On the Arrangement Complexity of Uniform Trees

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This paper studies the arrangement problem of uniform trees and shows that the arrangement complexity of a uniform tree is either $\Theta(1)$ or $\Omega((\lg n)^{\gamma})(\gamma > 0)$. It also presents a recursive algorithm to compute the optimal complete arrangements for $\Theta(1)$ arrangeable balanced uniform trees.

1 Introduction

A tree-like structured VLSI system is called a tree. Every node inside the tree implements a function and has several input lines and one output line. A uniform tree is a special one in which all nodes implement the same function. A tree is considered to be balanced if the difference between the length of the longest path and that of the shortest one from the root to the leaves is at most 1. The input lines corresponding to the leaves of the tree are called *primary input lines*, while the output line of the root node is called *primary output line*.

An *arrangement* set for a tree with n primary input lines consists of a number of n-component patterns such that by applying them to the primary input lines of the tree, the input and output sets of every node inside the tree have a *predefined property*. An *arrangement* problem is to assign an arrangement set to a tree. This is a more general combinational problem extracted from the assignment and the test problem related to the tree VLSI systems. Assignment and test are two basic problems in the field of VLSI system design. The study of these problems is useful for the design of easily testable VLSI systems and the generation of their optimal test sets, and therefore it is helpful to reduce the costs of the system's design and maintenance[Haye71,SK77, BeHa90, BeSp91, Wu92, WuSp92, Wu93, HoWu95].

If the predefined property for the arrangement is that *the input set to the node includes all possible input combinations*, the arrangement is just the assignment problem. The VLSI system test problem can be divided into two essential subproblems, test application and response observation. For the I_{DDQ} testing technique [FSTH90], it suffices to completely exercise all internal nodes of the network, i.e. apply all possible input combinations to each of the internal nodes. If the predefined property is that *the input and output sets of the node comprise a test set of the node*, the arrangement problem is exactly the test problem. Thus, the assignment and test problems are two instances of the arrangement problem.

The arrangement complexity of a given tree is defined as the minimum of the cardinalities of its arrangement sets. The extensive study of the arrangement complexity has theoretical as well as practical value.

This paper discusses the arrangement problem of the balanced uniform trees. It consists of four sections. In the next section, we use the X-category theory developed in [Hotz65, Hotz74] to define tree systems and their arrangement problems formally such that we can simplify our discussion and generalize the results easily. In section 3, we study the constant arrangeable tree families. Section 4 shows that the arrangement complexity of a uniform tree is either $\Theta(1)$ or $\Omega((\lg n)^{\gamma})(\gamma > 0)$. It explores that according to their arrangement complexities, the balanced uniform trees can be roughly divided into two classes, constant and polynomial in $\lg n$. There is nothing in between.

2 X-Category and Arrangement of Uniform Trees

In this section, we give a formal description of VLSI systems by using the X-category. It is shown that all combinational VLSI systems correspond to an X-category denoted by $\mathcal{C}$, and that the functions implemented by the elements in $\mathcal{C}$ together correspond to an X-category denoted by $\mathcal{K}$. Furthermore, we show that there is a functor from $\mathcal{C}$ to $\mathcal{K}$. Then, we can study some properties of elements of $\mathcal{C}$ in the field of $\mathcal{K}$ without concern of their concrete physical structures.

Let $T = \{t_1, t_2, ..., t_m\}$ be the set of the basic types of signals which can be transferred through a (symbolic) line or (symbolic) bus in VLSI systems. A signal type $t_i \in T$ consists of a number of signals called individuals(values). For instance, the 1-bit binary signal type includes logic 0 and logic 1 as its individuals, and a line of such a signal type can transfer both logic 0 and logic 1.

Let $T^* = \{a_1...a_k | a_1, ..., a_k \in T, k \in \mathrm{N}_0\}$ be the free monoid generated by T. The concatenation of $u, v \in T^*$ is $u \cdot v$, and $|a_1 \cdots a_k|$ denotes the length of the sequence $a_1 \cdots a_k (a_i \in T)$. We use the symbol ε to denote the empty word.

We use I_{t_i} to denote the set of all individuals of the signal type t_i and consider I_{uv} as $I_u \times I_v$ for $u, v \in T^*$.

Let $\mathcal{A}$ be the set of all building blocks of the VLSI systems. Every line of an element $F \in \mathcal{A}$ has a signal type. Two functions $Q, Z : \mathcal{A} \longrightarrow T^*$ are used to determine the input and output types of the building blocks in $\mathcal{A}$. For instance, the input type and output type of $F \in \mathcal{A}$ are $Q(F)$ and $Z(F)$, respectively. Given two building blocks $F, G \in \mathcal{A}$, we can construct a new building block by using the parallel operation " $\times$ ". The building block constructed in this way is illustrated by Fig. 1. In case the output type $Z(G)$ of G is equal to the input type $Q(F)$ of F, we can construct a new building block by using the sequential operation " $\circ$ ". Fig. 2 shows the new building block constructed by linking the ith output line of G directly to the ith input line of F.

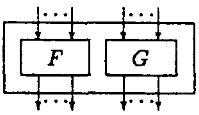

Fig. 1: $F \times G$

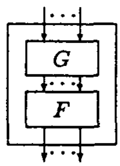

Fig. 2: $F \circ G$

For $F, G \in \mathcal{A}$, it holds

$$Q(F \times G) = Q(F) \cdot Q(G), \qquad Z(F \times G) = Z(F) \cdot Z(G),$$
$$Q(F \circ G) = Q(G), \qquad Z(F \circ G) = Z(F).$$

Let $\mathbf{B}_T = \{B_1, B_2, ...\}$ and $\mathbf{D}_T = \{L_u, D_u, V_{uv} \mid u, v \in T\}$ be two classes of basic building blocks. Assume that $|Q(B)| \in \mathrm{N}$(set of all positive integers) and $|Z(B)| = 1$ for every $B \in \mathbf{B}_T$. In other words, every basic building block B in $\mathbf{B}_T$ has only one output line and at least one input line. The input and output types of B are $Q(B)$ and $Z(B)$, and the input and output values of B are limited to $I_{Q(B)}$ and $I_{Z(B)}$, respectively. The elements in $\mathbf{D}_T$ can be illustrated by the following figures.

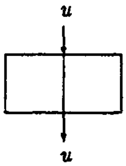

Fig. 3: L_u

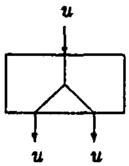

Fig. 4: D_u

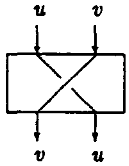

Fig. 5: V_{uv}

L_u denotes a line, D_u a fanout mechanism, and V_{uv} two crossing lines. For the three building blocks, we have $Q(L_u) = u$, $Z(L_u) = u$, $Q(D_u) = u$, $Z(D_u) = uu$, $Q(V_{uv}) = uv$, $Z(V_{uv}) = vu$.

Let $\mathbf{A} = \mathbf{B}_T \cup \mathbf{D}_T$. $\mathcal{A}$, the set of all building blocks of VLSI systems can be formally defined by
(1) $\mathcal{A} \supset \mathbf{A}$.
(2) $F, G \in \mathcal{A} \Longrightarrow F \times G \in \mathcal{A}$.
(3) $F, G \in \mathcal{A}$ and $Z(G) = Q(F) \Longrightarrow F \circ G \in \mathcal{A}$.

It has been shown that $\mathbf{C} = (T^*, \mathcal{A}, Q, Z, \circ)$ is a category, and that $\mathcal{C} = (T^*, \mathcal{A}, Q, Z, \circ, \times)$ is an X-category [Hotz65].

Suppose that t is the 1-bit binary signal type and $T = \{t\}$. If $\mathbf{D}_T = \{L_t, D_t, V_{tt}\}$ and $\mathbf{B}_T$ includes NOT, AND and OR gates, the X-category $\mathcal{C}$ defined above corresponds to the whole combinational VLSI system. $\mathcal{C}$ corresponds to the whole tree system if $\mathbf{D}_T$ does not include the fanout mechanism D_t.

Assume that a building block $F \in \mathcal{A}$ implements a function f and that its domain and codomain are denoted by $Q'(f)$ and $Z'(f)$. ($Q'(f) = I_{Q(F)}$ and $Z'(f) = I_{Z(F)}$). Let $\mathcal{T}^* = \{I_u | u \in T^*\}$. The set

$$\mathcal{F} = \{f \mid f : Q'(f) \to Z'(f) \ \wedge \ Q'(f), Z'(f) \in \mathcal{T}^*\}$$

includes all the functions implemented by the elements of $\mathcal{A}$.

We define two operations $\odot$ and $\otimes$ over $\mathcal{F}$. Given two functions f and g, $f \otimes g$ is a function from $Q'(f)Q'(g)$ to $Z'(f)Z'(g)$. In case $Q'(f) = Z'(g)$, $f \odot g$ is defined as a function from $Q'(g)$ to $Z'(f)$. It can be shown that $\mathbf{K} = (\mathcal{T}^*, \mathcal{F}, Q', Z', \odot)$ is a category and that $\mathcal{K} = (\mathcal{T}^*, \mathcal{F}, Q', Z', \odot, \otimes)$ is an X-category. In the following, we investigate the relationship between the two X-categories $\mathcal{C}$ and $\mathcal{K}$ by using *functors*.

Definition 1 (functor)
Given two categories $\mathbf{C} = (\mathrm{O(C)}, \mathrm{M(C)}, Q, Z, \circ)$ *and* $\mathbf{K} = (\mathrm{O(K)}, \mathrm{M(K)}, Q', Z', \odot)$ *and two mappings* $\phi_1 : \mathrm{O(C)} \to \mathrm{O(K)}$ *and* $\phi_2 : \mathrm{M(C)} \to \mathrm{M(K)}$, $\phi = (\phi_1, \phi_2)$ *is called a functor, provided that the following conditions are satisfied.*

1. $\forall F \in \mathrm{M(C)} \{Q'(\phi_2(F)) = \phi_1(Q(F)) \ \wedge \ Z'(\phi_2(F)) = \phi_1(Z(F))\}$.

2. $\forall F, G \in \mathrm{M(C)} \{Q(F) = Z(G) \Longrightarrow \phi_2(F \circ G) = \phi_2(F) \odot \phi_2(G)\}$.

3. $\forall u \in \mathrm{O(C)} \left\{\phi_2(\mathbf{1}_u) = \mathbf{1}_{\phi_1(u)}\right\}$.

We define the two mappings as
$\phi_1 : \ T^* \to \mathcal{T}^*, \quad \phi_1(u) = I_u, \quad u \in T^*,$
$\phi_2 : \ \mathcal{A} \to \mathcal{F}, \quad \phi_2(F) = f$ if $f : I_{Q(F)} \to I_{Z(F)}, \ F \in \mathcal{A}$

It is easy to check that:
1, $\forall F \in \mathcal{A} \{Q'(\phi_2(F)) = \phi_1(Q(F)) \ \wedge \ Z'(\phi_2(F)) = \phi_1(Z(F))\}$.
2, $\forall F, G \in \mathcal{A} \{Q(F) = Z(G) \Longrightarrow \phi_2(F \circ G) = \phi_2(F) \odot \phi_2(G)\}$.
3, $\forall u \in T^* \left\{\phi_2(\mathbf{1}_u) = \mathbf{1}_{\phi_1(u)}\right\}$.

Thus, $\phi = (\phi_1, \phi_2)$ is a functor from **C** to **K**, and a functor from $\mathcal{C}$ to $\mathcal{K}$ as well since

$$\forall F, G \in \mathcal{A}\, \{\phi_2(F \times G) = \phi_2(F) \otimes \phi_2(G)\}.$$

Thereafter, we use operators $\circ$ and $\times$ to replace $\odot$ and $\otimes$, and substitute the mappings Q and Z for Q' and Z' provided that no confusion can be caused. For the sake of simplicity, we often call a basic building block *cell* and use a lower case letter to represent a function implemented by a building block represented by the corresponding upper case letter. For example, b is used to represent the function implemented by $B \in \mathcal{A}$. L_t is used to denote a line transferring signals of type t. Furthermore, we use u to represent $\phi_1(u)$ ($u \in T^*$), namely the set of values of type u. For instance, we use the form $f : t^k \longrightarrow t$ to represent a function f from $\phi_1(t^k)$ to $\phi_1(t)$.

