1 Let $A = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \end{pmatrix}$. (10 pts each)

30 points

Solution.

(1) Find any particular solution for $A\mathbf{x} = b$.

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \therefore \mathbf{x}_p = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

- Any $\mathbf{x}_p$ satisfying $A\mathbf{x}_p = b$ will get full points. There is no partial points.

(2) Find a general solution for $A\mathbf{x} = \mathbf{0}$.
A series of row opertions gives

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

Hence the general solution for $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x}_h = t[1, 1, -1, -1, 1]^T$, where t is any real number.

- If your answer is wrong but you solved in the right way, you will get **(+5 points)**

(3) Using (a) and (b), find a general solution for $A\mathbf{x} = b$.
We can obtain a general solution for $A\mathbf{x} = b$ by adding a homogeneous solotion to a particular solution.

$$\therefore \mathbf{x} = \mathbf{x}_p + \mathbf{x}_h = [1, 1, 1, 1, 1]^T + t[1, 1, -1, -1, 1]^T \ (t \in \mathbb{R})$$

- If your answer is wrong but you obtained it by using $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$, you will get **(+5 points)**

- **(-2 points)** for each minor mistake.

2 Let $A = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$. (10 pts each)

30 points

(a) Find a sequence of elementary matrices $E_1, E_2, \ldots, E_k$ such that

$$E_k \ldots E_2 E_1 A = I.$$

Solution.

$$E_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad E_1 A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$$E_2 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad E_2 E_1 A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$$E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \qquad E_3 E_2 E_1 A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I \qquad \textbf{(+10 points)}$$

□

(b) Using the results in (a) find A^{-1}.

Solution. Since A is invertible and $A^{-1}A = I = E_3 E_2 E_1 A$, we have $A^{-1} = E_3 E_2 E_1$. **(+5 points)**

$$A^{-1} = \begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix} \qquad \textbf{(+5 points)}$$

□

(c) Find the solution for $A\mathbf{x} = b$.

Solution. Since A is invertible, $\mathbf{x} = A^{-1}b$. **(+2 points)**

$$\mathbf{x} = A^{-1}b = \begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \qquad \textbf{(+8 points)}$$

□

- In (a), each wrong elementary matrix makes subtract points 3. **(−3 points)** And only one correct elementary matrix gets only 3 points. **(+3 points)**
- In (a), other possible sequences of elementary matrices are graded similarly.
- In (b), if you didn't use the results in (a), you gets no points. **(−10 points)**

3 Let $A = \begin{pmatrix} -3 & 6 & 3 & -3 \\ 2 & -2 & 4 & -2 \\ 2 & -5 & -3 & 8 \\ 2 & -6 & 4 & 1 \end{pmatrix}$

30 points

(a) Find an LU-decomposition of A, where L is lower triangular and U is upper triangular.

(b) Find a lower triangular matrix L_1 with all diagonal entries equal to 1 and an upper triangular matrix U_1

(c) Show that such a decomposition in (b) is unique, i.e., if $L_1U_1 = L_2U_2$, where L_2 is a lower triangular matrix whose diagonal entries are equal to 1 and U_2 is upper triangular, then $L_1 = L_2$ and $U_1 = U_2$ (You can use the fact that A is invertible without a proof).

(a)

Solution.

$$A = \begin{pmatrix} -3 & 6 & 3 & -3 \\ 2 & -2 & 4 & -2 \\ 2 & -5 & -3 & 8 \\ 2 & -6 & 4 & 1 \end{pmatrix} \qquad I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$E_1A = \begin{pmatrix} 1 & -2 & -1 & 1 \\ 2 & -2 & 4 & -2 \\ 2 & -5 & -3 & 8 \\ 2 & -6 & 4 & 1 \end{pmatrix} \qquad E_1^{-1} = \begin{pmatrix} -3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Where E_1 is elementary matrix which makes first row of A multiplied by $-\frac{1}{3}$.

$$E_4E_3E_2E_1A = \begin{pmatrix} 1 & -2 & -1 & 1 \\ 0 & 2 & 6 & -4 \\ 0 & -1 & -1 & 6 \\ 0 & -2 & 6 & -1 \end{pmatrix} \qquad E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-4} = \begin{pmatrix} -3 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix}$$

Where E_2, E_3, E_4 are elementary matrices which make 2nd,3rd,4th row to be subtracted by 1st row times some constant which makes first elemnt of the 2nd,3rd,4th row to be zero respectively. Keep doing this procedure,

$$E_5 \dots E_1A = \begin{pmatrix} 1 & -2 & -1 & 1 \\ 0 & 1 & 3 & -2 \\ 0 & -1 & -1 & 6 \\ 0 & -2 & 6 & -1 \end{pmatrix} \qquad E_1^{-1} \dots E_5^{-1} = \begin{pmatrix} -3 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix}$$

$$E_7 \dots E_1 A = \begin{pmatrix} 1 & -2 & -1 & 1 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 12 & -5 \end{pmatrix} \qquad E_1^{-1} \dots E_7^{-1} = \begin{pmatrix} -3 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 2 & -1 & 1 & 0 \\ 2 & -2 & 0 & 1 \end{pmatrix}$$

