SCHOOL OF ENGINEERING

VANDERBILT UNIVERSITY

CS 3250
Algorithms
Graphs: Depth-First Search

Topological Sort

Announcements

- **HW2** is due Wednesday, February 7th by 9 AM.
- There are two parts:
  1. **Brightspace Hashing Quiz**. Formative assessment. Graded but not timed.
  2. **Gradescope Questions.**
     - For the first Gradescope question, you will need to reference the graph generator quiz on Brightspace to generate your random graph for Exercise #1.
     - If you haven't started the HW yet, get going.

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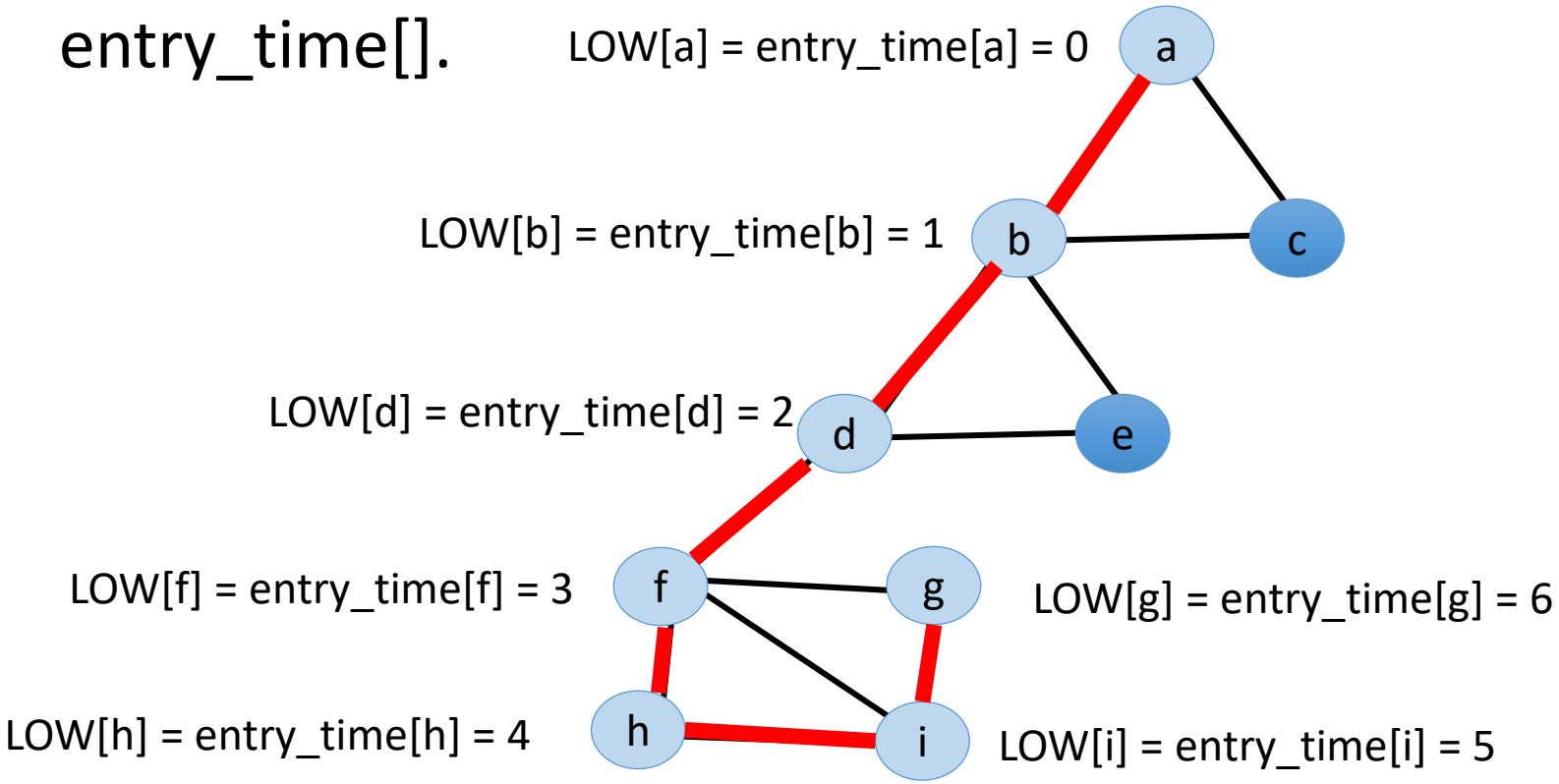
