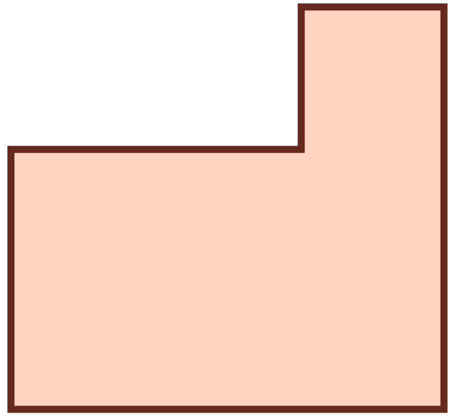
We will begin at 13:05 MMT

Read this problem while we wait…

In the picture, you are given a polygon all of whose interior angles are either 90 degrees or 270 degrees. Explicitly find a line through the polygon that divides the polygon into two pieces of equal area.



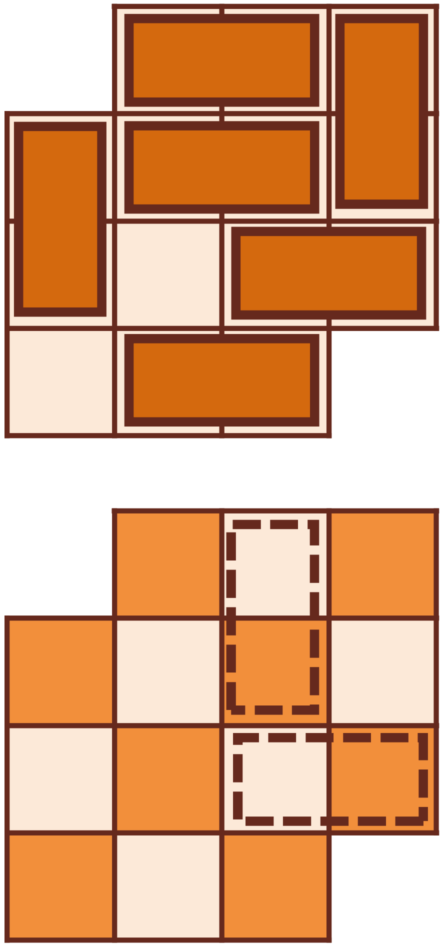
Record the meeting…

Content so far…

Lecture – 12

Colouring and WLOG

Classic Dominoes on Chessboard



Two squares in opposite corners of a 4×4 chessboard are cut out. Is it possible to tile the remaining board with 1×2 dominoes? You can rotate the dominoes.

The answer seems like a ‘no’.

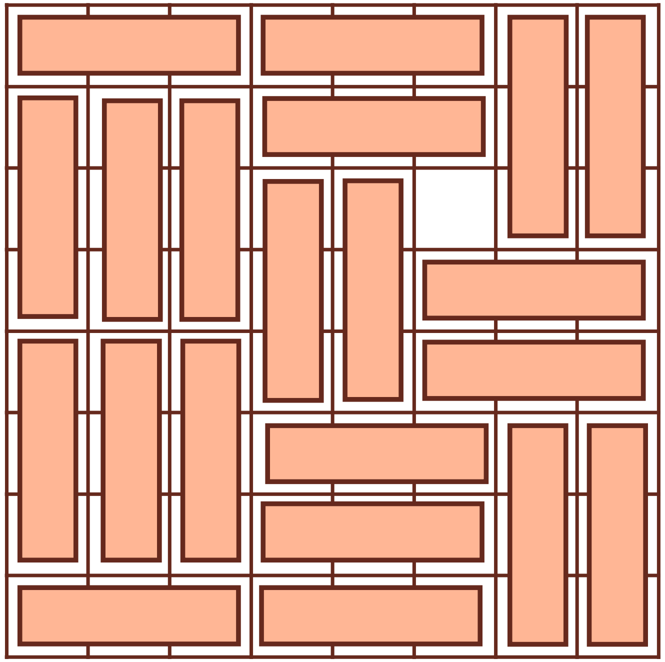
To see why, consider the chessboard colouring…

Each domino covers one cell of each colour. So, boards that can be tiled with dominoes have same number of cells of each colour.

