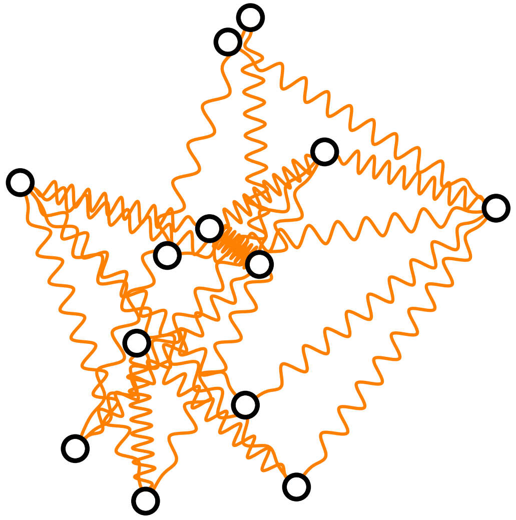
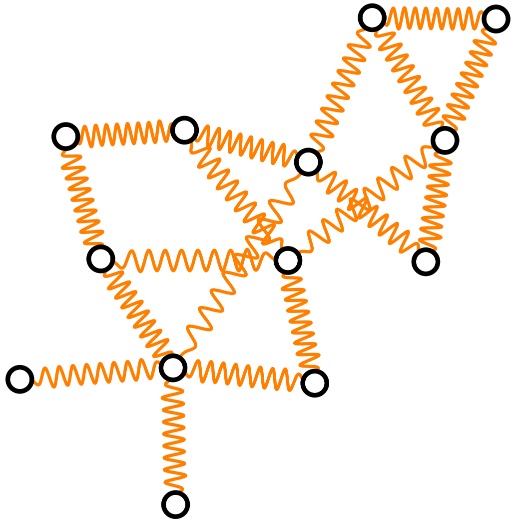
CS F402: Computational Geometry

Lecture 10:
GD - Force-Directed Drawing Algorithms-II



Siddharth Gupta

February 14+17+19, 2025



Variant by Fruchterman & Reingold

ForceDirected($G = (V, E)$, $p = (p_v)_{v \in V}$, $\varepsilon > 0$, $K \in \mathbb{N}$)
$t \leftarrow 1$
while $t < K$ **and** $\max_{v \in V} \|F_v(t)\| > \varepsilon$ **do**
  foreach $u \in V$ **do**
    $F_u(t) \leftarrow \sum_{v \in V} f_{\text{rep}}(u, v) + \sum_{uv \in E} f_{\text{attr}}(u, v)$
  foreach $u \in V$ **do**
    $p_u \leftarrow p_u + \delta(t) \cdot F_u(t)$
  $t \leftarrow t + 1$
return p

- **Repulsive forces**

repulsion constant (e.g. 2.0)

$$f_{\text{rep}}(u,v) = \frac{c_{\text{rep}}}{||p_v - p_u||^2} \cdot \overrightarrow{p_v p_u}$$

- **Attractive forces**

spring constant (e.g. 1.0)

$$f_{\text{spring}}(u,v) = c_{\text{spring}} \cdot \log \frac{||p_v - p_u||}{\ell} \cdot \overrightarrow{p_u p_v}$$

$$f_{\text{attr}}(u,v) = f_{\text{spring}}(u,v) - f_{\text{rep}}(u,v)$$

- **Resulting displacement vector**

$$F_u = \sum_{v \in V} f_{\text{rep}}(u,v) + \sum_{uv \in E} f_{\text{attr}}(u,v)$$

Notation.

- $||p_u - p_v||$ = Euclidean distance between u and v
- $\overrightarrow{p_u p_v}$ = unit vector pointing from u to v
- ℓ = ideal spring length for edges

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- **Repulsive forces**

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```
ForceDirected(G = (V, E), p = (p_v)_{v∈V}, ε > 0, K ∈ ℕ)
  t ← 1
  while t < K and max_{v∈V} ||F_v(t)|| > ε do
    foreach u ∈ V do
      F_u(t) ← Σ_{v∈V} f_rep(u, v) + Σ_{uv∈E} f_attr(u, v)
    foreach u ∈ V do
      p_u ← p_u + δ(t) · F_u(t)
    t ← t + 1
  return p
```

- **Repulsive forces**

$$f_{\text{rep}}(u,v) = \frac{\ell^2}{||p_v - p_u||} \cdot \overrightarrow{p_v p_u}$$

- **Attractive forces**

$$f_{\text{attr}}(u,v) = \frac{||p_v - p_u||^2}{\ell} \cdot \overrightarrow{p_u p_v}$$

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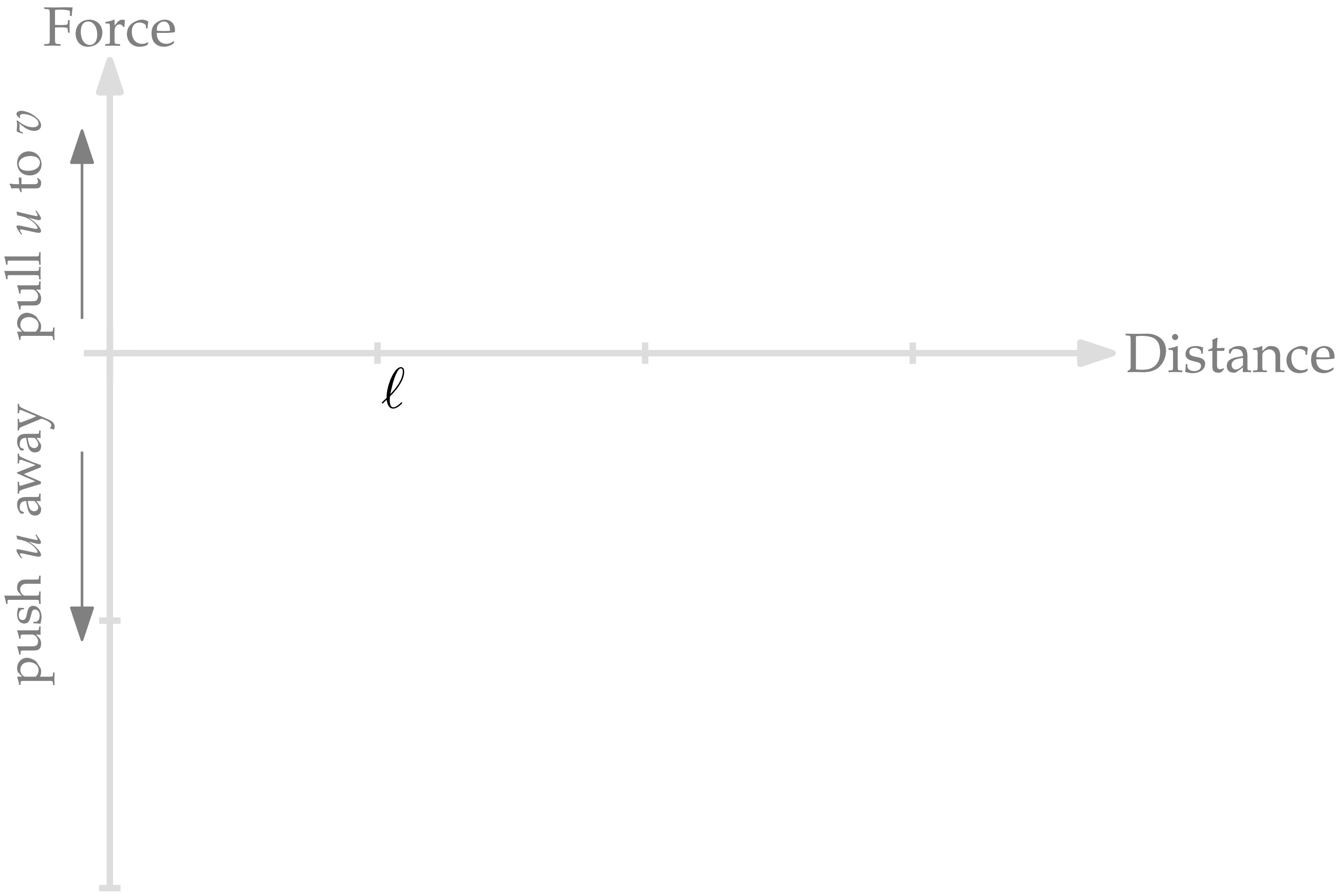
Fruchterman & Reingold – Force Diagram



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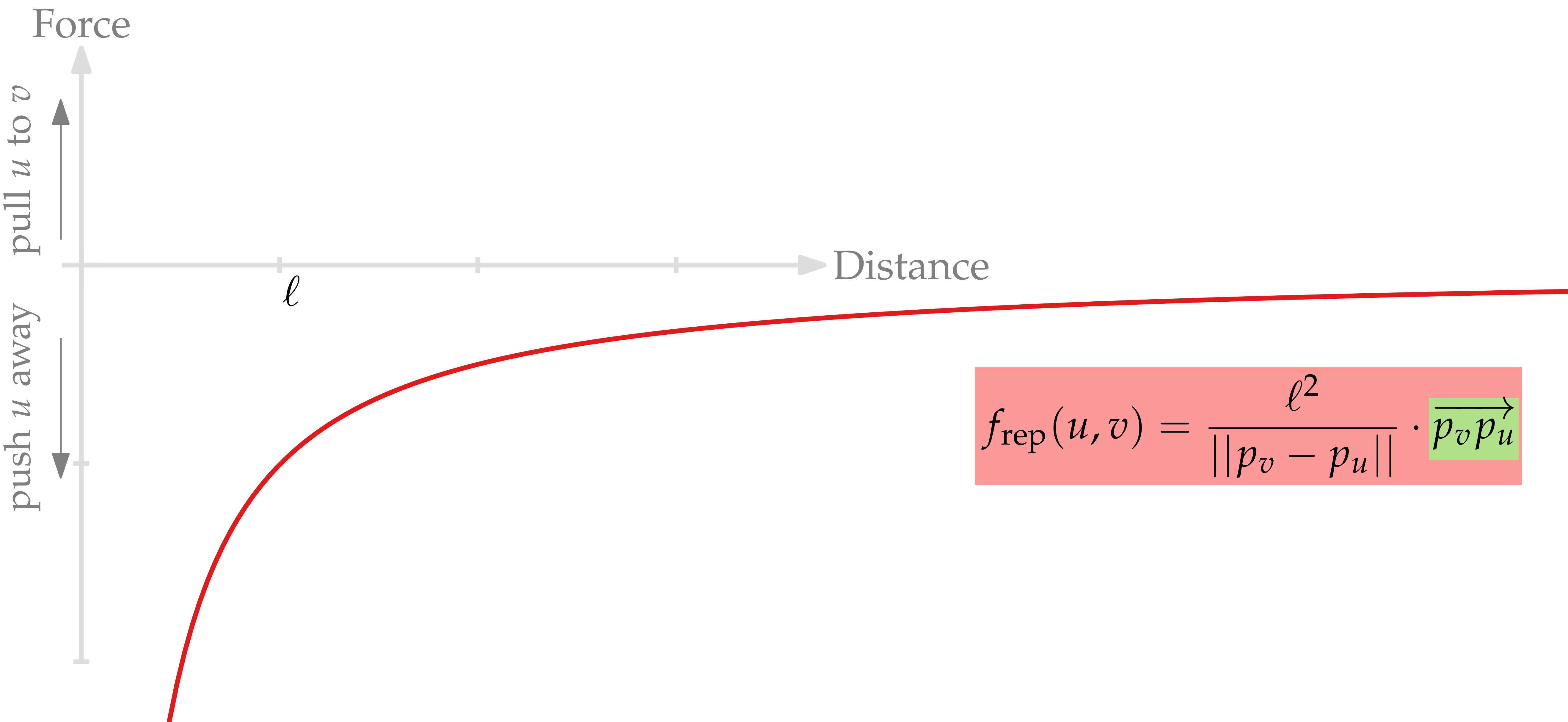

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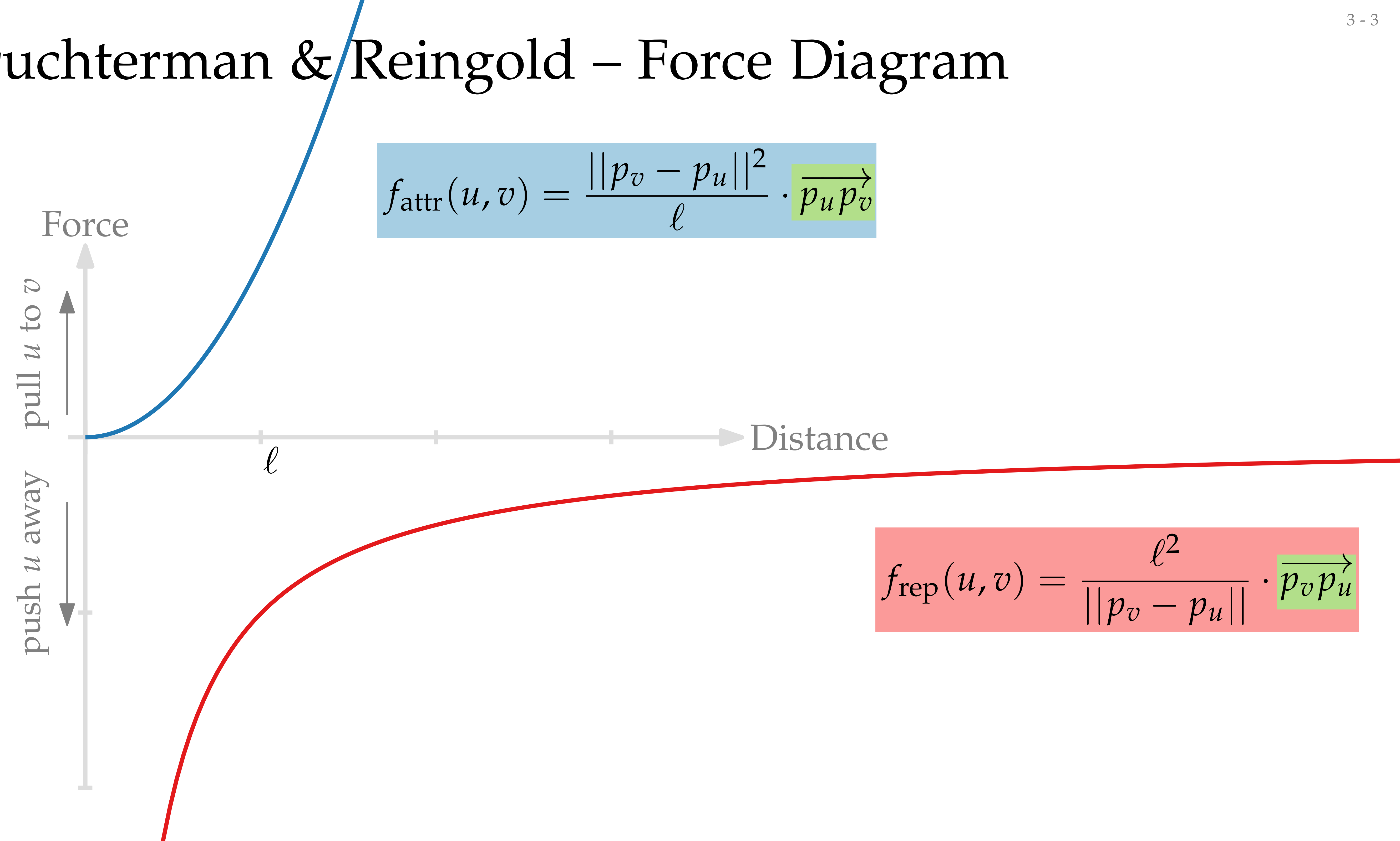

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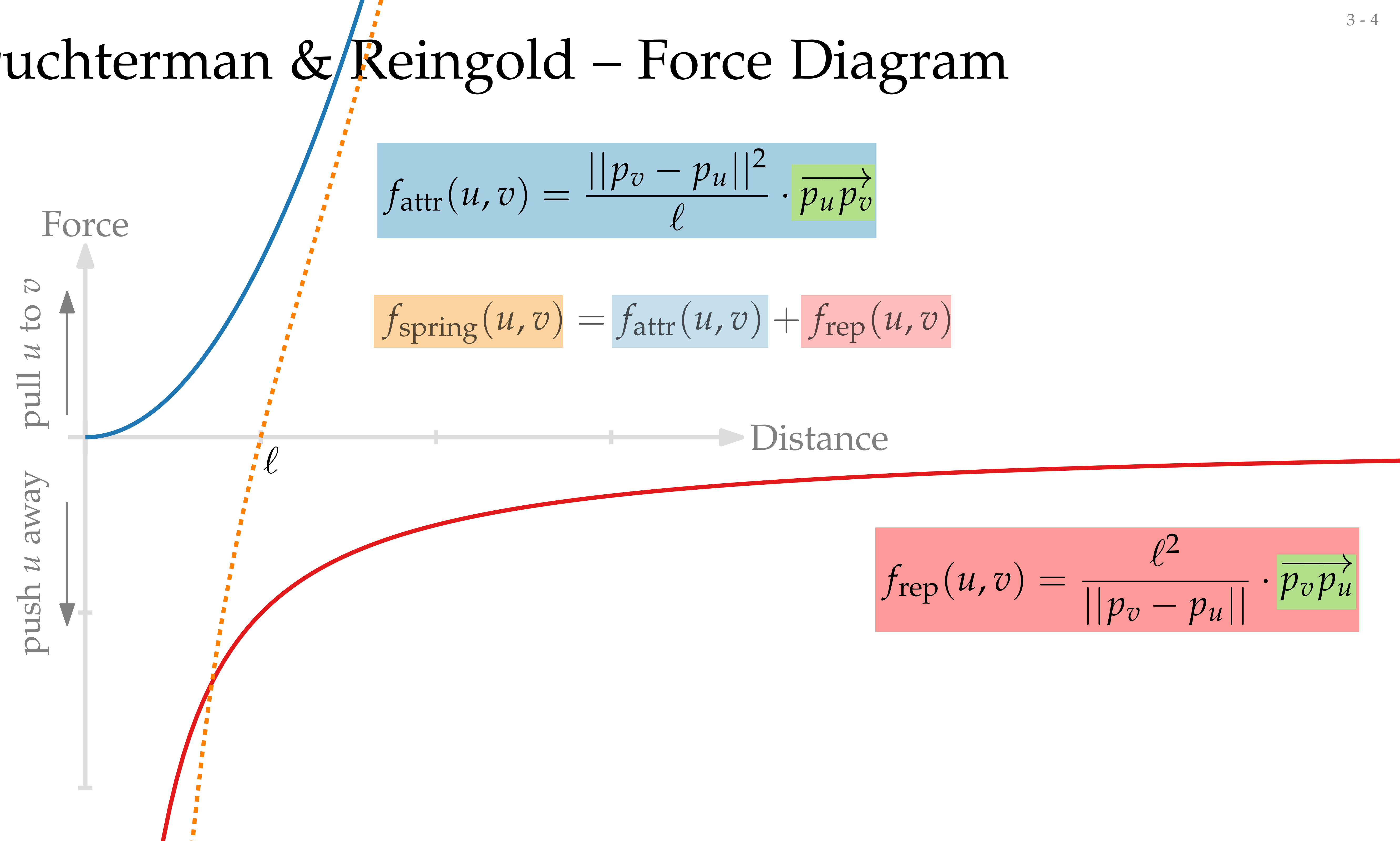
$$f_{\text{spring}}(u,v) = f_{\text{attr}}(u,v) + f_{\text{rep}}(u,v)$$



$$f_{\text{rep}}(u,v) = \frac{\ell^2}{||p_v - p_u||} \cdot \overrightarrow{p_v p_u}$$

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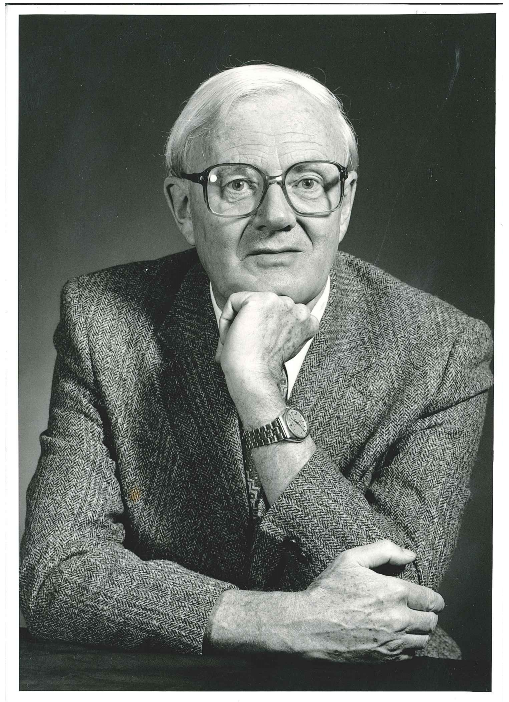

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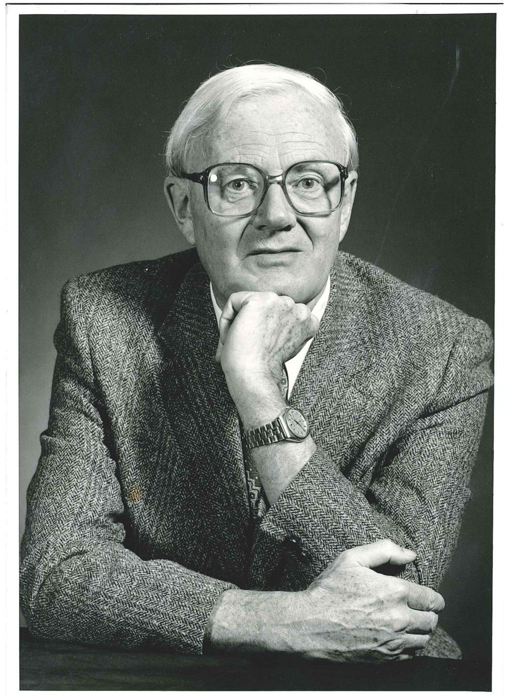

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HOW TO DRAW A GRAPH

By W. T. TUTTE

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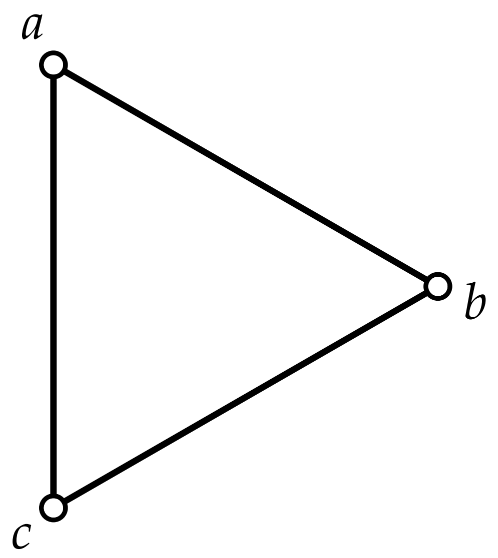

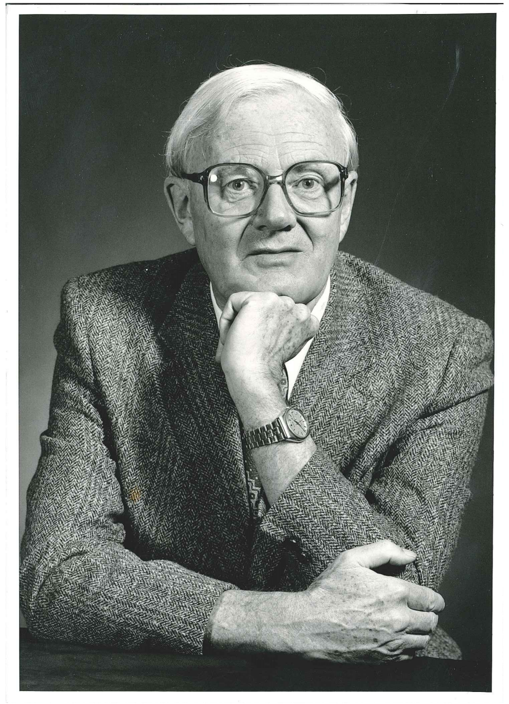

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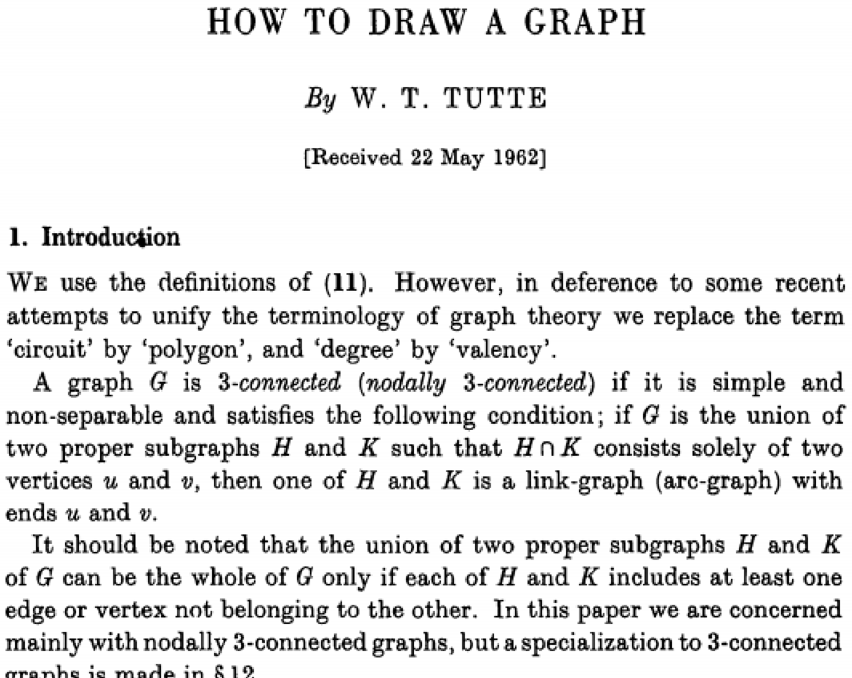

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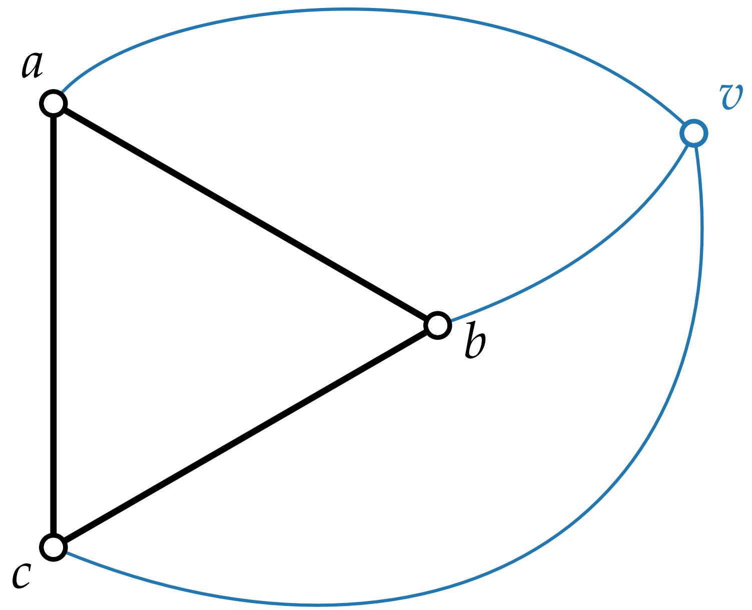

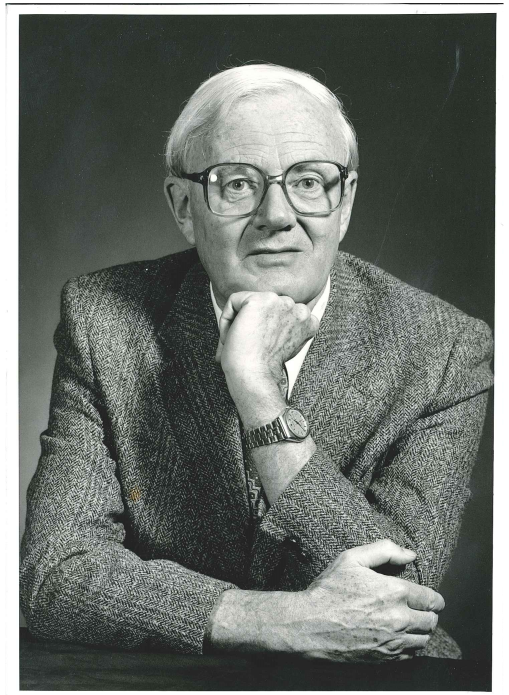

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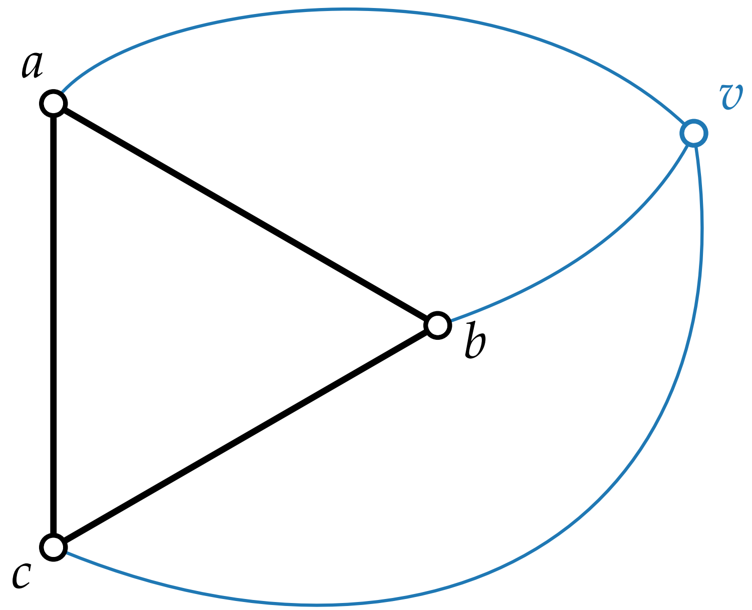
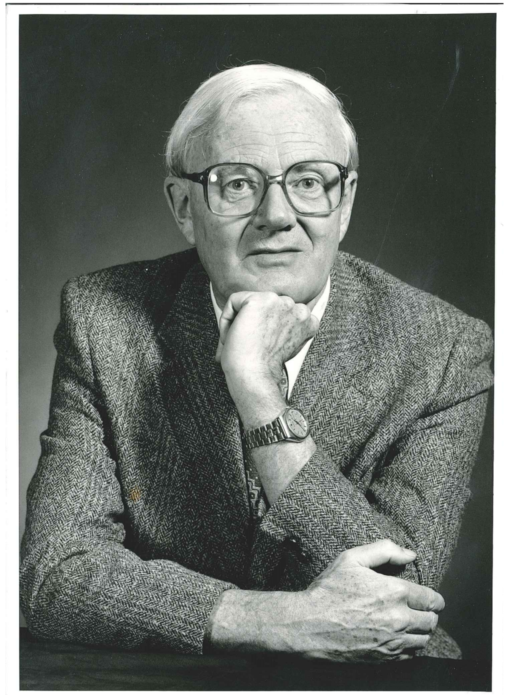
Where would you place v?





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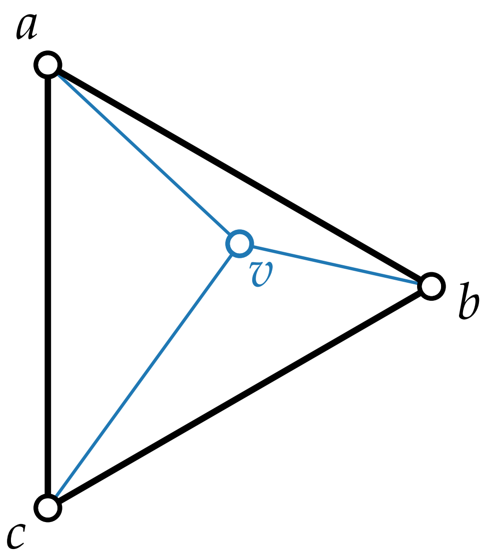

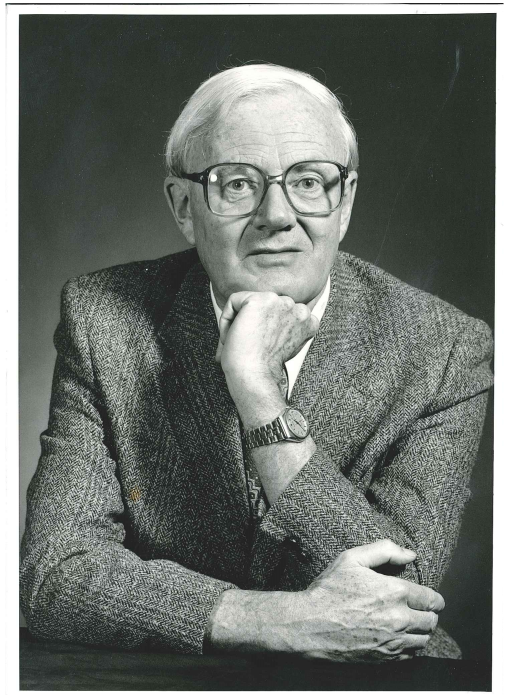

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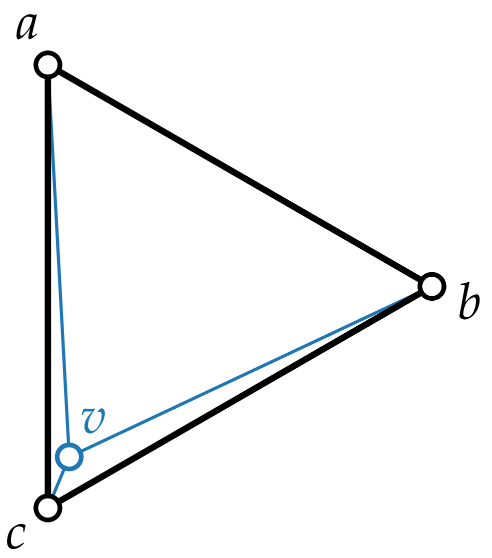

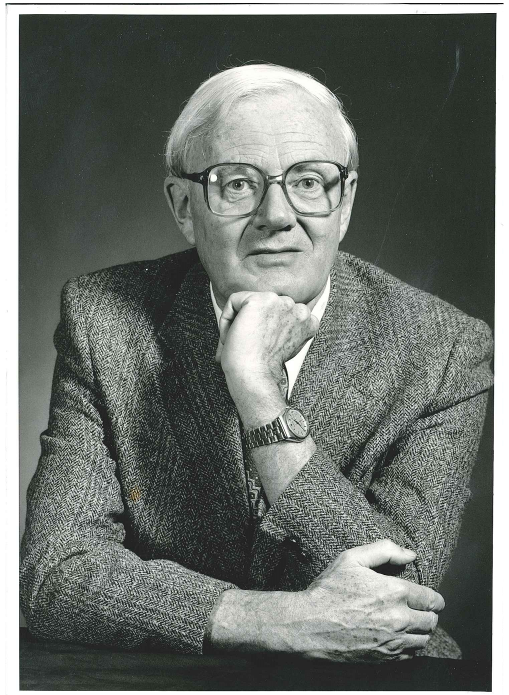

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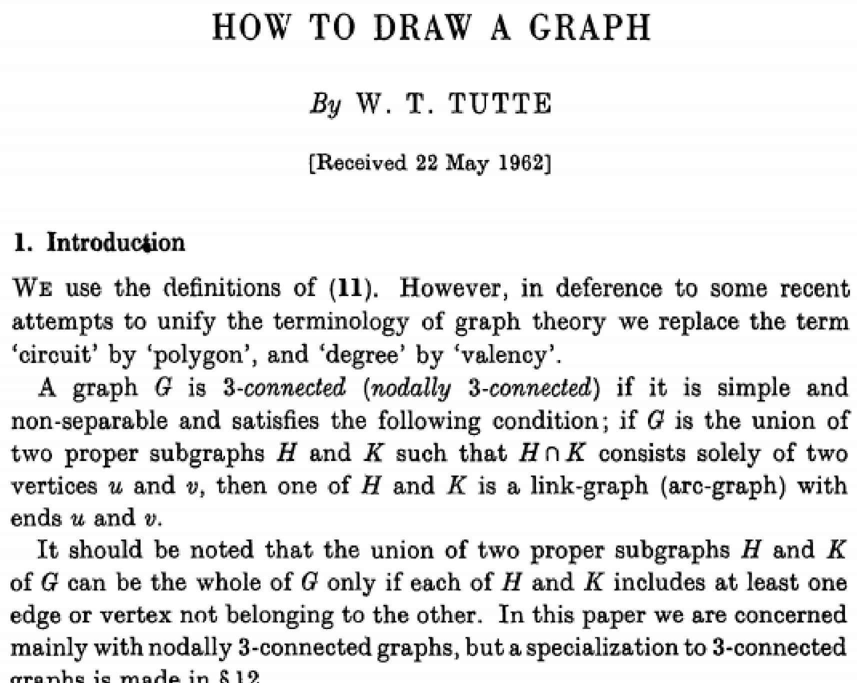

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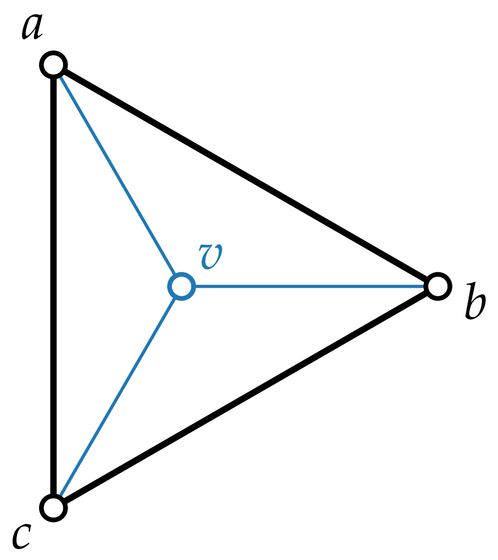

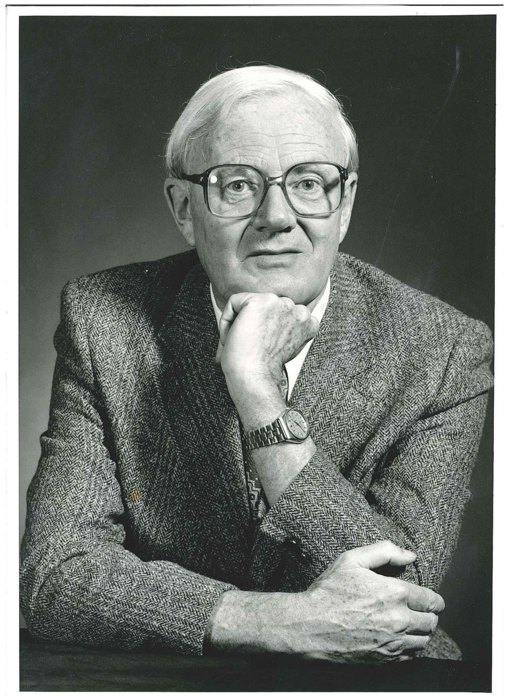

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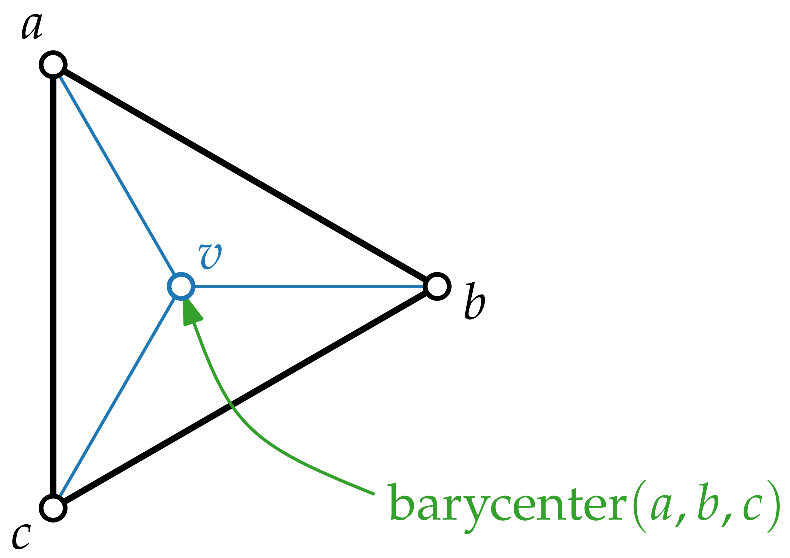

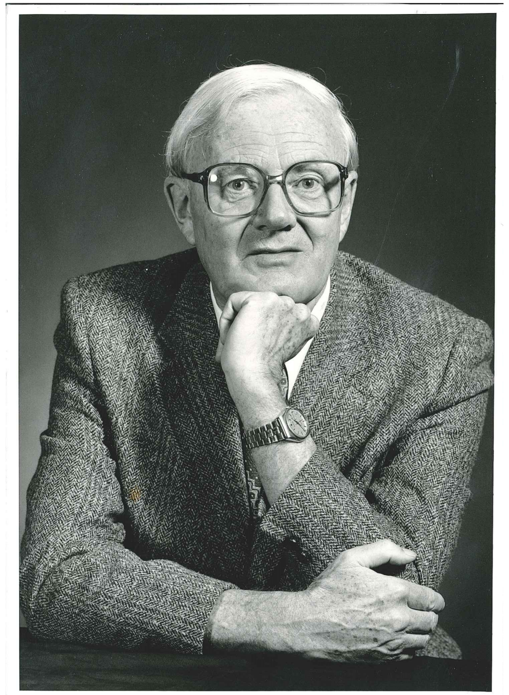

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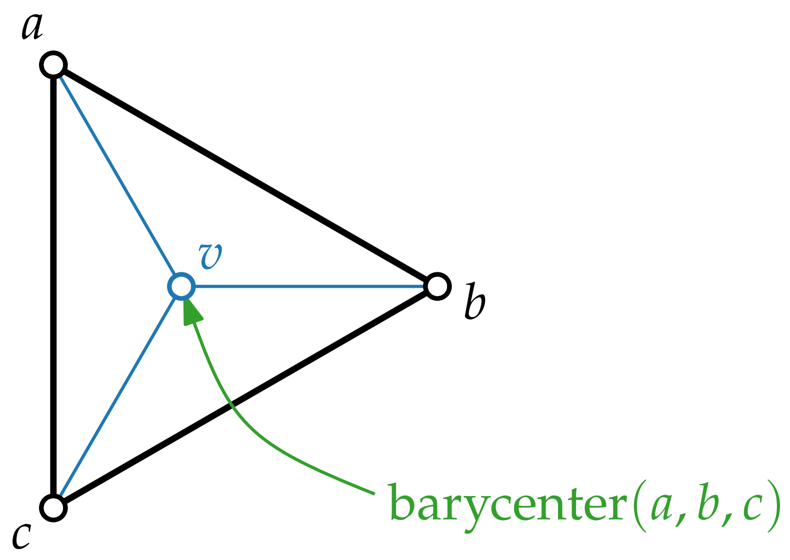
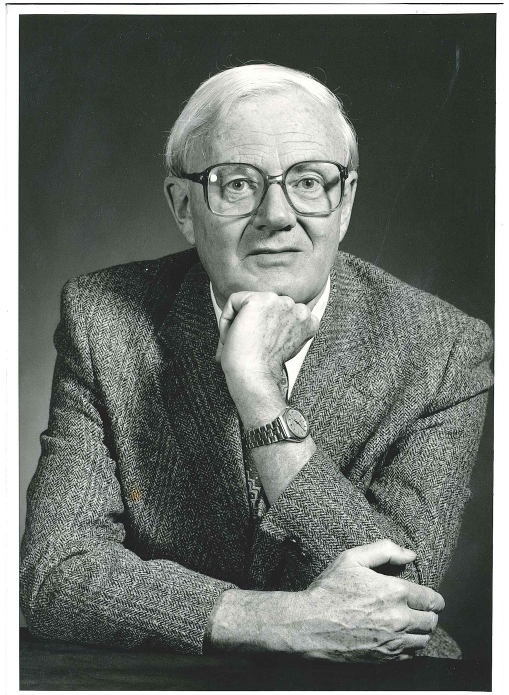


$$\text{barycenter}(x_1, \ldots, x_k) = \quad ?$$



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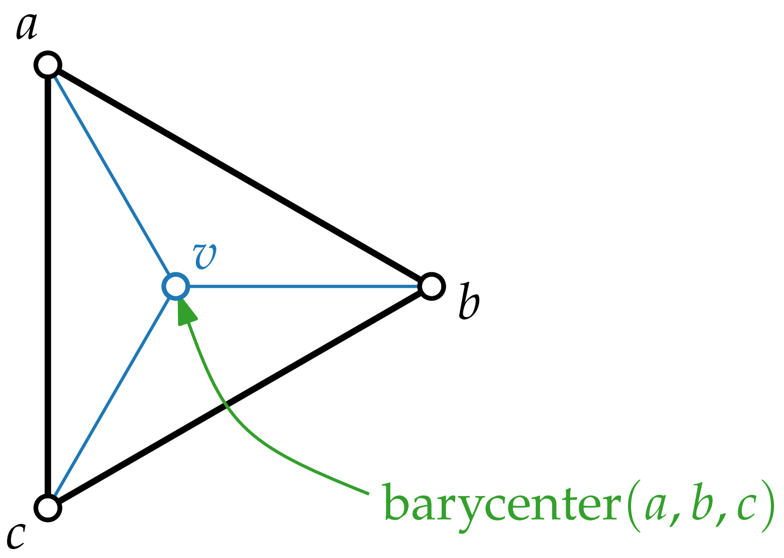
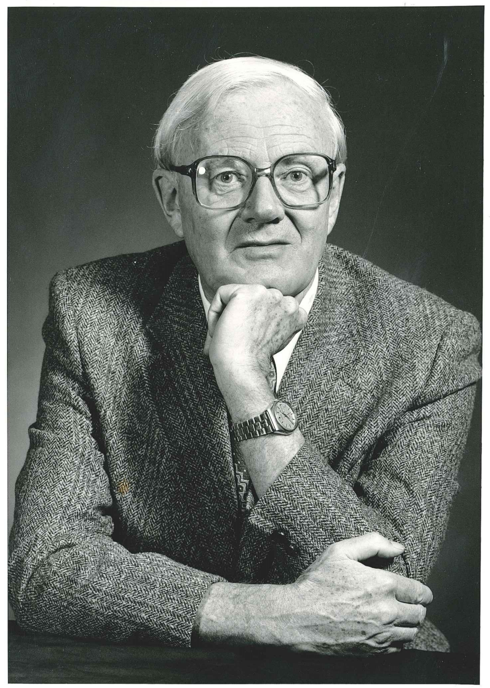


$$\text{barycenter}(x_1, \ldots, x_k) = \textstyle\sum_{i=1}^{k} x_i / k$$



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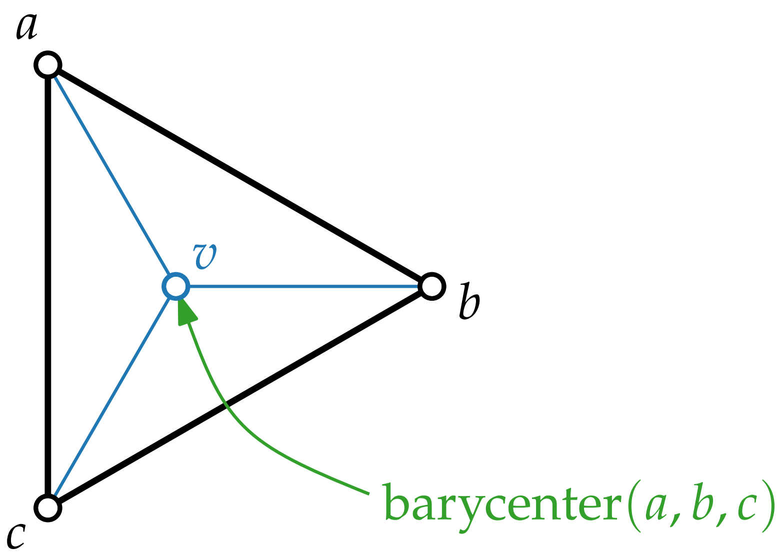
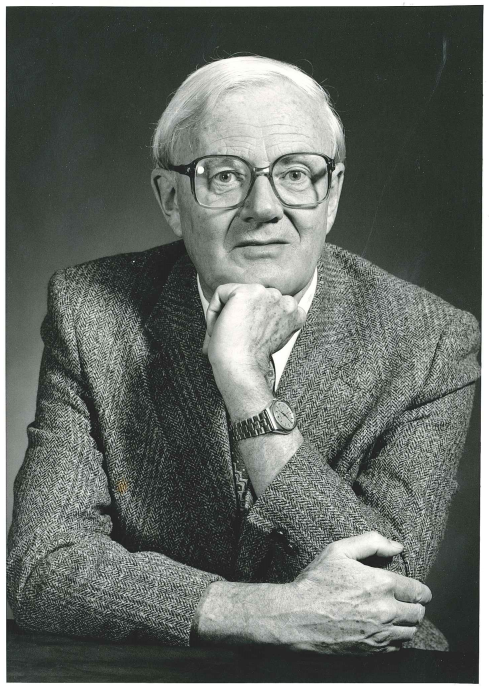


barycenter$(x_1, \ldots, x_k) = \sum_{i=1}^{k} x_i / k$

Idea.
Repeatedly place every vertex at barycenter of neighbors.



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$t \leftarrow 1$

while $t < K$ **and** $\max_{v \in V} \|F_v(t)\| > \varepsilon$ **do**

  foreach $u \in V$ **do**

    $F_u(t) \leftarrow \sum_{v \in V} f_{\mathrm{rep}}(u, v) + \sum_{uv \in E} f_{\mathrm{attr}}(u, v)$

  foreach $u \in V$ **do**

    $p_u \leftarrow p_u + \delta(t) \cdot F_u(t)$

  $t \leftarrow t + 1$

return p

Tutte's Forces

ForceDirected($G = (V, E)$, $p = (p_v)_{v \in V}$, $\varepsilon > 0$, $K \in \mathbb{N}$)

$t \leftarrow 1$

while $t < K$ **and** $\max_{v \in V} \|F_v(t)\| > \varepsilon$ **do**

 foreach $u \in V$ **do**

  $F_u(t) \leftarrow \sum_{v \in V} f_{\mathrm{rep}}(u, v) + \sum_{uv \in E} f_{\mathrm{attr}}(u, v)$

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  $p_u \leftarrow p_u + \cancel{\delta(t)}\, 1 \cdot F_u(t)$

 $t \leftarrow t + 1$

return p

Tutte's Forces

Goal.

$p_u = \text{barycenter}(\bigcup_{uv \in E} v)$

ForceDirected$(G = (V, E), p = (p_v)_{v \in V}, \varepsilon > 0, K \in \mathbb{N})$

$t \leftarrow 1$

while $t < K$ **and** $\max_{v \in V} \|F_v(t)\| > \varepsilon$ **do**

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$\text{barycenter}(x_1, \ldots, x_k) = \sum_{i=1}^{k} x_i / k$

Tutte's Forces

Goal.

$p_u = \text{barycenter}(\bigcup_{uv \in E} v)$

$= \sum_{uv \in E} p_v /$

ForceDirected($G = (V, E)$, $p = (p_v)_{v \in V}$, $\varepsilon > 0$, $K \in \mathbb{N}$)

$t \leftarrow 1$

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barycenter$(x_1, \ldots, x_k) = \sum_{i=1}^{k} x_i / k$

Tutte's Forces

Goal.

$p_u = \text{barycenter}(\bigcup_{uv \in E} v)$

$\quad = \sum_{uv \in E} p_v / \deg(u)$

ForceDirected($G = (V, E)$, $p = (p_v)_{v \in V}$, $\varepsilon > 0$, $K \in \mathbb{N}$)

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while $t < K$ **and** $\max_{v \in V} \|F_v(t)\| > \varepsilon$ **do**

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Tutte's Forces

Goal.

$$p_u = \text{barycenter}(\bigcup_{uv \in E} v)$$
$$= \sum_{uv \in E} p_v / \deg(u)$$

$$\boxed{F_u(t)} = \sum_{uv \in E} p_v / \deg(u) - p_u$$

ForceDirected($G = (V, E)$, $p = (p_v)_{v \in V}$, $\varepsilon > 0$, $K \in \mathbb{N}$)

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Tutte's Forces

Goal.

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$$= \sum_{uv\in E} (p_v - p_u) / \deg(u)$$

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$$p_u = \text{barycenter}\left(\bigcup_{uv \in E} v\right) = \sum_{uv \in E} p_v / \deg(u)$$

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barycenter$(x_1, \ldots, x_k) = \sum_{i=1}^{k} x_i / k$

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Goal.

$$p_u = \text{barycenter}(\textstyle\bigcup_{uv \in E} v)$$
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$$\boxed{F_u(t)} = \textstyle\sum_{uv \in E} p_v / \deg(u) - p_u$$
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$$\text{barycenter}(x_1, \ldots, x_k) = \textstyle\sum_{i=1}^{k} x_i / k$$

- **Repulsive forces**

$$\boxed{f_{\text{rep}}(u, v) = 0}$$

Tutte's Forces

Goal.

$$p_u = \text{barycenter}(\bigcup_{uv \in E} v) = \sum_{uv \in E} p_v / \deg(u)$$

$$F_u(t) = \sum_{uv \in E} p_v / \deg(u) - p_u = \sum_{uv \in E} (p_v - p_u) / \deg(u) = \sum_{uv \in E} ||p_u - p_v|| / \deg(u)$$

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$$\text{barycenter}(x_1, \ldots, x_k) = \sum_{i=1}^{k} x_i / k$$

- **Repulsive forces**

$$f_{\text{rep}}(u, v) = 0$$

- **Attractive forces**

$$f_{\text{attr}}(u, v) = \frac{1}{\deg(u)} \cdot ||p_u - p_v||$$

Tutte's Forces

Goal.

$$p_u = \text{barycenter}(\bigcup_{uv \in E} v)$$
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ForceDirected($G = (V, E)$, $p = (p_v)_{v \in V}$, $\varepsilon > 0$, $K \in \mathbb{N}$)

