Static Output Feedback Synthesis for MIMO LTI Constrained Positive Systems Using LP Approach

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***Abstract*—The aim of this article is to design static output feedback controllers for multi-input–multi-output (MIMO) linear time-invariant (LTI) systems, with the requirements of positivity and asymptotic stability to be guaranteed in closed-loop systems. By expressing the matrix product KC as a sum of rank one matrices, new necessary and sufficient conditions for the existence of a static output feedback controller are derived in terms of linear programming (LP) conditions. One of the advantages of the proposed LP method is that constraints on the system can be dealt with explicitly and easily. Thus, by following this LP approach, other problems are solved, for instance, stabilization with restricted sign control, stabilization with nonsymmetrical constrained control, and stabilization with state constraint. Furthermore, this proposed technique is also used to derive an iterative LP algorithm, which computes the gain matrix of the static output feedback controller. Finally, illustrative examples confirm the effectiveness and applicability of theoretical results, and they also show that the proposed LP method has advantages over existing approaches.**

***Index Terms*—Linear programming (LP), multiinput multioutput (MIMO) linear time-invariant (LTI) positive systems, stabilization with restricted sign control and nonsymmetrical constrained control, static output feedback controller.**

I. Introduction

Positive systems are dynamic systems that have nonnegative states whenever the initial conditions are nonnegative [2], [11], [19], [21]. The interest in positive systems is motivated by the possibility of using them in various fields of applications, such as congestion control [7], compartment models [17], economics [11], chemical networks [25], network communication [24], biological systems [15], and many more [6], [13].

The interest in the stabilization problems of positive systems has grown over the years. Various state feedback control problems have been tackled using algebraic approaches [2], [9], Gersgorin's theorem [18], linear matrix inequality (LMI)-based approach [5], [8], [14], and linear programming (LP)-based approach [5], [7], [22]. Note that when designing state feedback controllers, the system state variables must be available. However, in practice, it is not always possible to have access to all state variables, and only partial state variables can be measured. On the other hand, the static output feedback problem of positive systems is one of the most important open problems in control engineering. Simply stated, the problem is as follows: given an linear time-invariant (LTI) positive system, find a static output feedback controller such that the closed-loop system is positive and asymptotically stable, or determine that such a feedback does not exist.

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In the literature, there are a few attempts to solve this problem. The LP-based approach proposed in [1] solves the problem for single-input and/or single-output positive systems; however, the extension of this method to multi-input–multi-output (MIMO) positive systems is restricted to a rank one controller gain matrix. The design technique proposed in [23] can only be applied to a class of systems whose output or input matrix has a particular row or column echelon form, respectively. Also, iterative algorithm approaches have been proposed in [3], [12], and [26] to design a static output feedback controller for positive systems; however, the convergence of these algorithms to a numerical solution is not guaranteed; in addition, these algorithms will increase the computational burden.

The design of control laws for systems subject to constraints still presents a significant challenge. In general, the problem of designing controllers under constraints (structured or bounded gains) is an NP (nondeterministic polynomial-time)-hard problem to solve [4]. Instead, for positive LTI systems, this problem can be efficiently solved. Indeed, most of the previous results [7], [20], [22] dealt with constrained controllers based on state-feedback control law. However, the synthesis problem under state and/or control constraint by static output feedback control law is not fully investigated and is still not completely solved for positive LTI systems.

The main issue of this article is to solve the static output feedback problem for continuous-time MIMO linear positive systems. New necessary and sufficient conditions that guarantee the positivity and the asymptotic stability of the closed-loop system are established, and the desired gain matrix can be computed easily through the solution of the LP problem. Also, an iterative LP algorithm is proposed to solve the static output feedback problem. In comparison with previous works based on the LMI approach, the proposed LP approach is more simple and possesses a low computational complexity. Note that previous works based on the LMI approach cannot handle control with componentwise constraints. Instead, our LP-based approach can easily and explicitly handle the synthesis problems by static output feedback with respect to control componentwise constraints, such as componentwise lower and/or upper bounds on the controls. Unlike other conditions, which may require writing the output matrix in row/column echelon form, all the conditions proposed in this article are expressed in terms of the state-space matrices.

The rest of this article is organized as follows. Section II gives some preliminary results on positive systems. Section III presents the main results. Section IV provides examples to illustrate the applicability of the proposed approach. Finally, Section V concludes this article.

Notations: $\mathbb{R}$, $\mathbb{R}^n$, $\mathbb{R}^n_+$, and $\mathbb{R}^{n\times m}$ denote the sets of real numbers, n-dimensional vectors, nonnegative n-dimensional vectors, and $n \times m$ matrices, respectively. M^T denotes the transpose of the real matrix M. $M > 0$ or $v > 0$ means that each element of matrix M or vector v is positive. Similarly, $M \geq 0$ or $v \geq 0$ means that each element of matrix M or vector v is nonnegative. I denotes the identity matrix with appropriate dimension. e_i is the ith unit vector, i.e., the vector that is zero in all entries except the ith, at which it is 1. 1_n is the vector of size n with all elements equal to 1. $\text{diag}(\lambda)$ is a diagonal matrix with diagonal components, which are the elements of the vector λ.

II. Preliminaries

In this section, we recall some definitions and fundamental results for LTI positive systems with free input ($u \equiv 0$). These results will be used later to establish our main results.

Let us consider the following LTI system:

$$\begin{cases} \dot{x}(t) = Ax(t) \\ x(0) = x_0 \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ is the system state and $x_0 \in \mathbb{R}^n$ is the given initial condition.

Definition 1. [21]: LTI system (1) is said to be positive if for every nonnegative initial condition ($x_0 \in \mathbb{R}^n_+$), the trajectory of the state vector $x(t)$ remains in the closed positive orthant for all time ($t \geq 0$).

