
Estimating mutual information in high dimensions via classification error

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Abstract

1 Estimating the mutual information $I(X; Y)$ based on observations becomes sta-
2 tistically infeasible in high dimensions without some kind of assumption or prior.
3 One approach is to assume a parametric joint distribution on (X, Y) , but in many
4 applications, such a strong modeling assumption cannot be justified. Alternatively,
5 one can estimate the mutual information based the performance of a classifier
6 trained on the data. Existing methods include using the empirical mutual infor-
7 mation of the confusion matrix of the classifier, as well as an estimator based on
8 Fano’s inequality. However, both of these methods all produce an estimate which
9 is bounded by $\log(k)$, where k is the number of classes. This presents a substantial
10 limitation for classification-based approaches, since the number of repeats per
11 class must be large for the classifier to work well, hence limiting the number of
12 classes k that can be defined. In this paper, we construct a novel classification-
13 based estimator of mutual information which overcomes these limitations. Our
14 estimator is based on high-dimensional asymptotics: we show that in a particular
15 limiting regime, the mutual information is an invertible function of the expected
16 k -class Bayes error. While the theory is based on a large-sample, high-dimensional
17 limit, we demonstrate through simulations that our proposed estimator has superior
18 performance to the alternatives in problems of moderate dimensionality.

1 Introduction

19 **1 Introduction**
20 Mutual information $I(X; Y)$ is fundamentally a measure of dependence between random variables
21 X and Y , and is defined as

$$I(X; Y) = \int p(x, y) \log \frac{p(x, y)}{p(x)p(y)} dx dy.$$

22 In its original context of information theory, the mutual information describes the rate at which a
23 noisy communications channel Y can communicate bits from a source stream X , but by now, the
24 quantity $I(X, Y)$ has found many new uses in science and engineering. Mutual information is used
25 to test for conditional independence [1], to quantifying the information between a random stimulus
26 X and the signaling behavior of an ensembles of neurons, Y [2]; for use as an objective function for
27 training neural networks [3], for feature selection in machine learning, and even as an all-purpose
28 nonlinear measure of “correlation for the 21st century” [4]. What is common to all of these new
29 applications, and what differs from the original setting of Shannon’s theory of information, is that
30 the variables X and Y have unknown distributions which must be inferred from data. In the case
31 when X and Y are both low-dimensional, for instance, when summarizing the properties of a single
32 neuron in response to a single stimulus feature, $I(X; Y)$ can be estimated nonparametrically using a
33 reasonable number of observations. There exists a huge literature on nonparametric estimation of
34 entropy and mutual information, see [5] for a review.

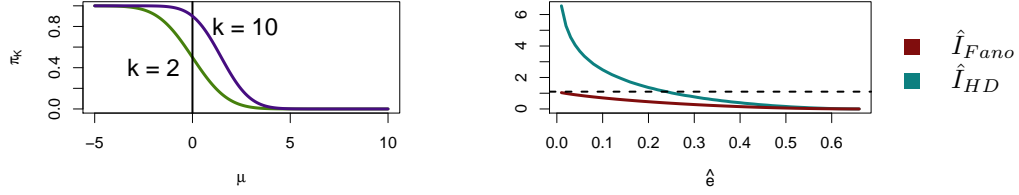


Figure 1: Left: The function $\pi_k(\mu)$ for $k = \{2, 10\}$. Right: \hat{I}_{HD} with \hat{I}_{Fano} as functions of \hat{e}_{gen} , for $k = 3$. While \hat{I}_{Fano} is bounded from above by $\log(k)$ (dotted line), \hat{I}_{HD} is unbounded.