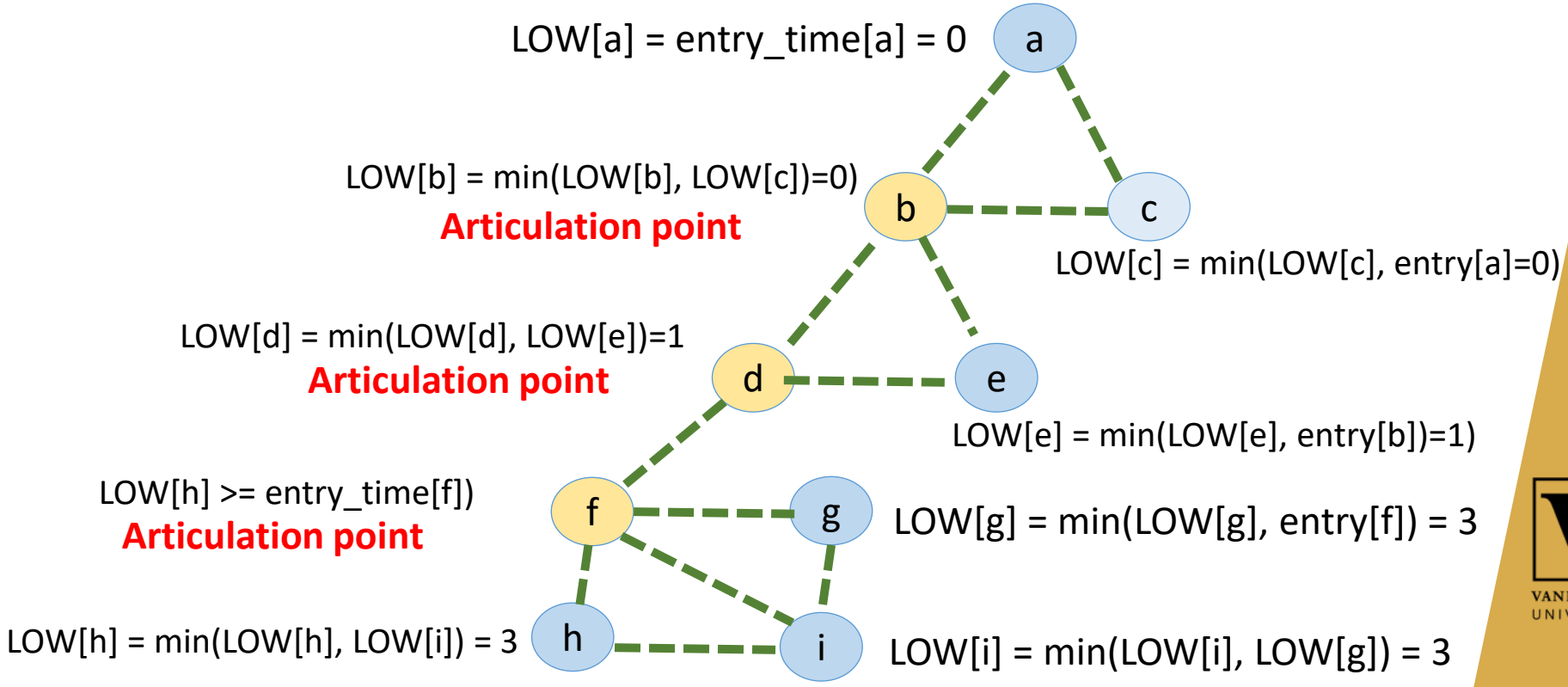
Recap: Tarjan's Articulation Point Algorithm

- **Walkthrough Step 1:** During DFS, let's pretend we discover the vertices in this order: a, b, d, f, h, i, g (I know, it's not lexicographic). As we **discover** each of those vertices, we initialize the LOW of each newly discovered vertex to its entry_time[].



Recap: Tarjan's Articulation Point Algorithm

Walkthrough Step 12: Backtrack from b and recognize that vertex a is the root of the DFS tree of discovery with only one child. Therefore vertex "a" is not an articulation point. DFS now complete and we have located all articulation points.



