Chapter 5

Scale Free Networks

Summary

- Scale Free Networks
- Power Law degree distribution
- Barabasi-Albert model
- Advanced: Alternative models

Reading

- Chapters 4 & 5 of Barabasi's book.

Example

World Wide Web

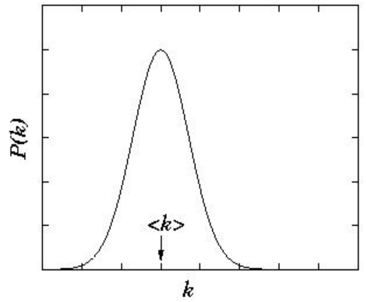
Nodes: WWW documents
Links: URL links

Over 3 billion documents

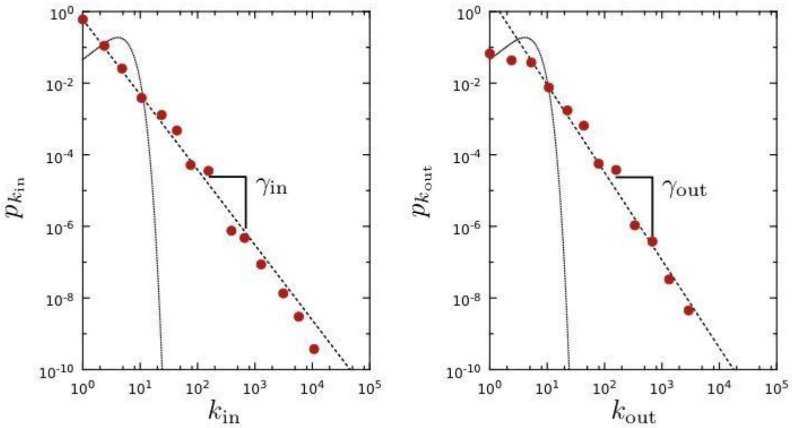
Data Collection:

web crawler collected all URL's found in a document and followed them recursively

R. Albert, H. Jeong, A-L Barabasi, Nature, 401 130 (1999).



Expected



Observed

Scale-Free

A network is called Scale-free when its degree distribution follows *(to some extent)* a Power-Law distribution:

$$P(k) \sim Ck^{-\gamma} = C\frac{1}{k^{\gamma}}$$

with γ called the exponent of the distribution

Discrete Formalism:

As node degrees are always positive integers, the discrete formalism captures the probability that a node has exactly k links:

$$p_k = Ck^{-\gamma}. \quad \sum_{k=1}^{\infty} p_k = 1$$

$$C\sum_{k=1}^{\infty} k^{-\gamma} = 1$$

$$C = \frac{1}{\sum_{k=1}^{\infty} k^{-\gamma}} = \frac{1}{\zeta(\gamma)}, \qquad p_k = \frac{k^{-\gamma}}{\zeta(\gamma)}$$

Interpretation p_k

Continuum Formalism:

In analytical calculations it is often convenient to assume that the degrees can take up any positive real value:

$$p(k) = Ck^{-\gamma} \qquad \int_{k_{\min}}^{\infty} p(k)dk = 1$$

$$C = \frac{1}{\int_{k_{\min}}^{\infty} k^{-\gamma}dk} = (\gamma - 1)k_{\min}^{\gamma-1}$$

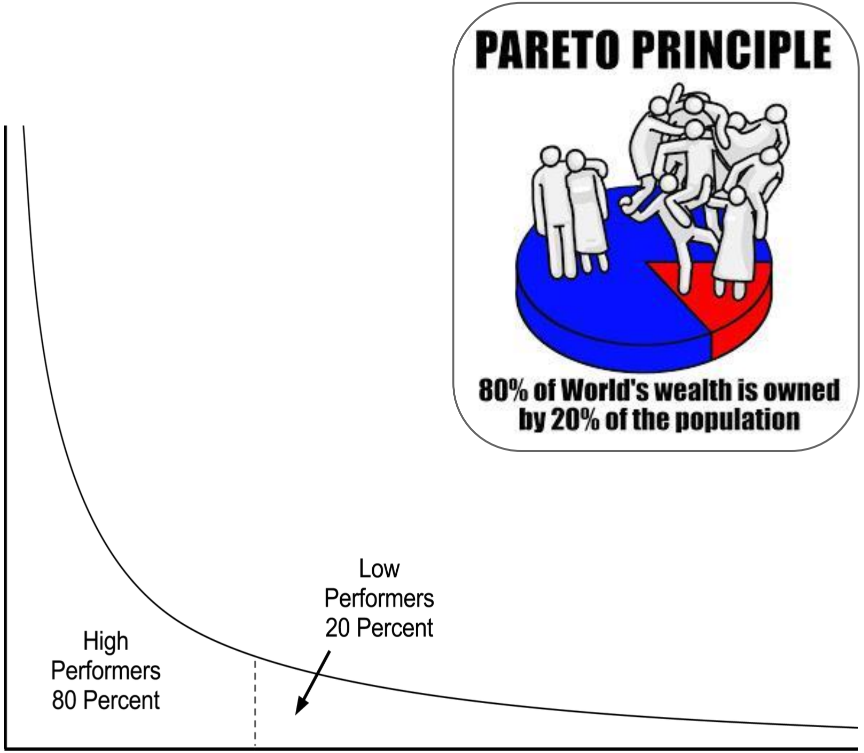
$$p(k) = (\gamma - 1)k_{\min}^{\gamma-1}k^{-\gamma}$$

