Upper and lower bounds on cdf of generalized non-central chi-squared

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1 Introduction

Let $Z \sim N(0, I_p)$, and let $\mu \in \mathbb{R}^p$ and Σ a positive semidefinite matrix. Define the generalized noncentral chi-squared distribution with noncentrality μ and shape Σ as the distribution of

$$Y = (Z + \mu)^T \Sigma (Z + \mu)$$

Let $V \Lambda V^T = \Sigma$ be the eigendecomposition of Σ, and let $\eta = V^T \Sigma$. Then

$$Y \stackrel{d}{=} (Z + \eta)^T \Lambda (Z + \eta) = \sim_{i=1}^p \lambda_i W_i$$

where $W_i \sim \chi_1^2(\eta_i^2)$. Recall that the mgf of the noncentral chi-sqaured with one df is given by

$$\mathbf{E}[e^{tW_i}] = \frac{\exp[\frac{\eta_i^2 t}{1-2t}]}{\sqrt{1-2t}}$$

It follows that the moment-generating function of Y is given by

$$\mathbf{E}[e^{tY}] = \prod_{i=1}^p \mathbf{E}[e^{\lambda_i t W_i}] = \prod_{i=1}^p \frac{\exp[\frac{\eta_i^2 \lambda_i t}{1-2t\lambda_i}]}{\sqrt{1-2t\lambda_i}}$$

2 Bound

We wish to bound the probability $\Pr[Y < x]$. We have

$$\begin{aligned}
\log \Pr[Y < x] &= \log \Pr[e^{tY} > e^{tx}] \text{ for } t < 0 \\
&\leq \log\left(\frac{\mathbf{E}[e^{tY}]}{e^{tx}}\right) \\
&= \log(\mathbf{E}[e^{tY}]) - tx \\
&= \left(\frac{-1}{2}\sum_{i=1}^{p} \log(1 - 2t\lambda_i)\right) + \left(\sum_{i=1}^{p} \frac{\eta_i^2 \lambda_i t}{1 - 2t\lambda_i}\right) - tx
\end{aligned}$$

Now consider minimizing the bound over t.