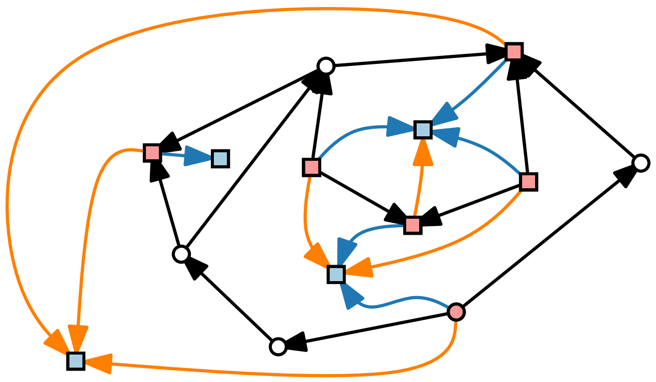
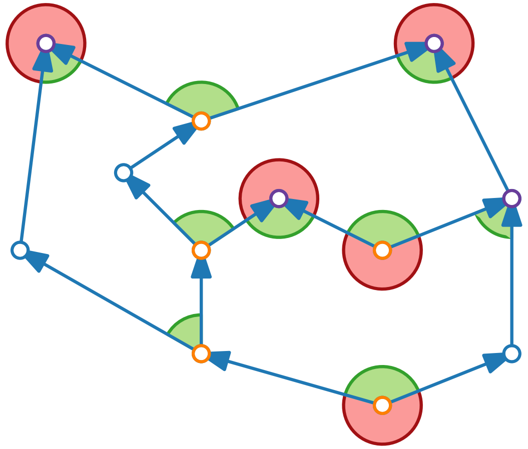
CS F402: Computational Geometry



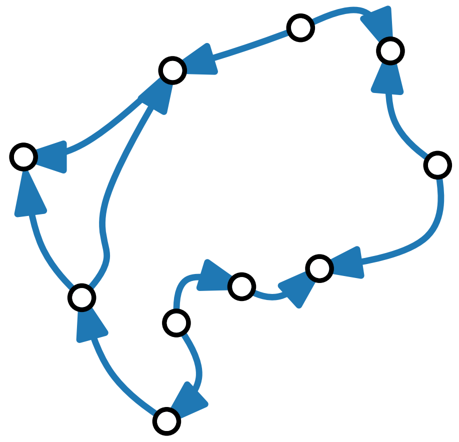
Lecture 14:
Upward Planar Drawings

Siddharth Gupta

April 28, 2025



Upward Planar Drawings – Motivation



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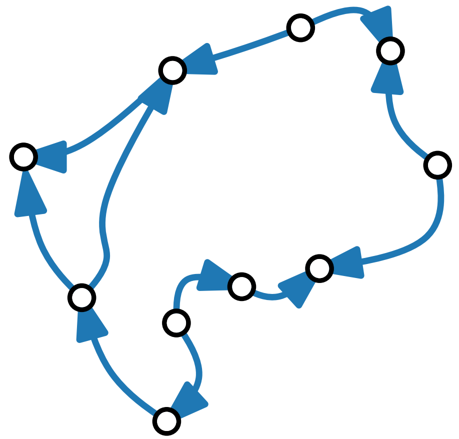
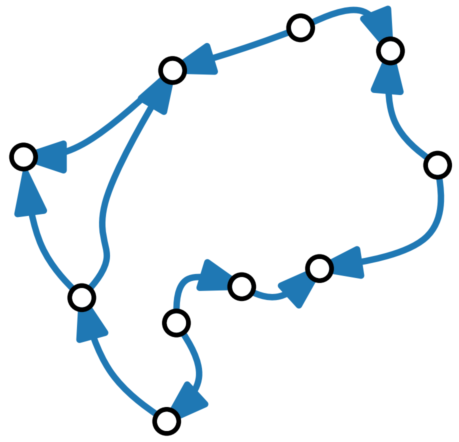
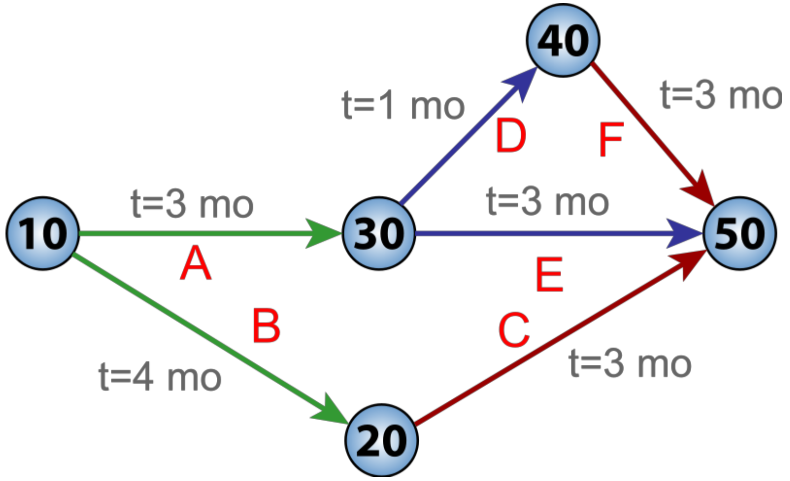


- What may the direction of edges in a digraph represent?

Upward Planar Drawings – Motivation



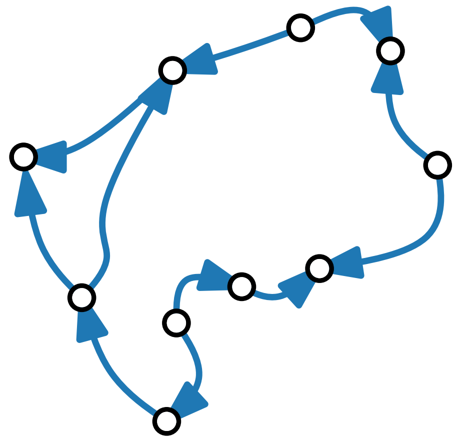
- What may the direction of edges in a digraph represent?
  - Time

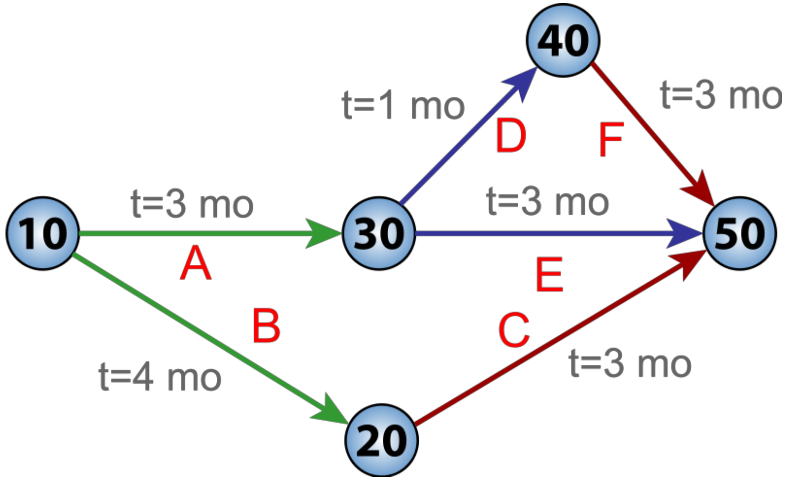


PERT diagram

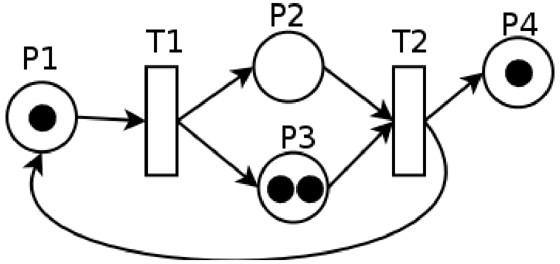
Upward Planar Drawings – Motivation

- What may the direction of edges in a digraph represent?
  - Time
  - Flow



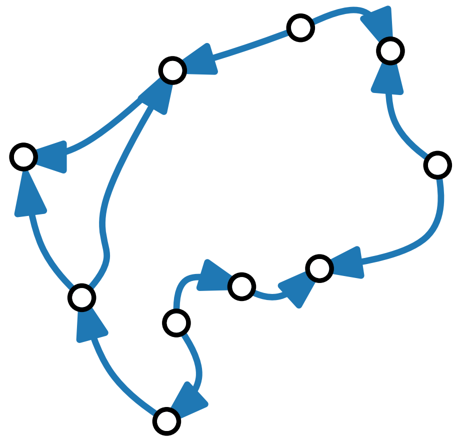


PERT diagram

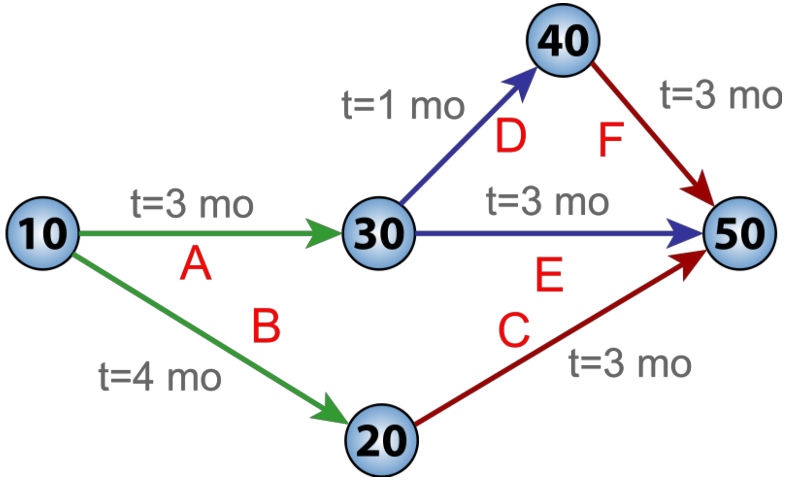


Petri net

Upward Planar Drawings – Motivation



- What may the direction of edges in a digraph represent?
  - Time
  - Flow
  - Hierarchy



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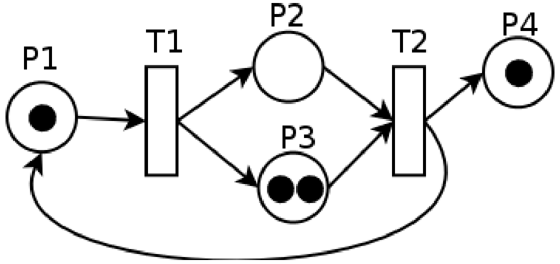

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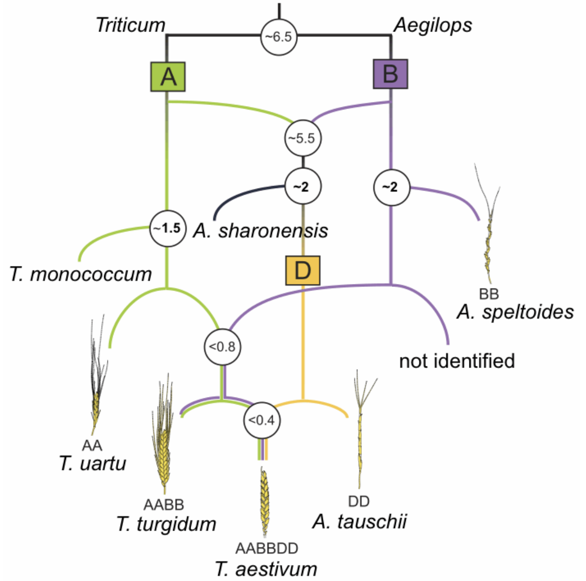
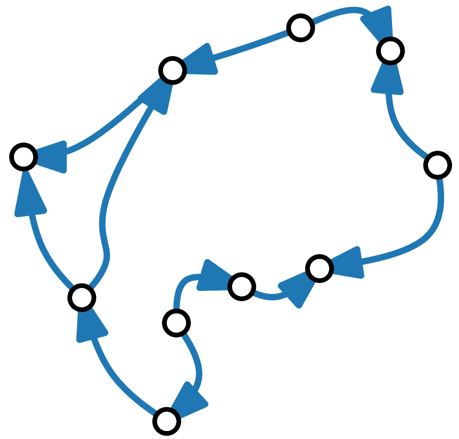
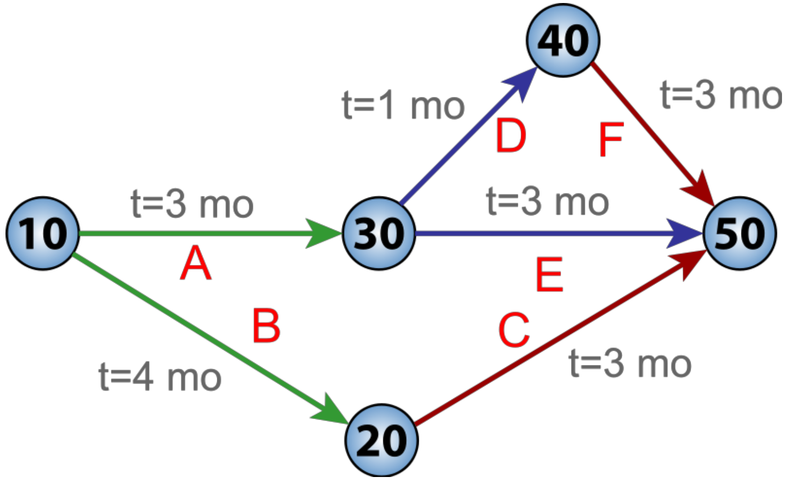


Phylogenetic network

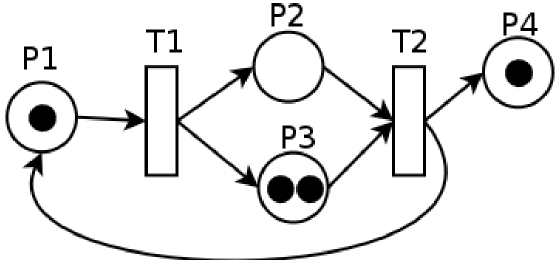
Upward Planar Drawings – Motivation

- What may the direction of edges in a digraph represent?
  - Time
  - Flow
  - Hierarchy
  - ...

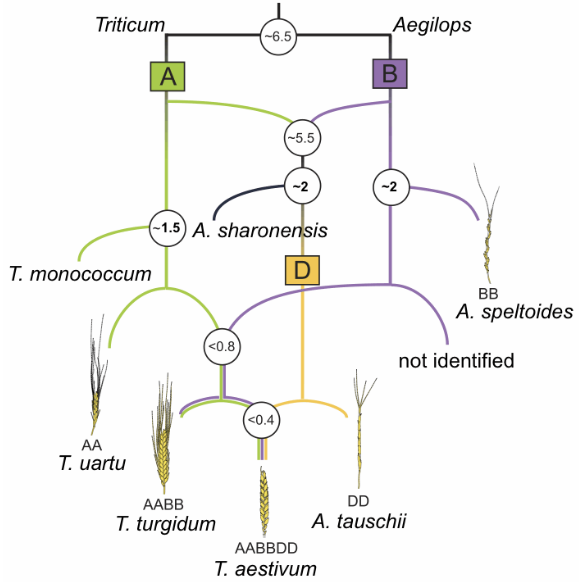




PERT diagram

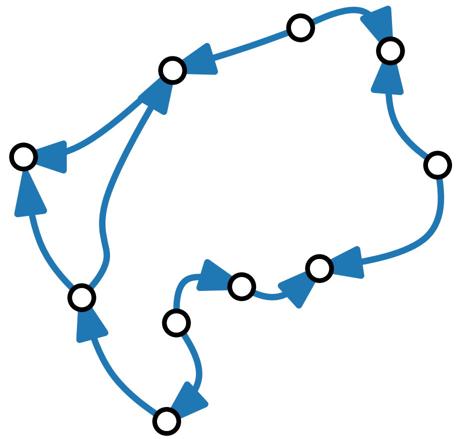


Petri net

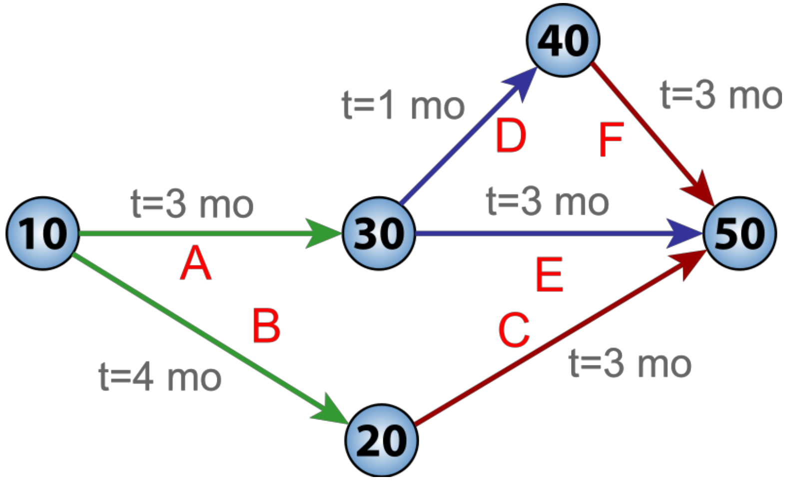


Phylogenetic network

Upward Planar Drawings – Motivation



- What may the direction of edges in a digraph represent?
  - Time
  - Flow
  - Hierarchy
  - ...
- Would be nice to have general direction preserved in drawing.



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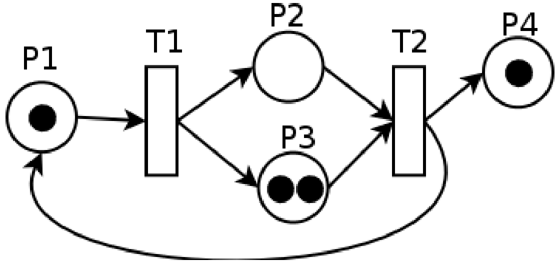

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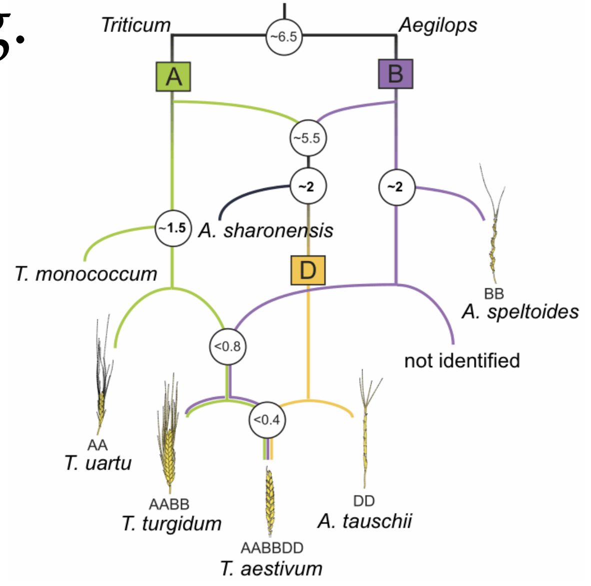
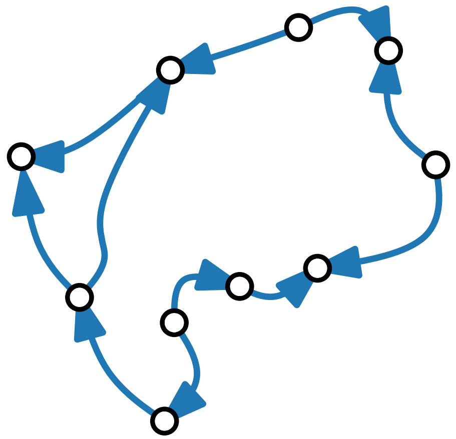


Phylogenetic network

Upward Planar Drawings – Definition

A directed graph $G = (V, E)$ is **upward planar** when it admits a drawing Γ that is



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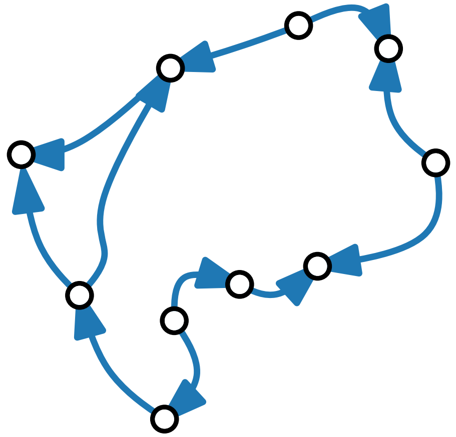

Upward Planar Drawings – Definition

A directed graph $G = (V, E)$ is **upward planar** when it admits a drawing Γ that is

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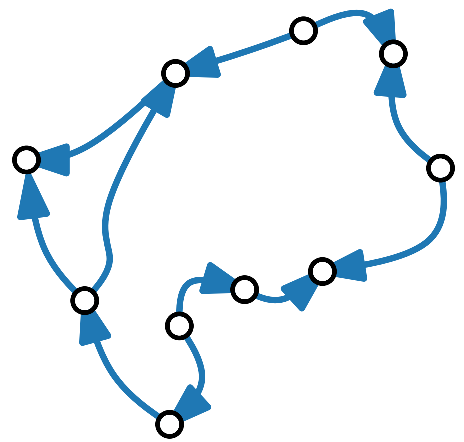
- where each edge is drawn as an upward, y-monotone curve.



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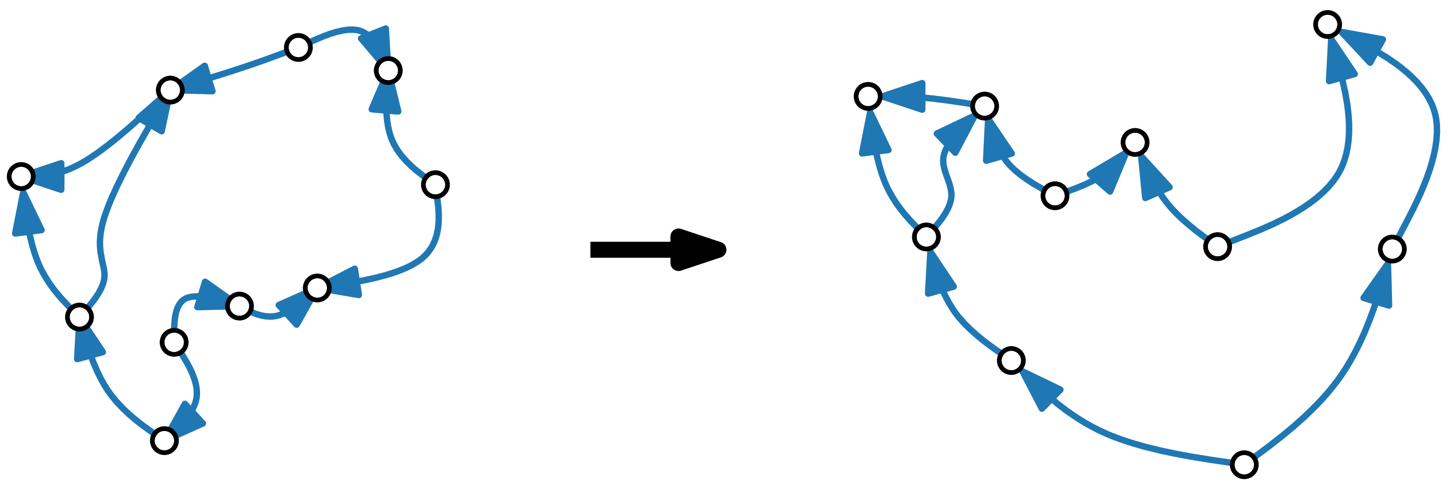
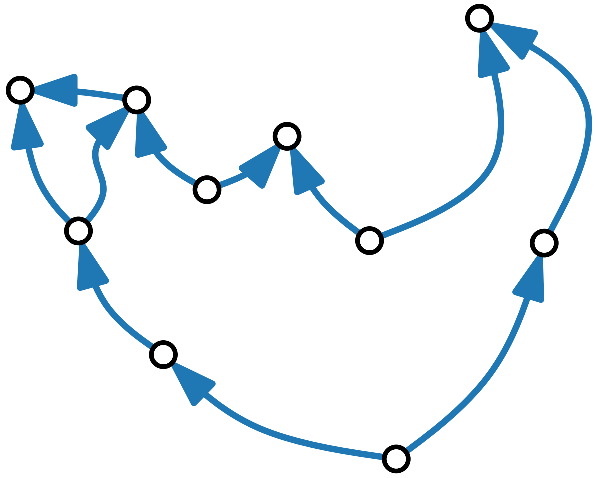


Upward Planarity – Necessary Conditions

- For a digraph G to be upward planar, it has to be:



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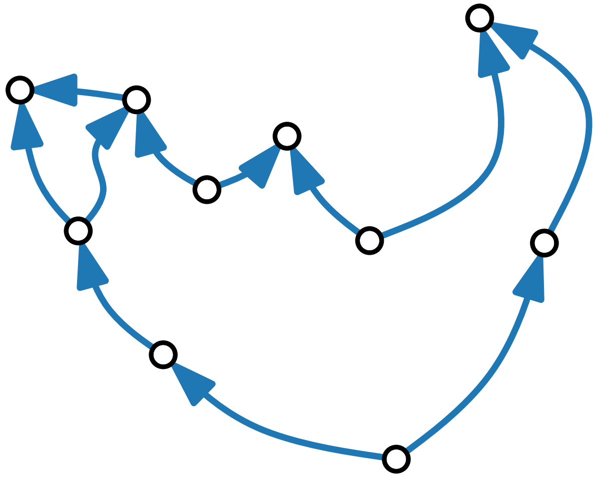

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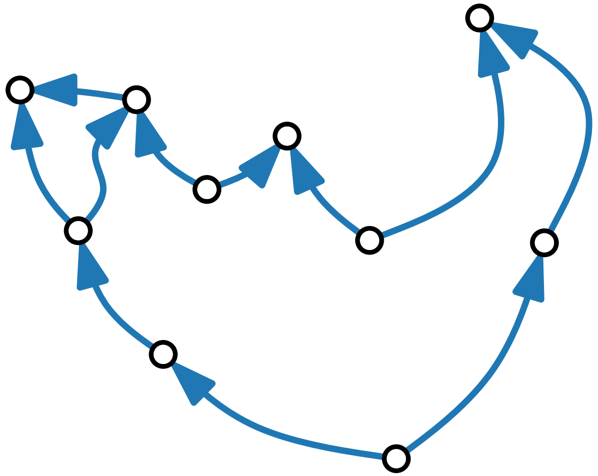

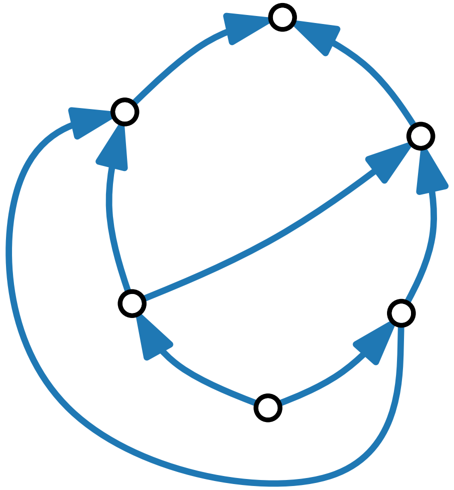

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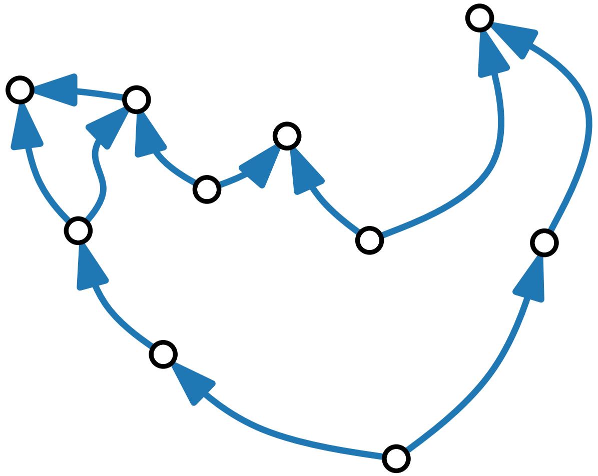

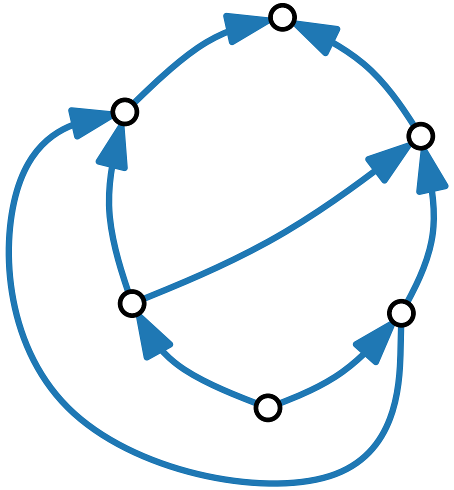

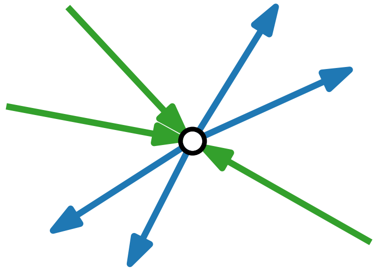

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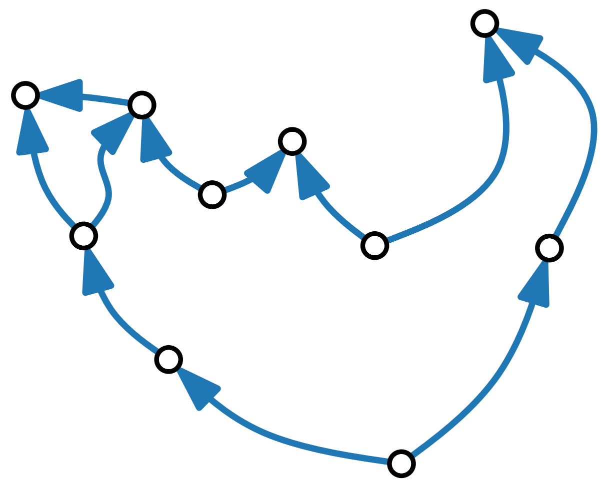

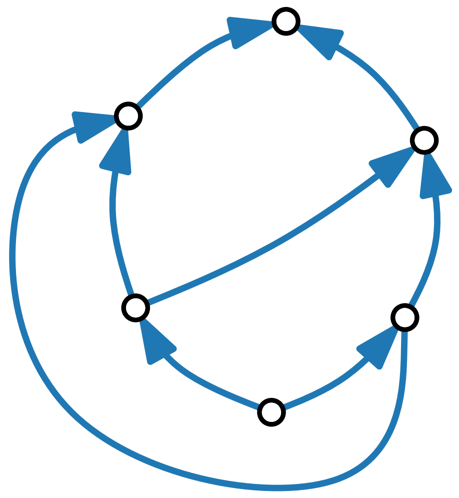

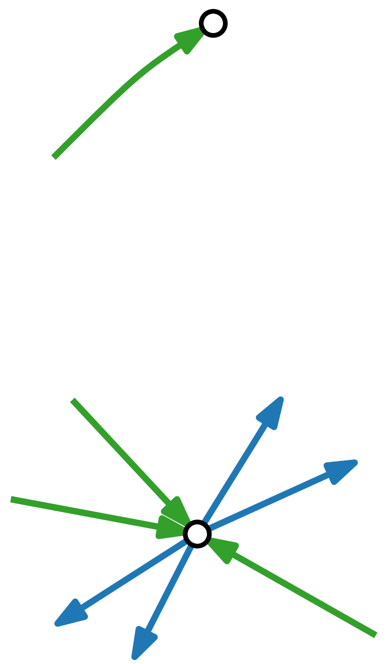

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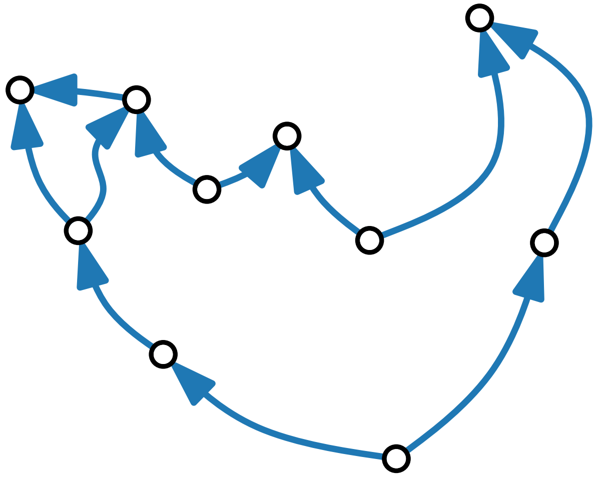

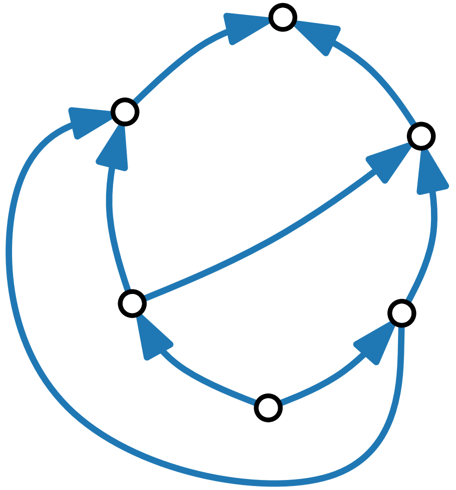

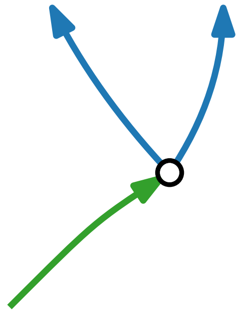

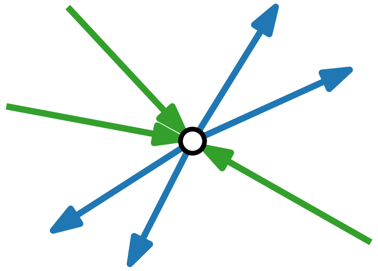

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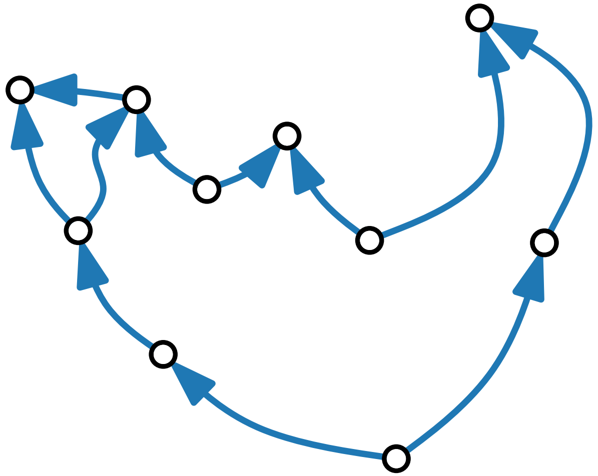

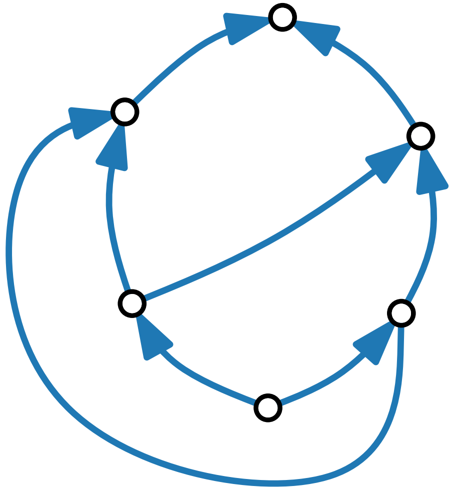

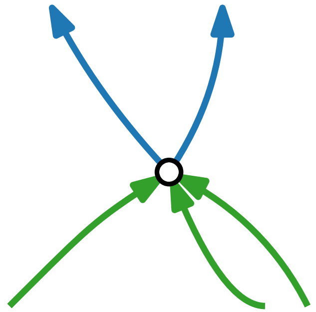

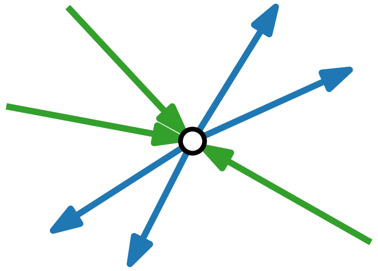

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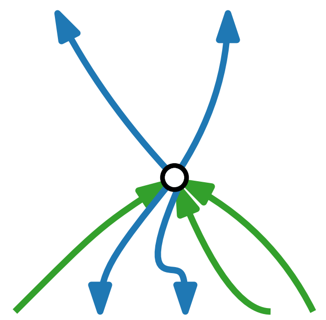

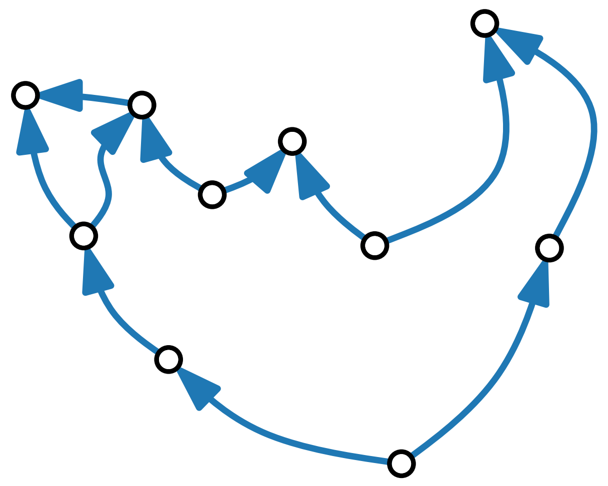

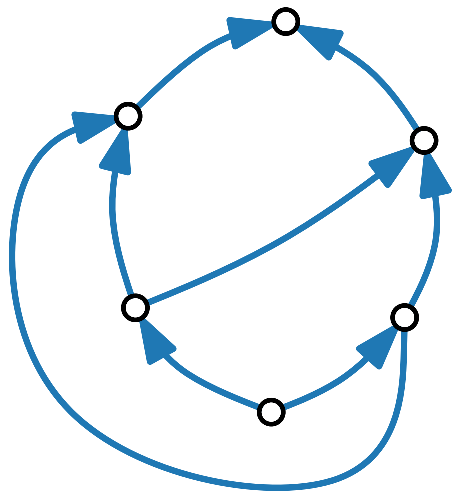

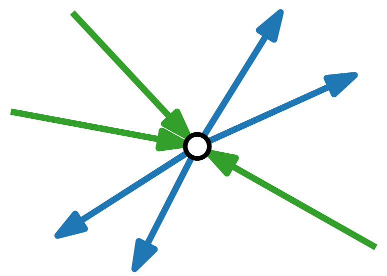

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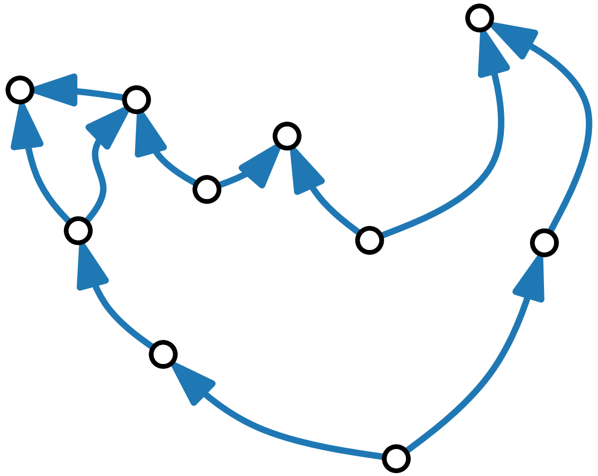

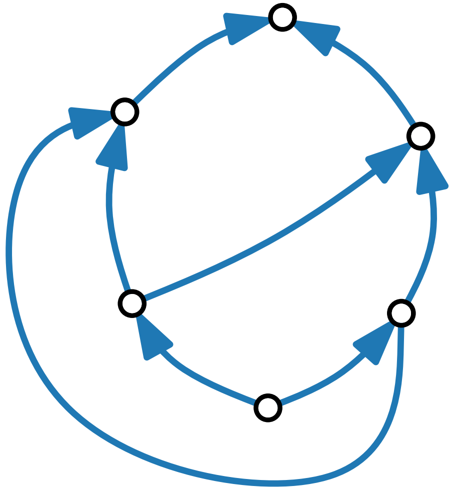

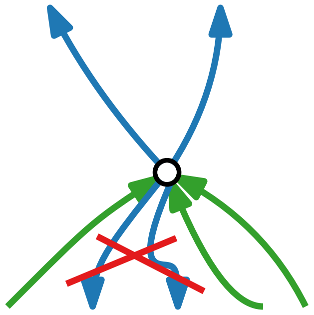

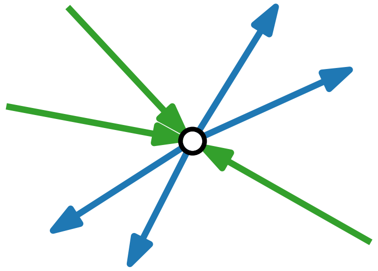

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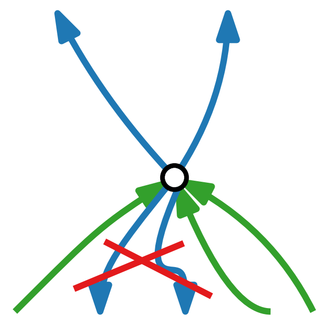

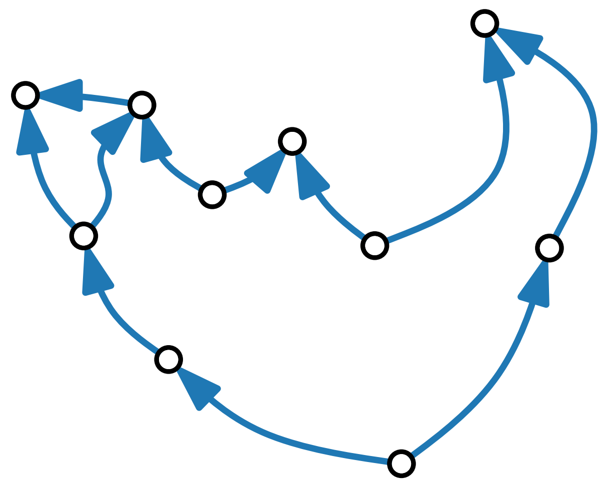

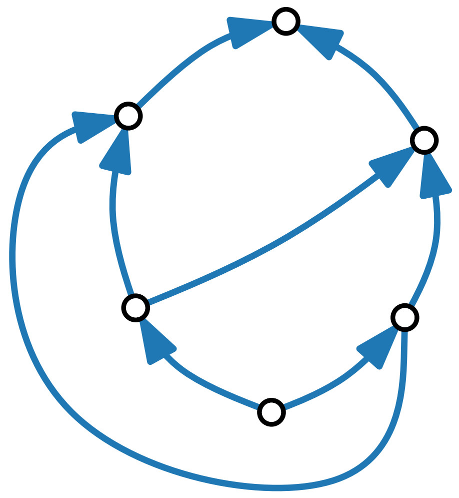

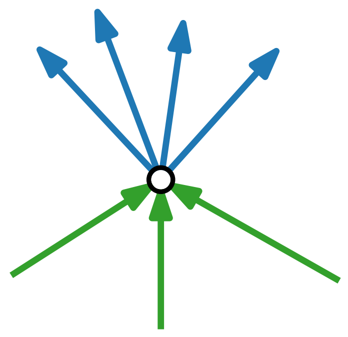

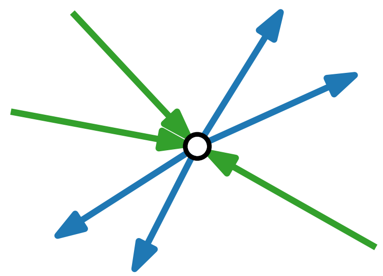

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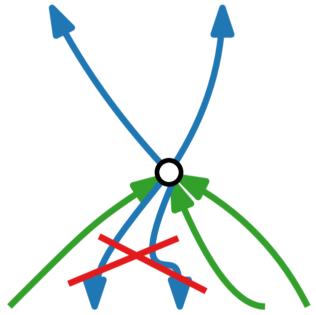
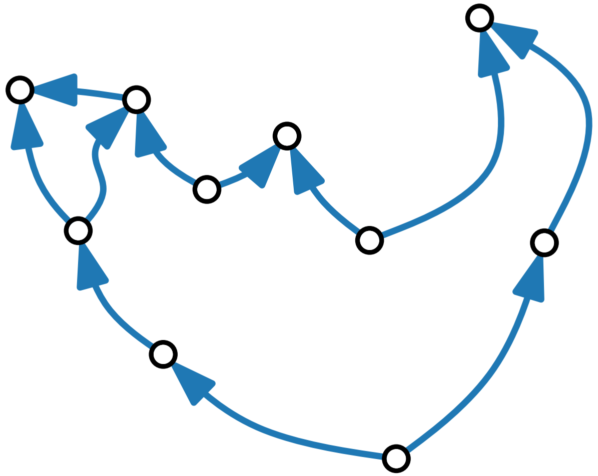
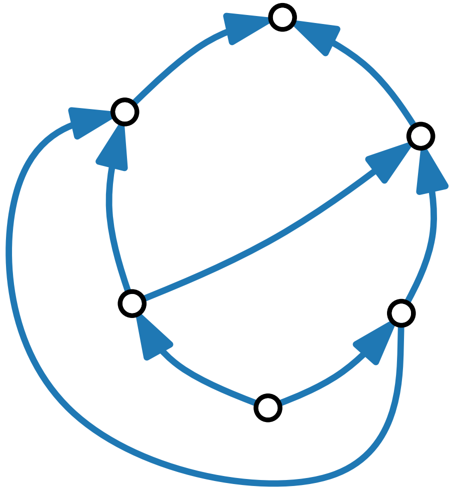
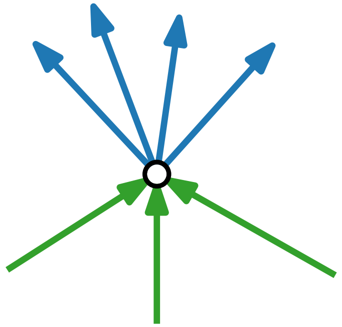


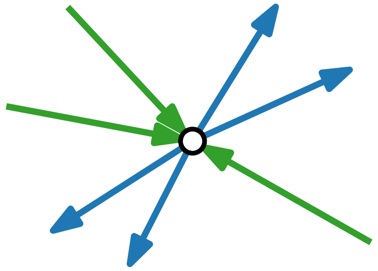




bimodal vertex



not bimodal



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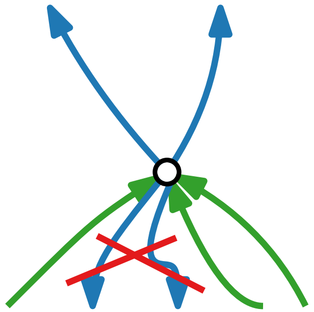

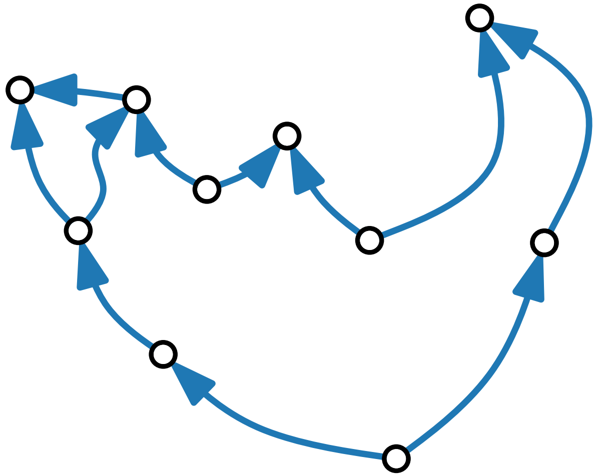

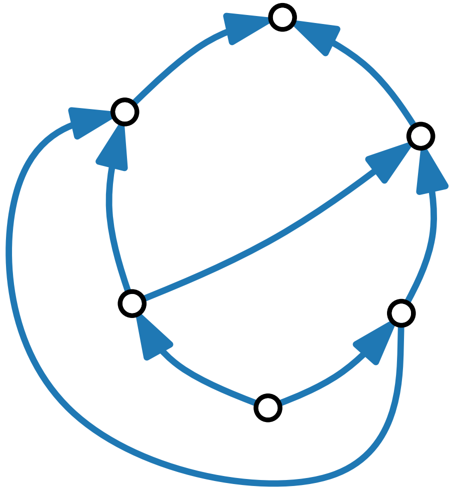

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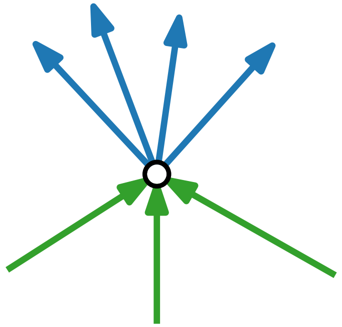

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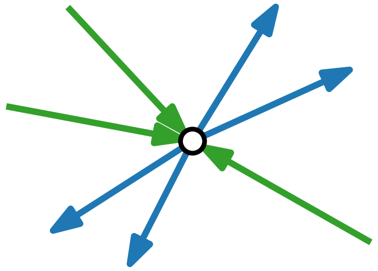
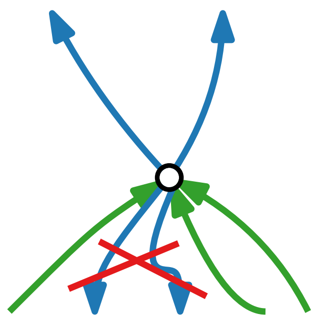
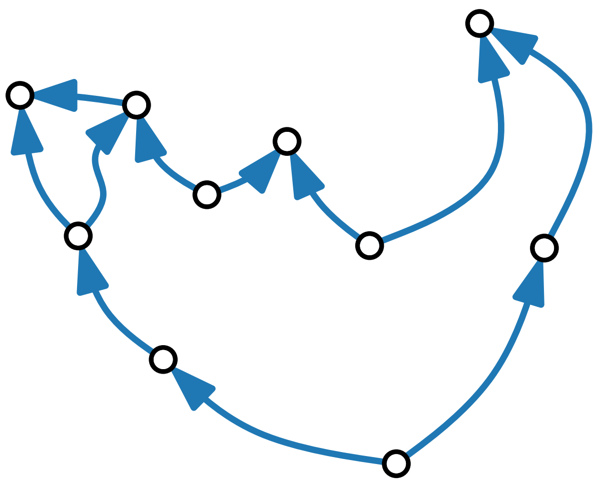
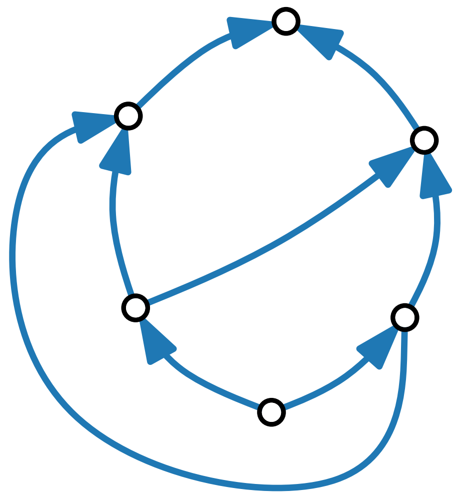


Upward Planarity – Necessary Conditions

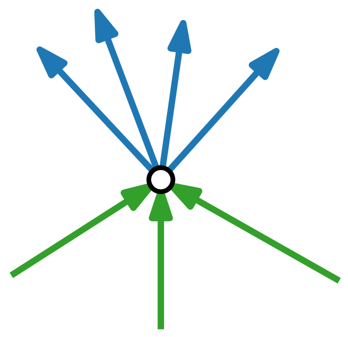
- For a digraph G to be upward planar, it has to be:
  - planar
  - acyclic
  - bimodal
- . . . but these conditions are *not sufficient*.



