

Matrix Computation Homework 1

Yifan Zhang 2025251018 zhangyf52025@shanghaitech.edu.cn

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Problem 1. (Discrete-Time LTI Systems)

1)

Solution.

According to the problem description, \mathbf{A} is a convolution matrix constructed from $h[n] = [h_0, h_1, h_2]$. $\mathbf{y} = \mathbf{A}\mathbf{x}$ describes the linear convolution $y[n] = x[n] * h[n]$.

$$y[n] = \sum_{k=0}^p h[k] \cdot x[n-k] = h_0x[n] + h_1x[n-1] + h_2x[n-2]$$

Since there is no noise, we can directly solve for h_0, h_1, h_2 using the first few outputs.

- **Solving for h_0 :**

$$y[0] = h_0x[0] + h_1x[-1] + h_2x[-2]$$

Since $x[n < 0] = 0$, we have:

$$y[0] = h_0x[0]$$

$$1 = h_0 \cdot 1 \implies \mathbf{h_0 = 1}$$

- **Solving for h_1 :**

$$y[1] = h_0x[1] + h_1x[0] + h_2x[-1]$$

$$y[1] = h_0x[1] + h_1x[0]$$

$$2 = (1) \cdot (0) + h_1 \cdot (1) \implies \mathbf{h_1 = 2}$$

- **Solving for h_2 :**

$$y[2] = h_0x[2] + h_1x[1] + h_2x[0]$$

$$4 = (1) \cdot (1) + (2) \cdot (0) + h_2 \cdot (1)$$

$$4 = 1 + h_2 \implies \mathbf{h_2 = 3}$$

We have obtained the channel impulse response $\mathbf{h} = [1, 2, 3]^T$.

Since

$$\mathbf{A} = \begin{pmatrix} h_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ h_1 & h_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ h_2 & h_1 & h_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & h_2 & h_1 & h_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & h_2 & h_1 & h_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & h_2 & h_1 & h_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & h_2 & h_1 & h_0 & 0 \\ 0 & 0 & 0 & 0 & 0 & h_2 & h_1 & h_0 \end{pmatrix}$$

Substituting $h_0 = 1, h_1 = 2, h_2 = 3$, we solve for \mathbf{A} as:

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 2 & 1 \end{pmatrix}$$

The full equation for $\mathbf{y} = \mathbf{Ax}$ is:

$$\begin{pmatrix} 1 \\ 2 \\ 4 \\ 2 \\ 4 \\ 3 \\ 5 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

b)

Solution.

Here, $\mathbf{y} = \mathbf{Ch}$, where \mathbf{C} is the convolution matrix constructed from the input \mathbf{x} , and $\hat{\mathbf{h}}$ is the channel response vector we want to estimate.

Constructing the matrix \mathbf{C}

It is obviously that:

$$\mathbf{C} = \begin{pmatrix} x[0] & x[-1] & x[-2] \\ x[1] & x[0] & x[-1] \\ x[2] & x[1] & x[0] \\ x[3] & x[2] & x[1] \\ x[4] & x[3] & x[2] \\ x[5] & x[4] & x[3] \\ x[6] & x[5] & x[4] \\ x[7] & x[6] & x[5] \end{pmatrix}$$

Substituting $\mathbf{x} = [1, 0, 1, 0, 1, 1, 0, 1]$ and $x[n < 0] = 0$:

$$\mathbf{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

Setting up the Normal Equations

The least-squares solution $\hat{\mathbf{h}}$ is given by the normal equations:

$$(\mathbf{C}^T \mathbf{C}) \hat{\mathbf{h}} = \mathbf{C}^T \mathbf{y}$$

Calculating $\mathbf{C}^T \mathbf{C}$

$$\mathbf{C}^T \mathbf{C} = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 1 & 3 \\ 1 & 4 & 1 \\ 3 & 1 & 4 \end{pmatrix}$$

Calculating $\mathbf{C}^T \mathbf{y}$

$$\mathbf{C}^T \mathbf{y} = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 4 \\ 2 \\ 4 \\ 3 \\ 5 \\ 4 \end{pmatrix} = \begin{pmatrix} 16 \\ 12 \\ 17 \end{pmatrix}$$

Solving for $\hat{\mathbf{h}}$

We now need to solve the linear system:

$$\begin{pmatrix} 5 & 1 & 3 \\ 1 & 4 & 1 \\ 3 & 1 & 4 \end{pmatrix} \begin{pmatrix} h_0 \\ h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} 16 \\ 12 \\ 17 \end{pmatrix}$$

$$\hat{\mathbf{h}} = \begin{pmatrix} h_0 \\ h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Problem 2. (Gram-Schmidt Procedure)

1)

Solution.

