SCHOOL OF ENGINEERING

VANDERBILT UNIVERSITY

CS 3250
Algorithms
Lecture 5

Graphs: BFS & Detecting Cycles

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Announcements

- **HW1B Due** Tuesday, September 12th by 9AM.
  - **Brightspace Quiz on Asymptotic Notation** – It’s tricky by design. Be sure to think before answering. You get (2) tries at it. Your highest score is recorded.
  - **Gradescope Written HW** – A few written questions on the topic of asymptotic analysis. Submitted to Gradescope as a PDF.

Data Structures for Graphs

1. Adjacency Matrix –

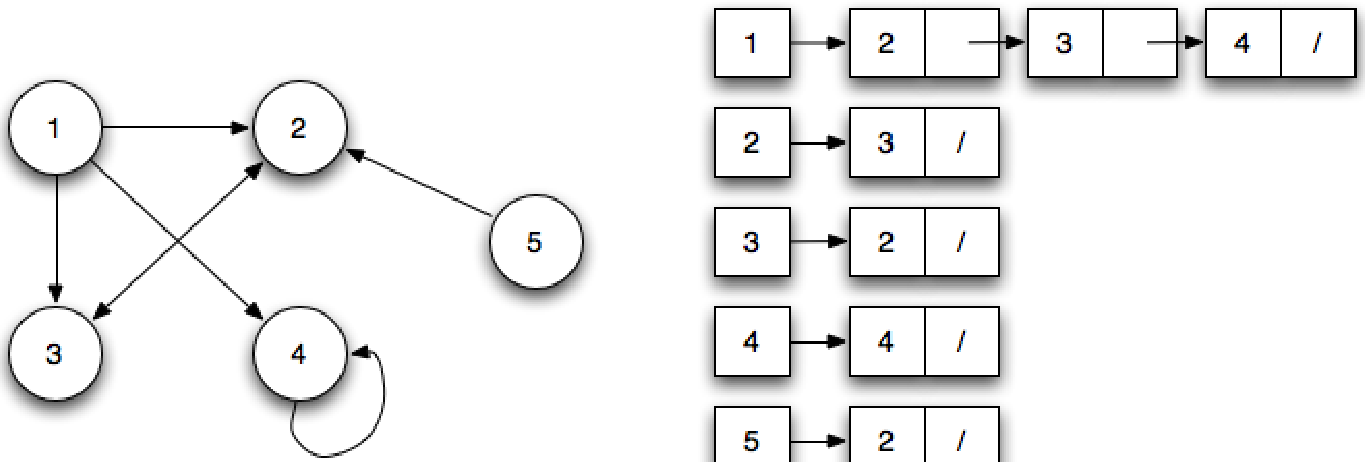
- **Space:** An array of n x n is needed $\Theta(n^2)$ regardless of a sparse or dense graph in terms of number of edges.
- **Speed:** Depends on the question being asked.
    - Is there an edge from vertex 4 to vertex 8? Fast. $\Theta(1)$
    - List all the neighbors of every vertex? Slow. $\Theta(n^2)$
    - Delete vertex 4? Slow. $O(n^2)$

	1	2	3	4	5
1	0	1	1	1	0
2	0	0	1	0	0
3	0	1	0	0	0
4	0	0	0	0	0
5	0	1	0	0	0

Data Structures for Graphs

There are two main data structures used to represent graph G = (V, E) which contains n vertices and m edges.

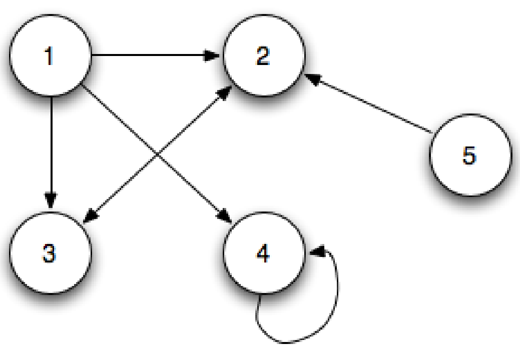
2. **Adjacency List** - An adjacency list consists of an array of |V| pointers, where the i^{th} element points to a linked list of all the edges incident to vertex i. For a **weighted graph** store the edge weight as an additional field in each edge node in the adjacency list.

