Assignment 1

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1. Prove the König theorem: Let G be bipartite, then cardinality of maximum matching = cardinality of minimum vertex cover.

Proof.

- We define L to be the left part of $G = (V, E)$ and R to be the right part, and suppose we have M to be the maximum matching. Start from a vertex in R that is not a vertex of any edge in M, we go through a path (not in M) $\rightarrow$ (in M) $\rightarrow$ (not in M) $\rightarrow$ (in M) $\ldots$ until the path cannot continue(alternating path).
- **All vertices in this path** form a vertex set U. **All edges in this path** form a edge set P. This path starts at a vertex in R and ends at a vertex in R. Firstly we prove that $m = (L \cap U) \cup (R \backslash U)$ is a **vertex cover**.
- We prove by contradiction. Suppose $e \in E$'s right vertex r is in $R \cap U$ and left vertex l is in $L \backslash U$. $e \notin M$ because it shares a vertex $r \in R \cap U$ with an edge $f \in M \cap P$. Then, since $e \notin M$, we have a path from f to e that becomes a part of an alternating path, which is a contradiction.
- Next we prove that $|m| = |M|$. For each $k \in M \cap P$, it has a vertex in $L \cap U$ corresponding to it. For each $k \in M \setminus P$, its right vertex must $\in R \setminus U$, or it would become part of the path.
- Finally, m must be the minimum cover. If we remove a vertex v from m, then $e \in M$ corresponding to v cannot be covered.

$\square$

2. Consider the algorithm **Negative-Dijkstra** for computing shortest paths through graphs with negative edge weights (but without negative cycles) Note that **Negative-Dijkstra** shifts all edge

Algorithm 1 Algorithm 1: Negative-Dijkstra(G,s)

$w^* \leftarrow$ minimum edge weight in G;
for $e \in E(G)$ **do**
  $w'(e) \leftarrow w(e) - w^*$
end for
$T \leftarrow$**Dijkstra**(G', s);
return weights of T in the original G;

weights to be non-negative(by shifting all edge weights by the smallest original value) and runs in $O(m + \log n)$ time.
Prove or Disprove: **Negative-Dijkstra** computes single-source shortest paths correctly in graphs with negative edge weights. To prove the algorithm correct, show that for all $u \in V$ the shortest $s - u$ path in the original graph is in T. To disprove, exhibit a graph with negative edges, with no negative cycles where **Negative-Dijkstra** outputs the wrong ”shortest” paths, and explain why the algorithm fails.

Disprove:

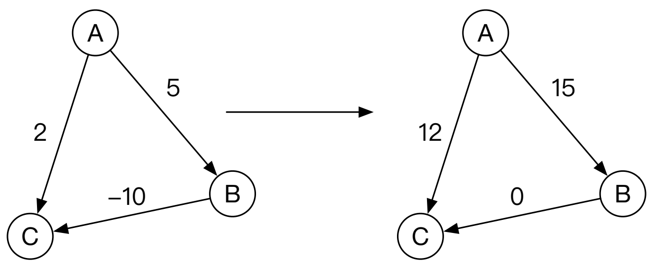

Figure 1: Negative to Non-negative Edges

See Figure 1, we shift the graph $G = \{w_{AB} = 5, w_{BC} = -10, w_{AC} = 2\}$ to $G' = \{w_{AB} = 15, w_{BC} = 0, w_{AC} = 12\}$.

- Use Dijkstra’s Algorithm on G', A is the source vertex. Then the shortest path from A to C is 12, after shifting back, it is -2 ($A \to C$).
- Use Dijkstra’s Algorithm on G, then the shortest path from A to C is -5 ($A \to B \to C$), not -2.
- Negative-Dijkstra fails because the **shifting** operation can shift different **weight** even for the two paths with the same length. The **more edges** a path has, the more *shift* it gets. It is unfair.

3. Consider a weighted, directed graph G with n vertices and m edges that have integer weights. A graph walk is a sequence of not-necessarily-distinct vertices $v_1, v_2, \dots, v_k$ such that each pair of consecutive vertices v_i, v_{i+1} are connected by an edge. This is similar to a path, except a walk can have repeated vertices and edges. The length of a walk in a weighted graph is the sum of the weights of the edges in the walk. Let s, t be given vertices in the graph, and L be a positive integer. We are interested counting the number of walks from s to t of length exactly L.

