1、根據 phasor addition rule，多個同 frequency 的弦波相加仍為一 frequency 相同的弦波

$$\left[\begin{array}{l} \text{phasor addition rule} \\ x(t) = \sum_{k=1}^{N} A_k \cos(w_0 t + \phi_k) = A\cos(w_0 t + \phi) \\ \text{其中 } A\cdot e^{j\phi} = \sum_{k=1}^{N} A_k e^{j\phi_k} \end{array}\right]$$

<pf>

$$x(t) = \sum_{k=1}^{N} A_k \cos(w_0 t + \phi_k) = \sum_{k=1}^{N} Re\{A_k e^{j(w_0 t + \phi_k)}\}$$

$$= Re\{(\sum_{k=1}^{N} A_k e^{j\phi_k}) e^{jw_0 t}\} = Re\{A e^{j\phi} \cdot e^{jw_0 t}\}$$

$$= A\cos(w_0 t + \phi) \quad \#$$

2. $x(t) = \frac{1}{T/2} t$, $0 < t < 0.02$, $T = 0.04$

$$a_0 = \frac{1}{T/2} \int_0^{T/2} x(t)\, dt$$

$$= \frac{1}{T/2} \cdot \frac{1}{T/2} \int_0^{T/2} t\, dt$$

$$= \frac{4}{T^2} \cdot \frac{t^2}{2} \Big|_0^{T/2}$$

$$= \frac{4}{T^2} \frac{T^2}{2\cdot 4} = \frac{1}{2} \quad \#$$

3.

$f(t) = e^{-at} \cdot u(t)$, $a > 0$

$$F(jw) = \int_{-\infty}^{\infty} f(t) \cdot e^{-jwt} dt$$

$$= \int_{-\infty}^{\infty} e^{-at} \cdot u(t) \cdot e^{-jwt} dt$$

$$= \int_0^{\infty} e^{-at} \cdot 1 \cdot e^{-jwt} dt$$

$$= \int_0^{\infty} e^{-(a+jw)t} dt$$

$$= \frac{-1}{(a+jw)} [e^{-(a+jw)t}]_0^{\infty}$$

$$= \frac{-1}{a+jw} [e^{-\infty} - e^0]$$

其中 $e^{-\infty} = \frac{1}{e^{\infty}} = \frac{1}{\infty} = 0$

$$= \frac{1}{a+jw}$$

$$\Rightarrow e^{-at} \cdot u(t) , a > 0 \overset{F}{\longleftrightarrow} \frac{1}{a+jw}$$

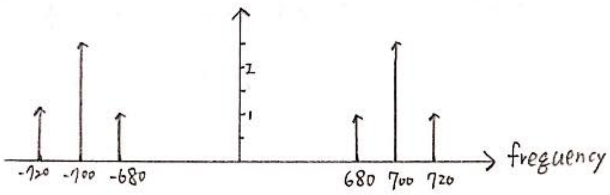
5.

$$x(t) = v(t)\cos(2\pi f_c t)$$

$$= [5 + 4\cos(40\pi t)] \cos(1400\pi t)$$

$$= [5 + 4[\tfrac{1}{2}(e^{j40\pi t} + e^{-j40\pi t})]][\tfrac{1}{2}(e^{j1400\pi t} + e^{-j1400\pi t})]$$

$$= \frac{5}{2}(e^{j1400\pi t} + e^{-j1400\pi t}) + e^{j1440\pi t} + e^{-j1360\pi t} + e^{j1360\pi t} + e^{-j1440\pi t}$$



* $f(x) = a \cdot e^{-\frac{(x-b)^2}{2c^2}}$ $(c>0)$ Gaussian Function

$F(\omega) = \int_{-\infty}^{\infty} f(t) \cdot e^{-j\omega t} dt$ Fourier Transformation.

