GEO

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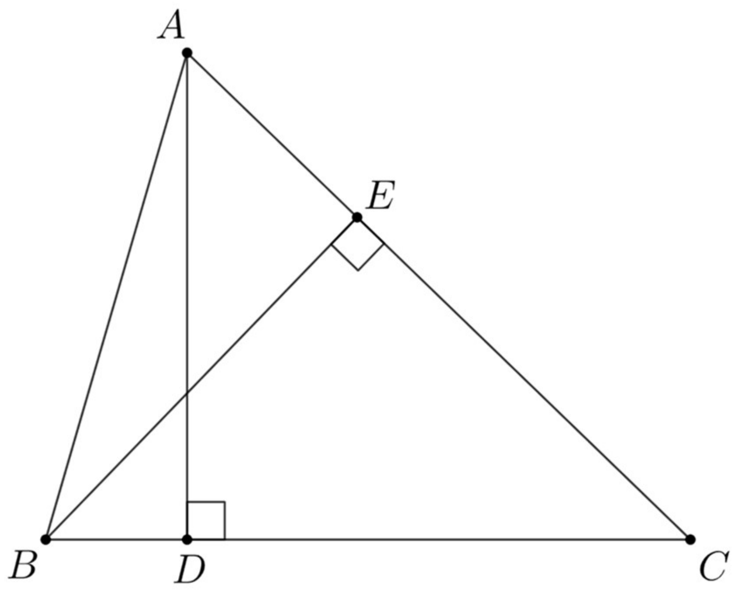
Video – 3

Equation Bash with Pythagoras

Q1. Finding the Altitude

Triangle ABC is an acute triangle with AB = 5, BC = 6 and AC = 7. Find the lengths of the altitudes from A to BC, and from B to AC.



Q1. Finding the Altitude

Solution

Let $BD = x$. Then, $CD = 6 - x$.

Then,

$AD^2 = AB^2 - BD^2 = 25 - x^2$,

$AD^2 = AC^2 - CD^2 = 49 - (6 - x)^2$.

Therefore,

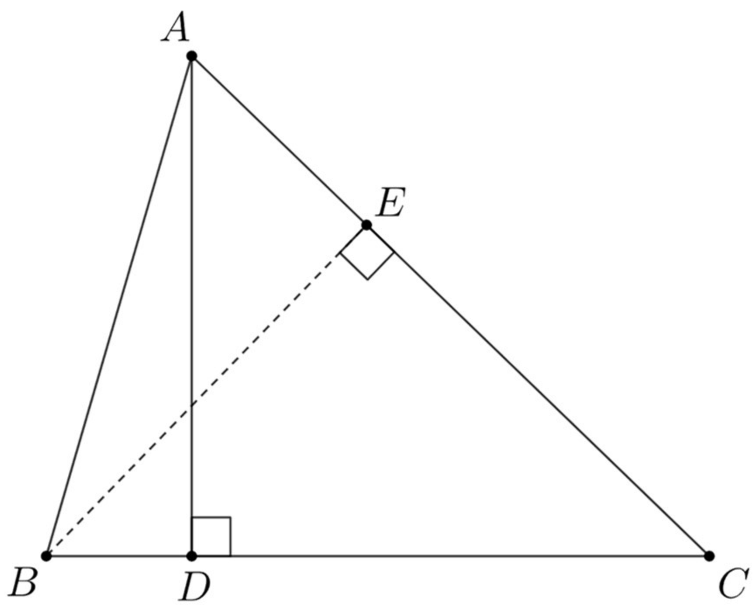
$25 - x^2 = 49 - (6 - x)^2$

$25 - x^2 = 49 - 36 + 12x - x^2$

$x = (25 + 36 - 49)/12 = 1$.

Hence, $AD^2 = 25 - 1 = 24$.

Therefore, $AD = \text{sqrt}(24) = 2\text{sqrt}(6)$.



Q1. Finding the Altitude

Solution (continued)

Note: We can use the same method to find BE.

That is, let AE = y, find equation in y, find y, then find BE.

