A tent space approach to mild solutions of Navier–Stokes equations with rough initial data

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Contents

Notation

$x \lesssim y$	There exists $c > 0$ such that $x \leq cy$
$A \subset B$	A is a subset of B (not necessarily proper subset)
$\mathbb{1}_E$	characteristic function of the set E
$\|A\|$	Lebesgue measure of a set A
$B(x,r)$	ball of radius r centred at x
dx	Lebesgue measure
$\mathrm{supp}(u)$	support of the function u
p'	$p/(p-1)$ whenever $p \in (1,\infty)$, $1' = \infty$, and $\infty' = 1$.
$\partial_i = \dfrac{\partial}{\partial x_i}$	the ith partial derivative
$\Delta = \partial_1^2 + \cdots + \partial_n^2$	Laplacian on $\mathbb{R}^n$
$\mathcal{R}_j = \dfrac{\partial_j}{\sqrt{\Delta}}$	Riesz transform
$\mathcal{L}(A,B)$	bounded operators from A into B, $\mathcal{L}(A) = \mathcal{L}(A,A)$
$\rho(T)$	resolvent set of T
$\sigma(T)$	spectrum of T
$S_{\mu+}$	the positive sector of angle μ, $\{z \in \mathbb{C} \setminus \{0\} : \|\mathrm{Arg}(z)\| < \mu\}$
D	the diagonal $\{(x,x) : x \in R^n\} \subset \mathbb{R}^{2n}$
$\mathbb{R}_+$	nonnegative real numbers $\{x \in \mathbb{R} : x \geq 0\}$
$\mathbb{K}$	either $\mathbb{R}$ or $\mathbb{C}$
$\|x\|$	$\sqrt{x_1^2 + \cdots + x_n^2}$ where $x = (x_1, \dots x_n) \in \mathbb{R}^n$
$C_c^k(\Omega)$	space of C^k functions with compact support in Ω
ℓ^p	space of $\mathbb{C}$ valued sequences with norm $\\|x\\|_{\ell^p} := \left(\sum_{j\geq 1} \|x_j\|^p\right)^{1/p}$
$L^p(X,\mu,\mathbb{F})$	Lebesgue space of functions with values in $\mathbb{F}$
L^p_{loc}	space of functions that lie in $L^p(K)$ for every compact $K \subset \mathbb{R}^n$
$W^{k,p}$	Sobolev spaces
H^p	real Hardy spaces
BMO	functions of bounded mean oscillation
$\mathcal{S}$	Schwartz functions
$\mathcal{S}'$	tempered distributions
$\mathcal{D}'$	distributions
M	centred Hardy–Littlewood maximal function
$\tilde{M}$	uncentered Hardy–Littlewood maximal function
$\mathcal{M}_\varphi$	smooth maximal function
$\mathcal{M}_\varphi^*$	nontangential smooth maximal function
$\mathcal{M}_A$	maximal regularity operator of A

Chapter 1

Introduction

In this thesis we examine the Navier–Stokes equations in n spatial dimensions in a rather restricted setting: we consider a viscous, homogeneous, incompressible fluid that fills the entire space and feels no external forces. The equations that describe the evolution of the velocity $\vec{u}(x,t)$ of such a fluid at time t and position x are

$$\begin{cases} \partial_t \vec{u} = \Delta \vec{u} - \mathbb{P}\nabla(\vec{u} \otimes \vec{u}), \\ \nabla \cdot \vec{u} = 0, \end{cases} \tag{1.1}$$

where the Leray projector $\mathbb{P}$ is defined by $\mathbb{P}\vec{f} := \vec{f} - \nabla \frac{1}{\Delta}(\nabla \cdot \vec{f})$.

The focus of this thesis is the existence of global mild solutions for the Navier–Stokes equations. We are interested in the Cauchy problem for the Navier–Stokes equations; given some initial velocity field $\vec{u}_0(x)$ we seek a solution $\vec{u}(x,t)$ that satisfies (1.1) in an appropriate mild sense with initial value $\vec{u}(x,0) = \vec{u}_0(x)$. That is, we look for a solution to the corresponding integral equation

$$\begin{cases} \vec{u} = e^{t\Delta}\vec{u}_0 - \displaystyle\int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla \cdot (\vec{u} \otimes \vec{u}) ds, \\ \nabla \cdot \vec{u}_0 = 0. \end{cases} \tag{1.2}$$

We solve (1.2) by looking for a fixed point of the transform

$$\vec{v} \mapsto e^{t\Delta}\vec{u}_0 - \int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla \cdot (\vec{v} \otimes \vec{v}) ds. \tag{1.3}$$

The Picard contraction principle then gives a solution of (1.3). Given a Banach space of initial data $\mathcal{E}$ and a space of solutions E such that $(x,t) \mapsto e^{t\Delta}u_0(x) \in E$ for all $u_0 \in \mathcal{E}$ the existence of solutions in E reduces to asking whether

$$B(\vec{u}, \vec{v}) := \int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla \cdot (\vec{u} \otimes \vec{v}) ds \tag{1.4}$$

defines a bounded bilinear operator from $E^n \times E^n$ to E^n.

In 2001 Koch and Tataru [KT01] proved well-posedness of Navier–Stokes equations in BMO^{-1} (the space of derivatives of functions with bounded mean oscillation) by showing boundedness of the bilinear transform (1.4) in an appropriate Banach space. The goal of this thesis is to establish their global existence result for data in BMO^{-1}. We show that for every sufficiently small $\vec{u}_0 \in (\mathrm{BMO}^{-1})^n$ there exists a mild solution $\vec{u}$ to the Navier–Stokes equations that lies in a certain subspace of the tent space $T^{\infty,2}$.

Theorem 1.1. *There exits $\epsilon > 0$ such that for all $\vec{u}_0 \in (\mathrm{BMO}^{-1})^n$ with $\|\vec{u}_0\|_{\mathrm{BMO}^{-1}} < \epsilon$ and $\nabla \cdot \vec{u}_0 = 0$ there exists a mild solution $\vec{u}$ of the Navier–Stokes equations on $\mathbb{R}^{n+1}_+ := \mathbb{R}^n \times \mathbb{R}_+$:*

$$\vec{u} = e^{t\Delta}\vec{u}_0 - \int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla(\vec{u} \otimes \vec{u})ds, \tag{1.5}$$

so that

(i) $\sup_{t>0}\sup_{x\in\mathbb{R}^n} |\sqrt{t}\vec{u}(x,t)| < \infty$,

(ii) $\sup_{t>0}\sup_{x\in\mathbb{R}^n} \left(t^{-n/2}\int_0^t\int_{B(x,\sqrt{t})} |e^{s\Delta}\vec{u}(x,s)|^2 ds\, dx\right)^{\frac{1}{2}} < \infty$,

(iii) $\lim_{t\to 0}\vec{u}(x,t) = \vec{u}_0(x)$ *for a.e.* $x \in \mathbb{R}^n$.

Chapter 2 begins by establishing the necessary background theory. The focus is on finding sufficient conditions to establish the boundedness, in several function spaces, of singular integral operators of the form

$$Tf(x) := \int_{\mathbb{R}^n} k(x,y)f(y)d\mu(y) \qquad \forall f \in \mathcal{S},\ x \notin \operatorname{supp}(f).$$

Here, $k : \mathbb{R}^n \times \mathbb{R}^n \setminus D \to \mathbb{C}$ is called the kernel of the operator T and is singular on the diagonal $D = \{(x,x) : x \in \mathbb{R}^n\}$.

The space of functions of bounded mean oscillation (BMO) is introduced in chapter 3. We begin by introducing the Hardy spaces and eventually characterise BMO as the dual of the Hardy space H^1; along the way we present several equivalent characterisations of both H^1 and BMO. The chapter ends by introducing tent spaces in parabolic scaling and pointing out connections to H^1 and BMO.

In chapter 4 we discuss the Cauchy problem, an abstract formulation of initial value problems. The aim of this chapter is to introduce the concept of maximal regularity and establish maximal regularity of the heat equation in the tent space $T^{\infty,2}$. This will allow us to treat the Navier–Stokes equations as a nonlinear perturbation of the heat equation on that space. This program is finally carried out in chapter 5. The proof of Theorem 1.1 we present is based on recent work by Auscher and Frey [AF13]. It differs from the original proof by Koch and Tataru [KT01] in that it relies more heavily on tent space theory and does not use the self-adjointness of the Laplacian. The hope is that a better understanding of this result will allow treatment of related problems such as stochastic Navier–Stokes equations.

Chapter 2

Singular integral operators

2.1 Introduction

This chapter is on the boundedness of singular integral operators, defined by

$$Tf(x) = \int_{\mathbb{R}^n} k(x,y)f(y)d\mu(y) \qquad \forall x \in \mathbb{R}^n \tag{2.1}$$

and all $f \in \mathcal{S}$. The kernel $k : \mathbb{R}^n \times \mathbb{R}^n \setminus D \to \mathbb{C}$ is singular on the diagonal $D = \{(x,x) : x \in \mathbb{R}^n\}$. This singularity means that (2.1) is a priori not well defined. We could treat k in a distributional sense, or as a principal value integral,

$$Tf(x) = \lim_{\epsilon \to 0} \int_{|x-y|>\epsilon} k(x,y)f(y)d\mu(y) \qquad \forall x \in \mathbb{R}^n, f \in \mathcal{S}.$$

An alternative treatment of singular operators begins by defining

$$Tf(x) = \int_{\mathbb{R}^n} k(x,y)f(y)d\mu(y) \qquad \forall f \in \mathcal{S}.$$

for $x \notin \mathrm{supp}(f)$. Note that this is is now well defined since we avoid integrating along the diagonal. In section 2.7 this is the approach that we take.

The operators we use in this thesis uses are generally nicely behaved in that they are L^2 bounded and have convolution kernels associated to them. This mitigates questions on the well definedness of (2.1) since we are in the realm of Fourier multipliers. Nevertheless, this chapter attempts to give a feel for some of the general theory.

2.2 Spatial assumptions

This chapter focuses on the boundedness of singular integral operators in L^p spaces. The framework in which we present these results follows that of [Ste93]. We work in the general framework of $\mathbb{R}^n$ equipped with a general metric and doubling measure. We will also make some extra technical assumptions, some of which could be removed, but for our purposes the return from doing so would be minimal. In fact, from chapter 3 onwards our spatial setting will be $\mathbb{R}^n$ with the usual Euclidean metric and Lebesgue measure.

Throughout this chapter we will consider the space $\mathbb{R}^n$ equipped with a metric d. We assume that we are given a nonnegative Borel measure μ with the property that $\mu(\mathbb{R}^n) > 0$. We define the ball of radius r centred at $x \in \mathbb{R}^n$ in the usual way

$$B(x, r) = \{y \in \mathbb{R}^n : d(x, y) < r\}$$

and write $V(x, r) = \mu(B(x, r))$ for the measure of this set. We will make two key assumptions about this measure. We assume that there exists $c_1, c_2 > 1$ so that for all $x, y \in \mathbb{R}^n$, $\delta > 0$

$$B(x, \delta) \cap B(y, \delta) \neq \emptyset \quad \text{implies} \quad B(y, \delta) \subset B(x, c_1\delta), \tag{2.2}$$

$$V(x, c_1\delta) \leq c_2 V(x, \delta). \tag{2.3}$$

Statement (2.2) gives us the engulfing property which is crucial in Vitali-type covering lemmas we prove later, while (2.3) captures the fact that μ is a "doubling" measure.

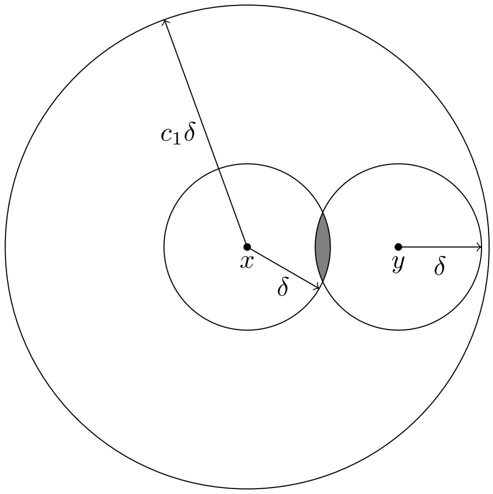


Figure 2.1: The engulfing property.

In addition to these properties it will be convenient to make two further assumptions about our space. These assumptions are technical and in many cases can be removed. For our purposes they will be useful. We assume,

$$\bigcap_{\delta>0} \overline{B}(x, \delta) = \{x\} \quad \text{and} \quad \bigcup_{\delta>0} B(x, \delta) = \mathbb{R}^n, \tag{2.4}$$

$$\begin{array}{l}\text{For each open set } U \subset \mathbb{R}^n \text{ and each } \delta > 0 \text{ the function} \\ x \mapsto \mu(\{B(x, \delta) \cap U\}) \text{ is continuous.}\end{array} \tag{2.5}$$

There are several simple consequences of these assumptions.

Proposition 2.1. $\mu(B) > 0$ *for any ball* B.

Proof. Suppose for a contradiction that there exists $\delta > 0$ and $x \in \mathbb{R}^n$ such that $\mu(B(x, \delta)) = 0$. Then by the additional assumption (2.4), subadditivity of the measure μ and the doubling property (2.3) we observe

$$\mu(\mathbb{R}^n) = \mu\left(\bigcup_{n\geq 0} B(x, c_1^n\delta)\right) \leq \sum_{n\geq 0} V(x, c_1^n\delta) \leq \sum_{n\geq 0} c_2^n V(x,\delta) = 0.$$

However, this contradicts the assumption on our measure that $\mu(\mathbb{R}^n) > 0$. $\square$

Definition 2.2. Suppose $\delta > 0$ and the function $f : \mathbb{R}^n \to \mathbb{R}$ is locally integrable. The mean value operator is defined by

$$A_\delta f(x) = \frac{1}{\mu(B(x,\delta))}\int_{B(x,\delta)} f(y)d\mu(y) \qquad \forall x \in \mathbb{R}^n.$$

Proposition 2.3. *For any locally integrable function f, and any $\delta > 0$, the function $x \mapsto A_\delta f(x)$ is continuous.*

Proof. Let $\epsilon, \delta > 0$ and fix $x, y \in \mathbb{R}^n$. Then,

$$\begin{aligned}
&|A_\delta f(y) - A_\delta f(x)| \\
&= \left|\frac{1}{V(y,\delta)}\int_{B(y,\delta)} f(z)d\mu(z) - \frac{1}{V(x,\delta)}\int_{B(x,\delta)} f(z)d\mu(z)\right| \\
&= \frac{1}{|V(x,\delta)V(y,\delta)|}\left|V(x,\delta)\int_{B(y,\delta)} f(z)d\mu(z) - V(y,\delta)\int_{B(x,\delta)} f(z)d\mu(z)\right| \\
&\leq \frac{1}{|V(x,\delta)V(y,\delta)|}\Bigg(\left|(V(x,\delta) - V(y,\delta))\int_{B(x,\delta)} f(z)d\mu(z)\right| \\
&\qquad + \left|V(x,\delta)\int_{B(x,\delta)\setminus B(y,\delta)} f(z)d\mu(z)\right| \\
&\qquad + \left|V(x,\delta)\int_{B(y,\delta)\setminus B(x,\delta)} f(z)d\mu(z)\right|\Bigg).
\end{aligned}$$

We argue that by choosing $d(x, y)$ sufficiently small, the quantity above is less that ϵ. We examine these three terms separately.

Indeed, by (2.5), since $|V(x,\delta) - V(y,\delta)| \to 0$ as $d(x,y) \to 0$, there exists γ_1 such that $d(x,y) < \gamma_1$ implies

$$\left|\int_{B(x,\delta)} f(z)d\mu(z)\right| |V(y,\delta) - V(x,\delta)| < \frac{\epsilon}{3}V(y,\delta)V(x,\delta).$$

Of course if $\displaystyle\int_{B(x,\delta)} f(z)d\mu(z) = 0$ then this statement is vacuous.

By (2.5) we know that $\mu(B(x,\delta)\setminus B(y,\delta))$ converges to 0 as $d(x,y)$ tends to 0. Since f is locally integrable $\int_{B(x,\delta)\setminus B(y,\delta)} f(z)d\mu(z) \to 0$ as $d(x,y) \to 0$. We thus pick γ_2 such that $d(x,y) < \gamma_2$ implies

$$\int_{B(x,\delta)\setminus B(y,\delta)} f(z)d\mu(z) \leq \frac{\epsilon V(x,\delta)}{3c_2} \leq \frac{\epsilon V(y, c_1\delta)}{3c_2} \leq \frac{\epsilon V(y,\delta)}{3}$$

by properties (2.2) and (2.3). Likewise, there exists γ_3 so that $d(x,y) < \gamma_3$ implies

$$\int_{B(y,\delta)\setminus B(x,\delta)} f(z)d\mu(z) \leq \frac{\epsilon V(x,\delta)}{3}.$$

Choosing $\gamma := \min\{\gamma_1, \gamma_2, \gamma_3\}$ we have that $d(x,y) < \gamma$ implies $|A_\delta f(y) - A_\delta f(x)| < \epsilon$. $\square$

2.3 Maximal functions

In this section we introduce both the standard and uncentered Hardy–Littlewood maximal functions. The average function from the previous section is deeply connected with these maximal functions.

Definition 2.4 (Hardy–Littlewood maximal function)**.** Let $f \in L^1_{\mathrm{loc}}(d\mu)$. Define the maximal function

$$Mf(x) := \sup_{\delta>0} \frac{1}{\mu(B(x,\delta))} \int_{B(x,\delta)} |f(y)| d\mu(y) = \sup_{\delta>0} A_\delta(|f|)(x) \qquad \forall x \in \mathbb{R}^n.$$

Analysis of the average function $A_\delta(f)$, such as the question of whether $(A_\delta f) \to f$ as $\delta \to 0$, will be dealt with by first studying this maximal function.

We now prove a finite version of the Vitali covering lemma. See for instance [SS05, pp.102–104]. This lemma will allow us to obtain some fundamental results about the average and maximal operators.

Lemma 2.5 (Vitali)**.** *Let $E \subset \mathbb{R}^n$ be a measurable set that is the union of a finite collection of balls $\{B_k\}_{k=1}^m$. Then there is a disjoint subcollection $B'_1, \dots, B'_{m'}$ so that*

$$\sum_{k=1}^{m'} \mu(B'_k) \geq \frac{1}{c_2} \mu(E).$$

Proof. Choose $B'_1 \in \{B_k\}_{k=1}^m$ to be a ball of maximal radius. Next choose B'_2 to be a ball of maximal radius which is disjoint to B'_1. Continue this process until it terminates. That is, construct a disjoint subcollection $B'_1, \dots, B'_{m'}$ with radii $\delta_1 \geq \dots \geq \delta_{m'}$. For each ball $B' = B'(x, \delta)$, define

$$B^* := \bigcup_\lambda B_\lambda,$$

where the union is taken over all balls B_λ meeting B' whose radius is $\delta_\lambda \leq \delta$. By (2.2) and then (2.3),

$$\mu(B^*) = \mu\left(\bigcup_\lambda B_\lambda\right) \leq \mu\left(\bigcup_\lambda B(x, c_1\delta_\lambda)\right) = \mu(B(x, c_1\delta)) \leq c_2\mu(B'). \tag{2.6}$$

Notice that B^*_1 is the union of all balls meeting B'_1 whose radius is at most δ_1. Likewise B^*_2 is the union of all balls meeting B'_2 with radius $\leq \delta_2$ and so on. It follows that $\bigcup_{k=1}^{m'} B^*_k$ contains the union of the initial collection of balls. By subadditivity,

$$\sum_{k=1}^{m'} \mu(B^*_k) \geq \mu\left(\bigcup_{k=1}^{m'} B^*_k\right) \geq \mu\left(\bigcup_{k=1}^{m} B_k\right) = \mu(E).$$

Since (2.6) tells us $\mu(B^*_k) \leq c_2\mu(B_k)$, the lemma follows. □

This lemma is useful when establishing some rudimentary properties of the maximal function.

Theorem 2.6. *Let $f : \mathbb{R}^n \to \mathbb{R}$.*

(a) If $f \in L^p$, $1 \leq p \leq \infty$, then $Mf(x)$ is finite for a.e. $x \in \mathbb{R}^n$.

(b) If $f \in L^1$, then for every $\alpha > 0$,

$$\mu(\{x : Mf(x) > \alpha\}) \leq \frac{c_2}{\alpha} \int_{\mathbb{R}^n} |f(y)| d\mu(y).$$

(c) If $f \in L^p$, $1 < p \leq \infty$, then $Mf \in L^p$ and

$$\|Mf\|_p \lesssim_{c_2,p} \|f\|_p \,.$$

Proof. We first consider the uncentred maximal function $\tilde{M}f$ defined by

$$\tilde{M}f(x) := \sup_{B \ni x} \frac{1}{\mu(B)} \int_B |f(y)| d\mu(y),$$

where the supremum is taken over all balls B containing x. Note the pointwise inequality $Mf(x) \leq \tilde{M}f(x)$. We begin by showing (b).

(b) Let $E_\alpha = \{x : \tilde{M}f(x) > \alpha\}$ and let $E \subset E_\alpha$ be any compact subset. For each $x \in E$ there exists $B_x \ni x$ such that

$$\int_{B_x} |f(y)| d\mu(y) > \alpha \mu(B_x). \tag{2.7}$$

The collection $\{B_x\}_{x \in E}$ thus covers E. By compactness of E we can select a finite subcover. Lemma 2.5 allows us to select a disjoint subcollection $B_1, \ldots, B_m$ with the property

$$\mu(E) \leq c_2 \sum_{k=1}^{m} \mu(B_k).$$

For any ball $B_k \subset E$ property (2.7) implies

$$\int_{B_k} |f(y)| d\mu(y) > \alpha \mu(B_k).$$

It follows that

$$\mu(E) \leq \frac{c_2}{\alpha} \sum_{k=1}^{m} \int_{B_k} |f(y)| d\mu(y) \leq \frac{c_2}{\alpha} \int_{\mathbb{R}^n} |f(y)| d\mu(y),$$

and conclusion (b) is proved.

Both (a) and (c) follow from (b). First we define

$$f_1(x) = \begin{cases} f(x) & |f(x)| > \alpha/2, \\ 0 & \text{otherwise.} \end{cases}$$

Note that

$$\begin{aligned} \tilde{M}f(x) &= \sup_{B \ni x} \frac{1}{\mu(B)} \int_B |f(y)| d\mu(y) \\ &= \sup_{B \ni x} \frac{1}{\mu(B)} \left(\int_{B \cap \{|f| > \frac{\alpha}{2}\}} |f(y)| d\mu(y) + \int_{B \cap \{|f| \leq \frac{\alpha}{2}\}} |f(y)| d\mu(y) \right) \\ &\leq \sup_{B \ni x} \frac{1}{\mu(B)} \left(\int_B |f_1(y)| d\mu(y) + \frac{\alpha}{2} \int_B d\mu(y) \right) \\ &= \tilde{M}f_1(x) + \frac{\alpha}{2}. \end{aligned}$$

Thus, $\{\tilde{M}(f) > \alpha\} \subset \{\tilde{M}(f_1) > \alpha/2\}$ and by (b)

$$\mu(\{x : \tilde{M}f(x) > \alpha\}) \leq \frac{c}{\alpha} \int_{\{x:|f(x)|>\frac{\alpha}{2}\}} |f(y)|d\mu(y).$$

However,

$$\int_{\mathbb{R}^n} |\tilde{M}f(y)|^p d\mu(y) = p \int_0^\infty \mu(\{\tilde{M}f > \alpha\})\alpha^{p-1}d\alpha,$$

which, by the above, is dominated by

$$cp \int_{\mathbb{R}^n} \left(\int_0^{2|f(y)|} \alpha^{p-2} d\alpha \right) |f(y)|d\mu(y) = \frac{cp}{p-1} 2^{p-1} \int_{\mathbb{R}^n} |f(y)|^p d\mu(y) < \infty,$$

proving (c). The case when $p = \infty$ is trivial, $\|Mf\|_\infty \leq |B|^{-1} \|f\|_\infty |B| = \|f\|_\infty$. The fact that (b) implies (a) is clear. Given $f \in L^p$ and $\epsilon > 0$ with $p \in (1, \infty]$ choose $\alpha > (c_2 \|f\|_{L^1(d\mu)})^{\frac{1}{p}}$ such that

$$\begin{aligned}
\mu(\{x : Mf(x) > \alpha\}) &= \mu\Big(\Big\{x : \sup_{B \ni x} \frac{1}{\mu(B)} \int_B |f(y)|d\mu(y) > \alpha\Big\}\Big) \\
&\leq \mu\Big(\Big\{x : \sup_{B \ni x} \mu(B)^{1/q-1} \left(\int_B |f(y)|^p d\mu(y) \right)^{\frac{1}{p}} > \alpha\Big\}\Big) \\
&= \mu(\{x : Mf^p(x) > \alpha^p\}) \\
&\leq \frac{c_2}{\alpha^p} \|f\|_{L^1(d\mu)} < \epsilon.
\end{aligned}$$

The first inequality is an application of Hölder's inequality where $\frac{1}{p} + \frac{1}{q} = 1$, while the second inequality comes from applying (c) with $f^p \in L^1$. Clearly, Mf is finite almost everywhere. For $p = 1$ it follows immediately from (b). □

Corollary 2.7 (Lebesgue differentiation theorem)**.** *If f is locally integrable with respect to $d\mu$, then for almost every $x \in \mathbb{R}^n$,*

$$\lim_{\delta \to 0} A_\delta f(x) = f(x).$$

Proof. By our assumptions we have that $\cap_{\delta>0}\overline{B}(x, \delta) = \{x\}$ for all $x \in \mathbb{R}^n$ and $\overline{B}(x, \delta)$ are compact for all $x \in \mathbb{R}^n$, $\delta > 0$. Then, for each x, the Euclidean diameters $|B(x, \delta)|$ tend to zero as $\delta \to 0$ which implies that $(A_\delta f)(x) \to f(x)$ whenever f is continuous at x. It follows that (2.7) holds for all continuous compactly supported functions. Since these functions are dense in L^1, given $f \in L^1_{\text{loc}}$ we consider a sequence $f_n \in C_c$ such that $f_n \to f$ in L^1. Fix $B(x, \delta) \subset \mathbb{R}^n$ and write

$$\begin{aligned}
\frac{1}{\mu(B)} \int_B f(y)d\mu(y) - f(x) &= \left(\frac{1}{\mu(B)} \int_B (f(y) - f_n(y))dy \right) \\
&\quad + \left(\frac{1}{\mu(B)} \int_B f_n(y)dy - f_n(x) \right) \\
&\quad + (f_n(x) - f(x))
\end{aligned}$$

The first term we bound by the maximal function

$$\left\| \frac{1}{\mu(B)} \int_B (f(y) - f_n(y))dy \right\|_{L^1} \leq \|M(f - f_n)\|_{L^1} \lesssim \|f - f_n\|_{L^1} \to 0$$

as $n \to \infty$ by Theorem 2.6. The third term tends to zero in L^1 by our assumption on f_n. Letting $\delta \to 0$ the second term tends to zero because the f_n are continuous. The left hand side converges to $A_\delta f(x) - f(x)$. □

Proposition 2.8. *The uncentred maximal function $\tilde{M}f$ is lower semi-continuous.*

Proof. As in the previous theorem, fix $\alpha > 0$ and let $E_\alpha = \{x : \tilde{M}f(x) > \alpha\}$. Provided E_α is non-empty, choose $x \in E_\alpha$ so that there exists a ball $B \ni x$ such that

$$\frac{1}{\mu(B)} \int_B |f(y)| d\mu(y) > \alpha.$$

For any $w \in B$ we have

$$\tilde{M}f(w) \geq \frac{1}{\mu(B)} \int_B |f(y)| d\mu(y) > \alpha$$

and so $w \in E_\alpha$ also. In particular, $B \subset E_\alpha$ and hence E_α is open. □

2.4 Calderón–Zygmund decomposition

The final step before establishing boundedness of singular integral operators is the development of some basic Calderón–Zygmund theory. The basic idea of this section is to decompose $f \in L^1(d\mu)$ as the sum of a good part g that is bounded and a bad part b that has mean zero and is supported in a set that is small in measure.

In order to do this we will first introduce another spatial result, namely a variant of the Whitney covering lemma. The idea of the Whitney covering lemma is to decompose a set into disjoint sets whose sizes are proportional to their distances to the boundary. Typically the decomposition is into cubes, see [Gra04, pp.A34] for instance, but here we present a decomposition into balls.

Lemma 2.9. *Given F, a closed non-empty set, there exists $c^{**} > c^* > 1$ and a collection of balls $\{B_k\}_{k=1}^\infty$ so that*

(a) The B_k are pairwise disjoint.

(b) $\displaystyle\bigcup_{k=1}^\infty B_k^* = O = F^c.$

(c) $B_k^{**} \cap F \neq \emptyset$ for each $k \in \mathbb{N}$.

where $B(x,\delta)^* = B(x, c^*\delta)$, $B(x,\delta)^{**} = B(x, c^{**}\delta)$.

Remark. From the above collection $\{B_k\}_{k=1}^\infty$ we can construct a collection $\{Q_k\}_{k=1}^\infty$ so that the Q_k are disjoint, $B_k \subset Q_k \subset B_k^*$ for all $k \in \mathbb{N}$ and $\bigcup_{k=1}^\infty Q_k = O$. For example, define

$$Q_k = B_k^* \cap \left(\bigcup_{j<k} Q_j\right)^c \cap \left(\bigcup_{j>k} B_j\right)^c, \quad k = 1, 2, \ldots$$

These Q_k then satisfy the requirements for the usual Whitney covering lemma.

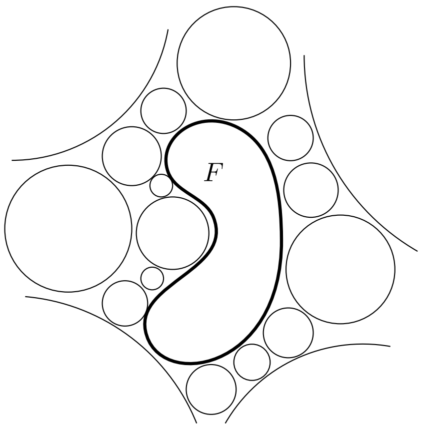


Figure 2.2: A Whitney style covering of $O = F^c$.

Proof. Fix $0 < \varepsilon < 1/2$ to be determined later and consider the covering

$$\{B(x, \varepsilon\delta(x))\}_{x\in O}$$

of O where $\delta(x) = d(x, F)$ is the distance from x to F. Consider the collection of all pairwise disjoint subcollections

$$\mathcal{C} = \{\{B(x, \varepsilon\delta(x)) : x \in O'\} : O' \subset O\}.$$

where the O' are chosen such each collection is pairwise disjoint. By Zorn's lemma, with the usual partial order '$\subset$', we have a maximal disjoint collection $\{B_k\}_{k=1}^{\infty}$. This set is countable since for every ball $B(x, \varepsilon\delta(x))$ we find a rational point $q \in \mathbb{Q}^n$ sufficiently close to x so that it is contained in the ball and thus provides an index for it. By construction, (a) holds.

Set $c^* = \varepsilon/2$, $c^{**} = 2\varepsilon$ such that

$$B_k^* = B(x_k, \delta(x_k)/2), \qquad B_k^{**} = B(x_k, 2\delta(x_k)).$$

Note that (c) holds by construction since $2\delta(x_k) = 2d(x_k, F) > d(x_k, F)$. It suffices to verify (b). Since $B_k^* = B(x_k, \delta(x_k)/2) \subset O$, it is immediately clear that $\cup_{k=1}^{\infty} B_k^* \subset O$. We want to show that by maximality, the collection $\{B_k\}_{k=1}^{\infty}$ covers O. Let $x \in O$ and notice that

$$B(x, \varepsilon\delta(x)) \cap B_k \neq \emptyset \quad \text{for some } k \in \mathbb{N}$$

otherwise we could add $B(x, \varepsilon\delta(x))$ to our (already maximal) collection.

Claim: $\delta(x_k) \geq \delta(x)/c_1$.
It follows from the claim, since $c_1\varepsilon\delta(x_k) > \varepsilon\delta(x)$, that

$$B(x, c_1\varepsilon\delta(x_k)) \cap B(x_k, \varepsilon\delta(x_k)) \neq \emptyset,$$

implying, since $c_1 > 1$,

$$B(x, c_1\varepsilon\delta(x_k)) \cap B(x_k, c_1\varepsilon\delta(x_k)) \neq \emptyset.$$

By the engulfing property,

$$B(x, \varepsilon c_1\delta(x_k)) \subset B(x_k, c_1^2\varepsilon\delta(x_k)) \implies x \in B(x_k, c_1^2\varepsilon\delta(x_k)).$$

We pick $\varepsilon = 1/c_1^2$ (which is less than $1/c_1$), so that

$$c^* = 1/(2c_1^2), \quad c^{**} = 2/c_1^2.$$

In particular,

$$x \in B(x_k, \delta(x_k)/2) = B_k^*. \qquad \square$$

Proof of Claim. Assume for a contradiction that $\delta(x_k) < \delta(x)/c_1$. Take $\varepsilon < 1/c_1$ such that

$$B(x_k, \delta(x_k)/c_1) \cap B(x, \delta(x)/c_1) \neq \emptyset.$$

Since $c_1 > 1$ and $\delta(x_k) < \delta(x)$,

$$B(x_k, \delta(x)/c_1) \cap B(x, \delta(x)/c_1) \neq \emptyset.$$

By the engulfing property (2.2),

$$B(x_k, \delta(x)/c_1) \subset B(x, \delta(x)).$$

Fix $\varepsilon < \delta(x)/c_1 - \delta(x_k)$ so that

$$\delta(x_k) + \varepsilon < \delta(x)/c_1$$

by the assumption. It follows that

$$B(x_k, \delta(x_k) + \varepsilon) \subset B(x, \delta(x)).$$

where $\varepsilon > 0$. This is a contradiction since $B(x, \delta(x)) \subset O$, while $B(x_k, \delta(x_k)+\varepsilon)$ meets O^c. $\square$

We now present the Calderón–Zygmund decomposition theorem.

Theorem 2.10. *Given $f \in L^1(d\mu)$ and $\alpha > 0$ with $\alpha > \frac{1}{\mu(\mathbb{R}^n)} \|f\|_{L^1}$, there exists a decomposition $f = g + b$ with $b = \sum_{k=1}^{\infty} b_k$ and a sequence of balls $\{B_k^*\}_{k=}^{\infty}$, so that*

(a) $|g(x)| \leq c\alpha$ for a.e. x,

(b) supp$(b_k) \subset B_k^*$,

$$\|b_k(x)\|_{L^1} \leq c\alpha\mu(B_k^*), \quad \text{and} \quad \int_{B_k^*} b_k(x) d\mu(x) = 0,$$

(c) $\sum_{k=1}^{\infty} \mu(B_k^*) \leq \frac{c}{\alpha} \|f\|_{L^1}$,

where c depends only on μ, and n.

Proof. Let $E_\alpha = \{x : \tilde{M}f(x) > \alpha\}$ where $\tilde{M}$ is the uncentred maximal function. $\tilde{M}f$ is lower semi-continuous by Proposition 2.8 so this set is open.

By Lemma 2.9 and the remark that follows it we have a collection of balls $\{B_k\}_{k=1}^{\infty}$, $\{B_k^*\}_{k=1}^{\infty}$, and cubes $\{Q_k\}_{k=1}^{\infty}$, where

$$B_k \subset Q_k \subset B_k^*, \quad \text{and } \bigcup_k Q_k = E_\alpha, \quad Q_k \text{ mutually disjoint.}$$

It follows that

$$\sum_{k=1}^{\infty} \mu(B_k) \leq \mu(E_\alpha).$$

We now define our 'good' function,

$$g(x) = \begin{cases} \frac{1}{\mu(Q_k)} \int_{Q_k} f(y) d\mu(y) & x \in Q_k \\ f(x) & x \notin E_\alpha \end{cases},$$

so that $f = g(x) + \sum_{k=1}^{\infty} b_k(x)$ where

$$b_k(x) = \mathbb{1}_{Q_k}(x) \left(f(x) - \frac{1}{\mu(Q_k)} \int_{Q_k} f(y) d\mu(y) \right).$$

We define b_k in this way so that they have mean zero,

$$\int_{B_k^*} b_k(x) d\mu(x) = \int_{Q_k} \left(f(x) - \frac{1}{\mu(Q_k)} \int_{Q_k} f(y) d\mu(y) \right) d\mu(x) = 0.$$

By Corollary 2.7 we have $|f(x)| \leq \alpha$ for a.e. $x \in E_\alpha^c = \left(\bigcup_{k=1}^{\infty} Q_k\right)^c$. Since the b_k are supported in Q_k, $f = g$ in E_α. That is, $|g(x)| \leq \alpha$ for a.e. $x \in E_\alpha^c$. For $x \in Q_k \subset E_\alpha$, g is just the average of f over Q_k. In particular for $x \in Q_k$,

$$|g(x)| \leq \frac{1}{\mu(Q_k)} \|f\|_{L^1} \leq c_k \alpha.$$

Because there are only finitely many Q_k we have $c > 1$ such that $|g| \leq c\alpha$ for all $x \in Q_k$. It follows, $\|g\|_{L^\infty} \leq c\alpha$.

We note further

$$\|b_k\|_{L^1} = \int_{Q_k} |b_k(x)| d\mu(x) \leq 2 \int_{Q_k} |f(x)| d\mu(x) \leq c\alpha\mu(Q_k) \leq c'\alpha\mu(B_k^*)$$

by the doubling property, thus proving (b).

Again by the doubling property,

$$\sum_{k=1}^{\infty} \mu(B_k^*) \leq c''\mu(E_\alpha)$$

By the maximal theorem, $\mu(E_\alpha) \leq \frac{c''}{\alpha} \|f\|_{L^1}$ thus proving (c).

Implicit in this proof was the assumption that E_α^c was non-empty. For the case where $E_\alpha = \mathbb{R}^n$, the maximal theorem tells us

$$\mu(\mathbb{R}^n) \leq \frac{c''}{\alpha} \|f\|_{L^1} (< \infty).$$

We construct the decomposition $f = g + b_1$ where

$$g = \frac{1}{\mu(\mathbb{R}^n)} \int_{\mathbb{R}^n} f(x) d\mu(x)$$

and b_1 is supported in the ball $B_1^* = \mathbb{R}^n$. Our assumption

$$\frac{1}{\mu(\mathbb{R}^n)} \int_{\mathbb{R}^n} |f| f\mu < \alpha$$

guarantees $|g| \leq \alpha$ as required. $\square$

2.5 Weak L^p spaces

We will now take an aside to discuss $L^{p,\infty}$, the weak L^p spaces. The important Marcinkiewicz interpolation theorem, to be introduced in the following section, is one major application of these spaces. We first define the distribution function as it relates to $L^{p,\infty}$. This section concludes with a comparison between $L^{p,\infty}$ and the familiar L^p spaces.

Definition 2.11. For a measurable function $f : \mathbb{R}^n \to \mathbb{R}$ the distribution function of f is defined on $[0, \infty)$ as

$$d_f(\alpha) = \mu(\{x \in \mathbb{R}^n : |f(x)| > \alpha\}).$$

The distribution function tells us how $|f|$ is distributed, but says nothing local about f. It does however tell us enough to compute $\|f\|_{L^p}$.

Proposition 2.12. *For $f \in L^p$ with $0 < p < \infty$,*

$$\|f\|_{L^p}^p = p \int_0^\infty \alpha^{p-1} d_f(\alpha) d\alpha.$$

Proof. By Fubini's theorem,

$$\begin{aligned} p \int_0^\infty \alpha^{p-1} d_f(\alpha) d\alpha &= p \int_0^\infty \alpha^{p-1} \int_{\mathbb{R}^n} \mathbb{1}_{|f|>\alpha}(x) d\mu(x) d\alpha \\ &= \int_{\mathbb{R}^n} \int_0^{|f(x)|} p\alpha^{p-1} d\alpha \, d\mu(x) \\ &= \int_{\mathbb{R}^n} |f(x)|^p d\mu(x) = \|f\|_{L^p}^p . \end{aligned}$$

□

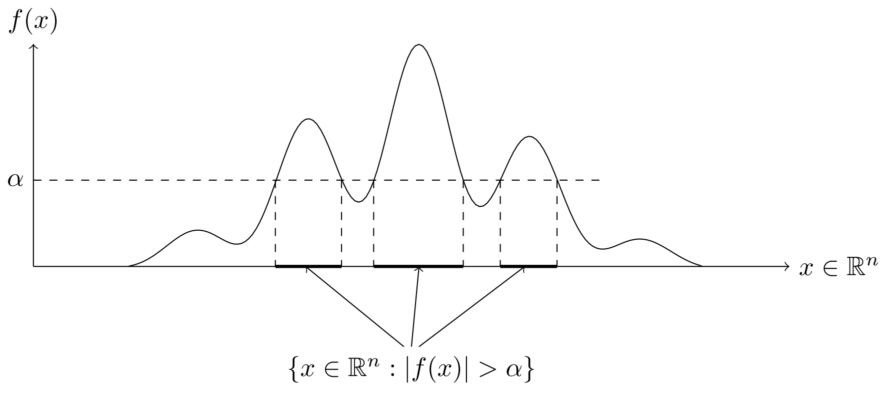


Figure 2.3: The function $d_f(\alpha)$ determines how the range of $|f|$ is distributed.

Definition 2.13. For $0 < p < \infty$ we define the weak L^p space, $L^{p,\infty}(d\mu)$, as the set of all measurable functions f such that

$$\|f\|_{L^{p,\infty}} := \inf \left\{ C > 0 : d_f(\alpha) \leq \frac{C^p}{\alpha^p} \text{ for all } \alpha > 0 \right\} \tag{2.8}$$

$$= \sup \left\{ \gamma d_f(\gamma)^{1/p} : \gamma > 0 \right\} < \infty. \tag{2.9}$$

Remark. Notice (2.8) and (2.9) are equivalent:

$$\begin{aligned}\inf\{C>0 : d_f(\alpha) \leq \frac{C^p}{\alpha^p} \text{ for all } \alpha > 0\} &= \inf\{C > 0 : \alpha d_f(\alpha)^{1/p} \leq C \text{ for all } \alpha > 0\} \\ &= \sup\{\alpha d_f(\alpha)^{1/p} : \alpha > 0\}\end{aligned}$$

by monotonicity of $x \mapsto x^p$ for $0 < p < \infty$.