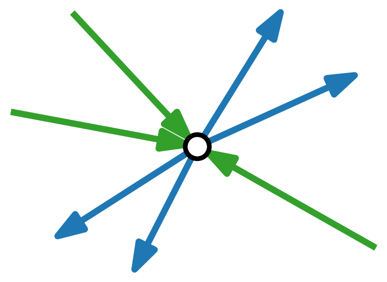




bimodal vertex



not bimodal



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acyclic digraph with
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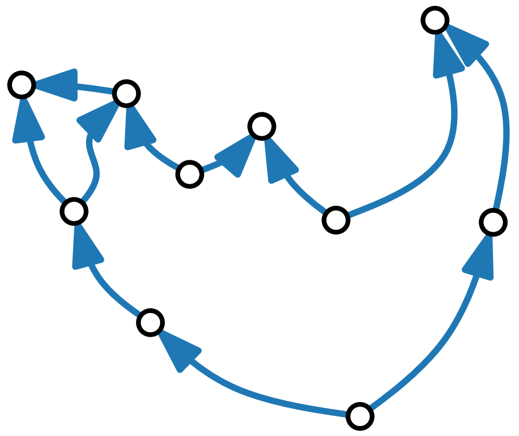

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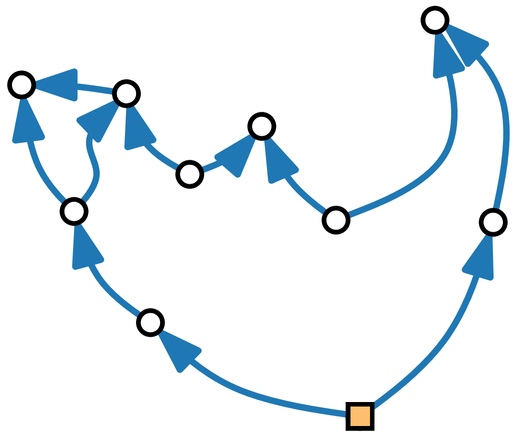

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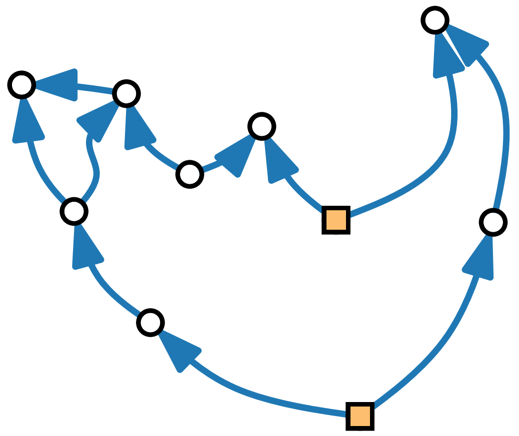

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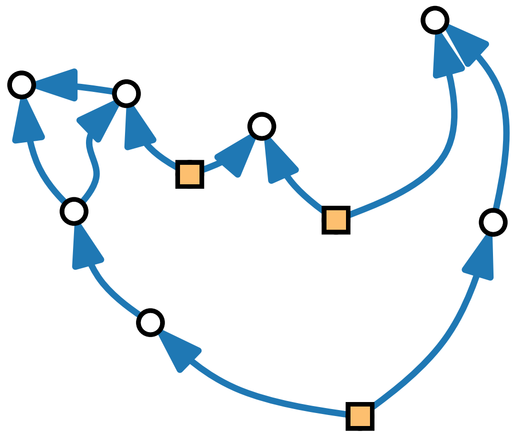

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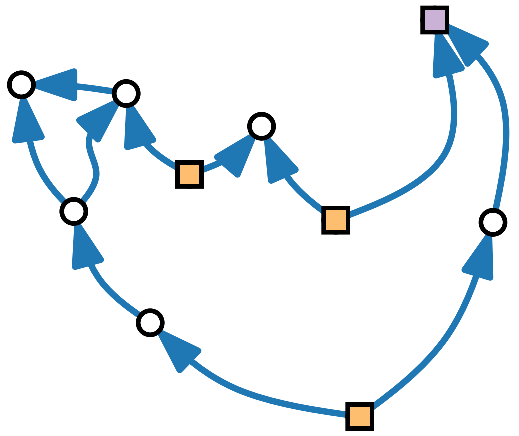

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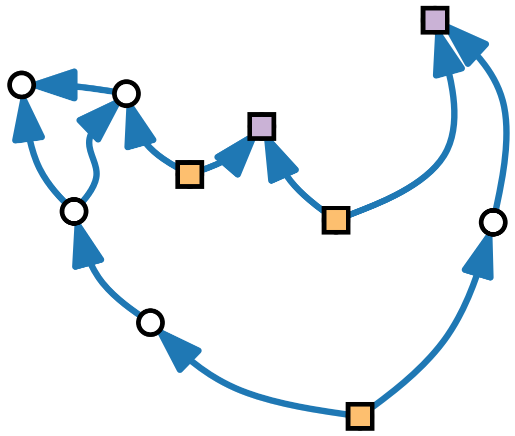

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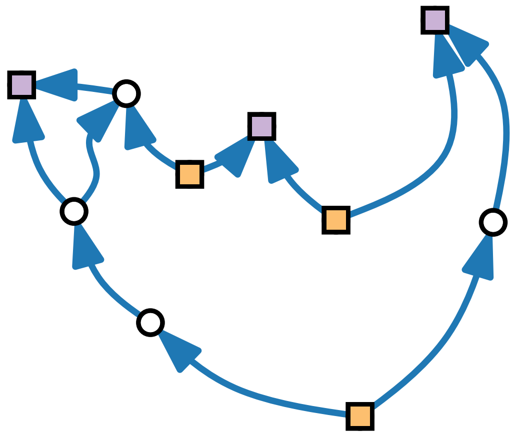

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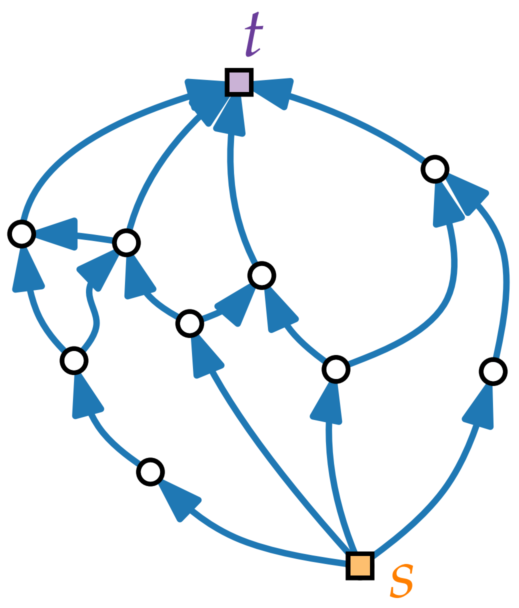

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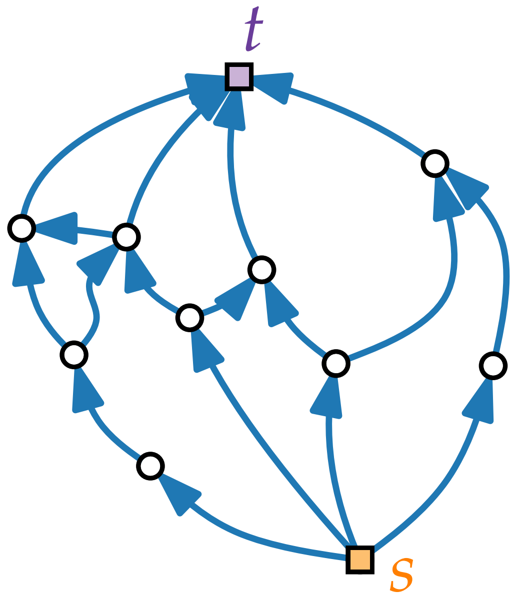

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Additionally:
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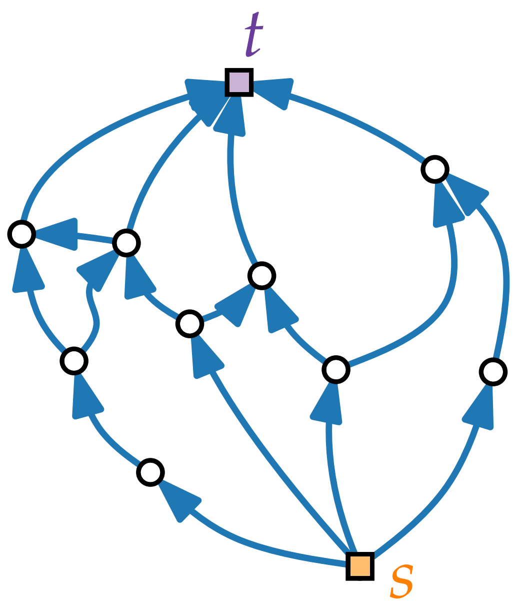

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or:
Edge (s,t) exists.

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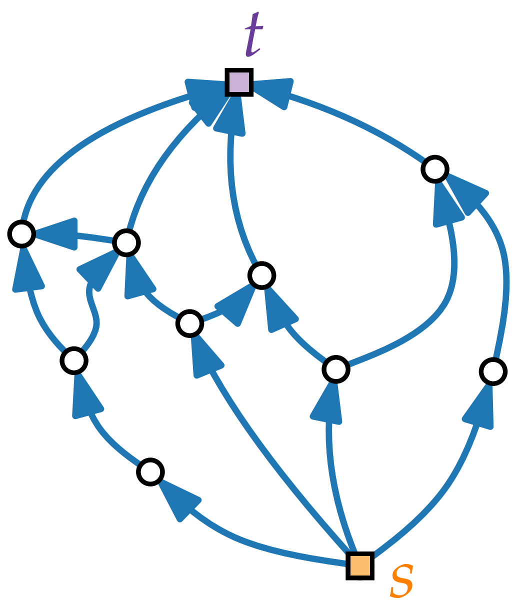

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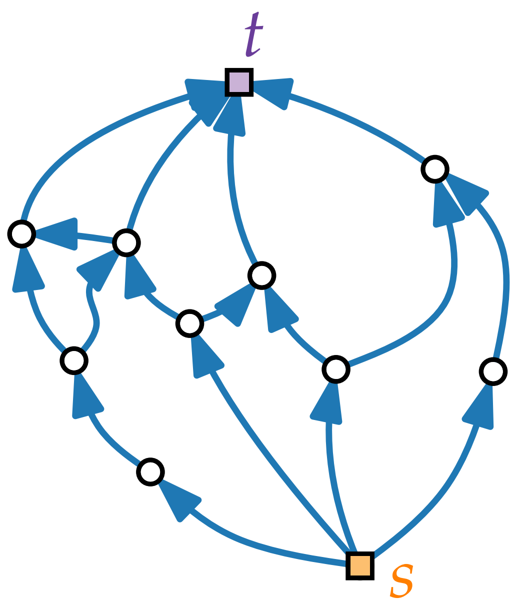

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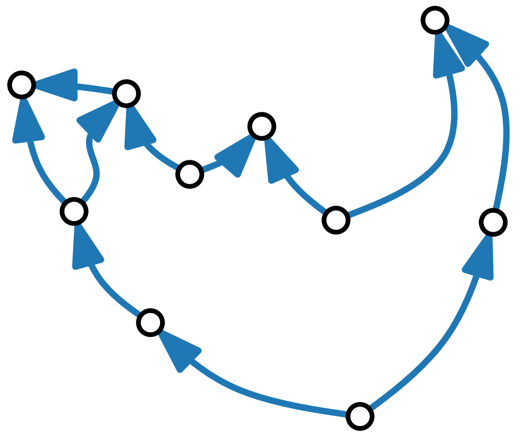

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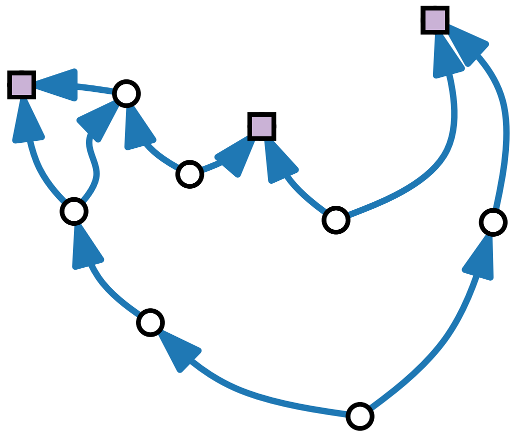

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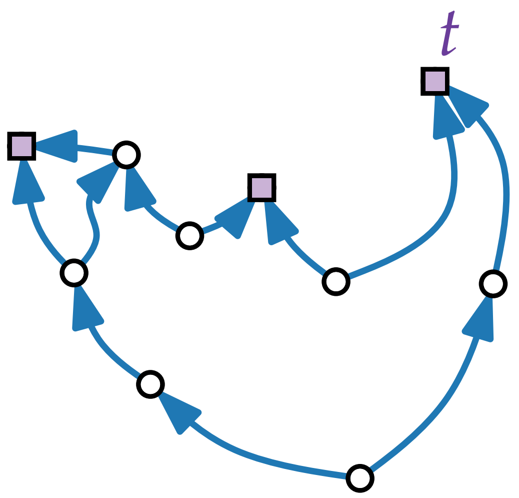

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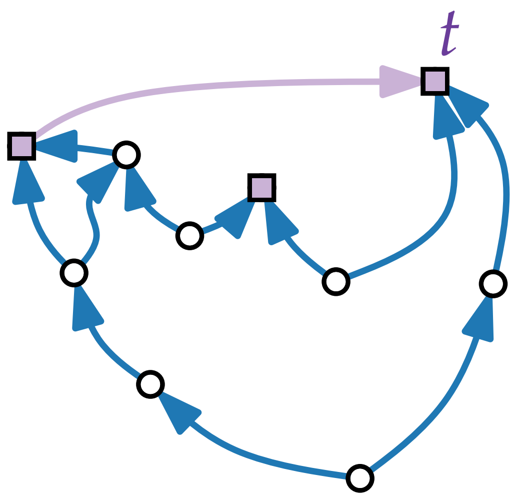

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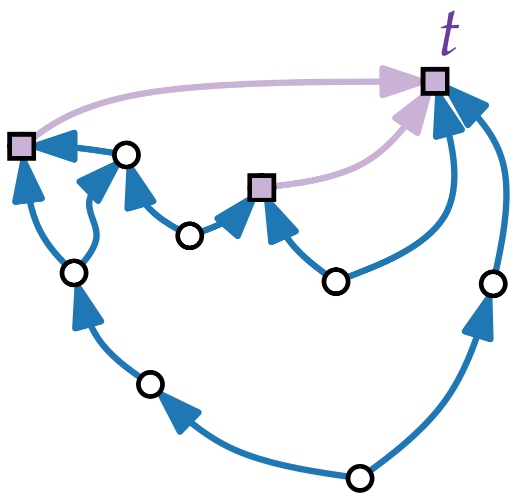

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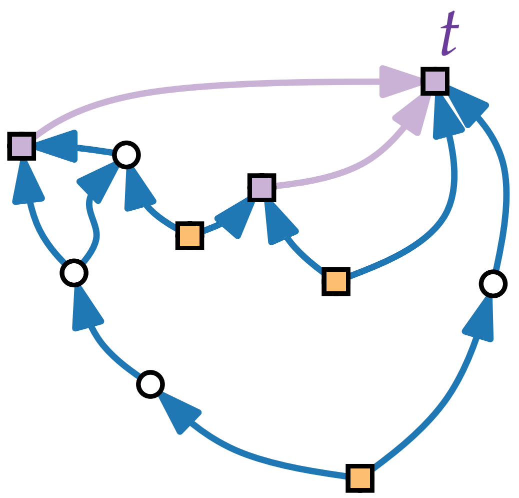

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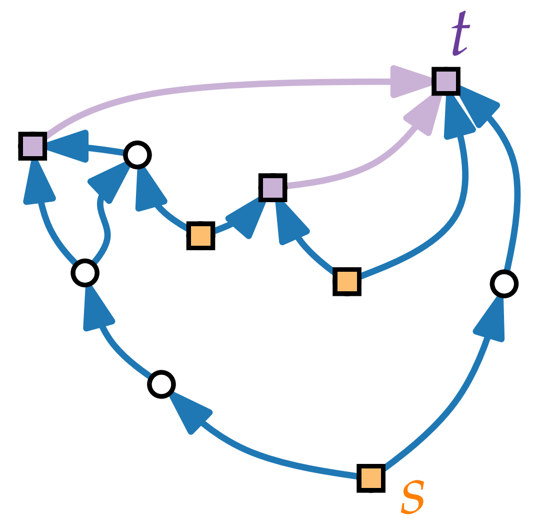

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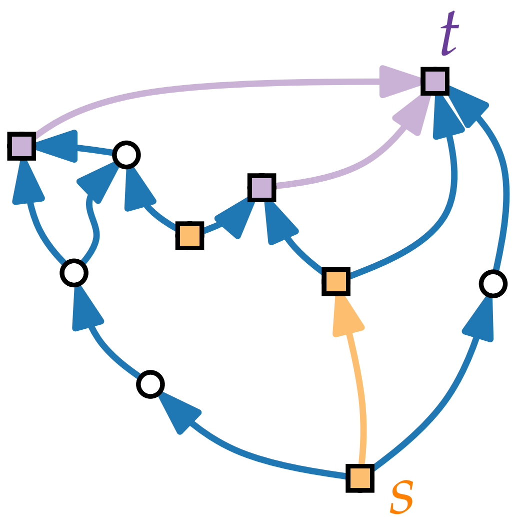

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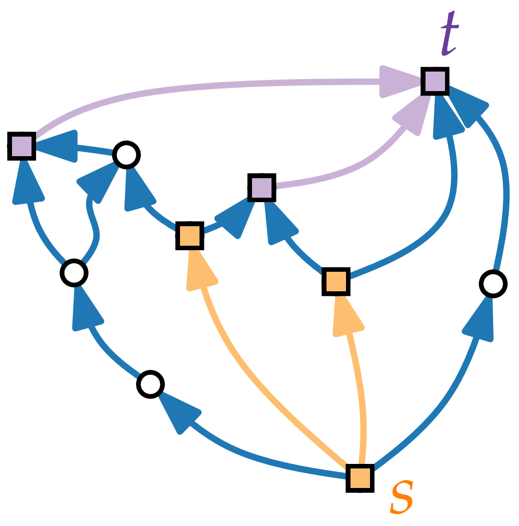

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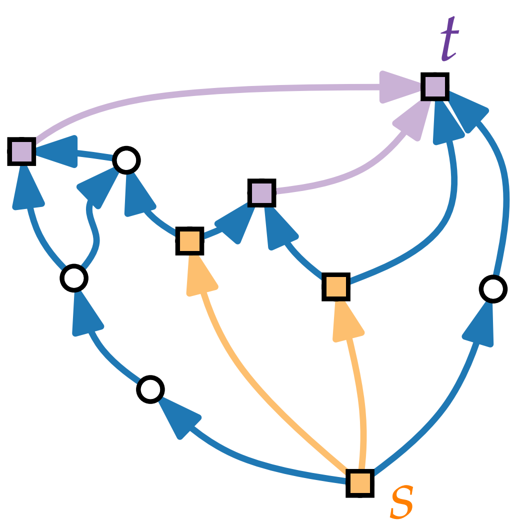

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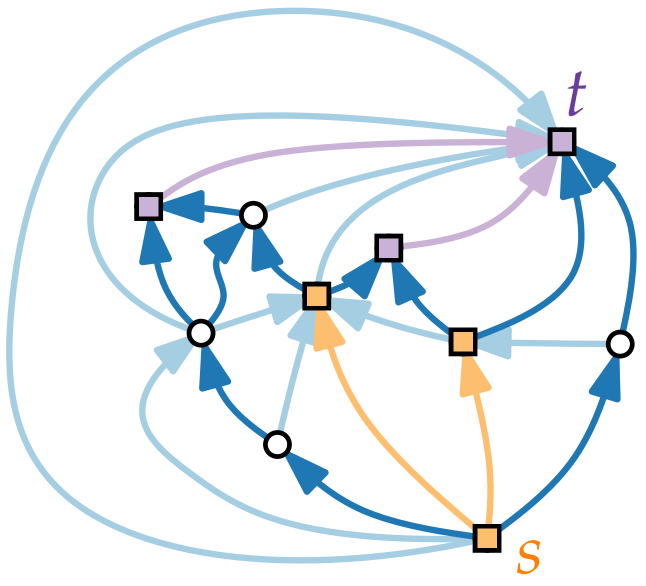

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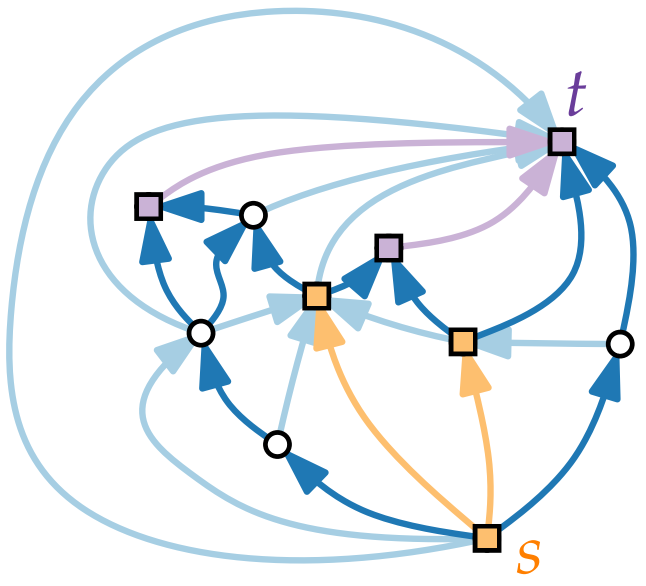

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Claim.
Can draw in prespecified triangle.

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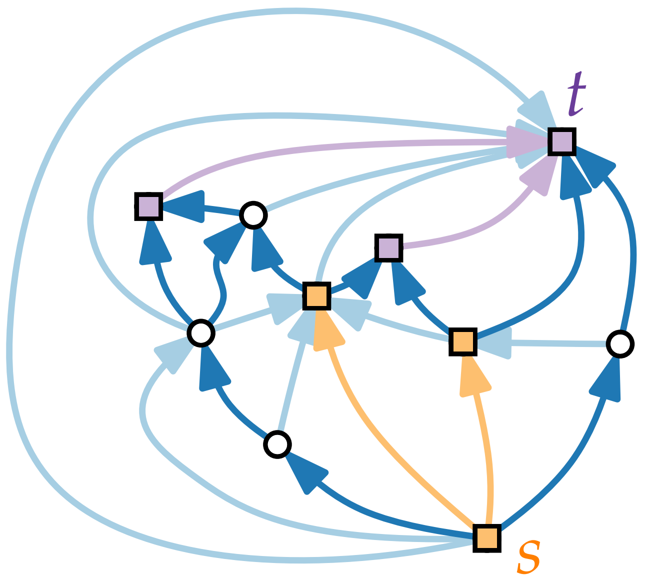

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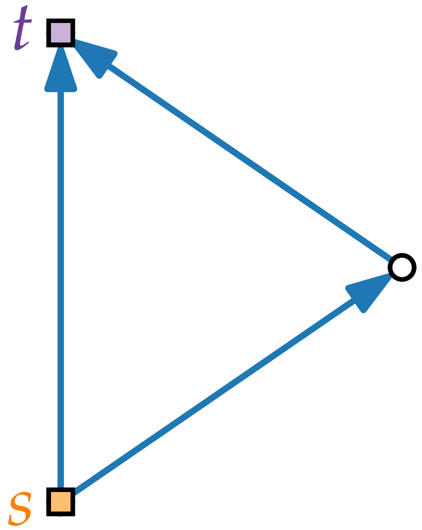

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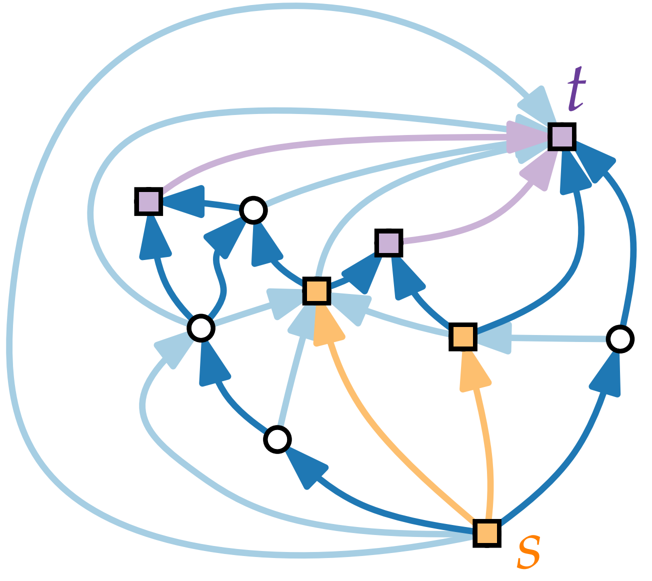

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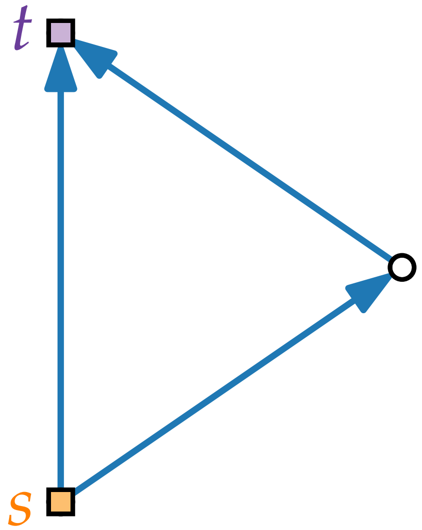
Induction on n.



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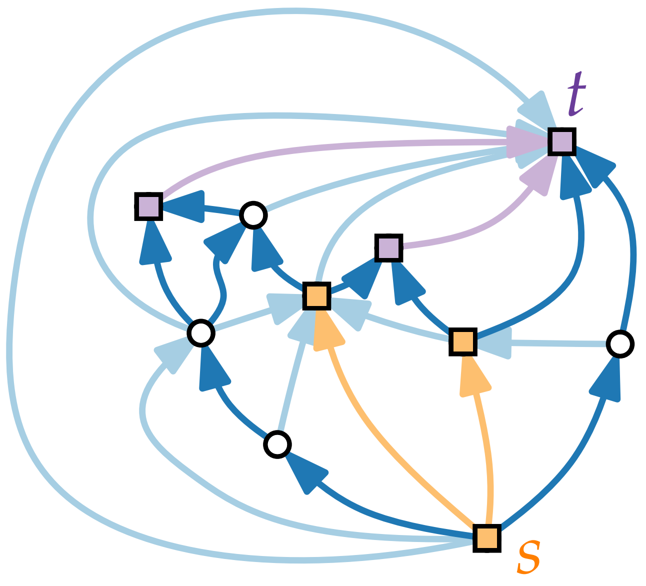

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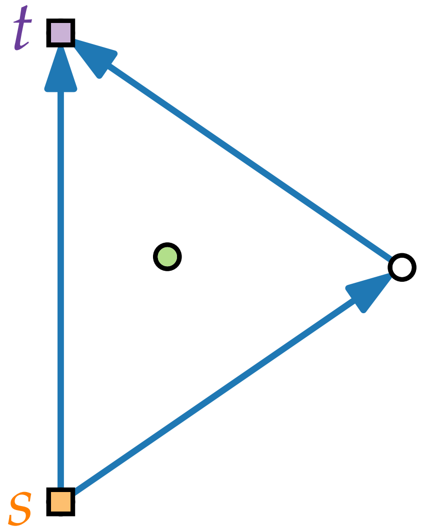

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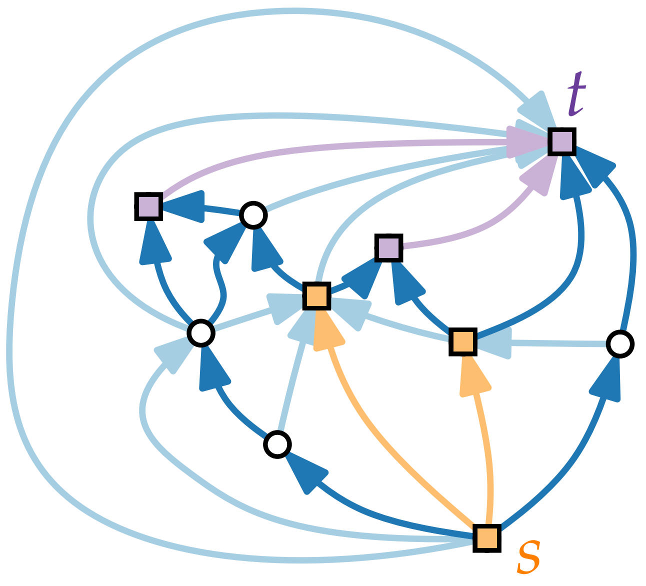

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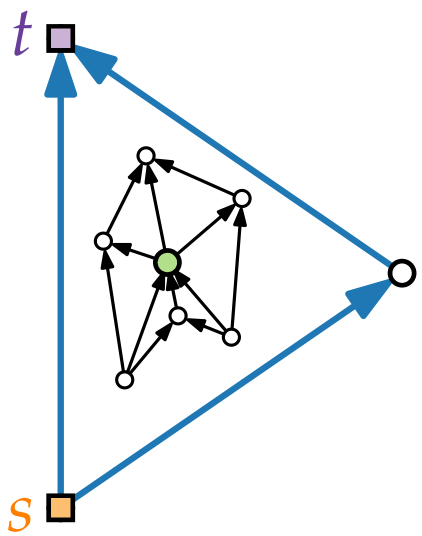

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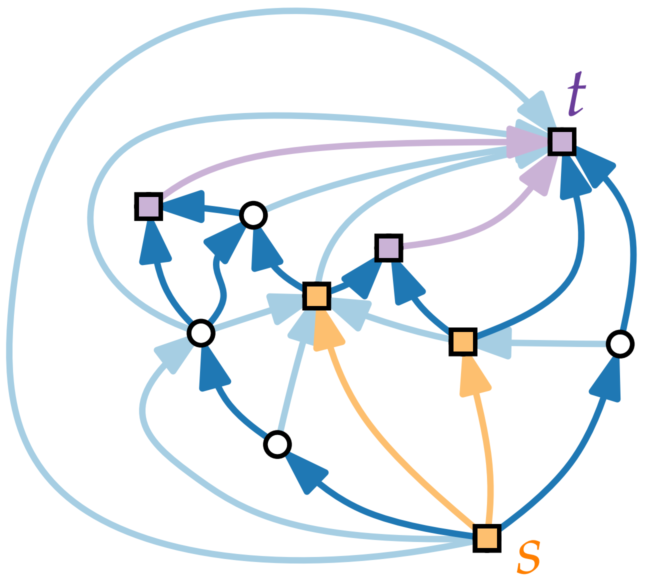

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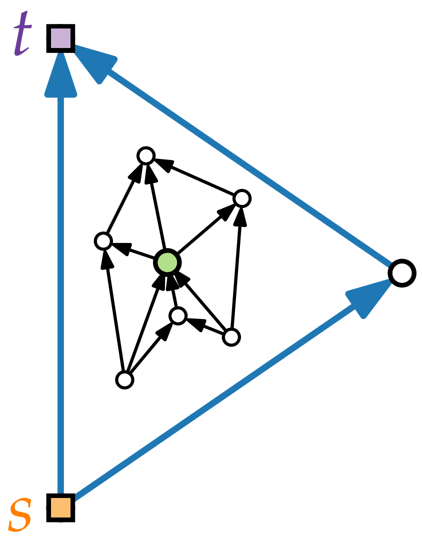
Case 1: chord



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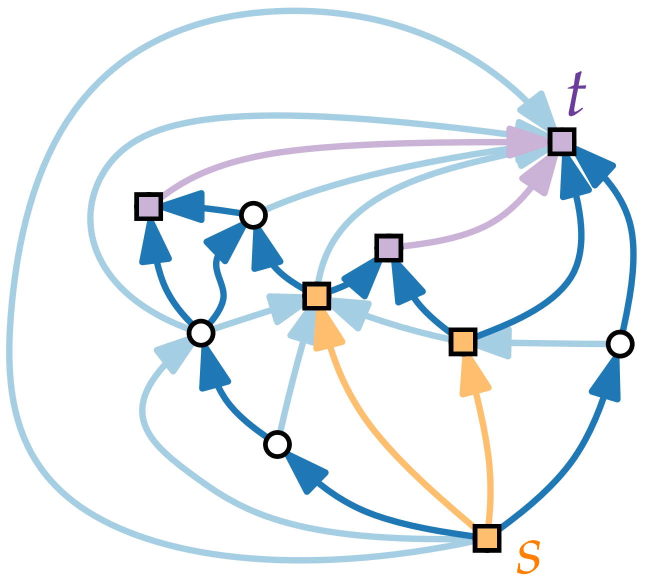

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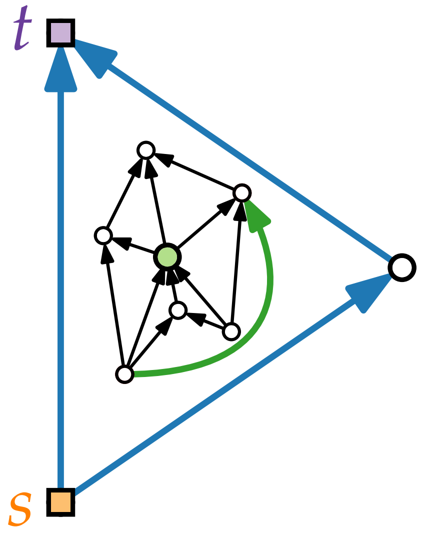

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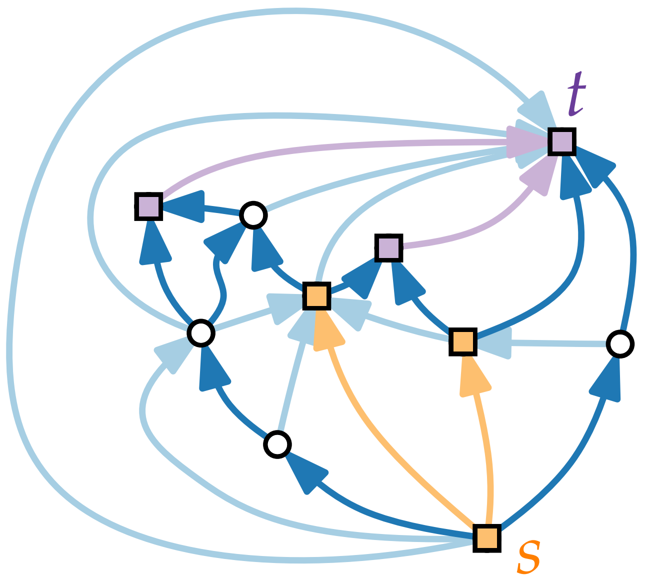

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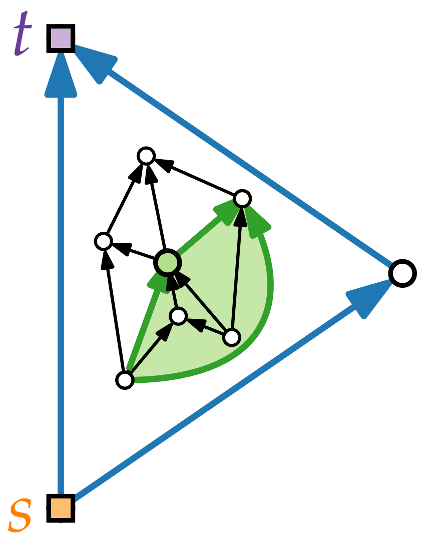

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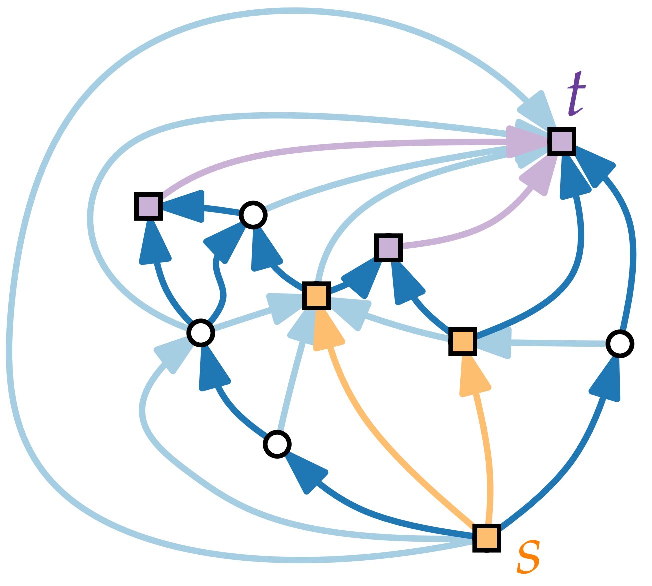

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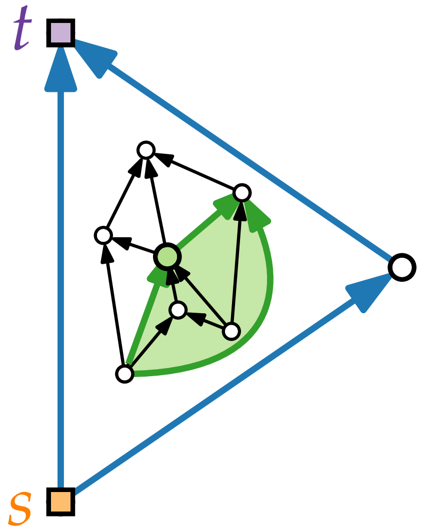

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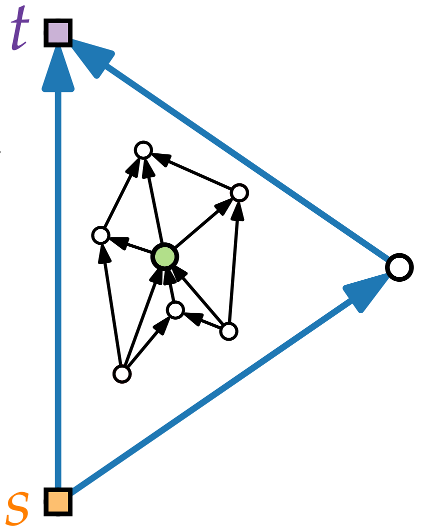

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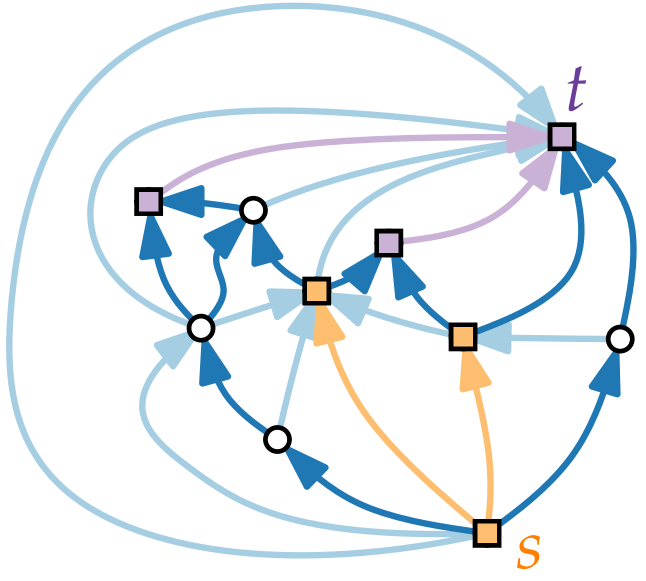

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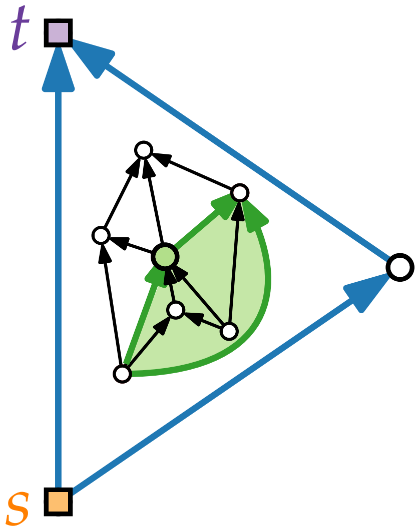

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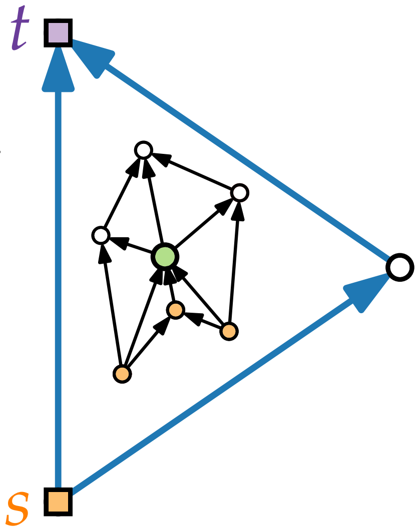

