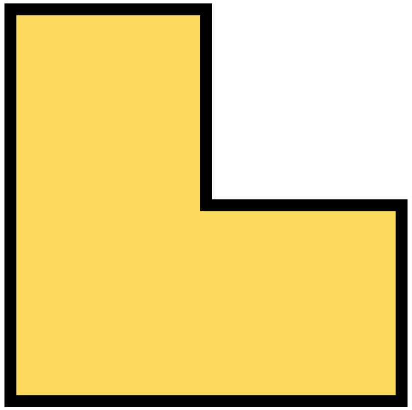
Let's begin at 07:03 PM

Try this problem in the mean time. :)

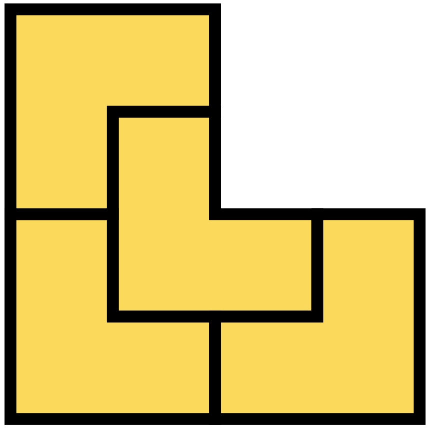
The figure on the right is obtained by putting together 3 unit squares in an L-shape.

- Can you divide the L-shape into 4 identical pieces?
- What about 64 identical pieces?



The answer will be posted when the notes are uploaded to the google classroom.

Answer to the Intro Problem



4 pieces

We can obtain 16 little L-shapes by dividing each of the four L-shapes into four pieces each.

Now, divide each of the 16 little L-shapes into four tiny L-shapes to get 64 tiny L-shapes in total.

If we continue this process, we can get 4^n pieces for any given positive integer n.

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Some Housekeeping

- Homework 1 is going to be due tomorrow midnight. Feel free to come and ask for hints and help if you get stuck on a question.
- In homework, you can submit multiple attempts. You do not need to submit the questions that you already got correct (google forms will not show your actual score then, but it's okay).
- We have recitation on Wednesday. Ako Naing Zaw Lu will cover some questions from homework 1.

Lesson – II

Permutations and Factorials

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Q1: Back to CARTS

Thiha wants to create a 3-letter word using the letters C, A, R, T, S without using the same letter more than once. How many words can he create?

$\boxed{\text{Answer: 60}} \leftarrow 5 \times 4 \times 3$

What about 4-letter words?

$\boxed{\text{Answer: 120}} \leftarrow 5 \times 4 \times 3 \times 2$

Well, this is easy. It's just multiplication principle.

C A R T S

Q1: Back to CARTS

Thiha wants to create a 10-letter word using each of the one-hundred different letters A_1, A_2, ..., A_{100} at most once. How many words can he create?

Well, still multiplication principle. The answer is

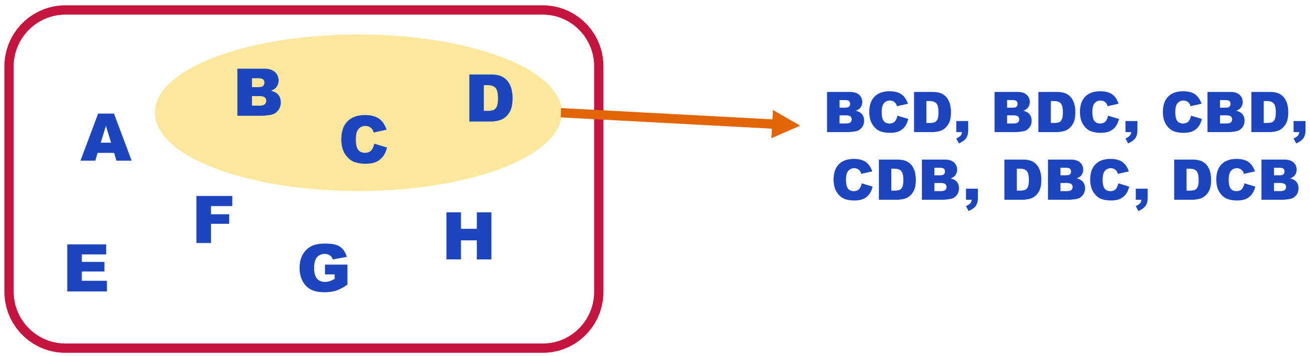
$$100 \times 99 \times 98 \times \cdots \times 91.$$

Permutations

A permutation of a set of objects is an arrangement of those objects in a row.