Remark. Chebyshev's inequality states

$$\alpha^p d_f(\alpha) \leq \int_{|f|>\alpha} |f(x)|^p d\mu(x).$$

Taking the supremum over $\alpha > 0$

$$\|f\|_{L^{p,\infty}}^p \leq \int_{\mathbb{R}^n} |f(x)|^p d\mu(x).$$

That is, $\|f\|_{L^{p,\infty}}^p \leq \|f\|_{L^p}^p$ so that $L^p \subset L^{p,\infty}$. In fact, L^p is a strict subset of $L^{p,\infty}$ when $1 < p < \infty$ as demonstrated in the following example.

Example 2.14. Consider the interval $[0,1]$ with the usual Lebesgue measure. Fix $1 < p < \infty$ and define $f(x) = x^{-1/p}$. This function is not in L^p, however

$$d_f(\alpha) = \mu(\{x \in [0,1] : x^{-1/p} > \alpha\}) = \mu((0, \alpha^{-p}) \cap [0,1]) = \begin{cases} \alpha^{-p} & \alpha > 1 \\ 1 & \alpha \leq 1 \end{cases},$$

and so $\|f\|_{L^{p,\infty}} = \sup\{\alpha d_f(\alpha)^{1/p}\} = 1$. That is, $f \in L^{p,\infty} \setminus L^p$.

Associated to the weak L^p spaces are the notions of weak and strong type operators.

Definition 2.15. Let $1 \leq p, q \leq \infty$. An operator T is said to be of weak type (p,q) if

$$\|Tf\|_{L^{q,\infty}} \lesssim \|f\|_{L^p},$$

and of strong type (p,q) if

$$\|Tf\|_{L^q} \lesssim \|f\|_{L^p}.$$

That is, an operator is of weak (strong) type (p,q) if it maps L^p to $L^{q,\infty}$ (L^q).

2.6 Interpolation

This section introduces the classical interpolation results for L^p. We present interpolation results of both real and complex flavours: namely the Marcinkiewicz and Riesz–Thorin interpolation results. The Marcinkiewicz theorem is given for operators acting on functions defined over $\mathbb{R}^n$ but the proof we give clearly extends to functions defined over a general measure space. See [Gra04, §1.3] for details.

Interpolation, at its core, is concerned with approximations on a space that lies in between two others. Such interpolation spaces extend far beyond the L^p scale that we mention here, see for instance [BL76].

2.6.1 Marcinkiewicz interpolation

Definition 2.16 (Sublinear operator)**.** Let T be an operator that acts on measurable complex-valued functions. We say that T is sublinear if

(i) $T(\gamma f) = \gamma T(f)$ for all $\gamma \in \mathbb{C}$ and all f.

(ii) $|T(f+g)| \leq |T(f) + T(g)|$ for all f and g

Theorem 2.17. *Suppose $0 < p_0 < p_1 \leq \infty$ and let T be a sublinear operator defined on the space $L^{p_0}(\mathbb{R}^n) \oplus L^{p_1}(\mathbb{R}^n)$ taking values in the space of measurable functions on $\mathbb{R}^n$. Assume that there exists $A_0, A_1 > 0$ such that*

$$\|T(f)\|_{L^{p_0,\infty}} \leq A_0 \|f\|_{L^{p_0}} \quad \text{for all } f \in L^{p_0}, \tag{2.10}$$

$$\|T(f)\|_{L^{p_1,\infty}} \leq A_1 \|f\|_{L^{p_1}} \quad \text{for all } f \in L^{p_1}. \tag{2.11}$$

Then for all $p_0 < p < p_1$ and all $f \in L^p$ we have

$$\|T(f)\|_{L^p} \leq A \|f\|_{L^p},$$

where the constant A is given by

$$A = 2\left(\frac{p}{p-p_0} + \frac{p}{p_1 - p}\right)^{\frac{1}{p}} A_0^{\frac{\frac{1}{p}-\frac{1}{p_1}}{\frac{1}{p_0}-\frac{1}{p_1}}} A_1^{\frac{\frac{1}{p_0}-\frac{1}{p}}{\frac{1}{p_0}-\frac{1}{p_1}}}. \tag{2.12}$$

Proof. We begin with the case $p_1 < \infty$. Fix $f \in L^p(\mathbb{R}^n)$ and $\alpha > 0$. We split $f = f_0^\alpha + f_1^\alpha$ where $f_0^\alpha \in L^{p_0}$ and $f_1^\alpha \in L^{p_1}$ by cutting at the height $\delta\alpha$ where δ is to be determined later,

$$f_0^\alpha(x) = \begin{cases} f(x) & |f(x)| > \delta\alpha \\ 0 & |f(x)| \leq \delta\alpha, \end{cases}$$

$$f_1^\alpha(x) = \begin{cases} 0 & |f(x)| > \delta\alpha \\ f(x) & |f(x)| \leq \delta\alpha. \end{cases}$$

Since $p_0 < p$ we have

$$\|f_0^\alpha\|_{L^{p_0}}^{p_0} = \int_{|f|>\delta\alpha} |f|^p |f|^{p_0-p} d\mu(x) \leq (\delta\alpha)^{p_0-p} \|f\|_{L^p}^p.$$

Similarly since $p < p_1$

$$\|f_1^\alpha\|_{L^{p_1}}^{p_1} = \int_{|f|\leq\delta\alpha} |f|^p |f|^{p_1-p} d\mu(x) \leq (\delta\alpha)^{p_1-p} \|f\|_{L^p}^p.$$

By sublinearity $|Tf| \leq |Tf_0^\alpha| + |Tf_1^\alpha|$, which implies

$$\{x : |Tf(x)| > \alpha\} \subset \{x : |Tf_0^\alpha(x)| > \alpha/2\} \cup \{x : |Tf_1^\alpha(x)| > \alpha/2\},$$

and therefore

$$d_{Tf}(\alpha) \leq d_{Tf_0}(\alpha/2) + d_{Tf_1}(\alpha/2).$$

Chebyshev's inequality then gives

$$\begin{aligned}
d_{Tf}(\alpha) &\le \frac{1}{(\alpha/2)^{p_0}} \int_{\{x:|Tf_0^\alpha(x)|>\alpha\}} |Tf_0^\alpha(x)|^{p_0} d\mu(x) + \frac{1}{(\alpha/2)^{p_1}} \int_{\{x:|Tf_1^\alpha(x)|>\alpha\}} |Tf_1^\alpha(x)|^{p_1} d\mu(x) \\
&\le \frac{A_0^{p_0}}{(\alpha/2)^{p_0}} \int_{\mathbb{R}^n} |f_0^\alpha(x)|^{p_0} d\mu(x) + \frac{A_1^{p_1}}{(\alpha/2)^{p_1}} \int_{\mathbb{R}^n} |f_1^\alpha(x)|^{p_1} d\mu(x) \\
&= \frac{A_0^{p_0}}{(\alpha/2)^{p_0}} \int_{\{x:|f(x)|>\delta\alpha\}} |f(x)|^{p_0} d\mu(x) + \frac{A_1^{p_1}}{(\alpha/2)^{p_1}} \int_{x:|f(x)|\le\delta\alpha} |f(x)|^{p_1} d\mu(x)
\end{aligned}$$

by the hypotheses (2.10) and (2.11). By Proposition 2.12,

$$\begin{aligned}
\|Tf\|_{L^p}^p &\le p(2A_0)^{p_0} \int_0^\infty \alpha^{p-1}\alpha^{-p_0} \int_{\{x:|f(x)|>\delta\alpha\}} |f(x)|^{p_0} d\mu(x) d\alpha \\
&\quad + p(2A_1)^{p_0} \int_0^\infty \alpha^{p-1}\alpha^{-p_1} \int_{\{x:|f(x)|\le\delta\alpha\}} |f(x)|^{p_1} d\mu(x) d\alpha \\
&\le p(2A_0)^{p_0} \int_{\mathbb{R}^n} |f(x)|^{p_0} \int_0^{\frac{1}{\delta}|f(x)|} \alpha^{p-1-p_0} d\alpha\, d\mu(x) \\
&\quad + p(2A_1)^{p_1} \int_{\mathbb{R}^n} |f(x)|^{p_1} \int_{\frac{1}{\delta}|f(x)|}^\infty \alpha^{p-1-p_1} d\alpha\, d\mu(x) \\
&= \left(\frac{p(2A_0)^{p_0}}{p-p_0} \frac{1}{\delta^{p-p_0}} + \frac{p(2A_1)^{p_1}}{p_1-p} \frac{1}{\delta^{p-p_1}} \right) \int_{\mathbb{R}^n} |f(x)|^p d\mu(x).
\end{aligned}$$

We pick $\delta > 0$ so that

$$(2A_0)^{p_0} \frac{1}{\delta^{p-p_0}} = (2A_1)^{p_1} \delta^{p_1-p}.$$

That is,

$$\|Tf\|_{L^p(Y)}^p \le A \, \|f\|_{L^p(X)}$$

where A is given above in (2.12).

The case for $p_1 = \infty$ is similar. We again write $f = f_0^\alpha + f_1^\alpha$ where

$$f_0^\alpha(x) = \begin{cases} f(x) & |f(x)| > \gamma\alpha \\ 0 & |f(x)| \le \gamma\alpha, \end{cases}$$

$$f_1^\alpha(x) = \begin{cases} 0 & |f(x)| > \gamma\alpha \\ f(x) & |f(x)| \le \gamma\alpha \end{cases}$$

Then, $\|T(f_1^\alpha)\|_{L^\infty} \le A_1 \|f_1^\alpha\|_{L^\infty} \gamma\alpha = \alpha/2$ if we choose $\gamma = 1/(2A_1)$. It follows that $\mu\{x : |Tf_1^\alpha(x)| > \alpha/2\} = 0$ and so

$$d_{Tf}(\alpha) = d_{Tf_0^\alpha}(\alpha/2).$$

Since T maps L^{p_0} to $L^{p_0,\infty}$, we have

$$d_{Tf_0^\alpha}(\alpha/2) \le \frac{(2A_0)^{p_0} \|f_0^\alpha\|_{L^{p_0}}^{p_0}}{\alpha^{p_0}} = \frac{(2A_0)^{p_0}}{\alpha^{p_0}} \int_{\{x:|f(x)|>\gamma\alpha\}} |f(x)|^{p_0} d\mu(x).$$

Now, Proposition 2.12 gives

$$\begin{aligned}
\|Tf\|_{L^p}^p &= p\int_0^\infty \alpha^{p-1} d_{Tf}(\alpha) d\alpha \\
&\leq p\int_0^\infty \alpha^{p-1}\frac{(2A_0)^{p_0}}{\alpha^{p_0}}\int_{\{x:|f(x)|>\alpha/(2A_1)\}} |f(x)|^{p_0} d\mu(x)\, d\alpha \\
&\leq p(2A_0)^{p_0}\int_{\mathbb{R}^n} |f(x)|^{p_0}\int_0^{2A_1|f(x)|} \alpha^{p-p_0-1} d\alpha\, d\mu(x) \\
&= \frac{p(2A_1)^{p-p_0}2(A_0)^{p_0}}{p-p_0}\int_{\mathbb{R}^n} |f(x)| d\mu(x).
\end{aligned}$$

This proves the theorem with the constant

$$A = 2\left(\frac{p}{p-p_0}\right)^{1/p} A_0^{\frac{p_0}{p}} A_1^{1-\frac{p_0}{p}}.$$

Note that this agrees with formula (2.12) when $p_1 = \infty$. □

2.6.2 Riesz–Thorin interpolation

The Riesz–Thorin interpolation differs from the Marcinkiewicz interpolation theorem in that it only applies to linear operators. By making stronger assumptions on the operator we are able to get a tighter approximation upon interpolating. Like the Marcinkiewicz theorem, this theorem also extends to operators acting on functions defined over a general measure spaces.

In some regards, the Riesz–Thorin theorem is a complex analogue of the Marcinkiewicz theorem. The standard proof which we present below requires that we first prove a simple lemma about analytic functions.

Lemma 2.18 (Hadamard's three lines lemma)**.** *Let F be analytic in the open strip $S = \{z \in \mathbb{C} : 0 < \mathrm{Re}(z) < 1\}$, continuous and bounded on its closure. Suppose there exists $B_0, B_1 > 0$ such that $|F(z)| \leq B_0$ when $\mathrm{Re}(z) = 0$ and $|F(z)| \leq B_1$ when $\mathrm{Re}(z) = 1$. Then $|F(z)| \leq B_0^{1-\theta}B_1^\theta$ when $\mathrm{Re}(z) = \theta$ for any $0 \leq \theta \leq 1$.*

Proof. Define analytic functions

$$G(z) = \frac{F(z)}{B_0^{1-z}B_1^z}, \quad \text{and} \quad G_n(z) = G(z)e^{(z^2-1)/n}.$$

Since F is bounded on the closed unit strip and $B_0^{1-z}B_1^z$ is bounded from below (since $B_0, B_1 > 0$) we conclude that G is bounded by some constant, say M, on the closed unit strip. Since,

$$|G_n(x+iy)| \leq Me^{-y^2/n}e^{(x^2-1)/n} \leq Me^{-y^2/n},$$

we find that $G_n(x+iy)$ converges to zero pointwise, and hence uniformly on the strip $0 \leq x \leq 1$ as $|y| \to \infty$. Choose y_0 such that $|y| > |y_0|$ implies $|G_n(x+iy)| \leq 1$ for $0 \leq x \leq 1$. By the maximum principle we find that $|G_n(z)| \leq 1$ in the rectangle $[0,1] \times [-|y_0|, |y_0|]$ and hence $|G_n(z)| \leq 1$ in the entire closed strip. Letting $n \to \infty$, $|G(z)| \leq 1$ in the whole closed strip and hence

$$|F(z)| \leq |B_0^{1-z}B_1^z| = \left|e^{\log(B_0)(1-z)+\log(B_1)z}\right| \leq B_0^{1-\theta}B_1^\theta$$

as required, where $\mathrm{Re}(z) = \theta$. □

Theorem 2.19. *Let T be a linear operator defined on the set of simple functions on $\mathbb{R}^n$ and taking values in the set of measurable functions on $\mathbb{R}^n$. Let $1 \leq p_0, p_1, q_0, q_1 \leq \infty$ and assume that*

$$\|T(f)\|_{L^{q_0}} \leq M_0 \|f\|_{L^{p_0}} \tag{2.13}$$

$$\|T(f)\|_{L^{q_1}} \leq M_1 \|f\|_{L^{p_1}} \tag{2.14}$$

for all simple functions $f : \mathbb{R}^n \to \mathbb{C}$, where Then for all $0 < \theta < 1$ we have

$$\|T(f)\|_{L^q} \leq M_0^{1-\theta} M_1^{\theta} \|f\|_{L^p} \tag{2.15}$$

for all simple functions $f : \mathbb{R}^n \to \mathbb{C}$.

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}. \tag{2.16}$$

Proof. Let the simple function $f : \mathbb{R}^n \to \mathbb{C}$ be defined by

$$f(x) = \sum_{k=1}^{m} a_k e^{i\alpha_k} \mathbb{1}_{A_k}(x)$$

where $a_k > 0$, α_k are real and $A_k \subset \mathbb{R}^n$ are pairwise disjoint with finite measure. We seek to control

$$\|Tf\|_{L^q} = \sup_g \left| \int_{\mathbb{R}^n} Tf(x) g(x) d\mu(x) \right|$$

where the supremum is taken over simple functions $g : \mathbb{R}^n \to \mathbb{C}$ with $\|g\|_{L^{q'}} \leq 1$. Write

$$g(x) = \sum_{j=1}^{n} b_j e^{i\beta_j} \mathbb{1}_{B_j}(x)$$

where $b_j > 0$, β_j are real and B_j are pairwise disjoint subsets of $\mathbb{R}^n$ with finite measure. Let

$$P(z) = \frac{p}{p_0}(1-z) + \frac{p}{p_1} z, \quad Q(z) = \frac{q'}{q_0'}(1-z) + \frac{q'}{q_1'} z.$$

for z in the closed strip $\overline{S} = \{z \in \mathbb{C} : 0 \leq \mathrm{Re}(z) \leq 1\} \subset \mathbb{C}$, where q', q_0', q_1' are dual exponents to q, q_0, q_1 respectively. Define,

$$F(z) = \int_{\mathbb{R}^n} Tf_z(x) g_z(x) d\mu(x),$$

where

$$f_z(x) = \sum_{k=1}^{m} a_k^{P(z)} e^{i\alpha_k} \mathbb{1}_{A_k}(x), \quad g_z(x) = \sum_{j=1}^{n} b_j^{Q(z)} e^{i\beta_j} \mathbb{1}_{B_j}(x).$$

By linearity,

$$F(z) = \sum_{k=1}^{m} \sum_{j=1}^{n} a_k^{P(z)} b_j^{Q(z)} e^{i\alpha_k} e^{i\beta_k} \int_{\mathbb{R}^n} T\mathbb{1}_{A_k}(x) \mathbb{1}_{B_j}(x) d\mu(x).$$

Note that F is analytic in $\overline{S}$ since $a_k, b_j > 0$.

When $\mathrm{Re}(z) = 0$ we have $P(z) = \frac{p}{p_0}$ such that

$$\|f_z\|_{L^{p_0}}^{p_0} = \int_{\mathbb{R}^n} \left| \sum_{k=1}^{m} a_k^{\frac{p}{p_0}} e^{i\alpha_k} \mathbb{1}_{A_k}(x) \right|^{p_0} dx = \|f\|_{L^p}^p$$

where equality holds by the disjointness of the A_k. Similarly, $\|g_z\|_{L^{q_0'}}^{q_0'} = \|g\|_{L^{q'}}^{q'}$.

Hölder's inequality gives

$$|F(z)| \leq \|Tf_z\|_{L^{q_0'}} \|g_z\|_{L^{q_0'}} \leq M_0 \|f_z\|_{L^{p_0}} \|g\|_{L^{q_0'}} = M_0 \|f\|_{L^p}^{\frac{p}{p_0}} \|g\|_{L^{q'}}^{\frac{q'}{q_0'}}$$

when $\mathrm{Re}(z) = 0$. Analogously, when $\mathrm{Re}(z) = 1$, we have $\|f_z\|_{L^{p_1}}^{p_1} = \|f\|_{L^p}^{p}$ and $\|g_z\|_{L^{q_1'}}^{q_1'} = \|g\|_{L^{q'}}^{q'}$ so that

$$|F(z)| \leq M_1 \|f\|_{L^p}^{\frac{p}{p_0}} \|g\|_{L^q}^{\frac{q'}{q_1'}}.$$

Applying Lemma 2.18 we find

$$F(z) \leq \left(M_0 \|f\|_{L^p}^{\frac{p}{p_0}} \|g\|_{L^{q'}}^{\frac{q'}{q_0'}} \right)^{1-\theta} \left(M_1 \|f\|_{L^p}^{\frac{p}{p_1}} \|g\|_{L^{q'}}^{\frac{q'}{q_1'}} \right)^{\theta} = M_0^{1-\theta} M_1^{\theta} \|f\|_{L^p} \|g\|_{L^{q'}},$$

when $\mathrm{Re}(z) = \theta$. Observe that $P(\theta) = Q(\theta) = 1$ and hence

$$F(\theta) = \int_{\mathbb{R}^n} Tf(x) g(x) d\mu(x).$$

Taking the supremum over all simple functions g with $\|g\|_{L^{q'}} \leq 1$ gives the required result. $\square$

Corollary 2.20. *By density T has a unique extension as a bounded operator from L^p to L^q for all p and q as in* (2.16).

2.7 Boundedness of singular integral operators

The main result for this chapter is a presentation of sufficient conditions for the boundedness of singular integral operators acting on L^p, In proving that a singular integral operator T is of strong type (p,p) we will make two core assumptions. The first is that we already have boundedness in L^q for some $q \geq p$. That is,

$$\|T(f)\|_q \leq A \|f\|_q, \quad \text{for all } f \in L^q \tag{2.17}$$

where $A < \infty$ is the L^q operator norm of T. The second assumption made is that away from the diagonal, $D = \{(x,x) : x \in \mathbb{R}^n\}$, the the kernel k is a measurable function that satisfies the cancellation property

$$\int_{B(y,c\delta)^c} |k(x,y) - k(x,\overline{y})| d\mu(x) \leq A \quad \text{whenever } y \in B(y,\delta) \tag{2.18}$$

for some constant $c > 0$ and all $y \in \mathbb{R}^n$, $\delta > 0$. We further assume some basic regularity of this kernel k. More precisely, we assume that for each $f \in L^q$ with compact support,

$$Tf(x) = \int_{\mathbb{R}^n} k(x,y) f(y) d\mu(y) \tag{2.19}$$

converges absolutely for almost every x not in the support of f.

Theorem 2.21. *Under the above assumptions* (2.17), (2.18) *and* (2.19), *T is bounded in L^p norm on $L^p \cap L^q$, when $1 < p < q$. That is,*

$$\|T(f)\|_p \leq A_p \|f\|_p \tag{2.20}$$

for $f \in L^p \cap L^q$, where the bound A_p depends only on the constant A appearing in (2.17) *and* (2.18), *and on p.*

The proof has two stages. After first establishing that the mapping $f \mapsto Tf$ is of weak type $(1,1)$, Marcinkiewicz interpolation extends the result to $p \in (1,q)$.

Proof. The first step is establishing that $f \mapsto Tf$ is of weak type $(1,1)$. That is, we want to show

$$\mu\{x : |Tf(x)| > \alpha\} \leq \frac{A^*}{\alpha} \int_{\mathbb{R}^n} |f(x)| d\mu(x) \tag{2.21}$$

for some $A^* > 0$, all $\alpha > 0$ and all $f \in L^1 \cap L^q$.

We can ignore the case $\alpha \leq \frac{1}{\mu(\mathbb{R}^n)} \|f\|_{L^1}$ since in this case (2.21) follows trivially. That is, $\mu\{x : |Tf(x)| > \alpha\} \leq \mu(\mathbb{R}^n) \leq \frac{1}{\alpha} \|f\|_{L^1}$.

With α fixed we invoke the Calderón–Zygmund decomposition $f = g + b$ at height $\gamma\alpha$ given by Theorem 2.10, where γ is some constant to be chosen later. In the spirit of the proof of the Marcinkiewicz interpolation theorem we compute

$$\begin{aligned}\mu\{x : |(Tf + Tb)(x)| > \alpha\} &\leq \mu\{x : |Tg(x)| + |Tb(x)| > \alpha\} \\ &\leq \mu\left(\{x : |Tg(x)| > \alpha/2\} \cup \{x : |Tb(x)| > \alpha/2\}\right) \\ &\leq \mu\{x : |Tg(x)| > \alpha/2\} + \mu\{x : |Tb(x)| > \alpha/2\}.\end{aligned}$$

We will show (2.21) by checking the Tg and Tb parts separately.

We begin with the case $q < \infty$,

$$\int_{\mathbb{R}^n} |g(x)|^q d\mu(x) = \int_{\left(\bigcup_{k\geq 1} B_k^*\right)^c} |g(x)|^q d\mu(x) + \int_{\bigcup_{k\geq 1} B_k^*} |g(x)|^q d\mu(x),$$

where $\operatorname{supp}(b) \subset \bigcup_{k\geq 1} B_k^*$ as in Theorem 2.10. On $(\bigcup_{k\geq 1} B_k^*)^c$, $g(x) = f(x)$, so

$$\int_{\left(\bigcup_{k\geq 1} B_k^*\right)^c} |g(x)|^q d\mu(x) = \int_{\left(\bigcup_{k\geq 1} B_k^*\right)^c} |f(x)||g(x)|^{q-1} d\mu(x) \leq (c\gamma\alpha)^{q-1} \int |f(x)| d\mu(x).$$

On $\bigcup_{k\geq 1} B_k^*$,

$$\int_{\mathbb{R}^n} |g(x)|^q d\mu(x) \leq (c\gamma\alpha)^q \mu\left(\bigcup_{k\geq 1} B_k^*\right) \leq (c\gamma\alpha)^q \sum_{k\geq 1} \mu(B_k^*) \leq \frac{c}{\gamma\alpha} (c\gamma\alpha)^q \|f\|_{L^1}.$$

Therefore, for $q < \infty$,

$$\|g\|_q^q \lesssim \alpha^{q-1} \|f\|_{L^1}.$$

Combining this with by Chebyshev's inequality and (2.17) gives

$$\mu\{x : Tg(x) > \alpha/2\} \leq (\alpha/2)^{-q} \|Tg\|_q^q \leq A(\alpha/2)^{-q} \|g\|_q^q \leq \frac{A'}{\alpha} \|f\|_{L^1}$$

The case when $q = \infty$ proceeds as follows. By Theorem 2.10, $\|g\|_\infty \leq c\gamma\alpha$. Choose $\gamma \geq 1/(2Ac)$ so that the set $\{x : |Tg(x)| > \alpha/2\}$ is empty and thus has measure zero.

We know that $b = \sum_{k\geq 1} b_k$ where each b_k is supported in the ball B_k^*. Since $f \in L^q$, $b_k \in L^q$ also. Let $\overline{y}_k$ denote the centre of B_k^* and $\overline{r}_k$ its radius. Let $B_k^{*\prime}$ denote $B(\overline{y}_k, c\overline{r}_k)$, that is, B_k^* with radius expanded by a factor of c. Hypothesis (2.18) implies

$$\int_{(B_k^{*\prime})^c} |k(x,y) - k(x,\overline{y}_k)|\, d\mu(x) \leq A.$$

When $x \notin B_k^{*\prime}$ we can use the integral representation (2.1) for Tb_k since b_k is supported in B_k^*;

$$\begin{aligned}\int_{\left(\bigcup_{k\geq 1} B_k^{*\prime}\right)^c} |Tb(x)|d\mu(x) &\leq \sum_{k\geq 1} \int_{(B_k^{*\prime})^c} |Tb_k(x)|d\mu(x)\\ &= \sum_{k\geq 1} \int_{(B_k^{*\prime})^c} \left| \int_{\mathbb{R}^n} k(x,y) b_k(y) d\mu(y) \right| d\mu(x).\end{aligned}$$

By the mean zero property of b_k this term is dominated by

$$\sum_{k\geq 1} \int_{(B_k^{*\prime})^c} \int_{B_k^*} |k(x,y) - k(x,\overline{y}_k)|\, |b_k(y)| d\mu(y) d\mu(x) \leq A \sum_{k\geq 1} \int_{B_k^*} |b_k(y)| d\mu(y).$$

By Theorem 2.10 this is bounded by

$$Ac\gamma\alpha \sum_{k\geq 1} \mu(B_k^*) \leq Ac^2 \int_{\mathbb{R}^n} |f(x)| d\mu(x).$$

That is,

$$\mu\left(\{x : |Tb(x)| > \alpha/2\} \cap (B_k^{*\prime})^c\right) \leq \frac{Ac^2}{\gamma\alpha} \int_{\mathbb{R}^n} |f(x)| d\mu(x).$$

Note that the component of this set removed by intersection is small,

$$\mu\left(\bigcup_{k\geq 1} B_k^{*\prime}\right) \leq \sum_{k\geq 1} \mu(B_k^{*\prime}) \leq c \sum_{k\geq 1} \mu(B_k^*) \leq \frac{c}{\gamma\alpha} \int_{\mathbb{R}^n} |f(x)| d\mu(x),$$

so that

$$\mu\{x : |Tb(x)| > \alpha/2\} \leq \frac{A''}{\alpha} \int_{\mathbb{R}^n} |f(x)| d\mu(x).$$

Combining these results for Tg and Tb we have (2.21), the mapping $f \mapsto Tf$ is of weak-type $(1,1)$. The Marcinkiewicz interpolation theorem, Theorem 2.17 then gives (2.20). Note that the constants A', A'', and A_p in (2.20), do not depend on the regularity of k. They are a functions of A from (2.17) and (2.18), the exponent p, and the constant c from Theorem 2.10. $\square$

Corollary 2.22. *The operator T in Theorem 2.21 has a unique extension to L^p for $1 < p < q$ that satisfies* (2.20) *and* (2.21)*.*

Proof. We note that $L^p \cap L^q$ is a dense linear subspace of L^p when $p < \infty$. Let f_n be a sequence in $L^p \cap L^q$ converging to f in L^p norm. For $p > 1$, by (2.20)

$$\|Tf_n - Tf_m\|_{L^p} \leq A_p \|f_n - f_m\|_{L^p} \to 0.$$

That is, the sequence Tf_n is Cauchy in L^p. When $p = 1$, Tf_n converges in measure by (2.21),

$$\mu\{x : (Tf_n - Tf)(x) > \alpha\} \leq \frac{A'}{\alpha} \|f_n - f\|_{L^1} \to 0.$$

$\square$

Remark. Often we can use duality to extend this boundedness to L^p for a wider range of p. In particular, suppose that T with kernel k is bounded on L^2 and satisfies the above requirements to give boundedness for $p \in (1, 2]$. The adjoint operator is bounded on $L^{p'}$ for $p' \in [2, \infty)$ and is given by

$$(T^* f)(x) = \int_{\mathbb{R}^n} \tilde{k}(x, y) f(y) d\mu(y)$$

where $\tilde{k}(x, y) = \overline{k(y, x)}$. For example if T is self-adjoint then we conclude that T is bounded on L^p for $p \in (1, \infty)$.

2.8 Calderón–Zygmund operators

We now turn our attention to a special class of operators that satisfy some of the requirements for L^p boundedness, $1 < p < \infty$, established in the previous section. As we soon see, the Calderón–Zygmund operators are well behaved class of L^2 bounded singular integral operators.

In this section we strengthen our spatial assumptions to $\mathbb{R}^n$ with the Lebesgue measure dx.

The diagonal, the subset of $\mathbb{R}^{2n}$ away from which which the associated kernels are nonsingular, we denote by $D = \{(x, x) : x \in \mathbb{R}^n\}$.

Definition 2.23. We say that a continuous function $K : D^c \to \mathbb{K}$ is a Calderón–Zygmund kernel of order $\alpha \in (0, 1]$ if there exists $C > 0$ such that for $(x, y) \in D^c$:

(i) $|K(x, y)| \leq \dfrac{C}{|x - y|^n}$.

(ii) For all $y' \in \mathbb{R}^n$ such that $|y - y'| \leq \frac{1}{2}|x - y|$,

$$|K(x, y) - K(x, y')| \leq C \frac{|y - y'|^\alpha}{|x - y|^{\alpha + n}}.$$

(iii) For all $x' \in \mathbb{R}^n$ such that $|x - x'| \leq \frac{1}{2}|x - y|$,

$$|K(x, y) - K(x', y)| \leq C \frac{|x - x'|^\alpha}{|x - y|^{\alpha + n}}.$$

We write $K \in \mathrm{CZK}_\alpha$.

Definition 2.24. We say that $K : D^c \to \mathbb{K}$ is the kernel associated to an operator $T \in \mathcal{L}(L^2(\mathbb{R}^n))$ if for all $f \in L^2(\mathbb{R}^n)$ with compact support,

$$Tf(x) = \int_{R^n} K(x, y) f(y) dy$$

for almost every x not in the support of f. We also define $\mathrm{CZO}_\alpha \subset \mathcal{L}(L^2(\mathbb{R}^n))$, the Calderón–Zygmund operators, the operators which are bounded on $L^2(\mathbb{R}^n)$ and have an associated Calderón–Zygmund kernel of order α.

Proposition 2.25. *If $\alpha = 1$ and K is differentiable the inequalities in (ii) and (iii) can be replaced respectively by*

$$|\nabla_y K(x, y)| \leq \frac{C}{2^{n+1}} \frac{1}{|x - y|^{n+1}} \quad \textit{and} \quad |\nabla_x K(x, y)| \leq \frac{C}{2^{n+1}} \frac{1}{|x - y|^{n+1}}.$$

Proof. By the mean value theorem there exists $c \in [0,1]$ such that

$$|K(x,y) - K(x',y)| = |x-x'| \cdot |\nabla_x K((1-c)x + cx', y)|. \tag{2.22}$$

Since $|x-x'| \leq \frac{1}{2}|x-y|$ we compute

$$|(1-c)x + cx' - y| \geq |x-y| - c|x-x'| \geq |x-y| - |x-x'| \geq \frac{1}{2}|x-y|$$

If we assume $|\nabla_x K(x,y)| \leq \frac{C}{2^{n+1}} \frac{1}{|x-y|^{n+1}}$ then (2.22) implies

$$|K(x,y) - K(x',y)| \leq \frac{C}{2^{n+1}} \frac{|x-x'|}{|(1-c)x + cx' - y|^{n+1}} \leq C \frac{|x-x'|}{|x-y|^{n+1}}.$$

The same result holds for $\nabla_y K(x,y)$. □

Proposition 2.26. *$T \in \mathrm{CZO}_\alpha$ if and only if $T^* \in \mathrm{CZO}_\alpha$.*

Proof. Let $f, g \in L^2(\mathbb{R}^n)$ with supports of f and g compact and disjoint. Then,

$$\langle T^* g, f\rangle = \langle g, Tf\rangle = \int_{\mathbb{R}^n} g(x) \int_{\mathbb{R}^n} \overline{K(x,y) f(y)} dy\, dx = \int_{\mathbb{R}^n} \overline{f(y)} \int_{\mathbb{R}^n} \overline{K(x,y)} g(x) dx\, dy$$

Since f was arbitrary

$$T^* g(y) = \int_{\mathbb{R}^n} \overline{K(x,y)} g(x) dx$$

for almost every y not in the support of g. That is, $T^* \in \mathrm{CZO}_\alpha$ with kernel $K^*(x,y) = \overline{K(y,x)}$. □

2.8.1 The Riesz transform

The Riesz transform is one important example of a Calderón–Zygmund operator. In some regards, the Riesz transforms can be thought of as a generalisation of the Hilbert transform to $\mathbb{R}^n$. We initially introduce the Riesz transform from an operator theoretic perspective before presenting the singular integral operator approach.

The Riesz transforms can be extended to fairly general frameworks, however the techniques required to treat such objects are beyond the scope of this thesis. For instance, the Riesz transforms are known to be L^p bounded, for p in an open neighbourhood about 2, on non-compact Riemannian manifolds whose heat kernels satisfy Gaussian estimates from above and below, see [ACDH04]. Similarly one can obtain the L^p boundedness for $1 < p < 2$ if we instead assume doubling in place of heat-kernel estimates, see for instance [CD99].

In spite of this, we will continue with our spatial assumptions of $\mathbb{R}^n$ equipped with Lebesgue measure dx.

Definition 2.27. We define the Riesz transform $R_j : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ by

$$\mathcal{R}_j = \frac{\partial_j}{\sqrt{-\Delta}}.$$

Note that the Riesz transforms are immediately L^2 bounded because we can view them as Fourier multipliers,

$$\widehat{\mathcal{R}_j f}(\xi) = -i\frac{\xi_j}{|\xi|}\widehat{f}(\xi) \qquad j = 1, \dots, n.$$

In particular, Plancherel's theorem gives $\|\mathcal{R}_j f\|_2 \le \|f\|_2$ for all $f \in L^2$.

Proposition 2.28. $\mathcal{R}_j \in \mathrm{CZO}_1$.

Proof. Consider the convolution kernel

$$K_j(x) = \text{p.v.}\frac{c_n x_j}{|x|^{n+1}},$$

for some $c_n > 0$ to be determined. Here p.v. denotes the principal value, the tempered distribution that results from considering the limit of the kernel truncated to $|x| > \epsilon$. That is, for all $\varphi \in \mathcal{S}$,

$$(K_j * \varphi)(y) = \lim_{\epsilon\to 0}\int_{|y-x|>\epsilon} \frac{c_n(y_j - x_j)}{|y-x|^{n+1}}\varphi(x)dx.$$

For $1 \le j \le n$ we have

$$\begin{aligned}\langle \widehat{K_j}, \varphi\rangle = \langle K_j, \widehat{\varphi}\rangle &= c_n \lim_{\epsilon\to 0}\int_{|\xi|>\epsilon}\widehat{\varphi}(\xi)\frac{\xi_j}{|\xi|^{n+1}}d\xi \\ &= c_n \lim_{\epsilon\to 0}\int_{\frac{1}{\epsilon}\ge|\xi|>\epsilon}\int_{\mathbb{R}^n} e^{-i\xi\cdot x}\varphi(x)dx\frac{\xi_j}{|\xi|^{n+1}}d\xi\end{aligned}$$

For $x \ne 0$ we let

$$I_\epsilon^j = c_n\int_{\frac{1}{\epsilon}\ge|\xi|>\epsilon}\frac{\xi_j}{|\xi|^{n+1}}e^{-i\xi\cdot x}d\xi.$$

We check that I_ϵ^j is uniformly bounded in x and ϵ, and $I_\epsilon^j \to \frac{-ix_j}{|x|}$ as $\epsilon \to 0$. Write $e^{-i\xi\cdot x} = \cos(\xi\cdot x) - i\sin(\xi\cdot x)$ and disregard the real part since it only contributes complex phase. Apply the change of coordinates $w = \frac{x}{|x|}$ and $\xi = |x|y$ so that

$$I_\epsilon^j = -ic_n\int_{\frac{\epsilon}{|x|}<|\xi|<\frac{1}{|x|\epsilon}}\frac{y_i}{|y|^{n+1}}\sin(y\cdot w)dy.$$

Evaluating this integral in polar coordinates

$$I_\epsilon^j = -ic_n\int_{S^{n-1}}\theta_j\left(\int_{\frac{\epsilon}{|x|}}^{\frac{1}{\epsilon|x|}}\frac{\sin(r\theta\cdot w)}{r}dr\right)d\theta.$$

Note that $|I_\epsilon^j|$ is uniformly bounded in ϵ and $|x|$ since

$$\left|\int_{\frac{\epsilon}{|x|}}^{\frac{1}{\epsilon|x|}}\frac{\sin(r\theta\cdot w)}{r}dr\right| \le 4\left|\int_{-\infty}^{\infty}\frac{\sin(r\theta\cdot w)}{r}dr\right| = 4\pi.$$

Furthermore,

$$\int_{\frac{\epsilon}{|x|}}^{\frac{1}{\epsilon|x|}}\frac{\sin(r\theta\cdot w)}{r}dr \to \int_0^\infty\frac{\sin(r\theta\cdot w)}{r}dr = \frac{\pi}{2}\mathrm{sign}(\theta\cdot w)$$

as $\epsilon \to 0$, so that

$$I_\epsilon^j \to -ic_n\frac{\pi}{2}\int_{S^{n-1}}\theta_j\,\mathrm{sign}(\theta\cdot w)d\theta.$$

Consider now the vector of such objects under the limit $\epsilon \to 0$,

$$\begin{aligned} I &:= \lim_{\epsilon\to 0}(I_\epsilon^1, \dots I_\epsilon^n) \\ &= -ic_n \frac{\pi}{2}\int_{S^{n-1}} \theta \operatorname{sign}(\theta\cdot w)d\theta \\ &= -ic_n \frac{\pi}{2}\int_{S^{n-1}} (\theta - (\theta\cdot w)w + (\theta\cdot w)w) \operatorname{sign}(\theta\cdot w)d\theta \\ &= -ic_n \frac{\pi}{2} w\int_{S^{n-1}} |\theta\cdot w| d\theta \end{aligned}$$

since $(\theta - (\theta\cdot w)w)\operatorname{sign}(\theta\cdot w)$ is an odd function with respect to the hyperplane perpendicular to w.

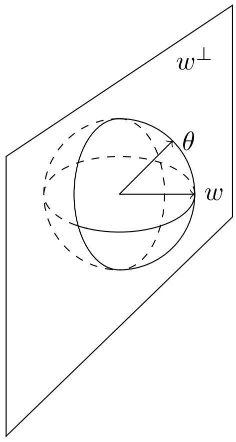


Figure 2.4: Antisymmetry of $(\theta - (\theta\cdot w)w)\operatorname{sign}(\theta\cdot w)$ about the hyperplane $w^\perp$.

By rotational symmetry we have

$$\int_{S^{n-1}} |\theta\cdot w|\, d\theta = \int_{S^{n-1}} |\theta_1|\, d\theta.$$

We thus define

$$c_n = \left(\frac{\pi}{2}\int_{S^{n-1}} |\theta_1| d\theta\right)^{-1}$$

so that

$$I = -iw = -i\frac{x}{|x|}.$$

To conclude notice that

$$\langle \widehat{K_j}, \varphi\rangle = \text{p.v.}\int_{\mathbb{R}^n} (-i)\frac{x_j}{|x|}\varphi(x)dx$$

for all $\varphi \in \mathcal{S}$. That is,

$$\widehat{K_j}(\xi) = -i\frac{\xi_j}{|\xi|}$$

for almost every $\xi \in \mathbb{R}^n$. In particular, K_j defines the (convolution) kernel of the Riesz transform.

It remains to check that $K_j(x-y) \in \mathrm{CZO}_1$. Let $x, y \in D^c$. Then

(i)
$$|K_j(x-y)| = \frac{x_j - y_j}{|x-y|^{n+1}} \leq \frac{1}{|x-y|^n}.$$

(ii) Let $|y-y'| \leq \frac{1}{2}|x-y|$. Then, $|x-y| \sim |x-y'|$ [1] so that
$$|K_j(x-y) - K(x-y')| \lesssim \frac{|y-y'|}{|x-y|^{n+1}}$$

(iii) Follows from (ii) since K_j is a convolution kernel. □

2.8.2 L^p boundedness of CZO_α operators

Definition 2.29. We say that $K \in L^1_{\mathrm{loc}}(D^c)$ is a Hörmander kernel if there exists $C > 0$ such that
$$\mathrm{esssup}_{(y,y') \in \mathbb{R}^{2n}} \int_{|x-y| \geq 2|y-y'|} |K(x,y) - K(x,y')| dx \leq C.$$

Note that the number 2 appearing in the domain of integration is arbitrary. Only the constant C changes if it is replaced by any $A > 1$.

