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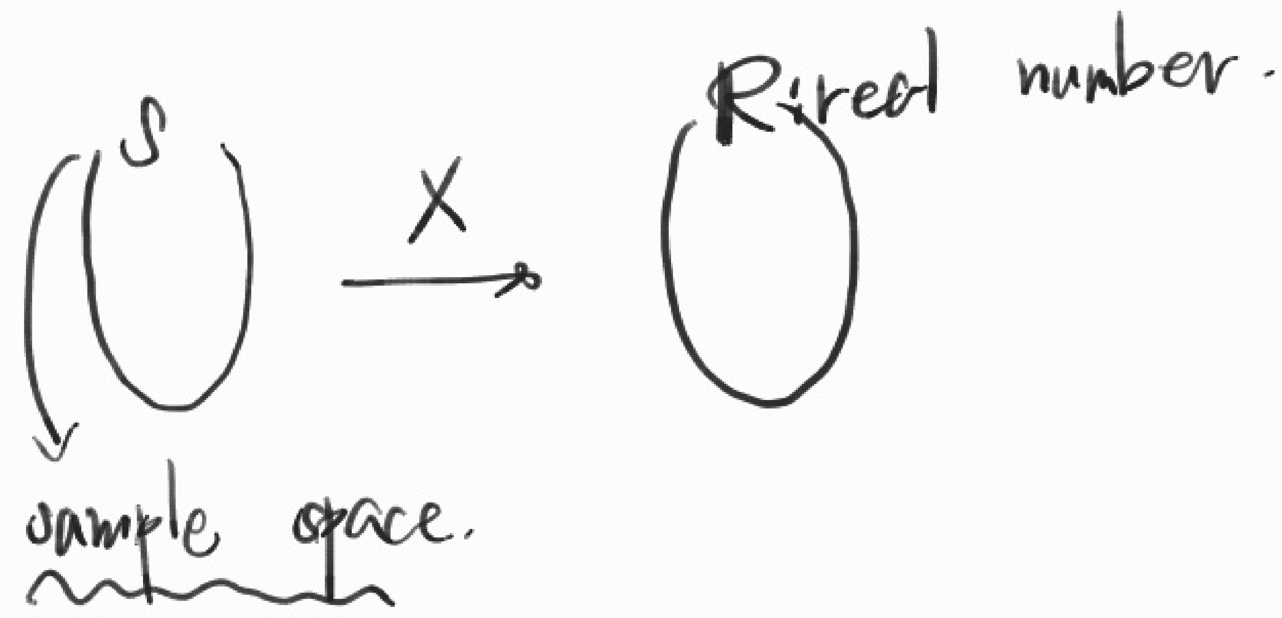
Ch 3. Random variables and probability distribution

Ch 4. Mathmatical expactation

What is a random variable?



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1. $f(x) \geq 0$

2. $\sum_x f(x) = 1$ 　　　　 2. $\int_{-\infty}^{\infty} f(x)\,dx = 1$

3. $P(X=a) = f(a)$ 　　　　 3. $P(a<X<b) = \int_a^b f(x)\,dx$

if X is descrete 　　　　 if X is continuous

$\boxed{S} \xrightarrow{X} \boxed{\mathbb{R}} \xrightarrow[f(a)]{P} \boxed{\text{probability}}$

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ex) 100 100

$S = \{(h,h), (h,t), (t,h), (t,t)\}$

$\downarrow$ X: the number of head coins

X: 2 1 1 0

$P(X=x)$: $\frac{1}{4} + \frac{1}{2} + \frac{1}{2} + 0 = 1$

#1. Let X be a random variable with probability density function (= pdf)

$$f(x) = \begin{cases} c(1-x^2), & -1<x<1 \\ 0 & , \text{ otherwise} \end{cases}$$

(a) What is the value of c?

