Matrix Computation Homework 1

Yifan Zhang 2025251018 zhangyf52025@shanghaitech.edu.cn

November 15, 2025

Problem 1. (Discrete-Time LTI Systems)

1)

Solution.

According to the problem description, $\mathbf{A}$ is a convolution matrix constructed from $h[n] = [h_0, h_1, h_2]$. $\mathbf{y} = \mathbf{Ax}$ describes the linear convolution $y[n] = x[n] * h[n]$.

$$y[n] = \sum_{k=0}^{p} h[k] \cdot x[n-k] = h_0 x[n] + h_1 x[n-1] + h_2 x[n-2]$$

Since there is no noise, we can directly solve for h_0, h_1, h_2 using the first few outputs.

- **Solving for** h_0:

$$y[0] = h_0 x[0] + h_1 x[-1] + h_2 x[-2]$$

  Since $x[n < 0] = 0$, we have:

$$y[0] = h_0 x[0]$$

$$1 = h_0 \cdot 1 \implies \mathbf{h_0 = 1}$$

- **Solving for** h_1:

$$y[1] = h_0 x[1] + h_1 x[0] + h_2 x[-1]$$

$$y[1] = h_0 x[1] + h_1 x[0]$$

$$2 = (1) \cdot (0) + h_1 \cdot (1) \implies \mathbf{h_1 = 2}$$

- **Solving for h_2:**

$$y[2] = h_0 x[2] + h_1 x[1] + h_2 x[0]$$
$$4 = (1)\cdot(1) + (2)\cdot(0) + h_2\cdot(1)$$
$$4 = 1 + h_2 \implies \mathbf{h_2 = 3}$$

We have obtained the channel impulse response $\mathbf{h} = [1, 2, 3]^T$.

Since

$$\mathbf{A} = \begin{pmatrix} h_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ h_1 & h_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ h_2 & h_1 & h_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & h_2 & h_1 & h_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & h_2 & h_1 & h_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & h_2 & h_1 & h_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & h_2 & h_1 & h_0 & 0 \\ 0 & 0 & 0 & 0 & 0 & h_2 & h_1 & h_0 \end{pmatrix}$$

Substituting $h_0 = 1, h_1 = 2, h_2 = 3$, we solve for $\mathbf{A}$ as:

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 2 & 1 \end{pmatrix}$$

The full equation for $\mathbf{y} = \mathbf{A}\mathbf{x}$ is:

$$\begin{pmatrix} 1 \\ 2 \\ 4 \\ 2 \\ 4 \\ 3 \\ 5 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

b)

Solution.

Here, $\mathbf{y} = \mathbf{C}\hat{\mathbf{h}}$, where $\mathbf{C}$ is the convolution matrix constructed from the input $\mathbf{x}$, and $\hat{\mathbf{h}}$ is the channel response vector we want to estimate.

Constructing the matrix C

It is obviously that:

$$\mathbf{C} = \begin{pmatrix} x[0] & x[-1] & x[-2] \\ x[1] & x[0] & x[-1] \\ x[2] & x[1] & x[0] \\ x[3] & x[2] & x[1] \\ x[4] & x[3] & x[2] \\ x[5] & x[4] & x[3] \\ x[6] & x[5] & x[4] \\ x[7] & x[6] & x[5] \end{pmatrix}$$

Substituting $\mathbf{x} = [1, 0, 1, 0, 1, 1, 0, 1]$ and $x[n < 0] = 0$:

$$\mathbf{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

Setting up the Normal Equations

The least-squares solution $\hat{\mathbf{h}}$ is given by the normal equations:

$$(\mathbf{C}^T\mathbf{C})\hat{\mathbf{h}} = \mathbf{C}^T\mathbf{y}$$

Calculating $\mathbf{C}^T\mathbf{C}$

$$\mathbf{C}^T\mathbf{C} = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 1 & 3 \\ 1 & 4 & 1 \\ 3 & 1 & 4 \end{pmatrix}$$

Calculating $\mathbf{C}^T\mathbf{y}$

$$\mathbf{C}^T\mathbf{y} = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 4 \\ 2 \\ 4 \\ 3 \\ 5 \\ 4 \end{pmatrix} = \begin{pmatrix} 16 \\ 12 \\ 17 \end{pmatrix}$$

Solving for $\hat{\mathbf{h}}$

We now need to solve the linear system:

$$\begin{pmatrix} 5 & 1 & 3 \\ 1 & 4 & 1 \\ 3 & 1 & 4 \end{pmatrix} \begin{pmatrix} h_0 \\ h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} 16 \\ 12 \\ 17 \end{pmatrix}$$

$$\hat{\mathbf{h}} = \begin{pmatrix} h_0 \\ h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Problem 2. (Gram-Schmidt Procedure)

1)

Solution.

Step 1: Compute $\mathbf{q}_1$

$$r_{11} = \|\mathbf{a}_1\|_2 = \sqrt{1^2 + 1^2 + 1^2 + 1^2} = 2$$

$$\mathbf{q}_1 = \frac{\mathbf{a}_1}{r_{11}} = \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{bmatrix}$$

Step 2: Compute $\mathbf{q}_2$

$$r_{12} = \mathbf{q}_1^T \mathbf{a}_2 = 0.5(0) + 0.5(2) + 0.5(1) + 0.5(1) = 2$$

$$\mathbf{v}_2 = \mathbf{a}_2 - r_{12}\mathbf{q}_1 = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 1 \end{bmatrix} - 2 \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$r_{22} = \|\mathbf{v}_2\|_2 = \sqrt{2}$$

$$\mathbf{q}_2 = \frac{\mathbf{v}_2}{r_{22}} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix}$$

Step 3: Compute $\mathbf{q}_3$

$$r_{13} = \mathbf{q}_1^T \mathbf{a}_3 = 3$$

$$r_{23} = \mathbf{q}_2^T \mathbf{a}_3 = \frac{1}{\sqrt{2}}(-3 + 1) = -\sqrt{2}$$

$$\mathbf{v}_3 = \mathbf{a}_3 - r_{13}\mathbf{q}_1 - r_{23}\mathbf{q}_2 = \begin{bmatrix} 3 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1.5 \\ 1.5 \\ 1.5 \\ 1.5 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.5 \\ -0.5 \\ -0.5 \end{bmatrix}$$

$$r_{33} = \|\mathbf{v}_3\|_2 = 1$$

$$\mathbf{q}_3 = \begin{bmatrix} 0.5 \\ 0.5 \\ -0.5 \\ -0.5 \end{bmatrix}$$

Result:

$$\mathbf{Q} = \begin{bmatrix} 0.5 & -1/\sqrt{2} & 0.5 \\ 0.5 & 1/\sqrt{2} & 0.5 \\ 0.5 & 0 & -0.5 \\ 0.5 & 0 & -0.5 \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} 2 & 2 & 3 \\ 0 & \sqrt{2} & -\sqrt{2} \\ 0 & 0 & 1 \end{bmatrix}$$

2)

Solution.

Initialization Let $\mathbf{v}_1 = \mathbf{a}_1, \mathbf{v}_2 = \mathbf{a}_2, \mathbf{v}_3 = \mathbf{a}_3$.

Step 1

$$r_{11} = \|\mathbf{v}_1\| = 2, \quad \mathbf{q}_1 = \mathbf{v}_1/2 = [0.5, 0.5, 0.5, 0.5]^T$$

Update $\mathbf{v}_2, \mathbf{v}_3$:

$$r_{12} = \mathbf{q}_1^T \mathbf{v}_2 = 2 \implies \mathbf{v}_2^{(2)} = \mathbf{v}_2 - r_{12}\mathbf{q}_1 = [-1, 1, 0, 0]^T$$

$$r_{13} = \mathbf{q}_1^T \mathbf{v}_3 = 3 \implies \mathbf{v}_3^{(2)} = \mathbf{v}_3 - r_{13}\mathbf{q}_1 = [1.5, -0.5, -0.5, -0.5]^T$$

Step 2

$$r_{22} = \|\mathbf{v}_2^{(2)}\| = \sqrt{2}, \quad \mathbf{q}_2 = \mathbf{v}_2^{(2)}/\sqrt{2} = [\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0]^T$$

Update $\mathbf{v}_3$:

$$r_{23} = \mathbf{q}_2^T \mathbf{v}_3^{(2)} = -\sqrt{2}$$

$$\mathbf{v}_3^{(3)} = \mathbf{v}_3^{(2)} - r_{23}\mathbf{q}_2 = [0.5, 0.5, -0.5, -0.5]^T$$

Step 3

$$r_{33} = \|\mathbf{v}_3^{(3)}\| = 1, \quad \mathbf{q}_3 = [0.5, 0.5, -0.5, -0.5]^T$$

Result:

$$\mathbf{Q} = \begin{bmatrix} 0.5 & -1/\sqrt{2} & 0.5 \\ 0.5 & 1/\sqrt{2} & 0.5 \\ 0.5 & 0 & -0.5 \\ 0.5 & 0 & -0.5 \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} 2 & 2 & 3 \\ 0 & \sqrt{2} & -\sqrt{2} \\ 0 & 0 & 1 \end{bmatrix}$$

3)

Solution.

Given $\mathbf{A}_2 = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4]$ with rank 3. We compute $\mathbf{Q} \in \mathbb{R}^{6\times 3}$ and $\mathbf{R} \in \mathbb{R}^{3\times 4}$.

Processing Columns

Vector $\mathbf{a}_1$:

$$\|\mathbf{a}_1\| = 2$$

$$\mathbf{q}_1 = [0.5, 0.5, 0, 0, 0.5, 0.5]^T, \quad r_{11} = 2$$

Vector $\mathbf{a}_2$:

$$r_{12} = \mathbf{q}_1^T \mathbf{a}_2 = 1$$

$$\mathbf{v}_2 = \mathbf{a}_2 - r_{12}\mathbf{q}_1 = [-0.5, 0.5, 0, 1, 0.5, -0.5]^T, \quad \|\mathbf{v}_2\| = \sqrt{2}$$

$$\mathbf{q}_2 = \frac{1}{\sqrt{2}}[-0.5, 0.5, 0, 1, 0.5, -0.5]^T, \quad r_{22} = \sqrt{2}$$

Vector $\mathbf{a}_3$:

$$r_{13} = \mathbf{q}_1^T \mathbf{a}_3 = 7, \quad r_{23} = \mathbf{q}_2^T \mathbf{a}_3 = \sqrt{2}$$

Since $\mathbf{v}_3 = \mathbf{a}_3 - 7\mathbf{q}_1 - \sqrt{2}\mathbf{q}_2 = \mathbf{0}$. Thus, no new basis vector is formed.

$$r_{14} = \mathbf{q}_1^T \mathbf{a}_4 = 1.5$$

$$r_{24} = \mathbf{q}_2^T \mathbf{a}_4 = \frac{1}{\sqrt{2}}(0.5) = \frac{1}{2\sqrt{2}}$$

$$\mathbf{v}_4 = \mathbf{a}_4 - 1.5\mathbf{q}_1 - \frac{1}{2\sqrt{2}}\mathbf{q}_2 = \frac{1}{8}[3, 1, 8, -2, 1, -5]^T$$

$$r_{34} = \|\mathbf{v}_4\| = \frac{\sqrt{26}}{4}$$

$$\mathbf{q}_3 = \frac{\mathbf{v}_4}{r_{34}} = \frac{1}{2\sqrt{26}}[3, 1, 8, -2, 1, -5]^T$$

Result:

$$\mathbf{Q} = \begin{bmatrix} 0.5 & \frac{-1}{2\sqrt{2}} & \frac{3}{2\sqrt{26}} \\ 0.5 & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{26}} \\ 0 & 0 & \frac{8}{2\sqrt{26}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{-2}{2\sqrt{26}} \\ 0.5 & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{26}} \\ 0.5 & \frac{-1}{2\sqrt{2}} & \frac{-5}{2\sqrt{26}} \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} 2 & 1 & 7 & 1.5 \\ 0 & \sqrt{2} & \sqrt{2} & \frac{\sqrt{2}}{4} \\ 0 & 0 & 0 & \frac{\sqrt{26}}{4} \end{bmatrix}$$

4)

Solution.

Since $A = QR, \quad Ax = b$, we can easily get $\mathbf{Rx} = \mathbf{Q}^T\mathbf{b}$:

$$\mathbf{Q}^T\mathbf{b} = \begin{bmatrix} \mathbf{q}_1^T\mathbf{b} \\ \mathbf{q}_2^T\mathbf{b} \\ \mathbf{q}_3^T\mathbf{b} \end{bmatrix} = \begin{bmatrix} 0.5(4) + 0 + 0.5(4) + 0 \\ \frac{-1}{\sqrt{2}}(4) + 0 + 0 + 0 \\ 0.5(4) + 0 - 0.5(4) + 0 \end{bmatrix} = \begin{bmatrix} 4 \\ -2\sqrt{2} \\ 0 \end{bmatrix}$$

Solving the system:

$$\begin{bmatrix} 2 & 2 & 3 \\ 0 & \sqrt{2} & -\sqrt{2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -2\sqrt{2} \\ 0 \end{bmatrix}$$

- $1 \cdot x_3 = 0 \implies x_3 = 0.$
- $\sqrt{2}x_2 - \sqrt{2}(0) = -2\sqrt{2} \implies x_2 = -2.$
- $2x_1 + 2(-2) + 3(0) = 4 \implies 2x_1 - 4 = 4 \implies 2x_1 = 8 \implies x_1 = 4.$

Solution: $\mathbf{x} = [4, -2, 0]^T$.

Problem 3. (Householder Reflection)

1)

Solution.

Step 1: First Householder Reflection (H_1)

We focus on the first column of A, $\mathbf{x} = A_1 = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$.

First, find its norm:

$$\|\mathbf{x}\|_2 = \sqrt{1^2 + (-2)^2 + 2^2} = 3$$

Next, we define the vector $\mathbf{v}_1$. The problem states to choose the sign $\mp$ to maximize $\|\mathbf{v}\|_2$.

- $\mathbf{v}_- = \mathbf{x} - \|\mathbf{x}\|_2\mathbf{e}_1 = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} - 3\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ 2 \end{bmatrix}$. $\|\mathbf{v}_-\|_2^2 = 4 + 4 + 4 = 12.$
- $\mathbf{v}_+ = \mathbf{x} + \|\mathbf{x}\|_2\mathbf{e}_1 = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix}$. $\|\mathbf{v}_+\|_2^2 = 16 + 4 + 4 = 24.$

So, $\mathbf{v}_1 = \begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix}$, and $\|\mathbf{v}_1\|_2^2 = 24$.

The Householder reflector is $H_1 = I - 2\frac{\mathbf{v}_1\mathbf{v}_1^T}{\|\mathbf{v}_1\|_2^2}$:

$$\mathbf{v}_1\mathbf{v}_1^T = \begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix} \begin{bmatrix} 4 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 16 & -8 & 8 \\ -8 & 4 & -4 \\ 8 & -4 & 4 \end{bmatrix}$$

$$H_1 = I - \frac{2}{24}\begin{bmatrix} 16 & -8 & 8 \\ -8 & 4 & -4 \\ 8 & -4 & 4 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -1 & 2 & -2 \\ 2 & 2 & 1 \\ -2 & 1 & 2 \end{bmatrix}$$

Step 2: Apply H_1 to A

We compute $A^{(1)} = H_1 A$:

$$A^{(1)} = \frac{1}{3}\begin{bmatrix} -1 & 2 & -2 \\ 2 & 2 & 1 \\ -2 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 & -3 \\ -2 & -5 & 20 \\ 2 & 8 & 3 \end{bmatrix} = \begin{bmatrix} -3 & -9 & 37/3 \\ 0 & 0 & 37/3 \\ 0 & 3 & 32/3 \end{bmatrix}$$

As expected, the first column below the diagonal is zero.

Step 3: Second Householder Reflection (H_2)

We now work on the 2×2 submatrix $\begin{bmatrix} 0 & 37/3 \\ 3 & 32/3 \end{bmatrix}$.

We want to zero out the $(2,1)$ entry of this submatrix.

Let $\mathbf{x}' = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$. $\|\mathbf{x}'\|_2 = 3$. We choose $\mathbf{v}' = \mathbf{x}' + \|\mathbf{x}'\|_2 \mathbf{e}_1 = \begin{bmatrix} 0 \\ 3 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$.
$\|\mathbf{v}'\|_2^2 = 9 + 9 = 18$.

The 2×2 reflector H' is:

$$H' = I - 2\frac{\mathbf{v}'\mathbf{v}'^T}{\|\mathbf{v}'\|_2^2} = I - \frac{2}{18}\begin{bmatrix} 3 \\ 3 \end{bmatrix}\begin{bmatrix} 3 & 3 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

This is a simple permutation matrix. We embed this into a 3×3 matrix H_2:

$$H_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

Step 4: Find R

The final upper triangular matrix R is $R = H_2 A^{(1)}$:

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} -3 & -9 & 37/3 \\ 0 & 0 & 37/3 \\ 0 & 3 & 32/3 \end{bmatrix} = \begin{bmatrix} -3 & -9 & 37/3 \\ 0 & -3 & -32/3 \\ 0 & 0 & -37/3 \end{bmatrix}$$

Step 5: Find Q

We have $R = H_2 A^{(1)} = H_2 H_1 A$. So, $A = (H_2 H_1)^{-1} R = (H_1^{-1} H_2^{-1}) R$.

Since H_1 and H_2 are orthogonal and symmetric,

$$A = (H_1 H_2) R$$

Thus, $Q = H_1 H_2$.

$$Q = \left(\frac{1}{3}\begin{bmatrix} -1 & 2 & -2 \\ 2 & 2 & 1 \\ -2 & 1 & 2 \end{bmatrix}\right)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -1 & 2 & -2 \\ 2 & -1 & -2 \\ -2 & -2 & -1 \end{bmatrix}$$

Conclusion An orthonormal basis for $R(A)$ is formed by the columns of Q:

$$\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\} = \left\{ \frac{1}{3}\begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix}, \frac{1}{3}\begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}, \frac{1}{3}\begin{bmatrix} -2 \\ -2 \\ -1 \end{bmatrix} \right\}$$

2)

Solution.

$$A^T A = (QR)^T (QR) = R^T Q^T Q R = R^T R$$

Therefore:

$$(A^T A)^{-1} = (R^T R)^{-1} = R^{-1} (R^T)^{-1} = R^{-1} (R^{-1})^T$$

We first need to find R^{-1} from $R = \begin{bmatrix} -3 & -9 & 37/3 \\ 0 & -3 & -32/3 \\ 0 & 0 & -37/3 \end{bmatrix}$.

We can find the inverse of this upper triangular matrix by back substitution or row reduction.

$$\left[\begin{array}{ccc|ccc} -3 & -9 & 37/3 & 1 & 0 & 0 \\ 0 & -3 & -32/3 & 0 & 1 & 0 \\ 0 & 0 & -37/3 & 0 & 0 & 1 \end{array}\right]$$

We can get $R^{-1} = \frac{1}{111}\begin{bmatrix} -37 & 111 & -133 \\ 0 & -37 & 32 \\ 0 & 0 & -9 \end{bmatrix}$.

Now we compute $(A^T A)^{-1} = R^{-1}(R^{-1})^T$:

$$(R^{-1})^T = \frac{1}{111}\begin{bmatrix} -37 & 0 & 0 \\ 111 & -37 & 0 \\ -133 & 32 & -9 \end{bmatrix}$$

$$\begin{aligned} (A^T A)^{-1} &= \frac{1}{111^2}\begin{bmatrix} -37 & 111 & -133 \\ 0 & -37 & 32 \\ 0 & 0 & -9 \end{bmatrix}\begin{bmatrix} -37 & 0 & 0 \\ 111 & -37 & 0 \\ -133 & 32 & -9 \end{bmatrix} \\ &= \frac{1}{12321}\begin{bmatrix} 1369 + 12321 + 17689 & -4107 - 4256 & 1197 \\ -4107 - 4256 & 1369 + 1024 & -288 \\ 1197 & -288 & 81 \end{bmatrix} \end{aligned}$$

Thus, the inverse of $A^T A$ is:

$$(A^T A)^{-1} = \frac{1}{12321}\begin{bmatrix} 31379 & -8363 & 1197 \\ -8363 & 2393 & -288 \\ 1197 & -288 & 81 \end{bmatrix}$$

Problem 4. (Givens Rotation)

Solution.

Step 1: Zero out A(4, 1)

We apply a rotation G_1 to zero out $A(4,1) = 2$ using $A(1,1) = 1$.

Let $a = 1$ and $b = 2$. $r = \sqrt{a^2 + b^2} = \sqrt{5}$, $c = a/r = 1/\sqrt{5}$, $s = b/r = 2/\sqrt{5}$

The rotation matrix G_1 is:

$$G_1 = \begin{bmatrix} c & 0 & 0 & s \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -s & 0 & 0 & c \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} & 0 & 0 & 2/\sqrt{5} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2/\sqrt{5} & 0 & 0 & 1/\sqrt{5} \end{bmatrix}$$

We compute $A_1 = G_1 A$:

$$A_1 = \begin{bmatrix} 1/\sqrt{5} & 0 & 0 & 2/\sqrt{5} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2/\sqrt{5} & 0 & 0 & 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 2 & 2 \\ 2 & 3 & 5 \end{bmatrix} = \begin{bmatrix} \sqrt{5} & 8/\sqrt{5} & 12/\sqrt{5} \\ 0 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & -1/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$$

Step 2: Zero out $\mathbf{A}_1$(3, 2)

Next, we apply a rotation G_2 to zero out $A_1(3,2) = 2$ using $A_1(2,2) = 1$.

Let $a = 1$ and $b = 2$.

$r = \sqrt{1^2 + 2^2} = \sqrt{5}$, $c = 1/\sqrt{5}$, $s = 2/\sqrt{5}$

$$G_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c & s & 0 \\ 0 & -s & c & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/\sqrt{5} & 2/\sqrt{5} & 0 \\ 0 & -2/\sqrt{5} & 1/\sqrt{5} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We compute $A_2 = G_2 A_1$:

$$A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/\sqrt{5} & 2/\sqrt{5} & 0 \\ 0 & -2/\sqrt{5} & 1/\sqrt{5} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{5} & 8/\sqrt{5} & 12/\sqrt{5} \\ 0 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & -1/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} = \begin{bmatrix} \sqrt{5} & 8/\sqrt{5} & 12/\sqrt{5} \\ 0 & \sqrt{5} & 4/\sqrt{5} \\ 0 & 0 & 2/\sqrt{5} \\ 0 & -1/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$$

Step 3: Zero out $\mathbf{A}_2$(4, 2)

We apply a rotation G_3 to zero out $A_2(4,2) = -1/\sqrt{5}$ using $A_2(2,2) = \sqrt{5}$.

Let $a = \sqrt{5}$ and $b = -1/\sqrt{5}$. $r = \sqrt{(\sqrt{5})^2 + (-1/\sqrt{5})^2} = \sqrt{26/5}$, $c = a/r = \sqrt{5}/\sqrt{26/5} = 5/\sqrt{26}$, $s = b/r = (-1/\sqrt{5})/\sqrt{26/5} = -1/\sqrt{26}$

$$G_3 = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & c & 0 & s\\ 0 & 0 & 1 & 0\\ 0 & -s & 0 & c\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 5/\sqrt{26} & 0 & -1/\sqrt{26}\\ 0 & 0 & 1 & 0\\ 0 & 1/\sqrt{26} & 0 & 5/\sqrt{26}\end{bmatrix}$$

We compute $A_3 = G_3 A_2$:

$$A_3 = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 5/\sqrt{26} & 0 & -1/\sqrt{26}\\ 0 & 0 & 1 & 0\\ 0 & 1/\sqrt{26} & 0 & 5/\sqrt{26}\end{bmatrix} \begin{bmatrix}\sqrt{5} & 8/\sqrt{5} & 12/\sqrt{5}\\ 0 & \sqrt{5} & 4/\sqrt{5}\\ 0 & 0 & 2/\sqrt{5}\\ 0 & -1/\sqrt{5} & 1/\sqrt{5}\end{bmatrix} = \begin{bmatrix}\sqrt{5} & 8/\sqrt{5} & 12/\sqrt{5}\\ 0 & \sqrt{26/5} & 19/\sqrt{130}\\ 0 & 0 & 2/\sqrt{5}\\ 0 & 0 & 9/\sqrt{130}\end{bmatrix}$$

Step 4: Zero out $\mathbf{A_3(4, 3)}$

Finally, we apply a rotation G_4 to zero out $A_3(4,3) = 9/\sqrt{130}$ using $A_3(3,3) = 2/\sqrt{5}$.

Let $a = 2/\sqrt{5}$ and $b = 9/\sqrt{130}$.

$r = \sqrt{(2/\sqrt{5})^2 + (9/\sqrt{130})^2} = \sqrt{37/26}$, $c = a/r = (2/\sqrt{5})/\sqrt{185/130} = 2\sqrt{26}/\sqrt{185}$, $s = b/r = (9/\sqrt{130})/\sqrt{185/130} = 9/\sqrt{185}$

$$G_4 = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & c & s\\ 0 & 0 & -s & c\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 2\sqrt{26}/\sqrt{185} & 9/\sqrt{185}\\ 0 & 0 & -9/\sqrt{185} & 2\sqrt{26}/\sqrt{185}\end{bmatrix}$$

We compute $R = G_4 A_3$:

$$R = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & c & s\\ 0 & 0 & -s & c\end{bmatrix} \begin{bmatrix}\sqrt{5} & 8/\sqrt{5} & 12/\sqrt{5}\\ 0 & \sqrt{26/5} & 19/\sqrt{130}\\ 0 & 0 & 2/\sqrt{5}\\ 0 & 0 & 9/\sqrt{130}\end{bmatrix} = \begin{bmatrix}\sqrt{5} & 8/\sqrt{5} & 12/\sqrt{5}\\ 0 & \sqrt{26/5} & 19/\sqrt{130}\\ 0 & 0 & \sqrt{185/130}\\ 0 & 0 & 0\end{bmatrix}$$

This gives our upper triangular matrix R.

Final Matrix R

Rationalizing the denominators, R is:

$$R = \begin{bmatrix}\sqrt{5} & \frac{8\sqrt{5}}{5} & \frac{12\sqrt{5}}{5}\\ 0 & \frac{\sqrt{130}}{5} & \frac{19\sqrt{130}}{130}\\ 0 & 0 & \frac{\sqrt{962}}{26}\\ 0 & 0 & 0\end{bmatrix}$$

Finding the Matrix Q

Now we find $Q = G_1^T G_2^T G_3^T G_4^T$.

$$G_1^T = \begin{bmatrix} 1/\sqrt{5} & 0 & 0 & -2/\sqrt{5} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2/\sqrt{5} & 0 & 0 & 1/\sqrt{5} \end{bmatrix}, \quad G_2^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/\sqrt{5} & -2/\sqrt{5} & 0 \\ 0 & 2/\sqrt{5} & 1/\sqrt{5} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$G_3^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5/\sqrt{26} & 0 & 1/\sqrt{26} \\ 0 & 0 & 1 & 0 \\ 0 & -1/\sqrt{26} & 0 & 5/\sqrt{26} \end{bmatrix}, \quad G_4^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{2\sqrt{26}}{\sqrt{185}} & \frac{-9}{\sqrt{185}} \\ 0 & 0 & \frac{9}{\sqrt{185}} & \frac{2\sqrt{26}}{\sqrt{185}} \end{bmatrix}$$

Final Matrix Q

Since $Q = G_1^T G_2^T G_3^T G_4^T$, we have:

$$Q = \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{130} & -18/\sqrt{962} & -4/\sqrt{37} \\ 0 & 5/\sqrt{130} & -19/\sqrt{962} & 4/\sqrt{37} \\ 0 & 10/\sqrt{130} & 14/\sqrt{962} & -1/\sqrt{37} \\ 2/\sqrt{5} & -1/\sqrt{130} & 9/\sqrt{962} & 2/\sqrt{37} \end{bmatrix}$$

Problem 5. (QR decomposition)

Proof.

Let $A = QR$ be the QR decomposition of the (square) matrix A.

Then, we can get that:

$$|\det(A)| = |\det(QR)| = |\det(Q)\det(R)| = |\det(Q)||\det(R)|$$

Since Q is orthogonal matrix, we have that $|\det(Q)| = 1$

Thus,

$$|\det(A)| = 1 \cdot |\det(R)| = |\det(R)|$$

Since R is an upper triangular matrix, its determinant is the product of its diagonal entries. Let r_{ii} be the diagonal entries of R.

$$\det(R) = r_{11}r_{22}\cdots r_{nn}$$

Therefore,

$$|\det(A)| = |r_{11}r_{22}\cdots r_{nn}| = |r_{11}||r_{22}|\cdots|r_{nn}|$$

From $A = QR$, we can look at the i-th column of A, which is $\mathbf{a}_i$. This column is the result of Q multiplying the i-th column of R, which we will call $\mathbf{r}_i$.

$$\mathbf{a}_i = Q\mathbf{r}_i$$

Thus,

$$\|\mathbf{a}_i\|_2^2 = \mathbf{a}_i^T \mathbf{a}_i = (Q\mathbf{r}_i)^T(Q\mathbf{r}_i) = \mathbf{r}_i^T Q^T Q \mathbf{r}_i = \mathbf{r}_i^T \mathbf{r}_i = \|\mathbf{r}_i\|_2^2$$

Thus, $\|\mathbf{a}_i\|_2 = \|\mathbf{r}_i\|_2$.

Because R is upper triangular, its i-th column $\mathbf{r}_i$ only has non-zero entries in the first i positions.

$$\mathbf{r}_i = \begin{pmatrix} r_{1i} \\ r_{2i} \\ \vdots \\ r_{ii} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

The squared norm of $\mathbf{r}_i$ is the sum of the squares of its components:

$$\|\mathbf{r}_i\|_2^2 = |r_{1i}|^2 + |r_{2i}|^2 + \cdots + |r_{ii}|^2$$

Since $\|\mathbf{a}_i\|_2 = \|\mathbf{r}_i\|_2$, we have:

$$\|\mathbf{a}_i\|_2^2 = |r_{1i}|^2 + |r_{2i}|^2 + \cdots + |r_{ii}|^2$$

Thus,

$$\|\mathbf{a}_i\|_2^2 \geq |r_{ii}|^2$$

Taking the square root of both sides:

$$\|\mathbf{a}_i\|_2 \geq |r_{ii}|$$

Above all,

$$|\det(A)| = |r_{11}||r_{22}|\cdots|r_{nn}| \leq \|\mathbf{a}_1\|_2\|\mathbf{a}_2\|_2\cdots\|\mathbf{a}_n\|_2$$

This completes the proof.

Problem 6. (The Application of QR Decomposition)

1)

Proof.

Since A_1 is nonsingular, it is obviously that A has full column rank, i.e., $\text{rank}(A) = n$.

For a matrix A with full column rank, the pseudoinverse $A^\dagger$ is given by the formula:

$$A^\dagger = (A^T A)^{-1} A^T$$

Since A_1 is nonsingular, A has full column rank, which implies that R is also nonsingular.

$$\begin{aligned}
A^\dagger &= \left((QR)^T (QR)\right)^{-1} (QR)^T \\
&= \left(R^T Q^T Q R\right)^{-1} (R^T Q^T) \\
&= \left(R^T R\right)^{-1} (R^T Q^T) \\
&= \left(R^{-1} (R^T)^{-1}\right) (R^T Q^T) \\
&= R^{-1} Q^T
\end{aligned}$$

The pseudoinverse is $A^\dagger = R^{-1} Q^T$.

2)

Proof.

From the partitioning, we have $A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} R$. This implies $A_1 = Q_1 R$.

Thus,

$$R = Q_1^{-1} A_1$$

Inverting this gives:

$$R^{-1} = (Q_1^{-1} A_1)^{-1} = A_1^{-1} (Q_1^{-1})^{-1} = A_1^{-1} Q_1$$

Then, focus on the $A^\dagger$

$$\begin{aligned}
\|A^\dagger\|_2 &= \|R^{-1} Q^T\|_2 \\
&\leq \|R^{-1}\|_2 \cdot \|Q^T\|_2 \\
&= \|R^{-1}\|_2 \\
&= \|A_1^{-1} Q_1\|_2 \\
&\leq \|A_1^{-1}\|_2 \cdot \|Q_1\|_2 \\
&= \|A_1^{-1}\|_2
\end{aligned}$$

This completes the proof.