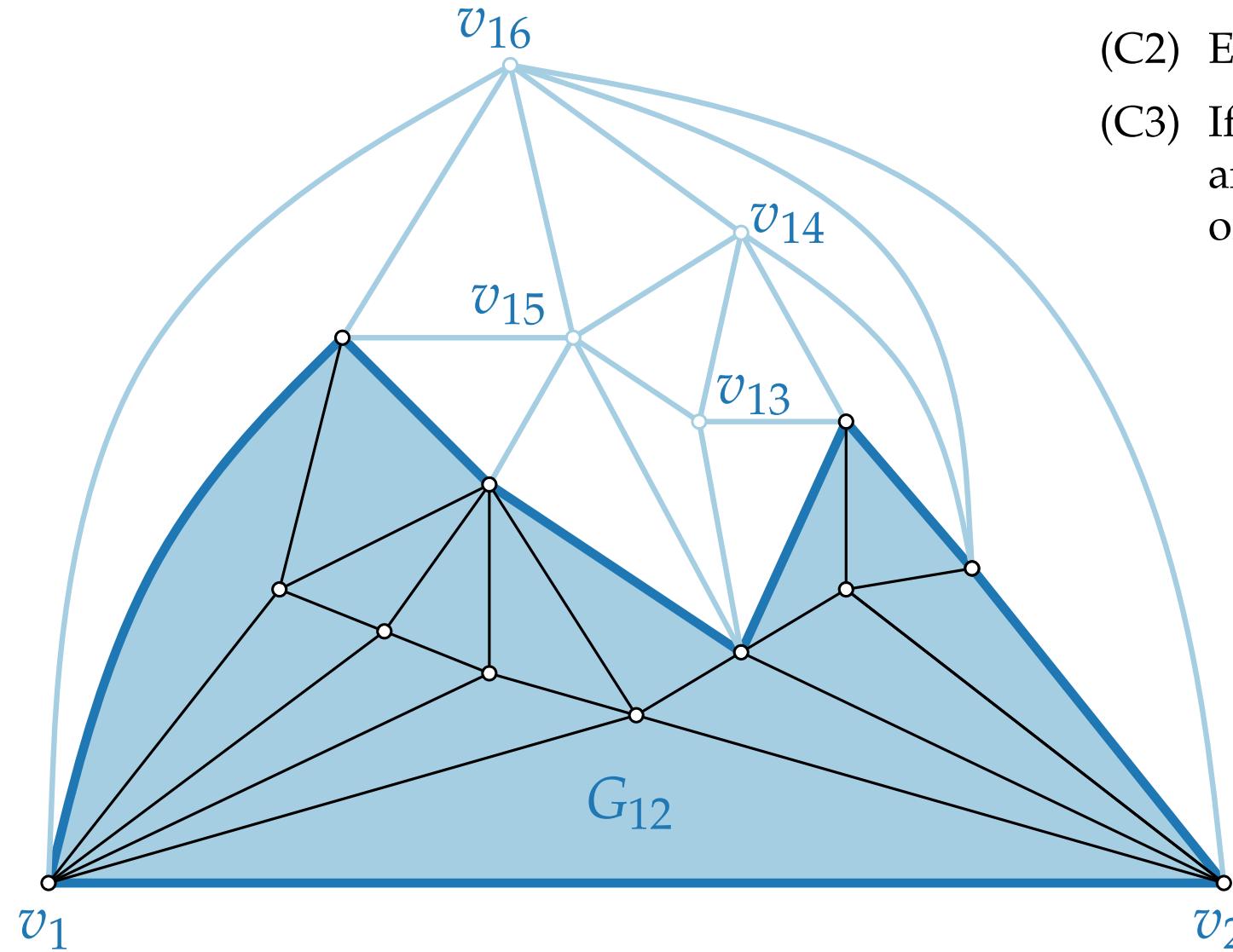
chord
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nonadjacent
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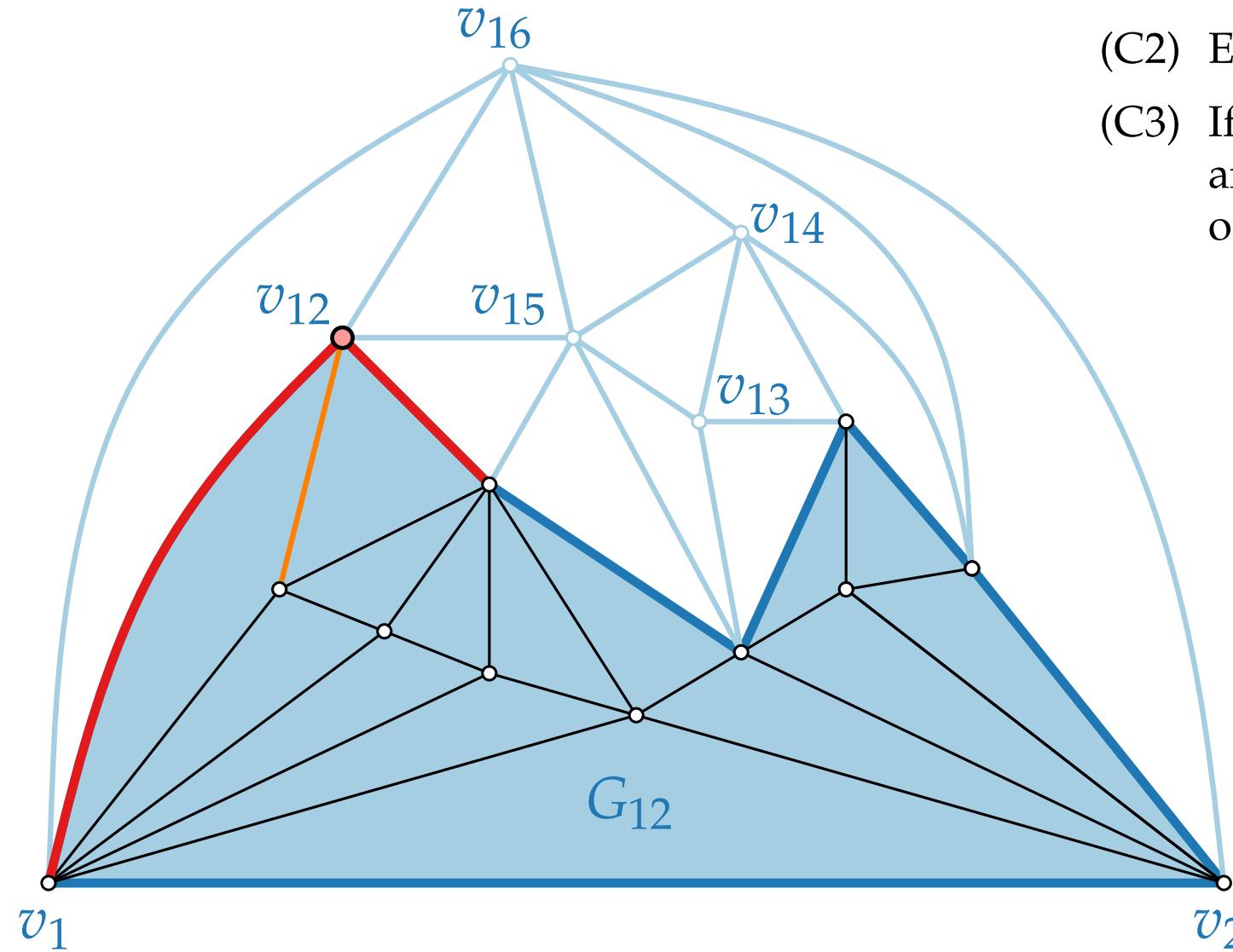
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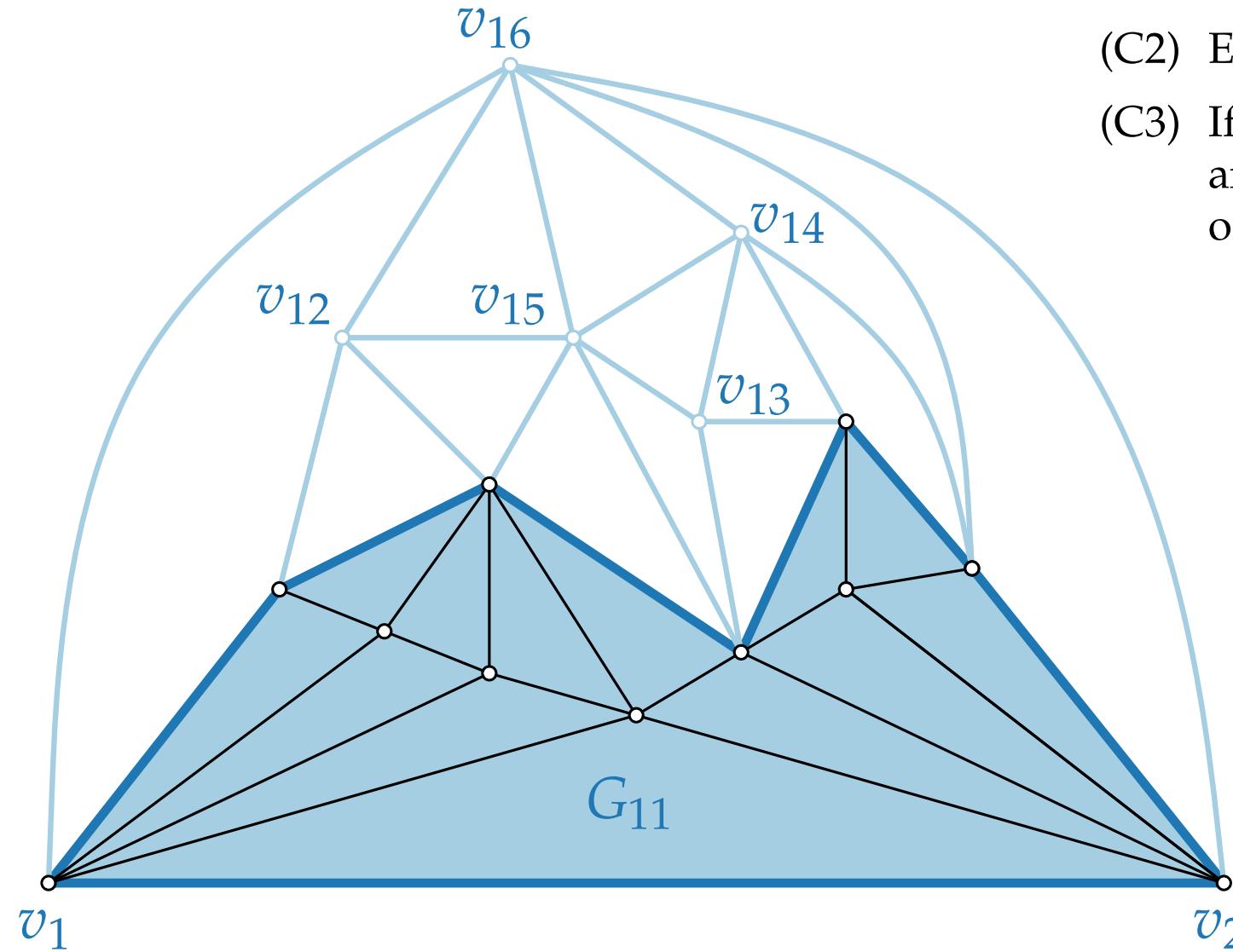
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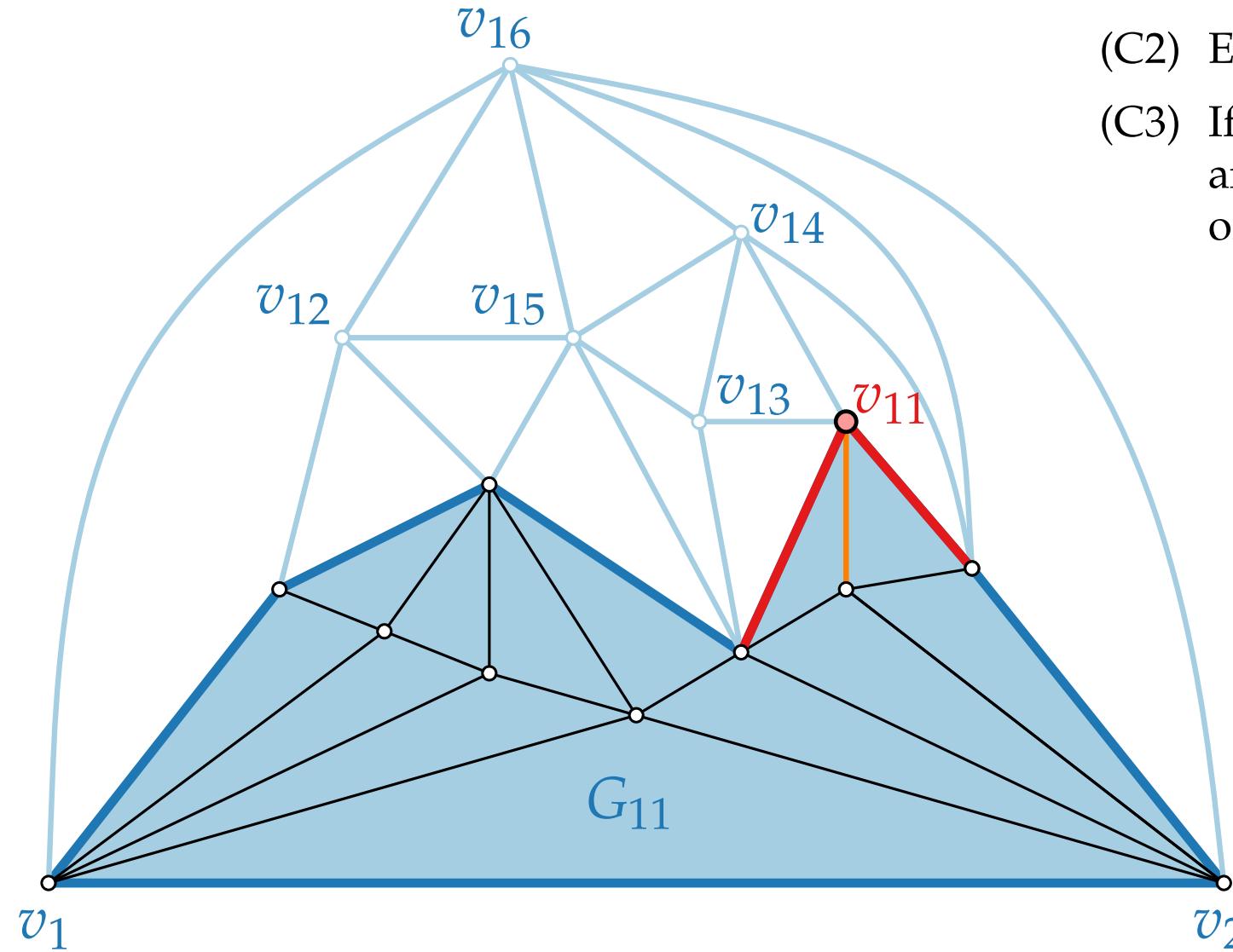
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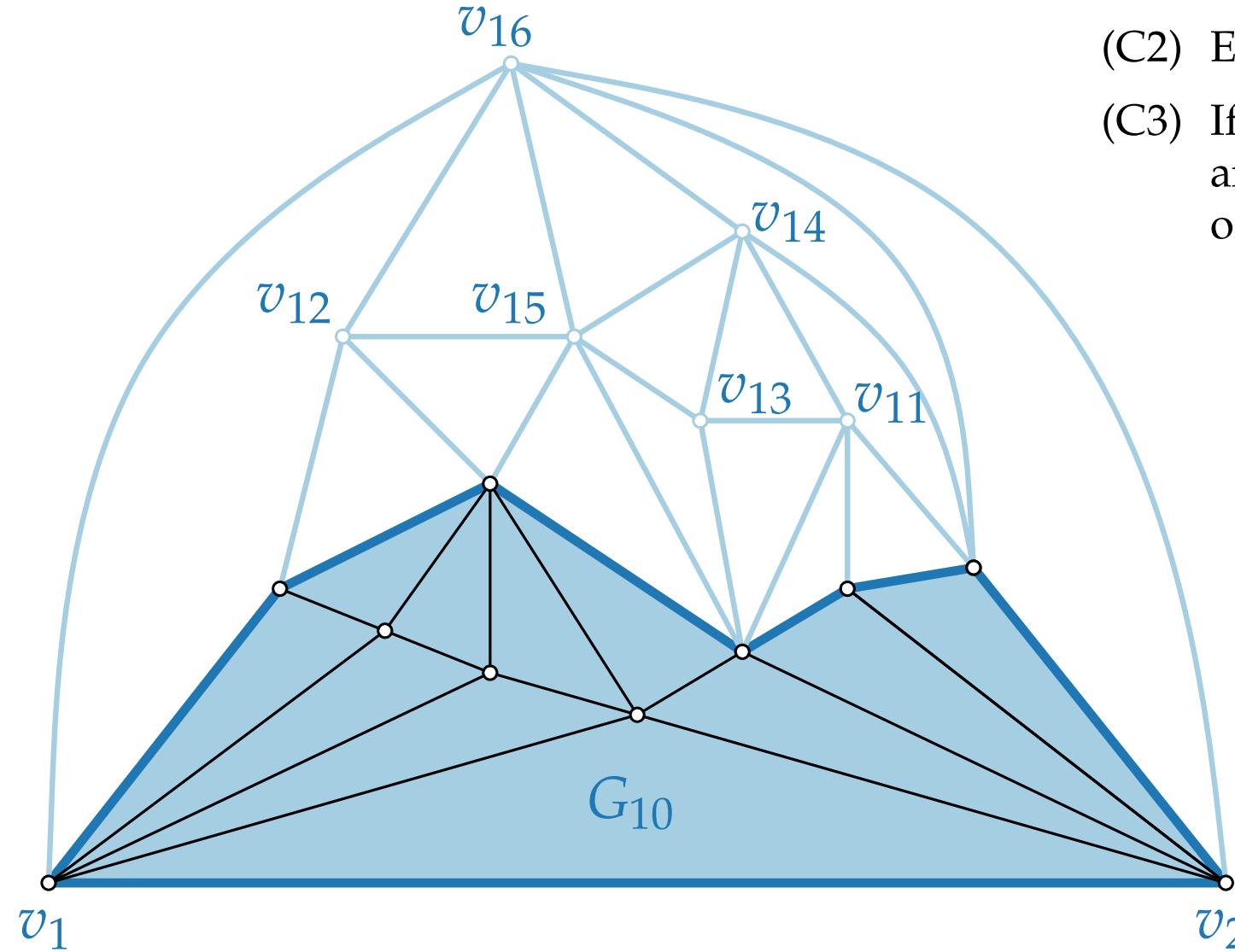
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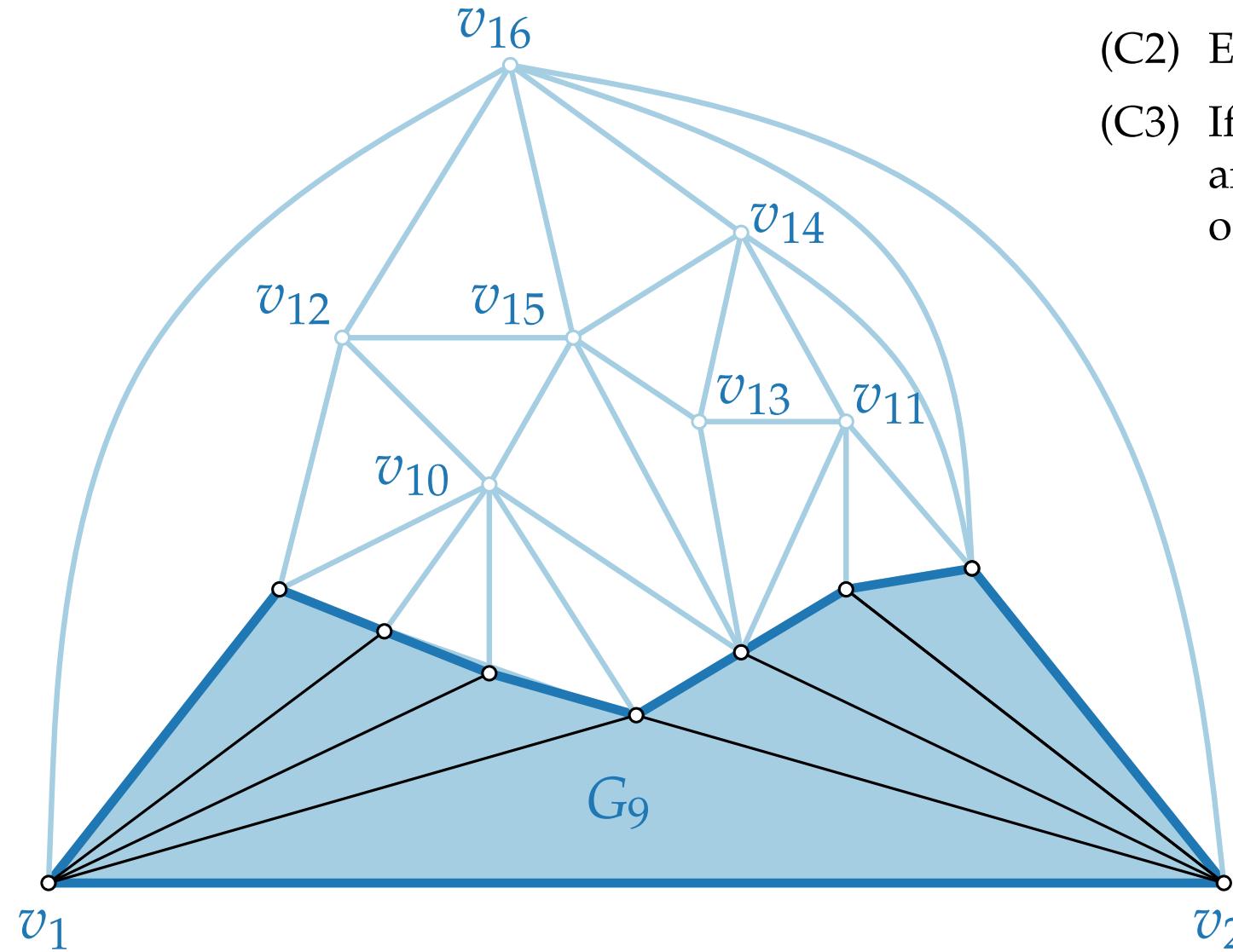
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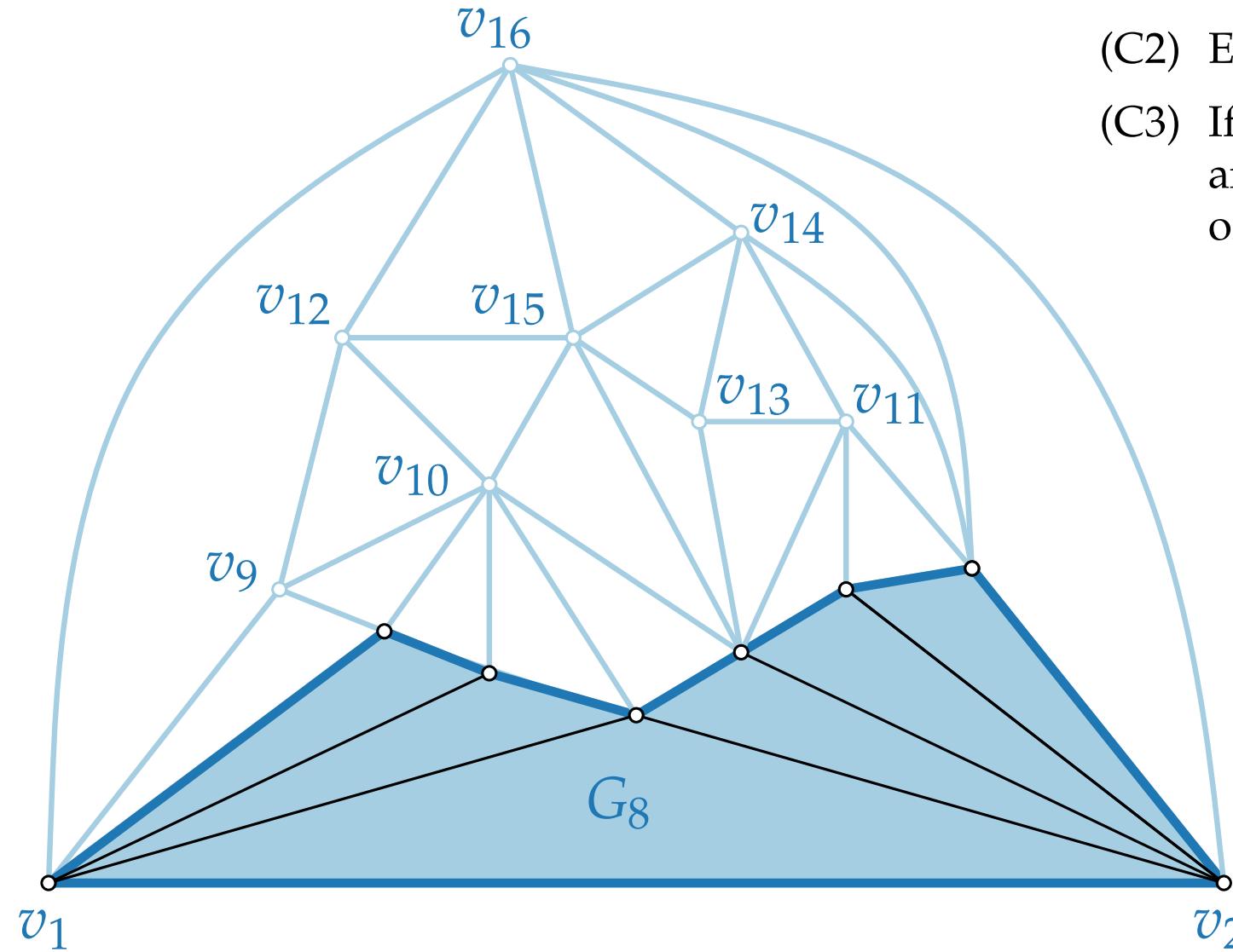
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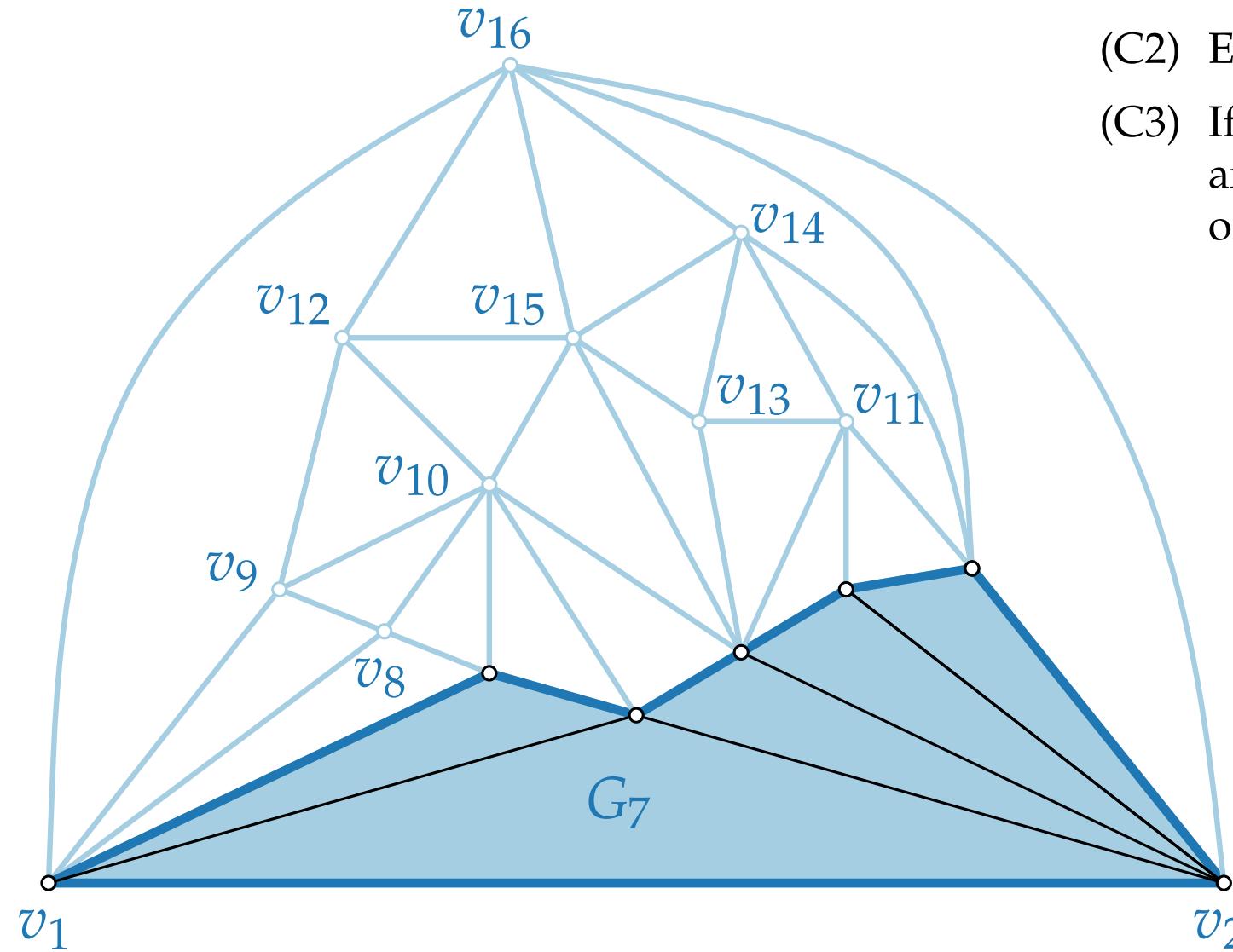
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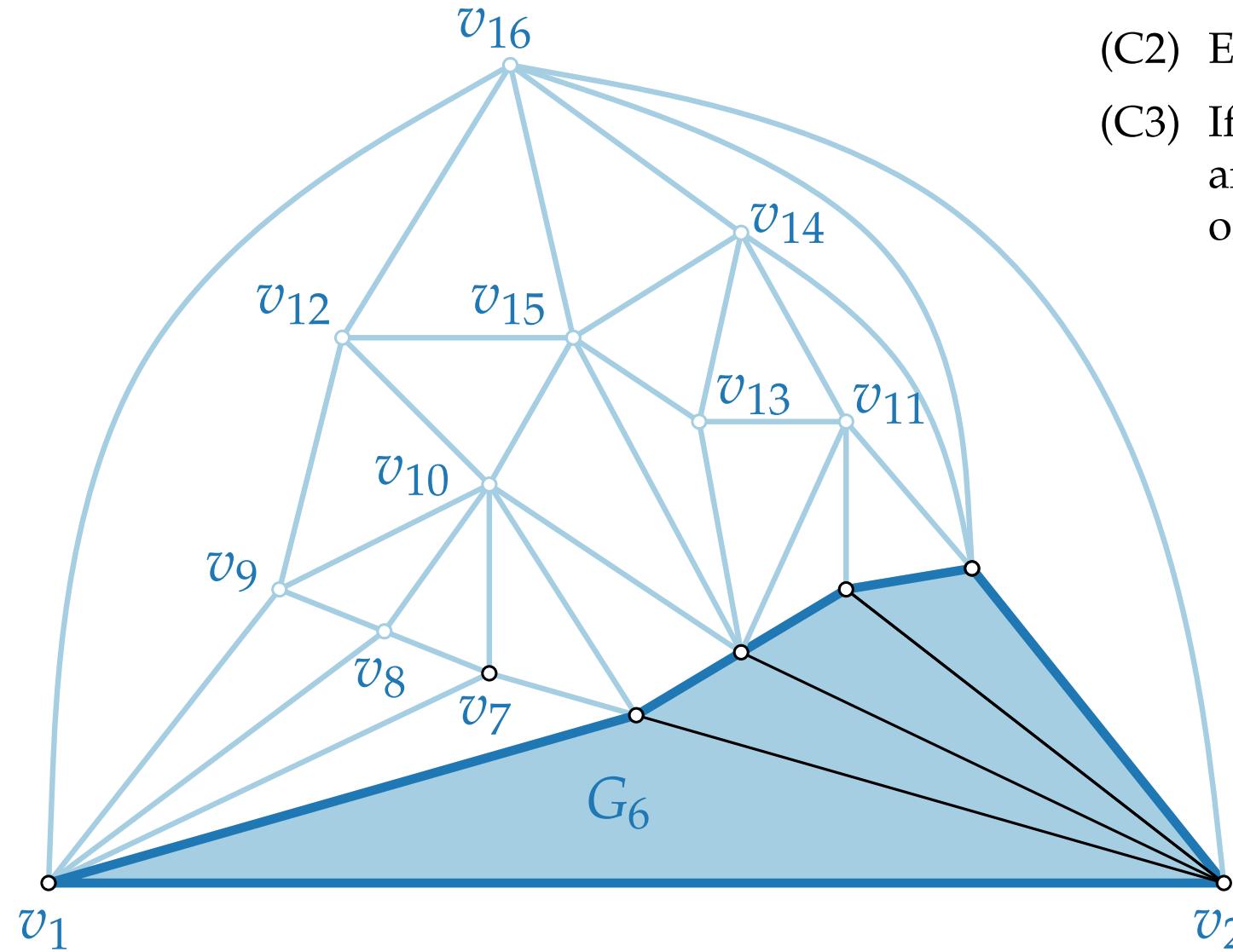
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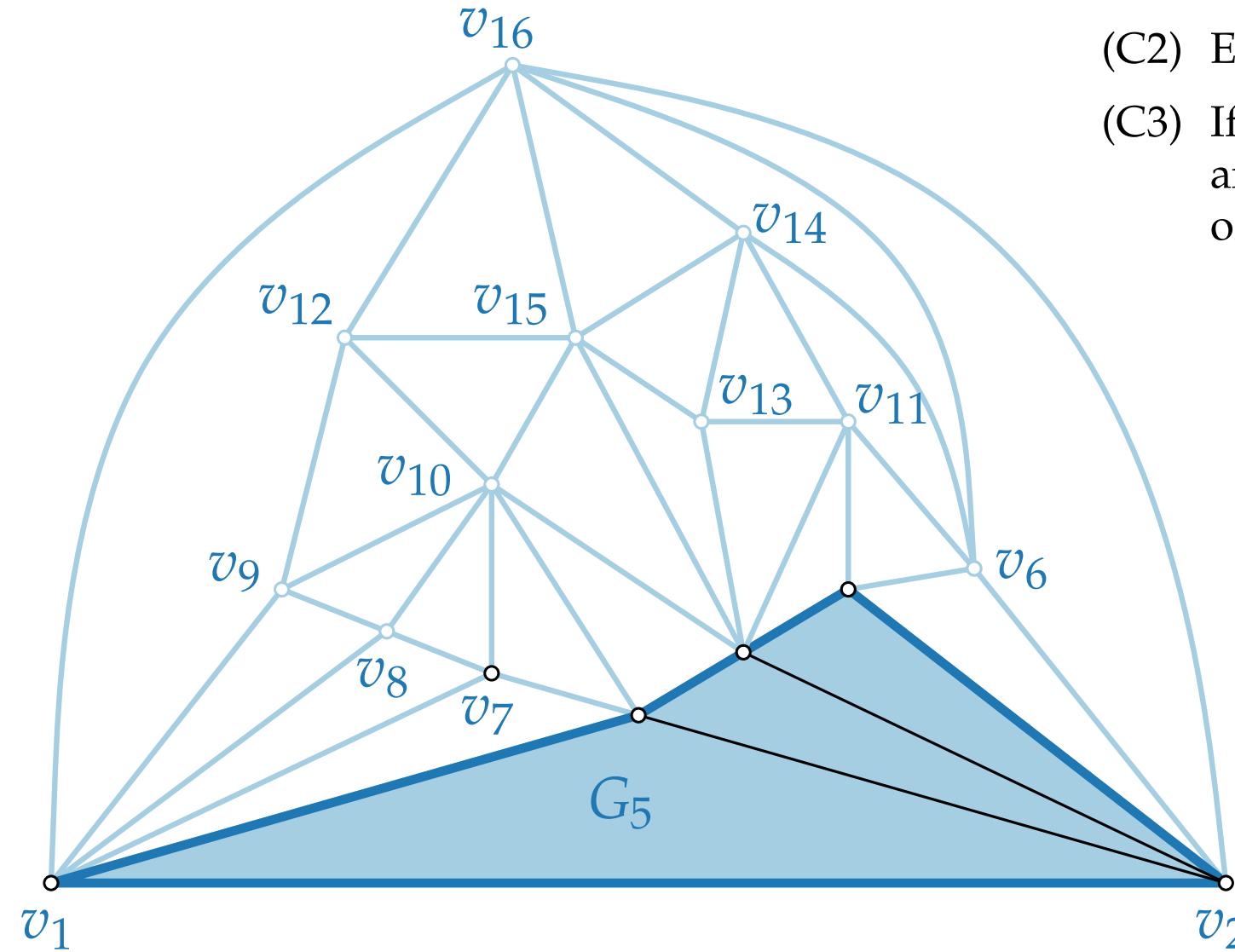
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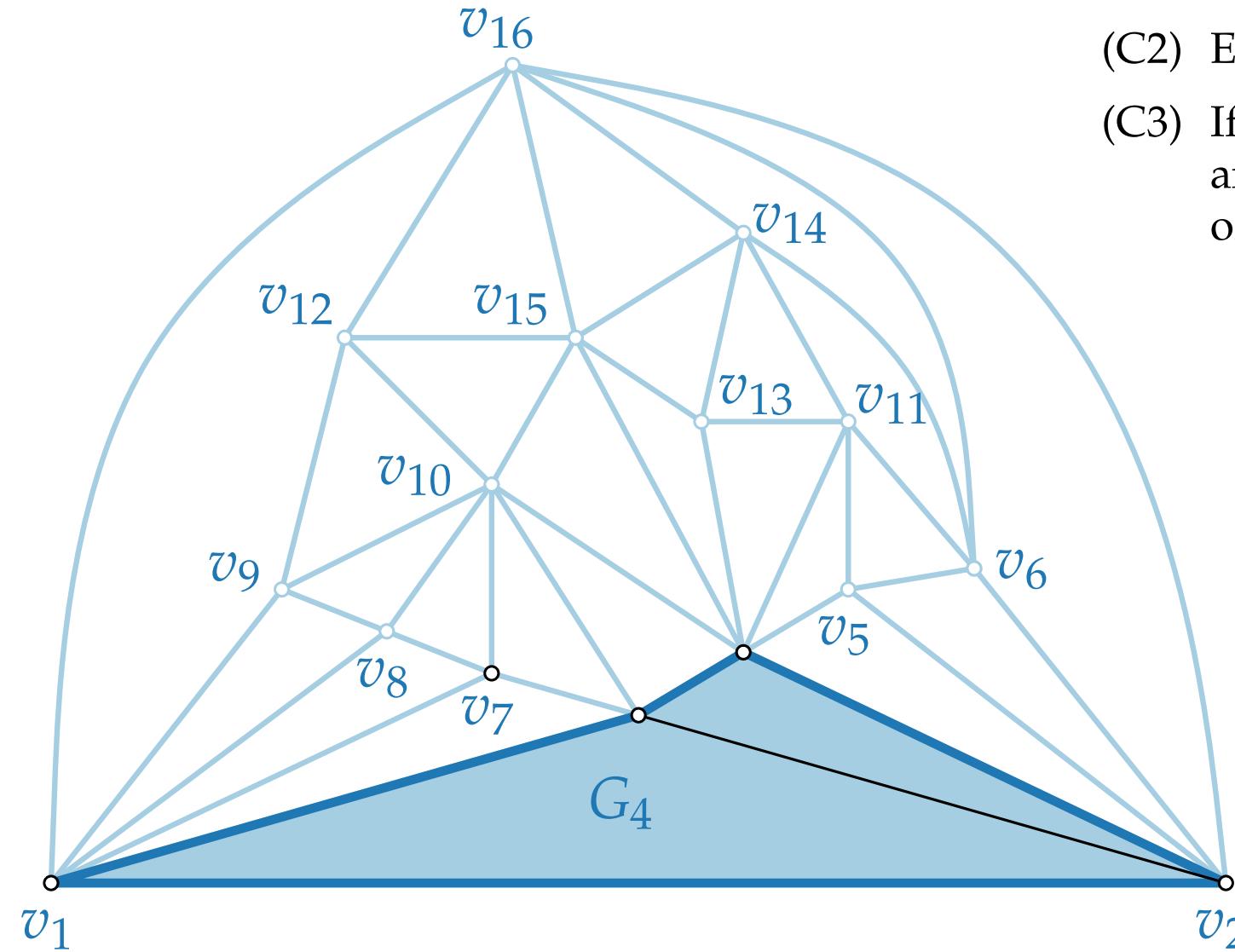
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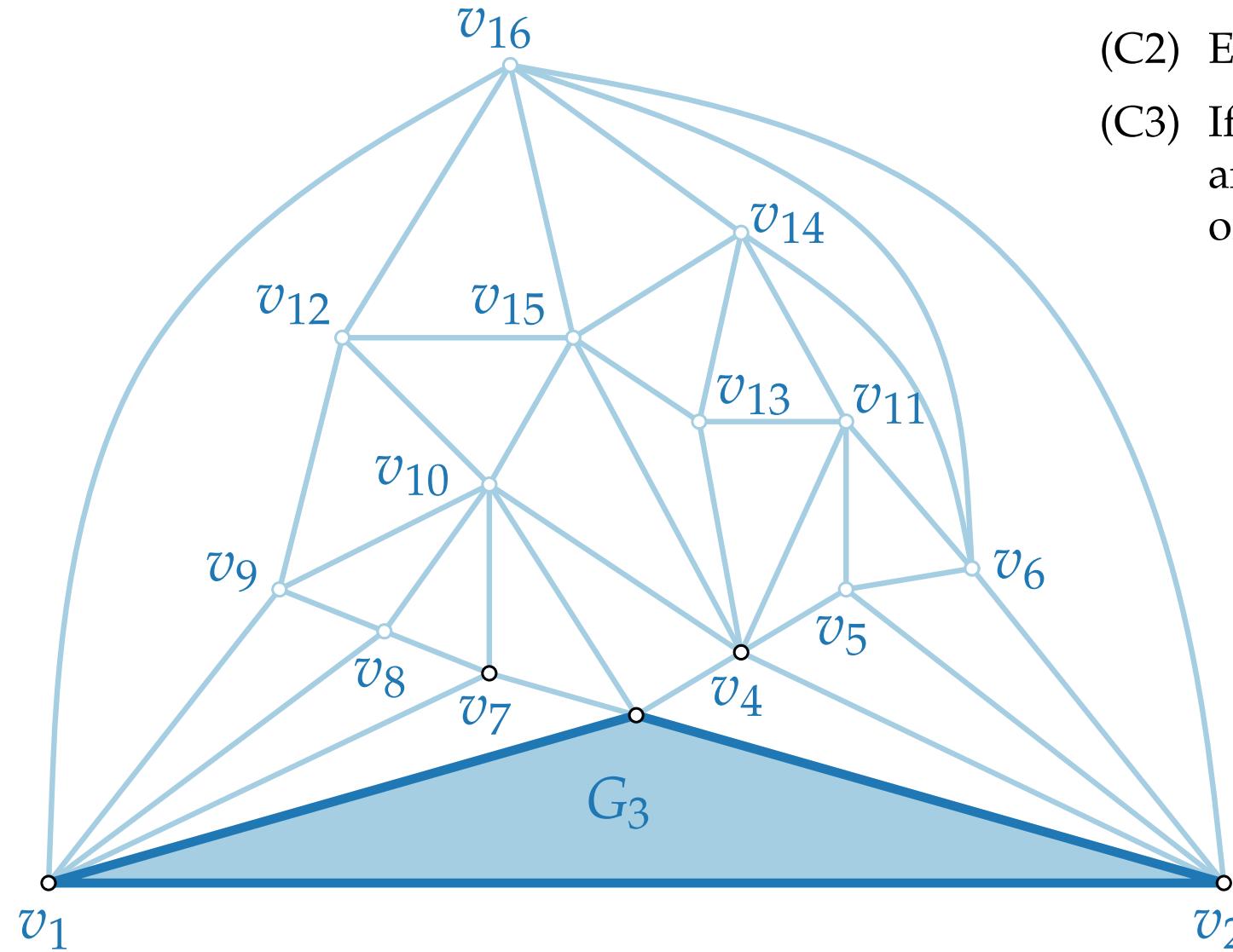
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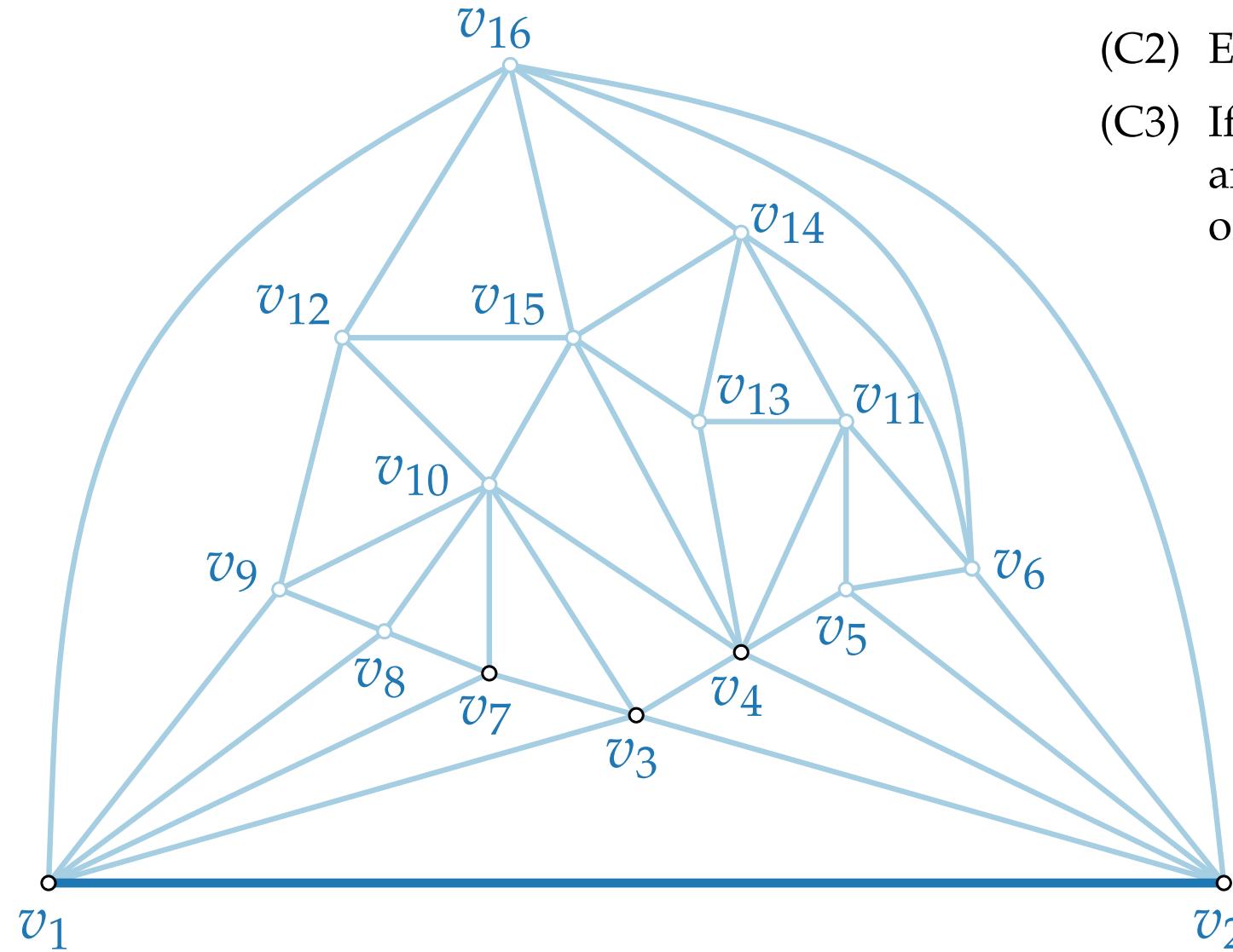
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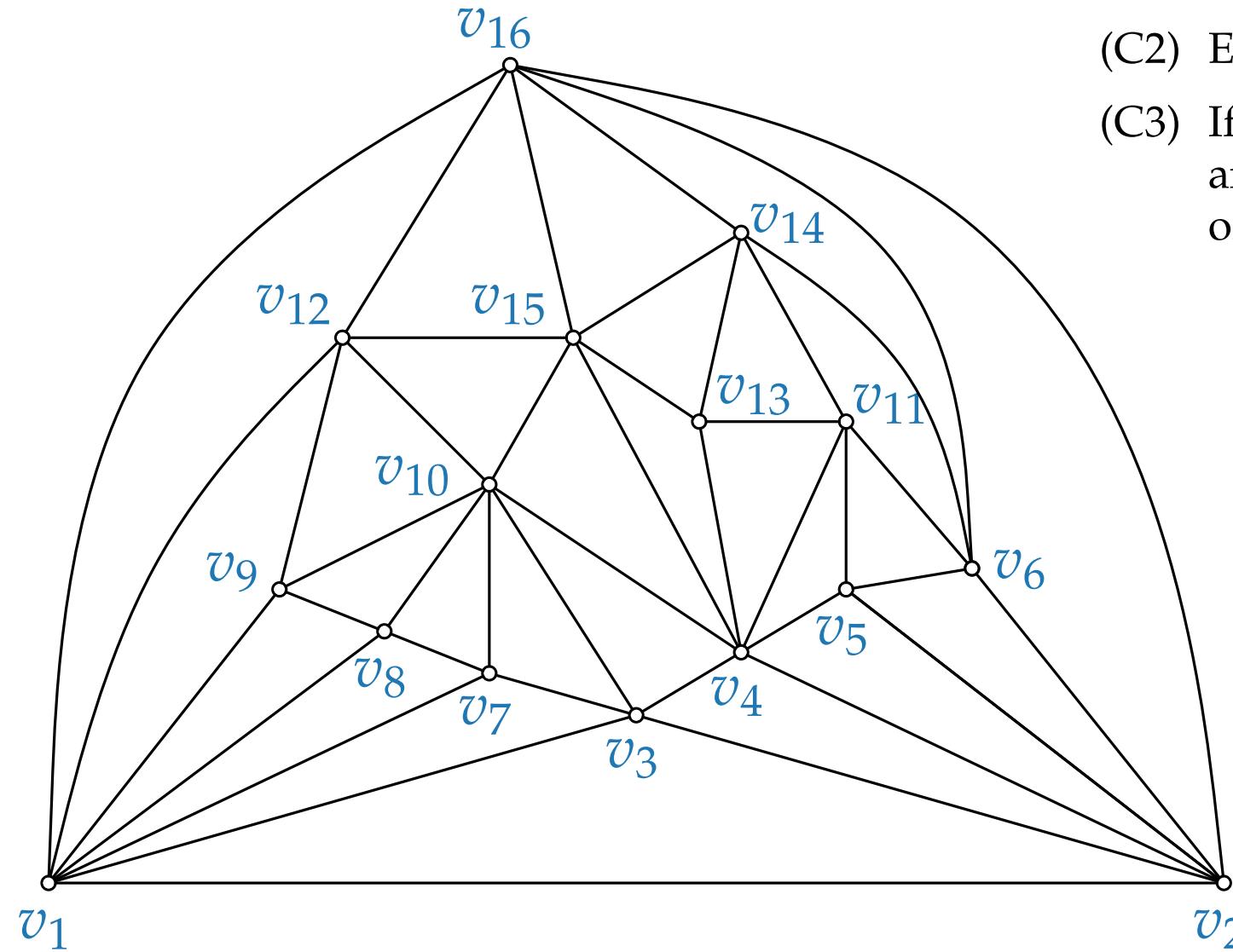
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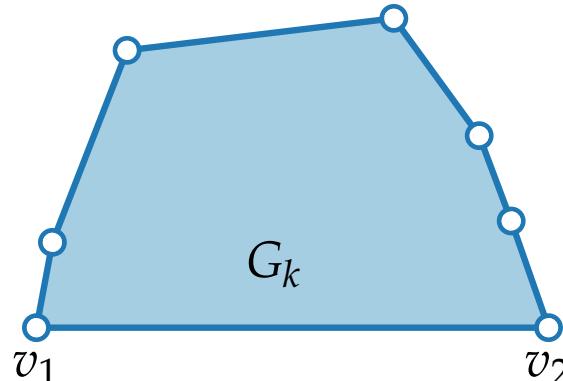
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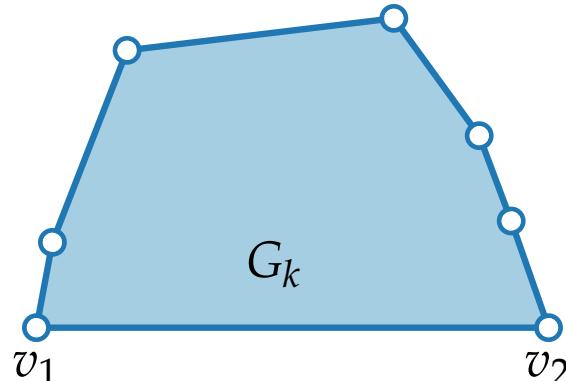
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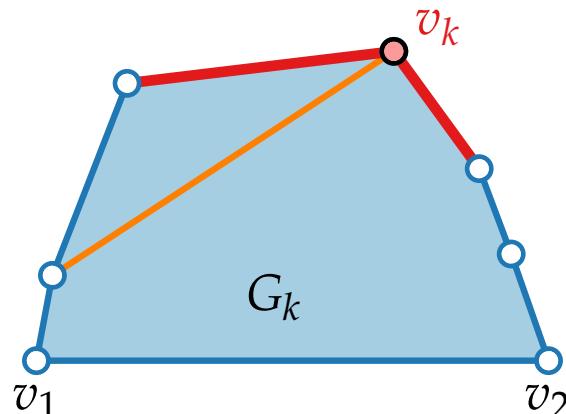
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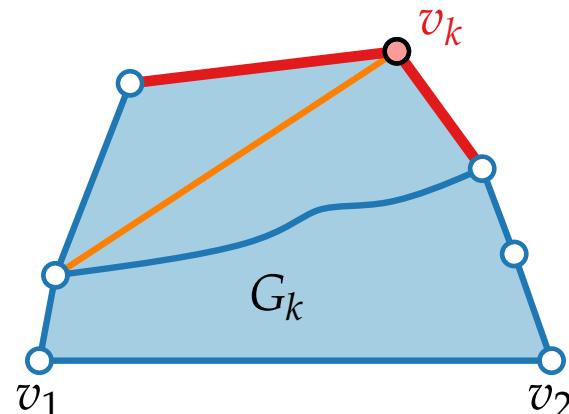
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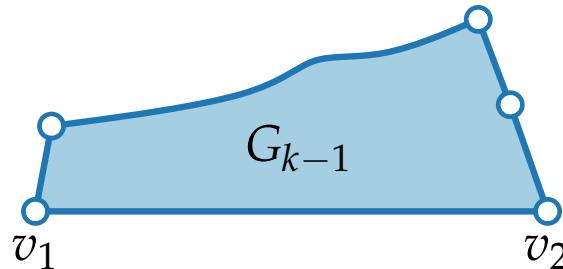
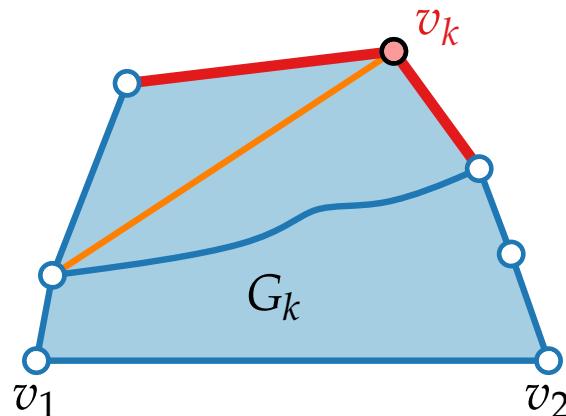
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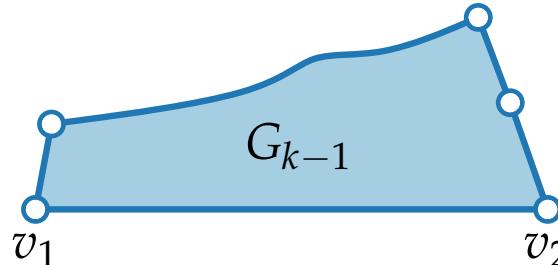
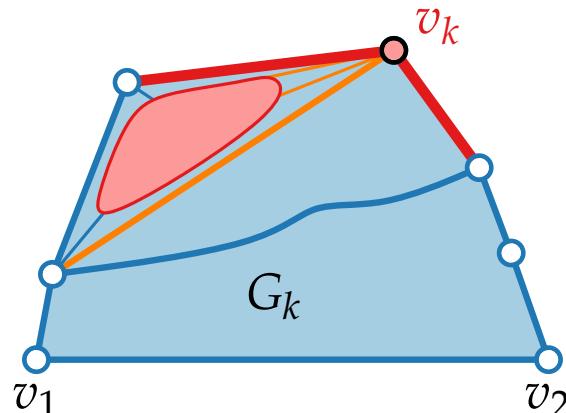
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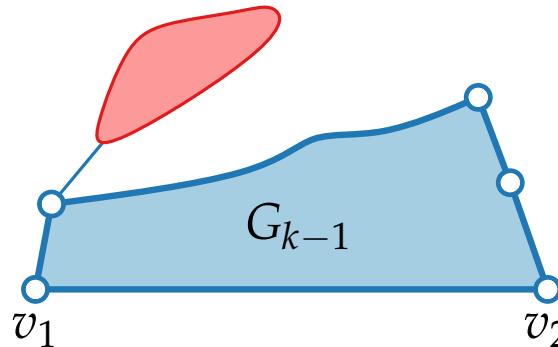
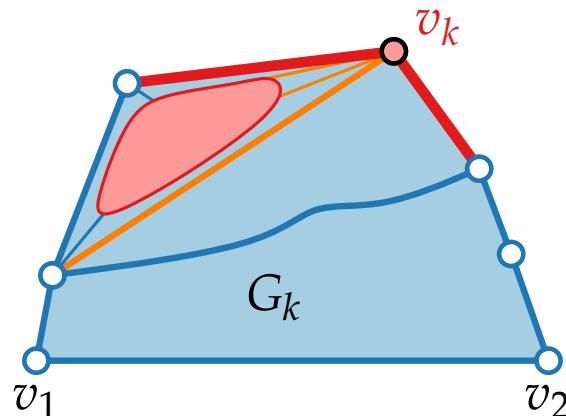
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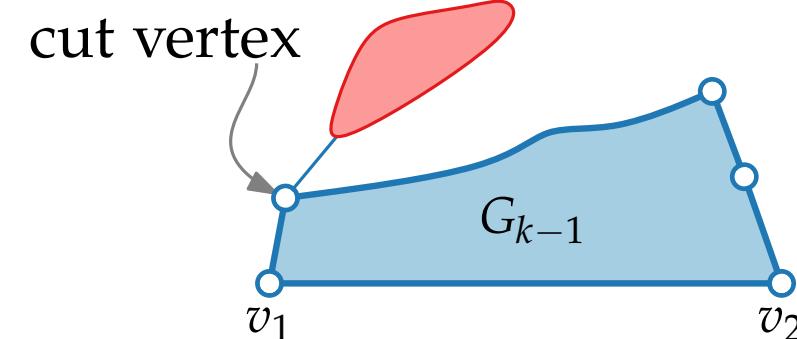
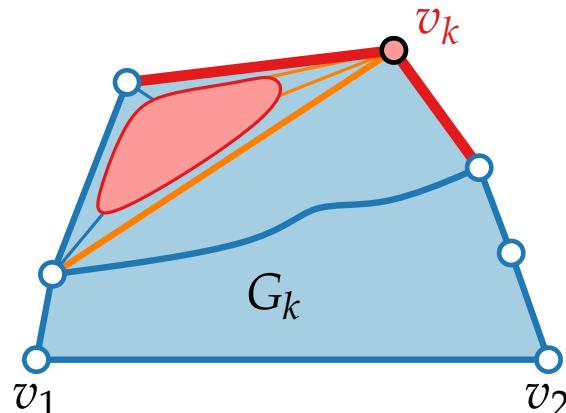
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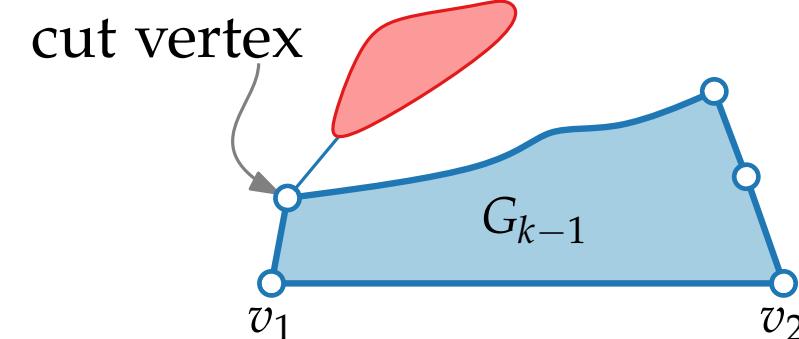
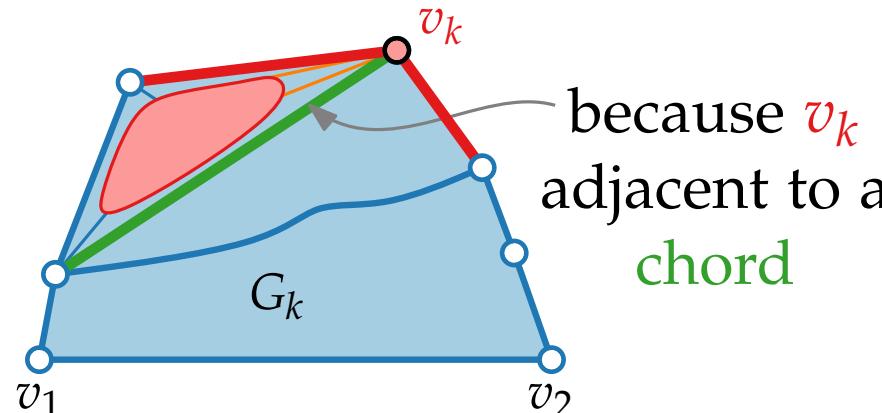
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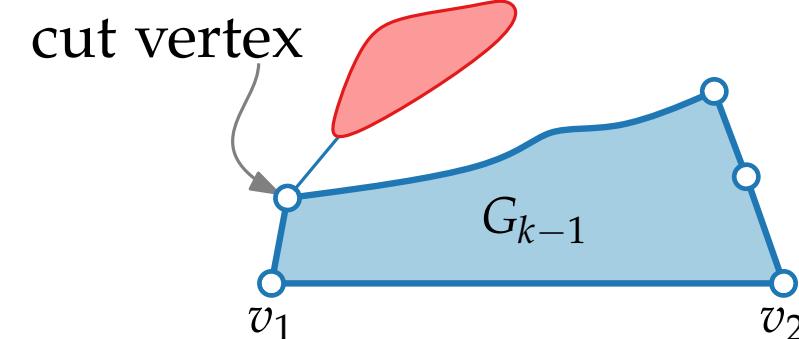
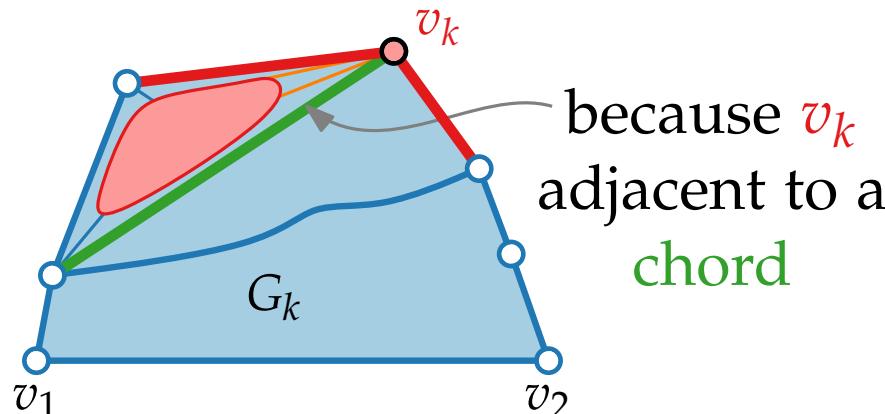
Have to show:

