Extended Balancing of Continuous LTI Systems: A Structure-Preserving Approach

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Abstract—In this article, we treat extended balancing for continuous-time linear time-invariant systems. We take a dissipativity perspective, thus, resulting in a characterization in terms of linear matrix inequalities. This perspective is useful for determining a priori error bounds. In addition, we address the problem of structure-preserving model reduction of the subclass of port-Hamiltonian systems. We establish sufficient conditions to ensure that the reduced-order model preserves a port-Hamiltonian structure. Moreover, we show that the use of extended Gramians can be exploited to get a small error bound and, possibly, to preserve a physical interpretation for the reduced-order model. We illustrate the results with a large-scale mechanical system example. Furthermore, we show how to interpret a reduced-order model of an electrical circuit again as a lower dimensional electrical circuit.

Index Terms—Error bound, extended Gramians, model reduction, port-Hamiltonian (PH) systems.

I. INTRODUCTION

BALANCING is a tool that is often used for model reduction purposes, giving rise to the balanced truncation methodology. This approach relies on realization theory, the observability, and controllability Gramians, and it is directly related to the concept of Hankel operator of a system. Since its introduction in the seminal work of Moore [17], balancing for stable linear systems has been extensively studied, in particular, a thorough exposition of this topic can be found in [1], while in [22] a brief tutorial is presented, which provides the basis for extending the results to nonlinear systems [10].

Balanced truncation, based on the use of standard observability and controllability Gramians, preserves some relevant properties of the original system, e.g., asymptotic stability, observability, and controllability. Furthermore, it is possible to establish an error bound, which is given in terms of the so-called Hankel singular values [11] corresponding to the truncated states. Nevertheless, in this standard formulation of balanced truncation, some original system properties, like passivity or particular structures, are not necessarily preserved. Another possible drawback of this approach occurs when the Hankel singular values are large, which originates a large error bound. To overcome this issue, the use of the so-called generalized Gramians for model reduction purposes was introduced in [12]. The generalized observability and controllability Gramians are solutions to the respective Lyapunov inequalities. This differs from the standard Gramians, which are given by the solutions of Lyapunov equations. In addition to stability, controllability, and observability, balanced truncation using generalized Gramians can preserve other properties, such as passivity, for the reduced-order model. Moreover, since the solutions of the involved Lyapunov inequalities are not unique, generalized Gramians can be used to obtain smaller error bounds [8], and in some cases, to preserve some particular structures for the reduced-order model [4].

A further extension of balanced truncation can be formulated by using the concept of extended Gramians, which was introduced in [21] for discrete-time systems; and a preliminary continuous-time counter part of these results was recently reported in [23]. The discrete-time and continuous-time methods are rather different, except from the fact that the disspativity theory plays a fundamental role in both to establish the error bound. In this approach, referred to as extended balancing, the Gramians are solutions to specific linear matrix inequalities (LMIs) and, in contrast to other balancing methods, the error bound is obtained by using dissipativity arguments [25] and not through a transfer function approach. Furthermore, this balancing method provides more degrees of freedom to impose certain structure to the reduced-order model, and can be potentially useful to improve the error bound.

In this article, we focus on the extended balanced truncation of continuous-time linear time-invariant (CTLTI) systems, where we are interested in the versatility of this methodology to preserve specific structures. In particular, we are interested in CTLTI port-Hamiltonian (PH) systems, which are suitable to represent a broad range of physical systems in several domains, e.g., RLC circuits and mechanical systems. These systems are *passive*, which is convenient for control and analysis purposes.

Moreover, the interconnection of two or more PH systems yields another PH system, which makes this modeling approach ideal to deal with large networks of physical systems. Some works where the problem of model reduction related to balancing of PH systems has been studied are [19] for LTI systems and [9] and [14] for the nonlinear case. With respect to those works, the novelty of this article lies in the use of extended balanced truncation to obtain the reduced-order model and, in contrast to the mentioned works, to establish an error bound. Moreover, this approach is fundamental to, under some conditions, preserve the physical interpretation of the reduced-order model.

This article aims to obtain a reduced-order model that approximates the behavior of the original system properly while preserving its PH structure. Towards this end, we first study extended balanced truncation for CTLTI systems, and then we focus on its application to CTLTI PH systems for structure preservation purposes. The main contributions of this article are as follows.

1) We recall the results from [23], and provide novel proofs for the error bound computation, which turn out to be rather different than in the discrete-time case [21].

2) We identify a family of generalized Gramians that are suitable for balanced truncation of CTLTI PH systems with PH structure preservation. To the best of our knowledge, the characterization of these solutions to the Lyapunov inequalities is new.

3) The use of extended balancing for PH structure preservation and as a tool to obtain a reduced-order system that approximates the original one with a small error bound. Moreover, we show with an illustrative example that this approach can be used to preserve more particular structures, like RLC circuits structure, and a physical interpretation for the reduced-order model.

The rest of this article is structured as follows. We provide the basic background in Section II. The fundamental notion of extended Gramians and the computation of the error bound are presented in Section III. In Section IV, we introduce the generalized and extended balancing of PH systems with structure preservation. We present two illustrative examples in Section V, where the use of extended Gramians in the second example allows us to preserve an even more particular structure than the PH one, that is, the reduced-order system is physically interpretable as an RLC circuit again. Finally, Section VI concludes this article.

Notation: We assume that all the matrices have exclusively real entries. The matrix $A \in \mathbb{R}^{n \times n}$ is said to be *positive semidefinite* if it is symmetric, and the inequality $x^\top A x \geq 0$ holds for all $x \in \mathbb{R}^n$ holds. Similarly, A is said to be a *positive definite* matrix if it is symmetric, and $x^\top A x > 0$ holds for all $x \in \mathbb{R}^n \backslash \{\mathbf{0}\}$. The identity matrix is denoted as I, when necessary, a subscript is added to indicate the dimension of the matrix. The symbol $\mathbf{0}$ denotes a matrix or vector whose entries are zeros. The set of positive real numbers is expressed as $\mathbb{R}_{>0}$, while the set of nonnegative real numbers is denoted by $\mathbb{R}_{\geq 0}$. Diagonal matrices are denoted as $\mathtt{diag}\{a_1, \ldots, a_n\}$, where $a_1, \ldots, a_n$ are the elements of the main diagonal of the matrix. Additionally, the symbol Λ is reserved for diagonal matrices with positive entries, that is, the square matrix $\Lambda \in \mathbb{R}^{n \times n}$ is given by $\Lambda = \mathtt{diag}\{\sigma_1, \ldots, \sigma_n\}$, where $\sigma_i \in \mathbb{R}_{>0}$, for $i = 1, \ldots, n$.

Block diagonal matrices are denoted as $\mathtt{block}\{A_1, \ldots, A_n\}$, where $A_1, \ldots, A_n$ are square matrices. The symbol U is reserved to orthogonal matrices, that is, $U U^\top = I$. Consider the vector $x \in \mathbb{R}^n$, then $\|x\|$ denotes the Euclidean norm of x, that is, $\|x\| = \sqrt{x^\top x}$. Consider a signal $e(t) \in \mathcal{L}_2^n$, then $\|e\|_2$ denotes the $\mathcal{L}_2$ norm of $e(t)$, given by $\|e\|_2 = (\int_0^\infty \|e(t)\|^2 \mathrm{d}t)^{\frac{1}{2}}$.

II. PRELIMINARIES

Consider a CTLTI system described as

$$\pm : \begin{cases} \dot{x} = & Ax + Bu \\ y = & Cx \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector for $m \leq n$, $u \in \mathbb{R}^m$ is the input vector, and $y \in \mathbb{R}^q$ denotes the output vector. Accordingly, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, and $C \in \mathbb{R}^{q \times n}$. Assume that the system (1) is *asymptotically* stable, thus, the so-called *generalized observability Gramians* $Q \in \mathbb{R}^{n \times n}$ are positive semi-definite solutions to the following Lyapunov inequality:

$$QA + A^\top Q + C^\top C \leq 0. \tag{2}$$

Analogously, the *generalized controllability Gramians* $\check{P} \in \mathbb{R}^{n \times n}$ are given by positive semidefinite solutions to

$$A\check{P} + \check{P}A^\top + BB^\top \leq 0. \tag{3}$$

In particular, when (2) and (3) are equalities, the matrices Q and $\check{P}$ are known as the *standard* observability and controllability Gramian, respectively. For further details, we refer the reader to [1].

A. Generalized Balanced Truncation for LTI

A CTLTI system is said to be *generalized balanced* if

$$Q = \check{P} = \Lambda_{QP} \tag{4}$$

where $\Lambda_{QP} > 0$ is a diagonal matrix, see the Notation section. Accordingly, balancing for LTI systems [17], relies on obtaining an invertible state transformation

$$\bar{x} = W_g^{-1} x \tag{5}$$

such that

$$W_g^{-1} \check{P} Q W_g = \Lambda_{QP}^2 \tag{6}$$

where we assume that the elements of $\Lambda_{QP} = \mathtt{diag}\{\sigma_1, \ldots, \sigma_n\}$ are ordered from largest to smallest, that is, $\sigma_i > \sigma_{i+1}$, for $i = 1, \ldots, n-1$. Model reduction based on balancing is carried out by truncating the states corresponding to the small elements of Λ_{QP}, i.e., if $\sigma_i >> \sigma_{i+1}$, then we set

$$\bar{x}_{i+1} = \cdots = \bar{x}_n = 0.$$

The error bound is given by the sum of the truncated singular values [11], i.e.,

$$\|\Sigma - \widehat{\Sigma}\|_\infty \leq 2 \sum_{j=i+1}^n \sigma_j \tag{7}$$

where $\widehat{\Sigma}$ corresponds to the realization of the reduced-order system and $\|\Sigma - \widehat{\Sigma}\|_\infty$ denotes the $\mathcal{H}_\infty$-norm of the error system.

For a more elaborated exposition of balancing and the corresponding reduced-order model properties, we refer the reader to [26]. At this point, we highlight that the error bound obtained through generalized balanced truncation is lower than the one obtained with the use of standard Gramians, for further details see [12].

III. EXTENDED BALANCED TRUNCATION

The generalized balanced truncation approach can be extended by considering the so-called *extended Gramians* instead of the generalized ones. This extension has two main advantages; on the one hand, the error bound can be reduced [20] for the discrete-time case. On the other hand, the use of extended Gramians provides extra degrees of freedom, which can be exploited to impose a certain structure on the reduced-order system.

In this section, we revisit and significantly improve the concept of extended balanced truncation for the continuous-time case, which was first introduced in [23]. Towards this end, we introduce the following assumption, which is necessary to establish the concept of extended Gramians.