But, it is not in this case. So, you cannot tile.

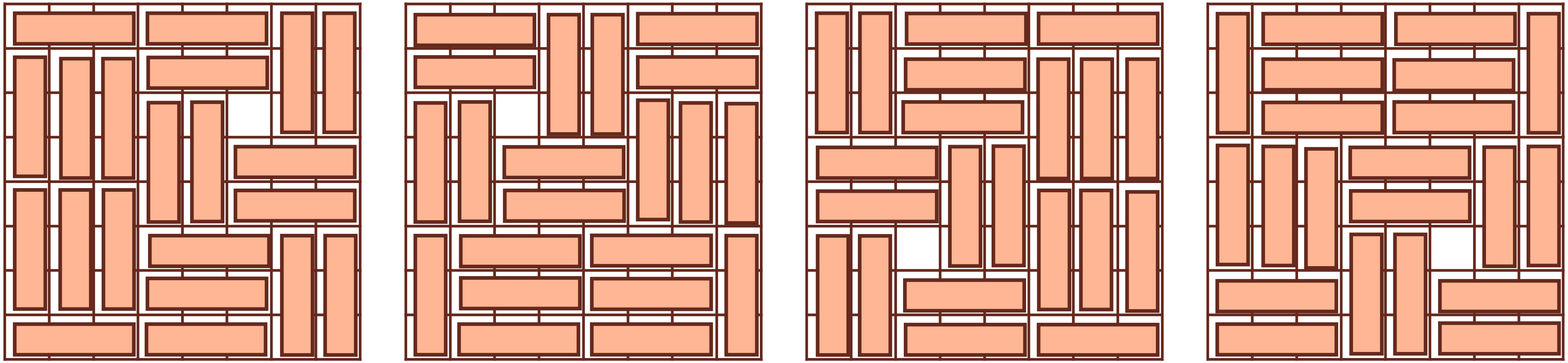
Where is the Missing Square? (Italy TST 1995)

An 8×8 board is tiled with 21 trominoes (3×1 tiles) so that exactly one square is not covered by a tromino. No two trominoes can overlap and no tromino can stick out of the board. Determine all possible positions of the square not covered by a tromino.



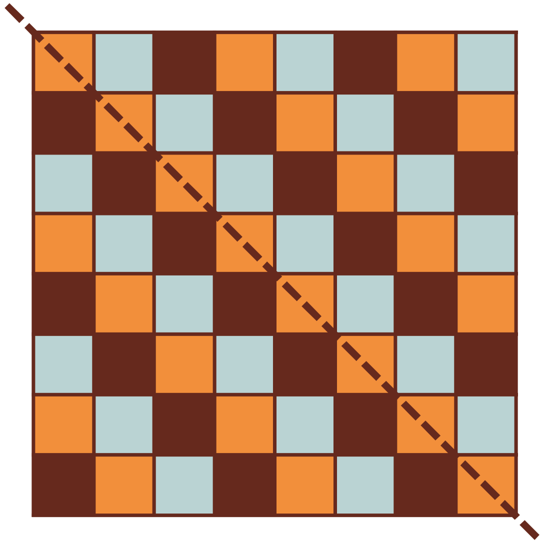
Symmetry!

Well, just by the example tiling, we can see that the following four cells can be free.



Where must the free cell be at?