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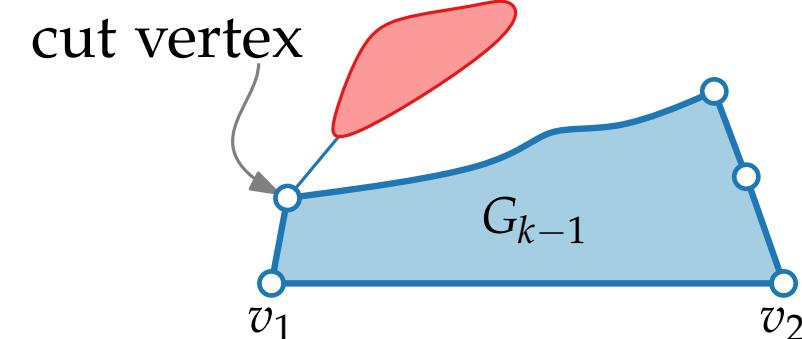
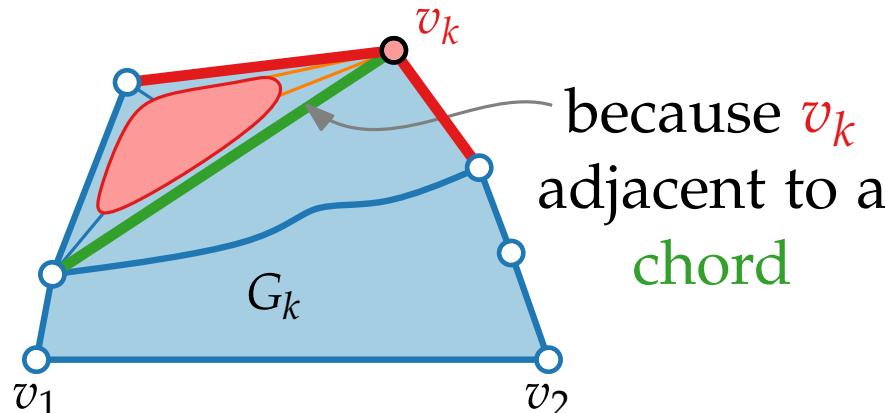
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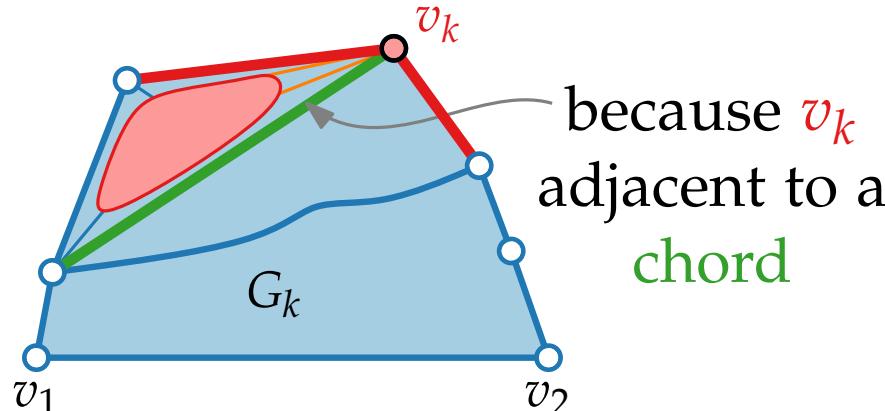
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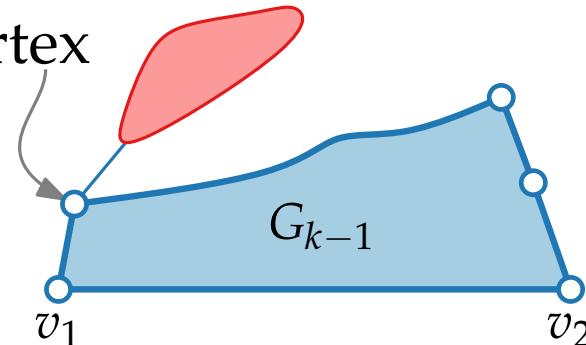
(C1) G_k biconnected and internally triangulated

(C2) (v_1, v_2) on outer face of G_k

(C3) $k < n \Rightarrow v_{k+1}$ in outer face of G_k , neighbors of v_{k+1} in G_k consecutive on boundary

Have to show:

- v_k not adjacent to chord is sufficient
- Such v_k exists



Canonical Order – Existence

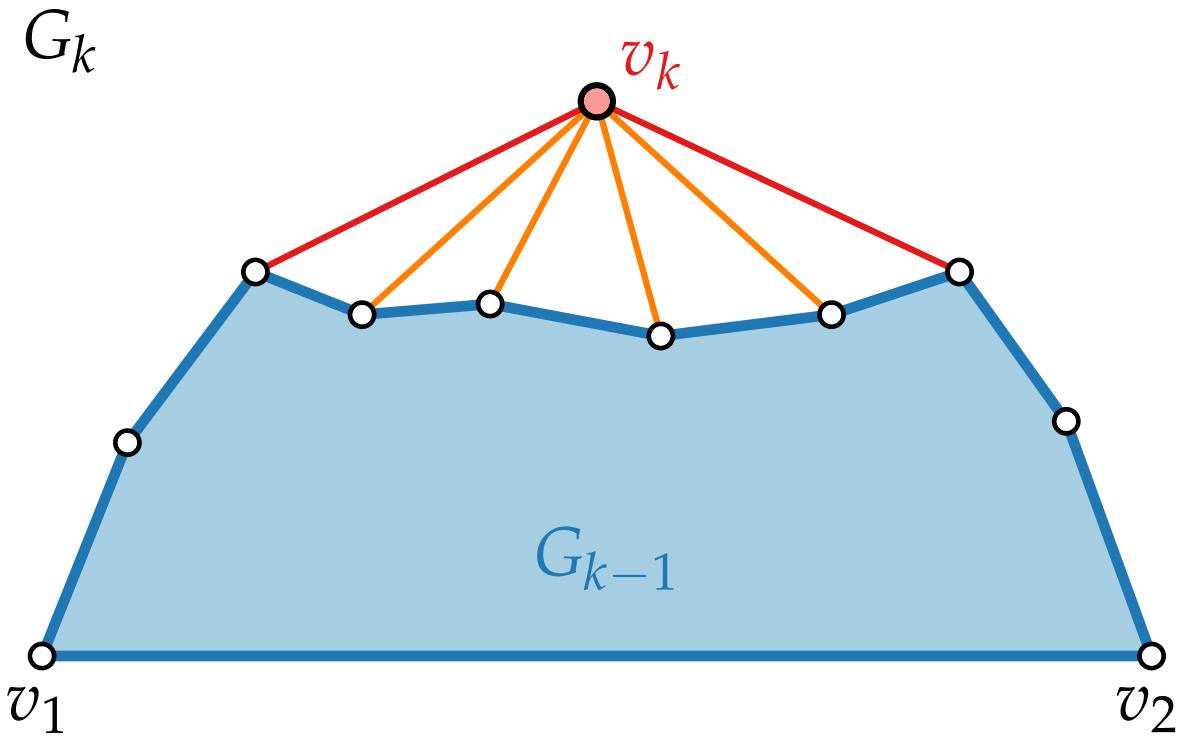
Claim 1.

If v_k is not adjacent to a chord,
then G_{k-1} is biconnected.

Canonical Order – Existence

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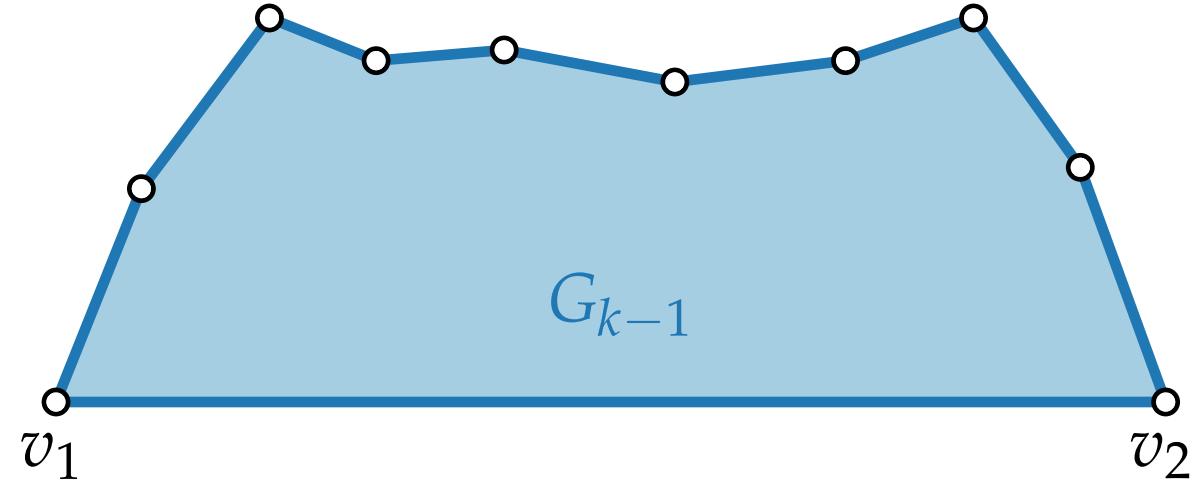
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Canonical Order – Existence

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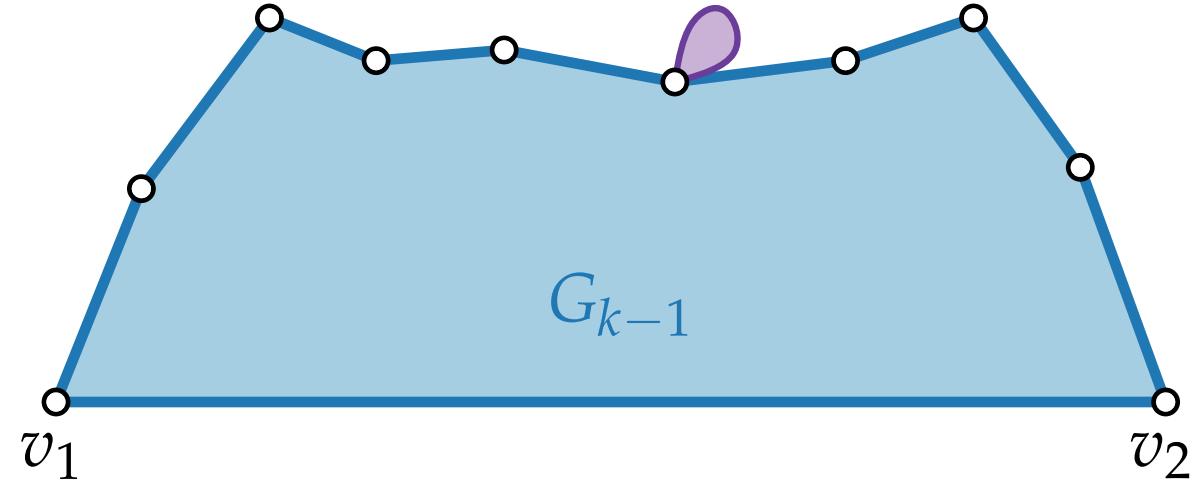
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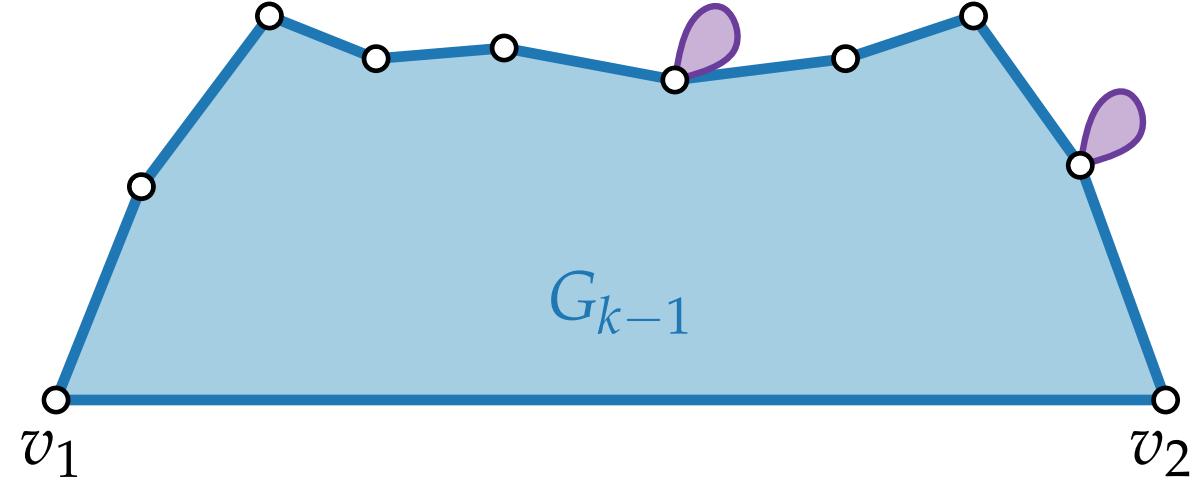
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Canonical Order – Existence

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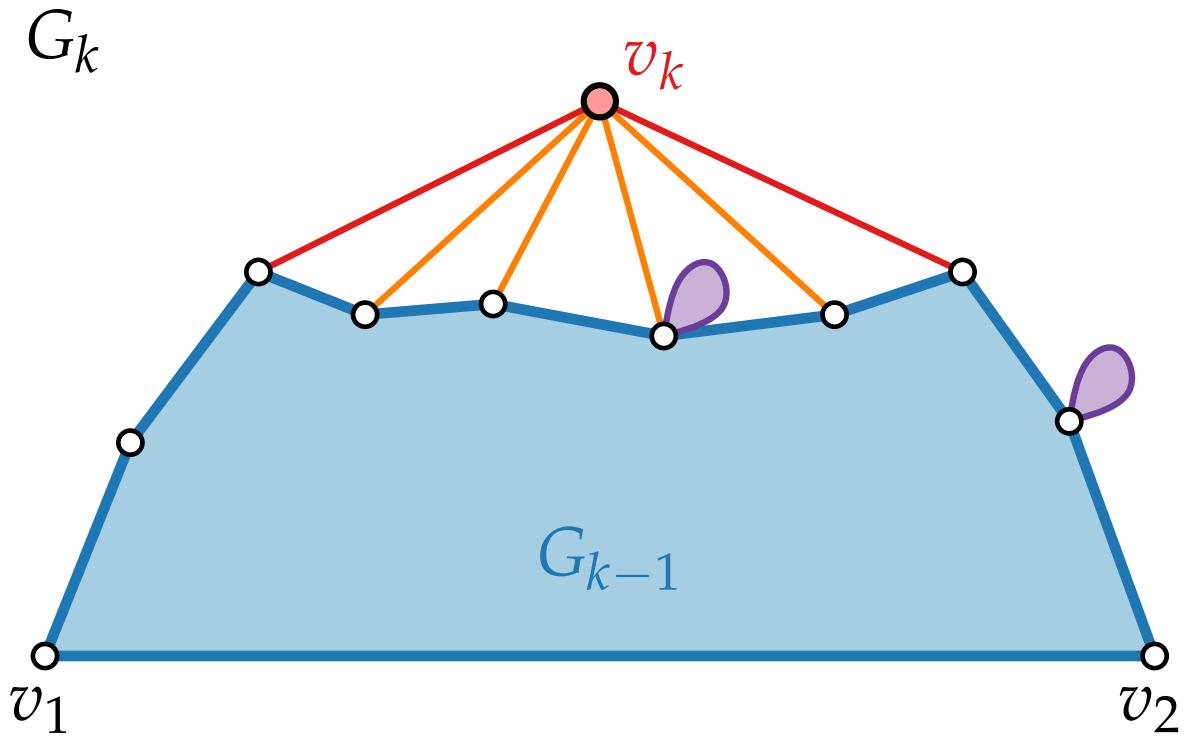
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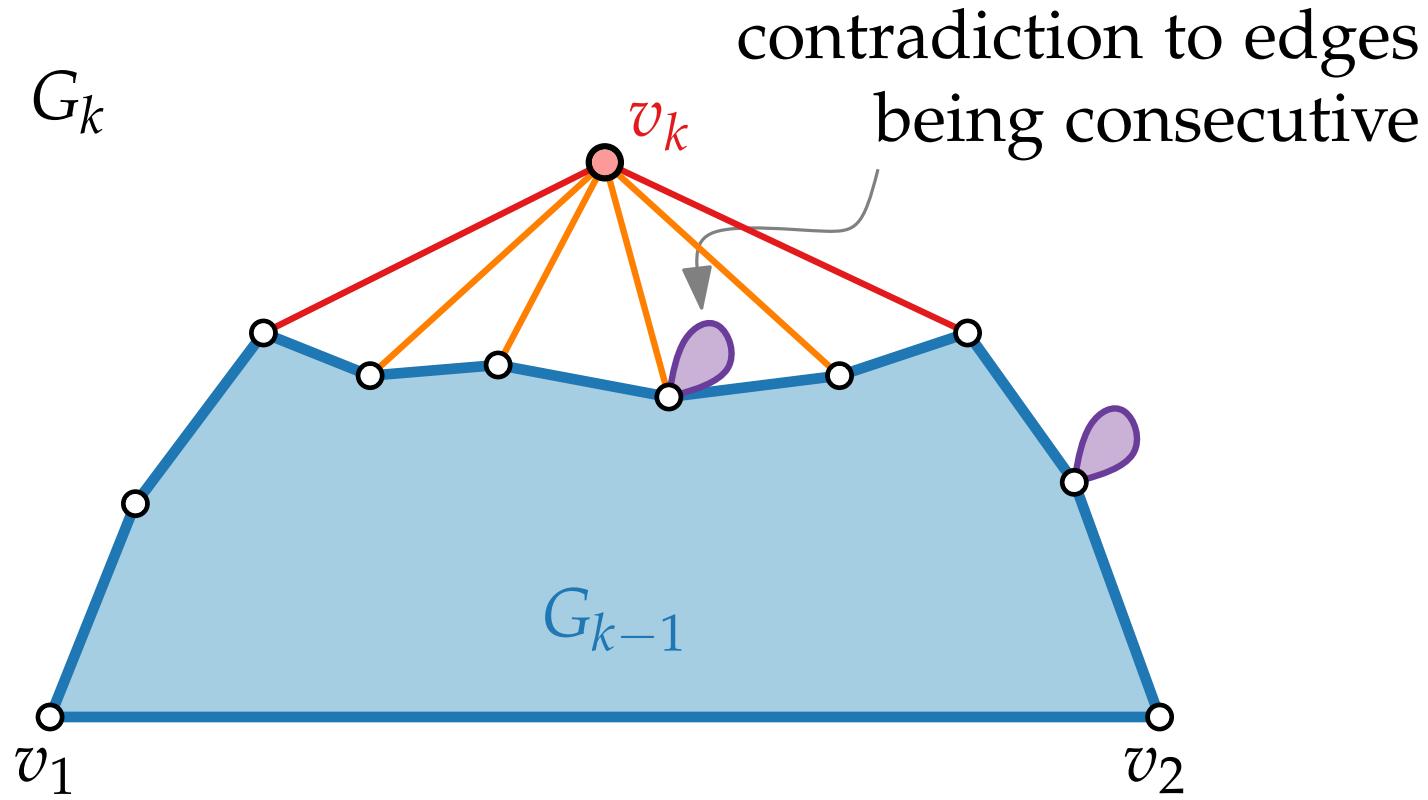
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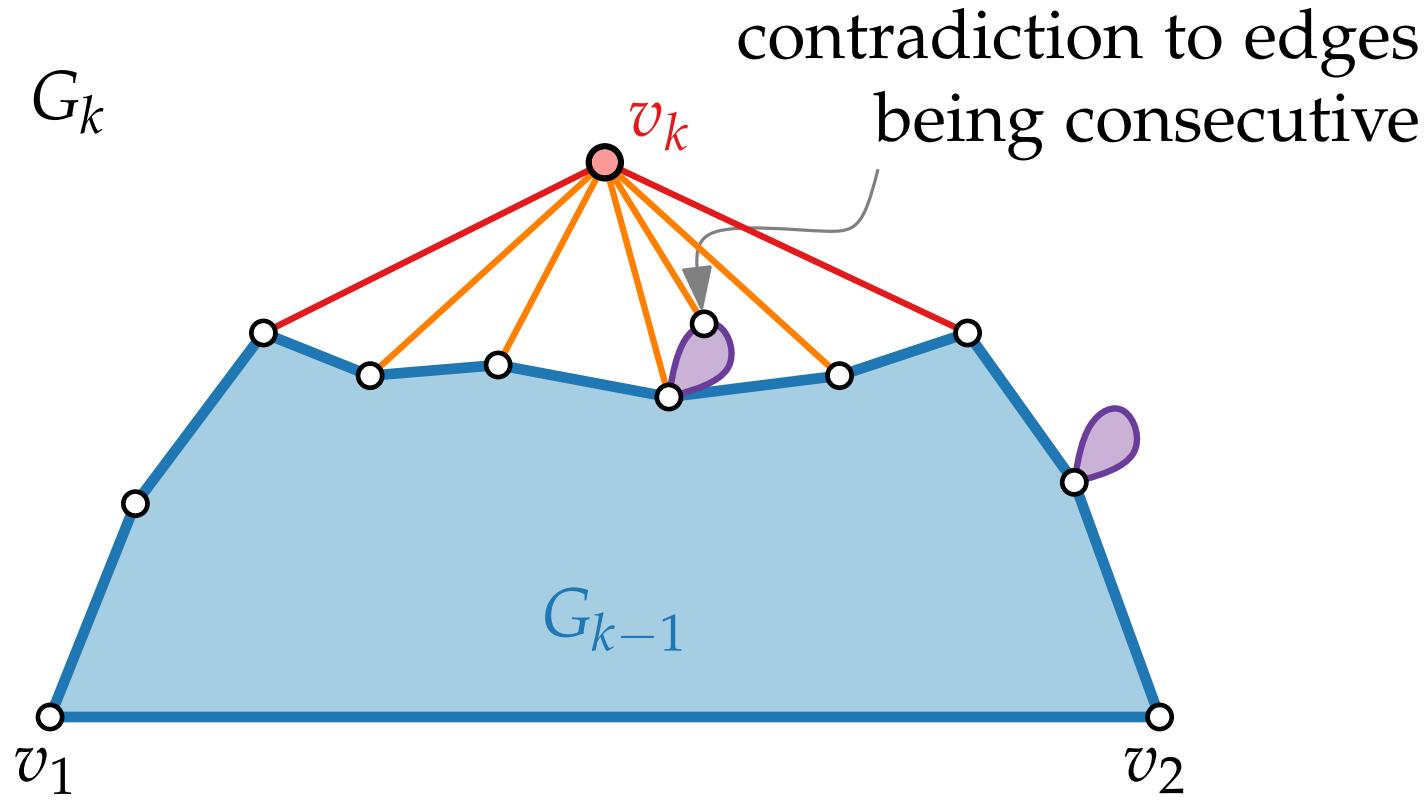
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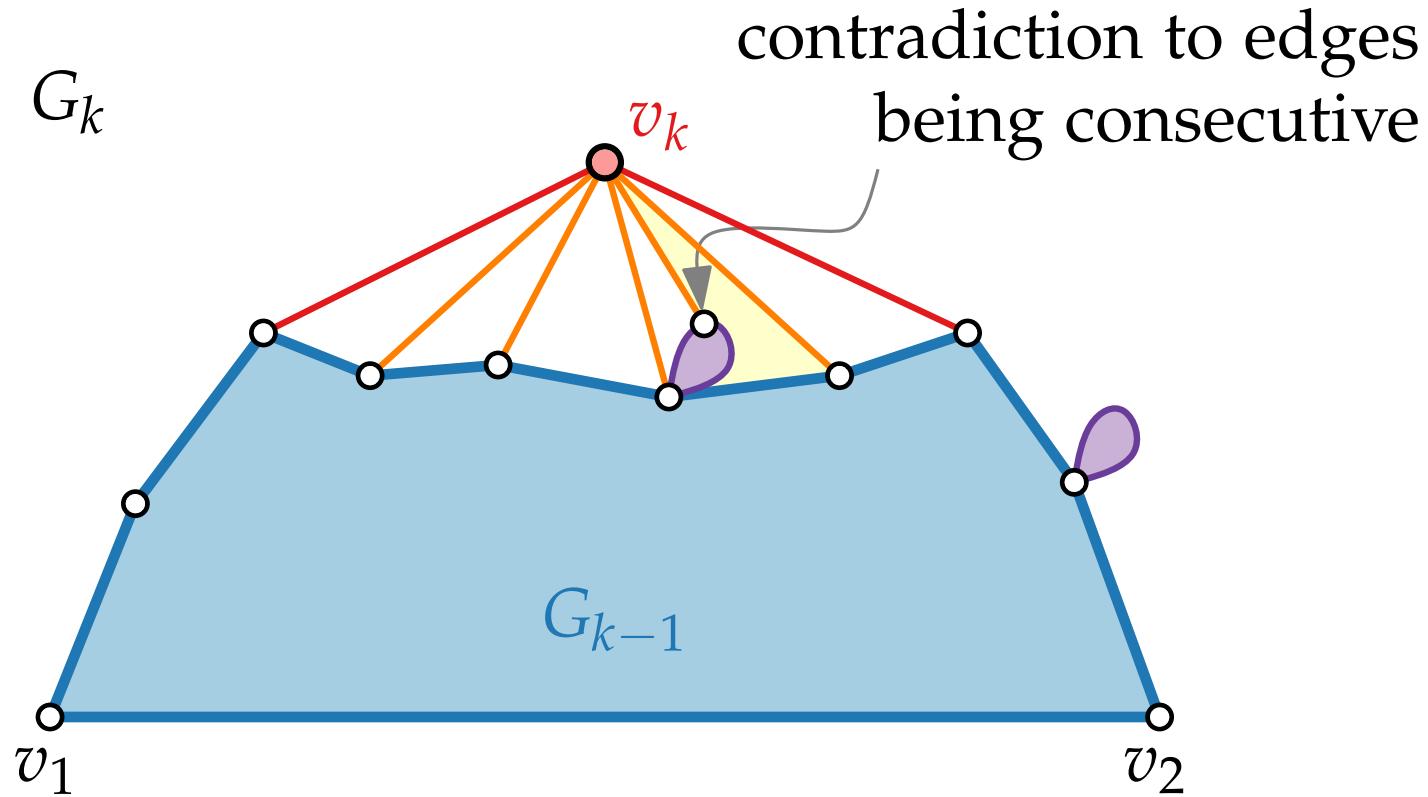
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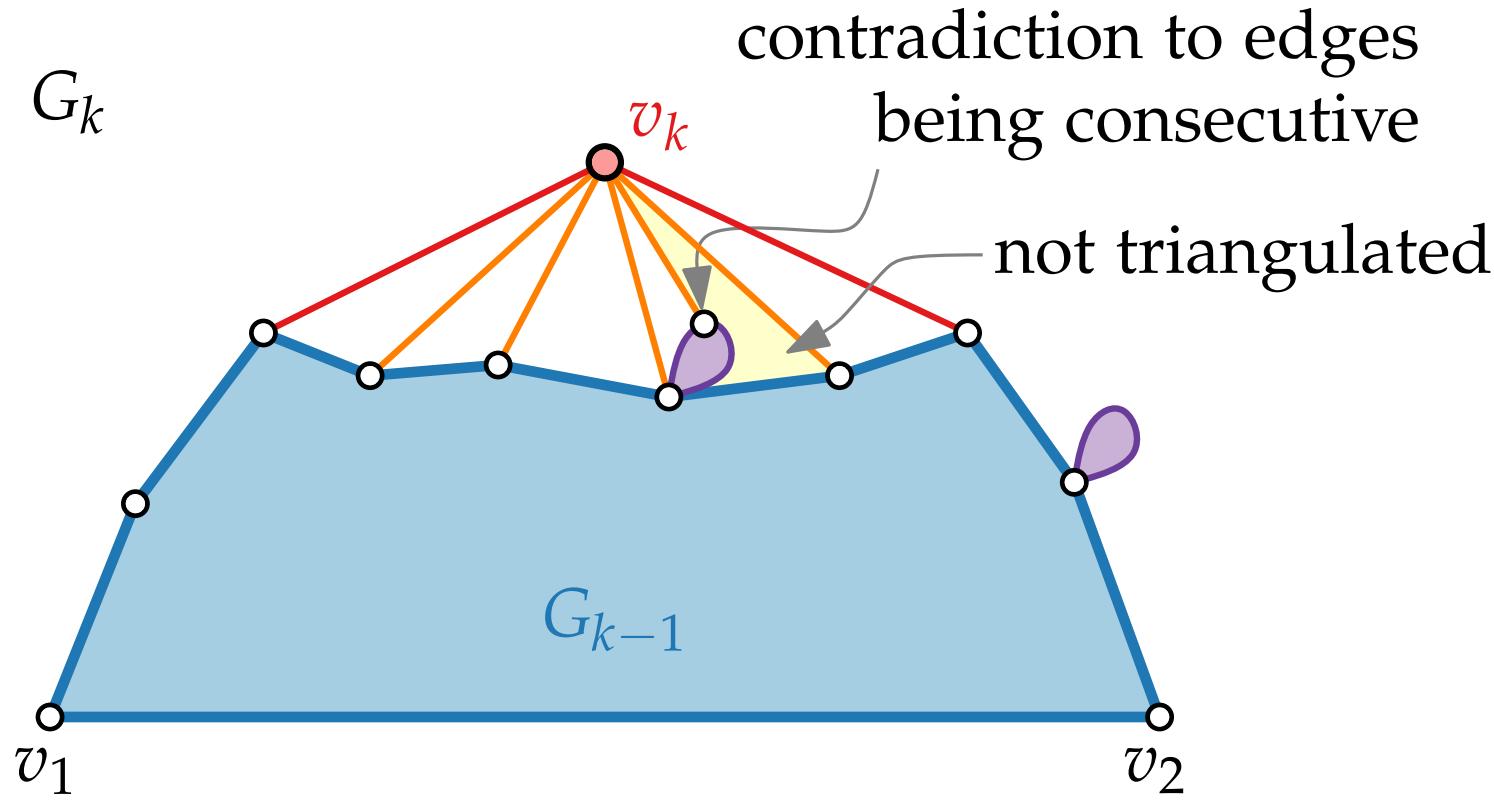
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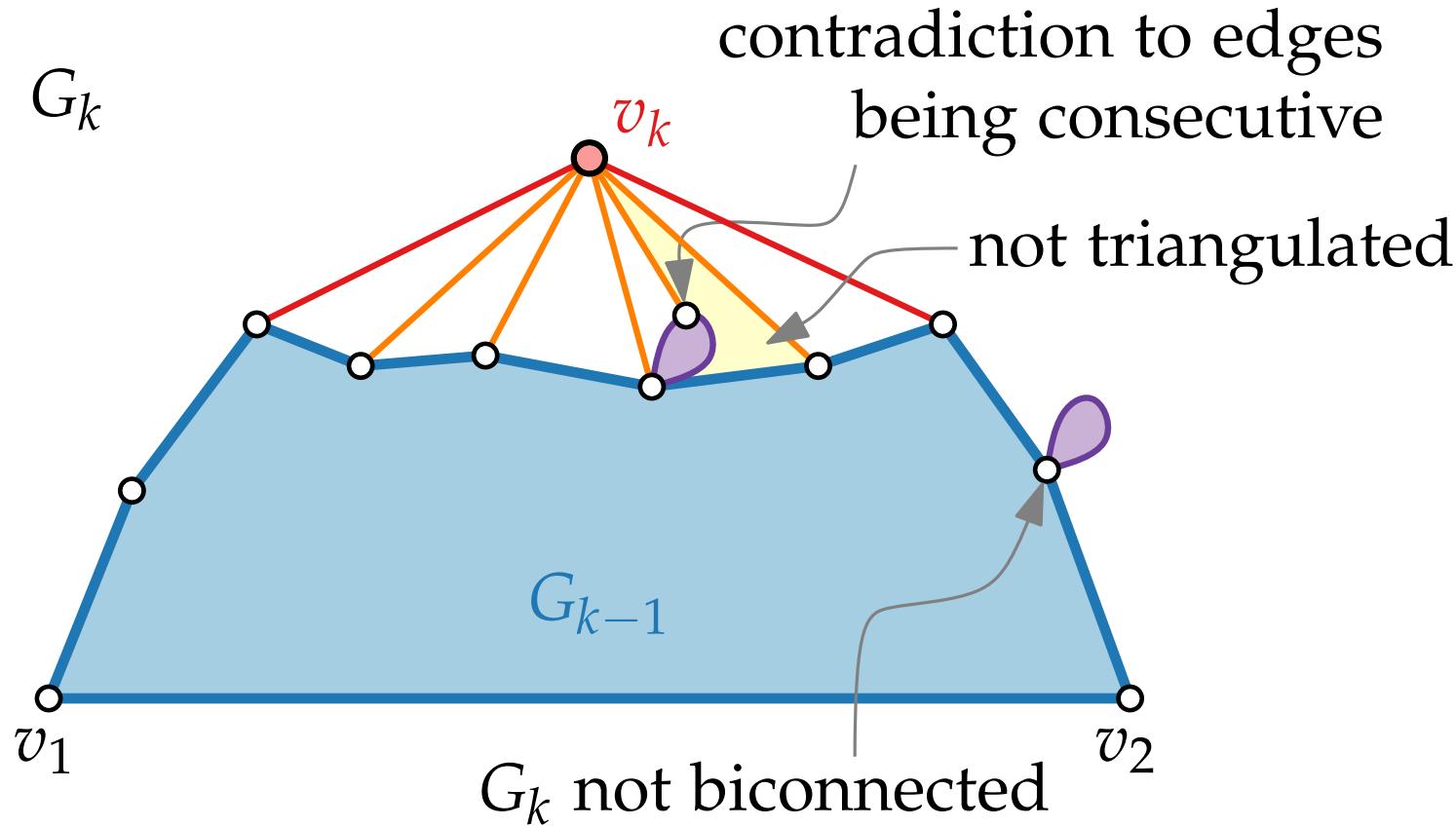
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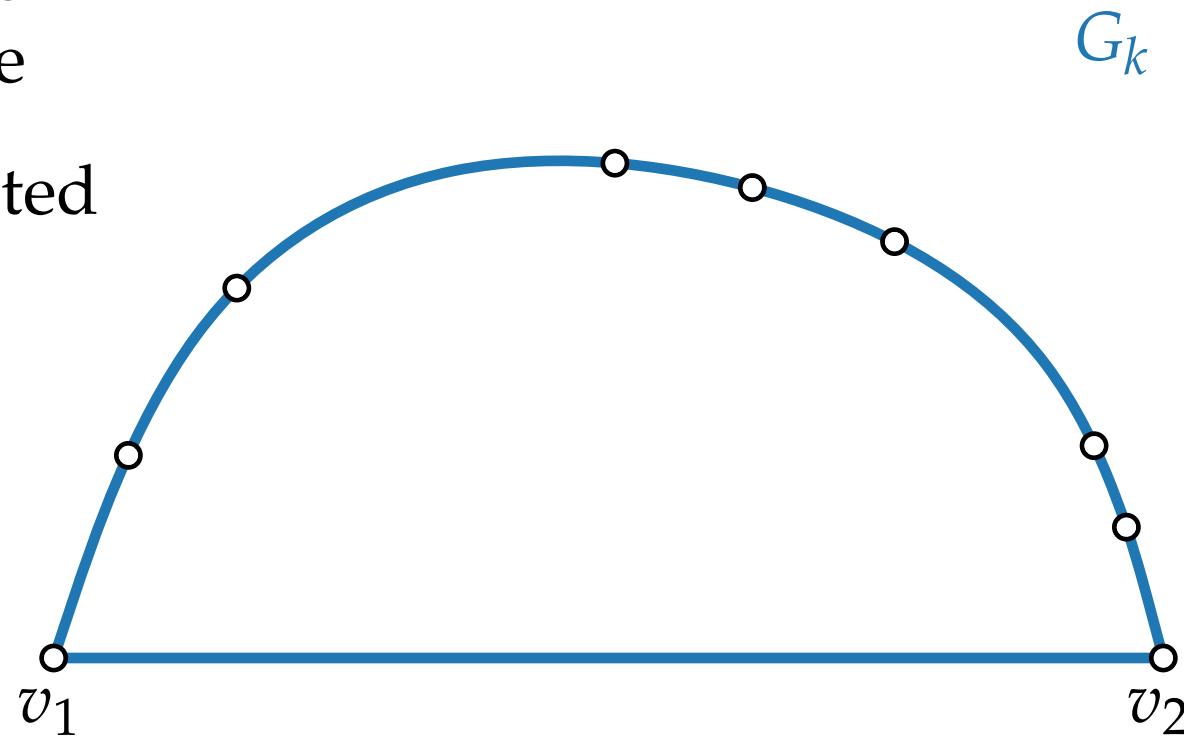
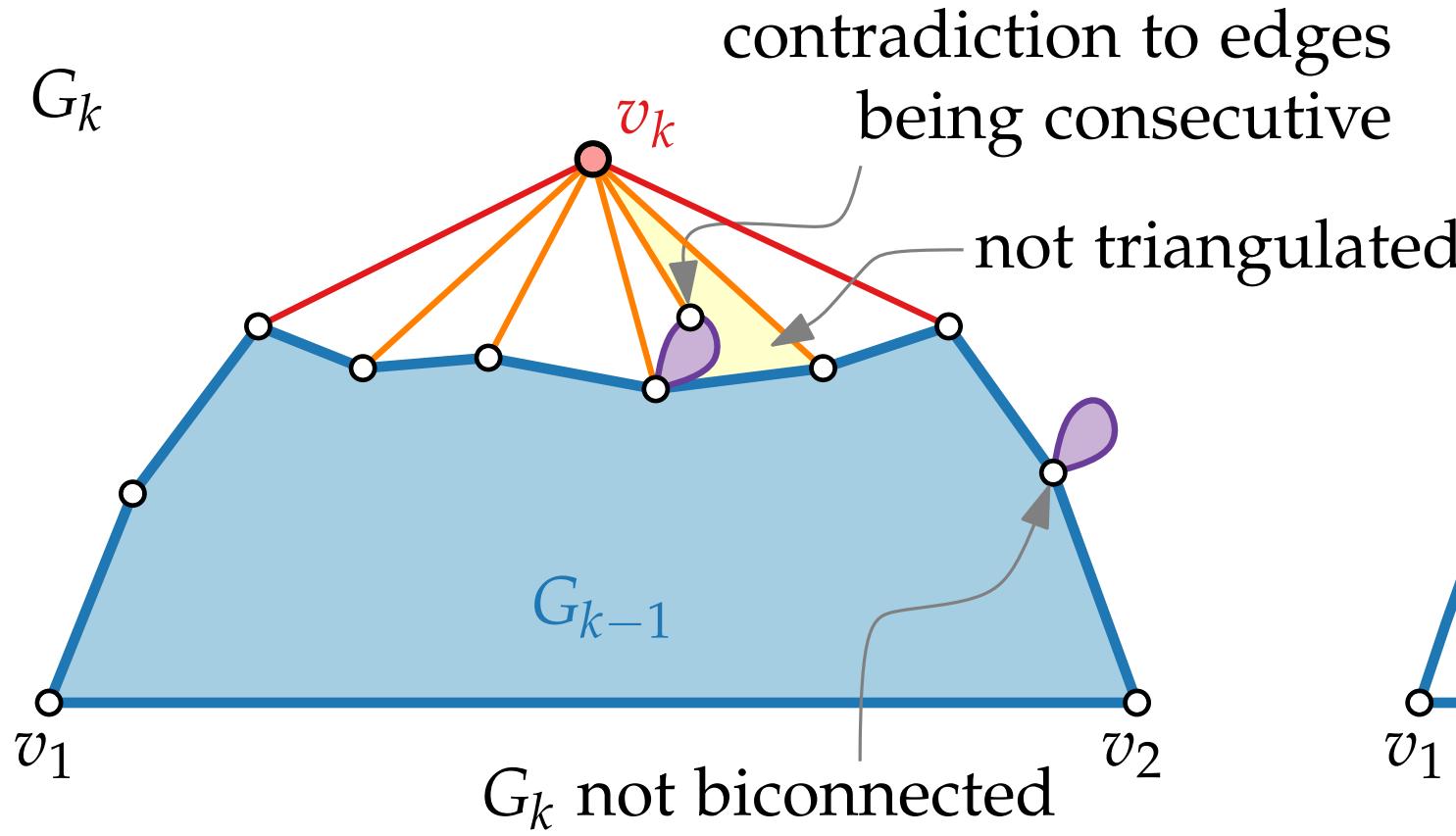
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There exists a vertex in G_k that is not adjacent to a chord as choice for v_k .



