Resource Bounded Randomness for Instantiating Cryptographic Security

Siddharth Namachivayam

Signatures

- Why do we sign messages? What is the social function of a signature?
- Signatures allow us to establish *public knowledge* of who said what.
- But how is public knowledge established?
- A good signature better be *unforgeable*.
- Extremely important for the operation of legal and financial contracts.

Cryptographic Security

- Unfortunately, traditional handwritten signatures can be easily forged.
- Modern cryptography provides a potential solution (e.g. RSA).
- But what guarantees does a signature scheme like RSA actually provide?
- Cryptographers appeal to a variety of different *security notions* to cash out in what sense a particular scheme is “unforgeable.”
- We now formalize what is meant by a signature scheme and present the security notion we will study.

Housekeeping

- $\{0,1\}^*$ denotes the set of finite binary strings.
- $\{0,1\}^\infty$ denotes the set of infinite binary sequences (Cantor space).
- A *variable-hash* is a function $H : \mathbb{N}^+ \times \{0,1\}^* \to \{0,1\}^*$ such that $\forall (n, x) \in \mathbb{N}^+ \times \{0,1\}^*, |H(n,x)| = n$.
- A *n-hash* is a function $h : \{0,1\}^* \to \{0,1\}^*$ such that $\forall x \in \{0,1\}^*, |h(x)| = n$.
- $H_n(\_)$ will be shorthand for $H(n, \_)$.
- $a^{-1} : \{0,1\}^* \to \mathbb{N}^+$ enumerates strings according to the lexicographic order.
- $b^{-1} : \mathbb{N}^+ \times \mathbb{N}^+ \to \mathbb{N}^+$ is the bijection $(m, n) \mapsto (m + n - 2)(m + n - 1)/2 + n$.
- $c : \mathbb{N}^+ \to \mathbb{N}^+ \times \{0,1\}^*$ is the bijection $n \mapsto (b_1(n), a(b_2(n)))$.
- We can establish a bijection between variable-hashes and elements of $\{0,1\}^\infty$ by mapping $H \mapsto H(c(1))H(c(2))H(c(3))\ldots$
- We will often identify variable-hashes with elements of $\{0,1\}^\infty$ using this bijection.

Signature Schemes

- A tuple $\Pi = (Gen, Sign, Vrfy)$ of algorithms is a *signature scheme* if:
  - #1- Gen:
    - Is a probabilistic algorithm.
    - Takes as input a positive natural n.
    - Outputs a public/private key pair (pk, sk) denoted by the r.v. $Gen(n)$.
    - Is polynomial time in n.

Signature Schemes

- A tuple $\Pi = (Gen, Sign, Vrfy)$ of algorithms is a *signature scheme* if:
  - #2- $Sign$:
    - Is a probabilistic algorithm.
    - Seeks oracle access to an n-hash h.
    - Takes as input a private key sk and a message m.
    - Outputs a signature σ denoted by the r.v. $Sign_{sk}^{h}(m)$.
    - Is polynomial time in n.

Signature Schemes

- A tuple $\Pi = (Gen, Sign, Vrfy)$ of algorithms is a *signature scheme* if:
  - #3- $Vrfy$:
    - Is a deterministic algorithm.
    - Seeks oracle access to an n-hash h.
    - Takes as input a public key pk, a message m, and a signature σ.
    - Outputs a single bit 1 or 0 denoted by $Vrfy^h_{pk}(m, \sigma)$.
    - Is polynomial time in n.

Signature Schemes

- A tuple $\Pi = (Gen, Sign, Vrfy)$ of algorithms is a *signature scheme* if:
  - #4- For every message m, for every variable-hash H, for every natural $n \in \mathbb{N}^+$, for every (pk, sk) in the support of $Gen(n)$ we have:

$$Vrfy_{pk}^{H_n}(m, Sign_{sk}^{H_n}(m)) = 1.$$

Honest Execution

- Step Zero: Everyone agrees to use a common security parameter n on a public variable-hash H.

