

# Eliciting Causal Bayesian Networks with Scoring Rules

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# Problem

- Scoring rules can elicit an agent's beliefs about probability distributions.
- How can we elicit an agent's beliefs about causal Bayesian networks?
- If we can:
  1. Credibly perform interventions
  2. Identify a causal Bayesian network from information about its interventional distributions

Then maybe we can combine scoring rules to discover the agent's beliefs...

# Causal Bayesian Networks

- Let  $V = \{V_1, \dots, V_K\}$  be a set of  $K$  *discrete* random variables each with full support and at least 2 distinct outcomes.
- For purposes of this talk, define a *causal Bayesian network*  $G$  over  $V$  to be a DAG whose vertices are variables in  $V$  such that its roots are mutually independent along with a set of ‘interventional’ distributions  $p_{\text{do}(X=x)}$  for each  $X \subseteq V$ .
- These interventional distribution satisfy the following compatibility condition:

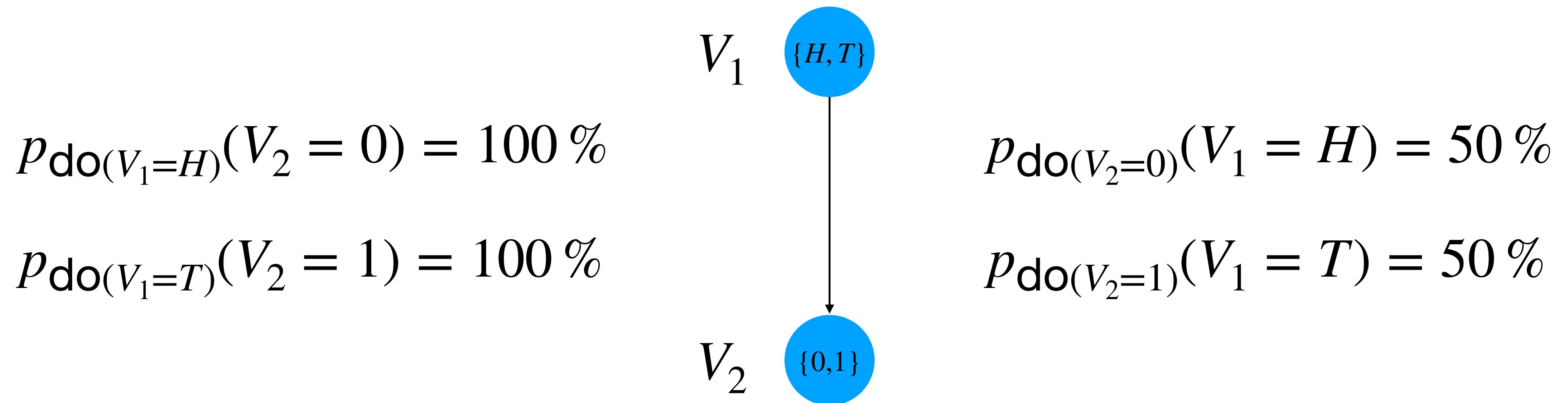
$$p_{\text{do}(X=x)}(V = v) = \begin{cases} \prod_{i: V_i \notin X} p(V_i = v_i \mid \text{PA}_i = \text{pa}_i) & v \text{ is consistent with } x \\ 0 & \text{otherwise} \end{cases}$$

where  $\text{PA}_i$  denotes the parents of  $V_i$  in  $G$  and  $\text{pa}_i$  denotes the values of  $\text{PA}_i$  in  $v$ .

# Example

$$p(V_1 = H, V_2 = 0) = 50 \%$$

$$p(V_1 = T, V_2 = 1) = 50 \%$$



# Setup

- $G$  is the causal Bayesian network over  $V$  believed by the agent.
- For any random variable  $R$ , we denote its outcomes by  $[R]$ .
- Given random variables  $X$  and  $Y$ , the agent's beliefs about the distribution of  $Y$  conditional on  $\text{do}(X = x)$  are denoted by  $b(Y \mid \text{do}(X = x))$ .
  - Equivalently, we will also write  $b(Y \mid \text{do}(x))$  or  $b_{Y \mid \text{do}(x)}$  when  $X$  is clear from context.
  - Additionally, we will write  $b(Y \mid \text{do}(X))$  when speaking of the function  $x \mapsto b_{Y \mid \text{do}(x)}$ .
- Finally, let  $b^* = \{b(V \mid \text{do}(X)) : X \subseteq V\}$  consist of all the agent's interventional beliefs.

# Identifiability

- Unfortunately  $G$  is not (in general) identifiable from  $b^*$ .
- Say for variables  $X, Y \in V$  that  $X$  has *zero direct effect* on  $Y$ , or equivalently, that  $\text{ZDE}(X, Y)$  holds iff:

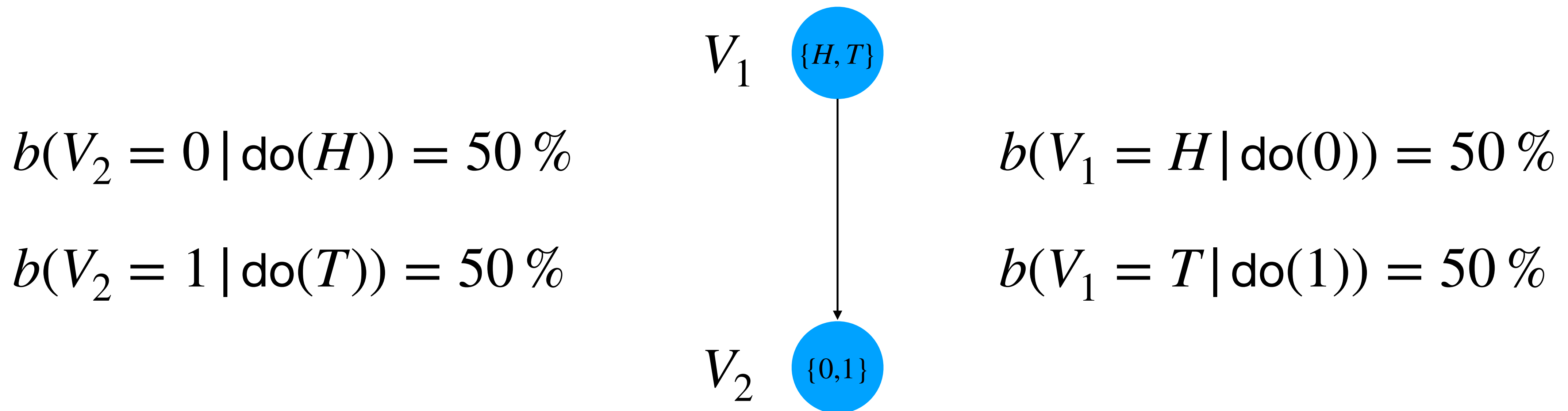
$$b(Y | \text{do}(V \setminus \{X, Y\})) = b(Y | \text{do}(V \setminus \{X, Y\}, X))$$

, i.e. intervening on  $X$  after intervening on  $V \setminus \{X, Y\}$  does not change the distribution of  $Y$ .

- Bareinboim, Brito, Pearl 2012 show that if  $G$  does not have an arrow going from  $X$  to  $Y$  then  $\text{ZDE}(X, Y)$ .
- However,  $\text{ZDE}(X, Y)$  does not imply that  $G$  does not have an arrow going from  $X$  to  $Y$ .

# Example

$V_1$  and  $V_2$  are fair iid Bernoullis



# ZDE Faithfulness

- We assume the agent's beliefs satisfy *ZDE faithfulness*, that is:

$\text{ZDE}(X, Y)$  holds iff  $G$  does not have an arrow going from  $X$  to  $Y$ .

- Given ZDE faithfulness,  $G$  is identifiable from  $b^*$ !
- Just need to know  $b(V_i \mid \text{do}(V \setminus \{V_i\}))$  for each  $i$ .



# Scoring Rules

- Suppose an agent's beliefs over  $[R] = \{1, \dots, n\}$  are given by  $\vec{b} \in \Delta^n$ .
- A *scoring rule* is a function  $s : \Delta^n \rightarrow \mathbb{R}^n$  that for all reports  $r \in \Delta^n$  returns a vector  $\vec{s}(r)$ .
- A *proper* scoring rule is a scoring rule  $s : \Delta^n \rightarrow \mathbb{R}^n$  such that  $\forall \vec{b} \in \Delta^n$ :

$$\vec{b} \in \operatorname{argmax}_{r \in \Delta^n} \vec{b} \cdot \vec{s}(r)$$

$$\max_{r \in \Delta^n} \vec{b} \cdot \vec{s}(r) \geq 0.$$

- A *strictly proper* scoring rule is a proper scoring rule  $s : \Delta^n \rightarrow \mathbb{R}^n$  such that  $\forall \vec{b} \in \Delta^n$ :

$$\{\vec{b}\} = \operatorname{argmax}_{r \in \Delta^n} \vec{b} \cdot \vec{s}(r).$$

# Mechanism Design

- Using a strictly proper scoring rule  $s^i : \Delta^{[V_i]} \rightarrow \mathbb{R}^{[V_i]}$ , the designer can  $\forall v_{-i} \in [V \setminus \{V_i\}]$  successfully elicit:

$$b(V_i \mid \text{do}(V \setminus \{V_i\} = v_{-i}))$$

by promising to interventionally realize  $v_{-i}$ .

- However, one might worry:
  1. Interventions are costly to perform
  2. The agent could learn from each experiment.

# Mechanism #1

- We can bypass both issues!
- The designer can promise to randomize uniformly over which of the:

$$Z = \sum_{i=1}^K |[V \setminus \{V_i\}]|$$

scoring rules will actually pay out and perform only its intervention.

# Objection #1

- The agent might incur a cognitive cost when processing each of the counterfactual conditions in mechanism #1.
- As  $Z$  increases, the agent's expected reward per bit of information on each scoring rule will shrink while processing costs stay fixed.
- This threatens incentive compatibility as the agent will just report their non-interventional beliefs for each scoring rule.

# Objection #1

- We can formalize this objection in the context of when each scoring rule  $s^i : \Delta^{|[V_i]|} \rightarrow \mathbb{R}^{|[V_i]|}$  is of the form:

$$s_v^i(r) = B \cdot \log_2 \left( \frac{r_v}{1/|[V_i]|} \right)$$

, i.e. a log scoring rule. Here, the designer's worst case loss is:

$$B \cdot \log_2(\max_{i \in K} |[V_i]|)$$

# Objection #1

- We can cash out the cost of processing the counterfactual condition in the scoring rule designed to elicit  $b(V_i | \text{do}(V \setminus \{V_i\} = v_{-i}))$  as:

$$C \cdot \text{KL}(b_{V_i | \text{do}(v_{-i})} || b(V_i))$$

- If the agent chooses to not process the counterfactual condition  $b(V_i)$  is reported instead.

# Objection #1

- One can show the expected benefit of processing the counterfactual condition for each scoring rule is:

$$\frac{B}{Z} \cdot \text{KL}(b_{V_i|\text{do}(v_{-i})} || b(V_i)).$$

- Hence, mechanism #1 will be incentive compatible iff  $B/Z > C$ .
- Furthermore, the infimum worst case loss for mechanism #1 to be incentive compatible is:

$$(C \cdot Z) \cdot \log_2(\max_{i \in K} |[V_i]|)$$

# Objection #1

- To illustrate the issue, consider when each  $[V_i] = \{0,1\}$ .
- The infimum worst case loss is:

$$C \cdot K \cdot 2^{K-1}.$$



# Lesson

- We need a mechanism that randomizes over fewer interventions.
- The Peter-Clark algorithm reconstructs a *partially oriented causal skeleton* from the joint distribution.
- This drastically reduces the number of interventions needed to identify  $G$ .
- Idea: Assume the agent's beliefs satisfy some kind of causal faithfulness!

# Causal Faithfulness

- $\forall A \subseteq V$  let  $\text{do}(A_r)$  randomize uniformly over all  $a \in A$ .
- Let  $G_A$  be the DAG obtained by removing all edges going into  $A$  from  $G$ .
- We say an agent's beliefs satisfy *causal faithfulness* if  $\forall A \subseteq V$  the intervention  $\text{do}(A_r)$  induces a distribution  $b(V \mid \text{do}(A_r))$  where:

$X$  and  $Y$  are dependent conditional on every subset  $V' \subseteq V \setminus \{X, Y\}$

iff

$X$  and  $Y$  are adjacent in  $G_A$ .

# Tests

- An *adjacency test* for  $X, Y \in V$  is an intervention  $\text{do}(A_r)$  such that  $X, Y \notin A$ .
- A *directional test* for  $X, Y \in V$  is an intervention  $\text{do}(A_r)$  such that  $X \in A$  or  $Y \in A$  but not both.
- Eberhardt, Glymour, Scheines 2005 note that as long as each pair of variables is subject to one adjacency test and one directional test we can orient all edges in  $G$ .

# Tests

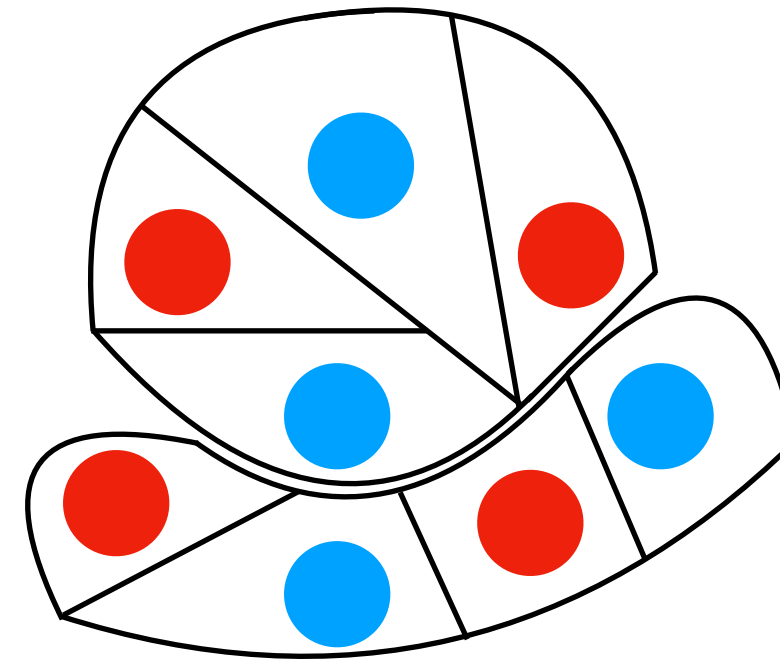
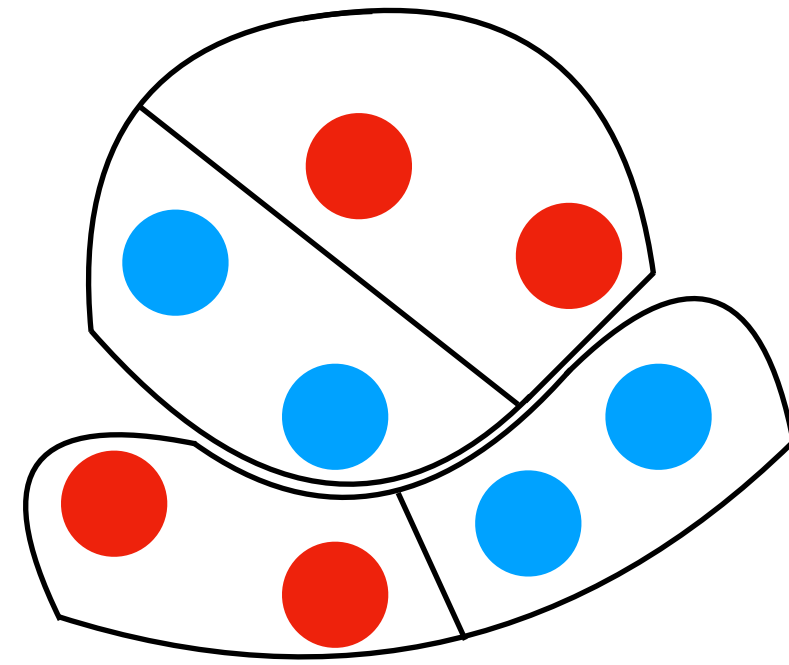
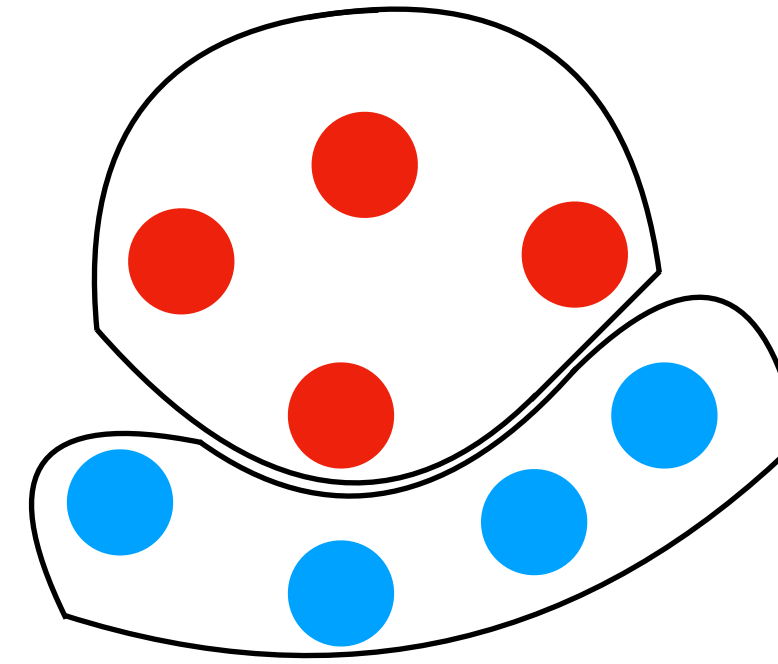
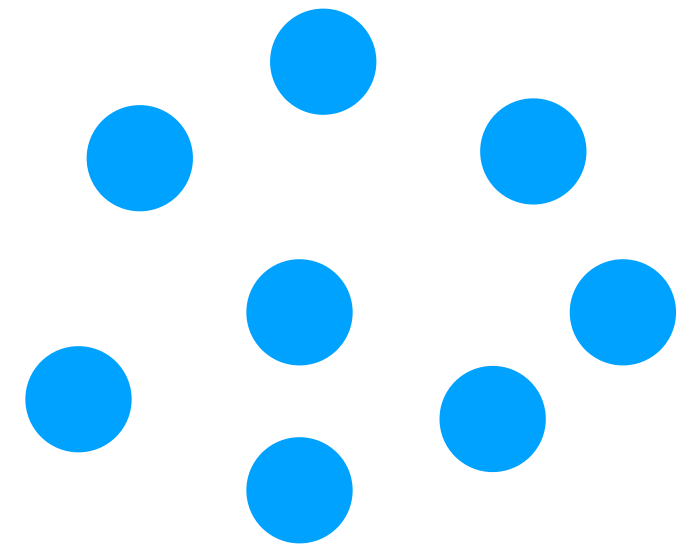
- Towards that end, they form a sequence of subsets  $A \subseteq V$  corresponding to interventions  $\text{do}(A_r)$  as follows:

$\text{seq}(V) :=$

1. If  $|V| = 1$  return  $\langle \emptyset \rangle$ .
2. Else, partition  $V$  into sets  $V^1$  and  $V^2$  where  $V^1 = \lfloor |V|/2 \rfloor$  and  $V^2 = \lceil |V|/2 \rceil$ .
3. Return  $\langle V^1 \rangle + (\text{seq}(V^1) \cup \text{seq}(V^2))$

where  $\cup$  first pads the shorter list with  $\emptyset$  at the end to make it equal in length to the longer list and takes their component-wise union.

# Tests



# Mechanism #2

- Using a strictly proper scoring rule  $s : \Delta^{[V]} \rightarrow \mathbb{R}^{[V]}$ , the designer can  $\forall A \in \text{seq}(V)$  successfully elicit:

$$b(V \mid \text{do}(A_r)).$$

- Thus the designer can promise to randomize uniformly over which of the  $|\text{seq}(V)| = \lceil 1 + \log_2(K) \rceil$  scoring rules will actually payout and perform only its intervention to elicit  $G$ .

# Mechanism #2

- Assuming each  $s : \Delta^{|[V]|} \rightarrow \mathbb{R}^{|[V]|}$  is a log scoring rule:

$$s_v(r) = B \cdot \log_2 \left( \frac{r_v}{1/|[V]|} \right)$$

we can show the infimum worst case loss for mechanism #2 to be incentive compatible is:

$$(C \cdot \lceil 1 + \log_2(K) \rceil) \cdot \log_2(|[V]|)$$

# Mechanism #2

- To see why this answers objection #1 consider when each  $[V_i] = \{0,1\}$ .
- The infimum worst case loss is now:

$$C \cdot K \cdot \lceil 1 + \log_2(K) \rceil$$

- Significant improvement from earlier:

$$C \cdot K \cdot 2^{K-1}$$

- In fact, no matter how many outcomes each  $V_i$  has we can always show:

$$\frac{\text{infimum worst case loss for mechanism \#2 to be IC}}{\text{infimum worst case loss for mechanism \#1 to be IC}} \leq \frac{\lceil 1 + \log_2(K) \rceil}{2^{K-1}}$$



# Objection #2

- The PC algorithm can reconstruct a partially oriented skeleton of  $S$  from a given joint distribution.
- Let  $S^*$  denote the set of vertices which belong to an unoriented edge in the skeleton.
- It suffices now to subject every pair of variables in  $S^*$  to a directional test.
- Wouldn't this allow us to randomize over even fewer scoring rules?

