

# Topological Semantics for Common Inductive Knowledge

Siddharth Namachivayam

Department of Philosophy, Carnegie Mellon University

## Abstract

Lewis' account of common knowledge in *Convention* describes the process which drives the generation of higher-order expectations as hinging upon agents' inductive standards and a shared witness. Subsequent formal treatments of common knowledge such as those in Aumann 1976 were largely deductive and took as primitives agents' information partitions over possible worlds. This project attempts to draw from insights in learning theory to provide a formal account of common inductive knowledge generated by a witness. Following Kelly's approach in *The Logic of Reliable Inquiry*, we take as primitives agents' information bases over possible worlds. After defining common inductive knowledge, we show how our semantics helps characterize the equilibria of an inductive coordination game.

## Notation

- $N$  is a set of agents and  $\Omega$  is a set of possible worlds.
- $\mathcal{E}_i$  is a basis over  $\Omega$  representing the possible information states of agent  $i$ .
- $\mathcal{T}_i$  is the topology generated by  $\mathcal{E}_i$ .
- $\mathcal{E}_{i|w}$  denotes those information states of  $\mathcal{E}_i$  which contain  $w$ .
- $\mathcal{T}_i|E$  denotes subspace topology of  $\mathcal{T}_i$  restricted to  $E$ .

## Difference Hierarchy and Learning

- $X \cap E$  is  $n$ -open in  $\mathcal{T}_i|E$  iff  $i$  can limit decide  $X$  in  $n$  mind switches starting with 'no' in light of evidence  $E$ .
- $X \cap E$  is  $n$ -closed in  $\mathcal{T}_i|E$  iff  $i$  can limit decide  $X$  in  $n$  mind switches starting with 'yes' in light of evidence  $E$ .
- $\text{Yes}_i^n(X|E)$  holds iff  $\exists k \leq n$  s.t.  $X \cap E$  is  $k$ -closed but not  $(k-1)$ -open in  $\mathcal{T}_i|E$  ( $0$ -open is  $\emptyset$  and  $0$ -closed is  $E$ ).

Define an agent's *switching tolerance*  $n_i$  to be the maximum number of times they are willing to switch their mind after first saying 'Yes' while limit deciding a proposition. Then:

- $\text{Yes}_i^{n_i}(X|E)$  holds iff  $i$  says 'yes' to  $X$  in light of evidence  $E$ .

