SCHOOL OF ENGINEERING

VANDERBILT UNIVERSITY

CS 3250
Algorithms
Graphs: Breadth-First Search
Lecture 6

Tree of Discovery & Bipartite Graphs

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Announcements

- **HW1 Grading**. Expect a 7–10 day turnaround.
- **HW2** will be released soon and due Wednesday, February 7th by 9 AM. It has two parts:
  1. **Brightspace Hashing Quiz**. Formative assessment. Graded but not timed.
  2. **Gradescope Written Questions.** Remember to keep your Gradescope answers:
     - Clear
     - Correct
     - Concise

Review: Breadth-First Search

- **When to use BFS:** BFS is a useful traversal method when you know the solution isn't far from the source vertex (e.g., friend suggestions on Facebook that are one-degree away from you).
- **Where BFS Shines:** Social-Networks, Shortest Path (unweighted), Spanning Tree (undirected).

Breadth-First Search: Pseudocode

```
BFS(G, s) [Initially all vertices undiscovered]
  Set start vertex s discovered and set parent to nil,
  Add s to the TO-DO list (enqueue)
  WHILE (there are vertices on the TO-DO list)
      v = take a vertex off the To-Do list (dequeue)
      Optional: Do early vertex processing
      FOR (all of v's neighbors w)
            IF (state[w] == UNDISCOVERED/white) THEN
                  Change status of vertex w to DISCOVERED/gray)
                  Add w to the TO-DO list (enqueue)
                  Optional: Record the level where w was discovered
                  Optional: Record the parent/predecessor of w, which is v
            Move to the next neighbor w of v
      LOOP
  Change status of vertex v to PROCESSED/black
  Optional: Do late vertex processing
  LOOP
END BFS
```


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Breadth-First Search: Manager Loop

BFSManager(G)

```
//housekeeping
FOR each vertex u in V(G)
  state[u] = undiscovered
  parent[u] = nil
 time = 0
END FOR
//catch all vertices
FOR each vertex u in V(G)
   IF (state[u] != discovered) THEN
        BFS(G, u)
END FOR
```

The manager loop for BFS ensures that if the graph is not connected, we still discover all vertices

Breadth-First Search: Detecting Cycles

- **Question:** Would our logic for detecting a cycle also work for determining cycles in a **directed** graph?
  A. Yes. We are experienced programmers.
  B. No. We are experienced programmers.
  C. I thought so, but now I'm not sure. I just can't figure out why.

Breadth-First Search: Detecting Cycles

- **Question:** Would this logic also work for determining cycles in a **directed** graph?
- **Answer:** No, it would not work to determine cycles in a directed graph. For a path to be a cycle you must be able to get back to the first vertex. The fact that a vertex has already been discovered by a someone else, does not mean you can get from that vertex back to whoever discovered it. The edge might only be directed toward the vertex and not back.

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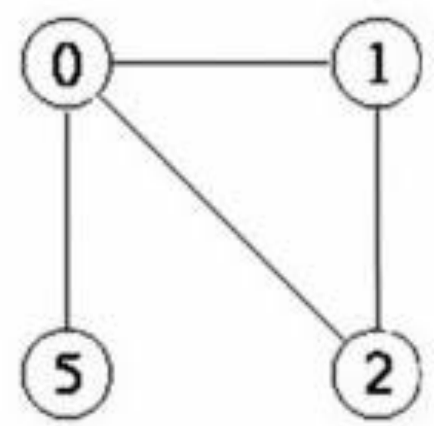
Review: Breadth-First Search

- **Question:** A BFS beginning at vertex 0 where ties are broken by the smallest numbered vertex will realize there's a cycle in the graph below when...
  A. vertex 2 tries to reach vertex 0.
  B. vertex 1 tries to reach vertex 2.
  C. vertex 1 tries to reach vertex 0.
  D. vertex 2 tries to reach vertex 1.
  E. None of the above