Proposition 2.30. *If $K \in \mathrm{CZK}_\alpha$ then K is Hörmander.*

Proof. Let $K \in \mathrm{CZK}_\alpha$. By property (ii)
$$\begin{aligned}
&\int_{|x-y| \geq 2|y-y'|} |K(x,y) - K(x,y')| dx \\
&\quad \leq \int_{|x-y| \geq 2|y-y'|} \frac{|y-y'|^\alpha}{|x-y|^{n+\alpha}} dx \\
&\quad \leq \sum_{j=1}^{\infty} \int_{2^j|y-y'| \leq |x-y| \leq 2^{j+1}|y-y'|} \frac{|y-y'|^\alpha}{|y-y|^{n+\alpha} 2^{-j(n+\alpha)}} dx \\
&\quad \lesssim \sum_{j=1}^{\infty} 2^{-j(n+\alpha)} \frac{|B(y, 2^j|y-y'|)|}{|y-y'|^n} \lesssim \sum_{j=1}^{\infty} 2^{-j\alpha}
\end{aligned}$$
which is summable since $\alpha > 0$. □

Theorem 2.31. *If $T \in \mathrm{CZO}_\alpha$ then T is of weak type $(1,1)$.*

Proof. By definition $T \in CZO_\alpha$ is bounded on L^2 and the Hörmander condition implies the required cancellation property for us to apply Theorem 2.21. □

Corollary 2.32. *If $T \in \mathrm{CZO}_\alpha$ then T is of strong type (p,p) for $p \in (1,\infty)$.*

Proof. Since T is of weak type $(1,1)$ and strong type $(2,2)$ we have that T is of strong type (p,p) for $p \in (1,2]$. Proposition 2.26 tells us that $T^* \in \mathrm{CZO}_\alpha$ and thus T^* is also of strong type (p',p') for $p' \in (1,2]$. Therefore T has a bounded extension to L^p for $p \in [2,\infty)$ also. □

[1] $\frac{3}{2}|x-y'| \geq |x-y'| + |y-y'| \geq |x-y| \geq |x-y'| - |y-y'| \geq \frac{1}{2}|x-y'|$

We can extend the notion of Calderón–Zygmund operators to Hilbert space valued functions. This modification turns out to be useful for the upcoming Littlewood–Paley theory.

Definition 2.33 (Hilbert CZK_α)**.** Let H_1, H_2 be separable Hilbert spaces. Then, we say K is a Hilbert valued Calderón–Zygmund kernel of order $\alpha \in (0,1]$ if for all $(x,y) \in D^c$, $K(x,y) \in \mathcal{L}(H_1, H_2)$ and there exists $C > 0$ such that

(i) $\|K(x,y)\|_{\mathcal{L}(H_1,H_2)} \leq \dfrac{C}{|x-y|^n}$.

(ii) For all $y' \in \mathbb{R}^n$ such that $|y-y'| \leq \frac{1}{2}|x-y|$

$$\|K(x,y) - K(x,y')\|_{\mathcal{L}(H_1,H_2)} \leq C\frac{|y-y'|^\alpha}{|x-y|^{\alpha+n}}.$$

(iii) For all $x' \in \mathbb{R}^n$ such that $|x-x'| \leq \frac{1}{2}|x-y|$

$$\|K(x,y) - K(x',y)\|_{\mathcal{L}(H_1,H_2)} \leq C\frac{|x-x'|^\alpha}{|x-y|^{\alpha+n}}.$$

We denote the set of all such kernels by $\mathrm{CZK}_\alpha(H_1, H_2)$.

Note that this is essentially the same definition as for the scalar case Definition 2.24 except with absolute value replaced by norm.

Definition 2.34 (Hilbert CZO_α)**.** Let H_1, H_2 be separable Hilbert spaces. We say that $T \in \mathcal{L}(L^2(\mathbb{R}^n, H_1); L^2(\mathbb{R}^n, H^2))$ is a Calderón–Zygmund operator of order α if it is associated with some $K \in \mathrm{CZO}_\alpha(H_1, H_2)$. That is for $f \in L^2(\mathbb{R}^n, H_1)$ compactly supported and almost every $x \in \mathrm{supp}(f)^c$,

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y) f(y) dy.$$

We denote the set of all such T by $\mathrm{CZO}_\alpha(H_1, H_2)$.

Proposition 2.35. *Let $T \in \mathrm{CZO}_\alpha(H_1, H_2)$. Then T extends to a bounded operator from $L^p(\mathbb{R}^n, H_1)$ to $L^p(\mathbb{R}^n, H_2)$ for $p \in (1, \infty)$.*

The proof is the same as in the scalar case.

2.8.3 Littlewood–Paley theory

The Littlewood–Paley theory is intimately related to the theory of singular integral operators with convolution kernels. Of particular interest is the fact that it gives sufficient conditions for bounded functions to be L^p multipliers. In this section we present an introduction to Littlewood–Paley theory in $\mathbb{R}^n$. The basic idea is to decompose the frequency space into 'dyadic' spherical shells.

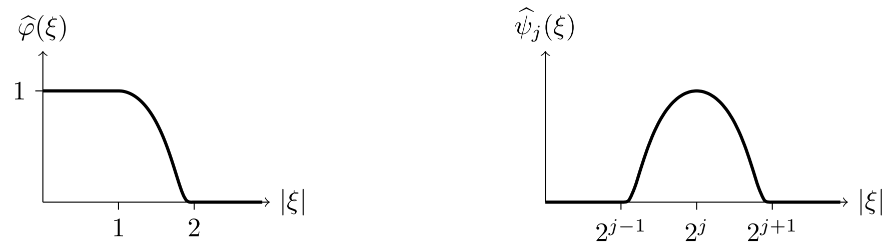
We begin by defining a function φ by specifying its Fourier transform. Fix $\widehat{\varphi} \in C_c^\infty(\mathbb{R}^n)$ such that $\widehat{\varphi}(\xi) = 1$ for $|\xi| \leq 1$, $\widehat{\varphi}(\xi) = 0$ for $|\xi| \geq 2$ and $0 \leq \widehat{\varphi}(\xi) \leq 1$ for $1 \leq |\xi| \leq 2$. In addition we define another function ψ in a similar way, $\widehat{\psi}(\xi) = \widehat{\varphi}(\xi) - \widehat{\varphi}(2\xi)$ so that we have the 'partitions of unity':

$$1 = \widehat{\varphi}(\xi) + \sum_{j=1}^{\infty} \widehat{\psi}(2^{-j}\xi) \quad \text{for all } \xi, \qquad \text{and} \qquad 1 = \sum_{j=-\infty}^{\infty} \widehat{\psi}(2^{-j}\xi) \quad \text{for all } \xi \neq 0.$$

We define $\psi_j(x) := 2^{-jn}\psi(2^{-j}x)$, and $\varphi_j(x) := 2^{-jn}\varphi(2^{-j}x)$ so that

$$\widehat{\psi}_j(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} 2^{-jn}\psi(2^{-j}x)dx = \widehat{\psi}(2^{-j}\xi)$$

are supported in the shells $2^{j-1} \leq |\xi| \leq 2^{j+1}$.



(a) $\widehat{\varphi}$ is 1 near the origin and compactly supported.

(b) $\widehat{\psi}_j$ are 'bumps' compactly supported in spherical shells.

Figure 2.5: Partition functions φ and ψ are defined through Fourier transforms.

We now define the partial sum operators S_j by

$$S_j(f) := f * \varphi_j$$

and the difference operators Δ_j by

$$\Delta_j(f) := S_j(f) - S_{j-1}(f) = f * \psi_j.$$

This leads to the corresponding operator partition identities

$$I = S_0 + \sum_{j=1}^{\infty} \Delta_j \qquad \text{and} \qquad I = \sum_{j=-\infty}^{\infty} \Delta_j.$$

Note that since $\widehat{\varphi}, \widehat{\psi} \in C_c^\infty \subset \mathcal{S}$, then $\varphi, \psi \in \mathcal{S}$ and if $f \in \mathcal{S}'$ then $\Delta_j(f), S_j(f)$ are well defined and

$$S_0(f) + \sum_{j=1}^{l} \Delta_j(f) = S_l(f) \to f \quad \text{as } l \to \infty$$

as distributions.

We say that the pieces $\Delta_j f$ are almost orthogonal in the sense that for $f \in L^2$, $|j-k| \geq 2$,

$$\langle \Delta_j f, \Delta_k f\rangle = \int_{\mathbb{R}^n} \Delta_j f(x)\overline{\Delta_k f}(x)dx = \int_{\mathbb{R}^n} \widehat{\Delta_j f}(\xi)\overline{\widehat{\Delta_k f}}(\xi)d\xi = 0.$$

Theorem 2.36 (Littlewood–Paley)**.** *There exist $C_1, C_2 > 0$ depending on n, φ and p such that*

$$C_1 \|f\|_p \leq \left\| \Big(\sum_{j\in\mathbb{Z}} |\Delta_j f|^2 \Big)^{\frac{1}{2}} \right\|_p \leq C_2 \|f\|_p \tag{2.23}$$

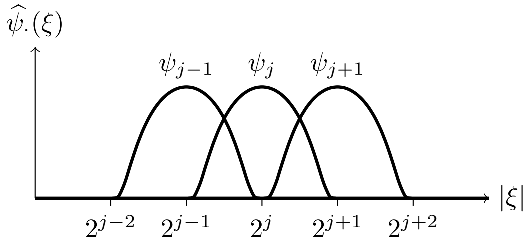


Figure 2.6: The Δ_j are almost orthogonal since the ψ_j have disjoint support when their indices differ by two or more.

To prove this theorem we begin by using Plancherel to establish a result for L^2 and then use the theory of Calderón–Zygmund operators to extend to L^p for $p \in (1, \infty)$.

Proof. Consider $T : L^2(\mathbb{R}^n, \mathbb{C}) \to L^2(\mathbb{R}^n, \ell^2)$ defined by $f \mapsto (\Delta_j f)_{j\in\mathbb{Z}}$. Note that $T \in \mathcal{L}(L^2(\mathbb{R}^n, \mathbb{C}); L^2(\mathbb{R}^n, \ell^2))$ since

$$\begin{aligned}\int_{\mathbb{R}^n} \|Tf(x)\|_{\ell^2}^2 \, dx &= \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^n} |\Delta_j f(x)|^2 dx \\ &= \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^n} |\widehat{\psi}_j(\xi)|^2 |\widehat{f}(\xi)|^2 d\xi \\ &= \int_{\mathbb{R}^n} m(\xi) |\widehat{f}(\xi)|^2 d\xi\end{aligned}$$

where $m(\xi) = \sum_{j\in\mathbb{Z}} |\widehat{\psi}_j(\xi)|^2$. Since the $\widehat{\psi}_j$ have almost disjoint supports, for any nonzero $\xi \in \mathbb{R}^n$,

$$m(\xi) \leq 2\|\widehat{\psi}\|_\infty^2.$$

It thus follows that

$$\int_{\mathbb{R}^n} \|Tf(x)\|_{\ell^2}^2 \, dx \leq 2 \left\|\widehat{\psi}\right\|_\infty^2 \int_{\mathbb{R}^n} |f(x)|^2 dx.$$

On the other hand, $\widehat{\psi(\xi)} = 1$ for $\frac{1}{2} \leq \xi \leq 4$ so $m(\xi) \geq 1$ for all nonzero $\xi \in \mathbb{R}^n$ so that

$$\int_{\mathbb{R}^n} \|Tf(x)\|_{\ell^2}^2 \, dx \geq \int_{\mathbb{R}^n} |f(x)|^2 dx.$$

The Littlewood–Paley estimate holds for $p = 2$.

We now check that $T \in \mathrm{CZO}_\alpha(\mathbb{C}, \ell^2)$. Define the kernel

$$K(x, y) = (K_j(x, y))_{j\in\mathbb{Z}} = (\psi_j(x - y))_{j\in\mathbb{Z}} \qquad \forall (x, y) \in D^c.$$

We note that

$$\|K(x, y)\|_{\mathcal{L}(\mathbb{C},\ell^2)}^2 = \sum_{j\in\mathbb{Z}} |K_j(x, y)|^2.$$

Since $\psi \in \mathcal{S}$ we have $C_\psi > 0$ such that $|\psi(x)| \leq C_\psi(1 + |x|)^{-M}$ for all $M > n$. It follows that

$$|\psi_j(x)| \leq \frac{C_\psi 2^{jn}}{(1 + 2^j|x|)^M} \quad \text{for all} \quad M > n,$$

so that

$$
\begin{aligned}
\sum_{j\in\mathbb{Z}} |K_j(x,y)|^2 &\le \sum_{j\in\mathbb{Z}} \frac{C_\psi^2 2^{2jn}}{(1+2^j|x-y|)^{2M}} \\
&\le \sum_{2^j|x-y|\le 1} \frac{C_\psi^2 2^{2jn}}{(1+2^j|x-y|)^{2M}} + \sum_{2^j|x-y|>1} \frac{C_\psi^2 2^{2jn}}{(1+2^j|x-y|)^{2M}} \\
&\le \sum_{2^j|x-y|\le 1} C_\psi^2 2^{2jn} + \sum_{2^j|x-y|>1} \frac{C_\psi^2 2^{2j(n-M)}}{|x-y|^{2M}} \\
&= \sum_{j\le -log_2(|x-y|)} C_\psi^2 2^{2jn} + \sum_{j>-\log_2(|x-y|)}^{\infty} \frac{C_\psi^2 2^{2j(n-M)}}{|x-y|^{2M}} \\
&= C_\psi^2 \frac{2^{2n(1-\log_2(|x-y|))}}{2^{2n}-1} + C_\psi^2 \frac{2^{2M+2(M-n)\log_2(|x-y|)}}{(2^n-2^M)|x-y|^{2M}} \\
&\le C_{\psi,n,M}^2 \frac{1}{|x-y|^{2n}}.
\end{aligned}
$$

We also estimate the gradient,

$$
|\nabla_x K(x,y)| \le C2^{j(n+1)}|\nabla\psi||2^j(x-y)| \le C_\psi 2^{j(n+1)}(1+2^j|x|)^{-M}
$$

where $M > n+1$ such that, as above,

$$
\|\nabla_x K(x,y)\|_{\mathcal{L}(\mathbb{C},\ell^2)} \lesssim \frac{1}{|x-y|^{n+1}}.
$$

The same holds for $\nabla_y K(x,y) = -\nabla_x K(x,y)$. Applying Proposition 2.25, it follows that $K \in \mathrm{CZK}_1$.

We now check that K is the kernel associated with T. Let $f \in C_c^\infty(\mathbb{R}^n)$ and $g \in C_c^\infty(\mathbb{R}^n, \ell^2)$ with supports of f and g disjoint. We compute

$$
\begin{aligned}
\int_{R^n} Tf(x)\cdot\overline{g(x)}dx &= \int_{\mathbb{R}^n} \sum_{j\in\mathbb{Z}} \Delta_j f(x)\overline{g_j(x)}dx \\
&= \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} K_j(x,y)f(y)dy\right)\overline{g_j(x)}dx \\
&= \int_{\mathbb{R}^n} \sum_{j\in\mathbb{Z}} \left(\int_{\mathbb{R}^n} K_j(x,y)\overline{g_j(x)}dx\right) f(y)dy \\
&= \int_{\mathbb{R}^n} \sum_{j\in\mathbb{Z}} \left(\int_{\mathbb{R}^n} K_j(x,y)\overline{g_j(x)}f(y)dy\right) dx \\
&= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} K(x,y)f(y)dy\right)\cdot\overline{g(x)}dx
\end{aligned}
$$

by Fubini's theorem. We know that $Tf \in L^2(\ell^2)$ and $\int_{\mathbb{R}^n} K(x,y)f(y)dy$ is ℓ^2 valued so that

$$
Tf(x) = \int_{\mathbb{R}^n} K(x,y)f(y)dy
$$

in $L^2_{\mathrm{loc}}(\mathrm{supp}(f)^c;\ell^2)$ and the above formula holds for almost every $x \in \mathrm{supp}(f)^c$.

We conclude that $T \in \mathrm{CZO}_1$ and thus extends to a bounded operator from $L^p(\mathbb{R}^n, \mathbb{C})$ to $L^p(\mathbb{R}^n, \ell^2)$ for $p \in (1, \infty)$, thus verifying the right hand side of (2.23).

It remains to prove the left hand side of the Littlewood–Paley estimate (2.23). We define $\tilde{\psi}$ by

$$\widehat{\tilde{\psi}}(\xi) = \begin{cases} \frac{\widehat{\psi}(\xi)}{\sum_{j\in\mathbb{Z}} |\widehat{\psi}_j(\xi)|^2} & \xi \in \mathbb{R}^n \setminus \{0\}, \\ 0 & \xi = 0, \end{cases}$$

so that for all $\xi \neq 0$

$$\sum_{j\in\mathbb{Z}} \widehat{\psi}_j(\xi)\widehat{\tilde{\psi}_j}(\xi) = 1,$$

and note that $\widehat{\tilde{\psi}} \in C_c^\infty \subset \mathcal{S}$. Let $f \in L^p(\mathbb{R}^n, \mathbb{C}) \cap L^2(\mathbb{R}^n, \mathbb{C})$ and $g \in C_c^\infty(\mathbb{R}^n)$. Then,

$$\begin{aligned} \int_{\mathbb{R}^n} f(x)\overline{g(x)}dx &= \int_{\mathbb{R}^n} \widehat{f}(\xi)\overline{\widehat{g}(\xi)}d\xi \\ &= \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^n} \widehat{\psi}_j \widehat{f}(\xi)\overline{\widehat{\tilde{\psi}_j}(\xi)\widehat{g}(\xi)}d\xi \\ &= \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^n} \Delta_j f(x)\overline{\tilde{\Delta}_j g(x)}dx \\ &= \int_{\mathbb{R}^n} \sum_{j\in\mathbb{Z}} \Delta_j f(x)\overline{\tilde{\Delta}_j g(x)}dx \\ &= \int_{\mathbb{R}^n} \langle Tf(x), \tilde{T}g(x)\rangle_{\ell^2} dx \end{aligned}$$

where $\tilde{\Delta}_j g = \psi_j * g$. From here we apply the Cauchy–Schwarz inequality in ℓ^2 and Hölder's inequality to see

$$\left|\int_{\mathbb{R}^n} f(x)g(x)dx\right| \leq \int_{\mathbb{R}^n} \|Tf(x)\|_{\ell^2}\|\tilde{T}g(x)\|_{\ell^2}dx \leq \Big\|\|Tf(x)\|_{\ell^2}\Big\|_p \Big\|\|\tilde{T}g(x)\|_{\ell^2}\Big\|_{p'}.$$

An argument identical to that used to prove the right hand side estimate yields

$$\Big\|\|\tilde{T}g\|_{\ell^2}\Big\|_{p'} \leq C_{\psi,n,p'}\Big\|g\Big\|_{p'}$$

such that

$$\left|\int_{\mathbb{R}^n} f(x)g(x)dx\right| \leq C_{\psi,n,p'}\Big\|\|Tf(x)\|_{\ell^2}\Big\|_p \Big\|g\Big\|_{p'}.$$

By duality of L^p and $L^{p'}$ and density of $C_c^\infty(\mathbb{R}^n)$ in L^p,

$$\|f\|_{L^p(\mathbb{R}^n;\mathbb{C})} = \sup_{\|g\|_{p'}\leq 1} |\langle f, g\rangle| \lesssim \|Tf\|_{L^p(\mathbb{R}^n;\ell^2)}.$$

By the density of $L^2(\mathbb{R}^n, \mathbb{C}) \cap L^p(\mathbb{R}^n, \mathbb{C})$ in $L^p(\mathbb{R}^n, \mathbb{C})$ we can remove the requirement that $f \in L^2(\mathbb{R}^n, \mathbb{C})$. $\square$

Chapter 3

Functions of bounded mean oscillation

This chapter is devoted to the functions of bounded mean oscillation (BMO). We begin with a rather indirect approach and first look at them them as the dual of the Hardy space H^1. We then look at the space BMO directly, through various equivalent characterisations. The final section deals with tent spaces, which although useful in their own right have an interesting relation to BMO.

3.1 Hardy spaces

There exist a plethora of equivalent characterisations for Hardy spaces of which we only present a select few. In the following section we identify BMO as the dual space of what is, for our purposes, the most important Hardy space H^1. The theory of Hardy spaces is well studied and many of the proofs in this section we defer to Stein [Ste93], however the proofs can be found in many places, see for instance [FS72] and [Gra04].

3.1.1 Maximal characterisation of H^p

In this section we present the maximal function characterisation of the Hardy spaces due to Fefferman and Stein [FS72].

Definition 3.1. Let $\varphi \in \mathcal{S}$ and let $v \in \mathcal{S}'$. We define the smooth maximal function of v with respect to φ by

$$\mathcal{M}_\varphi v(x) := \sup_{t>0} |(v * \varphi_t)(x)|,$$

and the nontangential maximal function

$$\mathcal{M}^*_\varphi v(x) := \sup_{|x-y|\leq t} (v * \varphi_t)(y),$$

for all $x \in \mathbb{R}^n$ where $\varphi_t(x) := t^{-n}\varphi(x/t)$.

Note the obvious pointwise inequality $\mathcal{M}_\varphi v(x) \leq \mathcal{M}^*_\varphi v(x)$ for all $x \in \mathbb{R}^n$.

The main result is that these maximal functions give equivalent characterisations of the Hardy space H^p.

Theorem 3.2. *Let $v \in \mathcal{S}'$ and $0 < p \leq \infty$. Then the following are equivalent:*

(i) There exists $\varphi \in \mathcal{S}$ with $\int_{\mathbb{R}^n} \varphi(x)\,dx \neq 0$ so that $\mathcal{M}_\varphi v \in L^p(\mathbb{R}^n)$.

(ii) There exists $\varphi \in \mathcal{S}$ with $\int_{\mathbb{R}^n} \varphi(x)\,dx \neq 0$ so that $\mathcal{M}^*_\varphi v \in L^p(\mathbb{R}^n)$.

A proof of this can be found in [Ste93, pp.92-98].

Definition 3.3. If any one of the properties from Theorem 3.2 holds we say that the tempered distribution $v \in \mathcal{S}'$ belongs to H^p.

We are most interested in the Hardy space H^1. When $p > 1$ one can show that $H^p = L^p$, see [Gra04, pp.450] for instance. On the other hand, $p < 1$ the Hardy spaces have some undesirable properties. For example, we define a family of seminorms $\|f\|_{H^p} := \|\mathcal{M}_\varphi f\|_{L^1}$ indexed by $\varphi \in \mathcal{S}$ that is only a norm when $p = 1$. We will eventually characterise BMO as the dual of the Hardy space H^1. For these reasons we will focus on the case $p = 1$.

3.1.2 Atomic decomposition of H^1

Although the atomic decomposition holds for H^p with $0 < p \leq 1$, we present here the case $p = 1$ only.

Definition 3.4. Let $1 < q \leq \infty$. A function a is called an L^q atom for H^1 if there exists a ball B such that

(i) a is supported in B,

(ii) $\left(\frac{1}{|B|}\int_B |a(x)|^q dx\right)^{\frac{1}{q}} \leq \frac{1}{|B|}$,

(iii) $\int_B a(x)dx = 0$.

When $q = \infty$, we interpret (ii) as $\|a\|_\infty < \frac{1}{|B|}$.

Notice that (i) and (ii) combine to give

$$\int_{\mathbb{R}^n} |a(x)|dx \leq |B|^{1-\frac{1}{q}} \left(\int_B |a(x)|^q dx\right)^{\frac{1}{q}} = |B| \left(\frac{1}{|B|}\int_B |a(x)|^q dx\right)^{\frac{1}{q}} \leq 1. \tag{3.1}$$

As one might expect from the name, an L^q atom a for H^1 is itself an element of H^1. This is formalised in the following proposition.

Proposition 3.5. *Let a be an L^q atom for H^1. Given a smooth function φ supported in $B(0,1)$, with $\int \varphi(x)\,dx \neq 0$, we have $\mathcal{M}_\varphi a \in L^1$.*

Proof. Let a be an L^q atom for H^1 that is supported in the ball $B = B(x_0, r_0)$ and φ as above. For $x \in B^* = B(x_0, 2r_0)$,

$$\begin{aligned}
\mathcal{M}_\varphi a(x) &= \sup_{t>0} \left| \int_{\mathbb{R}^n} a(y) t^{-n} \varphi\left(\frac{x-y}{t}\right) dy \right| \\
&\leq \sup_{t>0} \left(\int_B |a(y)|^q dy \right)^{\frac{1}{q}} \left(\int_B \left| t^{-n} \varphi\left(\frac{x-y}{t}\right) \right|^{q'} dy \right)^{\frac{1}{q'}} \\
&\leq |B|^{-1} \int_{\mathbb{R}^n} \left| t^{-n} \varphi\left(\frac{x-y}{t}\right) \right| dy \\
&= |B|^{-1} \int_{B(0,1)} |\varphi(z)|\, dz \\
&\lesssim_\varphi |B|^{-1}
\end{aligned}$$

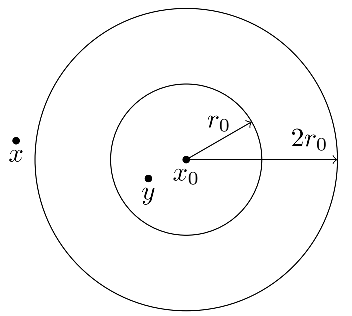
The first inequality is Hölder's inequality where $\frac{1}{q} + \frac{1}{q'} = 1$. We then use the fact that a is an atom and make the substitution $z = \frac{y-x}{t}$ concluding by smoothness of φ. Alternatively, for $x \notin B^*$,

$$(a * \varphi_t)(x) = \int_{\mathbb{R}^n} a(y) t^{-n} \varphi_t\Big(\frac{x-y}{t}\Big) dy = \int_{\mathbb{R}^n} a(y) \Big[t^{-n} \varphi\Big(\frac{x-y}{t}\Big) - t^{-n} \varphi\Big(\frac{x_0}{t}\Big) \Big] dy$$

by the mean-zero property of a. Since φ is smooth there exits $c > 0$ such that

$$|\varphi_t(x-y) - \varphi_t(x_0)| \leq c \frac{|y - x_0|}{t^{n+1}}.$$

We need only consider $y \in B(x_0, r_0)$ and $\frac{x-y}{t} \in B(0,1)$, that is, when $t > |y - x| > |y - x_0|$.



Since $|y - x_0| < r_0$,

$$|\varphi_t(x-y) - \varphi_t(x_0)| \leq c \frac{r_0}{|y - x_0|^{n+1}},$$

and this implies

$$\mathcal{M}_\varphi a(x) \lesssim |B|^{-1} \frac{r_0}{|y - x_0|^{n+1}}.$$

We conclude

$$\begin{aligned}
\|\mathcal{M}_\varphi a\|_{L^1} &= \int_{\mathbb{R}^n} |\mathcal{M}_\varphi a(x)|\, dx \\
&= \int_{B^*} |\mathcal{M}_\varphi a(x)|\, dx + \int_{(B^*)^c} |\mathcal{M}_\varphi a(x)|\, dx \\
&\lesssim \int_{B^*} |B|^{-1} dx + \int_{(B^*)^c} |B|^{-1} \left(\frac{r_0}{|x-x_0|^{n+1}} \right) dx \\
&\lesssim 1 + \int_{r_0}^{\infty} \frac{r_0}{r^2} dr \\
&\lesssim 1.
\end{aligned}$$

Thus any L^q atom for H^1 is in fact an element of H^1 by the maximal characterisation, Theorem 3.2. In fact the H^1 atoms are uniformly bounded, that is, independently of B. □

The main idea in this section is the atomic decomposition of H^1.

Theorem 3.6. *Given $f \in H^1$ there exists a collection of H^1 atoms $\{a_k\}_{k=1}^{\infty}$ and a sequence of complex-valued weights $\{\lambda_k\}_{k=1}^{\infty}$ such that*

$$f(x) = \sum_{k \geq 1} \lambda_k a_k(x)$$

for almost every x. Furthermore,

$$\sum_{k \geq 1} |\lambda_k| \leq c \|f\|_{H^1} .$$

The standard proof uses a Calderón–Zygmund decomposition to get the result for $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and then checks the density of $H^1 \cap L^1_{\text{loc}}$ in H^1. For a detailed proof of this theorem see [Ste93, pp.107-112].

The converse to this theorem also holds and so the atomic decomposition characterises H^1.

Proposition 3.7. *Given $\{a_k\}_{k=1}^{\infty}$ a collection of H^1 atoms and $\{\lambda_k\}_{k=1}^{\infty}$ a sequence of complex numbers with $\sum_{k \geq 1} |\lambda_k| < \infty$, there exists $f \in H^1$ such that*

$$f = \sum_{k \geq 1} \lambda_k a_k \tag{3.2}$$

converges in the sense of distributions with $\|f\|_{H^1} \leq c \sum_{k \geq 1} |\lambda_k|$.

Proof. The sum (3.2) is finite,

$$\mathcal{M} f = \mathcal{M} \sum_{k \geq 1} \lambda_k a_k \leq \sum_{k \geq 1} |\lambda_k| \mathcal{M} a_k.$$

Integrating this, and using the previous proposition,

$$\int_{\mathbb{R}^n} |\mathcal{M} f(x)| dx \leq \sum_{k \geq 1} |\lambda_k| \int_{\mathbb{R}^n} |\mathcal{M} a_k(x)| dx \lesssim \sum_{k \geq 1} |\lambda_k| < \infty.$$

That is, the sum (3.2) converges in H^1. □

3.1.3 Singular integrals and H^1

One of the important applications of H^1 theory is that it extends the L^p results for maximal functions and the boundedness of singular integrals. We will show that the results regarding the boundedness of singular integrals on L^p given in section 2.7 can be extended to H^1.

We return to the case of translation invariant, (convolution type) singular integrals. Beginning with an operator $T : L^2 \to L^2$ that commutes with translations we have a Fourier multiplier. That is, a bounded function m, so that

$$\widehat{Tf}(\xi) = m(\xi)\widehat{f}(\xi), \quad |m(\xi)| \leq A, \tag{3.3}$$

for $f \in L^2$. By Plancherel's theorem T is a bounded operator

$$\|Tf\|_{L^2} \leq A \|f\|_{L^2}. \tag{3.4}$$

Let K denote the tempered distribution with $\widehat{K} = m$ such that

$$Tf = f * K \qquad \forall f \in \mathcal{S}.$$

Analogously to the results of section 2.7,under certain conditions we are able to extend this to an operator on $H^1 \subset L^1$. We first assume, away from the origin, K is a locally integrable function so that the convolution can be represented in the usual (integral) form

$$Tf(x) = \int_{\mathbb{R}^n} K(x-y)f(y)dy \tag{3.5}$$

In addition to the L^2 boundedness (3.4) we assume that K has the cancellation property,

$$\int_{|x|>2|y|} |K(x-y) - K(x)|dy \leq A \tag{3.6}$$

for $y \neq 0$. Theorem 2.21 now applies and T satisfies the weak-type (1,1) inequality. The operator T extends to an operator defined on $H^1(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$. We formalise this discussion in the following theorem.

Theorem 3.8. *If T satisfies the assumptions* (3.3), (3.4), (3.5) *and* (3.6) *then it is bounded from H^1 to L^1. That is, there exists $A' > 0$ such that*

$$\|Tf\|_{L^1(\mathbb{R}^n)} \leq A' \|f\|_{H^1(\mathbb{R}^n)} \tag{3.7}$$

where A' depends only on the constant A coming from (3.4) *and* (3.6).

Proof. Begin by taking the atomic decomposition f into atoms as in Theorem 3.6. We will first verify (3.7) with L^2 atoms for H^1. Let a be an L^2 atoms for H^1. Since T is translation invariant, we can take a to be supported in some ball B of radius r centred at the origin. Let B^* denote the ball centred at the origin with radius $2r$. By (3.4),

$$\int_{B^*} |Ta(x)|^2 dx \leq A^2 \int_{B^*} |a(x)|^2 dx \leq \frac{A^2}{|B|}.$$

The Cauchy–Schwarz inequality and the fact that a is an L^2 atom gives,

$$\int_{B^*} |Ta(x)|dx \leq |B^*|^{1/2} \left(\int_{B^*} |Ta(x)|^2 dx \right)^{1/2} \leq A \left(\frac{|B^*|}{|B|} \right)^{1/2} = 2^{n/2} A$$

since $|B^*|/|B| > 1$. On the other hand if $x \notin B^*$ then by the representation (3.5)

$$Ta(x) = \int_{\mathbb{R}^n} K(x-y)a(y)dy = \int_{\mathbb{R}^n} (K(x-y) - K(x))\, a(y)dy$$

since a has the mean zero property. Notice that $|x| \geq 2r$ and $|y| \leq r$ so that we can apply (3.6) to give

$$\int_{(B^*)^c} |Ta(x)|dx \leq A \int_{\mathbb{R}^n} |a(x)|dx \leq A$$

by the properties of the decomposition. Combining this with the result above gives (3.7) for atoms.

It follows that the result extends to all of H^1. Let $f = \sum_{k\geq 1} \lambda_k a_k$ be an element of H^1. By Theorem 3.6,

$$\|Tf\|_{L^1} \leq \sum_{k\geq 1} |\lambda_k| \, \|Ta_k\|_{L^1} \leq A' \sum_{k\geq 1} |\lambda_k| \, \|a_k\|_{L^1} \leq A' \, \|f\|_{H^1} .$$ □

3.2 The space BMO

3.2.1 Classical BMO

The space of functions of bounded mean oscillation (BMO) was introduced by John and Nirenberg [JN61].

Definition 3.9. For a locally integrable function $f : \mathbb{R}^n \to \mathbb{C}$ set

$$\|f\|_{\mathrm{BMO}} := \sup_B \frac{1}{|B|} \int_B |f(x) - f_B|dx$$

where $f_B = \frac{1}{|B|}\int_B f(x)dx$ is the average of f over B. The supremum is taken over all balls $B \subset \mathbb{R}^n$. The function f is said to be of bounded mean oscillation if $\|f\|_{\mathrm{BMO}} < \infty$.

One of the main results of this chapter is the duality between the Hardy space H^1 and BMO. We begin by establishing a few basic properties of BMO.

Proposition 3.10. *If $\|f\|_{\mathrm{BMO}} = 0$ then f is a.e. equal to a constant.*

Proof. From the definition of the BMO norm, $f = f_B$ a.e. for all $B \subset \mathbb{R}^n$. Considering a sequence of balls $B_n = B(0,n)$ it follows $f(x) = f_{B_1}$ a.e. □

Proposition 3.11. $L^\infty \subset \mathrm{BMO}$

Proof. $\|f\|_{\mathrm{BMO}} = \sup_B \frac{1}{|B|} \int_B \left| f - \frac{1}{|B|}\int_B f dy \right| dx \leq \sup_B \frac{2}{B} \int_B |f|dx \leq 2\, \|f\|_\infty$. □

In fact, L^∞ is a proper subset of BMO as the following example demonstrates.

Example 3.12. The function $\log|x|$ is in $\mathrm{BMO}(\mathbb{R}^n) \setminus L^\infty(\mathbb{R}^n)$. To see this, first note that scaling $f(x) \mapsto f(\delta x)$, $\delta > 0$ preserves BMO norms,

$$\frac{1}{|B|} \int_B |f(\delta x) - f_{\delta B}|dx = \frac{1}{|\delta B|} \int_{\delta B} |f(x) - f_{\delta B}|dx.$$

Under these scalings, $\log|\delta x| = \log|x| + \log(\delta)$. It hence suffices to consider unit balls only. That is

$$\Big|\Big|\log|x|\Big|\Big|_{\mathrm{BMO}} = \sup_{x_0} \fint_{B(x_0,1)} \Big|\log|x| - \fint_{B(x_0,1)} \log|y|dy\Big|dx.$$

Notice

$$\Big|\log|x| - \fint_{B(x_0,1)} \log|y|dy\Big| \leq \Big|\log|x| - c\Big| + \fint_{B(x_0,1)} \Big|\log|y| - c\Big|dy$$

for all constants c. Averaging over $B(x_0,1)$,

$$\fint_{B(x_0,1)} \Big|\log|x| - \fint_{B(x_0,1)} \log|y|dy\Big|dx \leq 2\fint_{B(x_0,1)} \Big|\log|x| - c\Big|dx.$$

If $|x_0| \leq 2$ then take $c = 0$ and

$$\fint_{B(x_0,1)} \Big|\log|x|\Big|dx \leq \fint_{B(0,3)} \Big|\log|x|\Big|dx = C.$$

On the other hand, if $|x_0| > 2$ take $c = \log|x_0|$ so that

$$\fint_{B(x_0,1)} \Big|\log|x| - \log|x_0|\Big|dx = |x_0|^n \fint_{B(x_0,|x_0|)} \Big|\log\frac{|x|}{|x_0|}\Big|dx \leq (3/2)^n \log(3/2).$$

Combining both cases it is clear that $\Big|\Big|\log|x|\Big|\Big|_{\mathrm{BMO}} < \infty$.

3.2.2 Duality of H^1 and BMO

Having built up some preliminary properties about the spaces H^1 and BMO we come to our main theorem, namely that BMO is the dual to H^1. That is, each bounded linear functional l defined on H^1 can be expressed

$$l(f) = \int_{\mathbb{R}^n} f(x)b(x)dx, \quad f \in H^1 \tag{3.8}$$

for some $b \in \mathrm{BMO}$.

An important thing to notice is that the integral (3.8) does not converge absolutely. Indeed, we are integrating a possibly unbounded function against a function with possibly non-compact support. In this context integration becomes an abstract operation which will be handled by first considering a dense subspace of H^1.

Definition 3.13. We denote by H^1_0 the set of all finite linear combinations of L^2 atoms for H^1.

Note that in light of the atomic decomposition for H^1 this is a set of bounded, compactly supported functions.

Proposition 3.14. *H^1_0 is dense in H^1.*

Proof. Given $g \in H^1$ we have the atomic decomposition

$$g(x) = \sum_{k=1}^{\infty} \lambda_k a_k(x)$$

where $\lambda_k \in \mathbb{C}$ form an absolutely summable sequence and a_k are a sequence of atoms. We define the sequence $g_n \in H_0^1$ as the partial sum

$$g_n(x) = \sum_{k=1}^{n} \lambda_k a_k(x),$$

so that

$$\|g - g_n\|_{H^1} = \left\| \sum_{k=n}^{\infty} \lambda_k a(x) \right\|_{H^1} \leq \sum_{k=n}^{\infty} |\lambda_k| \to 0 \quad \text{as } n \to \infty$$

since λ_k is an absolutely summable sequence. □

For any $b \in \text{BMO}$, we define a linear functional l_b on H_0^1 by

$$l_b(f) = \int_{\mathbb{R}^n} f(x)b(x)dx. \tag{3.9}$$

This integral converges absolutely since f is bounded and compactly supported and b is locally (square) integrable. The fact that every bounded linear functional on H^1 takes this form on H_0^1 is a result of Theorem 3.16.

Before proving the duality, it will be useful to establish some simple properties about cutoff functions in BMO. Let b be a function in BMO and define the cutoff at $\pm k$ by

$$b^k(x) := \begin{cases} -k & b(x) < -k, \\ b(x) & -k \leq b(x) \leq k, \\ k & k \leq b(x). \end{cases}$$

Lemma 3.15. $\left\|b^k\right\|_{\text{BMO}} \leq \|b\|_{\text{BMO}}$.

Proof. This follows simply from the fact that BMO forms a Banach lattice. Since $\left||b| - |b_B|\right| \leq |b - b_B|$ it follows that $b \in \text{BMO}$ implies that $|b| \in \text{BMO}$ also. That is, if two functions f and g belong to BMO then so too do $\min(f, g)$ and $\max(f, g)$. To see this, define

$$\begin{aligned} \max(f, g) &= \frac{1}{2}\Big(f + g + |f - g|\Big), \\ \min(f, g) &= \frac{1}{2}\Big(f + g - |f - g|\Big). \end{aligned}$$

The cutoff function is in BMO since it can be expressed as

$$b^k = \max(\min(b, k), -k)$$

where the constant functions $\pm k$ are in BMO (with $\|\pm k\|_{\text{BMO}} = 0$). Combining these observations,

$$\begin{aligned} \left\|b^k\right\|_{\text{BMO}} &= \frac{1}{4}\Big\| \min(b, k) - k + |\min(b, k) + k| \Big\|_{\text{BMO}} \\ &\leq \frac{1}{4}\left(\Big\| \min(b, k) \Big\|_{\text{BMO}} + \Big\| |\min(b, k) + k| \Big\|_{\text{BMO}}\right) \\ &\leq \frac{1}{2}\Big\| \min(b, k) \Big\|_{\text{BMO}} \\ &\leq \frac{1}{2}\left(\Big\|b\Big\|_{\text{BMO}} + \Big\||b - k|\Big\|_{\text{BMO}}\right) \\ &\leq \Big\|b\Big\|_{\text{BMO}}. \end{aligned}$$

This is by several applications of the triangle inequality and the above results on the BMO norms of $\pm k$ and $|b|$.

We now come to the H^1–BMO duality, a result due to Fefferman and Stein [FS72]. $\square$

Theorem 3.16.

(a) *Suppose* $b \in \mathrm{BMO}$. *Then the linear functional defined by* (3.9) *has a unique bounded extension to* H^1 *with norm at most* $\|b\|_{\mathrm{BMO}}$.

(b) *For any bounded linear functional* l *on* H^1 *there exists* $b \in \mathrm{BMO}$ *such that for all* $f \in H^1_0$ *we have* $l(f) = l_b(f)$ *defined above and*

$$\|b\|_{\mathrm{BMO}} \leq \|l_b\| .$$

The key idea for (a) is to prove the inequality

$$\left| \int_{\mathbb{R}^n} f(x)b(x)dx \right| \lesssim \|f\|_{H^1} \|b\|_{\mathrm{BMO}} \tag{3.10}$$

for all $f \in H^1$, $b \in \mathrm{BMO}$. To prove (b), given l on H^1 we will use the Riesz representation theorem to give a locally defined functions which can be stitched together to construct $b \in \mathrm{BMO}$.

Proof.

(a) Let $b \in \mathrm{BMO}$. First consider the case where b is bounded. For $f \in H^1_0$ we have the linear functional

$$l_b(f) = \sum_{k=1}^{N} \int_{\mathbb{R}^n} \lambda_k a_k(x) b(x) dx.$$

Since the a_k have the mean zero property, and $\|a_k\|_\infty < |B_k|^{-1}$,

$$\begin{aligned} |l_b(f)| &= \left| \sum_{k=1}^{N} \int_{B_k} \lambda_k a_k(x) \left(b(x) - b_{B_k}\right) dx \right| \\ &\leq \sum_{k=1}^{N} |\lambda_k| \frac{1}{|B_k|} \int_{B_k} |b(x) - b_{B_k}|\, dx = \|f\|_{H^1} \|b\|_{\mathrm{BMO}} . \end{aligned}$$

For the case where b is unbounded, consider an approximating sequence of cutoff functions b^k. Due to the previous lemma (Lemma 3.15) and since $b^k \to b$ a.e. it follows by the dominated convergence theorem that $|l_b(f)| \leq \|b\|_{\mathrm{BMO}} \|f\|_{H^1}$. That is, $b \in \mathrm{BMO}$ gives a bounded linear functional on the dense subspace H^1_0 which extends by density.

(b) Suppose that l is a bounded linear functional on H^1_0. First fix $B \subset \mathbb{R}^n$ and define the space of square integrable functions supported on B,

$$L^2_B = \{f \in L^2 : \mathrm{supp}(f) \subset B\},$$

equipped with L^2 norm $\|g\|_{L^2(B)} = \left(\int_B |g(x)|^2 dx\right)^{1/2}$. Consider further, the linear subspace of functions with mean zero,

$$L^2_{B,0} = \left\{ f \in L^2_B : \int_B f(x)dx = 0 \right\}.$$

Note that every element $g \in L^2_{B,0}$ is a scalar multiple of an L^2 atom for H^1, say $g = ca$. By Hölder's inequality and then the maximal theorem,

$$\|g\|_{H^1} = c\,\|a\|^1_H = c\,\|Ma\|_{L^1} \le c|B|^{1/2}\,\|Ma\|_{L^2} \lesssim c|B|^{1/2}\,\|a\|_{L^2} = |B|^{1/2}\,\|g\|^2_L .$$

It is clear that l extends to a linear functional on $L^2_{B,0}$ with norm $\lesssim |B|^{1/2}$. By the Riesz representation theorem for the Hilbert space $L^2_{B,0}$, there exists an element $F^B \in L^2_{B,0}$ so that

$$l(g) = \int_B F^B(x)g(x)dx$$

for all $g \in L^2_{B,0}$ with

$$\left(\int_B |F^B(x)|^2dx\right)^{1/2} \lesssim |B|^{1/2}.$$

For each ball $B \subset \mathbb{R}^n$ we have an associated function F^B. In order to construct a single global function f we note that if $B_1 \subset B_2$ then since F^{B_1} and F^{B_2} give the same linear functional on $L^2_{B_1,0}$, the difference, $F^{B_1} - F^{B_2}$ must be constant on B_1. Since functions in $L^2_{B_i,0}$ have the mean zero property, we can modify F^{B_2} by adding this constant so that the functional remains unchanged and the functions agree on B_1.

Stitching these functions together gives f defined on all of $\mathbb{R}^n$,

$$f(x) = F^B(x) + c_B$$

for all $x \in B$. Hölder's inequality gives

$$\begin{aligned}\frac{1}{|B|}\int_B |f(x) - c_B|dx &\le \left(\frac{1}{|B|}\int_B |f(x) - c_B|^2dx\right)^2 \\ &= \frac{1}{|B|^{1/2}}\left(\int_B |F^B(x)|^2dx\right)^{1/2} \le c\end{aligned}$$

such that $f \in \mathrm{BMO}$ with $\|f\|_{\mathrm{BMO}} \le c$. By the mean zero property, we have

$$l(g) = \int_B f(x)g(x)dx$$

for all $g \in L^2_{B,0}$ for some B. By considering progressively larger balls we extend this to functional to all of H^1_0.

□

Corollary 3.17. *The dual of H^1 is* BMO*.*

Proof. Since H^1_0 is dense in H^1 the linear functional l_b has a unique bounded extension to H^1. Given $g \in H^1$, find a sequence $g_j \in H^1_0$ such that $g_j \to g$ in H^1. The estimate above gives

$$|l_b(g_j - g)| \le \|g_j - g\|_{H^1}\,\|b\|_{\mathrm{BMO}}$$

such that $l_b(g_j)$ forms a Cauchy sequence in $\mathbb{C}$ converging to $l_b(g)$. Suppose we have another sequence $g'_j \in H^1_0$ also converging to g in H^1. Then

$$|l_b(g_j - g'_j)| \le \left\|g_j - g'_j\right\|_{H^1}\|b\|_{\mathrm{BMO}} \to 0.$$

That is, $l_b(g'_j) \to l_b(g)$ also. □

3.3 Tent spaces

3.3.1 Tent spaces in parabolic scaling

In the next chapter we introduce Cauchy problems in n spatial dimensions and one time dimension. That is, we look for solutions to PDE that are defined on $\mathbb{R}^{n+1}_+ := \mathbb{R}^n \times \mathbb{R}_+$.

In the case of the Navier–Stokes equations we look to find the velocity field of a given fluid flowing in $\mathbb{R}^n$. Physical impositions place restrictions on what kinds of solutions are meaningful. In light of energy considerations it is natural that the solution should be locally L^2 integrable in space.

The second relevant consideration for Navier–Stokes equations is scale invariance. Upon making the rescaling $(x,t) \mapsto (\lambda x, \lambda^2 t)$ a solution to the Navier–Stokes equations should remain a solution. The tent spaces we define are scale invariant in that they do not change under this transformation.

Definition 3.18 (Tent spaces)**.** Let $1 \le q < \infty$. For $1 < p < \infty$, the tent space $T^{p,q}(\mathbb{R}^{n+1}_+)$ is the completion of $C_c^\infty(\mathbb{R}^{n+1}_+)$ such that

$$\|f\|_{T^{p,q}} := \left(\int_{\mathbb{R}^n} \left(\int_0^\infty \int_{B(x,\sqrt{t})} t^{-n/2} |f(y,t)|^q dy\, dt \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}} < \infty.$$

while for $p = \infty$, it is the completion of $C_c^\infty(\mathbb{R}^{n+1}_+)$

$$\|f\|_{T^{\infty,q}} := \sup_{B(x,\sqrt{t})} \left(t^{-n/2} \int_0^t \int_{B(x,\sqrt{t})} |f(y,s)|^q dy\, ds \right)^{\frac{1}{q}} < \infty.$$

where the supremum is taken over all balls $B(x,\sqrt{t}) \subset \mathbb{R}^n$.

The tent spaces were introduced by Coifman, Meyer and Stein in elliptic scaling, whereas the presentation here is with parabolic scaling. We have a connection between the two,

$$(x,t) \mapsto f(x,t) \in T^{p,q} \iff (x,t) \mapsto t f(x,t^2) \in T^{p,q}_{\text{ell}},$$

where $T^{p,q}_{\text{ell}}$ are the tent spaces with elliptic scaling as in [CMS85], [Ame14].

Although we have defined these spaces over $\mathbb{R}^n$ with Lebesgue measure, tent spaces can be defined much more generally. Amenta for instance [Ame14] defines tent spaces over metric measure spaces under assumptions related to the doubling condition introduced in chapter 2.

The reason that these spaces are called tent spaces is more clear in the elliptic scaling, however for $q < \infty$ we can still view these norms as integrals over parabolic cones $\Gamma(x)$ defined by

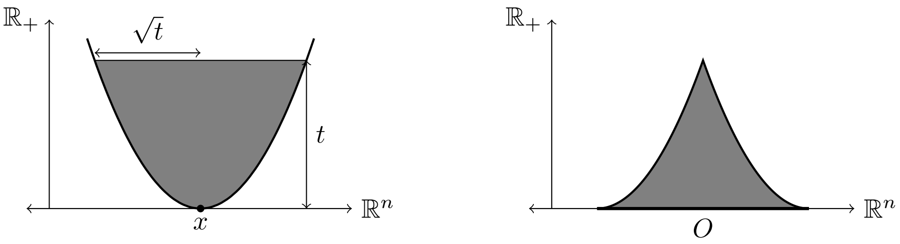
$$\Gamma(x) := \{(y,t) \in \mathbb{R}^{n+1}_+ : y \in B(x,\sqrt{t})\}.$$

For any $O \subset \mathbb{R}^n$ we define the cone over O to be

$$\Gamma(O) = \bigcup_{x \in O} \Gamma(x).$$

We also define the parabolic tent over O, as

$$T(O) := \Gamma(O^c)^c.$$



(a) $\Gamma(x)$. (b) $T(O)$.

Figure 3.1: Parabolic cones and tents.

In the case $q = \infty$ we are integrating over a parabolic cylinder $B(0, \sqrt{t}) \times (0, t)$ but we could take an equivalent definition using parabolic tents,

$$\|f\|_{T^{\infty,q}} := \sup_{B(x,\sqrt{t})} \left(t^{-n/2} \iint_{T(B)} |f(y,s)|^q dy\, ds \right)^{\frac{1}{q}} < \infty. \tag{3.11}$$

Lemma 3.19 (Averaging trick)**.** *Suppose $f \in L^2_{loc}(\mathbb{R}^{n+1}_+)$. Then*

$$\int_{\mathbb{R}^n} \int_0^\infty \int_{B(x,\sqrt{t})} t^{-n/2} f(y,t) dy\, dt\, dx = S_{n-1} \int_{\mathbb{R}^{n+1}_+} f(y,t) dy\, dt.$$

Proof. This is a straightforward application of Fubini–Tonelli's theorem.

$$\begin{aligned}
&\int_{\mathbb{R}^n} \int_0^\infty \int_{B(x,\sqrt{t})} t^{-n/2} f(y,t) dy\, dt\, dx \\
&\quad = \int_0^\infty \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \mathbb{1}_{B(y,\sqrt{t})}(x) dx\, t^{-n/2} f(y,t) dy\, dt \\
&\quad = S_{n-1} \int_0^\infty \int_{\mathbb{R}^n} t^{-n/2} t^{n/2} f(y,t) dy\, dt \\
&\quad = S_{n-1} \int_{\mathbb{R}^{n+1}_+} f(y,t) dy\, dt.
\end{aligned}$$

□

The constant S_{n-1} can be removed by if we define our tent spaces slightly differently. That is, by taking $|B(x, \sqrt{t})|^{-1}$ instead of $t^{-n/2}$ in the above definitions.

Example 3.20. It is clear that $T^{2,2}(\mathbb{R}^{n+1}_+) = L^2(\mathbb{R}^{n+1}_+)$,

$$\begin{aligned}
\|f\|^2_{T^{2,2}} &= \int_{\mathbb{R}^n} \int_0^\infty \int_{B(x,\sqrt{t})} t^{-n/2} |f(y,t)|^2 dy\, dt\, dx \\
&= S_{n-1} \int_{\mathbb{R}^{n+1}_+} |f(y,t)|^2 dy\, dt = S_{n-1} \|f\|^2_{L^2}.
\end{aligned}$$

Lemma 3.21. *For any compact $K \subset \mathbb{R}^{n+1}_+$ and $1 < p, q < \infty$,*

$$\|f\|_{L^q(K)} \lesssim_{K,p,q} \|\mathbb{1}_K f\|_{T^{p,q}} \lesssim_{K,p,q} \|f\|_{L^q(K)} . \tag{3.12}$$

Before proving this, we introduce another geometric idea, the (parabolic) shadow over a set $K \subset \mathbb{R}^{n+1}_+$,

$$S(K) := \{x \in \mathbb{R}^n : \Gamma(x) \cap K \neq \emptyset\}.$$

That is, $S(K)$ is the set of points in $x \in \mathbb{R}^n$ such that the parabolic cone based at x meets K.

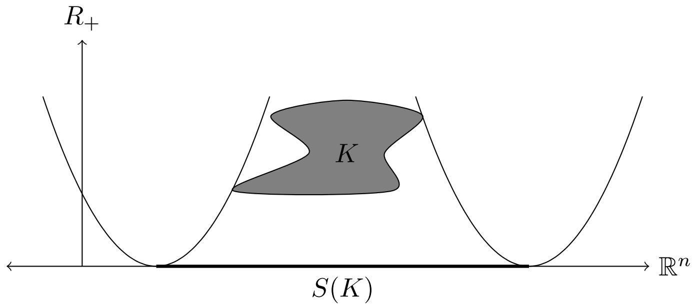


Figure 3.2: Parabolic shadow over a set.

Proof of Lemma 3.21. Let $K \subset \mathbb{R}^{n+1}_+$ be compact. For the case of $p \geq q$, by the averaging trick,

$$\begin{aligned}
\|f\|_{L^q(K)} &= \left(\int_{\mathbb{R}^{n+1}_+} \mathbb{1}_K |f(y,t)|^q dy\, dt \right)^{\frac{1}{q}} \\
&\lesssim \left(\int_{\mathbb{R}^n} \int_0^\infty \int_{B(x,\sqrt{t})} t^{-n/2} \mathbb{1}_K(y,t) |f(y,t)|^q dy\, dt\, dx \right)^{\frac{1}{q}} \\
&\leq \left(\int_{\mathbb{R}^n} |K|^{1/(p/q)'} \left(\int_0^\infty \int_{B(x,\sqrt{t})} t^{-n/2} \mathbb{1}_K(y,t) |f(y,t)|^q dy\, dt \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}} \\
&\lesssim_{K,p,q} \|\mathbb{1}_K f\|_{T^{p,q}} .
\end{aligned}$$

The second inequality is Hölder's inequality with exponents p/q and $(p/q)'$. On the other hand, when $p < q$, we use the Minkowski inequality

$$\begin{aligned}
\|f\|_{L^q(K)} &= \left(\int_{\mathbb{R}^{n+1}_+} \mathbb{1}_K |f(y,t)|^q dy\, dt \right)^{\frac{1}{q}} \\
&\lesssim \left(\int_{\mathbb{R}^n} \int_0^\infty \int_{B(x,\sqrt{t})} t^{-n/2} \mathbb{1}_K(y,t) |f(y,t)|^q dy\, dt\, dx \right)^{\frac{1}{q}} \\
&\leq \left(\int_{\mathbb{R}^n} \left(\int_0^\infty \int_{B(x,\sqrt{t})} t^{-n/2} \mathbb{1}_K(y,t) |f(y,t)|^q dy\, dt \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}} \\
&\lesssim_{K,p,q} \|\mathbb{1}_K f\|_{T^{p,q}} .
\end{aligned}$$

The second half of (3.12) follows similarly, For $p \geq q$, we use the Minkowski inequality

$$\begin{aligned}
\|\mathbb{1}_K f\|_{T^{p,q}} &= \left(\int_{\mathbb{R}^n} \left(\int_0^\infty \int_{B(x,\sqrt{t})} t^{-n/2} \mathbb{1}_K(y,t) |f(y,t)|^q dy\, dt \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}} \\
&\leq \left(\int_{\mathbb{R}^n} \left(\int_0^\infty \int_{B(x,\sqrt{t})} \mathbb{1}_K(y,t) t^{-n/2} |f(y,t)|^q dy\, dt \right) dx \right)^{\frac{1}{q}} \\
&\lesssim \left(\int_{\mathbb{R}^{n+1}_+} \mathbb{1}_K(y,t) |f(y,t)|^q dy\, dt \right)^{\frac{1}{q}} \\
&= \|f\|_{L^q(K)}
\end{aligned}$$

where the last inequality is by the averaging lemma. For $p < q$,

$$\begin{aligned}
\|\mathbb{1}_K f\|_{T^{p,q}} &= \left(\int_{\mathbb{R}^n} \left(\int_0^\infty \int_{B(x,\sqrt{t})} \mathbb{1}_K(y,t) |f(y,t)|^q dy\, dt \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}} \\
&\leq \left(\int_{S(K)} \left(\int_K |f(y,t)|^q dy\, dt \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}} \\
&\leq |S(K)|^{1/(q/p)'} \left(\int_{S(K)} \left(\int_K |f(y,t)|^q dy\, dt \right) dx \right)^{\frac{1}{q}} \\
&\lesssim_{p,q} \|f\|_{L^q(K)} .
\end{aligned}$$

The second inequality is Hölder's inequality over $S(K)$ with dual exponents $q/p > 1$ and $(q/p)'$. $\square$

The tent spaces also have nice duality properties.

Proposition 3.22. *Let $1 \leq p < \infty$, $1 < q < \infty$ with $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$. Then we have the following duality*

$$(T^{p,q}(K))' = T^{p',q'}(K)$$

under the $T^{2,2}/L^2$ inner product $\int_{\mathbb{R}^{n+1}} f(y,t) g(y,t) dy\, dt$.

Proof. We split into two cases, a midpoint result when $1 < p, q < \infty$ and an endpoint result when $p = 1$.

Midpoint result Let $f \in T^{p,q}$, $g \in T^{p',q'}$ with $1 < p, q < \infty$. Then,

$$\begin{aligned}
&\left| \int_{\mathbb{R}^{n+1}} f(y,s) g(y,s) dy\, ds \right| \\
&\quad \leq \int_{\mathbb{R}^n} \int_0^\infty \int_{B(x,\sqrt{s})} s^{-n/2} |f(y,s) g(y,s)| dy\, ds\, dx \\
&\quad = \int_{\mathbb{R}^n} \int_0^\infty \int_{B(x,\sqrt{s})} s^{-n/2q} |f(y,s)| s^{-n/2q'} |g(y,s)| dy\, ds\, dx \\
&\quad \leq \int_{\mathbb{R}^n} \left(\int_0^\infty \int_{B(x,\sqrt{s})} s^{-n/2} |f(y,s)|^q dy\, ds \right)^{\frac{1}{q}} \left(\int_0^\infty \int_{B(x,\sqrt{s})} s^{-n/2} |g(y,s)|^{q'} dy\, ds \right)^{\frac{1}{q'}} dx \\
&\quad \leq \|f\|_{T^{p,q}} \|g\|_{T^{p',q'}}
\end{aligned}$$

The first inequality is by the averaging trick, while the second is Hölder's inequality applied on $(0,\infty)\times B(x,\sqrt{s})$ and the third is Hölder's inequality on $\mathbb{R}^n$. That is, every element in $T^{p',q'}$ induces a bounded linear functional on $T^{p,q}$.

Conversely, we must show that all linear functionals on $T^{p,q}$ can be identified with elements of $T^{p',q'}$. We begin by assuming $1<p<q$. Suppose that l is a bounded linear functional on $T^{p,q}$. Let $K\subset\mathbb{R}^{n+1}_+$ be a compact set. If l is a bounded linear functional on $T^{p,q}$ then l induces a bounded linear functional on $L^q(K)$. That is, for all $f\in L^q$ supported in K,

$$l(f)=\int_{\mathbb{R}^{n+1}_+} f(y,s)g(y,s)dy\,ds$$

for some fixed $g\in L^{q'}(K)$. Taking an increasing family of K we can define g on all of $\mathbb{R}^{n+1}_+$. That is, $g\in L^{q'}_{\text{loc}}(\mathbb{R}^{n+1}_+)$ and the above formula holds for all compactly supported f. Define now, $g_K:=\mathbb{1}_K g$. It suffices to show that

$$\left(\int_{\mathbb{R}^n}\left(\int_0^\infty\int_{B(x,\sqrt{t})} t^{-n/2}|g_K(y,t)|^{q'}dy\,dt\right)^{\frac{p'}{q'}}dx\right)^{\frac{1}{p'}}\le C\,\|l\|$$

where the constant C is independent of K. Let r be the exponent dual to p'/q'. Note that $1<r<\infty$ since $p'>q'$. Then,

$$\begin{aligned}&\left(\int_{\mathbb{R}^n}\left(\int_0^\infty\int_{B(x,\sqrt{t})} t^{-n/2}|g_K(y,t)|^{q'}dy\,dt\right)^{\frac{p'}{q'}}dx\right)^{\frac{q'}{p'}}\\&\quad=\sup_\psi\int_{\mathbb{R}^n}\int_0^\infty\int_{B(x,\sqrt{t})} t^{-n/2}|g_K(y,t)|^{q'}dy\,dt\,\psi(x)dx\\&\quad=\sup_\psi\int_{\mathbb{R}^{n+1}_+}|g_K(y,t)|^{q'}M_t(\psi)dy\,dt\end{aligned}$$

where the supremum is taken over all nonnegative $\psi\in L^r(\mathbb{R}^n)$ for which $\|\psi\|_{L^r(\mathbb{R}^n)}\le 1$ and

$$M_t(\psi):=t^{-n/2}\int_{B(y,\sqrt{t})}\psi(x)dx.$$

Note that this last integral is just the L^p-L^q dual pairing $\langle f_K,g_K\rangle$ over $\mathbb{R}^{n+1}_+$ where

$$f_K:=\begin{cases}\overline{g_K(y,t)}^{q'/2}|g_K(y,t)|^{q'/2-1}M_t(\psi)(y). & g_K(y,t)\neq 0\\ 0 & g_K(y,t)=0\end{cases}$$

Notice that $f_K \in T^{p,q}$ since g_K has compact support in $\mathbb{R}^{n+1}_+$. That is, since $q(q'-1) = \frac{q'-1}{1-\frac{1}{q'}} = q'$,

$$\begin{aligned}
\|f_K\|^p_{T^{p,q}} &\leq \int_{\mathbb{R}^n} \left(\int_0^\infty \int_{B(x,\sqrt{t})} t^{-n/2} |g_K(y,t)|^{q'} M_t(\psi)^q dy\, dt \right)^{\frac{p}{q}} dx \\
&\leq \int_{\mathbb{R}^n} \left(\int_0^\infty \int_{B(x,\sqrt{t})} t^{-n/2} |g_K(y,t)|^{q'} \mathcal{M}\psi(x)^q dy\, dt \right)^{\frac{p}{q}} dx \\
&= \int_{\mathbb{R}^n} (\mathcal{M}\psi(x))^p \left(\int_0^\infty \int_{B(x,\sqrt{t})} t^{-n/2} |g_K(y,t)|^{q'} dy\, dt \right)^{\frac{p}{q}} dx \\
&\leq \|\mathcal{M}\psi\|^p_{L^r} \left(\int_{\mathbb{R}^n} \left(\int_0^\infty \int_{B(x,\sqrt{t})} t^{-n/2} |g_K(y,t)|^{q'} dy\, dt \right)^{\frac{r'p}{q}} dx \right)^{\frac{1}{r'}} \\
&\lesssim \|\psi\|^p_{L^r} \left(\int_{\mathbb{R}^n} \left(\int_0^\infty \int_{B(x,\sqrt{t})} t^{-n/2} |g_K(y,t)|^{q'} dy\, dt \right)^{\frac{r'p}{q}} dx \right)^{\frac{1}{r'}} \\
&\leq \left(\int_{\mathbb{R}^n} \left(\int_0^\infty \int_{B(x,\sqrt{t})} t^{-n/2} |g_K(y,t)|^{q'} dy\, dt \right)^{\frac{r'p}{q}} dx \right)^{\frac{1}{r'}}.
\end{aligned}$$

Notice however, $r'q'/q = p'$ since $1 < r < \infty$ implies that L^r is reflexive and because r was chosen dual to p'/q'. That is,

$$\|f_K\|^p_{T^{p,q}} \leq \left(\int_{\mathbb{R}^n} \left(\int_0^\infty \int_{B(x,\sqrt{t})} t^{-n/2} |g_K(y,t)|^{q'} dy\, dt \right)^{p'} dx \right)^{\frac{q'}{p'}},$$

so that $f \in T^{p,q}$ by Lemma 3.21. In particular,

$$\|g\|^{q'}_{T^{p'q'}} \leq \|l\| \, \|f\|_{T^{p,q}} \leq \|l\| \, \|g_K\|^{q'/q}_{T^{p',q'}} \lesssim \|l\| \, .$$

Since this estimate is independent of K we have shown $g \in T^{p',q'}$ thus completing the proof when $p < q$.

To check the case $p > q$, we check that $T^{p,q}$ is reflexive. In light of the Eberlein Šmulian theorem (see for instance [AK06, pp.24]) it suffices to check that every bounded sequence in $T^{p',q'}$ has a weakly convergent subsequence. Let f_n be a uniformly bounded sequence, that is, $\|f_n\| \leq 1$ for all $n \geq 1$. We select a subsequence f_{n_k} such that f_{n_k} converges weakly on $L^{q'}(K)$ for every compact $K \subset \mathbb{R}^{n+1}_+$. Let $l \in (T^{p',q'})'$. Since $p' < q'$ we have $g \in T^{p,q}$ such that $l(f) = \langle f, g \rangle$. For any given $\epsilon > 0$ choose $K_\epsilon \subset \mathbb{R}^{n+1}_+$ such that

$$\|g - \mathbb{1}_{K_\epsilon} g\|_{T^{p,q}} \leq \epsilon.$$

Thus,

$$l(f_{n_j}) - l(f_{n_k}) = \langle f_{n_j} - f_{n_k}, g \rangle = \langle f_{n_j} - f_{n_k}, g_K \rangle + \langle f_{n_j} - f_{n_k}, g - g_K \rangle.$$

The first term tends to zero by weak convergence and the second term is bounded by 2ϵ. Hence, $T^{p,q}$ is reflexive and has dual $T^{p',q'}$ for all $1 < p, q < \infty$.

Endpoint result Now consider the case $p = 1$. Let $f \in T^{1,q}$ and $g \in T^{\infty,q'}$ where $1 < q < \infty$. We will use the tent definition of $T^{\infty,q'}$ as in (3.11) as opposed to the definition with cylinders. We begin by defining truncated cones

$$\Gamma^h(x) := \{(y,t) \in \mathbb{R}^{n+1}_+ : t \in [0,h] \text{ and } y \in B(x,\sqrt{t})\}$$

and the associated Lusin operators,

$$\mathcal{A}^h_q f(x) := \left(\int_0^h t^{-n/2} \int_{B(x,\sqrt{t})} |f(y,t)|^q dy\, dt\right)^{\frac{1}{q}}.$$

Note that $\mathcal{A}^h_q$ are increasing in h. We define the stopping time $h(x)$ as

$$h_{g,q,M}(x) = \sup_{h>0}\left\{h > 0 : \mathcal{A}^h_q(g)(x) \le M \sup_{B(x_0,\sqrt{t})\ni x}\left(t^{-n/2}\int_{T(B(x_0,\sqrt{t}))} |f(y,s)|^q dy\, ds\right)^{\frac{1}{q}}\right\}.$$

By Lemma 3.23 there exists M sufficiently large such that

$$\int_0^\infty \int_{\mathbb{R}^n} |f(x)||g(x)| dy\, dt \lesssim \int_{\mathbb{R}^n}\int_0^{h(x)}\int_{B(x,\sqrt{t})} |f(y,t)||g(y,t)| dx\, dt\, dx$$

where $h(x) = h_{g,q,M}(x)$. By Hölder's inequality,

$$\begin{aligned}
&\int_{\mathbb{R}^n}\int_0^{h(x)}\int_{B(x,\sqrt{t})} |f(y,t)||g(y,t)| dx\, dt\, dx \\
&\quad \le \int_{\mathbb{R}^n} \mathcal{A}^{h(x)}_q f(x) \mathcal{A}^{h(x)}_{q'} g(x) dx \\
&\quad \le M \int_{\mathbb{R}^n} \mathcal{A}^{h(x)}_q \sup_{B(x_0,\sqrt{t})\ni x}\left(t^{-n/2}\iint_{T(B(x_0,\sqrt{t}))} |f(y,s)|^{q'} dy\, ds\right)^{\frac{1}{q'}} dx \\
&\quad \lesssim \|f\|_{T^{1,q}(\mathbb{R}^{n+1}_+)} \|g\|_{T^{\infty,q'}(\mathbb{R}^{n+1}_+)}.
\end{aligned}$$

Every $g \in T^{\infty,q'}$ induces a bounded linear functional on $T^{1,q}$ and thus $T^{\infty,q'} \subset (T^{1,q})'$.

Conversely suppose l is a bounded linear functional on $T^{1,q}$. As in the case $1 < p, q < \infty$ we construct a function $g \in L^{q'}_{\text{loc}}(\mathbb{R}^{n+1}_+)$ such that

$$l(f) = \int_0^\infty \int_{\mathbb{R}^n} f(y,t)\overline{g(y,t)} dy\, dt$$

for all $f \in T^{1,q}$ with compact support. We just need to check that $g \in T^{\infty,q'}$. Fix a ball $B \subset \mathbb{R}^n$. For all nonnegative $\psi \in L^q(T(B))$ with $\|\psi\|_{L^{q'}(T(B))} \le 1$ we have

$$\begin{aligned}
\|\psi\|_{T^{1,q}} &= \int_{S(T(B))}\left(\int_0^\infty t^{-n/2}\int_{B(x,\sqrt{t})} |\psi(y,t)|^q dy\, dt\right)^{\frac{1}{q}} dx \\
&\le |S(T(B))|^{\frac{1}{q'}}\left(\int_{S(T(B))}\left(\int_0^\infty t^{-n/2}\int_{B(x,\sqrt{t})} |\psi(y,t)|^q dy\, dt\right)^{\frac{q}{q}} dx\right)^{\frac{1}{q}} \\
&\le |S(T(B))|^{\frac{1}{q'}} \|\psi\|_{T^{q,q}} = |B|^{\frac{1}{q'}} \|\psi\|_{L^q(\mathbb{R}^{n+1}_+)} \le |B|^{\frac{1}{q'}}.
\end{aligned}$$

The last line used $T^{q,q} = L^q$. This result is a simple consequence on the averaging trick; see Example 3.20. We also used the fact that the shadow cast by a tent over a set O is just the set O. In particular $\psi \in T^{1,q}$ such that

$$l(\psi) = \int_{T(B)} \psi(y,t)\overline{g(y,t)}dy\,dt.$$

By the results established above,

$$\begin{aligned}
\|g\|_{T^{\infty,q'}} &= \sup_{B(x,\sqrt{t})\subset\mathbb{R}^n} \left(t^{-n/2} \iint_{T(B(x,\sqrt{t}))} |g(y,t)|^{q'} dy\,dt \right)^{\frac{1}{q'}} \\
&= \sup_{B(x,\sqrt{t})} (t^{-n/2})^{1/q'} \sup_{\psi} \int_{T(B(x,\sqrt{t}))} \psi(y,t)\overline{g(y,t)}dy\,dt \\
&= \sup_{B(x,\sqrt{t})} (t^{-n/2})^{1/q'} \sup_{\psi} l(\psi) \\
&\leq \sup_{B(x,\sqrt{t})} (t^{-n/2})^{1/q'} \sup_{\psi} \|l\|\,\|\psi\|_{T^{1,q}} \leq \|l\|
\end{aligned}$$

where the supremum is taken over $\|\psi\|_{L^q(T(B))} \leq 1$. This completes the proof that $(T^{1,q})' \subset T^{\infty,q'}$. $\square$

Lemma 3.23. *For each measurable function $g : \mathbb{R}^{n+1}_+ \to \mathbb{C}$ and each $q \in (1,\infty)$ there exists $M > 0$ such that whenever Φ is a nonnegative measurable function on $\mathbb{R}^{n+1}_+$ we have*

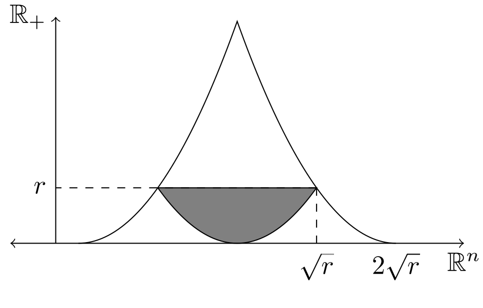
$$\int_{\mathbb{R}^{n+1}_+} \Phi(y,t)t^{n/2}dy\,dt \lesssim \int_{\mathbb{R}^n}\int_0^{h_{g,q,M}(x)} \int_{B(x,\sqrt{t})} \Phi(y,t)dy\,dt\,dx.$$

Proof. Fix $B = B(x_0,\sqrt{r}) \subset \mathbb{R}^n$. By Lemma 3.24 there exists M such that $|\{x \in B : h_{g,q,M}(x) \geq \sqrt{r}\}| \gtrsim r^{n/2}$. It follows by an application of Fubini–Tonelli's theorem,

$$\begin{aligned}
\int_{\mathbb{R}^{n+1}_+} \Phi(y,t)t^{n/2}dy\,dt &\lesssim \int_{\mathbb{R}^{n+1}_+} \Phi(y,t) \int_{\{x\in B(y,\sqrt{t}):h_{g,p,M}(x)\geq\sqrt{t}\}} dx\,dy\,dt \\
&= \int_{\mathbb{R}^n}\int_0^{h_{g,p,M}(x)} \int_{B(x,\sqrt{t})} \Phi(y,t)dy\,dt\,dx.
\end{aligned}$$ $\square$

Lemma 3.24. *With g, q defined as in Lemma 3.23 there exists $M > 0$ such that $|\{x \in B : h_{g,q,M}(x) \geq \sqrt{r}\}| \gtrsim r^{n/2}$.*

Proof. It is clear that we can enclose the truncated cone $\Gamma^r(x)$ in tent $T(B(x_0, 2\sqrt{r}))$.



It follows, again by the averaging lemma,

$$\begin{aligned}\int_B \mathcal{A}_q^r g(x)^q dx &= \int_B \iint_{\Gamma^r(x)} t^{-n/2}\mathbb{1}_{T(B(x_0,2\sqrt{r}))}(y,t)|g(y,t)|^q dy\,dt\,dx \\ &\leq \int_B \iint_{\Gamma(x)} t^{-n/2}\mathbb{1}_{T(B(x_0,2\sqrt{r}))}(y,t)|g(y,t)|^q dy\,dt\,dx \\ &\lesssim \iint_{T(B(z,2\sqrt{t}))} |g(y,t)|^q dy\,dt \\ &\lesssim r^{n/2} \inf_{x\in B} \sup_{B(z,\sqrt{t})\ni x} \left(t^{-n/2} \iint_{T(B(z,2\sqrt{t}))} |g(y,t)|^q dy\,dt \right).\end{aligned}$$

On the other hand for $M > 0$ to be chosen later, we estimate

$$\begin{aligned}&\int_B \mathcal{A}_q^r g(x)^q dx \\ &\geq M^q \inf_{x\in B} \sup_{B'\ni x} t^{-n/2} \iint_{T(B')} |g(y,t)|^q dy\,dt \\ &\quad\times \left| \left\{ x \in B : \mathcal{A}_q^r g(x) > M \inf_{x\in B} \sup_{B'\ni x} \left(t^{-n/2} \iint_{T(B')} |g(y,t)|^q dy\,dt \right)^{\frac{1}{q}} \right\} \right|.\end{aligned}$$

Combining this with the previous estimate

$$M^q \left| \left\{ x \in B : \mathcal{A}_q^r g(x) > M \inf_{x\in B} \sup_{B'\ni x} \left(t^{-n/2} \iint_{T(B')} |g(y,t)|^q dy\,dt \right)^{\frac{1}{q}} \right\} \right| \lesssim r^{n/2}.$$

It follows that

$$\begin{aligned}&\left| \left\{ x \in B : \mathcal{A}_q^r g(x) \leq M \inf_{x\in B} \sup_{B'\ni x} \left(t^{-n/2} \iint_{T(B')} |g(y,t)|^q dy\,dt \right)^{\frac{1}{q}} \right\} \right| \\ &\quad\geq \left| \left\{ x \in B : \mathcal{A}_q^r g(x) \leq M \sup_{B'\ni x} \left(t^{-n/2} \iint_{T(B')} |g(y,t)|^q dy\,dt \right)^{\frac{1}{q}} \right\} \right| \\ &\quad= |B| - \left| \left\{ x \in B : \mathcal{A}_q^r g(x) > M \sup_{B'\ni x} \left(t^{-n/2} \iint_{T(B')} |g(y,t)|^q dy\,dt \right)^{\frac{1}{q}} \right\} \right| \\ &\quad\geq \left(1 - \frac{C}{M^q}\right) r^{n/2}.\end{aligned}$$

for some constant $C > 0$. Choosing $M > C^{1/q}$,

$$\left| \left\{ x \in B : \mathcal{A}_q^r g(x) \leq M \sup_{B'\ni x} \left(t^{-n/2} \iint_{T(B')} |g(y,t)|^q dy\,dt \right)^{\frac{1}{q}} \right\} \right| \gtrsim r^{n/2}.$$

In fact we are done because It follows from the definition of $h_{g,q,M}$ that

$$\mathcal{A}_q^r g(x) \leq M \sup_{B'\ni x} \left(t^{-n/2} \iint_{T(B')} |g(y,t)|^q dy\,dt \right)^{\frac{1}{q}}$$

if and only if $h_{g,q,M} \geq r$ since $\mathcal{A}_q^h g$ is increasing in h. That is,

$$\left|\left\{x \in B : h_{g,q,M}(x) \geq \sqrt{r}\right\}\right| \gtrsim r^{n/2}. \qquad \square$$

The endpoint result above identifies $T^{\infty,q'}$ as the dual of $T^{1,q}$ for $1 < q < \infty$. This duality result is incomplete in the sense that there are several endpoints to be considered. We can extend our endpoint result to its endpoints. The case $p = q = 1$ is simple because we know $T^{1,1}(\mathbb{R}^{n+1}_+) = L^1(\mathbb{R}^{n+1}_+)$ has dual $L^\infty(\mathbb{R}^{n+1}_+)$. In order to look at the case $p = 1$, $q = \infty$ we must first introduce another tent space.

Definition 3.25. The tent space $T^{1,\infty}$ is the completion of $C_c^\infty(\mathbb{R}^{n+1}_+)$ such that

$$\|f\|_{T^{1,\infty}} := \int_{\mathbb{R}^n} \sup_{B(y,\sqrt{t})\ni x} |f(y,t)| dx < \infty.$$

Alternatively we write $f^*(x) := \sup_{B(y,\sqrt{t})} |f(y,t)|$ and express the above as $f \in T^{1,\infty}$ when $f^* \in L^1(\mathbb{R}^n)$.

Proposition 3.26. *Let $f \in T^{1,\infty}$ and $g \in T^{\infty,1}$. Then*

$$|\langle f, g\rangle| \lesssim \|f\|_{T^{1,\infty}} \|g\|_{T^{\infty,1}}.$$

Proof. Let $f \in T^{1,\infty}$ and $g \in T^{\infty,1}$. Define $O_\lambda = \{x \in \mathbb{R}^n : f^*(x) > \lambda\}$. Then

$$\begin{aligned} |\langle f, g\rangle| &= \int_0^\infty \int_{\mathbb{R}^n} |f(t,x)||g(t,x)| dx\, dt \\ &= \int_0^\infty \int_{\mathbb{R}^n} \int_0^\infty \mathbb{1}_{\{(t,x)\in\mathbb{R}^{n+1}_+ : |f(t,x)|>\lambda\}}(\lambda)|g(t,x)| d\lambda\, dx\, dt \\ &\leq \int_0^\infty \iint_{T(O_\lambda)} |g(t,x)| dx\, dt\, d\lambda. \end{aligned}$$

Indeed, if $(t,y) \in \mathbb{R}^{n+1}_+$ is such that $(t,y) \notin T(O_\lambda)$ then there exists $x \in \mathbb{R}^n$ such that $f^*(x) \leq \lambda$ and $y \in B(x,\sqrt{t})$. Therefore $f(t,y) \leq \lambda$, otherwise $x \in O_\lambda$.

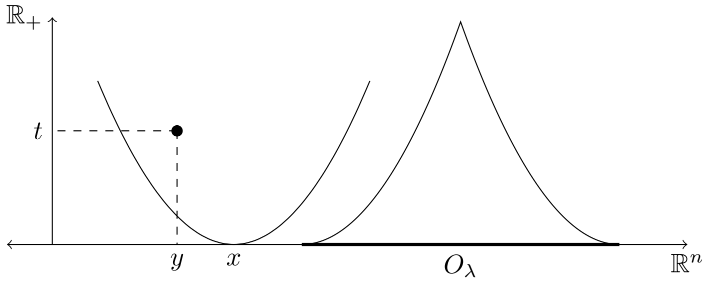


Figure 3.3: If $(y,t) \notin T(O_\lambda)$ then $f(t,y) \leq \lambda$.

Let $\{Q_j\}_{j=1}^N$ be a Whitney decomposition of O_λ, see Lemma 2.9 and the remarks that follow it. Let $(B(x_j, cr_j))_{j=1}^N$ be balls such that $Q_j \subset B(x_j, cr_j)$ for all $1 \leq j \leq N$, $c > 1$ where $r_j \sim \operatorname{diam}(Q_j)$. Notice that if $y \in Q_j$ then for sufficiently large c,

$$\operatorname{dist}(y, B(x_j, cr_j)^c) \gtrsim \operatorname{diam}(Q_j) \geq c \operatorname{dist}(y, O_\lambda^c) \geq c\sqrt{t}.$$

Pick $c > 1$ sufficiently large so that $T(O_\lambda) \subset \bigcup_{j=1}^N T(B(x_j, cr_j))$. We conclude,

$$\begin{aligned}
|\langle f, g\rangle| &\leq \int_0^\infty \iint_{T(O_\lambda)} |g(t,x)| dx\, dt\, d\lambda \\
&\lesssim \int_0^\infty \sum_{j=1}^N \iint_{T(B(x_j, cr_j))} |g(t,x)| dx\, dt\, d\lambda \\
&\lesssim \int_0^\infty \sum_{j=1}^N |Q_j| d\lambda \, \|g\|_{T^{\infty,1}} \\
&= \int_0^\infty |O_\lambda| d\lambda \, \|g\|_{T^{\infty,1}} \\
&= \int_{\mathbb{R}^n} |f^*(x)| dx \, \|g\|_{T^{\infty,1}} \\
&= \|f\|_{T^{1,\infty}} \|g\|_{T^{\infty,1}} .
\end{aligned}$$

□

3.3.2 Tent spaces and H^1

This section outlines the connection between tent spaces and the Hardy space H^1. This connection, rephrased in the modern language of tent spaces, is the heart of the seminal work by Fefferman and Stein [FS72]. The proofs will not be presented here.

In fact we have already seen one connection, the maximal characterisation of H^1. We reformulate the nontangential maximal function definition of H^1, Theorem 3.2, in the language of tent spaces.

Proposition 3.27. *Let $v \in H^1$. Then*

$$Q_\infty v(x,t) := e^{t\Delta} v(x) \in T^{1,\infty}$$

and $\|Q_\infty v\|_{T^{1,2}} \sim \|v\|_{H^1}$.

That is, we view the heat extension as an injection from H^1 into $T^{1,\infty}$. This observation is due to Fefferman and Stein [FS72].

An alternative definition of the Hardy space H^1 is the square function characterisation again due to Fefferman and Stein [FS72]. We present this characterisation in the language of tent space.

Proposition 3.28. *If $v \in H^1$ then*

$$Q_2 v(t,x) := \nabla e^{t\Delta} v(x) \in T^{1,2}$$

and $\|Q_2 v\|_{T^{1,2}} \sim \|v\|_{H^1}$. *Conversely, if $f \in T^{1,2}$ then*

$$S_2 f(x) := \frac{1}{2} \int_0^\infty \operatorname{div} e^{t\Delta} f(t,x) dt \in H^1$$

and $\|S_2 f\|_{H^1} \lesssim \|f\|_{T^{1,2}}$.

