Matrix Computation Homework 1

Yifan Zhang 2025251018 zhangyf52025@shanghaitech.edu.cn

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Problem 1. (Eigenvalue)

1)

Solution.

Let λ be an eigenvalue of the matrix A, and let v be the corresponding eigenvector. Then, we have:

$$Av = \lambda v$$

Since,

$$P(A) = A^3 - 2A^2 - A + 2I = 0$$

any eigenvalue λ of A must satisfy the corresponding scalar polynomial equation $P(\lambda) = 0$.

Substituting λ into the polynomial:

$$\lambda^3 - 2\lambda^2 - \lambda + 2 = 0$$

Then we can get that:

$$\begin{aligned} \lambda^2(\lambda - 2) - 1(\lambda - 2) &= 0 \\ (\lambda^2 - 1)(\lambda - 2) &= 0 \\ (\lambda - 1)(\lambda + 1)(\lambda - 2) &= 0 \end{aligned}$$

Thus,we can get that

$$\lambda \in \{1, -1, 2\}$$

Then, we can get the λ pairs:

λ_1	-1	-1	-1	-1	-1	-1	1	1	1	2
λ_2	-1	-1	-1	1	1	2	1	1	2	2
λ_3	-1	1	2	1	2	2	1	2	2	2

Table 1: λ pairs

2)

Solution.

Since the eigenvalues of A are:

$$\lambda_1 = 1, \quad \lambda_2 = -1, \quad \lambda_3 \in \{-1, 1, 2\}$$

we can get that the eigenvalues of $(A + I)$ are:

$$\lambda_1' = 1 + 1 = 2, \quad \lambda_2' = -1 + 1 = 0, \quad \lambda_3' \in \{0, 2, 3\}$$

Thus,

$$det(A + I) = \lambda_1' \cdot \lambda_2' \cdot \lambda_3' = 0$$

Problem 2. (Eigenvector and similarity)

1)

Solution.

Let v_3 be the eigenvector of A with corresponding eigenvalue λ_3.

Then

$$\begin{aligned} Av_3 &= \lambda_3 v_3 \\ A(k_1 v_1 + k_2 v_2) &= \lambda_3(k_1 v_1 + k_2 v_2) \\ \lambda_1 k_1 v_1 + \lambda_2 k_2 v_2 &= \lambda_3 k_1 v_1 + \lambda_3 k_2 v_2) \\ k_1(\lambda_1 - \lambda_3)v_1 + k_2(\lambda_2 - \lambda_3)v_2 &= 0 \end{aligned}$$

Since λ_1, λ_2 are distince eigenvalue, we can get that v_1, v_2 are linearly independent.

Thus,

$$k_1(\lambda_1 - \lambda_3) = 0, \quad k_2(\lambda_2 - \lambda_3) = 0$$

If $k_1 \neq 0$, then $\lambda_1 = \lambda_3$. Since $\lambda_1 \neq \lambda_2$, we can get that $\lambda_3 \neq \lambda_2$. Thus $k_2 = 0$.

If $k_2 \neq 0$, then $\lambda_3 = \lambda_2$. Since $\lambda_2 \neq \lambda_1$, we can get that $\lambda_3 \neq \lambda_1$. Thus $k_2 = 0$.

Above, $k_1 k_2 = 0$.

2)

Solution.

Two matrices are similar if and only if they have the same Jordan Normal Form.

The condition can only tell us the number of Jordan blocks associated with an eigenvalue, and the sum of the sizes of those blocks.

Counter Example:

Define:

Matrix $\mathbf{A}$ (Partition 2+2):

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Matrix $\mathbf{B}$ (Partition 3+1):

$$\mathbf{B} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

It is obviously that $\mathbf{A}$ and $\mathbf{B}$ satisfy the condition.

If $\mathbf{A}$ is similar with $\mathbf{B}$, we can easily get that $\mathbf{A}^2$ is similar with $\mathbf{B}^2$.

But

$$\mathbf{A}^2 = \mathbf{0}$$

$$\mathbf{B}^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \neq \mathbf{0}$$

Thus $\mathbf{A}$ is not similar with $\mathbf{B}$.

Problem 3. (Diagonalization)

1)

Solution.

Analysis of Matrix A

$$\begin{aligned}
\det(\lambda \mathbf{I} - \mathbf{A}) &= \det \begin{bmatrix} \lambda+2 & -1 & -1 \\ 0 & \lambda-2 & 0 \\ 4 & -1 & \lambda-3 \end{bmatrix} \\
&= (\lambda-2)\cdot \det \begin{bmatrix} 2+\lambda & -1 \\ 4 & \lambda-3 \end{bmatrix} \\
&= (\lambda-2)^2(\lambda+1)
\end{aligned}$$

The eigenvalues are:

- $\lambda_1 = 2$ with algebraic multiplicity 2.
- $\lambda_2 = -1$ with algebraic multiplicity 1.

Eigenspace for $\lambda_1 = 2$:

Fisrt we solve $(\mathbf{A} - 2\mathbf{I})\mathbf{v} = \mathbf{0}$.

$$\begin{bmatrix} -4 & 1 & 1 \\ 0 & 0 & 0 \\ -4 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We have two free variables (x and y).

- Setting $x = 1, y = 0 \implies z = 4$, we get $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$.
- Setting $x = 0, y = 1 \implies z = -1$, we get $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$.

Since we have 2 independent eigenvector, the geometry multiplicity is 2 and equal to algebraic multiplicity.

Eigenspace for $\lambda_2 = -1$:

Next, we solve $(\mathbf{A} + \mathbf{I})\mathbf{v} = \mathbf{0}$.

$$\begin{bmatrix} -1 & 1 & 1 \\ 0 & 3 & 0 \\ -4 & 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Setting $x = 1$, we get $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

Geometric multiplicity is 1 and equal to algebraic multiplicity.

Since geometric multiplicities equal algebraic multiplicities for all eigenvalues, **A is diagonalizable.**

Analysis of Matrix B

$$\begin{aligned}\det(\lambda\mathbf{I} - \mathbf{B}) &= \det\begin{bmatrix}\lambda - 2 & -4 & 0 \\ 0 & \lambda - 2 & 0 \\ 0 & -5 & \lambda - 2\end{bmatrix} \\ &= (\lambda - 2)^3\end{aligned}$$

Thus, eigenvalue $\lambda = 2$ has algebraic multiplicity 3.

Eigenspace for $\lambda = 2$:

We solve $(\mathbf{B} - 2\mathbf{I})\mathbf{v} = \mathbf{0}$.

$$\begin{bmatrix}0 & 4 & 0 \\ 0 & 0 & 0 \\ 0 & 5 & 0\end{bmatrix}\begin{bmatrix}x \\ y \\ z\end{bmatrix} = \begin{bmatrix}0 \\ 0 \\ 0\end{bmatrix}$$

We can easily get the basis vectors: $\mathbf{u}_1 = \begin{bmatrix}1 \\ 0 \\ 0\end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix}0 \\ 0 \\ 1\end{bmatrix}$. Thus, the geometric multiplicity is 2.

Since geometric multiplicity $\neq$ algebraic multiplicity, **B is NOT diagonalizable.**

2)

Based on question 1), it is obviously that

$$\mathbf{P} = \begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3\end{bmatrix} = \begin{bmatrix}1 & 0 & 1 \\ 0 & 1 & 0 \\ 4 & -1 & 1\end{bmatrix}$$

$$\mathbf{D} = \begin{bmatrix}2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1\end{bmatrix}$$

and $\mathbf{A} = \mathbf{PDP}^{-1}$

3)

Fisrt, we use Gaussian transformation to calculate $\mathbf{P}^{-1}$ and get

$$\mathbf{P}^{-1} = \begin{bmatrix} -1/3 & 1/3 & 1/3 \\ 0 & 1 & 0 \\ 4/3 & -1/3 & -1/3 \end{bmatrix}$$

$$\begin{aligned} \mathbf{A}^{100} &= (\mathbf{PDP}^{-1})^100 \\ &= \mathbf{PD}^{100}\mathbf{P}^{-1} \\ &= \mathbf{PD}^{100}\mathbf{P}^{-1} \\ &= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 4 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2^{100} & 0 & 0 \\ 0 & 2^{100} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1/3 & 1/3 & 1/3 \\ 0 & 1 & 0 \\ 4/3 & -1/3 & -1/3 \end{bmatrix} \\ &= \begin{bmatrix} \frac{4-2^{100}}{3} & \frac{2^{100}-1}{3} & \frac{2^{100}-1}{3} \\ 0 & 2^{100} & 0 \\ \frac{4(1-2^{100})}{3} & \frac{2^{100}-1}{3} & \frac{4\cdot 2^{100}-1}{3} \end{bmatrix} \end{aligned}$$

Problem 4. (Schur Decomposition)

1)

Solution.