- Step One: Signer announces the public key pk realized by $Gen(n)$ while keeping the secret key sk to themselves.

- Step Two: Signer chooses a desired message m and attaches the signature σ realized by $Sign_{sk}^{H_n}(m)$ to m before announcing it.

- Step Three: Verifiers trust the signer sent m iff running $Vrfy_{pk}^{H_n}(m, \sigma)$ yields 1.

Adversaries

- An algorithm $\mathscr{A}$ is an *adversary* if it:
  - Is a probabilistic algorithm.
  - Seeks oracle access to a n-hash h as well as a signature function $\Sigma : \{0,1\}^* \to \{0,1\}^*$.
  - Takes as input a public key pk.
  - Outputs a message/signature pair (m, σ)
  - Is polynomial time in n.

Adaptive Chosen-Message Attack

- Suppose everyone has agreed to use a common security parameter n and public variable-hash H.
- Given an adversary $\mathscr{A}$ and a signature scheme Π, define the probabilistic algorithm $SigForge_{\mathscr{A},\Pi}(n, H)$ as follows:
  - First store the pair (pk, sk) realized by $Gen(n)$.
  - Next $\mathscr{A}$ obtains oracle access to H_n as well as $Sign_{sk}^{H_n}(\_)$ and is given pk as input.
  - Let $\mathscr{Q}$ be the set of messages $\mathscr{A}$ queries to the oracle $Sign_{sk}^{H_n}(\_)$ during its execution.
  - After the adversary terminates and outputs (m, σ), check to see if $m \in \mathscr{Q}$. If so return 0.
  - Otherwise, check to see if $Vrfy_{pk}^{H_n}(m, \sigma) = 1$. If so return 1. Otherwise return 0.
- Roughly if $SigForge_{\mathscr{A},\Pi}(n, H) = 0$ then $\mathscr{A}$, *even after receiving examples of valid signatures on many other messages,* was not able to successfully forge a signature on a new message of its own choosing.

EUF-ACMA Security

- Say a signature scheme Π is "existentially unforgeable under an adaptive chosen-message attack" or, more simply, ***EUF-ACMA secure relative to a variable-hash*** H if for all adversaries $\mathscr{A}$ and all $d \in \mathbb{N}^+$, for sufficiently large n we have:

$$P(Sigforge_{\mathscr{A},\Pi}(n, H) = 1) \leq \frac{1}{n^d}.$$

- EUF-ACMA security relative to a variable-hash H is difficult to prove but it entails many other kinds of security.

The Random Oracle Model

- Now, say $\mathbb{H}$ is a r.v. which follows the uniform distribution over $\{0,1\}^\infty$.
- We can also view $\mathbb{H}$ as a r.v. uniformly distributed over all variable-hashes.
- A signature scheme Π is ***EUF-ACMA secure in the random oracle model*** if for all adversaries $\mathscr{A}$ and all $d \in \mathbb{N}^+$, for sufficiently large n we have:

$$P(Sigforge_{\mathscr{A},\Pi}(n, \mathbb{H}) = 1) \leq \frac{1}{n^d}.$$

- EUF-ACMA security in the random oracle model (ROM) is much easier to prove than EUF-ACMA security relative to a variable-hash H.
- Under standard assumptions, Bellare & Rogaway 1993 proved RSA-FDH is EUF-ACMA secure in the ROM.
- Still open whether a polynomial time variable-hash H exists relative to which RSA-FDH is EUF-ACMA secure...

Instantiating the ROM

- Question: If Π is proved EUF-ACMA secure in the ROM can we always find **some** variable-hash H relative to which Π is EUF-ACMA secure?

- Tadaki & Doi 2015: "**Yes!** Algorithmic randomness is the perfect tool."

- Follow Up: What about a *computable* H?

- Tadaki & Doi 2015: "We don't know…"

- This Talk 2025: "Yes! Resource bounded randomness is the perfect tool."

