NETWORK COMPLEXITY

by

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Introduction : The use of computers in executing algorithms always leads to the question of how "expensive" these algorithms are. This can mean, for example, the amount of computing time or storage space required by a given algorithm. Such questions are handled in complexity theory; their practical significance is apparent. Upon closer examination these questions are seen to be quite complicated, and everyday problems prove to be extremely difficult to solve.

The first attempt at developing a complexity theory for general computable functions began with the axiomatic approach of Blum*. In the form of the famous speed-up theorems this approach led, however, to disappointing results. Another approach by Schnorr** with respect to optimal Gödel numberings was also taken up by Hartmanis***, but appears have not to been further handled. In the direction of a general theory the most far reaching results have been presented by Strassen in his development of the degree-bound. Although this is sharp for several interesting special cases, in general, it underestimates the complexity of polynomials quite significantly.

For several reasons the complexity of Boolean networks has received special attention. Good lower bounds for these networks would also lead to good lower bounds for polynomials. Moreover, the results of Fischer and Schnorr**** show that beyond this, such results could also yield information about the complexity of general computable functions. Unfortunately, until now all of the efforts applied to the complexity of Boolean networks have led to only modest results. For example, see the report by Patterson [9], and also the papers by Paul [13] and Schnorr [15], which are quite complicated considering their results.

The goal of this paper is to examine complexity measures over an axiomatic basis. These measures include the complexity measures induced by

* Blum,M.:" A Machine Independent Theory of the Complexity of Recursive Functions", J.ACM 14,2 (1967), 322-336.

** Schnorr,C.P.: "Optimal Enumerations and Optimal Gödel Numberings",M.Syst.Th.8(1974)

*** Hartmanis,J.+ Baker,T.P.: "On Simple Gödel Numberings and Translations" in Automata, Languages and Programming, 2nd Coll. Univ. of Saarbrücken, 1974, LNCS 14.

**** Fischer,M.J.: Lectures on Network Complexity, pres.at the Univ. of Frankfurt,1974

Schnorr,C.P.: "The Network Complexity and the Turing Complexity of Finite Functions", Acta Informatica 7, (1976), 95-1o7.

cost functions like the size complexity, depth complexity, and breadth complexity studied in [16] and [9]. Also included are the applications of entropy [12] [18] and the degree-bound [17].

We develop this theory on the basis of categories with an added monoid multiplication. It will be examined under which conditions the cost-function-induced complexity measures can be approximated by general complexity measures. Methods will be developed for constructing complexity measures, and the conditions under which the entropy can be used for the definition of complexity measures will be given. Finally we will briefly mention complexity measures for monotone functions developed over monotone elements.

This paper has its basis in the extended abstract [5]. The formal, more complete, and extended version found in the current paper is primarily the work of the second author.

1. The Mathematical Representation of Switching Circuits.

In this paper the size complexity, depth complexity, and other complexity measures of Boolean functions represented by switching circuits will be examined. First of all we need a good mathematical representation of switching circuits.

The first idea in this direction is to represent a switching circuit by a digraph in which nodes are labeled with elementary switching elements and the edges represent wires of the circuit.

A more algebraic description of switching circuits is given in [4]. There the concept of so called X-categories is introduced. In [3] the X-categories are called "strict monoidial categories" and in [1] they are called "Kronecker categories".

1.1. Definition : An X-category X is a 3-tupel $(X, \times, \varepsilon)$ such that

(i) X is a category,

(ii) $\times : X \times X \to X$ is a covariant bifunctor (we write $f \times g$ instead of $\times(f,g)$ for morphisms $f, g \in Mor(X)$ and, analogously $u \times v$ for objects $u, v \in Ob(X)$),

(iii) $\varepsilon \in Ob(X)$ is a special element, and

(iv) $(Mor(X), \times, 1_\varepsilon)$ is a monoid with $\times$ as operation and $1_\varepsilon : \varepsilon \to \varepsilon$, the identity on ε, as unit element.

Property (ii) implies : If $f : u' \to u$, $f' : u'' \to u'$, $g : v' \to v$, $g' : v'' \to v'$ are morphisms in X, then $f \times g : u' \times v' \to u \times v$ and $f' \times g' : u'' \times v'' \to u' \times v'$, and we have the following equation :

$$(f \circ f') \times (g \circ g') = (f \times g) \circ (f' \times g').$$

Furthermore claim (ii) together with claim (iv) implies that $(\mathrm{Ob}(X), \times, \varepsilon)$ is also a monoid. Because of (iv) every X-category is a small category (a monoid is always a set).

1.2. Definition : Let X, Y be two X-categories. A functor $\Phi : X \to Y$ is called X-functor iff Φ regarded as a mapping $\mathrm{Mor}(X) \to \mathrm{Mor}(Y)$ is a monoid homomorphism.