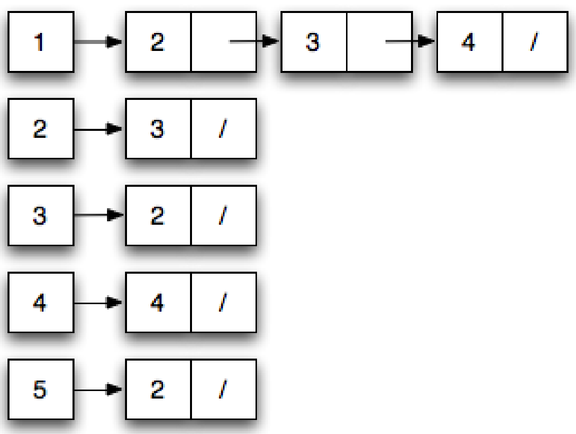


Data Structures for Graphs

2. Adjacency List –

- **Space:** $O(|V| + |E|) = O(V + E) = O(\max(V, E))$
  - Cardinality symbol usually not included.
- **Speed:** Depends on the question being asked.
  - Is there an edge from vertex 4 to vertex 8? $O(V)$ since a vertex might be adjacent to all other vertices.
  - List all the neighbors of every vertex? $\Theta(V + E)$





Adjacency Matrix vs. Adjacency List

Task	Winner
Does edge (x, y) exist?	Adjacency Matrix
What is the degree of vertex 4?	Adjacency List
Add/Delete an edge	Adjacency Matrix
Traverse the graph	Adjacency List
Better for most problems	Adjacency List

Graphs: Breadth-First Search

- Once you construct a graph, the next question becomes to do with the graph?
- Many questions are answered by exploring the graph.
- This idea of exploring a graph via its data structure is better known as a **traversal**.

Graphs: Breadth-First Search

- When traversing a graph, it's important to ensure the following:
  1. **Efficiency** – We don't waste time visiting places we've already explored.
  2. **Accuracy/Correctness** – We don't miss any places along the way (i.e., leave no stone unturned).

Graphs: Breadth-First Search

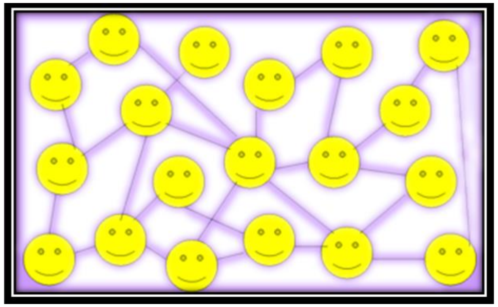
- One of the simplest graph traversals that can be used on a directed or undirected graph is known as **Breadth-First Search (BFS).** You saw it in 2201.
- **Analogy:** Imagine yourself as an explorer.
  - You **gradually** expand the frontier between discovered and undiscovered lands.
  - "lands" we discover and explore are the vertices
  - "paths" we take are the edges in the graph.

Graphs: Breadth-First Search

- Nobody (including your tech interviewer) will ever walk up to you and say, “I’ll give you a bag of cash if you can write a breadth-first search for me.”
- **Breadth-first search** shows up in numerous real-world applications (and job interview questions).
  - A user will describe a problem to you.
  - Your job is to recognize BFS is the right tool.

Graphs: Breadth-First Search in Action

- **Breadth-First Search in Action**
  - Shortest path in **unweighted** graph
  - Cycle detection in undirected graph
  - Connected components
  - Social Networks – Who are Kristin's friends?
  - GPS Navigation Systems – What interesting places are near my current location?
  - Search – How many sites link to this site?