Every building block $F \in \mathcal{A}$ implements a function $\phi_2(F) : Q(F) \rightarrow Z(F)$. For example, L_u realizes an identity on u, D_u a function from u to $u \times u$, and V_{uv} a function from $u \times v$ to $v \times u$. Suppose x and y are two individuals of type u and v, respectively, then

$$\phi_2(L_u)(x) = x, \quad \phi_2(D_u)(x) = xx \quad \text{and} \quad \phi_2(V_{uv})(xy) = yx.$$

For the rest of the paper, we assume that $T = \{t\}$, $\mathbf{B}_T = \{B\}$, $\mathbf{D}_T = \{L_u, V_{uv} | u, v \in T\}$, $Q(B) = \underbrace{t \cdots t}_{k}$ and $Z(B) = t$. We use $\mathcal{A}$ to denote the set of all uniform trees based on the unique basic cell B, and $T_b^{(n)}$ to denote a balanced uniform tree consisting of b-cells and having n primary input lines. Our discussion will deal with the arrangement problem for $T_b^{(n)}$.

For distinguishing from the Cartesian product t^l, we use $t^{(l)}$ to denote the set of all l-dimensional vectors consisting of elements of type t. Given a function $f : t^k \longrightarrow t$ and an integer $l \in \mathrm{N}$, we use $f^{(l)}$ to denote a vector function defined as follows:

$$f^{(l)}(\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_k) = \begin{pmatrix} f(v_{11}, v_{12}, \cdots, v_{1k}) \\ f(v_{21}, v_{22}, \cdots, v_{2k}) \\ \vdots \\ f(v_{l1}, v_{l2}, \cdots, v_{lk}) \end{pmatrix}, \qquad \vec{v}_i \in t^{(l)}.$$

Example 1: Function f_1(logical function NAND) is defined by

f_1	0	1
0	1	1
1	1	0

For $\vec{v}_0 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $f_1^{(3)}(\vec{v}_1, \vec{v}_2) = \begin{pmatrix} f_1(1,1) \\ f_1(1,0) \\ f_1(0,1) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \vec{v}_0$

Definition 2 (similar matrices)
Let $\vec{u}_i, \vec{v}_i \in t^{(l)} (i \in [1, k])$. Consider $(\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_k)$ and $(\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_k)$ as two $l \times k$ matrices. They are said to be similar, *denoted by $(\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_k) \sim (\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_k)$, if and only if they can be transformed into each other by row exchanges.* (It is easy to see that $\sim$ is an equivalence relation.)

For example, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$. For the arrangement of a uniform tree $T_b^{(n)}$, we are concerned about a predefined property which is expected to be satisfied for every b-cell in the tree. We use the *predicate* $\mathbf{P}_b$ to describe this property. The following is the formal definition of arrangement problem. An example following the definition gives a further explanation of it.

Definition 3 (arrangement, arrangement complexity)
Given a function $b : t^k \longrightarrow t$ and a predefined property P, we define the predicate $\mathbf{P}_b$ as

$$\mathbf{P}_b(\vec{v}_0, \vec{v}_1, \cdots, \vec{v}_k) = \begin{cases} \text{true} & : \quad \begin{array}{c} \vec{v}_0 = b^{(l)}(\vec{v}_1, \cdots, \vec{v}_k) \quad \textit{and} \\ (\vec{v}_0, \vec{v}_1, \cdots, \vec{v}_k) \quad \text{has property} \quad P \end{array} \\ \\ \text{false} & : \qquad\qquad \text{otherwise} \end{cases} \tag{1}$$

where l is an arbitrary integer in N, *and $\vec{v}_i \in t^{(l)}$ for $i \in [0, k]$. In addition, the predicate satisfies (2)*

$$(\vec{u}_0, \vec{u}_1, \cdots, \vec{u}_k) \sim (\vec{v}_0, \vec{v}_1, \cdots, \vec{v}_k) \Longrightarrow \mathbf{P}_b(\vec{u}_0, \vec{u}_1, \cdots, \vec{u}_k) = \mathbf{P}_b(\vec{v}_0, \vec{v}_1, \cdots, \vec{v}_k) \tag{2}$$

We call n l-dimensional vectors of $t^{(l)}$ (l n-component patterns of t^n) a complete arrangement set for $T_b^{(n)}$ if by applying them to the n primary input lines of $T_b^{(n)}$, $\mathbf{P}_b(\vec{v}_0, \vec{v}_1, \cdots, \vec{v}_k)$ is true for every b-cell in $T_b^{(n)}$. Here, $\vec{v}_0$ is the output vector and $\vec{v}_1$, ..., $\vec{v}_k$ are the input vectors of the cell.

The arrangement complexity l of $T_b^{(n)}$ is defined as the minimum of the dimensions of the complete arrangement sets for $T_b^{(n)}$.

For example, if we define the predicate $\mathbf{P}_b$ as the following

$$\mathbf{P}_b(\vec{v}_0, \vec{v}_1, \cdots, \vec{v}_k) = \begin{cases} \text{true} & : \quad \begin{array}{c} \vec{v}_0 = b^{(l)}(\vec{v}_1, \cdots, \vec{v}_k) \quad \text{and} \\ t^k \subset \{(v_{i1}, \cdots, v_{ik}) \,|\, i = 1, 2, ..., l\} \end{array} \\ \\ \text{false} & : \qquad\qquad \text{otherwise} \end{cases}$$

where $\vec{v}_i \in t^{(l)}$, then the input set to every b-cell in the tree includes t^k, namely all possible input combinations, and the above arrangement problem becomes the assignment problem[Wu93].

It can be shown that the arrangement problem becomes a test problem when we give another interpretation to the function b and the predicate $\mathbf{P}_b$[Wu92].

3 $\Theta(1)$ Arrangeable Trees

The arrangement complexity of $F \in \mathcal{A}$ depends on the property of $\phi_2(B)$, namely b. In the following, we show that one can construct a complete arrangement set of a constant size for every $F \in \mathcal{A}$ if the function b has a special property described later. In order to describe this property, we require a new symbol D_u^i which is defined for all $u \in T^*$ by

(1) $D_u^2 = D_u$, (2) $D_u^{i+1} = (D_u^i \times L_u) \circ D_u$.