Here we have written down an injection from H^1 to $T^{1,2}$ and a projection back from $T^{1,2}$ to H^1. In fact S_2 is a left inverse of Q_2. For all $v \in H^1$,

$$S_2 Q_2 v = \frac{1}{2}\int_0^\infty \operatorname{div} e^{t\Delta}\nabla e^{t\Delta} v(x)dt = \frac{1}{2}\int_0^\infty \Delta e^{2t\Delta} v(x)dt = v(x)$$

since

$$I = \int_0^\infty \Delta e^{t\Delta} dt,$$

in the strong operator topology, is a reproducing formula, see [McI10].

More generally:

Proposition 3.29. *Let $(\Psi_t)_{t\geq 0}$ be a family of operators defined as Fourier multipliers*

$$\widehat{\Psi_t f}(\xi) = \psi(t|\xi|^2)\widehat{f}(\xi) \qquad \forall \xi \in \mathbb{R}^n,\ t > 0 \tag{3.13}$$

where $\psi \in H^\infty(S_{\omega+})$ for some $\omega > 0$ and there exists $\alpha, \beta, C > 0$ such that

$$|\psi(\zeta)| \leq C \min(|\zeta|^\alpha, |\zeta|^{-\beta}) \qquad \forall \zeta \in S_{\omega+}.$$

Then the operator S defined by

$$SF(x) = \int_0^\infty \Psi_t F(t,\cdot)(x)dt$$

extends to a bounded operator from $T^{1,2}$ to H^1.

A special case is proven in [CMS85, Theorem 6] and the general case is proven, for instance, in [AMM13].

3.3.3 Tent spaces and BMO

Similarly the heat extension also defines an injection from BMO into the tent space $T^{\infty,2}$. This gives the Carleson measure characterisation of BMO.

Lemma 3.30. *The map,*

$$Qu(x,t) := e^{t\Delta}u(x) \qquad \forall x \in \mathbb{R}^n,\ t \geq 0,$$

extends to a bounded operator from BMO to $T^{\infty,2}$ and $\|Qv\|_{T^{\infty,2}} \sim \|v\|_{\mathrm{BMO}}$.

Chapter 4

Cauchy problems

4.1 Abstract Cauchy problems

In this chapter we consider Cauchy problems for abstract evolution equations and the related semigroup theory. Several notions of solutions will be given by first considering the homogeneous problem.

We discuss maximal regularity for parabolic PDE and the connection to boundedness of the maximal regularity operator. The main result is the boundedness of the maximal regularity operator on the tent space $T^{\infty,2}$. The chapter ends on a brief discussion of semilinear Cauchy problems.

Let X be a Banach space and $A : D(A) \subset X \to X$ a closed, densely defined, (unbounded) linear operator and let $f : \mathbb{R}_+ \to X$ be continuous. The inhomogeneous Cauchy problem is

$$\begin{cases} u'(t) + Au(t) = f(t) & t \geq 0 \\ u(0) = u_0 \end{cases} \tag{4.1}$$

where $u_0 \in X$ we call the initial value.

One of the simplest examples of a Cauchy problem is the homogeneous heat equation.

Example 4.1 (Homogeneous heat equation)**.** Take $A = -\Delta$ with domain $D(A) = W^{2,2}(\mathbb{R}^n)$ and $f = 0$.

$$\begin{cases} \partial_t u(x,t) = \Delta u(x,t) & t \geq 0, x \in \mathbb{R}^n \\ u(x,0) = u_0(x) \end{cases} . \tag{4.2}$$

It is well known that the heat equation admits a smooth solution for initial data $u_0 \in L^2(\mathbb{R}^n)$. In fact, the solution can be written down explicitly,

$$u(x,t) = \frac{1}{(4\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-|x-y|^2/4t} g(y) dy \tag{4.3}$$

for $x \in \mathbb{R}^n$, $t > 0$.

Just as we are concerned with weaker forms of solution to PDE, there are several notions of a solution $u : \mathbb{R}_+ \to X$ to the Cauchy problem (4.1). Before we discuss these however, we must first introduce some semigroup theory.

4.2 Semigroup theory

Definition 4.2. A semigroup of operators is a map $T : \mathbb{R}_+ \to \mathcal{L}(X)$ such that

(i) $T(0) = I$,

(ii) $T(t+s) = T(t)T(s)$.

Definition 4.3. We say that a linear operator $A : D(A) \subset X \to X$ is a generator for the semigroup T

$$D(A) = \left\{ x \in X : \lim_{h \downarrow 0} \frac{T(h)x - x}{h} \text{ exists} \right\}$$

and for all $x \in D(A)$,

$$Ax = \lim_{h \downarrow 0} \frac{T(h)x - x}{h}$$

Since we have specified the domain of A, the generator of a semigroup is unique. Conversely, given an operator A a natural question is whether $-A$ generates a unique semigroup. As, we show later, the answer to this question depends on on the continuity of the semigroup and whether A is densely defined.

Remark. It appears as if we are taking a derivative of the semigroup; as we will soon see, this insight is the correct one. Were we to possess an appropriate functional calculus for A, then connection between semigroups and generators would be explicit, if $-A$ is the generator of a semigroup T then $T(t) = e^{-tA} \in \mathcal{L}(X)$ for all $t \geq 0$.

Example 4.4. The unbounded operator $-A = \Delta$ with domain $W^{2,2}$ generates a semigroup, namely the heat semigroup $e^{t\Delta}$. One way to define this is through the H^∞ functional calculus,

$$e^{t\Delta} = \frac{1}{2\pi i} \oint_\gamma e^{-tz}(zI + \Delta)^{-1} dz$$

for $t > 0$, and $e^{t\Delta}\big|_{t=0} = I$. The contour γ encloses $\mathbb{R}_+ \subset \mathbb{C}$ (the spectrum of $-\Delta$), see [McI10] for details. See also [Kat95, pp.495] for the heat equation in $X = L^2(\mathbb{R}^3)$. Another possible definition is as a Fourier multiplier. For all $u \in L^2(\mathbb{R}^n)$,

$$\widehat{e^{t\Delta}u}(\xi) = e^{-t|\xi|^2}\widehat{u}(\xi) \qquad \forall \xi \in \mathbb{R}^n.$$

We now turn to the continuity of semigroups. There is a hierarchy of progressively weaker topologies for which we could define continuity of the semigroup. What follows is a presentation of a subset of this hierarchy.

Definition 4.5 (Uniformly continuous semigroup)**.** We say that a semigroup T is uniformly continuous if it is continuous in the operator norm topology. That is, for all $t \geq 0$, we require

$$\|T(t+h) - T(t)\|_{\mathcal{L}(X)} \to 0 \qquad \text{as } h \downarrow 0.$$

In some ways this is the natural topology to take on $\mathcal{L}(X)$, but it turns out to be too strong a topology. One way to demonstrate this is the following result.

Proposition 4.6. *$-A$ is the generator of a uniformly continuous semigroup if and only if $A \in \mathcal{L}(X)$.*

Proof. If A is bounded then by the power series functional calculus for instance

$$T(t) = e^{-tA} = \sum_{k\geq 0} \frac{(-tA)^k}{k!}.$$

Since $A \in \mathcal{L}(X)$, $D(A) = X$ and thus the semigroup is uniquely defined. It is an easy computation that T is a uniformly continuous semigroup with generator $-A$. Estimating the power series,

$$\|T(t+h) - T(t)\| \leq \|T(t)\|_{\mathcal{L}(X)} \|T(h) - I\|_{\mathcal{L}(X)} \leq h \|A\| e^{(t+h)\|A\|_{\mathcal{L}(X)}} \to 0$$

as $h \downarrow 0$ for all $t \geq 0$ and similarly,

$$\begin{aligned} \left\| \frac{T(t+h) - T(t)}{h} - T(t)A \right\|_{\mathcal{L}(X)} &\leq \|T(t)\|_{\mathcal{L}(X)} \left\| \frac{T(h) - I}{h} - A \right\|_{\mathcal{L}(X)} \\ &\leq \|T(t)\|_{\mathcal{L}(X)} \|A\|_{\mathcal{L}(X)} \|T(t) - I\|_{\mathcal{L}(X)} \to 0 \end{aligned}$$

as $t \downarrow 0$. That is, T is uniformly continuous with generator $-A$.

On the other hand, assuming T is a uniformly continuous semigroup,

$$\frac{1}{\lambda} \int_0^\lambda T(s) ds \xrightarrow{\mathcal{L}(X)} T(0) = I \qquad \text{as } \lambda \downarrow 0.$$

Pick $\lambda > 0$ small enough that

$$\left\| I - \frac{1}{\lambda} \int_0^\lambda T(s) ds \right\|_{\mathcal{L}(X)} < 1.$$

It follow that $\int_0^\lambda T(s)ds$ is invertible. We compute

$$\begin{aligned} \frac{1}{h}(T(h) - I) \int_0^\lambda T(s) ds &= \frac{1}{h} \left(\int_0^\lambda T(s+h) ds - \int_0^\lambda T(s) ds \right) \\ &= \frac{1}{h} \left(\int_\lambda^{\lambda+h} T(s) ds - \int_0^h T(s) ds \right) \end{aligned}$$

and thus,

$$\frac{1}{h}(T(h) - I) = \frac{1}{h} \left(\int_\lambda^{\lambda+h} T(s) ds - \int_0^h T(s) ds \right) \left(\int_0^\lambda T(s) ds \right)^{-1}.$$

Letting $h \downarrow 0$ it is clear that $\frac{1}{h}(T(h) - I)$ converges in norm to the bounded operator $A := (T(\lambda) - I) \left(\int_0^\lambda T(s) ds \right)^{-1}$, the unique generator of T. ☐

Definition 4.7 (Strongly continuous (C_0) semigroup)**.** We say that a semigroup is strongly continuous or C_0 if it is continuous in the strong operator topology. That is, for all $x \in X$, $t \geq 0$,

$$\|T(t+h)x - T(t)x\|_X \to 0 \qquad \text{as } h \downarrow 0.$$

This turns out to be a very useful definition. The topic of C_0 semigroups is a large field in its own right. The short monograph by Engel and Nagel [EN06] provides a good introduction.

An even weaker form of continuity is given by the weak operator topology.

Definition 4.8 (Weakly continuous semigroup)**.** A semigroup is weakly continuous if it is continuous in the weak operator topology. That is, if for all $x \in X, y \in X^*$,

$$\langle T(t+h)x, y\rangle - \langle T(t)x, y\rangle \xrightarrow{\mathbb{C}} 0 \qquad \text{as } h \downarrow 0.$$

It turns out however that this is not a useful concept since the weakly continuous semigroups turn out to be exactly the C_0 semigroups. See for instance [EN06, pp.5–6] Proposition §1.1.3.

The functional calculus required to produce C_0 semigroups turns out to be a little odd. The linear operator $-A$ generates a C_0 semigroup if it admits a half-plane functional calculus, that is, if the spectrum of $-A$ lies in the right half plane. See for instance [BHJ13]. The way this is more commonly presented however is the famous Hille–Yosida theorem discovered independently by Yosida [Yos48] and Hille [Hil50].

Theorem 4.9 (Hille–Yosida)**.** *Let A be a closed linear (unbounded) densely defined, operator. Then A generates a contraction C_0 semigroup if and only if $\rho(A) \subset \mathbb{R}_+ \subset \mathbb{C}$ and for every $\lambda > 0$,*

$$\left\|(\lambda I - A)^{-1}\right\|_{\mathcal{L}(X)} \leq \frac{1}{\lambda}$$

We also present a stronger generation theorem proven by Phillips [Phi52] slightly later in 1952.

Theorem 4.10 (Feller–Miyadera–Phillips)**.** *Let A be a closed, densely defined operator. Then $-A$ generates a strongly continuous semigroup if and only if there exists $M \geq 1$ and $\omega \in \mathbb{R}$ such that for every $\lambda > \omega$, $\lambda \in \rho(A)$ and*

$$\|((\lambda - \omega)(\lambda I - A))^n\| \leq M \qquad \textit{for all} \quad n \in \mathbb{N}.$$

Proofs of these theorems are readily available in many textbooks on semigroups. For a proof Theorem 4.9 see [Eva10, pp.439–441]. Alternatively, Pazy [Paz83, pp.12–13] and Engel and Nagel [EN00, pp.73–75] show that the closed and densely defined properties of A is actually implied by $-A$ generating a C_0 semigroup. Engel and Nagel [EN00, pp.77] also present a proof of Theorem 4.10 which is again slightly stronger than what we present here.

An important subclass of C_0 semigroups is the class of analytic semigroups restricted to the real axis.

As we will soon see, the functional calculus that is used to generate analytic semigroups is far nicer; it is the well known H^∞ functional calculus for sectorial operators.

Definition 4.11 (Analytic semigroup)**.** Let $S_{\mu+} \subset \mathbb{C}$ denote the sector of half angle $\mu < \frac{\pi}{2}$. A family of operators $(T(z))_{z \in S_{\mu+}} \subset \mathcal{L}(X)$ is called an analytic semigroup if:

(i) $T(0) = I$ and $T(z_1 + z_2) = T(z_1)T(z_2)$ for all $z_1, z_2 \in S_{\mu+}$,

(ii) The map $z \mapsto T(z)x$ is analytic in $S_{\mu+}$ for all $x \in X$,

(iii) $\lim_{S_{\mu'+} \ni z \to 0} T(z)x = x$ for all $x \in X$ and $0 < \mu' < \mu$.

If in addition,

(iv) $\|T(z)\|_{\mathcal{L}(X)}$ is bounded in $S_{\mu'+}$ for all $0 < \mu' < \mu$,

we say that $T(z)$ is a bounded analytic semigroup.

As mentioned earlier, the analytic semigroups have a connection with the holomorphic functional calculus. The following theorem solidifies this concept.

Theorem 4.12. *For a linear operator $A : D(A) \subset X \to X$, the following are equivalent*

(a) $-A$ generates a bounded analytic semigroup on $(T(z))_{S_{\mu+} \cup \{0\}} \subset \mathcal{L}(X)$,

(b) $-A$ generates a bounded C_0 semigroup T and there exists $C > 0$ such that

$$\left\|(zI - A)^{-1}\right\|_{\mathcal{L}(X)} \leq \frac{C}{|\operatorname{Im}(z)|},$$

for all $z \in \mathbb{C}$ such that $\operatorname{Re}(z) < 0$ and $\operatorname{Im}(z) \neq 0$,

(c) There exists $\omega \in (0, \frac{\pi}{2})$ such that $e^{\pm i\omega}A$ generates a bounded C_0 semigroup,

(d) $-A$ generates a bounded C_0 semigroup T such that for all $t > 0$ the range of $T(t)$ is contained in $D(A)$ and

$$\sup_{t>0} \|tAT(t)\| < \infty,$$

(e) A is ω-sectorial with $0 < \omega < \pi/2$.

A proof of this theorem can be found in [EN00, pp101–105], but note that Engel and Nagel take the semigroup convention for sectorial operators. We take the convention that A is sectorial when $\sigma(A) \subset S_{\mu+} \subset \mathbb{C} \setminus \mathbb{R}_-$, that is, when Engel and Nagel say that $-A$ is sectorial.

Remark. One important remark to make is that property (d) implies $\|t^n A^n T(t)\| < \infty$ for all $t > 0$ and all $n \in \mathbb{N}$. To see this suppose A were sectorial so that $\frac{1}{n}A$ is also sectorial. If $-A$ generates the C_0 semigroup T then $\frac{-1}{n}A$ generates the C_0 semigroup $t \mapsto T(t/n)$. It follows by property (d) applied to $T(t/n)$ that

$$\sup_{t>0} \|t^n A^n T(t)\| \leq \sup_{t>0} \|tAT(t/n)\|^n < \infty.$$

One way to look at the difference between C_0 and analytic semigroups is through spectral properties of their generators. C_0 semigroups are generated by operators with spectrum contained in the half plane, while analytic semigroups have spectrum in a sector of half-angle $0 < \omega < \pi/2$.

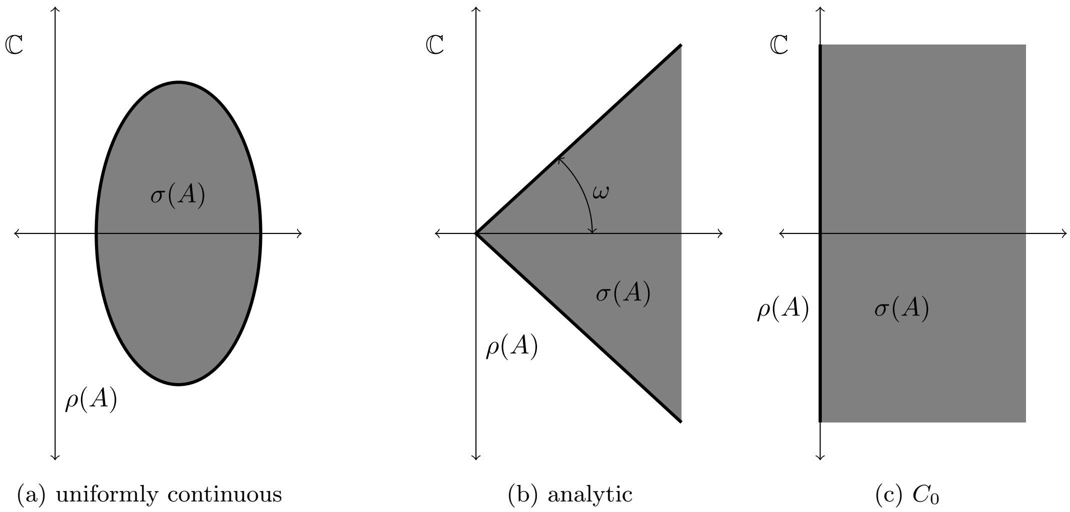


Figure 4.1: The spectrum of A determines the semigroup continuity.

This approach of viewing semigroup properties through spectral properties of their generators can be taken further. Engel and Nagel for instance [EN00, pp.96–123] extend this analysis to include eventually norm continuous semigroups and eventually differentiable semigroups among other special classes of semigroups.

Example 4.13. Consider again the heat equation on $X = L^2(\mathbb{R}^n)$.

Define the heat semigroup through the fundamental solution (4.3) of the heat equation

$$T(t)f(x) := \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-|x-y|^2/4t} f(y)dy \tag{4.4}$$

for all $f \in L^2(\mathbb{R}^n)$. Notice that this is nothing but a convolution with a Gaussian $\mu_t(x) = \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t}$. Young's inequality immediately tells us that T is not only bounded, but a contraction semigroup,

$$\|T(t)f\|_{L^2} \leq \|\mu_t\|_{L^1} \|f\|_{L^2} \leq \|f\|_{L^2}.$$

Upon taking a Fourier transform the semigroup becomes a multiplication operator,

$$\widehat{T(t)f} = \widehat{\mu_t} \cdot \widehat{f},$$

and since $\widehat{\mu_t}$ is another Gaussian, the semigroup becomes continuous. By Plancherel's theorem the semigroup is a C_0 contraction.

Alternatively, consider the heat semigroup through its generator $-A = \Delta$. The Hille–Yosida theorem, Theorem 4.9, gives the same result as above; T is a contraction C_0 semigroup because $A = -\Delta$ is sectorial with $\sigma(A) = \mathbb{R}_+$.

Even stronger than this, Theorem 4.12 tells us that the heat semigroup is a bounded analytic semigroup since $A = -\Delta$ is ω-sectorial for any $\omega > 0$. This property can also be shown directly by looking at the fundamental heat solution (4.4), see [BFCO12, Lecture 9].

4.3 Solutions to the Cauchy problems

Having established some basic semigroup results we return to the inhomogeneous Cauchy problem (4.1). Suppose that $-A$ generates a semigroup T and $u \in C^1(\mathbb{R}_+, X)$ is a solution to the Cauchy problem. Define the function $g(s) = T(t-s)u(s)$ so that[1]

$$\begin{aligned}\frac{dg}{ds} &= AT(t-s)u(s) + T(t-s)u'(s)\\ &= AT(t-s)u(s) - T(t-s)Au(s) + T(t-s)f(s)\\ &= T(t-s)f(s).\end{aligned}$$

Integrating from 0 up to t,

$$g(t) - g(0) = u(t) - T(t)u(0) = \int_0^t T(t-s)f(s)ds,$$

such that

$$u(t) = T(t)u_0 + \int_0^t T(t-s)f(s)ds. \tag{4.5}$$

In essence this section will show that if T is a semigroup with generator $-A$ then the above formula defines a solution to the inhomogeneous Cauchy problem. Something that must be considered however, is what type of solution the above formula gives.

A priori we have no reason why $t \mapsto u(t)$ need be continuously differentiable; in some cases differentiability may fail. Nevertheless, by weakening our notion of solution we still say that $u(t)$ defined in this way is a solution to the Cauchy problem.

4.3.1 Notions of solution

The obvious class of solution is when $t \mapsto u(t)$ is continuously differentiable, that is, a classical solution to the Cauchy problem.

Definition 4.14 (Classical solution)**.** We call $u \in C^1(\mathbb{R}_+, X)$ satisfying (4.1) a classical solution to the Cauchy problem.

Familiar from classical PDE theory, see for instance [Eva10], the notion of a weak solution can be extended to the abstract Cauchy problem.

Definition 4.15 (Weak solution)**.** We say that $u \in C(\mathbb{R}_+, X)$ is a weak solution to the Cauchy problem if for all $v^* \in D(A^*) \subset X^*$, the function $t \mapsto \langle u(t), v^* \rangle$ is absolutely continuous and

$$\frac{d}{dt}\langle u(t), v^* \rangle + \langle u(t), A^* v^* \rangle = \langle f(t), v^* \rangle$$

for almost every $t \geq 0$.

Weaker still is the notion of a mild solution.

[1]We use here the fact that $\frac{d}{dt}T(t) = -AT(t)$ for all $t \geq 0$ which is justified in Proposition 4.18.

Definition 4.16 (Mild solution)**.** If $-A$ is the generator of a C_0 semigroup T and $f \in L^1(\mathbb{R}_+, X)$, the function $u \in C(\mathbb{R}_+, X)$ given by

$$u(t) = T(t)u_0 + \int_0^t T(t-s)f(s)ds$$

is the mild solution of the Cauchy problem.

Remark. The calculation at the start of this section tells us immediately that classical solutions are always weak solutions, under the additional assumption that $f \in L^1(\mathbb{R}_+, X)$. The converse is not always true.

As we will soon demonstrate, the two terms in (4.5) solve distinct components of the Cauchy problem. The first term $w(t) = T(t)u_0$ solves the related homogeneous Cauchy problem

$$\begin{cases} u'(t) + Au(t) = 0 & t \geq 0, \\ u(0) = u_0. \end{cases} \tag{4.6}$$

while the second term $v(t) = \int_0^t T(t-s)f(s)ds$ solves the inhomogeneous Cauchy problem with zero initial data

$$\begin{cases} u'(t) + Au(t) = f(t) & t \geq 0, \\ u(0) = 0. \end{cases} . \tag{4.7}$$

Conversely, if $w(t)$ solves (4.6) and $v(t)$ solves (4.7) then their sum $u(t) = w(t) + v(t)$ will solve the full inhomogeneous Cauchy problem (4.1). It is for this reason that we consider these two problems separately.

4.3.2 Homogeneous Cauchy problem

We turn to the homogeneous Cauchy problem (4.6). The aim is to construct a semigroup solution and thus we begin by establishing some basic properties of C_0 semigroups. The first of these is an exponential growth estimate on the semigroup.

Proposition 4.17. *Let T be a C_0 semigroup. There exists constants $\omega \geq 0$ and $M \geq 1$ such that*

$$\|T(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t}$$

for all $t \geq 0$.

We show that T is bounded on a finite interval and then use the semigroup property to deduce the exponential growth estimate.

Proof. Define $M_n := \sup_{0 \leq t \leq \frac{1}{n}} \|T(t)\|_{\mathcal{L}(X)}$ for all $n \in \mathbb{N}$. If $M_n = \infty$ for all n then there exists $t_n \in [0, \frac{1}{n}]$ such that $\|T(t_n)\| \geq n$ so that $\|T(t_n)\| \to \infty$ as $n \to \infty$. By the uniform boundedness principle, there exists $x \in X$ such that $\lim_{n\to\infty} \|T(t_n)x\|_X = \infty$ but this contradicts the assumption that T is a C_0 semigroup since

$$\lim_{n\to\infty} \|T(t_n)x\| = \|T(0)x\|_X = \|x\|_X < \infty.$$

It follows that there exists $n \in \mathbb{N}$ such that $M := M_n < \infty$. Any $t > 0$ can be written as $t = \frac{k}{n} + \delta$ where $0 \le \delta \le \frac{1}{n}$ and $k \in \mathbb{N} \cup \{0\}$. It thus follows,

$$\|T(t)\| = \left\|T(1/n)^k T(\delta)\right\| \le \|T(1/n)\|^{k+1} \le M^{k+1} \le MM^{nt} = Me^{\omega t}$$

where $\omega = n \log M \ge 0$ since $T(0) = I$ has norm 1 means that $M \ge 1$. $\square$

The Cauchy problem can be thought of as the Banach space analogue of the simple ODE,

$$\begin{cases} u'(t) + au(t) = 0, \\ u(0) = u_0, \end{cases}$$

where $a \in \mathbb{C}$ is a constant. Likewise, the semigroup solution $u(t) = T(t)u_0$ is analogous to the ODE solution $u(t) = e^{-at}u_0$. In this context the exponential growth estimate is not totally surprising. We proceed by developing the core theory formalising the supposition that semigroups solve the homogeneous Cauchy problem.

Proposition 4.18. *Let $-A$ be the generator for a C_0 semigroup. Then,*

(a) For $x \in X$, $t \ge 0$,

$$\lim_{h \downarrow 0} \frac{1}{h} \int_t^{t+h} T(s)x \, ds = T(t)x.$$

(b) For $x \in X$, $t \ge 0$,

$$-A\left(\int_0^t T(s)x \, ds\right) = T(t)x - x.$$

(c) For $x \in D(A)$, $T(t)x \in D(A)$ for all $t \ge 0$ and

$$\frac{d}{dt}T(t)x = -AT(t)x = -T(t)Ax.$$

(d) For $x \in D(A)$, $t, s \ge 0$,

$$T(t)x - T(s)x = -\int_s^t T(\tau)Ax \, d\tau = -\int_s^t AT(\tau)x \, d\tau$$

These properties follow simply from the definition of a C_0 semigroup.

Proof. (a) Let $x \in X$. Then

$$\begin{aligned} \lim_{h \downarrow 0} \frac{1}{h} \int_t^{t+h} T(s)x \, ds &= \lim_{h \downarrow 0} \frac{1}{h} T(t) \int_0^h T(s)x \, ds \\ &= T(t) \frac{d^+}{dh} \int_0^h T(s)x ds\Big|_{h=0} \\ &= T(t)T(0)x = T(t)x \end{aligned}$$

by continuity of $t \mapsto T(t)x$ and the semigroup properties.

(b) Let $x \in X$, $h > 0$. Then

$$\begin{aligned} \frac{T(h) - I}{h} \int_0^t T(s)x \, ds &= \frac{1}{h} \int_0^t (T(s+h)x - T(s)x) ds \\ &= \frac{1}{h} \int_t^{t+h} T(s)x \, ds - \frac{1}{h} \int_0^h T(s)x \, ds. \end{aligned}$$

Letting $h \downarrow 0$ and using (a) and the definition of the generator gives the result.

(c) Let $x \in D(A)$, $h > 0$.

$$\left(\frac{T(h)-I}{h}\right) T(t)x = T(t)\left(\frac{T(h)-I}{h}\right) x.$$

Letting $h \downarrow 0$,

$$\frac{d^+}{dt} T(t)x = -AT(t)x = -T(t)Ax$$

and thus $T(t)x \in D(A)$ for all $t \geq 0$. To check the left derivative,

$$\lim_{h\downarrow 0}\left(\frac{T(t)x + T(t-h)x}{h} - T(t)Ax\right) = \lim_{h\downarrow 0} T(t-h)\left(\frac{T(h)x - x}{h} - Ax\right) + \lim_{h\downarrow 0}\left(T(t)Ax - T(t-h)Ax\right).$$

The first term is zero since $x \in D(A)$ and $\|T(t-h)\|_{\mathcal{L}(X)} \leq Me^{\omega(t-h)} \leq Me^{\omega t} < \infty$. The second term is also zero by continuity of $t \mapsto T(t)w$ where $w = Ax \in X$.

(d) Integrate (c) from s to t. □

These properties are enough to show the nice uniqueness properties relating semigroups and generators.

Proposition 4.19. *The generator of a strongly continuous semigroup is a closed and densely defined linear operator that uniquely determines the semigroup.*

Proof. Let T be a strongly continuous semigroup with generator $-A$. The generator is clearly linear, by linearity of the semigroup. To show it is closed consider a sequence $x_n \in D(A)$ such that $x_n \to x$ and $Ax_n \to y \in X$. Note that by property (b) and (c) of the previous proposition,

$$\frac{T(t)x_n - x_n}{t} = -\frac{1}{t}\int_0^t T(s)Ax_n\, ds,$$

but since $\|T(s)\| \leq Me^{\omega t}$ for $s \in [0,t]$, $T(s)Ax_n \to T(s)y$ uniformly as $n \to \infty$ such that

$$\frac{T(t)x - x}{t} = \frac{-1}{t}\int_0^t T(s)y\, ds.$$

Letting $t \downarrow 0$, we see $x \in D(A)$ and $Ax = y$. That is, A is a closed operator.

Property (b) also tells us $\int_0^t T(s)x\, ds \in D(A)$ for all $t \geq 0$. It follows by the C_0 property that,

$$\lim_{t\downarrow 0}\frac{1}{t}\int_0^t T(s)x\, ds = x$$

for every $x \in X$. That is, $D(A)$ is dense in X.

To show uniqueness of the generator, suppose that $S(t)$ is another semigroup with the same generator $-A$. For $x \in D(A), t > 0$ consider the map $\eta_x : [0,t] \to X$ defined by

$$\eta_x(s) := T(t-s)S(s)x.$$

For fixed s, the set

$$\left\{\frac{S(s+h)x - S(s)x}{h} : h \in (0,1]\right\} \cup \left\{ -AS(s)x\right\}$$

is compact. Now,

$$\frac{1}{h}(\eta_x(s+h) - \eta_x(s)) = T(t-s-h)\frac{1}{h}(S(s+h)x - S(s)x) + \frac{1}{h}(T(t-s-h) - T(t-s))S(s)x.$$

Letting $h \downarrow 0$, the left hand side converges to $\frac{d}{ds}\eta_x(s)$, the first term on the right converges to $-T(t-s)AS(s)x$ and the second to $AT(t-s)S(s)x$ by property (c) from the above proposition,

$$\frac{d}{ds}\eta_x(s) = -T(t-s)AS(s)x + A(t-s)T(t-s)T(t-s)S(s)x = 0.$$

Note however $\eta_x(0) = T(t)x$ and $\eta_x(t) = S(t)x$ such that

$$T(t)x = S(t)x$$

for all x in the dense domain $D(A)$. Hence $T(t) = S(t)$ for all $t \geq 0$. □

It should now be fairly evident if $-A$ generates a C_0 semigroup T then $u(t) = T(t)u_0$ is a classical solution to the homogeneous Cauchy problem. In particular, property (c) from the previous proposition gives

$$u'(t) = \frac{d}{dt}T(t)u_0 = -AT(t)u_0 = -Au(t)$$

and the first semigroup property gives

$$u(0) = T(0)u_0 = u_0.$$

The function $u(t)$ defined in this way is continuous where T is C_0. Moreover, it is continuously differentiable since

$$u'(t) = -T(t)Au_0 = T(t)w_0$$

is continuous since $w_0 = -Au_0 \in X$ and T is a C_0 semigroup.

These remarks are formalised in the following theorem. Under some mild conditions on A, we find a remarkably stronger result; asking whether the inhomogeneous Cauchy problem admits a classical solution is in direct correspondence with asking whether A generates a C_0 semigroup.

Theorem 4.20. *Let A be a densely defined, linear operator with non-empty resolvent set. The homogeneous Cauchy problem* (4.6) *has a unique classical solution for every $u_0 \in D(A)$ if and only if $-A$ generates a C_0 semigroup.*

For a proof of this result see [Paz83, pp.102–104].

4.3.3 Inhomogeneous Cauchy problem

Having established the conditions under which the homogeneous Cauchy problem is solvable, we turn to the inhomogeneous Cauchy problem with zero initial data (4.7).

Proposition 4.21. *Suppose $-A$ is the generator of a C_0 semigroup T and $f \in L^1(\mathbb{R}_+, X)$ is continuous. Let*

$$v(t) := \int_0^t T(t-s)f(s)ds.$$

The Cauchy problem (4.1) *has a classical solution $u(t)$ for every $u_0 \in D(A)$ provided one of the following hold:*

(a) $v(t) \in C^1(\mathbb{R}_+, X)$,

(b) $v(t) \in D(A)$ for $t \geq 0$ and $Av(t) \in C(\mathbb{R}_+, X)$.

Suppose that $-A$ generates a C_0 semigroup then $u(t)$ defined as in (4.5) *is a classical solution to* (4.1). *Moreover if u is a classical solution to* (4.1) *then $v(t)$ defined above satisfies (a) and (b).*

Proof. For $h > 0$,

$$\begin{aligned}\left(\frac{T(h)-I}{h}\right)v(t) &= \frac{1}{h}\int_0^t T(t+h-s)f(s)ds - \frac{1}{h}\int_0^t T(t-s)f(s)ds \\ &= \frac{v(t+h)-v(t)}{h} - \frac{1}{h}\int_t^{t+h} T(t+h-s)f(s)ds.\end{aligned}$$

By continuity of f, letting $h \downarrow 0$,

$$\frac{1}{h}\int_t^{t+h} T(t+h-s)f(s)ds = f(t).$$

If we assume (a) then this gives $v(t) \in D(A)$ for all $t > 0$ and

$$-Av(t) = v'(t) - f(t).$$

Clearly $v(0) = 0$ so that $u(t) = T(t)u_0 + v(t)$ is a classical solution (4.1) for all $u_0 \in D(A)$. Assuming (b) instead, the above calculation tells us $v(t)$ has a right derivative,

$$\frac{d^+}{dt}v(t) = -Av(t) + f(t).$$

Since $\frac{d^+}{dt}v(t)$ is continuous, as the sum of continuous functions, $v(t)$ is continuously differentiable. That is,

$$v'(t) = -Av(t) + f(t) \in C(\mathbb{R}_+, X).$$

As above, $u(t) = T(t)u_0 + v(t)$ is a classical solution to (4.1) for all $u_0 \in D(A)$.

On the other hand, if (4.1) has a solution u for some $u_0 \in D(A)$ then this solution is given by (4.5). It follows that $v(t) = u(t) - T(t)u_0$ is differentiable for $t > 0$ and

$$v'(t) = u'(t) + T(t)Au_0$$

is continuous for $t > 0$. That is, (a) holds. If $u_0 \in D(A)$ then $T(t)u_0 \in D(A)$ for $t \geq 0$ and thus $v(t) = u(t) - T(t)u_0 \in D(A)$ for $t \geq 0$. Also, $-Av(t) = -Au(t) + AT(t)u_0 = u'(t) - f(t) + T(t)Au_0$ is continuous for $t > 0$. That is, (b) holds too. $\square$

Corollary 4.22. *Let $-A$ generate the C_0 semigroup T. If $f \in C^1(\mathbb{R}_+, X)$ and $f, f' \in L^1(\mathbb{R}_+, X)$ then* (4.1) *has a classical solution $u(t)$ for every $u_0 \in D(A)$.*

Proof. We know,

$$v(t) = \int_0^t T(t-s)f(s)ds = \int_0^t T(s)f(t-s)ds.$$

It is clear that $v(t)$ is differentiable for $t > 0$,

$$v'(t) = T(t)f(0) + \int_0^t T(s)f'(t-s)ds = T(t)f(0) + \int_0^t T(t-s)f'(s)ds$$

which is continuous. We conclude by applying the previous theorem. □

4.4 Maximal regularity and PDE

We continue considering the inhomogeneous Cauchy problem with zero initial data. In the previous section we assumed $-A$ generated a C_0 semigroup $T(t)$ and $f \in L^1(\mathbb{R}_+, X)$ was continuous. In Corollary 4.22 we found that $f \in C^1(\mathbb{R}_+, X)$ guaranteed that the mild solution $u(t)$ was a classical solution provided f and f' have the appropriate integrability. In this section we derive slightly different results by weakening our assumptions on f and strengthening the assumptions on A.

As before, let X be a Banach space and $A : D(A) \subset X \to X$ a closed, densely defined (unbounded) operator. Let $f : \mathbb{R}_+ \to X$ be a measurable function and consider the inhomogeneous Cauchy problem with zero initial data,

$$\begin{cases} u'(t) + Au(t) = f(t) & t \geq 0, \\ u(0) = 0. \end{cases} \tag{4.8}$$

Definition 4.23. Let $p \in (1, \infty)$. We say that A has the (parabolic) maximal regularity property if there exists $C > 0$ such that for all $f \in L^p(X)$, there exists a unique $u \in L^p(\mathbb{R}_+, D(A))$ with $u' \in L^p(\mathbb{R}_+, X)$ that satisfies (4.8) for almost every $t \geq 0$ and

$$\|u'\|_{L^p(\mathbb{R}_+,X)} + \|Au\|_{L^p(\mathbb{R}_+,X)} \leq C\,\|f\|_{L^p(\mathbb{R}_+,X)} \tag{4.9}$$

As the following proposition and theorem demonstrate, there is a strong connection between maximal regularity and analytic semigroups.

Proposition 4.24. *Let X be a Banach space and let $A : D(A) \subset X \to X$ be an operator with maximal L^p regularity with $p \in (1, \infty)$. Then $-A$ generates a bounded analytic semigroup.*

Proof. By Theorem 4.12 it suffices to show that A is sectorial. To do this, we explicitly construct a resolvent and show resolvent bounds.

Let $z \in \mathbb{C}$ with $\mathrm{Re}(z) < 0$ and define

$$f_z(t) = \begin{cases} -e^{zt} & 0 \leq t \leq \frac{-1}{\mathrm{Re}(z)}, \\ 0 & t > \frac{-1}{\mathrm{Re}(z)}. \end{cases}$$

Let $x \in X$ and consider (4.8) with $f(t) = f_z(t)x$. By our assumption, we have a solution to this problem, say u_z.

Resolvent operator Define a family of operators $R_z : X \to X$ by

$$R_z x = \mathrm{Re}(z) \int_0^\infty e^{-zt} u_z(t)dt.$$

An integration by parts gives

$$R_z(A - zI)x = \mathrm{Re}(z) \int_0^\infty e^{-zt} A u_z(t)dt + \mathrm{Re}(z) \int_0^\infty e^{-zt} u_z'(t)dt.$$

Since u_z satisfies (4.8) we can simplify this as

$$R_z(zI - A)x = \mathrm{Re}(z) \int_0^\infty e^{-zt} f(t)xdt = -\mathrm{Re}(z) \int_0^{-1/\mathrm{Re}(z)} xdt = x.$$

That is, R_z defines a resolvent operator for A.

Resolvent bounds Hölder's inequality gives

$$\|R_z x\|_X \leq \mathrm{Re}(z) \|u_z\|_{L^p(\mathbb{R}_+,X)} \left\| t \mapsto e^{-zt} \right\|_{L^{p'}(\mathbb{R}_+,\mathbb{C})}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. The maximal regularity property of A gives

$$\|u_z\|_{L^p(\mathbb{R}_+,X)} \leq C \|f\|_{L^p(\mathbb{R}_+,X)} = C \|f_z\|_{L^p(\mathbb{R}_+,\mathbb{C})} \|x\|_X .$$

We calculate

$$\|f_z\|_{L^p(\mathbb{R}_+,\mathbb{C})} = \left(-\int_0^{-1/\mathrm{Re}(z)} e^{p\,\mathrm{Re}(z)t} dt \right)^{1/p} = \left(\frac{1 - e^{-p}}{p\,\mathrm{Re}(z)} \right)^{1/p},$$

$$\left\| t \mapsto e^{-zt} \right\|_{L^{p'}(\mathbb{R}_+,\mathbb{C})} = \left(\int_0^\infty e^{-p'\,\mathrm{Re}(z)t} dt \right)^{1/p'} = \left(\frac{1}{p'\,\mathrm{Re}(z)} \right)^{1/p'}.$$

Combining these with the above estimates gives

$$\|R_z x\|_X \leq C \frac{(1 - e^{-p})^{1/p}}{p^{1/p}(p')^{1/p'}} \|x\|_X .$$

An integration by parts yields the alternate expression

$$R_z x = \frac{\mathrm{Re}(z)}{z} \int_0^\infty e^{-tz} u_z'(t)dt.$$

Repeating the same argument as before, with $\|u_z'\|_{L^p(\mathbb{R}_+,X)} \leq C \|f\|_{L^p(\mathbb{R}_+,X)}$ from the maximal regularity property, we get

$$\|R_z x\|_X \leq \frac{C}{|z|} \frac{(1 - e^{-p})^{1/p}}{p^{1/p}(p')^{1/p'}} \|x\|_X .$$

We conclude that A is a sectorial operator. Indeed, we have resolvent bounds, and $\sigma(A) \subset \{z \in C : \mathrm{Re}(z) \geq 0\}$.

In order to apply Theorem 4.12 we need that A is ω-sectorial for $\omega < \pi/2$, yet we have thus far only show $\omega \leq \pi/2$. A small modification shows this. We show that if A has the L^p maximal regularity then so too does $A + \epsilon I$ for some $\epsilon > 0$ so that $A + \epsilon I$ is ω-sectorial with $\omega \leq \pi/2$. That is, A is ω'-sectorial with $\omega' < \pi/2$. □

A natural question to ask is whether the converse of this proposition is true. That is, if $-A$ generates a bounded analytic semigroup, does it have the maximal L^p regularity property for $p \in (1, \infty)$? In 1964 de Simon [dS64] answered this question for Hilbert spaces.

Theorem 4.25 (de Simon)**.** *Let H be a Hilbert space and $-A$ the generator of a bounded analytic semigroup in H. Then A has the maximal regularity property for all $p \in (1, \infty)$.*

This uses a simple extension of the Calderón–Zygmund theory developed in chapter 2 to operator valued kernels and Hilbert space valued functions. Note that the extension is not straightforward at all if the target space is a non-Hilbertian Banach space.

Proof. We realise $f \mapsto \left(t \mapsto \int_0^t T(t-s)f(s)ds\right)$ as a Hilbert space valued singular integral operator with (operator valued) convolution kernel. After establishing L^2 boundedness by Fourier multipliers, we apply the theory of singular integral operators to extend to L^p boundedness for $p \in (1, \infty)$.