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Recap: Tarjan's Articulation Point Algorithm

Execute DFS(v)

When vertex v is discovered, LOW[v] = entry_time[v] = time

time = time + 1

For each of v's neighbors, neigh

Total = O(V+E)

More informative: Θ(V+E)

    if neigh is undiscovered

        execute DFS(neigh)

        LOW[v] = min of {LOW[v], LOW[neigh]} (backing up)

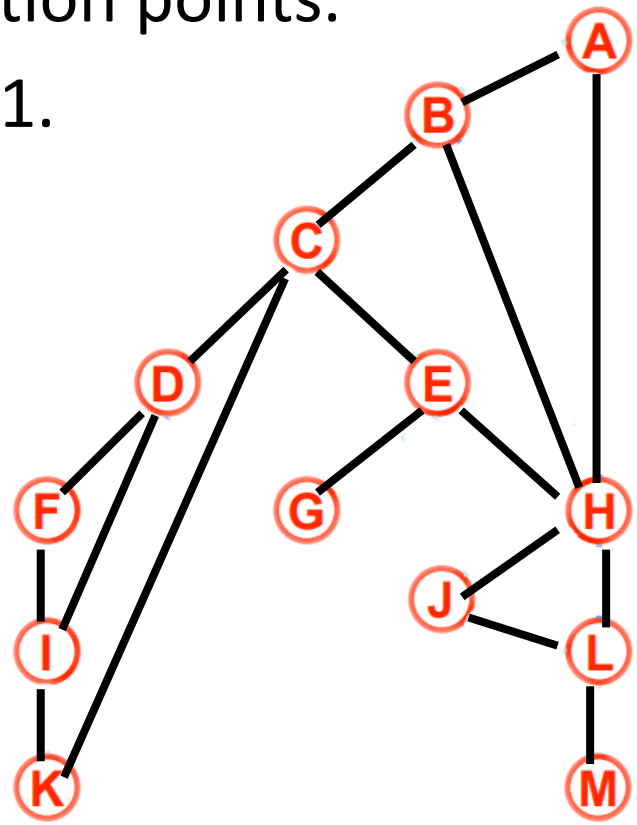
        if LOW[neigh] ≥ entry_time[v], v is articulation point

    else if (neigh is not v's parent but has been discovered)

        LOW[v] = min of {LOW[v] and entry_time[neigh]}

On Your Own: Tarjan's Algorithm

- Walk through and explain Tarjan's algorithm using the graph below starting at vertex A. When given a choice between 2 paths, choose the one that comes first lexicographically. Identify all articulation points.
- Start your clock at 1.