1.3. Definition :

(i) Let X be an X-category and $A \subset \mathrm{Mor}(X)$. A is called a free generating system for X iff the following universal property holds : If Y is an arbitrary X-category, $\Phi_1 : \mathrm{Ob}(X) \to \mathrm{Ob}(Y)$ a monoid homomorphism, and $\Phi_2 : A \to \mathrm{Mor}(Y)$ a mapping such that $\Phi_2(a) : \Phi_1(u) \to \Phi_1(v)$ for $a : u \to v \in A$ then there exists a uniquely determined X-functor $\Phi : X \to Y$ which is an extension of Φ_1 and Φ_2.

(ii) An X-category is called free iff it posses a free generating system.

1.4. Theorem and Notation : Given any free monoid O and an arbitrary set A together with two mappings $S, T : A \to O$ there exists a free X-category $F(A,O)$ which is unique up to functorial isomorphism with free generating system A and O as monoid of objects such that every $a \in A$ becomes a morphism $a : S(a) \to T(a)$ in $F(A,O)$.

For a proof see, for example, [1] or [7].

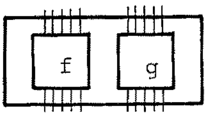
The following example shows us how a switching circuit may be represented by a free X-category.

1.5. Example : Let B be a basis generating all of the Boolean functions $\{\text{true}, \text{false}\}^n \to \{\text{true}, \text{false}\}^m$ for example, $B = \{\wedge, \vee, \neg\}$. In a switching circuit there are also some other elementary switching elements such as

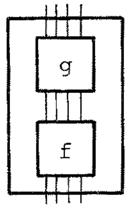
(i)	crossing of two wires	$\times$	denoted by c,
(ii)	branching of a single wire (diagonalization)	$\curlywedge$	denoted by d,
(iii)	Boolean constant wires	true $\top$ false $\top$	denoted by true and false,
(iv)	truncation of a wire	$\perp$	denoted by t.

As monoid O of objects we will use $\mathbb{N}_0$ with addition as operation. If f is a switching circuit we will interpret f as a morphism $f : u \to v$ with $u, v \in \mathbb{N}_0$, where u denotes the number of inputs and v the number of outputs of f. The free generating system A consists of the elements of B and c, d, true, false, t, where the mappings S, T are defined in the obvious way.

The operations $\times$ and $\circ$ of $F(A,O)$ are interpreted as follows : If f and g are two circuits $f\times g$ means the circuit built by drawing f on the left of g without connecting any wires. If f has u inputs and g has u outputs then $f\circ g$ is defined and means the circuit built by connecting the inputs of f pairwise with the outputs of g without changing their order.



$f\times g$



$f\circ g$

<u>1.6. Examples</u> (c.f.[4],[2]): Let M be an arbitrary set. Denote by C_M the following X-category :

$Ob(C_M) := \mathbb{N}_o$ together with + as monoid operation,

$Mor(C_M) := \{f : n \to m : f \text{ is a function from } M^n \text{ to } M^m\}$.

Thereby we identify M^o with the set $\{e\}$ consisting of a single element e. Identifying $M^o\times M^n$ and $M^n\times M^o$ with M^n we may define $f\times g$ for f, $g \in Mor(C_M)$ in the usual way. Then it is easily proved that $(Mor(C_M),\times)$ is a monoid and that C_M is indeed a (not free) X-category. As a special example we have the X-category $B := C_{\{true,\ false\}}$ of Boolean functions. If G is defined as in 1.5. we have an X-functor $I : G(\{\wedge,\vee,\neg\}) \to B$ such that for $f \in Mor(G(\{\wedge,\vee,\neg\}))$ $I(f)$ is the Boolean function represented by f.

As another example let R be an arbitrary ring and denote by P_R the subcategory of C_R consisting of all polynomial† functions $R^n \to R^m$ $(n, m \in \mathbb{N}_o)$. Then P_R is also an X-category. An interpretation $I : G(\{+,-,*\}) \to P_R$ is defined in the obvious way.

There is a difference between the representation of a switching circuit by a free X-category and the method discussed at the beginning of this section. The following two switching circuits have the same representation as digraphs but are different morphisms in $G(\{\wedge,\vee,\neg\})$:

† A function $f : R^n \to R^m$ is called polynomial iff there exists m polynomials $p_1,\dots,p_m \in R[x_1,\dots,x_n]$ such that $f(x_1,\dots,x_n) = (p_1(x_1,\dots,x_n),\dots,p_m(x_1,\dots,x_n))$ for all $(x_1,\dots,x_n) \in R^n$.

Because of this problem we will define special sorts of X-categories and use them for representing of switching circuits. First, Hotz has introduced such special X-categories, called the free D-categories (c.f.[4]).

<u>1.7. Definition</u> (c.f.[1]). Let X be an X-category. We call X an X-category with finite direct products (D-category for short) iff for every $u \in Ob(X)$ there are morphisms $t_u : u \to \varepsilon$ (called truncations) such that for any two objects $v, w \in Ob(X)$ the following diagram is a direct product diagram in X.

$$\begin{tikzcd} & v \times w \arrow[dl, "1_v \times t_w"'] \arrow[dr, "t_v \times 1_w"] & \\ v & & w \end{tikzcd}$$

If X is a D-category we introduce the following notations :

(a) Let $u \in Ob(X)$ and consider the following diagram in X :

$$\begin{tikzcd} & u \arrow[dl, "1_u"'] \arrow[d, dashed, "d_u"] \arrow[dr, "1_u"] & \\ u & u \times u \arrow[l, "1_u \times t_u"] \arrow[r, "t_u \times 1_u"'] & u \end{tikzcd}$$

There is a uniquely determined morphism $d_u : u \to u \times u$ (called diagonalization) which makes the diagram commutative :

(D2) $1_u = (1_u \times t_u) \circ d_u = (t_u \times 1_u) \circ d_u$.

(b) Let $u, v \in Ob(X)$ and consider the following diagram in X :

$$\begin{tikzcd} & u \times v \arrow[dl, "1_u \times t_v"'] \arrow[d, dashed, "c_{u,v}"] \arrow[dr, "t_u \times 1_v"] & \\ u & v \times u \arrow[l, "t_v \times 1_u"] \arrow[r, "1_v \times t_u"'] & v \end{tikzcd}$$

There exists a uniquely determined isomorphism $c_{u,v} : u \times v \to v \times u$ (called crossing) which makes the diagram commutative :

(D3) $1_u \times t_v = (t_v \times 1_u) \circ c_{u,v}$ and $t_u \times 1_v = (1_v \times t_u) \circ c_{u,v}$.

<u>1.8. Proposition :</u> Let X be a D-category. Then the following hold

(D0) $t_\varepsilon = 1_\varepsilon$ and $t_{u \times v} = t_u \times t_v$ for all $u, v \in Ob(X)$. Further if $f : u \to \varepsilon \in Mor(X)$ then $f = t_u$. This means that t_u is the only morphism $u \to \varepsilon$ in Mor(X).

(D1) Let $f : u \to v \times w$, $g : u \to v \times w$ be two morphisms in Mor(X). If

$$(1_v \times t_w) \circ f = (1_v \times t_w) \circ g \text{ and } (t_v \times 1_w) \circ f = (t_v \times 1_w) \circ g$$

then $f = g$.

(D4) $(1_u \times d_u) \circ d_u = (d_u \times 1_u) \circ d_u$ for all $u \in Ob(X)$.

(D5) $c_{u,u} \circ d_u = d_u$ for all $u \in Ob(X)$.

(D6) $c_{v,u} \circ c_{u,v} = 1_{u \times v}$ for all $u, v \in Ob(X)$.

(D7) $(1_u \times c_{u,v} \times 1_v) \circ (d_u \times d_v) = d_{u \times v}$ for all $u,v \in Ob(X)$.

(D8) $(1_v \times c_{u,w}) \circ (c_{u,v} \times 1_w) = c_{u,v \times w}$ for all $u, v, w \in Ob(X)$.

(D9) $(f \times f) \circ d_u = d_v \circ f$ for all morphisms $f : u \to v \in Mor(X)$.

(D1o) If $f : u \to r$ and $g : v \to s$ are morphisms in $Mor(X)$ then $c_{r,s} \circ (f \times g) = (g \times f) \circ c_{u,v}$.

For a proof see [1] or [4].

1.9. Definition :

(i) By K we will denote the category of all X-categories as objects together with all X-functors as morphisms†.

(ii) By D we will denote the full subcategory of K consisting of all D-categories as objects.

(iv) Let O be a fixed monoid (with $\times$ as operation). Then we will denote by K(O), D(O) the subcategories of K, D resp. consisting of X-categories X with $Ob(X) = O$ as objects and such X-functors which are the identity mapping $id : O \to O$ on the objects.

A functor $\Phi : X \to Y \in Mor(D)$ preserves crossings, truncations, diagonalizations, and finite direct products.

1.1o. Definition :

(i) Let X be a D-category and $A \subset Mor(X)$. A is called a free D-generating system for X iff the following universal property holds : If Y is an arbitrary D-category, $\Phi_1 : Ob(X) \to Ob(Y)$ a monoid homomorphism and $\Phi_2 : A \to Mor(Y)$ a mapping such that $\Phi_2(a) : \Phi_1(u) \to \Phi_1(v)$ if $a : u \to v \in A$, then there exists a uniquely defined X-functor $\Phi : X \to Y$ which is an extension of Φ_1 and Φ_2.

(ii) A is called free iff it possess a free D-generating system.