$$E_8 \dots E_1 A = \begin{pmatrix} 1 & -2 & -1 & 1 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 12 & -5 \end{pmatrix} \qquad E_1^{-1} \dots E_8^{-1} = \begin{pmatrix} -3 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 2 & -1 & 2 & 0 \\ 2 & -2 & 0 & 1 \end{pmatrix}$$

$$E_9 \dots E_1 A = \begin{pmatrix} 1 & -2 & -1 & 1 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -29 \end{pmatrix} \qquad E_1^{-1} \dots E_9^{-1} = \begin{pmatrix} -3 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 2 & -1 & 2 & 0 \\ 2 & -2 & 12 & 1 \end{pmatrix}$$

On the above procedure (or different LU decomposition method) and calculation: **(+5 points)**

$$U = E_{10} \dots E_1 A = \begin{pmatrix} 1 & -2 & -1 & 1 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad L = E_1^{-1} \dots E_{10}^{-1} = \begin{pmatrix} -3 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 2 & -1 & 2 & 0 \\ 2 & -2 & 12 & -29 \end{pmatrix}$$

For the answer: **(+5 points)**

□

(b)

Solution. We have a pair of L and U from (a) which satisfy $LU = A$. Let

$$D = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix}$$

be a diagonal matrix with nonzero diagonal entries. By the following facts

$$LD = \begin{pmatrix} -3a & 0 & 0 & 0 \\ 2a & 2b & 0 & 0 \\ 2a & -2b & 2c & 0 \\ 2a & -2b & 12c & -29d \end{pmatrix}$$

$$D^{-1} = \begin{pmatrix} 1/a & 0 & 0 & 0 \\ 0 & 1/b & 0 & 0 \\ 0 & 0 & 1/c & 0 \\ 0 & 0 & 0 & 1/d \end{pmatrix}$$

$$LU = LDD^{-1}U$$

On the using those facts: **(+2 points)**

If we put $a = -1/3, b = 1/2, c = 1/2, d = -1/29$

$$L_2 = LD = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2/3 & 1 & 0 & 0 \\ -2/3 & -1/2 & 1 & 0 \\ -2/3 & -1 & 6 & 1 \end{pmatrix}$$

$$U_2 = D^{-1}U = \begin{pmatrix} -3 & 6 & 3 & -3 \\ 0 & 2 & 6 & -4 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & -29 \end{pmatrix}$$

On the calculation: **(+3 points)**

On the answer: **(+5 points)**

(c)

Solution. Caution: Proving 'If LU decomposition is exist, then LU decomposition that L has all diagonal entries equal to 1 is unique' and proving 'If LDU decomposition is exist, then LDU decomposition is unique' are same process under the invertibility condition. Thus, You cannot use the property, 'When A has a row echelon form without row interchanging LDU decompositionm is unique', without proof. Silmilarly, someone used "rref is unique". This sentence is okay but problem was
"In $E_k E_{k-1} \dots E_1$ A=rref(A), $E_k E_{k-1} \dots E_1$ is unique so that our L and U are unique". In general case, the last assertion is wrong. For example,

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 3 \\ 7 & 3 & 0 & 9 \\ 8 & 3 & 0 & 9 \end{pmatrix} \quad \text{then} \quad U = \text{rref}(A) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

If the above assertion was right, then there must be a unique L such that $LU = A$. However,

$$LU = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 7 & 3 & 1 & 0 \\ 8 & 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 7 & 3 & 1 & 0 \\ 8 & 3 & 9 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 7 & 3 & 1 & 0 \\ 8 & 3 & k & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 3 \\ 7 & 3 & 0 & 9 \\ 8 & 3 & 0 & 9 \end{pmatrix} = A$$

k can be any number so it is a contradiction. Then you might say "rref of our A has nonzero diagonal so the assertion is right.", but you should prove why nonzero diagonal of rref assure such $E_k E_{k-1} \dots E_1$ is unique. Actually that was an essence of this problem.

⊙ Back to our solution.
Suppose we have two different such LU decomposition.

$$L_1 U_1 = A = L_2 U_2$$

And all the diagonal entries of L_1 and L_2 are 1. Since A is invertible, and determinants of L_1 and L_2 are 1 (nonzero).