First, we calculate the eigenvalue of A.

$$\det(\lambda I - A) = \lambda^3 - 6\lambda^2 + 11\lambda - 6 = (\lambda - 1)(\lambda - 2)(\lambda - 3)$$

Then we can get that

$$\begin{aligned} \lambda_1 &= 1 \\ \lambda_2 &= 2 \\ \lambda_3 &= 3 \end{aligned}$$

Next, we solve $(A - \lambda I)v = 0$ to get the eigenvector.

To λ_1, we get $v_1 = [1, 1, 1]^T$; To λ_2, we get $v_2 = [1, 2, 4]^T$; To λ_3, we get $v_3 = [1, 3, 9]^T$;

Since A have three distint eigenvalue, it have three linearly independent eigenvector. Then we can apply Gram-Schmidt to eigenvectors to get U.

To $\mathbf{u}_1$:

$$r_{11} = \|a_1\| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$
$$q_1 = \frac{a_1}{r_{11}} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

To $\mathbf{u}_2$:

$$r_{12} = \langle q_1, a_2 \rangle = \left(\frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \right) \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} = \frac{1}{\sqrt{3}}(1 + 2 + 4) = \frac{7}{\sqrt{3}}$$

$$w_2 = a_2 - r_{12} q_1 = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} - \frac{7}{\sqrt{3}} \left(\frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} - \begin{bmatrix} 7/3 \\ 7/3 \\ 7/3 \end{bmatrix} = \begin{bmatrix} -4/3 \\ -1/3 \\ 5/3 \end{bmatrix}$$

$$r_{22} = \|w_2\| = \sqrt{\left(-\frac{4}{3}\right)^2 + \left(-\frac{1}{3}\right)^2 + \left(\frac{5}{3}\right)^2} = \frac{1}{3}\sqrt{16 + 1 + 25} = \frac{\sqrt{42}}{3}$$
$$q_2 = \frac{w_2}{r_{22}} = \frac{3}{\sqrt{42}} \left(\frac{1}{3} \begin{bmatrix} -4 \\ -1 \\ 5 \end{bmatrix} \right) = \frac{1}{\sqrt{42}} \begin{bmatrix} -4 \\ -1 \\ 5 \end{bmatrix}$$

To $\mathbf{u}_3$:

$$r_{13} = \langle q_1, v_3 \rangle = \frac{1}{\sqrt{3}}(1 \cdot 1 + 1 \cdot 3 + 1 \cdot 9) = \frac{13}{\sqrt{3}}$$
$$r_{23} = \langle q_2, v_3 \rangle = \frac{1}{\sqrt{42}}(-4 \cdot 1 + (-1) \cdot 3 + 5 \cdot 9) = \frac{1}{\sqrt{42}}(-4 - 3 + 45) = \frac{38}{\sqrt{42}}$$
$$w_3 = v_3 - r_{13} q_1 - r_{23} q_2 = \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix} - \frac{13}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{38}{42} \begin{bmatrix} -4 \\ -1 \\ 5 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$$
$$r_{33} = \|w_3\| = \frac{1}{7}\sqrt{2^2 + (-3)^2 + 1^2} = \frac{1}{7}\sqrt{14} = \frac{\sqrt{14}}{7}$$
$$q_3 = \frac{w_3}{r_{33}} = \left(\frac{1}{7} \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} \right) \cdot \frac{7}{\sqrt{14}} = \frac{1}{\sqrt{14}} \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$$

Above,

$$Q = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{-4}{\sqrt{42}} & \frac{2}{\sqrt{14}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{42}} & \frac{-3}{\sqrt{14}} \\ \frac{1}{\sqrt{3}} & \frac{5}{\sqrt{42}} & \frac{1}{\sqrt{14}} \end{bmatrix}$$

$$R = \begin{bmatrix} \sqrt{3} & \frac{7}{\sqrt{3}} & \frac{13}{\sqrt{3}} \\ 0 & \frac{\sqrt{42}}{3} & \frac{38}{\sqrt{42}} \\ 0 & 0 & \frac{\sqrt{14}}{7} \end{bmatrix}$$

