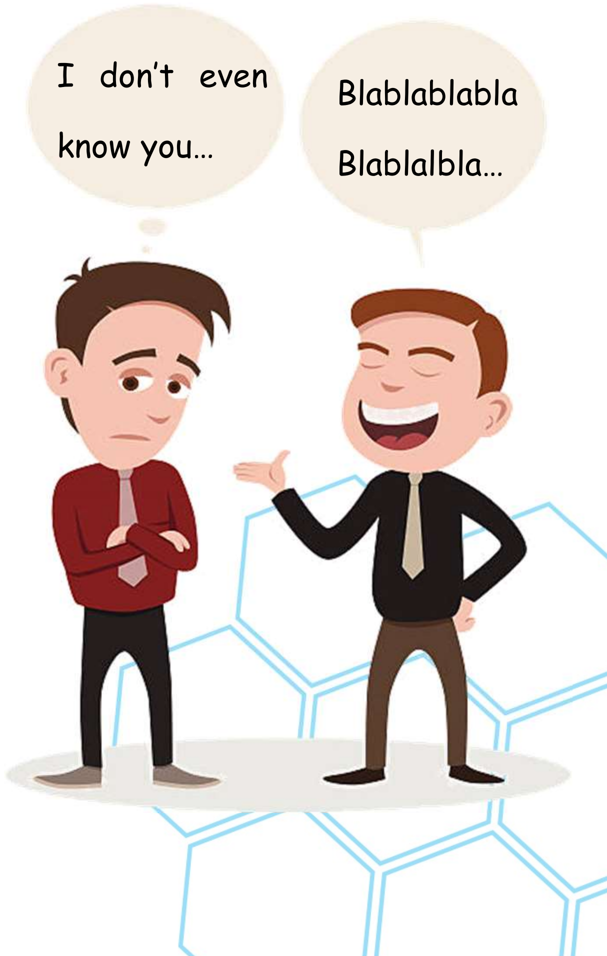
We will begin at 8:35 MMT

Read this problem while we wait…



In a society of finite number of people, some people are friends with each other. Friendship is mutual, that is if A is friends with B, then B is also friends with A. Suppose that everyone has at most k friends. Prove that it is possible to separate these people into $k+1$ rooms so that no one knows each other in their own room.

Record the meeting…

Some Housekeeping

- I put out an survey on how my class is going. Please fill it out if you haven't, it helps me a lot.

- Diamond-problems for Problem Set 2 will due tomorrow 3:00 PM. As usual, you may submit heart-problems at any point.

- I have put hints to all diamond problems in problem set 2. Check classroom announcements on when and how to use hints.

! Tomorrow's lecture will make use of counting, especially combinations (only basic ideas). So, if you are rusty on counting, you should give around 15 minutes to review your knowledge.

Content so far…

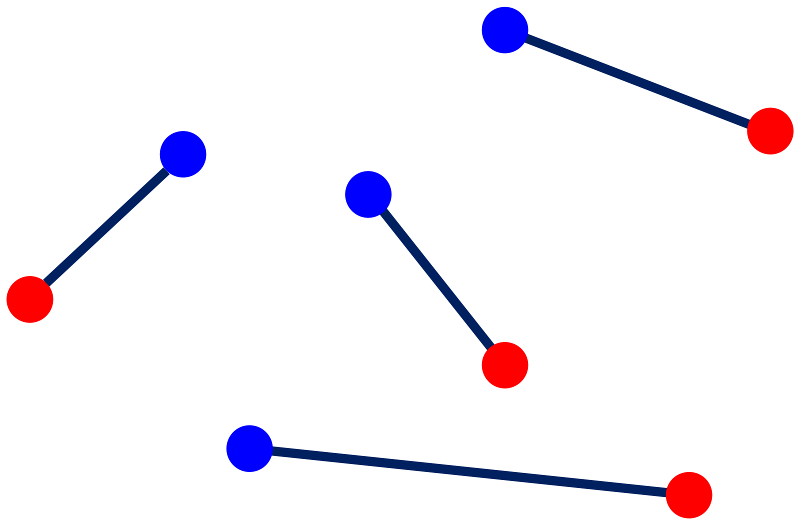
L1: Monovariants L2: Invariants L3: Alternating-variants	I	
L4: Inductive constructions L5: Greedy and RUST	II	
L6: Counting in two ways L7: Inequalities and bounding L8: Counting in graphs L9: Injections and bijections	III	
L10: Pigeonhole principle L11: Continuity and descent L12: Leveraging symmetry	IV	
L13: Combinatorial games	V	
L14: Combinatorial geometry	VI	
L15: Results in graph theory I L16: Results in graph theory II	VII	

