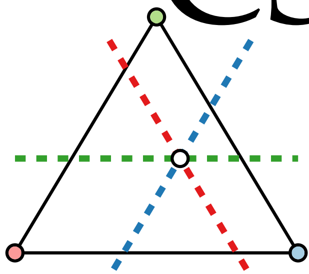
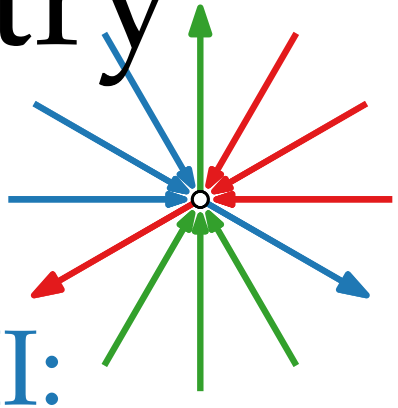
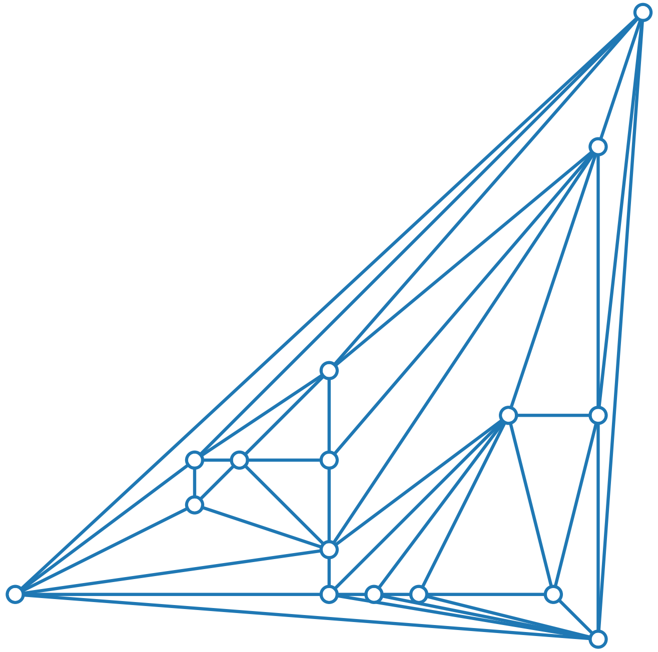
CS F402: Computational Geometry





Lecture 12:
Straight-Line Drawings of Planar Graphs II: Schnyder Realizer



Siddharth Gupta

March 21+26 + April 2+4, 2025

Planar Straight-Line Drawings

Theorem. [De Fraysseix, Pach, Pollack '90]
Every n-vertex planar graph has a planar straight-line drawing of size $(2n-4) \times (n-2)$.

Theorem. [Schnyder '90]
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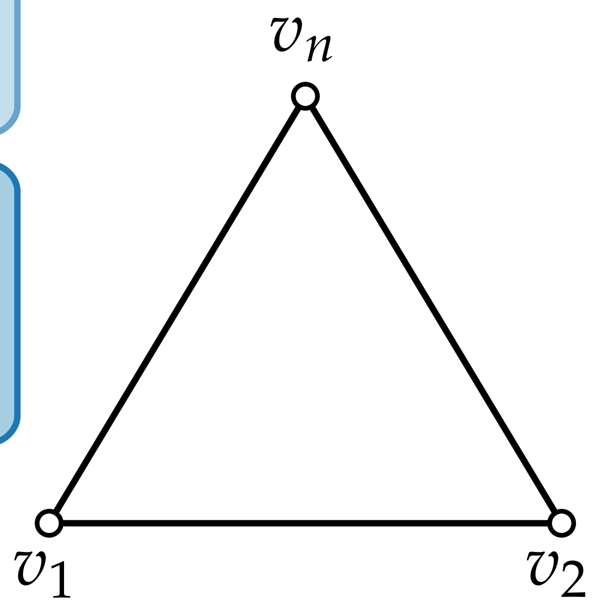

Idea.

- Fix outer triangle.

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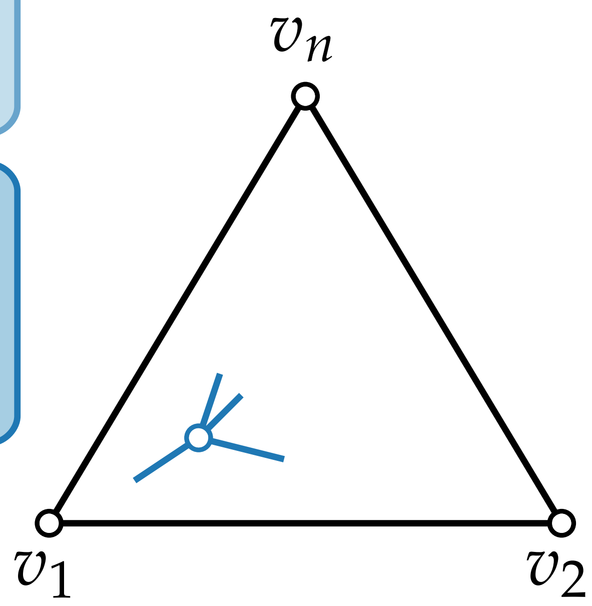

Idea.
- Fix outer triangle.
- Compute coordinates of inner vertices

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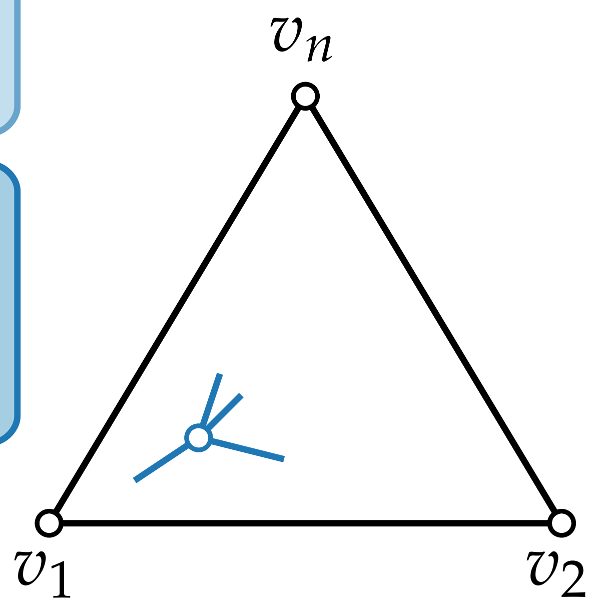

Idea.

- Fix outer triangle.
- Compute coordinates of inner vertices
  - based on outer triangle

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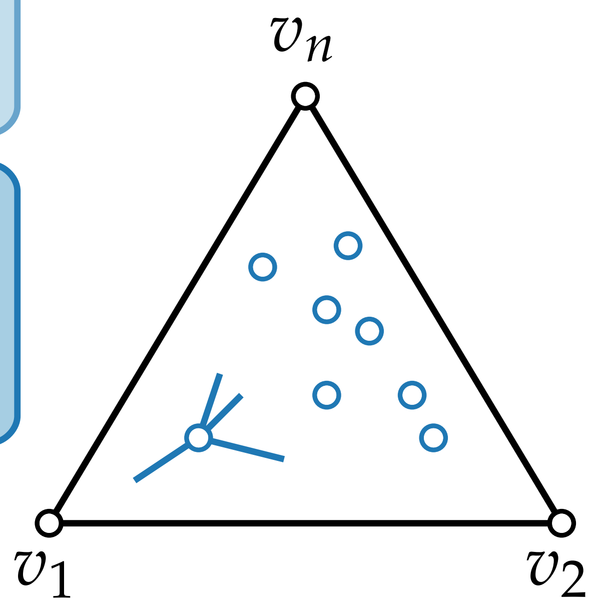

Idea.

- Fix outer triangle.
- Compute coordinates of inner vertices
  - based on outer triangle and
  - how much space there should be for other vertices

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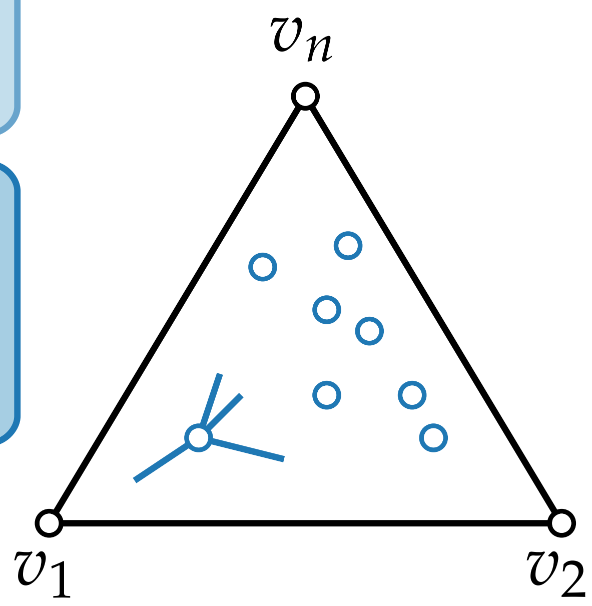

Idea.

- Fix outer triangle.
- Compute coordinates of inner vertices
  - based on outer triangle and
  - how much space there should be for other vertices
  - using weighted barycentric coordinates.

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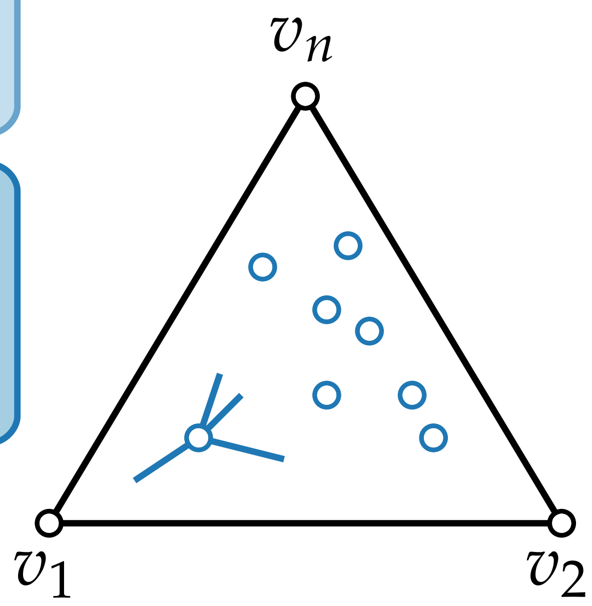

Idea.

- Fix outer triangle.
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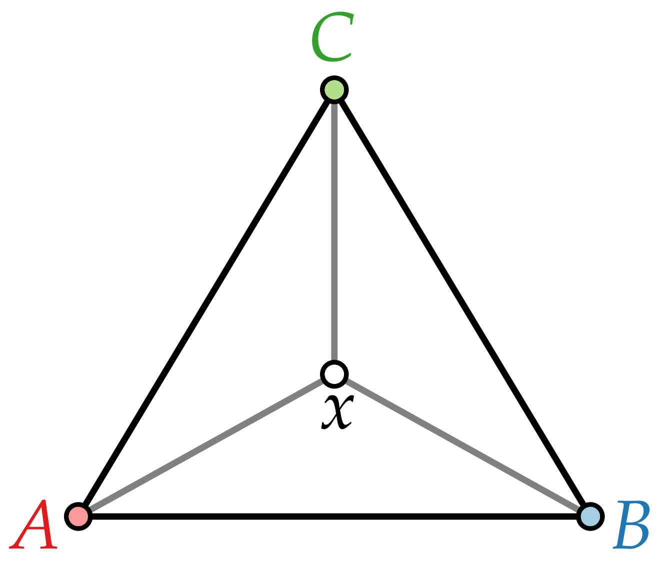
Barycentric Coordinates

Recall: barycenter$(x_1, \ldots, x_k) = \sum_{i=1}^{k} x_i / k$



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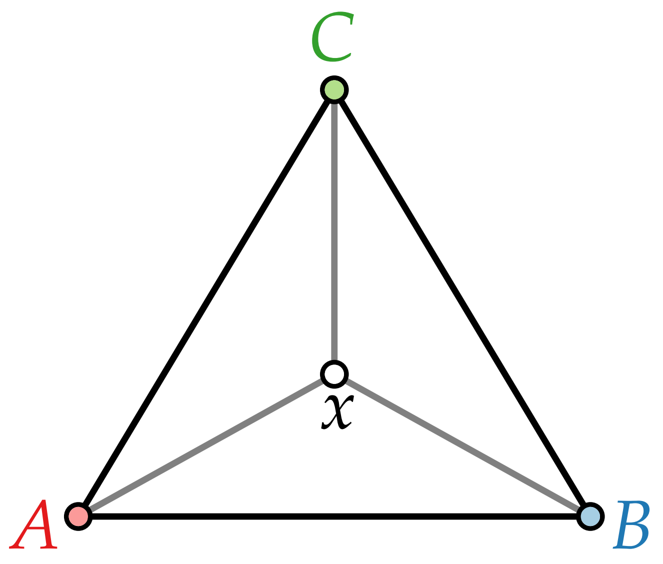

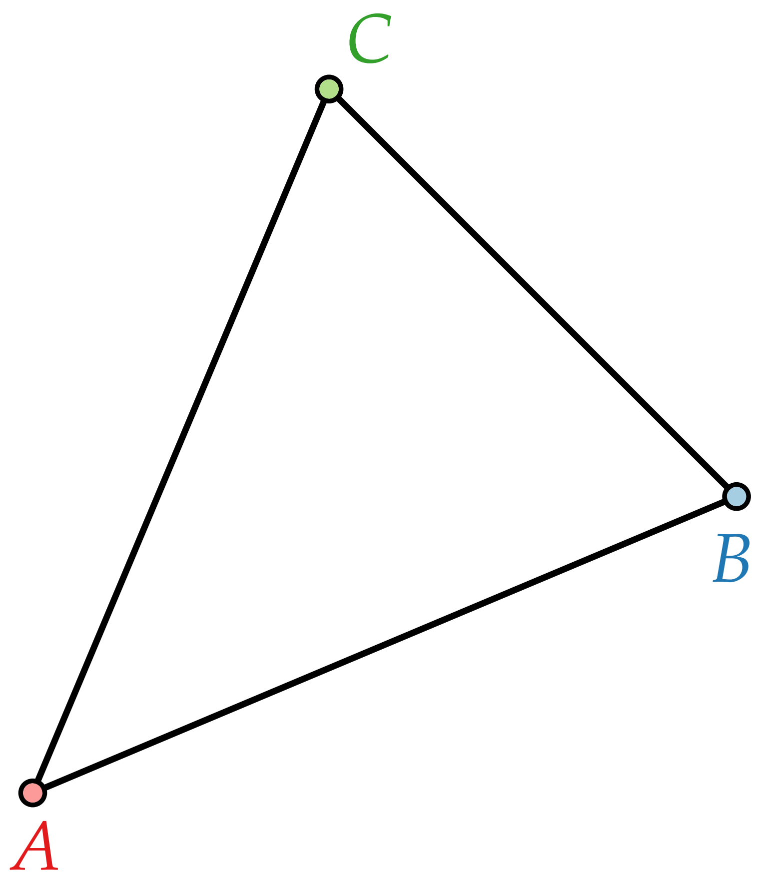

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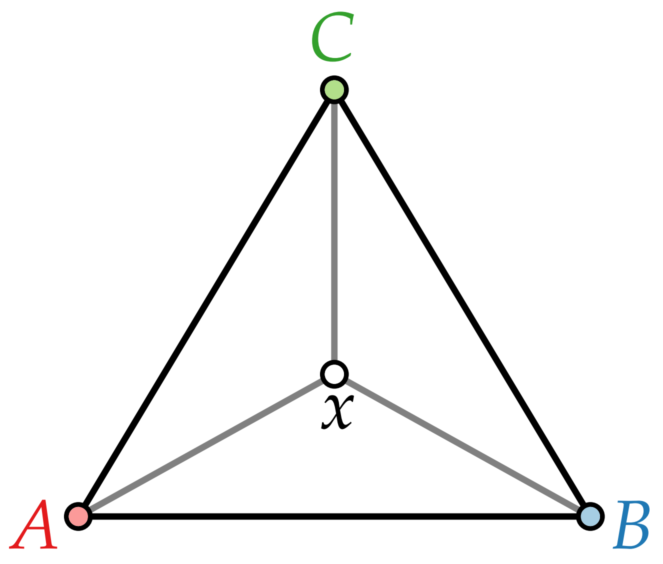
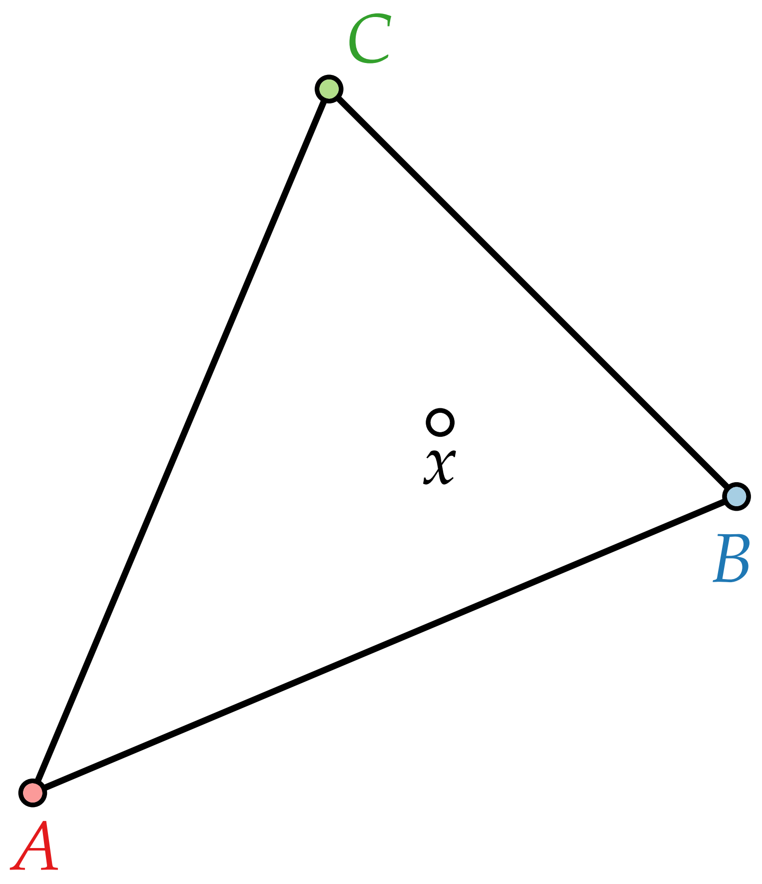
Recall: barycenter$(x_1, \ldots, x_k) = \sum_{i=1}^{k} x_i / k$

Let A, B, C form a triangle, let x lie inside $\triangle ABC$.





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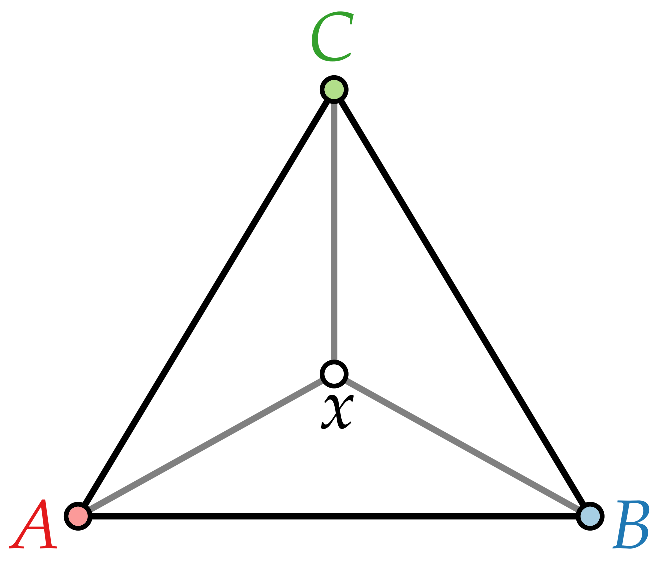

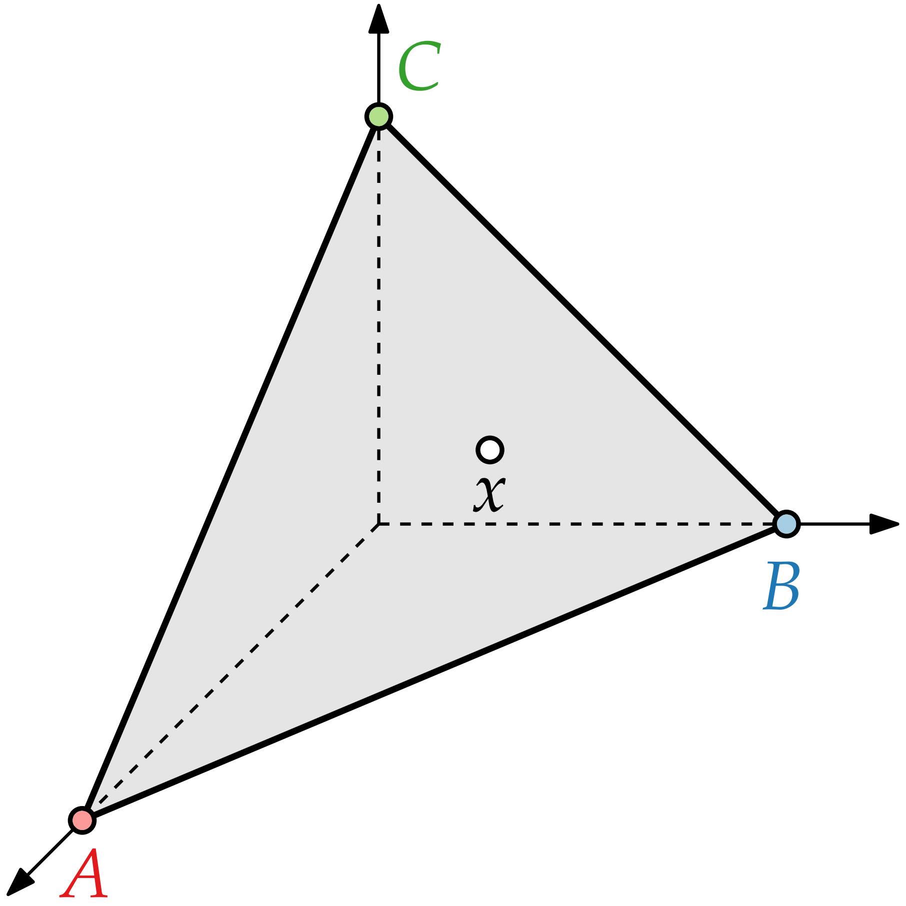

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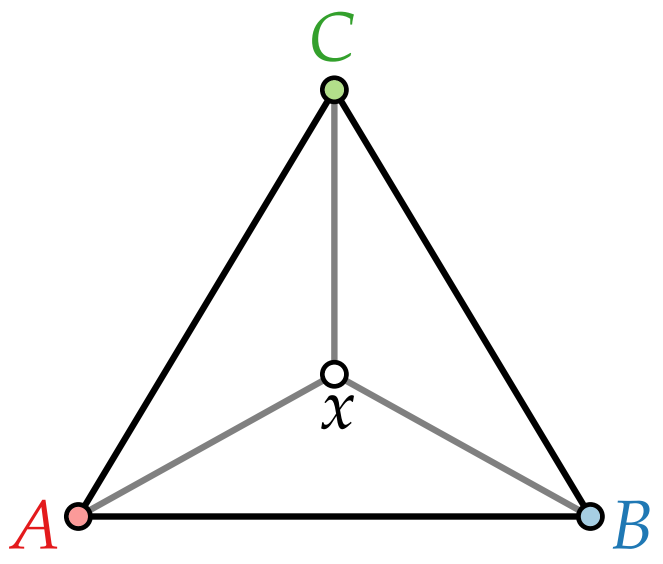

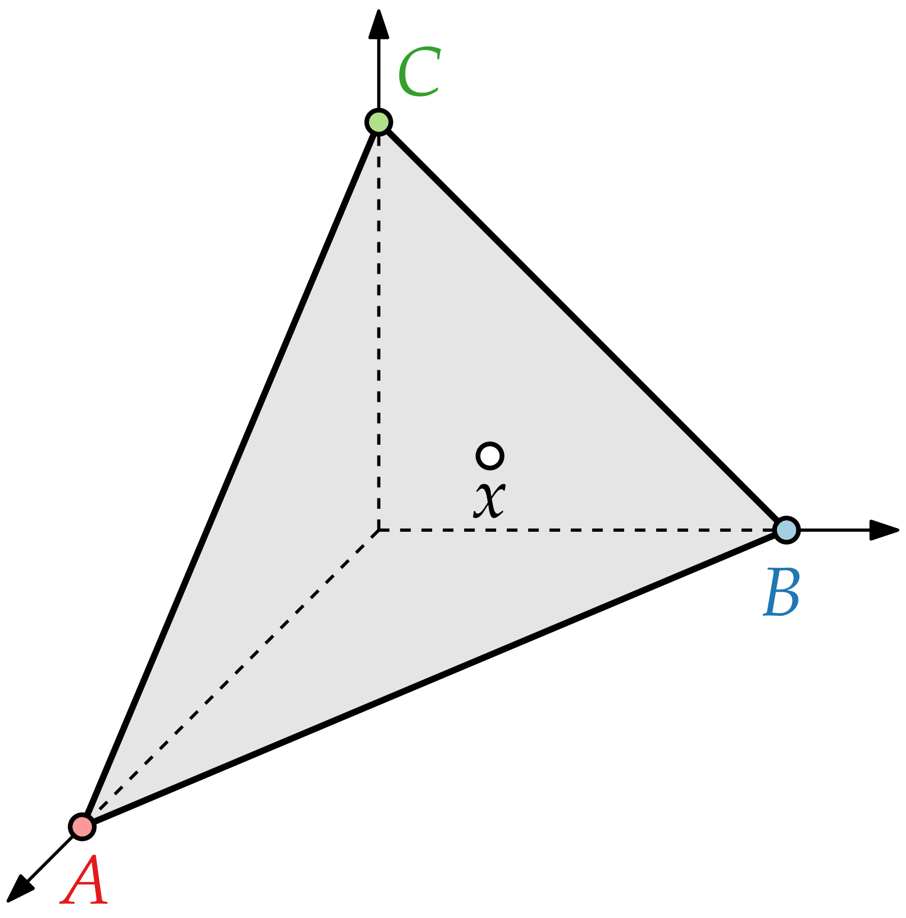

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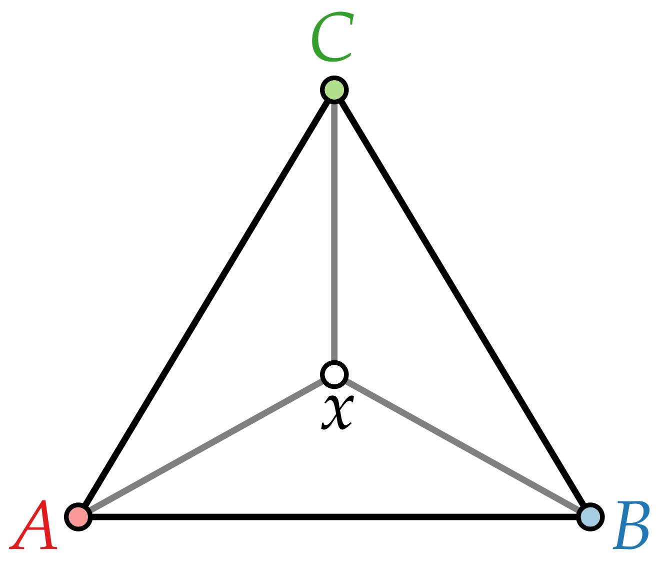

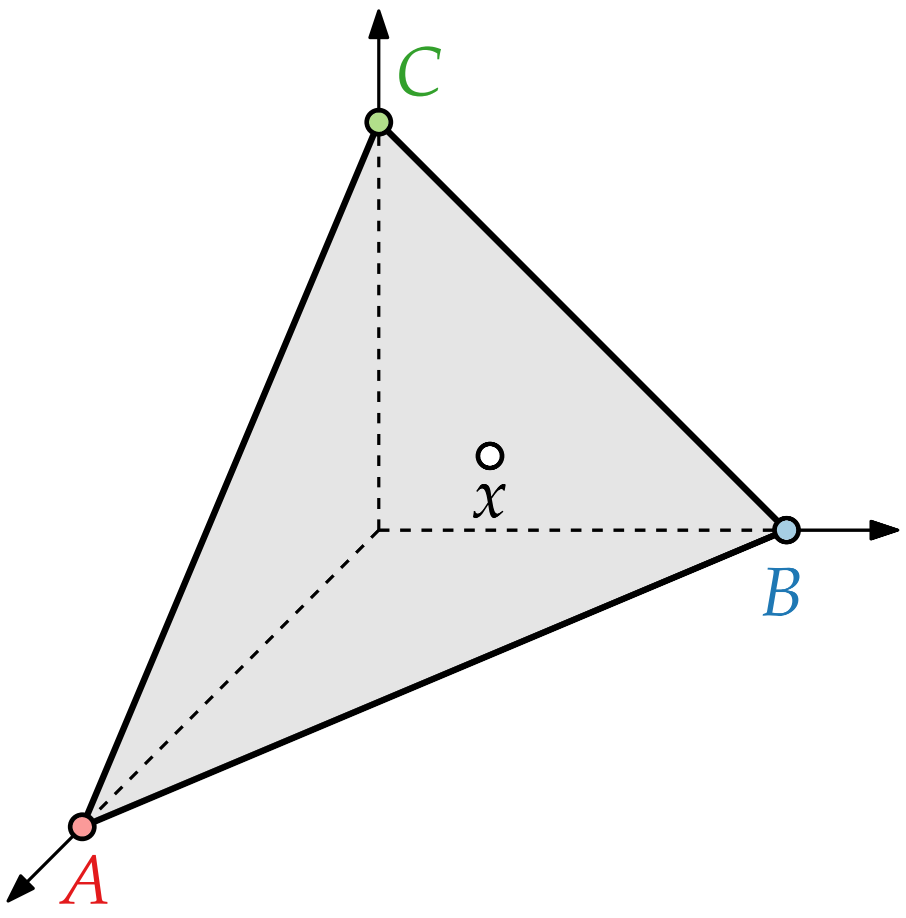

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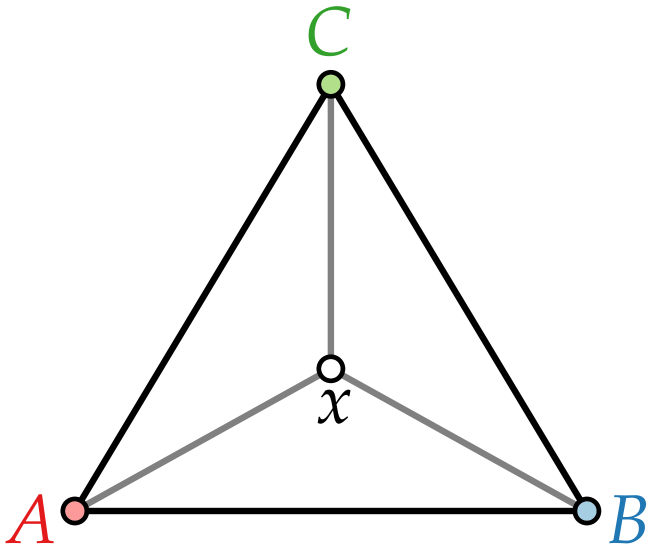

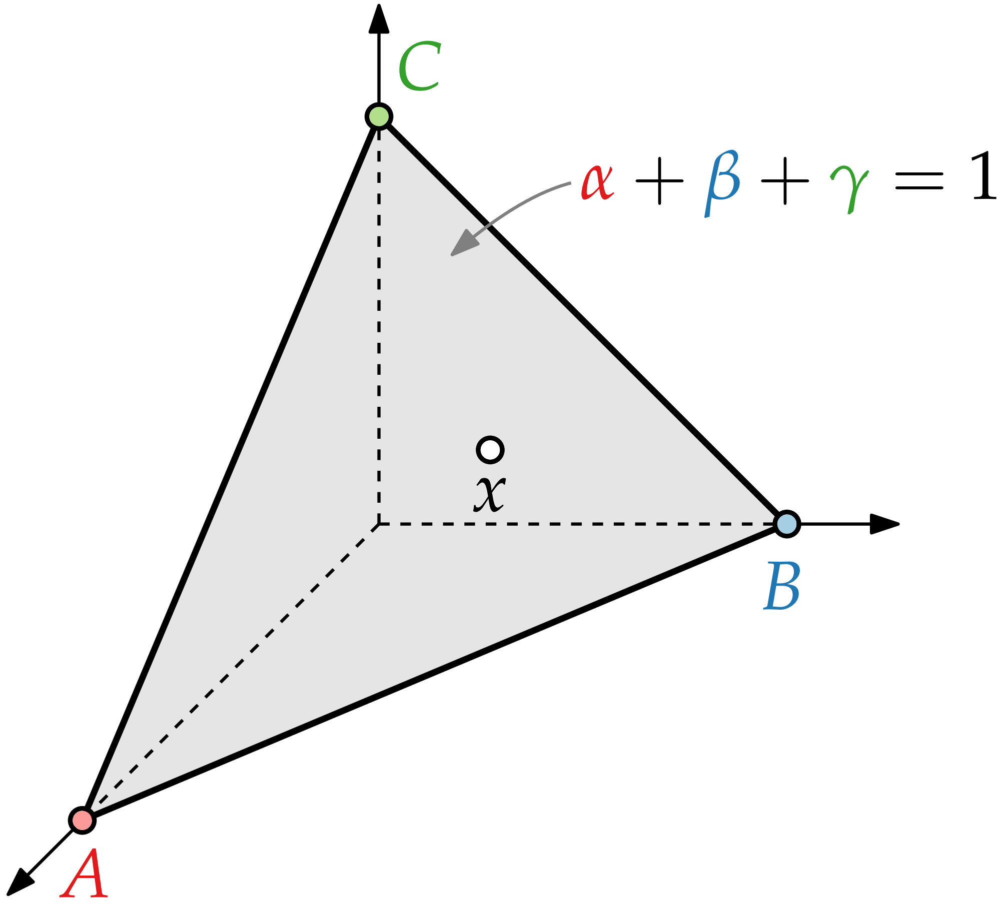

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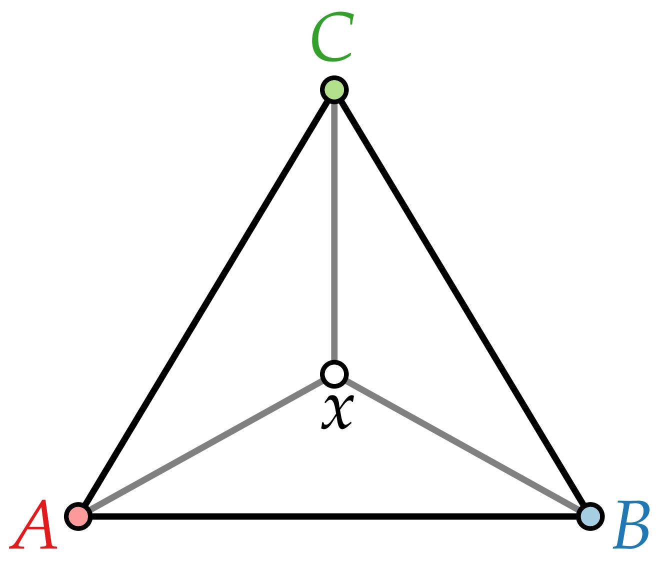

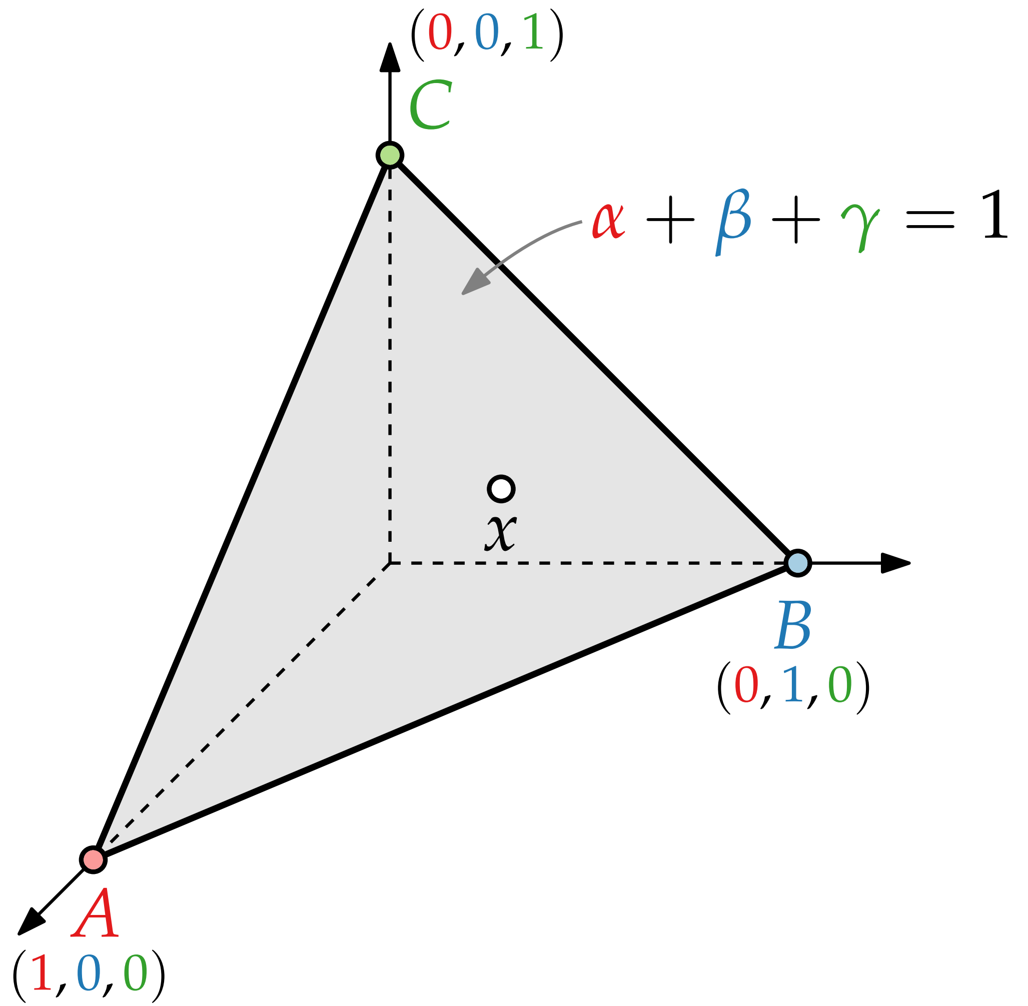

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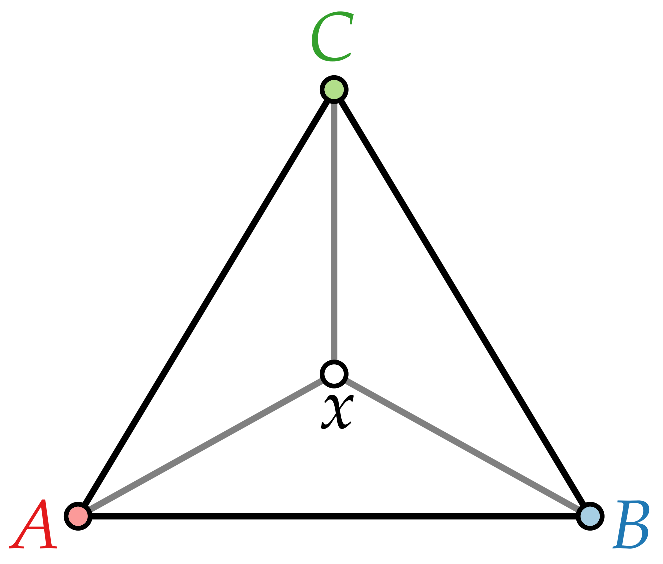

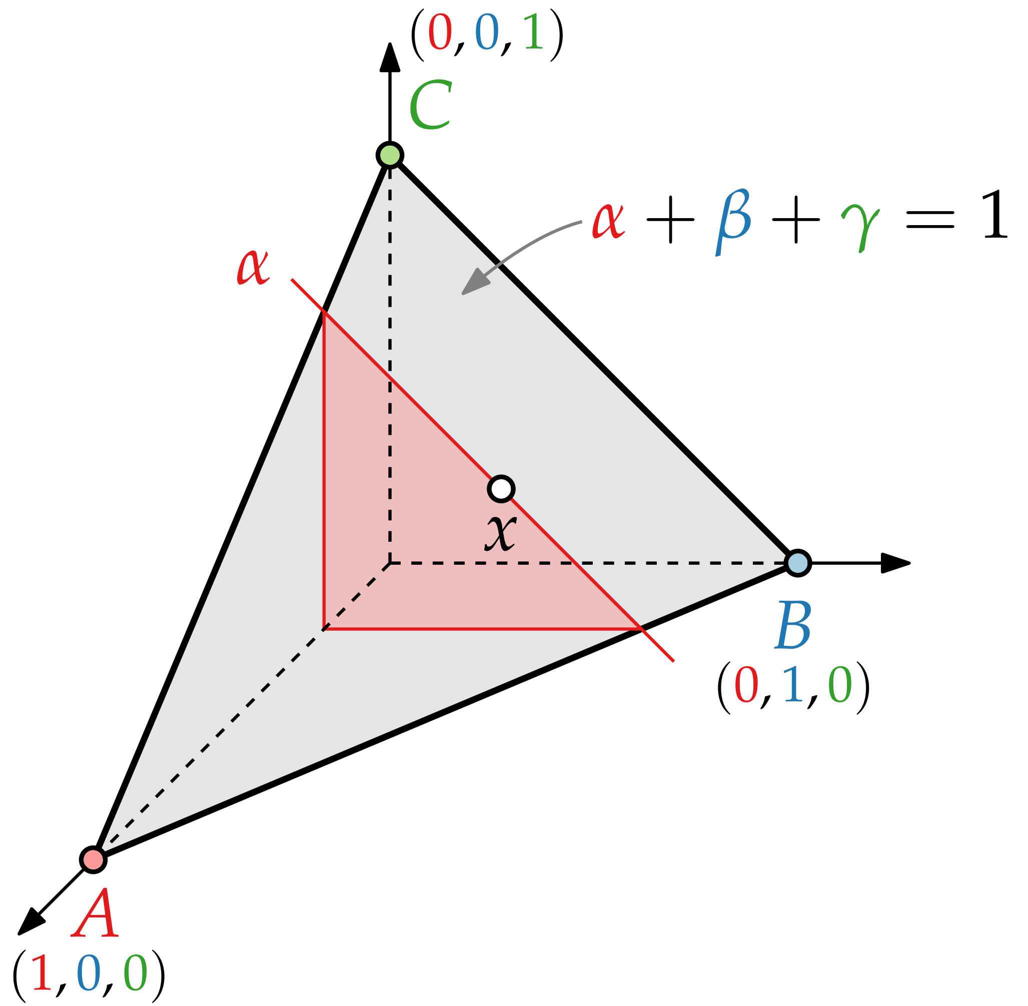

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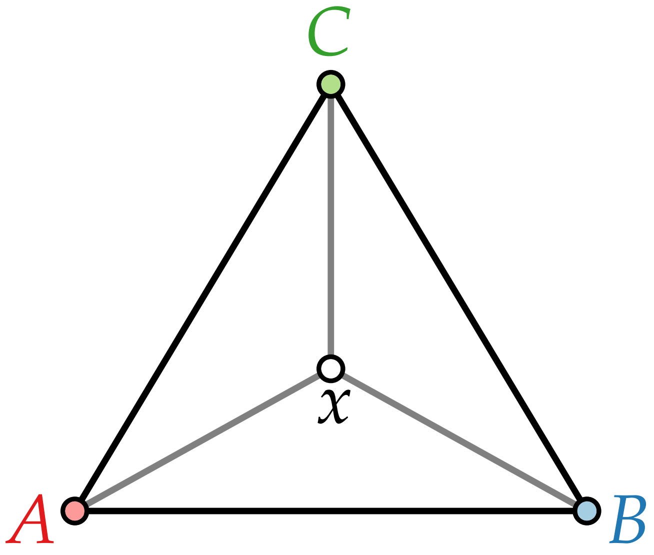

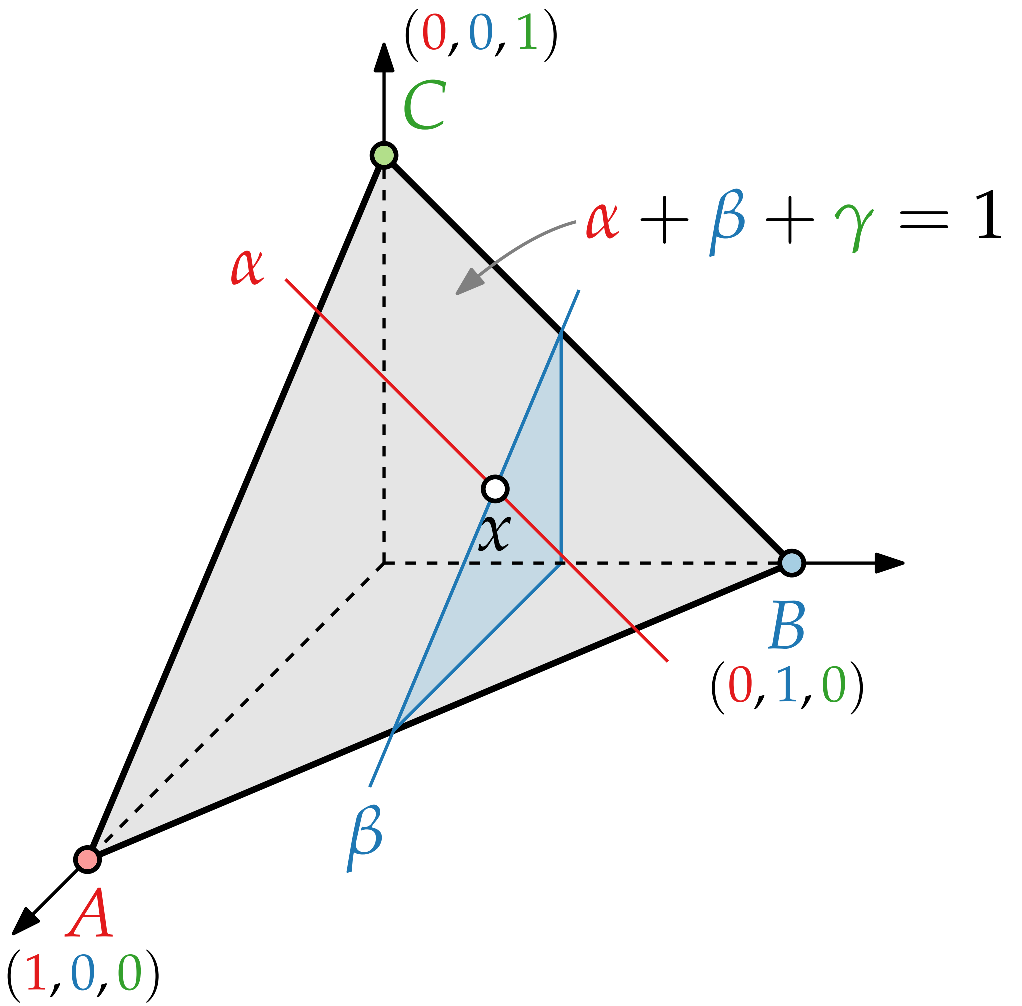

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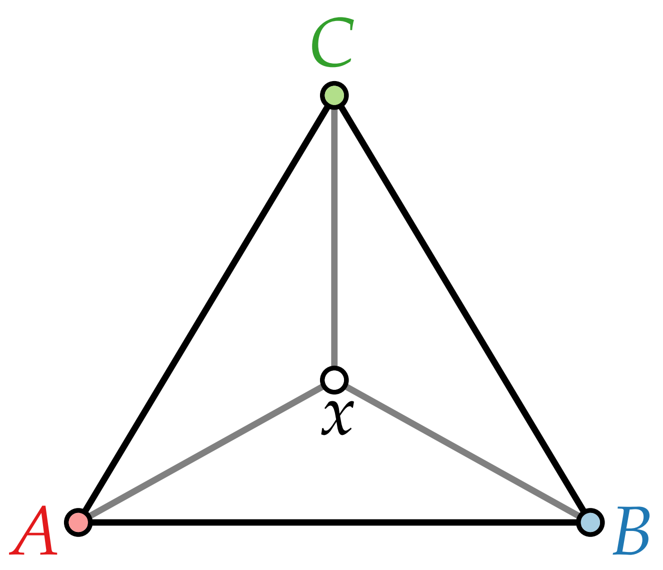

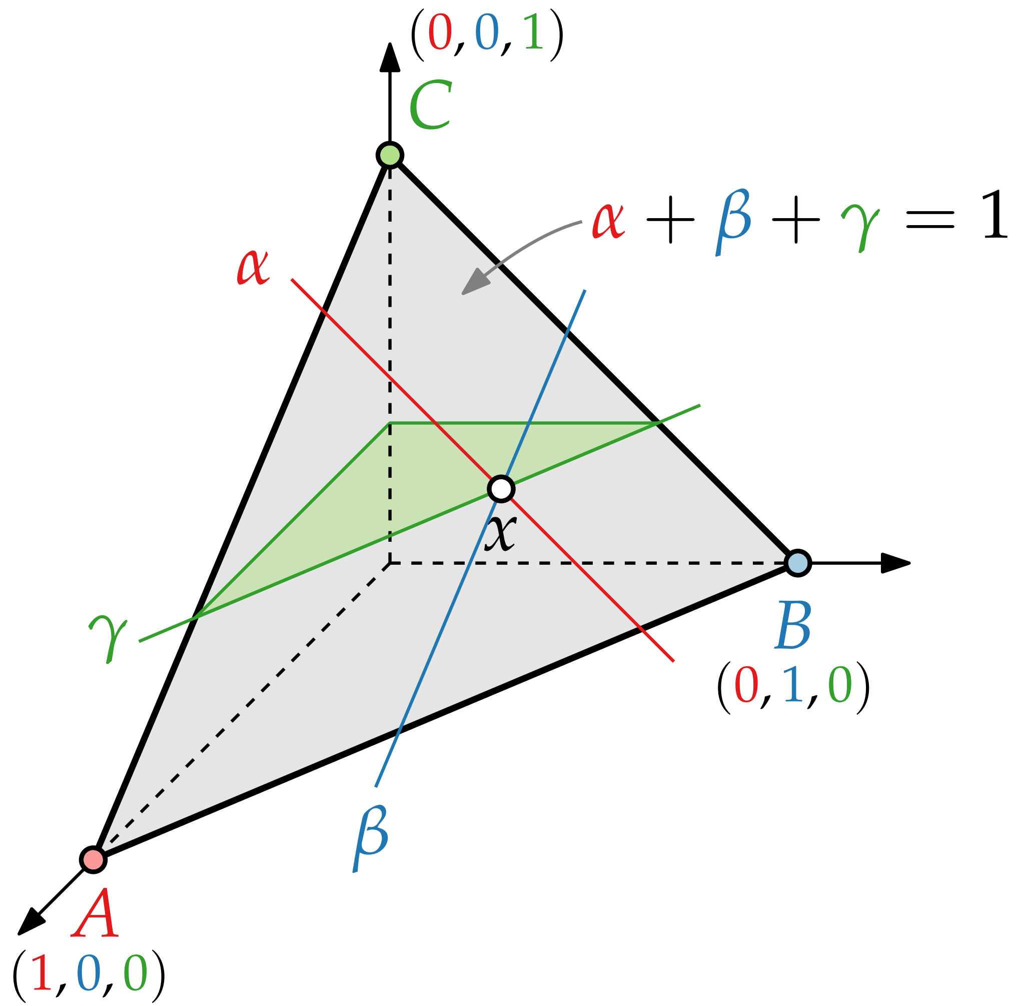

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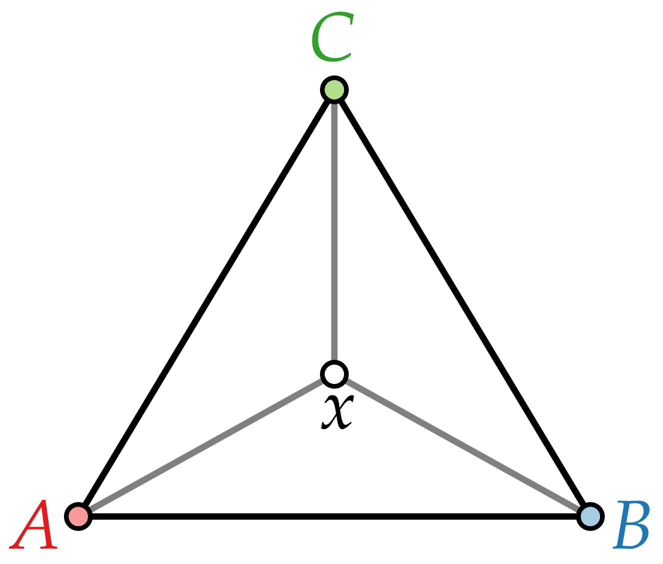

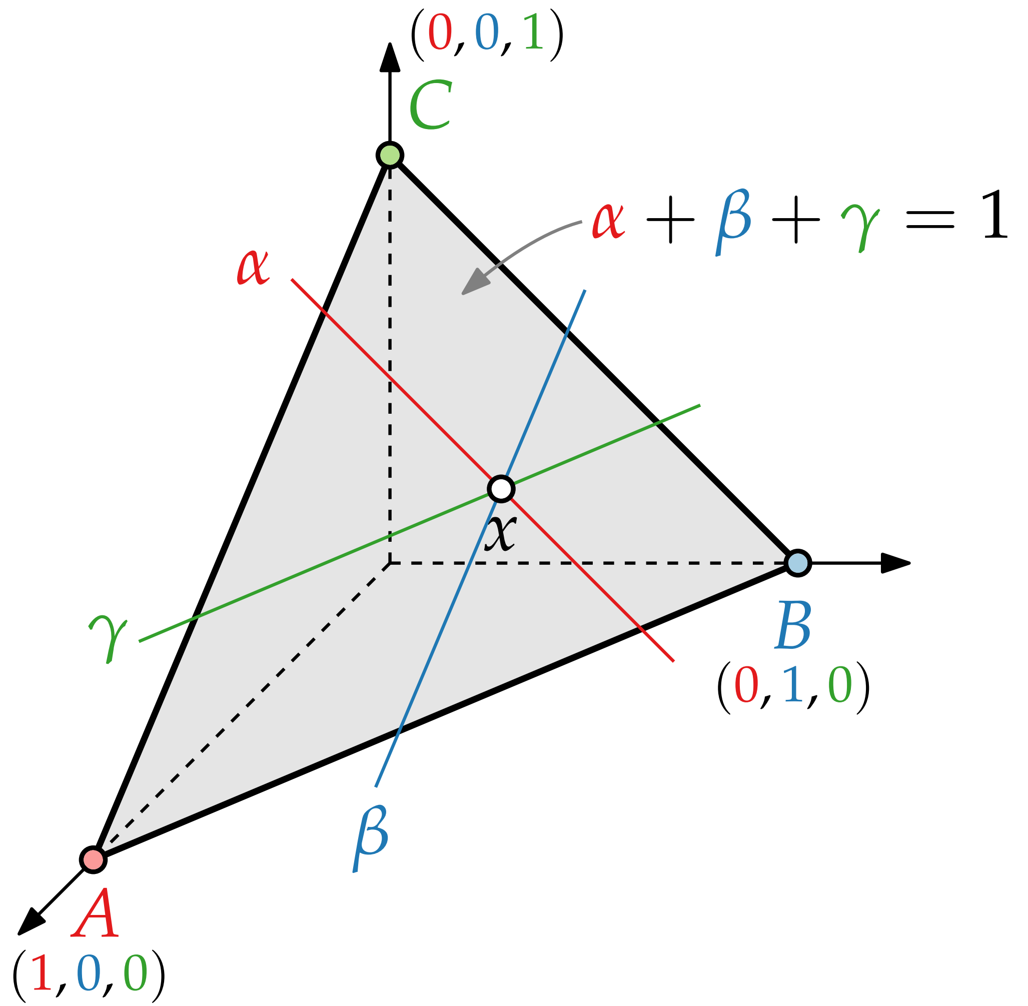

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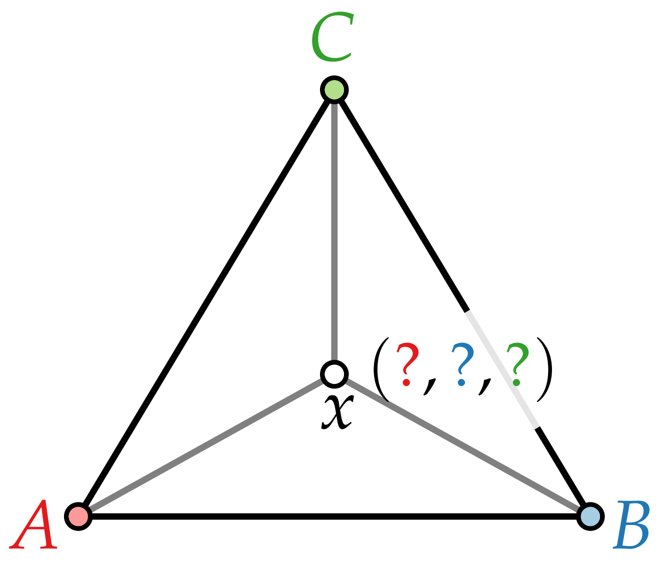

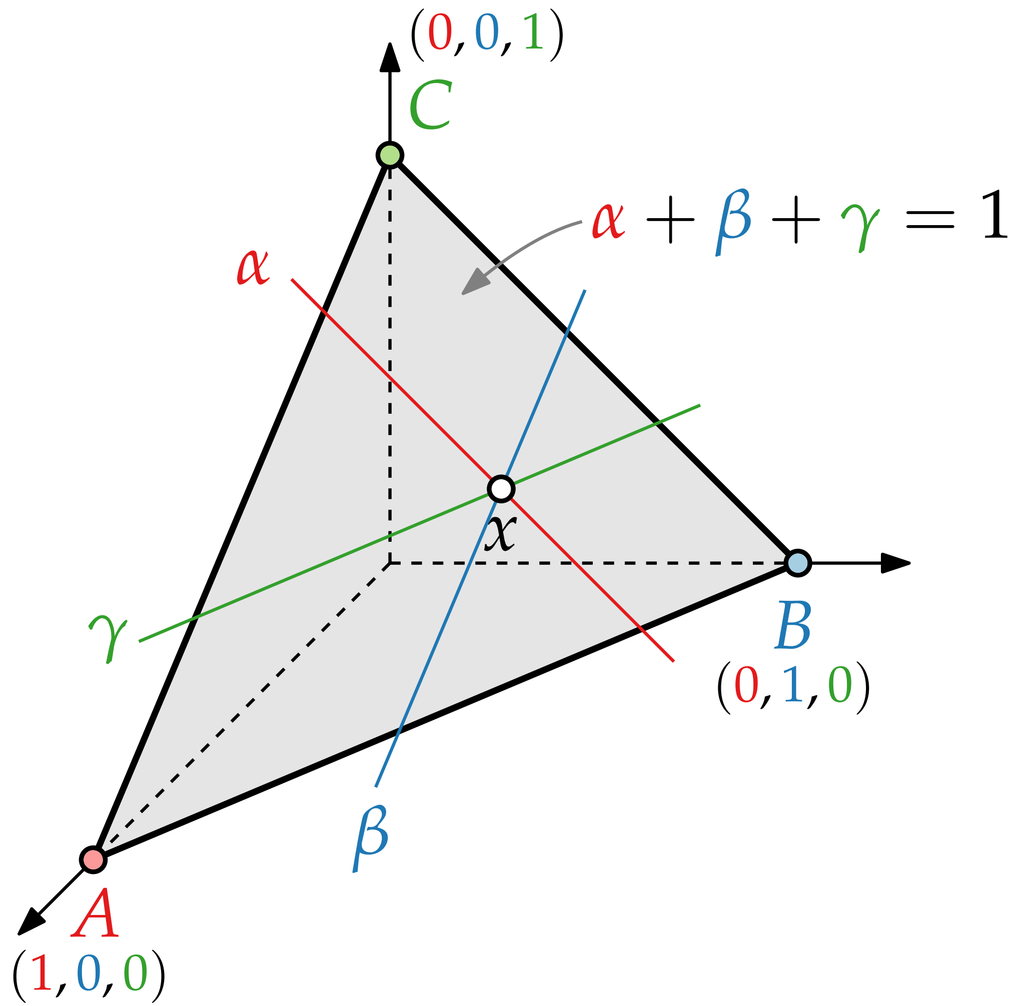

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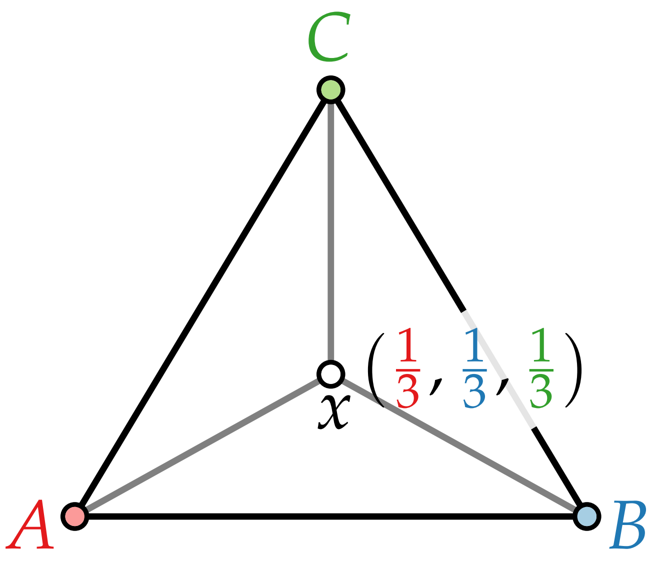

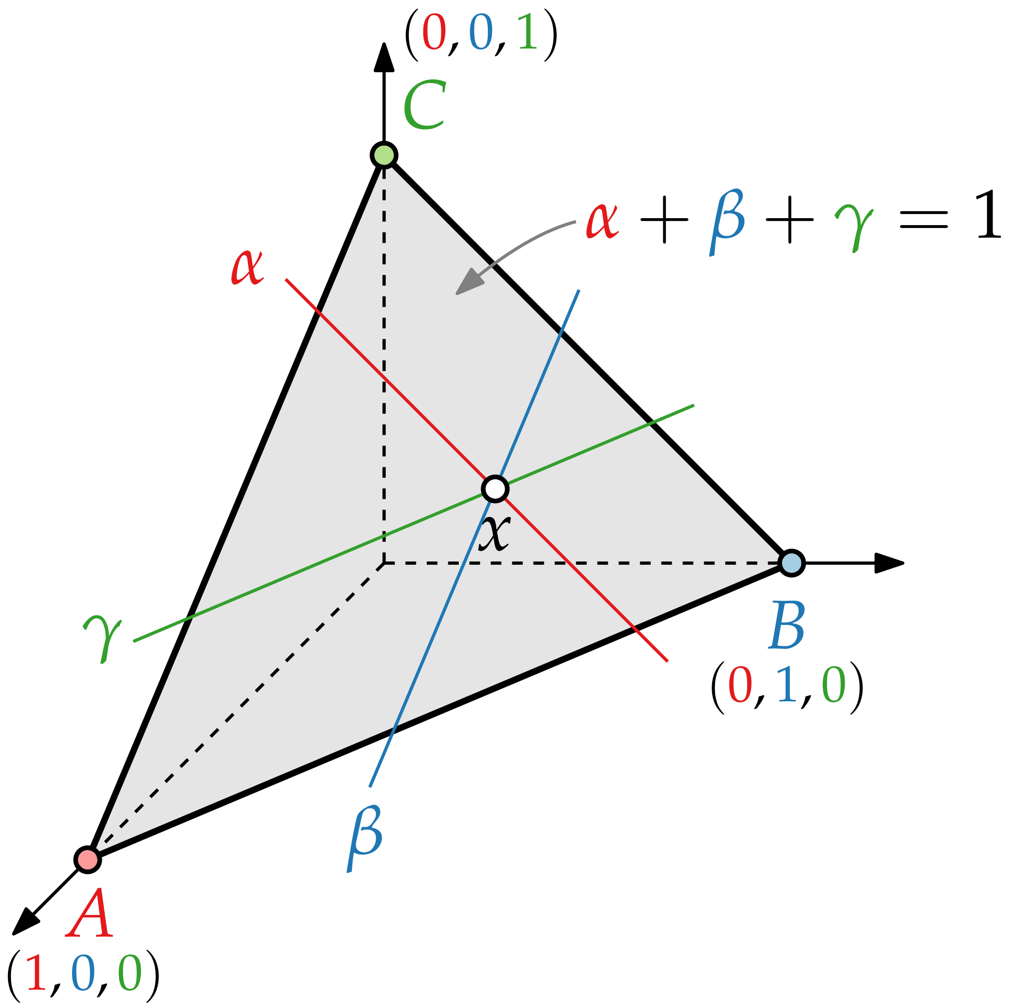

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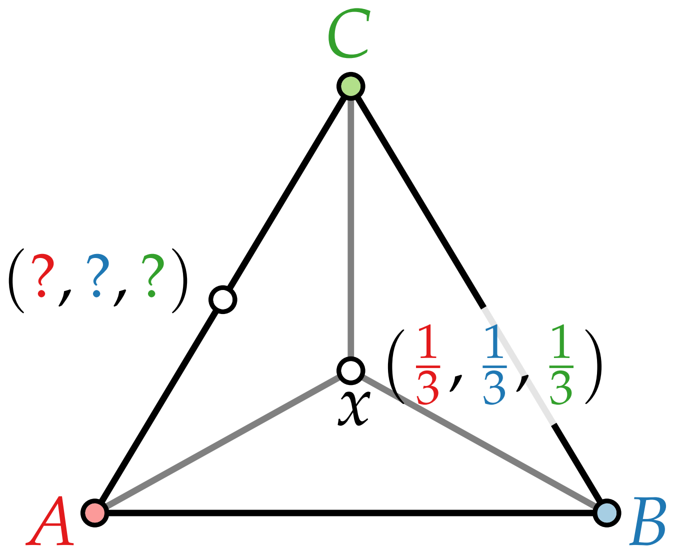

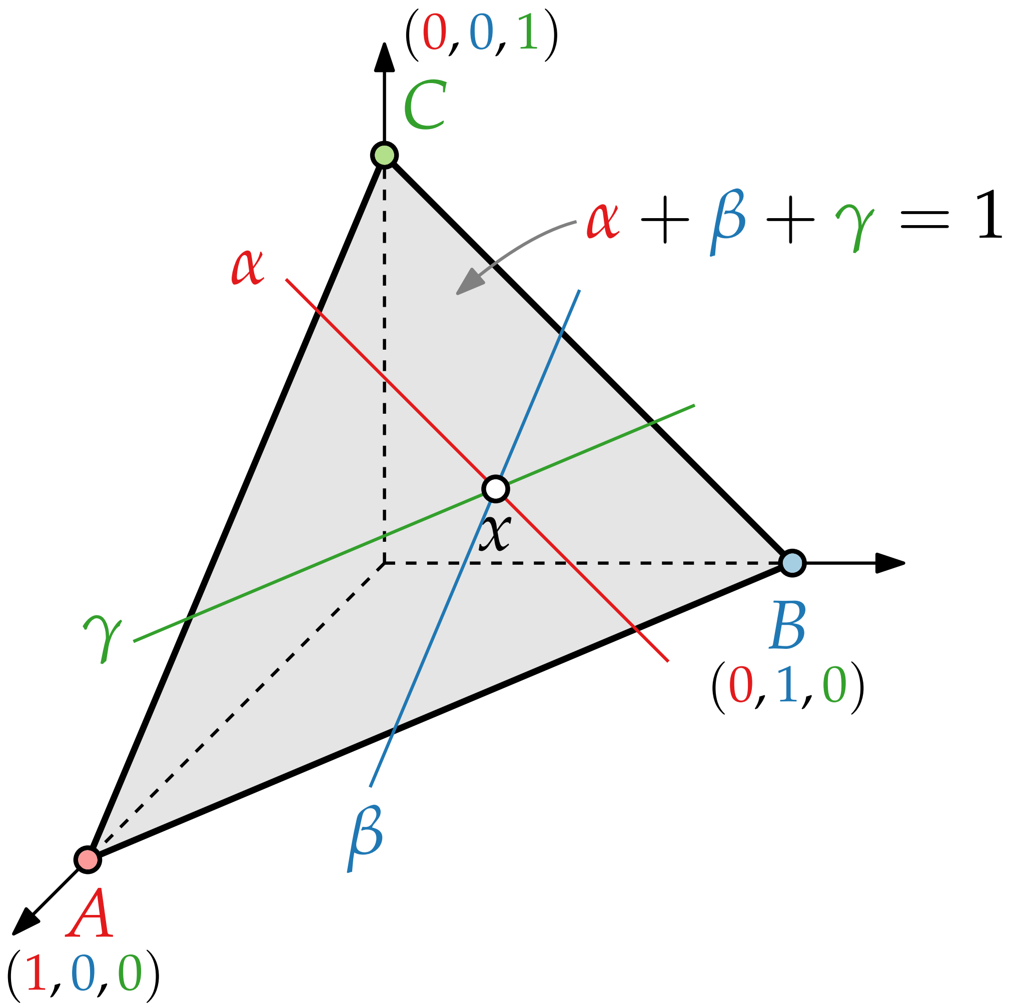

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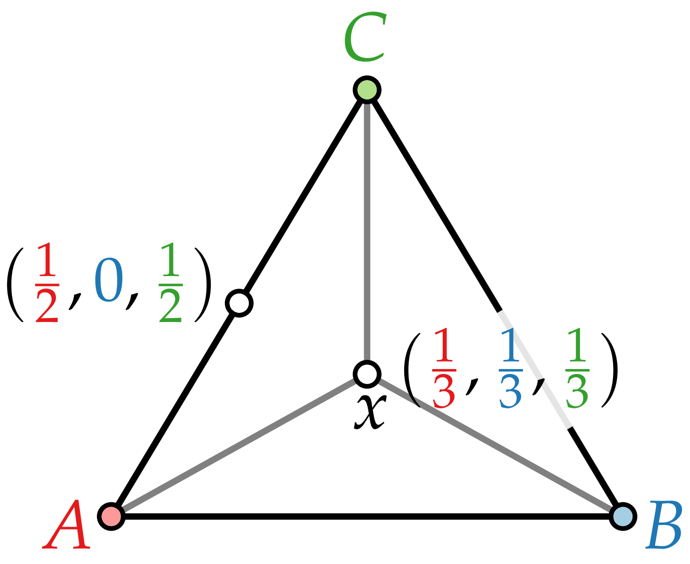
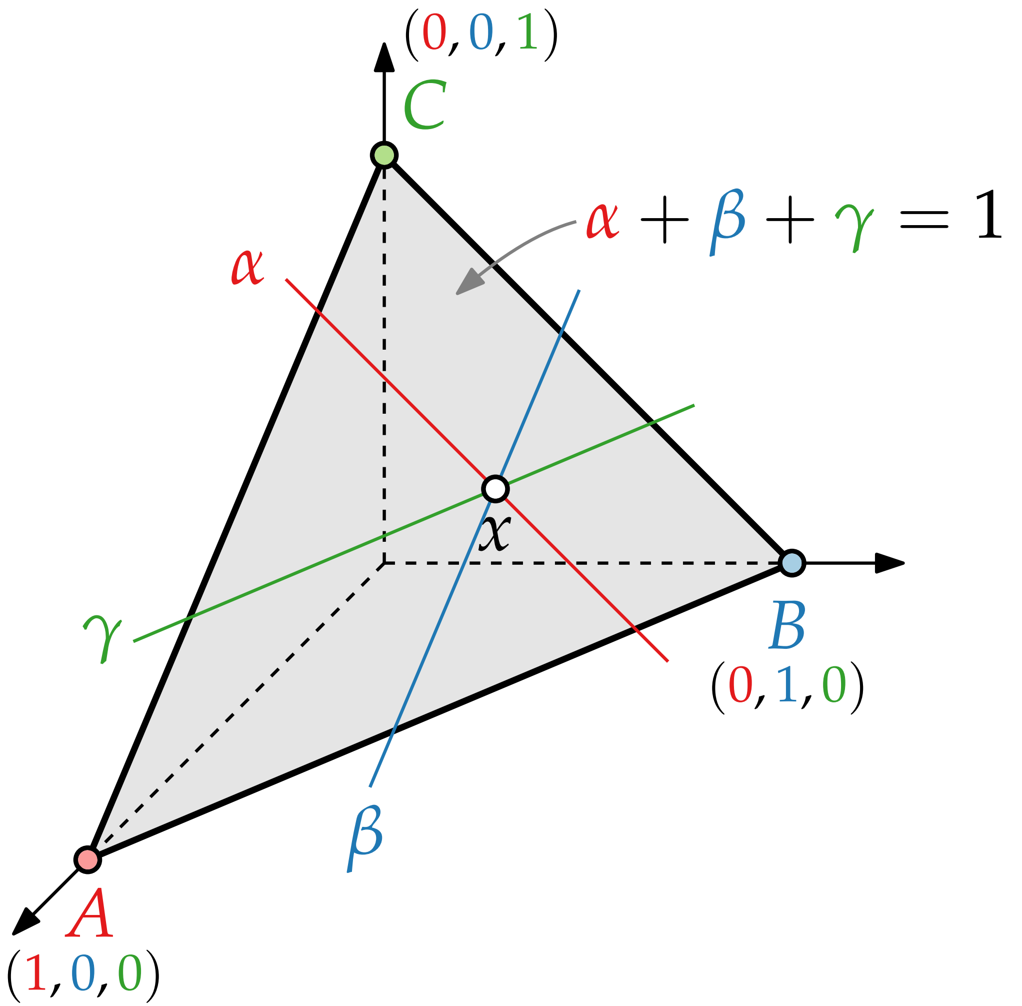


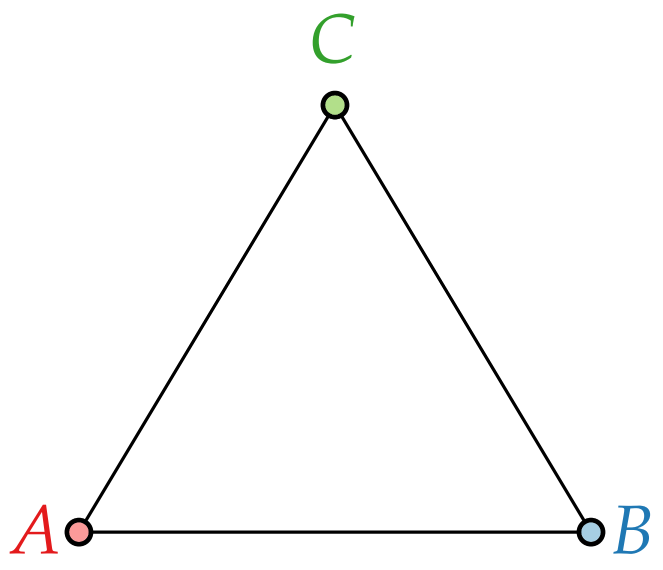


Barycentric Representation

A **barycentric representation** of a graph $G = (V, E)$ is an assignment of barycentric coordinates to the vertices of G:

$$f \colon V \to \mathbb{R}^3_{\geq 0}, v \mapsto (v_1, v_2, v_3)$$

with the following properties:



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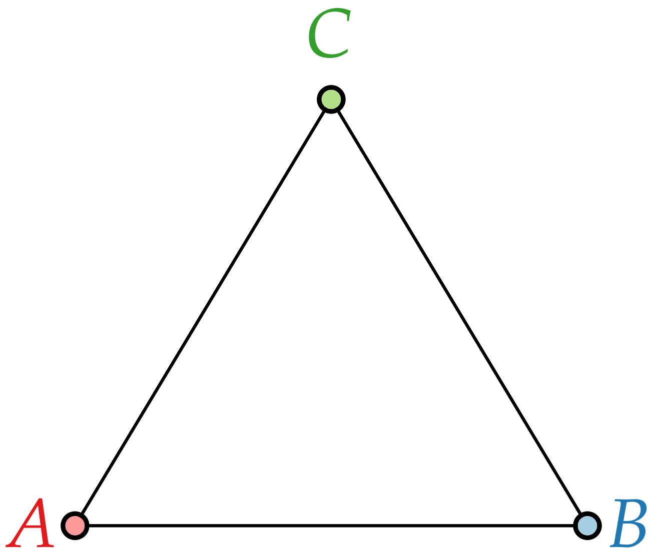
(B1) $v_1 + v_2 + v_3 = 1$ for all $v \in V$,



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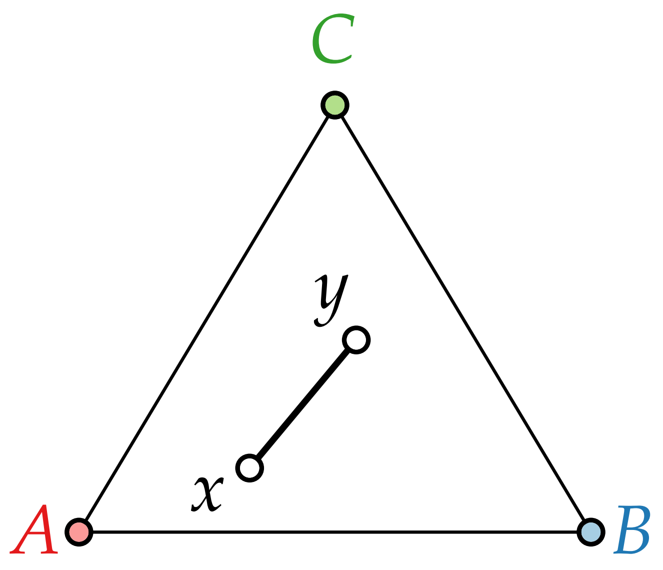

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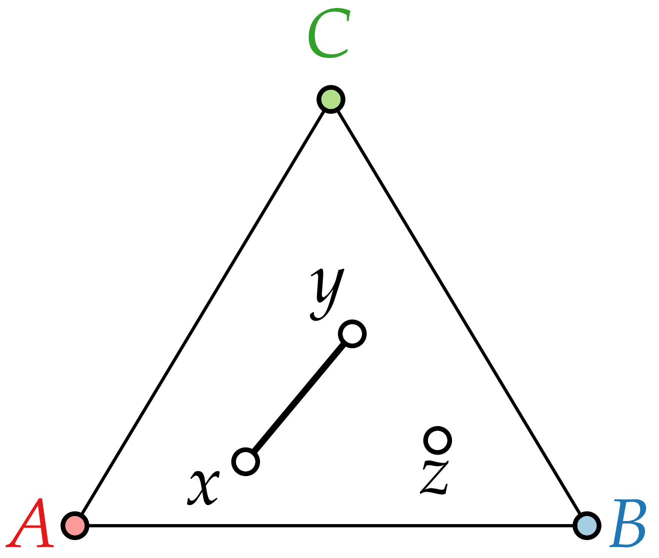

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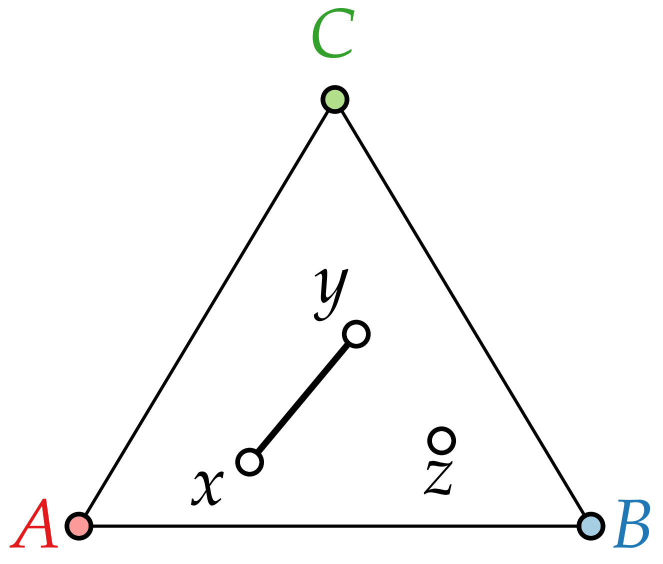

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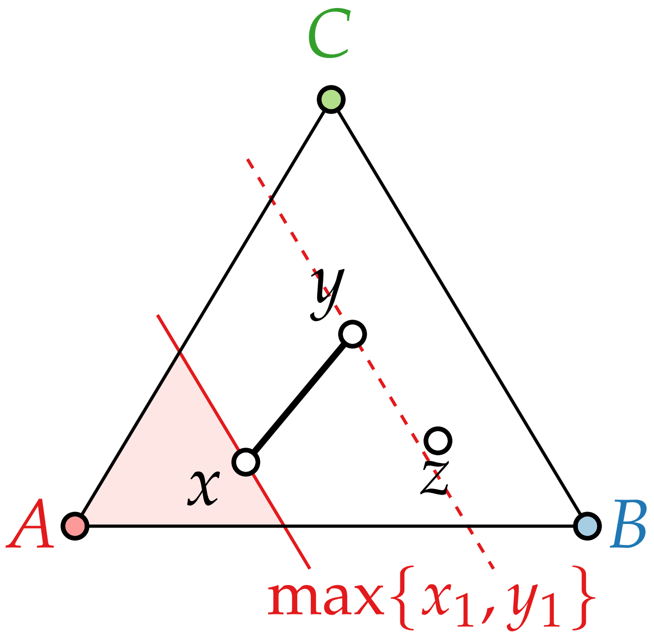

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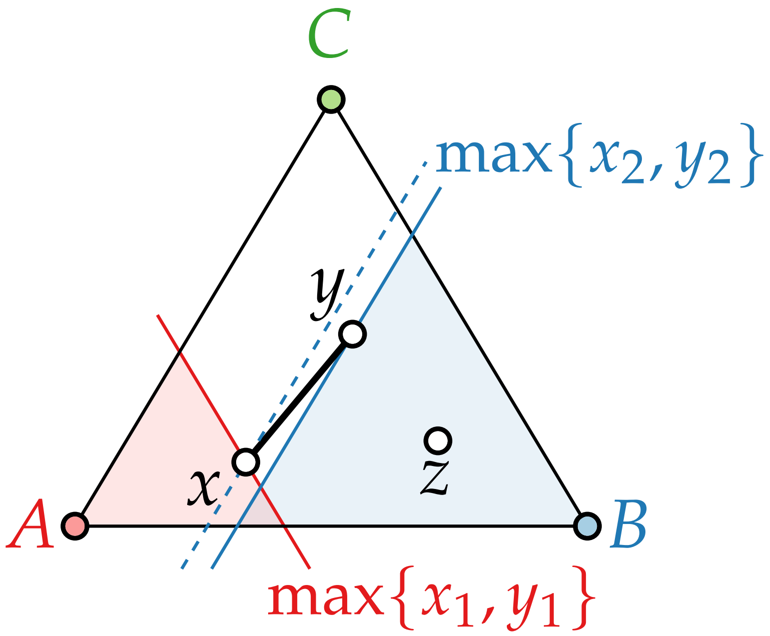

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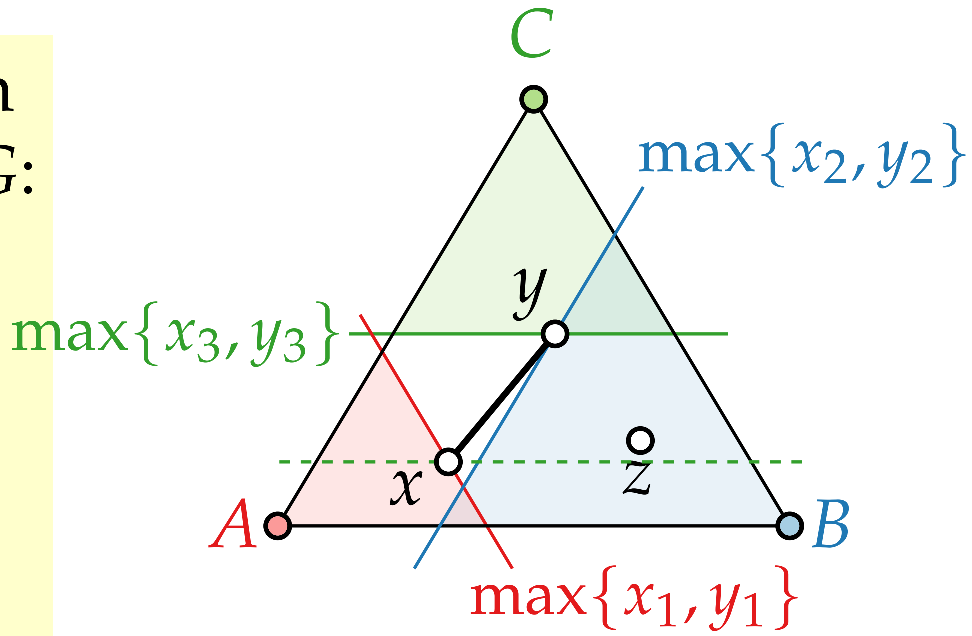

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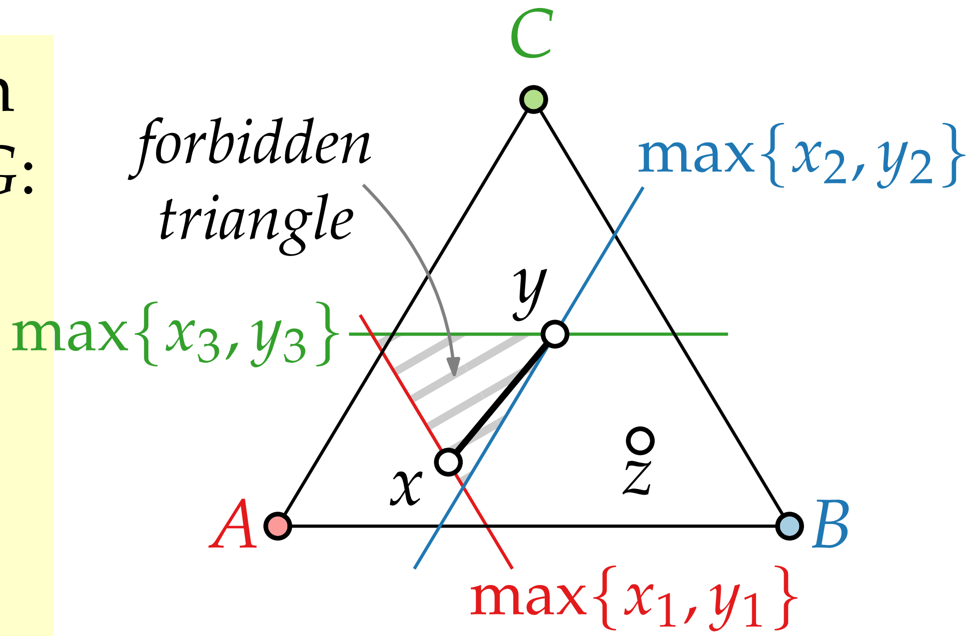

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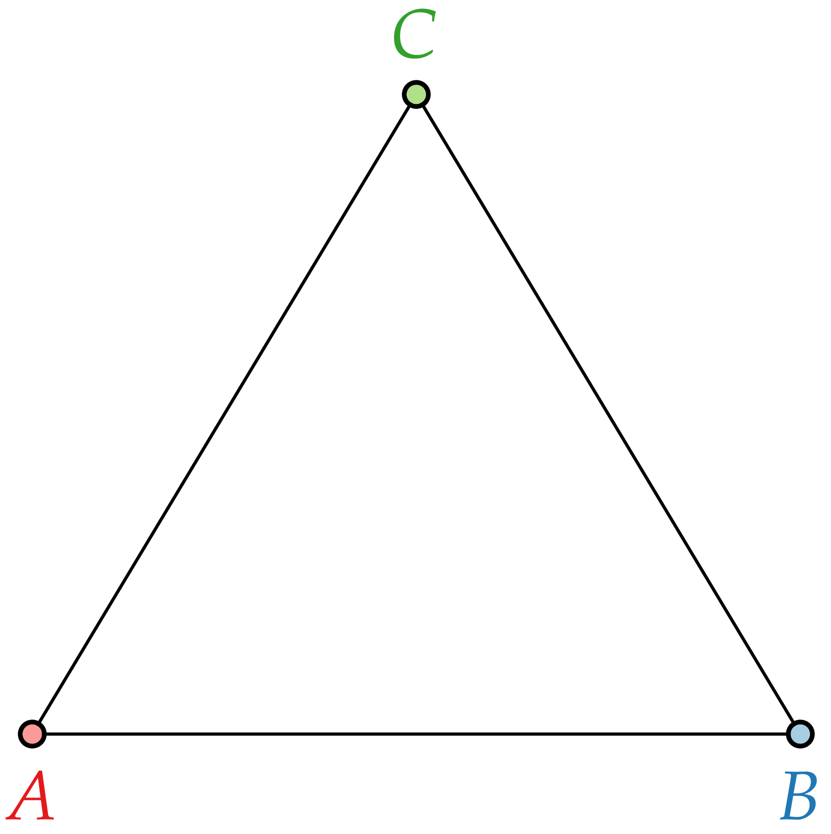

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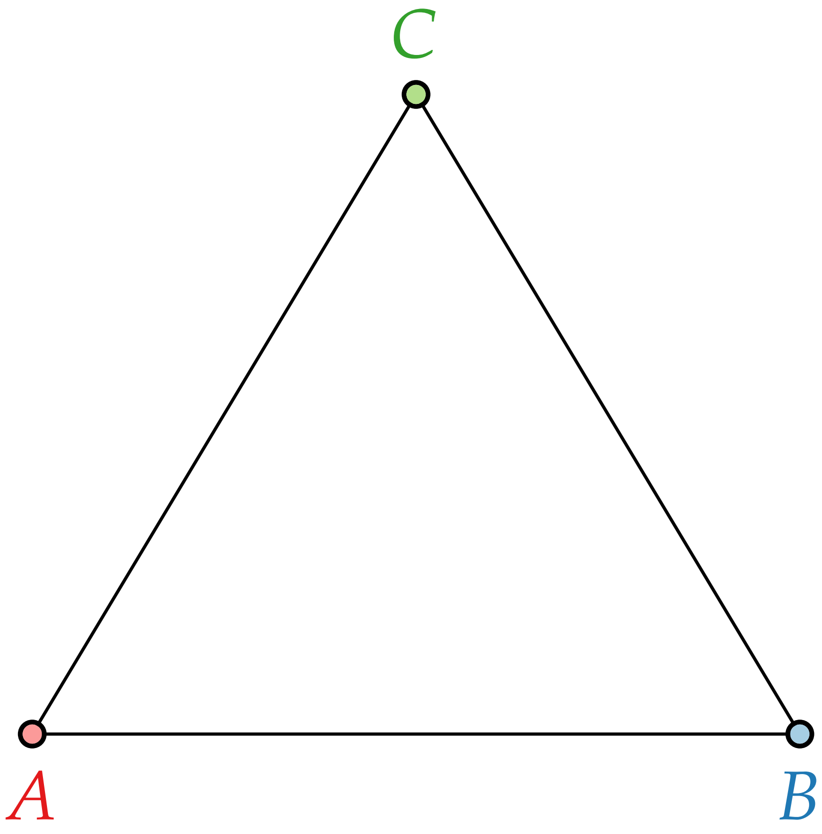

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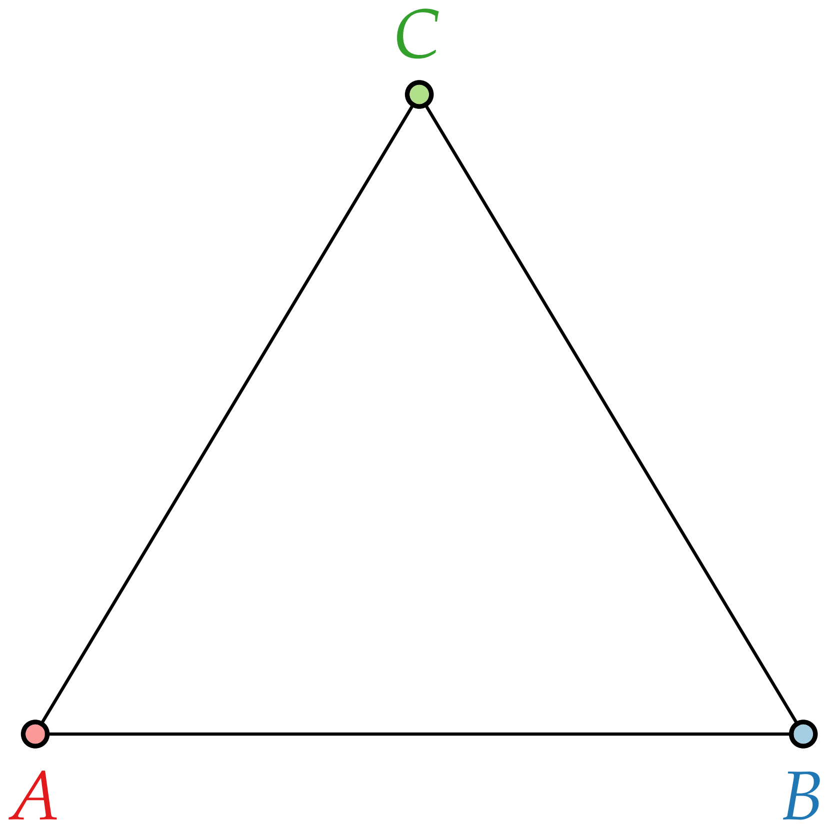

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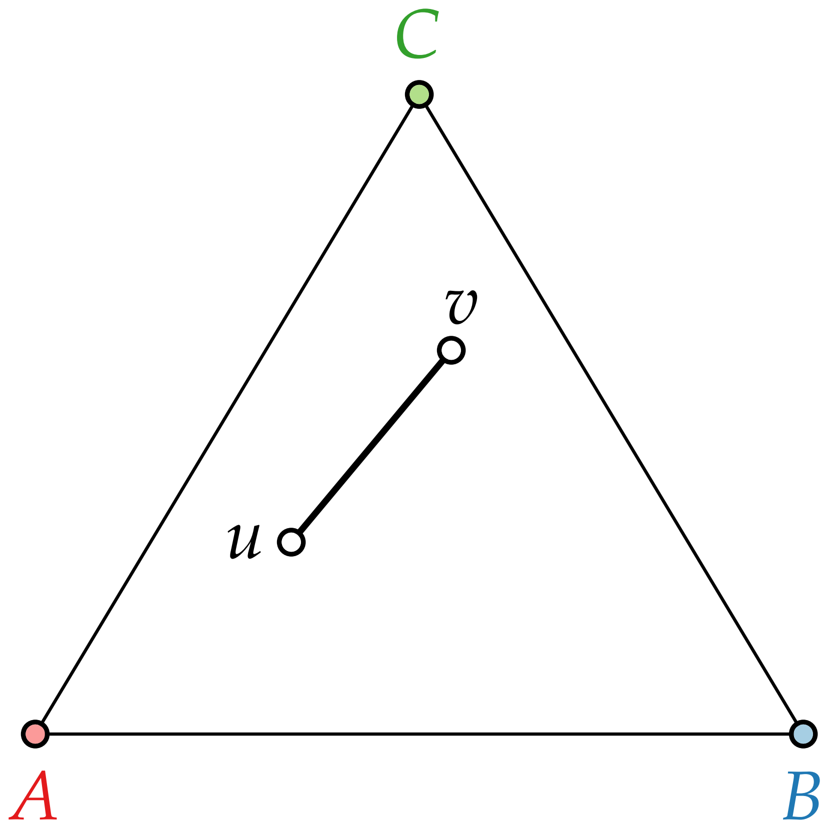

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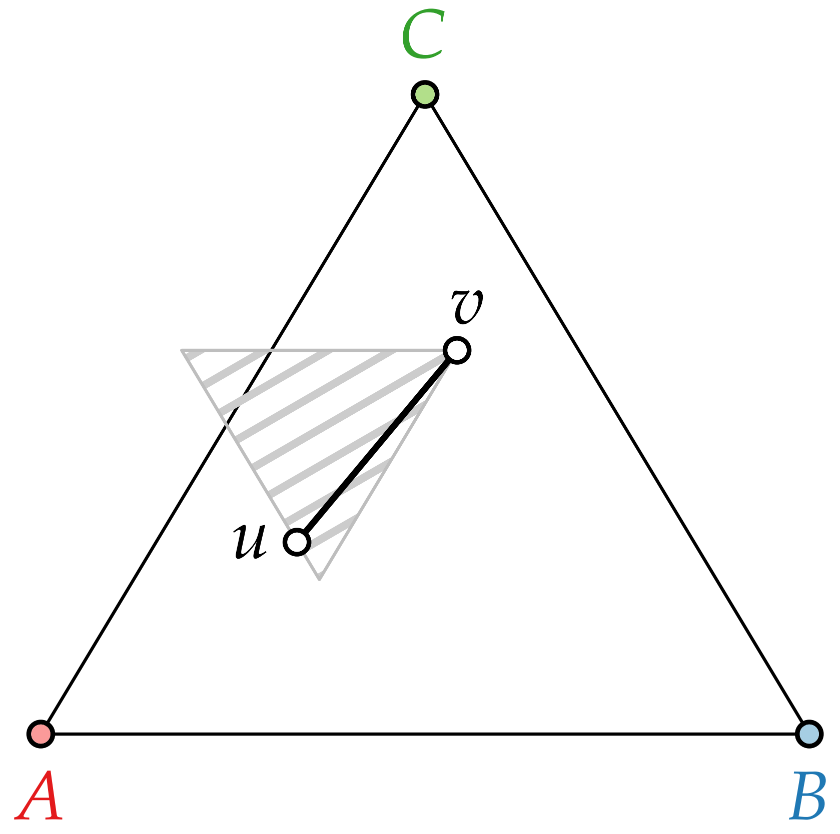

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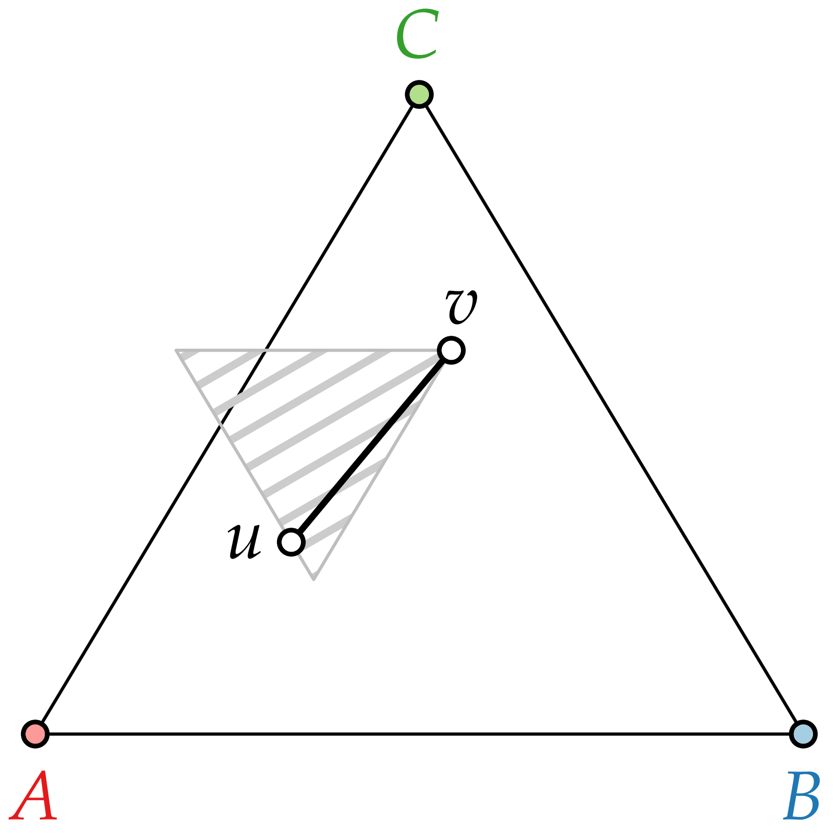

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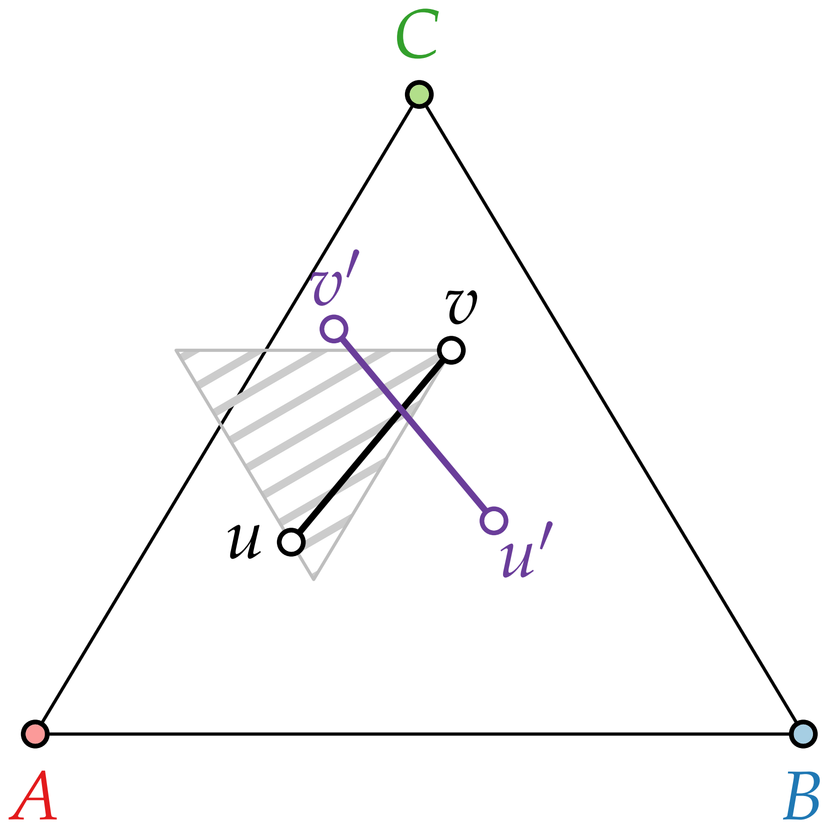

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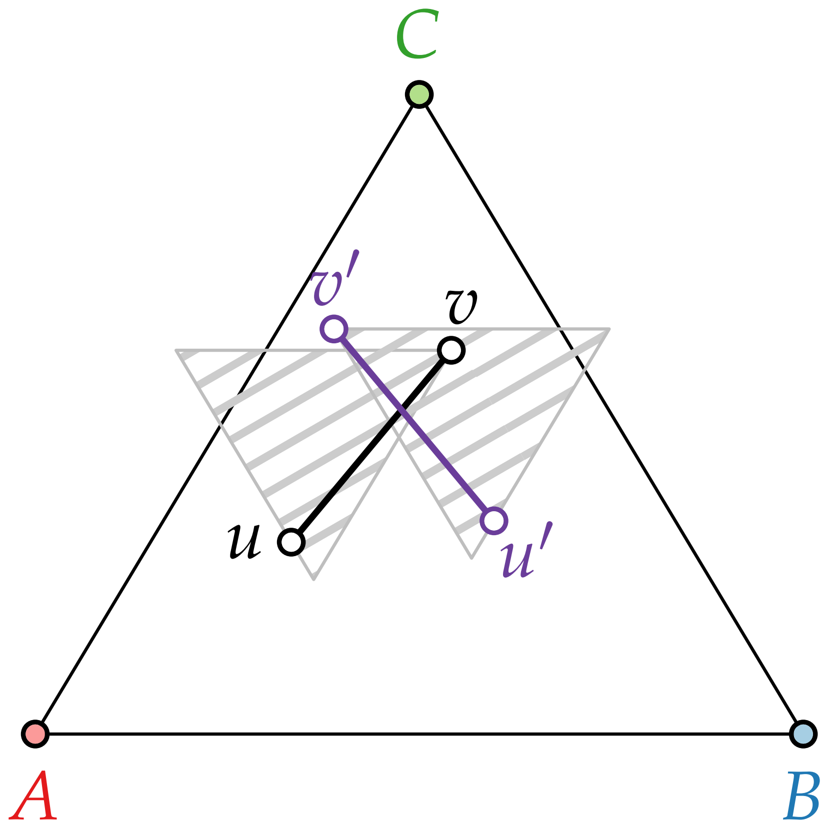

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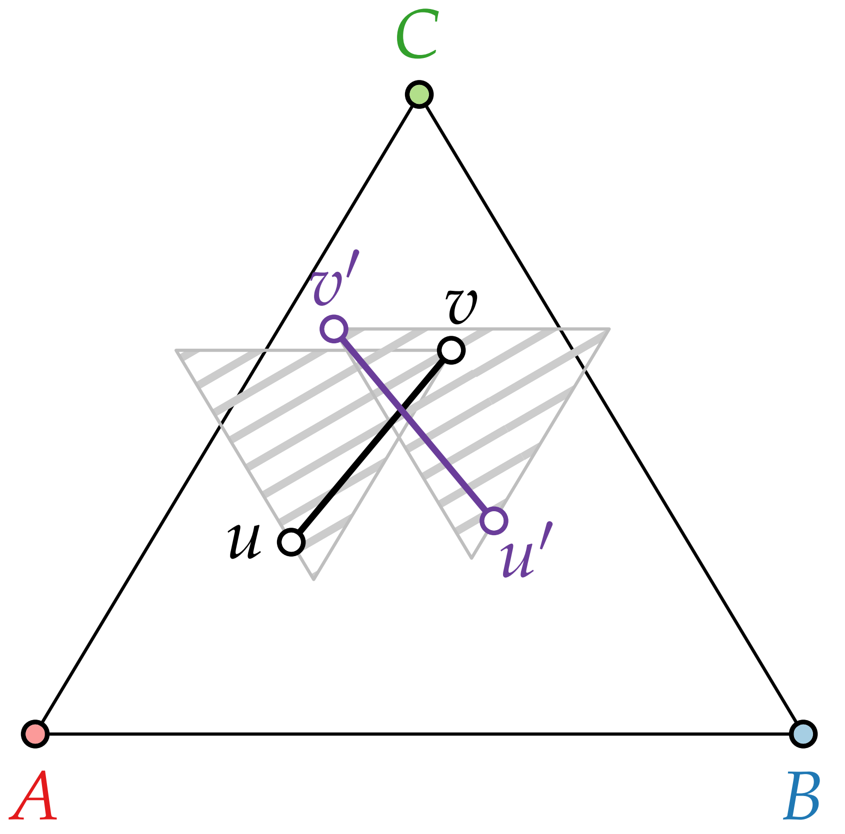

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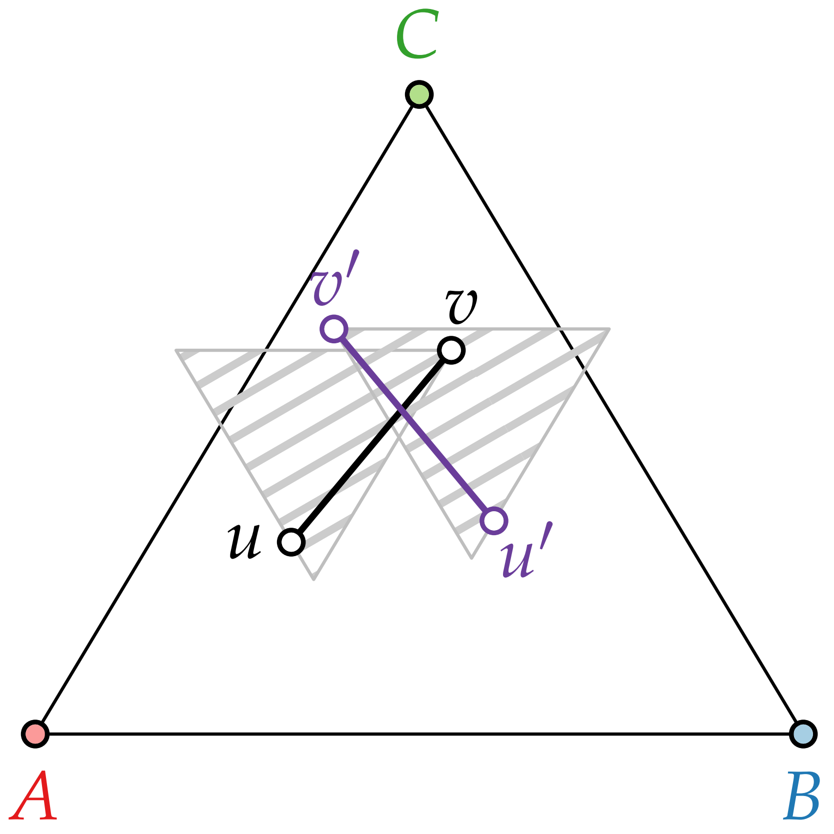

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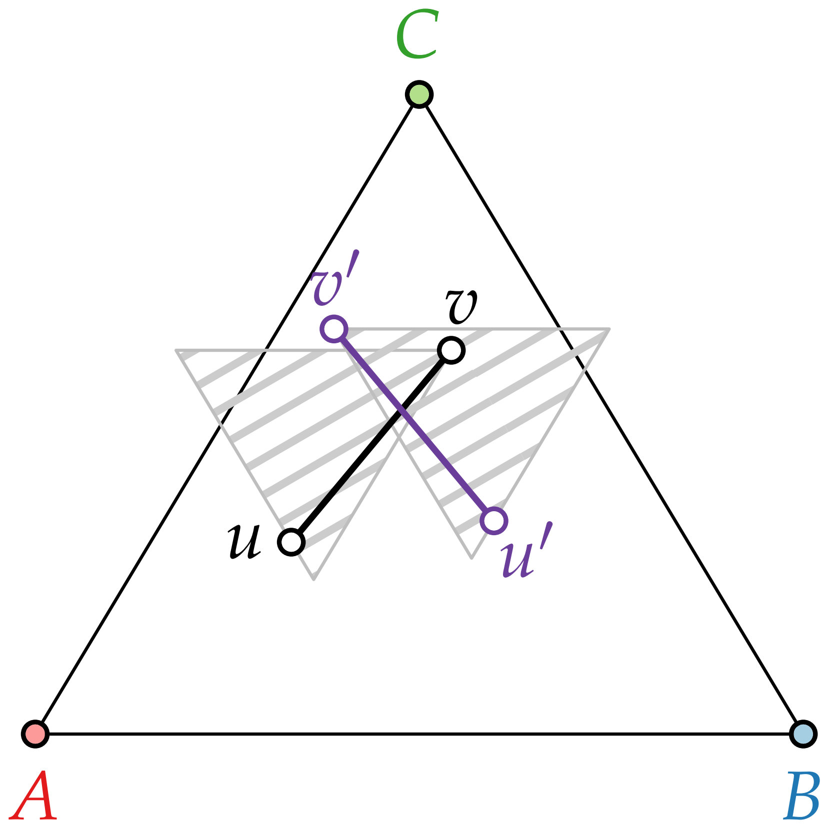

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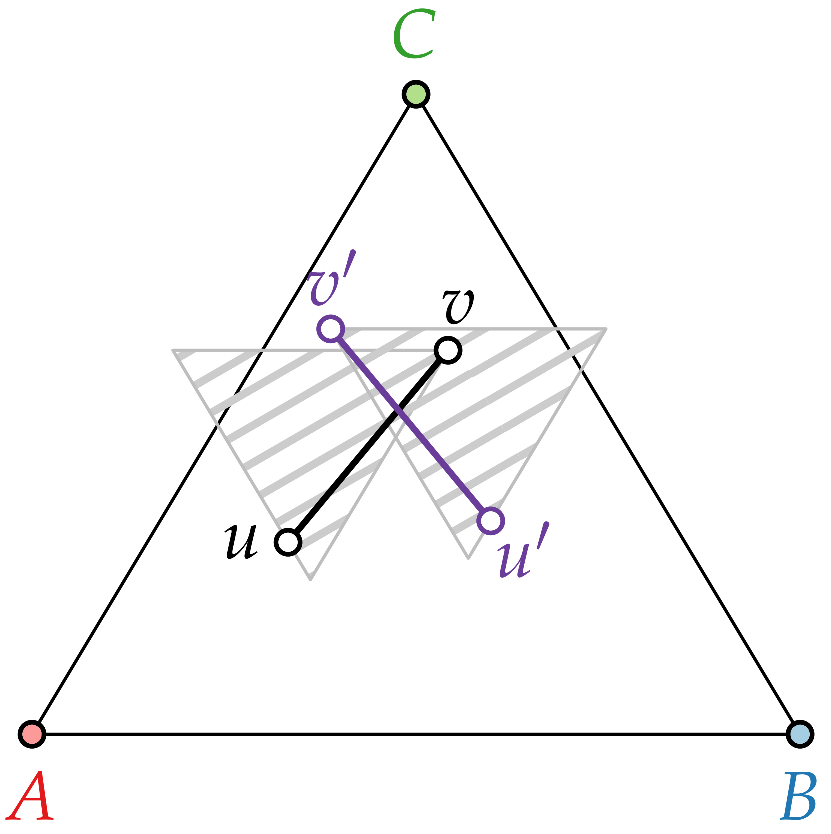

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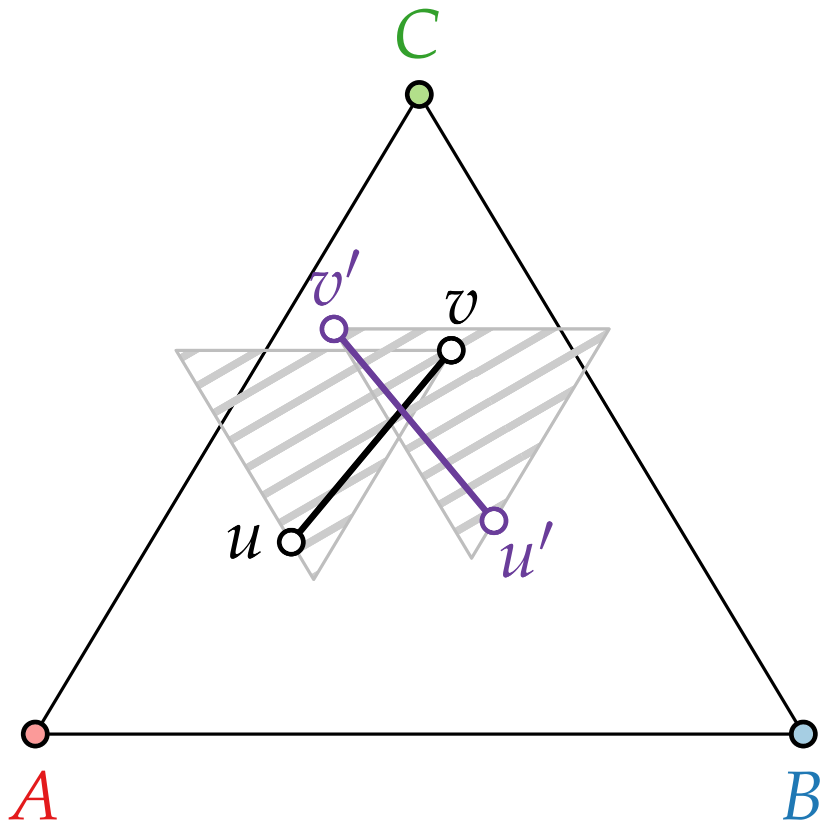

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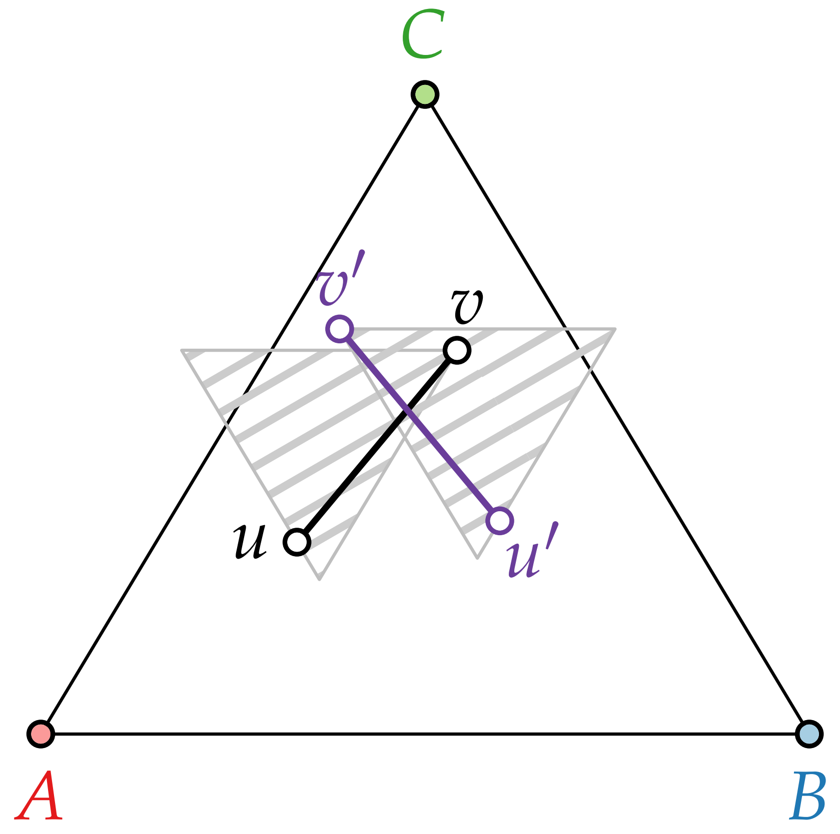

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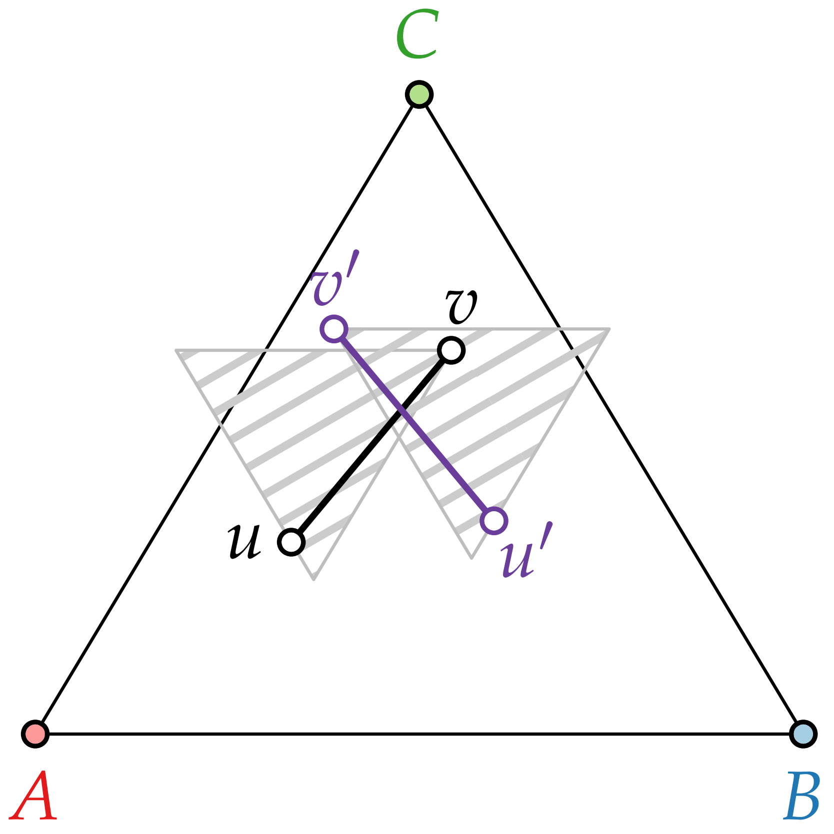