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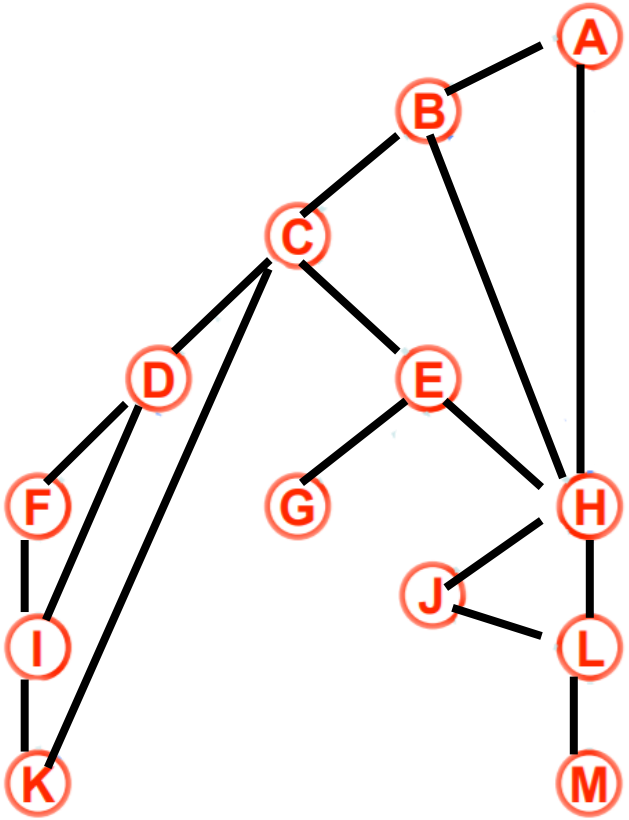
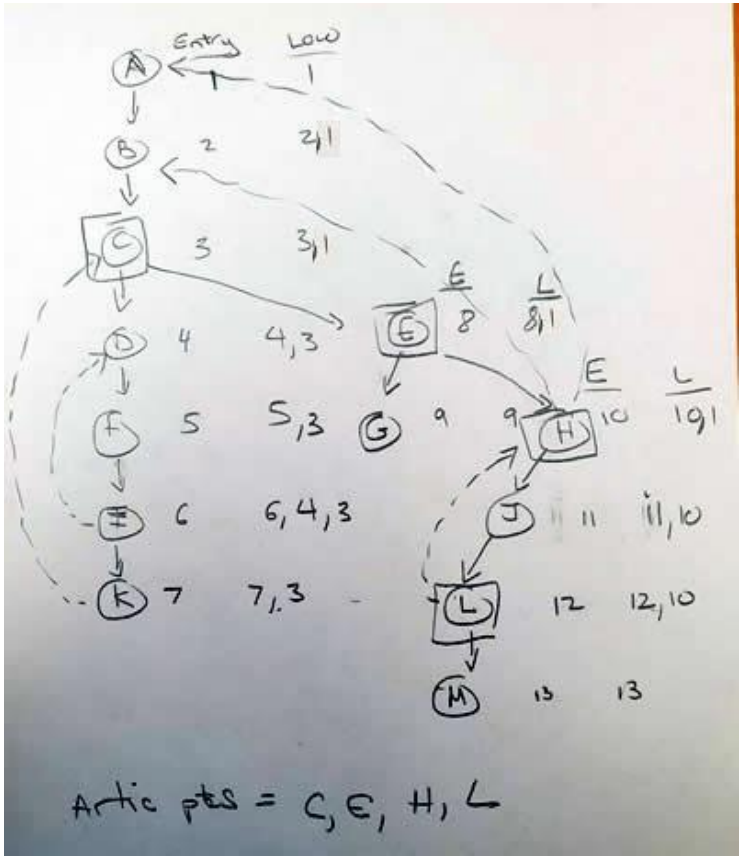
On Your Own: Tarjan's Algorithm

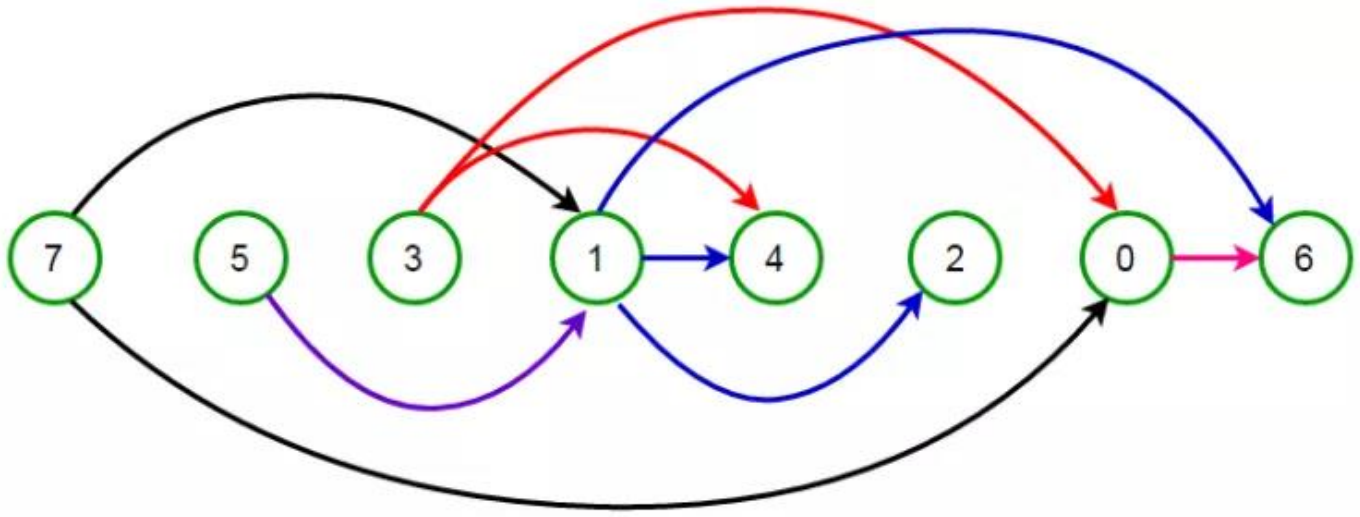
- Walk through and explain Tarjan's algorithm using the graph below starting at vertex A. When given a choice between 2 2 paths, choose the one that comes first lexicographically.