But, here is a faster trick:

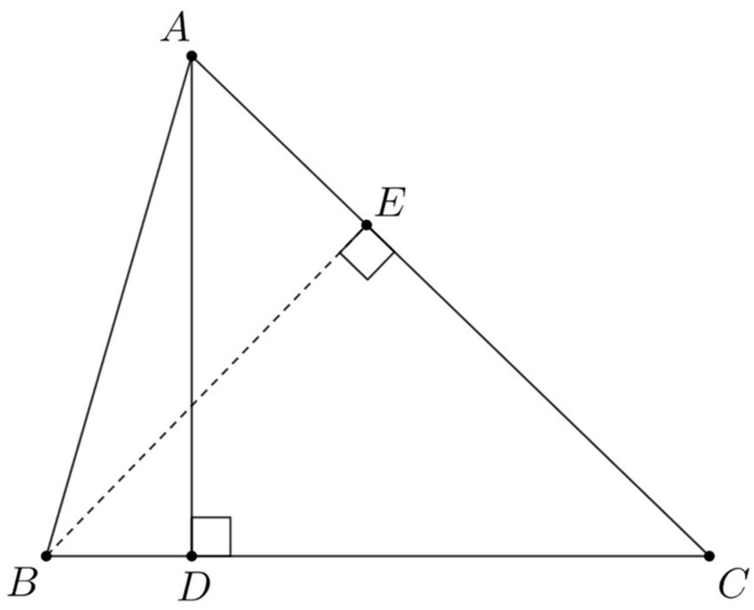
Area of ABC = AD $\times$ BC / 2.

Also, Area of ABC = BE $\times$ AC / 2.

Therefore, AD $\times$ BC = BE $\times$ AC.

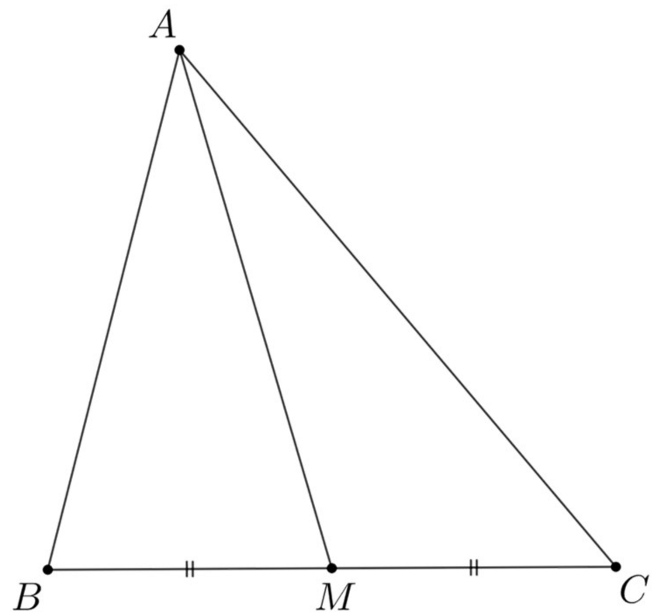
2sqrt(6) $\times$ 6 = BE $\times$ 7.

Hence, BE = 12sqrt(6)/7.



Q2. Finding the Median

Triangle ABC is an acute triangle with AB = 5, BC = 6 and AC = 7. Find the length of the median AM.



Q2. Finding the Median

Solution

Let AD be an altitude.

Strategy: Using the idea from previous problem, we can find BD and AD. From this, we can get AM.

Let BD = x.

Then, $AD^2 = 25 - x^2 = 49 - (6 - x)^2$.

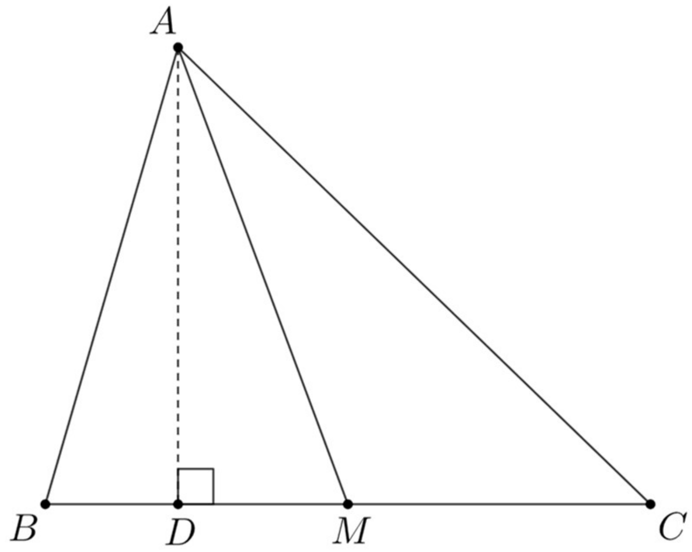
Solving for x gives x = 1.