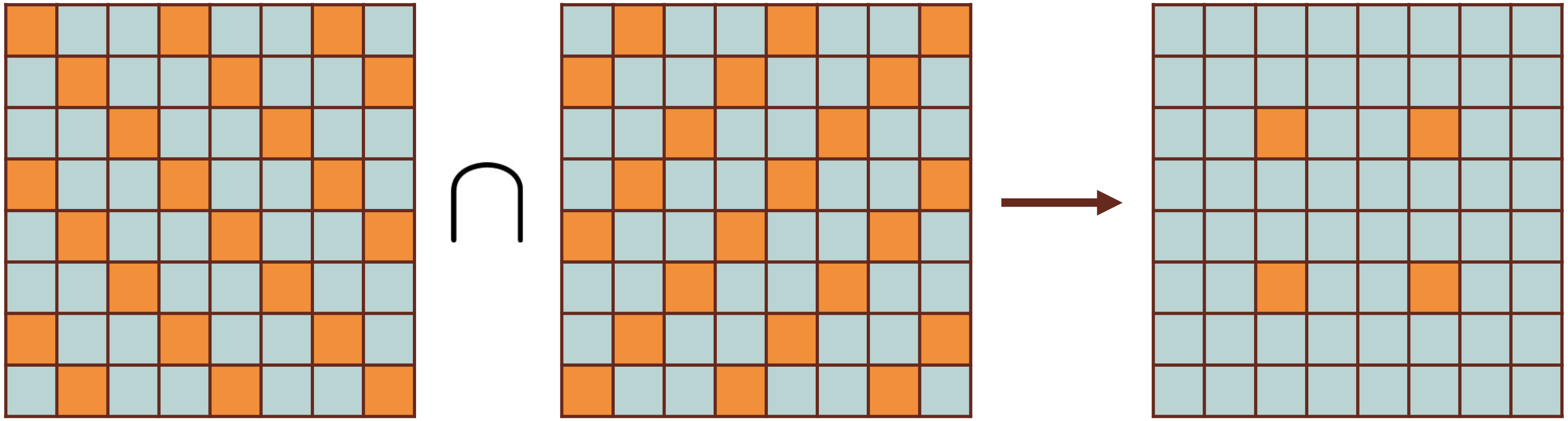
Colour alternatingly in three colours as follows:



- Each tromino covers one cell of each colour.
- By symmetry, there are equal number of browns and teals. There are more orange cells than brown and teal.
- Thus, the empty cell must be one of the oranges.

Even more symmetry!

The free cell still lies in orange cell if we rotate/reflect the board.



- So, the free cell must lie in the intersection of the orange cells in two different colourings.
- This shows that the free cells must lie in four of the cells shown.
- But, we have already seen that it is possible to have free cell at all of these cells. So, we are done!

Ants on a Stick

Finite number of ants sit on a stick of length 1 metre, each facing towards either end of the stick. When the ant collector Aunty clapped, all the ants begin to move in the direction they are facing with the constant speed of 1 metre per minute. Whenever two ants meet, they both turn around and keep going. Whenever an ant reaches an end of the stick, it falls off.

Show that all the ants will fall off the stick within one minute.

Examples

- This is obvious for one or two ants.
- What about three ants? Let’s take a moment to think.

Main Observation

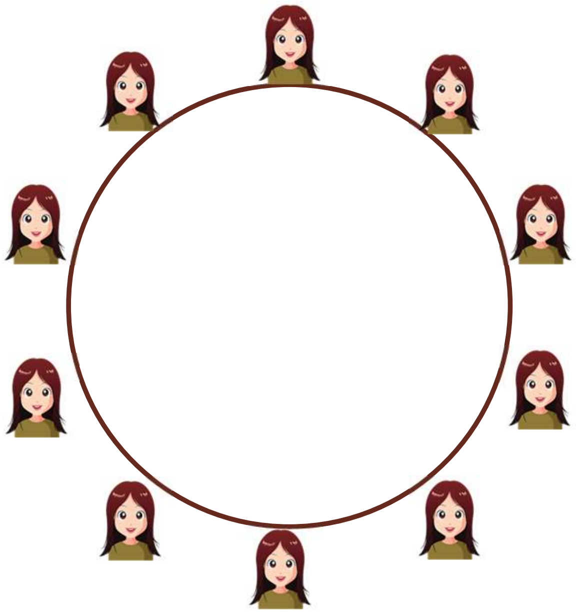
- We are only interested in the positions of the ants.
- So, we can assume that the ants never turn around, and walks through each other!
- This does not have any effect on the set of positions of the ants. What we are interested in is preserved!
- The conclusion should now be obvious.

Girls Revisited (IMOSL 1994/C5a)

1994 girls are seated in a circle. Initially, one girl is given n coins. In one move, each girl with 2 coins passes one coin to each of her two neighbors.