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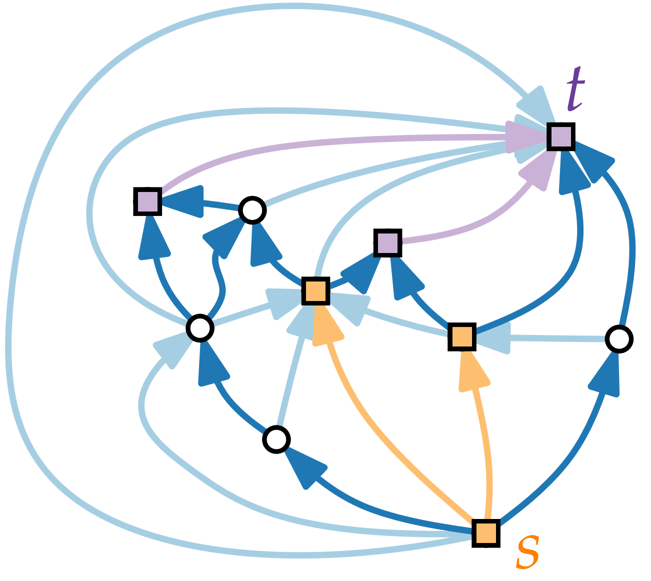

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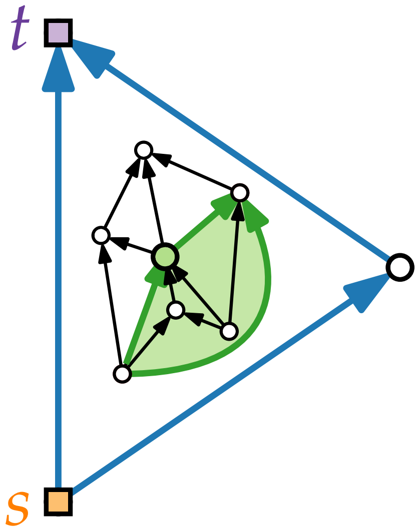

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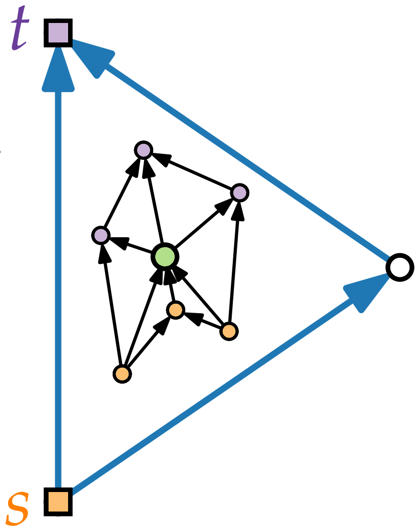

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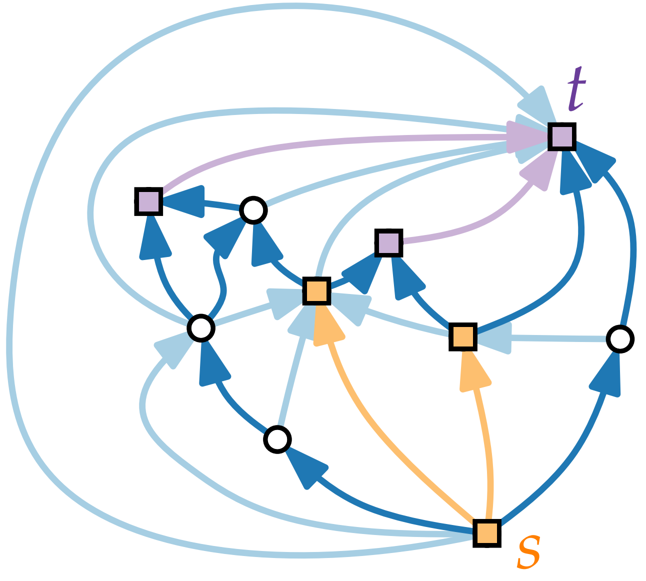

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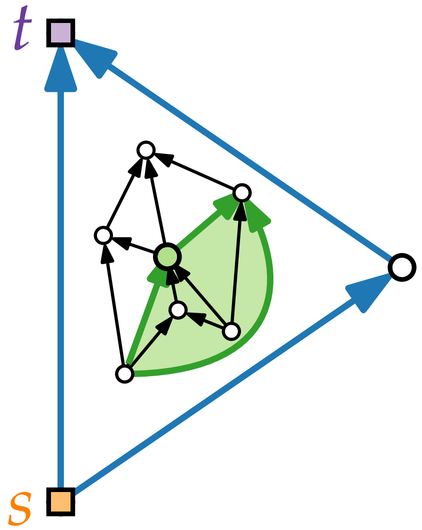

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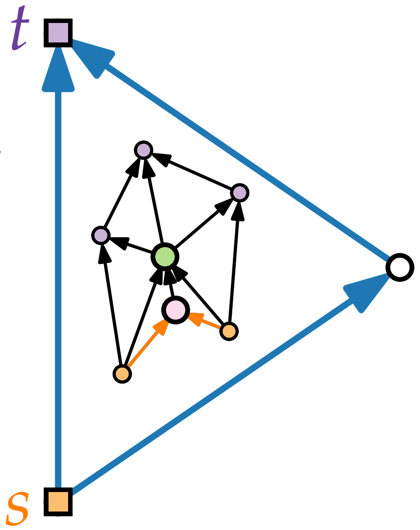

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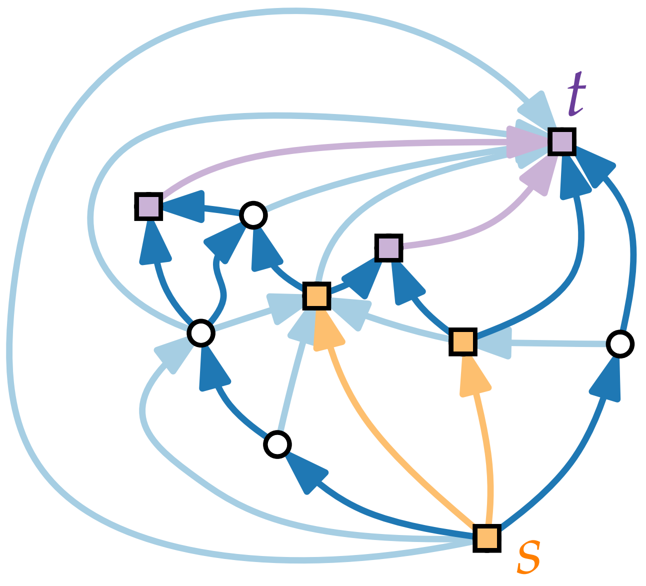

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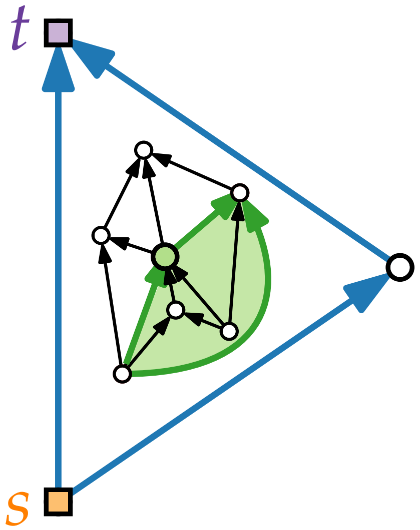

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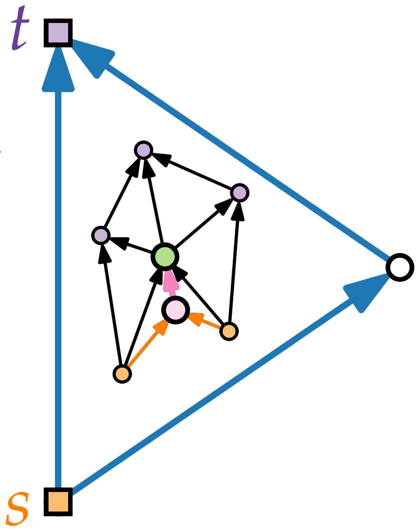

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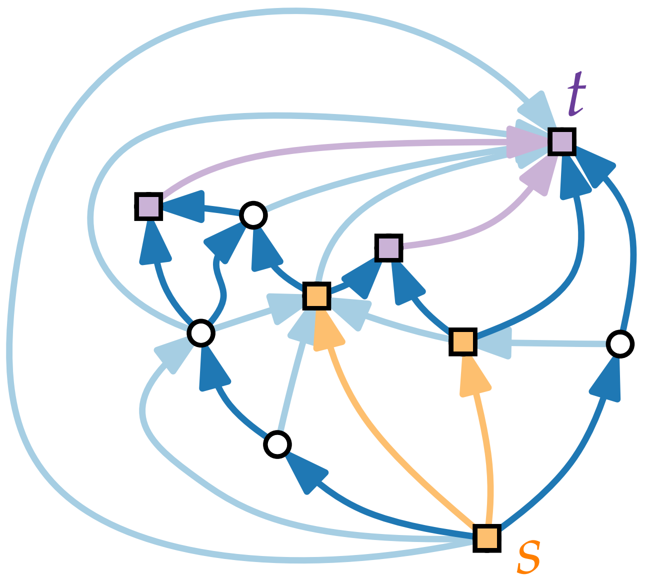

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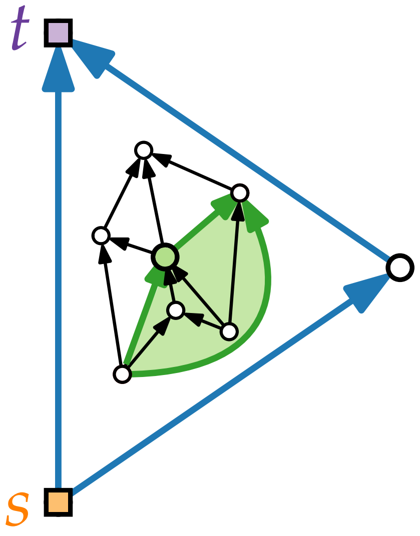

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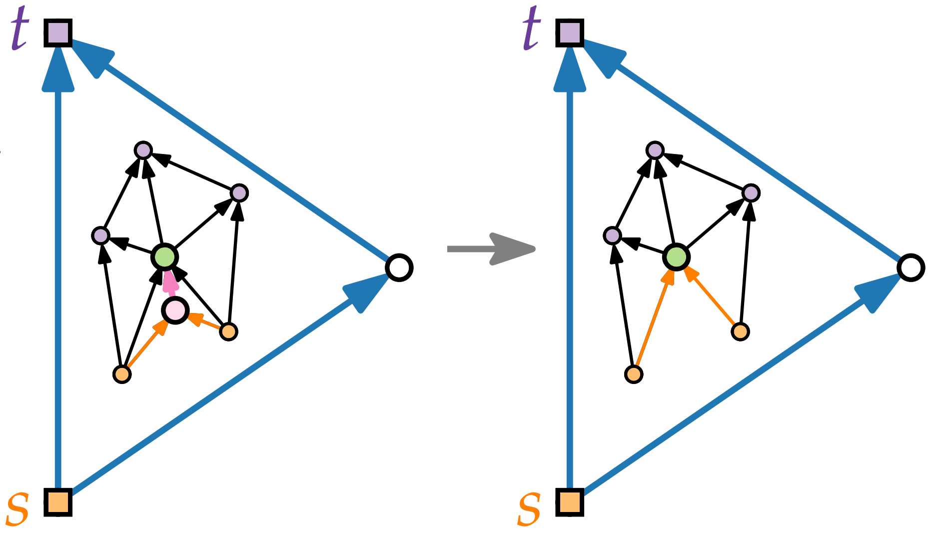

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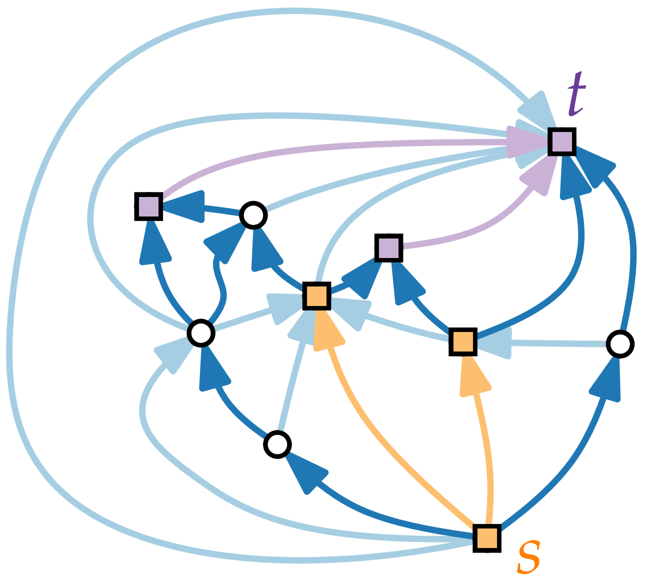

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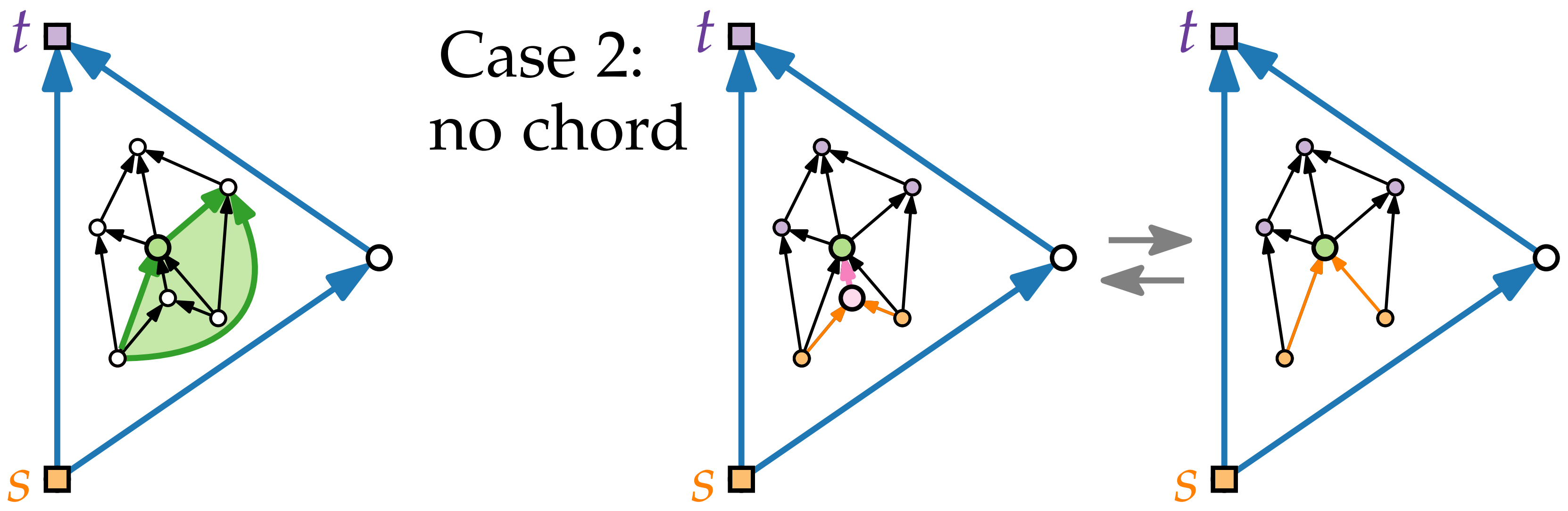

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Angles, Local Sources & Sinks

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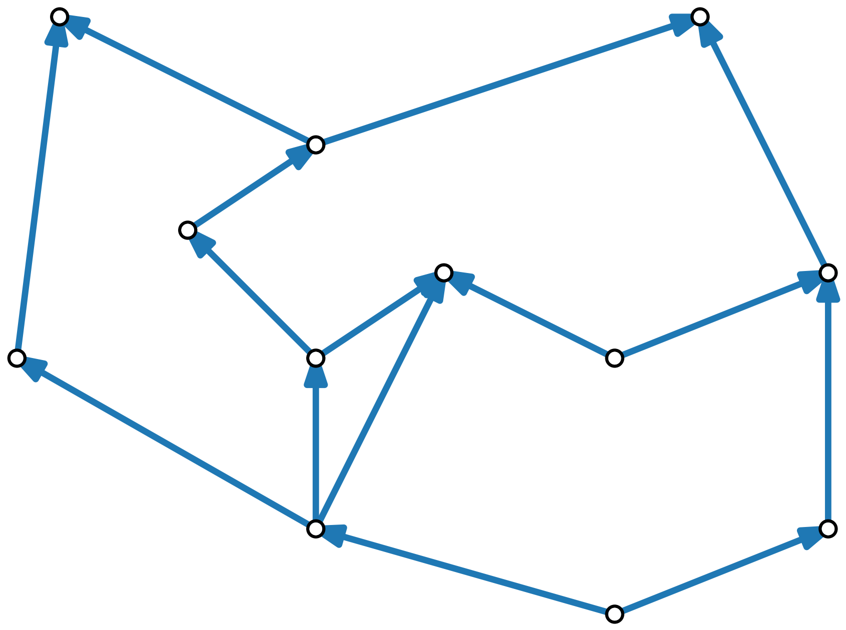

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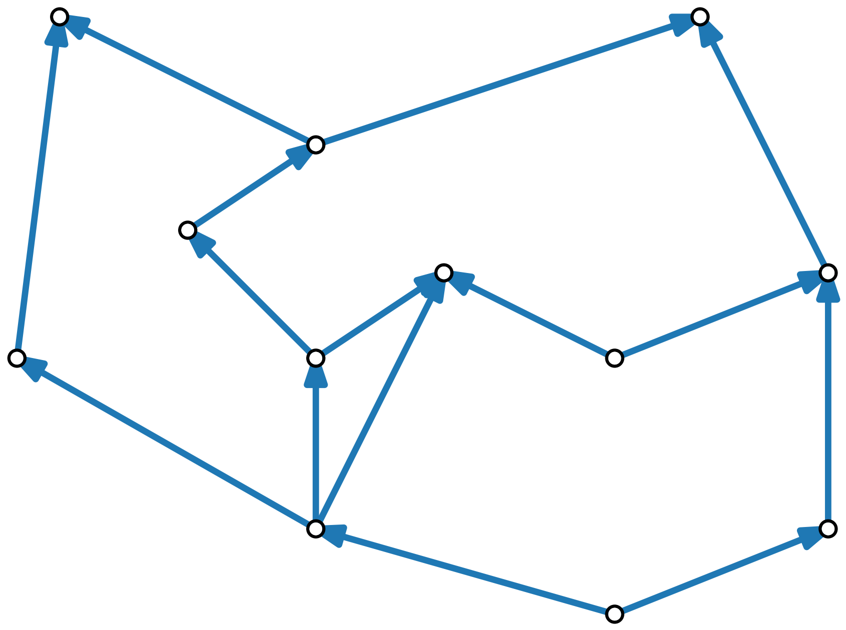

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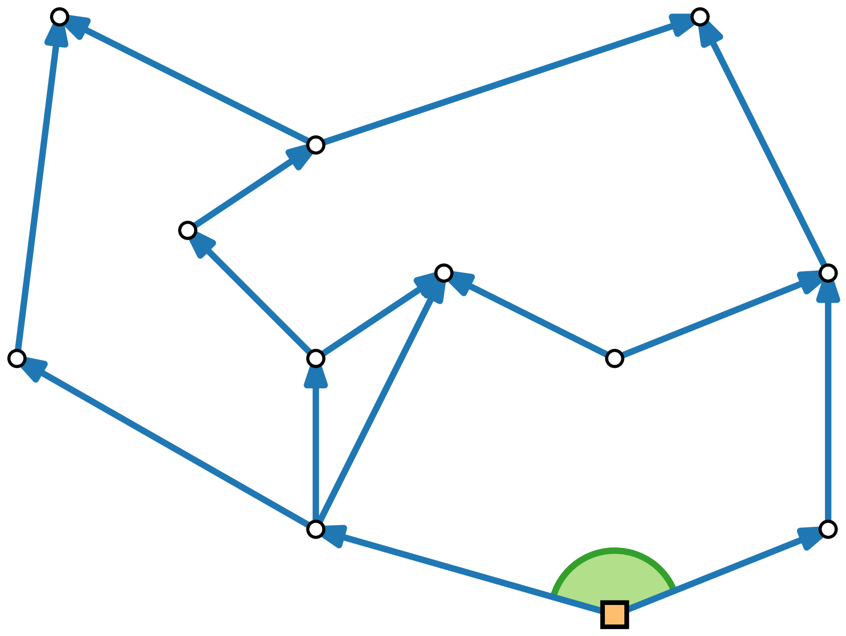

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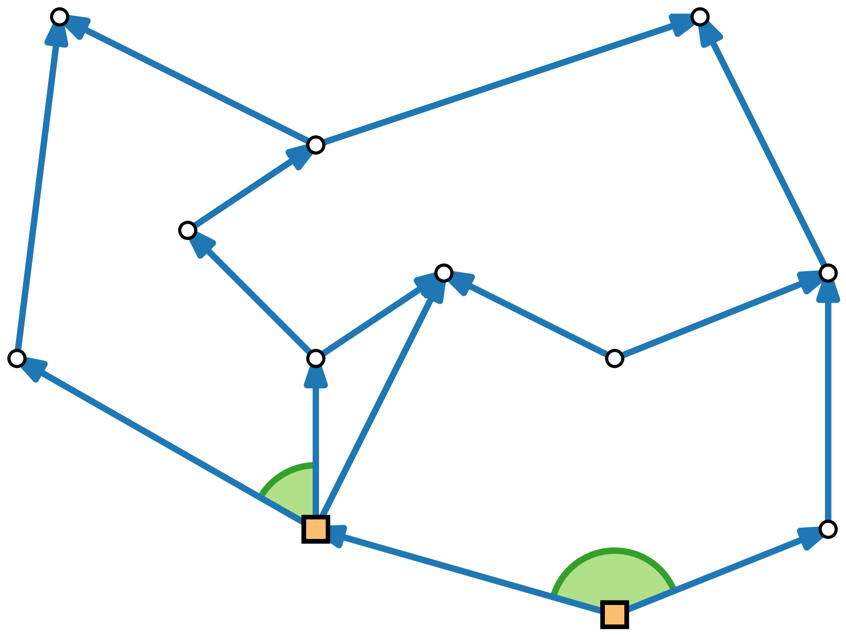

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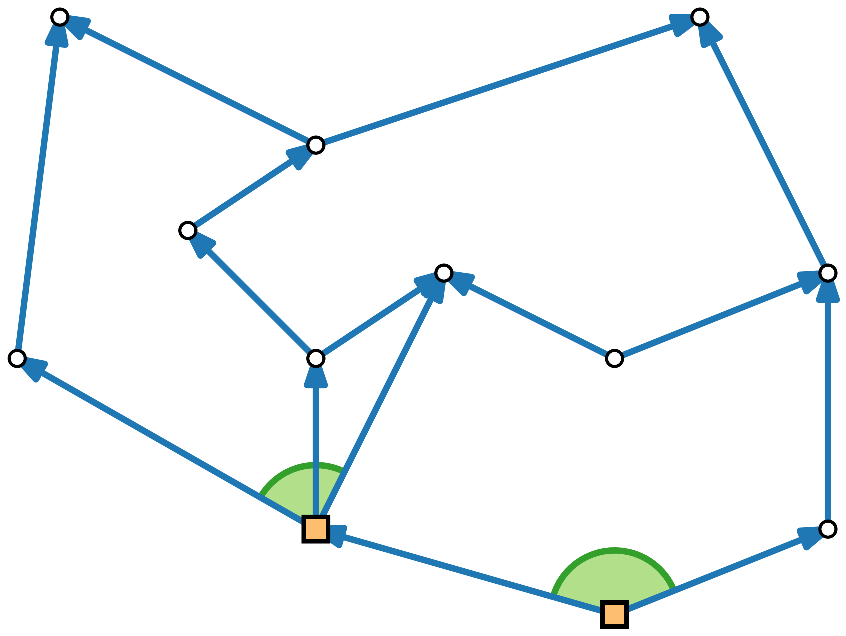

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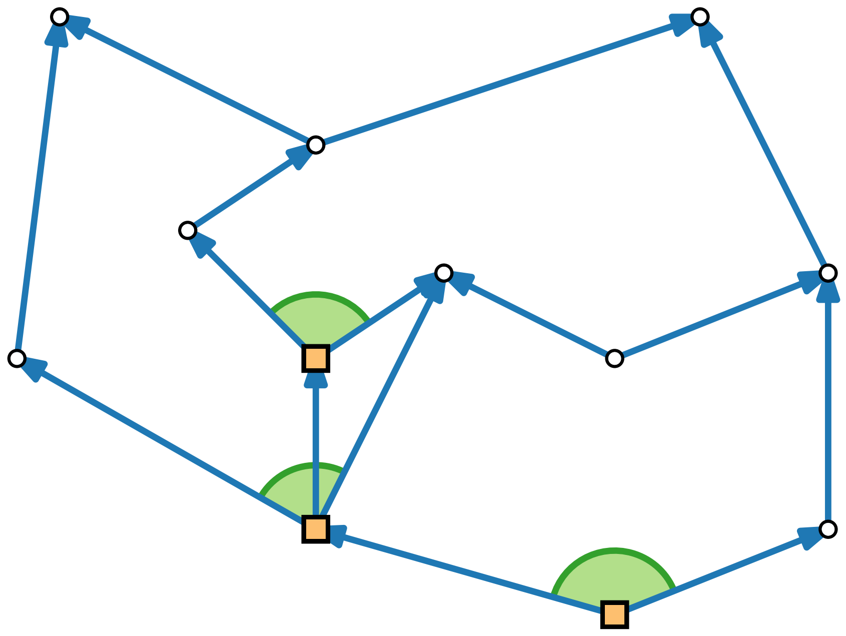

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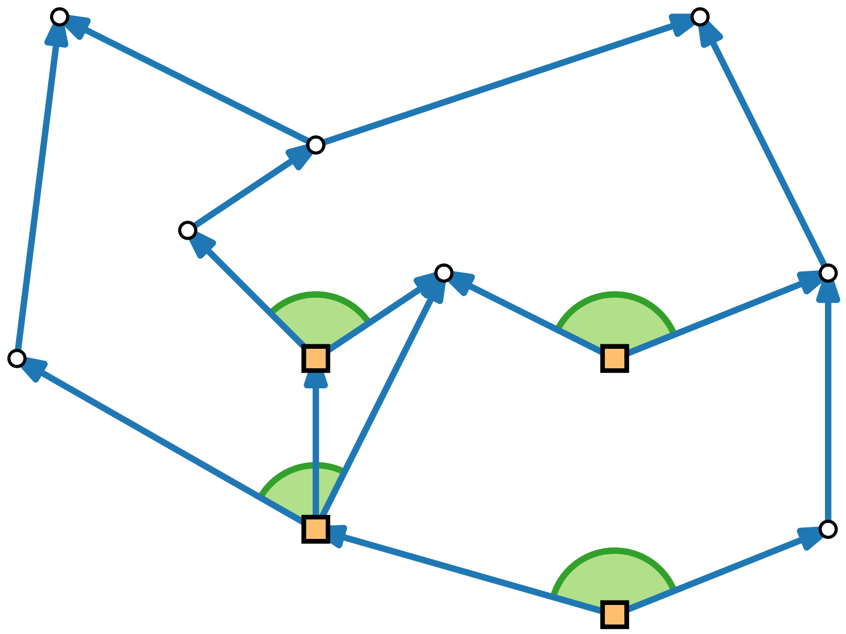

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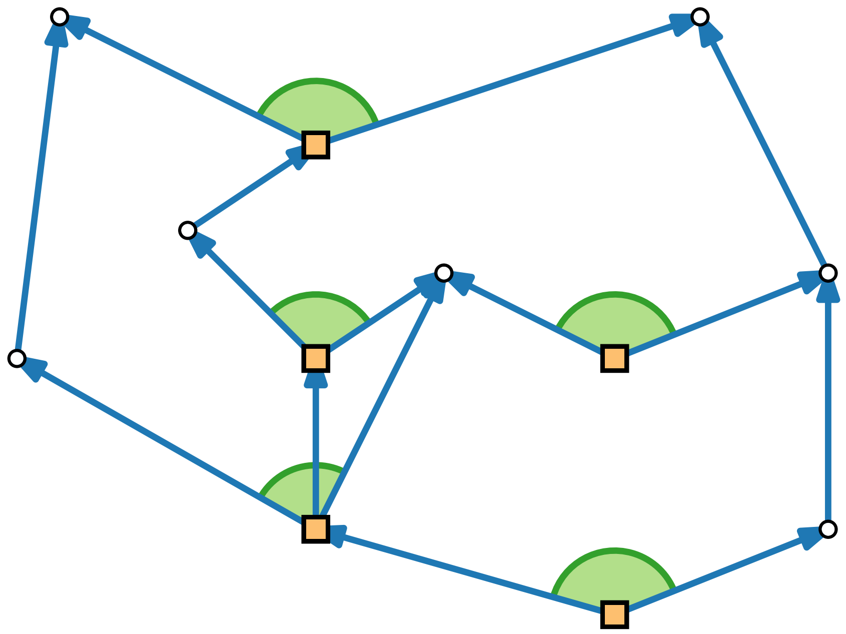

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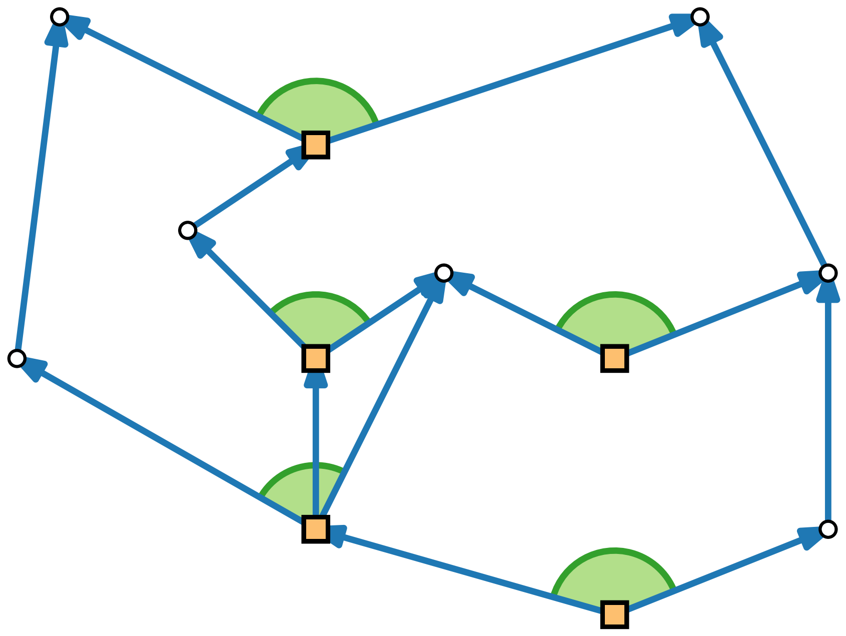

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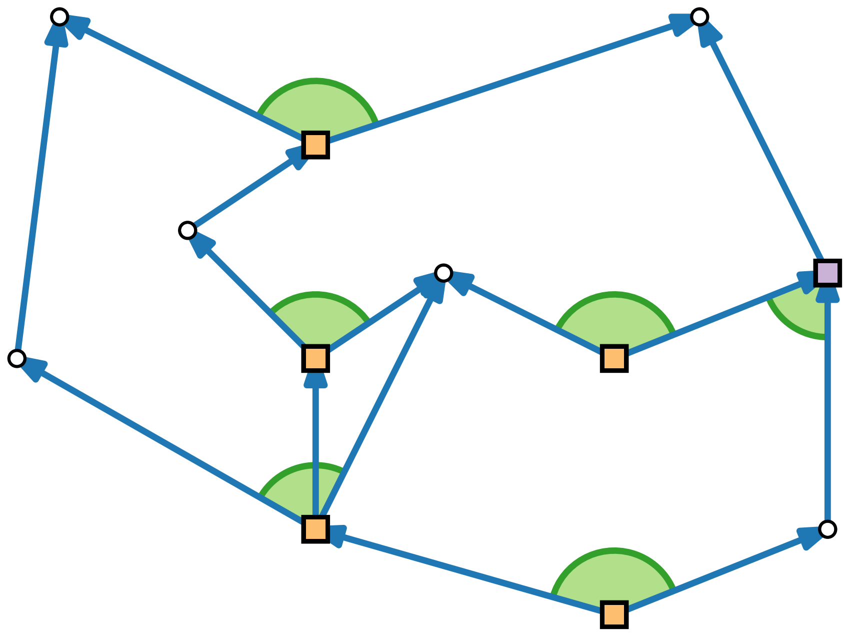

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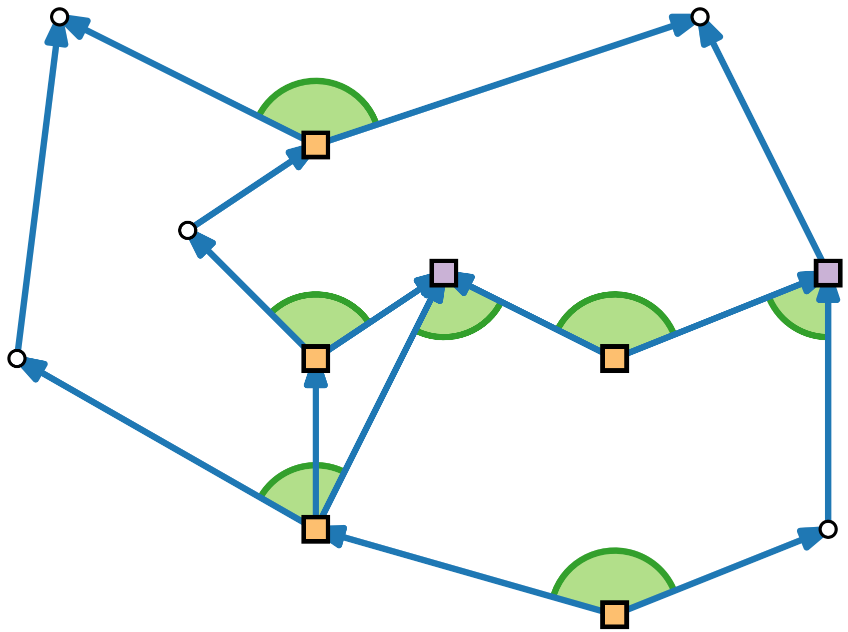

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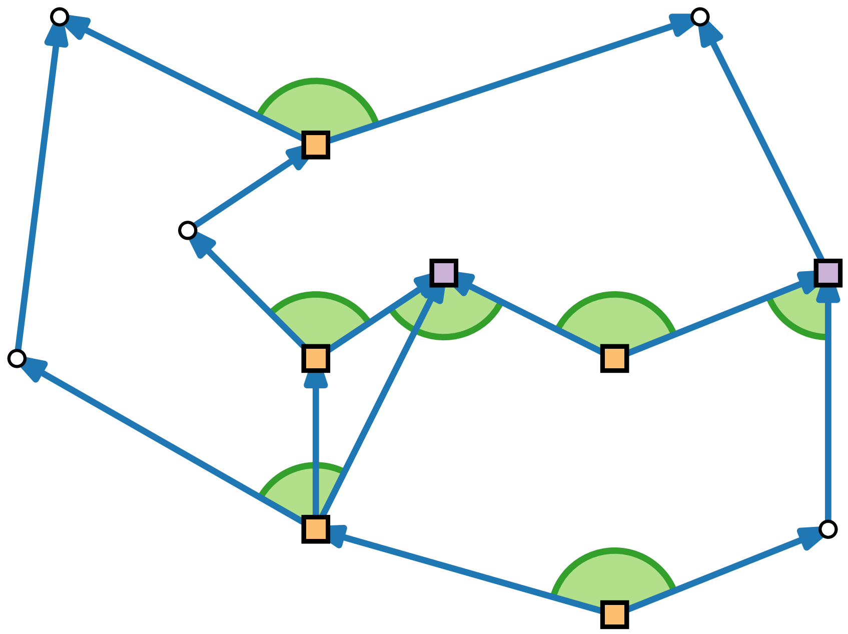

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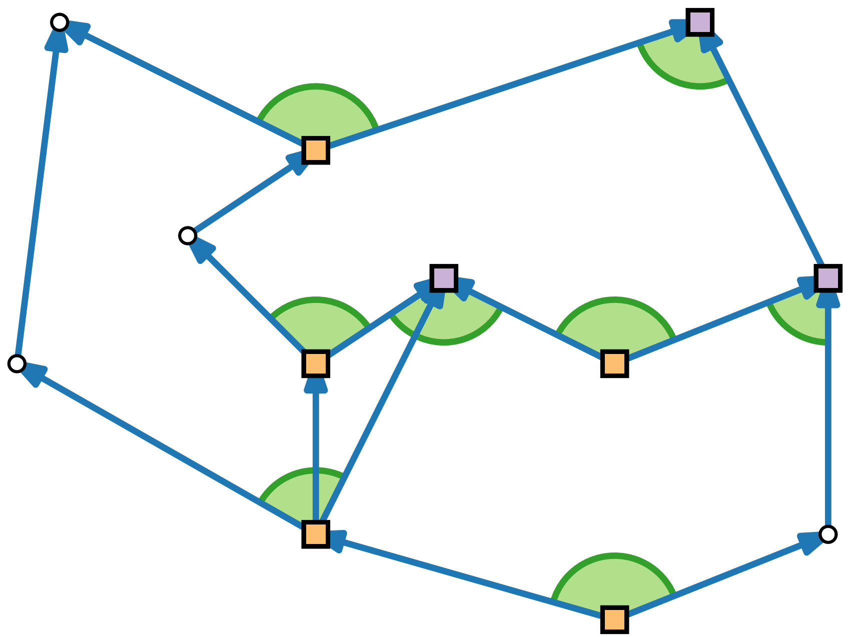

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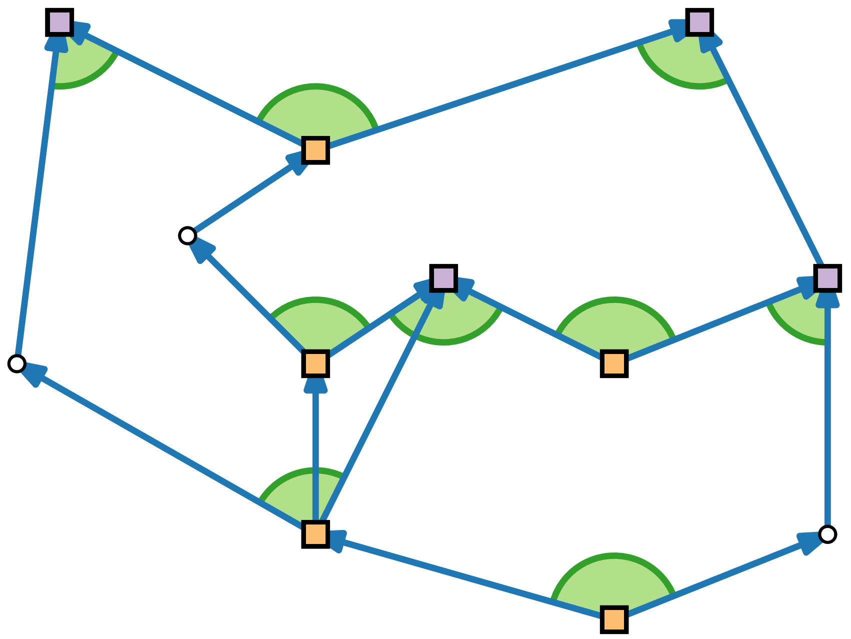

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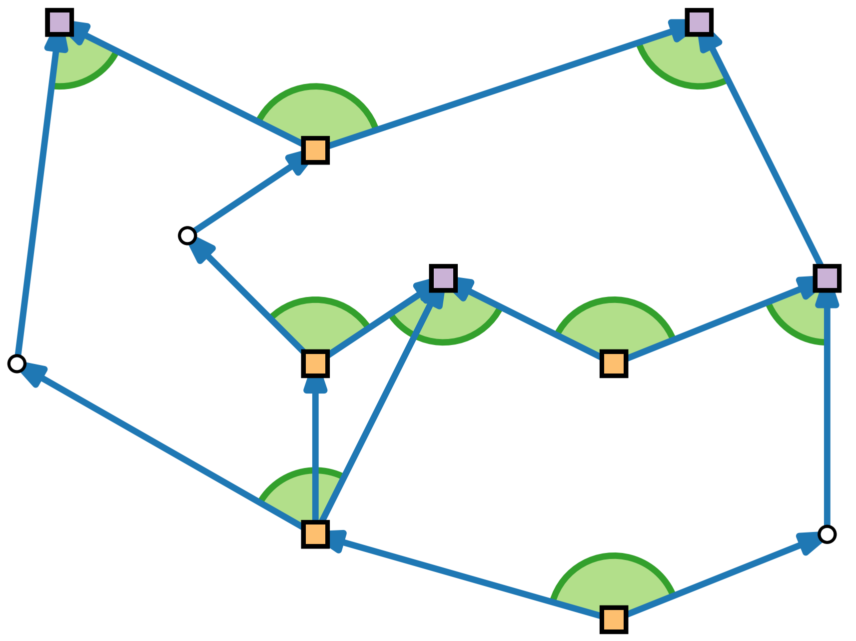

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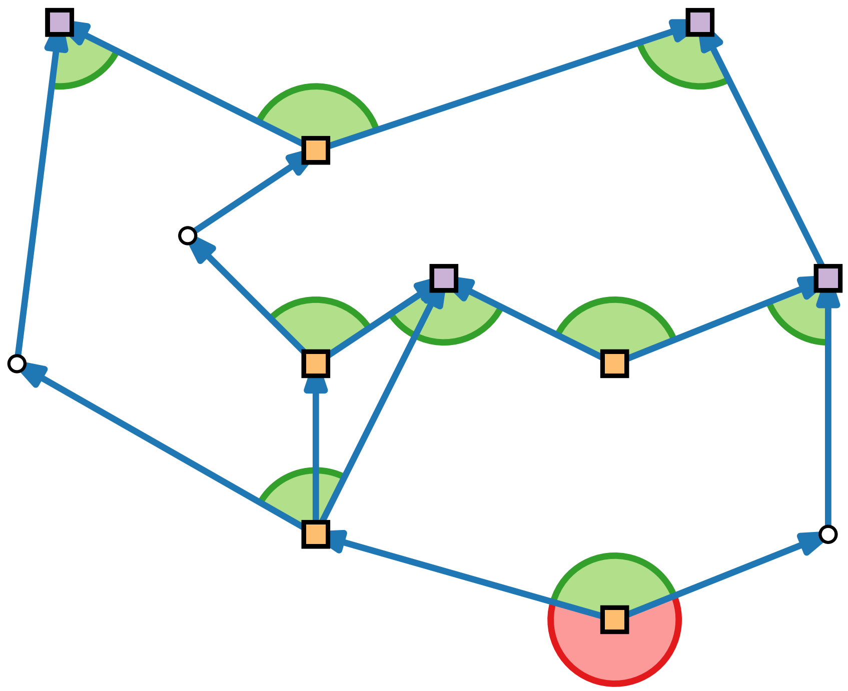

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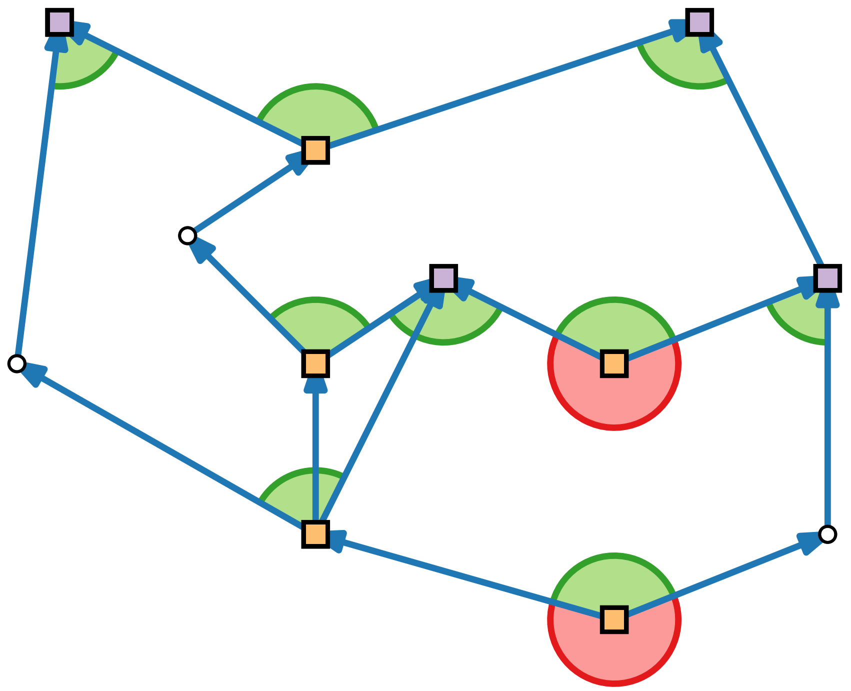

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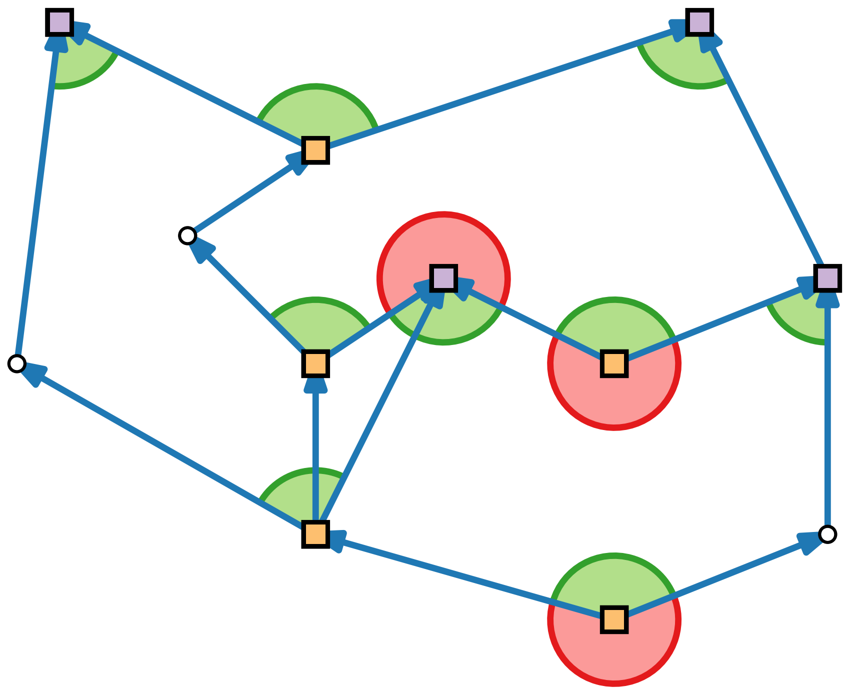