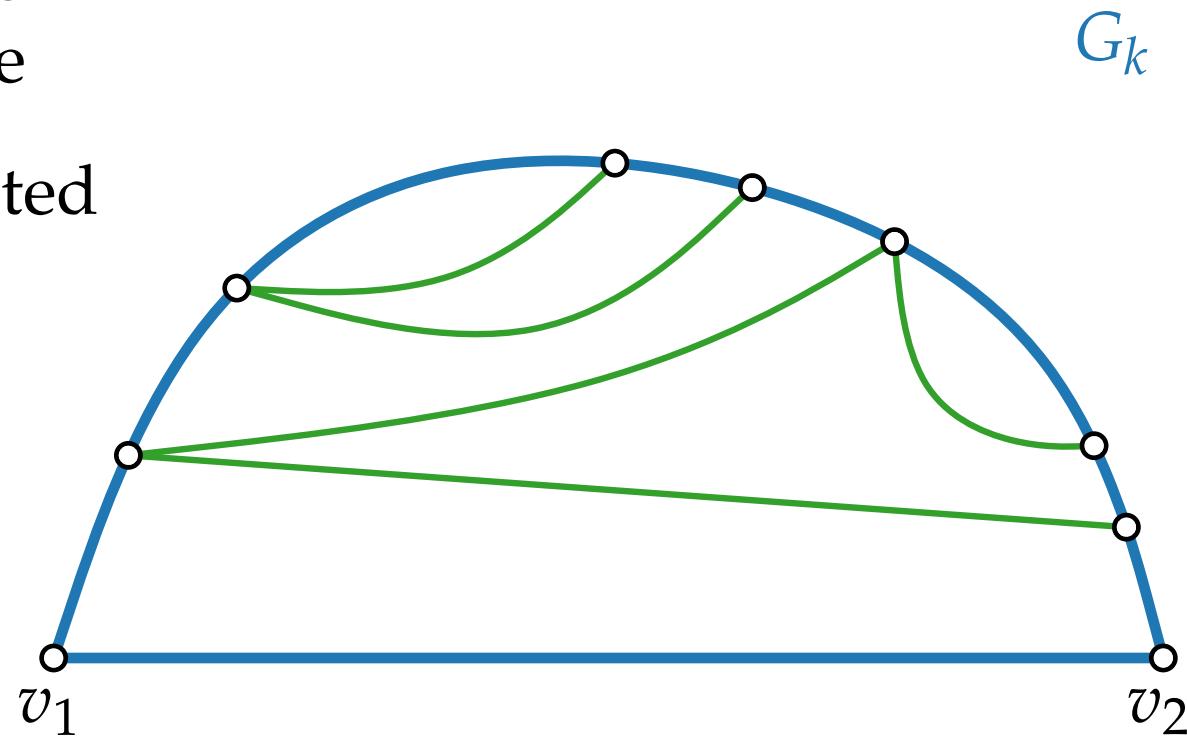
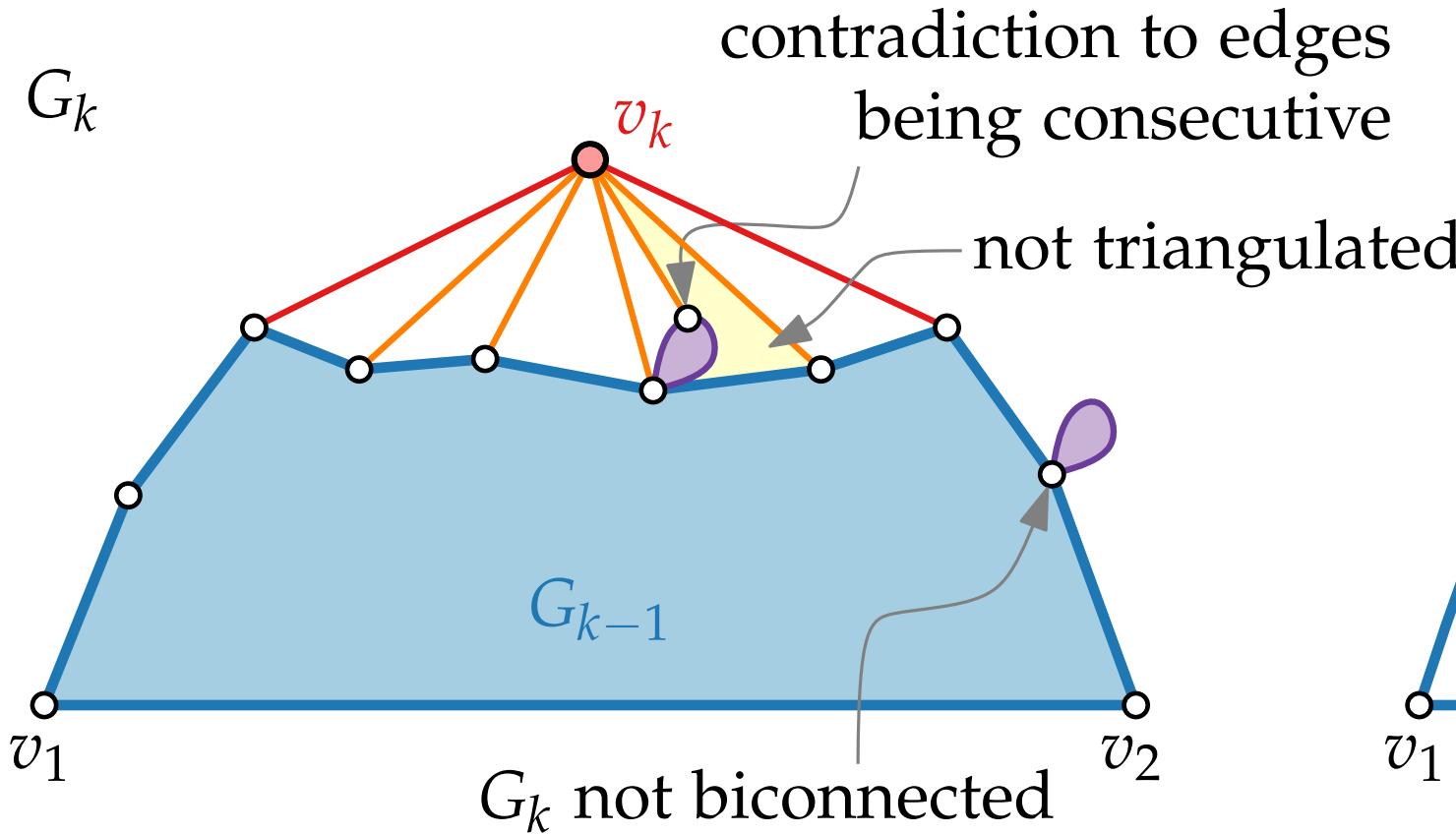
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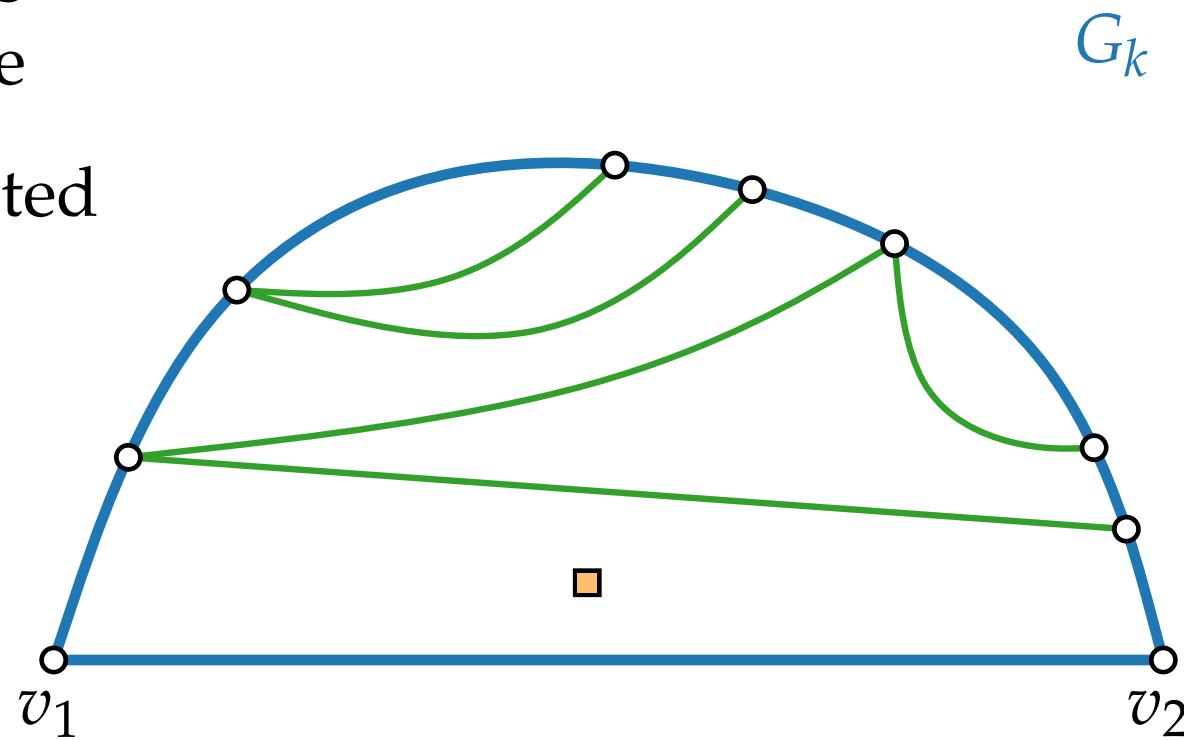
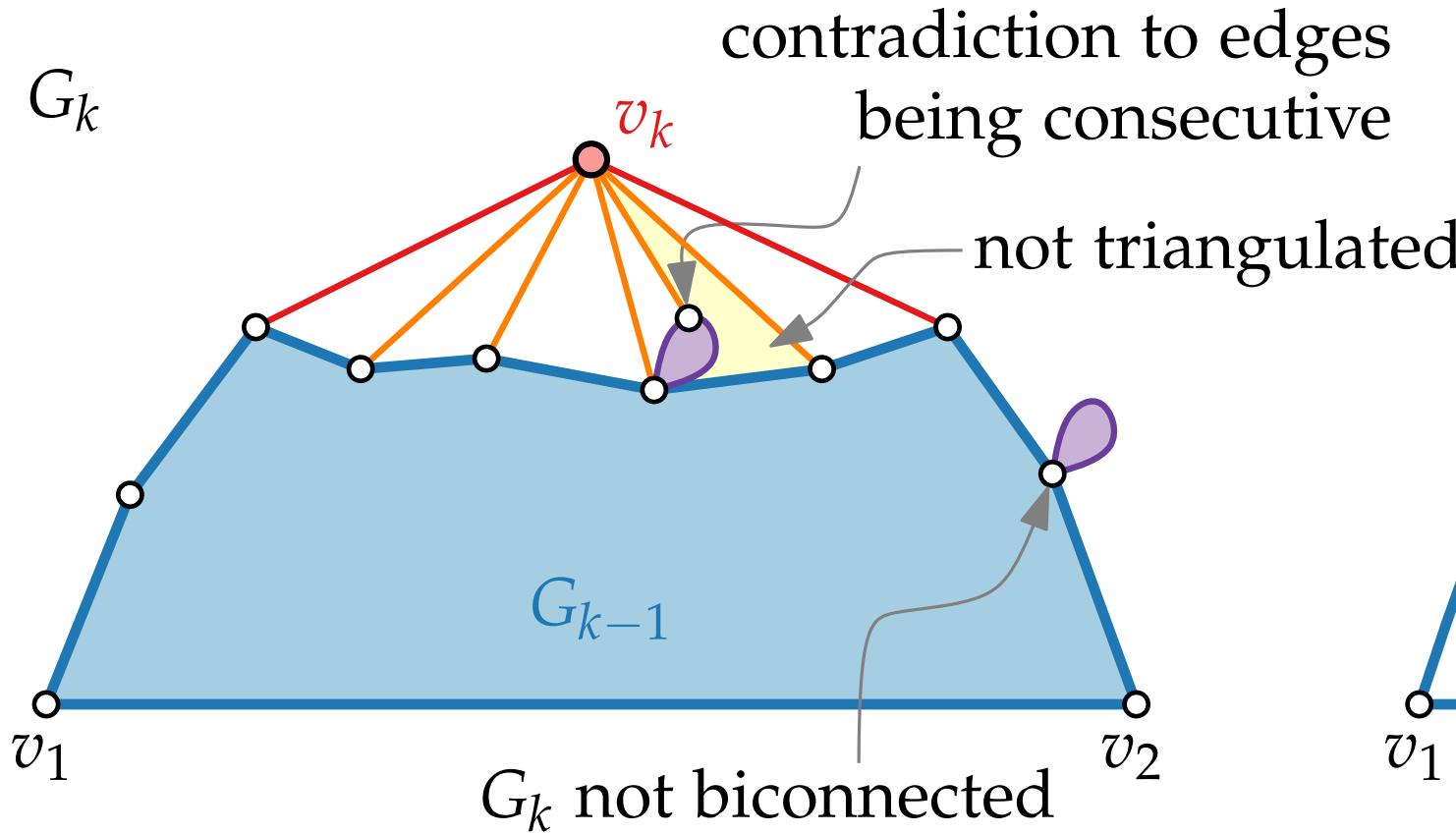
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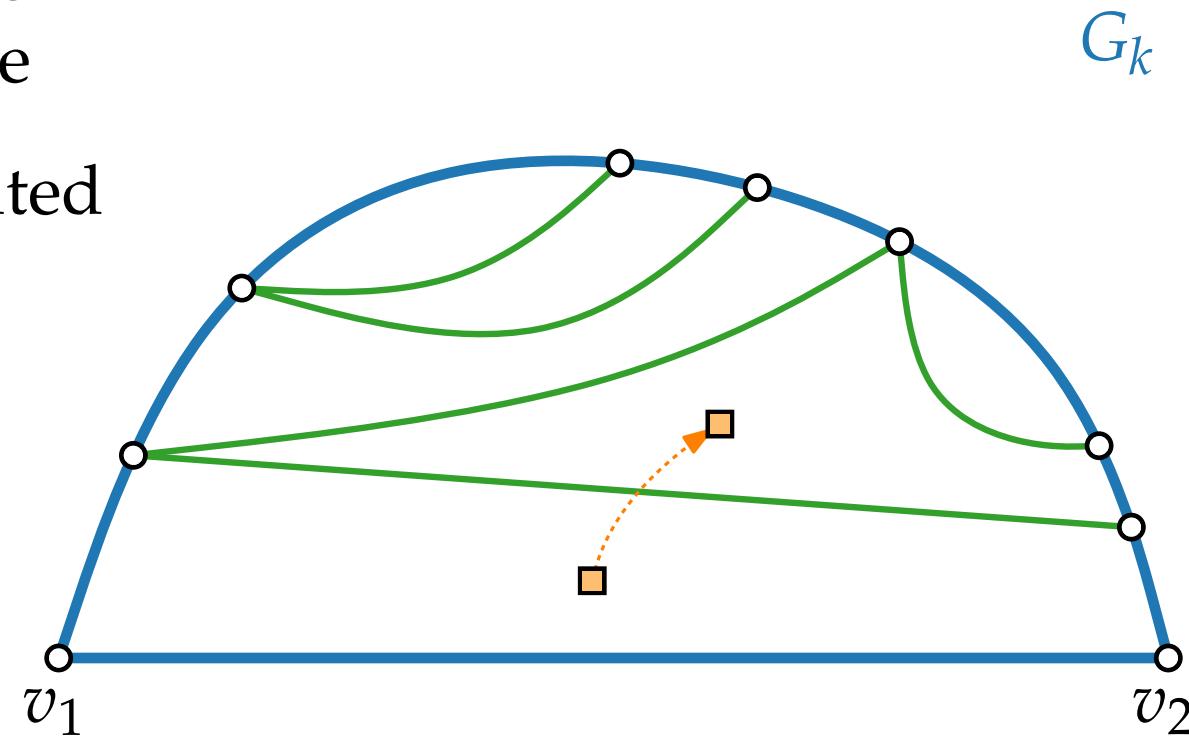
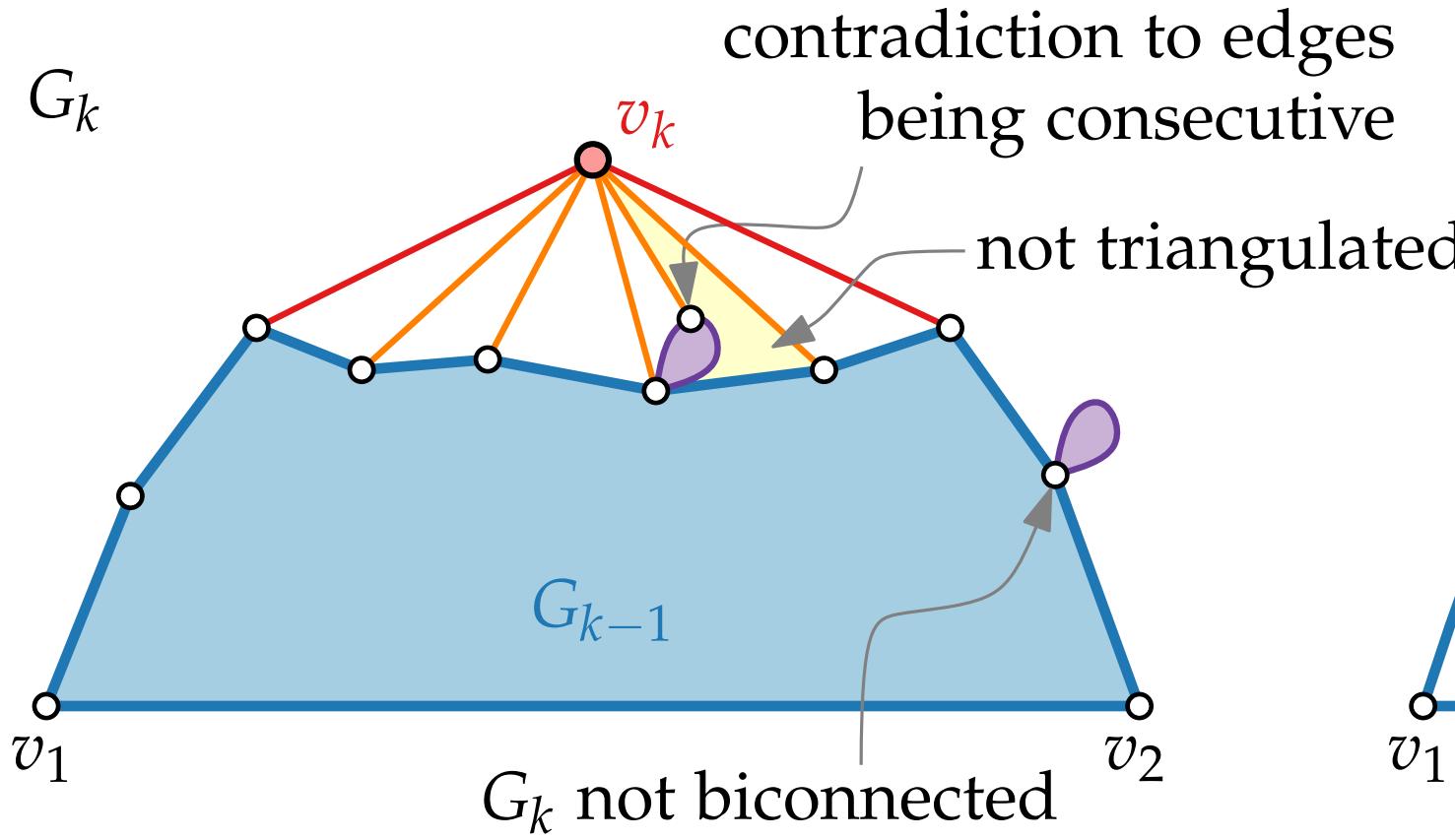
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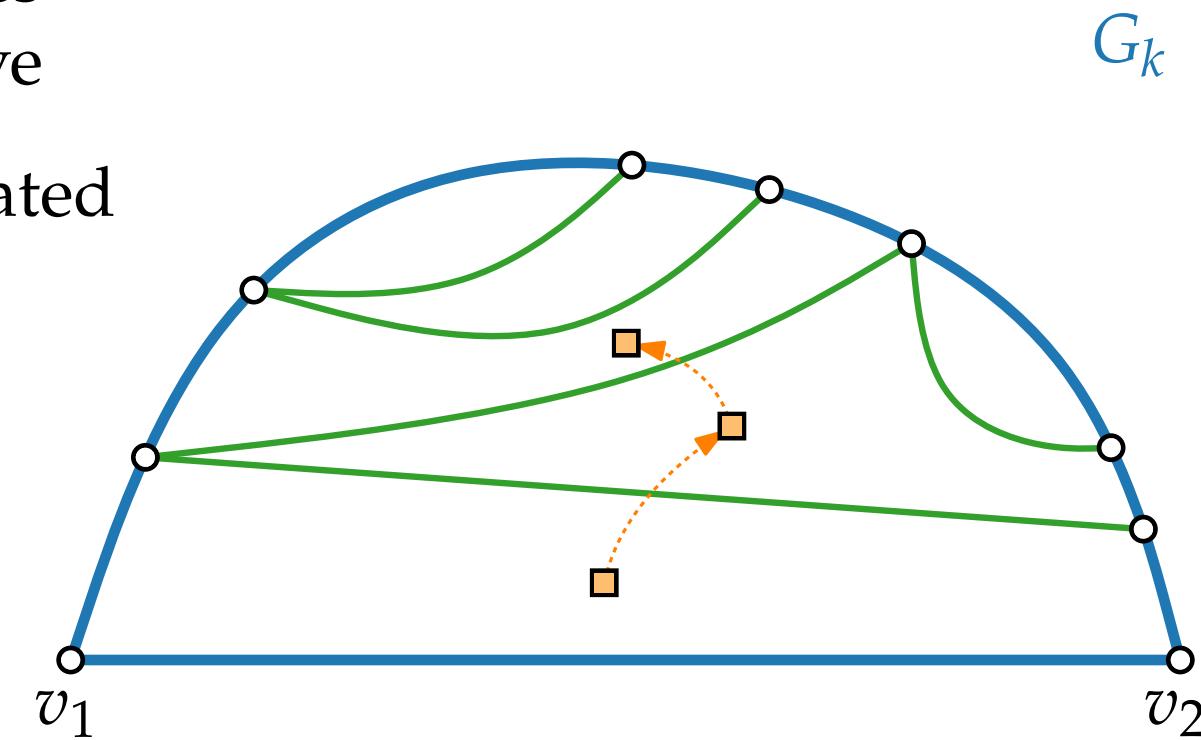
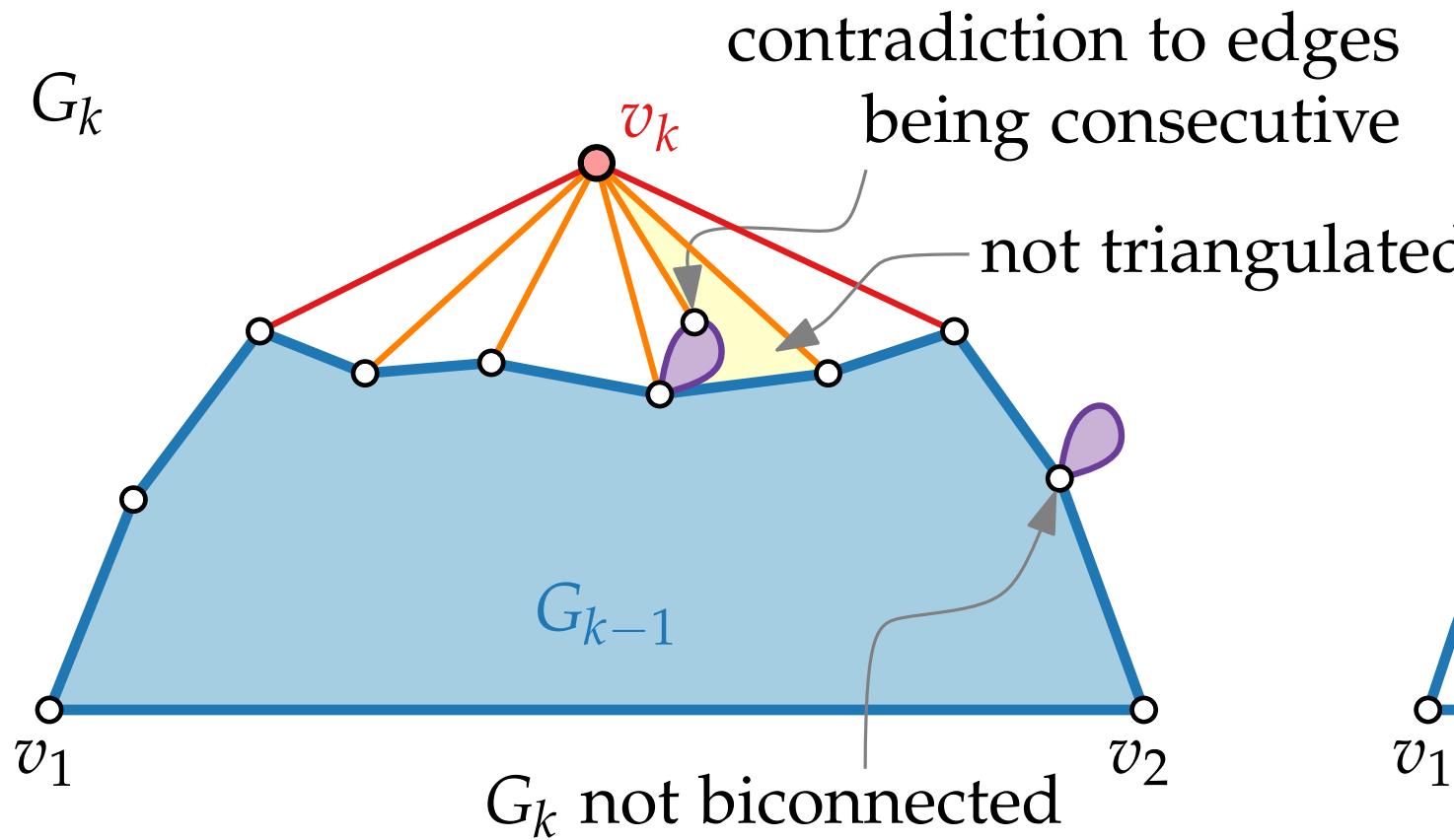
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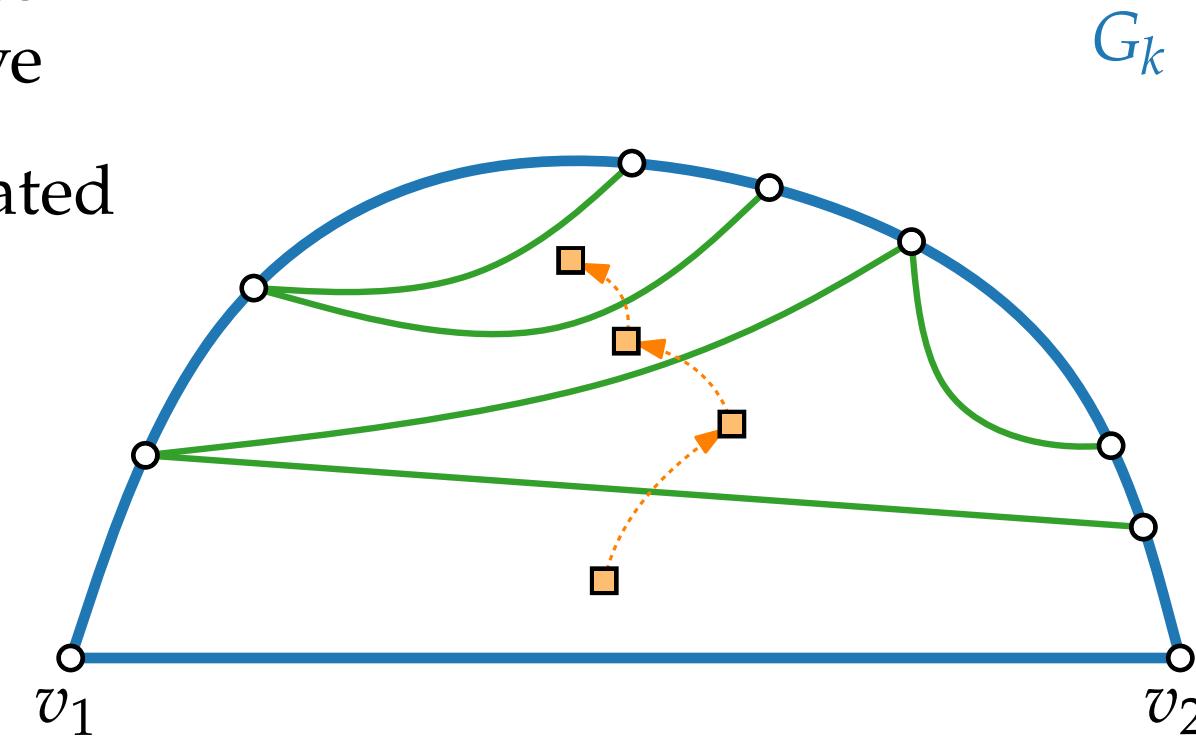
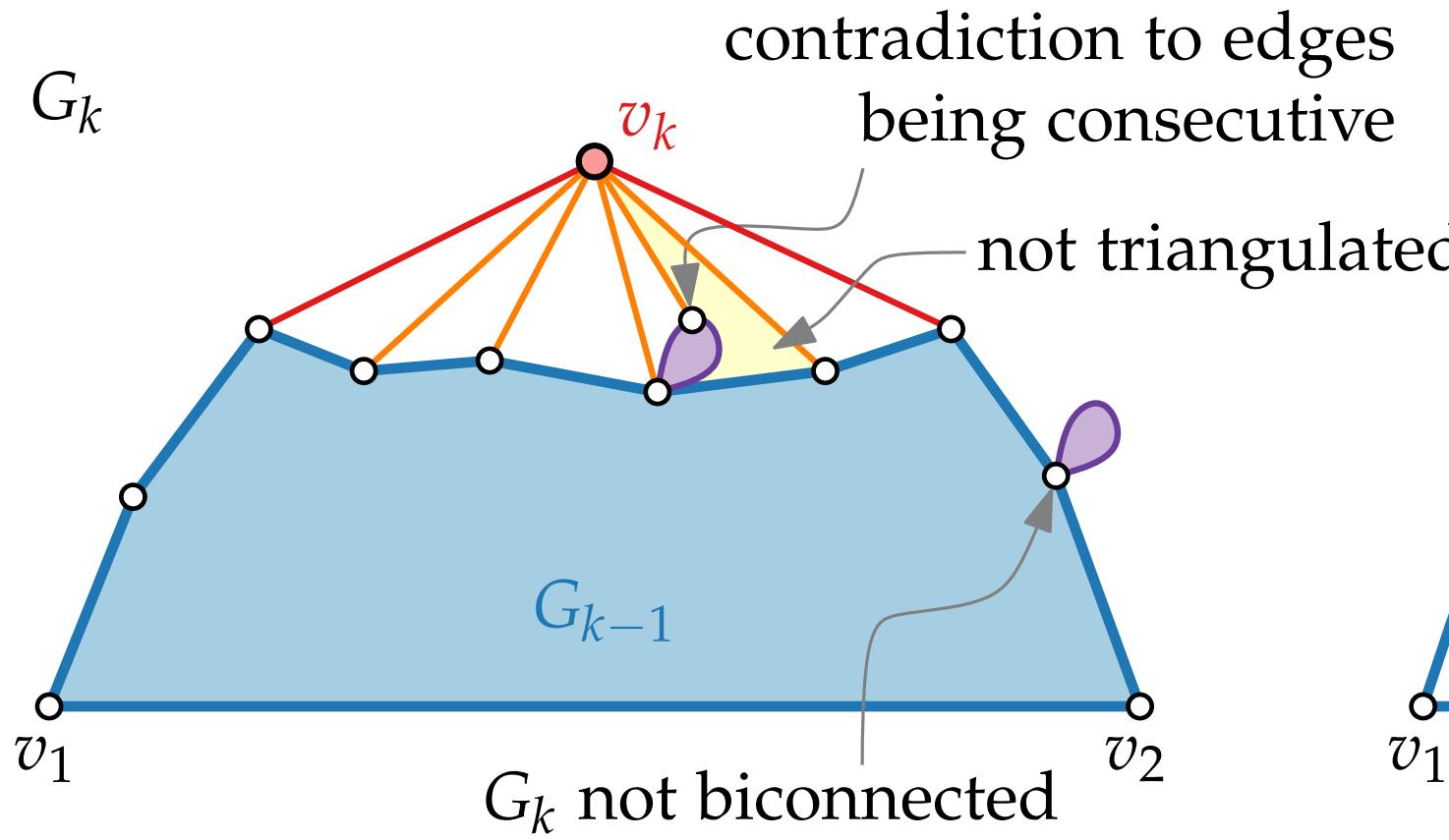
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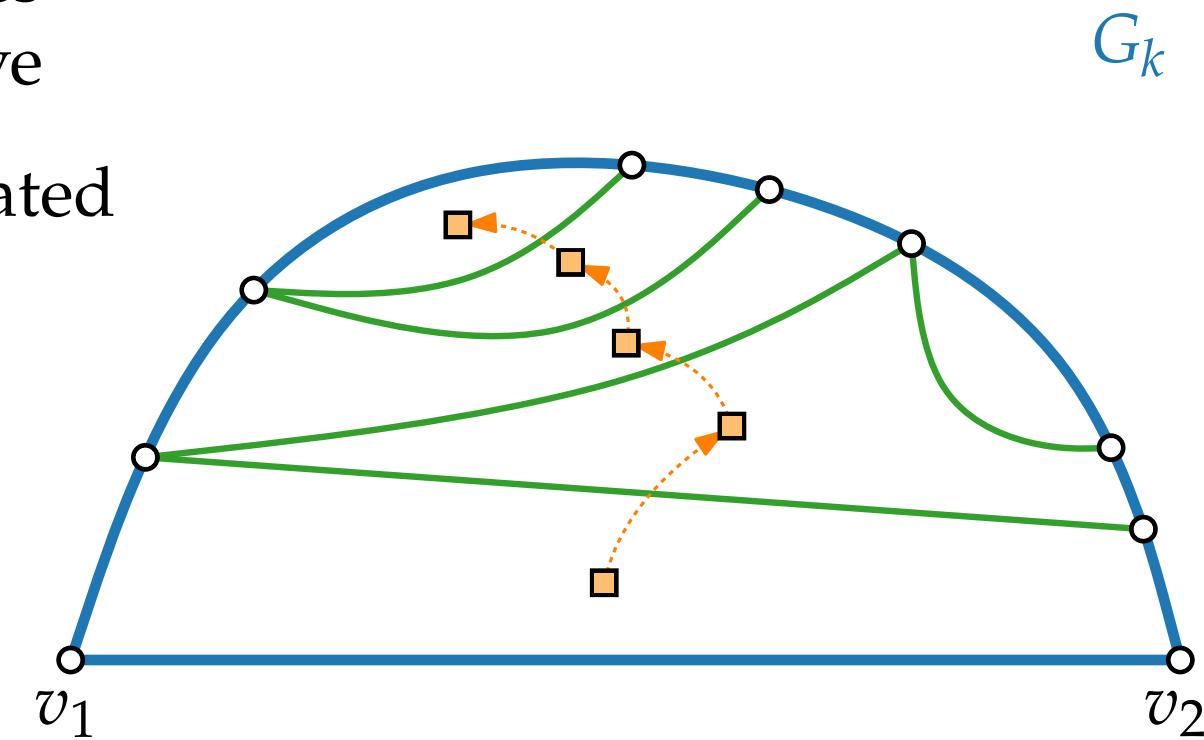
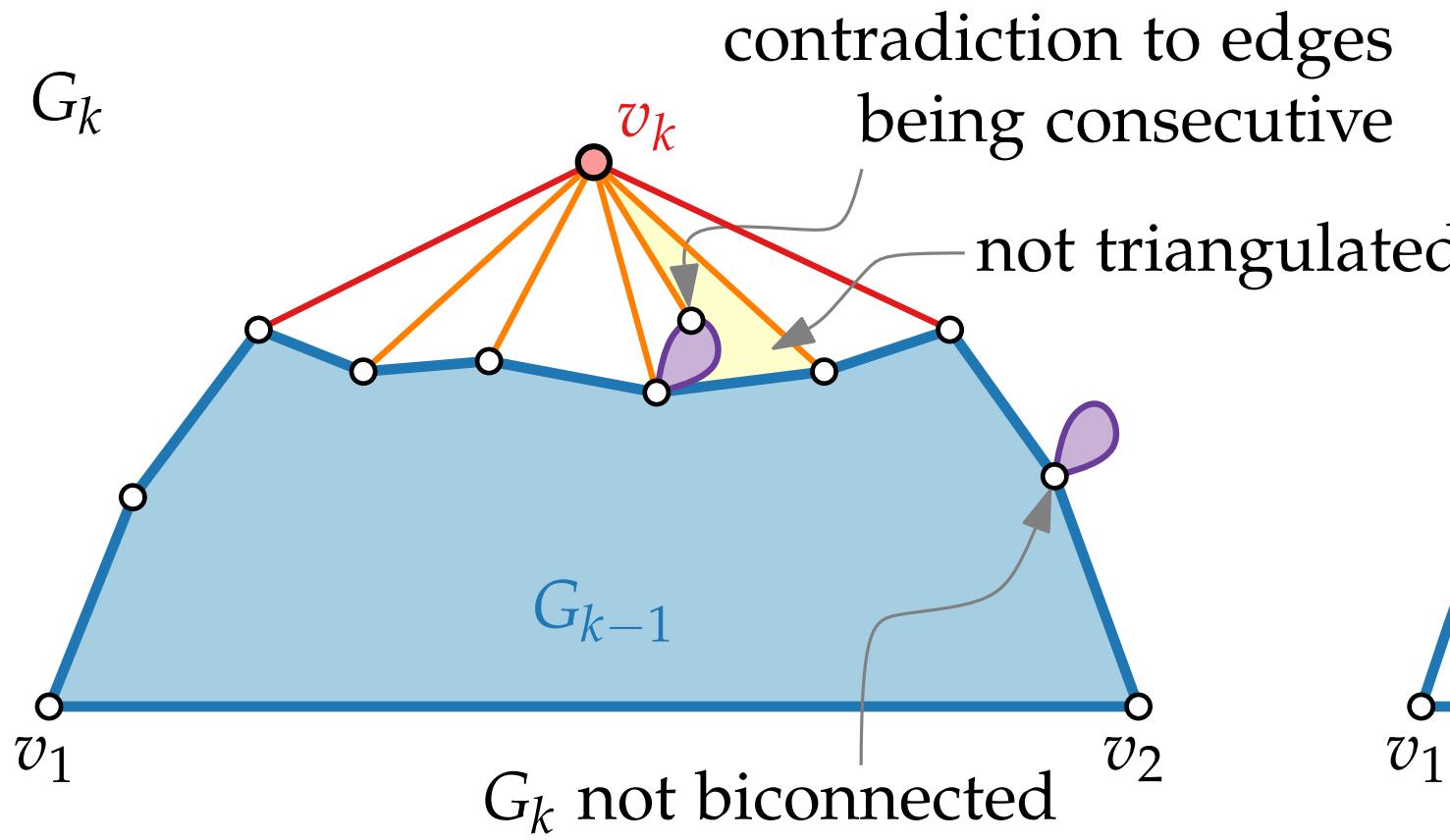
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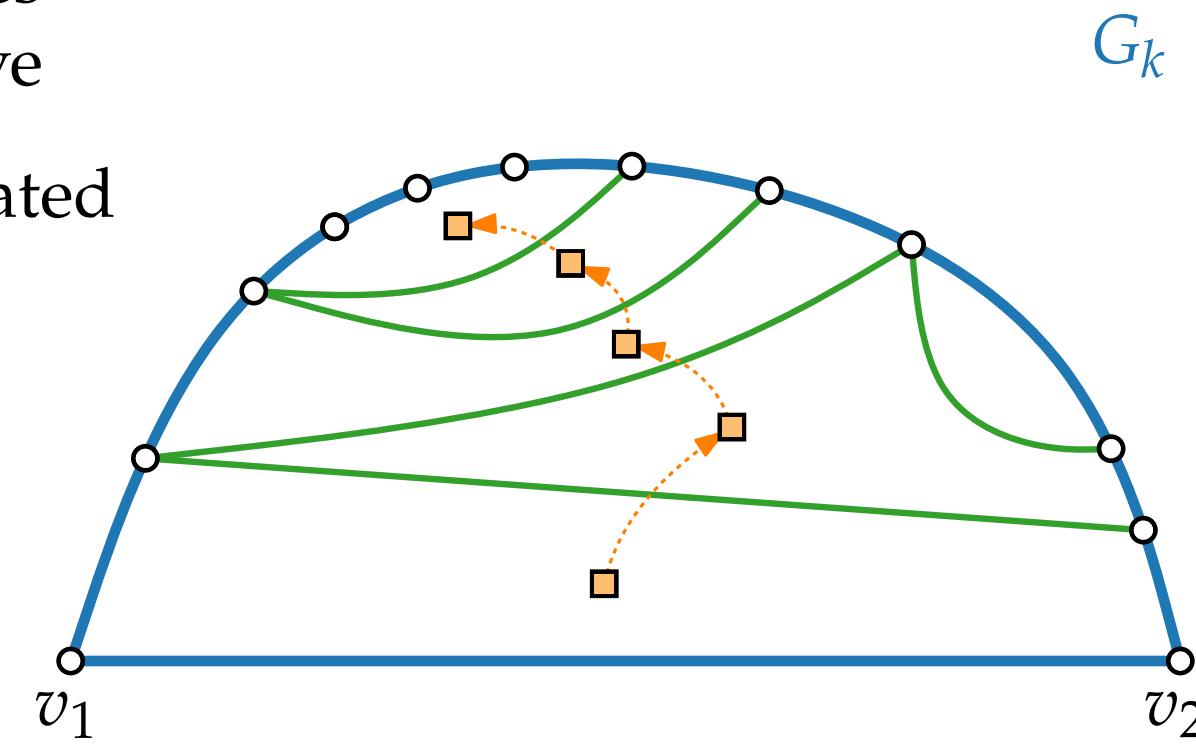
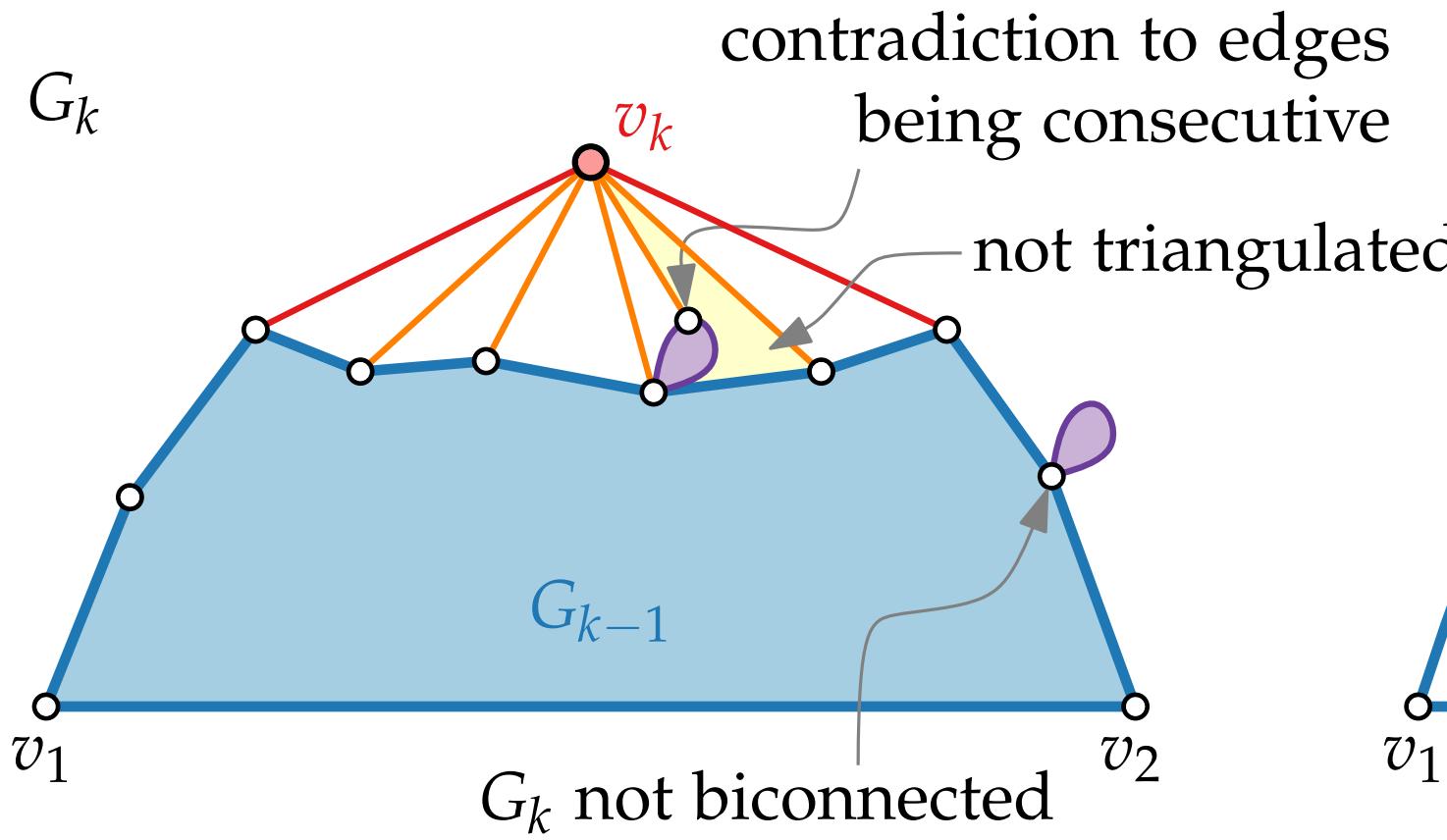
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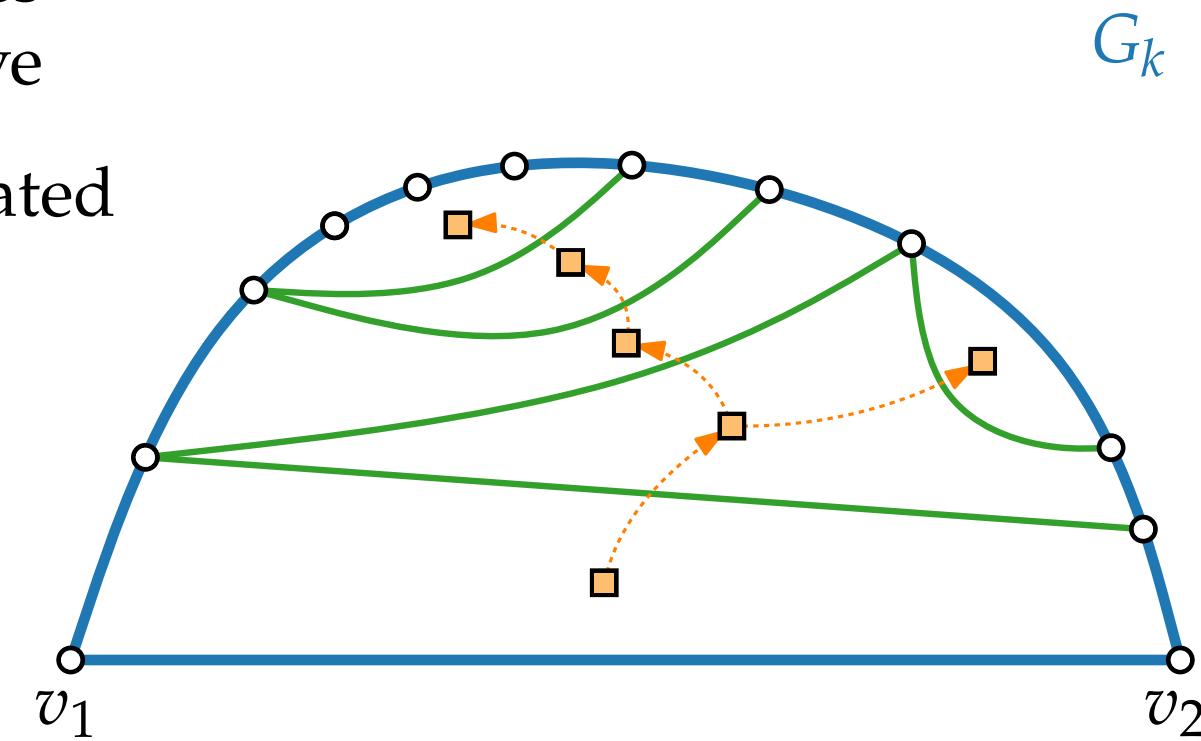
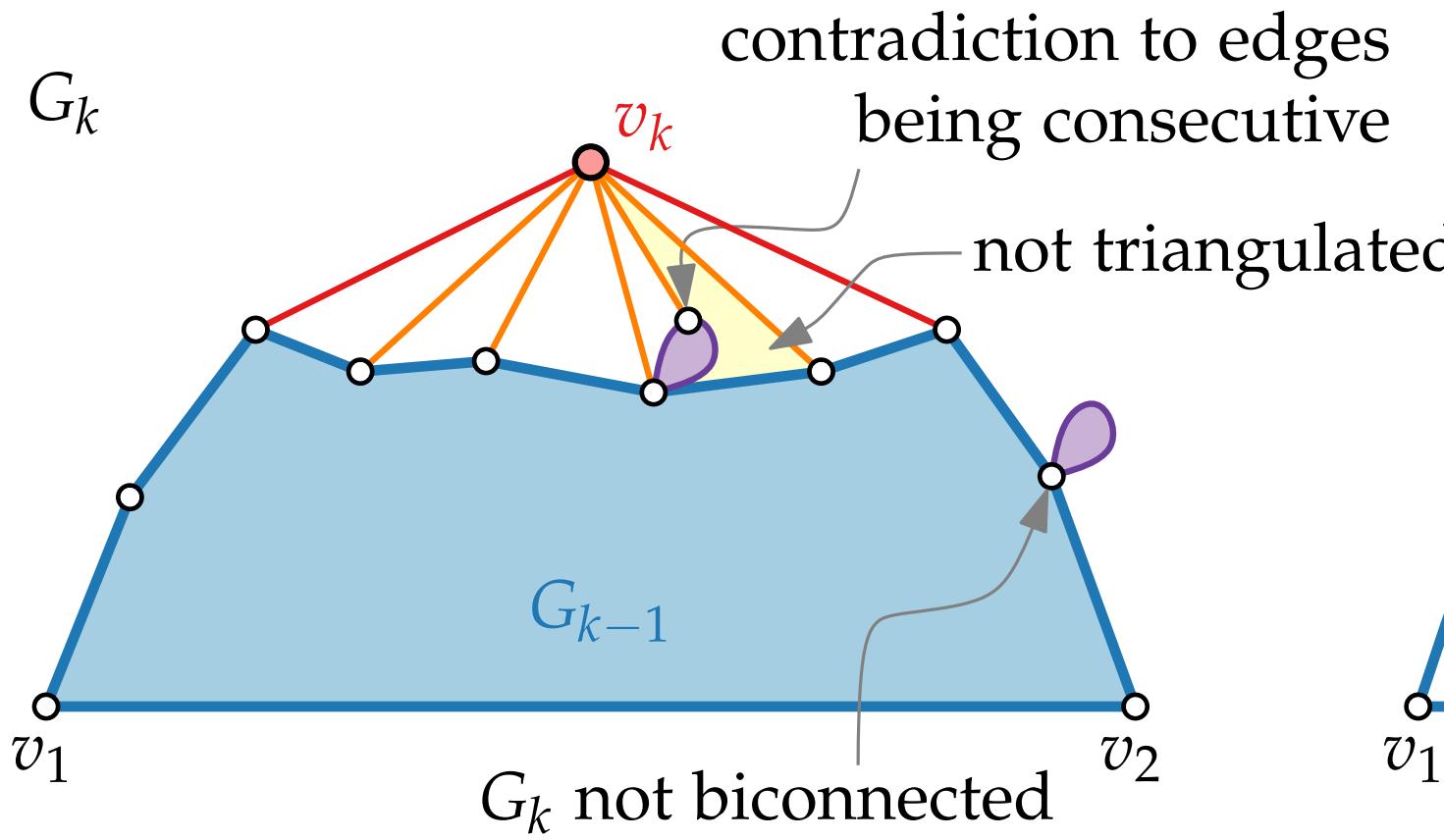
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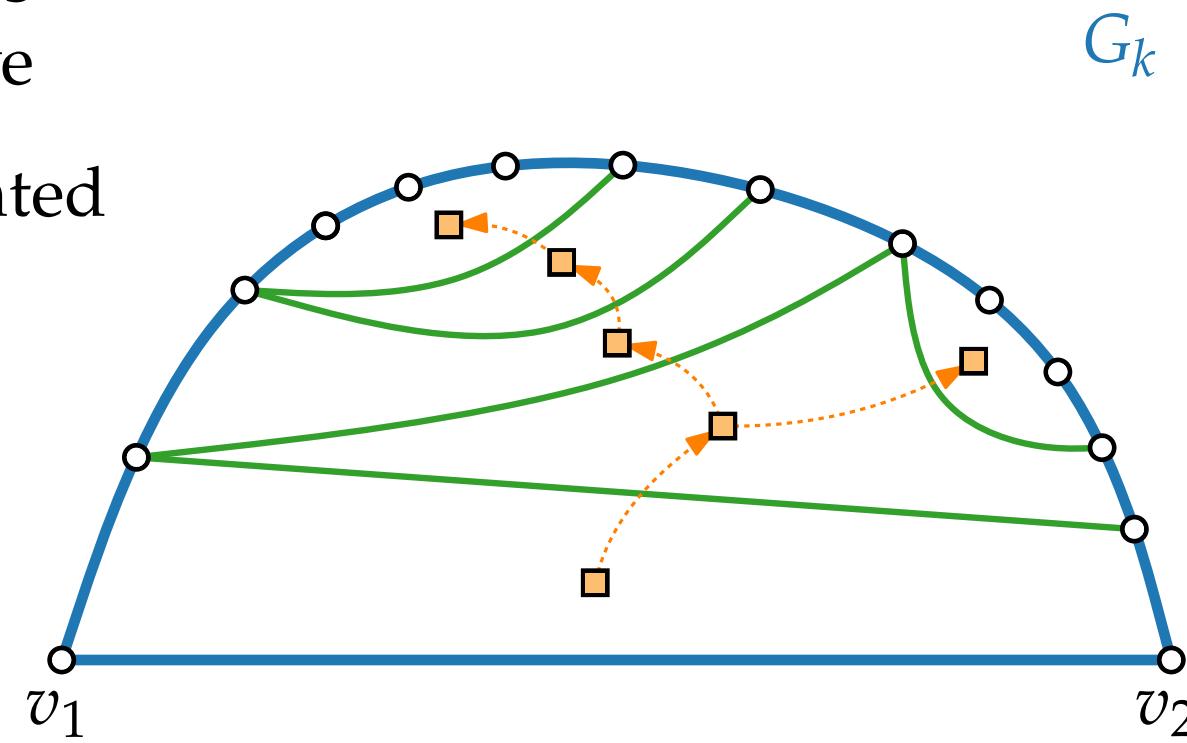
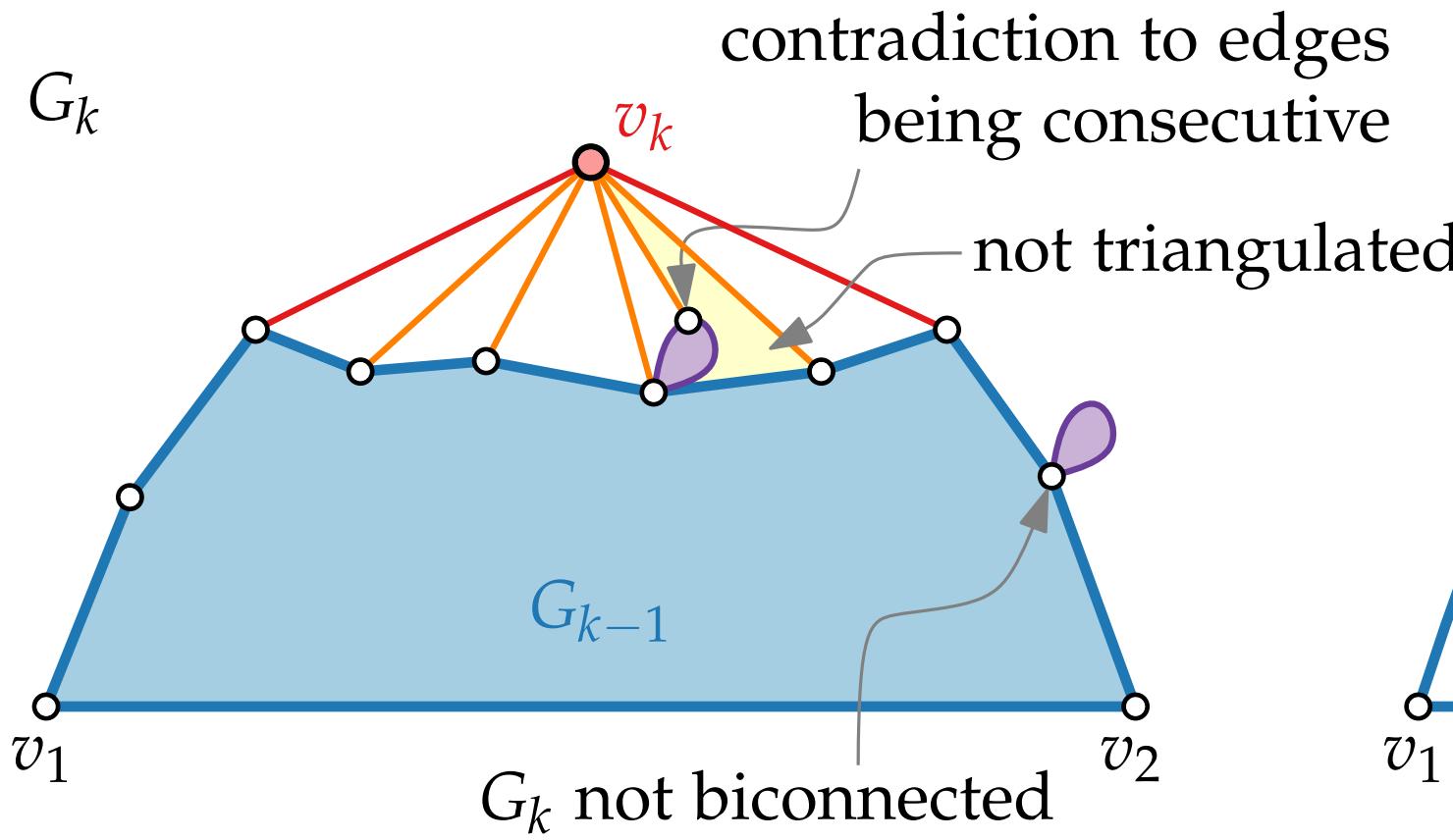
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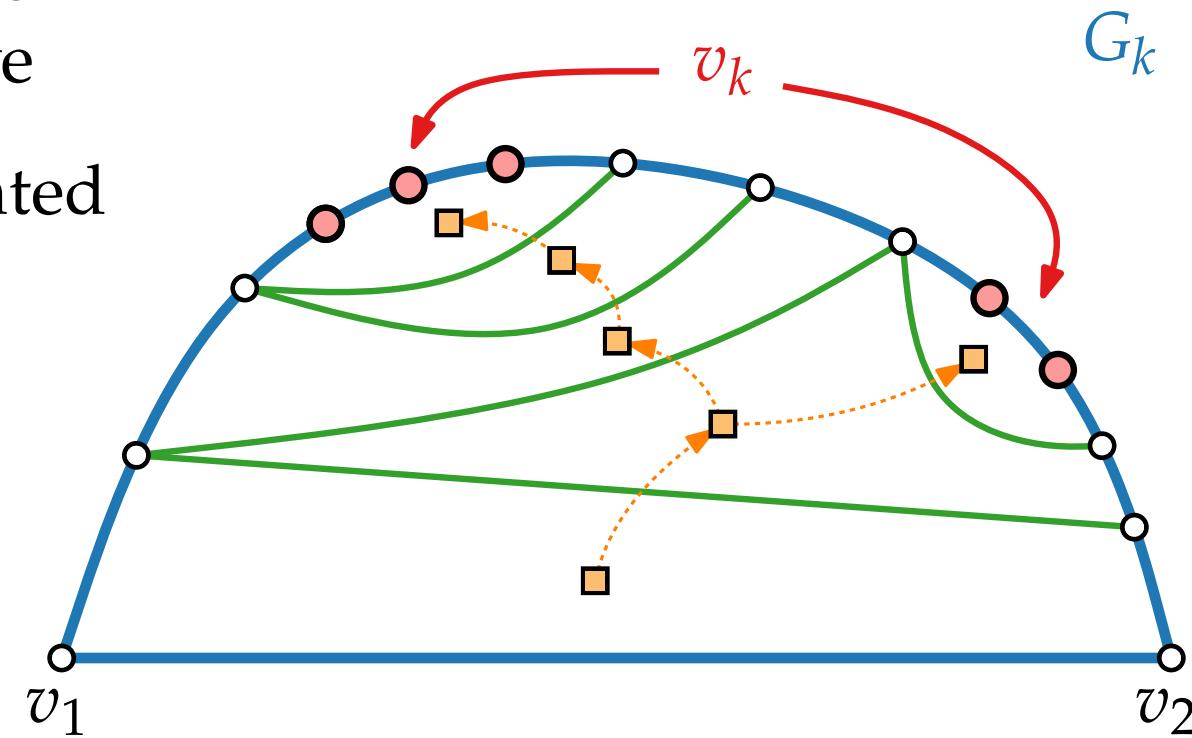
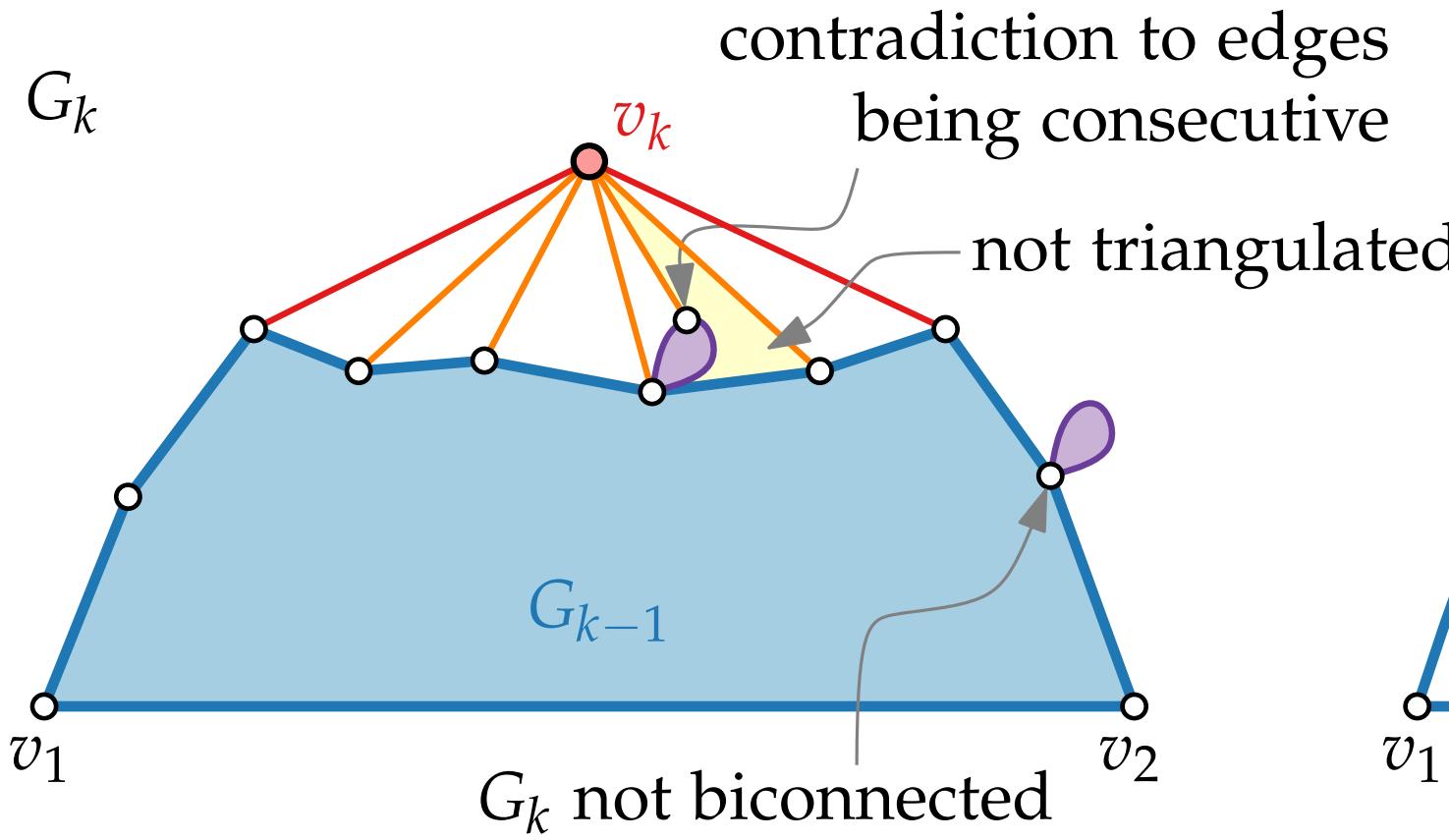
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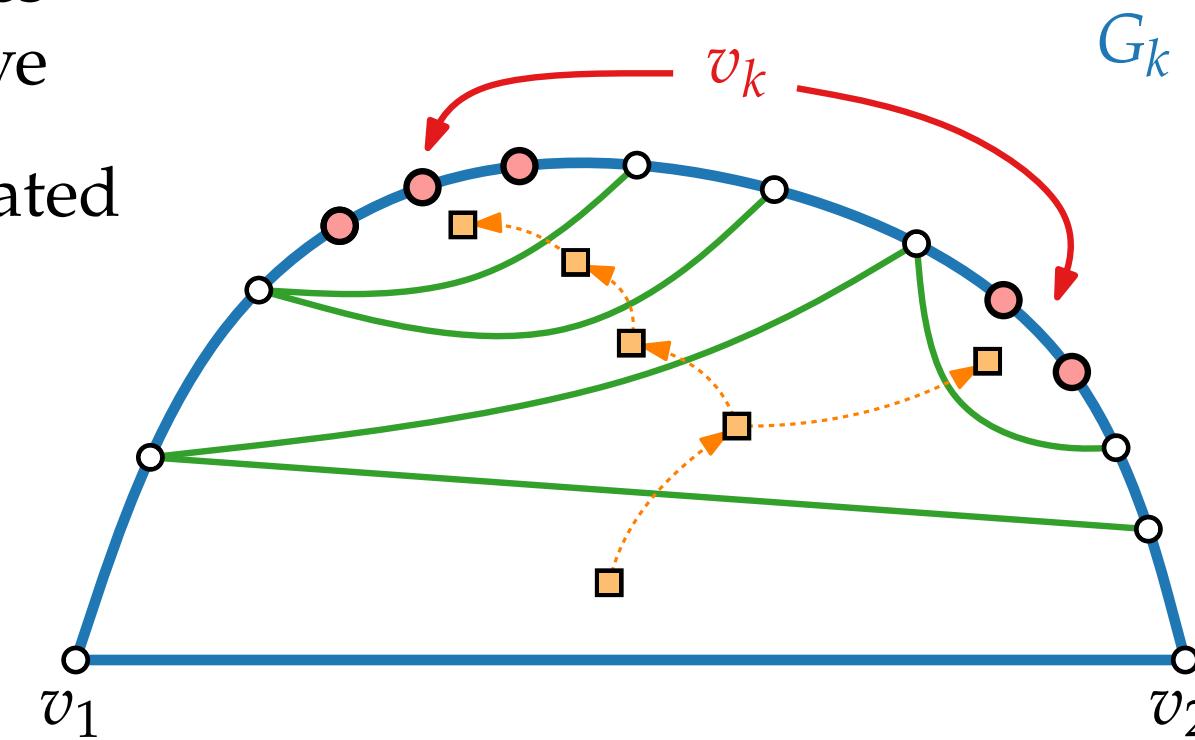
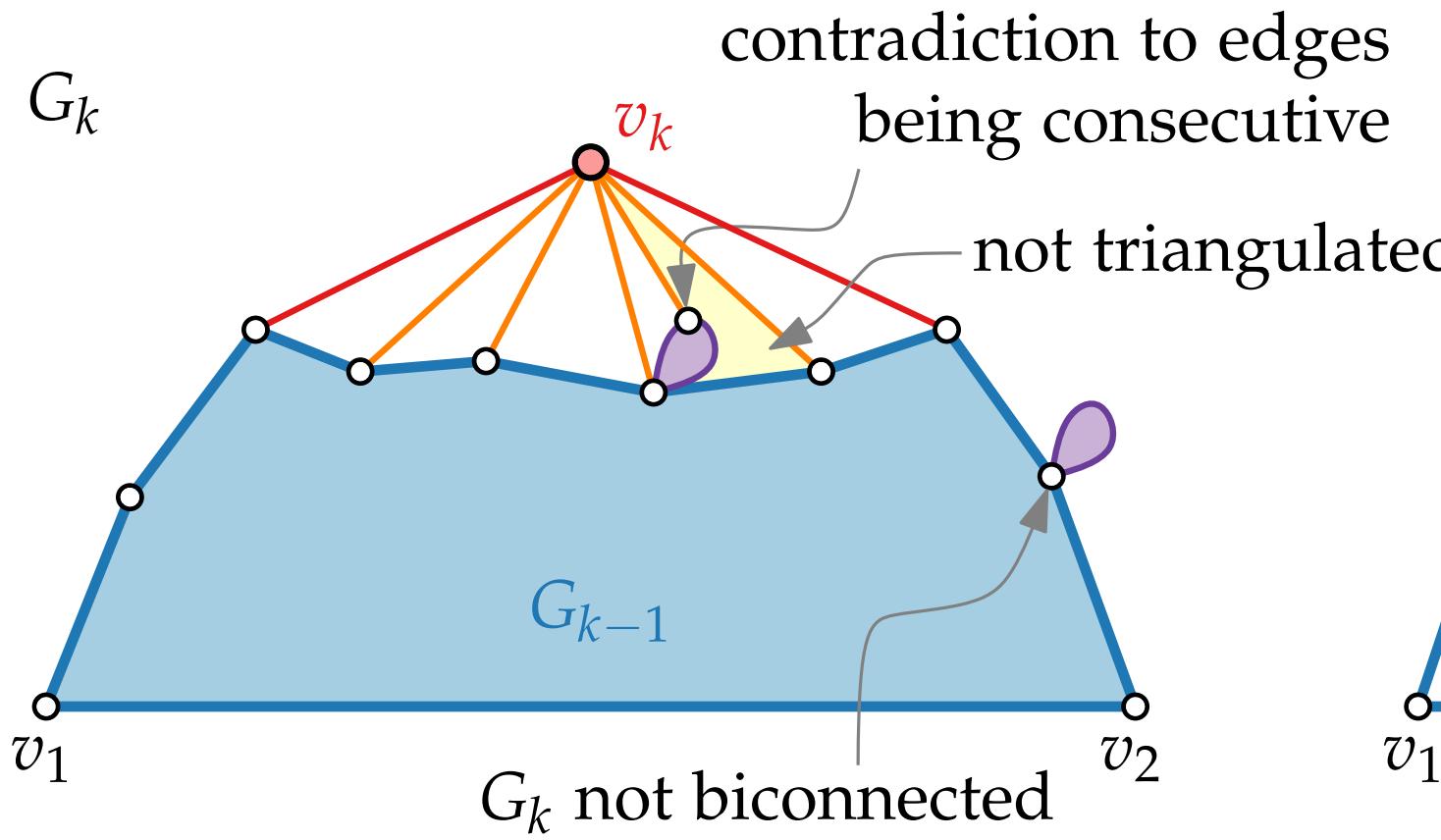
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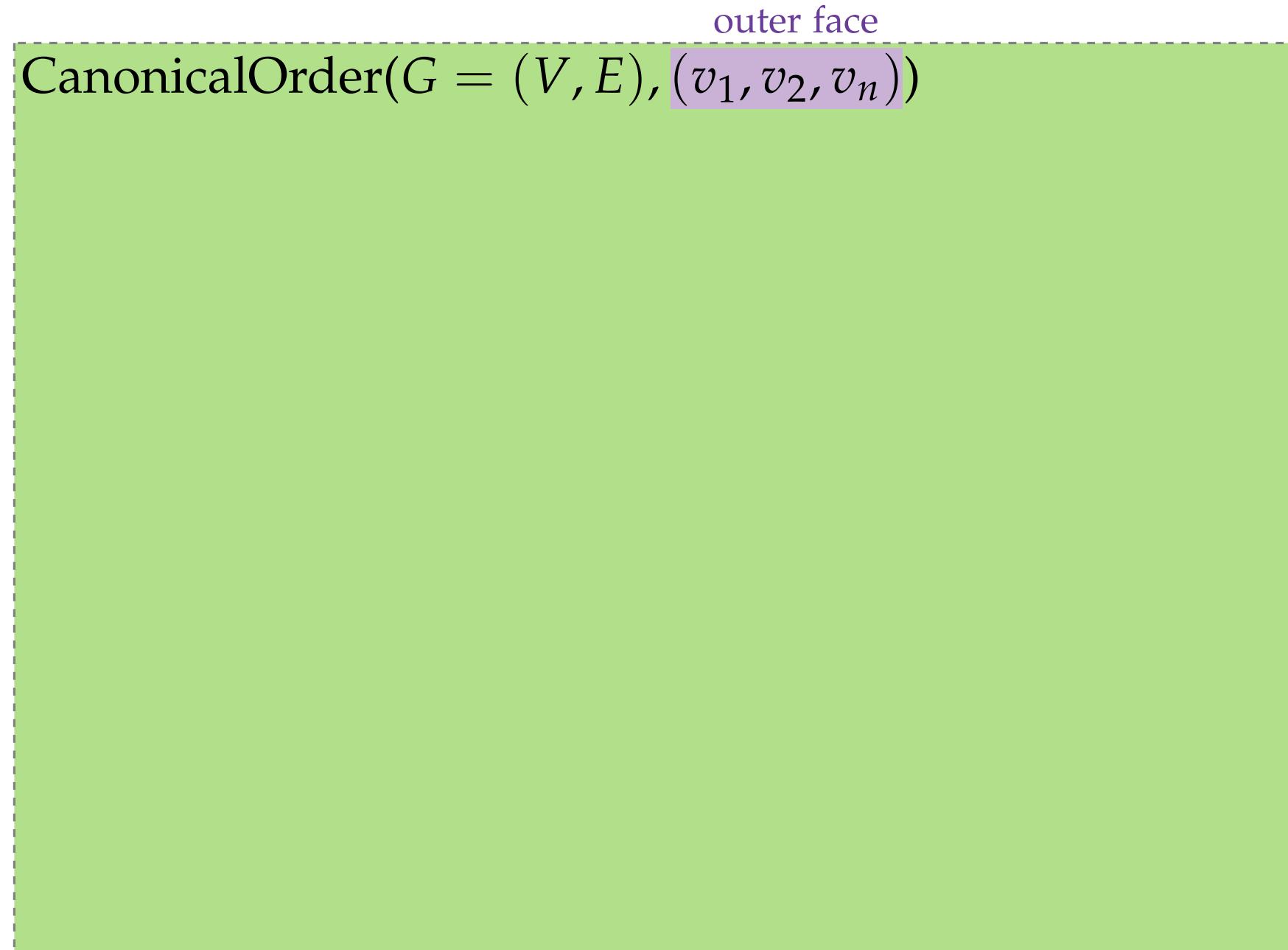
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Canonical Order – Implementation

```
CanonicalOrder(G = (V, E), (v1, v2, vn))
```

Canonical Order – Implementation



Canonical Order – Implementation

outer face

```
CanonicalOrder( $G = (V, E)$ ,  $(v_1, v_2, \dots, v_n)$ )
```

```
forall  $v \in V$  do
```

```
  └
```

d

Canonical Order – Implementation

outer face

```
CanonicalOrder( $G = (V, E)$ ,  $(v_1, v_2, v_n)$ )
```

```
forall  $v \in V$  do
  chords( $v$ )  $\leftarrow 0$ ;
```

d

Canonical Order – Implementation

outer face
CanonicalOrder($G = (V, E), (v_1, v_2, v_n)$)

forall $v \in V$ **do**
 └ chords(v) $\leftarrow 0$;

- chord(v): # chords adjacent to v

Canonical Order – Implementation

outer face

```
CanonicalOrder( $G = (V, E)$ ,  $(v_1, v_2, v_n)$ )
```

```
forall  $v \in V$  do
  chords( $v$ )  $\leftarrow 0$ ; out( $v$ )  $\leftarrow$  false;
```

- $\text{chord}(v)$: # chords adjacent to v

Canonical Order – Implementation

outer face

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- $\text{chord}(v)$: # chords adjacent to v
- $\text{out}(v)$ = true iff v is currently outer vertex

Canonical Order – Implementation

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outer face  
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Canonical Order – Implementation

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- $\text{chord}(v)$: # chords adjacent to v
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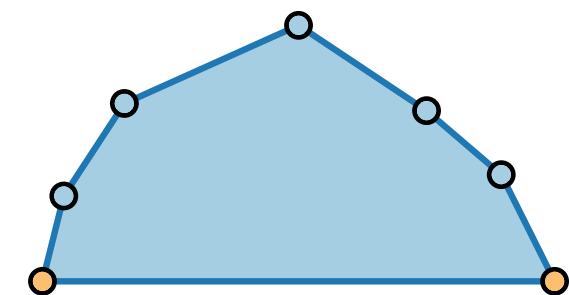
Canonical Order – Implementation

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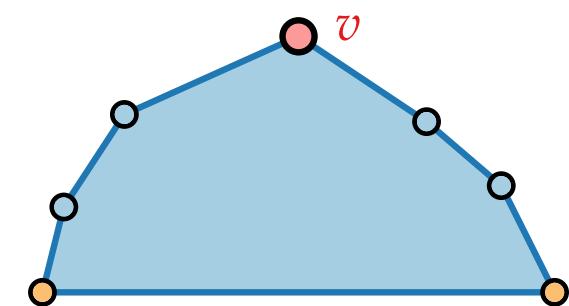
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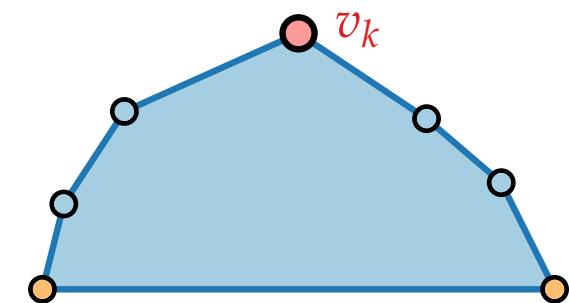
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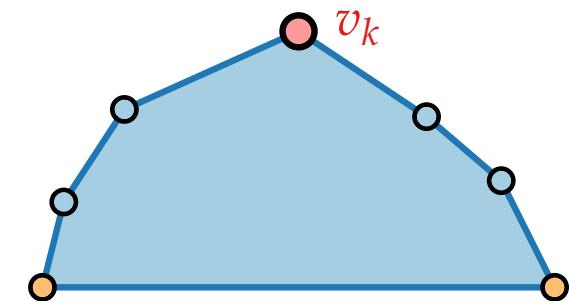


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  // Let  $w_1 = v_1, w_2, \dots, w_{t-1}, w_t = v_2$  denote the
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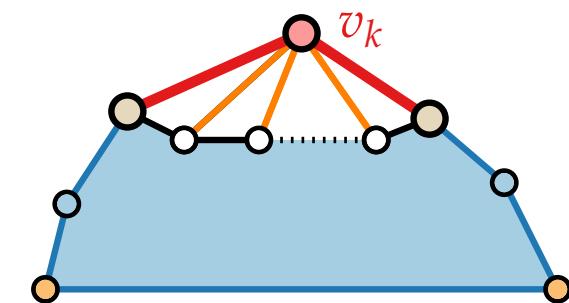


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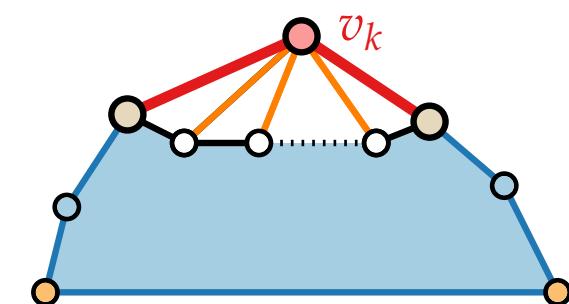


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  boundary of  $G_{k-1}$  in  $G_{k-1}$  and let  $w_p, \dots, w_q$  be the
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```

- chord(v): # chords adjacent to v
- out(v) = true iff v is currently outer vertex
- mark(v) = true iff v has received its number



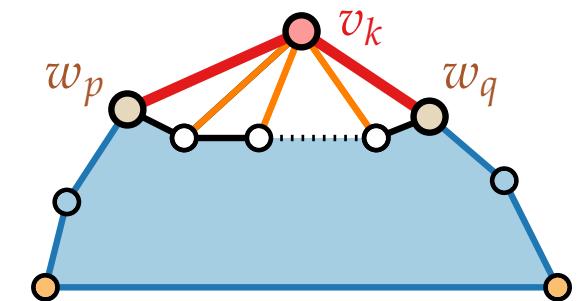
Canonical Order – Implementation

outer face

```

CanonicalOrder( $G = (V, E)$ ,  $(v_1, v_2, v_n)$ )
  forall  $v \in V$  do
    chords( $v$ )  $\leftarrow 0$ ; out( $v$ )  $\leftarrow$  false; mark( $v$ )  $\leftarrow$  false
  mark( $v_1$ ), mark( $v_2$ ), out( $v_1$ ), out( $v_2$ ), out( $v_n$ )  $\leftarrow$  true
  for  $k = n$  to 3 do
    choose  $v$  such that mark( $v$ ) = false, out( $v$ ) = true, and
    chords( $v$ ) = 0
     $v_k \leftarrow v$ ; mark( $v$ )  $\leftarrow$  true
    // Let  $w_1 = v_1, w_2, \dots, w_{t-1}, w_t = v_2$  denote the
    boundary of  $G_{k-1}$  in  $G_{k-1}$  and let  $w_p, \dots, w_q$  be the
    neighbors of  $v_k$ 
```

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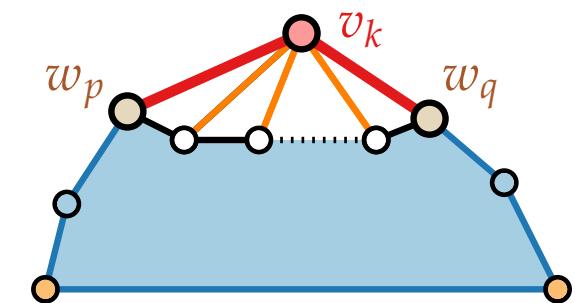
Canonical Order – Implementation

```

    outer face
CanonicalOrder( $G = (V, E), (v_1, v_2, v_n)$ )
forall  $v \in V$  do
   $\text{chords}(v) \leftarrow 0$ ;  $\text{out}(v) \leftarrow \text{false}$ ;  $\text{mark}(v) \leftarrow \text{false}$ 
 $\text{mark}(v_1), \text{mark}(v_2), \text{out}(v_1), \text{out}(v_2), \text{out}(v_n) \leftarrow \text{true}$ 
for  $k = n$  to 3 do
  choose  $v$  such that  $\text{mark}(v) = \text{false}$ ,  $\text{out}(v) = \text{true}$ , and
   $\text{chords}(v) = 0$ 
   $v_k \leftarrow v$ ;  $\text{mark}(v) \leftarrow \text{true}$ 
  // Let  $w_1 = v_1, w_2, \dots, w_{t-1}, w_t = v_2$  denote the
  boundary of  $G_{k-1}$  in  $G_{k-1}$  and let  $w_p, \dots, w_q$  be the
  neighbors of  $v_k$ 
   $\text{out}(w_i) \leftarrow \text{true}$  for all  $p < i < q$ 

```

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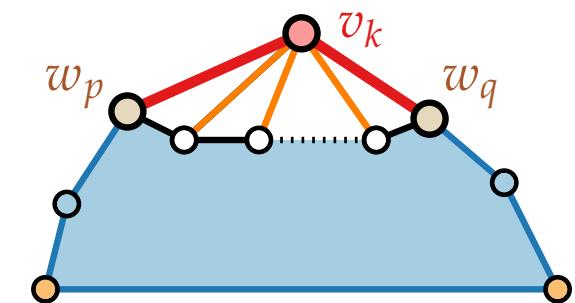
Canonical Order – Implementation

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mark( $v_1$ ), mark( $v_2$ ), out( $v_1$ ), out( $v_2$ ), out( $v_n$ )  $\leftarrow \text{true}$ 
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  update number of chords for  $w_i$ 
  and its neighbours

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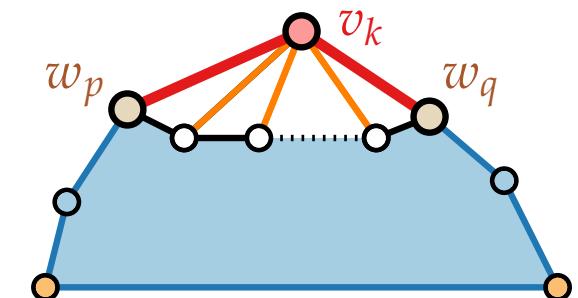
Canonical Order – Implementation

outer face

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CanonicalOrder( $G = (V, E)$ ,  $(v_1, v_2, v_n)$ )
  forall  $v \in V$  do
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  mark( $v_1$ ), mark( $v_2$ ), out( $v_1$ ), out( $v_2$ ), out( $v_n$ )  $\leftarrow$  true
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- chord(v): # chords adjacent to v
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Lemma.

Algorithm CanonicalOrder computes a canonical order of a plane graph in $\mathcal{O}(n)$ time.

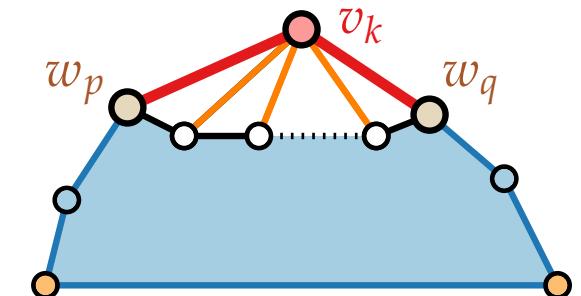
Canonical Order – Implementation

outer face

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    choose  $v$  such that mark( $v$ ) = false, out( $v$ ) = true, and
    chords( $v$ ) = 0                                // keep list with candidates
     $v_k \leftarrow v$ ; mark( $v$ )  $\leftarrow$  true
    // Let  $w_1 = v_1, w_2, \dots, w_{t-1}, w_t = v_2$  denote the
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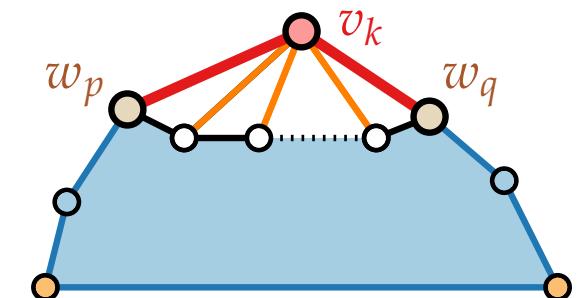
Canonical Order – Implementation

```

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  out( $w_i$ )  $\leftarrow \text{true}$  for all  $p < i < q$           //  $O(n)$  in total
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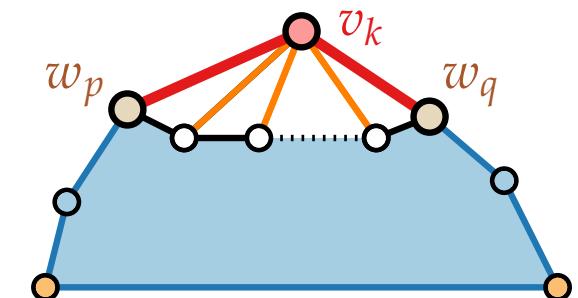
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  // neighbors of  $v_k$ 
  out( $w_i$ )  $\leftarrow \text{true}$  for all  $p < i < q$       //  $O(n)$  in total
  update number of chords for  $w_i$ 
  and its neighbours                            //  $O(m) = O(n)$  in total

```

- **chord**(v): # chords adjacent to v
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Lemma.

Algorithm CanonicalOrder computes a canonical order of a plane graph in $\mathcal{O}(n)$ time.

Shift Method – Idea

Drawing invariants:

G_{k-1} is drawn such that

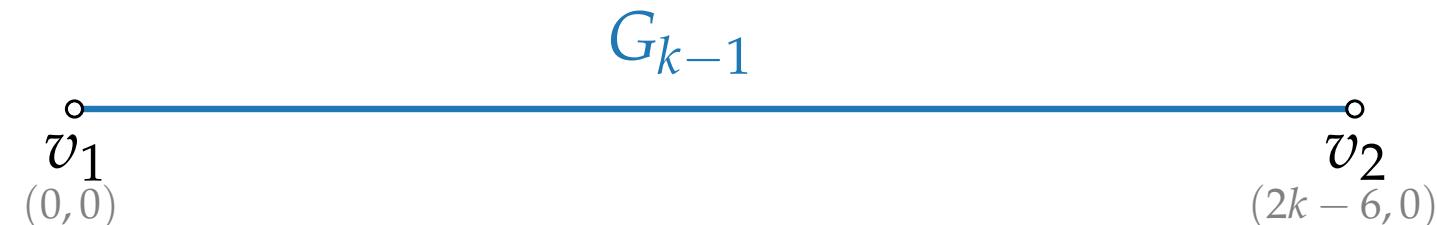
$$G_{k-1}$$

Shift Method – Idea

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- v_1 is on $(0, 0)$, v_2 is on $(2k - 6, 0)$,

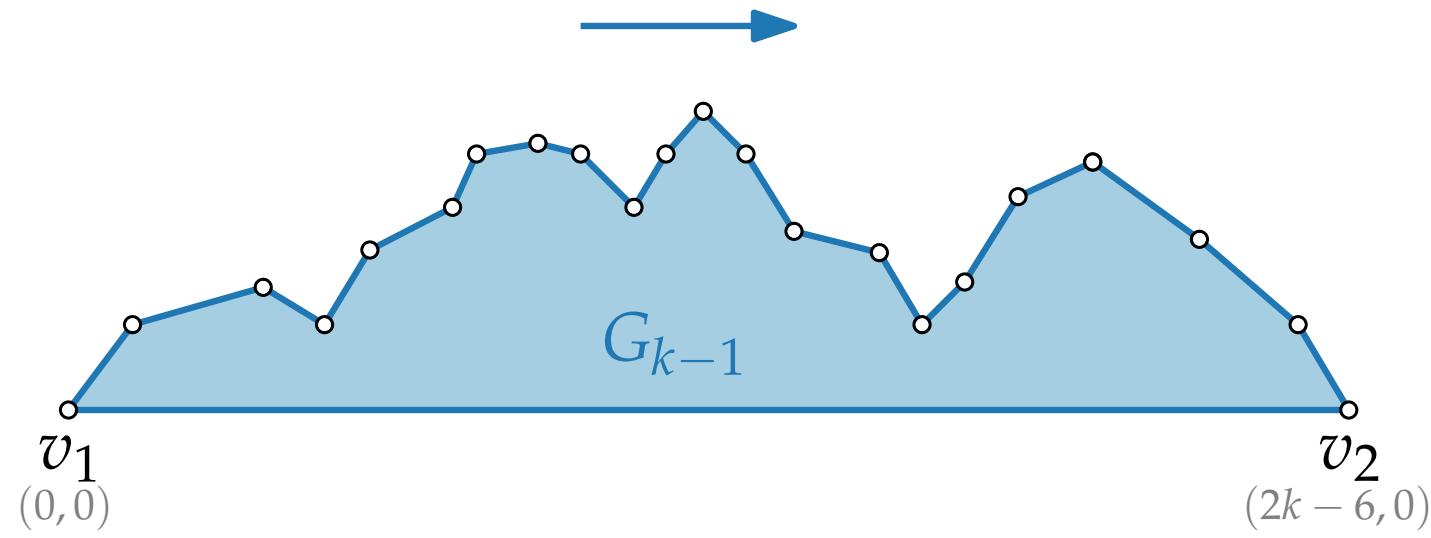


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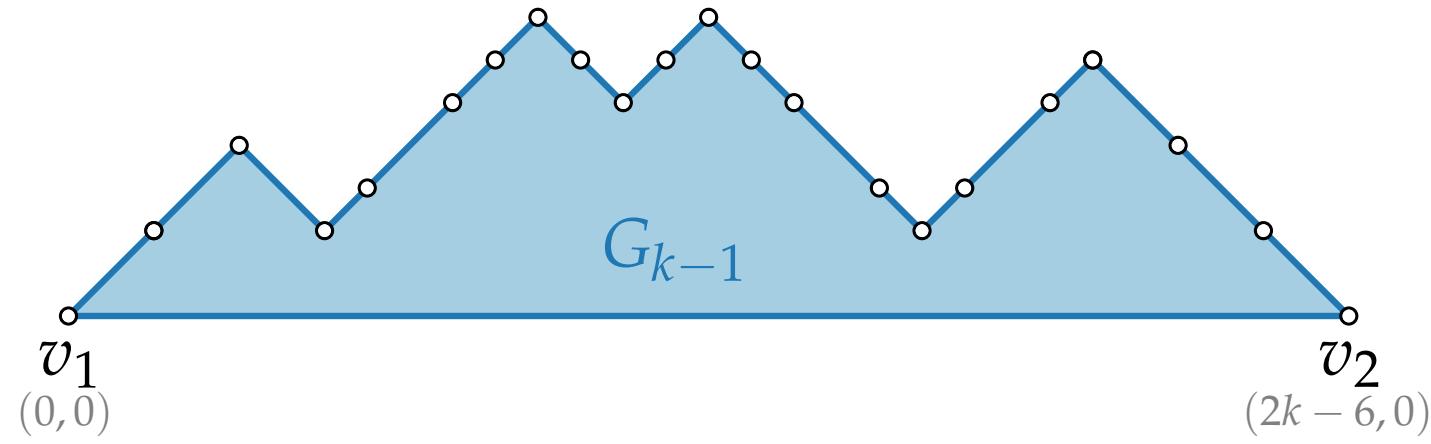


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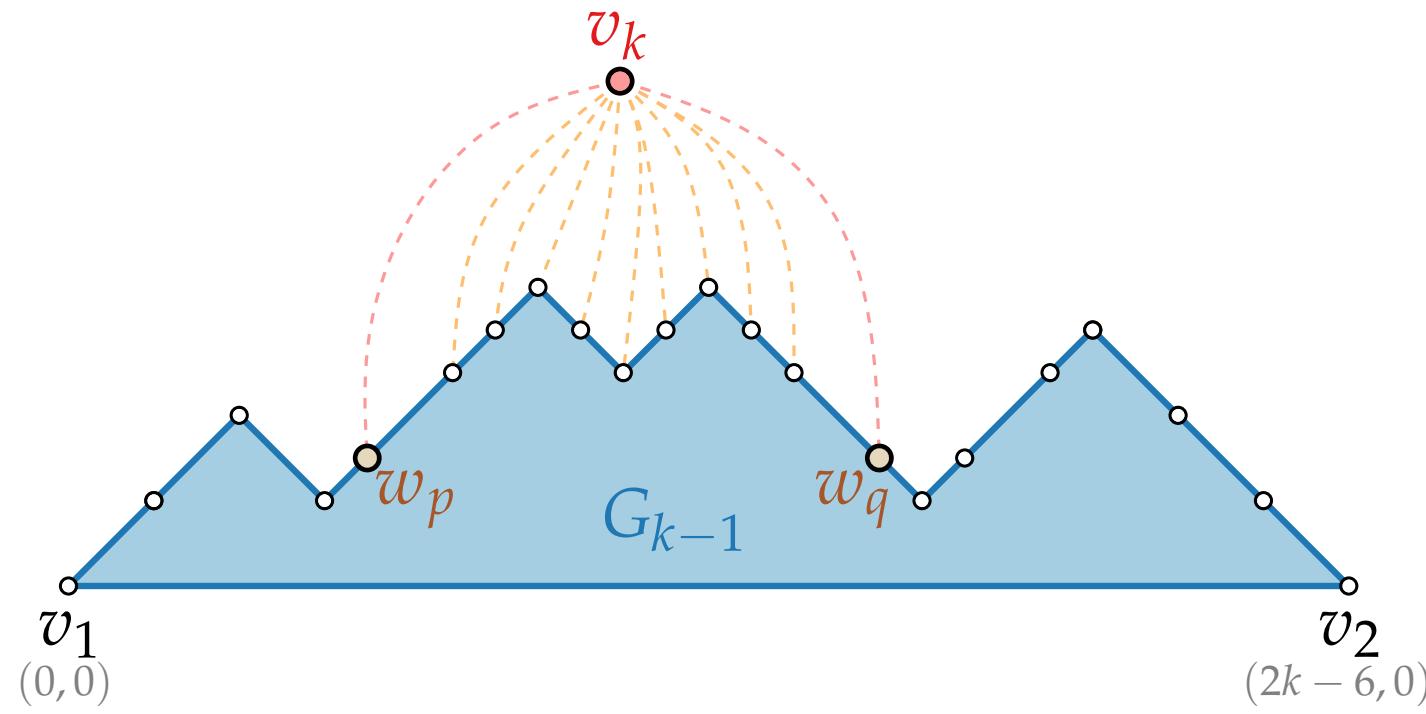


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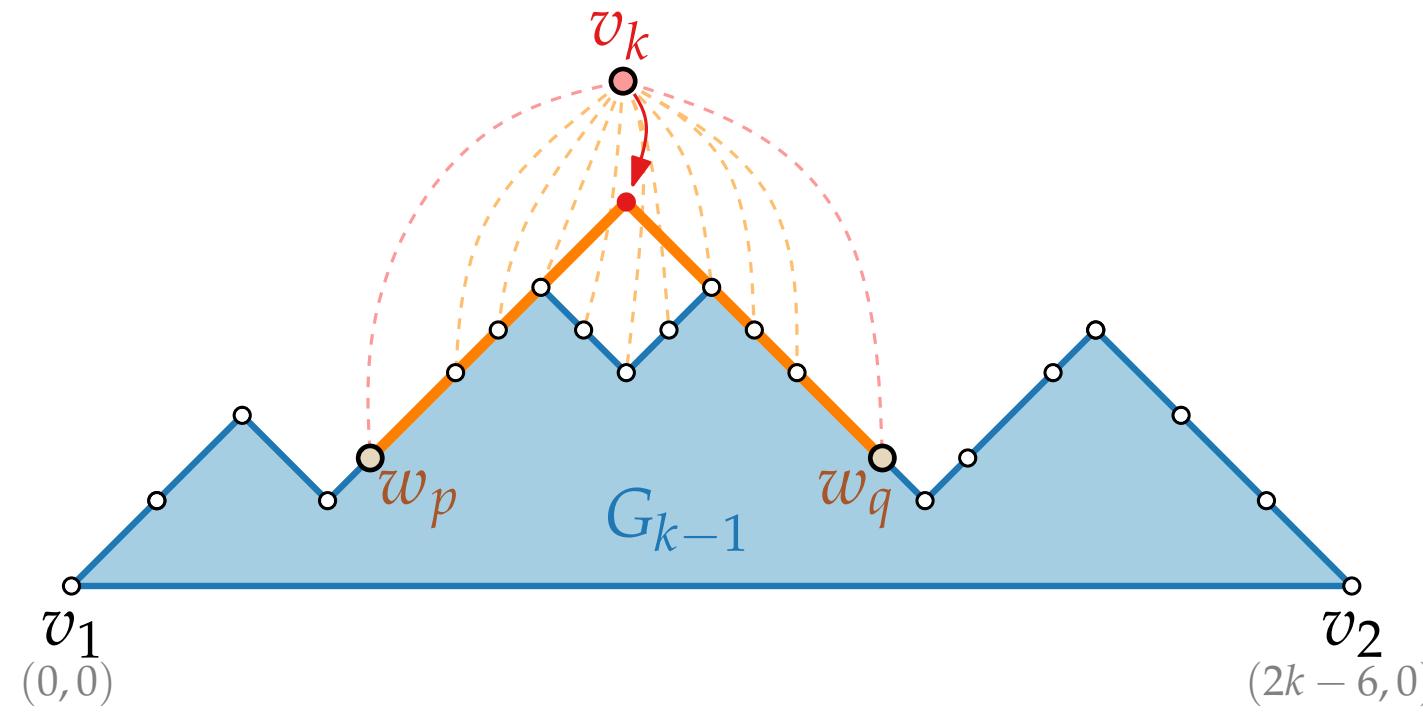


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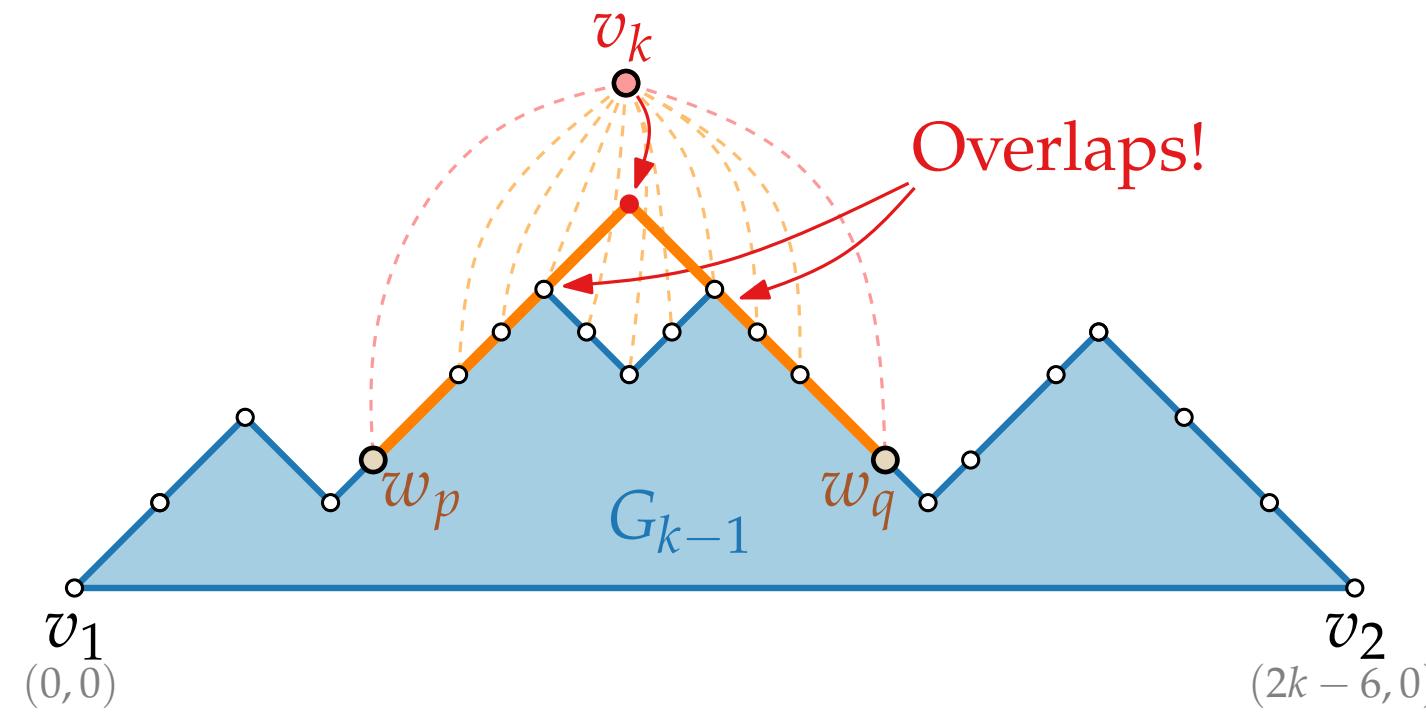


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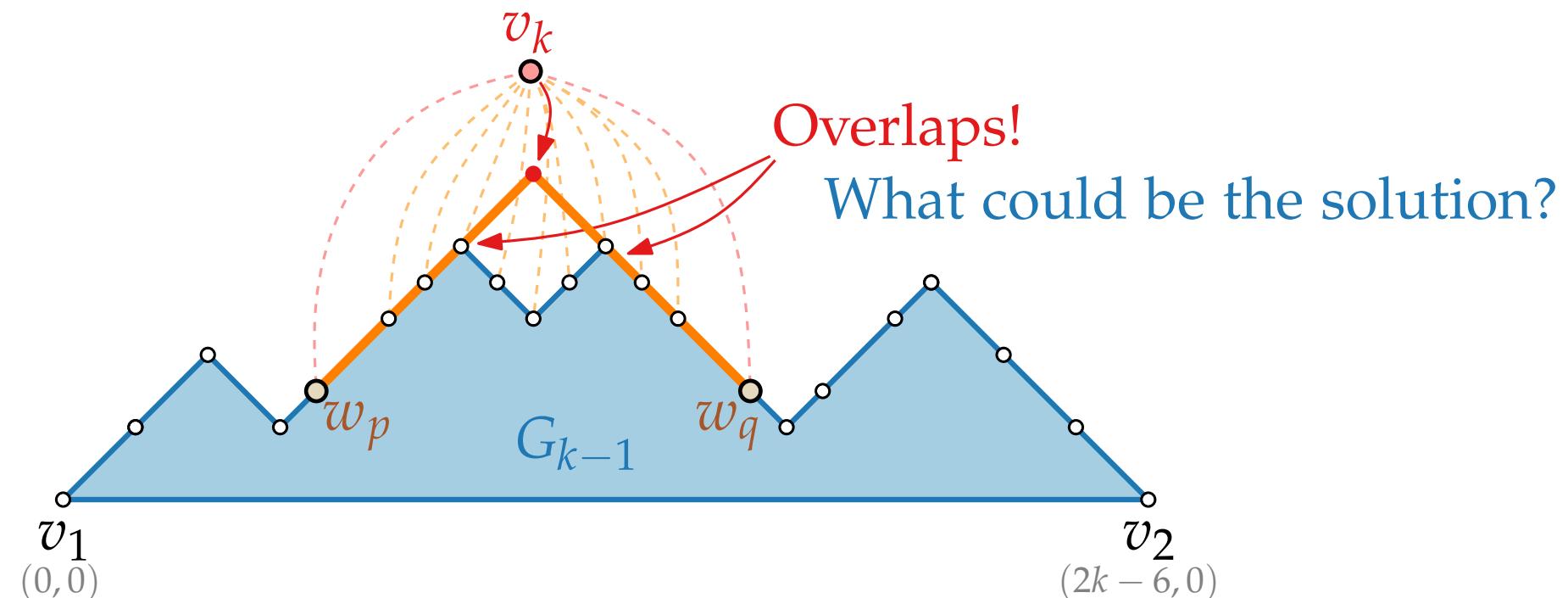


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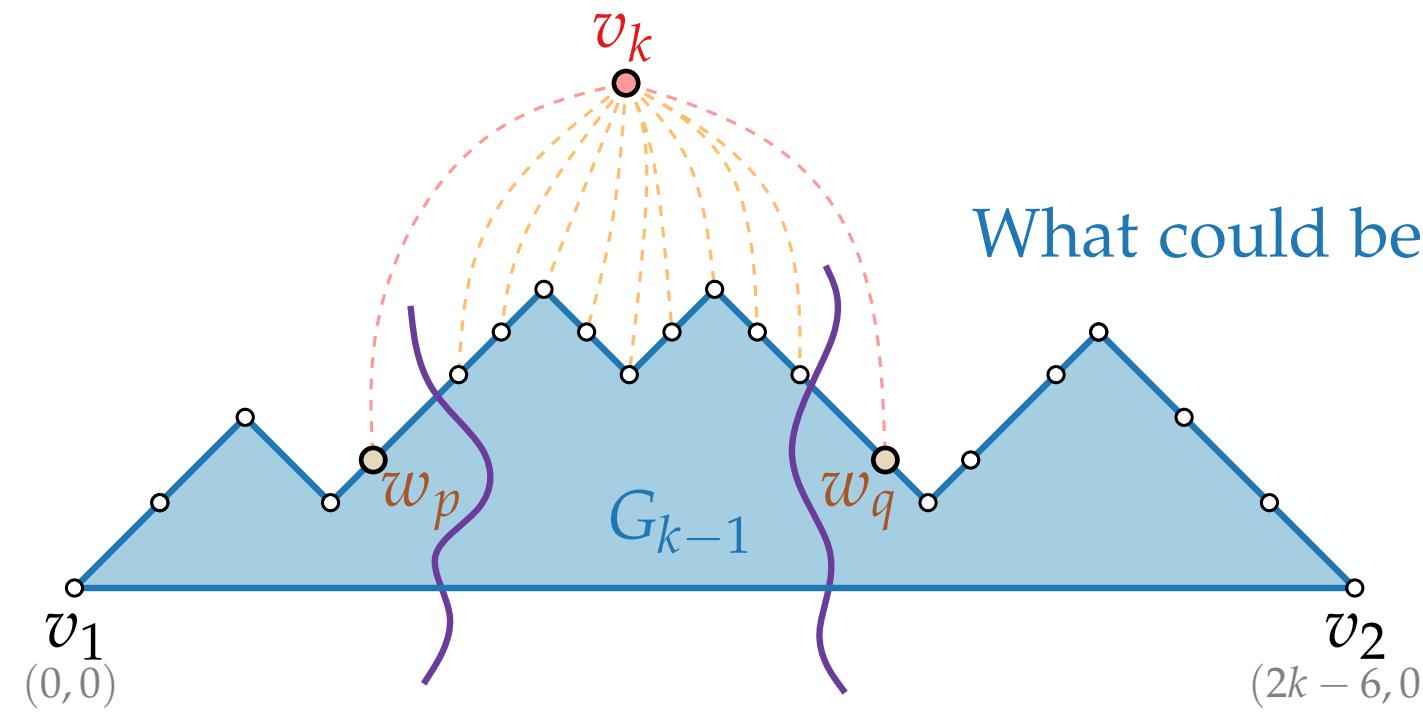


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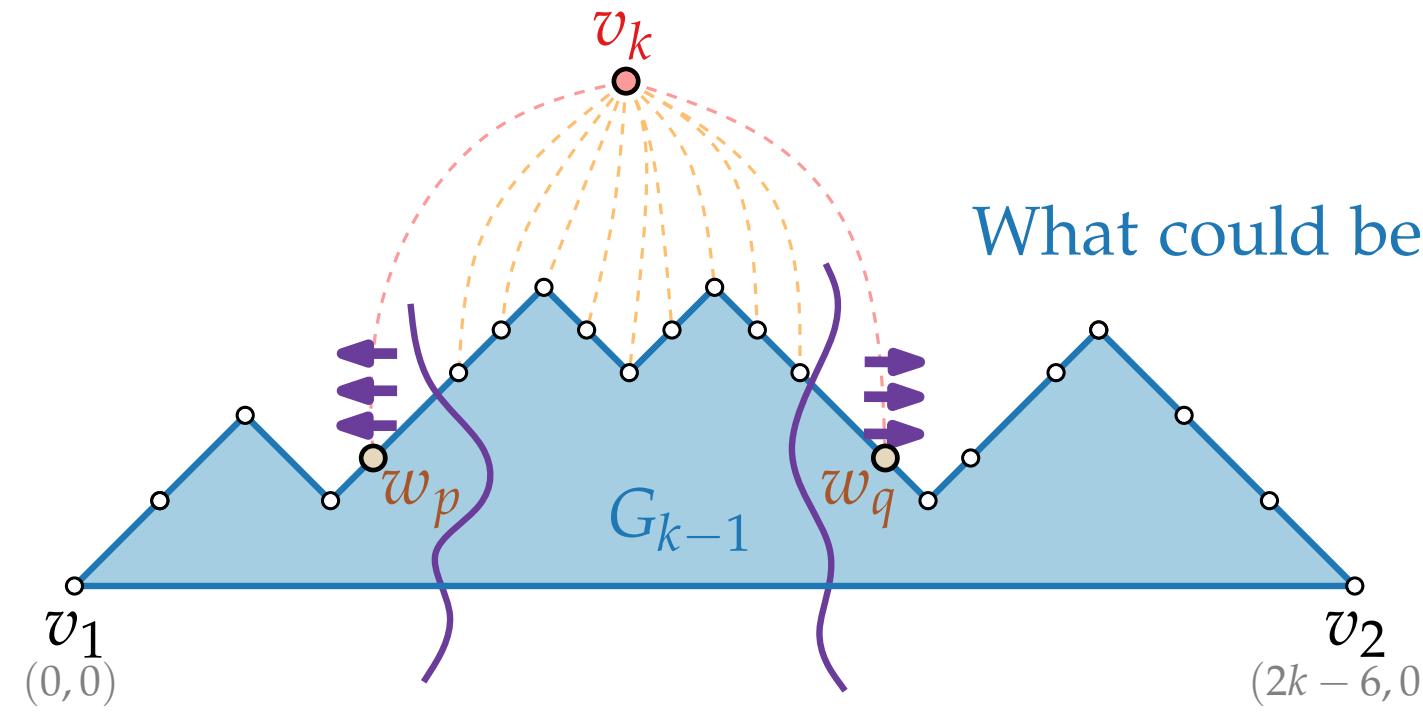
What could be the solution?

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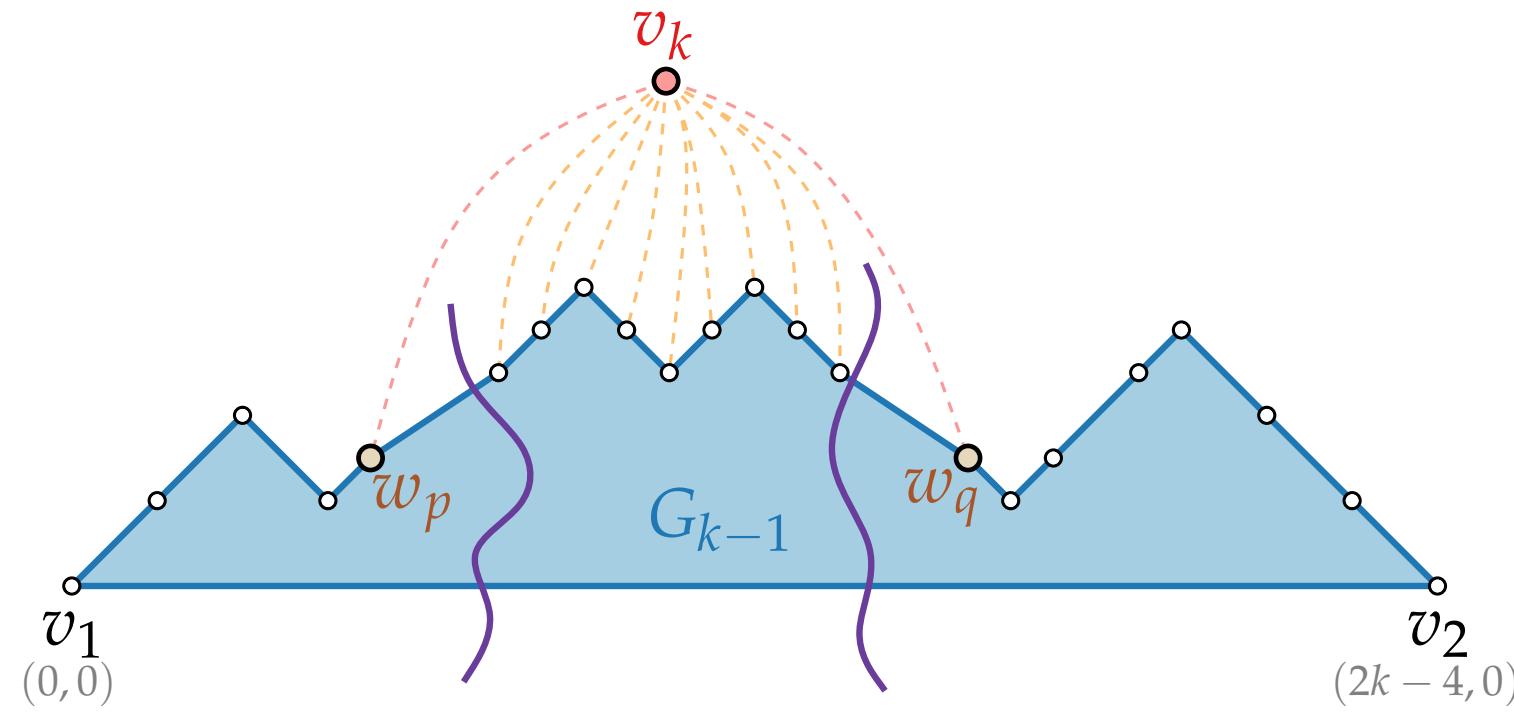
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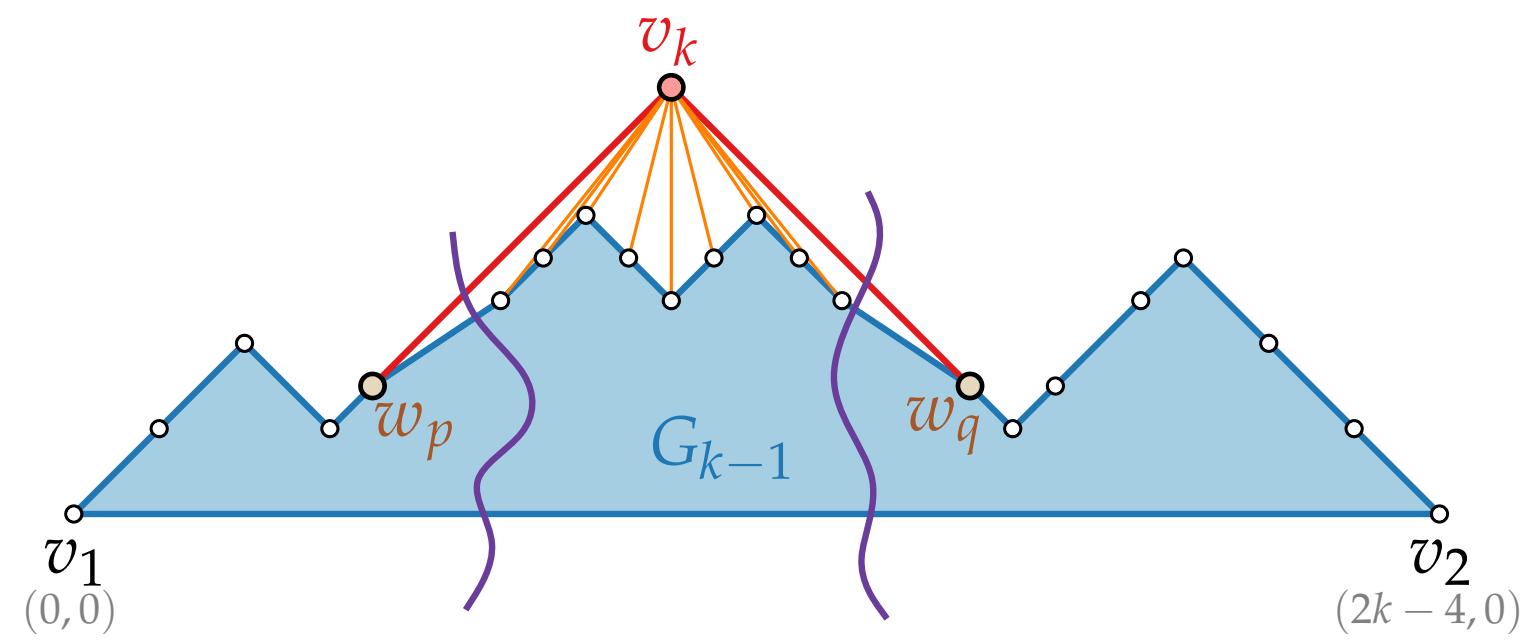


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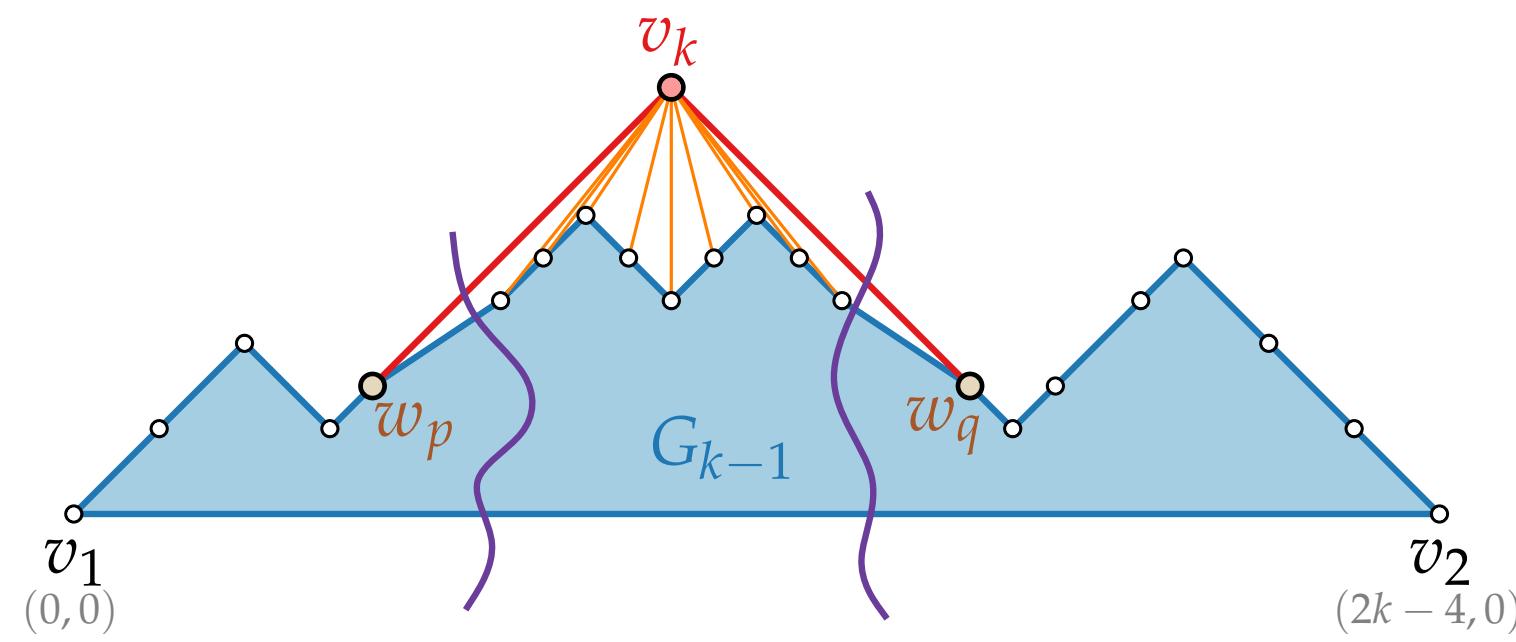
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Does v_k land on grid?



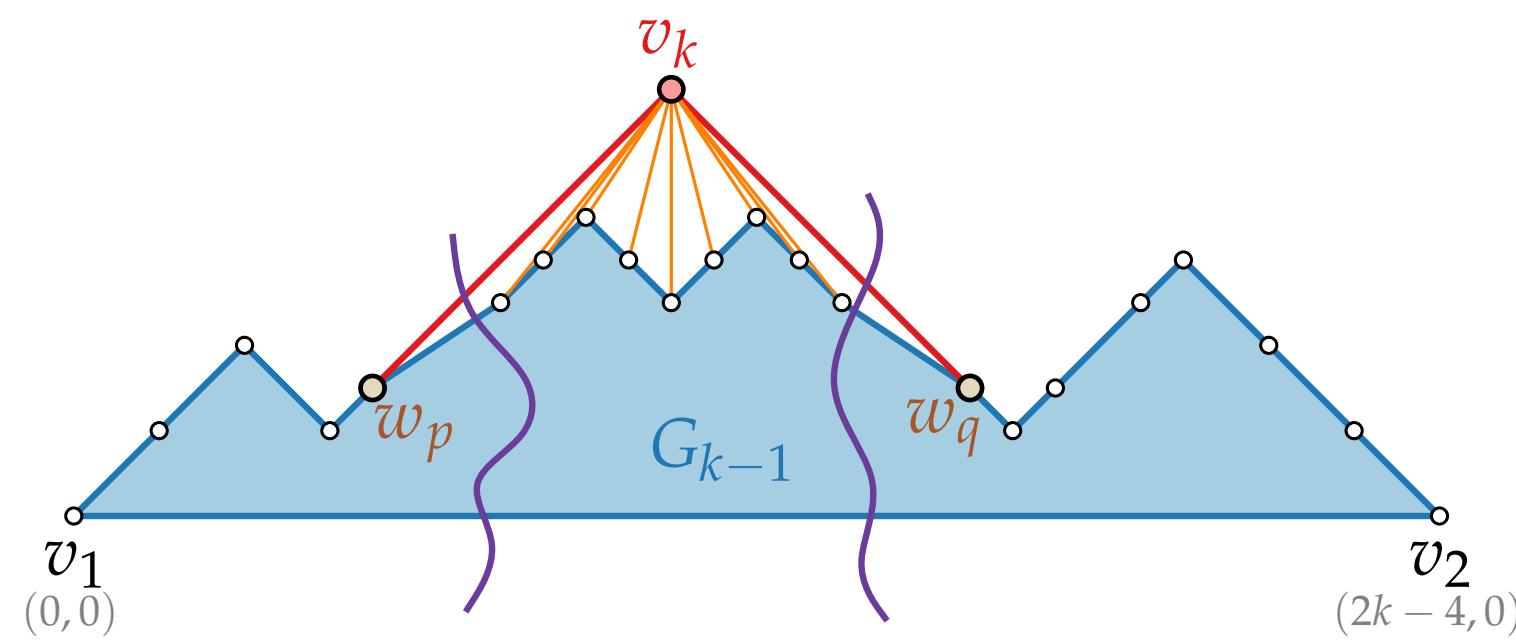
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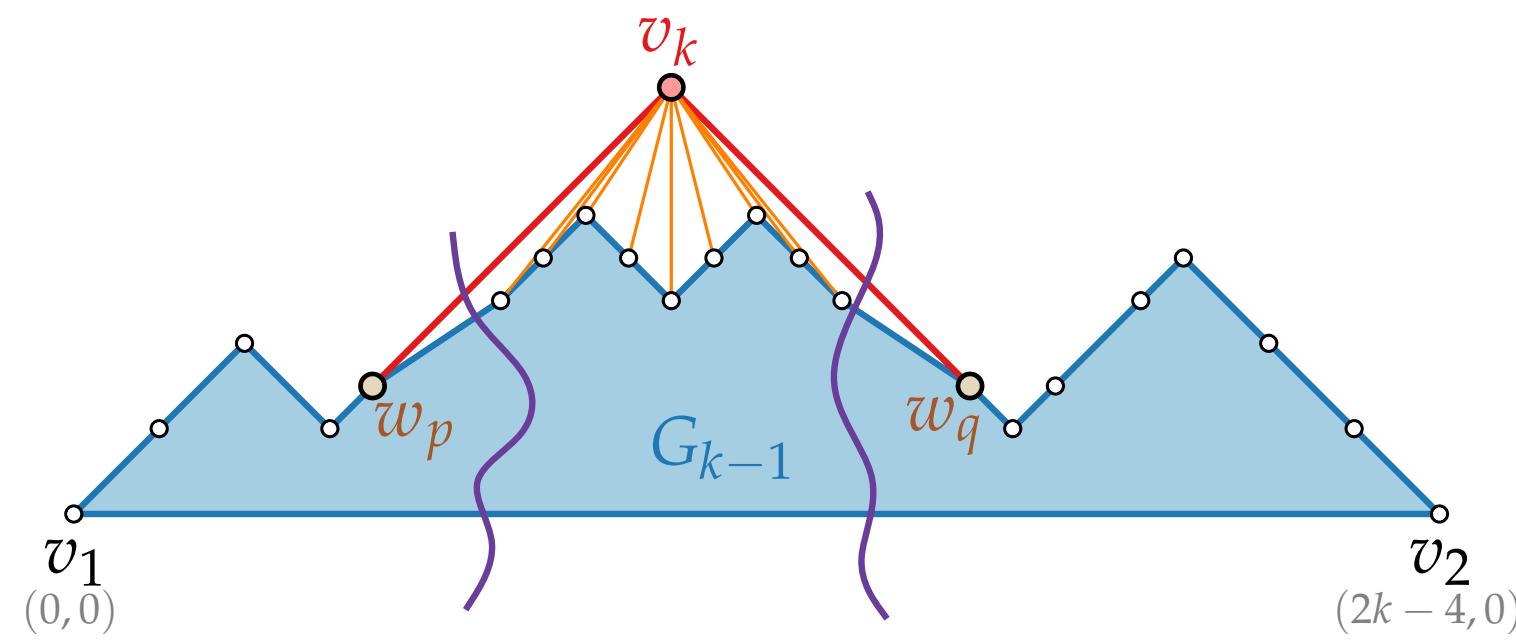
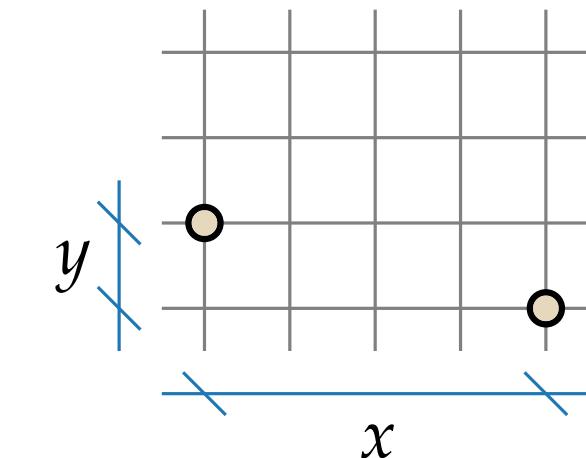
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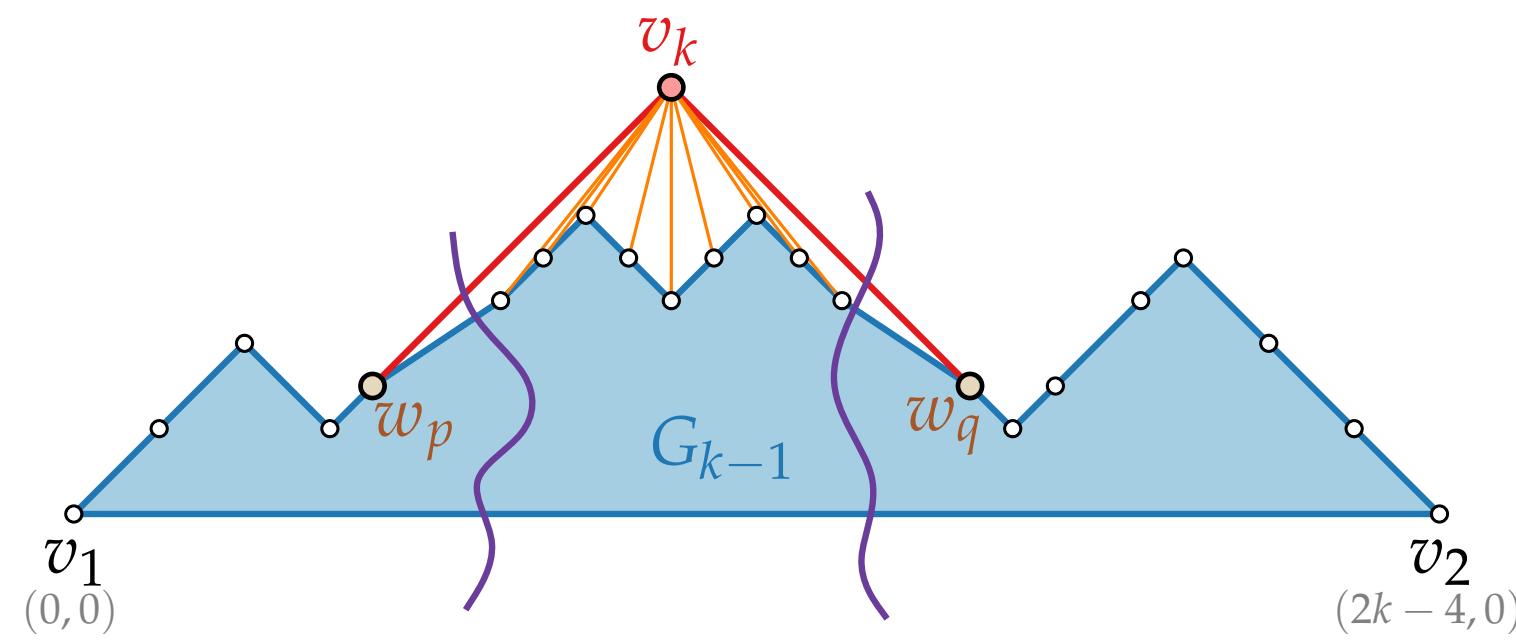
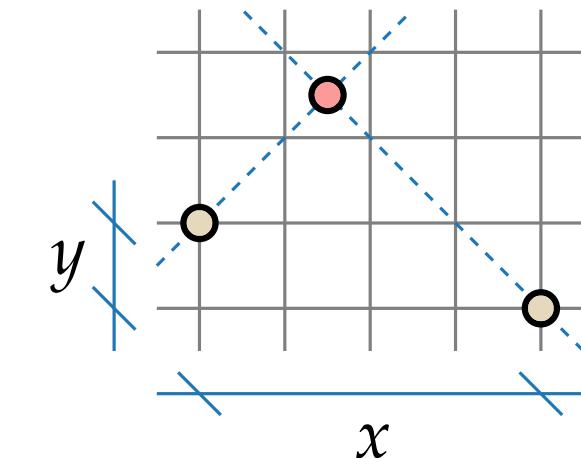
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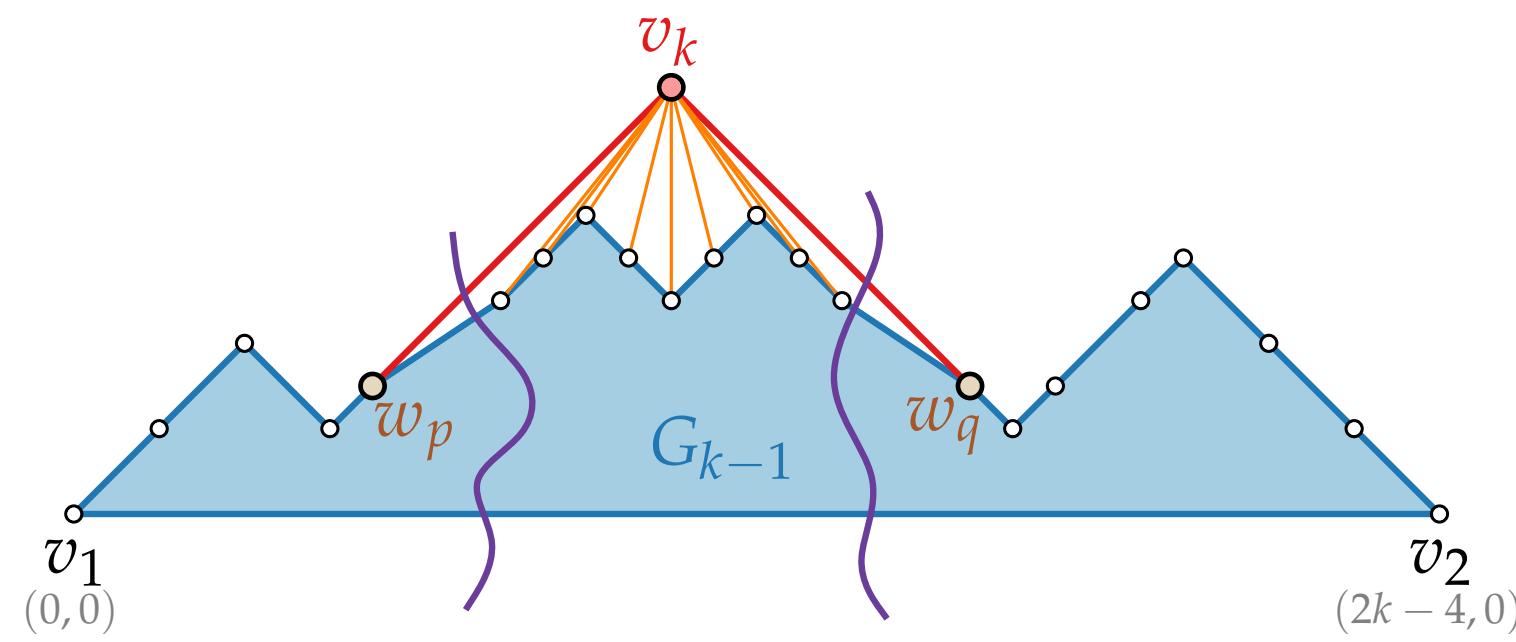
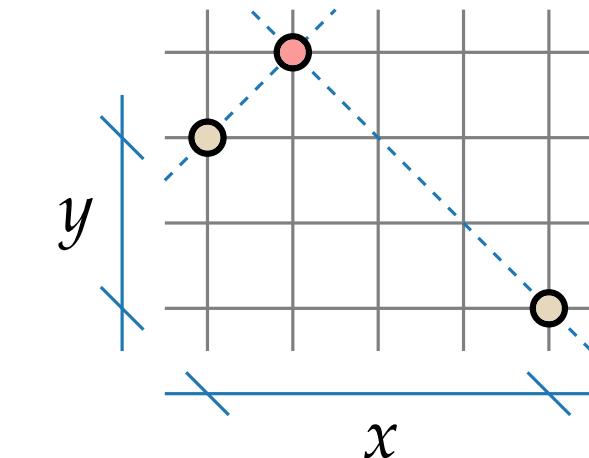
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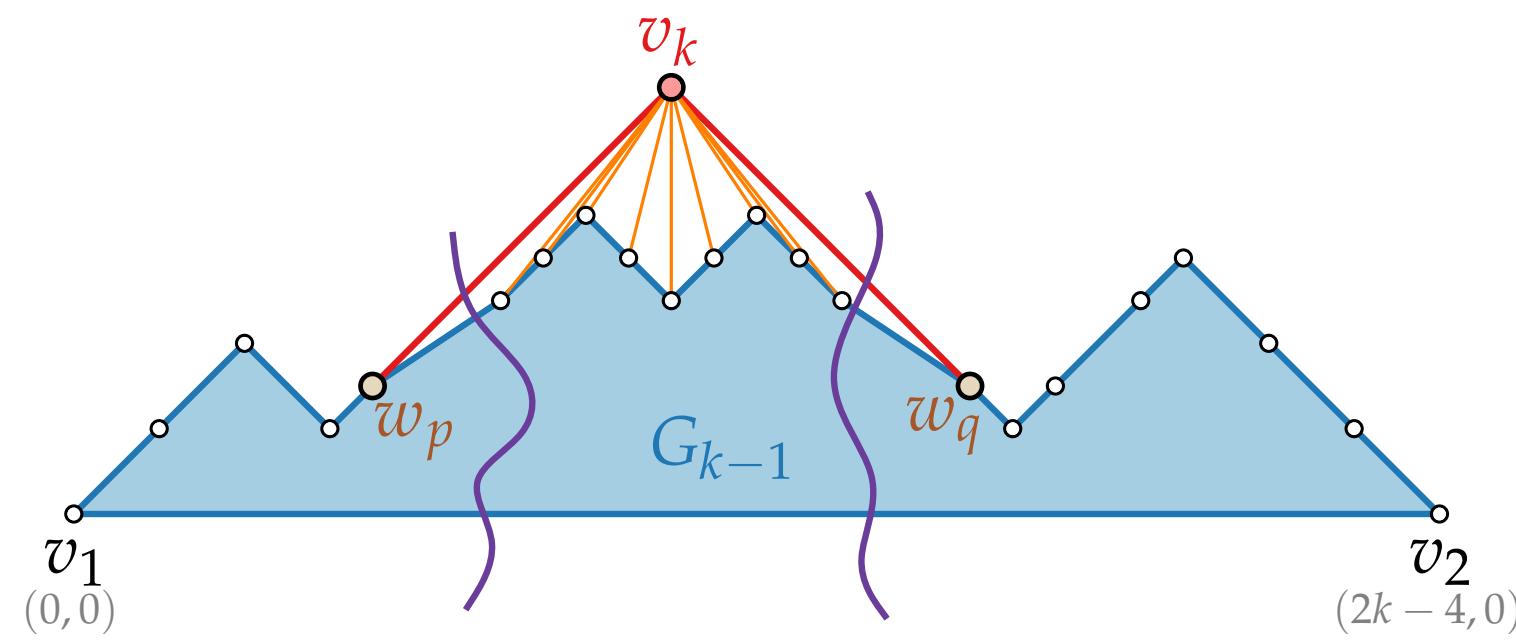
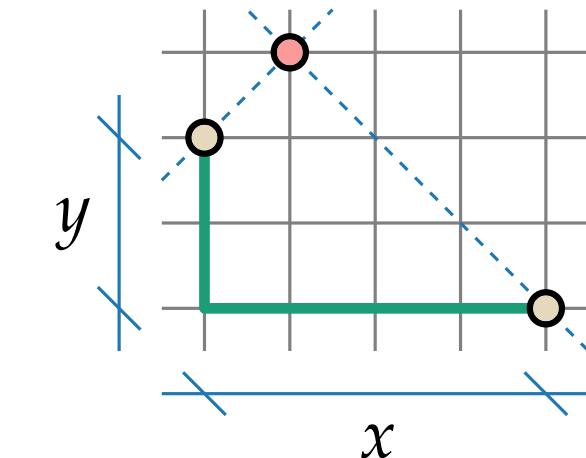
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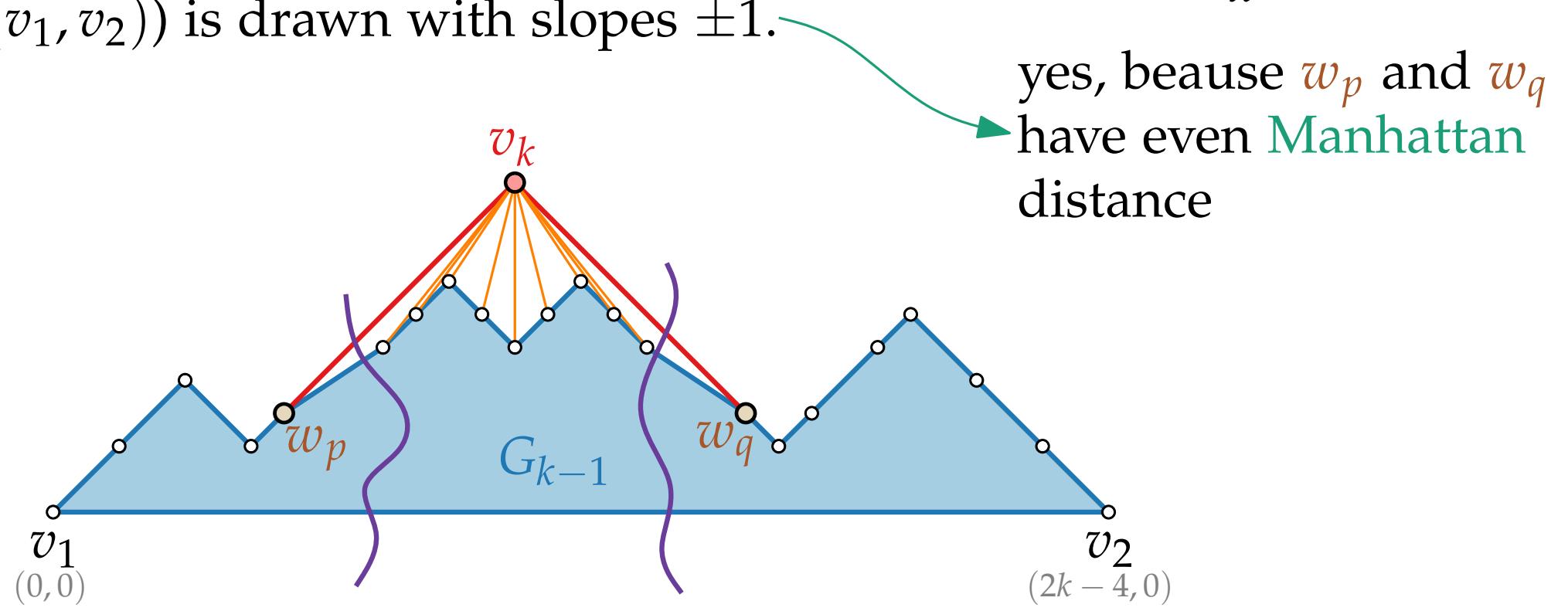