$$U_1 = L_1^{-1} A \qquad U_2 = L_2^{-1} A$$

are also invertible. Then

$$L_1^{-1} A U_2^{-1} = U_1 U_2^{-1} = L_1^{-1} L_2$$

Note that inverse of lower triagular matrix is also lower triangular matrix and matrix product of lower triangular matrix of lower triagular matrix is also lower triangular. Likewise, inverse of upper triagular matrix is also upper triangular matrix and matrix product of upper triangular matrix of upper triagular matrix is also upper triangular. $L_1^{-1} L_2$ is a lower triangular and $U_1 U_2^{-1}$ is an upper triangular. To be $L_1^{-1} L_2 = U_1 U_2^{-1}$, $L_1^{-1} L_2$ is lower and upper triangular which implies it is a diagonal matrix.
Since by the following property of lower triangular matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ a_1 & 1 & 0 & 0 \\ b_1 & c_1 & 1 & 0 \\ d_1 & e_1 & f_1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_2 & 1 & 0 & 0 \\ b_2 & c_2 & 1 & 0 \\ d_2 & e_2 & f_2 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ a_2 & 1 & 0 & 0 \\ b_2 & c_2 & 1 & 0 \\ d_2 & e_2 & f_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_3 & 1 & 0 & 0 \\ b_3 & c_3 & 1 & 0 \\ d_3 & e_3 & f_3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_4 & 1 & 0 & 0 \\ b_4 & c_4 & 1 & 0 \\ d_4 & e_4 & f_4 & 1 \end{pmatrix}$$

The diagonal entries of $L_1^{-1} L_2$ are all 1. According to the fact that $L_1^{-1} L_2$ was diagonal matrix.

$$L_1^{-1} L_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = I_4$$

So that $L_1 = L_2$. Since $U_1 U_2^{-1} = L_1^{-1} L_2 = I_4$, $U_1 = U_2$. In conclusion, such decomposition is unique. □

Solution. Another way to prove this problem is just solving the following

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ b & c & 1 & 0 \\ d & e & f & 1 \end{pmatrix} \begin{pmatrix} g & h & i & j \\ 0 & k & l & m \\ 0 & 0 & n & o \\ 0 & 0 & 0 & p \end{pmatrix} = \begin{pmatrix} -3 & 6 & 3 & -3 \\ 2 & -2 & 4 & -2 \\ 2 & -5 & -3 & 8 \\ 2 & -6 & 4 & 1 \end{pmatrix}$$

But you need to clarify right hand side of the equation. The one who just put arbitraly matrix on the right hand side lost some points because sometime this method would fail depends on the matrix on the right hand side.

And you should finish the calculation because we don't know whether the matrix A will justify all the variables on the left hand side. (The one who only mentioned about the number of equations and variables without solving actual equation will lose some points because depends on the right hand side, it fails sometime. So you need to show.) □

⊚ Criteria

- In (a),

  case 1: You have L and U and L*U=A (we checked all the product). **(+10 points)**

  case 2: You have wrong L and U but the procedure of decomposition was right without any calculation miss. **(−3 points)**

  case 3 : You have wrong L and U but the procedure was right and there were few calculation mistake **(−5 points)**

  case 4 : In case 3, there were many calculation mistake **(−8 points)**

  Otherwise, 0.

- In (b),

  case 1: You have 'the' L and U **(+10 points)**

  case 2: You have wrong L and U , but the procedure was right without any calculation mistake **(−3 points)**

  case 3 :You have wrong L and U with right procedure but there were few calculation mistake **(−5 points)** case 4 :With wrong L and U with right procudure but there were many calculation mistake **(−8 points)** Otherwise, 0.

- In (c),

  case 1: Use the property "If A has LDU decomposition then it is unique " without any proof and finish the answer of our problem **(−10 points)**

  case 2: Use the property "A has unique rref. And In $E_k \dots E_1 A = \text{rref}(A)$ then $E_k \dots E_1$ is unique " and finish the answer **(−10 points)**

  case 3: " $A^{-1}A = L_2U_2U_1^{-1}L_2^{-1}$ so $U_2U_1^{-1}$ must be identity." without any proof. **(−8 points)**

  case 4: The one who said "$L_1^{-1}L_2U_2 = U_1$ which means lower * upper =upper. So that $L_1^{-1}L_2$ is identity". However, lower*upper = upper also can be happened when lower is just any diagonal matrix. Therefore, this is wrong solution. **(−5 points)**

  case 5: The one who used the information that lower triangular (or upper triangular) matrix group is closed under matrix product and inverse (if it exists). **(+2 points)**

case 6: $\det(A) = \det(L_1U_1) = \det(L_2U_2) = \det(U_1) = \det(U_2)$ so $U_1 = U_2$. **(−10 points)**

case 7: Set up the matrix equation as above solution.And one solved some part of it. **(+3 points)** (depends on how much solved.)

case 8: Finished the matrix equation and so that proved there is unique solution. **(+10 points)**

case 9: Any verbal explanation without any mathematical reason or justification. **(−8 points)**

case 10: Any right proof **(+10 points)**

Including those case, any logical or mathematical error caused loss of points.

4 Answer the following questions.

10 points each

(a) Let $\mathbb{R}^n$=span$\{v_1, \cdots, v_n\}$ and let A be an $n \times n$ matrix. Show that $\mathbb{R}^n$=span$\{Av_1, \cdots, Av_n\}$ if and only if A is invertible.

(b) Let A be a symmetric $n \times n$ matrix. Prove that $(Au) \cdot v = u \cdot (Av)$ for any vectors $u, v \in \mathbb{R}^n$(here "$\cdot$" denotes the dot product).