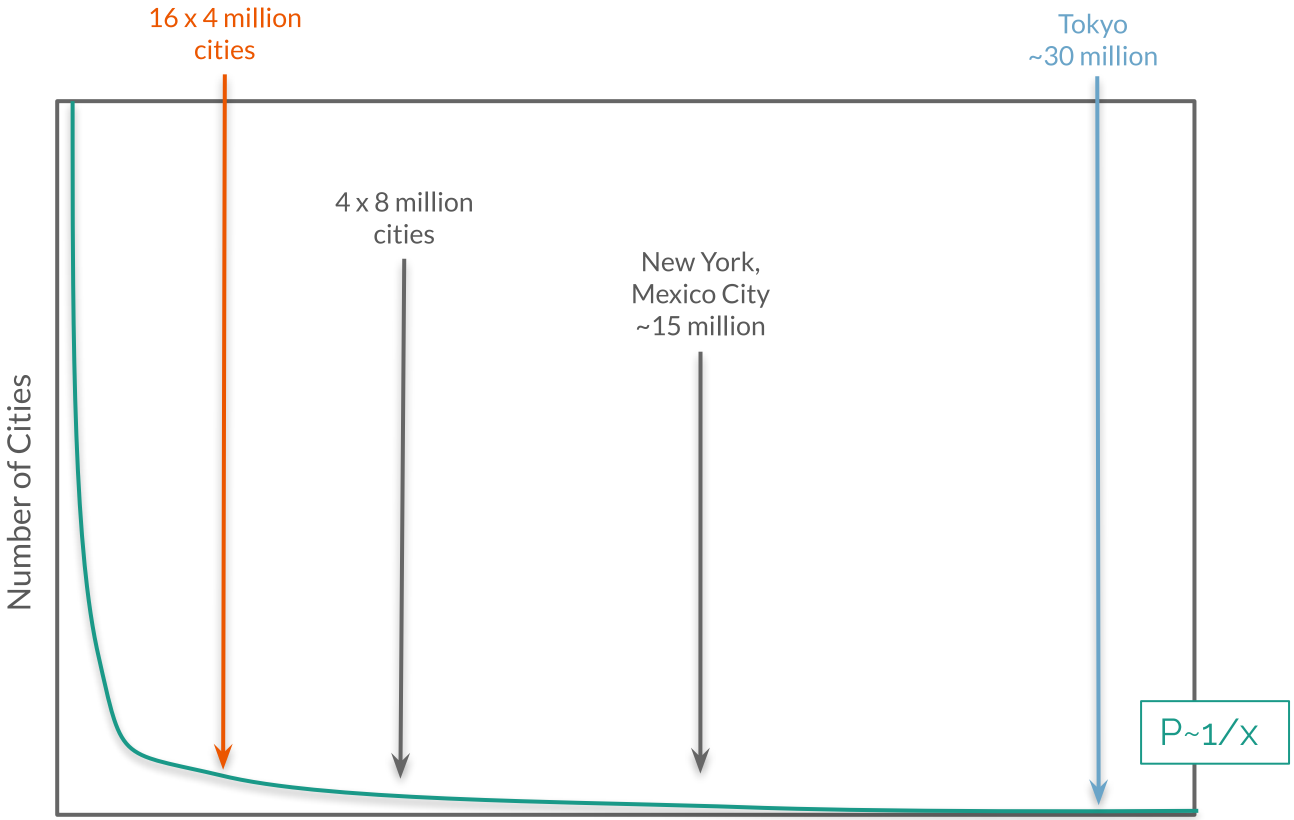
Interpretation $\int_{k_1}^{k_2} p(k)dk$

80/20 Rule

Vilfredo Federico Damaso Pareto (1848 – 1923)

Italian economist, political scientist and philosopher, who had important contributions to our understanding of income distribution and to the analysis of individuals choices.
A number of fundamental principles are named after him, like Pareto efficiency, Pareto distribution *(another name for a power-law distribution)*, the Pareto principle (or 80/20 law).





Sizes of Cities:
there is an equivalent number of people living in cities of all sizes!

Power-Law

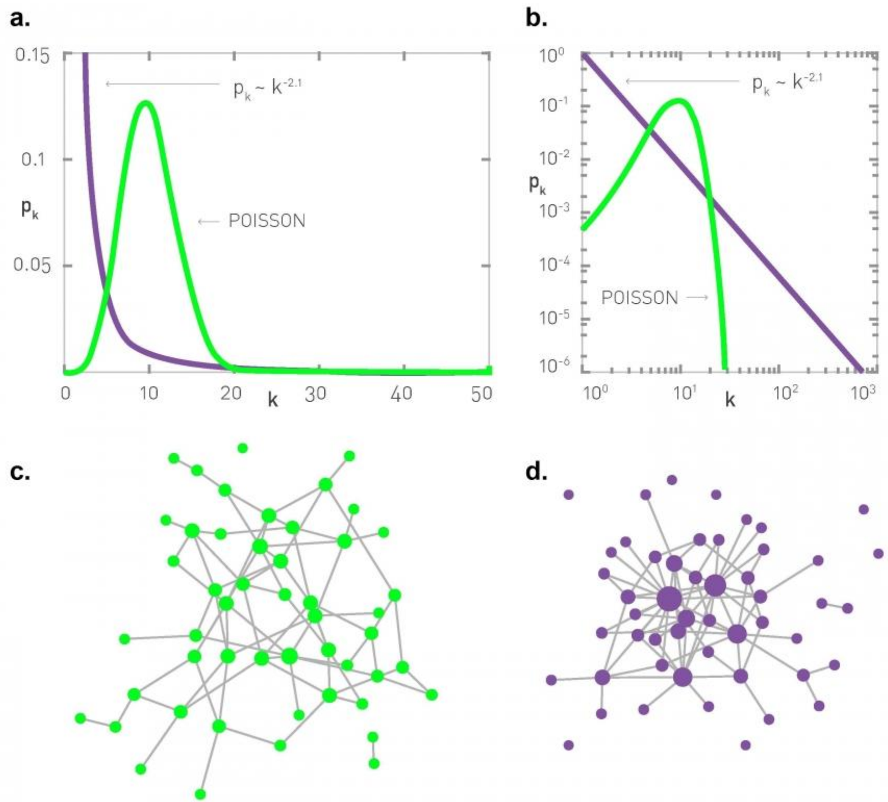
Hubs

The main difference between a random and a scale-free network comes in the *tail* of the degree distribution, representing the high-k region of p_k

For small k the power law is above the Poisson function, indicating that a scale-free network has a large number of small degree nodes, most of which are absent in a random network.