(b) What is the cumulative distribution function of X?

sol)

(a) $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = \int_{-1}^{1} c(1-x^2)\,dx = c\left[x - \frac{1}{3}x^3\right]_{-1}^{1}$

$$= c\left(1-\frac{1}{3}+1-\frac{1}{3}\right) = \frac{4}{3}c = 1$$

$\therefore\ c = \dfrac{3}{4}.$

(b) $\underset{\text{cdf}}{F(x)} := P(X \le x) = \displaystyle\int_{-1}^{x} f(t)\,dt$

$$= \int_{-1}^{x} \frac{3}{4}(1-t^2)\,dt$$

$$= \frac{3}{4}\left[t - \frac{t^3}{3}\right]_{-1}^{x} = \frac{3}{4}x - \frac{1}{4}x^3 + \frac{1}{2}$$

$$F(x) = \begin{cases} 0 & \text{if } x \le -1 \\ \frac{3}{4}x - \frac{1}{4}x^3 + \frac{1}{2} & \text{if } -1<x<1 \\ 1 & \text{if } x \ge 1 \end{cases}$$

#2. The random vector (X, Y) is said to be uniformly distributed over a region R in the plane if, for some constant c, its joint density is

$$f(x,y) = \begin{cases} c & \text{if } (x,y) \in R \\ 0 & \text{otherwise} \end{cases}$$

(a) Show that $1/c$ = area of region R

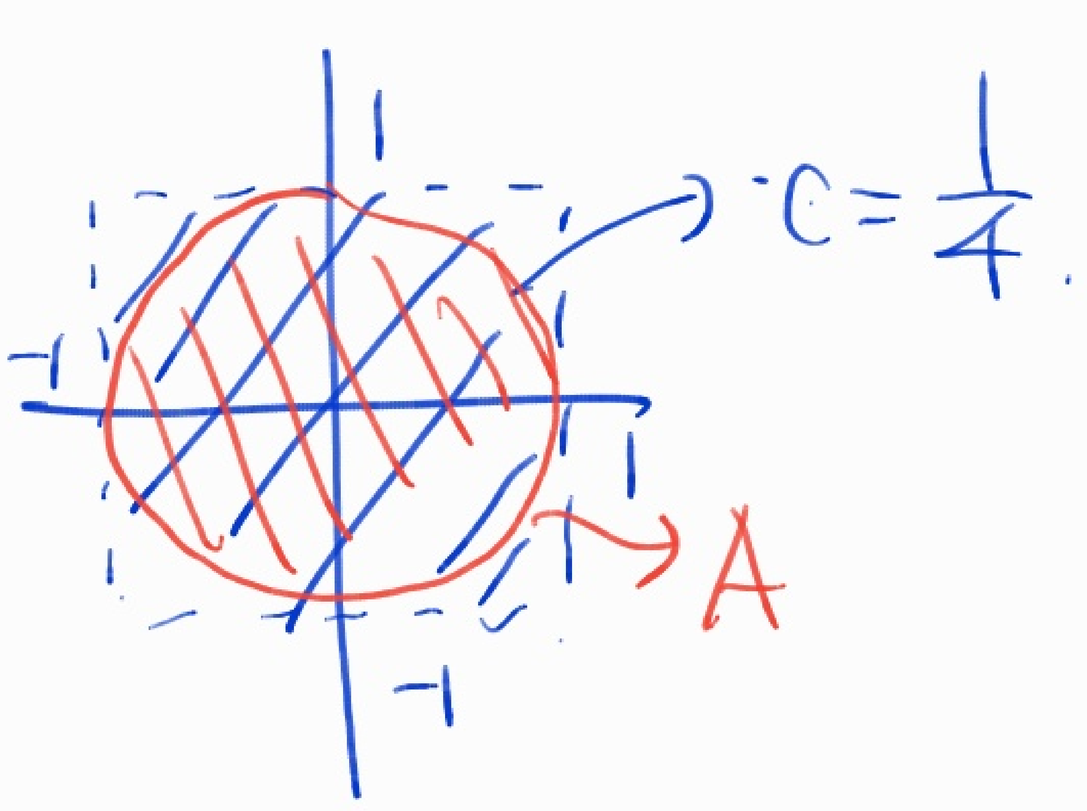
Suppose that (X, Y) is uniformly distributed over the square centered at $(0,0)$ and with sides of length 2.

