Topological Semantics for Common Inductive Knowledge

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Abstract

Lewis' account of common knowledge in *Convention* describes the process which drives the generation of higher-order expectations as hinging upon agents' inductive standards and a shared witness. Subsequent formal treatments of common knowledge such as those in Aumann 1976 were largely deductive and took as primitives agents' information partitions over possible worlds. This project attempts to draw from insights in learning theory to provide a formal account of common inductive knowledge generated by a witness. Following Kelly's approach in *The Logic of Reliable Inquiry*, we take as primitives agents' information bases over possible worlds. After defining common inductive knowledge, we show how our semantics helps characterize the equilibria of an inductive coordination game.

Notation

- N is a set of agents and Ω is a set of possible worlds.
- $\mathcal{E}_i$ is a basis over Ω representing the possible information states of agent i.
- $\mathcal{T}_i$ is the topology generated by $\mathcal{E}_i$.
- $\mathcal{E}_{i|w}$ denotes those information states of $\mathcal{E}_i$ which contain w.
- $\mathcal{T}_i|E$ denotes the subspace topology of $\mathcal{T}_i$ restricted to E.

Difference Hierarchy and Learning

- $X \cap E$ is n-open in $\mathcal{T}_i|E$ iff i can limit decide X in n mind switches starting with 'no' in light of evidence E.
- $X \cap E$ is n-closed in $\mathcal{T}_i|E$ iff i can limit decide X in n mind switches starting with 'yes' in light of evidence E.
- $\mathrm{Yes}_i^n(X|E)$ holds iff $\exists k \leq n$ s.t. $X \cap E$ is k-closed but not $(k-1)$-open in $\mathcal{T}_i|E$ (0-open is $\varnothing$ and 0-closed is E).

Define an agent's *switching tolerance* n_i to be the maximum number of times they are willing to switch their mind after first saying 'Yes' while limit deciding a proposition. Then:

