Chapter 4

It’s a Small World

Summary

- Six degrees of separation
- Watts-Strogatz model

Reading

- Chapter 20 of Kleinberg’s book.

History of

Six Degrees

Karinthy, Frigyes

1929:

Minden másképpen van (Everything is Different)

Láncszemek (Chains)

"Look, Selma Lagerlöf just won the Nobel Prize for Literature, thus she is bound to know King Gustav of Sweden, after all he is the one who handed her the Prize, as required by tradition. King Gustav, to be sure, is a passionate tennis player, who always participates in international tournaments. He is known to have played Mr. Kehrling, whom he must therefore know for sure, and as it happens I myself know Mr. Kehrling quite well."

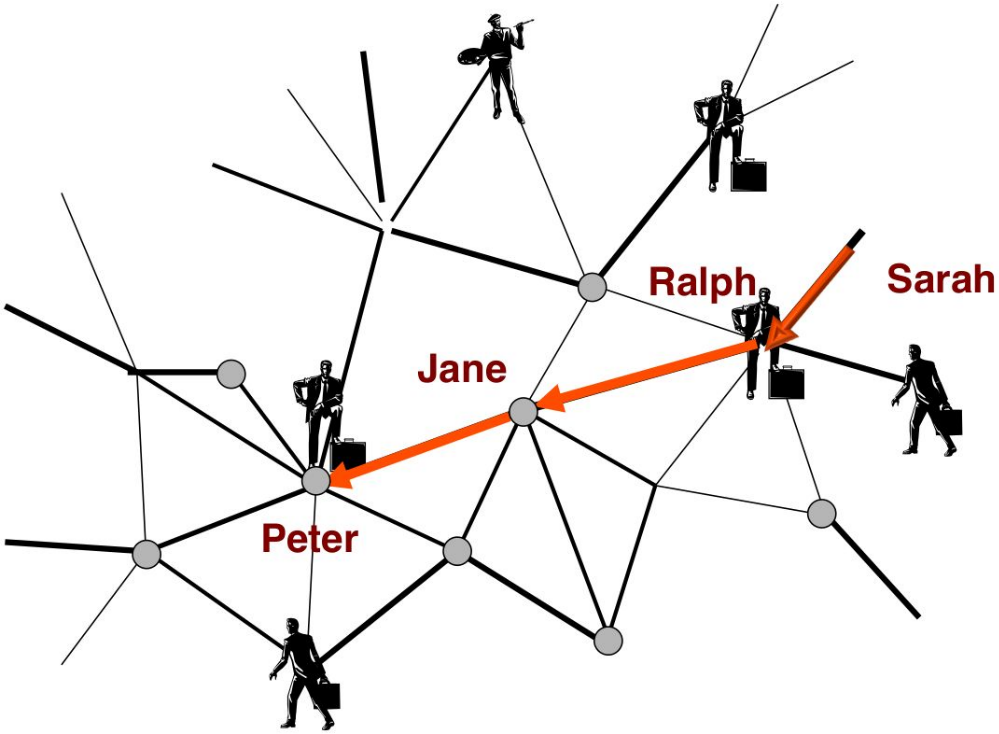
History of

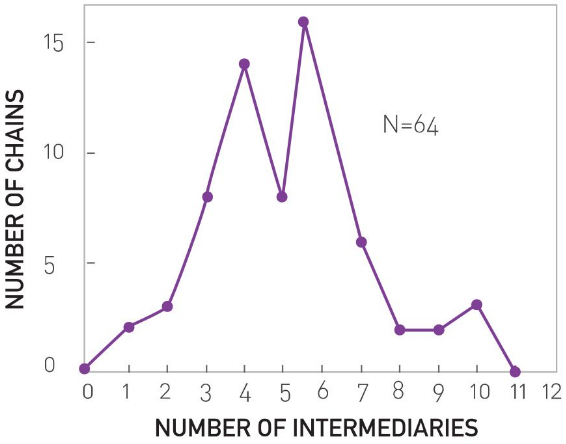
Six Degrees

*1967: **Stanley Milgram***

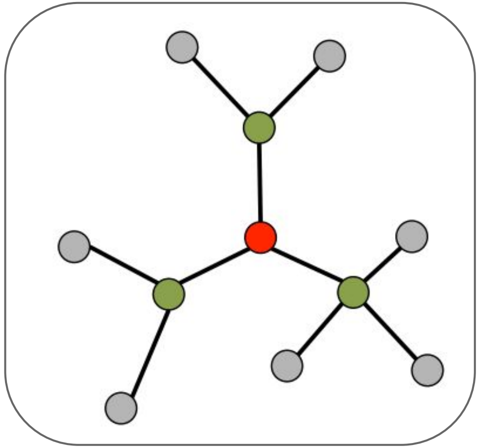
HOW TO TAKE PART IN THIS STUDY

1. *ADD YOUR NAME TO THE ROSTER AT THE BOTTOM OF THIS SHEET, so that the next person who receives this letter will know who it came from.*
2. *DETACH ONE POSTCARD. FILL IT AND RETURN IT TO HARVARD UNIVERSITY. No stamp is needed. The postcard is very important. It allows us to keep track of the progress of the folder as it moves toward the target person.*
3. *IF YOU KNOW THE TARGET PERSON ON A PERSONAL BASIS, MAIL THIS FOLDER DIRECTLY TO HIM (HER).*
   Do this only if you have previously met the target person and know each other on a first name basis.