1.11. Theorem and Notation : Given any free monoid O and an arbitrary set A together with two mappings $S, T : A \to O$ such that $\varepsilon \notin T(A)$ there exists a - up to an isomorphism in D - uniquely determined free D-category $F_D(A,O)$ with free D-generating system A and O as monoid of

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† Recall that X-categories are small categories.

objects such that every $a \in A$ becomes a morphism $a : S(a) \to T(a)$ $\in Mor(F_D(A,O))$.

For a proof see [1],[2] and [4].

For describing a switching circuit as a morphism in a free D-category it is not essential how the circuit paths are factored. However, because of (D9) the following two circuits also have the same description.

But if we are not interested in the outdegree of switching elements this is not a handicap.

The following definition generalizes the concept of Boolean and arithmetic functions in the sense of Strassen's Ω-algebras [16].

1.12. Definition : Let M be a set. A function $\omega : M^n \to M$ is called an n-ary operation on $M (n \in \mathbb{N})$. Let Ω be a set of operations on M together with a mapping $S : \Omega \to \mathbb{N}$ such that $\omega \in \Omega$ is an $S(\omega)$-ary operation on M. Now we define $A := M \cup \Omega$ and extend S to a mapping $S : A \to \mathbb{N}_o$ by setting $S(x) = 0$ for all $x \in M$. Further, we define another mapping $T : A \to \mathbb{N}_o$ such that $T(a) = 1$ for all $a \in A$. Using theorem 1.13 we get the free D-category $F_D(A,\mathbb{N}_o)$. Let C_M be defined as in example 1.6. and define an interpretation $I : F_D(A,\mathbb{N}_o) \to C_M$ as follows :

I is the identity mapping on the objects and

$$I(\omega)(x_1,\ldots,x_{S(\omega)}) = \omega(x_1,\ldots,x_{S(\omega)}) \quad \text{for all } \omega \in \Omega, \; x_1,\ldots,x_{S(\omega)} \in M,$$

$$I(x)(e) = x \quad \text{for all } x \in M.$$

This means that the elements of M are interpreted as constant functions $\{e\} \to M$. Now we denote by $P_{M,\Omega}$ the image of the functor I. Clearly $P_{M,\Omega}$ is a D-category.

2. Complexity Measures on X-categories.

In this section S denotes a fixed additive commutative ordered monoid with unit element 0 satisfying the following condition :

(P) $s \geq 0$ for all $s \in S$.

In this case we call S a positive monoid. For example S may be the additive semigroup $\mathbb{N}_0$ together with the natural order.

2.1. Definition : Let X be an X-category. A complexity measure on X with values in S is a function $c : \mathrm{Mor}(X) \to S$ satisfying the following axioms :

(C1) $c(1_u) = 0$ for all objects $u \in \mathrm{Ob}(X)$.

(C2) $c(f \circ g) \leq c(f) + c(g)$ for all $f, g \in \mathrm{Mor}(X)$ such that $f \circ g$ is defined.

(C3) $c(f \times g) \leq c(f) + c(g)$ for all $f, g \in \mathrm{Mor}(X)$.

In the future we will often denote a complexity measure c by $|\ldots|$.
If c satisfies the stronger condition

(C2') $c(f \circ g) \leq \mathrm{Max}(c(f), c(g))$ if $f \circ g$ is defined $(f, g \in \mathrm{Mor}(X))$ instead of (C2), we call c a breadth measure on X, and if c satisfies

(C3') $c(f \times g) \leq \mathrm{Max}(c(f), c(g))$ $(f, g \in \mathrm{Mor}(X))$
instead of (C3), we call c a depth measure on X.

If X is a D-category we call a complexity measure c on X a D-complexity measure iff

$$c(t_u) = c(d_u) = c(c_{u,v}) = 0 \text{ for all } u, v \in \mathrm{Ob}(X).$$

2.2. Examples : Let $X = F(A,O)$ $(X = F_S(A,O))$ be a free X-category and $L : A \to S$ an arbitrary function. We may interpret S as an X-category where $\circ$ and $\times$ both mean the addition + in S and $\circ$ is always defined. The monoid Ob(S) consists of a single element ε. Extending the function L to an X-functor

$$L : X \to S$$

gives us a complexity measure L on X with values in S for which the following equations hold :

(i) $L(1_u) = 0$ $(L(c_{u,v}) = 0$ in addition) $(u, v \in \mathrm{Ob}(X))$.

(ii) $L(f \circ g) = L(f) + L(g)$ if $f \circ g$ is defined $(f, g \in \mathrm{Mor}(X))$.

(iii) $L(f \times g) = L(f) + L(g)$ $(f, g \in \mathrm{Mor}(X))$.

Such a complexity measure on free X-categories is called a cost function. If X denotes the free X-category G(A) of switching circuits over the basis A and $L : A \to S$ is the cost of a single switching element (in A), then for a circuit $f \in \mathrm{Mor}(X)$ L(f) denotes the cost of this circuit.