With regard to the above definition, we present the following definitions, which are helpful in the rest of this article.

Definition 2. [21]: A real matrix $M \in \mathbb{R}^{n\times n}$ is said to be a Metzler matrix if all nondiagonal elements are nonnegative, that is, $m_{ij} \geq 0$, for all $i \neq j$.

Definition 3. [21]: A real matrix $M \in \mathbb{R}^{n\times m}$ is said to be nonnegative if all its elements are nonnegative.

Lemma 1. [21]: A real matrix $M \in \mathbb{R}^{n\times n}$ is Metzler if and only if there exists $\beta \in \mathbb{R}^+$ such that $M + \beta I$ is a nonnegative matrix, that is, $M + \beta I \geq 0$.

A necessary and sufficient condition for the characterization of the positivity of LTI system (1) is presented below.

Lemma 2. [21]: LTI system (1) is positive if and only if the system matrix $A \in \mathbb{R}^{n\times n}$ is a Metzler matrix.

Definition 4. [9]: A real matrix $M \in \mathbb{R}^{n\times n}$ is called a Hurwitz matrix if all its eigenvalues have a strictly negative real part.

Now, we present a necessary and sufficient LP condition for the asymptotic stability of positive LTI system (1).

Lemma 3. [22]: Assume that the LTI system (1) is positive, or equivalently, the matrix A is Metzler; then the following statements are equivalent.

i) The matrix A is Hurwitz.

ii) There exists a vector $\lambda \in \mathbb{R}^n$ such that

$$A\lambda < 0, \qquad \lambda > 0. \tag{2}$$

iii) There exists a vector $\nu \in \mathbb{R}^n$ such that

$$A^T\nu < 0, \qquad \nu > 0. \tag{3}$$

In order to derive our results on the existence of stabilizing constrained controls, we introduce by the following lemma a box-invariance principle for linear positive systems.

Lemma 4: Consider $x(t)$ the solution of the LTI system (1), then, for a given $\lambda > 0$, we have $0 \leq x(t) < \lambda \quad \forall t > 0$ for any initial condition satisfying $0 \leq x(0) \leq \lambda$ if and only if A is Metzler and $A\lambda < 0$.

Proof: Necessity: suppose that for any initial condition satisfying $0 \leq x(0) \leq \lambda$ the solution $x(t)$ to the autonomous system (1) is such that $0 \leq x(t) < \lambda \quad \forall t > 0$ holds. Hence, the autonomous system (1) is positive, or equivalently, by using Lemma 2, we have A is Metzler. Now, consider the case where $x(0) = \lambda$, then the right derivative of $x(t)$ at 0 is $\dot{x}^+(0) = Ax(0)$. Since $x(0) = \lambda$ and $x(t) < \lambda \quad \forall t > 0$, it is necessary that $\dot{x}^+(0) < 0$. Hence, $A\lambda < 0$.

Sufficiency: Assume that the matrix A is Metzler and there exist $\lambda > 0$ such that $A\lambda < 0$. Then, we consider the following solution of the autonomous system (1), $x(t) = e^{At}x(0) \quad \forall t \geq 0$. Due to the fact that $0 \leq x(0) \leq \lambda$, and by using $e^{At} > 0$ ($\forall t > 0$) (because A is a Metzler matrix), we have $0 \leq x(t) \leq e^{At}\lambda \quad \forall t \geq 0$. Now, by using the well-known formula $e^{At} = \sum_{q=0}^{\infty} \frac{1}{q!}(tA)^q$, we have $(e^{At} - I)\lambda = \sum_{q=1}^{\infty} \frac{1}{q!}(tA)^q\lambda = \int_0^t e^{As}dsA\lambda$, or equivalently, $e^{At}\lambda = \lambda + \int_0^t e^{As}dsA\lambda$. Thus, it is possible to deduce that $0 \leq x(t) \leq \lambda + \int_0^t e^{As}dsA\lambda \quad \forall t > 0$. This yields, by $A\lambda < 0$ and $e^{tA} \geq 0$, that for $0 \leq s \leq t$, the integral $\int_0^t e^{As}dsA\lambda$ is negative. Then, it follows that $0 \leq x(t) < \lambda \ \forall t > 0$. Thus, the proof is complete. ■

III. Main Results

In this section, we develop stabilization results for constrained MIMO positive LTI systems. We restrict our attention to output feedback controllers, and we develop easy computational conditions for the existence of control laws that respect the specified constraints. Also, we present an iterative LP algorithm to compute the static output feedback gain matrix.

A. Static Output Feedback Controller

In the following, we will first solve the stabilization problem by using an unconstrained static output feedback control law. Let us consider the following governed system:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \in \mathbb{R}^n_+ \\ y(t) = Cx(t) \end{cases} \tag{4}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ are the state, input, and output vectors, respectively. The matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $C \in \mathbb{R}^{p\times n}$ are given constant matrices.

The problem addressed in the following is that of designing a static output feedback controller of the form:

$$u(t) = Ky(t) = KCx(t), \qquad K \in \mathbb{R}^{m\times p} \tag{5}$$

for which the following resulting closed-loop system:

$$\dot{x}(t) = (A + BKC)x(t), \qquad x(0) = x_0 \tag{6}$$

is positive and asymptotically stable. In other words, the matrix $A + BKC$ must be Metzler and Hurwitz.

Next, we state the following result, which establishes necessary and sufficient conditions for the existence of static output feedback controller.