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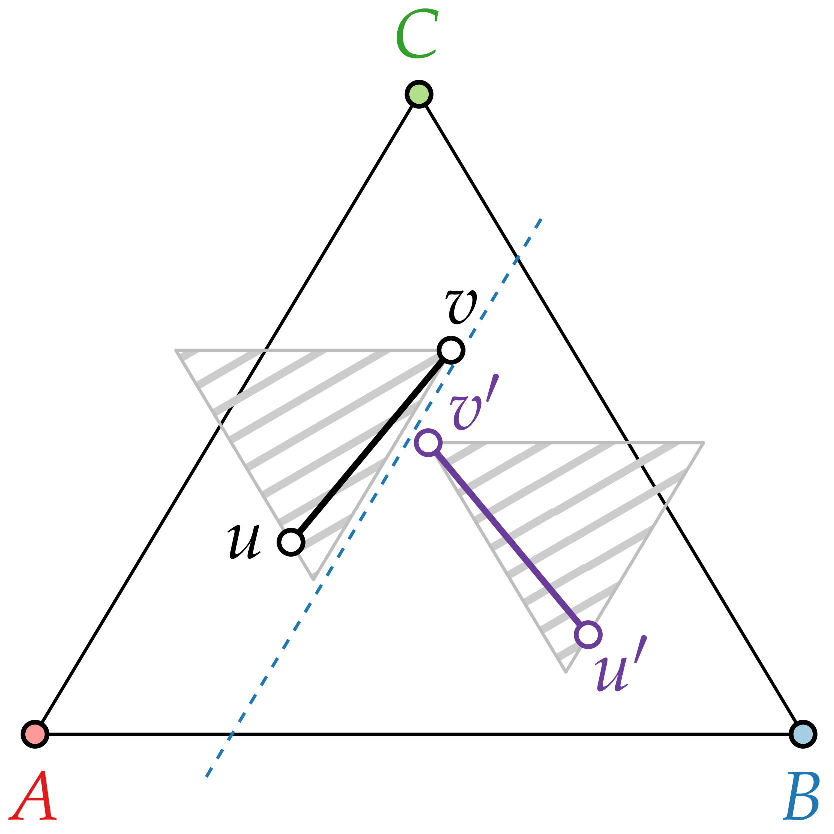

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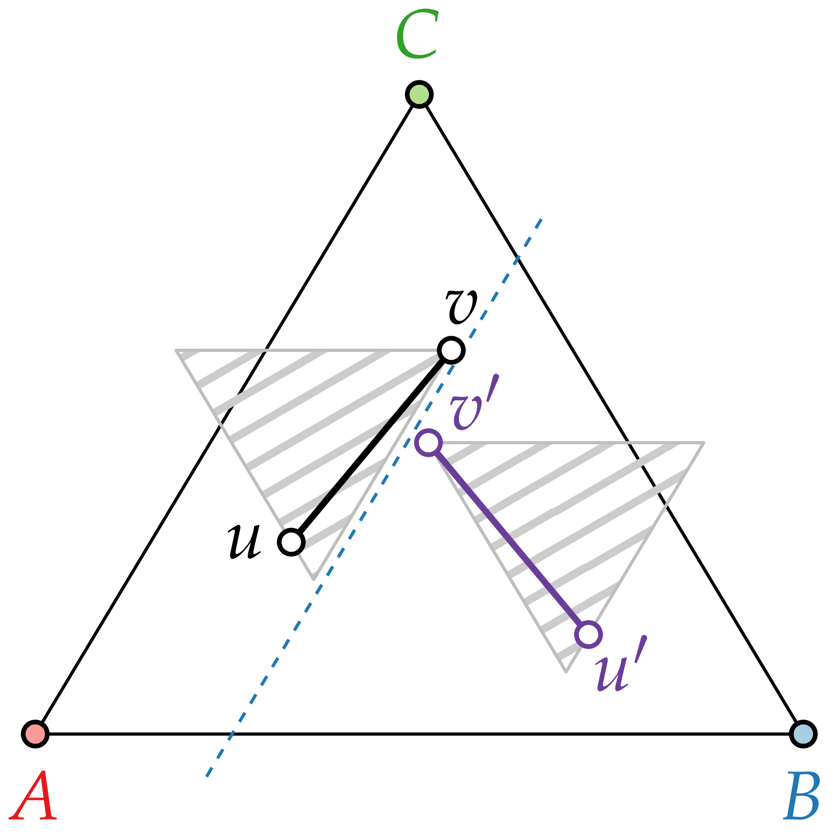

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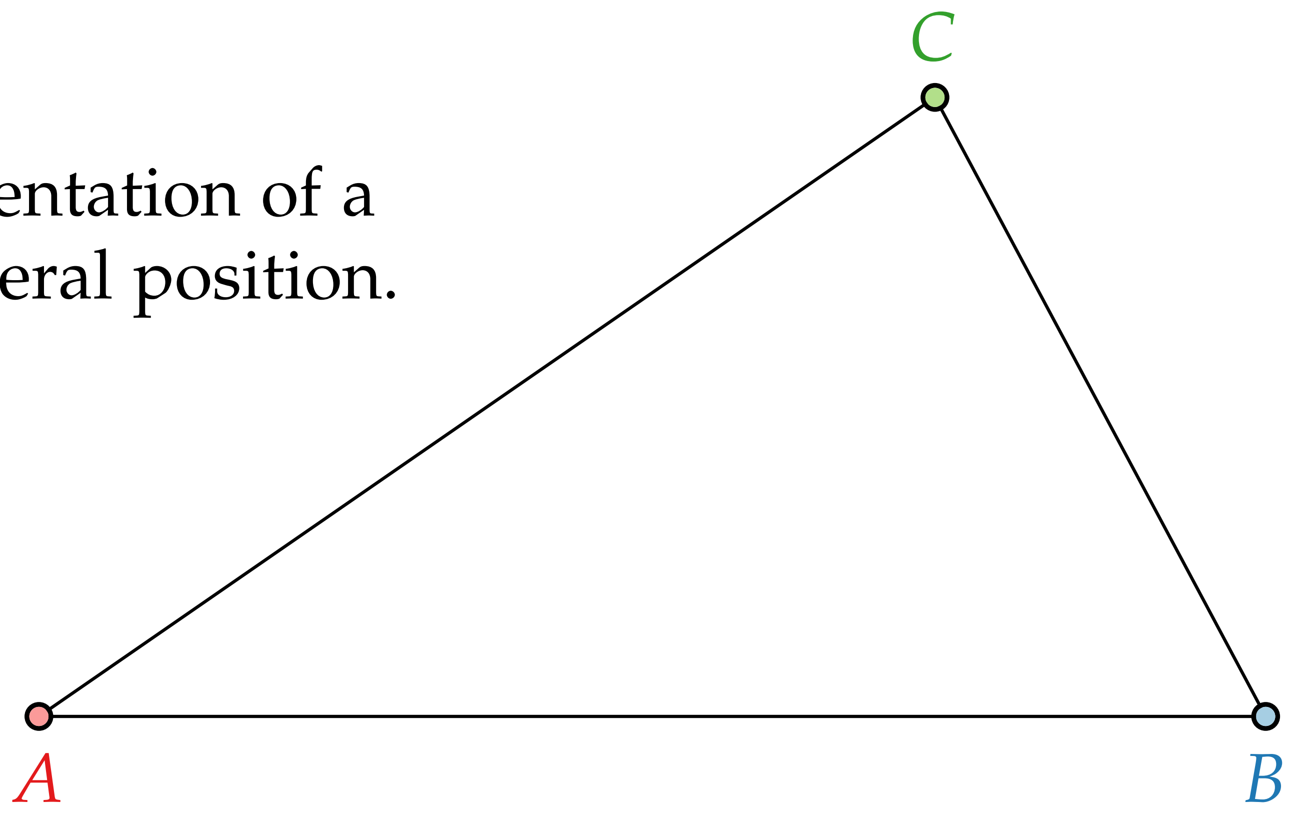

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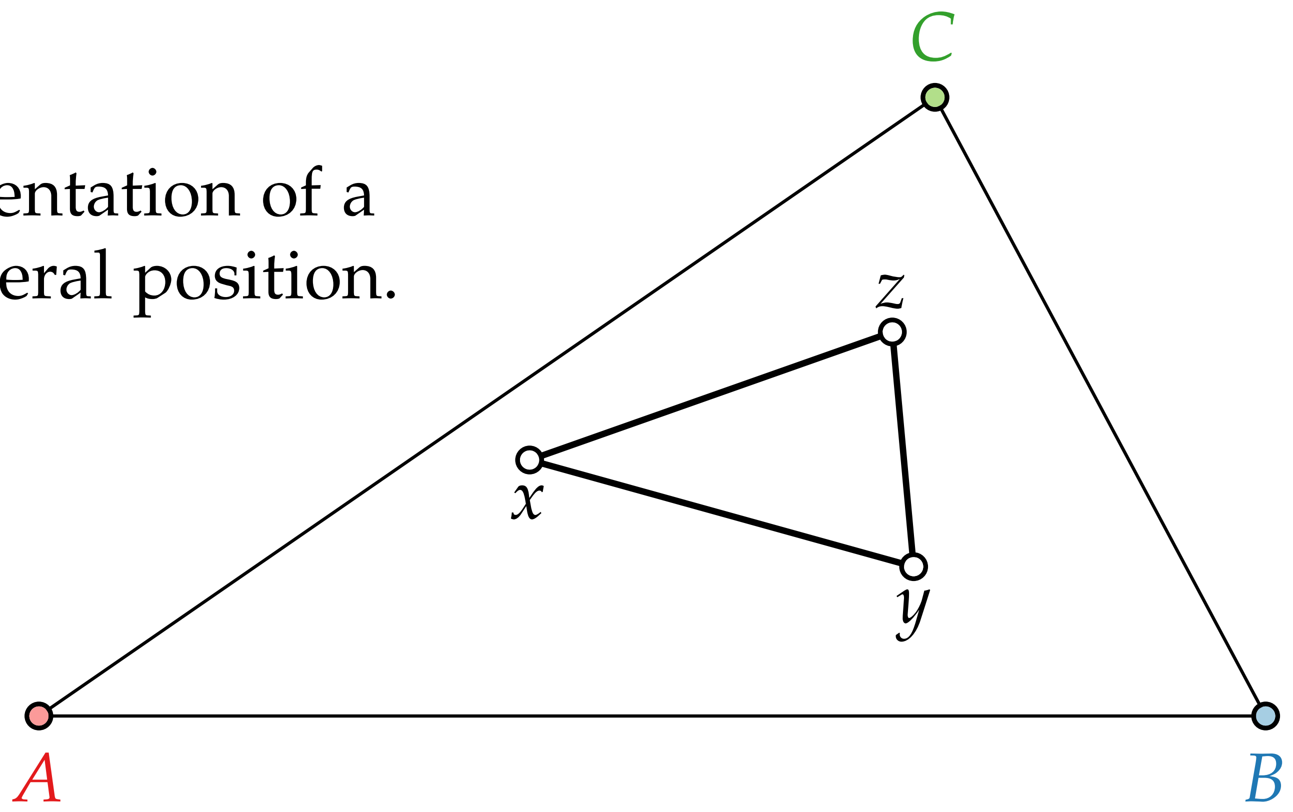

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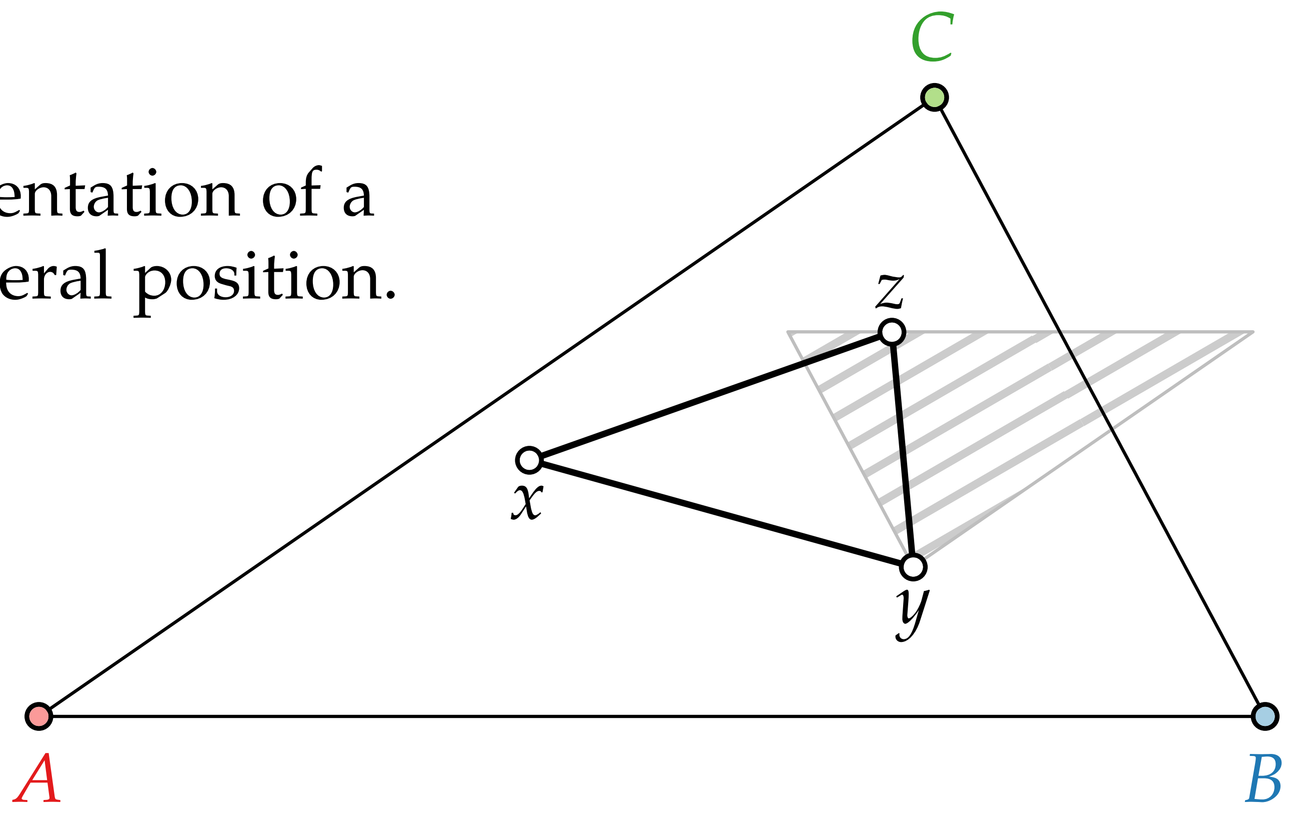

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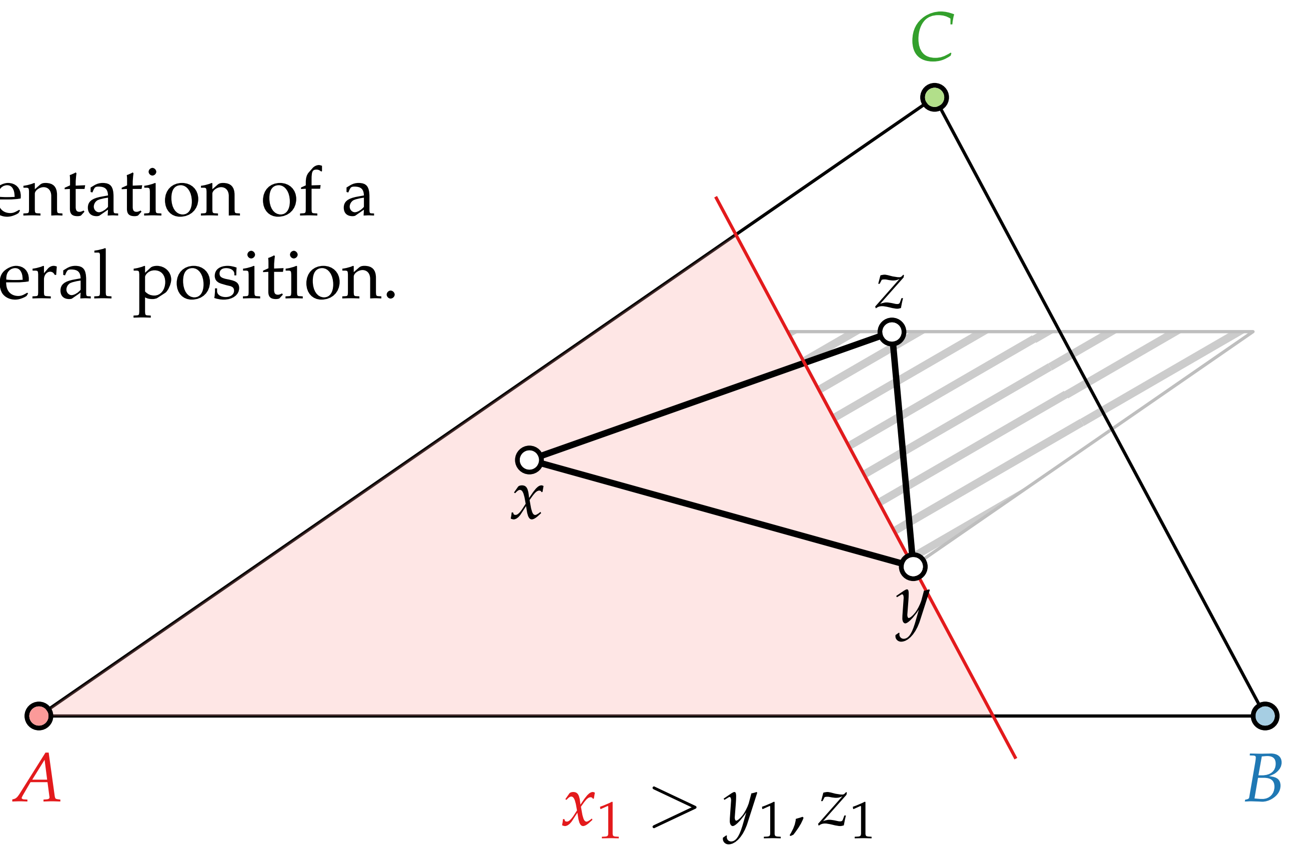

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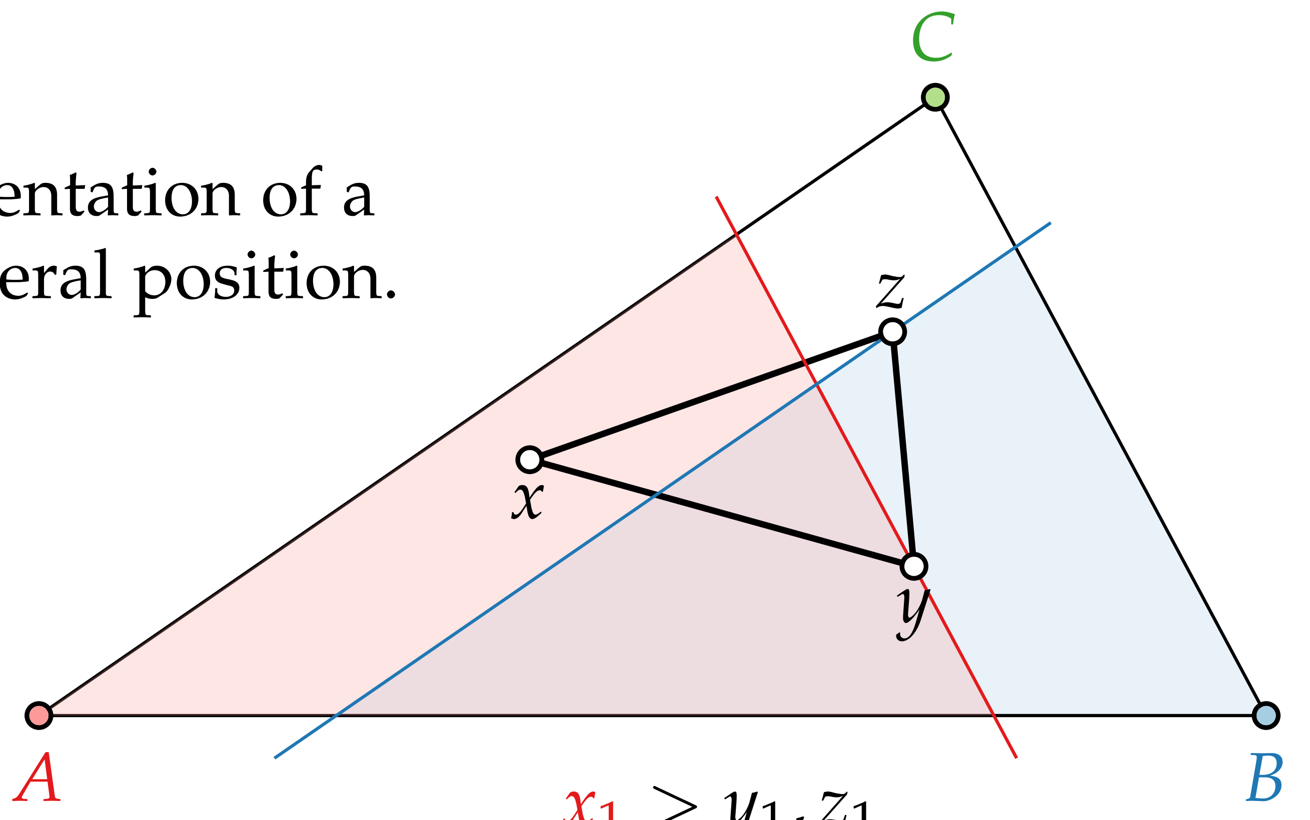

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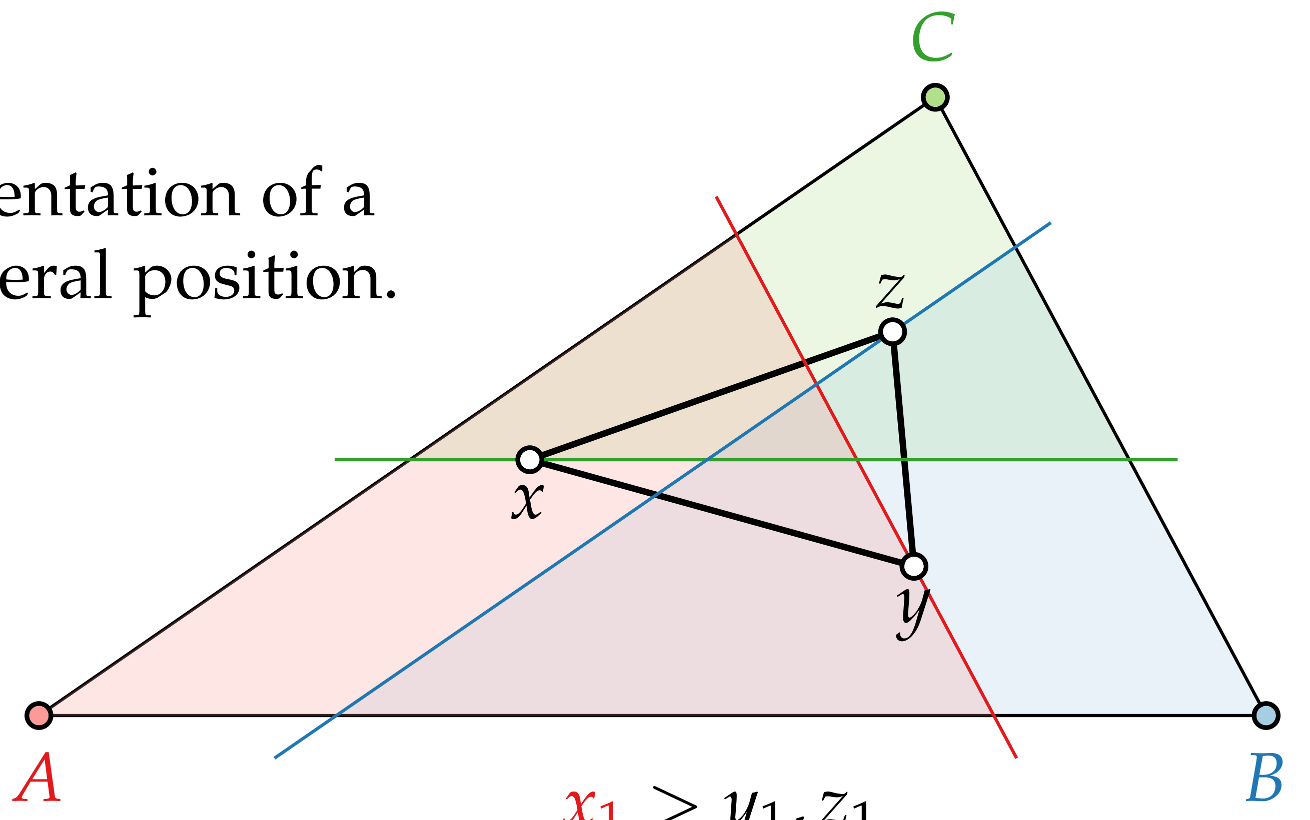

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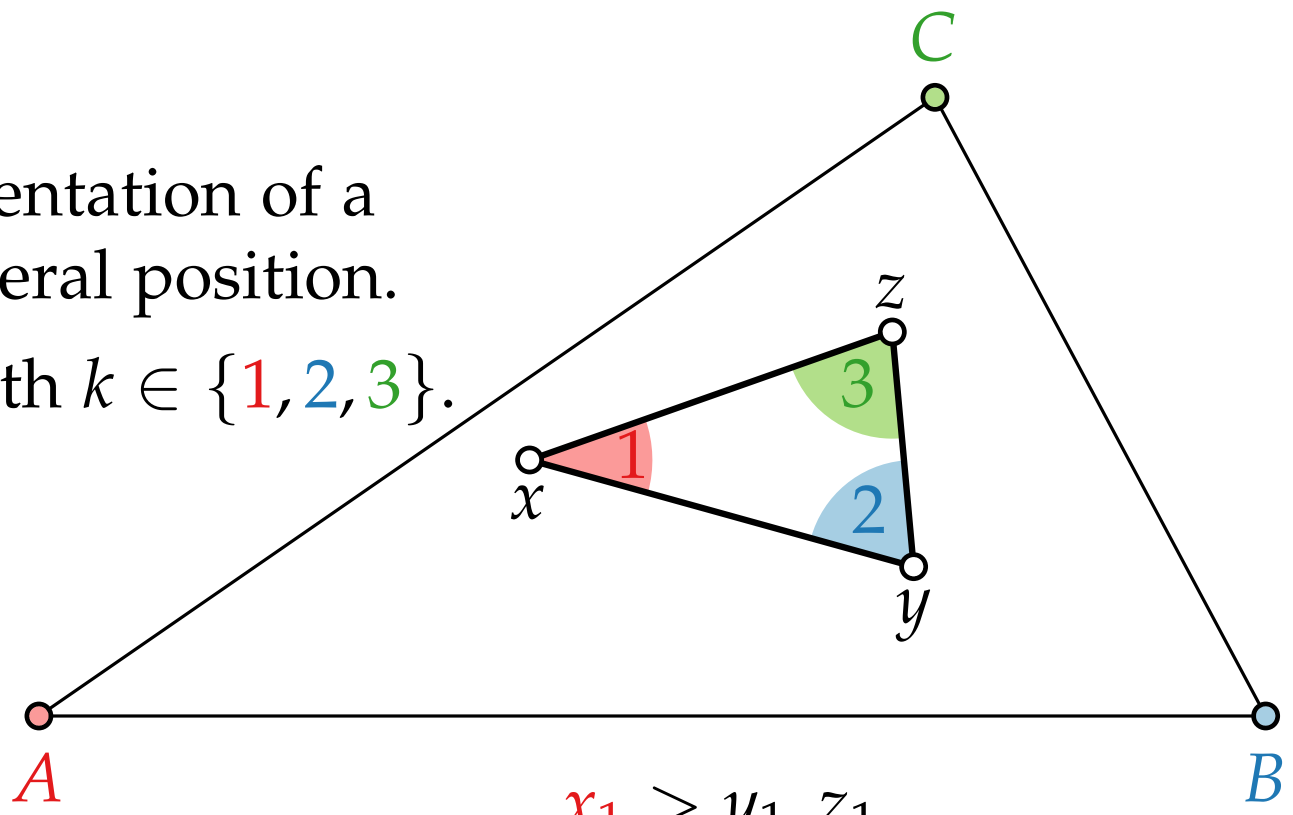

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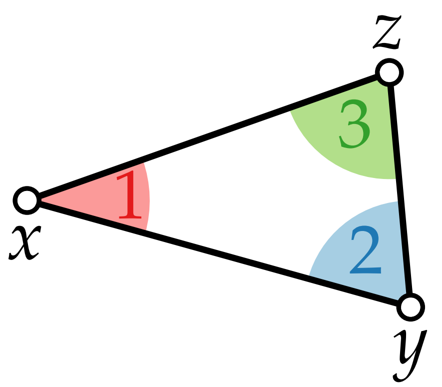

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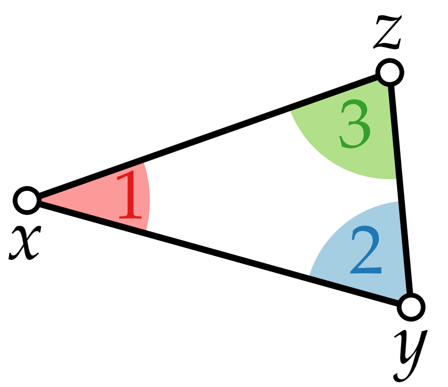

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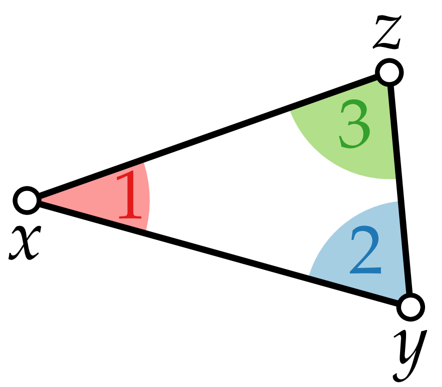

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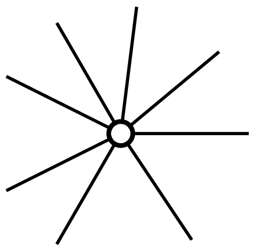
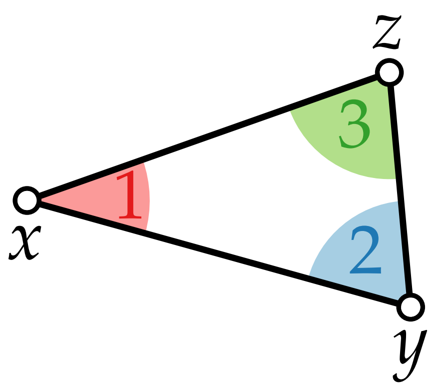


Schnyder Labeling

Let $\phi : v \mapsto (v_1, v_2, v_3)$ be a barycentric representation of a planar graph G and let $A, B, C \in \mathbb{R}^2$ be in general position.

We can label each angle in $\triangle xyz$ **uniquely** with $k \in \{1, 2, 3\}$.



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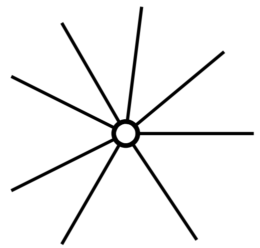

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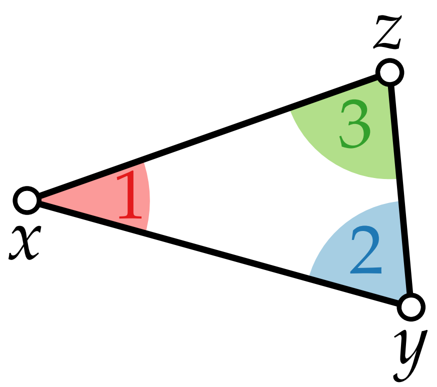

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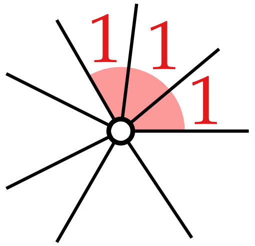

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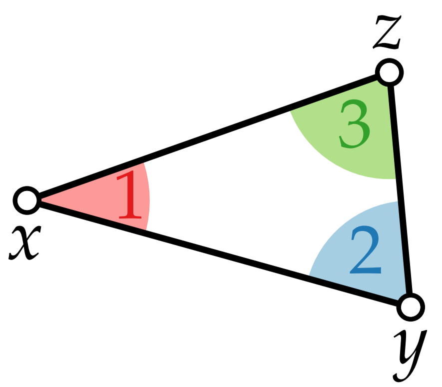

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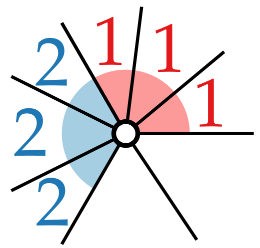
- followed by a nonempty interval of 2's



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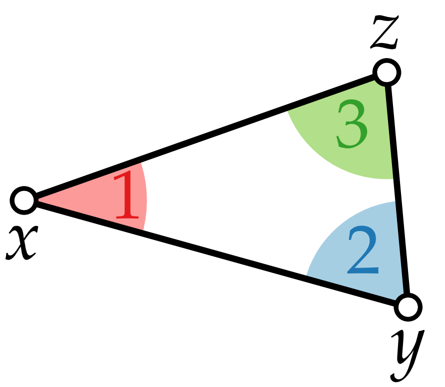

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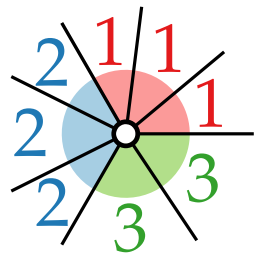
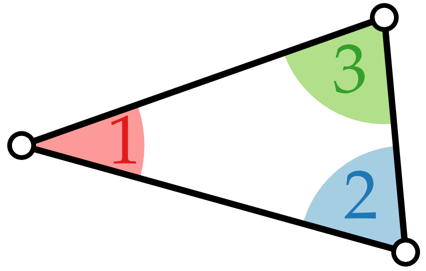
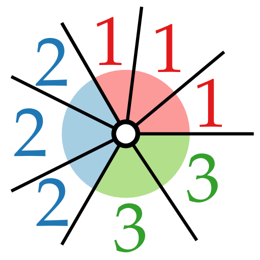


Schnyder Realizer

A Schnyder labeling induces an edge labeling.





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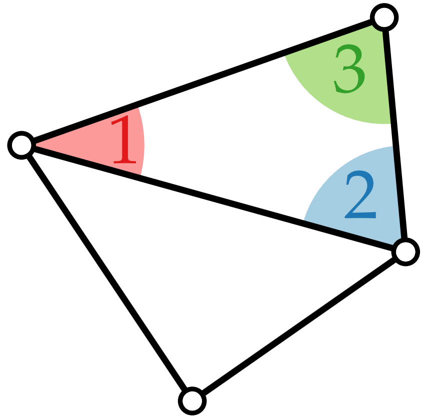

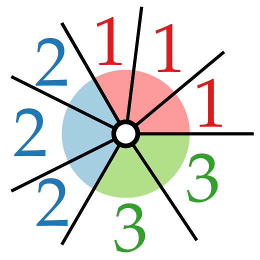

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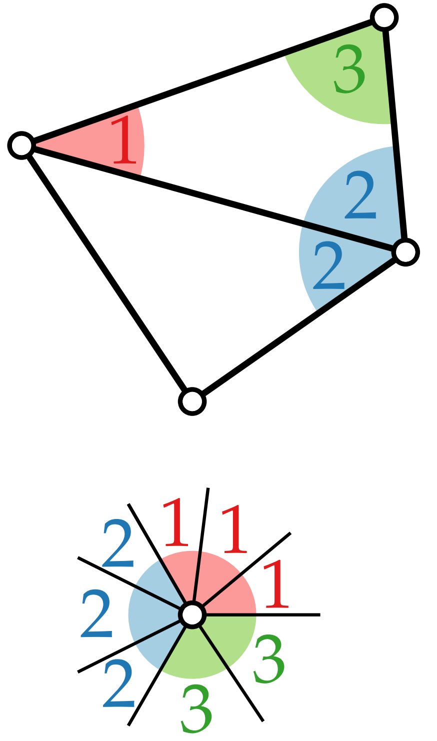

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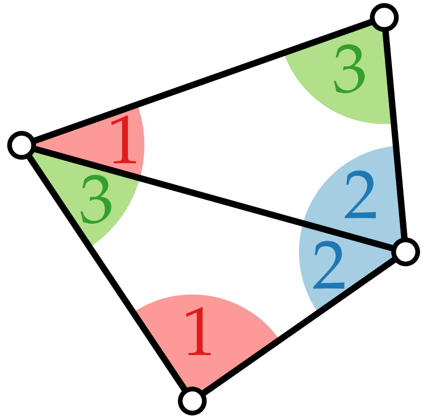

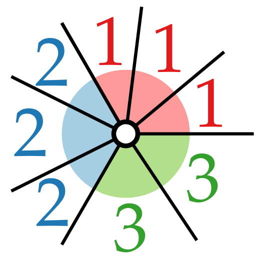

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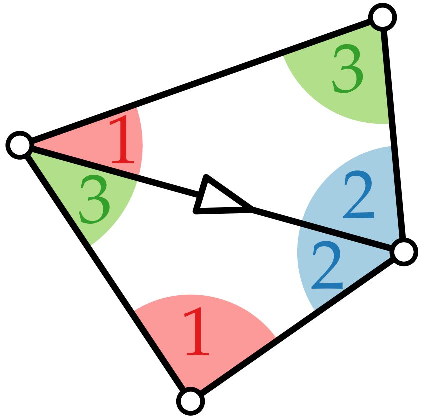

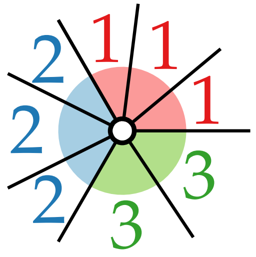

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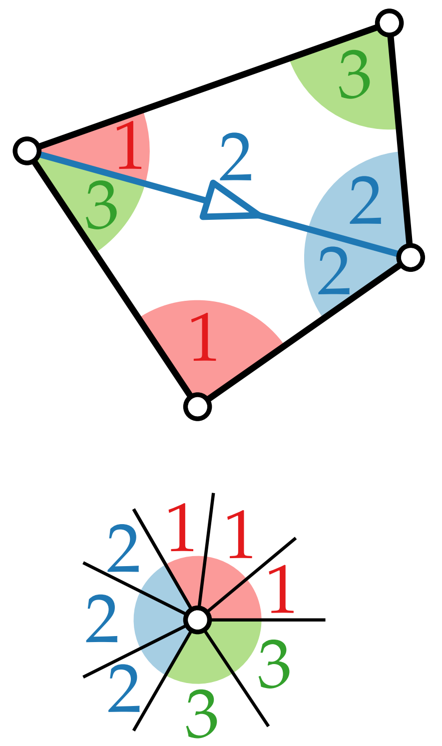

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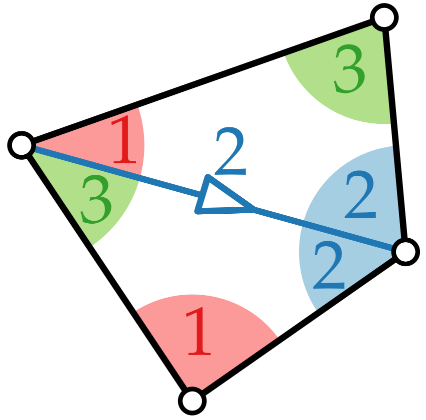

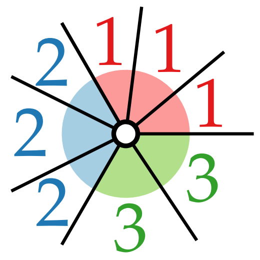

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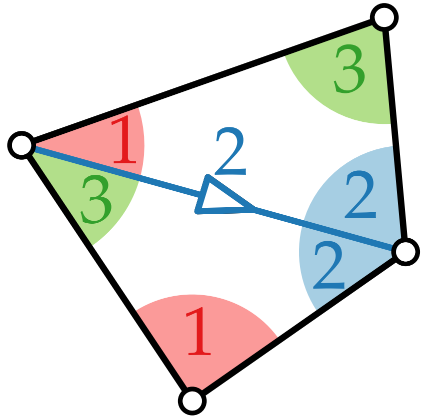

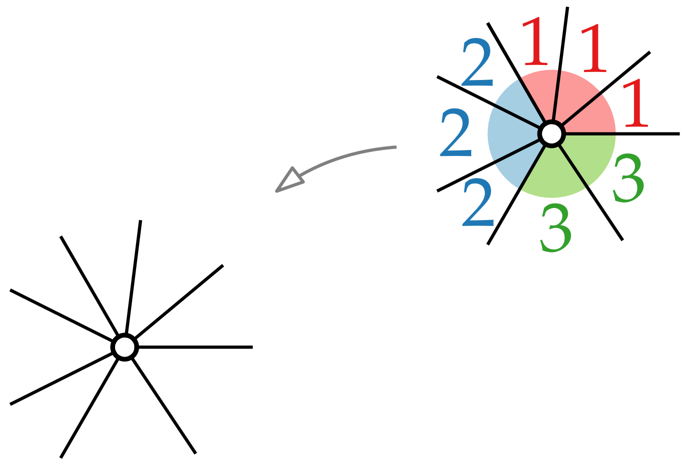

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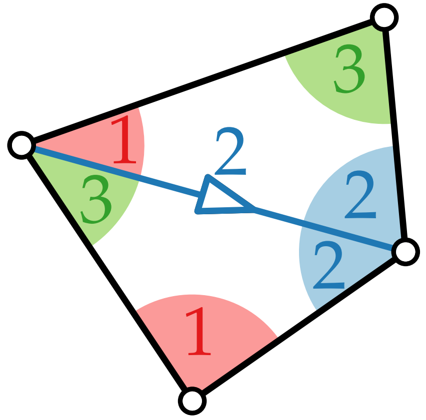

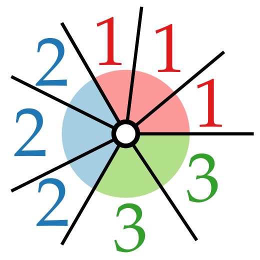

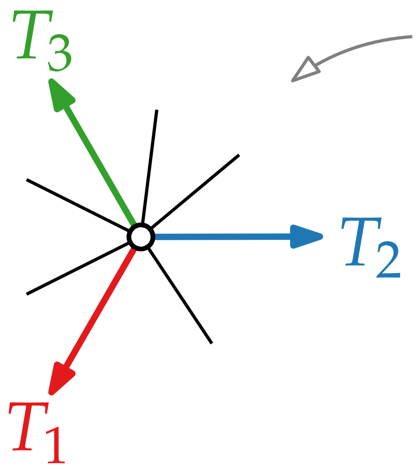

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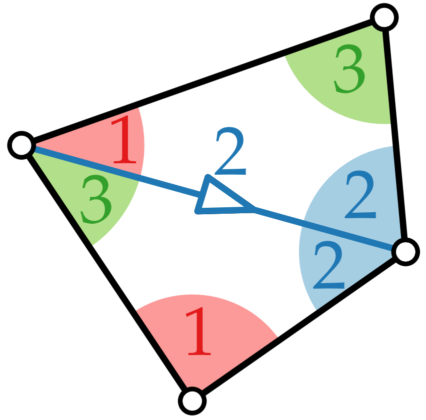

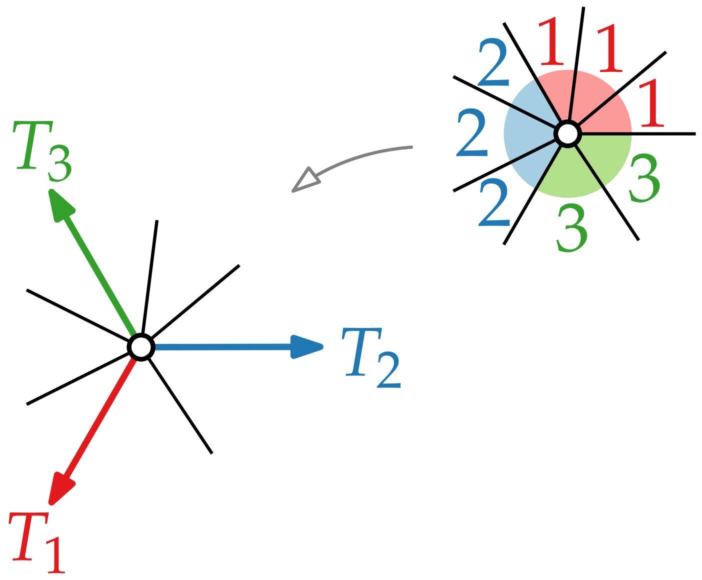

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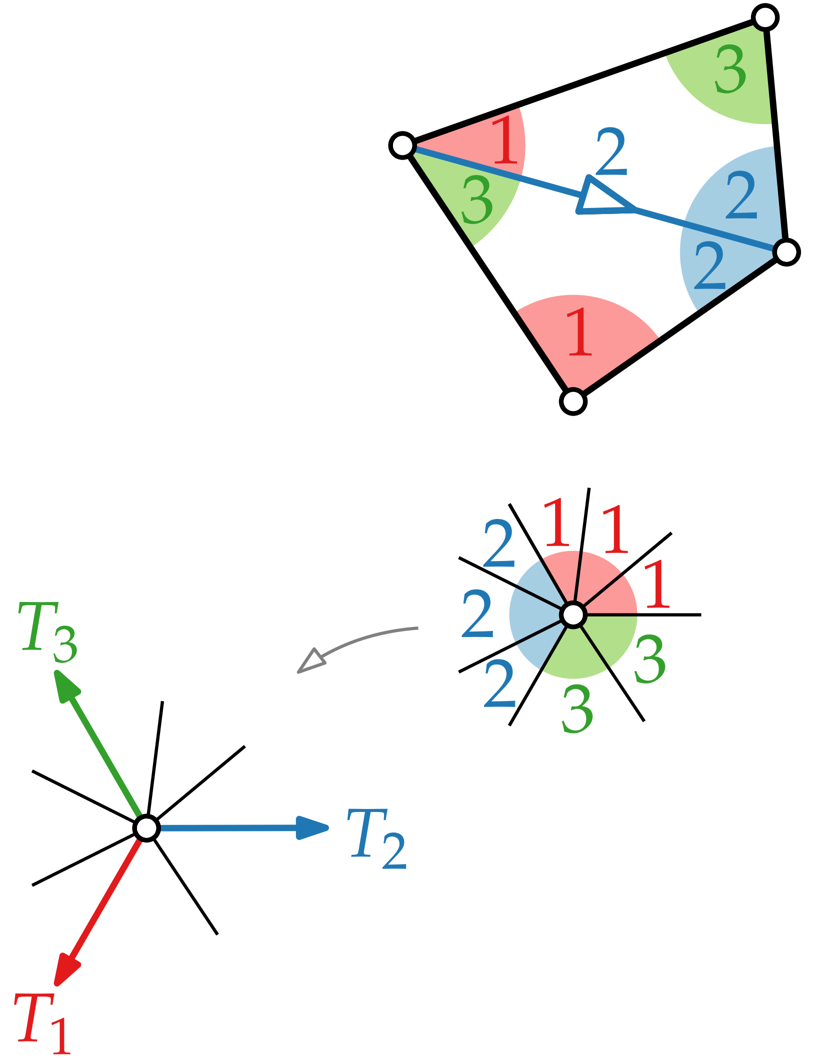
  leaving in T_1



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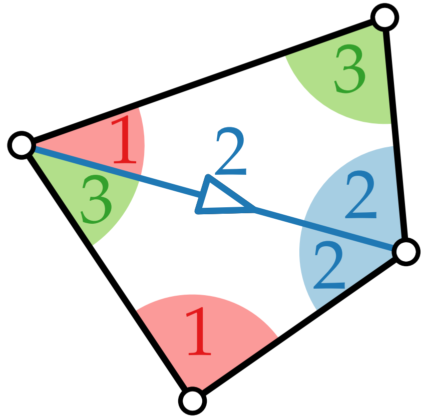

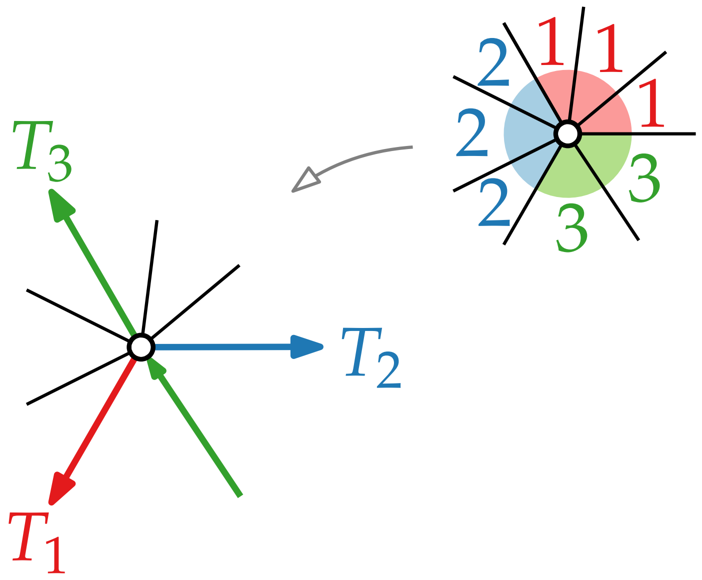

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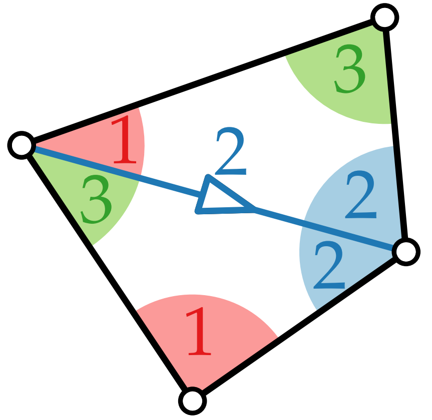

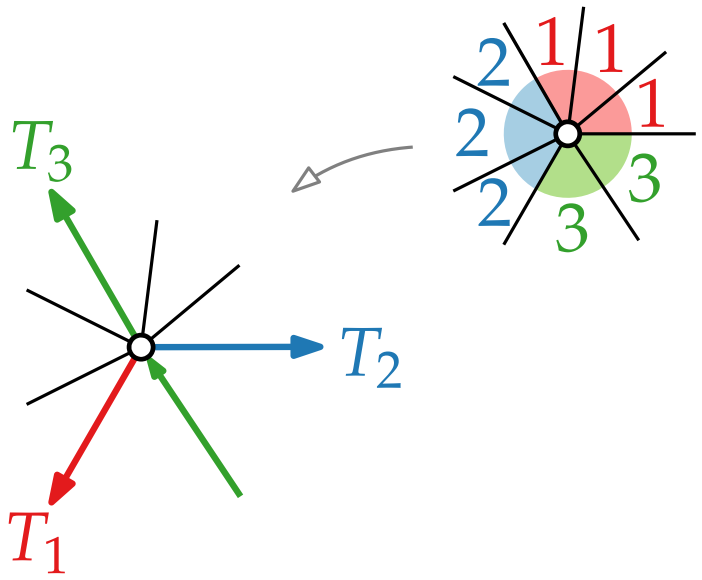

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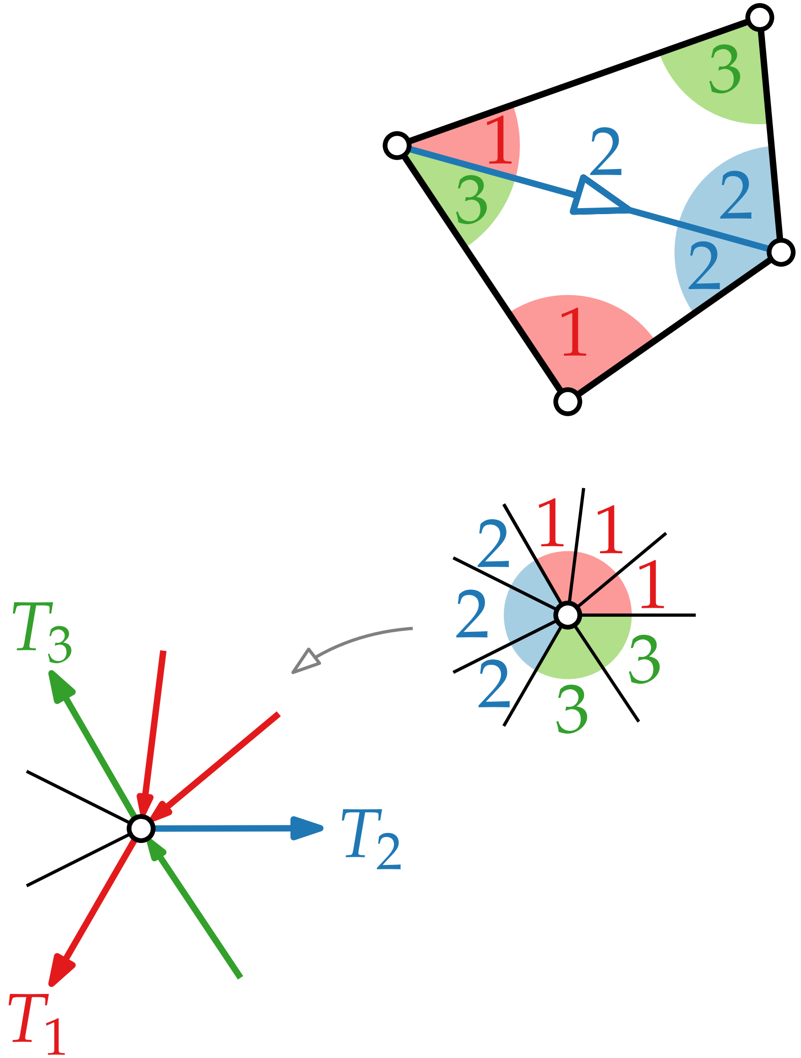

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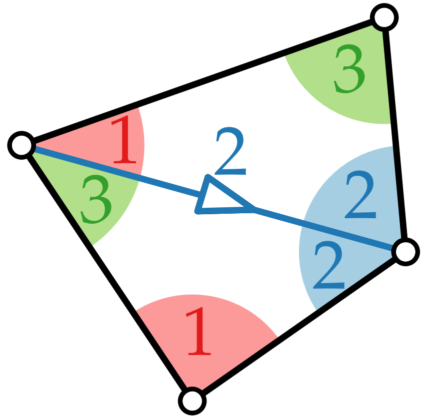

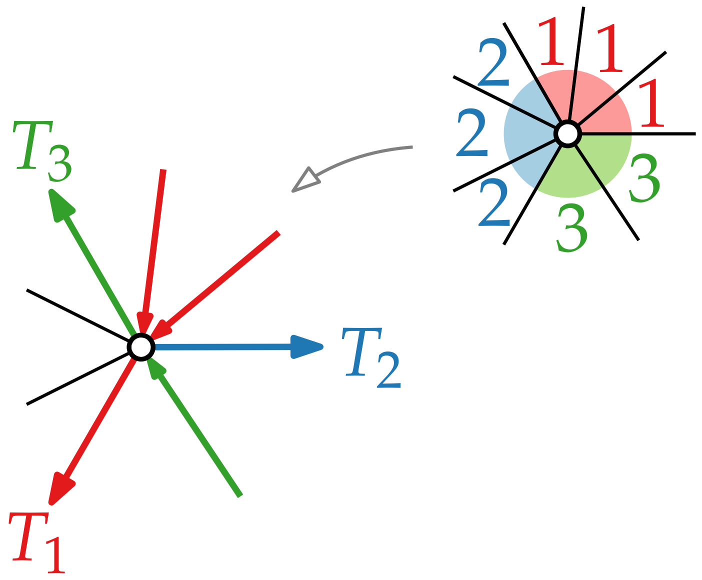

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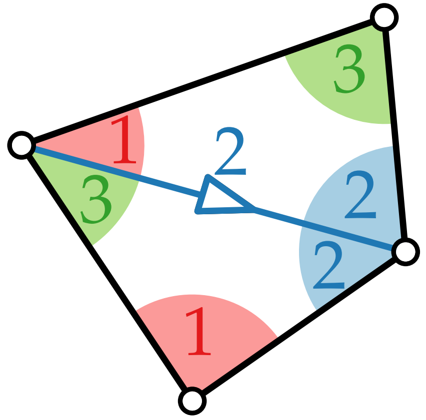
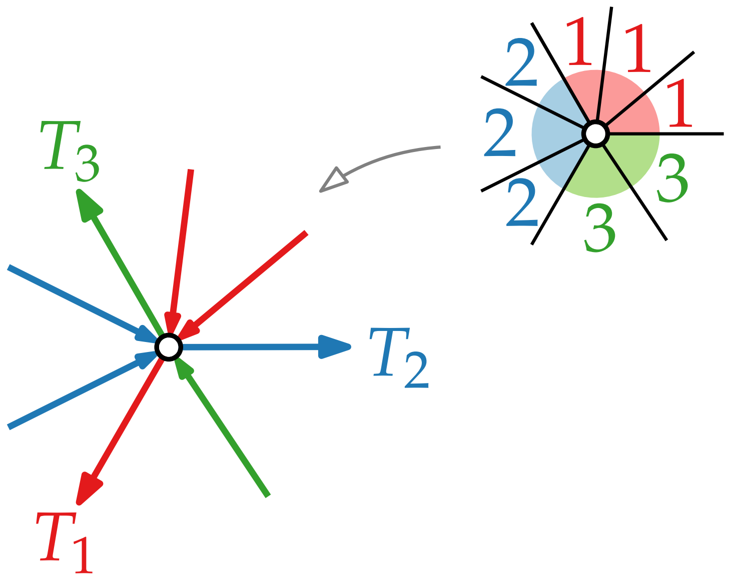
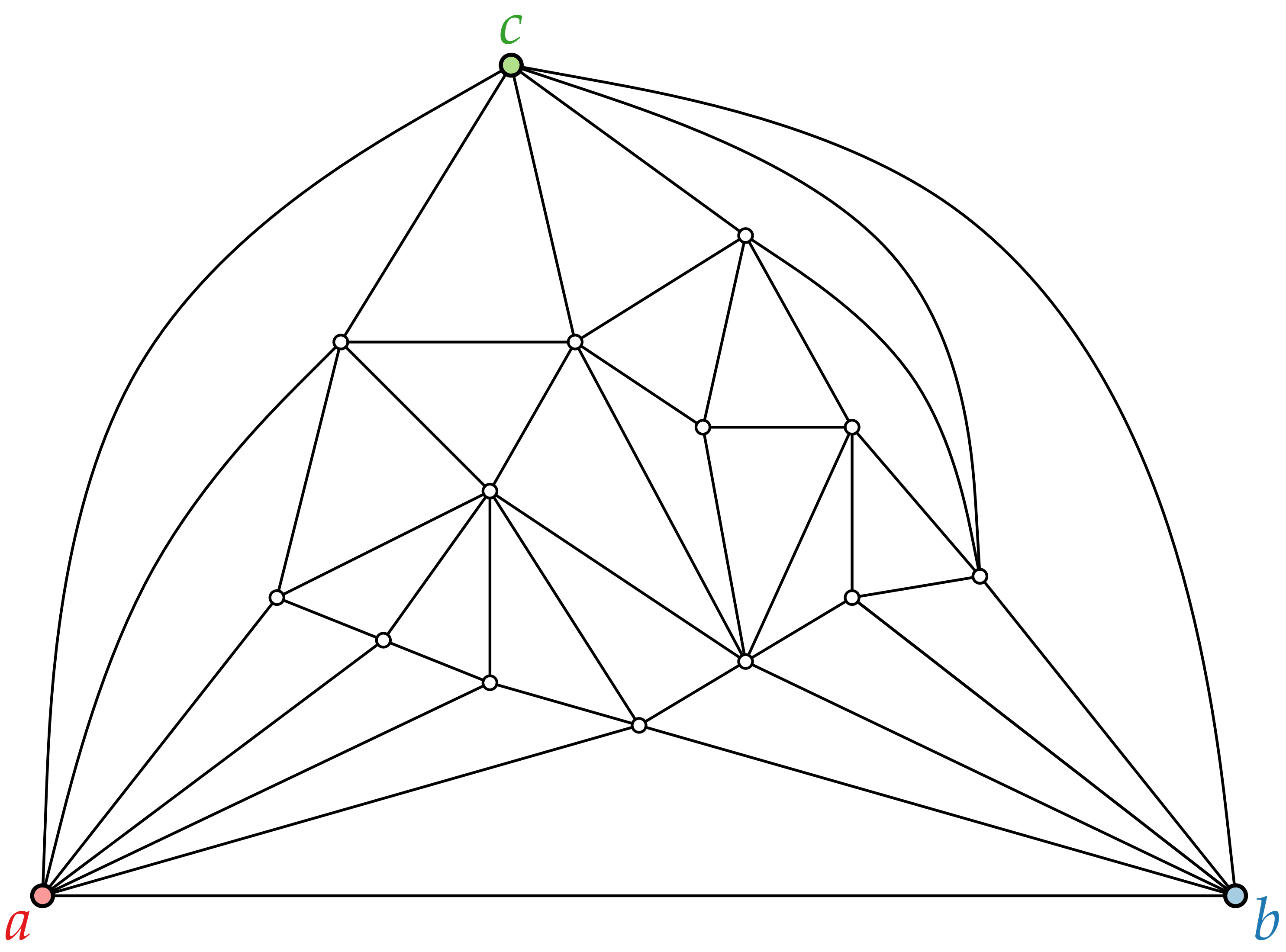
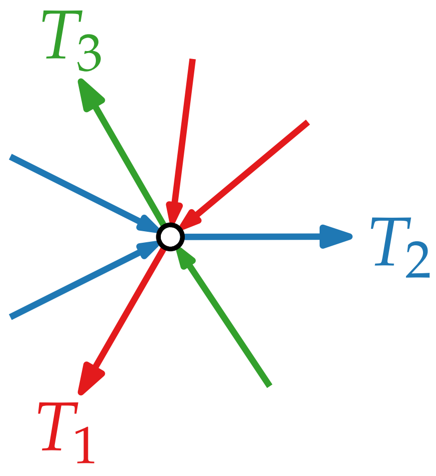




Schnyder Realizer – Example and Properties





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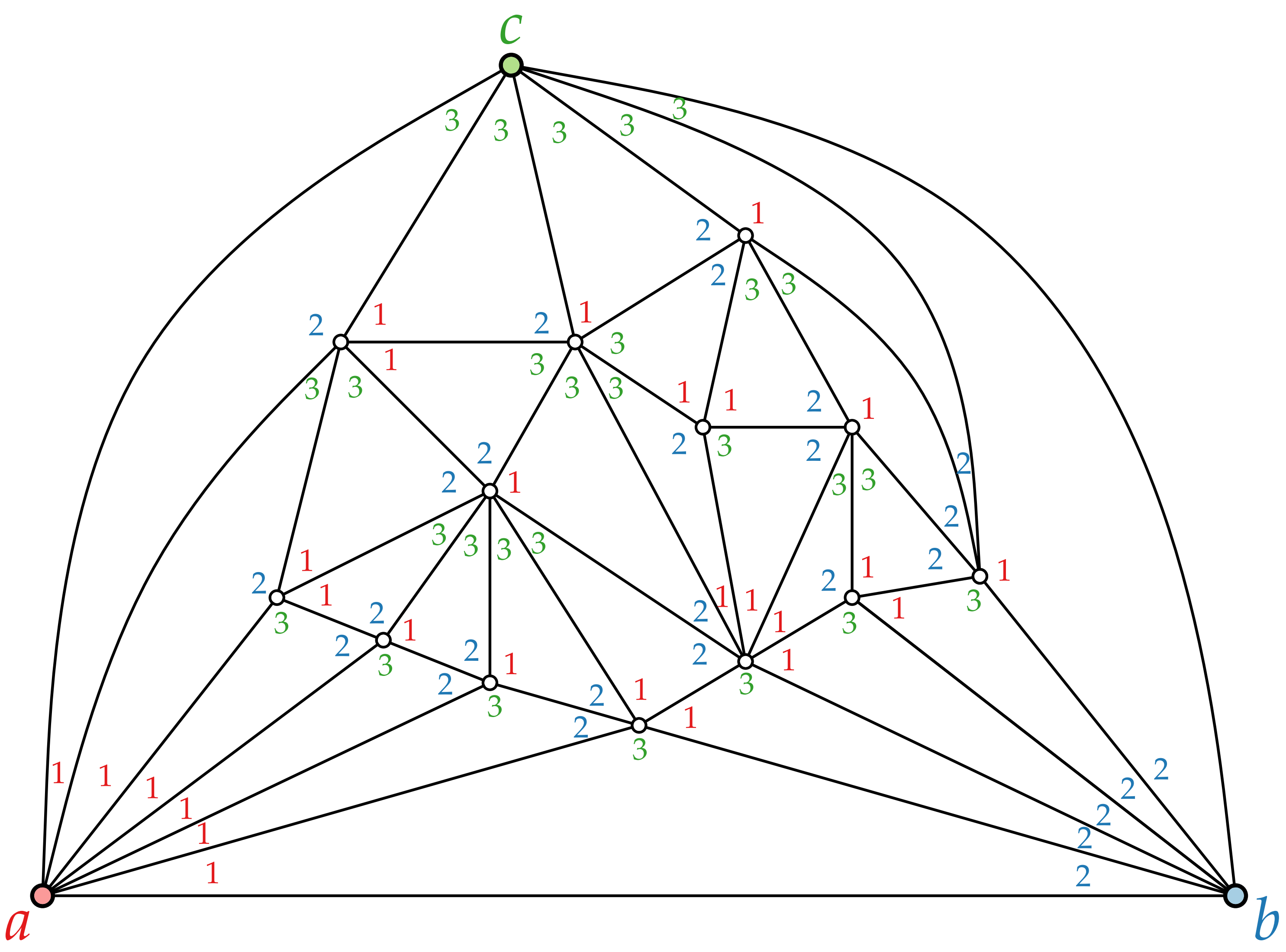

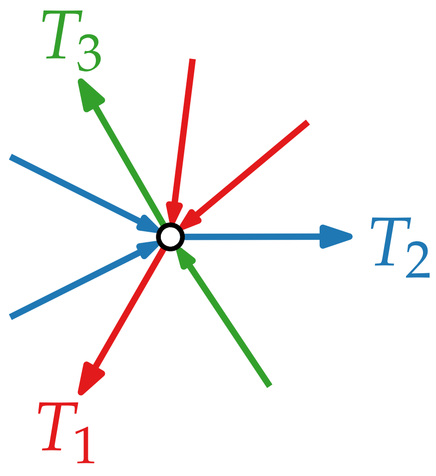

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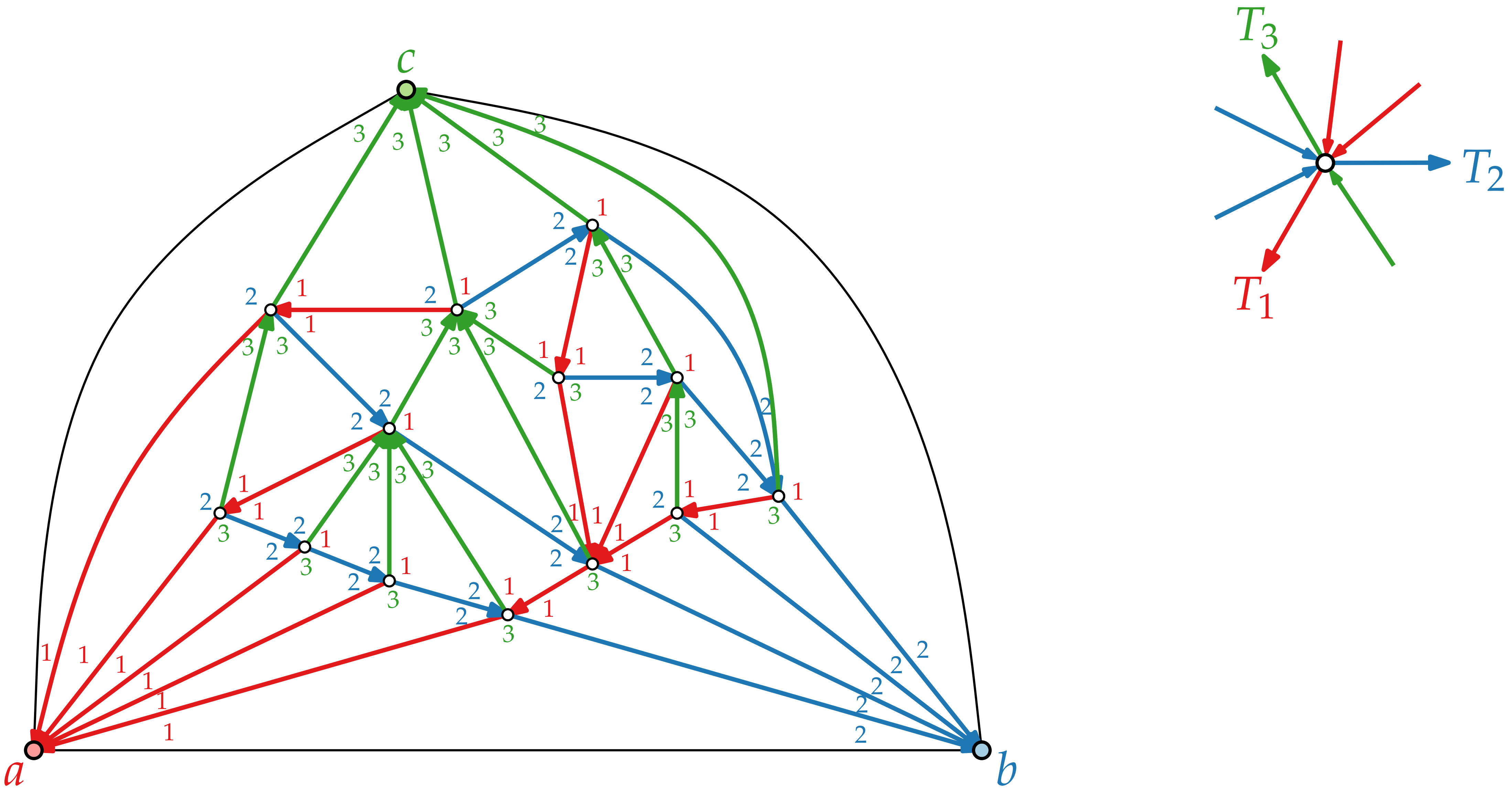

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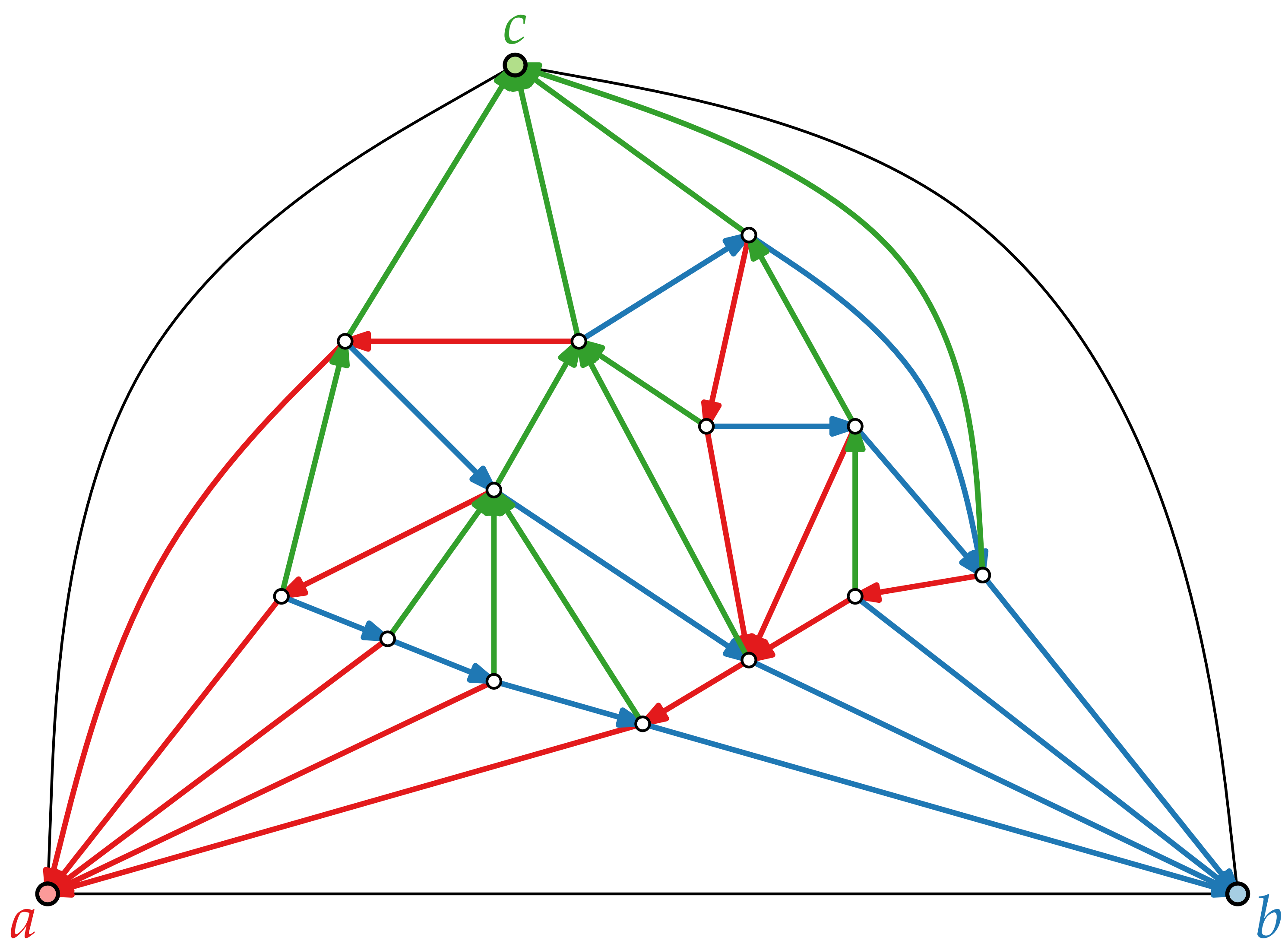

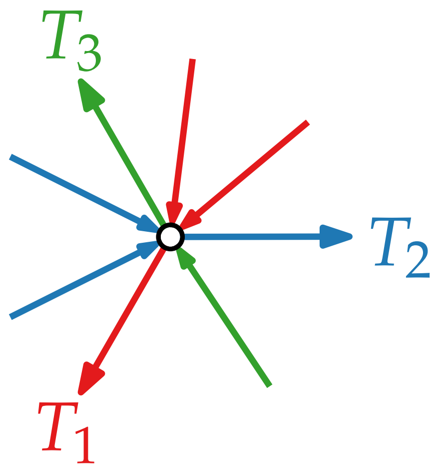

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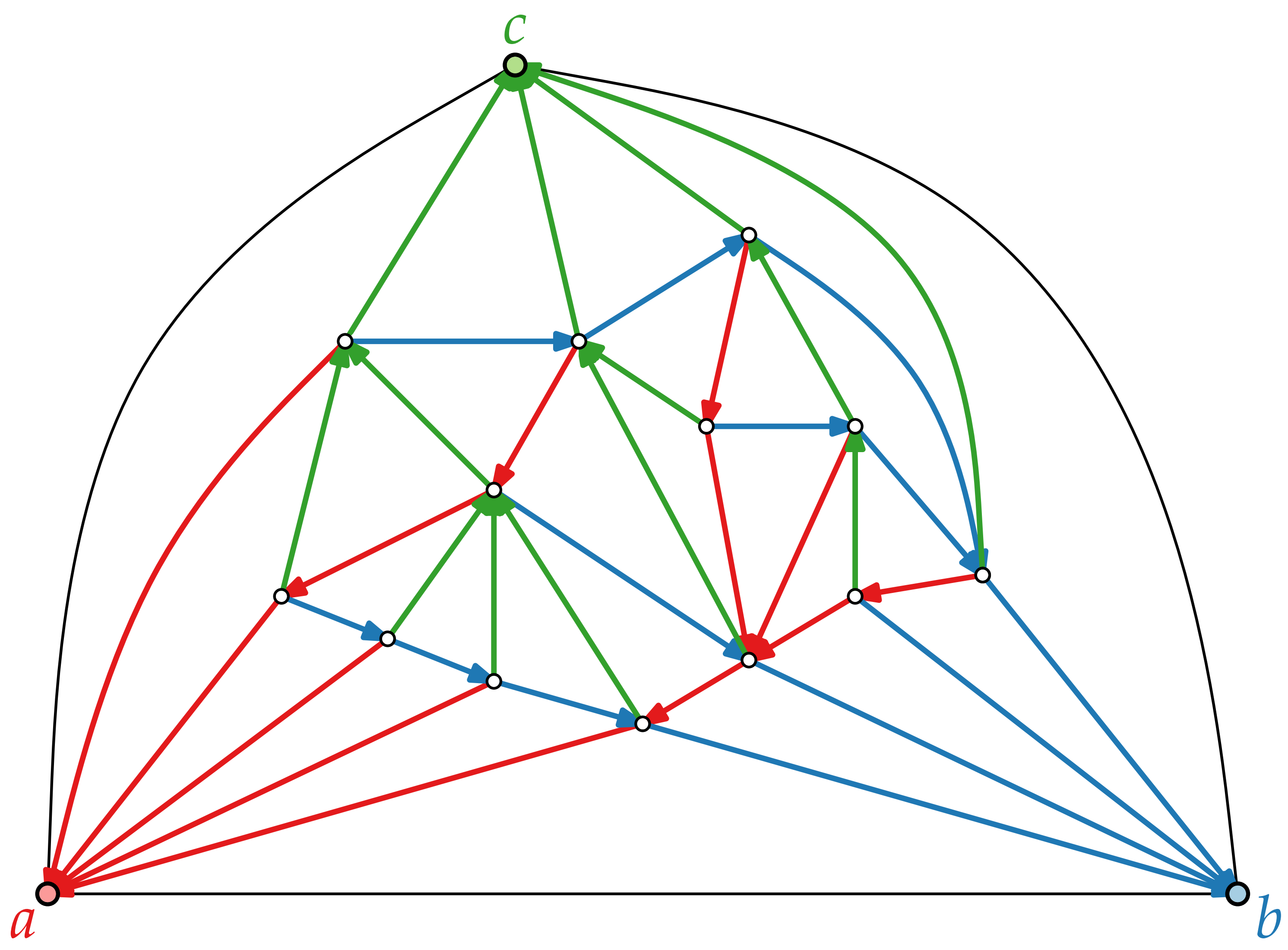

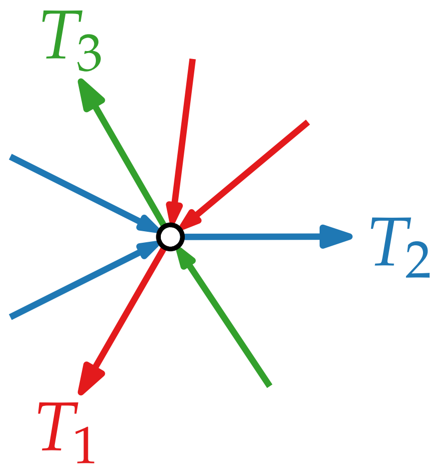

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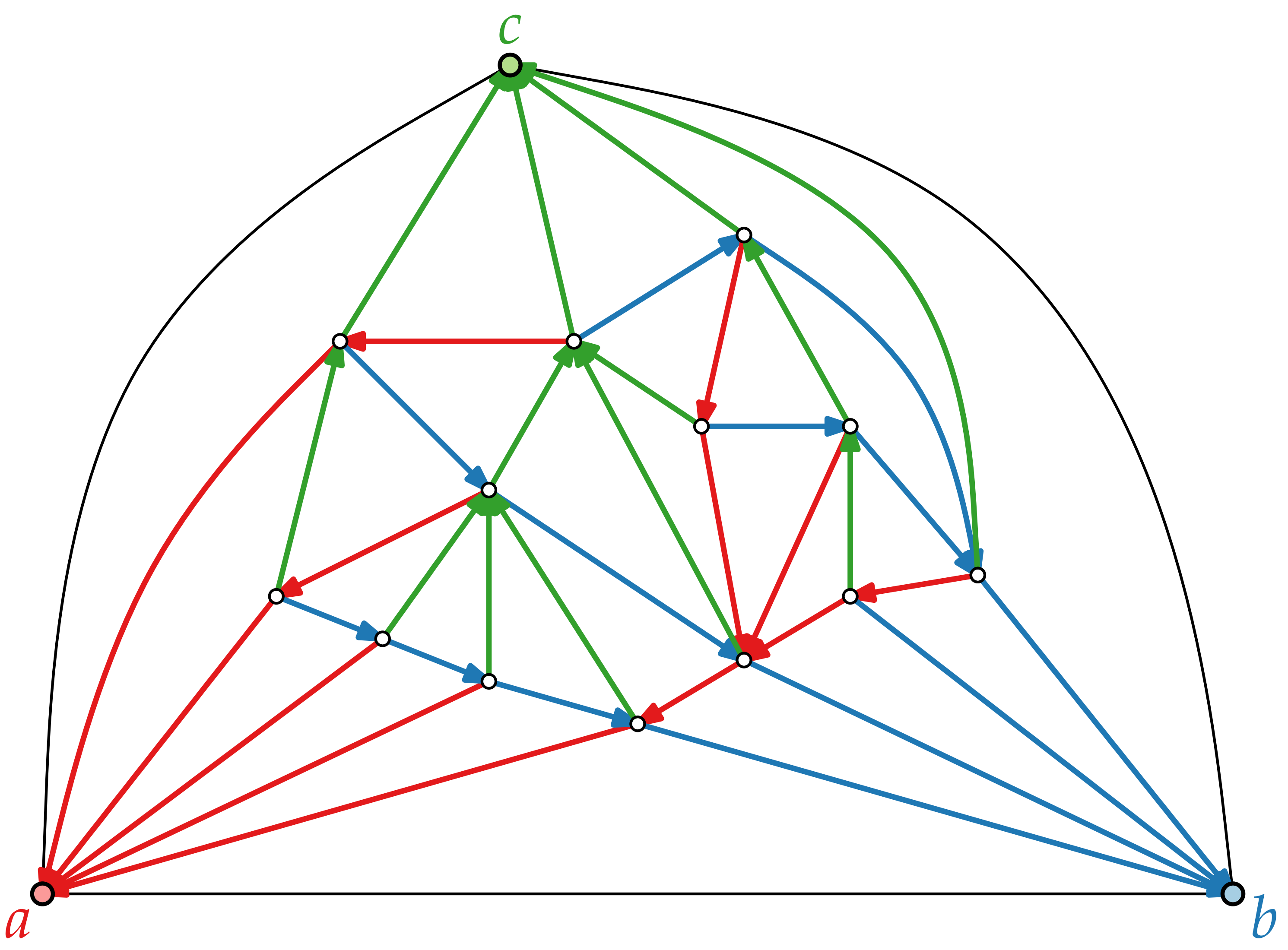

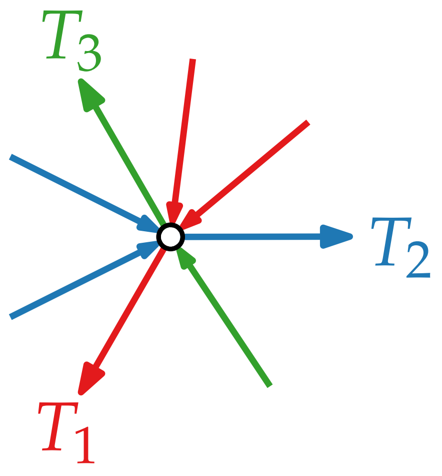

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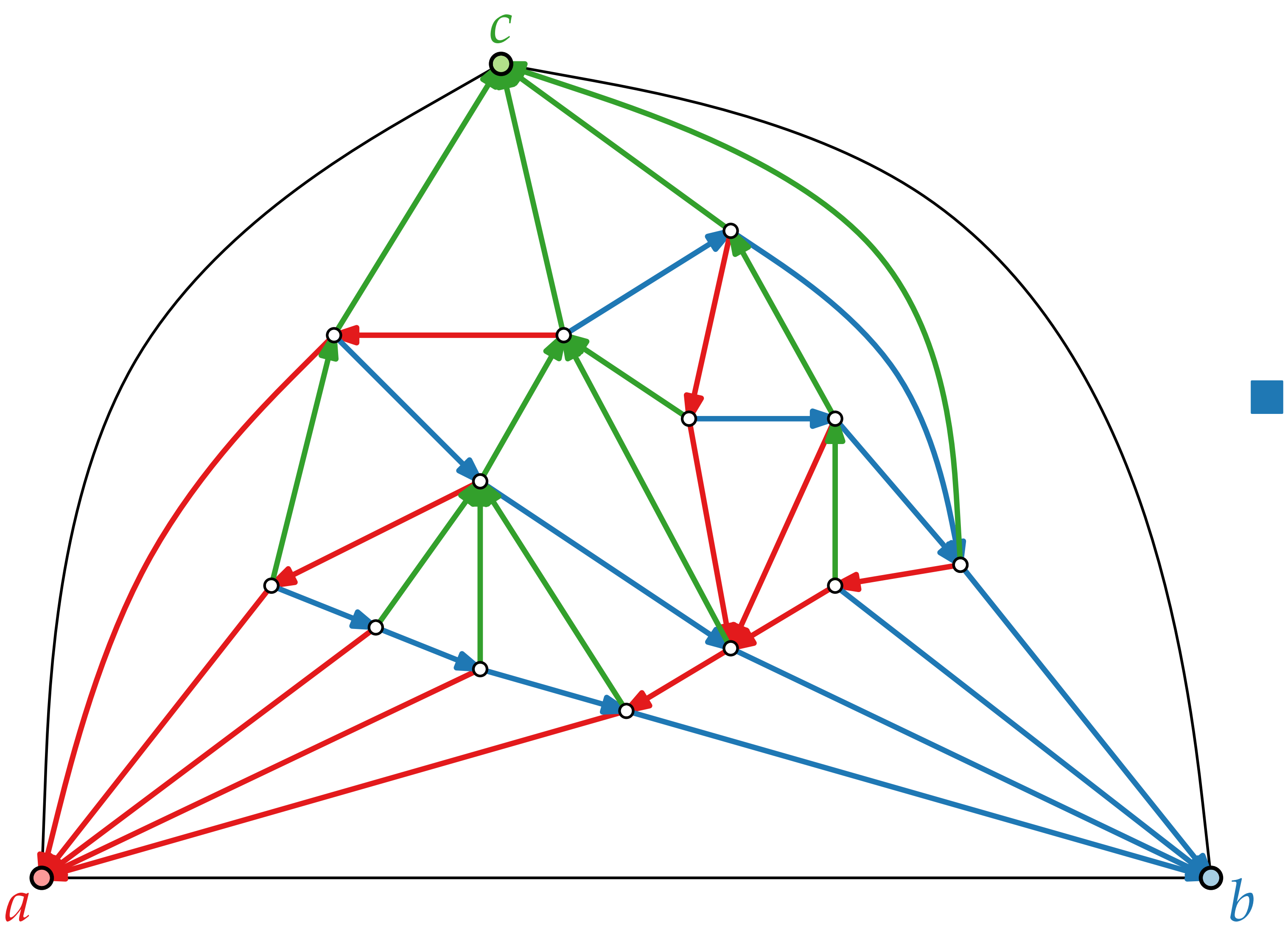

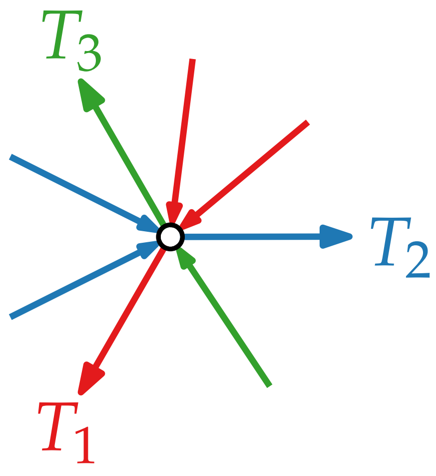

- All inner edges incident to a, b, and c are incoming in the same color.

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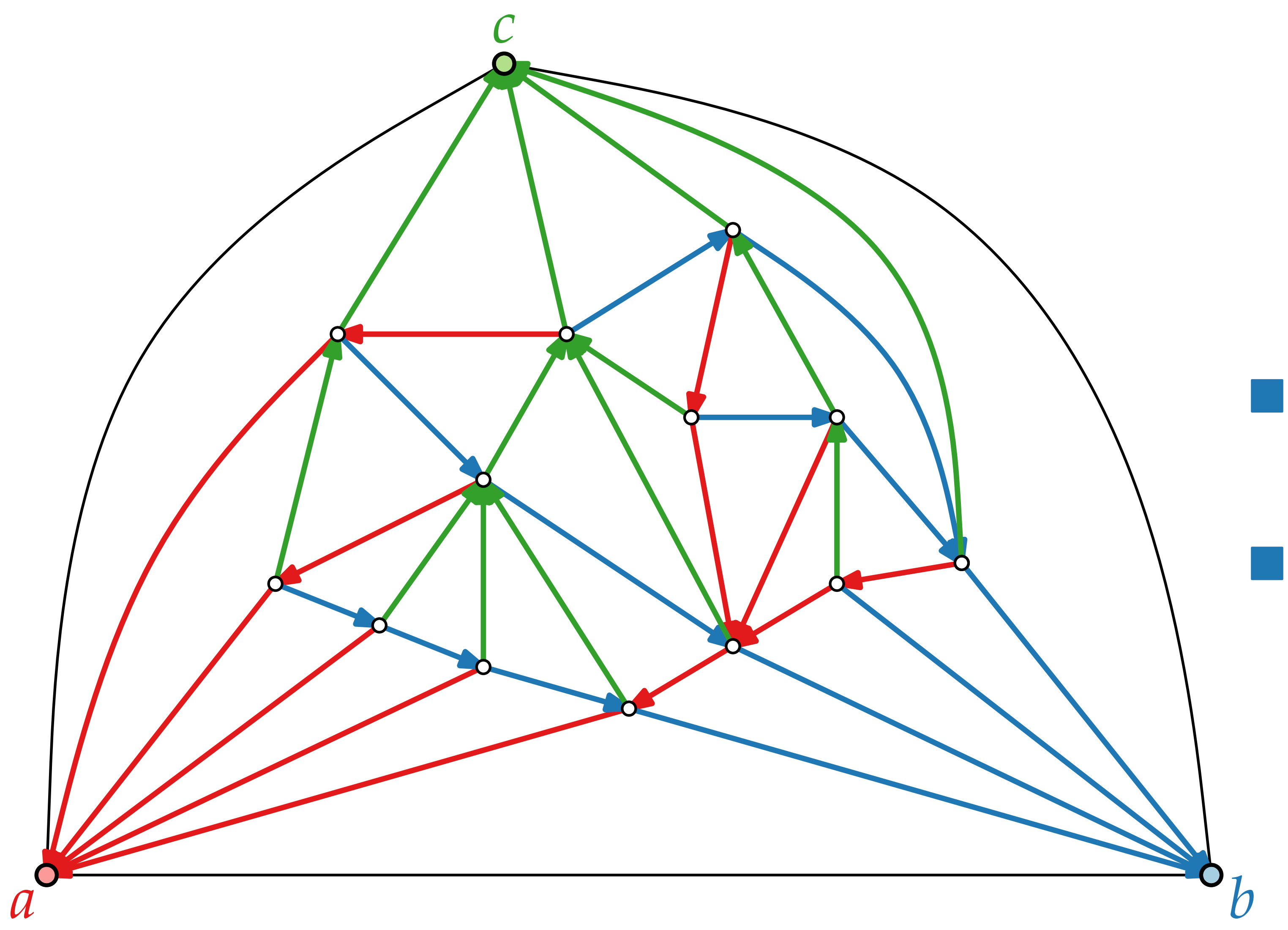

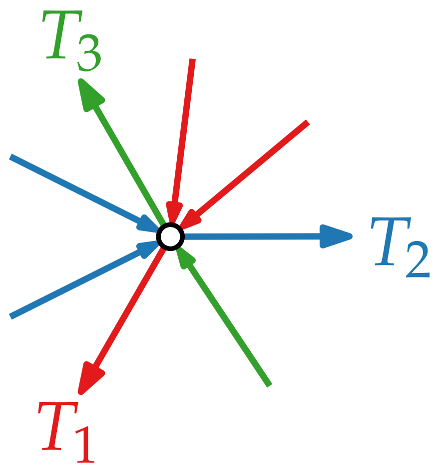

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- T_1, T_2, and T_3 are trees on all inner vertices and one outer vertex each (as its root).

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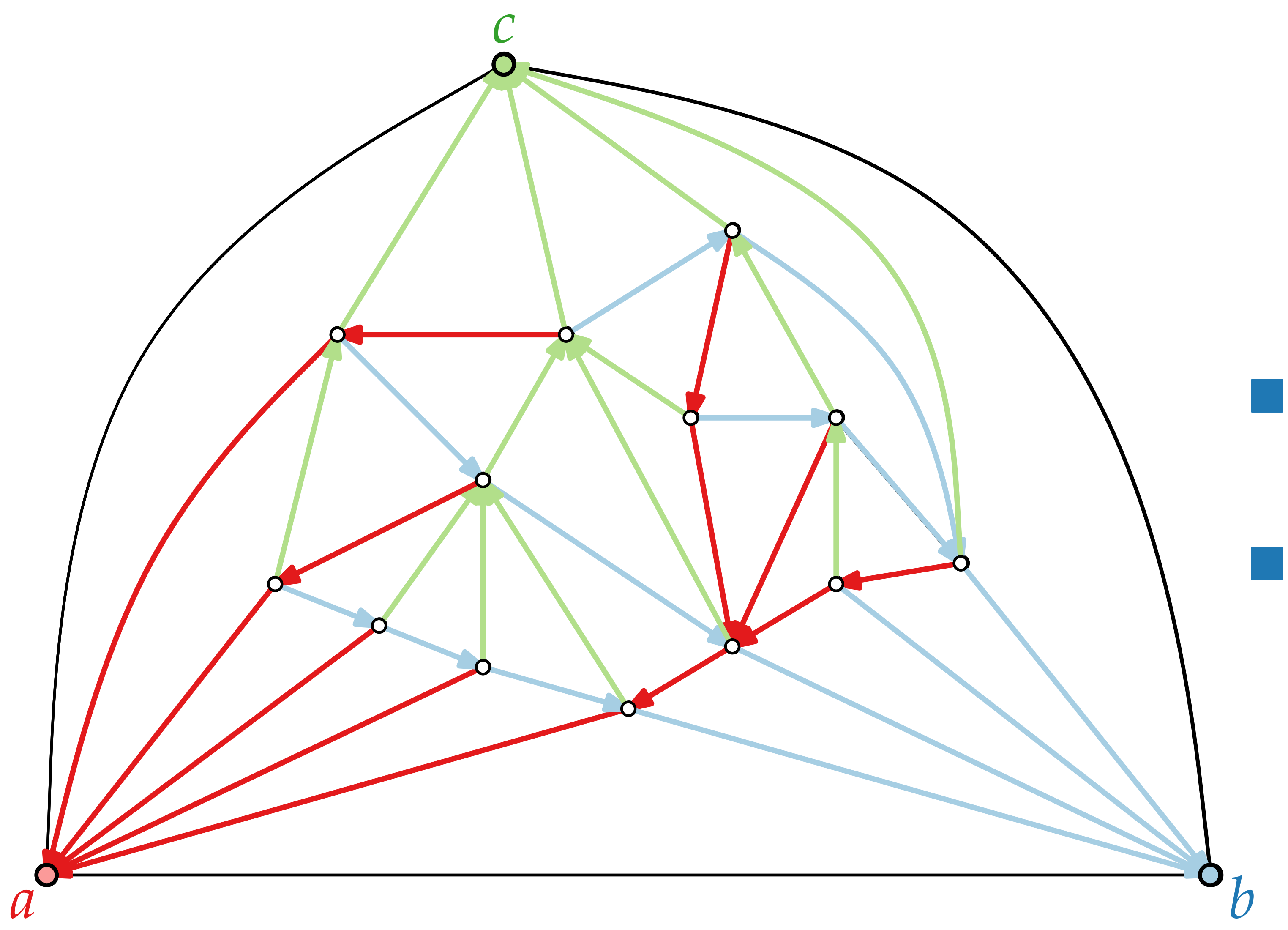

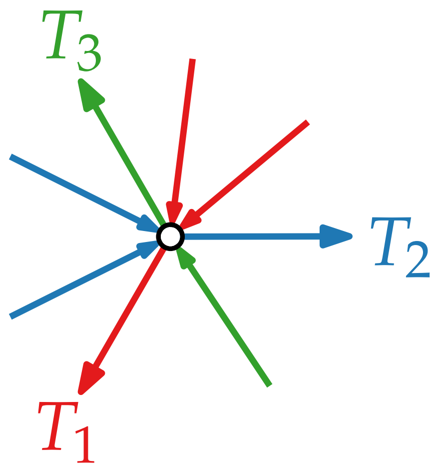

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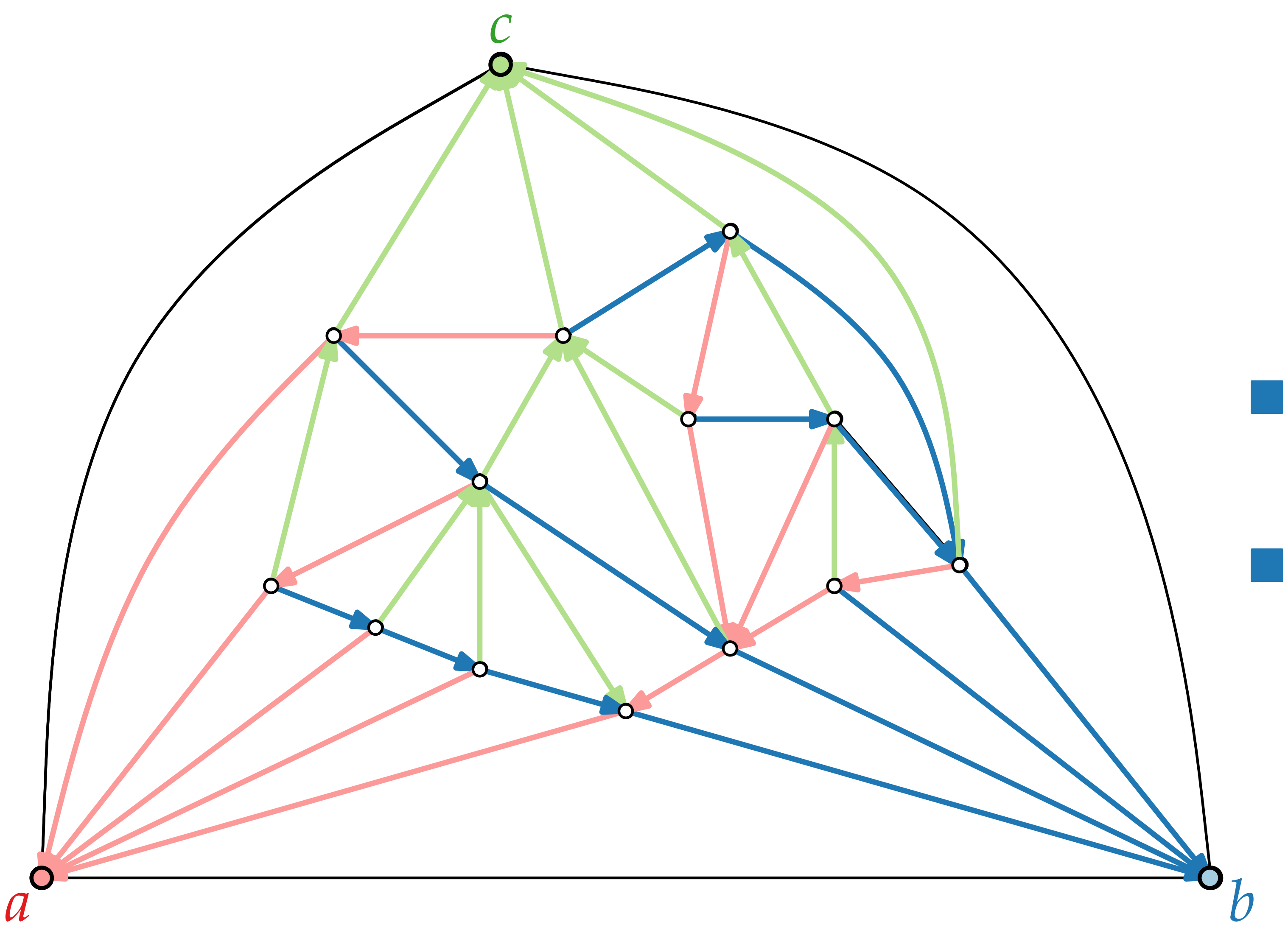

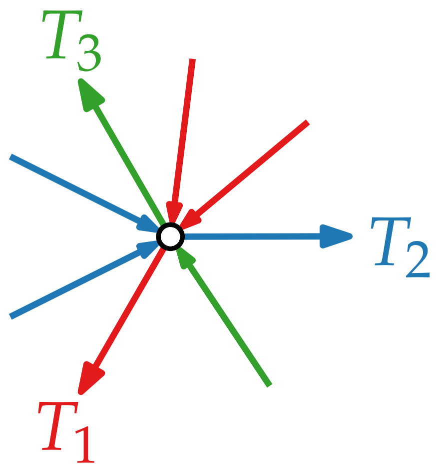

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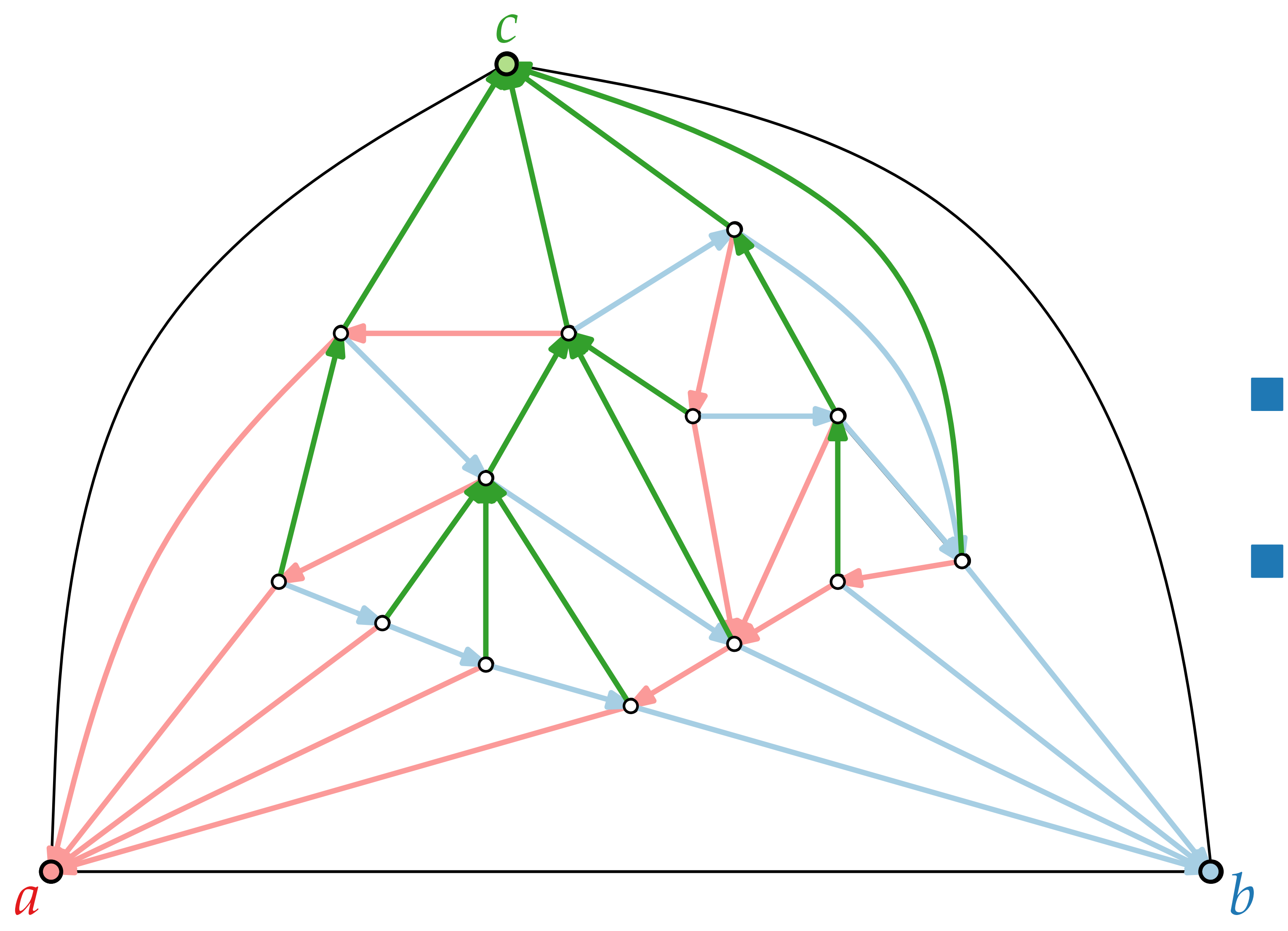

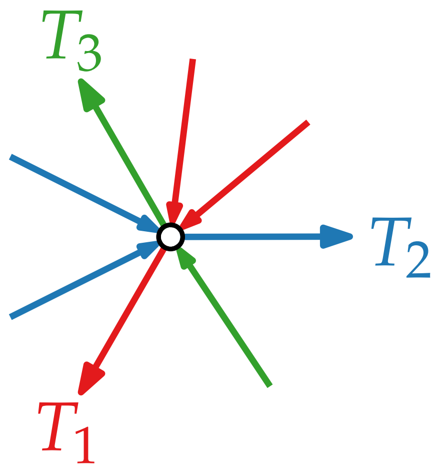

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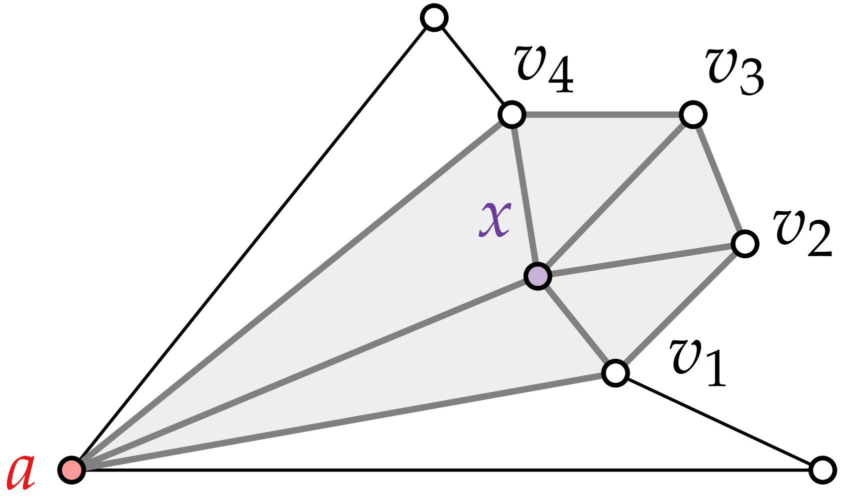

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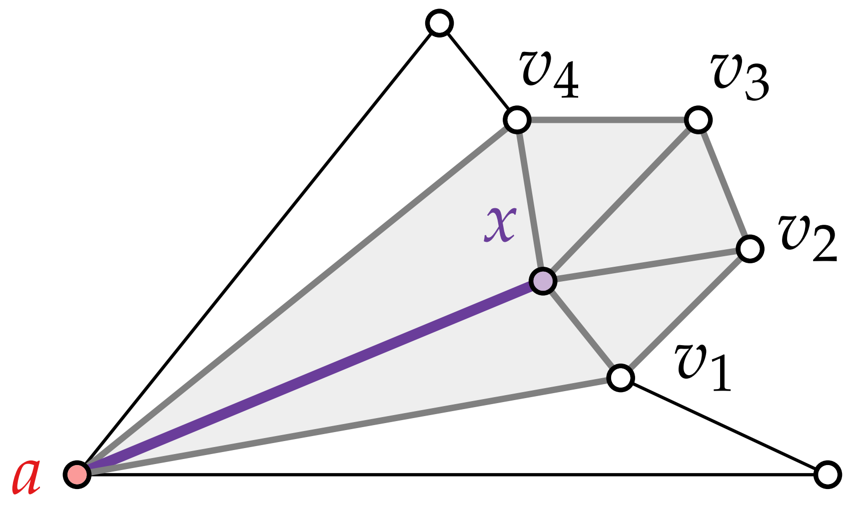

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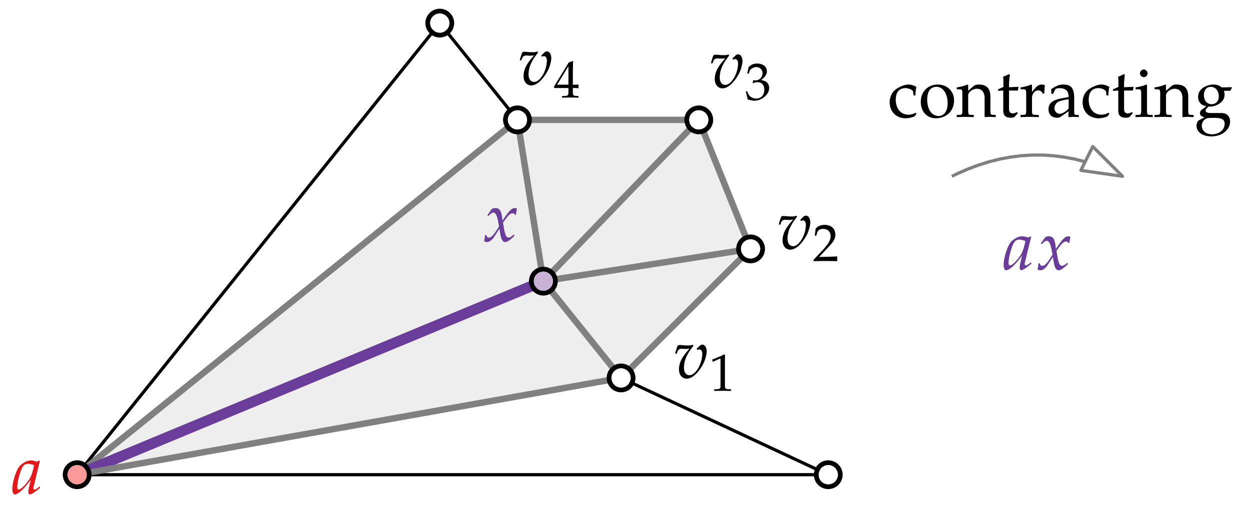

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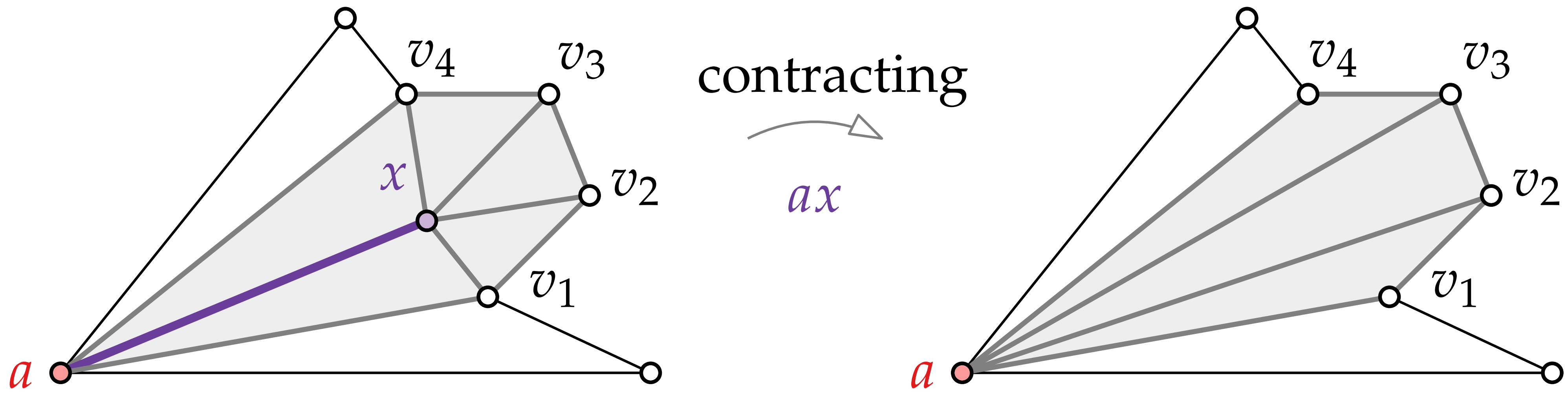

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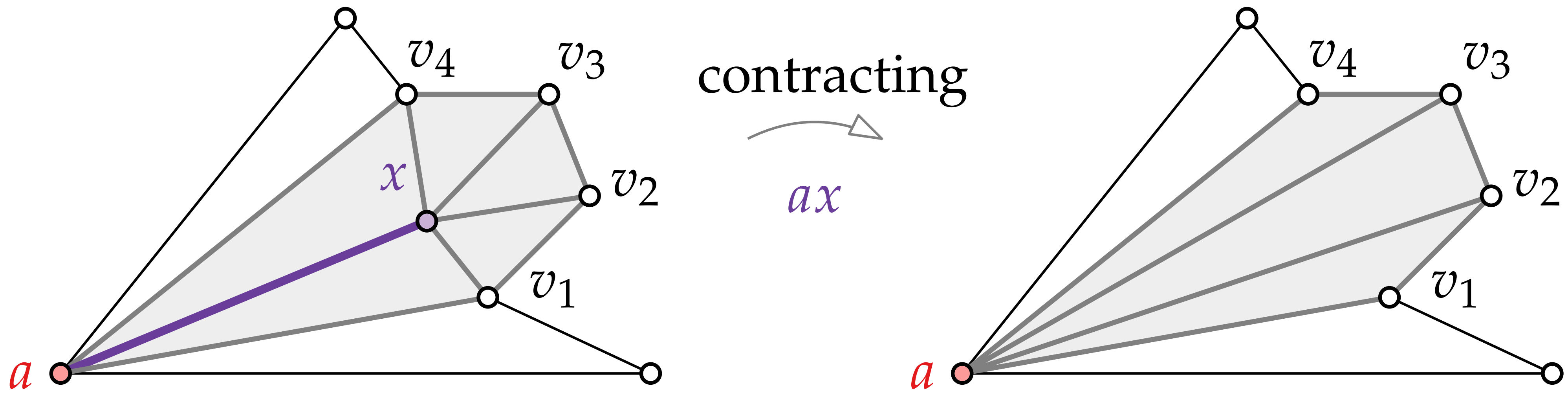

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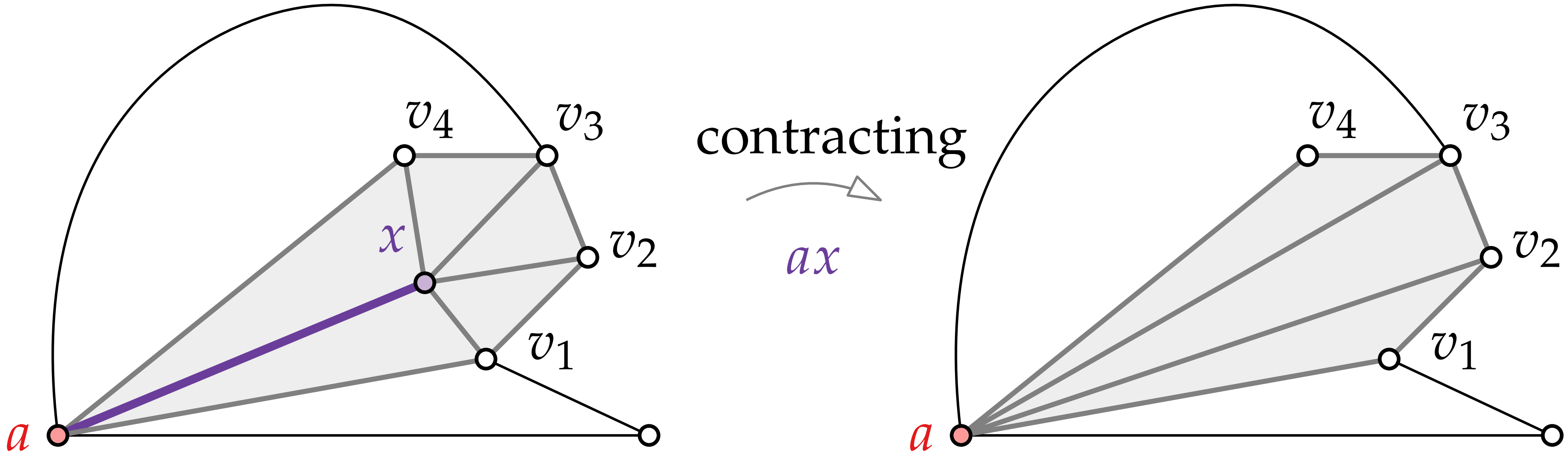

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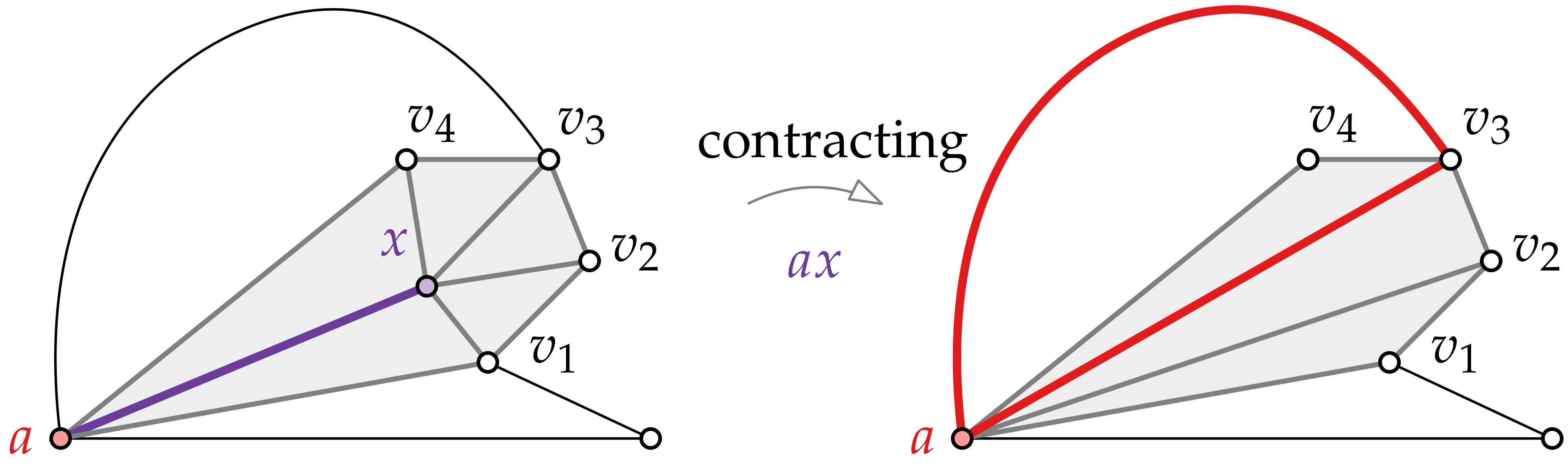

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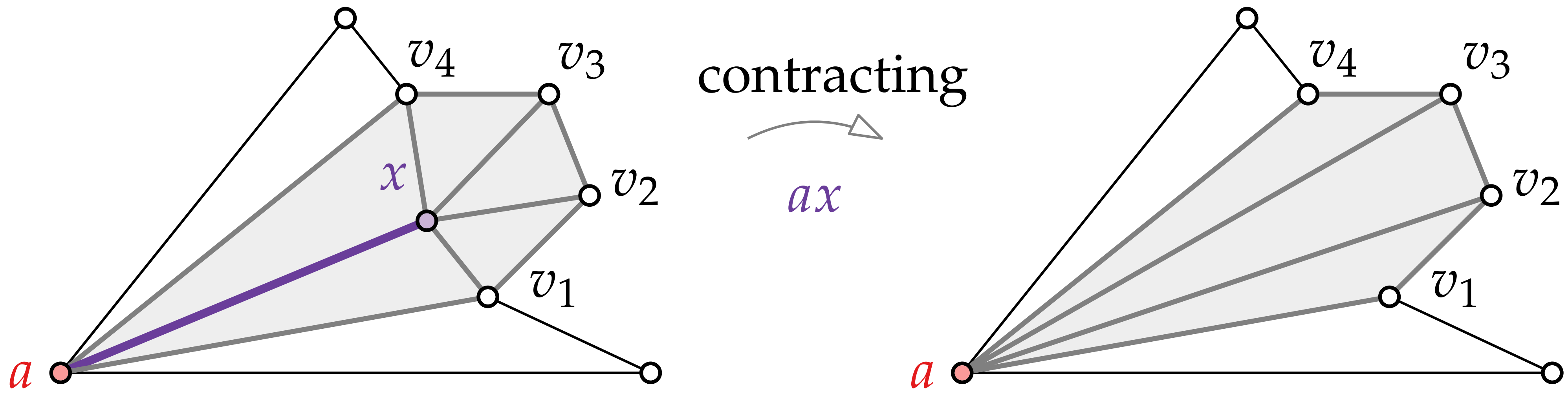