Instantiating the ROM

- Another Follow Up: What about a *polynomial time computable* H?
- Canetti, Goldreich, & Halevi 2002: "No!"
- This Talk 2025: "I would bet that there is an elementary f such that we can find a $O(f(n))$ time computable H."
- Final Question: What's the *best* complexity guarantee we can establish?
- This Talk 2025: "I don't know…perhaps you have intuitions!"

Algorithmic Randomness

- What makes an element of $\{0,1\}^\infty$ *random*?
- Given a r.v. like $\mathbb{H}$, probability theory helps us gauge the relative likelihood its outcomes.
- However, probability theory does not capture all our intuitions about randomness.
- Consider the following sequences:

10101010101010…

01001101110100…

- When pressed most would say the second one is ‘more random’ than the first. Why?
- The first has a rare property of never seeing two 0’s in a row (a probability 0 event) while the second one does not.
- So maybe a sequence is random iff one cannot specify a rare property it possesses.
- But then all sequences have the rare property of being themselves…so no sequence is random?

Algorithmic Randomness

- Algorithmic randomness responds as follows:

An element of $\{0,1\}^\infty$ *is random iff one cannot effectively specify a sequence of rarer and rarer properties that it possesses.*

- Many ways of cashing out this answer and traditional approaches take 'effectively' to mean 'computably.'

- As a result, traditional algorithmic randomness notions require that random elements of $\{0,1\}^\infty$ be uncomputable.

Tadaki & Doi's Meta-Question

- Cryptography asks: which variable-hashes H can instantiate the security provided by $\mathbb{H}$?

- Algorithmic randomness asks: which elements of $\{0,1\}^\infty$ are random realizations of $\mathbb{H}$?

- Tadaki & Doi 2015 ask: are these the same question?

Tadaki & Doi's Result

- Martin-Löf (ML) randomness is often taken to be the 'one true' algorithmic randomness notion.

- Tadaki & Doi 2015 show if a variable-hash H is Martin-Löf random then *every* scheme Π which is proved EUF-ACMA secure in the ROM is also EUF-ACMA secure relative to H.

- Note, H is *scheme agnostic*.

- Sadly however, Martin-Löf randoms are uncomputable.

- They note Schnorr randoms also work, but these too are uncomputable.

Tadaki & Doi's Conjecture

- Tadaki & Doi 2015 conjectured that every scheme Π which is proved EUF-ACMA secure in the ROM is also EUF-ACMA secure relative to some *computable* variable-hash H.
- I have recently shown that this conjecture is true using resource bounded randomness!

Resource Bounded Randomness

- Recall our pretheoretic definition of randomness:

An element of $\{0,1\}^\infty$ *is random iff one cannot effectively specify a sequence of rarer and rarer properties that it possesses.*

- Resource bounded randomness researchers say there is no real reason to cash out ‘effectively’ as ‘computably.’
- ‘Effectively’ could mean ‘primitive recursively’ or ‘exponential time computably’ or ‘polynomial space computably’ etc.
- This means, **resource bounded randomness notions can have computable instances**.
- I believe resource bounded randomness is the correct paradigm for studying the scheme agnostic instantiation of cryptographic security.

Primitive Recursive Schnorr Randomness

- A sequence of subsets $\{U_i\}_{i\in\mathbb{N}^+}$ of $\{0,1\}^\infty$ is *primitive recursively approximable* if there is an array of subsets $\{S_{i,j}\}_{(i,j)\in\mathbb{N}^+\times\mathbb{N}^+}$ of $\{0,1\}^*$ such that:

  - $\#1$- There is a primitive recursive function $f : \mathbb{N}^+ \times \mathbb{N}^+ \to \mathbb{N}^+$ so $\forall (i,j) \in \mathbb{N}^+ \times \mathbb{N}^+$ we have:

$$\max\{\,|w| : w \in S_{i,j}\} \leq f(i,j).$$

Primitive Recursive Schnorr Randomness

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  - $\#2$- The language $L = \{\langle w, i, j \rangle : w \in S_{i,j}\}$ is primitive recursive, i.e. there is a primitive recursive algorithm which decides membership in L.

Primitive Recursive Schnorr Randomness

- A sequence of subsets $\{U_i\}_{i \in \mathbb{N}^+}$ of $\{0,1\}^\infty$ is *primitive recursively approximable* if there is an array of subsets $\{S_{i,j}\}_{(i,j) \in \mathbb{N}^+ \times \mathbb{N}^+}$ of $\{0,1\}^*$ such that:

  - #3- $\forall i \in \mathbb{N}^+$ we have:

$$U_i = \bigcup_{j \in \mathbb{N}^+} [S_{i,j}]$$

where $[S_{i,j}]$ denotes all the infinite extensions of strings in $S_{i,j}$.

Primitive Recursive Schnorr Randomness

- A sequence of subsets $\{U_i\}_{i \in \mathbb{N}^+}$ of $\{0,1\}^\infty$ is *primitive recursively approximable* if there is an array of subsets $\{S_{i,j}\}_{(i,j) \in \mathbb{N}^+ \times \mathbb{N}^+}$ of $\{0,1\}^*$ such that:
  - #4- $\forall (i,j) \in \mathbb{N}^+ \times \mathbb{N}^+$ we have:

$$\lambda\left(U_i - \bigcup_{k=1}^{j} [S_{i,k}] \right) \leq 2^{-j}.$$

Primitive Recursive Schnorr Randomness

- A *primitive recursive Schnorr test* is a descending sequence of subsets $\{U_i\}_{i \in \mathbb{N}^+}$ of $\{0,1\}^\infty$ such that $\{U_i\}_{i \in \mathbb{N}^+}$ is primitive recursively approximable and $\lambda(U_i) \leq 2^{-i}$.

- A sequence $\alpha \in \{0,1\}^\infty$ *passes* a primitive recursive Schnorr test $\{U_i\}_{i \in \mathbb{N}^+}$ if $\alpha \notin \bigcap_{i \in \mathbb{N}^+} U_i$.

- A sequence $\alpha \in \{0,1\}^\infty$ is *primitive recursive Schnorr random* if it passes all primitive recursive Schnorr tests.

Proof Strategy

- Henceforth, suppose Π is EUF-ACMA secure in the ROM:
  - Π is *not* EUF-ACMA secure relative to H iff there exists an adversary $\mathscr{A}$ and $d \in \mathbb{N}^+$ so that for infinitely many n:

$$P(Sigforge_{\mathscr{A},\Pi}(n, H) = 1) > \frac{1}{n^d}.$$

  - We will show for all adversaries $\mathscr{A}$ and $d \in \mathbb{N}^{>1}$ there exists $N \in \mathbb{N}^+$ so that the sequence of sets:

$$U_i = \bigcup_{n \geq N+2^{i+1}} \left\{ H : \; P(Sigforge_{\mathscr{A},\Pi}(n, H) = 1) > \frac{1}{n^d} \right\}$$

    is a primitive recursive Schnorr test.

  - It follows that if H is primitive recursive Schnorr random then Π is EUF-ACMA secure relative to H.

Proof Strategy

- Lemma (Tadaki & Doi 2015)— Fix an adversary $\mathscr{A}$ and $d \in \mathbb{N}^{>1}$. There exists an $N \in \mathbb{N}^{+}$ so that $\forall M \geq N$:

$$\lambda\left(\bigcup_{n>M}\left\{H: \; P(Sigforge_{\mathscr{A},\Pi}(n,H)=1) > \frac{1}{n^d}\right\}\right) \leq \frac{2}{M}.$$

- Thus, if we define:

$$U_i = \bigcup_{n \geq N+2^{i+1}} \left\{H: \; P(Sigforge_{\mathscr{A},\Pi}(n,H)=1) > \frac{1}{n^d}\right\}$$

then:

$$\lambda\left(U_i\right) \leq \frac{2}{2^{i+1}+N} \leq 2^{-i}.$$

- Since $\{U_i\}_{i \in \mathbb{N}^+}$ is clearly descending, all that remains is to show $\{U_i\}_{i \in \mathbb{N}^+}$ is primitive recursively approximable.