For k in the vicinity of $\langle k \rangle$ the Poisson distribution is above the power law, indicating that in a random network there is an excess of nodes with degree $k \approx \langle k \rangle$

For large k the power law is above the Poisson curve, indicating that the probability of observing a high-degree node, or *hub*, is several orders of magnitude higher in a scale-free than in a random network



Example

Hubs

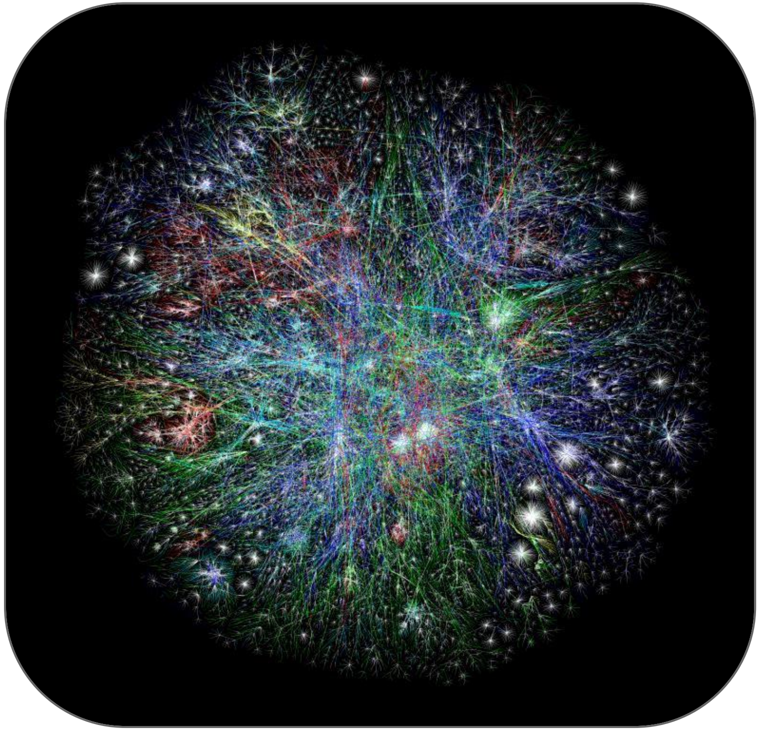
Let us use the WWW to illustrate the properties of the high-k regime.

The probability to have a node with k~100 is

- About $p_{100} \simeq 10^{-30}$ in a Poisson distribution
- About $p_{100} \simeq 10^{-4}$ if p_k follows a power law

Consequently, if the WWW were to be a random network, according to the Poisson prediction we would expect 10^{-18} k>100 degree nodes, or none.

For a power law degree distribution,
we expect about $N_{k>100} = 10^{9}$ k>100 degree nodes

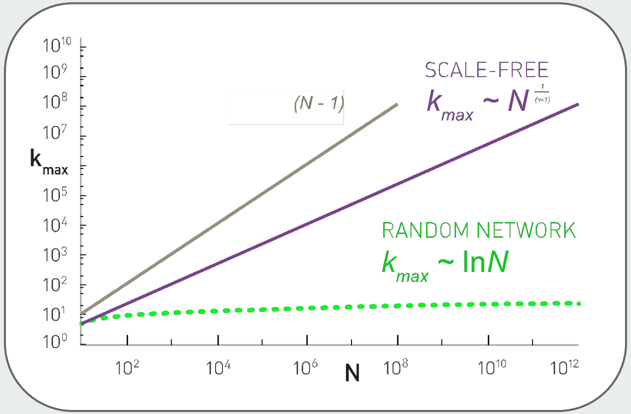


The biggest Hub

All real networks are finite

↓

We have an expected maximum degree, k_{max}



Estimating k_{max}

$$\int_{k_{max}}^{\infty} P(k)dk \approx \frac{1}{N}$$

the probability to have a node larger than k_{max} should not exceed the prob. to have one node, i.e. *1/N* fraction of all nodes

$$\int_{k_{max}}^{\infty} P(k)dk = (\gamma-1)k_{min}^{\gamma-1}\int_{k_{max}}^{\infty} k^{-\gamma}\, dk$$

$$= \frac{(\gamma-1)}{(-\gamma+1)}k_{min}^{\gamma-1}\left[k^{-\gamma+1}\right]_{k_{max}}^{\infty} = \frac{k_{min}^{\gamma-1}}{k_{max}^{\gamma-1}} \approx \frac{1}{N}$$

therefore,

$$k_{max} = k_{min}N^{\frac{1}{\gamma-1}}$$

Ultra Small World

Small World

$$\langle l \rangle \sim \begin{cases} const. & \gamma = 2 \\ \dfrac{\ln \ln N}{\ln(\gamma - 1)} & 2 < \gamma < 3 \\ \dfrac{\ln N}{\ln \ln N} & \gamma = 3 \\ \ln N & \gamma > 3 \end{cases}$$

Size of the biggest hub is of order O(N).
Most nodes can be connected within two layers of it, thus the average path length will be independent of the system size.

The average path length increases slower than logarithmically.
In a random network all nodes have comparable degree, thus most paths will have comparable length. In a scale-free network the vast majority of the path go through the few high degree hubs, reducing the distances between nodes.

Some key models produce $\gamma=3$, so the result is of particular importance for them.
This was first derived by Bollobas and collaborators for the network diameter in the context of a dynamical model, but it holds for the average path length as well.

The second moment of the distribution is finite, thus in many ways the network behaves as a random network. Hence the average path length follows the result that we derived for the random network model earlier.