A singular integral operator Let T denote the semigroup generated by $-A$. We have a solution to (4.8) given by

$$u(t) = \int_0^t T(s-t)f(s)ds \tag{4.10}$$

for all $t \geq 0$. Indeed, by the semigroup properties Proposition 4.18,

$$u'(t) = T(0)f(t) + \int_0^t T'(t-s)f(s)ds = f(t) - Au(t) \quad \text{and} \quad u(0) = 0.$$

It thus suffices to check the regularity of $Au(t)$. By (4.10) we have

$$Au(t) = \int_{\mathbb{R}} k(t-s)f(s)ds,$$

with a convolution kernel defined by

$$k(t) = \begin{cases} AT(t) & t > 0, \\ 0 & t \leq 0. \end{cases}$$

$\mathbf{L^2}$ boundedness Applying the Fourier transform in t,

$$\begin{aligned}
\widehat{Au(\tau)} &= \int_{\mathbb{R}} e^{-it\tau} Au(t)dt \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-it\tau} k(t-s)f(s)ds\, dt \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-i(t-s)\tau} k(t)f(s)ds\, dt \\
&= \int_0^\infty e^{-it\tau} T(t) \int_{\mathbb{R}} e^{-is\tau} Af(s)ds\, dt \\
&= (i\tau I + A)^{-1} A\widehat{f(\tau)} \\
&= A(i\tau I + A)^{-1} \widehat{f(\tau)}.
\end{aligned}$$

The second last step we used the integral representation of the resolvent justified by an integration by parts,

$$\begin{aligned}(i\tau I + A)&\int_0^\infty e^{-it\tau}T(t)dt \\ &= -\int_0^\infty (-i\tau)e^{-it\tau}T(t)dt + \int_0^\infty e^{-it\tau}AT(t)dt \\ &= -\int_0^\infty e^{-it\tau}AT(t)dt + \int_0^\infty e^{-it\tau}AT(t)dt - e^{-it\tau}T(t)\Big|_0^\infty = I.\end{aligned}$$

The uniform stability of the semigroup, the fact that $\lim_{t\to\infty}\|T(t)\| = 0$ requires $0 \notin \sigma(A)$; see for instance [EN00, pp.302]. By reflexivity of L^p and sectoriality of A we have the decomposition $L^p = \mathcal{N}(A) \oplus \overline{\mathcal{R}(A)}$ as in [McI10], and the maximal regularity property is clear on the null space $\mathcal{N}(A)$. Since $-A$ generates a bounded analytic semigroup,

$$\begin{aligned}\sup_{\tau\in\mathbb{R}} &\left\|A(i\tau I + A)^{-1}\right\|_{\mathcal{L}(H)} \\ &\le \sup_{\tau\in\mathbb{R}}\left\|(i\tau I + A)(i\tau I + A)^{-1}\right\|_{\mathcal{L}(H)} + \sup_{\tau\in\mathbb{R}}\left\|i\tau(i\tau I + A)^{-1}\right\|_{\mathcal{L}(H)} < \infty\end{aligned}$$

by Theorem 4.12.

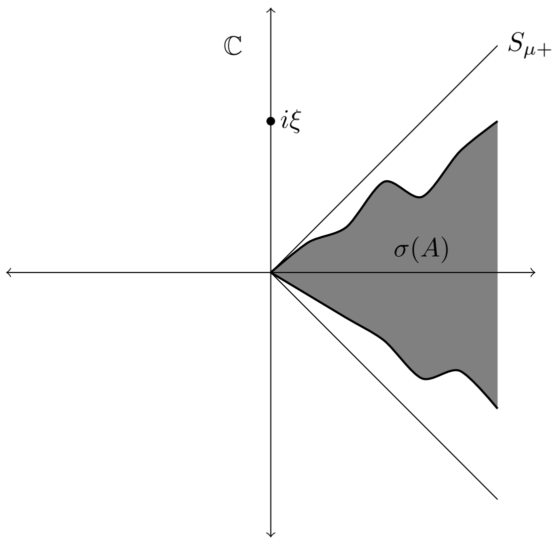


Figure 4.2: $\sigma(A) \subset S_{\mu+}$ for $0 < \mu < \frac{\pi}{2}$.

Applying Plancherel's theorem, $\|Au\|_{L^2(\mathbb{R};H)} \le c\,\|f\|_{L^2(\mathbb{R};H)}$ with $c > 0$ coming from the (finite) operator norm above.

$\mathbf{L^p}$ boundedness The kernel k is clearly measurable since $k \in L^1_{loc}(\mathbb{R} \setminus \{0\}; H)$. Fix $s \in \mathbb{R}$ and compute

$$\begin{aligned}\int_{|t|>2|s|} \|k(t-s) - k(t)\|_{\mathcal{L}(H)}\, dt &= \int_{|t|>2|s|} \left\| \int_t^{t-s} A^2 T(\tau) d\tau \right\|_{\mathcal{L}(H)} dt \\ &\leq C \int_{|t|>2|s|} \left| \int_t^{t-s} \frac{1}{\tau^2} d\tau \right| dt \\ &= C \int_{|t|>2|s|} \left| \frac{1}{t-s} - \frac{1}{t} \right| dt \\ &\leq C \log(2).\end{aligned}$$

The first equality comes from writing the semigroup as the integral of its derivative. The first inequality is from the semigroup property $\sup_t \|t^n A^n T(t)\|_{\mathcal{L}(H)} < \infty$ for all $n \in \mathbb{N}$, see the remark following Theorem 4.12. We conclude by a simple modification of Theorem 2.21.

By the preceding calculation,

$$\|u'\|_{L^p(\mathbb{R};H)} \leq \|Au\|_{L^p(\mathbb{R};H)} \leq (c + c') \|f\|_{L^p(\mathbb{R},H)} .$$

We thus conclude that A has the maximal L^p property. □

An open problem for several years was whether this result extended to all Banach spaces. Initial work by Coulhon and Lamberton [CL86] suggested that the result might be true, but Katlon and Lancien [KL00] eventually showed that in fact the result was false.

Theorem 4.26 (Kalton–Lancien, 2000)**.** *On every Banach lattice which is not isomorphic to a Hilbert space, there exists a generator of an analytic semigroup without maximal L^p regularity.*

4.5 Maximal regularity operator

Previously in this chapter we saw that if we possess a suitable functional calculus for the closed, linear, densely defined operator $A : D(A) \subset X \to X$ then the semigroup generated by $-A$ can be written in the form $T(t) = e^{-tA}$. In this section we use this viewpoint to further treat maximal regularity.

In the previous section the difficult part in checking the maximal regularity property (at least in the case of Hilbert spaces) was in establishing boundedness of the term

$$\int_0^t A e^{-(t-s)A} f(s) ds. \tag{4.11}$$

In this section we present a slightly different approach to maximal regularity which focuses on establishing the boundedness of the related maximal regularity operator.

Definition 4.27. Let A be a densely defined closed linear operator acting on $L^2(\mathbb{R}^n)$. Suppose further that $-A$ generates a bounded analytic semigroup e^{-tA}. We define the maximal regularity operator

$$(\mathcal{M}_A f)(t) = \int_0^t A e^{-(t-s)A} f(s) ds \tag{4.12}$$

for all $f \in L^2(\mathbb{R}_+, D(A))$.

The importance of establishing the boundedness of this operator is readily apparent. If (4.12) defines a bounded operator, then the desired boundedness of (4.11) follows directly and a mild solution u has the maximal regularity property.

In the previous section we considered maximal regularity on spaces $L^p(\mathbb{R}_+, X)$ where X is a Banach space. One advantage of this approach to maximal regularity is that is more readily extends to spaces which treat space and time together.

4.5.1 L^2 boundedness

In proving Theorem 4.25 we essentially proved that the maximal regularity operator was bounded on $L^p(\mathbb{R}_+, H)$. We present this result for $p = 2$ in the language of the maximal regularity operator.

Theorem 4.28. *Assume that $-A$ generates a bounded analytic semigroup in H. Then $\mathcal{M}_A$, initially defined on $L^2(\mathbb{R}_+, D(A))$ extends to a bounded operator on $L^2(\mathbb{R}_+, H)$.*

4.5.2 Tent space boundedness

An important result that we will make use of later is the boundedness of the maximal regularity operator on tent spaces. It was shown by Auscher, Monniaux and Portal [AMP12] that if the family $\left(tAe^{-tA}\right)_{t\geq 0}$ satisfies certain off-diagonal estimates then the maximal regularity operator (4.12) is bounded on the tent space $T^{p,2}$ for $p \in (1, \infty]$.

Definition 4.29. A family of bounded linear operators $(T_t)_{t\geq 0} \subset \mathcal{L}(L^2(\mathbb{R}^n))$ is said to satisfy off-diagonal estimates of order M if for all Borel sets $E, F \subset \mathbb{R}^n$, all $t > 0$ and all $f \in L^2(\mathbb{R}^n)$:

$$\|\mathbb{1}_E T_t \mathbb{1}_F f\|_{L^2} \lesssim \left(1 + \frac{\operatorname{dist}(E,F)^2}{t}\right)^{-M} \|\mathbb{1}_F f\|_{L^2}. \tag{4.13}$$

Theorem 4.30. *Let $p \in \left(\frac{2n}{n+1}, \infty\right] \cap (1, \infty]$ and $\tau = \min(p, 2)$. If $\left(tAe^{-tA}\right)_{t\geq 0}$ satisfy off-diagonal estimates of order $M > \frac{n}{2\tau}$, then $\mathcal{M}_A$ extends to a bounded operator on $T^{p,2}$.*

We will eventually use the result for the tent space $T^{\infty,2}$ and thus present the proof for the case $p = \infty$ only.

Proof. Pick a ball $B(x, \sqrt{r})$. Let

$$I^2 = \int_{B(x,\sqrt{r})} \int_0^r |(\mathcal{M}_A f)(y,t)|^2 dy\, dt.$$

It suffices to show that $I^2 \lesssim r^{\frac{n}{2}} \|f\|^2_{T^{\infty,2}}$. Consider the radially symmetric decomposition,

$$C_j(y,t) \begin{cases} B(y,t) & j = 0, \\ B(y, 2^j t) \setminus B(y, 2^{j-1}t) & j \neq 0, \end{cases}$$

and define

$$f_j(y,t) = f(y,t)\mathbb{1}_{C_j(x,4\sqrt{r})}(y)\mathbb{1}_{(0,r)}(t).$$

By the triangle inequality

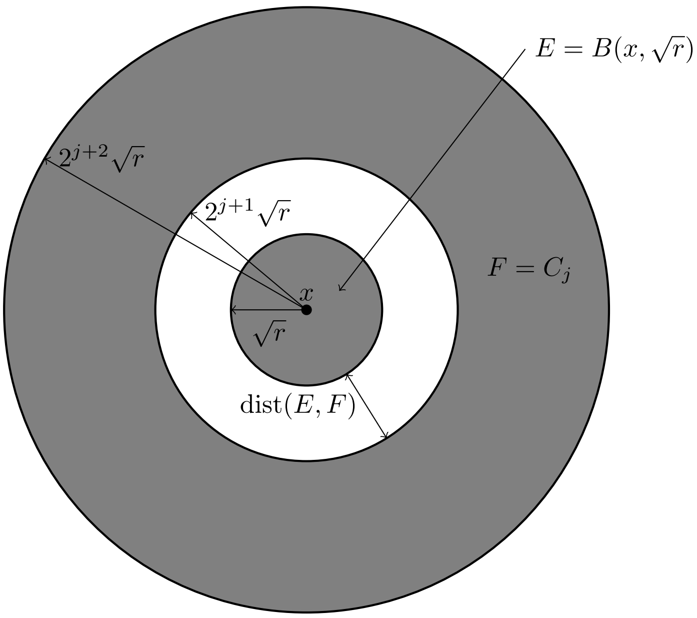


Figure 4.3: Decomposing $\mathbb{R}^n$ into concentric rings about x allows us to exploit off-diagonal bounds.

$$\begin{aligned} I &= \left(\int_{B(x,\sqrt{r})} \int_0^r \Big| \sum_{j\geq 0} \mathcal{M}_A(f_j)(y,t) \Big|^2 dt\, dy \right)^{\frac{1}{2}} \\ &\leq \sum_{j\geq 0} \left(\int_{B(x,\sqrt{r})} \int_0^r |\mathcal{M}_A(f_j)(y,t)|^2 dt\, dy \right)^{\frac{1}{2}} =: \sum_{j\geq 0} I_j. \end{aligned}$$

We apply the L^2 boundedness of $\mathcal{M}_A$ as in Theorem 4.28 and invoke Lemma 3.21 to obtain the estimate

$$I_0^2 \lesssim \int_{B(x,4\sqrt{r})} \int_0^r |f(y,t)|^2 dt\, dy \lesssim r^{\frac{n}{2}} \|f\|_{T^{\infty,2}}^2 .$$

For $j \geq 1$ we have a secondary decomposition by the triangle inequality,

$$\begin{aligned} I_j &= \left(\int_{B(x,\sqrt{r})} \int_0^r \left| \sum_{k\geq 0} \int_{2^{-k-1}t}^{2^{-k}t} Ae^{-(t-s)A} f_j(y,s) ds \right|^2 dt\, dy \right)^{\frac{1}{2}} \\ &\leq \sum_{k\geq 0} \left(\int_{B(x,\sqrt{r})} \int_0^r \left| \int_{2^{-k-1}t}^{2^{-k}t} Ae^{-(t-s)A} f_j(y,s) ds \right|^2 dt\, dy \right)^{\frac{1}{2}} =: \sum_{k\geq 0} I_{j,k}. \end{aligned}$$

We now estimate each $I_{j,k}$,

$$
\begin{aligned}
I_{j,k}^2 &\le \int_{B(x,\sqrt{r})} \int_0^r \left(\int_{2^{-k-1}t}^{2^{-k}t} \left| Ae^{-(t-s)A} f_j(y,s) ds \right| \right)^2 dt\, dy \\
&\lesssim \int_{B(x,\sqrt{r})} \int_0^r 2^{-k}t \int_{2^{-k-1}t}^{2^{-k}t} \left| Ae^{-(t-s)A} f_j(y,s) \right|^2 ds\, dt\, dy \\
&= \int_0^r 2^{-k}t \int_{2^{-k-1}t}^{2^{-k}t} \int_{B(x,\sqrt{r})} \left| (t-s)Ae^{-(t-s)A} f_j(y,s) \right|^2 dy \frac{ds}{(t-s)^2} dt
\end{aligned}
$$

where the second inequality follows by an application of Cauchy–Schwarz. The final equality is an application of Fubini's theorem. We apply the off diagonal estimate with $E = B(x, \sqrt{r})$ and $F = C_j(x, 4\sqrt{r}) = B(x, 2^{j+2}\sqrt{r}) \setminus B(x, 2^{j+1}\sqrt{r})$ where $\operatorname{dist}(E, F) = 2^{j+1}\sqrt{r} - \sqrt{r} \le 2^j\sqrt{r}$.

$$
\begin{aligned}
I_{j,k}^2 &\le \int_0^r 2^{-k}t \int_{2^{-k-1}t}^{2^{-k}t} \left(1 + \frac{2^{2j}r}{t-s}\right)^{-2M} \|f_j(\cdot,s)\|_2^2 \frac{ds\, dt}{(t-s)^2} \\
&= \int_0^{2^{-k}r} 2^{-k}t \int_{2^k s}^{2^{k+1}s} \left(1 + \frac{2^{2j}r}{t-s}\right)^{-2M} \|f_j(\cdot,s)\|_2^2 \frac{dt\, ds}{(t-s)^2}.
\end{aligned}
$$

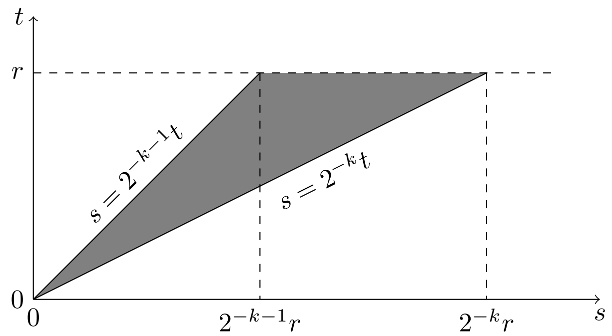


Figure 4.4: Changing the order of integration over a triangle.

For the $k = 0$ term we have $t \sim s$, so

$$
I_{j,0} \le 2^{-2jM} r^{-M} \left(\int_0^r \int_s^{2s} (t-s)^{2M-1} \|f_j(\cdot,s)\|_2^2 \, ds\, dt \right)^{\frac{1}{2}},
$$

and for $k \ge 1$, $t \sim t - s$[2] so that

$$
I_{j,k} \lesssim 2^{-k/2} 2^{-2jM} r^{-M} \left(\int_0^{2^{-k}r} \int_{2^k s}^{2^{k+1}s} t^{2M-1} \|f_j(\cdot,s)\|_2^2 \, ds\, dt \right)^{\frac{1}{2}}.
$$

[2]for $k = 0$, $\frac{t}{2} \le s \le t$, while for $k \ge 0$, $\frac{t}{2} \le t - 2^{-k}t \le t - s \le t$

We thus estimate

$$\begin{aligned}
I_j &\lesssim \sum_{k\geq 0} 2^{-k/2} 2^{-2jM} r^{-M} \left(\int_0^{2^{-k}r} (2^k s)^{2M} \|f_j(\cdot,s)\|_2^2 \, ds \right)^{\frac{1}{2}} \\
&\leq \sum_{k\geq 0} 2^{-k/2} 2^{-2jM} \left(\int_0^{2^{-k}r} \|f_j(\cdot,s)\|_2^2 \, ds \right)^{\frac{1}{2}} \\
&\leq \sum_{k\geq 0} 2^{-k/2} 2^{-2jM} \left(\int_0^{r} \|f_j(\cdot,s)\|_2^2 \, ds \right)^{\frac{1}{2}} \\
&\lesssim 2^{-2jM} \left(\int_0^{r} \|f_j(\cdot,s)\|_2^2 \, ds \right)^{\frac{1}{2}} .
\end{aligned}$$

That is,

$$\begin{aligned}
I_j^2 &\lesssim 2^{-4jM} \int_0^r \|f_j(\cdot,s)\|_2^2 \, ds = 2^{-4jM} \int_0^r \int_{C_j(x,4\sqrt{r})} |f(y,s)|^2 dy \, ds \\
&\lesssim 2^{-4jM} \int_0^r \int_{B(x,2^{j+2}\sqrt{r})} |f(y,s)|^2 dy \, ds \lesssim 2^{-4jM} \left(2^j \sqrt{r}\right)^n \|f\|_{T^{\infty,2}}^2 .
\end{aligned}$$

Now, the condition, $M > \frac{n}{4}$ ensures that the I_j are summable. We conclude $I^2 \leq r^{\frac{n}{2}} \|f\|_{T^{\infty,2}}^2$. □

The proof for $p \in (1,\infty)$ can be found in [AMP12]. Setting $\beta = 0$ and $m = 2$ in their paper gives the parabolic instance we are interested in.

4.6 Nonlinear Cauchy problems

Until this point we have considered linear Cauchy problems, both homogeneous and inhomogeneous. Just as the inhomogeneous Cauchy problem can be viewed as a perturbation of the homogeneous problem, so too the nonlinear problems we consider in this section can be viewed in a similar way. We will study the semilinear Cauchy problem

$$\begin{cases} u'(t) + Au(t) = f(u(t)), & t \geq 0 \\ u(0) = u_0 \end{cases} \tag{4.14}$$

where $-A$ is the generator of an analytic semigroup on a Banach space X and $f : X \to X$ is continuous.

There are of course other conditions we could assume on A and f; Pazy [Paz83, pp.184–194] for example assumes $-A$ generates a C_0 semigroup and $f(t,u)$ is Lipschitz in u. Alternatively one might also consider evolution equations in which A and f both depend on t.

Unlike the previous sections, the existence of a solution is not guaranteed and there are many examples of seemingly nice problems for which no smooth solution exists.

Example 4.31 (Semilinear heat equation)**.** Also called the (scalar) reaction-diffusion equation, see [Eva10, pp.5–6].

$$\begin{cases} u'(t) = \Delta u(t) + u(t)^2 & t > 0, \\ u(0) = u_0. \end{cases} \tag{4.15}$$

In many ways this appears to be a nice nonlinear problem. Both the operator $A = -\Delta$ and the nonlinearity $f(u) = u^2$ are well behaved, yet the behaviour of solutions is surprisingly bad. It is known that a solution u blows up in finite time even for small (nonzero) initial data, see [Eva10, pp.549–550].

We can extend the notions of solution to nonlinear Cauchy problems too.

Definition 4.32. A function $u \in C^1(\mathbb{R}_+, X)$ is a classical solution to the semilinear Cauchy problem if $u(t) \in D(A)$ for all $t \geq 0$ and u satisfies (4.14).

If $u \in C^1(\mathbb{R}_+, X)$ is a classical solution then the function $g(s) := T(t-s)u(s)$ is differentiable for $0 < s < t$ and

$$\begin{aligned} \frac{dg}{ds} &= AT(t-s)u(s) + T(t-s)u'(s) \\ &= AT(t-s)u(s) - T(t-s)Au(s) + T(t-s)f(u(s)) \\ &= T(t-s)f(u(s)). \end{aligned}$$

Integrating from 0 to t,

$$g(t) - g(0) = u(t) - T(t)u_0 = \int_0^t T(t-s)f(u(s))ds.$$

As for linear problems, this condition could be true even if u were not a classical solution.

Definition 4.33 (Mild solutions)**.** A continuous function $u : \mathbb{R}_+ \to X$ is called a mild solution of the semilinear Cauchy problem if it satisfies the integral equation

$$u(t) = T(t)u_0 + \int_0^t T(t-s)f(u(s))ds$$

where $T(t)$ is the semigroup generated by $-A$.

Example 4.34 (Navier–Stokes equations)**.** The Navier–Stokes equations are a semilinear system,

$$\begin{cases} \begin{cases} \vec{u}'(t) = \Delta \vec{u} - \nabla \cdot (\vec{u} \otimes \vec{u}) - \nabla p, \\ \nabla \cdot \vec{u} = 0, \end{cases} \\ \vec{u}(0) = \vec{u}_0, \end{cases}$$

where $\vec{u} : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is now a vector valued solution. We will henceforth focus on this particular example.

Chapter 5

Navier–Stokes equations

5.1 Introduction

The Navier–Stokes equations describe the motion of fluids. In n dimensions the equations are given by

$$\begin{cases} \partial_t \vec{u} = \Delta \vec{u} - \nabla \cdot (\vec{u} \otimes \vec{u}) - \nabla p, \\ \nabla \cdot \vec{u} = 0. \end{cases} \tag{5.1}$$

Expanding the notation, this is a system of $n+1$ equations

$$\begin{cases} \partial_t u_k = \Delta u_k - \sum_{l=1}^n \partial_l (u_l\, u_k) - \partial_k p \qquad\qquad k = 1, \dots, n, \\ \sum_{j=1}^n \partial_j u_j = 0. \end{cases}$$

The unknowns are $\vec{u} : \mathbb{R}_+^{n+1} \to \mathbb{R}^n$, the velocity field of the fluid, and the pressure $p : \mathbb{R}_+^{n+1} \to \mathbb{R}$.

This equation contains four components, each with physical interpretation. The time derivative of the velocity field $\partial_t \vec{u}$ is written as the sum of the diffusing viscous term $\Delta \vec{u}$, a transport term $\nabla \cdot (\vec{u} \otimes \vec{u})$ and the gradient of the pressure ∇p. The second equation, $\nabla \cdot \vec{u} = 0$, the divergence free condition, expresses that the fluid is incompressible.

We now present a heuristic calculation that demonstrates it is possible rewrite the Navier–Stokes equations in order to eliminate the pressure term. Taking the divergence of (5.1),

$$\sum_{k=1}^n \partial_k \partial_t u_k = \sum_{k=1}^n \partial_k \Delta u_k - \sum_{k=1}^n \sum_{l=1}^n \partial_k \partial_l (u_l u_k) - \Delta p. \tag{5.2}$$

Commuting the derivatives this becomes,

$$\Delta p = - \sum_{k=1}^n \sum_{l=1}^n \partial_k \partial_l (u_k u_l) = -\nabla \otimes \nabla \cdot (\vec{u} \otimes \vec{u}). \tag{5.3}$$

Introducing the Leray projection operator,

$$\mathbb{P} \vec{f} = \vec{f} - \nabla \frac{1}{\Delta} (\nabla \cdot \vec{f}), \tag{5.4}$$

we compute

$$\begin{aligned}\mathbb{P}\nabla\cdot(\vec{u}\otimes\vec{u}) &= \nabla\cdot(\vec{u}\otimes\vec{u}) - \nabla\frac{1}{\Delta}(\nabla\otimes\nabla)(\vec{u}\otimes\vec{u})\\ &= \nabla\cdot(\vec{u}\otimes\vec{u}) + \nabla\frac{1}{\Delta}\Delta p\\ &= \nabla\cdot(\vec{u}\otimes\vec{u}) + \nabla p,\end{aligned}$$

such that (5.1) can be rewritten

$$\begin{cases}\partial_t\vec{u} = \Delta\vec{u} - \mathbb{P}(\nabla\cdot(\vec{u}\otimes\vec{u})),\\ \nabla\cdot\vec{u} = 0.\end{cases} \tag{5.5}$$

Remark. The above derivation is largely unjustified. A priori, the Leray projection operator $\mathbb{P}$ is ill-defined since the the Laplacian need not be invertible. We must also define our solution space.

5.2 Solutions to the Navier–Stokes equations

We are interested in the Cauchy problem for the Navier–Stokes equations. Given an initial velocity field $\vec{u}_0$ we look for $\vec{u}$, a velocity field at later times.

Definition 5.1 (Weak solutions to Navier–Stokes equations)**.** A distribution vector field $\vec{u}\in(\mathcal{D}'(\mathbb{R}^{n+1}_+))^n$ is called a weak solution to the Navier–Stokes equations if

(a) $\vec{u}\in L^2_{\text{loc}}(\mathbb{R}^{n+1}_+)$,

(b) $\nabla\cdot\vec{u}=0$,

(c) there exists $p\in\mathcal{D}'(\mathbb{R}^{n+1}_+)$ such that $\partial_t\vec{u}=\Delta\vec{u}-\nabla\cdot(\vec{u}\otimes\vec{u})-\nabla p$ in $\mathcal{D}'$.

We are particularly interested in mild solutions to the Cauchy problem for Navier–Stokes equations. We look for a Banach spaces of initial data $\mathcal{E}$ and a space of solutions E such that for all $\vec{u}_0\in\mathcal{E}^n$, $e^{t\Delta}\vec{u}_0\in E^n$ and

$$\vec{u} = e^{t\Delta}\vec{u}_0 - \int_0^t e^{(t-s)\Delta}\mathbb{P}(\nabla\cdot(\vec{u}\otimes\vec{u}))ds$$

is a mild solution to the Navier–Stokes equations. In section 5.5 we prove a global existence result. We show that for sufficiently small initial data $\vec{u}_0\in(\text{BMO}^{-1})^n$, there exists a mild solution $\vec{u}$ that lies in a subspace of the tent space $T^{\infty,2}$.

Definition 5.2 (Mild solutions to Navier–Stokes equations)**.** We call $\vec{u}\in E^n$ a mild solution to the Navier–Stokes equations for initial data $\vec{u}_0\in\mathcal{E}^n$ if

$$\begin{cases}\vec{u} = e^{t\Delta}\vec{u}_0 - \int_0^t e^{(t-s)\Delta}\mathbb{P}(\nabla\cdot(\vec{u}\otimes\vec{u}))ds \quad \text{in } E^n,\\ \nabla\cdot\vec{u}_0 = 0.\end{cases} \tag{5.6}$$

We now introduce the uniformly locally L^p spaces which will be important when showing the equivalence of notions of solutions to Navier–Stokes equations.

Definition 5.3. The space L^p_{uloc} is the Banach space of measurable functions $f : \mathbb{R}^n \to \mathbb{C}$ such that

$$\|f\|_{L^p_{\mathrm{uloc}}} := \sup_{x_0 \in \mathbb{R}^n} \left(\int_{B(x_0,1)} |f(x)|^p \, dx \right)^{1/p} < \infty.$$

We are interested in solutions $\vec{u}$ that are uniformly locally square integrable in space and locally square integrable in time. That is, a measurable function $\vec{u}$ defined on $\mathbb{R}^{n+1}_+$ such that

$$\|\vec{u}\|_{L^2_{\mathrm{uloc},x} L^2_t} := \sup_{x_0 \in \mathbb{R}^n} \left(\int_{B(x_0,1)} \left(\int_0^\infty |f(t,x)|^2 dt \right) dx \right)^{1/2} < \infty.$$

In fact, the notions of weak and mild solution coincide on this space.

Theorem 5.4. *Let $\vec{u} \in \bigcap_{t_1 > 0} (L^2_{uloc,x} L^2_t((0,t_1) \times \mathbb{R}^n))^n$. Then $\vec{u}$ is a weak solution to the Navier–Stokes equations if and only if it is a mild solution.*

A proof of this theorem can be found in [LR02, pp.112–113].

5.3 The Leray projection operator

In the previous section we introduced the Leray projection operator

$$\mathbb{P}\vec{f} = \vec{f} - \nabla \frac{1}{\Delta} (\nabla \cdot \vec{f}).$$

The first thing to note about this operator is that it projects onto divergence free vector fields. If $\vec{u} : \mathbb{R}^n \to \mathbb{C}^n$ is a vector field in the domain of the Leray projector then

$$\nabla \cdot \mathbb{P}\vec{u} = \nabla \cdot \vec{u} - \nabla \cdot \nabla \frac{1}{\Delta} \nabla \vec{u} = \nabla \cdot \vec{u} - \nabla \cdot \vec{u} = 0.$$

The previous section gave the standard definition of the Leray projector. Here we present two alternate interpretations of the Leray projection operator.

5.3.1 Riesz transform interpretation

Recall the Riesz transforms

$$\mathcal{R}_j = \frac{\partial_j}{\sqrt{-\Delta}} \qquad 1 \le j \le n,$$

when acting on $f \in L^2(\mathbb{R}^n)$ can be expressed by Fourier multipliers

$$\widehat{\mathcal{R}_j f}(\xi) = \frac{i\xi_j}{|\xi|} \widehat{f}(\xi) \qquad 1 \le j \le n.$$

We thus have an equivalent definition for the Leray projection operator when acting on $L^2(\mathbb{R}^n)$,

$$\mathbb{P} = I + \mathcal{R} \otimes \mathcal{R} \tag{5.7}$$

where $\mathcal{R}$ is the vector of Riesz transforms,

$$(\mathbb{P}\vec{f})_{1 \le j \le n} = f_j + \sum_{k=1}^n \mathcal{R}_j \mathcal{R}_k f_k.$$

We can thus realise the Leray projector $\mathbb{P}$ on L^2 as the matrix valued Fourier multiplier

$$\mathrm{Id} - \frac{\xi \otimes \xi}{|\xi|^2} = \left(\delta_{jk} - |\xi|^{-2} \xi_j \xi_k \right)_{1 \le j,k \le n}.$$

5.3.2 The Oseen kernel

We are interested in mild solutions to the Navier–Stokes equations. That is, a function $\vec{u}$ satisfying

$$\vec{u} = e^{t\Delta}\vec{u_0} - \int_0^t e^{(t-s)\Delta}\mathbb{P}(\nabla\cdot(\vec{u}\otimes\vec{u}))ds.$$

We look at,

$$e^{t\Delta}\mathbb{P}\vec{f} = e^{t\Delta}\vec{f} - e^{t\Delta}\nabla\frac{1}{\Delta}(\nabla\cdot\vec{f}).$$

The heat semigroup term $e^{t\Delta}\vec{f}$ we will treat with the semigroup methods of chapter 4. For now we focus on the other term[1]

$$e^{t\Delta}\nabla\frac{1}{\Delta}(\nabla\cdot\vec{f}) = e^{t\Delta}\partial^j\frac{1}{\Delta}\partial_k f^k = \frac{1}{\Delta}\partial^j\partial_k e^{t\Delta}f^k =: O_{j,k,t}f^k.$$

Proposition 5.5. *For $1 \le j,k \le n$ and $t > 0$, the operator $O_{j,k,t} = \frac{1}{\Delta}\partial_j\partial_k e^{t\Delta}$ is a convolution operator,*

$$O_{j,k,t}f(x) = \int_{\mathbb{R}^n} K_{j,k,t}(x-y)f(y)dy$$

where the Oseen kernel $K_{j,k,t}$ satisfies $K_{j,k,t}(x) = t^{-n/2}K_{j,k}\left(t^{-1/2}x\right)$ for a smooth function $K_{j,k}$ such that

$$x \mapsto (1+|x|)^{n+|\alpha|}\,\partial^\alpha K_{j,k} \in L^\infty(\mathbb{R}^n) \tag{5.8}$$

for all multiindices $\alpha \in \mathbb{N}^n$.

This kernel was introduced in 1927 by Wilhelm Oseen [Ose27] when studying the Navier–Stokes equations in $\mathbb{R}^3$.

Proof. We write the kernel as a Fourier multiplier

$$\widehat{K_{j,k}} = \frac{\xi_j\xi_k}{|\xi|^2}e^{-|\xi|^2}.$$

Let $\alpha \in \mathbb{N}^n$ be a multiindex. We have the representation

$$\widehat{\partial^\alpha K_{j,k}} = (i)^{|\alpha|}\xi^\alpha\frac{\xi_j\xi_k}{|\xi|^2}e^{-|\xi|^2}.$$

where ξ^α is the vector $(\xi_1^{\alpha_1},\ldots,\xi_n^{\alpha_n})$. By the decay of $e^{-|\xi|^2}$ it follows that $\widehat{\partial^\alpha K_{j,k}} \in \mathcal{S}$ such that $\partial^\alpha K_{j,k} \in \mathcal{S}$ so that $\partial^\alpha K_{j,k} \in L^\infty$. We have thus verified (5.8) for $|x| < 1$. For $|x| \ge 1$ we use the Littlewood–Paley decomposition, Theorem 2.36 and write

$$K_{j,k} = (\mathrm{Id} - S_0)K_{j,k} + \sum_{l\le 0}\Delta_l K_{j,k},$$

where $S_0 f = f * \varphi$, $\Delta_l = f * \psi_l$ with $\psi_l(x) = 2^{-ln}\psi(2^{-l}x)$. Notice $(\mathrm{Id} - S_0)K_{j,k} \in \mathcal{S}$ since $K_{j,k} \in \mathcal{S}$ and

$$(I - \widehat{S_0)K_{j,k}}(\xi) = (1-\widehat{\varphi}(\xi))\widehat{K_{j,k}}(\xi) \in \mathcal{S}$$

since $\widehat{\varphi} \in C_c^\infty$. The other term can be written as

$$\widehat{\Delta_l K_{j,k}}(\xi) = \widehat{\psi_l}(\xi)\widehat{K_{j,k}}(\xi) = \widehat{\psi}(2^{-l}\xi)\frac{\xi_j\xi_k}{|\xi|^2}e^{-|\xi|^2}.$$

[1] Commuting the operators is justified by writing each as a Fourier multiplier.

We thus have

$$\Delta_l K_{j,k} = 2^{ln}\omega_{j,k,l}(2^l x) \quad \text{where} \quad \widehat{\omega}_{j,k,l}(\xi) = \widehat{\psi}(\xi)\frac{\xi_j\xi_k}{|\xi|^2}e^{-|2^l\xi|^2}.$$

Analogously, $\omega_{j,k,l} \in \mathcal{S}$ for all j,k,l and in fact these functions are uniformly bounded when we consider only $l \leq 0$. Since $\omega_{j,k,l} \in \mathcal{S}$ we have bounds on the Fréchet norms, see section B.2,

$$\begin{aligned}
\|\omega_{j,k,l}\|_{\alpha,N} &= \sup_{x\in\mathbb{R}^n}(1+|x|)^N|\partial^\alpha\omega_{j,k,l}| \\
&= \sup_{\xi\in\mathbb{R}^n}(1+|\xi|)^N|\partial^\alpha\widehat{\omega}_{j,k,l}| \\
&= \sup_{\xi\in\mathbb{R}^n}(1+|\xi|)^N\left|\partial^\alpha\widehat{\psi}(\xi)\frac{\xi_j\xi_k}{|\xi|^2}e^{-|2^l\xi|}\right| \\
&\leq C_{\alpha,N}
\end{aligned}$$

for all $N \in \mathbb{N}$ since the Fourier transform is an isomorphism on $\mathcal{S}$. It follows that

$$\begin{aligned}
|\partial_\alpha\Delta_l K_{j,k}| &= 2^{ln}|\partial_\alpha(\omega_{j,k,l}(2^l x))| \\
&= 2^{l(n+|\alpha|)}|(\partial_\alpha\omega_{j,k,l})(2^l x)| \\
&\leq (1+2^l|x|)^{-N}2^{l(n+|\alpha|)}C_{\alpha,N}.
\end{aligned}$$

If we consider now $N > n+|\alpha|$,

$$\begin{aligned}
|\partial^\alpha S_0 K_{j,k}| &\leq \sum_{l<0}|\partial_\alpha\Delta_l K_{j,k}| \\
&\lesssim \sum_{l<0}(1+2^l|x|)^{-N}2^{l(n+|\alpha|)} \\
&= \sum_{2^l|x|\leq 1}(1+2^l|x|)^{-N}2^{l(n+|\alpha|)} + \sum_{2^l|x|>1}(1+2^l|x|)^{-N}2^{l(n+|\alpha|)} \\
&\leq \sum_{l<-\log_2(|x|)}2^{l(n+|\alpha|)} + \sum_{-\log_2(|x|)<l<0}(2^l|x|)^{-N}2^{l(n+|\alpha|)} \\
&= \frac{2^{n+|\alpha|}|x|^{-(n+\alpha)}}{2^{n+|\alpha|}-1} + \frac{2^N(-1+|x|^{-n-\alpha+N})}{(2^N-2^{n+|\alpha|})|x|^N} \\
&\lesssim |x|^{-n-|\alpha|}.
\end{aligned}$$

□

5.3.3 Well definedness

The fact that we can write the Leray projector in terms of Riesz transforms shows that the Leray projector is well defined on $(L^2(\mathbb{R}^n))^n$. The Leray projector is actually well defined on even larger spaces. In addition to the well definedness of the Leray projector the following theorem justifies the elimination of pressure derivation presented at the beginning of this chapter.

Theorem 5.6 (Elimination of pressure). *Let $\vec{u} \in (L^2_{uloc,x}L^2_t(\mathbb{R}^{n+1}_+))^n$. Then,*

(i) $\mathbb{P}\nabla(\vec{u}\otimes\vec{u})$ is well defined in $(\mathcal{D}'(\mathbb{R}^{n+1}_+))^n$ and there exists $p \in \mathcal{D}'(\mathbb{R}^{n+1}_+)$ such that

$$\mathbb{P}\nabla(\vec{u}\otimes\vec{u}) = \nabla\cdot(\vec{u}\otimes\vec{u}) + \nabla p.$$

(ii) If $\vec{u}$ weak solution to the Navier–Stokes equations and for all $t_1 > t_0 > 0$ we have

$$\lim_{R\to\infty} \sup_{x_0\in\mathbb{R}^n} R^{-n} \int_{t_0}^{t_1} \int_{B(x_0,R)} |\vec{u}(x,t)|^2 dx\, dt = 0$$

then $\vec{u}$ is a mild solution to the Navier–Stokes equations.

A proof of this is given by [LR02, pp.109–110]. In some ways the well definedness of the Leray projector is a moot point since it is part of the proof of the Koch and Tataru result in section 5.5.

5.4 Picard fixed point method

We are interested in finding mild solutions to the Navier–Stokes equations. That is, solutions to the integral equation

$$\vec{u} = e^{t\Delta}\vec{u_0} - \int_0^t e^{(t-s)\Delta}\mathbb{P}(\nabla\cdot(\vec{u}\otimes\vec{u}))ds. \tag{5.9}$$

The approach we present here is a fixed point argument. We consider the bilinear transform

$$B(\vec{u},\vec{v}) = \int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla\cdot(\vec{u}\otimes\vec{v})ds. \tag{5.10}$$

Solving (5.9) reduces to finding a fixed point of the transform

$$\vec{v} \mapsto e^{t\Delta}\vec{u_0} - B(\vec{v},\vec{v}). \tag{5.11}$$

In order to find mild solutions we are interested in the boundedness of the bilinear transform (5.10). We begin by looking for a Banach space $\mathcal{E}$ of functions defined on $\mathbb{R}^{n+1}_+$ such that $B : \mathcal{E}^n \times \mathcal{E}^n \to \mathcal{E}^n$ is bounded. We also define the associated distribution space $E \subset \mathcal{S}'$ by $f \in E$ if and only if $f \in \mathcal{S}'$ and $e^{t\Delta}f \in \mathcal{E}$ for all $t > 0$.

The main result of this section is that we can prove the existence of mild solutions to the Navier–Stokes equations by showing that B is bounded.

Theorem 5.7 (The Picard contraction principle)**.** *Let $\mathcal{E} \subset L^2_{uloc,x}L^2_t(\mathbb{R}^{n+1}_+)$ be chosen such that the bilinear transform B is bounded. Then:*

(i) If $\vec{u} \in \mathcal{E}^n$ is a weak solution for the Navier–Stokes equations

$$\partial_t\vec{u} = \Delta\vec{u} - \mathbb{P}(\nabla\cdot(\vec{u}\otimes\vec{u})) \text{ and } \nabla\cdot\vec{u} = 0$$

then the associated initial value $\vec{u_0}$ is contained in the space E^n.

(ii) There exists $C > 0$ such that for $\vec{u_0} \in \mathcal{E}^n$ such that $\nabla\cdot\vec{u_0}$ and $\left\|e^{t\Delta}\vec{u_0}\right\|_{\mathcal{E}} < C$ there exists a mild solution $\vec{u} \in E^n$ of the Navier–Stokes equations associated with initial value $\vec{u_0}$,

$$\vec{u} = e^{t\Delta}\vec{u_0} - \int_0^t e^{(t-s)\Delta}\mathbb{P}(\nabla\cdot(\vec{u}\otimes\vec{u}))ds.$$

Proof.

(i) Let $\vec{u} \in \mathcal{E}^n$ be a solution to the differential form of Navier–Stokes equations. Since B maps $\mathcal{E}^n \times \mathcal{E}^n$ to $\mathcal{E}^n$, write $e^{t\Delta}\vec{u_0} = \vec{u} + B(\vec{u}, \vec{u})$ such that $e^{t\Delta}\vec{u_0} \in \mathcal{E}^n$. By the definition of E we have immediately that $\vec{u_0} \in E^n$.