Another important example is given in [5] : Let M be a finite set and X a sub-X-category of C_M such that Mor(X) consists only of isomorphisms (bijections) on C_M. Let $\Pi = (\pi_u)_{u \in \mathbb{N}_0}$ be a sequence of partitions such that

$$M^u = \bigcup_{\alpha \in \pi_u} \alpha \text{ where } \alpha \cap \beta = \emptyset \text{ for different } \alpha, \beta \in \pi_u.$$

Define an entropy function $H : \mathrm{Mor}(X) \to \mathbb{R}_0^+$ by

$$H_\Pi(f) := - \sum_\alpha \frac{\mathrm{card}(\alpha)}{\mathrm{card}(M^u)} \sum_\beta \frac{\mathrm{card}(f(\alpha) \cap \beta)}{\mathrm{card}(\alpha)} \log \frac{\mathrm{card}(f(\alpha) \cap \beta)}{\mathrm{card}(\alpha)}$$

for $f : u \to v \in \mathrm{Mor}(X)$ where α runs over π_u and β runs over π_v.

Then in [5] is proved that $H : \mathrm{Mor}(X) \to \mathbb{R}_0^+$ defined by

$$H(f) = \sup \{H_\Pi(f) : \Pi \text{ sequence of partitions as above}\}$$

is a complexity measure. This complexity measure is used to get lower bounds on the complexity of permutations (matrix transposition for example) and merging networks (c.f.[11] and [16]).

In algebra we define valuations on rings. Here, one of the axioms is $|x \circ y| = |x| \circ |y|$ where an equallity instead of an inequallity is postulated. Therefore in the theory of valuations on a ring or on a field we have a lot of results characterizing the set of possible valuations. But in our case we have the following difficulties :

<u>2.3. Remarks :</u> Let X be an X-category and $C : \mathrm{Mor}(X) \to S$ a complexity measure on X. If $\varphi : S \to S$ denotes an arbitrary monotone convex function with $\varphi(0) = 0$, then $\varphi \circ c : \mathrm{Mor}(X) \to S$ is also a complexity measure on X.

<u>2.4. Example :</u> If S is the positive semigroup $\mathbb{R}_0^+$, then $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ defined as $\varphi(x) = \log(1+x)$ is monotone and convex and satisfies $\varphi(0) = 0$. If we have one complexity measure $c : \mathrm{Mor}(X) \to S$ we are able to construct a lot of other complexity measures, such as $\varphi \circ c$, $\varphi \circ \varphi \circ c$, $\varphi \circ \varphi \circ \varphi \circ c$, ... which are all not very different from c.

<u>2.5. Theorem and Notation :</u> Let $\Phi : X \to Y$ be an X-functor defined over X-categories X and Y.

(i) If $c : \mathrm{Mor}(Y) \to S$ is a complexity measure on Y, then $\Phi^{-1}(c) : \mathrm{Mor}(X) \to S$ defined by

$$\Phi^{-1}(c)(f) := c(\Phi(f)) \qquad (f \in \mathrm{Mor}(X))$$

is a complexity measure on X.

(ii) Let S be such, that for every subset $M \subset S$ $\inf(M)$ exists respective to the order on S. If Φ considered as a mapping $\mathrm{Ob}(X) \to \mathrm{Ob}(Y)$ is bijective and $c : \mathrm{Mor}(X) \to S$ is a complexity measure on X, then $\Phi(c) : \mathrm{Mor}(Y) \to S$ defined by

$$\Phi(c)(f) := \inf \{c(f') : \Phi(f') = f\} \qquad (f \in \mathrm{Mor}(Y))$$

is a complexity measure on Y. If $\inf(\emptyset) = \infty$ does not exist in S but Φ is also surjective on the morphisms, then $\Phi(c)$ can be defined in the same way.

Proof : easy calculation. □

Because of theorem 2.5. we are able to define the size complexity of Boolean functions. Let $I : G(\{\wedge,\vee,\neg\}) \to B$ the interpretation defined in section 1. On $G(\{\wedge,\vee,\neg\})$ we are able to define a cost function L with values in $\mathbb{N}_o$ such that $L(d_u) = L(c_{u,v}) = L(t_u) = O(u, v \in N_o)$ and $L(\wedge) = L(\vee) = L(\neg) = 1$ (c.f. 2.2.). Then I(L) is the size complexity on B.