Constructing an Array

- Since $Sign$, $Vrfy$, and $\mathscr{A}$ are all polynomial time in n, we can upper bound the maximum of their running times by some polynomial $q(n)$.

- This also means for all variable-hashes H, we can upper bound the length of inputs given to H_n while running $SigForge_{\mathscr{A},\Pi}(n, H)$ by $q(n)$.

- Thus to determine whether:

$$P(Sigforge_{\mathscr{A},\Pi}(n, H) = 1) > \frac{1}{n^d}$$

it suffices for H_n to be defined only on inputs of length less than or equal to $q(n)$.

Constructing an Array

- Define the primitive recursive algorithm $reqLen(n)$ to return the minimum length required for a string w so that given any two infinite extensions w^1 and w^2, the n-hashes w_n^1 and w_n^2 are identical on all inputs of length less than or equal to $q(n)$.

- Next, define the primitive recursive algorithm $decideL(w, i, j)$ as follows:

  - For n in $[N + 2^{i+j}, N + 2^{i+j+1})$:

    - If $|w| = reqLen(n)$ and $P(Sigforge_{\mathscr{A},\Pi}(n, w\overline{0}) = 1) > \frac{1}{n^d}$, return 1.

  - Otherwise, return 0.

- Finally, define $S_{i,j}$ so that $w \in S_{i,j}$ iff $decideL(w, i, j) = 1$.

Constructing an Array

- Note:
  - #1- max$\{\,|w|\,:\,w \in S_{i,j}\} \leq$ max$\{reqLen(n)\,:\,n \in [N + 2^{i+j}, N + 2^{i+j+1})\}$.
  - #2- $decideL(w, i, j)$ decides whether $w \in S_{i,j}$.
  - #3- $U_i = \bigcup_{j \in \mathbb{N}^+} [S_{i,j}]$ as:

$$[S_{i,j}] = \bigcup_{n \in [N+2^{i+j}, N+2^{i+j+1})} \left\{ H :\ P(Sigforge_{\mathscr{A},\Pi}(n, H) = 1) > \frac{1}{n^d} \right\}.$$

  - #4- $\lambda\left(U_i - \bigcup_{k=1}^{j} [S_{i,k}]\right) \leq \lambda\left(U_{i+j}\right) \leq 2^{-(i+j)} \leq 2^{-j}$.
- In other words, $\{U_i\}_{i \in \mathbb{N}^+}$ is primitive recursively approximable!

Taking Stock

- Recap: we just showed that if a variable-hash H is primitive recursive Schnorr random then *every* scheme Π which is proved EUF-ACMA secure in the ROM is also EUF-ACMA secure relative to H.

- Why do we know there exist computable primitive recursive Schnorr randoms?

- To answer this we now present a *martingale characterization* of primitive recursive Schnorr randomness.

Martingale Characterization

- A function $m : \{0,1\}^* \to \mathbb{R}^{\geq 0}$ satisfies *the martingale property* if $m(\varepsilon) > 0$ and for all $w \in \{0,1\}^*$:

$$m(w) = \frac{m(w0) + m(w1)}{2}.$$

- Let $\mathbb{Q}_2^{\geq 0}$ denote the set of non-negative dyadic rationals.
- A *primitive recursive martingale* is a primitive recursive function $m : \{0,1\}^* \to \mathbb{Q}_2^{\geq 0}$ that satisfies the martingale property.
- An *order* $h : \mathbb{N} \to \mathbb{N}$ is an unbounded non-decreasing function.
- Given an order h, we define its *inverse* $\mathrm{inv}_h(n)$ to return the smallest k for which $h(k) \geq n$.
- A *true primitive recursive order* is an order such that both itself and its inverse are primitive recursive.