```
t ← 1
while t < K and max_{v∈V} ||F_v(t)|| > ε do
    foreach u ∈ V do
        F_u(t) ← Σ_{v∈V} f_rep(u,v) + Σ_{uv∈E} f_attr(u,v)
    foreach u ∈ V do
        p_u ← p_u + δ(t) 1 · F_u(t)
    t ← t + 1
return p
```

$\text{barycenter}(x_1, \ldots, x_k) = \sum_{i=1}^{k} x_i / k$

- **Repulsive forces**

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$$f_{\text{attr}}(u, v) = \frac{1}{\deg(u)} \cdot ||p_u - p_v||$$

Solution: $p_u = (0, 0)\ \forall u \in V$

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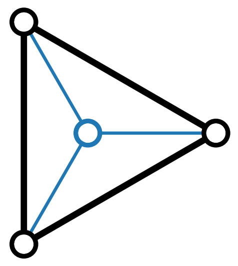

Tutte's Forces

Goal.

$$p_u = \text{barycenter}(\bigcup_{uv \in E} v) = \sum_{uv \in E} p_v / \deg(u)$$

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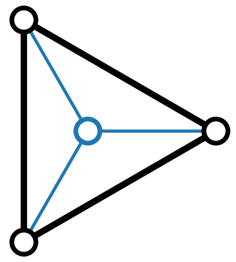

Fix coordinates
of outer face!

Tutte's Forces

Goal.

$$p_u = \text{barycenter}(\bigcup_{uv \in E} v)$$
$$= \sum_{uv \in E} p_v / \deg(u)$$

$$F_u(t) = \sum_{uv \in E} p_v / \deg(u) - p_u$$
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- **Repulsive forces**

$$f_{\text{rep}}(u, v) = 0$$

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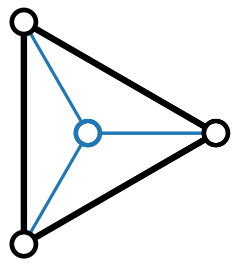
$$f_{\text{attr}}(u, v) = \begin{cases} 0 & u \text{ fixed} \\ \frac{1}{\deg(u)} \cdot ||p_u - p_v|| & \text{else} \end{cases}$$

Solution: $p_u = (0, 0)\ \forall u \in V$



Fix coordinates of outer face!

Tutte's Forces

Goal.

$$p_u = \text{barycenter}(\bigcup_{uv \in E} v)$$
$$= \sum_{uv \in E} p_v / \deg(u)$$

$$F_u(t) = \sum_{uv \in E} p_v / \deg(u) - p_u$$
$$= \sum_{uv \in E} (p_v - p_u) / \deg(u)$$
$$= \sum_{uv \in E} ||p_u - p_v|| / \deg(u)$$

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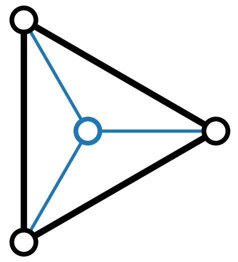

Fix coordinates of outer face!

Linear System of Equations

Goal.

$p_u = \text{barycenter}(\bigcup_{uv \in E} v) = \sum_{uv \in E} p_v / \deg(u)$

Linear System of Equations

Goal. $p_u = (x_u, y_u)$
$p_u = \text{barycenter}(\bigcup_{uv \in E} v) = \sum_{uv \in E} p_v / \deg(u)$

Linear System of Equations

Goal. $p_u = (x_u, y_u)$

$p_u = \text{barycenter}(\bigcup_{uv \in E} v) = \sum_{uv \in E} p_v / \deg(u)$

$x_u = \sum_{uv \in E} x_v / \deg(u)$

$y_u = \sum_{uv \in E} y_v / \deg(u)$

Linear System of Equations

Goal. $p_u = (x_u, y_u)$

$p_u = \text{barycenter}(\bigcup_{uv \in E} v) = \sum_{uv \in E} p_v / \deg(u)$

$x_u = \sum_{uv \in E} x_v / \deg(u) \quad \Leftrightarrow \deg(u) \cdot x_u = \sum_{uv \in E} x_v$

$y_u = \sum_{uv \in E} y_v / \deg(u) \quad \Leftrightarrow \deg(u) \cdot y_u = \sum_{uv \in E} y_v$

Linear System of Equations

Goal. $p_u = (x_u, y_u)$

$p_u = \text{barycenter}(\bigcup_{uv \in E} v) = \sum_{uv \in E} p_v / \deg(u)$

$$x_u = \sum_{uv \in E} x_v / \deg(u) \quad \Leftrightarrow \deg(u) \cdot x_u = \sum_{uv \in E} x_v \qquad \Leftrightarrow \deg(u) \cdot x_u - \sum_{uv \in E} x_v = 0$$

$$y_u = \sum_{uv \in E} y_v / \deg(u) \quad \Leftrightarrow \deg(u) \cdot y_u = \sum_{uv \in E} y_v \qquad \Leftrightarrow \deg(u) \cdot y_u - \sum_{uv \in E} y_v = 0$$

Linear System of Equations

Goal. $p_u = (x_u, y_u)$

$p_u = \text{barycenter}(\bigcup_{uv \in E} v) = \sum_{uv \in E} p_v / \deg(u)$

2 Systems of linear equations

$$x_u = \sum_{uv \in E} x_v / \deg(u) \quad \Leftrightarrow \deg(u) \cdot x_u = \sum_{uv \in E} x_v \quad \Leftrightarrow \deg(u) \cdot x_u - \sum_{uv \in E} x_v = 0$$

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Linear System of Equations

Goal. $p_u = (x_u, y_u)$

$p_u = \text{barycenter}(\bigcup_{uv \in E} v) = \sum_{uv \in E} p_v / \deg(u)$

$Ax = b$

2 Systems of linear equations

$x_u = \sum_{uv \in E} x_v / \deg(u) \quad \Leftrightarrow \deg(u) \cdot x_u = \sum_{uv \in E} x_v \quad \Leftrightarrow \deg(u) \cdot x_u - \sum_{uv \in E} x_v = 0$

$y_u = \sum_{uv \in E} y_v / \deg(u) \quad \Leftrightarrow \deg(u) \cdot y_u = \sum_{uv \in E} y_v \quad \Leftrightarrow \deg(u) \cdot y_u - \sum_{uv \in E} y_v = 0$

Linear System of Equations

Goal. $p_u = (x_u, y_u)$

$p_u = \text{barycenter}(\bigcup_{uv \in E} v) = \sum_{uv \in E} p_v / \deg(u)$

$Ax = b \quad Ay = b$

2 Systems of linear equations

$$x_u = \sum_{uv \in E} x_v / \deg(u) \quad \Leftrightarrow \deg(u) \cdot x_u = \sum_{uv \in E} x_v \quad \Leftrightarrow \deg(u) \cdot x_u - \sum_{uv \in E} x_v = 0$$

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Linear System of Equations

Goal. $p_u = (x_u, y_u)$

$p_u = \text{barycenter}(\bigcup_{uv \in E} v) = \sum_{uv \in E} p_v / \deg(u)$

$Ax = b \quad Ay = b \quad b = (0)_n$

2 Systems of linear equations

$x_u = \sum_{uv \in E} x_v / \deg(u) \;\Leftrightarrow\; \deg(u) \cdot x_u = \sum_{uv \in E} x_v \;\Leftrightarrow\; \boxed{\deg(u) \cdot x_u - \sum_{uv \in E} x_v = 0}$

$y_u = \sum_{uv \in E} y_v / \deg(u) \;\Leftrightarrow\; \deg(u) \cdot y_u = \sum_{uv \in E} y_v \;\Leftrightarrow\; \boxed{\deg(u) \cdot y_u - \sum_{uv \in E} y_v = 0}$

Linear System of Equations

Goal. $p_u = (x_u, y_u)$

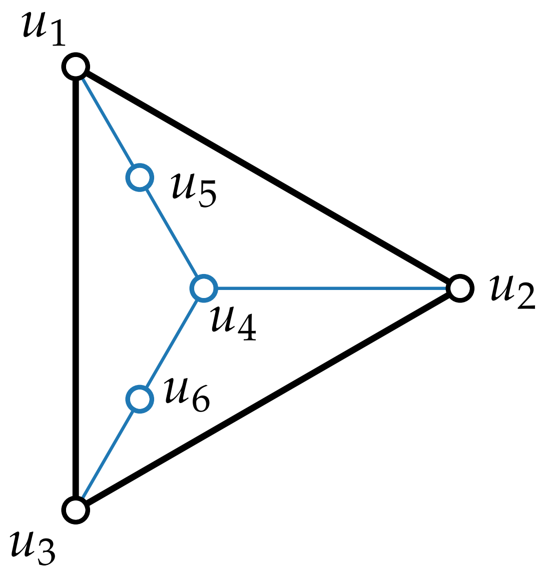
$p_u = \text{barycenter}(\bigcup_{uv \in E} v) = \sum_{uv \in E} p_v / \deg(u)$

$Ax = b \quad Ay = b \quad b = (0)_n$

2 Systems of linear equations

$x_u = \sum_{uv \in E} x_v / \deg(u) \quad \Leftrightarrow \deg(u) \cdot x_u = \sum_{uv \in E} x_v \quad \Leftrightarrow \deg(u) \cdot x_u - \sum_{uv \in E} x_v = 0$

$y_u = \sum_{uv \in E} y_v / \deg(u) \quad \Leftrightarrow \deg(u) \cdot y_u = \sum_{uv \in E} y_v \quad \Leftrightarrow \deg(u) \cdot y_u - \sum_{uv \in E} y_v = 0$



Linear System of Equations

Goal. $p_u = (x_u, y_u)$

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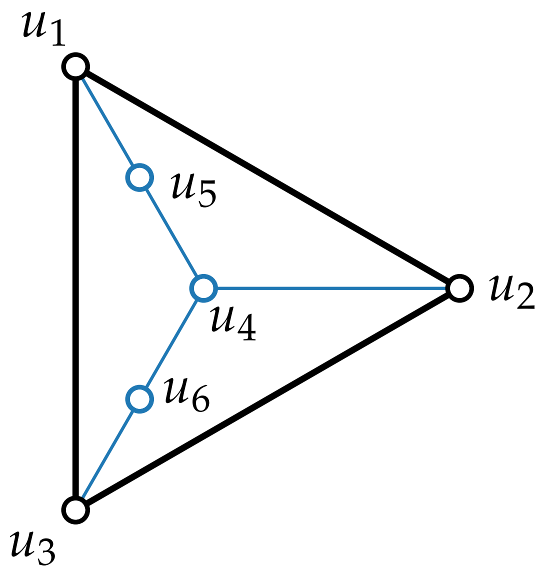
A



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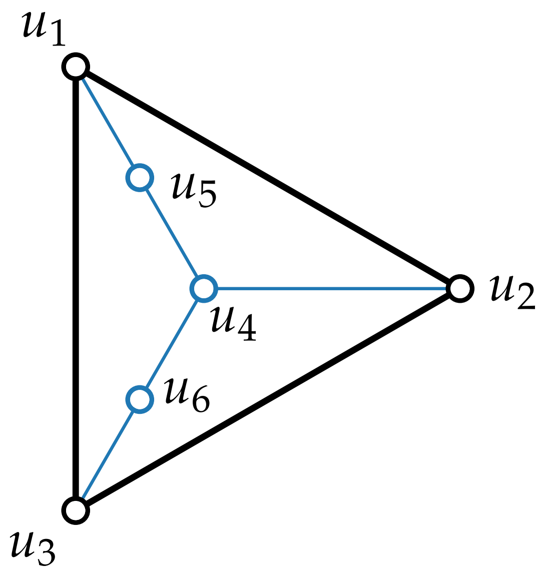

A

$$\begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} \begin{pmatrix} \\ \\ \\ \\ \\ \\ \end{pmatrix}$$

Linear System of Equations

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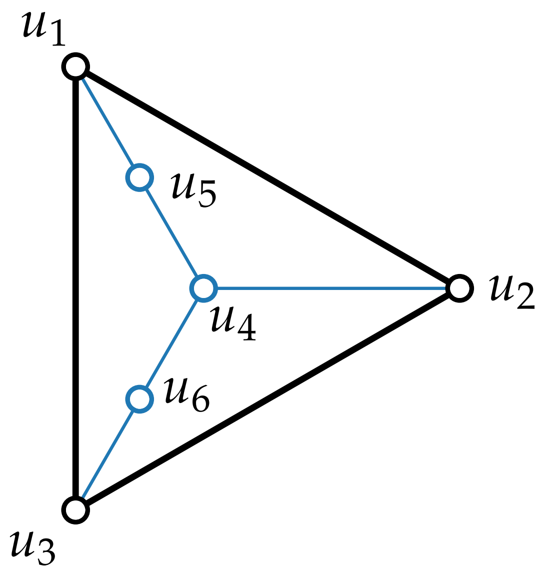

$$A \quad \begin{array}{c|cccccc} & u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \\ \hline u_1 & & & & & & \\ u_2 & & & & & & \\ u_3 & & & & & & \\ u_4 & & & & & & \\ u_5 & & & & & & \\ u_6 & & & & & & \end{array}$$

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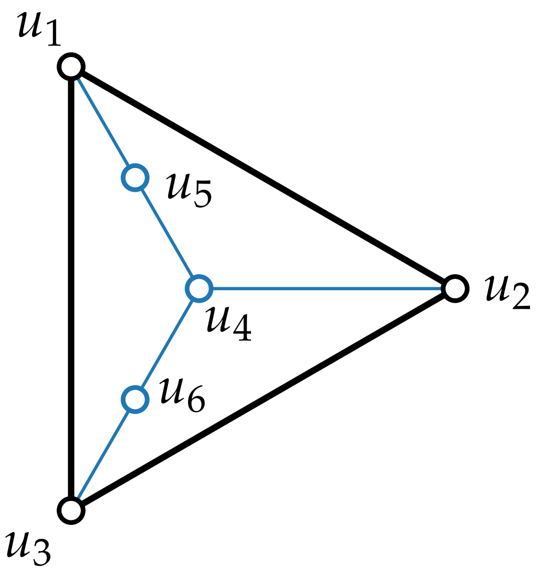