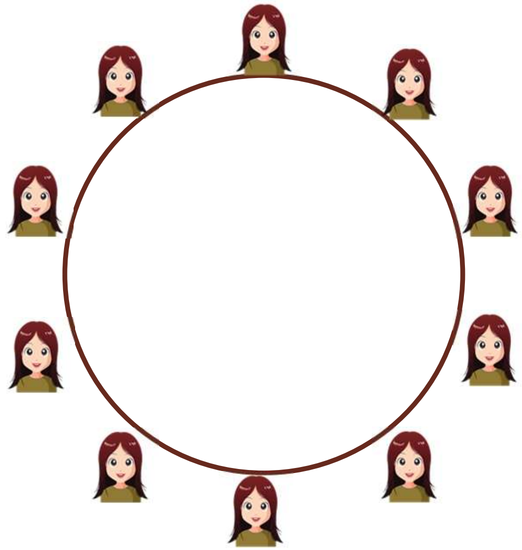
(a) Show that if $n < 1994$, the game must terminate.

(b) Show that if $n = 1994$, the game cannot terminate. ← We did this in lecture 2

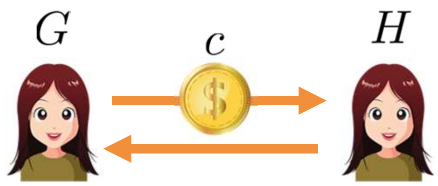


Awesome Solution

- Let $n < 1994$, suppose the game never terminates.
- Then, one of the girls will pass coins infinitely many times.
- Thus, all the girls will pass coins infinitely many times.



Awesome Idea: When a coin c is passed between the girls G and H for the first time, we can WLOG assume that they only keep coin c between them!



So, for any pair of adjacent girls, a coin will be eventually stuck between them.

This means that we need at least 1994 coins in the first place!

I have no idea how they came up with this solution.

What is WLOG?

- Sometimes, you can impose more conditions into a problem.
  - Example: 1994 Girls, Relatively prime assumptions in NT

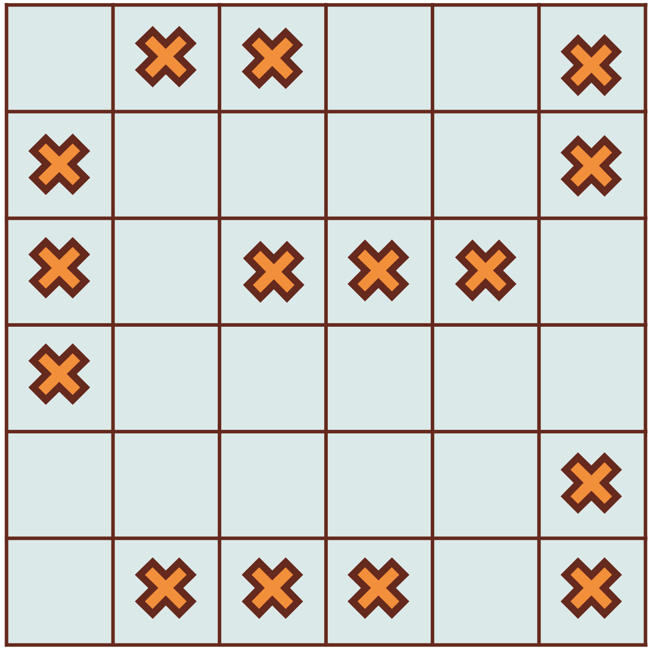
- Sometimes, you can just ignore some conditions given in a problem.
  - This is often dangerous if you have not gotten a sufficient progress in the problem.

- Sometimes, you can even change the rules in the problem!
  - Example: Ants, Inequality normalization

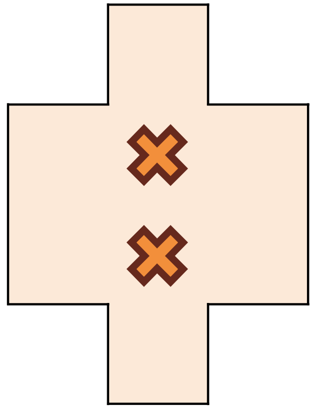
But, despite what you do, the problem itself does not change. Or the information governing what we want to show does not change.

Is this tiling? (IMO 1999/P3)

Let n be an even positive integer. We say that two different cells of an $n \times n$ board are neighboring if they have a common side. Find the minimal number of cells on the $n \times n$ board that must be marked so that any cell (marked or unmarked) has a marked neighboring cell.

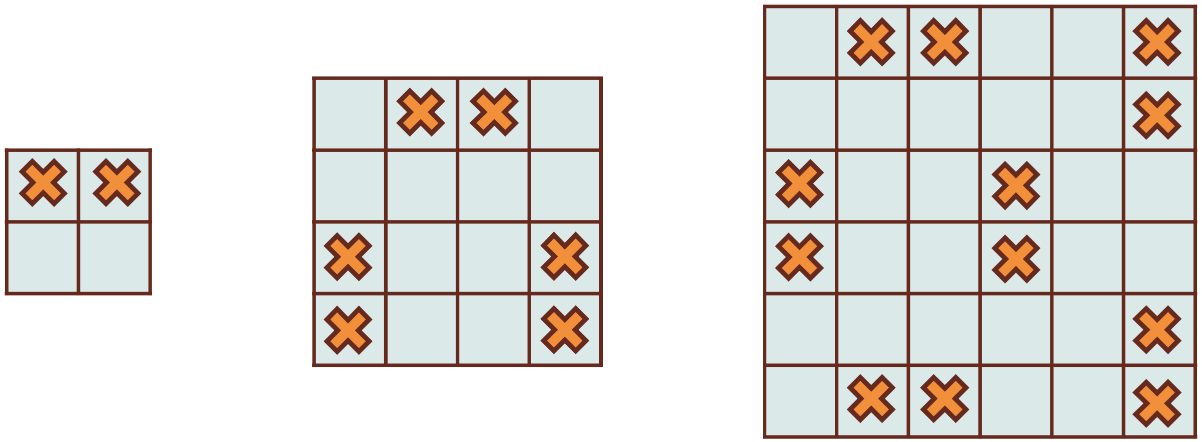


First Observations

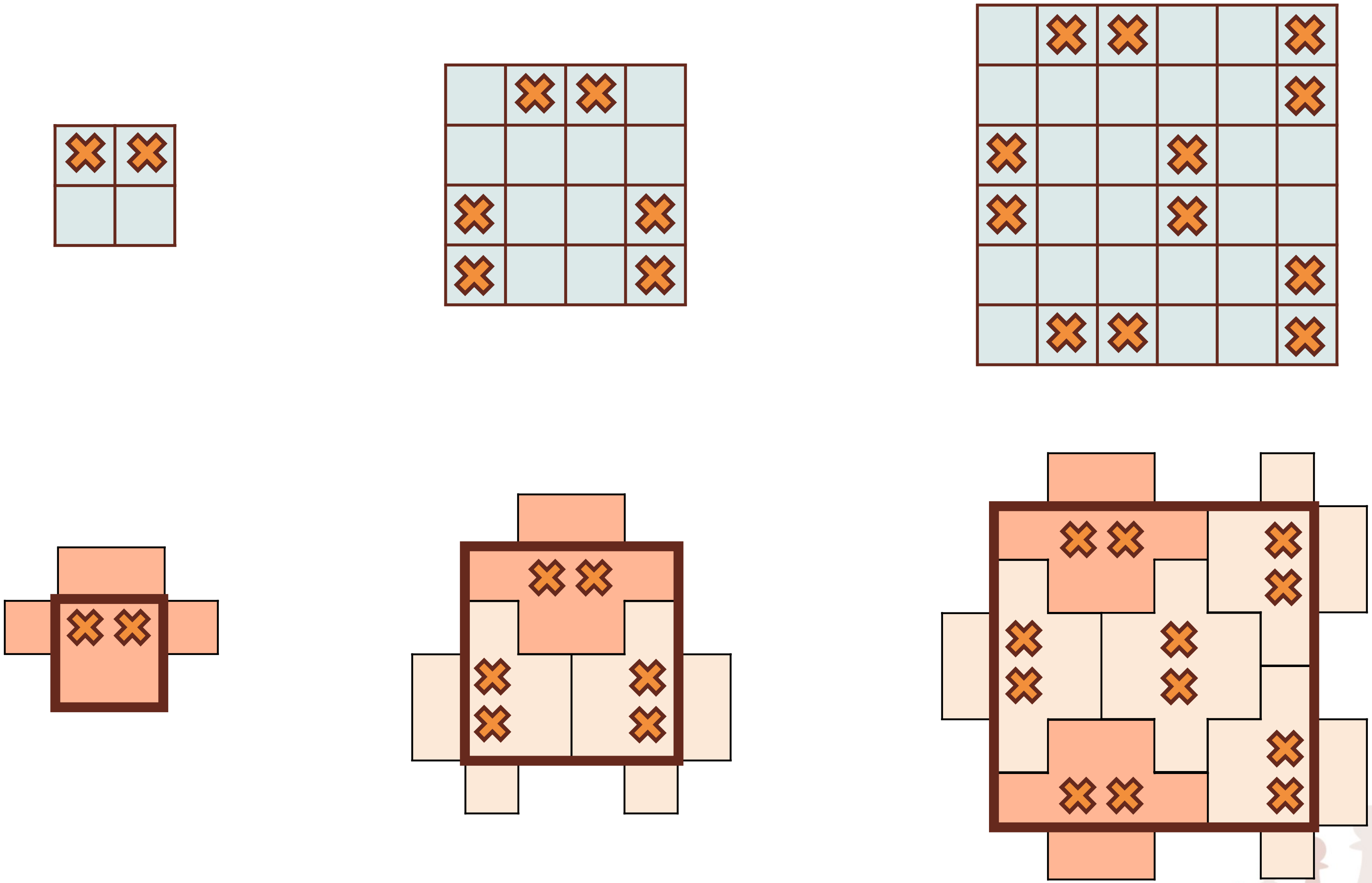


- All the marked cells should be adjacent to another marked cell. So, marked cells come in chunks of size at least 2.

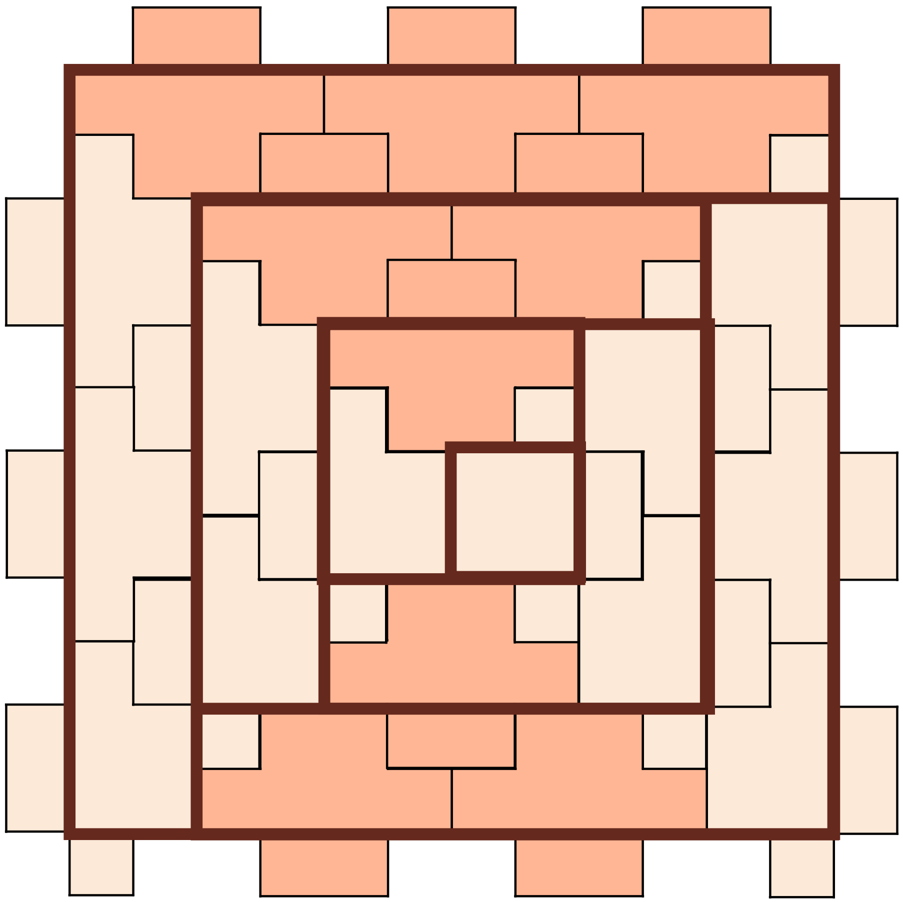
- If we are trying to minimize the number of marks, we should try with all chunks having size exactly 2. All the “auras” of these chunks should be disjoint.