- Assume all the edge weights are positive. Describe an algorithm that computes the number of graph walks from s to t of length exactly L in $O((n+m)L)$ time. Prove the correctness and analyze the running time

- Now assume all the edge weights are non-negative(but they can be 0), but there are no cycles consisting entirely of zero-weight edges. That is, for any cycle in the graph, at least one edge has a positive weight.
  Describe an algorithm that computes the number of graph walks from s to t of length exactly L in $O((n+m)L)$ time. Prove correctness and analyze running time.

Solution: We use Dynamic Programming to calculate the number of walks of length exactly L.

- Define $nw(v, l)$ to be the number of walks from vertex s to v of length exactly l (nw stands for **N**umber of **W**alks). We need to calculate $nw(t, L)$, where

$$nw(v, l) = \sum_{\{u|(u,v)\in E, w(u,v)\le l\}} nw(u, l - w(u, v))$$

Algorithm 2 Number of walks from s to t of length exactly L

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1: for v ∈ V do
2:     for l from 0 → L do
3:         nw(v, l) ← 0
4:     end for
5: end for
6:
7: nw(s, 0) ← 1
8: for l from 0 → L do
9:     for v ∈ V do
10:        for u ∈ {u | (u, v) ∈ E, w(u, v) ≤ l} do
11:            nw(v, l) ← nw(v, l) + nw(u, l − w(u, v))
12:        end for
13:    end for
14: end for
```

- For G with **zero weight** edges, we remove all edges in E that has positive weight and get G'. On line 1, when setting the initial value of matrix $w(v, l)$, instead of setting them all to 0, we first run the algorithm on G' to initialize the matrix $w(v, l)$.

- On line 9, it traverses **no more than** all vertices and edges, which needs $O(n+m)$ time. On line 8, it runs line 9 for L times. Finally, this algorithm runs in $O((n+m)L)$ time.

4. The diameter of a connected, undirected graph $G = (V, E)$ is the length (in number of edges) of the longest shortest path between two nodes. Show that is the diameter of a graph is d then there is some set $S \subseteq V$ with $|S| \leq n/(d-1)$ such that removing the vertices in S from the graph would break it into several disconnected pieces.

Proof.

We use **Menger's theorem:** Let G be a finite undirected graph and x and y two distinct vertices. Then the size of the minimum edge cut for x and y is equal to the maximum number of pairwise edge-independent paths from x to y.

- Suppose $x, y \in G$ is the two ends of the diameter of G. According to **Menger's theorem**, the size of the minimum edge cut S for x and y is equal to the maximum number of pairwise edge-independent paths from x to y.
- The maximum number of pairwise edge-independent paths from x to y is m, then $m = |S|$, and

$$m * (d-1) + 2 \leq n \implies |S| \leq (n-2)/(d-1) \implies |S| < n/(d-1)$$

- ,where removing the vertices in $S \subseteq V$ from the graph would break it into **two** disconnected pieces.

$\square$

5. Let G be a n vertices graph. Show that if every vertex in G has degree at least $n/2$, then G contains a Hamiltonian path.

Proof. This is **Dirac's theorem**, which is weaker than Ore's theorem. Now we just need to prove **Ore's theorem**: Let G be a (finite and simple) graph with $n \geq 3$ vertices. If $deg(u) + deg(v) \geq n$ for every pair of distinct non-adjacent vertices u and v of G, then G is Hamiltonian.

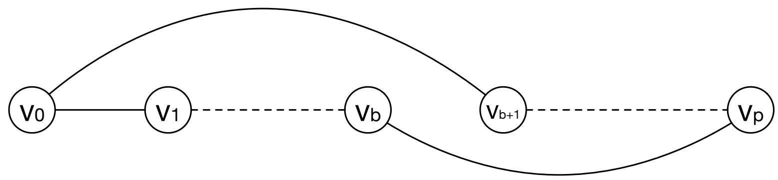

Figure 2: Ore's theorem

- Firstly, we prove that $G = (V, E)$ must be **connected** by contradiction. Suppose V_1 and V_2 are the two sets of disconnected vertices in V and $|V_1|+|V_2| = |V| = n$. WLOG, $|V_1| \leq n/2 \leq |V_2|$. Then, for two vertices $v_1, v_2 \in V_1$, we have $deg(v_1) + deg(v_2) \leq (n/2 - 1) * 2 < n$, which is a contradiction.