$$\begin{array}{c c} & \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\ \begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} & \begin{pmatrix} 3 & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{pmatrix} \end{array} \; A$$

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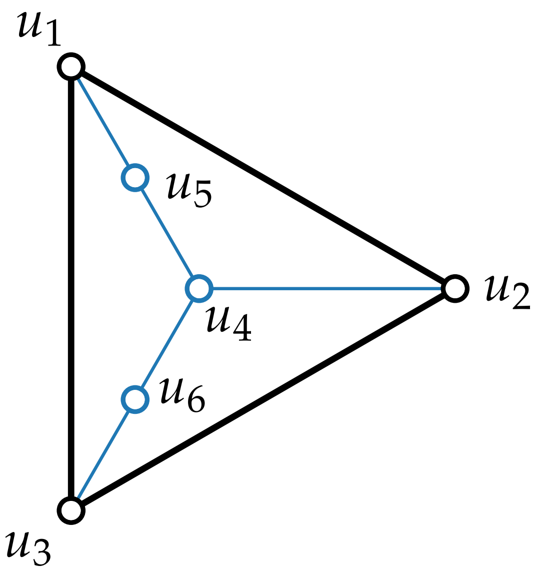

$$A: \quad \begin{array}{cc} & \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\ \begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} & \begin{pmatrix} 3 & -1 & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{pmatrix} \end{array}$$

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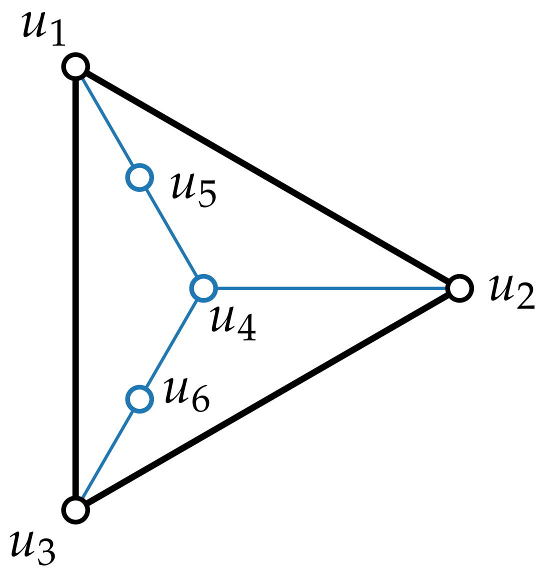

$$A: \quad \begin{array}{c|cccccc} & u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \\ \hline u_1 & 3 & -1 & -1 & & & \\ u_2 & & & & & & \\ u_3 & & & & & & \\ u_4 & & & & & & \\ u_5 & & & & & & \\ u_6 & & & & & & \end{array}$$

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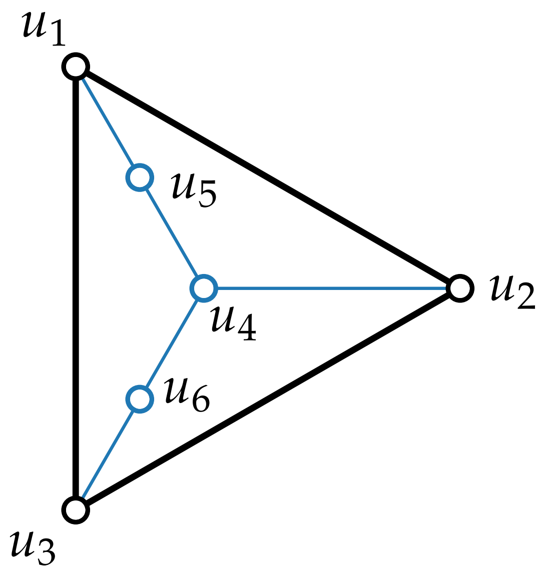

$$A \quad \begin{array}{c c} & \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\ \begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} & \begin{pmatrix} 3 & -1 & -1 & 0 & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{pmatrix} \end{array}$$

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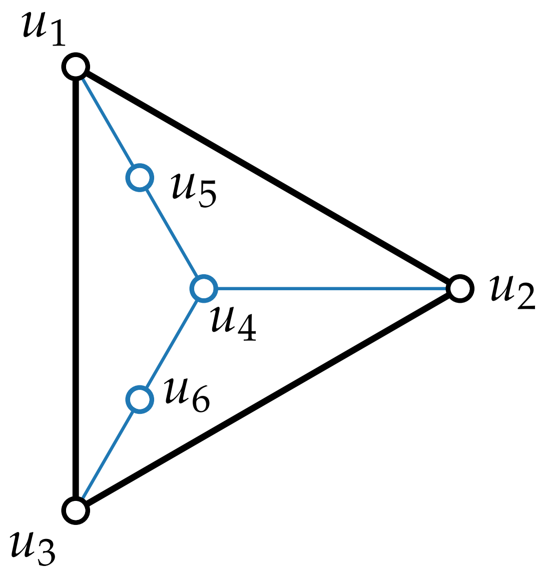

$$
\begin{array}{c|cccccc}
 & u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \\
\hline
u_1 & 3 & -1 & -1 & 0 & -1 & \\
u_2 & & & & & & \\
u_3 & & & & & & \\
u_4 & & & & & & \\
u_5 & & & & & & \\
u_6 & & & & & &
\end{array}
\quad A
$$

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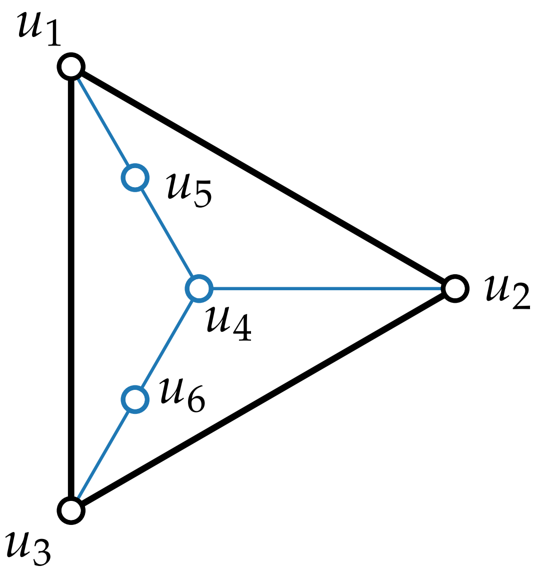

$$\begin{array}{c c} & \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \quad A \\ \begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} & \begin{pmatrix} 3 & -1 & -1 & 0 & -1 & 0 \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{pmatrix} \end{array}$$

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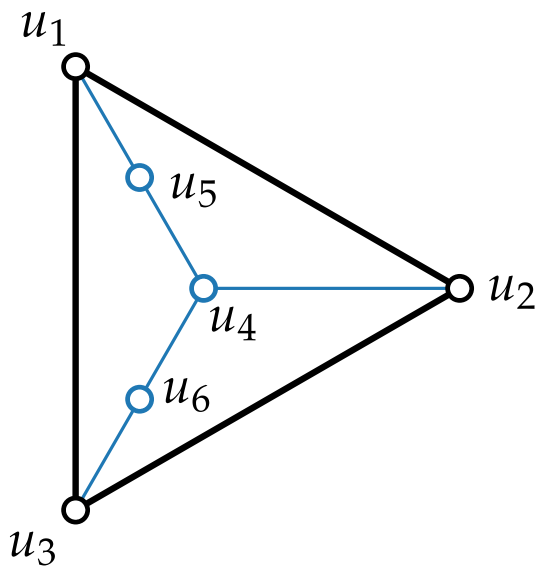

$$
\begin{array}{c c}
 & \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \quad A \\
\begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} &
\begin{pmatrix}
3 & -1 & -1 & 0 & -1 & 0 \\
 & 3 & & & & \\
 & & & & & \\
 & & & & & \\
 & & & & & \\
 & & & & &
\end{pmatrix}
\end{array}
$$

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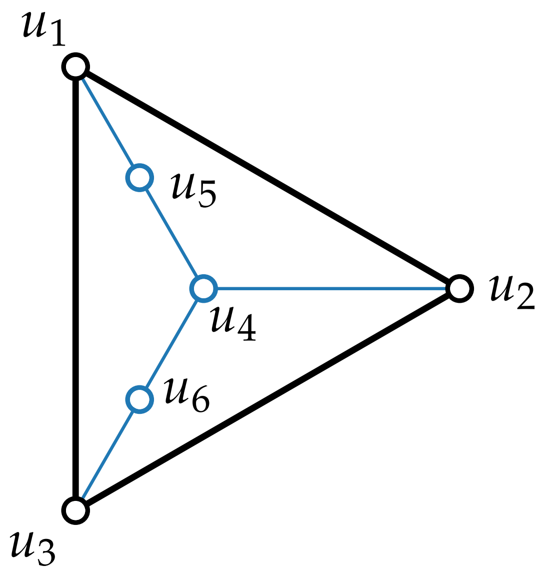

$$A: \quad \begin{array}{c|c} & \begin{matrix} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{matrix} \\ \begin{matrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{matrix} & \begin{pmatrix} 3 & -1 & -1 & 0 & -1 & 0 \\ -1 & 3 & -1 & -1 & 0 & 0 \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{pmatrix} \end{array}$$

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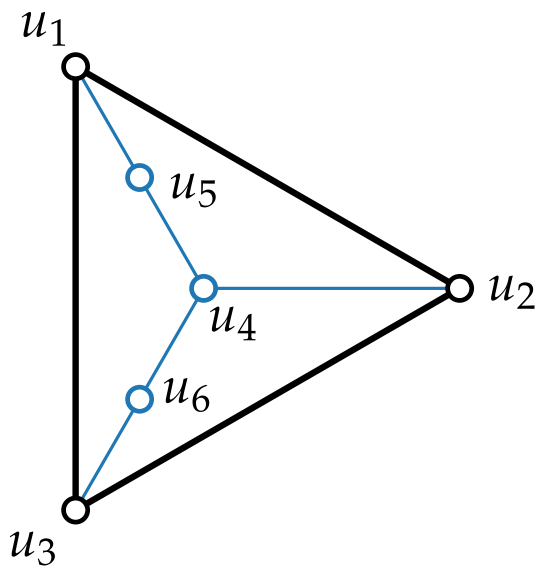

$$A \quad \begin{array}{c c} & \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\ \begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} & \begin{pmatrix} 3 & -1 & -1 & 0 & -1 & 0 \\ -1 & 3 & -1 & -1 & 0 & 0 \\ -1 & -1 & 3 & 0 & 0 & -1 \\ & & & & & \\ & & & & & \\ & & & & & \end{pmatrix} \end{array}$$

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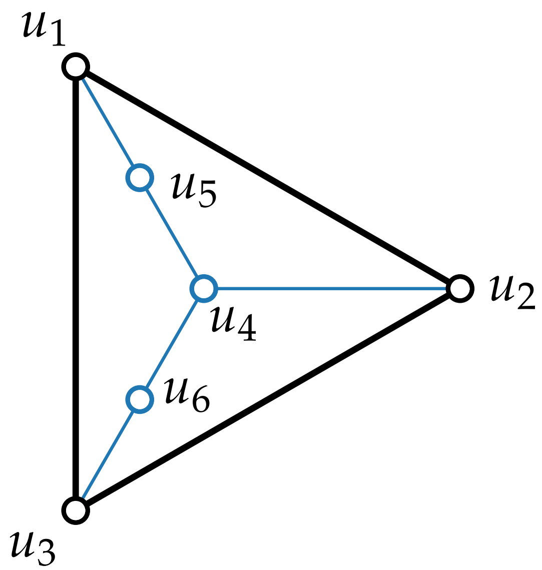

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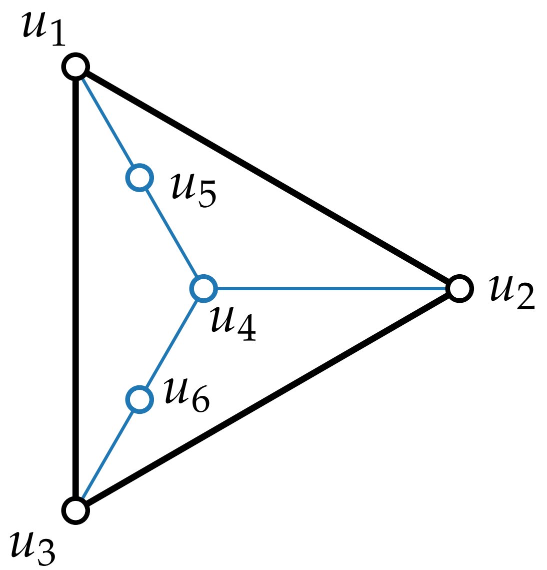

$$\begin{array}{c|c} & \begin{matrix} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{matrix} \quad A \\ \begin{matrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{matrix} & \begin{pmatrix} 3 & -1 & -1 & 0 & -1 & 0 \\ -1 & 3 & -1 & -1 & 0 & 0 \\ -1 & -1 & 3 & 0 & 0 & -1 \\ 0 & -1 & 0 & 3 & -1 & -1 \\ -1 & 0 & 0 & -1 & 2 & 0 \\ & & & & & \end{pmatrix} \end{array}$$

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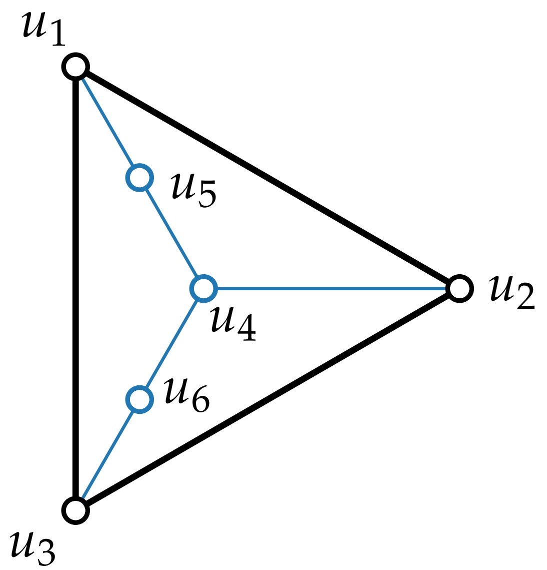

$$A = \begin{array}{c c} & \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\ \begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} & \begin{pmatrix} 3 & -1 & -1 & 0 & -1 & 0 \\ -1 & 3 & -1 & -1 & 0 & 0 \\ -1 & -1 & 3 & 0 & 0 & -1 \\ 0 & -1 & 0 & 3 & -1 & -1 \\ -1 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & -1 & 0 & 2 \end{pmatrix} \end{array}$$

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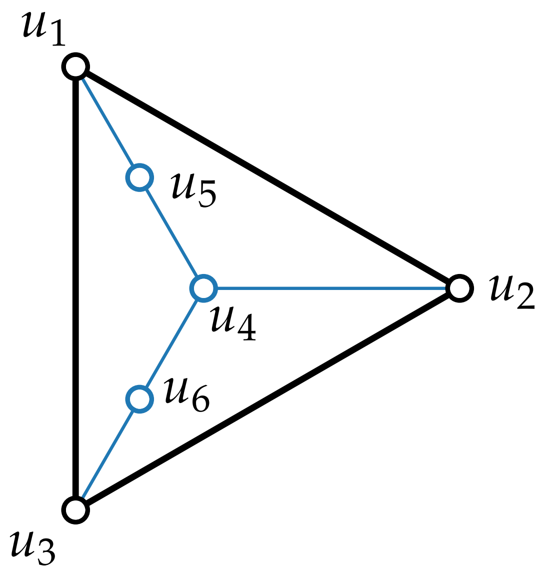

$$A = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} \begin{array}{c} \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\ \begin{pmatrix} 3 & -1 & -1 & 0 & -1 & 0 \\ -1 & 3 & -1 & -1 & 0 & 0 \\ -1 & -1 & 3 & 0 & 0 & -1 \\ 0 & -1 & 0 & 3 & -1 & -1 \\ -1 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & -1 & 0 & 2 \end{pmatrix} \end{array}$$

$A_{ii} = \deg(u_i)$

Linear System of Equations

Goal. $p_u = (x_u, y_u)$

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2 Systems of linear equations

$$x_u = \sum_{uv \in E} x_v / \deg(u) \quad \Leftrightarrow \deg(u) \cdot x_u = \sum_{uv \in E} x_v \quad \Leftrightarrow \boxed{\deg(u) \cdot x_u - \sum_{uv \in E} x_v = 0}$$

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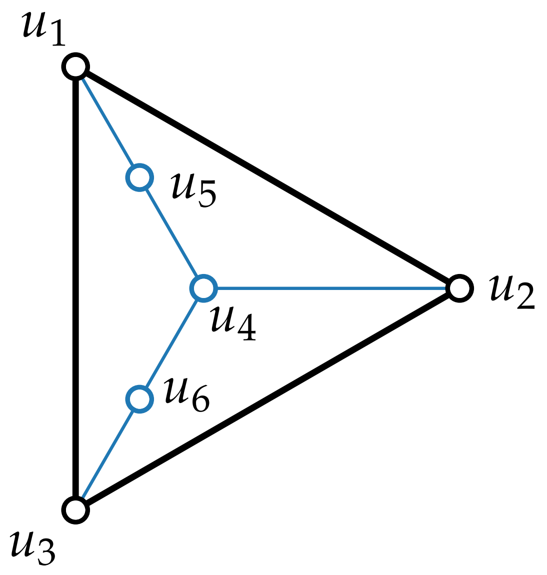

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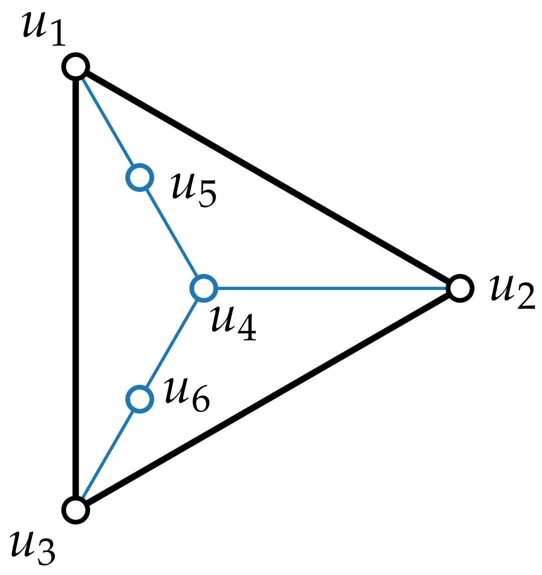

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Laplacian matrix of G

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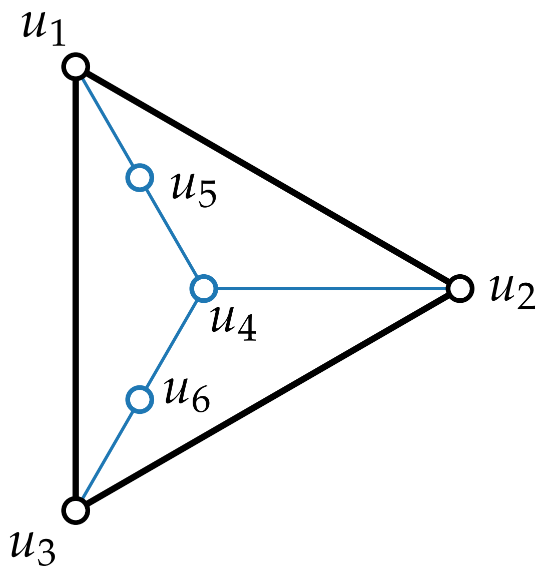

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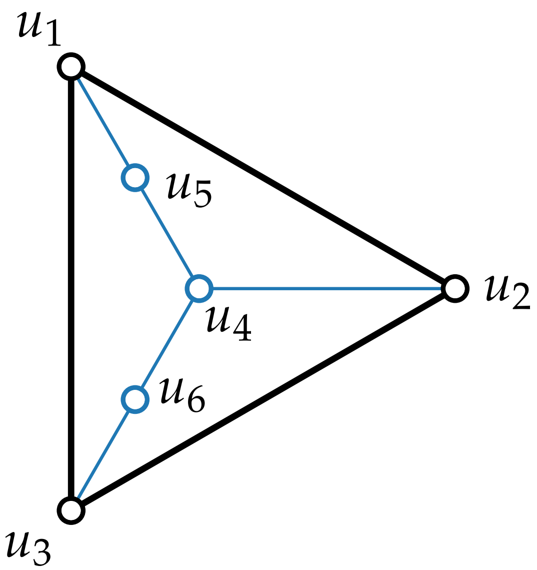

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3 & -1 & -1 & 0 & -1 & 0 \\
-1 & 3 & -1 & -1 & 0 & 0 \\
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variables, constraints, $\det(A) =$

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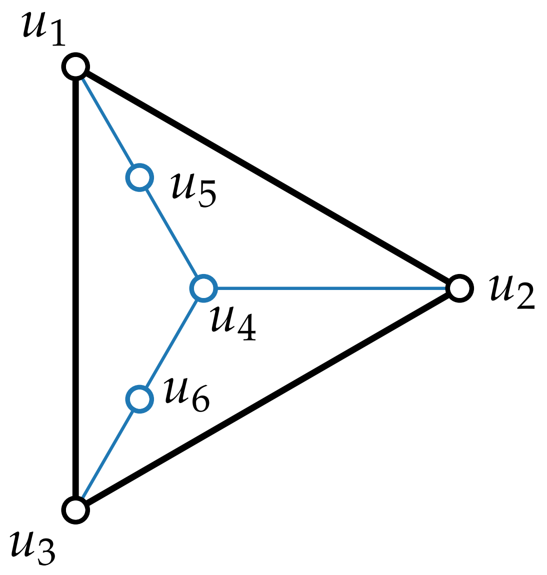

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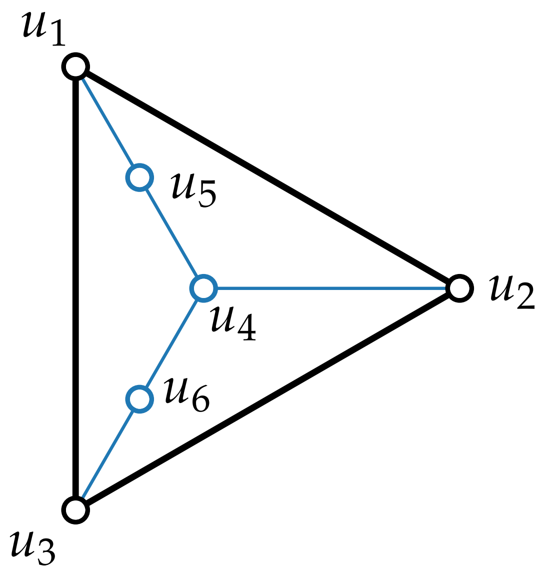

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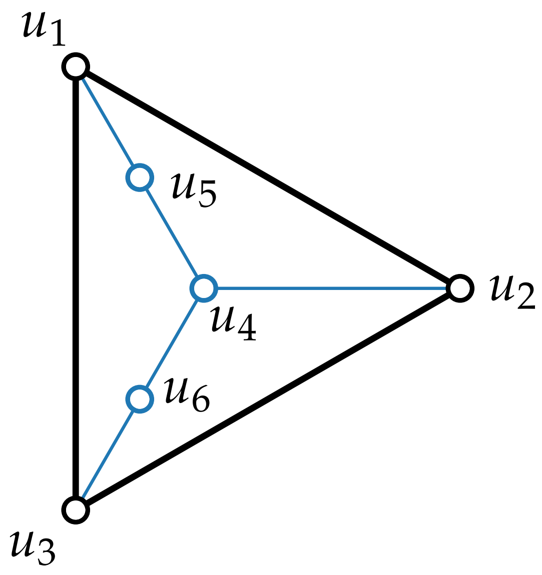

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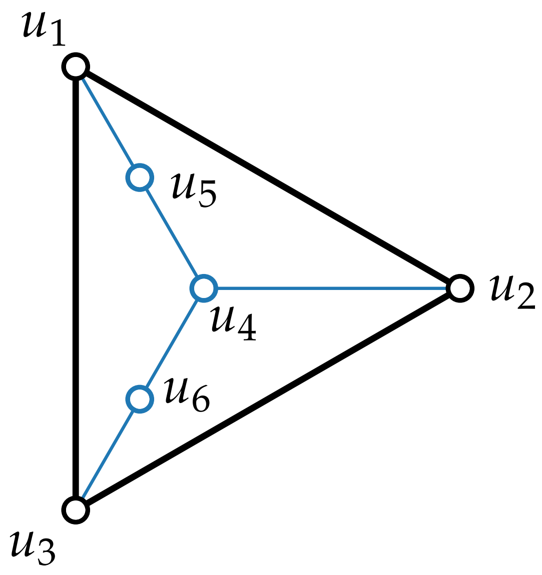

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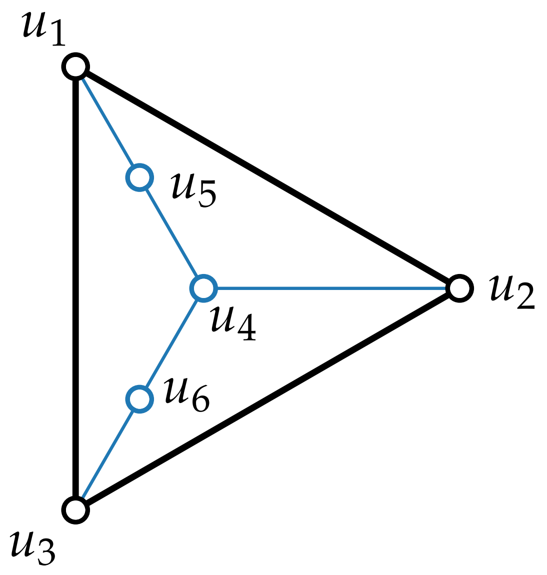

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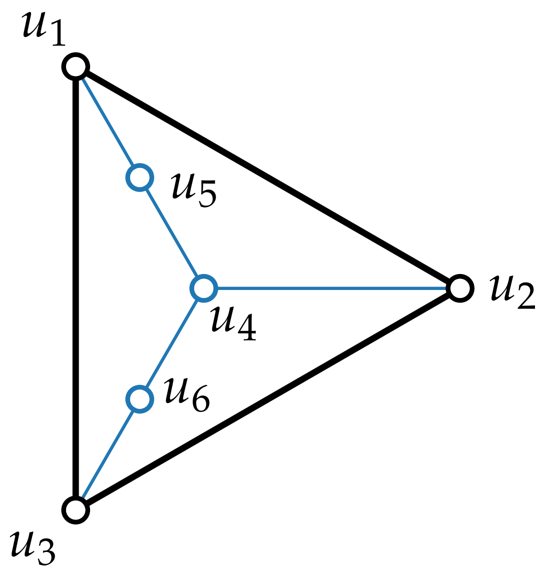

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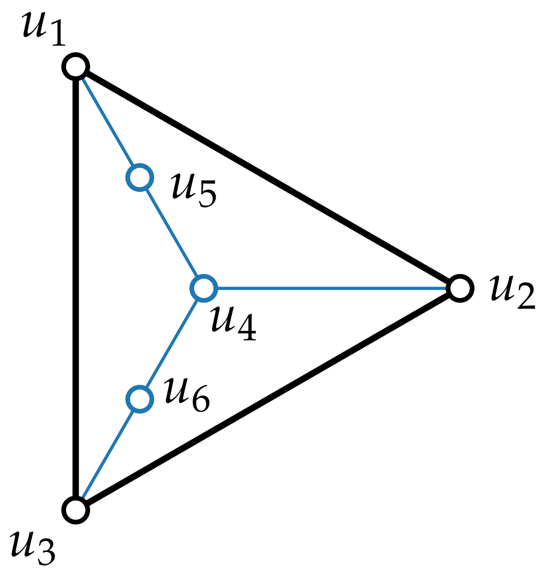

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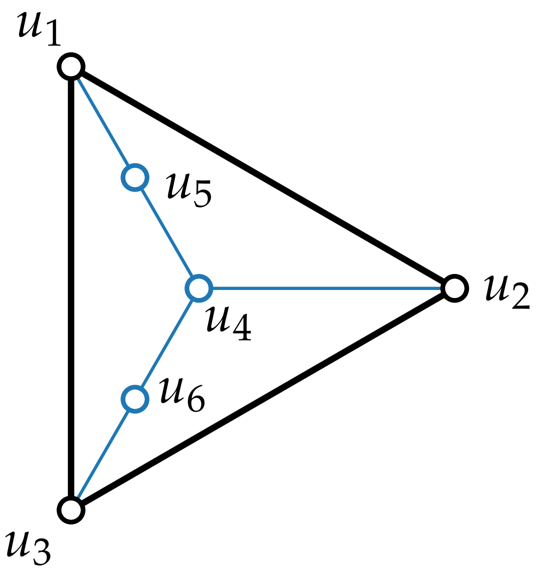