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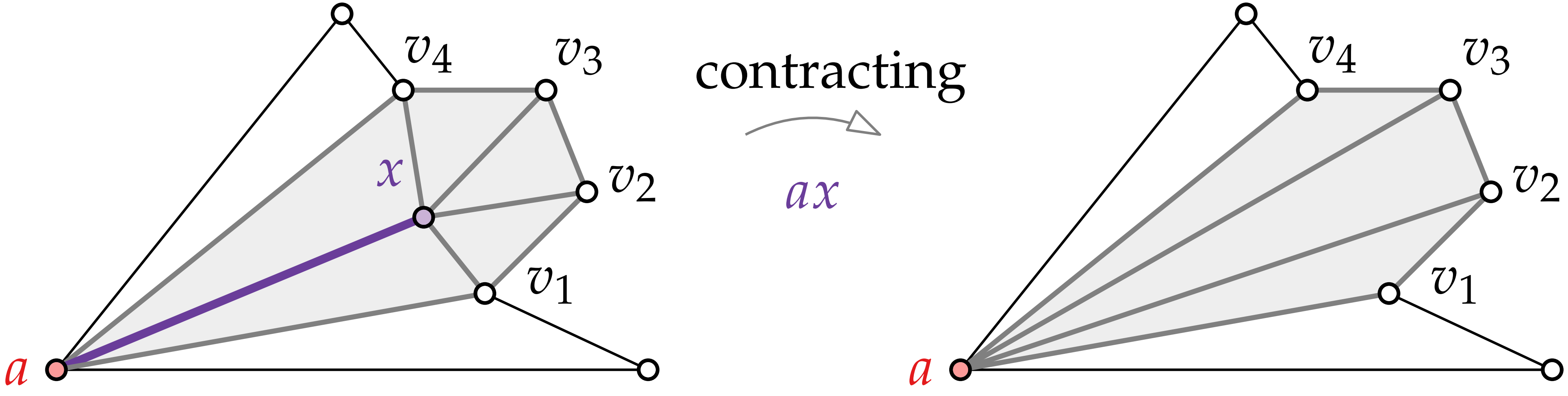

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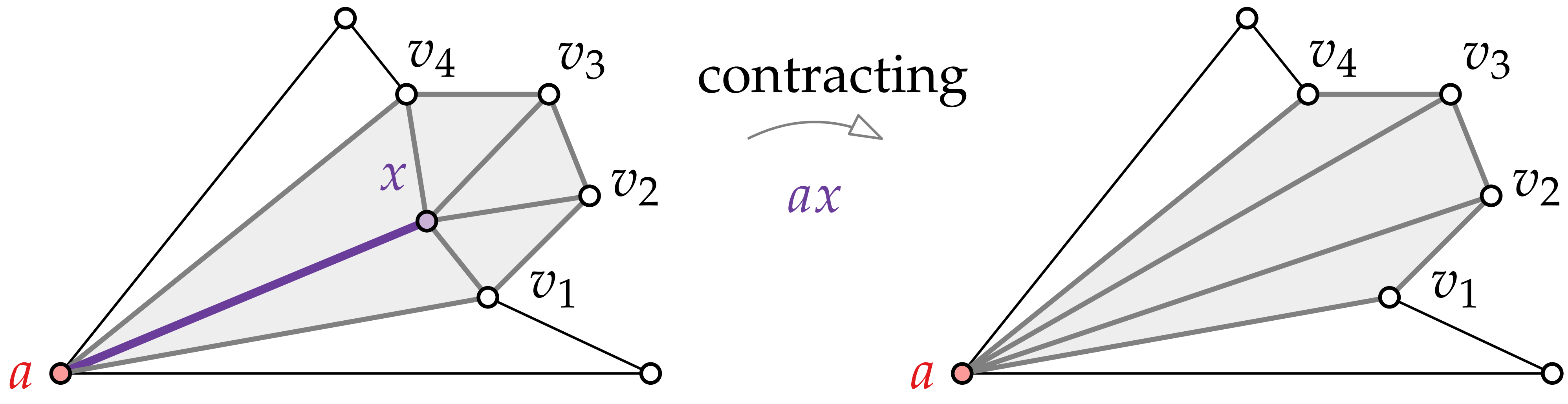

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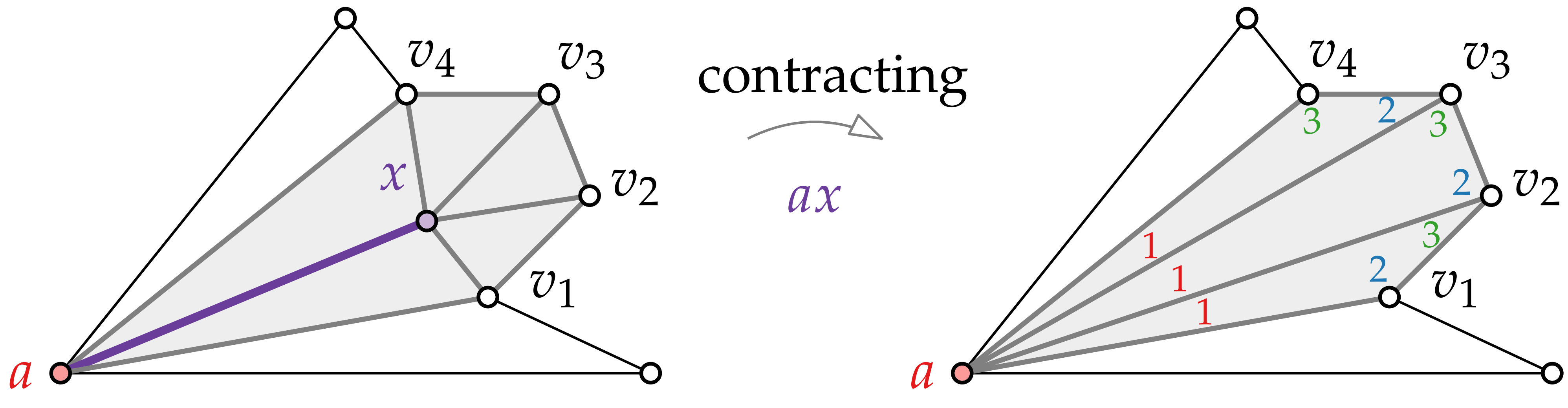

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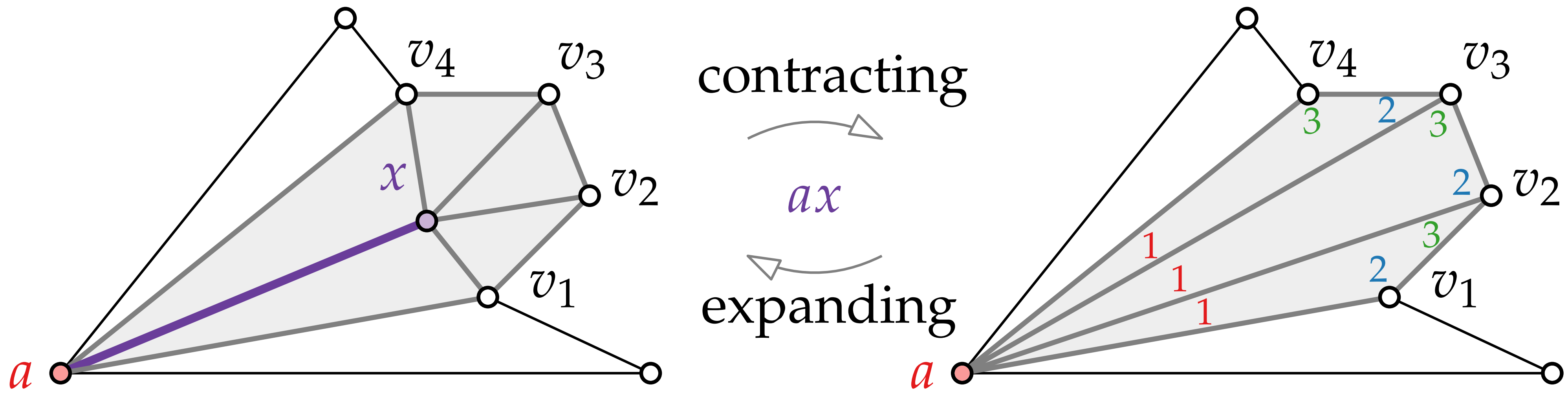

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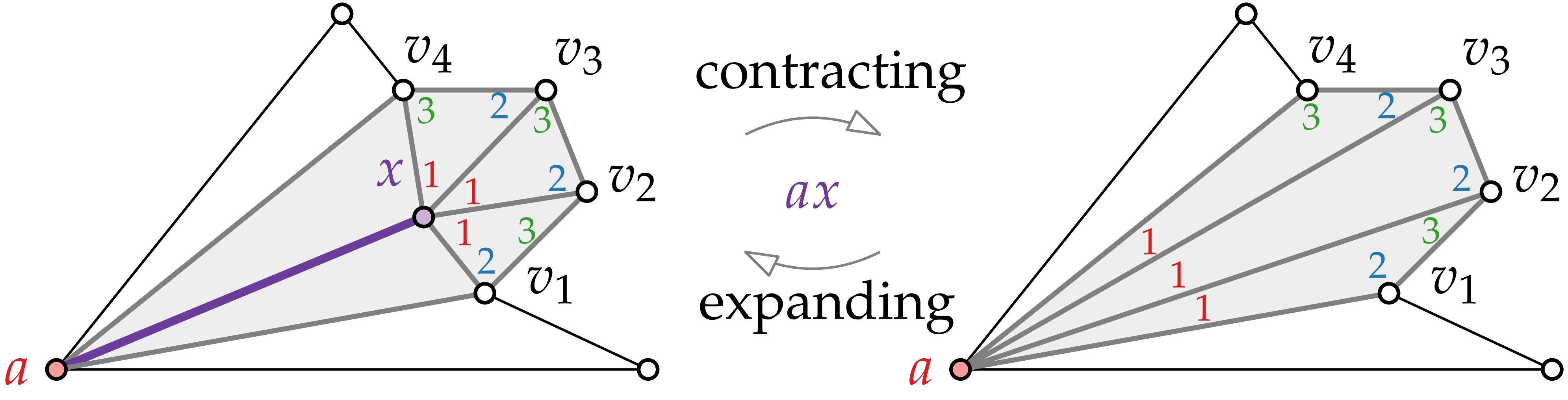

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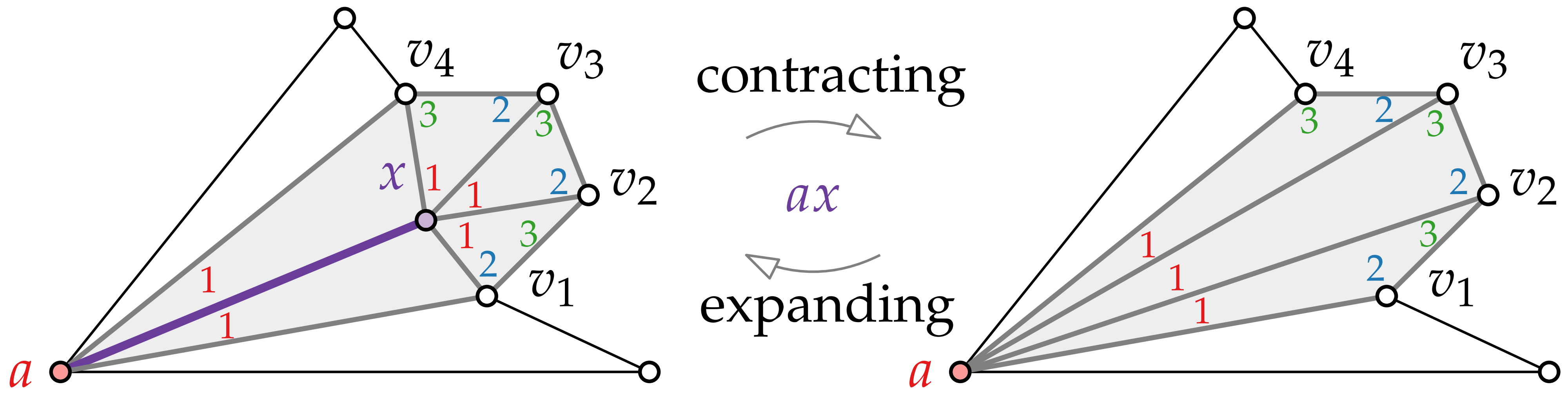

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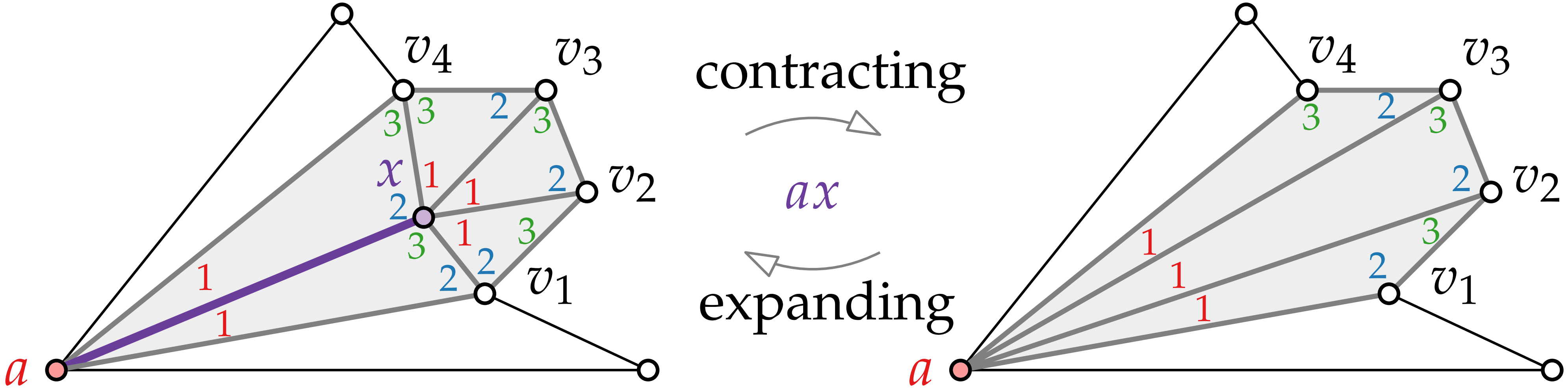

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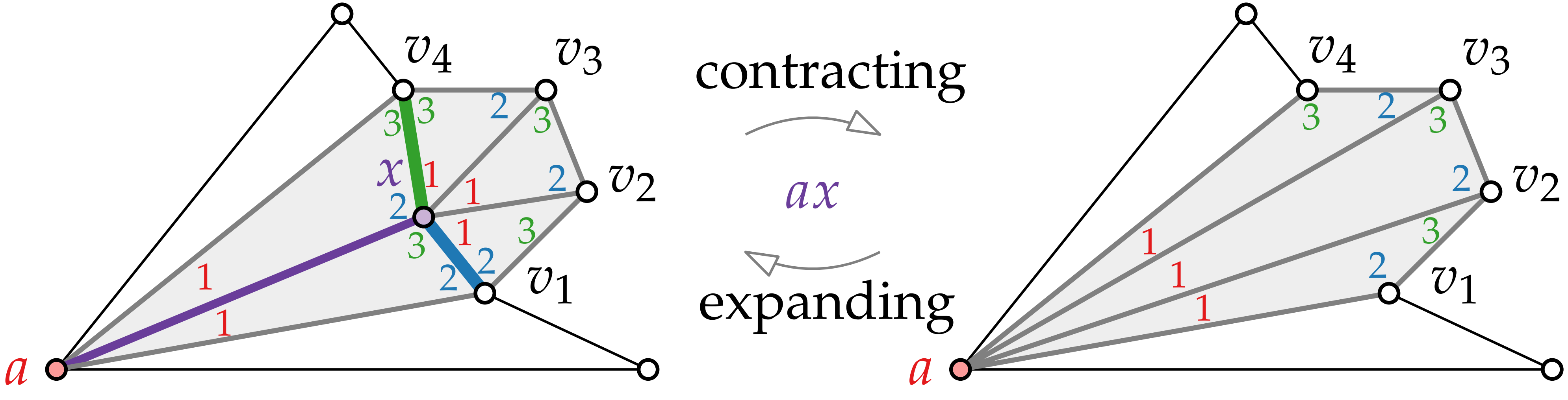

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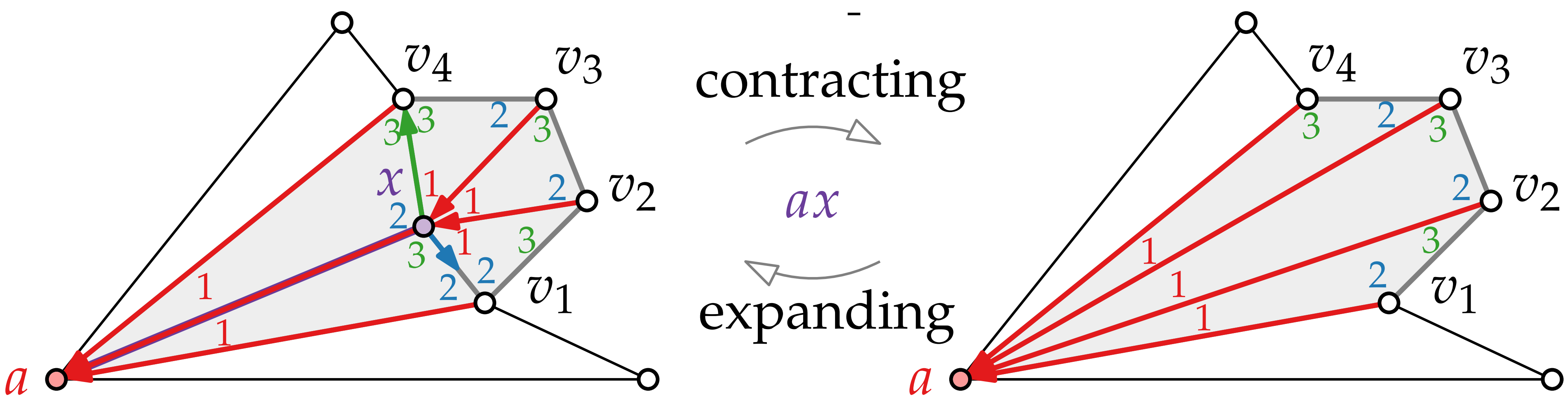

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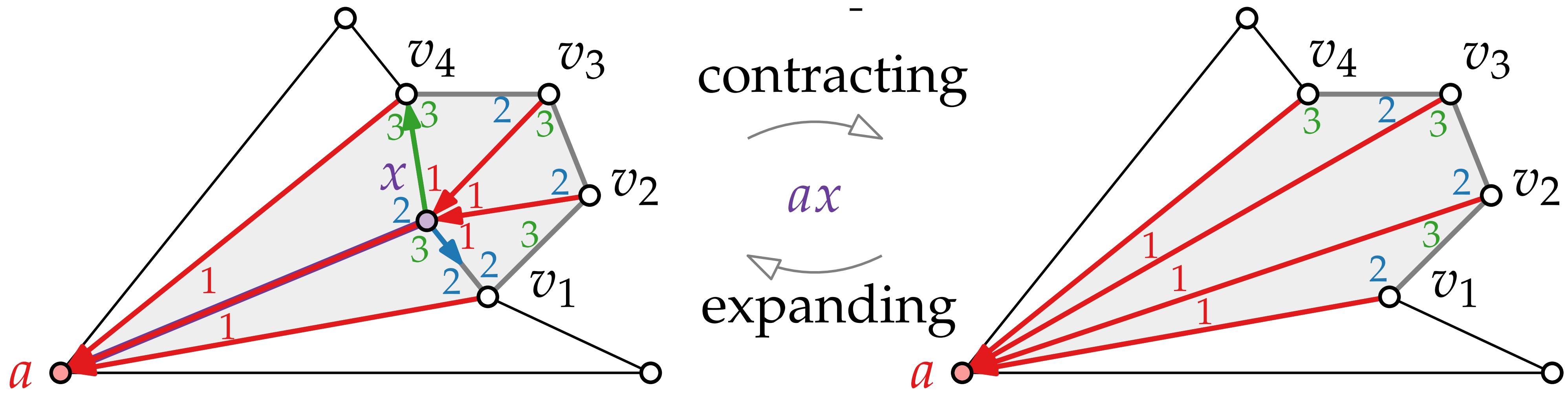
Constructive proof can be used as algorithm to compute a Schnyder labeling. It can be implemented in $\mathcal{O}(n)$ time.



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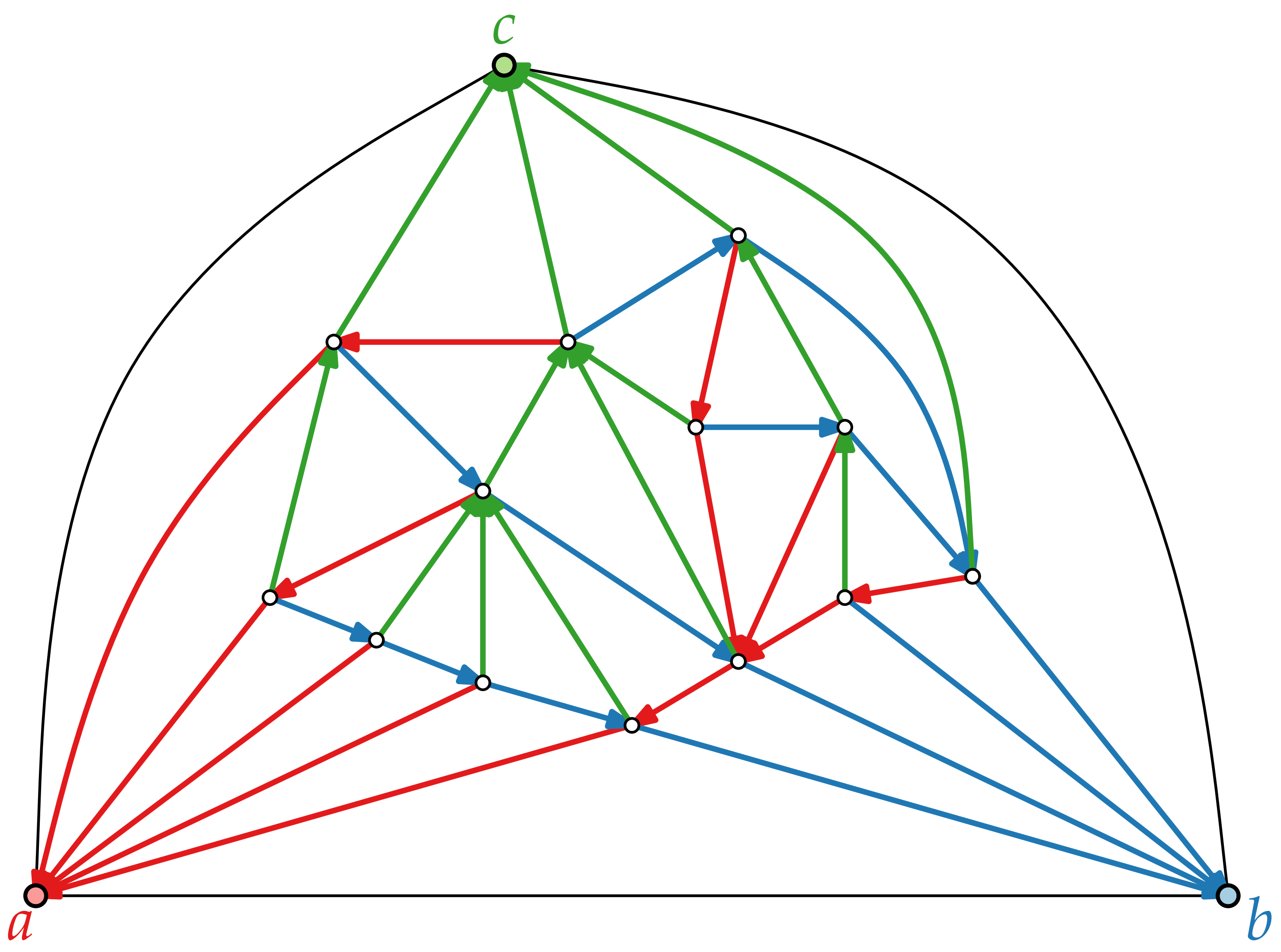

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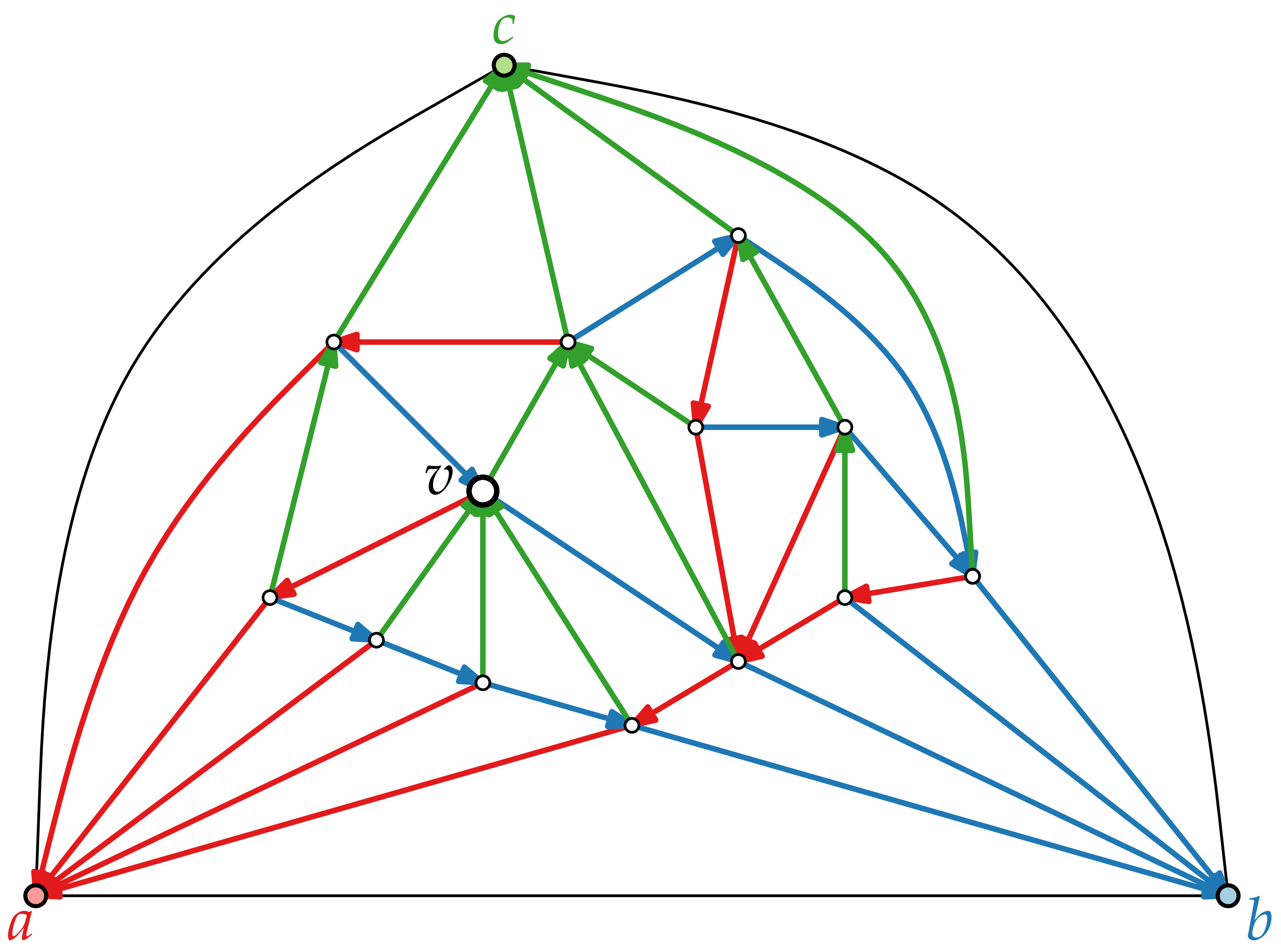

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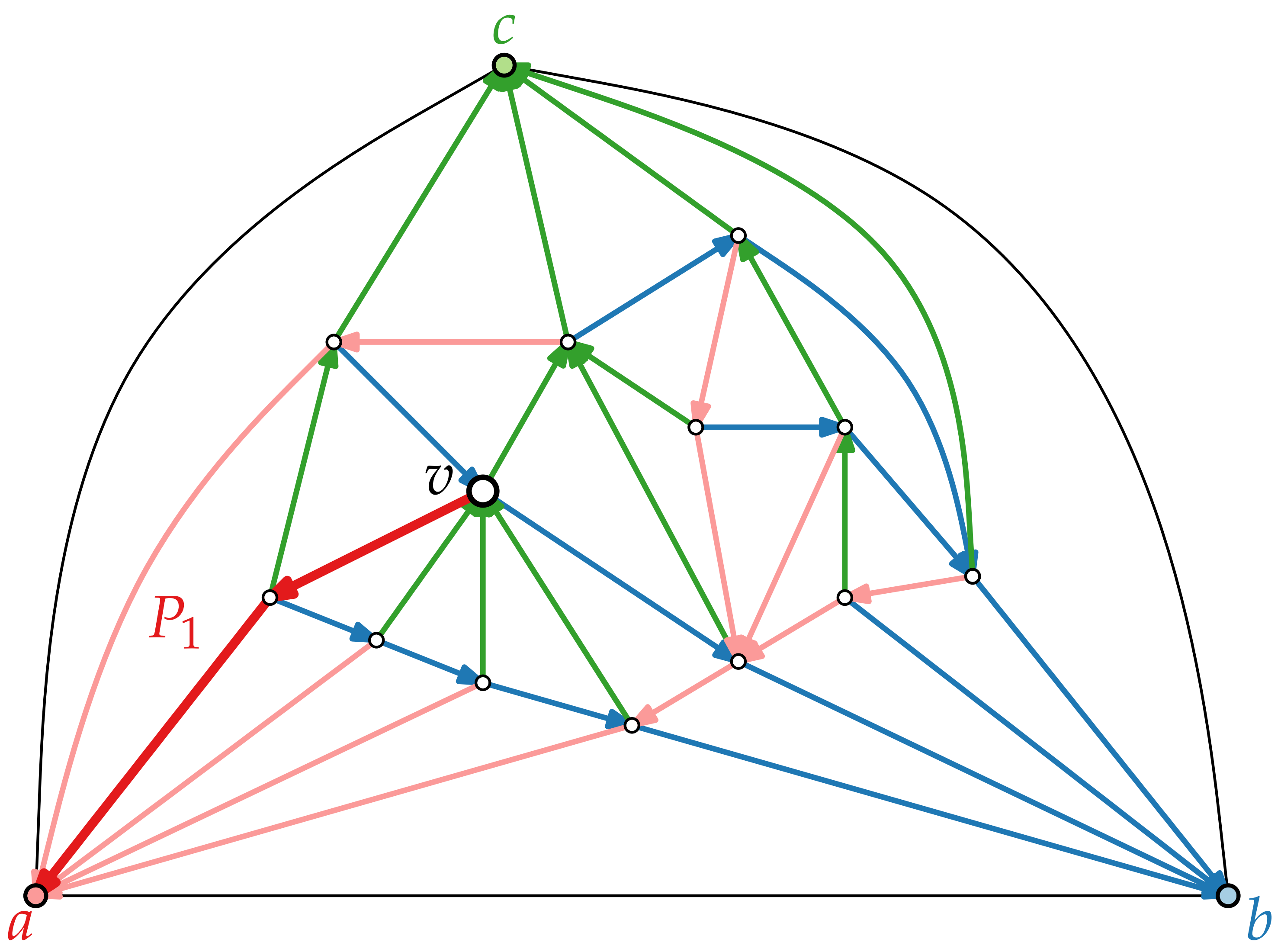

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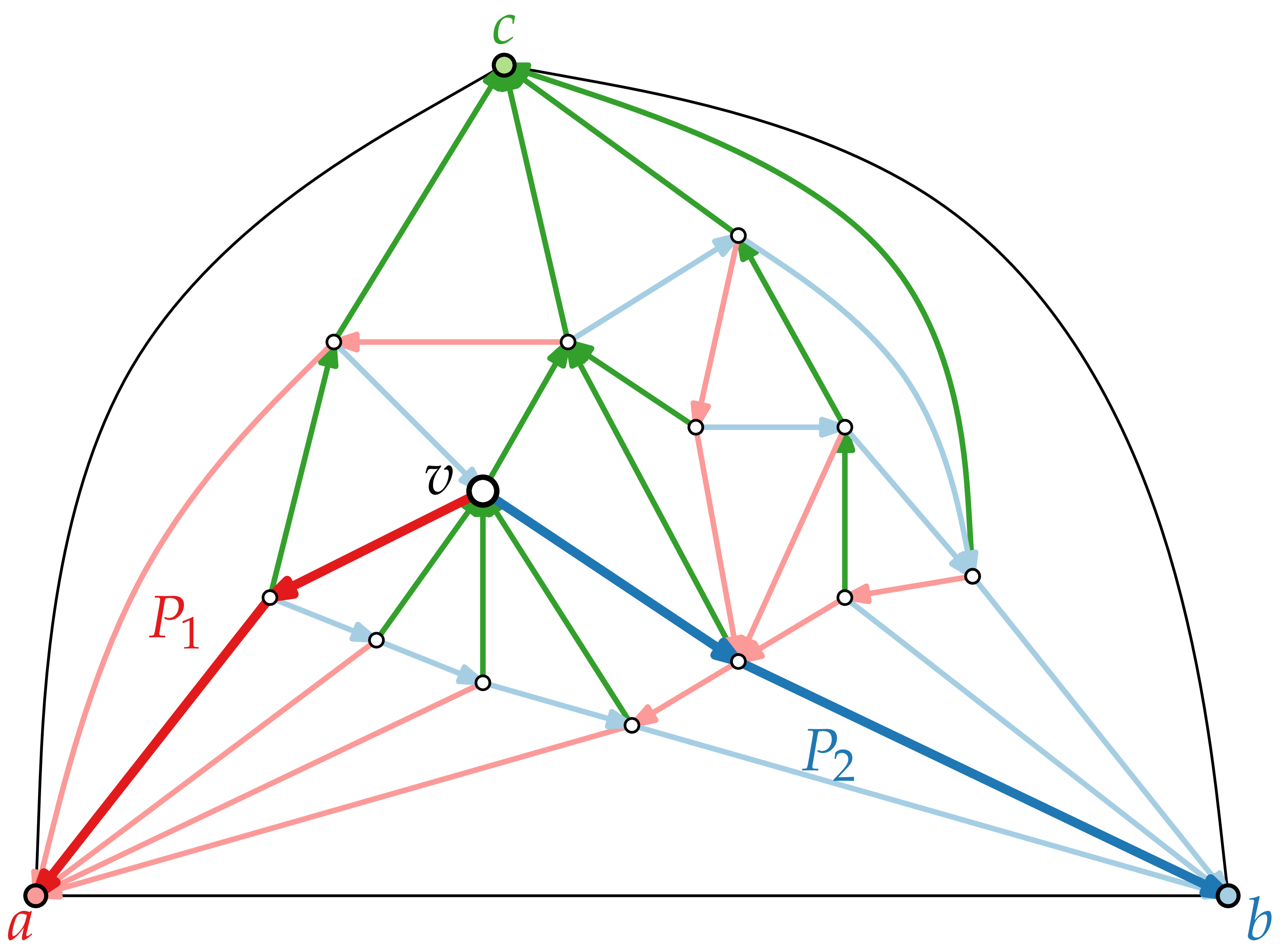

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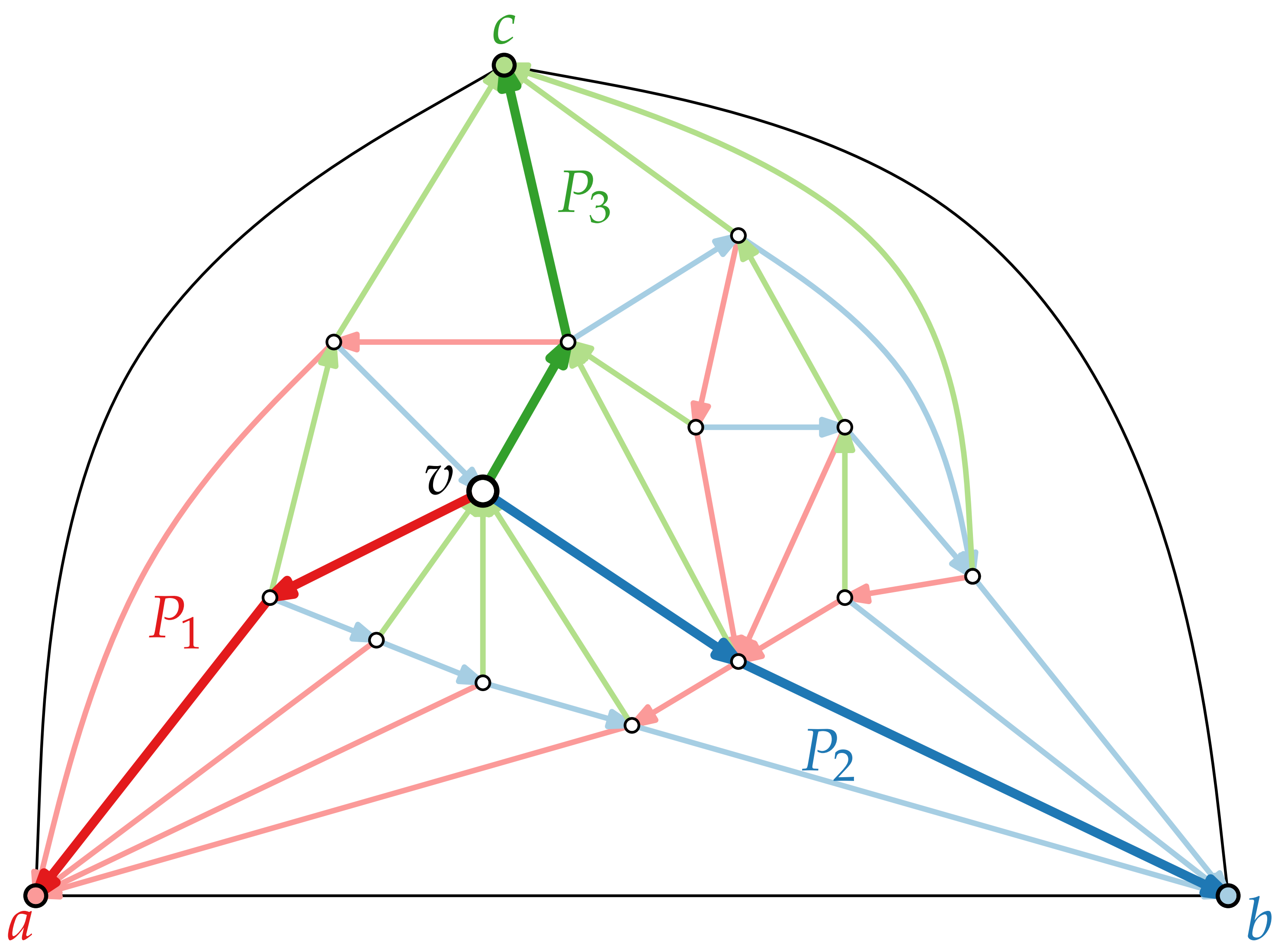

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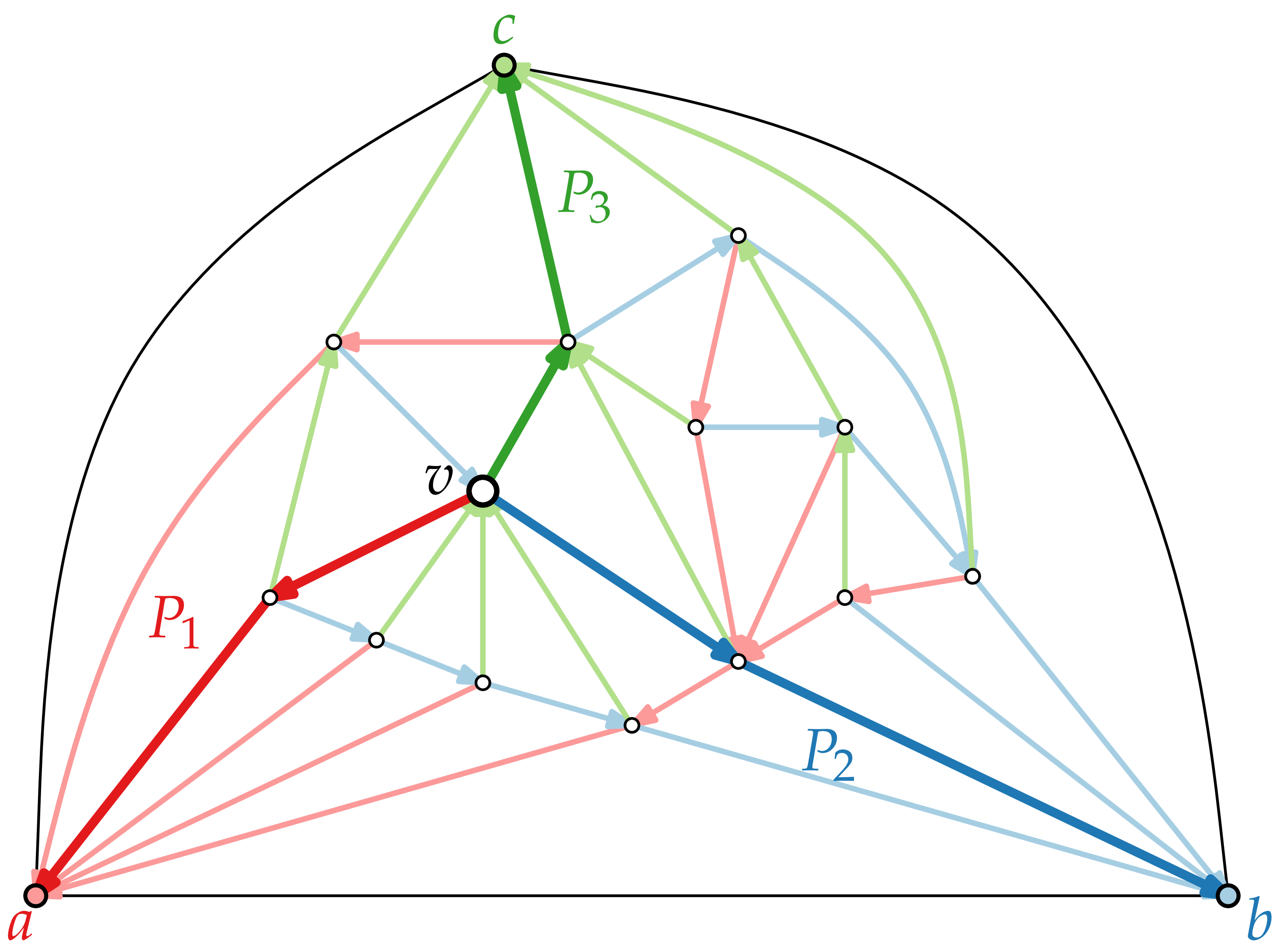

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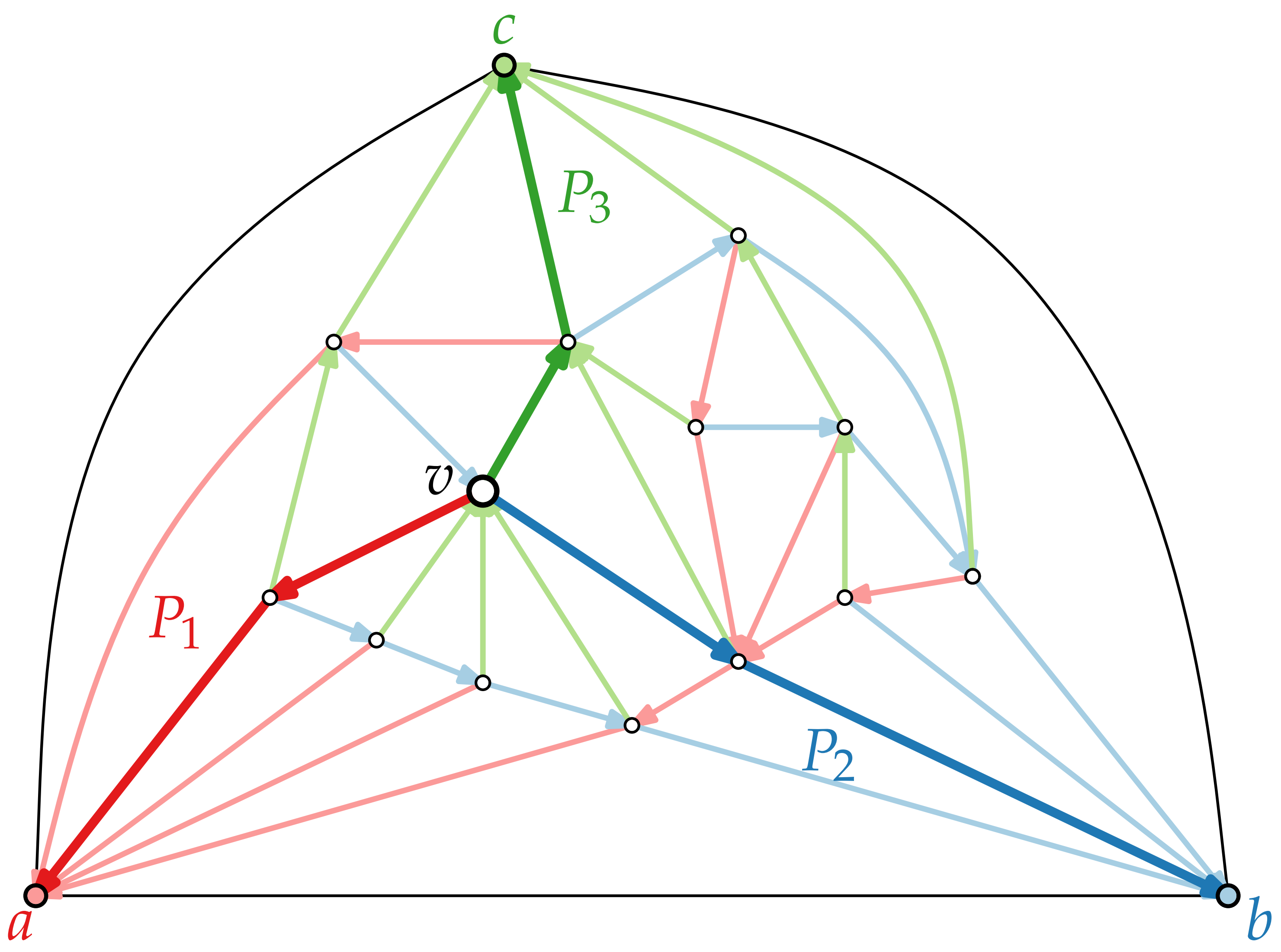

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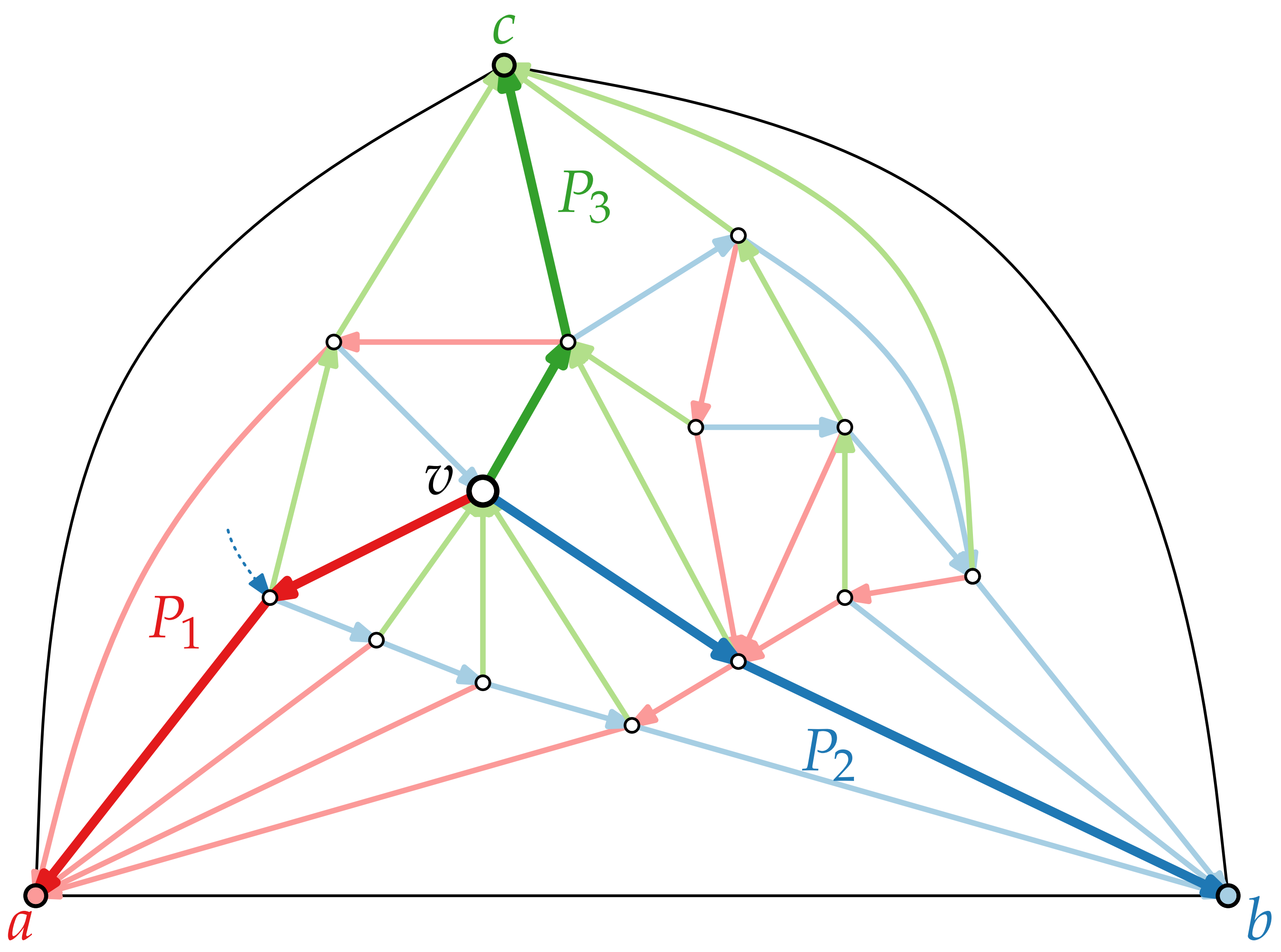

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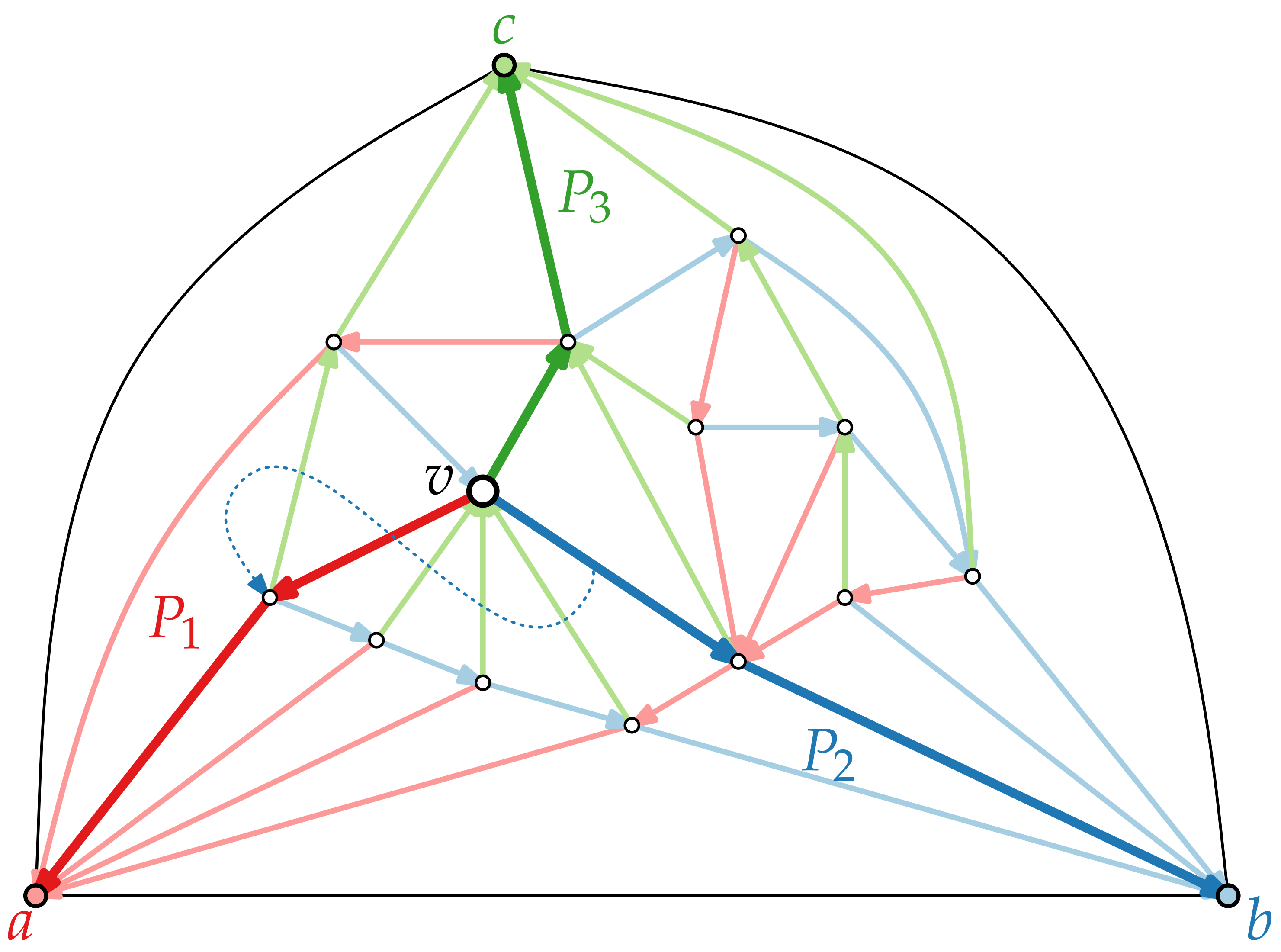

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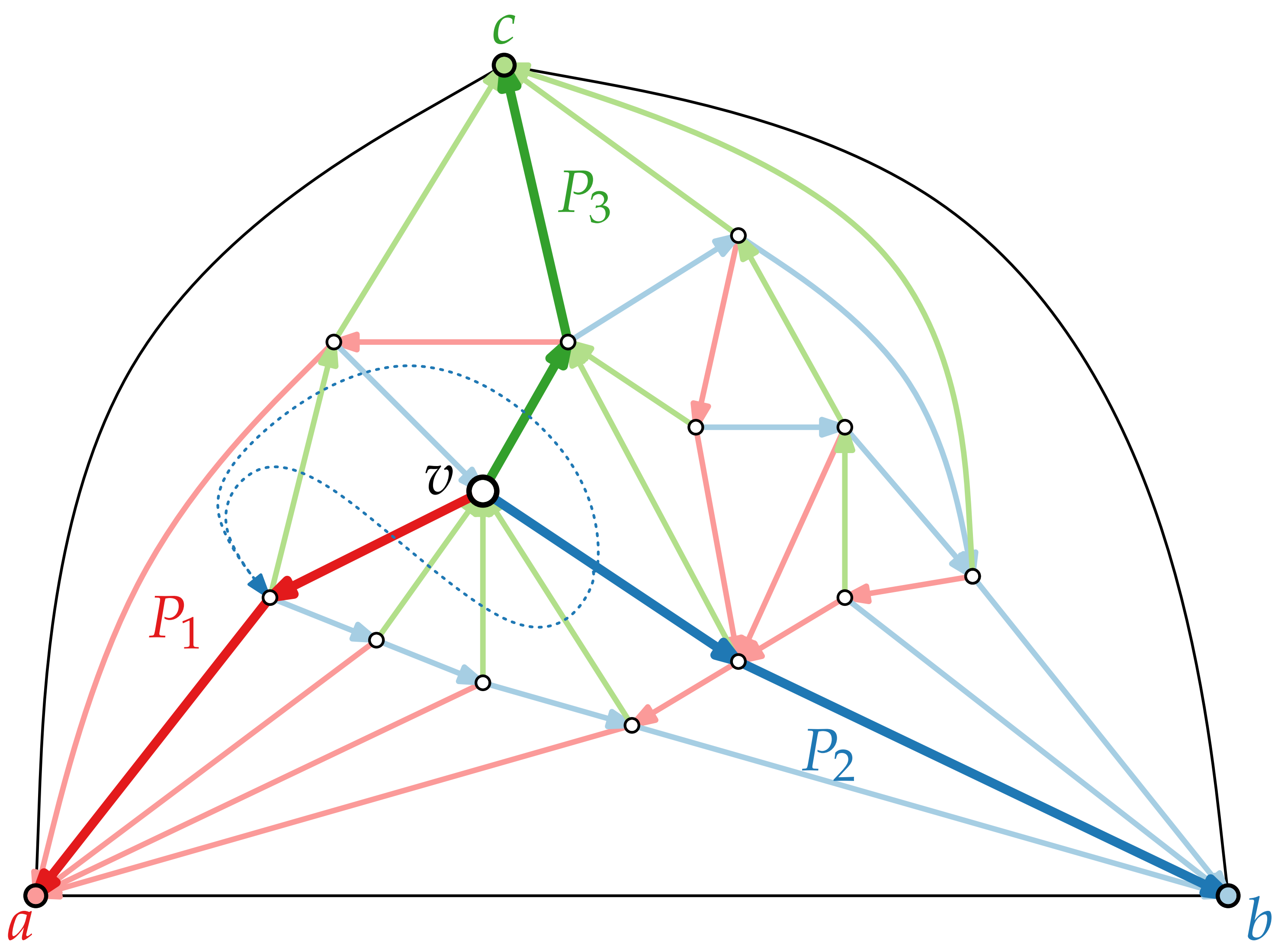

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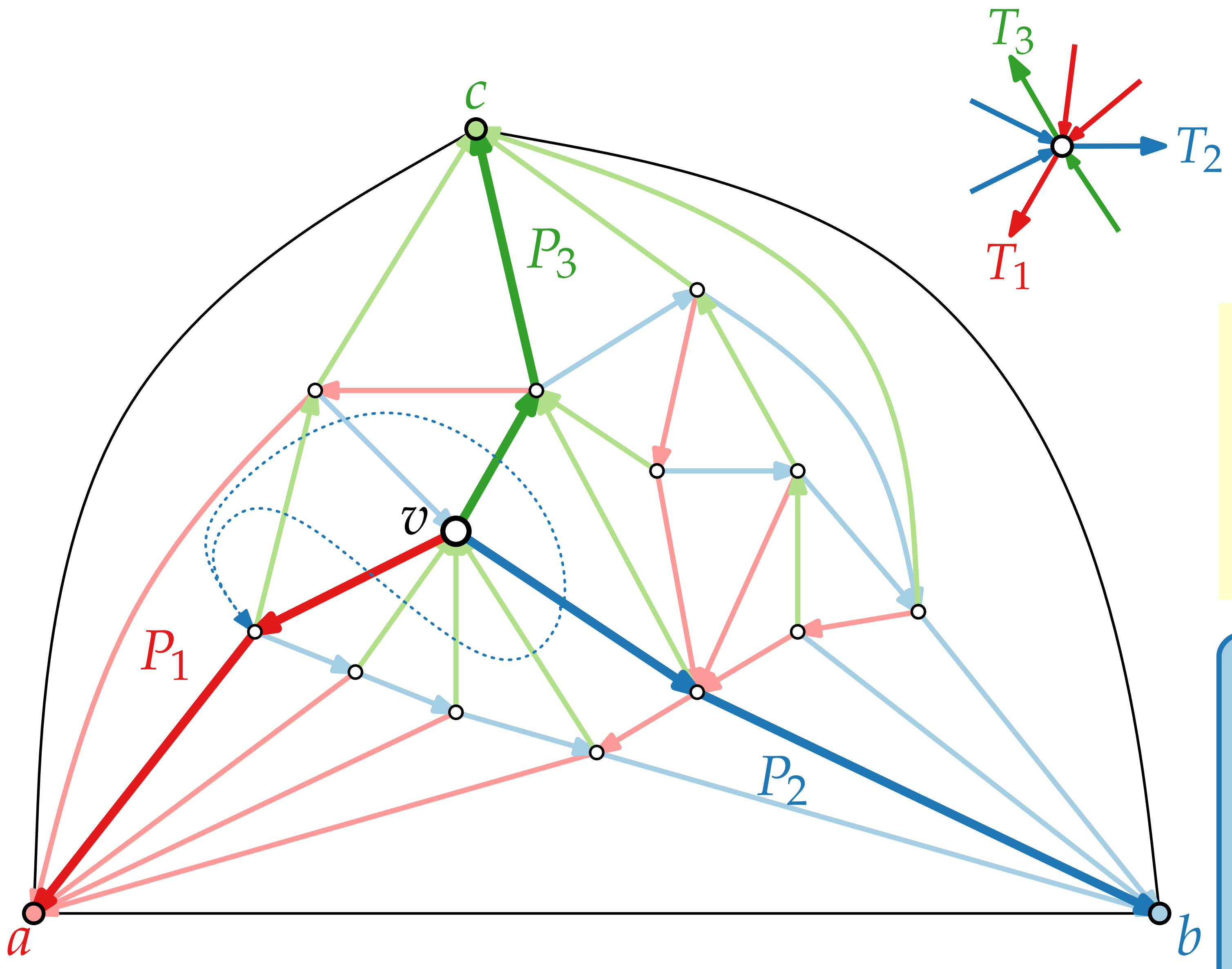

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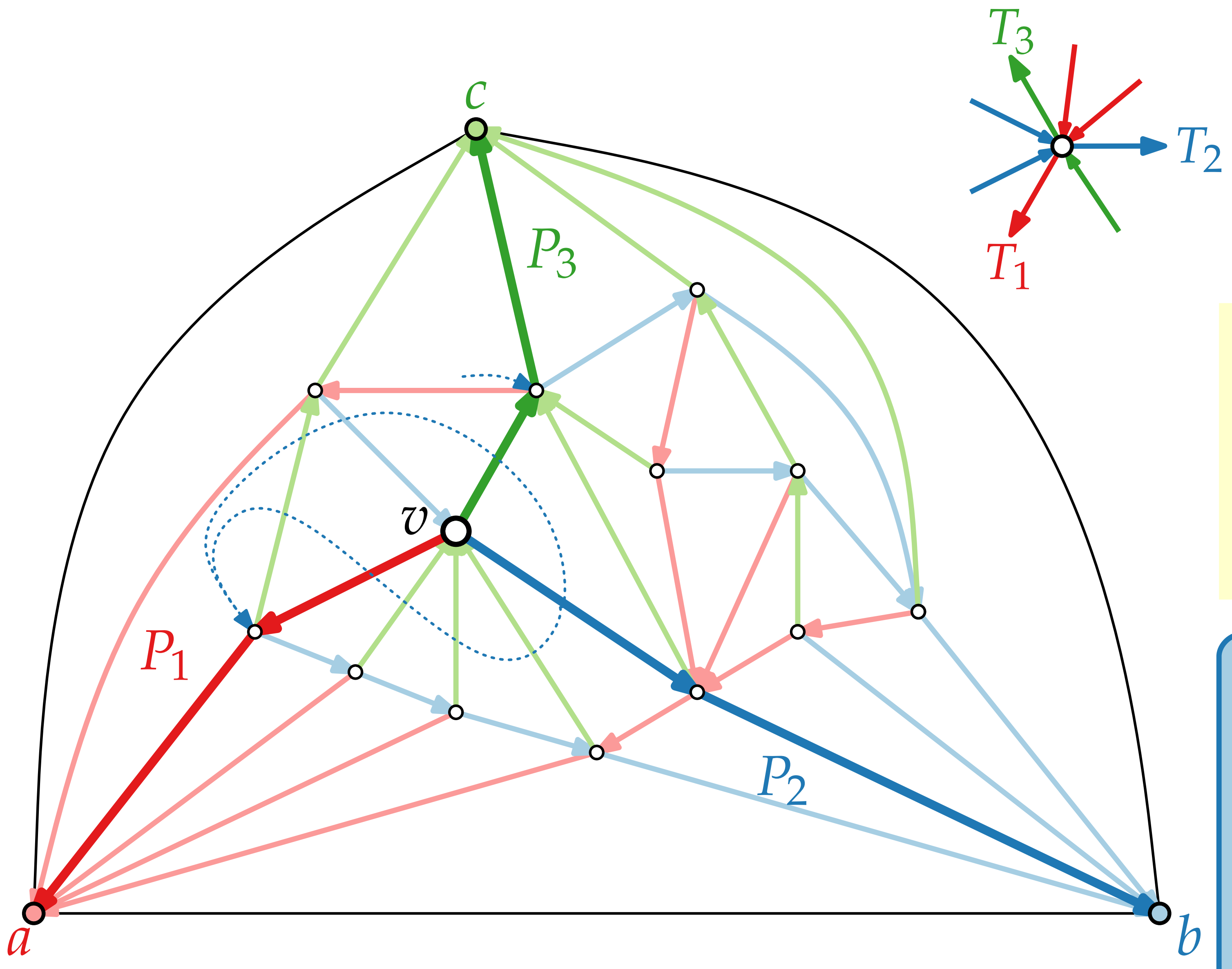

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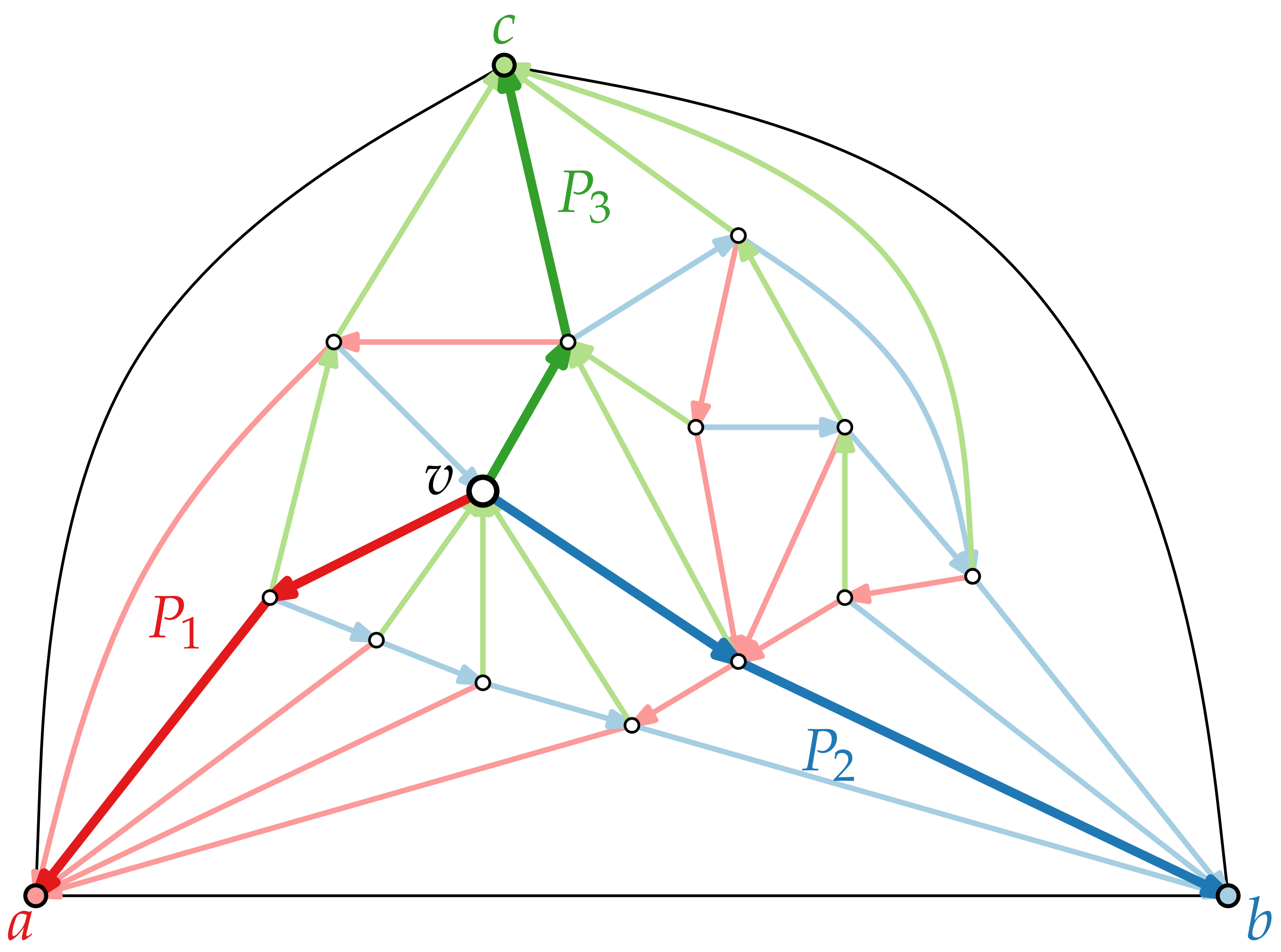

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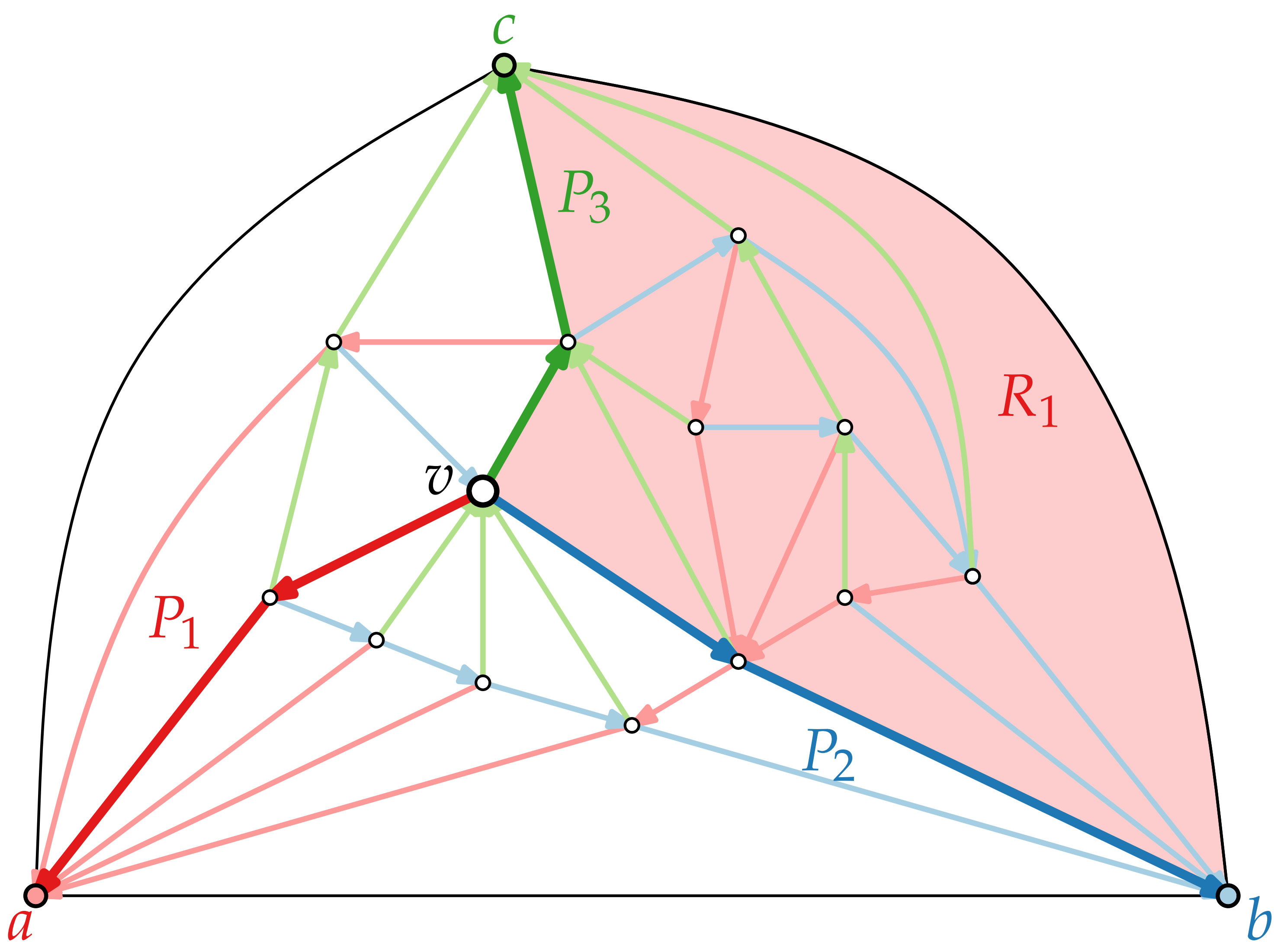

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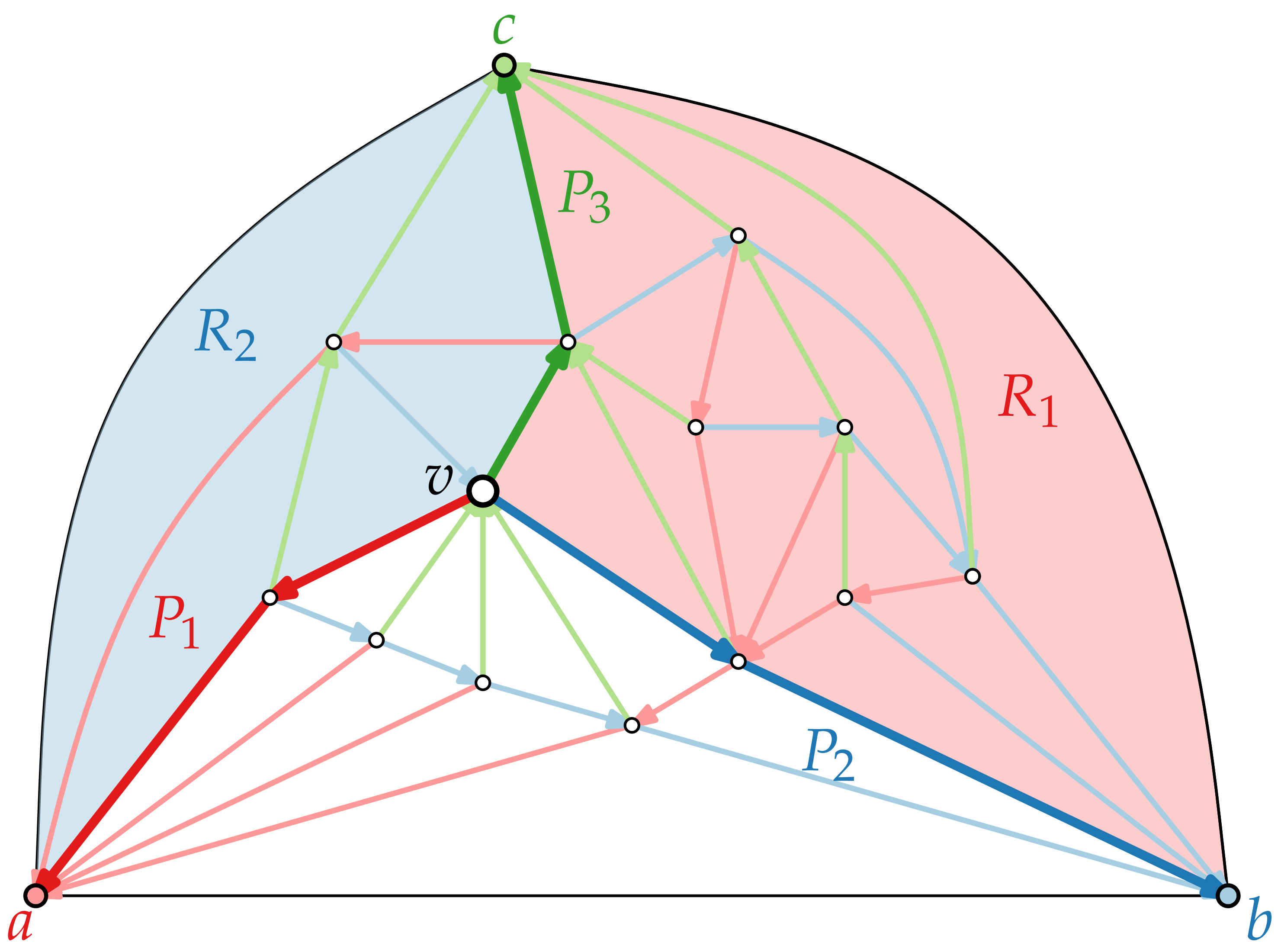

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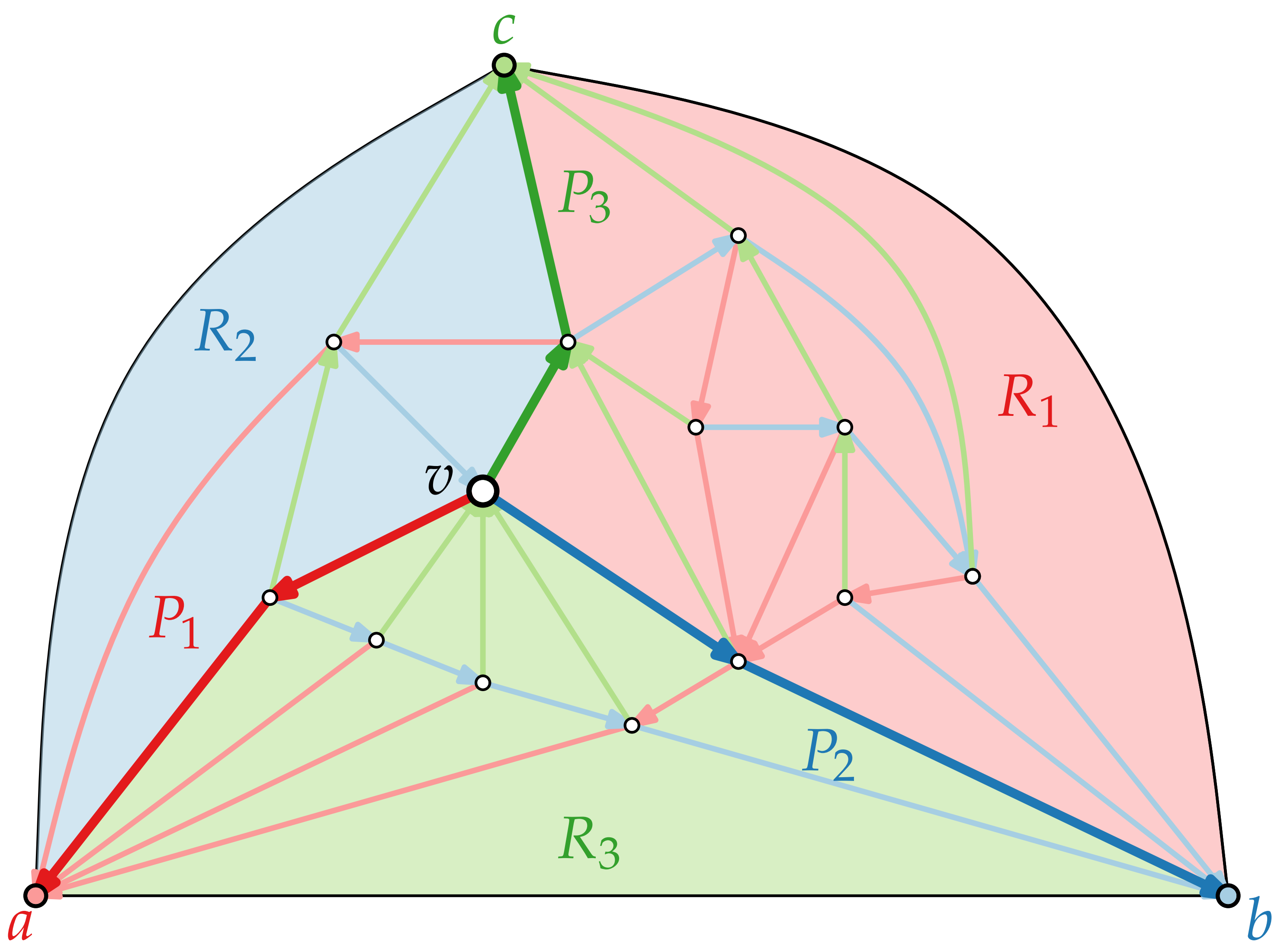

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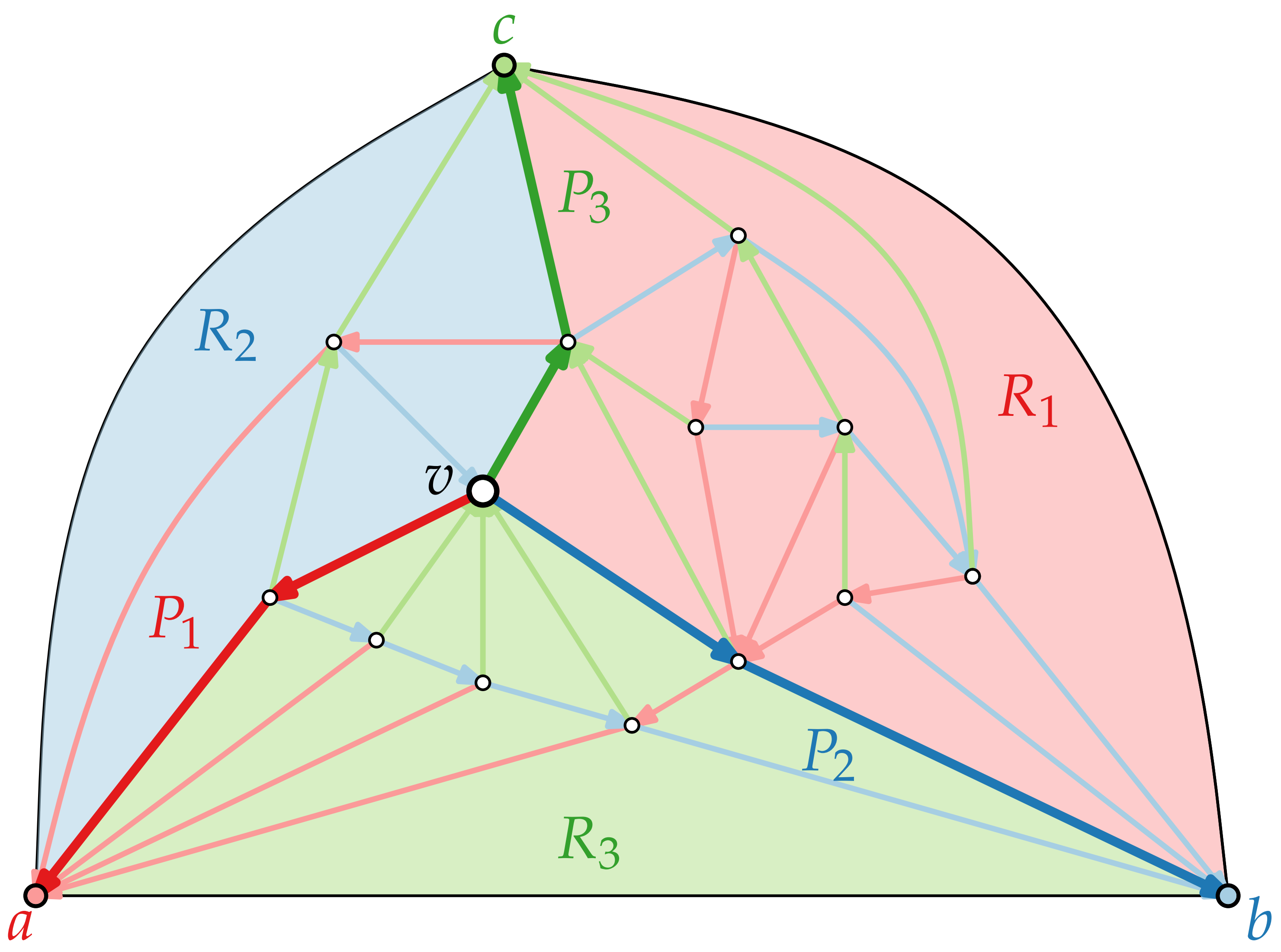

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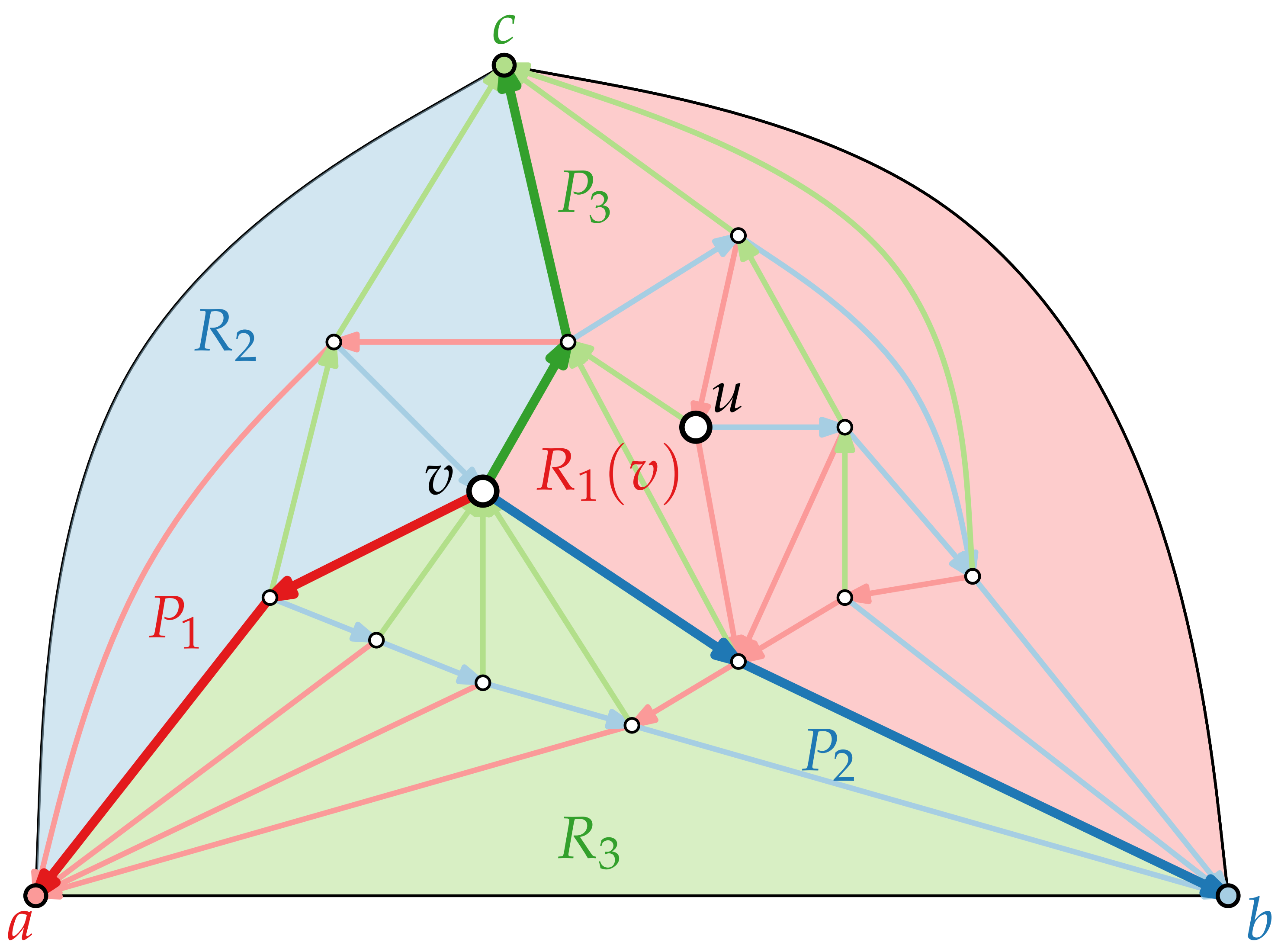

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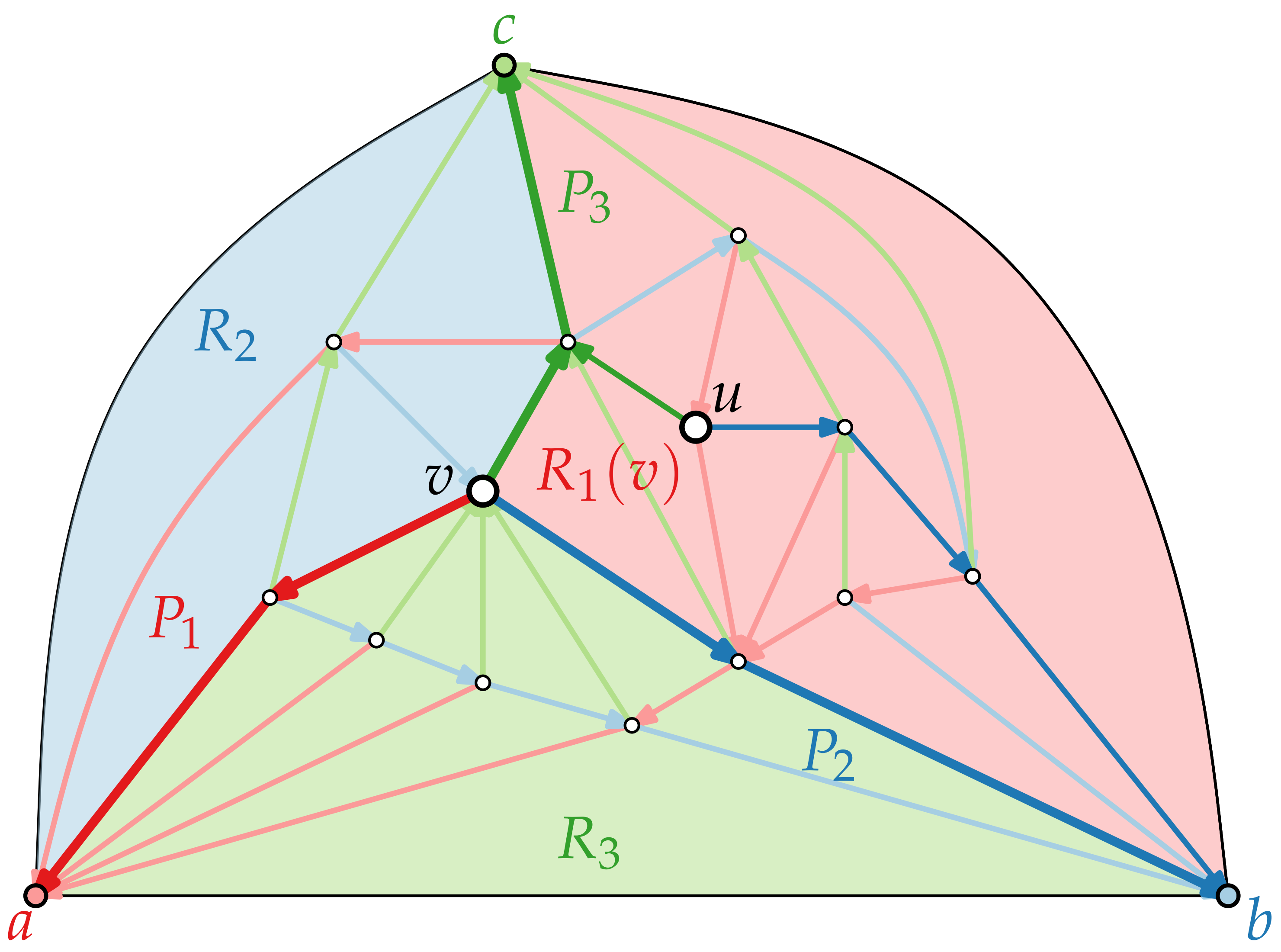

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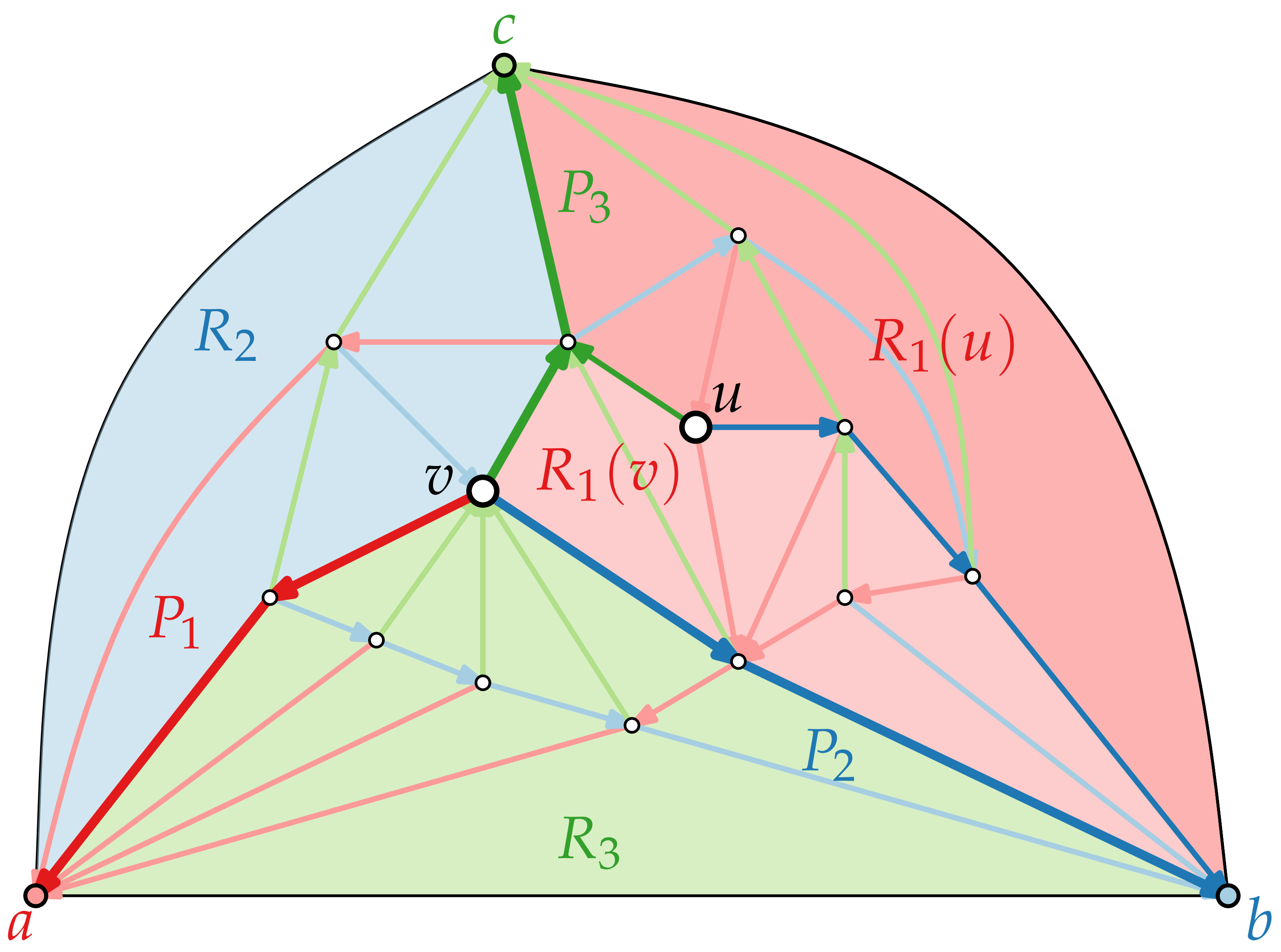
Schnyder Realizer – More Properties



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$P_i(v)$: path from v to root of T_i.
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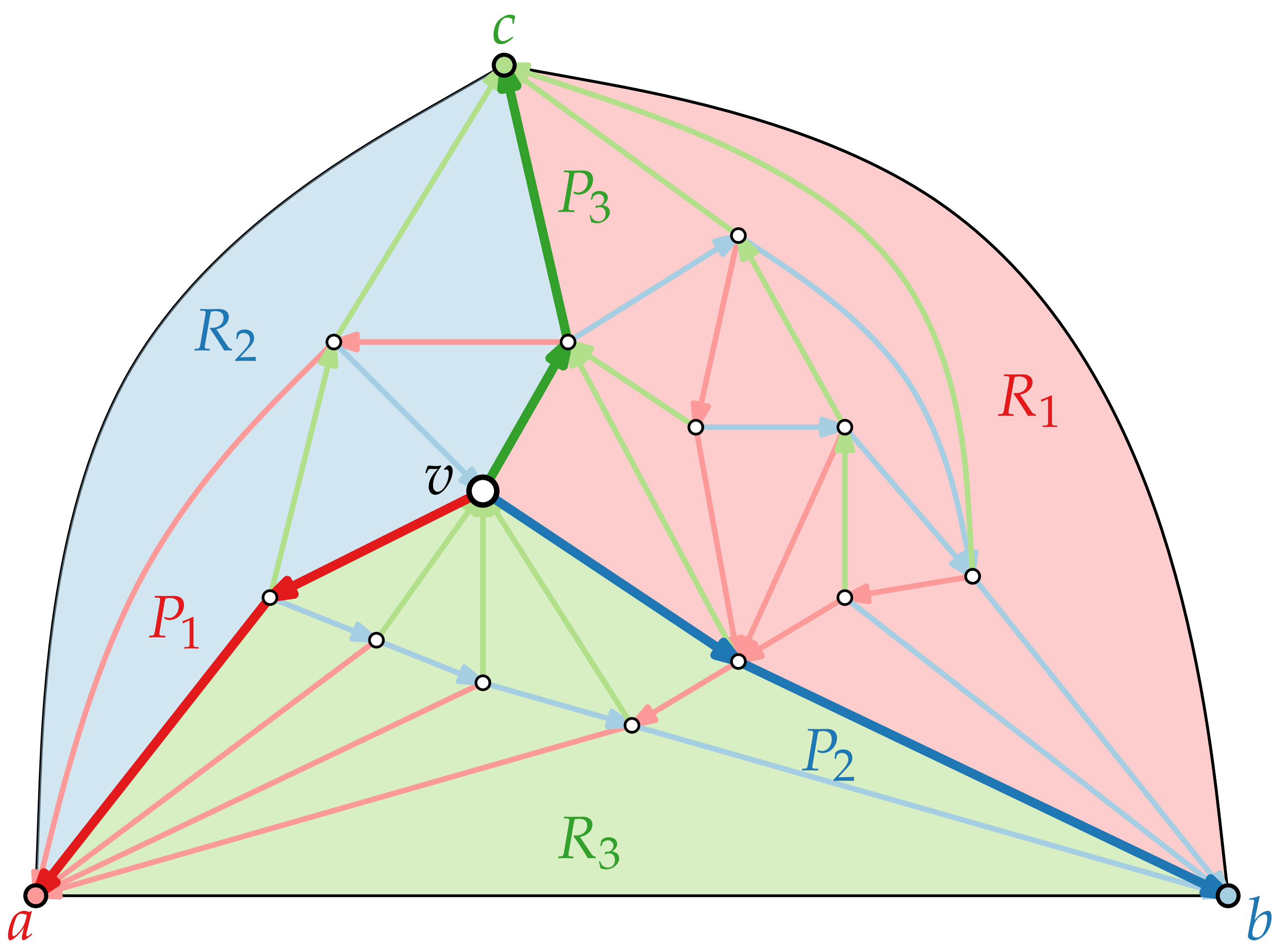

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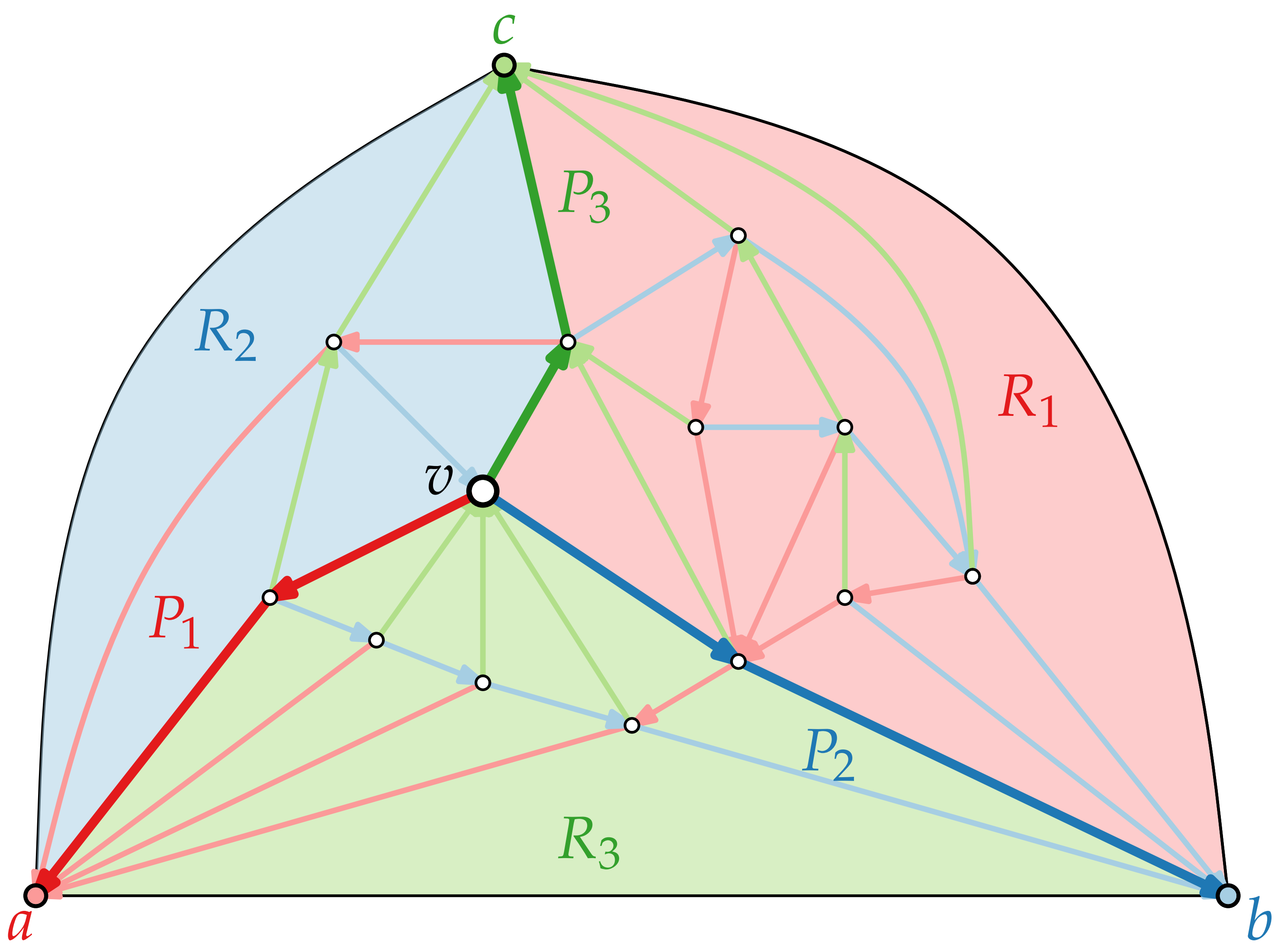

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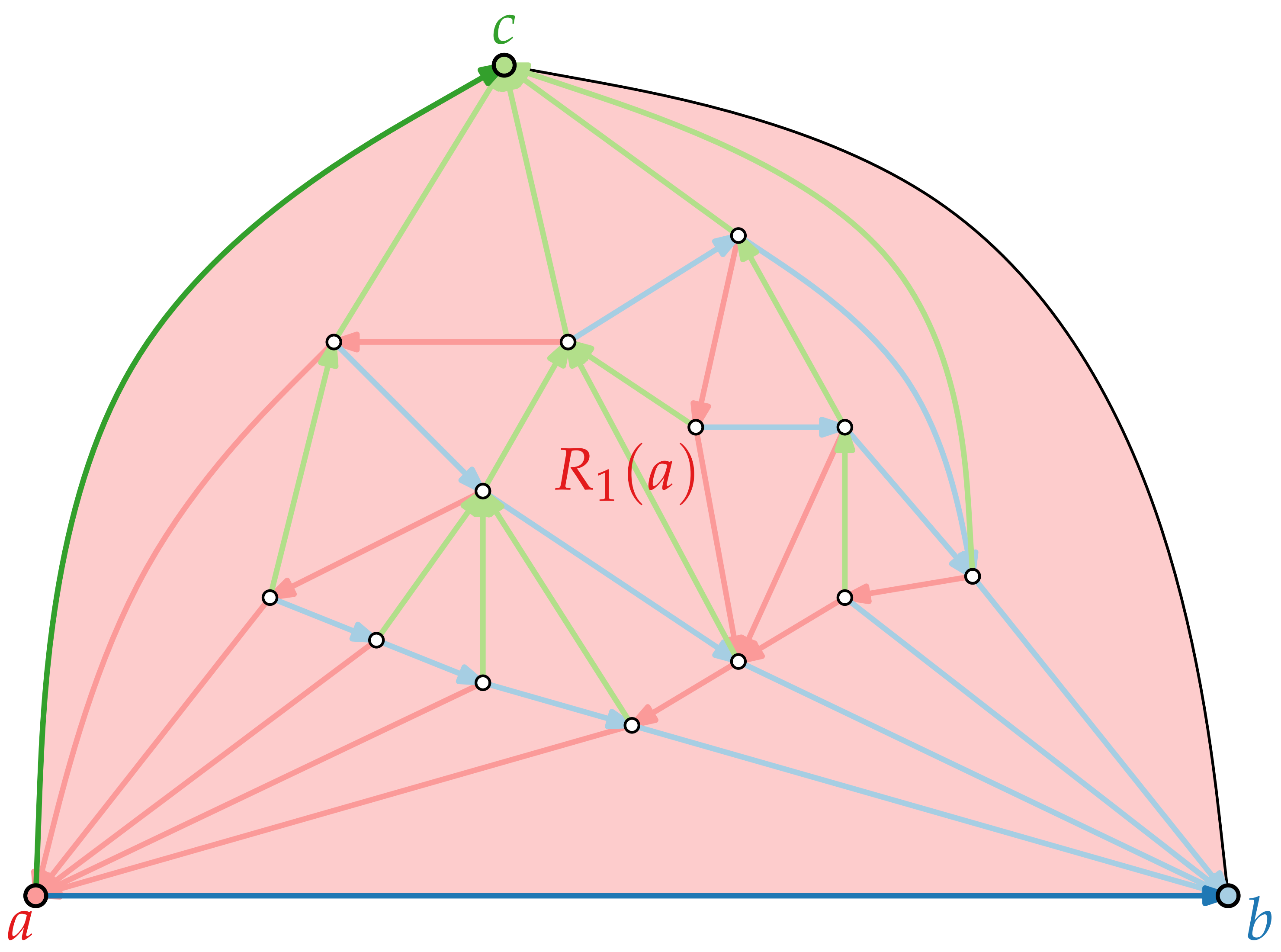

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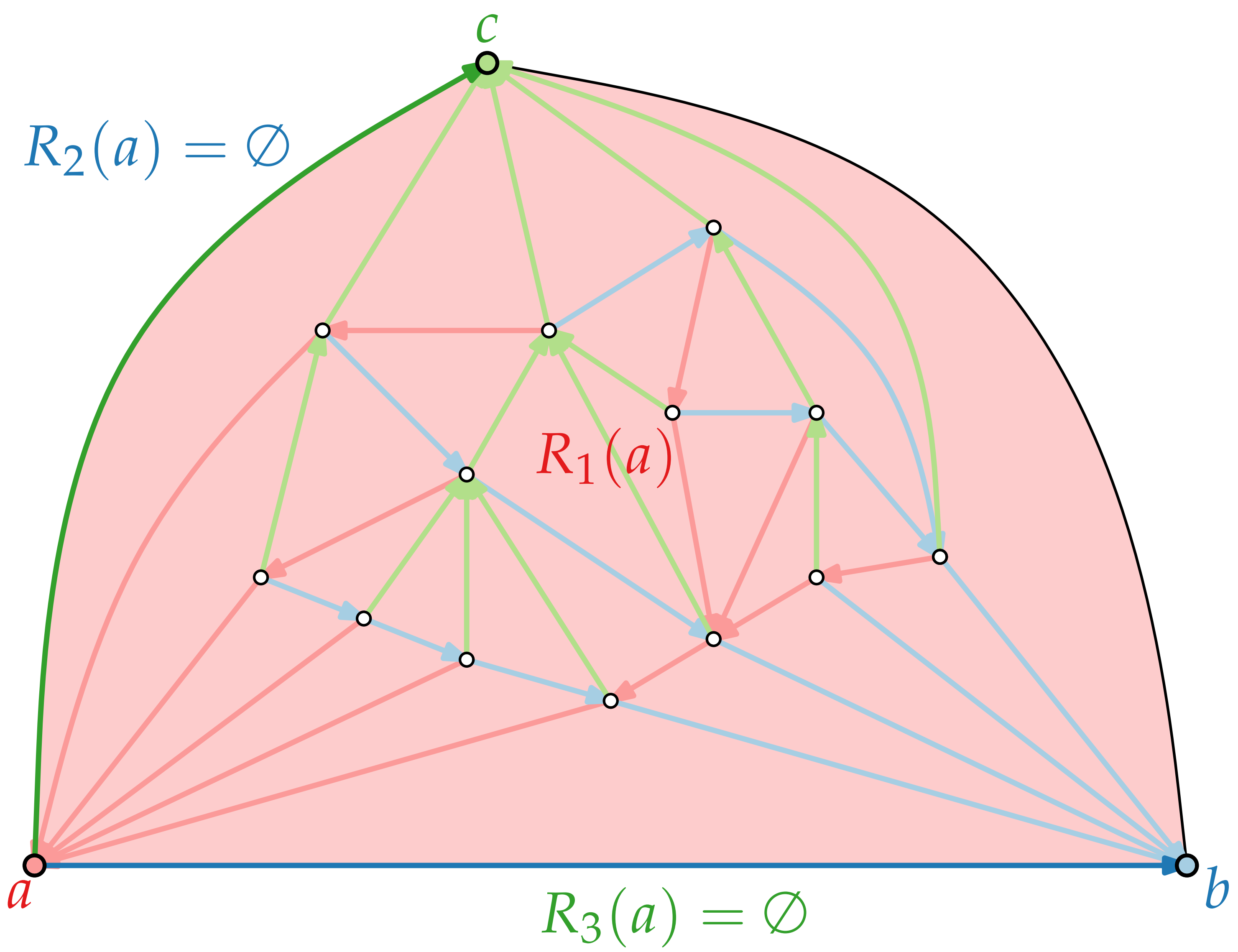

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Schnyder Drawing

Theorem. [Schnyder '89]
For a plane triangulation G, the mapping

$$f : v \mapsto (v_1, v_2, v_3) = \tfrac{1}{2n-5}(|R_1(v)|, |R_2(v)|, |R_3(v)|)$$

is a barycentric representation of G, which thus gives a planar straight-line drawing of G

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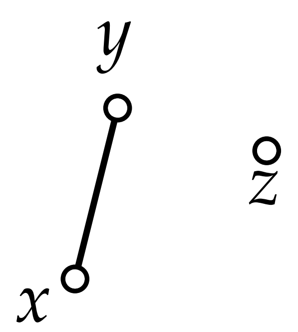

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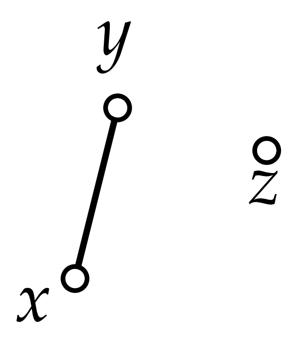

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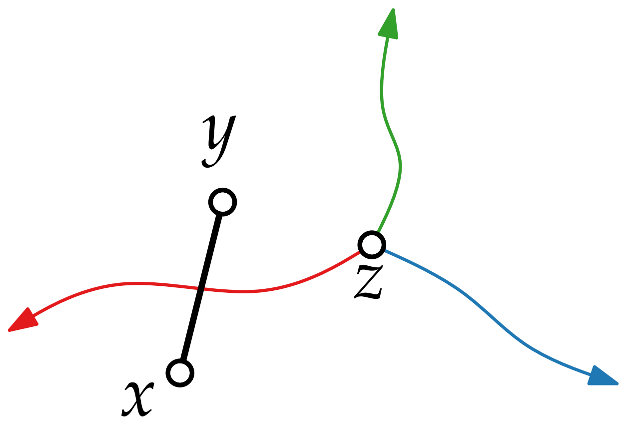

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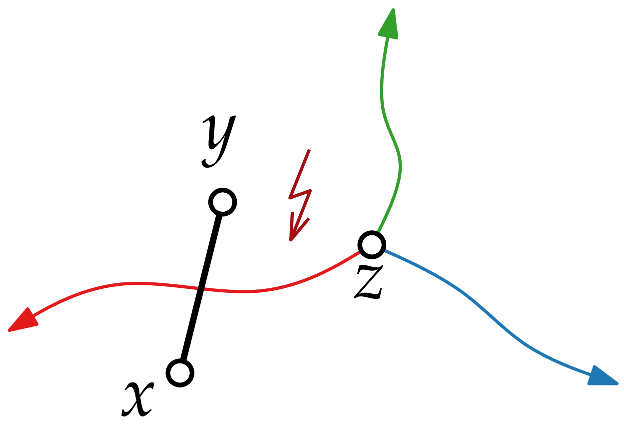

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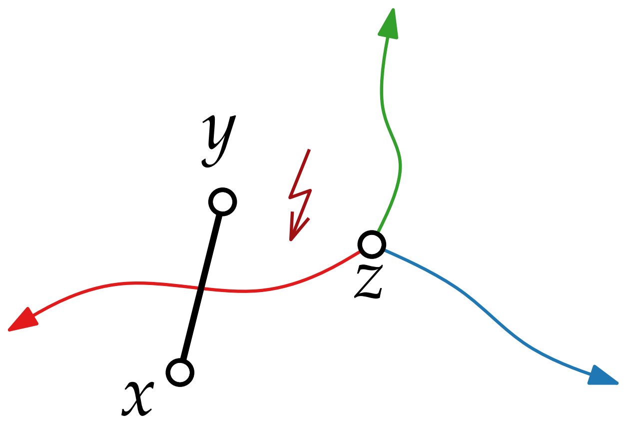

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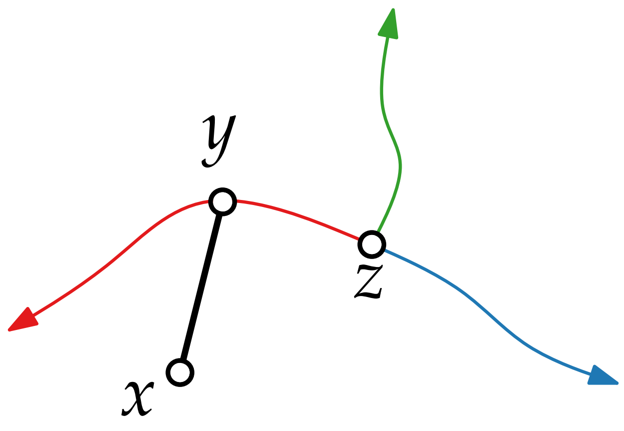

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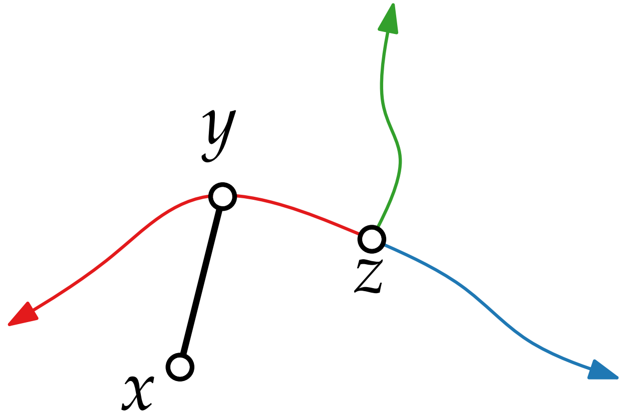

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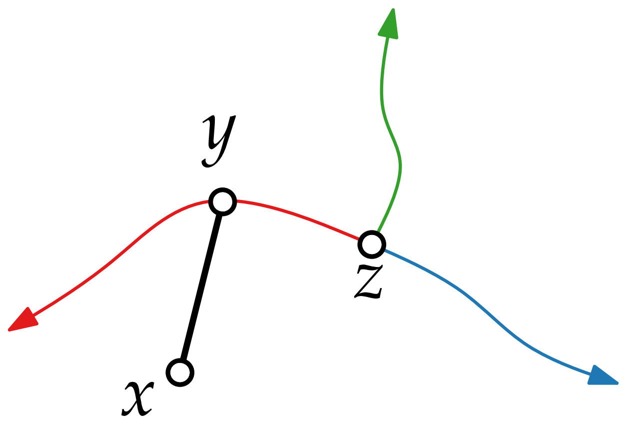

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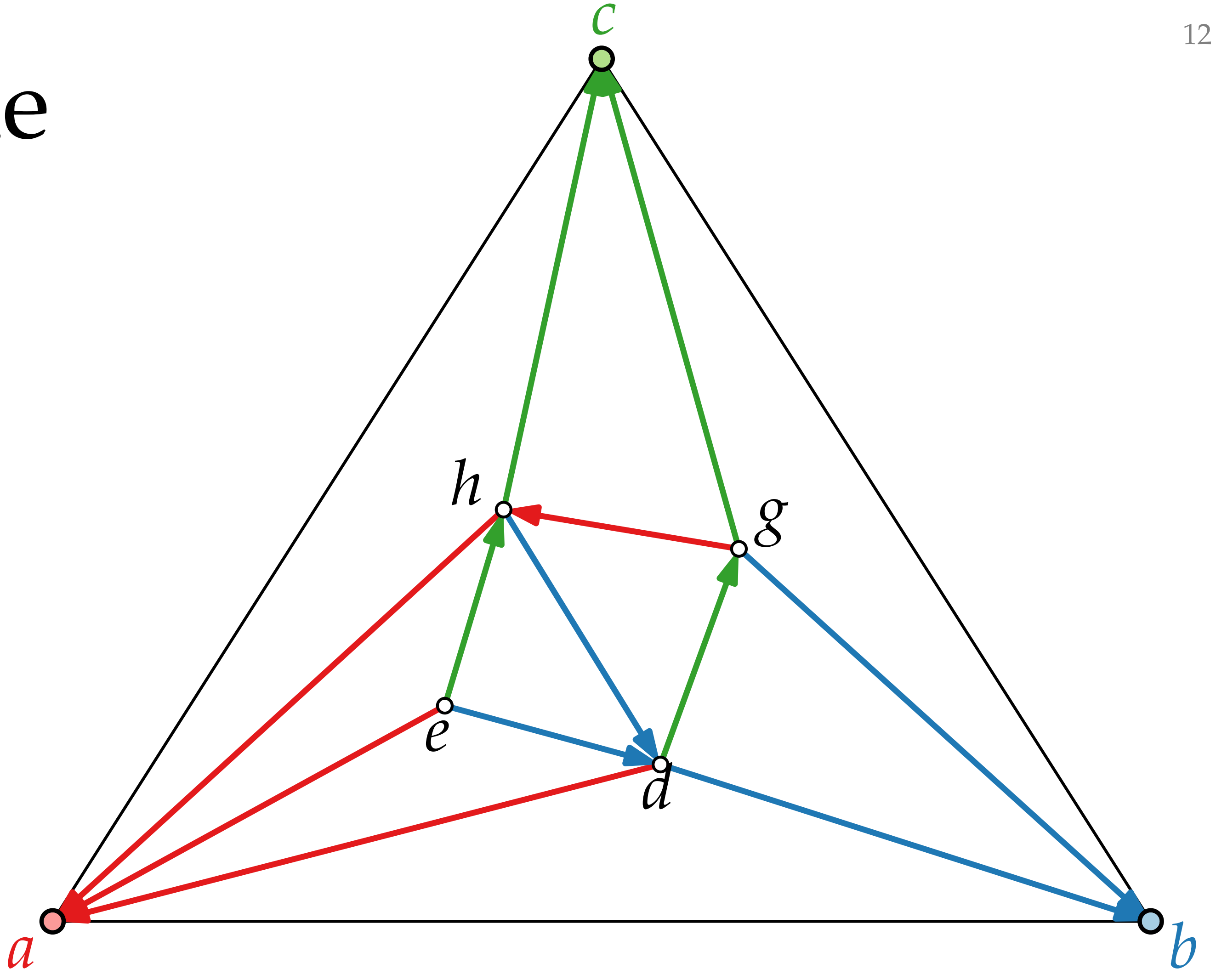
Schnyder Drawing – Example