Assumption 1: The solutions, Q and $\check{P}$, to the inequalities (2) and (3) are positive definite.

We stress the fact that if the system (1) is controllable and observable, then Assumption 1 holds. Nonetheless, this latter condition is sufficient but not necessary, thus, might be conservative. Moreover, if Assumption 1 is satisfied, then we can define

$$P := \check{P}^{-1}. \tag{8}$$

Note that P is a positive definite matrix. To ease the readability and simplify the notation of this section, we define the following matrices:

$$
\begin{aligned}
A_o &:= \alpha I_n + A \\
A_c &:= \beta I_n + A \\
X_o &:= -QA - A^\top Q - C^\top C \\
X_c &:= -PA - A^\top P - PBB^\top P
\end{aligned}
\tag{9}
$$

where P is defined in (8), and $\alpha > 0$ and $\beta \geq 0$. Note that, from (2) and (3), $X_o \geq 0$ and $X_c \geq 0$. The definition of extended Gramians is the starting point of the theory contained in the following sections of this article. These concepts were introduced for CTLTI systems without proofs in [23]. We present the, slightly altered, results and their corresponding proof as follows.
Extended Gramians. Consider the following two LMIs:

$$
\begin{bmatrix}
X_o & Q - A_o^\top S \\
Q - S^\top A_o & S + S^\top
\end{bmatrix} \geq 0
\tag{10}
$$

and

$$
\begin{bmatrix}
-PA - A^\top P & -P + A_c^\top T & -2PB \\
-P + T^\top A_c & T + T^\top & 2T^\top B \\
-2B^\top P & 2B^\top T & 4I_m
\end{bmatrix} \geq 0
\tag{11}
$$

with $A_o, X_o,$ and A_c defined as in (9), and $T, S \in \mathbb{R}^{n \times n}$. We call (10) and (11) the extended observability and controllability LMIs with extended observability Gramian (Q, S, α) and

extended inverse controllability Gramian (P, T, β), respectively. Now, we are in position to formulate the relation between the generalized observability Gramian and the extended observability Gramian.

Theorem 1: (Observability Gramians)
The inequality (2) has a solution $Q > 0$ if and only if the LMI (10) admits a solution (Q, S, α) with $Q > 0$, $(S + S^\top) \geq 0$, and α large enough. Moreover, if X_o, defined in (9), is positive definite, then there exist α and $S = S^\top > 0$ such that the LMI (10) holds.

Proof: Only if. Assume that (10) has a solution (Q, S, α), then multiplying (10) by $[I_n \ \mathbf{0}]$ from the left and by $[I_n \ \mathbf{0}]^\top$ from the right, it follows that (2) admits a solution $Q > 0$.
If: Assume there exists $Q > 0$ solving (2). Select $S = A_o^{-\top} Q$, with $-\alpha$ not an eigenvalue of A. Then, the off-diagonal blocks of (10) are zero. Furthermore

$$S + S^\top = A_o^{-\top} Q + Q A_o^{-1}.$$

Accordingly, we have the following equivalence:

$$
\begin{aligned}
0 &\leq S + S^\top \Longleftrightarrow \\
0 &\leq A_o^\top (S + S^\top) A_o = A_o^\top Q + Q A_o \\
&= 2\alpha Q - C^\top C - X_o.
\end{aligned}
\tag{12}
$$

Note that, since X_o does not depend on α, the inequality (12) holds for α large enough. Hence, there exist $Q > 0$ and $\alpha > 0$ such that the LMI (10) holds.
Symmetric S: Assume that $Q > 0$ and $X_o > 0$. Consider a symmetric matrix $\Gamma_o \in \mathbb{R}^{n \times n}$ verifying

$$\alpha Q + \Gamma_o > 0. \tag{13}$$

Select

$$S = Q(\alpha Q + \Gamma_o)^{-1} Q. \tag{14}$$

Hence, $S = S^\top > 0$. Now, multiply (10) by $\texttt{block}\{I_n, QS^{-1}\}$ from the left and by $\texttt{block}\{I_n, S^{-1}Q\}$ from the right, yielding

$$
\begin{bmatrix}
X_o & QS^{-1}Q - A_o^\top Q \\
QS^{-1}Q - QA_o & 2QS^{-1}Q
\end{bmatrix}
$$
$$
= \begin{bmatrix}
X_o & \Gamma_o - A^\top Q \\
\Gamma_o - QA & 2(\alpha Q + \Gamma_o)
\end{bmatrix} \geq 0.
\tag{15}
$$

Furthermore, the LMI (15) is equivalent through Schur complement to

$$2\alpha Q + 2\Gamma_o - \Theta_o \geq 0 \tag{16}$$

with

$$\Theta_o := (\Gamma_o - QA) X_o^{-1} (\Gamma_o - A^\top Q).$$

Note that there exists α, large enough, such that (16) is satisfied. This completes the proof. ∎
The results on generalized and extended observability Gramians have a controllability version as follows.
Theorem 2: (Controllability Gramians)
The inequality (3) has a solution $\check{P} > 0$ if and only if the LMI (11) has a solution (P, T, β) with $P > 0$. Furthermore, if X_c, defined in (9), is positive definite, then there exist $\beta > 0$ and $T = T^\top > 0$ such that the LMI (11) holds.

Proof: To simplify the notation, we define

$$Y_c := -P + (A_c^\top + PBB^\top)T.$$

Note that a Schur complement analysis yields that (11) is equivalent to the following LMI:

$$\begin{bmatrix} X_c & Y_c \\ Y_c^\top & T + T^\top - T^\top BB^\top T \end{bmatrix} \geq 0. \tag{17}$$

Only if: Assume that (11) admits a solution (P, T, β) with $P > 0$, thus, equivalently, (17) is satisfied. Multiplying the latter LMI by $[I_n \;\; \mathbf{0}]$ from the left and by $[I_n \;\; \mathbf{0}]^\top$ from the right, it follows that

$$
\begin{aligned}
& X_c \geq 0 \\
\Longleftrightarrow\; & -PA - A^\top P - PBB^\top P \geq 0 \\
\Longleftrightarrow\; & A\breve{P} + \breve{P}A^\top + BB^\top \leq 0
\end{aligned}
$$

where we used (8) to obtain the last inequality.

If: Assume there exists $\breve{P} > 0$ solution to (3). Fix[1] $T = P(\beta I_n - A)^{-1}$ with P defined in (8), then we get

$$
\begin{aligned}
Y_c &= -P + (A_c^\top + PBB^\top)P(\beta I_n - A)^{-1} \\
&= -P + (\beta P - PA - X_c)(\beta I_n - A)^{-1} \\
&= -X_c(\beta I_n - A)^{-1} \\
&= -X_c \breve{P} T
\end{aligned}
$$

and

$$
\begin{aligned}
T + T^\top - T^\top BB^\top T &= T^\top \left(T^{-1} + T^{-\top} - BB^\top \right) T \\
&= T^\top \breve{P} \left(2\beta P + X_c \right) \breve{P} T.
\end{aligned}
$$

Hence, the LMI (17) takes the form

$$\begin{bmatrix} X_c & -X_c\breve{P}T \\ T^\top \breve{P} X_c & T^\top \breve{P}\left(2\beta P + X_c\right)\breve{P}T \end{bmatrix} \geq 0. \tag{18}$$

Now, we multiply (18) by $\mathtt{block}\{I_n, PT^{-\top}\}$ from the left and by $\mathtt{block}\{I_n, T^{-1}P\}$ from the right, yielding

$$\begin{bmatrix} X_c & -X_c \\ -X_c & X_c \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 2\beta P \end{bmatrix} \geq 0$$

which holds for every $\beta \geq 0$.

Symmetric T: Assume that $P > 0$ and $X_c > 0$. Consider a symmetric matrix $\Gamma_c \in \mathbb{R}^{n \times n}$ verifying

$$\beta\breve{P} + \Gamma_c > 0. \tag{19}$$

Select

$$T = \left(\beta\breve{P} + \Gamma_c\right)^{-1}. \tag{20}$$

Hence, $T = T^\top > 0$. Multiply (17) by $\mathtt{block}\{I_n, T^{-\top}\}$ from the left and by $\mathtt{block}\{I_n, T^{-1}\}$ from the right, and substitute (20) to obtain

$$\begin{bmatrix} X_c & -P\Gamma_c + A^\top + PBB^\top \\ -\Gamma_c P + A + BB^\top P & 2(\beta\breve{P} + \Gamma_c) - BB^\top \end{bmatrix} \geq 0 \tag{21}$$

which is equivalent to

$$2\beta\breve{P} + 2\Gamma_c - BB^\top - \Theta_c \geq 0$$

[1] Since $\beta \geq 0$ and $\Re\{\lambda(A)\} < 0$, β is not an eigenvalue of A.

where

$$\Theta_c := (-\Gamma_c P + A + BB^\top P)X_c^{-1}(-P\Gamma_c + A^\top + PBB^\top).$$

Since Θ_c does not depend on β, it follows that the LMI (21), and in consequence the LMI (11), holds for $\beta > 0$ large enough. Accordingly, there exists $\beta > 0$ such that (19) and (21) are satisfied. This completes the proof. ∎

Remark 1: For clarity of presentation, we assume that $X_o > 0$ and $X_c > 0$ to prove the existence of symmetric solutions to (10) and (11), respectively. While these conditions are not restrictive, they can be relaxed to $X_o \geq 0$ and $X_c \geq 0$ by using generalized inverses. This, however, needs the introduction of the following conditions:

$$
\begin{aligned}
(I_n - X_o X_o^\dagger)(\Gamma_o - A^\top Q) &= \mathbf{0} \\
(I_n - X_c X_c^\dagger)(-P\Gamma_c + A^\top + PBB^\top) &= \mathbf{0}
\end{aligned} \tag{22}
$$

where $X_o^\dagger$ and $X_c^\dagger$ denote generalized inverses of X_o and X_c, respectively. Note that both expressions in (22) are satisfied if $X_o > 0$ and $X_c > 0$.

Remark 2: If $X_o > 0$ and $X_c > 0$, then the inequalities (2) and (3) can be expressed as Lyapunov equations. Moreover, since the system (1) is asymptotically stable, A is Hurwitz. This implies that the mentioned Lyapunov equations have positive definite solutions Q and $\breve{P}$. Accordingly, in this case, Assumption 1 is satisfied.

Remark 3: The symmetric matrices Γ_o and Γ_c provide degrees of freedom in the selection of the extended Gramians. These degrees of freedom can be used to improve the error bound, in case the Gramians are used for model reduction, see Section III-A, or to impose a desired structure to the reduced-order model as given in Section V.

For the model reduction application, we assume that the matrices S and T are symmetric. From Theorems 1 and 2, it is clear that this assumption is not necessary to ensure the existence of solutions to (10) and (11), but we need it for obtaining an error bound in Section III-A.