Martingale Characterization

- Theorem (Sureson 2017): A sequence $\alpha \in \{0,1\}^\infty$ is primitive recursive Schnorr random iff for every primitive recursive martingale m and every true primitive recursive order h we have that $m(\alpha \upharpoonright n) \geq 2^{h(n)}$ for only finitely many n.

Putnam's Ghost

- Fact: One can recursively enumerate all primitive recursive martingales.
- Let m_i denote the i-th primitive recursive martingale in this enumeration.
- Let $m^* = \sum_{i \in \mathbb{N}^+} 2^{-i} \cdot \frac{m_i}{m_i(\varepsilon)}$.
- m^* is a computable function which satisfies the martingale property.
- We recursively construct a computable sequence β on which m^* is bounded as follows:

$$\beta(n) = \begin{cases} 1 & m^*(\beta(0)\ldots\beta(n-1)0) \geq m^*(\beta(0)\ldots\beta(n-1)) \\ 0 & m^*(\beta(0)\ldots\beta(n-1)1) > m^*(\beta(0)\ldots\beta(n-1)) \end{cases}$$

- Since m^* is bounded on β, each m_i is also bounded on β.
- Therefore, β is a computable primitive recursive Schnorr random.

Interlude

- Resource bounded randomness allows us to study the scheme agnostic instantiation of cryptographic security.
- Useful for the randomness notion to admit a test characterization as well as a martingale characterization.
- Test characterizations allow us to easily argue that the randomness notion suffices for instantiating cryptographic security.
- Martingale characterizations allow us to easily argue that the randomness notion has efficiently calculable instances.
- Future work should see if our results can be extended beyond primitive recursive Schnorr randomness…perhaps **polynomial space randomness**?

Interlude

This would let us find an $O(f(n))$ time computable H!

(Where f is elementary)

Interlude

- Unfortunately, polynomial space randomness only has a martingale characterization.

- I have already shown that the result can be extended if polynomial space randomness implies an alternative candidate polynomial space randomness notion defined in terms of a test characterization.

- We now define the candidate polynomial space randomness notion and sketch an outline of why it also suffices.

Candidate Polynomial Space Randomness

- A sequence of subsets $\{U_i\}_{i\in\mathbb{N}^+}$ of $\{0,1\}^\infty$ is *candidate polynomial space approximable* if there is an array of subsets $\{S_{i,j}\}_{(i,j)\in\mathbb{N}^+\times\mathbb{N}^+}$ of $\{0,1\}^*$ such that:
  - $\#1$- There is a polynomial space function $f:\mathbb{N}^+\times\mathbb{N}^+\to\mathbb{N}^+$ so $\forall(i,j)\in\mathbb{N}^+\times\mathbb{N}^+$ we have:

$$\max\{\,|w|:w\in S_{i,j}\}\leq f(i,j).$$

Candidate Polynomial Space Randomness

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  - $\#2$- The language $L = \{\langle w, i, j \rangle : w \in S_{i,j}\}$ is polynomial space, i.e. there is a polynomial space algorithm which decides membership in L.

Candidate Polynomial Space Randomness

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    - $\#3$- Same as before.

Candidate Polynomial Space Randomness

- A sequence of subsets $\{U_i\}_{i\in\mathbb{N}^+}$ of $\{0,1\}^\infty$ is *candidate polynomial space approximable* if there is an array of subsets $\{S_{i,j}\}_{(i,j)\in\mathbb{N}^+\times\mathbb{N}^+}$ of $\{0,1\}^*$ such that:

  - #4- $\forall(i,j) \in \mathbb{N}^+ \times \mathbb{N}^+$ we have:

$$\lambda\left(U_i - \bigcup_{k=1}^{j}[S_{i,k}]\right) \leq \frac{1}{j}.$$

Candidate Polynomial Space Randomness

- A *candidate polynomial space test* is a descending sequence of subsets $\{U_i\}_{i\in\mathbb{N}^+}$ of $\{0,1\}^\infty$ such that $\{U_i\}_{i\in\mathbb{N}^+}$ is candidate polynomial space approximable and $\lambda(U_i) \leq \frac{1}{i}$.