Review: Breadth-First Search

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  A. vertex 2 tries to reach vertex 0.
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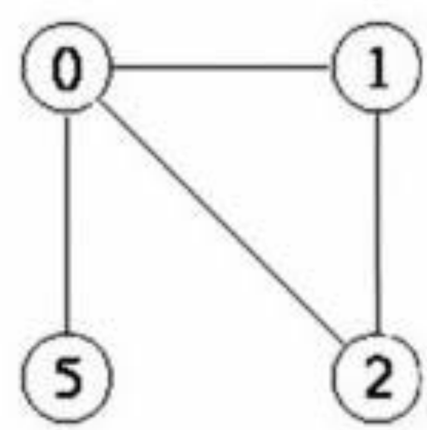
  C. vertex 1 tries to reach vertex 0.
  D. vertex 2 tries to reach vertex 1.
  E. None of the above



Breadth-First Search: Algorithm Analysis

- "optional" tasks have been removed since work can vary.

```
BFS(G, s) [Initially all vertices undiscovered]      ← O(V)
  Set start vertex s discovered and set parent to nil ← O(1)
  Add s to the TO-DO list (enqueue)                   ← O(1)
```

Pre-Work Housekeeping O(V)

```
  WHILE (there are vertices on the TO-DO list)        ← O(V)
     v = take a vertex off the To-Do list (dequeue)
     FOR (all of v's neighbors w)                     ← O(E)
           IF (state[w] == undiscovered) THEN
                 Change state[w] to discovered
                 Add w to the TO-DO list (enqueue)
           Move to the next neighbor w of v
     LOOP
  Change state[v] to explored/processed
  LOOP
  END BFS
```

O(V*E) ???

Breadth-First Search: Algorithm Analysis

- When doing analysis, it's important to THINK about how the algorithm works rather than just looking at the code.
- If you just look at the code, you might conclude $O(V*E)$ or V^2*E or even V^3 is a good bound. Yes, those are correct bounds, but they are sloppy.
- If we think about how the algorithm works, we can arrive at a tighter bound.

Breadth-First Search: Algorithm Analysis

- A "smarter" analysis yields the following:

```
BFS(G, s) [Initially all vertices undiscovered]      ← O(V)
  Set start vertex s discovered and set parent to nil ← O(1)
  Add s to the TO-DO list (enqueue)                   ← O(1)
```

Pre-Work Housekeeping

O(V)

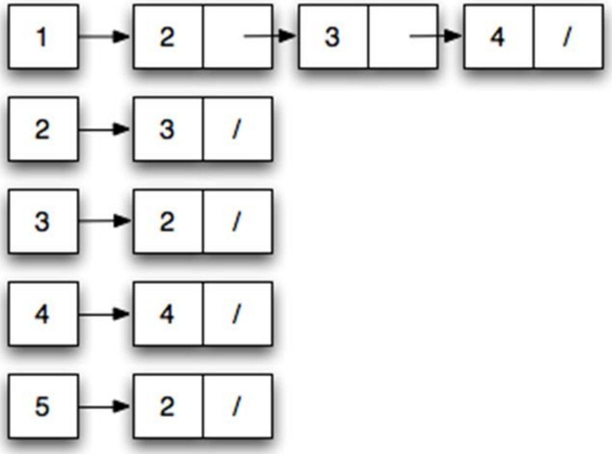
```
  WHILE (there are vertices on the TO-DO list)
     v = take a vertex off the To-Do list (dequeue)
     FOR (all of v's neighbors w) ← (*every edge looked at 1x or 2x)
           IF (state[w] == undiscovered) THEN
                 Change state[w] to discovered
                 Add w to the TO-DO list (enqueue)
           Move to the next neighbor w of v
     LOOP
  Change state[v] to explored/processed
  LOOP
END BFS
```

O(V+E)

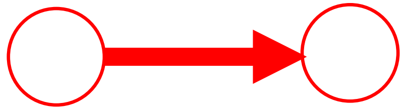
O(V + E)

Breadth-First Search: Data Structure Choice

- Let’s think visually…
- We will definitely go down the list of all vertices. No way to avoid that. That’s |V| work.
- We will definitely need to look at all the edges. No way to avoid that. That’s |E| work.
- In an undirected graph, we’ll see 2x the number of edges. That’s 2|E| work.



