Assignment 3

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1. Prove or disprove the following statement. If all capacities in a network are distinct, then there exists a unique flow function that gives the maximum flow.

Disprove:

- See Figure 1, each edge has a distinct capacity, but we have two flow functions f_1, f_2 that both give the maximum flow, where

$$f_1(s,A) = f_1(A,B) = f_1(B,t) = 2 \text{ and } f_1(e) = 0 \text{ for other edges,}$$
$$f_2(s,C) = f_2(C,B) = f_2(B,t) = 2 \text{ and } f_2(e) = 0 \text{ for other edges.}$$

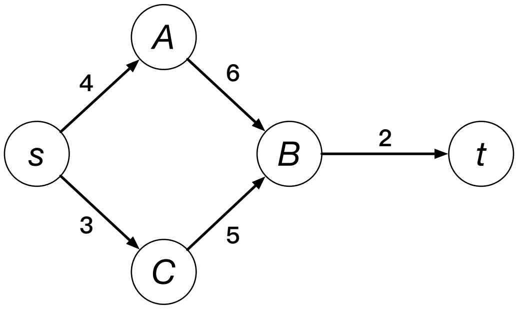

Figure 1: Graph with two maximum flow functions

2. An edge of a flow network is called **critical** if decreasing the capacity of this edge results in a decrease in the maximum flow value. Present an efficient algorithm that, given an s-t network G finds any critical edge in a network(assuming one exists).

Solution:

- If an edge is not filled to capacity in the maximum flow, then decreasing its capacity will not result in a decrease in the maximum flow value. Thus, a **critical** edge must be full edge in the maximum flow of the graph.
- We run Ford-Fulkerson algorithm on the graph G and get residual graph G_f. For each full edge (u, v), we do DFS from u to v on G_f. If there is a path P from u to v, then we can decrease the capacity of (u, v) and increase the same amount of flow on P so the maximum flow value does not change, thus (u, v) is not critical path.The following algorithm gives any critical edge:

Algorithm 1 Find any critical edge

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$G_f$ ←Ford-Fulkerson($G$)
for $(u_0, v_0) \in \{(u, v) | (u, v) \in G, (u, v) \notin G_f, (v, u) \in G_f\}$ do
    DFS($G_f, u_0$)
    if $u_0$ has no path to $v_0$ then
        return $(u_0, v_0)$
    end if
end for
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3. Let $G = (V, E)$ be an undirected weighted graph with two distinguished vertices $s, t \in V$. Give an efficient algorithm to find a minimum weight cut that separates s from t.

Solution:

- We transform G to directed graph $G' = (V, E')$. For all edge $(s, u) \in E$, we have $(s, u) \in E'$. For all edge $(v, t) \in E$, we have $(v, t) \in E'$. For all edge $(u, v) \in E$ where $u, v \notin \{s, t\}$, we have $(u, v), (v, u) \in E'$.
- We run Ford-Fulkerson algorithm on G', and we can get a minimum weight cut that separates s from t.

4. You are given a matrix with fractional elements between 0 and 1. The sum of all numbers in each row and in each column is integer. Prove that we can always round each element to 0 or 1 so that the sum of each row and each column remains unchanged and design a polynomial time algorithm to find such a rounding result.

Solution:

- For row $i, 1 \leq i \leq n$ of the matrix, we construct a row vertex R_i. For column $j, 1 \leq j \leq n$ of the matrix, we construct a column vertex C_j. Also construct a source s and a sink t.
- We construct a directed edge (s, R_i) with capacity of sum row i and a directed edge (C_j, t) with capacity of sum column j. For each R_i and C_j, we have a directed edge (R_i, C_j) with capacity 1, and its flow $0 \leq f(i,j) \leq 1$ represents the element $M(i,j)$ in the matrix.
- Then f is a maximum flow of the constructed graph, where the maximum flow value is the sum of M.
- We run Ford-Fulkerson algorithm on the constructed graph, and since all edges have integer capacity, we could find the maximum flow where $f^*(i,j) \in \{0,1\}$ in **polynomial** time, and thus we find a matrix $M^*(i,j) = f^*(i,j)$ with all elements $\in \{0,1\}$.

5. Suppose that, in addition to edge capacities, a flow network has **vertex capacities**. That is each vertex has a limit on how much flow can pass though. Show how to transform a flow network $G = (V, E)$ with vertex capacities into an equivalent flow network $G' = (V', E')$ without vertex capacities, such that a maximum flow in G' has the same value as a maximum flow in G. How many vertices and edges does G' have?

Solution:

- For each $v \in V$, we have $v_{in}, v_{out} \in V'$ and $(v_{in}, v_{out}) \in E'$. For each in edge $(a, v) \in E$, we have $(a, v_{in}) \in E'$, and for each out edge $(v, b) \in E$, we have $(v_{out}, b) \in E'$.
- Let $cap(v_{in}, v_{out}) = cap(v)$. Thus, the in/out flow of v never exceeds $cap(v)$. In this way, the maximum flow of $G' = (V', E')$ can be transformed into the maximum flow of G with **vertex capacities**.
- We have $|V'| = 2|V|$ and $|E'| = |E| + |V|$.

6. Consider a bipartite graph $G = (X \cup Y, E)$ with parts X and Y. Each part contains $2k$ vertices (i.e. $|X| = |Y| = 2k$). Suppose that $deg(u) \geq k$ for every $u \in X \cup Y$. Prove that G has a perfect matching.

Solution:

- For subset $S \subseteq L$ with $1 \leq |S| \leq k$, we have $N(S) \geq deg(u) \geq k \geq |S|$, where $u \in S$.
- For subset $S \subseteq L$ with $k < |S| \leq 2k$, consider any vertex $v \in Y \setminus N(S)$. Thus $N(v) \cap S = \varnothing$. Since $deg(v) \geq k$, $|N(v)| \geq k$. Then

$$2k = |Y| \geq |S| + |N(v)| > k + k = 2k$$

- which is a contradiction. Thus, we have $N(S) = Y$, and $|N(S)| = |Y| = 2k \geq |S|$.
- In conclusion, for all $S \subseteq L$, $|N(S)| \geq |S|$. According to Hall's Marriage Theorem, G has a perfect matching.

7. You are designing a experiment in which you want to measure certain properties $p_1, \ldots, p_n$ of a yeast culture. You have a set of tools $t_1, \ldots, t_m$ that can each measure a subset S_i of the properties. For example, tool t_i measures S_i may equal $\{p_7, p_8\}$. To be sure that your results are not due to noise or other artifact, you must measure every property at least k times using k different tools.

- Give a polynomial-time algorithm that decides whether the tools you have are sufficient to measure the desired properties the desired number of times.
- Suppose each tool t_i comes from manufacturer M_i and we have the additional constraint that the tools to test any property p_i can't all come from the same manufacturer. Give a polynomial-time algorithm to solve this problem.

Solution:

- We reduce the first problem to a maximum flow problem with lower bounds.
- We add vertices $p_1, \ldots, p_n$ and $t_1, \ldots, t_m$, and if tool t_j can measure property p_i, we add a directed edge (t_i, p_j).
- Add source s and edges (s, p_i), where $cap(s, p_i) = |S_i|$. Add sink t and edges (t_j, t), where $k \leq cap(t_j, t) \leq m$.
- We shift the capacity of (t_j, t) to $[0, m-k]$ and set $d(p_j) = k, d(t) = -nk$. Add sink s^* and t^*. For vertices with $d(v) < 0$, add edge (s', v) with capacity $-d(v)$. For vertices with $d(v) > 0$, add edge (v, t^*) with capacity $d(v)$.
- In this flow graph, if we could find a flow of value nk, the tools are sufficient to measure the desired properties the desired number of times.

- We reduce the second problem to a maximum flow problem.
- We add vertices $p_1, \ldots, p_n$ and $t_1, \ldots, t_m$. For each t_j, we add m vertices $M_{ij}, 1 \leq i \leq m, 1 \leq j \leq n$ to represent all m manufacturers, where $M_{i1} = M_{i2} = \cdots M_{in}$.
- Add edges (t_j, M_{ij}) with capacity 1 if tool t_j comes from manufacturer M_{ij}. Add edges $(M_{ij}, p_j), 1 \leq j \leq n$ with capacity $k-1$ to avoid using tools that come from all the same manufacturer. Add source and sink s, t. Add edges (s, t_j) with $+\infty$ capacity, add edges (p_i, t) with k capacity.
- If we could find a maximum flow of value nk, the tools are sufficient to measure the desired properties the desired number of times with the additional constraint. We could find this maximum flow with Ford-Fulkerson algorithm, which runs in polynomial time.

8. Consider a flow network $G = (V, E)$ with positive edge capacities $\{c(e)\}$. Let $f : E \to \mathbb{R}_{\geq 0}$ be a maximum flow in G, and G_f be the residual graph. Denote by S the set of nodes reachable from s in G_f and by T the set of nodes from which t is reachable in G_f. Prove that $V = S \cup T$ if and only if G has a **unique** s-t minimum cut.

Solution:

- First, we notice that $S \cap T = \varnothing$, since if a vertex $v \in S$ and $v \in T$, we could go from s through v to t in the residual graph, and increase the maximum flow, which is a contradiction.
- $\Rightarrow$: We have $C_1 = (S, V \setminus S)$ and $C_2 = (T, V \setminus T)$ to be minimum cuts of G. Since $V = S \cup T$ and $S \cap T = \varnothing$, we have $C_1 = C_2$, which means s-t minimum cut of G is unique.
- $\Leftarrow$: We prove by contradiction. Suppose $S \cup T \subset V$, and let

$$X = V \setminus (S \cup T), X \neq \varnothing$$

- Since $C_1 = (S \cup X, T)$ and $C_2 = (S, X \cup T)$ are two distinct minimum cut, we have a contradiction.