- Suppose path $P = v_0 v_1 \ldots v_p (p \leq n-1)$ is the **longest** path that does not go through a same vertex more than once. Suppose $v_{i_0}, v_{i_1}, \ldots, v_{i_k}$ are all vertices adjacent to v_0, then they are all on path P and $k \leq p$, **or** path $v_{i_x} v_0 v_1 \ldots v_p (0 \leq x \leq k)$ is longer than P, which is a contradiction. Thus, $deg(v_0) = k + 1$.

- Also, at least one of $v_{i_0}, v_{i_1}, \ldots, v_{i_k} (k \leq p)$ is adjacent to v_p. **Or**, $deg(v_p) \leq p - (k+1) \implies deg(v_0)+deg(v_p) \leq k+1+(p-(k+1)) = p \leq n-1$, which contradicts with $deg(u)+deg(v) \geq n$. Thus, $\exists v_b$ on path P that is both adjacent to v_0 and v_p. Since v_b and v_{b+1} are adjacent, so we have a Hamiltonian cycle $v_0 \ldots v_b v_p \ldots v_{b+1} v_0$ in Figure 2, which has a length of $p + 1$.

- Suppose $p+1 < n$, then $\exists v_m \in G$ is not in path P. Since G is **connected**, v_m can go to a vertex $v_n \in P$ through some path. If $0 \leq n \leq b$, then path

$$v_m \ldots v_n v_{n-1} \ldots v_0 v_{b+1} \ldots v_p v_{p-1} \ldots v_b v_{b-1} \ldots v_{n+1}$$

- becomes a at least $(p+1)$-long path, which contradicts with the longest path P. For $b+1 \leq n \leq p$ we have similar contradiction.

- Thus, we have $p + 1 \geq n \implies p \geq n - 1$, and $p \leq n - 1$ since $P \subseteq V$, finally $p = n - 1$, which means path P is a Hamiltonian path.

$\square$

6. Show how to find a minimal cut of a graph (not only the cost of minimum cut, but also the set of edges in the cut).

Solution1:

- Let $G = (V, E)$ be a weighted undirected graph. For two vertices $s, t \in V$, there are two possible situations: global minimum cut of G is also $s - t$ min-cut, or s and t belong to the same side of the global min-cut.
- In the latter situation, the global min-cut can be found by merging s and t. G becomes $G' = G \setminus \{s, t\} \cup \{st\}$, where st is a vertex representing merged s and t. If edge $s - t \in E$, then it disappears. For a vertex $v \in V$ that have an edge to both s and t, then

$$w(v, st) = w(v, s) + w(v, t)$$

- Return the cut-set of the final state. Run the algorithm recursively on G', and the min-cut of G' is equal to that of G.

Algorithm 3 Stoer-Wagner Algorithm

procedure MinCutPhase($G = (V, E), a$)
    $S \leftarrow \{a\}$
    while $|S| < |V|$ **do**
        $w(A, z) = \max\{w(A, y) \mid y \notin A\}$
        where $w(A, y)$ is the sum of the weights of all the edges between A and y.
        $S \leftarrow S \cup \{z\}$
        shrink G by merging the two vertices (s, t) added last.
        return {cut-of-phase, cut-set}=$w(A, last - two - vertices)$, cut-set
    end while
end procedure

procedure MinCut($G = (V, E), a$)
    while $|V| > 1$ **do**
        {cut-of-phase, cut-set}$\leftarrow MinCutPhase(G = (V, E), a)$

        if $mincut$ $>$cut-of-phase **then**
            $mincut \leftarrow$ cut-of-phase
            $mincutset \leftarrow$ cut-set
        end if
    end while
end procedure

Solution2:

- Let vertex s to be the source, we run Ford-Fulkerson Algorithm on G, and remove all edges that has the max flow to get G'.
- Run DFS to traverse G', mark all visited vertex to be in $V' \subseteq V$. Then all edges between vertex sets V' and $V \setminus V'$ form the minimum cut.