Breadth-First Search: Analysis Summary

- **Directed graph** 
  - Set up + Traverse each vertex/edge once
  - O(V) + E
- **Undirected graph** 
  - Set up + Traverse each edge twice
  - O(V) + 2E

Breadth-First Search: Analysis Summary

- **Overall Running Time**
  - $\Theta(V + E)$. Most authors simply write $O(V+E)$. Some say $O(\max(V, E))$.
  - Note that this could approach $O(V^2)$ if the graph is dense (has lots of edges).
  - If the graph is not dense, work is less than $O(V^2)$

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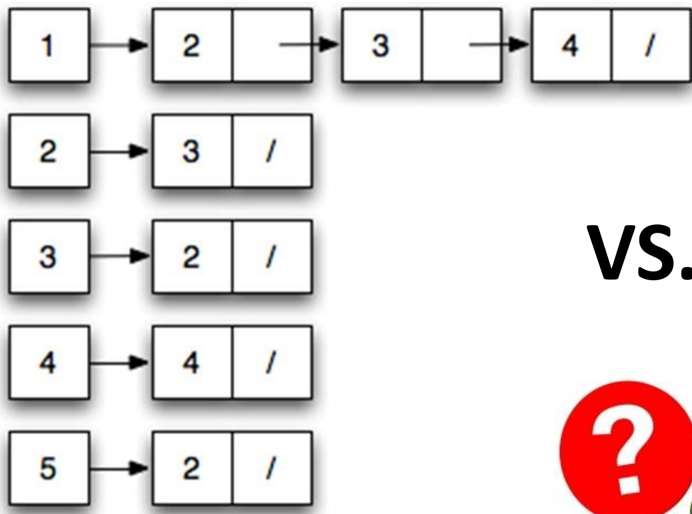
Breadth-First Search: Data Structure

- **When Choosing Your Data Structure:**

1. **Be efficient**. Always choose the most efficient data structure for the algorithm.
2. **Be obvious.** If your data structure is not obvious, you should explicitly state your data structure.
3. **Breadth-First Search.** Our BFS analysis was based on an adjacency list data structure.

Breadth-First Search: Data Structure Choice

- **Question:** What if we used an **adjacency matrix** instead of an adjacency list as our data structure? How does that effect our BFS analysis?



VS.

	1	2	3	4	5
1	0	1	1	1	0
2	0	0	1	0	0
3	0	1	0	0	0
4	0	0	0	0	0
5	0	1	0	0	0

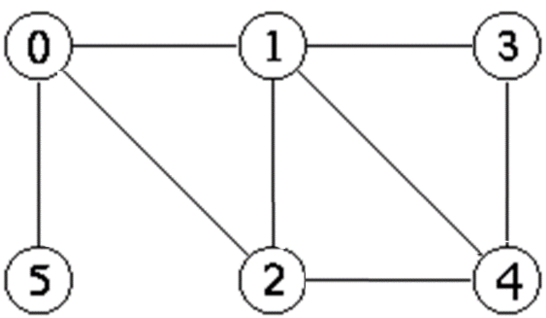
Breadth-First Search: Data Structure Choice

- **Question:** What if we used an **adjacency matrix** instead of an adjacency list as our data structure? How does that effect our BFS analysis?
- **Answer:** For each vertex visited, every other vertex needs to be checked for a "1" (in the same row). So, each vertex visited results in V checks. That means V checks for each vertex V which is O(V^2).