Shift Method – Idea

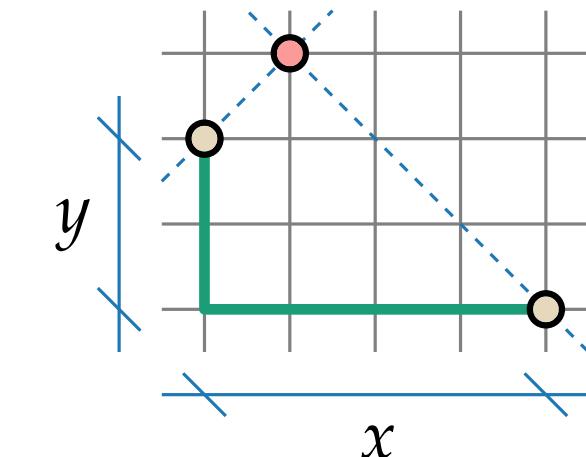
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Does v_k land on grid?



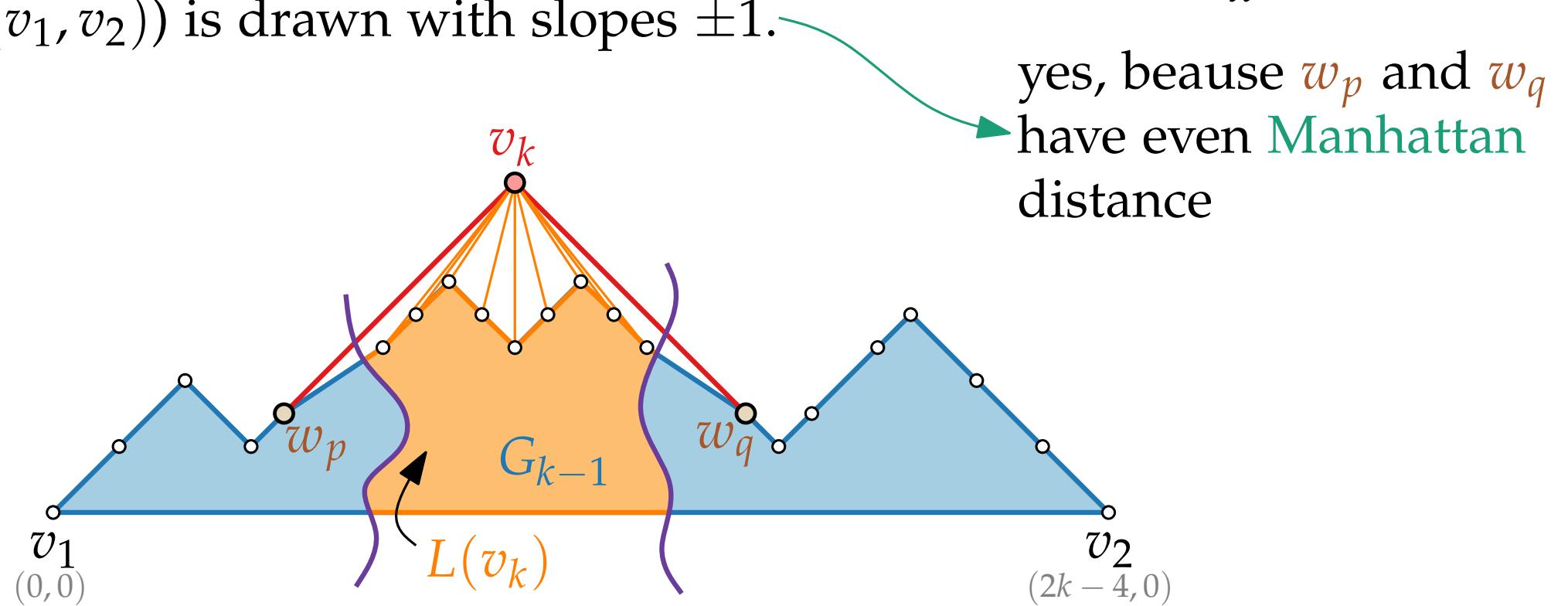
yes, because w_p and w_q have even Manhattan distance

Shift Method – Idea

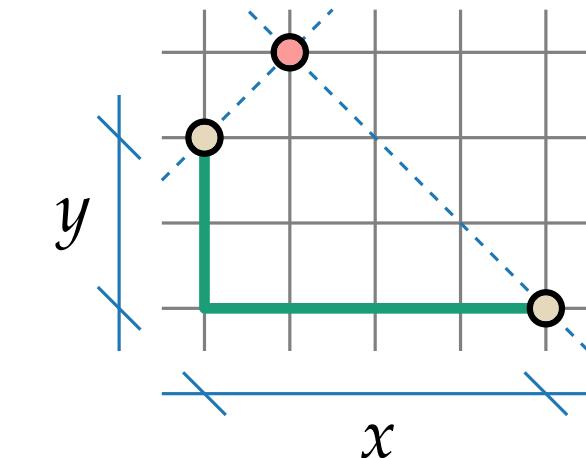
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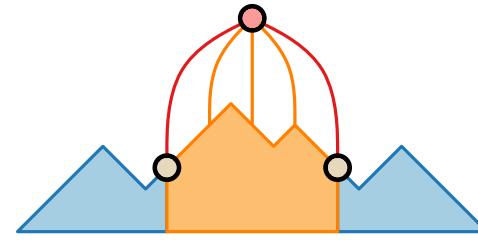


Does v_k land on grid?

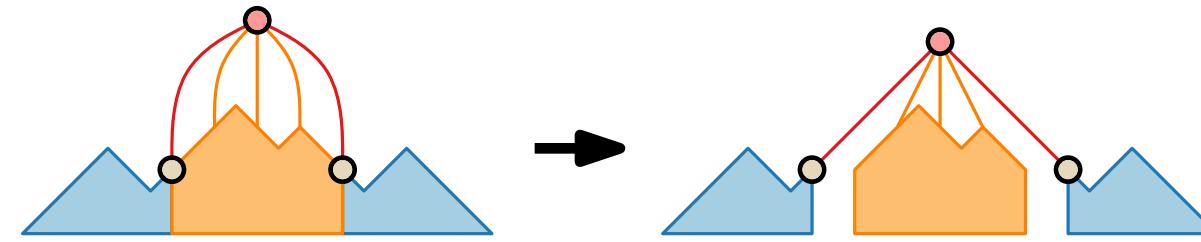


yes, because w_p and w_q have even Manhattan distance

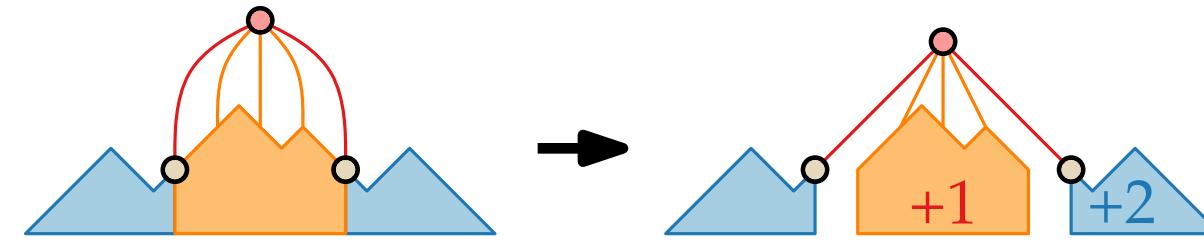
Shift Method – Example



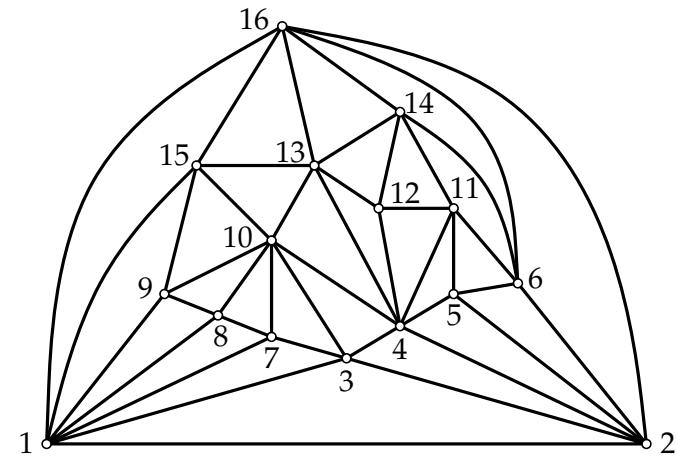
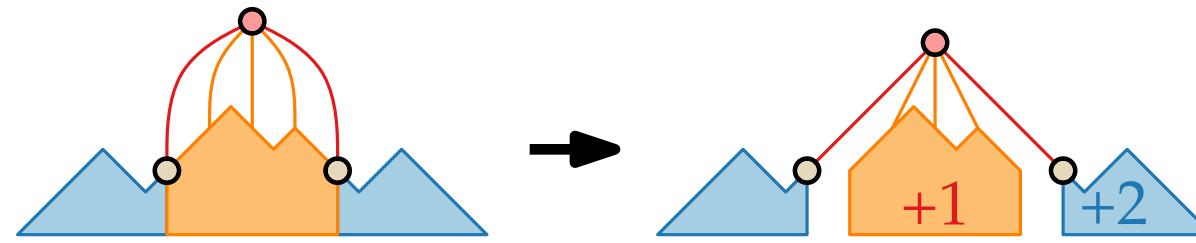
Shift Method – Example



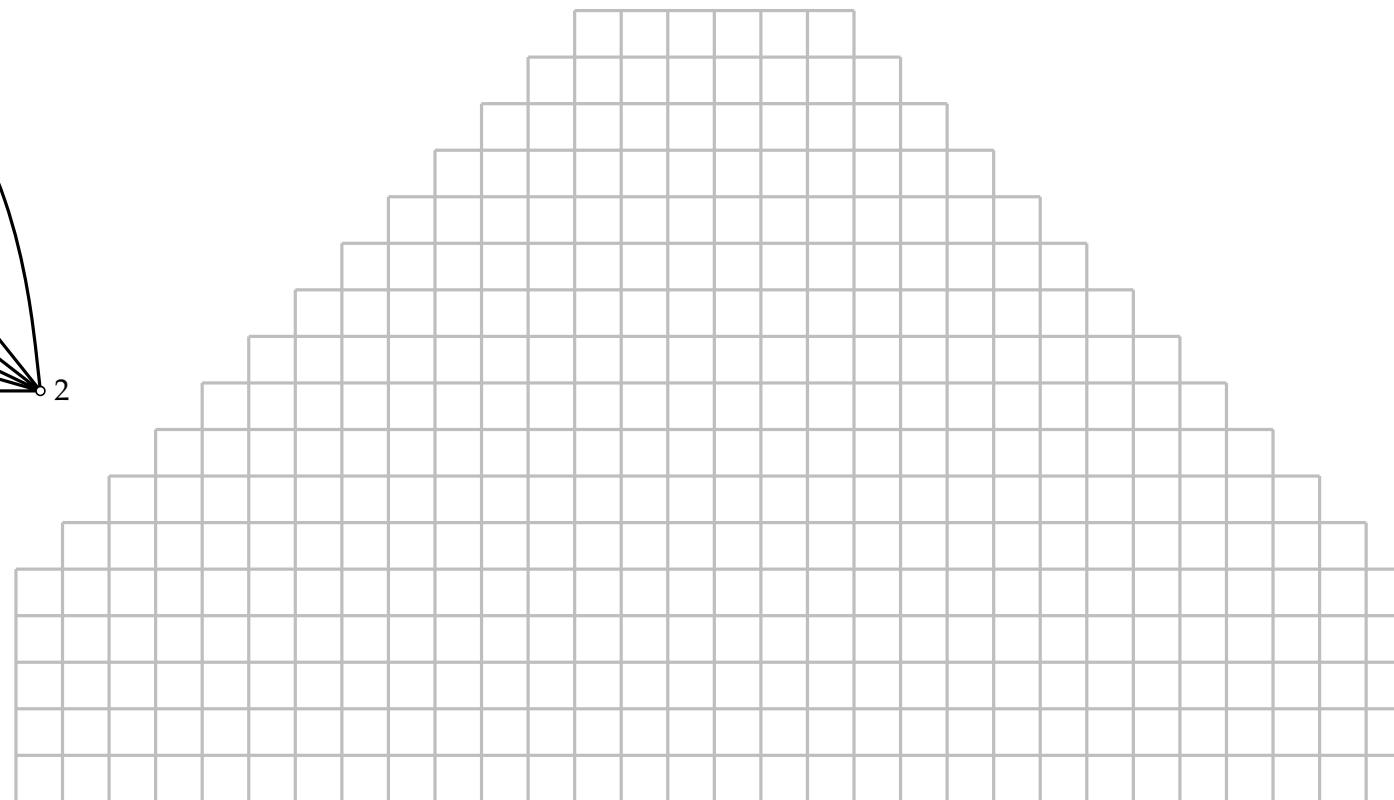
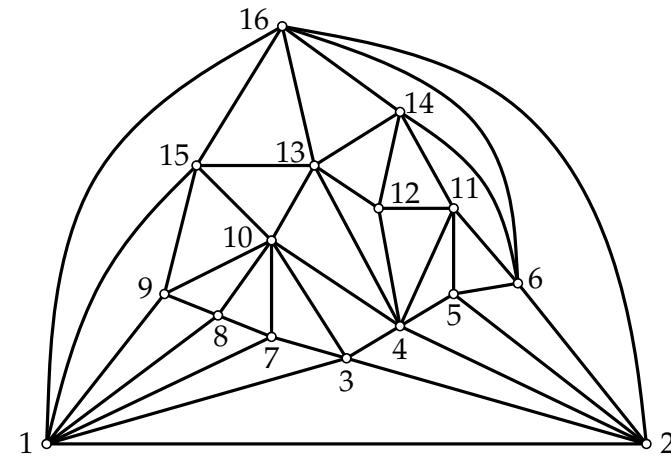
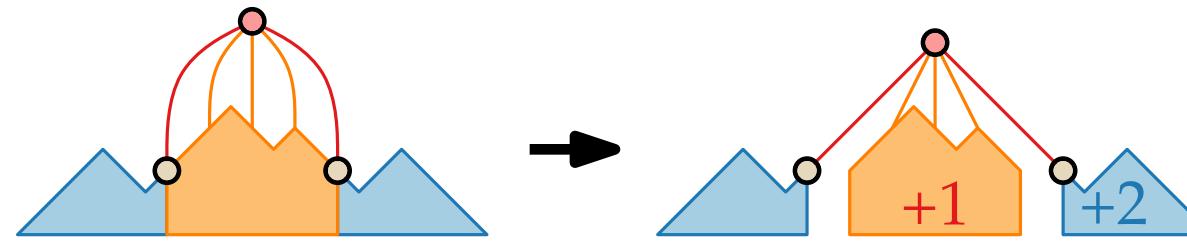
Shift Method – Example



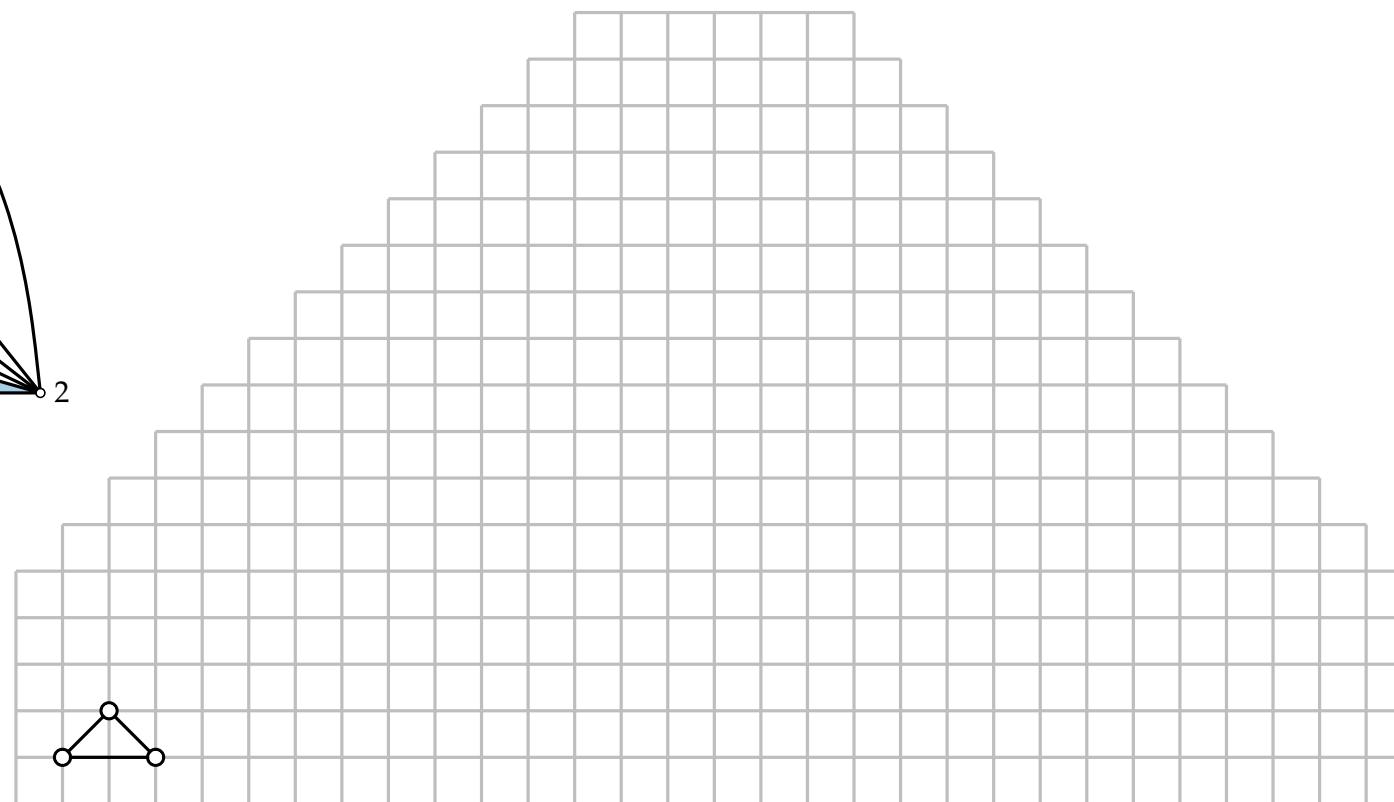
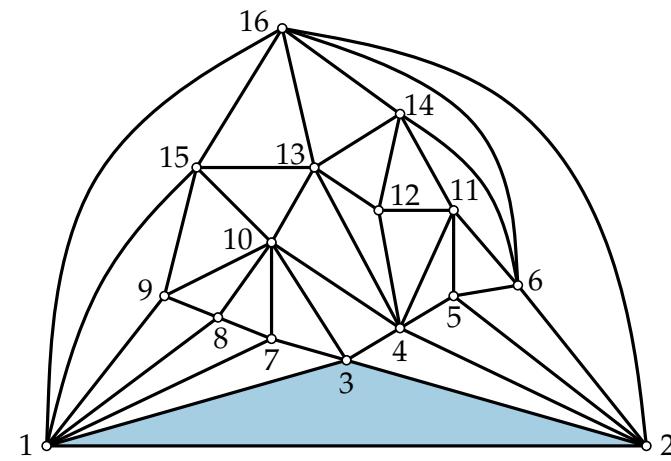
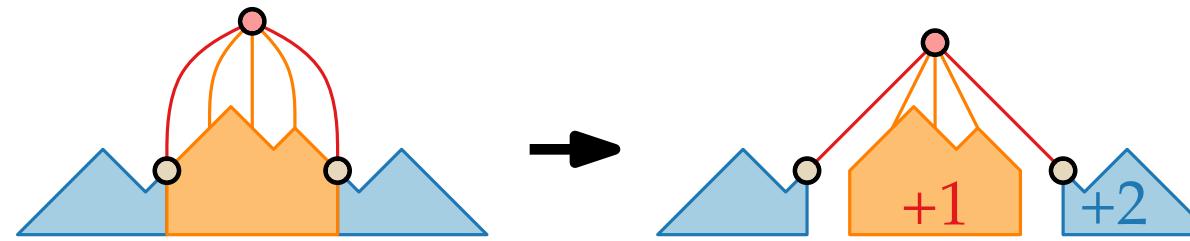
Shift Method – Example



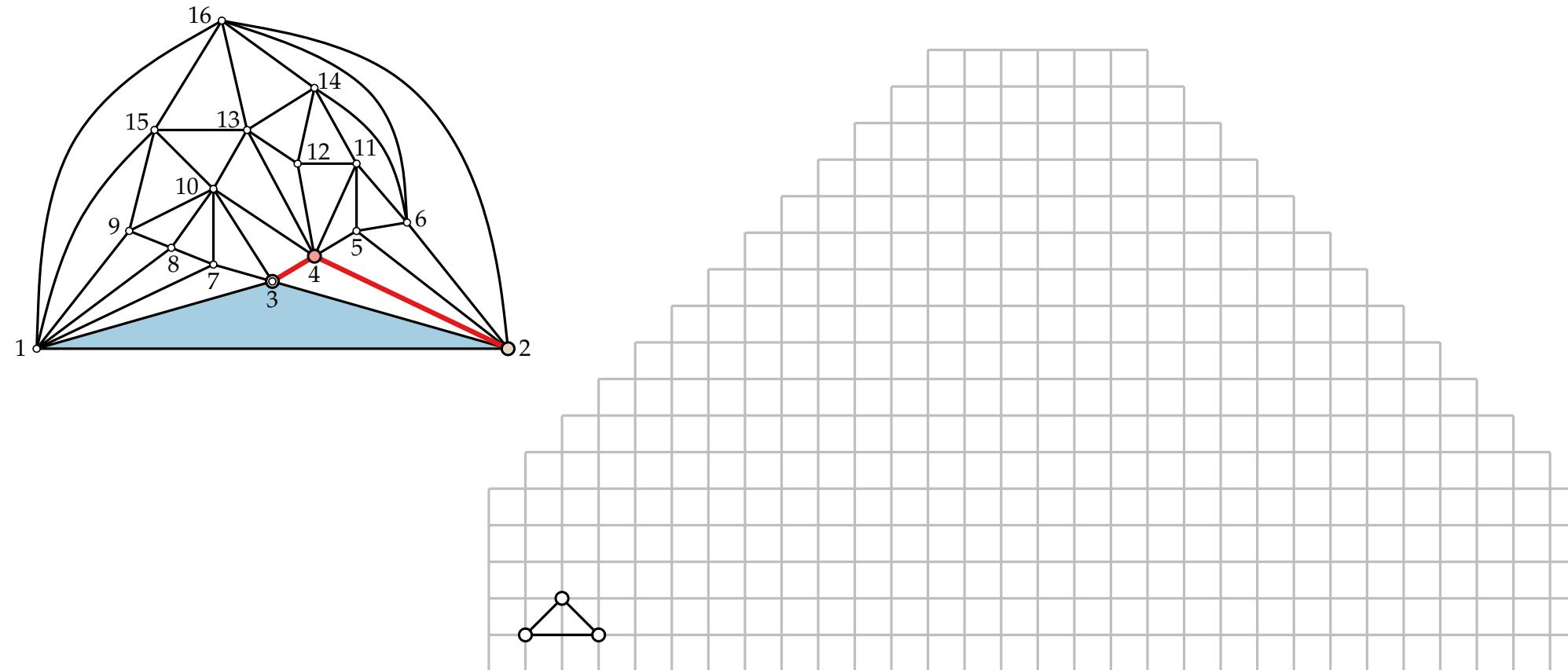
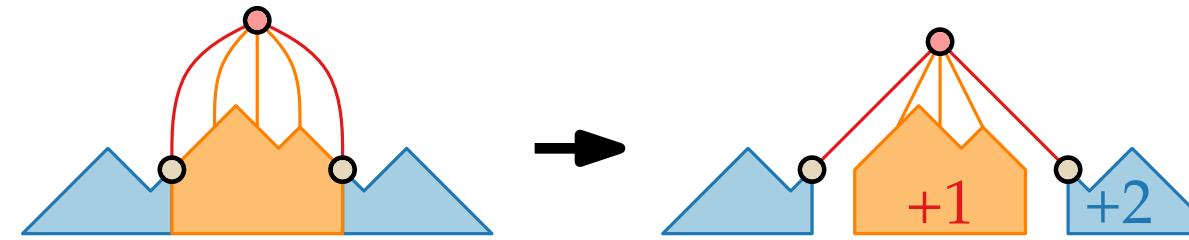
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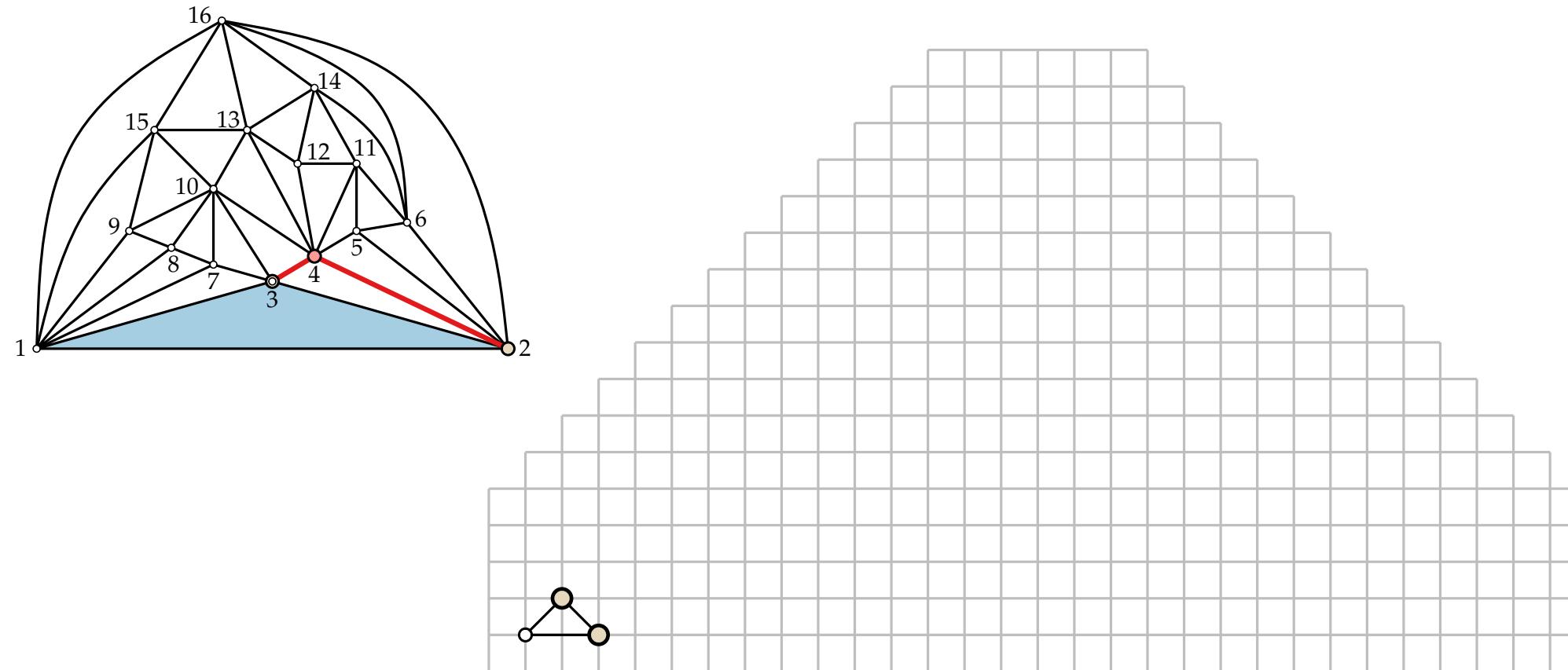
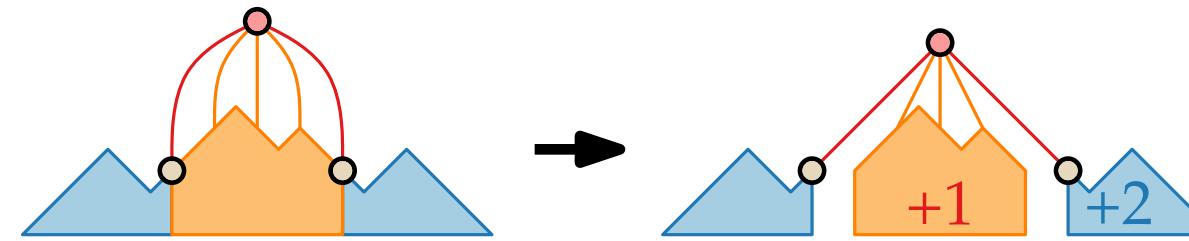
Shift Method – Example



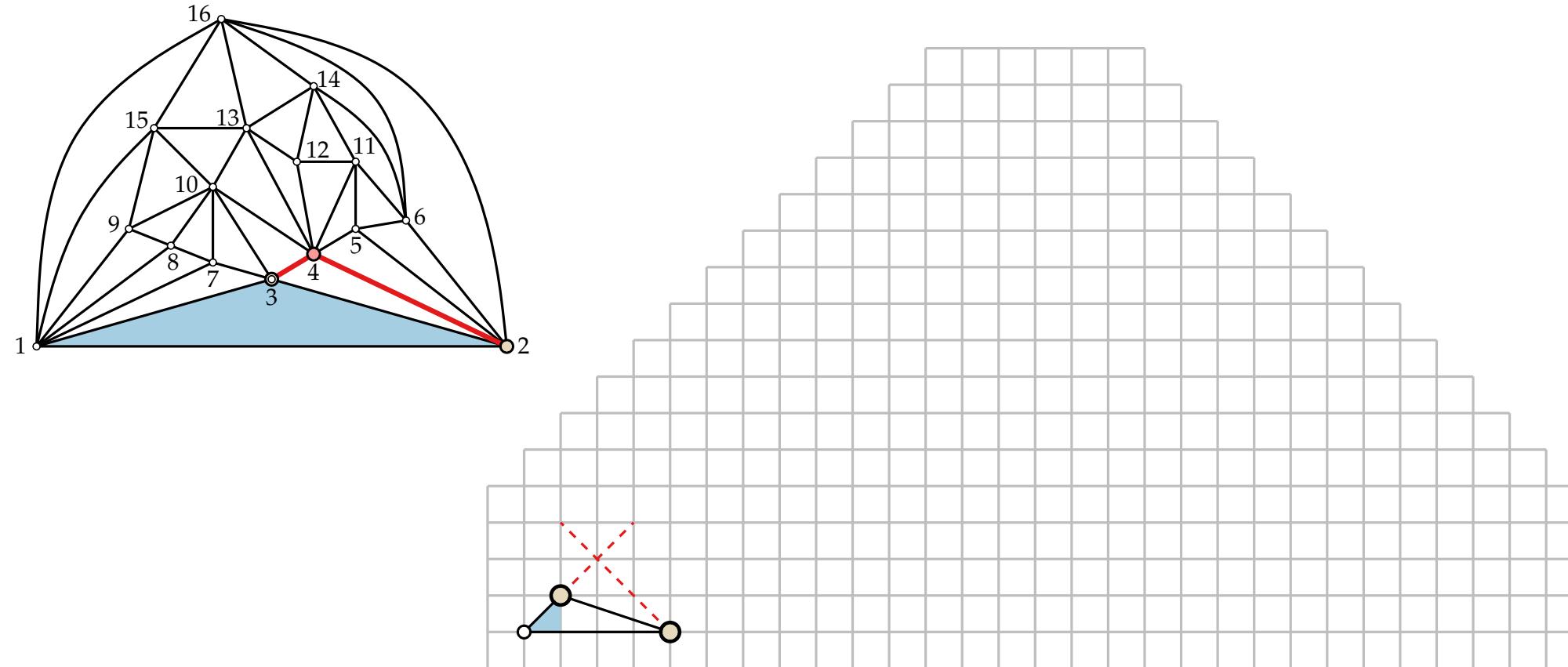
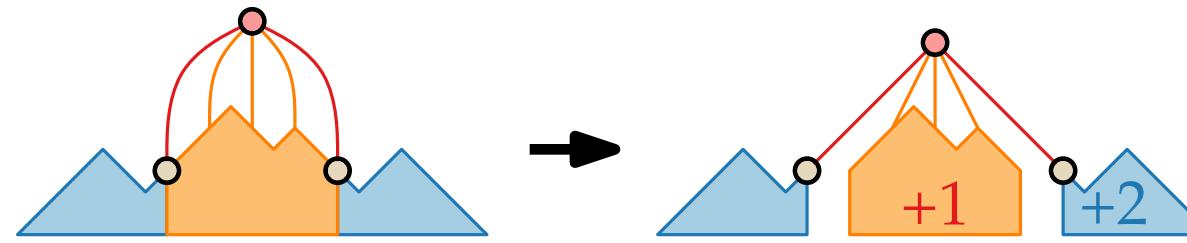
Shift Method – Example



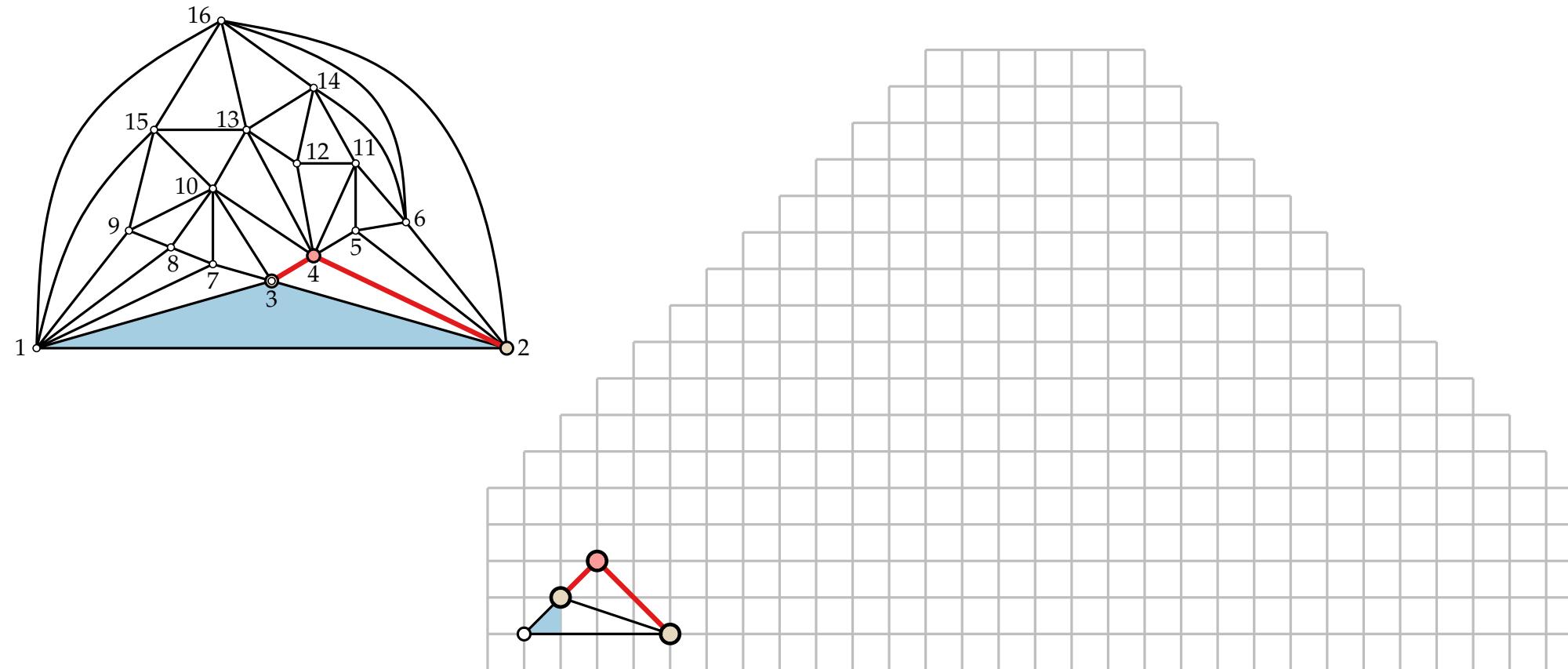
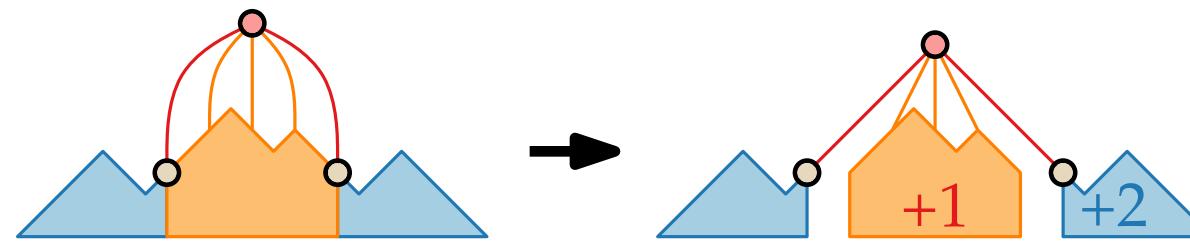
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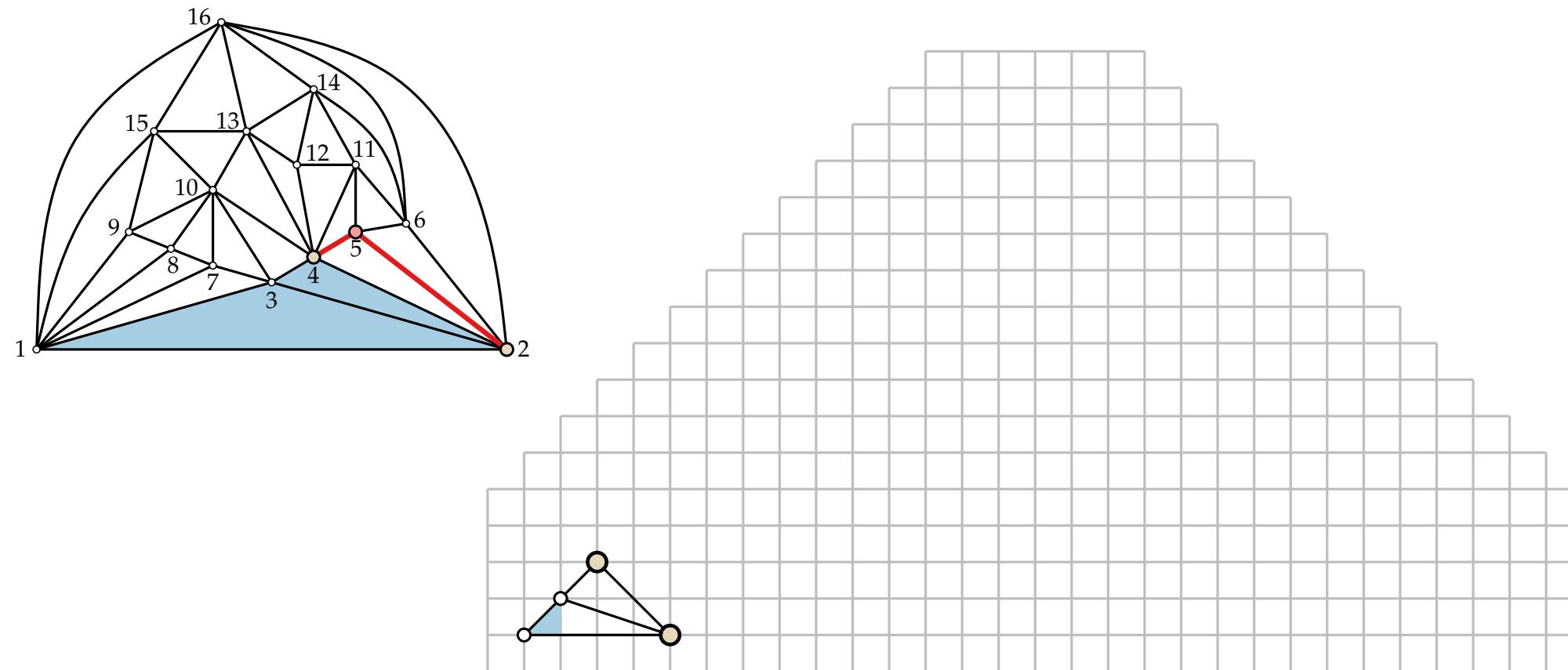
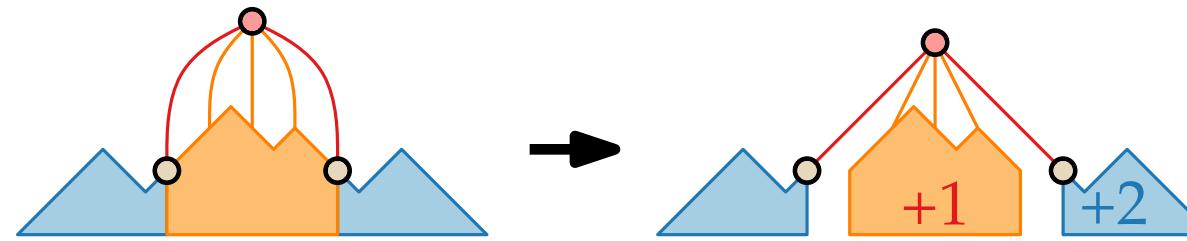
Shift Method – Example