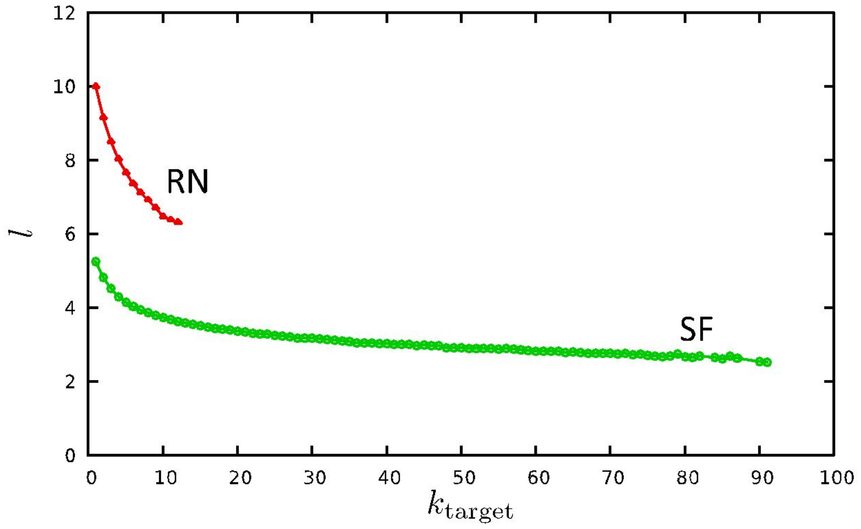
Small-World in Scale-Free networks

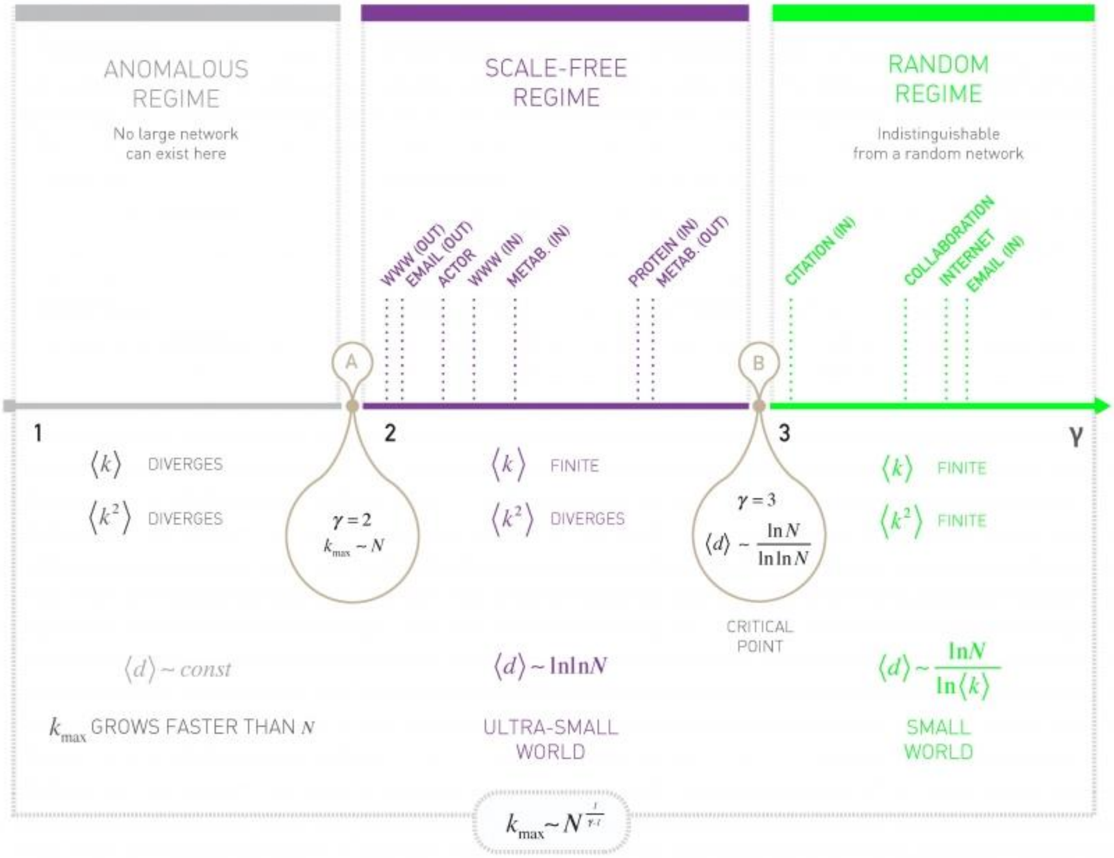
Bollobas, (1985), Newman (2001), Dorogovtsev et al (2002), Chung and Lu (2002), Bollobas (2002), Cohen (2003)

We are always close to the Hubs

"It's always easier to find someone who knows a famous or popular figure than some run-the-mill, insignificant person."

(Frigyes Karinthy, 1929)





Behavior of Scale-Free networks

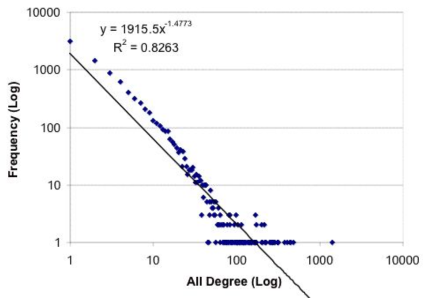
The Barabási-Albert model

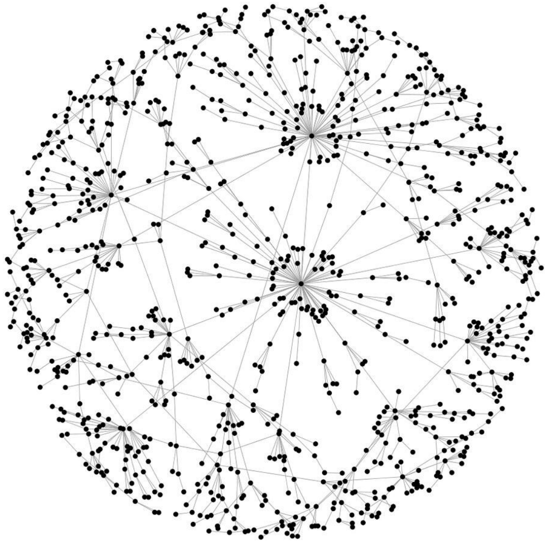
Modeling
Scale-Free Networks

1. Why does the random network model of Erdős and Rényi fail to reproduce the hubs and the power laws observed in many real networks?