Breadth-First Search: How It Works

- The key to any graph traversal is keeping track of where you are.
- **Question:** Which of the following do you think is **not** a "state" that needs to be remembered during a BFS traversal?
  - Undiscovered
  - Discovered
  - Soon to be processed
  - Processed

Breadth-First Search: How It Works

- **Question:** Which of the following do you think is **not** a "state" that needs to be remembered during a BFS traversal?
  - Undiscovered
  - Discovered
  - **Soon to be processed**
  - Processed
- We don't need to worry about soon be processed.

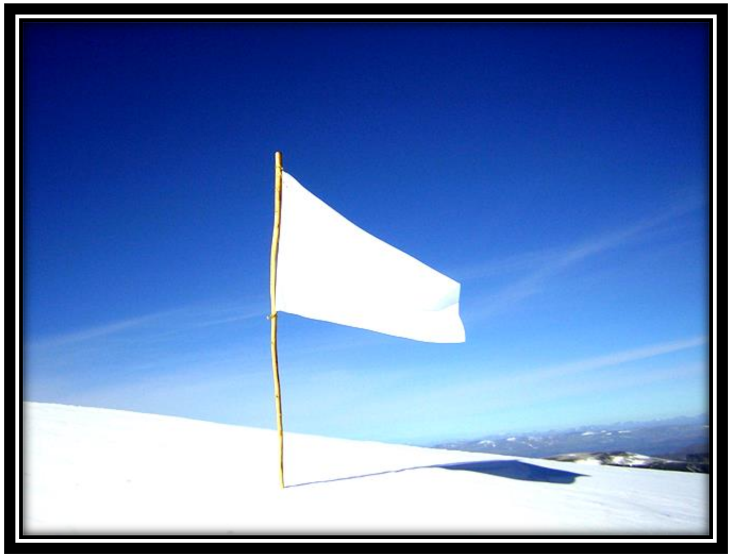
Breadth-First Search: How It Works

- The key to any graph traversal is:
  1. Marking each vertex when first **discovered**
  2. Knowing when each vertex is **finished/processed**.
- We can also leave a breadcrumb to remember our "parent" (the person who discovered us). This is optional but can provide useful information.

Breadth-First Search: How It Works

- We will label each vertex to be in one of three states (or colors).

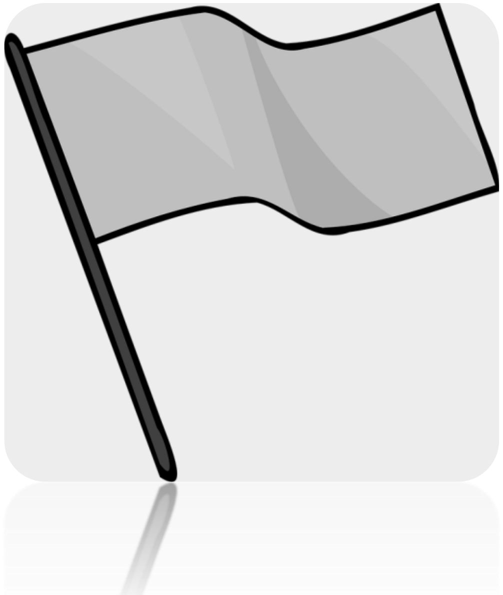
1. **Undiscovered (white)** – I haven't yet discovered this vertex. It is unchartered territory (initially all vertices are undiscovered).



Breadth-First Search: How It Works

- We will label each vertex to be in one of three states (or colors).

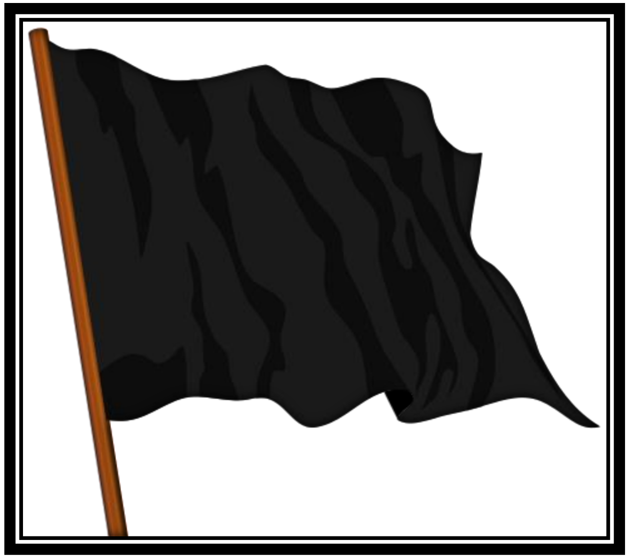
2. **Discovered but unexplored (gray)** – I have discovered this vertex and planted my gray flag here. However, I haven't thoroughly explored the area yet.



Breadth-First Search: How It Works

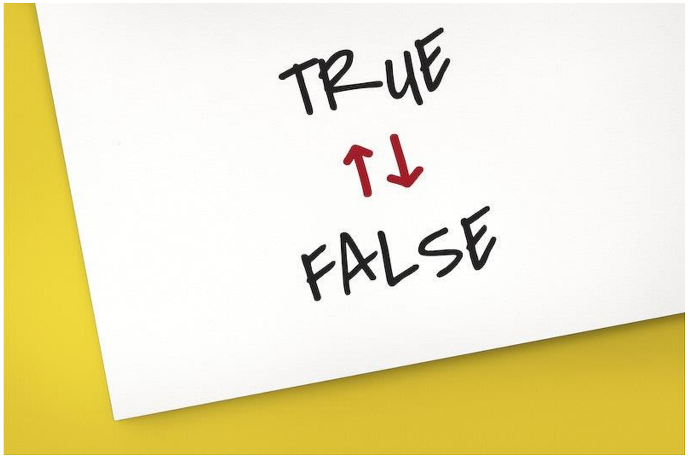
- We will label each vertex to be in one of three states (or colors).

3. **Processed/Explored (black)** – This vertex has not only been discovered, but I have also explored every aspect of this vertex, so I have finished processing it.



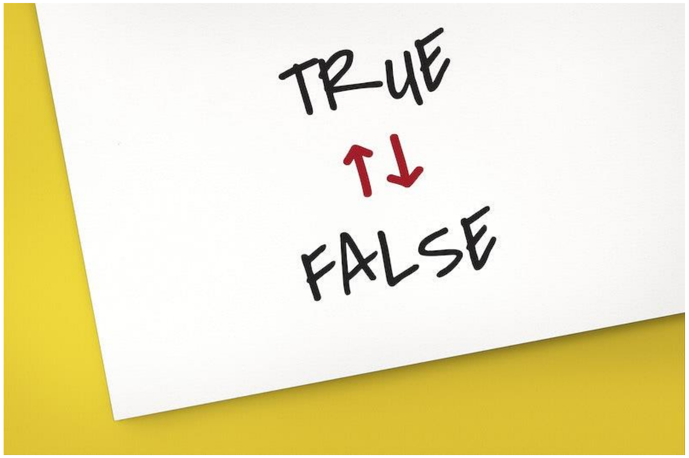
Breadth-First Search:

- **Question:** True/False. Every graph has a unique BFS traversal.



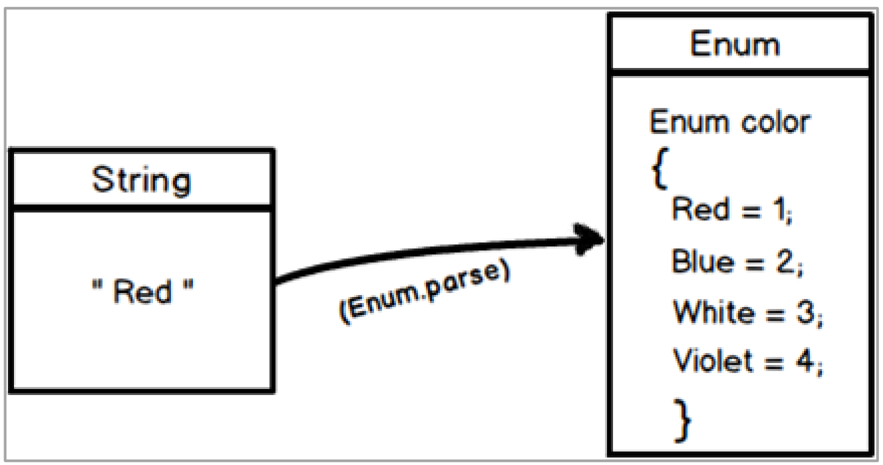
Breadth-First Search:

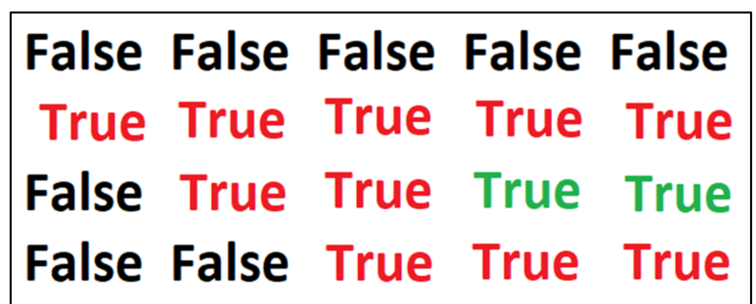
- **Question:** True/False. Every graph has a unique BFS traversal.
- **Answer:** False. It's quite possible to have multiple BFS traversals. That's why we often say on an exam or HW "when given a choice, choose the vertex that comes first alphabetically."



Breadth-First Search: Algorithm Details

- How should we implement these vertex states?
  - Some programmers use Enum types or Constants.
  - Some programmers use boolean arrays.
- The details do not matter provided they do not change the overall algorithm efficiency.





Breadth-First Search: Pseudocode

```
BFS(G, s) [Initially all vertices undiscovered]
  Set start vertex s discovered and set parent to nil,
  Add s to the TO-DO list (enqueue)
  WHILE (there are vertices on the TO-DO list)
     v = take a vertex off the To-Do list (dequeue)
     Optional: Do early vertex processing
     FOR (all of v's neighbors w)
           IF (state[w] == UNDISCOVERED/white) THEN
                 Change status of vertex w to DISCOVERED/gray)
                 Add w to the TO-DO list (enqueue)
                 Optional: Record the level where w was discovered
                 Optional: Record the parent/predecessor of w, which is v
           Move to the next neighbor w of v
     LOOP
  Change status of vertex v to PROCESSED/black
  Optional: Do late vertex processing
  LOOP
END BFS
```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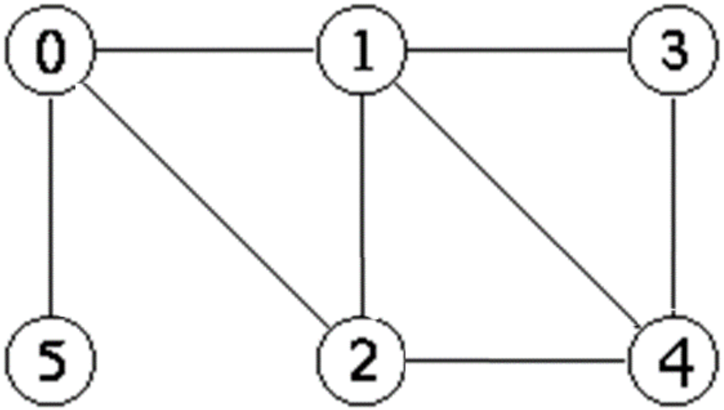
Example: Breadth-First Search

- **Example:** Perform a BFS on the undirected graph below starting at vertex 0. When given a choice between two vertices, choose the one that comes first numerically.