(ii) Since B is bounded from $\mathcal{E}^n \times \mathcal{E}^n$ to $\mathcal{E}^n$ we have $M > 0$ such that $\|B(\vec{v}, \vec{w})\|_{\mathcal{E}} \leq M \|\vec{v}\|_{\mathcal{E}} \|\vec{w}\|_{\mathcal{E}}$ for all $\vec{u}, \vec{v} \in \mathcal{E}^n$

Step 1 Fix $\delta < \frac{1}{4M}$ and suppose $\vec{u_0} \in E^n$ such that $\left\|e^{t\Delta}\vec{u_0}\right\|_{\mathcal{E}} \leq \delta/2$. Define the ball

$$B' = \{\vec{v} \in \mathcal{E}^n : \|\vec{v}\|_{\mathcal{E}} \leq \delta\}.$$

Observe that the mapping $F : \vec{v} \mapsto e^{t\Delta}\vec{u_0} - B(\vec{v}, \vec{v})$ is contractive on this ball. For $\vec{u}, \vec{v} \in B'$ we have

$$\begin{aligned} \|F(\vec{v}) - F(\vec{u})\|_{\mathcal{E}} &= \|B(\vec{v}, \vec{v}) - B(\vec{u}, \vec{u})\|_{\mathcal{E}} \\ &= \|B(\vec{v} - \vec{u}, \vec{v}) + B(\vec{u}, \vec{v} - \vec{u})\|_{\mathcal{E}} \\ &\leq \|B(\vec{v} - \vec{u}, \vec{v})\|_{\mathcal{E}} + \|B(\vec{u}, \vec{v} - \vec{u})\|_{\mathcal{E}} \\ &\leq M \|\vec{v} - \vec{u}\|_{\mathcal{E}} \left(\|\vec{v}\|_{\mathcal{E}} + \|\vec{u}\|_{\mathcal{E}}\right) \\ &\leq \frac{1}{2} \|\vec{v} - \vec{u}\|_{\mathcal{E}}. \end{aligned}$$

That is, F is a (local) contraction.

Step 2 Consider the sequence defined by recursively applying this contraction mapping

$$\vec{u}^{(0)} := 0, \qquad \vec{u}^{(m+1)} := F(\vec{u}^{(m)}).$$

We compute

$$\begin{aligned} \left\|\vec{u}^{(m+1)} - \vec{u}^{(m)}\right\|_{\mathcal{E}} &= \left\|F(\vec{u}^{(m)}) - F(\vec{u}^{(m-1)})\right\|_{\mathcal{E}} \\ &\leq \frac{1}{2} \left\|\vec{u}^{(m)} - \vec{u}^{(m-1)}\right\|_{\mathcal{E}} \\ &\vdots \\ &\leq 2^{-m} \left\|\vec{u}^{(1)} - u^{(0)}\right\|_{\mathcal{E}} = 2^{-m} \left\|e^{t\Delta}\vec{u}_0\right\|_{\mathcal{E}} \\ &\leq 2^{-(m+1)}\delta \end{aligned}$$

so that

$$\left\|\vec{u}^{(m)}\right\|_{\mathcal{E}} \leq \sum_{k=1}^{m} \left\|\vec{u}^{(k)} - \vec{u}^{(k-1)}\right\|_{\mathcal{E}} \leq \sum_{k=1}^{m} 2^{-k}\delta \leq \delta$$

for all $m \geq 0$. Furthermore, $\left(\vec{u}^{(m)}\right)_{m\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{E}$ and thus converge to $\vec{u}^{(\infty)}$ a fixed point of F. That is, $\vec{u}^{(\infty)}$ is a mild solution to the Navier–Stokes equations. □

In light of the Picard contraction principle, the problem of finding a mild solution to Navier–Stokes equations has been reduced to a question of the boundedness of a bilinear operator.

5.5 The Koch and Tataru result

Mild solutions were first constructed by Fujita and Kato [FK64] in the Sobolev spaces $W^{2,p}(\mathbb{R}^n)$ where $p \geq n/2 - 1$. In 1984 Kato [Kat84] proved the existence of mild solutions in L^p for $p \geq n$.

Following Kato's result, Giga and Miyakawa [GM89] and Taylor [Tay92] proved analogous results for initial data in Morrey spaces. Cannone [Can97] and Planchon [Pla96] obtained similar results for the Besov spaces $\dot{B}_p^{n/p-1,\infty}(\mathbb{R}^n)$ with $p \in (1, \infty)$.

In 2001 Koch and Tataru [KT01] demonstrated that the Navier–Stokes equations are well posed for data in the space BMO^{-1}. The space BMO^{-1} has a special role since it is the largest critical space for which Navier Stokes is known to be well posed. In 2008, Bourgain and Pavlović [BP08] solved the $p = \infty$ endpoint case of Cannone and Planchon by showing the ill posedness of Navier–Stokes equations in the Besov space $\dot{B}_\infty^{-1,\infty}$.

The spaces mentioned in this section are called critical spaces for the Navier–Stokes equations because they have the correct scaling. In fact we have the following embeddings, with BMO^{-1} sandwiched between well posed and ill posed Besov spaces,

$$W^{2,p|p\geq n/2-1} \hookrightarrow L^{p|p\geq n} \hookrightarrow \dot{B}_{p|p<\infty}^{n/p-1,\infty} \hookrightarrow \mathrm{BMO}^{-1} \hookrightarrow \dot{B}_\infty^{-1,\infty}. \tag{5.12}$$

See [Can04] for a proof of these embeddings into $\dot{B}_\infty^{-1,\infty}$.

Definition 5.8. We define BMO^{-1} as the space $f \in \mathcal{S}'$ for which

$$\|f\|_{\mathrm{BMO}^{-1}} := \sup_{t>0} \sup_{x\in\mathbb{R}^n} \left(\frac{1}{|B(x,\sqrt{t})|} \int_0^t \int_{B(x,\sqrt{t})} \left| e^{s\Delta} f(y) \right|^2 dy\, ds \right)^{\frac{1}{2}} < \infty.$$

Remark. Note the clear connection with tent spaces,

$$f \in \mathrm{BMO}^{-1} \iff (x,t) \mapsto e^{t\Delta} f(x) \in T^{\infty,2}$$

Proposition 5.9. *We have $f \in \mathrm{BMO}^{-1}$ if and only if there exists $f_1, \ldots, f_n \in \mathrm{BMO}$ with*

$$f = \sum_{i=1}^n \partial_i f_i.$$

This is proven in [LR02, pp.160–162] using Besov spaces. An alternative proof using tent spaces is possible. One direction is immediate from the square function characterisation of BMO while the reverse follows the lines of section 4.5.

We now state Koch and Tataru's existence result for global mild solutions to the Navier–Stokes equations.

Theorem 5.10 (Koch and Tataru)**.** *There exits a constant $\epsilon > 0$ such that for all $\vec{u}_0 \in (\mathrm{BMO}^{-1})^n$ with $\|\vec{u}_0\|_{\mathrm{BMO}^{-1}} < \epsilon$ and $\nabla \cdot \vec{u}_0 = 0$ there exists a weak solution $\vec{u}$ of the Navier–Stokes equations on $\mathbb{R}_+^{n+1}$:*

$$\begin{cases} \partial_t \vec{u} = \Delta \vec{u} - \mathbb{P}\nabla \cdot (\vec{u} \otimes \vec{u}), \\ \nabla \cdot \vec{u} = 0, \end{cases} \tag{5.13}$$

so that

(i) $\sup_{t>0} \sup_{x\in\mathbb{R}^n} |\sqrt{t}\vec{u}(x,t)| < \infty$,

(ii) $\sup_{t>0} \sup_{x\in\mathbb{R}^n} \left(t^{-n/2} \int_0^t \int_{B(x,\sqrt{t})} |e^{s\Delta}\vec{u}(x,s)|^2 ds\, dx\right)^{\frac{1}{2}} < \infty$,

(iii) $\lim_{t\to 0} \vec{u}(x,t) = \vec{u}_0(x)$ *for a.e.* $x \in \mathbb{R}^n$.

Suppose $\vec{u} \in (L^2_{\mathrm{uloc},x} L^2_t(\mathbb{R}^{n+1}_+))^n$ such that conditions (i) and (ii) hold. We then say that $\vec{u} \in \mathcal{E}^n$. Notice that condition (ii) is precisely a $T^{\infty,2}$ norm. We thus define the adapted value space $\mathcal{E}$ as the set of $f \in L^2_{\mathrm{uloc},x} L^2_t(\mathbb{R}^{n+1}_+)$ such that

$$\|f\|_{\mathcal{E}} := \|\sqrt{t}f\|_{L^\infty(\mathbb{R}^{n+1}_+)} + \|f\|_{T^{\infty,2}(\mathbb{R}^{n+1}_+)} < \infty.$$

The proof of Theorem 5.10 relies on the boundedness of B established in Theorem 5.11.

Proof of Theorem 5.10. By Theorem 5.11 in conjunction with the Picard contraction principle Theorem 5.7 we have existence of mild solutions. We conclude by the equivalence of weak and mild solutions Theorem 5.4. □

It suffices to prove the following.

Theorem 5.11. *The bilinear form* (5.10) *is bounded from* $\mathcal{E}^n \times \mathcal{E}^n$ *into* $\mathcal{E}^n$*.*

Proof.

Step 1 (Reduction to linear estimate).
Let $u, v \in \mathcal{E}$ and define $\alpha := u \otimes v$. We denote $|\alpha| = \|\alpha\|_{C^n\otimes\mathbb{C}^n} = \sum_{j,k=1}^n |u_j v_k|$. First observe that $\alpha \in T^{\infty,1}(\mathbb{R}^{n+1}_+; \mathbb{C}^n \otimes \mathbb{C}^n)$,

$$\begin{aligned}
\|\alpha\|_{T^{\infty,1}} &= \sup_{B(x,\sqrt{t})} t^{-n/2} \int_0^t \int_{B(x,\sqrt{t})} |\alpha(s,y)| dy\, ds \\
&\le \sum_{j,k=1}^n \sup_{B(x,\sqrt{t})} t^{-n/2} \int_0^t \int_{B(x,\sqrt{t})} |u_j(s,y) v_k(s,y)| dy\, ds \\
&\le \sum_{j,k=1}^n \sup_{B(x,\sqrt{t})} \left(t^{-n/2} \int_0^t \int_{B(x,\sqrt{t})} |u_j(s,y)|^2 dy\, ds\right)^{\frac{1}{2}} \left(t^{-n/2} \int_0^t \int_{B(x,\sqrt{t})} |v_k(s,y)|^2 dy\, ds\right)^{\frac{1}{2}} \\
&\le \sum_{j,k=1}^n \|u_j\|_{\mathcal{E}} \|v_k\|_{\mathcal{E}} \lesssim \|\vec{u}\|_{\mathcal{E}^n} \|\vec{v}\|_{\mathcal{E}^n}.
\end{aligned}$$

where the last step the uses the equivalence of finite dimensional norms. Similarly, $s^{1/2}\alpha(s,\cdot) \in T^{\infty,2}(\mathbb{R}^{n+1}_+; \mathbb{C}^n \otimes \mathbb{C}^n)$ since

$$\begin{aligned}
\left\|s^{1/2}\alpha\right\|_{T^{\infty,2}} &= \sup_{B(x,\sqrt{t})} \left(t^{-n/2} \int_0^t \int_{B(x,\sqrt{t})} s|\alpha(s,y)|^2 dy\, ds \right)^{\frac{1}{2}} \\
&\leq \sup_{B(x,\sqrt{t})} \left(t^{-n/2} \int_0^t \|s\alpha(s,\cdot)\|_{L^\infty} \left(\int_{B(x,\sqrt{t})} |\alpha(s,y)| dy \right) ds \right)^{\frac{1}{2}} \\
&\leq \sup_{t>0} \left\| t|\alpha(t,\cdot)| \right\|_{L^\infty}^{\frac{1}{2}} \cdot \sup_{B(x,\sqrt{t})} \left(t^{-n/2} \int_0^t \int_{B(x,\sqrt{t})} |\alpha(s,y)| dy\, ds \right)^{\frac{1}{2}} \\
&\leq \left(\sum_{j,k=1}^n \sup_{t>0} \left\| t u_j(t,\cdot) v_k(t,\cdot) \right\|_{L^\infty} \right)^{\frac{1}{2}} \left(\sum_{j,k=1}^n \|u_j\|_{\mathcal{E}} \|v_k\|_{\mathcal{E}} \right)^{\frac{1}{2}} \\
&\leq \left(\sum_{j,k=1}^n \sup_{t>0} \left\| t^{1/2} u_j(t,\cdot) \right\|_{L^\infty} \left\| t^{1/2} v_k(t,\cdot) \right\|_{L^\infty} \right)^{\frac{1}{2}} \left(\sum_{j,k=1}^n \|u_j\|_{\mathcal{E}} \|v_k\|_{\mathcal{E}} \right)^{\frac{1}{2}} \\
&\leq \sum_{j,k=1}^n \|u_j\|_{\mathcal{E}} \|v_k\|_{\mathcal{E}} \lesssim \|\vec{u}\|_{\mathcal{E}^n} \|\vec{v}\|_{\mathcal{E}^n}.
\end{aligned}$$

The final linearisation to notice is $s\alpha \in L^\infty(\mathbb{R}^{n+1}_+; \mathbb{C}^n \otimes \mathbb{C}^n)$,

$$\begin{aligned}
\|s\alpha(s,\cdot)\|_{L^\infty(\mathbb{R}^{n+1}_+)} &= \sum_{j,k=1}^n \|s u_k(s,\cdot) v_j(s,\cdot)\|_{L^\infty(\mathbb{R}^n)} \\
&\leq \sum_{j,k=1}^n \sup_{t>0} \left\| t^{1/2} u_j(t,\cdot) \right\|_{L^\infty(\mathbb{R}^n)} \sup_{t>0} \left\| t^{1/2} v_k(t,\cdot) \right\|_{L^\infty(\mathbb{R}^n)} \\
&\leq \sum_{j,k=1}^n \|u_j\|_{\mathcal{E}} \|v_k\|_{\mathcal{E}} \lesssim \|\vec{u}\|_{\mathcal{E}^n} \|\vec{v}\|_{\mathcal{E}^n}.
\end{aligned}$$

We define a new (linear) operator

$$\mathcal{A}(\alpha)(t,\cdot) := \int_0^t e^{(t-s)\Delta} \mathbb{P} \operatorname{div} \alpha(s,\cdot) ds.$$

In light of the above calculations, showing that the bilinear operator is bounded reduces to checking whether $\mathcal{A}$ is bounded. That is, it suffices to show

$$\left\| t^{1/2} \mathcal{A}(\alpha) \right\|_{L^\infty(\mathbb{R}^{n+1}_+; \mathbb{C}^n)} \lesssim \left\| \alpha \right\|_{T^{\infty,1}(\mathbb{R}^{n+1}_+; \mathbb{C}^n \otimes \mathbb{C}^n)} + \left\| s\alpha(s,\cdot) \right\|_{L^\infty(\mathbb{R}^{n+1}_+; \mathbb{C}^n \otimes \mathbb{C}^n)}, \tag{5.14}$$

$$\left\| \mathcal{A}(\alpha) \right\|_{T^{\infty,2}(\mathbb{R}^{n+1}_+; \mathbb{C}^n)} \lesssim \left\| \alpha \right\|_{T^{\infty,1}(\mathbb{R}^{n+1}_+; \mathbb{C}^n \otimes \mathbb{C}^n)} + \left\| s^{1/2}\alpha(s,\cdot) \right\|_{T^{\infty,2}(\mathbb{R}^{n+1}_+; \mathbb{C}^n \otimes \mathbb{C}^n)}. \tag{5.15}$$

Step 2 (L^∞ estimate).
We split the integral as

$$\mathcal{A}(\alpha)(t,\cdot) = \int_0^{\frac{t}{2}} e^{(t-s)\Delta} \mathbb{P} \operatorname{div} \alpha(s,\cdot) ds + \int_{\frac{t}{2}}^t e^{(t-s)\Delta} \mathbb{P} \operatorname{div} \alpha(s,\cdot) ds.$$

Expanding the Leray projector we write,

$$e^{(t-s)\Delta} \mathbb{P} \operatorname{div} \alpha(s,\cdot) = e^{(t-s)\Delta} \left(\operatorname{div} \alpha(s,\cdot) - \nabla \frac{1}{\Delta} \partial_j \partial_k \alpha^{jk}(s,\cdot) \right).$$

When $0 < s < t/2$,

$$\left| t^{1/2} \int_0^{\frac{t}{2}} e^{(t-s)\Delta} \mathbb{P} \operatorname{div} \alpha(\cdot, s) ds \right| = \left| t^{1/2} \int_0^{\frac{t}{2}} e^{(t-s)\Delta} \left(\operatorname{div} \alpha(s, \cdot) - \nabla \frac{1}{\Delta} \partial_j \partial_k \alpha^{jk}(s, \cdot) \right) ds \right|.$$

We recognise the Oseen kernel,

$$\begin{aligned} e^{(t-s)\Delta} \nabla \frac{1}{\Delta} \partial_j \partial_k \alpha^{jk} &= \nabla \frac{1}{\Delta} \partial_j \partial_k e^{(t-s)\Delta} \alpha^{jk} \\ &= \nabla O_{j,k,t-s} \alpha^{jk} \\ &= \nabla_x \int_{\mathbb{R}^n} K_{j,k,t-s}(x-y) \alpha^{jk}(s, y) dy \\ &= \int_{\mathbb{R}^n} \nabla_x (t-s)^{-n/2} K_{j,k} \left(\frac{x-y}{\sqrt{t-s}} \right) \alpha^{jk}(s, y) dy. \end{aligned}$$

Applying the Oseen kernel bounds Proposition 5.5,

$$e^{(t-s)\Delta} \nabla \frac{1}{\Delta} \partial_j \partial_k \alpha^{jk} \leq \int_{\mathbb{R}^n} (t-s)^{-n/2} (1 + |x-y|/\sqrt{t-s})^{-n-1} |\alpha(s, y)| dy.$$

The divergence only term can be estimated similarly. Using the decay of the heat kernel at ∞,

$$\begin{aligned} \left| e^{(t-s)\Delta} \operatorname{div} \alpha(s, x) \right| &\lesssim \int_{\mathbb{R}^n} (t-s)^{-n/2} e^{-(t-s)|x-y|^2} (x-y) \cdot \alpha(s, y) dy \\ &\lesssim \int_{\mathbb{R}^n} (t-s)^{-n/2} (1 + (t-s)^{-1/2} |x-y|)^{-n-1} |\alpha(s, y)| dy. \end{aligned}$$

Since $0 < s < t/2$ implies $t - s \sim t$,

$$\begin{aligned} \left| e^{(t-s)\Delta} \mathbb{P} \operatorname{div} \alpha(s, x) \right| &\lesssim \int_{\mathbb{R}^n} (t-s)^{-n/2} (1 + (t-s)^{-1/2} |x-y|)^{-n-1} \alpha(s, y) dy \\ &\lesssim \int_{\mathbb{R}^n} (\sqrt{t} + |x-y|)^{-n-1} \alpha(s, y) dy. \end{aligned}$$

In order to estimate this term we decompose $\mathbb{R}^n$ as a grid of disjoint cubes with side length $\sqrt{t}$. That is,

$$\begin{aligned} \left| e^{(t-s)\Delta} \mathbb{P} \operatorname{div} \alpha(s, x) \right| &\lesssim \sum_{k \in \mathbb{Z}^n} \int_{x-y \in \sqrt{t}(k+[0,1]^n)} (\sqrt{t} + |x-y|)^{-n-1} |\alpha(y, s)| dy \\ &\lesssim \sum_{k \in \mathbb{Z}^n} \int_{x-y \in \sqrt{t}(k+[0,1]^n)} t^{-n/2-1/2} (1 + |k|)^{-n-1} |\alpha(y, s)| dy. \end{aligned}$$

Multiplying by $t^{1/2}$, integrating from 0 up to $t/2$ and using the fact that every cube of side length $\sqrt{t}$ is contained in a ball of radius $\sqrt{t}$,

$$\begin{aligned} &\sup_{t>0} \left| t^{1/2} \int_0^{\frac{t}{2}} e^{(t-s)\Delta} \mathbb{P} \operatorname{div} \alpha(s, x) ds \right| \\ &\leq \sup_{t>0} \sum_{k \in \mathbb{Z}^n} \int_0^{\frac{t}{2}} t^{-n/2} \int_{x-y \in \sqrt{t}(k+[0,1]^n)} (1 + |k|)^{-n-1} |\alpha(y, s)| dy \\ &\leq \sum_{k \in \mathbb{Z}^n} (1 + |k|)^{-n-1} \|\alpha\|_{T^{\infty,1}} \\ &\lesssim \|\alpha\|_{T^{\infty,1}}. \end{aligned}$$

Now we look at $t/2 < s < t$. The first term we treat by looking at the kernel of the heat semigroup,

$$
\begin{aligned}
&\left| e^{(t-s)\Delta} \operatorname{div} \alpha(s,x) \right| \\
&\quad = \left| (4\pi(t-s))^{-n/2} \int_{\mathbb{R}^n} e^{-|x-y|^2/(4(t-s))} \operatorname{div} \alpha(s,y) dy \right| \\
&\quad = \left| (4\pi(t-s))^{-n/2} \int_{\mathbb{R}^n} e^{-|x-y|^2/(4(t-s))} (2(t-s))^{-1} (x-y) \cdot \alpha(s,y) dy \right| \\
&\quad \leq (4\pi(t-s))^{-n/2} \left(\int_{\mathbb{R}^n} e^{-|x-y|^2/(4(t-s))} (2s(t-s))^{-1} |x-y| dy \right) \|s\alpha(s,\cdot)\|_{L^\infty(\mathbb{R}^n)} \\
&\quad \lesssim \frac{1}{s\sqrt{t-s}} \int_{\mathbb{R}^n} e^{-|z|^2} |z| dz \, \|s\alpha(s,\cdot)\|_{L^\infty(\mathbb{R}^n)} \\
&\quad \lesssim \frac{1}{s\sqrt{t-s}} \|s\alpha(s,\cdot)\|_{L^\infty(\mathbb{R}^n, \mathbb{C}^n \otimes \mathbb{C}^n)} .
\end{aligned}
$$

The second term we treat similarly,

$$
\begin{aligned}
&\left| e^{(t-s)\Delta} \nabla \frac{1}{\Delta} (\nabla \otimes \nabla) \cdot \alpha(s,x) \right| \\
&\quad = \left| \int_{\mathbb{R}^n} (4\pi(t-s))^{n/2} e^{-(t-s)|x-y|^2} (x-y) \frac{(x-y) \otimes (x-y)}{|x-y|^2} |\alpha(s,y)| dy \right| \\
&\quad \leq \int_{\mathbb{R}^n} (4\pi(t-s))^{n/2} e^{-(t-s)|x-y|^2} |x-y| |\alpha(s,y)| dy \\
&\quad = \frac{1}{s\sqrt{t-s}} \int_{\mathbb{R}^n} e^{-z^2} z dz \, \|s\alpha(s,\cdot)\|_{L^\infty(\mathbb{R}^n);\mathbb{C}^n \otimes \mathbb{C}^n} \\
&\quad \lesssim \frac{1}{s\sqrt{t-s}} \|s\alpha(s,\cdot)\|_{L^\infty(\mathbb{R}^n);\mathbb{C}^n \otimes \mathbb{C}^n} .
\end{aligned}
$$

Multiplying these estimates by $t^{1/2}$, integrating from $t/2$ up to t and using $s > t/2$

$$
\begin{aligned}
\sup_{t>0} \left| t^{1/2} \int_{\frac{t}{2}}^t e^{(t-s)\Delta} \mathbb{P} \operatorname{div} \alpha(s,x) ds \right| &\lesssim \sup_{t>0} \int_{\frac{t}{2}}^t \frac{\sqrt{t}}{(t/2)\sqrt{t-s}} \|s\alpha(s,\cdot)\|_{L^\infty(\mathbb{R}^n, \mathbb{C}^n \otimes \mathbb{C}^n)} ds \\
&\lesssim \left(\sup_{t>0} \frac{1}{\sqrt{t}} \int_0^{\frac{t}{2}} \frac{du}{\sqrt{u}} \right) \|s\alpha(s,\cdot)\|_{L^\infty(\mathbb{R}^{n+1}_+, \mathbb{C}^n \otimes \mathbb{C}^n)} \\
&\lesssim \|s\alpha(s,\cdot)\|_{L^\infty(\mathbb{R}^{n+1}_+, \mathbb{C}^n \otimes \mathbb{C}^n)} .
\end{aligned}
$$

Step 3 ($T^{\infty,2}$ estimate).
We split the linear operator $\mathcal{A}$ into three parts.

$$\begin{aligned}
\mathcal{A}(\alpha)(t,\cdot) &= \int_0^t e^{(t-s)\Delta}\mathbb{P}\operatorname{div}\alpha(s,\cdot)\,ds \\
&= \int_0^t \left(e^{(t-s)\Delta} - e^{(t+s)\Delta}\right)\mathbb{P}\operatorname{div}\alpha(s,\cdot)\,ds + \int_0^t e^{(t+s)\Delta}\mathbb{P}\operatorname{div}\alpha(s,\cdot)\,ds \\
&= \int_0^t e^{(t-s)\Delta}\left(I - e^{2s\Delta}\right)\mathbb{P}\operatorname{div}\alpha(s,\cdot)\,ds \\
&\quad + \int_0^\infty e^{(t+s)\Delta}\mathbb{P}\operatorname{div}\alpha(s,\cdot)\,ds - \int_t^\infty e^{(t+s)\Delta}\mathbb{P}\operatorname{div}\alpha(s,\cdot)\,ds \\
&= \int_0^t e^{(t-s)\Delta}(-\Delta)(s(-\Delta))^{-1}\left(I - e^{2s\Delta}\right)s^{1/2}\mathbb{P}\operatorname{div}s^{1/2}\alpha(s,\cdot)\,ds \\
&\quad + \int_0^\infty e^{(t+s)\Delta}\mathbb{P}\operatorname{div}\alpha(s,\cdot)\,ds - \int_t^\infty e^{(t+s)\Delta}\mathbb{P}s^{-1/2}\operatorname{div}s^{1/2}\alpha(s,\cdot)\,ds \\
&=: \mathcal{A}_1(\alpha)(t,\cdot) + \mathcal{A}_2(\alpha)(t,\cdot) + \mathcal{A}_3(\alpha)(t,\cdot)
\end{aligned}$$

We now estimate each of these terms in turn.

Step 3(i) To treat $\mathcal{A}_1$ we use the boundedness of the maximal regularity operator. We check that $t(-\Delta)e^{t\Delta}$ has off-diagonal decay of order $M > n/4$. Let $E, F \subset \mathbb{R}^n$ be disjoint Borel sets.

$$\begin{aligned}
\left\|\mathbb{1}_E t(-\Delta)e^{t\Delta}\mathbb{1}_F f\right\|_{L^2(\mathbb{R}^n)}^2 &= \int_E \left|t(-\Delta)e^{t\Delta}\mathbb{1}_F(x)f(x)\right|^2 dx \\
&= \int_E \left|\int_{\mathbb{R}^n} t|x-y|^2 e^{-t|x-y|^2}\mathbb{1}_F(y)f(y)dy\right|^2 dx \\
&\leq \|g_t * f\|_{L^2(\mathbb{R}^n)}^2,
\end{aligned}$$

where

$$g_t(x) = \begin{cases} t|x|^2 e^{-t|x|^2} & |x| \geq \operatorname{dist}(E,F) \\ 0 & |x| < \operatorname{dist}(E,F) \end{cases},$$

since $x \in E$, $y \in F$ implies $|x-y| > \operatorname{dist}(E,F)$. It follows by Young's inequality,

$$\left\|\mathbb{1}_E t(-\Delta)e^{t\Delta}\mathbb{1}_F f\right\|_{L^2(\mathbb{R}^n)} \leq \|g\|_{L^\infty(\mathbb{R}^n)}\|f\|_{L^2(\mathbb{R}^n)}.$$

The g_t term is easy to estimate due to the fast decay of the Gaussian. Since re^{-r} attains a maximum at $r = 1$, that is, when $t = |x|^{-2} \leq \operatorname{dist}(E,F)^{-2}$,

$$t|x|^2 e^{-t|x|^2} \lesssim \left(1 + \frac{\operatorname{dist}(E,F)^2}{t}\right)^{-M}$$

for all $M \geq 1$. In particular for any $M > n/4$, we have off-diagonal bounds,

$$\left\|\mathbb{1}_E t(-\Delta)e^{t\Delta}\mathbb{1}_F f\right\|_{L^2(\mathbb{R}^n)} \lesssim \left(1 + \frac{\operatorname{dist}(E,F)^2}{t}\right)^{-M}\|f\|_{L^2(\mathbb{R}^n)}.$$

Invoking, Theorem 4.30, the maximal regularity operator is bounded on $T^{\infty,2}(\mathbb{R}^{n+1}_+,\mathbb{C})$. Applying this estimate componentwise in $\mathbb{C}^n$ proves boundedness of

$$\mathcal{M}_{-\Delta} : T^{\infty,2}(\mathbb{R}^{n+1}_+,\mathbb{C}^n) \to T^{\infty,2}(\mathbb{R}^{n+1}_+,\mathbb{C}^n)$$

$$(\mathcal{M}_{-\Delta}\vec{f})(t,\cdot) := \int_0^t -\Delta e^{(t-s)\Delta}\vec{f}(s,\cdot)ds.$$

Lemma 5.13 states that

$$\mathcal{T}F(s,x) := (s(-\Delta))^{-1}(I - e^{2s\Delta})s^{1/2}\mathbb{P}\operatorname{div} F(s,x) \qquad \forall (s,x) \in \mathbb{R}^{n+1}_+$$

is bounded on $T^{\infty,2}(\mathbb{R}^{n+1}_+;\mathbb{C}^n\otimes\mathbb{C}^n)$ to $T^{\infty,2}(\mathbb{R}^{n+1}_+;\mathbb{C}^n)$. We conclude,

$$\begin{aligned}
\|\mathcal{A}_1(\alpha)\|_{T^{\infty,2}(\mathbb{R}^{n+1}_+;\mathbb{C}^n)} &= \left\|\mathcal{M}_{-\Delta}T(s^{1/2}\alpha(s,\cdot))\right\|_{T^{\infty,2}(\mathbb{R}^{n+1}_+;\mathbb{C}^n)} \\
&\lesssim \left\|T_s(s^{1/2}\alpha(s,\cdot))\right\|_{T^{\infty,2}(\mathbb{R}^{n+1}_+;\mathbb{C}^n)} \\
&\lesssim \left\|s^{1/2}\alpha(s,\cdot)\right\|_{T^{\infty,2}(\mathbb{R}^{n+1}_+;\mathbb{C}^n\otimes\mathbb{C}^n)}.
\end{aligned}$$

Step 3(ii) By the duality property established in Proposition 3.22 it suffices to show boundedness of the adjoint of $\mathcal{A}_2$. Let $F \in T^{\infty,2}(\mathbb{R}^{n+1}_+;\mathbb{C}^n\otimes\mathbb{C}^n)$, $G \in T^{1,2}(\mathbb{R}^{n+1}_+;\mathbb{C}^n)$. An application of Fubini's theorem yields

$$\begin{aligned}
\langle \mathcal{A}_2F, G\rangle &= \int_{\mathbb{R}^n}\int_0^\infty (\mathcal{A}_2F)(t,y)\overline{G(t,y)}dt\,dy \\
&= \int_{\mathbb{R}^n}\int_0^\infty\int_0^\infty e^{(t+s)\Delta}\mathbb{P}\operatorname{div} F(s,y)ds\,\overline{G(t,y)}dt\,dy \\
&= \int_{\mathbb{R}^n}\int_0^\infty\int_0^\infty \operatorname{div} F(s,y)\overline{e^{(t+s)\Delta}\mathbb{P}G(t,y)}dt\,ds\,dy \\
&= \int_{\mathbb{R}^n}\int_0^\infty\int_0^\infty F(s,y)\overline{\nabla e^{(t+s)\Delta}\mathbb{P}G(t,y)}dt\,ds\,dy \\
&= \langle F, \int_0^\infty \nabla e^{(t+s)\Delta}\mathbb{P}G(s,y)dt\rangle \\
&= \langle F, \mathcal{A}_2^*G\rangle.
\end{aligned}$$

It is enough to show that

$$\begin{aligned}
&\mathcal{A}_2^* : T^{1,2}(\mathbb{R}^{n+1}_+;\mathbb{C}^n) \to T^{1,\infty}(\mathbb{R}^{n+1}_+;\mathbb{C}^n\otimes\mathbb{C}^n) \\
&(\mathcal{A}_2^*G)(s,\cdot) = e^{s\Delta}\int_0^\infty \nabla\mathbb{P}e^{t\Delta}G(t,\cdot)dt
\end{aligned}$$

is bounded. We show this by factoring $\mathcal{A}_2^*$ through the Hardy space $H^1(\mathbb{R}^{n+1}_+;\mathbb{C}^n\otimes\mathbb{C}^n)$. Define the operator

$$\begin{aligned}
&\mathcal{S} : T^{1,2}(\mathbb{R}^{n+1}_+;\mathbb{C}^n) \to H^1(\mathbb{R}^n;\mathbb{C}^n\otimes\mathbb{C}^n) \\
&(\mathcal{S}G)(\cdot) := \int_0^\infty \nabla\mathbb{P}e^{t\Delta}G(t,\cdot)dt.
\end{aligned}$$

This operator is bounded; see Lemma 5.14. Furthermore, for $h \in H^1(\mathbb{R}^n)$, $(s,x) \mapsto e^{s\Delta}h(x) \in T^{1,\infty}$ by the square function characterisation of H^1, see Proposition 3.28. Applying this estimate componentwise extends this estimate to $\mathbb{C}^n\otimes\mathbb{C}^n$ valued functions.

$$\begin{aligned}
\|(\mathcal{A}_2^*G)(s,y)\|_{T^{1,\infty}(\mathbb{R}^{n+1}_+;\mathbb{C}^n\otimes\mathbb{C}^n)} &= \left\|e^{s\Delta}(\mathcal{S}G(y))\right\|_{T^{1,\infty}(\mathbb{R}^{n+1}_+;\mathbb{C}^n\otimes\mathbb{C}^n)} \\
&\lesssim \|(\mathcal{S}G)(y)\|_{H^1(\mathbb{R}^n;\mathbb{C}^n\otimes\mathbb{C}^n)} \\
&\lesssim \|G(s,y)\|_{T^{1,\infty}(\mathbb{R}^{n+1}_+;\mathbb{C}^n)}.
\end{aligned}$$

Step 3(iii) The integral in $\mathcal{A}_3$ is nonsingular. It suffices to show that

$$R : T^{\infty,2}(\mathbb{R}^{n+1}_+; C^n \otimes \mathbb{C}^n) \to T^{\infty,2}(\mathbb{R}^{n+1}_+; \mathbb{C}^n)$$
$$(RF)(t,\cdot) := \int_t^\infty e^{(t+s)\Delta}\mathbb{P}s^{-1/2}\operatorname{div} F(s,\cdot)ds$$

is bounded since $\mathcal{A}_3(\alpha) = R(s^{1/2}\alpha(s,\cdot))$. This is proven in Lemma 5.15. $\square$

Lemma 5.12. *The operator*

$$T_s := (s(-\Delta))^{-1}(I - e^{2s\Delta})s^{1/2}\mathbb{P}\operatorname{div}$$

is uniformly bounded from $L^2(\mathbb{R}^n; \mathbb{C}^n \otimes \mathbb{C}^n)$ *to* $L^2(\mathbb{R}^n; \mathbb{C}^n)$.

Proof. For $\alpha \in L^2(\mathbb{R}^{n+1}_+; \mathbb{C}^n \otimes \mathbb{C}^n)$,

$$\widehat{T_s\alpha(s,\cdot)} = \frac{1}{s|\xi|^2}\left(1 - e^{-2s|\xi|^2}\right)s^{1/2}\left(I - \frac{\xi\otimes\xi}{|\xi|^2}\right)\xi\cdot\widehat{\alpha}(s,\xi).$$

That is, T_s is a Fourier multiplier with symbol,

$$\widehat{k_s}(\xi) = \frac{1}{s|\xi|^2}\left(1 - e^{-2s|\xi|^2}\right)s^{1/2}\left(I - \frac{\xi\otimes\xi}{|\xi|^2}\right)\xi\cdot$$

It suffices to show that the symbol is bounded. We define

$$g(z) = \frac{1}{\sqrt{z}}\left(1 - e^{-z^2}\right).$$

For $|z| \geq 1$, $g(z) \leq \frac{1}{\sqrt{z}} \leq 1$. When $0 < z \leq 1$ we use by L'Hôpital's rule to evaluate

$$\lim_{z\to 0} g(z) = \lim_{z\to 0}\frac{1 - e^{-z^2}}{\sqrt{z}} = \lim_{z\to 0} 4z^{3/2}e^{-z^2} = 0,$$

and thus g is bounded in this case too as a smooth function on the set $(0,1]$. Since

$$0 \leq \left|\frac{\xi\otimes\xi}{|\xi|^2}\right| \leq 1$$

for all $\xi \in \mathbb{R}^n$ we conclude that the symbol $\widehat{k_s}$ is uniformly (in s) bounded,

$$\left|\widehat{k_s}\right| = \left|g(s|\xi|^2)\frac{\xi\otimes\xi}{|\xi|^2}\right| \lesssim_g 1.$$ $\square$

Lemma 5.13. *The operator*

$$\mathcal{T}F(s,x) := T_sF(s,\cdot)(x) \qquad \forall (s,x) \in \mathbb{R}^{n+1}_+$$

with

$$T_s := (s(-\Delta))^{-1}(I - e^{2s\Delta})s^{1/2}\mathbb{P}\operatorname{div}$$

is bounded from $T^{\infty,2}(\mathbb{R}^{n+1}_+; \mathbb{C}^n \otimes \mathbb{C}^n)$ *to* $T^{\infty,2}(\mathbb{R}^{n+1}_+; \mathbb{C}^n)$.

Proof. We extend Lemma 5.12 to obtain estimates on the decay of the kernel k_s. Define $g : \mathbb{R}^n \to \mathbb{R}$ by

$$g_s(\xi) = g(s^{1/2}\xi) = \frac{1 - e^{2s|\xi|}}{s|\xi|^2} s^{1/2} \left(1 - \frac{\xi \otimes \xi}{|\xi|^2}\right) \xi.$$

A calculation shows that $g \in C^\infty(\mathbb{R}^n)$ so that for all $N \in \mathbb{N}$ there exists $C > 0$ such that

$$\left\| g^{(k)} \right\|_{L^\infty} \leq C \qquad k = 1, \dots N.$$

The kernel of T_s is given by the inverse Fourier transform of the symbol,

$$\breve{g}_s(x) = \int_{\mathbb{R}^n} g(s^{1/2}\xi) e^{i\xi \cdot x} d\xi = s^{-n/2} \int_{\mathbb{R}^n} g(\xi) e^{i\xi \cdot x/s^{1/2}} d\xi = s^{-n/2} \breve{g}(s^{-1/2}x).$$

We thus estimate the kernel,

$$|\breve{g}_s(x)| = |k_s(x)| \lesssim s^{-n/2} (s^{-1/2}|x|)^{-n-1} \qquad \forall x \in \mathbb{R}^n, |x| \geq s^{1/2}. \tag{5.16}$$

It follows from the kernel estimate (5.16) that T_s satisfies $L^2 - L^\infty$ off-diagonal bounds. Let $E, F \subset \mathbb{R}^n$ be two Borel sets.

$$\begin{aligned}
\|\mathbb{1}_E T_s \mathbb{1}_F\|_{\mathcal{L}(L^2;L^\infty)} &= \sup_{\|f\|_{L^2(\mathbb{R}^n)}=1} \left\| \int_{\mathbb{R}^n} \mathbb{1}_E(x) k_s(y) \mathbb{1}_F(x-y) f(x-y) dy \right\|_{L^\infty(dx)} \\
&\leq \sup_{\|f\|_{L^2(\mathbb{R}^n)}=1} \left\| \int_{\mathbb{R}^n} \mathbb{1}_E(x) |k_s(y)| \mathbb{1}_F(x-y) |f(x-y)| dy \right\|_{L^\infty(dx)} \\
&\leq \left\| \int_{\mathbb{R}^n} \mathbb{1}_E(x) |k_s(y)|^2 \mathbb{1}_F(x-y) dy \right\|_{L^\infty(dx)}^{\frac{1}{2}} \\
&\leq \left(\int_{|y|>\operatorname{dist}(E,F)} |k_s(y)|^2 dy \right)^{\frac{1}{2}} \\
&\lesssim \left(\int_{|y|>\operatorname{dist}(E,F)} \frac{s^{-n}}{(s^{-1/2}|y|)^{2n+2}} dy \right)^{\frac{1}{2}} \\
&\lesssim \left(\int_{\operatorname{dist}(E,F)}^\infty \frac{s^{-n}}{s^{-n-1} r^{n+3}} dr \right)^{\frac{1}{2}} \\
&= s^{1/2} \operatorname{dist}(E,F)^{-\frac{n}{2}-1}.
\end{aligned}$$

Fix $F \in T^{\infty,2}(\mathbb{R}^{n+1}_+; \mathbb{C}^n \otimes \mathbb{C}^n)$ and $(t,x) \in \mathbb{R}^{n+1}_+$ and define

$$F_j(s,y) := \begin{cases} \mathbb{1}_{B(x,2\sqrt{t})}(y) F(s,y) & j = 0 \\ \mathbb{1}_{B(x,2^{j+1}\sqrt{t}) \setminus B(x,2^j\sqrt{t})}(y) F(s,y) & j \geq 1. \end{cases}$$

The uniform boundedness of T_s on $L^2(\mathbb{R}^n; \mathbb{C}^n \otimes \mathbb{C}^n)$ established in Lemma 5.12 gives

$$\|T_s F_0(s,\cdot)\|_{L^2(B(x,\sqrt{t}))} \lesssim \|F_0(s,\cdot)\|_{L^2(\mathbb{R}^n)} = \|F(s,\cdot)\|_{L^2(B(x,2\sqrt{t}))}.$$

For $s < t$ and $j \geq 1$

$$\begin{aligned}
\|T_s F_j(s,\cdot)\|_{L^2(B(x,\sqrt{t}))} &= \left(\int_{B(x,\sqrt{t})} |T_s F_j(s,y)|^2 dy\right)^{\frac{1}{2}} \\
&\leq \left(\int_{B(x,\sqrt{t})} 1\, dy\right)^{\frac{1}{2}} \left\| |T_s F_j(s,\cdot)|^2 \right\|^{\frac{1}{2}}_{L^\infty(B(x,\sqrt{t}))} \\
&\lesssim t^{n/4} \|T_s F_j(s,\cdot)\|_{L^\infty(B(x,\sqrt{t}))} \\
&\lesssim t^{n/4} s^{1/2} \left(2^j t^{1/2}\right)^{-\frac{n}{2}-1} \|F_j(s,\cdot)\|_{L^2(B(x,\sqrt{t}))} \\
&= (s/t)^{1/2} 2^{-j(\frac{n}{2}+1)} \|F_j(s,\cdot)\|_{L^2(B(x,\sqrt{t}))} \\
&\lesssim 2^{-j(\frac{n}{2}+1)} \|F_j(s,\cdot)\|_{L^2(B(x,\sqrt{t}))} \\
&\leq 2^{-j(\frac{n}{2}+1)} \|F(s,\cdot)\|_{L^2(B(x,2^j\sqrt{t}))}.
\end{aligned}$$

The first inequality is an application of Hölder's inequality. We then apply the $L^2 - L^\infty$ off-diagonal bounds established above with

$$\operatorname{dist}(B(x,\sqrt{t}), B(x, 2^{j+1}\sqrt{t}) \setminus B(x, 2^j\sqrt{t})) = (2^j - 1)\sqrt{t} \lesssim 2^j\sqrt{t}$$

and finish by using $s < t$.