Step 1: Compute \mathbf{q}_1

$$r_{11} = \|\mathbf{a}_1\|_2 = \sqrt{1^2 + 1^2 + 1^2 + 1^2} = 2$$

$$\mathbf{q}_1 = \frac{\mathbf{a}_1}{r_{11}} = \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{bmatrix}$$

Step 2: Compute \mathbf{q}_2

$$r_{12} = \mathbf{q}_1^T \mathbf{a}_2 = 0.5(0) + 0.5(2) + 0.5(1) + 0.5(1) = 2$$

$$\mathbf{v}_2 = \mathbf{a}_2 - r_{12}\mathbf{q}_1 = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 1 \end{bmatrix} - 2 \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$r_{22} = \|\mathbf{v}_2\|_2 = \sqrt{2}$$

$$\mathbf{q}_2 = \frac{\mathbf{v}_2}{r_{22}} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix}$$

Step 3: Compute \mathbf{q}_3

$$r_{13} = \mathbf{q}_1^T \mathbf{a}_3 = 3$$

$$r_{23} = \mathbf{q}_2^T \mathbf{a}_3 = \frac{1}{\sqrt{2}}(-3 + 1) = -\sqrt{2}$$

$$\mathbf{v}_3 = \mathbf{a}_3 - r_{13}\mathbf{q}_1 - r_{23}\mathbf{q}_2 = \begin{bmatrix} 3 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1.5 \\ 1.5 \\ 1.5 \\ 1.5 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.5 \\ -0.5 \\ -0.5 \end{bmatrix}$$

$$r_{33} = \|\mathbf{v}_3\|_2 = 1$$

$$\mathbf{q}_3 = \begin{bmatrix} 0.5 \\ 0.5 \\ -0.5 \\ -0.5 \end{bmatrix}$$

Result:

$$\mathbf{Q} = \begin{bmatrix} 0.5 & -1/\sqrt{2} & 0.5 \\ 0.5 & 1/\sqrt{2} & 0.5 \\ 0.5 & 0 & -0.5 \\ 0.5 & 0 & -0.5 \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} 2 & 2 & 3 \\ 0 & \sqrt{2} & -\sqrt{2} \\ 0 & 0 & 1 \end{bmatrix}$$

2) QR Decomposition of \mathbf{A}_1 (Modified Gram-Schmidt)

Initialization Let $\mathbf{v}_1 = \mathbf{a}_1, \mathbf{v}_2 = \mathbf{a}_2, \mathbf{v}_3 = \mathbf{a}_3$.

Step 1

$$r_{11} = \|\mathbf{v}_1\| = 2, \quad \mathbf{q}_1 = \mathbf{v}_1/2 = [0.5, 0.5, 0.5, 0.5]^T$$

Update $\mathbf{v}_2, \mathbf{v}_3$:

$$r_{12} = \mathbf{q}_1^T \mathbf{v}_2 = 2 \implies \mathbf{v}_2^{(2)} = \mathbf{v}_2 - r_{12}\mathbf{q}_1 = [-1, 1, 0, 0]^T$$

$$r_{13} = \mathbf{q}_1^T \mathbf{v}_3 = 3 \implies \mathbf{v}_3^{(2)} = \mathbf{v}_3 - r_{13}\mathbf{q}_1 = [1.5, -0.5, -0.5, -0.5]^T$$

Step 2

$$r_{22} = \|\mathbf{v}_2^{(2)}\| = \sqrt{2}, \quad \mathbf{q}_2 = \mathbf{v}_2^{(2)}/\sqrt{2} = \left[\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0\right]^T$$

Update \mathbf{v}_3 :

$$r_{23} = \mathbf{q}_2^T \mathbf{v}_3^{(2)} = -\sqrt{2}$$

$$\mathbf{v}_3^{(3)} = \mathbf{v}_3^{(2)} - r_{23}\mathbf{q}_2 = [0.5, 0.5, -0.5, -0.5]^T$$

Step 3

$$r_{33} = \|\mathbf{v}_3^{(3)}\| = 1, \quad \mathbf{q}_3 = [0.5, 0.5, -0.5, -0.5]^T$$

Result:

$$\mathbf{Q} = \begin{bmatrix} 0.5 & -1/\sqrt{2} & 0.5 \\ 0.5 & 1/\sqrt{2} & 0.5 \\ 0.5 & 0 & -0.5 \\ 0.5 & 0 & -0.5 \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} 2 & 2 & 3 \\ 0 & \sqrt{2} & -\sqrt{2} \\ 0 & 0 & 1 \end{bmatrix}$$

3) QR Decomposition of \mathbf{A}_2

Given $\mathbf{A}_2 = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4]$ with rank 3. We compute $\mathbf{Q} \in \mathbb{R}^{6 \times 3}$ and $\mathbf{R} \in \mathbb{R}^{3 \times 4}$.

Processing Columns

Vector \mathbf{a}_1 :

$$\|\mathbf{a}_1\| = 2$$

$$\mathbf{q}_1 = [0.5, 0.5, 0, 0, 0.5, 0.5]^T, \quad r_{11} = 2$$

Vector \mathbf{a}_2 :

$$r_{12} = \mathbf{q}_1^T \mathbf{a}_2 = 1$$

$$\mathbf{v}_2 = \mathbf{a}_2 - r_{12}\mathbf{q}_1 = [-0.5, 0.5, 0, 1, 0.5, -0.5]^T, \quad \|\mathbf{v}_2\| = \sqrt{2}$$

$$\mathbf{q}_2 = \frac{1}{\sqrt{2}}[-0.5, 0.5, 0, 1, 0.5, -0.5]^T, \quad r_{22} = \sqrt{2}$$

Vector \mathbf{a}_3 :

$$r_{13} = \mathbf{q}_1^T \mathbf{a}_3 = 7, \quad r_{23} = \mathbf{q}_2^T \mathbf{a}_3 = \sqrt{2}$$

Since $\mathbf{v}_3 = \mathbf{a}_3 - 7\mathbf{q}_1 - \sqrt{2}\mathbf{q}_2 = \mathbf{0}$. Thus, no new basis vector is formed.

$$\begin{aligned} r_{14} &= \mathbf{q}_1^T \mathbf{a}_4 = 1.5 \\ r_{24} &= \mathbf{q}_2^T \mathbf{a}_4 = \frac{1}{\sqrt{2}}(0.5) = \frac{1}{2\sqrt{2}} \\ \mathbf{v}_4 &= \mathbf{a}_4 - 1.5\mathbf{q}_1 - \frac{1}{2\sqrt{2}}\mathbf{q}_2 = \frac{1}{8}[3, 1, 8, -2, 1, -5]^T \\ r_{34} &= \|\mathbf{v}_4\| = \frac{\sqrt{26}}{4} \\ \mathbf{q}_3 &= \frac{\mathbf{v}_4}{r_{34}} = \frac{1}{2\sqrt{26}}[3, 1, 8, -2, 1, -5]^T \end{aligned}$$

Result:

$$\mathbf{Q} = \begin{bmatrix} 0.5 & \frac{-1}{2\sqrt{2}} & \frac{3}{2\sqrt{26}} \\ 0.5 & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{26}} \\ 0 & 0 & \frac{2\sqrt{26}}{8} \\ 0 & \frac{1}{\sqrt{2}} & \frac{-2}{2\sqrt{26}} \\ 0.5 & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{26}} \\ 0.5 & \frac{-1}{2\sqrt{2}} & \frac{-5}{2\sqrt{26}} \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} 2 & 1 & 7 & 1.5 \\ 0 & \sqrt{2} & \sqrt{2} & \frac{\sqrt{2}}{4} \\ 0 & 0 & 0 & \frac{\sqrt{26}}{4} \end{bmatrix}$$

4)

Since $A = QR$, $Ax = b$, we can easily get $\mathbf{R}\mathbf{x} = \mathbf{Q}^T\mathbf{b}$:

$$\mathbf{Q}^T\mathbf{b} = \begin{bmatrix} \mathbf{q}_1^T \mathbf{b} \\ \mathbf{q}_2^T \mathbf{b} \\ \mathbf{q}_3^T \mathbf{b} \end{bmatrix} = \begin{bmatrix} 0.5(4) + 0 + 0.5(4) + 0 \\ \frac{-1}{\sqrt{2}}(4) + 0 + 0 + 0 \\ 0.5(4) + 0 - 0.5(4) + 0 \end{bmatrix} = \begin{bmatrix} 4 \\ -2\sqrt{2} \\ 0 \end{bmatrix}$$

Solving the system:

$$\begin{bmatrix} 2 & 2 & 3 \\ 0 & \sqrt{2} & -\sqrt{2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -2\sqrt{2} \\ 0 \end{bmatrix}$$

- $1 \cdot x_3 = 0 \implies x_3 = 0.$
- $\sqrt{2}x_2 - \sqrt{2}(0) = -2\sqrt{2} \implies x_2 = -2.$
- $2x_1 + 2(-2) + 3(0) = 4 \implies 2x_1 - 4 = 4 \implies 2x_1 = 8 \implies x_1 = 4.$

Solution: $\mathbf{x} = [4, -2, 0]^T$.

Problem 3. (Householder Reflection)

1)

Step 1: First Householder Reflection (H_1)

We focus on the first column of A , $\mathbf{x} = A_1 = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$.

First, find its norm:

$$\|\mathbf{x}\|_2 = \sqrt{1^2 + (-2)^2 + 2^2} = 3$$

Next, we define the vector \mathbf{v}_1 . The problem states to choose the sign \mp to maximize $\|\mathbf{v}\|_2$.

$$\begin{aligned} \bullet \quad \mathbf{v}_- &= \mathbf{x} - \|\mathbf{x}\|_2 \mathbf{e}_1 = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} - 3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ 2 \end{bmatrix}. \quad \|\mathbf{v}_-\|_2^2 = 4 + 4 + 4 = 12. \\ \bullet \quad \mathbf{v}_+ &= \mathbf{x} + \|\mathbf{x}\|_2 \mathbf{e}_1 = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix}. \quad \|\mathbf{v}_+\|_2^2 = 16 + 4 + 4 = 24. \end{aligned}$$

So, $\mathbf{v}_1 = \begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix}$, and $\|\mathbf{v}_1\|_2^2 = 24$.

The Householder reflector is $H_1 = I - 2 \frac{\mathbf{v}_1 \mathbf{v}_1^T}{\|\mathbf{v}_1\|_2^2}$:

$$\begin{aligned} \mathbf{v}_1 \mathbf{v}_1^T &= \begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix} \begin{bmatrix} 4 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 16 & -8 & 8 \\ -8 & 4 & -4 \\ 8 & -4 & 4 \end{bmatrix} \\ H_1 &= I - \frac{2}{24} \begin{bmatrix} 16 & -8 & 8 \\ -8 & 4 & -4 \\ 8 & -4 & 4 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} -1 & 2 & -2 \\ 2 & 2 & 1 \\ -2 & 1 & 2 \end{bmatrix} \end{aligned}$$

Step 2: Apply H_1 to A

We compute $A^{(1)} = H_1 A$:

$$A^{(1)} = \frac{1}{3} \begin{bmatrix} -1 & 2 & -2 \\ 2 & 2 & 1 \\ -2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & -3 \\ -2 & -5 & 20 \\ 2 & 8 & 3 \end{bmatrix} = \begin{bmatrix} -3 & -9 & 37/3 \\ 0 & 0 & 37/3 \\ 0 & 3 & 32/3 \end{bmatrix}$$

As expected, the first column below the diagonal is zero.

Step 3: Second Householder Reflection (H_2)

We now work on the 2×2 submatrix $\begin{bmatrix} 0 & 37/3 \\ 3 & 32/3 \end{bmatrix}$.

We want to zero out the $(2, 1)$ entry of this submatrix.

Let $\mathbf{x}' = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$. $\|\mathbf{x}'\|_2 = 3$. We choose $\mathbf{v}' = \mathbf{x}' + \|\mathbf{x}'\|_2 \mathbf{e}_1 = \begin{bmatrix} 0 \\ 3 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$.
 $\|\mathbf{v}'\|_2^2 = 9 + 9 = 18$.

The 2×2 reflector H' is:

$$H' = I - 2 \frac{\mathbf{v}' \mathbf{v}'^T}{\|\mathbf{v}'\|_2^2} = I - \frac{2}{18} \begin{bmatrix} 3 \\ 3 \end{bmatrix} \begin{bmatrix} 3 & 3 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

This is a simple permutation matrix. We embed this into a 3×3 matrix H_2 :

$$H_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

Step 4: Find R

The final upper triangular matrix R is $R = H_2 A^{(1)}$:

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} -3 & -9 & 37/3 \\ 0 & 0 & 37/3 \\ 0 & 3 & 32/3 \end{bmatrix} = \begin{bmatrix} -3 & -9 & 37/3 \\ 0 & -3 & -32/3 \\ 0 & 0 & -37/3 \end{bmatrix}$$

Step 5: Find Q

We have $R = H_2 A^{(1)} = H_2 H_1 A$. So, $A = (H_2 H_1)^{-1} R = (H_1^{-1} H_2^{-1}) R$.

Since H_1 and H_2 are orthogonal and symmetric,

$$A = (H_1 H_2) R$$

Thus, $Q = H_1 H_2$.

$$Q = \left(\frac{1}{3} \begin{bmatrix} -1 & 2 & -2 \\ 2 & 2 & 1 \\ -2 & 1 & 2 \end{bmatrix} \right) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} -1 & 2 & -2 \\ 2 & -1 & -2 \\ -2 & -2 & -1 \end{bmatrix}$$

Conclusion An orthonormal basis for $R(A)$ is formed by the columns of Q :

$$\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\} = \left\{ \frac{1}{3} \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix}, \frac{1}{3} \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}, \frac{1}{3} \begin{bmatrix} -2 \\ -2 \\ -1 \end{bmatrix} \right\}$$

2)

$$A^T A = (QR)^T (QR) = R^T Q^T Q R = R^T R$$

Therefore:

$$(A^T A)^{-1} = (R^T R)^{-1} = R^{-1} (R^T)^{-1} = R^{-1} (R^{-1})^T$$

We first need to find R^{-1} from $R = \begin{bmatrix} -3 & -9 & 37/3 \\ 0 & -3 & -32/3 \\ 0 & 0 & -37/3 \end{bmatrix}$.

We can find the inverse of this upper triangular matrix by back substitution or row reduction.

$$\left[\begin{array}{ccc|ccc} -3 & -9 & 37/3 & 1 & 0 & 0 \\ 0 & -3 & -32/3 & 0 & 1 & 0 \\ 0 & 0 & -37/3 & 0 & 0 & 1 \end{array} \right]$$

We can get $R^{-1} = \frac{1}{111} \begin{bmatrix} -37 & 111 & -133 \\ 0 & -37 & 32 \\ 0 & 0 & -9 \end{bmatrix}$.

Now we compute $(A^T A)^{-1} = R^{-1} (R^{-1})^T$:

$$(R^{-1})^T = \frac{1}{111} \begin{bmatrix} -37 & 0 & 0 \\ 111 & -37 & 0 \\ -133 & 32 & -9 \end{bmatrix}$$

$$\begin{aligned} (A^T A)^{-1} &= \frac{1}{111^2} \begin{bmatrix} -37 & 111 & -133 \\ 0 & -37 & 32 \\ 0 & 0 & -9 \end{bmatrix} \begin{bmatrix} -37 & 0 & 0 \\ 111 & -37 & 0 \\ -133 & 32 & -9 \end{bmatrix} \\ &= \frac{1}{12321} \begin{bmatrix} 1369 + 12321 + 17689 & -4107 - 4256 & 1197 \\ -4107 - 4256 & 1369 + 1024 & -288 \\ 1197 & -288 & 81 \end{bmatrix} \end{aligned}$$

Thus, the inverse of $A^T A$ is:

$$(A^T A)^{-1} = \frac{1}{12321} \begin{bmatrix} 31379 & -8363 & 1197 \\ -8363 & 2393 & -288 \\ 1197 & -288 & 81 \end{bmatrix}$$

Problem 4. (Givens Rotation)

Step 1: Zero out $A(4, 1)$

We apply a rotation G_1 to zero out $A(4, 1) = 2$ using $A(1, 1) = 1$.

Let $a = 1$ and $b = 2$. $r = \sqrt{a^2 + b^2} = \sqrt{5}$, $c = a/r = 1/\sqrt{5}$, $s = b/r = 2/\sqrt{5}$

The rotation matrix G_1 is:

$$G_1 = \begin{bmatrix} c & 0 & 0 & s \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -s & 0 & 0 & c \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} & 0 & 0 & 2/\sqrt{5} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2/\sqrt{5} & 0 & 0 & 1/\sqrt{5} \end{bmatrix}$$

We compute $A_1 = G_1 A$:

$$A_1 = \begin{bmatrix} 1/\sqrt{5} & 0 & 0 & 2/\sqrt{5} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2/\sqrt{5} & 0 & 0 & 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 2 & 2 \\ 2 & 3 & 5 \end{bmatrix} = \begin{bmatrix} \sqrt{5} & 8/\sqrt{5} & 12/\sqrt{5} \\ 0 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & -1/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$$

Step 2: Zero out $A_1(3, 2)$

Next, we apply a rotation G_2 to zero out $A_1(3, 2) = 2$ using $A_1(2, 2) = 1$.

Let $a = 1$ and $b = 2$.

$r = \sqrt{1^2 + 2^2} = \sqrt{5}$, $c = 1/\sqrt{5}$, $s = 2/\sqrt{5}$

$$G_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c & s & 0 \\ 0 & -s & c & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/\sqrt{5} & 2/\sqrt{5} & 0 \\ 0 & -2/\sqrt{5} & 1/\sqrt{5} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We compute $A_2 = G_2 A_1$:

$$A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/\sqrt{5} & 2/\sqrt{5} & 0 \\ 0 & -2/\sqrt{5} & 1/\sqrt{5} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{5} & 8/\sqrt{5} & 12/\sqrt{5} \\ 0 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & -1/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} = \begin{bmatrix} \sqrt{5} & 8/\sqrt{5} & 12/\sqrt{5} \\ 0 & \sqrt{5} & 4/\sqrt{5} \\ 0 & 0 & 2/\sqrt{5} \\ 0 & -1/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$$

Step 3: Zero out $A_2(4, 2)$

We apply a rotation G_3 to zero out $A_2(4, 2) = -1/\sqrt{5}$ using $A_2(2, 2) = \sqrt{5}$.

Let $a = \sqrt{5}$ and $b = -1/\sqrt{5}$. $r = \sqrt{(\sqrt{5})^2 + (-1/\sqrt{5})^2} = \sqrt{26/5}$, $c = a/r = \sqrt{5}/\sqrt{26/5} = 5/\sqrt{26}$, $s = b/r = (-1/\sqrt{5})/\sqrt{26/5} = -1/\sqrt{26}$

$$G_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c & s & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -s & 0 & c \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5/\sqrt{26} & -1/\sqrt{26} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1/\sqrt{26} & 0 & 5/\sqrt{26} \end{bmatrix}$$

We compute $A_3 = G_3 A_2$:

$$A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5/\sqrt{26} & 0 & -1/\sqrt{26} \\ 0 & 0 & 1 & 0 \\ 0 & 1/\sqrt{26} & 0 & 5/\sqrt{26} \end{bmatrix} \begin{bmatrix} \sqrt{5} & 8/\sqrt{5} & 12/\sqrt{5} \\ 0 & \sqrt{5} & 4/\sqrt{5} \\ 0 & 0 & 2/\sqrt{5} \\ 0 & -1/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} = \begin{bmatrix} \sqrt{5} & 8/\sqrt{5} & 12/\sqrt{5} \\ 0 & \sqrt{26/5} & 19/\sqrt{130} \\ 0 & 0 & 2/\sqrt{5} \\ 0 & 0 & 9/\sqrt{130} \end{bmatrix}$$

Step 4: Zero out $A_3(4, 3)$

Finally, we apply a rotation G_4 to zero out $A_3(4, 3) = 9/\sqrt{130}$ using $A_3(3, 3) = 2/\sqrt{5}$.

Let $a = 2/\sqrt{5}$ and $b = 9/\sqrt{130}$.

$$r = \sqrt{(2/\sqrt{5})^2 + (9/\sqrt{130})^2} = \sqrt{37/26}, \quad c = a/r = (2/\sqrt{5})/\sqrt{185/130} = 2\sqrt{26}/\sqrt{185}, \quad s = b/r = (9/\sqrt{130})/\sqrt{185/130} = 9/\sqrt{185}$$

$$G_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2\sqrt{26}/\sqrt{185} & 9/\sqrt{185} \\ 0 & 0 & -9/\sqrt{185} & 2\sqrt{26}/\sqrt{185} \end{bmatrix}$$

We compute $R = G_4 A_3$:

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{bmatrix} \begin{bmatrix} \sqrt{5} & 8/\sqrt{5} & 12/\sqrt{5} \\ 0 & \sqrt{26/5} & 19/\sqrt{130} \\ 0 & 0 & 2/\sqrt{5} \\ 0 & 0 & 9/\sqrt{130} \end{bmatrix} = \begin{bmatrix} \sqrt{5} & 8/\sqrt{5} & 12/\sqrt{5} \\ 0 & \sqrt{26/5} & 19/\sqrt{130} \\ 0 & 0 & \sqrt{185/130} \\ 0 & 0 & 0 \end{bmatrix}$$

This gives our upper triangular matrix R .

Final Matrix R

Rationalizing the denominators, R is:

$$R = \begin{bmatrix} \sqrt{5} & \frac{8\sqrt{5}}{5} & \frac{12\sqrt{5}}{5} \\ 0 & \frac{\sqrt{130}}{5} & \frac{19\sqrt{130}}{130} \\ 0 & 0 & \frac{\sqrt{962}}{26} \\ 0 & 0 & 0 \end{bmatrix}$$

Finding the Matrix Q

Now we find $Q = G_1^T G_2^T G_3^T G_4^T$.

$$G_1^T = \begin{bmatrix} 1/\sqrt{5} & 0 & 0 & -2/\sqrt{5} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2/\sqrt{5} & 0 & 0 & 1/\sqrt{5} \end{bmatrix}, \quad G_2^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/\sqrt{5} & -2/\sqrt{5} & 0 \\ 0 & 2/\sqrt{5} & 1/\sqrt{5} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$G_3^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5/\sqrt{26} & 0 & 1/\sqrt{26} \\ 0 & 0 & 1 & 0 \\ 0 & -1/\sqrt{26} & 0 & 5/\sqrt{26} \end{bmatrix}, \quad G_4^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{2\sqrt{26}}{\sqrt{185}} & \frac{-9}{\sqrt{185}} \\ 0 & 0 & \frac{9}{\sqrt{185}} & \frac{2\sqrt{26}}{\sqrt{185}} \end{bmatrix}$$