The logical structure of D_u^{i+1} is illustrated by Fig. 6

In the rest of the paper, we use σ to denote the type of $\underbrace{t \cdots t}_{k}$.

Definition 4 (b-**stable set)**
A set $S \subset t^k$ is b-stable *if and only if there are bijective mappings $\pi_1, ..., \pi_k : S \to S$ such that $(b \circ \pi_1 \times \cdots \times b \circ \pi_k) \circ D_\sigma^k(S) = S$.*

The following lemma states that if the set $S \in t^k$ is b-stable, we can construct an identity mapping for the set S, and the mapping includes the function b as its component.

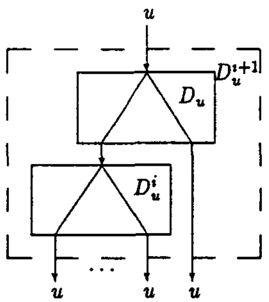

Fig. 6 D_u^{i+1}

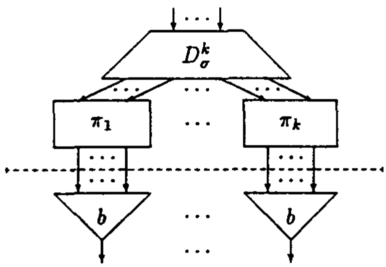

Fig. 7 f_b

Lemma 1 *If set $S \subset t^k$ is b-stable, there are k bijective mappings $\pi_1, ..., \pi_k : S \to S$ such that*

$$\underbrace{(b \times \cdots \times b)}_{k} \circ (\pi_1 \times \cdots \times \pi_k) \circ D_\sigma^k \tag{3}$$

is the identity on S.

Proof: Suppose $S \subset t^k$ is b-stable and $(b \circ \pi_1 \times \cdots \times b \circ \pi_k) \circ D_\sigma^k$ is a bijective mapping from S to S. Define a bijective mapping $\pi_0 : S \to S$ as the inverse mapping of $(b \circ \pi_1 \times \cdots \times b \circ \pi_k) \circ D_\sigma^k$. Hence,

$$(b \circ \pi_1 \times \cdots \times b \circ \pi_k) \circ D_\sigma^k \circ \pi_0|_S \quad = \quad id|_S,$$

where $id|_S$ is the identity on S. Because $D_\sigma^k \circ \pi_0 = \underbrace{(\pi_0 \times \cdots \times \pi_0)}_{k} \circ D_\sigma^k$, it holds

$$\begin{aligned} (b \circ \pi_1 \times \cdots \times b \circ \pi_k) \circ D_\sigma^k \circ \pi_0 &= (b \circ \pi_1 \times \cdots \times b \circ \pi_k) \circ \underbrace{(\pi_0 \times \cdots \times \pi_0)}_{k} \circ D_\sigma^k \\ &= (b \times \cdots \times b) \circ (\pi_1 \circ \pi_0 \times \cdots \times \pi_k \circ \pi_0) \circ D_\sigma^k. \end{aligned}$$

By substituting $\pi_i \circ \pi_0$ with π_i, we obtain the lemma.

Q.E.D.

We use f_b to denote the mapping defined by (3). Fig. 7 illustrates the structure of f_b. It consists of two parts. The upper part $(\pi_1 \times \cdots \times \pi_k) \circ D_\sigma^k$ is a fanout mechanism, and it has one input line and k output lines of type σ (k input lines and k^2 output lines of type t) and implements a mapping from t^k to t^{k^2}. The lower part $\underbrace{(b \times \cdots \times b)}_{k}$ consists of k parallel b cells. A b-cell can be considered as a uniform tree with k input lines of type t (an input line of type σ) and an output line of type t. Then, $\underbrace{(b \times \cdots \times b)}_{k}$ can be considered as a building block made up of k uniform trees. It has k input lines and one output line of type σ and implements a mapping from t^{k^2} to t^k. The two parts together realize the indentity mapping on t^k according to Lemma 1. The idea

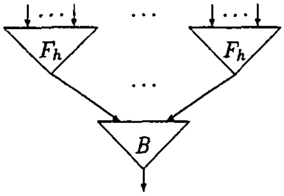

Fig. 8 F_{h+1}

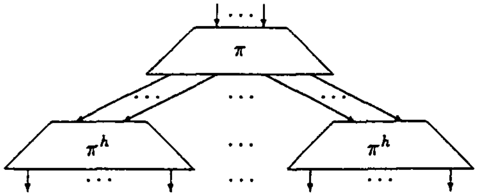

Fig. 9 π^{h+1}

here is to construct a building block which includes uniform trees and realizes the identity on a set S. By applying S to the building block, S is assigned to every uniform tree inside the building block. In the following we use this idea again to construct a family of identity mappings $F^h(h \in \mathbf{N})$. F^h includes h-level balanced uniform trees based on b.

We use π to denote $(\pi_1 \times \cdots \times \pi_k) \circ D^k_\sigma$. Thus, f_b represents $\underbrace{(b \times \cdots \times b)}_{k} \circ \pi$. At first, we give the recursive definitions of the balanced uniform trees F_h and the fanout mechanisms $\pi^h(h \in \mathbf{N})$.

The recursive definition for F_h is the following:

$$F_0 = L_t \tag{4}$$

$$F_{h+1} = B \circ \underbrace{(F_h \times \cdots \times F_h)}_{k}, \quad h \in \mathbf{N} \tag{5}$$

Fig. 8 shows F_{h+1}. It is easy to see that F_h $(h \in \mathbf{N})$ is a balanced uniform tree of level h. The π^h-family is defined as follows:

$$\pi^0 = L_t \tag{6}$$

$$\pi^{h+1} = \underbrace{(\pi^h \times \cdots \times \pi^h)}_{k} \circ \pi, \quad h = 1, 2, \ldots \tag{7}$$

Fig. 9 shows π^{h+1}. Using F_h and π^h we can define the F^h-family by

$$F^h = \underbrace{(F_h \times \cdots \times F_h)}_{k} \circ \pi^h, \qquad h \in \mathbf{N} \tag{8}$$

To put it differently, it holds

$$F^h = \underbrace{(F_h \circ \pi^{h-1} \times \cdots \times F_h \circ \pi^{h-1})}_{k} \circ \pi \tag{9}$$

Fig. 10 shows F^h.

