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1      **TRUNCATED QR FACTORIZATION WITH PIVOTING IN MIXED  
2      PRECISION\***

3      ALFREDO BUTTARI<sup>†</sup>, THEO MARY<sup>‡</sup>, AND ANDRÉ PACTEAU\*

4      **Abstract.** Low-rank approximations are widely used to reduce the memory footprint and operational complexity of numerous linear algebra algorithms in scientific computing and data analysis.  
5      In some of our recent work we have demonstrated that low-rank approximations can be stored using multiple arithmetic precisions to further reduce the storage and execution time. In this work  
6      we present a method that can produce this mixed-precision representation directly; this relies on  
7      a mixed-precision truncated rank-revealing QR (RRQR) factorization with pivoting. We present a  
8      floating-point error analysis and provide bounds on the error of the approximation demonstrating  
9      that the use of multiple precisions does not alter the overall accuracy. Finally, we present experimental  
10     results showing the execution time reduction for the cases where either classical or randomized  
11     pivoting are used.  
12

14     **Key words.** Mixed-precision, QR factorization, low-rank approximations

15     **MSC codes.** 65F55, 65G50

16     **1. Introduction.** Alongside many classical applications in data analysis, the use  
17     of low-rank approximations in computing has become increasingly popular in recent  
18     years due to their effectiveness in reducing both the memory consumption and the  
19     operational complexity of numerous linear algebra algorithms such as linear system  
20     solvers [2, 24, 13]. Essentially, these rely on the idea that a matrix  $A \in \mathbb{R}^{m \times n}$  can be  
21     approximately represented as a product  $XY^T$  where  $X \in \mathbb{R}^{m \times k}$  and  $Y \in \mathbb{R}^{n \times k}$  are  
22     matrices of rank  $k$  such that

$$23 \quad \|A - XY^T\| \leq \varepsilon \|A\|$$

24     in some norm, where  $\varepsilon$  is a prescribed accuracy tolerance; if  $k$  is sufficiently small  
25     compared to  $m$  and  $n$ , this approximation can be used to reduce the storage and  
26     operational complexity of operations on  $A$  at the cost of a controlled loss of accuracy.  
27     For a given  $\varepsilon$ , an optimal approximation can be obtained by computing the singular  
28     value decomposition of  $A$  and dropping all the singular values smaller than  $\varepsilon$  and  
29     the corresponding left and right singular vectors [12]. Nevertheless, this method  
30     is rarely used in practice due to its high cost and low efficiency. Instead, other  
31     methods are preferred which are sufficiently accurate and robust in practice, and  
32     more computationally efficient and/or scalable in a parallel setting; among these, we  
33     can cite rank-revealing QR factorizations [6] or randomized approaches [22].

34     Concurrently, low-precision floating-point arithmetic units have become increasingly available and supported not only in specialized computing platforms (such as  
35     GPUs) but also in commodity CPUs. This is partly due to the recent explosion of artificial intelligence and machine learning algorithms in science and engineering which  
36     work remarkably well with low (e.g., 16-bit) or even very low (8-bit) precisions. As  
37     a result, new floating-point arithmetic formats have been proposed and, sometimes,  
38     standardized, such as binary16 or BFloat16, which can achieve higher performance  
39

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41 than the traditional 32-bit or 64-bit floating-point formats. Despite their wide adoption  
 42 in machine learning, these low-precision formats cannot be straightforwardly employed in applications which require relatively high accuracy; this has reignited the  
 43 interest of the scientific computing community around mixed-precision algorithms.  
 44 These combine multiple arithmetic formats in order to achieve provably accurate results while maximizing the use of low-precision units in order to improve performance.  
 45 In particular, in our recent work [1] we have demonstrated that low-rank approximations can be stored in a mixed-precision format which can then be used in a direct  
 46 method to solve linear systems of equations in a backward stable way. This approach  
 47 amounts to partitioning the columns of  $X$  and  $Y$  into block-columns such that the  
 48 overall accuracy of the approximation

$$52 \quad (1.1) \quad \|A - X^1 Y^{1T} - \cdots - X^p Y^{pT}\| \leq \varepsilon \|A\|$$

53 is still satisfied when the  $X^i$  and  $Y^i$  data are stored, from left to right, using different  
 54 arithmetics of decreasing precision. Intuitively, this can be explained by the fact  
 55 that the rightmost columns of  $X$  and  $Y$  are associated with small singular values  
 56 which carry little information and play a minor role in the error of the low-rank  
 57 approximation. This splitting is dictated by the spectrum of  $A$ , the number and  
 58 accuracy of the  $p$  available precisions and the threshold  $\varepsilon$ . In the same work we have  
 59 demonstrated that this mixed-precision format can be used to reduce the execution  
 60 time of matrix factorizations without harming the backward stability. The format  
 61 in equation (1.1) can straightforwardly be obtained by first computing the low-rank  
 62 approximation fully in high precision and then by appropriately casting the block-  
 63 columns of  $X$  and  $Y$  into lower precisions; nevertheless, using this naive approach  
 64 leads to poor performance in contexts where computing the high precision low-rank  
 65 approximation is the bottleneck.

66 In the present work, we propose a mixed-precision pivoted QR factorization that  
 67 can directly compute the mixed-precision low-rank approximation of a matrix  $A$  such  
 68 that the condition in equation (1.1) is satisfied. We make the following contributions:

- 69 1. We present, in [section 3](#), a rounding error analysis of the Householder QR  
 70 factorization where the arithmetic precision is gradually reduced in the course  
 71 of the algorithm. We produce a theoretical upper bound on the error demon-  
 72 strating that, if these changes of precision are appropriately operated, the  
 73 accuracy of the resulting low-rank approximation can be made the same as  
 74 in the case where only high-precision arithmetic is used.
- 75 2. We present, in [section 4](#), two mixed-precision QR factorization algorithms  
 76 that rely, respectively, on the Businger-Golub and randomized pivoting.
- 77 3. Finally, in [section 5](#), we present an experimental evaluation of these algo-  
 78 rithms which validates the theoretical findings and demonstrates the higher  
 79 performance of the mixed-precision algorithms with respect to their full high-  
 80 precision counterparts.

81 A mixed-precision algorithm for computing a mixed-precision low-rank approxi-  
 82 mation of a matrix  $A$  was recently proposed by Connolly *et al.* [7]. This, however,  
 83 relies on the randomized range finder method [22] and the resulting low-rank ap-  
 84 proximation is stored in high precision despite the computations being carried in  
 85 mixed-precision.

86 **2. Background.** In the reminder of this document, we will use the following  
 87 notation. Upper-case and lower-case roman letters denote, respectively, matrices and

vectors, while Greek letters denote scalars. Subscripts will be used to denote matrix or vector indices and superscripts to denote data associated with different steps of some algorithm or sequence of operations. To keep the notation simple, when there is no ambiguity, we will denote a column of a matrix using the corresponding lower-case letter with a subscript indicating the column index; for example, the  $j$ -th column of matrix  $A$  will be denoted  $a_j$ .

**2.1. Householder QR and error analysis.** The Householder QR factorization [19] of an  $m \times n$  matrix  $A$  proceeds in  $n$  steps where, at each step  $k$ , an elementary reflector  $H^k$  is computed such that all the subdiagonal coefficients in column  $k$  are annihilated; as a result, the  $A$  matrix is reduced into an upper triangular matrix  $R$  and the product of the reflectors defines the orthogonal  $Q$  factor:

$$H^n \cdots H^1 A = Q^T A = R, \text{ where } H^k = (I - \tau^k v^k (v^k)^T).$$

Here the  $Q$  matrix is never explicitly computed but implicitly represented by the  $v^k$  vectors and  $\tau^k$  scalars; furthermore, the first  $k - 1$  coefficients in vector  $v^k$  are null. The  $v^k$  and  $\tau^k$  vectors can be computed in different ways, for example, to prevent cancellations; we refer the reader to the article by Lehouc [21] for an exhaustive discussion of this argument. Our work relies on the error analysis of Higham [16] (Chapter 19) which assumes that these are computed following the convention used in the LINPACK and LAPACK libraries.

It must be noted that, for the sake of performance, these Householder transformations are not computed and applied to  $A$  individually but, rather, in blocks of size  $b \ll n$  using the  $WY$  representation of Schreiber *et al.* [25]; this allows for using Level-3 BLAS operations for most of the computations which results in a much faster execution due to an efficient use of cache memories.