Depth-First Search: Topological Sort

- Another popular use of DFS is Topological Sort which is often used as the basis in task scheduling.
- A **topological sort** is a linear ordering of the vertices in a graph G such that for every **directed** edge uv, vertex u comes before v in the ordering.



Topological Order

Depth-First Search: Topological Sort

- **Analogy:** What is an ordering in which you can take all the CS core classes for the CS degree?
  - CS1101 → CS 2201 → CS 2212 → CS 3250 → CS 3270
- Can you think of another valid ordering?
  - CS1101 → CS 2212 → CS 2201 → CS 3250 → CS 3270
- What about this ordering?
  - CS2201 → CS 2212 → CS 1101 → CS 3250 → CS 3270

Depth-First Search: Topological Sort

- **Which of the following is true about a graph that contains a topological ordering?**

A. There must be a vertex with no incoming edges.

B. The graph must be sparse.

C. The graph must be a DAG.

D. Both A and C

E. Both A and B

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Depth-First Search: Topological Sort

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Depth-First Search: Topological Sort

- **Is it true that every DAG must have at least one valid topological ordering?**

A. No. It's only true that if there is a topological ordering the graph is a DAG. It's not necessarily true that every DAG has a topological order.

B. Yes. Every topological ordering must be a DAG and every DAG has a topological order.

C. Maybe. Some topological orderings might be on an undirected graph.

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Depth-First Search: Topological Sort

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Topological Sort: Observations

- Every DAG has a topological ordering.
- **Think about it…**
- This must be true. After all, let's say you were trying to complete the CS core requirements and Data Structures was a pre-requisite to Discrete Structures which was a pre-requisite to Algorithms.
- What happens if the CS department passes a new requirement that you cannot not take Data Structures unless you first complete Algorithms?

Topological Sort: Observations

Some observations:

1. A graph has a topological ordering IFF the graph is a DAG (there cannot be any cycles in the graph).
2. There must be at least one "source vertex" with **no incoming edges**
3. There must be at least one one "sink vertex" with **no outgoing edges**.
4. There can be more than one valid topological ordering for a given graph G.

Topological Sort: Observations

- **Claim:** There must be at least one "sink vertex" with no outgoing edges in any DAG.
- **Proof (BWOC):** Suppose there isn't. This means in an arbitrary graph with n vertices, I can pick any arbitrary vertex B and begin exploring from it (I must be able to explore from it since it's not a sink vertex). Let's follow an edge from vertex B to another vertex C. C can't be a sink vertex either so let's explore again to another vertex D. I can continue this process for all n vertices. But if I follow n arcs, I will have seen n+1 vertices. This means via the **pigeon-hole principle** I must have seen some vertex twice, which in turn means this graph has a cycle and is not a DAG. In other words, there must be at least one sink vertex in any DAG.■

Topological Sort: Observations

Proof (Inductive): Every DAG has a topological ordering

- **Base case:** n = 2; n is number of vertices in Graph G. Since the edge is directed, the graph must have a topological ordering from target vertex to source vertex.
- **Inductive hypothesis:** Assume our base case holds for k vertices. In other words, a DAG on k vertices has a topological ordering. We will prove that the hypothesis holds for k+1 vertices. That is, a DAG G on k+1 nodes has a topological ordering.
- Given an arbitrary graph G with k+1 vertices that is a DAG, find an arbitrary vertex v in G where v is a node with no outgoing edges. Remove v to create G', a graph on k vertices.

Topological Sort: Observations

Proof (con't): Every DAG has a topological ordering

- It must be the case that G' is a DAG as well. After all, if you have a DAG and you **remove** a vertex, you cannot create a cycle. So, G' is a DAG on k vertices and it has a topological ordering by the inductive hypothesis
- We can concatenate Topological ordering (G') + v to get a topological ordering of G.
- QED

That’s All For Now…

- Coming to a Slideshow Near You Soon…
  1. Kosaraju’s Algorithm
  2. Minimal Spanning Trees

That’s All For Now