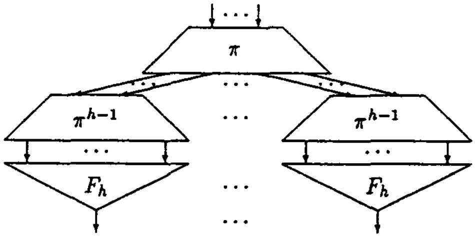

Fig. 10 F^h

According to the above definition, we have

$$\phi_2(F^1) = \phi_2((\underbrace{B \times \cdots \times B}_{k}) \circ \pi) = f_b.$$

Let $f^h = \phi_2(F^h)$ for $h \in \mathrm{N}$. Then, f^h is a mapping from S to S and it is a generalization of f_b. The following lemma holds.

Lemma 2 *$f^h|_S = id|_S$ holds for all $h \in \mathrm{N}$.*

Proof: We prove this lemma by using induction on the parameter h. For $h = 1$, we have $f^1 = f_b$ and $f^1|_S = id|_S$. Suppose $f^{l-1}|_S = id|_S$ holds. According to the above definition

$$\begin{aligned} F_h \circ \pi^{h-1} &= B \circ (\underbrace{F_{h-1} \times \cdots \times F_{h-1}}_{k}) \circ \pi^{h-1} \\ &= B \circ F^{h-1}, \qquad\qquad h \in \mathrm{N}. \end{aligned}$$

Hence

$$\begin{aligned} F^l &= (\underbrace{F_l \times \cdots \times F_l}_{k}) \circ \pi^l \\ &= (\underbrace{F_l \times \cdots \times F_l}_{k}) \circ (\underbrace{\pi^{l-1} \times \cdots \times \pi^{l-1}}_{k}) \circ \pi \\ &= (\underbrace{F_l \circ \pi^{l-1} \times \cdots \times F_l \circ \pi^{l-1}}_{k}) \circ \pi \\ &= (\underbrace{B \circ F^{l-1} \times \cdots \times B \circ F^{l-1}}_{k}) \circ \pi \\ &= (\underbrace{B \times \cdots \times B}_{k}) \circ (\underbrace{F^{l-1} \times \cdots \times F^{l-1}}_{k}) \circ \pi. \end{aligned}$$

Thus

$$\begin{aligned} f^l|_S &= \phi_2(F^l)|_S \\ &= (\underbrace{b \times \cdots \times b}_{k}) \circ (\underbrace{f^{l-1} \times \cdots \times f^{l-1}}_{k}) \circ \pi|_S \\ &= (\underbrace{b \times \cdots \times b}_{k}) \circ \pi|_S \\ &= f^1|_S \\ &= id|_S. \end{aligned}$$

Thus, $f^h|_S = id|_S$ holds for all $h \in \mathrm{N}$.

Q.E.D.

Theorem 1 *If $S \subset t^k$ is b-stable and a minimal arrangement set for cell B, $\#S$ is an upper bound to the minimum of the cardinalities of the complete arrangement sets for each of the trees in $\mathcal{A}$.*

Proof: Given a tree $\mathcal{T}' \in \mathcal{A}$, one can always find a balanced tree $\mathcal{T} \in \mathcal{A}$ such that $\mathcal{T}'$ can be embedded into $\mathcal{T}$. The arrangement complexity of $\mathcal{T}$ is an upper bound for that of $\mathcal{T}'$. We can construct a complete arrangement set of a constant size for every tree in $\mathcal{A}$, if we can construct a complete arrangement set of a constant size for every balanced tree in $\mathcal{A}$. In other words, all trees in $\mathcal{A}$ are constant arrangeable if all balanced trees in $\mathcal{A}$ are constant arrangeable.

A balanced tree $F_{h+1} \in \mathcal{A}$ has the structure $B \circ \underbrace{(F_h \times \cdots \times F_h)}_{k}$. Suppose $S \subset t^k$ is b-stable and an arrangement set for cell B. In the above, we showed that for every $h \in \mathrm{N}$, it holds

$$\phi_2(F^h) = \phi_2(\underbrace{(F_h \times \cdots \times F_h)}_{k} \circ \pi^h|_S) = id|_S$$

and $\underbrace{(F_h \times \cdots \times F_h)}_{k} \circ \pi^h(S) = S$. This indicates that when $\pi^h(S)$ is applied to the primary input lines of F_{h+1}, the vectors applied to an arbitrary cell B inside F_{h+1} comprise an arrangement set. Thus $\pi^h(S)$ is just a complete arrangement set for F_{h+1} and $\#S$ is an upper bound to the minimum of the cardinalities of the complete arrangement sets for F_{h+1}.

Q.E.D.

Corollary 1 *If $S \subset t^k$ is b-stable and an optimal arrangement set for cell B, $\pi^h(S)$ is an optimal complete arrangement set for F_{h+1}.*

Theorem 2 *The arrangement complexity of $T_b^{(n)}$ is $\Theta(1)$, if there exist an $i \in \mathrm{N}$, a subset $S \subset t^{k^i}$ and k^i bijective mappings $\pi_1, ..., \pi_{k^i} : S \longrightarrow S$ such that S is a complete arrangement set for F_i and it holds*

$$\underbrace{(F_i \times \cdots \times F_i)}_{k^i} \circ \underbrace{(\pi_1 \times \cdots \times \pi_{k^i})}_{k^i} \circ D_\sigma^{k^i}(S) = S.$$

Proof: Suppose there are an $i \in \mathrm{N}$, a subset $S \subset t^{k^i}$, and k^i bijective mappings $\pi_1, ..., \pi_{k^i} : S \longrightarrow S$ such that S is a complete arrangement set for F_i and

$$\underbrace{(F_i \times \cdots \times F_i)}_{k^i} \circ \underbrace{(\pi_1 \times \cdots \times \pi_{k^i})}_{k^i} \circ D_\sigma^{k^i}(S) = S. \tag{10}$$