Lecture – 5

Greedy and RUST

Putnam Problem

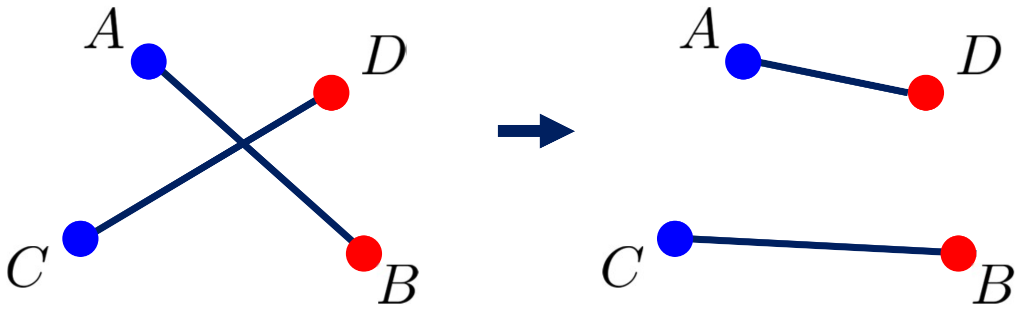
There are n blue points and n red points in the Euclidean plane and no three of them are collinear. Show that we can draw n non-intersecting segments joining distinct red and distinct blue points.



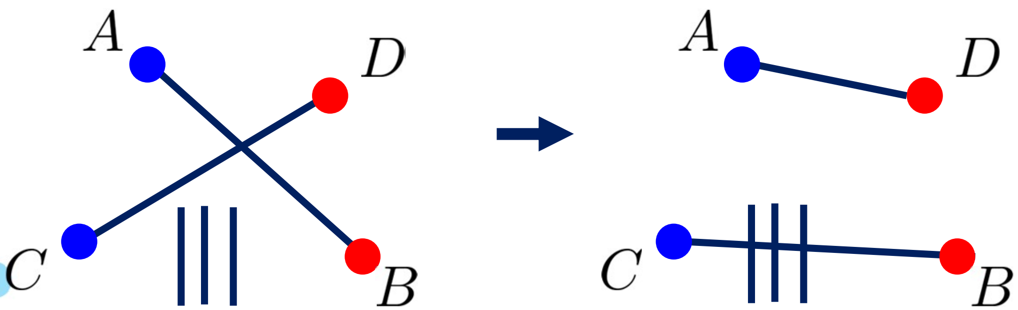
Dumb(?) Solution…

- Just draw n segments forgetting about the intersections…

- When you have an intersecting segments, say AB and CD, then resolve it by replacing them with AD and BC.