Thus,

$$U = Q = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{-4}{\sqrt{42}} & \frac{2}{\sqrt{14}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{42}} & \frac{-3}{\sqrt{14}} \\ \frac{1}{\sqrt{3}} & \frac{5}{\sqrt{42}} & \frac{1}{\sqrt{14}} \end{bmatrix} = \begin{bmatrix} 0.5774 & -0.6172 & 0.5345 \\ 0.5774 & -0.1543 & -0.8018 \\ 0.5774 & 0.7715 & 0.2673 \end{bmatrix}$$

$$T = R\Lambda R^{-1} = \begin{bmatrix} 1 & \frac{7}{\sqrt{14}} & \frac{49}{\sqrt{42}} \\ 0 & 2 & \frac{19}{\sqrt{3}} \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1.0000 & 1.8708 & 7.5607 \\ 0 & 2.0000 & 10.9697 \\ 0 & 0 & 3.0000 \end{bmatrix}$$

2)

Solution.

Let $Q^H BQ = C$. Then,

$$\begin{aligned} Q^H ABQ &= Q^H BAQ \\ Q^H AQQ^H BQ &= Q^H BQQ^H AQ \\ TC &= CT \end{aligned}$$

Since T is upper triangular matrix, use the hint and we can get that C is upper triangular matrix.

Problem 5. (Variational Characterizations)

Solution.

Section 1: Fisrt, we prove $\|\mathbf{A}\|_2 = \max_{\|\mathbf{z}\|_2=1} \|\mathbf{Az}\|_2$

According to the definition of $\|\mathbf{A}\|_2$:

$$\|\mathbf{A}\|_2 = \sup_{\mathbf{x}\neq 0} \frac{\|\mathbf{Ax}\|_2}{\|\mathbf{x}\|_2}$$

Let $\mathbf{x} = c \cdot \mathbf{z}$, and we can get that

$$\begin{aligned} \|\mathbf{A}\|_2 &= \sup_{\|\mathbf{z}\|_2=1} \frac{\|c \cdot \mathbf{Az}\|_2}{\|c \cdot \mathbf{z}\|_2} \\ &= \sup_{\|\mathbf{z}\|_2=1} \frac{|c| \cdot \|\mathbf{Az}\|_2}{|c| \cdot \|\mathbf{z}\|_2} \\ &= \sup_{\|\mathbf{z}\|_2=1} \frac{\|\mathbf{Az}\|_2}{\|\mathbf{z}\|_2} \end{aligned}$$

Section 2: Next, we prove $\max_{\|\mathbf{z}\|_2=1} \|\mathbf{A}\mathbf{z}\|_2 = \max_{\|\mathbf{z}\|_2=\|\mathbf{y}\|_2=1} |\mathbf{y}^T\mathbf{A}\mathbf{z}|$

We need to prove

$$\max_{\|\mathbf{z}\|_2=1} \|\mathbf{A}\mathbf{z}\|_2 \geq \max_{\|\mathbf{z}\|_2=\|\mathbf{y}\|_2=1} |\mathbf{y}^T\mathbf{A}\mathbf{z}|$$

and

$$\max_{\|\mathbf{z}\|_2=\|\mathbf{y}\|_2=1} |\mathbf{y}^T\mathbf{A}\mathbf{z}| \geq \max_{\|\mathbf{z}\|_2=1} \|\mathbf{A}\mathbf{z}\|_2$$

step 1:

To solve $\max_{\|\mathbf{z}\|_2=1} \|\mathbf{A}\mathbf{z}\|_2 \geq \max_{\|\mathbf{z}\|_2=\|\mathbf{y}\|_2=1} |\mathbf{y}^T\mathbf{A}\mathbf{z}|$:

$$\begin{aligned}\max_{\|\mathbf{z}\|_2=\|\mathbf{y}\|_2=1} |\mathbf{y}^T\mathbf{A}\mathbf{z}| &\leq \max_{\|\mathbf{z}\|_2=\|\mathbf{y}\|_2=1} \|\mathbf{y}\|_2\|\mathbf{A}\mathbf{z}\|_2 \\ &= \max_{\|\mathbf{z}\|_2=1} \|\mathbf{A}\mathbf{z}\|_2\end{aligned}$$

Thus, it satisfy this condition.

step 2:

To solve $\max_{\|\mathbf{z}\|_2=\|\mathbf{y}\|_2=1} |\mathbf{y}^T\mathbf{A}\mathbf{z}| \geq \max_{\|\mathbf{z}\|_2=1} \|\mathbf{A}\mathbf{z}\|_2$:

Let

$$\mathbf{y} = \frac{\mathbf{A}\mathbf{z}}{\|\mathbf{A}\mathbf{z}\|_2}$$

$$|\mathbf{y}^T\mathbf{A}\mathbf{z}| = |\left(\frac{\mathbf{A}\mathbf{z}}{\|\mathbf{A}\mathbf{z}\|_2}\right)^T \mathbf{A}\mathbf{z}| = |\frac{(\mathbf{A}\mathbf{z})^T(\mathbf{A}\mathbf{z})}{\|\mathbf{A}\mathbf{z}\|_2}| = \|\mathbf{A}\mathbf{z}\|_2$$

Since there exist a $\mathbf{y}$ to make $\max_{\|\mathbf{z}\|_2=\|\mathbf{y}\|_2=1} |\mathbf{y}^T\mathbf{A}\mathbf{z}| \geq \max_{\|\mathbf{z}\|_2=1} \|\mathbf{A}\mathbf{z}\|_2$, we can get that:

$$\max_{\|\mathbf{z}\|_2=\|\mathbf{y}\|_2=1} |\mathbf{y}^T\mathbf{A}\mathbf{z}| \geq \max_{\|\mathbf{z}\|_2=1} \|\mathbf{A}\mathbf{z}\|_2$$

Problem 6. (Power Iteration)

1)

Solution.

Matrix A

Iteration 1 ($k = 1$):

$$\tilde{\mathbf{v}}^{(1)} = A\mathbf{v}^{(0)} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$\|\tilde{\mathbf{v}}^{(1)}\|_\infty = 3 \implies \mathbf{v}^{(1)} = \frac{1}{3} \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\lambda^{(1)} = \frac{(A\mathbf{v}^{(1)})^T \mathbf{v}^{(1)}}{\|\mathbf{v}^{(1)}\|_2^2} = \frac{[1, 2] \cdot [0, 1]}{0^2 + 1^2} = \frac{2}{1} = \mathbf{2.0000}$$

Iteration 2 ($k = 2$):

$$\tilde{\mathbf{v}}^{(2)} = A\mathbf{v}^{(1)} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\|\tilde{\mathbf{v}}^{(2)}\|_\infty = 2 \implies \mathbf{v}^{(2)} = \frac{1}{2} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$$

$$\lambda^{(2)} = \frac{[2, 2.5] \cdot [0.5, 1]}{0.5^2 + 1^2} = \frac{1 + 2.5}{1.25} = \frac{3.5}{1.25} = \mathbf{2.8000}$$

Iteration 3 ($k = 3$):

$$\tilde{\mathbf{v}}^{(3)} = A\mathbf{v}^{(2)} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2.5 \end{bmatrix}$$

$$\|\tilde{\mathbf{v}}^{(3)}\|_\infty = 2.5 \implies \mathbf{v}^{(3)} = \frac{1}{2.5} \begin{bmatrix} 2 \\ 2.5 \end{bmatrix} = \begin{bmatrix} 0.8 \\ 1 \end{bmatrix}$$

$$\lambda^{(3)} = \frac{[2.6, 2.8] \cdot [0.8, 1]}{0.8^2 + 1^2} = \frac{2.08 + 2.8}{1.64} = \frac{4.88}{1.64}$$

Iteration 4 ($k = 4$):

$$\tilde{\mathbf{v}}^{(4)} = A\mathbf{v}^{(3)} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0.8 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.6 \\ 2.8 \end{bmatrix}$$

$$\|\tilde{\mathbf{v}}^{(4)}\|_\infty = 2.8 \implies \mathbf{v}^{(4)} = \frac{1}{2.8} \begin{bmatrix} 2.6 \\ 2.8 \end{bmatrix} \approx \begin{bmatrix} 0.9286 \\ 1 \end{bmatrix}$$

$$\lambda^{(4)} \approx \frac{[2.8571, 2.9286] \cdot [0.9286, 1]}{0.9286^2 + 1^2} \approx \frac{5.5816}{1.8623} \approx \mathbf{2.9972}$$