(c) Let $u = (1, 1, 0)$ and $w = (1, 2, 3)$ be two vectors in $\mathbb{R}^3$. Let $V = \{v \in \mathbb{R}^3 | (u \times v) \cdot w = 0\}$. Find a spanning set of V.

Solution. (a) Note that $B = [v_1 \quad v_2 \quad \cdots \quad v_n]$ is invertible since the equation $Bx = b$ is consistent for all $b \in \mathbb{R}^n$ **(+3 points)**, so $\det(B) \neq 0$. $AB = [Av_1 \quad Av_2 \quad \cdots \quad Av_n]$ is invertible if and only if the equation $ABx = b$ is consistent for all $b \in \mathbb{R}^n$, and it is equivalent to that span$\{Av_1, \cdots, Av_n\}$ = col$(AB) = \mathbb{R}^n$ **(+5 points)**. Since $\det(B) \neq 0$, $\det(AB) = \det(A)\det(B) \neq 0$ if and only if $\det(A) \neq 0$, which is equivalent to that A is invertible **(+2 points)**.

□

- The answer without justifications gets no points.
- If you only prove one direction you get only 5 points. **(−5 points)**
- The answer with vague arguments or symbols without definitions gets no points.

(b) We can see that the vectors u, v as $n \times 1$ matrices. Note that $(Au) \cdot v = v^T(Au) = (v^TA)u = (A^Tv)^Tu = u \cdot (A^Tv)$ **(+5 points)**. Since A is symmetric, $u \cdot (A^Tv) = u \cdot (Av)$ **(+5 points)**. □

- The answer without justification gets no points.

(c) Let $v = (v_1, v_2, v_3)$. Note that

$$(v \times u) \cdot w = \begin{vmatrix} v \\ u \\ w \end{vmatrix} = \begin{vmatrix} v_1 & v_2 & v_3 \\ 1 & 1 & 0 \\ 1 & 2 & 3 \end{vmatrix} = 3v_1 - 3v_2 + v_3 \text{ (+5 points)}.$$

So $V = \{(v_1, v_2, v_3) \in \mathbb{R}^3 : 3v_1 - 3v_2 + v_3 = 0\} = \{(1, 0, -3)v_1 + (0, 1, 3)v_2 : v_1, v_2 \in \mathbb{R}\}$. Hence the spanning set of V is $\{(1, 0, -3), (0, 1, 3)\}$ **(+5 points)**. □

- If you don't find 'spanning set', you get no points. **(−5 points)**
- The wrong answer gets no points. **(−5 points)**

5 Let $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{pmatrix}$. (15 pts each)

30 points

(a) Calculate $\det(A)$ via row reduction.

(b) Solve the linear system $Ax = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ using Cramer's rule.

Solution. (a) Note that row replacement doesn't change the determinant. So, using just row replacement, we get

$$\det(A) = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 5 & 9 \\ 0 & 3 & 9 & 19 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 3 & 10 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1. \textbf{ (+10 points)}$$

Here we use the fact that the determinant of a triangular matrix is the product of its diagonal entries. Hence $\det(A) = 1$. **(+5 points)**

- In this problem, you must use the row reduction. If you don't use it, for example, using the cofactor expansion only, then you get no points. There are also no points even if you write the correct answer.
- If you use row reduction at least once, then there is no penalty.
- If you make errors in computation, then you get -5 points of penalty.

(b) Let $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$. Then since $\det(A) = 1$, so by Cramer's rule,

$$x_1 = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 0 & 3 & 6 & 10 \\ 0 & 4 & 10 & 20 \end{vmatrix} = -\begin{vmatrix} 1 & 1 & 1 \\ 3 & 6 & 10 \\ 4 & 10 & 20 \end{vmatrix} = -(20 - 20 + 6) = -6, \textbf{ (+3 points)}$$

$$x_2 = \begin{vmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 3 & 4 \\ 1 & 0 & 6 & 10 \\ 1 & 0 & 10 & 20 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 6 & 10 \\ 1 & 10 & 20 \end{vmatrix} = (20 - 10 + 4) = 14, \textbf{ (+3 points)}$$

$$x_3 = \begin{vmatrix} 1 & 1 & 0 & 1 \\ 1 & 2 & 1 & 4 \\ 1 & 3 & 0 & 10 \\ 1 & 4 & 0 & 20 \end{vmatrix} = -\begin{vmatrix} 1 & 1 & 1 \\ 1 & 3 & 10 \\ 1 & 4 & 20 \end{vmatrix} = -(20-10+1) = -11, \ \textbf{(+3 points)}$$

$$x_4 = \begin{vmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 1 \\ 1 & 3 & 6 & 0 \\ 1 & 4 & 10 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 3 & 6 \\ 1 & 4 & 10 \end{vmatrix} = (6-4+1) = 3. \ \textbf{(+3 points)}$$

Hence $x = \begin{pmatrix} -6 \\ 14 \\ -11 \\ 3 \end{pmatrix}$. **(+3 points)**

□

- In this problem, you must use the Cramer's rule. If you don't use it, for example, using the adjoint matrix or augmented matrix, then you get no points. There are also no points even if you write the correct answer.
- If you don't write your answer in complete form, then you take some penalty.
- If you make errors in computation of x_i, then you get no point not only in that part but also in the final answer.
- If you get the wrong answer in (a), then you can't also derive the correct solution in here. So you get some penalty automatically. But if you compute $\det(A_i)$ correctly, then you get some points, less than a half.