2. Why do so different systems as the WWW or the cell converge to a similar scale-free architecture?

Hubs represent the most striking difference between a random and a scale-free network.



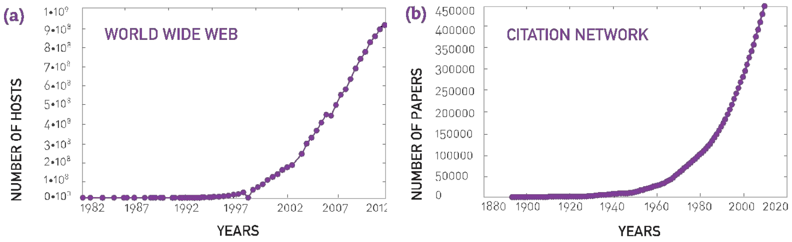


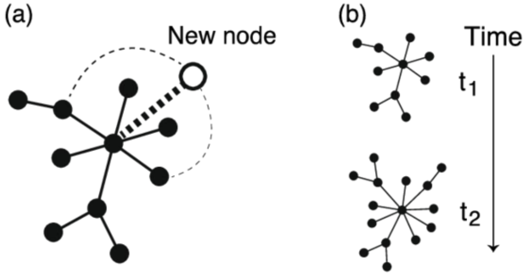
Growth and Preferential Attachment

The random network model differs from real networks in two important characteristics:

Growth:
While the random network model assumes that the number of nodes is fixed (time invariant), real networks are the result of a growth process that continuously increases.

Preferential Attachment:
While nodes in random networks randomly choose their interaction partner, in real networks new nodes prefer to link to the more connected nodes.





BA model

1. Networks continuously expand by the addition of new nodes
   WWW : addition of new documents

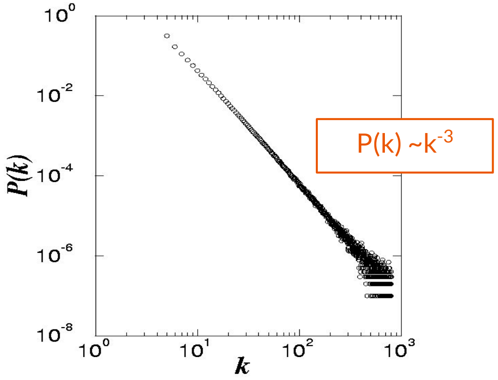
2. New nodes prefer to link to highly connected nodes.
   WWW : linking to well known sites

*Barabási & Albert, Science **286**, 509 (1999)*

1. Start with m_0 connected nodes
2. At each timestep add a new node with m links that connect it to nodes already in the network
3. The probability $\Pi(k)$ that on of the links connects to node i depends on the degree k_i of i

$$\Pi(k_i) = \frac{k_i}{\Sigma_j k_j}$$

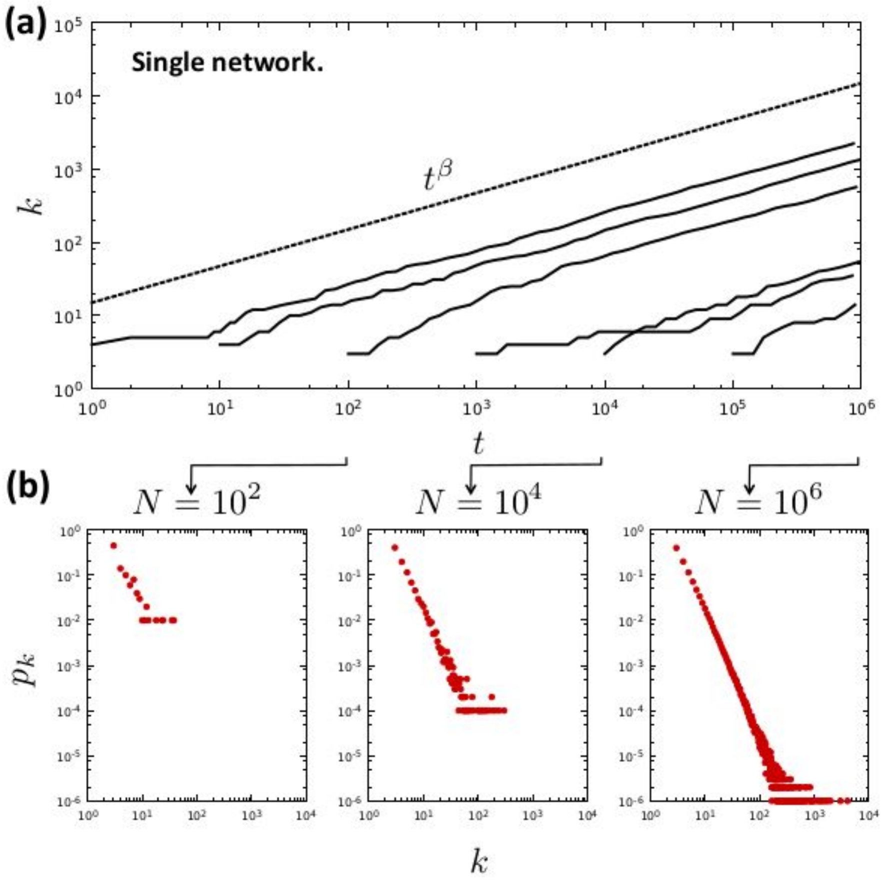
The emerging network will be scale-free with degree exponent $\gamma=3$ independently from the choice of m



BA model

- The degree of each node increases as power-law with exponent ½
- The earlier a node was added the larger its degree
  (due to its arrival time, not because of faster growth)