However, the sample complexity for nonparametric estimation grows exponentially with the dimension, rendering such methods ineffective in applications with high-dimensional data [5]. One such application includes multivariate pattern analysis (MVPA), an area of neuroscience research pioneered by Haxby [6], which studies how entire regions of the human brain respond to stimuli, using functional magnetic resonance imaging (fMRI) data; in MVPA studies, the input X could be a natural image parameterized by $p = 10000$ image features, while the output Y is a $q = 20000$ -dimensional vector of brain activation features obtained from the fMRI scan. In problems of such dimensionality, one can tractably estimate mutual information by assuming a multivariate Gaussian model: however, this approach essentially assumes a linear relationship between the input and output, and hence fails to quantify nonlinear dependencies. Rather than assuming a full parametric generative model, one can empirically select a good *discriminative* model by using machine learning. Treves [7] first proposed using the empirical mutual information of the classification matrix in order to obtain a lower bound of the mutual information $I(X; Y)$; this confusion-matrix-based lower bound has subsequently enjoyed widespread use in the MVPA literature [8]. But even earlier than this, the idea of linking classification performance to mutual information can be found in the beginnings of information theory: after all, Shannon’s original motivation was to characterize the minimum achievable error probability of a noisy communication channel. More explicitly, Fano’s inequality provides a lower bound on mutual information in relation to the optimal prediction error, or Bayes error. Therefore, one can construct an estimator based on Fano’s inequality, \hat{I}_{Fano} . In either case, any method which derives an estimate of mutual information from classification performance may be considered a *discriminative* estimation procedure, in contrast to the *parametric* and *nonparametric* classes of estimation procedures.

We derive a new discriminative estimator by exploiting an assumption on the random sampling of the classes (described in section 1.1) and also a universality property that arises in high-dimensions. This universality phenomenon allows us to establish a relationship between the mutual information $I(X; Y)$ and the k -class average Bayes error, $e_{ABE,k}$. In short, we will identify a function π_k (which depends on k),

$$e_{ABE,k} \approx \pi_k(\sqrt{2I(X; Y)}) \quad (1)$$

and that this approximation becomes accurate under a limit where $I(X; Y)$ is small relative to the dimensionality of X , and under the condition that the components of X are approximately independent. The function π_k is given by

$$\pi_k(c) = 1 - \int_{\mathbb{R}} \phi(z - c) \Phi(z)^{k-1} dz.$$

This formula is not new to the information theory literature: it appears as the error rate of an orthogonal constellation [13]. What is surprising is that the same formula can be used to approximate the error rate in much more general class of classification problems¹—this is precisely the universality result which provides the basis for our proposed estimator.

Figure 1 displays the plot of π_k for several values of k . For all values of k , $\pi_k(\mu)$ is monotonically decreasing in μ , and tends to zero as $\mu \rightarrow \infty$, which is what we expect since if $I(X; Y)$ is large, then the average Bayes error should be small. Another intuitive fact is that $\pi_k(0) = 1 - \frac{1}{k}$, since after all, an uninformative response cannot lead to above-chance classification accuracy.

¹An intuitive explanation for this fact is that points from any high-dimensional distribution lie in an orthogonal configuration with high probability.

In this paper, we argue for the advantages of our method in comparison to alternative discriminative estimators under the assumption that the discriminative model approximates the Bayes rule. While this is an unrealistic assumption, it simplifies the theoretical discussion, and allows us to clearly discuss the principles behind our method. We outline our framework in the following section.

1.1 Discriminative estimators of mutual information

In many applications, the discriminative approach takes an advantageous middle ground between the two extremes of nonparametric and parametric approaches for estimating mutual information. In neuroimaging data, we lack prior knowledge for specifying parametric models, and the data is too high-dimensional for nonparametric approaches, but we have a sufficient idea of the general “structure” in the data to achieve above-chance classification rates.

Five steps are required to implement discriminative estimation of mutual information. First, one must define a classification task. The kinds of tasks that can be defined depend on the sampling scheme used to collect the data. Second, one chooses a classifier \mathcal{F} . Third, the classifier is trained on a training subset of the data to obtain a classification rule f . Fourthly, the performance of the rule f is evaluated on the held-out test set. Finally, the performance metrics of the classifier are converted into an estimate of mutual information. In this paper we are mostly concerned with the final step: how to convert measures of classification performance into estimates of mutual information.