	1	2	3	4	5
1	0	1	1	1	0
2	0	0	1	0	0
3	0	1	0	0	0
4	0	0	0	0	0
5	0	1	0	0	0

Breadth-First Search: Tree of Discovery

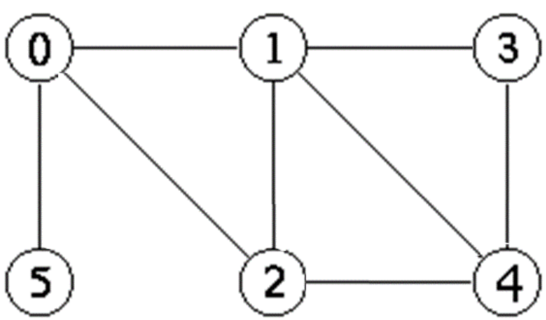
- While traversing a graph, a **spanning tree** is naturally formed known as the “**Tree of Discovery**” or **“Traversal Tree.”**
- I’ll use a black arrow to indicate who discovered a vertex first and a red dashed line from u to v to indicate u stumbled onto an already discovered, but unprocessed vertex, v (i.e., via a new edge).



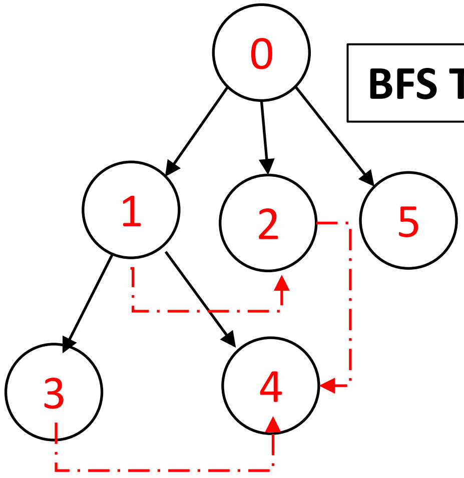
Initial Graph

Breadth-First Search: Tree of Discovery

- Here is the BFS "**Tree of Discovery**" or **"Traversal Tree"** for our undirected graph.
- A black arrow indicates who discovered a vertex first and a red dashed line from u to v indicates u stumbled onto an already discovered, but unprocessed vertex, v (i.e., via a new edge).
- Notice the Tree of Discovery uses arrows.



Initial Graph

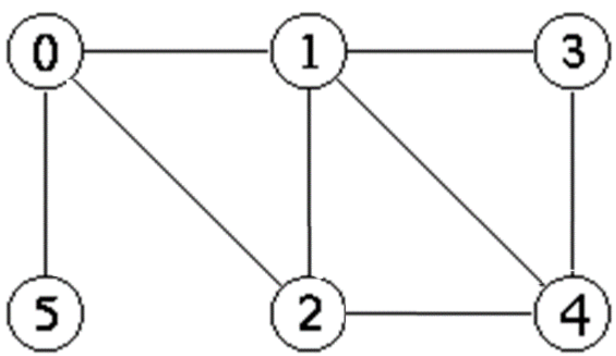


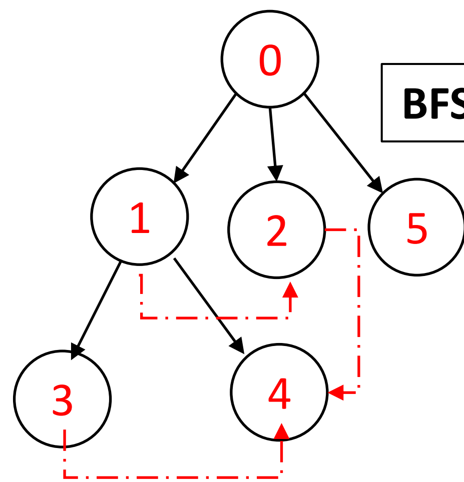
BFS Tree of Discovery

Breadth-First Search: Tree of Discovery

- **Fun Fact:** The BFS tree of discovery for an **unweighted** graph contains the shortest distance from a vertex to any other vertex from the graph.
- **Question:** How to find this shortest path?
- **Answer:** Keep track of the parent