We now demonstrate the $T^{\infty,2}$ boundedness of T_s,

$$\begin{aligned}
\|T_s F(s,\cdot)\|_{T^{\infty,2}(\mathbb{R}^{n+1}_+,\mathbb{C}^n)} &= \sup_{B(x,\sqrt{t})} \left(t^{-\frac{n}{2}} \int_0^t \|T_s F(s,\cdot)\|^2_{L^2(B(x,\sqrt{t}))}\, ds\right)^{\frac{1}{2}} \\
&\leq \sum_{j\geq 0} \sup_{B(x,\sqrt{t})} \left(t^{-\frac{n}{2}} \int_0^t \|T_s F_j(s,\cdot)\|^2_{L^2(B(x,\sqrt{t}))}\, ds\right)^{\frac{1}{2}} \\
&\lesssim \sum_{j\geq 0} 2^{-j(\frac{n}{2}+1)} \sup_{B(x,\sqrt{t})} \left(t^{-\frac{n}{2}} \int_0^t \|F(s,\cdot)\|^2_{L^2(B(x,2^j\sqrt{t}))}\, ds\right)^{\frac{1}{2}} \\
&= \sum_{j\geq 0} 2^{-j} \sup_{B(x,2^j\sqrt{t})} \left(\left(2^j\sqrt{t}\right)^{-n} \int_0^t \|F(s,\cdot)\|^2_{L^2(B(x,2^j\sqrt{t}))}\, ds\right)^{\frac{1}{2}} \\
&\lesssim \sum_{j\geq 0} 2^{-j} \|F\|_{T^{\infty,2}(\mathbb{R}^{n+1}_+;\mathbb{C}^n\otimes\mathbb{C}^n)} \\
&\lesssim \|F\|_{T^{\infty,2}(\mathbb{R}^{n+1}_+;\mathbb{C}^n\otimes\mathbb{C}^n)}.
\end{aligned}$$
$\square$

The first inequality is an application of the triangle inequality. The second is by the previous estimate and we then rearrange to obtain the $T^{\infty,2}$ norm.

Lemma 5.14. *The operator*

$$\mathcal{S}G(\cdot) := \int_0^\infty \nabla \mathbb{P} e^{t\Delta} G(t,\cdot) dt$$

is bounded from $T^{1,2}(\mathbb{R}^{n+1}_+;\mathbb{C}^n)$ to $H^1(\mathbb{R}^n;\mathbb{C}^n\otimes\mathbb{C}^n)$.

Proof. We examine the kernel of $\nabla \mathbb{P} e^{t\Delta}$.

$$\begin{aligned}\nabla \mathbb{P} e^{t\Delta} G &= \nabla e^{t\Delta} G - (\nabla \otimes \nabla)\frac{1}{\Delta} e^{t\Delta} \nabla \cdot G \\ &= \partial_i e^{t\Delta} \delta_{j,k} G^k - \partial_i \frac{1}{\Delta} \partial_j \partial_k e^{t\Delta} G^k \\ &= \partial_i \left(e^{t\Delta} \delta_{j,k} - O_{j,k,t} \right) G^k \\ &= \int_{\mathbb{R}^n} \partial_i \left(\frac{1}{(4\pi t)^{n/2}} e^{-t|x-y|^2} \delta_{j,k} - K_{j,k,t}(x-y) \right) G^k(y) dy.\end{aligned}$$

The heat kernel decays rapidly so the Oseen kernel bounds Proposition 5.5 give decay of order $n+1$ at ∞ of this kernel. The result from chapter 3 on the tent space characterisation of H^1, Proposition 3.29 tells us that this operator is bounded. $\square$

Lemma 5.15. *The operator*

$$RF(t, \cdot) := \int_t^\infty K(t,s) F(s, \cdot) ds,$$

with $K(t,s) := e^{(t+s)\Delta} \mathbb{P} s^{-1/2} \operatorname{div}$ *for* $s, t > 0$ *is bounded from* $T^{\infty,2}(\mathbb{R}^{n+1}_+; \mathbb{C}^n \otimes \mathbb{C}^n)$ *to* $T^{\infty,2}(\mathbb{R}^{n+1}_+; \mathbb{C}^n)$.

Proof. We begin with an $L^2(\mathbb{R}^n)$ estimate on $K(t,s)$. As in Lemma 5.12 we show that $K(t,s)$ has a bounded symbol.

$$\|K(t,s)F(s,\cdot)\|_{L^2(\mathbb{R}^n;\mathbb{C}^n)} \le \left\| e^{-(t+s)|\xi|^2} s^{-1/2} |\xi| \right\|_{L^\infty(\mathbb{R}^{n+1}_+; \mathbb{C}\otimes\mathbb{C}^n)} \|F(s,\cdot)\|_{L^2(\mathbb{R}^{n+1}_+;\mathbb{C}^n)},$$

We compute

$$\frac{d}{dr} e^{-(t+s)r^2} r = e^{-(t+s)r^2}(1 - 2(t+s)r^2) = 0 \implies r = (2(t+s))^{-1/2}$$

which implies

$$e^{-(t+s)r^2} r \le e^{-1/2}(2(t+s)^{-1/2}) \qquad \forall r \ge 0.$$

such that

$$\|K(t,s)\|_{\mathcal{L}(L^2,L^2)} \le C s^{-1/2}(t+s)^{-1/2}. \tag{5.17}$$

We want to extend this estimate to $L^2(\mathbb{R}^{n+1}_+)$. Let $\beta \in (-\frac{1}{2}, 0)$ and define

$$p(t) := t^\beta, \qquad k(t,s) := \mathbb{1}_{(t,\infty)}(s) \|K(t,s)\|_{\mathcal{L}(L^2,L^2)}.$$

By (5.17) we have

$$\int_0^\infty k(t,s) p(t)^2 dt \lesssim \int_0^s s^{-1/2} t^{-1/2} t^{2\beta} dt \lesssim s^{2\beta} = p(s)^2$$

since $0 < t < s$ implies $(t+s)^{-1/2} < 2^{-1/2} t^{-1/2}$. Likewise

$$\int_0^\infty k(t,s) p(s)^2 ds \lesssim \int_t^\infty s^{-1/2} s^{-1/2} s^{2\beta} ds \lesssim t^{2\beta} = p(t)^2$$

since $0 < t < s$ implies $(t+s)^{-1/2} < 2^{-1/2} s^{-1/2}$. Applying Schur's lemma for positive operators, Lemma A.2, the operator

$$\Big(t \mapsto f(t)\Big) \mapsto \left(t \mapsto \int_t^\infty \|K(t,s)\|_{\mathcal{L}(L^2,L^2)} f(s) ds\right)$$

is bounded from $L^2(\mathbb{R}_+)$ to $L^2(\mathbb{R}_+)$. We now show the $L^2(\mathbb{R}^{n+1}_+) = T^{2,2}(\mathbb{R}^{n+1}_+)$ boundedness. Let $F \in T^{2,2}(\mathbb{R}^{n+1}_+; \mathbb{C}^n \otimes \mathbb{C}^n)$, $G \in T^{2,2}(\mathbb{R}^{n+1}_+; \mathbb{C}^n)$.

$$\begin{aligned}
\left|\langle RF, G\rangle_{T^{2,2}(\mathbb{R}^{n+1}_+;\mathbb{C}^n)}\right| &= \left|\int_0^\infty \int_{\mathbb{R}^n} \int_t^\infty K(t,s)F(s,x)ds\,\overline{G(t,x)}dx\,dt\right| \\
&= \left|\int_0^\infty \int_t^\infty \int_{\mathbb{R}^n} K(t,s)F(s,x)\,\overline{G(t,x)}dx\,ds\,dt\right| \\
&\le \int_0^\infty \int_t^\infty \|K(t,s)F(s,\cdot)\|_{L^2(\mathbb{R}^n)} \|G(t,\cdot)\|_{L^2(\mathbb{R}^n)}\,ds\,dt \\
&\le \int_0^\infty \int_t^\infty \|K(t,s)\|_{\mathcal{L}(L^2)} \|F(s,\cdot)\|_{L^2(\mathbb{R}^n)}\,ds\,\|G(t,\cdot)\|_{L^2(\mathbb{R}^n)}\,dt \\
&\lesssim \int_0^\infty \|F(s,\cdot)\|_{L^2(\mathbb{R}^n)}\,ds\,\|G(t,\cdot)\|_{L^2(\mathbb{R}^n)}\,dt \\
&\le \|F\|_{T^{2,2}(\mathbb{R}^{n+1}_+;\mathbb{C}^n\otimes\mathbb{C}^n)} \|G\|_{T^{2,2}(\mathbb{R}^{n+1}_+;\mathbb{C}^n)}\,.
\end{aligned}$$

Taking the supremum over G with $\|G\|_{T^{2,2}} \le 1$ gives the result,

$$\|RF\|_{T^{2,2}(\mathbb{R}^{n+1}_+;\mathbb{C}^n)} \lesssim \|F\|_{T^{2,2}(\mathbb{R}^{n+1}_+;\mathbb{C}^n\otimes\mathbb{C}^n)}\,.$$

We write $K(t,s)$ as

$$K(t,s) = s^{-1/2}\left(e^{(t+s)\Delta} - O_{j,k,t+s}\right)\operatorname{div}\cdot$$

where

$$O_{j,k,t} = \frac{1}{\Delta}\partial_j\partial_k e^{t\Delta}$$

is the operator associated with the Oseen kernel $K_{j,k,t}$. An integration by parts yields,

$$\begin{aligned}
O_{j,k,t}\partial_l\alpha^{lk} &= \int_{\mathbb{R}^n} K_{k,j,t}(x-y)\partial_{y_l}\alpha^{lk}(y)dy \\
&= -\int_{\mathbb{R}^n} \partial_{y_l}K_{j,k,t}(x-y)\alpha^{lk}(y)dy = \partial_l K_{j,k,t} * \alpha^{lk}.
\end{aligned}$$

The decay of the Oseen kernel Proposition 5.5 kills the boundary terms and can further be used to estimate this expression. Letting $y = (t+s)^{-1/2}x$, the chain rule gives

$$\begin{aligned}
\partial_{y_l}K_{j,k}((t+s)^{-1/2}x) &= (t+s)^{1/2}\partial_{x_l}K_{j,k}((t+s)^{-1/2}x) \\
&= (t+s)^{n/2}(t+s)^{1/2}\partial_{x_l}(t+s)^{-n/2}K_{j,k}((t+s)^{-1/2}x) \\
&= (t+s)^{n/2}(t+s)^{1/2}\partial_{x_l}K_{j,k,t+s}(x).
\end{aligned}$$

By Proposition 5.5,

$$|\partial_l K_{j,k,t+s}| \lesssim (t+s)^{-n/2-1/2}(1+(t+s)^{-1/2}|x|)^{-n-1}.$$

Likewise for the heat semigroup part,

$$\begin{aligned}
e^{(t+s)\Delta}\delta_{jk}\partial_l\alpha^{lk}(x) &= \int_{\mathbb{R}^n} e^{-(t+s)|x-y|^2}\delta_{jk}\partial_{x_l}\alpha^{lk}(y)dy \\
&= -2(t+s)^{1/2}\int_{\mathbb{R}^n} e^{-(t+s)|x-y|^2}(t+s)^{1/2}(x_l-y_l)\delta_{jk}\alpha^{lk}(y)dy.
\end{aligned}$$

Since this convolution kernel is Schwartz we have order $n+1$ decay at ∞,[2]

$$\left|e^{-(t+s)|x|^2}(t+s)^{1/2}x_l\delta_{jk}\right| \lesssim C(t+s)^{-n/2-1/2}(1+(t+s)^{-1/2}|x|)^{-n-1}.$$

[2]In fact we have much more but $n+1$ is enough.

Combining the estimates we deduce that $K(s,t)$ is an integral operator whose kernel $k_{t,s}$ has the decay property

$$|k_{t,s}(x)| \lesssim s^{-1/2}(t+s)^{-1/2}(t+s)^{-n/2}\left(1+(t+s)^{-1/2}|x|\right)^{-n-1} \qquad \forall x\in\mathbb{R}^n. \tag{5.18}$$

From here we can deduce $L^2 - L^\infty$ off-diagonal bounds. Let $E, F \subset \mathbb{R}^n$ be two Borel sets. Then,

$$\|\mathbb{1}_E K(t,s)\mathbb{1}_F F(s,\cdot)\|_{L^\infty} \le \left\|\int_{\mathbb{R}^n} |1_E(x)k_{t,s}(x-y)\mathbb{1}_F(y)|^2 dy\right\|_{L^\infty}^{\frac12} \|F(s,\cdot)\|_{L^2}.$$

If $x\in E$ and $y\in F$, then $|x-y|\ge \operatorname{dist}(E,F)$ and

$$\begin{aligned}
\|\mathbb{1}_E K(t,s)\mathbb{1}_F\|_{\mathcal{L}(L^2,L^\infty)} &\lesssim \left\|\int_F \frac{s^{-1}(t+s)^{-n-1}}{\left(1+(t+s)^{-1/2}|\cdot-y|\right)^{2n+2}}dy\right\|_{L^\infty(E)}^{\frac12}\\
&\le \beta(s,t)\left\|\int_F \frac{1}{\left(1+(t+s)^{-1/2}|\cdot-y|\right)^{n+1}}dy\right\|_{L^\infty(E)}^{\frac12}\\
&\le \beta(s,t)\left(\int_{\mathbb{R}^n} \frac{1}{\left(1+(t+s)^{-1/2}|y|\right)^{n+1}}dy\right)^{\frac12} =: \beta(s,t)I
\end{aligned}$$

where

$$\beta(s,t) := s^{-1/2}(t+s)^{-1/2}(t+s)^{-n/2}\left(1+(t+s)^{-1/2}\operatorname{dist}(E,F)\right)^{-\frac n2-\frac12}$$

and the (square of) the above integral can be estimated

$$\begin{aligned}
I^2 &\lesssim \int_0^\infty \frac{r^{n-1}}{(1+(t+s)^{-1/2}r)^{n+1}}dr\\
&\le \int_0^{\sqrt{t+s}} \frac{r^{n-1}}{(1+(t+s)^{-1/2}r)^{n+1}}dr + \int_{\sqrt{t+s}}^\infty \frac{r^{n-1}}{(1+(t+s)^{-1/2}r)^{n+1}}dr\\
&\le \int_0^{\sqrt{t+s}} (t+s)^{n/2+1/2}r^{-2}dr + \int_{\sqrt{t+s}}^\infty r^{n-1}dr \lesssim (t+s)^{n/2}.
\end{aligned}$$

Combining this estimate with $\beta(t,s)$,

$$\|\mathbb{1}_E K(t,s)\mathbb{1}_F\|_{\mathcal{L}(L^2,L^\infty)} \lesssim s^{-1/2}(t+s)^{-1/2}(t+s)^{-n/4}\left(1+(t+s)^{-1/2}\operatorname{dist}(E,F)\right)^{-\frac n2-\frac12} \tag{5.19}$$

for all Borel $E, F\subset\mathbb{R}^n$.

The operator R now satisfies the assumptions of [AKMP12, Theorem 4.1(b)] where $\beta = 0$ and $m = 1$ which gives the result. $\square$

Appendix A

Schur's lemma

A.1 Schur's lemma

Schur's lemma provides sufficient conditions for the a linear operator to be bounded on L^p.

Lemma A.1. *Let (X, μ) and (Y, ν) be two σ-finite measure spaces. Suppose that a function $K(x, y)$ locally integrable on $X \times Y$ satisfies*

$$\sup_{x \in X} \int_Y |K(x, y)| d\nu(y) = A < \infty,$$
$$\sup_{y \in Y} \int_X |K(x, y)| d\mu(x) = B < \infty.$$

Then for $1 \leq p \leq \infty$ the operator T defined by

$$Tf(x) := \int_Y K(x, y) f(y) d\nu(y)$$

extends to a bounded operator from $L^p(Y)$ to $L^p(X)$ with norm $A^{1-\frac{1}{p}} B^{\frac{1}{p}}$.

Proof. The first condition tells us T maps $L^\infty(Y)$ to $L^\infty(X)$ with norm A. The second condition tells us that T maps $L^1(Y)$ to $L^1(X)$ with norm B,

$$\begin{aligned} \|Tf\|_{L^1(X)} &= \int_X \left| \int_Y K(x, y) f(y) d\mu(y) \right| d\nu(x) \\ &\leq \int_X \int_Y |K(x, y)| \, |f(y)| d\nu(x) \, d\mu(y) \\ &= \int_Y \left(\int_X |K(x, y)| d\nu(x) \right) |f(y)| d\mu(y) \leq B \, \|f\|_{L^1(Y)} \, . \end{aligned}$$

Invoking Riesz–Thorin interpolation Theorem 2.19, we conclude that T maps $L^p(Y)$ to $L^p(X)$ with bound $A^{1-\frac{1}{p}} B^{\frac{1}{p}}$. ☐

A.2 Schur's lemma for positive operators

By assuming that the kernel K is positive we are able to strengthen the above result. We have the following result about the L^p boundedness of the corresponding positive operator.

Lemma A.2. *Let (X,μ) and (Y,ν) be two σ-finite measure spaces, where μ and ν are positive measures. Let $1<p<\infty$, $0<A<\infty$ and suppose the $K(x,y)$ is a nonnegative measurable function on $X\times Y$. Define the linear operator*

$$Tf(x)=\int_Y K(x,y)f(y)d\nu(y)$$

and the transpose operator

$$T^t g(y)=\int_X K(x,y)f(x)d\mu(x).$$

In addition, assume that there exists a compactly supported, bounded, and positive ν-a.e. function h_1 such that $T(h_1)>0$ μ-a.e. The following are equivalent

(i) T is a bounded operator from $L^p(Y)$ into $L^p(X)$ with $\|T\|_{\mathcal{L}(L^p(Y);L^p(X))}\leq A$,

(ii) For all $B>A$ there exists a measurable function h on Y with $h, T(h)$ positive ν-a.e. such that

$$T^t\left(T(h)^{\frac{p}{p'}}\right)\leq B^p h^{\frac{p}{p'}},$$

(iii) For all $B>A$ there exists measurable functions u on X and v on Y that are positive μ and ν a.e. respectively such that

$$T(u^{p'})\leq Bv^{p'},$$
$$T^t(v^p)\leq Bu^p.$$

Proof.

(ii) ⇒ (iii) Define $u=(Bh)^{\frac{1}{p'}}$ and $v=(T(h))^{\frac{1}{p'}}$ such that

$$T(u^{p'})=T(Bh)=Bv^{p'}.$$

Using the inequality in (ii),

$$T^t(v^p)=T^t(T(h)^{\frac{p}{p'}})\leq B^p h^{\frac{p}{p'}}=B^{p-\frac{p}{p'}}u^p=Bu^p.$$

(iii) ⇒ (i) Fix $g\in L^{p'}(X)$. Hölder's inequality with the product measure $K(x,y)d\nu(x)d\mu(y)$ defined on $X\times Y$

$$\begin{aligned}\left|\int_X Tf(x)g(x)d\nu(x)\right| &\leq \int_X\int_Y\left|K(x,y)f(y)\frac{v(x)}{u(y)}g(x)\frac{u(y)}{v(x)}\right|d\nu(x)d\mu(y)\\ &\leq \left\|f(y)\frac{v(x)}{u(y)}\right\|_{L^p(K(x,y)d\nu(x)d\mu(y))}\left\|g(x)\frac{u(y)}{v(x)}\right\|_{L^{p'}(K(x,y)d\nu(x)d\mu(y))}.\end{aligned}$$

We now estimate, using Fubini–Tonelli theorem and the hypothesis,

$$\begin{aligned}\left\|f(y)\frac{v(x)}{u(y)}\right\|_{L^p(K(x,y)d\nu(x)\mu(y))} &= \left(\int_X\int_Y|f(y)|^p\frac{v(x)^p}{u(y)^p}K(x,y)d\nu(x)d\mu(y)\right)^{\frac{1}{p}}\\ &= \left(\int_Y\int_X K(x,y)v(x)^p\int_X\frac{|f(y)|^p}{u(y)^p}K(x,y)d\mu(y)d\mu(x)\right)^{\frac{1}{p}}\\ &\leq \left(B\int_Y u(y)^p\frac{|f(y)|^p}{u(y)^p}d\mu(y)\right)^{\frac{1}{p}}\\ &= B^{\frac{1}{p}}\|f\|_{L^p(Y)}.\end{aligned}$$

Likewise,

$$\begin{aligned}
\left\| g(x)\frac{u(y)}{v(x)} \right\|_{L^{p'}(K(x,y)d\nu(x)\mu(y))} &= \left(\int_X \int_Y |g(x)|^{p'} \frac{u(y)^{p'}}{v(x)^{p'}} K(x,y) d\nu(x) d\mu(y) \right)^{\frac{1}{p'}} \\
&= \left(\int_X \frac{|g(x)|^{p'}}{v(x)^{p'}} \int_Y u(y)^p K(x,y) d\nu(x) d\mu(y) \right)^{\frac{1}{p'}} \\
&\leq \left(B \int_X \frac{|g(x)|^{p'}}{v(x)^{p'}} v(y)^{p'} d\mu(y) \right)^{\frac{1}{p'}} \\
&= B^{\frac{1}{p'}} \|g\|_{L^{p'}(X)} .
\end{aligned}$$

Combining these estimates

$$\left| \int_X Tf(x) g(x) d\nu(x) \right| \leq B^{\frac{1}{p}+\frac{1}{p'}} \|f\|_{L^p(Y)} \|g\|_{L^{p'}(X)} .$$

Taking the supremum over all such $g \in L^{p'}(X)$ with norm 1, we have

$$\|T\|_{\mathcal{L}(L^p(Y);L^p(X))} \leq B$$

for all $B > A$. That is, the operator norm above must be bounded by A.

(i) $\Rightarrow$ (ii) Without loss of generality assume that $A = 1$ so that $B > 1$. Define $S : L^p(Y) \to L^p(Y)$ by

$$Sf(y) := \left(T^t (Tf)^{\frac{p}{p'}} \right)^{\frac{p}{p'}} (y).$$

Notice,

$$f_1 \leq f_2 \implies Tf_1 \leq Tf_2 \implies Sf_1 \leq Sf_2$$

and

$$\|f\|_{L^p} \leq 1 \implies \|Tf\|_{L^p} \leq 1 \implies \|Sf\|_{L^p} \leq 1.$$

Define a sequence of functions $(h_n)_{n\geq 0}$ acting on Y as follows. We choose h_0 such that $h_0 > 0$ μ-a.e. as in the hypothesis of the theorem. Upon multiplying by a appropriate constant assume $\|h_0\|_{L^p} \leq B^{-p'}(B^{p'} - 1)$. We define the h_n inductively,

$$h_{n+1} = h_0 + B^{-p'} S(h_n)$$

such that $h_n \leq h_{n+1}$ and $\|h_n\|_{L^p} \leq 1$ for all $n \geq 0$. We now define

$$h(x) := \lim_{n\to\infty} h_n(x).$$

We have $h \geq h_0 > 0$ ν-a.e. while Fatou's lemma gives $h < \infty$ ν-a.e. The dominated convergence theorem tells us $h_n \to h$ in $L^p(Y)$. By the hypothesis (i) it follows that $Th_n \to Th$ in $L^p(X)$ which in turn implies $(Th_n)^{\frac{p}{p'}} \to (Th)^{\frac{p}{p'}}$ in $L^{p'}(X)$. The hypothesis (i) also implies that T^t is bounded on $L^{p'}$ with norm 1. The same argument as we've seen above gives that

$$S(h_n) \to S(h) \quad \text{in} \quad L^p(Y).$$

We then extract a subsequence n_k for which $S(h_{n_k}) \to S(h)$ a.e. in Y, but the sequence h_n is increasing and thus $S(h_n) \to S(h)$ μ-a.e. in Y. Notice,

$$h = \lim_{n\to\infty} h_n = h_0 + B^{-p'} \lim_{n\to\infty} S(h_n) = h_0 + B^{-p'} S(h).$$

It follows that $S(h) \le B^{p'} h$ ν-a.e. which upon raising to the power $\frac{p}{p'}$ proves the estimate in (ii). We conclude by noting that $T(h) \ge T(h_1) > 0$ μ-a.e. $\square$

Appendix B

Distribution theory

B.1 Fréchet spaces

Definition B.1. Let X be a vector space over the scalar field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and let $\mathcal{I}$ be a topology on X. We say that $(X, \mathcal{I})$ is a topological vector space over $\mathbb{K}$ if:

(i) X is Hausdorff,

(ii) $(x, y) \mapsto x + y$ is continuous on $X \times X$ into X,

(iii) $(\lambda, x) \mapsto \lambda x$ is continuous on $\mathbb{K} \times X$ into X.

Remark. The condition that X is Hausdorff can be dropped to give a more general definition of a topological vector space. For our purposes however, we will always assume X is Hausdorff.

Definition B.2. A topological vector space X is called a Fréchet space when the following properties hold:

(i) X is metrisable with a translation invariant metric,

(ii) X is complete,

(iii) X is locally convex.

By locally convex we mean that for every $x \in X$ and all neighbourhoods $U \ni x$ there exists a convex sub-neighbourhood V. That is $x \in V \subset U$ where V is convex.

Theorem B.3. *A topological vector space X is a Fréchet space if and only if:*

(i) it is a Hausdorff space,

(ii) its topology can be induced by a family of seminorms $\|\cdot\|_k$, $k = 0, 1, 2, \ldots$,

(iii) it is complete with respect to this family of seminorms.

Remark. A topology is induced from a family of seminorms as follows. We say that $U \subset X$ is open if and only if for every $u \in U$ there exists $\epsilon > 0$ and $K \geq 0$ such that

$$\{v : \|v - u\|_k < \epsilon \text{ for all } k \leq K\}$$

is a subset of U.

Example B.4 ($C^k(\Omega)$)**.** Let $\Omega \subset \mathbb{R}^n$. Then $C^k(\Omega)$, k times differentiable functions mapping Ω into $\mathbb{R}$ is a Fréchet space with seminorms

$$\|f\|_{m,K} = \sup_{|\alpha|<m} \sup_{x\in K} |\partial^\alpha f(x)|,$$

where m is an integer $\leq k$ (or simply nonnegative in the case where $k = \infty$), $K \subset \Omega$ is compact and $\alpha \in \mathbb{N}^n$ is multiindex. Note that we have the induced C^k topology, or topology of uniform convergence on compact subsets of the functions and their derivatives of order $\leq k$.

B.2 Schwartz functions

Of particular importance in the theory of distributions is the Fréchet space of Schwartz functions.

Definition B.5. The Schwartz space $\mathcal{S}$ is the space of functions $f \in C^\infty(\mathbb{R}^n)$ such that for all $\alpha, \beta \in \mathbb{N}^n$ there exists $C_{\alpha,\beta} \geq 0$ such that

$$\|f\|_{\alpha,\beta} := \sup_{x\in\mathbb{R}^n} |x^\alpha \partial^\beta f(x)| = C_{\alpha,\beta} < \infty.$$

We call $\|\cdot\|_{\alpha,\beta}$ the Schwartz seminorms.

Under this family of seminorms it is clear that the Schwartz functions form a Fréchet space.

One possible alternative characterisation of the Schwartz functions is as follows. A C^∞ function is in $\mathcal{S}$ if and only if for all positive integers N and all multiindices α there exists $C_{\alpha,N}$ such that

$$|\partial^\alpha f(x)| \leq C_{\alpha,N}(1 + |x|)^{-N}.$$

B.3 Distributions

Distributions generalise functions in the sense that they often come about as the dual of functions, but are not themselves functions. The basic idea is that we represent a distribution as a linear functional on some space of test functions.

Definition B.6. A distribution on U is a continuous linear functional $T : C_c^\infty(U) \to \mathbb{C}$. That is, an element in the dual space of smooth functions with compact support in U. We denote the space of distributions $\mathcal{D}'$.

Example B.7. Consider the Heaviside function $H : \mathbb{R} \to \mathbb{R}$,

$$H(x) = \begin{cases} 1 & x > 0, \\ 0 & x \leq 0. \end{cases}$$

It is clear that this doesn't have a derivative function. that is, there is no function $v : \mathbb{R} \to \mathbb{R}$ satisfying

$$\int_{\mathbb{R}} H(x) \frac{d}{dx}\varphi(x)dx = \int_{\mathbb{R}} v(x)\varphi(x)dx \text{ for all } \varphi \in C_c^\infty(\mathbb{R}).$$

There is however a linear functional that has this property, namely the Dirac delta distribution.

B.4 Tempered distributions

Related to the space of distributions is the space of tempered distributions. We take all continuous linear functionals acting on a larger space of functions to get a smaller space of distributions.

Definition B.8. $\mathcal{S}'(\mathbb{R}^n)$ (also written $\mathcal{S}'$) is the vector space of continuous linear functionals on $\mathcal{S}(\mathbb{R}^n)$. The elements of $\mathcal{S}'$ are called tempered (or temperate) distributions.

Definition B.9. We say that a tempered distribution $v \in \mathcal{S}'$ is bounded if $v * \varphi \in L^\infty(\mathbb{R}^n)$ for all $\varphi \in \mathcal{S}$.

Example B.10. Recall the Poisson kernel

$$P(x) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}\left(1+|x|^2\right)^{\frac{n+1}{2}}}.$$

For $t > 0$ define $P_t(x) = t^{-n}P(t^{-1}x)$. If v is a bounded tempered distribution then $v * P_t$ is well defined as a convergent integral (see the following remark) since $P \in L^1(\mathbb{R}^n)$ implies $P_t \in L^1(\mathbb{R}^n)$. In fact, $v * P_t$ can be identified with a bounded function. Write $1 = \hat{\phi}(\xi) + \eta(\xi)$ where $\hat{\phi}$ has compact support and η is a smooth function vanishing in a neighbourhood of the origin. The function Ψ defined by $\hat{\Psi}(\xi) = e^{-2\pi|\xi|}\eta(\xi)$ is a Schwartz function. Moreover,

$$\hat{P}_t(\xi) = e^{-2\pi t|\xi|} = e^{-2\pi t|\xi|}\hat{\phi}(\xi) + \hat{\Psi}(t\xi).$$

That is, $\hat{P}_t \in \mathcal{S}$ such that $v * P_t = (v * \phi_t) * P_t + v * \Psi_t$. The important thing to notice is that $v * \phi_t$ and $v * \Psi_t$ are bounded functions and hence $v * P_t$ is a bounded function too.

Remark. If v is any bounded distribution and $h \in L^1(\mathbb{R}^n)$ then the convolution $v * h$ can always be defined as a distribution.

$$\langle v * h, \varphi\rangle = \langle v * \tilde{\varphi}, \tilde{h}\rangle = \int_{\mathbb{R}^n} (v * \tilde{\varphi})(x)\tilde{h}(x)dx$$

where $\tilde{\phi}(x) = \phi(-x)$. This integral is convergent since v a bounded distribution means $v * \varphi \in L^\infty(\mathbb{R}^n)$.

The Fourier transform is well defined on $\mathcal{S}$, however we can extend the Fourier transform to the tempered distributions as follows.

Definition B.11. Let $u \in \mathcal{S}'$. We define the Fourier transform $\widehat{u}$ as the distribution

$$\langle \widehat{u}, f\rangle = \langle u, \widehat{f}\rangle$$

for all $f \in \mathcal{S}$.

Bibliography

[AA11] Pascal Auscher and Andreas Axelsson. Remarks on Maximal Regularity. In *Parabolic Problems*, volume 80 of *Progress in Nonlinear Differential Equations and Their Applications*, pages 45–55. Springer Basel, 2011.

[ABHN11] Wolfgang Arendt, Charles J.K. Batty, Matthias Hieber, and Frank Neubrander. *Vector-valued Laplace Transforms and Cauchy Problems*. Monographs in Mathematics. Springer, 2011.

[ACDH04] Pascal Auscher, Thierry Coulhon, Xuan Thinh Duong, and Steve Hofmann. Riesz Transform on Manifolds and Heat Kernel Regularity. *Annales Scientifiques de l'École Normale Supérieure*, 37(6):911–957, 2004.

[AF13] Pascal Auscher and Dorothee Frey. A new proof for Koch and Tataru's result on the well-posedness of Navier–Stokes equations in BMO^{-1}. arXiv eprint 1310.3783, 2013.

[AK06] Fernando Albiac and Nigel J. Kalton. *Topics in Banach Space Theory*. Number v. 10 in Graduate Texts in Mathematics. Springer, 2006.

[AKMP12] Pascal Auscher, Christoph Kriegler, Sylvie Monniaux, and Pierre Portal. Singular integral operators on tent spaces. *Journal of Evolution Equations*, 12(4):741–765, 2012.

[Ame14] Alex Amenta. Tent spaces over metric measure spaces under doubling and related assumptions. In *Operator Theory in Harmonic and Non-commutative Analysis*, volume 240 of *Operator Theory: Advances and Applications*, pages 1–29. Springer International Publishing, 2014.

[AMM13] Pascal Auscher, Alan McIntosh, and Andrews J. Morris. Calderón reproducing formulas and applications to Hardy spaces. *Revista Matematica Iberoamericana*, page to appear, 2013. arXiv eprint 1304.0168.

[AMP12] Pascal Auscher, Sylvie Monniaux, and Pierre Portal. The maximal regularity operator on tent spaces. *Communications on Pure and Applied Analysis*, 11(6):2213–2219, 2012.

[Aus12] Pascal Auscher. *Real Harmonic Analysis / lectures by Pascal Auscher with the assistance of Lashi Bandara*. Griffin Press, 2012.

[BFCO12] András Bátkai, Bálint Farkas, Petra Csomós, and Alexander Ostermann. Operator Semigroups for Numerical Analysis. `https://isem-mathematik.uibk.ac.at/isemwiki/index.php/`, 2011/2012. 15th Internet Seminar.

[BHJ13] Charles Batty, Markus Haase, and Mubeen Junaid. The holomorphic functional calculus approach to operator semigroups. *Acta Scientiarum Mathematicarum*, 79(1):289–323, 2013.

[BL76] J. Bergh and J. Löfström. *Interpolation Spaces: An Introduction*. Grundlehren der mathematischen Wissenschaften. Springer, 1976.

[BP08] Jean Bourgain and Nataša Pavlović. Ill-posedness of the Navier–Stokes equations in a critical space in 3D. *Journal of Functional Analysis*, 255(9):2233–2247, 2008.

[Bro64] Felix E. Browder. Non–Linear Equations of Evolution. *Annals of Mathematics*, 80(3):485–523, 1964.

[Can97] Marco Cannone. A generalization of a theorem by Kato on Navier–Stokes equations. *Revista Mathematica Iberoamericana*, 13(3):771–831, 1997.

[Can04] Marco Cannone. Harmonic analysis tools for solving the incompressible Navier-Stokes equations. *Handbook of mathematical fluid dynamics*, 3:161–244, 2004.

[CD99] Thierry Coulhon and Xuan Thinh Duong. Riesz Transforms for $1 \leq p \leq 2$. *Transactions of the American Mathematical Society*, 351(3):1151–1169, 1999.

[CL86] Thierry Coulhon and Damien Lamberton. Régularité L^p pour les équations d'évolution. *Séminaire d'Analyse Fonctionelle*, 1985/1986(26):155–165, 1986.

[CMS85] Ronald Coifman, Yves Meyer, and Elias Stein. Some new function spaces and their applications to Harmonic analysis. *Journal of Functional Analysis*, 62(2):304–335, 1985.

[dS64] Luciano de Simon. Un'applicazione della teoria degli integrali singolari allo studio delle equazioni differenziali lineari astratte del primo ordine. *Rendiconti del Seminario Matematico della Universitá di Padova*, 34:205–235, 1964.

[EN00] Klaus-Jochen Engel and Rainer Nagel. *One–Parameter Semigroups for Linear Evolution Equations*. Graduate Texts in Mathematics. Springer, 2000.

[EN06] Klaus-Jochen Engel and Rainer Nagel. *A Short Course on Operator Semigroups*. Universitext. Springer, 2006.

[Eva10] Lawrence C. Evans. *Partial Differential Equations*. Graduate Studies in Mathematics. American Mathematical Society, 2010.

[FK64] Hiroshi Fujita and Tosio Kato. On the Navier-Stokes initial value problem. I. *Archive for rational mechanics and analysis*, 16(4):269–315, 1964.

[FS72] C. Fefferman and E. M. Stein. H^p spaces of several variables. *Acta Mathematica*, 129(3–4):137–193, 1972.

[GM89] Yoshikazu Giga and Tetsuro Miyakawa. Navier–Stokes flow in $\mathbb{R}^3$ with measures as initial vorticity and Morrey spaces. *Communications in Partial Differential Equations*, 14(5):577–618, 1989.

[GNR03] Gisele Ruiz Goldstein, Ranier Nagel, and Silvia Romanelli. *Evolution Equations*. Lecture Notes in Pure and Applied Mathematics. Taylor & Francis, 2003.

[Gra04] Loukas Grafakos. *Classical and Modern Fourier Analysis*. Pearson/Prentice Hall, Upper Saddle River, N.J, 2004.

[Gru09] Gerd Grubb. *Distributions and Operators*. Graduate Texts in Mathematics. Springer, Copenhagen Denmark, 2009.

[Hil50] Einar Hille. On the differentiability of semi-group operators. *Acta Scientiarum Mathematicarum*, 12:19–24, 1950.

[Hun14] John K. Hunter. Notes on Partial Differential Equations. `https://www.math.ucdavis.edu/~hunter/pdes/pdes.html`, 2014.

[JN61] Fritz John and Louis Nirenberg. On functions of bounded mean oscillation. *Communications on Pure and Applied Mathematics*, 14(3):415–426, 1961.

[Kat84] Tosio Kato. Strong L^p Solutions of the Navier–Stokes Equation in R^m, with Applications to Weak Solutions. *Mathematishe Zeitschrift*, 187(4):471–480, 1984.

[Kat95] Tosio Kato. *Pertubation Theory for Linear Operators*. Classics in Mathematics. Springer, 1995.

[KL00] N.J. Kalton and G. Lancien. A solution to the problem of L^p–maximal regularity. *Mathematische Zeitschrift*, 235(3):559–568, 2000.

[KT01] Herbert Koch and Daniel Tataru. Well–posedness for the Navier–Stokes Equations. *Advances in Mathematics*, 157(1):22–35, 2001.

[LR02] P.G. Lemarié-Rieusset. *Recent developments in the Navier–Stokes problem*. Chapman & Hall/CRC Research Notes in Mathematics Series. Taylor & Francis, 2002.

[McI10] Alan McIntosh. Operator Theory - Spectral and functional Calculi. `http://maths-people.anu.edu.au/~alan/lectures/optheory.pdf`, 2010. Lecture notes taken by Lashi Bandara.

[Mon09] Sylvie Monniaux. Maximal regularity and applications to PDEs. `http://junon.u-3mrs.fr/monniaux/berlinlecture.pdf`, May 2009.

[Ose27] CW Oseen. Neuere methoden und ergebnisse in der hydrodynamik, akad. *Verlagsgesellschaft, Leipzig*, 1927.

[Paz83] Amnon Pazy. *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Applied Mathematical Sciences. Springer New York, 1983.

[Phi52] Ralph S Phillips. On the generation of semigroups of linear operators. *Pacific Journal of Mathematics*, 2(3):343–369, 1952.

[Pla96] F. Planchon. Global strong solutions in Sobolev or Lebesgue spaces to the incompressible Navier–Stokes equations in $\mathbb{R}^3$. *Annals of the Herni Poincare Institute. Nonlinear analysis*, 13(3):319–336, 1996.

[SS05] Elias Stein and Rami Shakarchi. *Real Analysis*. Princeton lectures in Analysis. Princeton University Press, 2005.

[Ste56] Elias M. Stein. Interpolation of linear operators. *Transactions of the American Mathematical Society*, 83:482–492, 1956.

[Ste93] Elias Stein. *Harmonic Analysis : Real–Variable Methods, Orthogonality, and Oscillatory Integrals*. Princeton University Press, Princeton, N.J, 1993.

[SW99] Helmut H. Schaefer and Manfred P.H. Wolff. *Topological Vector Spaces*. Graduate Texts in Mathematics. Springer New York, 1999.

[Tao09] Terence Tao. 245C, Notes 1: Interpolation of L∧p spaces. `http://terrytao.wordpress.com/2009/03/30/245c-notes-1-interpolation-of-lp-spaces/`, 2009.

[Tay92] Michael E. Taylor. Analysis on Morrey Spaces and Applications to Navier–Stokes and Other Evolution Equations. *Communications in Partial Differential Equations*, 17(9–10):1407–1456, 1992.

[Tre13] F. Treves. *Topological Vector Spaces, Distributions and Kernels*. Dover Books on Mathematics. Dover Publications, 2013.

[Yos48] Kosaku Yosida. On the differentiability and the representation of one-parameter semi-group of linear operators. *Journal of the Mathematical Society of Japan*, 1(1):15–21, 09 1948.