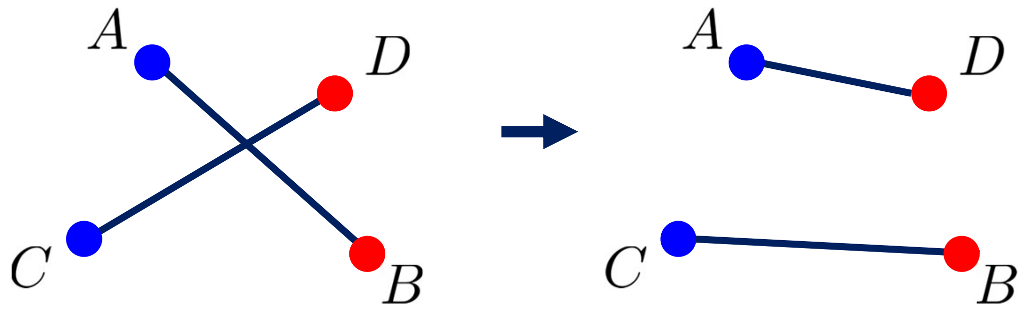
- Does this decrease the number of intersections? If yes, then we are done.



Unfortunately, no. :(

Dumb(?) Solution

Can you see something that decreases in this picture?



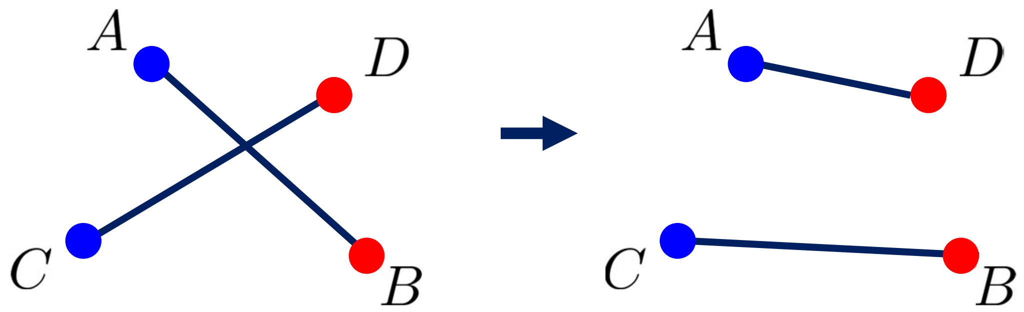
Answer: Total length of the segments

So… if we keep resolving the intersections, we will eventually come to a situation in which we cannot do any resolves. This means that the segments no longer intersect each other!

An Alternative Writeup

- Connect red points to blue points so the total length of the segments is as small as possible.

- We will show that our segments do not intersect. Suppose to the contrary that AB and CD intersect as follows.



- Then, we may replace them by AD and BC, thereby reducing the total length of the segments. But, this is contradiction as our construction have the smallest total length.

Repeat Until Stuck (RUST) Strategy

Do something repeatedly until you cannot do it anymore.