Theorem 1: The resulting closed-loop system (6) is positive and asymptotically stable with the static output feedback controller (5) if and only if the following LP problem in the variables $\lambda \in \mathbb{R}^n$, $\mu_i \in \mathbb{R}^m$, and $\beta_i \in \mathbb{R}^+$, $i = 1, \ldots, p$ is feasible:

$$\begin{cases} A\lambda + B\left(\sum_{i=1}^{p} \mu_i\right) < 0 \\ \lambda > 0 \\ e_i^T C\lambda > 0, \quad i = 1, \ldots, p \\ \dfrac{1}{p}Ae_i^T C\lambda + B\mu_i e_i^T C + \beta_i I \geq 0, \quad i = 1, \ldots, p. \end{cases} \tag{7}$$

Moreover, the desired gain matrix can be obtained as follows:

$$K = \sum_{i=1}^{p} \frac{\mu_i e_i^T}{e_i^T C\lambda} \tag{8}$$

where λ, μ_i, and β_i, $i = 1, \ldots, p$ are any feasible solution to the LP problem (7).

Proof. (Sufficiency): Assume that the LP problem (7) is feasible. Thus, one can define the gain K as $K = \sum_{i=1}^{p} \frac{\mu_i e_i^T}{e_i^T C\lambda}$, and we show that the resulting closed-loop system (6) is positive and asymptotically stable. For this, note that the LP conditions (7) are equivalent to

$$\begin{cases}(A+BKC)\lambda < 0\\ \lambda > 0\\ \dfrac{1}{p}A + B\dfrac{\mu_i e_i^T}{e_i^T C\lambda}C + \dfrac{\beta_i}{e_i^T C\lambda}I \geq 0, \quad i=1,\ldots,p.\end{cases} \tag{9}$$

By using Lemma 1 we have $\frac{1}{p}A + B\frac{\mu_i e_i^T}{e_i^T C\lambda}C, i=1,\ldots,p$ are Metzler then $\left(\frac{1}{p}+\cdots+\frac{1}{p}\right)A + \sum_{i=1}^p B\frac{\mu_i e_i^T}{e_i^T C\lambda}C = A+BKC$ is Metzler. Therefore, by using Lemma 2 and Lemma 3, we can conclude that the closed-loop system (6) is positive and asymptotically stable.

(Necessity): Assume that the closed-loop system (6) is positive and asymptotically stable. Thus, by applying Lemma 3 there exists λ such that $(A+BKC)\lambda < 0$ and $\lambda > 0$, or equivalently $A\lambda + B\left(\sum_{i=1}^p k_i e_i^T\right)C\lambda < 0$, with k_i is the ith column of the gain matrix K. Since $e_i^T C\lambda \neq 0, i=1,\ldots,p$, then by using the change of variable $\mu_i = k_i e_i^T C\lambda$, $i=1,\ldots,p$, we obtain $A\lambda + B\left(\sum_{i=1}^p \mu_i\right) < 0$ and $\lambda > 0$. On the other hand, the closed-loop system (6) is positive. Thus, by using Lemma 2 we have $A+BKC$ is Metzler. Note that $A+BKC = A + \sum_{i=1}^p Bk_i e_i^T C$ is a Metzler matrix if the matrices $\frac{1}{p}A + Bk_i e_i^T C$, $i=1,\ldots,p$ are Metzler, or equivalently, by using Lemma 1 with the change of variable $\mu_i = k_i e_i^T C\lambda$, $i=1,\ldots,p$, we obtain $\frac{1}{p}Ae_i^T C\lambda + B\mu_i e_i^T C + \beta_i I \geq 0$, $e_i^T C\lambda > 0$, $\beta_i \geq 0$, $i = 1,\ldots,p$, and the proof is complete. ■

Remark 1: The output stabilization problem has been considered in [1] and [27]. The proposed approach given in [27] is restricted to systems that are positive in open-loop. Also, [27, Thm. 1], presents just sufficient but not necessary conditions for the existence of the desired controller. In [1], it is shown that this important problem is completely solved for SISO positive systems. However, the extension of the proposed method given in [1] to MIMO systems is restricted to rank one controller gain matrix; also, [1, Thm. 3.8] does not provide particular properties of the parameter ν. Since $\nu = 0$ is a valid choice for Theorem 3.8, and then the resulting LP is always infeasible. In contrast, there are stabilizable systems, but the designs given in [1] and [27] do not give a feasible solution. Thus, the issue of designing a static output feedback controller for MIMO positive systems remains an open problem.

Remark 2: The result given in Theorem 1 divides the matrix product KC into a sum of p rank one matrices as follows: $KC = \sum_{i=1}^p k_i c_i^*$, where $k_i = \frac{\mu_i}{e_i^T C\lambda}$ is the ith vector column of the matrix K and $c_i^* = e_i^T C$ is the ith vector line of the matrix C. Moreover, without loss of generality, the gain matrix K can also be expressed as

$$K = \sum_{i=1}^p \frac{\mu_i e_i^T}{e_i^T C\lambda} = Z\mathrm{diag}\left(\begin{bmatrix}e_1^T C\lambda & \cdots & e_p^T C\lambda\end{bmatrix}^T\right)^{-1}$$

where $Z = [\mu_1 \quad \ldots \quad \mu_p] \in \mathbb{R}^{m\times p}$. Thus, an other equivalent result to Theorem 1 can be derived in the variables λ, Z, and β_i, $i=1,\ldots,p$.

Remark 3: Theorem 1 presents new necessary and sufficient conditions for the existence of desired static output feedback controllers. All the conditions given in problem (7) are all convex in the vector variables, and they can be solved as an LP problem. For the implementation of the LP problem (7), we can use a small scalar variable $\epsilon > 0$, in order to deal with the strict negativity of the first constraint in this LP problem. Moreover, this small variable can be minimized, and then, the following optimized LP problem can be solved:

$$\begin{aligned}&\min_{\lambda,\mu_i,\beta_i,\epsilon} \ \epsilon\\ &\text{subject to}\\ &\begin{cases}A\lambda + B\left(\sum_{i=1}^p \mu_i\right) \leq -\epsilon 1_n\\ \lambda > 0\\ e_i^T C\lambda > 0, \quad i=1,\ldots,p\\ \dfrac{1}{p}Ae_i^T C\lambda + B\mu_i e_i^T C + \beta_i I \geq 0, \quad i=1,\ldots,p.\end{cases}\end{aligned} \tag{10}$$

Therefore, the above LP problem (10) can be readily computed by using standard numerical LP software.

