7) Consider a two-way ANOVA model with fixed effects

$$Y_{ijk} = \mu + \alpha_i + \beta_j + \epsilon_{ijk}$$

$i = 1, \ldots, a$, $j = 1, \ldots, b$, $k = 1, \ldots, n$

where $\{\alpha_i\}$ satisfies that $\sum_i^a \alpha_i = 0$, $\{\beta_j\}$ satisfies that $\sum_j^b \beta_j = 0$, and $\{\epsilon_{ijk}\}$ are iid $N(0, \sigma^2)$. Derive the least squares estimators for the above model

use criterion function

Criterion function: $\sum_i \sum_j \sum_k (Y_{ijk} - \hat{\mu}_{..} - \hat{\alpha}_i - \hat{\beta}_j)^2$

$$\frac{\partial}{\partial \mu}(\text{Criterion function}) = \sum_i \sum_j \sum_k 2(Y_{ijk} - \hat{\mu}_{..} - \hat{\alpha}_i - \hat{\beta}_j)(-1) \overset{\text{set equal to } 0}{=} 0$$

$$\Rightarrow \sum_i \sum_j \sum_k Y_{ijk} = \sum_i \sum_j \sum_k \hat{\mu}_{..} + \cancel{\sum_i \sum_j \sum_k \hat{\alpha}_i} + \cancel{\sum_i \sum_j \sum_k \hat{\beta}_j}$$

by constraints $\sum_j^b \beta_j = 0$ & $\sum_i^a \alpha_i = 0$

$$\Rightarrow \sum_i \sum_j \sum_k Y_{ijk} = \sum_i \sum_j \sum_k \hat{\mu}_{..} = \hat{\mu}_{..} N$$

$$\Rightarrow \boxed{\hat{\mu}_{..} = \frac{\sum_i \sum_j \sum_k Y_{ijk}}{N} = \bar{Y}_{...}}$$

$i = i'$ $\quad \frac{\partial}{\partial \alpha_{i'}} \sum_{i}^{a} \sum_{j}^{b} \sum_{k}^{n} (Y_{ijk} - \hat{\mu}_{\cdot} - \hat{\alpha}_{i} - \hat{\beta}_{j})^2$

set each to 0

$$\Rightarrow -2 \sum_{i}^{a} \sum_{j}^{b} \sum_{k}^{n} Y_{ijk} - \hat{\mu}_{\cdot\cdot} - \hat{\alpha}_{i'} - \hat{\beta}_{j} = 0$$

by constant

$$\Rightarrow \sum_{i}^{a} \sum_{j}^{b} \sum_{k}^{n} Y_{ijk} = \sum_{i}^{a} \sum_{j}^{b} \sum_{k}^{n} \hat{\mu}_{\cdot\cdot} + \sum_{i}^{a} \sum_{j}^{b} \sum_{k}^{n} \hat{\alpha}_{i'}$$

$$\Rightarrow \bar{Y}_{i\cdot\cdot} \cancel{bn} = \cancel{bn} \hat{\mu}_{\cdot\cdot} + \hat{\alpha}_{i'} \cancel{bn}$$

$$\bar{Y}_{i\cdot\cdot} = \bar{Y}_{\cdot\cdot\cdot} + \hat{\alpha}_{i'}$$

$$\Rightarrow \boxed{\bar{Y}_{i'\cdot\cdot} - \bar{Y}_{\cdot\cdot\cdot} = \hat{\alpha}_{i'}}$$

$j = j'$

$$\frac{\partial}{\partial \hat{\beta}_{j'}} \sum_{i}^{a} \sum_{j}^{b} \sum_{k}^{n} \left(Y_{ijk} - \hat{\mu}_{..} - \hat{\alpha}_i - \hat{\beta}_{j'}\right)^2$$

$$\Rightarrow -2 \sum_{i}^{a} \sum_{j}^{b} \sum_{k}^{n} Y_{ijk} - \hat{\mu}_{..} - \hat{\alpha}_i - \hat{\beta}_{j'} = 0$$

by $\sum_{i}^{a} \alpha_i = 0$

$$\Rightarrow \sum_{i}^{a} \sum_{j}^{b} \sum_{k}^{n} Y_{ijk} - \sum_{i}^{a} \sum_{j}^{b} \sum_{k}^{n} \hat{\mu}_{..} - \sum_{i}^{a} \sum_{j}^{b} \sum_{k}^{n} \hat{\beta}_{j'} = 0$$