TO DO LIST

STATE

PROCESSED:

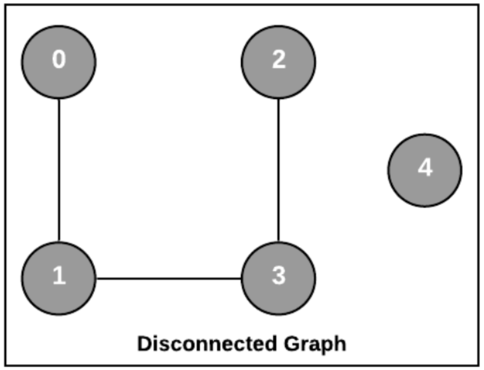
 This Photo courtesy of Zybooks

Breadth-First Search:

- **Question:** Does our breadth-first search algorithm work correctly for any undirected graph G?
  A. Yes of course. We’re experienced programmers.
  B. Definitely not. We’re experienced programmers.
  C. I thought it did, but now I'm not sure.

Breadth-First Search:

- **Answer: Definitely not**. Our current BFS algorithm does not work correctly for all types of undirected graphs. Consider the disconnected graph G shown below and a BFS traversal that starts at vertex 0. Our current algorithm would never find vertex 4.
- We need to add a controlling loop to ensure we hit all vertices in the graph.

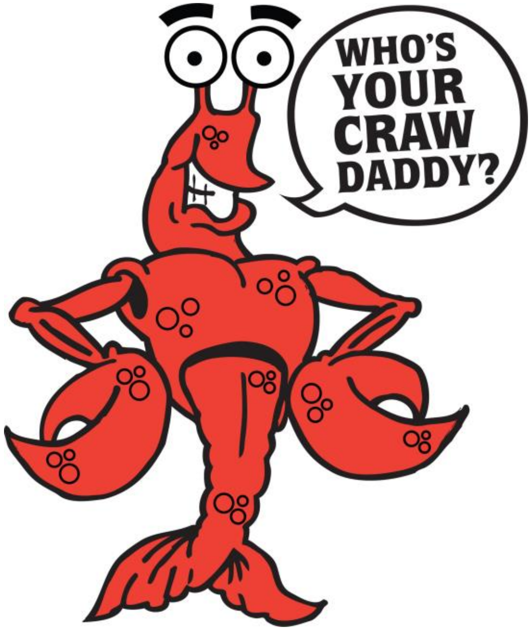


Breadth-First Search: Optional Tasks

- You may decide to perform additional work during your graph traversal depending on your needs.
- This is fine provided you do not sacrifice overall algorithm efficiency.
- Here are some tasks commonly performed with BFS.

Breadth-First Search: Optional Tasks

- **Task: Who's Your Daddy?**
  - **What:** Who discovered us (i.e., our "parent")?
  - **When:** Helpful if you need to report the shortest path from one vertex to another.



Breadth-First Search: Optional Tasks

- **Task: Level Up!**
  - **What:** Remember every vertex's distance from the starting vertex (e.g., "six degrees of separation").
  - **When:** Helpful if you need to determine the shortest path from one vertex to another.

Breadth-First Search: Optional Details

Placeholders for tasks in our BFS algorithm:

1. **processVertexEarly(v)** – Optional method you can include if you need to do pre-work before exploring a vertex.
2. **processVertexLate(v)** – Optional method you can include if you need to do post-work after exploring a vertex.
3. **processEdge (v, w)** – Optional method you can include if you need to do any work while processing an edge.

Thinking About Breadth-First Search

Now, let's THINK about the algorithm as an algorist might.

- **Question:** Given an undirected graph, G, how many times can we discover a particular vertex?

Thinking About Breadth-First Search

Let’s THINK about the algorithm as an algorist might.

- **Question:** Given an undirected graph, G, how many times can we discover a particular vertex?
- **Answer: Once.** I can only discover it once at which point, I plant my flag in it. If I (or anyone else) sees that land again, they will see my flag and know it’s already been discovered.