Final Matrix Q

Since $Q = G_1^T G_2^T G_3^T G_4^T$, we have:

$$Q = \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{130} & -18/\sqrt{962} & -4/\sqrt{37} \\ 0 & 5/\sqrt{130} & -19/\sqrt{962} & 4/\sqrt{37} \\ 0 & 10/\sqrt{130} & 14/\sqrt{962} & -1/\sqrt{37} \\ 2/\sqrt{5} & -1/\sqrt{130} & 9/\sqrt{962} & 2/\sqrt{37} \end{bmatrix}$$

Problem 5. (QR decomposition)

This inequality is known as **Hadamard's inequality**. We will prove it using the QR decomposition, as suggested by the problem context.

1. QR Decomposition

Let $A = QR$ be the QR decomposition of the (square) matrix A .

- Q is an $n \times n$ orthogonal matrix, which means $Q^T Q = I$ and $Q^{-1} = Q^T$.
- R is an $n \times n$ upper triangular matrix.

2. Determinant of A

We begin by taking the absolute value of the determinant of A :

$$|\det(A)| = |\det(QR)| = |\det(Q) \det(R)| = |\det(Q)| |\det(R)|$$

3. Determinant of Q

Since Q is orthogonal, we know $Q^T Q = I$. Taking the determinant of both sides:

$$\det(Q^T Q) = \det(I)$$

$$\det(Q^T) \det(Q) = 1$$

Since $\det(Q^T) = \det(Q)$, this gives $(\det(Q))^2 = 1$, which implies $|\det(Q)| = 1$.

4. Simplifying the Determinant

Substituting $|\det(Q)| = 1$ back into our equation from Step 2, we get:

$$|\det(A)| = 1 \cdot |\det(R)| = |\det(R)|$$

5. Determinant of \mathbf{R}

Since R is an upper triangular matrix, its determinant is the product of its diagonal entries. Let r_{ii} be the diagonal entries of R .

$$\det(R) = r_{11}r_{22} \cdots r_{nn}$$

Therefore,

$$|\det(A)| = |r_{11}r_{22} \cdots r_{nn}| = |r_{11}||r_{22}| \cdots |r_{nn}|$$

6. Relating R to A 's Columns

From $A = QR$, we can look at the i -th column of A , which is \mathbf{a}_i . This column is the result of Q multiplying the i -th column of R , which we will call \mathbf{r}_i .

$$\mathbf{a}_i = Q\mathbf{r}_i$$

Now, let's find the 2-norm (Euclidean length) of \mathbf{a}_i . Because Q is an orthogonal matrix, it preserves the 2-norm (i.e., it is an isometry).

$$\|\mathbf{a}_i\|_2^2 = \mathbf{a}_i^T \mathbf{a}_i = (Q\mathbf{r}_i)^T (Q\mathbf{r}_i) = \mathbf{r}_i^T Q^T Q \mathbf{r}_i = \mathbf{r}_i^T I \mathbf{r}_i = \mathbf{r}_i^T \mathbf{r}_i = \|\mathbf{r}_i\|_2^2$$

Thus, $\|\mathbf{a}_i\|_2 = \|\mathbf{r}_i\|_2$.

7. Norm of R 's Columns

Because R is upper triangular, its i -th column \mathbf{r}_i only has non-zero entries in the first i positions.

$$\mathbf{r}_i = \begin{pmatrix} r_{1i} \\ r_{2i} \\ \vdots \\ r_{ii} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

The squared norm of \mathbf{r}_i is the sum of the squares of its components:

$$\|\mathbf{r}_i\|_2^2 = |r_{1i}|^2 + |r_{2i}|^2 + \cdots + |r_{ii}|^2$$

8. The Key Inequality

From Step 6 and Step 7, we can equate the squared norms:

$$\|\mathbf{a}_i\|_2^2 = |r_{1i}|^2 + |r_{2i}|^2 + \cdots + |r_{ii}|^2$$

Since all the terms in the sum are non-negative (as they are squares), this sum must be greater than or equal to any single term in it. In particular, it must be greater than or equal to the last non-zero term, $|r_{ii}|^2$.

$$\|\mathbf{a}_i\|_2^2 \geq |r_{ii}|^2$$

Taking the square root of both sides (and knowing norms are non-negative):

$$\|\mathbf{a}_i\|_2 \geq |r_{ii}|$$

This holds for all $i = 1, \dots, n$.

