Information Theory Notes

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These are preliminary notes.

1 Classification in high-dimension, fixed SNR regime

We observe a data point y_* which belongs to one of K classes. The distribution in the ith class is $N(\mu_i, \Omega)$. We have another dataset with r repeats per class, which we use to estimate the centroids μ_i: we obtain estimates $\hat{\mu}_i \sim N(\mu_i, r^{-1}\Omega)$. The class centroids were originally drawn i.i.d. from a multivariate normal $N(0, I)$. Furthermore Ω is unknown and have to be estimated as well: assume we have obtained estimate $\hat{\Omega}$ via some method. Without loss of generality, take the Kth class to be the true class of y_*. Write $\hat{\mu}_* = \hat{\mu}_K$.

The classification rule is given by

$$\text{Estimated class} = \text{argmin}_i \ (y_* - B\hat{\mu}_i)^T A (y_* - B\hat{\mu}_i)$$

where A and B are matrices based on $\hat{\Omega}$. The Bayes rule is given by

$$A_{Bayes} = (I + \Omega - (I + r^{-1}\Omega)^{-1})^{-1}$$

$$B_{Bayes} = (I + r^{-1}\Omega)^{-1}.$$

The "plug-in" estimates of A and B are

$$A = (I + \hat{\Omega} + (I + r^{-1}\hat{\Omega})^{-1})^{-1}$$

$$B = (I + r^{-1}\hat{\Omega})^{-1}.$$

Note that

$$(y_* - B\hat{\mu}_i)^T A (y^* - B\hat{\mu}_i) = ||A^{1/2} y_* - A^{1/2} B \hat{\mu}_i||^2.$$

Therefore the classification rule is

$$\text{Estimated class} = \text{argmin}_i \; Z_i,$$

where

$$Z_i = ||A^{1/2} y_* - A^{1/2} B \hat{\mu}_i||^2.$$

We have

$$\begin{bmatrix} A^{1/2} y \\ A^{1/2} B \hat{\mu}_* \\ A^{1/2} B \hat{\mu}_i \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} A^{1/2}(I+\Omega)A^{1/2} & A^{1/2} B A^{1/2} & 0 \\ & A^{1/2} B (I + \frac{\Omega}{r}) B A^{1/2} & 0 \\ & & A^{1/2} B (I + \frac{\Omega}{r}) B A^{1/2} \end{bmatrix} \right)$$

Therefore

$$\mathbf{E} Z_i = \begin{cases} \text{tr}[A(I + \Omega + (B(I + r^{-1}\Omega)B))] & \text{for } i \neq K \\ \text{tr}[A(I + \Omega + (B(I + r^{-1}\Omega)B) - 2B)] & \text{for } i = K \end{cases},$$

$$\text{Cov}(Z_i, Z_j) = \begin{cases} \text{tr}[A(I + \Omega)]^2 & \text{for } i \neq j \neq K \\ \text{tr}[A(I + \Omega - B)]^2 & \text{for } i = K, j \neq K \\ \text{tr}[A(I + \Omega + B(I + r^{-1}\Omega)B)]^2 & \text{for } i = j \neq K \\ \text{tr}[A(I + \Omega + B(I + r^{-1}\Omega)B - 2B)]^2 & \text{for } i = j = K \end{cases}.$$