Initial Graph

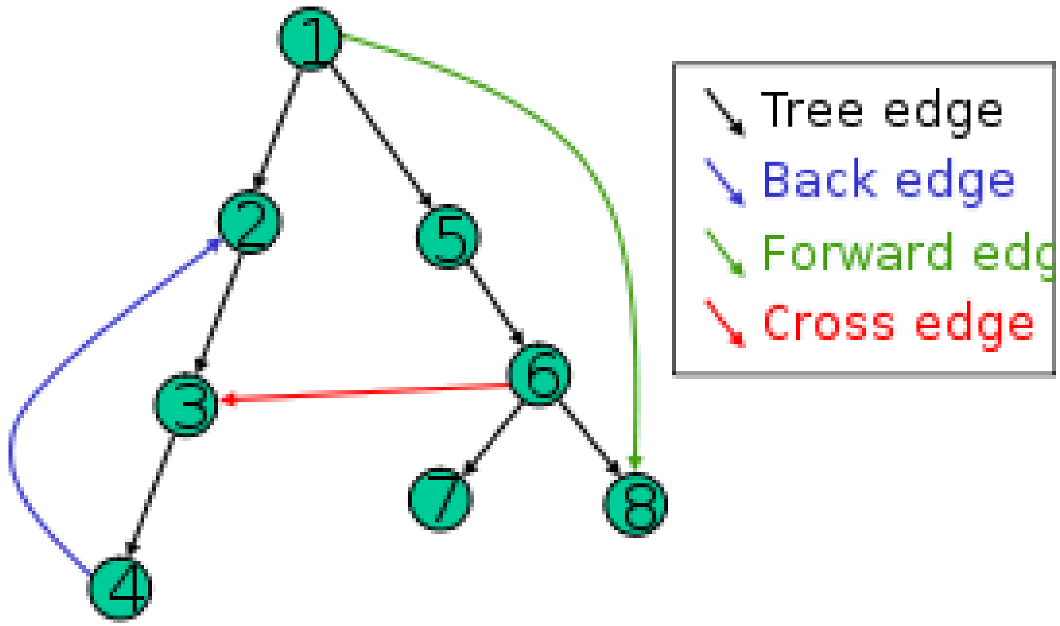




BFS Traversal Tree

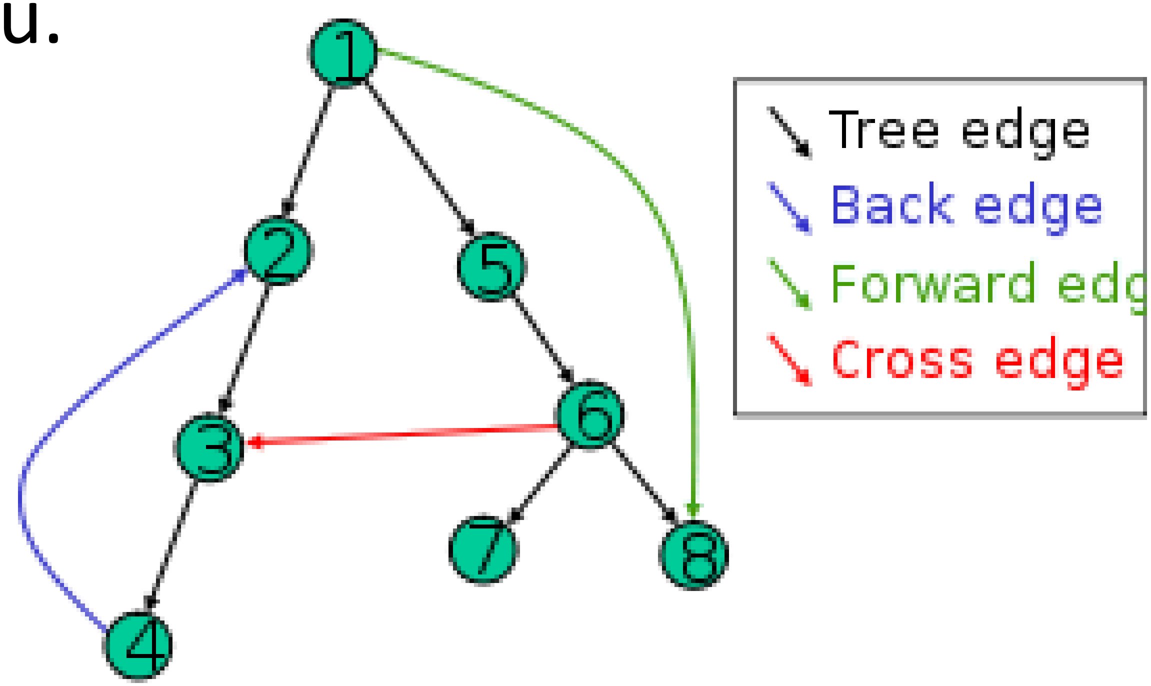
Breadth-First Search: Tree of Discovery

- The tree of discovery in a graph traversal has some useful properties. We need some additional terminology to talk about them:
  - A **tree edge** is an edge present in the tree of discovery after applying BFS to the graph.



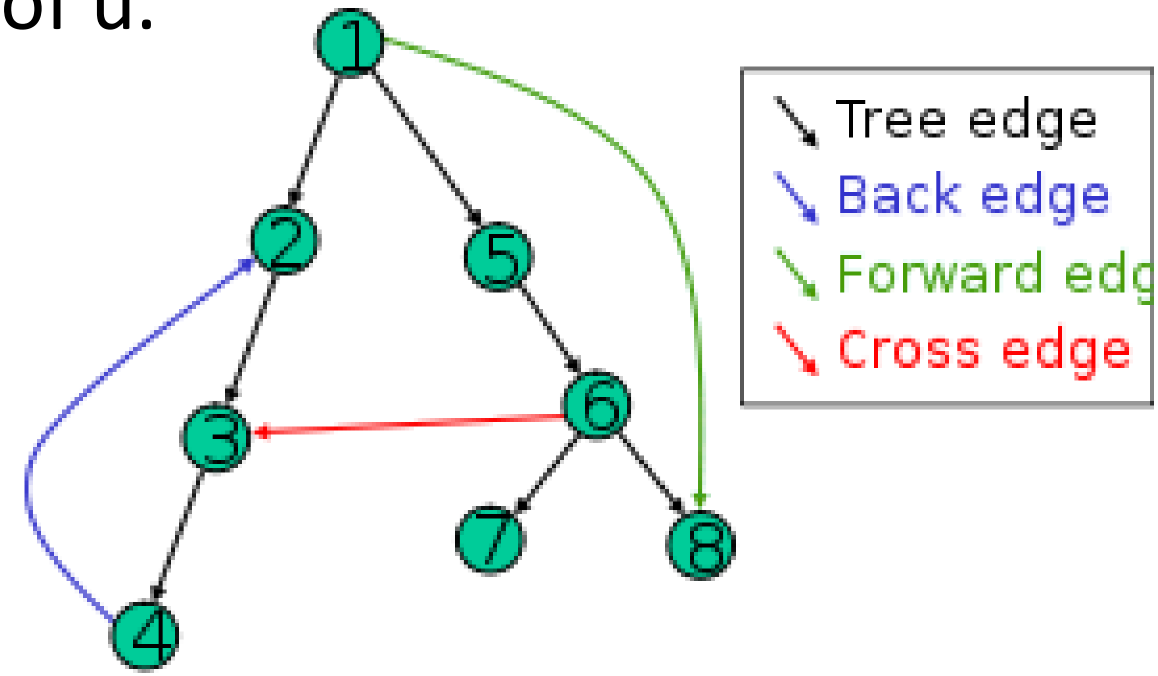
Breadth-First Search: Tree of Discovery

- The tree of discovery in a graph traversal has some useful properties. We need some terminology to talk about them:
  - A **back edge** is an edge (u, v) such that v is an ancestor of u.