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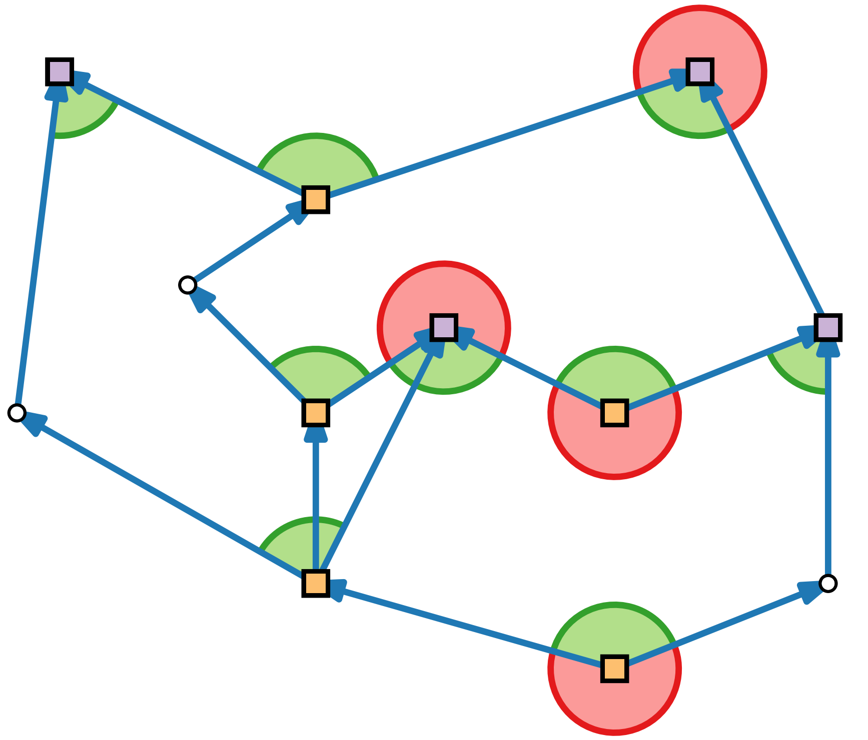

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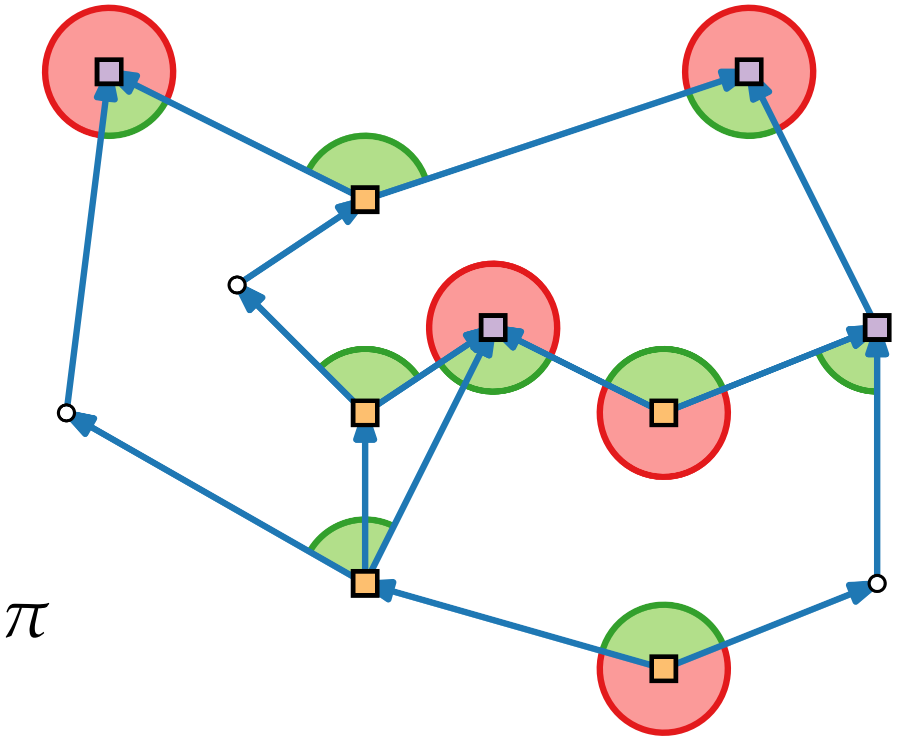

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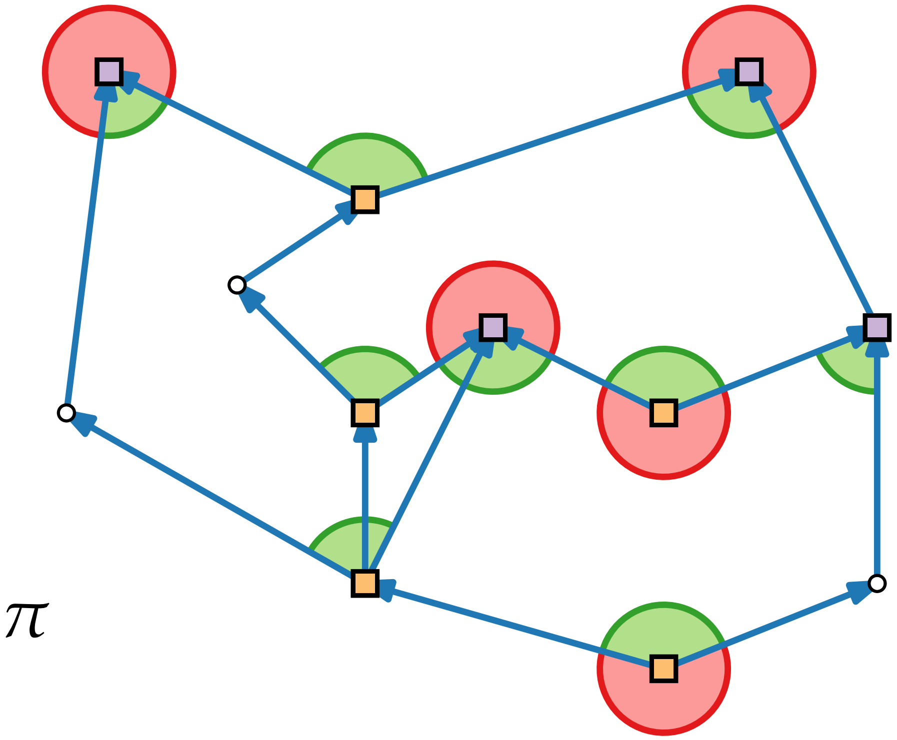

Definitions.

- A vertex v is a **local source** wrt a face f if v has two outgoing edges on ∂f.
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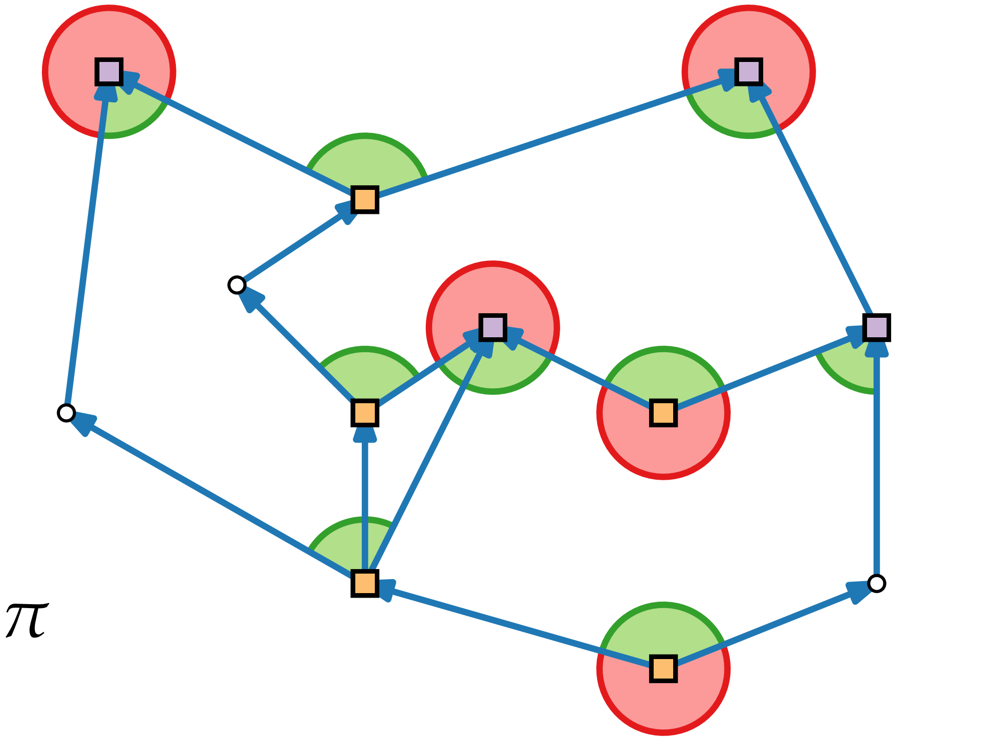

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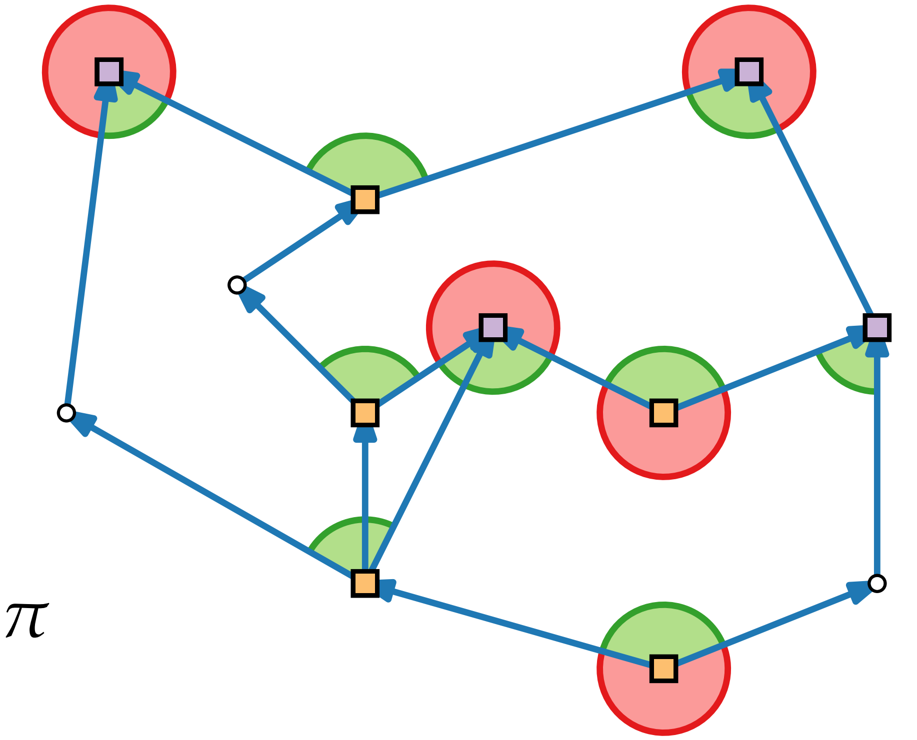

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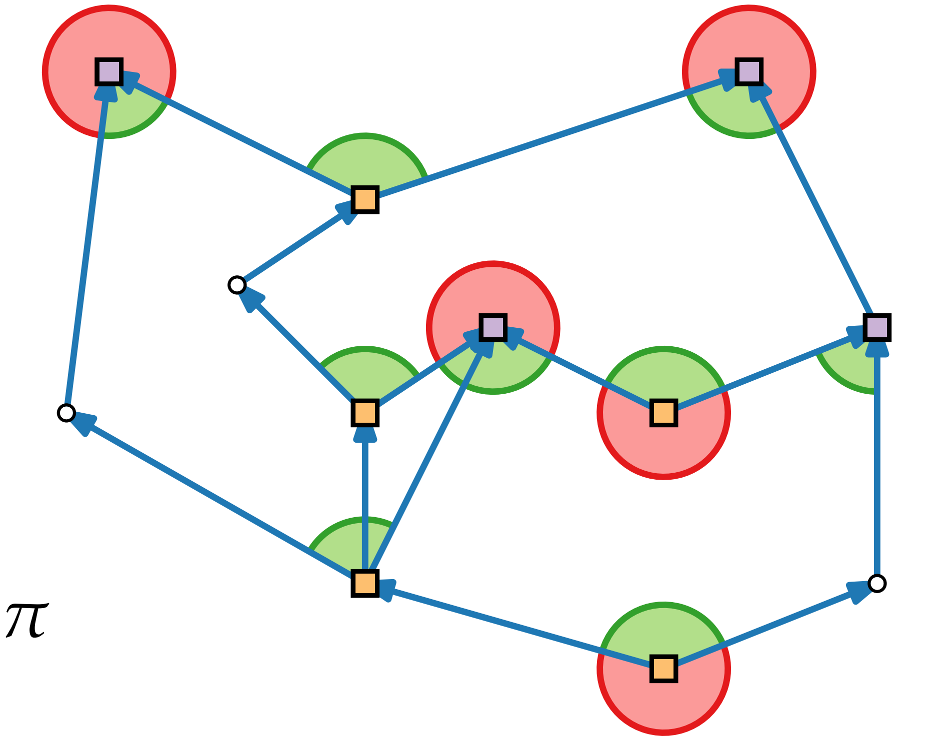

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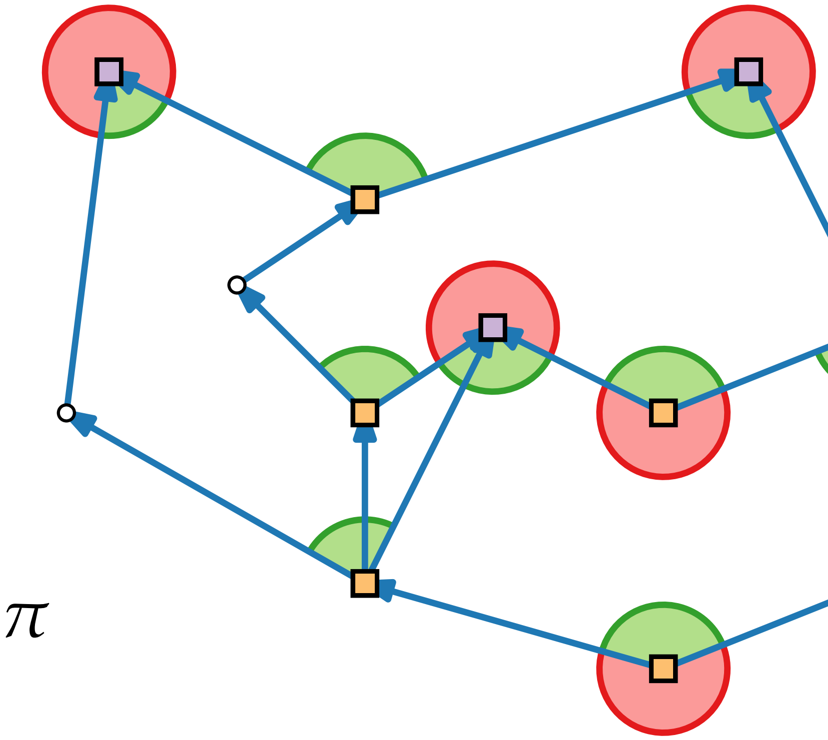
Angles, Local Sources & Sinks



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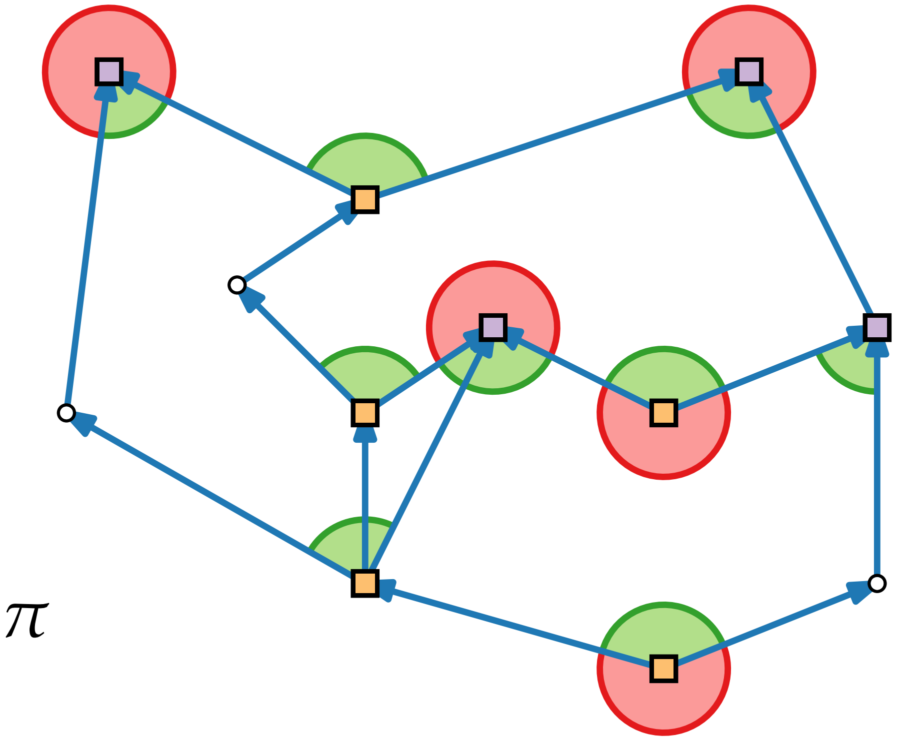
Angles, Local Sources & Sinks



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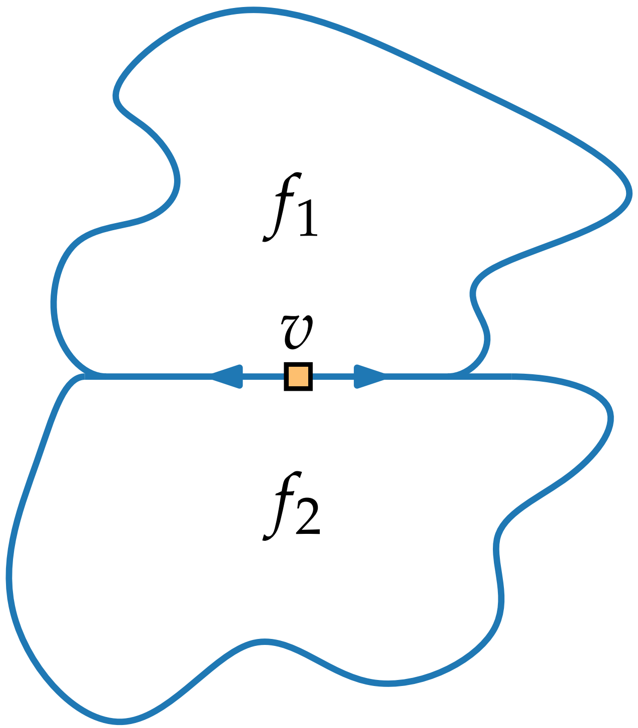
Lemma 1.
$L(f) + S(f) = 2A(f)$

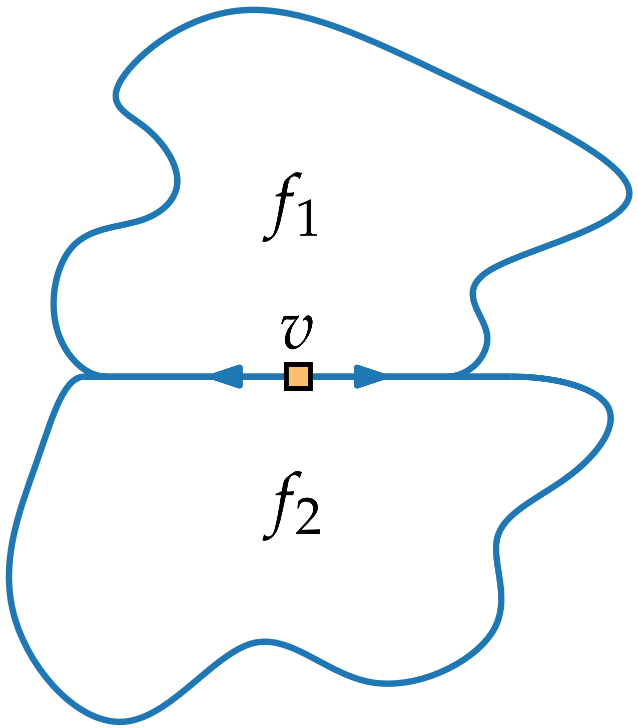
Assignment Problem

- Vertex v is a global source.



Assignment Problem

- Vertex v is a global source.
- At which face does v have a **large** angle?



Angle Relations

Lemma 2.

$$L(f) - S(f) = \begin{cases} -2, & f \neq f_0 \\ +2, & f = f_0 \end{cases}$$

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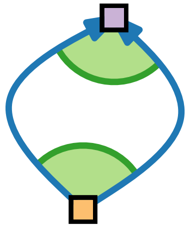

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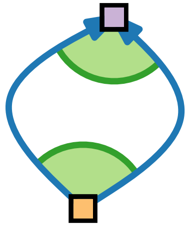


$\Rightarrow S(f) = 2$

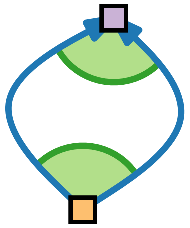
Angle Relations

Lemma 2.
$$\color{red}{L(f)} - \color{green}{S(f)} = \begin{cases} -2, & f \neq f_0 \\ +2, & f = f_0 \end{cases}$$

- $L(f) \geq 1$

Proof by induction.

- $L(f) = 0$



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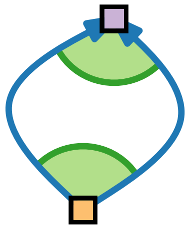

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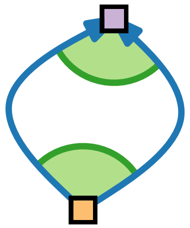

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Split f with edge from a large angle at a “low” sink u to

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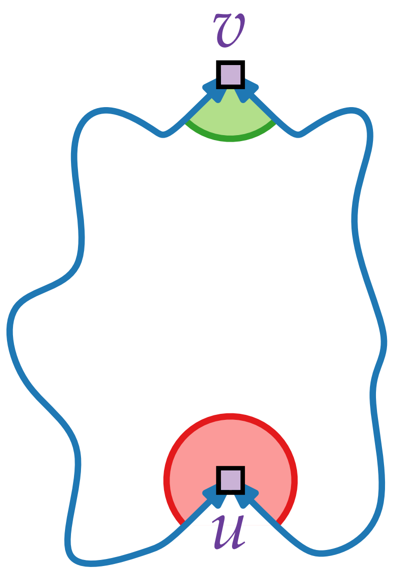

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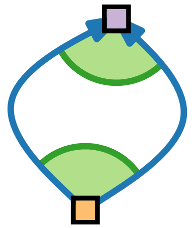
- $L(f) = 0$  $\Rightarrow S(f) = 2$

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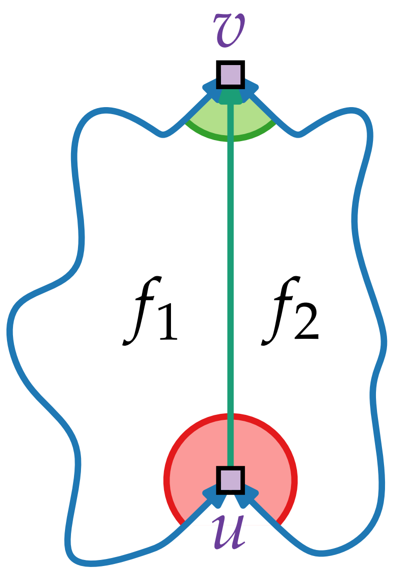

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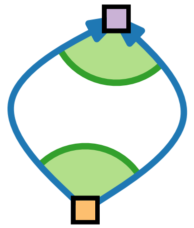

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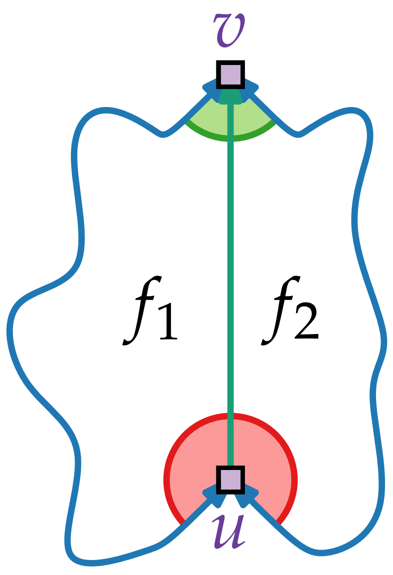

$L(f) - S(f)$

Angle Relations

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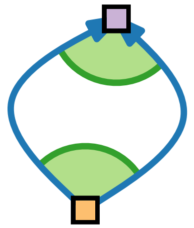

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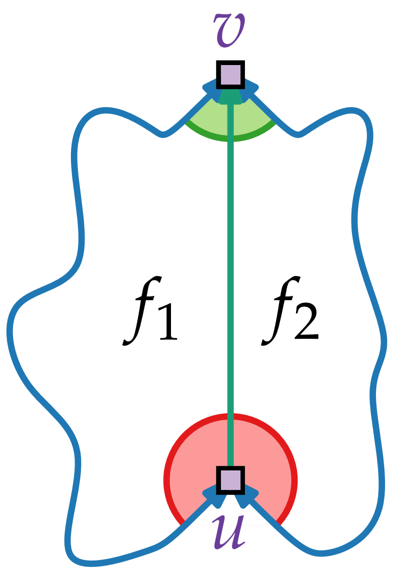

$$L(f) - S(f) = L(f_1) + L(f_2) + 1$$

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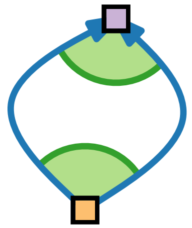

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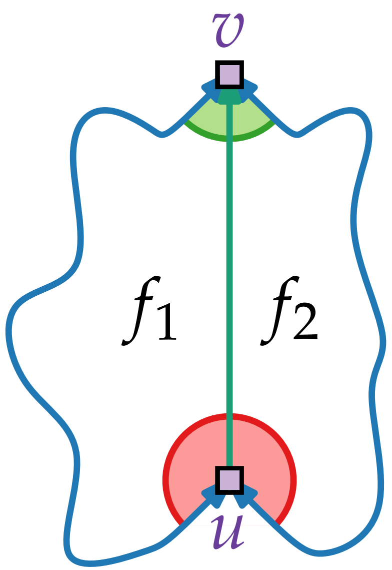

$$L(f) - S(f) = L(f_1) + L(f_2) + 1 - (S(f_1) + S(f_2) - 1)$$

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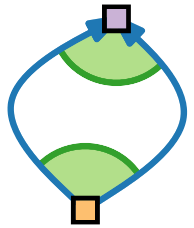

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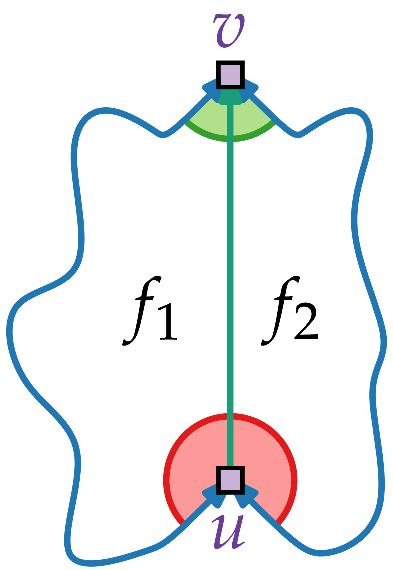

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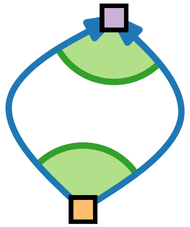
- $L(f) = 0$  $\Rightarrow S(f) = 2$
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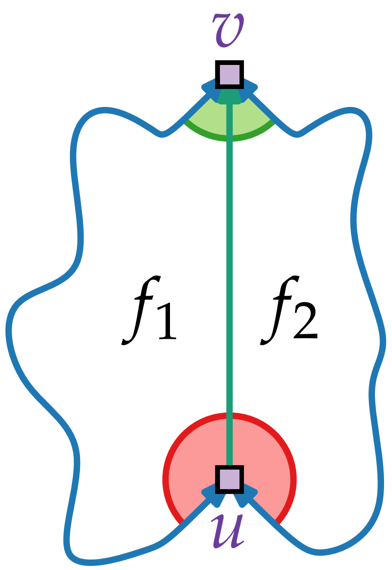

$$\begin{aligned} L(f) - S(f) &= \underbrace{L(f_1)}_{-2} + \underbrace{L(f_2)}_{-2} + 1 \\ &\quad - (S(f_1) + S(f_2) - 1) \\ &= -2 \end{aligned}$$

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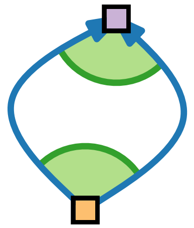

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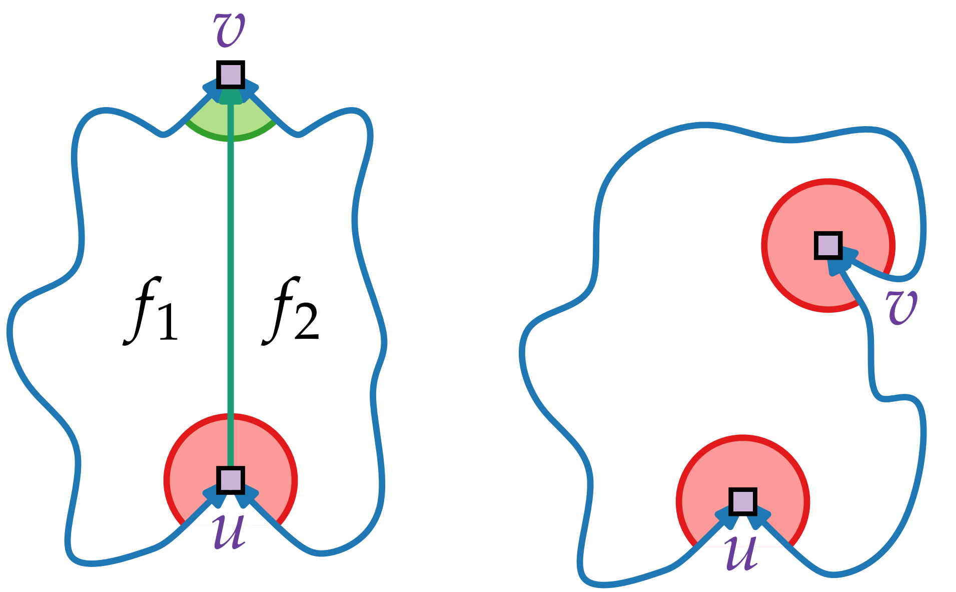

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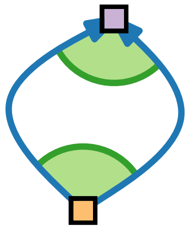
- $L(f) = 0$  $\Rightarrow S(f) = 2$

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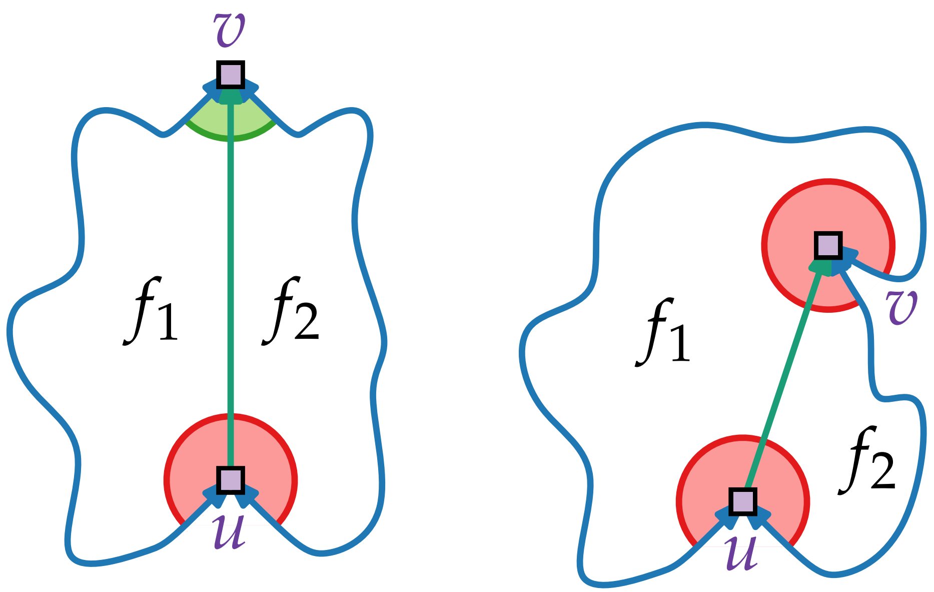

$$\begin{aligned} L(f) - S(f) &= \overset{-2}{L(f_1) + L(f_2)} + 1 \\ &\quad - (S(f_1) + S(f_2) - 1) \\ &= -2 \end{aligned}$$

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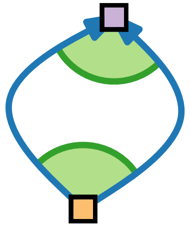
- $L(f) = 0$  $\Rightarrow S(f) = 2$

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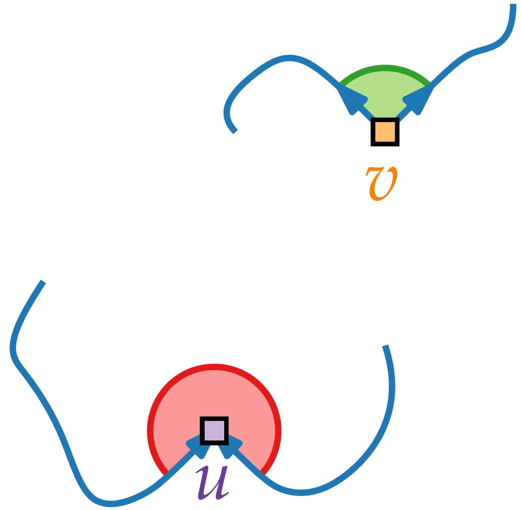

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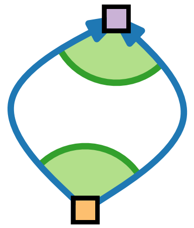

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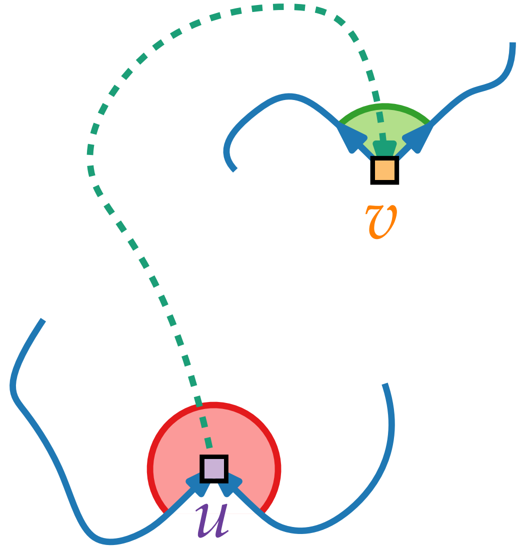

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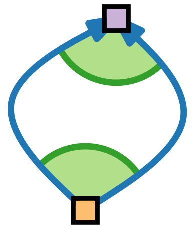
- $L(f) = 0$  $\Rightarrow S(f) = 2$

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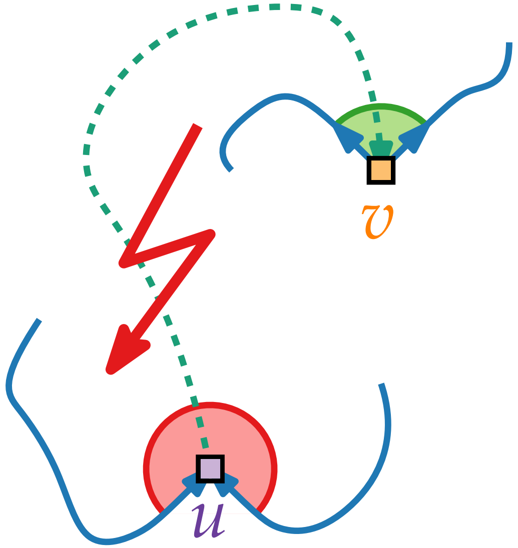

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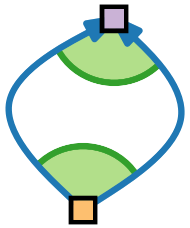
- $L(f) = 0$  $\Rightarrow S(f) = 2$

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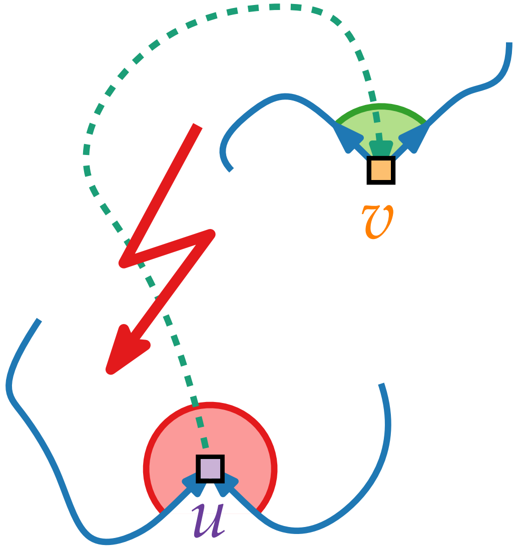

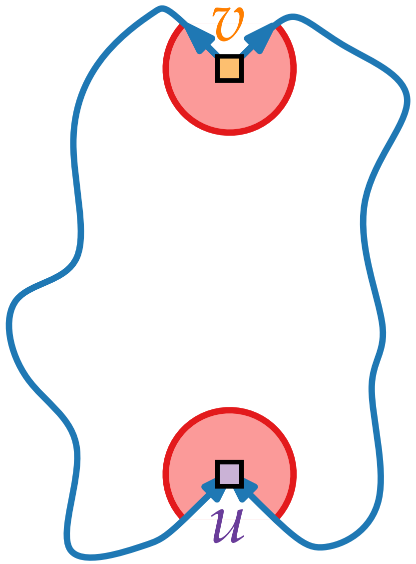

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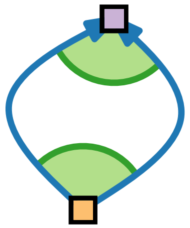
- $L(f) = 0$  $\Rightarrow S(f) = 2$
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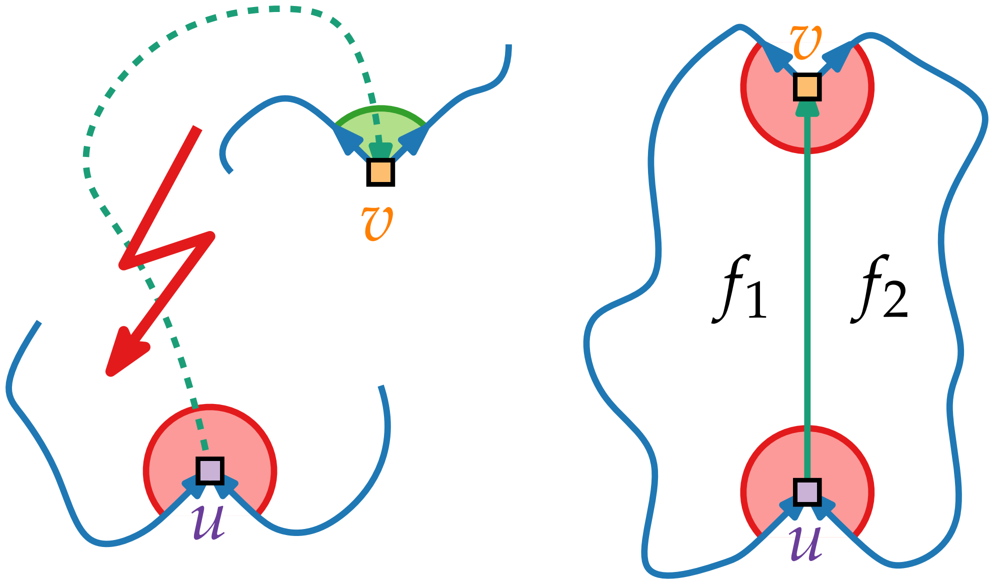

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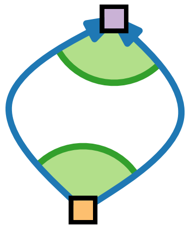

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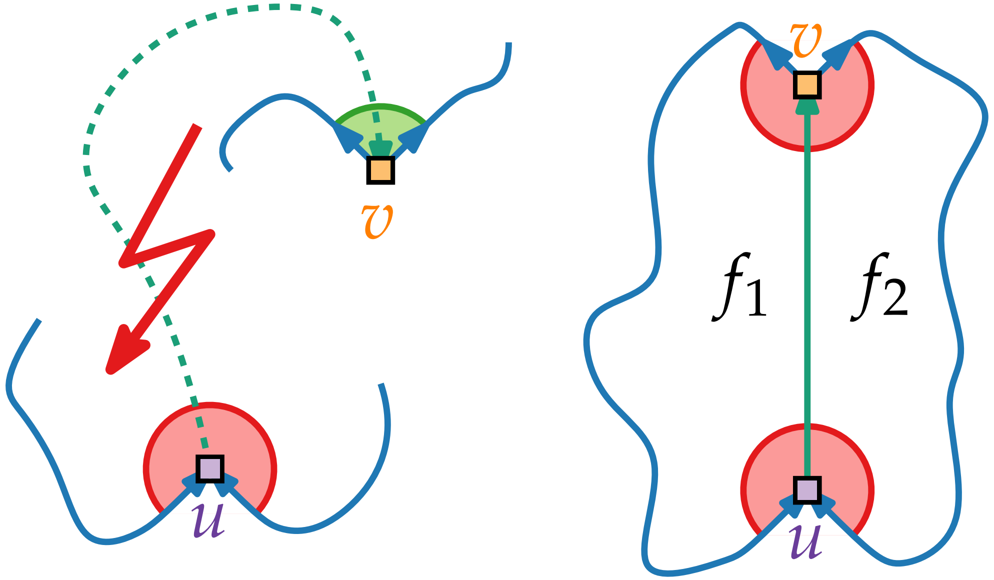

$L(f) - S(f)$

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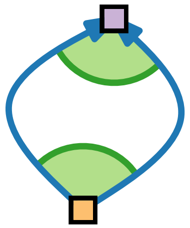

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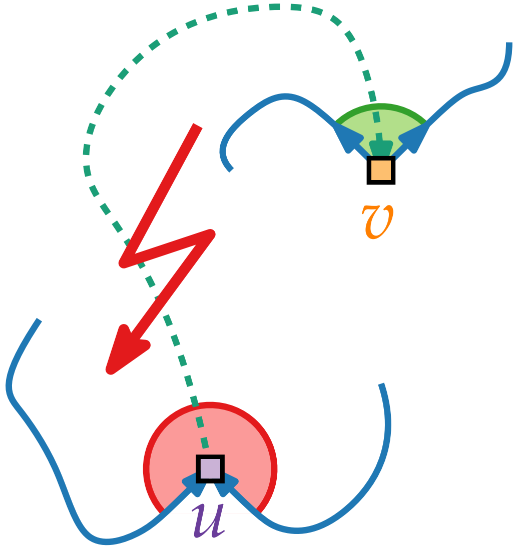

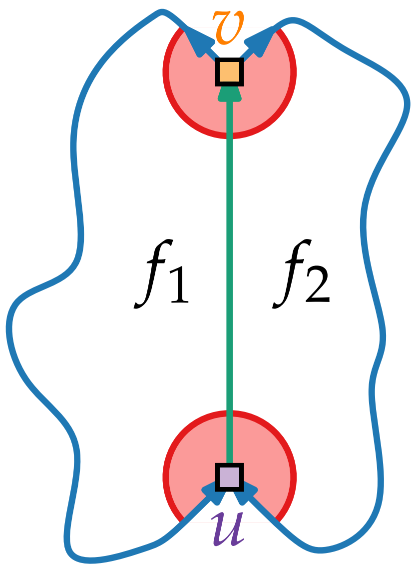

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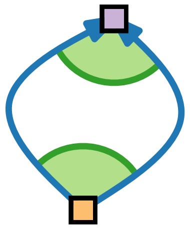

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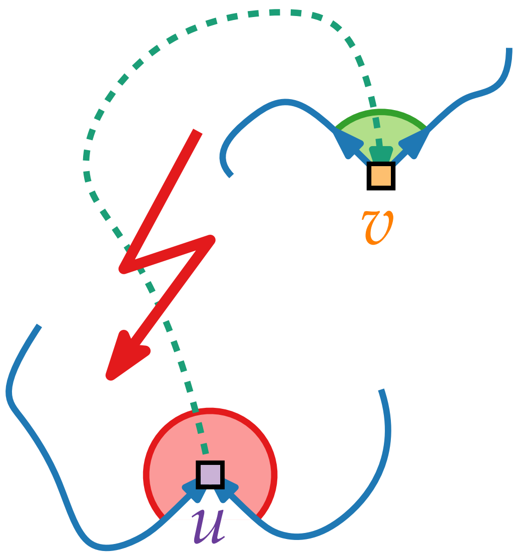

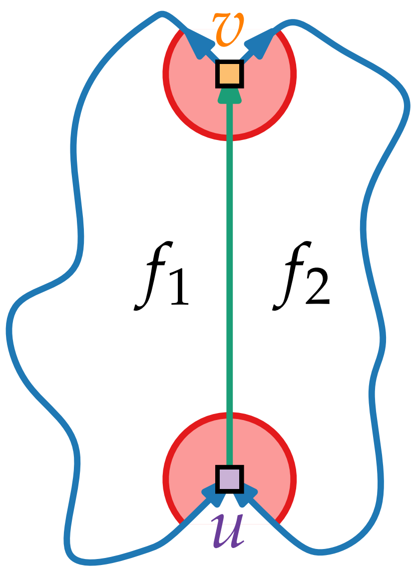

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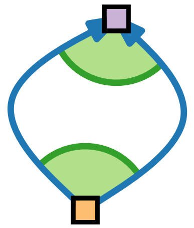
- $L(f) = 0$  $\Rightarrow S(f) = 2$