Thus, MD = 3 − 1 = 2.

And, $AD^2 = 25 - 1 = 24$.

Therefore, $AM^2 = 24 + 4 = 28$.

Hence, AM = 2sqrt(7).

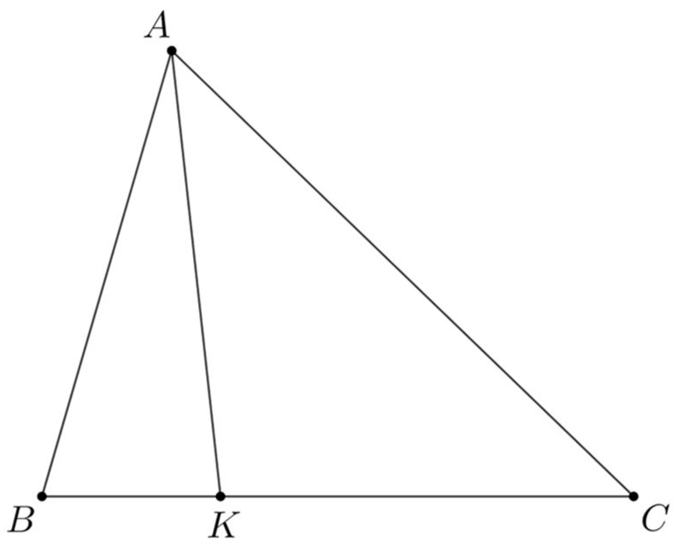


Finding Cevians

Suppose that we know the lengths of the sides of triangle ABC. Let K be a special point on BC, and suppose we know BK and KC. For example, we can do this if K is midpoint of BC.

Recipe to find AK

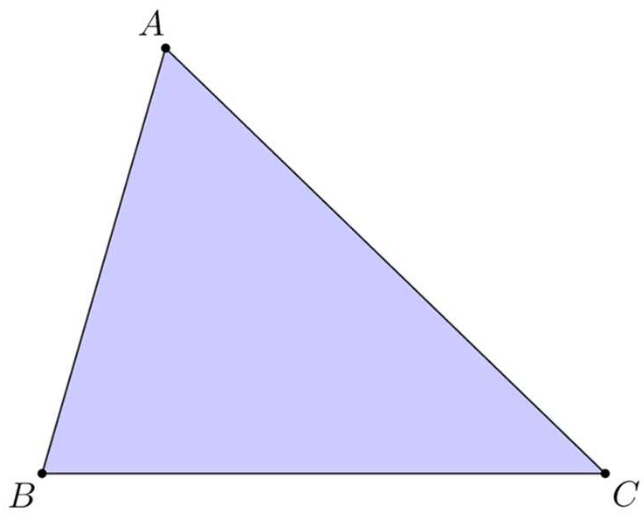
1. Let AD be the altitude.
2. Let BD = x. Use $AD^2 = AB^2 - BD^2 = AC^2 - CD^2$ to get (linear) equation in x.
3. Solve for x to get BD and AD.
4. So, we can find KD.
5. Use Pythagoras to get AK.



Heron's Formula

We can "skip" having to solve for x every time by knowing the area formula. Because then, we can find the altitude by

altitude = 2 × area / base.