Let us assume that the variables X, Y have a joint distribution F , and that one can define a conditional distribution of Y given X , $Y|X \sim F_X$, and let G denote the marginal distribution of X . We consider two different types of sampling procedures:

- *pair sampling*: For $i = 1, \dots, n$, the data (X^i, Y^i) are sampled i.i.d. from the joint distribution of (X, Y) .
- *stratified sampling*: For $j = 1, \dots, k$, sample i.i.d. *exemplars* $X^{(1)}, \dots, X^{(k)} \sim G$. For $i = 1, \dots, n$, draw Z^i iid from the uniform distribution on $1, \dots, k$, then draw Y^i from the conditional distribution $F_{X^{(Z^i)}}$.

Pair sampling occurs in observational studies, where one observes both X and Y externally. On the other hand, stratified sampling is more commonly seen in controlled experiments, where an experimenter chooses an input X to feed into a black box, which outputs Y . An example from fMRI studies is an experimental design where the subject is presented a stimulus X , and the experimenter measures the subject’s response via the brain activation Y .²

Given data from either pair sampling or stratified sampling, one can define various *classification tasks*. Here, the point is to use classification as a tool for extracting information about the relationship between X and Y . As such, it is up to us to define the classification tasks of interest. For instance, one can define tasks which either classify Y based on X , or classify X based on Y ; without loss of generality, we henceforth consider the latter. In the case of continuous X , we can define an arbitrary number of classes k by specifying a partition on the space of X . That is, one can define a *class function* $Z : X \rightarrow \{1, \dots, k\}$, and consider the problem of classifying Z given Y . A classification rule is any (possibly stochastic) mapping $f : \mathcal{Y} \rightarrow \{1, \dots, k\}$, where \mathcal{Y} is a superset of the support of Y . The *generalization error* of the classification rule is $e_{gen}(f) = Pr[f(Y) \neq Z]$. The Bayes error is the generalization error of the optimal classification rule, $e_{Bayes}(f) = \inf_f e_{gen}(f)$. We call such a classification task a *partition-based* classification task.

The freedom to choose the partition Z may be more of a curse than a blessing when it is unclear how to choose an appropriate partition on the support of X . If stratified sampling is employed, one can define an *exemplar-based* classification task which avoids having to specify a partition. One defines the *class function* Z by

$$Z : \{X^{(1)}, \dots, X^{(k)}\} \rightarrow \{1, \dots, k\},$$

$$Z(X^{(i)}) = i \text{ for } i = 1, \dots, k.$$

²Note the asymmetry in our definition of stratified sampling: our convention is to take X to be the variable preceding Y in causal order. Such causal directionality constrains the stratified sampling to have repeated X rather than repeated Y values, but has no consequence for the mutual information $I(X; Y)$, which is a symmetric function.

118 Note that the domain of Z is restricted to the set of observed exemplars $X^{(1)}, \dots, X^{(k)}$. The loss
 119 function is not well-defined when X lies outside the set of exemplars, so it is natural to define the
 120 generalization error by

$$e_{gen}(f) = \frac{1}{k} \sum_{i=1}^k \Pr[f(Y) \neq Z | X = X^{(i)}]. \quad (2)$$

121 Indeed, in experiments where stratified sampling is used, this is the most commonly employed notion
 122 of generalization error [9]. In an exemplar-based classification, there is no need to specify an arbitrary
 123 partition on the input space, but now the k classes will now be *randomly* defined. One consequence is
 124 that the Bayes error e_{Bayes} is a random variable: when the sampling produces k similar exemplars,
 125 e_{Bayes} will be higher, and when the sampling produces well-separated exemplars e_{Bayes} may be
 126 lower. Therefore, in stratified sampling, it is useful to consider the *average Bayes error*,

$$e_{ABE,k} = \mathbf{E}_{X^{(1)}, \dots, X^{(k)}}[e_{Bayes}], \quad (3)$$

127 where the expectation is taken over the joint distribution of $X^{(1)}, \dots, X^{(k)} \stackrel{iid}{\sim} G$.