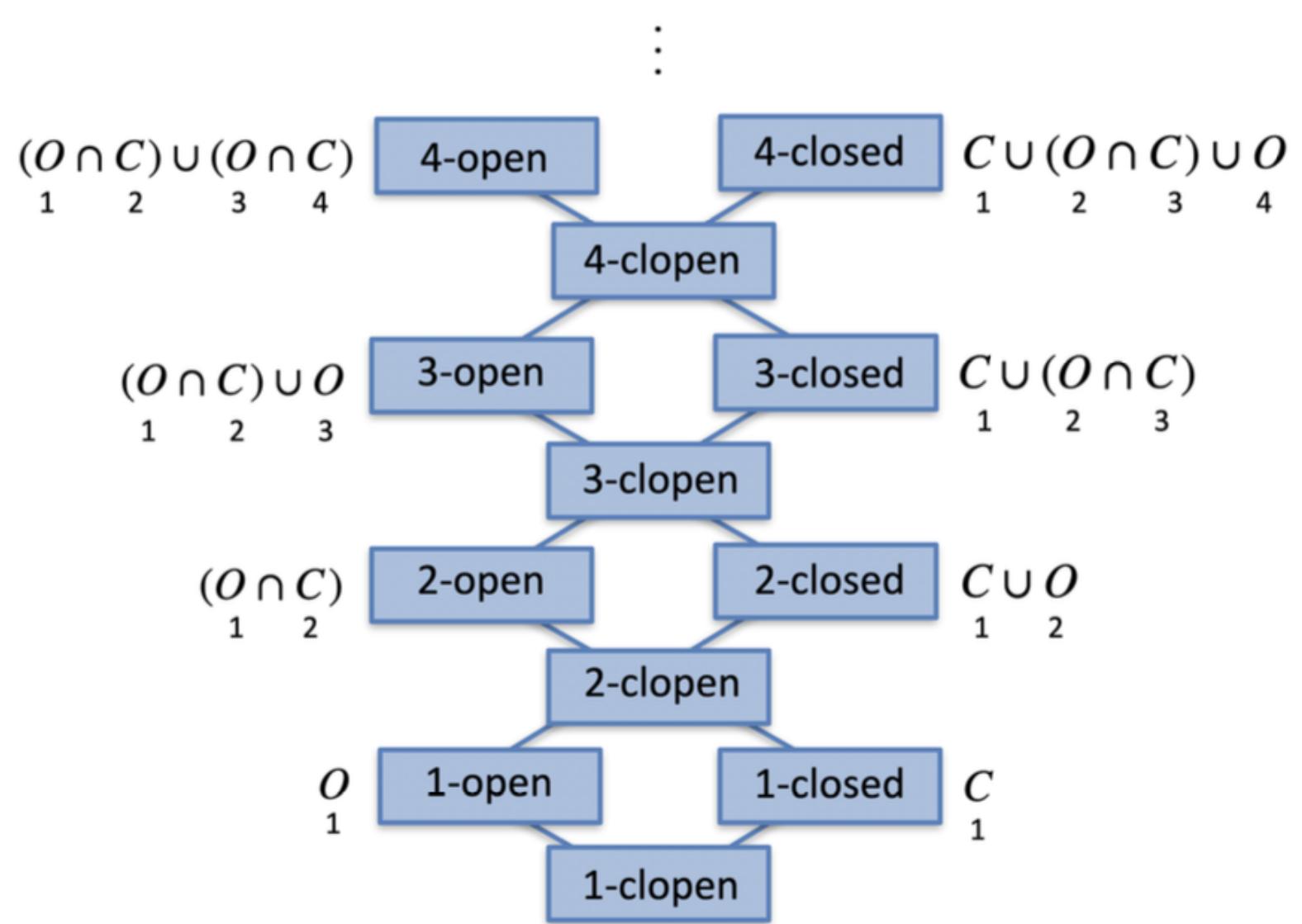


Figure 1: Topological difference hierarchy

## Epistemic Operators

The below list provides a gloss for each of the epistemic operators in our language:

- $\mathbb{R}_i W$ :  $i$  has reason simpliciter to believe  $W$ .
- $\mathbb{I}_{i|W} P$ :  $W$  indicates to  $i$  that  $P$ .
- $\mathbb{B}_{i|W} P$ :  $i$  has  $W$  as reason to believe  $P$ .
- $\mathbb{E}_W P$ : Everyone has  $W$  as reason to believe  $P$ .
- $\mathbb{G}_W P$ :  $W$  generates common inductive knowledge of  $P$ .
- $\mathbb{C}P$ :  $P$  is common inductive knowledge.

## Topological Semantics

The semantics of our epistemic operators are as follows:

- $w \in \mathbb{R}_i W$  iff  $\exists E \in \mathcal{E}_{i|w}$ ,  $\text{Yes}_i^{n_i}(W|E)$ .
- $w \in \mathbb{I}_{i|W} P$  iff  $\forall E \in \mathcal{E}_{i|w}$ ,  $\text{Yes}_i^{n_i}(W|E) \rightarrow W \cap E \subseteq P$ .
- $w \in \mathbb{B}_{i|W} P$  iff  $w \in \mathbb{R}_i W \cap \mathbb{I}_{i|W} P$ .
- $w \in \mathbb{E}_W P$  iff  $w \in \bigcap_{i \in N} \mathbb{B}_{i|W} P$ .
- $w \in \mathbb{E}_W^1 P$  iff  $w \in \mathbb{E}_W P$ .
- $w \in \mathbb{E}_W^n P$  iff  $w \in \mathbb{E}_W \mathbb{E}_W^{n-1} P$ .
- $w \in \mathbb{G}_W P$  iff  $w \in \bigcap_{n \in \mathbb{N}^+} \mathbb{E}_W^n P$ .
- $w \in \mathbb{C}P$  iff  $\exists W \subseteq \Omega$ ,  $w \in W \cap \mathbb{G}_W P$ .

## Conclusion

This project provides a topological semantics for capturing how common inductive knowledge is generated by a witness. Our basic epistemic operators are directly motivated by Lewis' analysis of common knowledge in *Convention*. We find that the resulting semantics elegantly characterizes the equilibria of an inductive coordination game which is closely related to structure we call the multitopological difference lattice. Going forward, we hope to develop a sound and complete proof system to accompany our semantics. Such an effort will not only help prove and verify new theorems about our particular language, but it will also furnish a more systematic understanding of how conventions in general may come to be formed.

## Additional Information

Please contact me if you wish to see a current draft.

## References

- [1] David Lewis. *Convention: A Philosophical Study*. Wiley-Blackwell, 1st edition, 1969.
- [2] Robert Aumann. Agreeing to disagree. *The Annals of Statistics*, 4(52):1236–1239, November 1976.
- [3] Kevin Kelly. *The Logic of Reliable Inquiry*. Oxford University Press, 1st edition, 1996.

## Acknowledgements

I would like to thank Adam Bjorndahl and Kevin Kelly for advising this project. Preliminary progress was presented in 2025 at the Indian Statistical Institute's conference on Recent Trends in Logic and Game Theory as well as the Progic Conference Series' 12th Workshop on Combing Logic and Probability. This work also received financial support in the summer of 2024 from Carnegie Mellon University's Institute for Complex Social Dynamics.

## Contact Information

- Email: snamachi@andrew.cmu.edu
- Department: <https://www.cmu.edu/dietrich/philosophy/people/phd/siddharth-namachivayam.html>