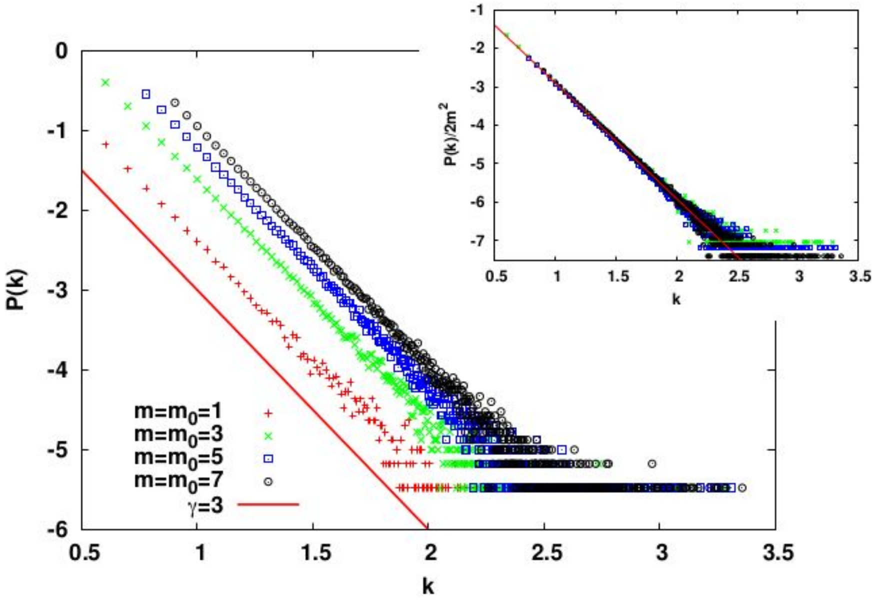
*Barabási & Albert, Science **286**, 509 (1999)*



BA model

- The degree exponent is independent of m
- The degree exponent is stationary in time and the degree distribution is time independent
- The exponent is compatible to the exponents of real networks

*Barabási & Albert, Science **286**, 509 (1999)*



$$\langle l \rangle \sim \begin{cases} const. & \gamma = 2 \\ \dfrac{\ln\ln N}{\ln(\gamma - 1)} & 2 < \gamma < 3 \\ \dfrac{\ln N}{\ln\ln N} & \gamma = 3 \\ \ln N & \gamma > 3 \end{cases}$$

Ultra Small World

$const.$, $\gamma = 2$: Size of the biggest hub is of order O(N).
Most nodes can be connected within two layers of it, thus the average path length will be independent of the system size.

$\dfrac{\ln\ln N}{\ln(\gamma - 1)}$, $2 < \gamma < 3$: The average path length increases slower than logarithmically.
In a random network all nodes have comparable degree, thus most paths will have comparable length. In a scale-free network the vast majority of the path go through the few high degree hubs, reducing the distances between nodes.

$\dfrac{\ln N}{\ln\ln N}$, $\gamma = 3$: Some key models produce γ=3, so the result is of particular importance for them.
This was first derived by Bollobas and collaborators for the network diameter in the context of a dynamical model, but it holds for the average path length as well.

Small World

$\ln N$, $\gamma > 3$: The second moment of the distribution is finite, thus in many ways the network behaves as a random network. Hence the average path length follows the result that we derived for the random network model earlier.

BA model: Path Length

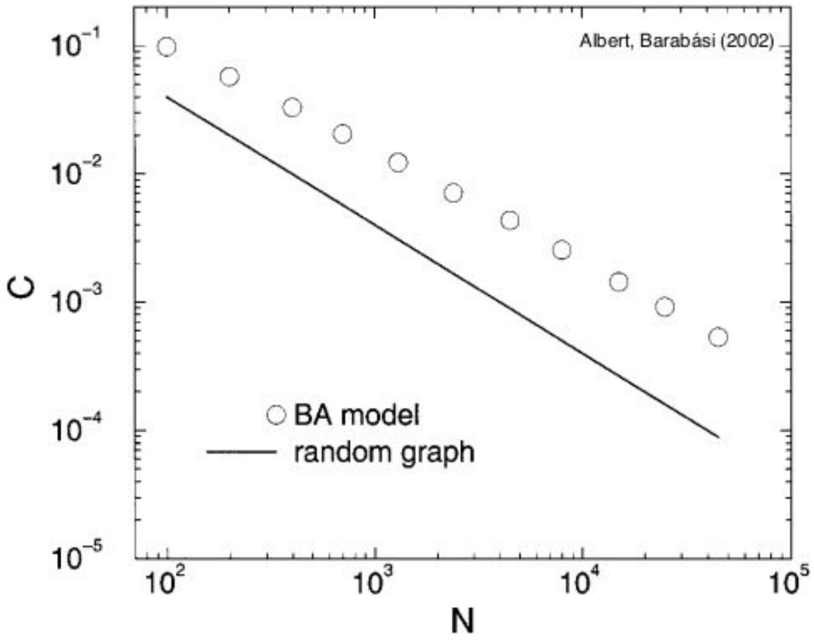
Bollobas, (1985), Newman (2001), Dorogovtsev et al (2002), Chung and Lu (2002), Bollobas (2002), Cohen (2003)

BA model

Clustering Coefficient

The clustering coeff. decreases with the system size as

$$C = \frac{m}{4} \frac{(\ln N)^2}{N}$$



Due to its definition the BA model induces non-trivial degree correlation

$$n_{kl} \simeq k^{-2} l^{-2}$$

Summarizing...