Remark 4: Note that no condition is imposed on A, B, and C for system (4), which means that the original system (4) is not necessarily positive. Thus, the proposed conditions of Theorem 1 can be applied to problems where the open-loop system (4) is not necessarily positive. Therefore, the static output feedback controller is designed not only to stabilize the system but also to render the closed-loop system positive. Moreover, our result given in Theorem 1 does not impose any restriction on the rank of the gain matrix (8).

Remark 5: The third and fourth conditions in the LP problem (7) can be replaced by the following conditions:

$$\begin{cases}e_i^T C\lambda < 0, \quad i=1,\ldots,p\\ \dfrac{1}{p}Ae_i^T C\lambda + B\mu_i e_i^T C + \beta_i I \leq 0, \quad i=1,\ldots,p.\end{cases}$$

if the LP problem (7) is not feasible.

Remark 6: The result given in Theorem 1 can be easily extended to the case where we look for a sign-restricted control signal (nonnegative or nonpositive). Note that the control law (5) is nonnegative (or nonpositive) if and only if $KC = \sum_{i=1}^p \frac{\mu_i e_i^T}{e_i^T C\lambda}C$ is a nonnegative (or nonpositive) matrix. Thus, to design a nonnegative (or nonpositive) static output feedback controller, we need just to add the following conditions: $\mu_i e_i^T C \geq 0, i=1,\ldots,p$ (or $\mu_i e_i^T C \leq 0, i=1,\ldots,p$) in the LP problem (7).

Due to the fact that the matrix $A+BKC$ is Metzler and Hurwitz if and only if its transpose $A^T + C^T K^T B^T$ is also Metzler and Hurwitz. The following result is a dual formulation of Theorem 1.

Theorem 2: The resulting closed-loop system (6) is positive and asymptotically stable with the static output feedback controller (5) if and only if the following LP problem in the variables $\lambda \in \mathbb{R}^n$, $\mu_i \in \mathbb{R}^p$, and $\beta_i \in \mathbb{R}^+$, $i=1,\ldots,m$ is feasible:

$$\begin{cases}A^T\lambda + C^T\left(\sum_{i=1}^m \mu_i\right) < 0\\ \lambda > 0\\ e_i^T B^T\lambda > 0, \quad i=1,\ldots,m\\ \dfrac{1}{m}A^T e_i^T B^T\lambda + C^T\mu_i e_i^T B^T + \beta_i I \geq 0, \quad i=1,\ldots,m.\end{cases} \tag{11}$$

Moreover, the desired gain matrix can be obtained as follows:

$$K^T = \sum_{i=1}^m \frac{\mu_i e_i^T}{e_i^T B^T\lambda} \tag{12}$$

where λ, μ_i, and β_i, $i=1,\ldots,m$ are any feasible solution to the above LP problem.

Proof: The proof of Theorem 2 can be obtained by following the Proof of Theorem 1, and it is omitted. ■

B. Controller With Asymmetric Bounds Constraint

Control engineering often faces the simultaneous concerns of avoiding constraints violations and instability. For instance, these concerns drive the designer toward using the available actuator power with regards to the limited range of operations in order to achieve a specific performance of the transient response of the system, and indeed, this may need a high amount of energy. Since the actuators cannot drive unlimited energy to the controlled plant, the control action might not be as high as it is needed. This fact must be taken into account by imposing specified bounds on control values. Indeed, these control values can be simultaneously upper and lower bounded. In practice, the problem at hand concerns the static output feedback control law, which takes both positive and negative values.

Next, our objective is to determine a constrained static output feedback controller of the form

$$-\underline{u} \leq u(t) = KCx(t) \leq \overline{u} \tag{13}$$

where $\underline{u} > 0$ and $\overline{u} > 0$ such that the closed-loop system (6) is positive and asymptotically stable. Then, for this situation, the following result gives sufficient conditions.

Theorem 3: For given control bounds $\overline{u} > 0$ and $\underline{u} > 0$, if the following LP problem in the variables $\lambda \in \mathbb{R}^n$, $\overline{\mu}_i \in \mathbb{R}^m$, $\underline{\mu}_i \in \mathbb{R}^m$ and $\beta_i \in \mathbb{R}^+$, $i = 1, \dots, p$:

$$\begin{cases} A\lambda + B\left(\sum_{i=1}^{p} \overline{\mu}_i - \underline{\mu}_i\right) < 0 \\ \lambda > 0 \\ \overline{\mu}_i \geq 0, \quad i = 1, \dots, p \\ \underline{\mu}_i \geq 0, \quad i = 1, \dots, p \\ e_i^T C\lambda > 0, \quad i = 1, \dots, p \\ \overline{\mu}_i e_i^T C \geq 0, \quad i = 1, \dots, p \\ \underline{\mu}_i e_i^T C \geq 0, \quad i = 1, \dots, p \\ \sum_{i=1}^{p} \overline{\mu}_i \leq \overline{u} \\ \sum_{i=1}^{p} \underline{\mu}_i \leq \underline{u} \\ \dfrac{1}{p} A e_i^T C\lambda + B(\overline{\mu}_i - \underline{\mu}_i) e_i^T C + \beta_i I \geq 0, \quad i = 1, \dots, p \end{cases} \tag{14}$$

is feasible, then there exists an asymmetric static output feedback control law (13) such that the closed-loop system (6) is positive and asymptotically stable for any initial condition satisfying $0 \leq x(0) \leq \lambda$. Moreover, the desired gain matrix K can be computed as follows:

$$K = \sum_{i=1}^{p} \frac{(\overline{\mu}_i - \underline{\mu}_i) e_i^T}{e_i^T C\lambda} \tag{15}$$

where λ, $\overline{\mu}_i$, $\underline{\mu}_i$, and β_i, $i = 1, \dots, p$ are any feasible solution to the above LP problem.