In the extended balancing approach, we balance S and T^{-1} to establish the error bound. Consequently, a CTLTI system is said to be *extended balanced* if

$$S = T^{-1} = \Lambda_{ST} \tag{23}$$

where Λ_{ST} is a diagonal matrix, see the Notation section. Hence, we look for an invertible state transformation

$$\bar{x} = W_e^{-1} x \tag{24}$$

such that

$$W_e^{-1} T^{-1} S W_e = \Lambda_{ST}^2. \tag{25}$$

Similar to Section II-A, we assume that the elements of the diagonal matrix Λ_{ST} are ordered from largest to smallest. Hence, the order of the CTLTI system is reduced by truncating the states that correspond to the smallest elements of the aforementioned matrix.

The discrete-time version of the LMIs (10) and (11) can be found in [5] and [6]. While, a thorough exposition of extended balanced truncation for discrete-time linear time-invariant systems is given in [20] and [21].

A. Computation of the Error Bound

One of the appealing features of the balanced truncation approach is the possibility of establishing a clear error bound. For the generalized balanced truncation case, the inequality (7) establishes the error bound, which is customarily obtained through the analysis in the frequency domain of the original system and the reduced-order one [11], [26]. An alternative methodology to establish the error bound is to propose a storage function for the error system and use dissipativity arguments. Some works that have investigated this method are [15], [16], [20], and [21] for discrete-time systems, and [25] and [23] for continuous-time systems. In this section, we propose a storage function, different from the one used in [23], to compute the error bound for the extended balancing of CTLTI systems. To this end, we assume that the linear transformation W_e, such that (25) holds, is known. Then, we introduce the following state-space systems:

$$\bar{\Sigma} : \begin{cases} \dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u \\ \bar{y} = \bar{C}\bar{x} \end{cases} \tag{26}$$

$$\Sigma_r : \begin{cases} \dot{x}_r = \bar{A}x_r + \bar{B}u + v(t) \\ y_r = \bar{C}x_r \end{cases} \tag{27}$$

where $\bar{x}$ is defined as in (24), $v(t) \in \mathbb{R}^n$ is an external signal, and $x_r \in \mathbb{R}^n$ is an auxiliary state, and

$$\bar{A} := W_e^{-1}AW_e, \quad \bar{B} := W_e^{-1}B, \quad \bar{C} := CW_e. \tag{28}$$

Now, we split $\bar{x}$ into two parts, namely

$$\bar{x} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} \tag{29}$$

where $\bar{x}_1 \in \mathbb{R}^k$ is the part of the state to be preserved after the reduction of the model and $\bar{x}_2 \in \mathbb{R}^\ell$, with $\ell := n - k$, is the part to be truncated. Accordingly, the matrices given in (28) can be expressed as follows:

$$\bar{A} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} \bar{B}_1 \\ \bar{B}_1 \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} \bar{C}_1 & \bar{C}_2 \end{bmatrix}$$

with

$$\bar{A}_{11} \in \mathbb{R}^{k \times k}, \quad \bar{A}_{12} \in \mathbb{R}^{k \times \ell}, \quad \bar{A}_{21} \in \mathbb{R}^{\ell \times k}, \quad \bar{A}_{22} \in \mathbb{R}^{\ell \times \ell}$$
$$\bar{B}_1 \in \mathbb{R}^{k \times m}, \quad \bar{B}_2 \in \mathbb{R}^{\ell \times m}, \quad \bar{C}_1 \in \mathbb{R}^{q \times k}, \quad \bar{C}_2 \in \mathbb{R}^{q \times \ell}.$$

Thus, the truncation of the state $\bar{x}_2$ leads to the following reduced-order model:

$$\widehat{\Sigma} : \begin{cases} \dot{\hat{x}} = \widehat{A}\hat{x} + \widehat{B}u \\ \hat{y} = \widehat{C}\hat{x} \end{cases} \tag{30}$$

where

$$\hat{x} = \bar{x}_1, \quad \widehat{A} := \bar{A}_{11}, \quad \widehat{B} := \bar{B}_1, \quad \widehat{C} := \bar{C}_1.$$

Now, inspired by the ideas presented in [25], the approach adopted in [20] and [21] for discrete-time, and in [23] for continuous-time, we propose a storage function that is instrumental to establish the error bound. Towards this end, we first introduce the following definitions to simplify the notation of this section:

$$\bar{Q} := W_e^\top QW_e, \quad \bar{P} := W_e^\top PW_e \tag{31}$$
$$z_o := \bar{x} - x_r, \quad z_c := \bar{x} + x_r.$$

where P is defined as in (8). The following proposition introduces a storage function suitable to establish an error bound by using dissipativity arguments and the LMIs (10) and (11).

Proposition 1: Consider the systems Σ, $\bar{\Sigma}$, and Σ_r given in (1), (26), and (27), respectively. Assume that the triplet (Q, S, α) solves the LMI (10) and the triplet (P, T, β) solves the LMI (11). Consider the storage function

$$\mathcal{S}(z_o, z_c) = z_o^\top \bar{Q}z_o + \sigma_n^2 z_c^\top \bar{P}z_c \tag{32}$$

where σ_n is the nth element in the main diagonal of Λ_{ST}, and z_o and z_c are defined in (31). Then,

$$\dot{\mathcal{S}} \leq 4\sigma_n^2\|u\|^2 - \|y - y_r\|^2 \\ + 2\left[\sigma_n^2(\beta z_c + \dot{z}_c)^\top \Lambda_{ST}^{-1} - (\alpha z_o + \dot{z}_o)^\top \Lambda_{ST}\right]v. \tag{33}$$

Proof. Note that

$$\dot{\mathcal{S}} = 2z_o^\top \bar{Q}\dot{z}_o + 2\sigma_n^2 z_c^\top \bar{P}\dot{z}_c. \tag{34}$$

Define the vectors

$$\xi_o := \begin{bmatrix} W_e z_o \\ W_e v \end{bmatrix}, \quad \xi_c := \begin{bmatrix} W_e z_c \\ W_e v \\ u \end{bmatrix}.$$

Multiply the LMI (10) by $\xi_o^\top$ from the left and by ξ_o from the right, yielding

$$2\left[v^\top - z_o^\top(\alpha I_n + \bar{A}^\top)\right]\Lambda_{ST}v \\ + z_o^\top\left[W_e^\top X_o W_e z_o + 2\bar{Q}v\right] \geq 0 \\ \iff -2(\dot{z}_o + \alpha z_o)^\top \Lambda_{ST}v \\ + z_o^\top\left[W_e^\top X_o W_e z_o + 2\bar{Q}v\right] \geq 0 \\ \iff -2(\dot{z}_o + \alpha z_o)^\top \Lambda_{ST}v - z_o^\top \bar{C}^\top \bar{C}z_o \\ + 2z_o^\top \bar{Q}\left[v - \bar{A}z_o\right] \geq 0 \\ \iff -2(\dot{z}_o + \alpha z_o)^\top \Lambda_{ST}v - \|y - y_r\|^2 - 2z_o^\top \bar{Q}\dot{z}_o \geq 0 \tag{35}$$

where we used the facts

$$\dot{z}_o = \bar{A}z_o - v \\ \bar{C}z_o = y - y_r.$$

Note that (35) implies that

$$2z_o^\top \bar{Q}\dot{z}_o \leq -2(\dot{z}_o + \alpha z_o)^\top \Lambda_{ST}v - \|y - y_r\|^2. \tag{36}$$

Now, multiply the LMI (11) by $\xi_c^\top$ from the left and by ξ_c from the right, to obtain

$$-2z_c^\top \bar{P}\left[\bar{A}z_c + 2\bar{B}u + v\right] + 4\|u\|^2 \\ + 2\left[z_c^\top(\beta I_n + \bar{A}^\top) + v^\top + 2\,u^\top \bar{B}^\top\right]\Lambda_{ST}^{-1}v \geq 0 \\ \iff -2z_c^\top\left(\bar{A}z_c + 2\bar{B}u + v\right) + 4\|u\|^2 \\ + 2(\dot{z}_c + \beta z_c)^\top \Lambda_{ST}^{-1}v \geq 0 \\ \iff 4\|u\|^2 + 2(\dot{z}_c + \beta z_c)^\top \Lambda_{ST}^{-1}v \geq 2z_c^\top \bar{P}\dot{z}_c \tag{37}$$

where we used that

$$\dot{z}_c = \bar{A}z_c + 2\bar{B}u + v.$$

The proof is completed by substituting (36) and (37) in (34) to obtain (33). ∎

In order to establish the error bound, we propose a particular selection of the signal $v(t)$ that allows us to compare the behavior of systems (26) and (27).

Lemma 1: Consider $\ell = 1$. Assume that systems (27) and (30) are initially at rest. Consider the partition $x_r = [x_{r_1}^\top \ x_{r_2}^\top]^\top$, with $x_{r_1} \in \mathbb{R}^{n-1}$ and $x_{r_2} \in \mathbb{R}$. Choose

$$v(t) = \begin{bmatrix} \mathbf{0} \\ A_v x_{r_2}(t) - \bar{A}_{21} x_{r_1}(t) - \bar{B}_2 u(t) \end{bmatrix} \qquad (38)$$

with $A_v \in \mathbb{R}$ such that $\bar{A}_{22} + A_v \neq 0$. Then, $\hat{y}(t) = y_r(t)$, and $x_{r_2}(t) = 0$ for every $t \geq 0$.

Proof: To establish the proof, replace (38) in (27) to obtain

$$\begin{aligned} \dot{x}_{r_1} &= \bar{A}_{11} x_{r_1} + \bar{A}_{12} x_{r_2} + \bar{B}_1 u \\ \dot{x}_{r_2} &= \left(\bar{A}_{22} + A_v\right) x_{r_2}. \end{aligned} \qquad (39)$$

Since $x_r(0) = \mathbf{0}$, from (39) we have the following chain of implications:

$$\begin{aligned} \dot{x}_{r_2} = 0 \ \forall \ t \geq 0 &\implies x_{r_2}(t) = 0 \ \forall \ t \geq 0 \\ &\implies \dot{x}_{r_1} = \bar{A}_{11} x_{r_1} + \bar{B}_1 u. \end{aligned} \qquad (40)$$
∎

Since $\hat{x}(0) = \mathbf{0}$, the last expression of (40) implies that $\hat{x}(t) = x_{r_1}(t)$ for all $t \geq 0$. Hence

$$y_r = \bar{C}_1 x_{r_1} = \hat{C}\hat{x} = \hat{y}.$$

Using the results of Proposition 1 and Lemma 1, the following Lemma establishes an error bound for the case $\ell = 1$, that is, when only one state is truncated.