Matrix B

Iteration 1 ($k = 1$)**:**

$$\tilde{\mathbf{v}}^{(1)} = B\mathbf{v}^{(0)} = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$$

$$\|\tilde{\mathbf{v}}^{(1)}\|_\infty = 7 \implies \mathbf{v}^{(1)} = \frac{1}{7} \begin{bmatrix} 4 \\ 7 \end{bmatrix} = \begin{bmatrix} 4/7 \\ 1 \end{bmatrix}$$

$$\lambda^{(1)} = \frac{(B\mathbf{v}^{(1)})^T \mathbf{v}^{(1)}}{\|\mathbf{v}^{(1)}\|_2^2} = \frac{[29/7, 32/7] \cdot [4/7, 1]}{(4/7)^2 + 1^2} = \frac{68}{13}$$

Iteration 2 ($k = 2$)**:**

$$\tilde{\mathbf{v}}^{(2)} = B\mathbf{v}^{(1)} = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 4/7 \\ 1 \end{bmatrix} = \begin{bmatrix} 29/7 \\ 32/7 \end{bmatrix}$$

$$\|\tilde{\mathbf{v}}^{(2)}\|_\infty = \frac{32}{7} \implies \mathbf{v}^{(2)} = \frac{7}{32} \begin{bmatrix} 29/7 \\ 32/7 \end{bmatrix} = \begin{bmatrix} 29/32 \\ 1 \end{bmatrix}$$

$$\lambda^{(2)} = \frac{[154/32, 157/32] \cdot [29/32, 1]}{(29/32)^2 + 1^2} = \frac{1898}{373}$$

Iteration 3 ($k = 3$)**:**

$$\tilde{\mathbf{v}}^{(3)} = B\mathbf{v}^{(2)} = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 29/32 \\ 1 \end{bmatrix} = \begin{bmatrix} 154/32 \\ 157/32 \end{bmatrix}$$

$$\|\tilde{\mathbf{v}}^{(3)}\|_\infty = \frac{157}{32} \implies \mathbf{v}^{(3)} = \frac{32}{157} \begin{bmatrix} 154/32 \\ 157/32 \end{bmatrix} = \begin{bmatrix} 154/157 \\ 1 \end{bmatrix}$$

$$\lambda^{(3)} = \frac{[779/157, 782/157] \cdot [154/157, 1]}{(154/157)^2 + 1^2} = \frac{48548}{9673}$$

Iteration 4 ($k = 4$)**:**

$$\tilde{\mathbf{v}}^{(4)} = B\mathbf{v}^{(3)} = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 154/157 \\ 1 \end{bmatrix} = \begin{bmatrix} 779/157 \\ 782/157 \end{bmatrix}$$

$$\|\tilde{\mathbf{v}}^{(4)}\|_\infty = \frac{782}{157} \implies \mathbf{v}^{(4)} = \frac{157}{782} \begin{bmatrix} 779/157 \\ 782/157 \end{bmatrix} = \begin{bmatrix} 779/782 \\ 1 \end{bmatrix} \approx \begin{bmatrix} 0.9962 \\ 1 \end{bmatrix}$$

$$\lambda^{(4)} = \frac{[3904/782, 3907/782] \cdot [779/782, 1]}{(779/782)^2 + 1^2} = \frac{1219298}{243673} \approx \mathbf{5.0038}$$

2)

Matrix A

The characteristic equation is $\det(A - \lambda I) = 0$:

$$(2-\lambda)(2-\lambda) - 1 = 0 \implies \lambda = 3, 1$$

Thus, the eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = 1$.

$$\text{Ratio } \left|\frac{\lambda_2}{\lambda_1}\right| = \frac{1}{3}$$

Matrix B

The characteristic equation is $\det(B - \lambda I) = 0$:

$$(2-\lambda)(4-\lambda) - 3 = 0 \implies (\lambda - 5)(\lambda - 1) = 0$$

Thus, the eigenvalues are $\lambda_1 = 5$ and $\lambda_2 = 1$.

$$\text{Ratio } \left|\frac{\lambda_2}{\lambda_1}\right| = \frac{1}{5}$$

The convergence rate of the Power Method is determined by ratio, $|\lambda_2/\lambda_1|$. A smaller ratio indicates faster convergence.

Since the ratio for Matrix B is smaller than the ratio for Matrix A, we expect that **Matrix B converges faster**.