Proof: Take any λ, $\overline{\mu}_i$, $\underline{\mu}_i$, and β_i, $i = 1, \dots, p$ that solve the LP problem (14) and define $K = \sum_{i=1}^{p} \frac{(\overline{\mu}_i - \underline{\mu}_i) e_i^T}{e_i^T C\lambda}$. From Lemma 4, the trajectory of the system (6) is such that $0 \leq x(t) < \lambda$ from any initial condition satisfying $0 \leq x(0) \leq \lambda$. Using this fact and recalling the inequalities $\sum_{i=1}^{p} \overline{\mu}_i \leq \overline{u}$, $e_i^T C\lambda > 0$, $\overline{\mu}_i e_i^T C \geq 0$, and $\underline{\mu}_i e_i^T C \geq 0$, for $i = 1, \dots, p$, we have $\overline{\mu}_i e_i^T Cx(t) \leq \overline{\mu}_i e_i^T C\lambda$, for $i = 1, \dots, p$, or equivalently, $\frac{(\overline{\mu}_i - \underline{\mu}_i) e_i^T Cx(t)}{e_i^T C\lambda} \leq \frac{\overline{\mu}_i e_i^T C\lambda}{e_i^T C\lambda}$, for $i = 1, \dots, p$, then $\sum_{i=1}^{p} \frac{(\overline{\mu}_i - \underline{\mu}_i) e_i^T Cx(t)}{e_i^T C\lambda} = KCx(t) \leq \sum_{i=1}^{p} \overline{\mu}_i \leq \overline{u}$, finally, it is easy to see that the static output feedback control $u(t) = KCx(t)$ is such that $u(t) \leq \overline{u}$ for any initial condition satisfying $0 \leq x(0) \leq \lambda$. Similarly, the static output feedback control $u(t) = KCx(t)$ is such that $-\underline{u} \leq u(t)$ for any initial condition satisfying $0 \leq x(0) \leq \lambda$. The rest of the proof follows the same arguments of Theorem 1 by using the fact that any gain matrix can be expressed as the difference of two positive matrices. ■

C. Controller With State Constraint

The previous results can be easily extended to another kind of constraint, such as state constraints (bounds on the state). Let consider the LTI system (4) with the following box:

$$0 \leq x(t) \leq \overline{x} \tag{16}$$

where $\overline{x} > 0$ is a given bound on the state variables.

In what follows, it is shown how to solve the following problem: Given $\overline{x} > 0$, find a static output feedback control law $u(t) = Ky(t) = KCx(t)$ and the corresponding set of initial conditions, such that the closed-loop system (6) is positive, asymptotically stable, and the state is constrained to evolve in the box (16).

Theorem 4: For given bound $\overline{x} > 0$ on the state variables, if the following LP problem in the variables $\lambda \in \mathbb{R}^n$, $\mu_i \in \mathbb{R}^m$ and β_i, $i = 1, \dots, p$:

$$\begin{cases} A\lambda + B\left(\sum_{i=1}^{p} \mu_i\right) < 0 \\ \lambda > 0 \\ \lambda \leq \overline{x} \\ e_i^T C\lambda > 0, \quad i = 1, \dots, p \\ \dfrac{1}{p} A e_i^T C\lambda + B\mu_i e_i^T C + \beta_i I \geq 0, \quad i = 1, \dots, p \end{cases} \tag{17}$$

is feasible, then there exists a static output feedback control (5), such that the closed-loop system (6) is positive, asymptotically stable, and the state is constrained to evolve in the box (16) for any initial condition within the box $0 \leq x(0) \leq \lambda$. Moreover, the gain matrix K is given as follows:

$$K = \sum_{i=1}^{p} \frac{\mu_i e_i^T}{e_i^T C\lambda} \tag{18}$$

where λ, μ_i, and β_i, $i = 1, \dots, p$ are any feasible solution to the above LP problem.

Proof: It suffices to show that the state is such that $x(t) \leq \overline{x}$. For this, take any λ, μ_i, and β_i, $i = 1, \dots, p$ that solve the LP problem (17) and define $K = \sum_{i=1}^{p} \frac{\mu_i e_i^T}{e_i^T C\lambda}$. From Lemma 4, the trajectory of the system (6) is such that $0 \leq x(t) < \lambda \quad \forall t > 0$ from any initial condition satisfying $0 \leq x(0) \leq \lambda$. Using this fact and recalling the inequality $\lambda \leq \overline{x}$, we have $0 \leq x(t) \leq \lambda \leq \overline{x} \quad \forall t \geq 0$. The rest of the proof follows the same argument as in the proof of Theorem 1, so it is omitted. ■

Remark 7: Of course, one can also combine Theorem 3 with Theorem 4 to have state and control constraints.

Remark 8: Note that the results proposed in this letter are a continuation of our work presented in [5] and [7]. Indeed, the full-information case ($C = I$) that corresponds to the state feedback controller can be easily derived, and the obtained results are equivalent to the results given in [5]. Therefore, the proposed controller framework has a unified property.