Throughout the rest of this section we will consider only X-categories in K(O) for a fixed monoid O. Further we will assume $S = \mathbb{R}_o^+ \cup \{\infty\}$, which we may also consider as a semiring. Then we define :

2.6. Definition : We denote by C the following category :

$$\begin{aligned} Ob(C) &:= \{(X,c) : X \in Ob(K(O)) \text{ and } c \text{ is a complexity measure on } X \text{ with values in } \mathbb{R}_o^+ \cup \{\infty\}\} \\ Mor(C) &:= \{\Phi : (X,c) \to (X',c') : \Phi : X \to X' \in Mor(K(O)) \text{ and } \exists\, \lambda \in \mathbb{R}^+ \text{ such that } c'(\Phi(f)) \leq \lambda\circ c(f) \text{ for all } f \in Mor(X)\}. \end{aligned}$$

If id : $X \to X$ is the identity functor ($X \in K(O)$) and c_1, c_2 are two complexity measures on X, then we will write

$$c_1 \succeq c_2 \quad \text{iff id} : (X,c_1) \to (X,c_2) \in Mor(X).$$

Further

$$c_1 \approx c_2 \quad \text{iff} \quad c_1 \succeq c_2 \text{ and } c_1 \preceq c_2.$$

2.7. Proposition : Let $\Phi : X \to Y$ be a morphism in the category K(O).

(i) If c_1, c_2 are complexity measures on Y and $c_1 \succeq c_2$, then we have $\Phi^{-1}(c_1) \succeq \Phi^{-1}(c_2)$ on X.

(ii) If c_1, c_2 are complexity measures on X and $c_1 \succeq c_2$, then we have $\Phi(c_1) \succeq \Phi(c_2)$ on Y.

In the following we will prove some simulation theorems using the following definition :

2.8. Definition : Let c be a complexity measure on an X-category X with values in $\mathbb{R}_o^+ \cup \{\infty\}$. Then c is called non degenerate iff for all $f \in Mor(X)$ $c(f) \neq \infty$.

2.9. Proposition : Let $X = F(A,O)$ be a free X-category with finite generating system A, c_1 a cost function on X, and c_2 an arbitrary complexity measure on X. Assume that c_1, c_2 both are non degenerate. If $c_1(a) = 0 \Rightarrow c_2(a) = 0 \ \forall\, a \in A$ then $c_1 \succeq c_2$.

Proof : Let $\lambda := \operatorname{Max} \{\frac{c_2(a)}{c_1(a)} : a \in A$ such that $c_1(a) \neq 0\}$. This maximum exists since A is finite and we have $0 < \lambda < \infty$ since c_1, c_2 both are non degenerate. Now $c_2(f) \leq \lambda \circ c_1(f)$ can easily be proved by induction on the length of a sequential representation (c.f. [7]) of $f \in \operatorname{Mor}(X)$. □

2.1o. First Simulation Theorem : Let X be a finitely generated free X-category and c_1, c_2 two non degenerate cost functions on X with $c_1(f) = 0 \Leftrightarrow c_2(f) = 0$ for all $f \in \operatorname{Mor}(X)$. Then we have $c_1 \approx c_2$.

Proof : Since every cost function is a complexity measure, we have from 2.9. that $c_1 \succeq c_2$ and $c_2 \succeq c_1$. Therefore $c_1 \approx c_2$. □

2.11. Theorem (Maximality of the Size Complexity) (c.f.[5]). Let $\Phi : X \to Y \in \operatorname{Mor}(K(O))$ where X is a finitely generated free X-category and Φ is surjective on the morphisms. Further, let c_1 be a non degenerate cost function on X and c_2 be an arbitrary non degenarete complexity measure on Y such that $c_1(f) = 0 \Rightarrow c_2\ (\Phi(f)) = 0$ for all $f \in \operatorname{Mor}(X)$. Then $c_2 \preceq \Phi(c_1)$. In the case $\Phi = I$, $X = G(\{\wedge,\vee,\neg\})$, and $Y = B$ this means : Every non degenerate complexity measure c on B with $c(d_u) = c(t_u) = c(c_{u,v}) = 0$ for all u, $v \in \operatorname{Ob}(B) = \mathbb{N}_0$ is (without a constant factor λ) a lower bound for the size complexity in B. Further, two non degenerate size complexity measures c_1 and c_2 on B are equivalent $(c_1 \approx c_2)$ iff $c_1(f) = 0 \Leftrightarrow c_2(f) = 0$ for all $f \in \operatorname{Mor}(X)$.

Proof : $\Phi^{-1}(c_2)$ is a non degenerate complexity measure on X with $c_1(f) = 0 \Rightarrow \Phi^{-1}(c_2)(f) = c_2(\Phi(f)) = 0$ for all $f \in \operatorname{Mor}(X)$. Then 2.9. implies that $\Phi^{-1}(c_2) \preceq c_1$ and because of 2.7. we get $\Phi(\Phi^{-1}(c_2)) \preceq \Phi(c_1)$. Now we will prove that under the supposition that Φ, regarded as a mapping $\Phi : \operatorname{Mor}(X) \to \operatorname{Mor}(Y)$, is surjective, we have $c_2 = \Phi(\Phi^{-1}(c_2))$. Let $f \in \operatorname{Mor}(Y)$. Then we have

$$\begin{aligned}\Phi(\Phi^{-1}(c_2))(f) &= \inf\{\Phi^{-1}(c_2)(f') : \Phi(f') = f\}\\ &= \inf\{c_2(\Phi(f')) : \Phi(f') = f\}\\ &= c_2(f)\end{aligned}$$

since there exists an $f' \in \operatorname{Mor}(X)$ with $\Phi(f') = f$. □

2.12. Open Problem : Find complexity measures which can easily be computed, but still give good lower bounds on the size complexity.