(b) Show that X, Y are independent, with each being distributed uniformly over $(-1,1)$

(c) What is the probability that (X, Y) lies in the circle of radius 1 centered at the origin?

sol) $\iint_R c\,dx\,dy = c\iint_R dx\,dy = c(\text{Area of } R) = 1$

$\therefore$ Area of $R = 1/c$



(b) $\boxed{f(x,y) = f(x)f(y)}$

$\quad \frac{1}{4}$

$f(x) = \int_{-1}^{1} \frac{1}{4}\,dy = \frac{1}{2}$

$f(y) = \int_{-1}^{1} \frac{1}{4}\,dx = \frac{1}{2}$

(c) $P\{X^2+Y^2 \le 1\} = \iint_A \frac{1}{4}\,dA = \frac{1}{4}\iint_A dA = \frac{\pi}{4}$

#3. If X and Y have joint density function

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{3}(x+y), & \text{if } 0<x<1,\ 0<y<2 \\ 0 & \text{otherwise} \end{cases}$$

(a) $E[XY]$

(b) $E[X]$

The mean of X is

$E[X] = \sum_x x f(x)$ if X is discrete

$E[X] = \int_{-\infty}^{\infty} x f(x) dx$ if X is continuous

Thm

$$E[g(X)] = \sum_x g(x) f(x) \qquad \int_{-\infty}^{\infty} g(x) f(x) dx.$$

If we have joint probability distribution $f(x,y)$, then

$$E[g(X,Y)] = \sum_x \sum_y g(x,y) f(x,y), \quad \iint g(x,y) f(x,y) dx dy$$

(a) $$E[XY] = \int_0^1 \int_0^2 xy \frac{1}{3}(x+y) dy dx = \frac{1}{3}\int_0^1 \left(2x^2 + \frac{8}{3}x\right) dx$$

$$= \frac{2}{3}$$

(b) $$E[X] = \int_0^1 x f_X(x) dx = \int_0^1 x\left(\int_0^2 \frac{1}{3}(x+y) dy\right) dx = \frac{5}{9}$$