128 Unless expert knowledge is available, it is usually necessary to choose the function f in a data-
 129 dependent way in order to obtain a reasonable classification rule. We use the terminology *classifier*
 130 to refer to any algorithm which takes data as input, and produces a classification rule f as output.
 131 Mathematically speaking, the classifier is a functional which maps a set of observations to a classifica-
 132 tion rule, $\mathcal{F} : \{(x^1, y^1), \dots, (x^m, y^m)\} \mapsto f(\cdot)$. The data $(x^1, y^1), \dots, (x^m, y^m)$ used to obtain the
 133 classification rule is called *training data*. When the goal is to obtain *inference* about the generalization
 134 error e_{gen} of the classification rule f , it becomes necessary to split the data into two independent
 135 sets: one set to train the classifier, and one to evaluate the performance. The reason that such a
 136 splitting is necessary is because using the same data to test and train a classifier introduces significant
 137 bias into the empirical classification error [10]. One creates a *training set* consisting of r_1 repeats
 138 per class, $S_{train} = \{(x^{(i)}, y^{(i),j})\}_{i=1, j=1}^{k, r_1}$, and a *test set* consisting of the remaining $r_2 = r - r_1$
 139 repeats, $S_{test} = \{(x^{(i)}, y^{(i),j})\}_{i=1, j=r_1+1}^{k, r}$. The classification rule is obtained via $f = \mathcal{F}(S_{train})$, and
 140 the performance of the classifier is evaluated by predicting the classes of the test set. The results of
 141 this test are summarized by a $k \times k$ *confusion matrix* M with $M_{ij} = \sum_{\ell=r_1+1}^r I(f(y^{(i),\ell}) = j)$. The
 142 i, j th entry of M counts how many times a output in the i th class was classified to the j th class. The
 143 *test error* is the proportion of off-diagonal terms of M , $e_{test} = \frac{1}{kr} \sum_{i \neq j} M_{ij}$, and is an unbiased
 144 estimator of e_{gen} . However, in small sampling regimes the quantity e_{test} may be too variable to use
 145 as an estimator of e_{gen} . We recommend the use of Bayesian smoothing, defining an α -smoothed
 146 estimate $\hat{e}_{gen,\alpha}$ by $\hat{e}_{gen,\alpha} = (1 - \alpha)e_{test} + \alpha \frac{k-1}{k}$, which takes a weighted average of the unbiased
 147 estimate e_{test} , and the natural prior of *chance classification*.

148 We define a discriminative estimator to be a function which maps the misclassification matrix to
 149 a positive number, $\hat{I} : \mathbb{N}^{k \times k} \rightarrow \mathbb{R}$. We are aware of the following examples of discriminative
 150 estimators: (1) estimators derived from using Fano's inequality, and (2) the empirical information of
 151 the confusion matrix, as introduced by Treves [7].

152 Fano's inequality can be easily adapted to yield a discriminative estimator. The original inequality
 153 reads

$$H(Z|Y) \leq H(e_{Bayes}) + e_{Bayes} \log ||\mathcal{Z}| - 1|$$

154 where $H(e)$ is the entropy of a Bernoulli random variable with probability e . Replacing $H(Z|Y)$
 155 with $H(X|Y)$ and replacing e_{Bayes} with $\hat{e}_{gen,\alpha}$, we get the estimator

$$\hat{I}_{Fano}(M) = \log(K) - \hat{e}_{gen,\alpha} \log(K - 1) + \hat{e}_{gen,\alpha} \log(p) + (1 - \hat{e}_{gen,\alpha}) \log(1 - \hat{e}_{gen,\alpha}).$$

156 Meanwhile, the confusion matrix estimator computes

$$\hat{I}_{CM}(M) = \frac{1}{k^2} \sum_{i=1}^k \sum_{j=1}^k \log \frac{M_{ij}}{r/k},$$

157 which is the empirical mutual information of the discrete joint distribution $(Z, f(Y))$.

158 As noted in the literature [8], such discriminative approaches tend to underestimate the mutual
 159 information. Two sources of "information loss" are (1) the fact that a continuous input variable X is

160 discretized into k classes, and (2) that the performance of any classifier trained from data can at best
 161 given an *upper bound* to the error of the best classification rule: the Bayes error. Dealing with the
 162 problem of estimating the Bayes error is beyond the scope of the current paper, but we will argue that
 163 the stratified sampling design overcomes some of the limitations imposed by the discretization of X .

