MAS201
Differential Equations and Applications

Section 1.2. Initial-Value Problems

Jongho Park

Department of Mathematical Sciences, KAIST

March 4, 2021

$\boxed{\begin{array}{c}\text{nth-order DE} \\ \text{n-parameter family}\end{array}}$ ← requires additional conditions to determine the parameters

Initial-value problems (IVPs)

$$\begin{cases} \dfrac{d^n y}{dx^n} = f(x, y, y', \cdots, y^{(n-1)}) \quad \text{on } I \quad (\text{s.t. } x_0 \in I) \\ y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \cdots, \quad y^{(n-1)}(x_0) = y_{n-1} \end{cases}$$

← initial conditions (IC)

e.g.

- First-order IVP $\begin{cases} \dfrac{dy}{dx} = f(x, y) \quad \text{on } I \\ y(x_0) = y_0 \end{cases}$

- Second-order IVP $\begin{cases} \dfrac{d^2 y}{dx^2} = f(x, y, y') \quad \text{on } I \\ y(x_0) = y_0, \quad y'(x_0) = y_1 \end{cases}$

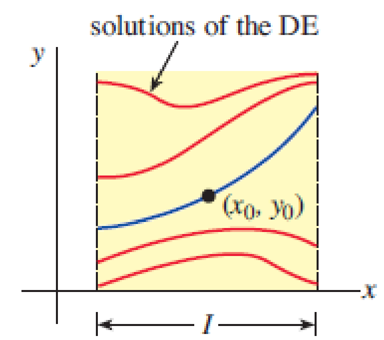


FIGURE 1.2.1 First-order IVP

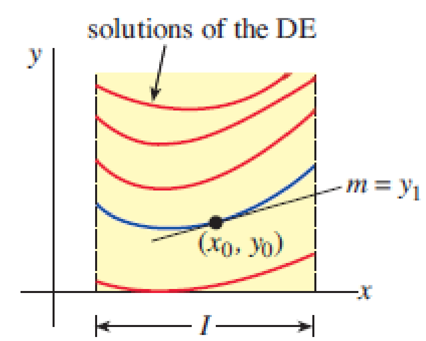


FIGURE 1.2.2 Second-order IVP

Why do we consider IVPs ?

- mathematical viewpoint : In order to determine n unknown parameters, we need n additional conditions (intuitively).
- physical viewpoint : DE describes "change" of a state.

  $\Rightarrow$ An "initial" state is needed to describe a whole phenomenon.

e.g. Phenomenon : An object moves by an external force.

$\Rightarrow$ Governing DE : Newton's equation of motion
IC : initial position of the object

e.g. It is known that $y = ce^x$ solves the DE $y' = y$ on $(-\infty, \infty)$.