6 Let $A = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$ be a skew-symmetric matrix of size 3×3. Determine the characteristic polynomial of A, all real eigenvalues of A and for each such eigenvalue the full eigenspace (depending on a, b, c). (30 pts)

30 points

Solution. Use the cofactor expansion (you can use other alternative ways).

$$\det(\lambda I - A) = \lambda(\lambda^2 + a^2 + b^2 + c^2) \quad \textbf{(+10 points)} \tag{1}$$

We suppose a, b and c are reals so it follows that $a^2 + b^2 + c^2 \geq 0$.
Thus, the real eigenvalues of A is 0. **(+10 points)**
If $a^2 + b^2 + c^2 = 0$, $A = 0$. Thus, the eigenspace corresponding to 0 is $\mathbb{R}^3$. **(+5 points)**

Otherwise, the eigenspace is $\{t(c, -b, a) : t \in \mathbb{R}\}$. This is the solution space of the following equations. **(+5 points)**

$$\begin{cases} ay + bz = 0 \\ -ax + cz = 0 \\ -bx - cy = 0 \end{cases} \tag{2}$$

□

Detailed criteria

- If you didn't know the meaning of the characteristic polynomial, you have no point.
- Wrong characteristic polynomials: **(−25 points)**.
- Wrong (real) eigenvalues: **(−20 points)**.
- You didn't specify the real eigenvalue(s): **(−15 points)**.
- You didn't consider the case $a = b = c = 0$ separately: **(−5 points)**.
- If a, b or c is a denominator in your eigenspace expression, you should consider the case that a, b or c is zero separately. If you didn't, **(−3 points)**.
- Any mistake in calculating the eigenspace: **(−3 points)** or **(−5 points)**
- **Your argument is logically wrong but the answer is correct by chance: No point**

7 Let $A = \begin{pmatrix} 3 & 2 \\ -2 & a \end{pmatrix}$ where $a \geq 0$.

40 points

(7-1) Find the characteristic polynomial of A. (10 points)

Solution.

$$\det(\lambda I - A) = \lambda^2 - (a+3)\lambda + (3a+4) \quad \textbf{(+10 points)} \tag{3}$$

□

Detailed criteria

- If you didn't know the meaning of the characteristic polynomial, you have no point.
- Any mistake in the calculation: **(−5 points)**.
- **Your argument is logically wrong but the answer is correct by chance: No point**

(7-2) Find all possible choices of a that make A to have real eigenvalues. (10 points)

Solution. Use the discriminant.

$$(a+3)^2 - 4(3a+4) = (a-7)(a+1) \geq 0 \quad \textbf{(+5 points)} \tag{4}$$

Since $a \geq 0$, we have $a \geq 7$ so that A have real eigenvalues. **(+5 points)** □

Detailed criteria

- Any mistake in the calculation: **(−5 points)**.
- Your computation is correct but the condition $a \geq 0$ is not considered: **(−2 points)**.
- $a = 7$ is omitted: **(−1 points)**.
- **Your argument is logically wrong but the answer is correct by chance: No point**

(7-3) Find the choice of a that make A to have one repeated real eigenvalue. Find the corresponding eigenvalue of A. (10 points)

Solution. Use the discriminant.

$$(a+3)^2 - 4(3a+4) = (a-7)(a+1) = 0 \tag{5}$$

Since $a \geq 0$, we have $a = 7$ so that A have one repeated real eigenvalue. **(+5 points)**

$$\det(\lambda I - A) = (\lambda - 5)^2 \tag{6}$$

The repeated real eigenvalue is 5. **(+5 points)** □

Detailed criteria

- Any mistake in calculating "a": **(−8 points)**.
- You didn't compute the eigenvalue: **(−5 points)**.
- Correct a but any mistake in calculating the eigenvalue: **(−3 points)**.
- Your computation is correct but the condition $a \geq 0$ is not considered: **(−2 points)**.
- **Your argument is logically wrong but the answer is correct by chance: No point**

(7-4) Express the corresponding eigenspace of A. (10 points)

Solution.

$$3x + 2y = 5x, \tag{7}$$
$$-2x + 7y = 5y. \tag{8}$$

In other words,

$$3x - 3y = 0, \tag{9}$$
$$-2x + 2y = 0. \tag{10}$$

Therefore, the eigenspace is $\{t(1,1) : t \in \mathbb{R}\}$. □

Detailed criteria

- If you didn't know the meaning of the eigenspace, you have no point.
- Any mistake in the computation: **(−8 points)**.
- Your computation is correct but the condition $a \geq 0$ is not considered: **(−2 points)**.
- **Your argument is logically wrong but the answer is correct by chance: No point**

8(a) 10 points

Consider the matrix $A = \begin{pmatrix} a & 0 & -\frac{\sqrt{3}}{2} \\ 0 & b & 0 \\ \frac{\sqrt{3}}{2} & 0 & c \end{pmatrix}$.