Heron's Formula

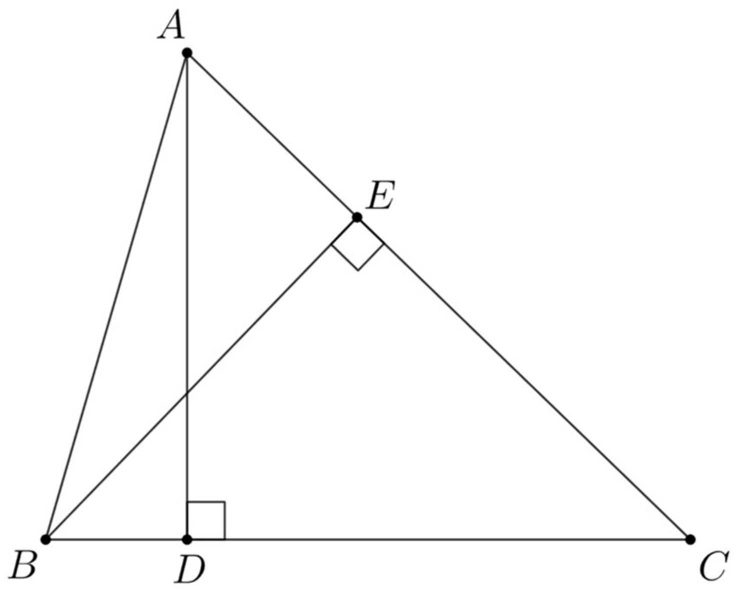
In triangle ABC, let BC = a, CA = b and AB = c. Then, area of triangle ABC is equal to

$$\sqrt{s(s-a)(s-b)(s-c)}$$

where s = (a + b + c)/2 is the semiperimeter.

Q1*. Finding the Altitude (redo)

Triangle ABC is an acute triangle with AB = 5, BC = 6 and AC = 7. Find the lengths of the altitudes from A to BC, and from B to AC.



Solution

Let s = (5 + 6 + 7)/2 = 9.

So, s(s – a)(s – b)(s – c) = 9 × 4 × 3 × 2.

Hence, area of ABC = sqrt(9 × 4 × 3 × 2) = 6sqrt(6).

So,

AD = 2 × 6sqrt(6) / 6 = 2sqrt(6).

BE = 2 × 6sqrt(6) / 7 = 12sqrt(6)/7.

Heron's Formula Proof

The idea is very simple. It is exactly the same with Q1, but we use a, b, c instead of 5, 6, 7.

Let $BD = x$. Then, $BC = a - x$.

Then,

$AD^2 = AB^2 - BD^2 = c^2 - x^2$.

$AD^2 = AC^2 - CD^2 = b^2 - (a - x)^2$.

Thus, $c^2 - x^2 = b^2 - (a - x)^2$

$c^2 - x^2 = b^2 - a^2 + 2ax - x^2$.

Thus, $x = (a^2 + c^2 - b^2)/2a$.

So, $a^2x^2 = (a^2 + c^2 - b^2)^2/4$

$(\text{Area of } ABC)^2$

$= a^2 \times AD^2/4$

$= a^2(c^2 - x^2)/4$

$= (a^2c^2 - (a^2 + c^2 - b^2)^2/4)/4$

$= (4a^2c^2 - (a^2 + c^2 - b^2)^2)/16$

$= (2ac + a^2 + c^2 - b^2)(2ac - a^2 - c^2 + b^2)/16$

$= ((a + c)^2 - b^2)(b^2 - (a - c)^2)/16$

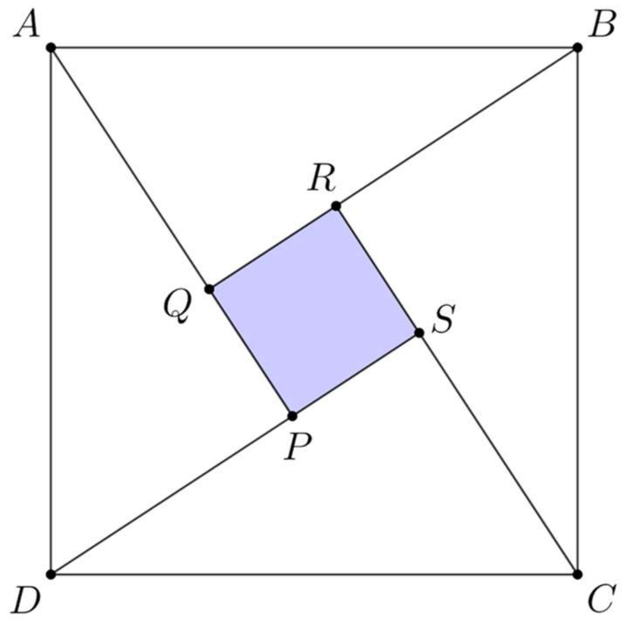
$= (a + c + b)(a + c - b)(b + a - c)(b - a + c)/16$

$= s(s - a)(s - b)(s - c)$.