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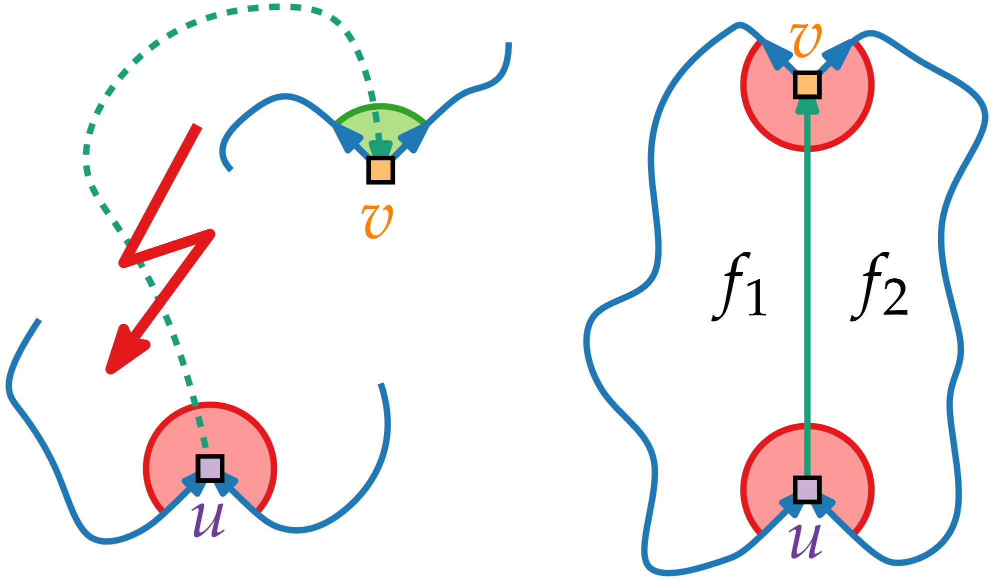

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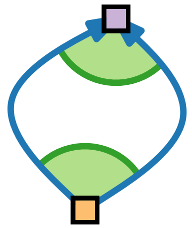
- $L(f) = 0$  $\Rightarrow S(f) = 2$

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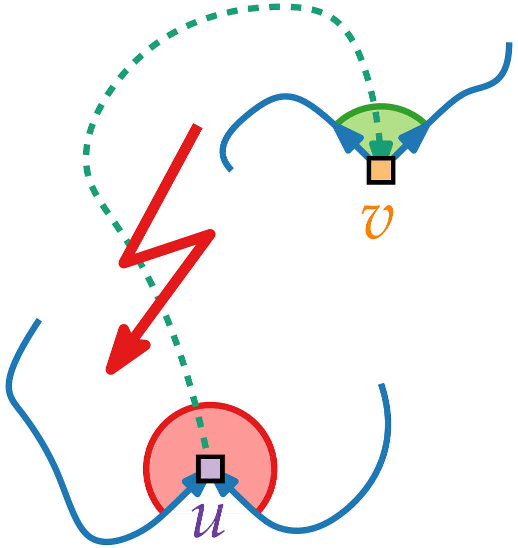

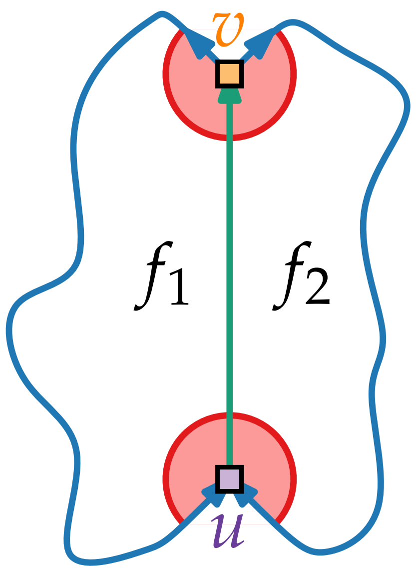

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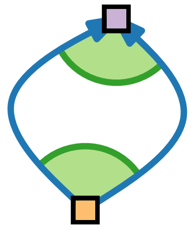
- $L(f) = 0$  $\Rightarrow S(f) = 2$
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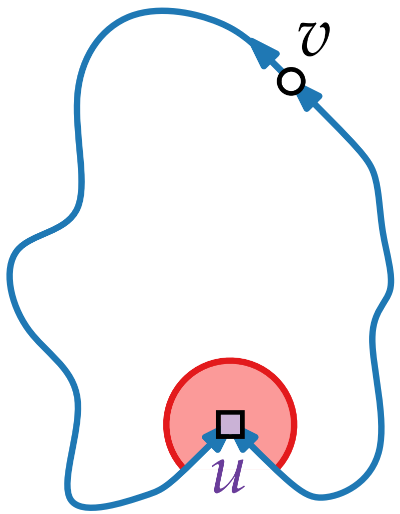

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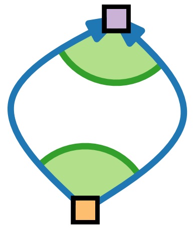

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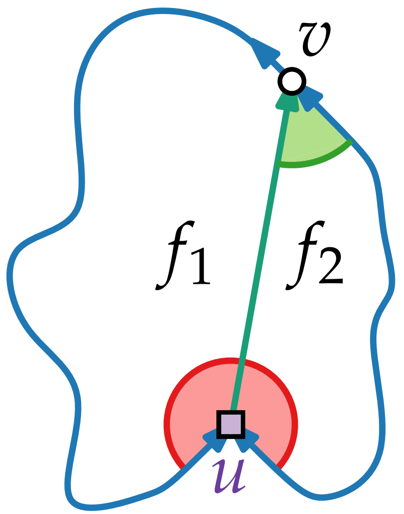

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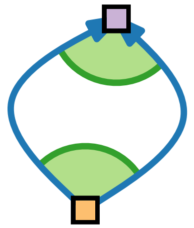

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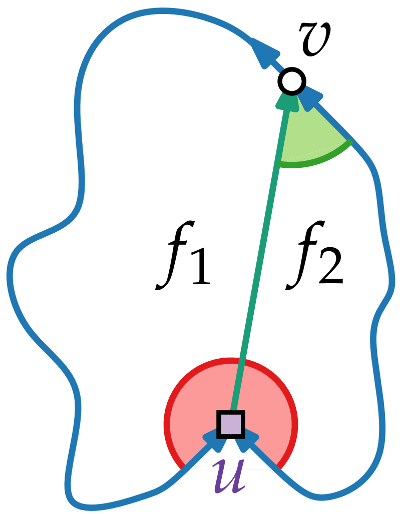

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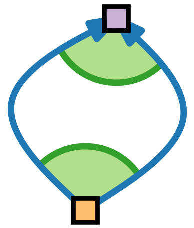

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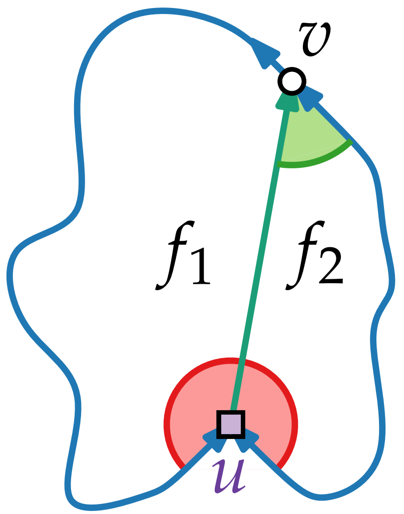

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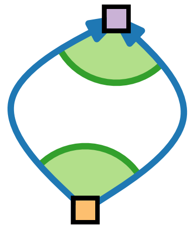

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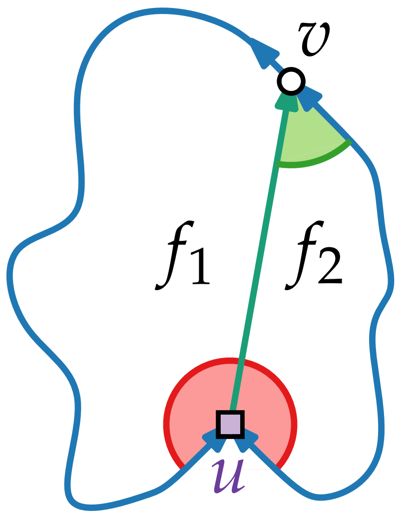

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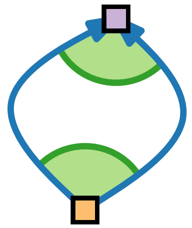
- $L(f) = 0$  $\Rightarrow S(f) = 2$

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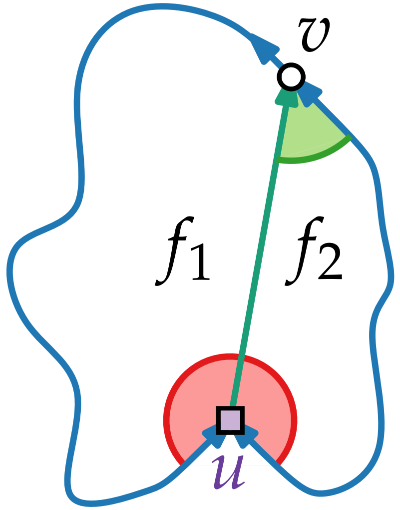

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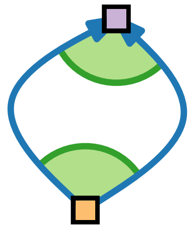

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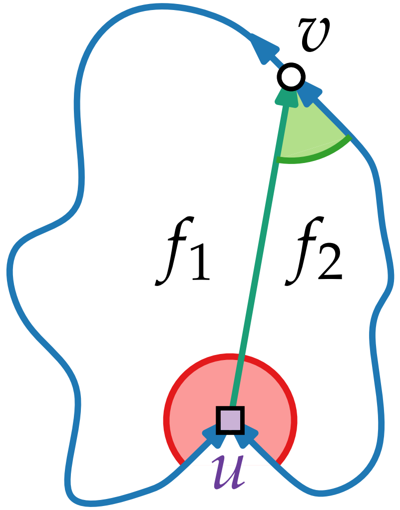

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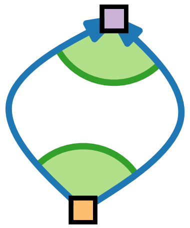

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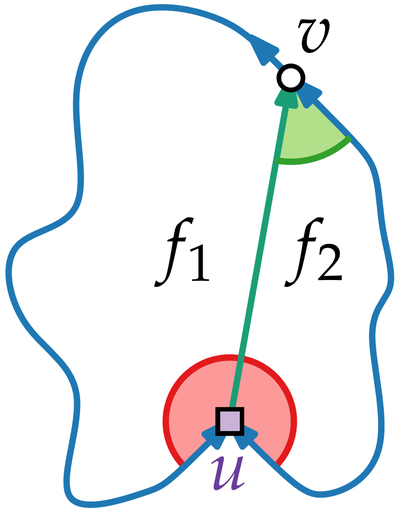

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- Otherwise "high" source u exists.

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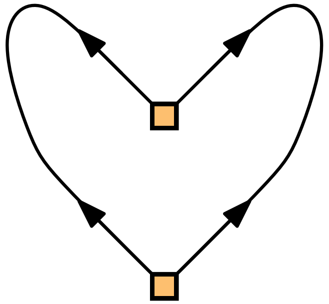

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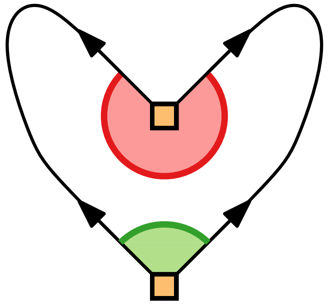

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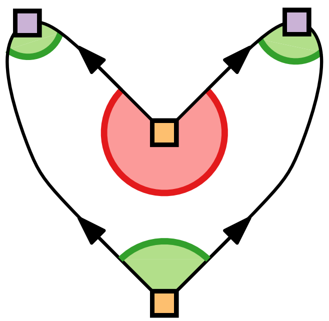

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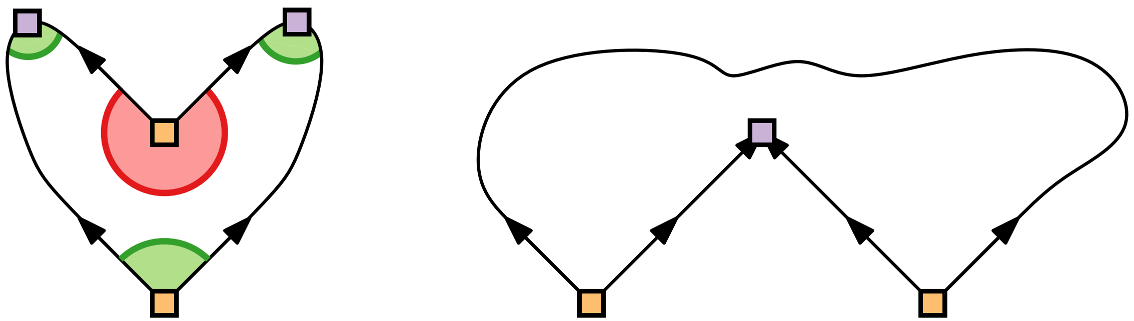

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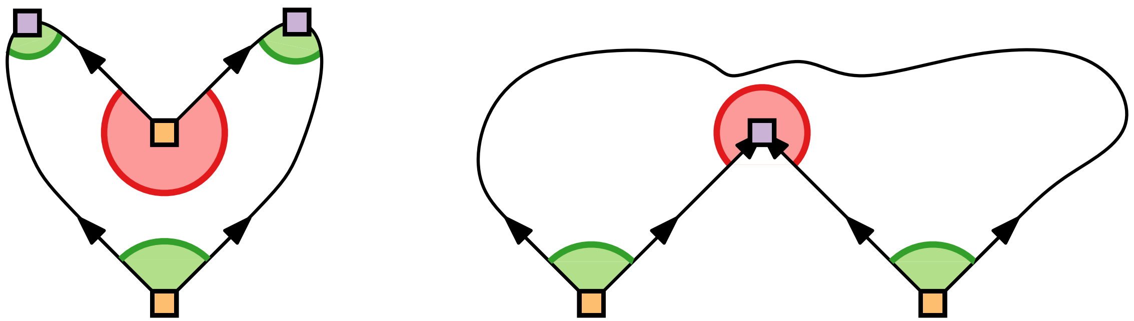

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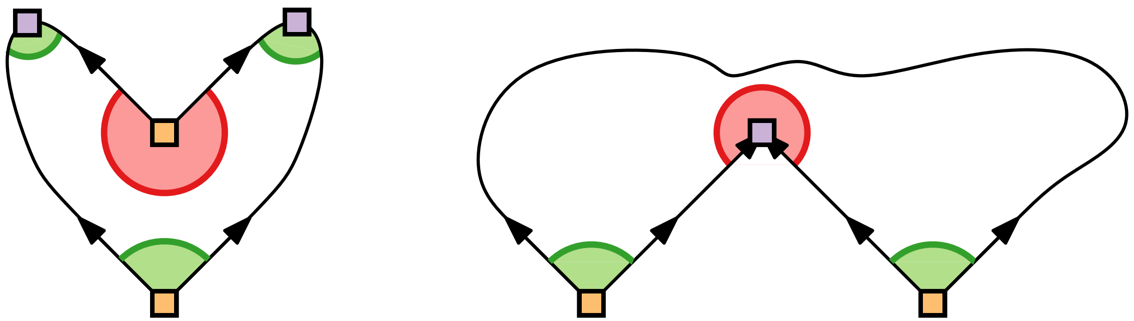

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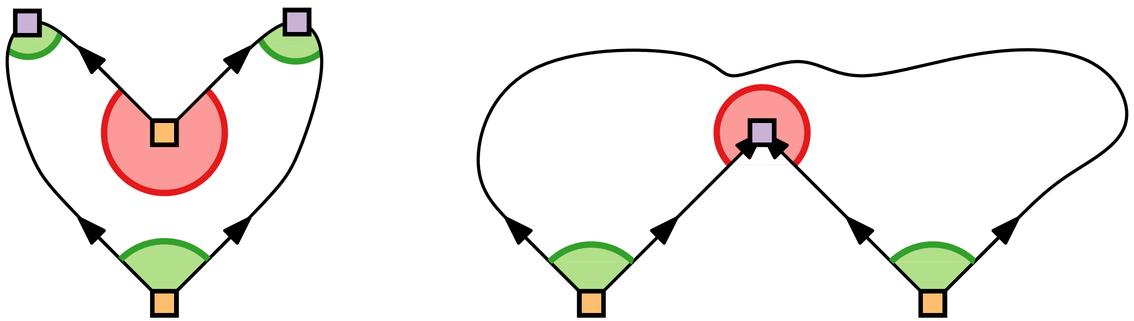

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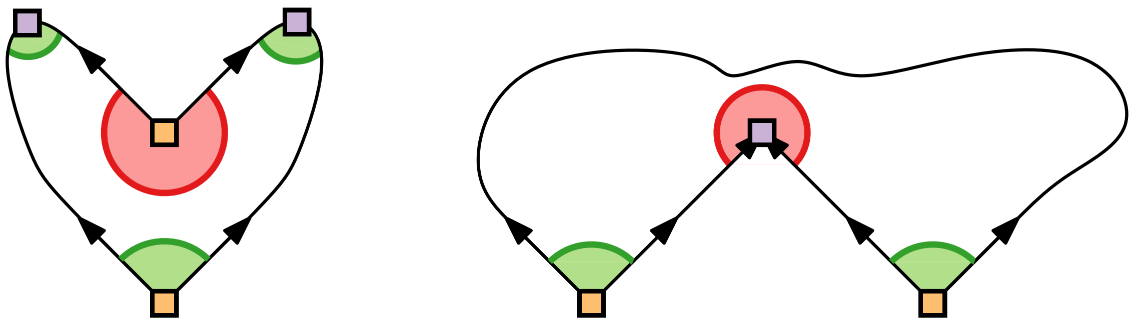

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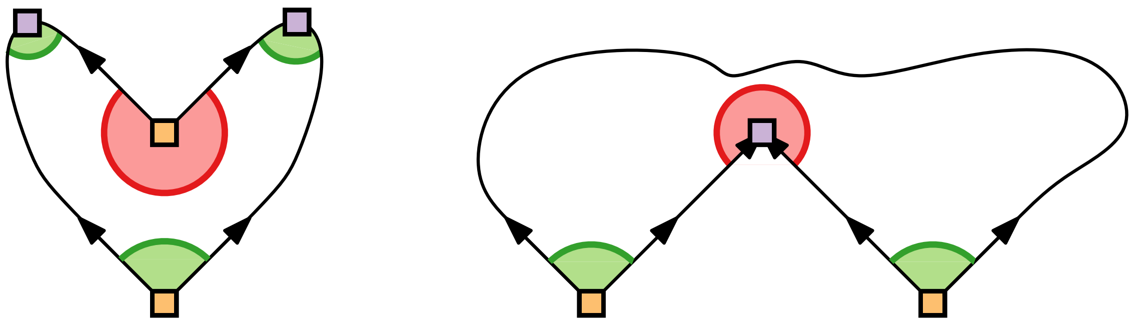

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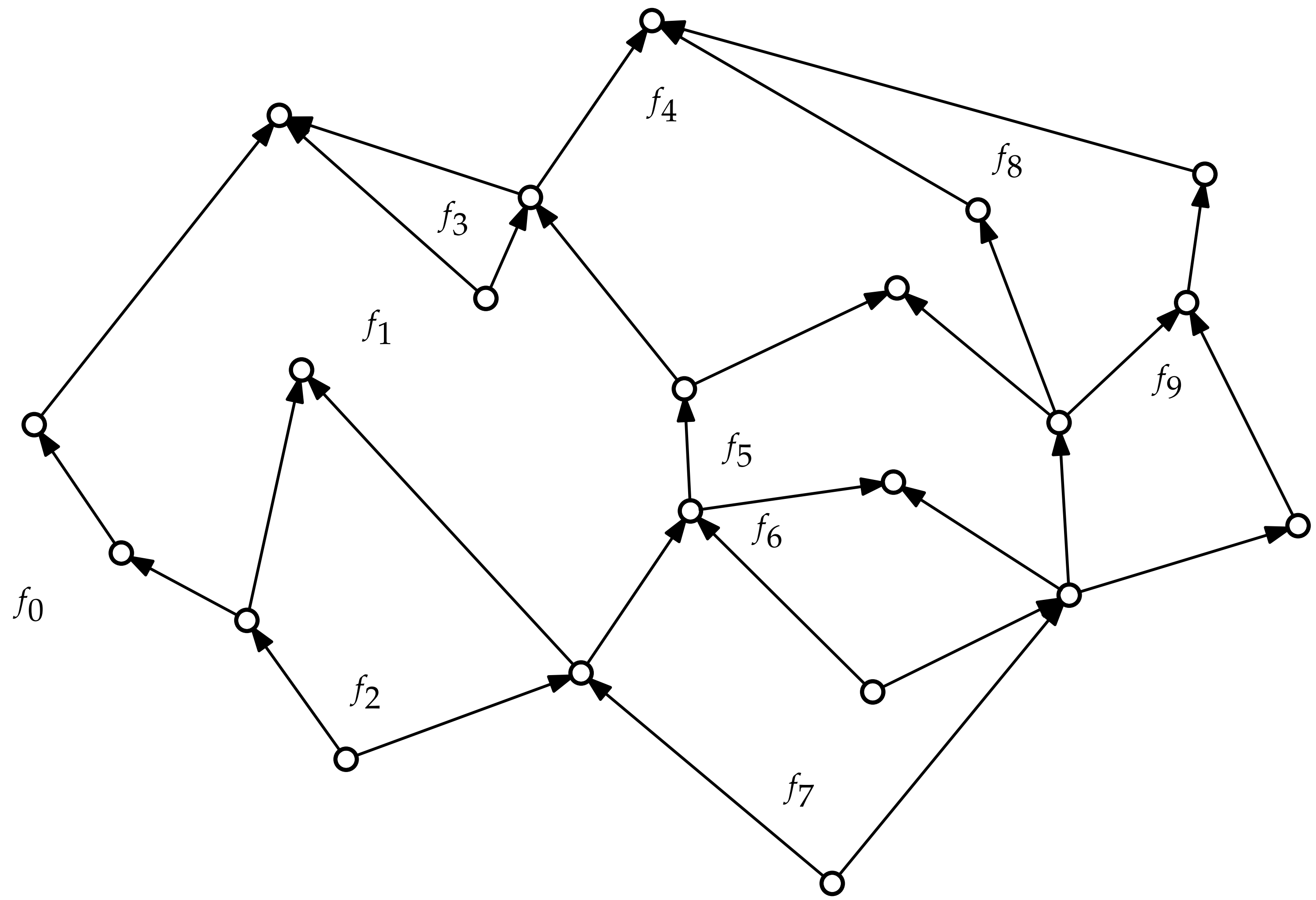

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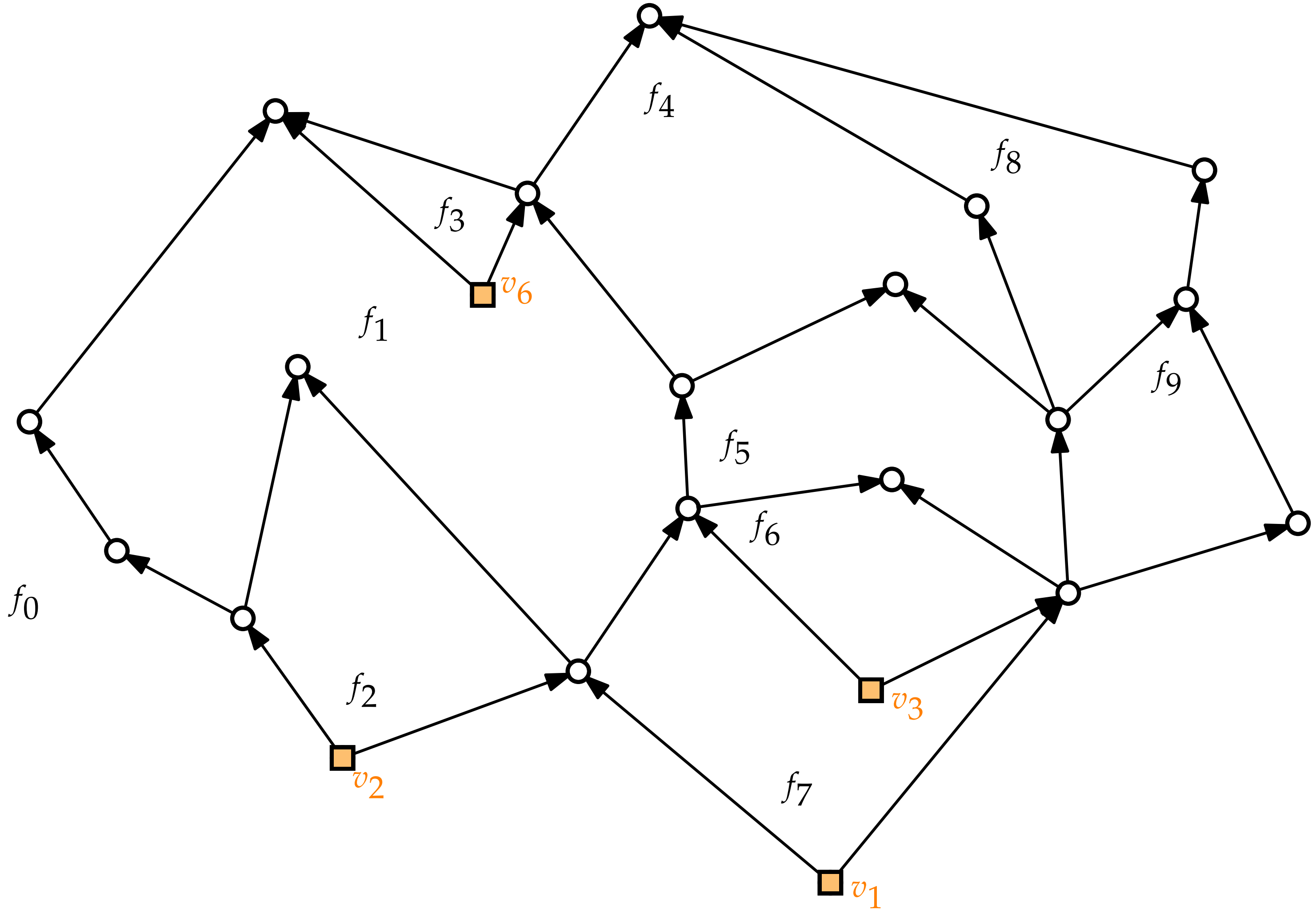

global sources

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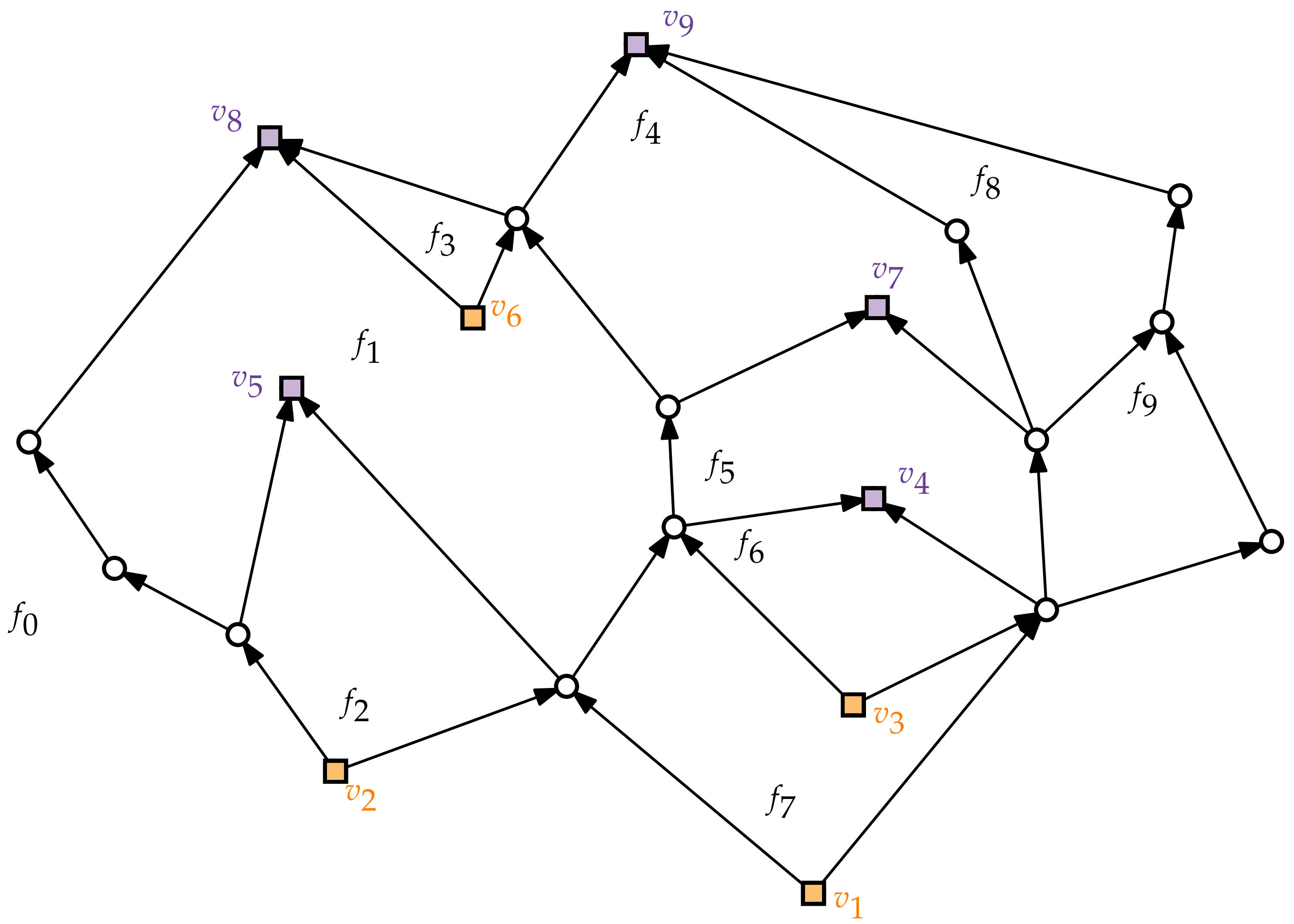

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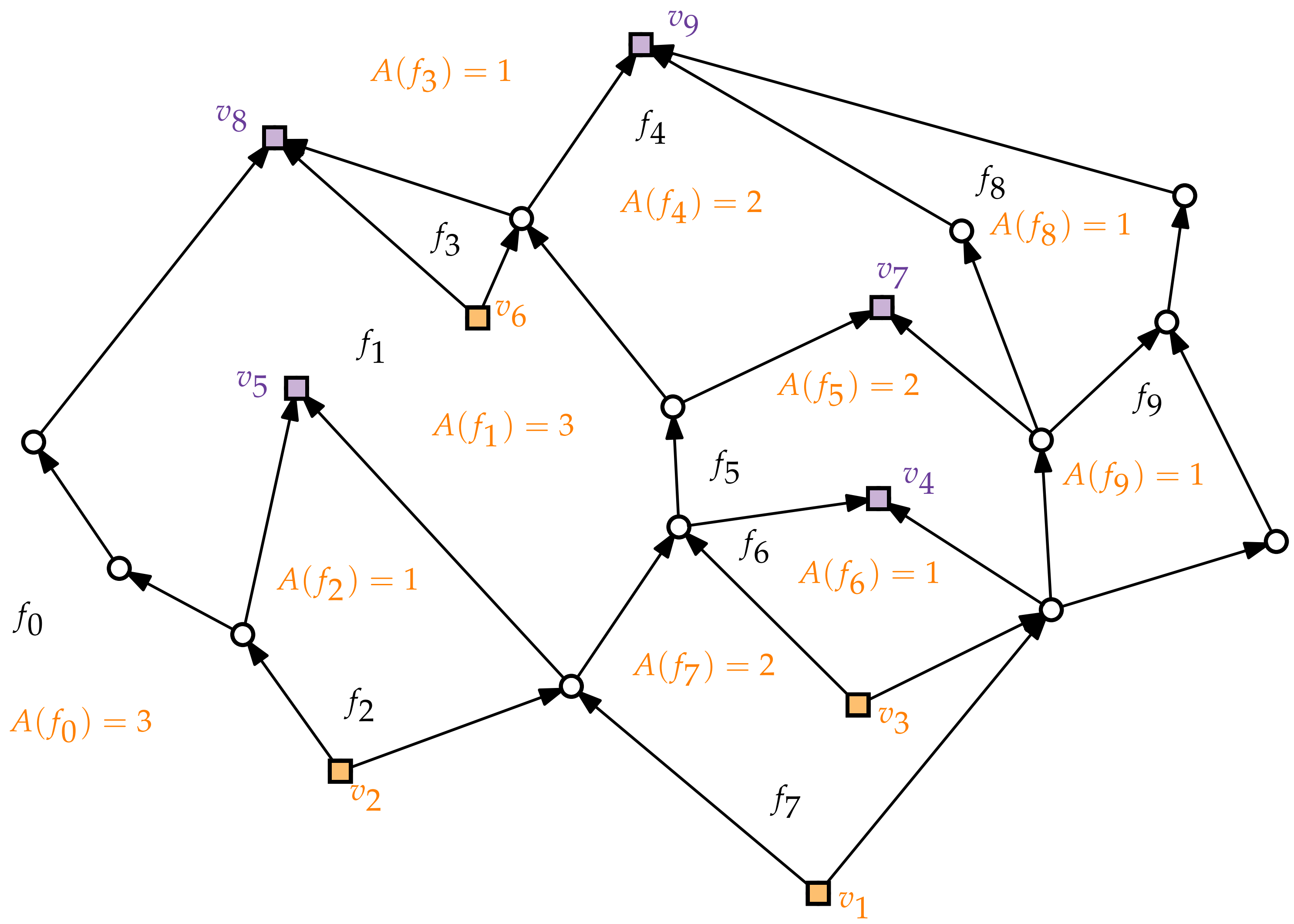

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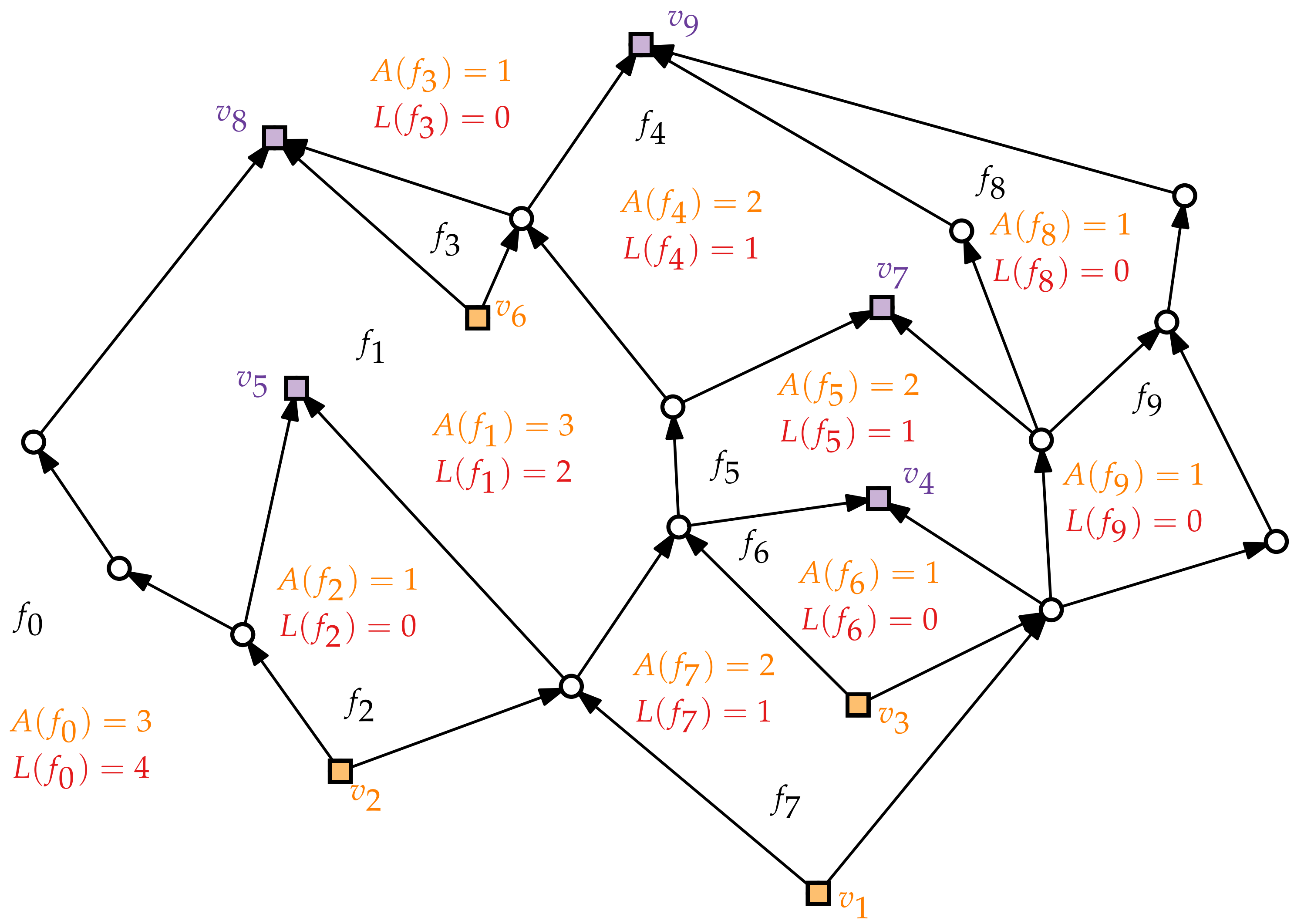

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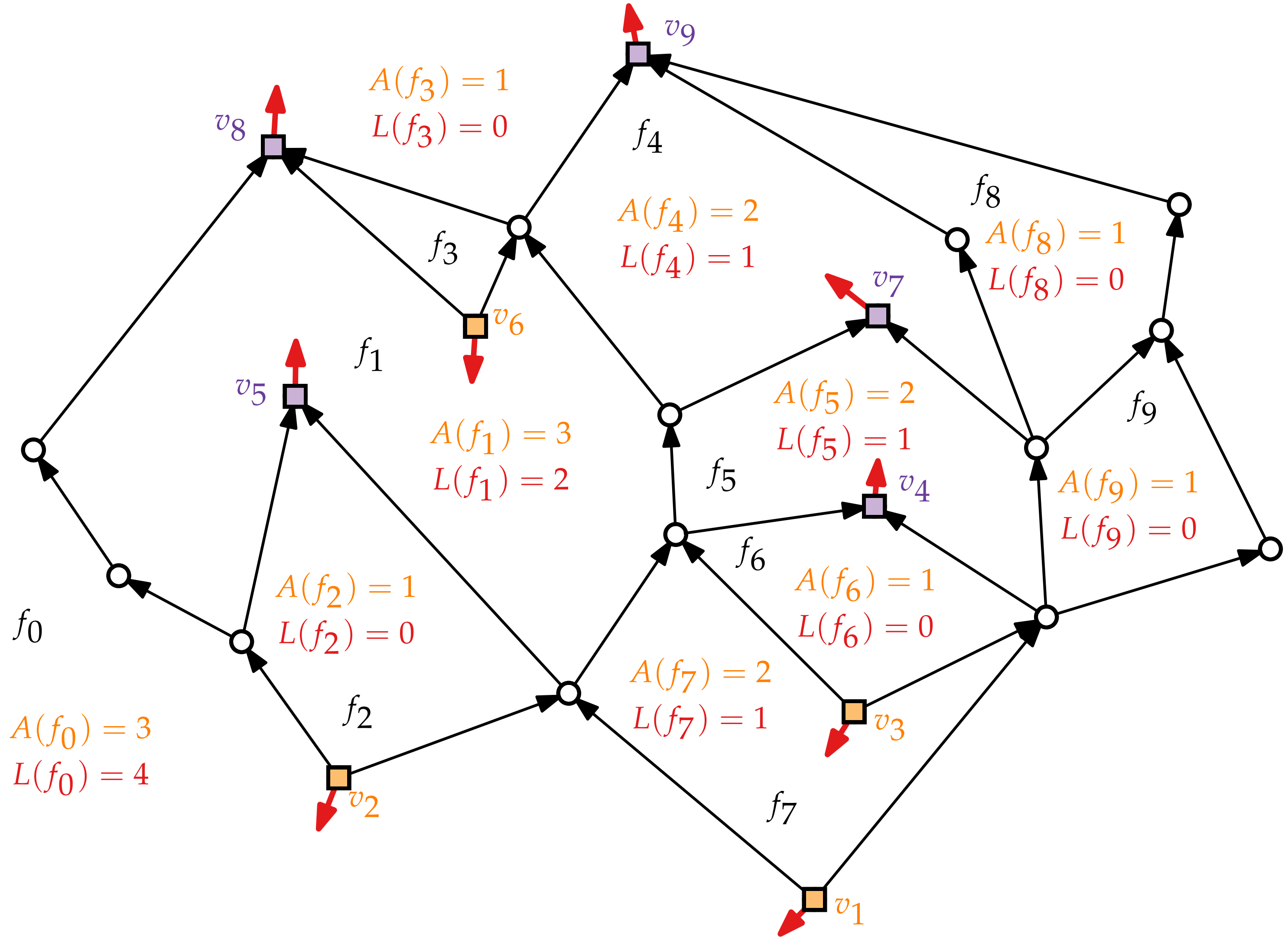

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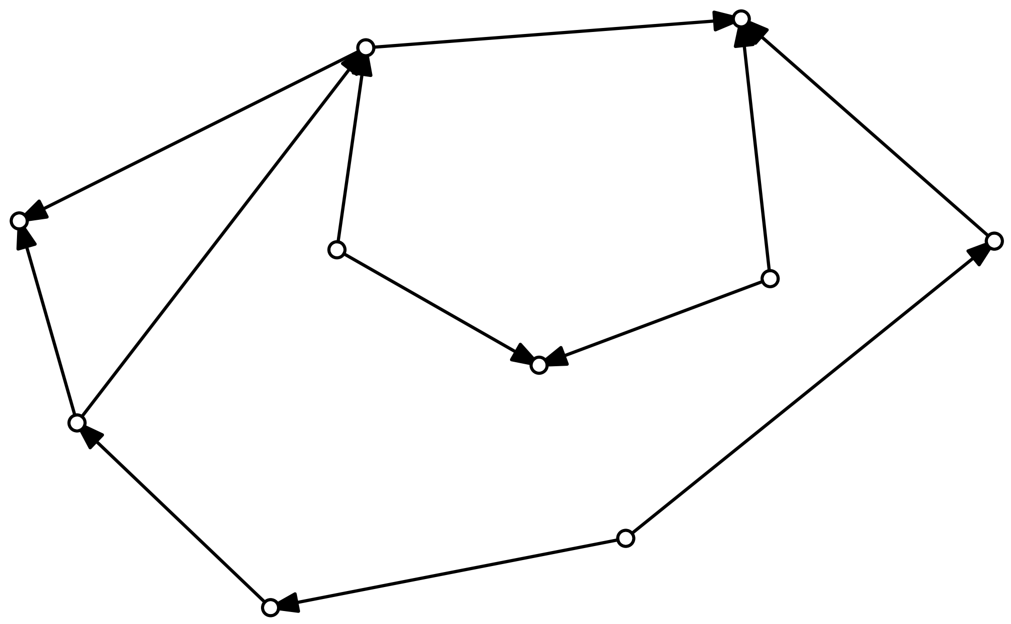

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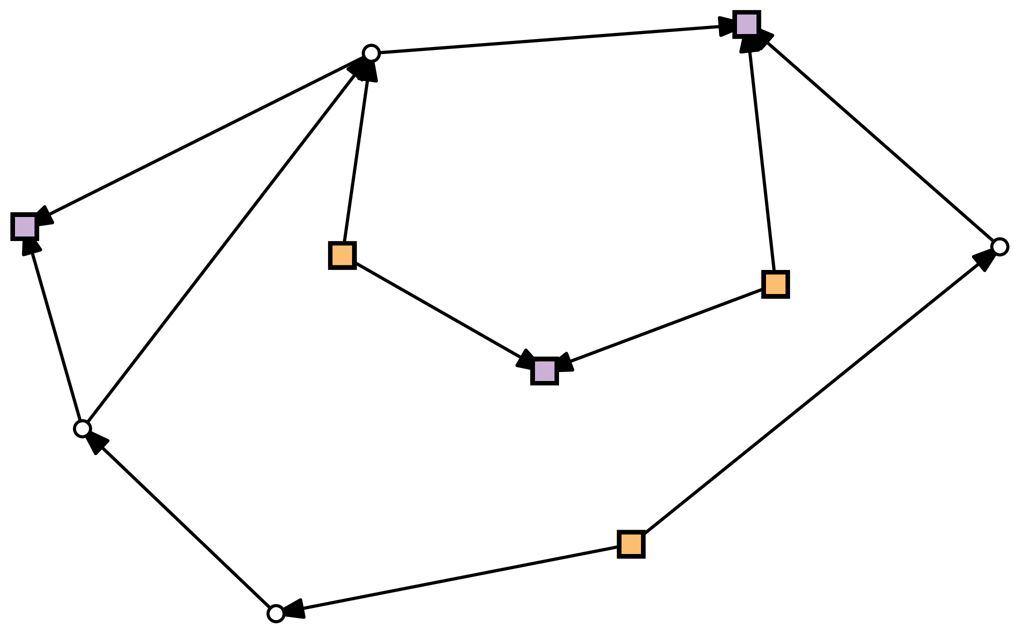