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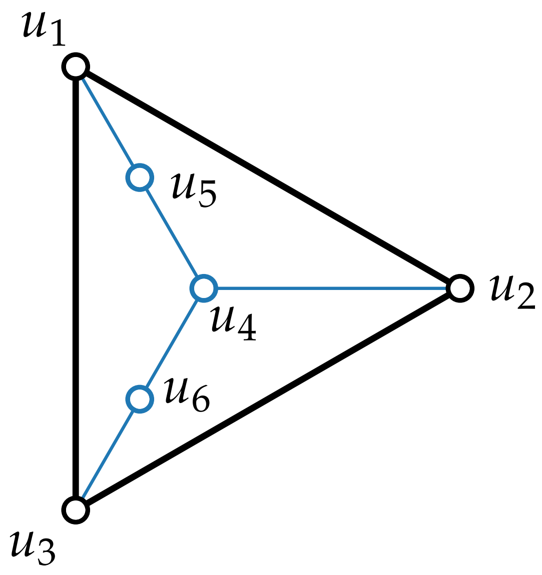


$$A = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} \begin{array}{c} \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\ \begin{pmatrix} 3 & -1 & -1 & 0 & -1 & 0 \\ -1 & 3 & -1 & -1 & 0 & 0 \\ -1 & -1 & 3 & 0 & 0 & -1 \\ 0 & -1 & 0 & 3 & -1 & -1 \\ -1 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & -1 & 0 & 2 \end{pmatrix} \end{array}$$

Laplacian matrix of G

$A_{ii} = \deg(u_i)$

$A_{ij, i \neq j} = \begin{cases} -1 & u_i u_j \in E \\ 0 & u_i u_j \notin E \end{cases}$

k variables, k constraints, $\det(A) > 0$

k = #free vertices

$\Rightarrow$ no unique solution ☹

Linear System of Equations

Goal. $p_u = (x_u, y_u)$

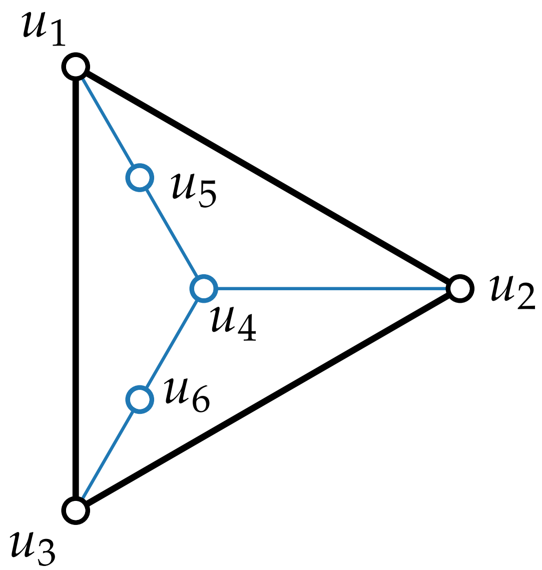
$p_u = \text{barycenter}(\bigcup_{uv \in E} v) = \sum_{uv \in E} p_v / \deg(u)$

$Ax = b \quad Ay = b \quad b = (0)_n$

2 Systems of linear equations

$x_u = \sum_{uv \in E} x_v / \deg(u) \iff \deg(u) \cdot x_u = \sum_{uv \in E} x_v \iff \deg(u) \cdot x_u - \sum_{uv \in E} x_v = 0$

$y_u = \sum_{uv \in E} y_v / \deg(u) \iff \deg(u) \cdot y_u = \sum_{uv \in E} y_v \iff \deg(u) \cdot y_u - \sum_{uv \in E} y_v = 0$



$$A = \begin{array}{c|c} & \begin{matrix} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{matrix} \\ \begin{matrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{matrix} & \begin{pmatrix} 3 & -1 & -1 & 0 & -1 & 0 \\ -1 & 3 & -1 & -1 & 0 & 0 \\ -1 & -1 & 3 & 0 & 0 & -1 \\ 0 & -1 & 0 & 3 & -1 & -1 \\ -1 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & -1 & 0 & 2 \end{pmatrix} \end{array}$$

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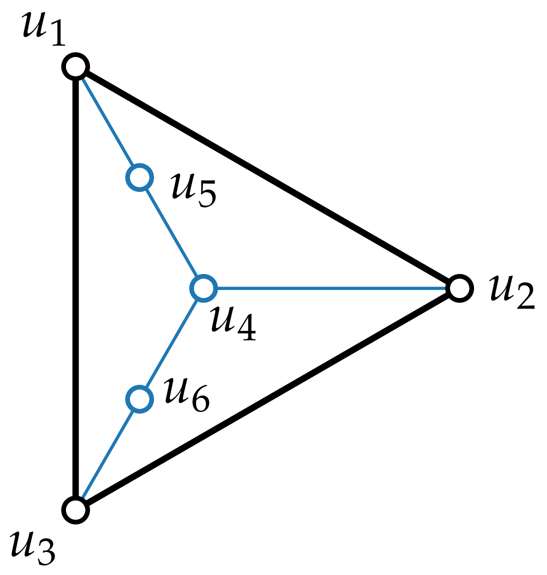
Linear System of Equations

Goal. $p_u = (x_u, y_u)$
$p_u = \text{barycenter}(\bigcup_{uv \in E} v)$

Theorem.
Tutte's barycentric algorithm admits a unique solution.
It can be computed in polynomial time.

$$x_u = \sum_{uv\in E} x_v / \deg(u) \quad \Leftrightarrow \deg(u)\cdot x_u = \sum_{uv\in E} x_v \quad \Leftrightarrow \deg(u)\cdot x_u - \sum_{uv\in E} x_v = 0$$
$$y_u = \sum_{uv\in E} y_v / \deg(u) \quad \Leftrightarrow \deg(u)\cdot y_u = \sum_{uv\in E} y_v \quad \Leftrightarrow \deg(u)\cdot y_u - \sum_{uv\in E} y_v = 0$$



$$\begin{array}{c|c} & \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\ \begin{array}{c} u_1\\u_2\\u_3\\u_4\\u_5\\u_6 \end{array} & \begin{pmatrix} 3 & -1 & -1 & 0 & -1 & 0 \\ -1 & 3 & -1 & -1 & 0 & 0 \\ -1 & -1 & 3 & 0 & 0 & -1 \\ 0 & -1 & 0 & 3 & -1 & -1 \\ -1 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & -1 & 0 & 2 \end{pmatrix} \end{array} \; A$$

Laplacian matrix of G

$$A_{ii} = \deg(u_i)$$
$$A_{ij,i\neq j} = \begin{cases} -1 & u_iu_j \in E \\ 0 & u_iu_j \notin E \end{cases}$$

k variables, k constraints, $\det(A) > 0$

k = #free vertices $\Rightarrow$ unique solution 😀

Linear System of Equations

Theorem. **Tutte drawing**
Tutte's barycentric algorithm admits a unique solution.
It can be computed in polynomial time.

Goal. $p_u = (x_u, y_u)$
$p_u = \text{barycenter}(\bigcup_{uv \in E} v)$

$x_u = \sum_{uv \in E} x_v / \deg(u) \quad \Leftrightarrow \deg(u) \cdot x_u = \sum_{uv \in E} x_v \quad \Leftrightarrow \deg(u) \cdot x_u - \sum_{uv \in E} x_v = 0$

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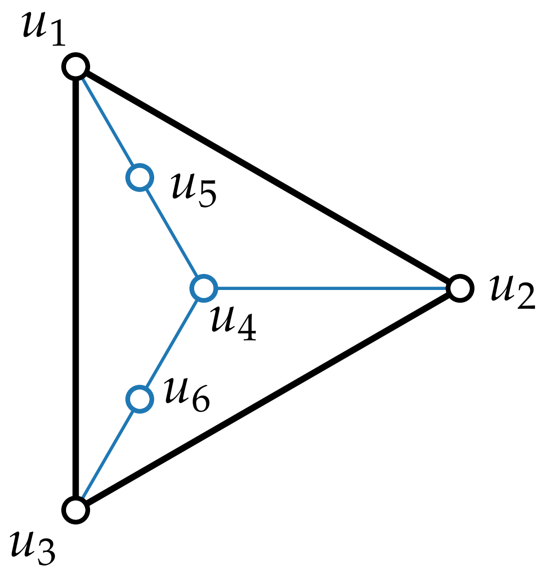

$$A = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} \begin{array}{c} \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\ \begin{pmatrix} 3 & -1 & -1 & 0 & -1 & 0 \\ -1 & 3 & -1 & -1 & 0 & 0 \\ -1 & -1 & 3 & 0 & 0 & -1 \\ 0 & -1 & 0 & 3 & -1 & -1 \\ -1 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & -1 & 0 & 2 \end{pmatrix} \end{array}$$

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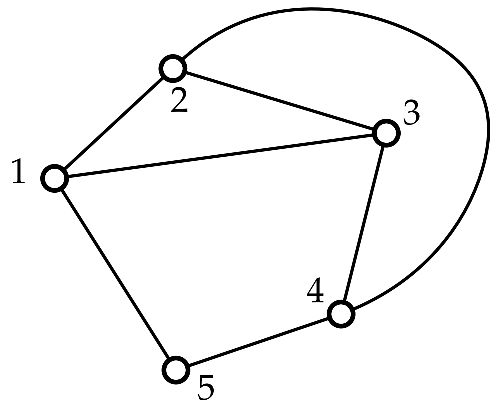
$\Rightarrow$ unique solution 😃

3-Connected Planar Graphs

planar: G can be drawn in such a way that no edges cross each other

connected: There is a u-v-path for every $u, v \in V$



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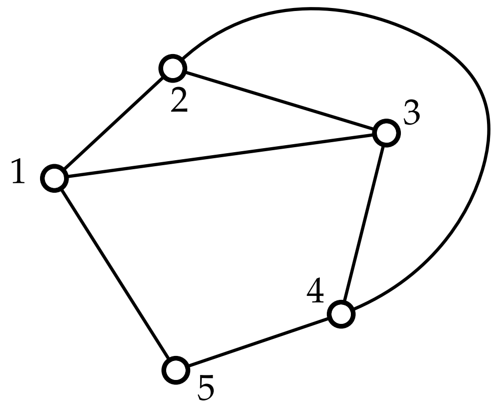

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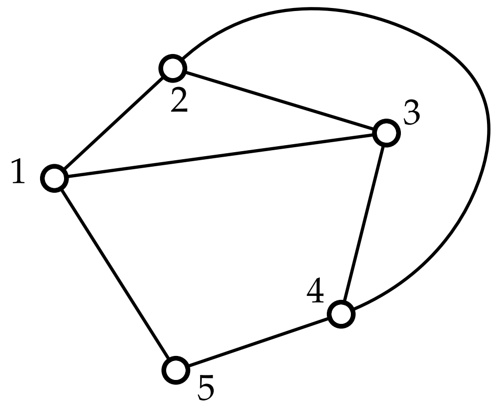

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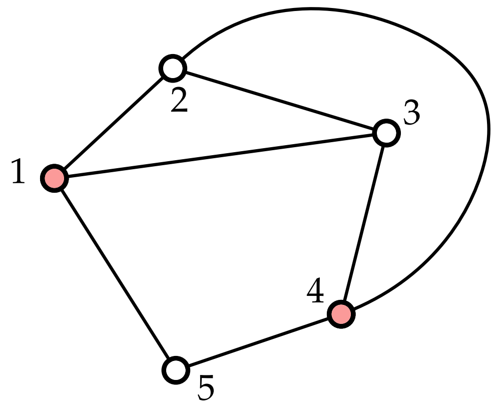

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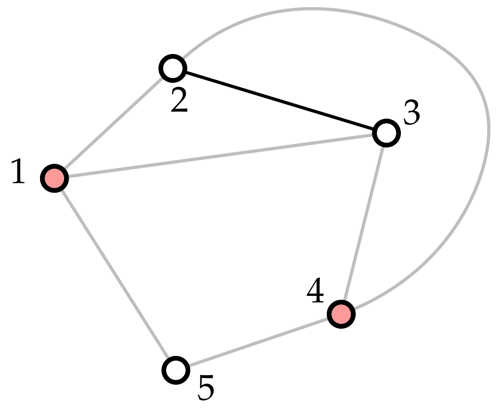

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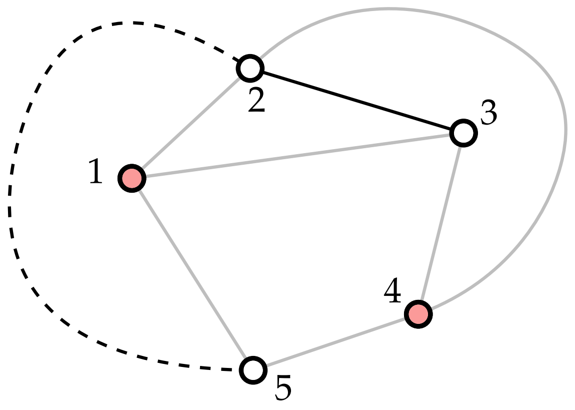

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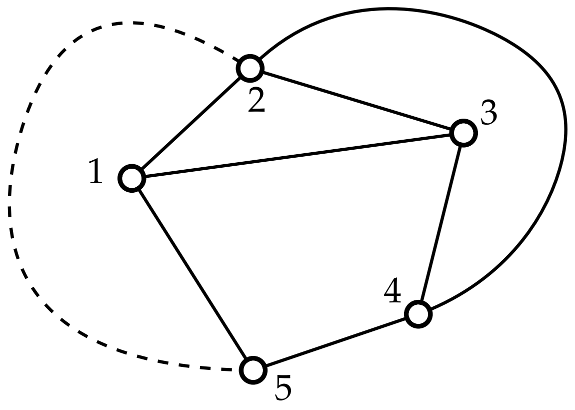

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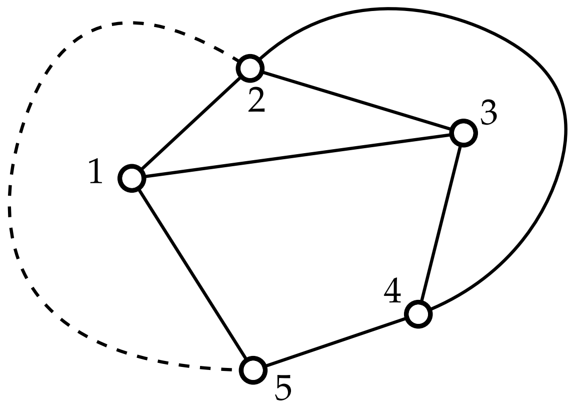

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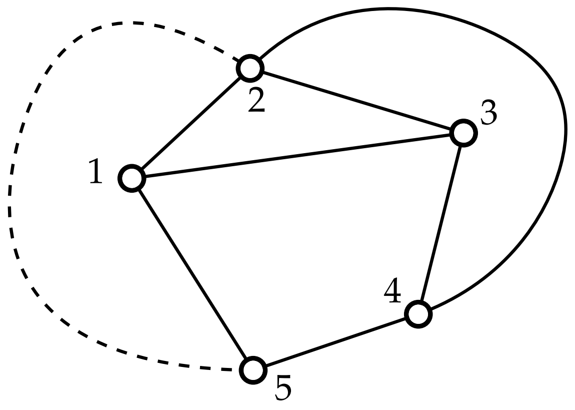

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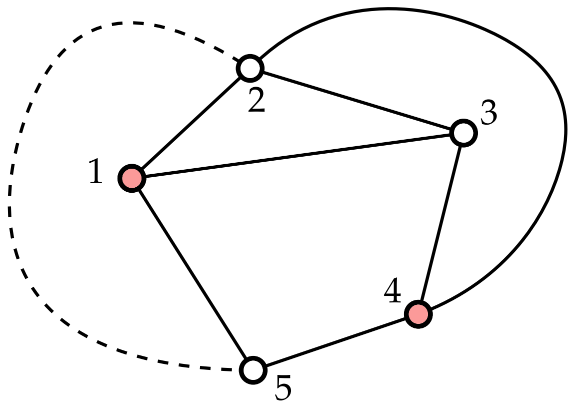

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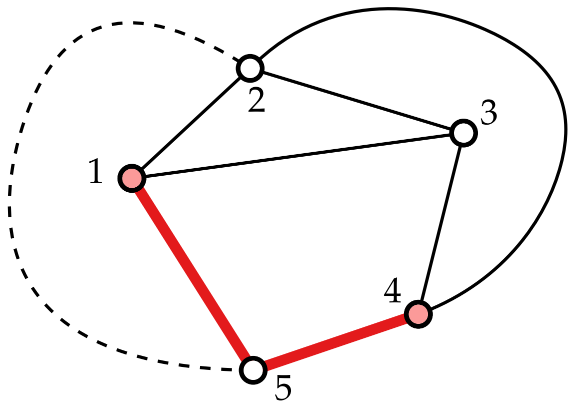

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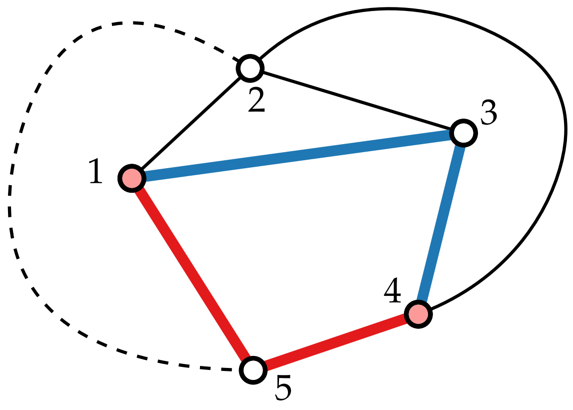

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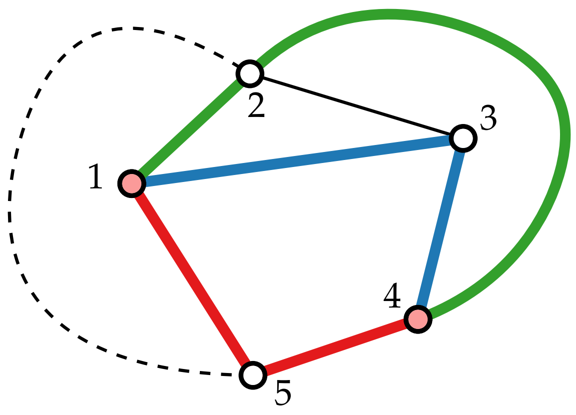

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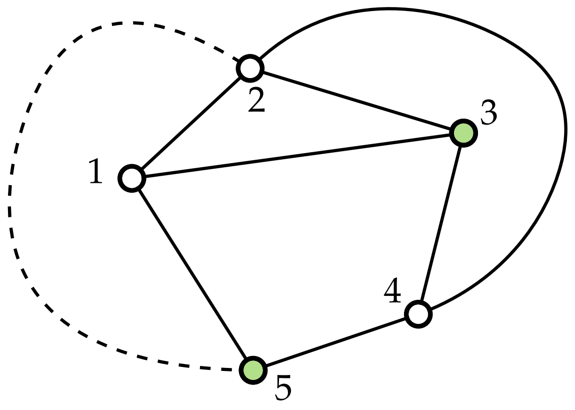

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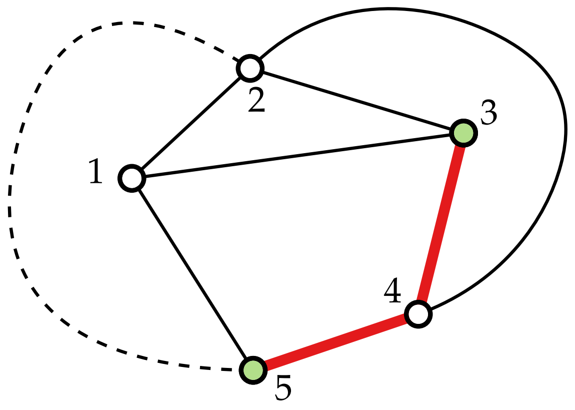

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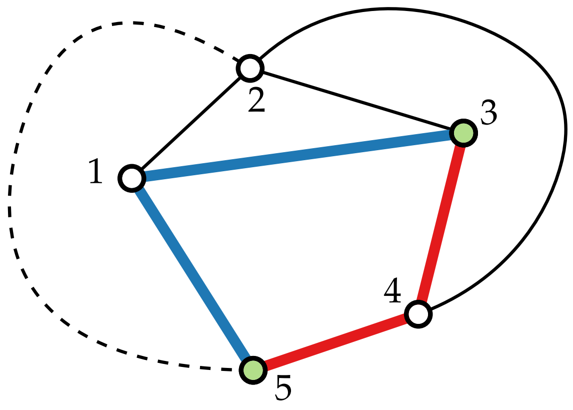

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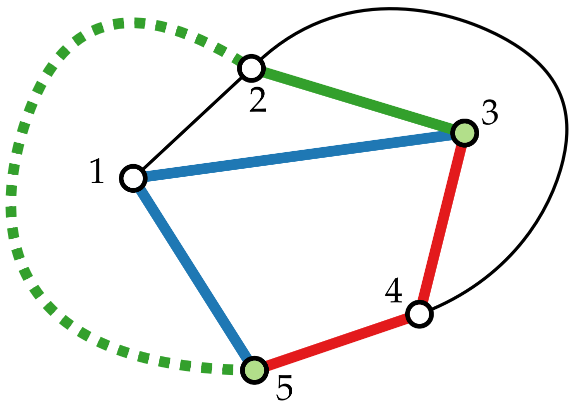

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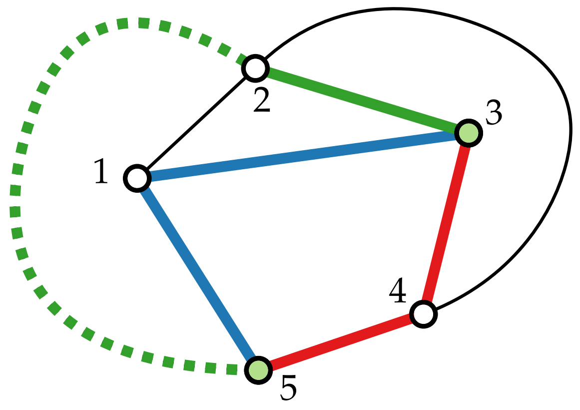
Theorem. [Whitney 1933]
Every 3-connected planar graph has a unique planar embedding.



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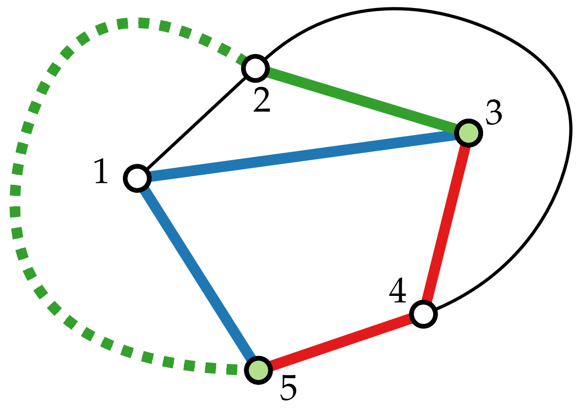

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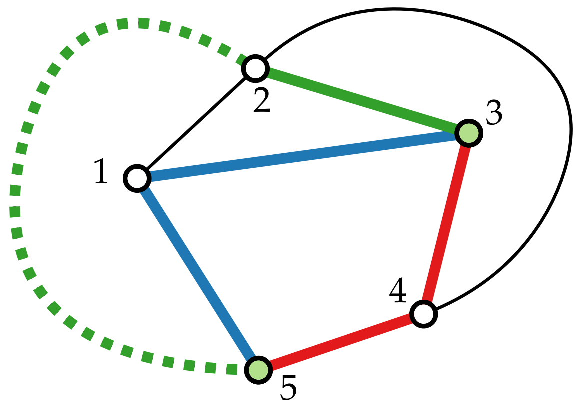
Γ_1, Γ_2 embeddings of G



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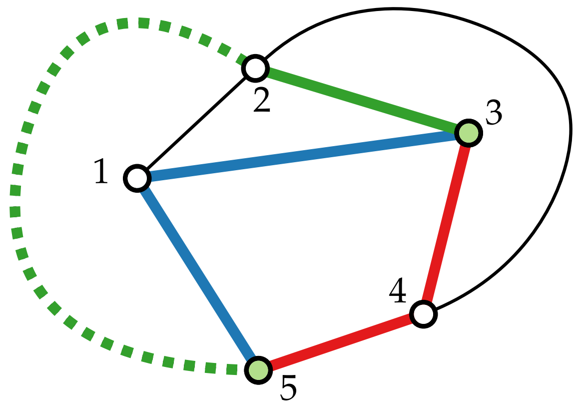
C face of Γ_2, but not Γ_1



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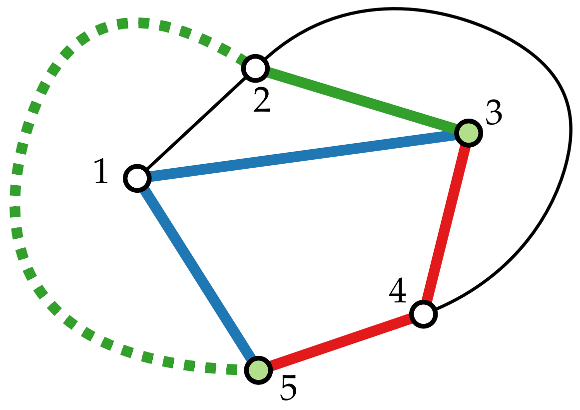

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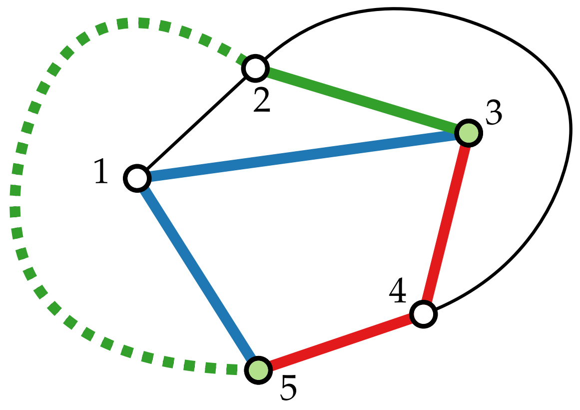

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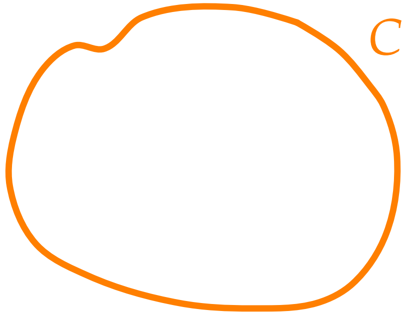

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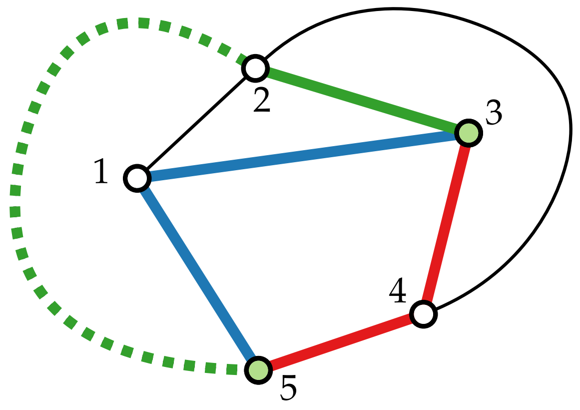