Let $\kappa = k^i$ and $g = \phi_2(F_i)$. Based on (10), S is g-stable. Let $l = \#S$. We define

$$\mathbf{P}_g(\vec{v}_0, \vec{v}_1, \cdots, \vec{v}_\kappa) = \begin{cases} \text{true} & : \begin{array}{c} \vec{v}_0 = g^{(l)}(\vec{v}_1, \cdots, \vec{v}_\kappa) \text{ and} \\ (\vec{v}_1, \cdots, \vec{v}_\kappa) \text{ is a complete arrangement for } F_i \end{array} \\ \\ \text{false} & : \quad \text{otherwise} \end{cases}$$

where $\vec{v}_i \in t^{(l)}$. As mentioned above, S is g-stable. Based on Theorem 1, we can state that $\#S$ is an upper bound to the minimum of the cardinalities of the arrangement sets for $T_g^{(n)}$ and that the arrangement complexity of $T_b^{(n)}$ is $\Theta(1)$.

Q.E.D.

The proofs of Theorem 1 and 2 are constructive. In fact, we can derive algorithms to construct a minimal complete arrangement set for the balanced uniform tree based on function b.

4 The Jump from $\Theta(1)$ to $\Omega((\lg n)^\gamma)$

In this section, we give another criteria of $\Theta(1)$ arrangeable uniform trees and show that the arrangement complexity of $T_b^{(n)}$ is either $\Theta(1)$ or $\Omega((\lg n)^\gamma)(\gamma > 0)$.

We assign every line and b-cell in $T_b^{(n)}$ a unique level. The levels will be arranged in ascending order from the primary output to the primary inputs of $T_b^{(n)}$. The primary output is assigned level 0. A b-cell and all its input lines are assigned level $k+1$, if its output line is in level k. $T_b^{(n)}$ is said to be of k-level, if it has k levels.

According to the definition of the complete arrangement set, we can define a predicate $\mathbf{P}_b$ based on the given function $b : t^k \longrightarrow t$ and the predefined property P, such that n l-dimensional vectors of $t^{(l)}$ comprise a complete arrangement set for $T_b^{(n)}$ if and only if by applying them to the n primary inputs lines of $T_b^{(n)}$, $\mathbf{P}_b(\vec{v}_0, \vec{v}_1, \cdots, \vec{v}_k)$ is true for every b-cell inside $T_b^{(n)}$. Here, $\vec{v}_0$ is the output vector and $\vec{v}_1, \cdots, \vec{v}_k$ are input vectors of that b-cell.

Lemma 3 *$T_b^{(n)}$ is $\Theta(1)$ arrangeable if there are an $l \in \mathrm{N}$ and a set $W \subset t^{(l)}$ such that for every $\vec{v}_0 \in W$, there are k vectors $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_k$ in W and $\mathbf{P}_b(\vec{v}_0, \vec{v}_1, \cdots, \vec{v}_k)$ is true. Put it formally:*

$$\exists l \in \mathrm{N}\ \exists W \subset t^{(l)}\ \forall \vec{v}_0 \in W\ \exists \vec{v}_1, \cdots, \vec{v}_k \in W\ \{\mathbf{P}_b(\vec{v}_0, \vec{v}_1, \cdots, \vec{v}_k)\} \tag{11}$$

Proof: Suppose $\exists l \in \mathrm{N}\ \exists W \subset t^{(l)}\ \forall \vec{v}_0 \in W\ \exists \vec{v}_1, \cdots, \vec{v}_k \in W\ \{\mathbf{P}_b(\vec{v}_0, \vec{v}_1, \cdots, \vec{v}_k)\}$. We prove that for every N-level $T_b^{(n)}(n \leq k^N)$, there are n vectors $\vec{v}_1, \cdots, \vec{v}_k$ in W and they comprise a complete arrangement set for $T_b^{(n)}$. This can be done by induction on the number of the levels of $T_b^{(n)}$.

In the case $N = 1$, the tree has only one cell. We choose $\vec{v}_0 \in W$ arbitrarily, then determine k vectors $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_k \in W$ such that $\mathbf{P}_b(\vec{v}_0, \vec{v}_1, \cdots, \vec{v}_k)$ is true. It is clear that $(\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_k)$ is an arrangement set for cell B, a tree having only one cell.

Assume that for $N = i$, the vectors $\vec{v}_{1,0}, \vec{v}_{2,0}, \cdots, \vec{v}_{k^i,0} \in W$ comprise a complete arrangement set for $T_b^{(k^i)}$. Suppose $T_b^{(k^{i+1})}$ is of $(i+1)$ levels and is constructed by connecting every primary input line in $T_b^{(k^i)}$ to the output line of a b-cell. According to the assumption,

$$\forall j \in [1, k^i]\ \exists \vec{v}_{j,0}, \vec{v}_{j,1}, \cdots, \vec{v}_{j,k} \in W\ \{\mathbf{P}_b(\vec{v}_{j,0}, \vec{v}_{j,1}, \cdots, \vec{v}_{j,k})\}.$$

Hence, $(\vec{v}_{j,0}, \vec{v}_{j,1}, \cdots, \vec{v}_{j,k})$ is an arrangement set for a b-cell. When $\vec{v}_{j,1}, \vec{v}_{j,2}, \cdots, \vec{v}_{j,k}$ are applied to the k input lines of the cell B directly linked to the jth input line in the level i, the vector offered to this input line is just $\vec{v}_{j,0}$. Thus, we can state that the k^{i+1} l-dimensional vectors $\vec{v}_{1,1}, \vec{v}_{1,2}, \cdots, \vec{v}_{1,k}, \vec{v}_{2,1}, \cdots, \vec{v}_{k^i,k}$ comprise a complete arrangement set for $T_b^{(k^{i+1})}$.

Q.E.D.