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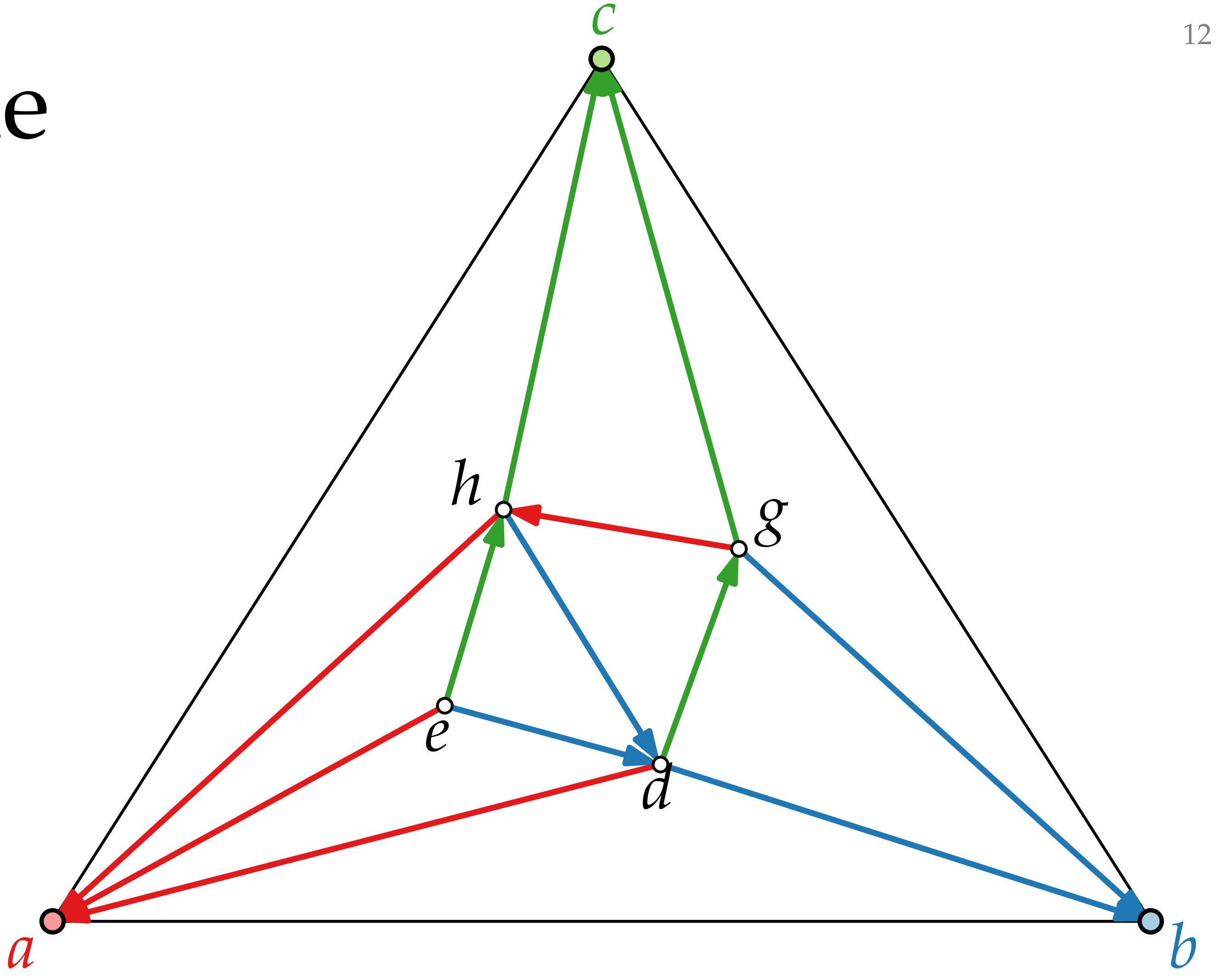
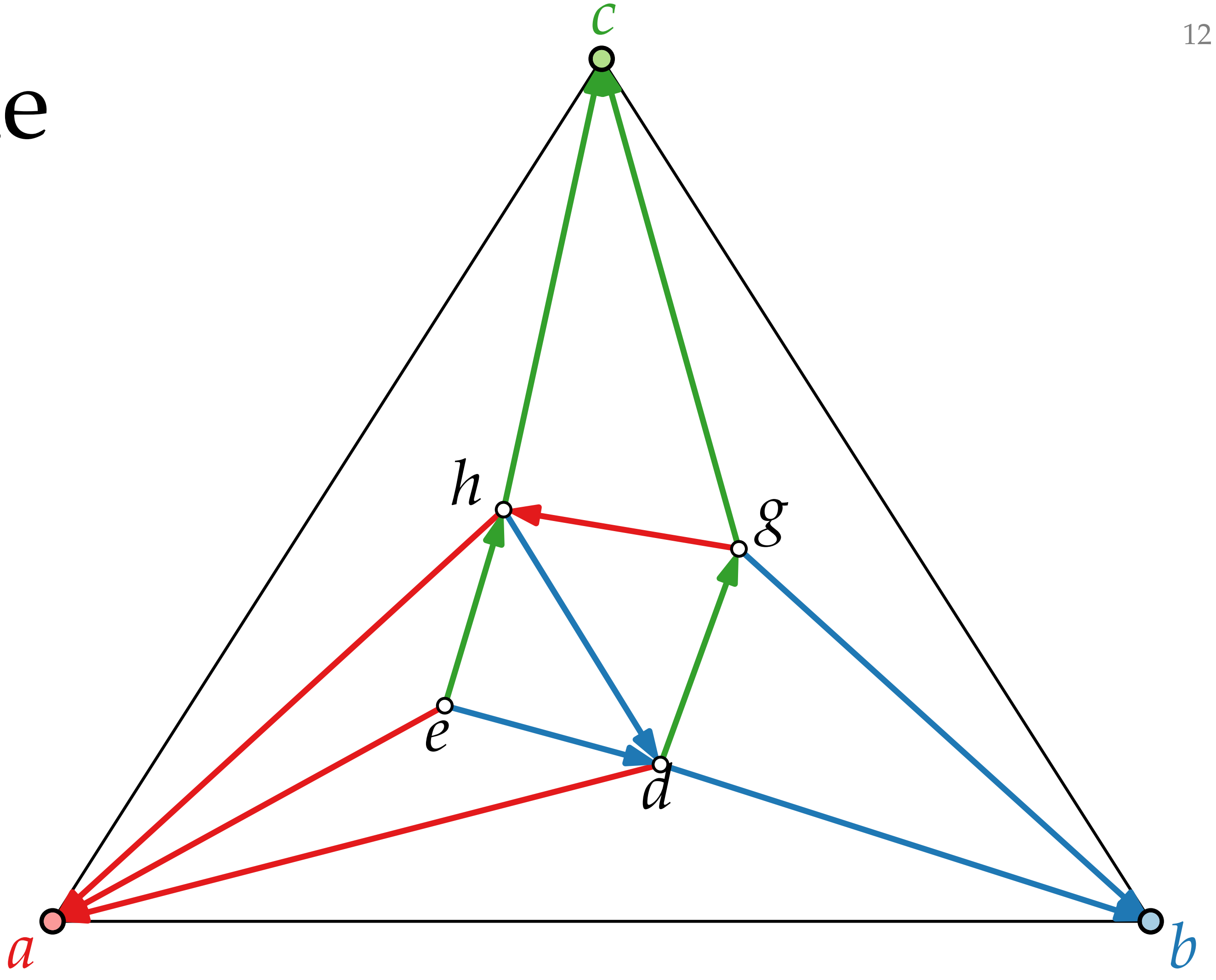


$n = 7,\, 2n - 5 = 9$

Schnyder Drawing – Example



$n = 7, 2n - 5 = 9$

$f(a) = (9, 0, 0)$

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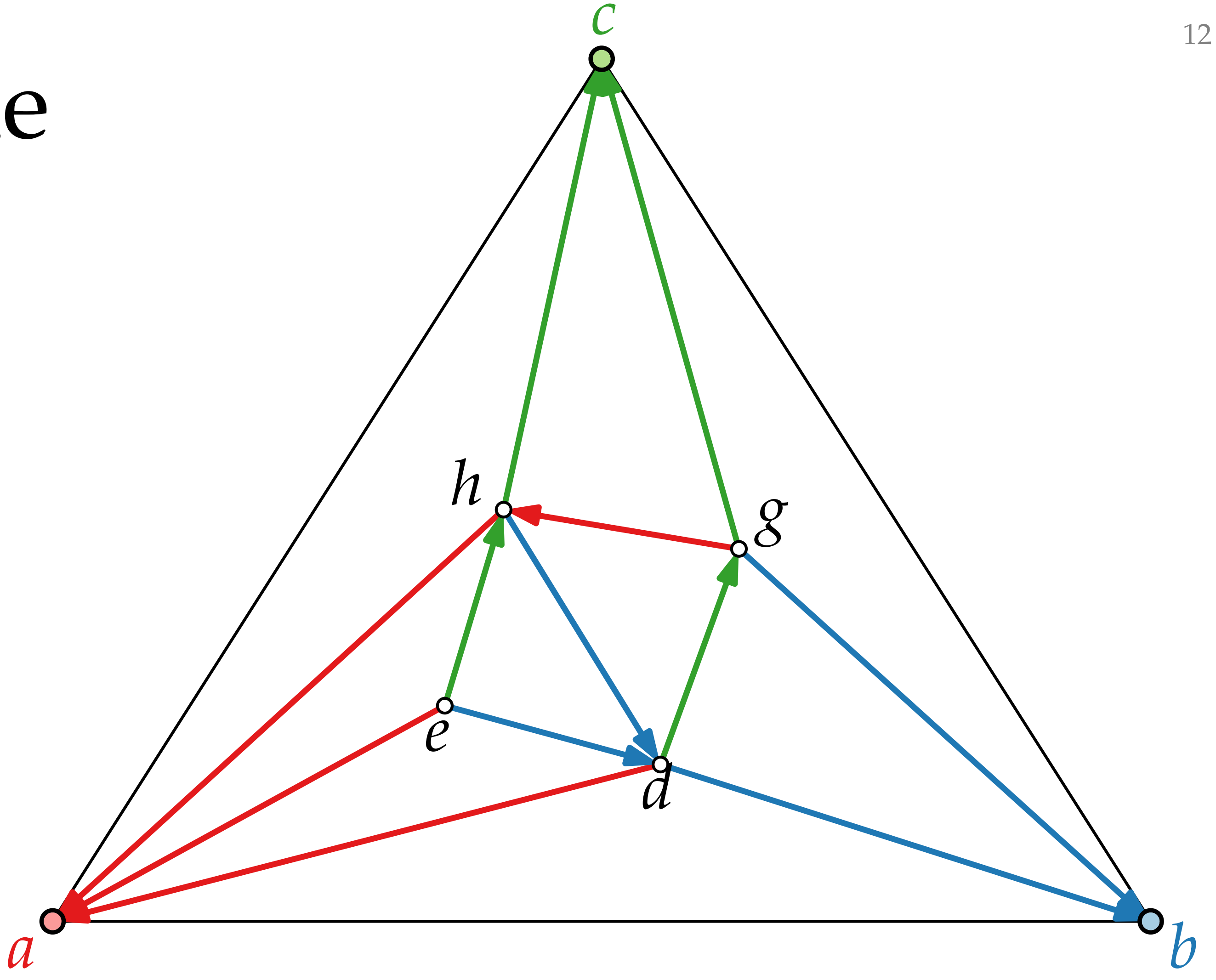

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$f(a) = (9, 0, 0)$

$f(b) = (0, 9, 0)$

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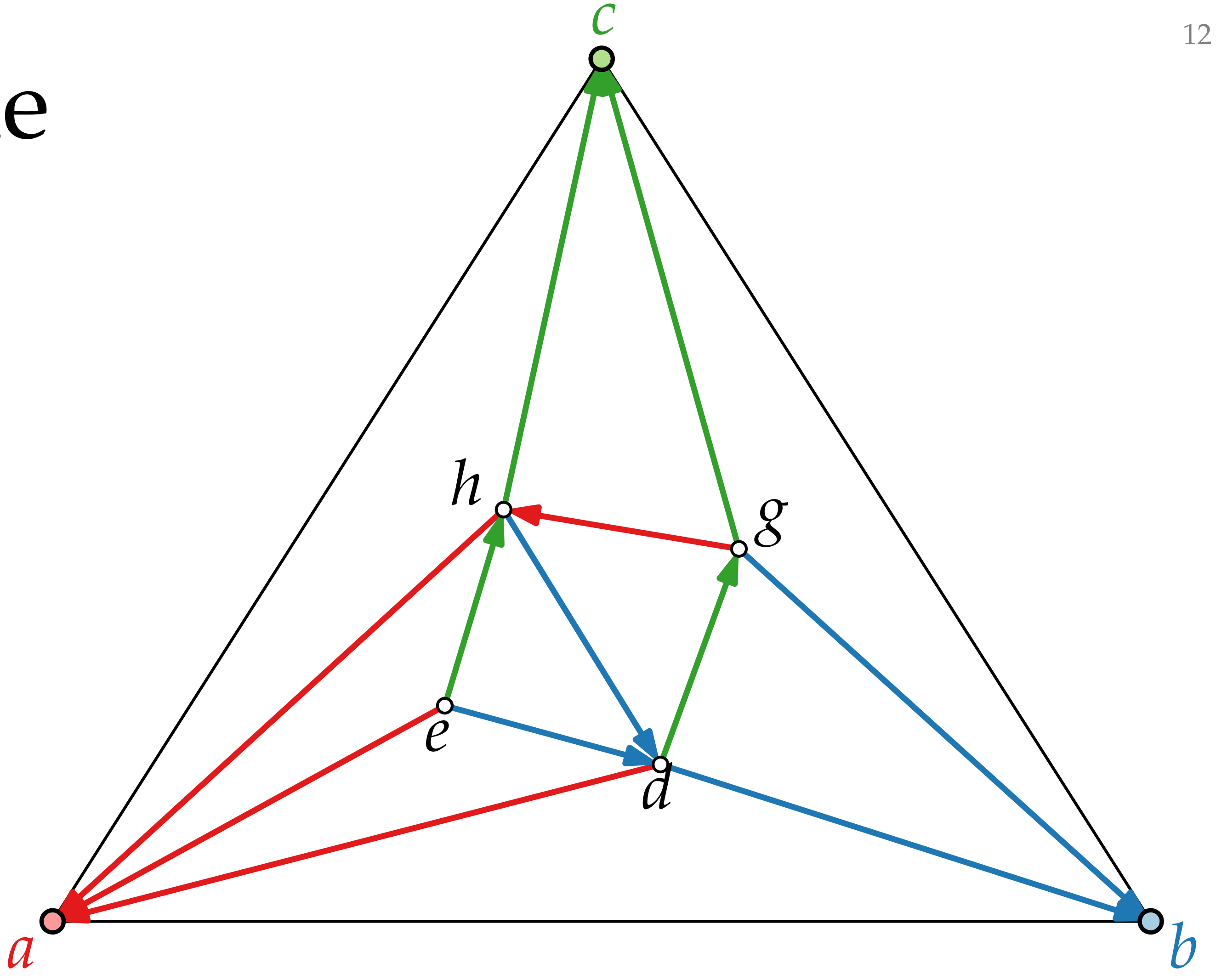

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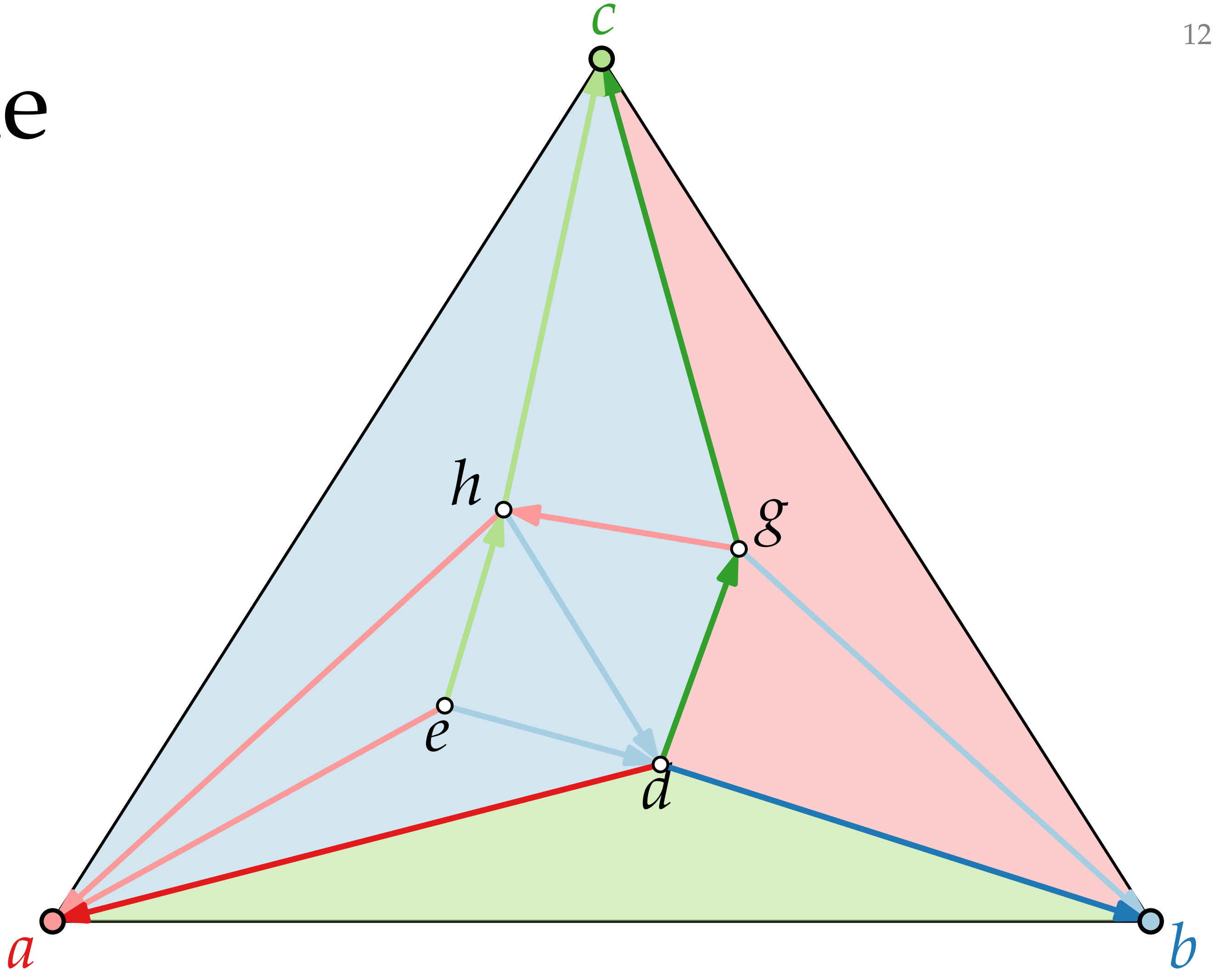

$n = 7, 2n - 5 = 9$ $\quad f(d) = (2, 6, 1)$

$f(a) = (9, 0, 0)$

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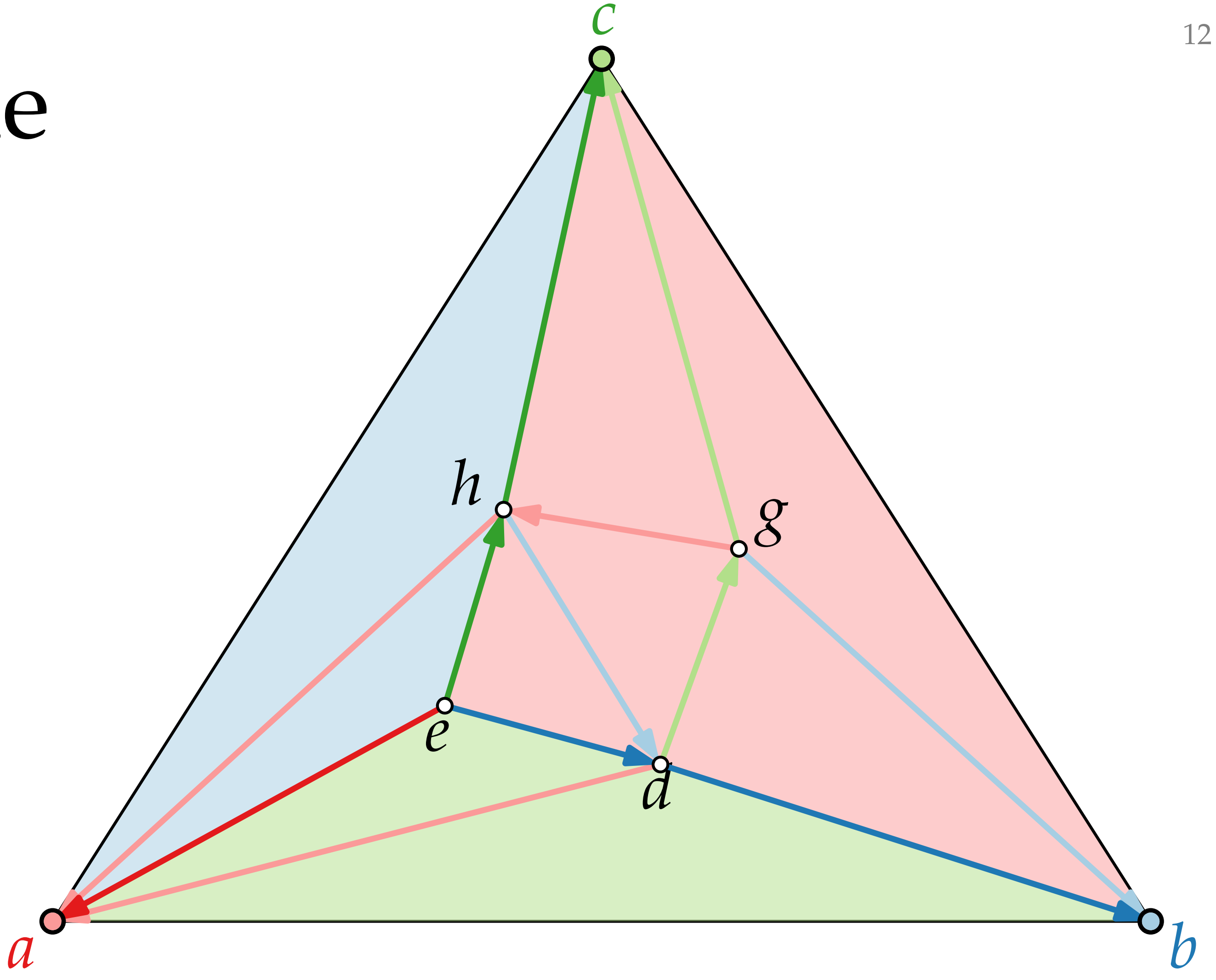

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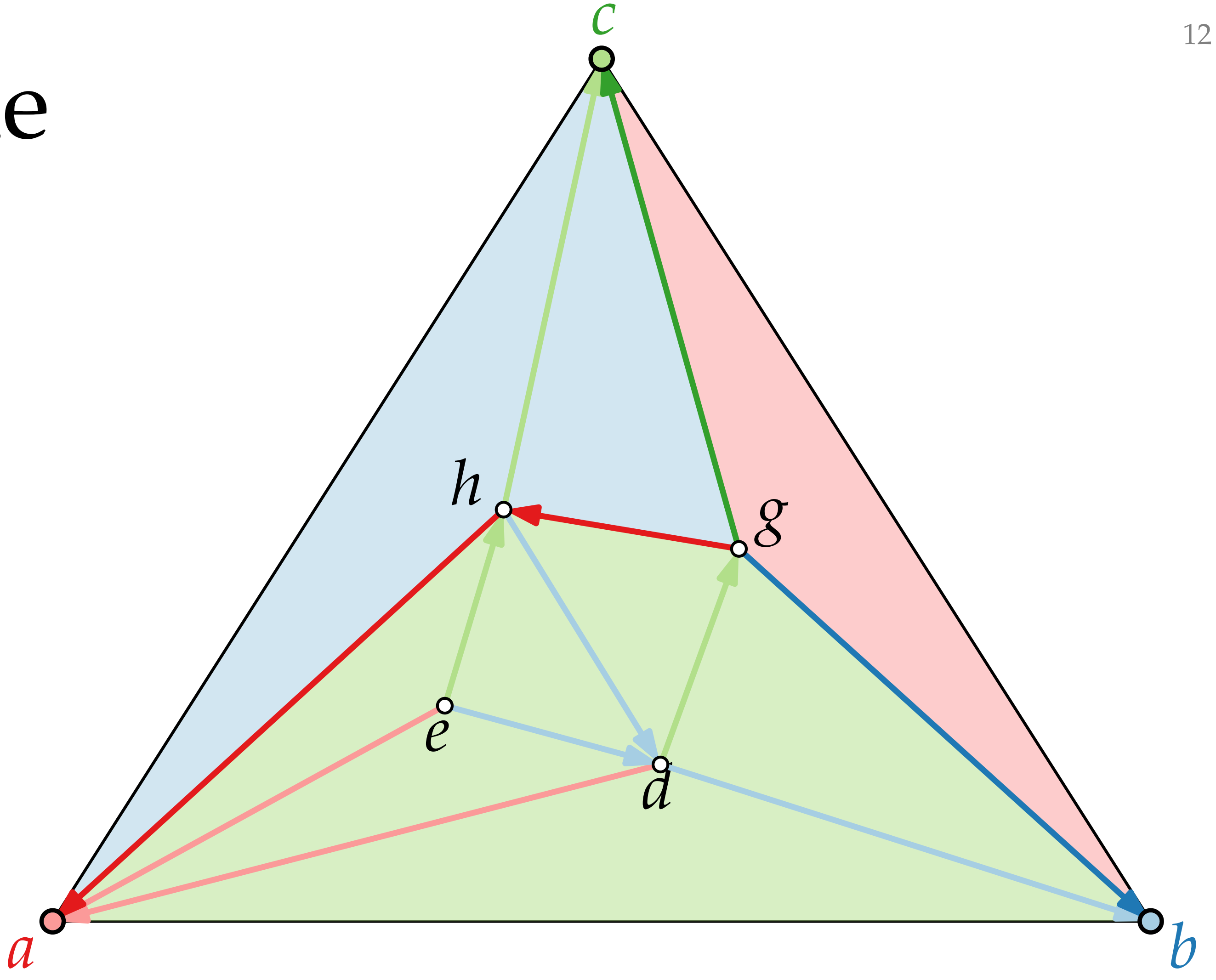

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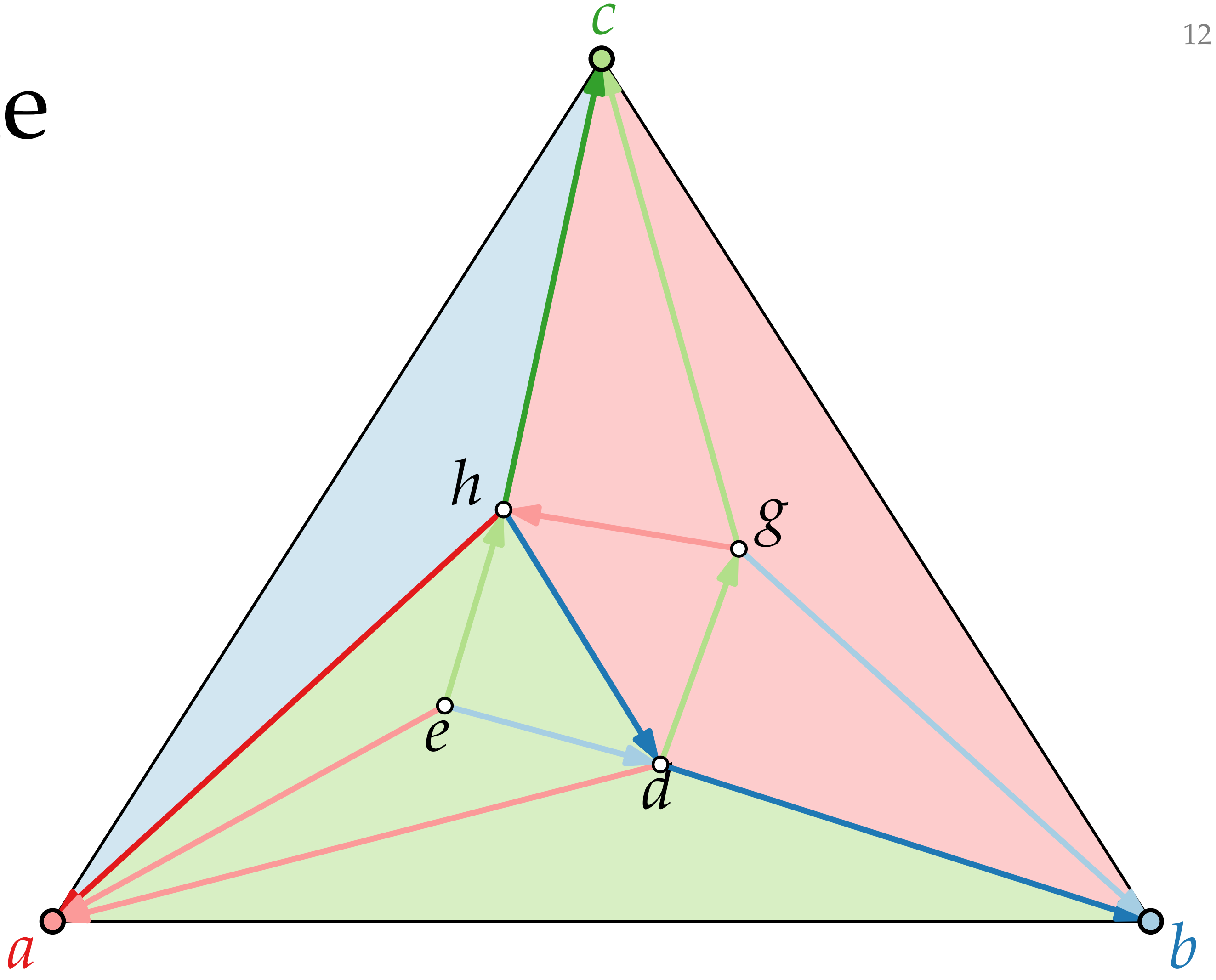

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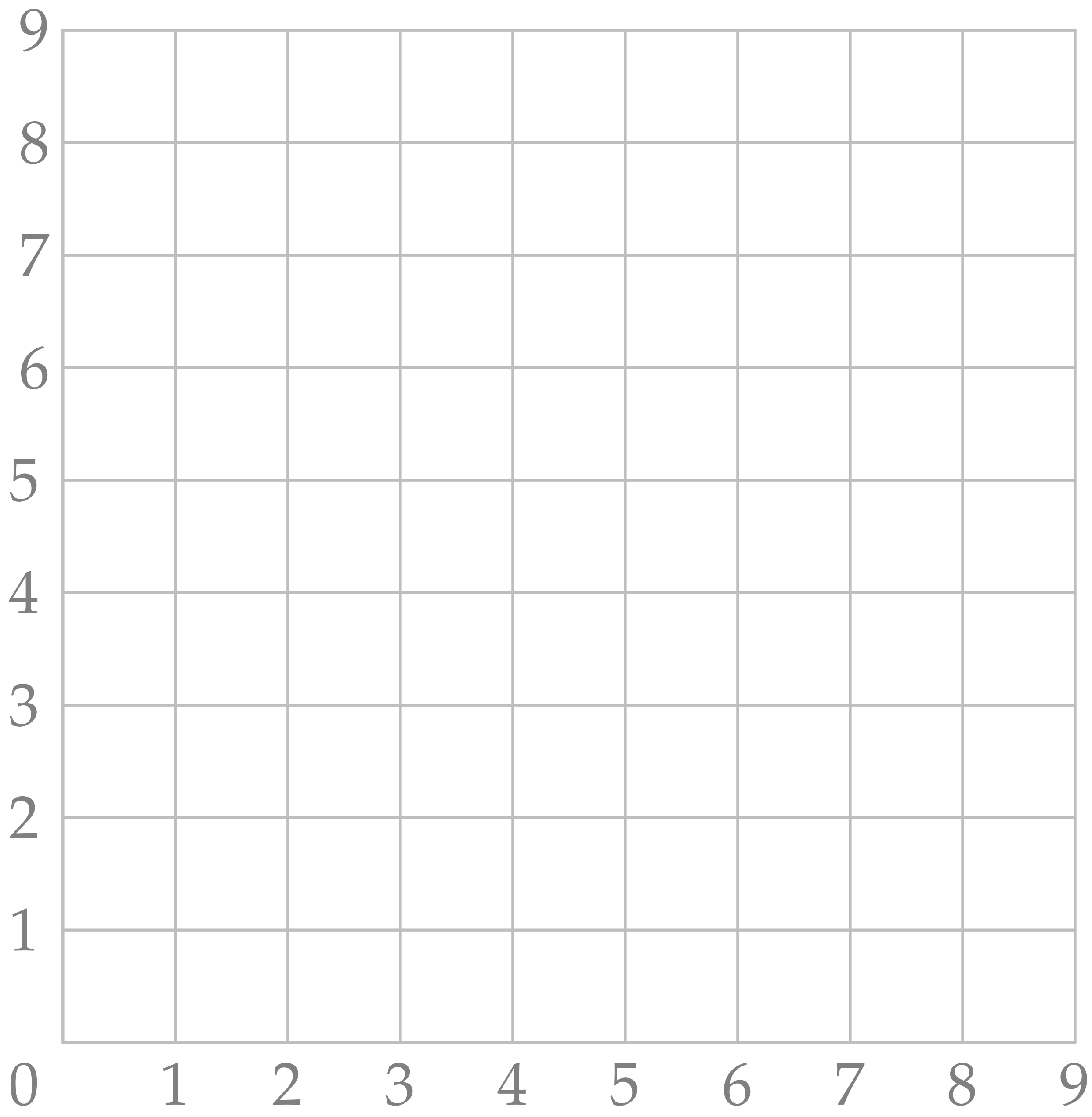

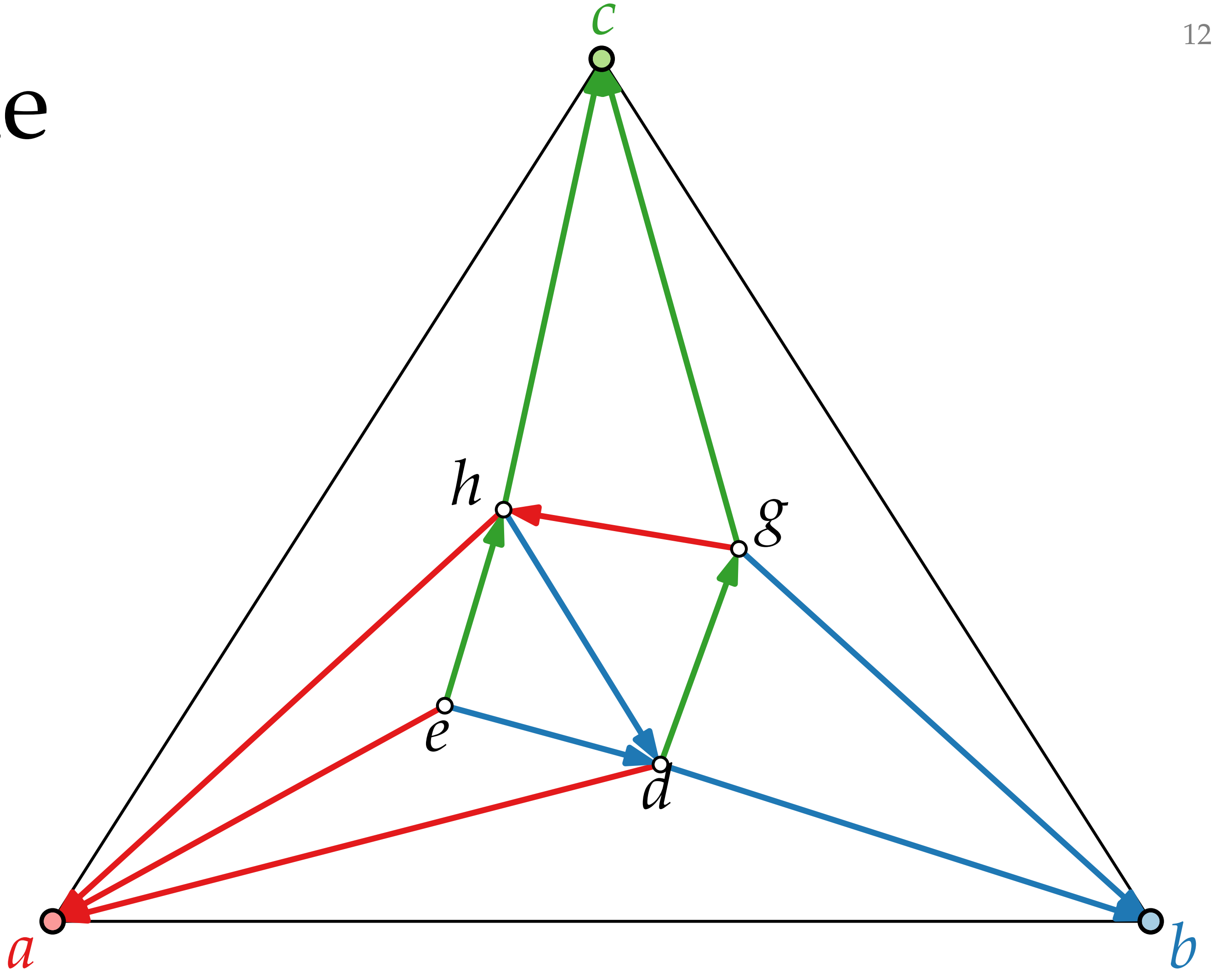

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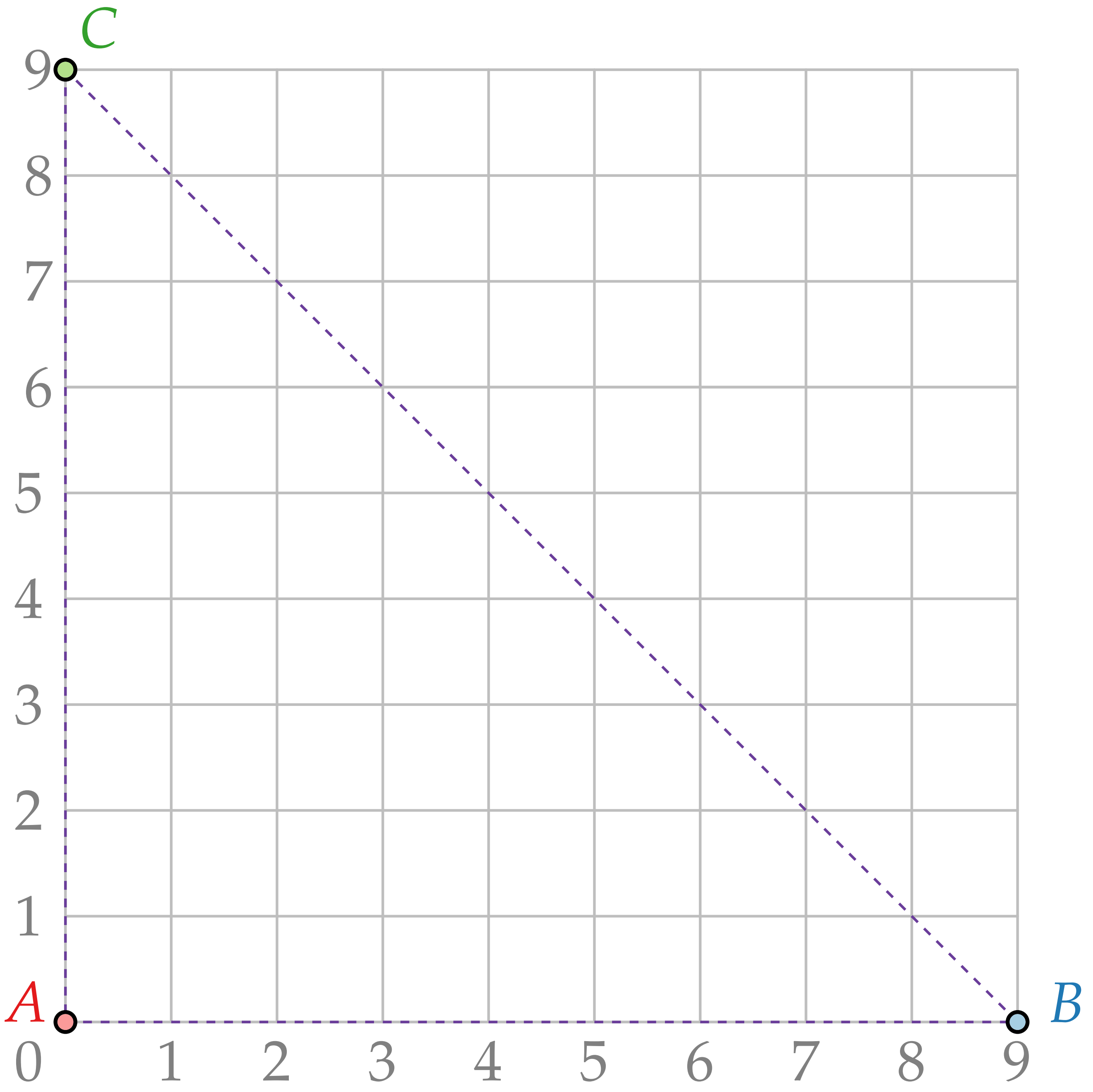

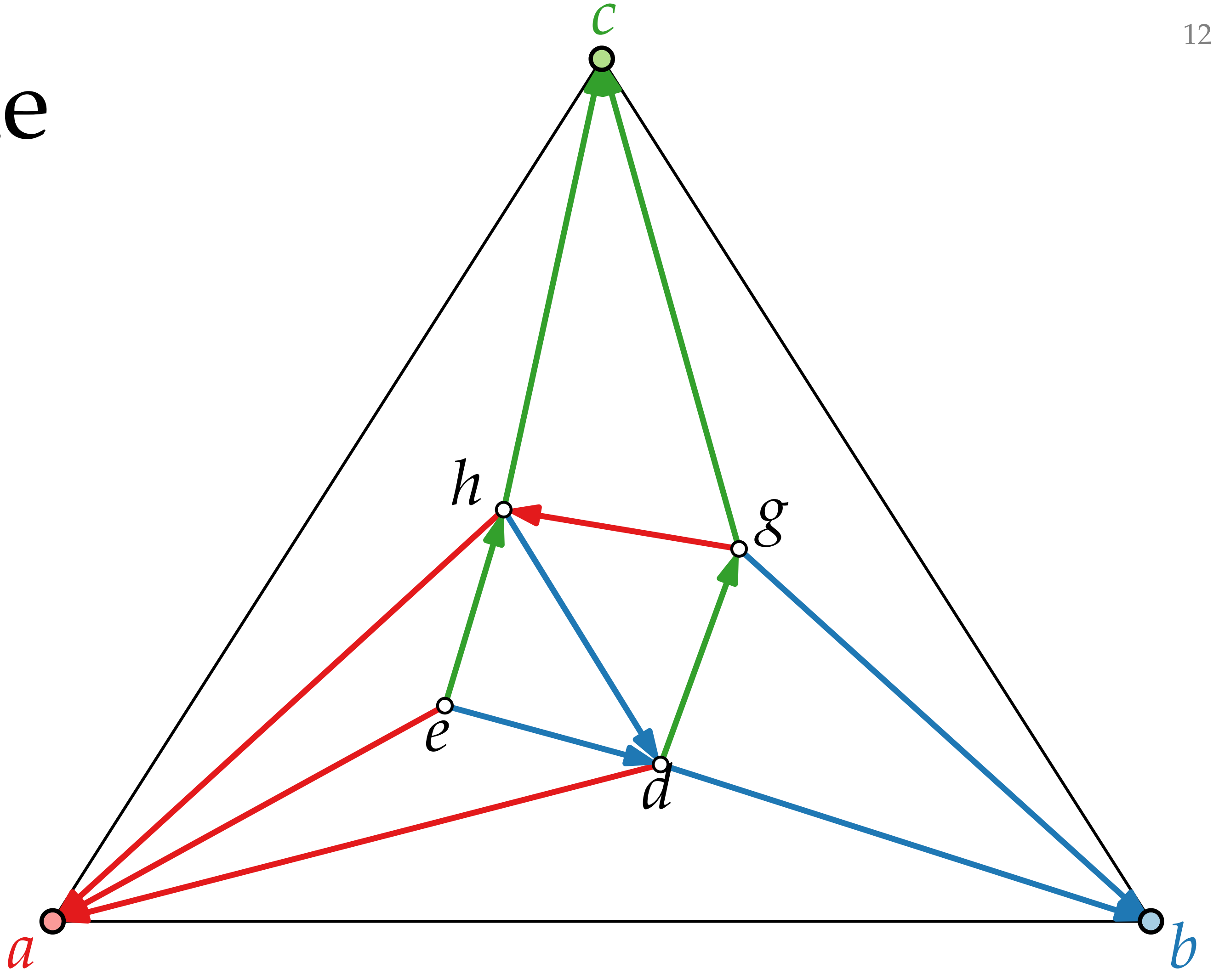

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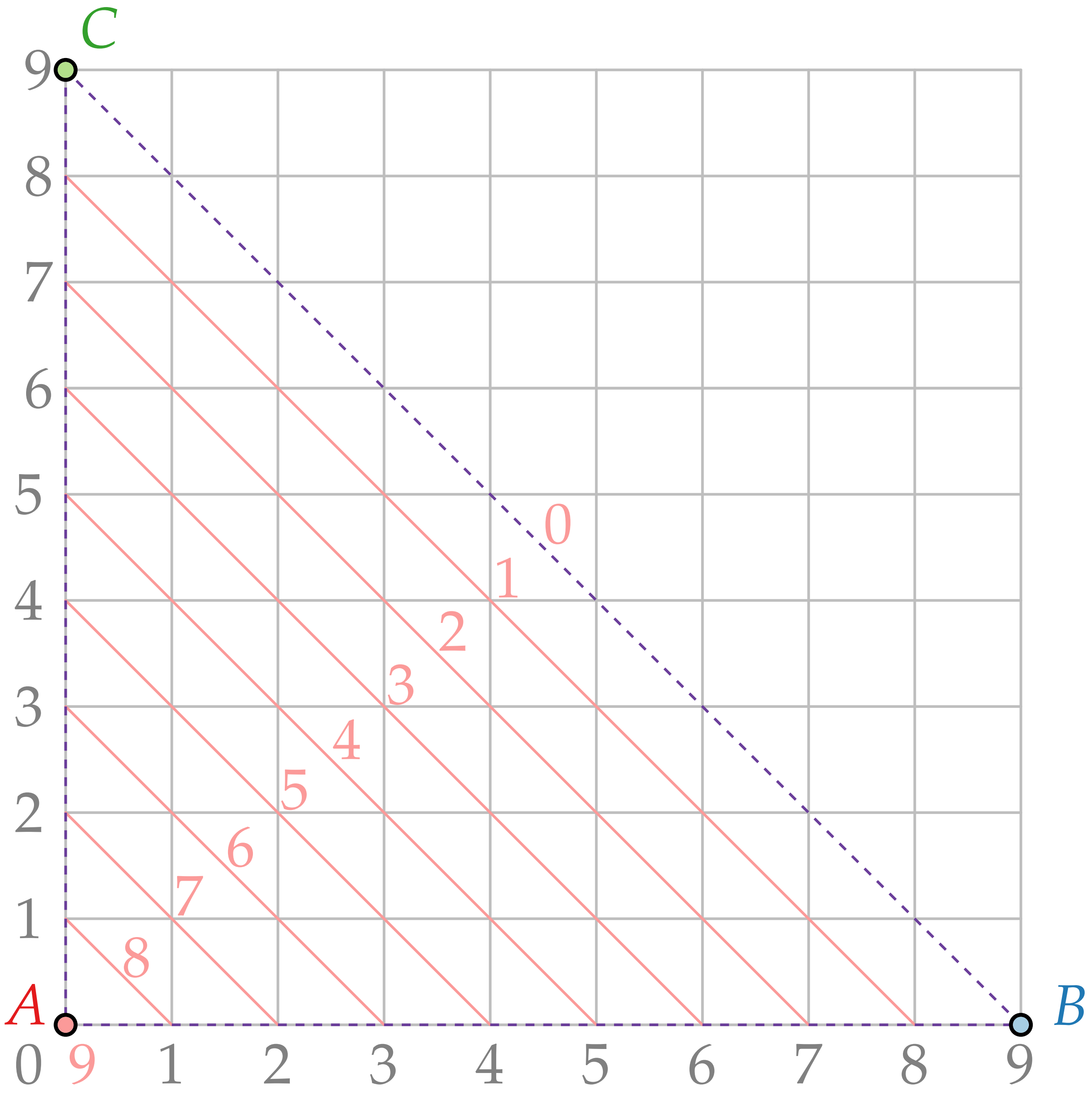

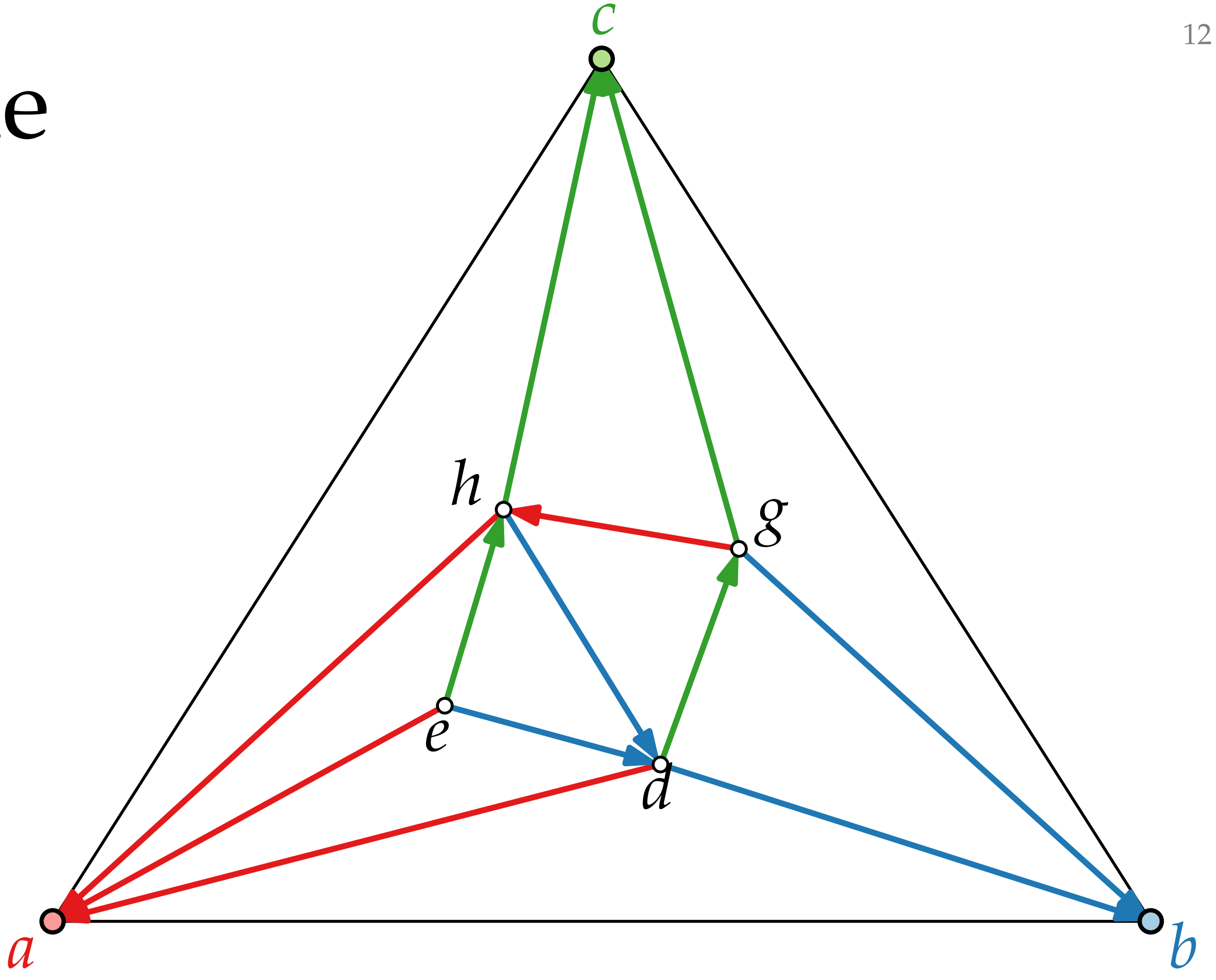

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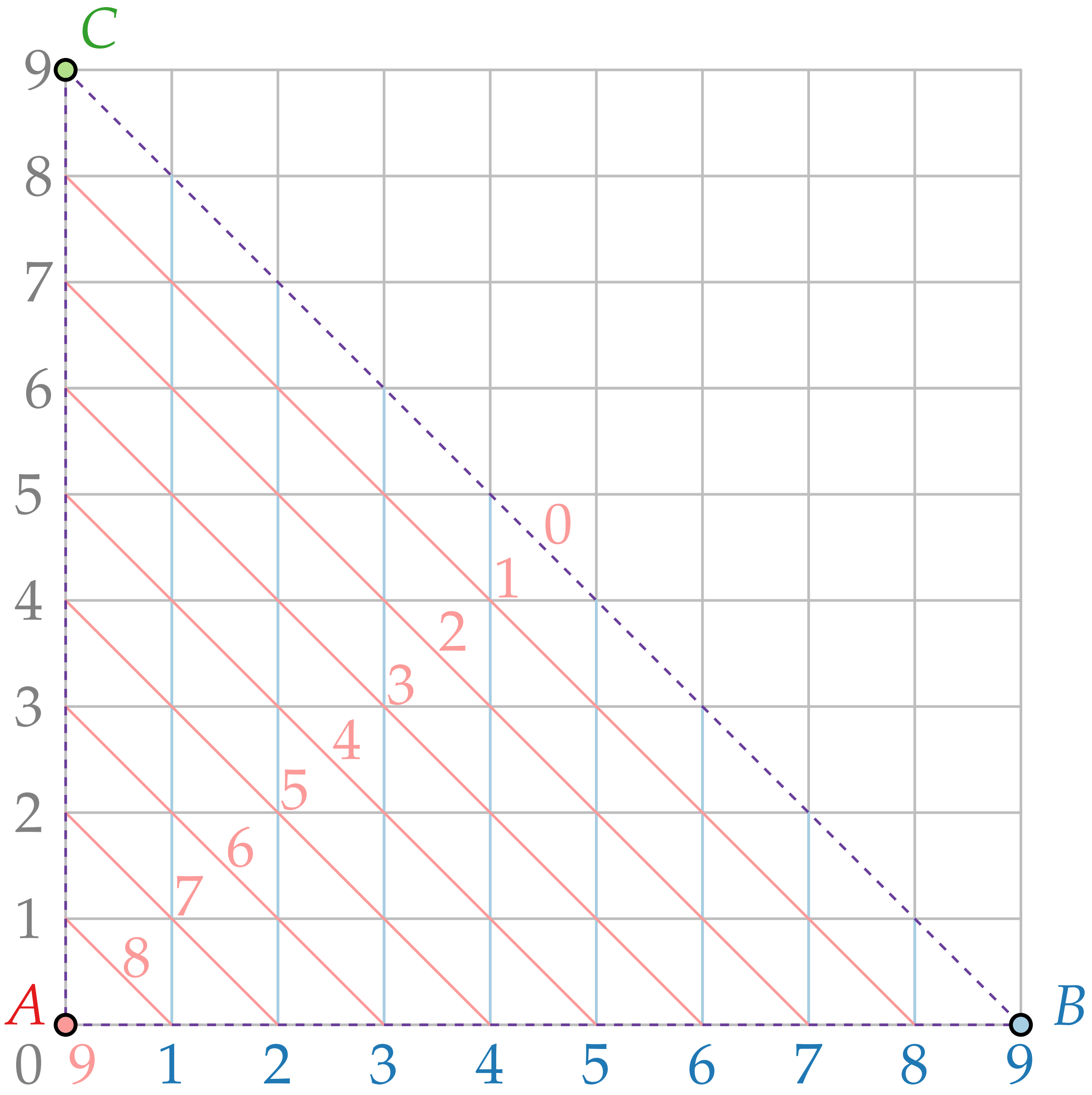

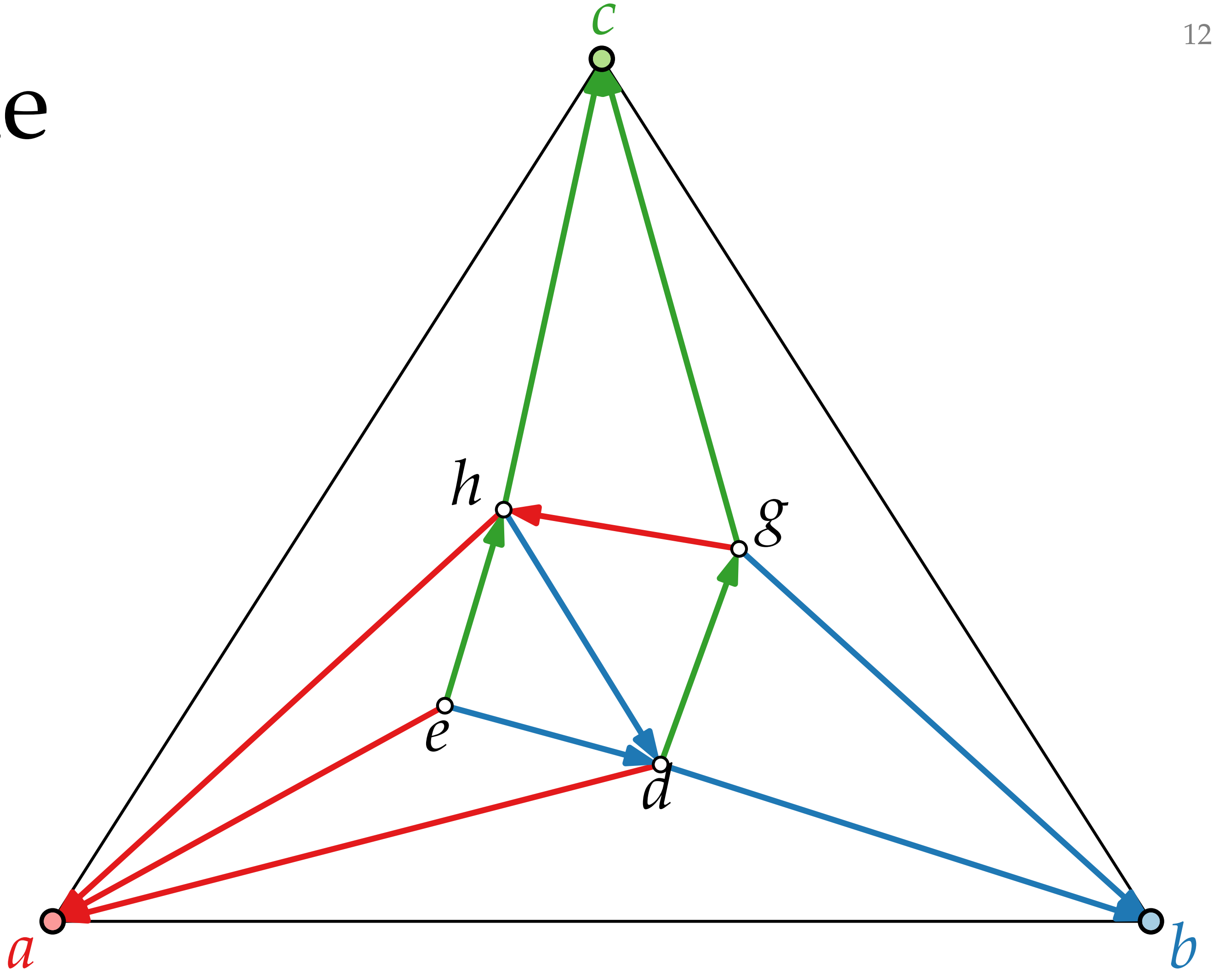

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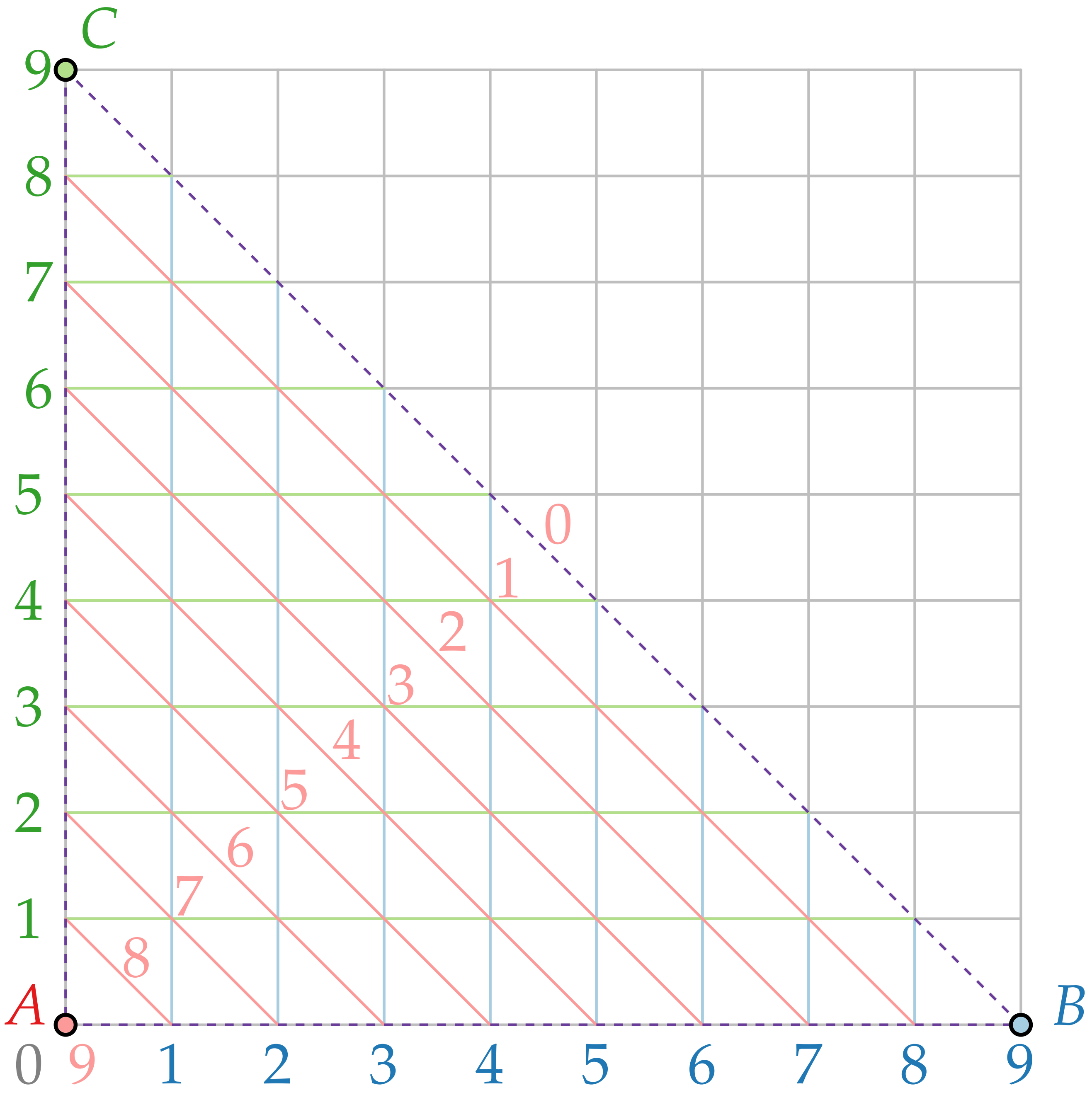

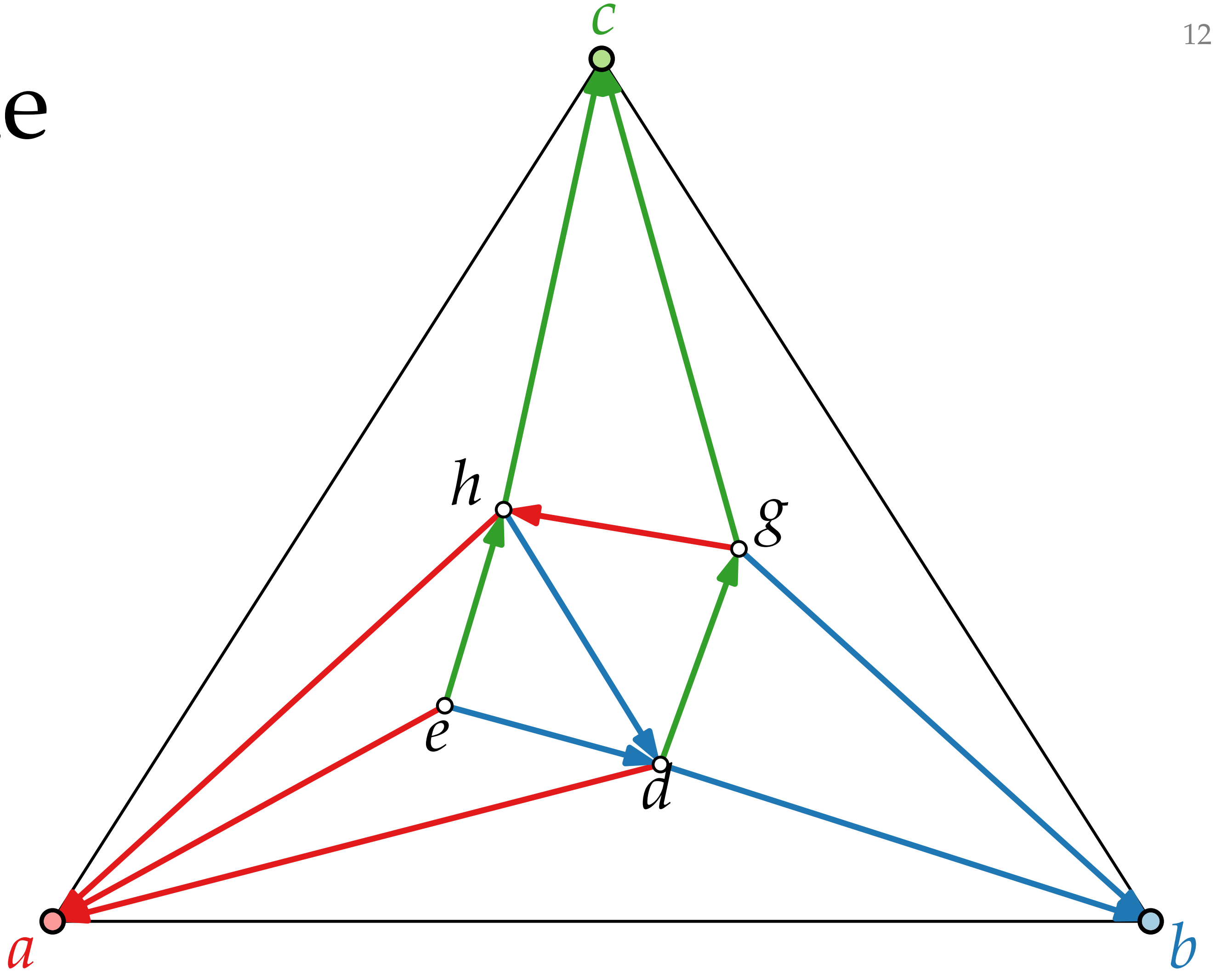

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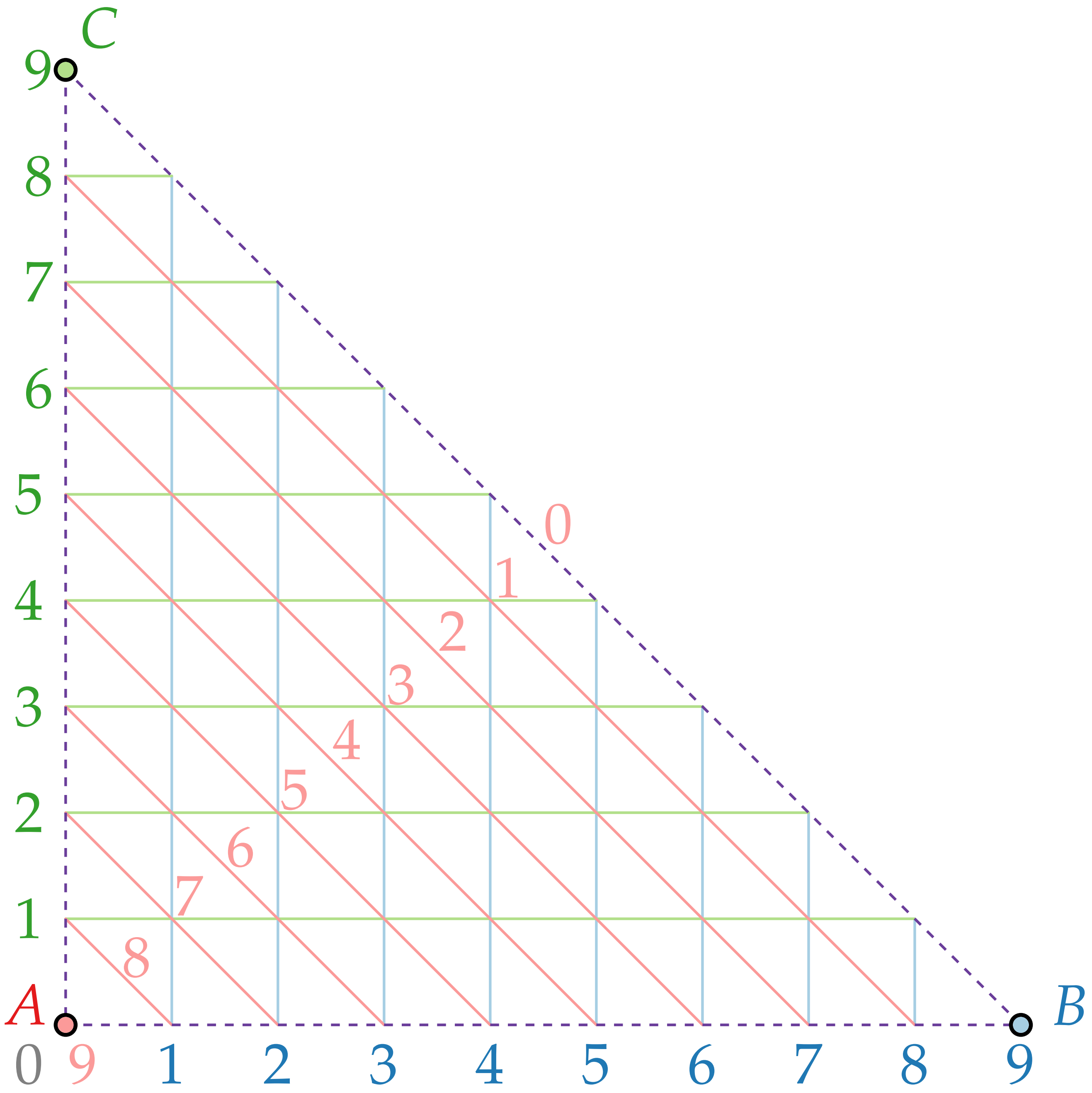

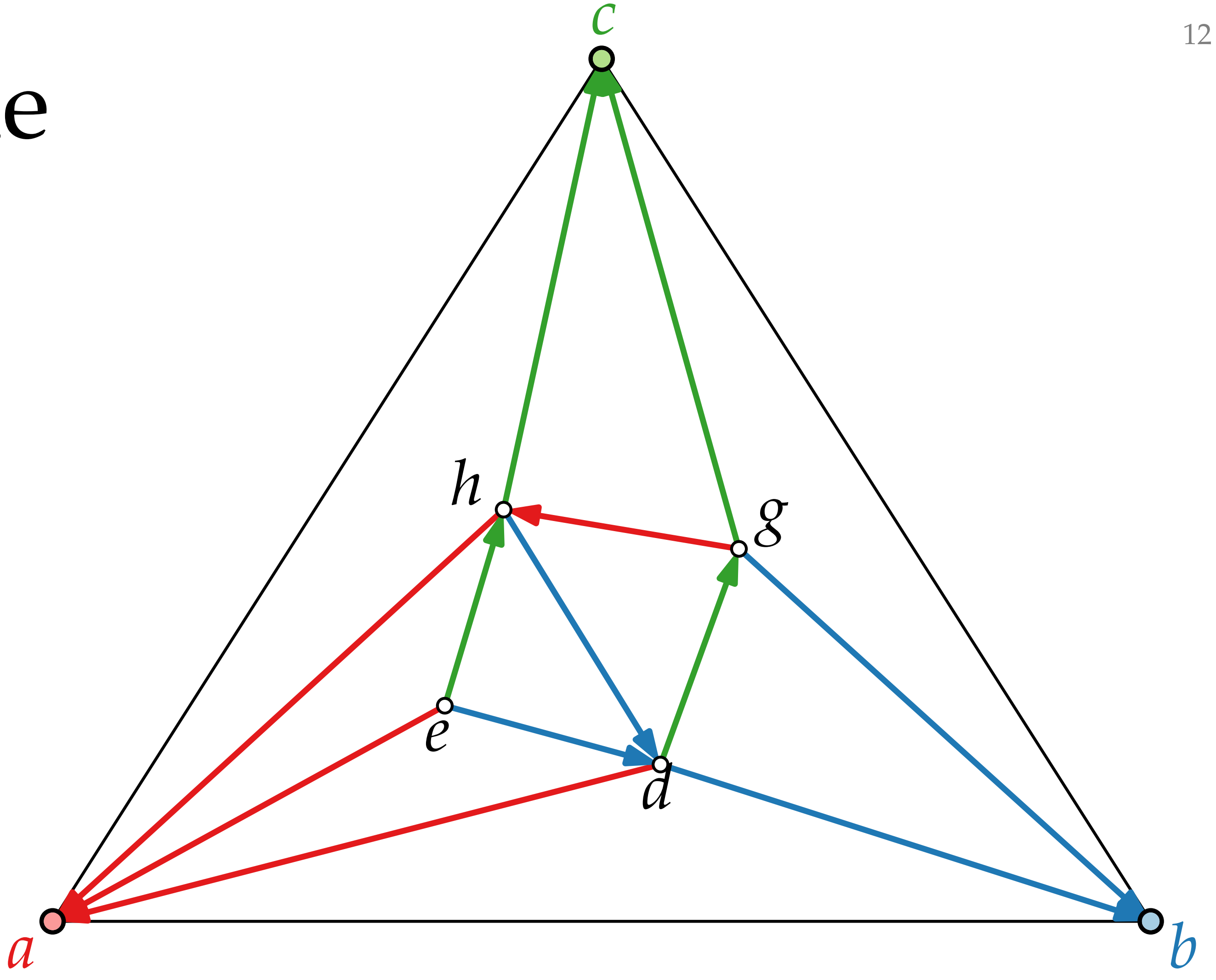

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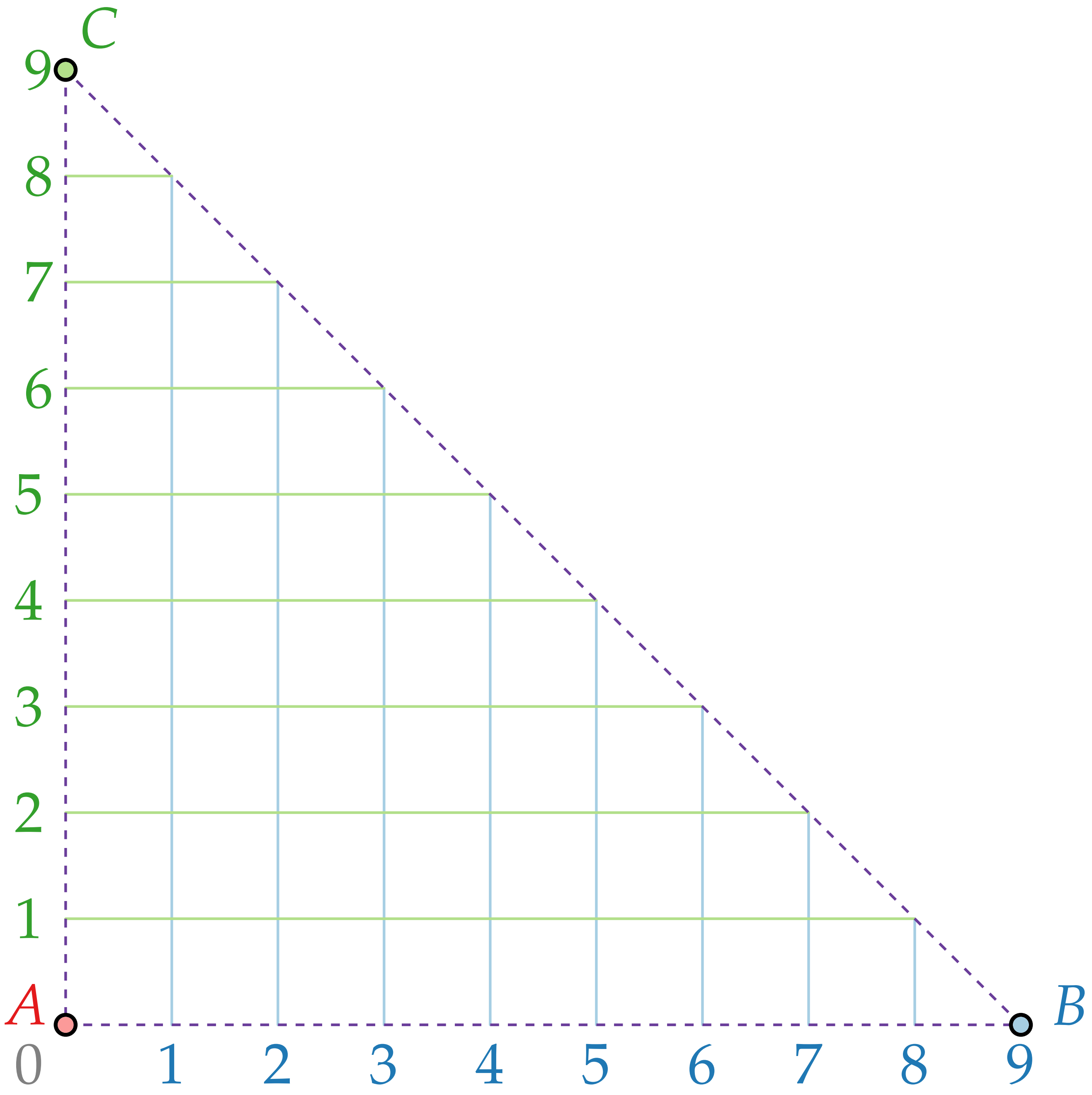

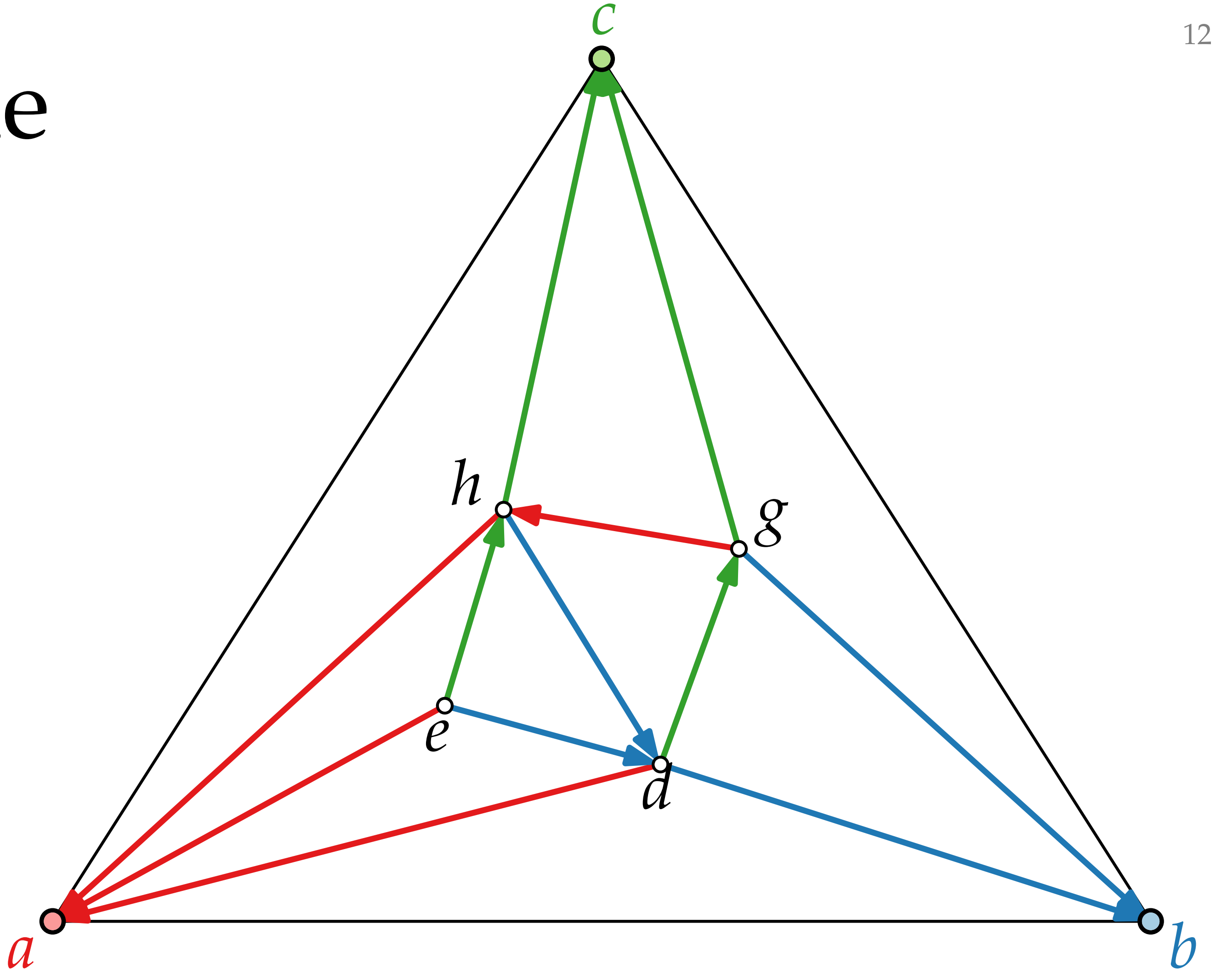

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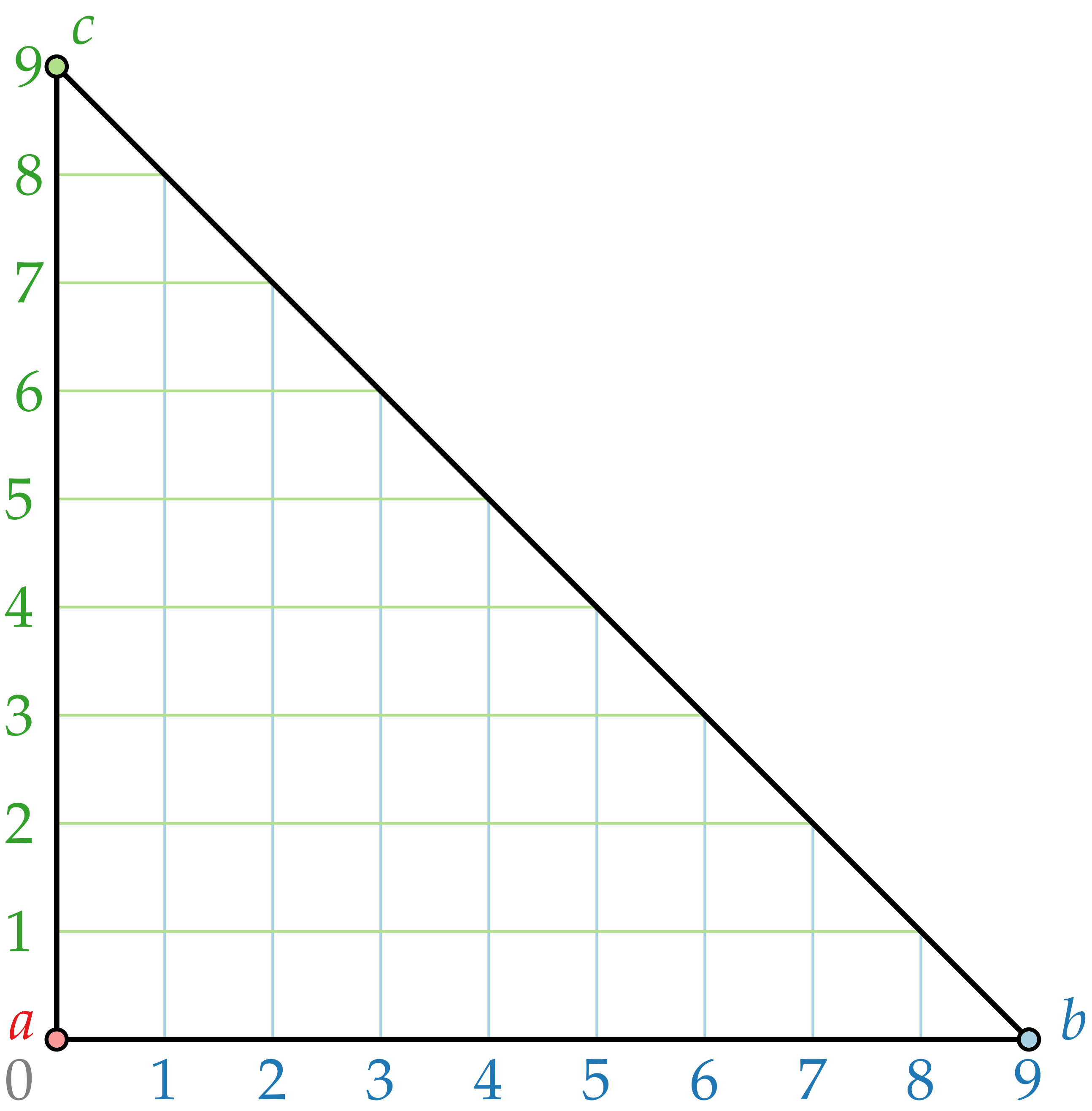

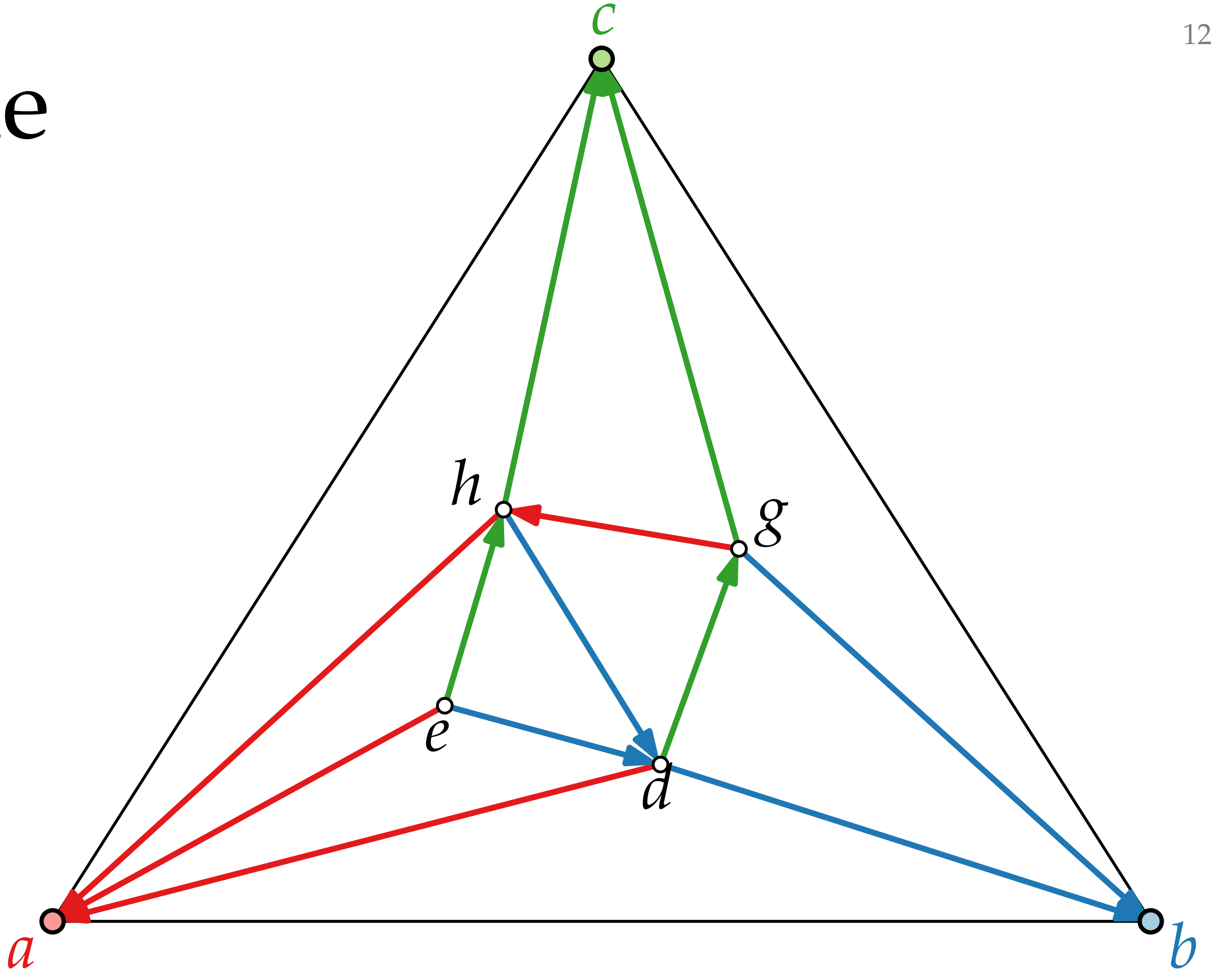

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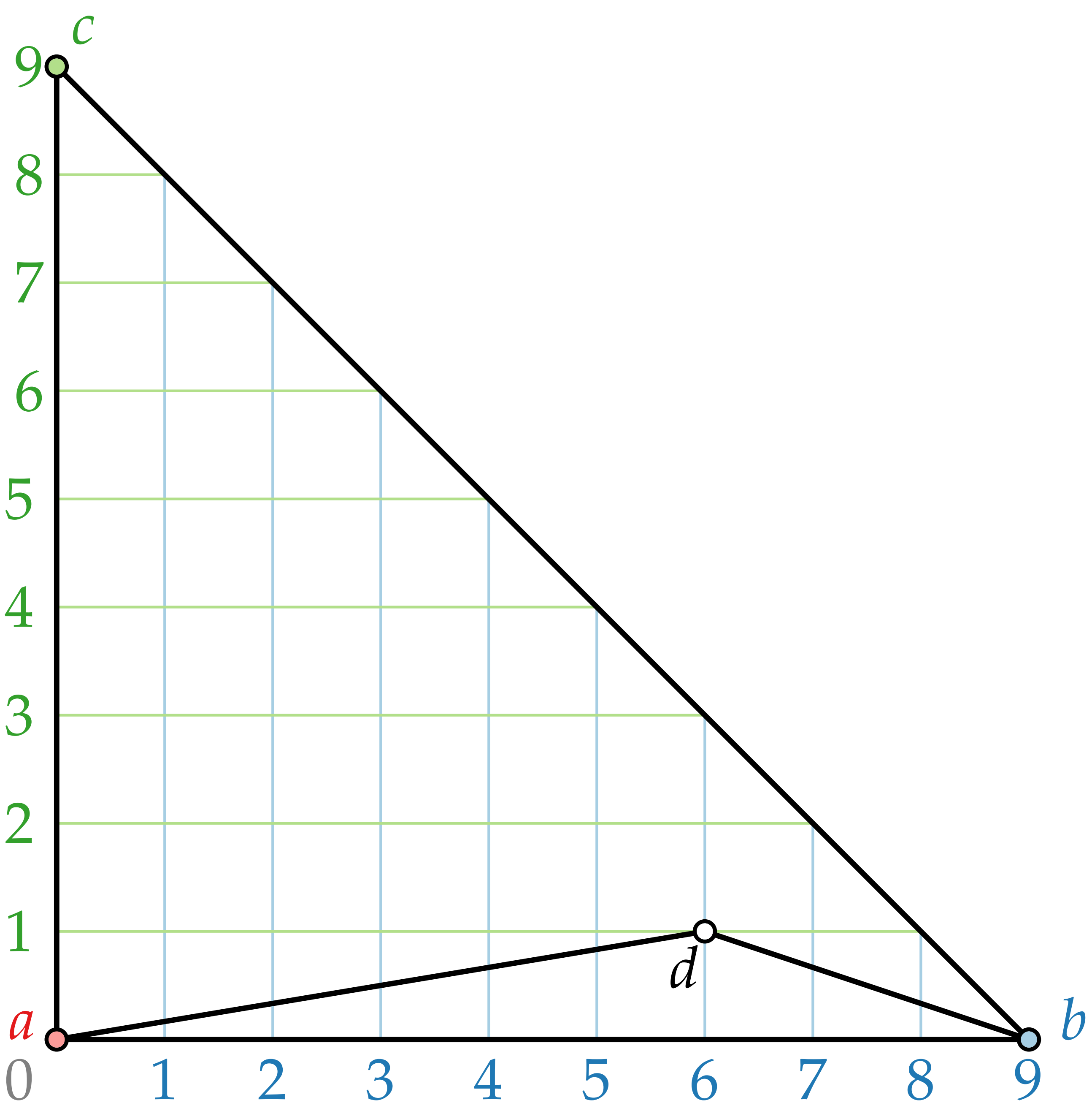

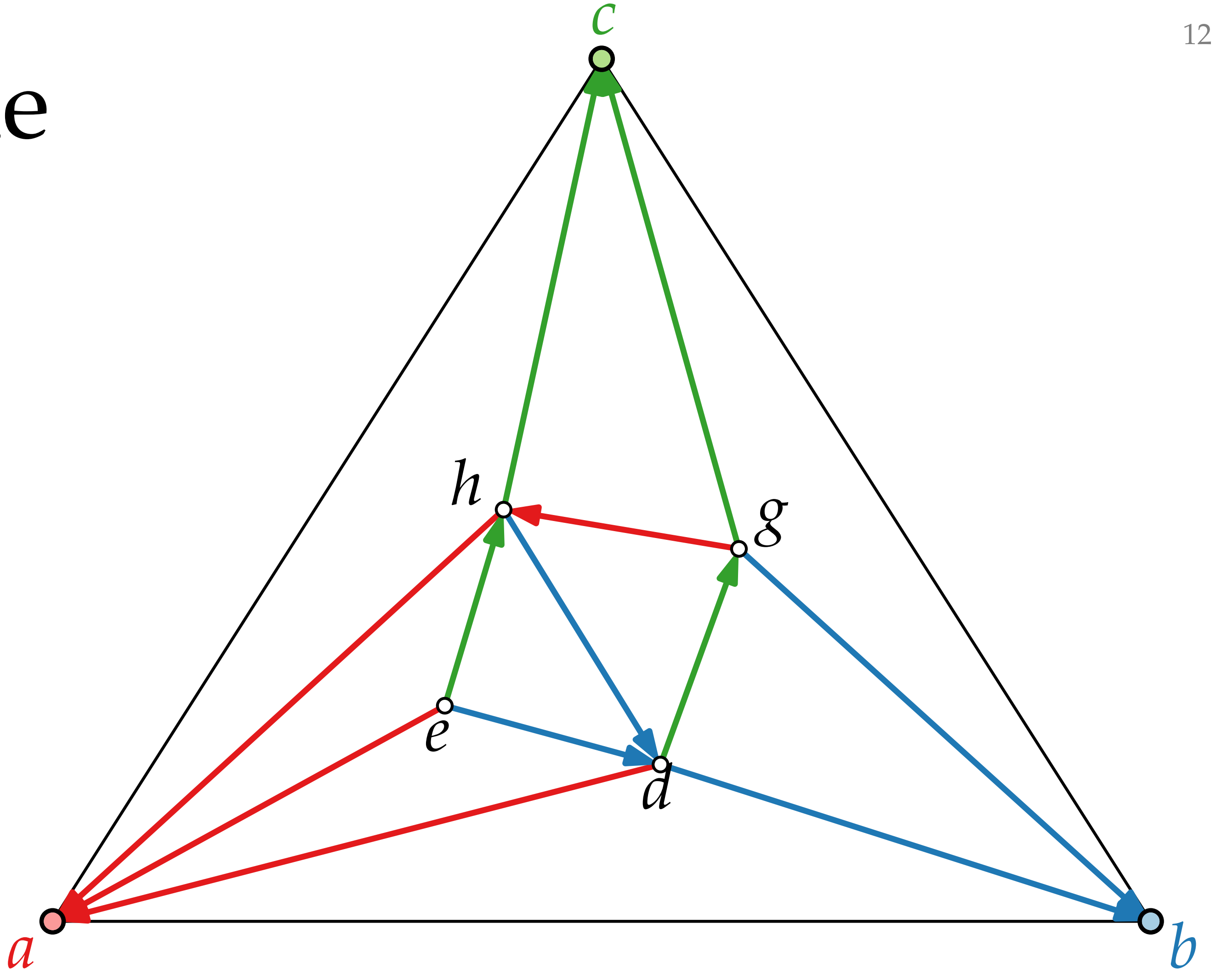

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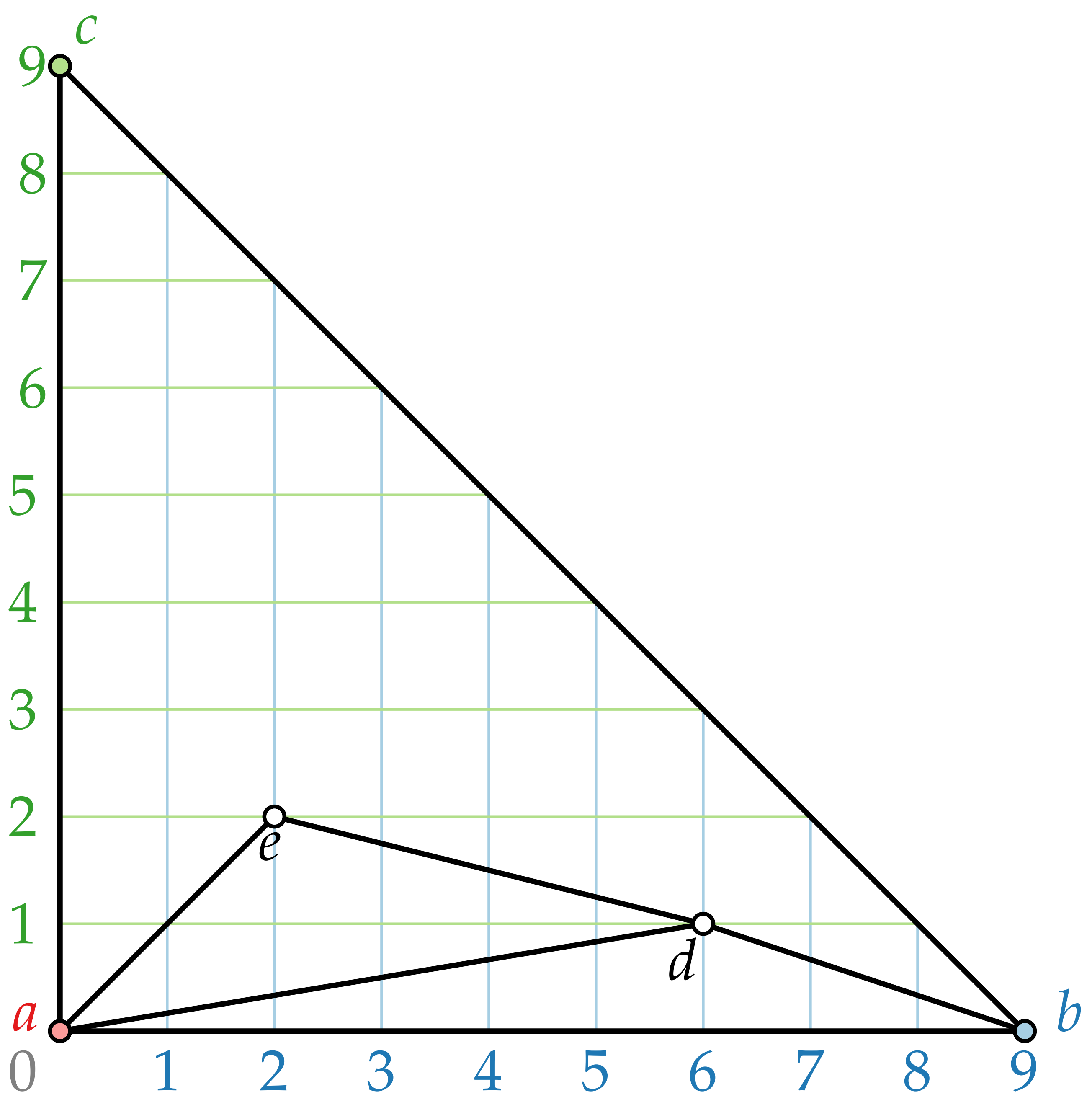

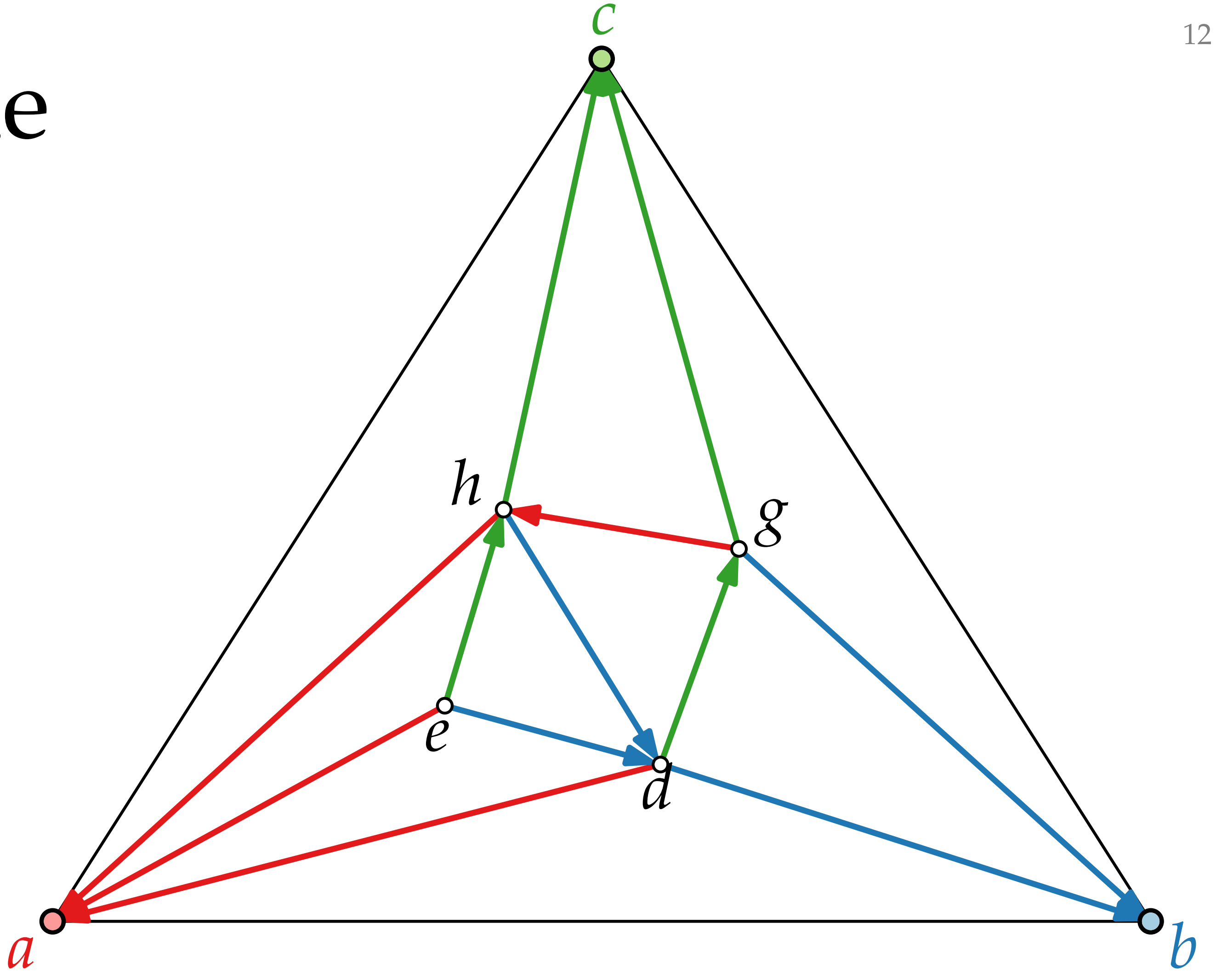

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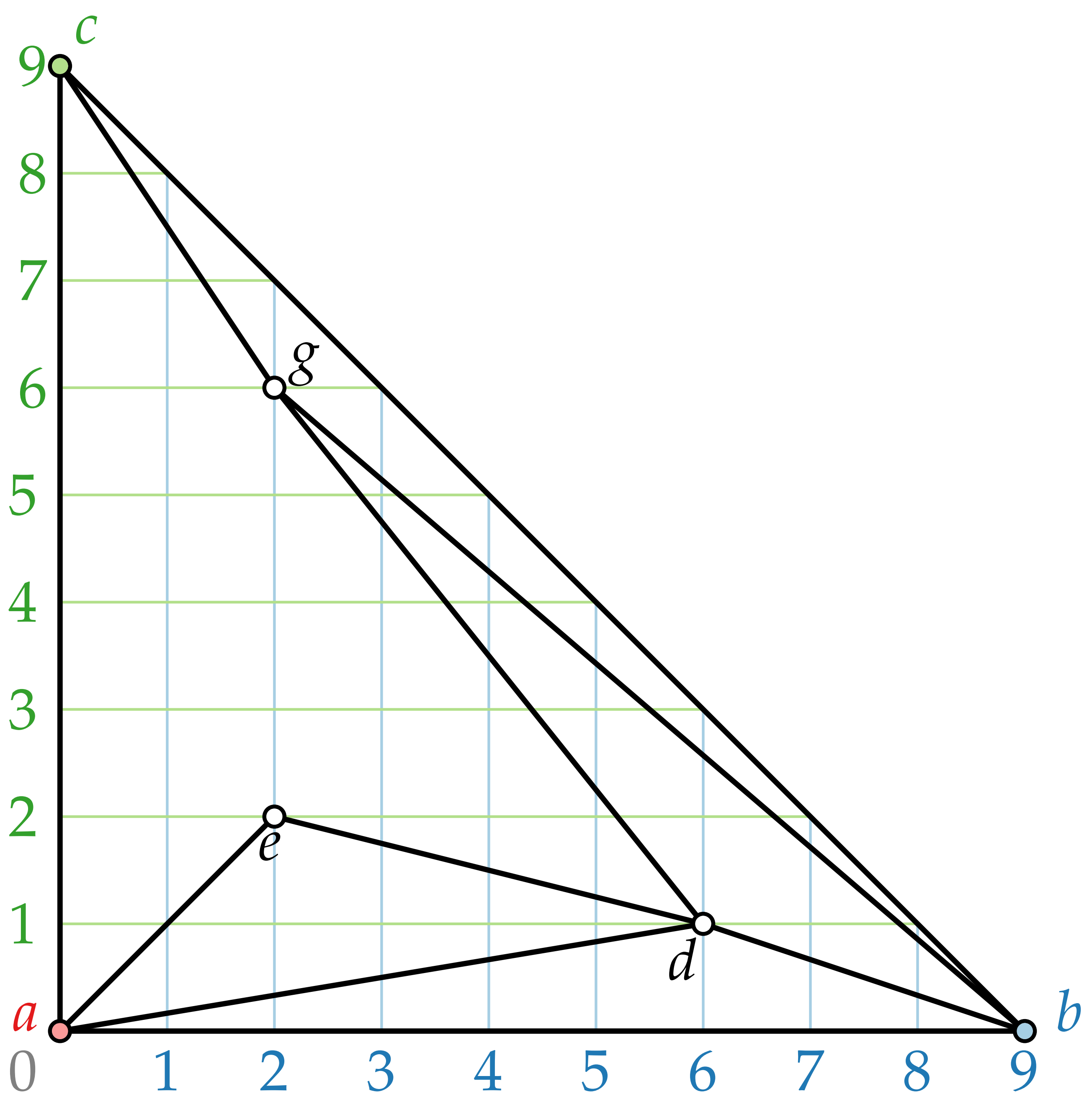

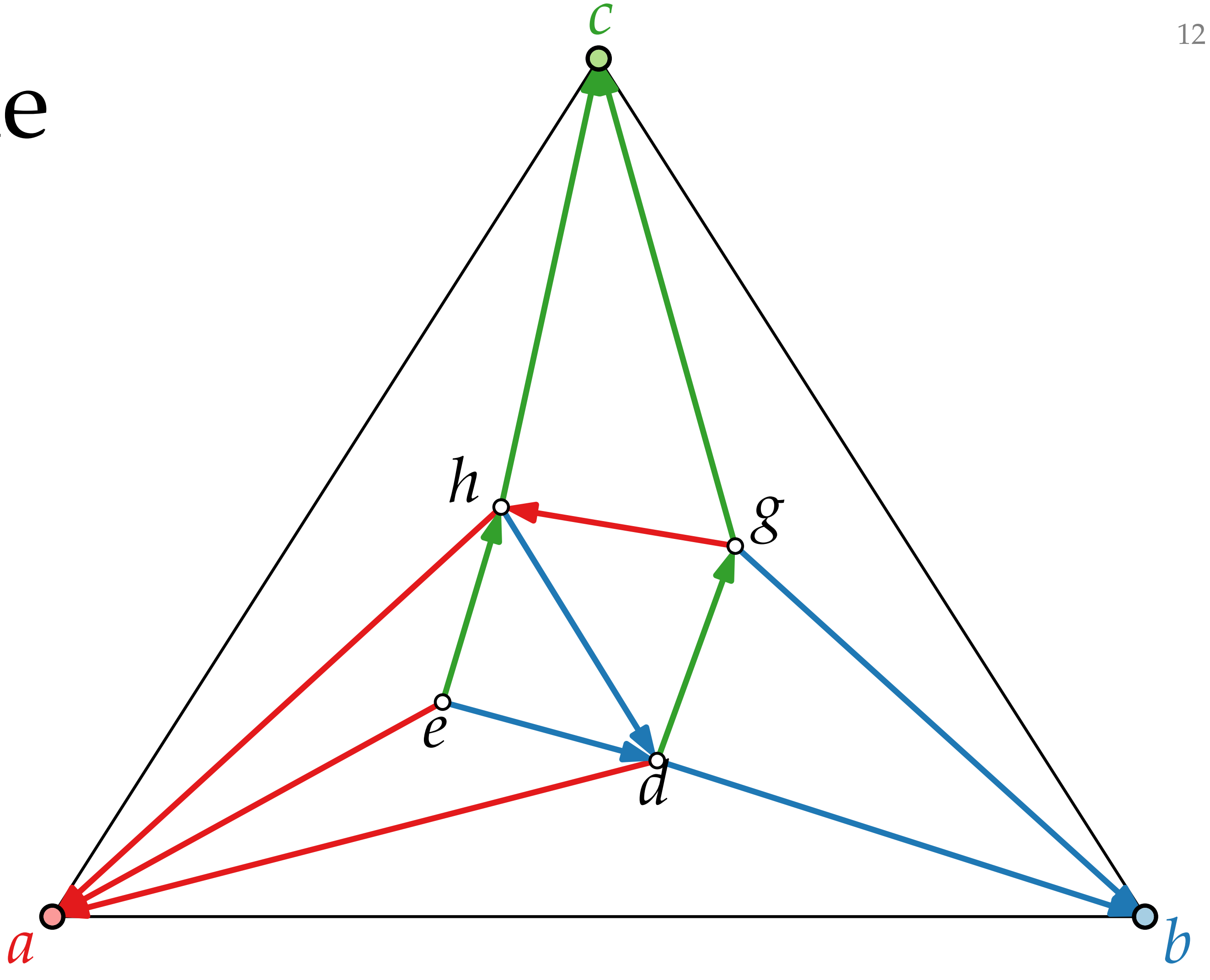

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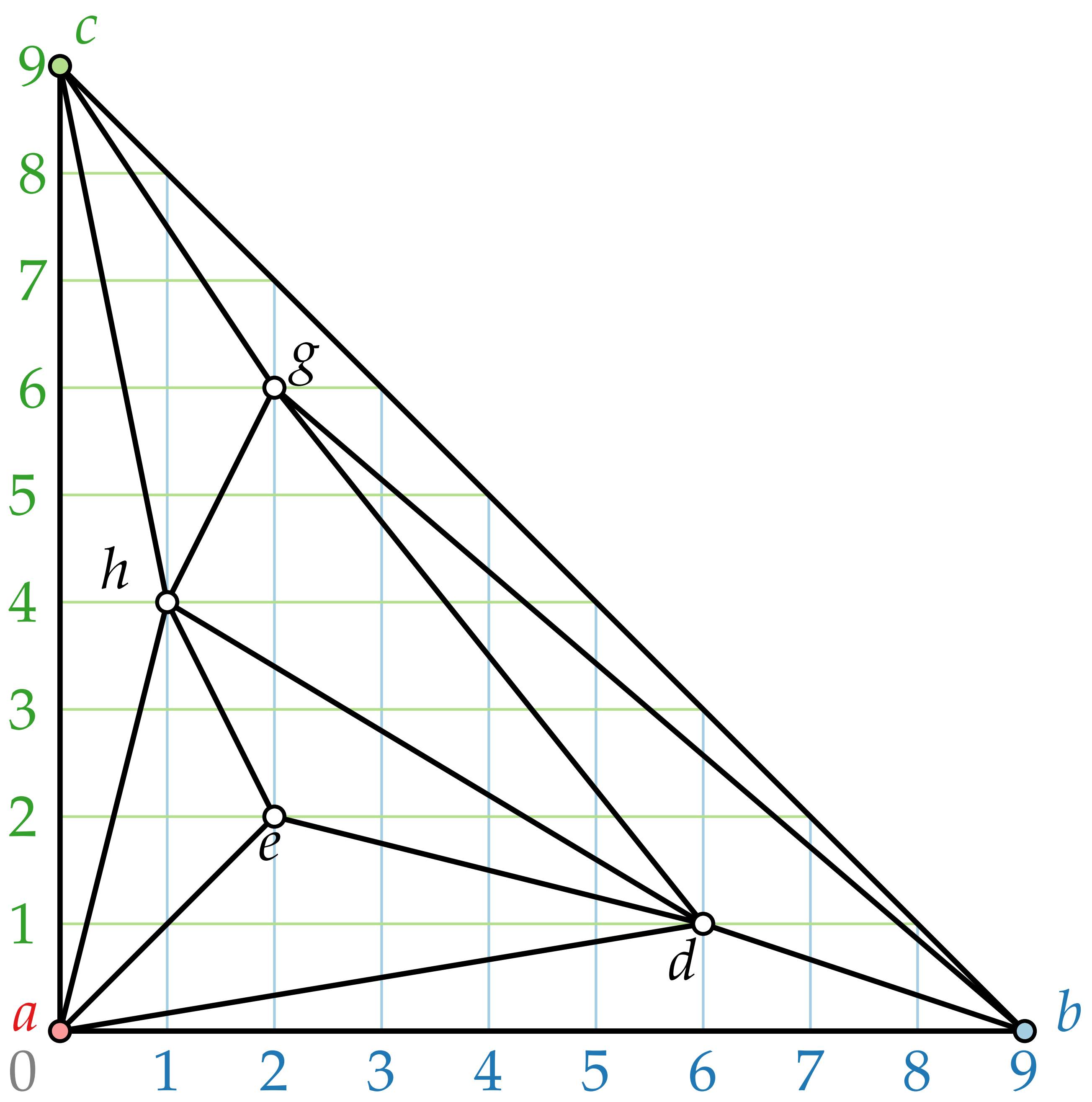

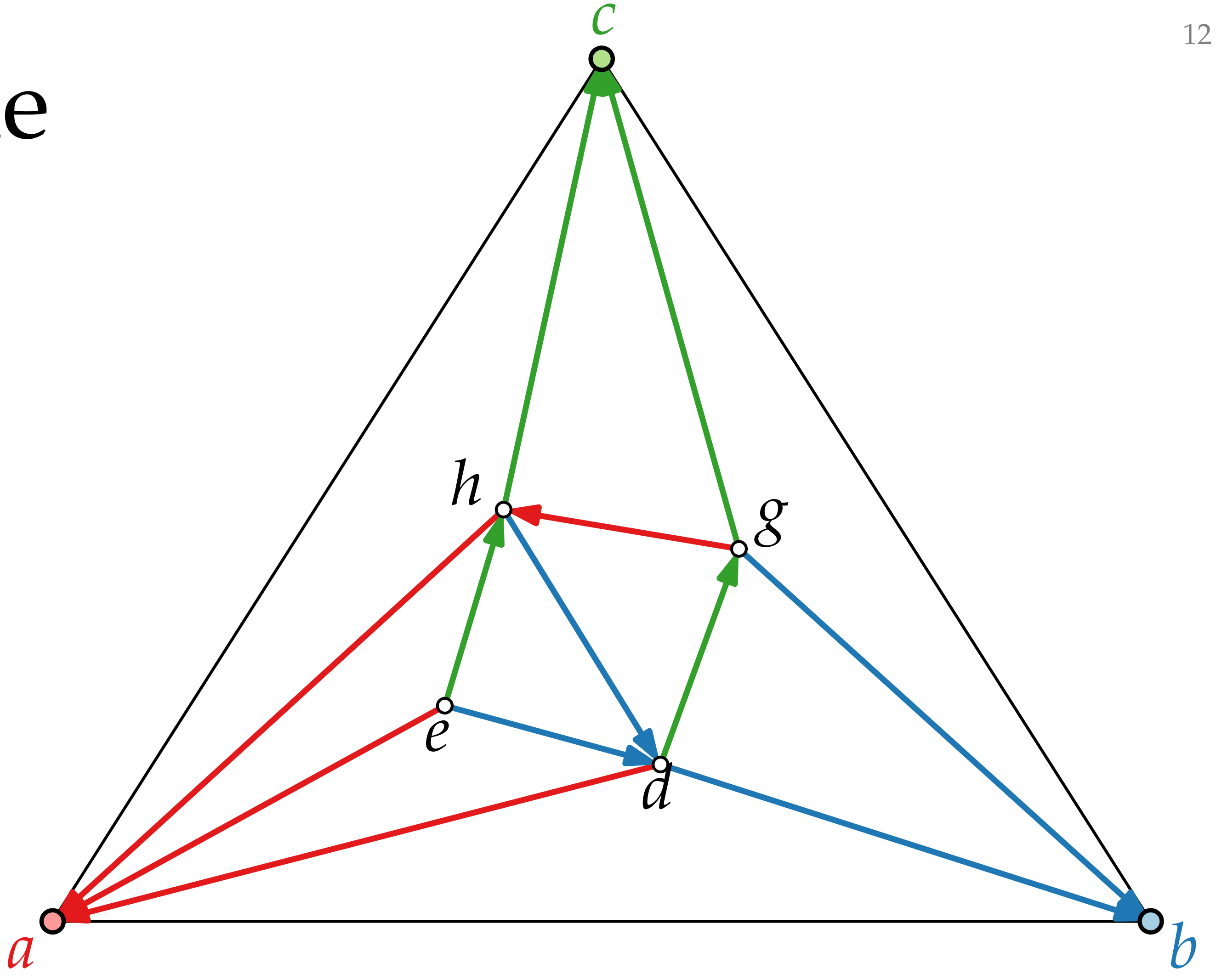

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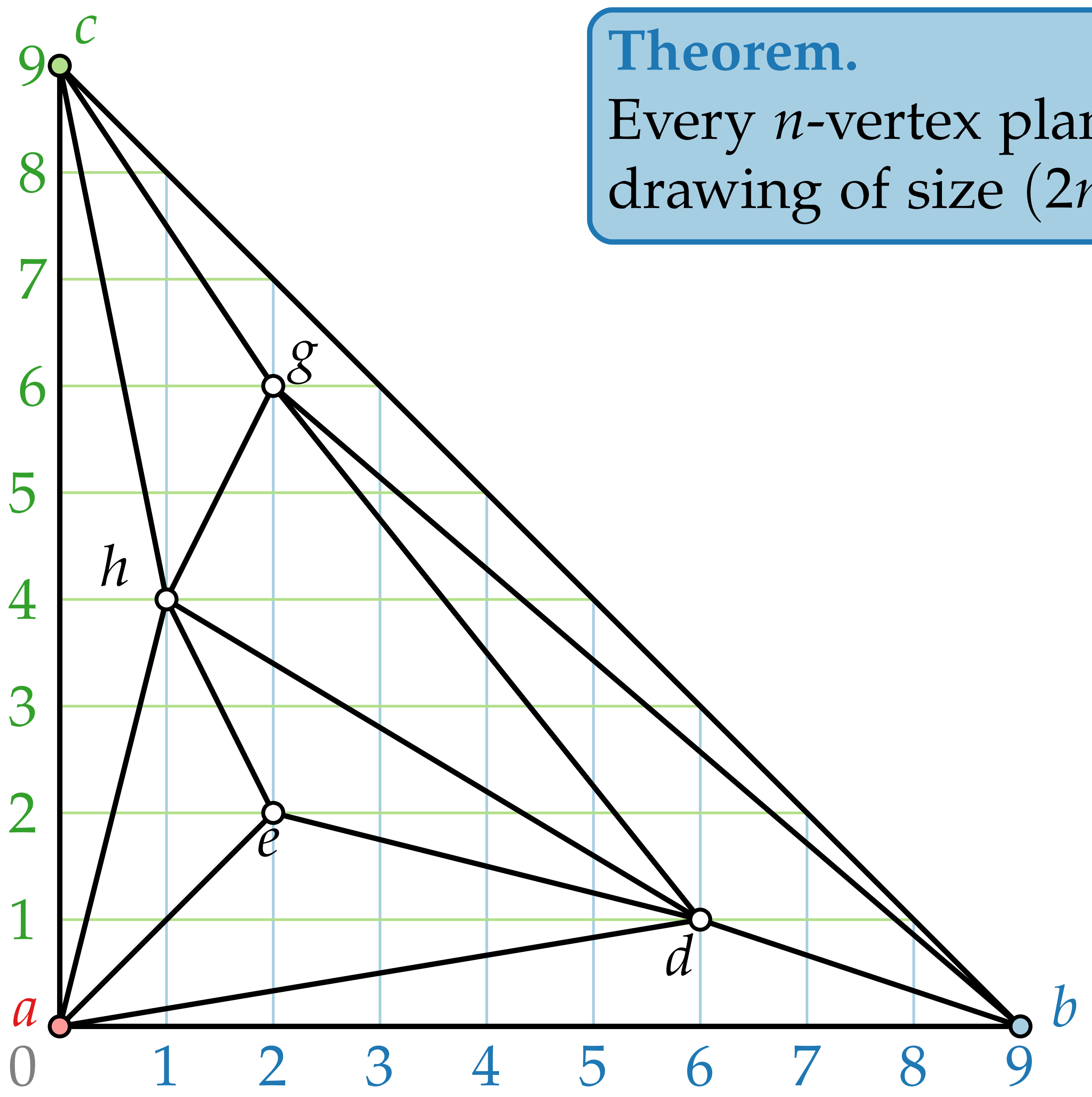
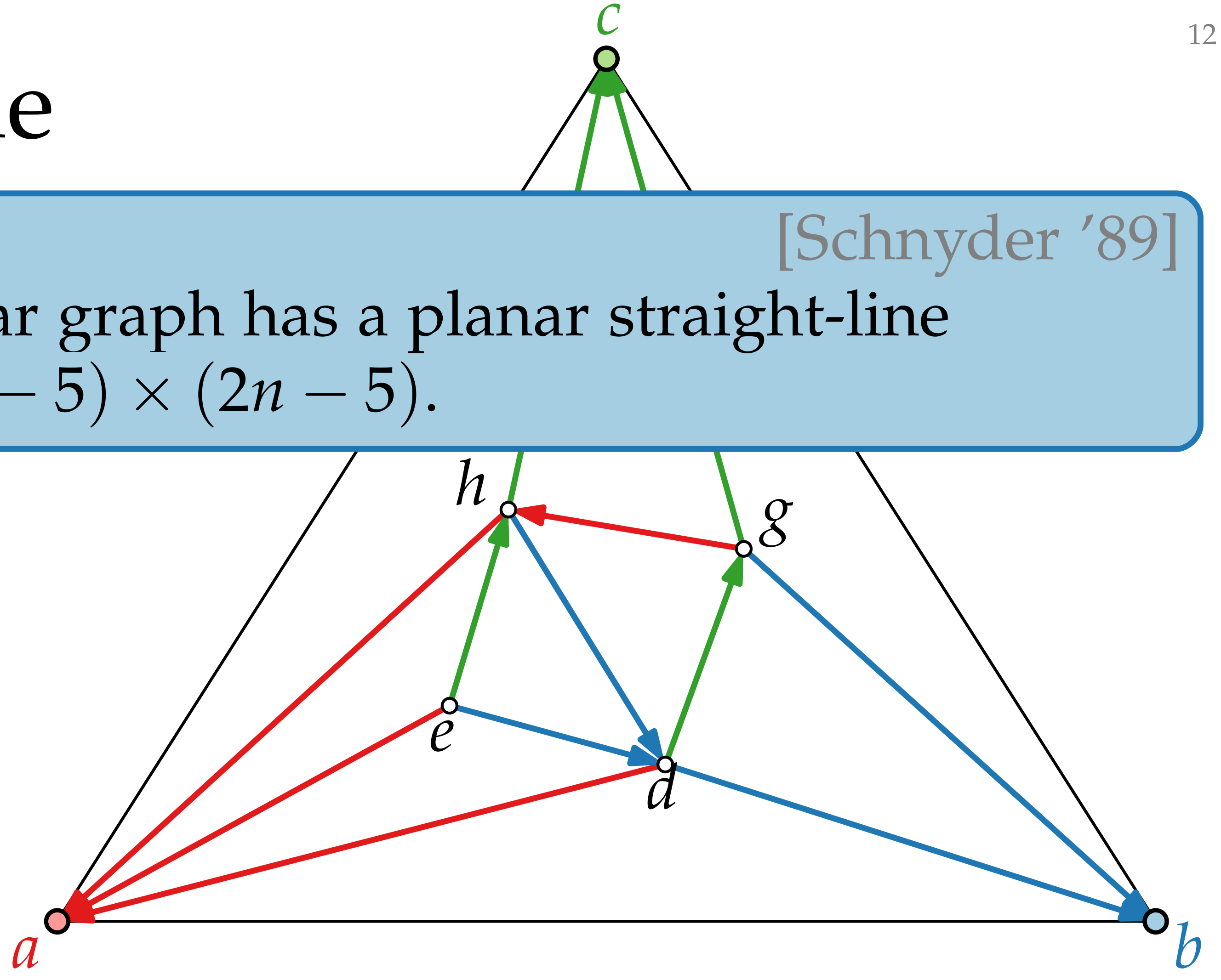


Theorem. [Schnyder '89]
Every n-vertex planar graph has a planar straight-line drawing of size $(2n-5) \times (2n-5)$.