7. Let $G(V, E)$ be a connected undirected graph with a weight $w(e) > 0$ for each edge $e \in E$. For any path $P_{u,v} =< u, v_1, v_2, \ldots, v_r, v >$ between two vertices u and v in G,let $\beta(P_{u,v})$ denote the maximum weight of an edge in $P_{u,v}$. We refer to $\beta(P_{u,v})$ as the **bottleneck weight** of $P_{u,v}$. Define

$$\beta^*(u, v) = \min\{\beta(P_{u,v}) : P_{u,v} \text{ is a path between } u \text{ and } v\}.$$

Give a polynomial algorithm to find $\beta^*(u, v)$ for each pair of vertices u and v in V and a proof of the correctness of the algorithm.

Solution:

- We say a **minimum bottleneck spanning tree (MBST)** is a spanning tree that minimizes the most expensive edge. First we prove that **MST** must be a MBST. If not, consider the most expensive edge e_1 of MST and most expensive edge e_2 of MBST, we have $w(e_1) > w(e_2)$. Then, we replace e_1 from MST with e_2, then we have a spanning tree that has smaller weight than MST, which is a contradiction.
- Similarly, we can prove by contradiction that a minimum bottleneck path must be on **MST**. So, we first get MST by running **Prim Algorithm**. Then, for each pair of vertices u and v, we do a DFS from u to v to find the minimum bottleneck path, then we get $\beta^*(u, v)$.

8. Let $G = (V, E)$ be a directed graph. Give a linear-time algorithm that given G, a node $s \in V$ and an integer k decides whether there is a walk in G starting at s that visits at least k distinct nodes.
Solution:

- Use Tarjan Algorithm to get all **Strongly Connected Components (SCC)** of G. In A **SCC** there is a path that goes through each vertex in SCC for at least once.
- Define $G' = (V', E')$ where each $v \in V'$ corresponds to an SCC of G, and for each edge $(u \to v) \in E'$, $w(u \to v) = |GCC(v)|$, where **GCC(v)** is the GCC in G corresponding to v.
- Suppose s is in $GCC(s')$. We do a DFS starting at s' in G'. We set $w \leftarrow 0$. When getting to a vertex t', $w \leftarrow w + w(t')$. If $w >= k$, return $true$. If DFS is done, return $false$.
- Time complexity is $O(|V| + |E|)$, which is linear to V and E.

9. Minimum Bottleneck Spanning Tree: Given a connected graph G with positive edge costs, find a spanning tree that minimizes the most expensive edge.

Solution:

Algorithm 4 Camerini's Algorithm

procedure MBST($G = (V, E)$)
    if $|E| = 1$ **then**
        return E
    end if

    $E_1 = \{e \in E \mid w(e) > w_m\}$, $E_2 = \{e \in E \mid w(e) \leq w_m\}$, $A = (E_1, V_1)$, $B = (E_2, V_2)$
    $|F| \leftarrow$ spanning tree (forest) of B

    if then$|F| = 1$
        return MBST(B)
    else
        $V' \leftarrow$ contract each connected component in B into a single vertex
        $E' \leftarrow$ edges in A that connect the connected components in B
        return $F \cup (V', E')$
    end if
end procedure

- Suppose we need to find an **MBST** (Minimum Bottleneck Spanning Tree) in a undirected, connected, positive edge-weighted graph $G = (V, E)$. We have Camerini's Algorithm to find such a MBST in $O(n)$ time, which works as follows.
- Find the **median** edge weight w_m in E, then partition E into $E_1 = \{e \in E \mid w(e) > w_m\}$ and $E_2 = \{e \in E \mid w(e) \leq w_m\}$.
- Define F to be the spanning tree forest of E_2. If $|F| = 1$ then run the former step on B.
- Else, contract each connected component in B to a single vertex. Create a new graph by using all vertices from the contraction (V'), and all edges in A that connects the connected components in B (E'). We have $|V'| = |F|$, $G' = (V', E')$, MBST is $F \cup MBST(G')$.
- This algorithm works in

$$O(E/2) + O(E/4) + O(E/8) + \ldots = O(E)$$

- time.