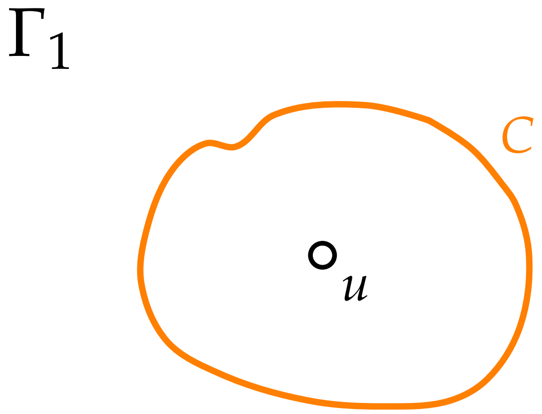

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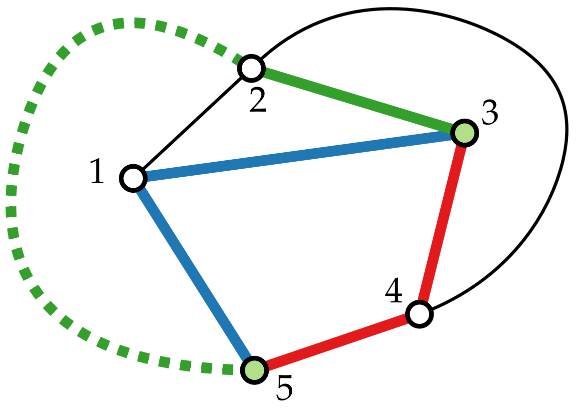

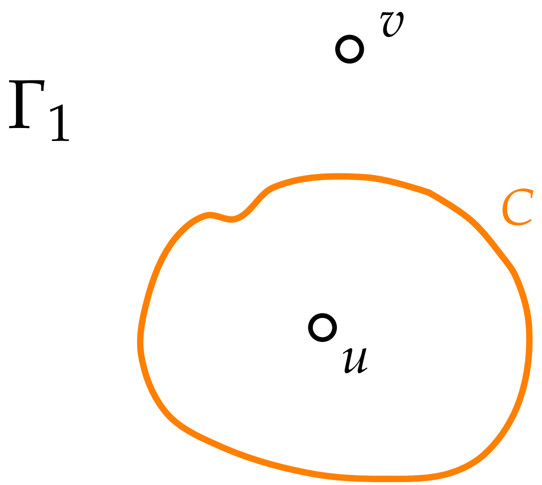

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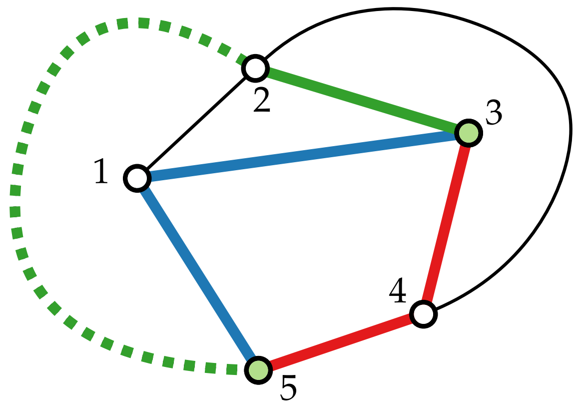

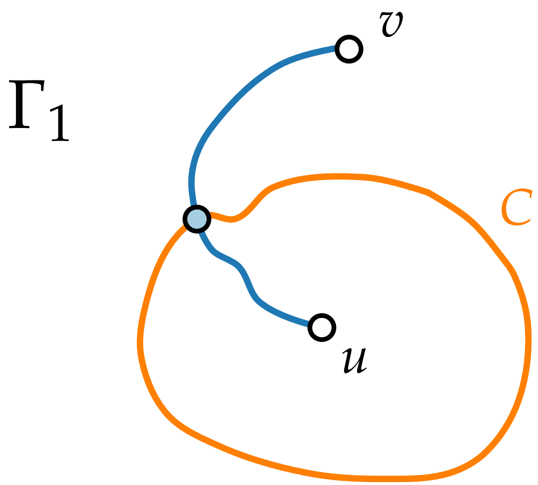

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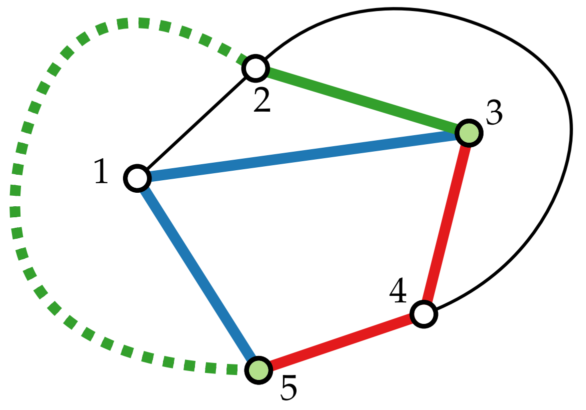

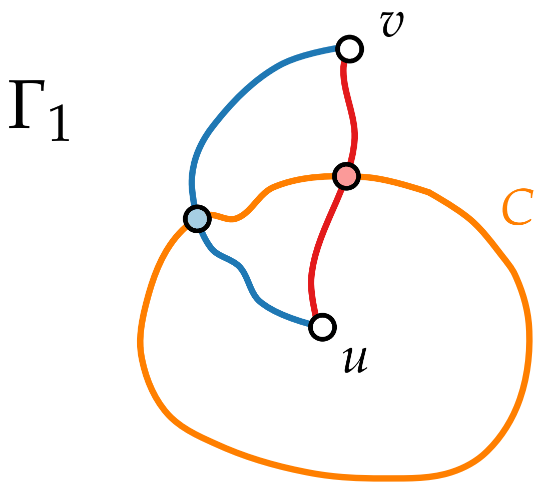

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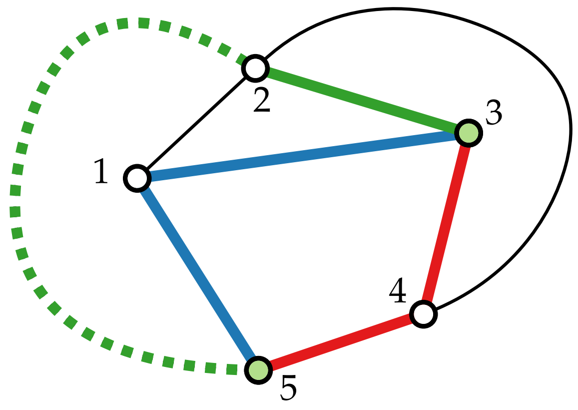

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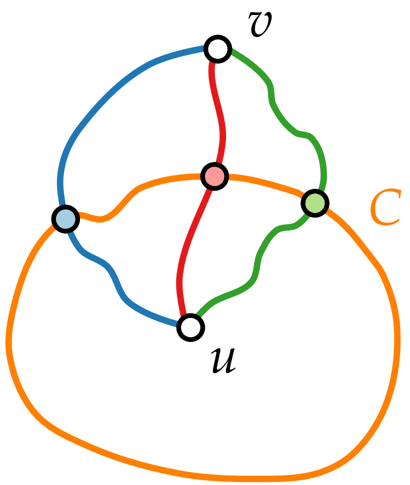

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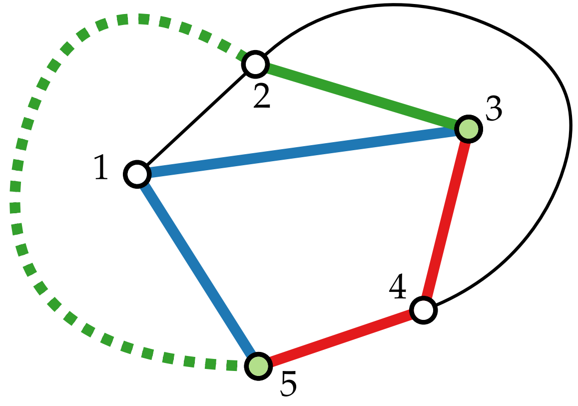

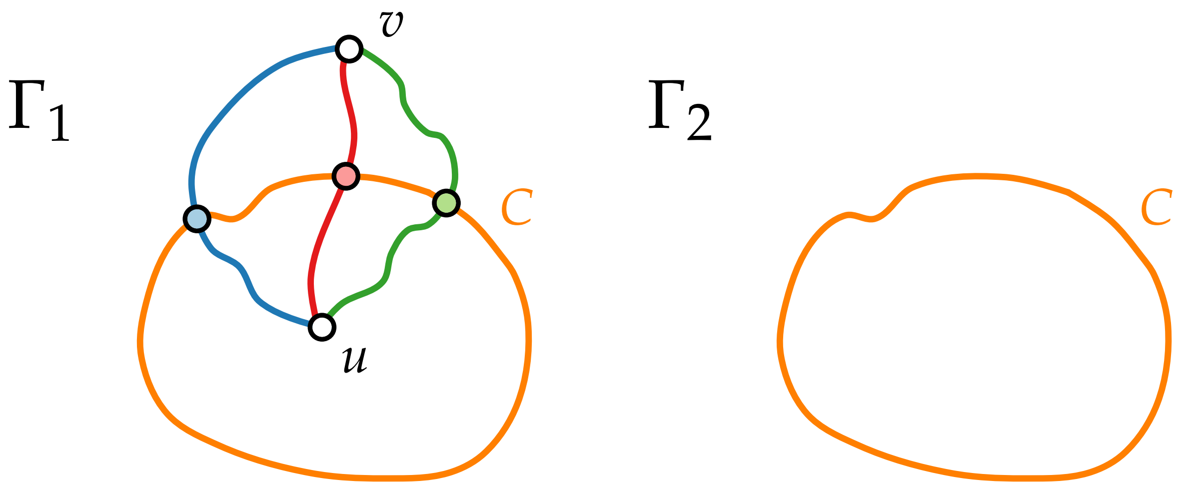

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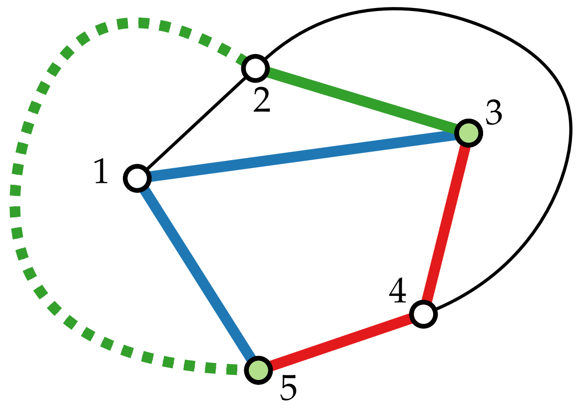

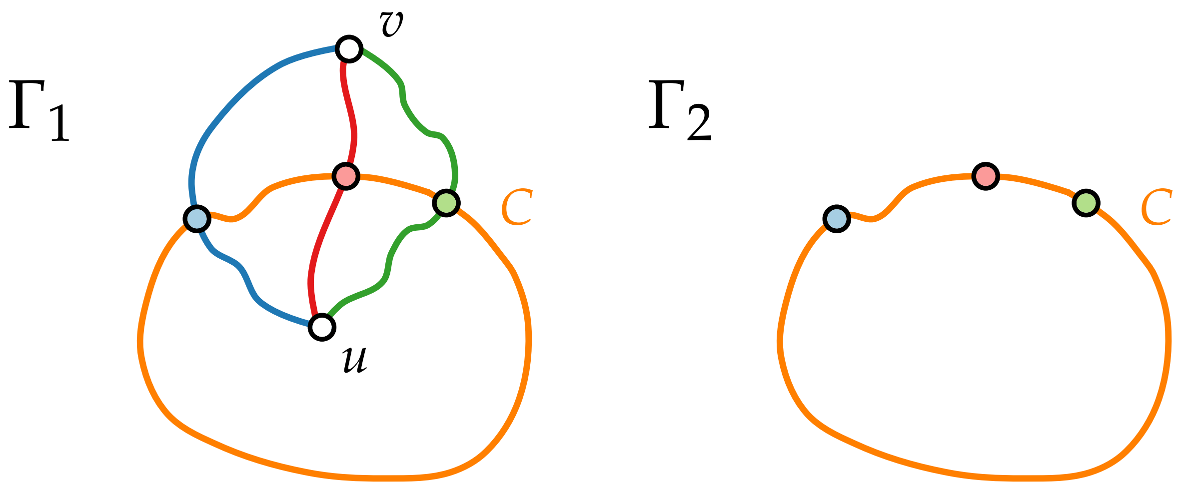

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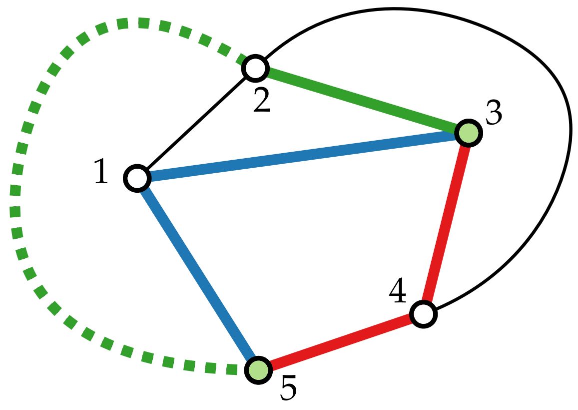

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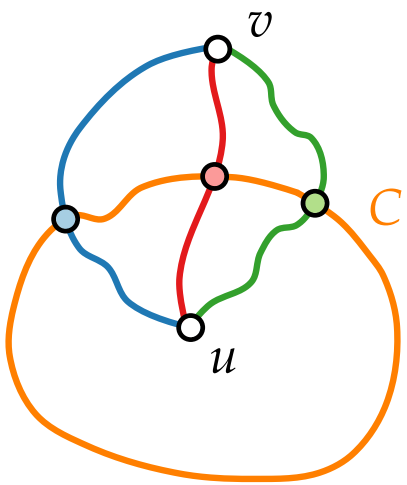

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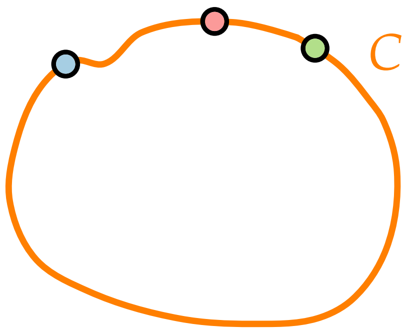

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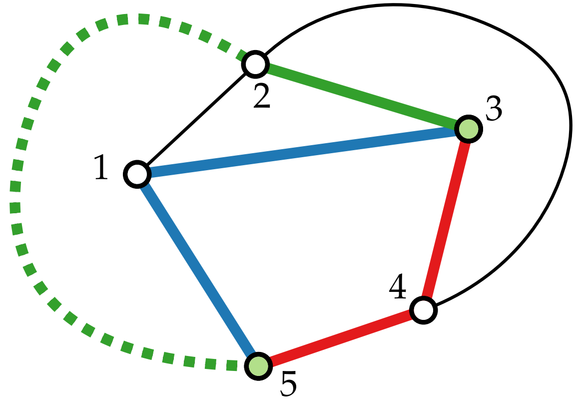

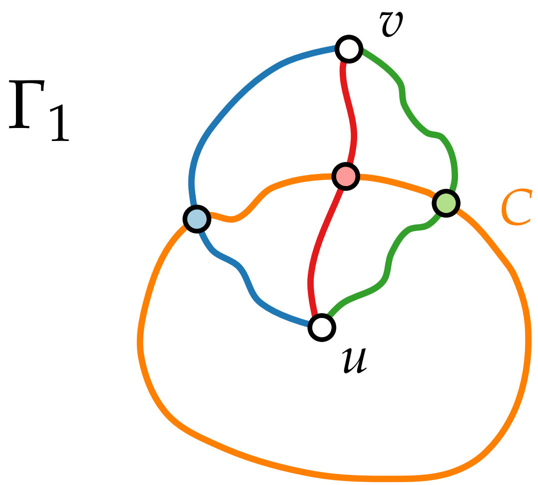

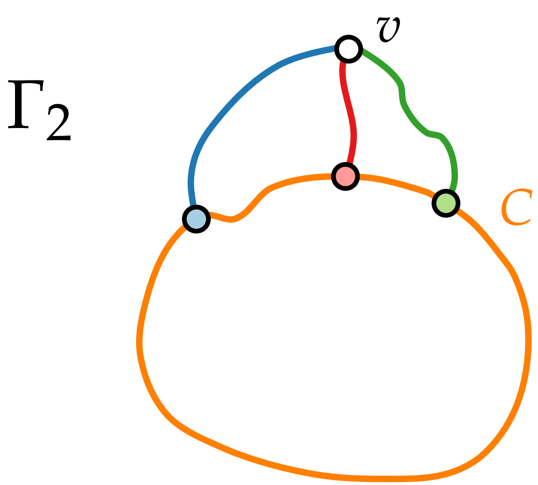

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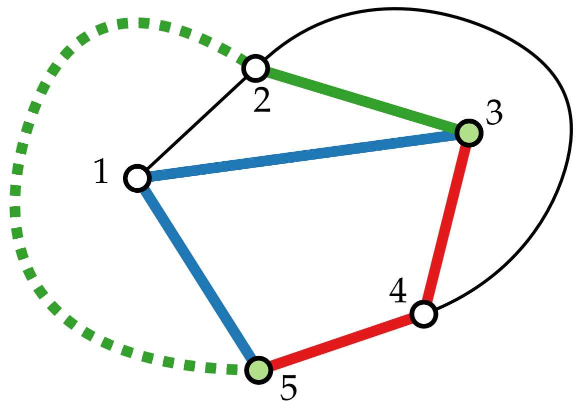

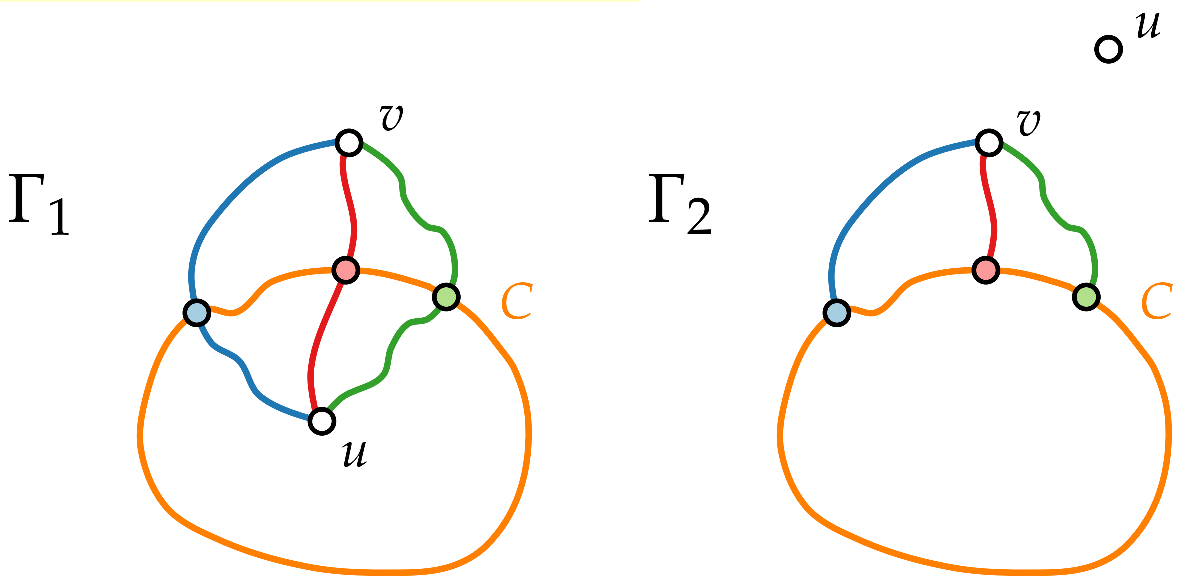

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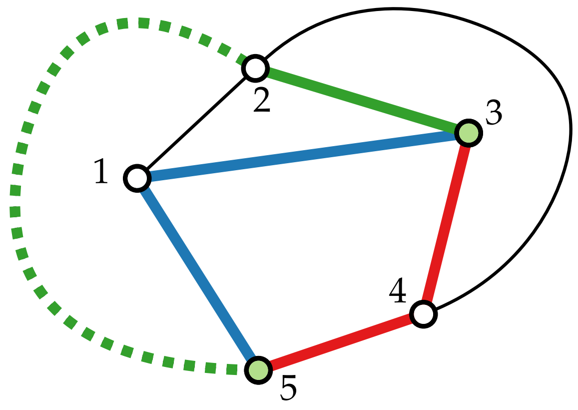

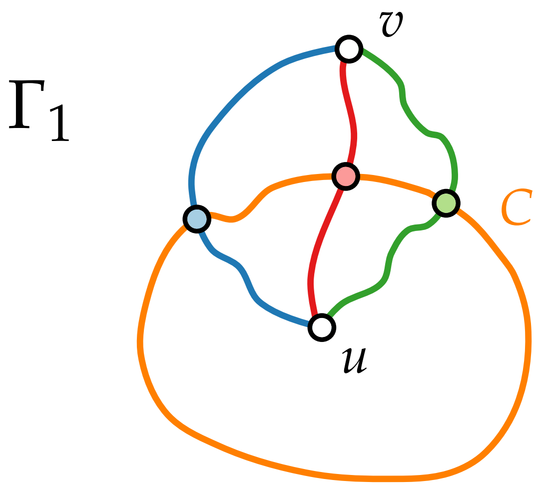

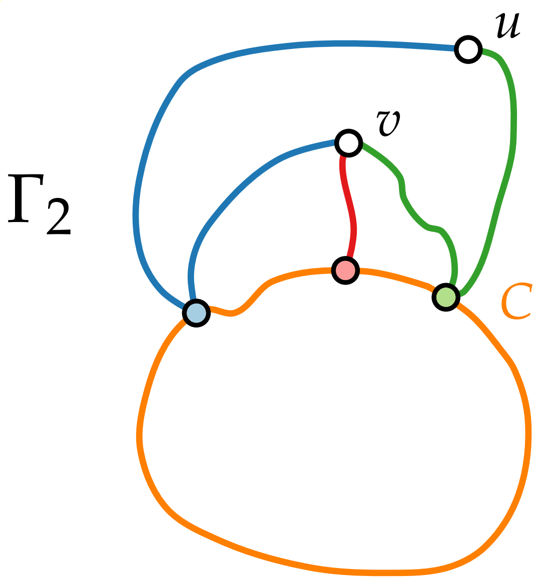

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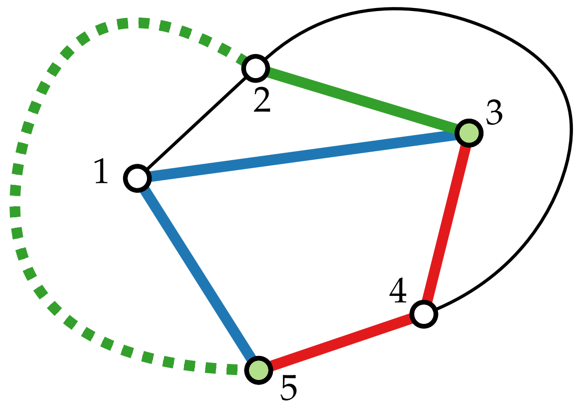

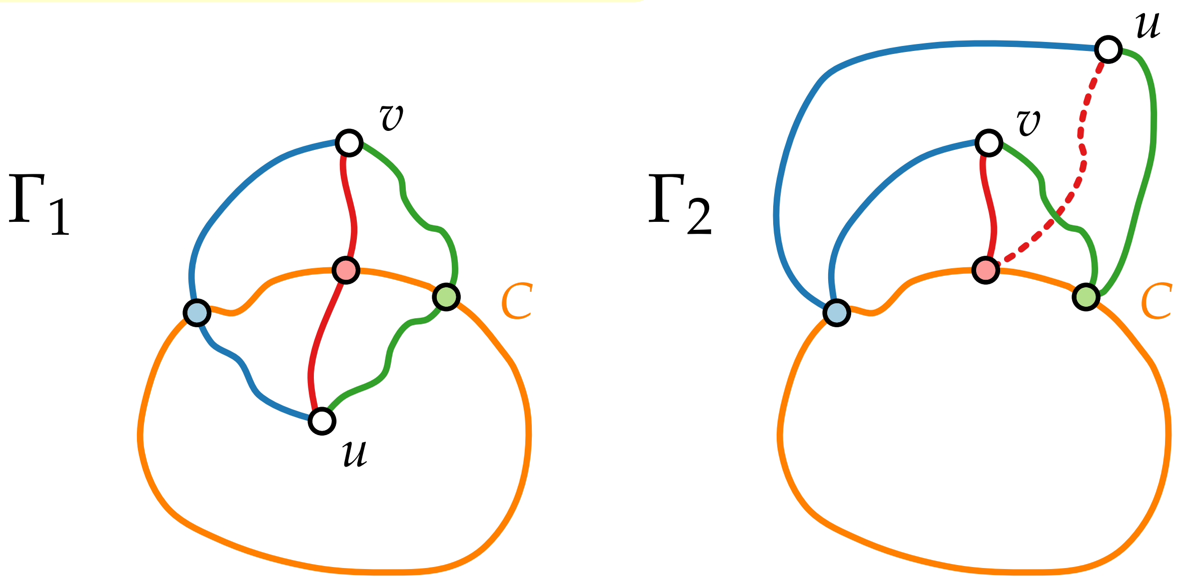

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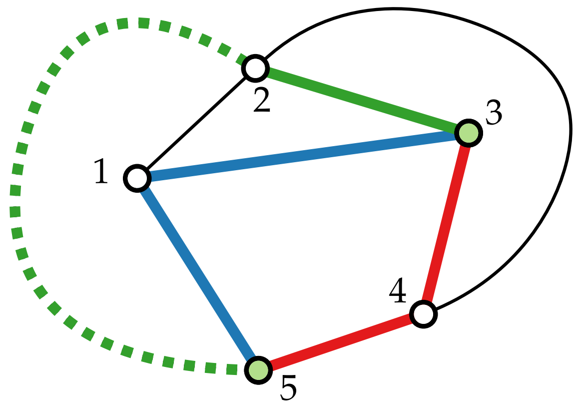

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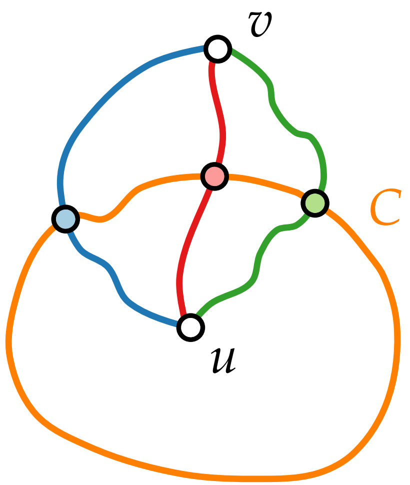

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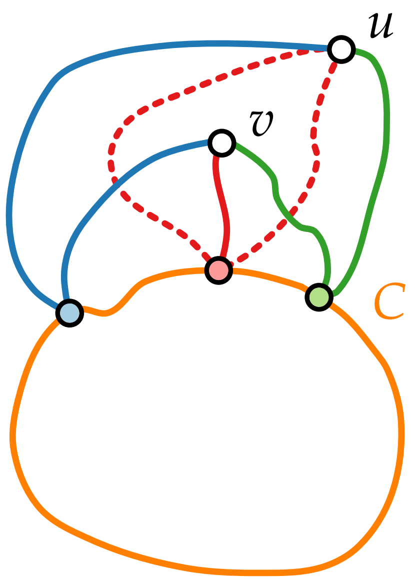

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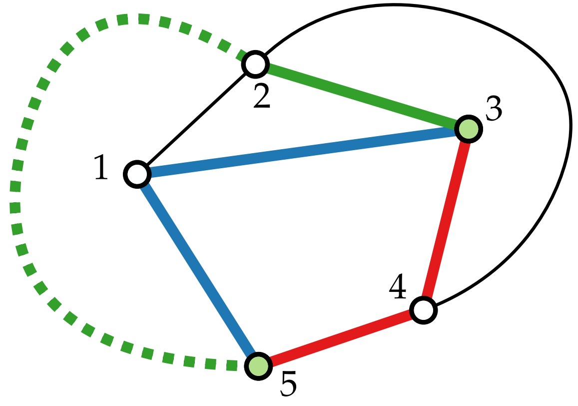

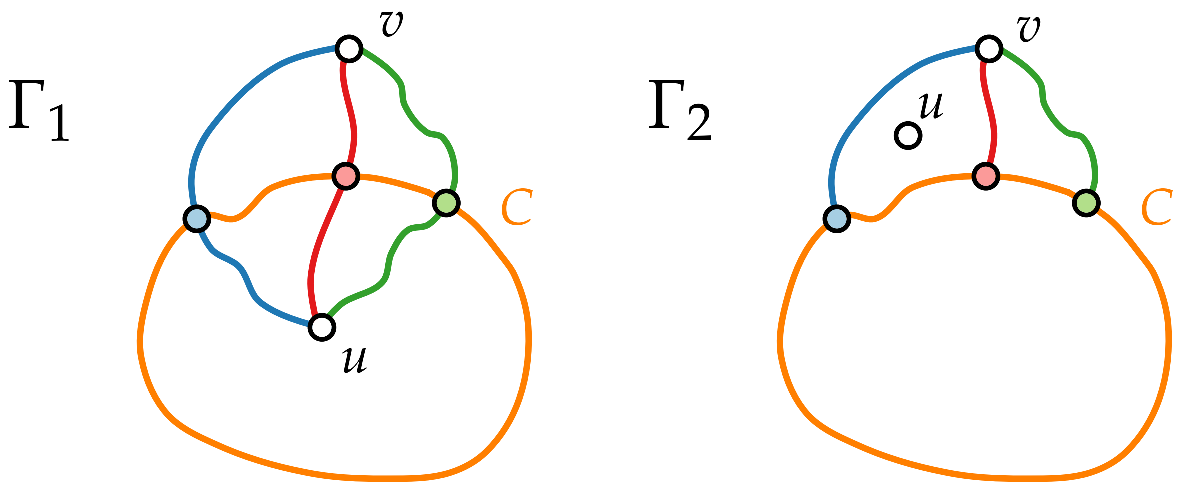

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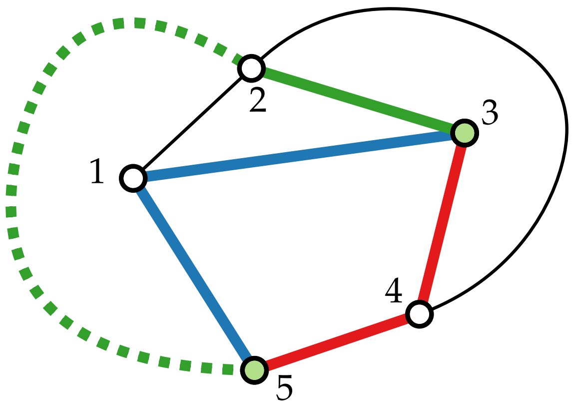

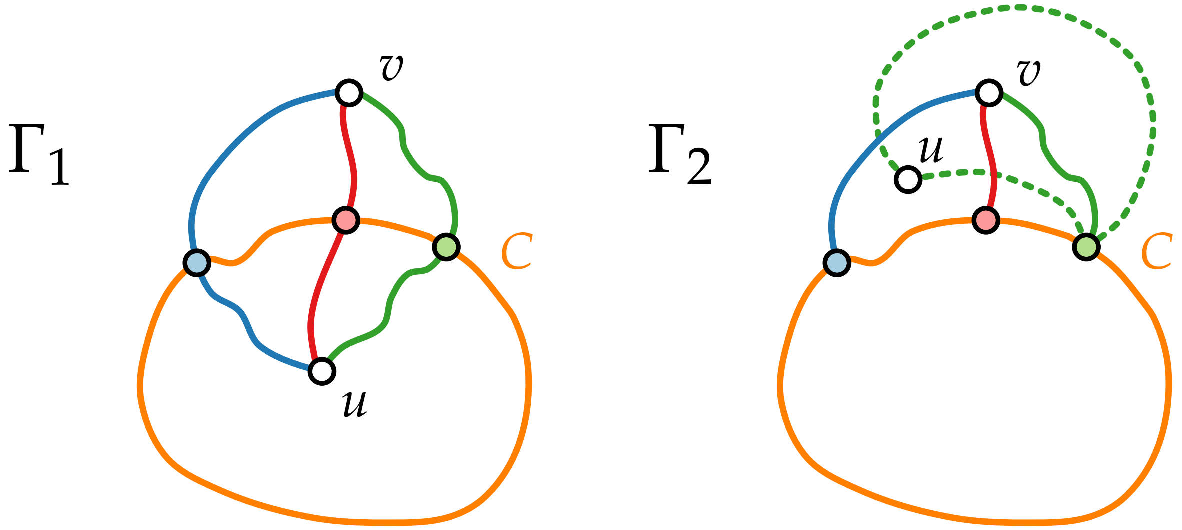

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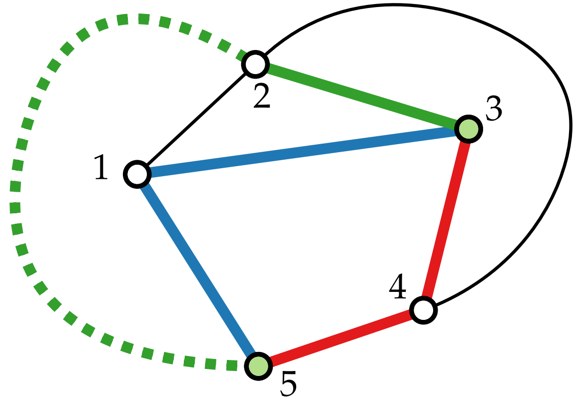

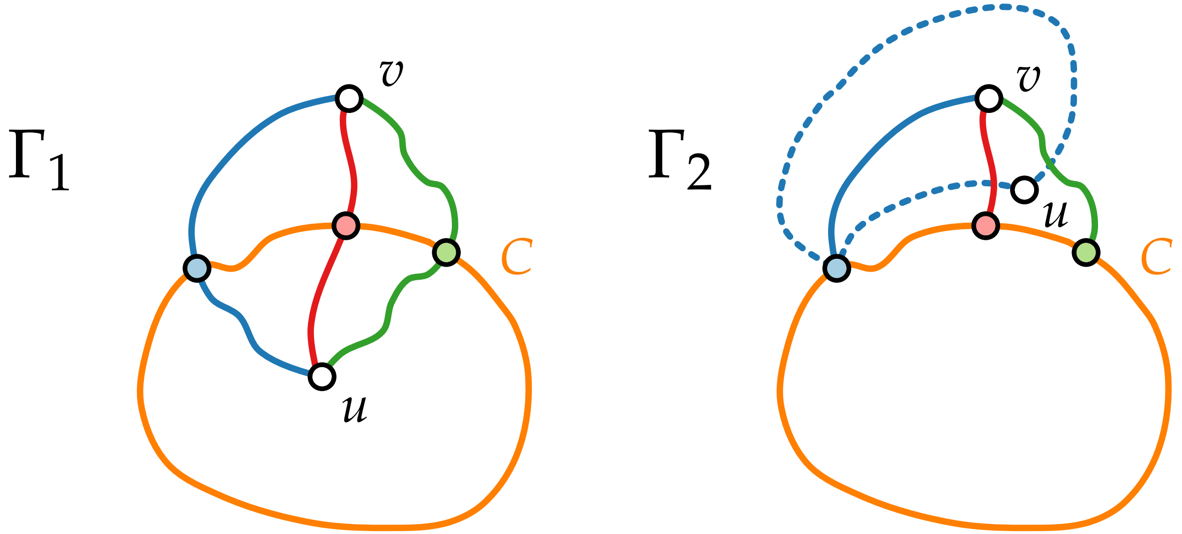

Tutte's Theorem

Theorem. [Tutte 1963]

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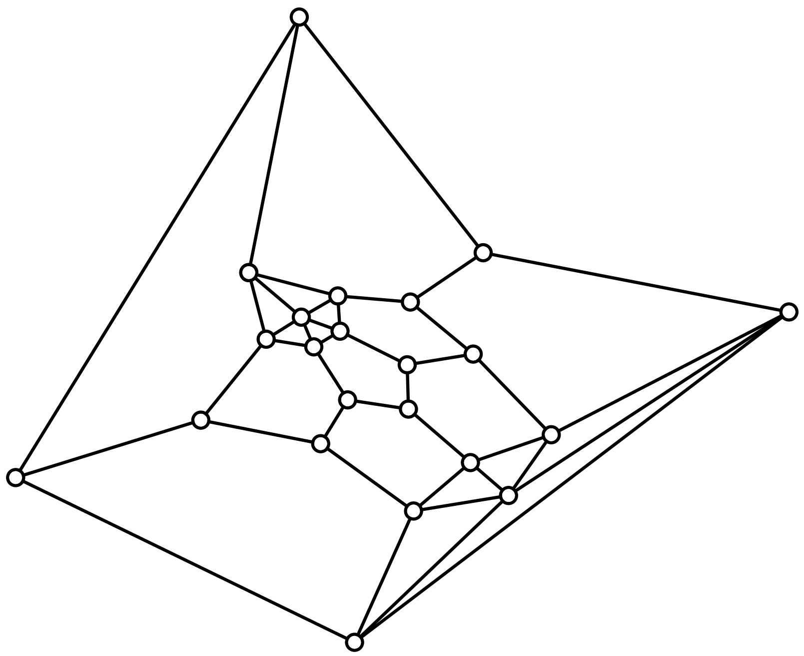

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Theorem. [Tutte 1963]

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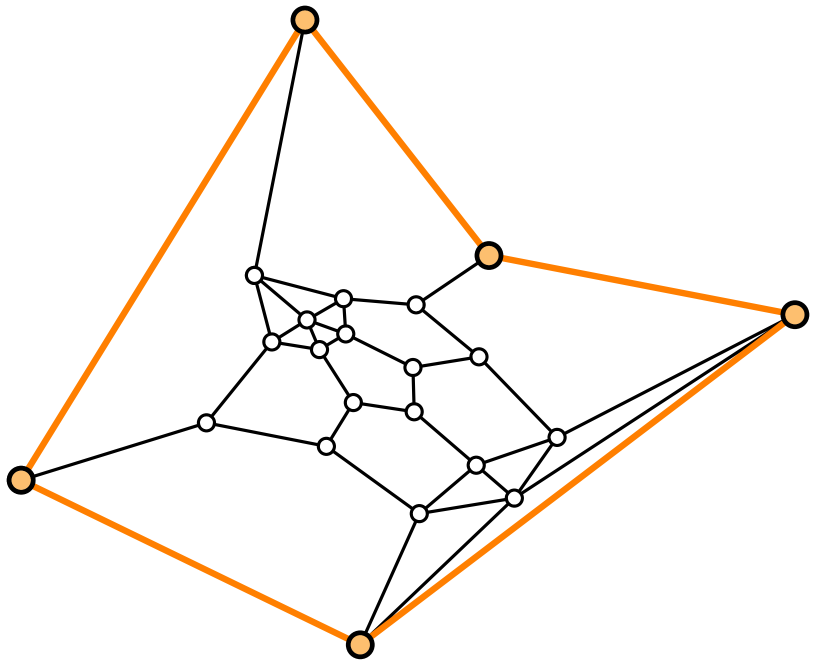

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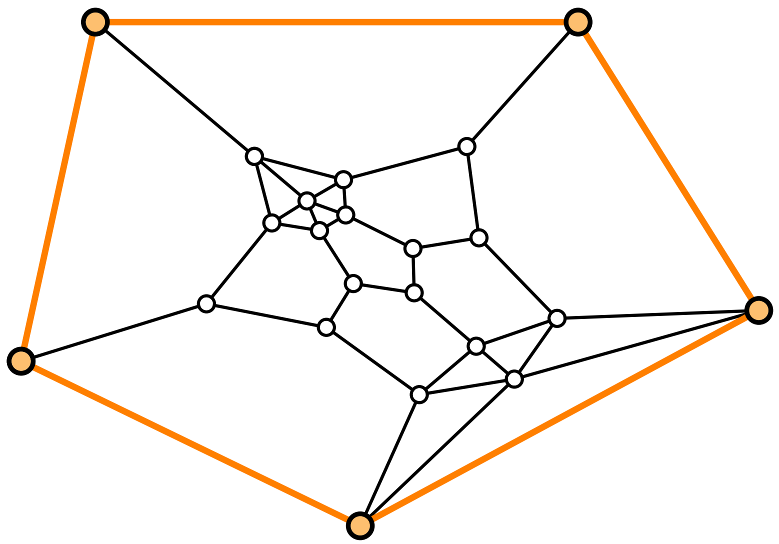

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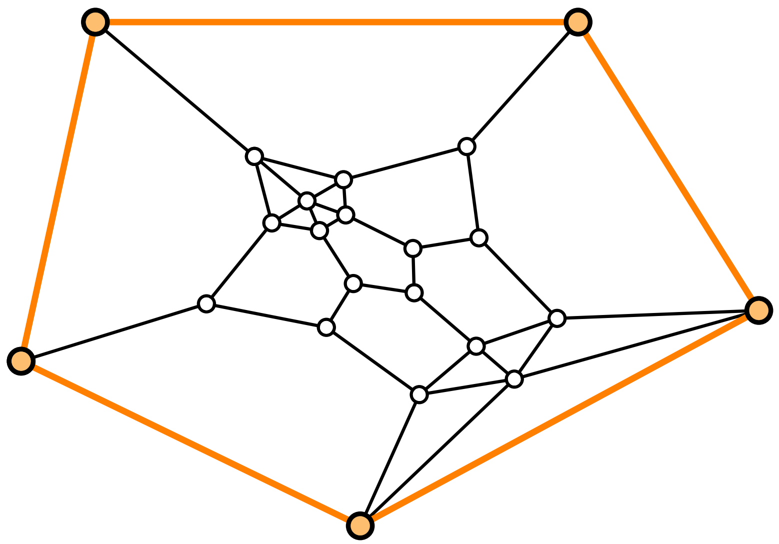

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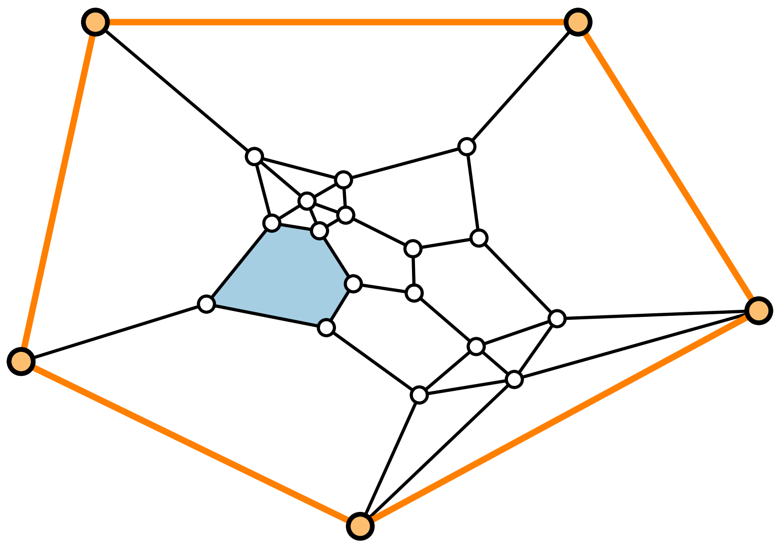
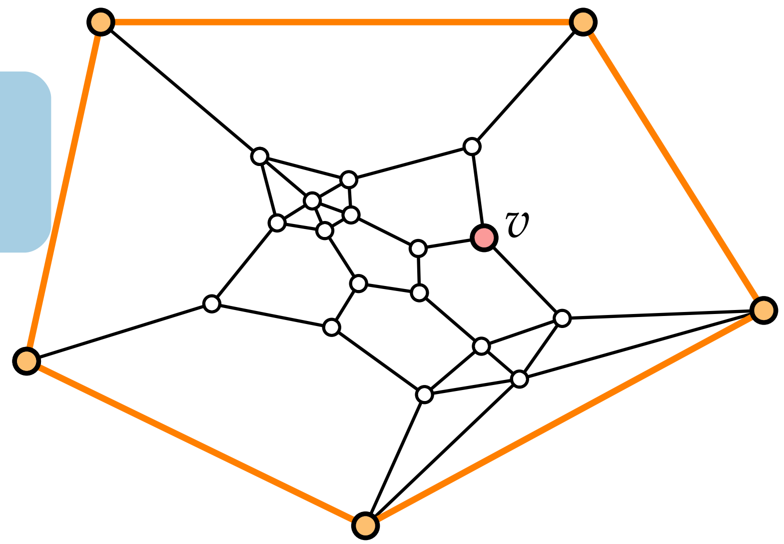


Properties of Tutte Drawings

Property 1. Let $v \in V$ free



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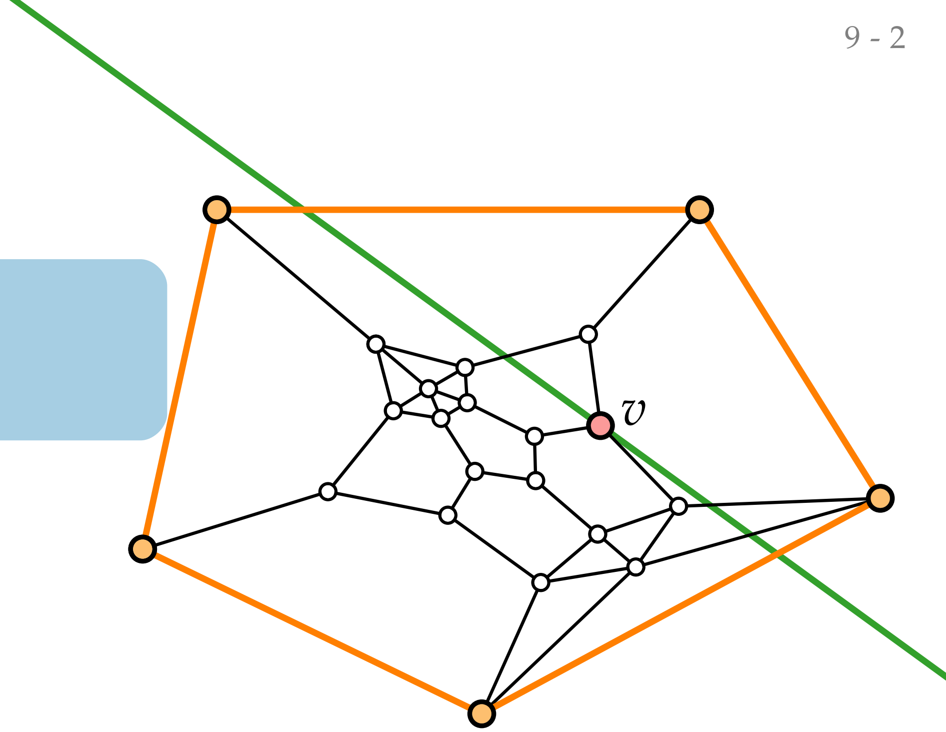

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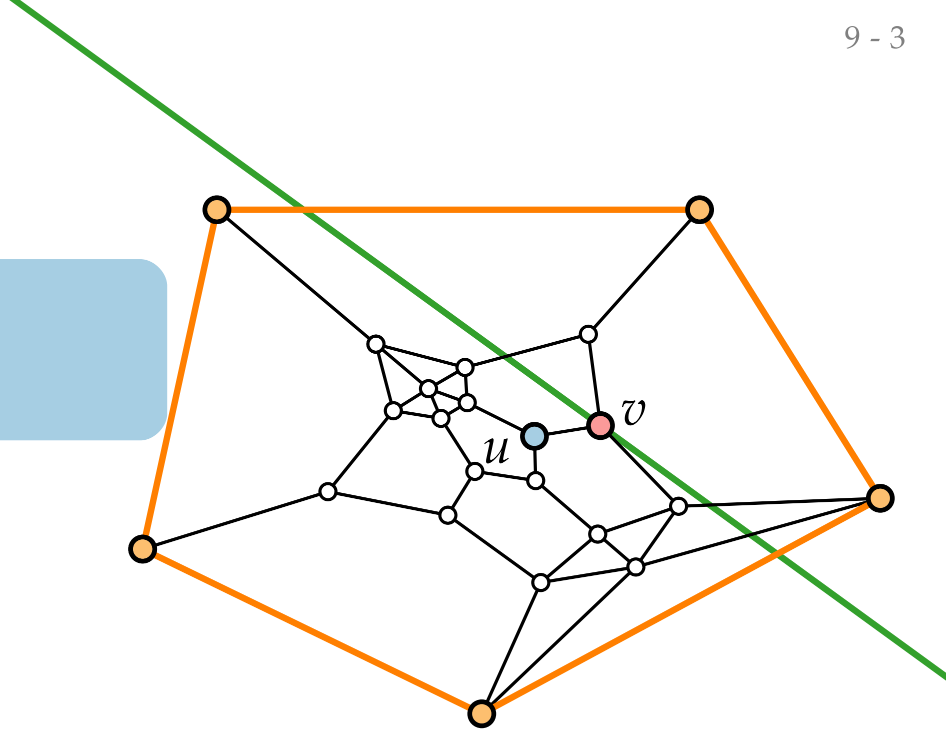
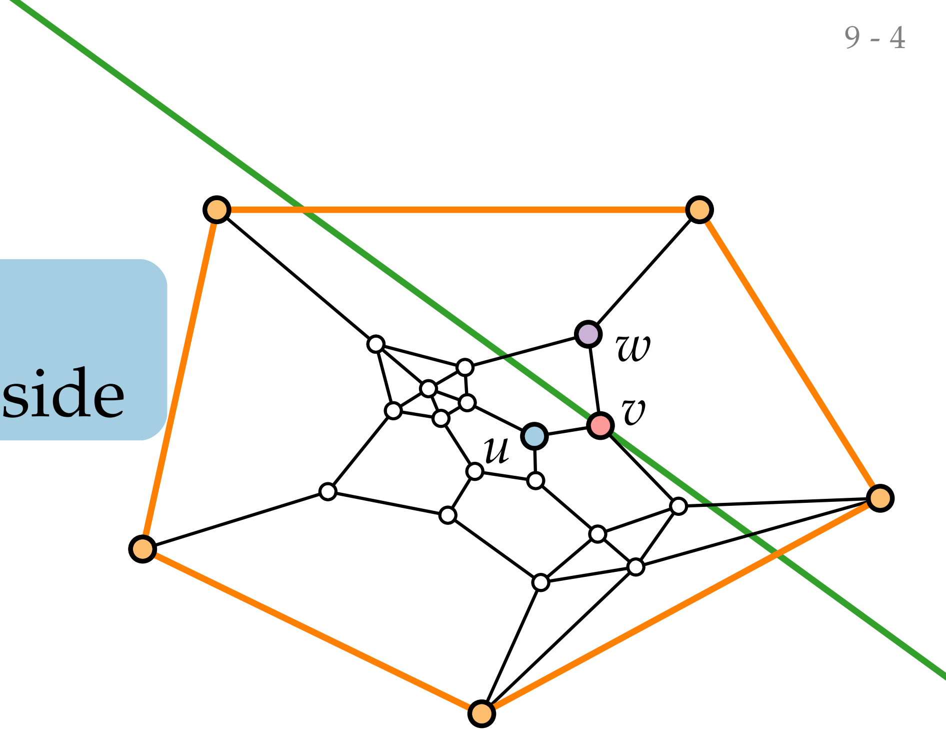
$\exists uv \in E$ on one side of ℓ



Properties of Tutte Drawings

Property 1. Let $v \in V$ free, ℓ line through v.
$\exists uv \in E$ on one side of $\ell \Rightarrow \exists vw \in E$ on other side



Properties of Tutte Drawings

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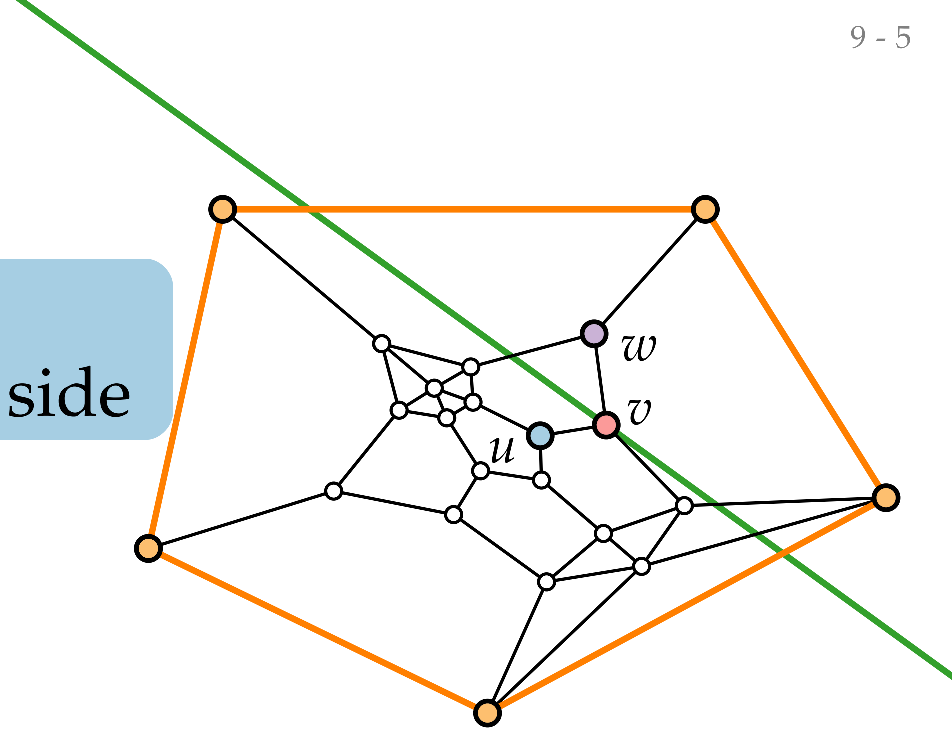
Otherwise, all forces to same side . . .



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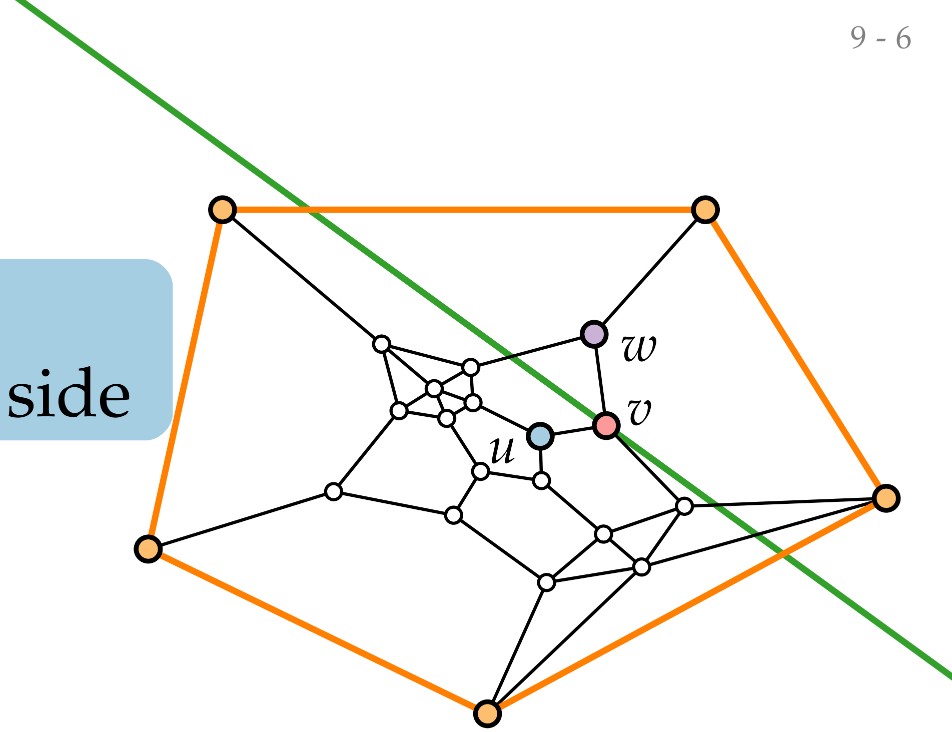
Property 2. All free vertices lie inside C.



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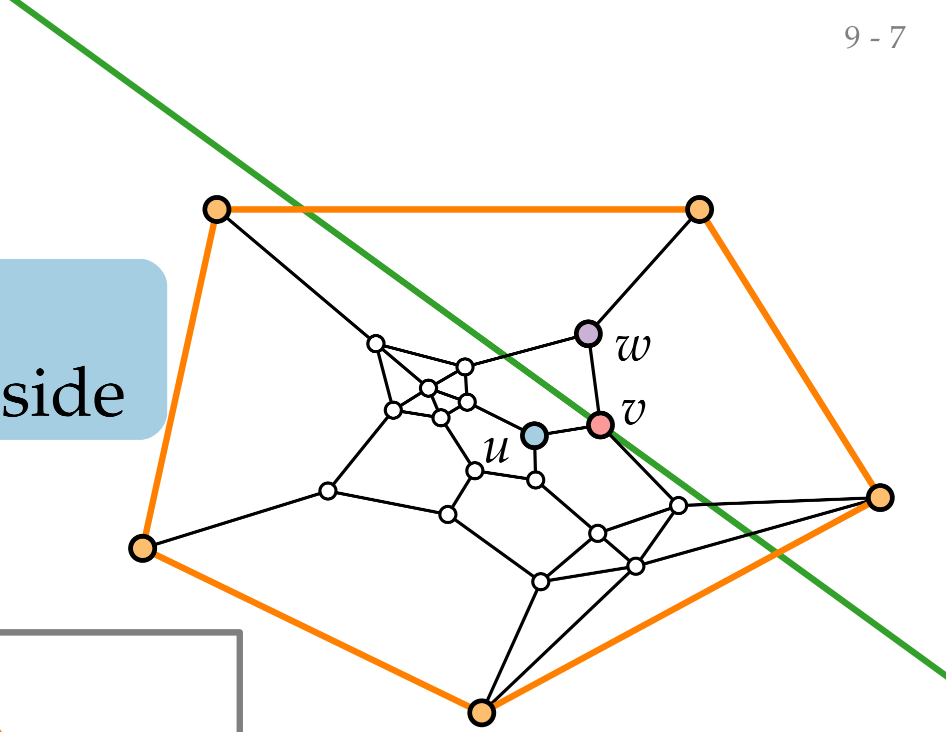

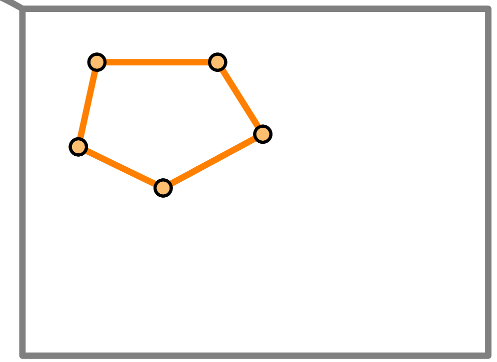

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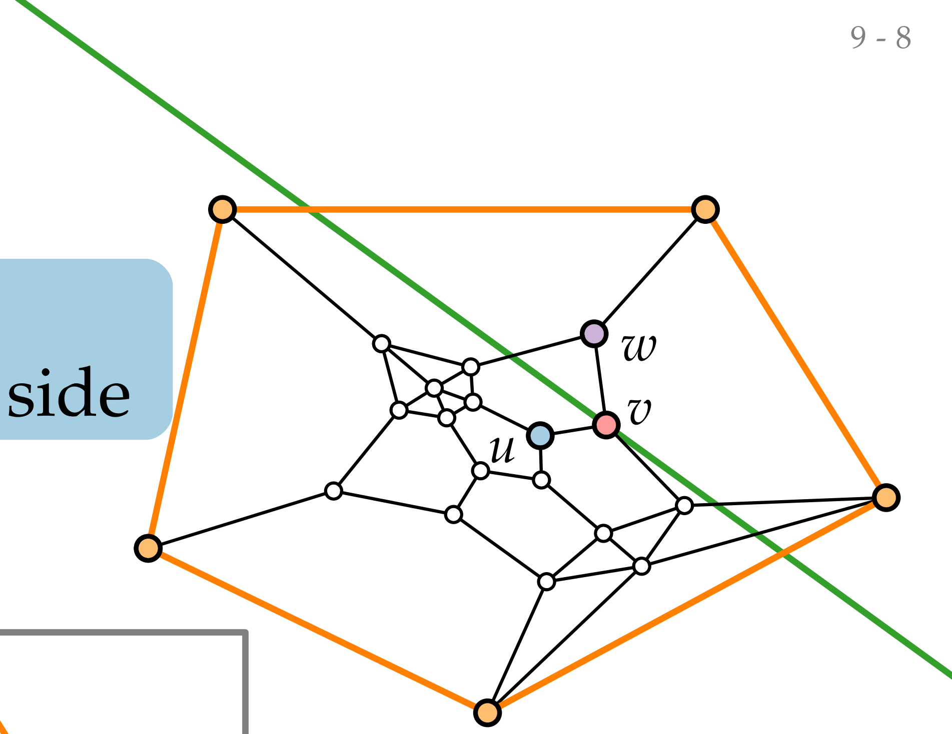

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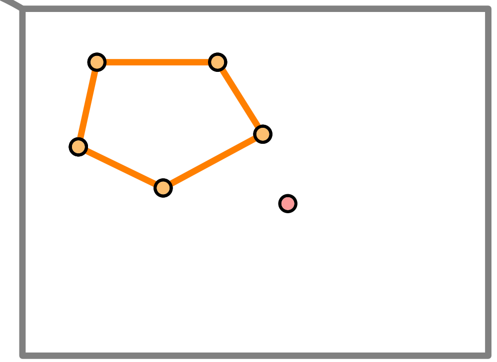

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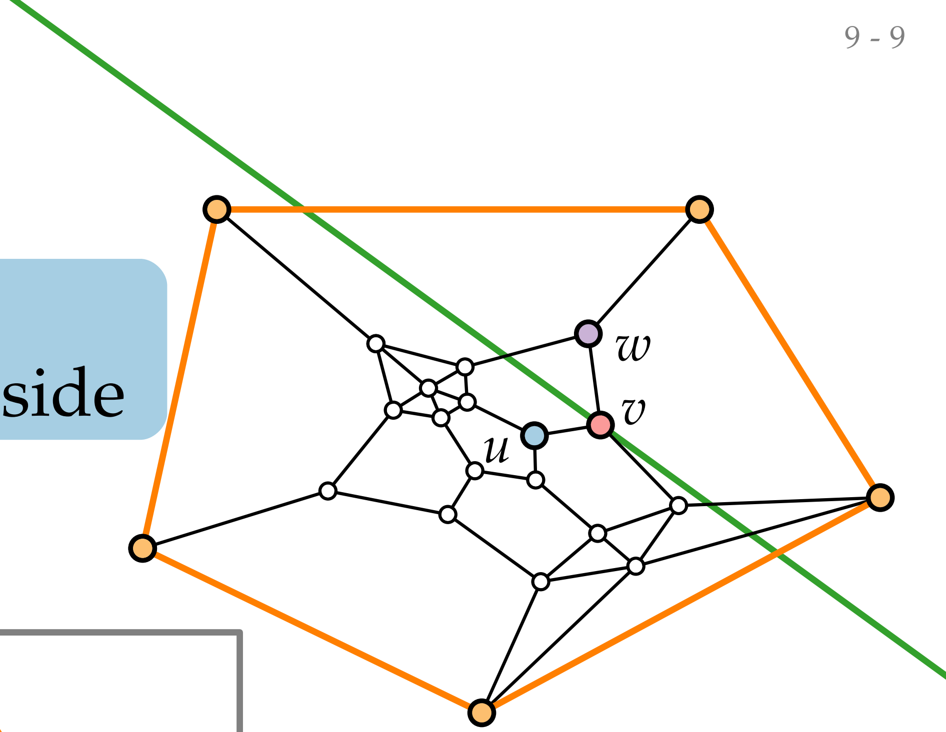

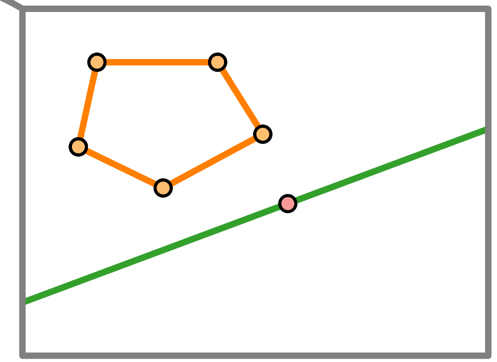

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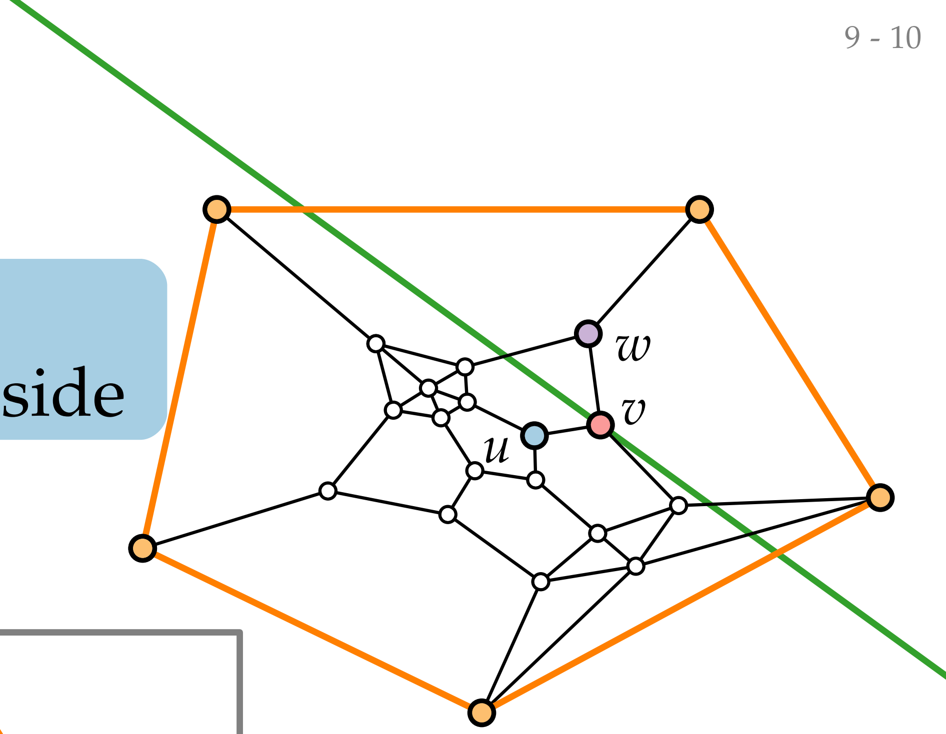

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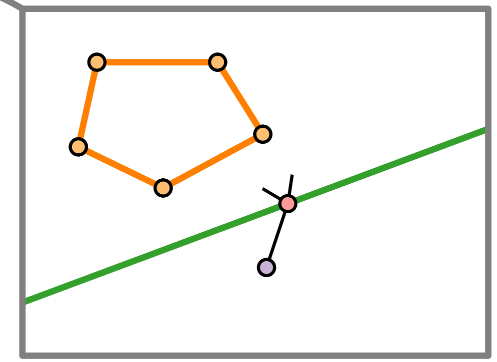

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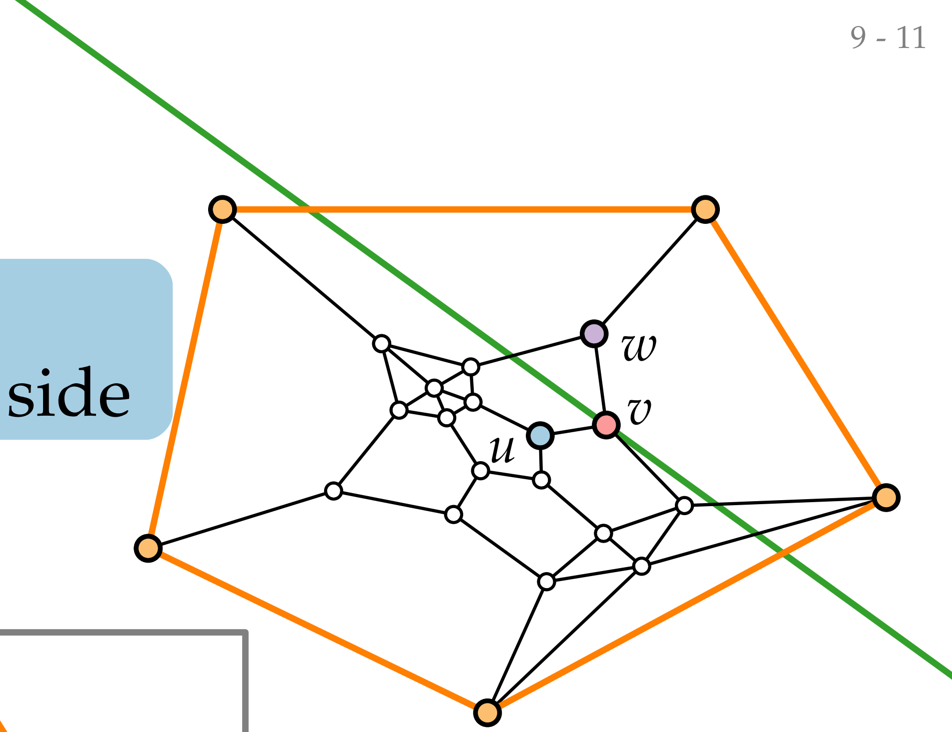

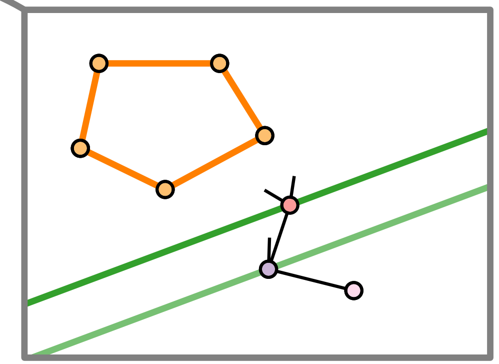

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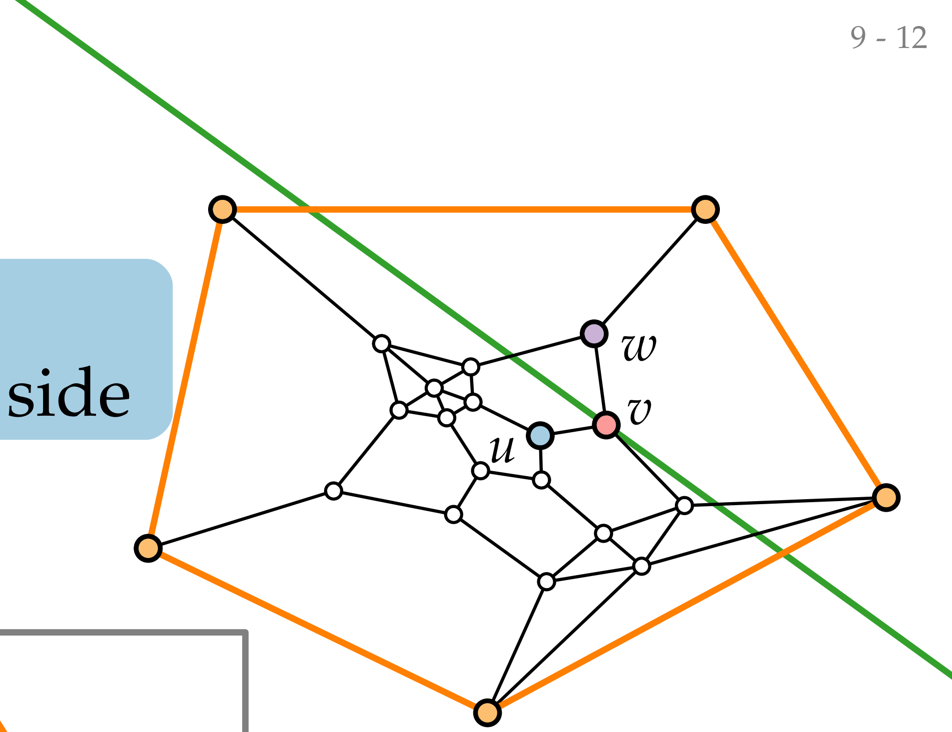

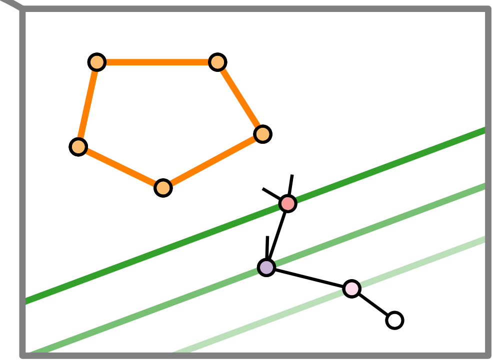

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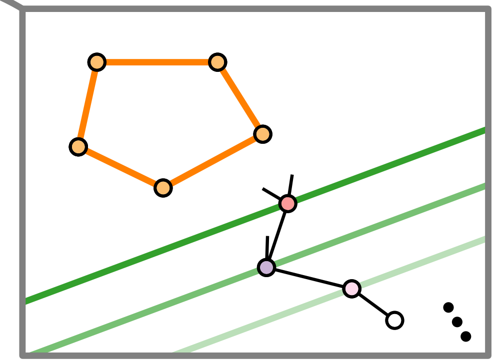

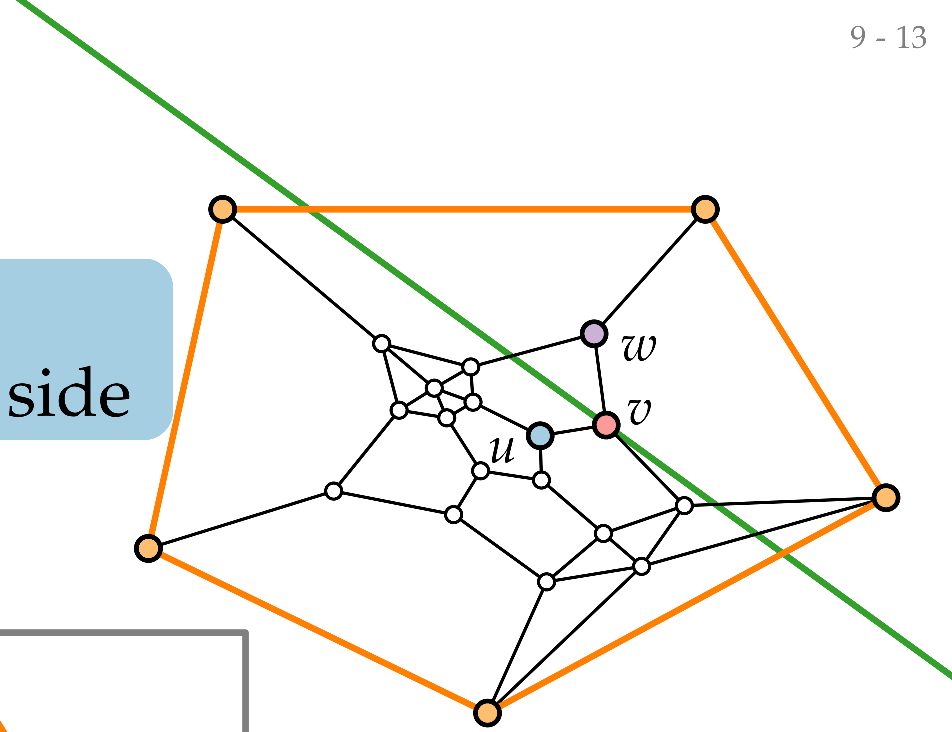

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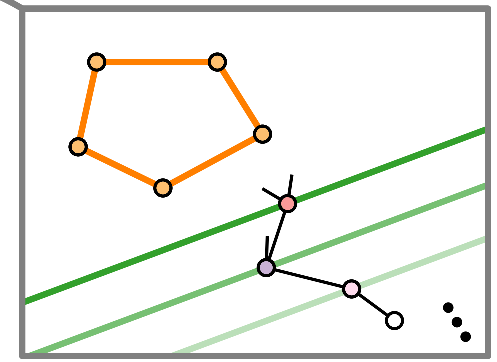

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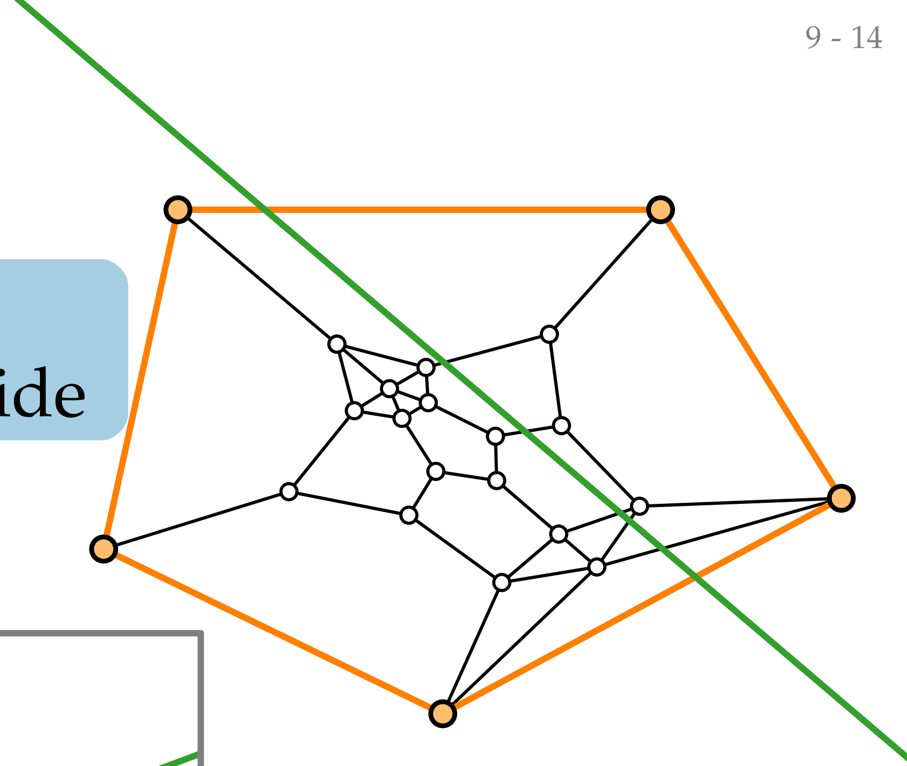

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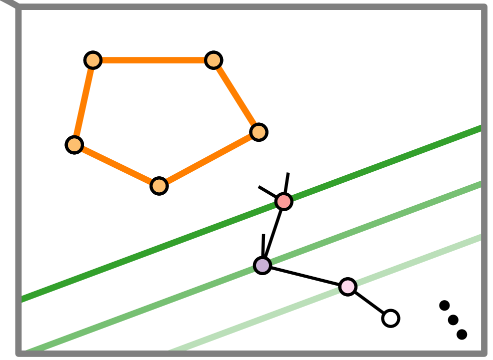
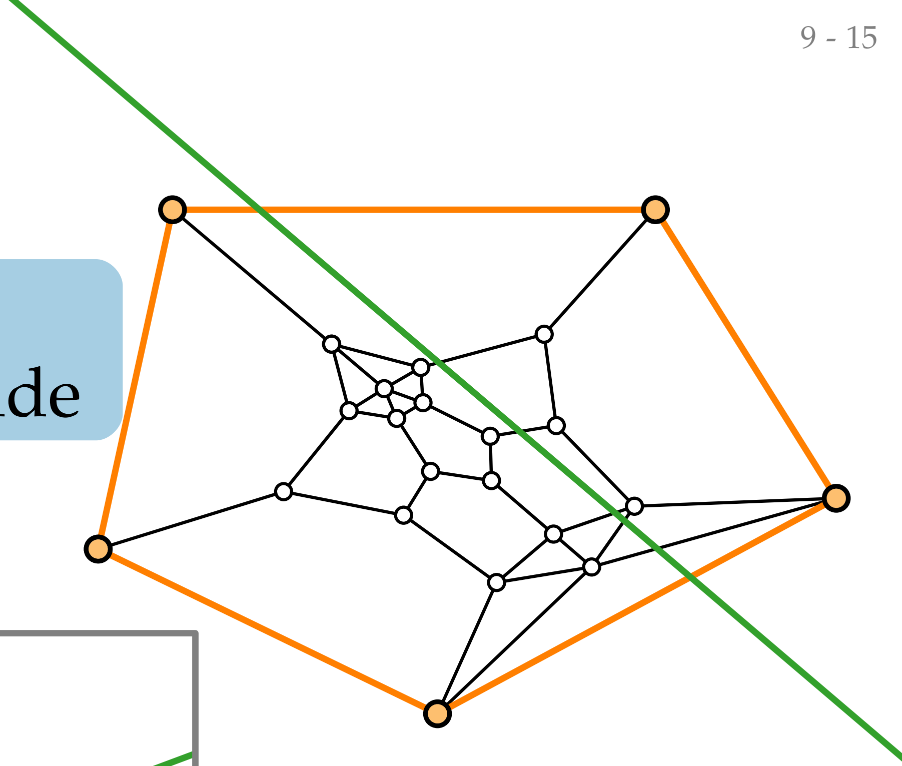
Let V_ℓ be all vertices on one side of ℓ.





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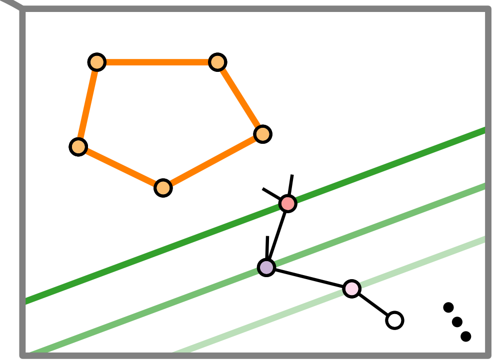

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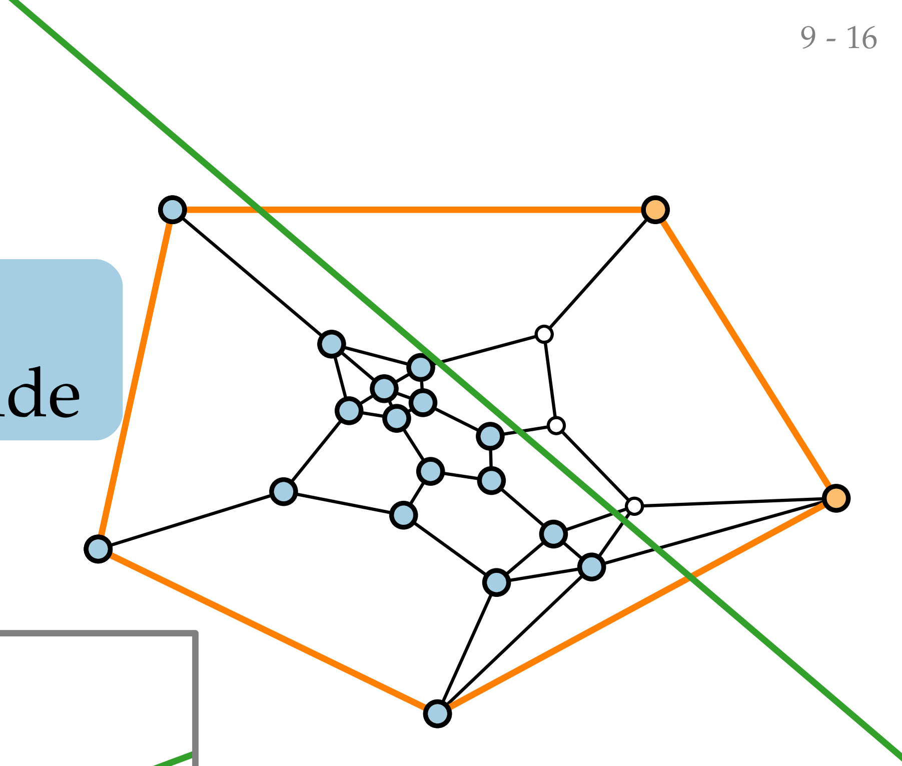

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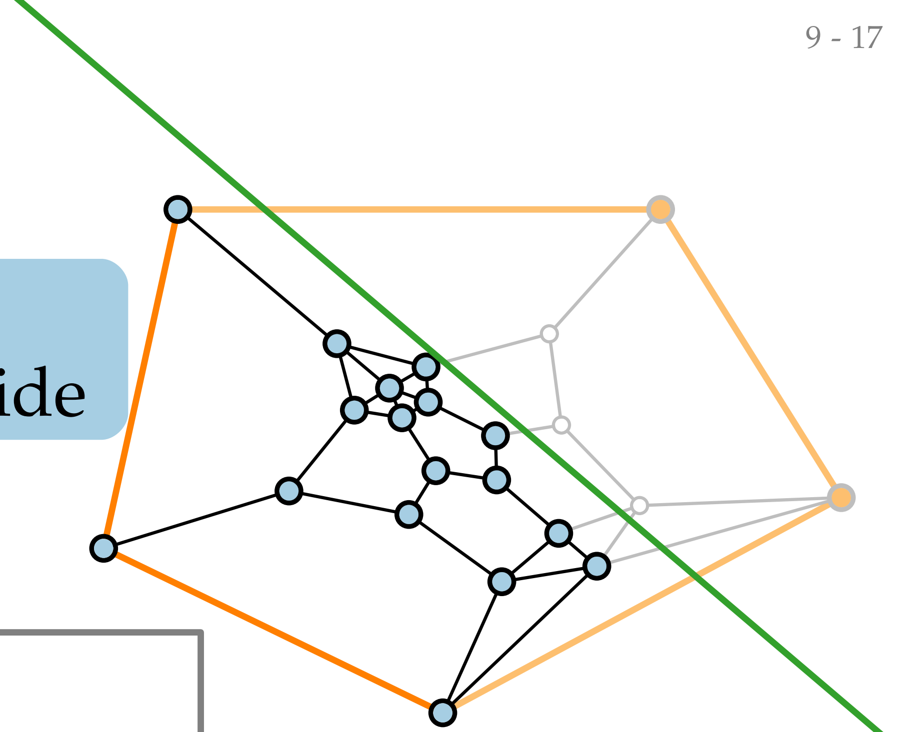

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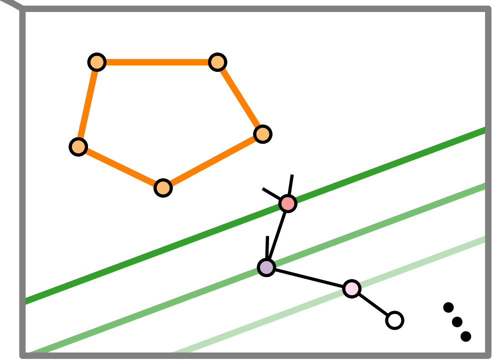

Property 3. Let ℓ be any line.
Let V_ℓ be all vertices on one side of ℓ.
Then $G[V_\ell]$ is connected.

Properties of Tutte Drawings

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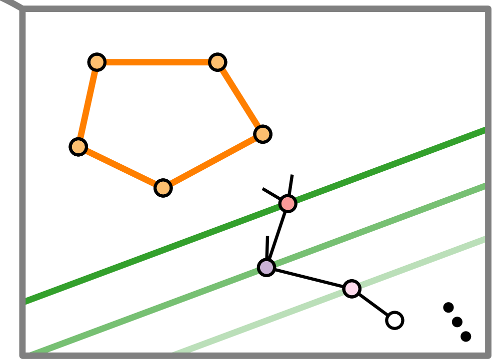

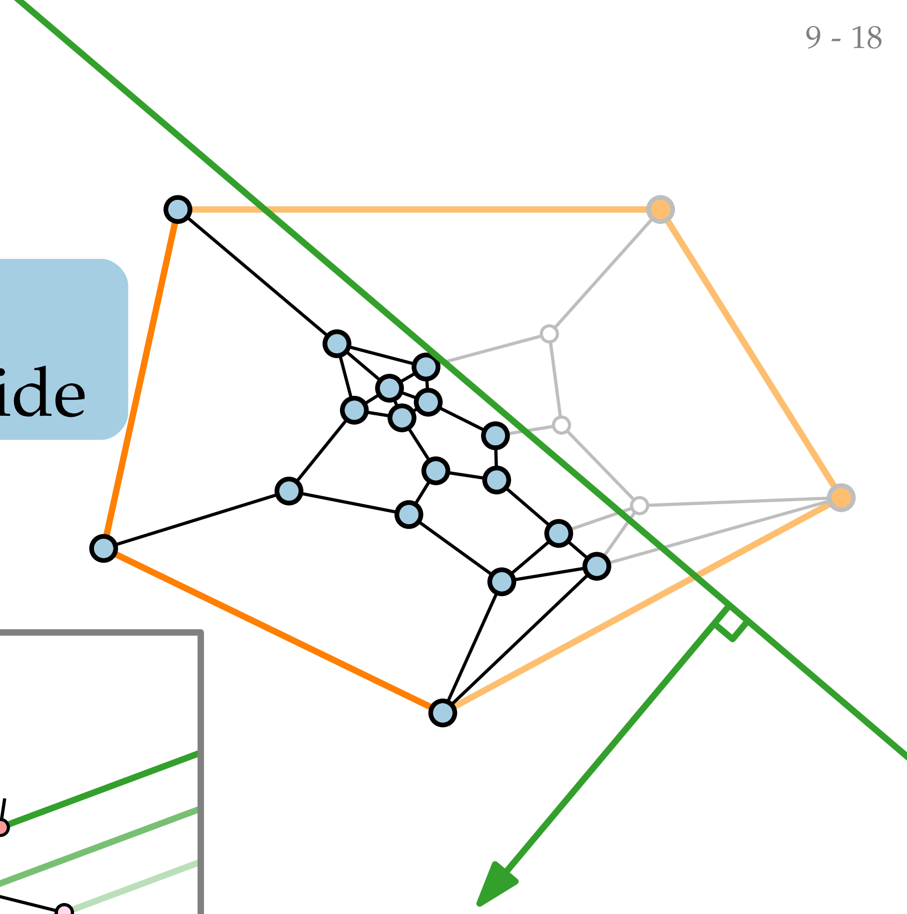

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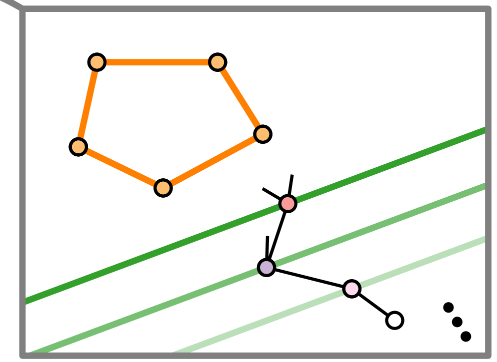

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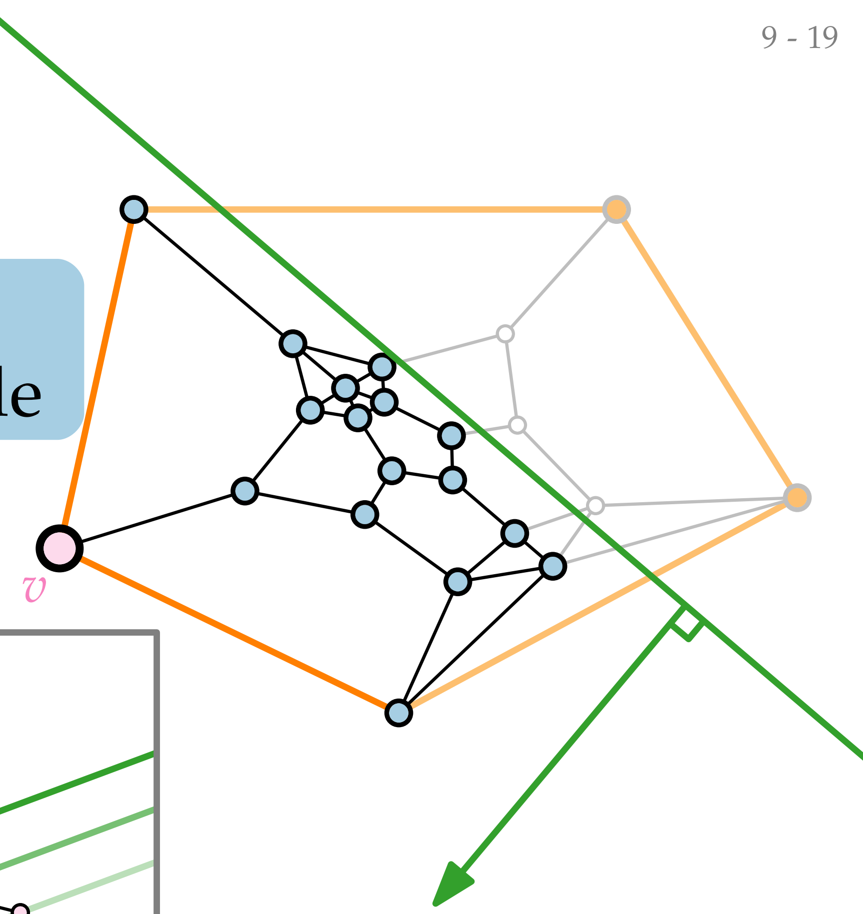
v furthest away from ℓ



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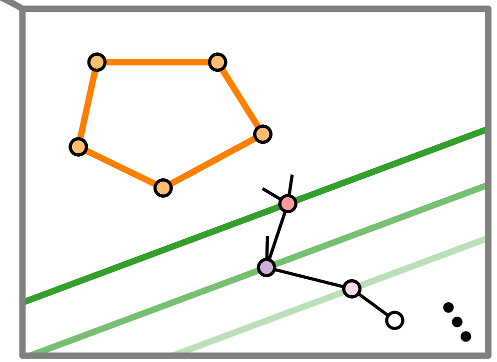