- A sequence $\alpha \in \{0,1\}^\infty$ *passes* a candidate polynomial space test $\{U_i\}_{i\in\mathbb{N}^+}$ if $\alpha \notin \bigcap_{i\in\mathbb{N}^+} U_i$.

- A sequence $\alpha \in \{0,1\}^\infty$ is *candidate polynomial space random* if it passes all candidate polynomial space tests.

Proof Strategy

- We will show if H is candidate polynomial space random then Π is EUF-ACMA secure relative to H.
- Lemma (Tadaki & Doi 2015)— Fix an adversary $\mathscr{A}$ and $d \in \mathbb{N}^{>1}$. There exists an $N \in \mathbb{N}^{+}$ so that $\forall M \geq N$:

$$\lambda\left(\bigcup_{n>M}\left\{H:\ P(Sigforge_{\mathscr{A},\Pi}(n,H)=1) > \frac{1}{n^d}\right\}\right) \leq \frac{2}{M}.$$

- Thus, if we define:

$$U_i = \bigcup_{n \geq N+2i}\left\{H:\ P(Sigforge_{\mathscr{A},\Pi}(n,H)=1) > \frac{1}{n^d}\right\}$$

then:

$$\lambda\left(U_i\right) \leq \frac{2}{2i+N} \leq \frac{1}{i}.$$

- Since $\{U_i\}_{i\in\mathbb{N}^+}$ is clearly descending, all that remains is to show $\{U_i\}_{i\in\mathbb{N}^+}$ is candidate polynomial space approximable.

Constructing an Array

- Define $reqLen(n)$ as before. It is polynomial space.
- Next, define the polynomial space algorithm $decideL(w, i, j)$ as follows:
  - For n in $[N + 2(i + j), N + 2(i + j) + 1]$:
    - If $|w| = reqLen(n)$ and $P(Sigforge_{\mathscr{A},\Pi}(n, w\overline{0}) = 1) > \dfrac{1}{n^d}$, return 1.
  - Otherwise, return 0.
- Finally, define $S_{i,j}$ so that $w \in S_{i,j}$ iff $decideL(w, i, j) = 1$.

Constructing an Array

- Note:
  - #1- max$\{\,|w|\,:\,w\in S_{i,j}\}\leq$ max$\{reqLen(n)\,:\,n\in[N+2(i+j),N+2(i+j)+1]$.
  - #2- $decideL(w,i,j)$ decides whether $w\in S_{i,j}$.
  - #3- $U_i=\bigcup_{j\in\mathbb{N}^+}[S_{i,j}]$ as:

$$[S_{i,j}]=\bigcup_{n\in[N+2(i+j),N+2(i+j)+1]}\left\{H:\ P(Sigforge_{\mathscr{A},\Pi}(n,H)=1)>\frac{1}{n^d}\right\}.$$

  - #4- $\lambda\left(U_i-\bigcup_{k=1}^{j}[S_{i,k}]\right)\leq\lambda\left(U_{i+j}\right)\leq\frac{1}{i+j}\leq\frac{1}{j}$.
- In other words, $\{U_i\}_{i\in\mathbb{N}^+}$ is candidate polynomial space approximable!

The Missing Piece

- A *polynomial space martingale* is a polynomial space function $m : \{0,1\}^* \to \mathbb{Q}_2^{\geq 0}$ that satisfies the martingale property.

- **Conjecture:** Take a sequence $\alpha \in \{0,1\}^\infty$. If for every polynomial space martingale m we have that $\limsup_{n \to \infty} m(A \upharpoonright n) < \infty$, i.e. α is polynomial space random, then α is candidate polynomial space random.

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