Our error analysis of section 3 will use the standard model of floating-point arithmetic [16, sect. 2.2]. We put a hat on variables to denote that they represent computed quantities. For any integer  $k$ , we define

$$\gamma_k = \frac{ku}{1 - ku}$$

where  $u$  denotes the unit roundoff of the employed arithmetic; a superscript on  $\gamma$  denotes that  $u$  carries that superscript; thus  $\gamma_k^f = \frac{ku^f}{(1 - ku^f)}$ , for example. We also use the notation  $\tilde{\gamma}_k = \gamma_{\eta k}$  to hide modest constants  $\eta$ .

Some modern computing devices are equipped with vector or matrix multiplication units where products are done in low precision (for example 16-bit) and accumulation in high precision (for example, 32-bit). For these devices, the error bounds of inner product-based computations can be reduced; see [3] for a detailed analysis of the mixed precision matrix multiply–accumulate available on Google TPUs, NVIDIA GPUs, etc. The reduction of the error bound however only affects the constant part of the bound: thus, a bound  $nu_{\text{low}}$  becomes  $2u_{\text{low}} + nu_{\text{high}}$  if the accumulation is done in precision  $u_{\text{high}}$  instead of  $u_{\text{low}}$ . This could be taken into account in our analysis of section 3 by accordingly reducing the precision switch threshold. The use of such hardware would thus in principle allow for a greater use of lower precisions. We do not explore this option in our work.

Note, also, that in writing  $\gamma_n$  we implicitly assume  $nu \ll 1$  for all the employed precisions which might not hold for relatively large values of  $n$  and  $u$ . In practice, though, the actual error rarely attains this worst-case bound, due to several factors.

133 First, hardware with very low (16 or 8-bit) precisions typically use high precision  
 134 accumulation; as mentioned above, this reduces the constant in the bound [3]. Second,  
 135 blocked algorithms benefit from a reduction of the error bound by a factor proportional  
 136 to the block size  $b$  [16, p.64], [4]. Third, probabilistic effects usually allow for replacing  
 137 the constants by their square root [18]. We refer to [17] for a thorough discussion on  
 138 why extreme-scale low-precision computations are much more accurate than what  
 139 worst-case analysis would suggest.

140 We will make extensive use of the results in lemmas 19.1, 19.2 and 19.3 and  
 141 theorem 19.4 of Higham [16] which we will report below (with some slight adaptations  
 142 and notation changes) for the sake of self-completeness prior to extending them to the  
 143 case where the factorization is truncated and conducted using multiple precisions.

144 LEMMA 2.1 (19.1 and 19.2 of Higham [16]). *Assume a Householder transforma-*  
 145 *tion  $H$  is computed and applied to a vector  $b$  using precision  $u$ . The computed result*  
 146  *$\hat{y}$  satisfies*

$$147 \quad \hat{y} = (H + \Delta H)b, \quad \|\Delta H\|_F \leq \tilde{\gamma}_m.$$

148 The previous lemma demonstrates that computing and applying a Householder  
 149 transformation in finite precision is a backward stable operation. The next one demon-  
 150 strates that this property also holds when multiple transformations of this type are  
 151 applied to a vector.

152 LEMMA 2.2 (19.3 of Higham [16]). *Consider the sequence of transformations*

$$153 \quad b^i = H^i b^{i-1}, \quad b^0 = b, \quad i = 1, \dots, k.$$

154 The computed  $\hat{b}^k$  satisfies

$$155 \quad \hat{b}^k = H^k \cdots H^1(b + \Delta b) = Q^T(b + \Delta b), \quad \|\Delta b\|_2 \leq \tilde{\gamma}_{mk} \|b\|_2.$$

156 A Householder QR factorization simply consists in applying to a matrix  $A$  a  
 157 sequence of transformations that annihilate all the subdiagonal coefficients one column  
 158 at a time; therefore, Lemma 2.2 applies on all columns, which leads to the following  
 159 theorem.

160 THEOREM 2.3 (19.4 of Higham [16]). *Let  $\hat{R} \in \mathbb{R}^{m \times n}$  be the computed upper*  
 161 *trapezoidal QR factor of  $A \in \mathbb{R}^{m \times n}$  ( $m \geq n$ ) obtained via the Householder QR algo-*  
 162 *rithm. Then there exists an orthogonal  $Q \in \mathbb{R}^{m \times m}$  such that*

$$163 \quad A + \Delta A = Q \hat{R}$$

164 where

$$165 \quad \|\Delta a_j\|_2 \leq \tilde{\gamma}_{mn} \|a_j\|_2, \quad j = 1, \dots, n.$$

166 The latter theorem simply says that one such  $Q$  matrix exists. In practice, it is  
 167 more useful to have a bound using the actually computed  $\hat{Q}$ , as stated in the next  
 168 Theorem.

169 THEOREM 2.4 (from Higham [16]). *Let  $\hat{R} \in \mathbb{R}^{m \times n}$  and  $\hat{Q} \in \mathbb{R}^{m \times m}$  be, re-*  
 170 *spectively, the computed upper trapezoidal and orthogonal QR factors of  $A \in \mathbb{R}^{m \times n}$*   
 171 *( $m \geq n$ ) obtained via the Householder QR algorithm. Then*

$$172 \quad (2.1) \quad \|A - \hat{Q} \hat{R}\|_F \leq \sqrt{n} \tilde{\gamma}_{mn} \|A\|_F.$$

173     *Proof.* It must be noted that  $\widehat{Q}$  is obtained by applying the sequence of trans-  
 174     formations to the identity matrix; therefore, Theorem 2.3 can be used to derive the  
 175     following bound

$$176 \quad \widehat{Q} = Q(I + \Delta I), \quad \|\Delta i_j\|_2 \leq \tilde{\gamma}_{mn}$$

177     where  $\Delta i_j$  denoted the  $j$ -th column of matrix  $\Delta I$ ; this implies

$$178 \quad (2.2) \quad \|Q - \widehat{Q}\|_F \leq \sqrt{n} \tilde{\gamma}_{mn}.$$

179     Now, using equation (2.2) and Theorem 2.3

$$\begin{aligned} \left\| \left( A - \widehat{Q} \widehat{R} \right)_j \right\|_2 &= \left\| \left( A - Q \widehat{R} \right)_j + \left( \left( Q - \widehat{Q} \right) \widehat{R} \right)_j \right\|_2 \\ &\leq \tilde{\gamma}_{mn} \|a_j\|_2 + \sqrt{n} \tilde{\gamma}_{mn} \|\widehat{r}_j\|_2 \\ &= \tilde{\gamma}_{mn} \|a_j\|_2 + \sqrt{n} \tilde{\gamma}_{mn} \|Q \widehat{r}_j\|_2 \\ &= \tilde{\gamma}_{mn} \|a_j\|_2 + \sqrt{n} \tilde{\gamma}_{mn} \|a_j + \Delta a_j\|_2 \\ &\leq (1 + \sqrt{n} + \sqrt{n} \tilde{\gamma}_{mn}) \tilde{\gamma}_{mn} \|a_j\|_2 \\ &= \sqrt{n} \tilde{\gamma}_{mn} \|a_j\|_2 \end{aligned}$$

180     which implies the result. Note that, in the last step, one second-order term was  
 181     dropped and the “1” was absorbed by  $\tilde{\gamma}_{mn}$ .  $\square$

182     **2.2. QR factorization with Businger-Golub pivoting.** A QR factorization  
 183     capable of revealing the rank of a matrix (RRQR for Rank-Revealing QR) can be  
 184     obtained using column pivoting as in the method proposed by Businger *et al.*[5].  
 185     Essentially, at each step  $k$  of the QR factorization, this method permutes the columns  
 186     of the trailing submatrix such that the pivotal column  $k$  is the one that has maximum  
 187     2-norm. This implies that the diagonal coefficients of the  $R$  factor are of non-increasing  
 188     absolute value and that the following property holds:

$$190 \quad (2.3) \quad AP = QR \text{ where } |R_{k,k}| \geq \|R_{k:m,j}\|_2 \quad j = k, \dots, n.$$

191     Note that explicitly recomputing the norm of all the columns in the trailing  
 192     submatrix not only is expensive but completely prevents the use of blocking (and, thus,  
 193     of Level-3 BLAS operations) because this requires entirely updating all the remaining  
 194     columns after every elimination step. This problem can be partially overcome taking  
 195     advantage of the fact that, once all the column norms of the original  $A$  matrix have  
 196     been computed, these do not have to be recomputed at every step but can be cheaply  
 197     updated. Essentially, at step  $k$  of the factorization, the norm of column  $j$  in the  
 198     trailing submatrix is updated by subtracting the freshly computed  $R_{k,j}$  coefficient.  
 199     This approach is shown in Algorithm 2.1 which we refer to as QRCP; here we have  
 200     assumed that  $V$  is a matrix containing all the computed  $v^k$  vectors in its columns, that  
 201     householder( $x, k$ ) is a function which computes and applies a Householder reflection  
 202     that annihilates the bottom  $m - k + 1$  coefficients of a vector  $x$  of size  $m$  and that  $W$   
 203     is a workspace. Although, in this algorithm, a large portion of computations is still  
 204     done using Level-2 BLAS operations, some Level-3 can be used (in line 13).