4. *IF YOU DO NOT KNOW THE TARGET PERSON ON A PERSONAL BASIS, DO NOT TRY TO CONTACT HIM DIRECTLY. INSTEAD, MAIL THIS FOLDER (POST CARDS AND ALL) TO A PERSONAL ACQUAINTANCE WHO IS MORE LIKELY THAN YOU TO KNOW THE TARGET PERSON. You may send the folder to a friend, relative or acquaintance, but it must be someone you know on a first name basis.*





Milgram Experiment



nr. of nodes at distance one (d=1)	$N(u)_1 =< k >$
nr. of nodes at distance two (d=2)	$N(u)_2 =< k >^2$
nr. of nodes at distance d (d=d)	$N(u)_d =< k >^d$

$$N = 1 + \langle k \rangle + \langle k \rangle^2 + \ldots + \langle k \rangle^{d_{\max}} = \frac{\langle k \rangle^{d_{\max}+1} - 1}{\langle k \rangle - 1} \approx \langle k \rangle^{d_{\max}}$$

$$d_{\max} = \frac{\log N}{\log \langle k \rangle}$$

$$< d >= \frac{\log N}{\log (k)}$$

We will call the small world phenomena the property that the average path length or the diameter depends logarithmically on the system size.

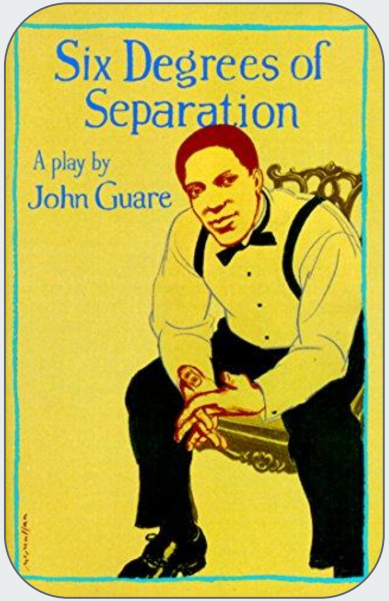
"Small" means that ⟨d⟩ is proportional to log N, rather than N.

The 1/log⟨k⟩ term implies that denser the network, the smaller will be the distance between the nodes.

Small World Phenomena

History of

Six Degrees



"Everybody on this planet is separated by only six other people.

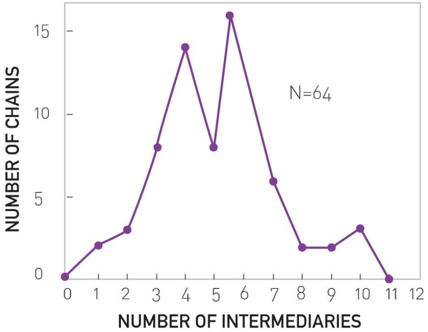
Six degrees of separation.
Between us and everybody else on this planet.
The president of the United States.
A gondolier in Venice.... It's not just the big names.
It's anyone.
A native in a rain forest.
A Tierra del Fuegan.
An Eskimo.
I am bound to everyone on this planet by a trail of six people.

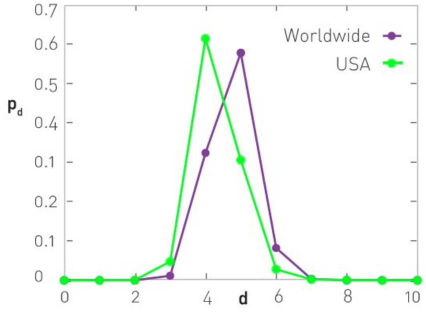
It's a profound thought.
How every person is a new door, opening up into other worlds."

Three, Four or Six degrees?

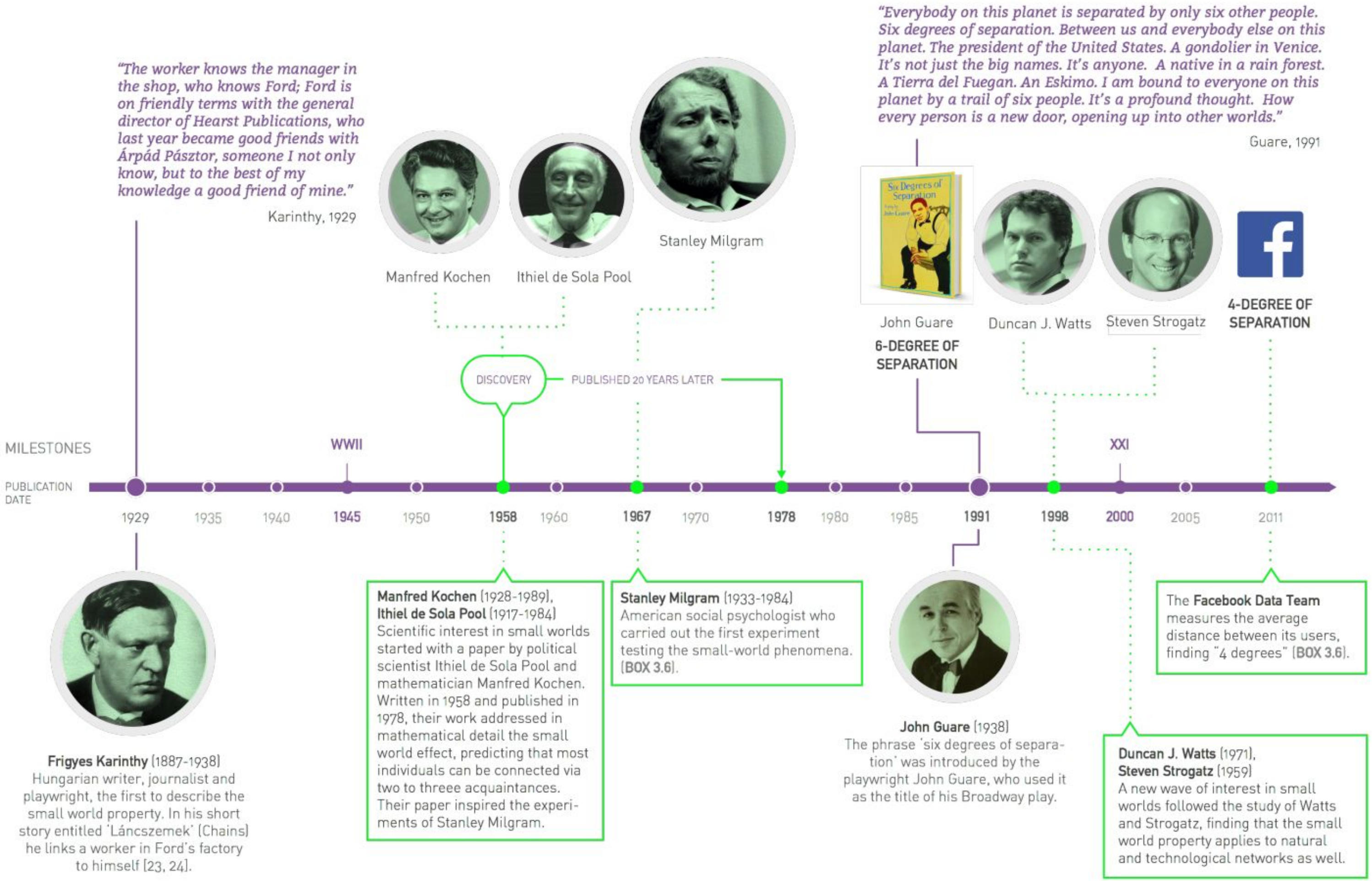
For the globe's social networks:

$\langle k \rangle \approx 10^3$
$N \approx 7 \times 10^9$ (for the world's population)





$$< d > = \frac{\ln(N)}{\ln\langle k \rangle} = 3.28$$



"The worker knows the manager in the shop, who knows Ford; Ford is on friendly terms with the general director of Hearst Publications, who last year became good friends with Árpád Pásztor, someone I not only know, but to the best of my knowledge a good friend of mine."
Karinthy, 1929
Manfred Kochen
Ithiel de Sola Pool
Stanley Milgram
"Everybody on this planet is separated by only six other people. Six degrees of separation. Between us and everybody else on this planet. The president of the United States. A gondolier in Venice. It's not just the big names. It's anyone. A native in a rain forest. A Tierra del Fuegan. An Eskimo. I am bound to everyone on this planet by a trail of six people. It's a profound thought. How every person is a new door, opening up into other worlds."
Guare, 1991
Six Degrees of Separation
John Guare
6-DEGREE OF SEPARATION
Duncan J. Watts
Steven Strogatz
4-DEGREE OF SEPARATION
DISCOVERY
PUBLISHED 20 YEARS LATER
MILESTONES
WWII
XXI
PUBLICATION DATE
1929
1935
1940
1945
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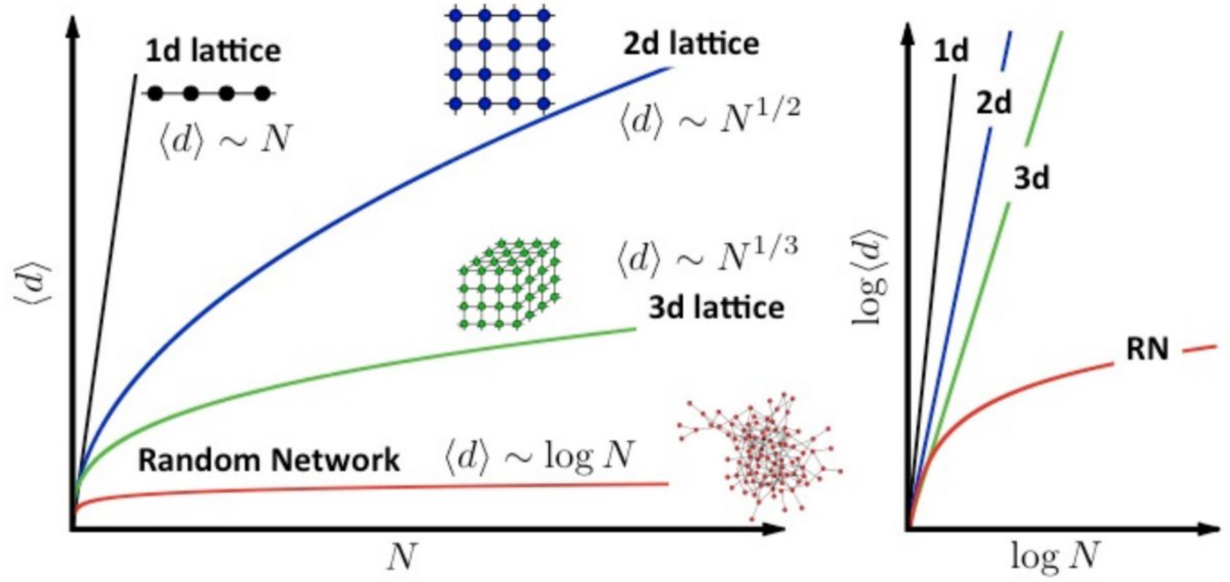
Frigyes Karinthy (1887-1938)
Hungarian writer, journalist and playwright, the first to describe the small world property. In his short story entitled 'Láncszemek' (Chains) he links a worker in Ford's factory to himself [23, 24].
Manfred Kochen (1928-1989), Ithiel de Sola Pool (1917-1984)
Scientific interest in small worlds started with a paper by political scientist Ithiel de Sola Pool and mathematician Manfred Kochen. Written in 1958 and published in 1978, their work addressed in mathematical detail the small world effect, predicting that most individuals can be connected via two to threee acquaintances. Their paper inspired the experiments of Stanley Milgram.
Stanley Milgram (1933-1984)
American social psychologist who carried out the first experiment testing the small-world phenomena. (BOX 3.6).
John Guare (1938)
The phrase 'six degrees of separation' was introduced by the playwright John Guare, who used it as the title of his Broadway play.
The Facebook Data Team measures the average distance between its users, finding "4 degrees" (BOX 3.6).
Duncan J. Watts (1971), Steven Strogatz (1959)
A new wave of interest in small worlds followed the study of Watts and Strogatz, finding that the small world property applies to natural and technological networks as well.