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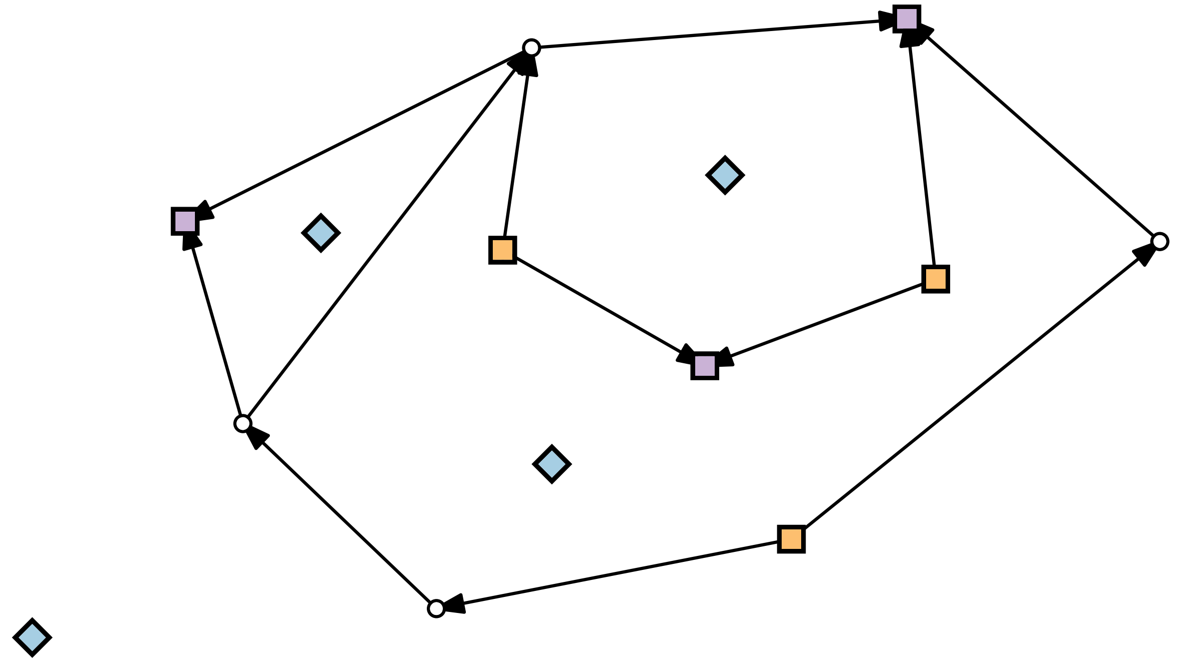

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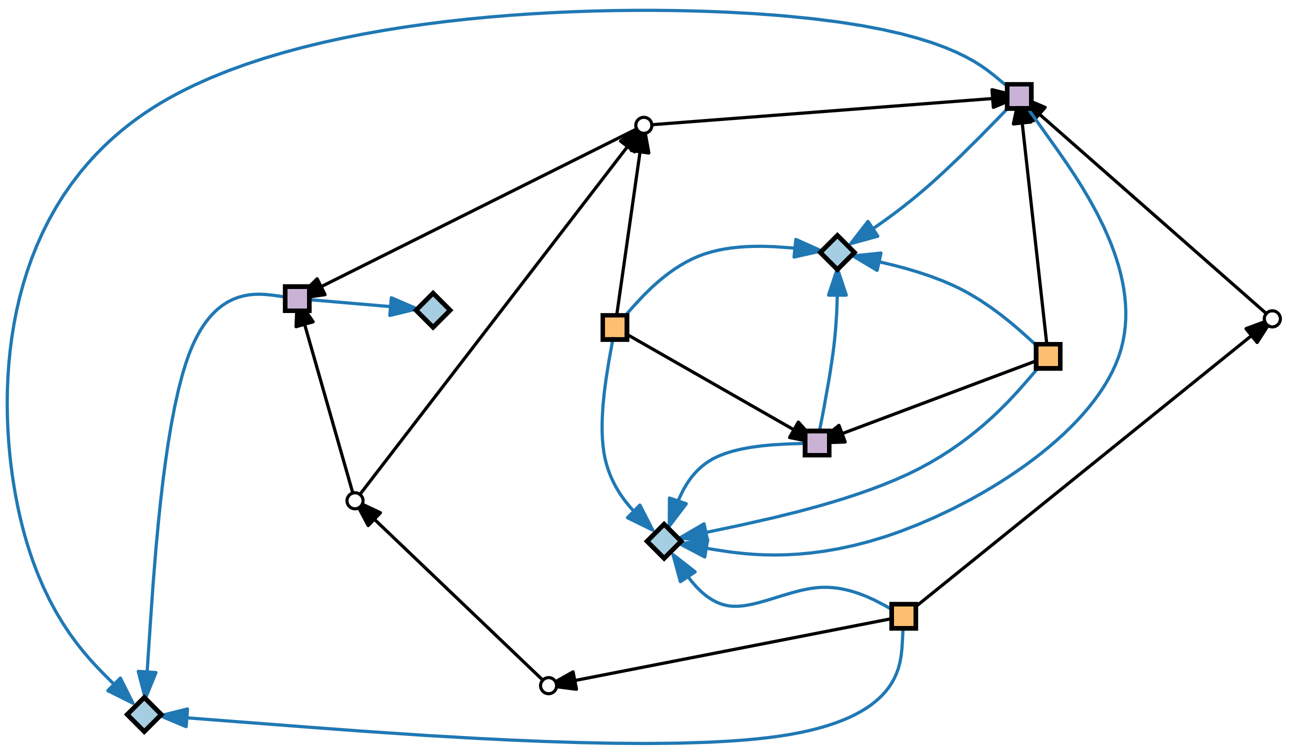

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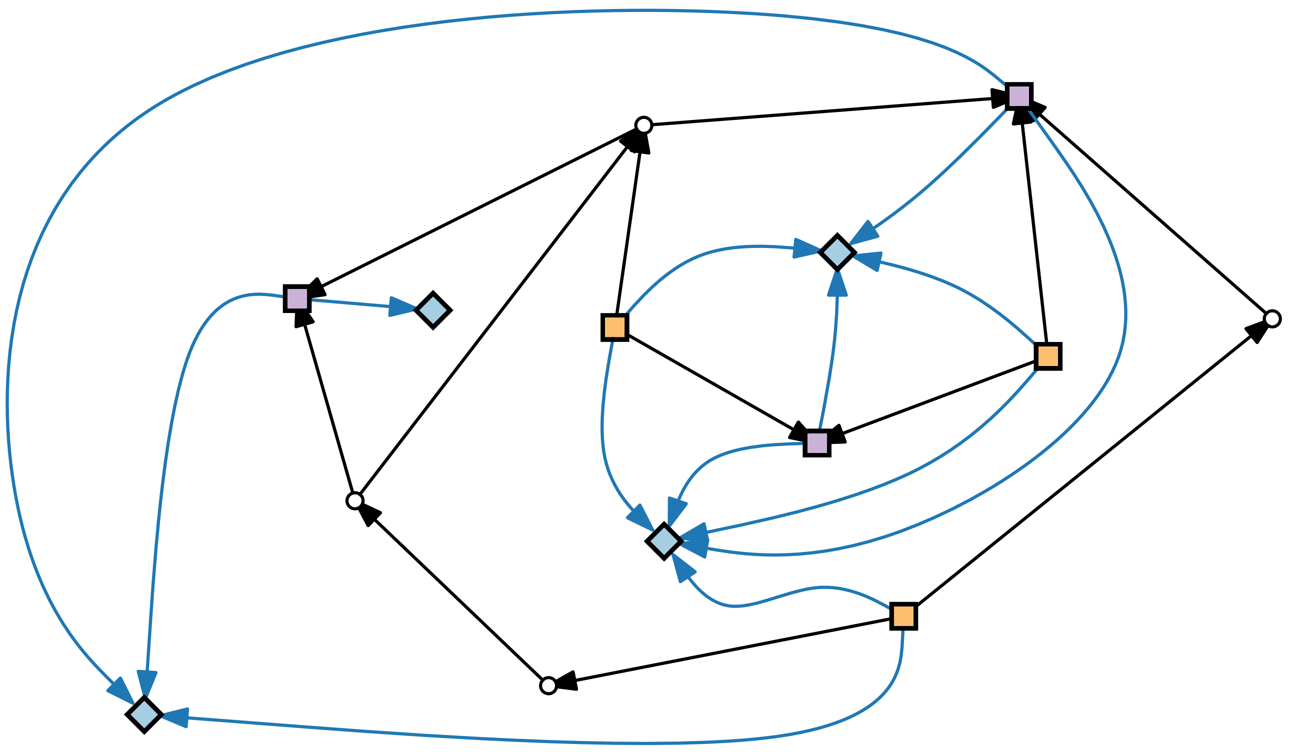

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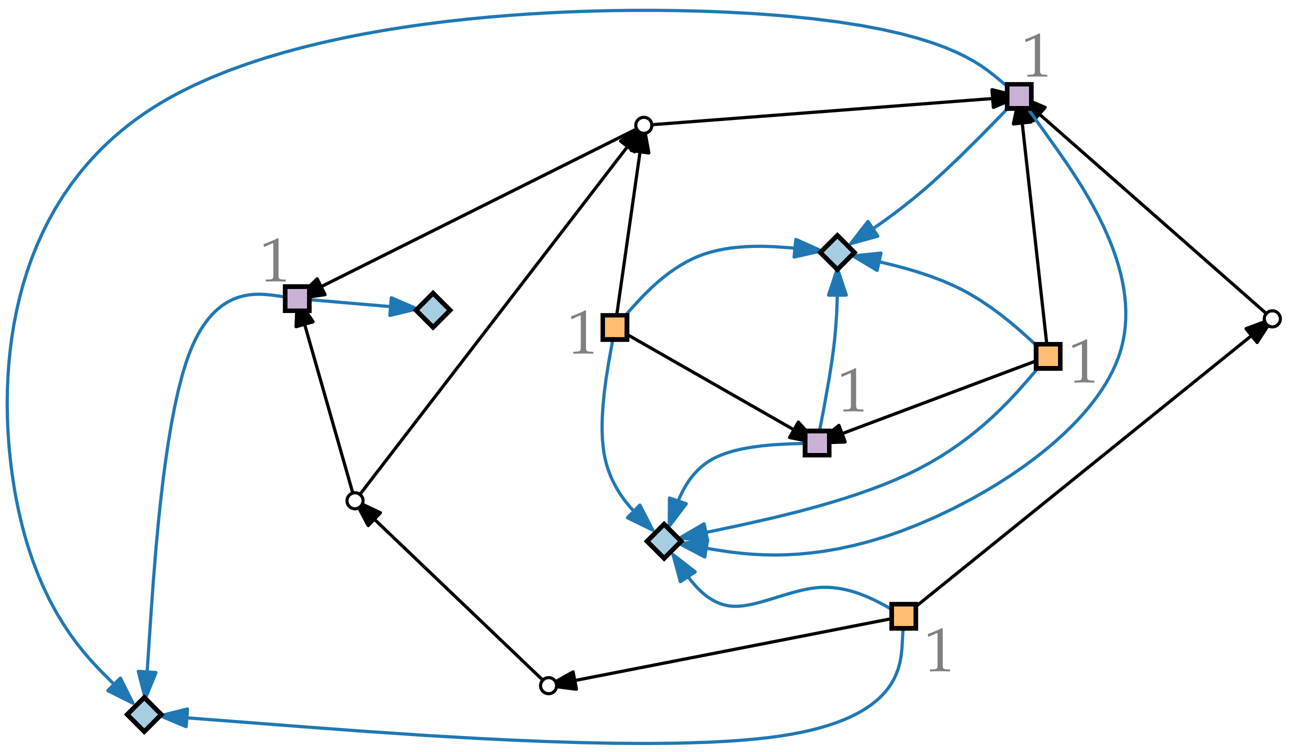

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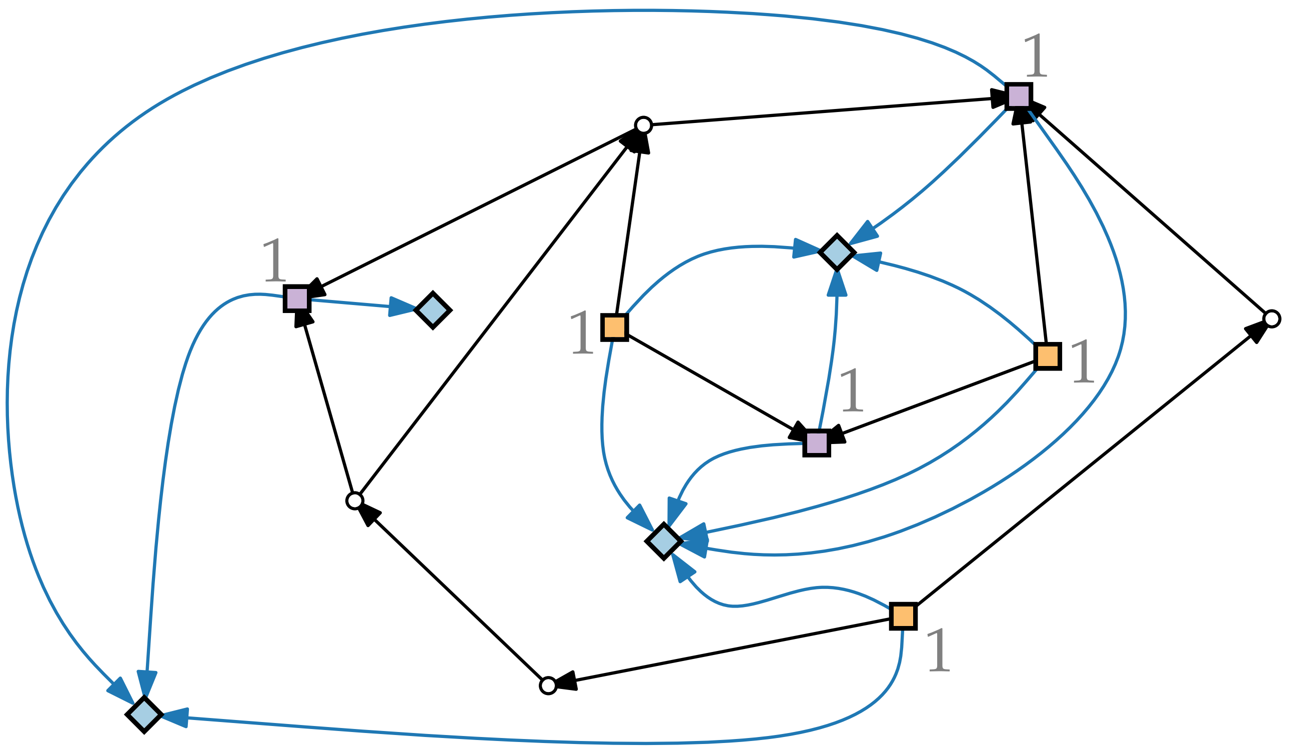

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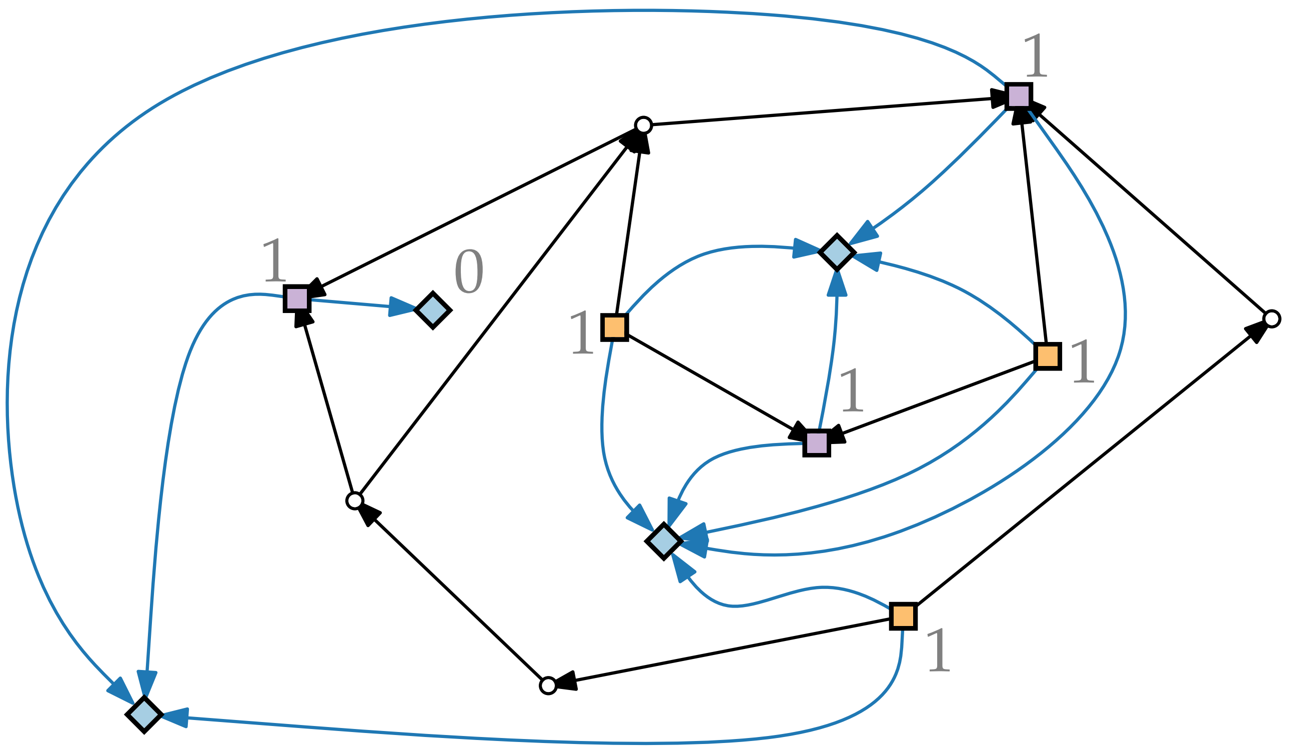

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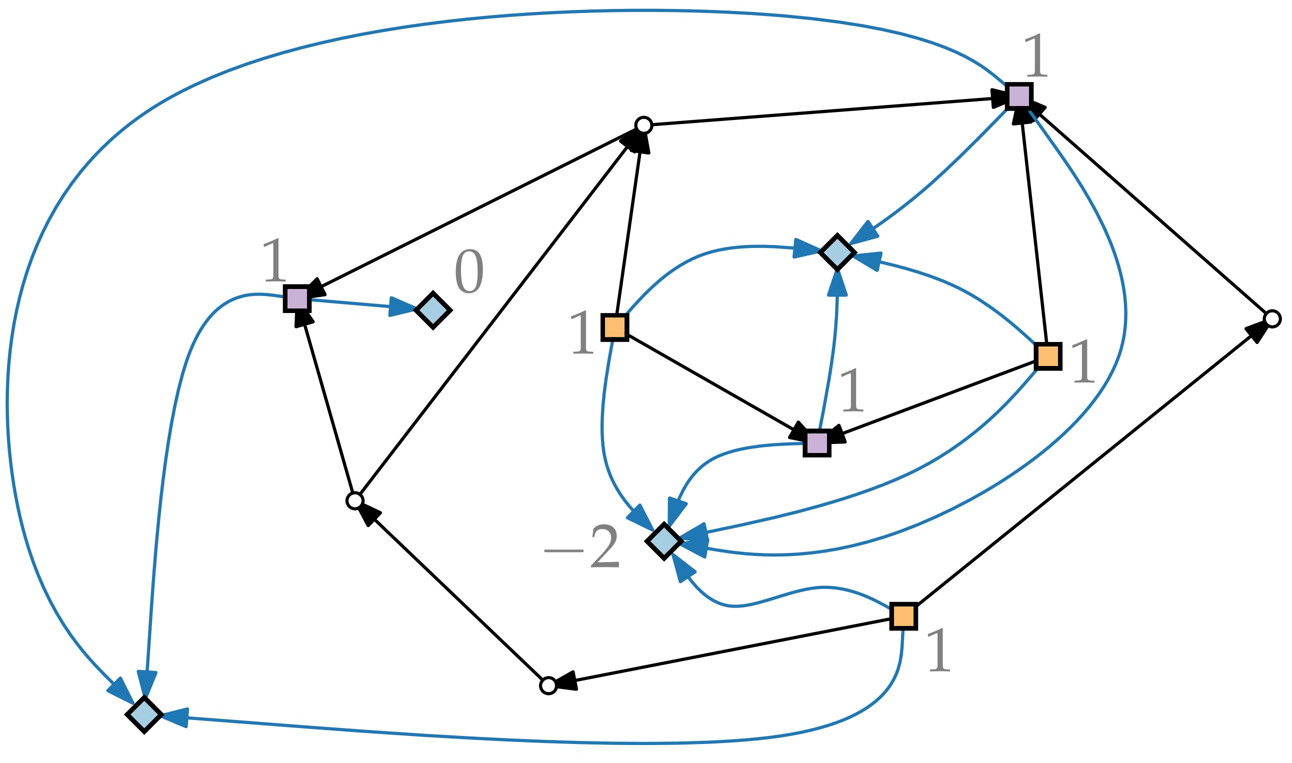

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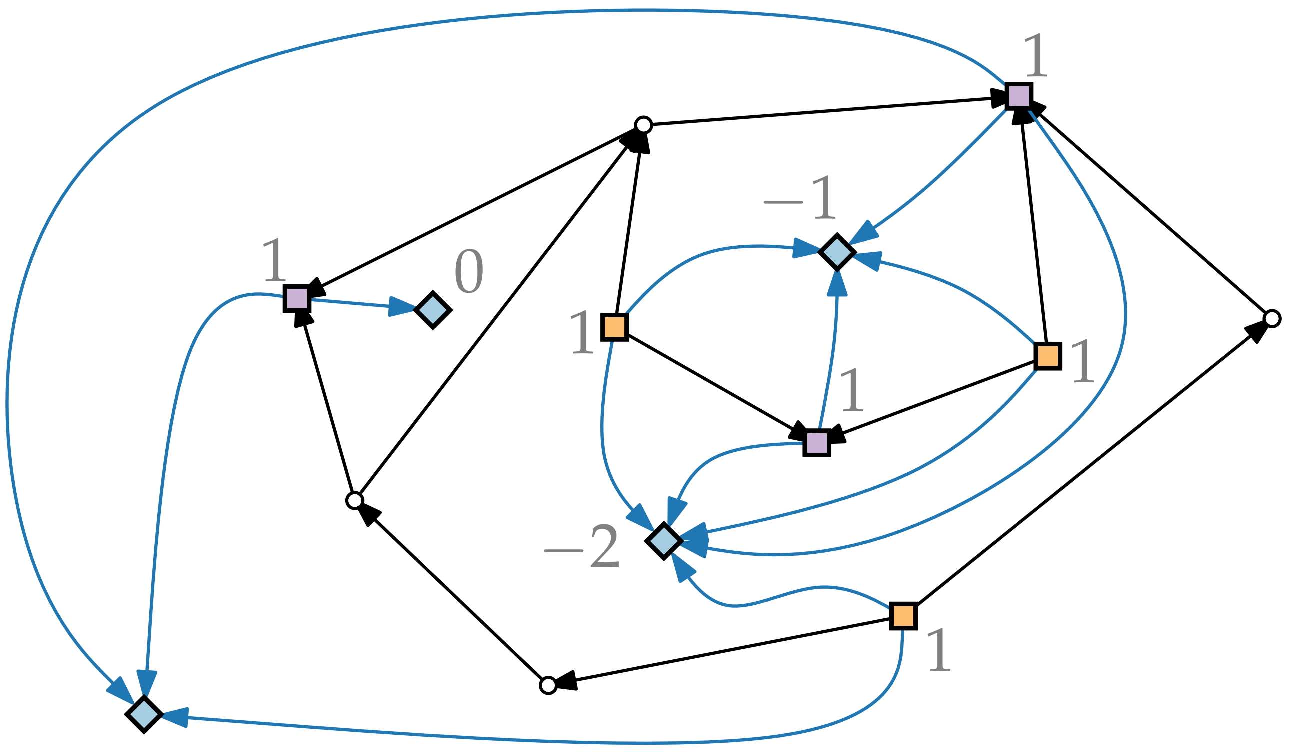

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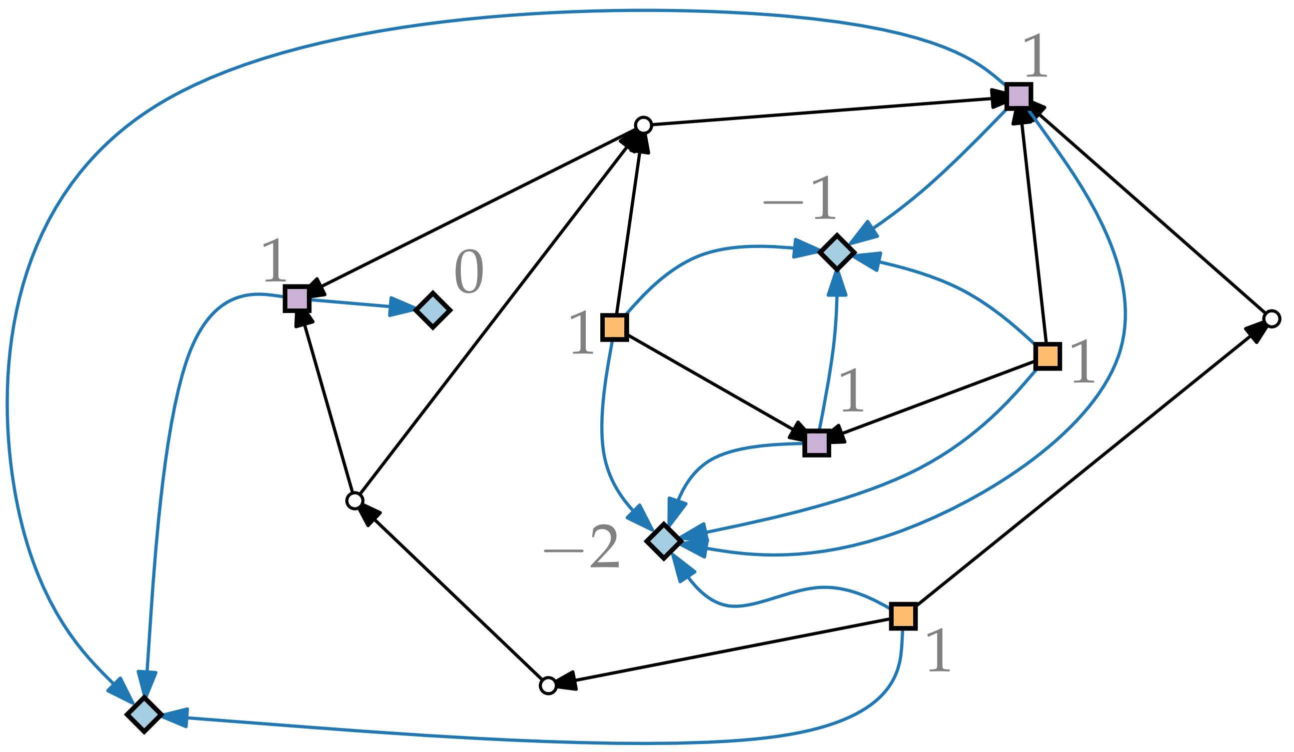

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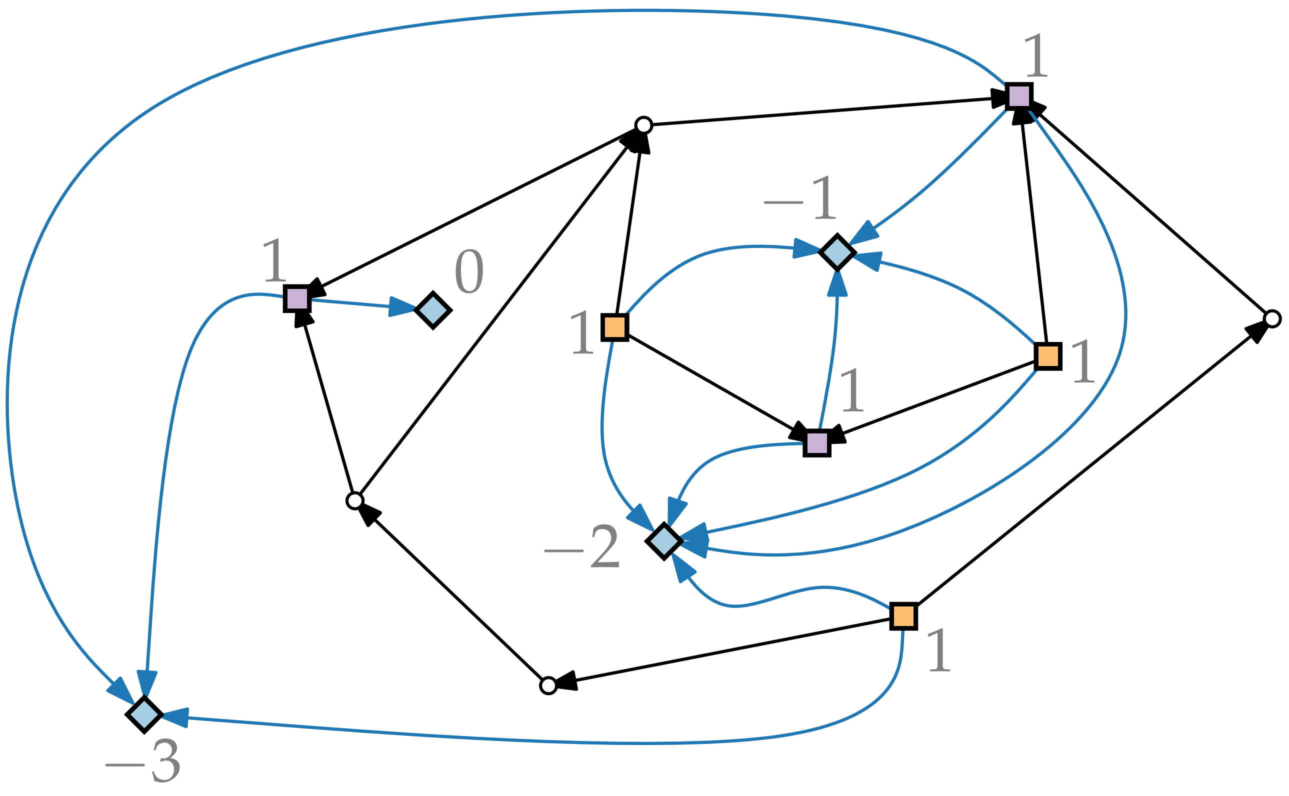

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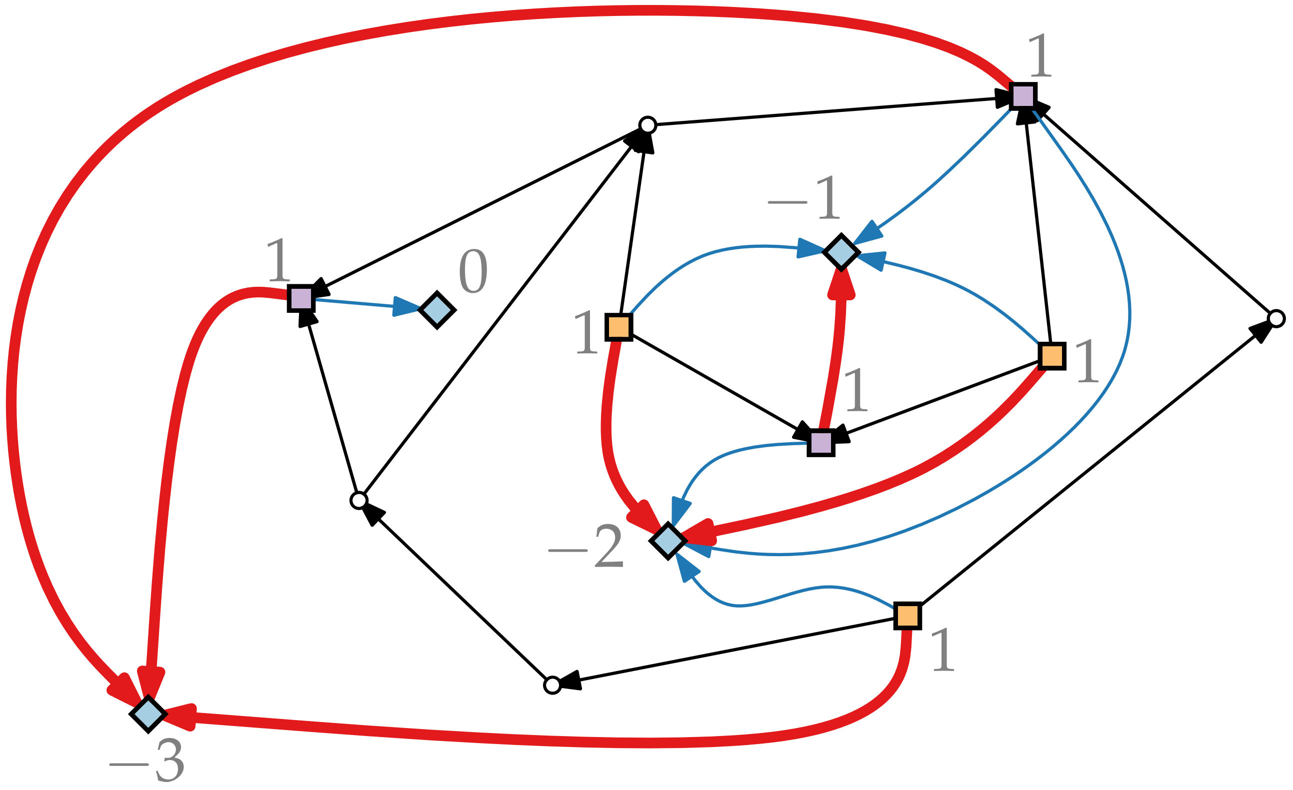

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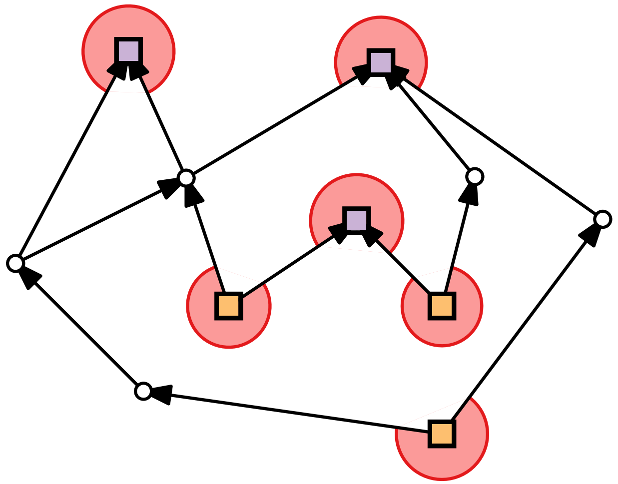

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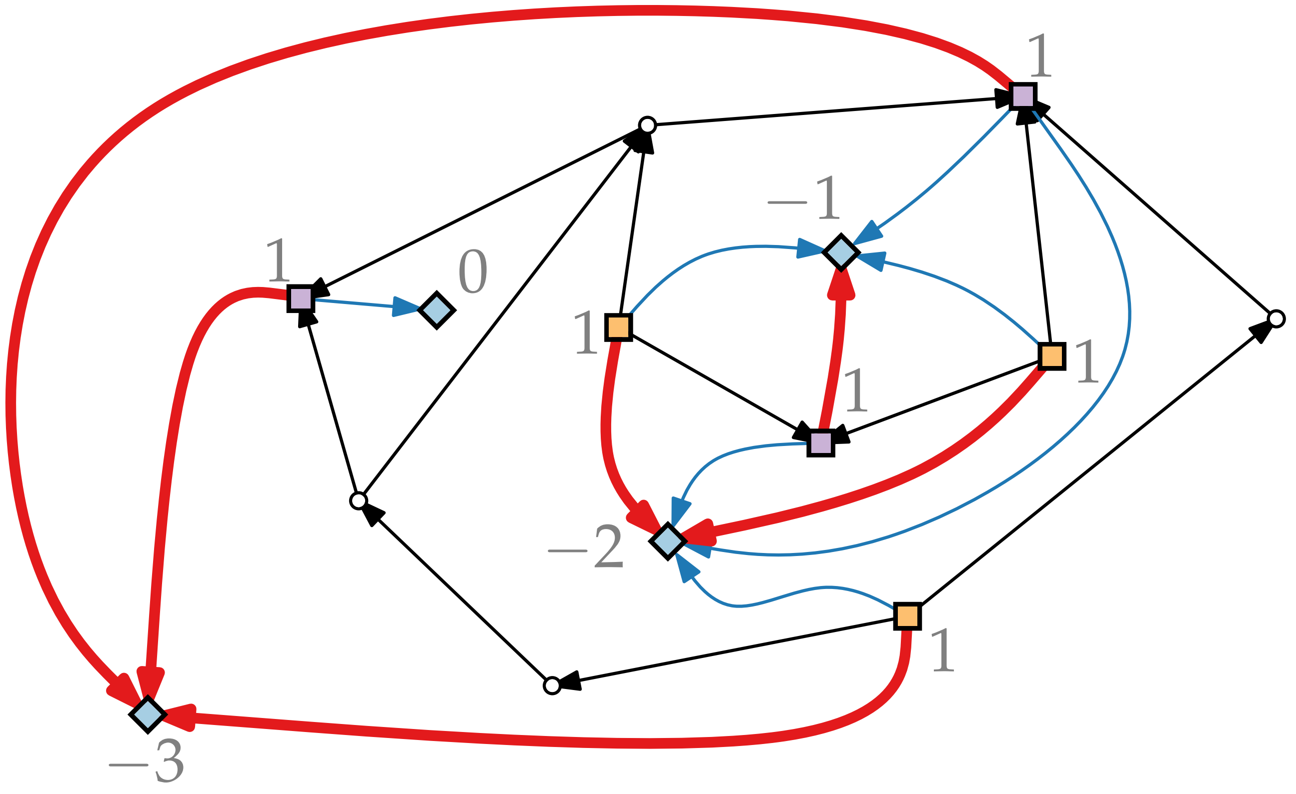

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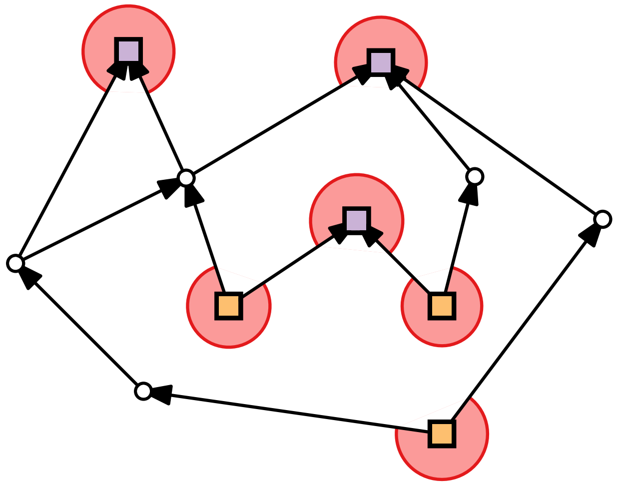

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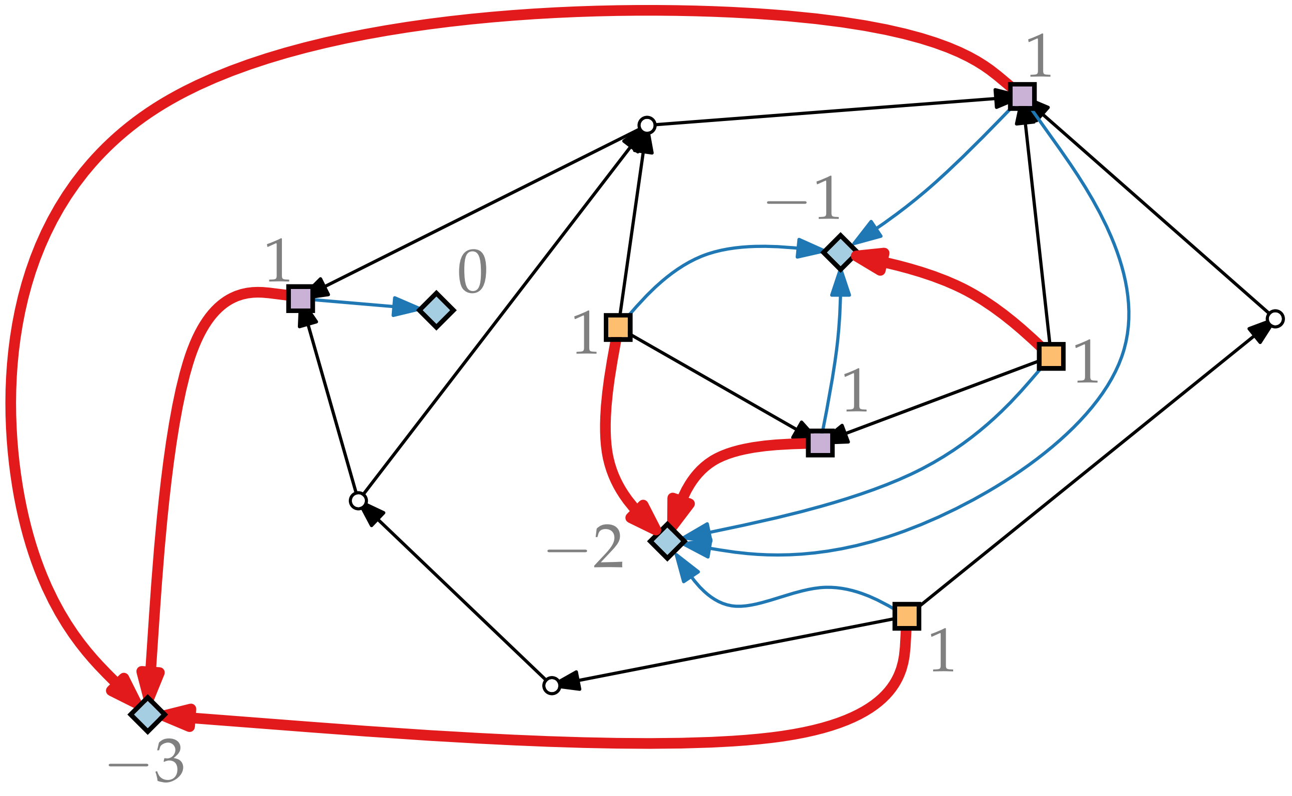

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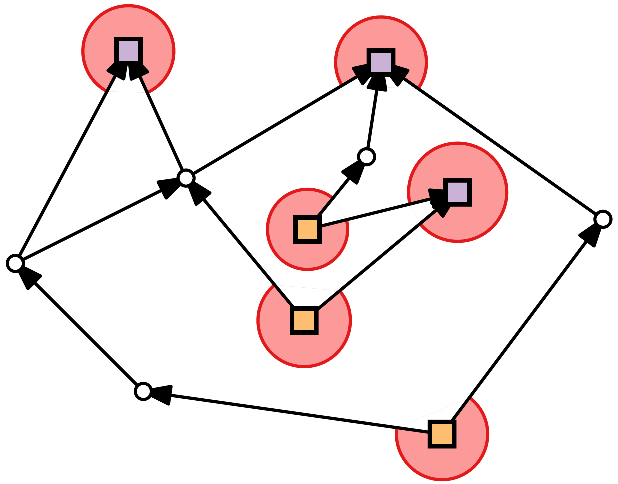

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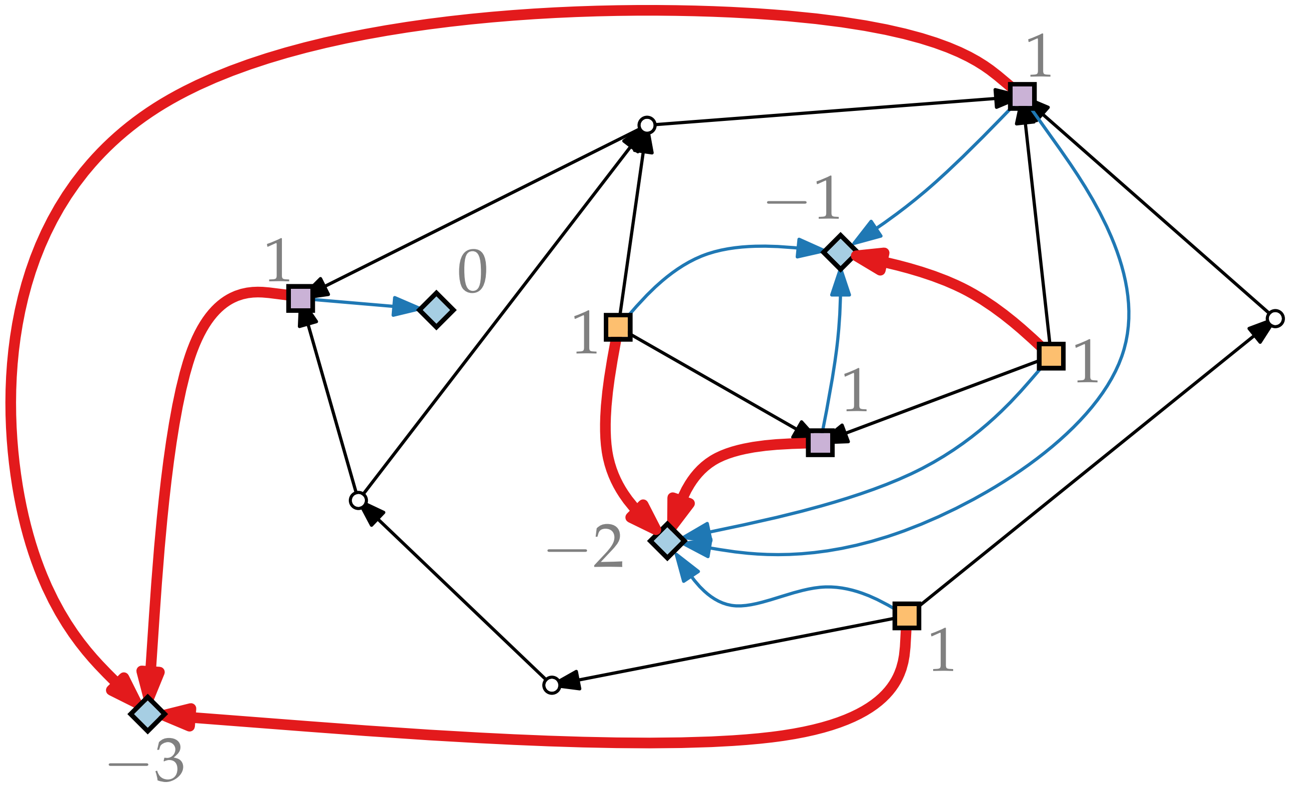

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Theorem 1. [Kelly 1987, Di Battista & Tamassia 1988]
[...] G is upward planar
$\Leftrightarrow$ G is the spanning subgraph of a planar st-digraph.