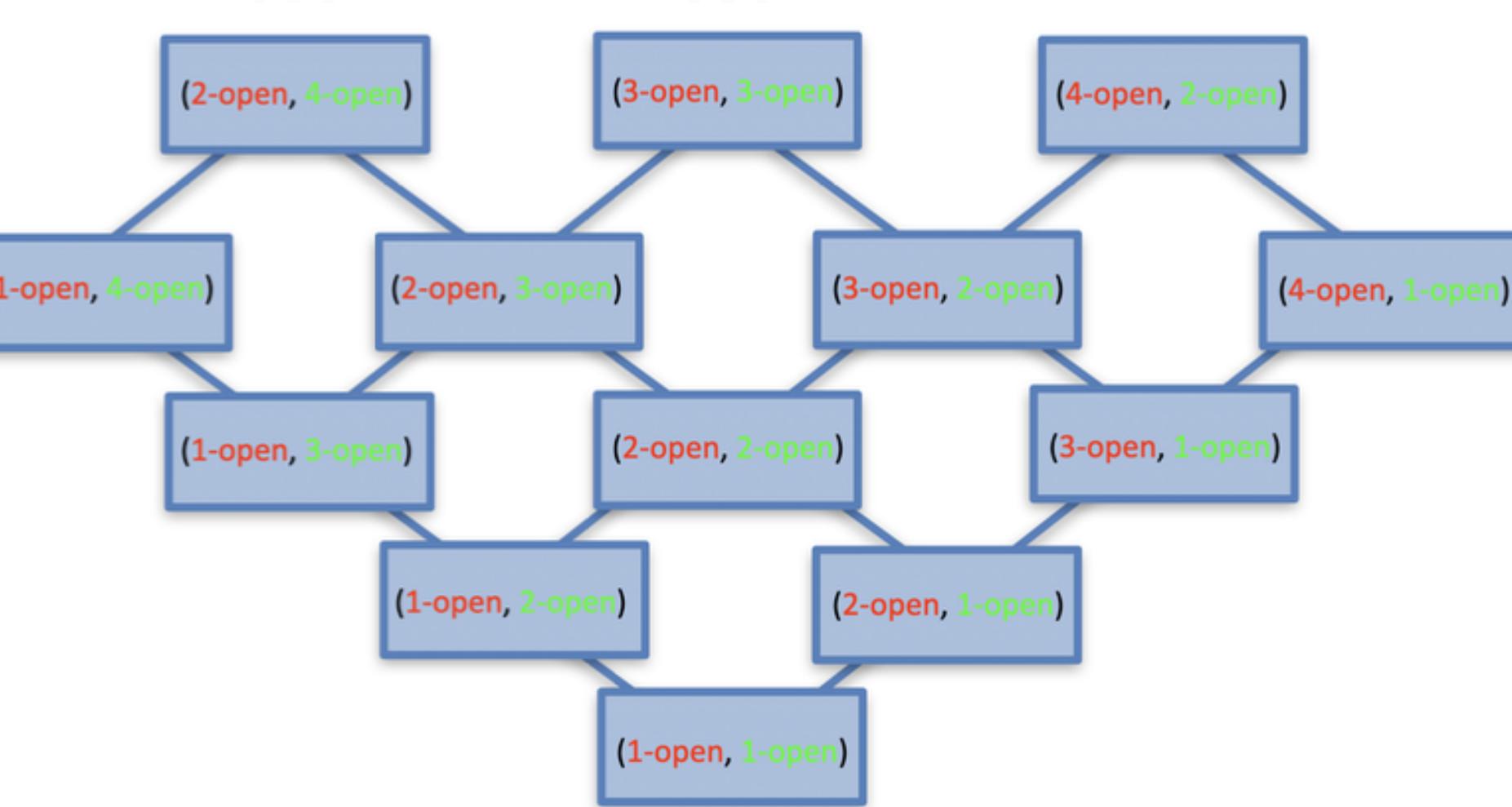


Figure 2: Multitopological difference lattice for  $a$  and  $b$

## Results

**Theorem 1:**  $X$  is a subset of  $P$  which is  $(n_i + 1)$ -open in each  $\mathcal{T}_i$  iff it is a fixed point of the map  $X \mapsto X \cap \mathbb{G}_X P$ . Further  $\forall W \subseteq \Omega$ ,  $W \cap \mathbb{G}_W P$  is itself such a fixed point. This means:

- The map  $X \mapsto X \cap \mathbb{G}_X P$  can be used to select any subset of  $P$  which is  $(n_i + 1)$ -open in each  $\mathcal{T}_i$ .
- $\mathbb{C}P$  consists of those worlds where agents converge to reporting 'yes' in *some* equilibrium of our game.

**Theorem 2:**  $\mathbb{C}P$  does not depend on agents' switching tolerances so long as each  $n_i > 0$ . To be clear:

- Agents might converge to reporting 'yes' on a new *set* in some equilibrium by increasing their switching tolerances.
- However, somewhat counterintuitively, there will be no new *worlds* in any of these sets.



In equilibrium,  $\forall i, j \in N$ , we have  $\sigma_{s_i}^{-1}(\text{Yes}) = \sigma_{s_j}^{-1}(\text{Yes}) \subseteq P$ . Moreover, there exists a strategy profile such that  $\sigma_{s_i}^{-1}(\text{Yes}) = X$  iff  $X$  is  $(n_i + 1)$ -open in  $\mathcal{T}_i$ . Thus :

- Equilibria are characterized by agents choosing some subset  $X$  of  $P$  which is  $(n_i + 1)$ -open in each  $\mathcal{T}_i$ .

But what are these subsets?