Therefore, Area of $ABC = \text{sqrt}(s(s - a)(s - b)(s - c))$.

Q3. Four Right Triangles in Square

In the figure, ABCD and PQRS are squares. Suppose that AB = 10 and PS = 2. Find the length of AQ.



Q3. Four Right Triangles in Square

Solution

Triangles AQB, BRC, CSD and DPA are congruent (by AAS).

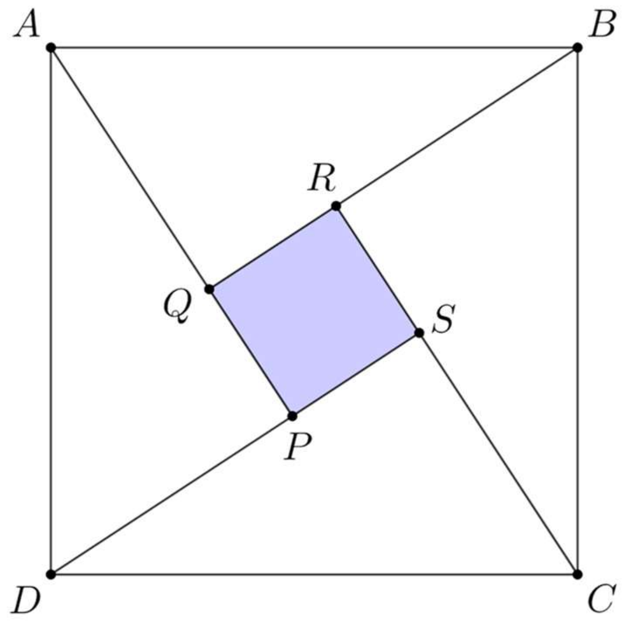
Let AQ = x. Then, BR = x.

In right triangle AQB,

$AB^2 = AQ^2 + QB^2$

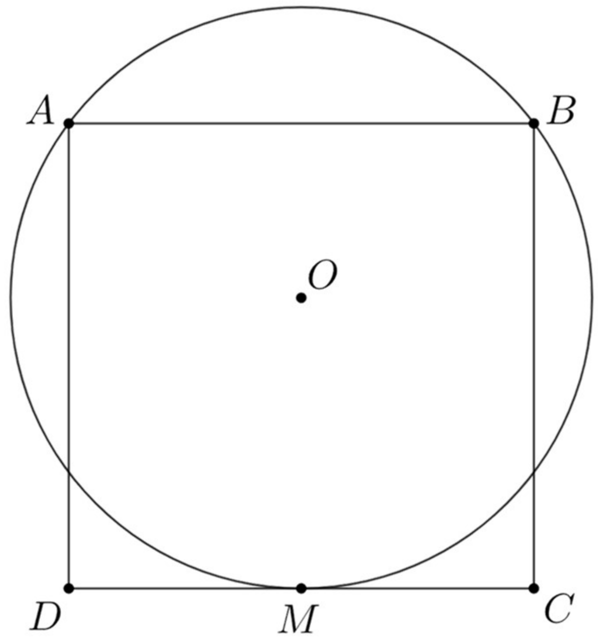
$100 = x^2 + (x + 2)^2$.

Solving for x gives x = 6.



Q4. Circle and Square

In the figure ABCD is a square and M is midpoint of CD. A circle centred at O passes through A, B and tangent to CD at M. If AB = 2, find the radius of the circle.



Q4. Circle and Square

Solution

Since OM $\perp$ CD and M is midpoint, the line OM is the line of symmetry of ABCD.

Let OM cut AB at N. Then, NA = NB and $\angle$ANO = 90° by symmetry.