Why are small worlds surprising? Surprising compared to what?!

Watts-Strogatz Model

A model for the Small-World phenomena

One of the first paper on network science…

Real world network observations lead to a contradiction w.r.t. ER graphs:

- High clustering coefficient and
- Short distances

Duncan Watts

Steve Strogatz

NATURE | VOL 393 | 4 JUNE 1998

Collective dynamics of 'small-world' networks

Duncan J. Watts* & Steven H. Strogatz

Department of Theoretical and Applied Mechanics, Kimball Hall, Cornell University, Ithaca, New York 14853, USA

Networks of coupled dynamical systems have been used to model biological oscillators[1–4], Josephson junction arrays[5,6], excitable media[7], neural networks[8–10], spatial games[11], genetic control networks[12] and many other self-organizing systems. Ordinarily, the connection topology is assumed to be either completely regular or completely random. But many biological, technological and social networks lie somewhere between these two extremes.

Table 1 Empirical examples of small-world networks

	L_{actual}	L_{random}	C_{actual}	C_{random}	N
Film actors	3.65	2.99	0.79	0.00027	22500
Power grid	18.7	12.4	0.080	0.005	4941
C. elegans	2.65	2.25	0.28	0.05	282

Clustering vs. Interconnectedness

Random networks:

- Logarithmically short distance among nodes

$$d = \frac{\log N}{\log\langle k\rangle}$$

- Vanishing clustering coefficient for large size

$$C_i \equiv \frac{1}{N}\langle k\rangle = p$$

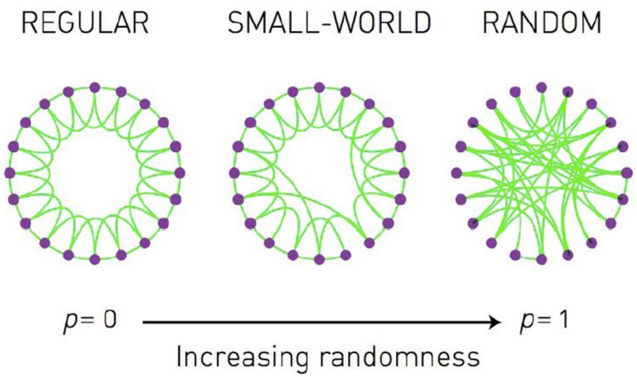
Real Networks:

High Clustering & Short Distances

Network	Size	$\langle k\rangle$	ℓ	ℓ_{rand}	C	C_{rand}
WWW, site level, undir.	153 127	35.21	3.1	3.35	0.1078	0.00023
Internet, domain level	3015–6209	3.52–4.11	3.7–3.76	6.36–6.18	0.18–0.3	0.001
Movie actors	225 226	61	3.65	2.99	0.79	0.00027
LANL co-authorship	52 909	9.7	5.9	4.79	0.43	1.8×10^{-4}
MEDLINE co-authorship	1 520 251	18.1	4.6	4.91	0.066	1.1×10^{-5}
SPIRES co-authorship	56 627	173	4.0	2.12	0.726	0.003
NCSTRL co-authorship	11 994	3.59	9.7	7.34	0.496	3×10^{-4}
Math. co-authorship	70 975	3.9	9.5	8.2	0.59	5.4×10^{-5}
Neurosci. co-authorship	209 293	11.5	6	5.01	0.76	5.5×10^{-5}
E. coli, substrate graph	282	7.35	2.9	3.04	0.32	0.026
E. coli, reaction graph	315	28.3	2.62	1.98	0.59	0.09
Ythan estuary food web	134	8.7	2.43	2.26	0.22	0.06
Silwood Park food web	154	4.75	3.40	3.23	0.15	0.03
Words, co-occurrence	460.902	70.13	2.67	3.03	0.437	0.0001
Words, synonyms	22 311	13.48	4.5	3.84	0.7	0.0006
Power grid	4941	2.67	18.7	12.4	0.08	0.005
C. Elegans	282	14	2.65	2.25	0.28	0.05

From Regular Lattices to Random Networks

A model to capture
large clustering coefficient and
short distances observed in real networks.