- Specifying IC $y(0) = 3$ determines c : $y(0) = ce^0 = 3$

  $\Leftrightarrow c = 3$ //

- Specifying IC $y(1) = -2$ determines c : $y(1) = ce^1 = -2$

  $\Leftrightarrow c = -\frac{2}{e}$ //

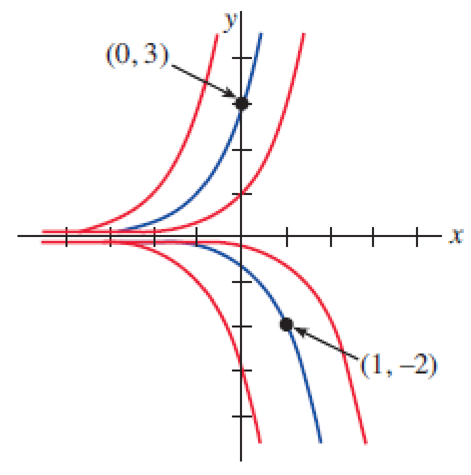
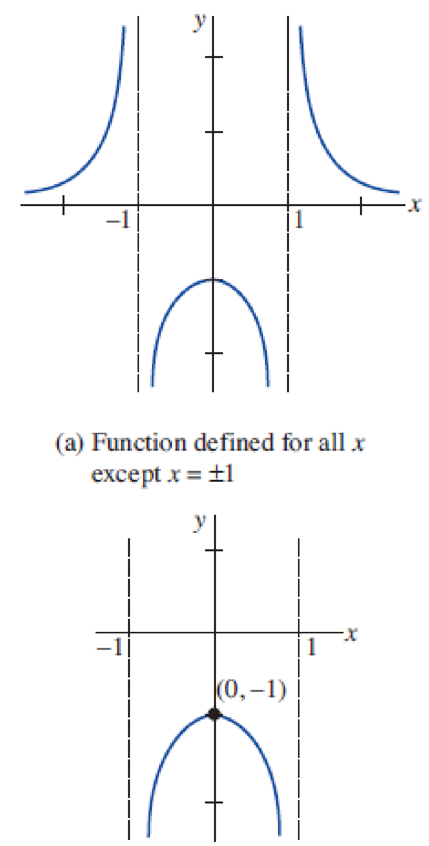


FIGURE 1.2.3 Solutions of IVPs in Example 1

e.g. It is known that $y = \frac{1}{x^2+c}$ solves the DE $y' + 2xy^2 = 0$.

- Setting $c = -1$ yields a particular solution $y = \frac{1}{x^2-1}$.
  $\Rightarrow$ (largest) intervals of definition : $(-\infty, -1), (-1, 1), (1, \infty)$ //
- If we impose the IC $y(0) = -1$, then we have $c = -1$.
  $\Rightarrow$ (largest) interval of definition : $(-1, 1) \ni 0$ //



(a) Function defined for all x except $x = \pm 1$

(b) Solution defined on interval containing $x = 0$

FIGURE 1.2.4 Graphs of function and solution of IVP in Example 2

Thm 1.2.1 (Existence/Uniqueness theorem for first-order IVPs)

We consider a first-order IVP $y' = f(x,y)$, $y(x_0) = y_0$.
Let $R = [a,b] \times [c,d] \subset \mathbb{R}^2$ s.t. (x_0, y_0) is contained in the interior of R. If both f and $\frac{\partial f}{\partial y}$ are continuous on R, then there exists some interval $I_0 = (x_0 - h, x_0 + h) \subset [a,b]$ $(h > 0)$ s.t. the IVP admits a unique solution on I_0.

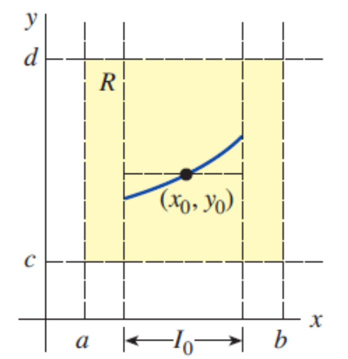


FIGURE 1.2.6 Rectangular region R

$\Rightarrow$ A "good" ODE with an IC admits a unique solution locally!

e.g.

- $\begin{cases} y' = -2xy^2 \\ y(0) = -1 \end{cases}$ $f(x,y) := -2xy^2$ satisfies that f and $\frac{\partial f}{\partial y}$ are continuous everywhere.

  $\Rightarrow$ By Theorem 1.2.1, a unique solution is ensured.

  Indeed, $y = \frac{1}{x^2-1}$ uniquely solves the IVP on $(-1,1) \ni 0$ //

- $\begin{cases} y' = xy^{\frac{1}{2}} \\ y(0) = 0 \end{cases}$ $f(x,y) := xy^{\frac{1}{2}} \Rightarrow \frac{\partial f}{\partial y}$ is not continuous near $(0,0)$.

  $\Rightarrow$ We cannot utilize Theorem 1.2.1 !

  One can verify that both $y = 0$ and $y = \frac{1}{16}x^4$ solve the IVP //