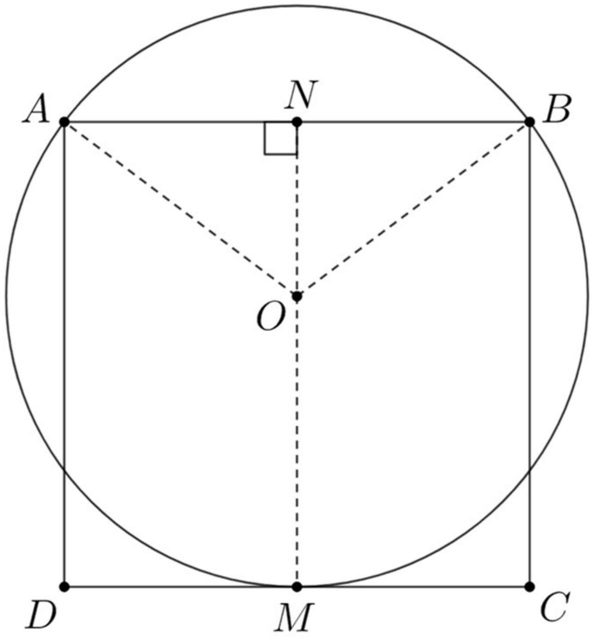
Let R be the radius.

OA = R, ON = 2 − R, AN = 1.

Therefore,

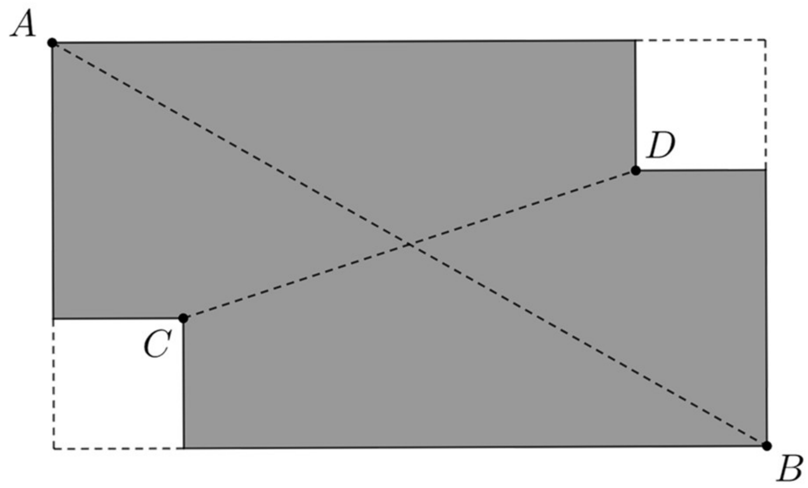
$R^2 = 1^2 + (2 - R)^2$.

Solving for R gives R = 5/4.



Q5. Cutting a Metal Sheet

Two unit squares are cut from a rectangualar metal sheet at the corners as shown in the figure. Suppose that AB = 8 and CD = 4sqrt(2). What is the area of the original metal sheet?



Q5. Cutting a Metal Sheet

Solution

Let width = x and height = y.

From $AB^2 = 64$ and $CD^2 = 32$, we get:

$x^2 + y^2 = 64$ and $(x - 2)^2 + (y - 2)^2 = 32$.

Simplifying the second equality:

$x^2 - 4x + y^2 - 4y + 8 = 32$

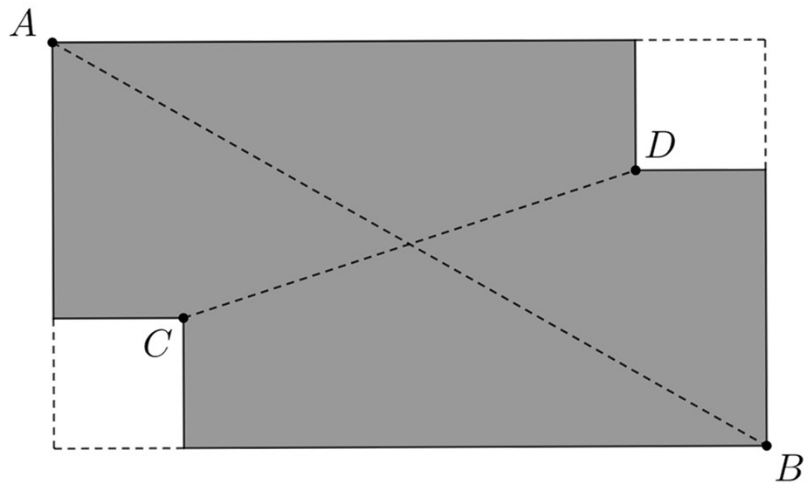
$64 - 4x - 4y + 8 = 32$ (because $x^2 + y^2 = 64$)

$x + y = 10$.