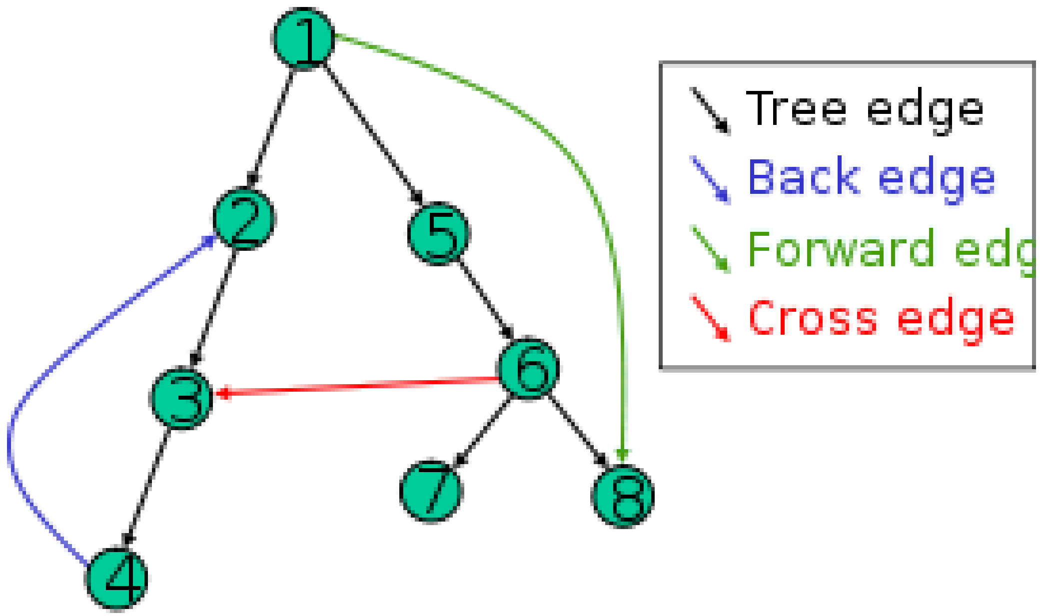
Breadth-First Search: Tree of Discovery

- The tree of discovery in a graph traversal has some useful properties. We need some terminology to talk about them:
  - A **forward edge** is an edge (u, v) such that v is a descendant of u.



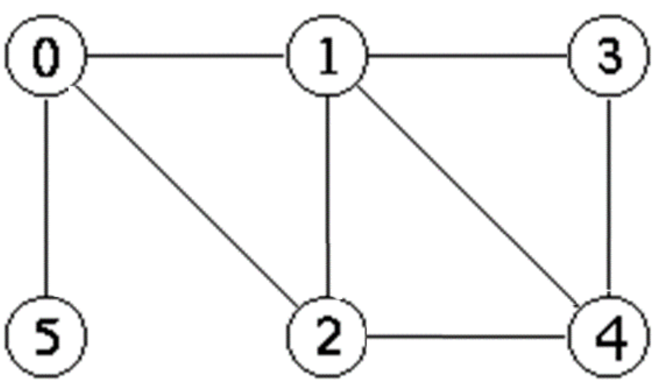
Breadth-First Search: Tree of Discovery

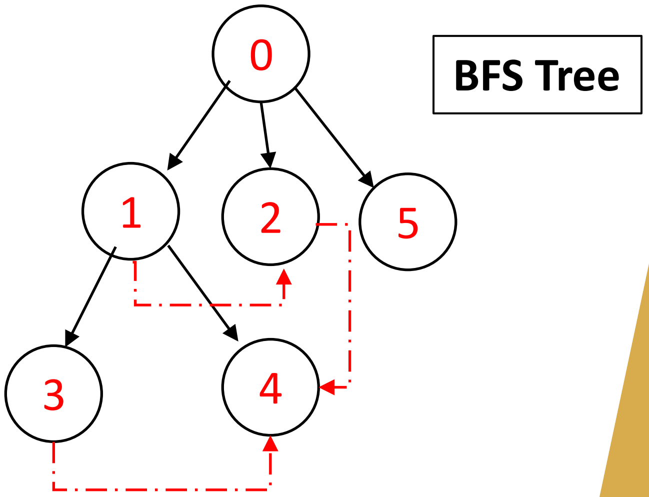
- The tree of discovery in a graph traversal has some useful properties. We need some terminology to talk about them:
  - A **cross edge** is an edge (u, v) connecting two nodes that do not have any ancestor/descendant relationship.