Most of the well known examples of such complexity measures which are easily computed are depth measures and therefore not good lower bounds for the size complexity. Another example which yields nonlinear lower bounds is the entropy function (c.f. 2.2.).

2.13. Theorem : Let X be an X-category and $r : Mor(X) \to \mathbb{R}_0^+ \cup \{\infty\}$ an arbitrary function satisfying $r(1_u) = 0$ for all $u \in Ob(X)$.

(i) $c_r : Mor(X) \to \mathbb{R}_0^+ \cup \{\infty\}$ defined by

$$c_r(f) := \sup\{r((1_u \times f \times 1_v) \circ h) - r(h) : u, v \in Ob(X),$$
$$h \in Mor(X) \text{ such that } (1_u \times f \times 1_v) \circ h \text{ is defined}\}$$

is a complexity measure on X satisfying $c_r(f) \geq r(f)$ for all $f \in Mor(X)$.

(ii) ${}_rc : Mor(X) \to \mathbb{R}_0^+ \cup \{\infty\}$ defined by

$${}_rc(f) := \sup r(h \circ (1_u \times f \times 1_v)) - r(h) : u, v \in Ob(X),$$
$$h \in Mor(X) \text{ such that } h \circ (1_u \times f \times 1_v) \text{ is defined}$$

is a complexity measure on X satisfying ${}_rc(f) \geq r(f)$ for all $f \in Mor(X)$.

A proof is given in [5]. □

2.14. Remarks :

(i) If $r : Mor(X) \to \mathbb{R}_0^+ \cup \{\infty\}$ is already a complexity measure on X then $c_r = {}_rc = r$.

(ii) Let A be a finite generating system for X. If $|\ldots|$ denotes the size complexity on X and $r : Mor(x) \to \mathbb{R}_0^+ \cup \{\infty\}$ satisfies

(a) $r(1_u) = 0$ for all $u \in Ob(X)$,

(b) $r(1_u \times f \times 1_v) \leq r(f)$ for all $u, v \in Ob(X)$, $f \in Mor(X)$,

(c) $r(a \circ h) \leq |a| + r(h)$ for all $a \in A$, $h \in Mor(X)$

then we have $c_r \preceq |\ldots|$.

Proof :

(i) $r(1_u \times f \times 1_v) \circ h) - r(h) \leq r(f)$ since r is a complexity measure.

(ii) Let $m := \max\{|a| : a \in A\}$. If $f \in Mor(X)$ has a representation of length l respective to A then $c_r(f) \leq l \cdot m$. This can easily be proved by induction on l using the suppositions (b) and (c). It follows that c_r is non degenerate and therefore (by theorem 2.11) $c_r \preceq |\ldots|$. □

The second remark in 2.14 is often used to get lower bounds for the size complexity. Examples are Strassen's degree bound [17] or Paul's 2.5 n lower bound [13].

The following is analogous to a theorem of Strassen (c.f.[16]).

2.15. Second Simulation Theorem (c.f.[5]) :

Consider the following diagram in C :

$$\begin{tikzcd}
(X,c) \arrow[r, "\chi"] \arrow[d, "\Phi"'] & (X',c') \arrow[d, "\Phi'"] \\
(Y,\Phi(c)) & (Y',\Phi'(c'))
\end{tikzcd}$$

and assume that there exists an X-functor $\Psi : Y \to Y'$ such that the following diagram in $K(O)$ is commutative :

$$\begin{tikzcd}
X \arrow[r, "\chi"] \arrow[d, "\phi"'] & X' \arrow[d, "\phi'"] \\
Y \arrow[r, "\Psi"'] & Y'
\end{tikzcd}$$

Then ϕ is a morphism in C too; that is $\Psi^{-1}(\Phi'(c')) \leq \phi(c)$.

Proof :

$$\begin{aligned}
\Phi'(c')(\Psi(f)) &= \inf \{c'(f') : \Phi'(f') = \Psi(f)\} \\
&\leq \inf \{c'(\chi(g)) : (\Phi' \circ \chi)(g) = \Psi(f)\} \\
&\qquad \text{(since } B \subset A \text{ implies } \inf(A) \leq \inf(B)) \\
&= \inf \{c'(\chi(g)) : (\Psi \circ \Phi)(g) = \Psi(f)\} \\
&\leq \inf \{\lambda \cdot c(g) : \Psi(\Phi(g)) = \Psi(f)\} \\
&\qquad \text{(since } \chi \in \mathrm{Mor}(c)) \\
&\leq \lambda \cdot \inf \{c(g) : \Phi(g) = f\} \\
&\qquad \text{(since } B \subset A \text{ implies } \inf(A) \leq \inf(B)) \\
&= \lambda \cdot \Phi(c)(f). \quad \square
\end{aligned}$$

3. The $\mathbb{R}^+$-Module of all the Complexity Measures on a Fixed X-Catgeory.

In this section O will be a fixed finitely generated free monoid and all of the complexity measures considered here will have values in the semiring $\mathbb{R}_o^+$.