Thinking About Breadth-First Search

Questions: Now, THINK about the algorithm as an algorist might. Given an undirected graph, G:

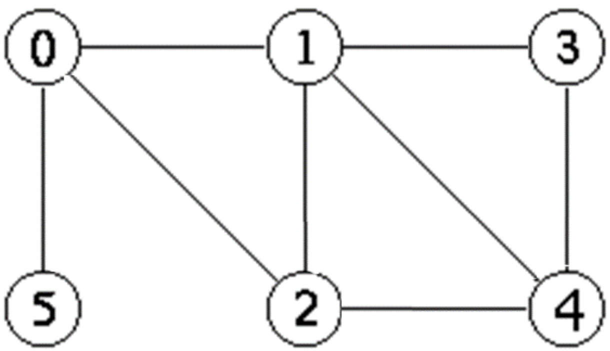
1. ~~How many times can we discover a vertex?~~
2. Could a neighbor of the active vertex already be…
   a. Discovered but not processed (gray)?
   b. Discovered and processed (black)?
3. What color are the vertices on the TO-DO queue?
4. Can there be more than one copy of a vertex on the queue?
5. How can you tell if there's a cycle in an undirected graph?

Thinking About Breadth-First Search

Questions: Now, THINK about the algorithm as an algorist might. Given an undirected graph, G:

2. **Could a neighbor of the active vertex be…**
    a. Discovered but not processed (gray)?

Answer: Yes. In the graph below, consider vertex 1 which we discover while actively working on vertex 0, but will not be processed until later in the BFS.

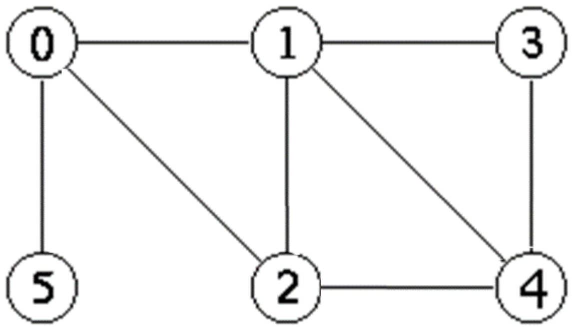


Thinking About Breadth-First Search

Questions: Now, THINK about the algorithm as an algorist might. Given an undirected graph, G:

2. Could a neighbor of the active vertex be…
   b. Discovered and processed (black)?

Answer: Yes, of course. At some point during the BFS, the vertex 0 will be flagged as both discovered and then processed.

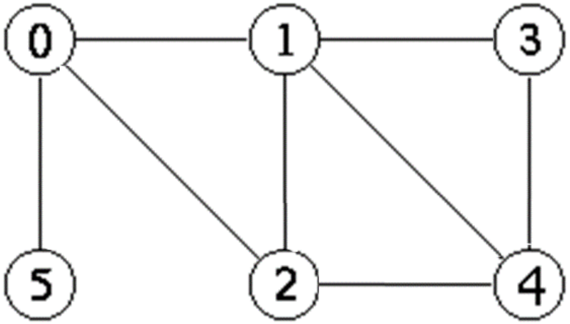


Thinking About Breadth-First Search

Questions: Now, THINK about the algorithm as an algorist might. Given an undirected graph, G:

3. What color are the vertices on the TO-DO queue?

Answer: They are gray because they are discovered but not processed.

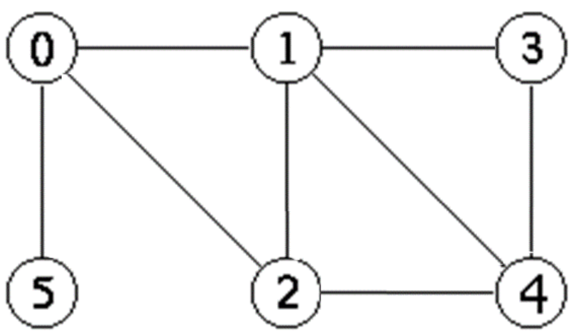


Thinking About Breadth-First Search

Questions: Now, THINK about the algorithm as an algorist might. Given an undirected graph, G:

