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1. 根據 phasor addition rule，多個同 frequency 的弦波相加仍為一 frequency 相同的弦波

$$\left[\begin{array}{l} \text{phasor addition rule} \\ x(t)=\sum_{k=1}^{N} A_k\cos(w_0t+\phi_k)=A\cos(w_0t+\phi) \\ \text{其中 } A\cdot e^{j\phi}=\sum_{k=1}^{N} A_k e^{j\phi_k} \end{array}\right]$$

<pf>

$$x(t)=\sum_{k=1}^{N} A_k\cos(w_0t+\phi_k)=\sum_{k=1}^{N}\mathrm{Re}\{A_k e^{j(w_0t+\phi_k)}\}$$

$$=\mathrm{Re}\{(\sum_{k=1}^{N}A_k e^{j\phi_k})e^{jw_0t}\}=\mathrm{Re}\{Ae^{j\phi}\cdot e^{jw_0t}\}$$

$$=A\cos(w_0t+\phi)\ \#$$

2. $x(t)=\frac{1}{T/2}t,\ 0<t<0.02,\ T=0.04$

$$a_0=\frac{1}{T/2}\int_0^{T/2}x(t)\,dt$$

$$=\frac{1}{T/2}\cdot\frac{1}{T/2}\int_0^{T/2}t\,dt$$

$$=\frac{4}{T^2}\cdot\frac{t^2}{2}\Big|_0^{T/2}$$

$$=\frac{4}{T^2}\,\frac{T^2}{2\cdot4}=\frac{1}{2}\ \#$$

3.

$$f(t)=e^{-at}\cdot u(t),\ a>0$$

$$F(jw)=\int_{-\infty}^{\infty}f(t)\cdot e^{-jwt}dt$$

$$=\int_{-\infty}^{\infty}e^{-at}\cdot u(t)\cdot e^{-jwt}dt$$

$$=\int_0^{\infty}e^{-at}\cdot1\cdot e^{-jwt}dt$$

$$=\int_0^{\infty}e^{-(a+jw)t}dt$$

$$=\frac{-1}{(a+jw)}\left[e^{-(a+jw)t}\right]_0^{\infty}$$

$$=\frac{-1}{a+jw}\left[e^{-\infty}-e^{0}\right]$$

$$\text{其中 } e^{-\infty}=\frac{1}{e^{\infty}}=\frac{1}{\infty}=0$$

$$=\frac{1}{a+jw}$$

$$\Rightarrow e^{-at}\cdot u(t),\ a>0 \overset{F}{\longleftrightarrow} \frac{1}{a+jw}$$

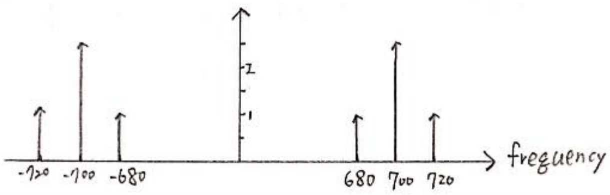
5.

$$x(t)=v(t)\cos(2\pi f_c t)$$

$$=[5+4\cos(40\pi t)]\cos(1400\pi t)$$

$$=[5+4[\tfrac{1}{2}(e^{j40\pi t}+e^{-j40\pi t})]][\tfrac{1}{2}(e^{j1400\pi t}+e^{-j1400\pi t})]$$

$$=\frac{5}{2}(e^{j1400\pi t}+e^{-j1400\pi t})+e^{j1440\pi t}+e^{-j1360\pi t}+e^{j1360\pi t}+e^{-j1440\pi t}$$



④

* $f(x) = a \cdot e^{-\frac{(x-b)^2}{2c^2}}$ $(c>0)$ Gaussian Function

$F(w) = \int_{-\infty}^{\infty} f(t) \cdot e^{-jwt} dt$ Fourier Transformation.

$\tau(\alpha) = \int_0^{\infty} z^{\alpha-1} e^{-z} dz$, $z \in Z$, $Re(z) > 0$ Gamma Function

Let $t = x - b$, $f(t) = a \cdot e^{-\frac{t^2}{2c^2}}$

$$
\begin{aligned}
F(w) &= \int_{-\infty}^{\infty} f(t) \cdot e^{-jwt} dt \\
&= a \cdot \int_{-\infty}^{\infty} e^{-(\frac{t^2}{2c^2} + jwt)} dt \\
&= a \int_{-\infty}^{\infty} e^{-\frac{1}{2c^2}(t^2 + 2jc^2wt)} dt \\
&= a \int_{-\infty}^{\infty} e^{\frac{-1}{2c^2}[t^2 + 2jc^2wt + (jc^2w)^2 - (jc^2w)^2]} dt \\
&= a \int_{-\infty}^{\infty} e^{\frac{-1}{2c^2}[(t + jc^2w)^2 + c^4w^2]} dt, \quad j^2 = -1 \\
&= a \cdot e^{-\frac{c^4w^2}{2c^2}} \int_{-\infty}^{\infty} e^{-\frac{(t+jc^2w)^2}{2c^2}} dt \\
&= a \cdot e^{-\frac{c^2}{2}w^2} \int_{-\infty}^{\infty} e^{-(\frac{t}{\sqrt{2}c} + \frac{j}{\sqrt{2}}cw)^2} dt
\end{aligned}
$$

$dz = \frac{1}{\sqrt{2}c} dt$

令 $z = \frac{t}{\sqrt{2}c} + \frac{j}{\sqrt{2}}cw$, $z \to Z$

$$
\left[
\begin{aligned}
&\text{Gamma Function } \begin{cases} \tau(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x} dx, \ Re(x) > 0 \\ \tau(\frac{1}{2}) = \sqrt{\pi} \end{cases} \\
&x = z^2, \ dx = 2z\,dz, \ \tau(\alpha) = \int_0^{\infty} z^{2\alpha-2} \cdot e^{-z^2} \cdot 2z\,dz \\
&\qquad\qquad\qquad\qquad\qquad = 2\int_0^{\infty} z^{2\alpha-1} \cdot e^{-z^2} dz \\
&\alpha = \tfrac{1}{2}, \ \tau(\tfrac{1}{2}) = \sqrt{\pi} = 2\int_0^{\infty} z^{2\alpha-1} \cdot e^{-z^2} dz
\end{aligned}
\right]
$$

當 $\alpha = \frac{1}{2}$, $\tau(\frac{1}{2}) = \sqrt{\pi} = 2\int_0^{\infty} z^0 \cdot e^{-z^2} dz$

$$
\begin{aligned}
F(w) &= a \cdot e^{-\frac{c^2}{2}w^2} \int_{-\infty}^{\infty} e^{-z^2} \cdot \sqrt{2}c\,dz \\
&= a \cdot \sqrt{2}c \cdot e^{-\frac{c^2}{2}w^2} \cdot 2\int_0^{\infty} z^0 \cdot e^{-z^2} dz \\
&= a \cdot 2\sqrt{2}c \cdot e^{-\frac{c^2}{2}w^2} \int_0^{\infty} z^0 \cdot e^{-z^2} dz \\
&= a \cdot 2\sqrt{2} \cdot c \cdot e^{-\frac{c^2}{2}w^2} \cdot \frac{\sqrt{\pi}}{2} \\
&= \sqrt{2\pi} \cdot a \cdot c \cdot e^{-\frac{c^2}{2}w^2}
\end{aligned}
$$

其中 $\sqrt{2\pi} \cdot a \cdot c$ 是常數, $e^{-\frac{c^2}{2}w^2}$ 為一 Gaussian