Information Theory Notes

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These are preliminary notes.

1 Classification in high-dimension, fixed SNR regime

We observe a data point y_* which belongs to one of K classes. The distribution in the ith class is $N(\mu_i, \Omega)$. We have another dataset with r repeats per class, which we use to estimate the centroids μ_i: we obtain estimates $\hat{\mu}_i \sim N(\mu_i, r^{-1}\Omega)$. The class centroids were originally drawn i.i.d. from a multivariate normal $N(0, I)$. Furthermore Ω is unknown and have to be estimated as well: assume we have obtained estimate $\hat{\Omega}$ via some method. Without loss of generality, take the Kth class to be the true class of y_*. Write $\hat{\mu}_* = \hat{\mu}_K$.

The classification rule is given by

$$\text{Estimated class} = \text{argmin}_i \ (y_* - B\hat{\mu}_i)^T A (y_* - B\hat{\mu}_i)$$

where A and B are matrices based on $\hat{\Omega}$. The Bayes rule is given by

$$A_{Bayes} = (I + \Omega - (I + r^{-1}\Omega)^{-1})^{-1}$$

$$B_{Bayes} = (I + r^{-1}\Omega)^{-1}.$$

The "plug-in" estimates of A and B are

$$A = (I + \hat{\Omega} + (I + r^{-1}\hat{\Omega})^{-1})^{-1}$$

$$B = (I + r^{-1}\hat{\Omega})^{-1}.$$

Note that

$$(y_* - B\hat{\mu}_i)^T A(y^* - B\hat{\mu}_i) = ||A^{1/2}y_* - A^{1/2}B\hat{\mu}_i||^2.$$

Therefore the classification rule is

$$\text{Estimated class} = \text{argmin}_i\ Z_i,$$

where

$$Z_i = ||A^{1/2}y_* - A^{1/2}B\hat{\mu}_i||^2.$$

We have

$$\begin{bmatrix} A^{1/2}y \\ A^{1/2}B\hat{\mu}_* \\ A^{1/2}B\hat{\mu}_i \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} A^{1/2}(I+\Omega)A^{1/2} & A^{1/2}BA^{1/2} & 0 \\ & A^{1/2}B(I+\frac{\Omega}{r})BA^{1/2} & 0 \\ & & A^{1/2}B(I+\frac{\Omega}{r})BA^{1/2} \end{bmatrix}\right)$$

Therefore

$$\mathbf{E}Z_i = \begin{cases} \text{tr}[A(I+\Omega+(B(I+r^{-1}\Omega)B))] & \text{for } i \neq K \\ \text{tr}[A(I+\Omega+(B(I+r^{-1}\Omega)B)-2B)] & \text{for } i = K \end{cases},$$

$$\text{Cov}(Z_i, Z_j) = \begin{cases} 2\text{tr}[A(I+\Omega)]^2 & \text{for } i \neq j \neq K \\ 2\text{tr}[A(I+\Omega-B)]^2 & \text{for } i = K, j \neq K \\ 2\text{tr}[A(I+\Omega+B(I+r^{-1}\Omega)B)]^2 & \text{for } i = j \neq K \\ 2\text{tr}[A(I+\Omega+B(I+r^{-1}\Omega)B-2B)]^2 & \text{for } i = j = K \end{cases}.$$

1.1 Ω known, fixed r

Suppose

$$\hat{\Omega} = \Omega.$$

Then,

$$\mathbf{E}Z_i = \begin{cases} p + 2\text{tr}[AB] & \text{for } i \neq K \\ p & \text{for } i = K \end{cases},$$

and

$$\text{Cov}(Z_i, Z_j) = \begin{cases} 2\text{tr}[I+AB]^2 & \text{for } i \neq j \neq K \\ 2p & \text{for } i = K, j \neq K \\ \text{tr}[I+2AB]^2 & \text{for } i = j \neq K \\ 2p & \text{for } i = j = K \end{cases}.$$

Let $\gamma_i(r)$ denote the eigenvalues of AB: we have

$$\gamma_i(r) = \frac{1}{(\omega_i^2 + \omega_i)/r + \omega_i}$$

where ω_i are the eigenvalues of Ω.

The misclassification probability is

$$\mathrm{MC} = 1 - \Pr[N(\mu(r), \sigma^2(r)) < M_{K-1}]$$

where M_{K-1} is the maximum of $K-1$ independent standard normal variates, and

$$\mu(r) = \frac{2\sum_{i=1}^p \gamma_i}{\sqrt{\sum_{i=1}^p 6\gamma_i^2 + 4\gamma_i}},$$

$$\sigma^2(r) = \frac{\sum_{i=1}^p \gamma_i^2 + 2\gamma_i}{\sum_{i=1}^p 3\gamma_i^2 + 2\gamma_i}.$$

Under the condition that

$$\mathrm{tr}[\Omega^{-1}] \to \text{const.}, \ \mathrm{tr}[\Omega^{-2}] \to 0$$

we have

$$\mu(r) \to \sqrt{\sum_{i=1}^p \gamma_i(r)},$$

$$\sigma^2(r) \to 1.$$

1.2 Known Ω, growing r

Assume that Ω/p has a limiting spectrum, i.e. defining

$$H^{(p)} = \sum_{i=1}^p \frac{1}{p}\delta_{\omega_i/p}$$

we have

$$H^{(p)} \to H$$

for some probability measure H on $\mathbb{R}^+$.

Also, suppose that $\frac{r}{p} \to \kappa$. Then, define

$$
\begin{aligned}
\mu^*(\kappa) = \lim_{p\to\infty} \mu(r) &= \lim_{p\to\infty} \frac{2\sum_{i=1}^p \gamma_i}{\sqrt{\sum_{i=1}^p 6\gamma_i^2 + 4\gamma_i}}, \\
&= \lim_{p\to\infty} \frac{2p\int \frac{1}{\frac{p^2\nu^2+p\nu}{r}+p\nu} dH(\nu)}{\sqrt{p\int 6\left(\frac{1}{\frac{p^2\nu^2+p\nu}{r}+p\nu}\right)^2 + 4\left(\frac{1}{\frac{p^2\nu^2+p\nu}{r}+p\nu}\right) dH(\nu)}} \\
&= \lim_{p\to\infty} \sqrt{p\int \frac{1}{\frac{p^2\nu^2+p\nu}{r}+p\nu} dH(\nu)} \\
&= \lim_{p\to\infty} \sqrt{\int \frac{1}{\frac{p\nu^2}{r}+(1+1/r)\nu} dH(\nu)} \\
&= \sqrt{\int \lim_{p\to\infty} \frac{1}{\frac{p\nu^2}{r}+(1+1/r)\nu} dH(\nu)} \\
&= \sqrt{\int \frac{1}{\frac{\nu^2}{\kappa}+\nu} dH(\nu)},
\end{aligned}
$$

while for similar reasons

$$\lim_{p\to\infty} \sigma^2(r) = 1.$$

Thus, we have $\mu^*(\kappa)$ increasing in κ, and tending to

$$\mu^*(\infty) = \int \nu^{-1} H(d\nu)$$

as $\kappa \to \infty$.

Therefore, we get the following result.

Theorem. *Let H be a positive measure. For Ω known, if $p\Omega$ has limiting spectrum H, and $\lim r/p = \kappa$, then the asymptotic misclassification rate $MC^* = \lim_{p\ to\infty} MC$ is a function only of κ, is increasing in κ, and is bounded between*

$$MC^*(0) = 1 - \Pr[N(0,1) < M_{K-1}] < 1$$

and

$$MC^*(\infty) = 1 - \Pr[N(\mu^*(\infty),1) < M_{K-1}] > 0,$$

where

$$\mu^*(\infty) = \int \nu^{-1} H(d\nu).$$

To summarize, the misclassification rate with a fixed number of repeats r converges to a constant, $\mathrm{MC}^*(0) < 1$, while the Bayes misclassification rate is given by $\mathrm{MC}^*(\infty) > 0$. To get a asymptotically constant misclassification rate between $\mathrm{MC}^*(0)$ and $\mathrm{MC}^*(\infty)$, one needs a number of repeats r which is of order $O(p)$.

2 Appendix

2.1 Gaussian min probs

Define

$$f_{ng}(\mu, \sigma^2, K) = \Pr[\sigma Z_* + \mu < \max_{i=1}^{K} Z_i]$$

for $Z_*, Z_1, \ldots, Z_K$ i.i.d normal.

Suppose

$$\begin{bmatrix} y_* \\ y_1 \\ \vdots \\ y_{K-1} \end{bmatrix} \sim N\left(\begin{bmatrix} a \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} b & c & \ldots & c \\ c & d & \ldots & e \\ \ldots & \ldots & \ddots & \vdots \\ c & e & \ldots & d \end{bmatrix} \right).$$

where $d > e > \frac{c^2}{b}$.

Then

$$\Pr[y_* < \min_{i=1}^{K-1} y_i] = 1 - f_{ng}\left(-\frac{a}{\sqrt{d-e}}, \frac{b+e-2c}{d-e}, K-1 \right).$$