**Algorithm 2.1** Blocked QR factorization with Businger-Golub pivoting (QRCP).

---

```

1: Input:  $A \in \mathbb{R}^{m \times n}$ 
2: Let:  $\eta_j = \|A_{:,j}\|_2, j = 1, \dots, n, P = I$ 
3: for  $j = 1 : b : n$  do
4:   for  $k = j : j + b - 1$  do
5:     Find  $i \in k, \dots, n$  such that  $\eta_i$  is maximal
6:     Swap columns  $k$  and  $i$  in  $A$  and  $P$ 
7:      $A_{k:m,k} = A_{k:m,k} - V_{k:m,j:k-1}W_{j:k-1,k}$ 
8:      $v^k, \tau^k = \text{householder}(A_{:,k}, k)$ 
9:      $W_{k+1,k:n} = \tau^k(v^k)^T A_{:,k+1:n} + \tau^k(v^k)^T V_{:,j:k-1}W_{j:k-1,k+1:n}$ 
10:     $A_{k,k+1:n} = A_{k,k+1:n} - V_{k,j:k-1}W_{j:k-1,k+1:n}$ 
11:     $\eta_i = \sqrt{\eta_i^2 - A_{k,i}^2}, i = k + 1, \dots, n$ 
12:   end for
13:    $A_{j+b:m,j+b:n} = A_{j+b:m,j+b:n} - V_{j+b:m,j:j+b-1}W_{j:j+b-1:j+b:n}$ 
14: end for
15: Output:  $Q = \prod_{k=1,n} (I - \tau^k v^k (v^k)^T), R = \text{triu}(A), P$ 

```

---

205 It must be noted that the column norm update in line 11 of [Algorithm 2.1](#) is a  
 206 very delicate step when carried in finite precision as it may be subject to severe cancellations.  
 207 This problem might be overcome using the approach proposed by Drmač[10]  
 208 where if, in line 11 the norm of some columns drops by a value larger than a prescribed  
 209 threshold which depends on the arithmetic unit roundoff, the algorithm breaks out of  
 210 the inner loop of line 4, fully updates the trailing submatrix (line 13) and explicitly  
 211 recomputes the norm of the problematic columns. This approach might reduce the  
 212 portion of Level-3 BLAS operations but renders the method robust. [Algorithm 2.1](#)  
 213 with this updating technique is implemented in the LAPACK `_GEQP3` routine.

214 **2.3. QR factorization with randomized pivoting.** In practice, [Algorithm 2.1](#)■  
 215 achieves very poor performance and parallel scalability because a large portion of com-  
 216 putations in [Algorithm 2.1](#) are of Level-2 BLAS type and because of the numerous  
 217 communications that the Businger-Golub pivoting requires. For this reason alterna-  
 218 tive pivoting techniques have been proposed in the literature that aim at overcoming  
 219 these drawbacks. Multiple methods proposed in the literature [11, 23, 8, 9] rely on  
 220 an approach where, at every step of the factorization, not one but  $b$  (the panel size)  
 221 selected columns of the trailing submatrix are moved upfront, eliminated and the cor-  
 222 responding transformations applied at once using Level-3 BLAS operations as in the  
 223 WY technique described above. These methods essentially differ in the way these  $b$   
 224 columns are selected. In the approach proposed by Duersch *et al.* [11] and Martins-  
 225 son *et al.* [23], illustrated in [Algorithm 2.2](#), this selection is done using randomized  
 226 sampling, that is, the trailing submatrix is left-multiplied by a i.i.d. Gaussian matrix  
 227  $\Omega \in \mathcal{N}(0, 1)^{(b+p) \times (m-j)}$  which produces a sample matrix  $S$ ; here we assume that  $j-1$   
 228 columns of  $A$  have already been eliminated and  $p$  is an *oversampling* parameter of  
 229 moderate value (less than ten). Because of the properties that connect  $S$  to the trail-  
 230 ing submatrix, the “important”  $b$  columns can be selected by applying QRCP to the  
 231 sample matrix  $S$ . This drastically improves the amount of level-3 BLAS operations  
 232 because  $S$  has a much smaller row-dimension than the trailing submatrix. As a matter  
 233 of fact, the sample matrix  $S$  does not have to be recomputed at every factorization  
 234 step but it can be computed only once and cheaply updated [11, 23].

**Algorithm 2.2** Blocked QR factorization with randomized pivoting (QRRP).

---

```

1: Input:  $A \in \mathbb{R}^{m \times n}$ 
2: Let:  $P = I$ ,  $\Omega \in \mathcal{N}(0, 1)^{(b+p) \times (m-j)}$ ,  $S = \Omega A$ 
3: for  $j = 1 : b : n$  do
4:    $\tilde{Q}, \tilde{R}, \tilde{P} = \text{QRCP}(S_{:,j:n})$ 
5:    $A_{:,j:n} = A_{:,j:n} \tilde{P}$ ,  $P_{:,j:n} = P_{:,j:n} \tilde{P}$ 
6:    $Q^j, R^j = QR(A_{j:m,j:j+b-1})$ 
7:    $A_{j:m,j+b:n} = (Q^j)^T A_{j:m,j+b:n}$ 
8:   Update  $S$ 
9: end for
10: Output:  $Q = \prod Q^j$ ,  $R = \text{triu}(A)$ ,  $P$ 

```

---

235 [26] demonstrate that the property of equation (2.3) does not apply formally to  
 236 the result of QRRP but holds in a probabilistic sense. Given  $\varepsilon$ ,  $\Delta \in (0, 1)$  and an  
 237 oversampling parameter  $p \geq \lceil \frac{4}{\varepsilon^2 - \varepsilon^3} \log(\frac{2nk}{\Delta}) \rceil$  the following property

238 (2.4) 
$$|R_{k,k}| \geq \sqrt{\frac{1-\varepsilon}{1+\varepsilon}} \|R_{k:m,j}\|_2, \quad i+1 \leq j \leq n$$

239 holds with probability at least  $1 - \Delta$ .

240 **3. Error analysis.** Let us assume that we have  $p$  precisions such that the re-  
 241 spective unit roundoff satisfy

242 
$$u^1 \leq u^2 \leq \cdots \leq u^p$$

243 and for each precision  $i$  a sequence of  $k^i$  transformations are computed and applied  
 244 to matrix  $A$  such that

245 
$$\sum_{i=1}^p k^i = k \leq n.$$

246 Therefore, first  $k^1$  transformations are computed and applied with precision  $u^1$ , then  
 247  $k^2$  with precision  $u^2$  and so forth:

248 (3.1) 
$$\overbrace{\left( H^{p,k^p} \cdots H^{p,1} \right)}^{u^p} \cdots \overbrace{\left( H^{1,k^1} \cdots H^{1,1} \right)}^{u^1} A = Q^T A = \begin{bmatrix} R \\ A^{p+1} \end{bmatrix}.$$

249 Here we use an extra superscript to denote the precision at which each transformation  
 250 is computed and applied. Furthermore, we denote as  $A^{i+1}$  the trailing submatrix after  
 251  $K^i$  of the above transformations are applied where

252 
$$K^i = \sum_{j=1}^i k^j.$$

253 Note that, because we are interested in a truncated QR factorization,  $k$  might be  
 254 smaller than  $n$ , in which case  $A^{p+1}$  corresponds to a  $(m - k) \times n$  matrix with the  
 255 rightmost  $n - k$  columns being non-zero.

256 Each  $A^i$  matrix has  $m - K^{i-1}$  rows and  $n$  columns, the first  $K^{i-1}$  columns being  
 257 equal to zero;  $A^1$  corresponds to the initial matrix  $A$ . The  $Q$  and  $R$  matrices in

258 equation (3.1) can be split in block-columns and block-rows, respectively, such that

$$259 \quad (3.2) \quad A = [Q_1 \cdots Q_p Q_{p+1}] \begin{bmatrix} R_1 \\ \vdots \\ R_p \\ A^{p+1} \end{bmatrix}.$$

260 Note that each  $Q_i$  is a  $m \times k^i$  submatrix of  $Q$  and is the result of computations  
261 carried in precisions 1 through  $i$ . Equivalently,  $R_i$  is a  $k^i \times n$  submatrix of  $R$  and  
262 is the result of computations carried in precisions 1 through  $i$ . Our mixed-precision  
263 low-rank approximation of  $A$  is obtained by dropping  $Q_{p+1}$  and  $A^{p+1}$  in the above  
264 equation.

265 Our analysis will rely on the observation that in the QR factorization the  $i$ -th of  
266 such transformations has the following structure

$$267 \quad \begin{bmatrix} I^{i-1} \\ & \bar{H}^i \end{bmatrix}$$

268 where  $I^{i-1}$  is the identity matrix of size  $i-1$ ; this is because the  $i$ -th transformation  
269 is computed so as to annihilate all the subdiagonal coefficients in the  $i$ -th column  
270 of  $A$  and implies that the application of such transformation will only concern the  
271 bottom  $m-i+1$  rows of the matrix. In the analysis of Higham [16] this property  
272 was not used because in the case where a single precision is used it does not yield  
273 any significant improvement of the bounds. For our analysis, it is not necessary to  
274 consider the structure of each single transformation but, rather, it is enough to note  
275 that all transformations at precision  $i$  have the same structure

$$276 \quad \begin{bmatrix} I^{K^{i-1}} \\ & \bar{H}^{i,j} \end{bmatrix}, \quad j = 1, \dots, k^i.$$

277 LEMMA 3.1 (equivalent of Lemma 2.2 in mixed precision). *Consider the sequence  
278 of transformations*

$$279 \quad b^{K^i} = \prod_{j=1}^{k^i} H^{i,j} b^{K^{i-1}}, \quad b^0 = b, \quad i = 1, \dots, p, \quad K^p = k.$$

280 where all transformations  $H^{i,j}$ ,  $j = 1, \dots, k^i$  are computed and applied in precision  $u^i$   
281 and have the following structure

$$282 \quad (3.3) \quad H^{i,j} = \begin{bmatrix} I^{K^{i-1}} \\ & \bar{H}^{i,j} \end{bmatrix}.$$

283 The computed  $\hat{b}^k$  satisfies

$$284 \quad \hat{b}^k = Q^T(b + \Delta b), \quad \|\Delta b\|_2 \leq \sum_{i=1}^p \tilde{\gamma}_{mk^i}^i \left\| b_{K^{i-1}+1:m}^{K^{i-1}} \right\|_2.$$

285 *Proof.* The result follows by using Lemma 2.2 “in packets” where, in each packet,  
286 the Householder transformations have the structure of equation (3.3). Each packet  
287 of  $k^i$  transformations at precision  $u^i$  only concerns rows  $K^{i-1}+1, \dots, m$  of  $b^{K^{i-1}}$   
288 and, by Lemma 2.2, introduces an error bounded by  $\tilde{\gamma}_{mk^i}^i \left\| b_{K^{i-1}+1:m}^{K^{i-1}} \right\|_2$ . Note that it  
289 would be more appropriate to use  $\tilde{\gamma}_{(m-K^{i-1})k^i}^i$  but we use  $m$  instead of  $m - K^{i-1}$  to  
290 keep the notation simple.  $\square$

291 Based on [Lemma 3.1](#), we are now ready to derive a columnwise error bound for  
 292 a truncated QR factorization in mixed precision.

293 **LEMMA 3.2** (equivalent of [Theorem 2.3](#) with truncation and mixed precision).  
 294 Assume that a truncated QR factorization is computed such that  $k \leq n$  Householder  
 295 transformations are computed and applied to a matrix  $A \in \mathbb{R}^{m \times n}$  using  $p$  different  
 296 precisions of increasing unit roundoff  $u^i$ . Let  $k^i$  be the number of transformations that  
 297 are computed using precision  $i$ . Then there exist matrices  $Q_1, \dots, Q_{p+1}$  such that the  
 298 computed  $\hat{R}^i$  and  $\hat{A}^{p+1}$  satisfy

$$299 \quad (3.4) \quad (A + \Delta A) = [Q_1 \cdots Q_p Q_{p+1}] \begin{bmatrix} \hat{R}_1 \\ \vdots \\ \hat{R}_p \\ \hat{A}^{p+1} \end{bmatrix}, \quad \|\Delta a_j\|_2 \leq \sum_{i=1}^p \tilde{\gamma}_{mk^i}^i \|a_j^i\|_2$$

300 and, consequently,

$$301 \quad (3.5) \quad \left\| \left( A - \sum_{i=1}^p Q_i \hat{R}_i \right)_j \right\|_2 \leq \|a_j^{p+1}\|_2 + \sum_{i=1}^p \tilde{\gamma}_{mk^i}^i \|a_j^i\|_2.$$

302 *Proof.* Equation (3.4) straightforwardly results from the application of [Lemma 3.1](#)  
 303 to the case where the sequence of Householder transformations is computed so as to  
 304 annihilate all the subdiagonal coefficients of  $A$  and, therefore, have the structure  
 305 defined in equation (3.3). Equation (3.5), instead, follows from the observation that

$$306 \quad \hat{a}_j^{p+1} = a_j^{p+1} + \Delta a_j^{p+1}, \quad \|\Delta a_j^{p+1}\|_2 \leq \sum_{i=1}^p \tilde{\gamma}_{mk^i}^i \|a_j^i\|_2. \quad \square$$

307 **THEOREM 3.3** (equivalent of [Theorem 2.4](#) with truncation and mixed precision).  
 308 Assume that a truncated QR factorization is computed such that  $k \leq n$  Householder  
 309 transformations are computed and applied to a matrix  $A \in \mathbb{R}^{m \times n}$  using  $p$  different  
 310 precisions of increasing unit roundoff  $u^i$ . Let  $k^i$  be the number of transformations that  
 311 are computed using precision  $i$ . The computed  $\hat{R}_i$  and  $\hat{Q}_i$  satisfy

$$312 \quad (3.6) \quad \left\| A - \sum_{i=1}^p \hat{Q}_i \hat{R}_i \right\|_F \leq \|A^{p+1}\|_F + \sum_{i=1}^p \sqrt{k^i} \tilde{\gamma}_{mk^i}^i \|A^i\|_F.$$

313 *Proof.* Following the same path that led us to equation (2.2), in the case where  
 314 multiple precisions are used we obtain

$$315 \quad \left\| Q_i - \hat{Q}_i \right\|_F \leq \sqrt{k^i} \sum_{j=1}^i \tilde{\gamma}_{mk^j}^j = \sqrt{k^i} \tilde{\gamma}_{mk^i}^i$$

316 which allows us to derive a bound on the quality of the approximation using the

317 actually computed  $\widehat{Q}_i$  and  $\widehat{R}_i$

$$\begin{aligned}
 \left\| \left( A - \sum_{i=1}^p \widehat{Q}_i \widehat{R}_i \right)_j \right\|_2 &= \left\| \left( A - \sum_{i=1}^p Q_i \widehat{R}_i \right)_j + \sum_{i=1}^p \left( (Q_i - \widehat{Q}_i) \widehat{R}_i \right)_j \right\|_2 \\
 &\leq \left\| \left( A - \sum_{i=1}^p Q_i \widehat{R}_i \right)_j \right\|_2 + \sum_{i=1}^p \|Q_i - \widehat{Q}_i\|_F \left\| \left( \widehat{R}_i \right)_j \right\|_2 \\
 &\leq \left\| a_j^{p+1} \right\|_2 + \sum_{i=1}^p \tilde{\gamma}_{mk^i}^i \|a_j^i\|_2 + \sum_{i=1}^p \sqrt{k^i} \tilde{\gamma}_{mk^i}^i \left\| \left( \widehat{R}_i \right)_j \right\|_2 \\
 &\leq \left\| a_j^{p+1} \right\|_2 + \sum_{i=1}^p \sqrt{k^i} \tilde{\gamma}_{mk^i}^i \|a_j^i\|_2
 \end{aligned}$$

□

318

319 which implies the result.

320 **4. Pivoted QR factorization in mixed precision.** Based on the theoretical  
 321 findings of the previous section, we are now ready to formulate a truncated rank-  
 322 revealing QR factorization in mixed precision. We will first introduce two truncated  
 323 QR factorization algorithms based, respectively, on the Businger-Golub and random-  
 324ized pivoting, and then a mixed-precision algorithm that can use either of these.

325 Specifically, the truncated mixed-precision algorithm relies on the use of the error  
 326 bound in equation (3.6) to gradually switch to lower precision and, eventually, to halt  
 327 the factorization as soon as the prescribed accuracy  $\varepsilon$  is reached. The terms of this  
 328 bound, however, are difficult, in practice, to compute accurately. In order to make  
 329 this bound more usable in practice, we can proceed to some simplifications. First of  
 330 all we can ignore the constants related to the accumulation of rounding errors, which  
 331 are often pessimistic [15]; this amounts to replacing each  $\tilde{\gamma}_{mk^i}^i$  with the unit roundoff  
 332 of the corresponding arithmetic precision  $u^i$ . Second, because in the course of the  
 333 factorization it is not known beforehand how many transformations will be computed  
 334 with each precision  $i$ , we will replace  $k^i$  with  $n - K^{i-1}$ . The error bound thus becomes

$$335 \quad (4.1) \quad \left\| A - \sum_{i=1}^p \widehat{Q}_i \widehat{R}_i \right\|_F \leq \|A^{p+1}\|_F + \sum_{i=1}^p \sqrt{n - K^{i-1}} u^i \|A^i\|_F.$$

336 Assuming that a low-rank approximation of relative accuracy  $\varepsilon$  is to be computed

$$337 \quad \left\| A - \sum_{i=1}^p \widehat{Q}_i \widehat{R}_i \right\|_F \leq \|A^{p+1}\|_F + \sum_{i=1}^p \sqrt{n - K^{i-1}} u^i \|A^i\|_F \leq \varepsilon \|A\|_F$$

338 and, once again, ignoring the constants, it will be enough to ensure that all the terms  
 339 in the bound are smaller than or equal to  $\varepsilon \|A\|_F$ . That is to say, it will be possible  
 340 to switch from precision  $i$  to precision  $i + 1$  at step  $j$  of the factorization when

$$341 \quad (4.2) \quad \sqrt{n - j} u^{i+1} \|A_{j:m,j:n}\|_F \leq \varepsilon \|A\|_F$$

342 and the factorization can be truncated at step  $k$  when

$$343 \quad (4.3) \quad \|A_{k:m,k:n}\|_F \leq \varepsilon \|A\|_F.$$

344 It must be noted that our error analysis does not make any assumption on how the  
 345 pivoting is done. Actually, for our method to work, it is only required that the norm

346 of the trailing submatrix decreases in the course of the factorization which happens  
 347 even in the case where no pivoting is applied. Obviously, the use of pivoting will  
 348 lead to a faster decay of the trailing submatrix norm and, consequently, to an earlier  
 349 truncation and change of precisions.

350 **4.1. Truncated QR factorization with Businger-Golub pivoting.** The  
 351 two criteria in equations (4.2) and (4.3) can be straightforwardly used to halt [Algorithm 2.1](#)  
 352 because the Frobenius norm of the trailing submatrix is readily available as  
 353 the 2-norm of part of the vector storing the  $\eta_i$ , that is, the 2-norm of the corresponding  
 354 columns. The resulting algorithm is presented in [Algorithm 4.1](#) and amounts to a  
 355 simple modification of [Algorithm 2.1](#) where the factorization is interrupted as soon as  
 356 either equation (4.2) or (4.3) is satisfied. It must be noted that, when the truncation  
 357 happens because equation (4.2) is satisfied, it is implicitly assumed that the factorization  
 358 will be continued using a lower precision (see the details in [subsection 4.3](#))  
 359 and, therefore, the trailing submatrix update on line 17 is necessary. If, instead, the  
 360 factorization is interrupted because equation (4.3) is satisfied, the trailing submatrix  
 361 is discarded and therefore need not be updated; for the sake of conciseness we do not  
 362 include this optimization in [Algorithm 4.1](#) although it is implemented in the code we  
 363 used for the experimental evaluation of [subsection 5.2](#).

---

**Algorithm 4.1** Truncated blocked QR factorization with Businger-Golub pivoting (TQRCP).

---

```

1: Input:  $A \in \mathbb{R}^{m \times n}$ ,  $\varepsilon_t$ ,  $\varepsilon_p$ 
2: Let:  $\eta_j = \|A_{:,j}\|_2$ ,  $j = 1, \dots, n$ ,  $P = I$ 
3: for  $j = 1 : b : n$  do
4:   for  $k = j : j + b - 1$  do
5:     if  $\|\eta_{k:n}\|_2 \leq \varepsilon_t$  or  $\sqrt{n-j} \|\eta_{k:n}\|_2 \leq \varepsilon_p$  then
6:        $b = k - j$ ;  $k = k - 1$ 
7:       goto 17 and break  $j$  loop
8:     end if
9:      $\arg \min_i \{\eta_i, i = k : n\}$ 
10:    Swap column  $k$  and  $i$  in  $A$  and  $P$ 
11:     $A_{k:m,k} := V_{k:m,j:k-1} W_{j:k-1,k}$ 
12:     $v^k, \tau^k = \text{householder}(A_{:,k})$ 
13:     $W_{k+1,k:n} = \tau^k (v^k)^T A_{:,k+1:n} + \tau^k (v^k)^T V_{:,j:k-1} W_{j:k-1,k+1:n}$ 
14:     $A_{k,k+1:n} = A_{k,k+1:n} - V_{k,j:k-1} W_{j:k-1,k+1:n}$ 
15:     $\eta_i = \sqrt{\eta_i^2 - A_{k,i}^2}$ ,  $i = k + 1, \dots, n$ 
16:  end for
17:   $A_{j+b:m,j+b:n} := V_{j+b:m,j:j+b-1} W_{j:j+b-1:j+b:n}$ 
18: end for
19: Output:  $Q_{1:m,1:k} = \prod_{i=1,k} (I - \tau^i v^i (v^i)^T)$ ,  $R = \text{triu}(A_{1:k,1:n})$ ,  $P$ ,  $k$ 
```

---

364 It must be noted that, thanks to the property in equation (2.3), the Frobe-  
 365 nius norm of the trailing submatrix at step  $k$  can be bounded using  $|R_{k,k}|$ , that  
 366 is,  $\|A_{k:n,k:n}\|_F \leq \sqrt{n-k} |R_{k,k}|$ ; this criterion, however, can largely overestimate the  
 367 norm of the trailing submatrix and result in a late switch of precision, or truncation  
 368 leading to sub-optimal performance.

369 **4.2. Truncated QR factorization with randomized pivoting.** In the case  
 370 of the QR factorization with randomized pivoting in [Algorithm 2.2](#), the norm of the

371 trailing submatrix cannot be easily computed. Nevertheless, it is possible to check for  
 372 precision switch and truncation within the QRCP factorization of the sample matrix  
 373 based on the norm of the trailing submatrix of the sample. According to Theorem 3.1  
 374 by [11], after  $i$  transformations the trailing 2-norm squared of the sample, that is,  
 375 the norm of the columns in the trailing submatrix of the sample, can be written as a  
 376 factor of the actual trailing column norms. Assuming  $S$  is a sample of row-dimension  
 377  $l = b + p$  for a matrix  $A$  and  $S^i$  and  $A^i$  the corresponding trailing submatrices  
 378 after  $i$  Householder transformations,  $\|s_j^i\|_2^2 = x_j \|a_j^i\|_2^2$  where  $x_j$  has a truncated chi-  
 379 squared distribution with  $l - i$  degrees of freedom. Although the expectation of  $x_j$  can  
 380 theoretically be smaller than  $l - i$ , in practice we have found that assuming  $x_j = l - i$   
 381 or, more conservatively,  $x_j = p$  works very well on our experimental test set (see  
 382 experiments in subsection 5.2). Alternatively, the property in equation (2.4) can be  
 383 used to obtain a probabilistic bound on the norm of the trailing submatrix but this  
 384 would likely result in an excessively pessimistic criterion.

385 Based on this discussion, we propose the truncated QR factorization with ran-  
 386 domized pivoting in Algorithm 4.2; this relies on Algorithm 4.1 for the factorization  
 387 of the sample matrix.

---

**Algorithm 4.2** Truncated blocked QR factorization with randomized pivoting (TQRPP).

---

```

1: Input:  $A \in \mathbb{R}^{m \times n}$ ,  $\varepsilon_t$ ,  $\varepsilon_p$ 
2: Let:  $P = I$ ,  $\Omega \in \mathcal{N}(0, 1)^{(b+p) \times (m-j)}$ ,  $S = \Omega A$ 
3: for  $j = 1 : b : n$  do
4:    $\tilde{Q}, \tilde{R}, \tilde{P}, \tilde{k} = \text{TQRCP}(S_{:,j:n}, \sqrt{p}\varepsilon_t, \sqrt{p}\varepsilon_p)$ 
5:    $A_{:,j:n} = A_{:,j:n}\tilde{P}$ ,  $P_{:,j:n} = P_{:,j:n}\tilde{P}$ 
6:    $Q^j, R^j = QR(A_{j:m,j:\tilde{k}-1})$ 
7:    $A_{j:m,j+\tilde{k}:n} = (Q^j)^T A_{j:m,j+\tilde{k}:n}$ 
8:   if  $\tilde{k} \leq b$  then
9:     break
10:   end if
11:   Update  $S$ 
12: end for
13: Output:  $Q_{:,1:k} = \prod Q^j$ ,  $R = \text{triu}(A_{1:k,1:n})$ ,  $P$ ,  $k = j + \tilde{k}$ 
```

---

388 **4.3. Truncated QR factorization in mixed precision.** The truncated QR  
 389 factorization in mixed precision is illustrated in Algorithm 4.3. Essentially, assuming  
 390  $p$  precisions are to be used, the algorithm proceeds in  $p$  iterations where, at iteration  
 391  $i$ , the current trailing submatrix  $A^i$  is factorized in precision  $u^i$  using either Algo-  
 392 rithm 4.1 or Algorithm 4.2 and, then, the resulting trailing submatrix  $A^{i+1}$  is cast  
 393 into precision  $u^{i+1}$ .

394 **5. Numerical experiments.** In this section we present an experimental eval-  
 395 uation of the theoretical results and algorithms presented in the previous sections.  
 396 This evaluation is twofold. First, in subsection 5.1 we validate the correctness of  
 397 the error analysis in section 3 using up to three different precisions, that is, double  
 398 (fp64), single (fp32) and BFloat16 (bf16). Second, in subsection 5.2, we assess the  
 399 performance of the mixed-precision variants compared with their high-precision coun-  
 400 terpart; for these experiments, we will use precisions for which support is available in  
 401 commodity CPUs and optimized LAPACK and BLAS libraries, namely double and

**Algorithm 4.3** Truncated RRQR factorization in mixed precision (MPTRRQR).

---

```

1: Input:  $A \in \mathbb{R}^{m \times n}$ ,  $\varepsilon$ ,  $[u^1, \dots, u^p]$ 
2: Let:  $A^1 = A$ ,  $u^{p+1} = \infty$ 
3: for  $i = 1, \dots, p$  do
4:   Set working precision to  $u^i$ 
5:    $Q^i, R^i, P^i, k^i = \text{TQR}^*P(A^i, \varepsilon \|A\|_F, \varepsilon \|A\|_F / u^{i+1})$ 
6:   if  $i < p$ ,  $A^{i+1} = \text{cast}(A_{k^i+1:, k^i+1:}^i, u^{i+1})$ 
7: end for

```

---

402 single precision. All these codes, along with instructions on how to reproduce our  
 403 experimental results, are available in a public Git repository<sup>1</sup>.

404 Experiments were conducted on the following problems:

- 405 • **randsvd** [16]: random matrices with geometrically distributed singular values between 1 and  $10^{-16}$ ;
- 406 • **gravity** [14]: Discretization of a 1-D model problem in gravity surveying;
- 407 • **heat** [14]: inverse heat equation;
- 408 • **kahan** [20]: this is a triangular matrix with columns of decreasing norms; therefore, the pivoted QR factorization does not perform any computations.
- 409 To avoid this trivial behavior, we scaled the columns of this matrix in such a way that column  $j$  is multiplied by  $(1 - \tau)^{n-j}$  with  $\tau = 10^{-10}$  and  $n$  being the number of columns in the matrix;
- 410 • **phillips** [14]: Phillips' famous test problem;

411 for all problems, only square matrices were generated of varying sizes.

412    **5.1. Theory validation with a Julia prototype.** In this section we present  
 413 an experimental analysis aiming at validating the theoretical analysis of [section 3](#). For  
 414 this purpose, we have implemented a prototype using the Julia language which provides  
 415 half precision, namely, the BFloat16 arithmetic through the BFloat16s<sup>2</sup> package.  
 416 Because the only purpose of this prototype is to validate the theoretical results, it  
 417 does not actually implement [Algorithm 4.1](#) and [4.2](#) but, instead, proceeds through  
 418 the following steps:

- 419 1. Fully factorizes the input matrix  $A$  using Householder QR with Businger-Golub pivoting (i.e., [Algorithm 2.1](#));
- 420 2. explicitly permutes the input matrix  $\tilde{A} = AP$  using the permutation resulting from the previous step;
- 421 3. computes the  $k^i$  based on criteria in equations [\(4.2\)](#) and [\(4.3\)](#) using, instead of  $A$ , the  $R$  factor resulting from step 1;
- 422 4. factorizes  $\tilde{A}$  using [Algorithm 4.3](#) with standard, unpivoted QR on line 5.

423 Note that this procedure is consistent with the fact that our error analysis does not  
 424 make any assumptions on whether and how the pivoting is done, as explained in  
 425 [section 4](#).

426    Table 1 and Table 2 show experimental results for the randsvd and phillips matrices,  
 427 respectively, of size 2048. In the first column we report the value of the chosen  
 428  $\varepsilon$  threshold, in the second, the measured error when the factorization is computed en-  
 429 tirely in double precision, in the third the error bound computed using equation [\(4.1\)](#)  
 430 assuming three precisions are used, in the fourth the actual error and in the fifth, sixth

---

<sup>1</sup><https://gitlab.com/mpqr/mpqr>

<sup>2</sup><https://github.com/JuliaMath/BFloat16s.jl>

and seventh, respectively, the number of transformations computed in fp64, fp32, and bfloat16; the last column shows the truncation point which corresponds to the sum of the previous three columns and which we have reported for convenience. Note that the truncation happens on the same step both in full double precision and mixed precision. The following conclusions can be drawn by these results. First, the error bound slightly exceeds but closely tracks the  $\varepsilon$  threshold; this is perfectly expected because in our error analysis some constants were ignored. Second, the actual error never exceeds the bound, which validates our rounding error analysis. Third, as the  $\varepsilon$  threshold grows, an increasingly large amount of factorization steps are computed using lower-precision arithmetics; clearly, as  $\varepsilon$  exceeds  $u^s \sqrt{n}$ , fp64 is not needed anymore and all computations are done in fp32 and bf16. All these observations are confirmed by experiments conducted on the other matrices of our test set which we omit for the sake of space.

	fp64		fp64/fp32/bf16				
$\varepsilon$	Error	Bound	Error	fp64	fp32	bf16	Trunc.
1.0e-14	9.80e-15	3.42e-14	1.04e-14	1183	597	117	1897
1.0e-12	9.72e-13	2.40e-12	1.04e-12	928	609	99	1636
1.0e-10	9.79e-11	2.27e-10	1.02e-10	669	617	88	1374
1.0e-08	9.75e-09	2.18e-08	1.03e-08	403	629	79	1111
1.0e-06	9.78e-07	2.17e-06	1.02e-06	124	650	75	849
1.0e-05	9.75e-06	1.81e-05	1.02e-05	0	644	72	716
1.0e-04	9.73e-05	1.53e-04	1.01e-04	0	513	70	585
1.0e-03	9.74e-04	1.51e-03	1.02e-03	0	378	69	447
1.0e-02	9.75e-03	1.53e-02	1.02e-02	0	242	67	309

TABLE 1  
Error and use of precisions for the randsvd matrix of size 2048.

	fp64		fp64/fp32/bf16				
$\varepsilon$	Error	Bound	Error	fp64	fp32	bf16	Trunc.
1.0e-14	1.20e-15	2.44e-13	8.36e-14	1849	198	1	2048
1.0e-12	1.20e-15	2.02e-12	6.50e-13	774	1271	3	2048
1.0e-10	4.07e-11	2.76e-10	5.54e-11	140	1848	58	2046
1.0e-08	9.92e-09	2.75e-08	1.13e-08	24	1242	482	1748
1.0e-06	9.93e-07	2.89e-06	1.05e-06	6	284	151	441
1.0e-05	9.83e-06	2.59e-05	1.02e-05	0	116	71	187
1.0e-04	9.55e-05	2.14e-04	1.05e-04	0	50	25	75
1.0e-03	9.12e-04	2.33e-03	1.11e-03	0	21	9	30
1.0e-02	7.62e-03	2.20e-02	1.19e-02	0	10	4	14

TABLE 2  
Error and use of precisions for the phillips matrix of size 2048.

In Figure 1 we report results of an image compression experiment for an image of size  $1600 \times 1057$ . In this figure we compare the original image (*top-left*) with images that are compressed using a truncated QR factorization in full fp32 (*top-right*), fp32/bf16 (*bottom-left*) and full bf16 (*bottom-right*) and then reconstraucted

455 in fp32. Here, the truncation threshold  $\varepsilon$  was set to 0.04 and thus only fp32 and  
 456 bf16 arithmetics were used; the truncation happened on column 191 in all the three  
 457 reconstructed images. It can clearly be seen that using bf16 only it is not possible to  
 458 achieve a satisfactory compression. Both the other two approaches, instead, achieve a  
 459 satisfactory result although, in the mixed-precision case, only 12 out of 190 columns  
 460 are computed and stored in fp32 and the rest in bf16 which might result in considerable  
 461 time savings especially on modern GPUs where bf16 computations are much faster  
 462 than fp32 ones.

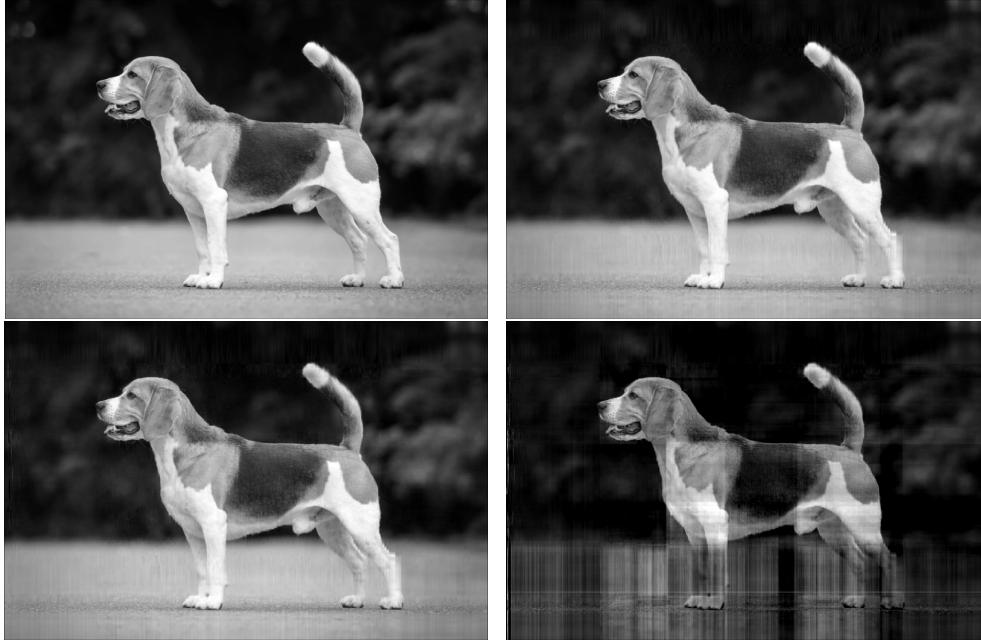


FIG. 1. *Image compression. Original image (top-left); reconstructed image after compression in full fp32 (top-right), in fp32/bf16 (bottom-left) and full bf16 (bottom-right).  $\varepsilon$  was set to 0.04 and the truncation happened at column 190 but in the mixed-precision case only 12 transformations were computed in fp32 and the rest in bf16. Image size is 1057 × 1600.*

463     **5.2. Performance analysis with optimized code.** In this section we evaluate  
 464 the performance improvement brought by the use of the mixed-precision algorithm  
 465 proposed in subsection 4.3 using both Businger-Golub and randomized pivoting com-  
 466 pared with the full, high-precision corresponding variants. For these experiments  
 467 we have implemented Algorithm 4.1, Algorithm 4.2 and Algorithm 4.3 in the For-  
 468 tran language. The code for Algorithm 4.1 is a straightforward adaptation of the  
 469 DGEQP3/SQEQP3 and DLAQPS/SLAQPS routines in LAPACK. Algorithm 4.2,  
 470 instead, is implemented with an entirely new code. Both codes heavily rely on some  
 471 advanced LAPACK and BLAS routines for which no low-precision (e.g., fp16 or bf16)  
 472 implementation is currently available neither for CPUs or GPUs. For this reason, our  
 473 performance analysis is limited to variants that employ fp64 and fp32.

474     Our experiments were conducted on an AMD Zen3 EPYC 7763 processor, using  
 475 the BLAS and LAPACK routines in the Intel MKL 2020 package and the GNU  
 476 gfortran 11.2 Fortran compiler. All experiments are sequential; although parallelism  
 477 could be straightforwardly used within BLAS operations, we have chosen not to do

478 so because this does not bring any specific insight on the behavior of the proposed  
 479 methods.

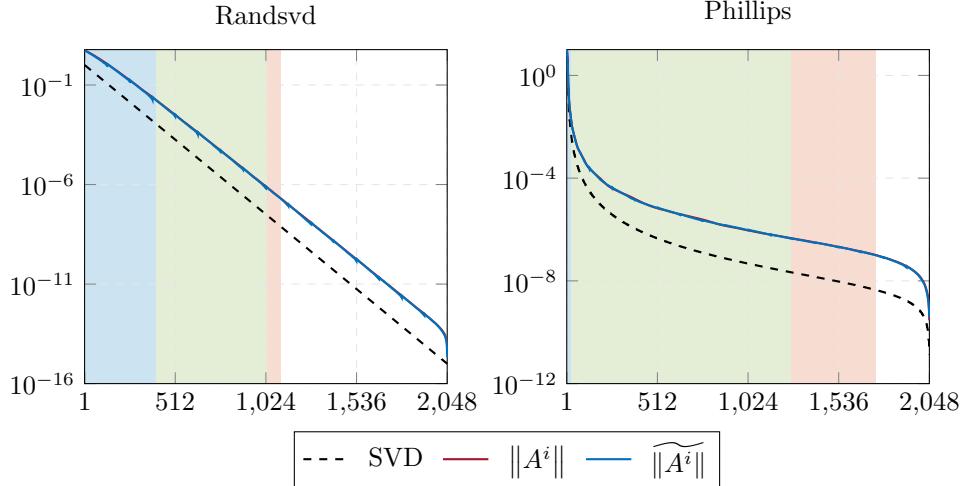


FIG. 2. The black curves show the spectrum of the randsvd and phillips (left and right, respectively) matrices of size 2048. The red and blue curves show the trailing submatrix norm and its estimate as computed in the factorization with Businger-Golub and randomized pivoting, respectively; the two curves are indistinguishable at low resolution and therefore only the blue one is visible. The vertical bands show number of transformations computed in fp64, fp32 and bf16, respectively, for  $\varepsilon = 10^{-8}$  and correspond to the values reported in Tables 1 and 2.

480 As a further validation of the numerical behavior of our methods, in Figure 2, we  
 481 report the spectrum (in black) of the randsvd and phillips matrices of size 2048 along  
 482 with the two criteria that are used within the factorization to decide when to switch  
 483 precision and when to halt the process. Namely, in red we report the norm of the  
 484 trailing submatrix as computed within the factorization with Businger-Golub pivoting  
 485 and, in blue, its estimate computed within the factorization with randomized pivoting,  
 486 as described in the end of section 4.2. These two curves are almost superposed, which  
 487 means that the estimate is very accurate; this is actually the case for all of our test  
 488 problems. Upon a closer inspection, our experiment reveal that the value computed  
 489 in the factorization with randomized pivoting slightly underestimates the norm of the  
 490 trailing submatrix which translates into the fact that the precision switch and the  
 491 truncation happen, on average, slightly earlier than in the case where Businger-Golub  
 492 pivoting is used. When only two precisions (double and single) are used, on all our  
 493 test matrices, the median underestimation of the precision switch and the rank is  
 494 below 1% and in the worst case less than 4%. Despite this fact, the final low-rank  
 495 approximation accuracy obtained by the factorization with randomized pivoting was  
 496 always consistent with the error bound and essentially the same as that obtained in  
 497 the case where Businger-Golub pivoting is used; as explained in the end of section 4.2,  
 498 a more conservative criterion can be used in the case of randomized pivoting which can  
 499 offer better robustness although it would likely lead to an excessively late precision  
 500 switch and truncation.

501 Figure 3 shows, for the matrices in our test set, the execution time for double-  
 502 precision (dark color) and mixed single/double-precision (light color) truncated QR  
 503 factorization algorithms with Businger-Golub (yellow) and randomized (blue) pivoting

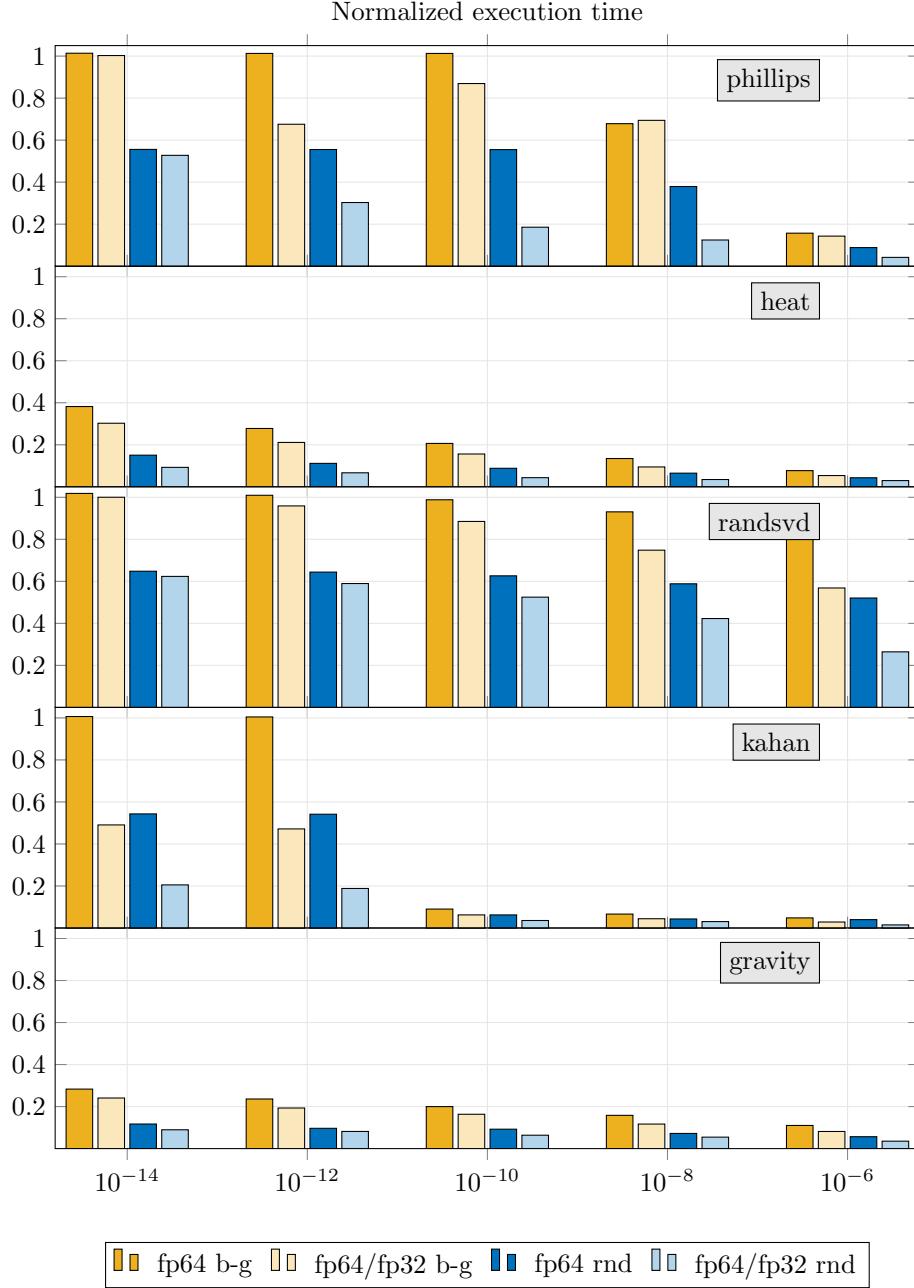


FIG. 3. Execution times for double-precision (dark color) and mixed single/double-precision (light color) truncated QR factorization algorithms with Businger-Golub (yellow) and randomized (blue) pivoting with respect to the truncation threshold  $\varepsilon$ . All values are normalized to the execution time of the corresponding full double-precision QR factorization with Businger-Golub pivoting. All the matrices are of size 8192.

504 with respect to the truncation threshold  $\varepsilon$ . For each matrix, the values are normalized  
 505 to the execution time of the corresponding full double-precision QR factorization with

506 Businger-Golub pivoting.

507 The behavior of the proposed algorithms varies considerably across the test prob-  
508 lems. Nevertheless, some conclusions can be drawn.

509 Obviously, the benefit of using the mixed-precision algorithms heavily depends  
510 on the spectrum of the problem and the distribution of its singular values which ultil-  
511 mately determines the ratio of operations that are done in double and single precision.  
512 As the truncation threshold increases, this ratio normally evolves favorably making  
513 the potential benefit of mixed precision higher. This is clearly visible on the randsvd  
514 matrix. Although this trend also applies to the other matrices, the execution time  
515 does not always evolve correspondingly. This is due to the fact that the ratio of double  
516 and single-precision operations alone does not entirely describe performance but other  
517 factors must be taken into account. One important factor is the arithmetic intensity  
518 of operations. Despite the fact that a fixed panel size is chosen for all algorithms, as  
519 explained in subsection 2.2, a panel reduction may be interrupted if the norm of some  
520 columns drops beyond a prescribed value which eventually reduces the granularity of  
521 operations and, consequently, their speed. This happens in a hardly predictable way  
522 and may adversely affect the speed of computation in either double or single precision.  
523 This behavior is clearly visible on the phillips matrix comparing the results obtained  
524 on the Businger-Golub case with  $\varepsilon = 10^{-12}$  and  $\varepsilon = 10^{-8}$ : in the second case, the  
525 single to double-precision operations ratio is more favorable but a large number of  
526 restarts happens during the single-precision computations. As a result, the mixed-  
527 precision algorithm is slightly slower than the double-precision one, whereas, in the  
528 first case, it is 33% faster. Note that in the variant with randomized pivoting, restarts  
529 happen in the pivoted factorization of the sample matrix; however, this operation only  
530 accounts for a small fraction of the overall operational complexity and, therefore, there  
531 restarts have a limited impact. Note that, on standard CPUs, single-precision com-  
532 putations are expected to be twice as fast as double-precision ones. For CPU-bound  
533 operations, this is mainly related to the use of vector units (one vector instruction can  
534 do twice as many fp32 operations as fp64 ones) whereas for memory-bound operations  
535 this is due to a better use of the memory bandwidth (twice as many fp32 coefficients  
536 can be transferred as fp64 in the same time). Nevertheless, this assumption does not  
537 take into account other factors related to the use of cache memories. Although we  
538 cannot assume that the original matrix fits into cache, at some point of the factor-  
539 ization the trailing submatrix, whose size is smaller and smaller, will; when single  
540 precision is used, this will happen at an earlier step of the factorization with respect  
541 to the double-precision case. Although we haven't conducted dedicated experiments  
542 to validate this effect, we speculate that it can explain the fact that in some cases the  
543 mixed-precision algorithm is more than twice as fast as the full double-precision one  
544 (for example the randomized algorithm on the phillips matrix at  $\varepsilon = 10^{-10}$  or the  
545 kahan matrix at  $\varepsilon = 10^{-12}$ ).

546 **6. Conclusions and future work.** In this work we have introduced a mixed-  
547 precision truncated Householder QR factorization where the arithmetic precision of  
548 computations is gradually reduced as the norm of the trailing submatrix decreases.  
549 We presented an error analysis that results in an error bound demonstrating that if  
550 these changes of precision are appropriately operated, the resulting mixed-precision  
551 low-rank representation has the same accuracy as in the case where all computations  
552 are done in high precision.

553 Based on our theoretical findings, we have presented two Householder QR factor-  
554 ization algorithms based on Businger-Golub and randomized pivoting, respectively.

555 Pivoting is not necessary for the mixed-precision algorithm to work because this simply  
 556 relies on the assumption that the Frobenius norm of the trailing submatrix decreases,  
 557 which holds true regardless of pivoting; nevertheless, if pivoting is applied, the  
 558 trailing submatrix norm decays much faster, which ultimately leads to more compact  
 559 low-rank representations and more efficient use of low-precision arithmetics.

560 We have presented a twofold experimental analysis. First we focused on validating  
 561 the theoretical analysis. We did this using a prototype written in the Julia  
 562 language where we could use up to three different precisions. The corresponding  
 563 experiments validate the presented theoretical analysis. Second, we evaluated the  
 564 performance of the two proposed mixed-precision algorithms using double and single-  
 565 precision arithmetics. Our experimental results on synthetic matrices with different  
 566 spectra demonstrate that the mixed-precision algorithms can achieve better perfor-  
 567 mance than the full double-precision counterparts, sometimes exceeding a factor of  
 568 two.

569 Some opportunities can be identified for pushing the presented ideas further.  
 570 First, the performance of the mixed-precision algorithms must be evaluated using  
 571 even lower-precision arithmetic such as Float16 or BFloat16 which, on some hard-  
 572 ware, achieve much higher performance than single precision; these are supported in  
 573 hardware on modern CPUs and GPUs but the corresponding LAPACK and BLAS  
 574 libraries are still lacking, which prevents us from implementing the mixed-precision  
 575 algorithms. Second, we must investigate whether and how our approach can be used  
 576 with other pivoting strategies such as tournament pivoting which might be better  
 577 suited to parallel implementations. Finally, we would like to study scaling algorithms  
 578 to prevent issues related to overflow and underflow in low-precision computations.

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<sup>3</sup><https://www.plafrim.fr>

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