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Weak Barycentric Representation

A **weak barycentric representation** of a graph $G = (V, E)$ is an assignment of barycentric coordinates to V:

$$\phi\colon V \to \mathbb{R}^3_{\geq 0}, v \mapsto (v_1, v_2, v_3)$$

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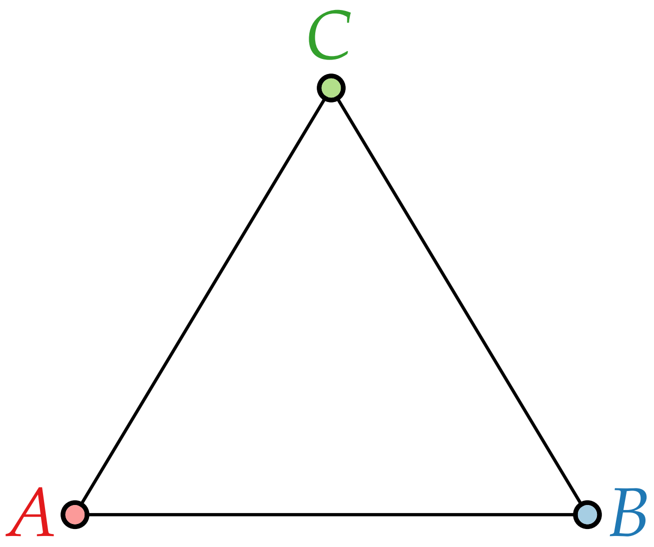

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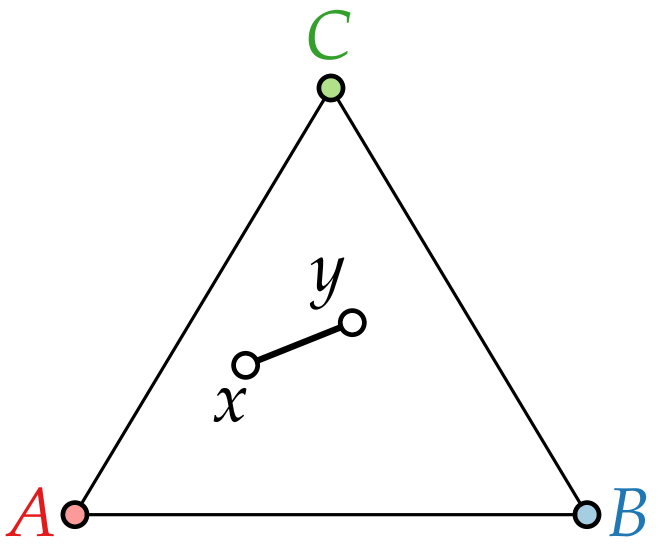

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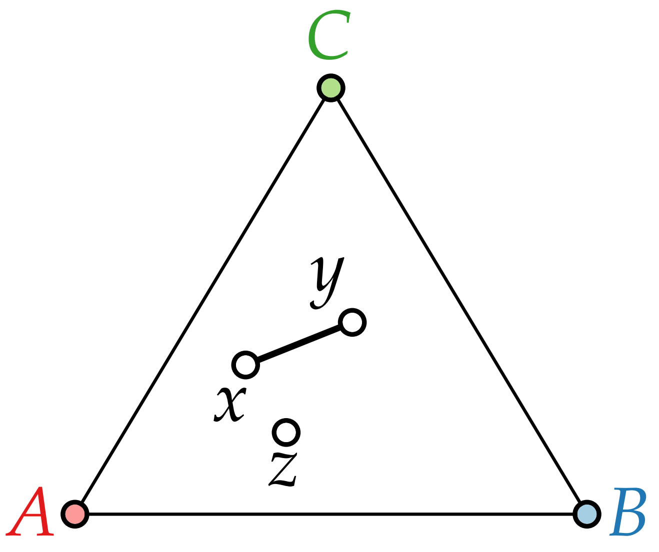

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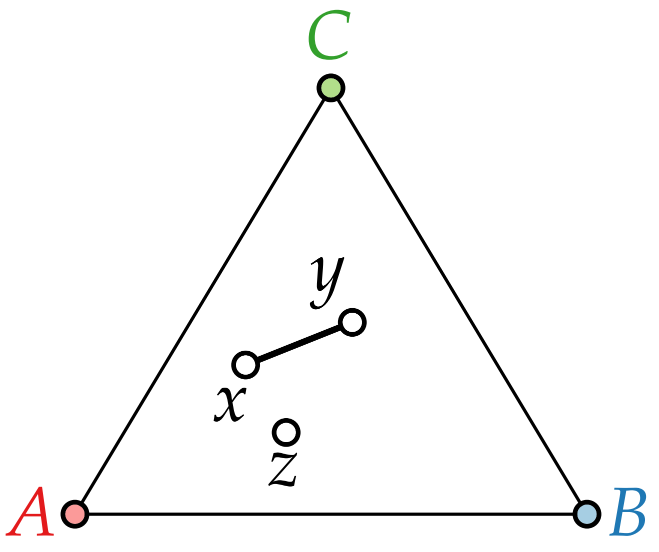

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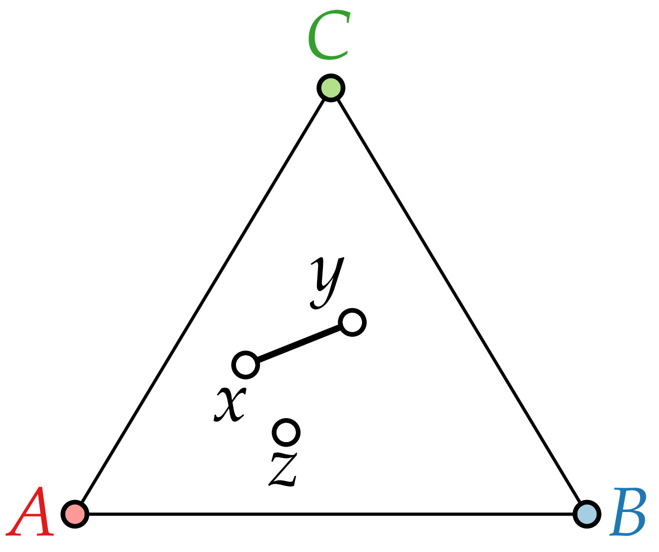

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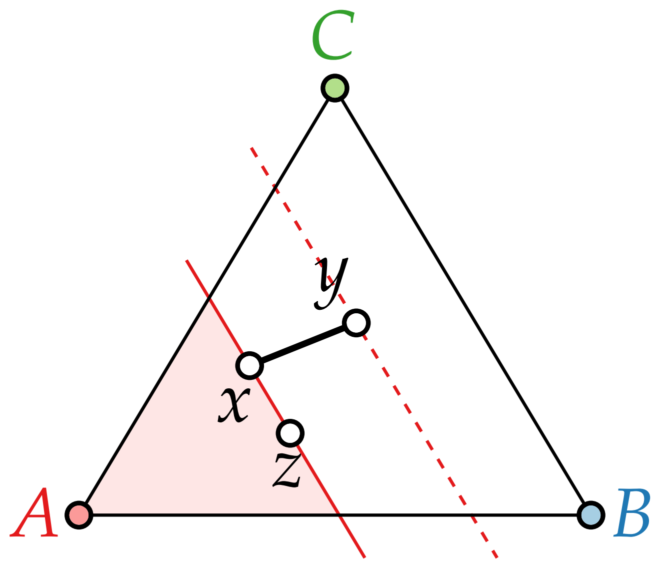

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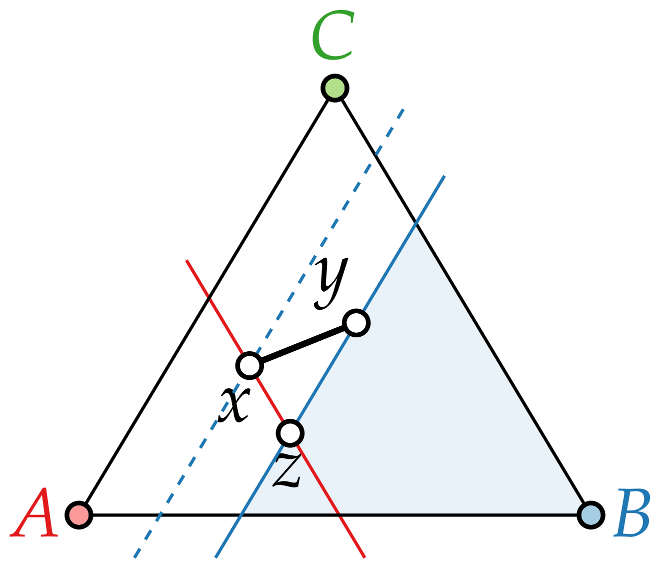

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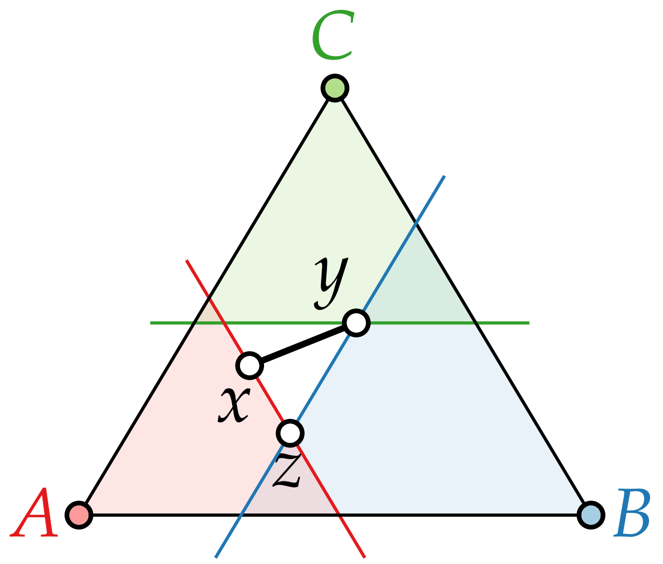

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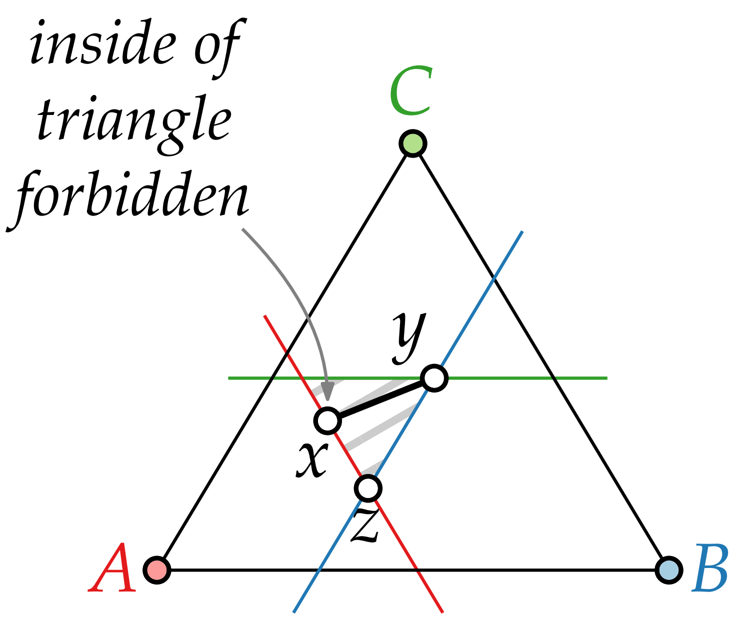

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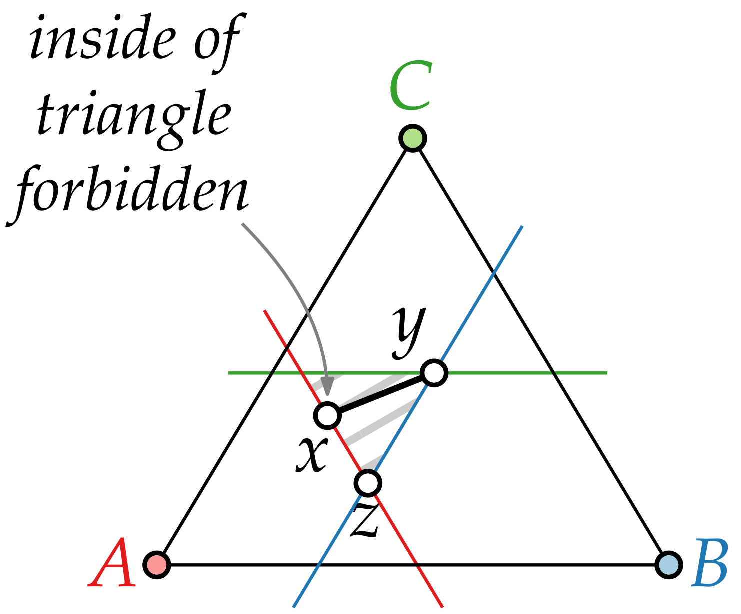
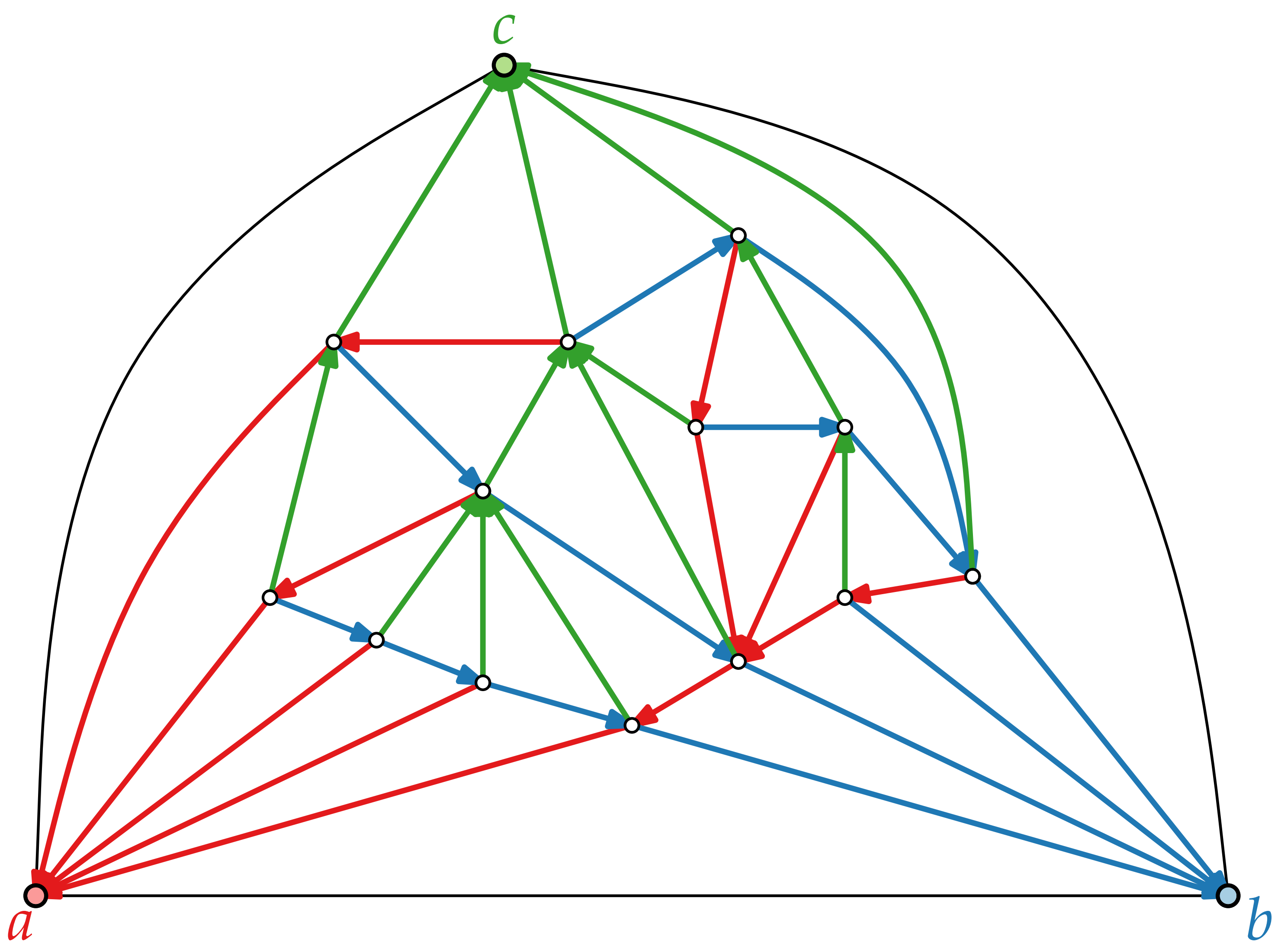


Lemma.
For a weak barycentric representation $\phi : v \mapsto (v_1, v_2, v_3)$ and a triangle A, B, C, the mapping

$$f : v \in V \mapsto v_1 A + v_2 B + v_3 C$$

gives a **planar** drawing of G inside $\triangle ABC$.

Counting Vertices



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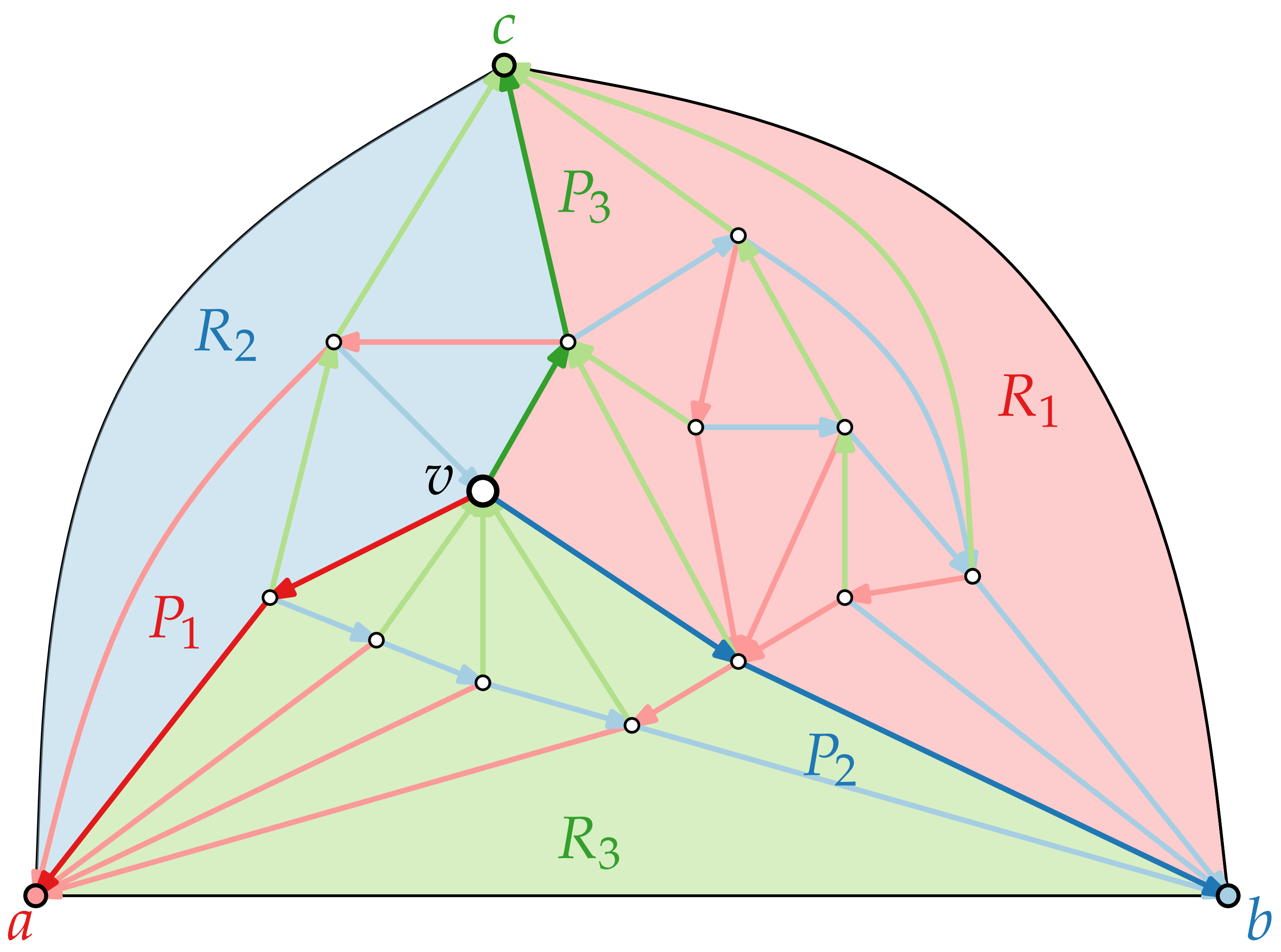

$P_i(v)$: path from v to root of T_i.
$R_1(v)$: set of faces contained in P_2, bc, P_3.
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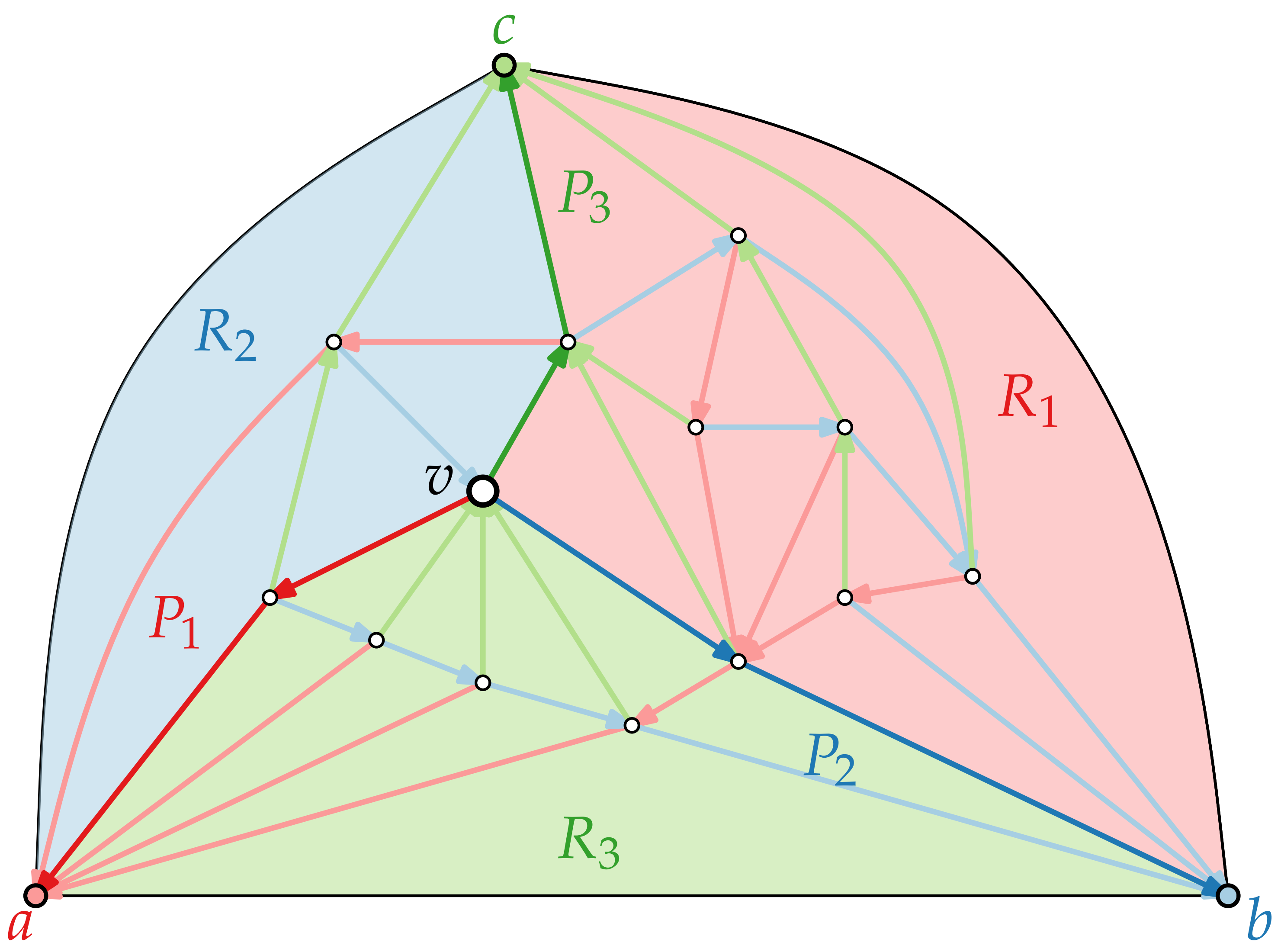

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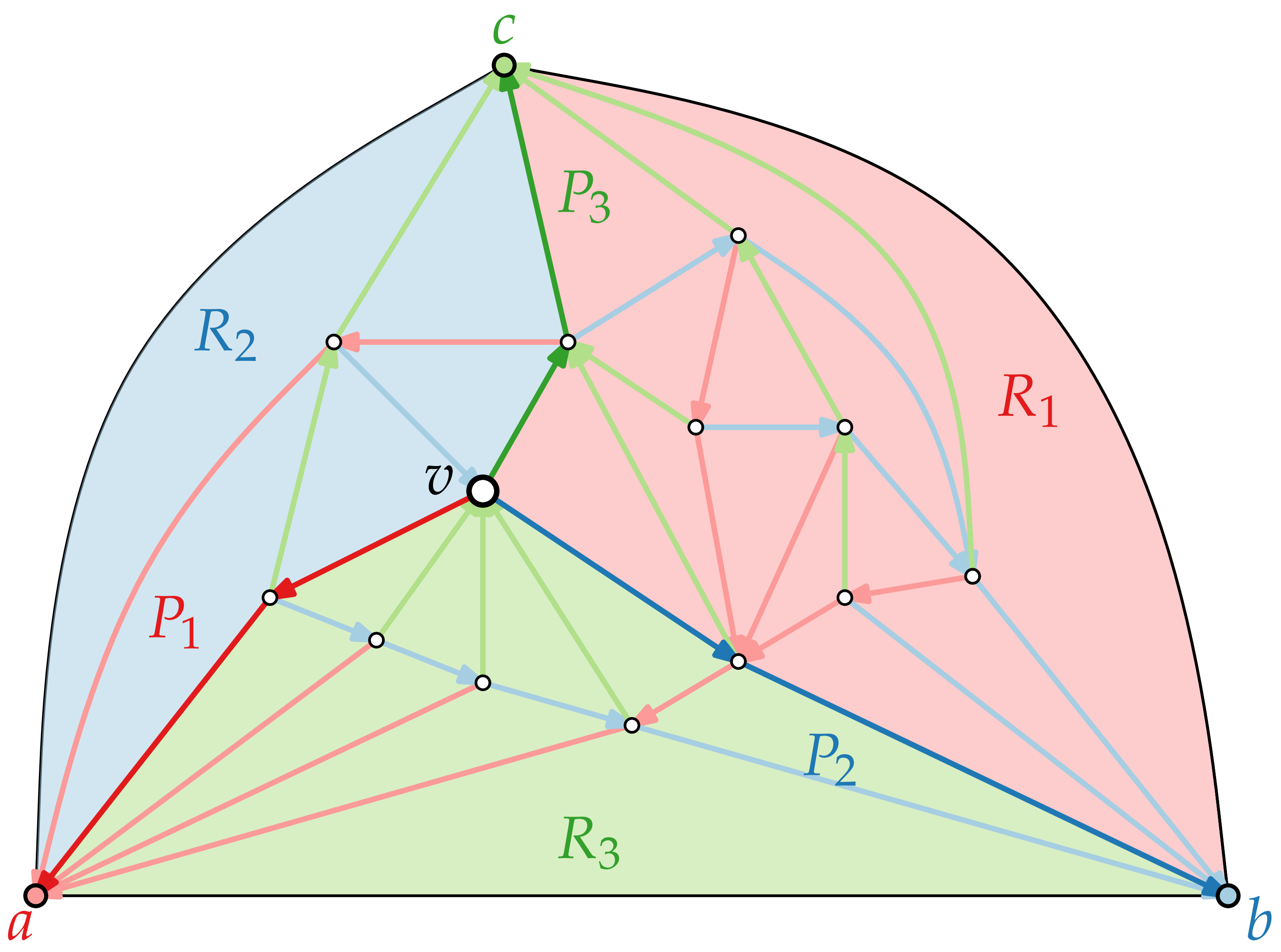

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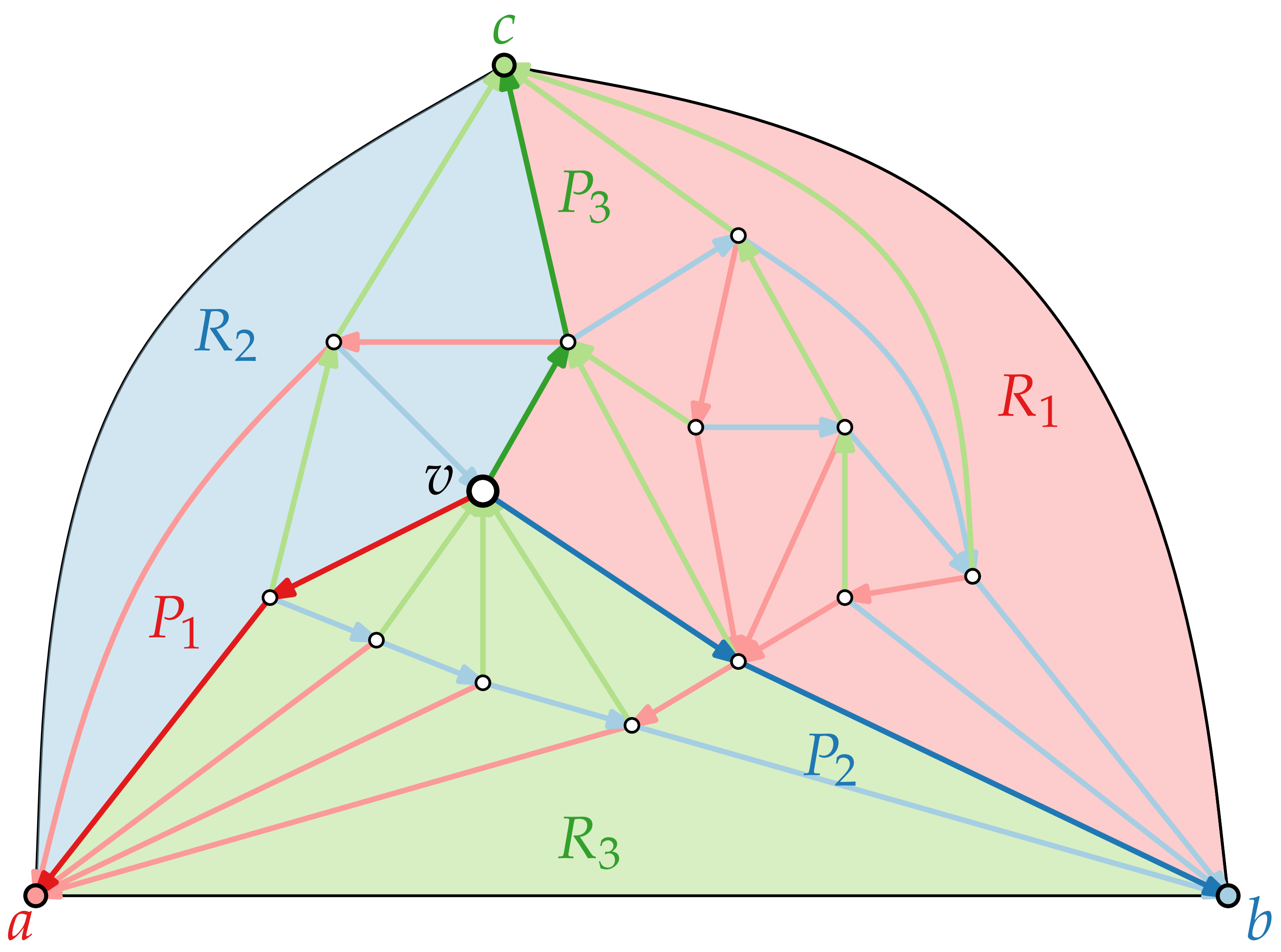

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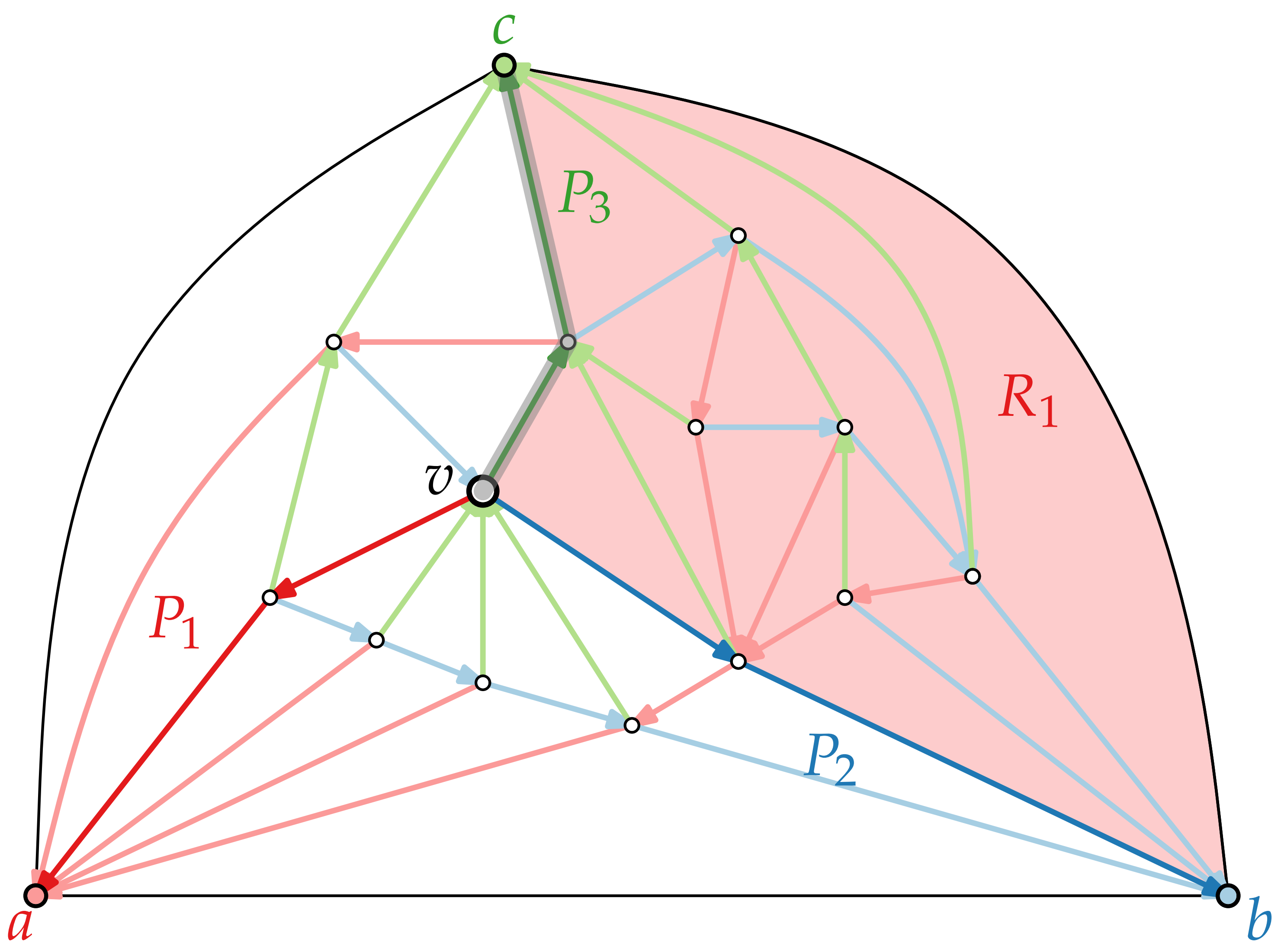

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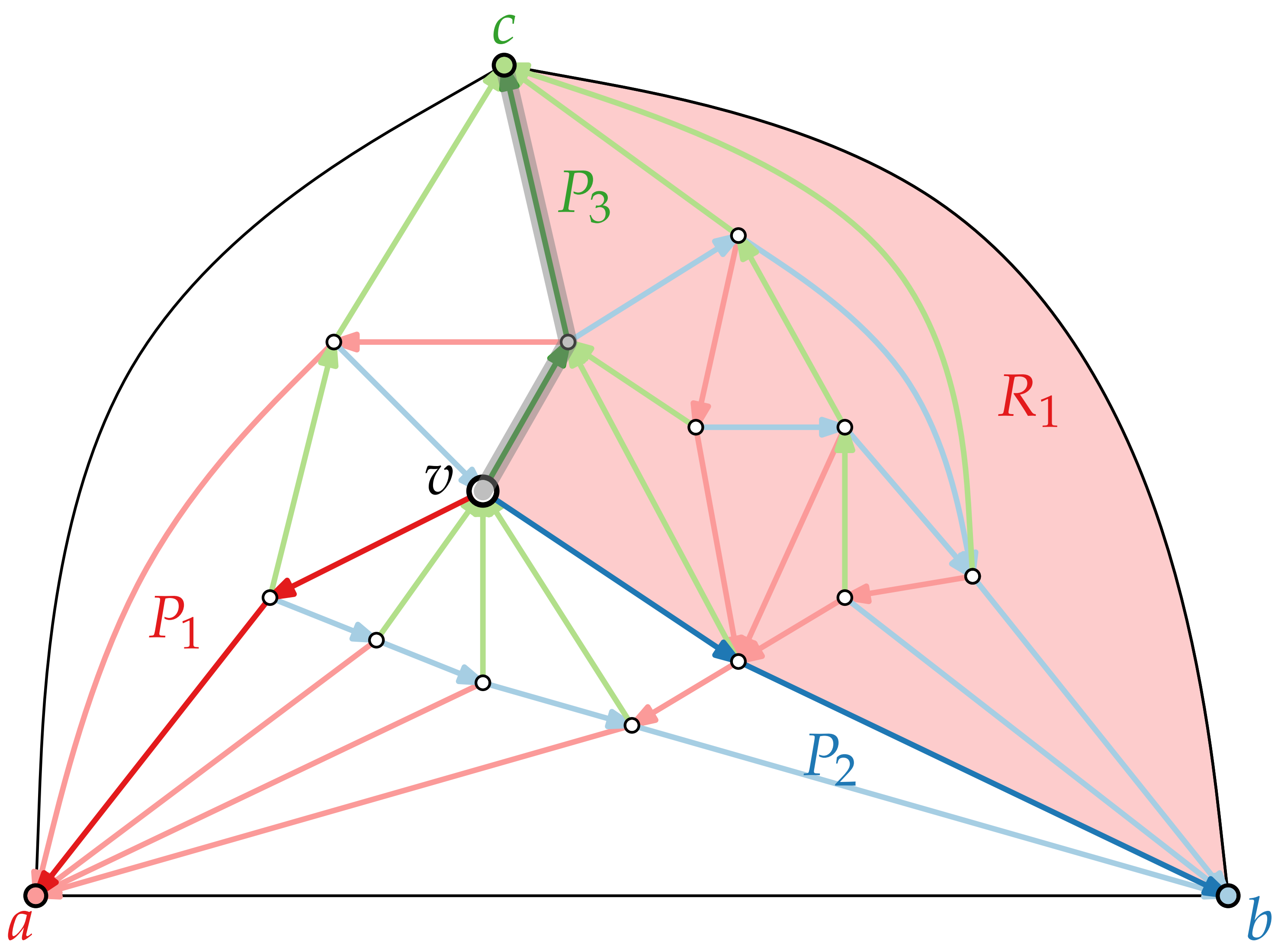

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$v_1 = 10 - 3 = 7$

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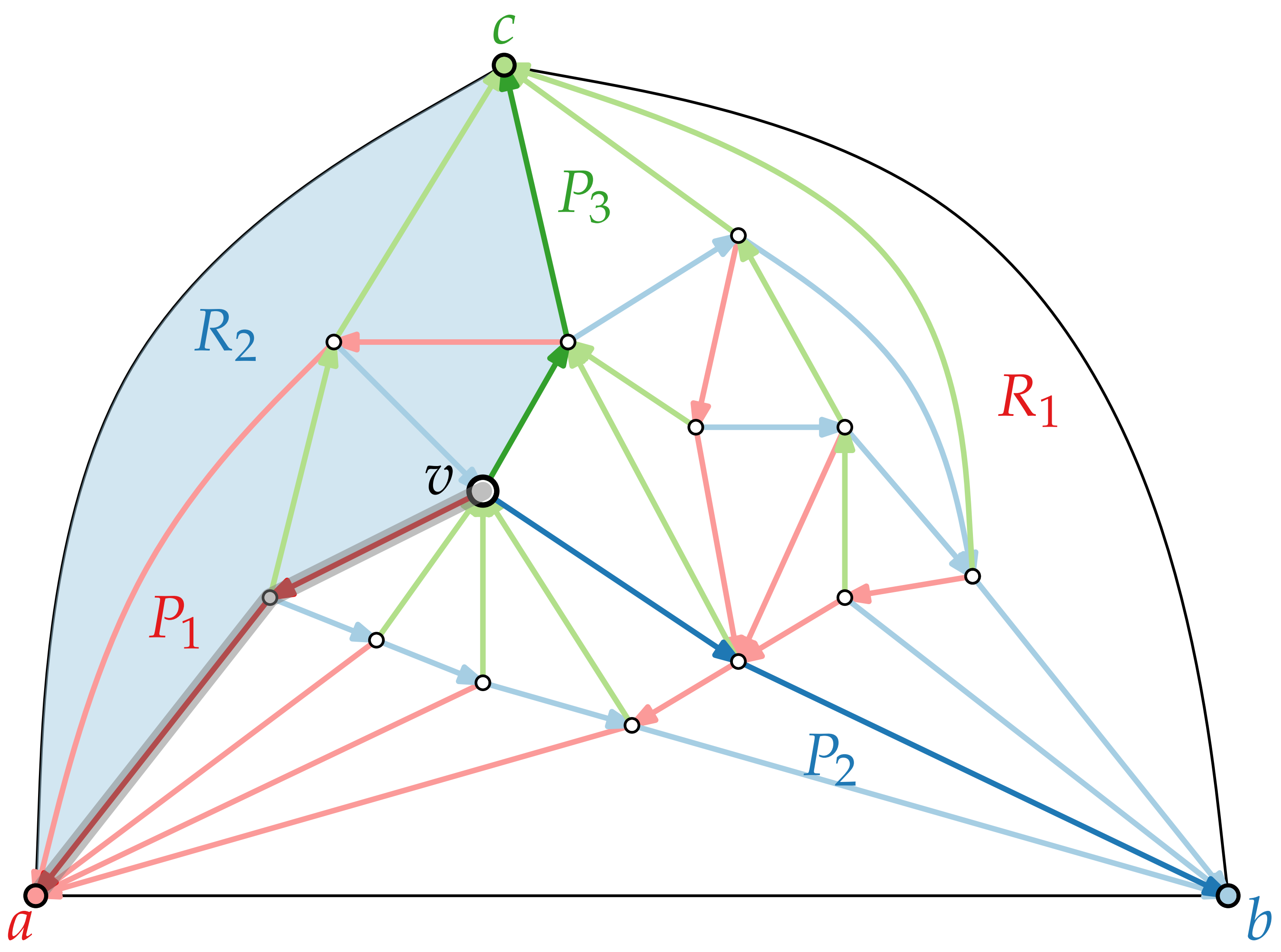

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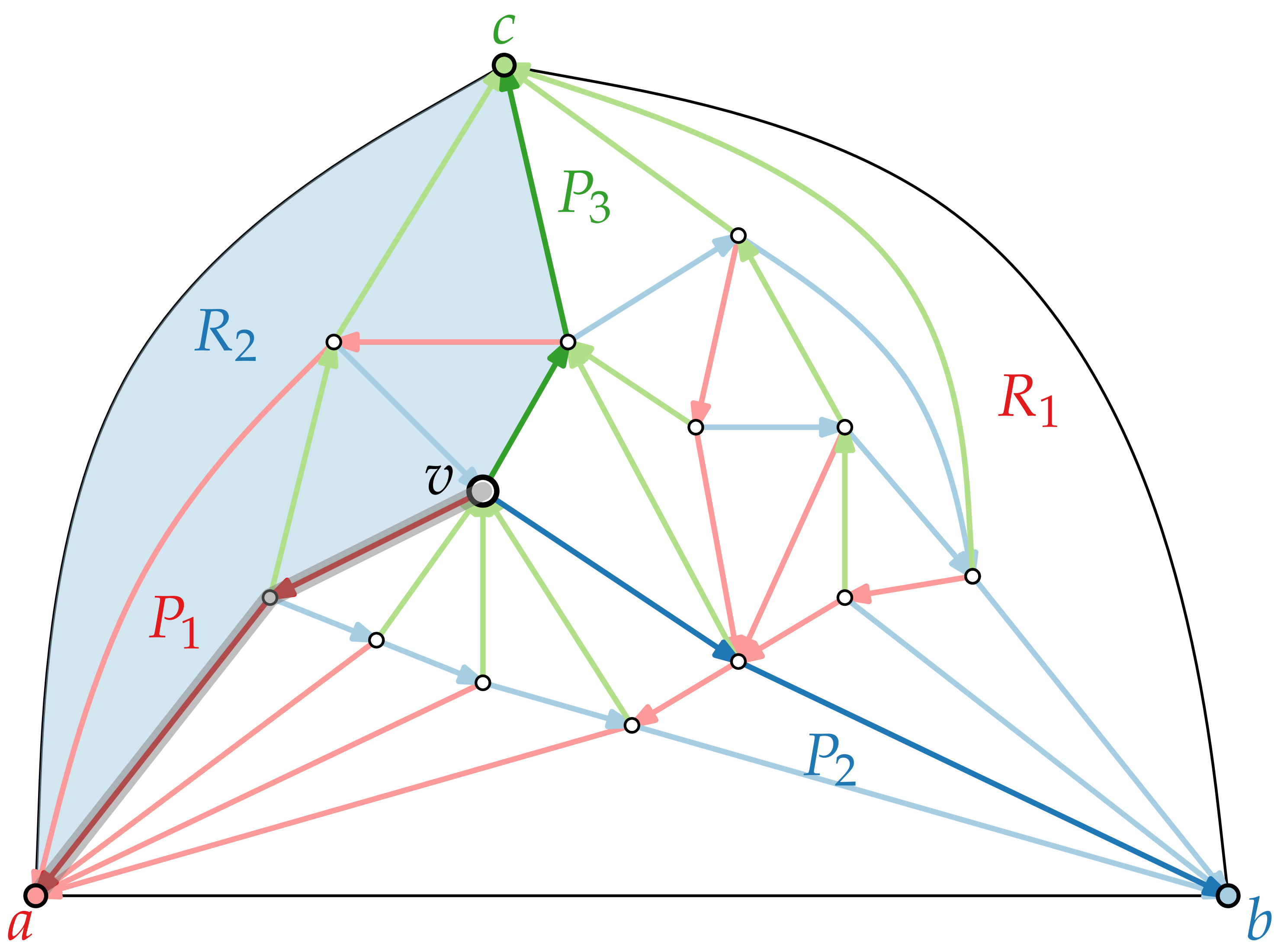
$v_2 =$

Counting Vertices



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$v_1 = 10 - 3 = 7$

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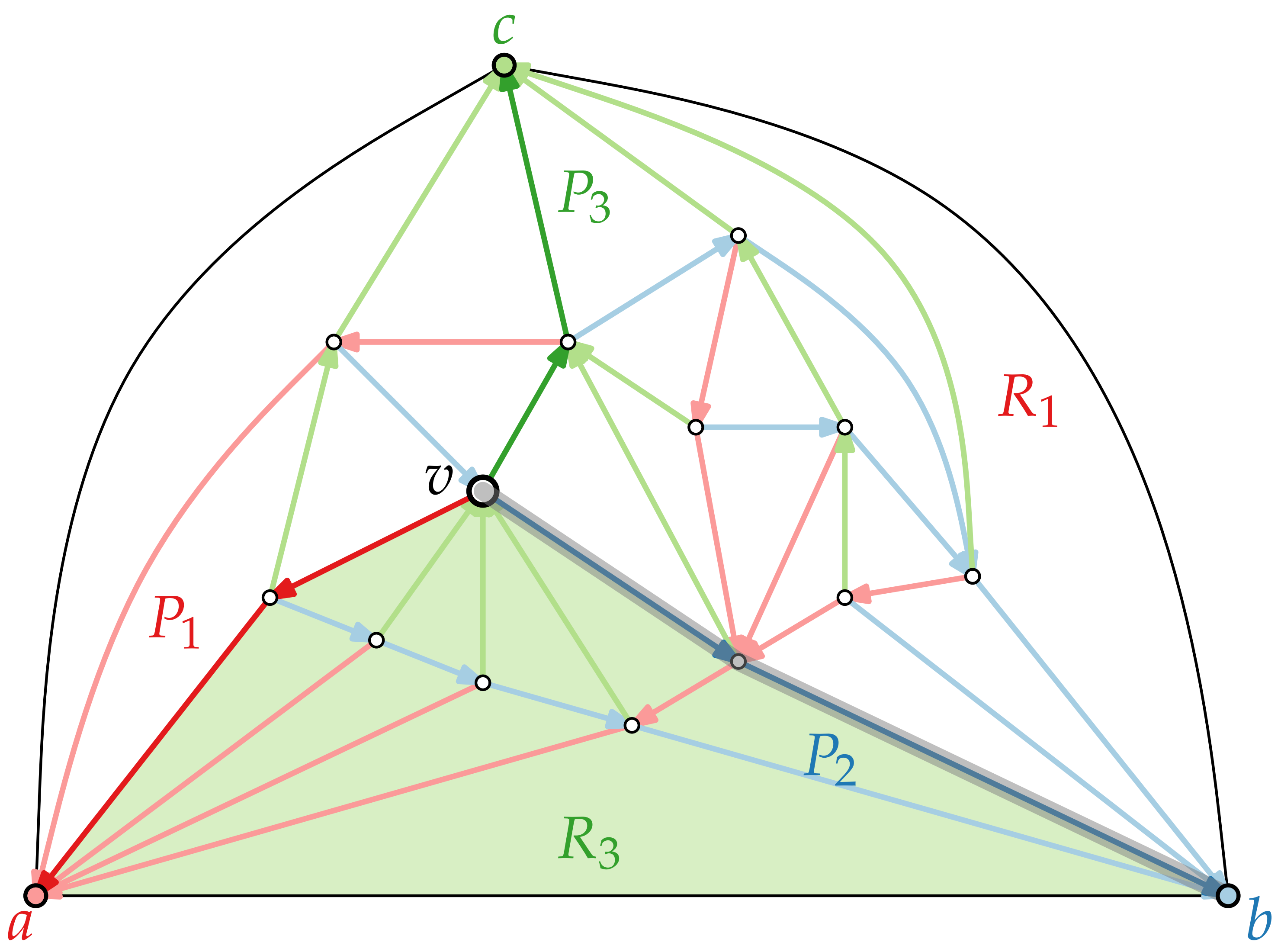
Counting Vertices



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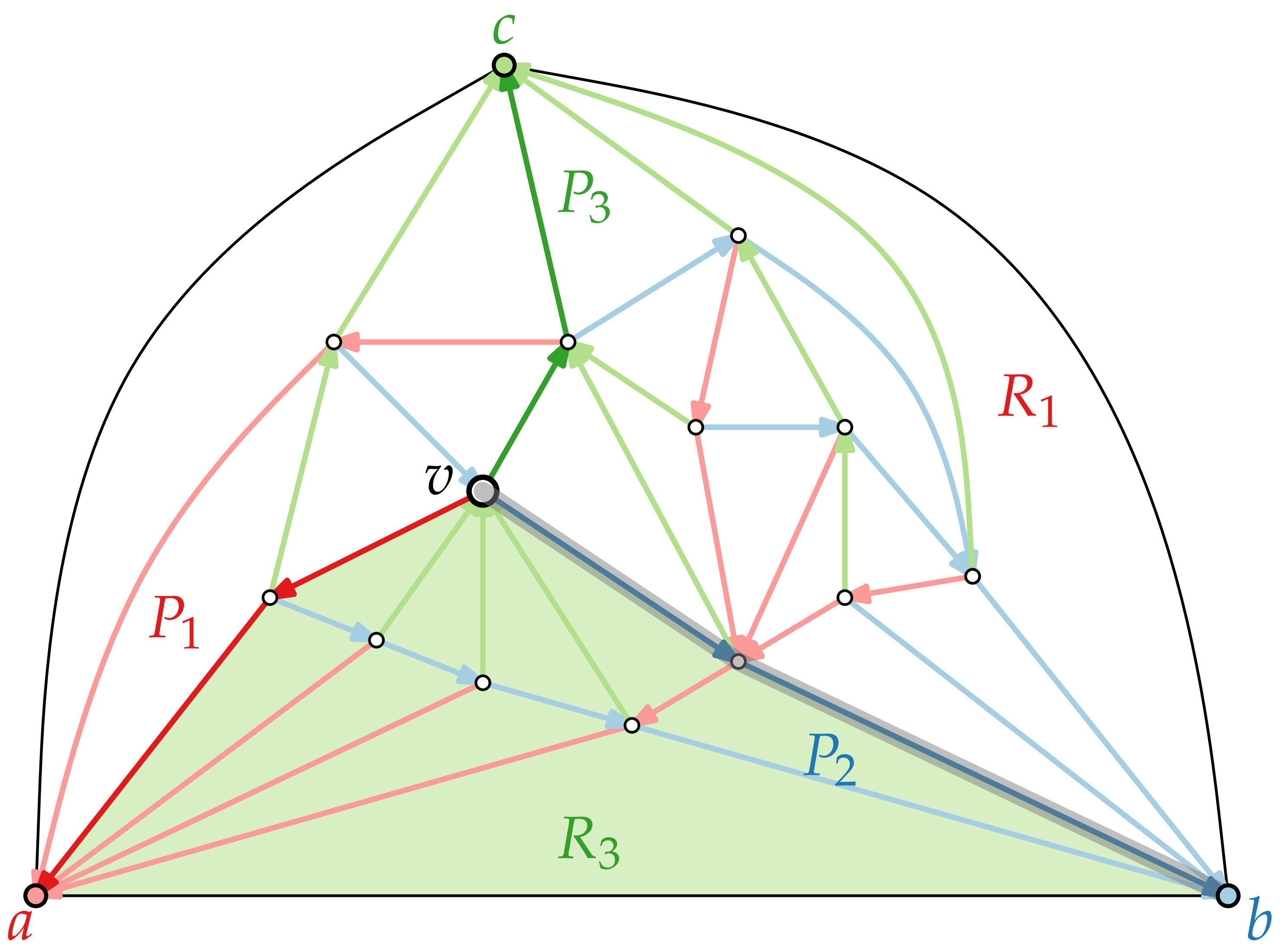

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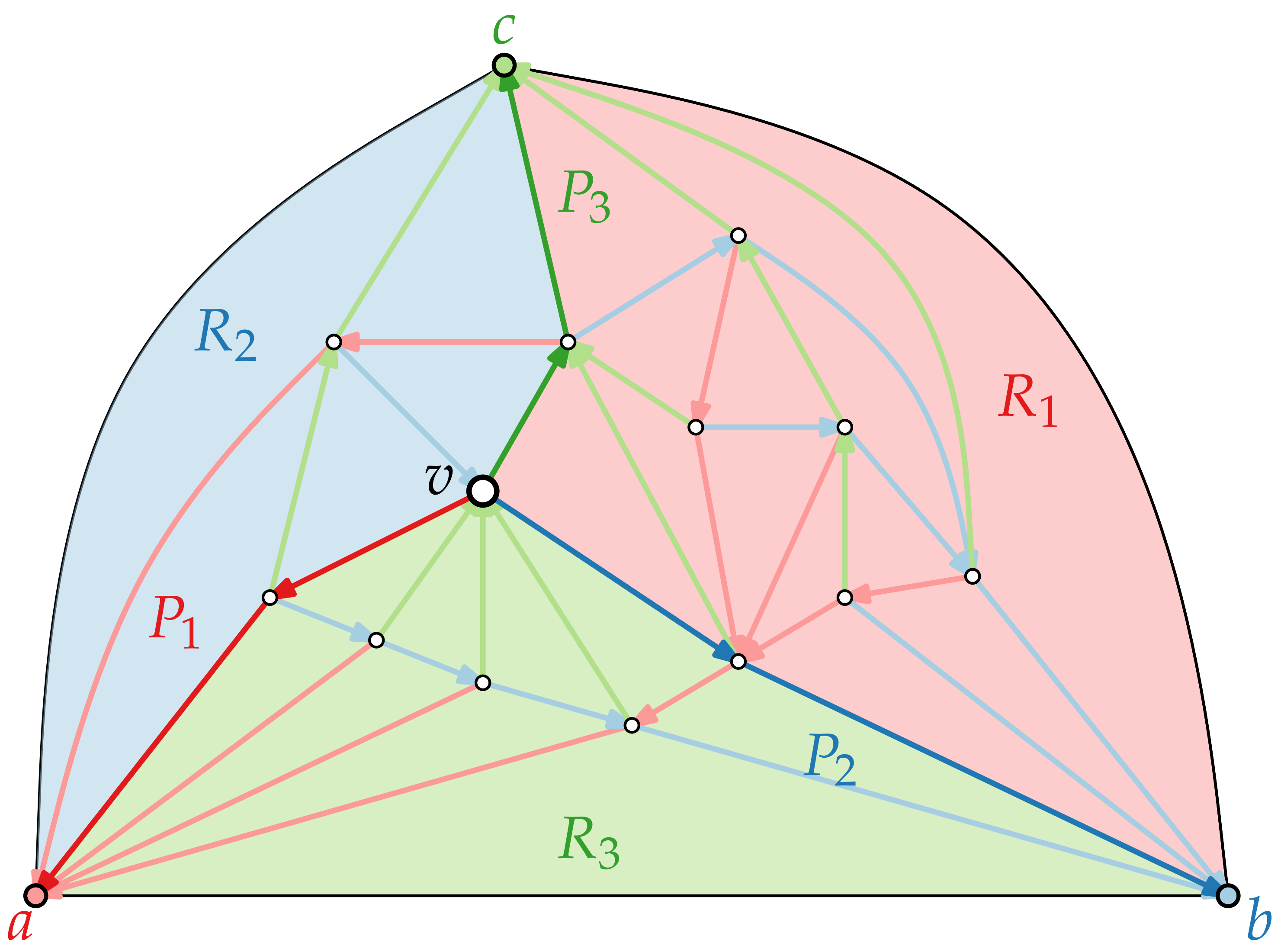

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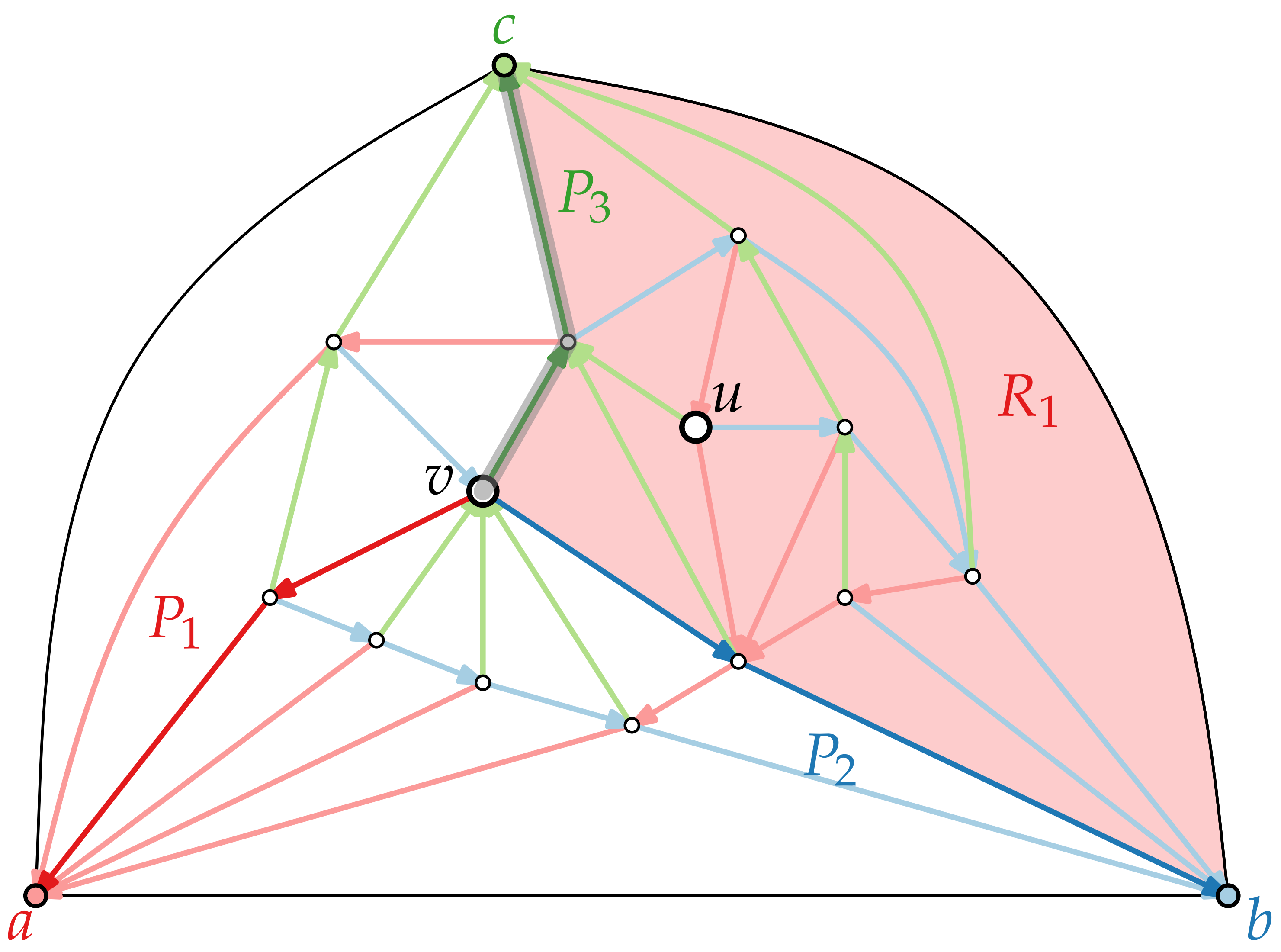

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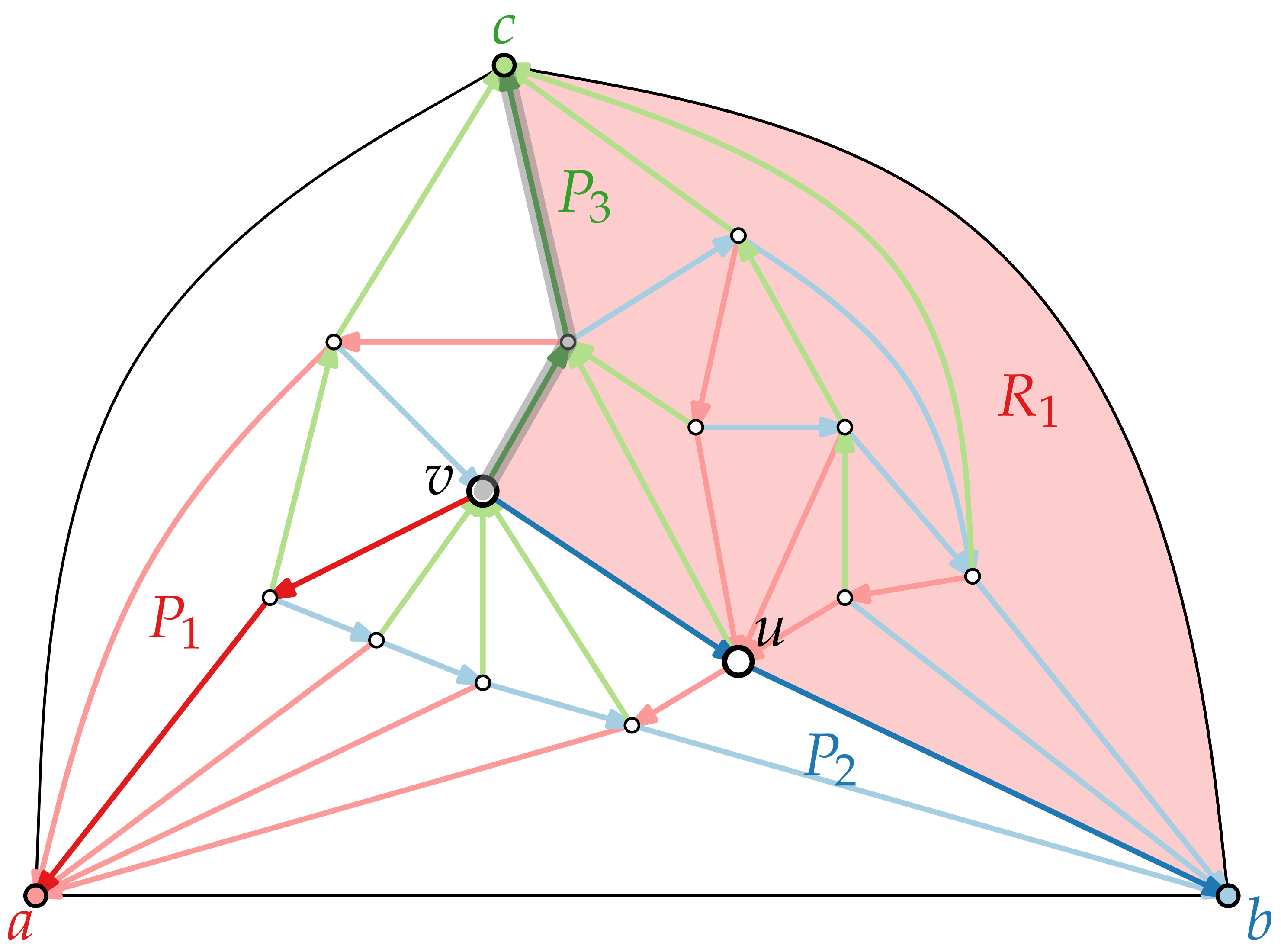

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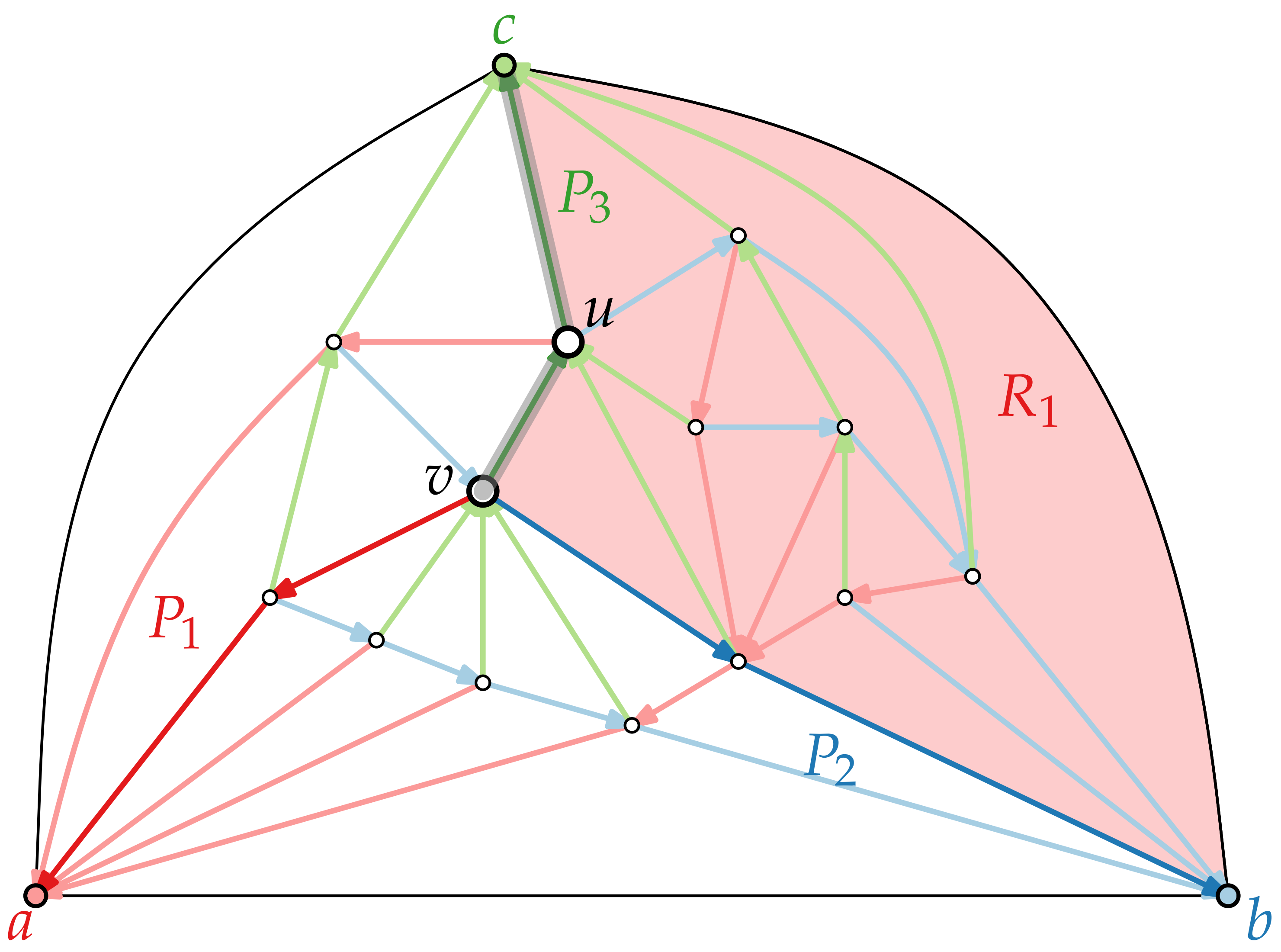

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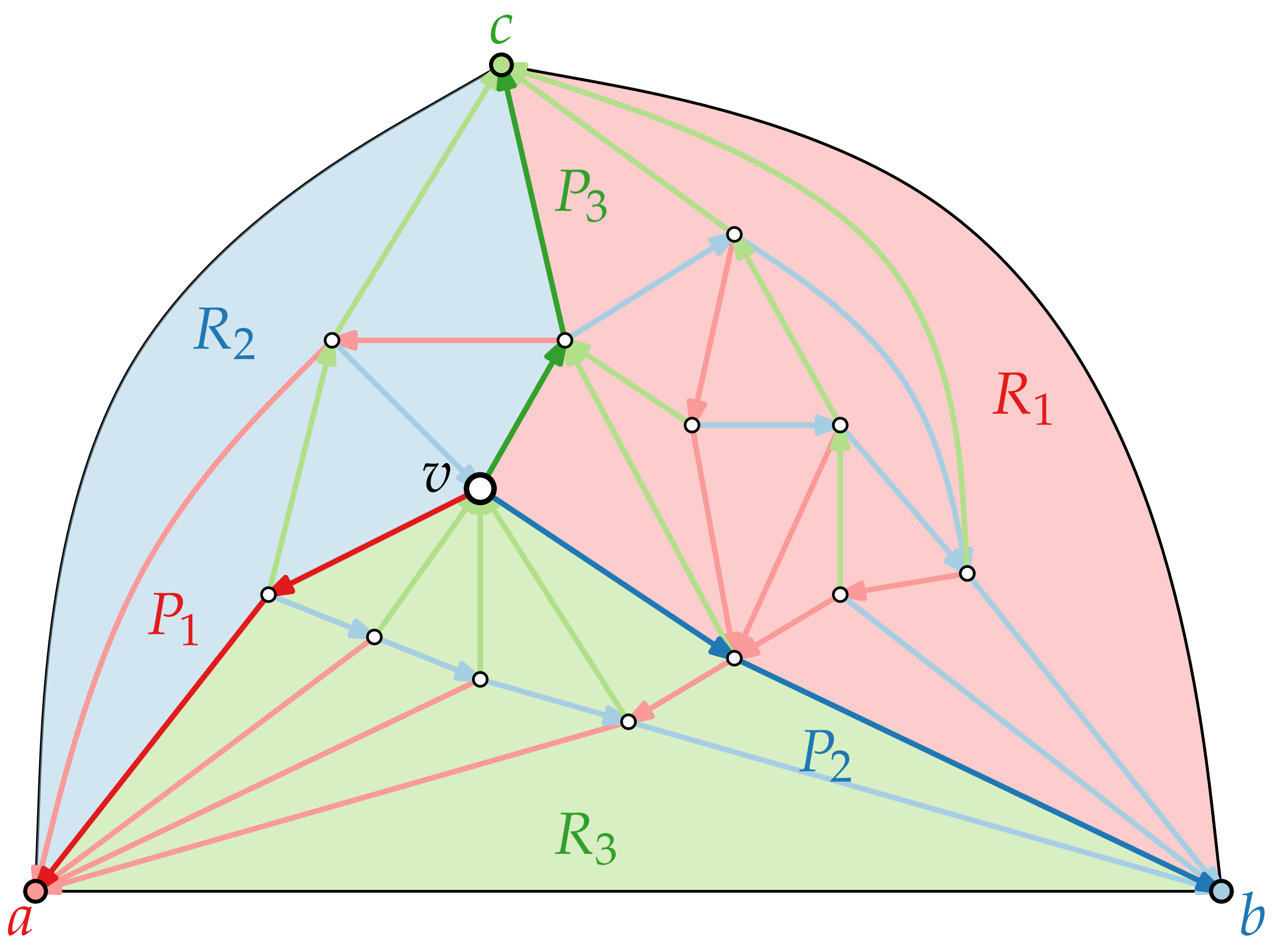

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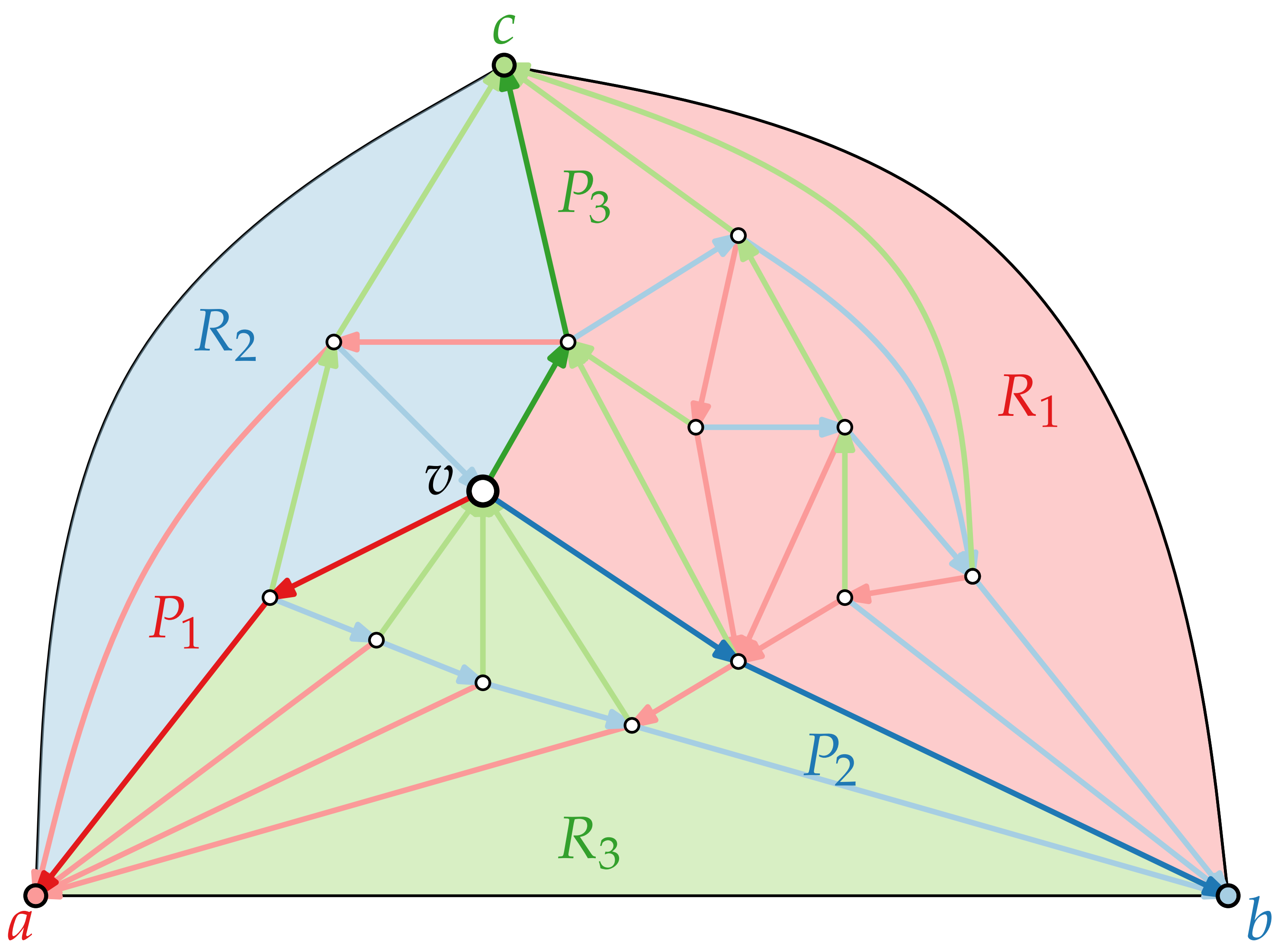

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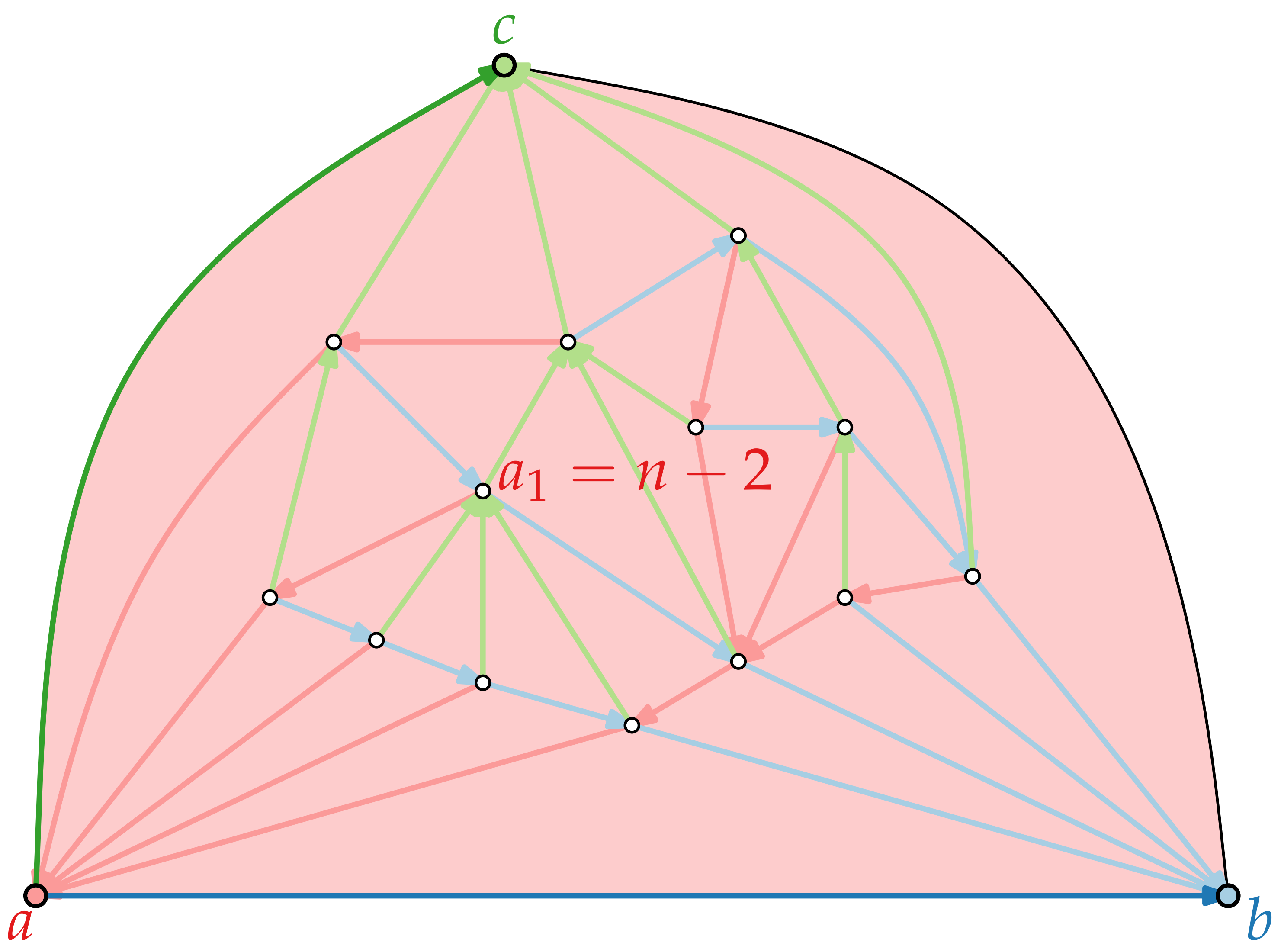

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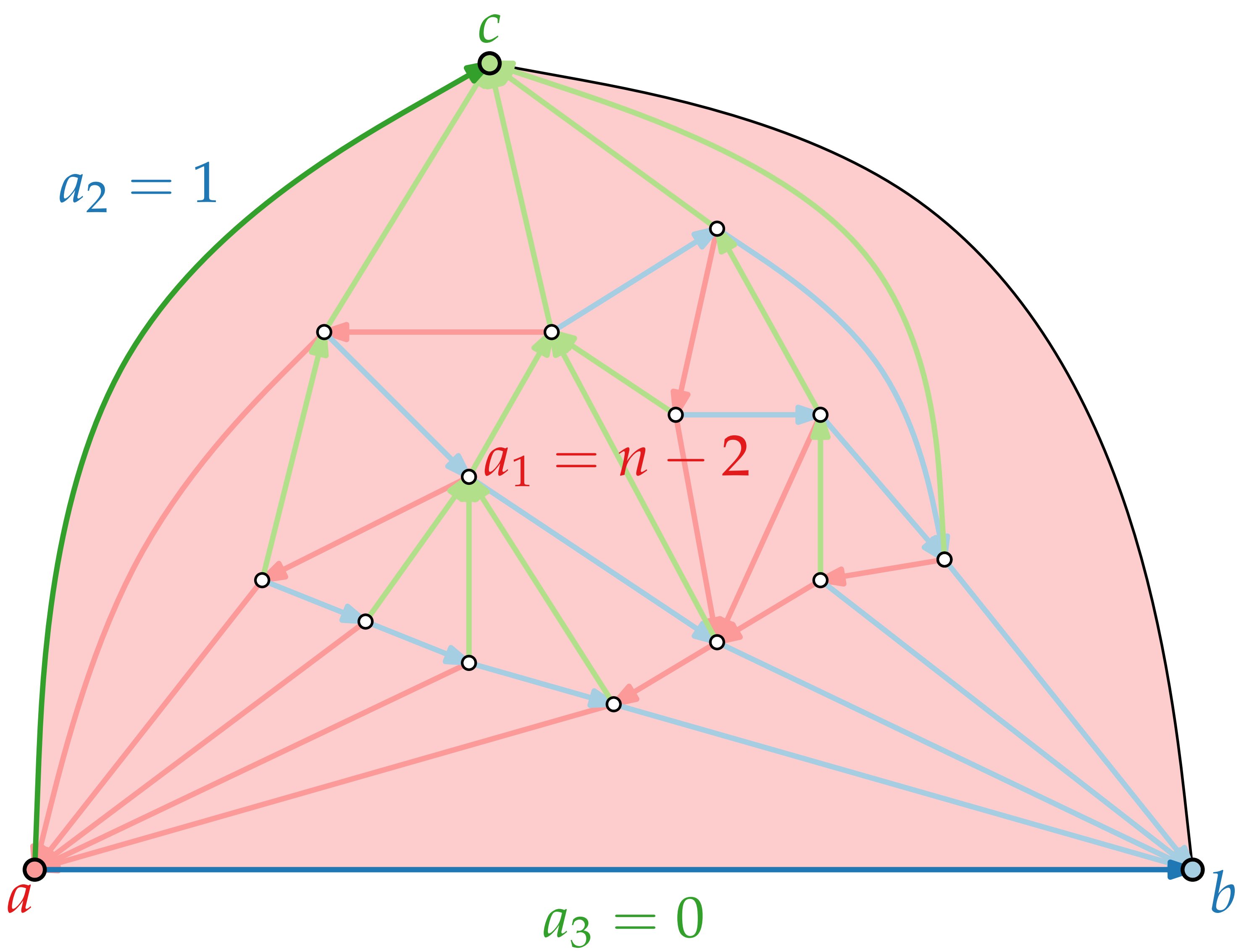

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Schnyder Drawing*

Set $A = (0,0)$, $B = (n-1,0)$, and $C = (0,n-1)$.

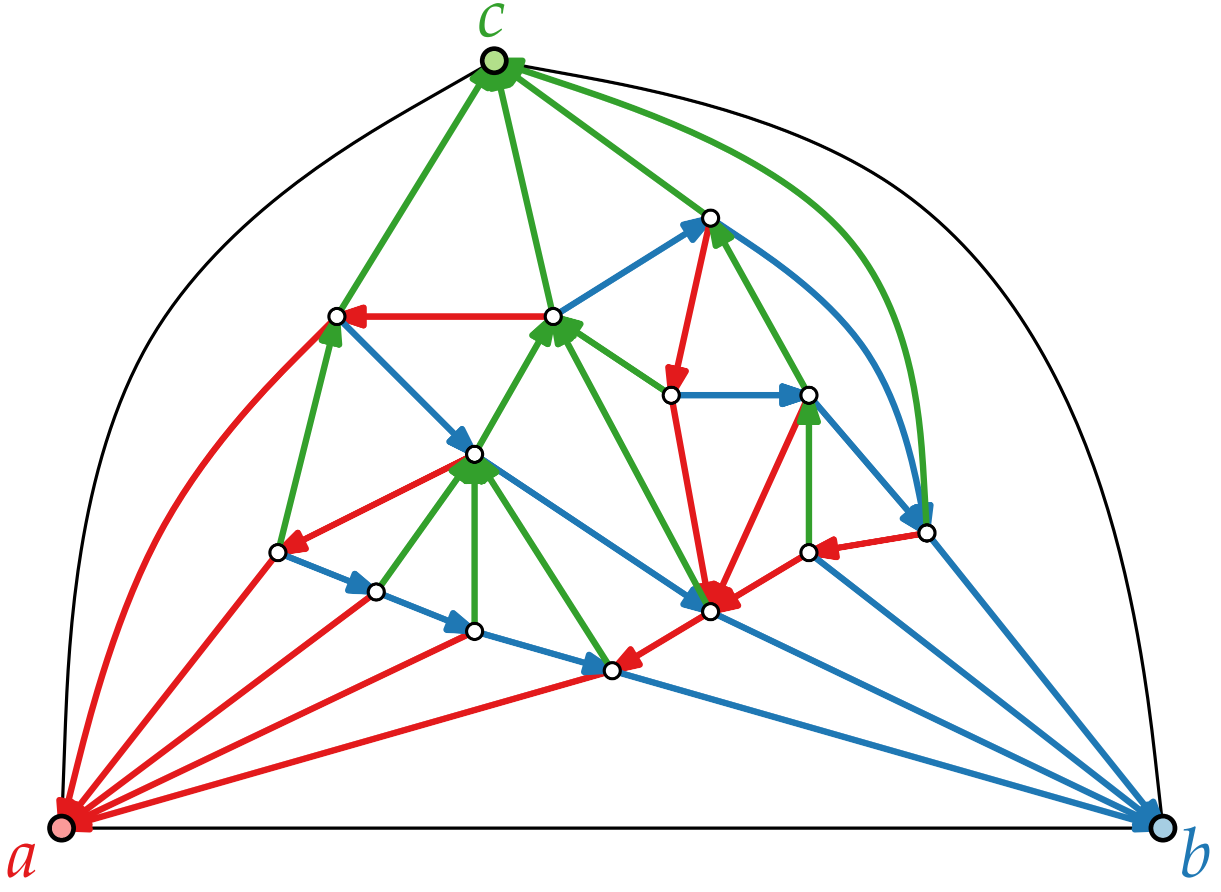
Theorem. [Schnyder '90]

For a plane triangulation G, the mapping

$$f : v \mapsto \frac{1}{n-1}(v_1, v_2, v_3)$$

is a barycentric representation of G, which thus gives a planar straight-line drawing of G on the $(n-2) \times (n-2)$ grid.

Schnyder Drawing* – Example



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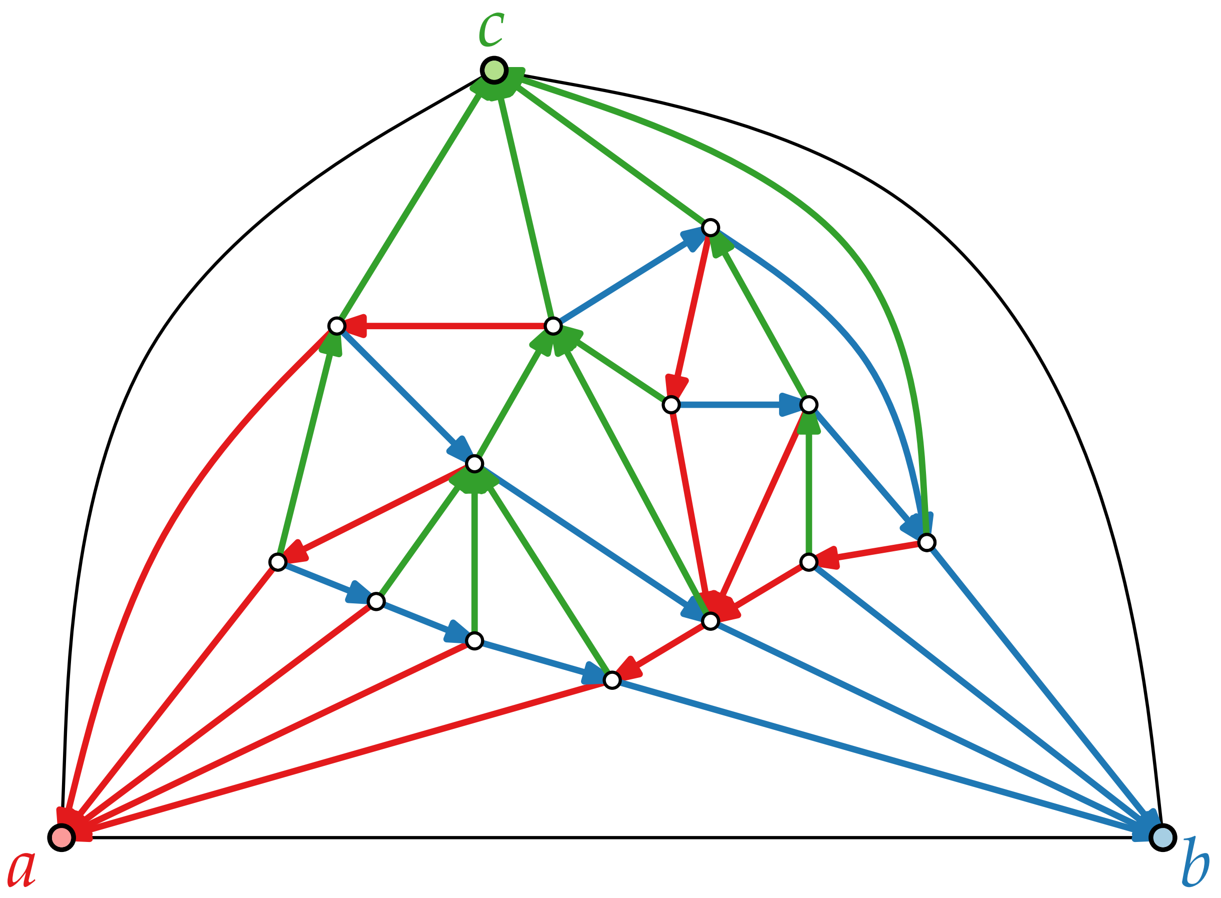

$n = 16, n - 2 = 14$

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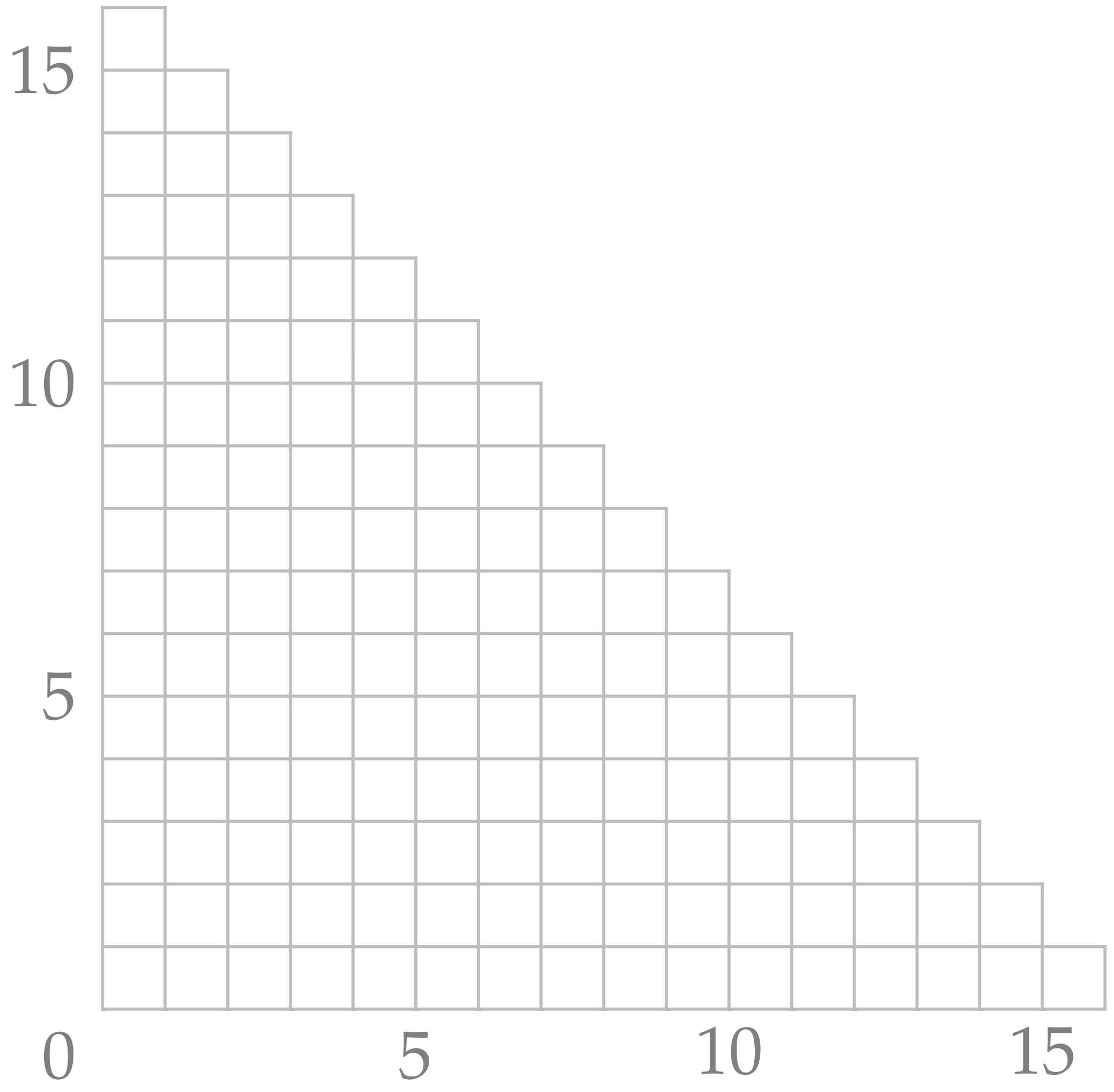

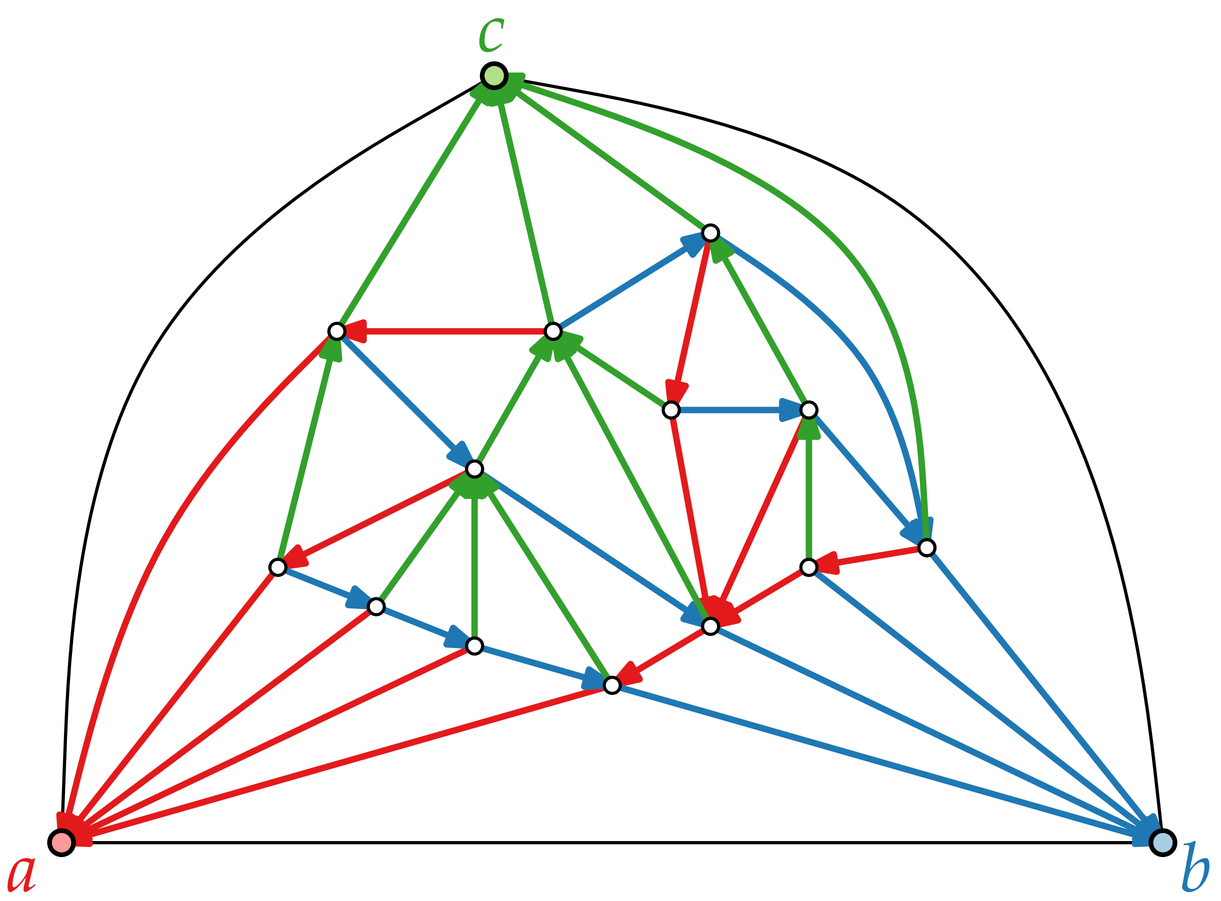

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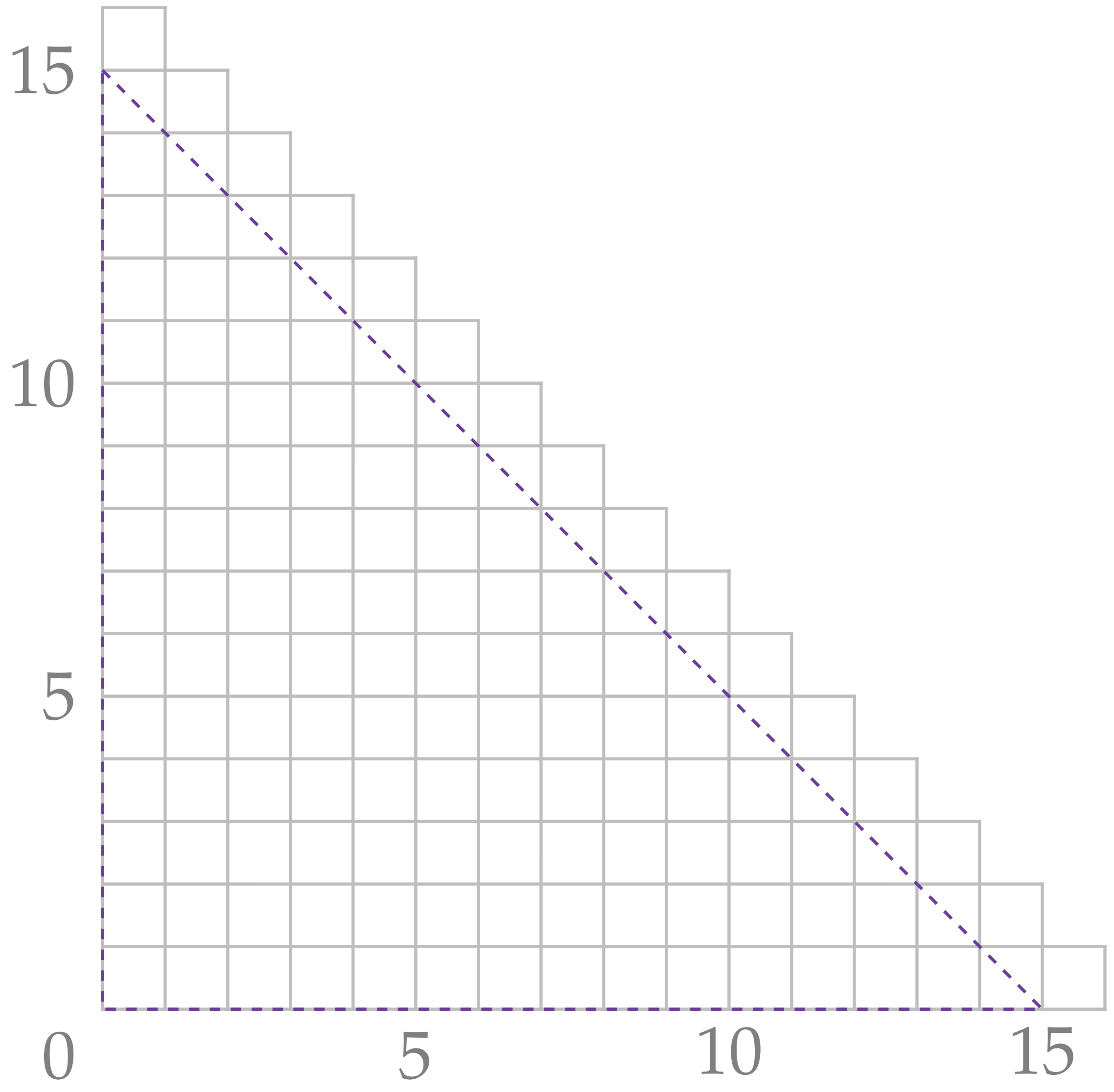

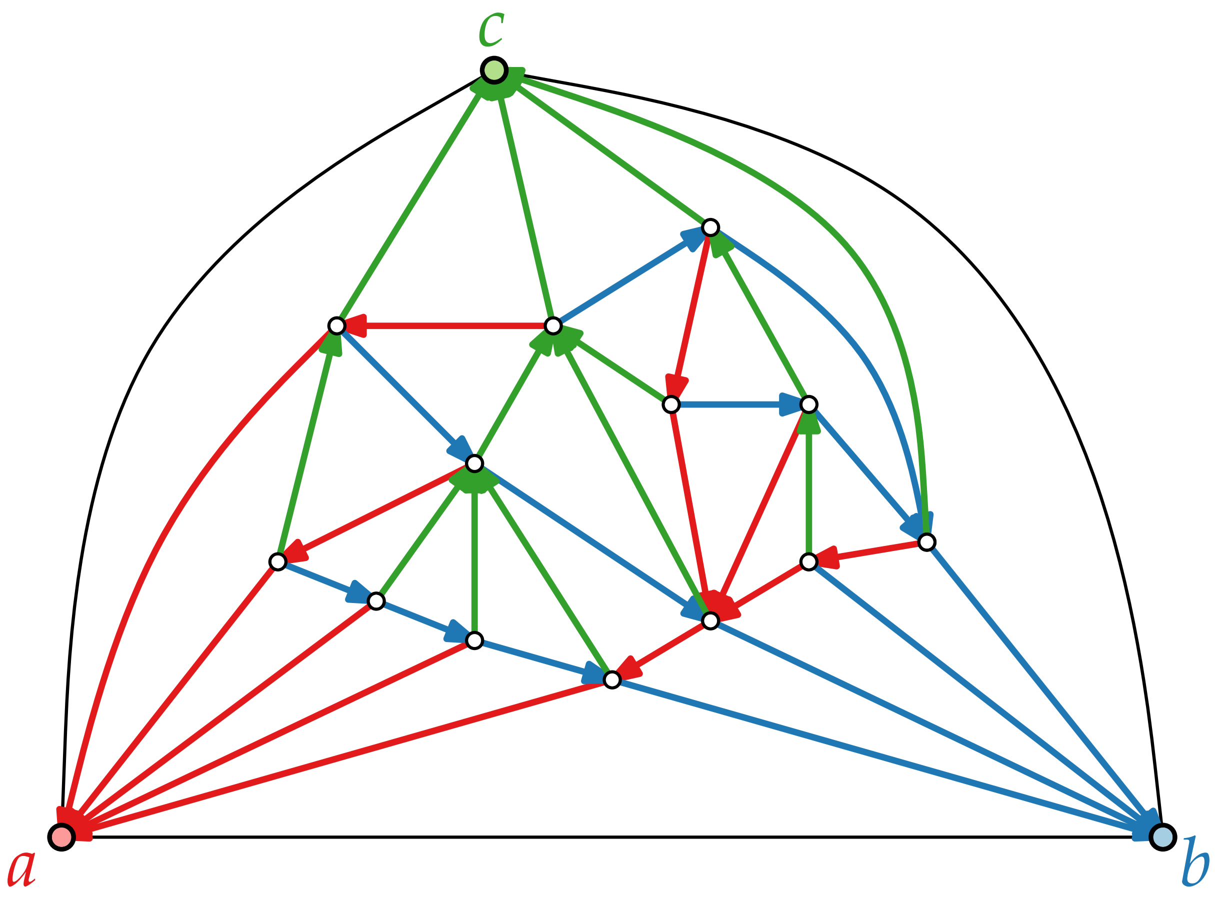

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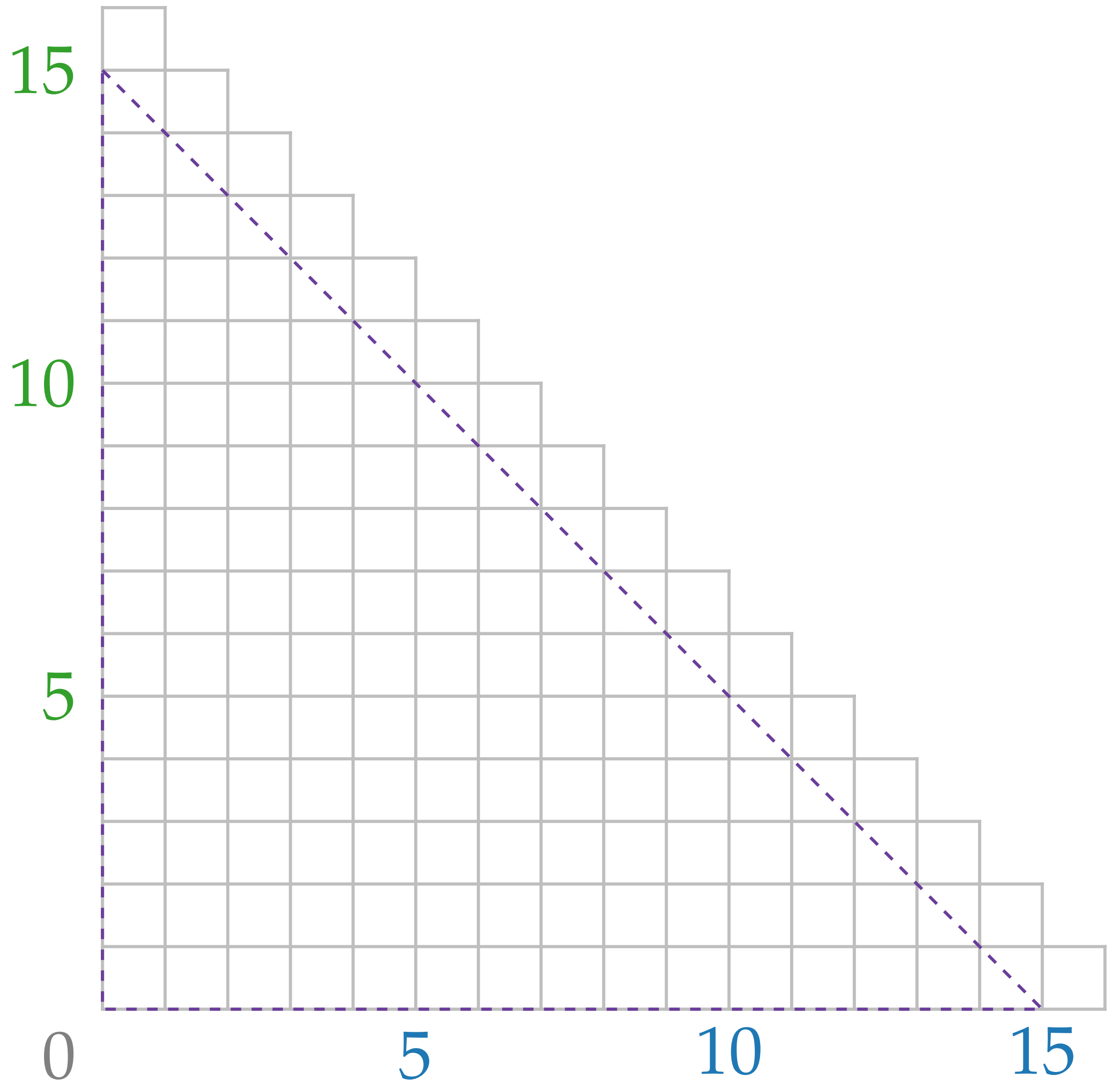

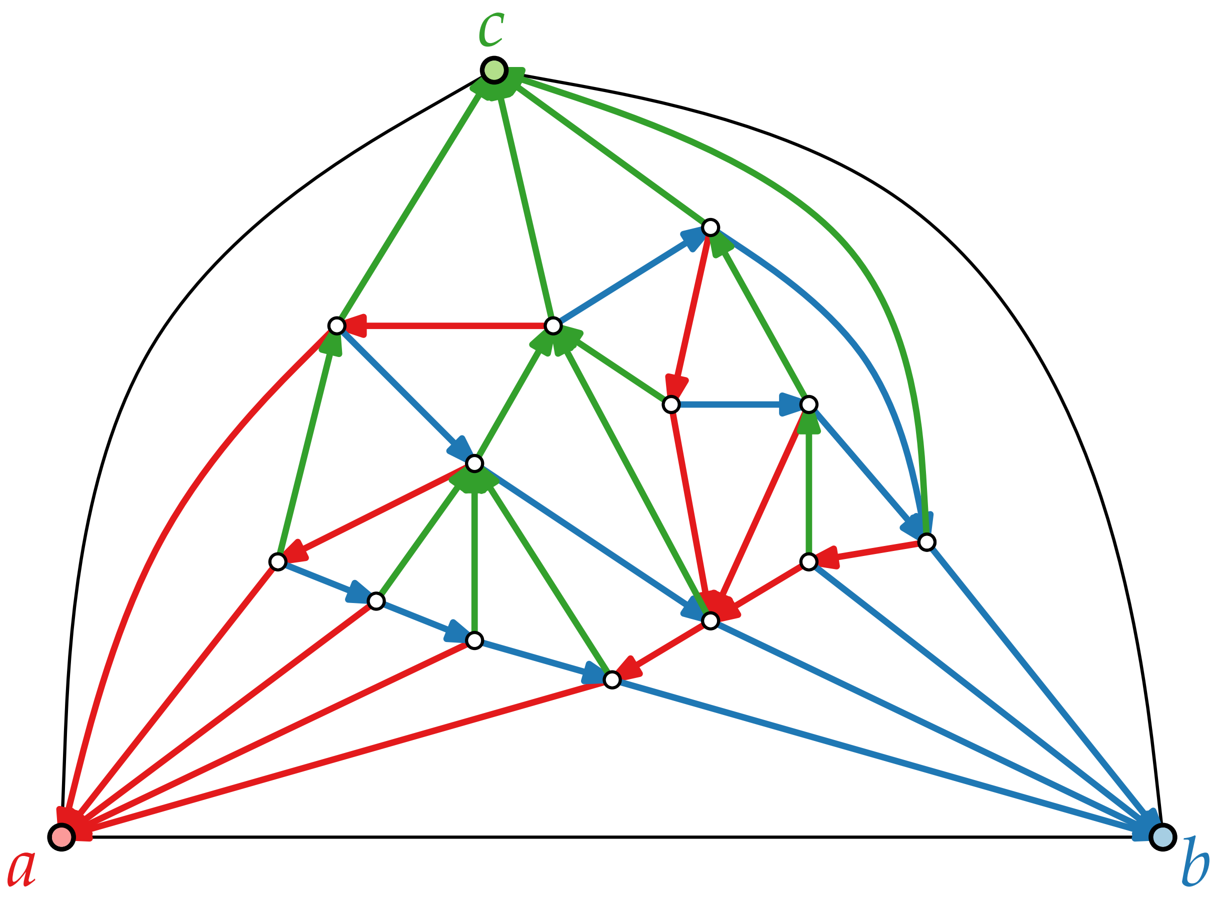

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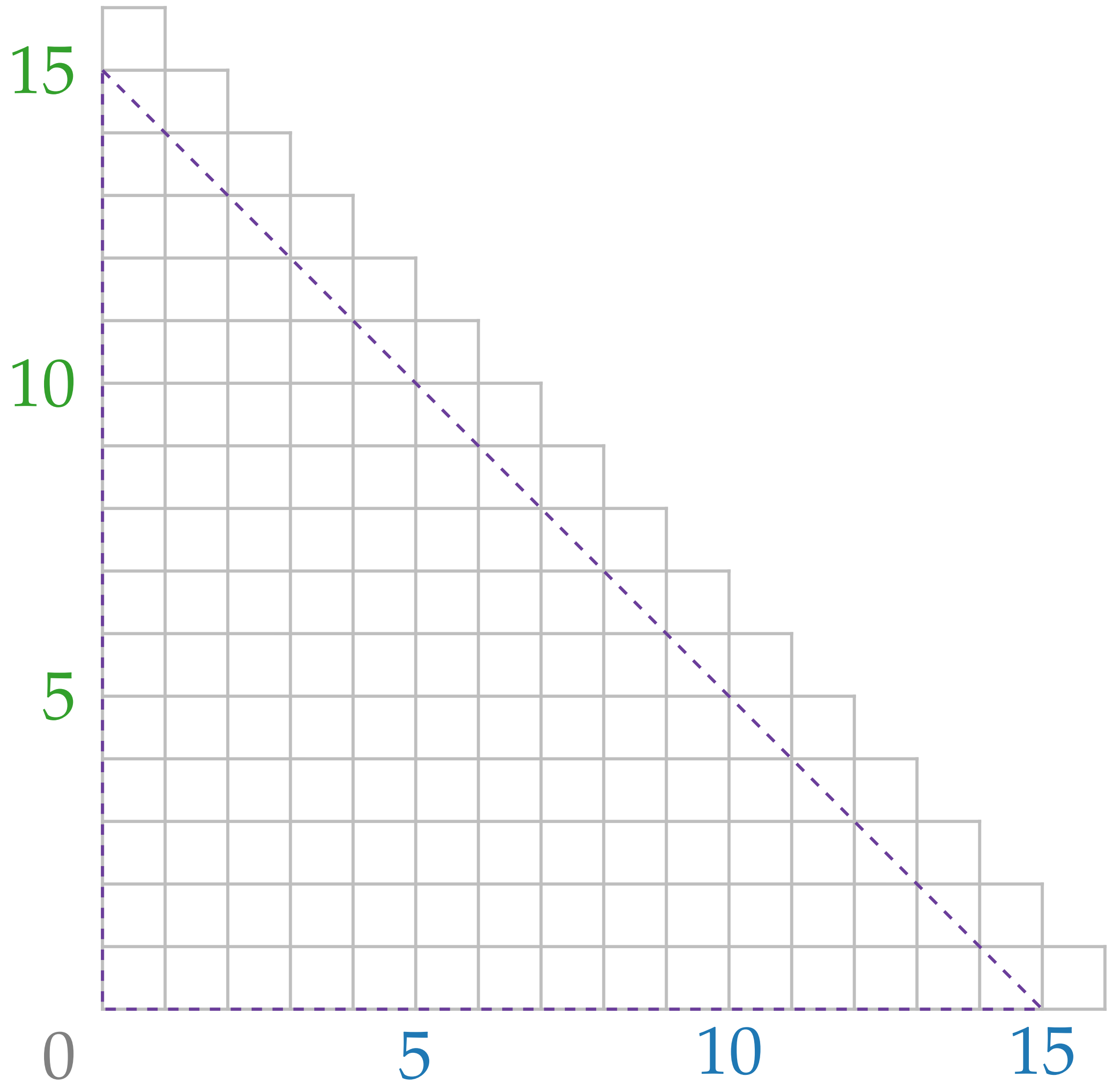