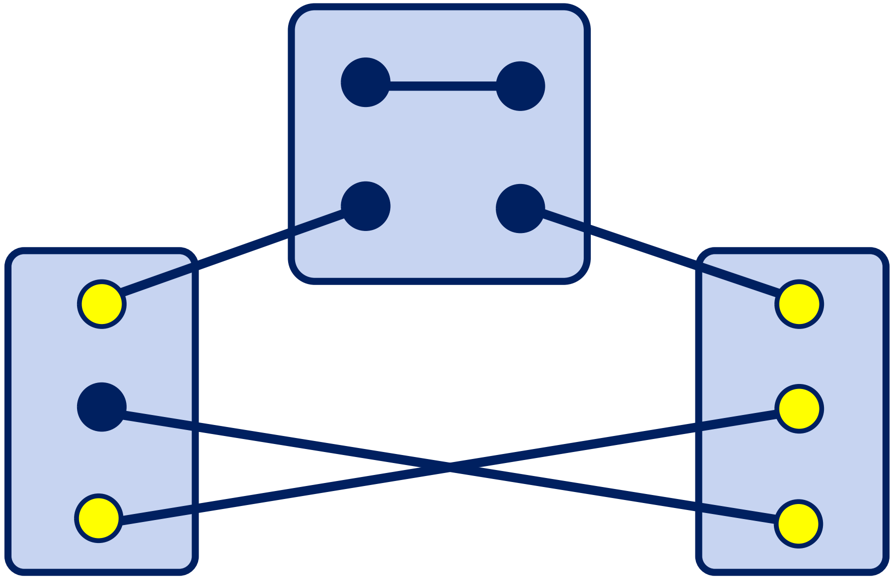
These strategies are often the same

Taking the Extremal Object

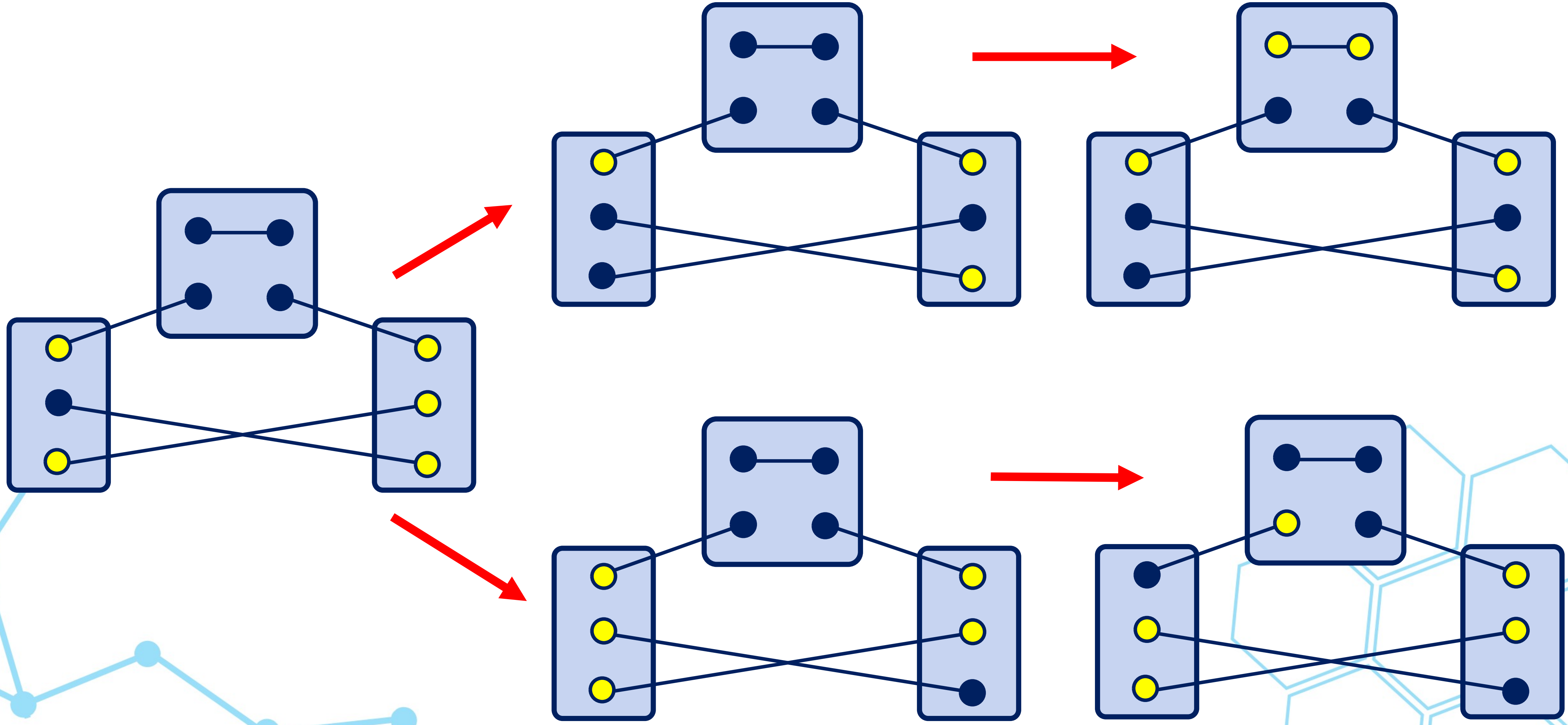
Pick a situation that maximizes/minimizes some quantity.

Partially Lit Rooms (IMOSL 2007/C1)

A house has an even number of lamps distributed among its rooms in such a way that there are at least three lamps in every room. Each lamp shares a switch with exactly one other lamp, not necessarily form the same room. Each change in the switch shared by two lamps changes their states simultaneously. Prove that for every initial state of the lamps, there exists a sequence of changes in some of the switches at the end of which each room contains lamps which are on as well as which are off.



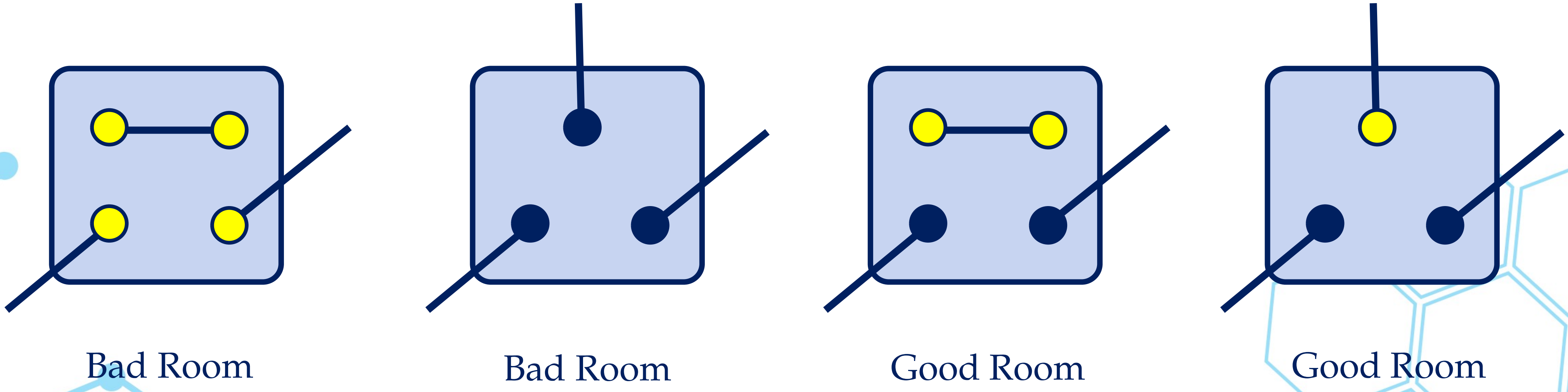
Example



Reducing our task…

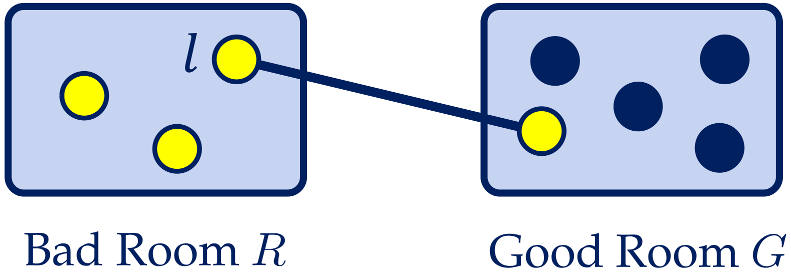
- Call a room bad if all of its lamps are in same state, call it good otherwise.
- We want to make every room good.
- To Do: Just find a way to reduce the number of bad rooms!!!

If we can do this,
we are done.



How to remove bad rooms

- Suppose we have a bad room R.
- If R has a lamp connected to a lamp in another bad room, we are done!
- So… suppose that R is only connected to good rooms.



- Let l be a lamp in R and let l's partner be in good room G. Then, all lamps in G other than l's partner should be in the same state or else we are done! In this case (and only in this case), the 'badness' is transferred to G.