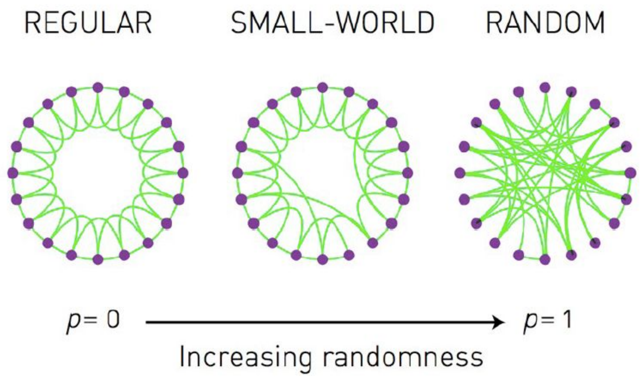


Fixed parameters:
- n - system size
- K - initial coordination number

Variable parameters:
- p - rewiring probability

From Regular Lattices to Random Networks

A model to capture
large clustering coefficient and
short distances observed in real networks.



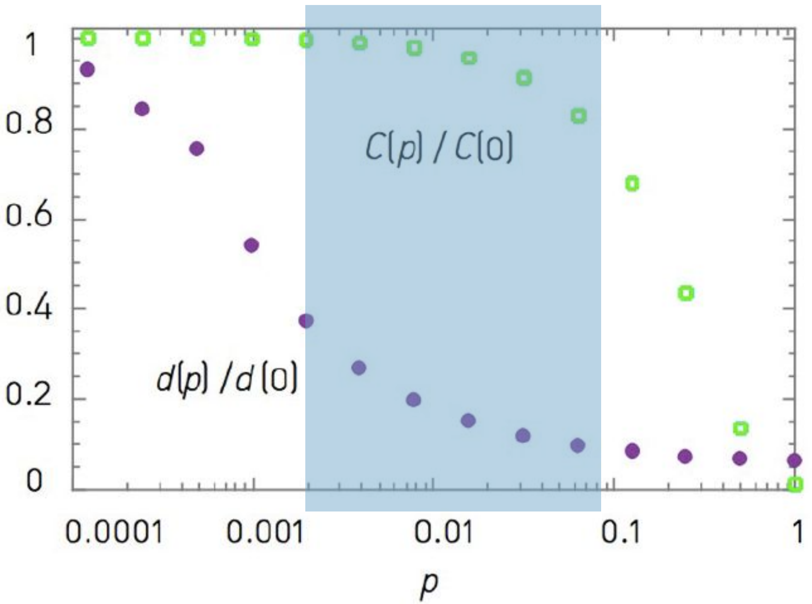
Algorithm:

1. Start with a ring lattice with n nodes in which every node is connected to its first K neighbors (K/2 on either side)
2. Randomly rewire each edge of the lattice with probability p such that self-connections are excluded.

From
Regular Lattices
to Random Networks

By varying p the network can be transformed from a completely ordered (p=0) to a completely random (p=1) structure

Small-World regime



n and K are chosen:

$$n >> K >> ln(n) >> 1$$

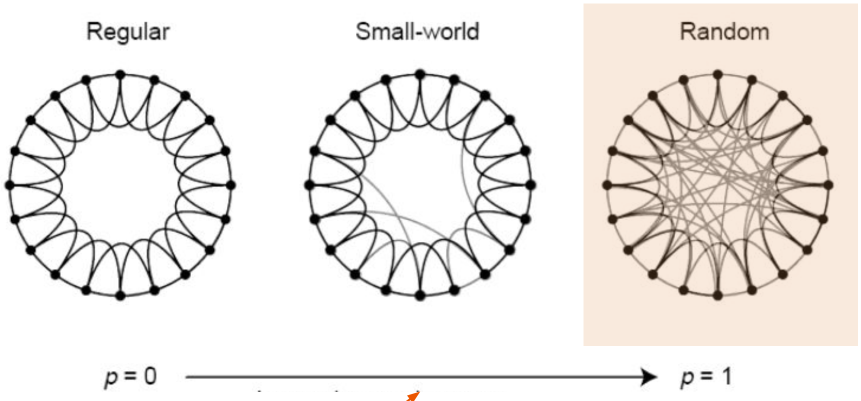
thus the random graph remains connected

$$K >> ln(n)$$

Measuring
Watts-Strogatz Graphs

Watts-Strogatz model

Alternative definitions



Definition 1:

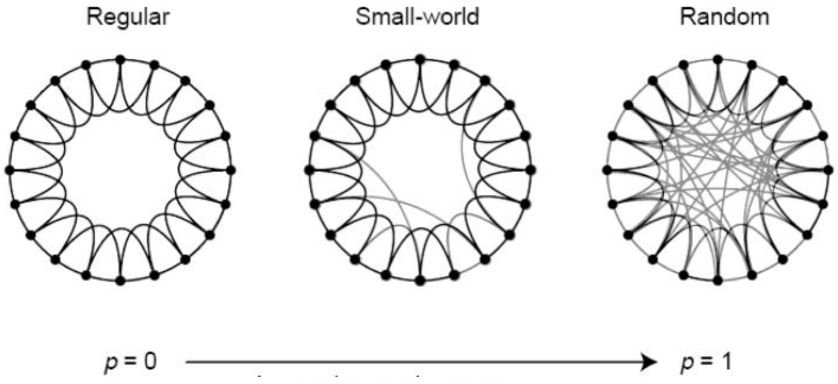
1. Start with a ring lattice with N nodes in which every node is connected to its first K neighbours (K/2 on either side).
2. Randomly rewire each edge of the lattice with probability p such that self-connections and duplicate edges are excluded.

Definition 2:

1. Start with a ring lattice with N nodes in which every node is connected to its first K neighbours (K/2 on either side).
2. For every edge in the network add an additional edge with independent probability p, connecting two nodes selected uniformly at random .

Definition 2

Global Clustering Coefficient



p=0

regular ring with constant clustering:

$$C = \frac{3(K-2)}{4(K-1)}$$

- $0 \leqq C \leqq 3/4$
- independent of n

p>0

we can count triangles and tuples

$$C = \frac{\frac{1}{4}NK(\frac{1}{2}K-1)\times 3}{\frac{1}{2}NK(K-1)+NK^2p+\frac{1}{2}NK^2p^2} = \frac{3(K-2)}{4(K-1)+8Kp+4Kp^2}$$

- Independent of n
- if $p \to 0$ it recovers the ring value
- if $p \to 1$ it well approximates 1

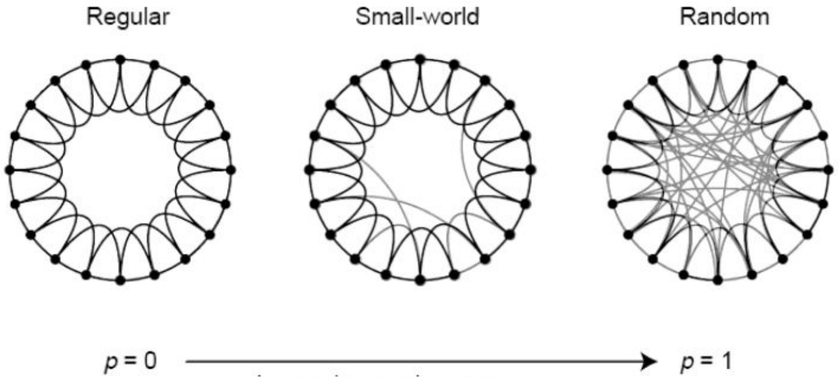
Definition 2

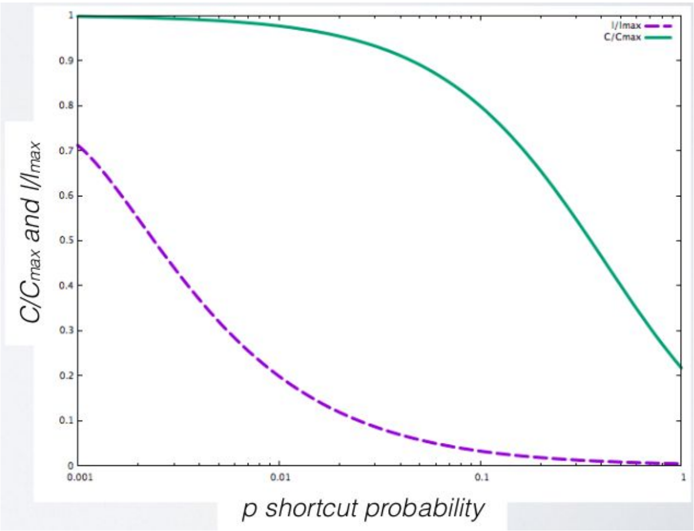
Average path length

No closed form solution

From numerical simulations we can approximate it as:

$$l = \frac{\ln(nKp)}{K^2 p}$$



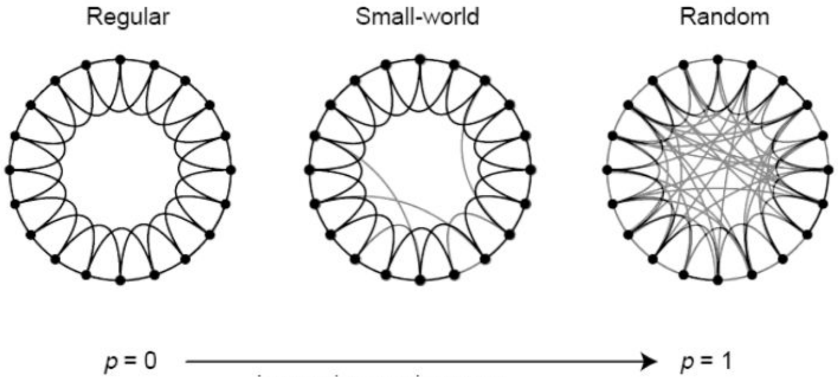


Definition 2

Degree Distribution

p=0
each node has the same degree K
(Dirac delta function)

p>0
approximates a Poisson distribution just like a random network



p>0

- each node has degree K + shortcut links
- Number of shortcut edges:

$$s = \frac{1}{2} NK \times p$$

- Each node will have on average Kp number of shortcuts
- The degree distribution is

$$P(k) = e^{-Kp} \frac{(Kp)^{(k-K)}}{(k-K)!}$$

if k≥K and P(k)=0 if k<K

Summarizing...

W-S Networks
in a Nutshell

Degree Distribution	$e^{-Kp}\frac{(Kp)^{(k-K)}}{(k-K)!}$
Clustering	$\frac{3(K-2)}{4(K-1)+8Kp+4Kp^2}$
Path length	$\frac{\ln(nKp)}{K^2p}$

Network	Degree Distribution	Path Length	Clustering Coefficient
Real-world networks	Broad	Short	Large
ER graphs	Poissonian	Short	Small
Configuration model	Custom, can be broad	Short	Small
Watts & Strogatz (in SW regime)	Poissonian	Short	Large

9 January 2020

· 5h

in maths, you study things often named after mathematicians: "the Bernoulli distribution", "the Markov inequality" etc

in undergrad, almost all these names are of dead guys with greyscale wikipedia pics

3 years ago I studied something called "the Watts-Strogatz model" (1/2)

2 3 30

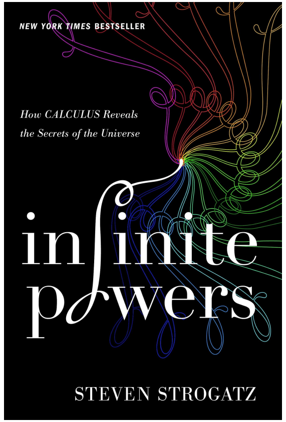
so naturally, I filed "Strogatz" under the "Dead Greyscale Mathematicians" category in my head.

well, it turns out he's actually alive, in colour, and on twitter! but I still do a double-take when I see his tweets — in my head, Strogatz is still dead😂

sorry @stevenstrogatz!!

Steven Strogatz @stevenstrogatz · 3h

Well, I did have a cold a few weeks ago. But I'm not dead yet!



Strogatz, Steven. *Infinite Powers: How Calculus Reveals the Secrets of the Universe*. Houghton Mifflin Harcourt, 2019.

Remember:
Network Science is a pretty recent (and alive) discipline...

Chapter 4

Conclusion

Take Away Messages

1. Small diameters and high clustering coefficient deeply characterize social networks topologies

Suggested Readings

- Chapter 20 of Kleinberg's book

What's Next

Chapter 5:
Scale Free Networks

Notebook

Chapter 4: It's a Small World
https://github.com/sna-unipi/SNA_lectures_notebooks