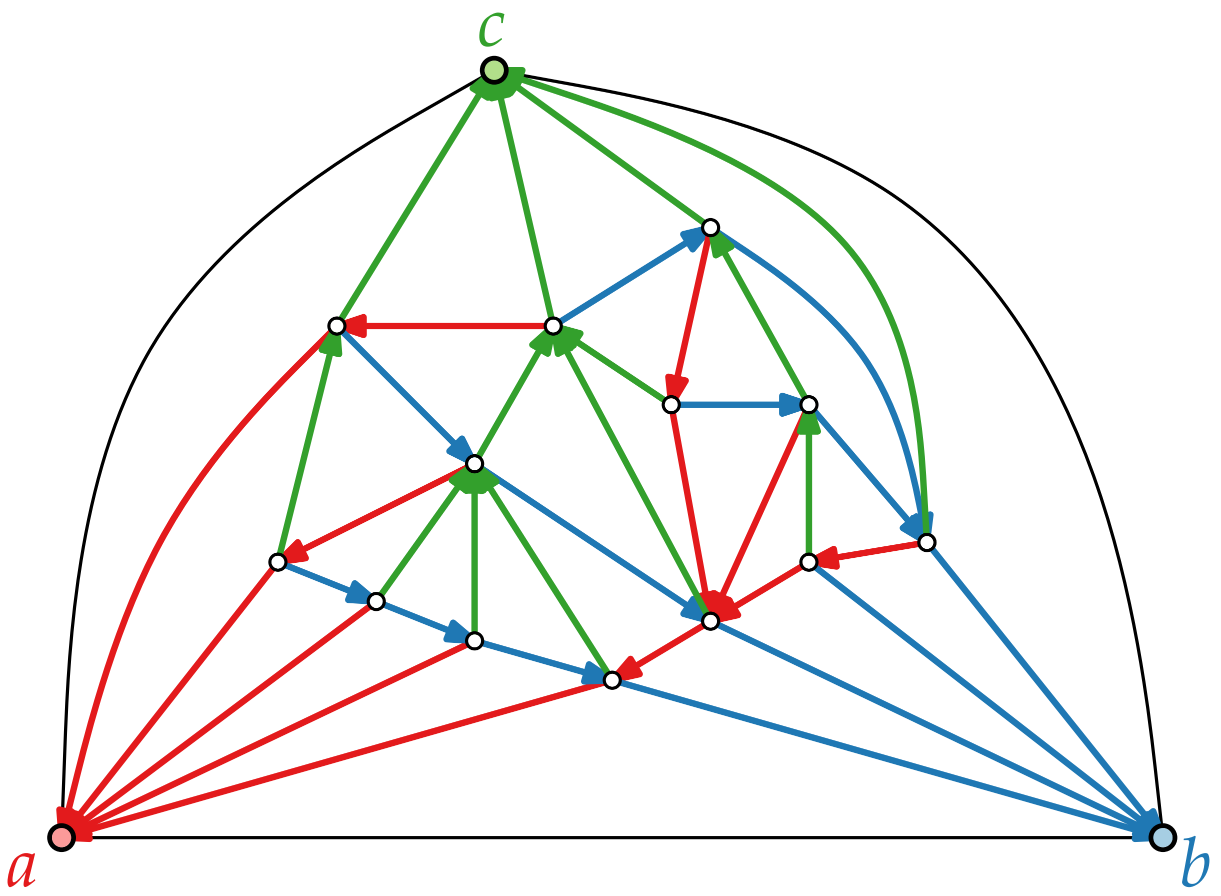

$n = 16, n - 2 = 14$

$f(a) = (n - 2, 1, 0)$

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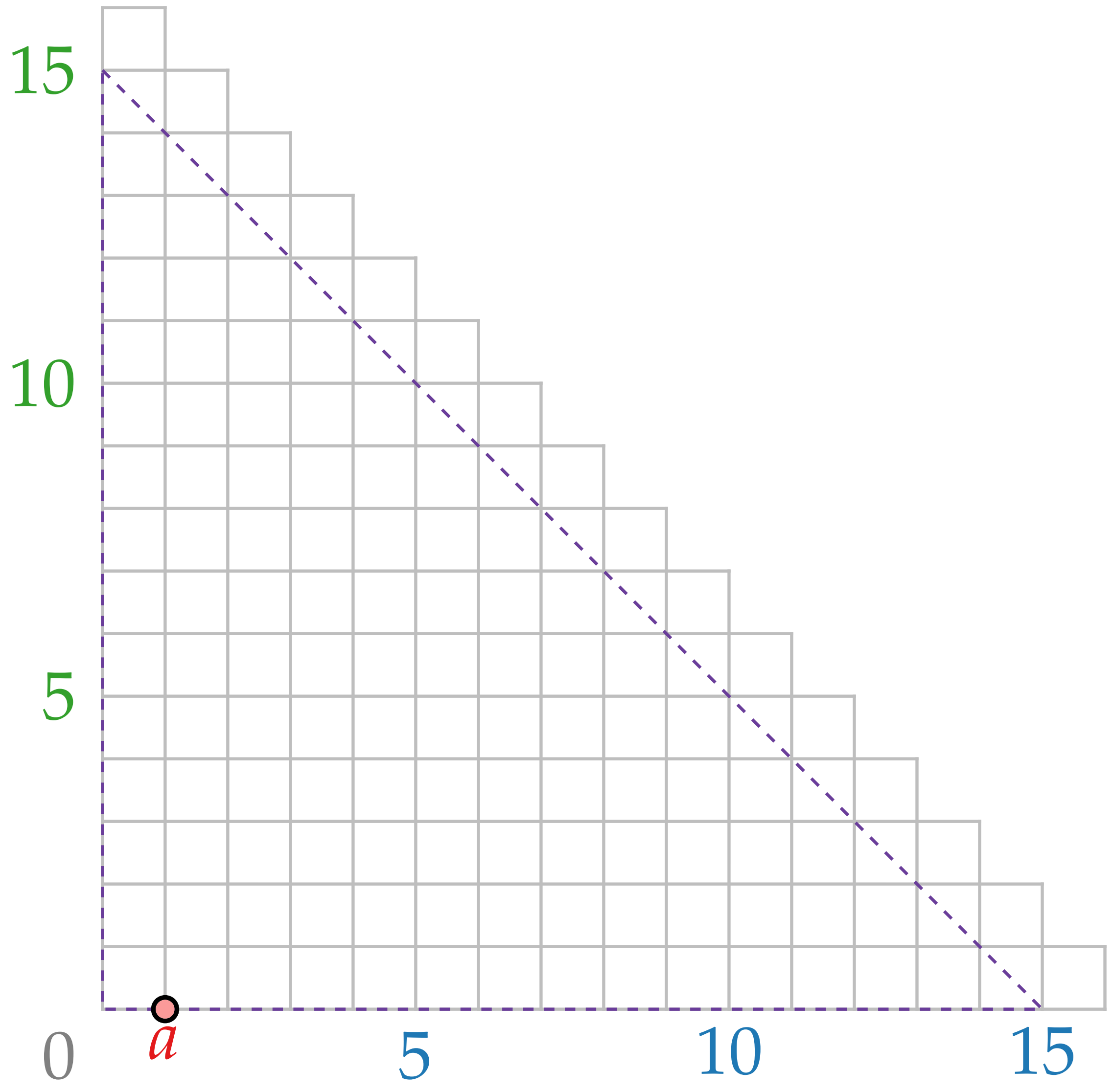

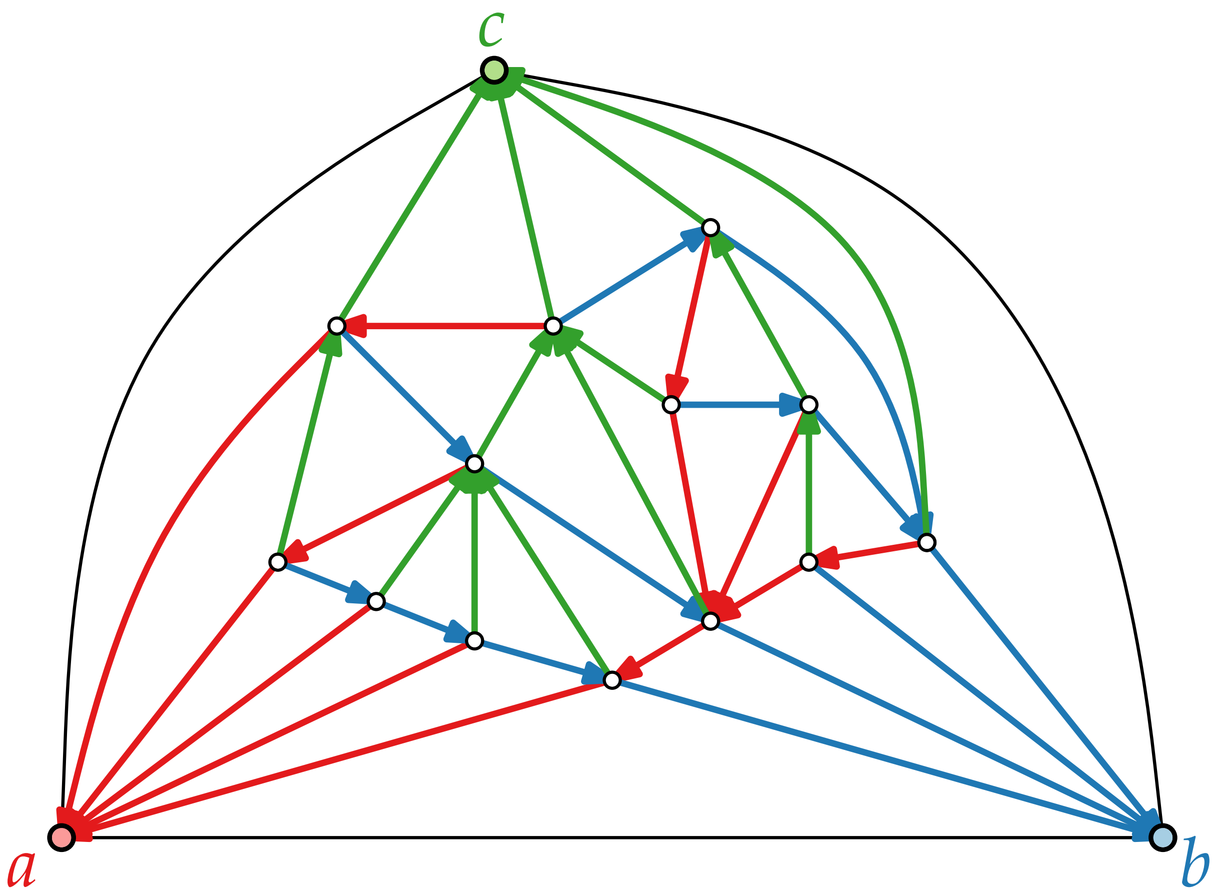

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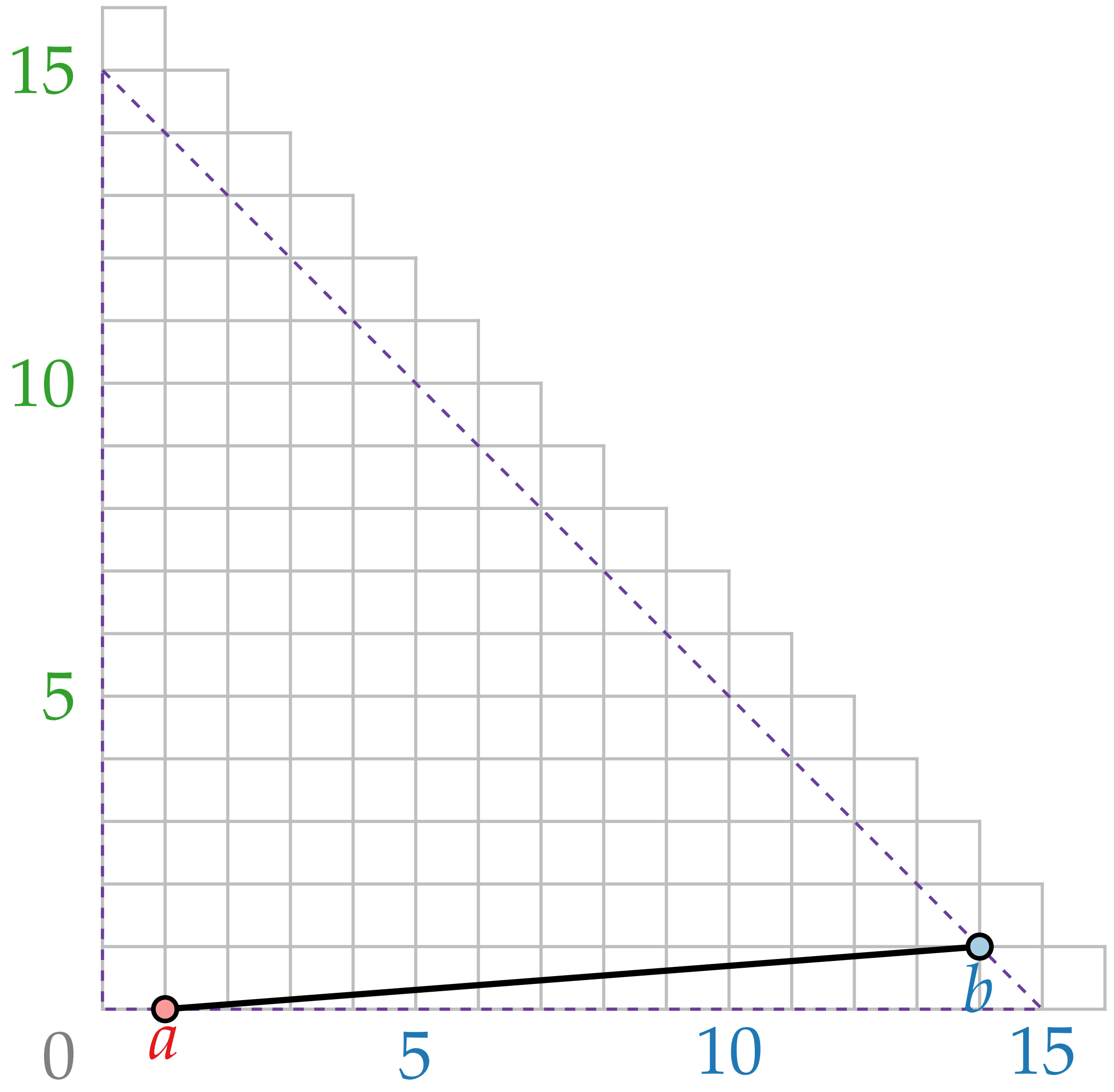

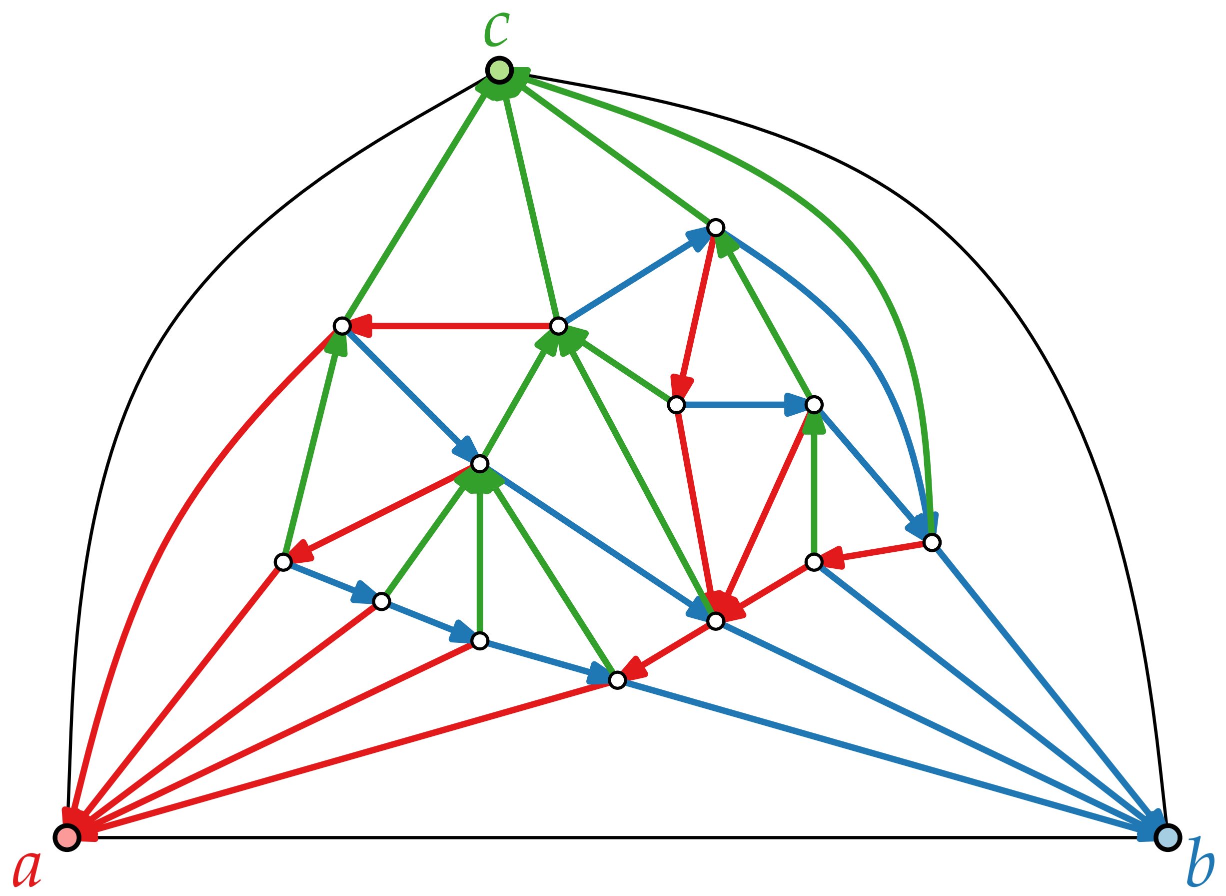

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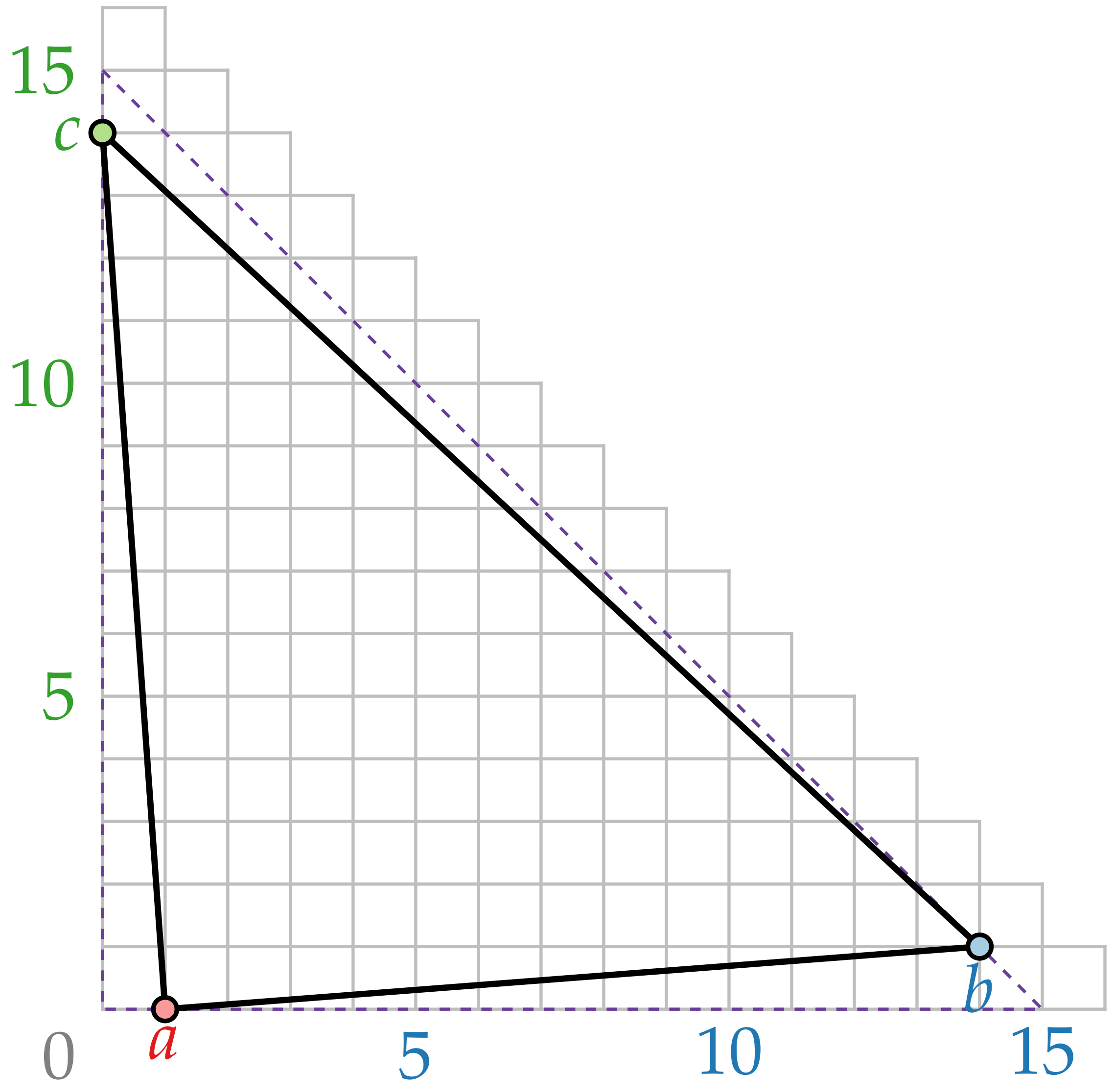

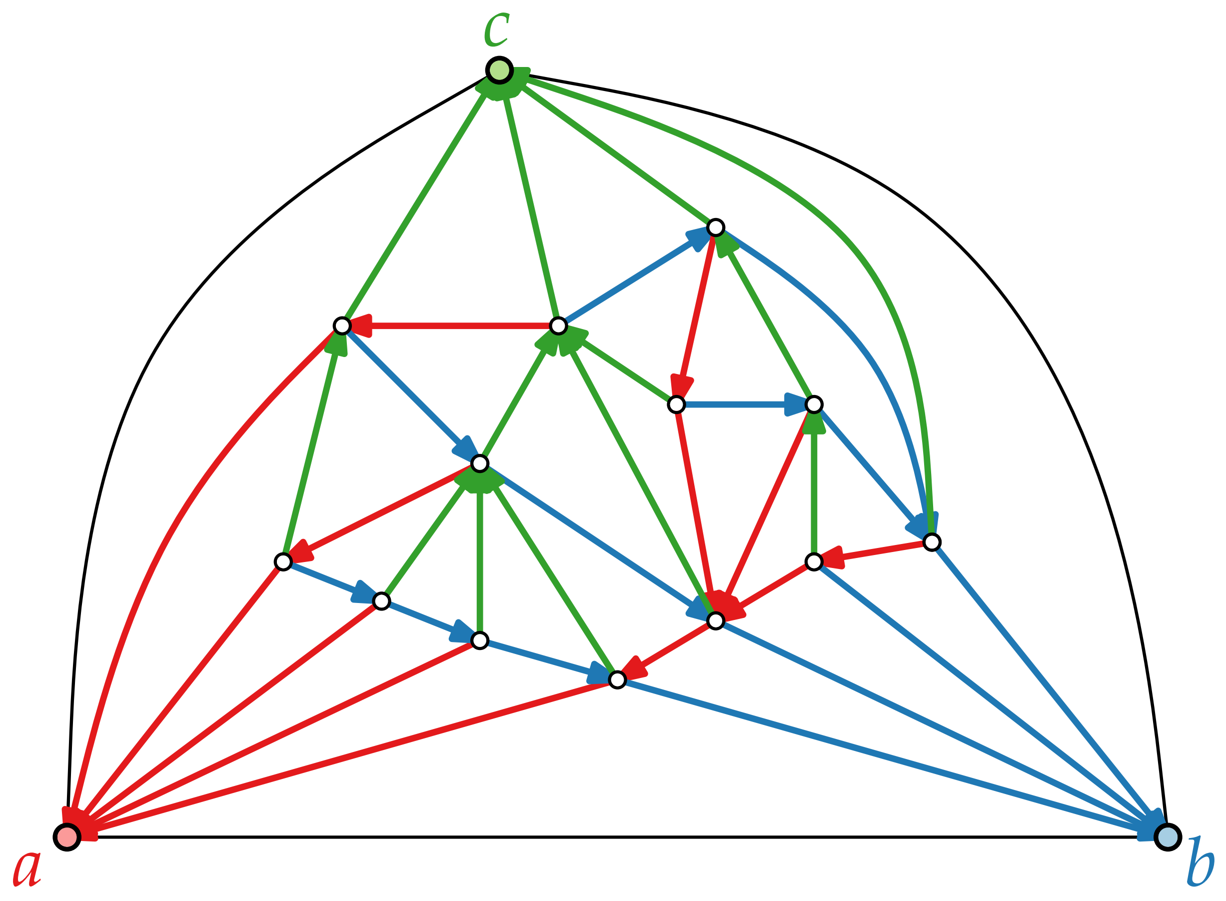

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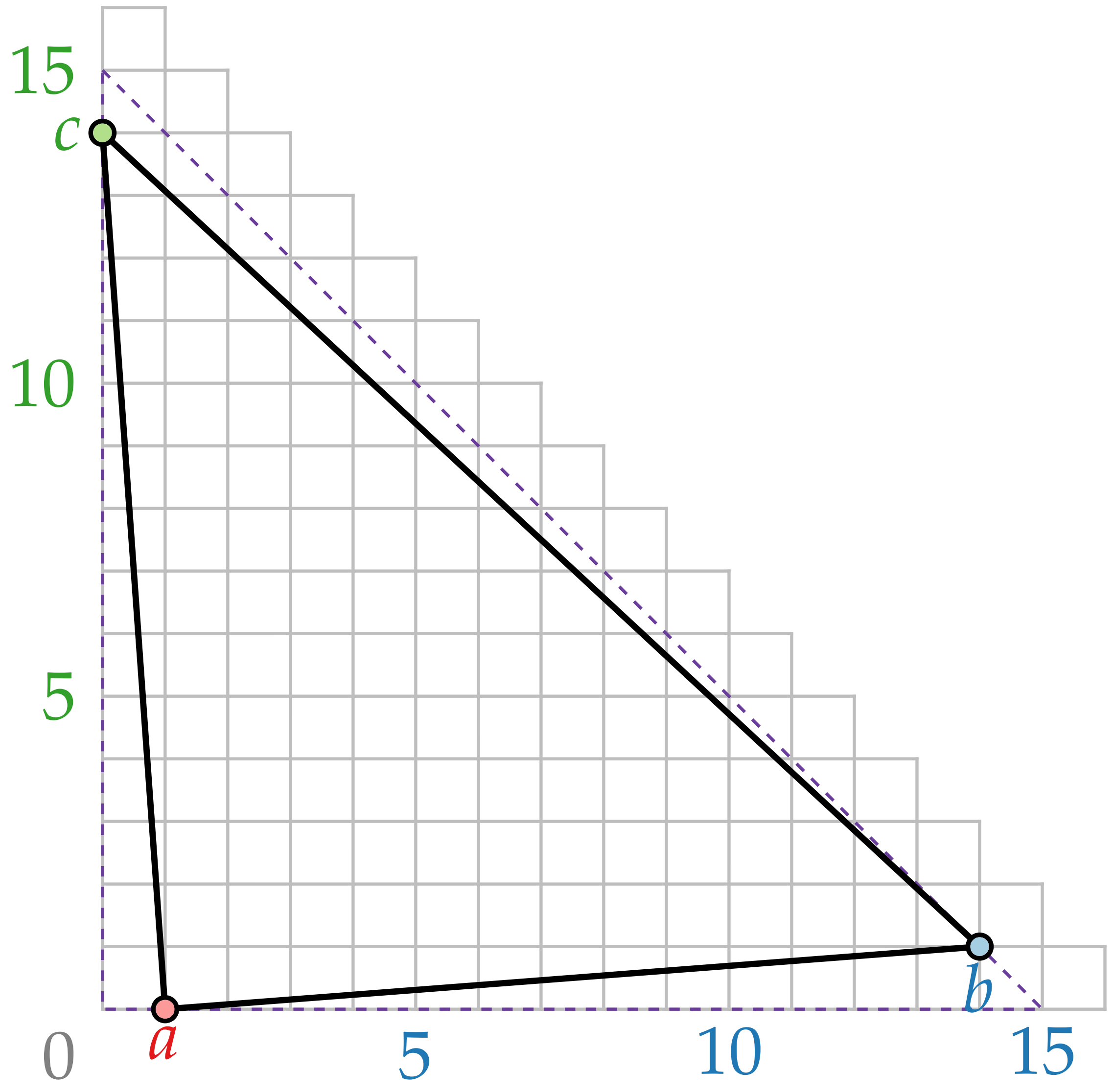

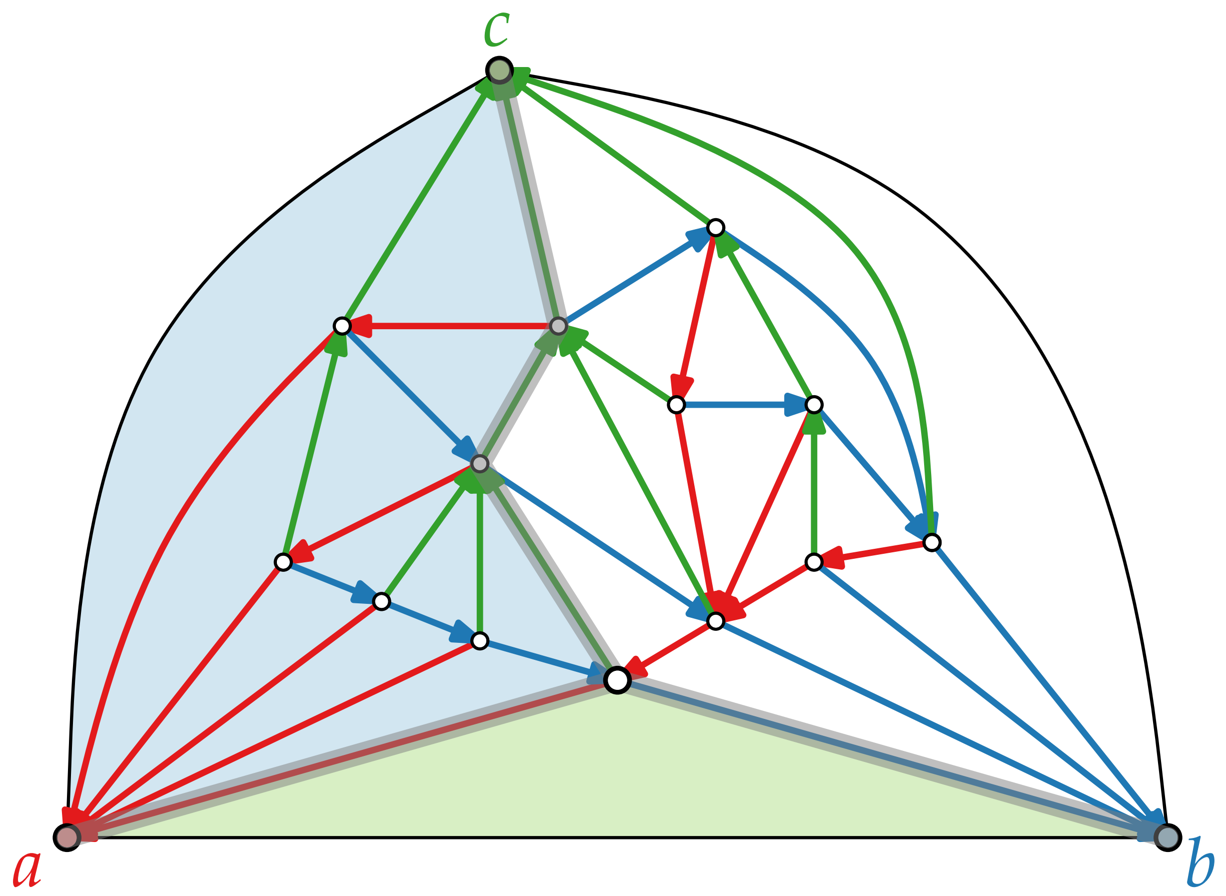

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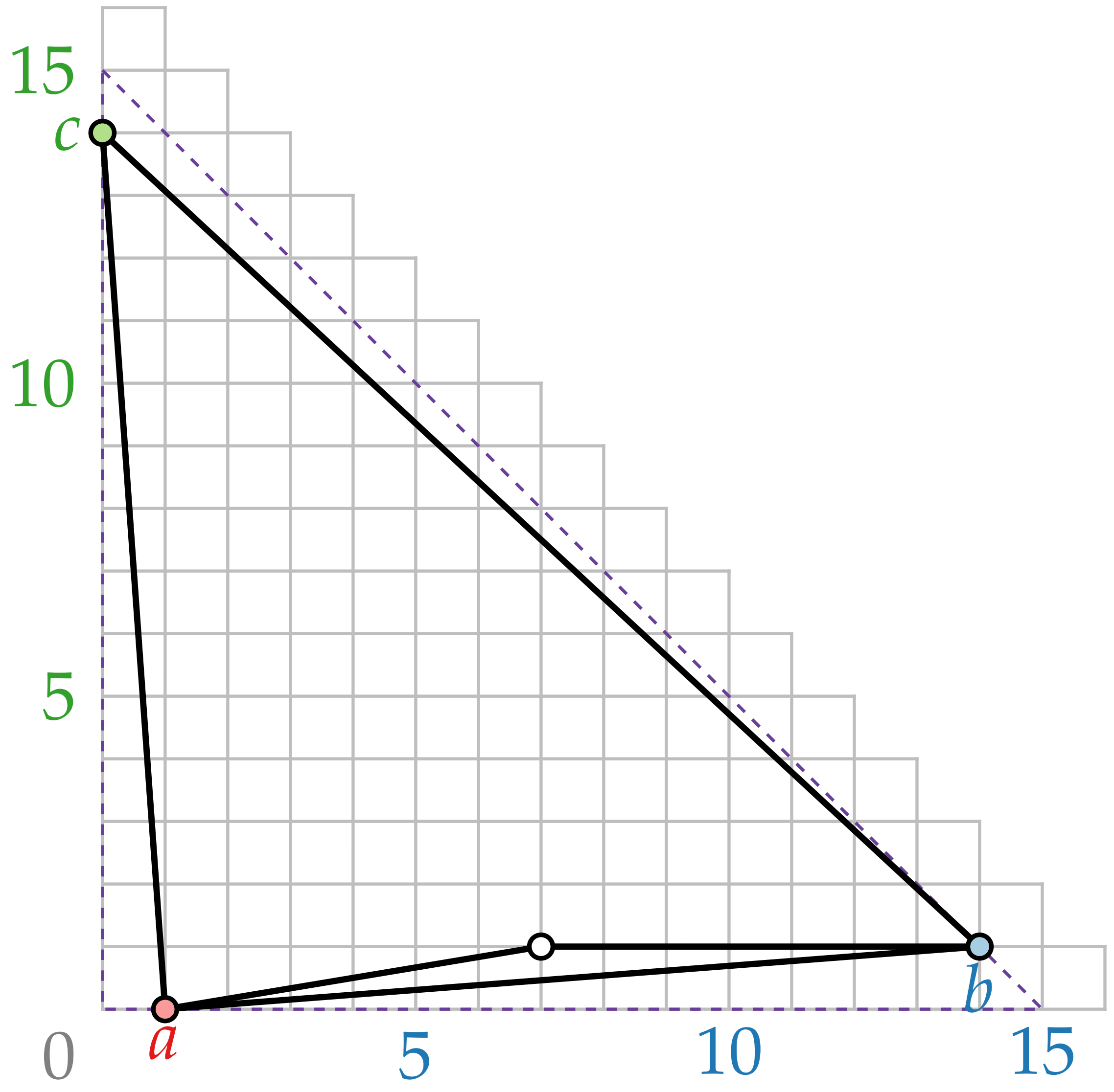

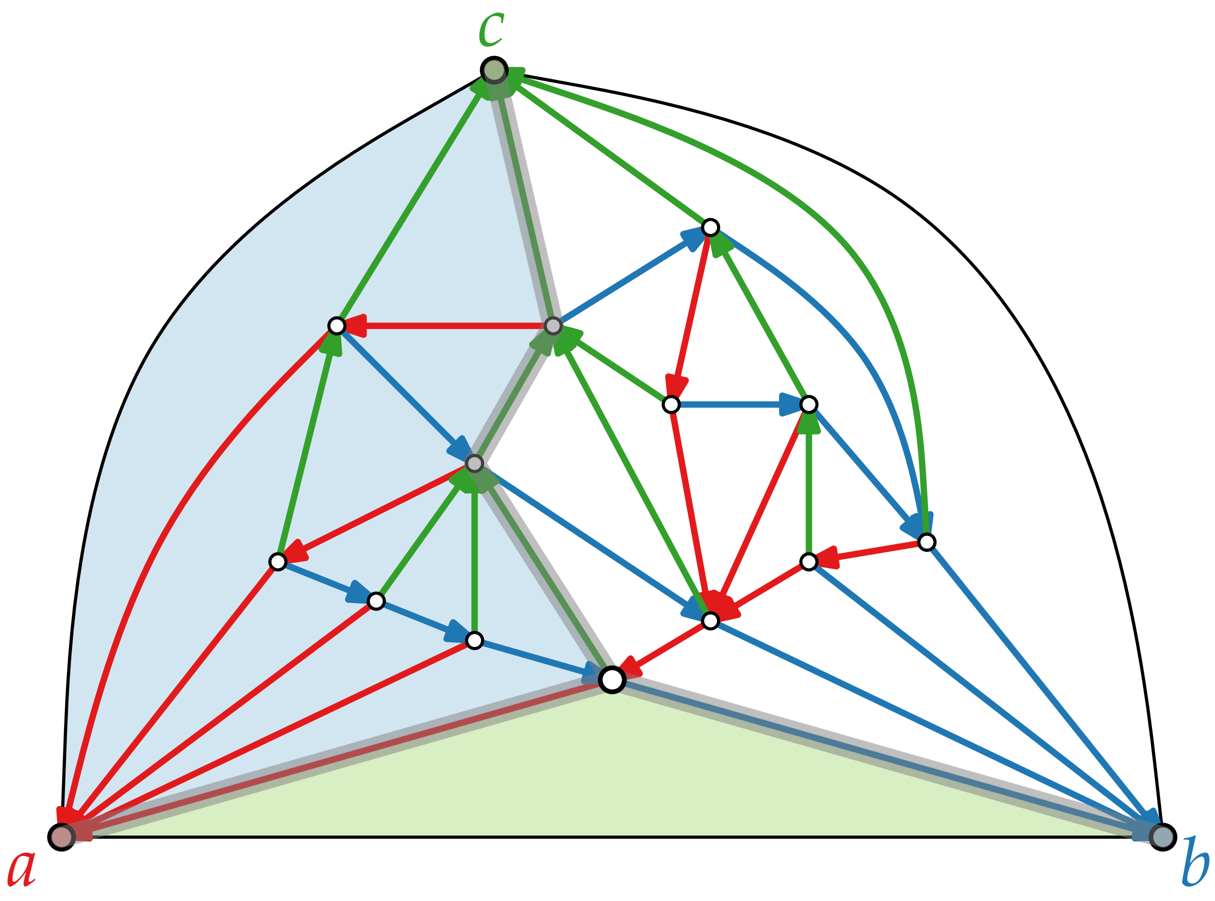

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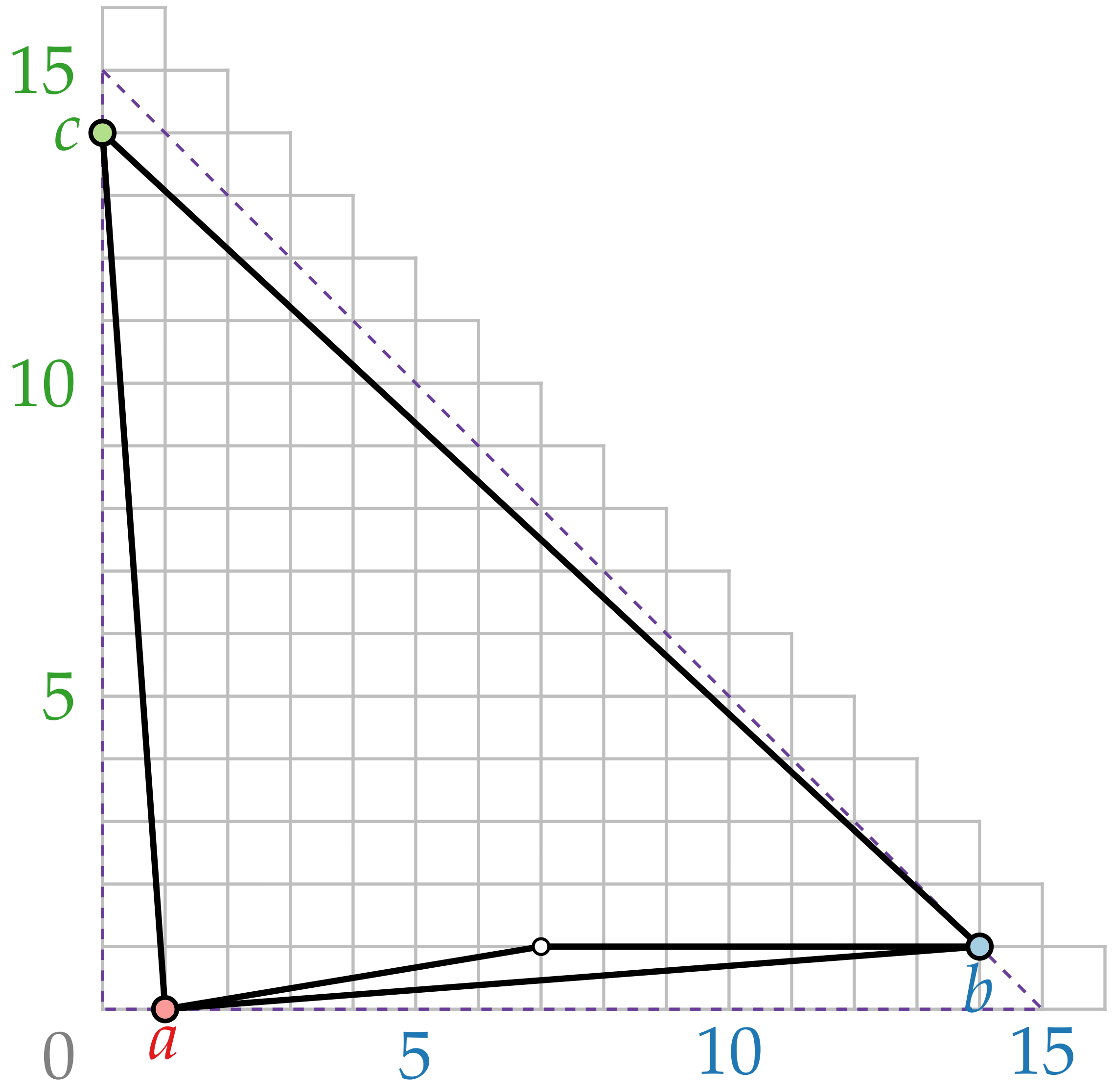

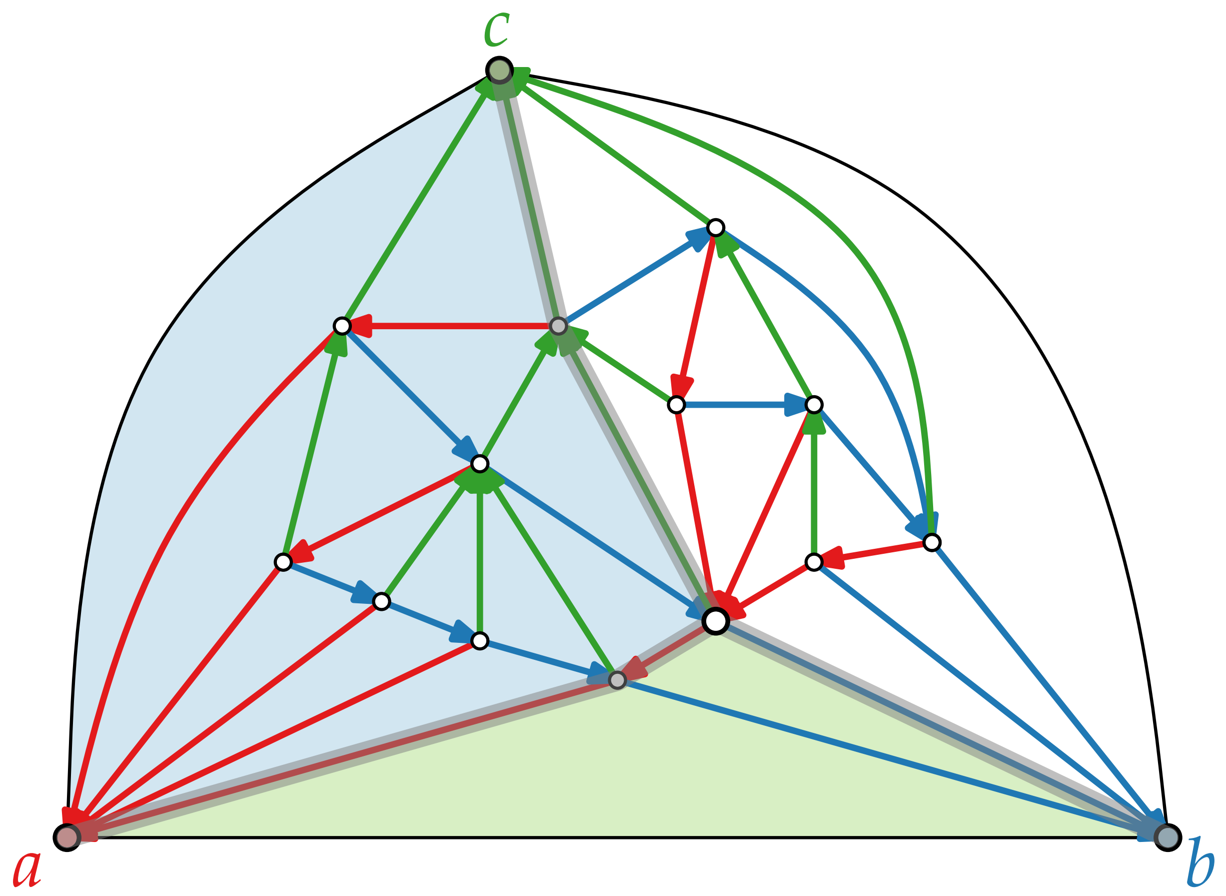

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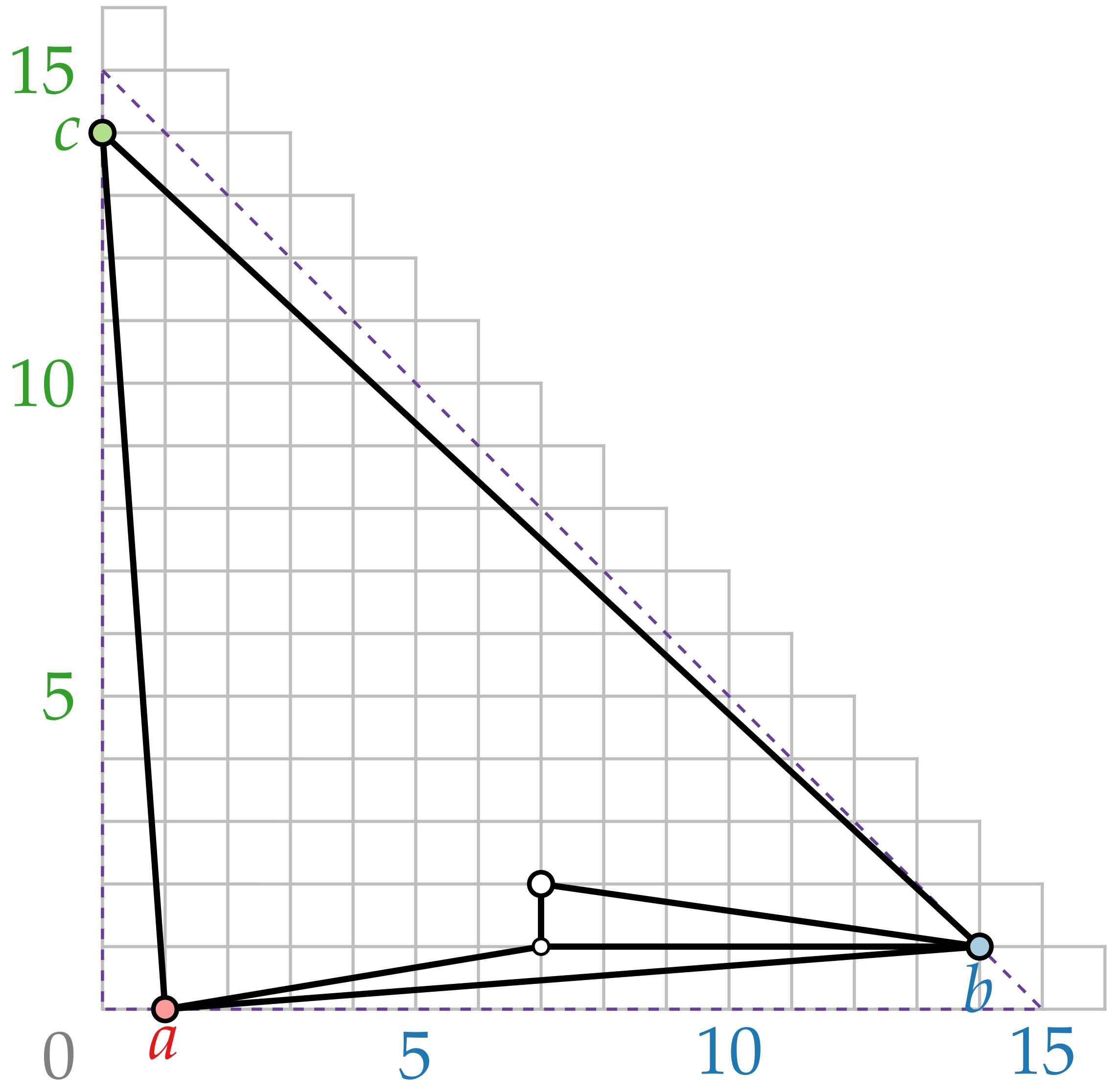

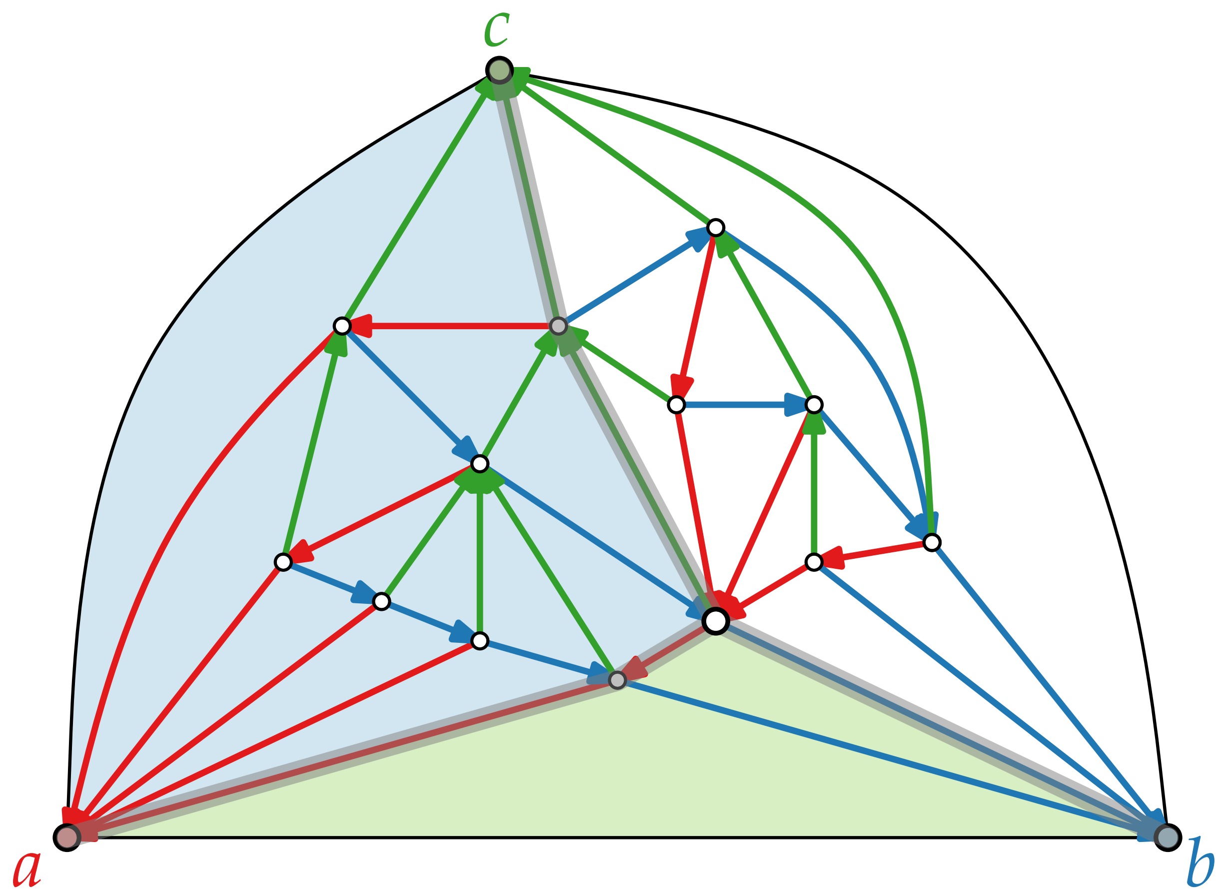

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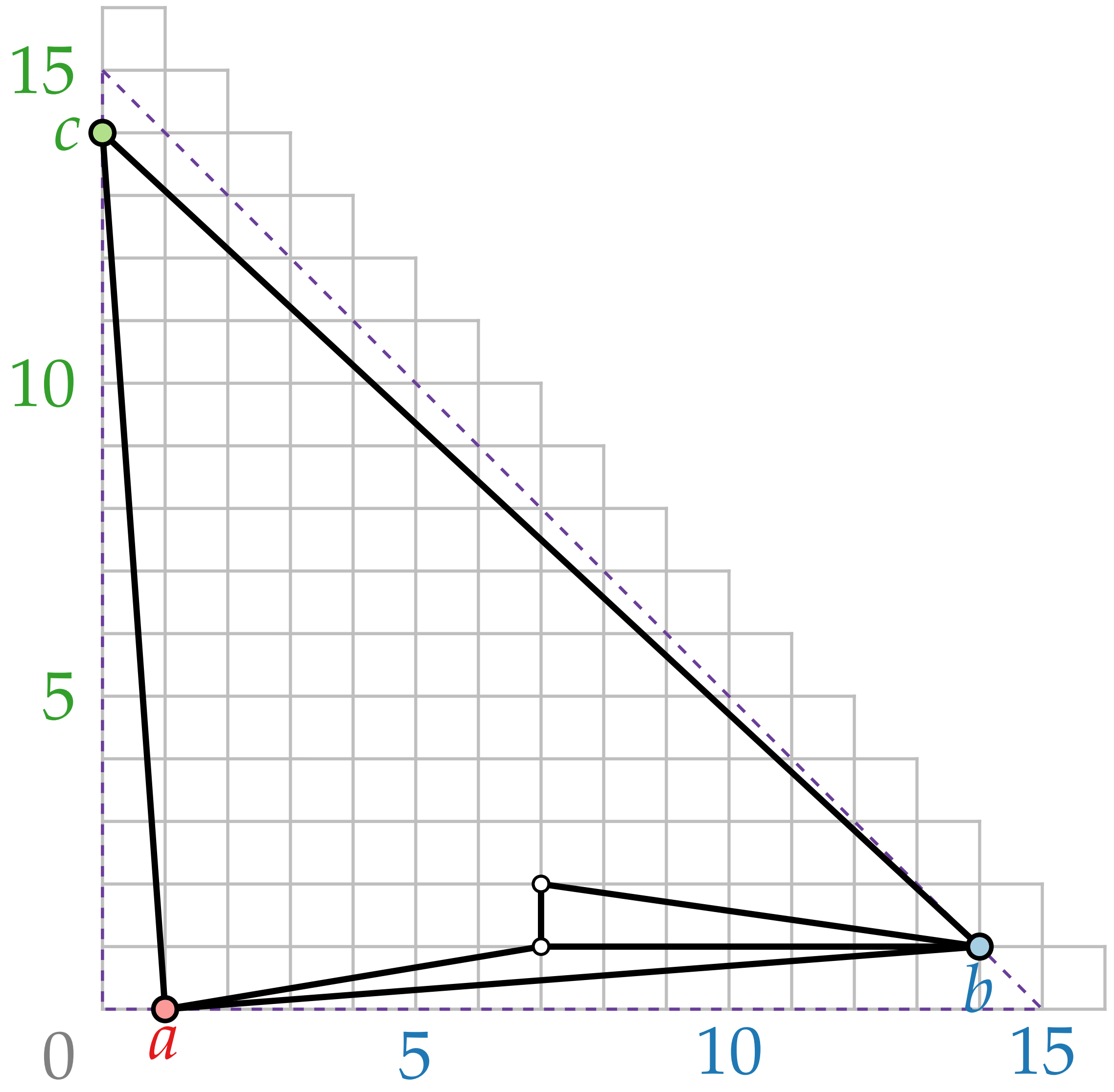

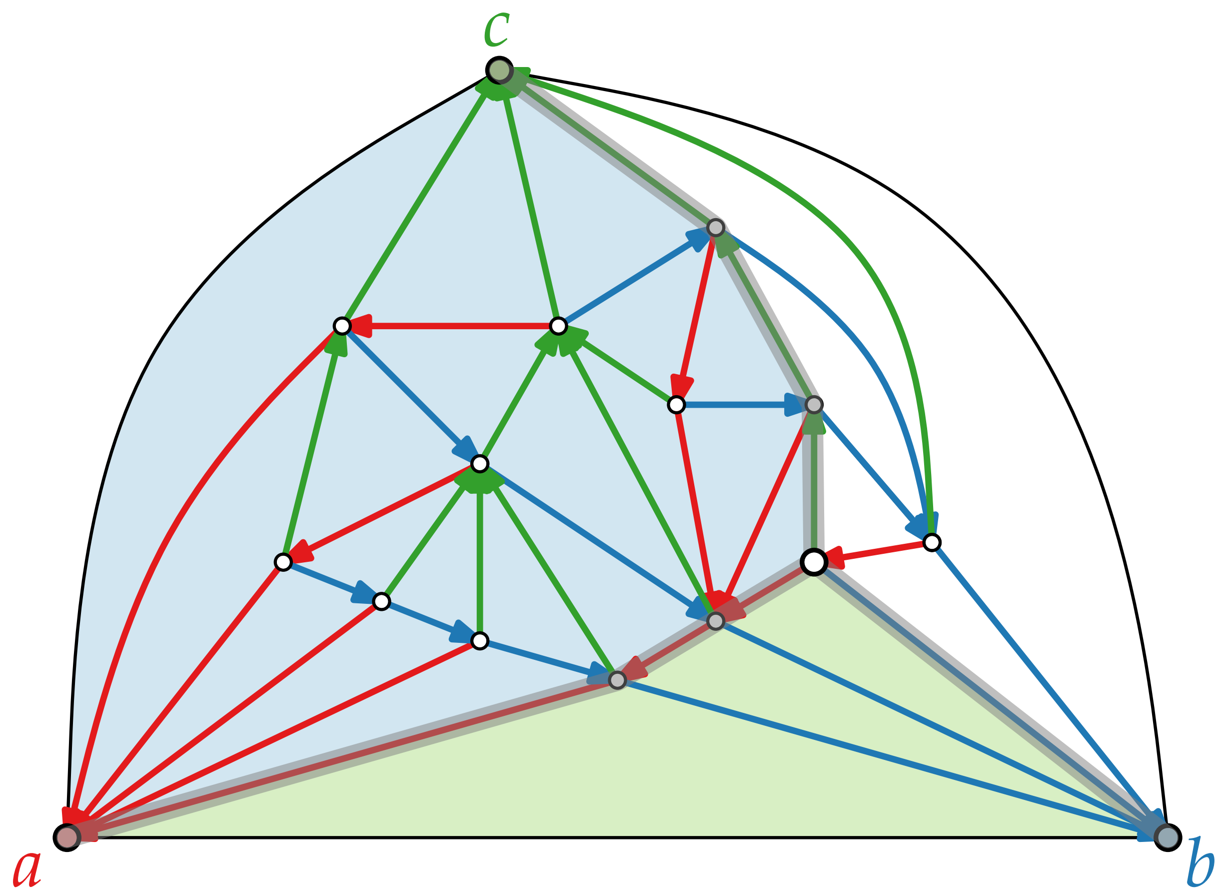

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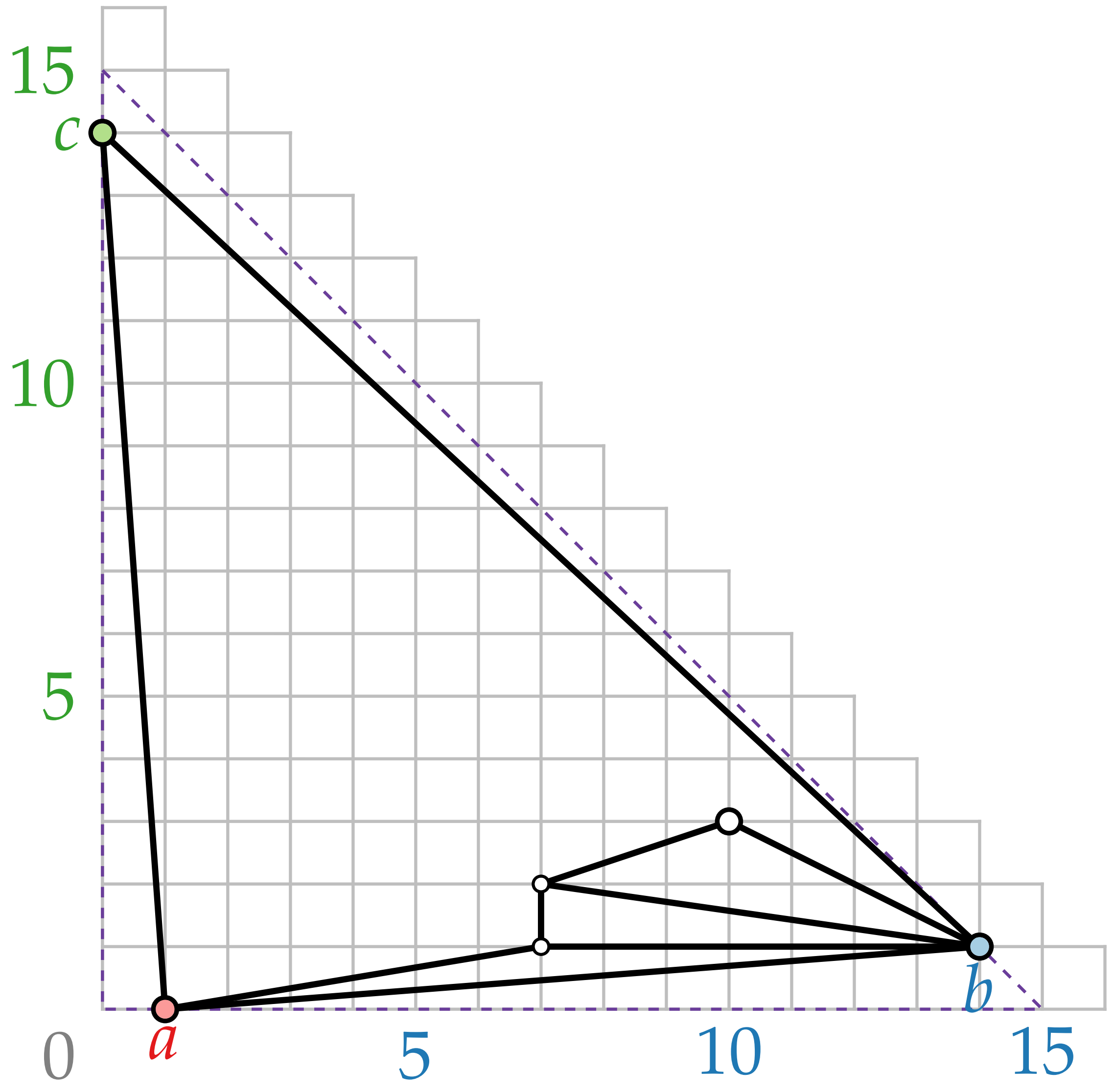

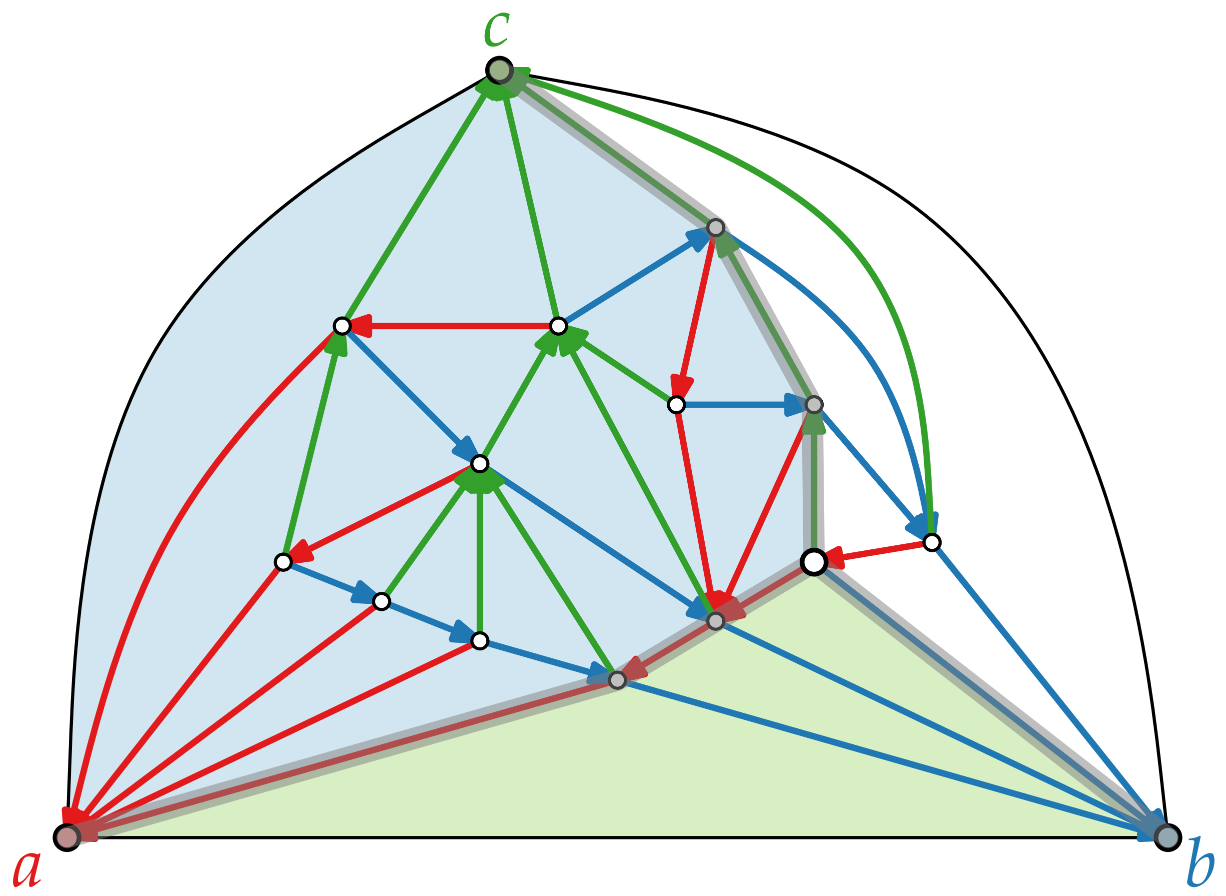

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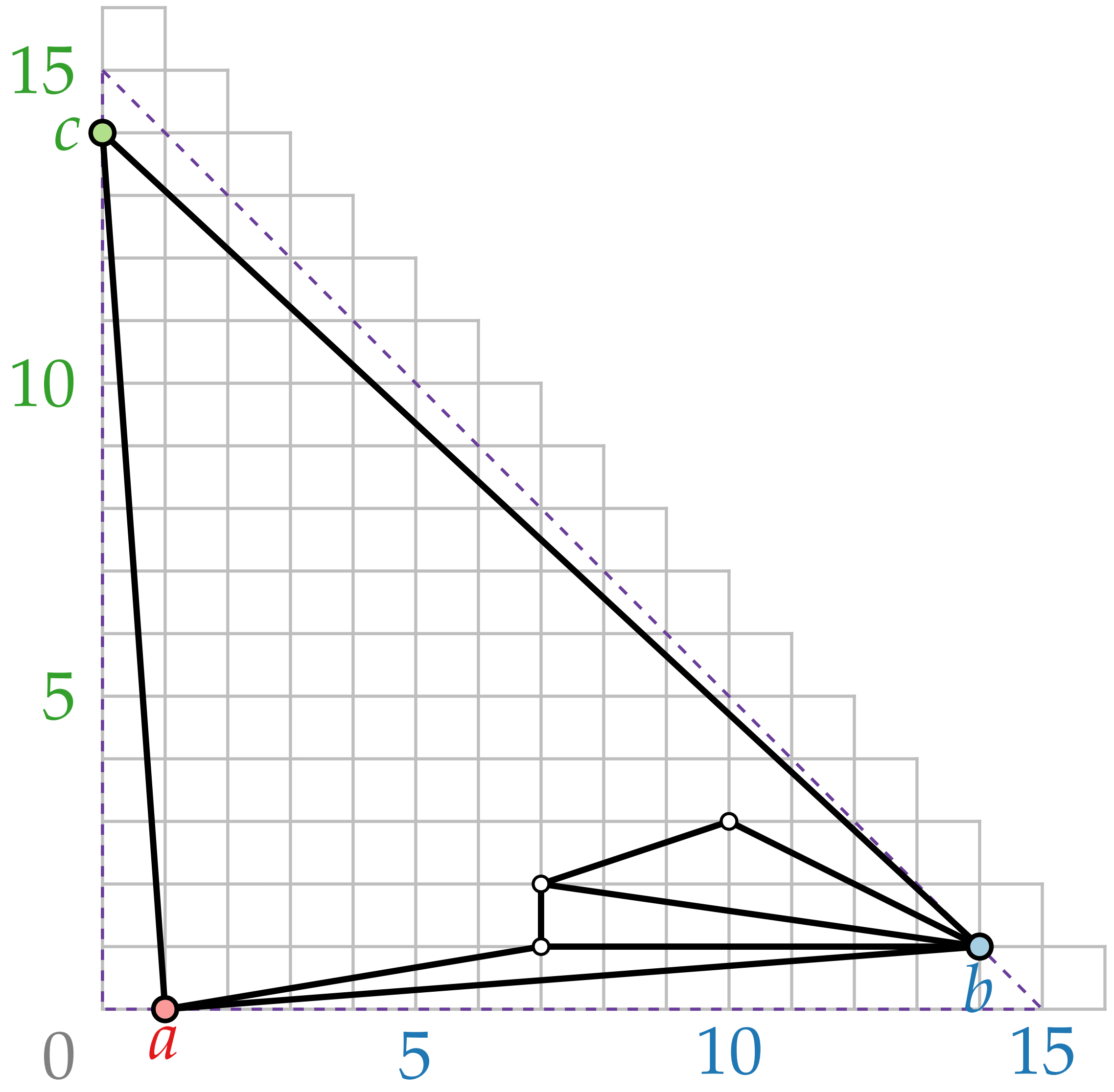

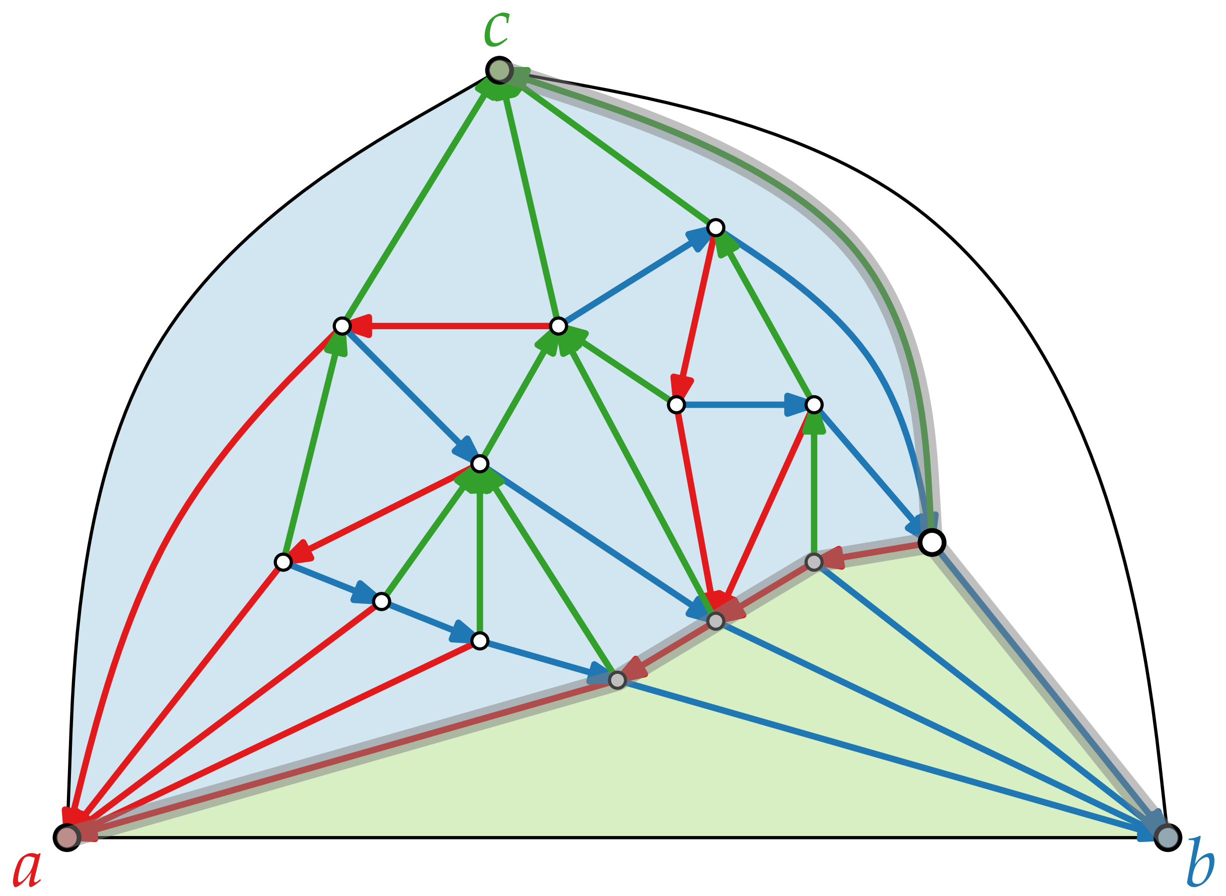

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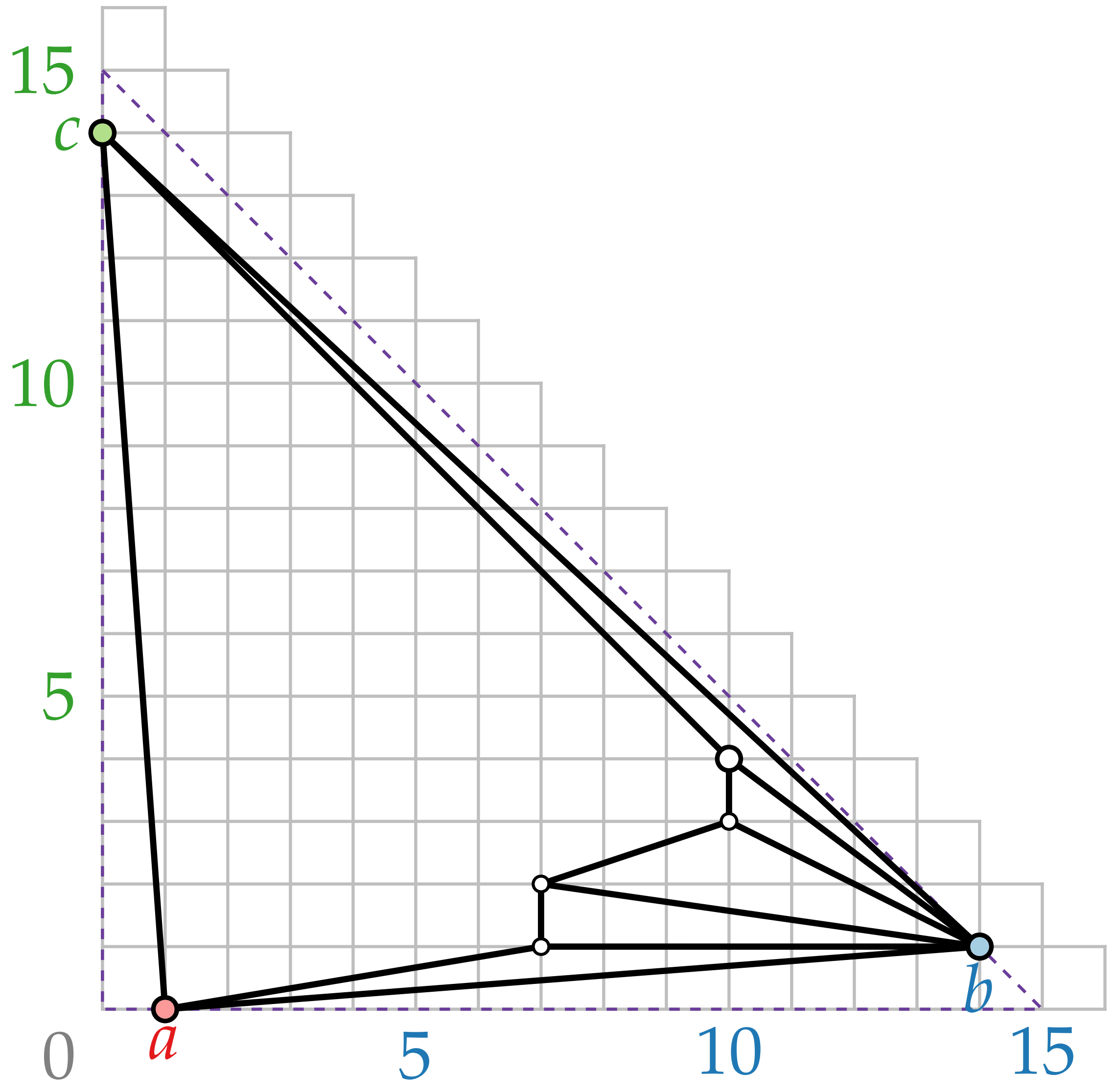

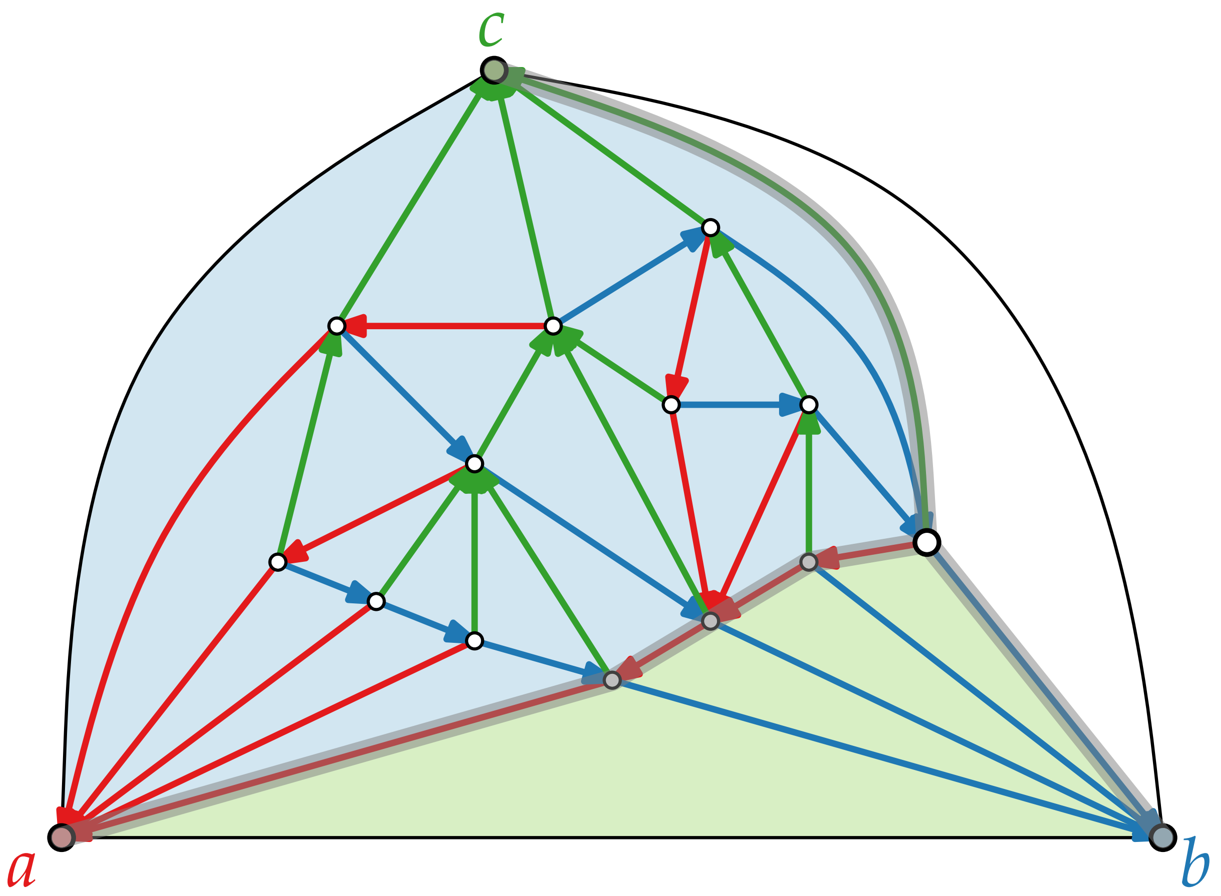

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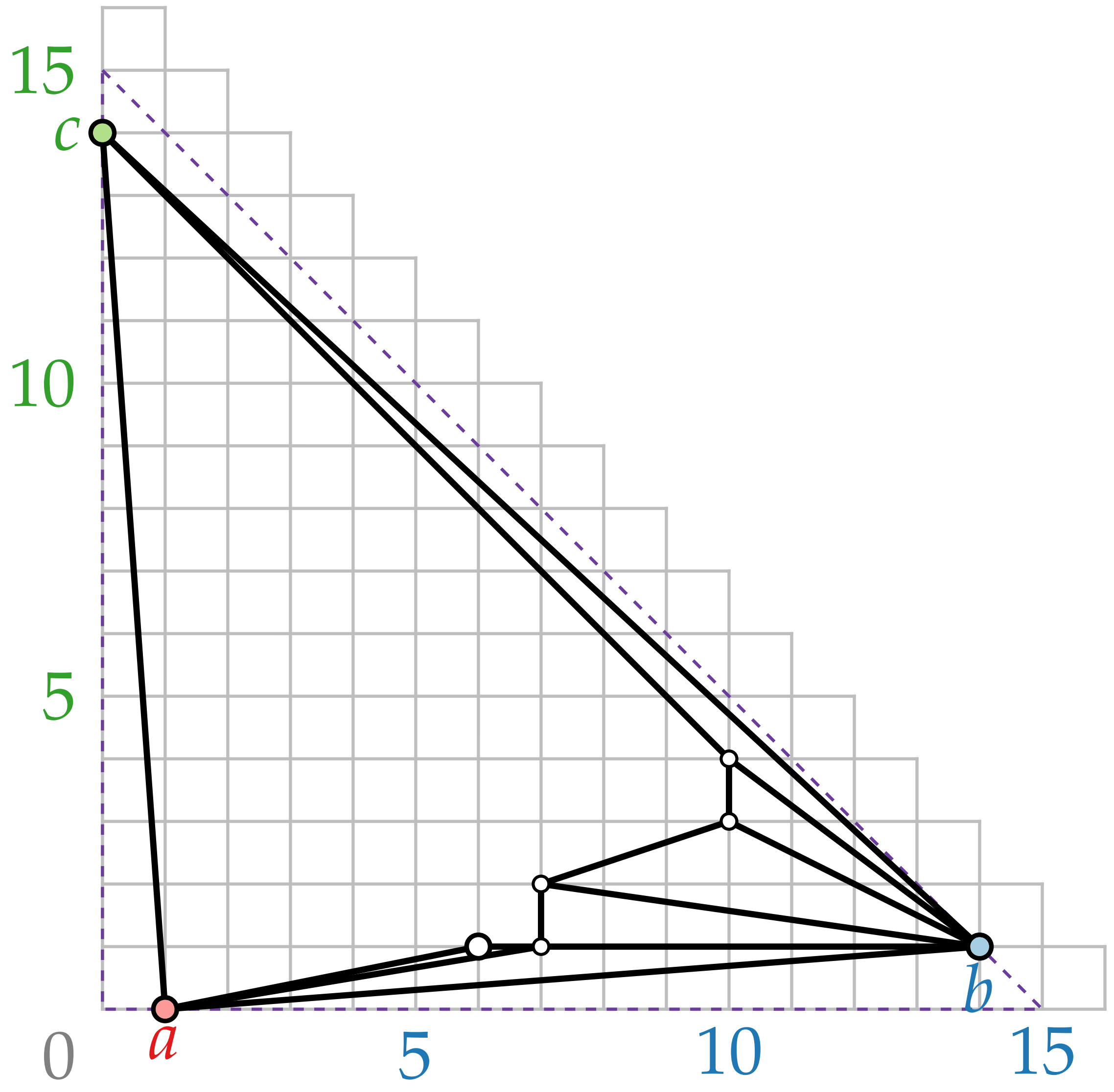

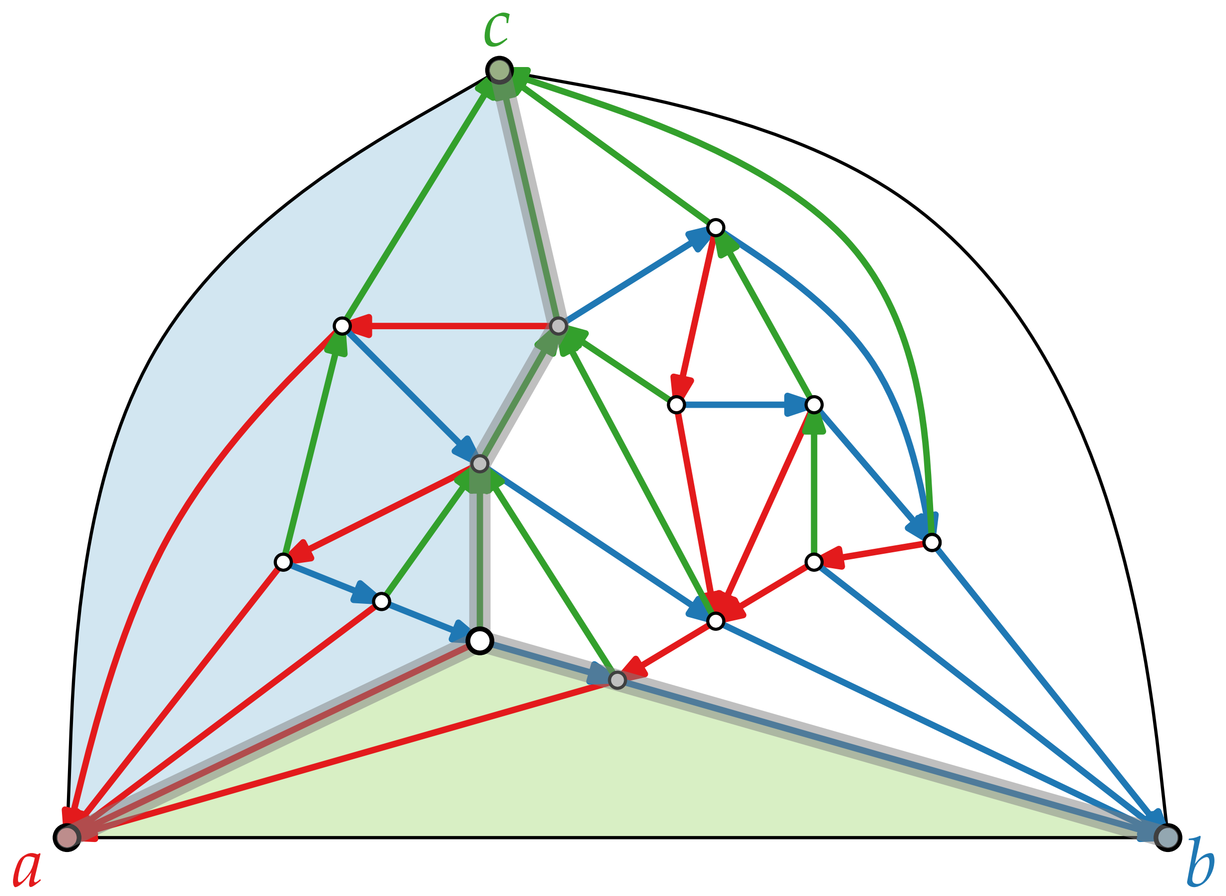

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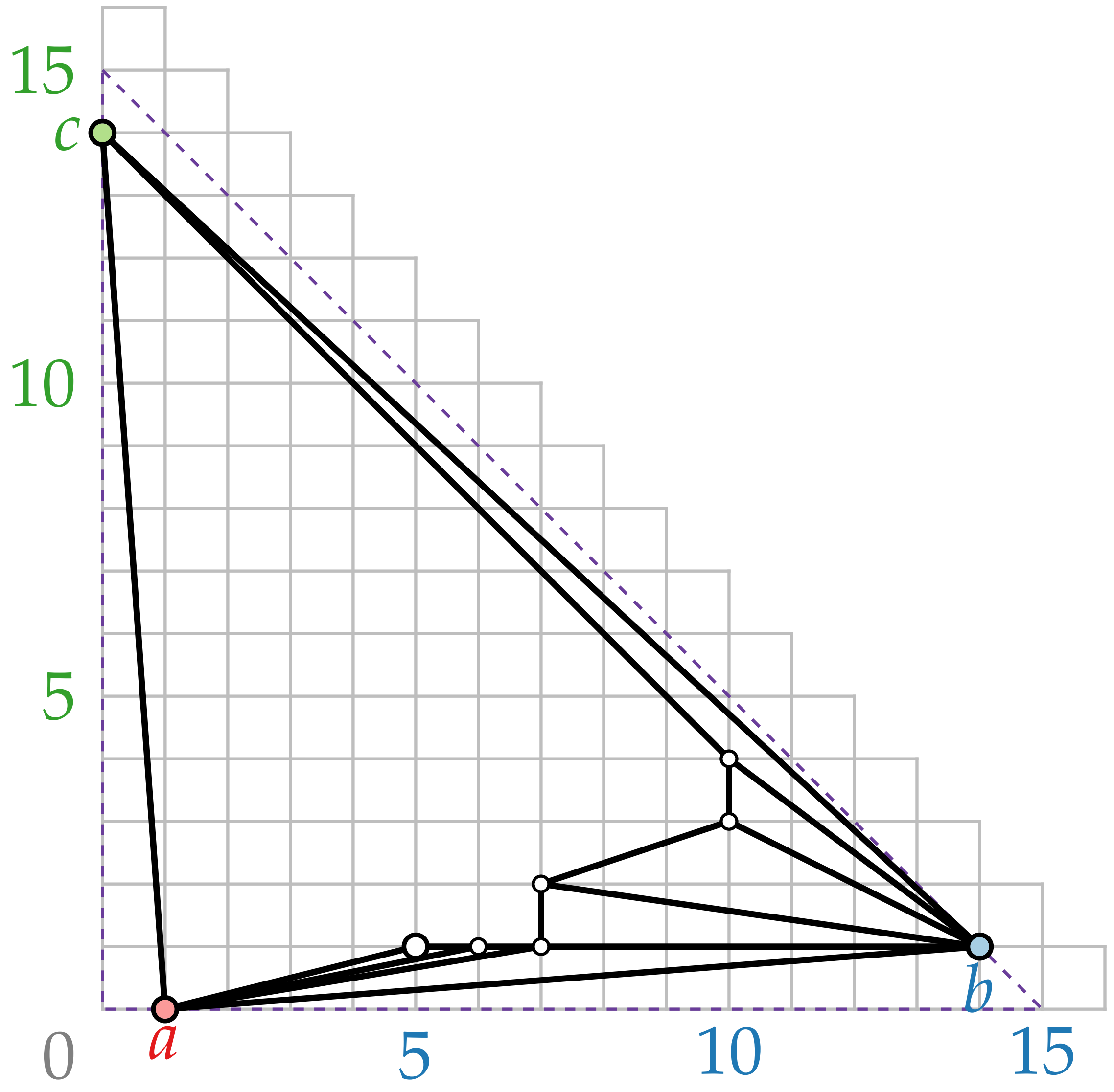
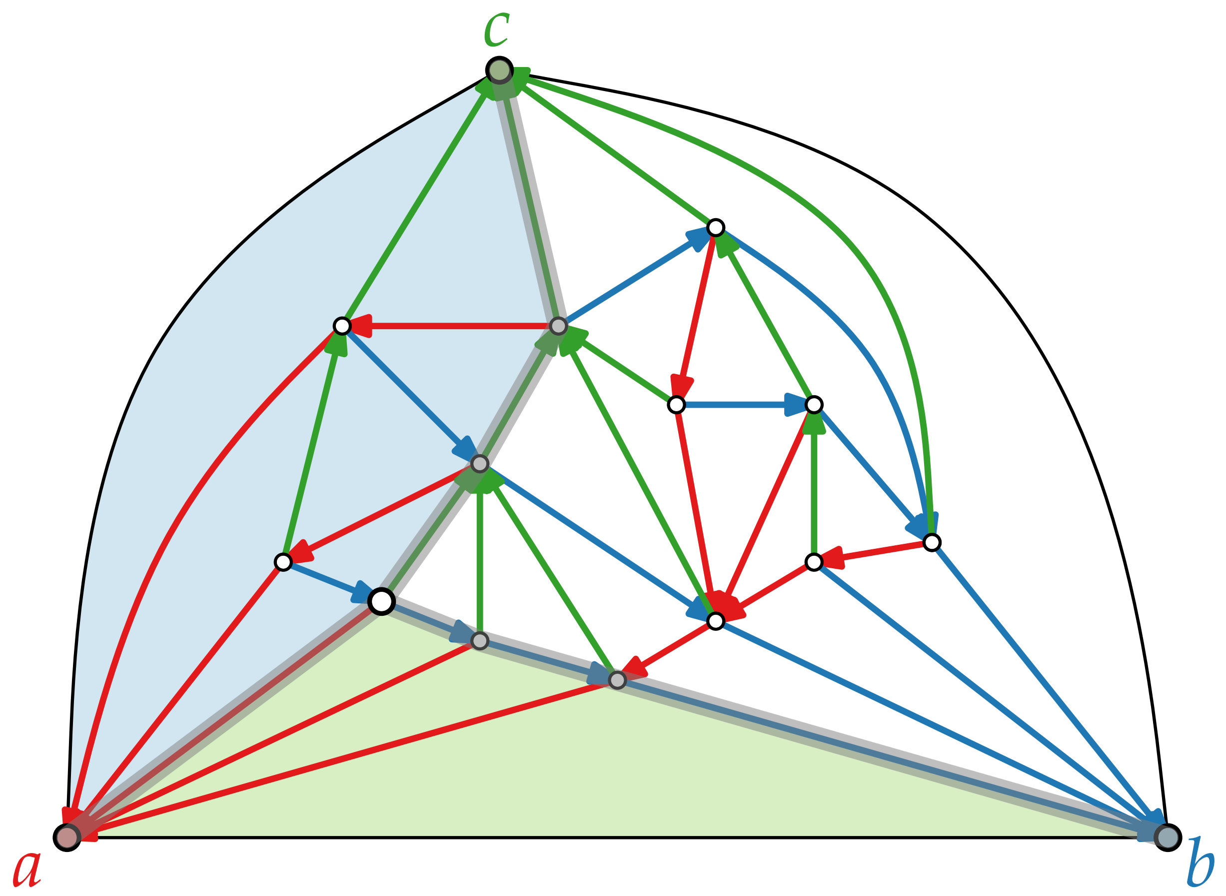
Schnyder Drawing* – Example





$n = 16, n - 2 = 14$

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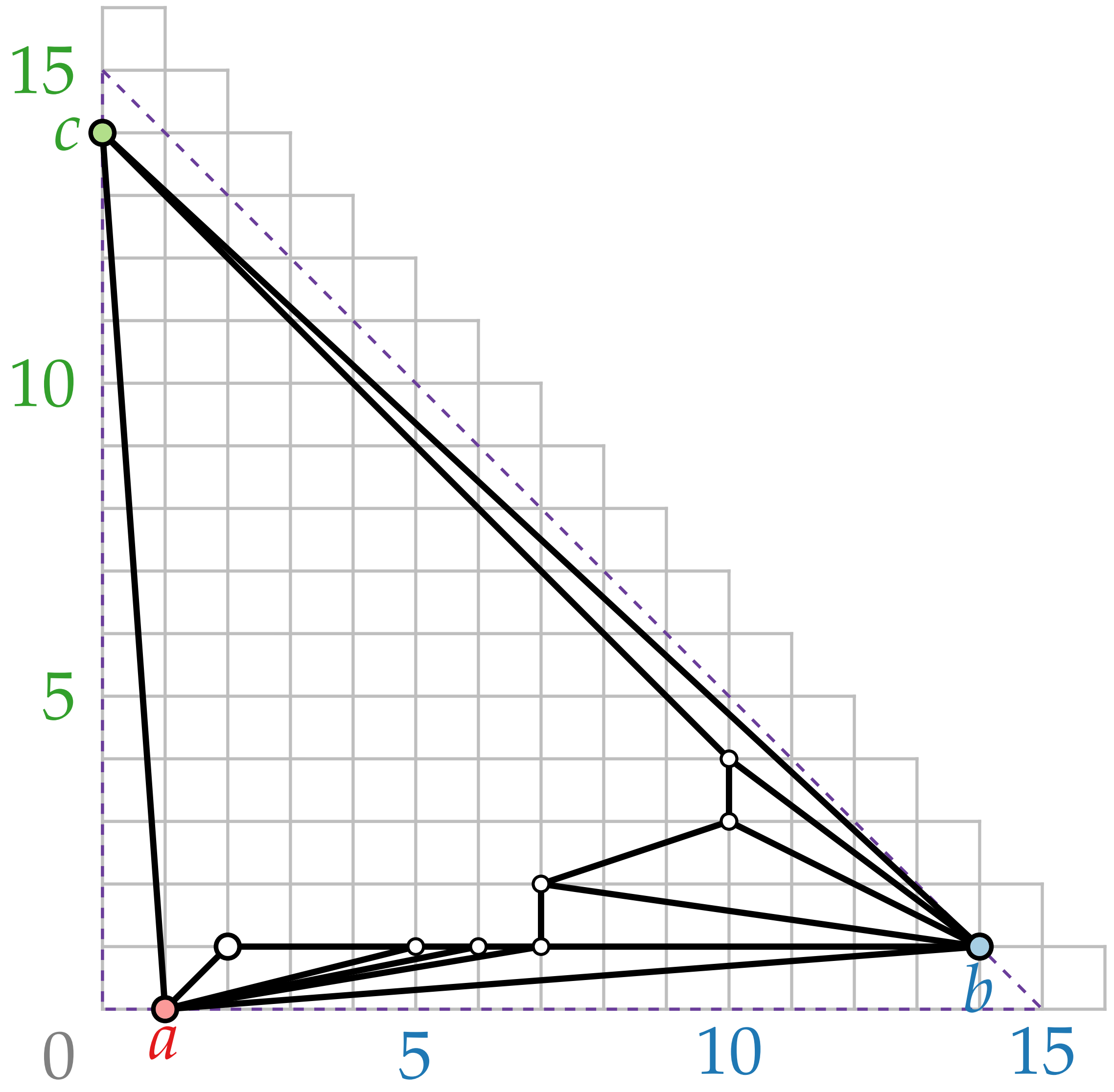

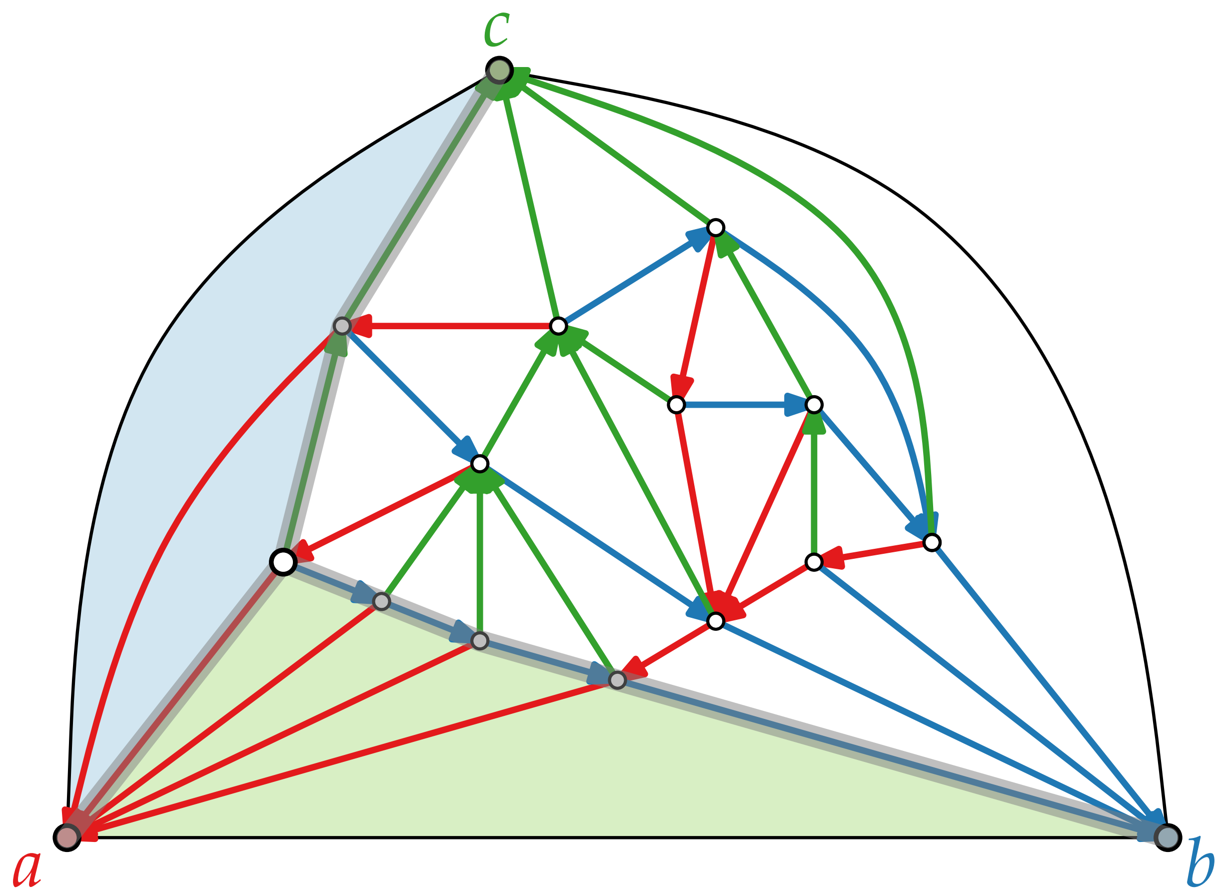

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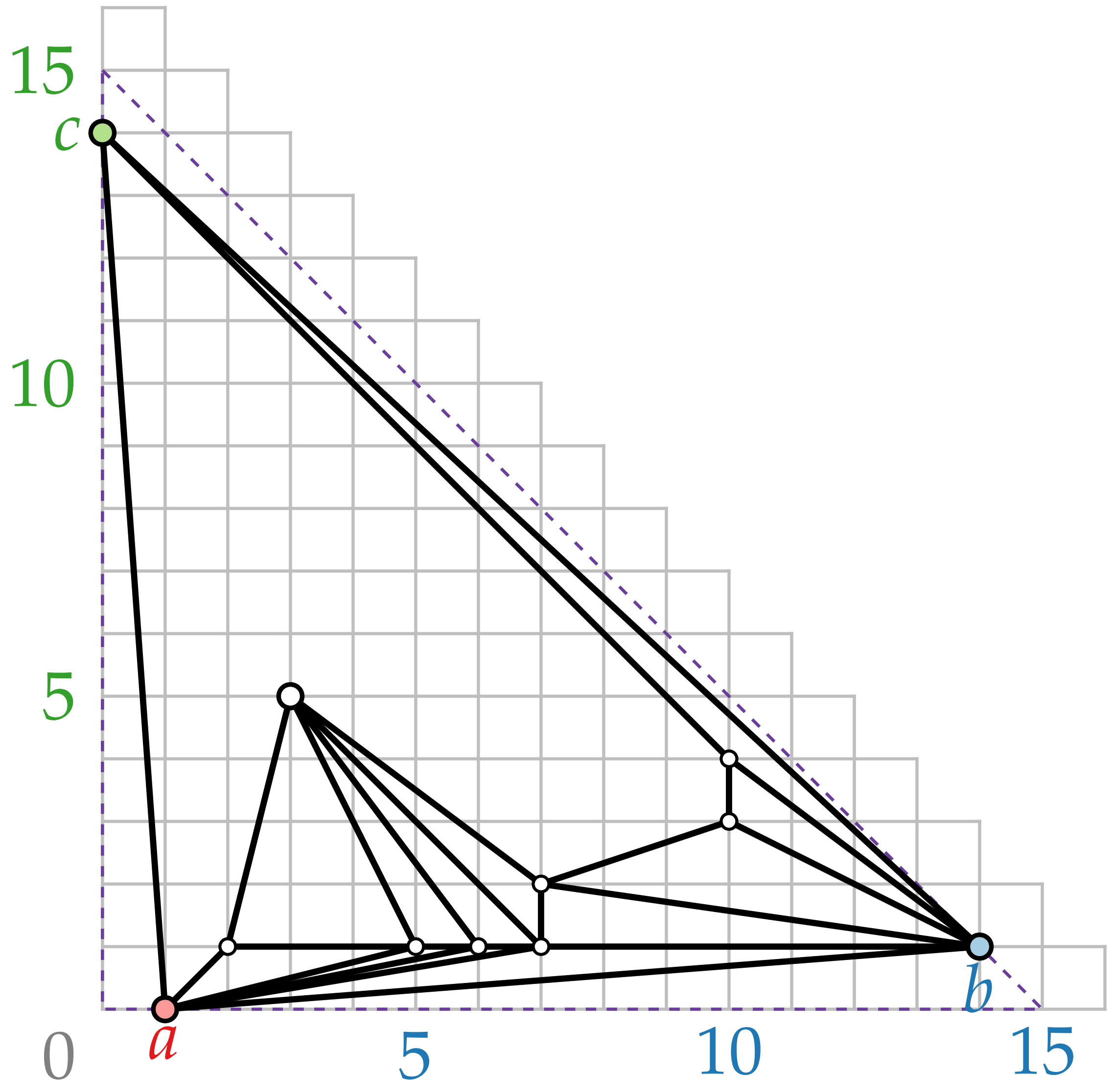

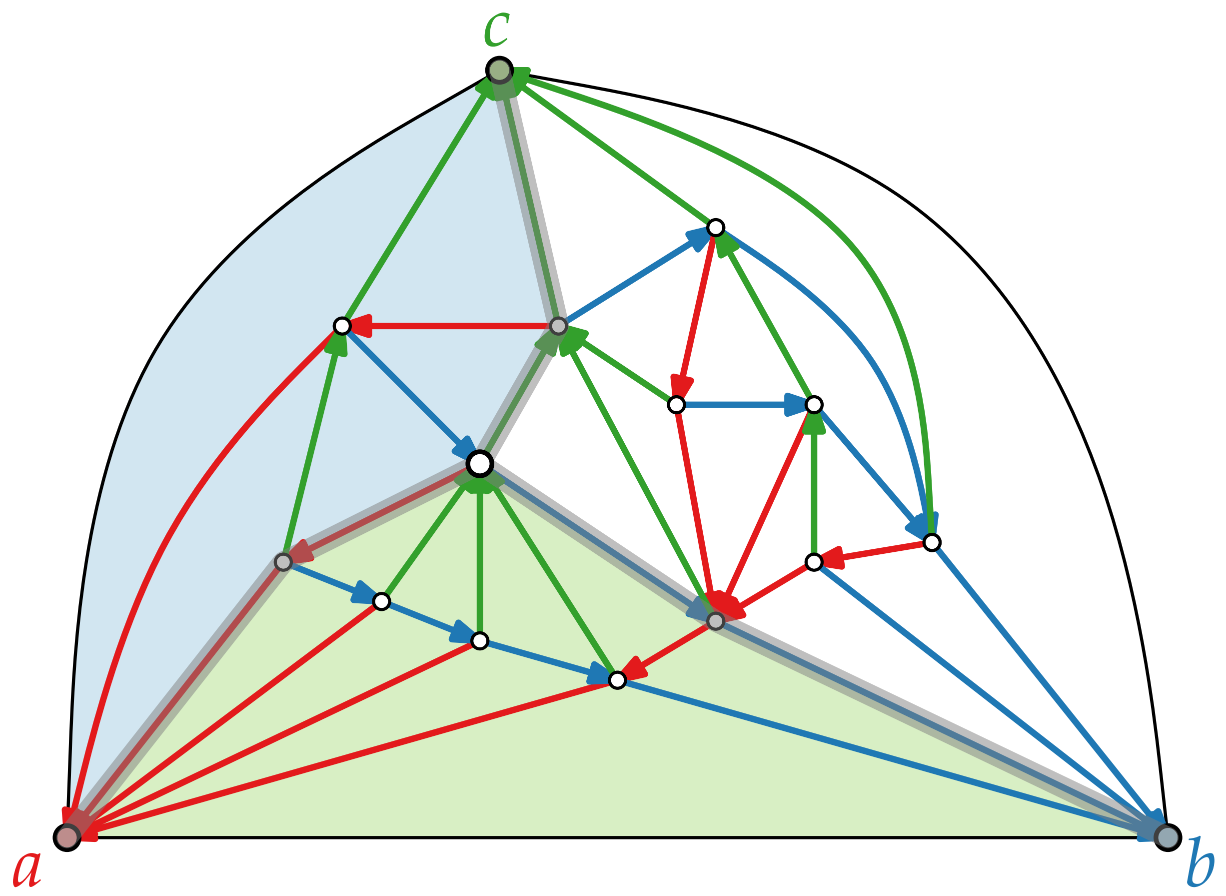

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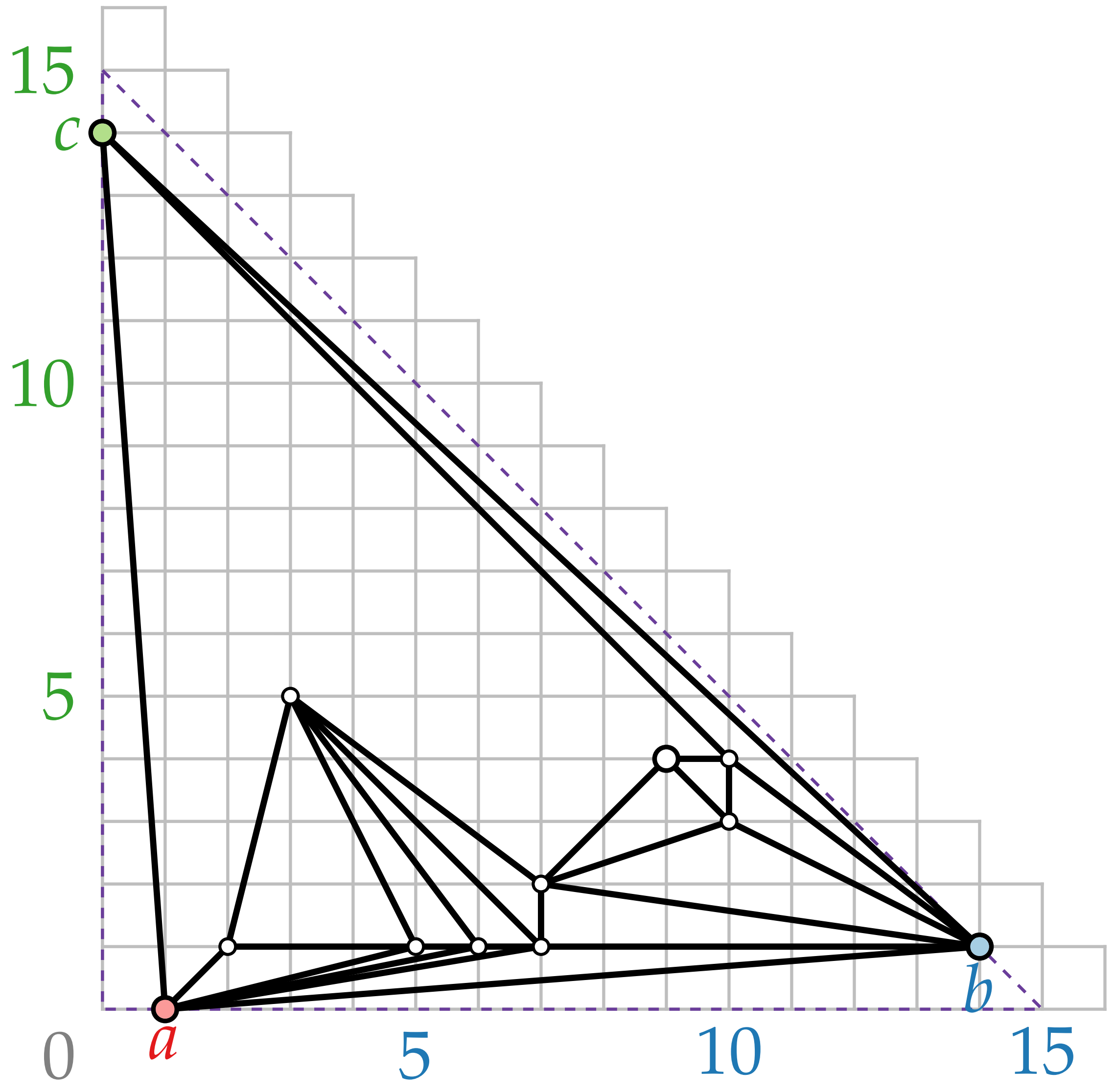