$$\Rightarrow \bar{Y}_{.j.}\, an - an\hat{\mu}_{..}\ \hat{\beta}_{j'}\, an = 0$$

$$\Rightarrow \boxed{\bar{Y}_{.j'.} - \bar{Y}_{...} = \hat{\beta}_{j'}}$$

8) Consider the following models

$$Y_{i,j,k} = \mu_{i,j} + \varepsilon_{i,j,k},$$

$k=1,\dots,n,\ i=1,\dots,a,\ j=1,\dots,b$

and

$$Y_{i,j,k} = \sum_{l=1}^{a}\sum_{m=1}^{b} \beta_{l,m} X_{l,m;i,j,k} + \varepsilon_{i,j,k}$$

$k=1,\dots,n,\ i=1,\dots,a,\ j=1,\dots,b$

where $\{\varepsilon_{i,j,k}\}$ are iid $N(0,\sigma^2)$

and $X_{l,m;i,j,k} = 1$ when

$(l,m)=(i,j)$ and

$X_{l,m;i,j,k} = 0$ otherwise.

Express

$\{\beta_{l,m}: l=1,\dots,a;\ m=1,\dots,b\}$

using $\{\mu_{i,j}: i=1,\dots,a;\ j=1,\dots,b\}$

$$\mu_{i,j} + \cancel{\varepsilon_{i,j,k}} = \sum_{l=1}^{a}\sum_{m=1}^{b} \beta_{l,m} X_{l,m;i,j,k} + \cancel{\varepsilon_{i,j,k}}$$

when $(l,m) = (i,j)$

$$\mu_{i,j} = \beta_{i,j}$$

or

$$\mu_{i,j} = \beta_{l,m}$$

$SSE = \sum_{i=1}^{n} e_i^2$, $SSR = \sum_{i=1}^{n} (\hat{Y}_i - \bar{Y})^2$

9) Hint:

$\hat{\beta}_{OLS} = (X^T X)^{-1} X^T Y.$

We then have $\hat{Y} = X(X^T X)^{-1} X^T Y \equiv P_x Y$

and $e = Y - \hat{Y} = Y - X\hat{\beta} = (I - P_x) Y$

Therefore, we can write down the two sums of squares as

$SSE = e^T e = Y^T (I - P_x) Y$

$SSR = Y^T (P_x - \underbrace{P_1}_{\frac{1}{n} J_n}) Y$ where $J_n = \begin{array}{c} n \times n \\ \begin{bmatrix} 1 & \cdots \\ 1 & \ddots \end{bmatrix} \end{array}$

where $P_1 = 1(1^T 1)^{-1} 1^T = 11^T / n_T$, $1 = (1, \ldots, 1)^T_{n_T}$

$P_x, P_1, I - P_x, P_x - P_1$ are orthogonal projects

Properties of orthogonal projectors:

$P_x^T = P_x$, $P_x^T P_x = P_x$, $P_x^2 = P_x$

X - design matrix

basis of where we project onto?

$X = \begin{array}{c} n \times 2 \\ \begin{bmatrix} 1 & X_1 \\ 1 & X_2 \\ \vdots & \vdots \\ 1 & X_n \end{bmatrix} \end{array}$

projection matrix maps a vector of <u>response variables</u> to a vector of <u>fitted variable</u>

what properties / always at right angle?

Try to prove $Cov((I-P_X)Y, (P_X-P_1)Y) = 0$

We will need $P_X P_1 = P_1$ where we use the fact that the first column of design matrix X is

$(1, \dots, 1)^T_{n_T}$. $C(P) \in C(X)$

<u>Prove</u>: $P_X P_1 = P_1$

Note: P_X orthogonally projects onto $Col(X)$ span of columns of X

P_1 orthogonally projects onto $span\{(1, \dots, 1)\} \subset Col(X)$

So $P_X P_1 = P_1$. $\square$

Also, $(I-P_X)(P_X-P_1) = (I-P_X)P_X - (I-P_X)P_1$

$= (P_X - P_X) - (P_1 - \cancel{P_X P_1}) = 0$
$\quad P_1$

So $Cov((I-P_X)Y, (P_X-P_1)Y) = (I-P_X)(P_X-P_1)Cov(Y,Y) = 0$

& under normality assumption this means they are 0. $\checkmark$