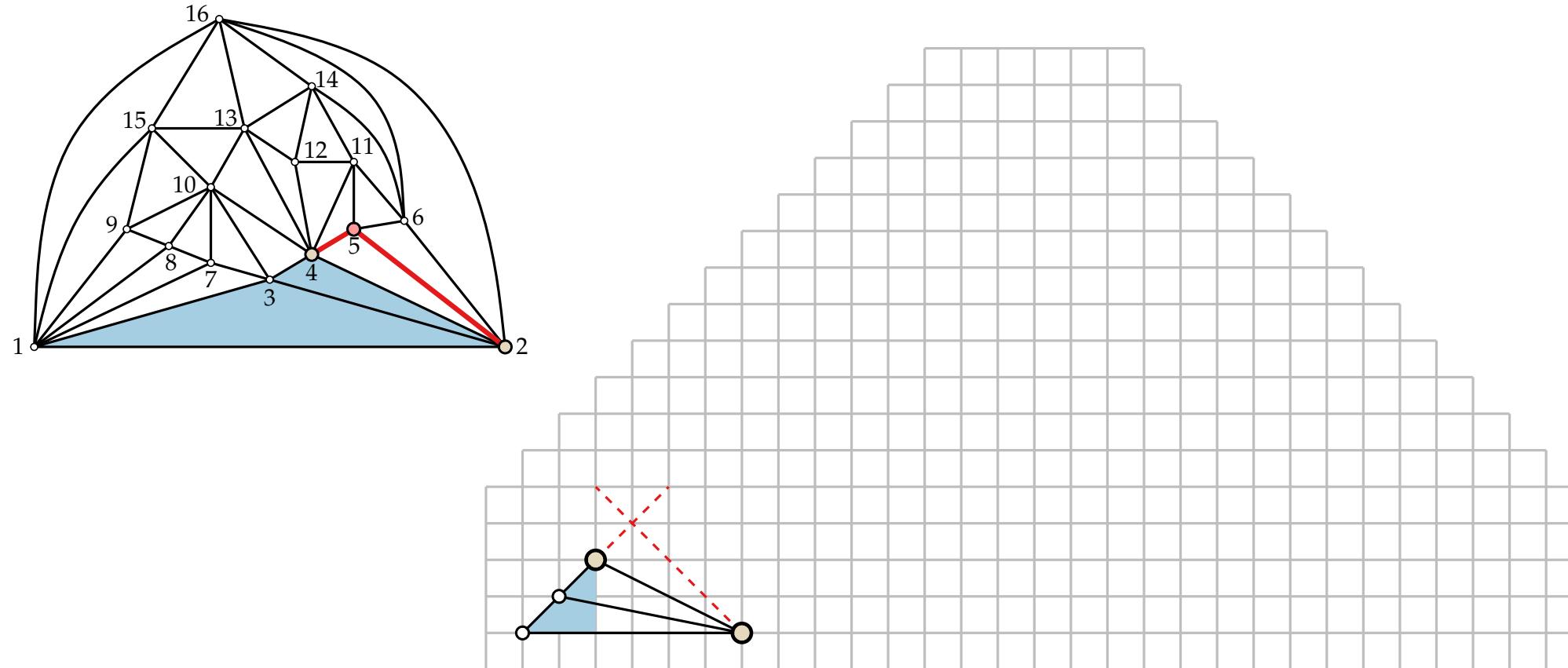
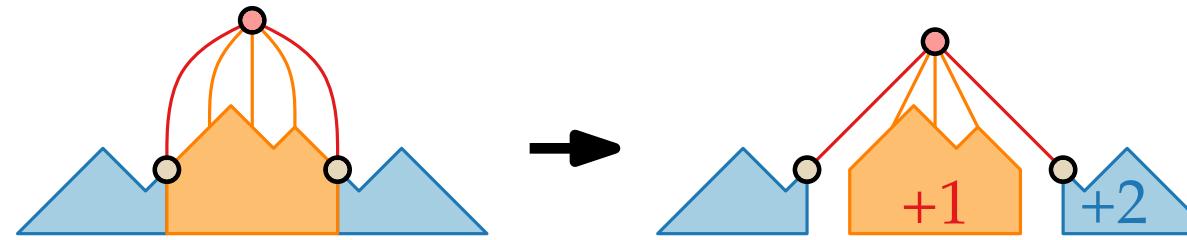
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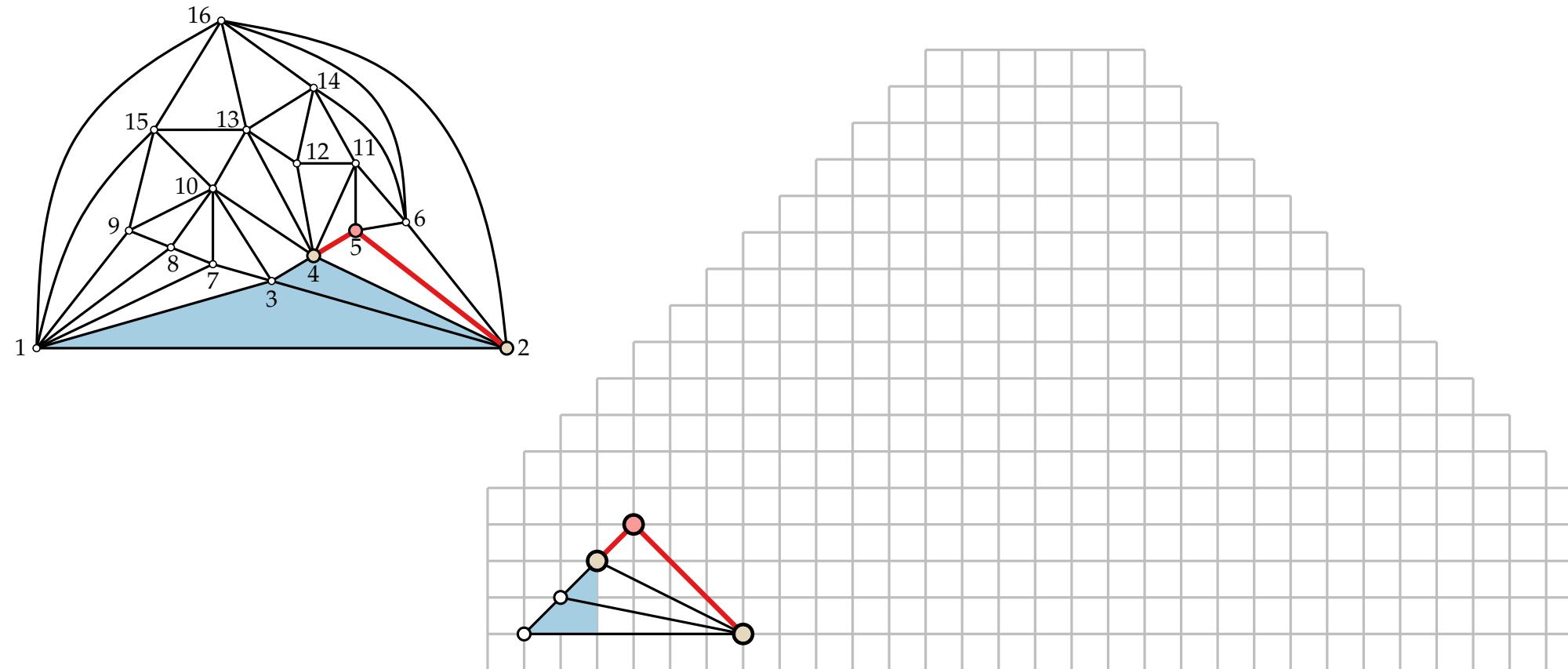
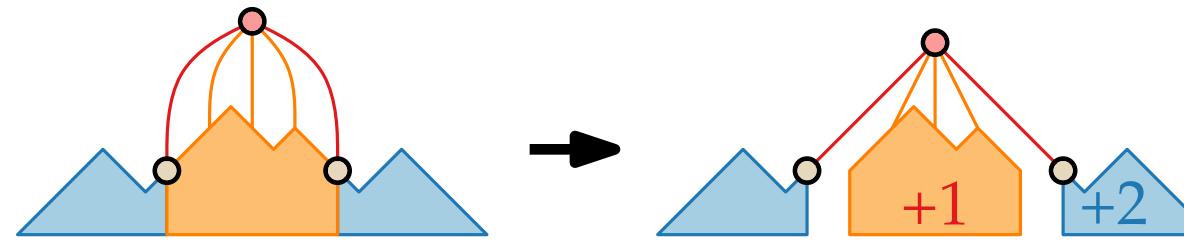
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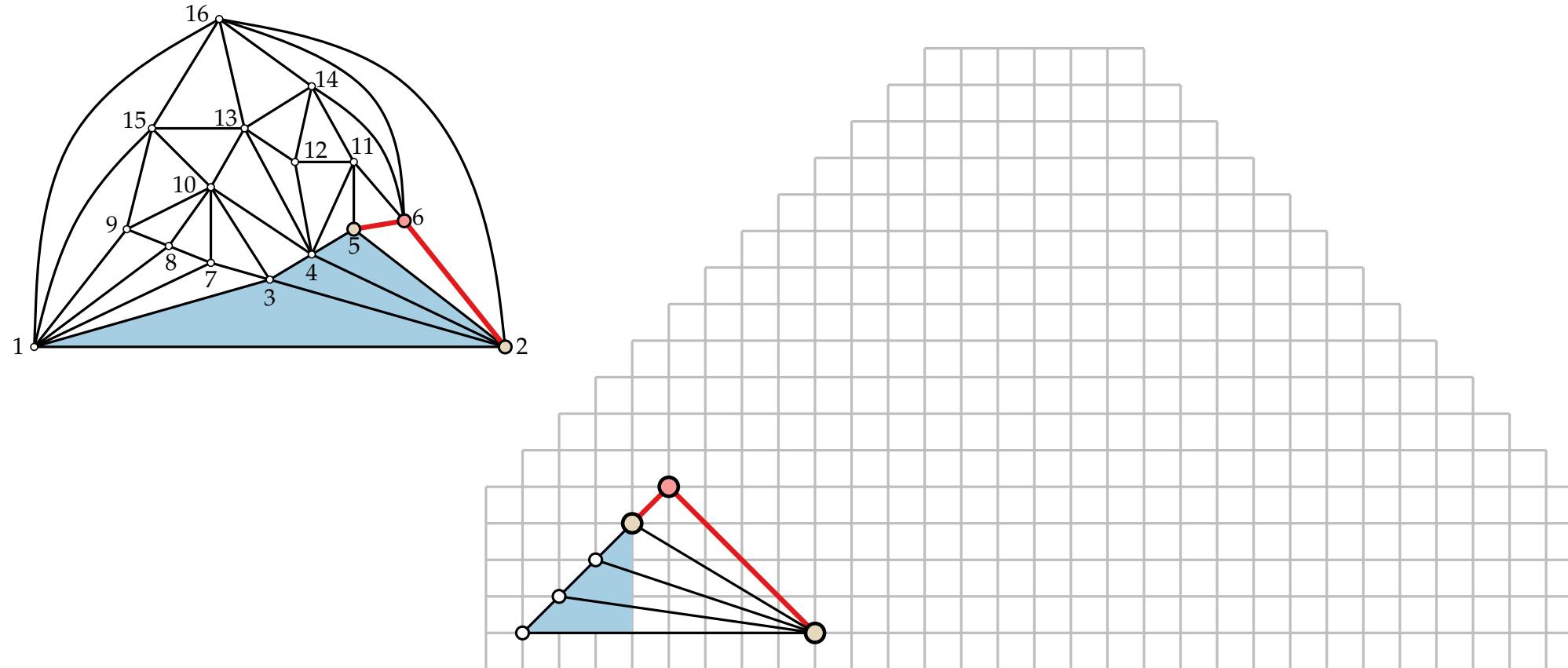
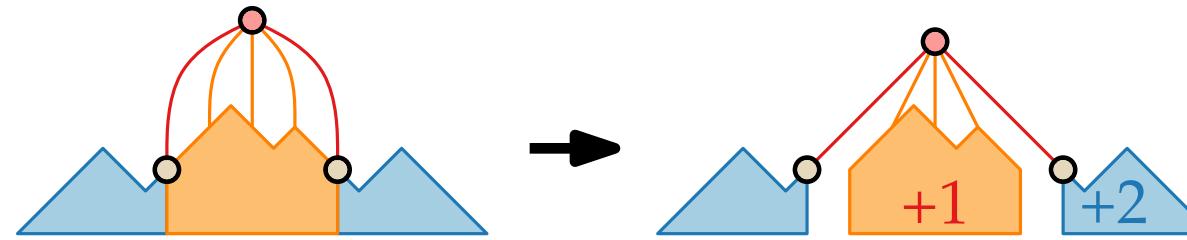
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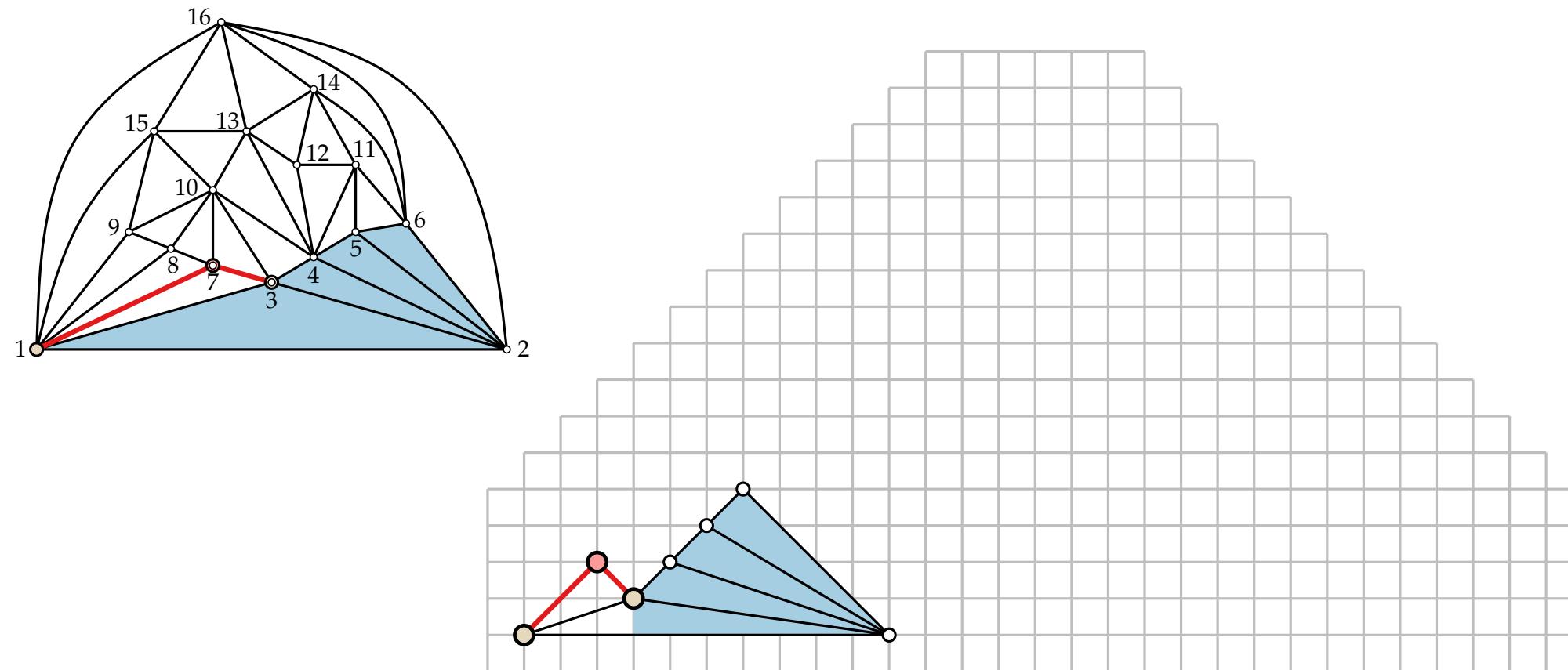
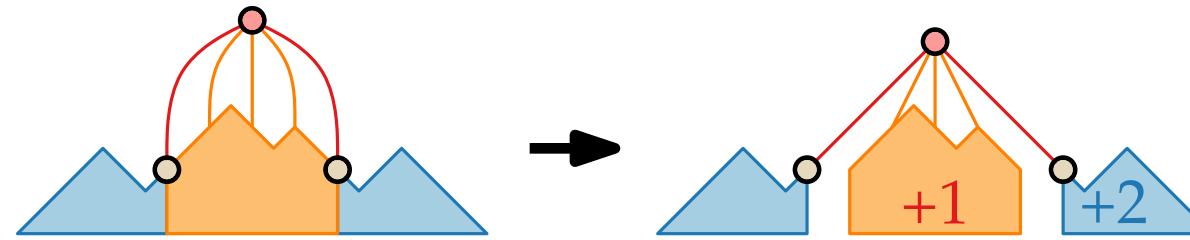
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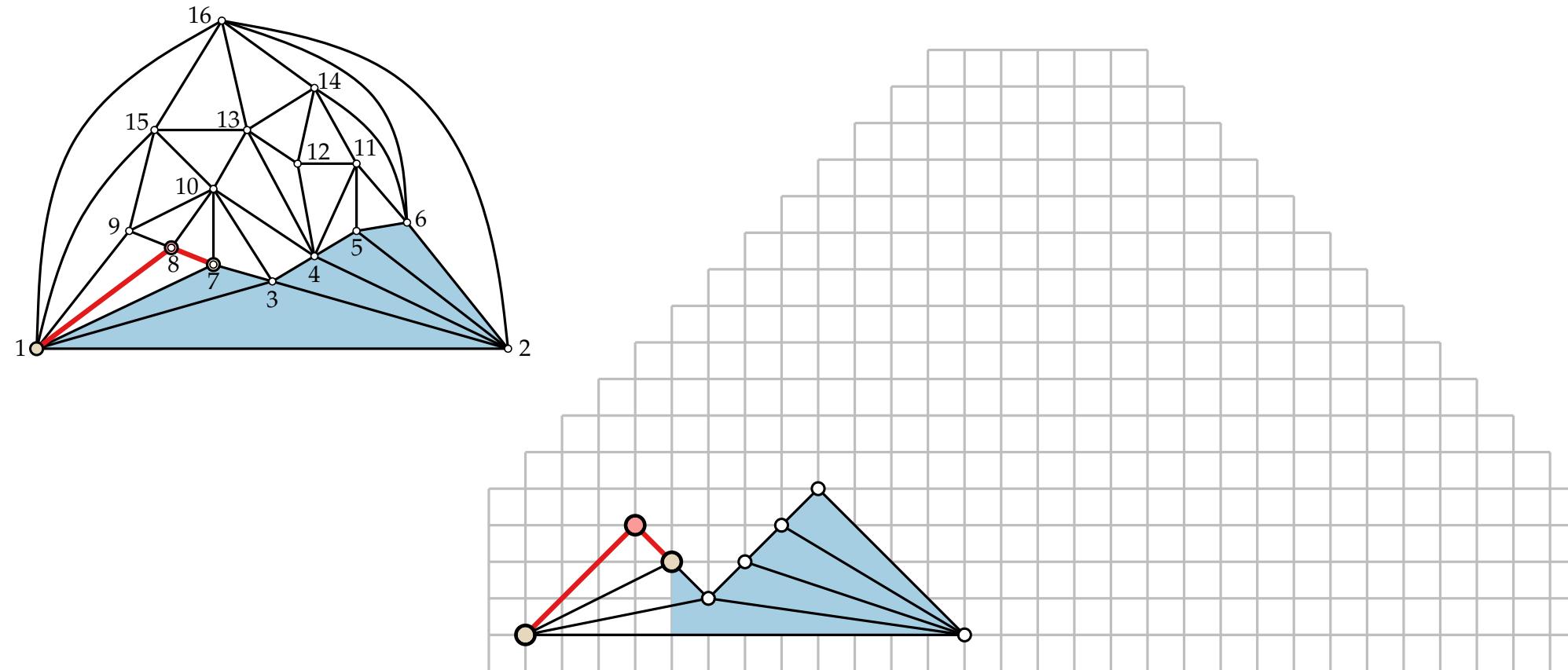
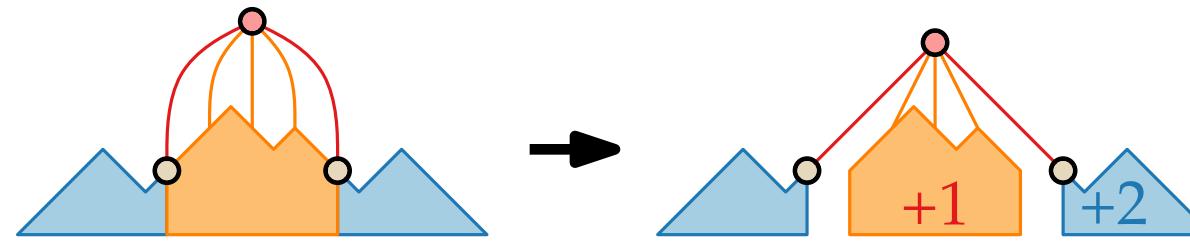
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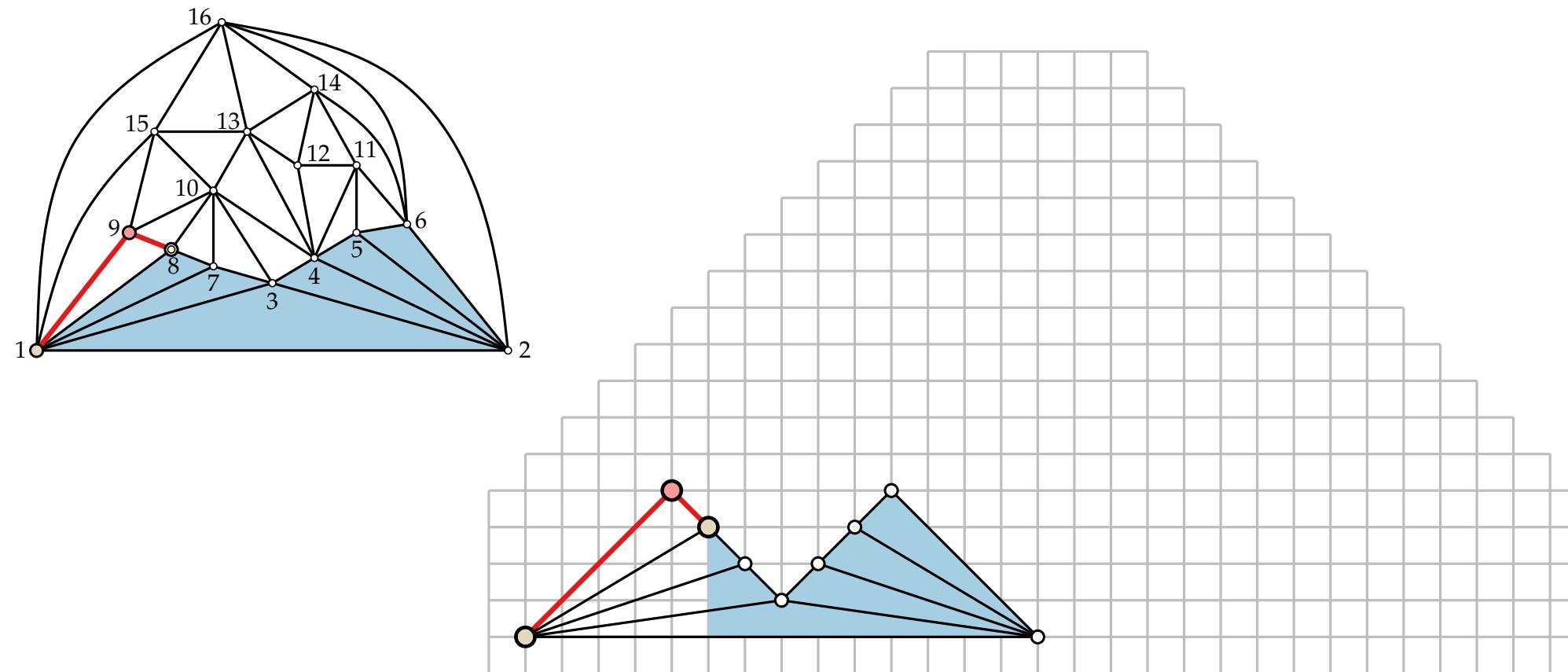
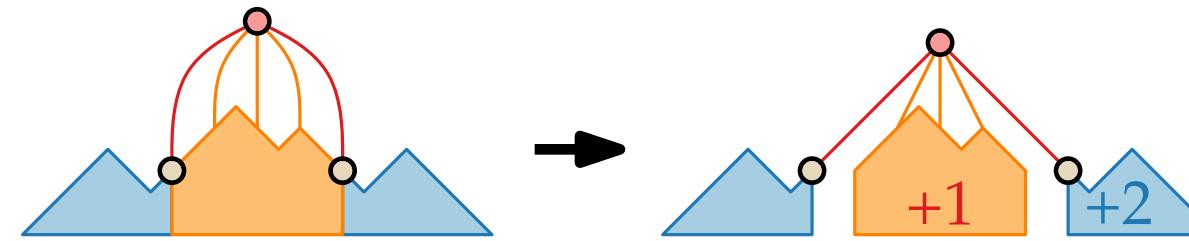
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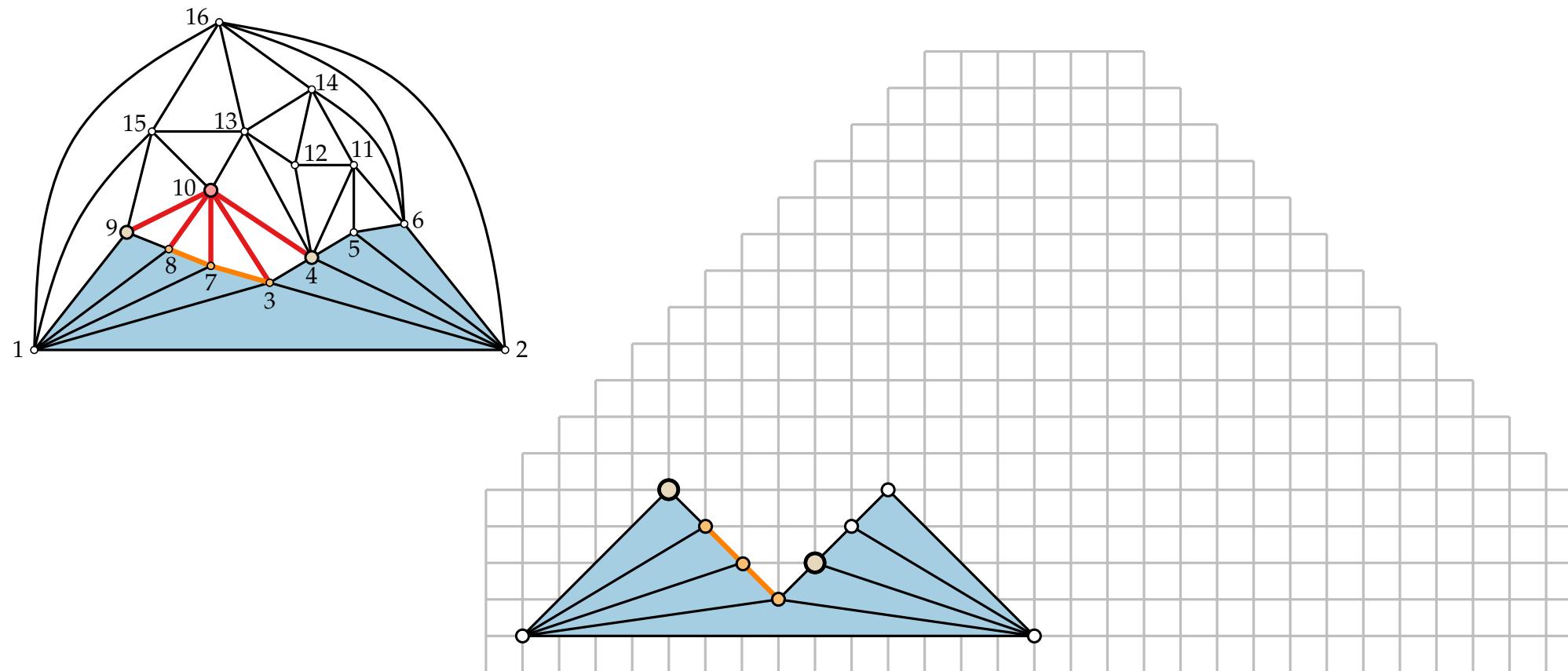
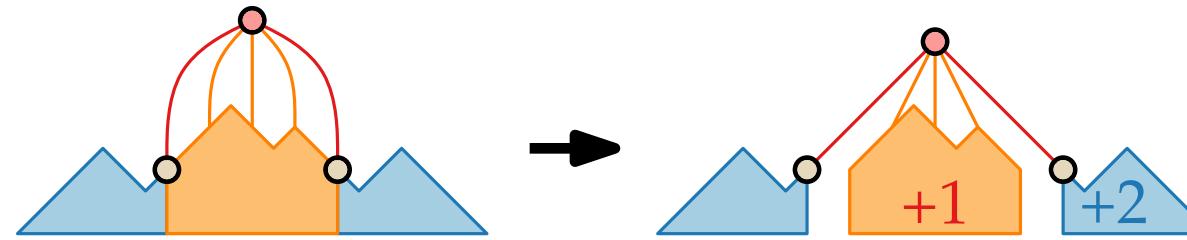
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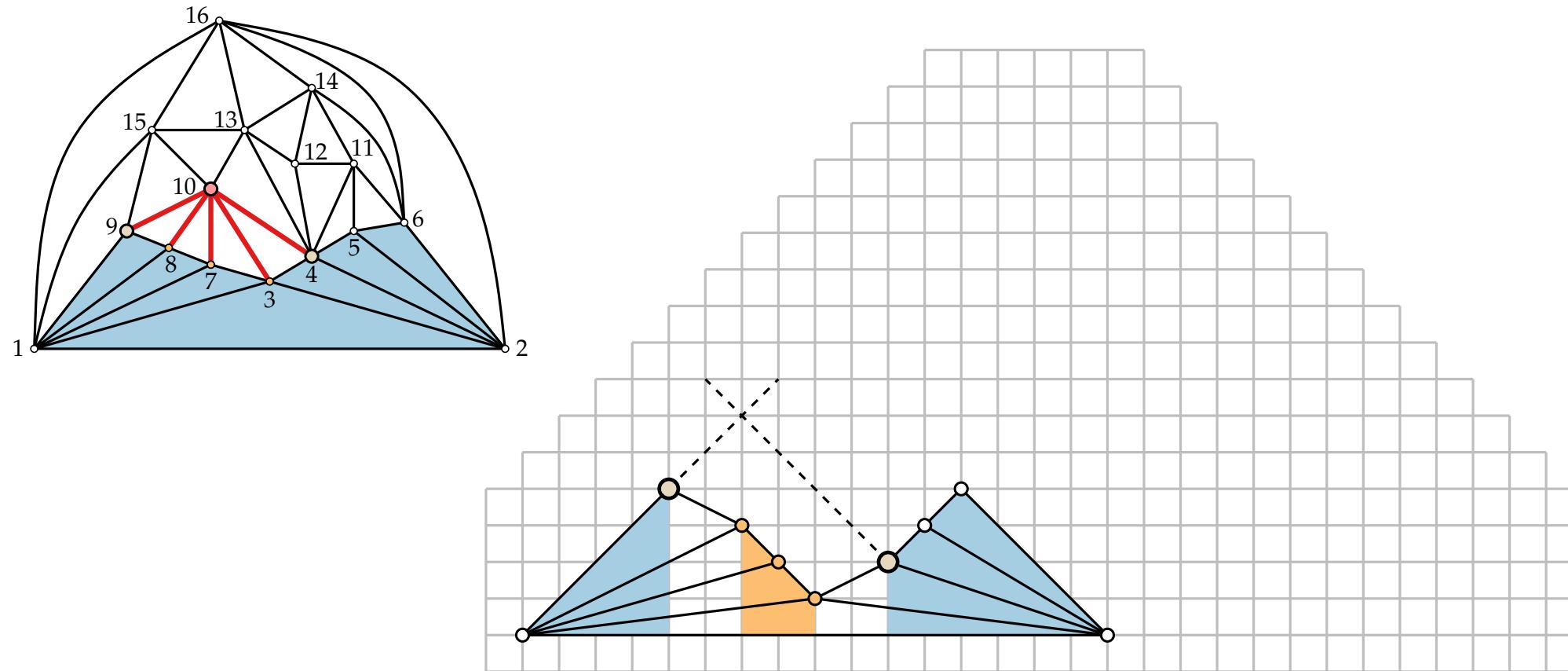
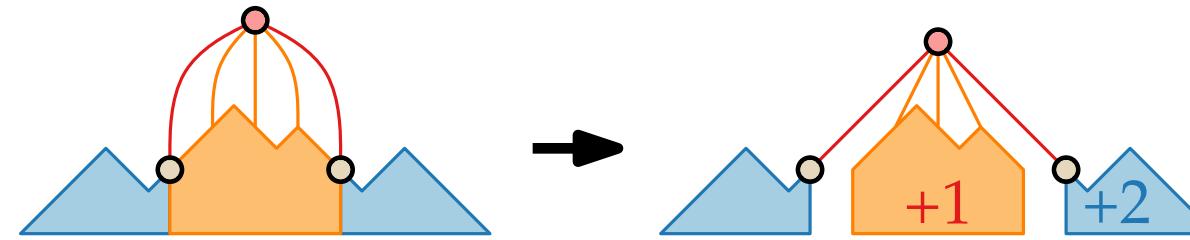
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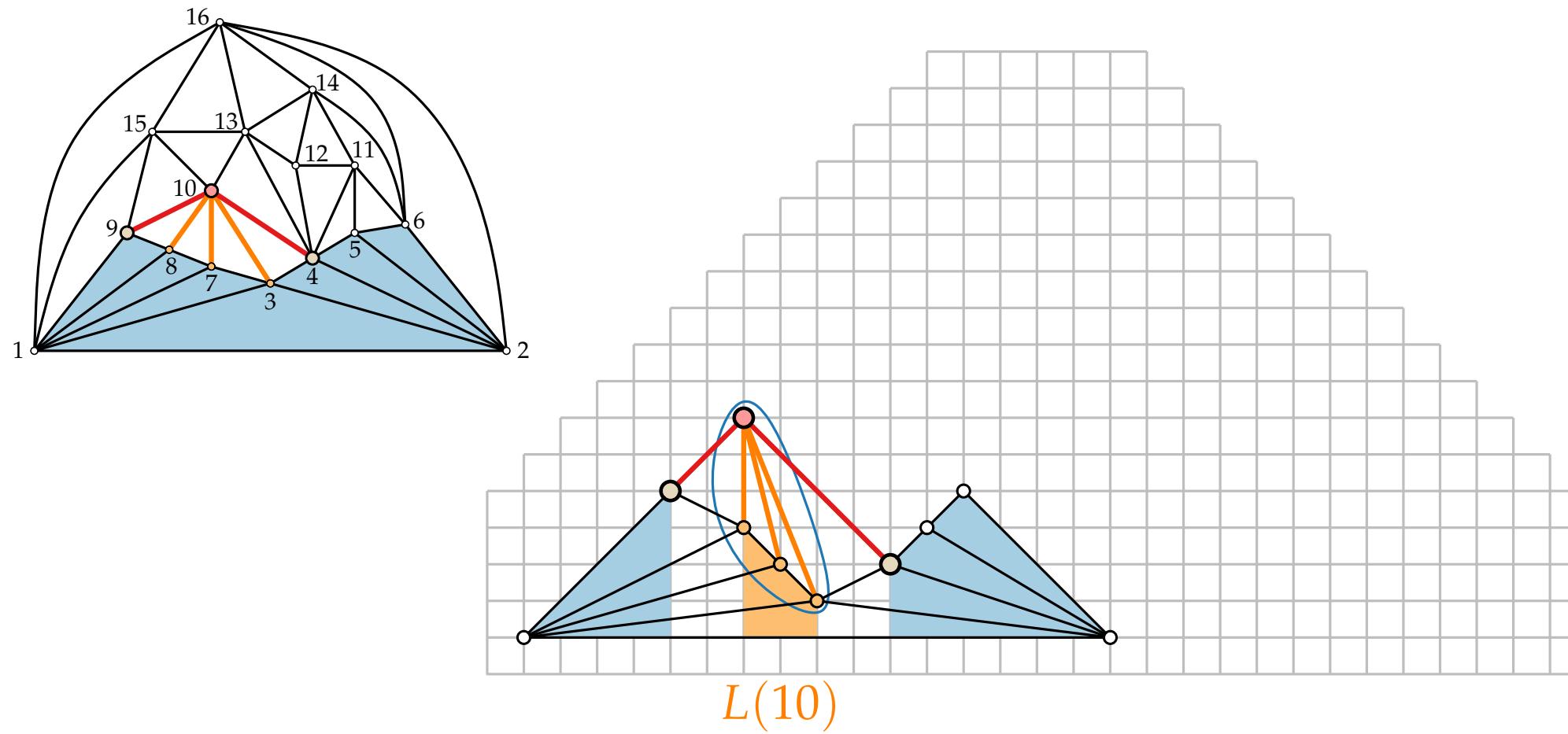
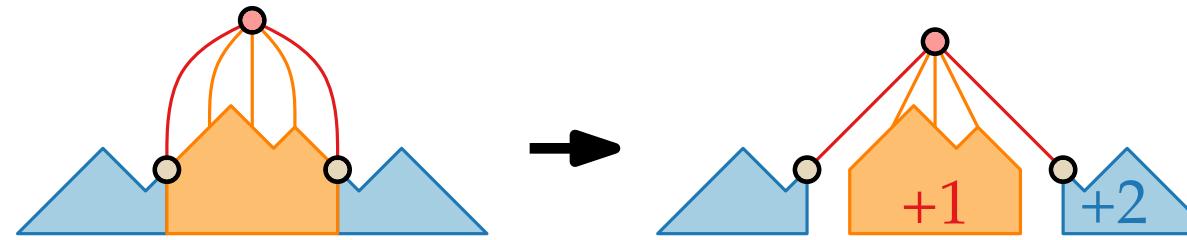
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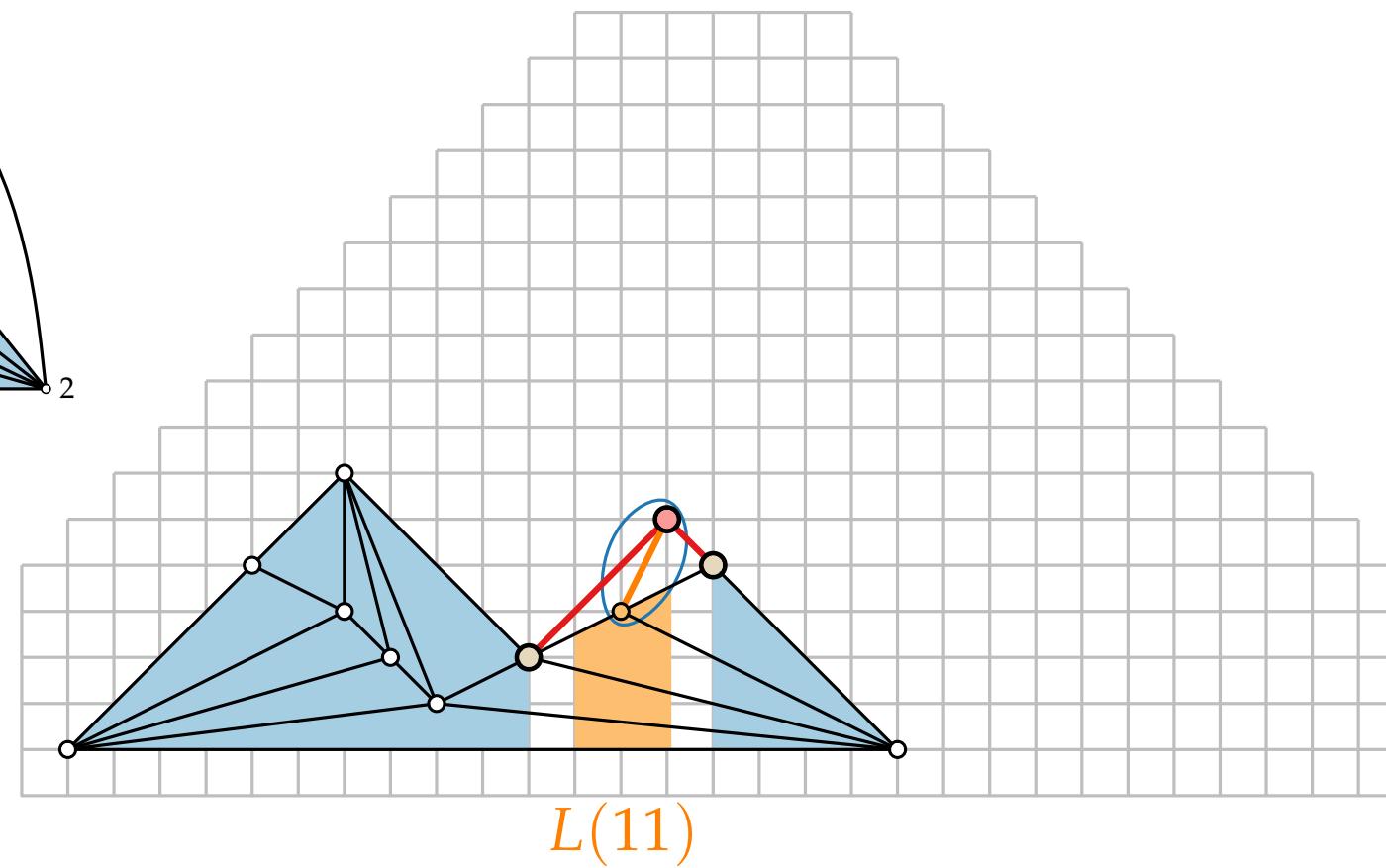
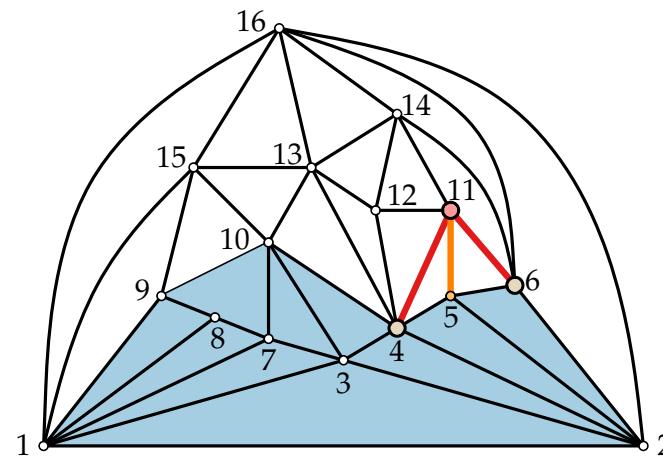
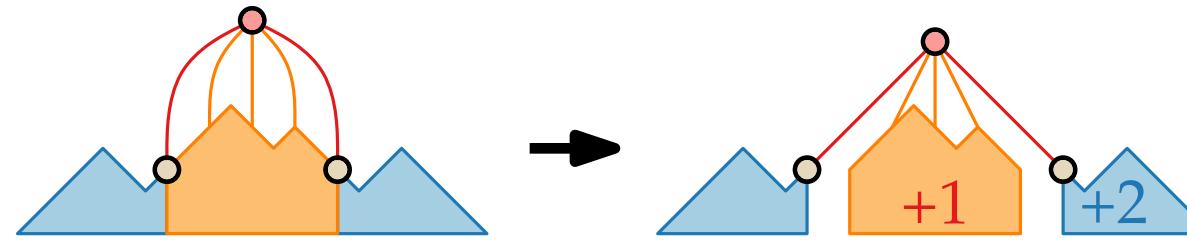
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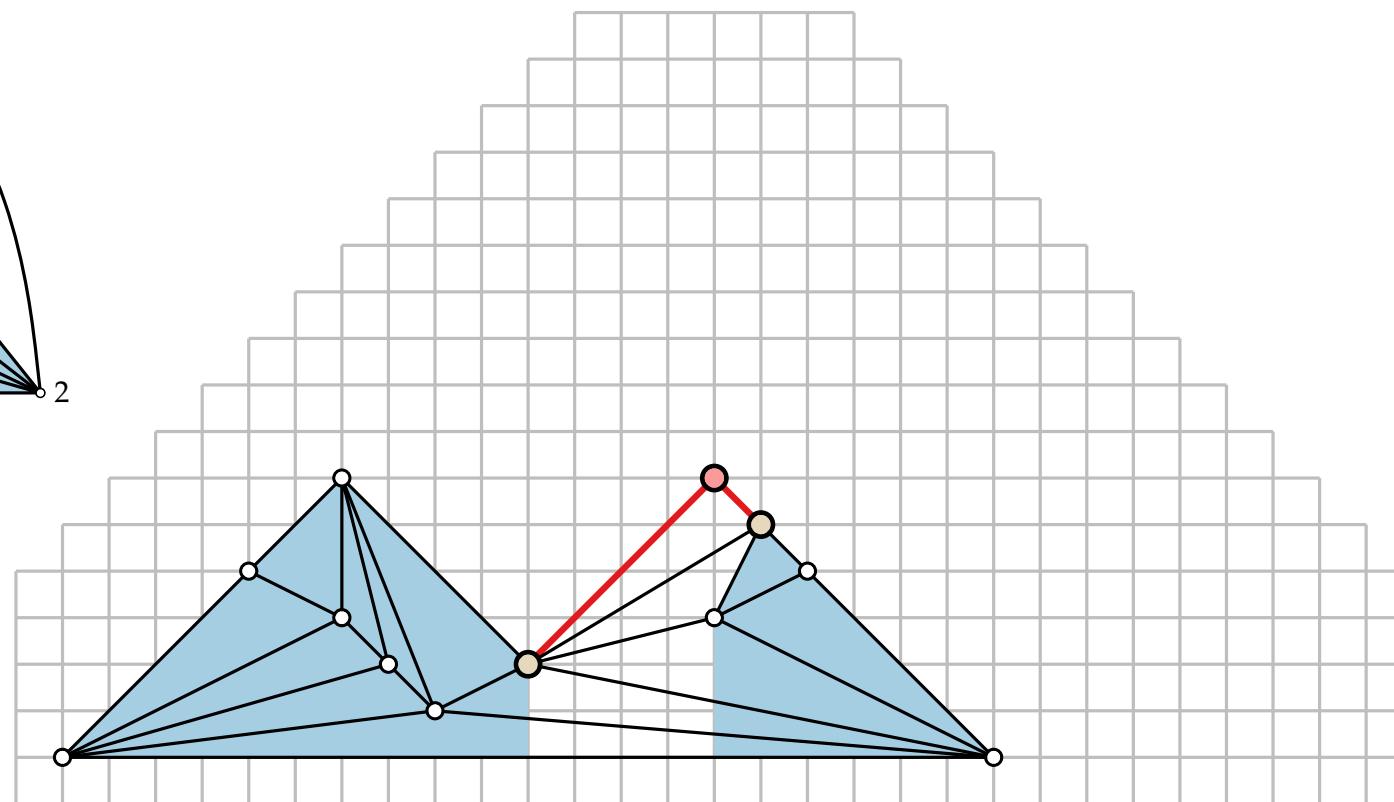
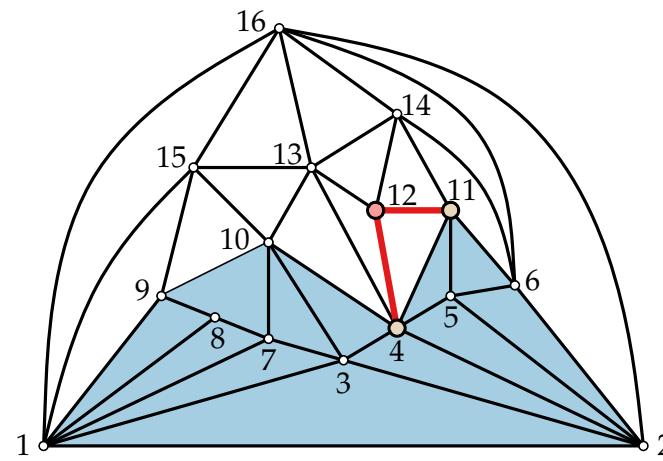
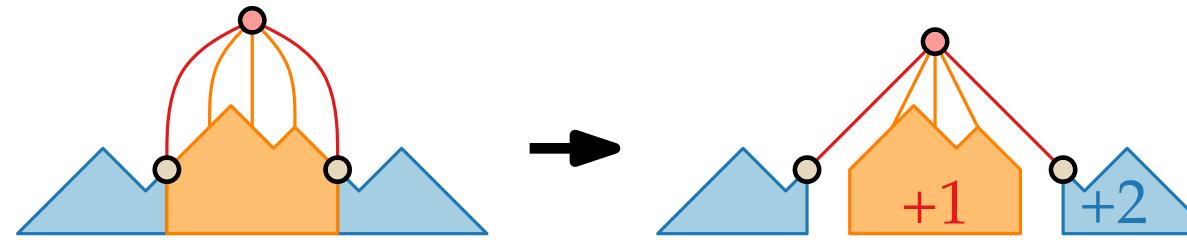
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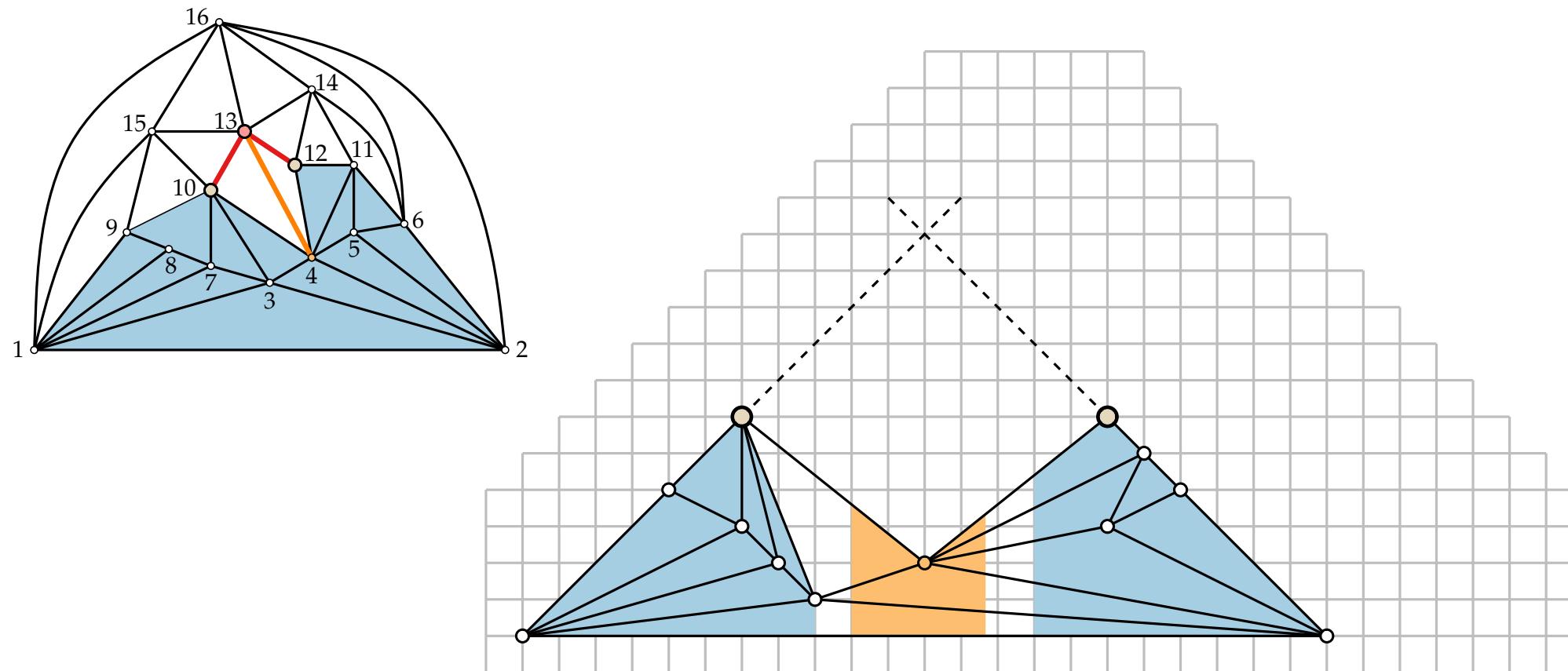
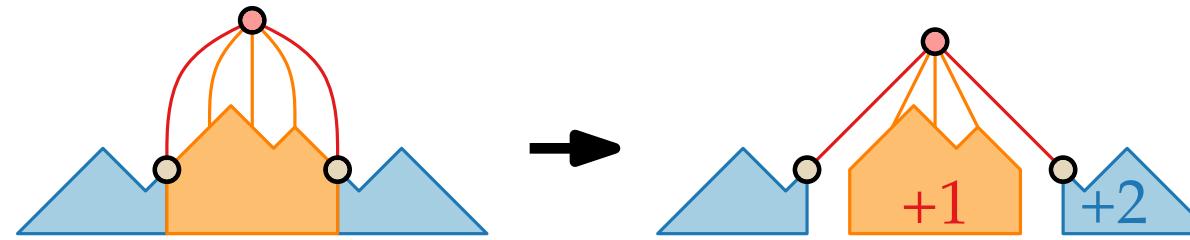
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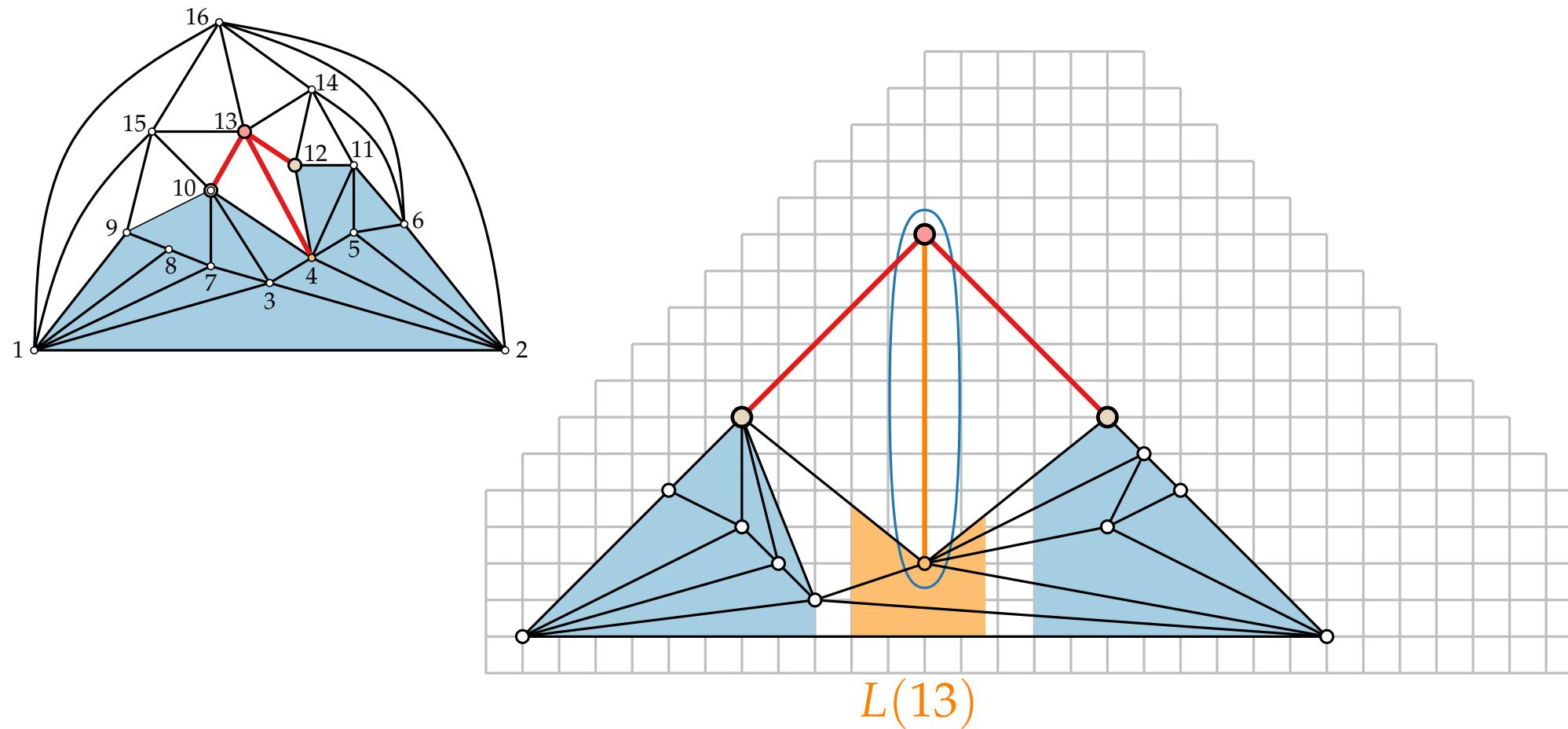
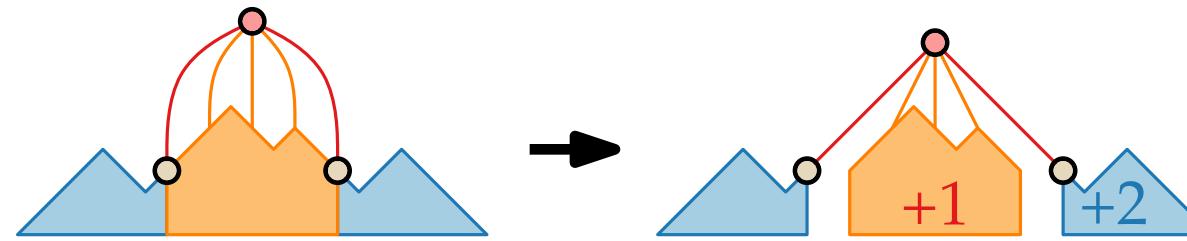
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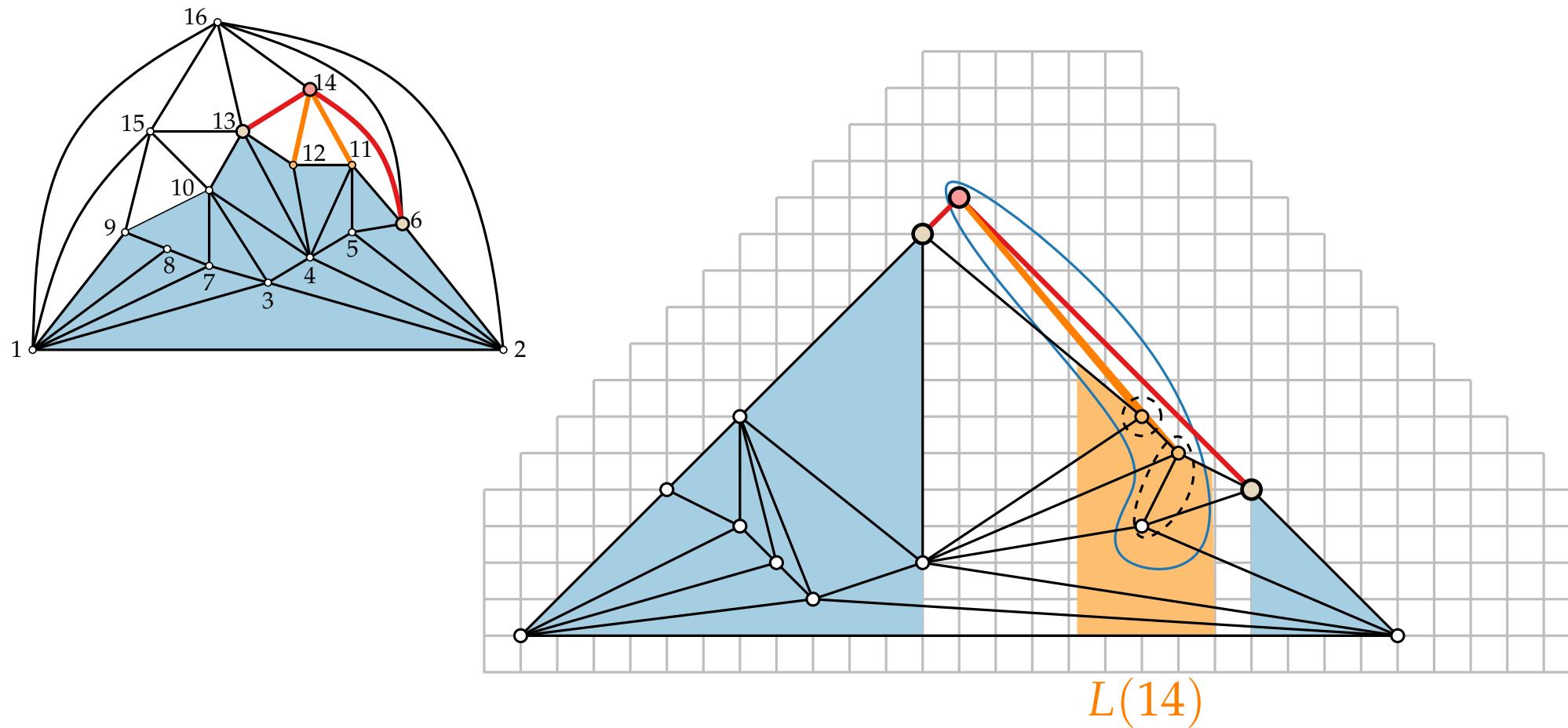
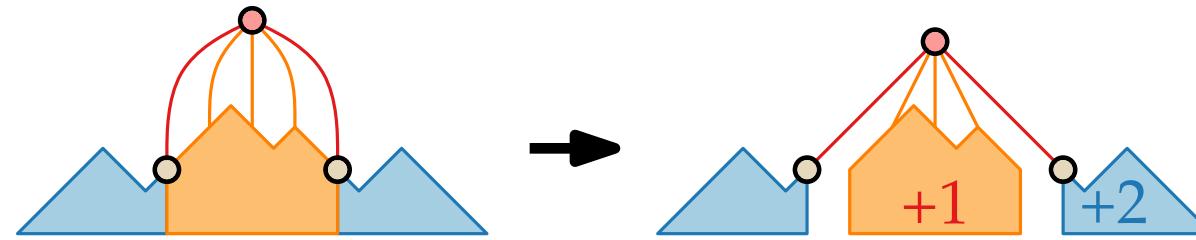
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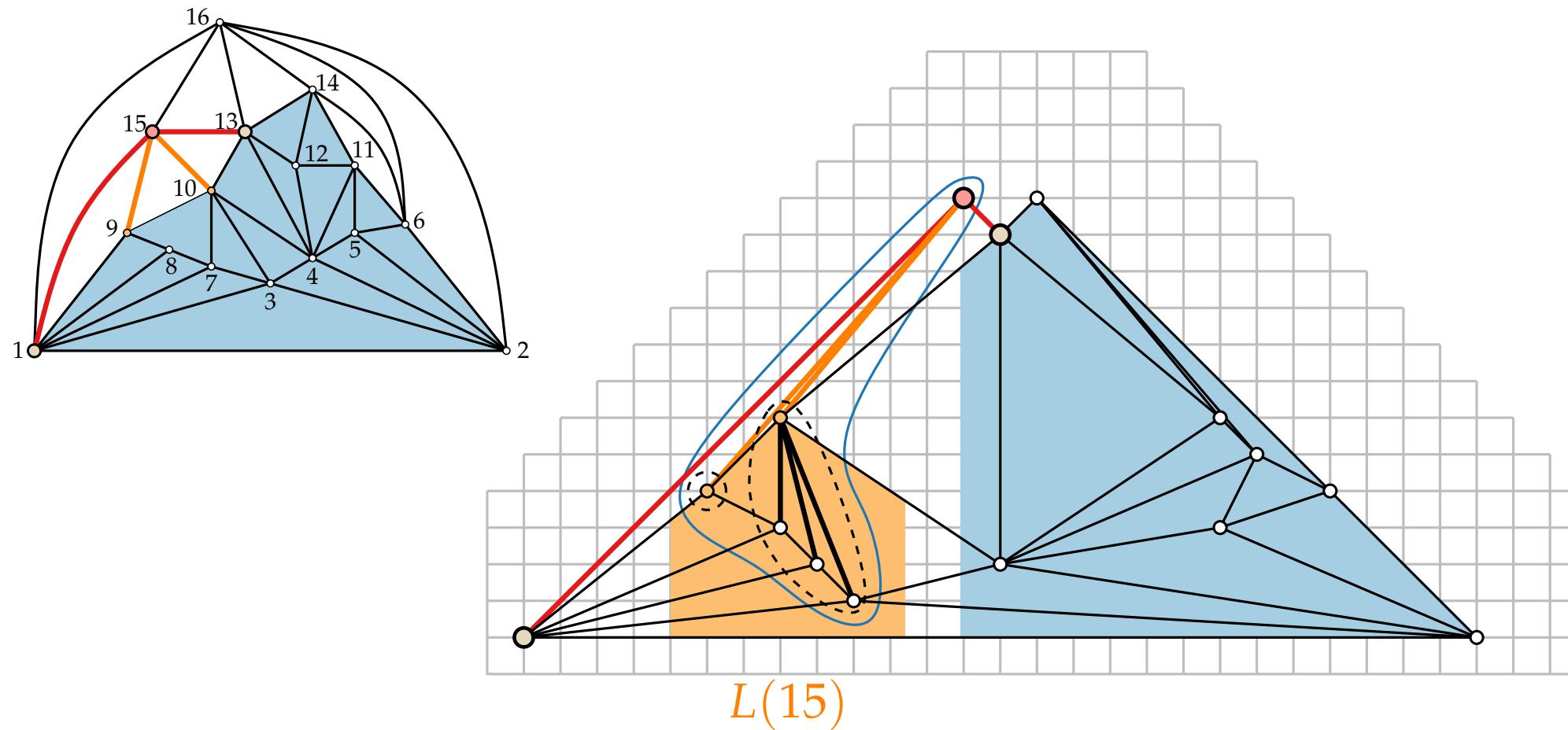
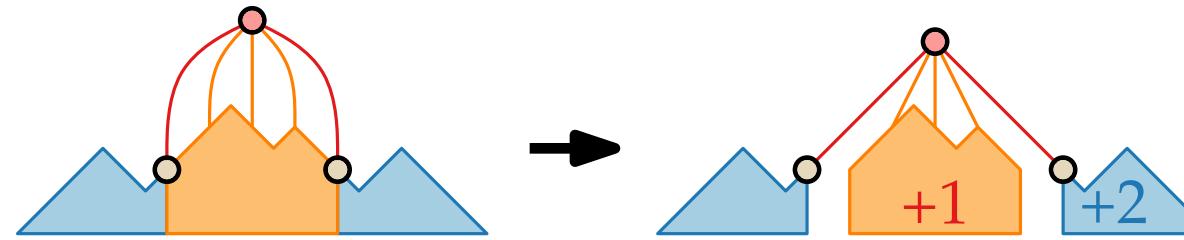
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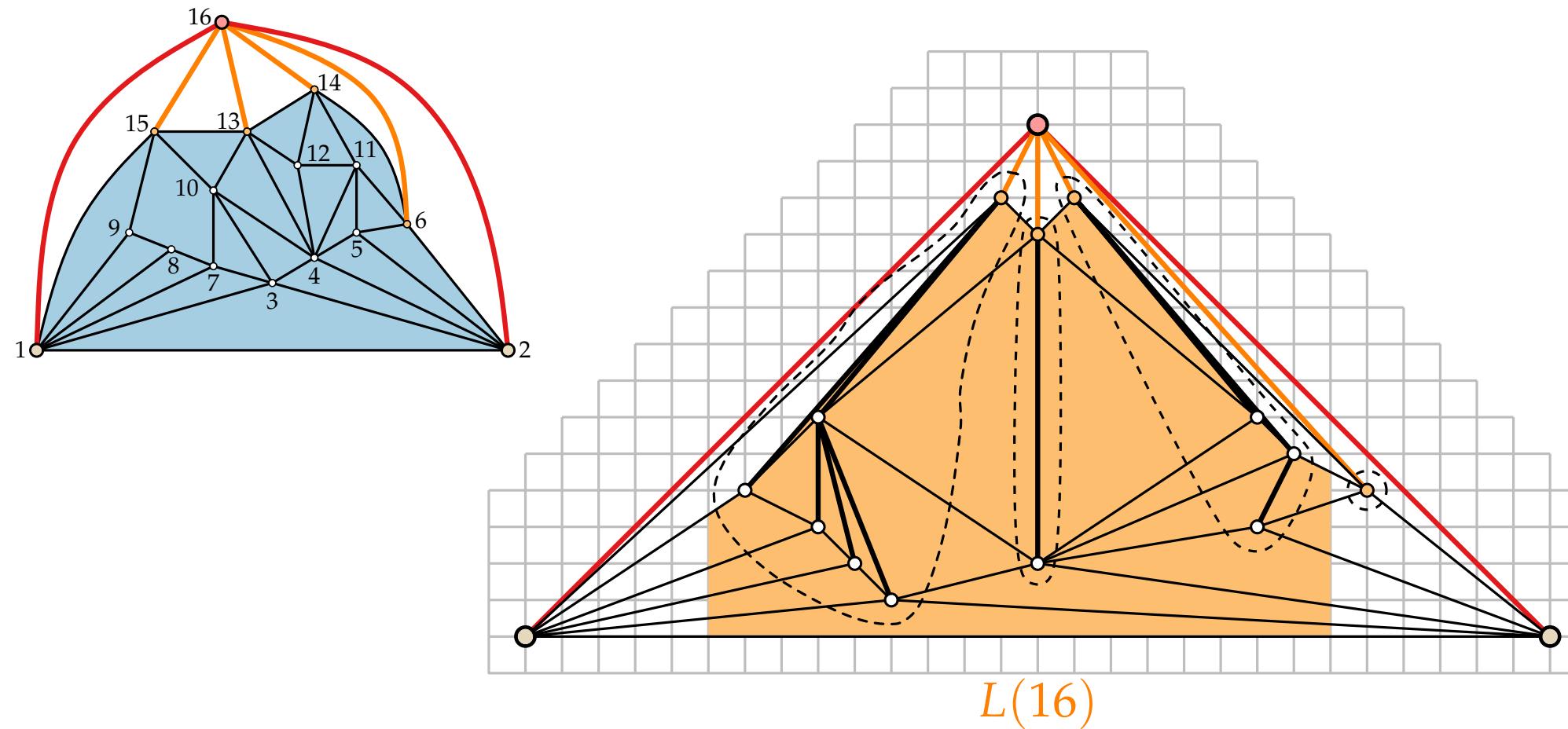
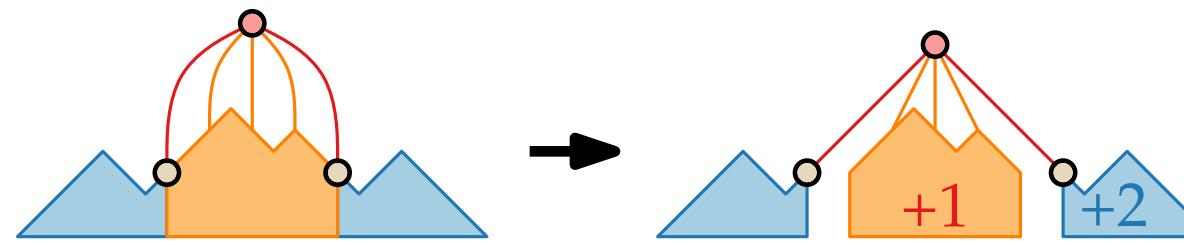
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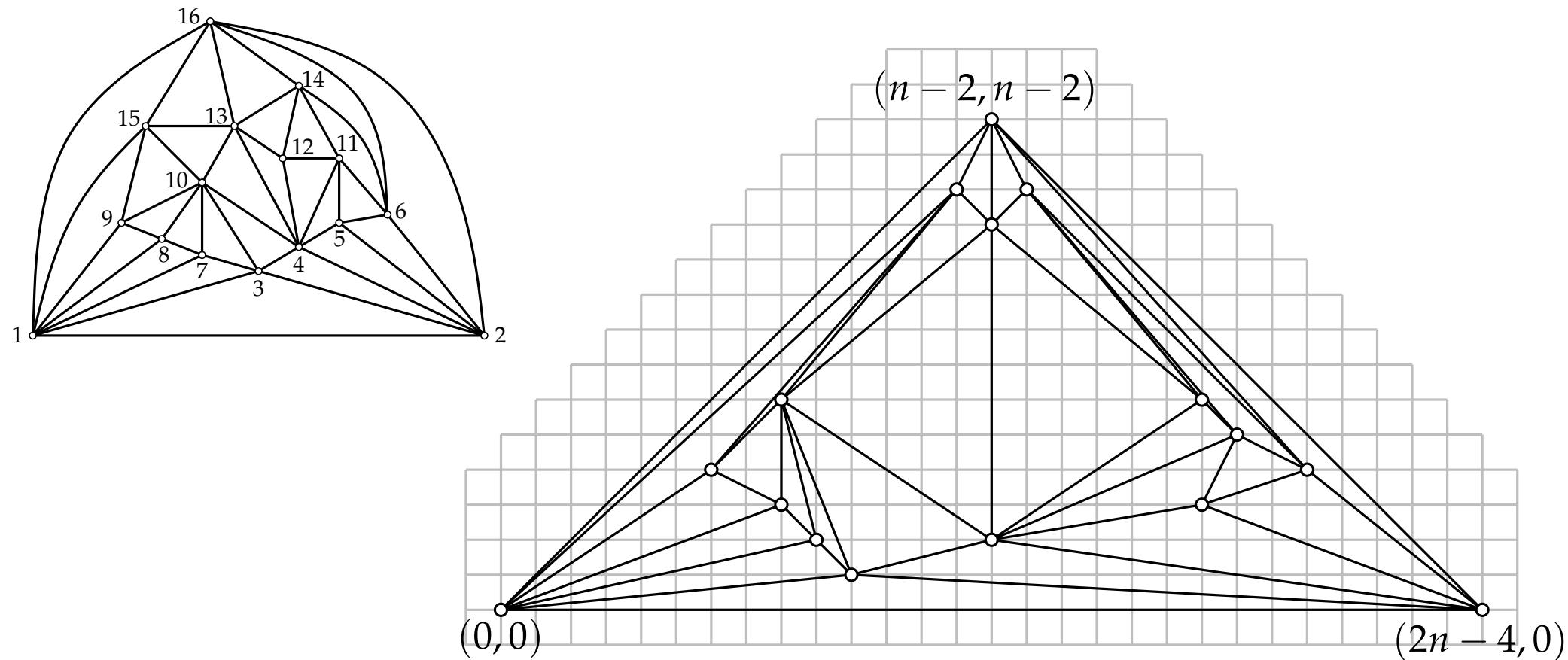
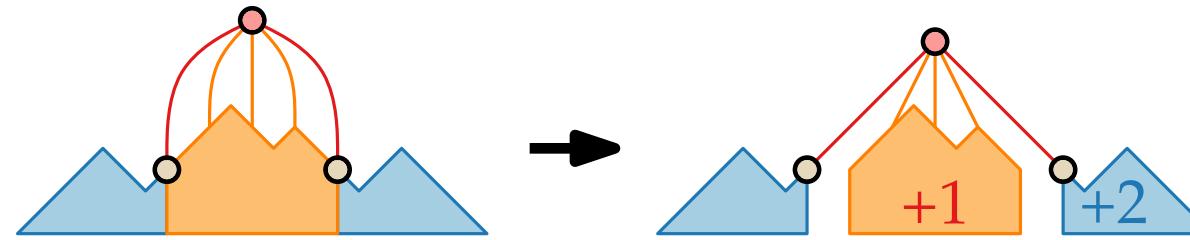
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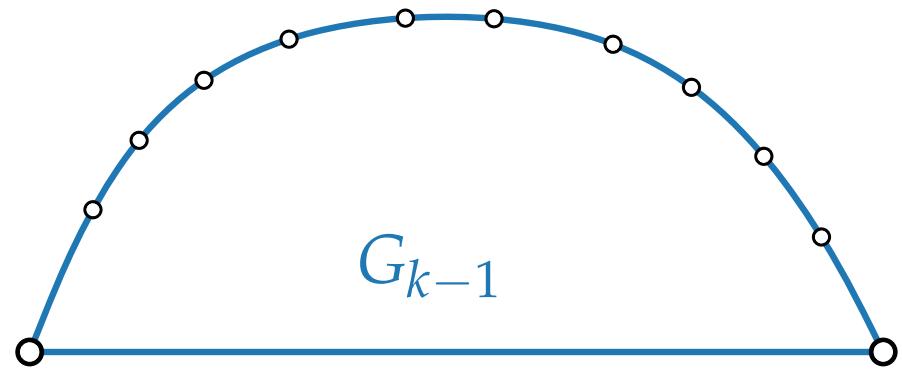
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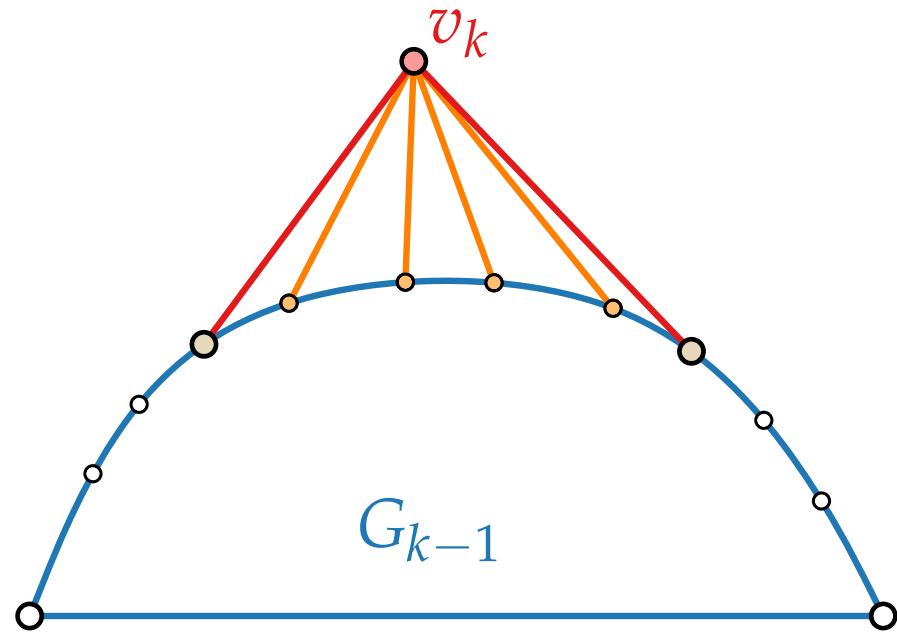
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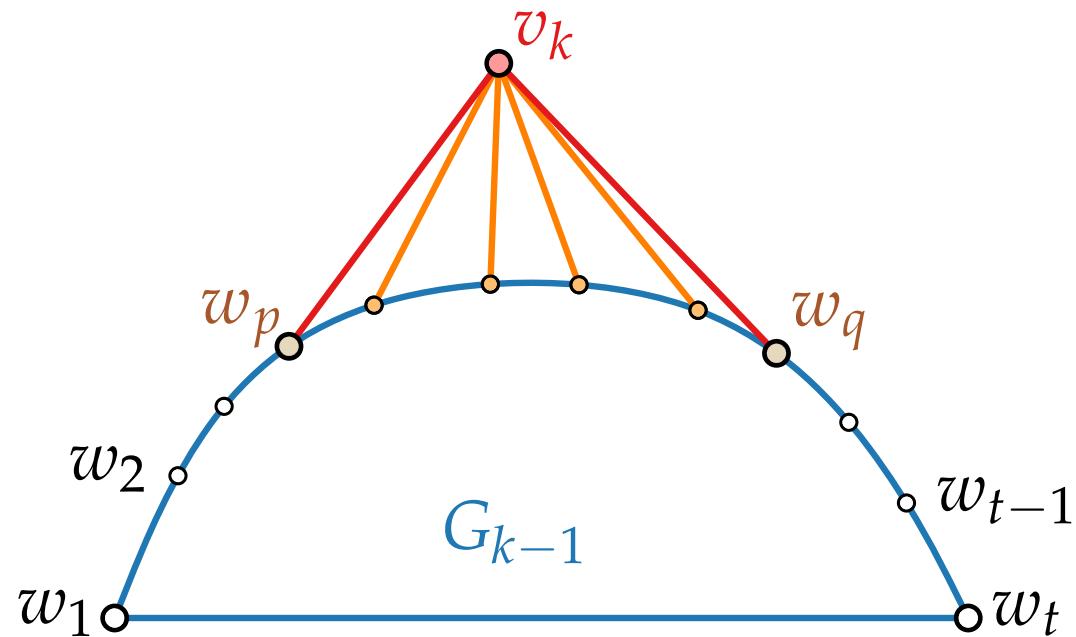
Shift Method – Planarity



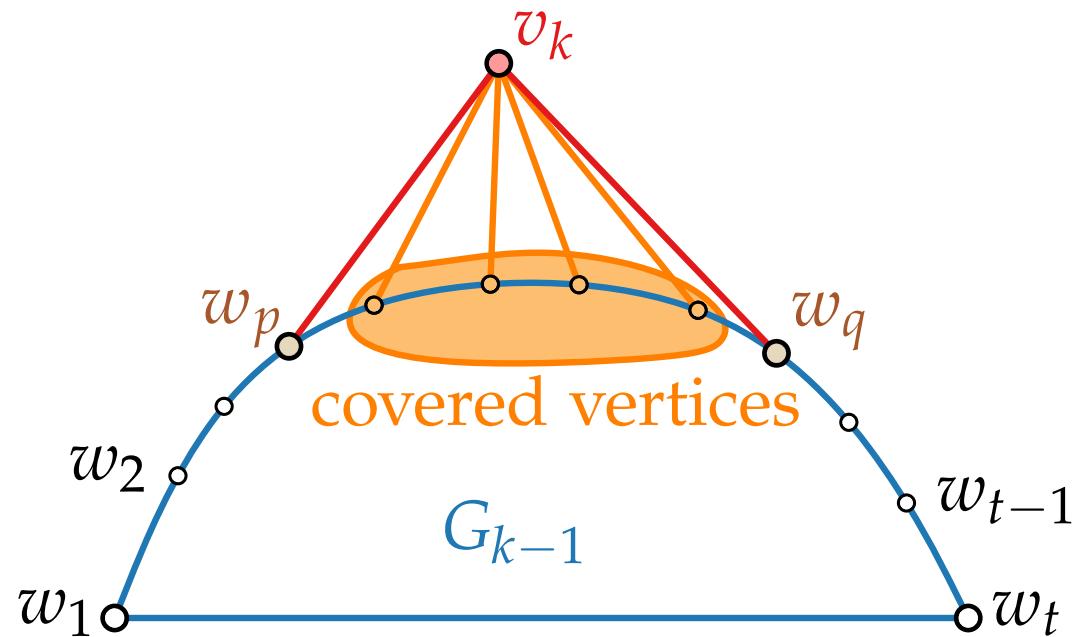
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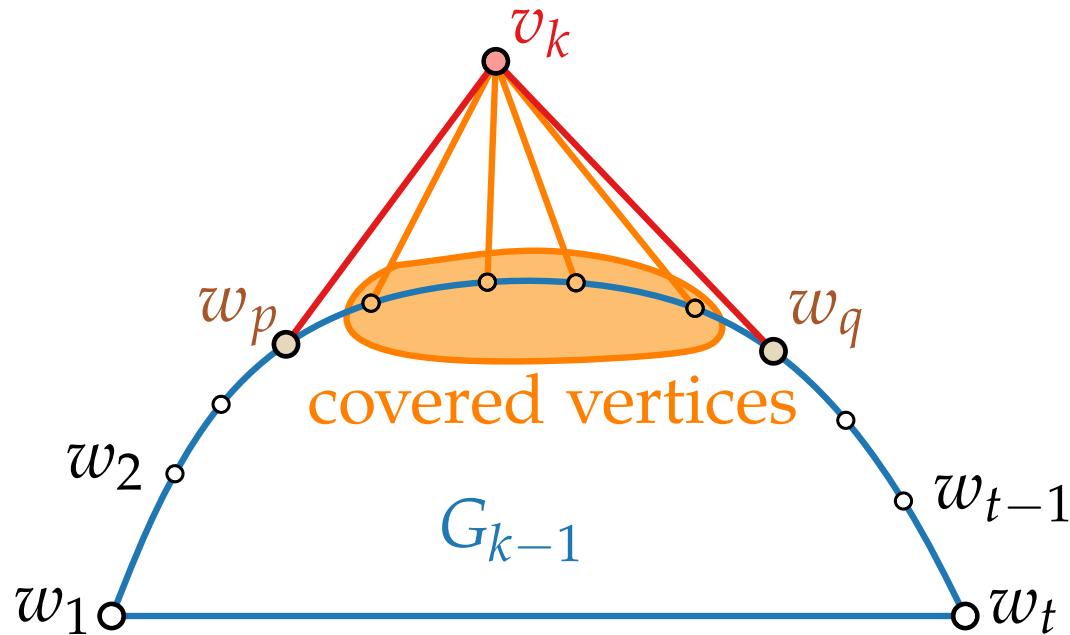
Shift Method – Planarity



Shift Method – Planarity

Observations.

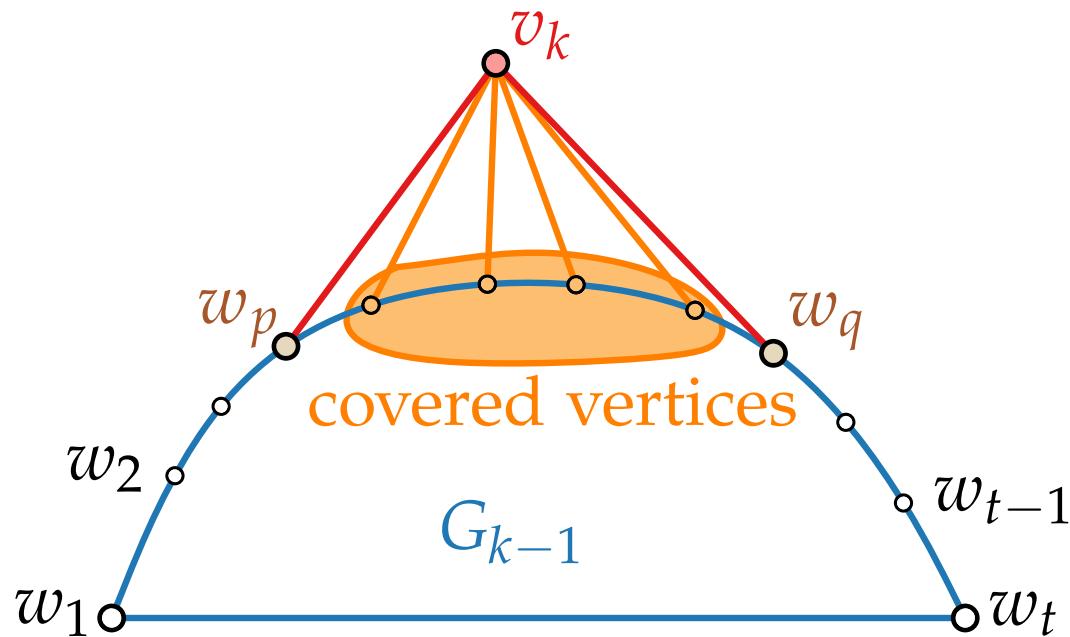
- Each internal vertex is **covered** exactly once.



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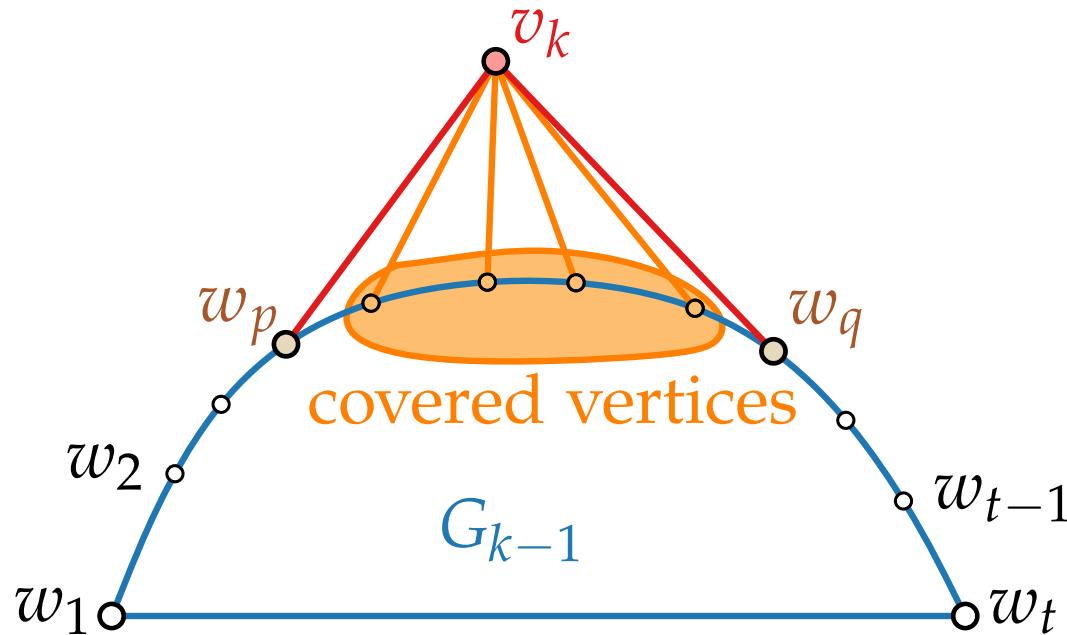
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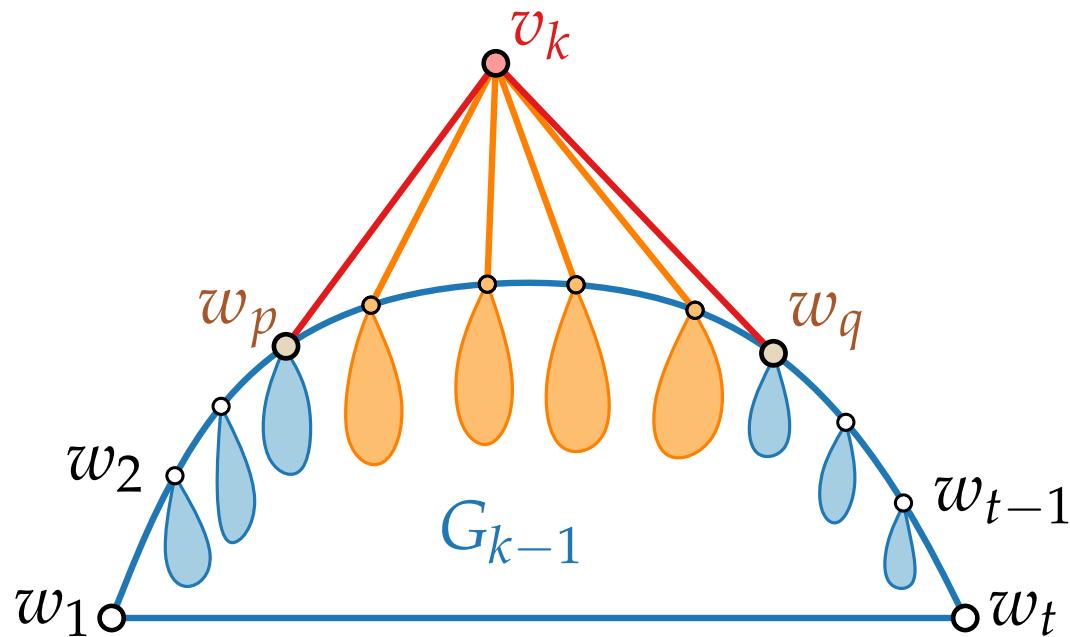
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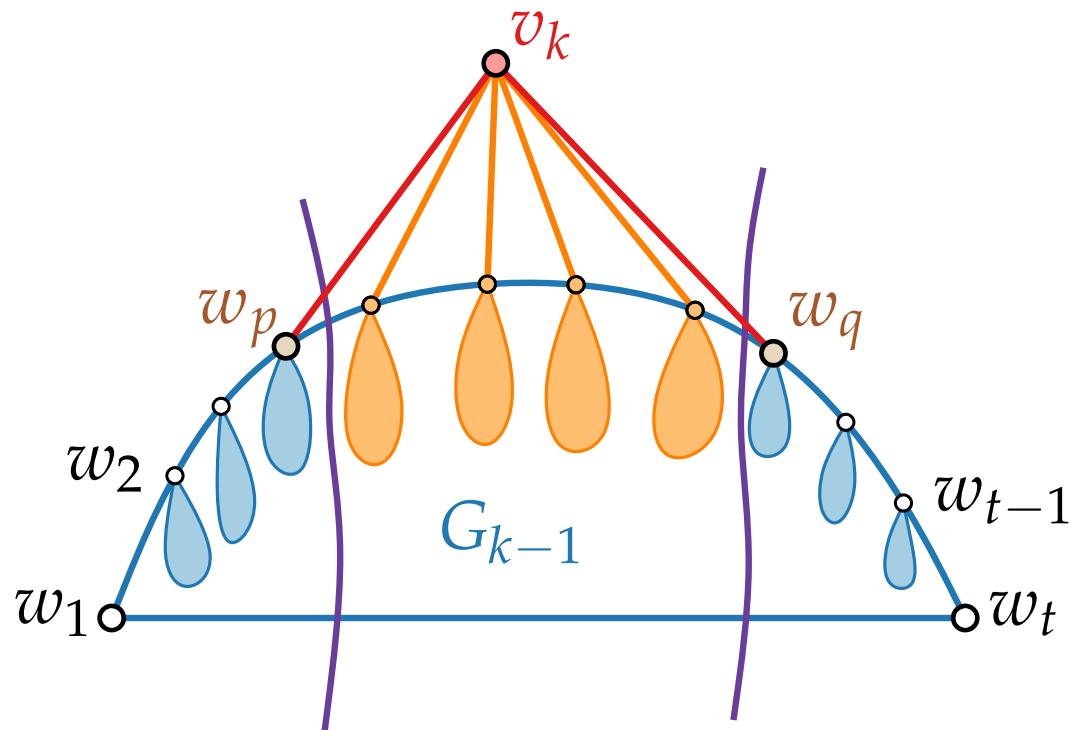
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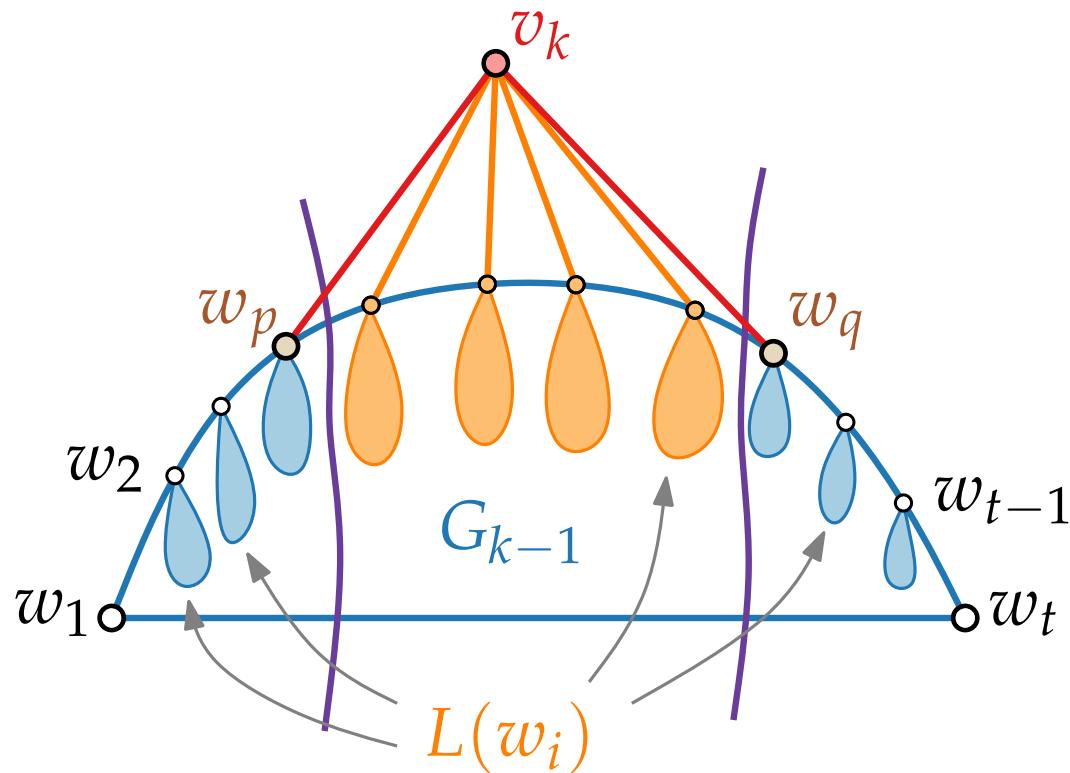
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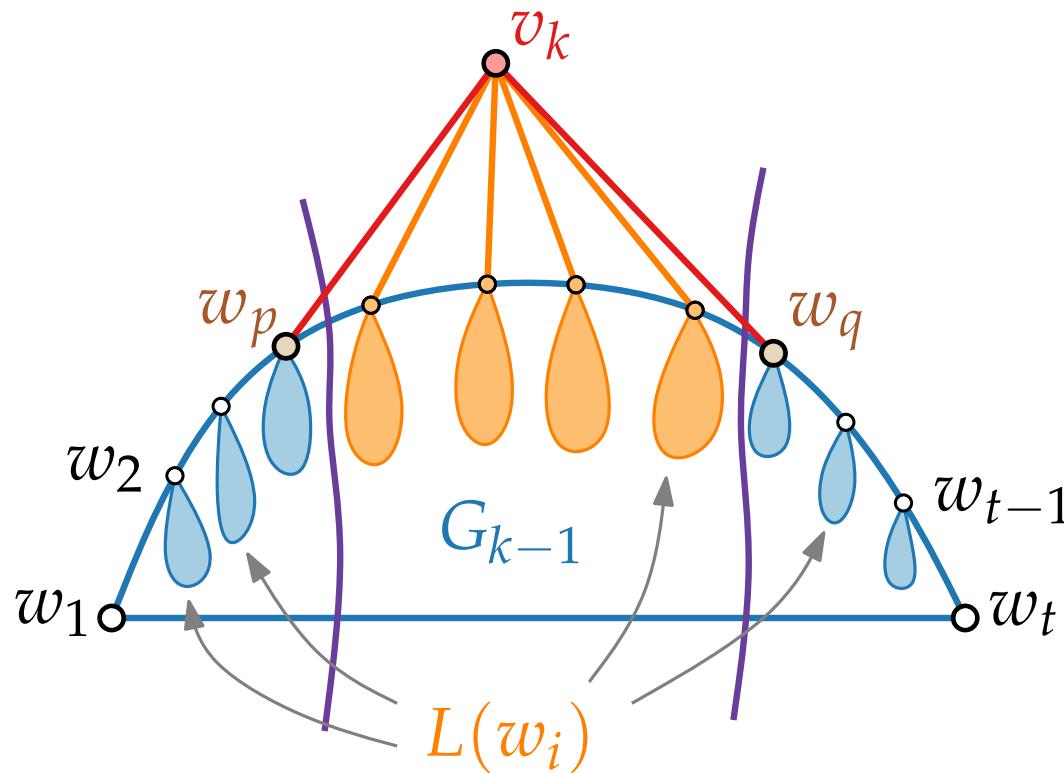
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Lemma.

Let $0 < \delta_1 \leq \delta_2 \leq \dots \leq \delta_t \in \mathbb{N}$, such that $\delta_q - \delta_p \geq 2$ and even.



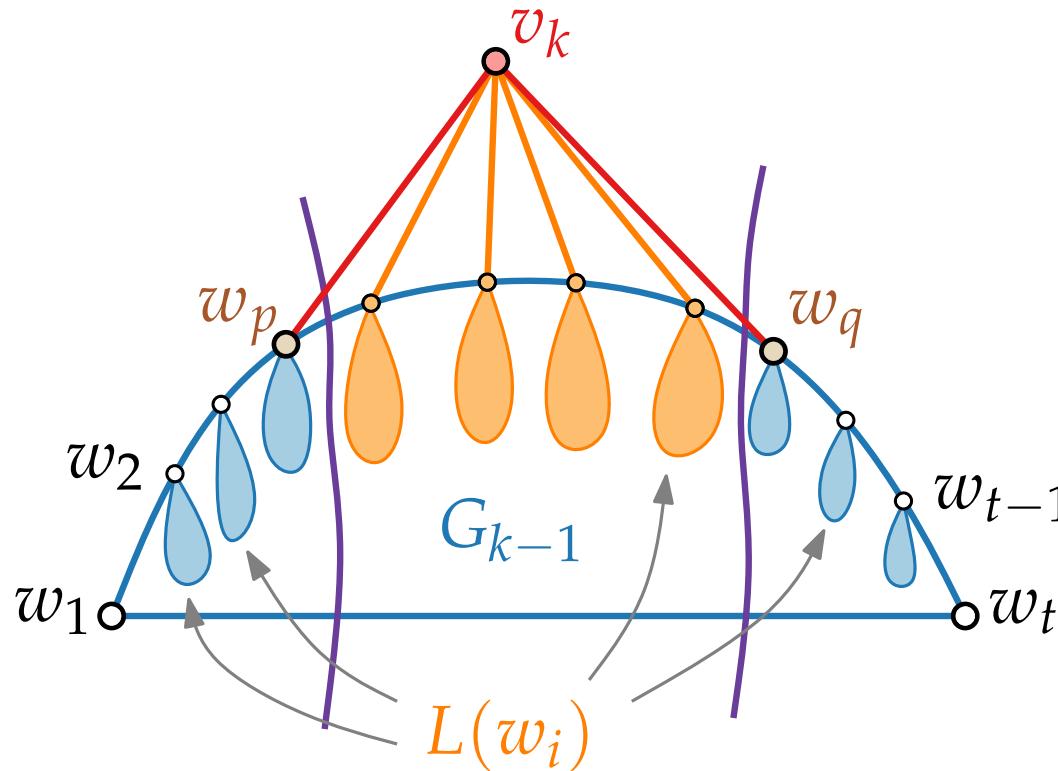
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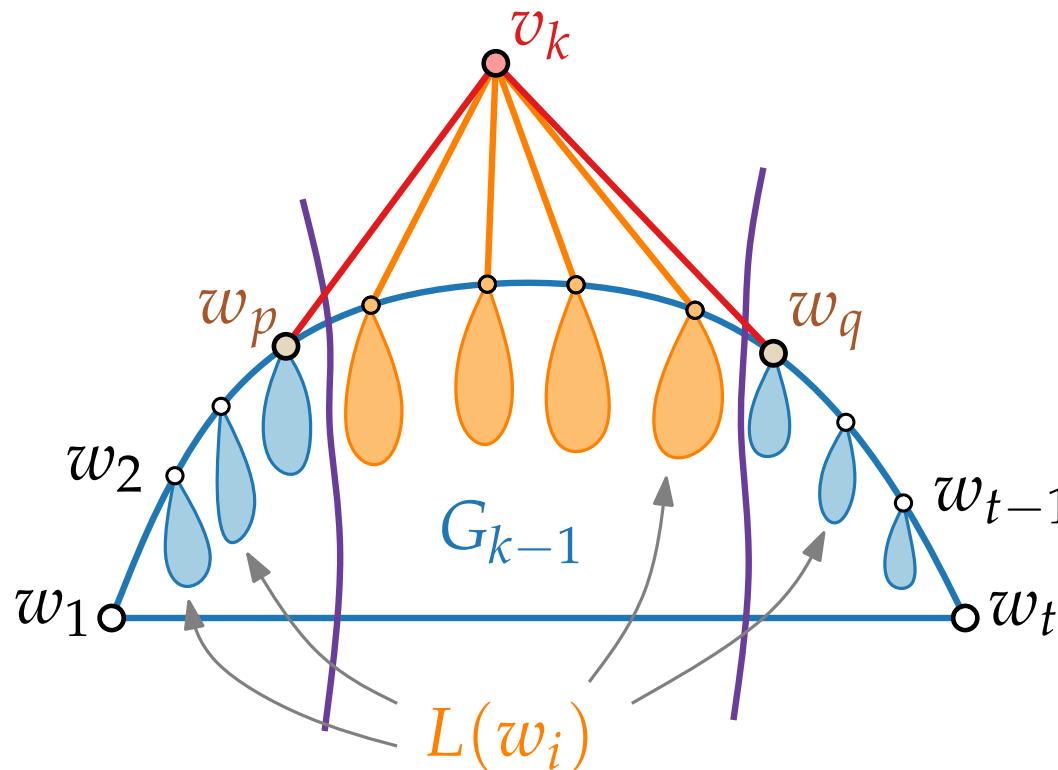
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Proof by induction:

If G_{k-1} is drawn planar and straight-line, then so is G_k .

Shift Method – Pseudocode

Let v_1, \dots, v_n be a canonical order of G

for $i = 1$ to 3 **do**

[

for $i = 4$ to n **do**

[

Shift Method – Pseudocode

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$L(v_i) \leftarrow \{v_i\}$

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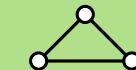
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for $i = 1$ to 3 **do**

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$P(v_1) \leftarrow (0, 0); P(v_2) \leftarrow (2, 0), P(v_3) \leftarrow (1, 1)$

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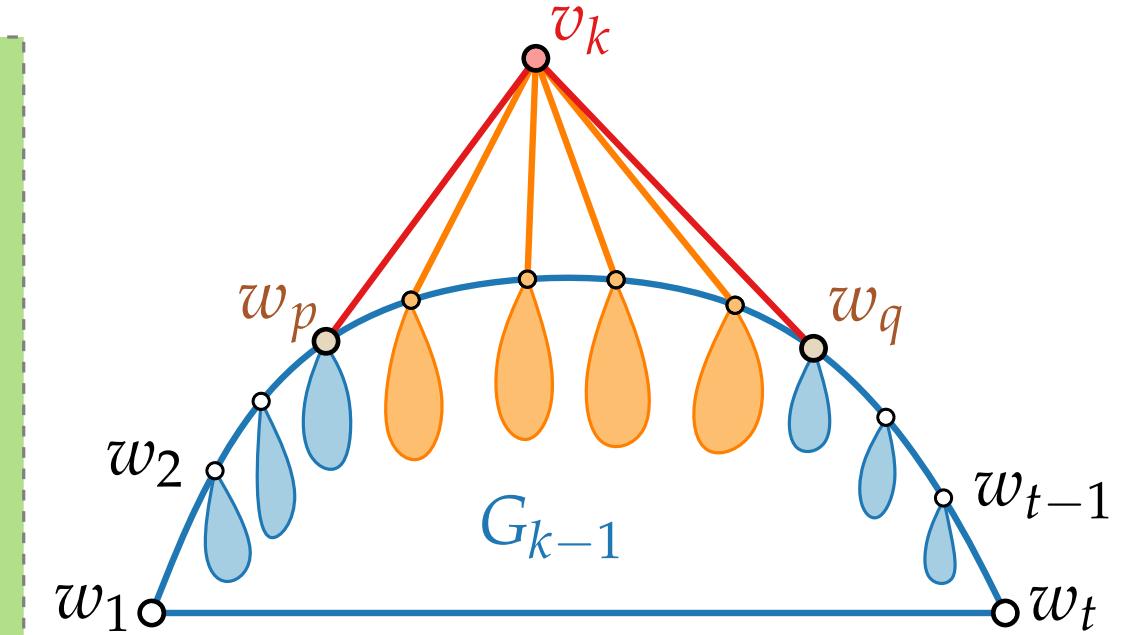
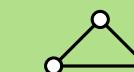
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for $i = 4$ to n **do**

 Let $w_1 = v_1, w_2, \dots, w_{t-1}, w_t = v_2$
 denote the boundary of G_{i-1}
 and let w_p, \dots, w_q be the neighbours of v_i



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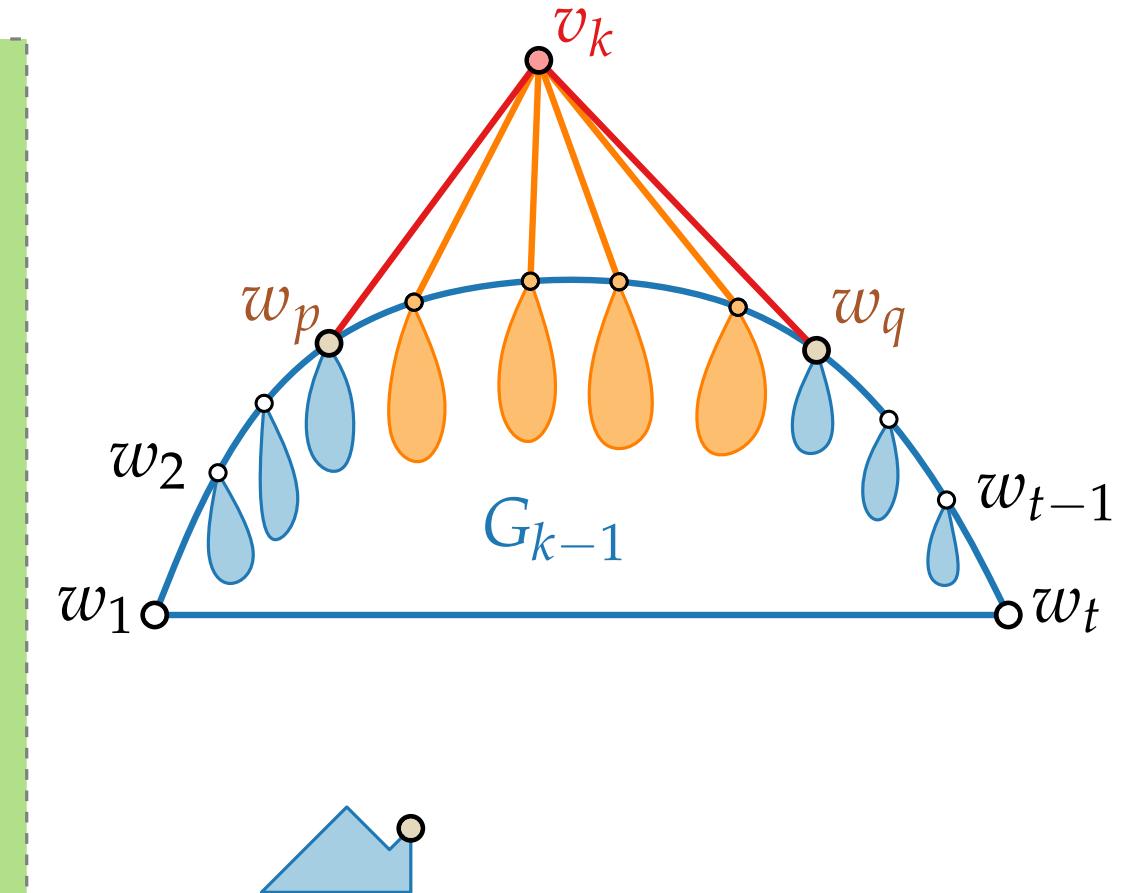
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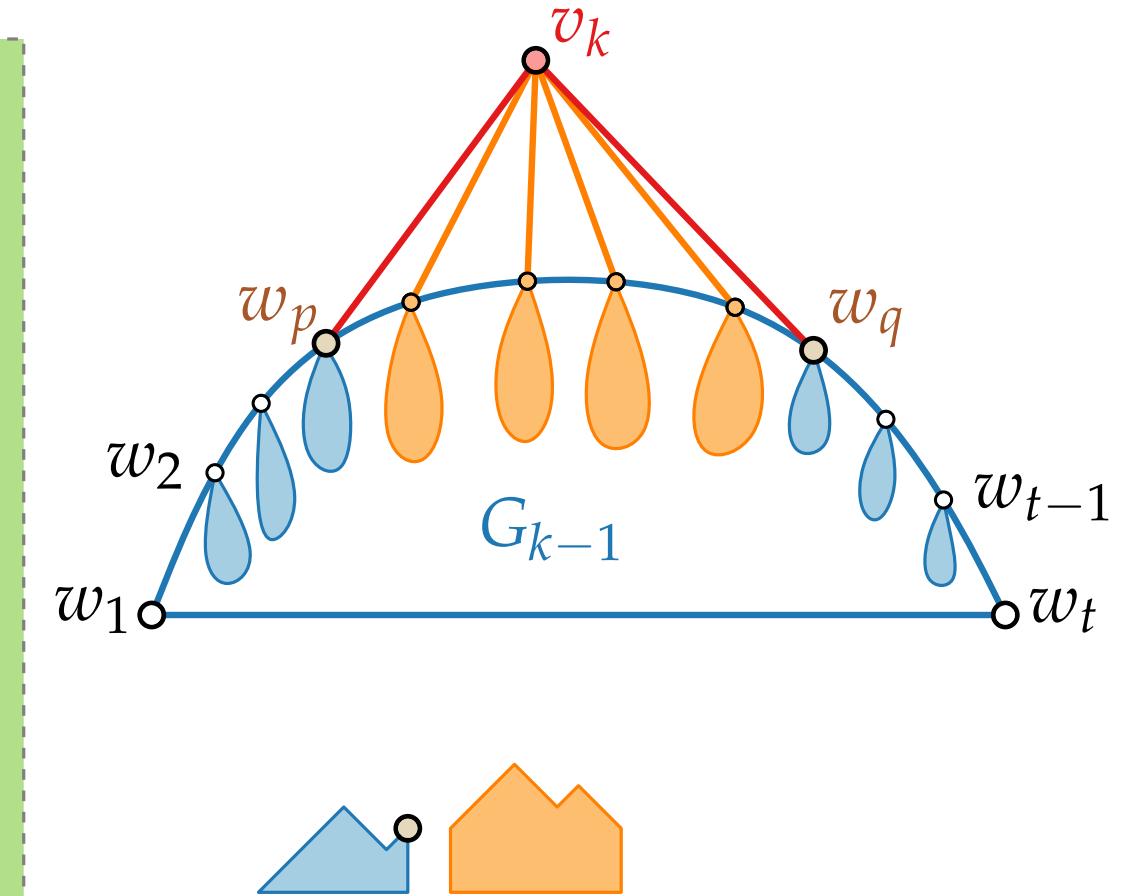
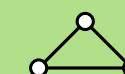
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for $\forall v \in \bigcup_{j=p+1}^{q-1} L(w_j)$ **do**

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Shift Method – Pseudocode

Let v_1, \dots, v_n be a canonical order of G

for $i = 1$ to 3 **do**

$L(v_i) \leftarrow \{v_i\}$

$P(v_1) \leftarrow (0, 0); P(v_2) \leftarrow (2, 0), P(v_3) \leftarrow (1, 1)$

for $i = 4$ to n **do**

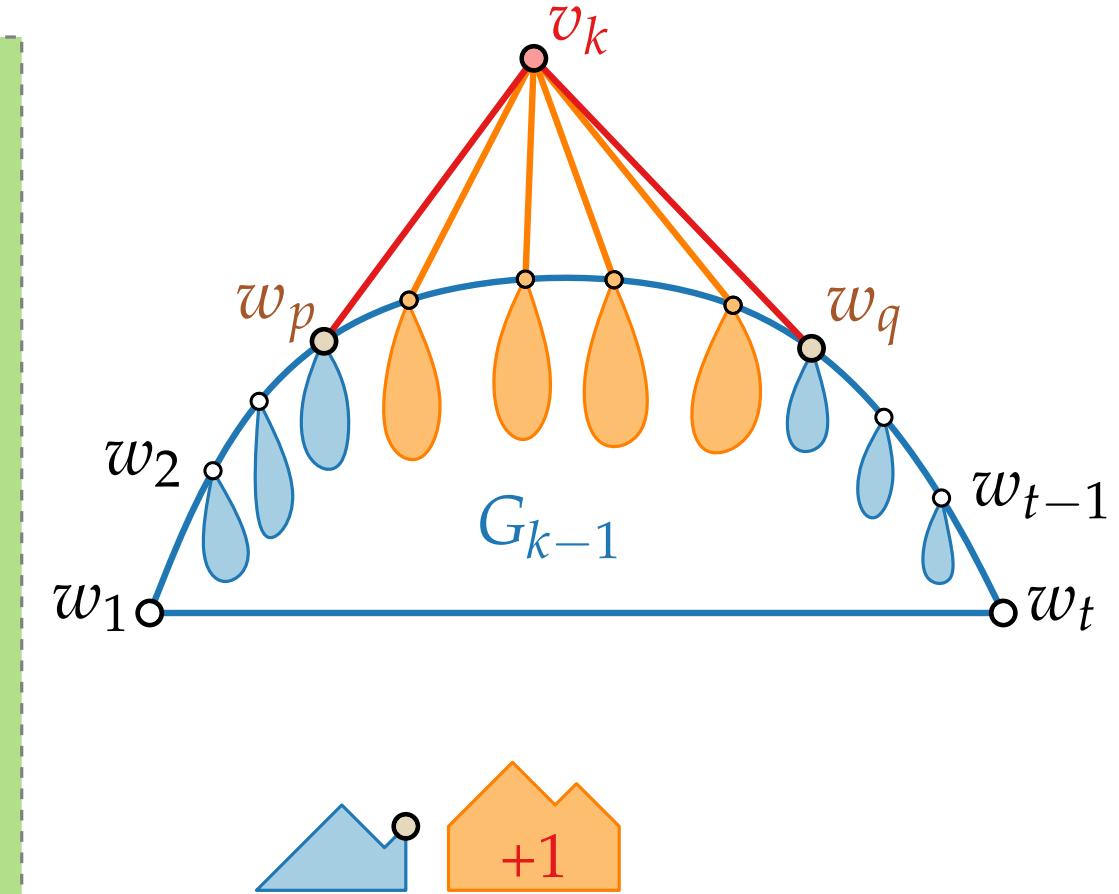
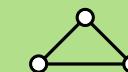
 Let $w_1 = v_1, w_2, \dots, w_{t-1}, w_t = v_2$

 denote the boundary of G_{i-1}

 and let w_p, \dots, w_q be the neighbours of v_i

for $\forall v \in \bigcup_{j=p+1}^{q-1} L(w_j)$ **do**

$x(v) \leftarrow x(v) + 1$



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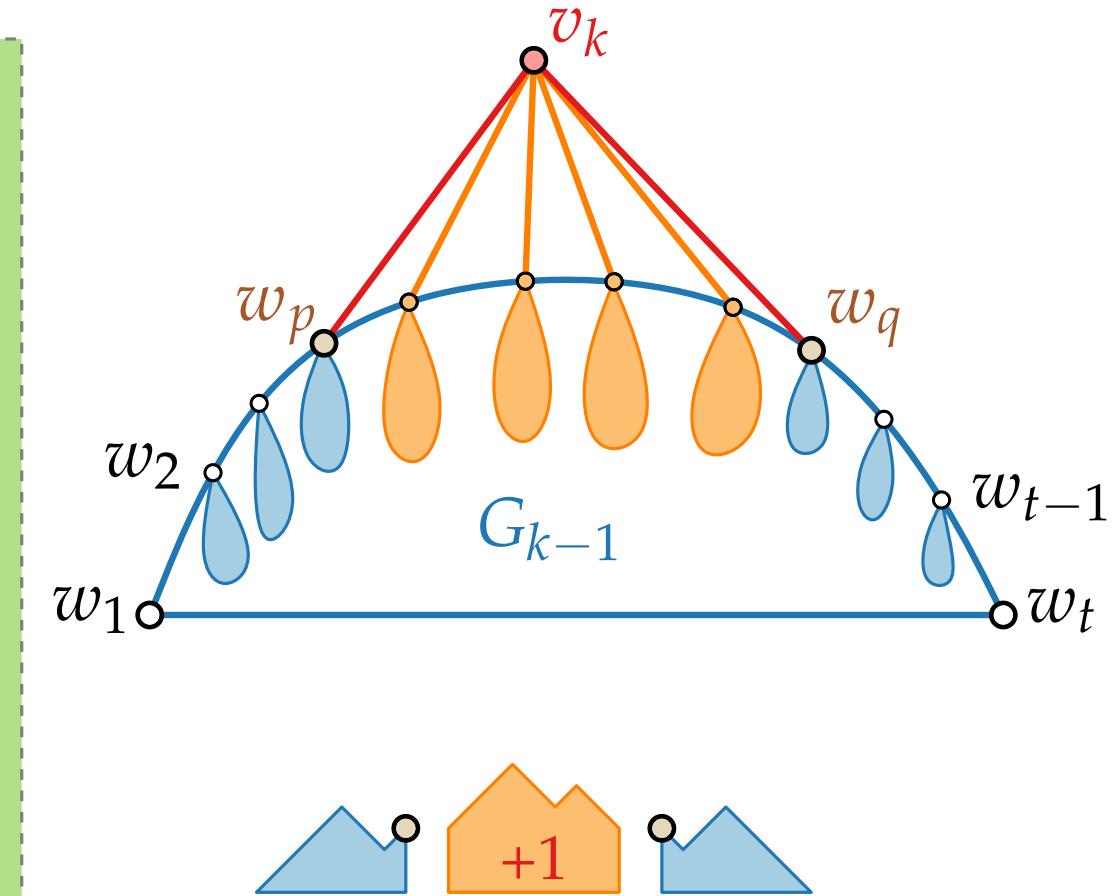
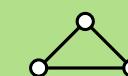
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for $\forall v \in \bigcup_{j=p+1}^{q-1} L(w_j)$ **do**

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for $\forall v \in \bigcup_{j=q}^t L(w_j)$ **do**



Shift Method – Pseudocode

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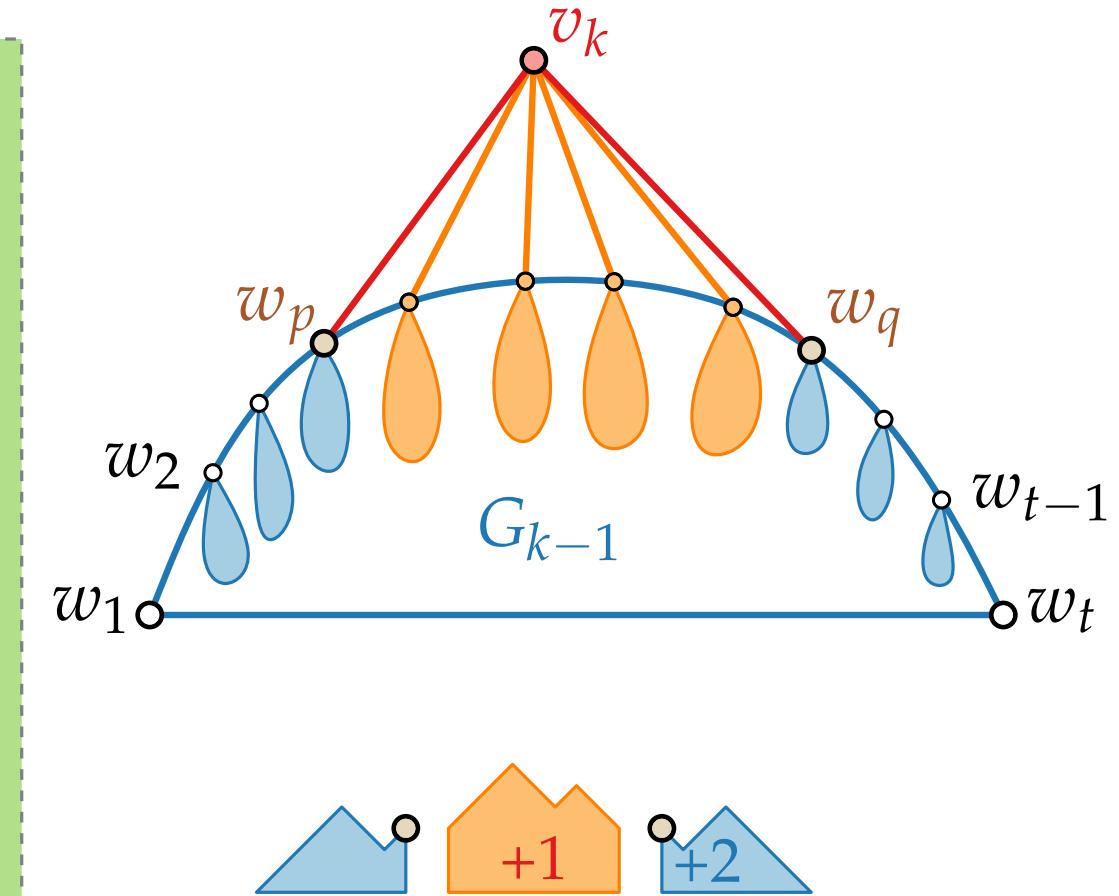
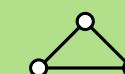
 and let w_p, \dots, w_q be the neighbours of v_i

for $\forall v \in \cup_{j=p+1}^{q-1} L(w_j)$ **do**

$x(v) \leftarrow x(v) + 1$

for $\forall v \in \cup_{j=q}^t L(w_j)$ **do**

$x(v) \leftarrow x(v) + 2$



Shift Method – Pseudocode

Let v_1, \dots, v_n be a canonical order of G

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for $i = 4$ to n **do**

 Let $w_1 = v_1, w_2, \dots, w_{t-1}, w_t = v_2$

 denote the boundary of G_{i-1}

 and let w_p, \dots, w_q be the neighbours of v_i

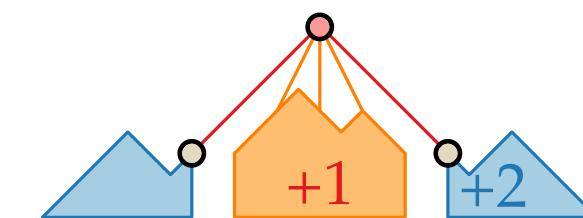
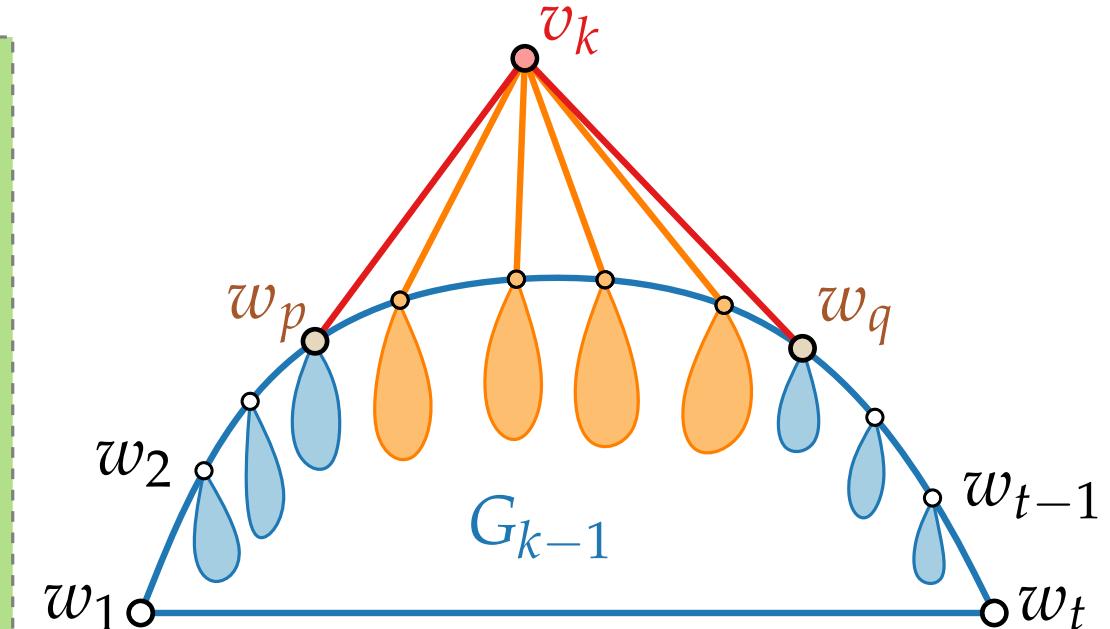
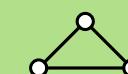
for $\forall v \in \bigcup_{j=p+1}^{q-1} L(w_j)$ **do**

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$x(v) \leftarrow x(v) + 2$

$P(v_i) \leftarrow \text{intersection of } +1/-1 \text{ diagonals}$
 through $P(w_p)$ and $P(w_q)$



Shift Method – Pseudocode

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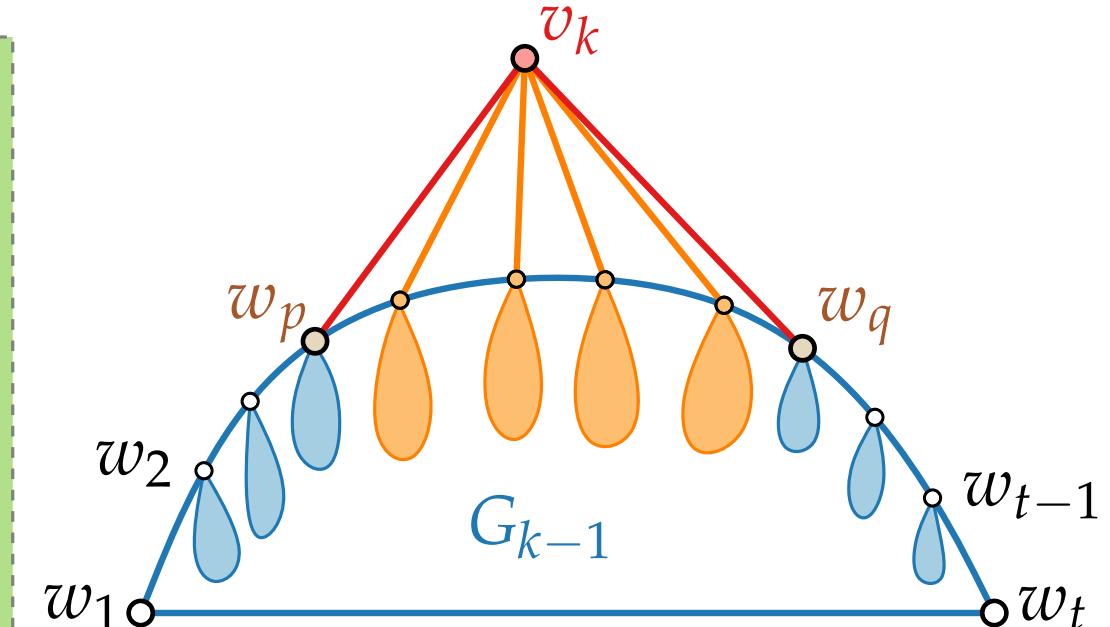
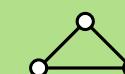
$x(v) \leftarrow x(v) + 1$

for $\forall v \in \cup_{j=q}^t L(w_j)$ **do**

$x(v) \leftarrow x(v) + 2$

$P(v_i) \leftarrow$ intersection of $+1/-1$ diagonals
 through $P(w_p)$ and $P(w_q)$

$L(v_i) \leftarrow \cup_{j=p+1}^{q-1} L(w_j) \cup \{v_i\}$



Shift Method – Pseudocode

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for $\forall v \in \cup_{j=p+1}^{q-1} L(w_j)$ **do**

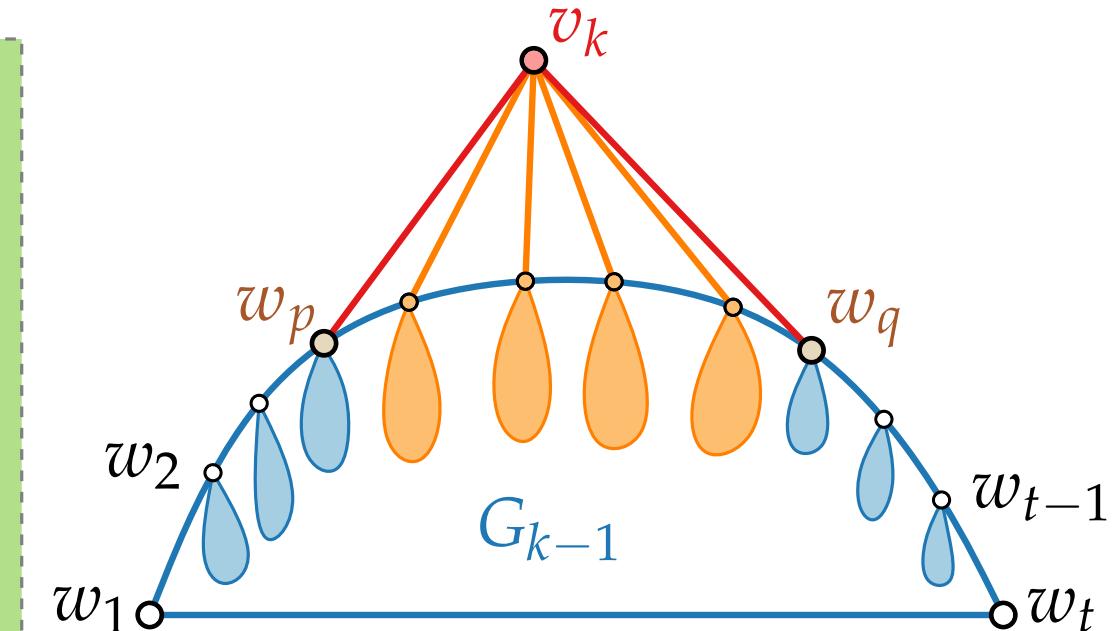
$x(v) \leftarrow x(v) + 1$

for $\forall v \in \cup_{j=q}^t L(w_j)$ **do**

$x(v) \leftarrow x(v) + 2$

$P(v_i) \leftarrow$ intersection of $+1/-1$ diagonals
 through $P(w_p)$ and $P(w_q)$

$L(v_i) \leftarrow \cup_{j=p+1}^{q-1} L(w_j) \cup \{v_i\}$



Running Time?