Auras as tiles



General Construction



In the construction to the left, we have $n = 12$.

If we write $n = 2k$, then we can break our grid into giant L-shapes of sizes

$$2k,\ 2k-2,\ 2k-4,\ \ldots\ ,\ 4,\ 2.$$

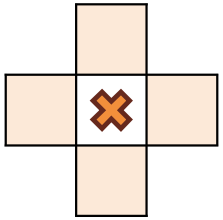
There are exactly m marked squares in the L-shape of size m.

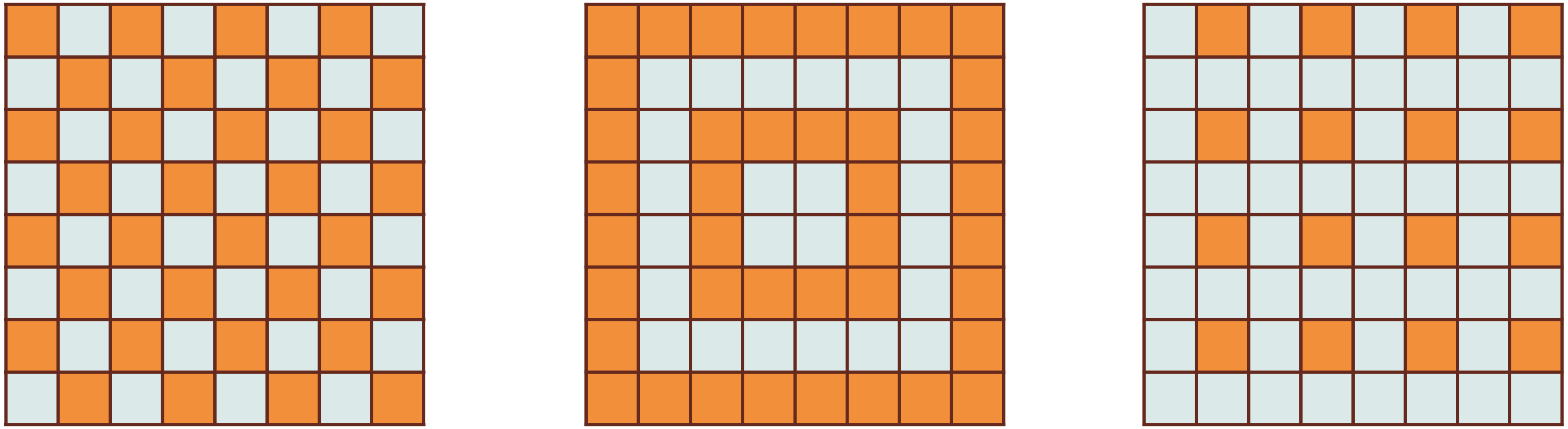
Thus, number of marked cells we used this way is

$$2k + 2(k-1) + 2(k-2) + \cdots + 2 = k(k+1)$$

The Upper Bound

One of the following colourings behave nicely when we put the ‘four neighbors diamond’ on the board.