Chase down the bad room

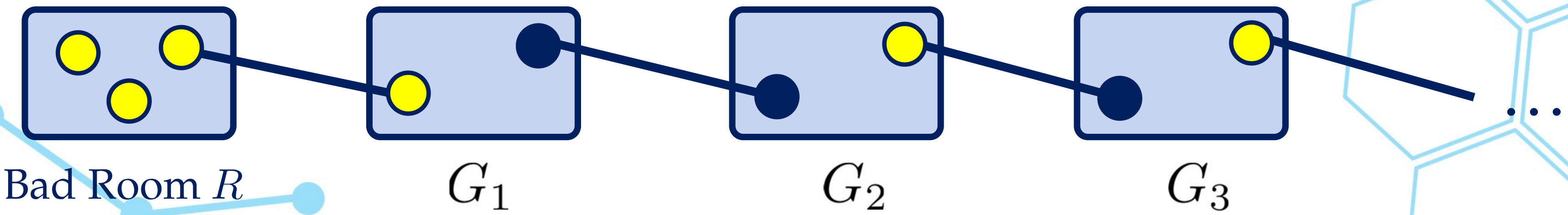
- RUST Strategy: Pick a bad room R and switch any lamp from it. Then,
  - Case 1: the number of bad rooms decreases (in which case we repeat) or
  - Case 2: we make R good while make a good room G bad. In this case, our RUST on G.

Don't switch the previous lamp

- To Prove: Case 2 cannot happen forever (i.e. badness cannot be transferred forever).

- Suppose to the contrary that our RUST gets stuck in Case 2.

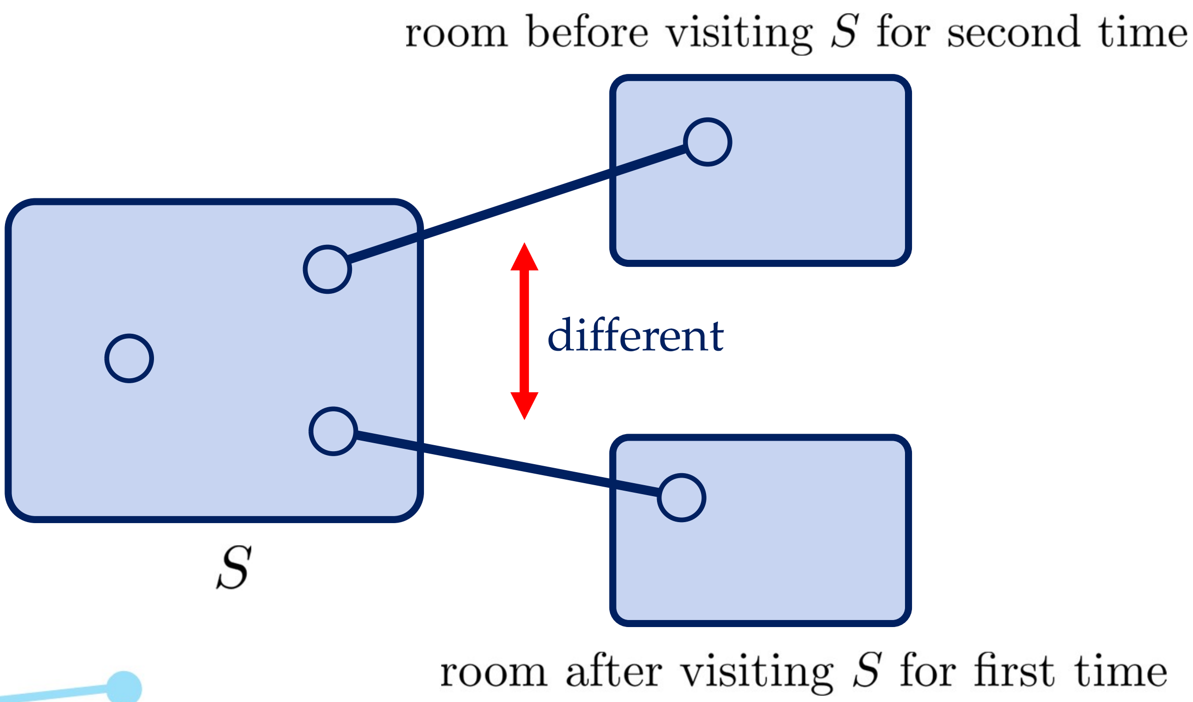
- Then, we have an infinite sequence of rooms $G_1, G_2, G_3, \ldots$ in which we switch a lamp.