Lemma 2: Consider the balanced system (26) with extended observability Gramian $(\bar{Q}, \Lambda_{ST}, \alpha)$, and inverse extended controllability Gramian $(\bar{P}, \Lambda_{ST}^{-1}, \beta)$, where $\alpha = \beta$ and $\ell = 1$. Assume that systems (50), (30), and (27) are initially at rest and select v as in (38). Then

$$\|\Sigma - \widehat{\Sigma}\|_\infty \leq 2\sigma_n.$$

Proof: Define

$$v_2 := A_v x_{r_2} - \bar{A}_{21} x_{r_1} - \bar{B}_2 u.$$

Hence, we can rewrite (38) as follows:

$$v = \begin{bmatrix} \mathbf{0} \\ v_2 \end{bmatrix}.$$

On the other hand, from Lemma 1, we have that

$$x_r = \begin{bmatrix} \hat{x} \\ 0 \end{bmatrix}, \ y_r = \hat{y}.$$

Therefore, since $\alpha = \beta$, we get

$$\begin{aligned} (\alpha z_o + \dot{z}_o)^\top \Lambda_{ST} v &= \sigma_n (\alpha \bar{x}_2 + \dot{\hat{x}}_2) v_2 \\ &= \sigma_n^2 \left(\beta z_c + \dot{z}_c\right)^\top \Lambda_{ST}^{-1} v. \end{aligned} \qquad (41)$$

Now, consider the storage function $\mathcal{S}(z_o, z_c)$, given in (32). Then, substituting (41) in (33), its derivative along the trajectories reduces to

$$\dot{\mathcal{S}} \leq 4\sigma_n^2 \|u\|^2 - \|y - \hat{y}\|^2 \qquad (42)$$

where we used (41). Moreover, integrating (42) from 0 to ∞, yields

$$0 \leq 4\sigma_n^2 \|u\|_2^2 - \|y - \hat{y}\|_2^2$$

which implies

$$\frac{\|y - \hat{y}\|_2}{\|u\|_2} \leq 2\sigma_n. \qquad (43)$$

We recall, see [1], that the $\mathcal{H}_\infty$-norm of the error system satisfies the following relationship:

$$\|\Sigma - \widehat{\Sigma}\|_\infty = \sup \frac{\|y - \hat{y}\|_2}{\|u\|_2} \qquad (44)$$

for $u \in \mathcal{L}_2$, $\|u\|_2 \neq 0$. Therefore, from (43) and (44), we get

$$\|\Sigma - \widehat{\Sigma}\|_\infty \leq 2\sigma_n$$

which completes the proof. ∎

Now, we are in the position to present the main result of this article in terms of the error bound for model reduction of CTLTI systems based on extended balanced truncation.

Theorem 3: Consider the balanced system (26) with extended observability Gramian $(\bar{Q}, \Lambda_{ST}, \alpha)$ and inverse extended controllability Gramian $(\bar{P}, \Lambda_{ST}^{-1}, \beta)$, where $\alpha = \beta$ and

$$\Lambda_{ST} = \mathtt{diag}\{\sigma_1, \ldots, \sigma_n\}.$$

Consider the truncated kth order system (30). Then, the error bound is given by the following inequality:

$$\|\Sigma - \widehat{\Sigma}\|_\infty \leq 2 \sum_{j=k+1}^{n} \sigma_j. \qquad (45)$$

Proof: To establish the proof, apply iteratively Lemma 2. ∎

Similar to the discrete-time results reported in [20] and [21], the error bound (45) is obtained by proposing a storage function and using dissipativity arguments, as in [25]. This procedure contrasts to the traditional analysis using transfer functions.

Note that the extended Gramians depend on the generalized ones and the parameters α and β, and the symmetric matrices Γ_o and Γ_c verifying (13) and (19), respectively. The following proposition establishes that for any given generalized Gramians an appropriate selection of Γ_o and Γ_c ensures that the error bound obtained via extended balanced truncation is smaller than that the one obtained via generalized balanced truncation.

Proposition 2: Given the *generalized* observability and controllability Gramians Q and $\breve{P}$, there exist matrices T and S, and constants α and β such that extended balanced truncation guarantees a smaller error bound for the reduced-order system than generalized balanced truncation.

Proof: The error bound associated with generalized balanced truncation is determined by the diagonal matrix Λ_{QP}, verifying (4). Similarly, the error bound associated with extended balanced truncation depends on the diagonal matrix Λ_{ST}, which satisfies (23). Hence, the resulting error bound from the extended balanced truncation approach is smaller if

$$\Lambda_{ST} < \Lambda_{QP}. \qquad (46)$$

To prove that the degrees of freedom in extended balancing can be chosen such that (46) holds, select S and T as in (14) and (20), respectively. Consider $\alpha = \beta$, and fix

$$\Gamma_o = \varepsilon_o Q, \quad \Gamma_c = -\varepsilon_c \breve{P} \tag{47}$$

with $0 < \varepsilon_o$ and $0 < \varepsilon_c < \alpha$. Therefore, from (14) and (20), we get that

$$T^{-1}S = \frac{\alpha - \varepsilon_c}{\alpha + \varepsilon_o} \breve{P} Q.$$

Accordingly, the linear transformation W_g such that (6) holds, also satisfies

$$W_g^{-1} T^{-1} S W_g = \frac{\alpha - \varepsilon_c}{\alpha + \varepsilon_o} \Lambda_{QP}^2 = \Lambda_{ST}^2. \tag{48}$$

Since $\frac{\alpha - \varepsilon_c}{\alpha + \varepsilon_o} < 1$, the inequality (46) is satisfied. ∎

Remark 4: If the matrices Γ_o and Γ_c are chosen as zero in (14), (20), and $\alpha = \beta$, then $S = \frac{1}{\alpha} Q$ and $T = \frac{1}{\alpha} P$. Hence, $Q\breve{P} = ST^{-1}$ and $\Lambda_{QP} = \Lambda_{ST}$. Accordingly, the error bound obtained via extended balancing coincides with the error bound obtained from the generalized balancing approach. Moreover, the reduced-order model obtained from both methods is the same.

IV. BALANCING OF CTLTI PH SYSTEMS

The PH framework has been proven suitable to capture physical phenomenon in different domains while preserving conservation laws [7], [24]. In this framework, it is possible to represent large-scale networks of complex physical systems and, at the same time, underscore the roles of the energy, the interconnection pattern, and the dissipation in the behavior of such systems. Moreover, the passivity property of these systems can be straightforwardly proven by selecting the Hamiltonian function as the storage function. Thus, given the possible physical interpretation of the PH models and their geometrical properties; this framework is appealing from both points of view: the theoretical and the practical one. Therefore, preserving the PH structure for the reduced-order model is interesting for analysis purposes, and might be useful to give an interpretation of the behavior of the reduced-order system. This section addresses the model reduction problem of CTLTI PH systems while preserving the PH structure for the reduced-order system. Furthermore, in some cases, more particular structures than the PH one are preserved, endowing the reduced-order model with a more specific physical interpretation.

A. CTLTI PH Systems

The representation of a CTLTI PH system is given by

$$\Sigma_H : \begin{cases} \dot{x} &= (J - R)Hx + Bu \\ y &= B^\top Hx \\ \mathcal{H}(x) &= \frac{1}{2} x^\top Hx \end{cases} \tag{49}$$

where $x \in \mathbb{R}^n$ is the state vector, $u, y \in \mathbb{R}^m$ are the input and output vectors, respectively, $\mathcal{H}(x)$ represents the Hamiltonian of the system with $H = H^\top > 0$, and $R = R^\top \geq 0$ and $J = -J^\top$ represent the dissipation and the interconnection matrix, respectively. In order to simplify notation, we define $F := J - R$.

The objective of this article is twofold; on the one hand, we aim to balance system (49) and obtain a lower order model. On the other hand, we want the reduced model to have a PH structure, because of the interpretation and the interconnection properties of this kind of systems. Towards this end, we assume that system (49) is asymptotically stable, and we look for an invertible linear transformation W that balances the system. Such transformation is given by $W = W_g$ in the generalized case, while in the extended case we have $W = W_e$. Then, we write the dynamics of the balanced system as follows:

$$\bar{\Sigma}_H : \begin{cases} \dot{\bar{x}} &= \bar{F} \bar{H} \bar{x} + \bar{B} u \\ \bar{y} &= \bar{B}^\top \bar{H} \bar{x} \end{cases}$$

where

$$\bar{F} := W^{-1} F W^{-\top}, \quad \bar{H} := W^\top H W, \quad \bar{B} := W^{-1} B.$$

Hence, if we split $\bar{x}$ as in (29), the balanced system can be expressed as

$$\bar{\Sigma}_H : \begin{cases} \begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \end{bmatrix} &= \begin{bmatrix} \bar{F}_{11} & \bar{F}_{12} \\ \bar{F}_{21} & \bar{F}_{22} \end{bmatrix} \begin{bmatrix} \bar{H}_{11} & \bar{H}_{12} \\ \bar{H}_{12}^\top & \bar{H}_{22} \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} + \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix} u \\ \bar{y} &= \begin{bmatrix} \bar{B}_1^\top & \bar{B}_2^\top \end{bmatrix} \begin{bmatrix} \bar{H}_{11} & \bar{H}_{12} \\ \bar{H}_{12}^\top & \bar{H}_{22} \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} \end{cases} \tag{50}$$

with

$$\begin{aligned} \bar{F}_{11}, \bar{H}_{11} &\in \mathbb{R}^{k \times k}, & \bar{F}_{22}, \bar{H}_{22} &\in \mathbb{R}^{\ell \times \ell}, & \bar{F}_{12}, \bar{H}_{12} &\in \mathbb{R}^{k \times \ell} \\ \bar{B}_1 &\in \mathbb{R}^{k \times m}, & \bar{F}_{21} &\in \mathbb{R}^{\ell \times k}, & \bar{B}_2 &\in \mathbb{R}^{\ell \times m}. \end{aligned}$$

Problem Formulation for PH Systems: Given the system (49), find an invertible linear transformation W, that performs the balancing of the system and at the same time satisfies

$$\bar{H}_{12} = \mathbf{0}. \tag{51}$$

Note that, if (51) holds, the truncation leads to the following reduced-order system:

$$\widehat{\Sigma}_H : \begin{cases} \dot{\hat{x}} &= \bar{F}_{11} \bar{H}_{11} \hat{x} + \bar{B}_1 u \\ \hat{y} &= \bar{B}_1^\top \bar{H}_{11} \hat{x} \\ \mathcal{H}(\hat{x}) &= \frac{1}{2} \hat{x}^\top \bar{H}_{11} \hat{x} \end{cases} \tag{52}$$

which is another CTLTI PH system, with $\hat{x} = \bar{x}_1$. Therefore, it follows that one solution to the problem of model reduction with PH structure preservation takes place when the Hamiltonian matrix of the balanced system, $\bar{H}$, is diagonal. In such case, our problem is reduced to the simultaneous diagonalization of three matrices, namely, (Q, P, H) or (S, T, H).

Remark 5: The complete diagonalization of H is not necessary. In fact, a block diagonalization that ensures (51) is enough to preserve the PH structure. Nevertheless, if H is not a diagonal matrix, then it is necessary to know the dimension of the part of the state to be truncated.

The subsequent sections of this article are devoted to the identification of a transformation W that balances the system and ensures that (51) is satisfied.

B. Generalized Balancing of CTLTI PH Systems

In this section, we study the generalized balancing method for CTLTI PH systems, which is the starting point of extended

balancing of CTLTI PH studied in Section IV-C. Further, we provide sufficient conditions to ensure the existence of a transformation W_g that complies with the requirements established in Section IV-A. To this end, we revisit the following theorem, which establishes necessary and sufficient conditions for the existence of a transformation that diagonalizes simultaneously three matrices when at least one of them has definite sign.

Theorem 4 ([18]): Let $L, M,$ and N be symmetric matrices. In the case of at least one fixed-sign quadratic form (e.g., M positive definite), the condition

$$LM^{-1}N = NM^{-1}L \qquad (53)$$

is necessary and sufficient for the existence of a linear invertible congruent transformation W that diagonalizes simultaneously $L, M,$ and N.

For the proof and further details about Theorem 4, we refer the reader to [18] and [3]. For a thorough exposition on simultaneously diagonalizable matrices, we refer the reader to [13], Chapter 4. In generalized balancing of CTLTI PH systems, the condition (53) takes the form

$$H\breve{P}^{-1}Q = Q\breve{P}^{-1}H. \qquad (54)$$

Accordingly, we look for Q and $\breve{P}$ verifying (2) and (3), respectively, such that (54) holds. A trivial solution to this problem takes place when Q or $\breve{P}$ coincides with the (scaled) Hamiltonian matrix H or its inverse. This idea was studied in [9] and [14], among other works; and for the sake of completeness, the following proposition identifies a class of CTLTI PH systems, for which the (scaled) Hamiltonian matrix, or its inverse, solves the inequalities (2) and (3).

Proposition 3: Consider $\delta \in \mathbb{R}_{>0}$. Assume that the system (49) is asymptotically stable. If the following condition holds:

$$2\delta R - BB^{\top} \geq 0. \qquad (55)$$

Then, $Q = \delta H$ solves (2), and $\breve{P} = \delta H^{-1}$ is a solution to (3).

Proof: To establish the proof, note that for CTLTI PH systems, (2) and (3) take the forms

$$QFH + HF^{\top}Q + HBB^{\top}H \leq 0 \qquad (56)$$

$$FH\breve{P} + \breve{P}HF^{\top} + BB^{\top} \leq 0 \qquad (57)$$

respectively. Hence, substituting $Q = \delta H$ in (56), we obtain

$$\begin{aligned} 0 &\geq \delta HFH + \delta HF^{\top}H + HBB^{\top}H \\ &= H(BB^{\top} - 2\delta R)H \\ \Longleftrightarrow 0 &\leq 2\delta R - BB^{\top}. \end{aligned}$$

On the other hand, replacing $\breve{P} = \delta H^{-1}$ in (57), we have

$$\begin{aligned} 0 &\geq \delta F + \delta F^{\top} + BB^{\top} \\ &= -2\delta R + BB^{\top} \\ \Longleftrightarrow 0 &\leq 2\delta R - BB^{\top}. \quad \blacksquare \end{aligned}$$

Condition (55) is satisfied by systems that have dissipation in all the input channels, e.g., fully damped mechanical systems. Nonetheless, R and B are system parameters, thus, it might happen that condition (55) is not satisfied by the system (49). In order to overcome this issue, we state following two propositions

to identify generalized Gramians, such that the triplet $(Q, \breve{P}, H)$ verifies (54) and solves the Lyapunov inequalities (56) and (57). These propositions represent the main result of this article in terms of generalized balancing with PH structure preservation.

Proposition 4: Let $\breve{P}$ be a solution to (57). Consider a full rank matrix $\phi_P \in \mathbb{R}^{n \times n}$ verifying the following:

$$\begin{aligned} \breve{P} &= \phi_P^{\top}\phi_P \\ \phi_P H \phi_P^{\top} &= U_{HP}\Lambda_{HP}U_{HP}^{\top} \end{aligned}$$

where U_{HP} is an orthogonal matrix and Λ_{HP} is a diagonal matrix, whose entries are the singular values of $\phi_P H \phi_P^{\top}$, see the notation at the end of Section I. Define the matrices

$$\begin{aligned} \mathcal{F}_c &:= U_{HP}^{\top}\phi_P^{-\top}F\phi_P^{-1}U_{HP} \\ \mathcal{B}_c &:= U_{HP}^{\top}\phi_P^{-\top}B. \end{aligned} \qquad (58)$$

Assume that

$$-\Lambda_{QP}^2\Lambda_{HP}^{-1}\mathcal{F}_c - \mathcal{F}_c^{\top}\Lambda_{HP}^{-1}\Lambda_{QP}^2 - \mathcal{B}_c\mathcal{B}_c^{\top} \geq 0 \qquad (59)$$

holds for a diagonal matrix Λ_{QP}. Hence, (56) is solved by

$$Q = \phi_P^{-1}U_{HP}\Lambda_{QP}^2U_{HP}^{\top}\phi_P^{-\top}. \qquad (60)$$

Moreover, the transformation

$$W_{gc} = \phi_P^{\top}U_{HP}\Lambda_{QP}^{-\frac{1}{2}} \qquad (61)$$

balances the system and diagonalizes H.

Proof: To establish the proof, we define

$$\mathcal{X}_o := -\Lambda_{QP}^2\Lambda_{HP}^{-1}\mathcal{F}_c - \mathcal{F}_c^{\top}\Lambda_{HP}^{-1}\Lambda_{QP}^2 - \mathcal{B}_c\mathcal{B}_c^{\top}.$$

Note that, if (59) holds, we have the following chain of implications:

$$\begin{aligned} \mathcal{X}_o &\geq 0 \\ \Longleftrightarrow \phi_P^{-1}U_{HP}\Lambda_{HP}\mathcal{X}_o\Lambda_{HP}U_{HP}^{\top}\phi_P^{-\top} &\geq 0 \\ \Longleftrightarrow -QFH - HF^{\top}Q - HBB^{\top}H &\geq 0 \\ \Longleftrightarrow QFH + HF^{\top}Q + HBB^{\top}H &\leq 0 \end{aligned}$$

where we used (58) and (60). Moreover

$$\begin{aligned} W_{gc}^{\top}QW_{gc} &= \Lambda_{QP} \\ W_{gc}^{-1}\breve{P}W_{gc}^{-\top} &= \Lambda_{QP} \\ W_{gc}^{\top}HW_{gc} &= \Lambda_{QP}^{-1}\Lambda_{HP}. \end{aligned}$$

This completes the proof. $\blacksquare$

The following proposition is the dual version of Proposition 4 and relaxes condition (55), in this case, for a given generalized observability Gramian Q.

Proposition 5: Let Q be a solution to (56). Consider a full rank matrix $\phi_Q \in \mathbb{R}^{n \times n}$ verifying the following:

$$\begin{aligned} Q &= \phi_Q^{\top}\phi_Q \\ \phi_Q^{-\top}H\phi_Q^{-1} &= U_{HQ}\Lambda_{HQ}U_{HQ}^{\top}. \end{aligned}$$

Define the matrices

$$\begin{aligned} \mathcal{F}_o &:= U_{HQ}^{\top}\phi_Q F\phi_Q^{\top}U_{HQ} \\ \mathcal{B}_o &:= U_{HQ}^{\top}\phi_Q B. \end{aligned} \qquad (62)$$

Assume that

$$-\mathcal{F}_o\Lambda_{HQ}\Lambda_{QP}^2 - \Lambda_{QP}^2\Lambda_{HQ}\mathcal{F}_o^{\top} - \mathcal{B}_o\mathcal{B}_o^{\top} \geq 0 \qquad (63)$$

holds for a diagonal matrix Λ_{QP}. Hence, (57) is solved by

$$\breve{P} = \phi_Q^{-1} U_{HQ} \Lambda_{QP}^2 U_{HQ}^\top \phi_Q^{-\top}. \tag{64}$$

Moreover, the transformation

$$W_{go} = \phi_Q^{-1} U_{HQ} \Lambda_{QP}^{\frac{1}{2}} \tag{65}$$

balances the system and diagonalizes H.

Proof: Define

$$\mathcal{X}_c := -\mathcal{F}_o \Lambda_{HQ} \Lambda_{QP}^2 - \Lambda_{QP}^2 \Lambda_{HQ} \mathcal{F}_o^\top - \mathcal{B}_o \mathcal{B}_o^\top.$$

Therefore, if (63) is satisfied, we have

$$\begin{aligned}
\mathcal{X}_c &\geq 0 \\
\Longleftrightarrow \phi_Q^{-1} U_{HQ} \mathcal{X}_c U_{HQ}^\top \phi_Q^{-\top} &\geq 0 \\
\Longleftrightarrow -FH\breve{P} - \breve{P}HF^\top - BB^\top &\geq 0 \\
\Longleftrightarrow FH\breve{P} + \breve{P}HF^\top + BB^\top &\leq 0
\end{aligned}$$

where we used (62) and (64). To complete the proof, note that

$$\begin{aligned}
W_{go}^\top Q W_{go} &= \Lambda_{QP} \\
W_{go}^{-1} \breve{P} W_{go}^{-\top} &= \Lambda_{QP} \\
W_{go}^\top H W_{go} &= \Lambda_{HQ} \Lambda_{QP}.
\end{aligned}$$

∎

In Propositions 4 and 5, the condition (55) is relaxed by imposing a particular structure to the generalized observability and controllability Gramians, respectively. Such structure depends on the Hamiltonian matrix, however, it is less restrictive than (55). Indeed, if this latter condition is satisfied, then (59) and (63) hold. Using the results presented in this section, we studied following extended balancing of CTLTI PH systems. As was mentioned in Section III, the use of extended Gramians can be advantageous for different purposes, for instance, to obtain a lower error bound or to impose a more particular structure to the reduced-order model.

C. Extended Balancing of CTLTI PH Systems

Similar to the generalized balancing case, in this section, we provide sufficient conditions for the existence of a linear transformation W_e that balances the system and diagonalizes the Hamiltonian matrix. Towards this end, we introduce following two propositions that provide a suitable transformation W_e. Such propositions constitute the main result of this article, regarding extended balancing with PH structure preservation.

Proposition 6: Let $\breve{P}$ be a solution to (57) such that $X_c > 0$. Select β and Γ_c such that (19) holds and T, defined in (20), solves the LMI (11). Consider a full rank matrix $\phi_T \in \mathbb{R}^{n \times n}$ verifying the following:

$$\begin{aligned}
T^{-1} &= \phi_T^\top \phi_T \\
\phi_T H \phi_T^\top &= U_{HT} \Lambda_{HT} U_{HT}^\top.
\end{aligned}$$

Define the matrices

$$\begin{aligned}
\mathcal{F}_{ec} &:= U_{HT}^\top \phi_T^{-\top} F \phi_T^{-1} U_{HT} \\
\mathcal{B}_{ec} &:= U_{HT}^\top \phi_T^{-\top} B.
\end{aligned} \tag{66}$$

Assume that

$$-\Lambda_{QT}^2 \Lambda_{HT}^{-1} \mathcal{F}_{ec} - \mathcal{F}_{ec}^\top \Lambda_{HT}^{-1} \Lambda_{QT}^2 - \mathcal{B}_{ec} \mathcal{B}_{ec}^\top > 0 \tag{67}$$

holds for a diagonal matrix Λ_{QT}. Then, (56) is solved by

$$Q = \phi_T^{-1} U_{HT} \Lambda_{QT}^2 U_{HT}^\top \phi_T^{-\top}. \tag{68}$$

Select α such that the matrix

$$S = \frac{1}{\alpha + \varepsilon_o} Q \tag{69}$$

with $\varepsilon_o \geq 0$, solves the LMI (10). Then, the invertible transformation

$$W_{ec} = \sqrt[4]{\alpha + \varepsilon_o} \phi_T^\top U_{HT} \Lambda_{QT}^{-\frac{1}{2}} \tag{70}$$

balances the system and diagonalizes H.

Proof: Define

$$\mathcal{X}_{eo} := -\Lambda_{QT}^2 \Lambda_{HT}^{-1} \mathcal{F}_{ec} - \mathcal{F}_{ec}^\top \Lambda_{HT}^{-1} \Lambda_{QT}^2 - \mathcal{B}_{ec} \mathcal{B}_{ec}^\top.$$

Then, the inequality (67) is satisfied if and only if

$$\begin{aligned}
\mathcal{X}_{eo} &> 0 \\
\phi_T^{-1} U_{HT} \Lambda_{HT} \mathcal{X}_{eo} \Lambda_{HT} U_{HT}^\top \phi_T^{-\top} &> 0 \\
\Longleftrightarrow X_o &> 0 \\
\Longleftrightarrow QFH + HF^\top Q + HBB^\top H &< 0 \\
\Longleftrightarrow QFH + HF^\top Q + HBB^\top H &\leq 0
\end{aligned}$$

where we used

$$A = FH. \tag{71}$$

Select Γ_o as in (47), with $0 \leq \varepsilon_o$. Hence, for α large enough, the selection of S given in (69) solves the LMI (10).

To establish the last part of the proof, define

$$\Lambda_{ST} := \frac{1}{\sqrt{\alpha + \varepsilon_o}} \Lambda_{QT}.$$

Note that

$$W_{ec}^{-1} T^{-1} S W_{ec} = \Lambda_{ST}^2$$

$$W_{ec}^\top H W_{ec} = \Lambda_{HT} \Lambda_{ST}^{-1}.$$

∎

The following proposition is the dual version of Proposition 6.

Proposition 7: Let Q be a solution to (56) such that $X_o > 0$. Select α and Γ_o, such that (13) holds and S, defined in (14), solves the LMI (10). Consider a full rank matrix $\phi_S \in \mathbb{R}^{n \times n}$ verifying the following:

$$\begin{aligned}
S &= \phi_S^\top \phi_S \\
\phi_S^{-\top} H \phi_S^{-1} &= U_{HS} \Lambda_{HS} U_{HS}^\top.
\end{aligned}$$

Define the matrices

$$\begin{aligned}
\mathcal{F}_{eo} &:= U_{HS}^\top \phi_S F \phi_S^\top U_{HS} \\
\mathcal{B}_{eo} &:= U_{HS}^\top \phi_S B.
\end{aligned} \tag{72}$$

Assume

$$-\mathcal{F}_{eo} \Lambda_{HS} \Lambda_{SP}^2 - \Lambda_{SP}^2 \Lambda_{HS} \mathcal{F}_{eo}^\top - \mathcal{B}_{eo} \mathcal{B}_{eo}^\top > 0 \tag{73}$$

holds for a diagonal matrix Λ_{SP}. Thus, (57) is solved by

$$\breve{P} = \phi_S^{-1} U_{HS} \Lambda_{SP}^2 U_{HS}^\top \phi_S^{-\top}. \tag{74}$$

Select β such that the matrix

$$T^{-1} = (\beta - \varepsilon_c) \breve{P} \tag{75}$$

with $0 \leq \varepsilon_c < \beta$, solves the LMI (11). Then,

$$W_{eo} = \sqrt[4]{\beta - \varepsilon_c}\phi_S^{-1}U_{HS}\Lambda_{SP}^{\frac{1}{2}} \tag{76}$$

balances the system and diagonalizes H.

Proof: Define

$$\mathcal{X}_{eo} := -\mathcal{F}_{eo}\Lambda_{HS}^2\Lambda_{SP}^2 - \Lambda_{SP}^2\Lambda_{HS}\mathcal{F}_{eo}^\top - \mathcal{B}_{eo}\mathcal{B}_{eo}^\top.$$

Hence, if (73) holds, we have the following chain of implications:

$$\mathcal{X}_{eo} > 0$$
$$\iff \phi_S^{-1}U_{HS}\mathcal{X}_{co}U_{HS}^\top\phi_S^{-\top} > 0$$
$$\iff -FH\breve{P} - \breve{P}HF^\top - BB^\top > 0.$$

Moreover,

$$-FH\breve{P} - \breve{P}HF^\top - BB^\top > 0$$
$$\implies \begin{cases} FH\breve{P} + \breve{P}HF^\top + BB^\top \leq 0 \\ X_c > 0 \end{cases}$$

where we used (71). Select Γ_c, as in (47), with $0 \leq \varepsilon_c < \alpha$. Accordingly, for β large enough, the selection of T given in (75) solves the LMI (11).

To establish the last part of the proof, define

$$\Lambda_{ST} := \sqrt{\beta - \varepsilon_c}\Lambda_{SP}.$$

Note that

$$W_{eo}^{-1}T^{-1}SW_{eo} = \Lambda_{ST}^2$$

$$W_{eo}^\top HW_{eo} = \Lambda_{HS}\Lambda_{ST}. \qquad \blacksquare$$

We remark that Γ_o and Γ_c are degrees of freedom in the selection of S and T, respectively. These matrices can be selected in order to improve the error bound or preserve more particular structures as is given in Section V.

The following algorithm summarizes the extended balancing method for PH systems with structure preservation.

S1) Find a positive definite matrix $\breve{P}$ such that

$$FH\breve{P} + \breve{P}HF^\top + BB^\top < 0.$$

S2) Consider, $A = FH$. Propose a symmetric matrix Γ_c and a constant $\beta > 0$ such that T, defined as in (20), solves the LMI (11).

S3) If the inequality (55) holds, select $Q = \delta_o H$ and $S = \frac{1}{\alpha+\varepsilon_o}Q$, with $\delta_o, \varepsilon_o > 0$. Otherwise, use the result of Proposition 6.

S4) Check if the proposed S solves the LMI (10), with $\alpha = \beta$. If not, return to S2 and propose a larger β.

S5) Find the linear transformation W_e such that (25) holds.

Note that a similar algorithm starting for the proposition of Q can be straightforwardly obtained.

V. EXAMPLES

In this section, we present two examples to illustrate the applicability of the results reported in previous sections. Both examples represent physical systems. The first one is a larger scale mass–spring–damper mechanical system, where we preserve the PH structure. While, the second example represents a smaller scale RLC circuit network, where we illustrate how to preserve the RLC structure in addition to the PH one.

A. Mechanical System

Consider the mechanical system shown in Fig. 1, which consists of 200 masses, 198 linear dampers, and 200 linear springs. This system can be represented in the PH framework as follows:

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{0} & I_{200} \\ -I_{200} & -R_2 \end{bmatrix}}_{F}\underbrace{\begin{bmatrix} K & \mathbf{0} \\ \mathbf{0} & M^{-1} \end{bmatrix}}_{H}\begin{bmatrix} q \\ p \end{bmatrix} + \underbrace{\begin{bmatrix} \mathbf{0} \\ G \end{bmatrix}}_{B}u, \tag{77}$$

where $q, p \in \mathbb{R}^{200}$, $M \in \mathbb{R}^{200 \times 200}$ is a positive definite diagonal matrix, $K \in \mathbb{R}^{200 \times 200}$ is positive definite, $R_2 \in \mathbb{R}^{200 \times 200}$ is a positive semidefinite matrix that contains the information of the dampers, and $G = [1 \; \mathbf{0}]^\top$.

The objective is to reduce the order of the model and ensure that the PH structure is preserved. Note that, independently of $\delta > 0$, this system does not satisfy the condition (55). Thus, the Hamiltonian matrix cannot be proposed as a generalized Gramian.

To illustrate the results of Section IV, we obtain PH reduced-order models via generalized balanced truncation and extended balanced truncation. Then, we compare the results obtained from both techniques. For the generalized balancing case, we proceed as follows.

1) We consider a positive definite matrix $\breve{X}_c$. Then, using MATLAB, we solve the Lyapunov equation

$$FH\breve{P} + \breve{P}HF^\top + BB^\top + \breve{X}_c = 0$$

to obtain the generalized controllability Gramian $\breve{P}$.

2) We solve the LMI (59). Then, we propose Q and W_{gc} as in Proposition 4.

3) We use W_{gc}, defined in (61), to balance the system.

For the extended balancing, we follow the algorithm provided at the end of Section IV. To this end, we consider the same $\breve{P}$ as in the generalized balancing method. To address S2, we propose $\beta > 0$ such that the LMI (11) is satisfied by $T = \frac{1}{\beta}P$. Then, we consider $\Gamma_c = -\varepsilon_c\breve{P}$, with $0 < \varepsilon_c < \beta$, such that the LMI (11) is satisfied by

$$T = \frac{1}{\beta - \varepsilon_c}P.$$

For S3, we look for a diagonal solution to (67), and we propose Q as in (68). Then, we select S as in (69). The rest of the algorithm is straightforward, where we use W_{ec}, defined in (70), to balance the system.

