Information Theory Notes

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These are preliminary notes.

1 Classification in high-dimension, fixed SNR regime

We observe a data point y_* which belongs to one of K classes. The distribution in the ith class is $N(\mu_i, \Omega)$. We have another dataset with r repeats per class, which we use to estimate the centroids μ_i: we obtain estimates $\hat{\mu}_i \sim N(\mu_i, r^{-1}\Omega)$. The class centroids were originally drawn i.i.d. from a multivariate normal $N(0, I)$. Furthermore Ω is unknown and have to be estimated as well: assume we have obtained estimate $\hat{\Omega}$ via some method. Without loss of generality, take the Kth class to be the true class of y_*. Write $\hat{\mu}_* = \hat{\mu}_K$.

The classification rule is given by

$$\text{Estimated class} = \text{argmin}_i \ (y_* - B\hat{\mu}_i)^T A (y_* - B\hat{\mu}_i)$$

where A and B are matrices based on $\hat{\Omega}$. The Bayes rule is given by

$$A_{Bayes} = (I + \Omega - (I + r^{-1}\Omega)^{-1})^{-1}$$

$$B_{Bayes} = (I + r^{-1}\Omega)^{-1}.$$

The "plug-in" estimates of A and B are

$$A = (I + \hat{\Omega} + (I + r^{-1}\hat{\Omega})^{-1})^{-1}$$

$$B = (I + r^{-1}\hat{\Omega})^{-1}.$$

Note that

$$(y_* - B\hat{\mu}_i)^T A(y^* - B\hat{\mu}_i) = ||A^{1/2}y_* - A^{1/2}B\hat{\mu}_i||^2.$$

Therefore the classification rule is

$$\text{Estimated class} = \text{argmin}_i \; Z_i,$$

where

$$Z_i = ||A^{1/2}y_* - A^{1/2}B\hat{\mu}_i||^2.$$

We have

$$\begin{bmatrix} A^{1/2}y \\ A^{1/2}B\hat{\mu}_* \\ A^{1/2}B\hat{\mu}_i \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} A^{1/2}(I+\Omega)A^{1/2} & A^{1/2}BA^{1/2} & 0 \\ & A^{1/2}B(I+\frac{\Omega}{r})BA^{1/2} & 0 \\ & & A^{1/2}B(I+\frac{\Omega}{r})BA^{1/2} \end{bmatrix} \right)$$

Therefore

$$\mathbf{E}Z_i = \begin{cases} \text{tr}[A(I+\Omega+(B(I+r^{-1}\Omega)B))] & \text{for } i \neq K \\ \text{tr}[A(I+\Omega+(B(I+r^{-1}\Omega)B)-2B)] & \text{for } i = K \end{cases},$$

$$\text{Cov}(Z_i, Z_j) = \begin{cases} \text{tr}[A(I+\Omega)]^2 & \text{for } i \neq j \neq K \\ \text{tr}[A(I+\Omega-B)]^2 & \text{for } i = K, j \neq K \\ \text{tr}[A(I+\Omega+B(I+r^{-1}\Omega)B)]^2 & \text{for } i = j \neq K \\ \text{tr}[A(I+\Omega+B(I+r^{-1}\Omega)B-2B)]^2 & \text{for } i = j = K \end{cases}.$$

2 Appendix

2.1 Gaussian min probs

Define

$$F(\alpha, \beta, K) = \Pr[\alpha Z_* + \beta < \min_{i=1}^{K-1} Z_i]$$

for $Z_*, Z_1, \ldots, Z_{K-1}$ i.i.d normal, hence

$$F(\alpha, \beta, K) = \int_{\mathbb{R}} (1 - \Phi(\alpha z + \beta))^{K-1} d\Phi(z).$$

Suppose

$$\begin{bmatrix} y_* \\ y_1 \\ \vdots \\ y_{K-1} \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} b & c & \dots & c \\ c & d & \dots & e \\ \dots & \dots & \ddots & \vdots \\ c & e & \dots & d \end{bmatrix}\right).$$

where $d > e > \frac{c^2}{b}$.

Then

$$\Pr[y_* + a < \min_{i=1}^{K-1} y_i] = F\left(\sqrt{\frac{b + e - 2c^2/b - 2c}{d - e}}, \frac{a}{\sqrt{d - e}}, K\right).$$