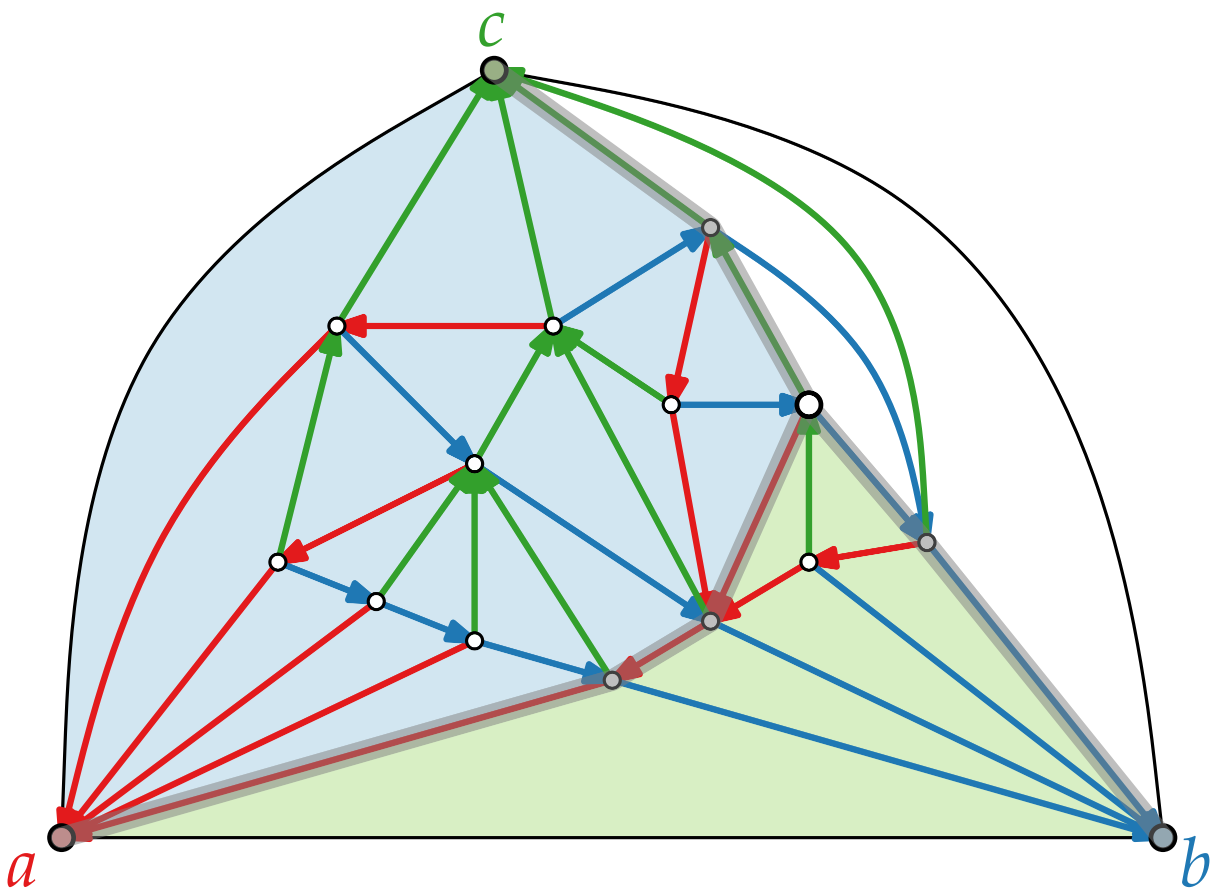

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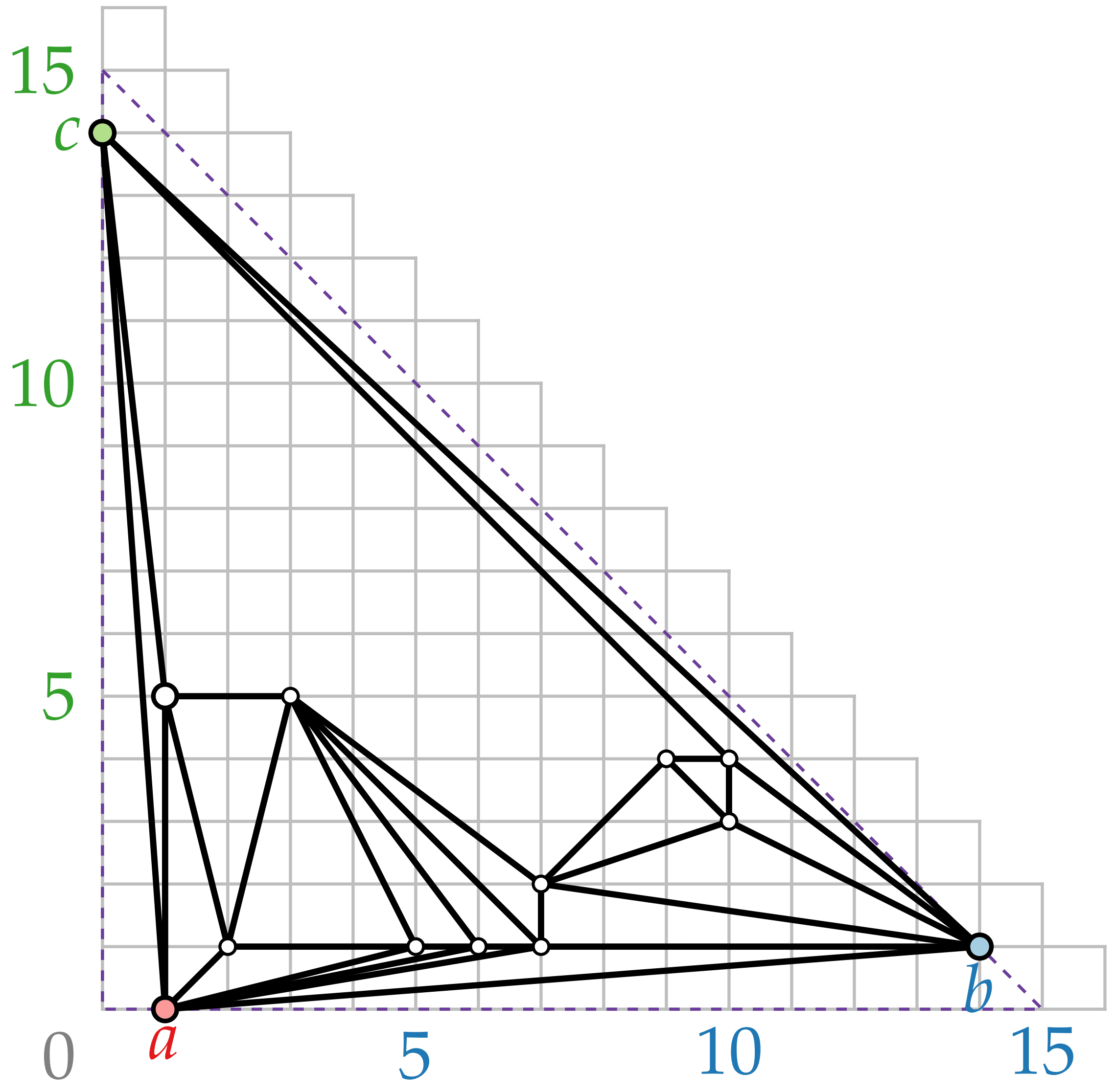

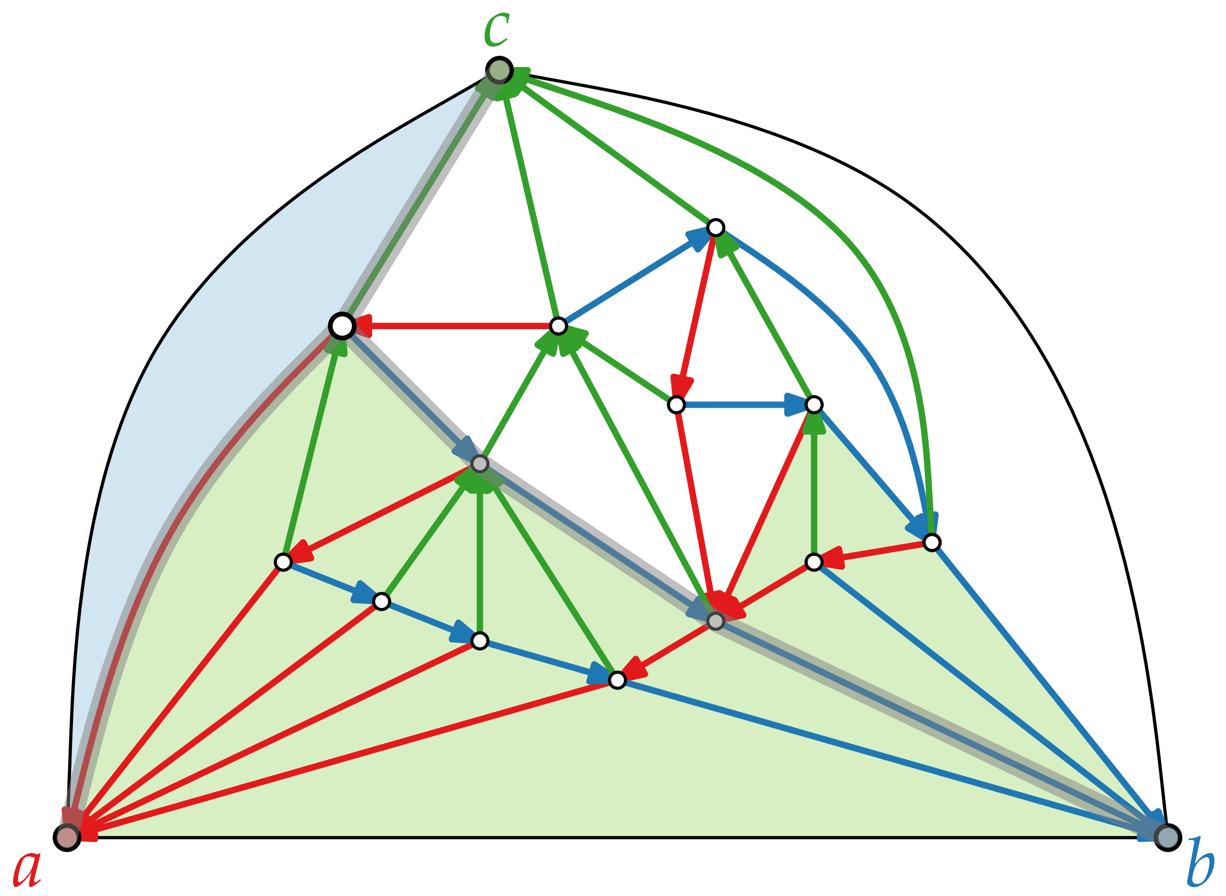

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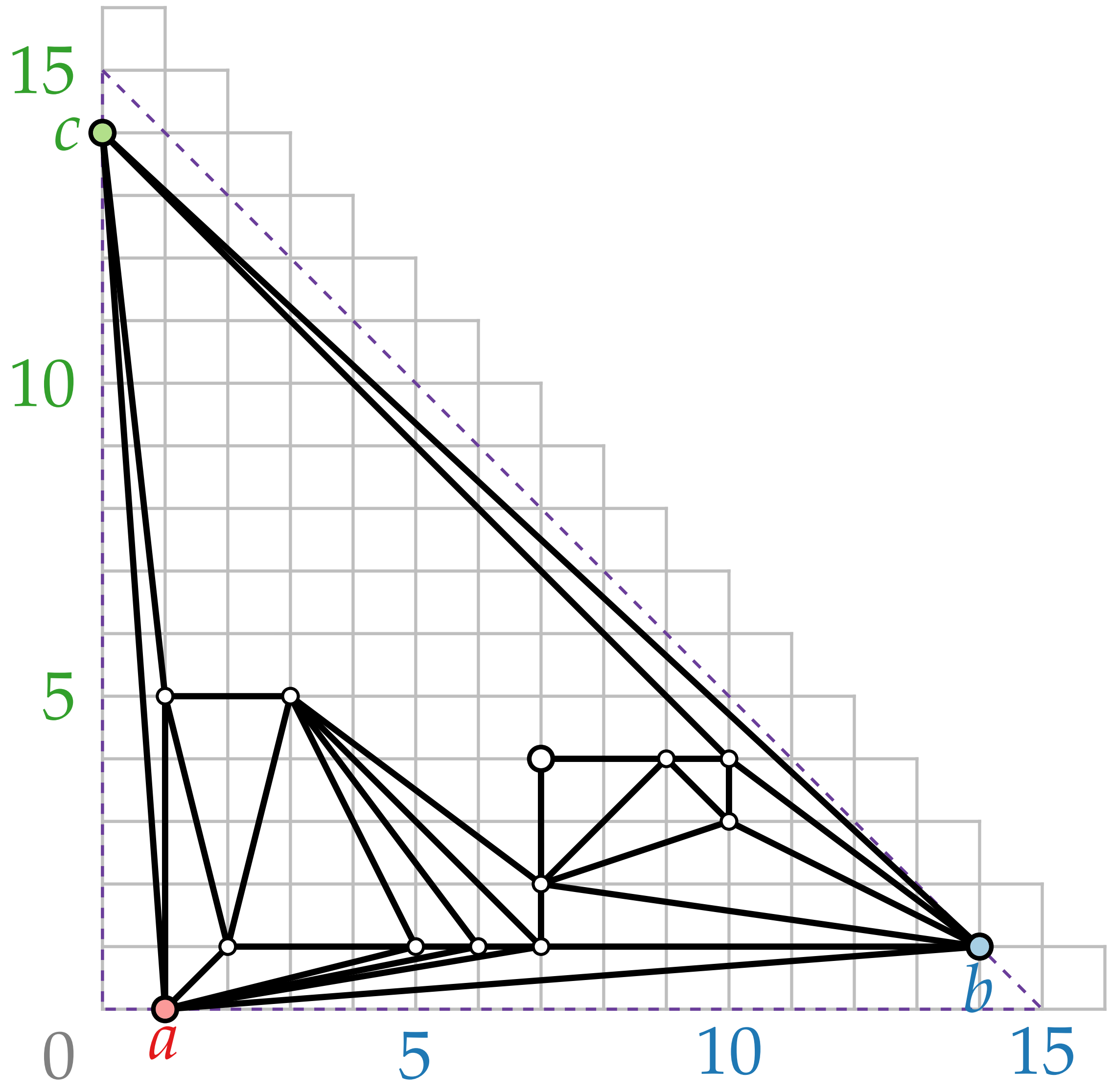

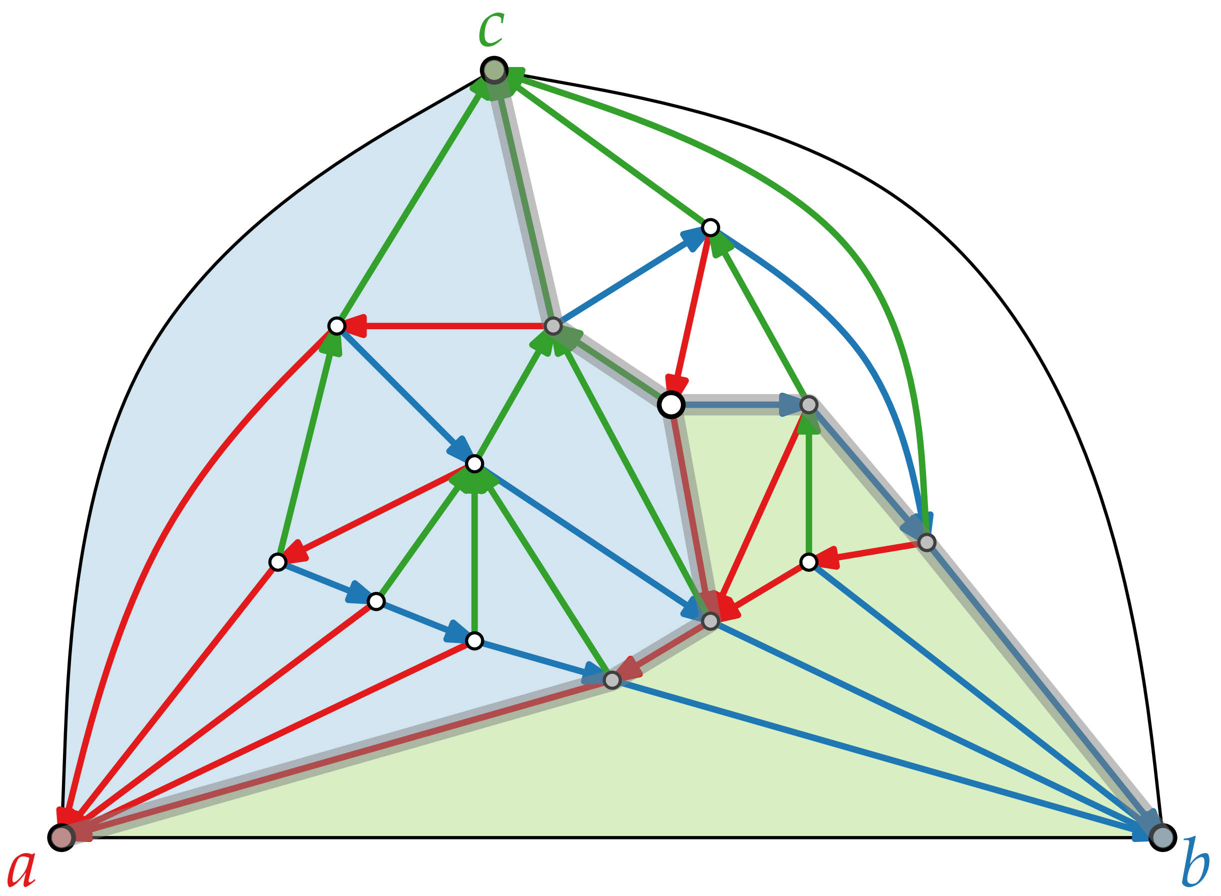

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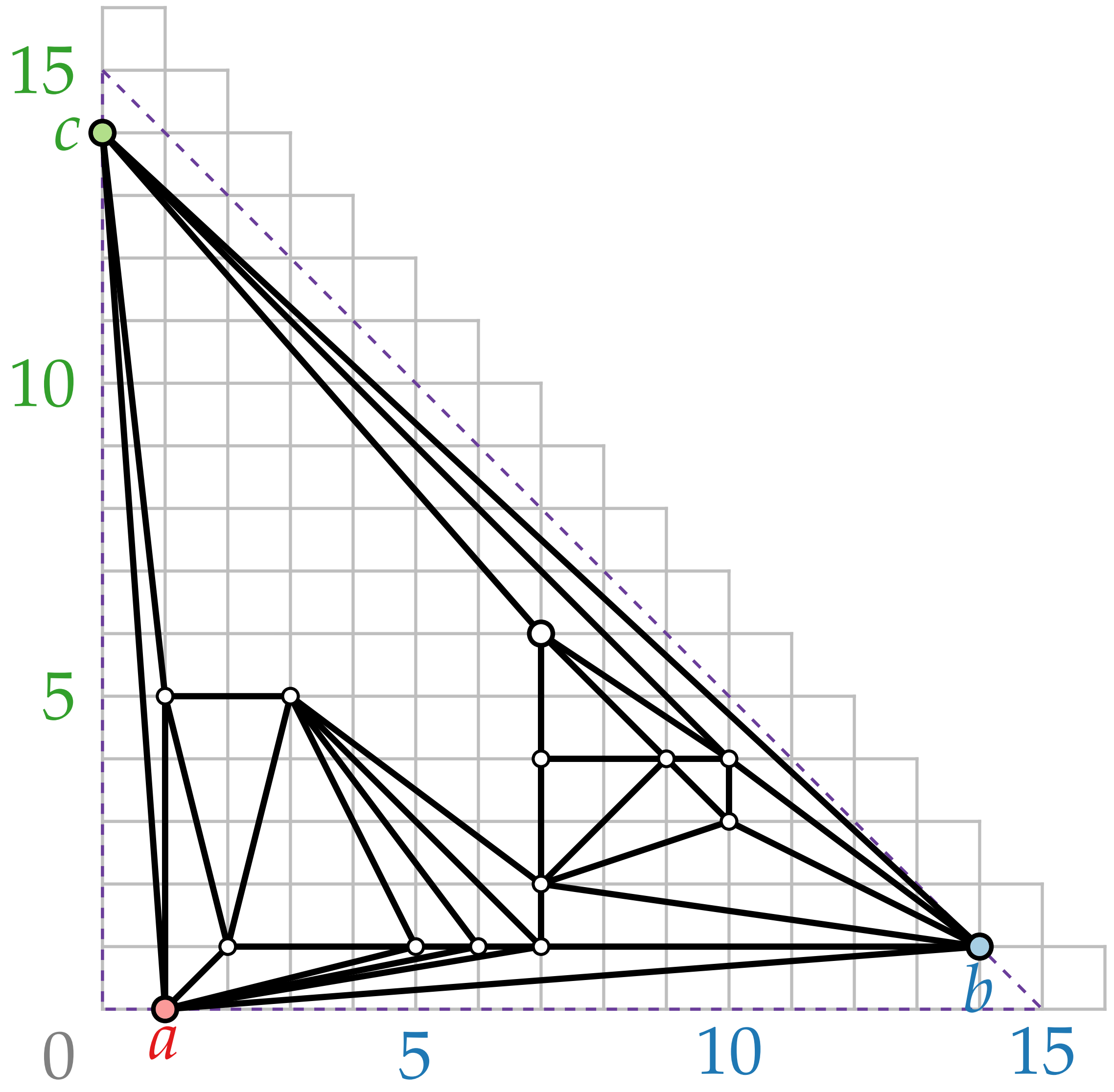

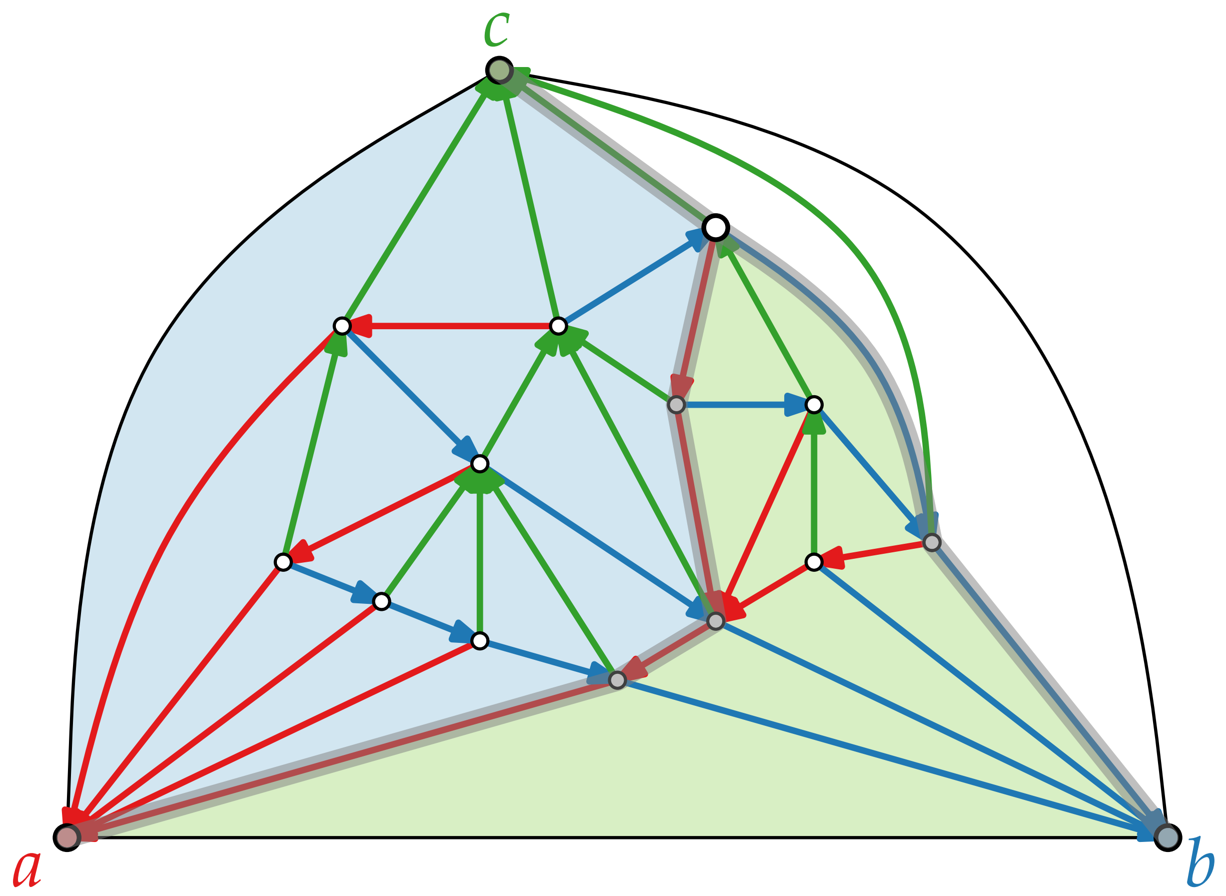

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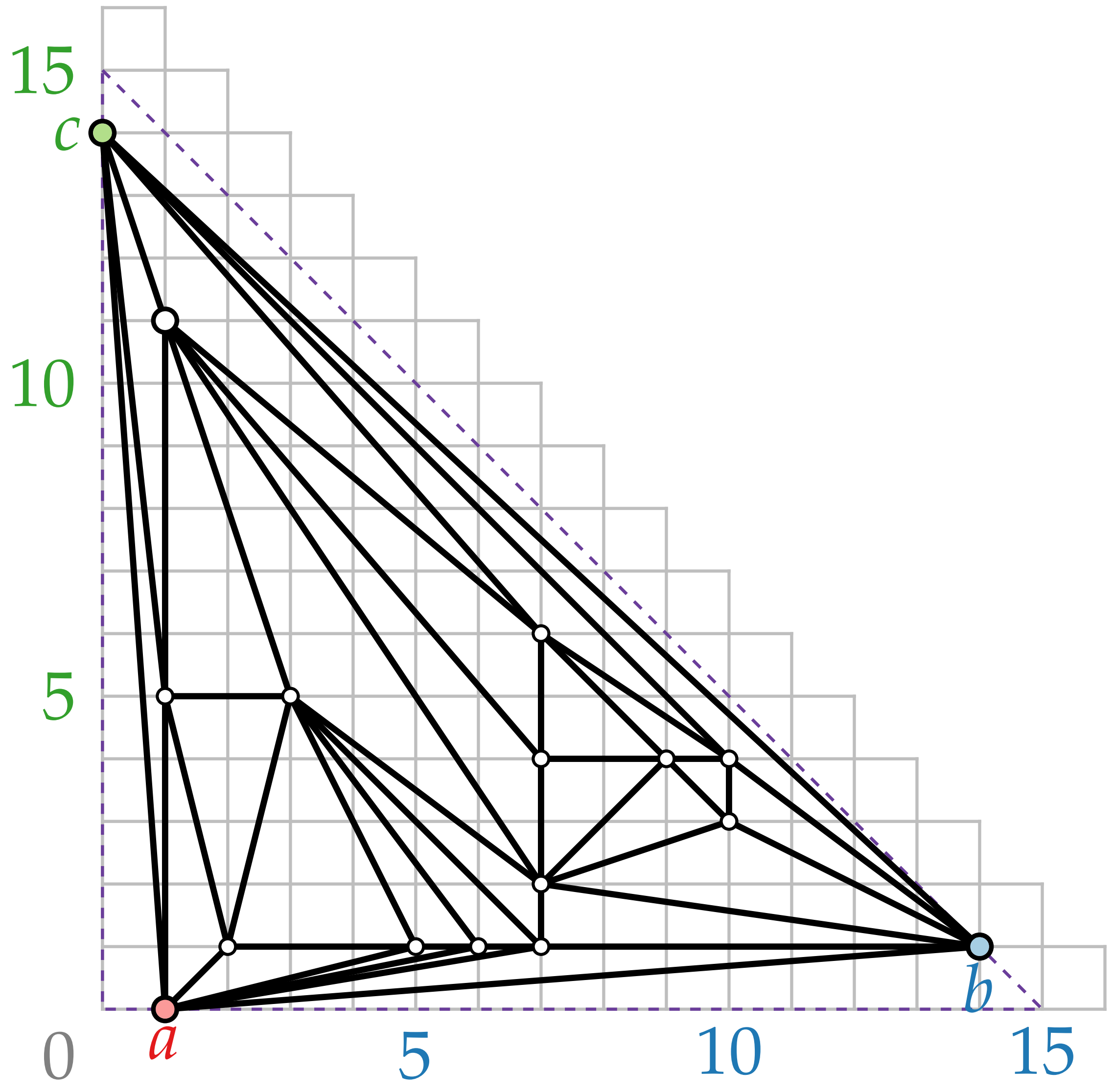

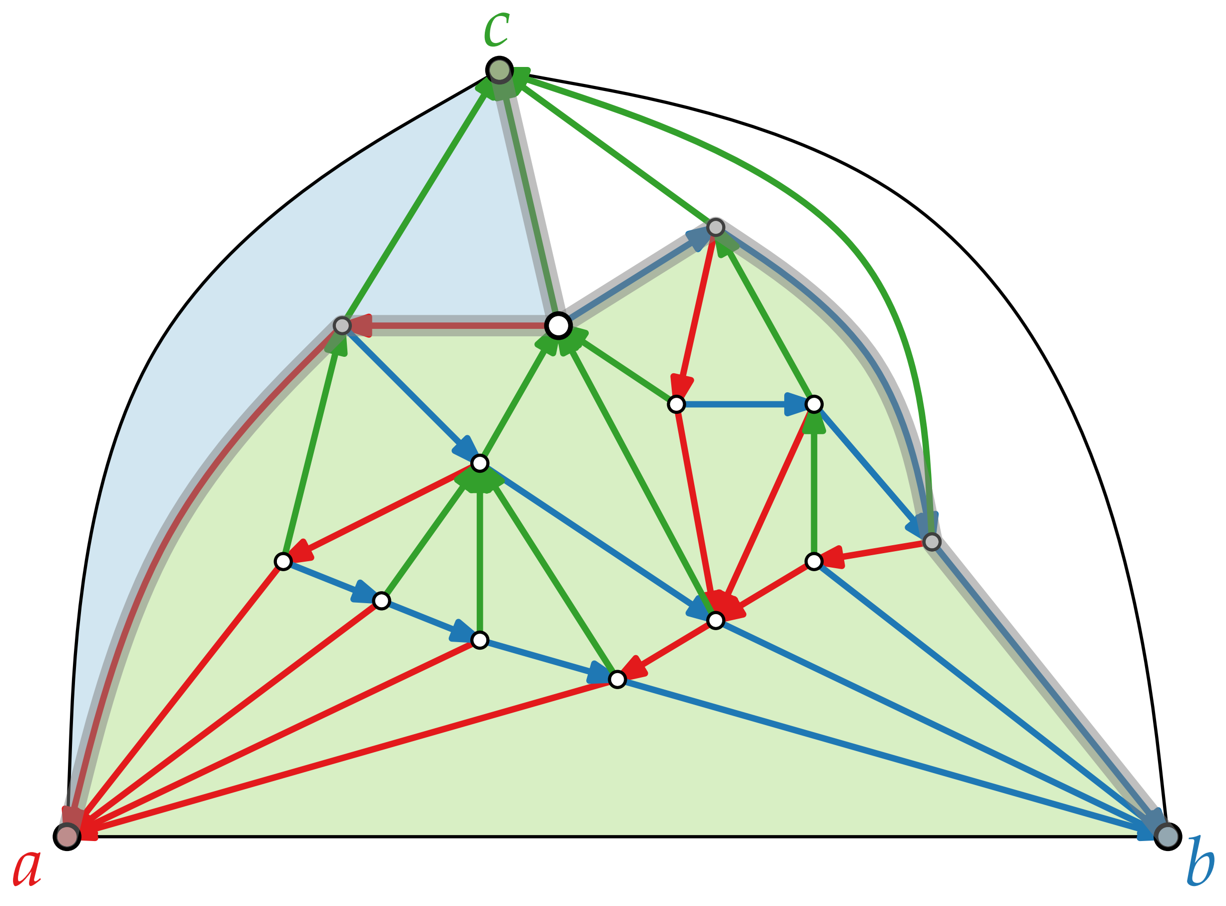

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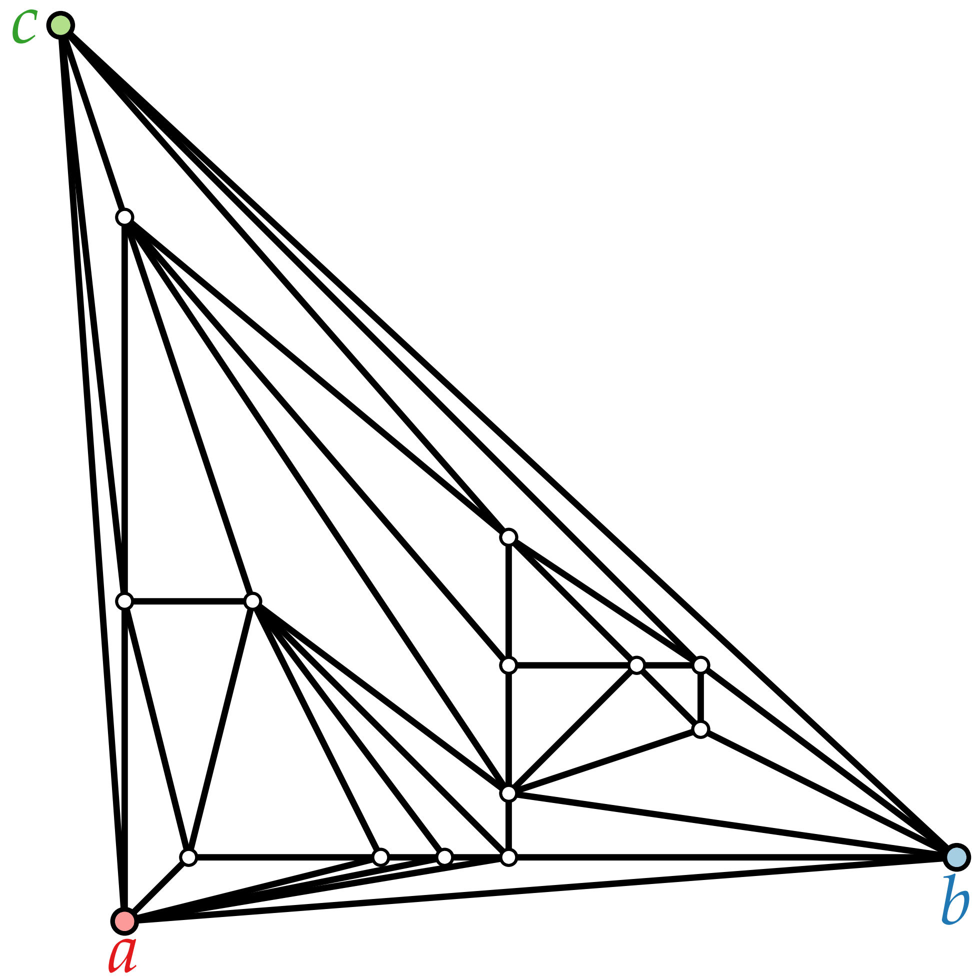

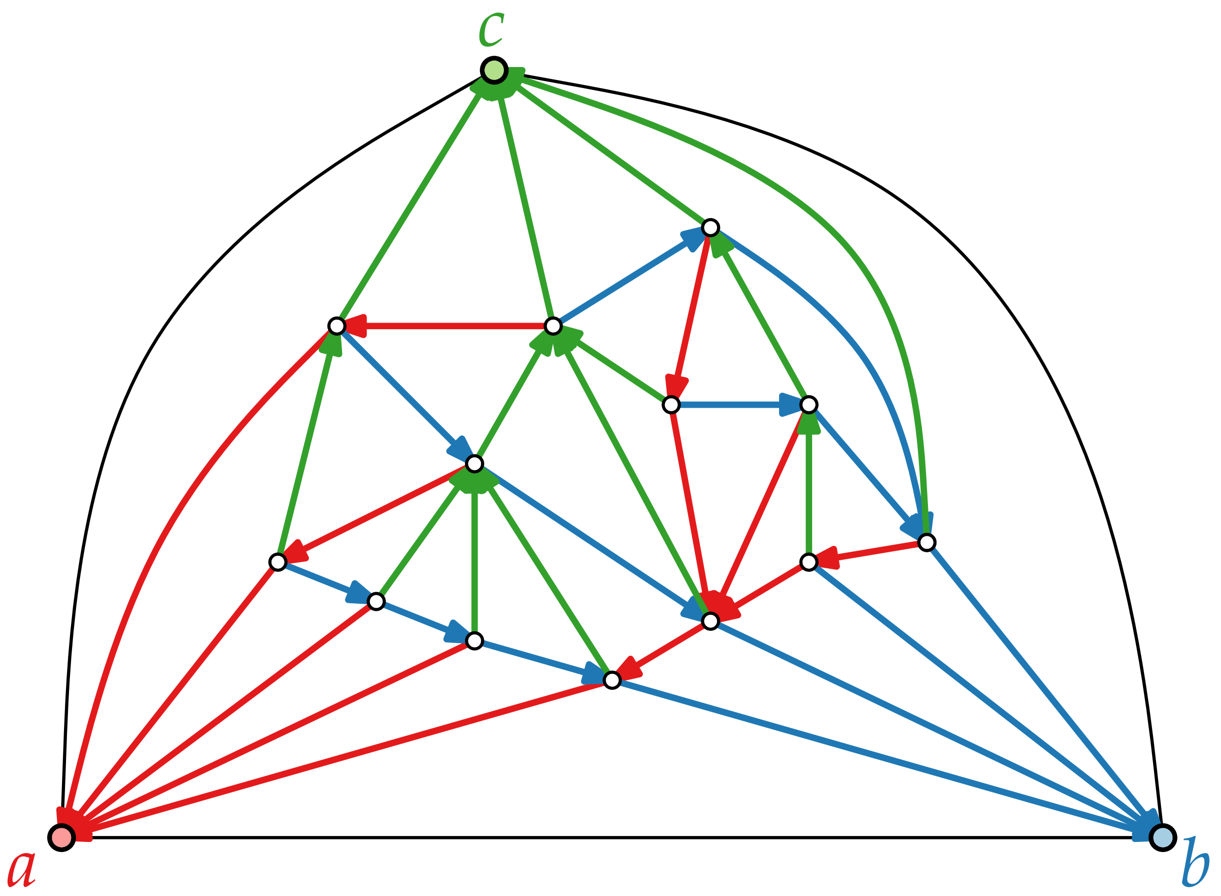

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Results & Variations

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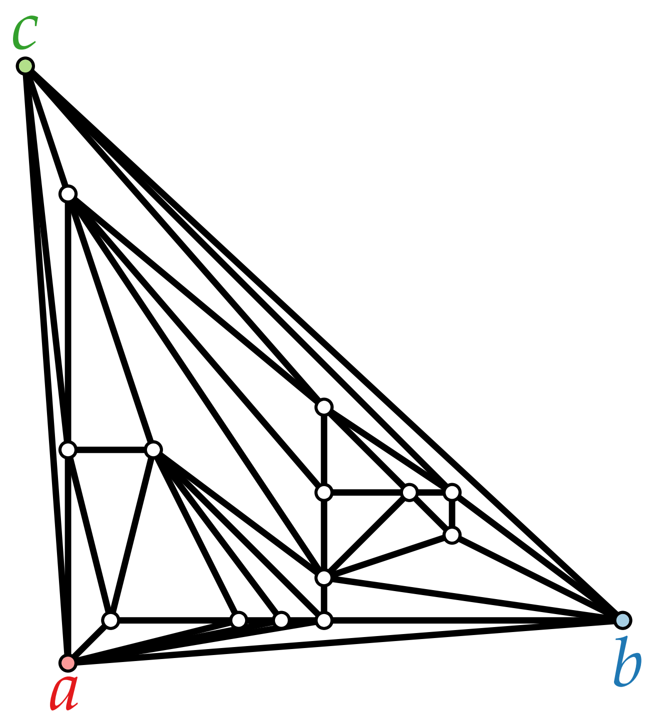

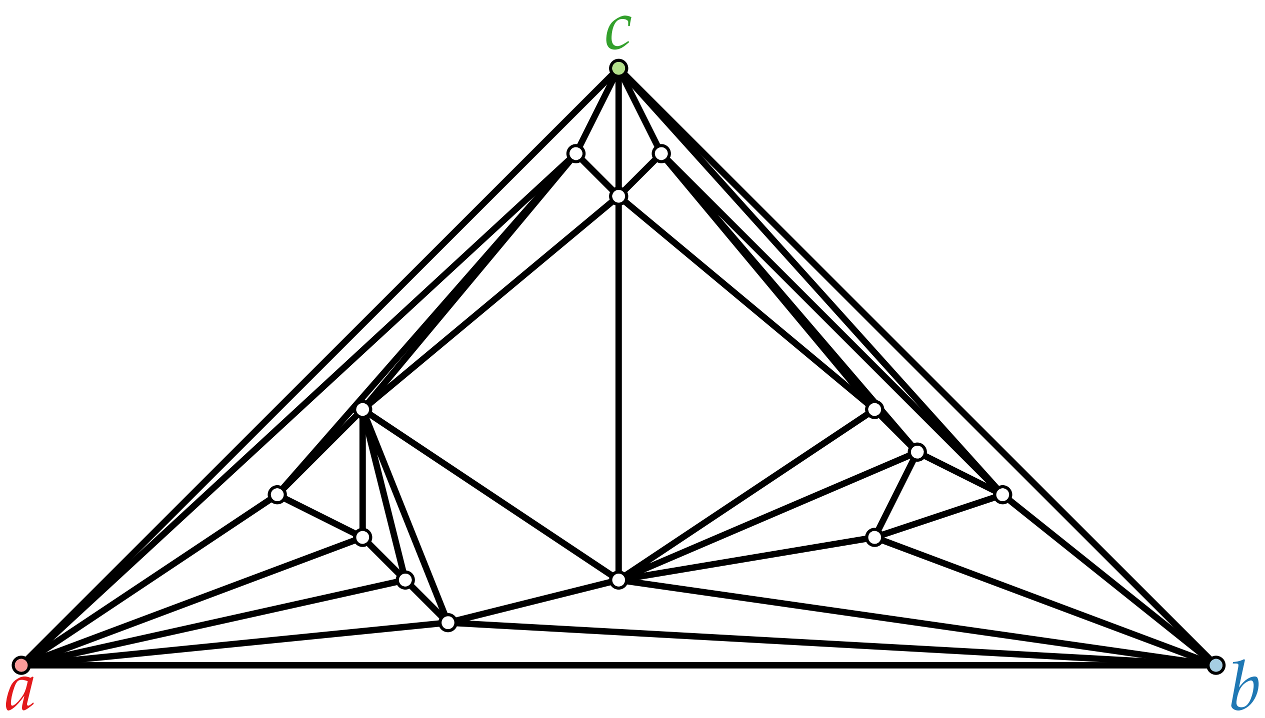

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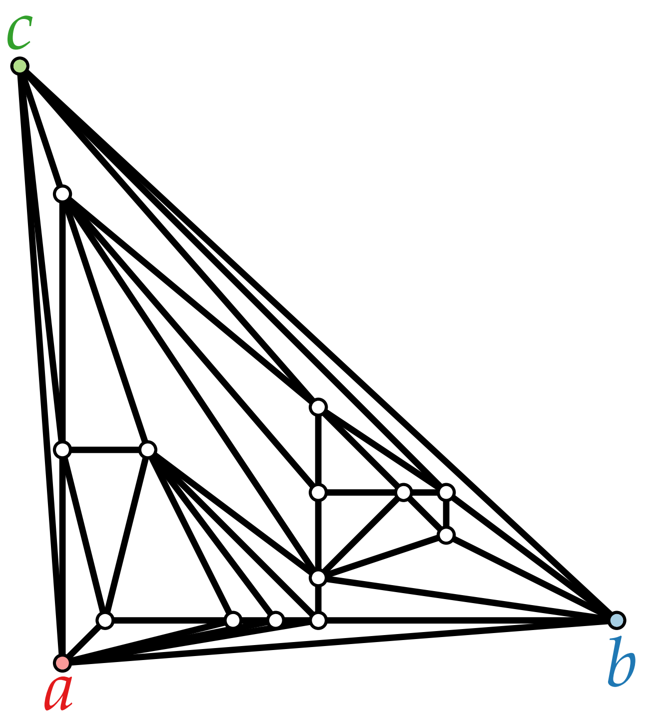

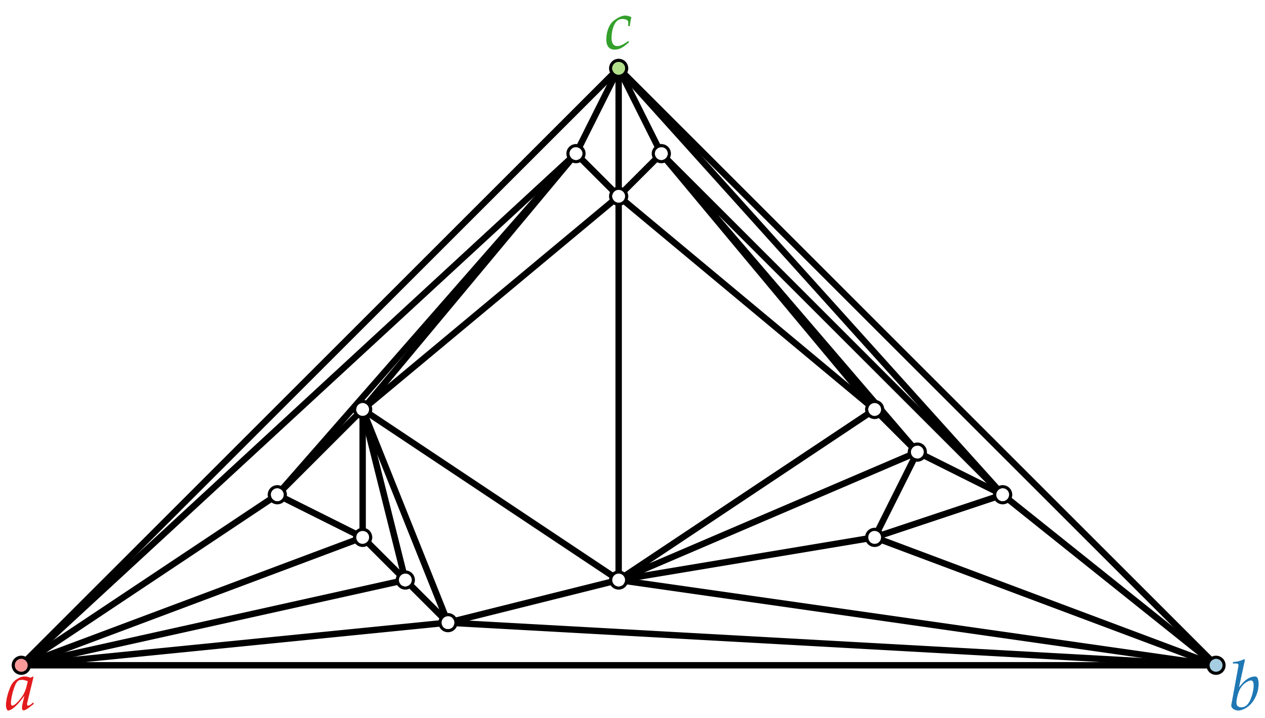

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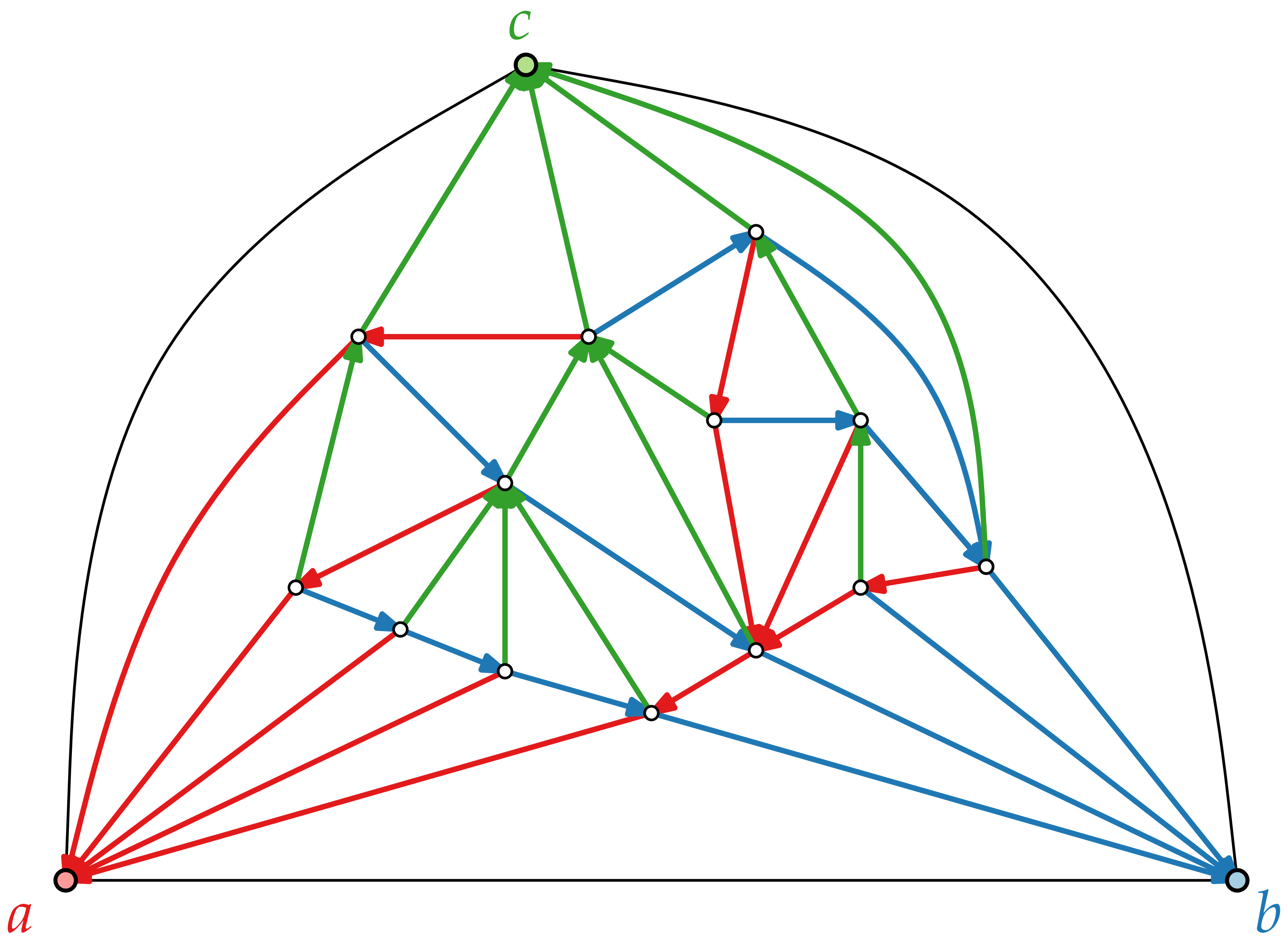
Schnyder Realizer $\rightarrow$ Canonical Order



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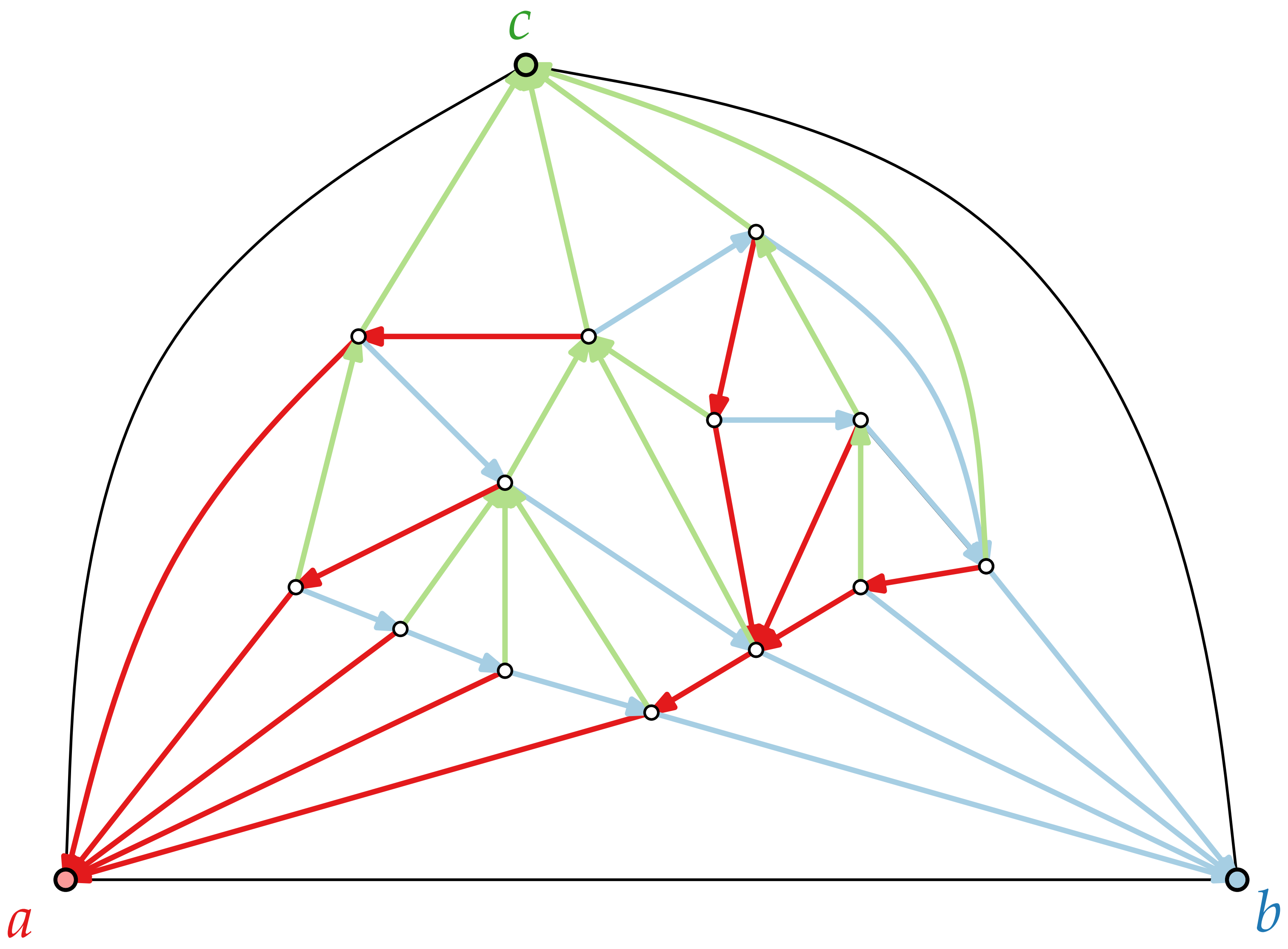

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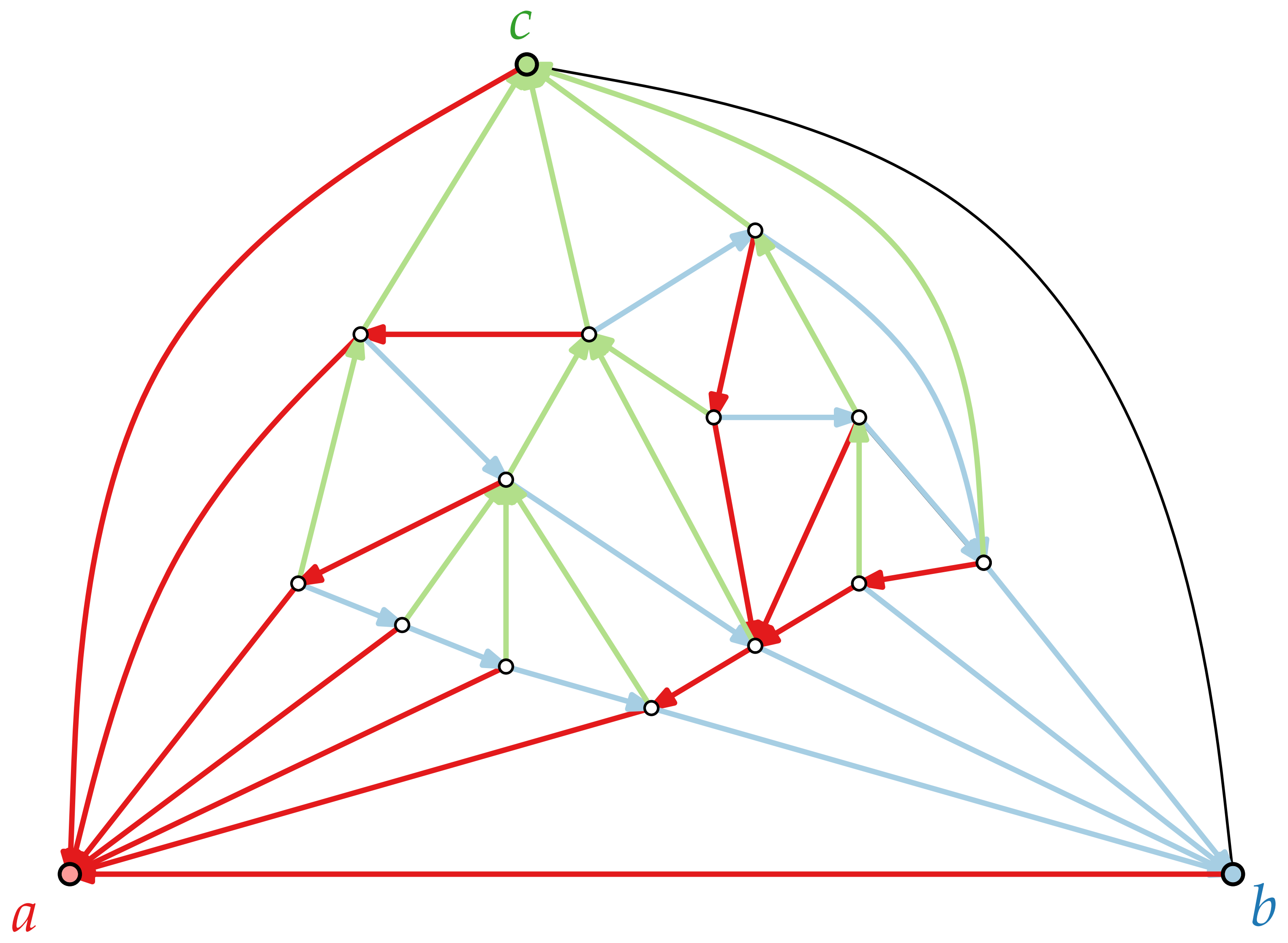

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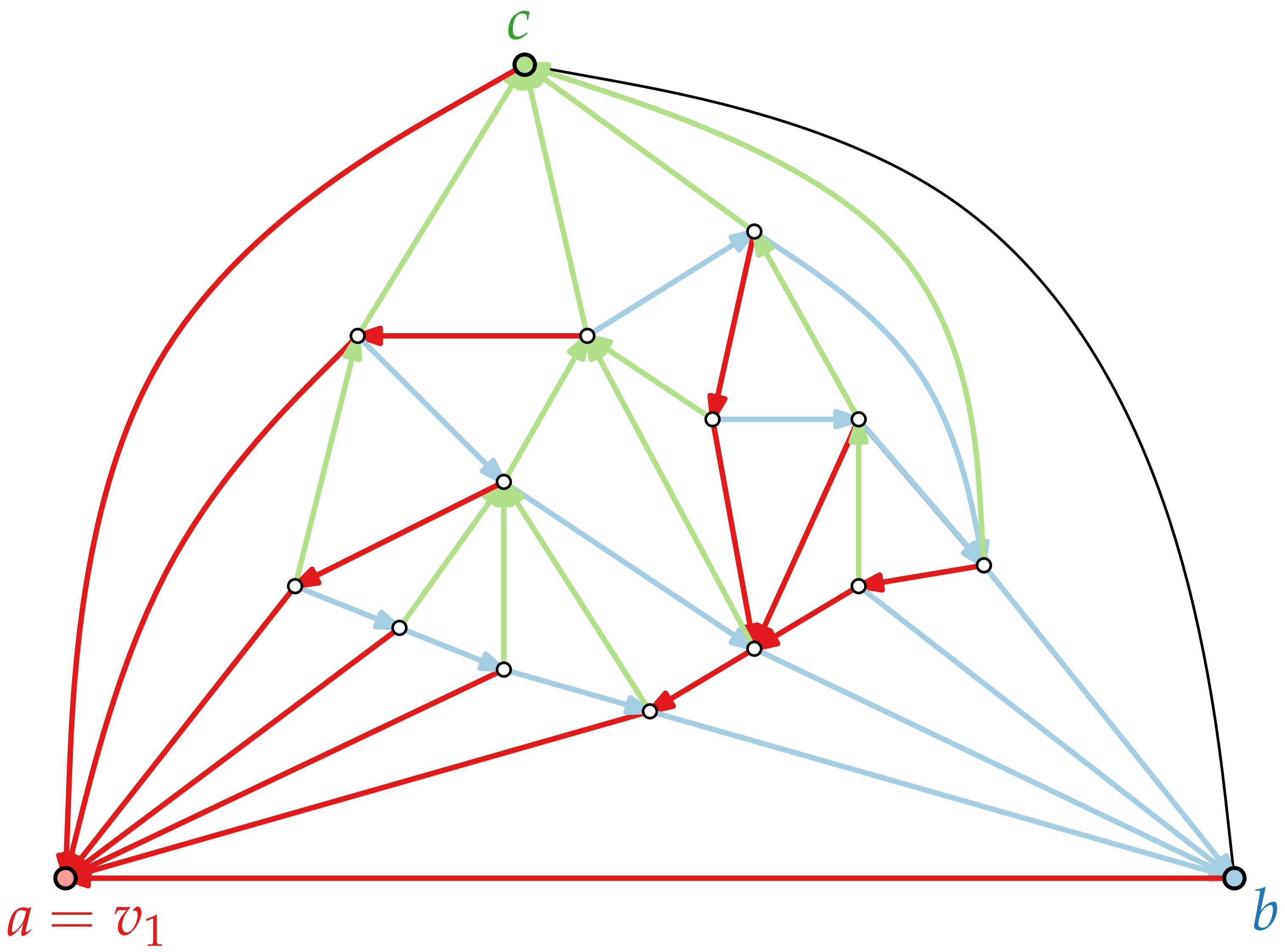

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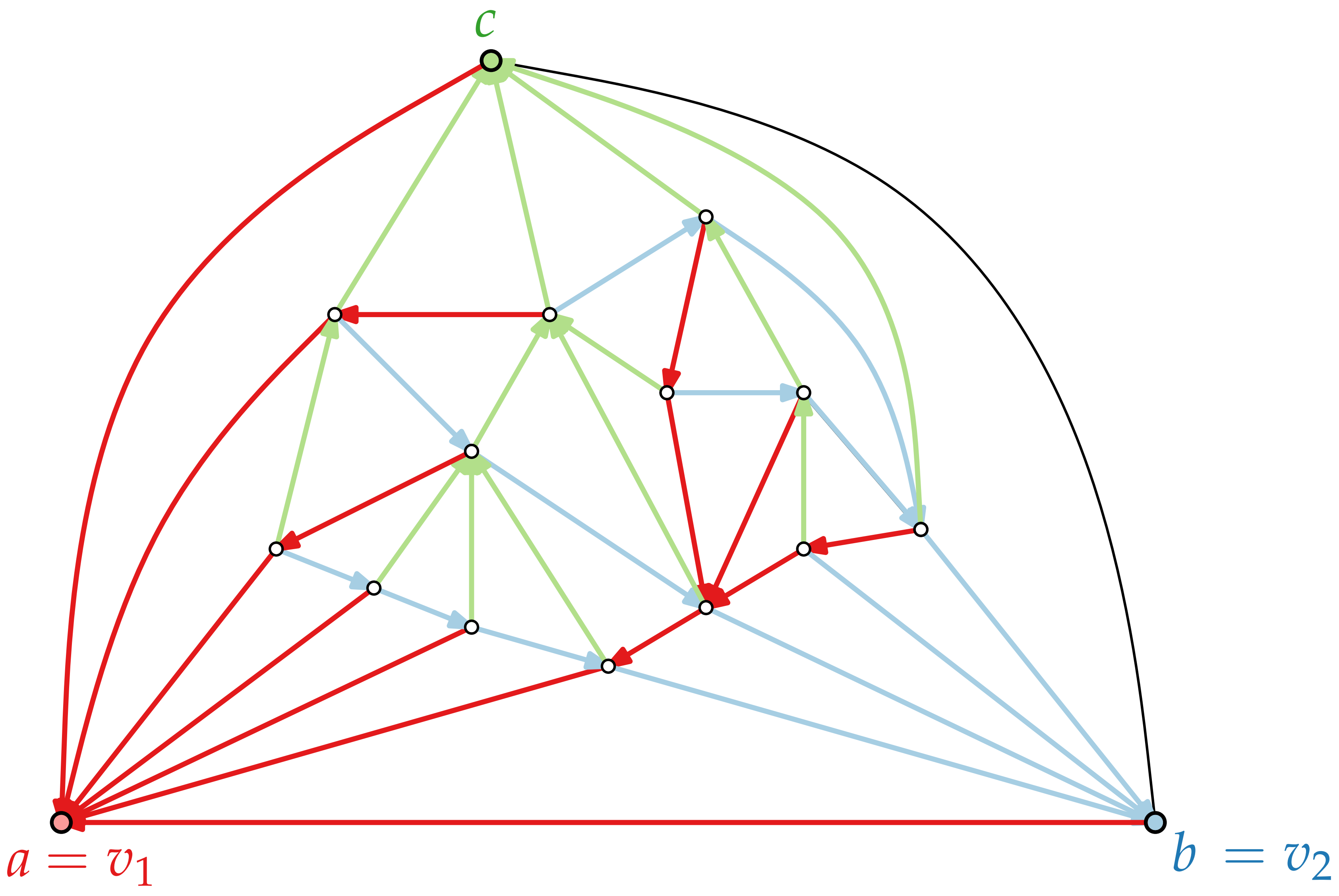

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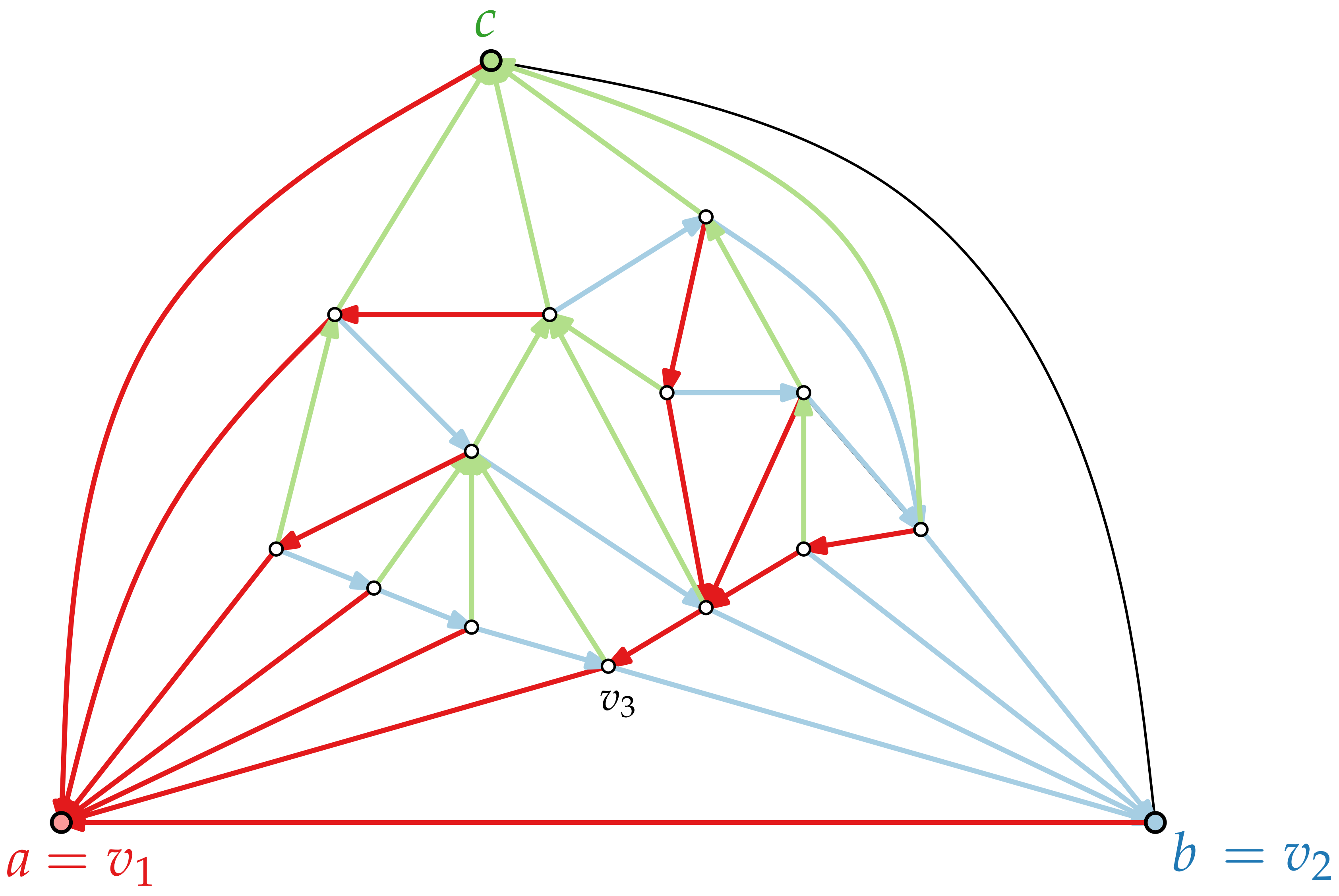

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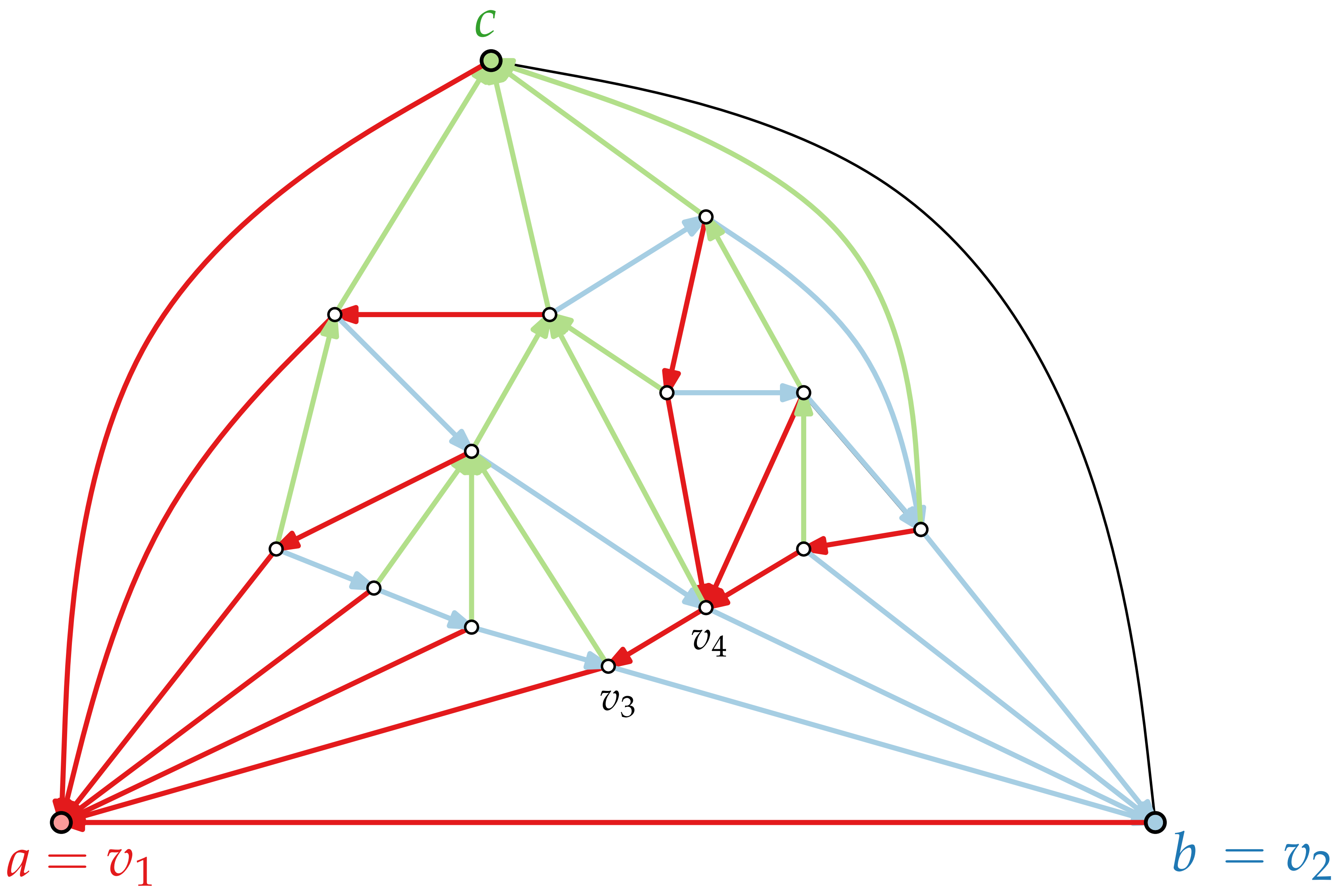

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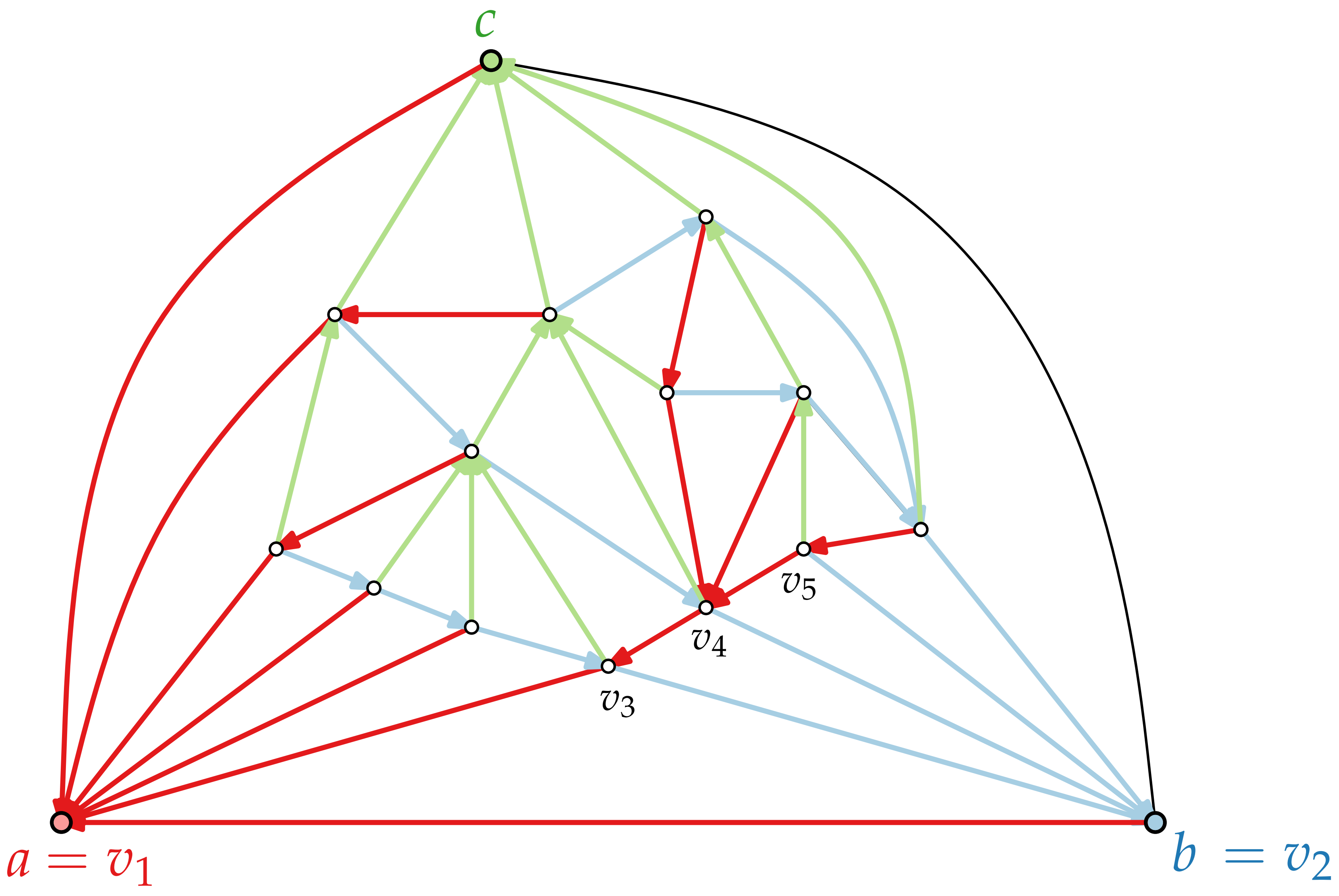

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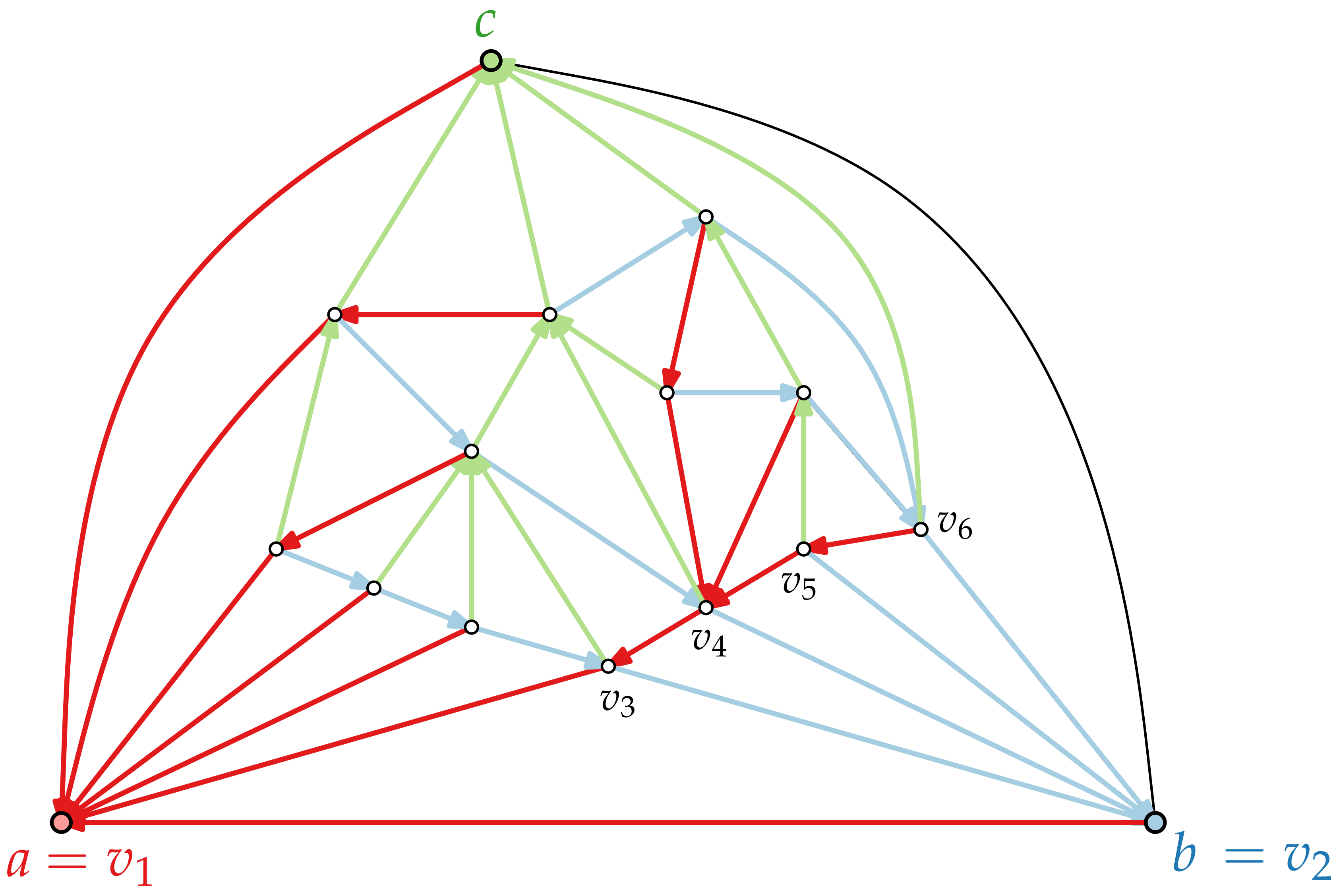

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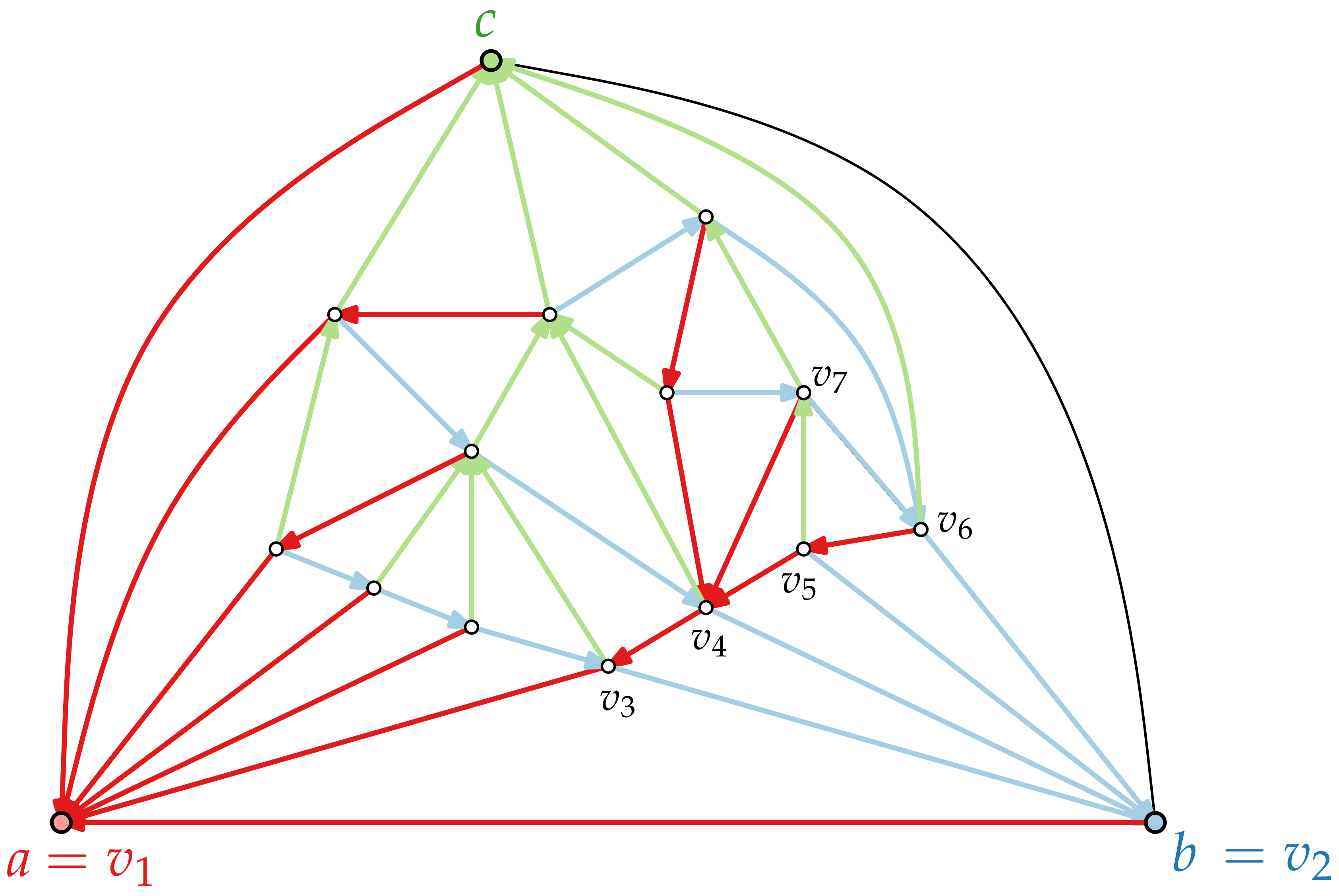

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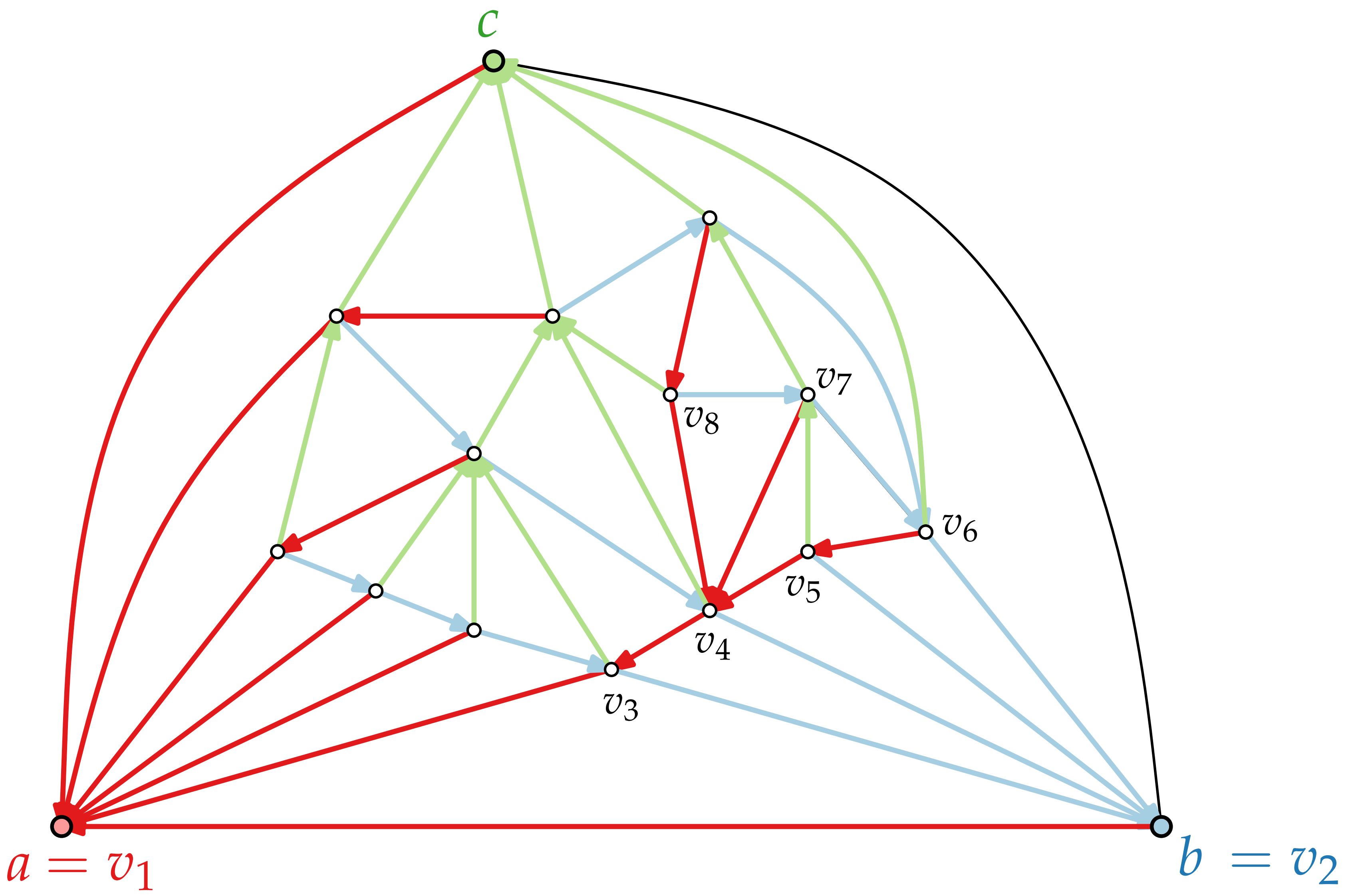

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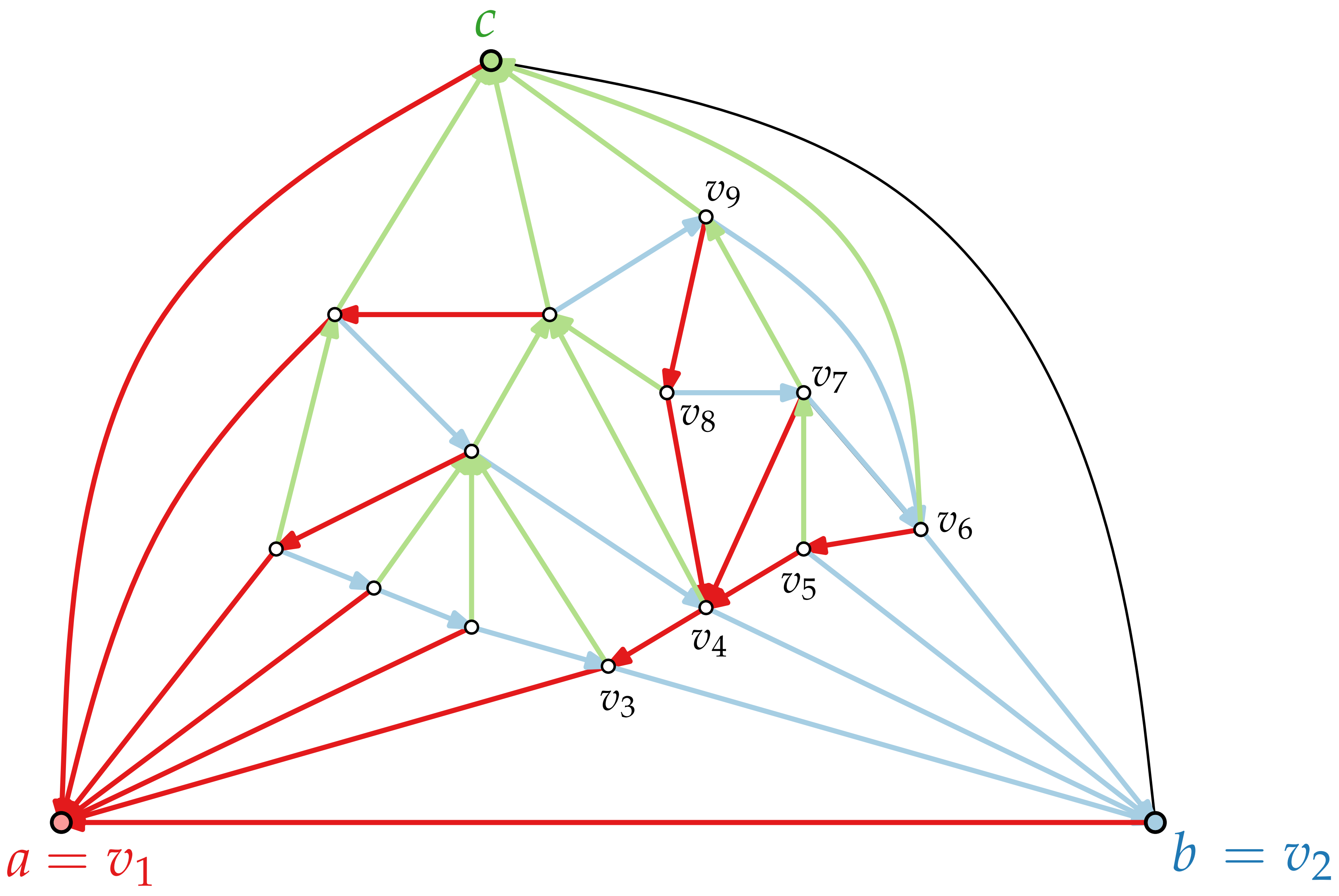

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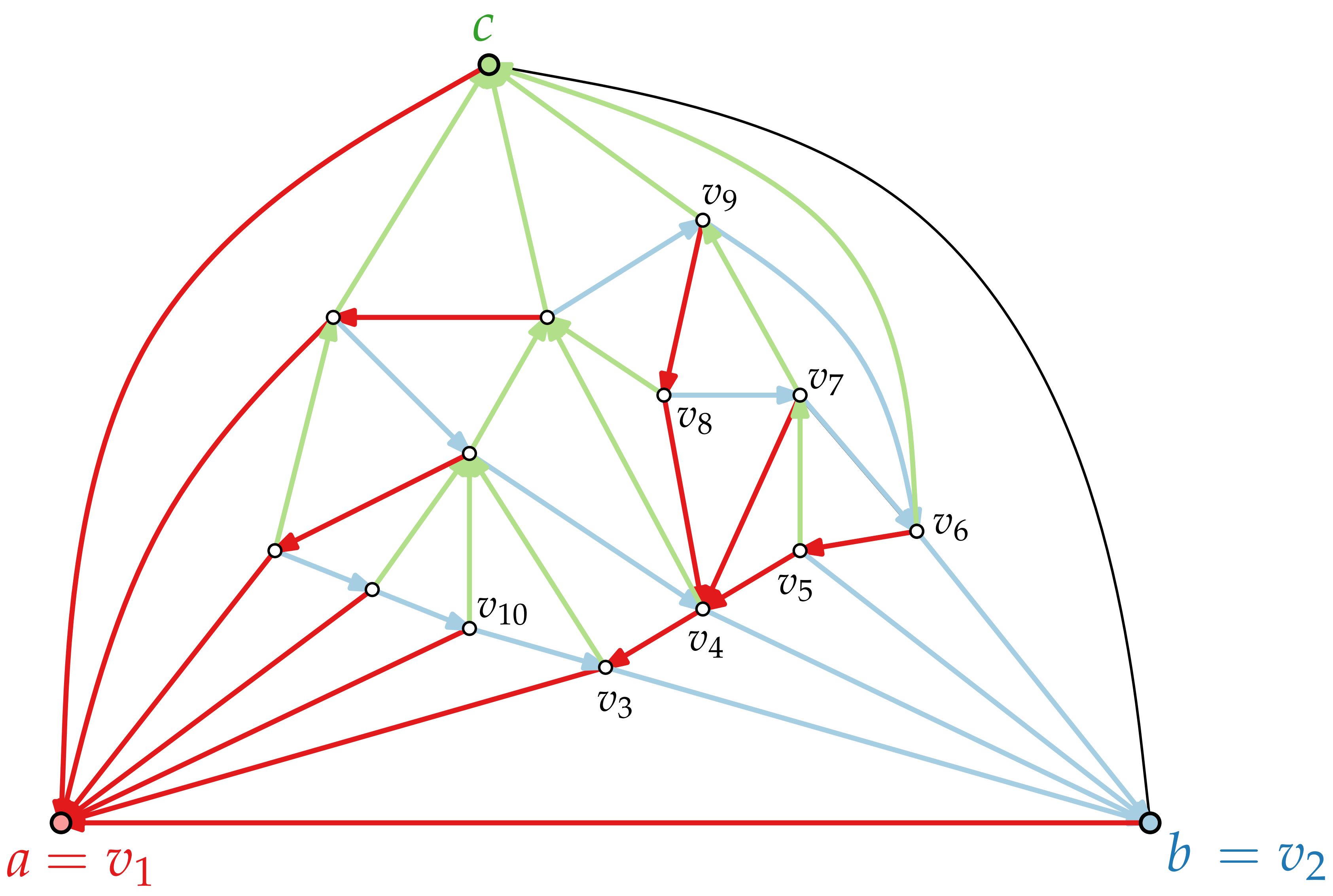

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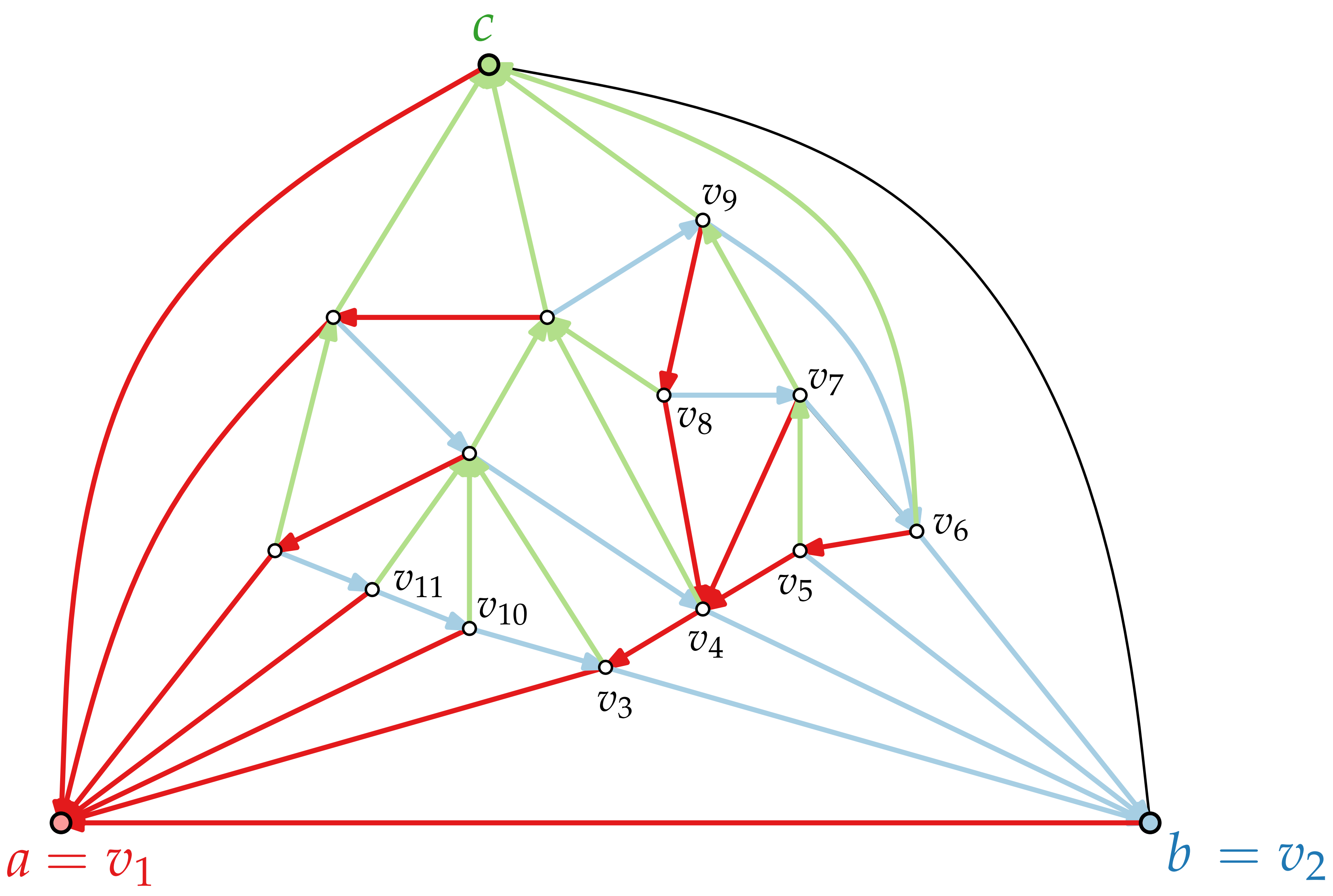

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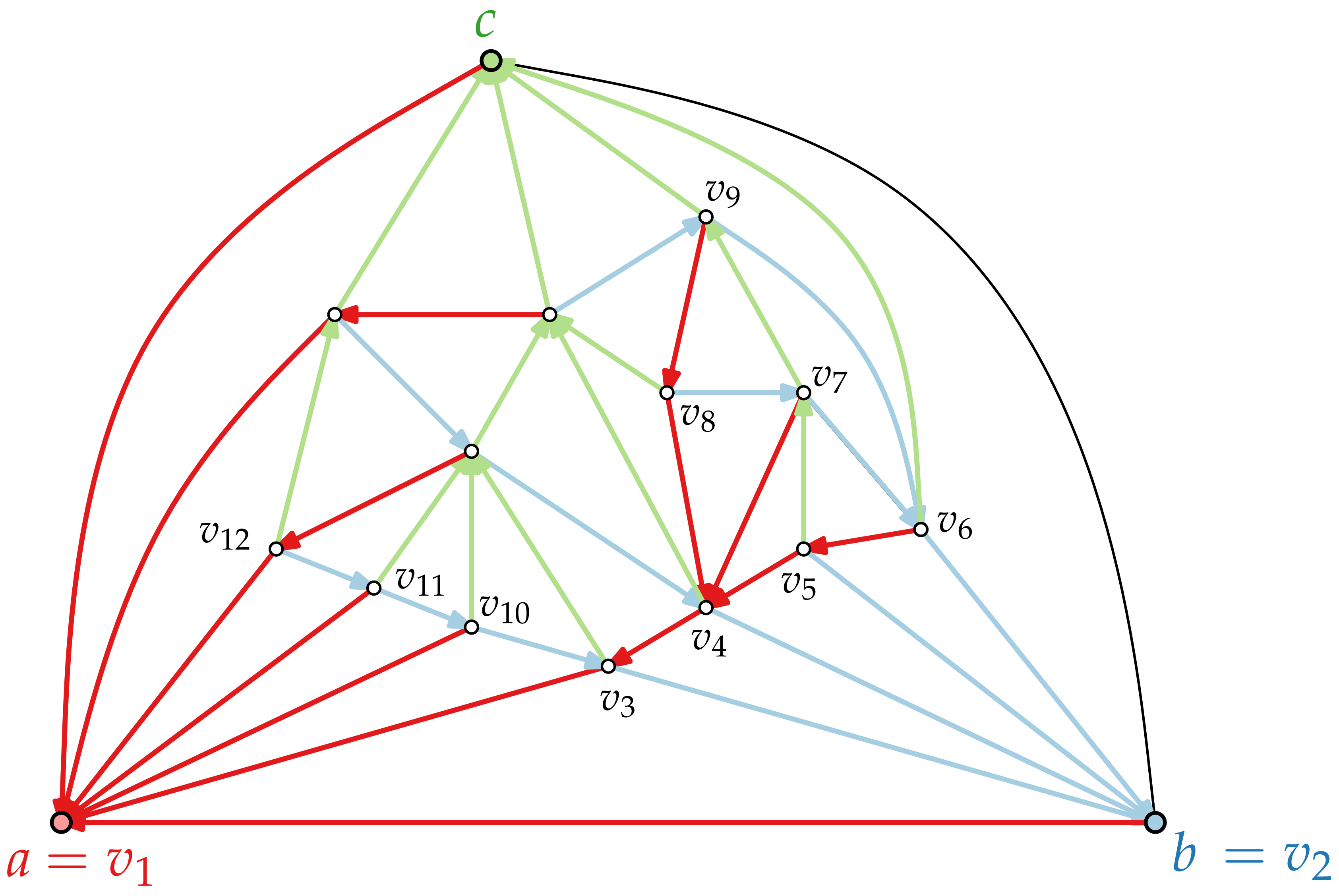

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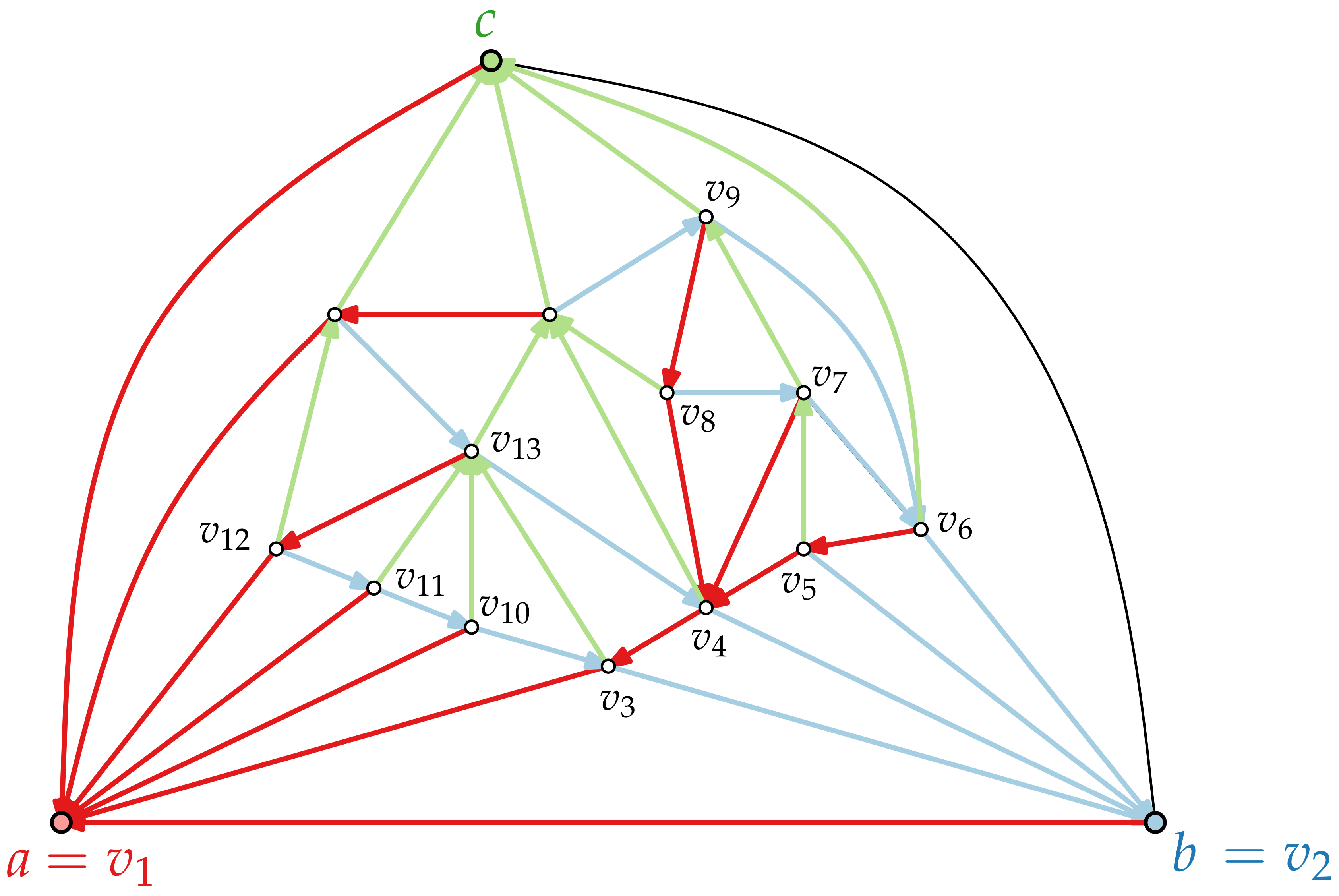

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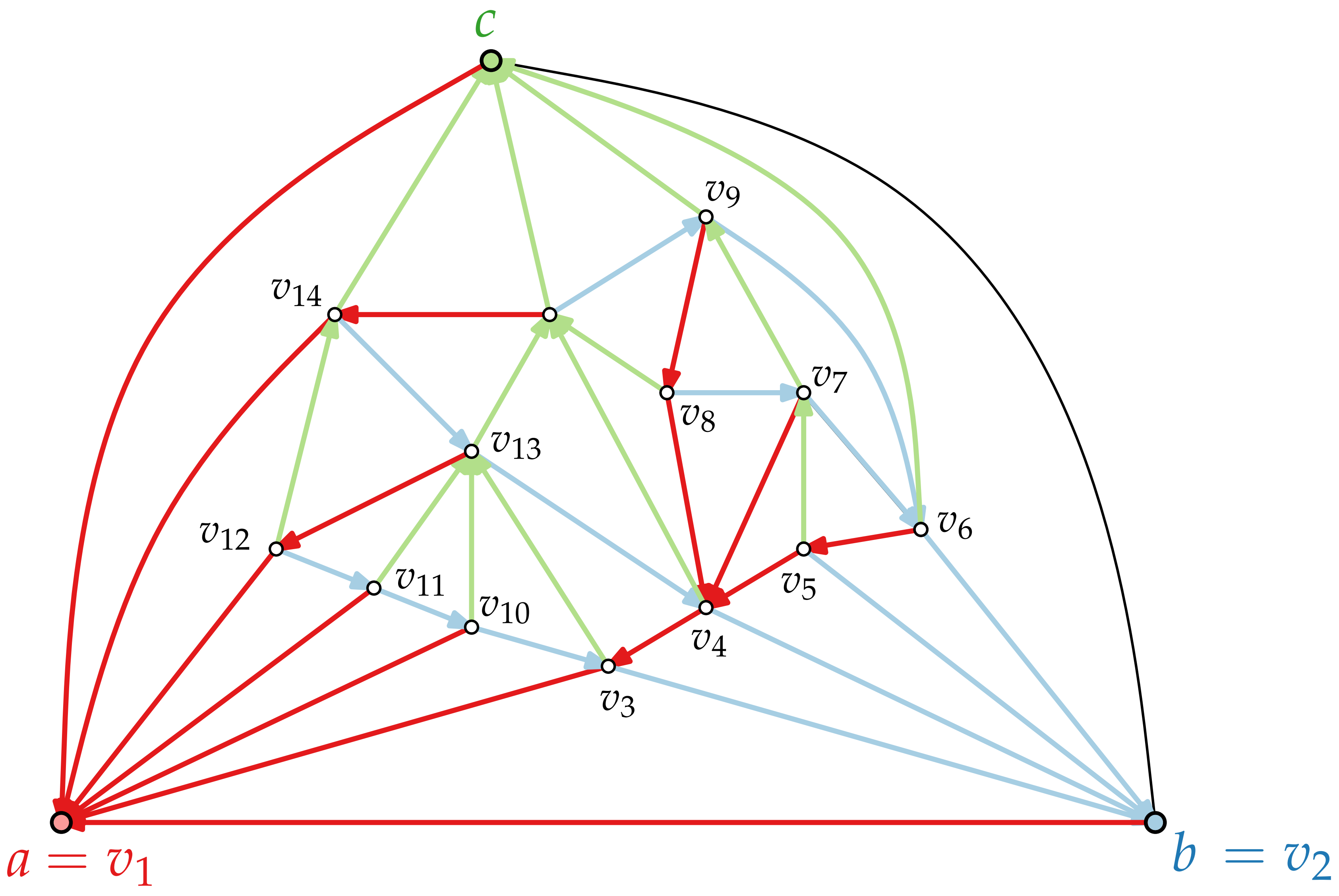

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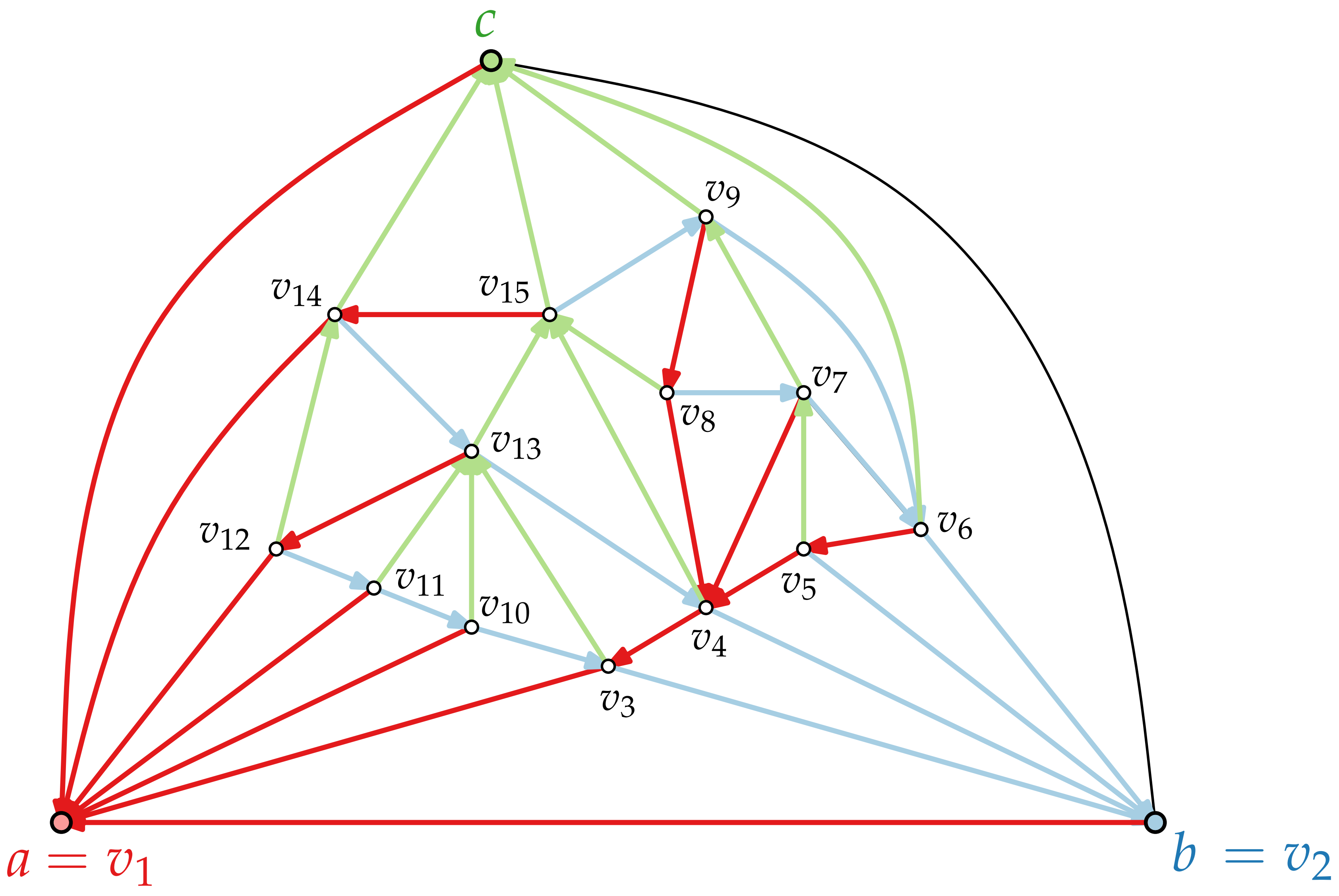

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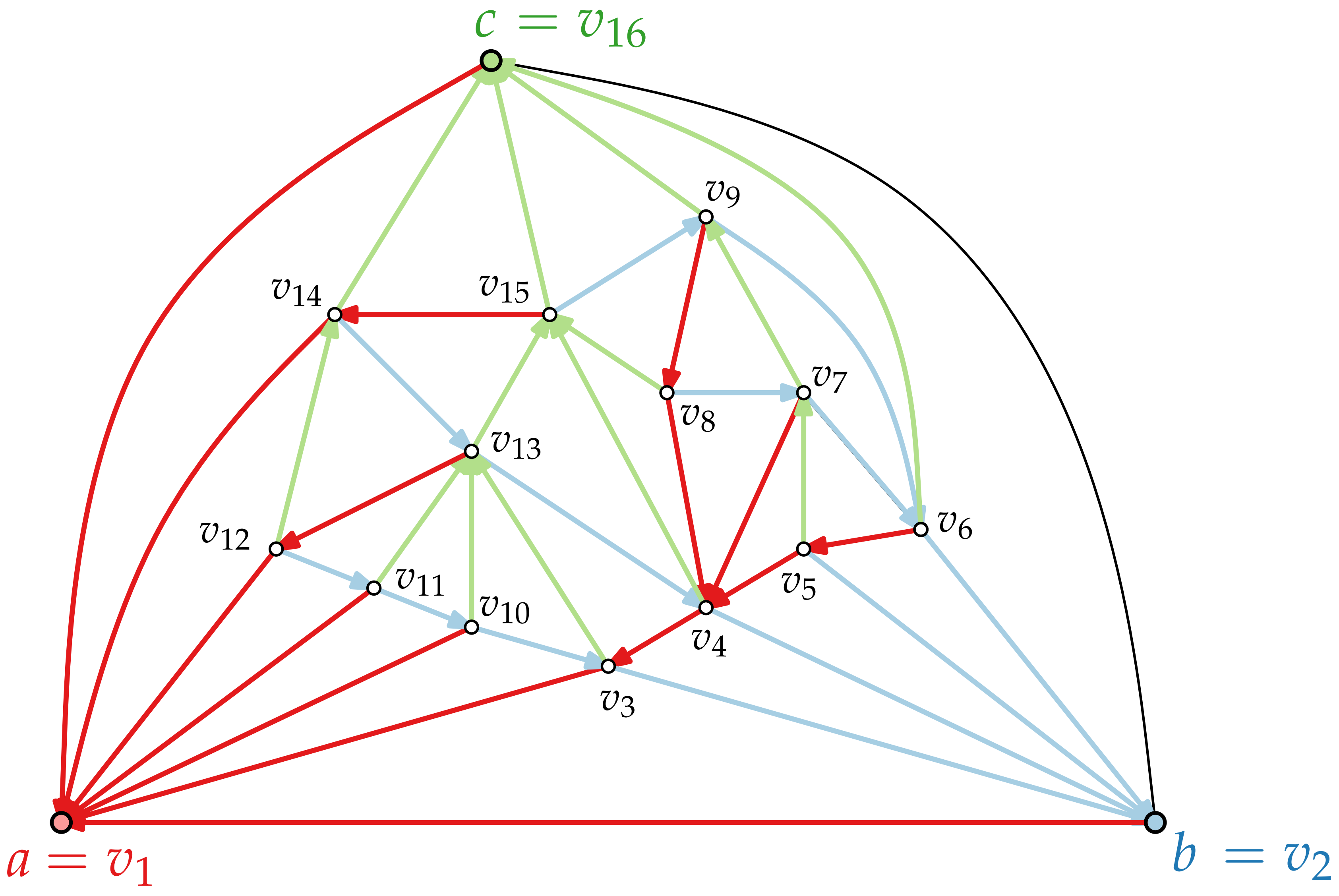

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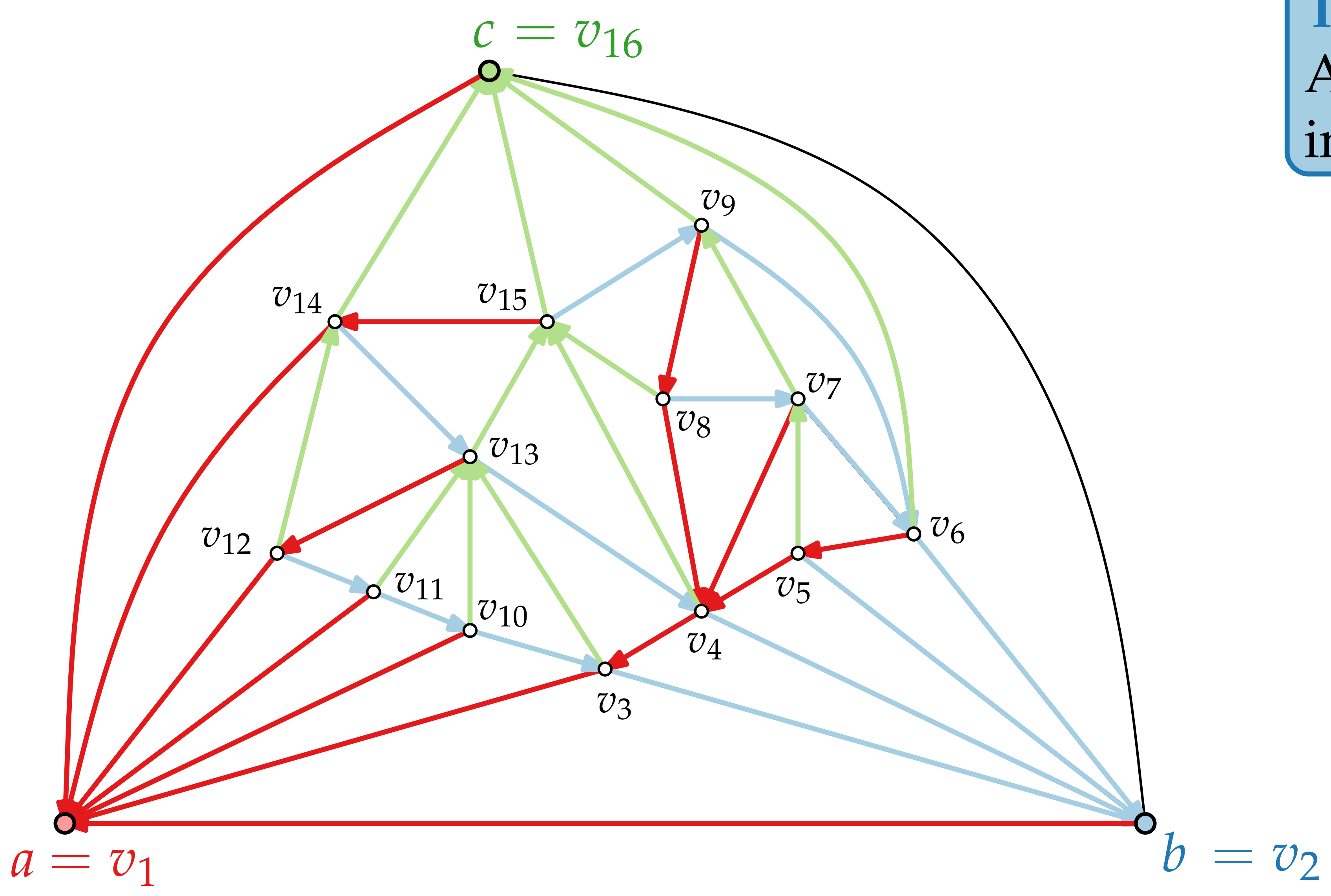

Theorem.
A ccw pre-order traversal on T_i induces a canonical order.

