SI231B: Matrix Computations, 2025 Fall

Homework Set #3

Acknowledgements:

1) Deadline: **2025-11-29 23:59:59**
2) Please submit the PDF file to gradescope. Course entry code: N2382J.
3) You have 5 “free days” in total for all late homework submissions.
4) If your homework is handwritten, please make it clear and legible.
5) All your answers are required to be in English.
6) Write down the major steps for deriving the solution; otherwise you may loss points.

Problem 1. (Eigenvalue) (15 points)

Let $\mathbf{A} \in \mathbb{R}^{3\times 3}$ be a real symmetric matrix satisfying the matrix equation $\mathbf{A}^3 - 2\mathbf{A}^2 - \mathbf{A} + 2\mathbf{I} = \mathbf{0}$.

1) Find all eigenvalues of matrix $\mathbf{A}$. (10 points)
2) Suppose $\mathbf{A}$ has two eigenvalues at 1 and -1, compute $\det(\mathbf{A} + \mathbf{I})$.(5 points)

Solution:

Problem 2. (Eigenvector and similarity) (15 points)

1) Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ have two distinct eigenvalues λ_1 and λ_2 with corresponding eigenvectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ respectively. Prove that: If $k_1\boldsymbol{v}_1 + k_2\boldsymbol{v}_2$ (where the scalars k_1, k_2 are not both zero) is an eigenvector of $\mathbf{A}$, then $k_1k_2 = 0$. (7 points)
2) Suppose two matrices have the same characteristic polynomial, determinant, rank, nullity, trace, eigenvalues, algebraic multiplicity, geometric multiplicity. Are they similar? Justify your answer. (8 points)

Solution:

Problem 3. (Diagonalization) (20 points)

1) Determine if each of the following matrices is diagonalizable.

$$\mathbf{A} = \begin{bmatrix} -2 & 1 & 1 \\ 0 & 2 & 0 \\ -4 & 1 & 3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 2 & 4 & 0 \\ 0 & 2 & 0 \\ 0 & 5 & 2 \end{bmatrix}$$

(10 points)

2) If it is diagonalizable, diagonalize the matrix using a similarity transformation. (5 points)

3) Compute $\mathbf{A}^{100}$. (5 points)

Solution:

Problem 4. (Schur Decomposition) (20 points)

1) Let

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{bmatrix}.$$

Compute the real Schur decomposition $A = UTU^T$, where U is a real orthogonal matrix and T is upper triangular. **The results are reported rounded to four decimal places.** (12 points)

2) Suppose $A \in \mathbb{C}^{n\times n}$ has n distinct eigenvalues. Show that if $Q^H AQ = T$ is its Schur decomposition and $AB = BA$, then $Q^H BQ$ is upper triangular. (Hint: let $T \in \mathbb{C}^{n\times n}$ be an upper triangular matrix whose diagonal entries $t_{11}, \ldots, t_{nn}$ are distinct. If $C \in \mathbb{C}^{n\times n}$ satisfies $TC = CT$, then C must be upper triangular.) (8 points)

Solution:

Problem 5. (Variational Characterizations) (10 points)

For matrix $\mathbf{A} \in \mathbb{R}^{r\times r}$, prove $||\mathbf{A}||_2 = \max_{||\mathbf{z}||_2=1} ||\mathbf{A}\mathbf{z}||_2 = \max_{||\mathbf{z}||_2=||\mathbf{y}||_2=1} |\mathbf{y}^T\mathbf{A}\mathbf{z}|$.

Solution:

Problem 6. (Power Iteration) (20 points)

Consider

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, B = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}.$$

1) Apply the power method with infinite-norm normalization to each matrix (Compute only four steps), starting from the initial vector $\mathbf{x}^{(0)} = \begin{bmatrix} -1 & 2 \end{bmatrix}^T$. Note: To implement the infinite-norm normalization, replace the normalization step $\mathbf{v}^{(k)} = \frac{\tilde{\mathbf{v}}^{(k)}}{\|\tilde{\mathbf{v}}^{(k)}\|_2}$ used in the slides with $\mathbf{v}^{(k)} = \frac{\tilde{\mathbf{v}}^{(k)}}{\|\tilde{\mathbf{v}}^{(k)}\|_\infty}$, where $\|\cdot\|_\infty$ denotes the infinity norm. $\mathbf{v}^{(k)}$ approximates a dominant eigenvector at iteration k and $\lambda^{(k)} = (A\mathbf{v}^{(k)})^T\mathbf{v}^{(k)}/||\mathbf{v}^{(k)}||_2^2$. **The results are reported rounded to four decimal places.** (10 points)
2) Compute the ratios λ_2/λ_1 for A and B, where λ_1 and λ_2 denote the eigenvalues of largest and second-largest magnitude, respectively. For which matrix do you expect faster convergence of the power method? Explain your reason. (10 points)

Solution: