

SI231B: Matrix Computations, 2025 Fall

Homework Set #3

Acknowledgements:

- 1) Deadline: **2025-11-29 23:59:59**
 - 2) Please submit the PDF file to [gradescope](#). Course entry code: N2382J.
 - 3) You have 5 “free days” in total for all late homework submissions.
 - 4) If your homework is handwritten, please make it clear and legible.
 - 5) All your answers are required to be in English.
 - 6) Write down the major steps for deriving the solution; otherwise you may loss points.
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Problem 1. (Eigenvalue) (15 points)

Let $\mathbf{A} \in \mathbb{R}^{3 \times 3}$ be a real symmetric matrix satisfying the matrix equation $\mathbf{A}^3 - 2\mathbf{A}^2 - \mathbf{A} + 2\mathbf{I} = \mathbf{0}$.

- 1) Find all eigenvalues of matrix \mathbf{A} . (10 points)
- 2) Compute $\det(\mathbf{A} + \mathbf{I})$. (5 points)

Solution:

Problem 2. (Eigenvector and similarity) (15 points)

- 1) Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ have two distinct eigenvalues λ_1 and λ_2 with corresponding eigenvectors \mathbf{v}_1 and \mathbf{v}_2 respectively.
Prove that: If $k_1\mathbf{v}_1 + k_2\mathbf{v}_2$ (where the scalars k_1, k_2 are not both zero) is an eigenvector of \mathbf{A} , then $k_1k_2 = 0$.
(7 points)
- 2) Suppose two matrices have the same characteristic polynomial, determinant, rank, nullity, trace, eigenvalues, algebraic multiplicity, geometric multiplicity. Are they similar? Justify your answer. **(8 points)**

Solution:

Problem 3. (Diagonalization)

(20 points)

- 1) Determine if each of the following matrices is diagonalizable.

$$\mathbf{A} = \begin{bmatrix} -2 & 1 & 1 \\ 0 & 2 & 0 \\ -4 & 1 & 3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 2 & 4 & 0 \\ 0 & 2 & 0 \\ 0 & 5 & 2 \end{bmatrix}$$

(10 points)

- 2) If it is diagonalizable, diagonalize the matrix using a similarity transformation. (5 points)
3) Compute \mathbf{A}^{100} . (5 points)

Solution:

Problem 4. (Schur Decomposition) (20 points)

1) Let

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{bmatrix}.$$

Compute the real Schur decomposition $A = UTU^T$, where U is a real orthogonal matrix and T is upper triangular. **The results are reported rounded to four decimal places. (12 points)**

- 2) Suppose $A \in \mathbb{C}^{n \times n}$ has n distinct eigenvalues. Show that if $Q^H A Q = T$ is its Schur decomposition and $AB = BA$, then $Q^H B Q$ is upper triangular. (Hint: let $T \in \mathbb{C}^{n \times n}$ be an upper triangular matrix whose diagonal entries t_{11}, \dots, t_{nn} are distinct. If $C \in \mathbb{C}^{n \times n}$ satisfies $TC = CT$, then C must be upper triangular.) **(8 points)**

Solution:

Problem 5. (Variational Characterizations) (10 points)

For matrix $\mathbf{A} \in \mathbb{R}^{r \times r}$, prove $\|\mathbf{A}\|_2 = \max_{\|\mathbf{z}\|_2=1} \|\mathbf{A}\mathbf{z}\|_2 = \max_{\|\mathbf{z}\|_2=\|\mathbf{y}\|_2=1} |\mathbf{y}^T \mathbf{A}\mathbf{z}|$.

Solution:

Problem 6. (Power Iteration)

(20 points)

Consider

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, B = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}.$$

- 1) Apply the power method with infinite-norm normalization to each matrix (Compute only four steps), starting from the initial vector $\mathbf{x}^{(0)} = \begin{bmatrix} -1 & 2 \end{bmatrix}^T$. Note: To implement the infinite-norm normalization, replace the normalization step $\mathbf{v}^{(k)} = \frac{\tilde{\mathbf{v}}^{(k)}}{\|\tilde{\mathbf{v}}^{(k)}\|_2}$ used in the slides with $\mathbf{v}^{(k)} = \frac{\tilde{\mathbf{v}}^{(k)}}{\|\tilde{\mathbf{v}}^{(k)}\|_\infty}$, where $\|\cdot\|_\infty$ denotes the infinity norm. $\mathbf{v}^{(k)}$ approximates a dominant eigenvector at iteration k and $\lambda^{(k)} = (\mathbf{A}\mathbf{v}^{(k)})^T \mathbf{v}^{(k)} / \|\mathbf{v}^{(k)}\|_2^2$. **The results are reported rounded to four decimal places.** (10 points)
- 2) Compute the ratios λ_2/λ_1 for A and B , where λ_1 and λ_2 denote the eigenvalues of largest and second-largest magnitude, respectively. For which matrix do you expect faster convergence of the power method? Explain your reason. (10 points)

Solution: