Assignment 4

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1. Transform following problems into linear programming

(1)

$$\begin{aligned} \text{maximize} \quad & 5x + 2y \\ \text{subject to} \quad & 0 \leq x \leq 20 \\ & |x - y| \leq 5 \end{aligned}$$

(2)

$$\begin{aligned} \text{maximize} \quad & min(x1, x2, x3) \\ \text{subject to} \quad & x1 + x2 + x3 = 15 \end{aligned}$$

Solution:

(1)

$$\begin{aligned} \text{maximize} \quad & 5x + 2y \\ \text{subject to} \quad & x \leq 20 \\ & x \geq 0 \\ & x - y \leq 5 \\ & x - y \geq -5 \end{aligned}$$

(2)

$$\begin{aligned} \text{maximize} \quad & x1 \\ \text{subject to} \quad & x1 + x2 + x3 = 15 \\ & x1 \leq x2 \\ & x1 \leq x3 \end{aligned}$$

2. Given a graph G, each vertex v_i has a profit p_i and each edge e_{ij} has a cost c_{ij}. Define the profit of a cycle is the total profits of all the vertices in the cycle, and the cost of a cycle is the total costs of all the edges in the cycle. We need to find a simple cycle in G which contains a given vertex v_0, and maximize the profit of it within the cost bound B. Write the linear programming formulation of this problem.

Solution:

- We have x_i for each $v_i \in V$, if v_i is in the simple cycle, $x_i = 1$, else $x_i = 0$. We have y_{ij} for each $e_{ij} \in E$, if e_{ij} is in the simple cycle, $y_{ij} = 1$, else $y_{ij} = 0$.

$$
\begin{aligned}
\text{maximize} \quad & \sum_{v_i \in V} x_i * p_i \\
\text{subject to} \quad & x_0 = 1 \\
& x_i \leq 1, x_i \geq 0 \quad v_i \in V \\
& y_{ij} \leq 1, y_{ij} \geq 0 \quad e_{ij} \in E \\
& \sum_{e_{ij} \in E} y_{ij} * c_{ij} \leq B \\
& \sum_{e_{ij} \in E} y_{ij} \geq 2x_i \quad v_i \in V \\
& \sum_{e_{ij} \in E} y_{ij} \leq 2 \quad v_i \in V \\
& x_i + x_j \geq 2y_{ij} \quad e_{ij} \in E
\end{aligned}
$$

3. Consider the following optimization problem

$$\begin{array}{ll} \text{minimize} & f_0(\mathbf{x}) \\ \text{subject to} & f_i(\mathbf{x}) \leq 0, \quad i = 1, \ldots, m \end{array}$$

with variable $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbf{R}^n$. Note that the program may not be linear. The Lagrangian $L : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}$ associated with the program is defined as

$$L(\mathbf{x}, \lambda) = f_0(\mathbf{x}) + \sum_{i=1}^{m} \lambda_i f_i(\mathbf{x})$$

where $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbf{R}^m$

Define the Lagrange dual function $g : \mathbf{R}^m \to \mathbf{R}$ as the minimum value of the Lagrangian over x : for $\lambda \in \mathbf{R}^m$,

$$g(\lambda) = \inf_{\mathbf{x}} L(\mathbf{x}, \lambda)$$

We write $\lambda \geq 0$ if $\lambda_i \geq 0$ for all $1 \leq i \leq m$ and let p^* be the optimal value of original program.

Show that:

$$g(\lambda) \leq p^*, \text{ for every } \lambda \geq 0$$

Solution:

- Since $\lambda_i \geq 0$ and $f_i \leq 0$, we have
$$\sum_{i=1}^{m} \lambda_i f_i(x) \leq 0$$
.
- Let $f_0(x^*) = p^*$, we have

$$g(\lambda) = \inf_{\mathbf{x}} L(\mathbf{x}, \lambda) \leq \mathbf{x}^* L(\mathbf{x}^*, \lambda) = f_0(x^*) + \sum_{i=1}^{m} \lambda_i f_i(x) \leq p^*$$

4. Consider the following optimization problem

$$\text{maximize } g(\lambda)$$
$$\text{subject to } \lambda \geq 0$$

Show that if the program in previous exercise is a linear program, then this program is its dual program.

Solution:

- We have

$$\begin{aligned} L(\mathbf{x}, \lambda) &= c^T\mathbf{x} + \lambda_1^T(b - Ax) + \lambda_2^T * (-\mathbf{x}) \\ &= (c^T - \lambda_1^T A - \lambda_2^T)\mathbf{x} + \lambda_1^T b \\ &= (c - A^T\lambda_1 - \lambda_2)^T * \mathbf{x} + \lambda_1 b^T \end{aligned}$$

- Then

$$g(\lambda) = \inf_{\mathbf{x}} L(\mathbf{x}, \lambda) = \begin{cases} \lambda_1 b^T, & c - A^T\lambda_1 - \lambda_2 = 0 \\ -\infty, & otherwise \end{cases} \tag{1}$$

- Thus, the above problem can be transformed into

$$\begin{aligned} \text{maximize} \quad & \lambda_1 b^T \\ \text{subject to} \quad & c - A^T\lambda_1 - \lambda_2 = 0 \\ & \lambda_1, \lambda_2 \geq 0 \end{aligned}$$

- which can be transformed into

$$\begin{aligned} \text{maximize} \quad & \lambda_1 b^T \\ \text{subject to} \quad & c - A^T\lambda_1 \geq 0 \\ & \lambda_1 \geq 0 \end{aligned}$$

5. Prove the complementary slackness property of linear programs.

Proof. The complementary slackness theorem is: Assume LP problem (P) has a solution x^* and its dual problem (D) has a solution y^*.

- (1) If $x_j^* > 0$, then the j-th constraint in (D) is binding.
- (2) If the j-th constraint in (D) is not binding, then $x_j^* = 0$.
- (3) If $y_j^* > 0$, then the i-th constraint in (P) is binding.
- (4) If the i-th constraint in (P) is not binding, then $y_j^* = 0$.
- Suppose problem P is

$$\begin{aligned} \text{maximize} \quad & \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

- and its dual problem D

$$\begin{aligned} \text{minimize} \quad & \mathbf{y}^T\mathbf{b} \\ \text{subject to} \quad & \mathbf{y}^T\mathbf{A} \geq \mathbf{c}^T \\ & \mathbf{y} \geq \mathbf{0} \end{aligned}$$

- Suppose $\mathbf{x}^*$ and $\mathbf{y}^*$ are the optimum solution of problem P and D, respectively. Then according to problem D, we have
$$\mathbf{y}{*}^T\mathbf{A} \geq \mathbf{c}^T$$
- Since $\mathbf{x}^* \geq \mathbf{0}$, multiply x^*, we have
$$\mathbf{y}{*}^T\mathbf{A}\mathbf{x}^* \geq \mathbf{c}^T\mathbf{x}^*$$
- According to problem P, we have
$$\mathbf{y}{*}^T\mathbf{b} \geq \mathbf{y}{*}^T\mathbf{A}\mathbf{x}^*$$
- Then
$$\mathbf{y}{*}^T\mathbf{b} \geq \mathbf{y}{*}^T\mathbf{A}\mathbf{x}^* \geq \mathbf{c}^T\mathbf{x}^*$$
- According to the Strong Duality Theorem, we have
$$\mathbf{y}{*}^T\mathbf{b} = \mathbf{y}{*}^T\mathbf{A}\mathbf{x}^* = \mathbf{c}^T\mathbf{x}^*$$
- which implies
$$\mathbf{y}{*}^T(\mathbf{A}\mathbf{x}^* - \mathbf{b}) = 0, (\mathbf{y}{*}^T\mathbf{A} - \mathbf{c}^T)\mathbf{x}^* = 0$$
- The former inequality implies (1),(2) of the complementary slackness property, the latter inequality implies (3),(4) of the complementary slackness property.

$\square$

6. Write the dual problem of:

$$\begin{array}{ll} \text{maximize} & c^T x \\ \text{subject to} & A_1 x \leq b_1 \\ & A_2 x \geq b_2 \\ & A_3 x = b_3 \end{array}$$

Solution:

- The problem can be rewritten as

$$\begin{array}{lr} \text{maximize} & c^T(x_1 - x_2) \\ \text{subject to} & A_1(x_1 - x_2) \leq b_1 \\ & -A_2(x_1 - x_2) \leq -b_2 \\ & A_3(x_1 - x_2) \leq b_3 \\ & -A_3(x_1 - x_2) \leq -b_3 \\ & x_1, x_2 \geq 0 \end{array}$$

- Then its dual is

$$\begin{array}{lr} \text{minimize} & b_1^T y_1 - b_2^T y_2 + b_3^T y_3 \\ \text{subject to} & A_1^T y_1 - A_2^T y_2 + A_3^T y_3 - A_3^T y_4 = c^T \\ & y_1, y_2, y_3, y_4 \geq 0 \end{array}$$

7. Prove the König theorem: Let G be bipartite, then cardinality of maximum matching = cardinality of minimum vertex cover.

Solution:

- The maximum matching problem can be written as the following linear program

$$\begin{array}{lll} \text{maximize} & \displaystyle\sum_{e \in E} x_e & \\ \text{subject to} & \displaystyle\sum_{e=(u,v) \in E} x_e \leq 1, & v \in V \\ & x_e \geq 0, & e \in E \end{array}$$

- Consider its dual LP

$$\begin{array}{lll} \text{minimize} & \displaystyle\sum_{v \in V} y_v & \\ \text{subject to} & y_u + y_v \geq 1, & e = (u,v) \in E \\ & y_v \geq 0, & v \in V \end{array}$$

- Which means the minimum vertex cover problem. According to the Strong Duality Theorem, then **minimize** $\sum_{v \in V} y_v =$ **maximize** $\sum_{e \in E} x_e$, we have cardinality of maximum matching = cardinality of minimum vertex cover.

8. Show that the dual of the dual of a linear program is the primal linear program.
Solution:

- Suppose the problem P is

$$\begin{array}{ll} \text{maximize} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{array}$$

- Then its dual problem D is

$$\begin{array}{ll} \text{minimize} & \mathbf{y}^T\mathbf{b} \\ \text{subject to} & \mathbf{y}^T\mathbf{A} \geq \mathbf{c}^T \\ & \mathbf{y} \geq \mathbf{0} \end{array}$$

- Transpose the above formula

$$\begin{array}{ll} \text{minimize} & \mathbf{b}^T\mathbf{y} \\ \text{subject to} & \mathbf{A}^T\mathbf{y} \geq \mathbf{c} \\ & \mathbf{y} \geq \mathbf{0} \end{array}$$

- which can be rewritten as

$$\begin{array}{ll} \text{maximize} & -\mathbf{b}^T\mathbf{y} \\ \text{subject to} & -\mathbf{A}^T\mathbf{y} \leq -\mathbf{c} \\ & \mathbf{y} \geq \mathbf{0} \end{array}$$

- Then the dual problem of the above problem is

$$\begin{array}{ll} \text{minimize} & -\mathbf{c}^T\mathbf{z} \\ \text{subject to} & -\mathbf{A}\mathbf{z} \geq -\mathbf{b} \\ & \mathbf{z} \geq \mathbf{0} \end{array}$$

- Which can be rewritten as

$$\begin{array}{ll} \text{maximize} & \mathbf{c}^T\mathbf{z} \\ \text{subject to} & \mathbf{A}\mathbf{z} \leq \mathbf{b} \\ & \mathbf{z} \geq \mathbf{0} \end{array}$$

- Let $z = x$, then the above problem becomes problem P.