For illustration purposes, we consider that the masses vary between 0.4 and $0.6[kg]$, spring constants between 0.9 and $1.1[kg/s^2]$, and damping coefficients between 1.8 and $2.2[kg/s]$. We select a matrix $\breve{X}_c$ that guarantees a small trace of $\breve{P}$ without causing numerical errors for the LMI solver of MATLAB. Here, we omit the matrices involved in the balancing processes due to their large dimension. The data of this example
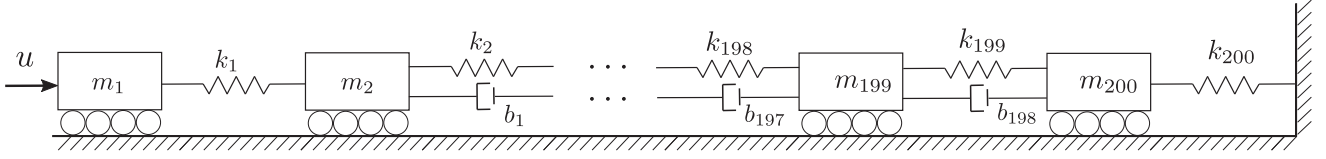
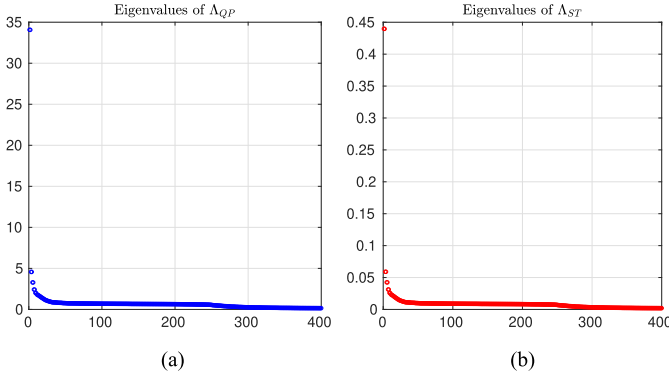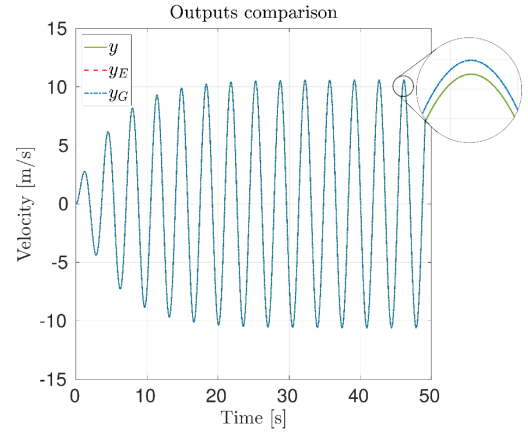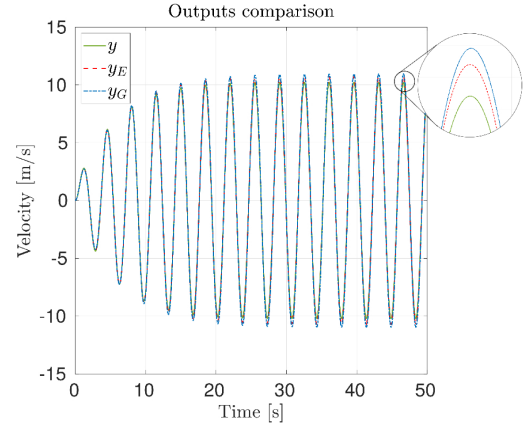Fig. 1. Mass–spring–damper network.

(a)

(b)

Fig. 2. Singular values resulting from generalized balancing and extended balancing. (a) Generalized balancing. (b) Extended balancing.

can be found in [2]. Fig. 2 shows the singular values obtained from both balancing techniques, where it is evident that the singular values corresponding to extended balancing are considerably smaller for all $n = 1, \ldots, 400$. Thus, a smaller error bound is expected for extended balanced truncation. Moreover, the last 100 singular values are much smaller than the first 300 ones in both cases. Consequently, we anticipate that a reduced-order system of dimension 300 can offer an appropriate approximation of the original system.

To compare the performance of both model reduction techniques, we present reduced-order models of dimensions 300, 200, and 100, obtained through each balanced truncation approach. To this end, we perform simulations using MATLAB, under initial conditions zero and inputs of the form $u = 2\sin(\omega t)$. The frequency ω is chosen as the peak frequency of the error system obtained via generalized balanced truncation. To present the results, we adopt the following notation: EB stands for extended balancing and GB stands for generalized balancing. The dimension of the reduced-order system is denoted by k. The output of the original system is represented by y, the output of the reduced-order system obtained via generalized balanced truncation is denoted by y_G, and the output of the reduced-order system obtained via extended balanced truncation is given by y_E.

The results of the simulations are shown in Figs. 3–6 and given in Table I. While the error bound obtained via extended balancing is considerably smaller in all the cases, for $k = 300$, the performance of both methodologies is similar. This can be observed in Fig. 3, where the behavior of the reduced-order system outputs is very similar to the behavior of the original system output. Moreover, the $\mathcal{H}_\infty$-norm of both error systems coincides as given in Table I. However, we observe in Figs. 4–6

Fig. 3. Outputs comparison for $k = 300$.

Fig. 4. Outputs comparison for $k = 200$.

that the extended balanced truncation approach approximates better the original system as the number of truncated states increases.

B. RLC Circuit

Consider the RLC network shown in Fig. 7, which admits a PH representation of the form (49) with

$$J = \begin{bmatrix} \mathbf{0} & J_1 \\ -J_1^\top & \mathbf{0} \end{bmatrix}, \quad R = \begin{bmatrix} R_C^{-1} & \mathbf{0} \\ \mathbf{0} & R_L \end{bmatrix}$$

$$H = \operatorname{diag}\left\{ \frac{1}{C_1}, \frac{1}{C_2}, \frac{1}{C_3}, \frac{1}{C_4}, \frac{1}{C_5}, \frac{1}{L_1}, \frac{1}{L_2}, \frac{1}{L_3}, \frac{1}{L_4}, \frac{1}{L_5} \right\}$$
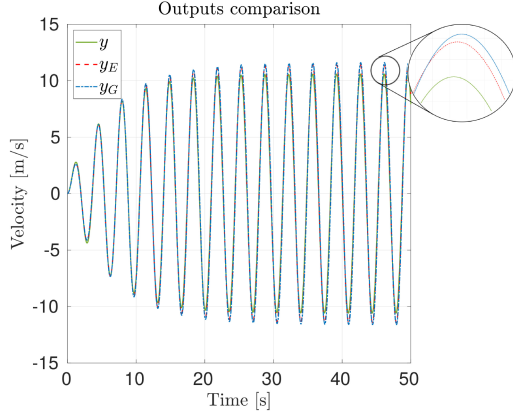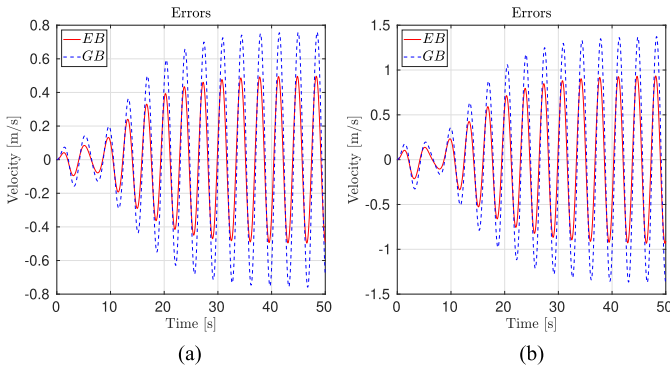
Fig. 5.   Outputs comparison for $k = 100$.

Fig. 6.   Comparison of the errors obtained from both techniques. (a) $k = 200$. (b) $k = 100$.

TABLE I
$\mathcal{H}_\infty$-NORM OF THE ERROR SYSTEM AND SUM OF THE SINGULAR VALUES

k	ω	GB $\mathcal{H}_\infty$	GB $2 \sum_{j=k+1}^{n} \sigma_j$	EB $\mathcal{H}_\infty$	EB $2 \sum_{j=k+1}^{n} \sigma_j$
300	1.8078	0.0120	38.3047	0.0120	0.4943
200	1.7864	0.3801	140.0848	0.2492	1.7635
100	1.8081	0.6898	276.8439	0.4701	3.4875

$$R_C = \mathtt{diag}\{R_{C_1}, R_{C_2}, R_{C_3}, R_{C_4}, R_{C_5}\}$$

$$R_L = \mathtt{diag}\{R_{L_1}, R_{L_2}, R_{L_3}, R_{L_4}, R_{L_5}\}$$

$$J_1 = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} \mathbf{0}_5 \\ 1 \\ \mathbf{0}_4 \end{bmatrix}$$

where x_i are the charges in the capacitors and x_{5+i} denote the fluxes in the inductors, for $i = 1, \ldots, 5$.

The objective is to reduce the order of the model and obtain a PH system that can be interpreted as an RLC circuit. Hence, we require that the reduced PH system has a diagonal damping matrix, and the interconnection matrix must be skew-symmetric and block antidiagonal. Note that this is a more particular structure

TABLE II
PARAMETERS OF THE RLC NETWORK

R_{C_1}	$270[\Omega]$	C_1	$2.2[mF]$
R_{C_2}	$1[k\Omega]$	C_2	$1[mF]$
R_{C_3}	$330[\Omega]$	C_3	$3.3[mF]$
R_{C_4}	$1.5[k\Omega]$	C_4	$15[\mu F]$
R_{C_5}	$220[\Omega]$	C_5	$4.7[\mu F]$
R_{L_1}	$4.7[\Omega]$	L_1	$10[mH]$
R_{L_2}	$3.9[\Omega]$	L_2	$4.3[mH]$
R_{L_3}	$2.2[\Omega]$	L_3	$2.7[mH]$
R_{L_4}	$2.74[\Omega]$	L_4	$6.2[\mu H]$
R_{L_5}	$3.92[\Omega]$	L_5	$3[\mu H]$

than the PH structure given in (49). We stress that the matrices J, R, and H can be decomposed into block matrices whose dimension depends on the number of inductors and capacitors, i.e., 5. Moreover, H is already diagonal. Thus, a block diagonal transformation W ensures that $\bar{H}$ remains diagonal, and the block structure that determines the RLC architecture of the system is not affected. Note that the damping matrix R has full rank. Hence, we can select

$$Q = \delta_o H, \quad \check{P} = \delta_c H^{-1}$$

where δ_o and δ_c are positive constants such that (55) holds. Therefore, both generalized Gramians are diagonal, and the resulting transformation W_g does not modify the structure of the original system. Nevertheless, the Hankel singular values are given by $\Lambda_{QP} = \sqrt{\delta_o \delta_c} I_n$. Since all the entries of Λ_{QP} are equal, the criterion of truncating the states related to the smallest singular values is impractical, and further information is required to decide which states can be removed. To deal with this situation, we adopt the extended balancing approach. In particular, we want to have a significant contrast among the entries of Λ_{ST}, which provides information about which states can be truncated without affecting the response of the reduced-order system significantly. To this end, we follow the algorithm provided at the end of Section IV with a minor modification, i.e., since the generalized Gramians are diagonal, Γ_c can be chosen as a diagonal matrix with nonpositive entries and Γ_o can be selected as a diagonal matrix with nonnegative entries. This selection improves the error bound and provides the desired contrast among the singular values. Moreover, the matrices H, T, and S are diagonal. As a result, W_e is a block diagonal matrix. Thus, we can express the matrices W_e and Λ_{ST} as follows:

$$W_e = \mathtt{block}\{W_1, W_2\}$$
$$\Lambda_{ST} = \mathtt{block}\{\Lambda_{ST_1}, \Lambda_{ST_2}\}$$
$$\Lambda_{ST_i} = \mathtt{diag}\{\sigma_{i_1}, \ldots, \sigma_{i_5}\}, \quad i = 1, 2.$$

At this point, we make following three observations regarding the preservation of the RLC structure.