$\Gamma(\alpha) = \int_0^{\infty} z^{\alpha-1} e^{-z} dz$, $z \in Z$, $Re(z) > 0$ Gamma Function

Let $t = x - b$, $f(t) = a \cdot e^{-\frac{t^2}{2c^2}}$

$$
\begin{aligned}
F(\omega) &= \int_{-\infty}^{\infty} f(t) \cdot e^{-j\omega t} dt \\
&= a \cdot \int_{-\infty}^{\infty} e^{-\left(\frac{t^2}{2c^2} + j\omega t\right)} dt \\
&= a \int_{-\infty}^{\infty} e^{-\frac{1}{2c^2}(t^2 + 2jc^2\omega t)} dt \\
&= a \int_{-\infty}^{\infty} e^{\frac{-1}{2c^2}\left[t^2 + 2jc^2\omega t + (jc^2\omega)^2 - (jc^2\omega)^2\right]} dt \\
&= a \int_{-\infty}^{\infty} e^{\frac{-1}{2c^2}\left[(t + jc^2\omega)^2 + c^4\omega^2\right]} dt, \quad j^2 = -1 \\
&= a \cdot e^{-\frac{c^4\omega^2}{2c^2}} \int_{-\infty}^{\infty} e^{-\frac{(t + jc^2\omega)^2}{2c^2}} dt \qquad dz = \frac{1}{\sqrt{2}c} dt \\
&= a \cdot e^{-\frac{c^2}{2}\omega^2} \int_{-\infty}^{\infty} e^{-\left(\frac{t}{\sqrt{2}c} + \frac{j}{\sqrt{2}}c\omega\right)^2} dt, \quad 令\ z = \frac{t}{\sqrt{2}c} + \frac{j}{\sqrt{2}}c\omega,\ z \to Z
\end{aligned}
$$

$$
\left[
\begin{aligned}
&\text{Gamma Function} \begin{cases} \Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x} dx, \ Re(x) > 0 \\ \Gamma(\tfrac{1}{2}) = \sqrt{\pi} \end{cases} \\
&x = z^2,\ dx = 2z\,dz,\ \Gamma(\alpha) = \int_0^{\infty} z^{2\alpha-2} \cdot e^{-z^2} \cdot 2z\,dz \\
&\qquad\qquad\qquad\qquad\quad = 2\int_0^{\infty} z^{2\alpha-1} \cdot e^{-z^2} dz \\
&\alpha = \tfrac{1}{2},\ \Gamma(\tfrac{1}{2}) = \sqrt{\pi} = 2\int_0^{\infty} z^{2\alpha-1} \cdot e^{-z^2} dz
\end{aligned}
\right]
$$

当 $\alpha = \frac{1}{2}$, $\Gamma(\frac{1}{2}) = \sqrt{\pi} = 2\int_0^{\infty} z^0 \cdot e^{-z^2} dz$

$$
\begin{aligned}
F(\omega) &= a \cdot e^{-\frac{c^2}{2}\omega^2} \int_{-\infty}^{\infty} e^{-z^2} \cdot \sqrt{2}c\,dz \\
&= a \cdot \sqrt{2}c \cdot e^{-\frac{c^2}{2}\omega^2} \cdot 2\int_0^{\infty} z^0 \cdot e^{-z^2} dz \\
&= a \cdot 2\sqrt{2}c \cdot e^{-\frac{c^2}{2}\omega^2} \int_0^{\infty} z^0 \cdot e^{-z^2} dz \\
&= a \cdot 2\sqrt{2} \cdot c \cdot e^{-\frac{c^2}{2}\omega^2} \cdot \frac{\sqrt{\pi}}{2} \\
&= \sqrt{2\pi} \cdot a \cdot c \cdot e^{-\frac{c^2}{2}\omega^2}
\end{aligned}
$$

其中 $\sqrt{2\pi} \cdot a \cdot c$ 是常数，$e^{-\frac{c^2}{2}\omega^2}$ 為一 Gaussian