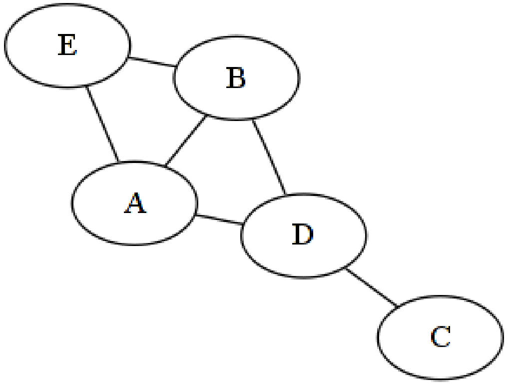
4. Can there be more than one copy of a vertex on the queue?

Answer: No. Once a vertex is placed on the queue, it is marked gray/discovered. As we've previously noted, a vertex can only be discovered once.



Breadth-First Search: Detecting Cycles

- **Question:** 5. How can we detect a cycle in an **undirected** graph?
- **Answer:** If we encounter a vertex during traversal that is in the discovered state, but "we" didn't discover it, we've found a cycle (i.e., got to same vertex via a different edge).



Breadth-First Search: Detecting Cycles

- Let’s expand on this idea and modify our BFS to detect cycles in a **connected**, **undirected** graph.
- Hmm…what do we need to do and where?
  - If we are looking at an edge that has not been processed, we should ask whether this edge leads to a vertex that has already been discovered by someone else.
  - If that vertex has already been discovered by someone else, then there must be two ways to get to it.
  - That means we just found a cycle.

Breadth-First Search: Detecting Cycles

```
BFS(G, s) [Initially all vertices undiscovered]
 Set start vertex s discovered and set parent to nil,
 Add s to the TO-DO list (enqueue)
 WHILE (there are vertices on the TO-DO list)
    v = take a vertex off the To-Do list (dequeue)
    FOR (all of v's neighbors w)
          IF (state[w] != PROCESSED) THEN
                processEdge(v, w)
          IF (state[w] == UNDISCOVERED) THEN
                Change status of vertex w to DISCOVERED)
                Add w to the TO-DO list (enqueue)
                Record the parent/predecessor of w, which is v
          Move to the next neighbor w of v
    LOOP
 Change status of vertex v to PROCESSED
 LOOP
END BFS
```

Breadth-First Search: Detecting Cycles

- Now for the important piece…

```
processEdge(x, y)
   IF ((state[y] == DISCOVERED) THEN
        PRINT "Cycle found:” + x + "," + y)
END Method
```

- **Question:** Would our current code also work for determining cycles in a **directed** graph?

Breadth-First Search: Detecting Cycles

- **Question:** Would our current code also work for determining cycles in a **directed** graph?
- **Answer:** No, it would not work to determine cycles in a directed graph. For a path to be a cycle you must be able to get back to the first vertex. The fact that a vertex has already been discovered by a someone else, does not mean you can get from that vertex back to whoever discovered it. The edge might only be directed toward the vertex and not back.

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Breadth-First Search: Manager Loop

BFSManager(G)

```
//housekeeping
FOR each vertex u in V(G)
  state[u] = undiscovered
  parent[u] = nil
 time = 0
END FOR
//catch all vertices
FOR each vertex u in V(G)
   IF (state[u] != discovered) THEN
        BFS(G, u)
END FOR
```

The manager loop for BFS ensures that if the graph is not connected, we still discover all vertices

Breadth-First Search Summary

- **When to use BFS:** BFS is a useful traversal method when you know the solution isn't far from the source vertex (e.g., friend suggestions on Facebook that are one-degree away from you).
- **Where BFS Shines:** Shortest Path (unweighted), Spanning Tree (undirected), Social-Networks.

That’s All For Now…

- Coming to a Slideshow Near You Soon…
  1. BFS Tree of Discovery
  2. Analysis of the BFS algorithm
  3. Applications of BFS algorithm

That’s All For Now