# Mechanism #3

- Step #1- The agent reports to a scoring rule  $s : \Delta^{[V]} \rightarrow \mathbb{R}^{[V]}$  which pays out in case the designer performs no intervention.
- Step #2- The designer:
  - Reconstructs a partially oriented skeleton  $S$  from the report in step #1.
  - Computes  $\text{seq}(S^*)$  and  $\forall A \in \text{seq}(S^*) \setminus \{\emptyset\}$ , allows the agent to report to a scoring rule  $s : \Delta^{[V]} \rightarrow \mathbb{R}^{[V]}$  which pays out in case the designers performs the intervention  $\text{do}(A_r)$ .
- Step #3- The designer randomizes uniformly over which of the  $|\text{seq}(S^*)|$  interventions to perform and pays out the agent.

# Objection #3

- While  $|\text{seq}(S^*)| \leq |\text{seq}(V)|$ , IC cannot be established.
- Agents can leverage their knowledge that the PC algorithm determines which interventions the designer will randomize over.
- Namely, if they know more about certain interventions than others, they may prefer the PC algorithm to output a skeleton different from the one they truly believe.
- So, mechanism #3 must assume agents are *myopic* to guarantee IC, i.e. agents ignore profits from future actions at each stage of the game.

# Possible Solution

- Maybe we can mitigate the incentive to lie in step #1 by making the reward sizes for the scoring rules in step #2 small in comparison.
- Formally, suppose the scoring rule in step 1,  $s^{t_1} : \Delta^{|[V]|} \rightarrow \mathbb{R}^{|[V]|}$ , is of the form:

$$s_v^{t_1}(r) = B_1 \cdot \log_2 \left( \frac{r_v}{1/|[V]|} \right)$$

while the scoring rules in step 2,  $s^{t_2} : \Delta^{|[V]|} \rightarrow \mathbb{R}^{|[V]|}$ , are of the form:

$$s_v^{t_2}(r) = B_2 \cdot \log_2 \left( \frac{r_v}{1/|[V]|} \right)$$

# Possible Solution

- $B_2 \cdot \log_2(|[V]|)$  is the maximum profit an agent can obtain from a scoring rule used in step #2.
- $B_1 \cdot \text{KL}(b(V) || r)$  is the cost of reporting  $r$  instead of  $b(V)$  to the scoring rule in step #1.
- In equilibrium then:

$$\text{KL}(b(V) || r) \leq \frac{B_2}{B_1} \cdot \log_2(|[V]|).$$

- Since making  $B_1 \gg B_2$  implies  $r \approx b(V)$ , we might hope for *approximate* IC.

# Possible Solution

- Without additional assumptions, this may be no more than a pipe dream.
- The causal skeleton output by the PC algorithm could be highly sensitive to the input joint distribution.
- But even if we could find suitable assumptions, the presence of processing costs prevents us from decreasing  $B_2$  arbitrarily.
- Therefore, such an approach would likely increase  $B_1$  so high that the designer's worst case loss is greater than in mechanism #2.

# Summary

Mechanism	Identification Assumptions	Myopic IC	IC	Inf Worst Case Loss (Log Rules)
Mechanism #1	ZDE faithfulness	Yes	Yes	$(C \cdot Z) \cdot \log_2(\max_{i \in K}  [V_i] )$
Mechanism #2	Causal faithfulness	Yes	Yes	$(C \cdot \lceil 1 + \log_2(K) \rceil) \cdot \log_2( [V] )$
Mechanism #3	Causal faithfulness	Yes	No	$(C \cdot  \text{seq}(S^*) ) \cdot \log_2( [V] )$

# References

- Bareinboim, Brito, Pearl: [https://link.springer.com/chapter/10.1007/978-3-642-29449-5\\_1](https://link.springer.com/chapter/10.1007/978-3-642-29449-5_1)
- Spirtes, Glymour, Scheines: <https://direct.mit.edu/books/monograph/2057/Causation-Prediction-and-Search>
- Eberhardt, Glymour, Scheines: <https://arxiv.org/pdf/1207.1389>