9. Final Assembly

We now return to our determinant equation from Step 5 and apply the inequality from Step 8 for each term in the product.

$$|\det(A)| = |r_{11}| |r_{22}| \cdots |r_{nn}|$$

Since $\|\mathbf{a}_1\|_2 \geq |r_{11}|$, $\|\mathbf{a}_2\|_2 \geq |r_{22}|$, and so on, we can substitute them:

$$|\det(A)| \leq \|\mathbf{a}_1\|_2 \|\mathbf{a}_2\|_2 \cdots \|\mathbf{a}_n\|_2$$

This completes the proof. \square

Problem 6. (The Application of QR Decomposition)

1)

We are given $A \in \mathbb{R}^{m \times n}$ with $m \geq n$. We are also given that the $n \times n$ submatrix A_1 is nonsingular. This implies that A has full column rank, i.e., $\text{rank}(A) = n$.

For a matrix A with full column rank, the pseudoinverse (or left inverse) A^\dagger is given by the formula:

$$A^\dagger = (A^T A)^{-1} A^T$$

We are instructed to use the Thin QR Decomposition, $A = QR$, where $Q \in \mathbb{R}^{m \times n}$ has orthonormal columns and $R \in \mathbb{R}^{n \times n}$ is upper triangular. The property of Q having orthonormal columns means that $Q^T Q = I_n$, where I_n is the $n \times n$ identity matrix.

Since A_1 is nonsingular, A has full column rank, which implies that R is also nonsingular (invertible).

Now, we substitute $A = QR$ into the formula for A^\dagger :

$$\begin{aligned} A^\dagger &= ((QR)^T (QR))^{-1} (QR)^T \\ &= (R^T Q^T Q R)^{-1} (R^T Q^T) \end{aligned}$$

Since $Q^T Q = I_n$, the expression simplifies:

$$\begin{aligned} A^\dagger &= (R^T (I_n) R)^{-1} (R^T Q^T) \\ &= (R^T R)^{-1} (R^T Q^T) \end{aligned}$$

Using the property $(AB)^{-1} = B^{-1}A^{-1}$ for invertible matrices A and B :

$$A^\dagger = (R^{-1}(R^T)^{-1})(R^T Q^T)$$

The $(R^T)^{-1}$ and R^T terms cancel out, as $(R^T)^{-1}R^T = I_n$:

$$\begin{aligned} A^\dagger &= R^{-1}((R^T)^{-1}R^T)Q^T \\ &= R^{-1}(I_n)Q^T \\ &= R^{-1}Q^T \end{aligned}$$

The pseudoinverse is $A^\dagger = R^{-1}Q^T$.

2)

We will use the result from Part 1, the partitioning of A and Q , and the properties given in the hint.

From the partitioning, we have $A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} R$. This implies $A_1 = Q_1 R$.

Since A_1 is nonsingular (given) and R is nonsingular (from Part 1), Q_1 must also be nonsingular. We can therefore write:

$$R = Q_1^{-1}A_1$$

Inverting this gives:

$$R^{-1} = (Q_1^{-1}A_1)^{-1} = A_1^{-1}(Q_1^{-1})^{-1} = A_1^{-1}Q_1$$

Now, let's start with the expression for A^\dagger from Part 1 and find its spectral norm (2-norm):

$$\|A^\dagger\|_2 = \|R^{-1}Q^T\|_2$$

Using the submultiplicative property of matrix norms ($\|XY\|_2 \leq \|X\|_2\|Y\|_2$):

$$\|A^\dagger\|_2 \leq \|R^{-1}\|_2 \cdot \|Q^T\|_2$$

The hint provides as property (a) that $\|Q^T\|_2 = 1$. Substituting this, we get:

$$\|A^\dagger\|_2 \leq \|R^{-1}\|_2 \tag{1}$$

Now, let's use the expression we found for R^{-1} in terms of A_1 and Q_1 :

$$\|R^{-1}\|_2 = \|A_1^{-1}Q_1\|_2$$

Again, using the submultiplicative property:

$$\|R^{-1}\|_2 \leq \|A_1^{-1}\|_2 \cdot \|Q_1\|_2$$

The problem's hint provides that $\|Q_1\|_2 \leq 1$. Substituting this, we get:

$$\|R^{-1}\|_2 \leq \|A_1^{-1}\|_2 \cdot 1 = \|A_1^{-1}\|_2 \quad (2)$$

Finally, we combine inequalities (??) and (??):

$$\|A^\dagger\|_2 \leq \|R^{-1}\|_2 \quad \text{and} \quad \|R^{-1}\|_2 \leq \|A_1^{-1}\|_2$$

By transitivity, this proves the desired result:

$$\|A^\dagger\|_2 \leq \|A_1^{-1}\|_2$$