164 1.2 The problem of undersampling

165 Stratified sampling is commonly used in neuroscience experiments, and furthermore, it is often
 166 the case that the number of classes k is constrained to be small due to practical and statistical
 167 considerations [11]. When $\log(k) \ll I(X; Y)$, we say that the data is *undersampled*: meaning that
 168 even if the total number of observations is large (due to having many repeats per class,) the diversity
 169 of the exemplar set is poor.

170 The parametric approach is relatively robust to undersampling of the exemplars, since k exemplars
 171 suffice to identify a $p < k$ -parameter model. Gastpar et al. [11], studied the nonparametric estimator

$$\hat{I}_0 = \hat{H}(Y) - \frac{1}{k} \sum_{i=1} \hat{H}(Y|X^{(i)}),$$

172 where \hat{H} is an estimator for the entropy. Gastpar et al. showed that \hat{I}_0 is biased downwards due
 173 to undersampling of the exemplars: to counteract this bias, they introduce the anthropic correction
 174 estimator \hat{I}_α . However, without a principled approach to choose the parameter $\alpha \in (0, 1]$, \hat{I}_α could
 175 still vastly underestimate or overestimate the mutual information.

176 Meanwhile, the discriminative estimators \hat{I}_{CM} and \hat{I}_{Fano} perform quite badly in the undersampled
 177 regime. It is easy to show that $\log(k)$ is a tight upper bound for both \hat{I}_{CM} and \hat{I}_{Fano} . As $I(X; Y)$
 178 exceeds $\log(k)$, the estimate \hat{I} can no longer approximate $I(X; Y)$, even up to a constant factor.
 179 On the other hand, the following worst-case example shows that discriminative estimators cannot
 180 overcome this $\log(k)$ barrier in general. Let X and Y have joint density $p(x, y) = \frac{1}{k} I(\lfloor kx \rfloor = \lfloor ky \rfloor)$
 181 on the unit square. Under partition-based classification, if we set $Z(x) = \lfloor kx \rfloor + 1$, then no
 182 errors are made under the Bayes rule. We therefore have a joint distribution which maximizes any
 183 reasonable discriminative estimator but has *finite* information $I(X; Y) = \log(k)$. The consequence
 184 of this is that under partition-based classification, we cannot hope to distinguish distributions with
 185 $I(X; Y) > \log(k)$. The situation is more promising if we specialize to stratified sampling: in the
 186 same example, a Bayes of zero is no longer likely due to the possibility of exemplars being sampled
 187 from the same bin ('collisions')—we obtain an approximation to the average Bayes error through a
 188 Poisson sampling model: $e_{ABE, k} \approx \frac{1}{e} \sum_{j=1}^{\infty} \frac{1}{j(j!)} = 0.484$. Recognizing that the natural setting for
 189 discriminative estimation is for high-dimensional data, we specialize further to consider deriving
 190 an estimator under a *high-dimensional* regime. The extra assumption of high dimensionality allows
 191 an extreme degree of control over the average Bayes error—it reduces the average Bayes error to a
 192 function of the mutual information!

193 In section 2 we present an asymptotic setting intended to capture the notion of high dimensionality;
 194 namely, one where the number of classes is fixed, and where the information $I(X; Y)$ remains fixed,
 195 while the dimensionality of the input X and output Y both grow to infinity. We make a number of
 196 additional regularity conditions to rule out scenarios where (X, Y) is really less “high-dimensional”
 197 than it appears, since most of the variation is captured a low-dimensional manifold³. In section 2.1
 198 we present our key result, which links the asymptotic average Bayes error to the mutual information;
 199 in section 2.2 we apply this result to derive our proposed estimator, \hat{I}_{HD} (where HD stands for
 200 “high-dimensional.”) Section 3 presents simulation results, and section 4 concludes. All proofs are
 201 given in the supplement.

³In situations where (X, Y) lie on a manifold, one could effectively estimate mutual information by would
 be to combining dimensionality reduction with nonparametric information estimation [12].

2 Theory

2.1 Universality result

We obtain the universality result in two steps. First, we link the average Bayes error to the moments of some statistics Z_i . Secondly, we use Taylor approximation in order to express $I(X; Y)$ in terms of the moments of Z_i . Connecting these two pieces yields the formula (1).

Let us start by rewriting the average Bayes error:

$$e_{ABE,k} = \Pr[p(Y|X_1) \leq \max_{j \neq 1} p(Y|X_j) | X = X_1].$$

Defining the statistic $Z_i = \log p(Y|X_i) - \log p(Y|X_1)$, where $Y \sim p(y|X_1)$, we obtain $e_{ABE} = \Pr[\max_{j > 1} Z_j > 0]$. The key assumption we need is that Z_2, \dots, Z_k are asymptotically multivariate normal. If so, the following lemma allows us to obtain a formula for the misclassification rate.

Lemma 1. *Suppose (Z_1, Z_2, \dots, Z_k) are jointly multivariate normal, with $\mathbf{E}[Z_1 - Z_i] = \alpha$, $\text{Var}(Z_1) = \beta \geq 0$, $\text{Cov}(Z_1, Z_i) = \gamma$, $\text{Var}(Z_i) = \delta$, and $\text{Cov}(Z_i, Z_j) = \epsilon$ for all $i, j = 2, \dots, k$, such that $\beta + \epsilon - 2\gamma > 0$. Then, letting*

$$\mu = \frac{\mathbf{E}[Z_1 - Z_i]}{\sqrt{\frac{1}{2}\text{Var}(Z_i - Z_j)}} = \frac{\alpha}{\sqrt{\delta - \epsilon}},$$

$$\nu^2 = \frac{\text{Cov}(Z_1 - Z_i, Z_1 - Z_j)}{\frac{1}{2}\text{Var}(Z_i - Z_j)} = \frac{\beta + \epsilon - 2\gamma}{\delta - \epsilon},$$

we have

$$\begin{aligned} \Pr[Z_1 < \max_{i=2}^k Z_i] &= \Pr[W < M_{k-1}] \\ &= 1 - \int \frac{1}{\sqrt{2\pi\nu^2}} e^{-\frac{(w-\mu)^2}{2\nu^2}} \Phi(w)^{k-1} dw, \end{aligned}$$

where $W \sim N(\mu, \nu^2)$ and M_{k-1} is the maximum of $k-1$ independent standard normal variates, which are independent of W .

To see why the assumption that Z_2, \dots, Z_k are multivariate normal might be justified, suppose that X and Y have the same dimensionality d , and that joint density factorizes as

$$p(x^{(j)}, y) = \prod_{i=1}^d p_i(x_i^{(j)}, y_i)$$

where $x_i^{(j)}, y_i$ are the i th scalar components of the vectors $x^{(j)}$ and y . Then,

$$Z_i = \sum_{m=1}^d \log p_m(y_m | x_m^{(i)}) - \log p_m(y_m | x_m^{(1)})$$

where $x_{i,j}$ is the i th component of x_j . The d terms $\log p_m(y_m | x_{m,i}) - \log p_m(y_m | x_{m,1})$ are independent across the indices m , but dependent between the $i = 1, \dots, k$. Therefore, the multivariate central limit theorem can be applied to conclude that the vector (Z_2, \dots, Z_k) can be scaled to converge to a multivariate normal distribution. While the componentwise independence condition is not a realistic assumption, the key property of multivariate normality of (Z_2, \dots, Z_k) holds under more general conditions, and appears reasonable in practice.

It remains to link the moments of Z_i to $I(X; Y)$. This is accomplished by approximating the logarithmic term by the Taylor expansion

$$\log \frac{p(x, y)}{p(x)p(y)} \approx \frac{p(x, y) - p(x)p(y)}{p(x)p(y)} - \left(\frac{p(x, y) - p(x)p(y)}{p(x)p(y)} \right)^2 + \dots$$

A number of assumptions are needed to ensure that needed approximations are sufficiently accurate; and additionally, in order to apply the central limit theorem, we need to consider a *limiting sequence* of problems with increasing dimensionality. We now state the theorem.