Corollary 2 *$T_b^{(n)}$ is $\Theta(1)$ arrangeable if*

$$\exists l \in \mathrm{N}\ \exists W' \subset t^{(l)}\ \forall \vec{u}_0 \in W'\ \exists \vec{v}_0, \vec{v}_1, \cdots, \vec{v}_k \in W'\ \{\mathbf{P}_b(\vec{v}_0, \vec{v}_1, \cdots, \vec{v}_k) \wedge \vec{u}_0 \sim \vec{v}_0\} \tag{12}$$

Proof: Given a set $W' \subset t^{(l)}(l \in \mathrm{N})$, we can always induce a set W such that

$$\forall \vec{v} \in t^{(l)}\ \{\exists \vec{u} \in W'\ \{\vec{u} \sim \vec{v}\} \Longrightarrow \vec{v} \in W\}\ \wedge\ \forall \vec{v} \in W\ \exists \vec{u} \in W'\ \{\vec{u} \sim \vec{v}\}.$$

The set W includes every vector which is *similar* to a vector in W'. It is obvious that such a W satisfies (11) if W' fulfills (12).

Q.E.D

Assume that $T_b^{(n)}$ consists of b-cells $C_{1,1}, C_{2,1}, C_{2,2}, ..., C_{k,m}, ...$, and cell $C_{i,j}$ is the jth cell in the ith level of $T_b^{(n)}$. Let A denote n vectors in $t^{(l)}$ and apply A to this tree. Let W denote the corresponding set including A and all other vectors delivered to other lines in every level of $T_b^{(n)}$. We use (A, i, j, m) to represent the corresponding vector applied to the mth input line of $C_{i,j}$, and $(A, i, j, 0)$ to represent the vector delivered to the output line of $C_{i,j}$.

Given a complete arrangement set A for an N-level $T_b^{(n)}$, we determine N sets in the following way:

$$W_s(A) := \left\{(A,i,j,m) \mid i \in [1,s], j \in [1,k^{i-1}], m \in [0,k]\right\}, \quad s \in [1,N] \tag{13}$$

$W_s(A)$ includes all vectors delivered to a line in level i ($i \in [1,s]$) and the vector delivered to the primary output line. According to Definition 2, $\sim$ is an equivalence relation. $W_s(A)$ can be partitioned into equivalence classes according to the equivalence relation $\sim$ and we use $\#W_s(A)/_\sim$ to denote the number of equivalence classes in $W_s(A)$. The following observation is obvious:

Observation 1 *Assume A to be a complete arrangement set for an N-level $T_b^{(n)}$. Then it holds*

1. $\forall s \in [2,N]\left\{W_{s-1}(A) \subseteq W_s(A) \subseteq t^{(l)}\right\}$.
2. $\forall s \in [2,N]\{1 \le \#W_{s-1}(A)/_\sim \le \#W_s(A)/_\sim\}$.

Lemma 4 *Assume A to be a complete arrangement set for an N-level $T_b^{(n)}$. Then, $T_b^{(n)}$ is $\Theta(1)$ arrangeable if $\#W_s(A)/_\sim = \#W_{s-1}(A)/_\sim$ for an $s \in [2,N]$.*

Proof: Assume A to be a complete arrangement set for an N-level $T_b^{(n)}$. Suppose $\#W_s(A)/_\sim = \#W_{s-1}(A)/_\sim$ for an $s \in [2,N]$. Based on Observation 1, $W_{s-1}(A) \subseteq W_s(A)$ for all $s \in [2,N]$. According to the definition of $W_s(A)$, we can state that

$$\forall \vec{u}_0 \in W_s(A)\ \exists \vec{v}_0, \vec{v}_1, \cdots, \vec{v}_k \in W_s(A)\,\{\mathbf{P}_b(\vec{v}_0, \vec{v}_1, \cdots, \vec{v}_k) \wedge \vec{u}_0 \sim \vec{v}_0\}.$$

Based on Corollary 2, $T_b^{(n)}$ is $\Theta(1)$ arrangeable.

Q.E.D.

Lemma 5 *For every complete arrangement set A for an N-level $T_b^{(n)}$, it holds*

$$\forall s \in [1,N]\,\{\#W_s(A)/_\sim \ge s\} \tag{14}$$

if $T_b^{(n)}$ is not $\Theta(1)$ arrangeable.

Proof: Suppose $T_b^{(n)}$ is not $\Theta(1)$ arrangeable. According to Observation 1 and Lemma 4,

$$(\#W_1(A)/_\sim \ge 1) \wedge (\forall s \in [2,N]\,\{\#W_s(A)/_\sim > \#W_{s-1}(A)/_\sim\}).$$

Therefore, $\#W_s(A)/_\sim \ge s$.

Q.E.D.

Given an arbitrary $l \in N$, we can partition $t^{(l)}$ into a number of equivalence classes according to the equivalence relation $\sim$. Let $\#t^{(l)}/_\sim$ denote the number of equivalence classes of $t^{(l)}$.

Observation 2 *For every N-level $T_b^{(n)}$, its complete arrangement A consists of vectors in $t^{(l)}$ for an $l \in \mathrm{N}$ and satisfies*

$$1 \le \#W_N(A)/_\sim \le \#t^{(l)}/_\sim \tag{15}$$

Theorem 3 *$T_b^{(n)}$ is $\Theta(1)$ arrangeable if and only if there are an $l \in \mathrm{N}$ and a set $W \subset t^{(l)}$ such that*

$$\forall \vec{u}_0 \in W\ \exists \vec{v}_0, \vec{v}_1, \cdots, \vec{v}_k \in W\,\{\mathbf{P}_b(\vec{v}_0, \vec{v}_1, \cdots, \vec{v}_k)\ \wedge\ \vec{u}_0 \sim \vec{v}_0\} \tag{16}$$

Proof: The *if* part follows immediately from Corollary 2. Assume $T_b^{(n)}$ to be $\Theta(1)$ arrangeable. There is a constant $l \in \mathrm{N}$, and one can determine a complete arrangement set A, which is made up of vectors in $t^{(l)}$, to an arbitrary $T_b^{(n)}$. Suppose $\lceil \lg n \rceil = N$ and $N > \#t^{(l)}/_\sim$. Since

$$\forall s \in [1,N]\left\{\#W_s(A)/_\sim \le \#t^{(l)}/_\sim < N\right\},$$

there must be such an $s \in [2, N]$ that $\#W_s(A)/_\sim = \#W_{s-1}(A)/_\sim$ and

$$\forall \vec{u}_0 \in W_s(A)\ \exists \vec{v}_0, \vec{v}_1, \cdots, \vec{v}_k \in W_s(A)\ \{\mathbf{P}_b(\vec{v}_0, \vec{v}_1, \cdots, \vec{v}_k)\ \wedge\ \vec{u}_0 \sim \vec{v}_0\}$$

Q.E.D

Now, we show that the arrangement complexity of $T_b^{(n)}$ is either $\Theta(1)$ or $\Omega((\lg n)^\gamma)(\gamma > 0)$. In other words, there is a jump from $\Theta(1)$ to $\Omega((\lg n)^\gamma)$.