- $\mathrm{Yes}_i^{n_i}(X|E)$ holds iff i says 'yes' to X in light of evidence E.
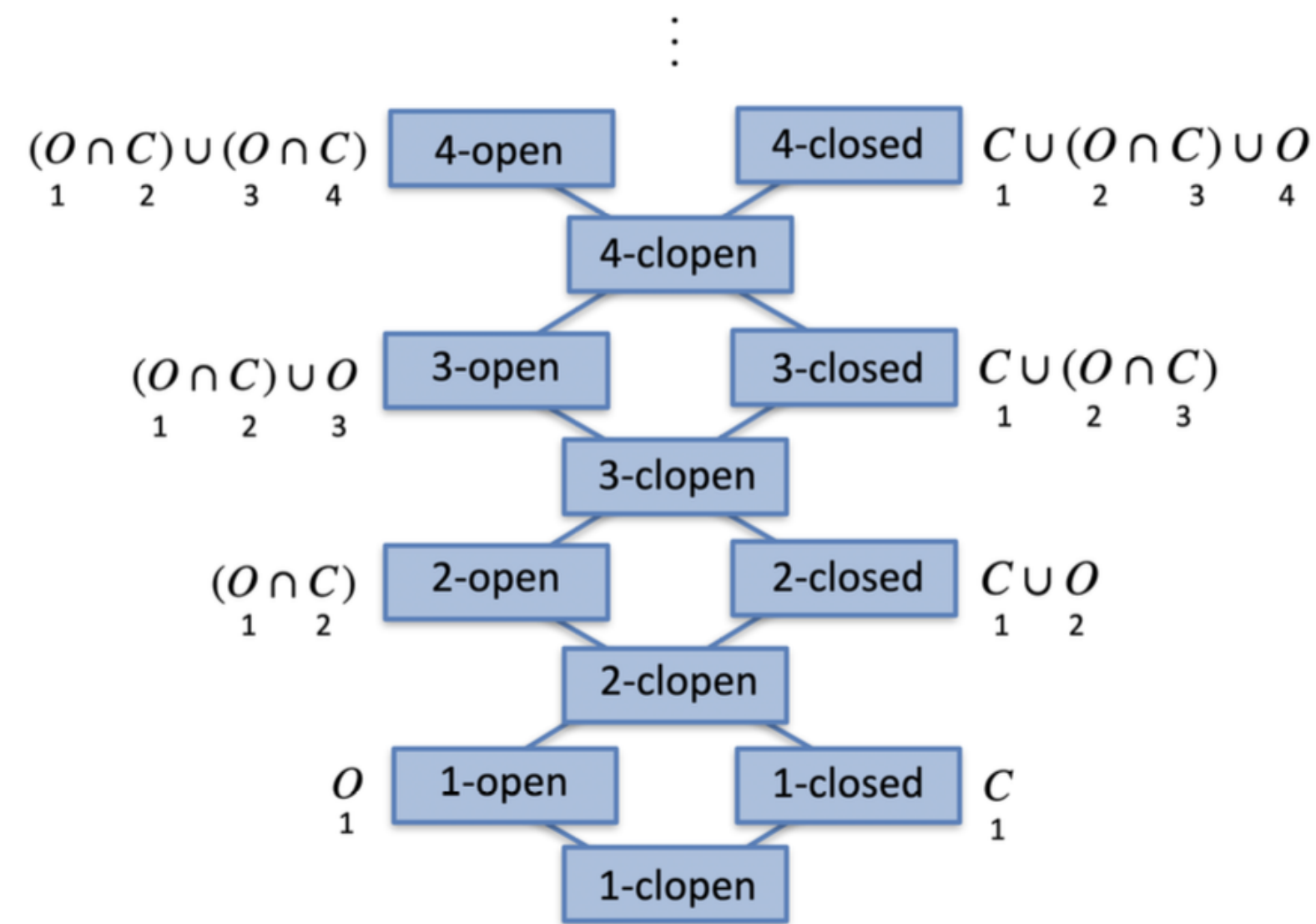
$\vdots$

Figure 1: Topological difference hierarchy

Epistemic Operators

The below list provides a gloss for each of the epistemic operators in our language:

- $\mathbb{R}_i W$: i has reason simpliciter to believe W.
- $\mathbb{I}_{i@W} P$: W indicates to i that P.
- $\mathbb{B}_{i@W} P$: i has W as reason to believe P.
- $\mathbb{E}_W P$: Everyone has W as reason to believe P.
- $\mathbb{G}_W P$: W generates common inductive knowledge of P.
- $\mathbb{C} P$: P is common inductive knowledge.

An Inductive Coordination Game

Two agents a and b are deciding whether to report a proposition $P \subseteq \Omega$ is true or defer judgement:

- a has switching tolerance n_a and b has switching tolerance n_b.
- If P is true, a and b want to both eventually converge on reporting P is true.
- If P is false, or only one of a or b converges on reporting P is true, then anyone who converges on reporting P is true is perpetually shamed.

Our agents want to find joint strategies that are guaranteed to protect both of them from being perpetually shamed!

Strategies

Each agent i's strategy s_i is an n_i-switching method m_i where:

- A *method* for agent i is a map $m_i : \mathcal{E}_i \to \{\mathrm{Yes}, ?\}$.
- An *t-switching sequence* for m_i is a finite downward sequence $E_0 \supseteq ... \supseteq E_t$ of information states in $\mathcal{E}_i$ such that $m_i(E_{2k}) = \mathrm{Yes}$ and $m_i(E_{2k+1}) = ?$.
- An *n-switching method* for i has no t-switching sequences for any $t > n$.

We let $\sigma_{s_i}(w)$ denote the output that s_i converges to in world w.

Payoffs

Given a strategy profile $s = (s_i)_{i \in N}$, define agent i's payoff at world w be given by:

$$u_{i|w}(s) = \begin{cases} 1 & w \in P \wedge \forall j \in N, \sigma_{s_j}(w) = \mathrm{Yes}; \\ 0 & \sigma_{s_i}(w) = ?; \\ -\infty & \sigma_{s_i}(w) = \mathrm{Yes} \wedge (w \notin P \vee \exists j \in N, \sigma_{s_j}(w) = ?) \end{cases}$$

Equilibria

In equilibrium, $\forall i, j \in N$, we have $\sigma_{s_i}^{-1}(\mathrm{Yes}) = \sigma_{s_j}^{-1}(\mathrm{Yes}) \subseteq P$. Moreover, there exists a strategy profile such that $\sigma_{s_i}^{-1}(\mathrm{Yes}) = X$ iff X is $(n_i + 1)$-open in $\mathcal{T}_i$. Thus :

- Equilibria are characterized by agents choosing some subset X of P which is $(n_i + 1)$-open in each $\mathcal{T}_i$.
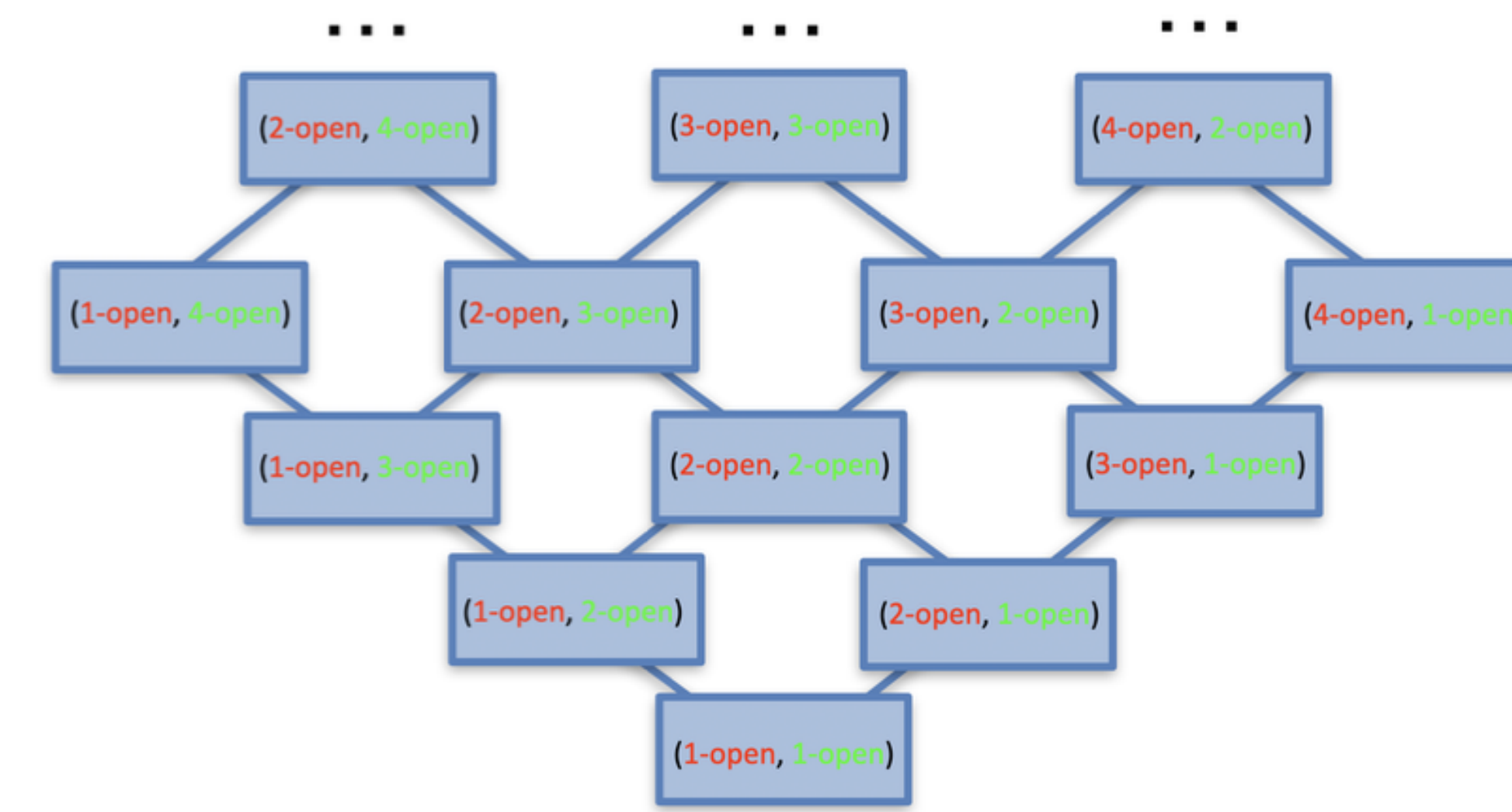
But what are these subsets?