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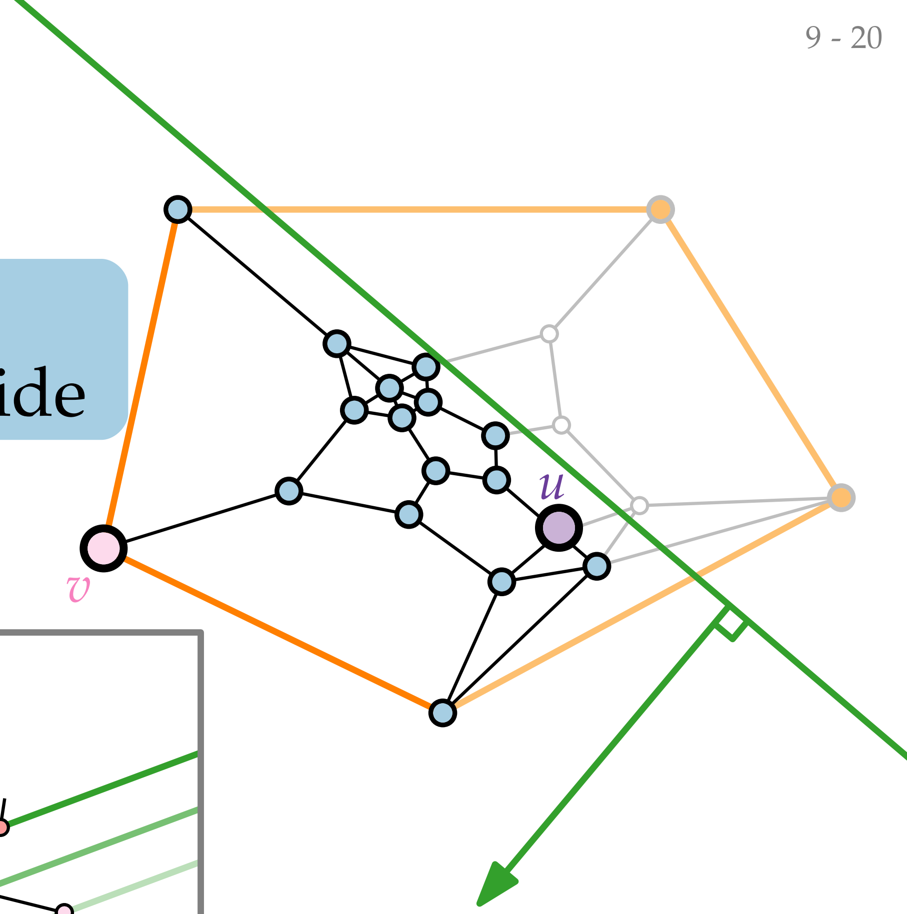
Let V_ℓ be all vertices on one side of ℓ.
Then $G[V_\ell]$ is connected.

v furthest away from ℓ
Pick any vertex u



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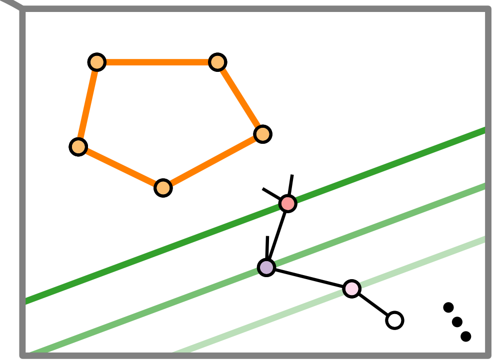

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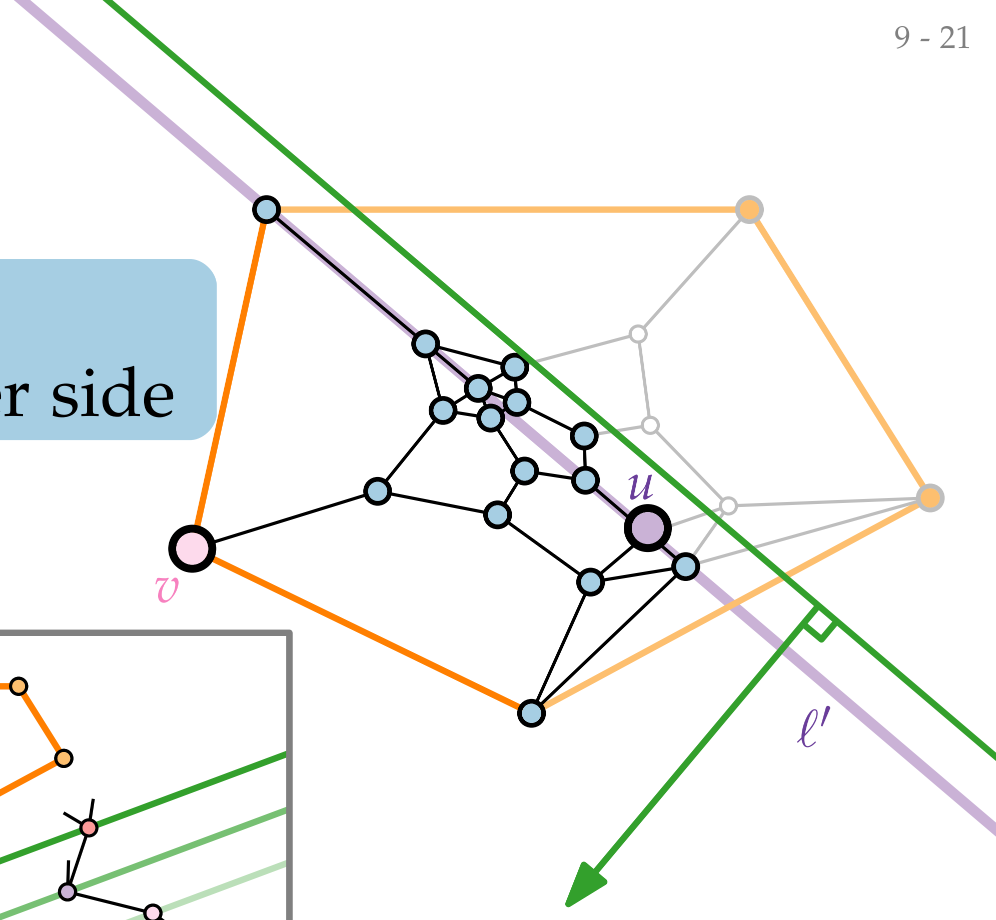
Let V_ℓ be all vertices on one side of ℓ.
Then $G[V_\ell]$ is connected.

v furthest away from ℓ
Pick any vertex u, ℓ' parallel to ℓ throught u



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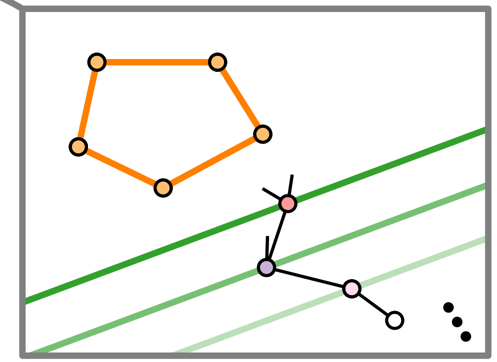

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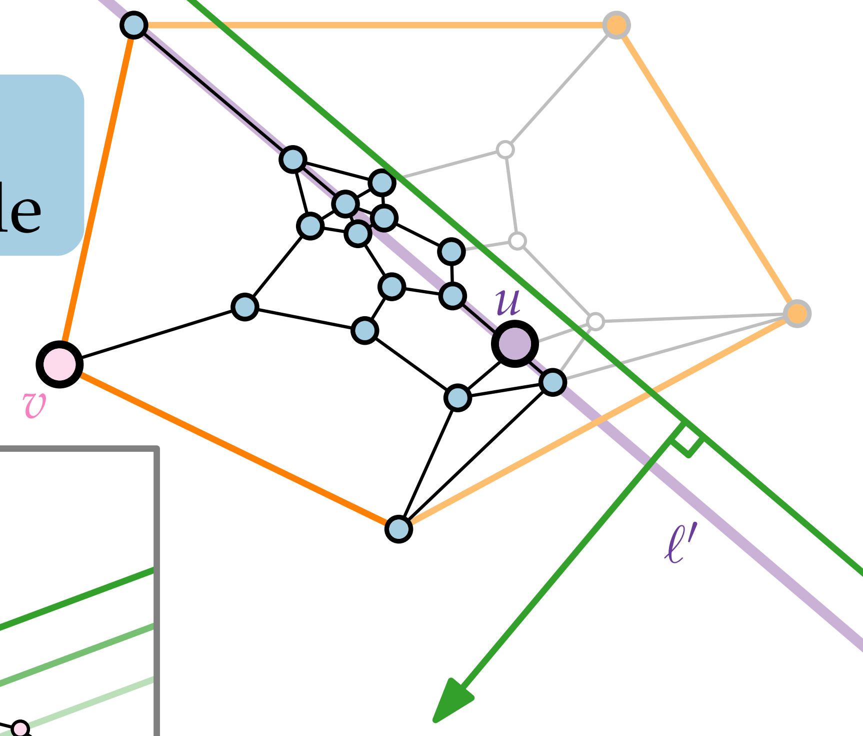
v furthest away from ℓ
Pick any vertex u, ℓ' parallel to ℓ throught u
G connected, v not on ℓ'



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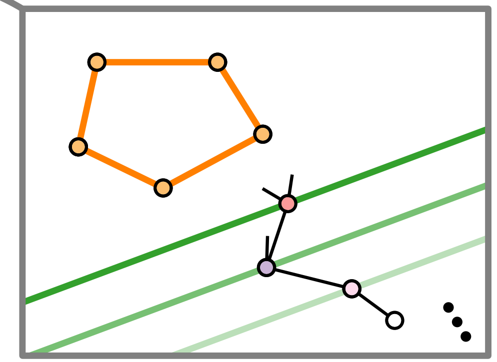

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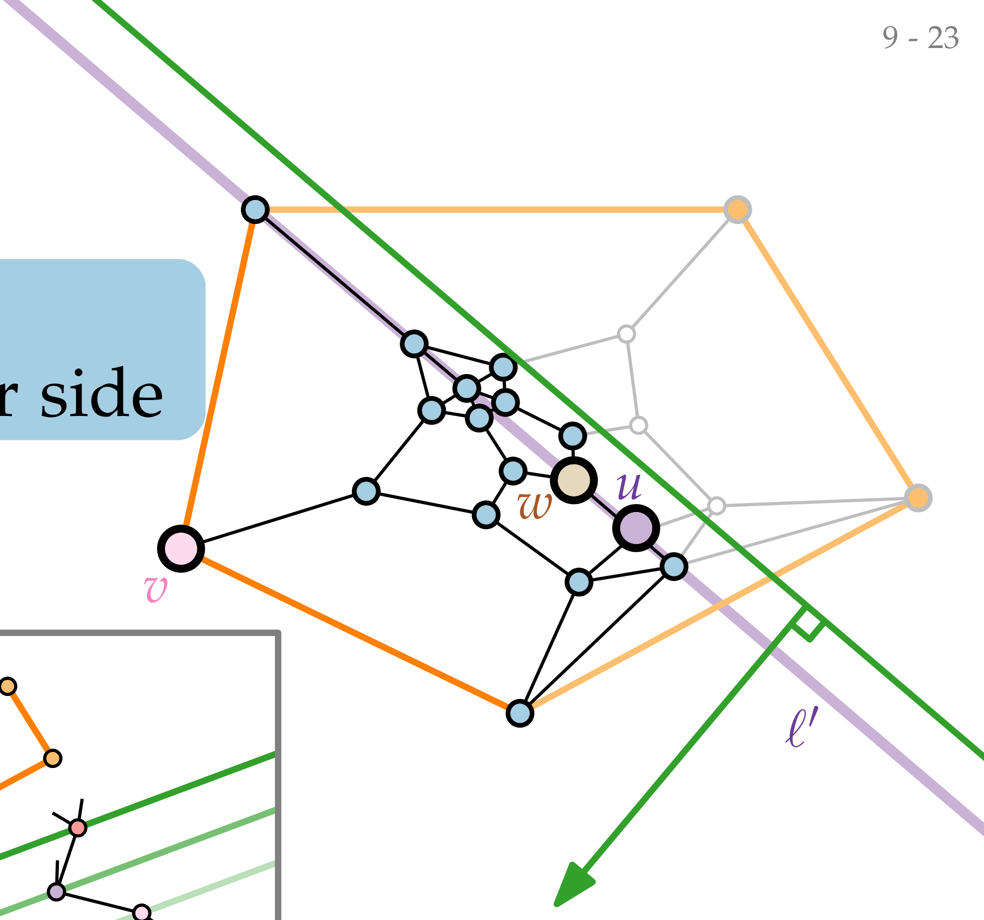
G connected, v not on $\ell' \Rightarrow \exists w$ on ℓ' with neighbor further away from ℓ



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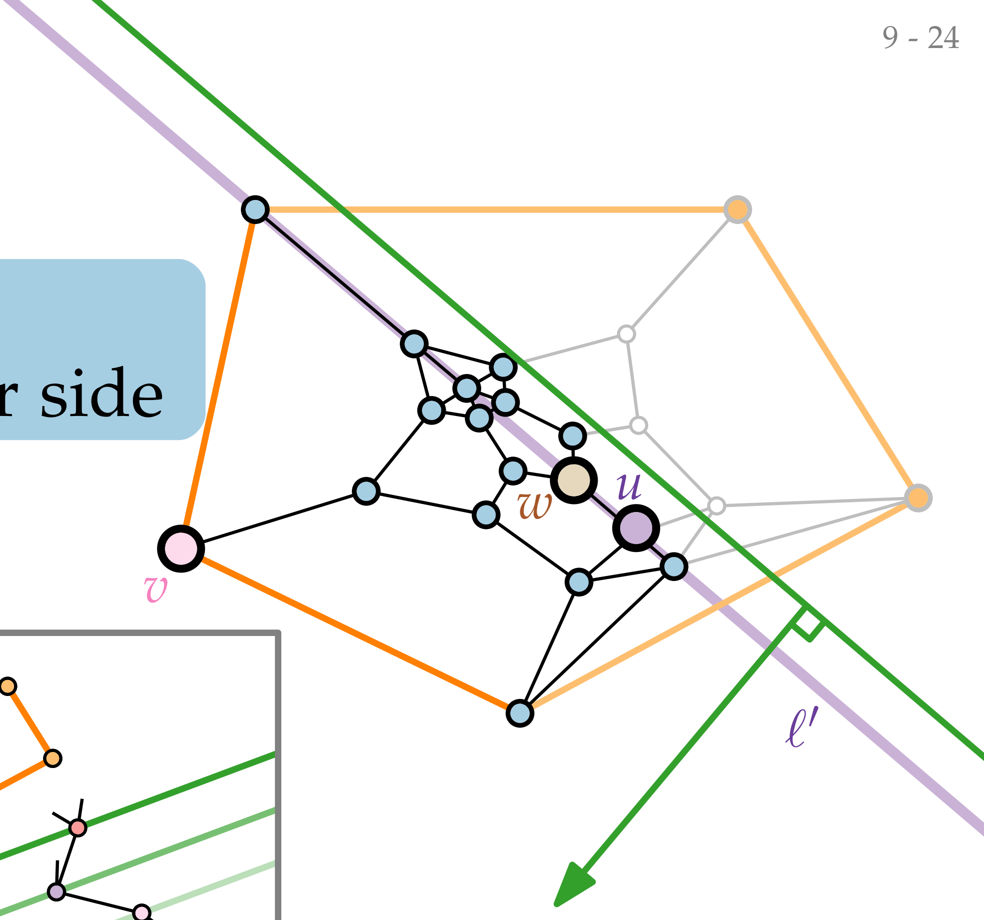

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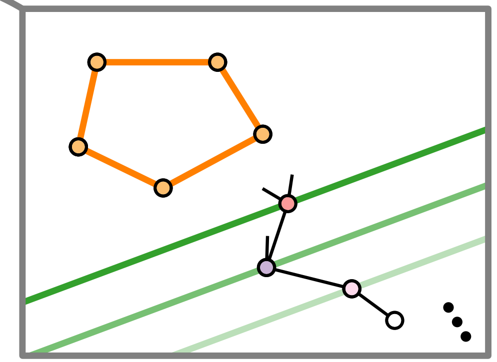

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v furthest away from ℓ
Pick any vertex u, ℓ' parallel to ℓ throught u
G connected, v not on $\ell' \Rightarrow \exists w$ on ℓ' with neighbor further away from ℓ
$\Rightarrow \exists$ path from u to v

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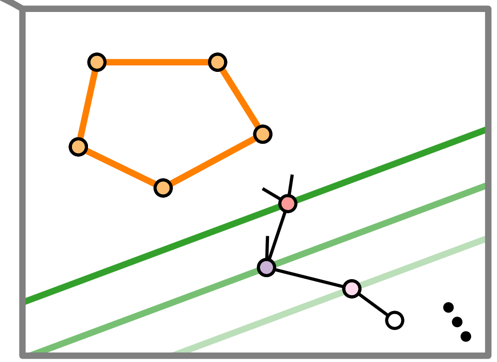

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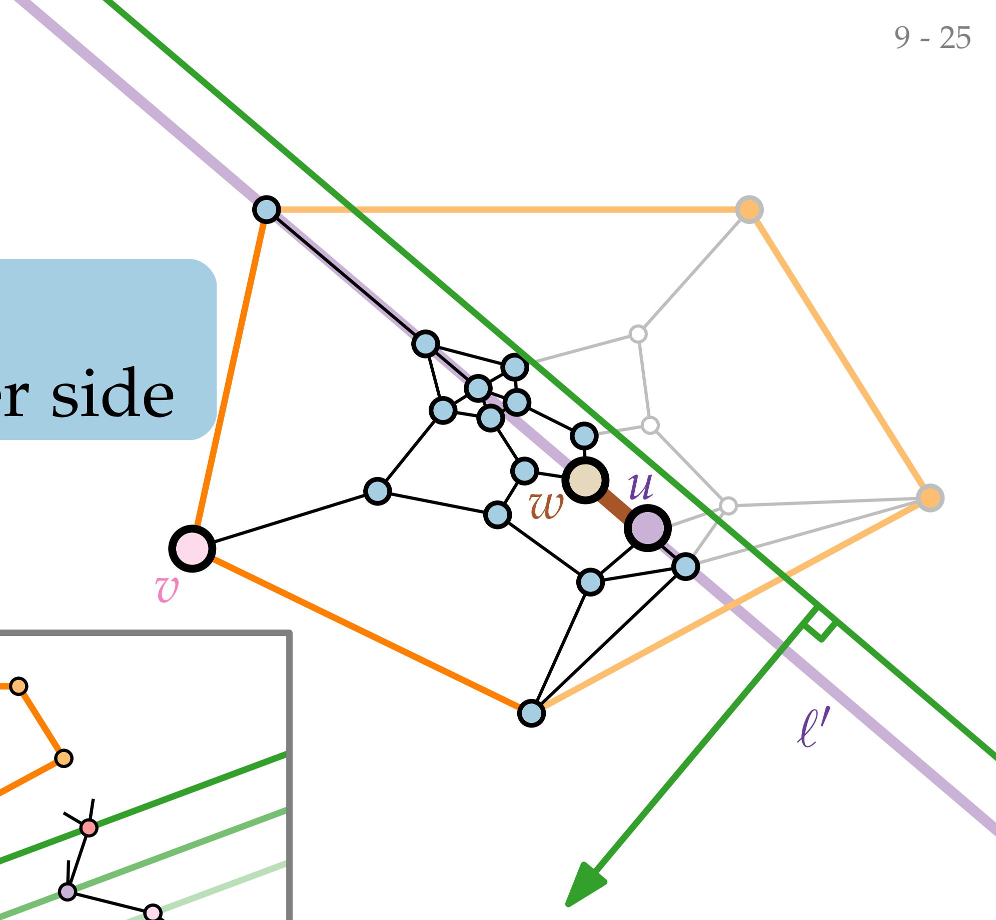

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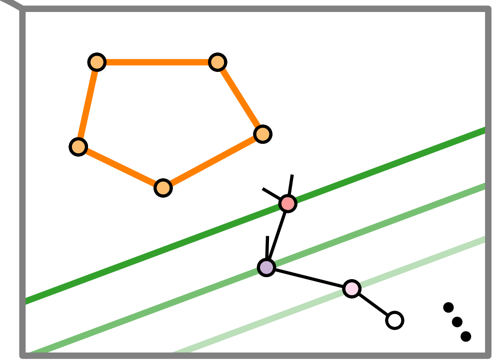

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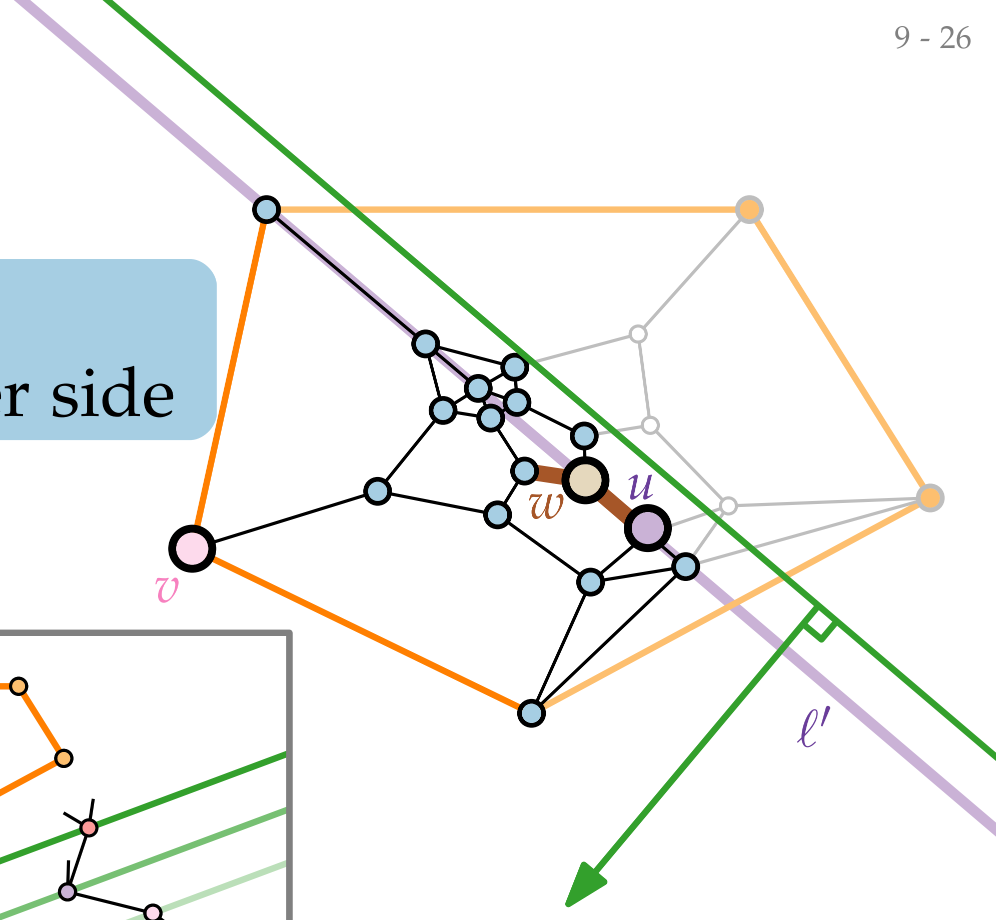

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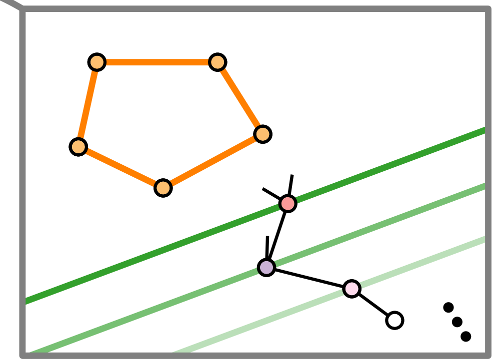

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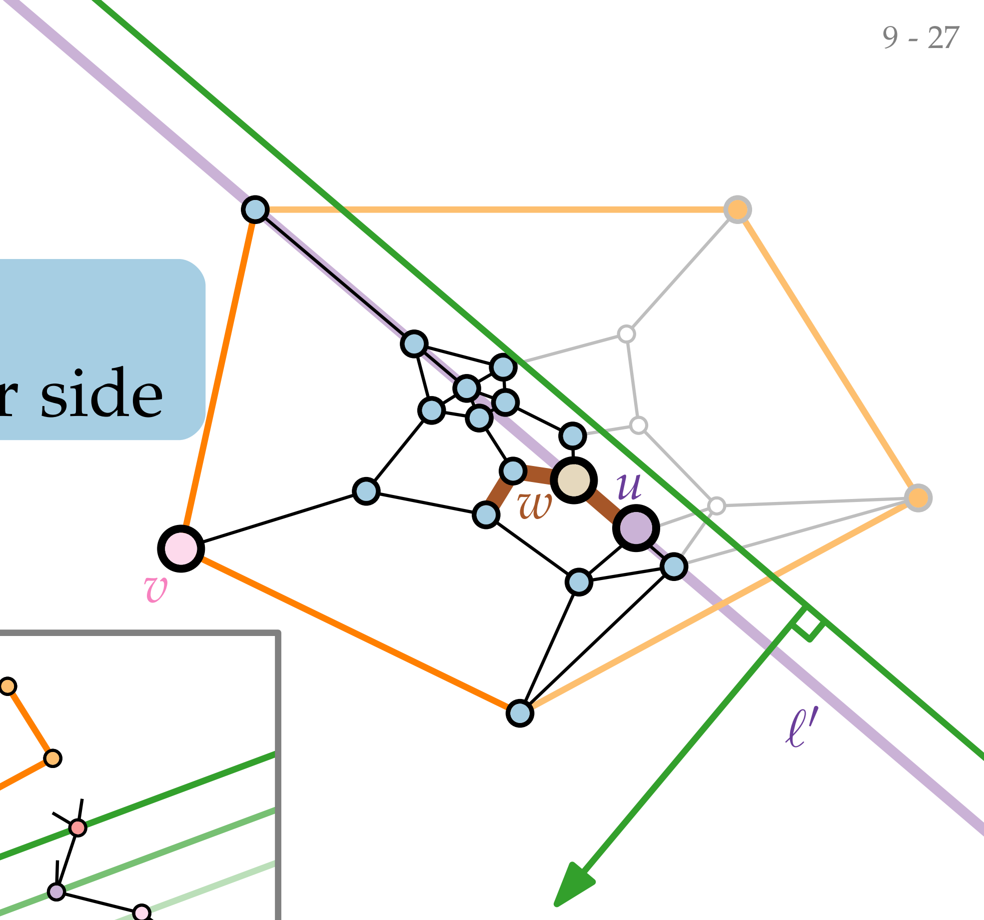

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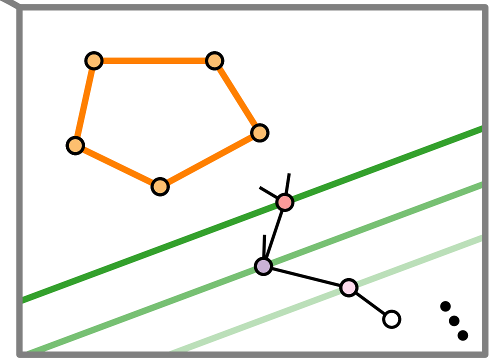

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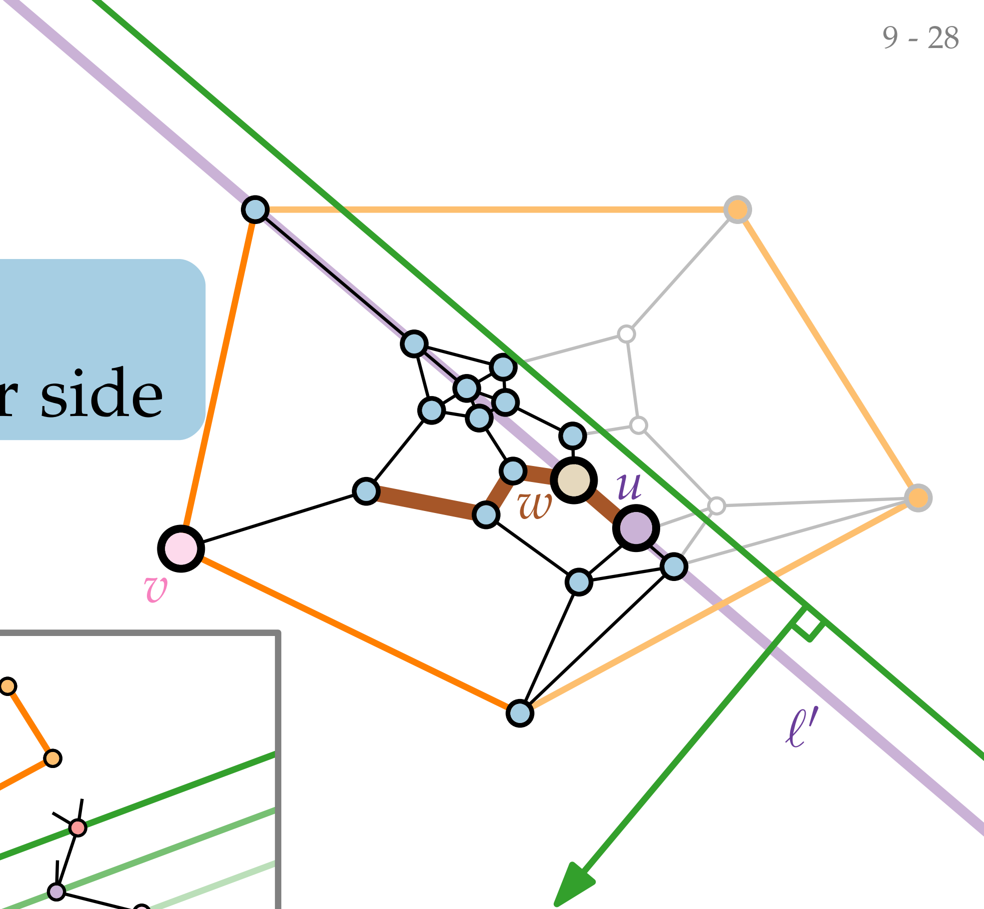

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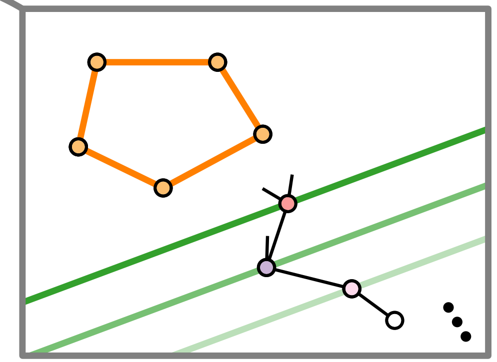

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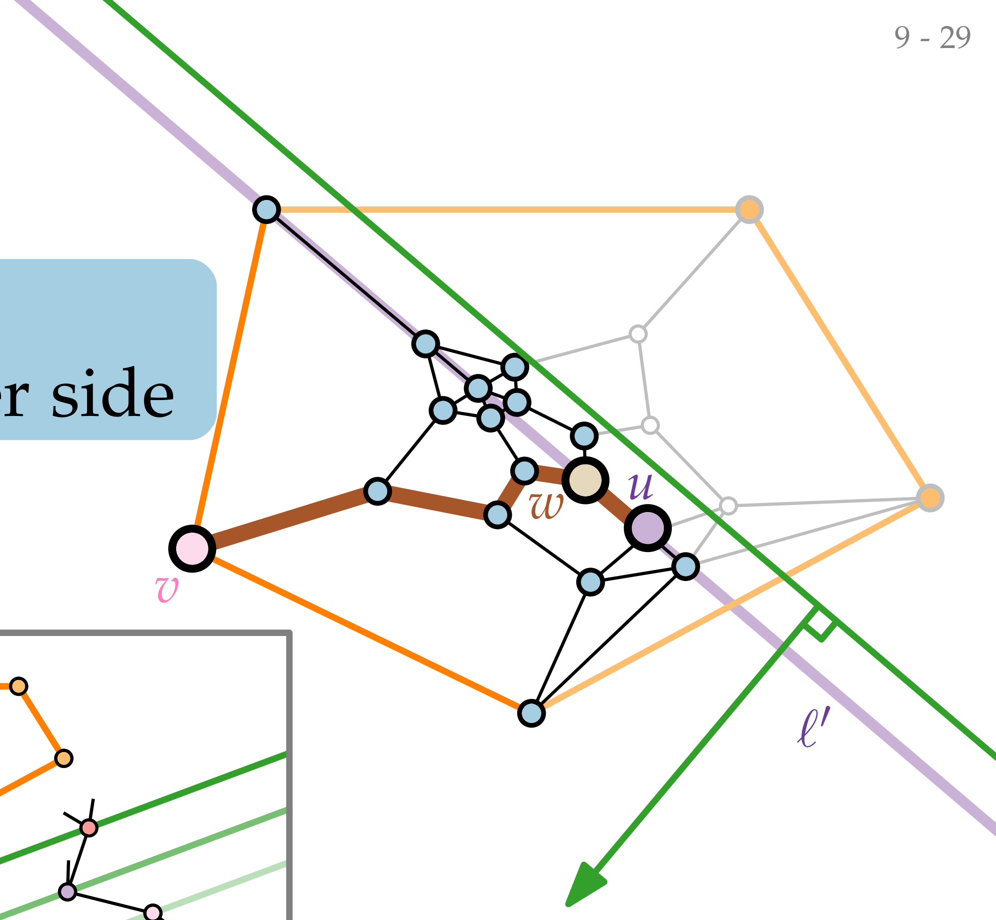
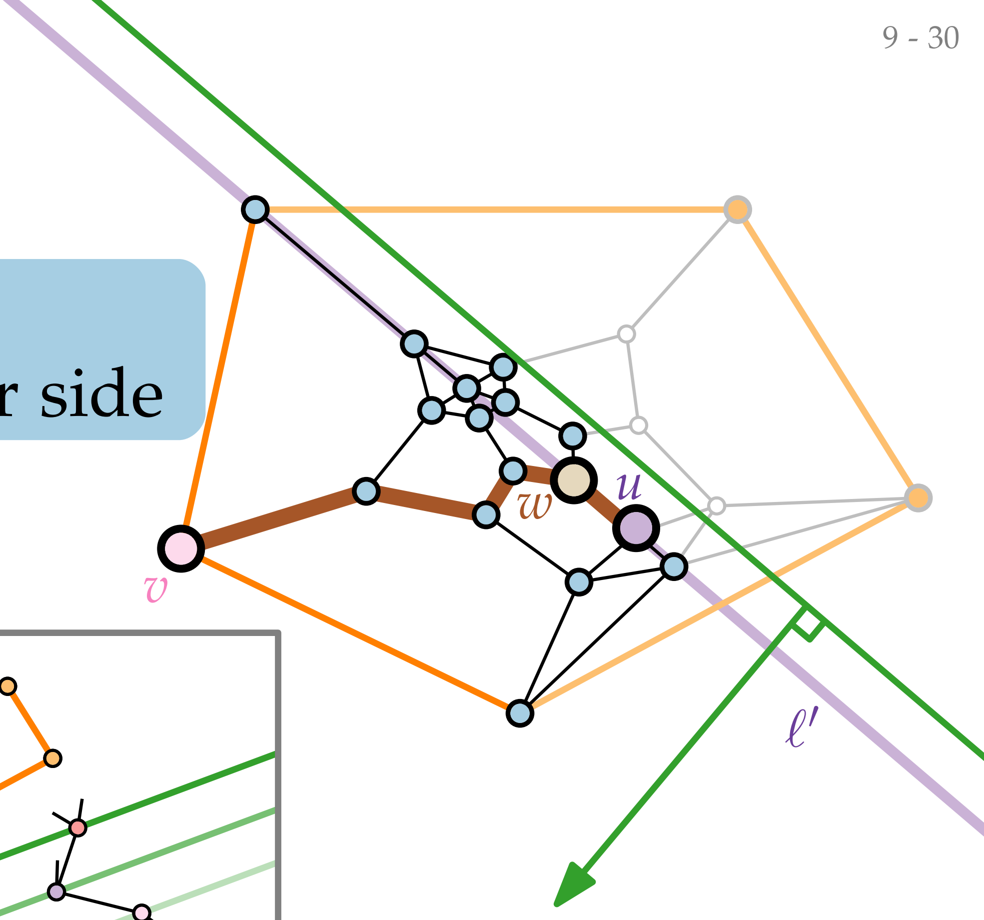


Properties of Tutte Drawings



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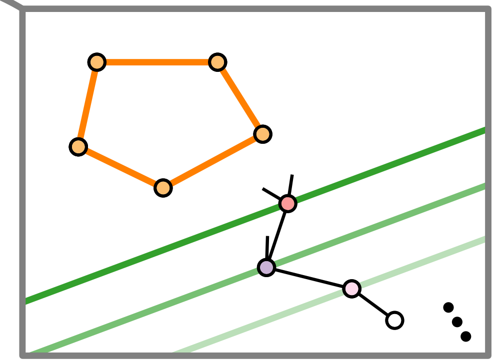

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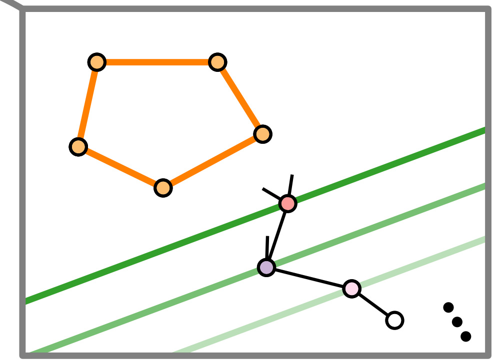

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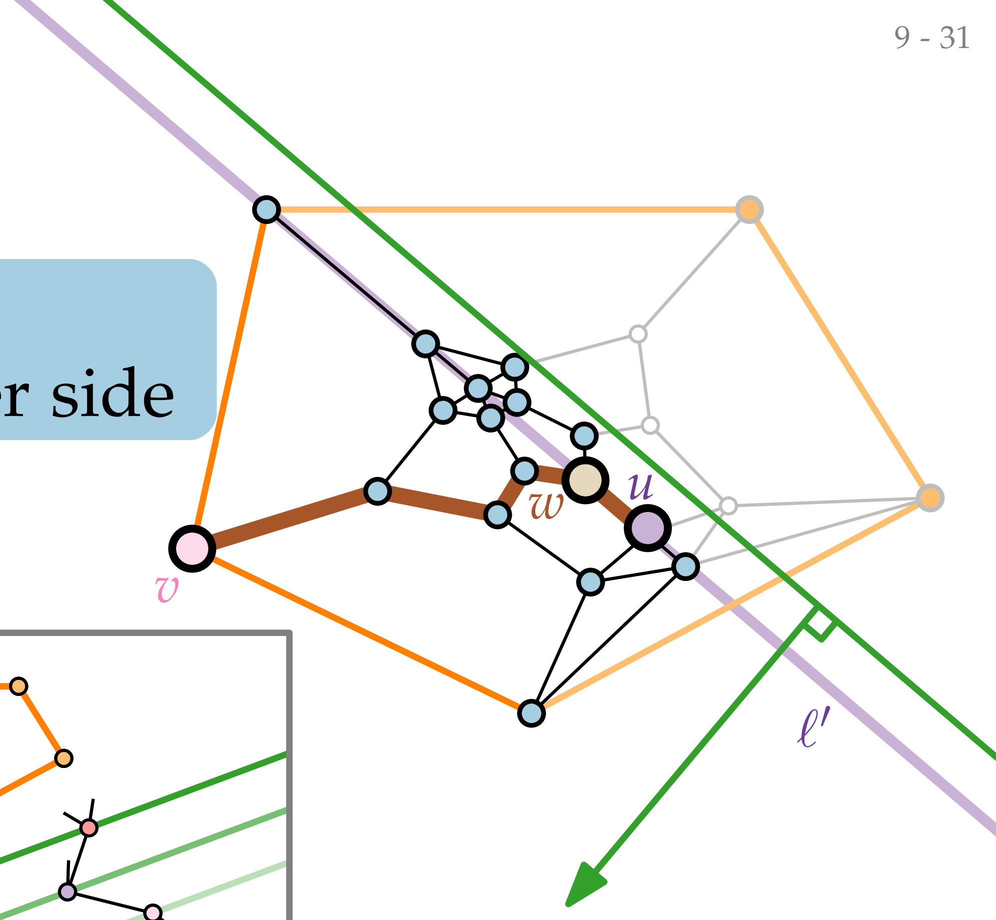

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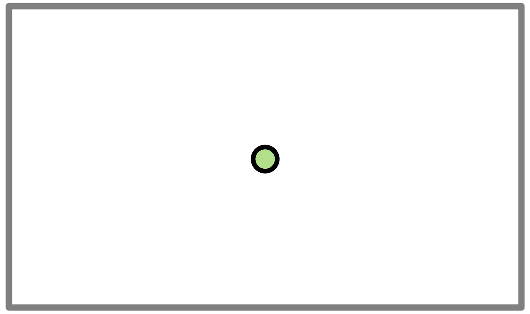

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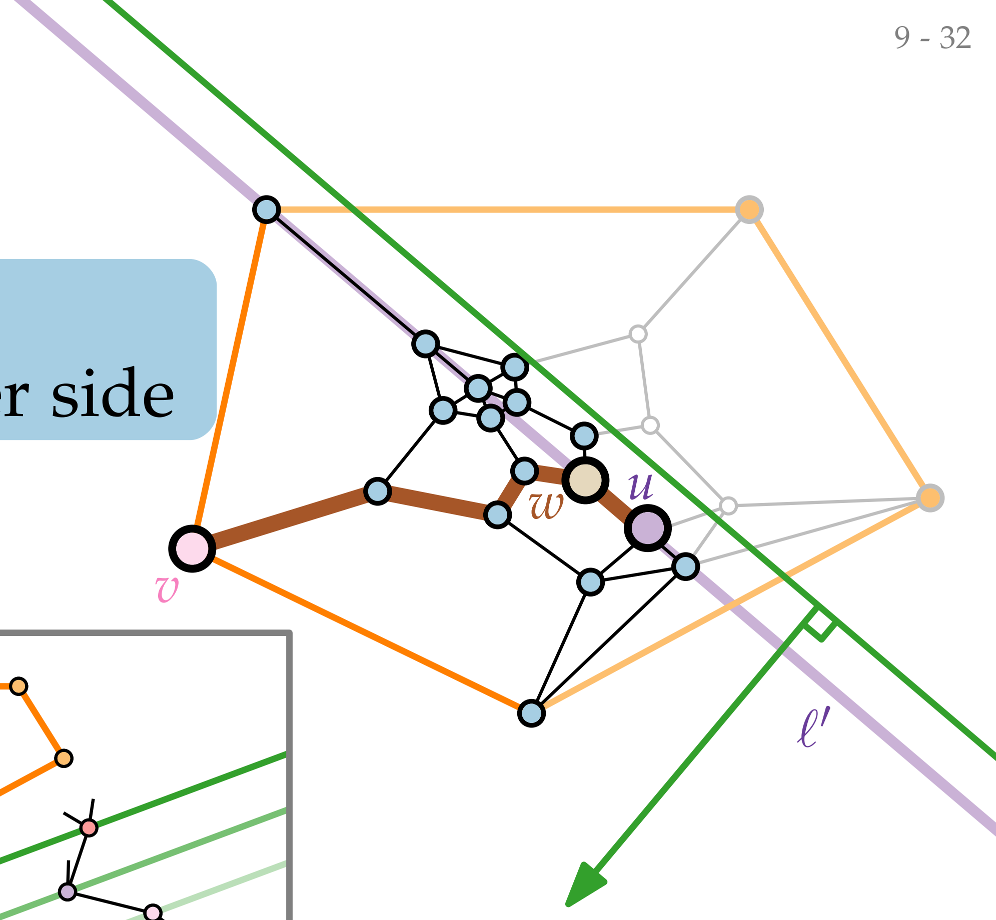

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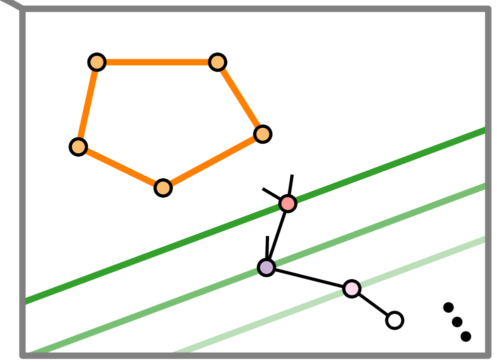

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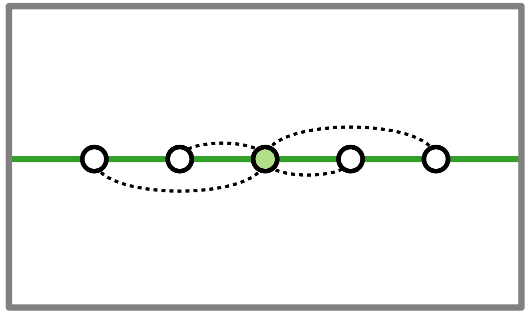

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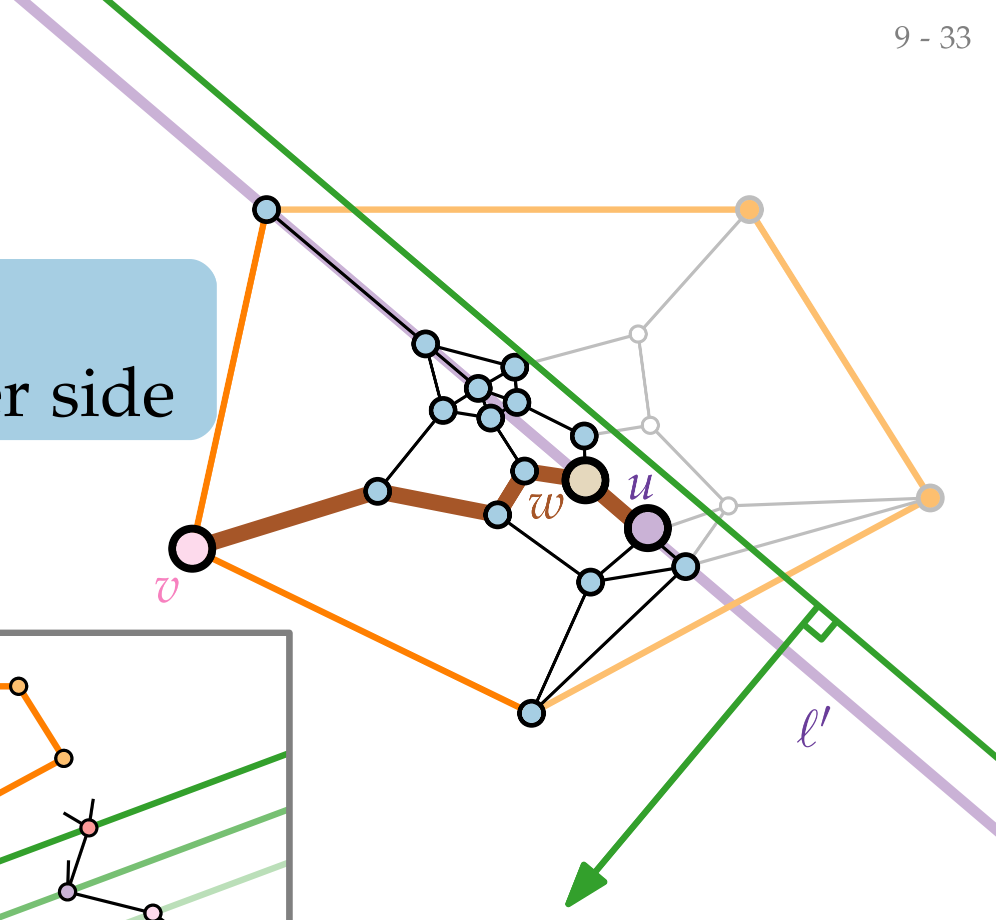

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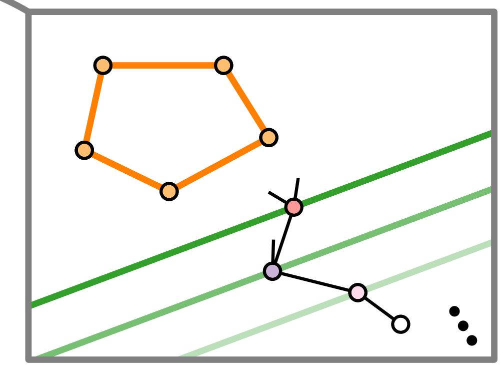

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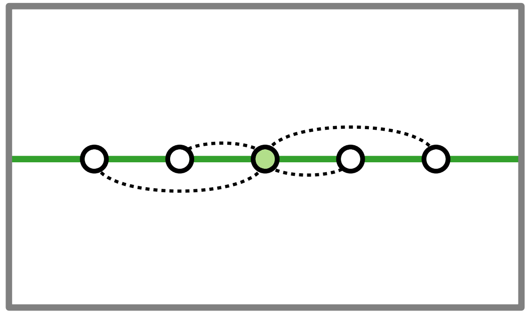

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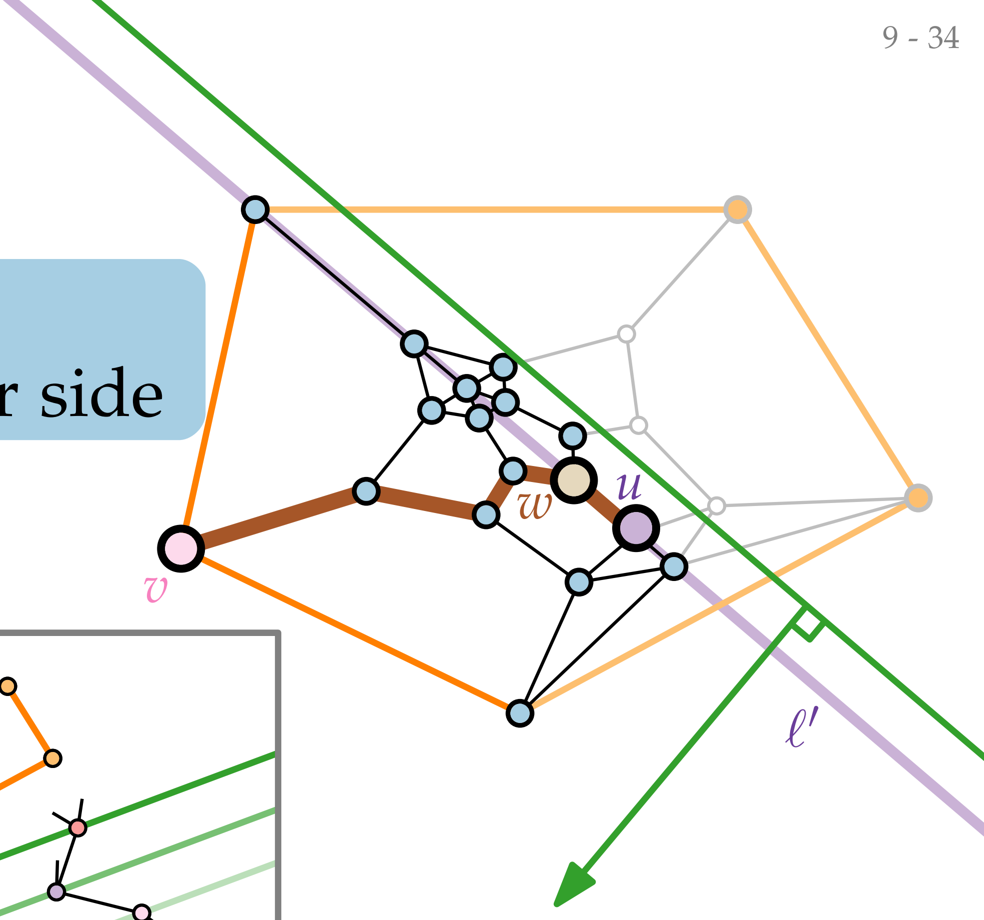

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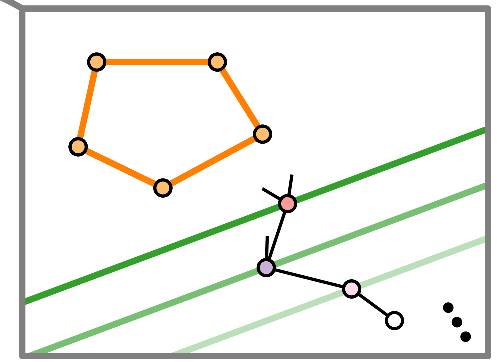

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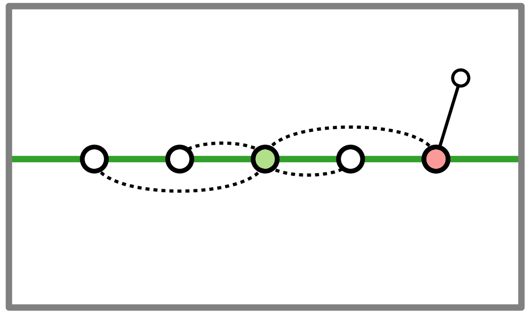

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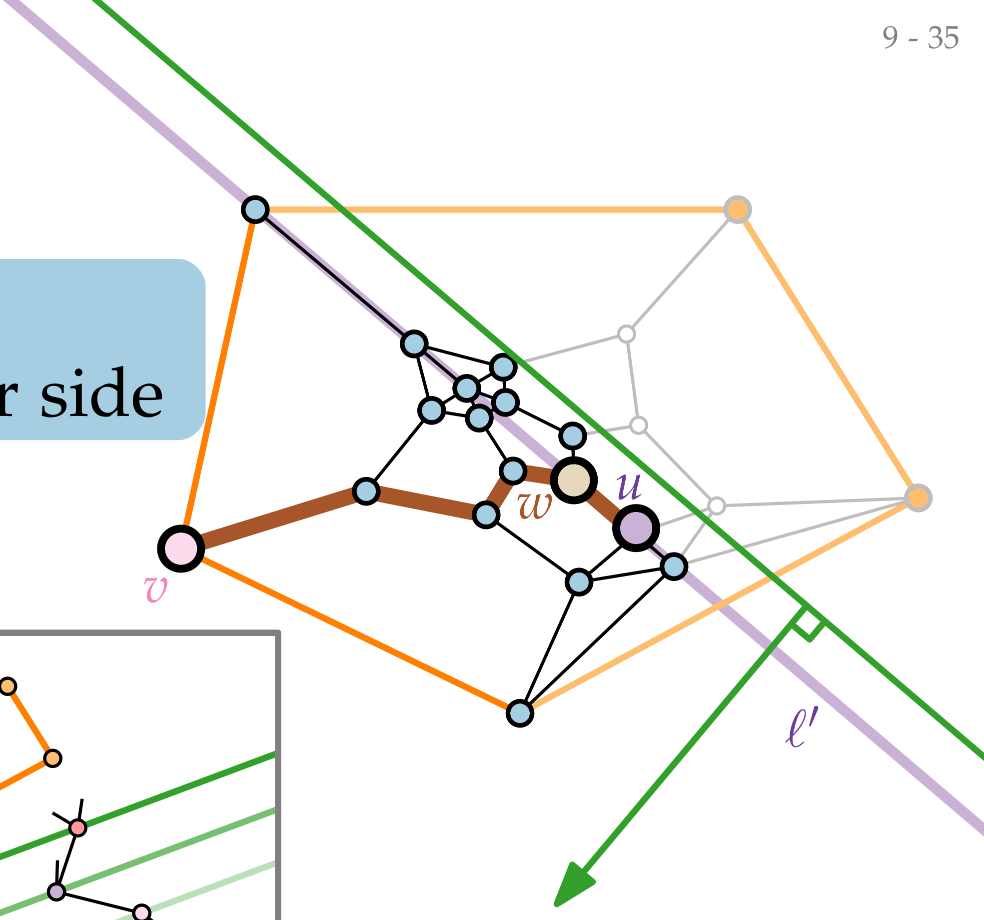

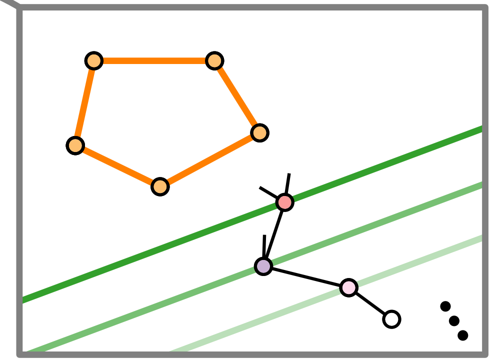

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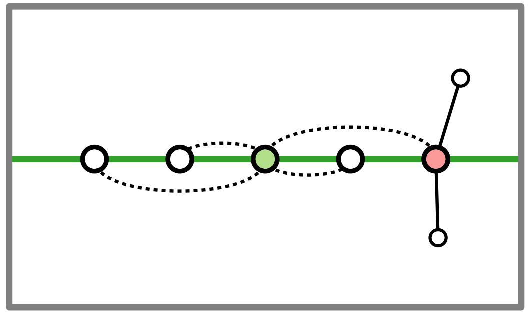

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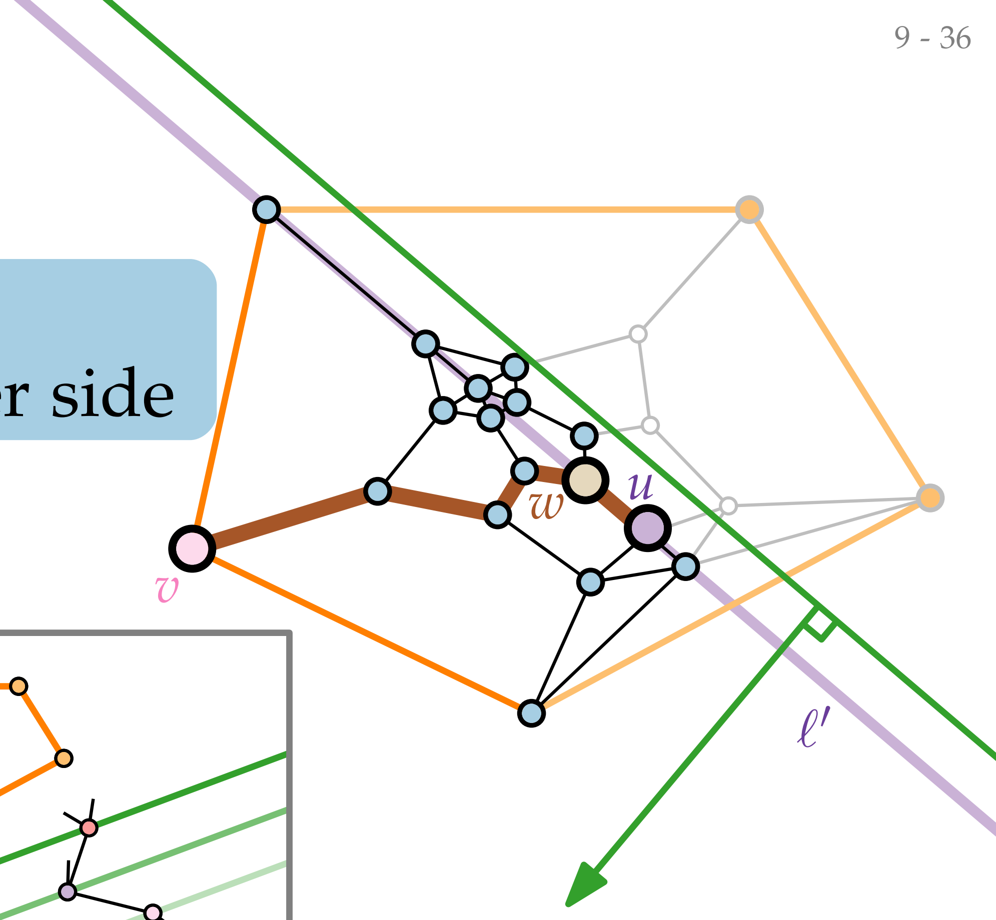

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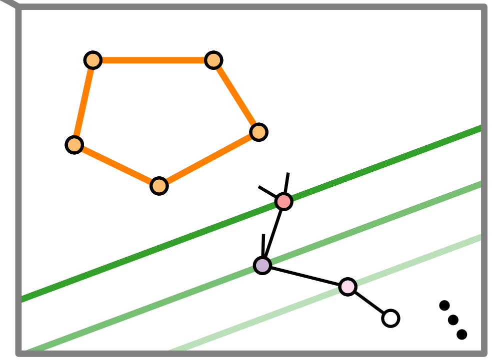

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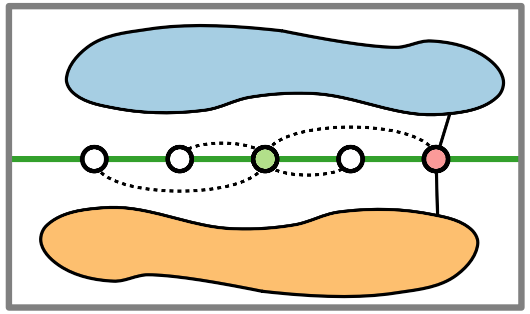

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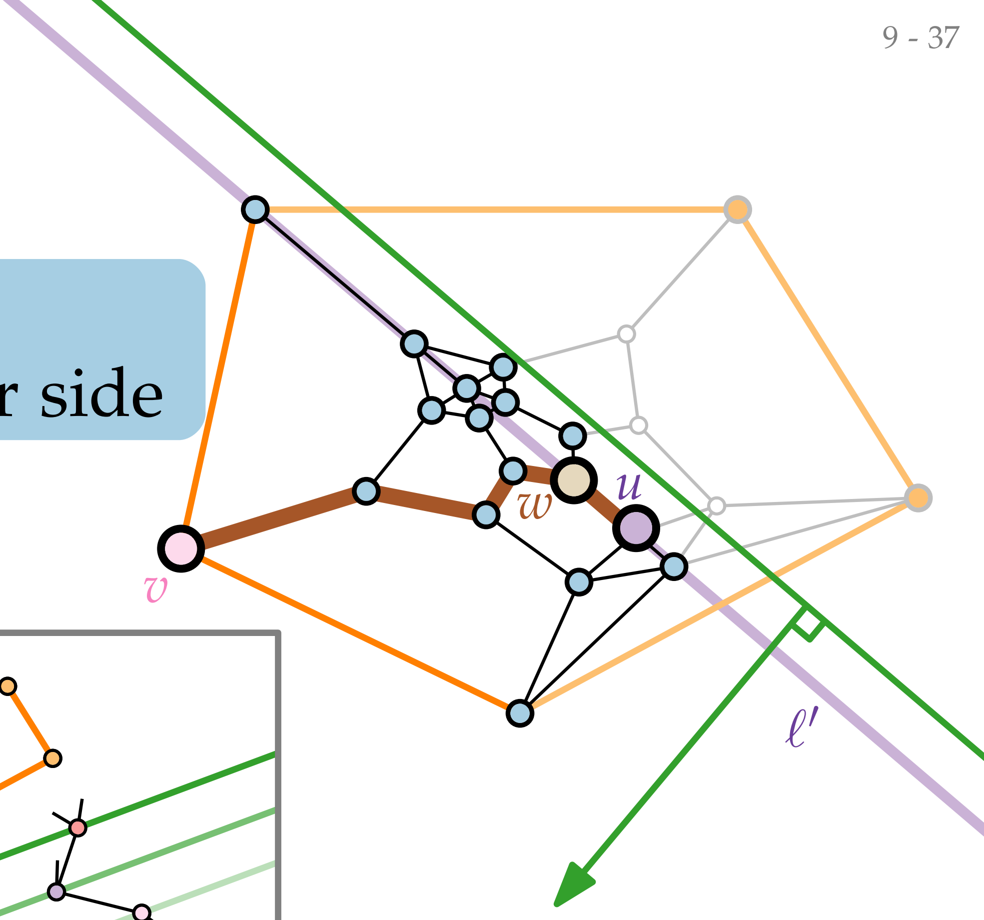

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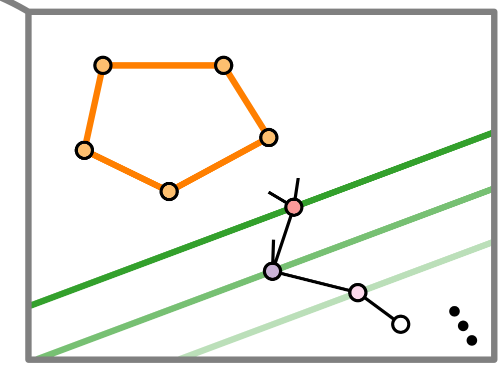

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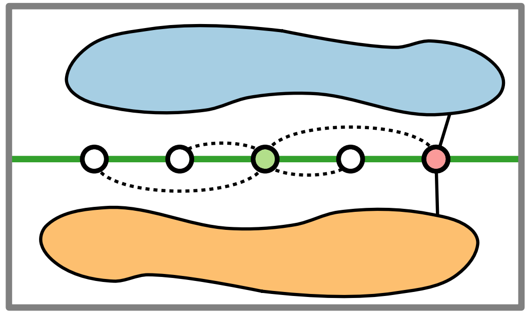

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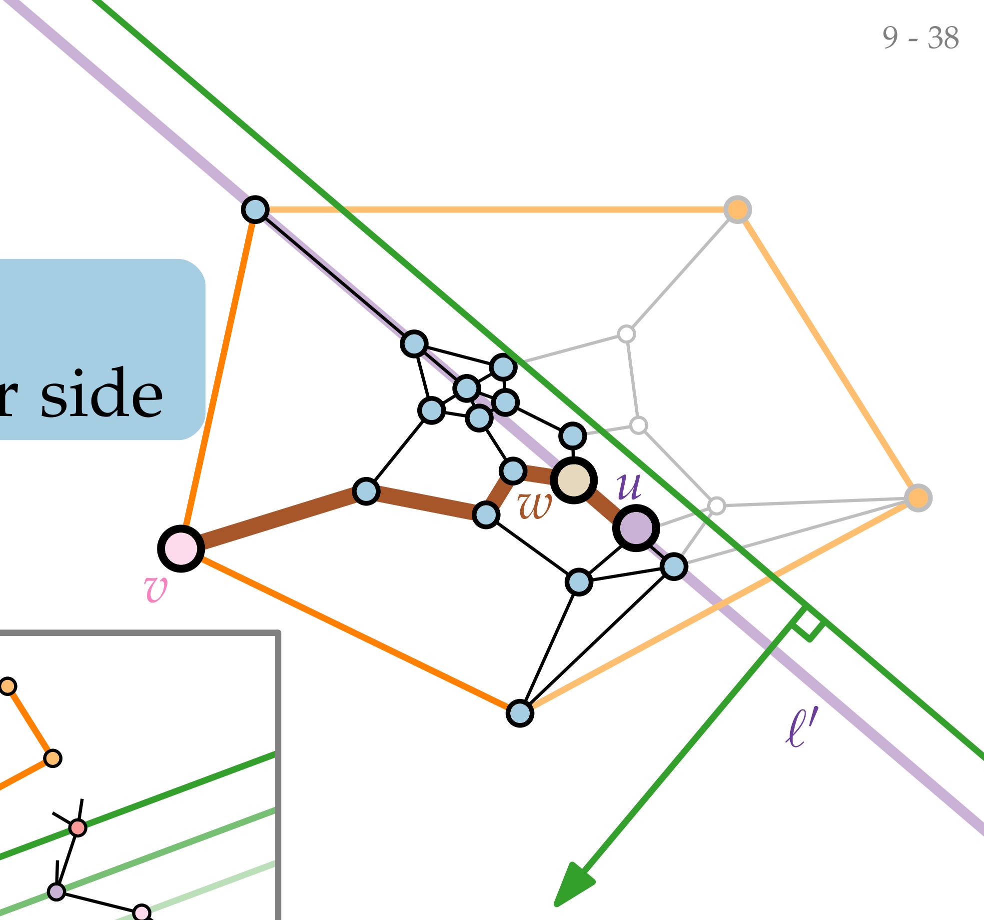

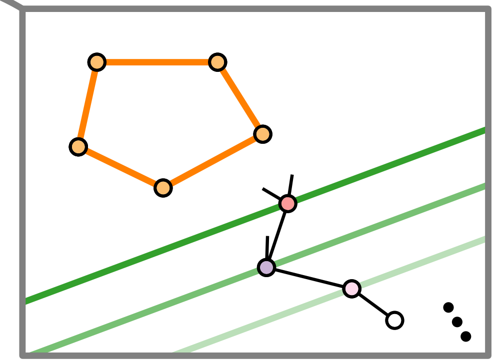

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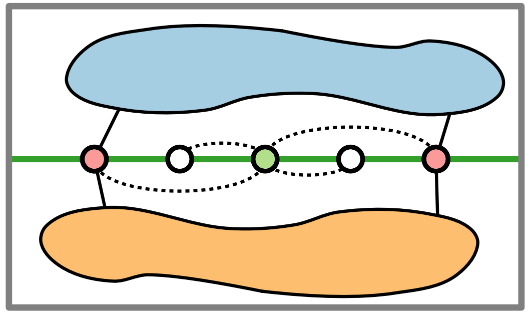

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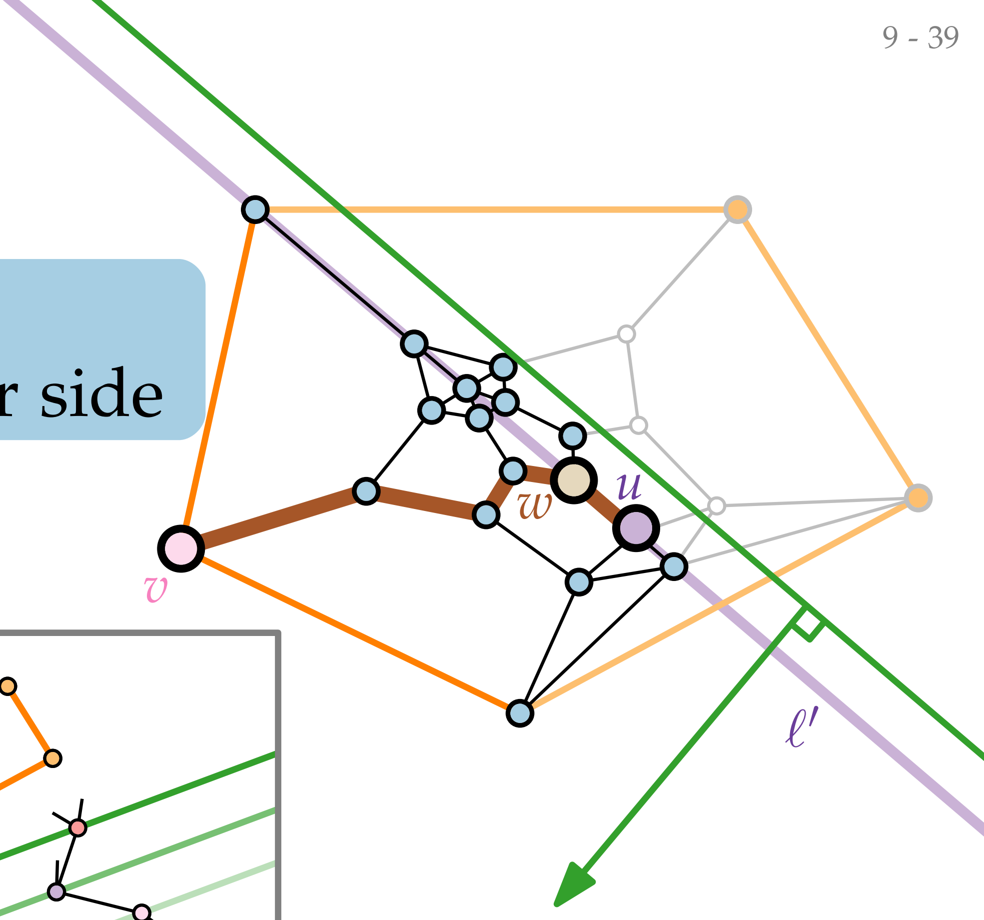

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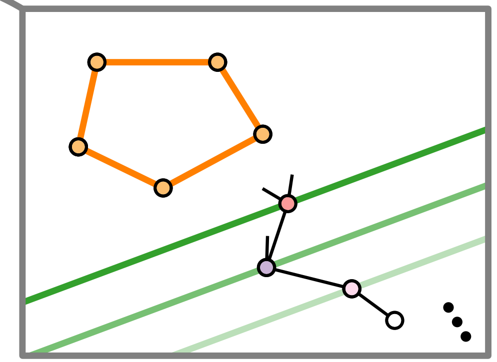

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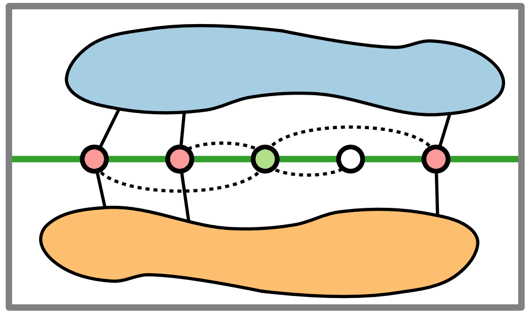

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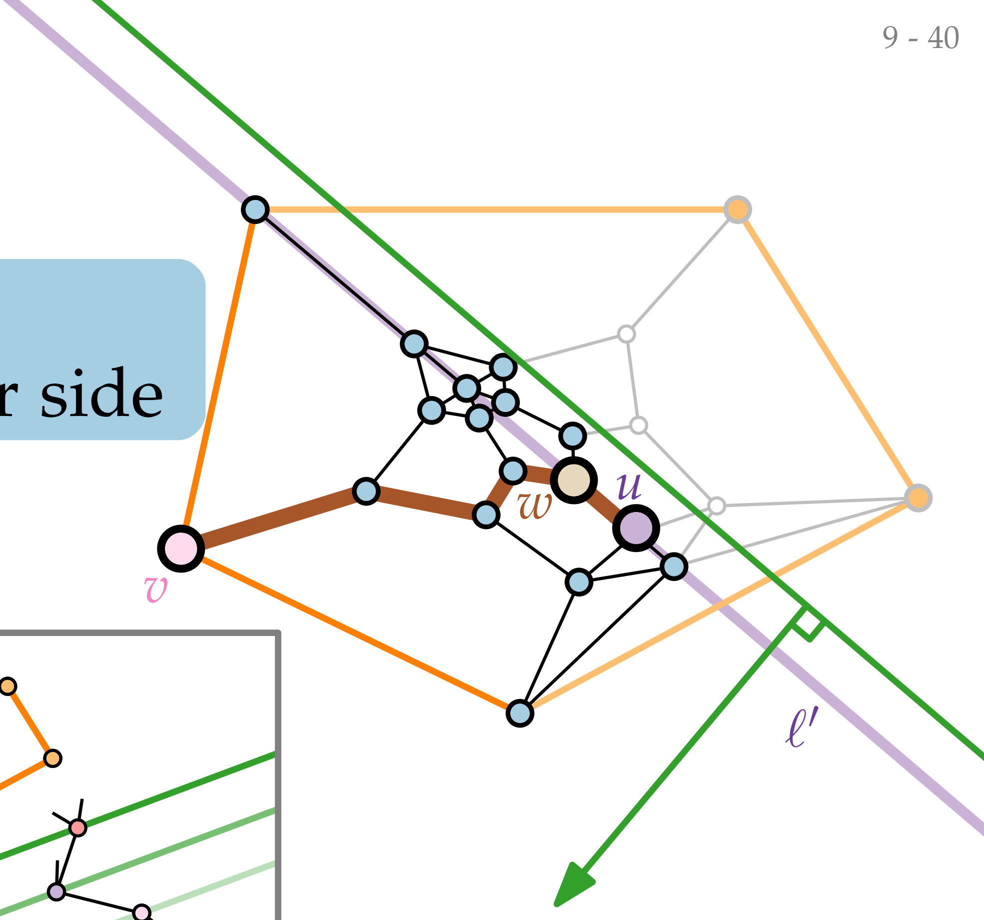

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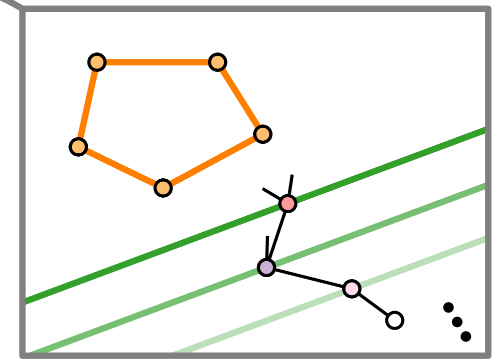

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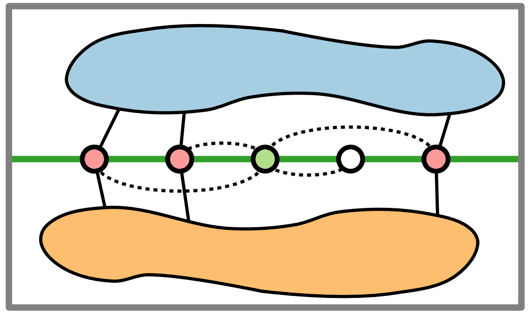

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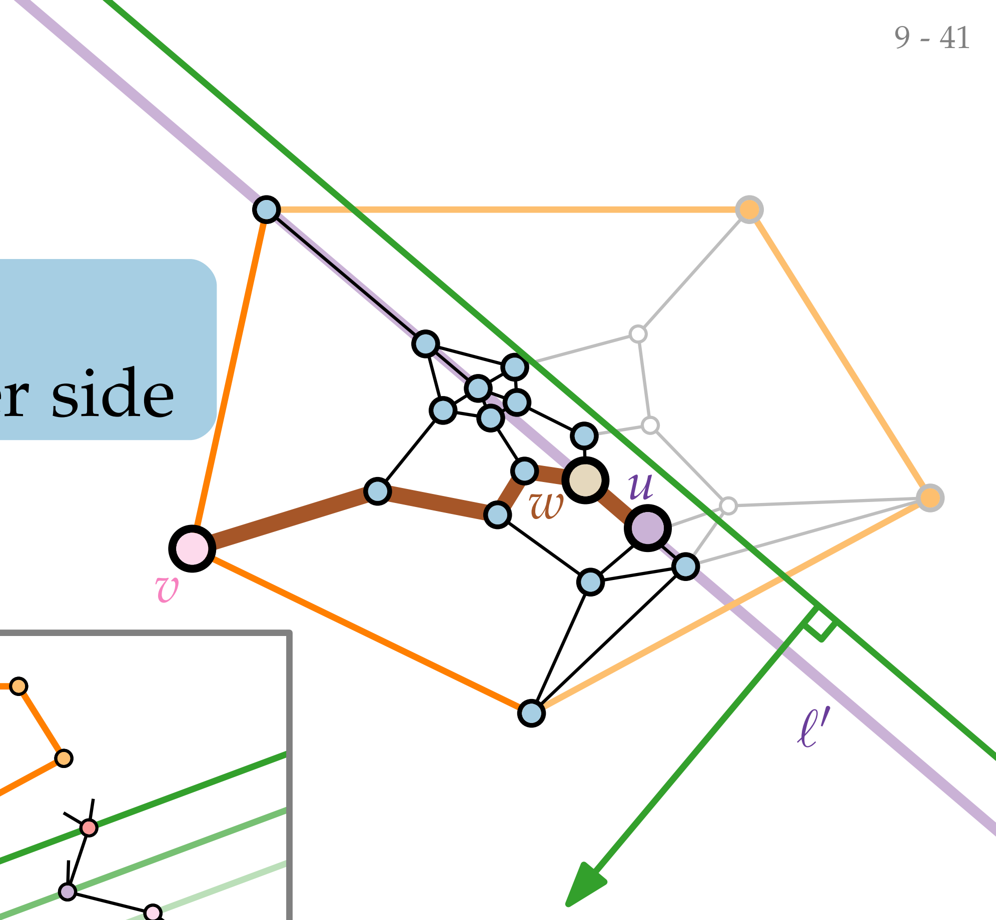

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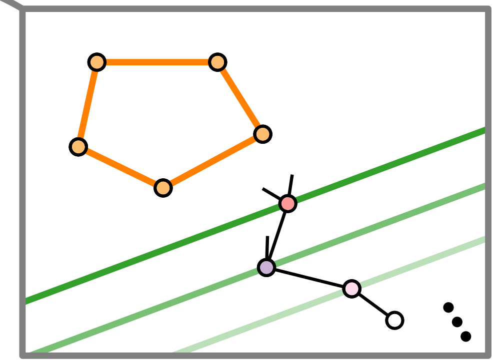

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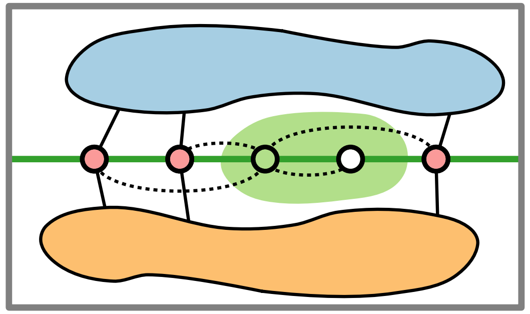

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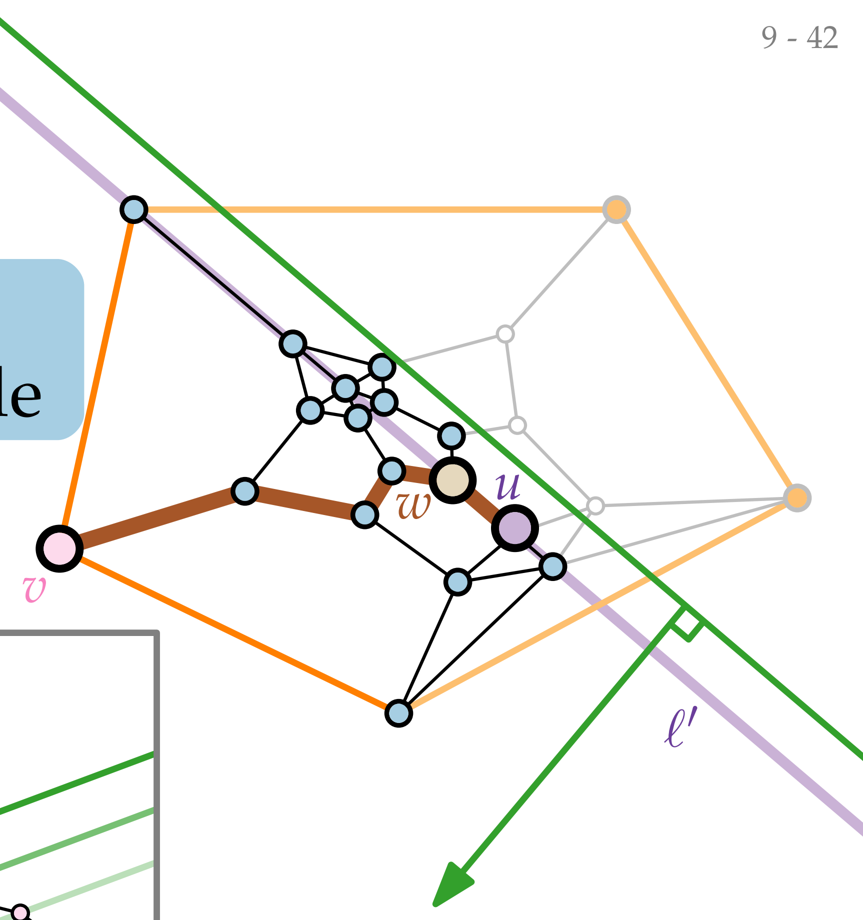

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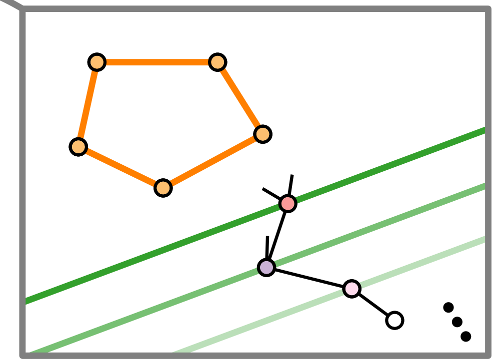

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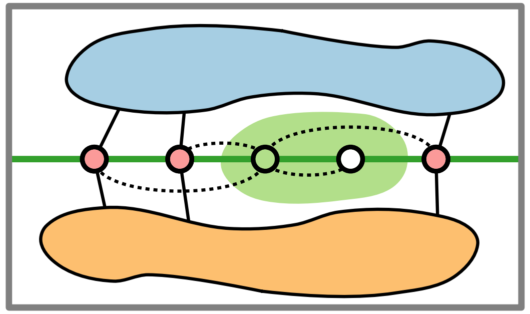

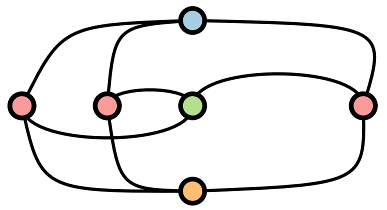

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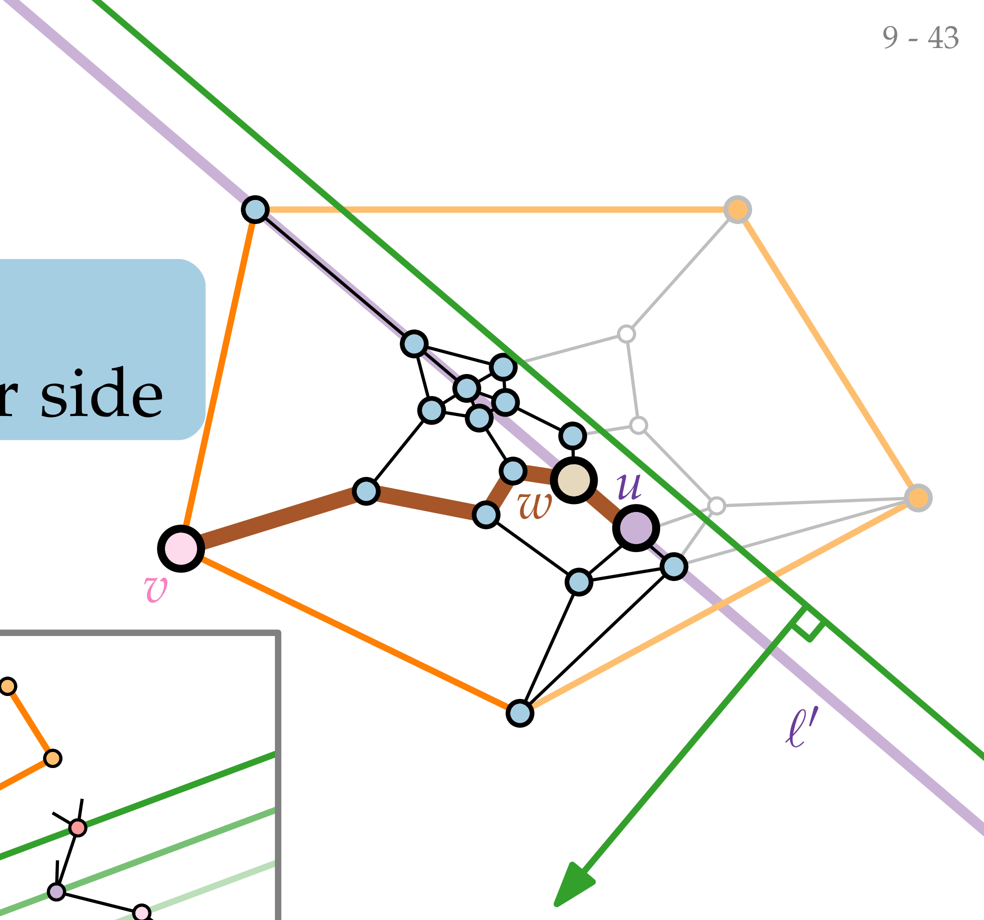

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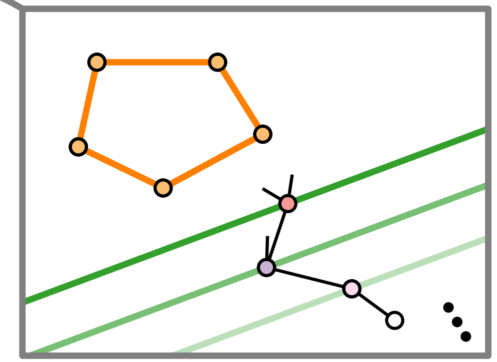

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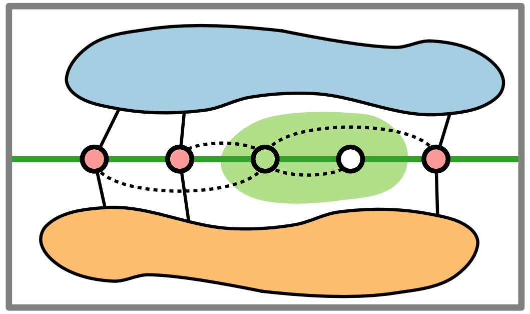

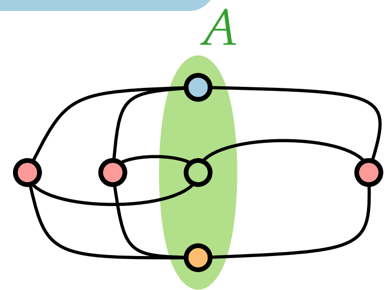

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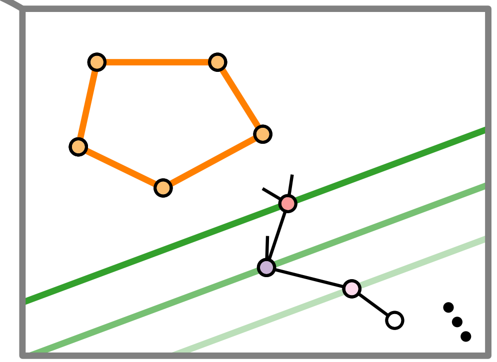

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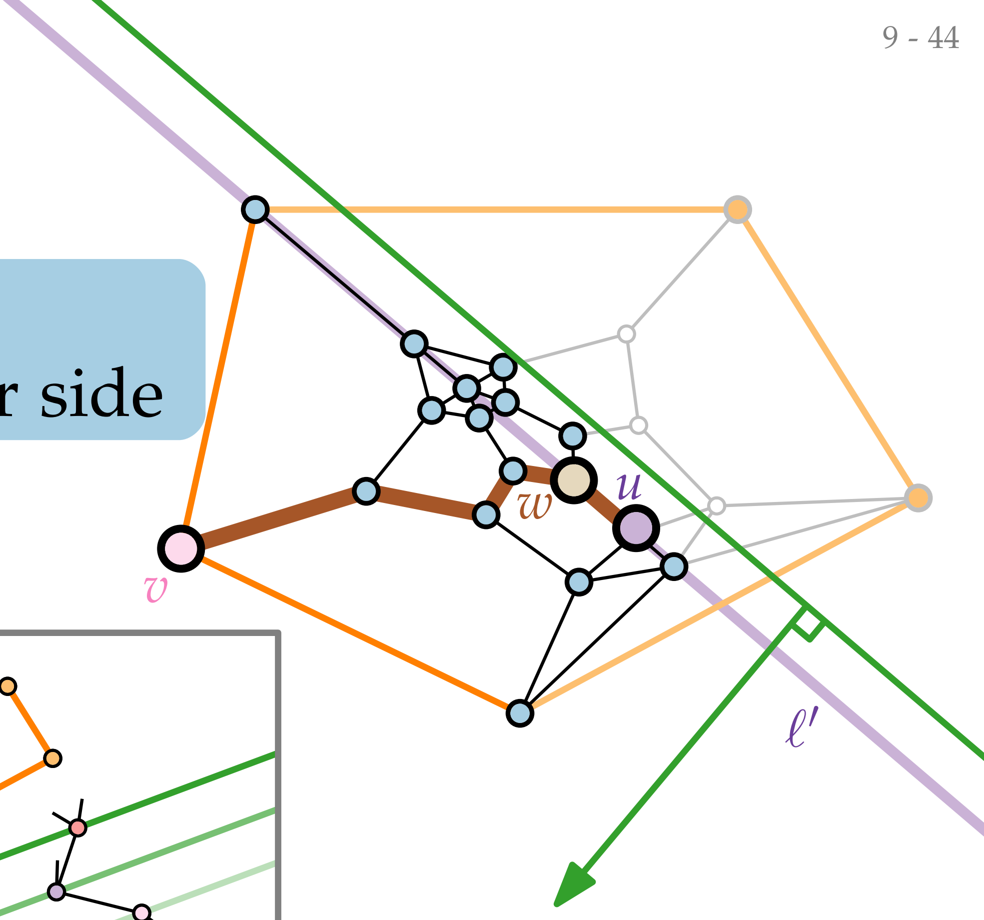
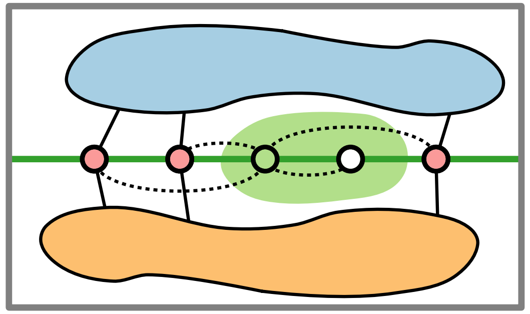
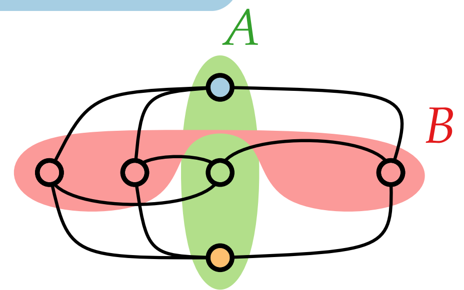


v furthest away from ℓ
Pick any vertex u, ℓ' parallel to ℓ throught u
G connected, v not on $\ell' \Rightarrow \exists w$ on ℓ' with neighbor further away from ℓ
$\Rightarrow \exists$ path from u to v

Property 4. No vertex is collinear with all of its neighbors.

Not all vertices collinear
G 3-connected
$\Rightarrow K_{3,3}$ minor





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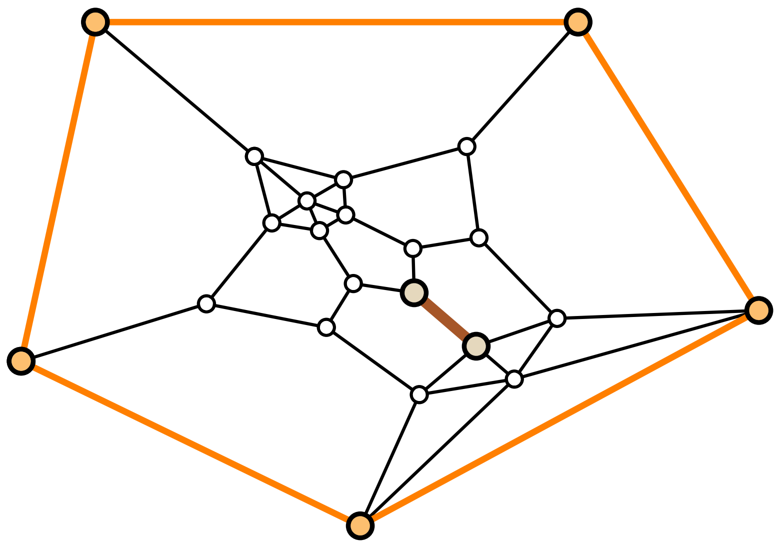

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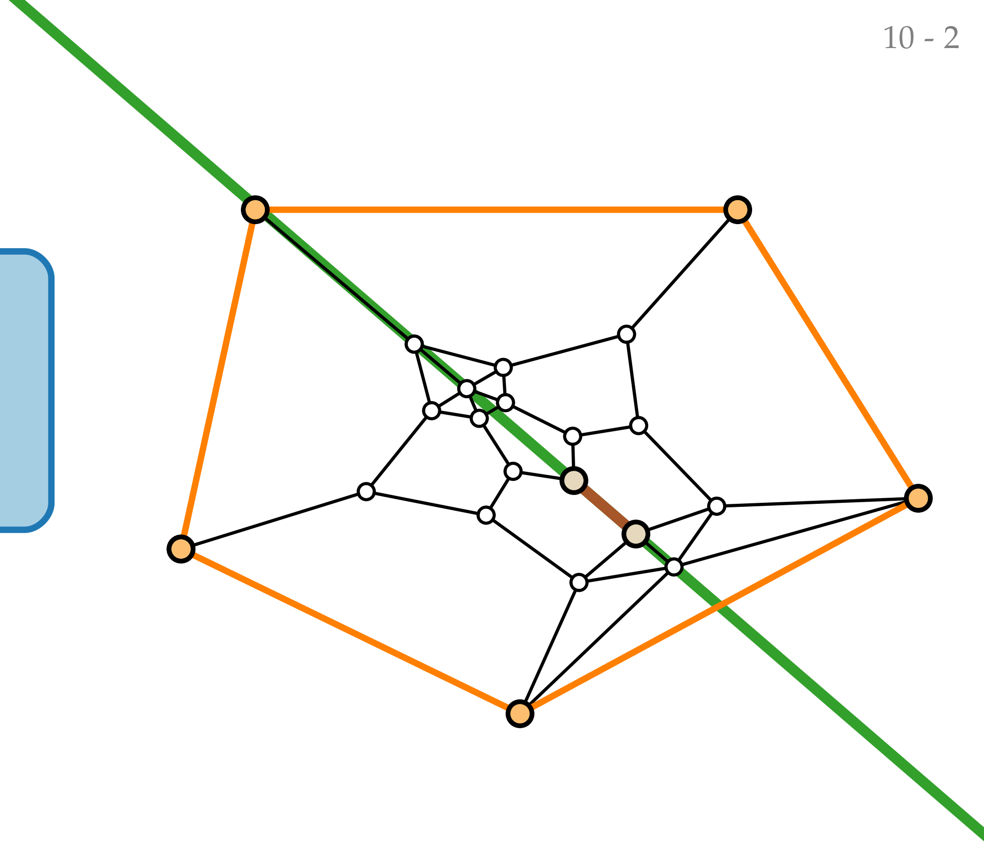

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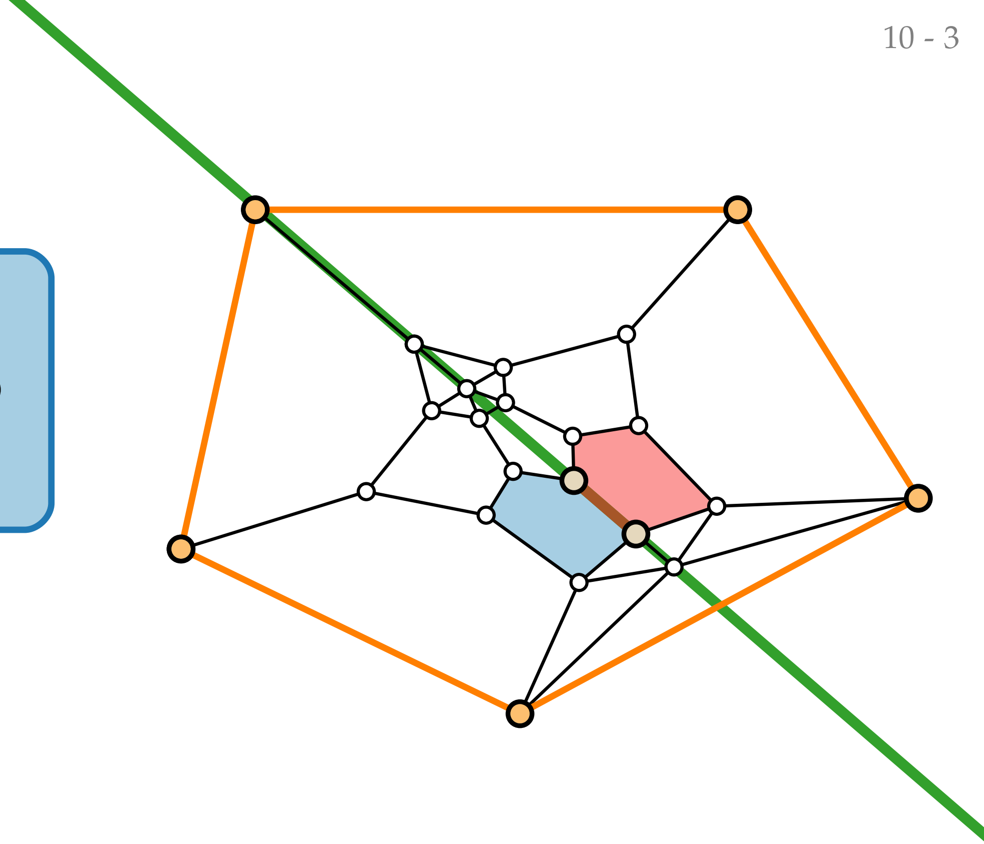

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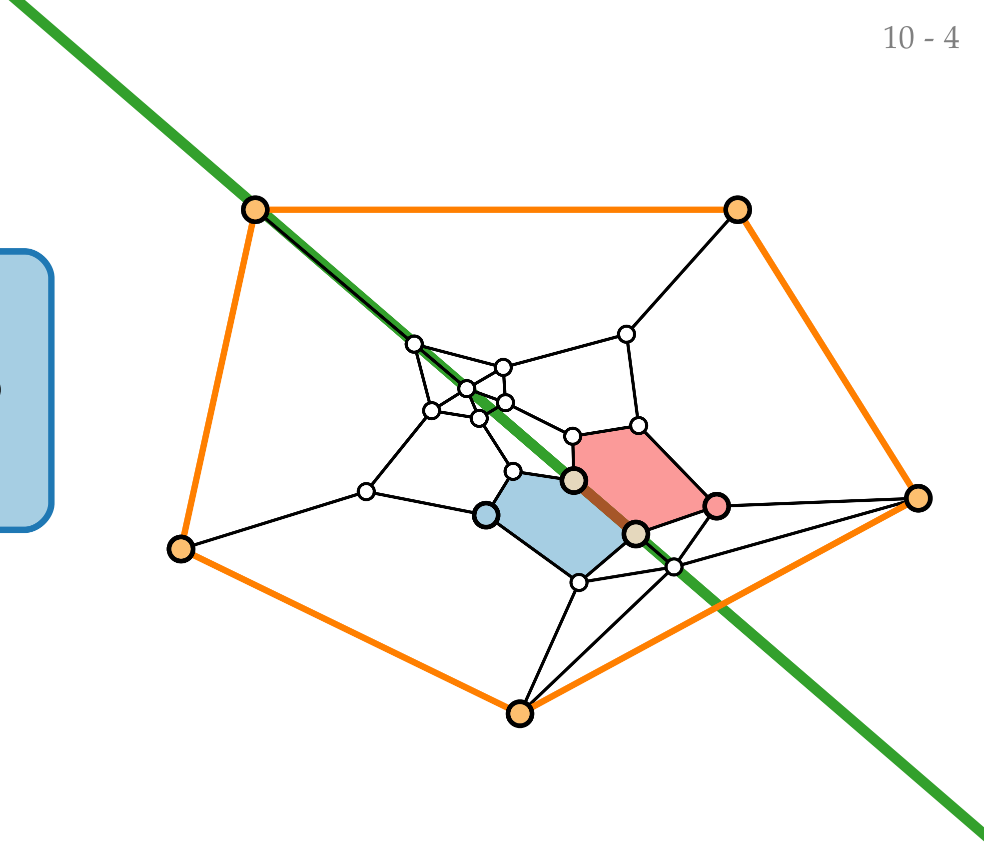

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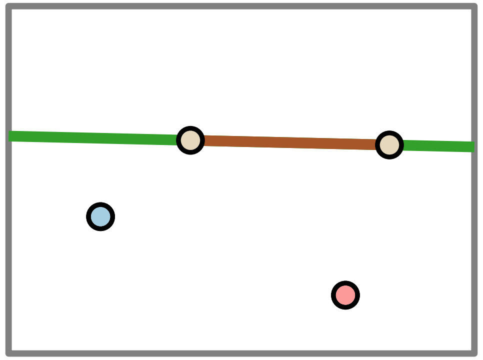

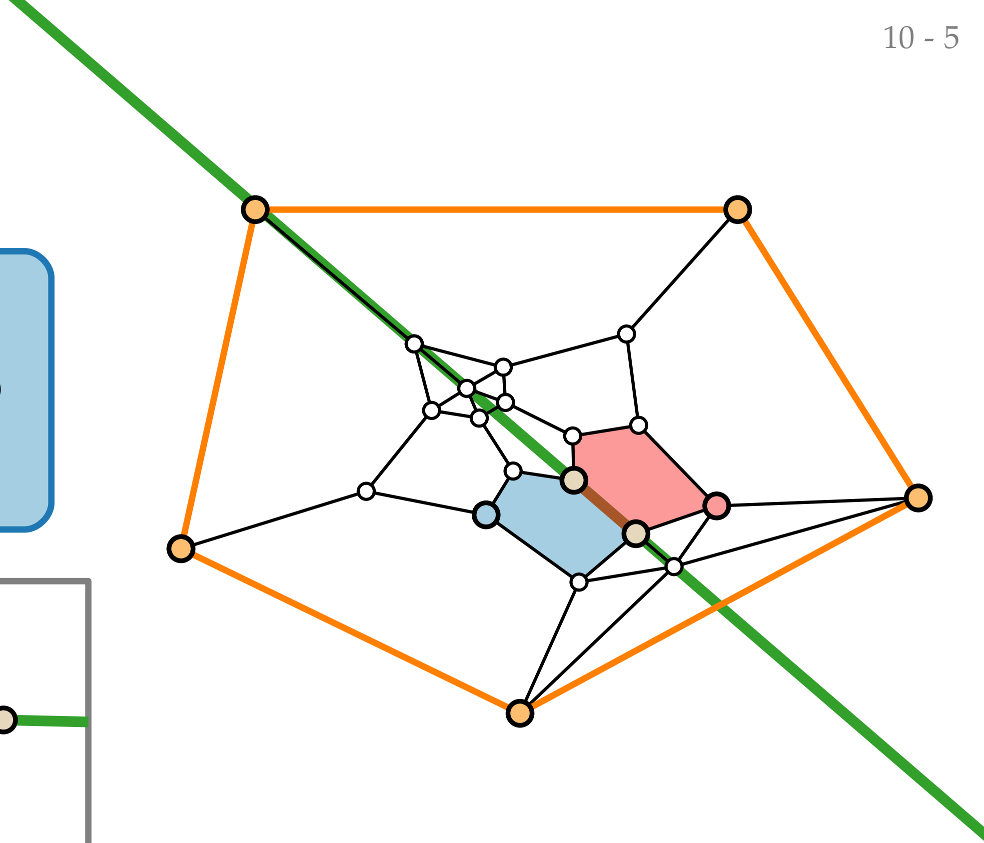

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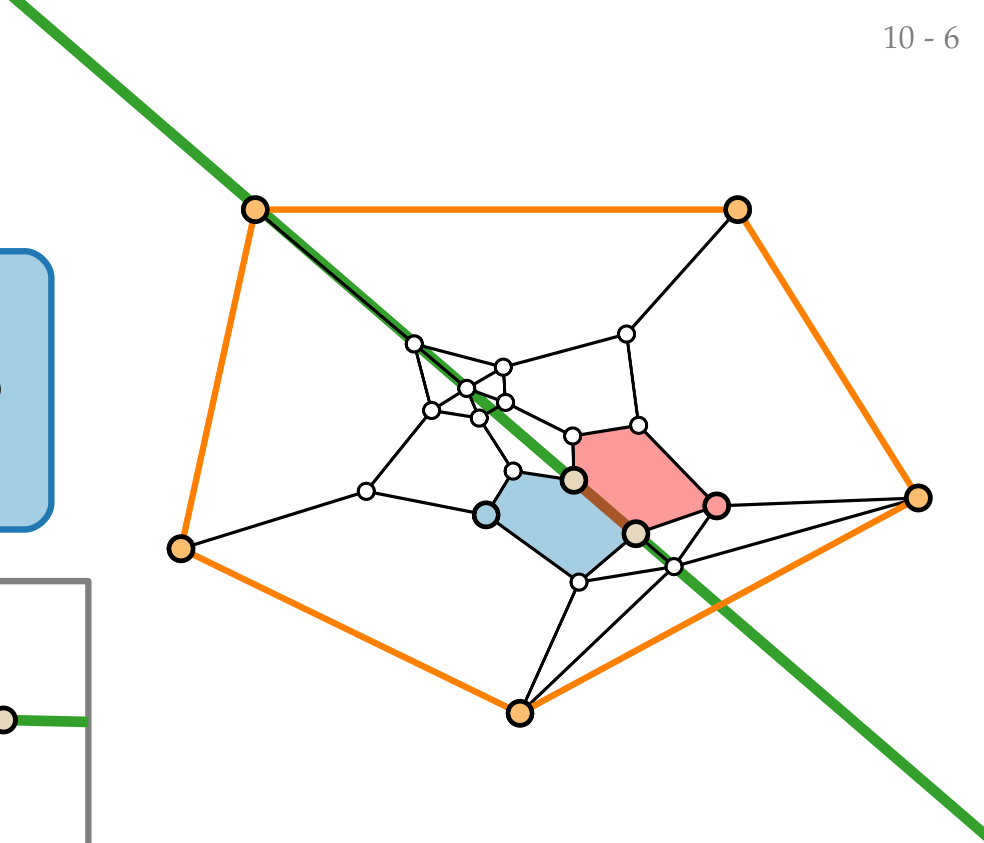

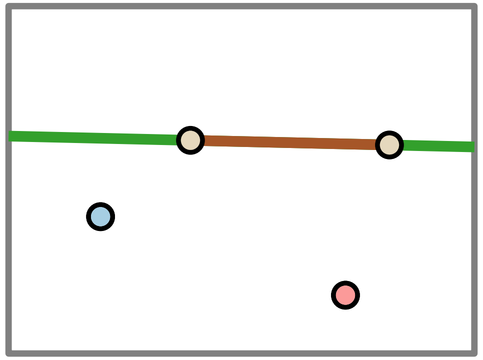

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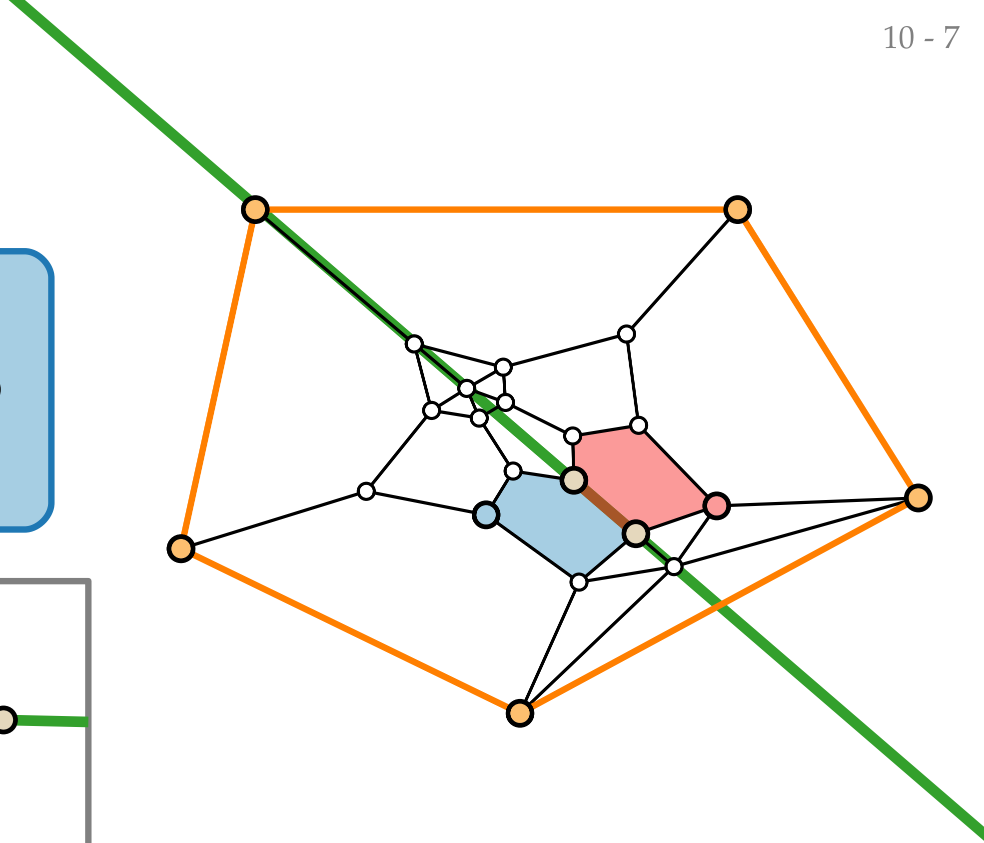

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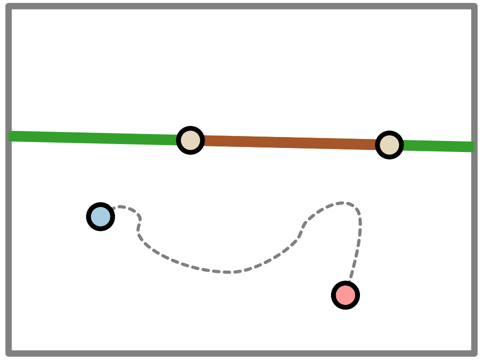

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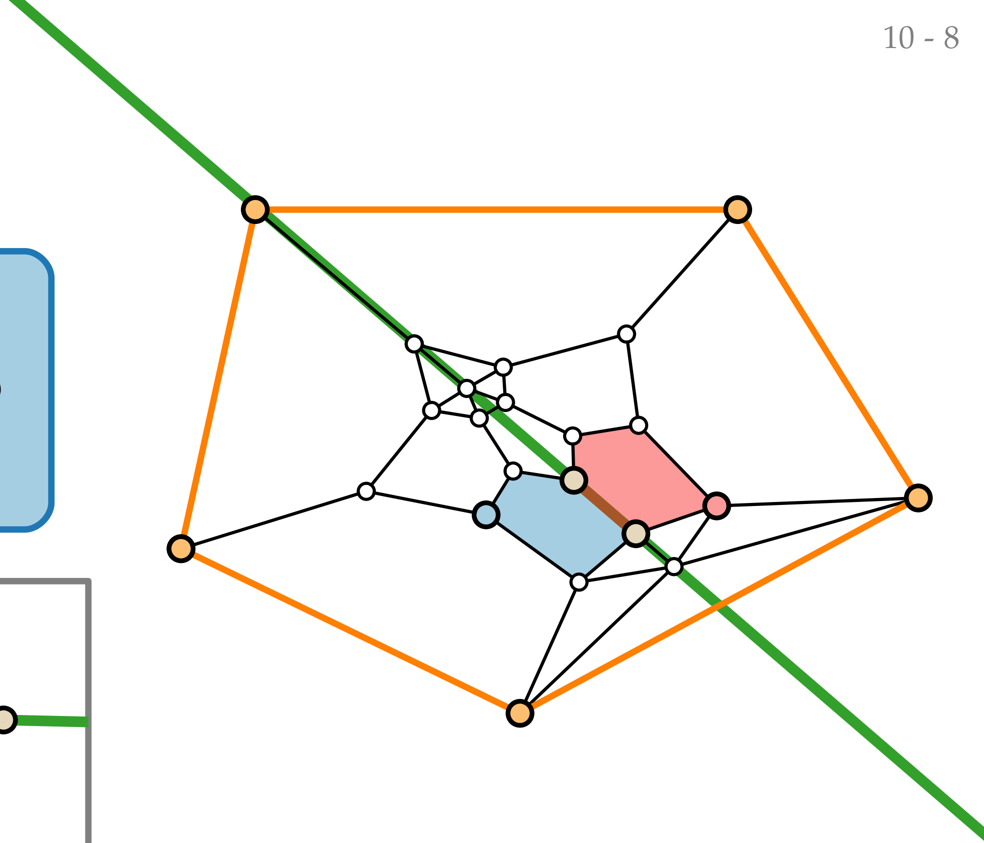

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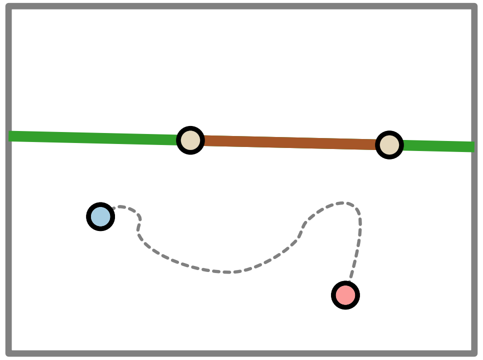

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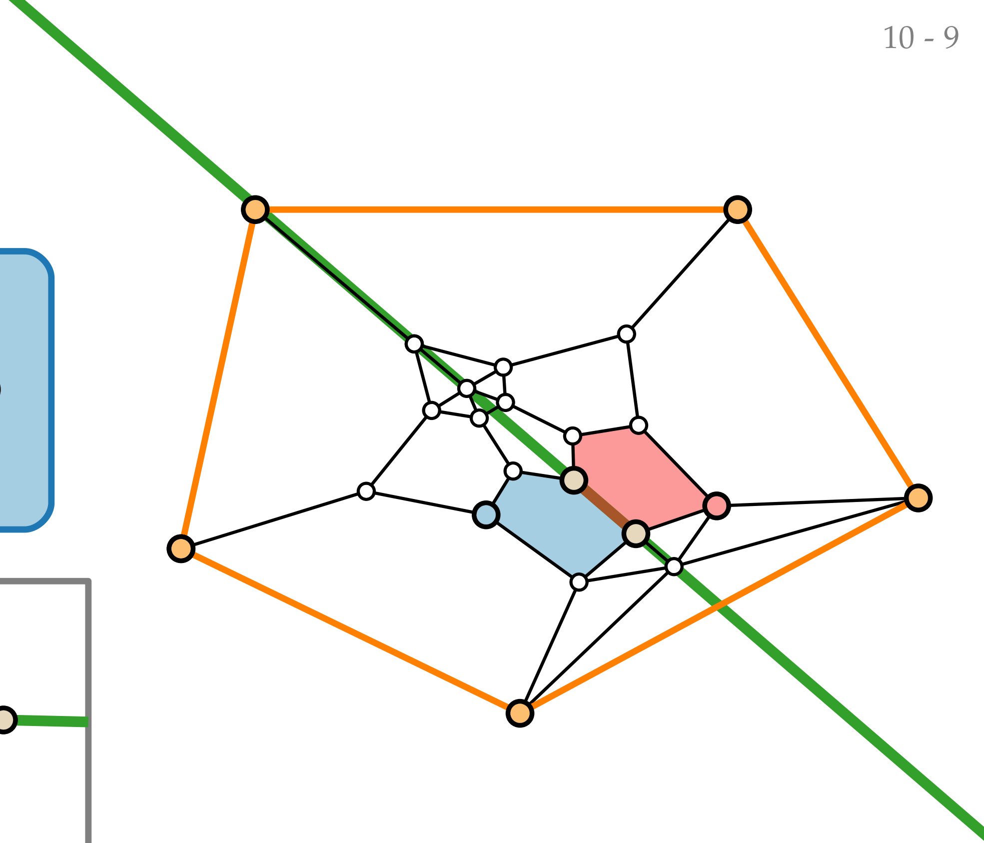

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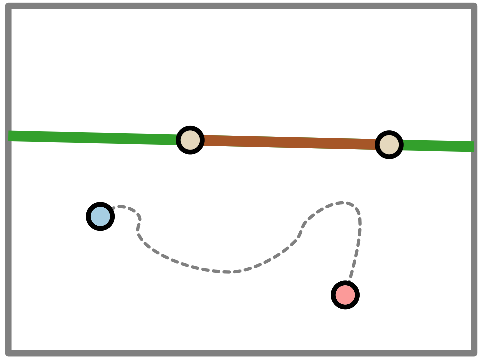

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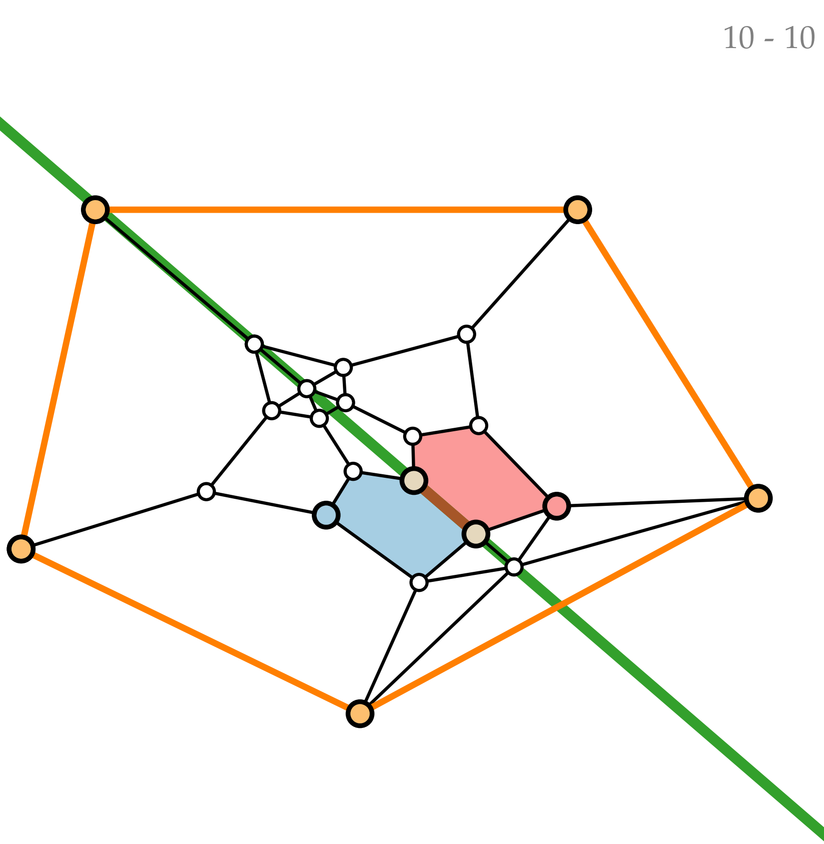

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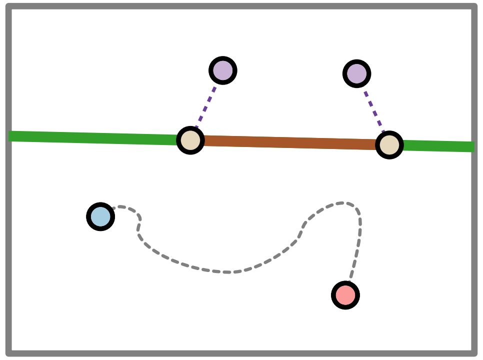

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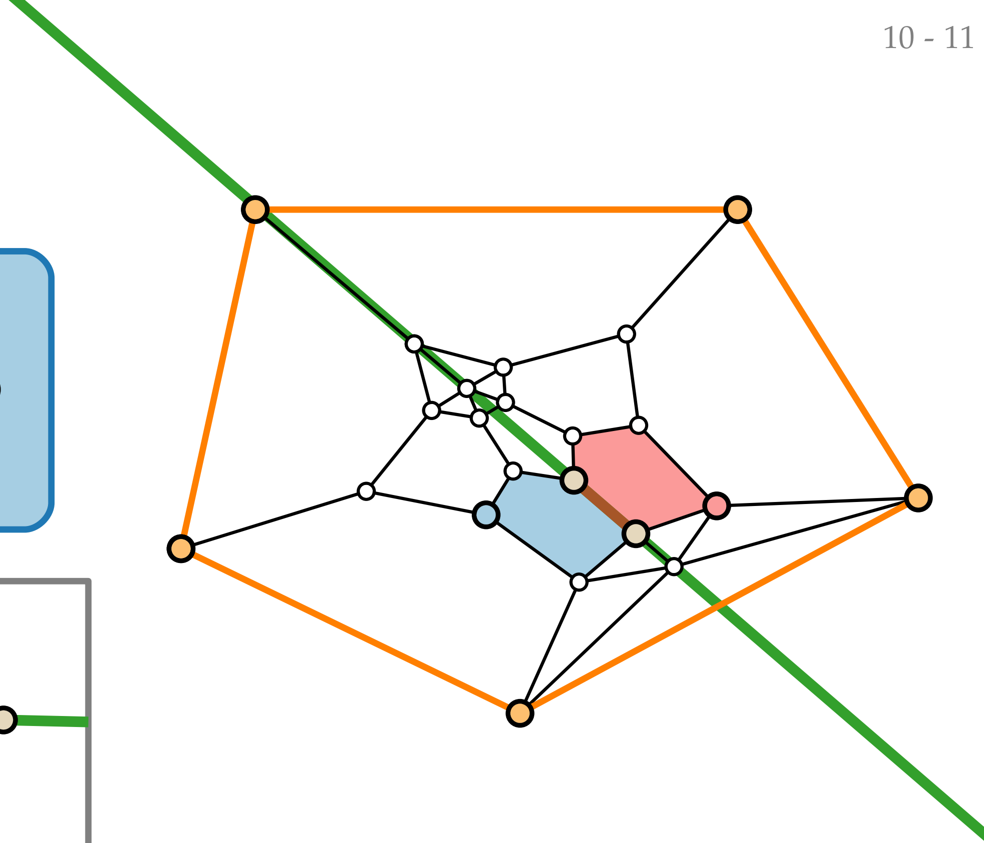

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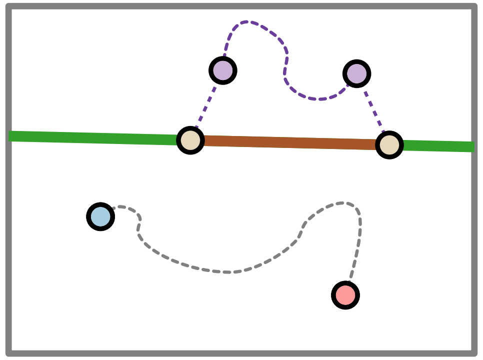

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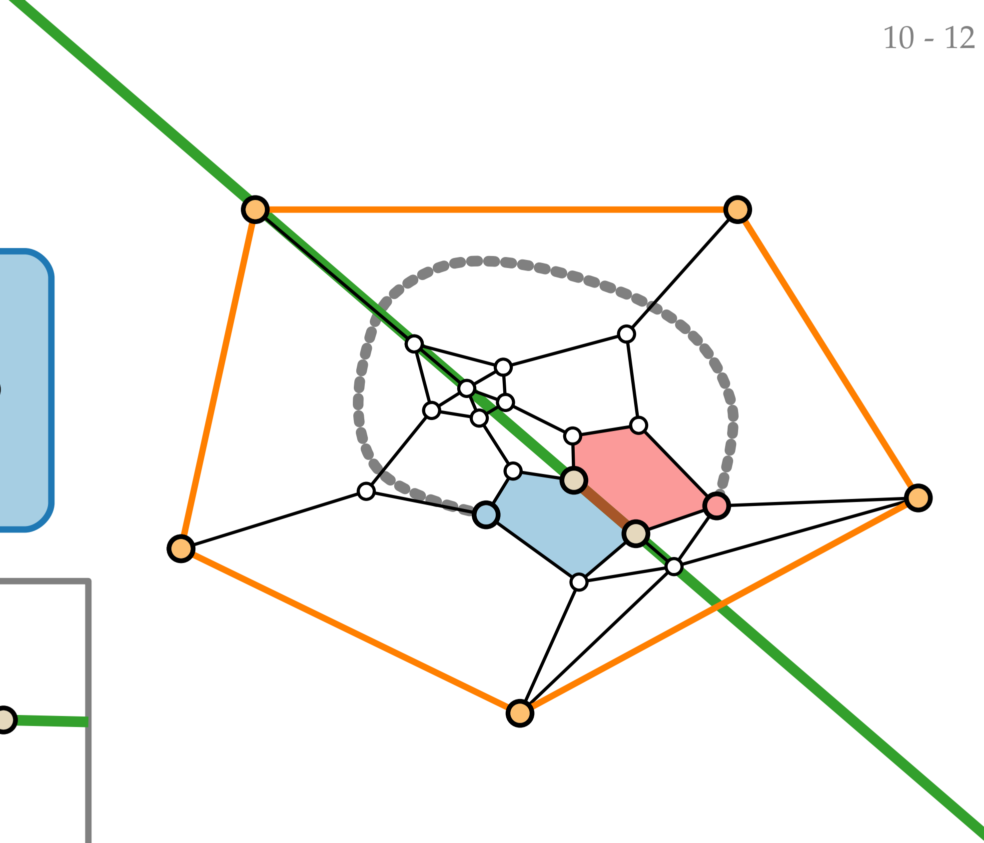

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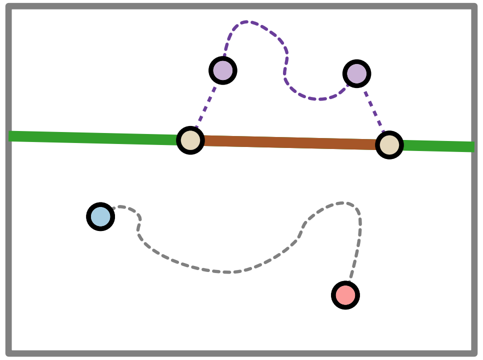

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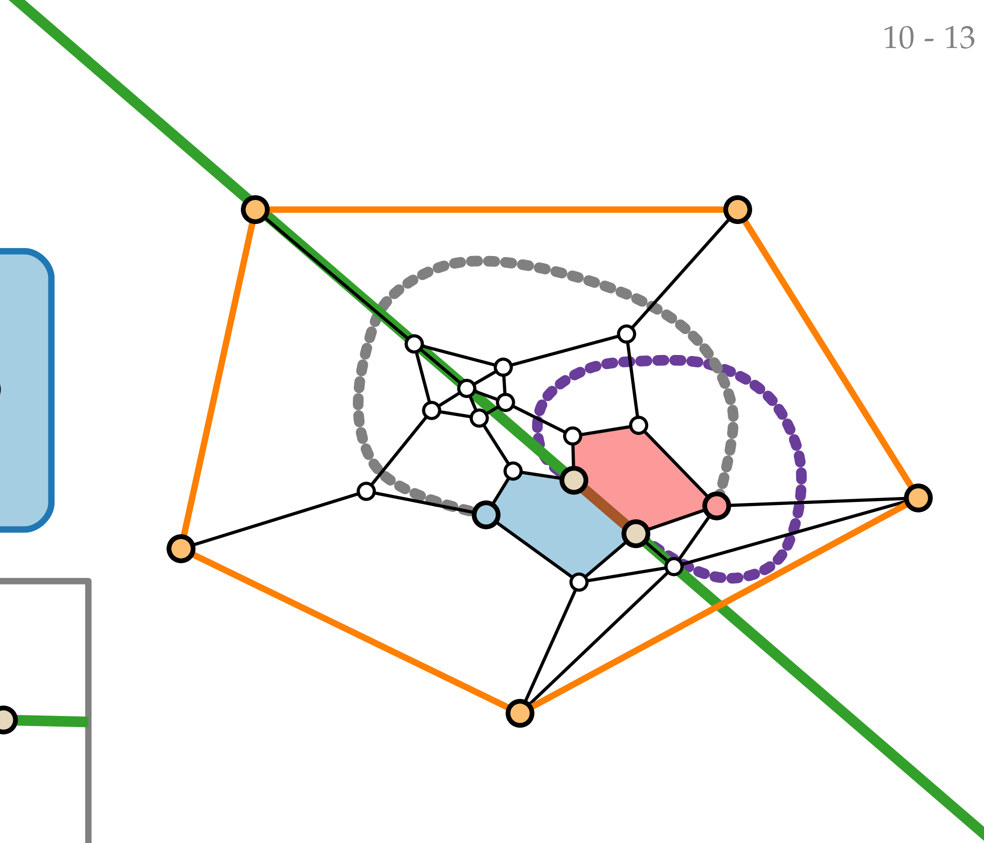

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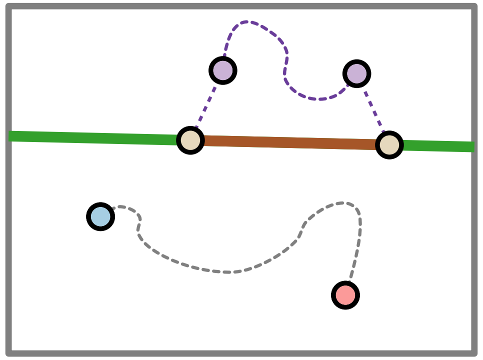

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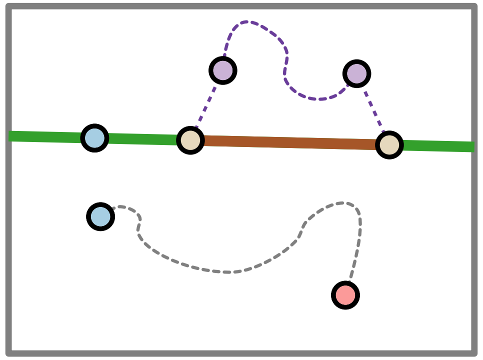

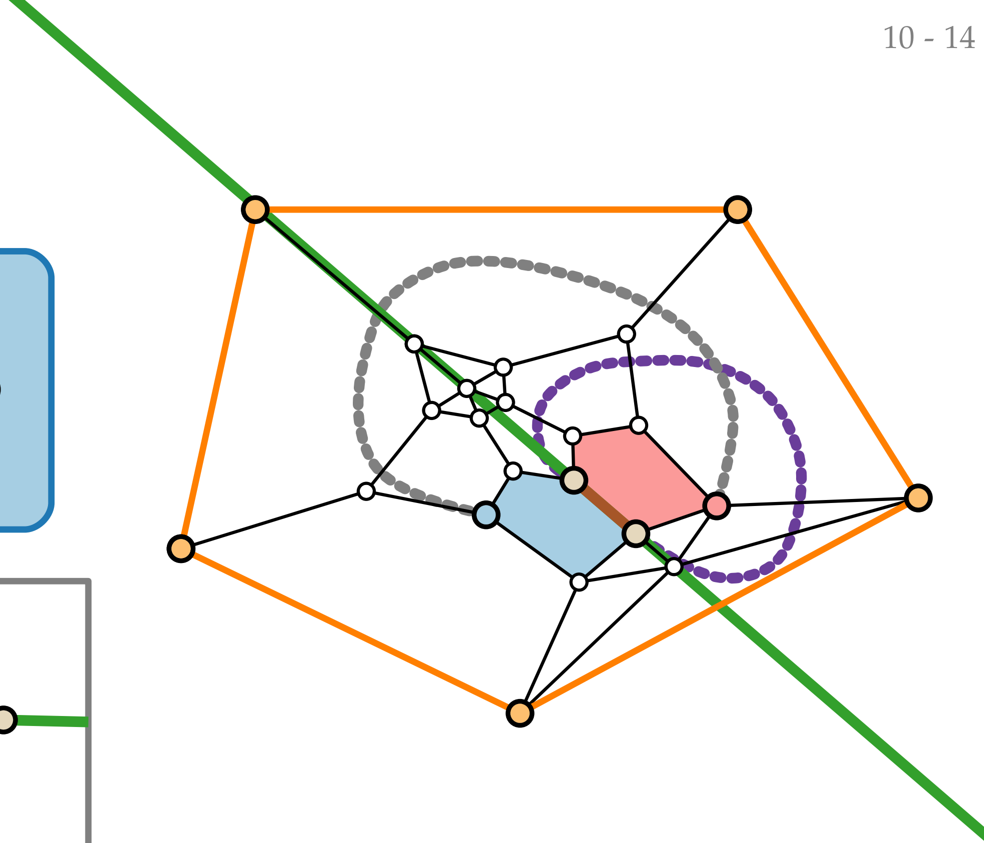

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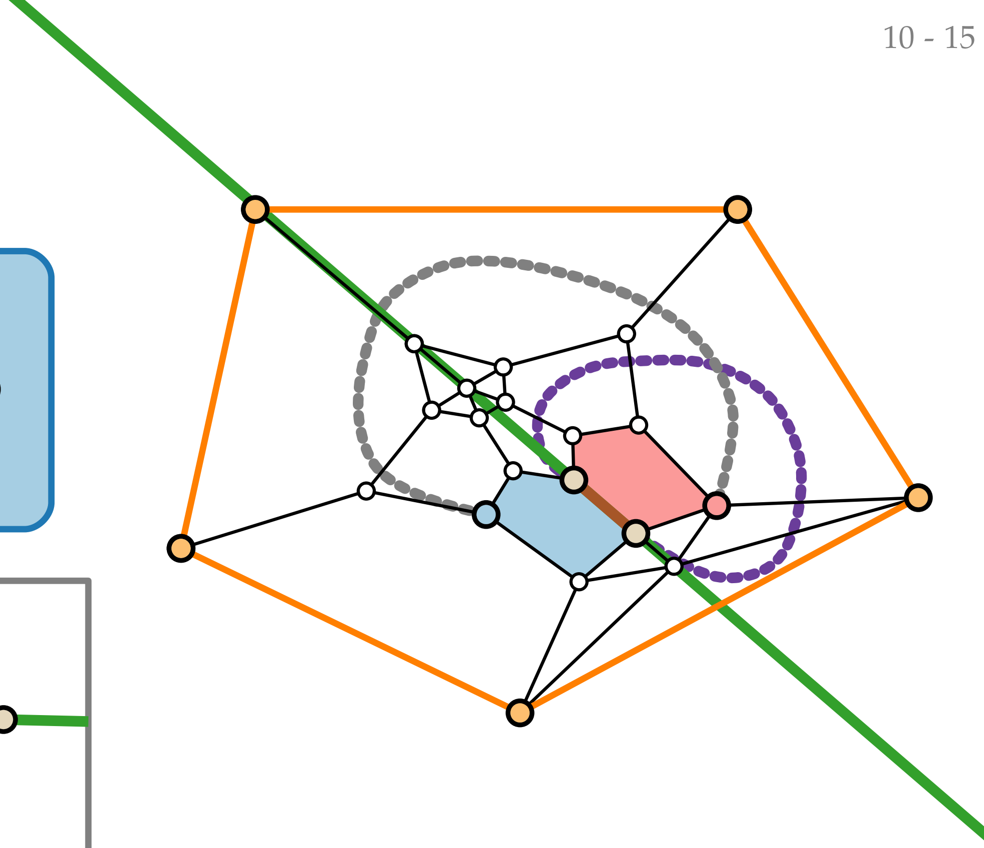

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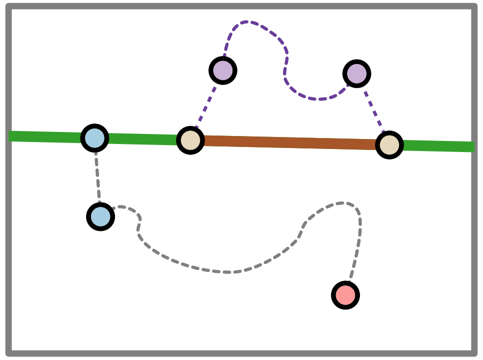

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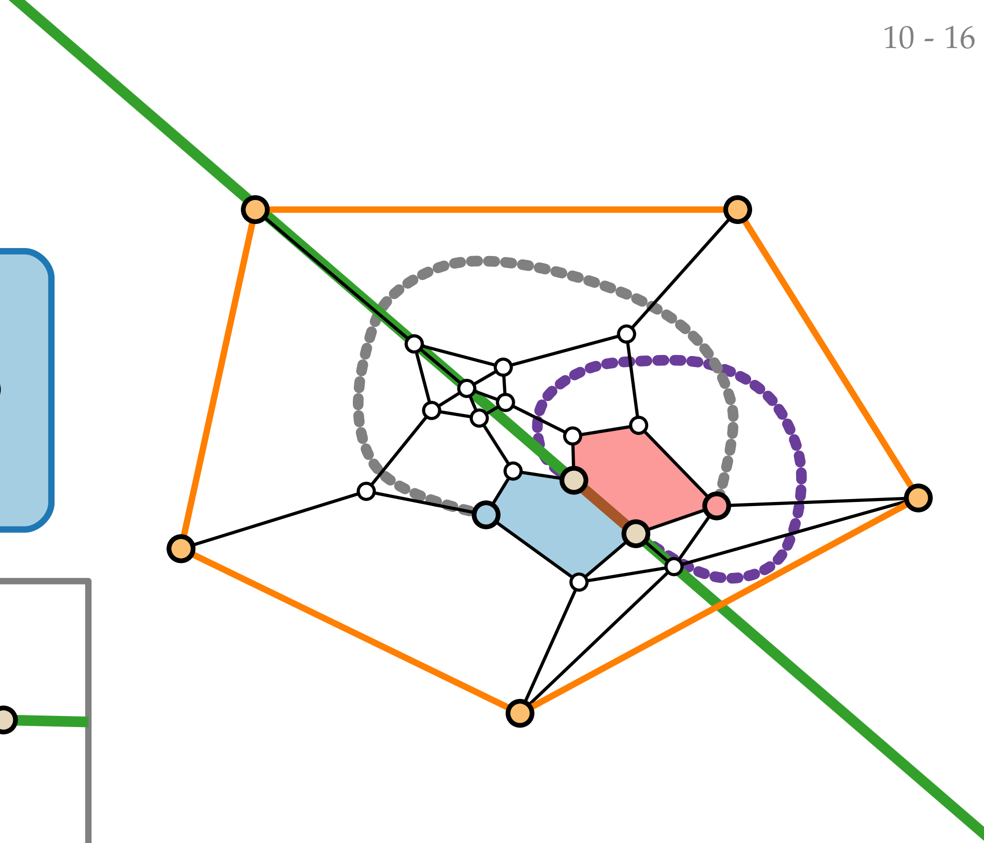

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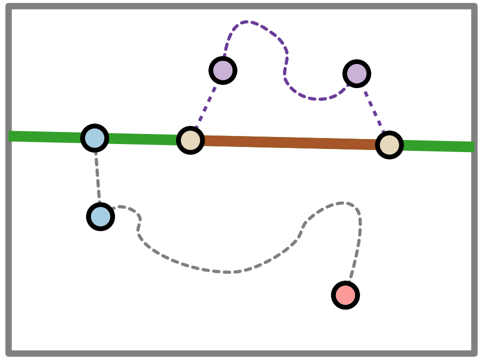

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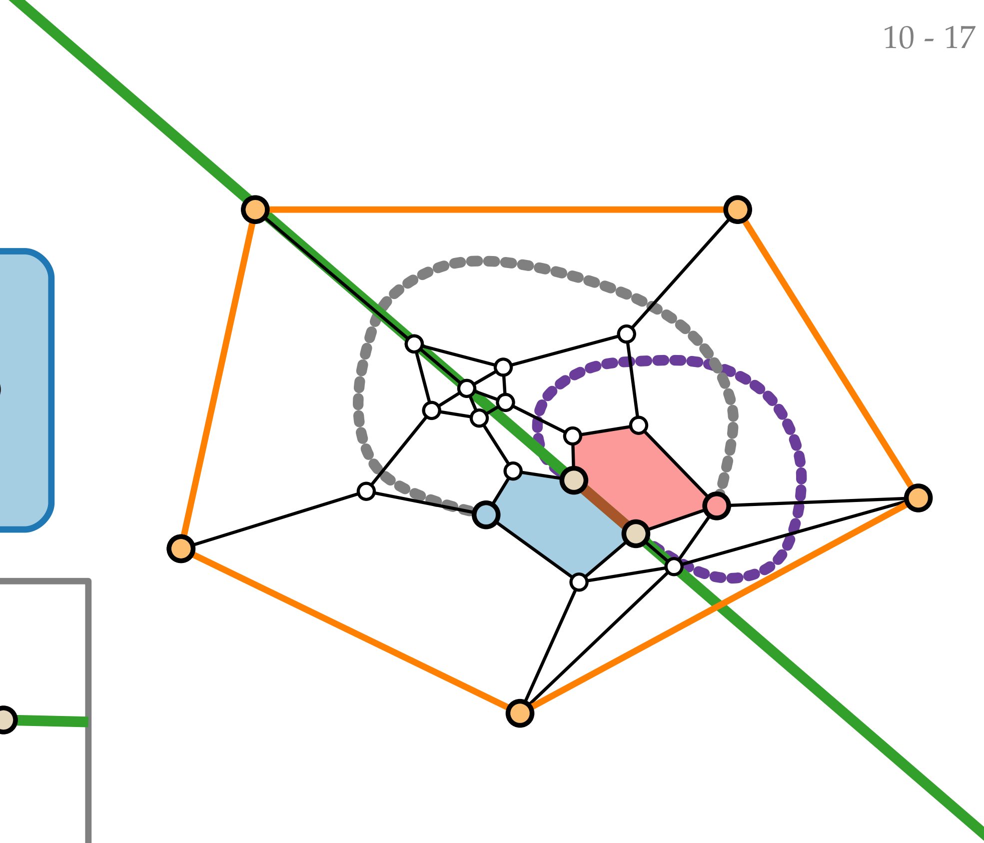

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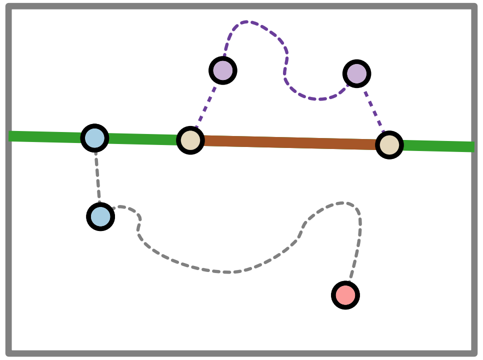

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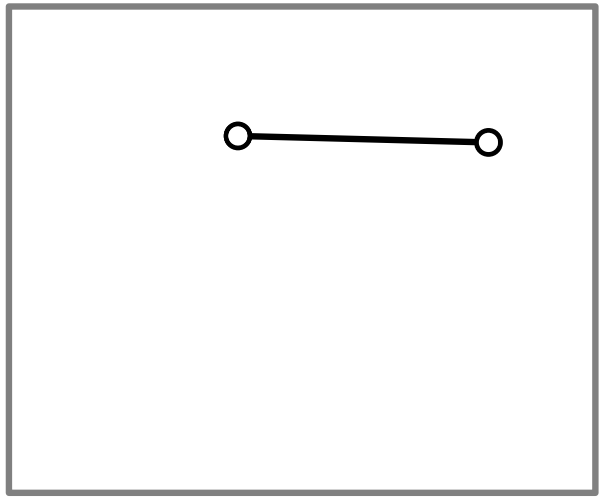

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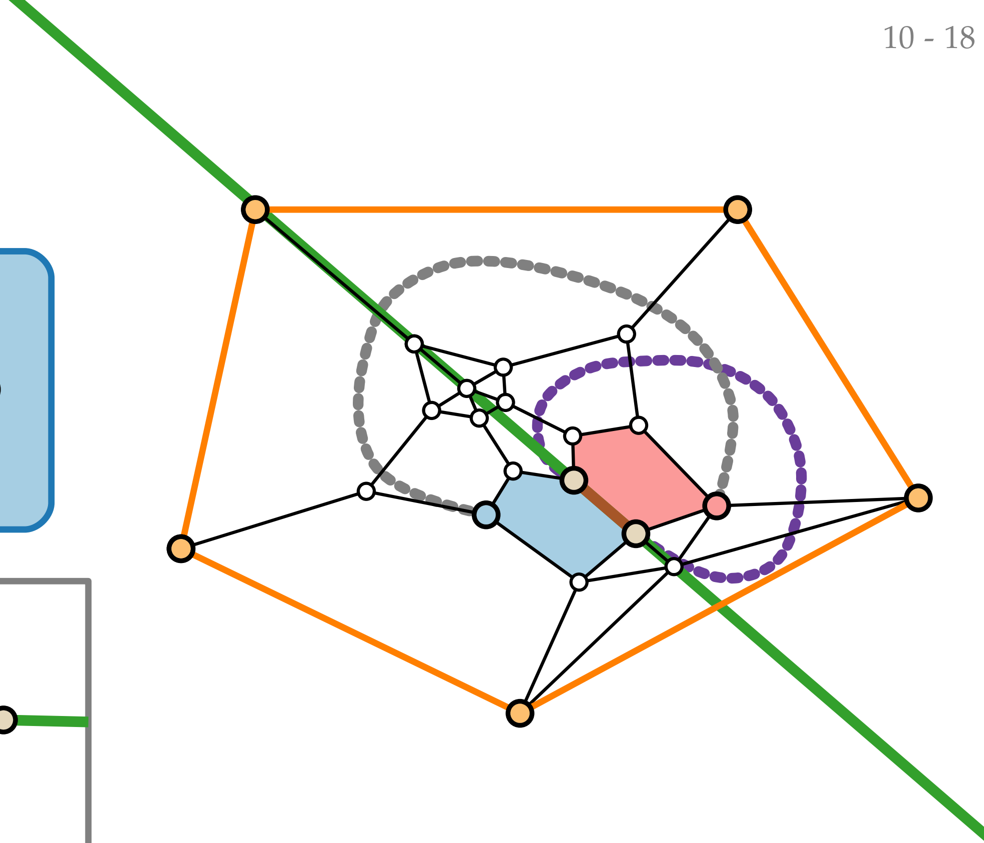

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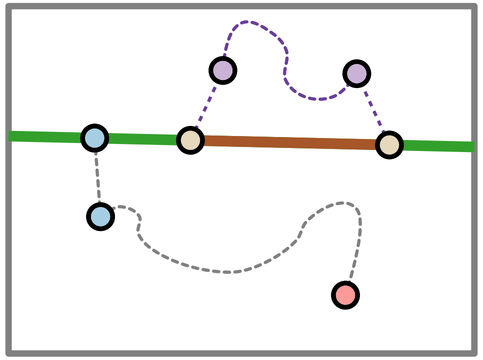

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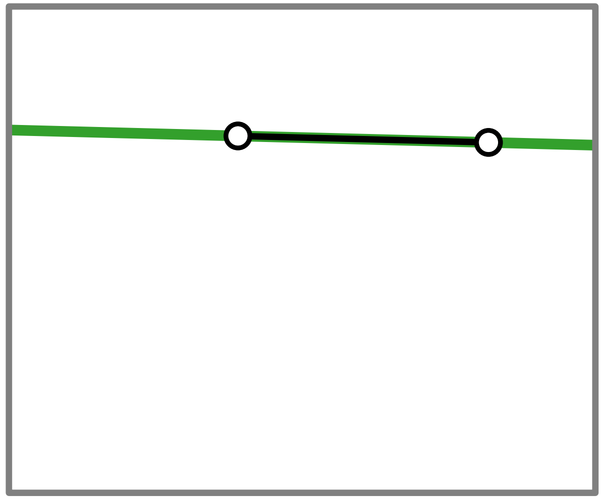

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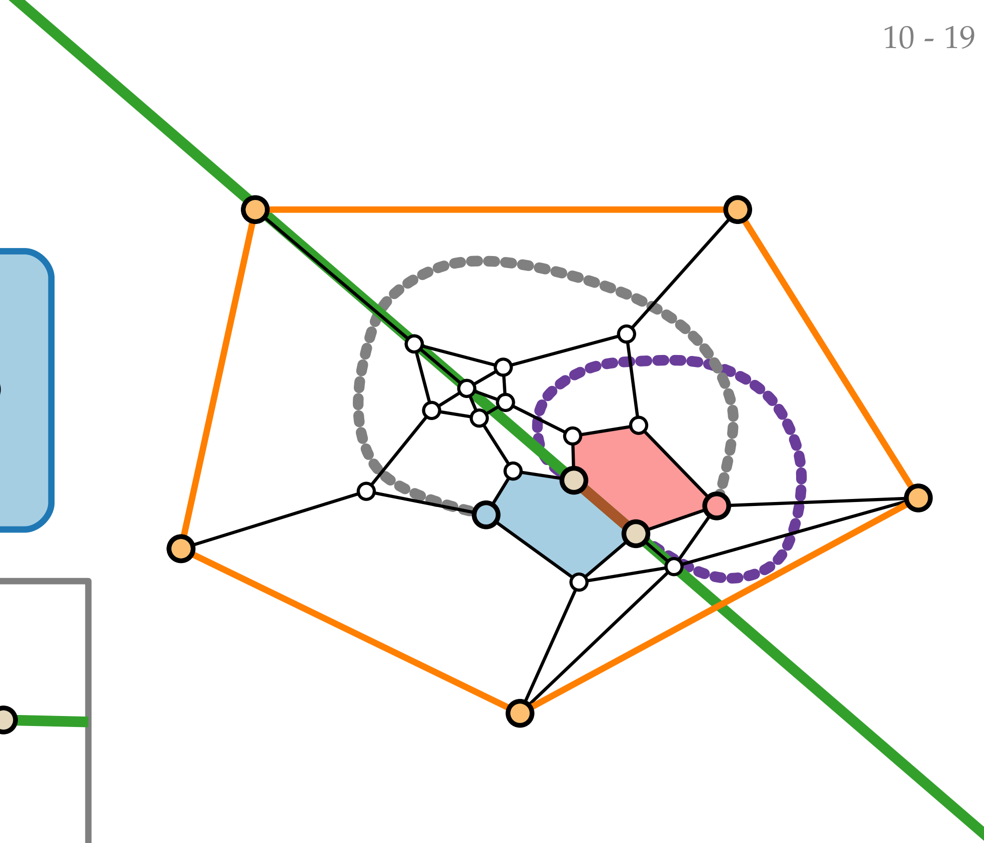

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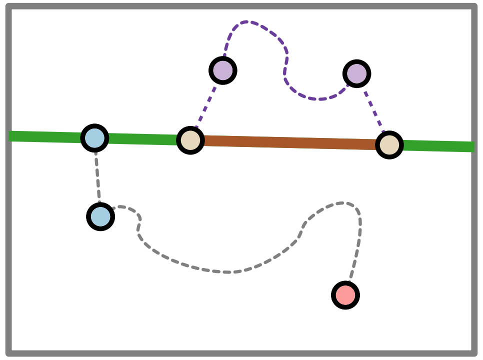

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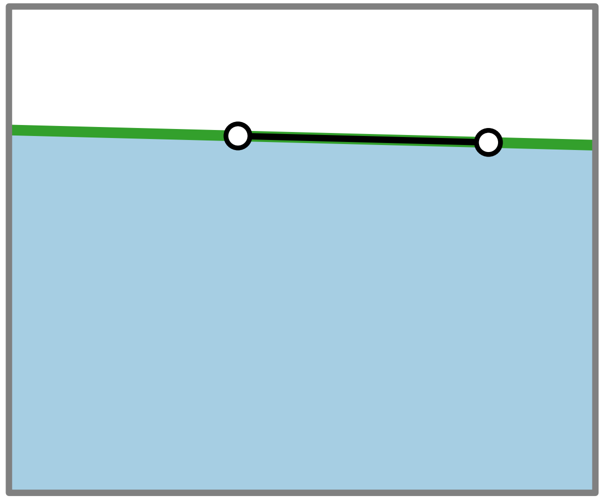

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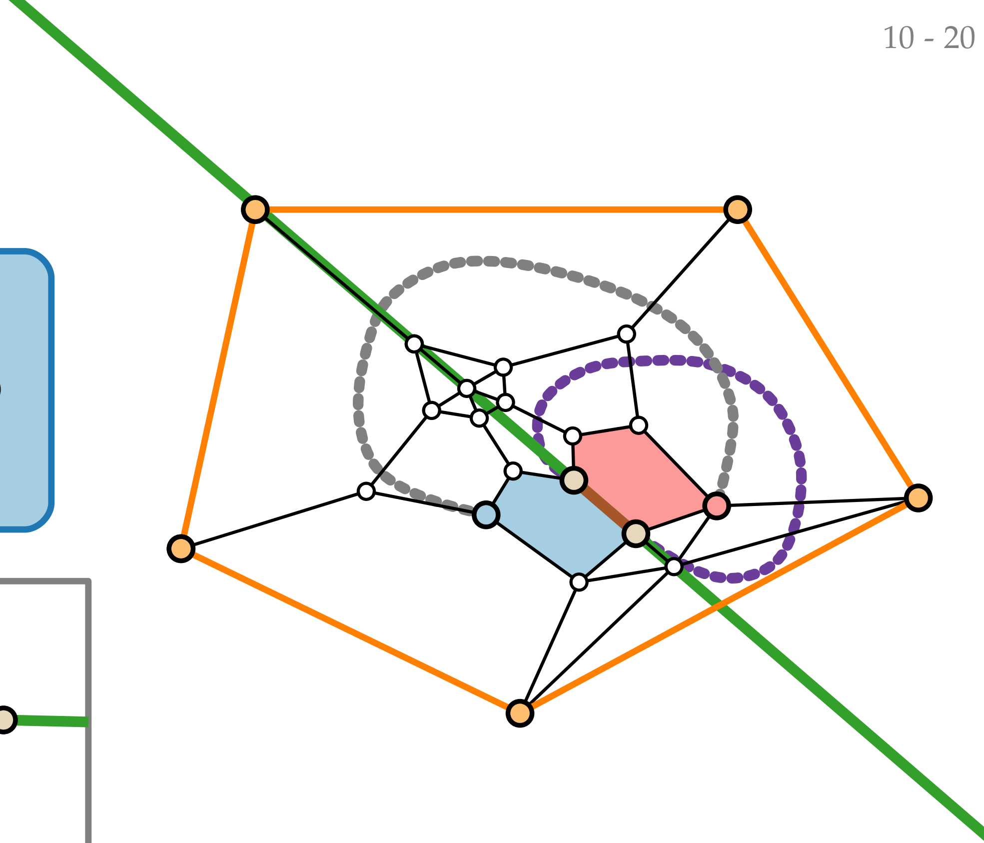

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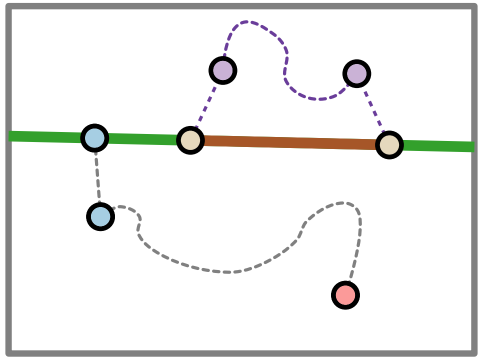

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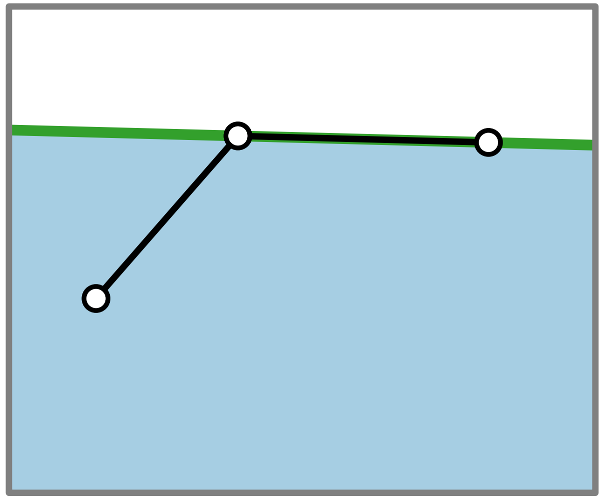

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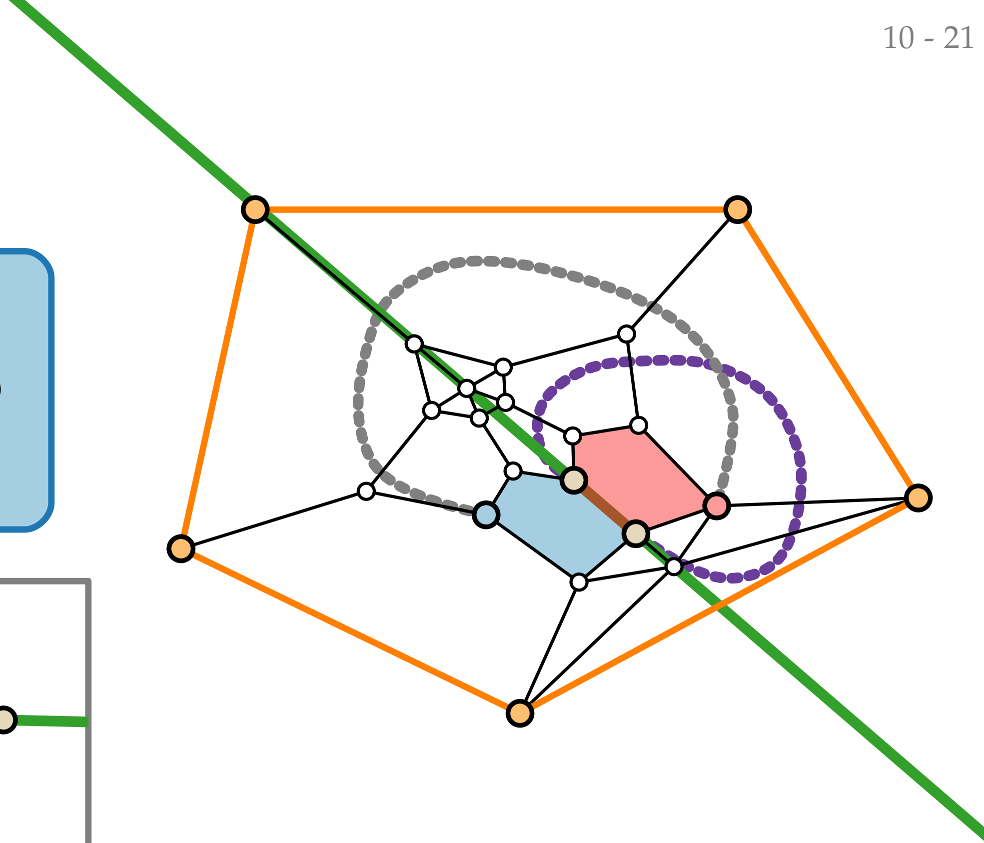

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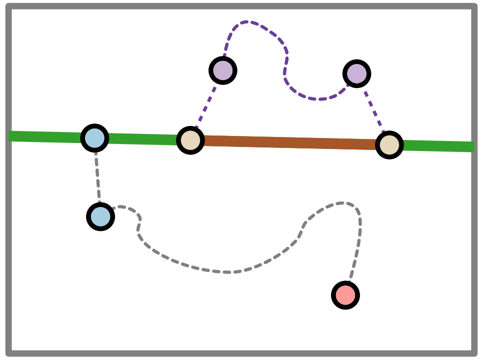

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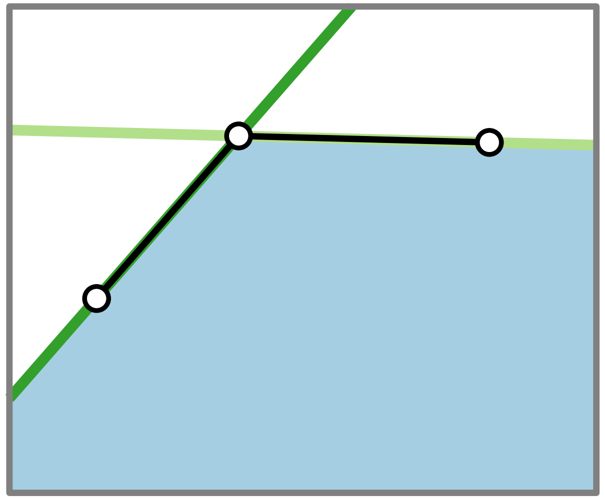

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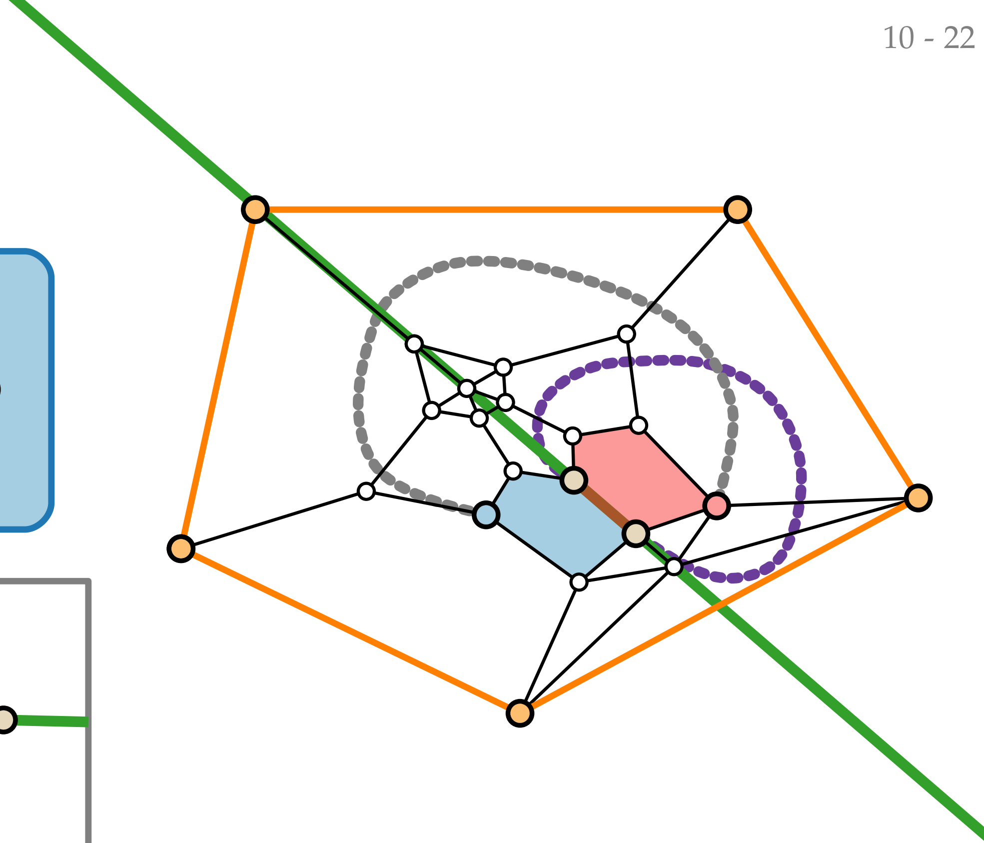

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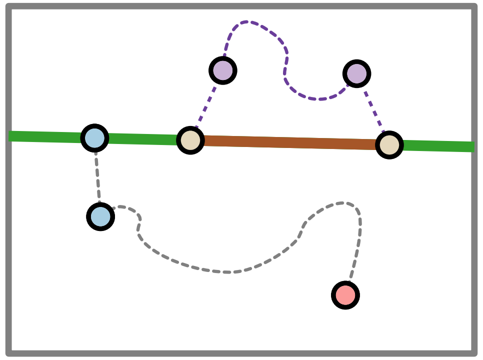

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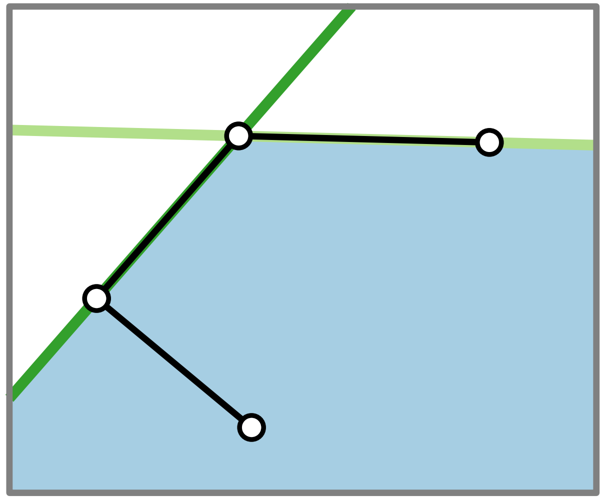

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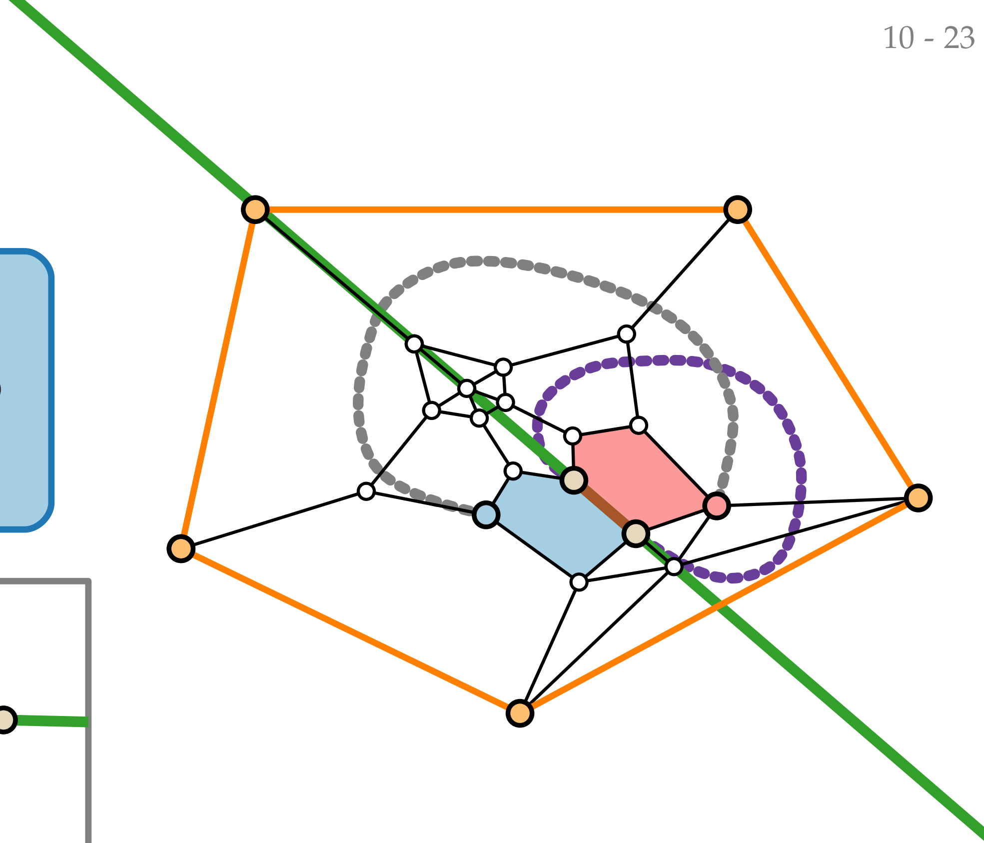

Lemma. Let $uv \in E$ be a non-boundary edge, ℓ line through uv. Then the two faces f_1, f_2 incident to uv lie completely on opposite sides of ℓ.

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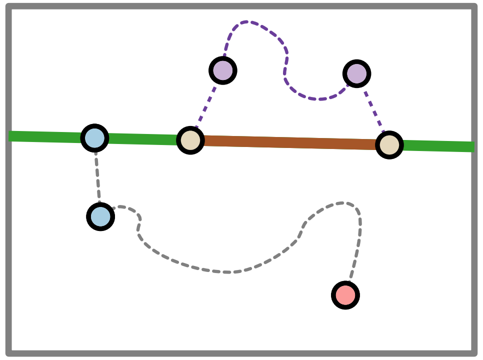
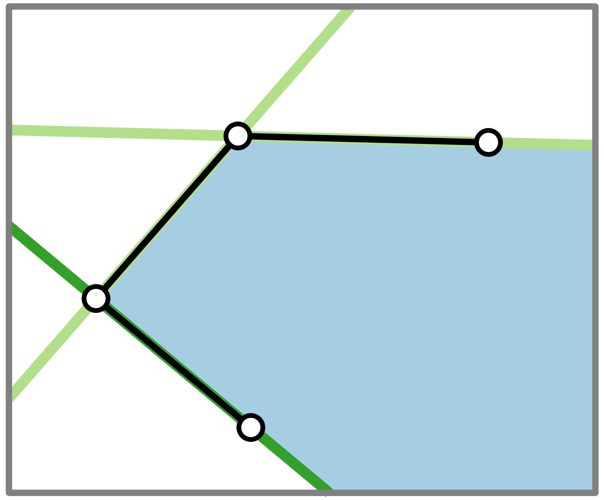
Then $G[V_\ell]$ is connected.



Property 4. No vertex is collinear with all of its neighbors.

Lemma. All faces are strictly convex.



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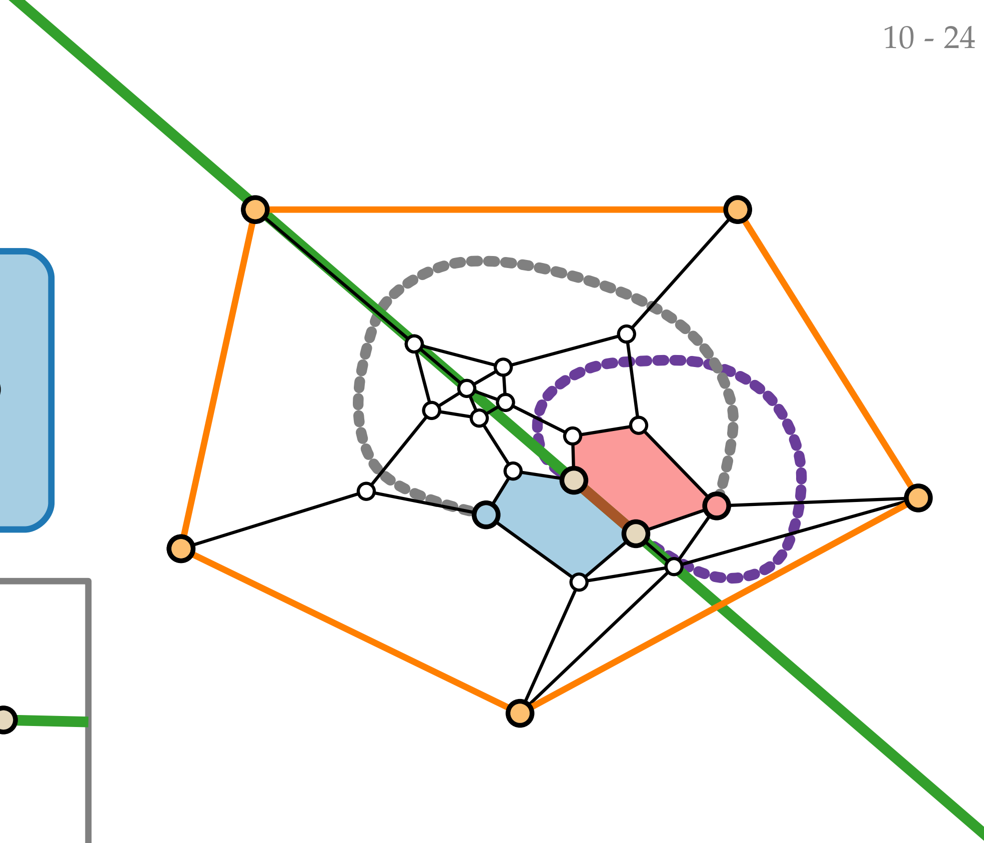

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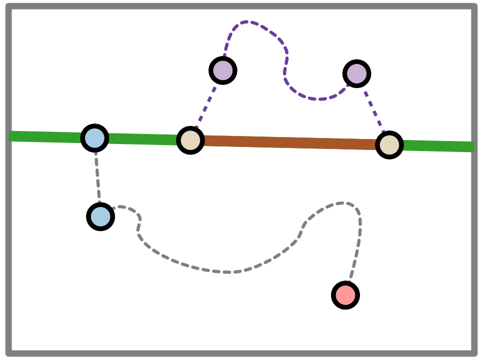

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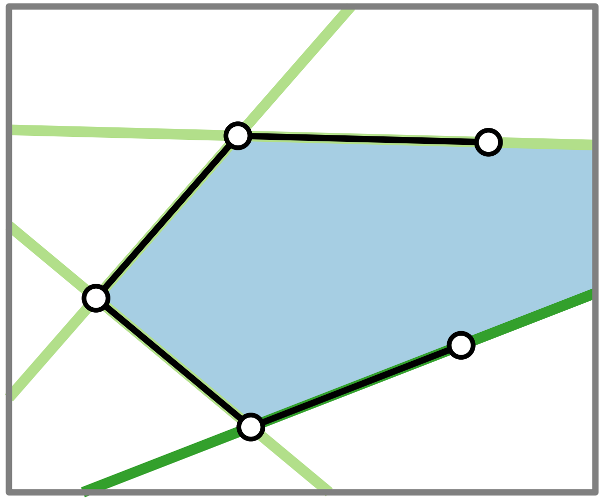

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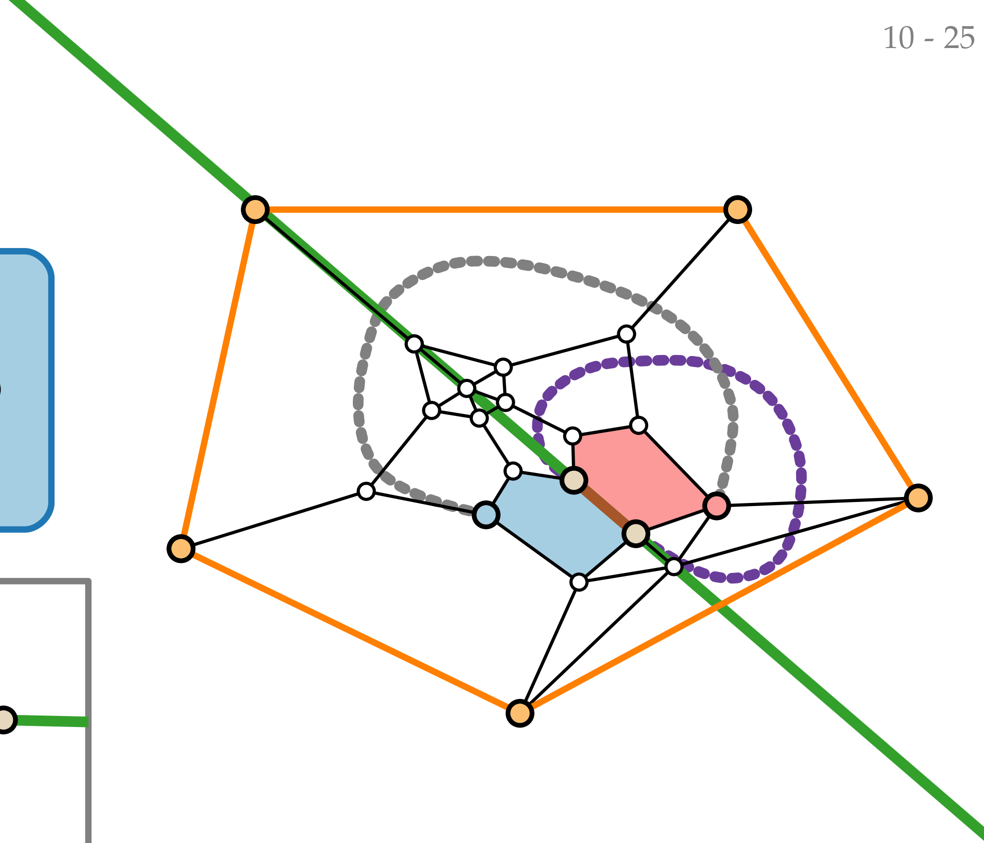

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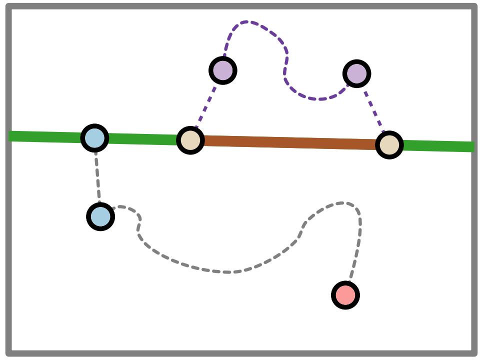

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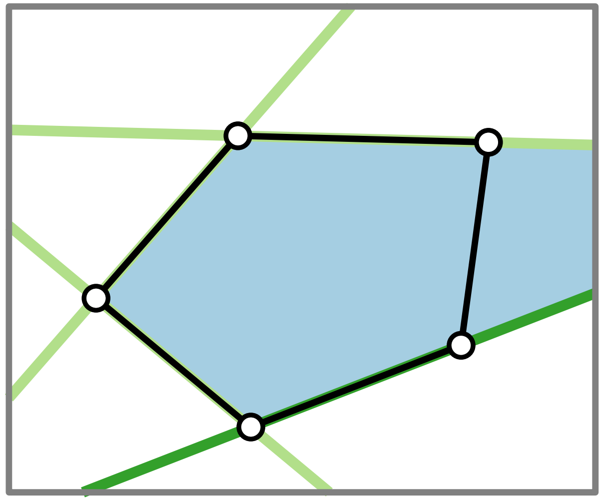

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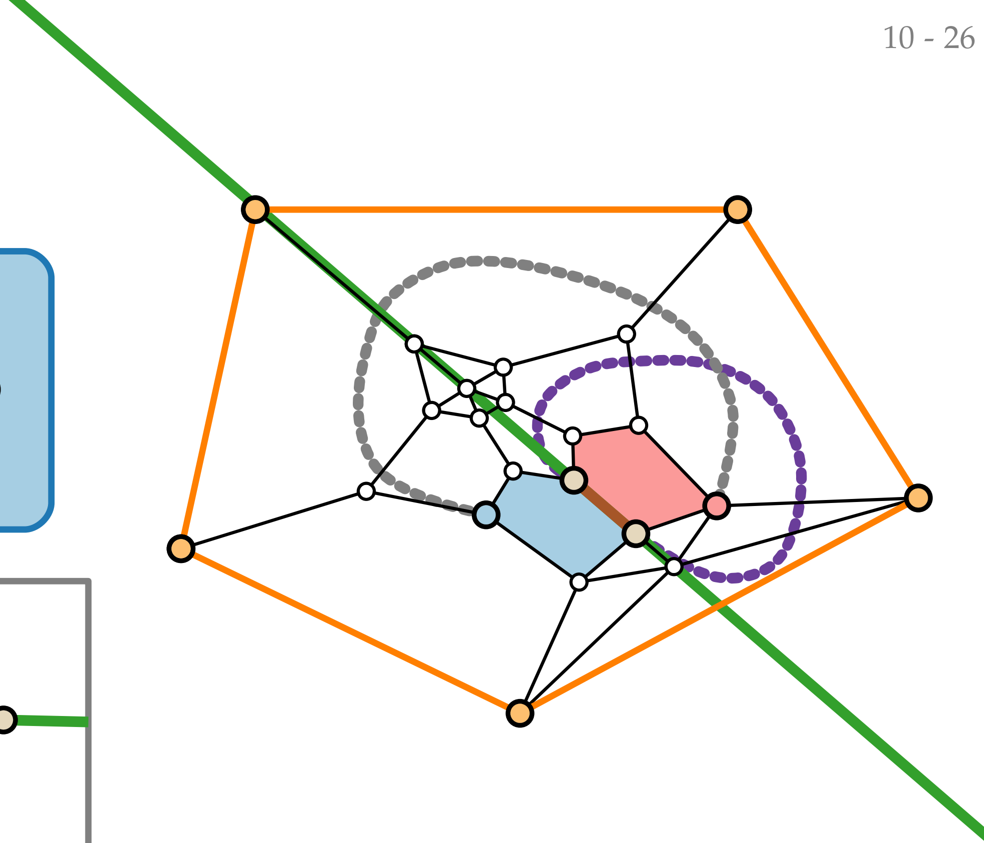

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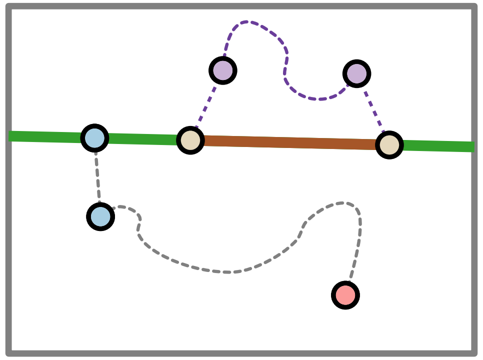

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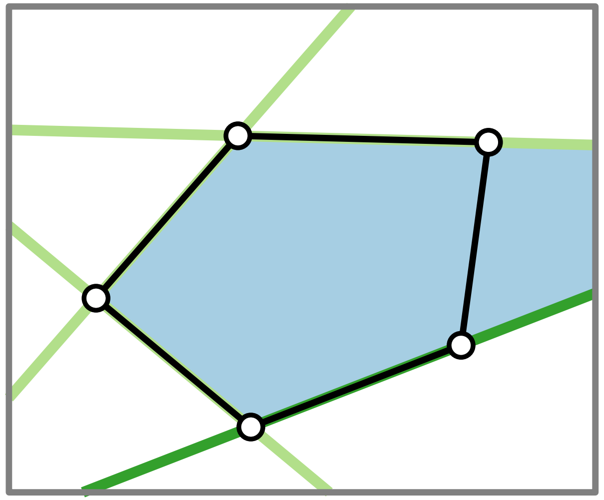

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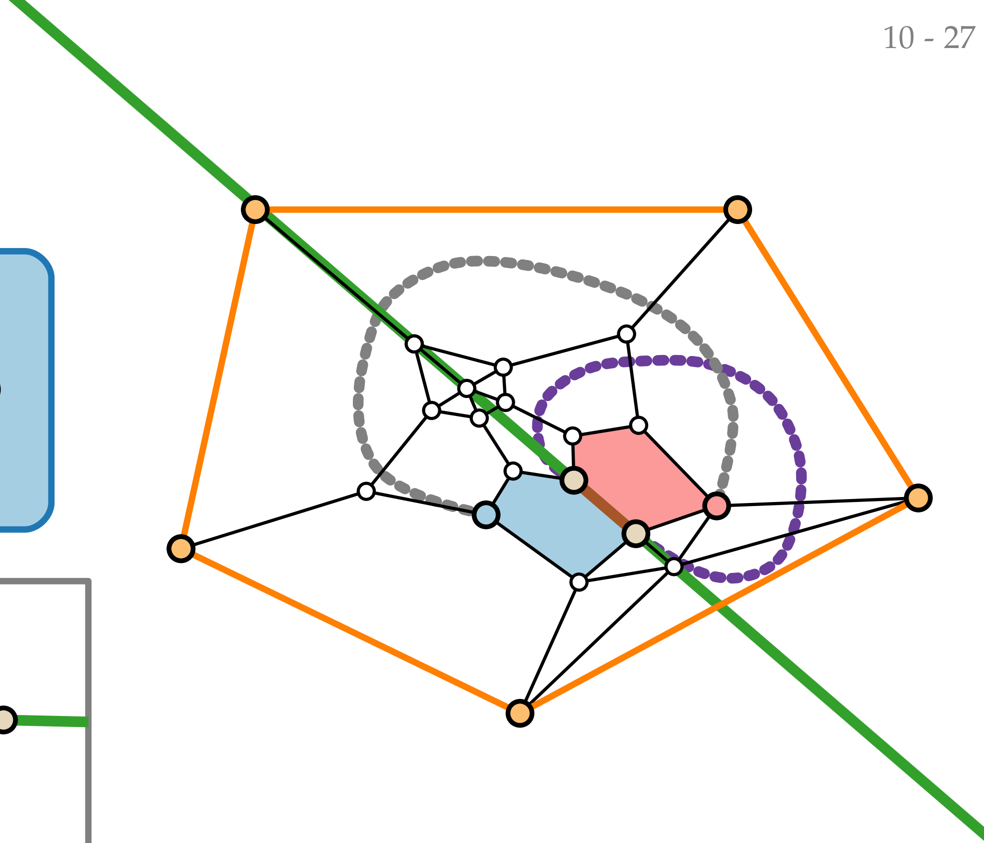