Theorem 4 *The arrangement complexity of $T_b^{(n)}$ is either $\Theta(1)$ or $\Omega((\lg n)^\gamma)$.*

Proof: Suppose the arrangement complexity of $T_b^{(n)}$ is not $\Theta(1)$ and A is a complete arrangement set for $T_b^{(n)}$. Let $s = \lceil \lg n \rceil$. Based on Lemma 5, we have

$$\#t^{(l)}/_\sim \geq \#W_s(A)/_\sim \geq s \geq \lg n$$

It is not hard to see that $\#t^{(l)}/_\sim$ is equal to the number of ways of inserting $l-1$ spaces into the sequence of $|t|$ 1's (see Lemma 5, in Wu93). We can show that

$$\begin{aligned} \#t^{(l)}/_\sim &= \binom{l+|t|-1}{|t|-1} \\ l^{|t|} &> \binom{l+|t|-1}{|t|-1} \end{aligned}$$

for $l > 1$. This means that $l \geq s^{\frac{1}{|t|}} \geq (\lg n)^{\frac{1}{|t|}}$. Hence, we have the theorem.

Q.E.D

A function f is said to be commutative, if for every permutation $(q_1, q_2, ..., q_k)$ of $(1, 2, ..., k)$, it holds $\mathbf{P}_b(\vec{v}_0, \vec{v}_1, \vec{v}_2, \cdots, \vec{v}_k) = \mathbf{P}_b(\vec{v}_0, \vec{v}_{q_1}, \vec{v}_{q_2}, \cdots, \vec{v}_{q_k})$.

Theorem 5 *For a commutative function b, the arrangement complexity of $T_b^{(n)}$ is $\Theta(1)$ if and only if there exist an $i \in \mathrm{N}$, a subset $S \subset t^{k^i}$ and k^i bijective mappings $\pi_1, ..., \pi_{k^i} : S \to S$ such that S is a complete arrangement set for F_i and*

$$\underbrace{(F_i \times \cdots \times F_i)}_{k^i} \circ \underbrace{(\pi_1 \times \cdots \times \pi_{k^i})}_{k^i} \circ D_\sigma^{k^i}(S) = S$$

where σ denotes the type of S.

Proof: Suppose f is commutative. The *if* part immediately follows from Theorem 2. We only have to treat the *only if* part.

In the same way used to analyze the assignment complexity of balanced uniform trees based on commutative functions(see Theorem 6 in Wu93), we can show that the arrangement complexity of $T_b^{(n)}$ is $\Theta(1)$ only if

$$\exists l \in \mathrm{N}\ \exists \vec{v}_0, \vec{v}_1, \vec{v}_2, \cdots, \vec{v}_k \in t^{(l)}\ \{\mathbf{P}_b(\vec{v}_0, \vec{v}_1, \vec{v}_2, \cdots, \vec{v}_k)\ \wedge\ \forall j \in [1, k]\{\vec{v}_j \sim \vec{v}_0\}\}. \quad (17)$$

Let l be the minimum of all integers satisfying (17). Then, there are an $i \in \mathrm{N}$ and k^i vectors $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_{k^i}$ in $t^{(l)}$ such that they are similar to each other and comprise a complete arrangement set for an i-level balanced uniform tree F_i. If we consider $(\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_{k^i})$ as an $l \times k^i$ matrix, it contains no duplicate rows.

Let $\kappa = k^i$ and $v_{m,j}$ denote the mth component of $\vec{v}_j$. Define

$$S = \{(v_{m,1}, v_{m,2}, \cdots, v_{m,\kappa}) \mid m \in [1, l]\}.$$

Suppose that the output vector is $\vec{v}_0$ when S is applied to F_i. According to the assumption, it holds

$$\forall j \in [1, \kappa]\{(\vec{v}_j \rightsquigarrow \vec{v}_0)\}.$$

To put it differently, $\vec{v}_0$ can be transformed to every vector $\vec{v}_j (j \in [1, \kappa])$ by exchanging its component positions. This means that there are κ bijective mappings $\pi_1, ..., \pi_\kappa : S \rightarrow S$ such that

$$\underbrace{(F_i \times \cdots \times F_i)}_{k'} \circ \underbrace{(\pi_1 \times \cdots \times \pi_{k'})}_{k'} \circ D_\sigma^\kappa(S) = S$$

where σ denotes the type of S. S is a complete arrangement set for F_i and the arrangement complexity of F_h is $\Theta(1) (h = i \times j, j \in \mathrm{N})$.

For an arbitrarily given $T_b^{(n)}$, we can always find an F_h such that F_h covers $T_b^{(n)}$. Thus, the arrangement complexity of $T_b^{(n)}$ is $\Theta(1)$.

Q.E.D

Conclusions

In this paper, we studied the arrangement complexity of uniform trees which can be used as mathematical models for information systems in different fields. In these trees, the information is received from the leaves, then transformed and propagated to their root. The arrangement complexity of a given tree is a measure of the minimal quantum of information required to be input to the leaves such that the input and output information of each of the nodes of the tree satisfies a predefined property. This paper shows that the arrangement complexity of a balanced uniform tree is either $\Theta(1)$ or $\Omega((\lg n)^\gamma)(\gamma > 0)$.

Suppose that every node of a given tree implements a mapping from t to t^k, and the information is received from the root of the tree, then transformed and propagated to the leaves. In contrast to the trees studied in this paper, such a tree can be considered as *contravariant tree*. It is worthwhile to study the arrangement complexity of the contravariant trees and the structure of their arrangement complexity.

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