1) To preserve the RLC structure, it is necessary to ensure that W is a block diagonal matrix.
2) We truncate the states related to the entries of Λ_{ST} in pairs, that is, one state related to one element of Λ_{ST_1} and one state related to one entry from Λ_{ST_2}. A physical interpretation of this approach is that we are removing the same number of inductors and capacitors.
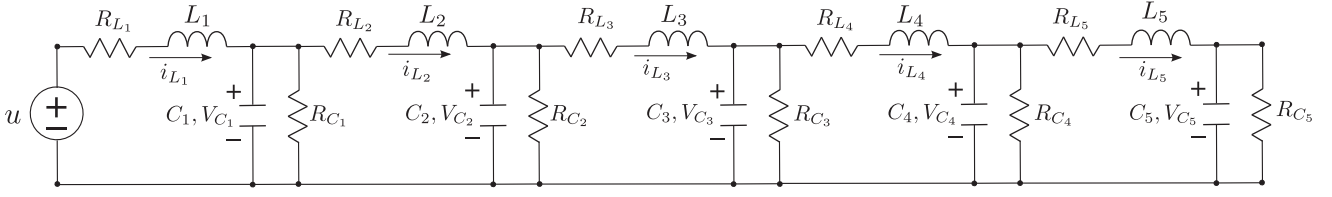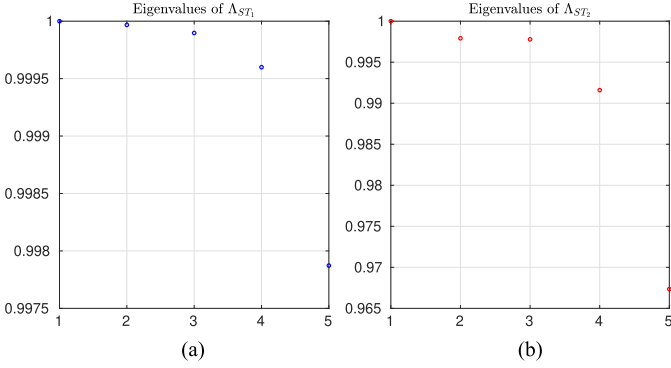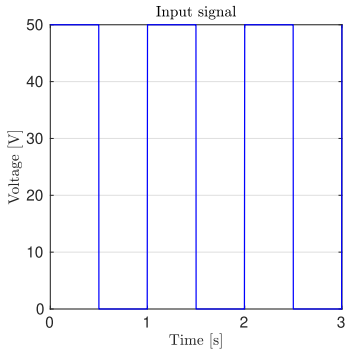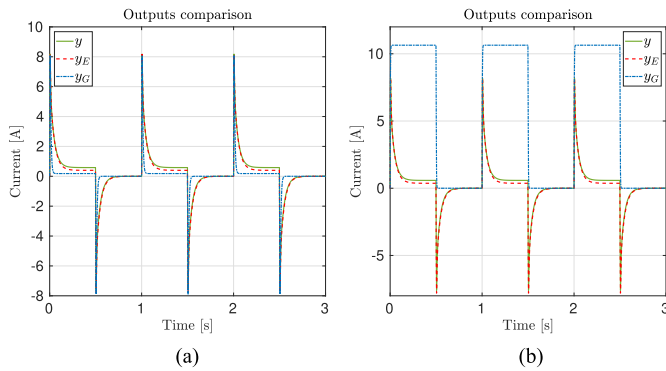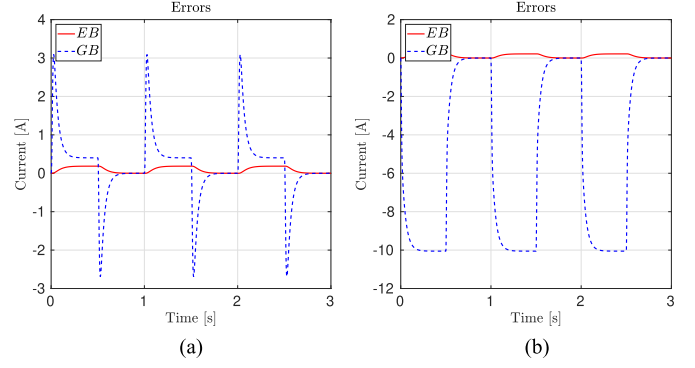
Fig. 7. RLC network.

Fig. 8. Normalized singular values resulting from extended balancing. (a) Normalized singular values related to the capacitors. (b) Normalized singular values related to the inductors.

Fig. 11. Errors from different reduced-order models. (a) $k = 8$. (b) $k = 6$.

Fig. 9. Input signal u.

Fig. 10. Outputs comparison for different reduced-order models. (a) $k = 8$. (b) $k = 6$.

3) By fixing Γ_o and Γ_c different from zero, we ensure that the entries of Λ_{ST} are different. Then, we can truncate the states related to the smallest entries of each submatrix Λ_{ST_i}.

To illustrate the methodology, we consider the values in Table II. Due to space constraints, we omit the matrices involved in the extended balancing procedure. The corresponding data can be found in [2]. Fig. 8 shows the normalized singular values of the balanced system, where we observe that the last two singular values of each block Λ_{ST_i} are smaller than the first three. Consequently, we expect that, in the extended balancing approach, the reduced-order models obtained by truncating σ_{i_5}, or σ_{i_4} and σ_{i_5}, approximate the original system properly. We carry out simulations to compare the behavior of the original system with the reduced-order systems of dimensions $k = 8$ and $k = 6$ obtained via both balancing methodologies. To this end, we consider the input shown in Fig. 9 and initial conditions equal to zero. Figs. 10 and 11 show the comparison of the outputs and the errors, respectively, where EB stands for extended balancing, GB stands for generalized balancing, the output of the original system is represented by y, the output of the reduced-order system obtained via generalized balanced truncation is denoted by y_G, and the output of the reduced-order system obtained via extended balanced truncation is given by y_E. Using MATLAB, we compute the $\mathcal{H}_\infty$-norm for each case and methodology. The corresponding values are given in Table III, where it is evident that the extended balanced truncation approach exhibits a better performance.

To illustrate that the RLC structure is preserved, we consider the reduced-order model of dimension $k = 6$, i.e., we truncate
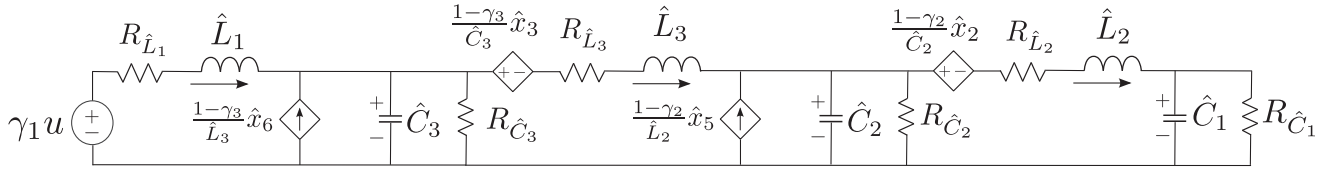
Fig. 12. Reduced RLC network.

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TABLE III
$\mathcal{H}_\infty$-NORM OF THE ERROR SYSTEM

k	GB	EB
8	0.0747	0.0037
6	0.2011	0.0043

the states related to σ_{i_4} and σ_{i_5}. Hence, the reduced-order model admits a PH representation with

$$R_r = \texttt{block}\{R_{C_r}^{-1}, R_{L_r}\}, \quad J_r = \begin{bmatrix} \mathbf{0} & J_{1_r} \\ -J_{1_r}^\top & \mathbf{0} \end{bmatrix}$$

$$H_r = \texttt{diag}\left\{ \frac{1}{C_{1_r}}, \frac{1}{C_{2_r}}, \frac{1}{C_{3_r}}, \frac{1}{L_{1_r}}, \frac{1}{L_{2_r}}, \frac{1}{L_{3_r}} \right\}$$

$$R_{C_r} := \texttt{diag}\left\{ R_{C_{1_r}}, R_{C_{2_r}}, R_{C_{3_r}} \right\}$$

$$R_{L_r} := \texttt{diag}\left\{ R_{L_{1_r}}, R_{L_{2_r}}, R_{L_{3_r}} \right\}$$

$$J_{1_r} = \begin{bmatrix} 1 & -\gamma_2 & 0 \\ 0 & 1 & -\gamma_3 \\ 0 & 0 & 1 \end{bmatrix}, \quad B_r = \begin{bmatrix} \mathbf{0}_3 \\ \gamma_1 \\ \mathbf{0}_2 \end{bmatrix}.$$

Moreover, the reduced-order model admits the RLC realization shown in Fig. 12, where the states $\hat{x}_i$ represent the charges in the capacitors and $\hat{x}_{i+3}$ denote the fluxes in the inductors, for $i = 1, 2, 3$.

VI. CONCLUSION

In this article, we have provided sufficient conditions to preserve the PH structure for reduced-order models obtained via generalized and extended balanced truncation of CTLTI PH systems. Moreover, we have shown how to exploit the degrees of freedom in extended balancing to obtain a lower error bound than the one obtained via generalized balancing. Additionally, we have illustrated with an example that more particular structures, such as physical ones, can be preserved via extended balanced truncation.

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