Here, we can solve for x and y if we want. But, here is a faster way:

$(x + y)^2 = 100$ and thus $x^2 + y^2 + 2xy = 100$.

From $x^2 + y^2 = 64$, we get $xy = 18$.

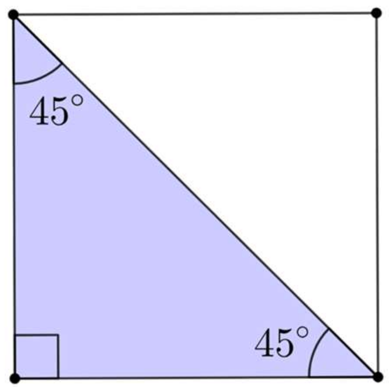


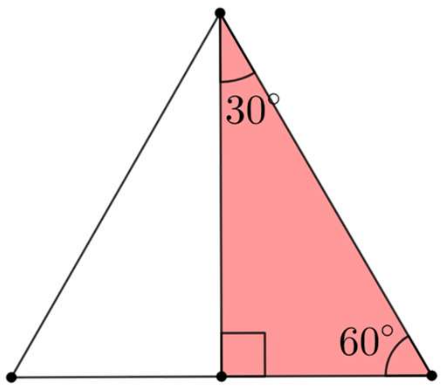
Special Right Triangles

In right triangles with angles 45-45-90, ratio of side lengths is 1 : 1 : sqrt(2).

In right triangles with angles 30-60-90, ratio of side lengths is 1 : sqrt(3) : 2.

Note: So, in a 45-45-90 or 30-60-90 triangle, we can find all the side-lengths just by knowing one.

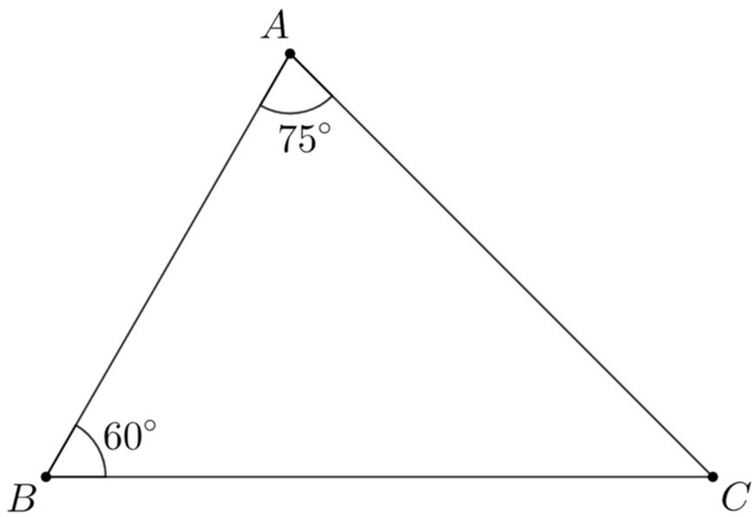




Q6. 45-60-75 Triangle

In the figure, ABC is a triangle with $\angle A = 75°$ and $\angle B = 60°$.

Find AB : BC : CA.



Q6. 45-60-75 Triangle

Solution

Draw the altitude AD.

Then, ABD is 30-60-90 triangle and ACD is 45-45-90.

Let BD = k.

Then, AB = 2k and AD = sqrt(3)k.

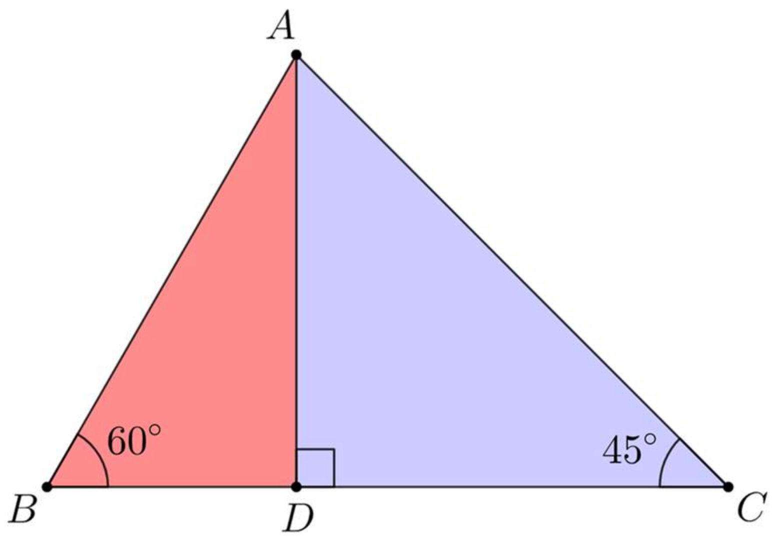
So, CD = sqrt(3)k and AC = sqrt(2) $\times$ sqrt(3)k = sqrt(6)k.