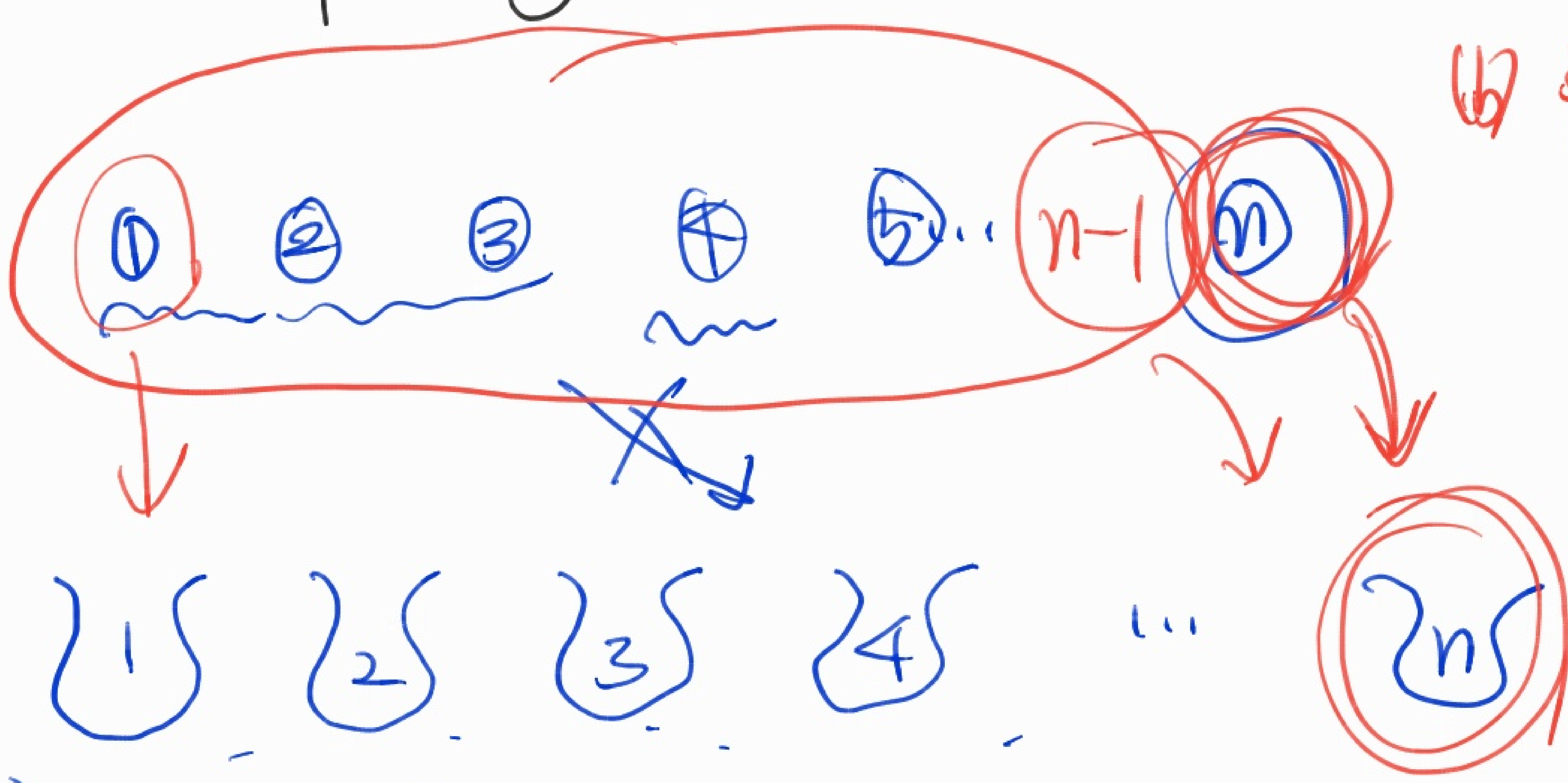
#4. A total of n balls, numbered 1 through n, are put into n urns, also numbered 1 through n in such a way that ball i is equally likely to go into any of the urns $1, 2, \dots, i$. Find

(a) the expected number of urns that are empty

(b) the probability that none of the urns is empty

(b) $p = \frac{1}{n} \cdot \frac{1}{n-1} \cdots \frac{1}{2} \cdot 1$

$= \frac{1}{n!}$



ex) 4th ball can only go into one of 1, 2, 3, 4 urns

That is, probability that 4th ball doesn't go to 4th urn is $\frac{3}{4}$ $\left(= 1 - \frac{1}{4}\right)$

$E[X_i] = P(X_i)$

$= 1 \cdot 1 \cdot 1 \cdot \frac{i-1}{\cancel{i}} \cdot \frac{\cancel{i}}{\cancel{i+1}} \cdots \frac{\cancel{n-1}}{n}$

$= \frac{i-1}{n}$

$\begin{cases} X_i = 1 & \text{if } i\text{-th urn empty} \\ X_i = 0 & \text{otherwise} \end{cases}$ Let $X = \sum_{i=1}^{n} X_i =$

$E[X] = E[X_1] + E[X_2] + \cdots + E[X_n] = \sum_{i=1}^{n} E[X_i] =$

$= \sum_{i=1}^{n} P(X_i) = \sum_{i=1}^{n} \frac{i-1}{n} = \frac{1}{n}(0 + \cdots + n-1) = \frac{1}{\cancel{n}} \frac{(n-1)\cancel{n}}{2}$

$= \frac{n-1}{2}$