The equations of active inference

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ABOUT THIS BOOK

The active inference literature presents numerous equations used for modeling action and perception in continuous and discrete state-spaces. However, the proofs of these equations are either scattered in the literature (usually in supplemental material) or left as an exercise for the reader. The equations in the literature often use very different notational conventions making it difficult to make connections to other equations in the literature.

While preparing *Fundamentals of Active Inference*, the author (Sanjeev Namjoshi) derived many of these equations and consolidated those found in other papers and supplemental materials.[1] This book gathers all of these equations to provide a single compendium of proofs for every equation in the literature. This work was heavily edited and greatly improved by the work of Fraser Patterson who ensured the proofs were easy to follow and sound.

As this text is intended to serve as more of a reference than a textbook the reader is assumed to have basic familiarity with the active inference literature. Nevertheless, an overview of the basics is presented in the first chapter to orient the reader. The notational conventions are described in detail in the appendix.

[1] The vast majority of equations in *Fundamentals of Active Inference* are available in this text. The remaining proofs, largely for equations that are related to more conventional statistics or machine learning concepts, are presented in full on the author's website: https://www.sanjeevnamjoshi.com/faif/proofs.

CONTENTS

CHAPTER 1

FUNDAMENTALS

Nearly every equation in the active inference literature may be obtained by following the steps:

1. Define the probability distributions and factorizations of the generative model.
2. Use generative model factors to determine probabilistic form of variational free energy.
3. Replace probabilistic form of variational free energy with equations for corresponding probability distributions and simplify the equation.
4. Take the partial derivative of variational free energy with respect to a variable of interest.
5. Set this partial derivative equal to zero and solve for the variable of interest.

In this chapter we introduce the basic concepts at play in active inference and give a more detailed explanation of how these steps are applied.

1 Generative models

2 Bayes' theorem

3 Hidden state estimation (MAP)

4 Hidden state estimation (MLE)

CHAPTER 2

VARIATIONAL BAYESIAN INFERENCE

CHAPTER 3

PREDICTIVE CODING

1 The generative model for predictive coding

2 Variational free energy under Dirac δ assumptions

3 Update equation for variational density mean (perception)

4 Update equation for prior mean (perception)

5 Update equation for prior variance (attention)

6 Update equation for likelihood variance (attention)

7 Update equation for parameters (learning)

8 Variational free energy update rules

9 Hierarchical predictive coding

CHAPTER 4

CONTINUOUS STATE-SPACE ACTIVE INFERENCE

1 The basic generative model for continuous state-space active inference

2 Variational free energy under the Laplace/Quadratic approximation

(univariate)

3 Variational free energy under the Laplace/Quadratic approximation (multivariate)

4 Variational free energy under the Laplace/Quadratic approximation and diagonal covariance matrices

5 The generalized state-space model

(including generalized prediction errors)

6 Variational free energy for the generalized state-space model and diagonal covariance matrices

7 Variational free energy for the generalized state-space model using the generalized precision matrix

(derive precision matrix)

8 Gradient flow of hidden states

1. Univariate 2. Multivariate 3. Generalized

9 Gradient flow of autonomous states

1. Univariate 2. Multivariate 3. Generalized

10 Gradient flow of action

(univariate and multivariate)

11 The complete generalized state-space model (with first and second-order parameters)

12 Variational free energy for the complete generalized state-space model

13 Gradient flows for first and second-order parameters

14 Solving the initial value problem

(Euler, Ozaki, etc.)

15 State-dependent precision

16 Hierarchical models

CHAPTER 5

EXPECTED FREE ENERGY AND DISCRETE VARIATIONAL FREE ENERGY

1. Policy-dependent 2. Policy-independent

13 Matrix forms of expected free energy

14 Origins of expected free energy

CHAPTER 6

DISCRETE STATE-SPACE ACTIVE INFERENCE

CHAPTER 7

EXTENSIONS

1 **Action-perception divergence**

2 **Renyi bound**

3 **Sophisticated inference**

4 **Federated belief sharing**

5 **Inductive planning**

6 **Renormalizing generative models**

7 **Hybrid generative models**

CHAPTER 8

VARIATIONS ON FREE ENERGY

1 Generalized free energy

2 Free energy of the future

3 Predicted free energy

4 Free energy of the expected future

5 Bethe free energy

6 Constrained Bethe free energy

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