Theorem 1. *Let $p^{[d]}(x, y)$ be a sequence of joint densities for $d = 1, 2, \dots$. Further assume that*

- 233 A1. $\lim_{d \rightarrow \infty} I(X^{[d]}; Y^{[d]}) = \iota < \infty$.
 234 A2. *There exists a sequence of scaling constants $a_{ij}^{[d]}$ and $b_{ij}^{[d]}$ such that the random vector*
 235 *$(a_{ij}^{[d]} \ell_{ij}^{[d]} + b_{ij}^{[d]})_{i,j=1,\dots,k}$ converges in distribution to a multivariate normal distribution.*
 236 A3. *Define*

$$u^{[d]}(x, y) = \log p^{[d]}(x, y) - \log p^{[d]}(x) - \log p^{[d]}(y).$$

237 *There exists a sequence of scaling constants $a^{[d]}, b^{[d]}$ such that*

$$a^{[d]} u^{[d]}(X^{(1)}, Y^{(2)}) + b^{[d]}$$

238 *converges in distribution to a univariate normal distribution.*

239 A4. *For all $i \neq k$,*

$$\lim_{d \rightarrow \infty} \text{Cov}[u^{[d]}(X^{(i)}, Y^{(j)}), u^{[d]}(X^{(k)}, Y^{(j)})] = 0.$$

240 *Then for $e_{ABE,k}$ as defined above, we have*

$$\lim_{d \rightarrow \infty} e_{ABE,k} = \pi_k(\sqrt{2\iota})$$

241 *where*

$$\pi_k(c) = 1 - \int_{\mathbb{R}} \phi(z - c) \Phi(z)^{k-1} dz$$

242 *where ϕ and Φ are the standard normal density function and cumulative distribution function,*
 243 *respectively.*

244 Assumptions A1-A4 are satisfied in a variety of natural models. One example is a multivariate
 245 Gaussian sequence model where $X \sim N(0, \Sigma_d)$ and $Y = X + E$ with $E \sim N(0, \Sigma_e)$, where Σ_d and
 246 Σ_e are $d \times d$ covariance matrices, and where X and E are independent. Then, if $d\Sigma_d$ and Σ_e have
 247 limiting spectra H and G respectively, the joint densities $p(x, y)$ for $d = 1, \dots$, satisfy assumptions
 248 A1 - A4. Another example is the multivariate logistic model, which we describe in section 3. We
 249 further discuss the rationale behind A1-A4 in the supplement, along with the detailed proof.

250 2.2 High-dimensional estimator

251 The estimator we propose is

$$\hat{I}_{HD}(M) = \frac{1}{2}(\pi_k^{-1}(\hat{e}_{gen,\alpha}))^2,$$

252 obtained by inverting the relation (1), then substituting the estimate $\hat{e}_{gen,\alpha}$ for the $e_{ABE,k}$. As such,
 253 our estimator can be directly compared to the \hat{I}_{Fano} , since both are functions of $\hat{e}_{gen,\alpha}$ (Figure 1.)

254 For sufficiently high-dimensional problems, \hat{I}_{HD} can accurately recover $I(X; Y) > \log k$, supposing
 255 also that the classifier \mathcal{F} consistently estimates the Bayes rule. The number of observations needed
 256 depends on the convergence rate of \mathcal{F} and also the complexity of estimating $e_{gen,\alpha}$. Therefore,
 257 without making assumptions on \mathcal{F} , the sample complexity is at least exponential in $I(X; Y)$. This
 258 is because when $I(X; Y)$ is large relative to $\log(k)$, the Bayes error $e_{ABE,k}$ is exponentially small.
 259 Hence $O(1/e_{ABE,k})$ observations in the test set are needed to recover $e_{ABE,k}$ to sufficient precision.
 260 While the sample complexity exponential in $I(X; Y)$ is by no means ideal, by comparison, the
 261 nonparametric estimation approaches have a complexity exponential in the dimensionality. Hence,
 262 \hat{I}_{HD} is favored over nonparametric approaches in settings with high dimensionality and low signal-
 263 to-noise ratio.