Find all possible choices of scalars a, b, and c that make A an orthogonal matrix.

Solution 1. A is an orthogonal matrix if and only if column vectors of A are orthonormal. Therefore we get $a^2 + (\frac{\sqrt{3}}{2})^2 = 1$, $b^2 = 1$, $(-\frac{\sqrt{3}}{2})^2 + c^2 = 1$, $(a, 0, \frac{\sqrt{3}}{2}) \cdot (-\frac{\sqrt{3}}{2}, 0, c) = 0$. **(+5 points)**
Thus, $a = c = \pm\frac{1}{2}$, $b = \pm 1$. **(+5 points)**

□

Solution 2. A is an orthogonal matrix if and only if $AA^T = I$. Therefore

$$\begin{pmatrix} a & 0 & -\frac{\sqrt{3}}{2} \\ 0 & b & 0 \\ \frac{\sqrt{3}}{2} & 0 & c \end{pmatrix} \begin{pmatrix} a & 0 & \frac{\sqrt{3}}{2} \\ 0 & b & 0 \\ -\frac{\sqrt{3}}{2} & 0 & c \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Then we get $a^2 + (\frac{\sqrt{3}}{2})^2 = 1$, $b^2 = 1$, $(-\frac{\sqrt{3}}{2})^2 + c^2 = 1$, $(a, 0, \frac{\sqrt{3}}{2}) \cdot (-\frac{\sqrt{3}}{2}, 0, c) = 0$. **(+5 points)**
Thus, $a = c = \pm\frac{1}{2}$, $b = \pm 1$. **(+5 points)**

□

- If you just use norm of column vectors of A is 1,then you get no points.

8(b) 10 points

Find the choice of a positive scalar a (*i.e.*, $a > 0$) and scalars b, c that make A a rotation matrix.

Solution. 3x3 matrix A is a rotation matrix if A is an orthogonal matrix with $det(A) = 1$. **(+5 points)**
Since $a > 0$, $a = \frac{1}{2}$. Hence $c = \frac{1}{2}$. $det(A) = (-1)^{(2+2)} \cdot b \cdot \begin{vmatrix} a & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & c \end{vmatrix} = b(ac + \frac{3}{4}) = b = 1$.
Thus, $a = c = \frac{1}{2}$, $b = 1$. **(+5 points)**

□

- The answer without justifications(why $b = 1$?) gets no points.

8(c) 10 points

Find its axis of rotation.

Solution. To find the axis of rotation we must solve the linear system $(I - A)x = 0$. **(+5 points)**

$$\begin{pmatrix} \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \\ 0 & 0 & 0 \\ -\frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

you can easily check that $x = z = 0$.
Thus, axis of rotation is the line through the origin that passes through the point (0,1,0). **(+5 points)**

□

Solution. For any nonzero vector $x \in \mathbf{R}^3$ that is not perpendicular to the axis of rotation, $v = Ax + A^T x + [1 - tr(A)]x$ is nonzero and is along the axis of rotation when x has its initial point at the origin. **(+5 points)**

$$v = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \\ -\frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Thus, axis of rotation is the line through the origin that passes through the point (0,1,0). **(+5 points)**

□

- If you don't verify why $b = 1$ in (b), then **(−3 points)**

8(d) Find its orientation and angle.

10 points

Solution. Choose any vector w in xz plane. Let $w = (1, 0, 0)$, then $Aw = (\frac{1}{2}, 0, \frac{\sqrt{3}}{2})$.

$$cos\theta = \frac{Aw \cdot w}{||Aw||\,||w||} = \frac{\frac{1}{2}}{1 \cdot 1} = \frac{1}{2} \quad \text{or} \quad cos\theta = \frac{tr(A) - 1}{2} = \frac{1}{2} \ \textbf{(+4 points)}$$

Since $0 \le \theta \le \pi$, $\theta = \frac{\pi}{3}$. **(+1 points)**

$$w \times Aw = (1, 0, 0) \times (\frac{1}{2}, 0, \frac{\sqrt{3}}{2}) = (0, -\frac{\sqrt{3}}{2}, 0) \ \textbf{(+4 points)}$$

Thus, an orientation is $(0, -\frac{\sqrt{3}}{2}, 0)$ or any formula $(0, \lambda, 0)$ with $\lambda < 0$. **(+1 points)**

□

- If you don't verify why $b = 1$ in (b), then **(−3 points)**

9 Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be a linear transformation satisfying

20 points

$$T(1,-1,0) = (4,2,1,0), \quad T(2,2,-1) = (1,3,0,6), \quad T(-1,-1,1) = (-1,3,-3,0).$$

Find the standard matrix $[T]$ for T.