D. Iterative LP Algorithm

The result given in Theorem 1 can also be solved in two steps. Indeed, the feasibility of the first two inequalities of LP problem (7) guarantees the asymptotic stability of the closed-loop system (6) under the assumption that $A + BKC$ is Metzler. Also, the positive vector λ obtained defines a linear Lyapunov function for the closed-loop system (6). Moreover, in order to guarantee the positivity and the asymptotic stability of the closed-loop system (6), the matrix $(A + BKC)$ must be Metzler and Hurwitz; then, by using Lemma 2 and Lemma 3, the following inequalities:

$$\begin{cases} A + BKC + \beta I \geq 0 \\ A\lambda + BKC\lambda < 0 \end{cases} \tag{19}$$

need to be verified for given $\lambda > 0$ and $\beta \geq 0$.

Finally, the controller design method is given by the following two steps.

Step 1: Solve the first two inequalities of LP problem (7). Search for a feasible solution of λ and $\mu = \sum_{i=1}^{p} \mu_i$.

Step 2: Solve LP problem (19) to find K by using the positive vector λ obtained in Step 1.

Based on the above arguments, we propose the following iterative algorithm, for given state-space matrices A, B, and C.

Iterative LP algorithm:

Step I: Set $j = 1$ and solve the following LP problem in the variables $\lambda \in \mathbb{R}^n$, $\mu \in \mathbb{R}^m$, and $\epsilon \in \mathbb{R}$:

$$\begin{aligned} &\epsilon_0 = \min_{\lambda,\mu,\epsilon} \epsilon \\ &\text{subject to:} \\ &\begin{cases} A\lambda + B\mu \leq -\epsilon 1_n \\ \lambda > 0. \end{cases} \end{aligned} \tag{20}$$

If feasible, go to the next step. Otherwise, this algorithm fails to find a solution to the output stabilization problem. Stop.

Step II: Fix λ and solve the following LP problem in the variables $K \in \mathbb{R}^{m\times p}$, $\beta \in \mathbb{R}$, and $\epsilon \in \mathbb{R}$:

$$\begin{aligned} &\epsilon_j = \min_{K,\beta,\epsilon} \epsilon \\ &\text{subject to:} \\ &\begin{cases} A\lambda + BKC\lambda \leq -\epsilon 1_n \\ A + BKC + \beta I \geq 0. \end{cases} \end{aligned} \tag{21}$$

If feasible and K satisfies that $A + BKC$ is Metzler and Hurwitz, stop. Otherwise, go to the next step.

Step III: Fix K and solve the following LP problem in the variables $\lambda \in \mathbb{R}^n$, $\beta \in \mathbb{R}$, and $\epsilon \in \mathbb{R}$:

$$\begin{aligned} &\epsilon_j = \min_{\lambda,\beta,\epsilon} \epsilon \\ &\text{subject to:} \\ &\begin{cases} A\lambda + BKC\lambda \leq -\epsilon 1_n \\ \lambda > 0 \\ A\mathrm{diag}(\lambda) + BKC\mathrm{diag}(\lambda) + \beta I \geq 0 \end{cases} \end{aligned} \tag{22}$$

If feasible, set $j = j + 1$ and go to Step II. Otherwise, this algorithm fails to find a solution to the output stabilization problem. Stop.

IV. Illustrative Examples

In this section, several illustrative examples have been considered to demonstrate the effectiveness and the applicability of the proposed method compared to the existing ones.

A. Example 1

To illustrate the applicability of Theorem 1 for MIMO continuous-time positive linear system, consider the state-space matrices given in [27]

$$A = \begin{bmatrix} -0.5 & 0 & 0.4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.2 \end{bmatrix}.$$

Note that the design technique of [1] cannot be applied to the above system.

By solving the optimized LP problem (10), we get

$$\lambda = \begin{bmatrix} 169.0130 \\ 129.7622 \\ 127.5925 \end{bmatrix}, \qquad \mu_1 = \begin{bmatrix} 37.4081 \\ 48.1470 \\ -96.9209 \end{bmatrix}$$

$$\mu_2 = \begin{bmatrix} -84.1004 \\ 54.2542 \\ 56.9069 \end{bmatrix}, \qquad \mu_3 = \begin{bmatrix} 3.5702 \\ -161.8675 \\ 10.0660 \end{bmatrix}$$

$$\beta_1 = 129.0295, \qquad \beta_2 = 136.8086$$

$$\beta_3 = 119.7134, \qquad \epsilon = 1.1433 \times 10^{-9}.$$

Then the static output feedback gain is

$$K = \begin{bmatrix} 0.4427 & -0.6481 & 0.1399 \\ 0.5697 & 0.4181 & -6.3431 \\ -1.1469 & 0.4385 & 0.3945 \end{bmatrix}.$$

It is worth mentioning that the LP problem (7) of theorem 1 is feasible; the closed-loop system matrix $A + BKC = \begin{bmatrix} -1.0735 & 0.4385 & 0.4789 \\ 0.2213 & -0.6481 & 0.0280 \\ 0.2849 & 0.4181 & -1.2686 \end{bmatrix}$ is Metzler, which means that the closed-loop system is positive. Also, the set of the eigenvalues of the closed-loop system matrix $A + BKC$ is $\{-0.3636, -1.5260, -1.1006\}$; this means that the matrix $A + BKC$ is Hurwitz; then, the closed-loop system is asymptotically stable. In addition, the rank of the static output feedback gain is 3 (full rank).

By comparing the number of decision variables and the convergence speed of our approach and the approach given in [27] for this example. We can note that the number of decision variables of our approach is 15 and those of [27] is 16, which means that our approach has a lower number of decision variables, and this fact is very important for large-scale positive systems. Also, the set of eigenvalues of the approach given in [27] is $\{-0.7036, -0.4743, -0.0984\}$. By comparing this set of eigenvalues with the set of eigenvalues of our approach given above, we can remark that the closed-loop system given by our approach has a faster convergence speed than that given by the approach in [27].