So what ?

- Then, we have an infinite sequence of rooms $G_1, G_2, G_3, \ldots$ in which we switch a lamp. So, we must repeatedly do operation on some room S infinitely many times.

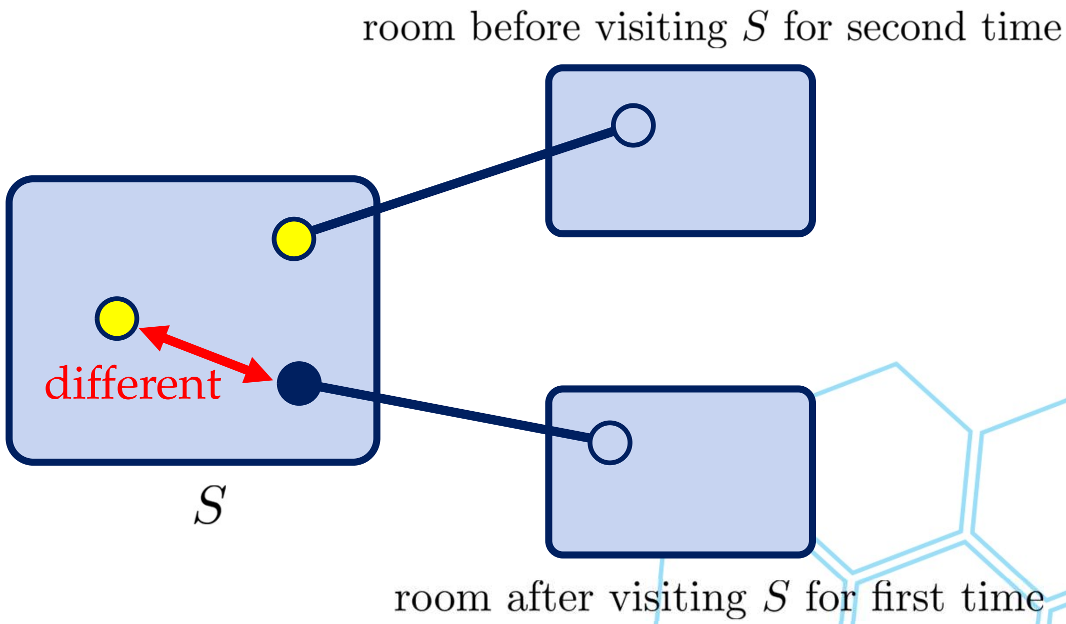
- Just look at the first time we operate on S versus the second time.



So what ?

- Right after visiting S, the lamp that is switched in S has different state with every other lamp in S.
- So, if we switch the lamp in the room before visiting S for the second time, then S does not become bad.
- Thus, 'badness' is not transferred, contradiction i.e. Case 2 didn't happen.
- This is contradiction. So, Case 2 must eventually stop.

Right after visiting S for 1st time



Repeat Until Stuck (RUST) Strategy

Do something repeatedly until you cannot do it anymore.

- If you get stuck, you can get some property because you get stuck.
- If you didn't get stuck, you can still get some property because you didn't get stuck.

Greedy Algorithm

Just try to be greedy. Take as many as you can. Try to get close to the goal as much as you can. Put as many conditions as you can.