Solution 1. Let $[T] = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{pmatrix}$. Then using the given codition, we have

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{pmatrix} \begin{pmatrix} 1 & 2 & -1 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 1 & -1 \\ 2 & 3 & 3 \\ 1 & 0 & -3 \\ 0 & 6 & 0 \end{pmatrix}.$$ **(+5 points)**

So it is enough to compute the inverse of $\begin{pmatrix} 1 & 2 & -1 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$. To compute its inverse, use augmented matrix then

$$\begin{pmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 4 & -2 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 4 & -2 & 1 & 1 & 0 \\ 0 & 0 & 1/2 & 1/4 & 1/4 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 4 & -2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1/2 & 1/2 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 & 3/2 & 1/2 & 2 \\ 0 & 4 & 0 & 2 & 2 & 4 \\ 0 & 0 & 1 & 1/2 & 1/2 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 1/2 & -1/2 & 0 \\ 0 & 1 & 0 & 1/2 & 1/2 & 1 \\ 0 & 0 & 1 & 1/2 & 1/2 & 2 \end{pmatrix}.$$

So its inverse is $\begin{pmatrix} 1/2 & -1/2 & 0 \\ 1/2 & 1/2 & 1 \\ 1/2 & 1/2 & 2 \end{pmatrix}$. **(+10 points)**

Therefore

$$[T] = \begin{pmatrix} 4 & 1 & -1 \\ 2 & 3 & 3 \\ 1 & 0 & -3 \\ 0 & 6 & 0 \end{pmatrix} \begin{pmatrix} 1/2 & -1/2 & 0 \\ 1/2 & 1/2 & 1 \\ 1/2 & 1/2 & 2 \end{pmatrix} = \begin{pmatrix} 2 & -2 & -1 \\ 4 & 2 & 9 \\ -1 & -2 & -6 \\ 3 & 3 & 6 \end{pmatrix}.$$ **(+5 points)**

□

- If you get the wrong answer but don't show how to compute the inverse of the matrix, then you take big penalty or almost no points.
- If you write the answer in not complete form, then it also gives some penalty.
- If you make errors in computation of the inverse, then you get -5 points of penalty in generally. But if your errors are serious, then you get -10 point of penalty or more than -10.

- If you write your answer in 3×4 matrix form, then you get at least -5 points of penalty. Note that the standard matrix is computed by the column form.

Solution 2. If we find $T(1,0,0)$, $T(0,1,0)$ and $T(0,0,1)$, then the standard matrix for T is given by $[T] = \begin{pmatrix} T(1,0,0) & T(0,1,0) & T(0,0,1) \end{pmatrix}$ where each vector is represented in column form. So it is enough to find $T(1,0,0)$, $T(0,1,0)$ and $T(0,0,1)$. **(+5 points)**
Since T is a linear transformation, $T(u+cv) = T(u) + cT(v)$ for any vectors u, v and scalar c. So, $T(2,2,-1) + T(-1,-1,1) = T(1,1,0) = (0,6,-3,6)$ and $T(1,-1,0) + T(1,1,0) = T(2,0,0) = (4,8,,-2,6)$, so $T(1,0,0) = (2,4,-1,3)$. Use this, we can also compute $T(0,1,0) = T(1,1,0) - T(1,0,0) = (-2,2,-2,3)$ and $T(0,0,1) = T(-1,-1,1) + T(1,0,0) + T(0,1,0) = (-1,9,-6,6)$. **(+10 points)**
Therefore

$$[T] = \begin{pmatrix} T(1,0,0) & T(0,1,0) & T(0,0,1) \end{pmatrix} = \begin{pmatrix} 2 & -2 & -1 \\ 4 & 2 & 9 \\ -1 & -2 & -6 \\ 3 & 3 & 6 \end{pmatrix}. \textbf{ (+5 points)}$$

□

- If you get the wrong answer but don't show how to compute the columns of $[T]$, then you take big penalty or almost no points.
- If you wrtie the answer in not complete form, then it also gives some penalty.
- If you make errors in computation, then you get -5 points of penalty in generally. But if your errors are serious, then you get -10 point of penalty or more than -10.
- If you write your answer in 3×4 matrix form, then you get at least -5 points of penalty. Note that the standard matrix is computed by the column form.

10 Suppose a 3×3 matrix B has the characteristic polynomial $\lambda^3 + 3\lambda^2 - 8\lambda - 4$.

10 points each

(a) Show that B is non-singular (invertible).

Solution. Let $p(\lambda) = \det(\lambda I - B) = \lambda^3 + 3\lambda^2 - 8\lambda - 4$.

Then $\det(B) = (-1)^3 \times p(0) = 4$. **(+5 points)**

Since $\det(B) \neq 0$, B is non-singlular. **(+5 points)**

□

- The answer without justifications gets no points.
- Calculating with specific B gets no points.

(b) Find $\text{tr}(B^2)$.

Solution. $\lambda^3 + 3\lambda^2 - 8\lambda - 4 = (\lambda - 2)(\lambda^2 + 5\lambda + 2)$.

Therefore eigenvalues of B are $\lambda_1 = 2$, $\lambda_2 = \frac{-5+\sqrt{17}}{2}$, and $\lambda_3 = \frac{-5-\sqrt{17}}{2}$.