Topological Semantics

The semantics of our epistemic operators are as follows:

- $w \in \mathbb{R}_i W$ iff $\exists E \in \mathcal{E}_{i|w}, \mathrm{Yes}_i^{n_i}(W|E)$.
- $w \in \mathbb{I}_{i@W} P$ iff $\forall E \in \mathcal{E}_{i|w}, \mathrm{Yes}_i^{n_i}(W|E) \to W \cap E \subseteq P$.
- $w \in \mathbb{B}_{i@W} P$ iff $w \in \mathbb{R}_i W \cap \mathbb{I}_{i@W} P$.
- $w \in \mathbb{E}_W P$ iff $w \in \bigcap_{i \in N} \mathbb{B}_{i@W} P$.
- $w \in \mathbb{E}_W^1 P$ iff $w \in \mathbb{E}_W P$.
- $w \in \mathbb{E}_W^n P$ iff $w \in \mathbb{E}_W \mathbb{E}_W^{n-1} P$.
- $w \in \mathbb{G}_W P$ iff $w \in \bigcap_{n \in \mathbb{N}^+} \mathbb{E}_W^n P$.
- $w \in \mathbb{C} P$ iff $\exists W \subseteq \Omega, w \in W \cap \mathbb{G}_W P$.

Figure 2: Multitopological difference lattice for a and b

Results

Theorem 1: X is a subset of P which is $(n_i + 1)$-open in each $\mathcal{T}_i$ iff it is a fixed point of the map $X \mapsto X \cap \mathbb{G}_X P$. Further $\forall W \subseteq \Omega$, $W \cap \mathbb{G}_W P$ is itself such a fixed point. This means:

- The map $X \mapsto X \cap \mathbb{G}_X P$ can be used to select any subset of P which is $(n_i + 1)$-open in each $\mathcal{T}_i$.
- $\mathbb{C} P$ consists of those worlds where agents converge to reporting 'yes' in *some* equilibrium of our game.

Theorem 2: $\mathbb{C} P$ does not depend on agents' switching tolerances so long as each $n_i > 0$. To be clear:

- Agents might converge to reporting 'yes' on a new *set* in some equilibrium by increasing their switching tolerances.
- However, somewhat counterintuitively, there will be no new *worlds* in any of these sets.

Conclusion

This project provides a topological semantics for capturing how common inductive knowledge is generated by a witness. Our basic epistemic operators are directly motivated by Lewis' analysis of common knowledge in *Convention*. We find that the resulting semantics elegantly characterizes the equilibria of an inductive coordination game which is closely related to structure we call the multitopological difference lattice. Going forward, we hope to develop a sound and complete proof system to accompany our semantics. Such an effort will not only help prove and verify new theorems about our particular language, but it will also furnish a more systematic understanding of how conventions in general may come to be formed.

Additional Information

Please contact me if you wish to see a current draft.

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Acknowledgements

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