Graphs: Breadth-First Search Trees

- What kinds of edges exist in our BFS tree below?
  - **Tree edges** – (0, 1), (0, 2), (1, 4), etc. are tree edges formed during the natural traversal of BFS.
  - **What about the dashed lines?**
    - The edges (1, 2), (3, 4) and (2, 4) are **cross edges**





Graphs: Breadth-First Search Trees

- Aside from tree edges, what other kind of edges are possible in any **undirected** BFS tree of discovery?
  - ❑ Cross Edges only
  - ❑ Back Edges only
  - ❑ Forward Edges only
  - ❑ A combination of all the above

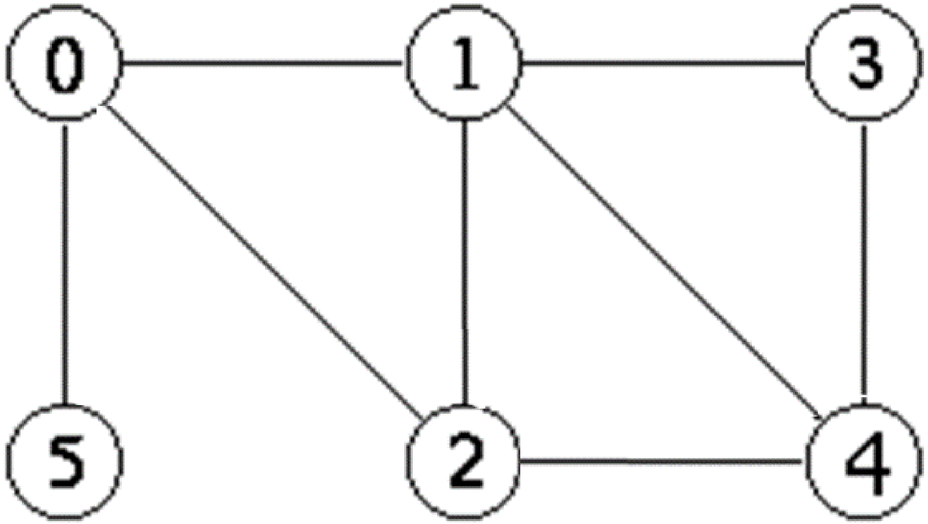
Graphs: Breadth-First Search Trees

- Aside from tree edges, what other kind of edges are possible in any undirected BFS tree of discovery?
  - ❑ **Cross Edges only**
  - ❑ Back Edges only
  - ❑ Forward Edges only
  - ❑ A combination of the above

- This is a subtle observation. Let’s talk about why it’s true.

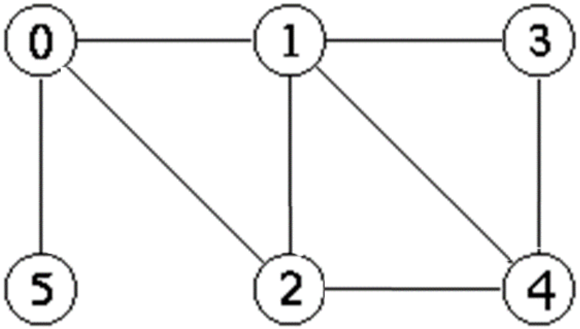
Graphs: Breadth-First Search Trees

- **Fun fact:** If you perform a BFS of our **undirected** graph G, **all edges** are either **tree edges** or **cross-edges**.
- Why aren't other kinds of edges possible?



Graphs: Breadth-First Search Trees

- **Fun Fact:** In a BFS of an **undirected** graph, all edges are **tree edges** or **cross-edges**.
- **How to think about it:** Consider our graph below. In order to have a backedge from 2 to 0, it would have to be the case that 0 would have discovered 1, and 1 discovered 2, and then 2 attempts to rediscover 0. In BFS because we explore all neighbors immediately. 0 always discovers 2 via a tree edge.



That’s All For Now…

- Coming to a Slideshow Near You Soon…
  1. Depth-First-Search
  2. Topological Sort

That’s All For Now