Note that eigenvalues of B^2 are $\lambda_1^2, \lambda_2^2, \lambda_3^2$. **(+5 points)**

Thus we have $\text{tr}(B^2) = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 25$. **(+5 points)** □

- The answer without justifications gets no points.
- Calculating with specific B gets no points.

(c) Find $\det(\mathrm{adj}(B))$.

Solution. We have $B \times \mathrm{adj}(B) = \det(B) \times I$. **(+4 points)**

By calculating determinants of both sides, we have

$$\det(B) \times \det(\mathrm{adj}(B)) = \det(\det(B) \times I) = \{\det(B)\}^n \times \det(I),$$

when B is $n \times n$ matrix.

Therefore, $\det(\mathrm{adj}(B)) = \{\det(B)\}^{n-1}$ **(+4 points)** $= 4^{3-1} = 16$. **(+2 points)** □

- The answer without justifications gets no points.
- Calculating with specific B gets no points.

(d) Find $\det(B^{-1} + I_3)$.

Solution. Note that $\det(B^{-1} + I_3) = \det(B^{-1}) \times \det(I_3 + B)$.

We have $\det(B^{-1}) = \{\det(B)\}^{-1} = \frac{1}{4}$. **(+4 points)**

We also have $\det(I + B) = (-1)^3 \det(-I - B) = -p(-1) = -6$. **(+4 points)**

Thus $\det(B^{-1} + I_3) = \frac{1}{4} \times (-6) = -\frac{3}{2}$. **(+2 points)** □

Solution. Note that $\frac{1}{\lambda_i}$ is eigenvalues of B^{-1} and let x be its eigenvector.

Then $(B^{-1} + I_3)x = \frac{1}{\lambda_i}x + x = (\frac{1}{\lambda_i} + 1)x$.

Therefore $\frac{1}{\lambda_i} + 1$ are eigenvalues of $B^{-1} + I_3$. **(+6 points)**

Since determinant is a product of eigenvalues, we have $\det(B^{-1} + I_3) = -\frac{3}{2}$. **(+4 points)**

□

- The answer without justifications gets no points.
- Calculating with specific B gets no points.
- Any calculation without the first equation gets no points.

11

5 points each

(a) Let matrices A and B be given by

```
>> A = [2, 3; -1, 0];
>> B = [-3, 2; 0, 5];
```

Write the results of the MATLAB commands `A*B` and `A.*B`.

(b) Let the MATLAB function `trans_T` produce a three-dimensional vector as an output when it take a three-dimensional vector as an input, that is, `trans_T` is a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$. We want to find the standard matrix A of the linear transformation `trans_T`. Fill in the blanks (1) - (3) :

```
% To construct a 3 by 3 identity matrix.
dim = 3;    % dimension is 3.
id_mat = eye((___(1)___));
% To allocate a space to store the standard matrix A.
A = zeros(dim,dim);
% For each k = 1:dim, compute k-th column of the matrix A.
for k = 1:dim
    e_k = id_mat((___(2)___));
    A((___(2)___)) = trans_T((___(3)___));
end
% display the matrix A in the command window.
disp(A);
```

(c) We want to draw a graph of the function $f : \mathbb{R}^2 \to \mathbb{R}$, where

$$f(x, y) = xy^2 + 2x^2y, \qquad -3 \le x,\, y \le 3.$$

Fill in the blanks (4) and (5).

```
% In the given domain, construct equally-spaced vectors x and y
% whose increment is 0.1.
x = (___(4)___);
y = x;
% To create a grid as two matrices X and Y.
[X, Y] = (___(5)___)(x,y);
% To assign the function value to Z.
fxy = X .* Y.^2 + 2 * X.^2 .* Y;
figure;     % Make a new figure window.
% To plot the graph of fxy = xy^2 + 2yx^2.
mesh(X,Y,fxy);
```

Solution. (a) `A*B` $= \begin{bmatrix} -6 & 19 \\ 3 & -2 \end{bmatrix}$ and `A.*B` $= \begin{bmatrix} -6 & 6 \\ 0 & 0 \end{bmatrix}$

(b1) `dim` *or* `dim,dim` *or* `3,3` *or* `3`

(b2) `:,k` *or* `1:end, k` *or* `1:3, k` *or* `1:dim, k` *or* `[1:3], k` *or* `[1:dim], k` *or* `[1:3], [k]` *or* `[1:dim], [k]`

(b3) `e_k` *or* `e_k'` *or* `id_mat(:, k)` *or* `id_mat(1:3, k)` *or* `id_mat(1:dim, k)` *or* `id_mat(k, :)`

(c4) `-3:0.1:3` *or* `linspace(-3,3,61)` *or* `[-3:0.1:3]` *or* `(-3:0.1:3)` *or* `[linspace(-3,3,61)]` *or* `(linspace(-3,3,61))`

(c5) `meshgrid` *or* `MESHGRID` *or* `meshgrid(x, y)`
(This was a question asking for the '**NAME**' of the MATLAB function you should use, so some exceptional answers were also admitted.)

There are no partial points. □