Hence,

AB : BC : CA

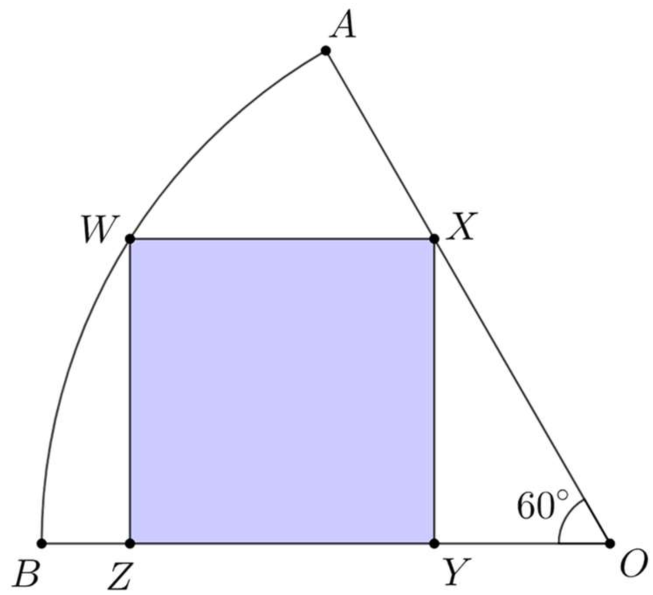
= 2k : (k + sqrt(3)k) : sqrt(6)k

= 2 : (1 + sqrt(3)) : sqrt(6).



Q7. Square in 60° Sector

In the figure, OAB is a sector with $\angle AOB = 60°$ and WXYZ is a square. Suppose that the area of WXYZ is 9. Find the area of the sector.



Q7. Square in 60° Sector

Solution

$WZ = ZY = 3$.

Since XYO is 30-60-90 triangle and $XY = 3$,

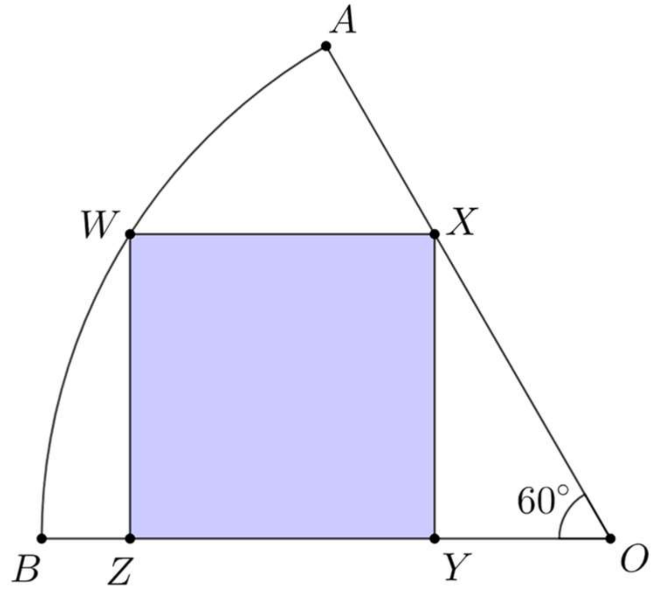
we get $OY = 3/\text{sqrt}(3) = \text{sqrt}(3)$.

Thus,

$OW^2 = 3^2 + (3 + \text{sqrt}(3))^2 = 21 + 6\text{sqrt}(3)$.

So, area of the sector is

$\pi \times OW^2 / 6 = (3.5 + \text{sqrt}(3))\pi$



That's it for this video.

We both earned our rest.

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