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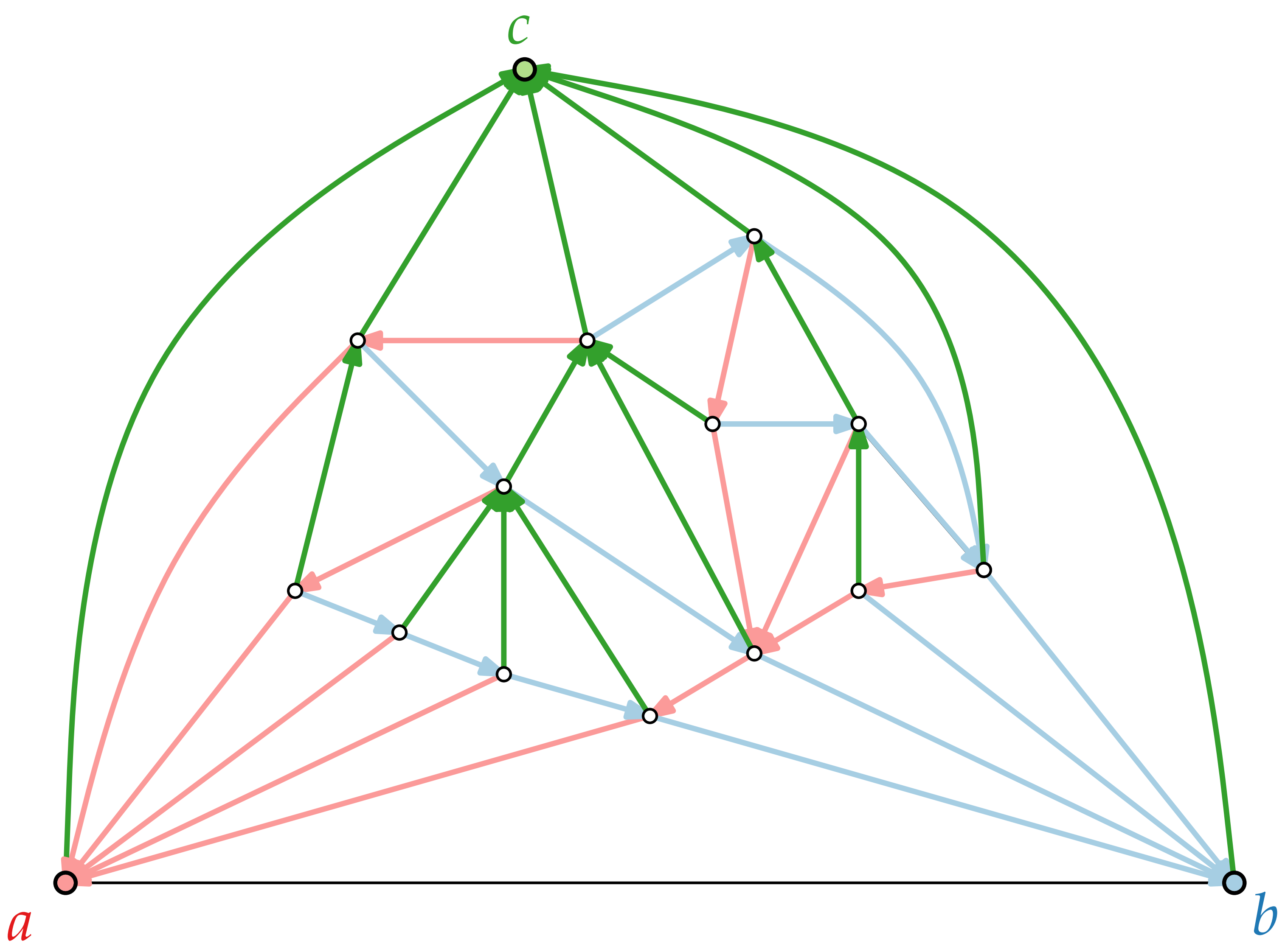

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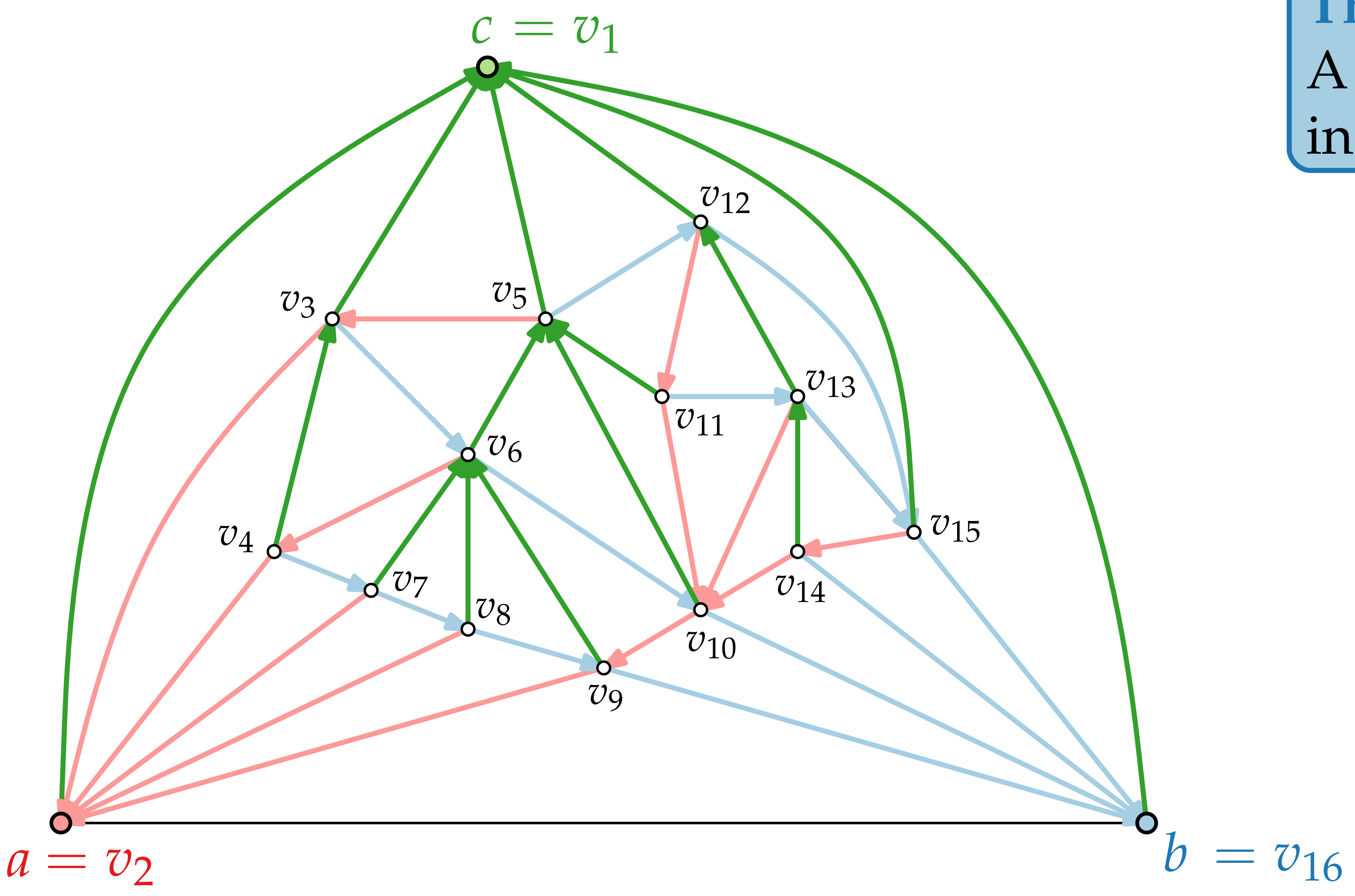

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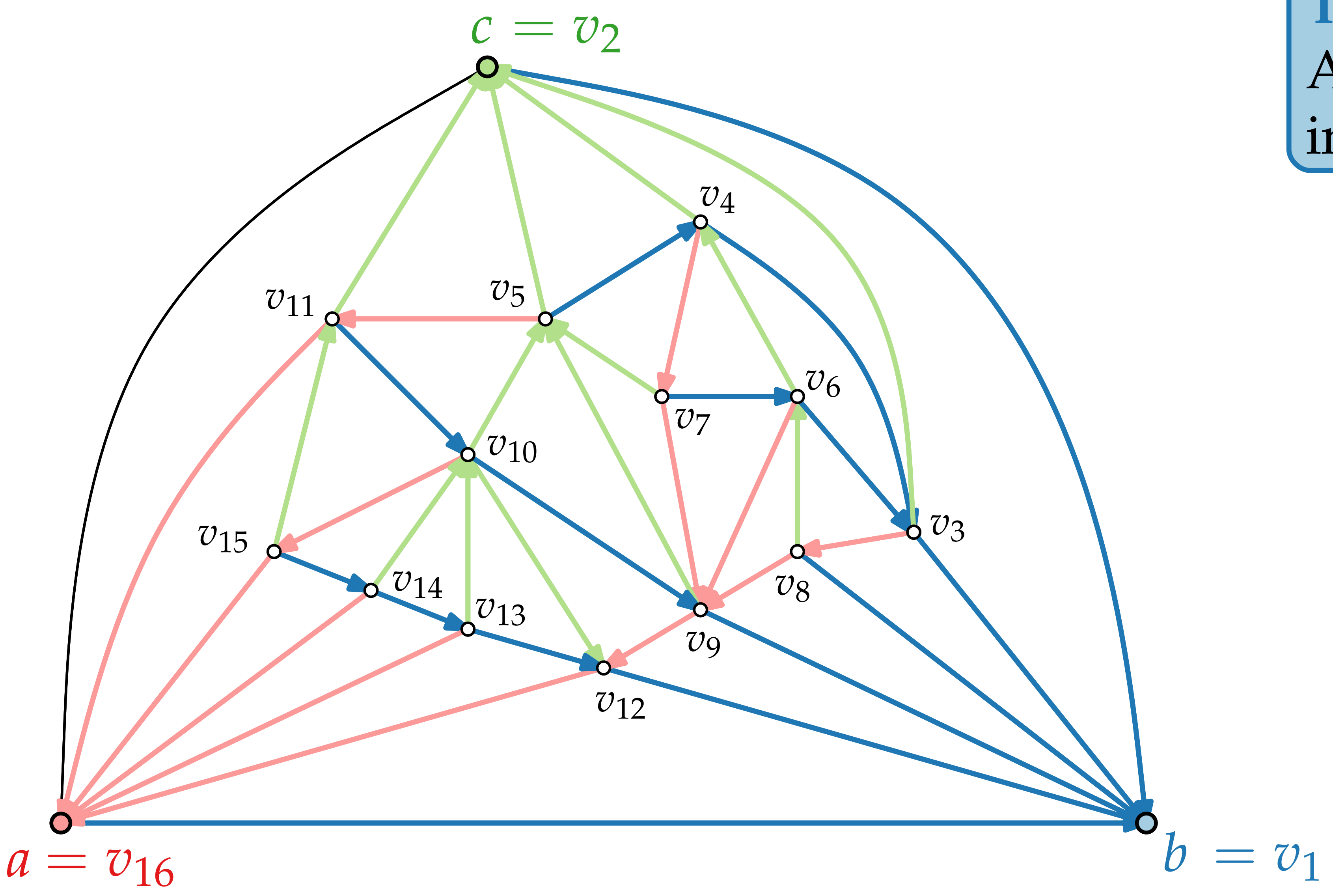

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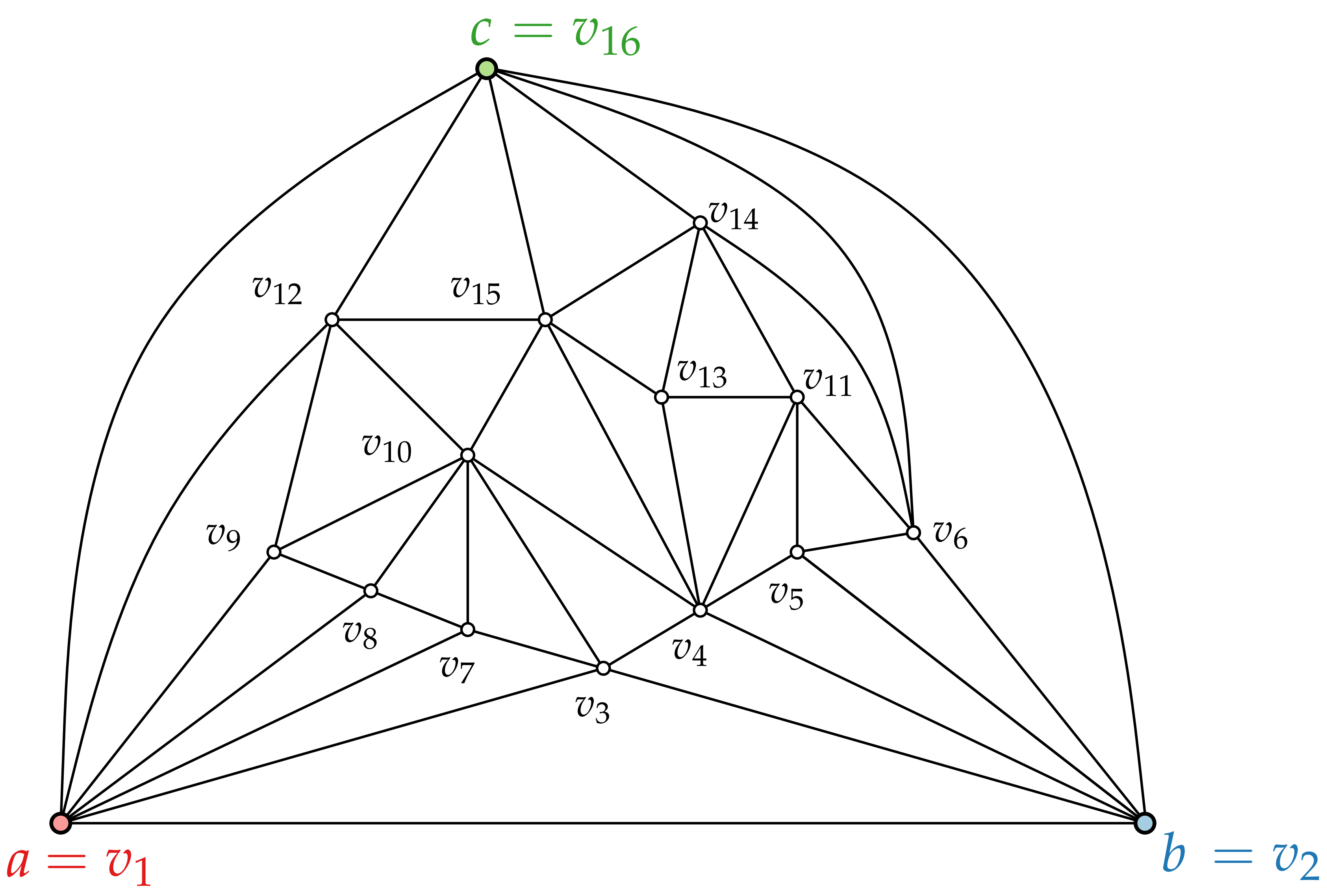

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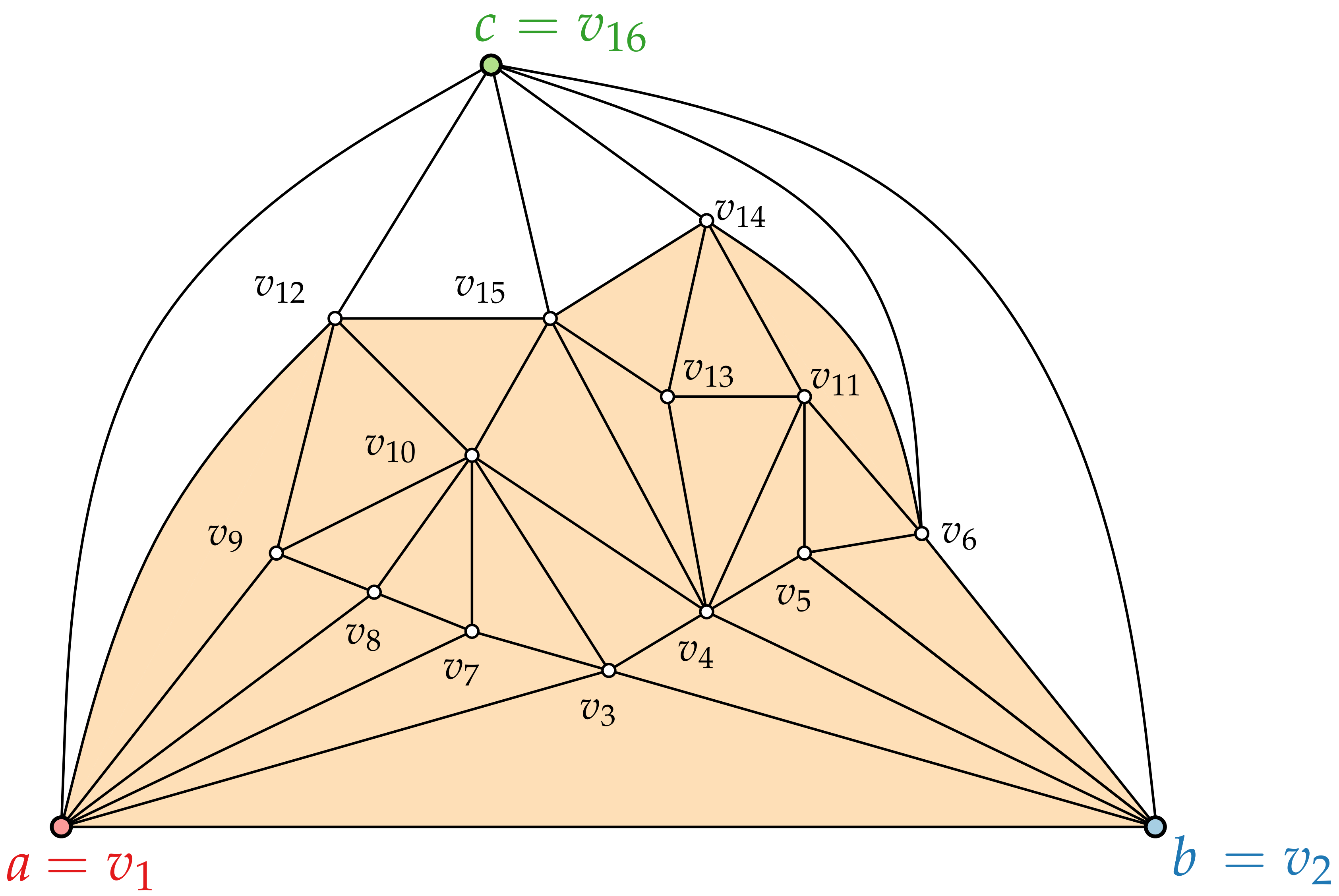

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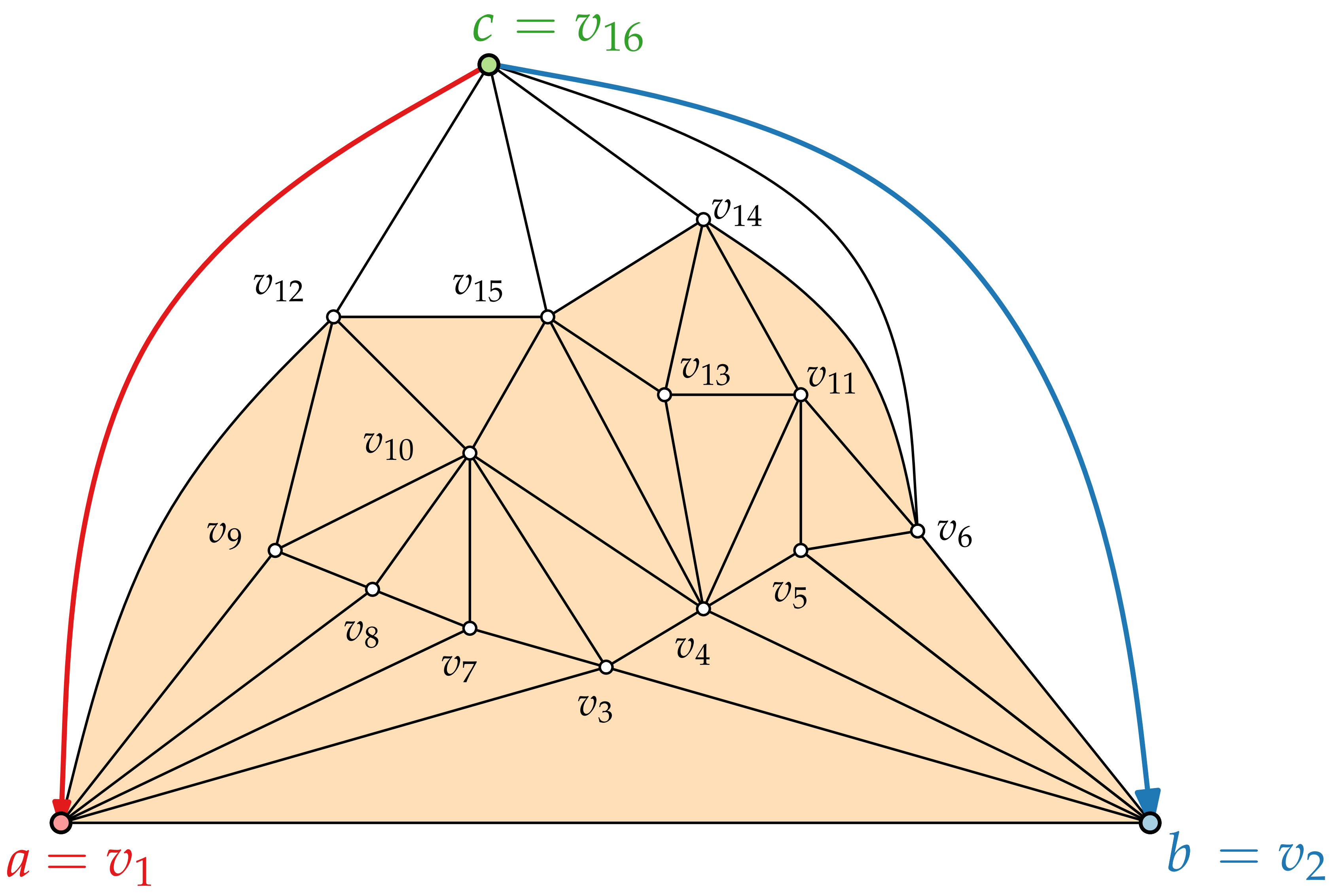

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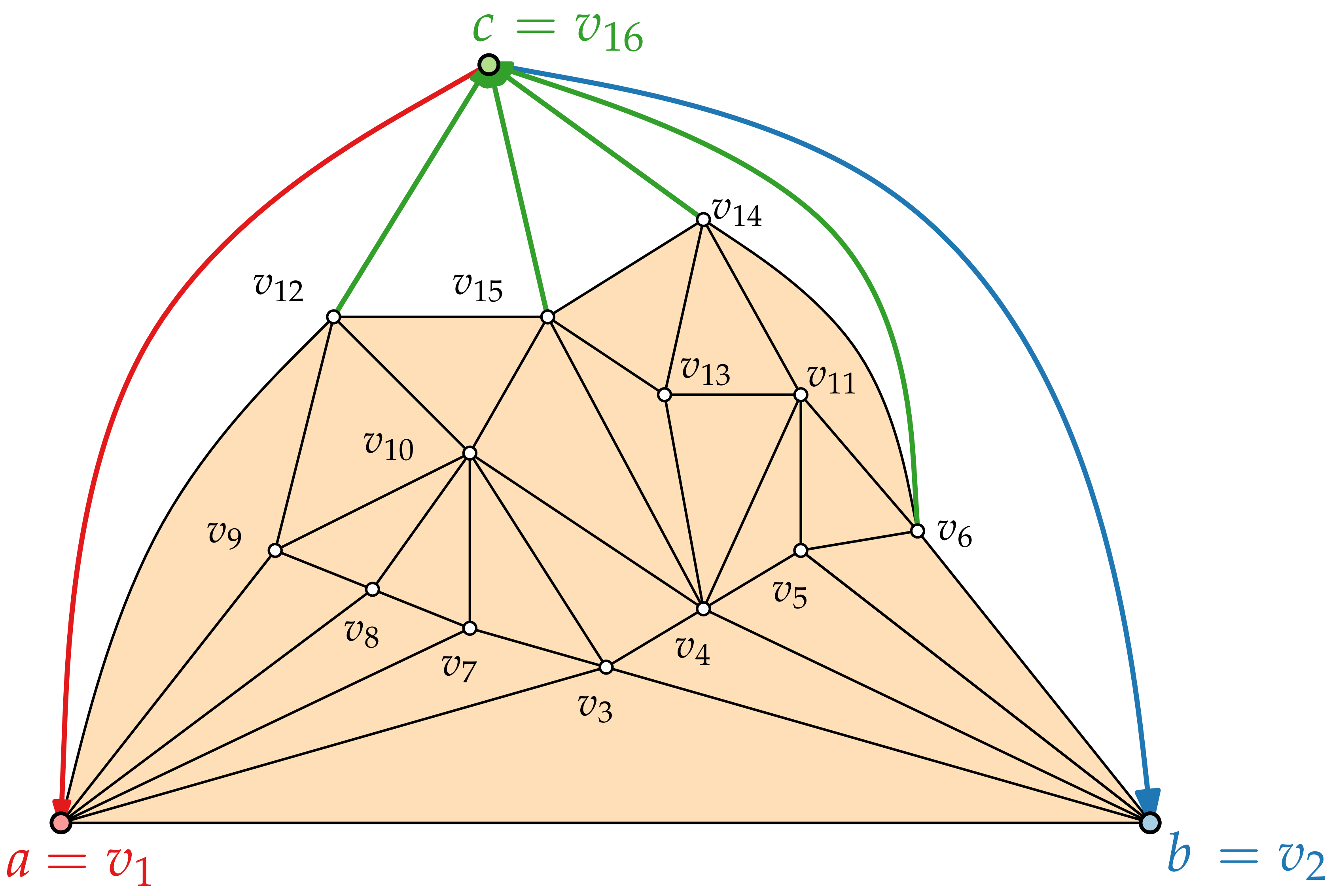

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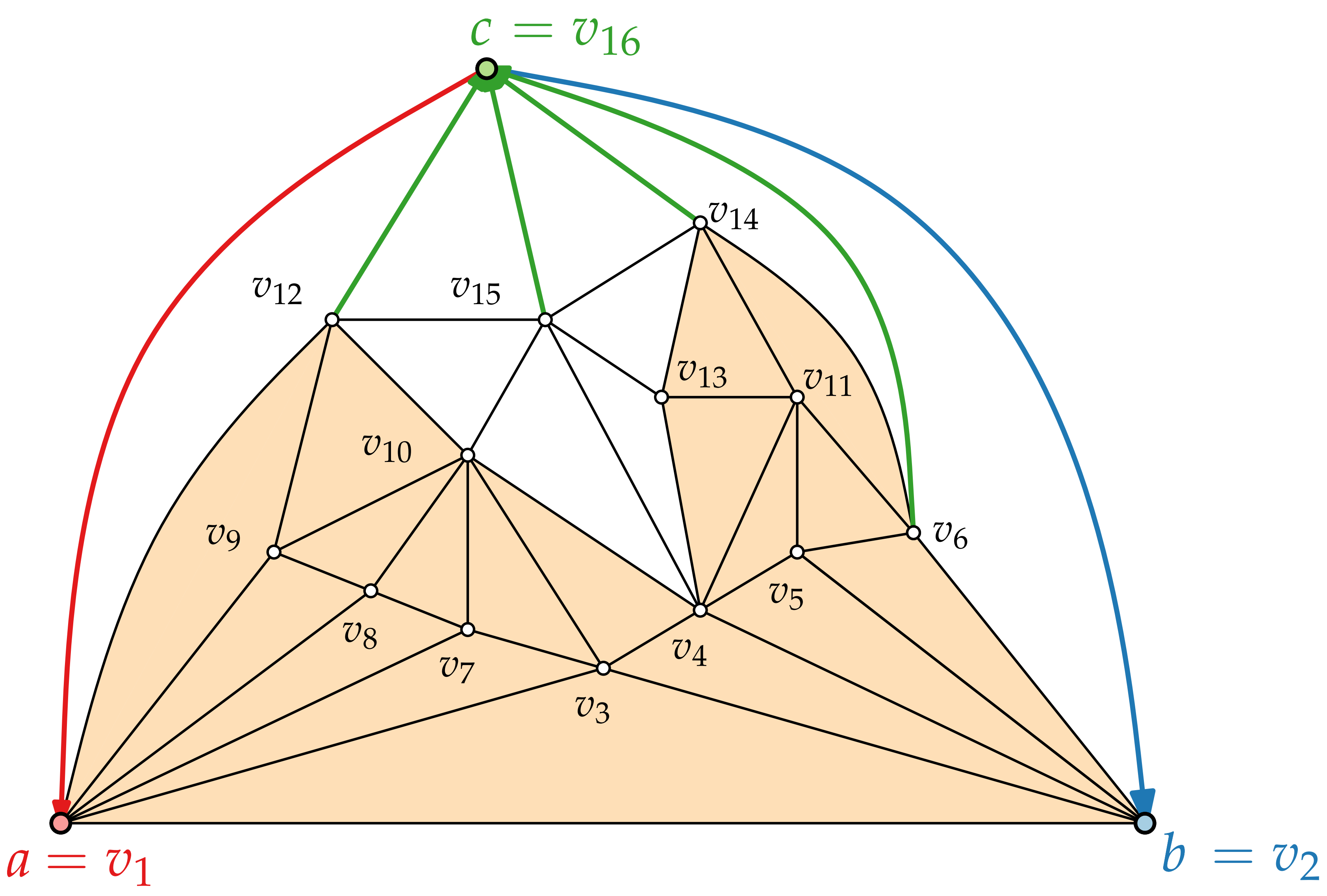

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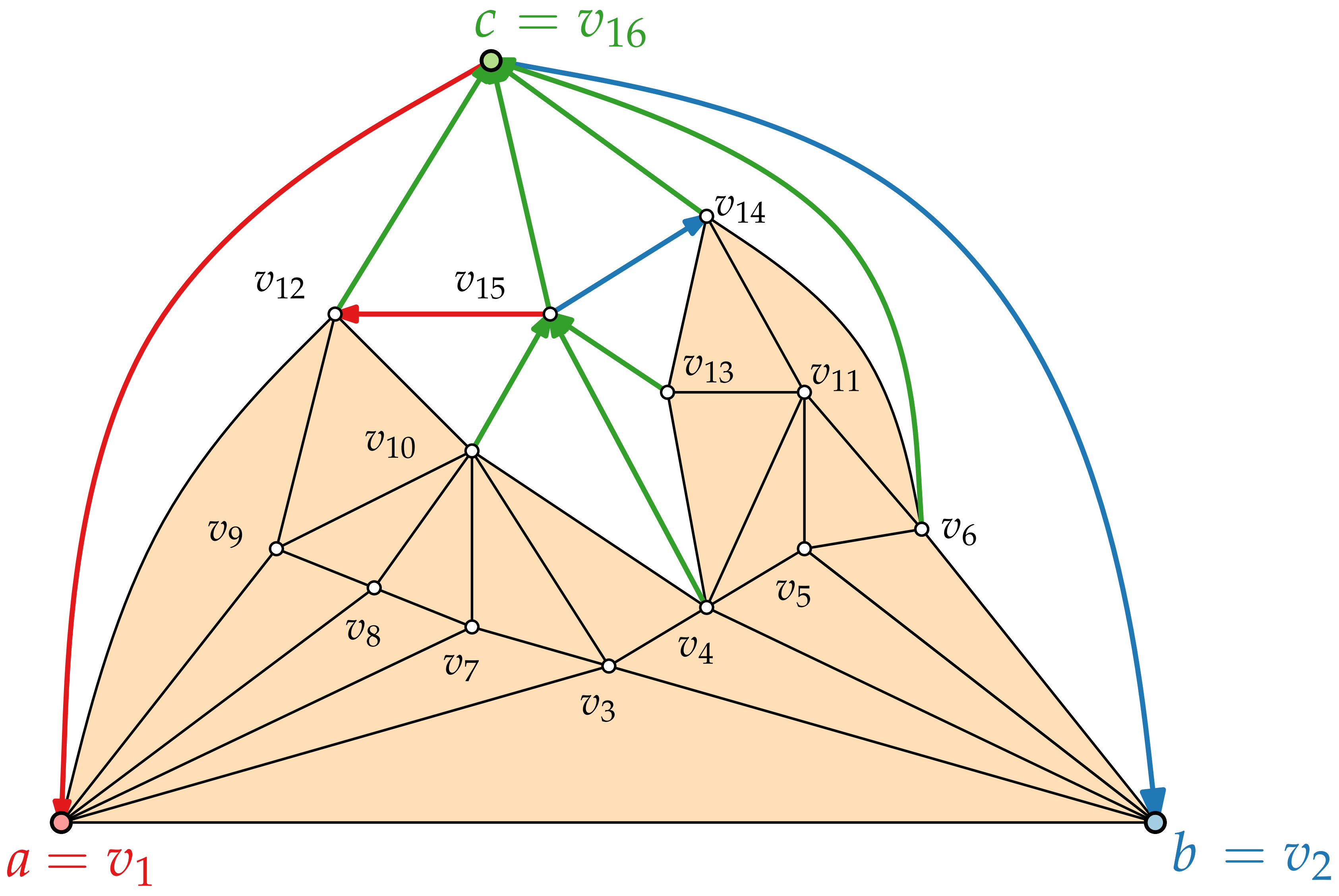

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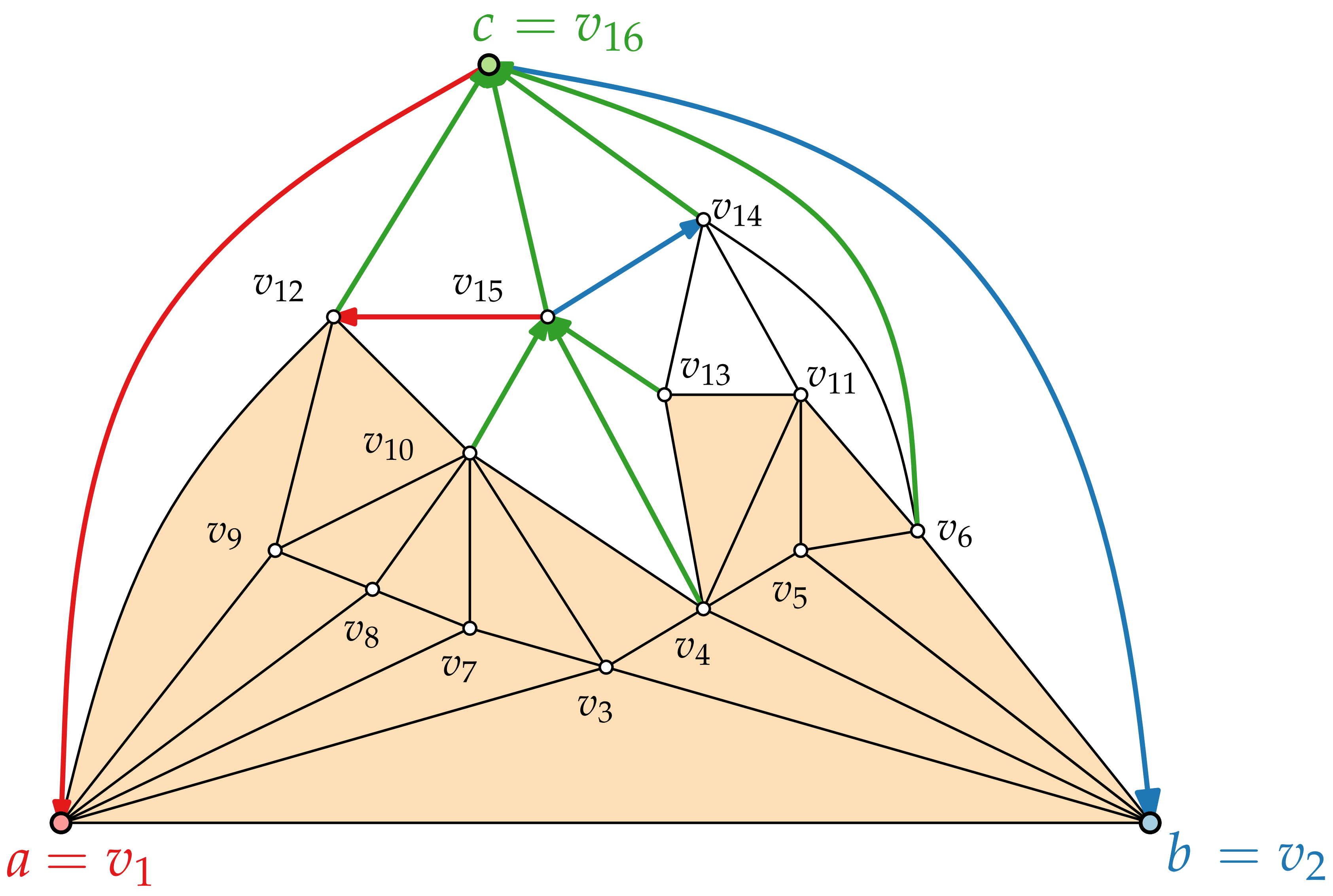

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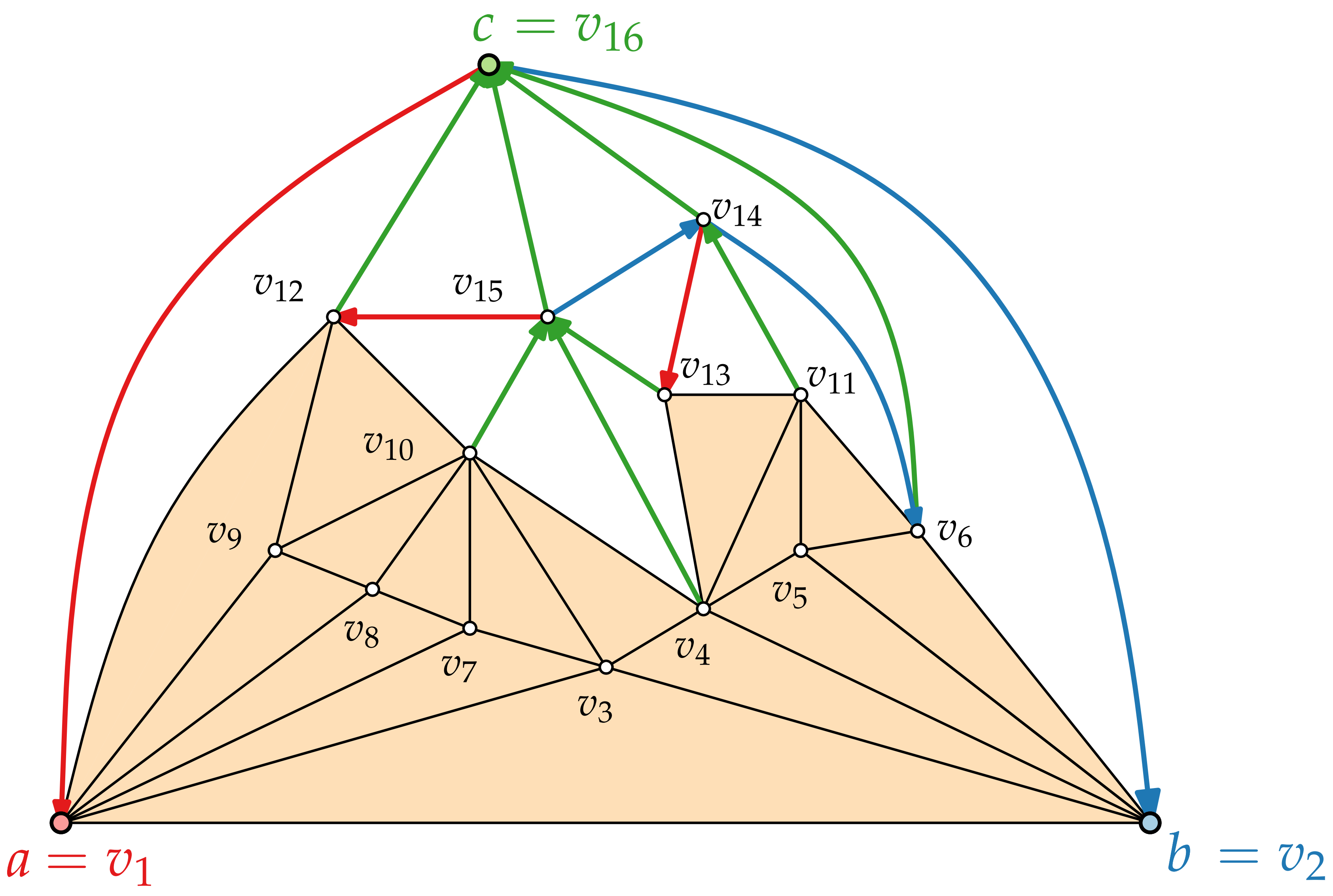

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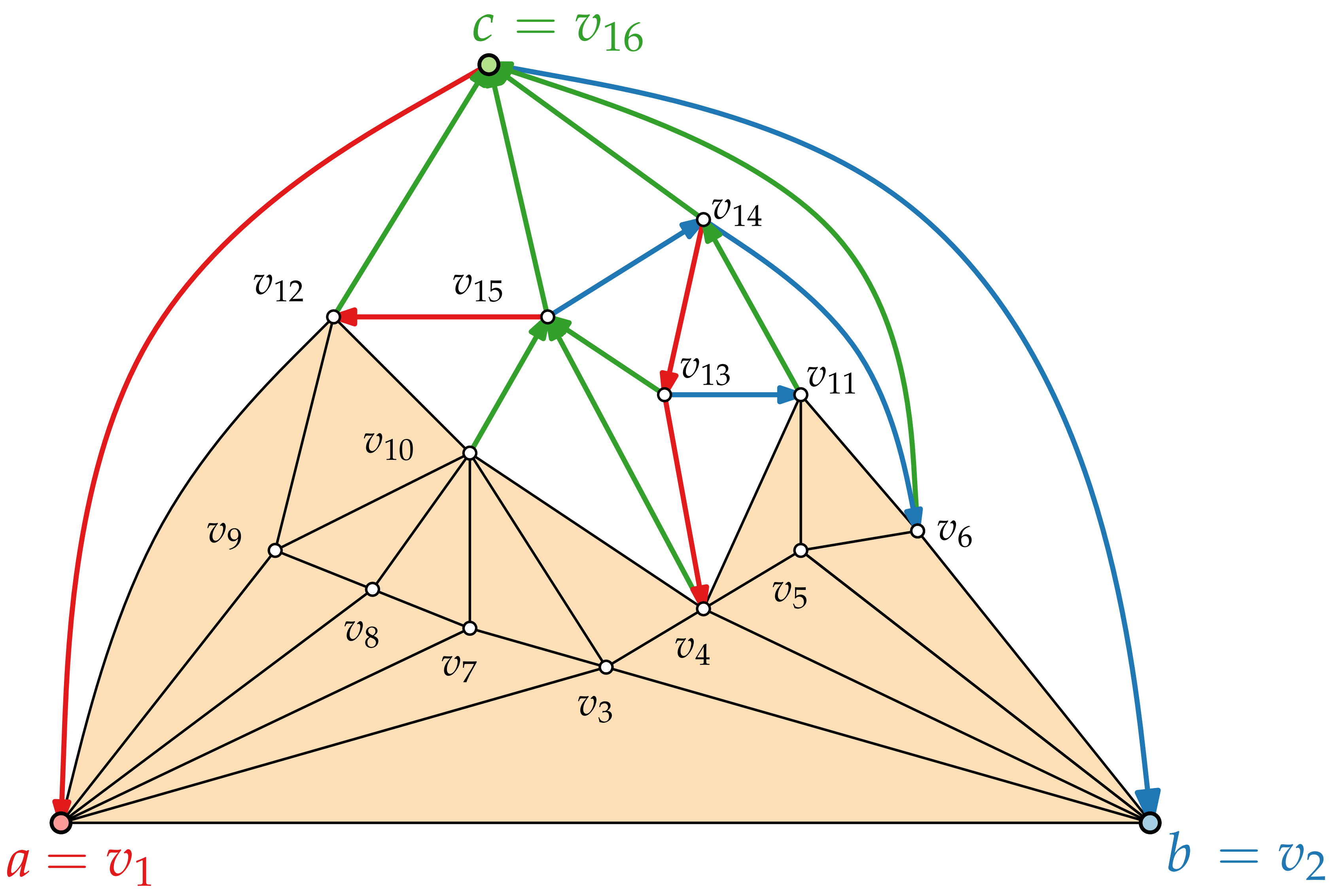

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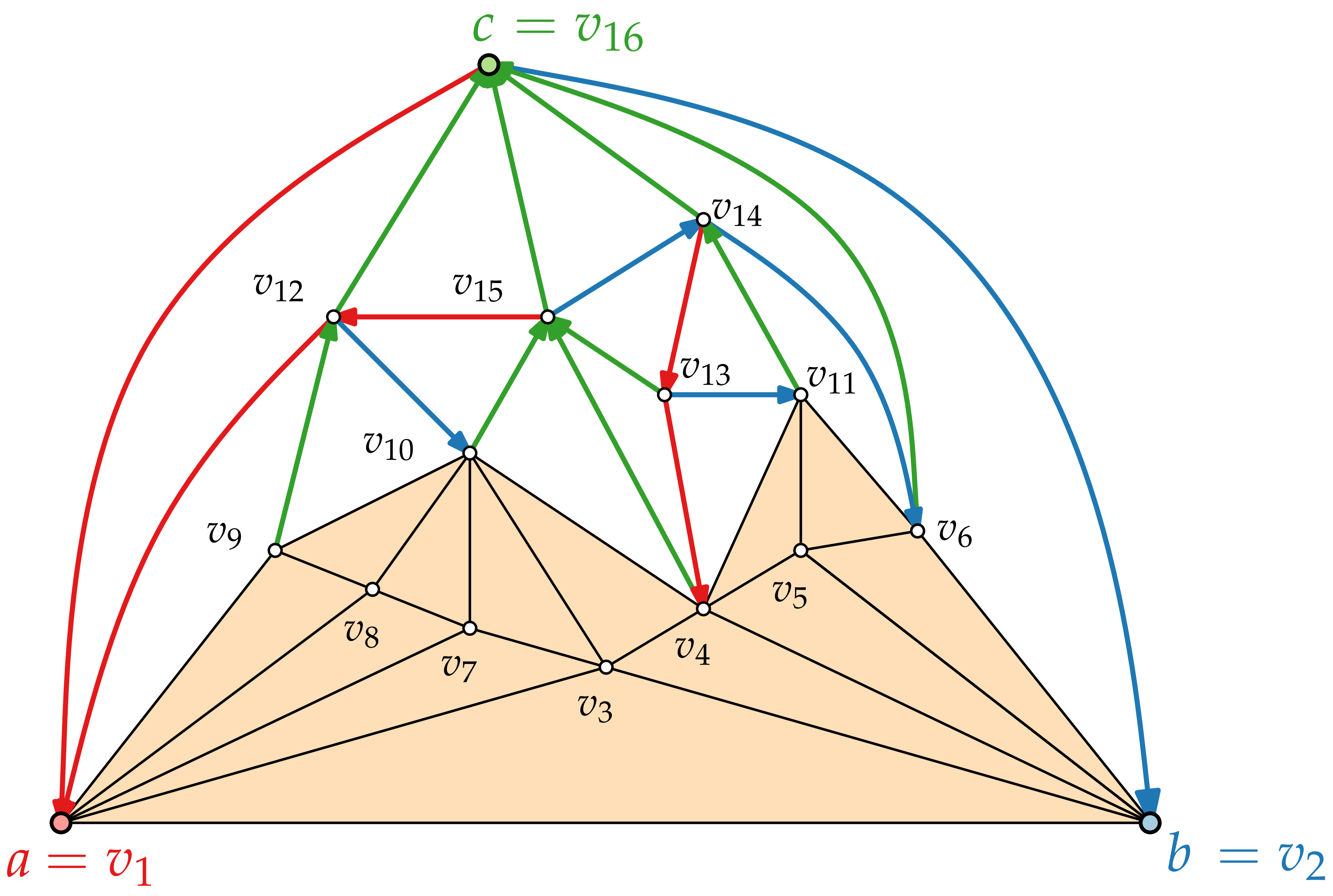

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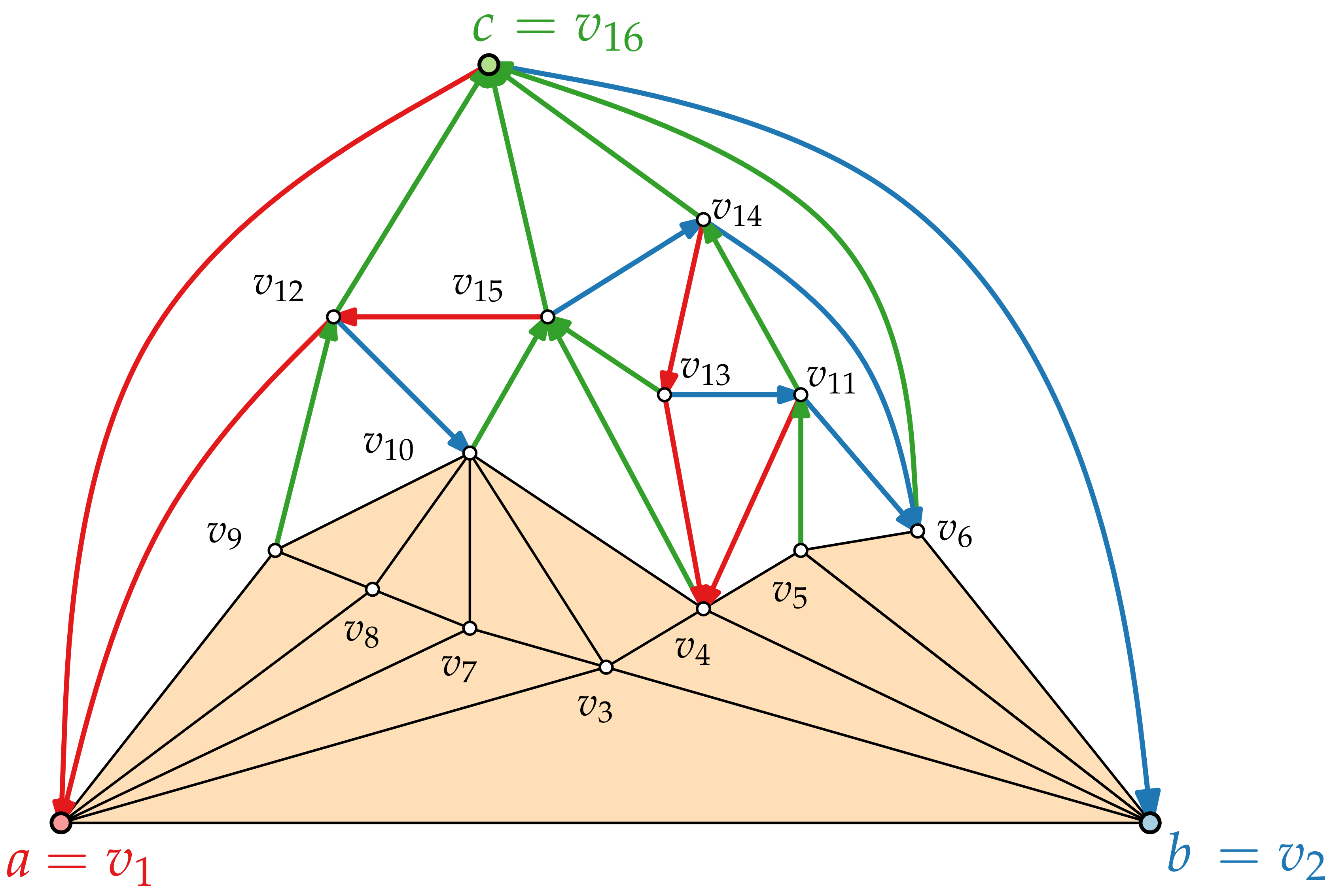

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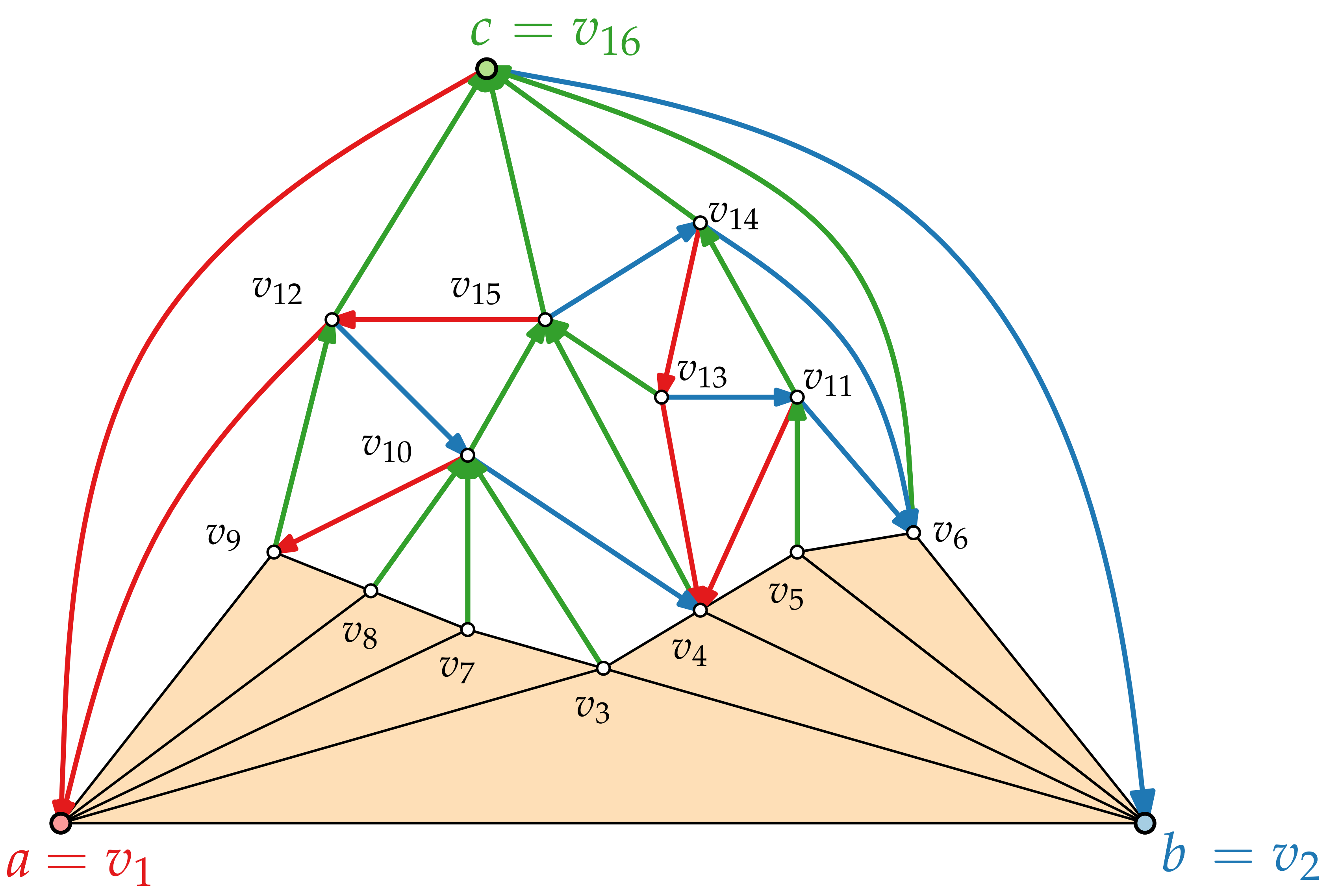

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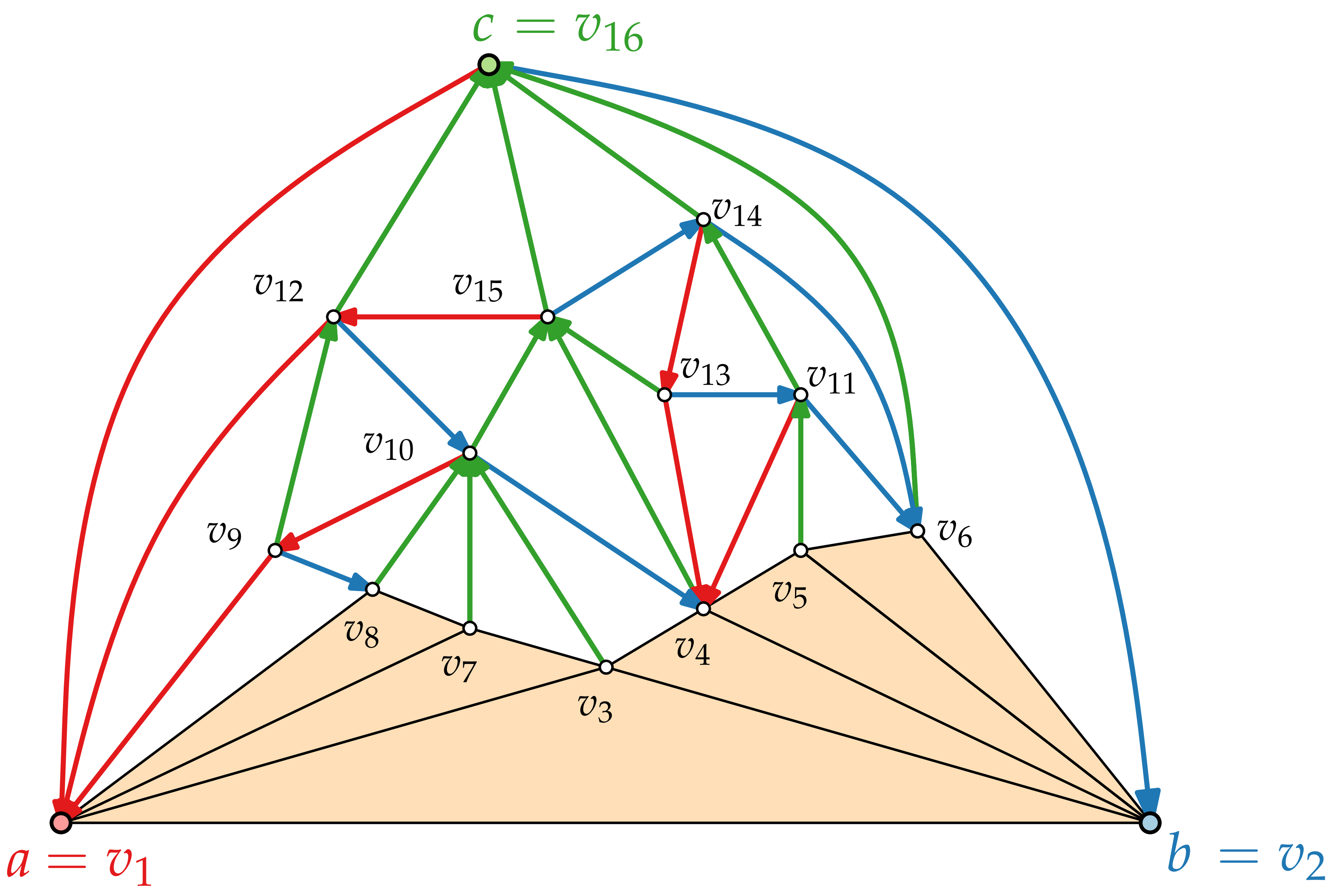

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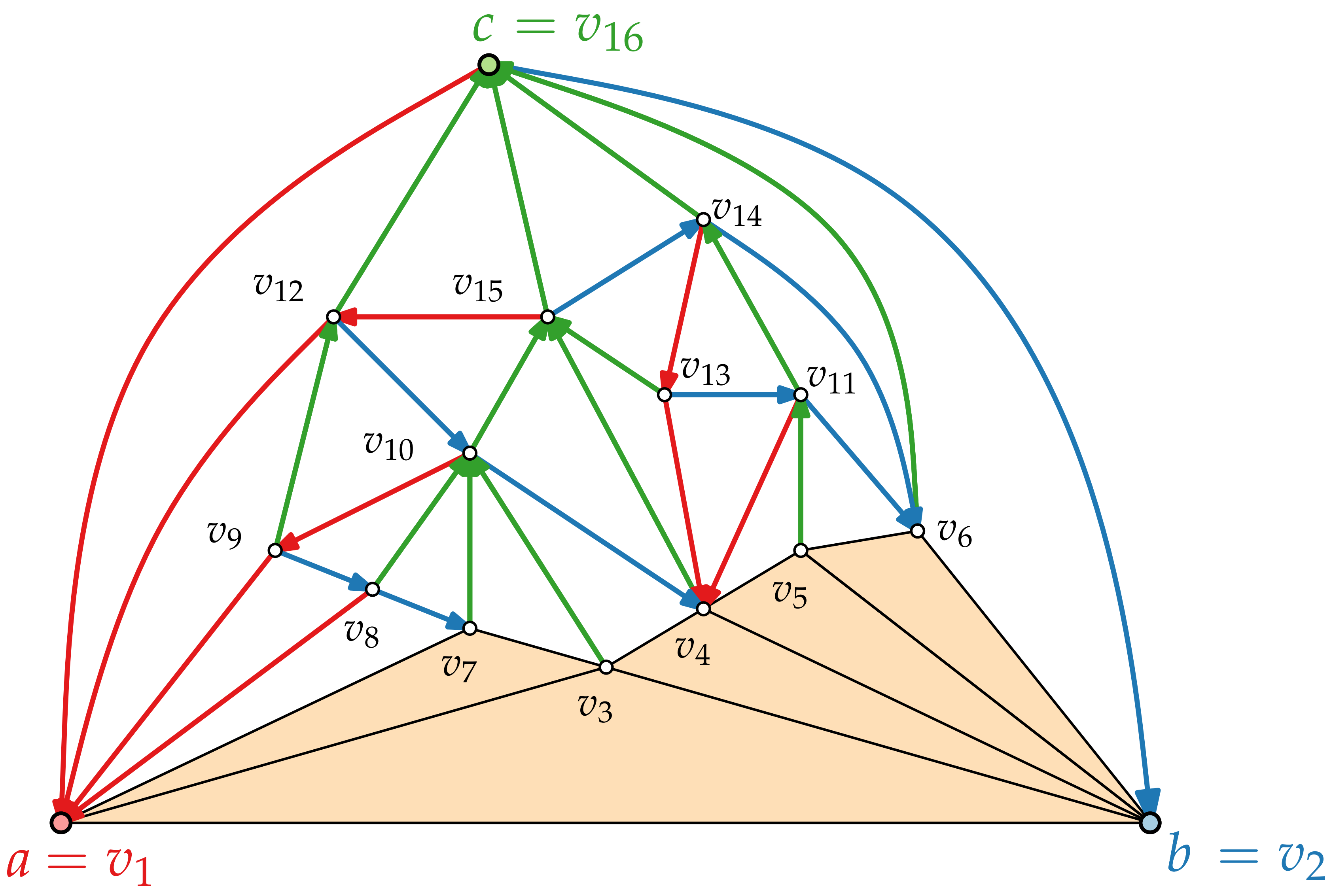
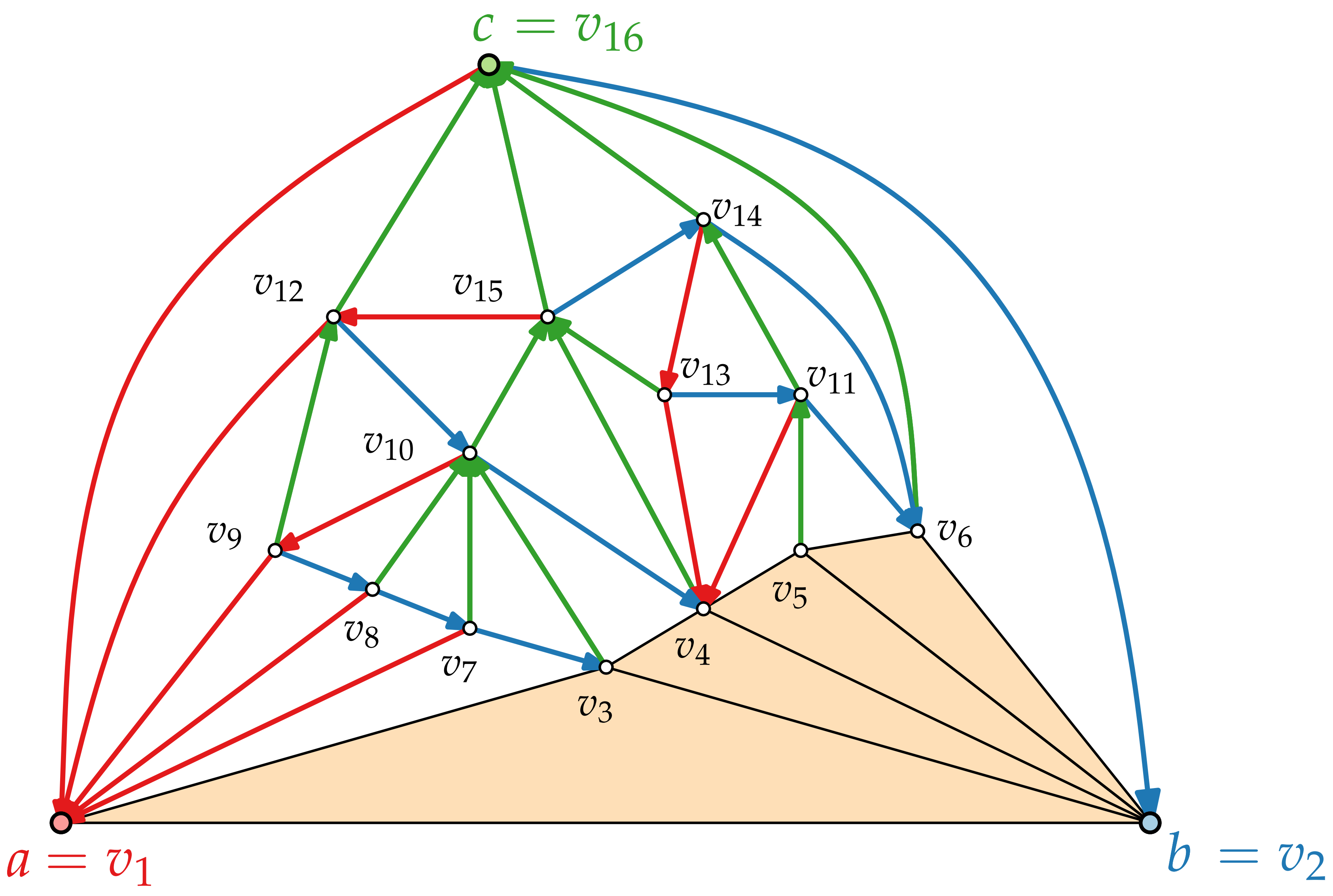
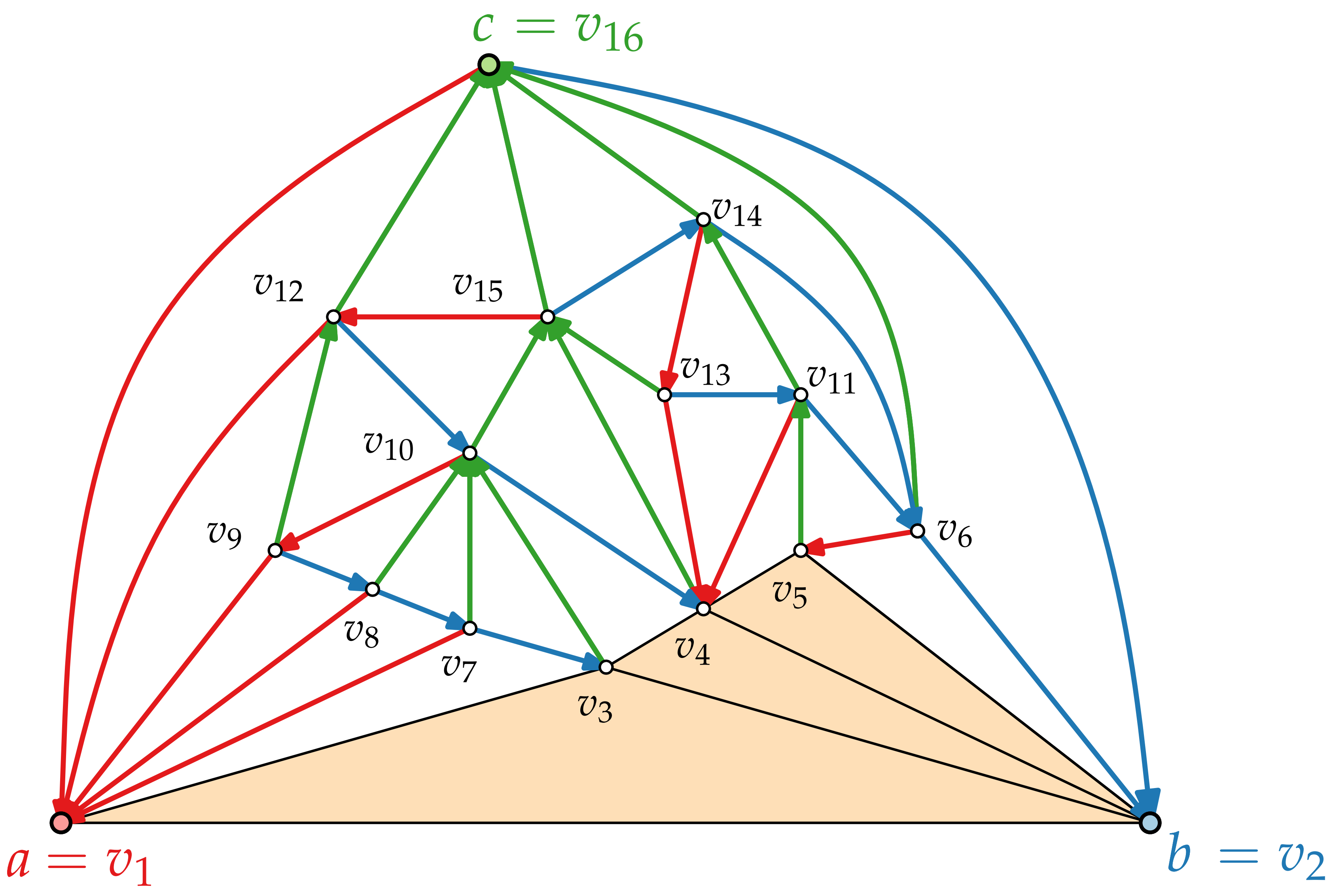


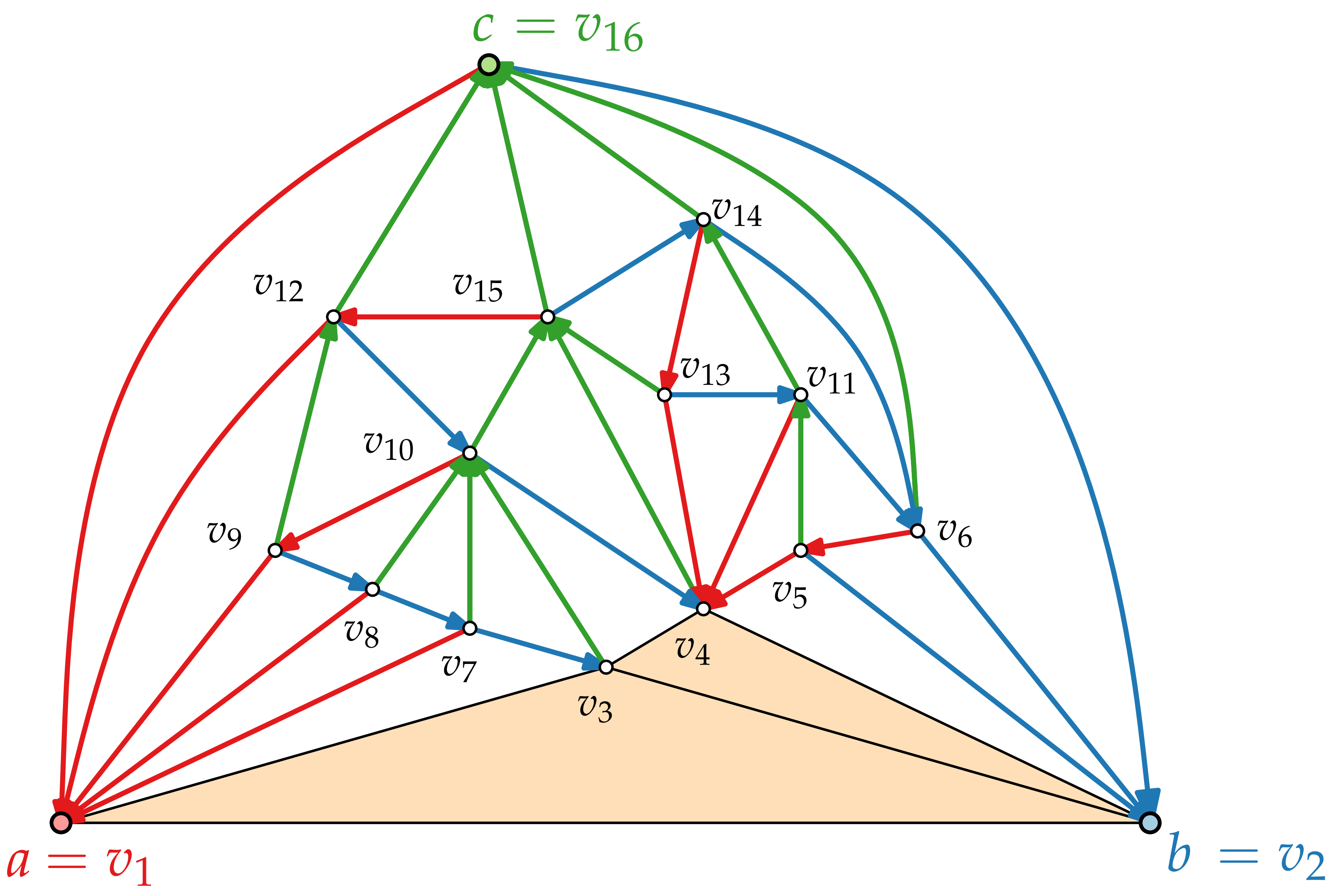
Canonical Order → Schnyder Realizer



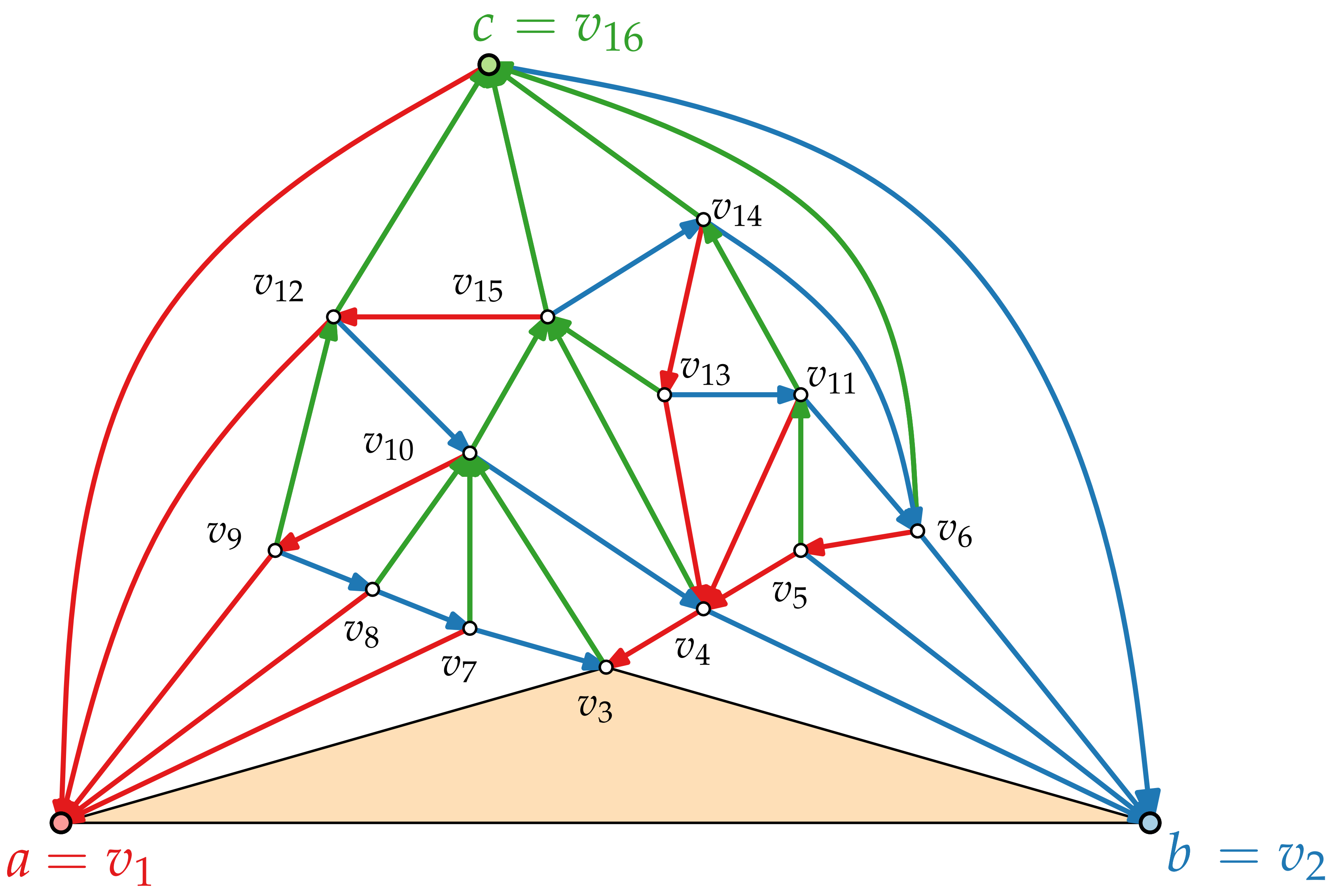
Canonical Order → Schnyder Realizer



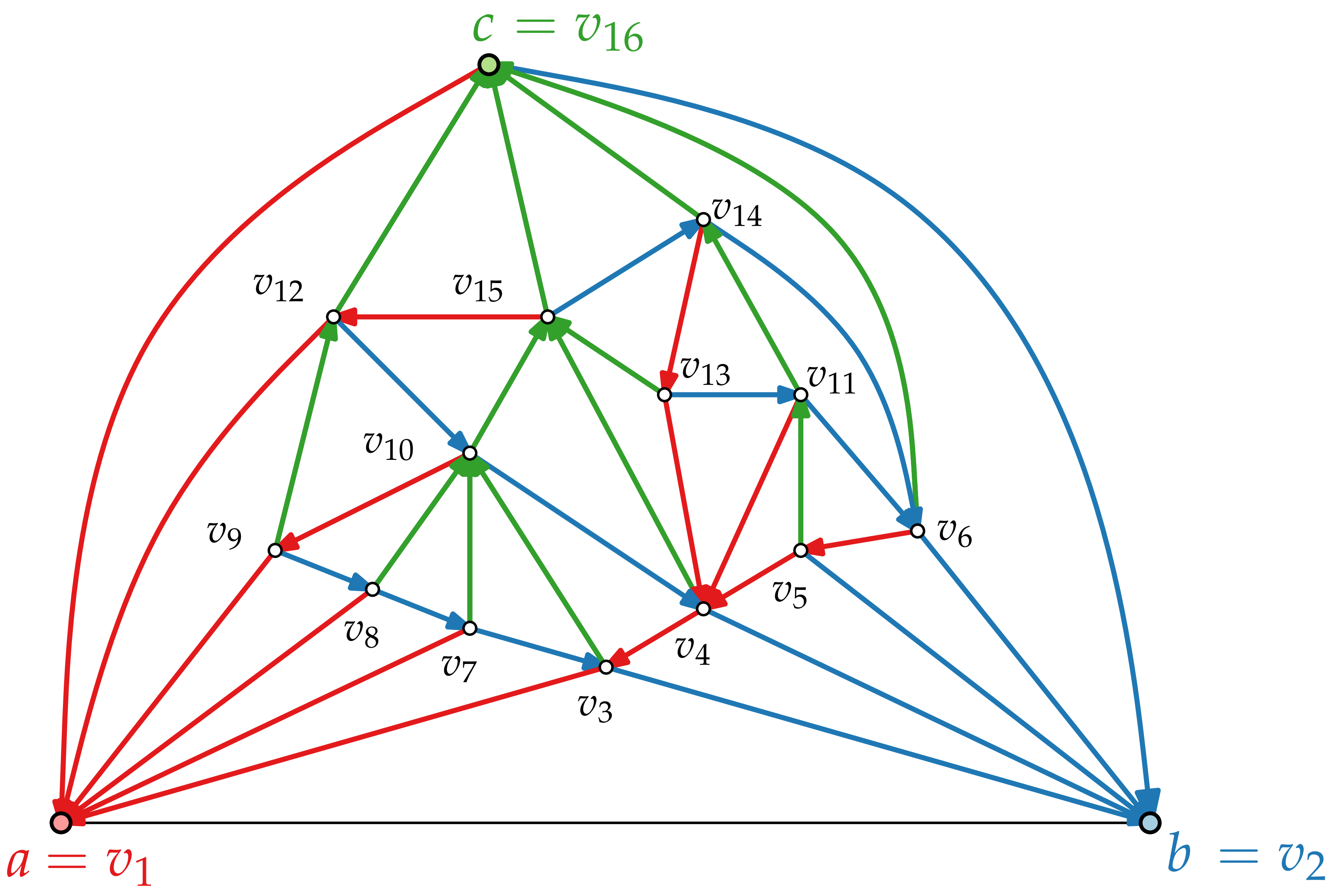
Canonical Order $\rightarrow$ Schnyder Realizer



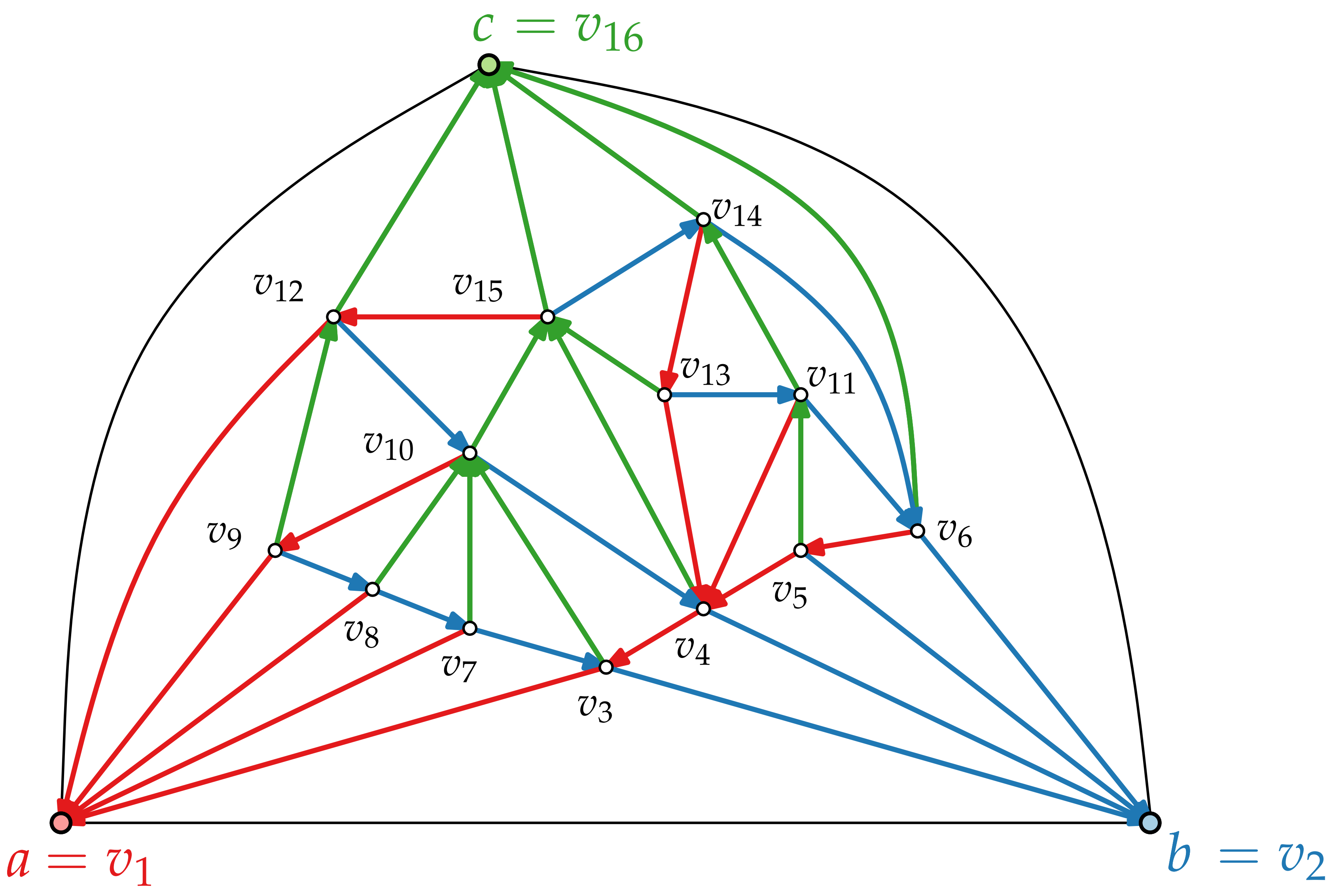
Canonical Order → Schnyder Realizer



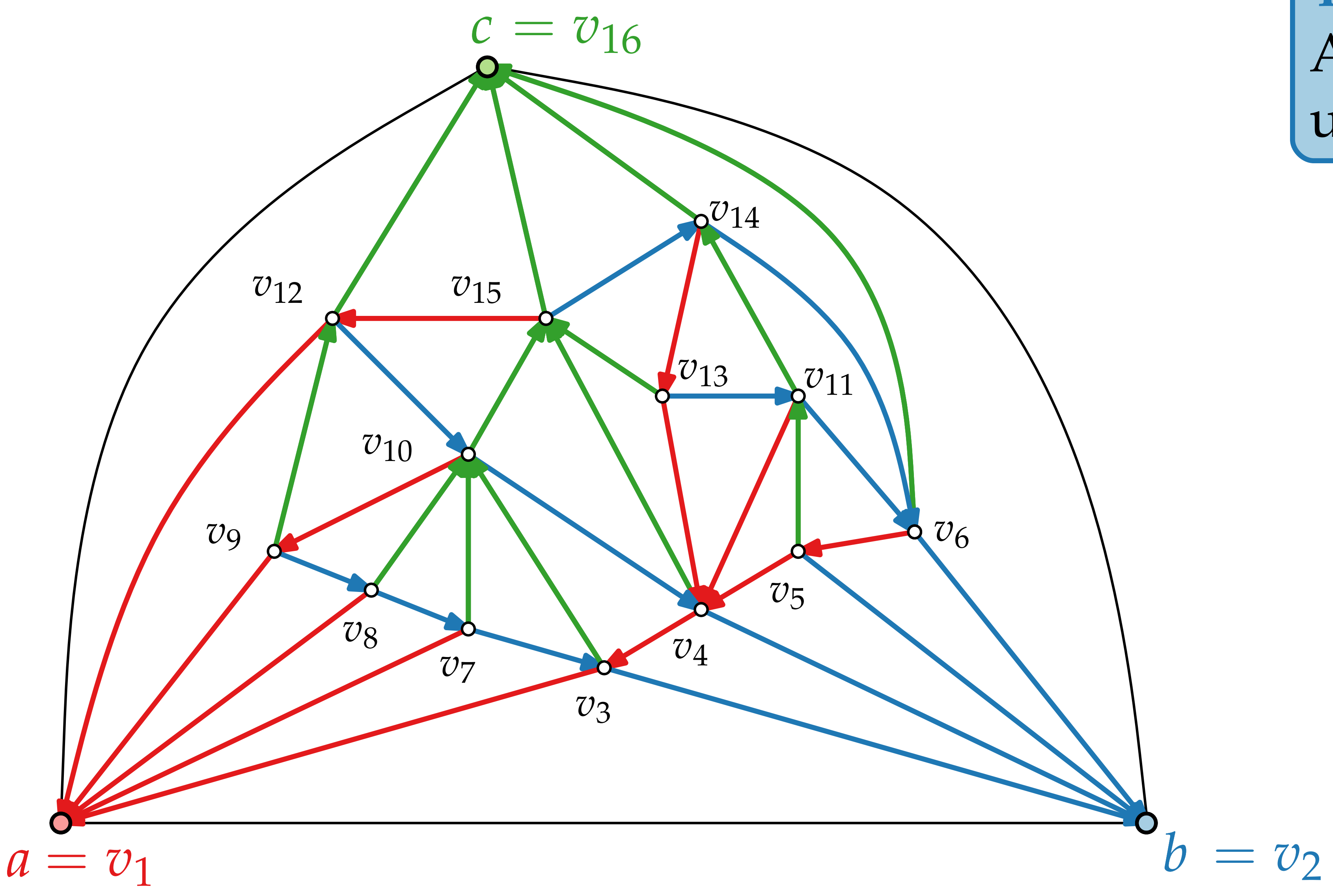
Canonical Order → Schnyder Realizer



Canonical Order → Schnyder Realizer

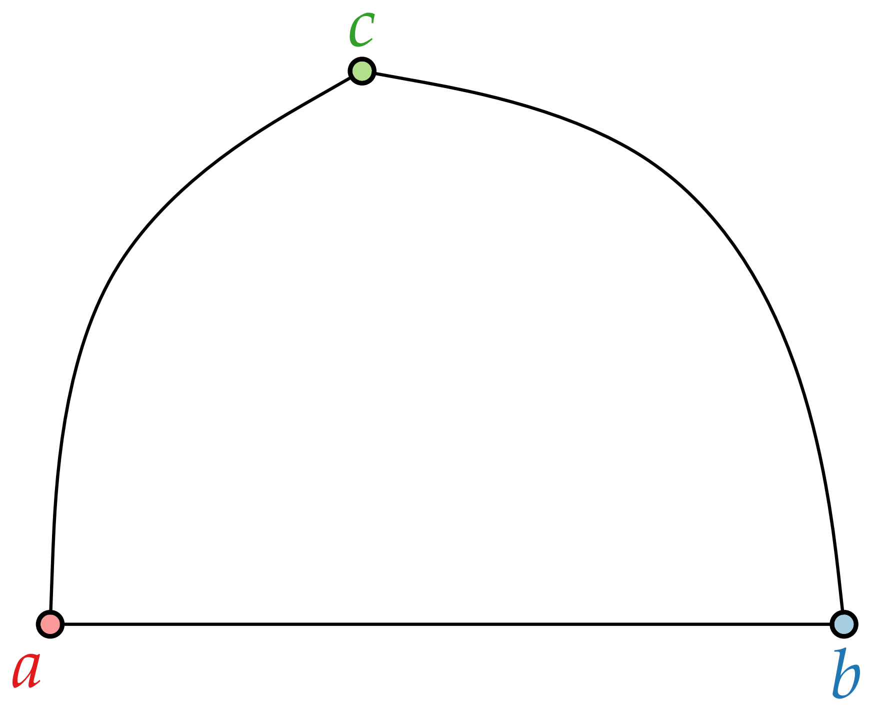


Canonical Order → Schnyder Realizer



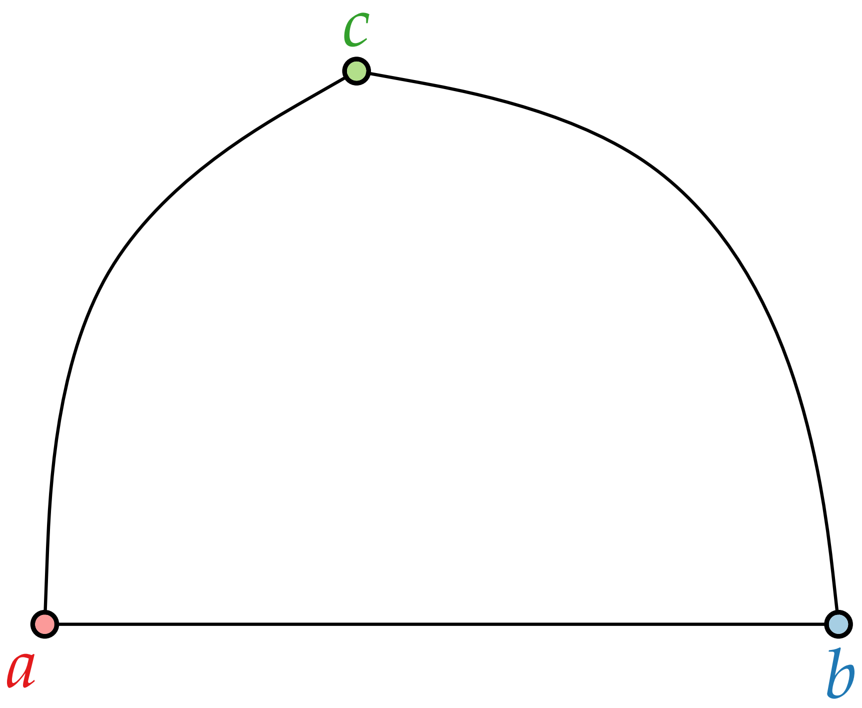
Theorem.
A canonical order induces a unique Schnyder Realizer.

Linear Time Computation



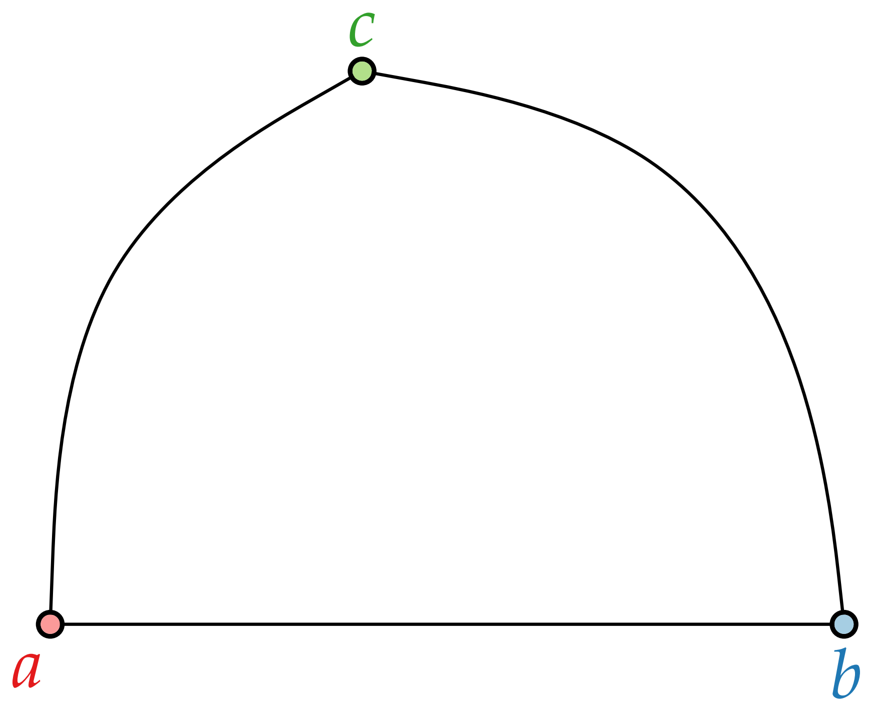
Linear Time Computation

- Compute Canonical Order



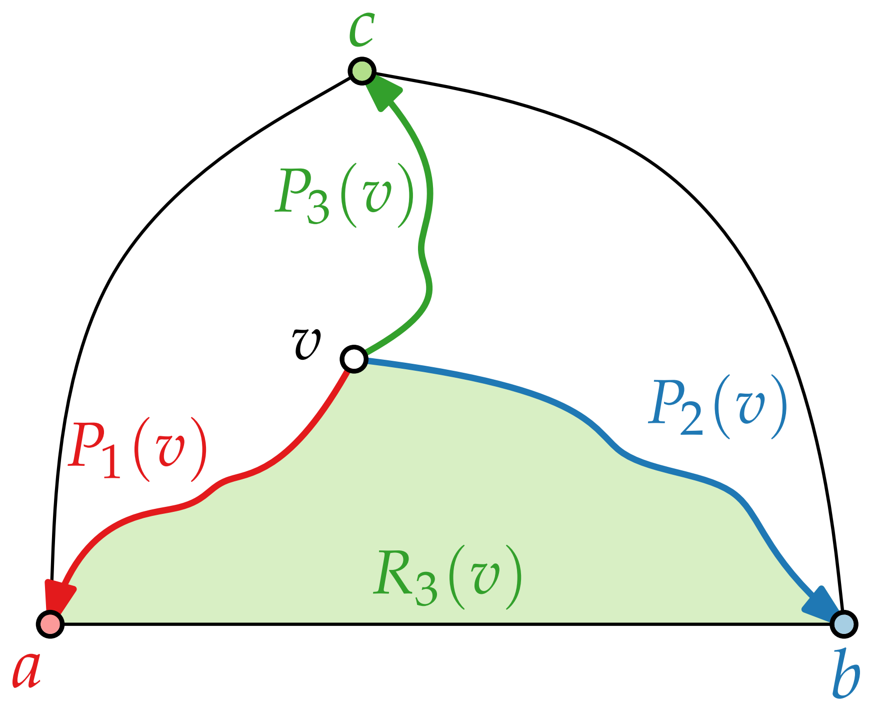
Linear Time Computation

- Compute Canonical Order
- Compute Schnyder Realizer



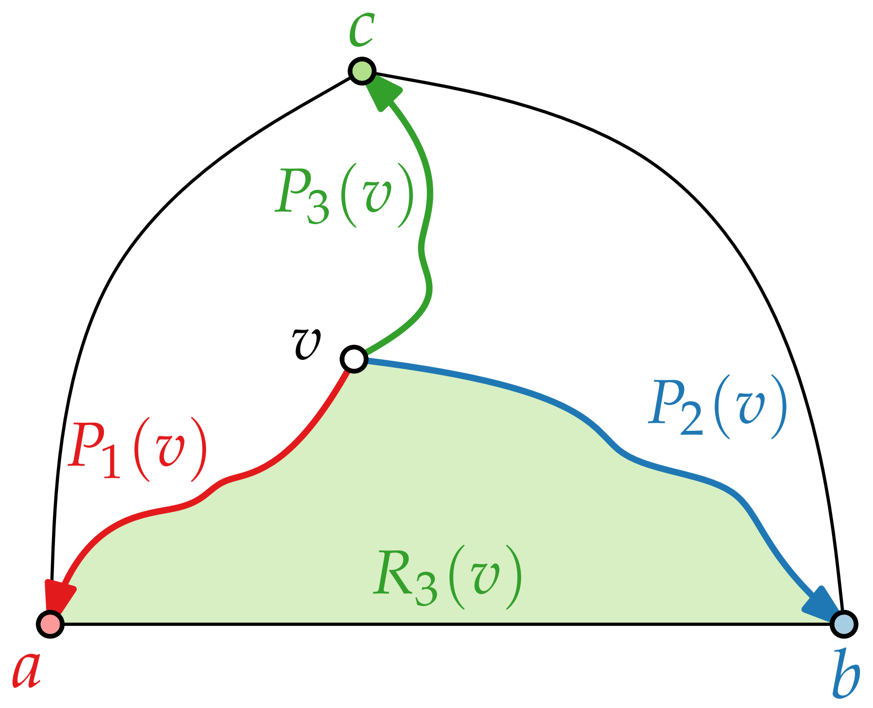
Linear Time Computation

- Compute Canonical Order
- Compute Schnyder Realizer



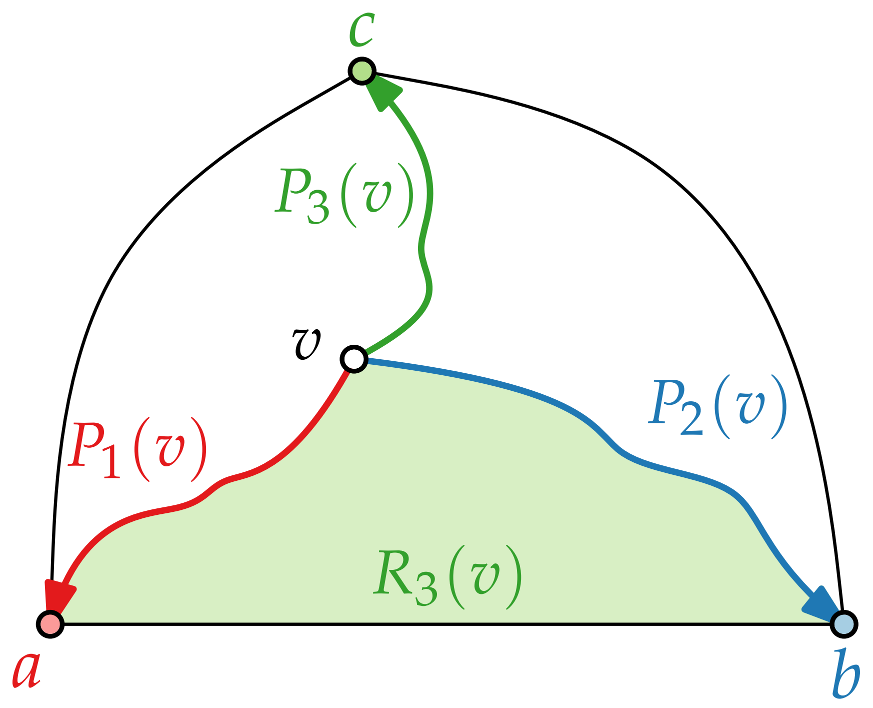
Linear Time Computation

- Compute Canonical Order
- Compute Schnyder Realizer
- Goal: $v_i = |V(R_i(v))| - |P_{i-1}(v)|$



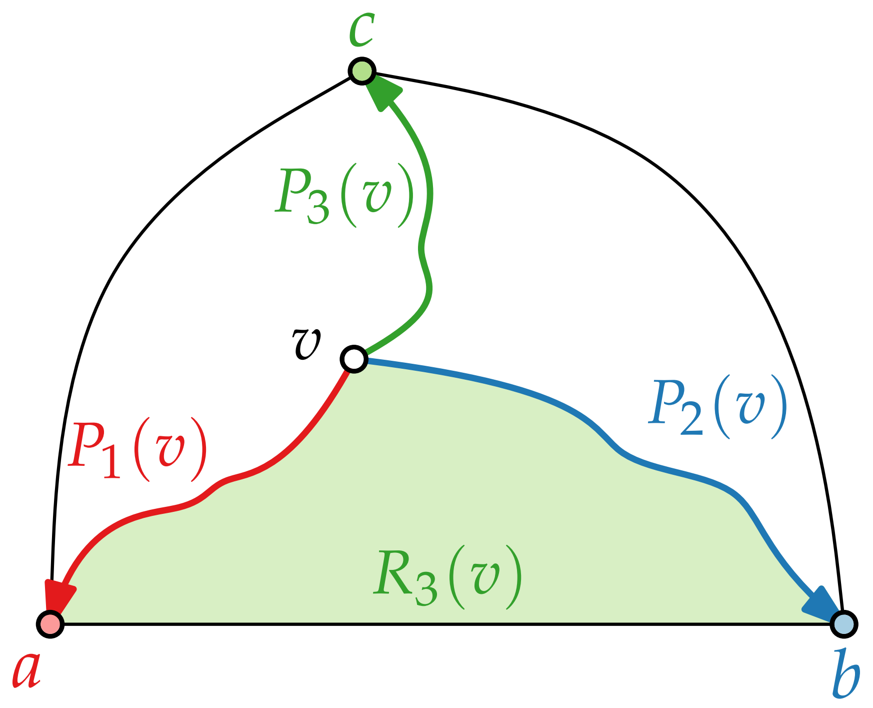
Linear Time Computation

- Compute Canonical Order
- Compute Schnyder Realizer
- Goal: $v_i = |V(R_i(v))| - |P_{i-1}(v)|$
- With traversal of each T_i, compute:



Linear Time Computation

- Compute Canonical Order
- Compute Schnyder Realizer
- Goal: $v_i = \boxed{|V(R_i(v))| - |P_{i-1}(v)|}$
- With traversal of each T_i, compute:
  - The number of vertices in $P_i(v)$



Linear Time Computation

- Compute Canonical Order
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- Goal: $v_i = |V(R_i(v))| - |P_{i-1}(v)|$
- With traversal of each T_i, compute:
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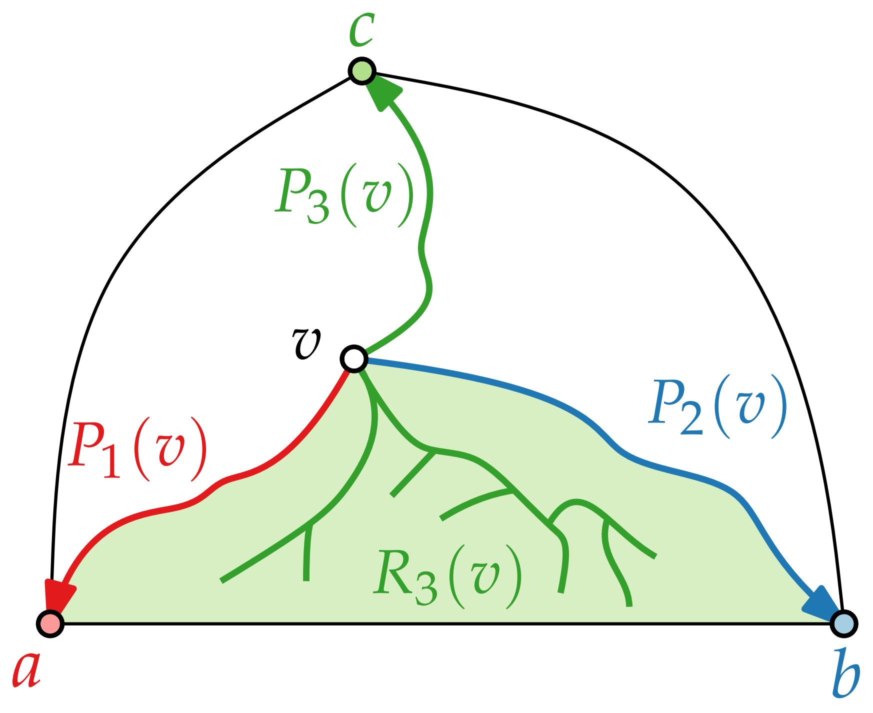
  - The number of vertices in the subtree $T_i(v)$ of T_i rooted at v



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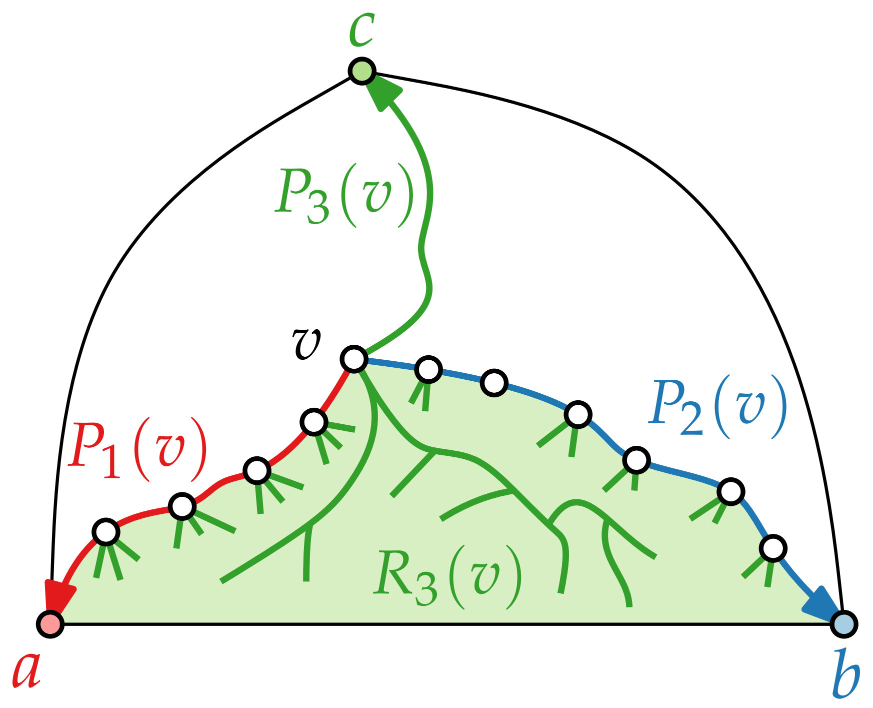

Linear Time Computation

- Compute Canonical Order
- Compute Schnyder Realizer
- Goal: $v_i = |V(R_i(v))| - |P_{i-1}(v)|$
- With traversal of each T_i, compute:
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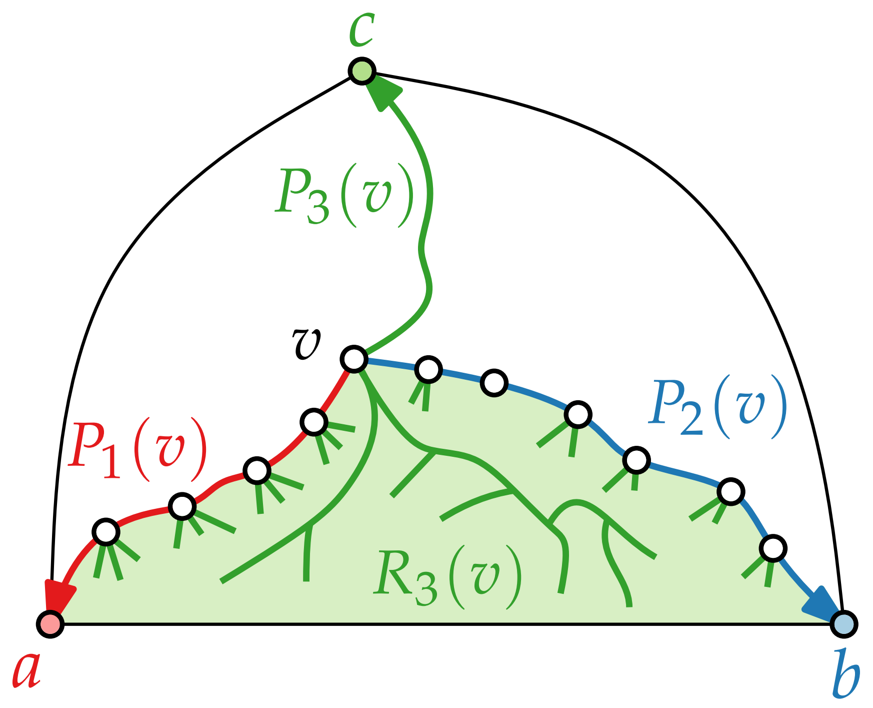
  - The number of vertices in the subtree $T_i(v)$ of T_i rooted at v
- $|V(R_i(v))| =$



Linear Time Computation

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- Goal: $v_i = \boxed{|V(R_i(v))| - |P_{i-1}(v)|}$
- With traversal of each T_i, compute:
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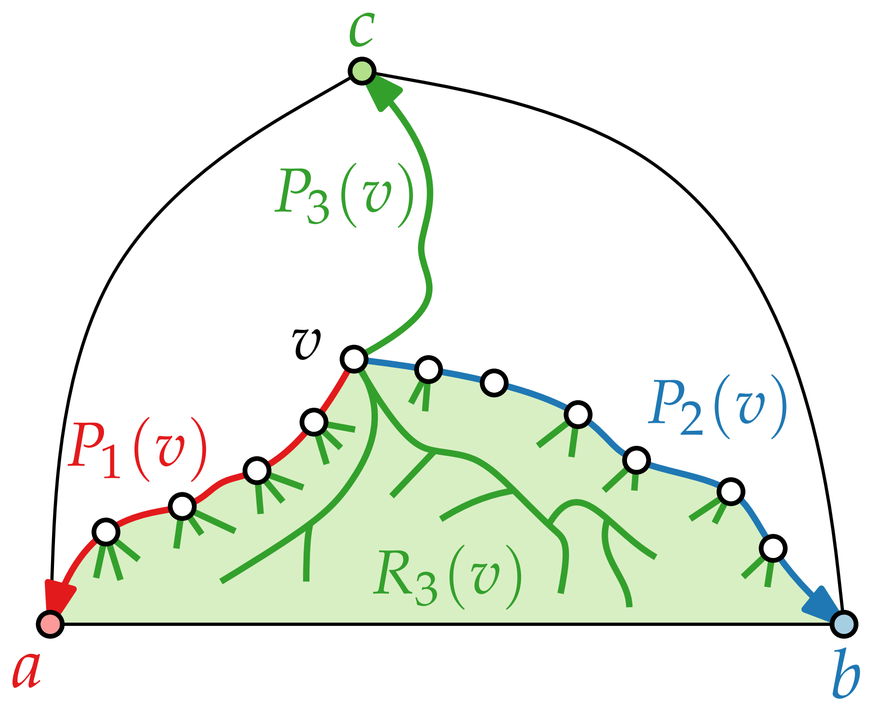
  - The number of vertices in the subtree $T_i(v)$ of T_i rooted at v
- $|V(R_i(v))| = \sum_{u \in P_{i+1}(v)} |T_i(u)|$



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- With traversal of each T_i, compute:
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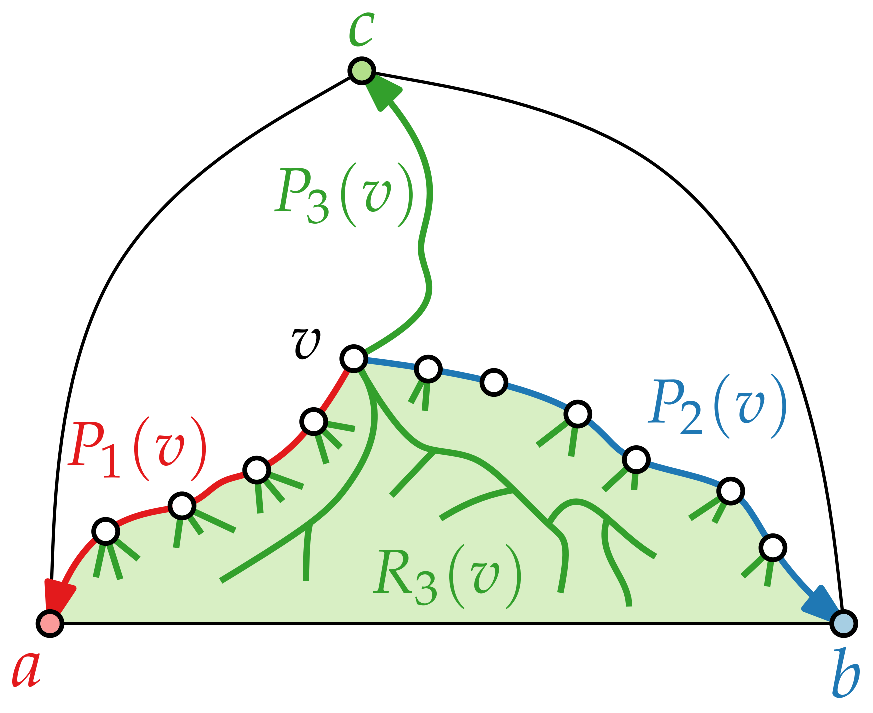
  - The number of vertices in the subtree $T_i(v)$ of T_i rooted at v
- $|V(R_i(v))| = \sum_{u \in P_{i+1}(v)} |T_i(u)| + \sum_{u \in P_{i-1}(v)} |T_i(u)|$



Linear Time Computation

- Compute Canonical Order
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- Goal: $v_i = \boxed{|V(R_i(v))| - |P_{i-1}(v)|}$
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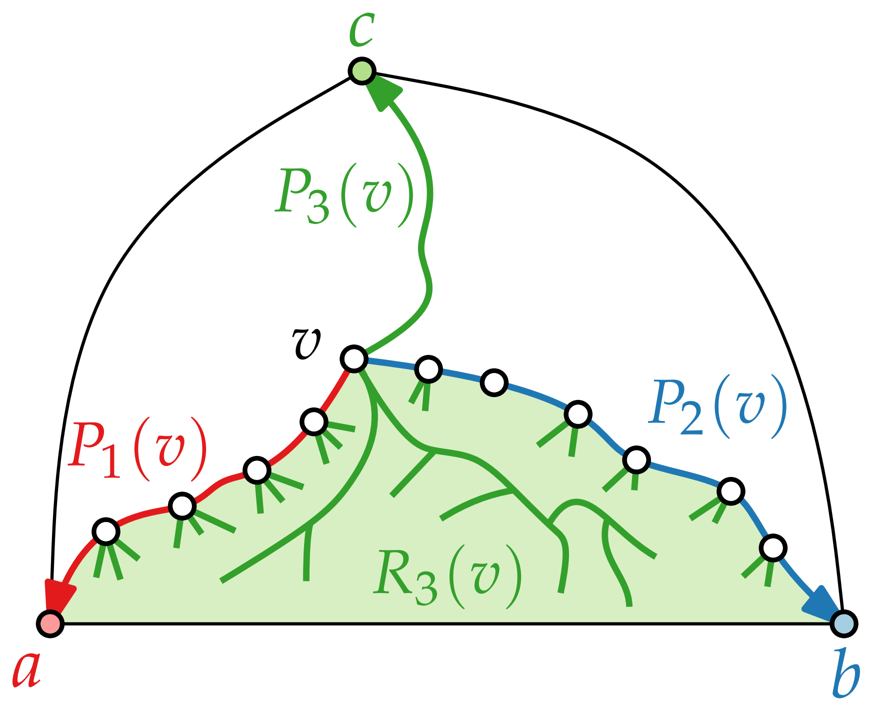
- $|V(R_i(v))| = \sum_{u \in P_{i+1}(v)} |T_i(u)| + \sum_{u \in P_{i-1}(v)} |T_i(u)| - T_i(v)$



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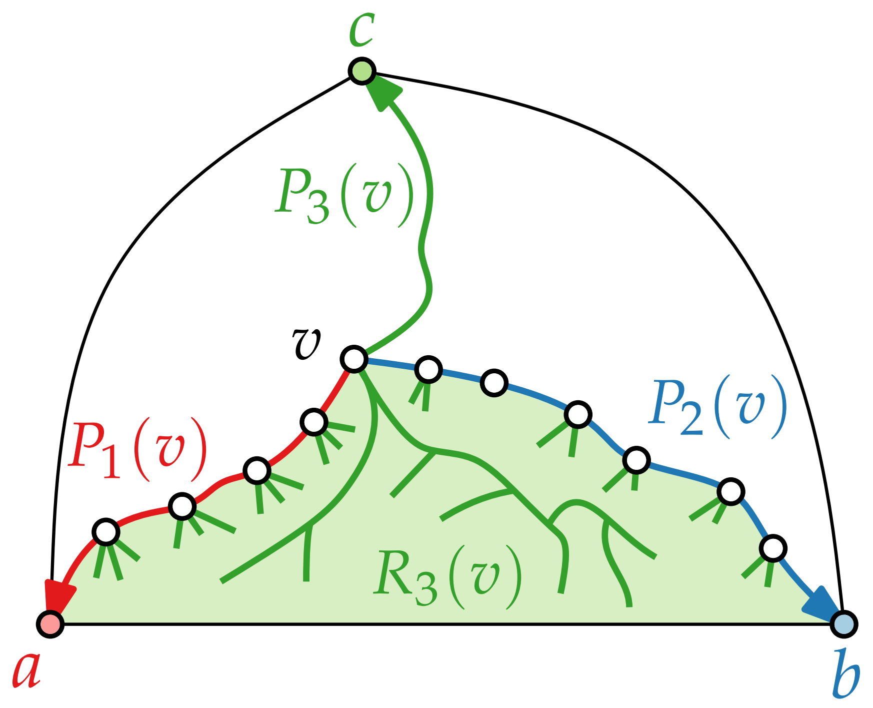
- Compute these sums in six tree traversals



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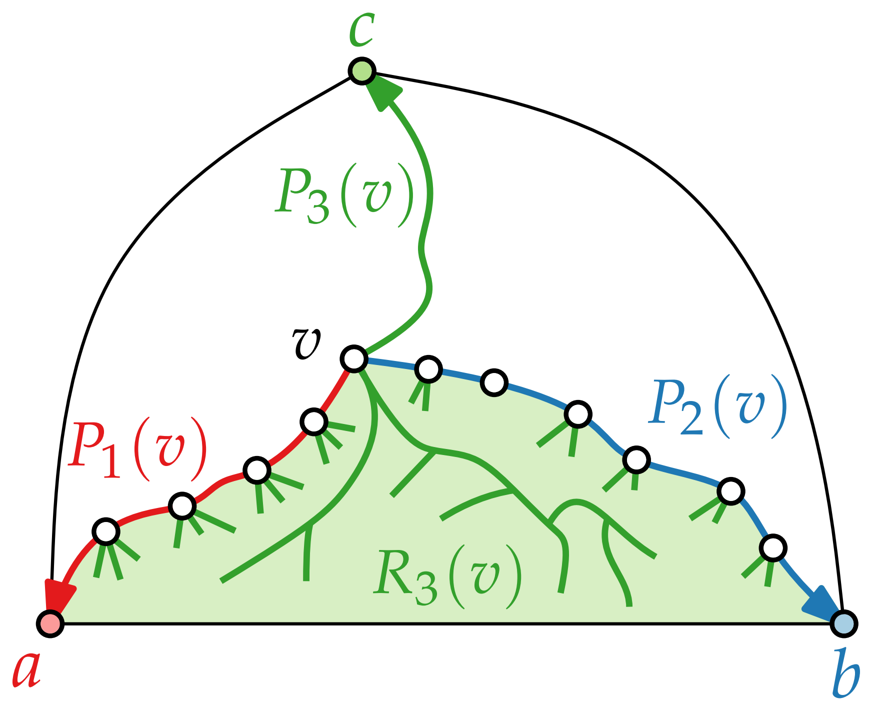

- $|V(R_i(v))| = \sum_{u \in P_{i+1}(v)} |T_i(u)| + \sum_{u \in P_{i-1}(v)} |T_i(u)| - T_i(v)$
- Compute these sums in six tree traversals

Theorem. [Schnyder '90]
Every n-vertex planar graph has a planar straight-line drawing of size $(n-2) \times (n-2)$. Such a drawing can be computed in $O(n)$ time.