3.1. Proposition and Notation (c.f.[5]) : Let $X \in Ob(K(O))$ be an X-category. We denote by

$$C(X) := \{c : Mor(x) \to \mathbb{R}_o^+ : c \text{ is a complexity measure on } X\}$$

the set of all complexity measures on X. Let $c_1, c_2 \in K(O)$ and $\lambda \in \mathbb{R}_o^+$. Then the following holds :

(i) $c_1 + c_2 \in C(X)$ where $(c_1+c_2)(f) := c_1(f) + c_2(f)$ for all $f \in Mor(X)$.

(ii) $\lambda \cdot c_1 \in C(X)$ where $(\lambda \circ c_1)(f) := \lambda \cdot c_1(f)$ for all $f \in Mor(X)$.

Thus $C(X)$ is a module over the semiring $\mathbb{R}_o^+$.

Proof : elementary calculation. $\square$

It is harder to classify the complexity measures on a free X-category than on a special X-category. For example, D-complexity measures on a D_-category Y have properties which are not satisfied by all complexity measures on the free X-category X generating Y, even if they have zerovalues on the switching elements interpreted as crossings, truncations, and diagonalizations.

3.2. Theorem : Let $|\dots|$ be a D-complexity measure on a D-category X with $Ob(X) = \mathbb{N}_0$. Then the following holds :

(i) $|f\times g| = |g\times f|$ for all $f : u \to u'$, $g : v \to v' \in Mor(X)$.

(ii) $|f| \leq |f\times g|$ for all $f : u \to u'$, $g : v \to v' \in Mor(X)$, $u \neq 0$.

(iii) $|1_u \times f \times 1_v| = |f|$ for all $f : w \to w' \in Mor(X)$, $w \neq 0$.

Proof :

(i) $|f\times g| = |c_{v',u'} \circ (g\times f) \circ c_{u,v}| \leq |g\times f|$ for $|c_{u,v}| = |c_{v',u'}| = 0$, and vice versa.

(ii) Let $h_v : 1 \to v$ be defined inductively as follows ($v \in \mathbb{N}$) :
$h_1 := 1_1$,
$h_{v+1} : = (1_1 \times h_v) \circ d_1$.
Then $(1_1 \times t_v) \circ h_{1+v} = 1_1$ and therefore
$f = (1_{u'} \times t_{v'}) \circ (f\times g) \circ (t_{u-1} \times h_{1+v})$ $(u-1 \in \mathbb{N}_0$ since $u \neq 0) \Rightarrow$
$|f| \leq |f\times g|$ by (C2).

(iii) $|f| \leq |1_u \times f \times 1_v| \leq |f|$ by (ii) and (C3). □

3.2. (iii) gives an answer to a question in [5], p. 4o8 f.

Let $X \in Ob(K(O))$ and $c \in C(X)$ such that c is not bounded above by a constant. If $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is the function $\varphi(x) = \log(1+x)$ defined in 2.4. then the complexity measures c, $\varphi \circ c$, $\varphi\circ\varphi\circ c, \dots$ are pairwise inequivalent. Furthermore they are algebraically independent over R. If we call the maximal number of algebraically independent elements in an $\mathbb{R}_0^+$-module the dimension of that module, C(X) has in general infinite dimension.
As noted in example 2.4. above we will now define strict complexity measures.

3.3. Definition : Let X be an X-category and c a complexity measure on X. c is called a strict complexity measure iff it satisfies

(CS3) $c(f\times g) = c(f) + c(g)$ for all $f, g \in Mor(X)$.

Since it seems to be very difficult to get good lower bounds on the size complexity of Boolean functions we consider monotone Boolean functions. Let B^m denote the subcategory of B having only monotone Boolean functions as morphisms. Clearly B^m is a D-category. Let

$$I : G(\{\wedge, \vee\}) \to B^m$$

be the natural interpretation. If L is the usual cost function then I(L) is a strict complexity measure (for a proof see [6] or [12]).

From example 2.4. it is clear that there exists complexity measures which are not strict.

In [12] Paul has shown that for every $\varepsilon > 0$ there are morphisms $f \in Mor(B)$ such that $|f \times f| \leq (1+\varepsilon) \circ |f|$, where $|\ldots|$ denotes the size complexity on B. It follows that even the size complexity must not be a strict complexity measure.

3.4. Open Problems :

1. Give a characterization of all strict complexity measures on a fixed X-category.
2. Find a definition of "monotone X-categories" as a generalization of B^m.
3. Find a definition of "monotone complexity measures" on a (monotone) X-category.

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