Main difficulty: Proving that it actually works (if it works).

Swiss TST 2019

In a $2 \times n$ array, we have non-negative reals such that the sum of the numbers in each of the n columns is 1. Show that one can select one number in each column such that the sum of the selected numbers in each row is at most $(n+1)/4$.

0.8	0.25	0.2	1	0.2	0.6	0.25	0.6
0.2	0.75	0.8	0	0.8	0.4	0.75	0.4

Easy Cases

What if every number is the same?

0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5

In this case, we should pick half from top row, and half form bottom row. Then,

$$\text{top row sum} = 0.5 \times \left\lceil \frac{n}{2} \right\rceil \longleftarrow$$ This is equal to $\frac{n+1}{4}$ if n is odd.

$$\text{bot row sum} = 0.5 \times \left\lfloor \frac{n}{2} \right\rfloor$$

Most straightforward greedy algorithm

The most straightforward way is to pick the smaller number in each column.

This does not work. Why?

0.49	0.49	0.49	0.49	0.49	0.49	0.49	0.49
0.51	0.51	0.51	0.51	0.51	0.51	0.51	0.51

We can trick the greedy algorithm to keep picking from same row…

Explicitly saying, top row sum can get as close as we want to $\frac{1}{2}n$.

Fixing our greed

Idea: Do the same as before, but stop as soon as you are going to exceed $(n+1)/4$.

Heuristic: We should prioritize smaller numbers (this way, large numbers are penalized, too!)

0.8	0.25	0.2	1	0.2	0.6	0.25	0.6
0.2	0.75	0.8	0	0.8	0.4	0.75	0.4

Now, we get the fixed greedy algorithm:

- Look at the top row, keep picking the smallest number remaining until you cannot pick anymore (that is, if the sum is going to exceed $(n+1)/4$).
- Pick the remaining numbers from the bottom row.

Put numbers into the game

Let's prove that our fixed greedy algorithm works.

Let the numbers in the top row (arranged in increasing order) be $a_1, a_2, \ldots, a_n$. And let the number under each a_i be b_i. Suppose our algorithm picked $a_1, a_2, \ldots, a_k, b_{k+1}, \ldots, b_n$.

a_7	a_6	a_4	a_8	a_3	a_5	a_1	a_2
b_7	b_6	b_4	b_8	b_3	b_5	b_1	b_2

Then, we have the following inequalities.

$$a_1 + \cdots + a_k \leq \frac{n+1}{4}$$

$$a_1 + \cdots + a_{k+1} > \frac{n+1}{4}$$

$$a_1 \leq a_2 \leq \cdots \leq a_n$$

Already used (to control the first row)

Haven't used yet!

Put numbers into the game

Then, we have the following inequalities.

$$a_1 + \cdots + a_k \leq \frac{n+1}{4}$$

$$\boxed{a_1 + \cdots + a_{k+1} > \frac{n+1}{4}}$$

$$\boxed{a_1 \leq a_2 \leq \cdots \leq a_n}$$

To Prove: $b_{k+1} + \cdots + b_n \leq \dfrac{n+1}{4}$

Proof: Since $b_1 \geq b_2 \geq \cdots \geq b_n$, we just need to show that $(n-k)b_{k+1} \leq \dfrac{n+1}{4}$.

On the other hand, from $a_1 + \cdots + a_{k+1} > \dfrac{n+1}{4}$, we have $a_{k+1} > \dfrac{n+1}{4(k+1)}$.

Therefore, we have $1 - \dfrac{n+1}{4(k+1)} > b_{k+1}$ and hence $(n-k)\left(1 - \dfrac{n+1}{4(k+1)}\right) > (n-k)b_{k+1}$.

So, it suffices to check whether or not

$$\boxed{(n-k)\left(1 - \frac{n+1}{4(k+1)}\right) \leq \frac{n+1}{4}}$$

Easy to check

for all $k = 1,2, \ldots, n$.