BA Networks
in a Nutshell

Number of nodes	$N = t$
Number of links	$N = mt$
Average degree	$\langle k \rangle = 2m$
Degree Distribution	$P(k) \sim Ck^{-\gamma}$
Clustering	$\frac{m}{4}\frac{(\ln N)^2}{N}$
Path length	$\frac{\ln N}{\ln \ln N}$

Network	Degree Distribution	Path Length	Clustering Coefficient
Real-world networks	Broad	Short	Large
ER graphs	Poissonian	Short	Small
Configuration model	Custom, can be broad	Short	Small
Watts & Strogatz (in SW regime)	Poissonian	Short	Large
Barabasi Albert (Scale-Free)	Power-Law	Short	Rather Small

Advanced Topics:

- **Scale-Free an open Debate**
- **Vertex copying and Holme-Kim models**

The Scale-Free debate

Scale-free networks are rare

Anna D. Broido & Aaron Clauset

Nature Communications **10**, Article number: 1017 (2019)

Love *is All You Need*

Clauset's fruitless search for scale-free networks

March 6, 2018

Rare and everywhere: Perspectives on scale-free networks

Petter Holme

Nature Communications **10**, Article number: 1016 (2019) | Cite this article

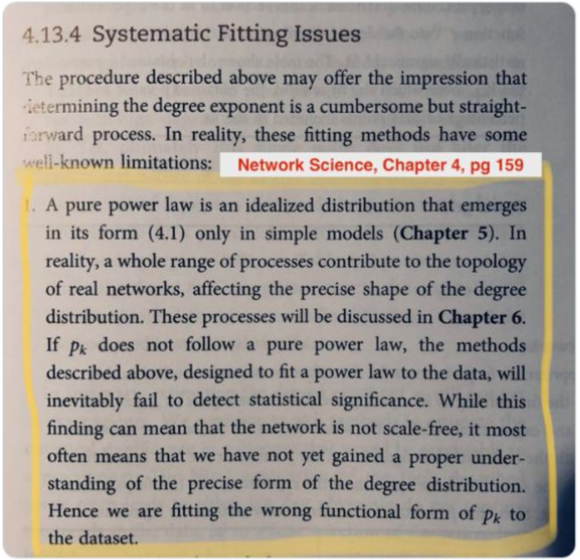
Are real networks really Scale Free ?

- In most real networks, the scale free stands only for a range of degrees, i.e., between a minimum degree and maximum degree different than those observed (cut-offs)

- Some other distributions, in particular log-normal distributions, might “look like” power-law

Albert-László Barabási
@barabasi

@aaronclauset Every 5 years someone is shocked to re-discover that a pure power law does not fit many networks. True: Real networks have predictable deviations. Hence forcing a pure power law on these is like...fitting a sphere to the cow. Sooner or later the hoof will stick out.



4.13.4 Systematic Fitting Issues

The procedure described above may offer the impression that determining the degree exponent is a cumbersome but straightforward process. In reality, these fitting methods have some well-known limitations:

Network Science, Chapter 4, pg 159

1. A pure power law is an idealized distribution that emerges in its form (4.1) only in simple models (**Chapter 5**). In reality, a whole range of processes contribute to the topology of real networks, affecting the precise shape of the degree distribution. These processes will be discussed in **Chapter 6**. If p_k does not follow a pure power law, the methods described above, designed to fit a power law to the data, will inevitably fail to detect statistical significance. While this finding can mean that the network is not scale-free, it most often means that we have not yet gained a proper understanding of the precise form of the degree distribution. Hence we are fitting the wrong functional form of p_k to the dataset.

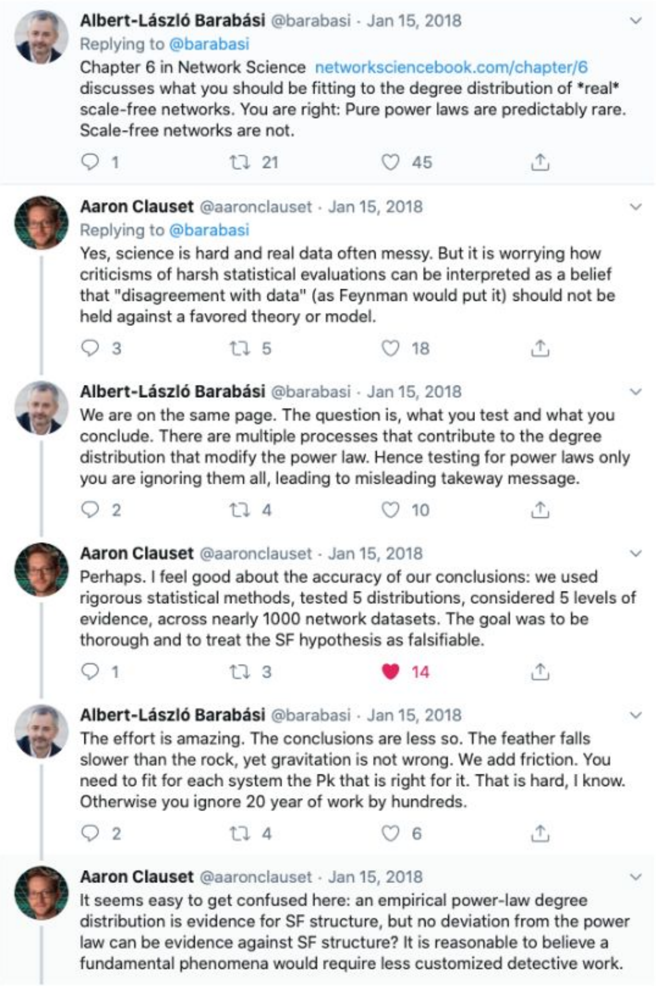
Rigorous statistical tests show that observed degree distributions are not compatible with a power law distribution (high p-values)

Compared with different distributions, in particular log-normal, most degree distributions are more likely to be generated by something else than power laws.

Networks are real objects, not mathematical abstraction, therefore they are sensible to noise (real life limits...)