B. Example 2

Let the state-space representation of a nonpositive unstable open-loop LTI MIMO system [3] whose state-space matrices are given as below

$$A = \begin{bmatrix} -0.15 & 1.90 & 1.55 \\ 0.50 & -0.30 & 0.10 \\ 0.20 & 0.50 & -2.55 \end{bmatrix}$$

$$B = \begin{bmatrix} 0.55 & -0.64 & 0.16 \\ 1.69 & 0.38 & 0 \\ 0.59 & -1.50 & 1.31 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Note that the technique of [27] cannot be applied to this example because this approach can only be applied to positive open-loop systems. Also, the design of [23] cannot be applied directly to this example since the output matrix C is not in row/column echelon form. Moreover, the LP method of [1] can only be applied to MIMO positive systems with rank one gain matrices. Furthermore, the proposed LMI-based iterative algorithm of [3] increases the execution time and the computational burden.

We have applied the proposed LP problem (10). Then, the following feasible solution is obtained:

$$\lambda = \begin{bmatrix} 149.6628 \\ 87.2597 \\ 64.0582 \end{bmatrix}, \quad \mu_1 = \begin{bmatrix} -115.6600 \\ 358.5139 \\ 495.6806 \end{bmatrix}$$

$$\mu_2 = \begin{bmatrix} -13.7215 \\ 64.0628 \\ 48.8182 \end{bmatrix}, \quad \beta_1 = 359.5028$$

$$\beta_2 = 193.2124, \quad \epsilon = 2.0320 \times 10^{-9}.$$

Thus, the gain matrix of the static output feedback controller is

$$K = \begin{bmatrix} -0.4882 & -0.2142 \\ 1.5132 & 1.0001 \\ 2.0922 & 0.7621 \end{bmatrix}$$

and its rank is 2 (full rank).

Finally, we can check that the closed-loop system is positive and asymptotically stable, since the matrix $A+BKC = \begin{bmatrix} -1.0522 & 0.9978 & 0.9141 \\ 0.2500 & -0.5500 & 0.1180 \\ 0.3829 & 0.6829 & -3.1781 \end{bmatrix}$ is Metzler, and its eigenvalues are $\{-3.3349, -1.3160, -0.1294\}$.

This example proves that the proposed LP approach is applicable for MIMO LTI systems without any restrictions on the state-space matrices or on the rank of the controller gain matrix.

C. Example 3

To show that our approach is easier compared with the approach given in [26], which is based on an iterative linear matrix inequality (ILMI) algorithm. We consider another unstable LTI system given in Example 1 of [26] whose state-space matrices are

$$A = \begin{bmatrix} -2 & 1 \\ 2 & -0.9 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 \end{bmatrix}.$$

Applying Theorem 1, the following feasible solution is obtained: $\lambda^T = [13.4752 \quad 31.6375]$, $\mu = -7.5760$, $\beta = 198.7949$, and $\epsilon = 4.6457 \times 10^{-11}$.

Thus, the gain of the static output feedback controller is $K = -0.0987$. It is verified that the matrix $A+BKC$ is Hurwitz and Metzler.

By comparing our design method with the design given in [26, Example 1], we can see that the number of decision variables (4) in our designed method is less than the number of decision variables (6) in the design of [26]. Also, the designed method of [26] uses a fixed matrix parameter, which may be difficult to choose. In addition, the average calculation time is 0.512 s for our design and 2.962 s for the design given in [26]. Finally, this example proves that the proposed LP approach has less computational complexity and average calculation time than the iterative LMI approach proposed in [26].

D. Example 4

We consider the following unstable MIMO LTI system:

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ -1 & -3 & 1 & 0 \\ -4 & 2 & -1 & 1 \\ 2 & 0 & -1 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 3 & -1 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 2 & 4 & 5 \\ 4 & 5 & 3 & 2 \\ 2 & 4 & 2 & 5 \end{bmatrix}.$$

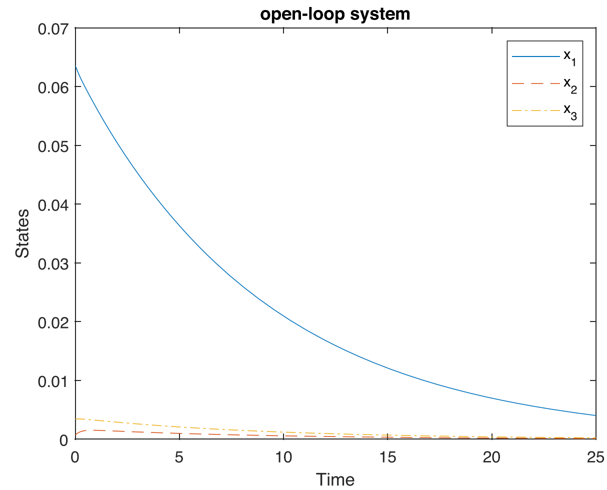

Fig. 1. Response of the open-loop system.

Note that the ILMI algorithm given in [26] cannot give a feasible solution to the above system. For this reason, we apply our iterative LP algorithm presented in this letter. Then, the following gain matrix is obtained in the first iteration:

$$K = \begin{bmatrix} 0.1747 & 0.0733 & -0.4028 \\ -0.2925 & 0.0881 & 0.1280 \\ -66.3658 & -30.6735 & 93.2983 \\ -70.6919 & 15.9231 & 37.8378 \end{bmatrix}$$

It is verified that the matrix $A+BKC$ is Metzler and Hurwitz.