Shift Method – Pseudocode

Let v_1, \dots, v_n be a canonical order of G

for $i = 1$ to 3 **do**

$L(v_i) \leftarrow \{v_i\}$

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for $i = 4$ to n **do**

Let $w_1 = v_1, w_2, \dots, w_{t-1}, w_t = v_2$

denote the boundary of G_{i-1}

and let w_p, \dots, w_q be the neighbours of v_i

for $\forall v \in \cup_{j=p+1}^{q-1} L(w_j)$ **do** $// \mathcal{O}(n^2)$ *in total*

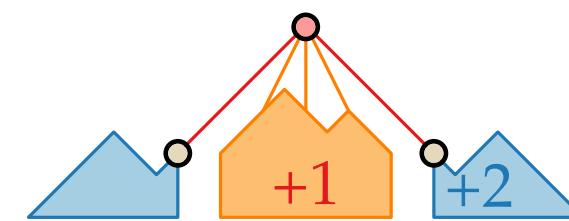
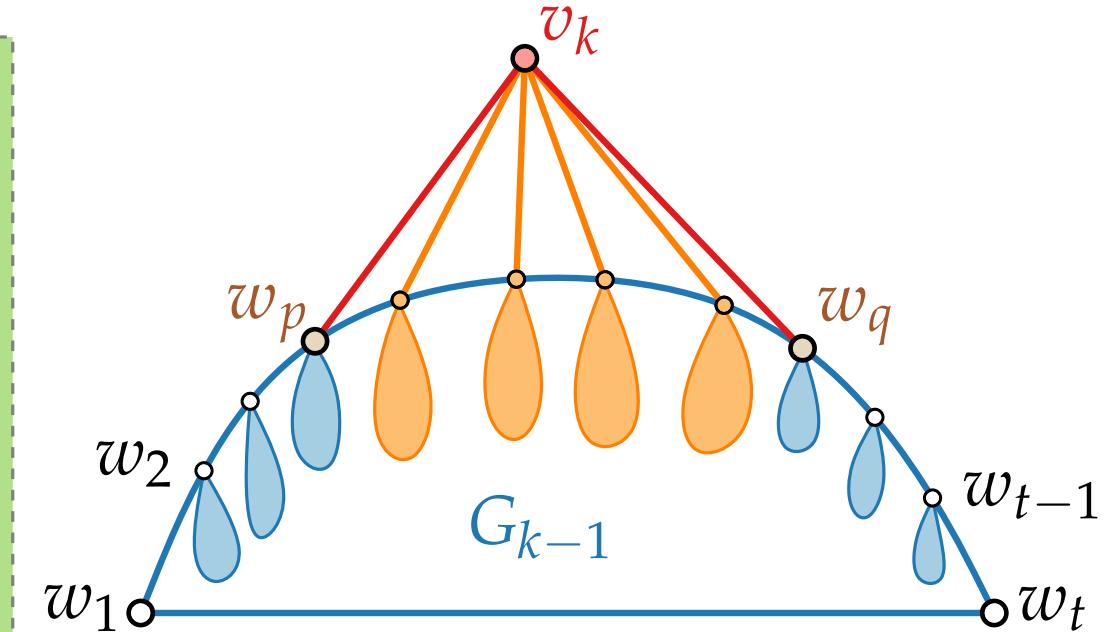
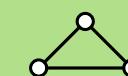
$$x(v) \leftarrow x(v) + 1$$

for $\forall v \in \cup_{i=a}^t L(w_i)$ **do** $\text{// } O(n^2) \text{ in total}$

$x(v) \leftarrow x(v) + 2$

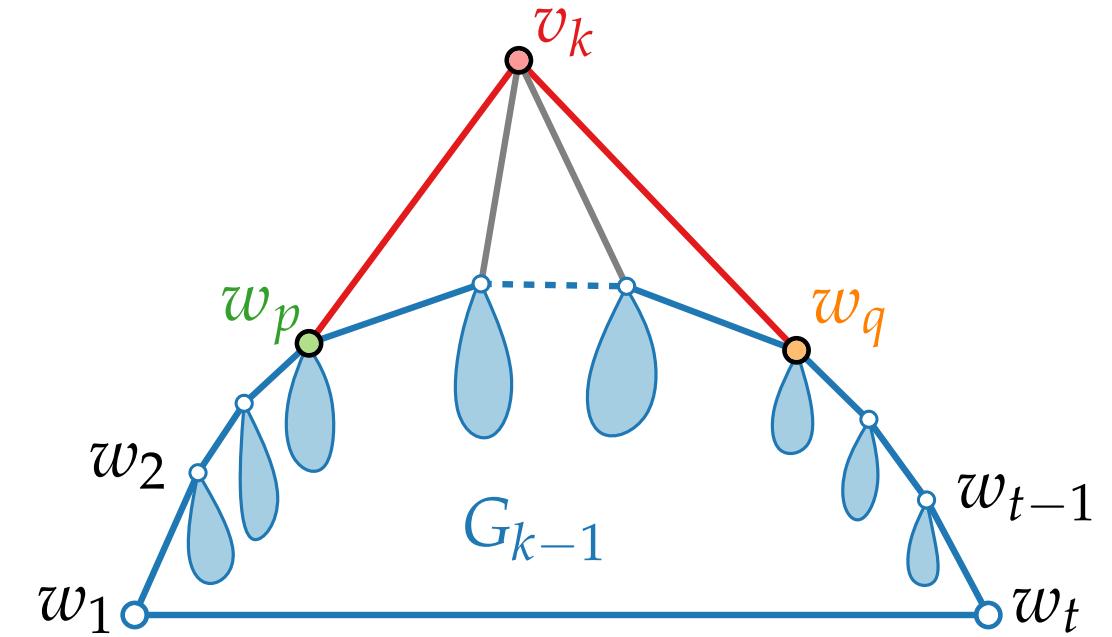
$P(\textcolor{red}{v}_i) \leftarrow$ intersection of $+1/-1$ diagonals
through $P(\textcolor{brown}{w}_p)$ and $P(\textcolor{brown}{w}_q)$

$$L(v_i) \leftarrow \cup_{j=p+1}^{q-1} L(w_j) \cup \{v_i\}$$



Running Time?

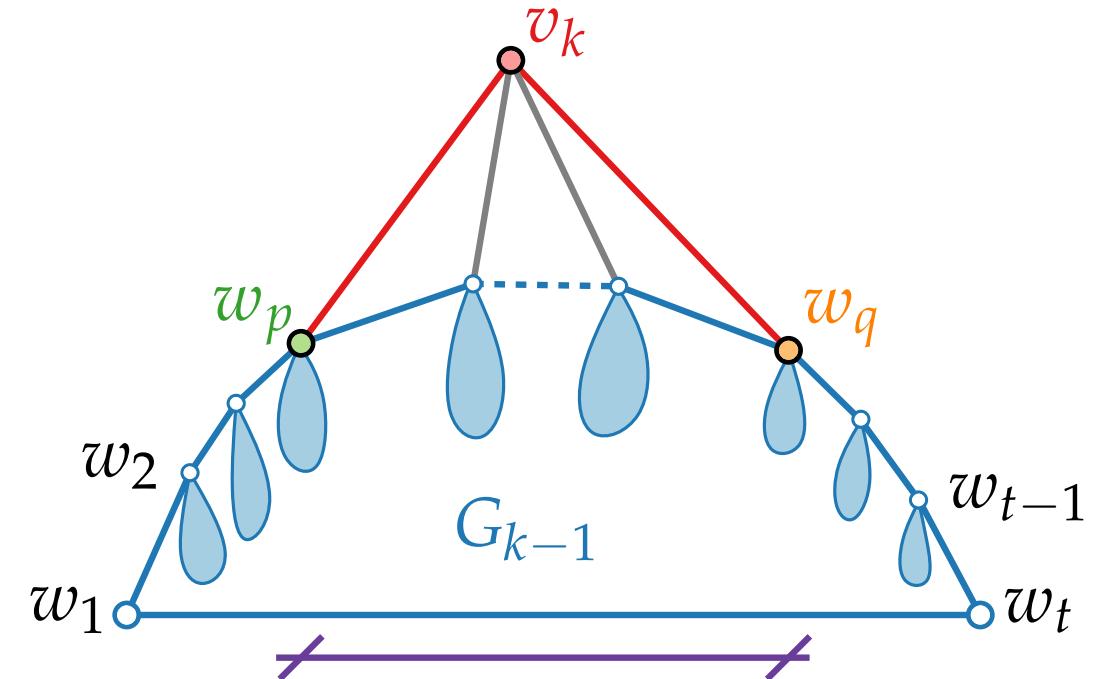
Shift Method – Linear Time Implementation



Shift Method – Linear Time Implementation

Idea 1.

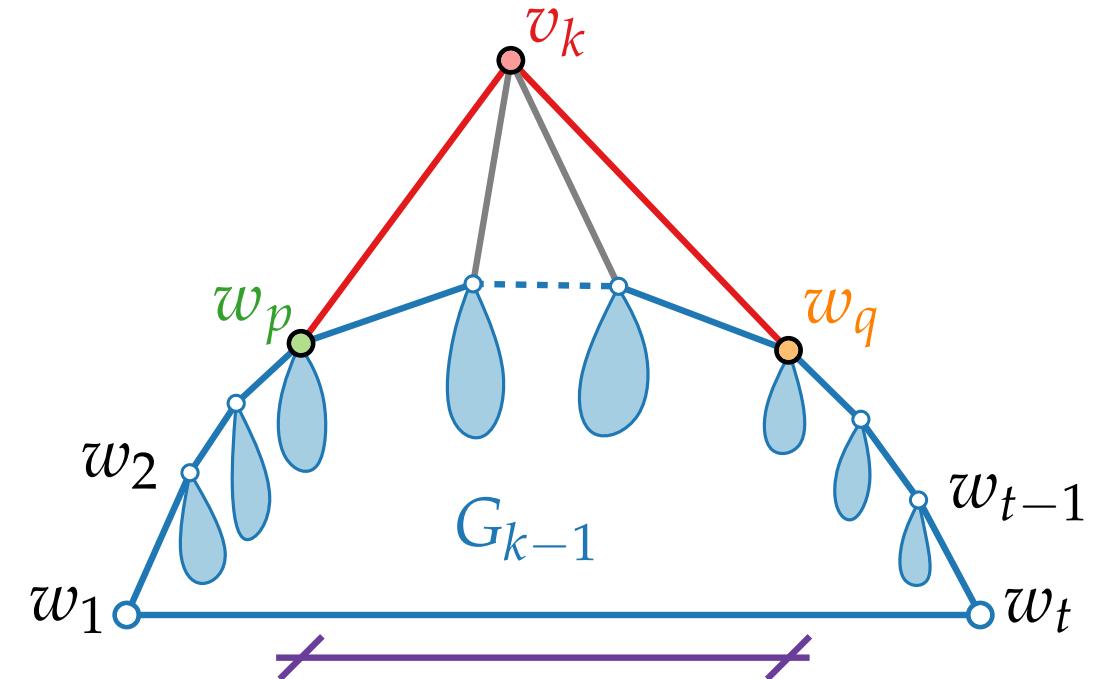
To compute $x(v_k)$ & $y(v_k)$,
we only need $y(w_p)$ and $y(w_q)$ and $x(w_q) - x(w_p)$



Shift Method – Linear Time Implementation

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To compute $x(v_k)$ & $y(v_k)$,
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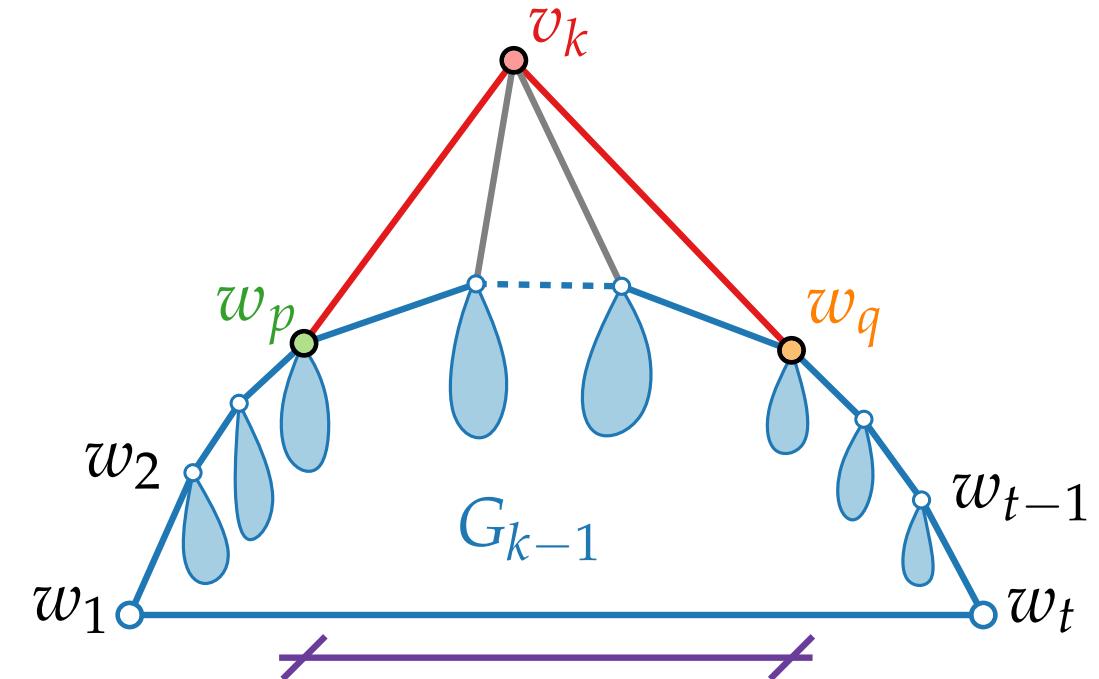
$$(1) \quad x(v_k) =$$

$$(2) \quad y(v_k) =$$

Shift Method – Linear Time Implementation

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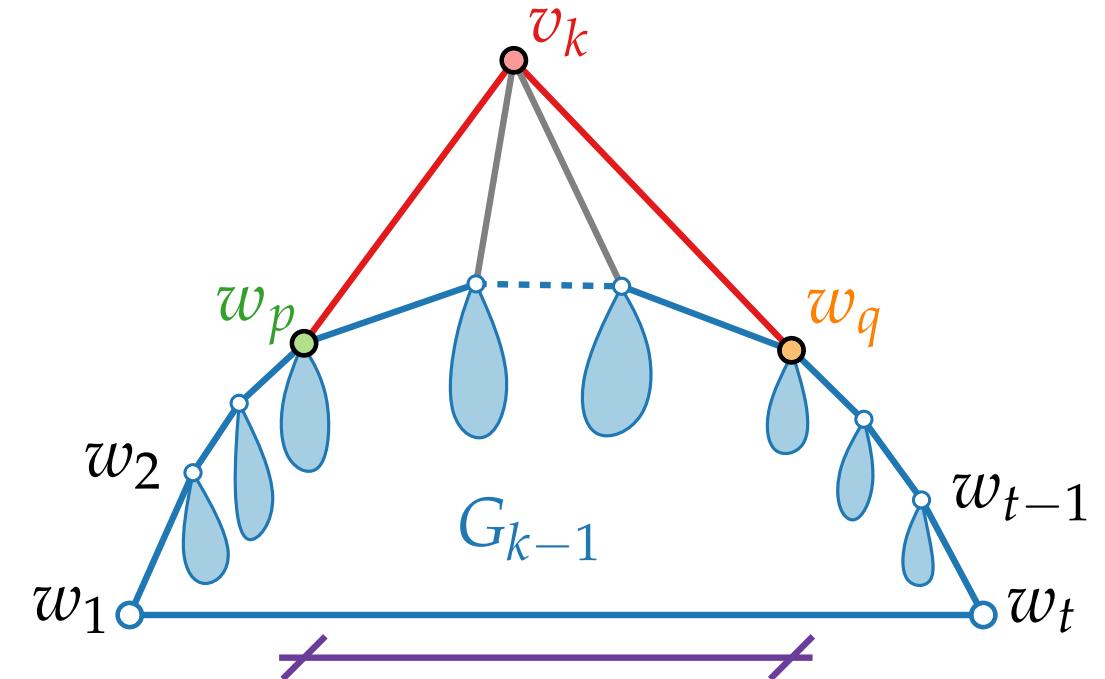
$$(1) \quad x(v_k) = \frac{1}{2}(x(w_q) + x(w_p) + y(w_q) - y(w_p))$$

$$(2) \quad y(v_k) =$$

Shift Method – Linear Time Implementation

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To compute $x(v_k)$ & $y(v_k)$,
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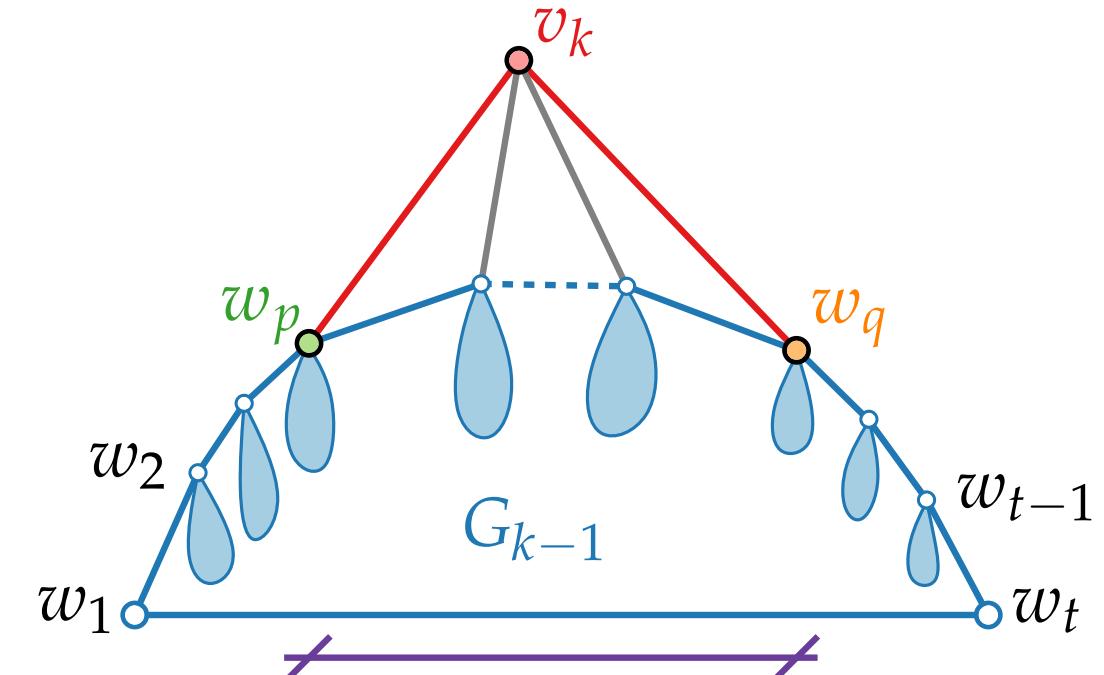
Shift Method – Linear Time Implementation

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To compute $x(v_k)$ & $y(v_k)$,
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Idea 2.

Instead of storing explicit x-coordinates,
we store x-distances.



$$(1) \quad x(v_k) = \frac{1}{2}(x(w_q) + x(w_p) + y(w_q) - y(w_p))$$

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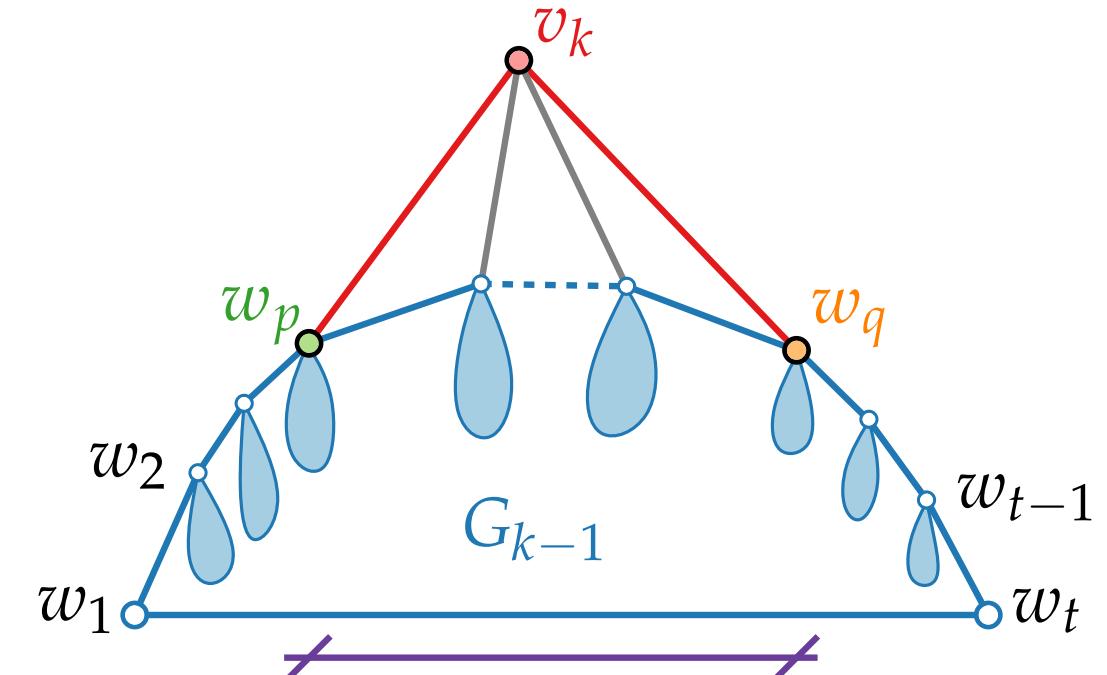
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$$(3) \quad x(v_k) - x(w_p) =$$

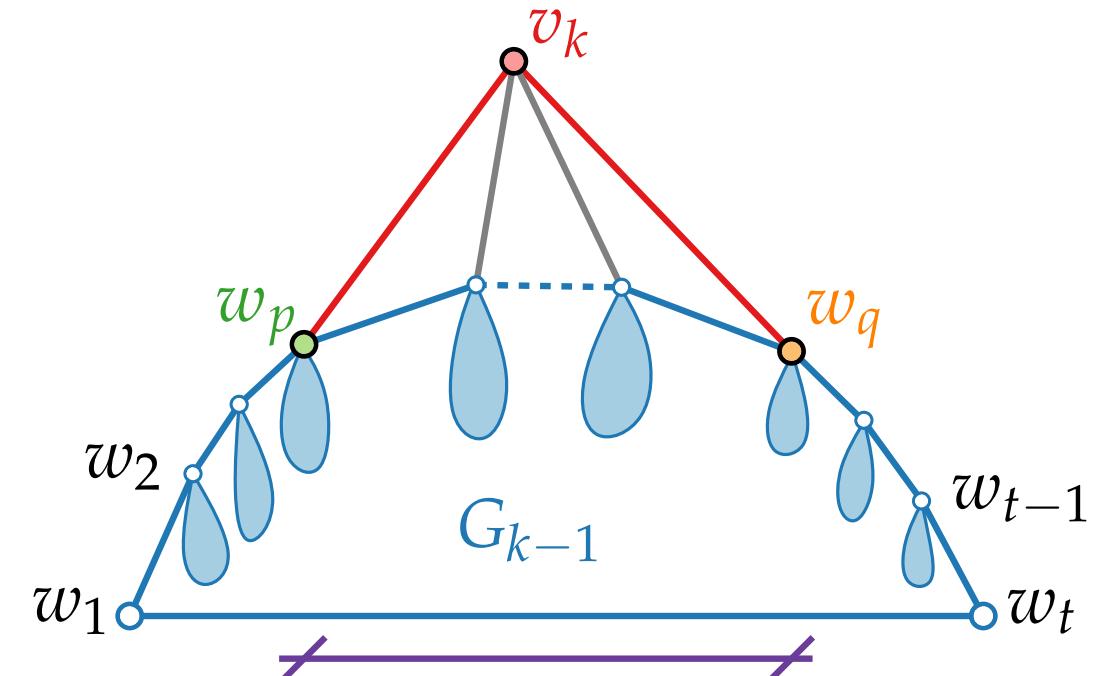
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Shift Method – Linear Time Implementation

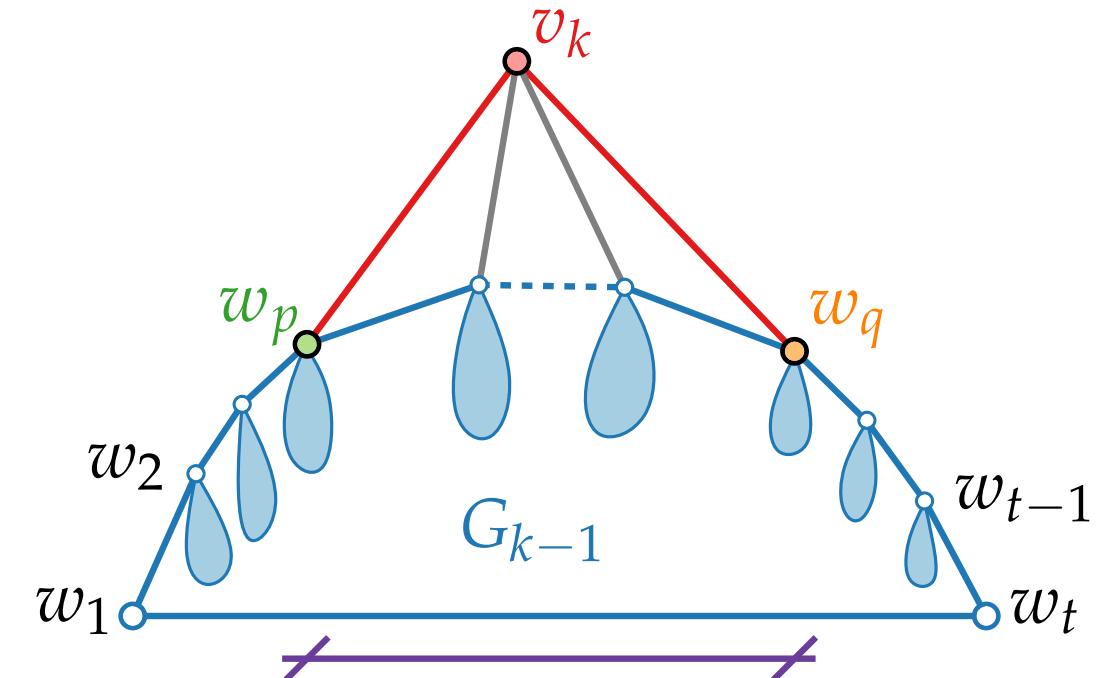
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Idea 2.

Instead of storing explicit x-coordinates,
we store x-distances.

After x distance for v_n computed, use
preorder traversal to compute all
x-coordinates.



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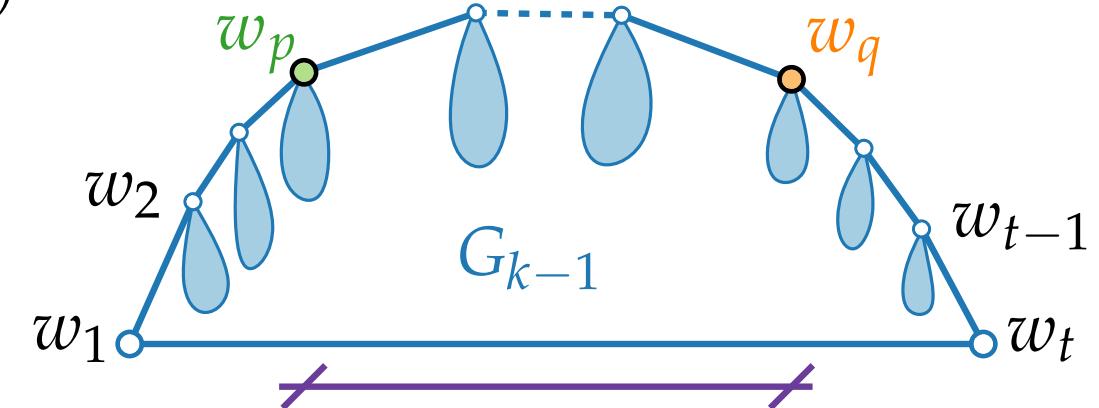
$$(3) \quad x(v_k) - x(w_p) = \frac{1}{2}(x(w_q) - x(w_p) + y(w_q) - y(w_p))$$

Shift Method – Linear Time Implementation

Relative x-distance tree.

For each vertex v store

- x-offset $\Delta_x(v)$ from parent
- y-coordinate $y(v)$



$$(1) \quad x(v_k) = \frac{1}{2}(x(w_q) + x(w_p) + y(w_q) - y(w_p))$$

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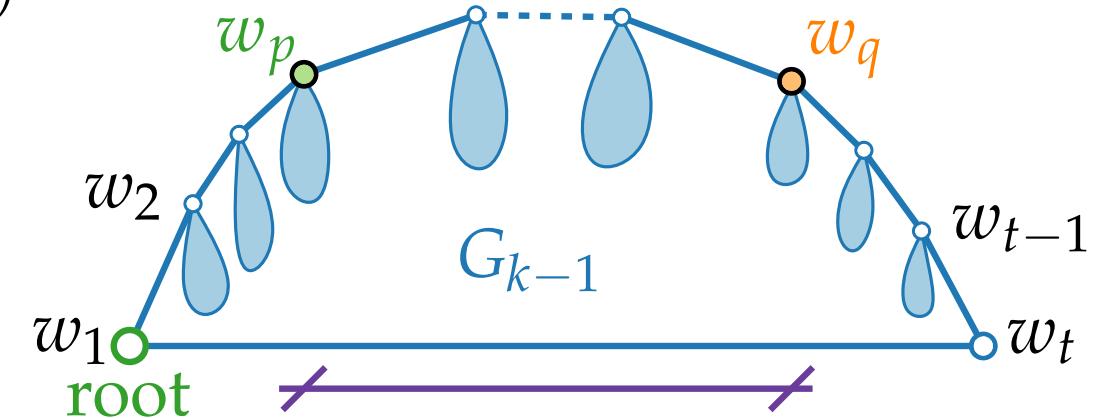
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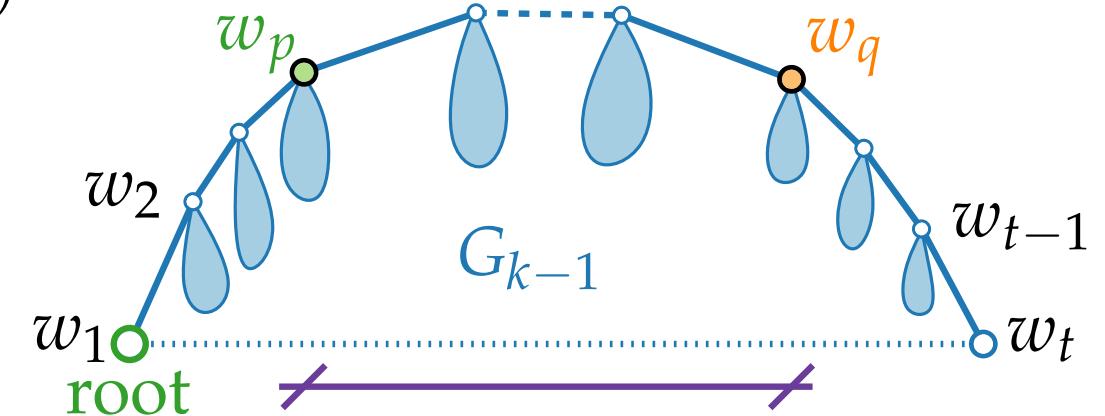
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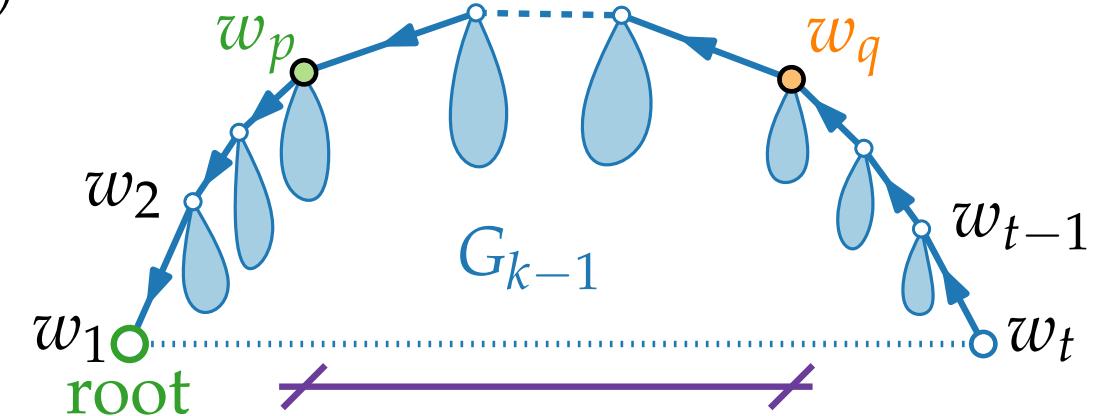
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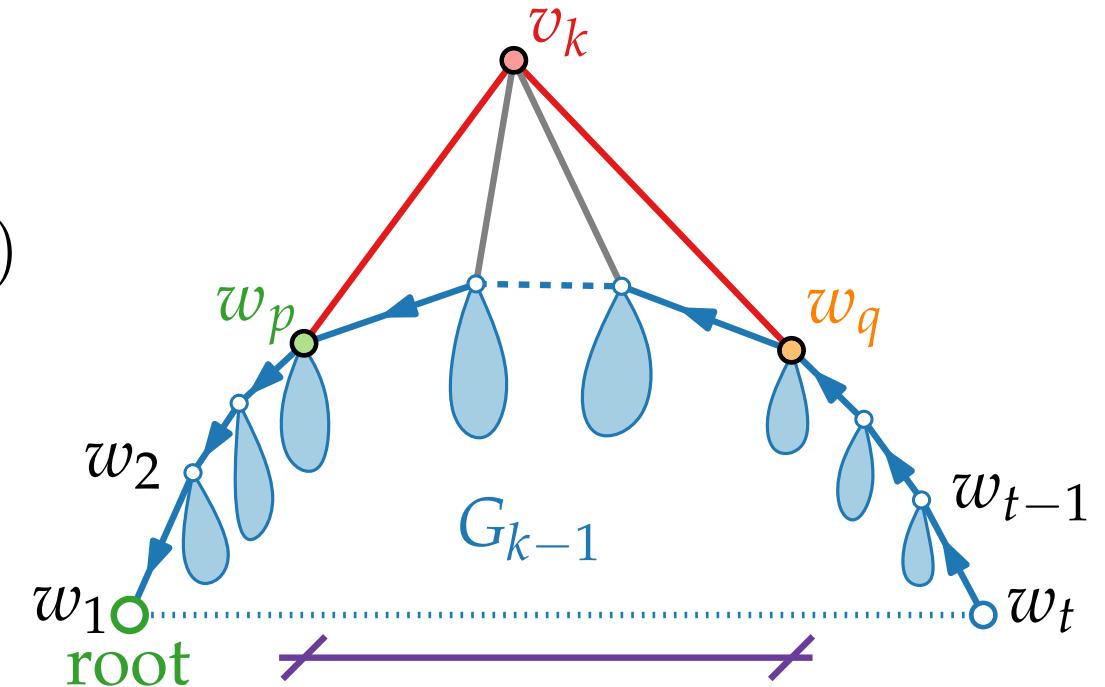
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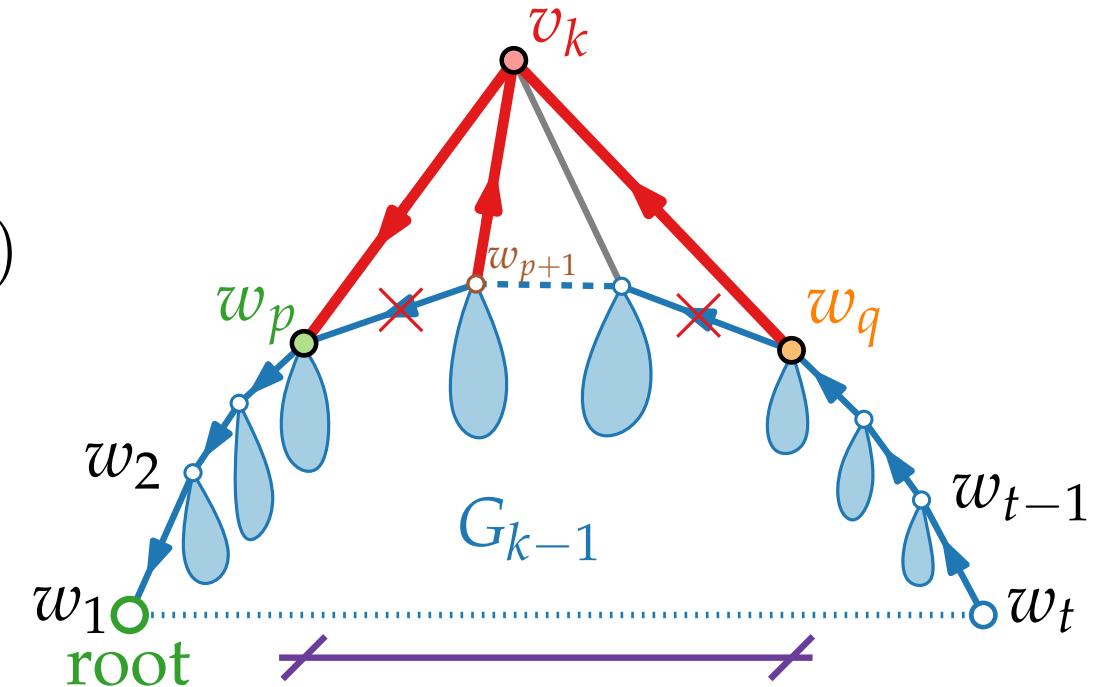
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Shift Method – Linear Time Implementation

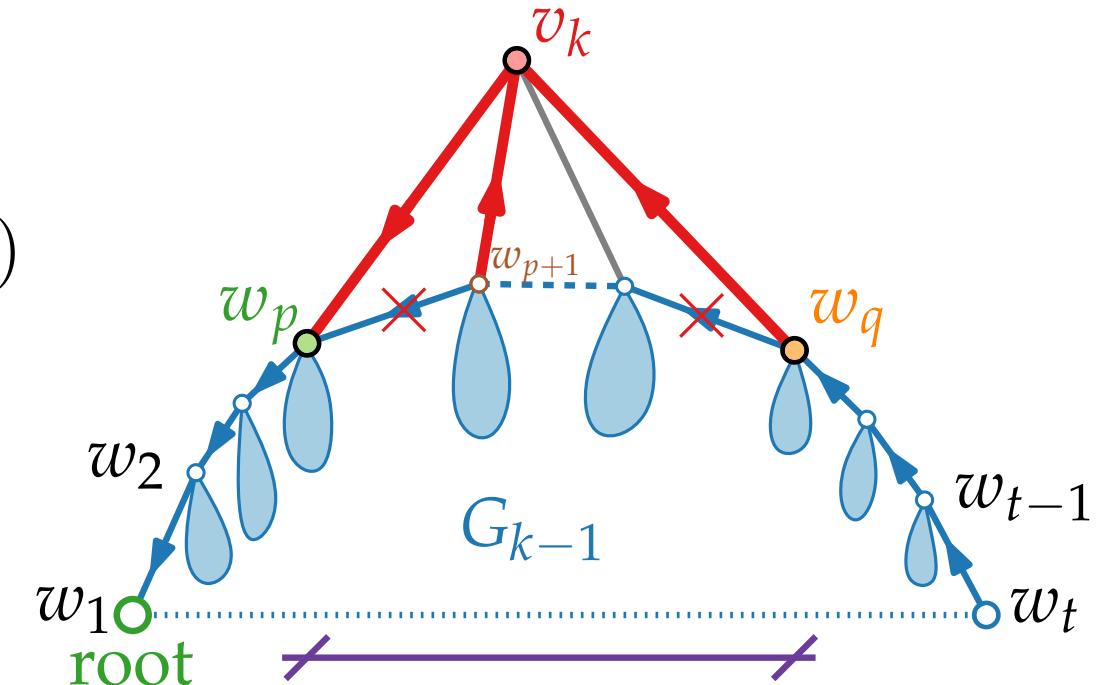
Relative x-distance tree.

For each vertex v store

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- y-coordinate $y(v)$

Calculations.

- $\Delta_x(w_{p+1})++, \Delta_x(w_q)++$



$$(1) \quad x(v_k) = \frac{1}{2}(x(w_q) + x(w_p) + y(w_q) - y(w_p))$$

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Shift Method – Linear Time Implementation

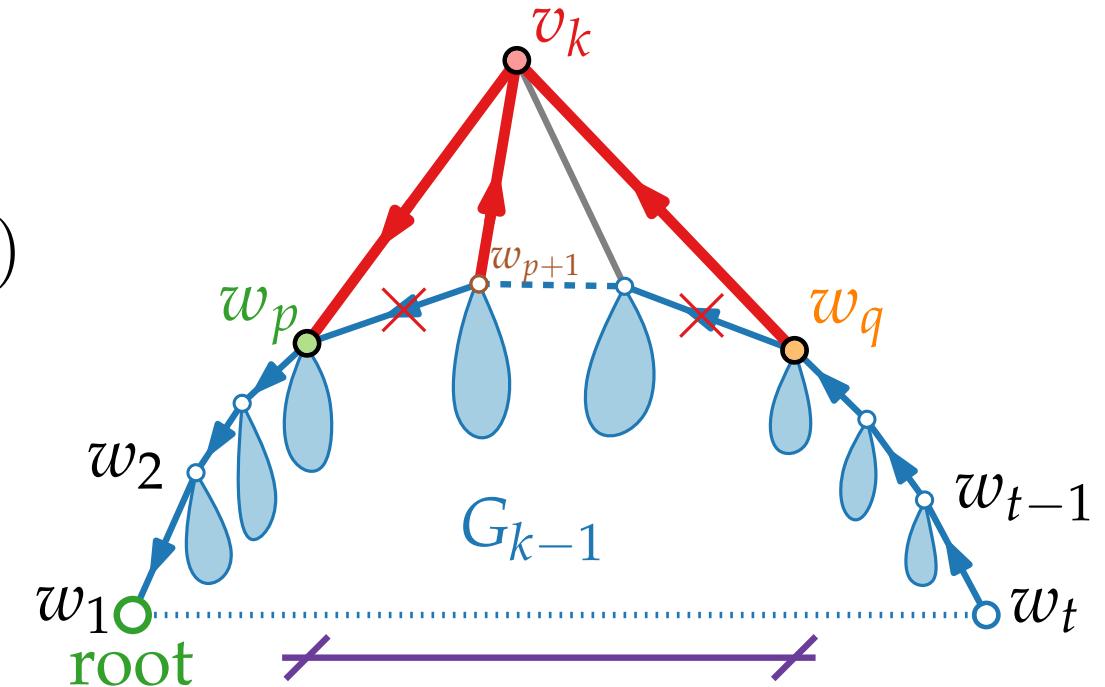
Relative x-distance tree.

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- y-coordinate $y(v)$

Calculations.

- $\Delta_x(w_{p+1})++, \Delta_x(w_q)++$
- $\Delta_x(w_p, w_q) = \Delta_x(w_{p+1}) + \dots + \Delta_x(w_q)$



$$(1) \quad x(v_k) = \frac{1}{2}(x(w_q) + x(w_p) + y(w_q) - y(w_p))$$

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Shift Method – Linear Time Implementation

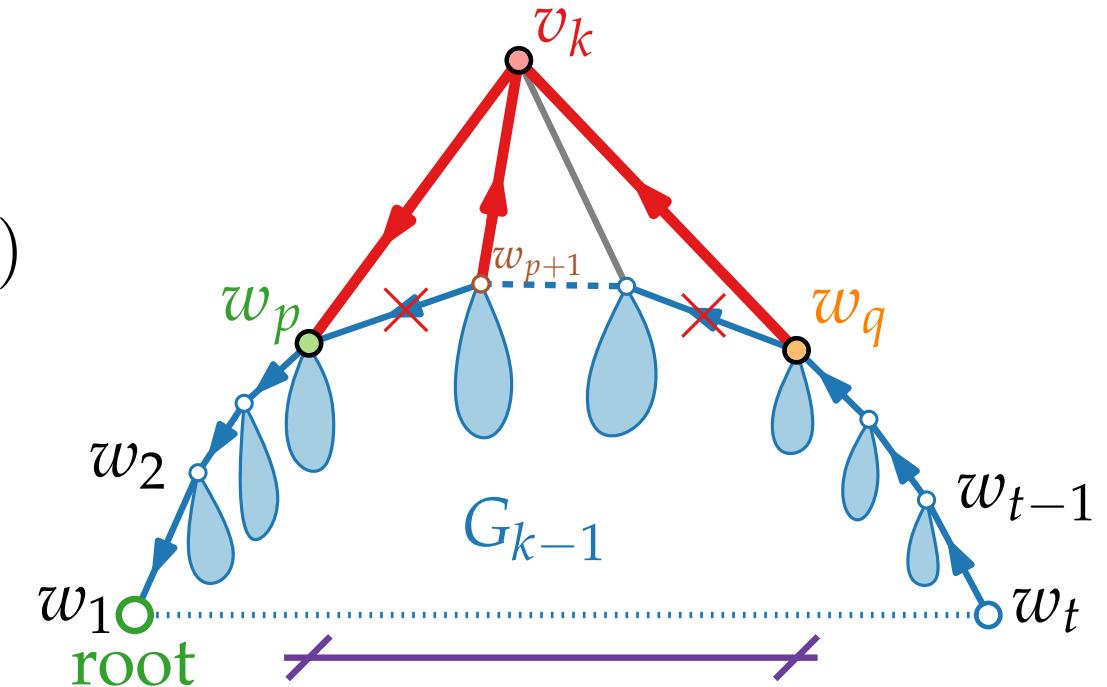
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- $\Delta_x(w_{p+1})++, \Delta_x(w_q)++$
- $\Delta_x(w_p, w_q) = \Delta_x(w_{p+1}) + \dots + \Delta_x(w_q)$
- $\Delta_x(v_k)$ by (3)



$$(1) \quad x(v_k) = \frac{1}{2}(x(w_q) + x(w_p) + y(w_q) - y(w_p))$$

$$(2) \quad y(v_k) = \frac{1}{2}(x(w_q) - x(w_p) + y(w_q) + y(w_p))$$

$$(3) \quad x(v_k) - x(w_p) = \frac{1}{2}(x(w_q) - x(w_p) + y(w_q) - y(w_p))$$

Shift Method – Linear Time Implementation

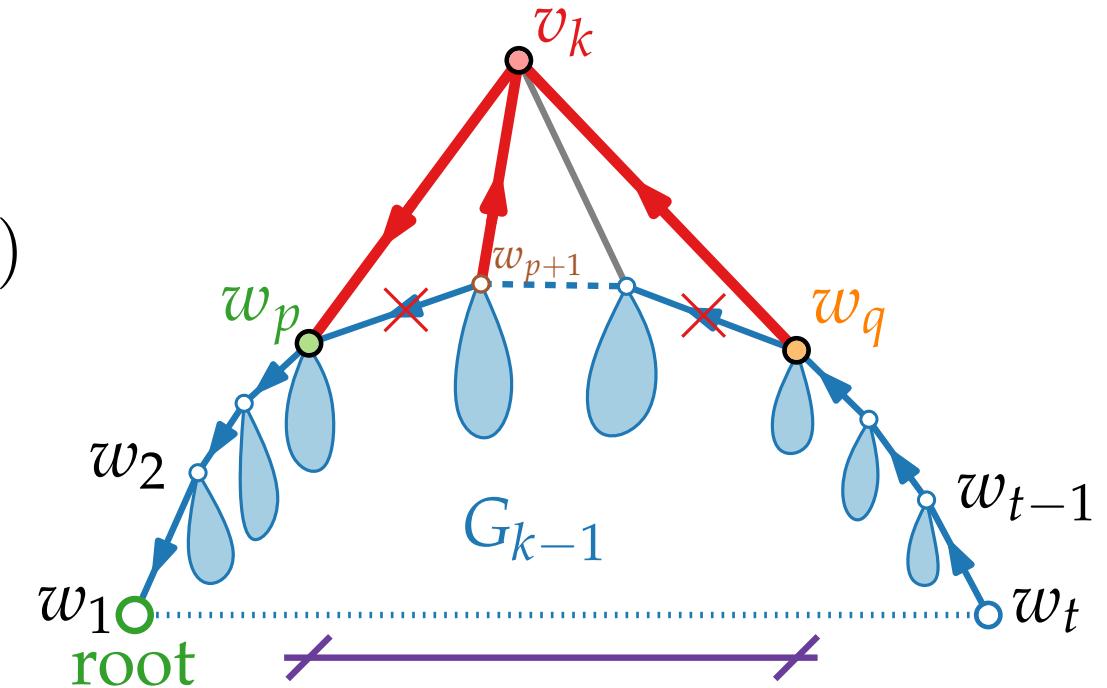
Relative x-distance tree.

For each vertex v store

- x-offset $\Delta_x(v)$ from parent
- y-coordinate $y(v)$

Calculations.

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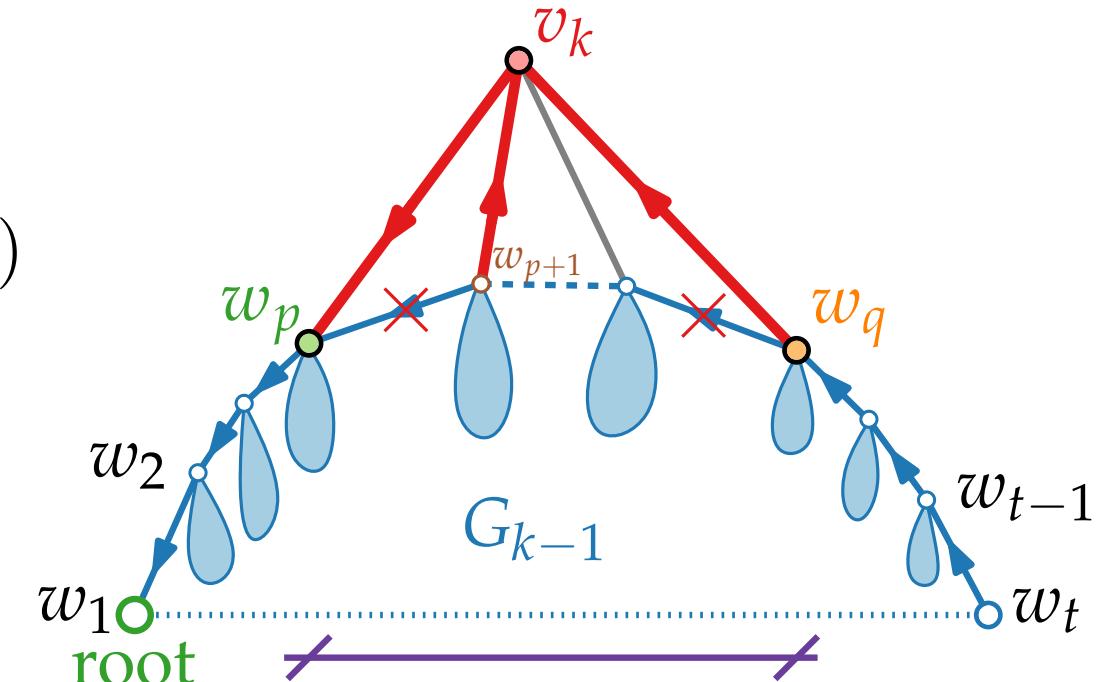
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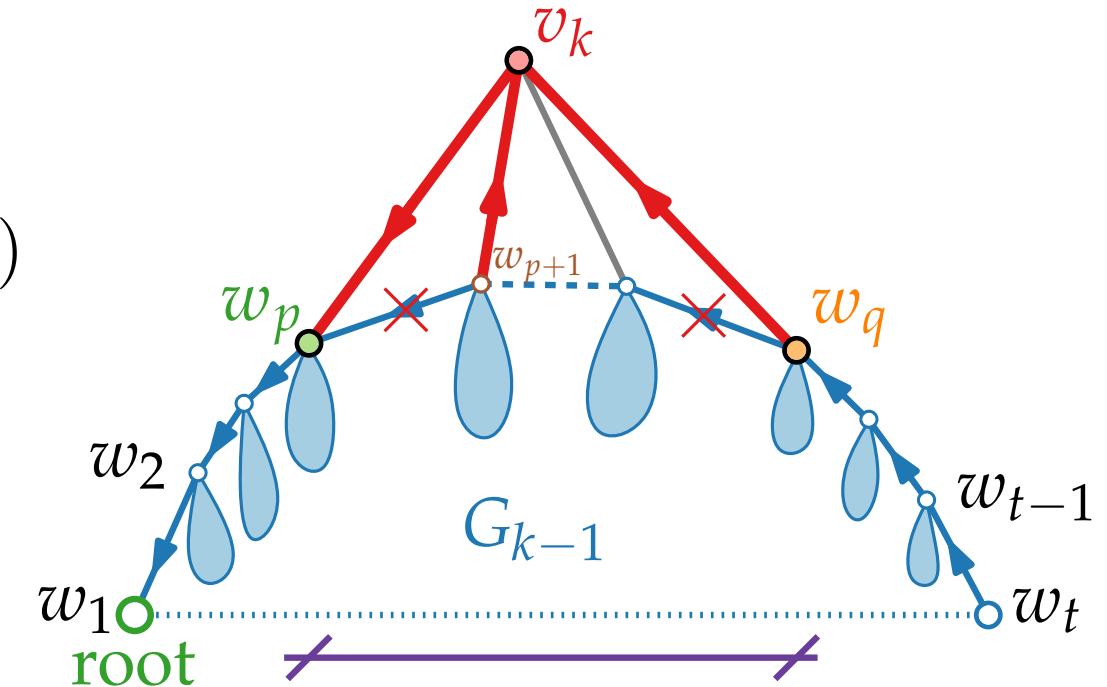
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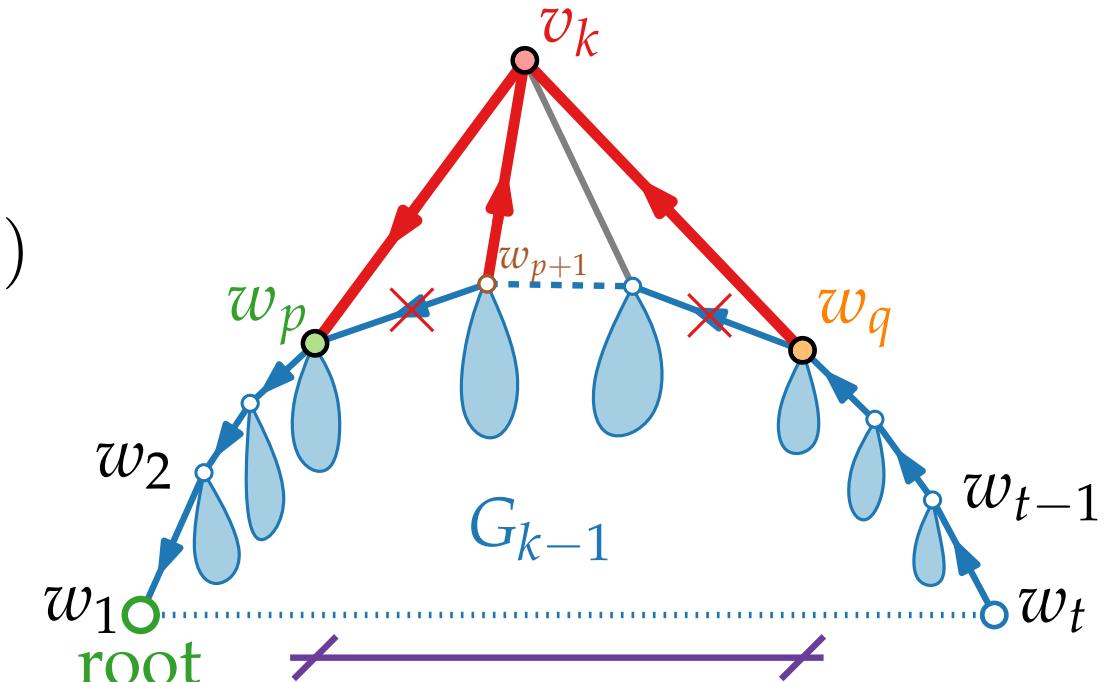
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$\mathcal{O}(n)$ in total

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Result & Variations

Theorem.

[De Fraysseix, Pach, Pollack '90]

Every n -vertex planar graph has a planar straight-line drawing of size $(2n - 4) \times (n - 2)$. Such a drawing can be computed in $O(n)$ time.

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[Kant '96]

Every n -vertex 3-connected planar graph has a planar straight-line drawing of size $(2n - 4) \times (n - 2)$ where all faces are drawn convex. Such a drawing can be computed in $O(n)$ time.

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Theorem.

[Brandenburg '08]

Every n -vertex planar graph has a planar straight-line drawing of size $\frac{4}{3}n \times \frac{2}{3}n$. Such a drawing can be computed in $O(n)$ time.