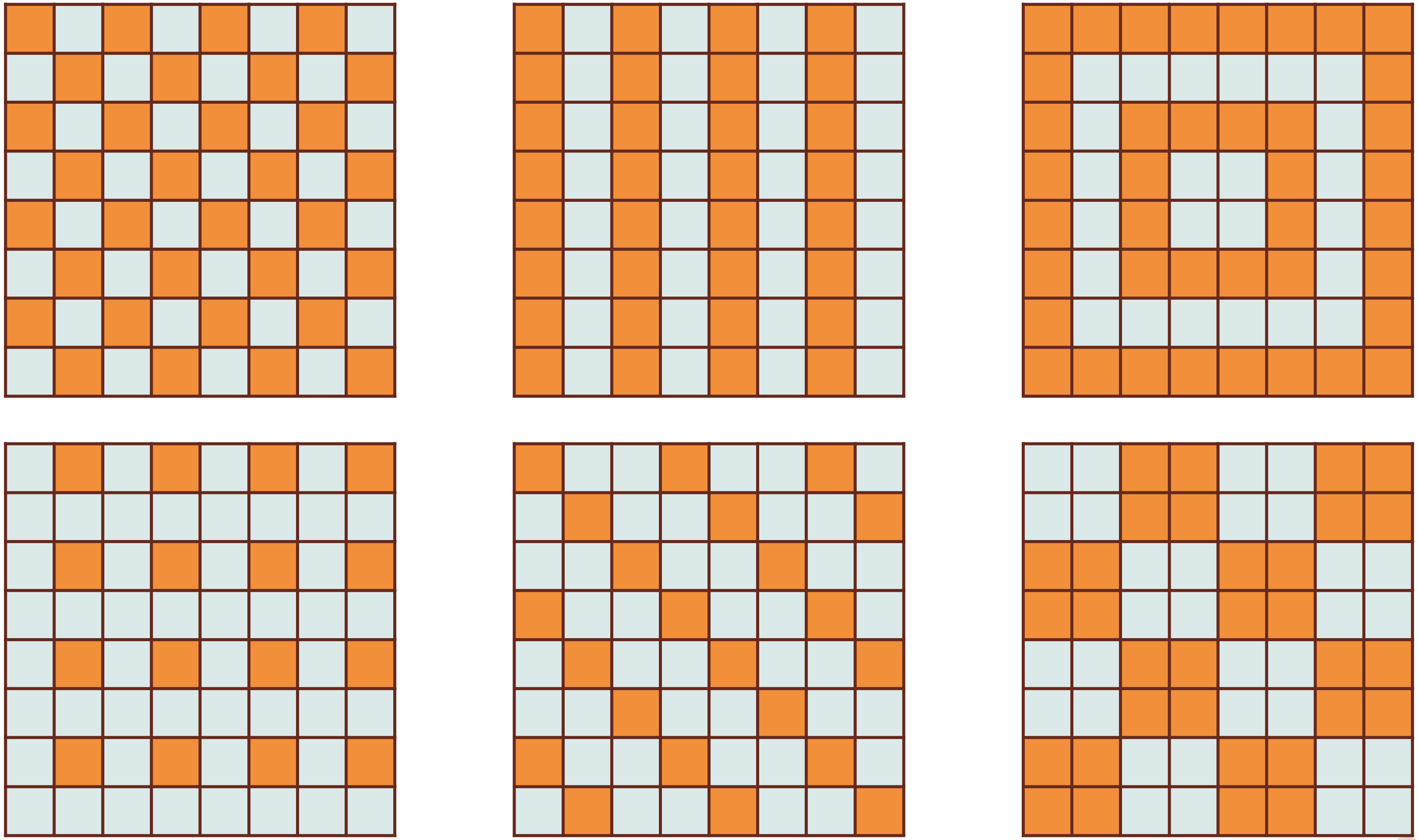




Answer: The middle colouring. Every diamond will cover exactly two orange squares!!

Since there are exactly $2k(k+1)$ orange squares, we must use at least $k(k+1)$ markings. So, the construction we found is indeed the best construction.

Many colourings to try out



That’s it for this lecture.

See you tomorrow!