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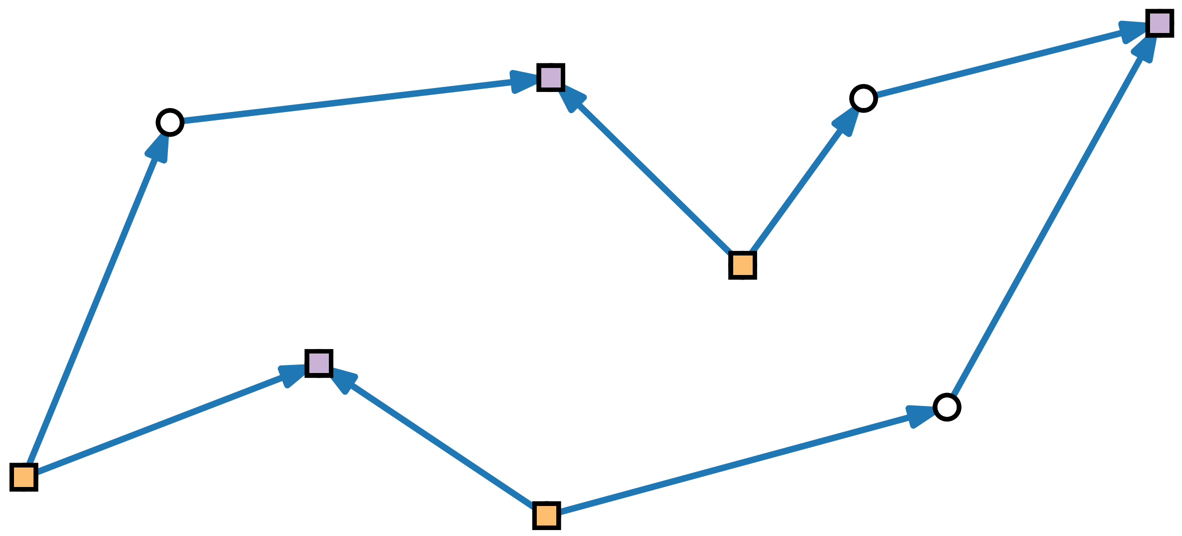

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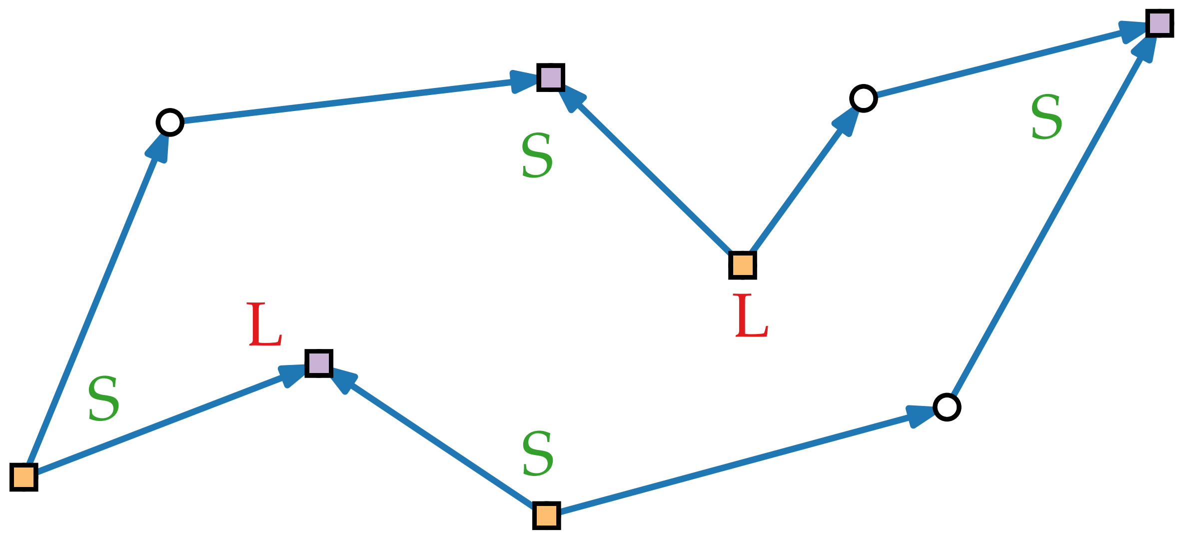

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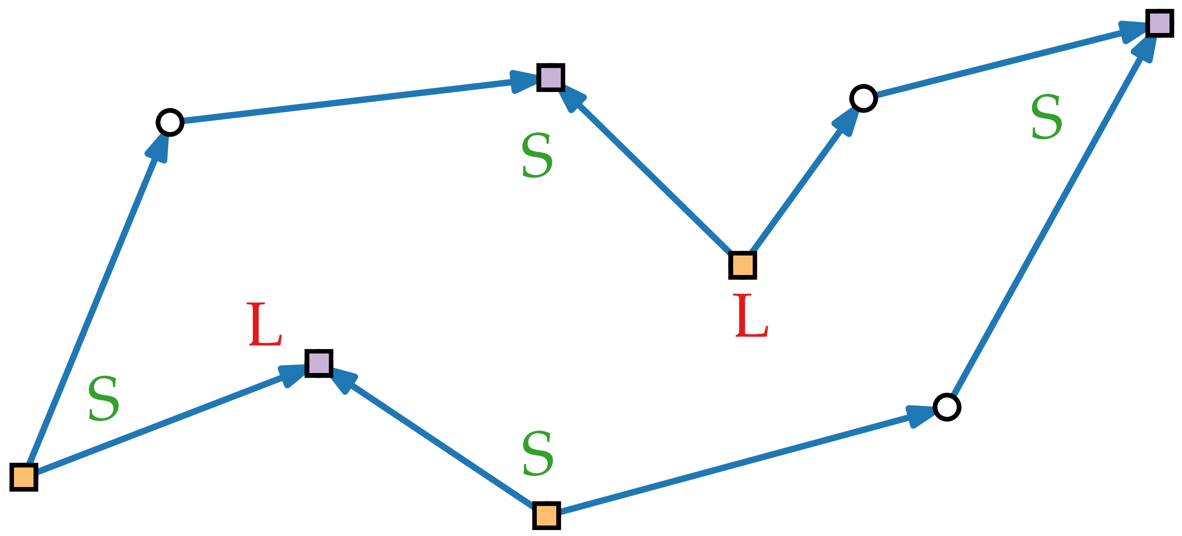

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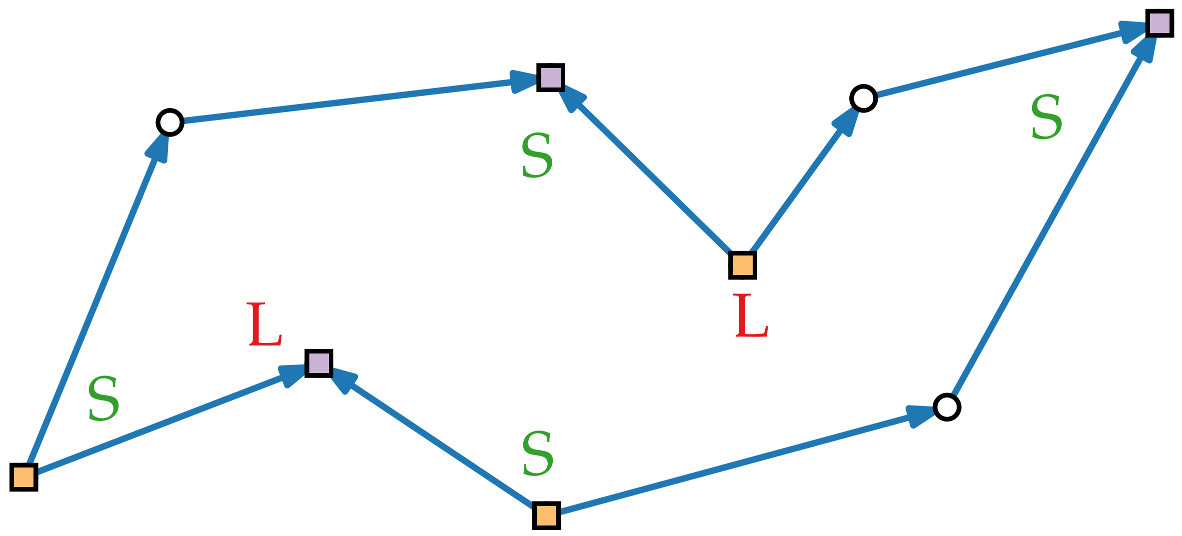

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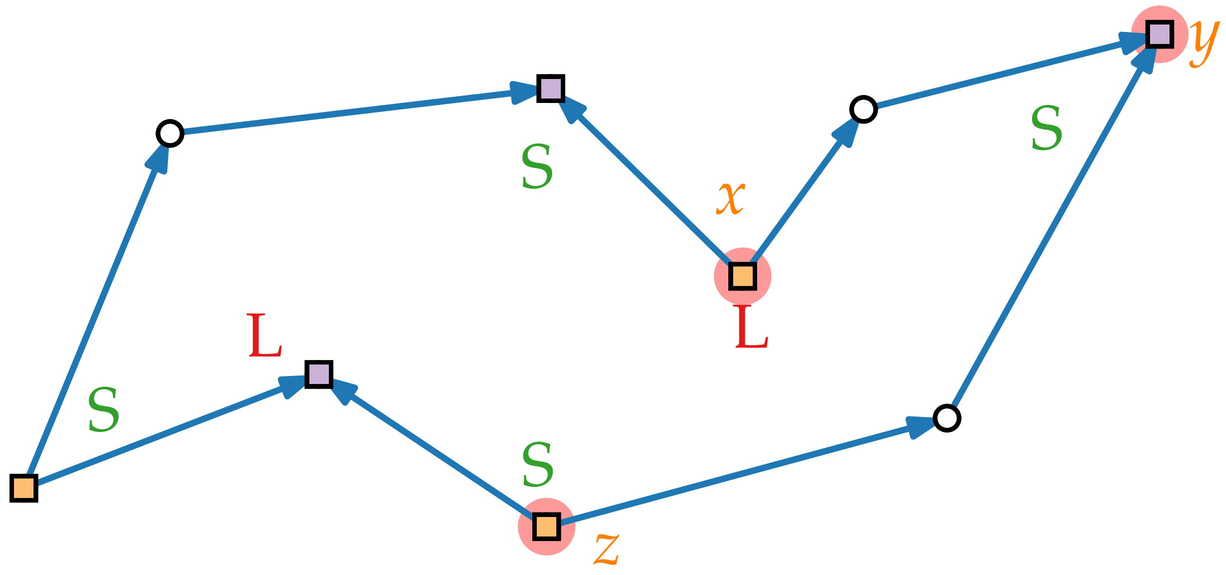

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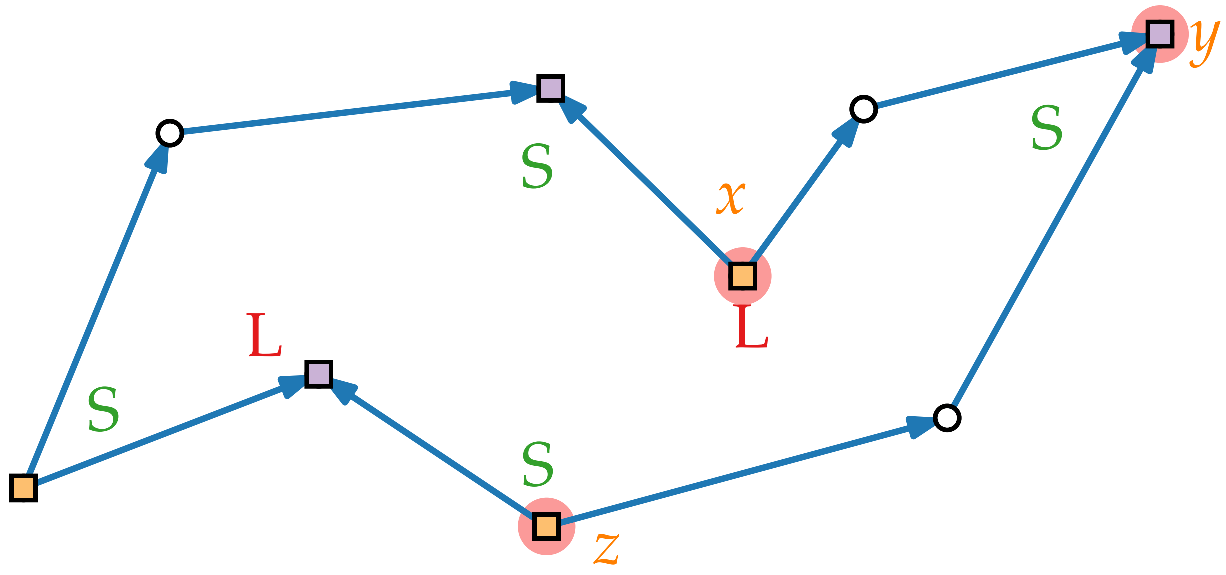

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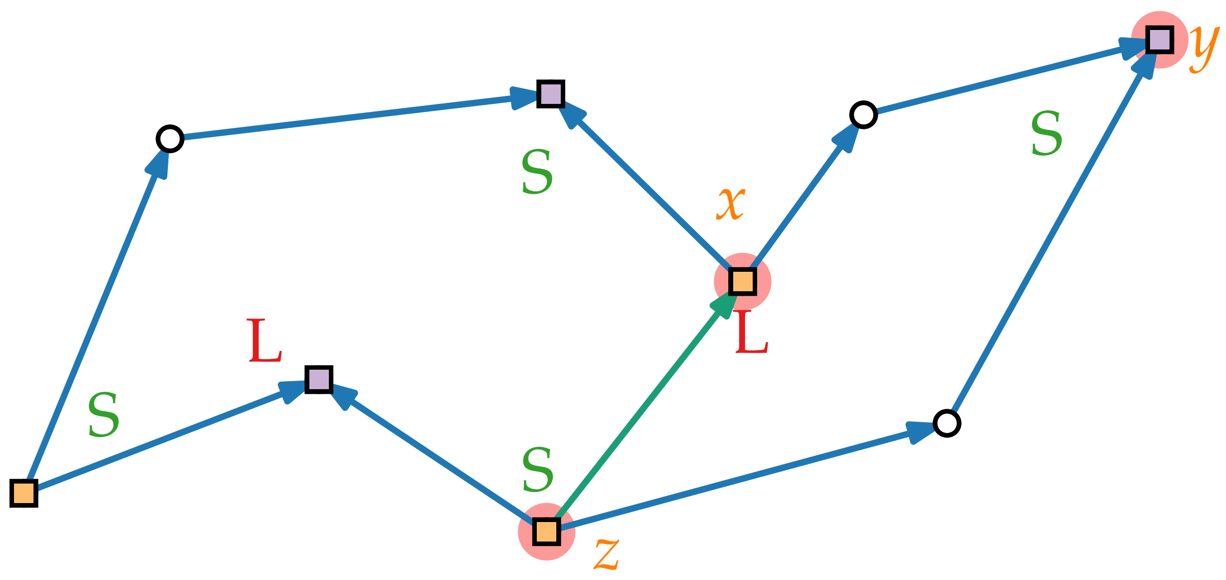

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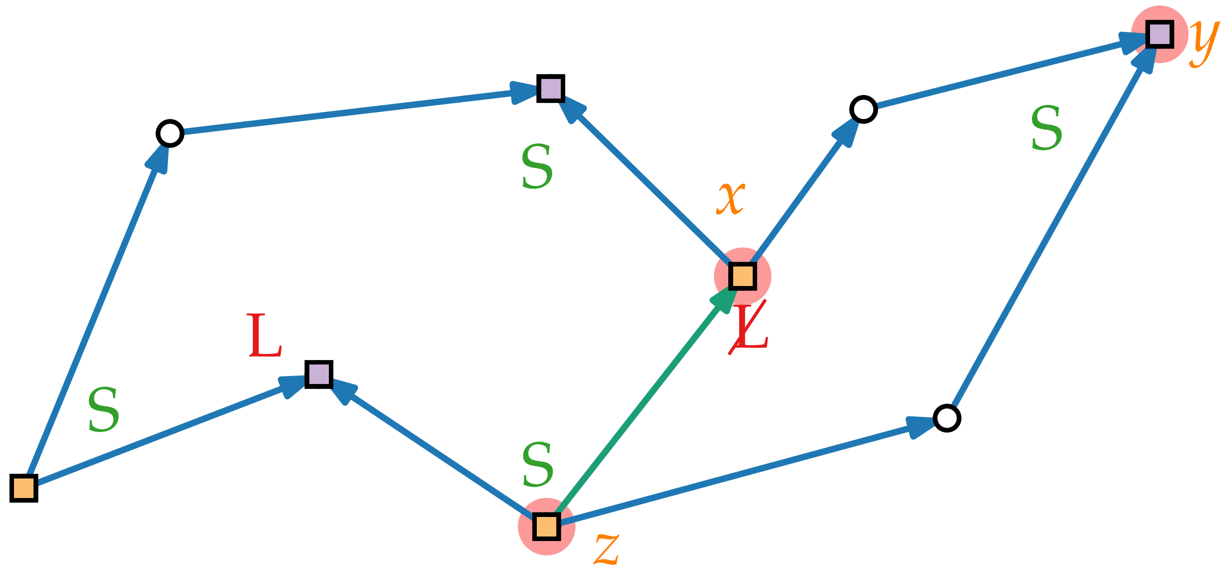

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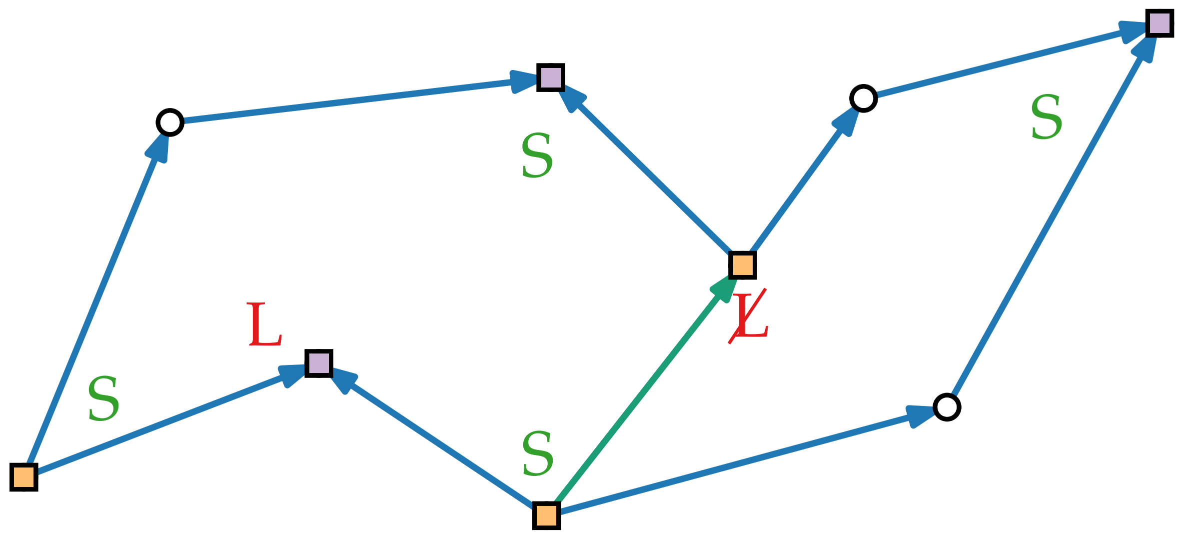

Refinement Algorithm – $\Phi, F, f_0 \to$ st-digraph

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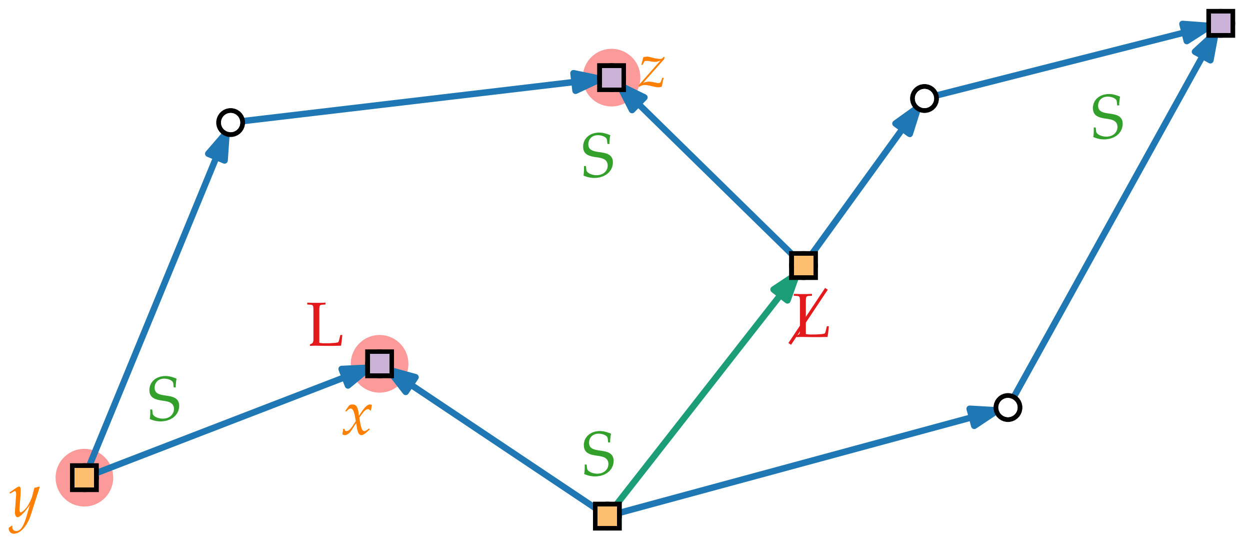

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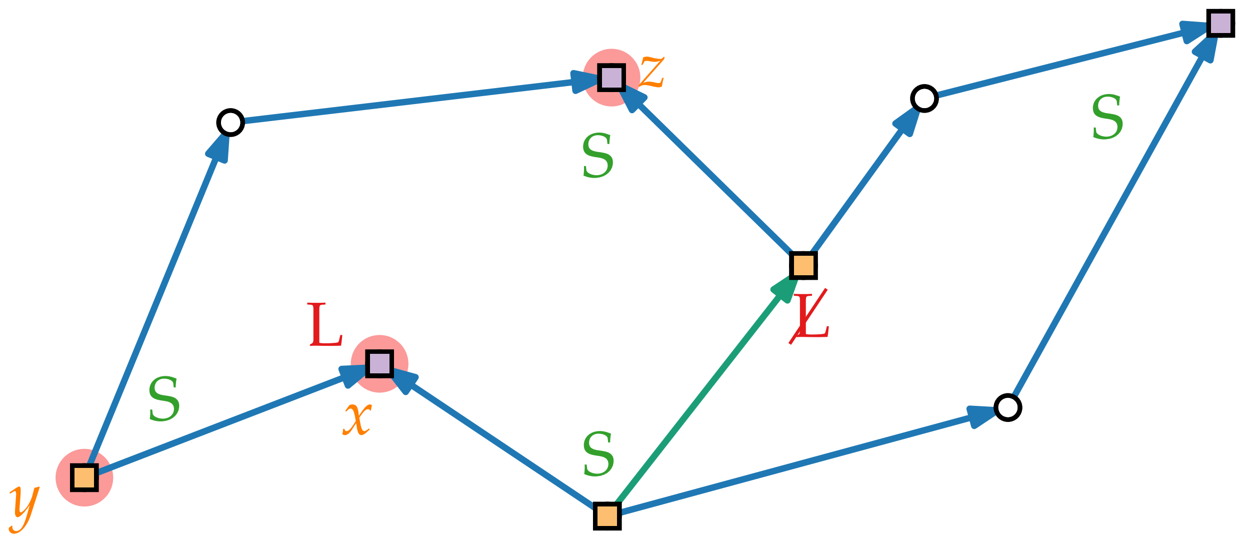

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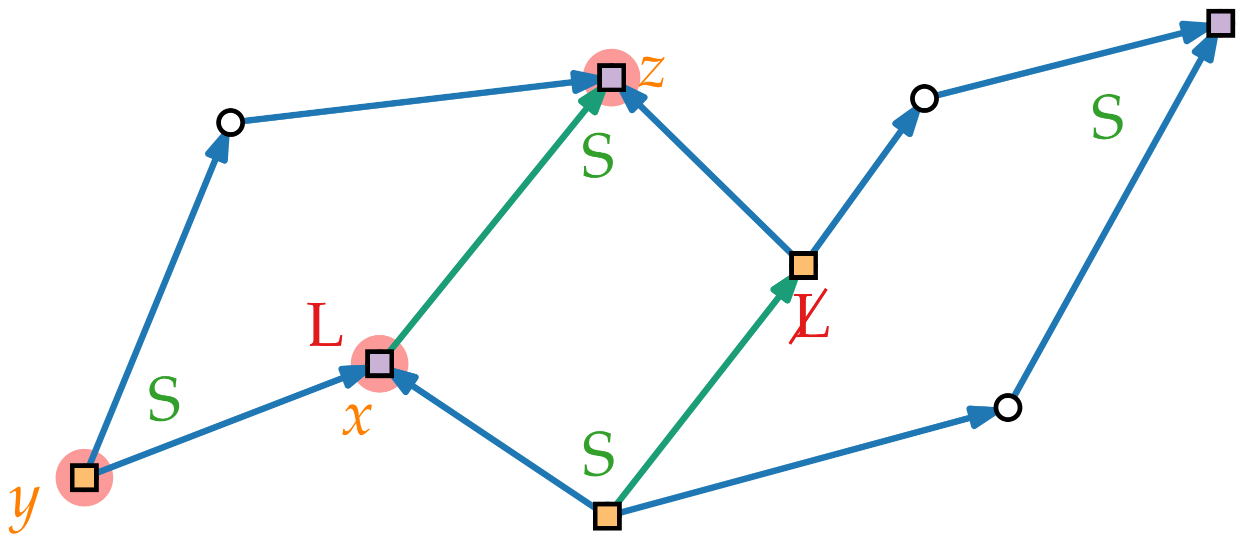

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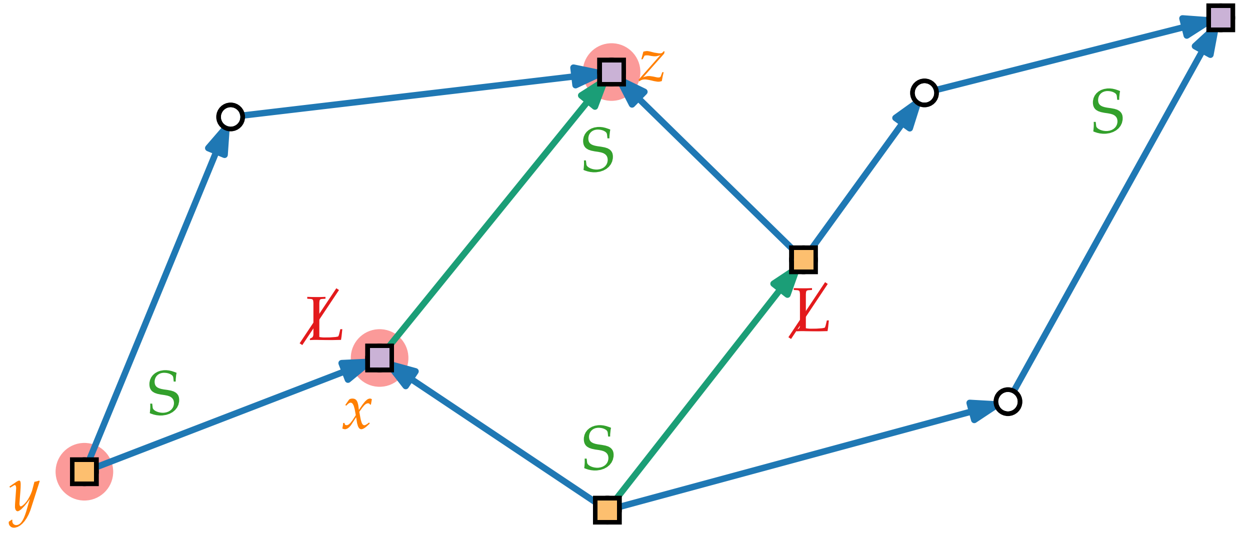

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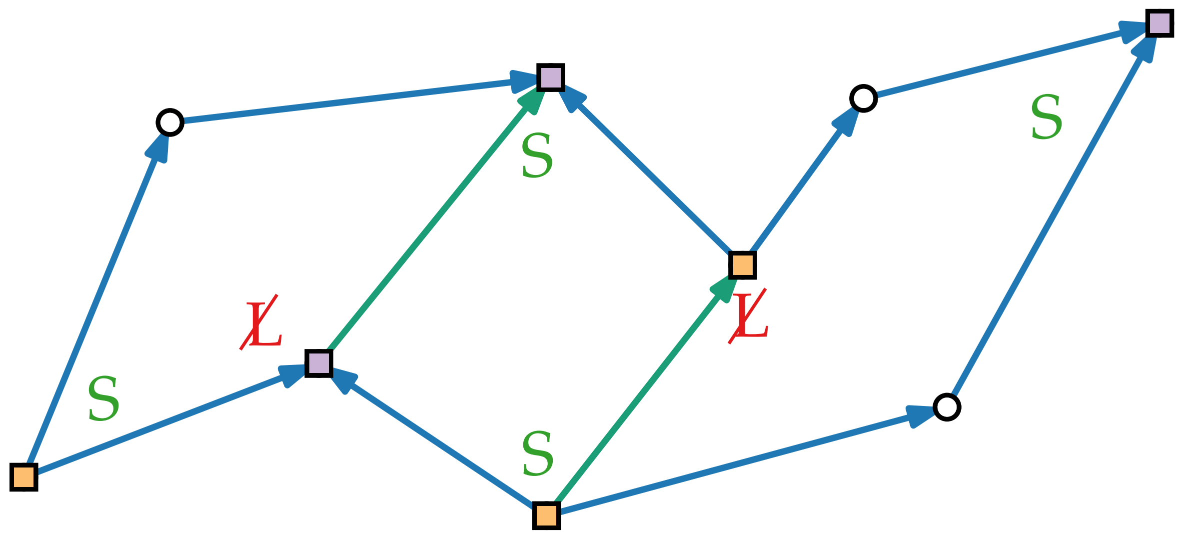

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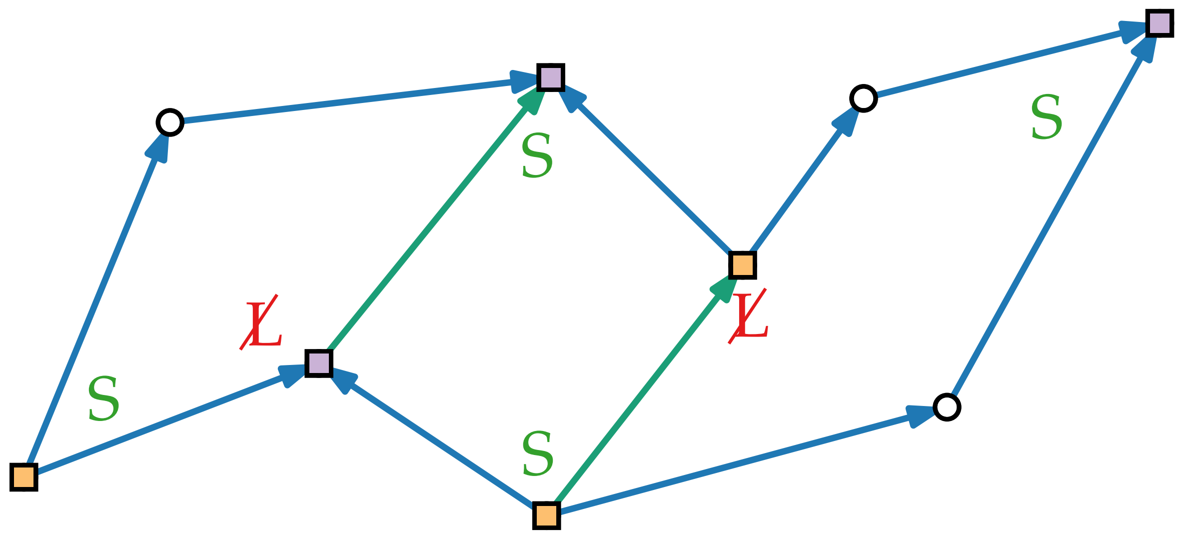

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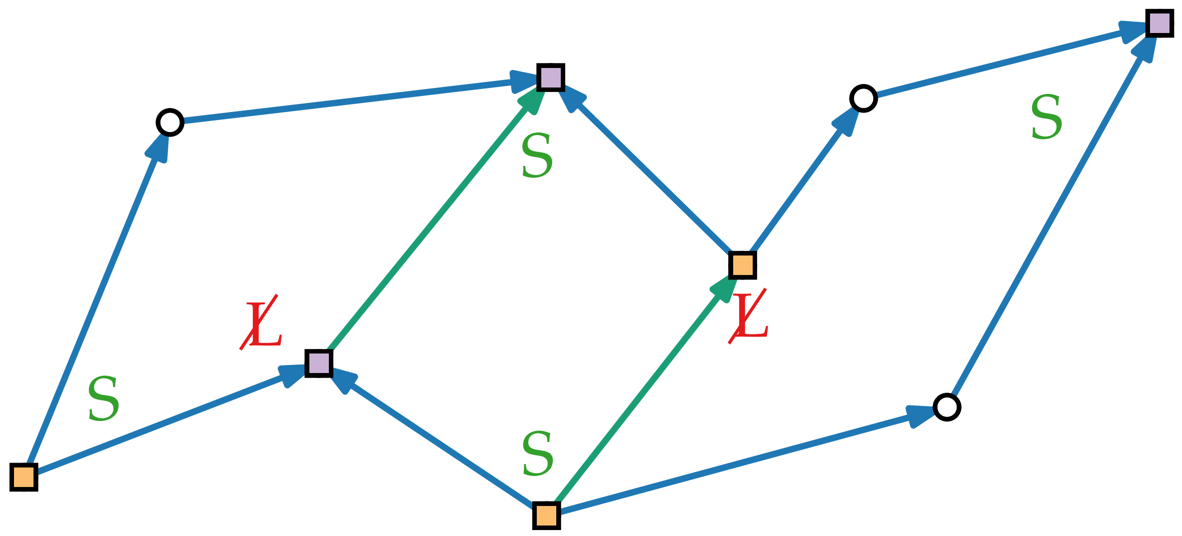

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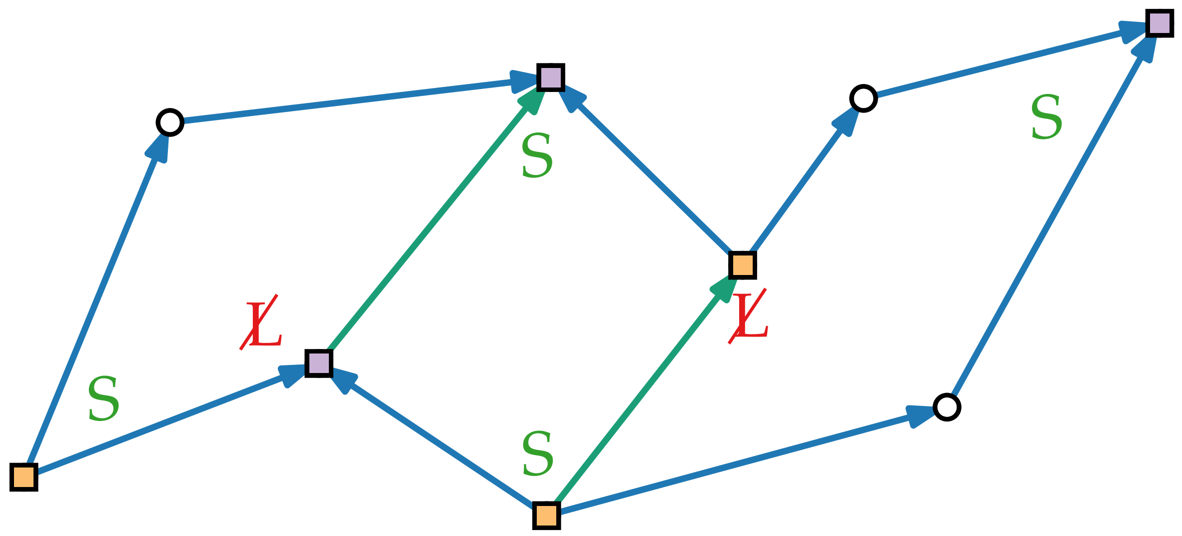

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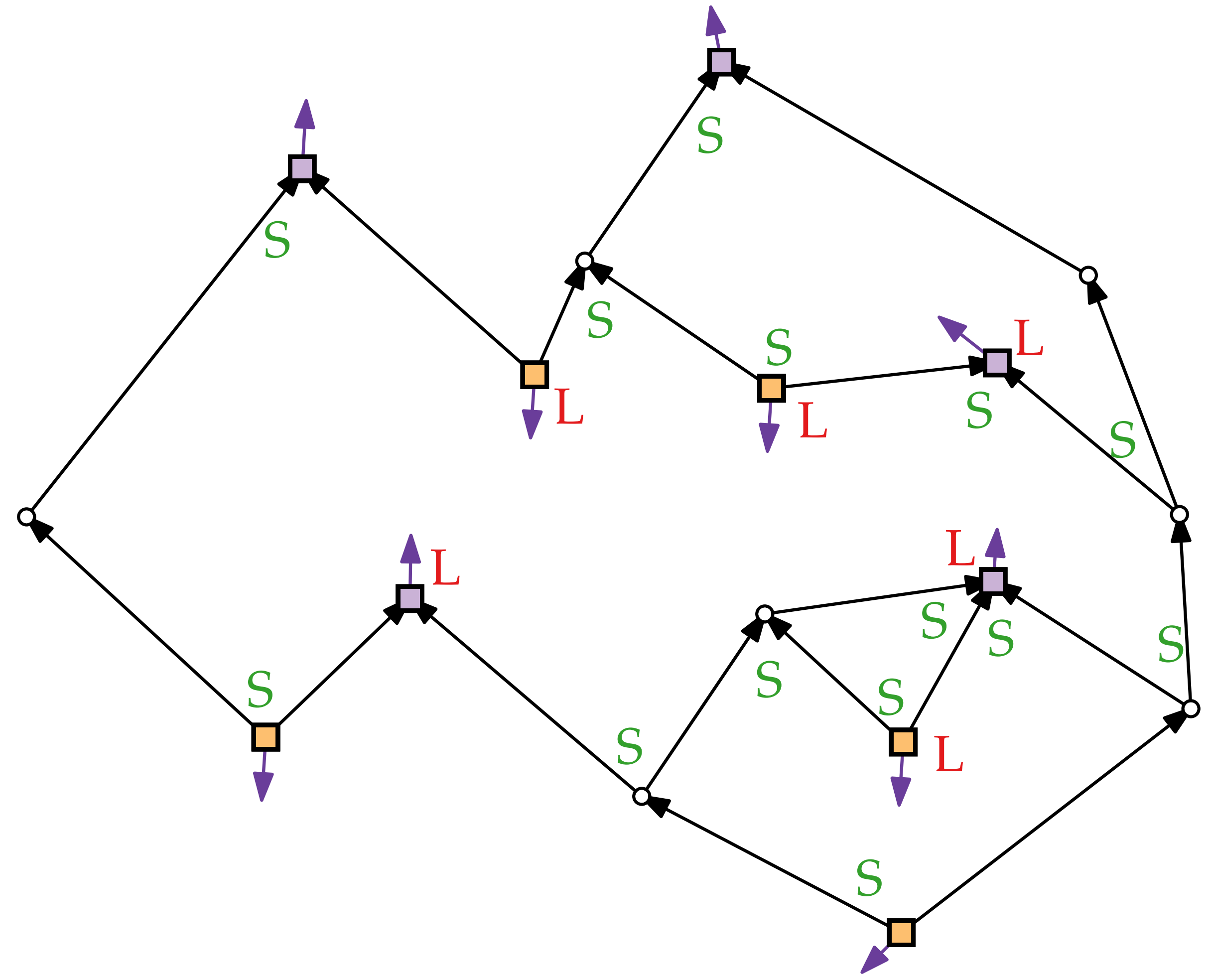

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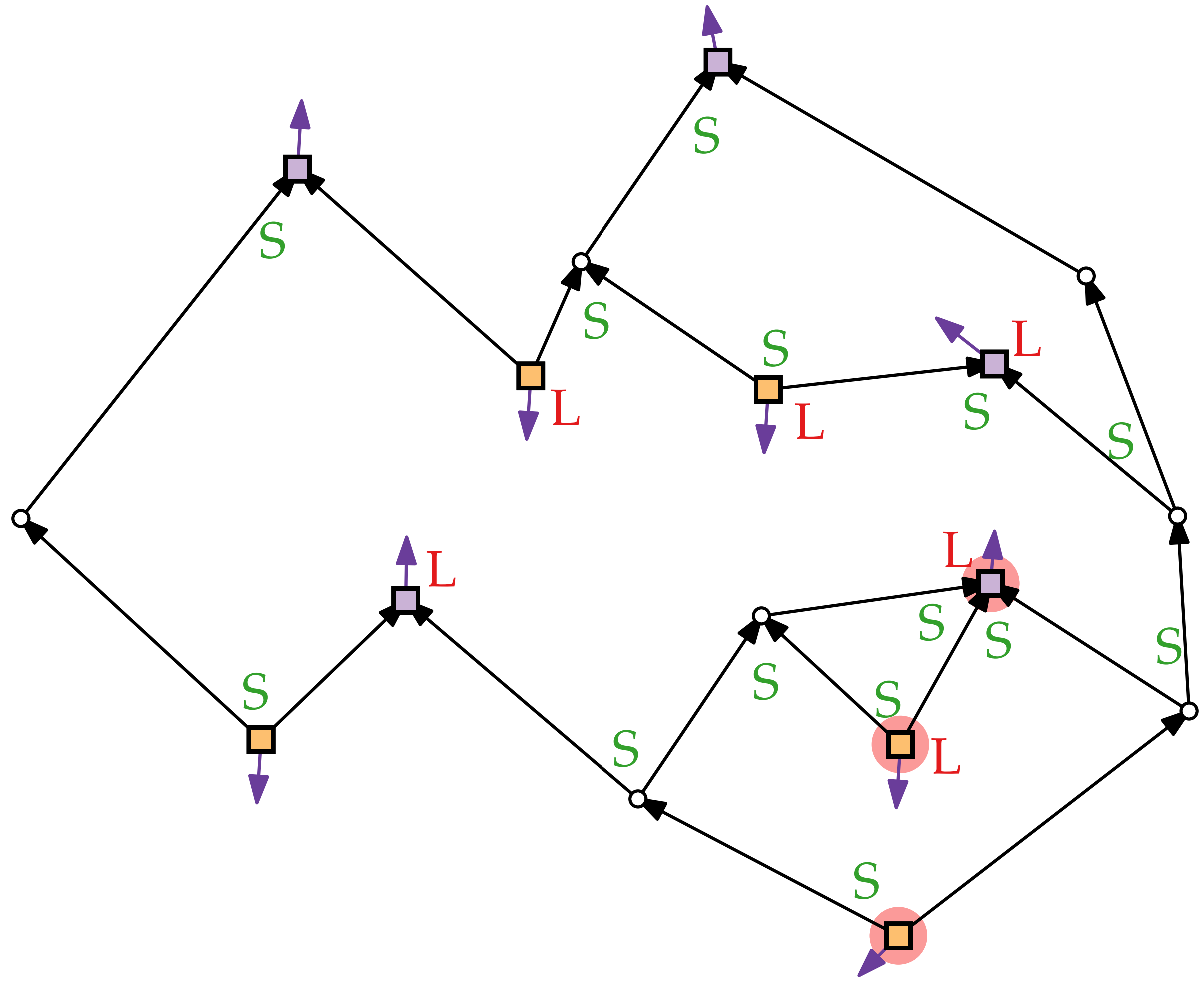

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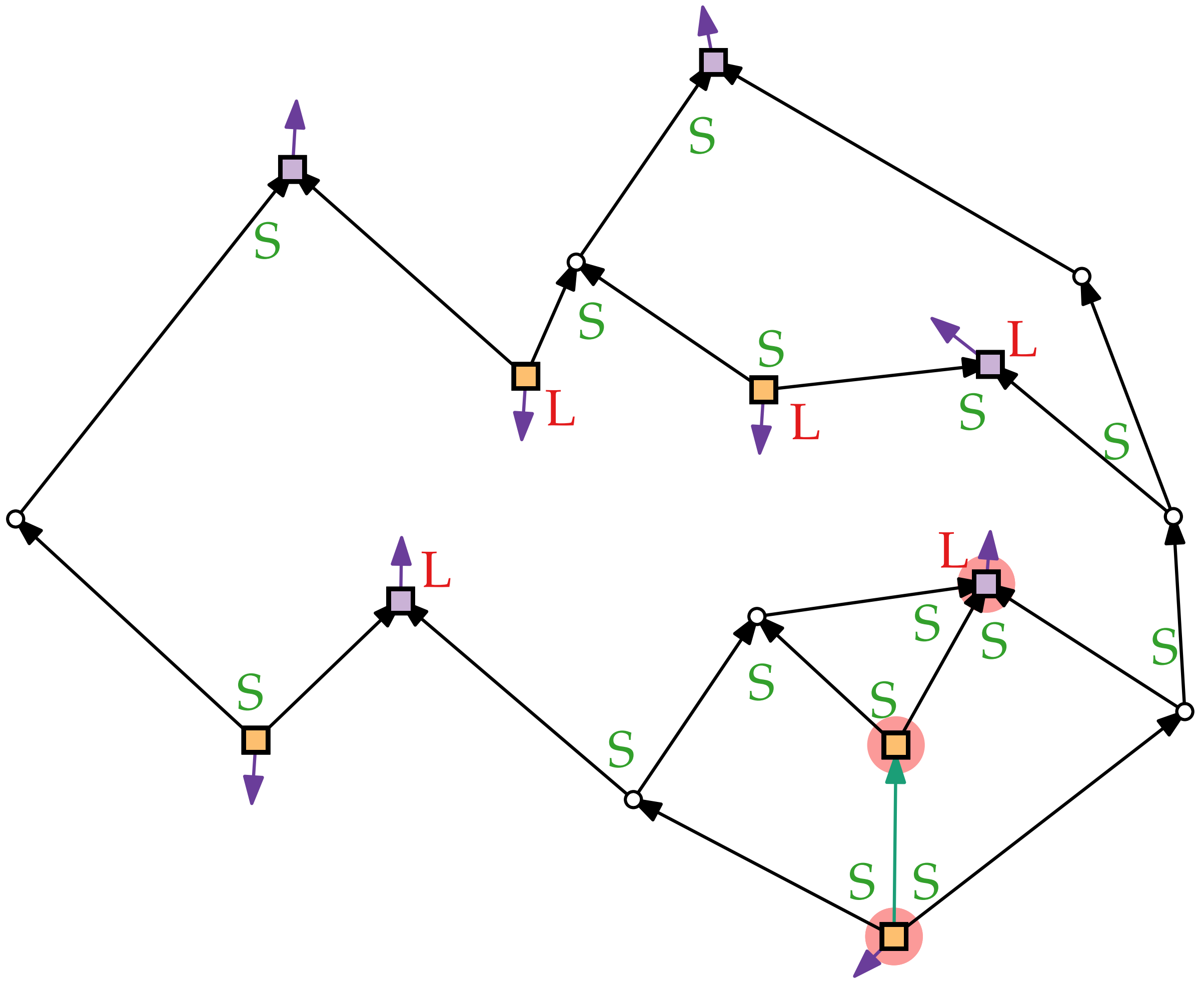