Remark 9: In the literature of positive systems [5], [6], [7], [16], [22], it has been verified that LP is more effective than LMI. The authors in [3] and [26] proposed LMI-based iterative algorithms to solve the static output feedback control of positive systems. Note that LMI-based iterative algorithms proposed in [3] and [26], used slack matrix variables in order to solve this problem. For instance, if we suppose that the system dimension is n, the number of inputs is m, and the number of outputs is p, the slack matrix variable in the design of [26] will contain $m \times n$ parameters, and the selection of such a matrix decides the success of the proposed algorithm. On the other hand, our iterative LP algorithm does not use any slack variables. Thus, it is clear that the proposed iterative LP algorithm is simpler than iterative LMI algorithms given in [3] and [26].

E. Example 5

Next, we consider a mammillary model from [10] described by the following differential equation:

$$\dot{x}(t) = \begin{bmatrix} -(a_{21}+a_{31}) & a_{12} & a_{13} \\ a_{21} & -(a_{02}+a_{12}) & 0 \\ a_{31} & 0 & -(a_{03}+a_{13}) \end{bmatrix} x(t) + \begin{bmatrix} b_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & b_3 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} a_c & 0 & 0 \end{bmatrix} x(t)$$

where $a_{ij} > 0$ represent flow rates.

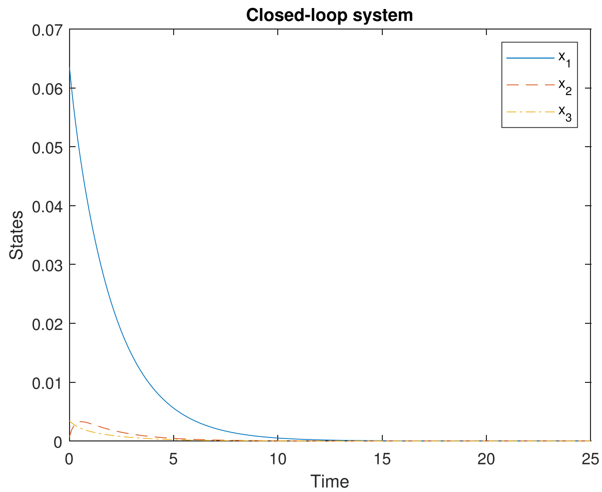

Fig. 2. Response of the closed-loop system.

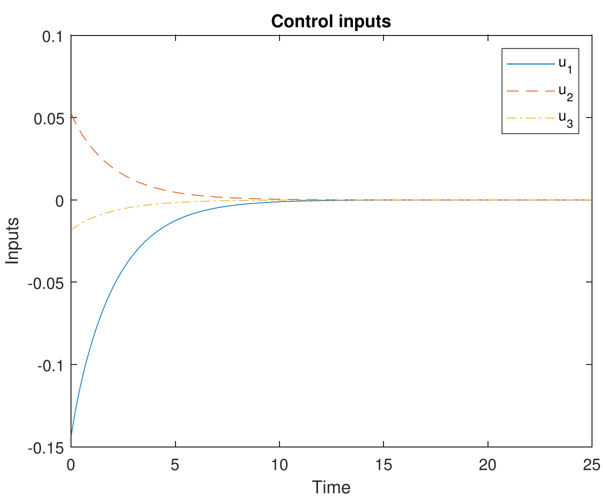

Fig. 3. Control input signals.

For the numerical simulation, it is assumed that the system's parameters are known, taking the following values: $a_{21} = 0.1$, $a_{12} = 2.0$, $a_{13} = 1.5$, $a_{31} = 0.15$, $a_{02} = 1.8$, $a_{03} = 1.23$, $b_1 = 0.2$, $b_2 = 0.2$, $b_3 = 0.2$, $a_c = 1$.

We would like to stabilize this system by designing an output feedback controller with asymmetric bounds based on Theorem 4. Hence, we have selected the control bounds as $\overline{u}^T = [0.6 \quad 0.4 \quad 0.1]$, and $\underline{u}^T = [0.5 \quad 0.2 \quad 0.3]$. The LP problem (14) is feasible with the following solution: $\lambda = \begin{bmatrix} 0.0735 \\ 0.0107 \\ 0.0134 \end{bmatrix}, \overline{\mu}_1 = \begin{bmatrix} 0.1778 \\ 0.1677 \\ 0.0800 \end{bmatrix}, \underline{\mu}_1 = \begin{bmatrix} 0.3439 \\ 0.1068 \\ 0.1012 \end{bmatrix}$, $\beta = 0.3321$ and $\epsilon = 1.6528 \times 10^{-11}$.

Therefore, we have obtained the following gain matrix:

$$K = \begin{bmatrix} -2.2596 \\ 0.8277 \\ -0.2890 \end{bmatrix}.$$

Simulation results for the open-loop, closed-loop systems, and input control signals starting from the initial condition $x(0)^T = [0.0635 \quad 0.0007 \quad 0.0034] \leq \lambda^T$ can be seen in Figs. 1, 2, and 3, respectively. From Fig. 2, we can see that the closed-loop system is positive and asymptotically stable. Also, the control input signals fulfill the desired asymmetric bounds, as shown in Fig. 3.

V. Conclusion

In this article, an easy LP-based approach that solves the static output feedback control synthesis has been investigated. New necessary and sufficient conditions that ensure the positivity and the asymptotic stability in closed-loop system are presented. In addition, the proposed method has been extended to synthesis problems with constraints on the static output feedback control law and/or on the state variables. Moreover, all the proposed results can also be applied to synthesis problems with unconstrained or constrained state feedback control laws in the case where the state variables are available. All the proposed conditions are expressed in terms of the state-space matrices, and they are solvable in terms of simple LP problems. Furthermore, based on Theorem 1, an iterative LP algorithm is presented to compute the gain matrix of the static output feedback controller. The effectiveness and the applicability of this LP-based synthesis method have been demonstrated on several illustrative examples.

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