H N T E I A

Suppose the set has n elements. For a positive integer $1 \leq r \leq n$, such an arrangement using r objects from the set is called an r-permutation.



Permutations of Distinct Objects

Number of r-permutations of n different objects is denoted by P_r^n .

By multiplication principle, formula for P_r^n is

$$P_r^n = \underbrace{n \times (n-1) \times (n-2) \times \cdots}_{\text{up to } r \text{ terms}}$$

For example,

- $P_4^9 = 9 \times 8 \times 7 \times 6$
- $P_7^7 = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$ $\longleftarrow$ This is also written 7! (will see later)

Q2. Permutation Symbol Practice

Write down the number of ways to do each of the following in permutation symbol.

- Ways to arrange 5 out of 10 students want in a line. ⟵ P_5^{10}
- Ways to arrange letters of OLYMPIAD. ⟵ P_8^8
- Ways to seat 7 students on a bus with 20 seats. ⟵ P_7^{20}
- Ways to arrange men-women marriages between 5 men and 5 women. ⟵ P_5^5

Q3. Mie Mie Reads Books

Mie Mie has seven different books, three of which are her favourite. She wants to choose five of those books and read one per week. She wants to read all her favourite books first. In how many can she make her reading schedule? **Answer: 72** ← $P_3^3 \times P_2^4$

Solution

1 2 3 4 5 6 7

__ __ __ __ __

Decision 1: Permute her favourite books, ← P_3^3

Decision 2: Permute the remaining books. ← P_2^4

Q4. Three Teams Standing

There are ten students in a class. Three of them are from red team, four from yellow team and three from green team. Teacher Moe wants them to stand in a line. In how many ways can she do this if

(a) there are no restrictions? $\boxed{\text{Answer: 3628800}}$ $\leftarrow$ P^{10}_{10}

(b) students from the red team must stand together?

$\boxed{\text{Answer: 241920}}$ $\leftarrow$ $P^8_8 \times P^3_3$

(c) students must be together with their team members?

$\boxed{\text{Answer: 5184}}$ $\leftarrow$ $P^3_3 \times P^3_3 \times P^4_4 \times P^3_3$

Q4. Three Teams Standing

(b) Students from the red team must stand together?

$\boxed{\text{R R R}}$ $\boxed{\text{Y}}$ $\boxed{\text{Y}}$ $\boxed{\text{Y}}$ $\boxed{\text{Y}}$ $\boxed{\text{G}}$ $\boxed{\text{G}}$ $\boxed{\text{G}}$

Decision 1: Permute the boxes, ← P_8^8

Decision 2: Permute the R's. ← P_3^3

(c) Students must be together with their team members?

$\boxed{\text{R R R}}$ $\boxed{\text{Y Y Y Y}}$ $\boxed{\text{G G G}}$

Decision 1: Permute the boxes, ← P_3^3

Decision 2, 3, 4: Permute the R's, Y's and G's. ← P_3^3, P_4^4, P_3^3

Q5. Friends at the Movies

Five friends Aye Aye, Bo Bo, Chaw Chaw, Dar Dar and Ei Ei went to watch movies. They bought tickets for five consecutive seats. In how many ways can they sit if

(a) there are no restrictions? $\boxed{\text{Answer: 120}}$ ← P_5^5

(b) Dar Dar wants to sit together with Ei Ei?

$\boxed{\text{Answer: 48}}$ ← $P_4^4 \times P_2^2$

(c) Dar Dar and Ei Ei sits together, but Aye Aye and Bo Bo do not sit together?

$\boxed{\text{Answer: 24}}$ ← $P_4^4 \times P_2^2 - P_3^3 \times P_2^2 \times P_2^2$

Q5. Friends at the Movies

(b) Dar Dar wants to sit with Ei Ei?

$$\boxed{A}\ \boxed{B}\ \boxed{C}\ \boxed{D \quad E}$$

$$P_4^4 \times P_2^2$$

(c) Dar Dar and Ei Ei sits together, but Aye Aye and Bo Bo do not sit together?

Strategy: Compute the ways for (A, B) to sit together and (D, E) to sit together.

Subtract from the answer of part b.

$$\boxed{A \quad B}\ \boxed{C}\ \boxed{D \quad E}$$

$$P_3^3 \times P_2^2 \times P_2^2$$

Let's have a break!

We will continue after 5 minutes.

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Q6. Model Line-up

Khin San is orgainizing a fashion show where 10 male models and 5 female models are to be arranged in a line-up. In how many ways can Khin San do this if she does not want female models to be consecutive?

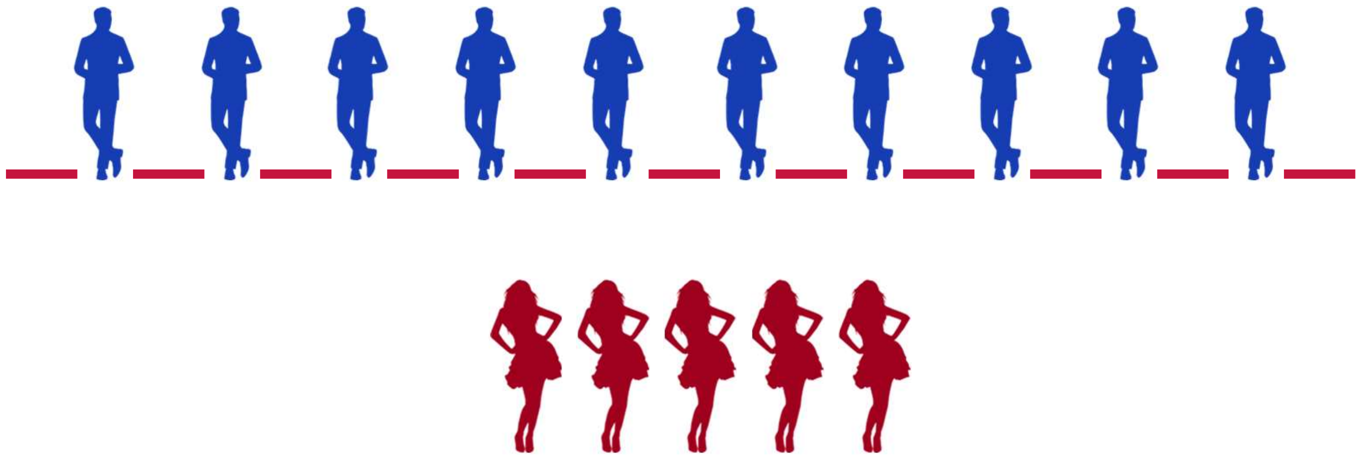
$\boxed{\text{Answer: 201180672000}} \leftarrow P^{10}_{10} \times P^{11}_{5}$

Q6. Model Line-up

Solution

Decision 1: Arrange 10 male models in a row. ← P^{10}_{10}

This creates 11 “gaps” for female models.



Decision 2: Arrange 5 female models in a row. ← P^{11}_{5}

Q7. My Name

In how many ways can we arrange the letters of the word HEINTA so that

(a) the vowels stay together and the consonants stay together?

$$\boxed{\text{Answer: 72}} \longleftarrow P_2^2 \times P_3^3 \times P_3^3$$

(b) no two consonants are adjacent to each other?

$$\boxed{\text{Answer: 144}} \longleftarrow P_3^3 \times P_3^4$$

Q7. My Name

(a) the vowels stay together and the consonants stay together?

$\boxed{H \quad N \quad T}$ $\boxed{E \quad I \quad A}$

$$P_2^2 \times P_3^3 \times P_3^3$$

(b) no two consonants are adjacent to each other?

__ E __ I __ A __

$$P_3^3 \times P_3^4$$

Some Common Tricks

Starting with, Ending with Trick: Just put the objects that you start/end at the start/end.

1 2 3 4 5 6 7

Box Together Trick: Put the objects that you wish to be together in a box. Put other objects in boxes of their own. First permute the boxes, then permute inside each box.

$\boxed{H\ N\ T}\boxed{E\ I\ A}$

Separation Trick: First place the "separator" objects. Then, put the objects to be separated in the gaps formed.

Factorials

Let n be a positive integer. Then, we define $n!$ (read n-factorial) to be

$$n! = 1 \times 2 \times 3 \times 4 \times \cdots \times n.$$

Some small factorials (it's worth it to memorize):

$$1! = 1 \qquad 2! = 2 \qquad 3! = 6 \qquad 4! = 24$$
$$5! = 120 \qquad 6! = 720 \qquad 7! = 5040$$

Note: Factorials grow very very fast! Much faster than exponential.

Zero Factorial

The “pattern” of the factorials is as follows:

$$1! \xrightarrow{\times 2} 2! \xrightarrow{\times 3} 3! \xrightarrow{\times 4} 4! \xrightarrow{\times 5} 5! \xrightarrow{\times 6}$$

This is the same as saying the following:

$$0! \xleftarrow{\div 1} 1! \xleftarrow{\div 2} 2! \xleftarrow{\div 3} 3! \xleftarrow{\div 4} 4! \xleftarrow{\div 5} 5! \xleftarrow{\div 6}$$

The only way to keep the “pattern” going is to define $0!$ as $1!/1$ which is 1.

Therefore, we define $0! = 1$.

Q8. Factorial Practice

Let's try to write the following products in terms of factorials:

- $1 \times 2 \times 3 \times 4 \times 5 \times 6 \longleftarrow 6!$
- $5 \times 6 \times 7 \times 8 \times 9 \longleftarrow \dfrac{9!}{4!}$
- $13 \times 14 \times 15 \times \cdots \times 100 \longleftarrow \dfrac{100!}{12!}$
- $\dfrac{5 \times 6 \times 7}{1 \times 2 \times 3} \longleftarrow \dfrac{7!}{4! \times 3!}$

Another Formula for Permutations

Recall that $P_r^n = \underbrace{n \times (n-1) \times (n-2) \times \cdots}_{\text{up to } r \text{ terms}}$

So, in terms of factorials,

$$P_r^n = \frac{n!}{(n-r)!}$$

For example,

- $P_3^8 = 8 \times 7 \times 6 = \dfrac{8!}{(8-3)!}$
- $P_4^{10} = 10 \times 9 \times 8 \times 7 = \dfrac{10!}{(10-4)!}$

Q10. A Factorial Identity

Prove that $1^{2023} \times 2^{2022} \times 3^{2021} \times \cdots \times 2023^{1} = 1! \times 2! \times 3! \times \cdots \times 2023!$.

Solution Consider the following arrangement of numbers:

$$\begin{array}{cccccc} 1 & & & & & \\ 1 & 2 & & & & \\ 1 & 2 & 3 & & & \\ 1 & 2 & 3 & 4 & & \\ & \vdots & \vdots & & & \\ 1 & 2 & 3 & 4 & \cdots & 2023 \end{array} \qquad \begin{array}{cccccc} 1 & & & & & \\ 1 & 2 & & & & \\ 1 & 2 & 3 & & & \\ 1 & 2 & 3 & 4 & & \\ & \vdots & \vdots & & & \\ 1 & 2 & 3 & 4 & \cdots & 2023 \end{array}$$

Both LHS and RHS are equal to the product of all those numbers in the triangular arrangement.

Well, I guess we both earned our rest!

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