264 3 Simulation

265 We compare the discriminative estimators \hat{I}_{CM} , \hat{I}_{Fano} , \hat{I}_{HD} with a nonparametric estimator \hat{I}_0 in the
 266 following simulation. We generate data according to a multiple-response logistic regression model,
 267 where $X \sim N(0, I_p)$, and Y is a binary vector with conditional distribution

$$Y_i | X = x \sim \text{Bernoulli}(x^T B_i)$$

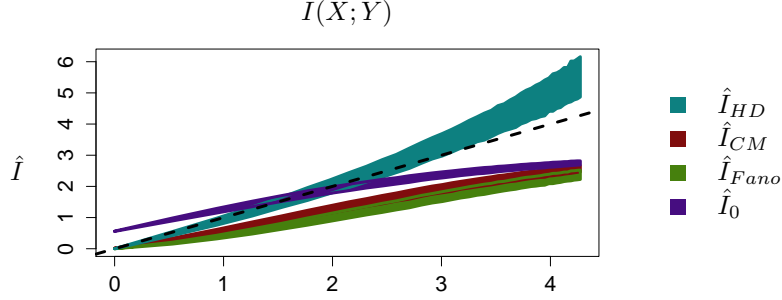


Figure 2: Sampling distributions of \hat{I} for data generated from the multiple-response logistic model. $p = q = 10$; $k = 20$; $B = sI_{10}$, where $s \in [0, \sqrt{200}]$; and $r = 1000$.

where B is a $p \times q$ matrix. One application of this model might be modeling neural spike count data Y arising in response to environmental stimuli X [14]. We choose the naive Bayes for the classifier \mathcal{F} : it is consistent for estimating the Bayes rule. In Figure 2 we show the sampling distributions of the four estimators as $I(X; Y)$ is varied in the interval $[0, 4]$. We see that \hat{I}_{CM} , \hat{I}_{Fano} , and \hat{I}_0 indeed begin to asymptote as they approach $\log(k) = 2.995$. In contrast, \hat{I}_{HD} remains a good approximation of $I(X; Y)$ within the range, although it begins to overestimate at the right endpoint.

4 Discussion

Discriminative estimators of mutual information have the potential to estimate mutual information in high-dimensional data without resorting to fully parametric assumptions. However, a number of practical considerations also limit their usage. First, one has to find a good classifier \mathcal{F} for the data: techniques for model selection can be used to choose \mathcal{F} from a large library of methods. However, there is no way to guarantee how well the chosen classifier approximates the optimal classification rule. Secondly, one has to estimate the generalization error from test data: the complexity of estimating e_{gen} could become the bottleneck when e_{gen} is close to 0. Thirdly, for previous estimators \hat{I}_{Fano} and \hat{I}_{CM} , the ability of the estimator to distinguish high values of $I(X; Y)$ is limited by the number of classes k . Our estimator \hat{I}_{HD} is subject to the first two limitations, along with any conceivable discriminative estimator, but overcomes the third limitation under the assumption of stratified sampling and high dimensionality.

The assumption of stratified sampling is satisfied through appropriate design. However, the assumption of high dimensionality is much more difficult to check: having a high-dimension response Y does not suffice, since Y could lie close to a low-dimensional manifold. One useful diagnostic is to subsample within the classes collected and check that \hat{I}_{HD} does not systematically increase or decrease with the number of classes k . Based on simulations, \hat{I}_{HD} could either overestimate or underestimate the mutual information when the high-dimensional assumption is violated. The assumption of approximating the Bayes rule is impractical to check, as any nonparametric estimate of the Bayes error requires exponentially many observations. Hence, while the present paper studies the ‘best-case’ scenario where the model is well-specified, it is even more important to understand the robustness of our method in the more realistic case where the model is misspecified. We leave this question to future work.

In our simulation experiment, our proposed estimator is seen to outperform existing estimators, but it remains to assess the utility of our estimation procedure in a real-world example. In a forthcoming work, we apply our framework to evaluate visual encoding models in human fMRI data.

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