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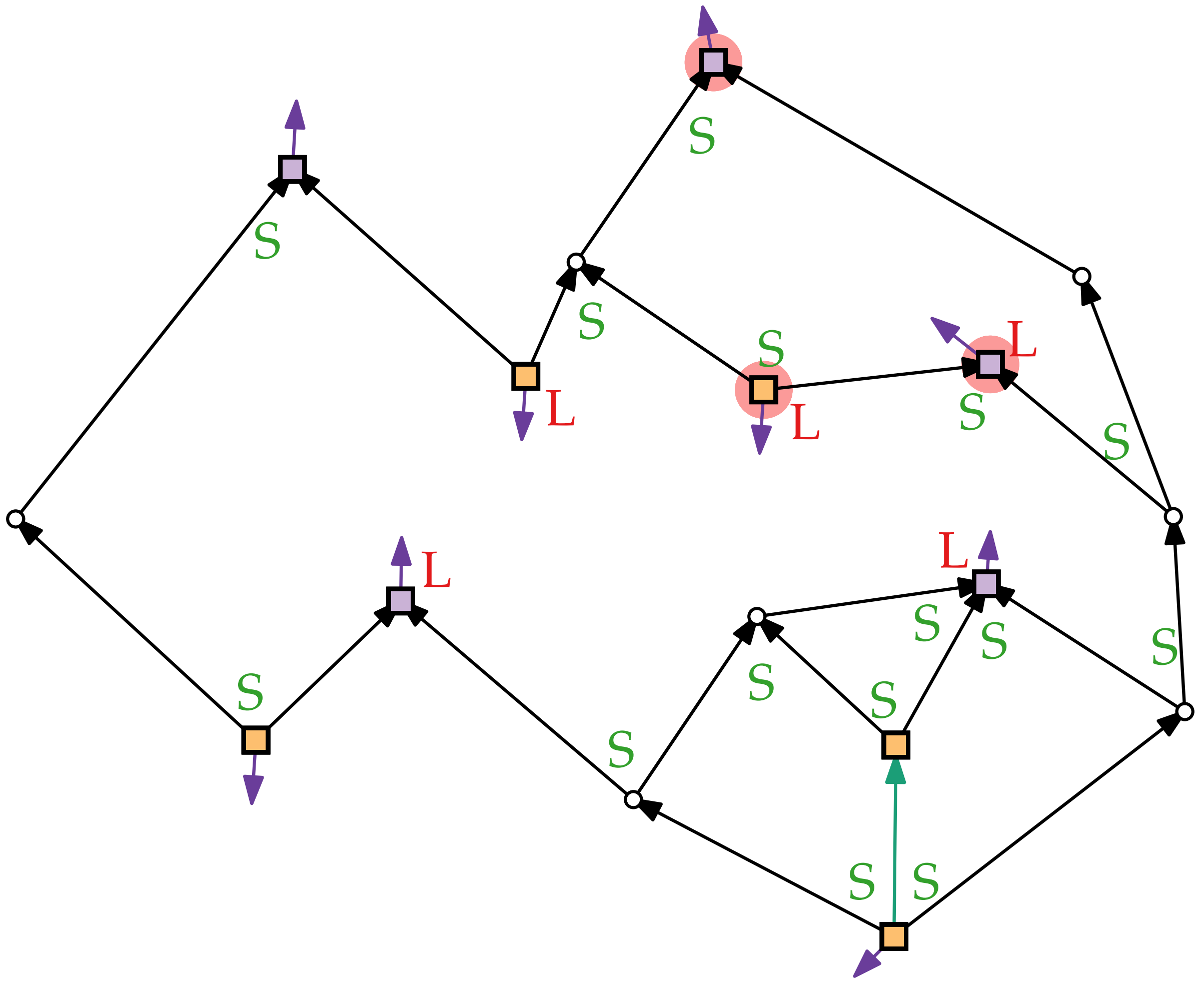

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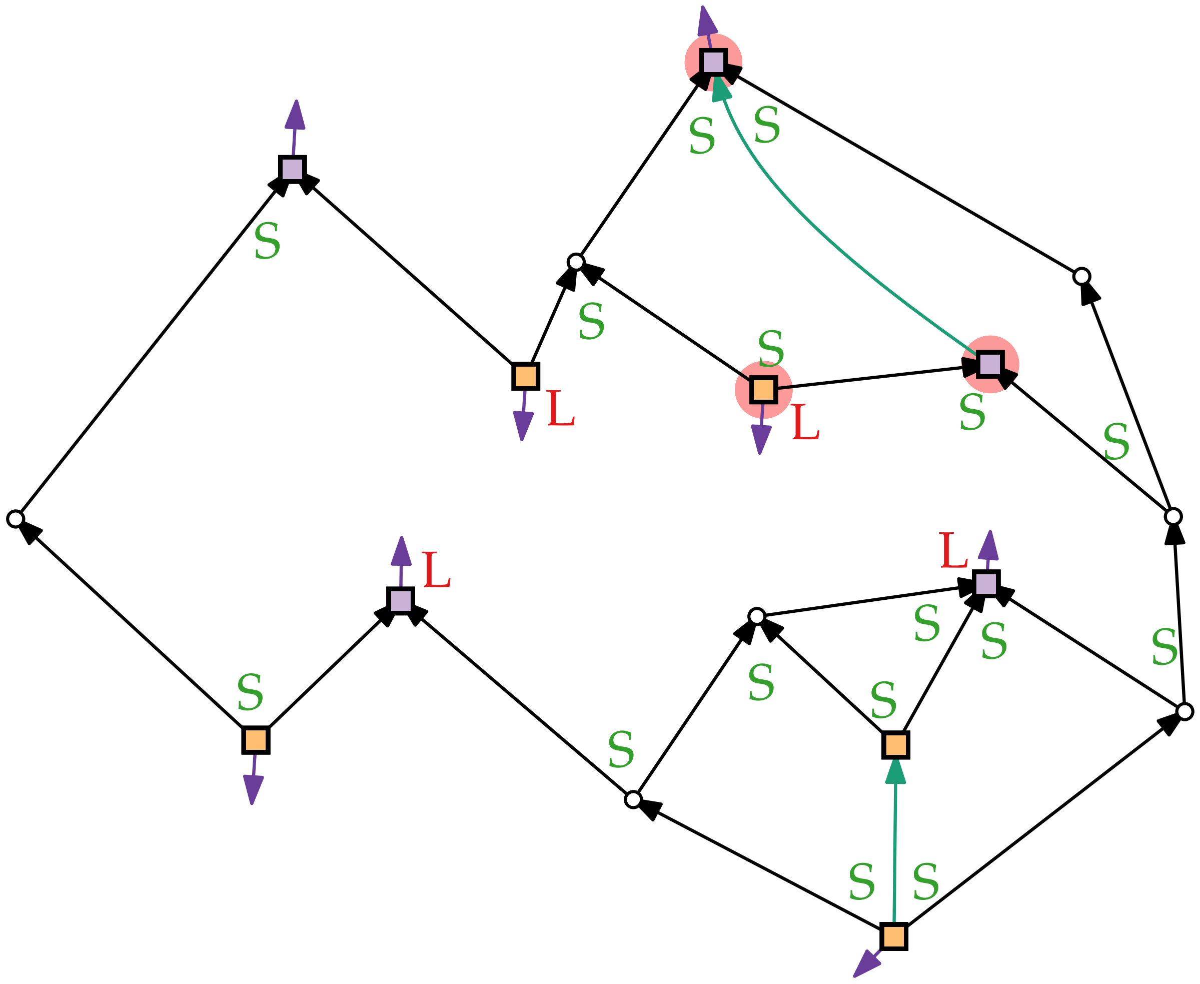

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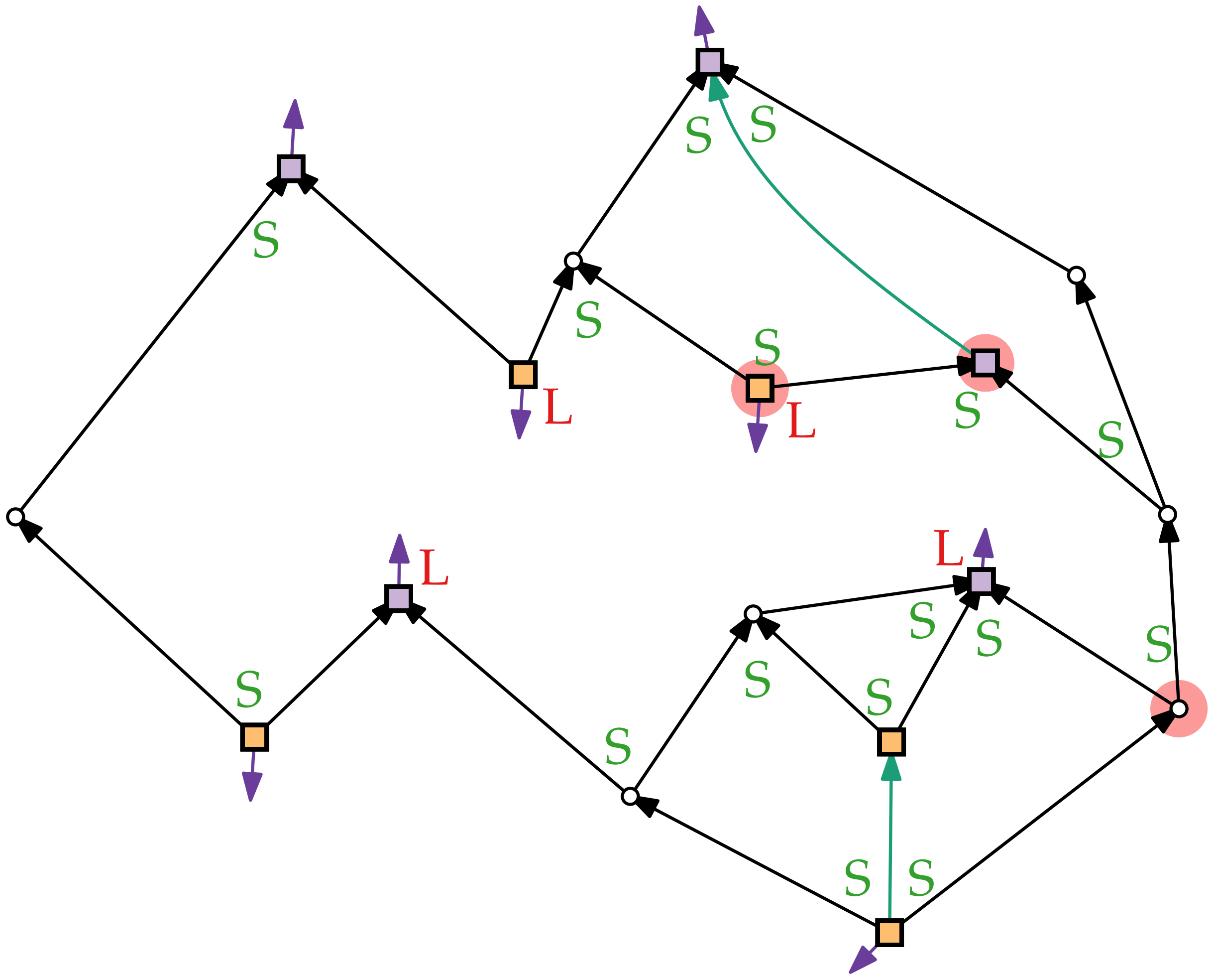

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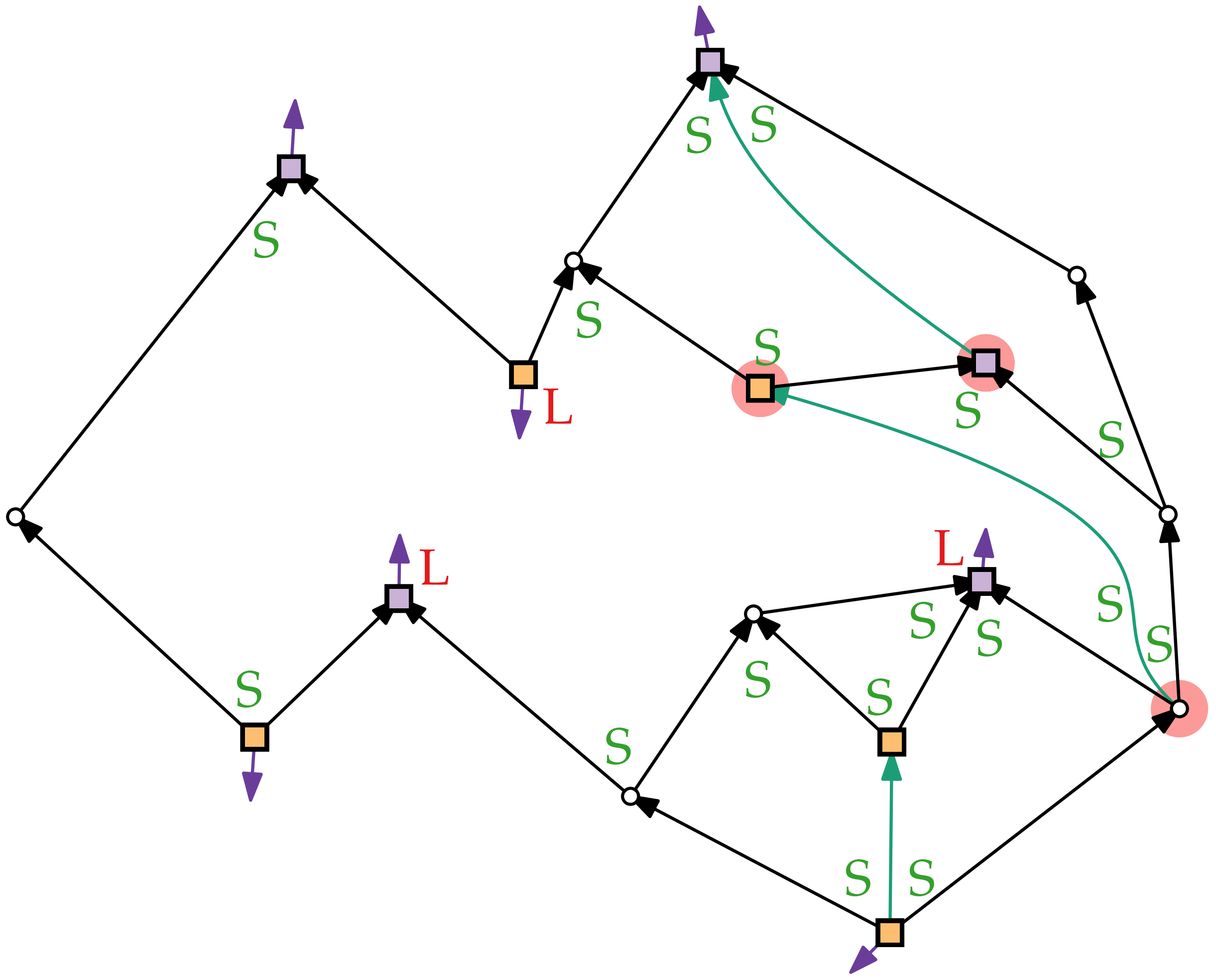

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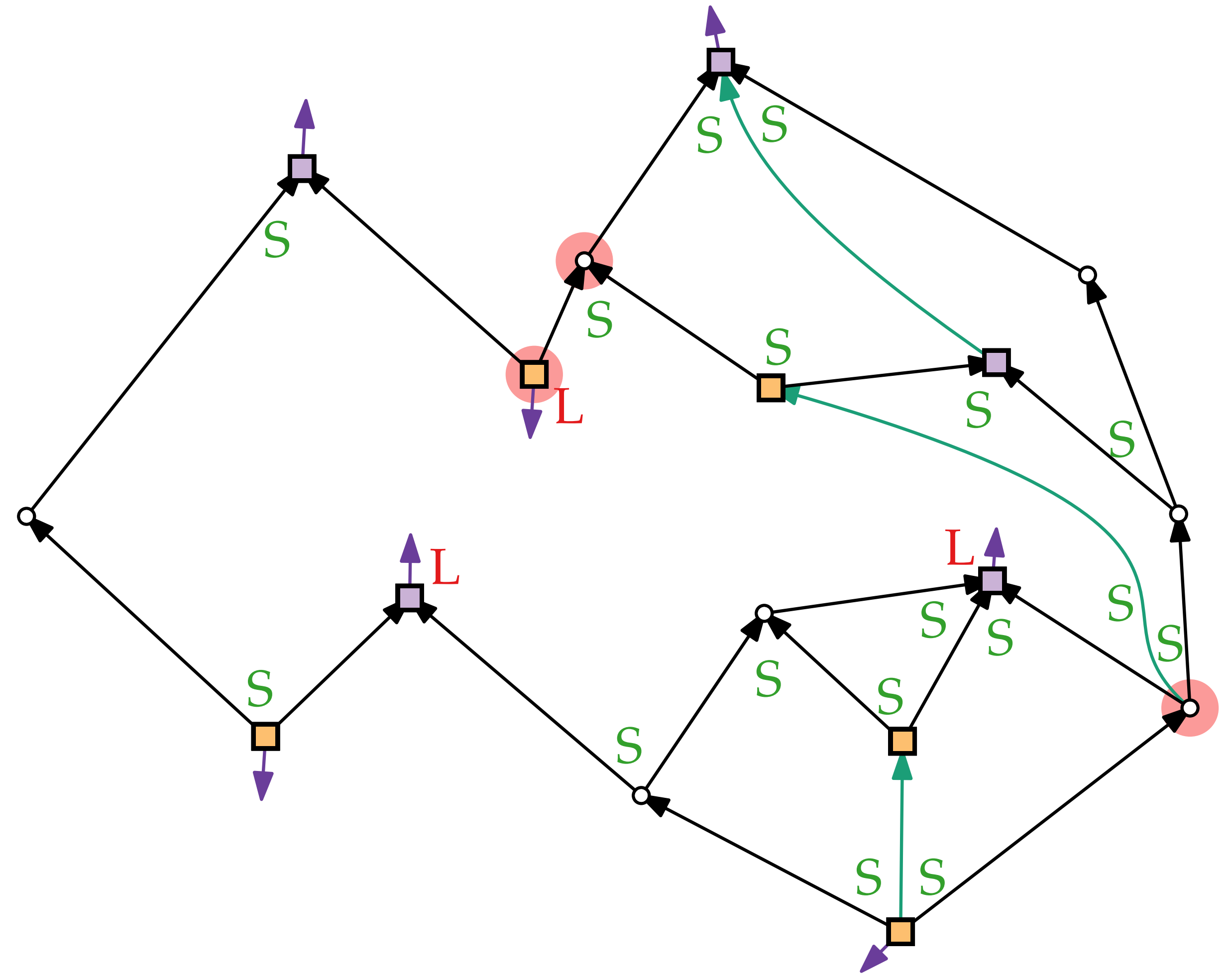

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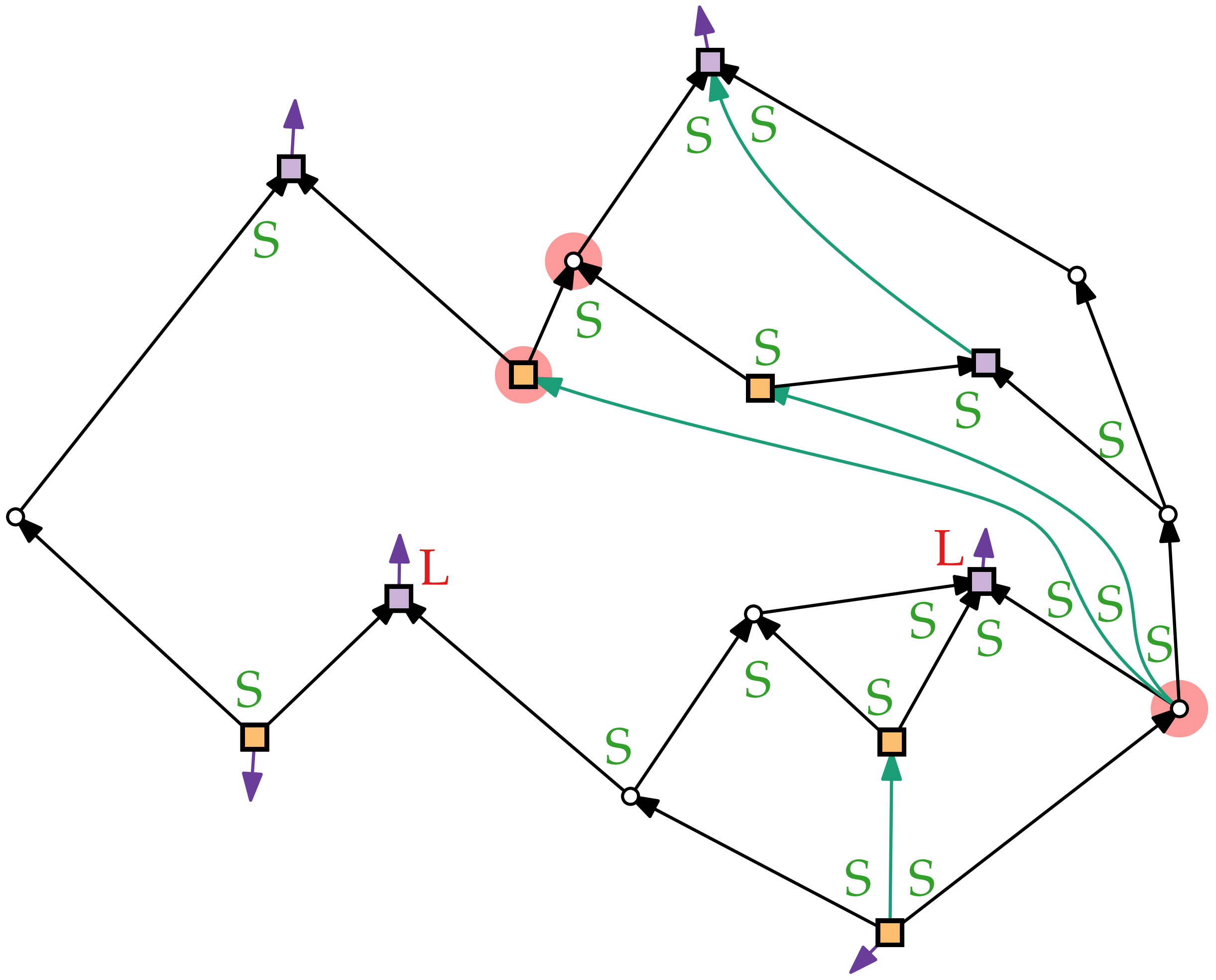

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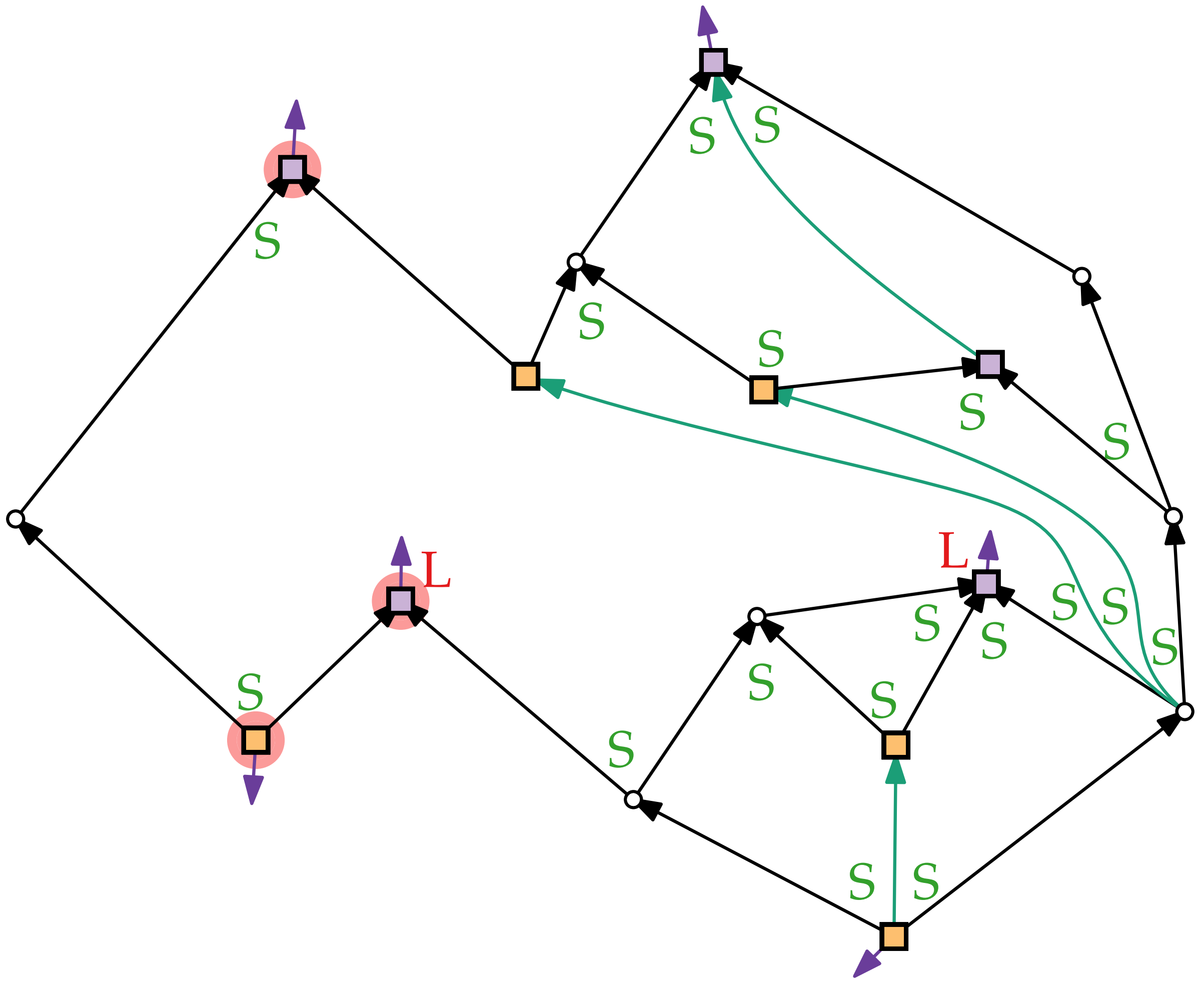

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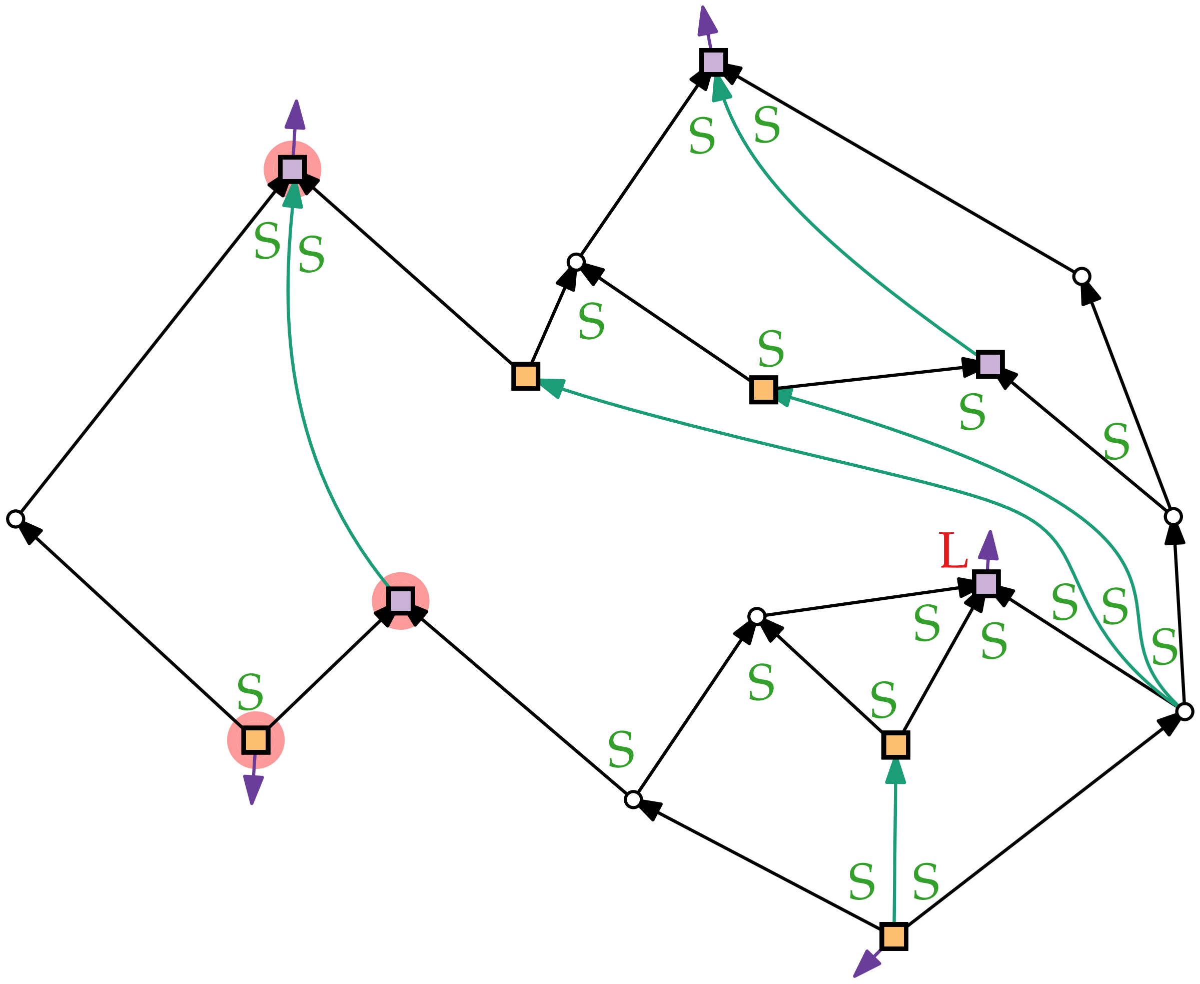

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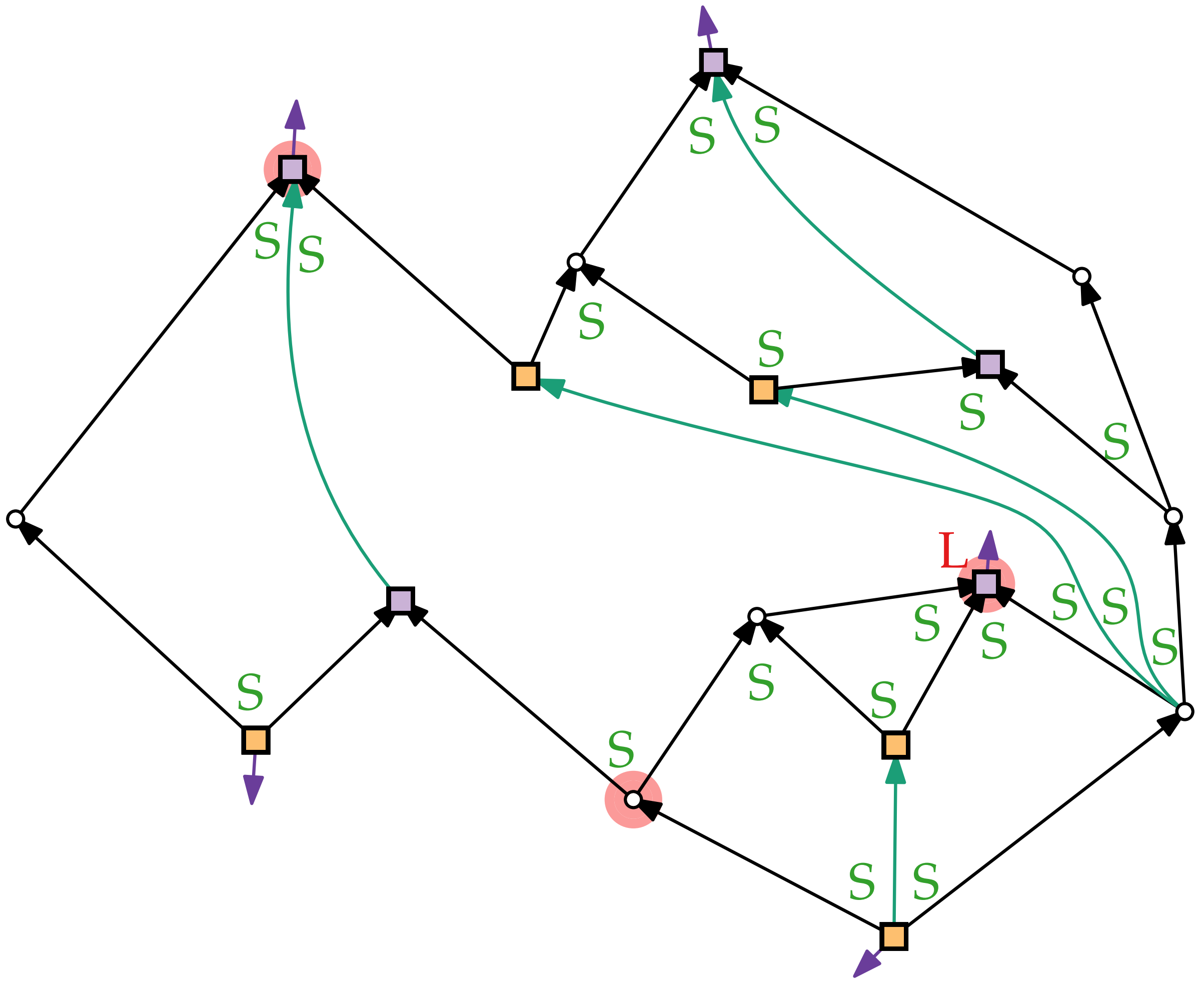

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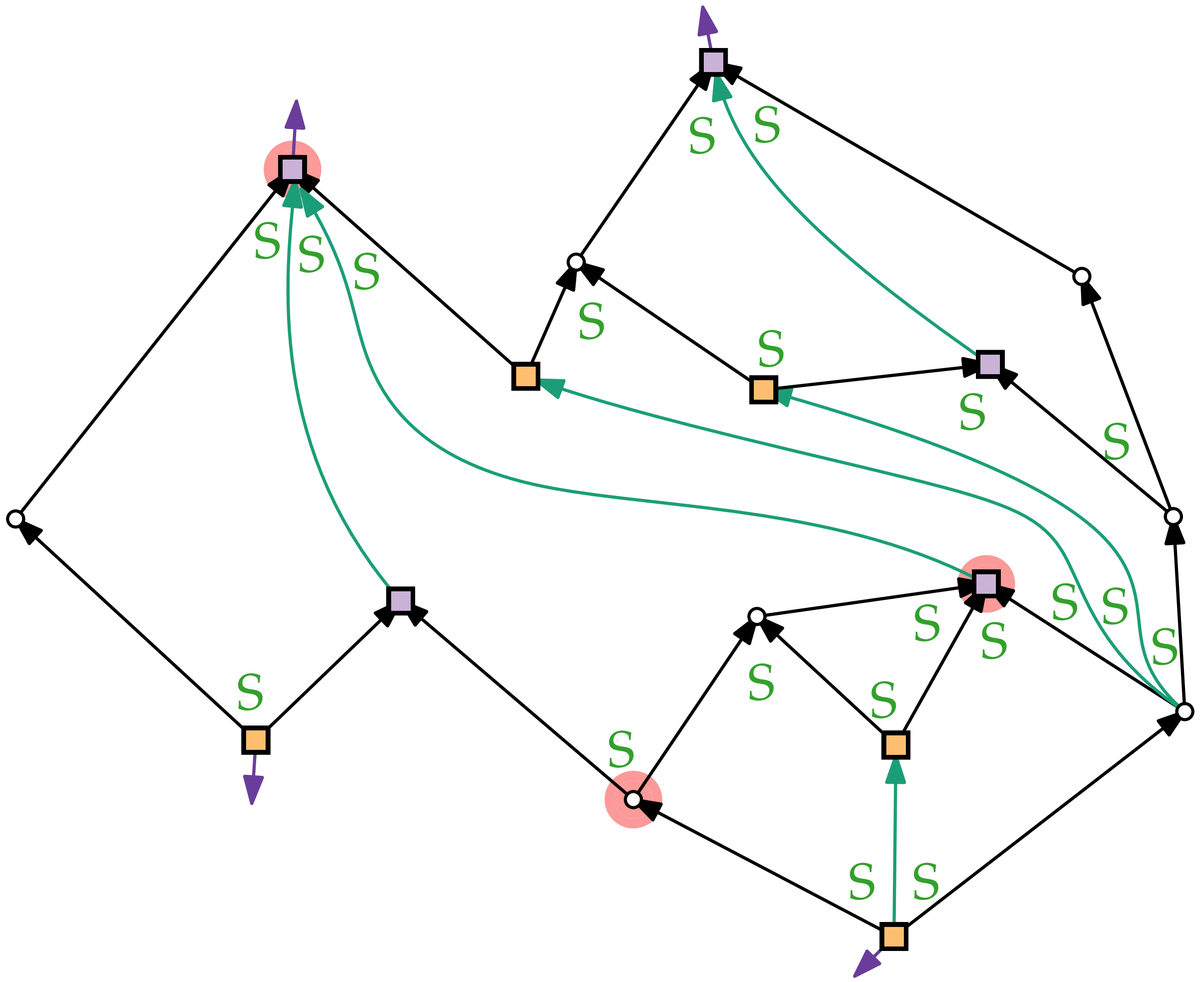

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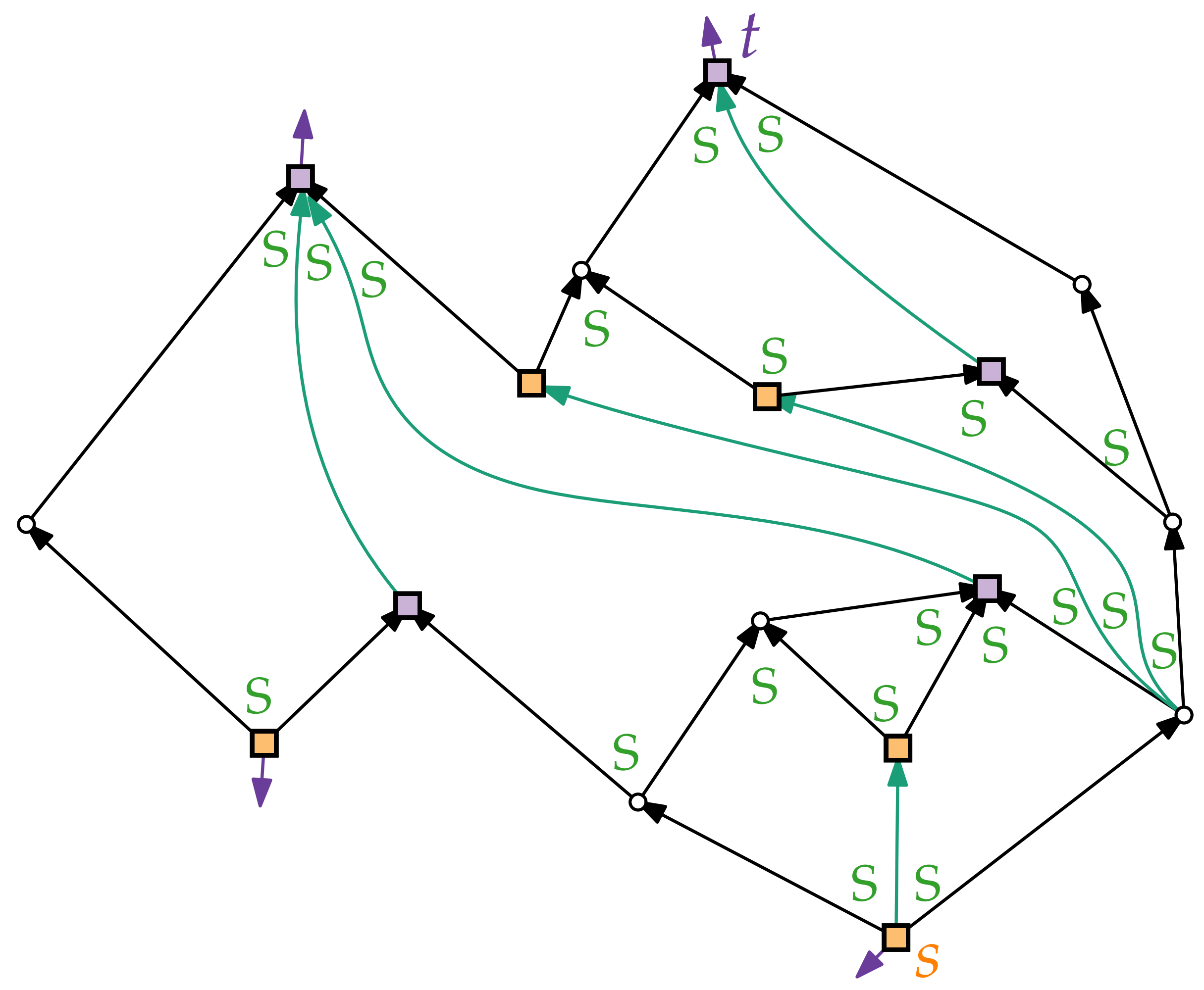

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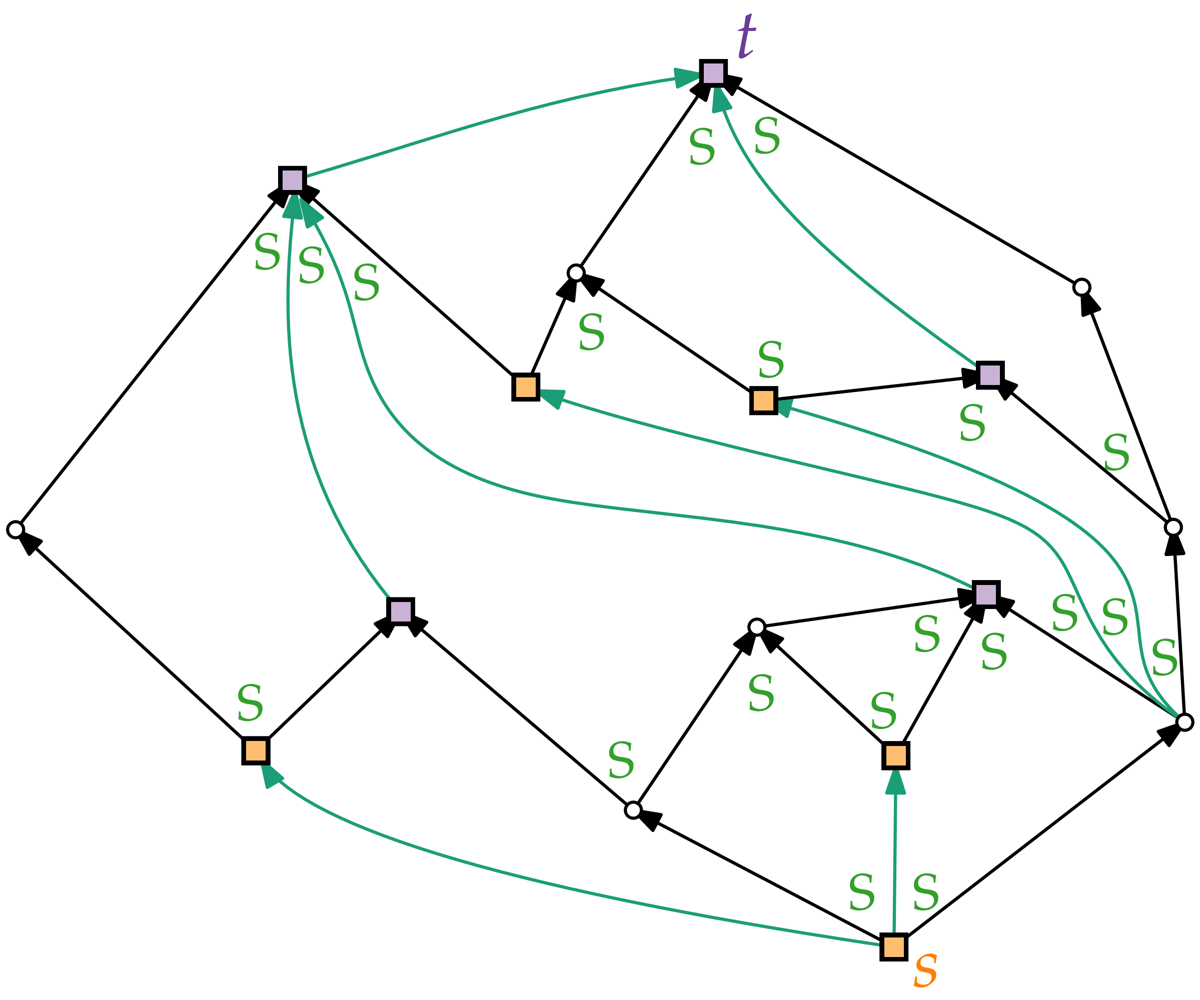

Result Upward Planarity Test

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- Many related concepts have been studied: quasi-planarity, upward drawings of mixed graphs, upward planarity on cylinder/torus, …