Power law is a good, simple model of degree distributions of a class of networks

20 years of fruitful research based on this model



Albert-László Barabási @barabasi · Jan 15, 2018
Replying to @barabasi
Chapter 6 in Network Science networksciencebook.com/chapter/6 discusses what you should be fitting to the degree distribution of *real* scale-free networks. You are right: Pure power laws are predictably rare. Scale-free networks are not.
1 · 21 · 45

Aaron Clauset @aaronclauset · Jan 15, 2018
Replying to @barabasi
Yes, science is hard and real data often messy. But it is worrying how criticisms of harsh statistical evaluations can be interpreted as a belief that "disagreement with data" (as Feynman would put it) should not be held against a favored theory or model.
3 · 5 · 18

Albert-László Barabási @barabasi · Jan 15, 2018
We are on the same page. The question is, what you test and what you conclude. There are multiple processes that contribute to the degree distribution that modify the power law. Hence testing for power laws only you are ignoring them all, leading to misleading takeway message.
2 · 4 · 10

Aaron Clauset @aaronclauset · Jan 15, 2018
Perhaps. I feel good about the accuracy of our conclusions: we used rigorous statistical methods, tested 5 distributions, considered 5 levels of evidence, across nearly 1000 network datasets. The goal was to be thorough and to treat the SF hypothesis as falsifiable.
1 · 3 · 14

Albert-László Barabási @barabasi · Jan 15, 2018
The effort is amazing. The conclusions are less so. The feather falls slower than the rock, yet gravitation is not wrong. We add friction. You need to fit for each system the Pk that is right for it. That is hard, I know. Otherwise you ignore 20 year of work by hundreds.
2 · 4 · 6

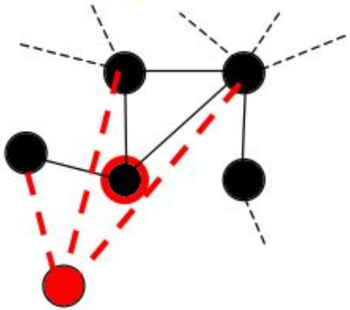
Aaron Clauset @aaronclauset · Jan 15, 2018
It seems easy to get confused here: an empirical power-law degree distribution is evidence for SF structure, but no deviation from the power law can be evidence against SF structure? It is reasonable to believe a fundamental phenomena would require less customized detective work.

Alternatives

Vertex-Copying model

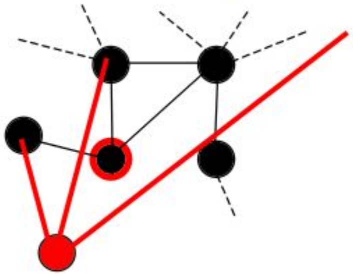
How to provide a local explanation to preferential attachment?

1. Take a small seed network
2. Pick a random vertex
3. Make a copy of it
4. with probability p move each edge of the copy to point to a random vertex
5. Repeat 2-4 until the network reach the desired size



Copy a vertex

Pewire edges with *p*



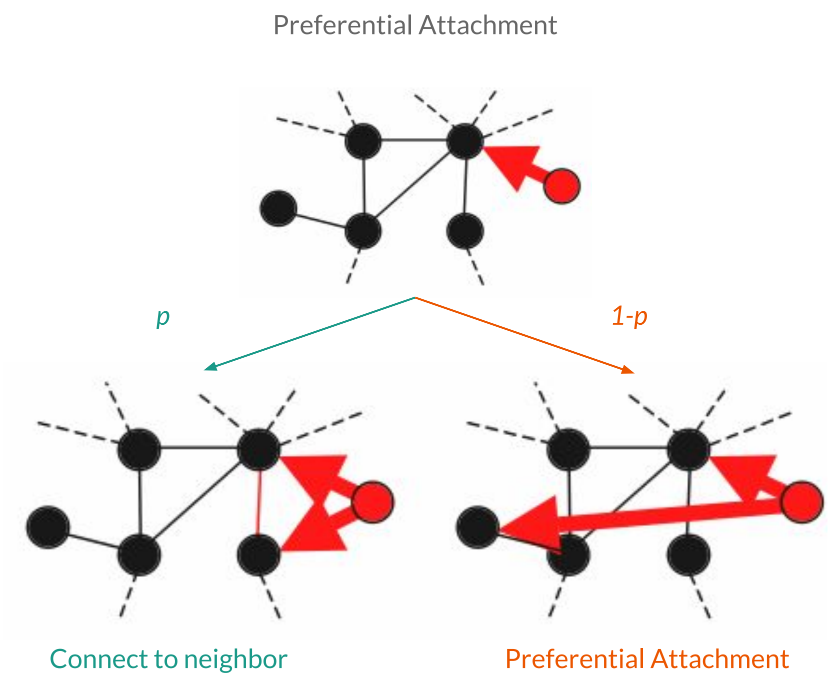
Asymptotically scale-free with exponent $\gamma \geq 3$

Alternatives

Holme-Kim model

How to get a more realistic clustering coefficient?

1. Take a small seed network
2. Create a new vertex with m edges
3. Connect the first of the m edges to existing vertices with a probability proportional to their degree k
4. With probability p, connect the next edge to a random neighbor of the vertex of step 3, otherwise repeat 3
5. Repeat 2-4 until the network reach the desired size



$$C(k) \propto \frac{1}{k}$$

For large N the clustering more realistic!
This type of clustering is found in many real-world networks.

Network models
in a Nutshell

"All models are wrong, but some are useful"

- ER models and Configuration models are used as reference models in a very large number of applications

- WS, BA are more *"making a point"* type models: simple processes can explain some non-trivial properties of networks, unfound in random networks.

- Correlation is not causation.
  Are these simple processes the "cause"?
  Maybe, maybe not, sometimes...

Network	Degree Distribution	Path Length	Clustering Coefficient
Real-world networks	Broad	Short	Large
ER graphs	Poissonian	Short	Small
Configuration model	Custom, can be broad	Short	Small
Watts & Strogatz (in SW regime)	Poissonian	Short	Large
Barabasi Albert (Scale-Free)	Power-Law	Short	Rather Small
Other models	Power-law	Short	Large

Chapter 5

Conclusion

Take Away Messages

1. Real world networks have heavy tailed degree distributions
2. Scale-Free networks
3. Ultra Small-world phenomena
4. BA models scale-free with γ=3
5. Additional models explains local behaviours, clustering coeff., ...

Suggested Readings

- Chapters 4 & 5 of Barabasi's book
- Chapter 18 of Kleinberg's book

What's Next

Chapter 6:
Centrality & Tie Strength

Notebook

Chapter 5: Scale Free Networks
https://github.com/sna-unipi/SNA_lectures_notebooks