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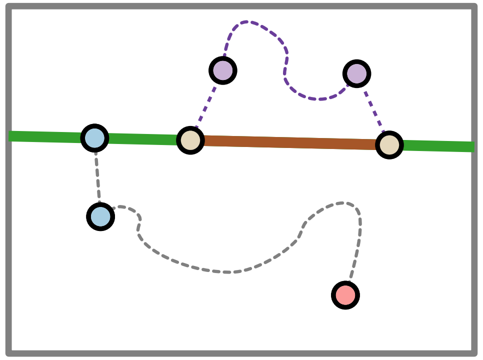

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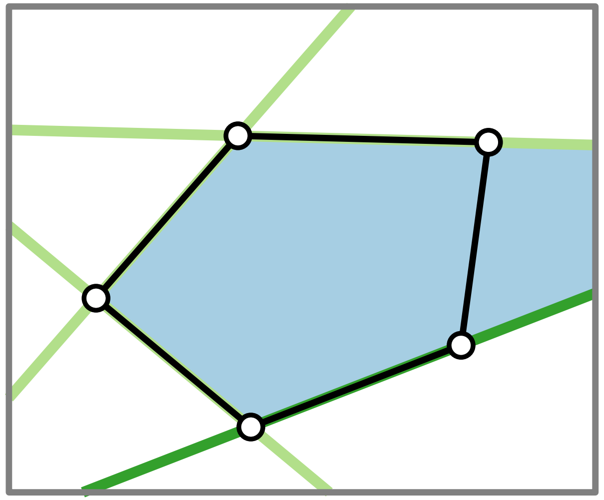

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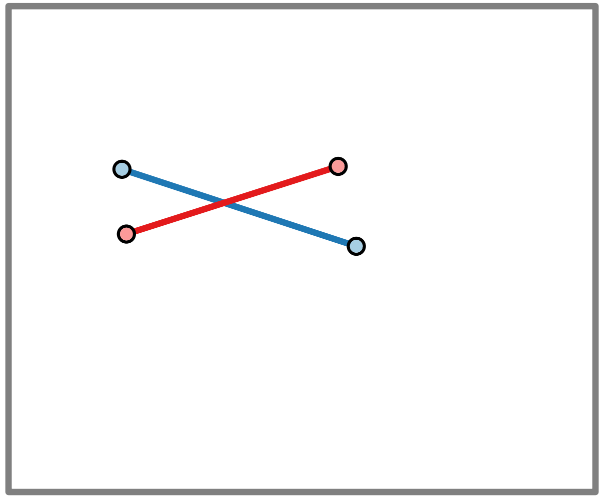

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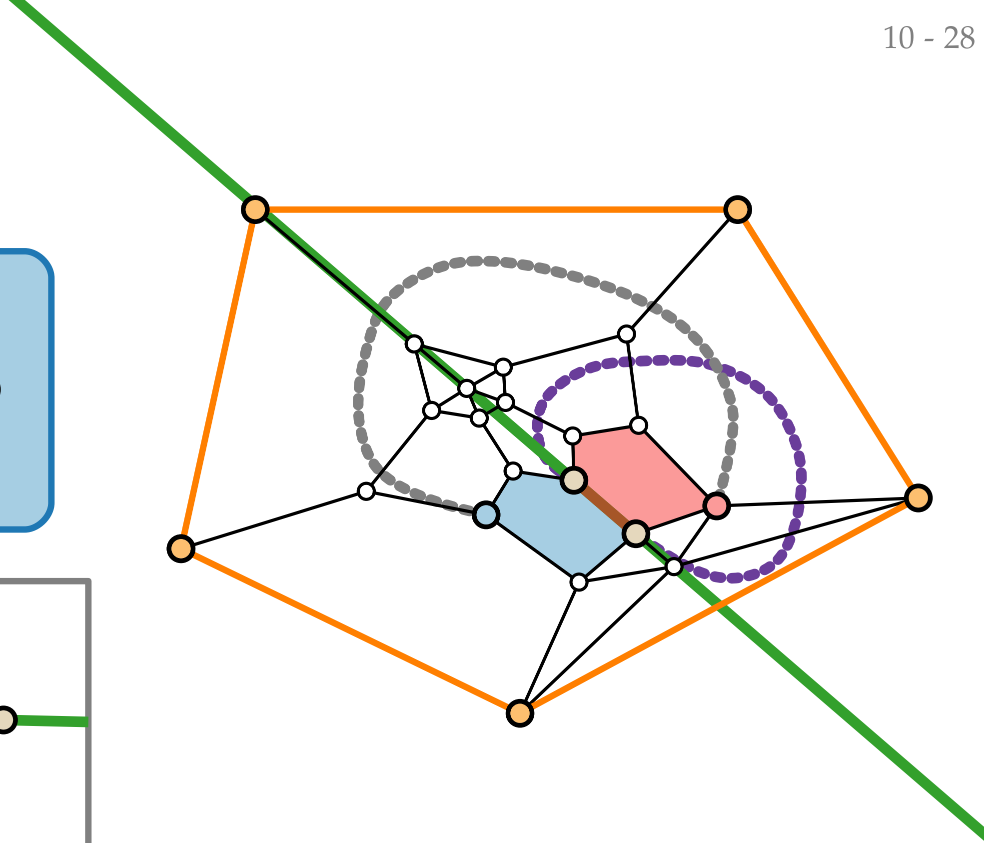

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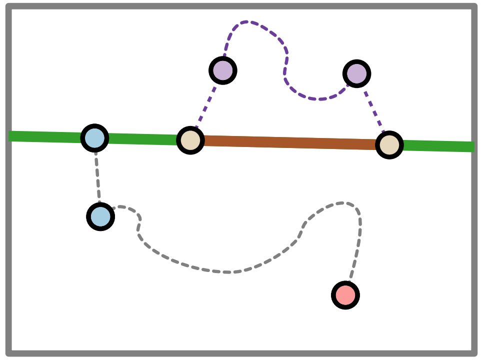

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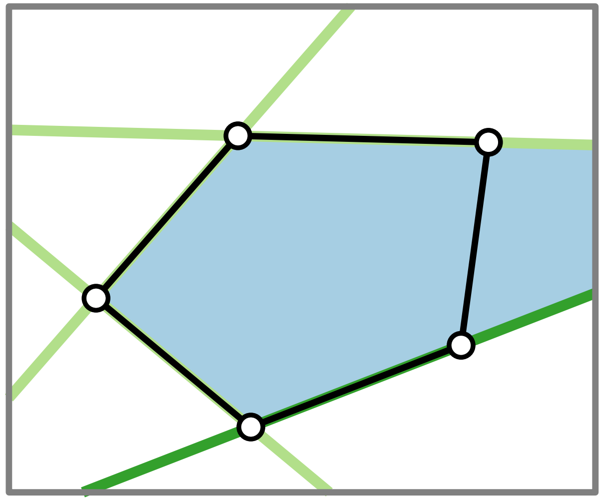

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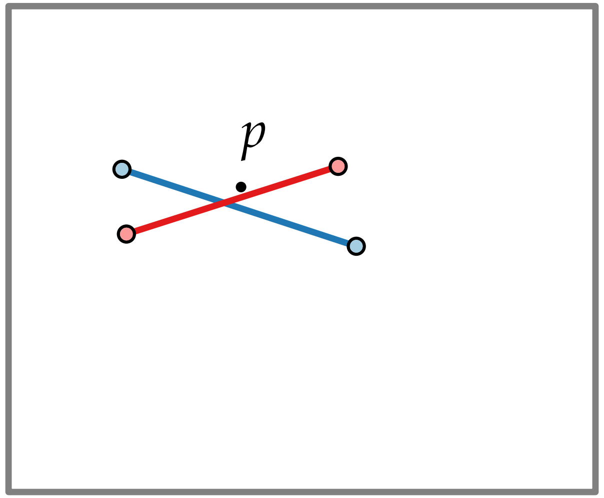

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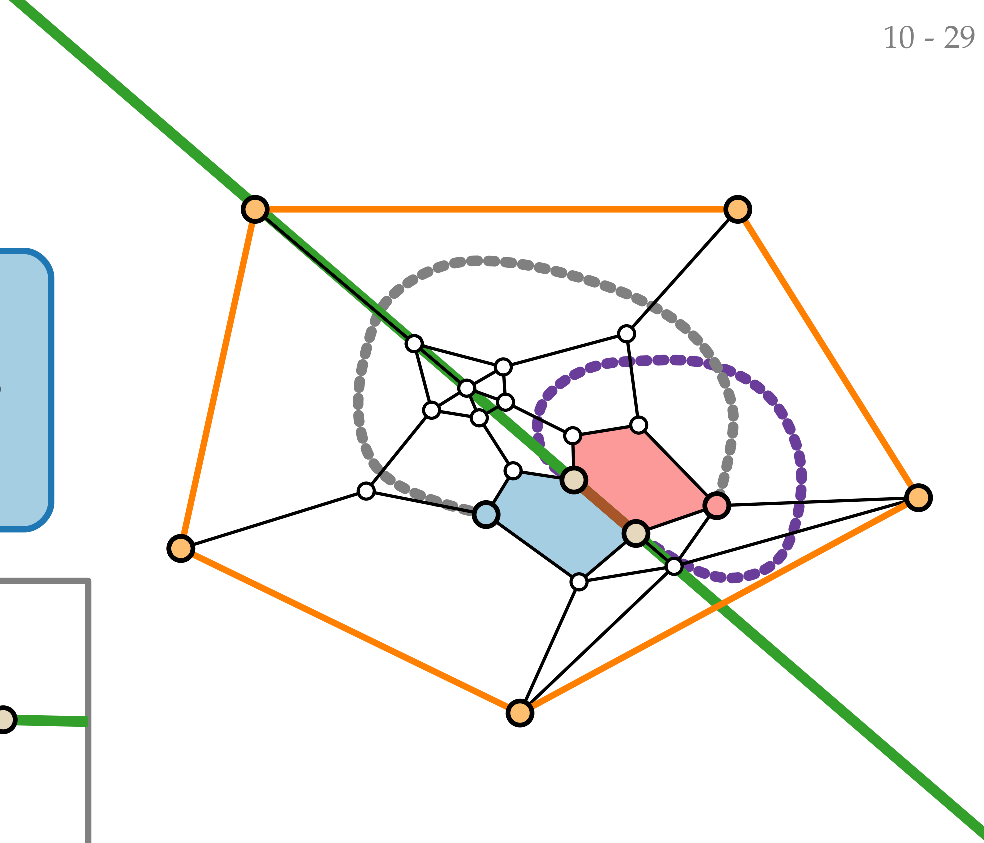

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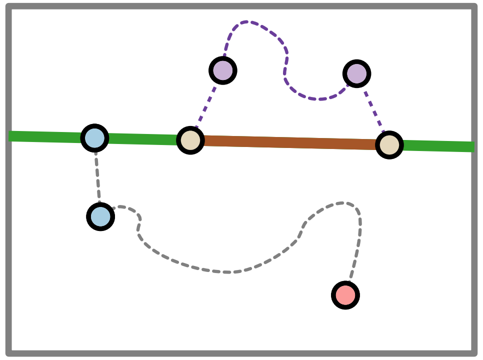

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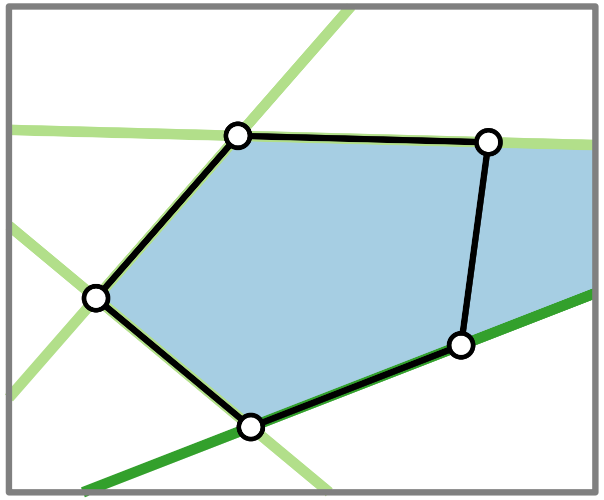
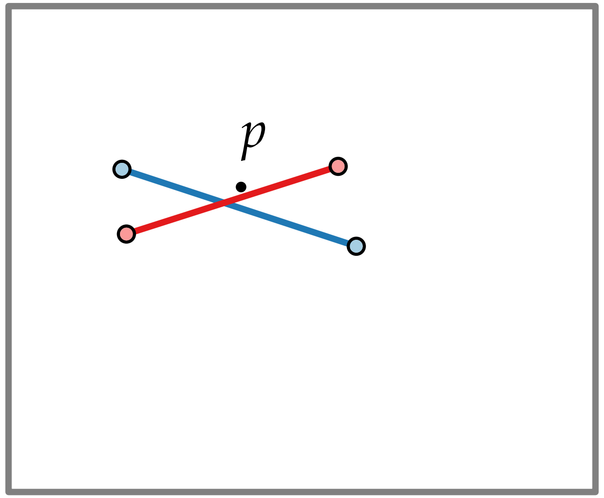


p inside two faces



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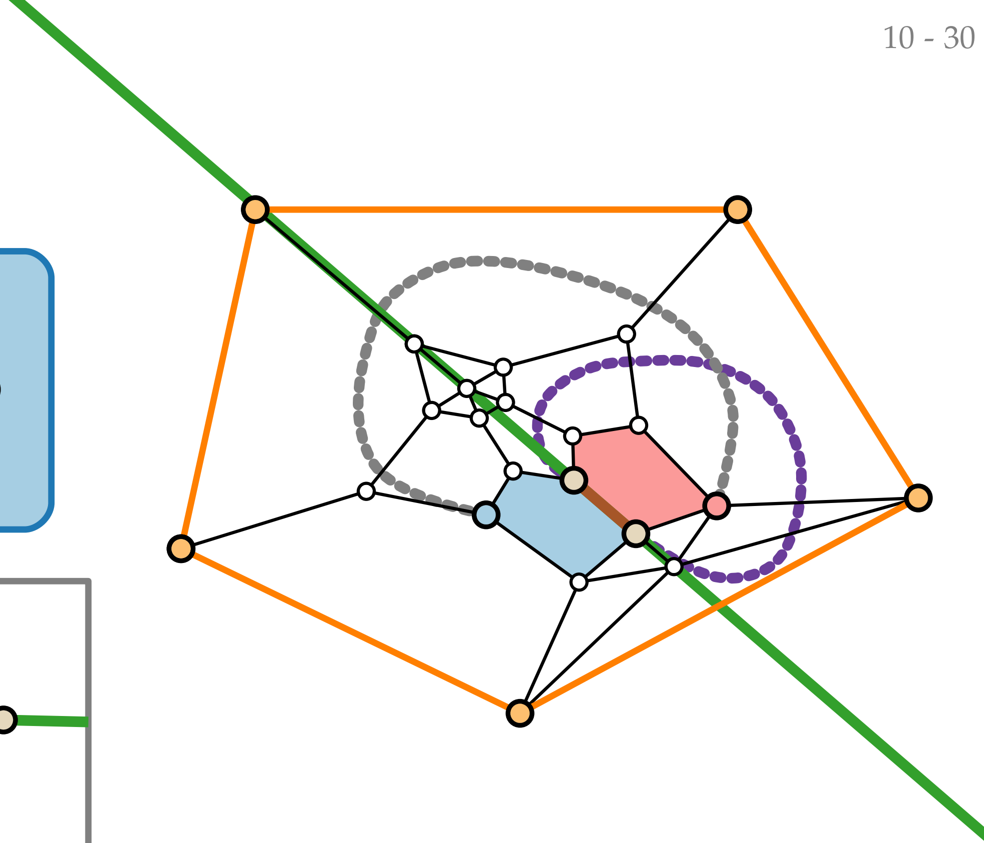

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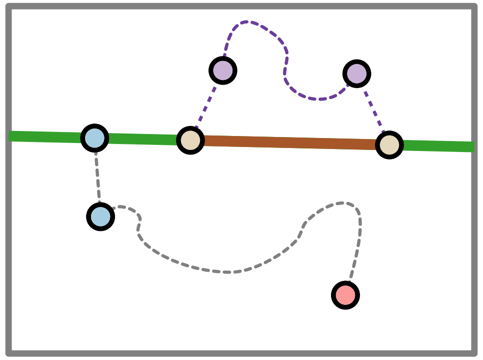

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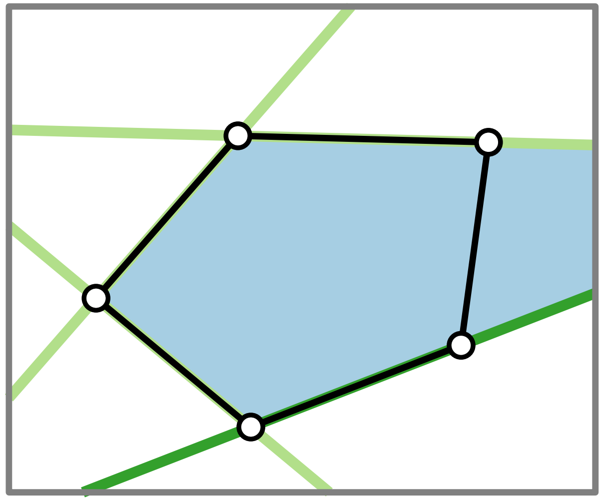

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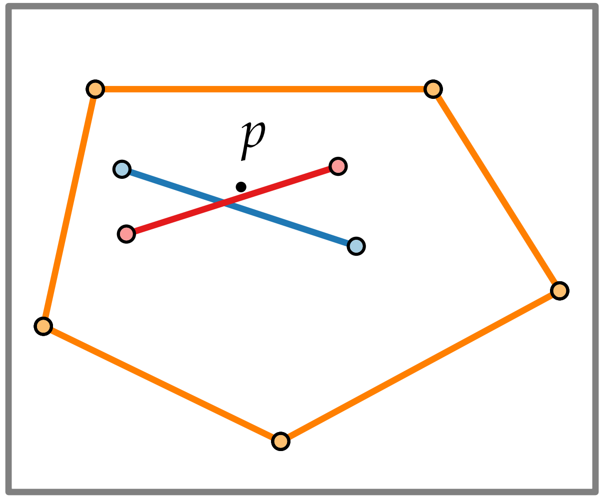

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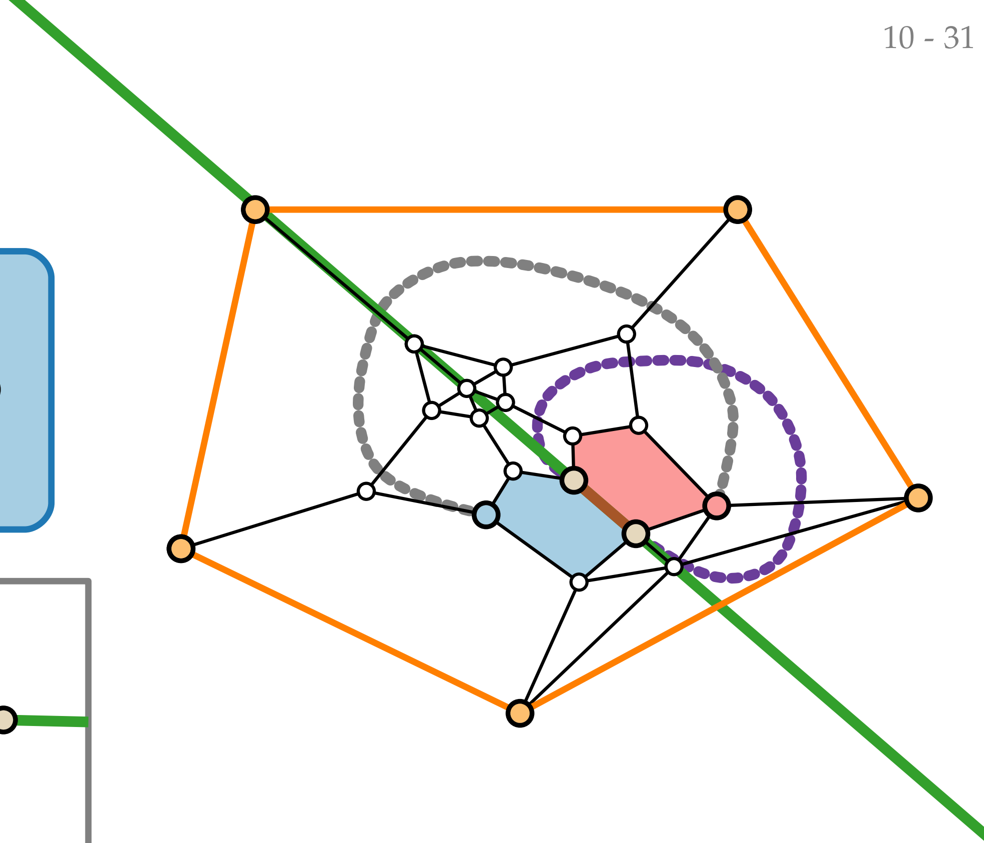

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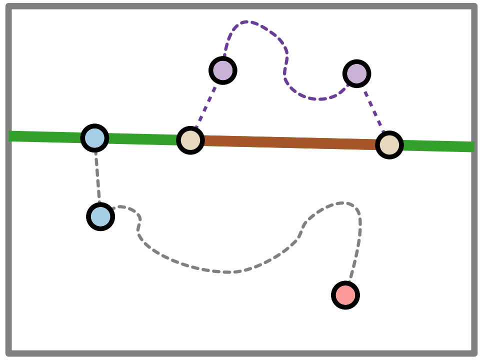

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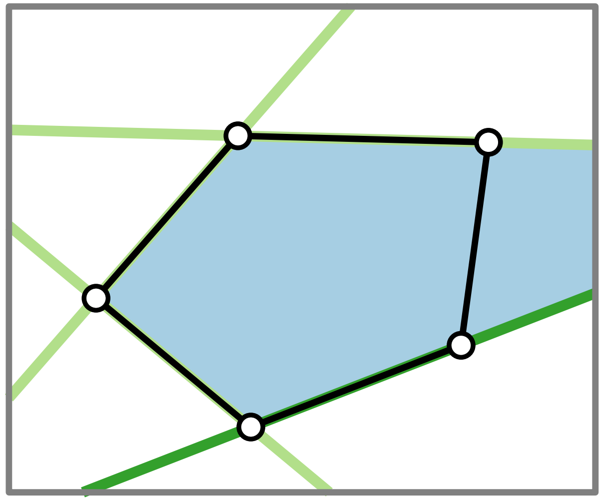

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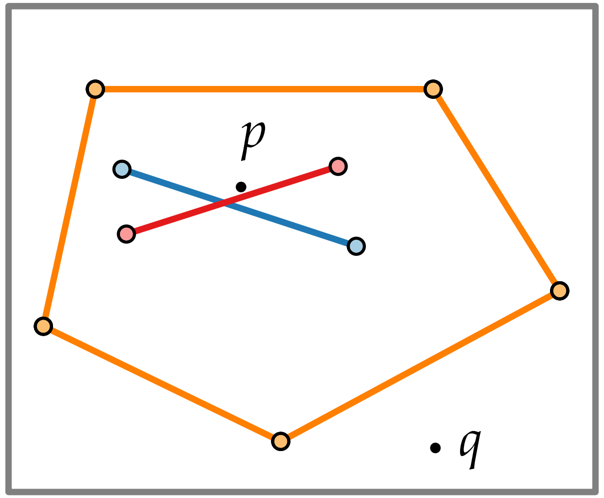

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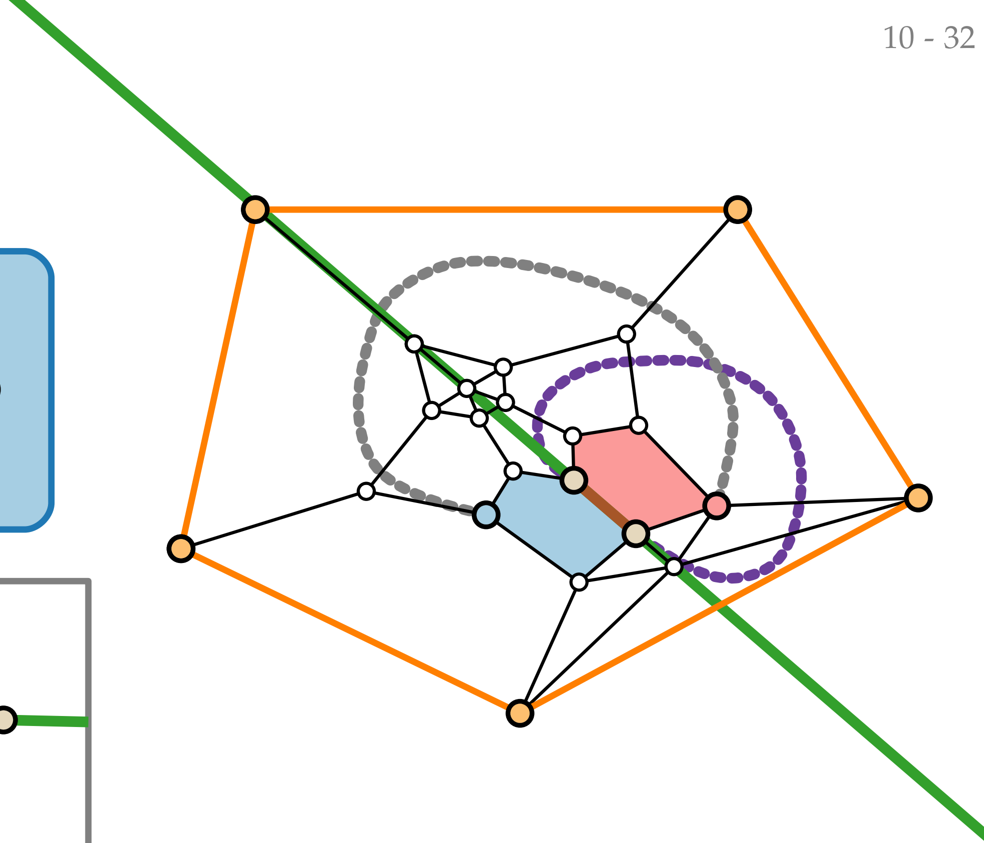

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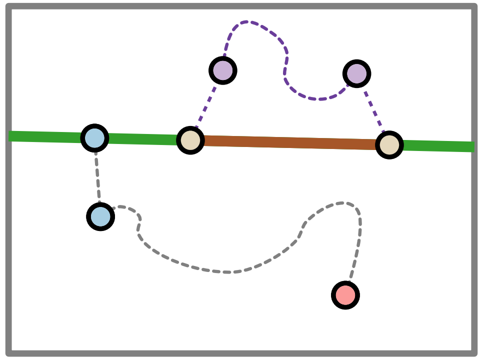

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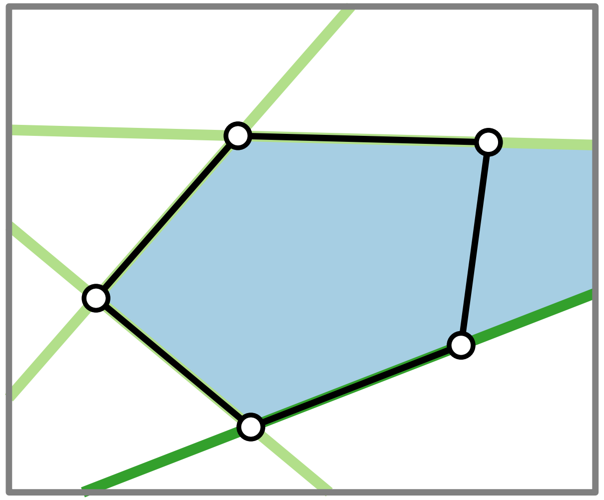

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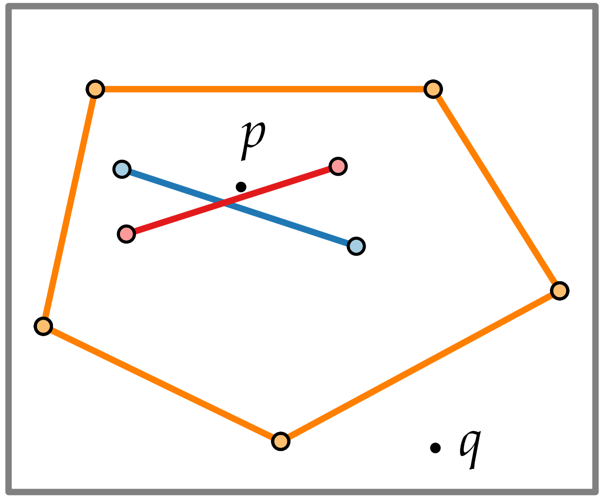

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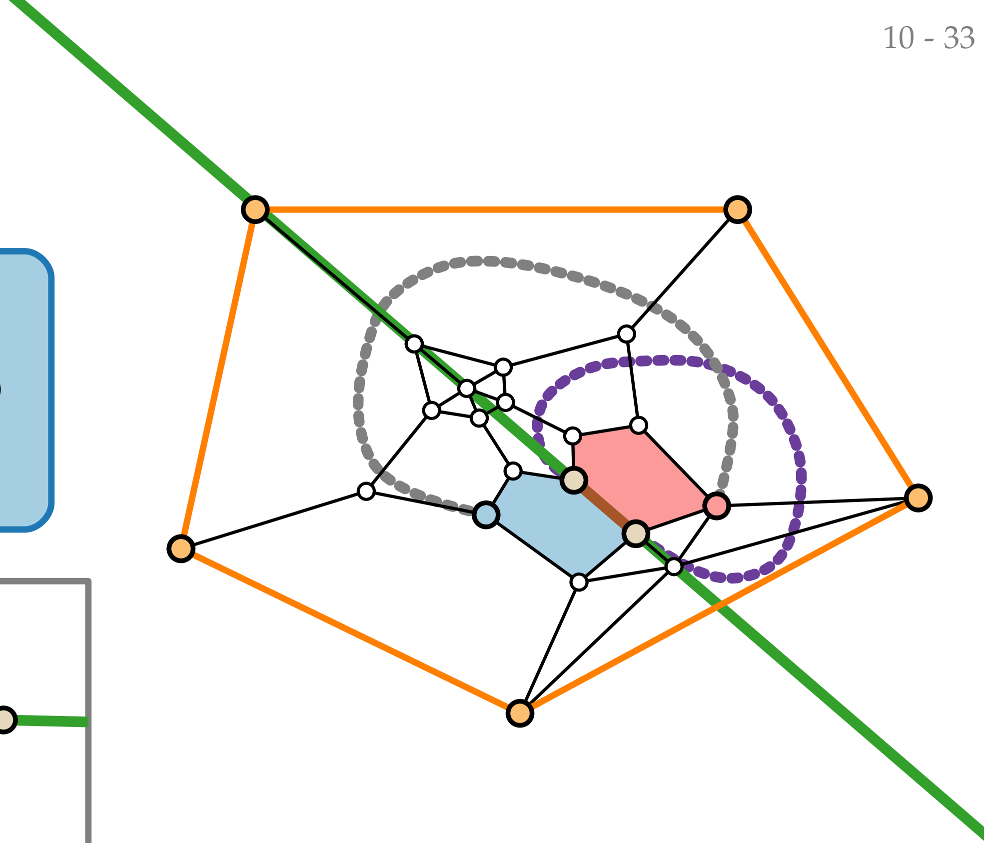

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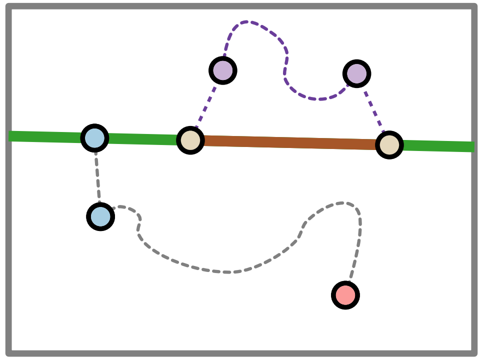

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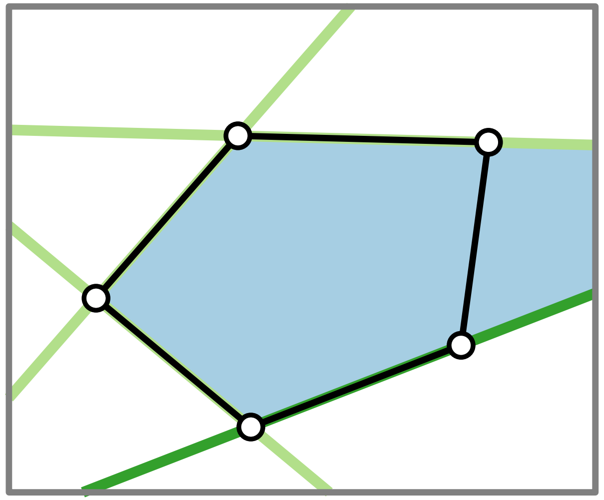

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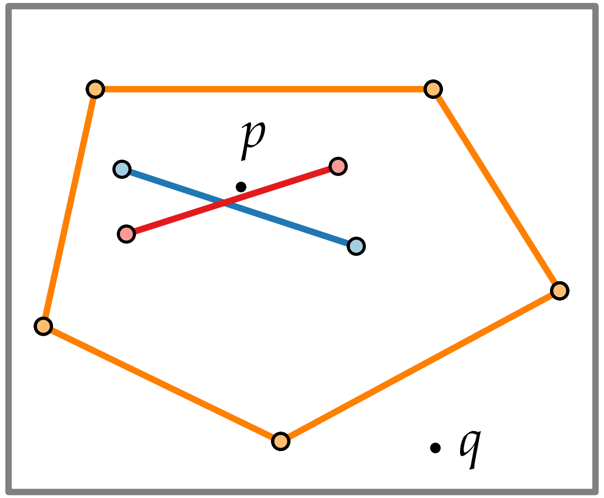

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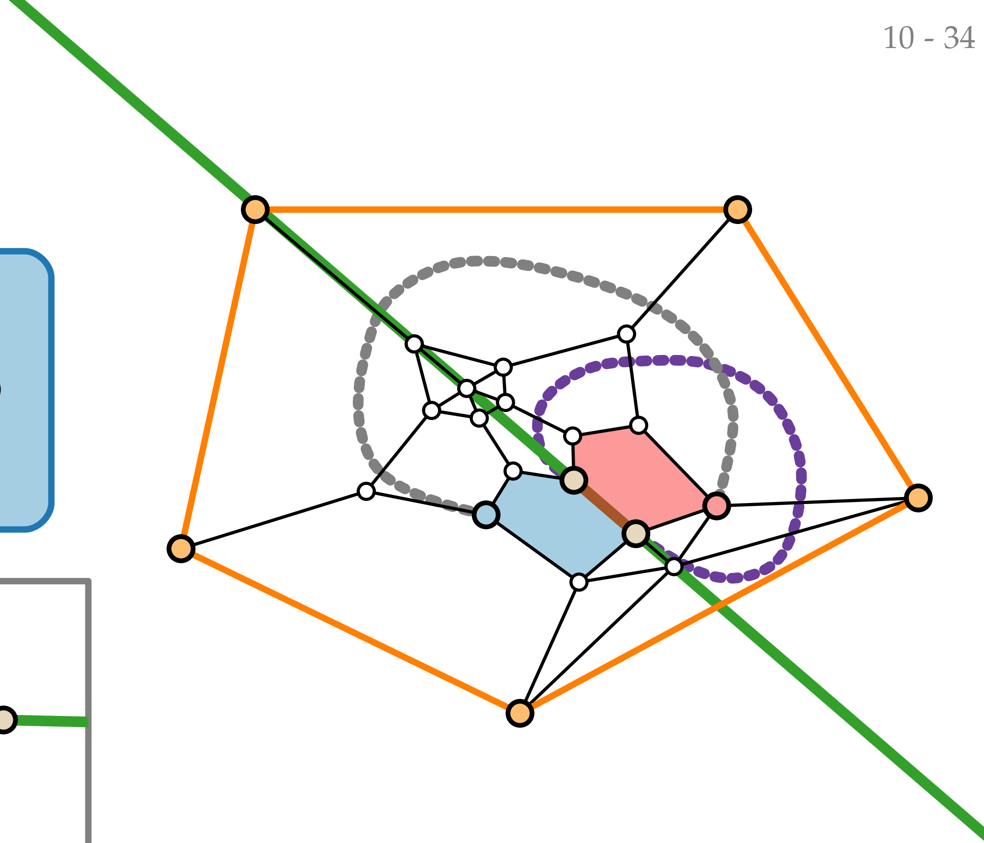

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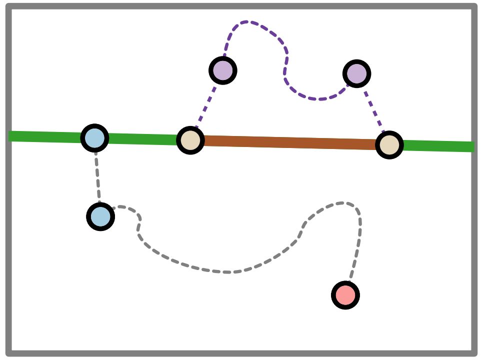

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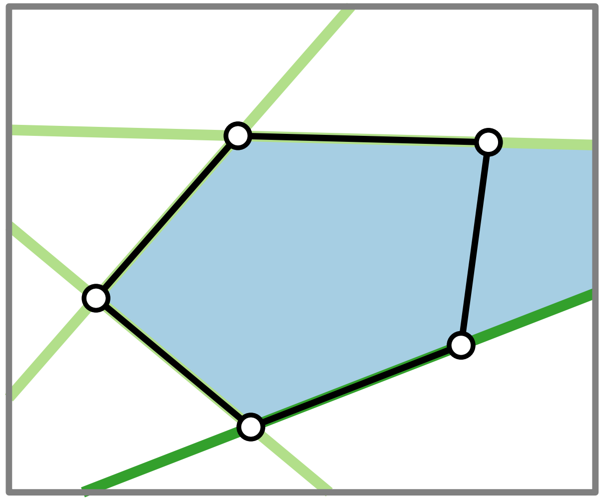

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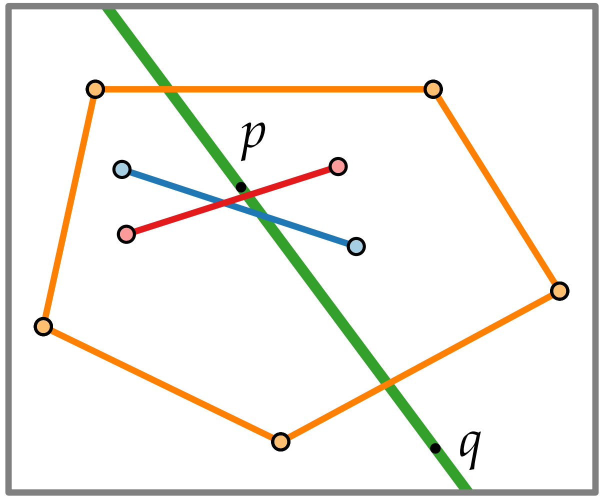

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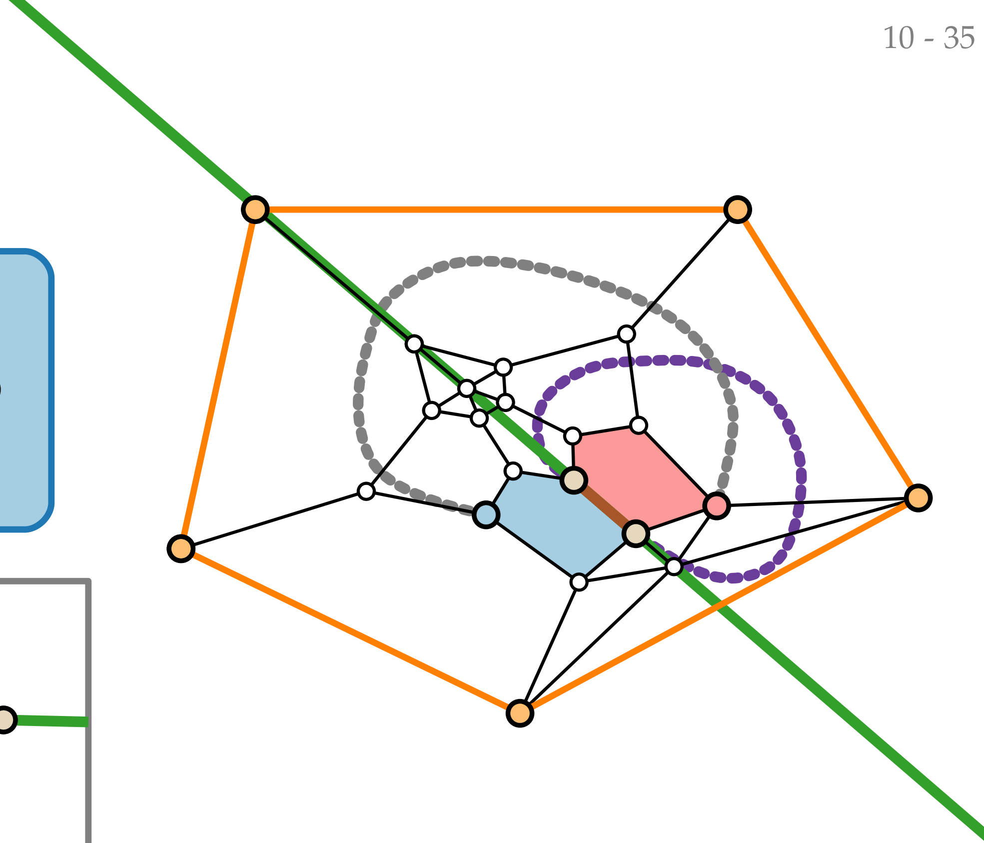

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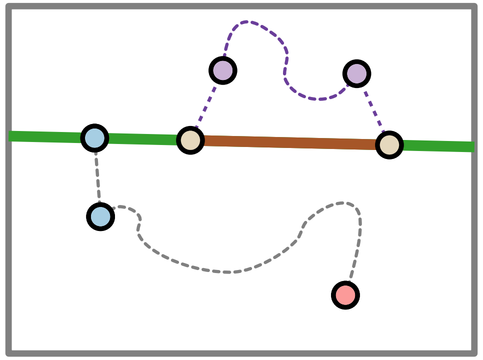

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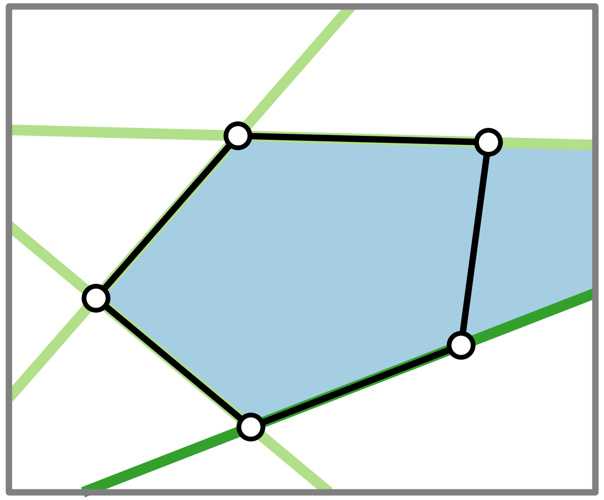

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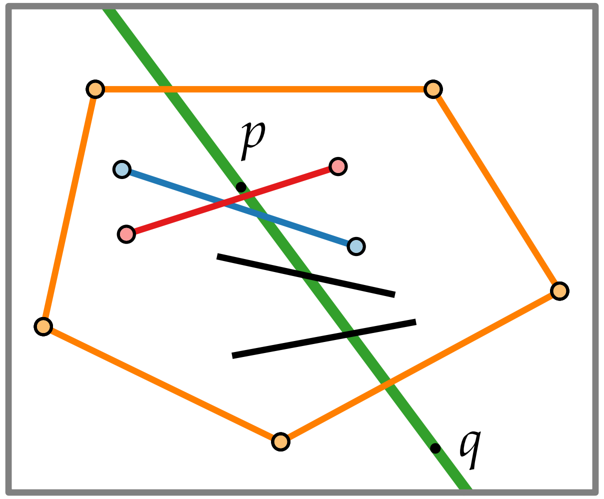

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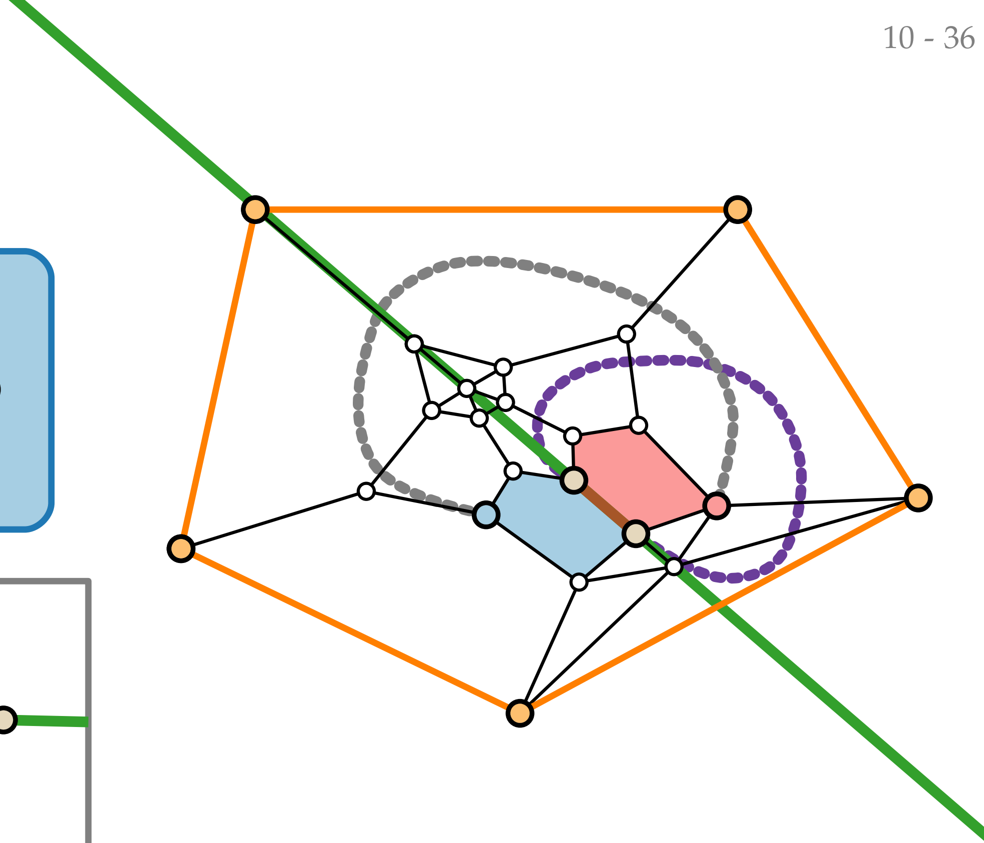

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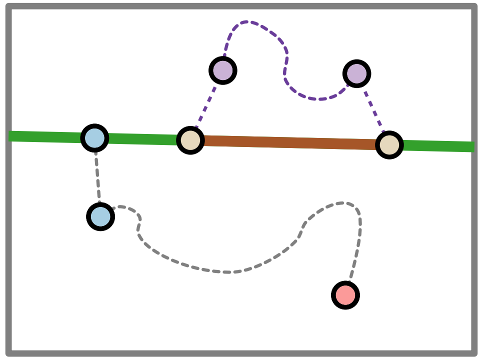

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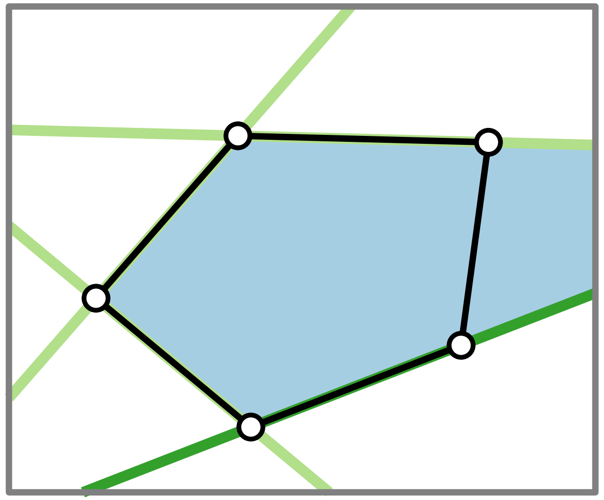

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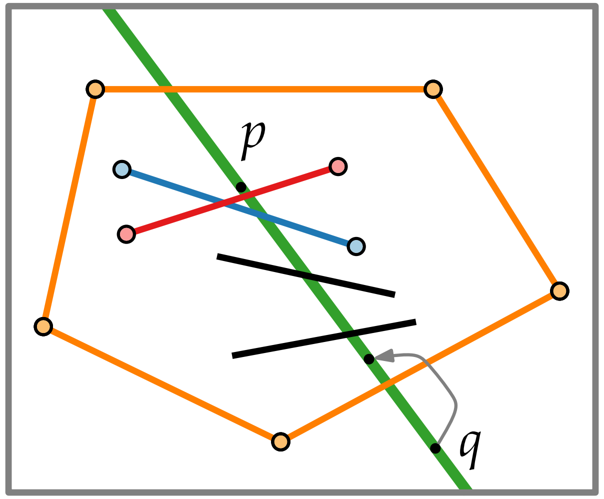
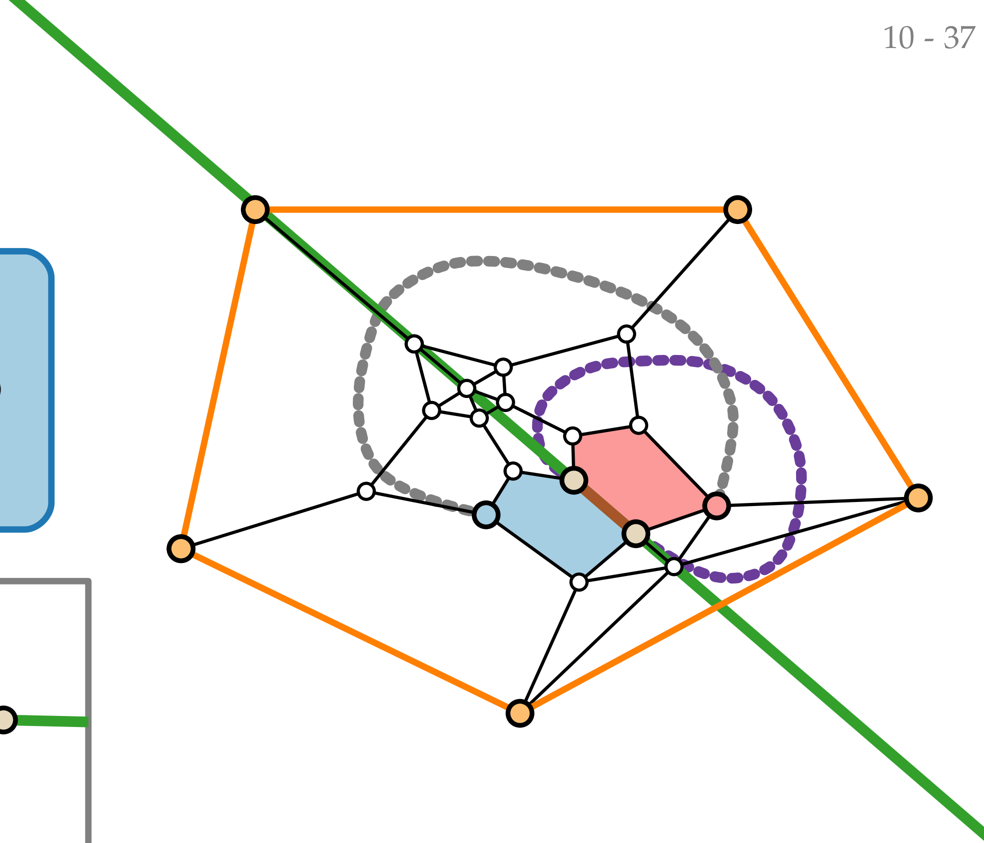


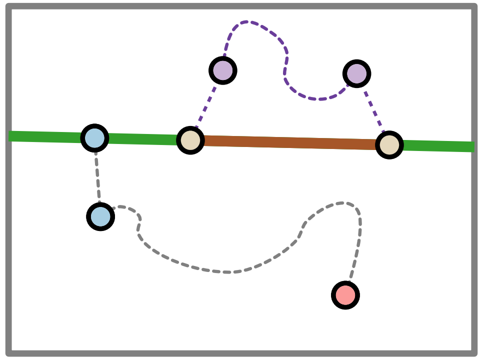
Proof of Tutte's Theorem



Lemma. Let $uv \in E$ be a non-boundary edge, ℓ line through uv. Then the two faces f_1, f_2 incident to uv lie completely on opposite sides of ℓ.

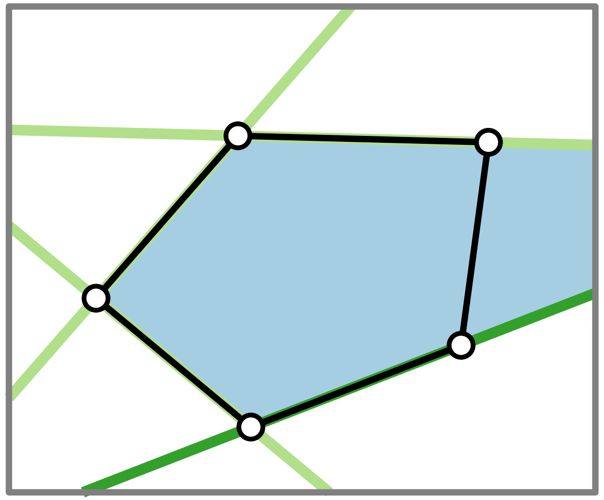
Property 1. Let $v \in V$ free, ℓ line through v.
$\exists uv \in E$ on one side of $\ell \Rightarrow \exists vw \in E$ on other side

Property 3. Let ℓ be any line.
Let V_ℓ be all vertices on one side of ℓ.
Then $G[V_\ell]$ is connected.



Property 4. No vertex is collinear with all of its neighbors.

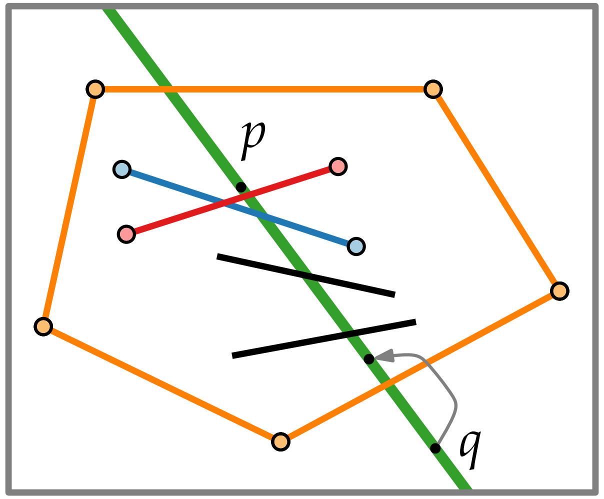
Lemma. All faces are strictly convex.



Property 2. All free vertices lie inside C.

p inside two faces
$\Rightarrow q$ in one face
jumping over edge
$\rightarrow$ #faces the same

Lemma. The drawing is planar.



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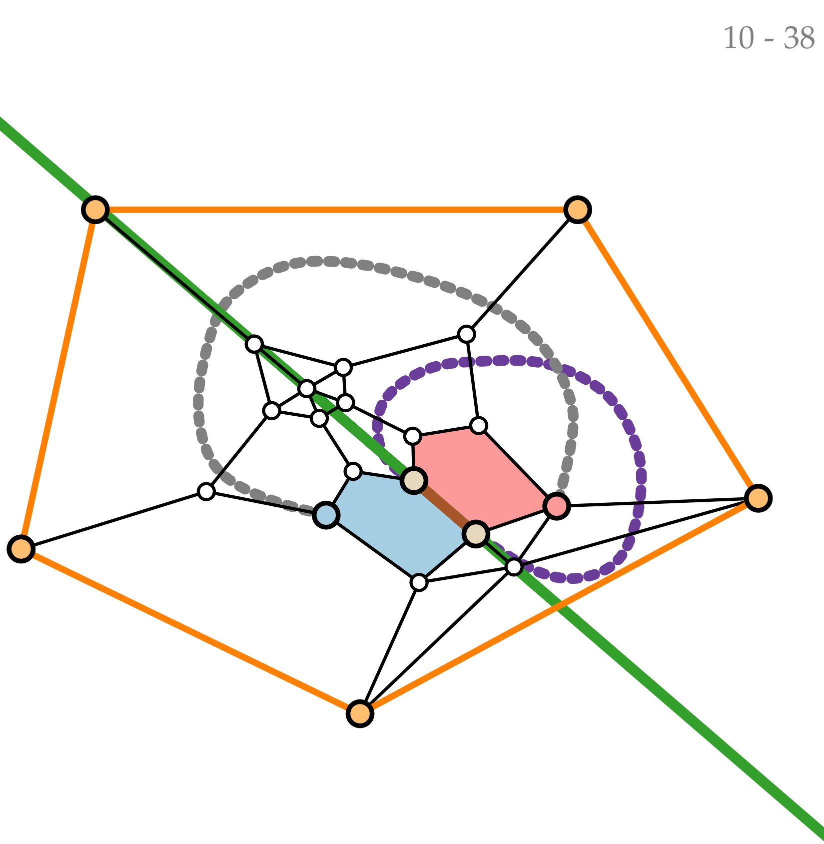

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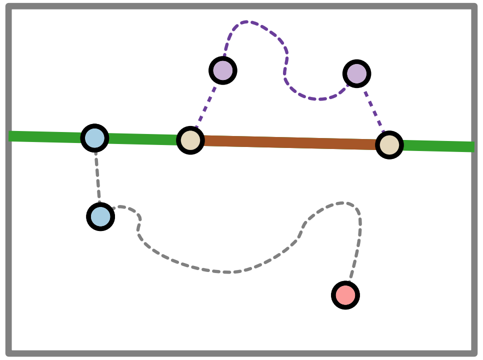

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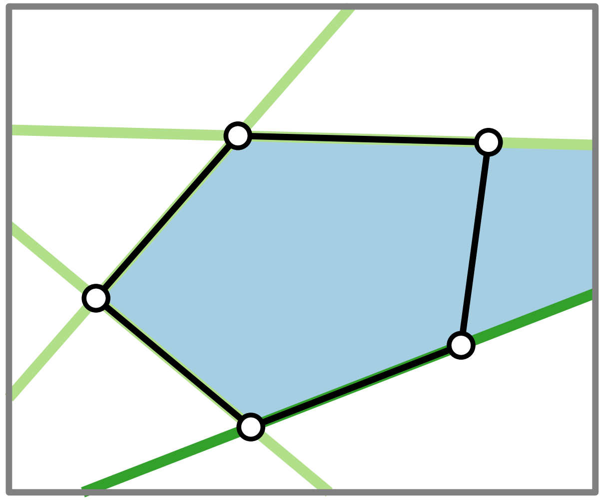

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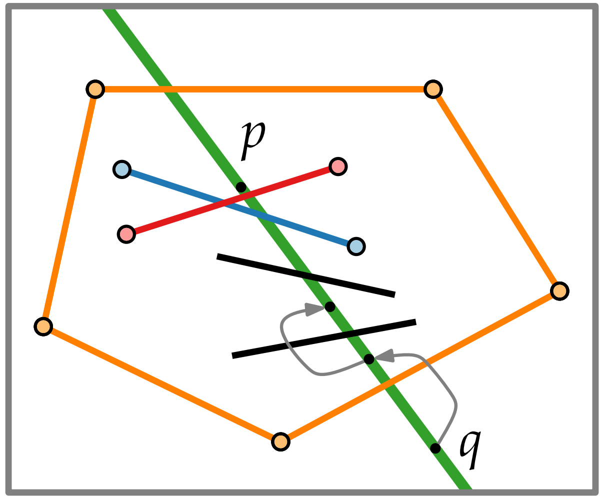

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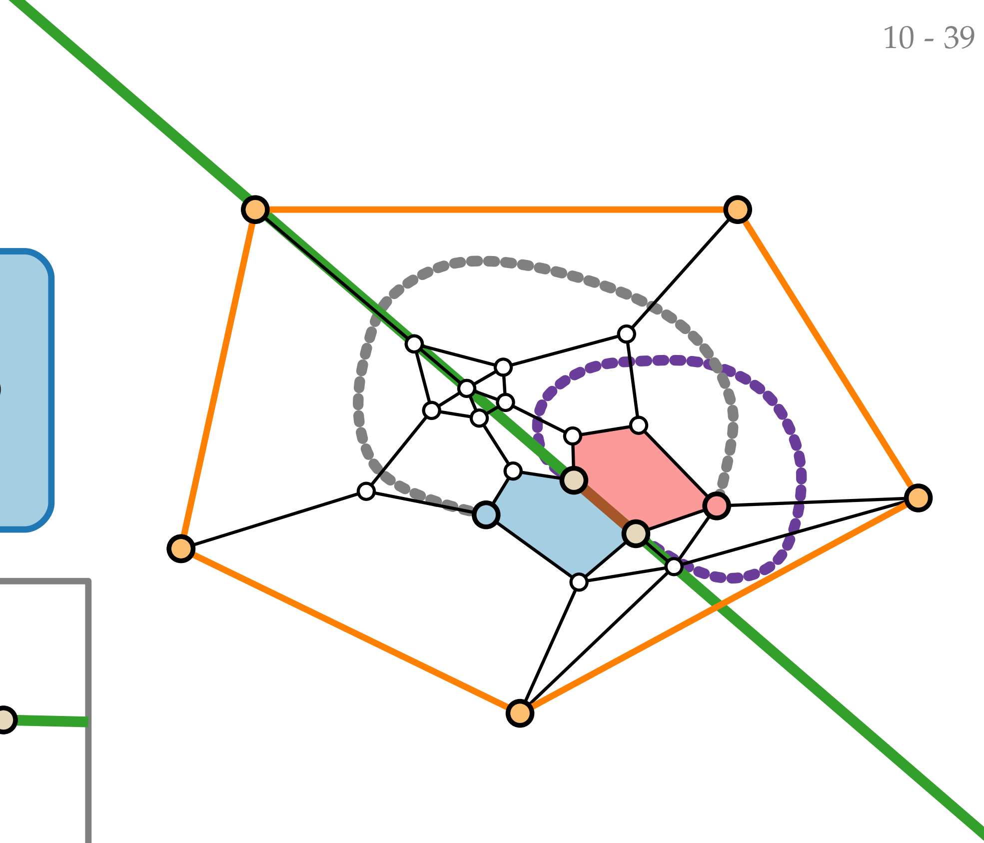

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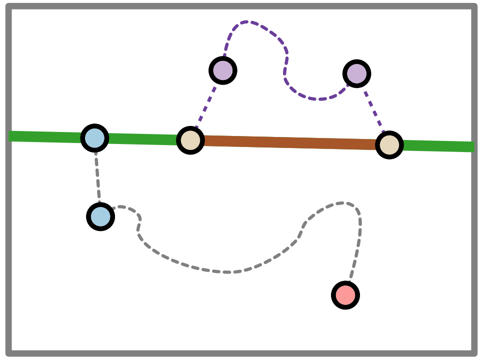

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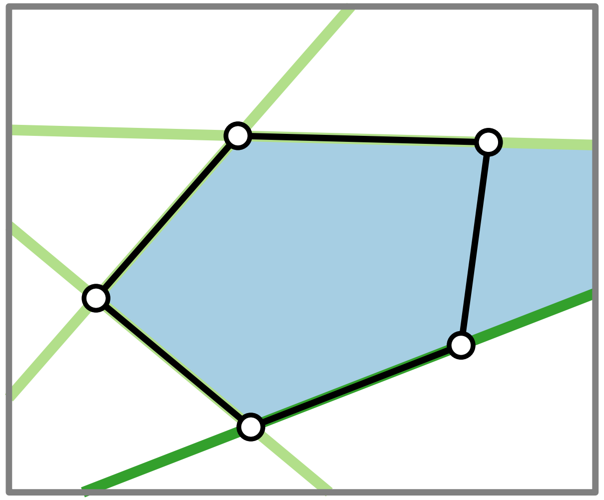

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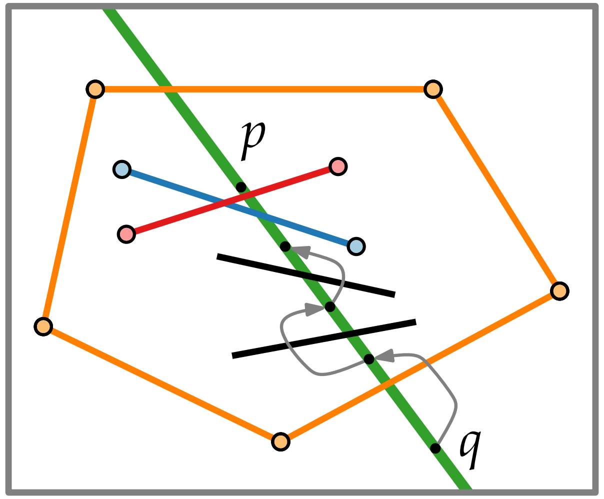

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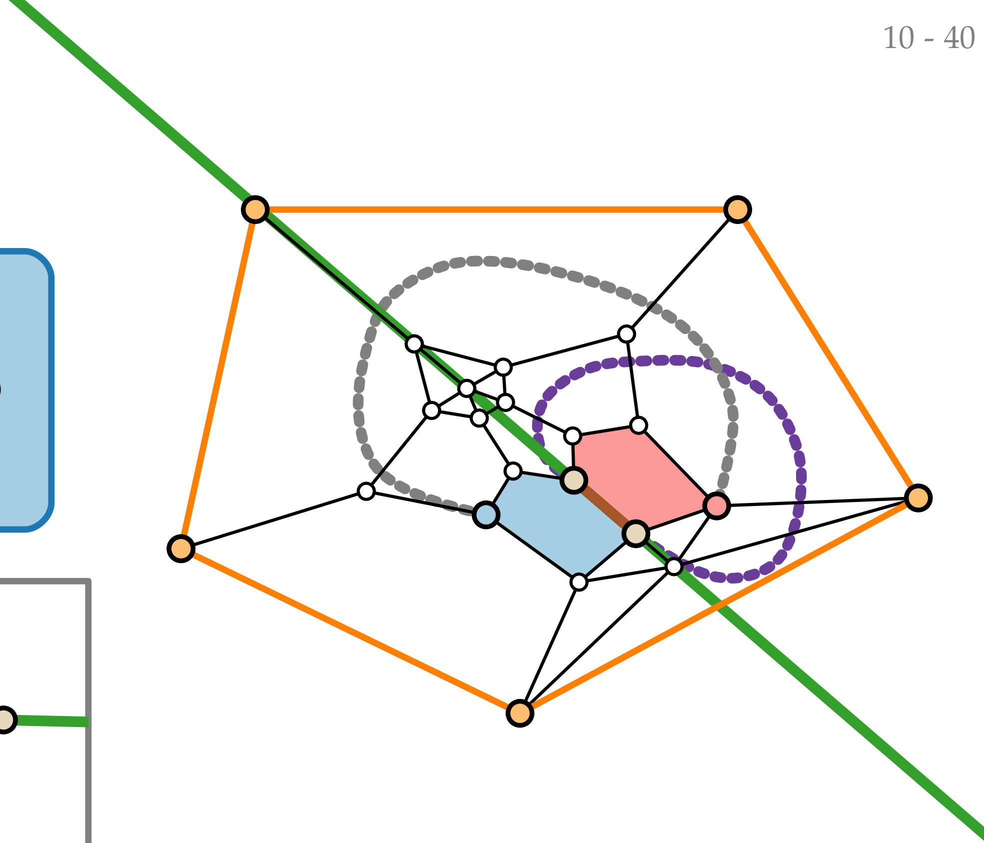

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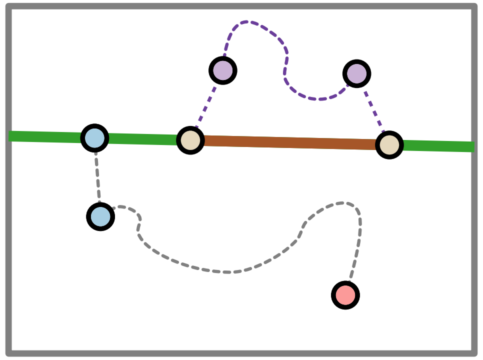

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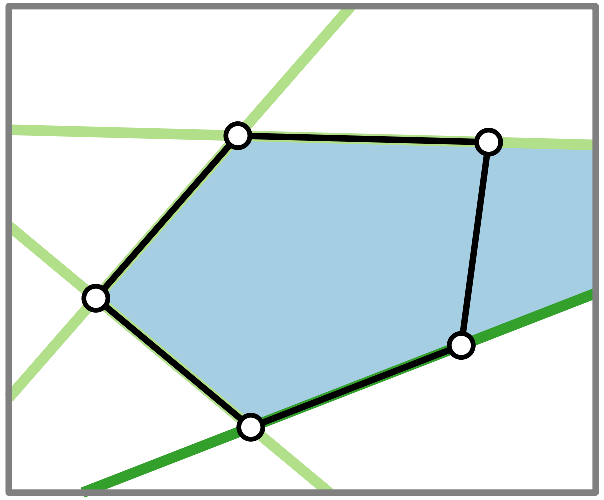

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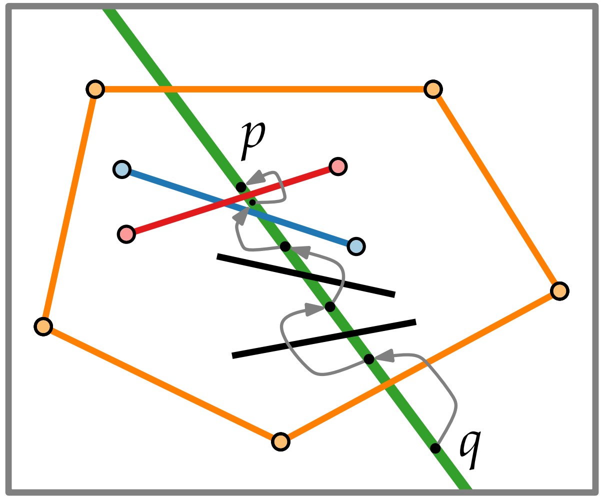

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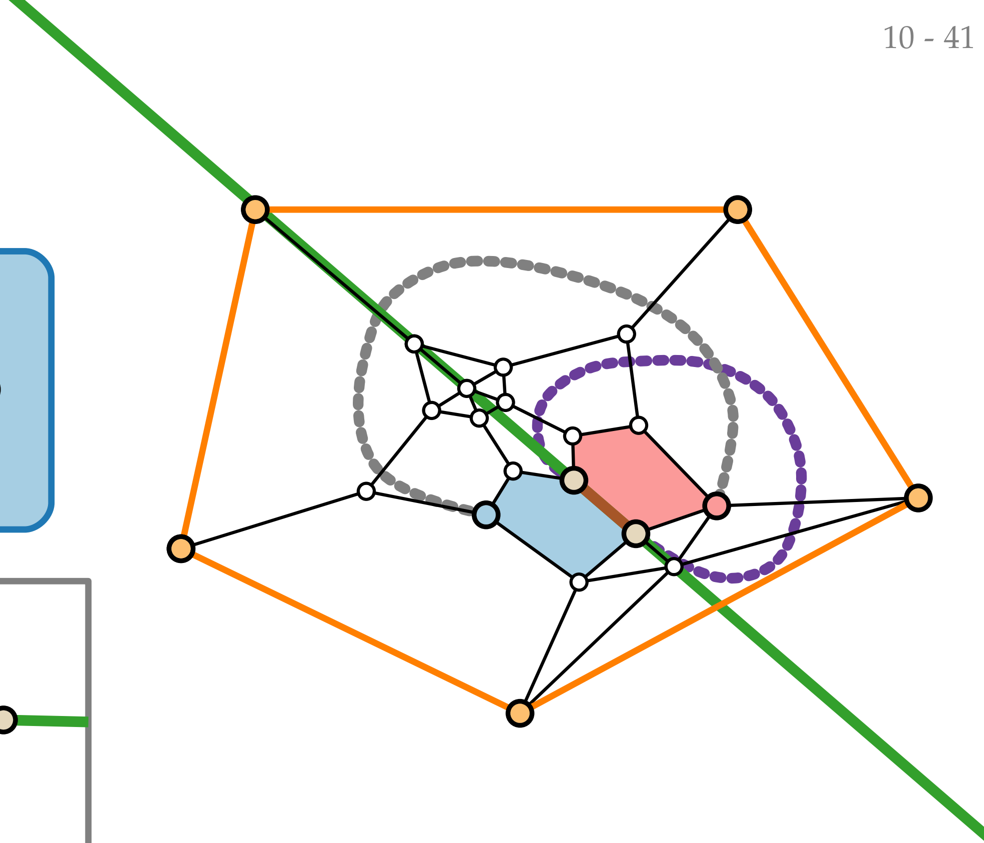

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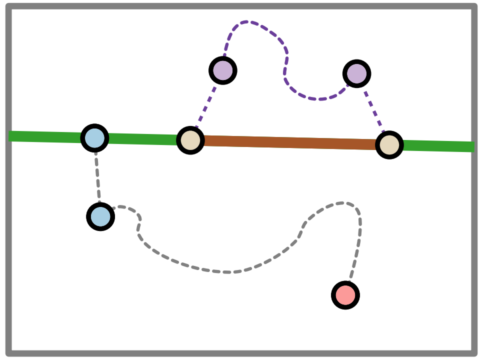

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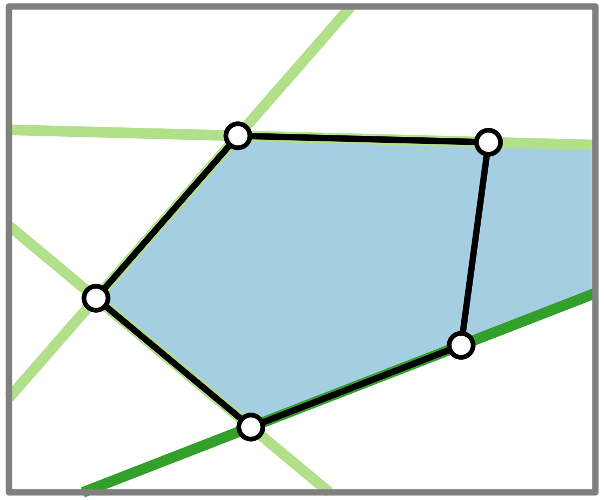

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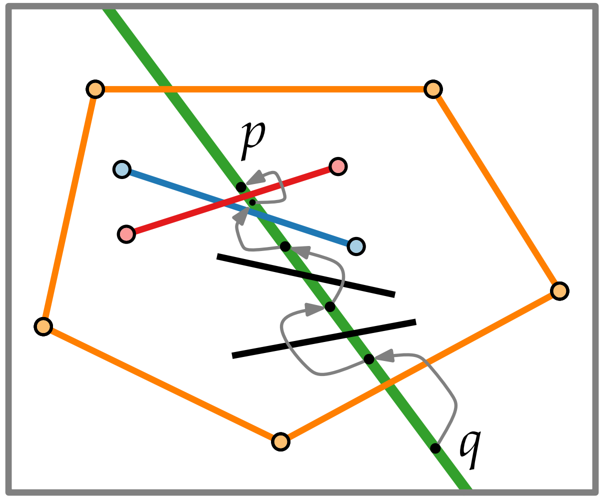

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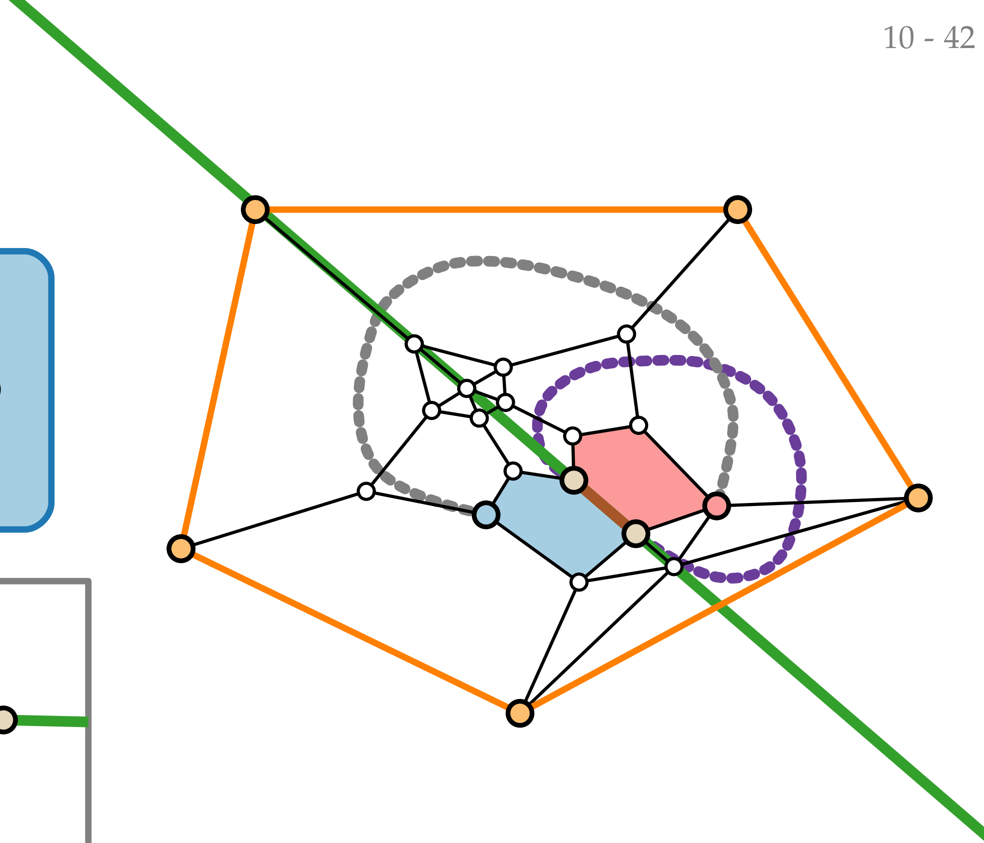

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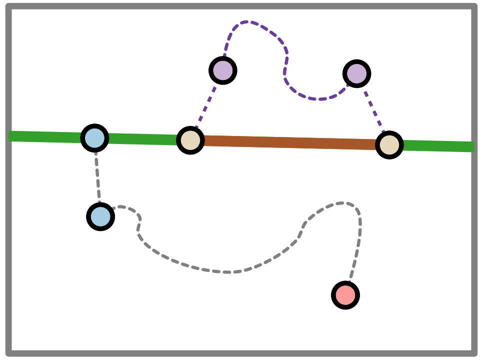

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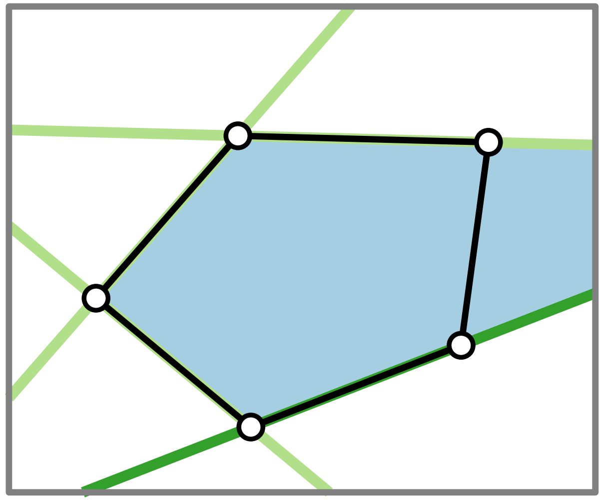

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