Supervised Evaluation of Representations

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Stanford University

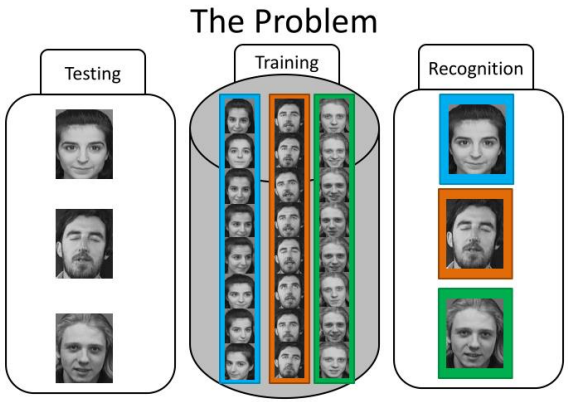
May 2, 2017

Section 1

Randomized Classification

Facial recognition problem

Let's say I have a database of users, $i = 1, \ldots, k$, with photos $\vec{z}_j$ that are labelled.



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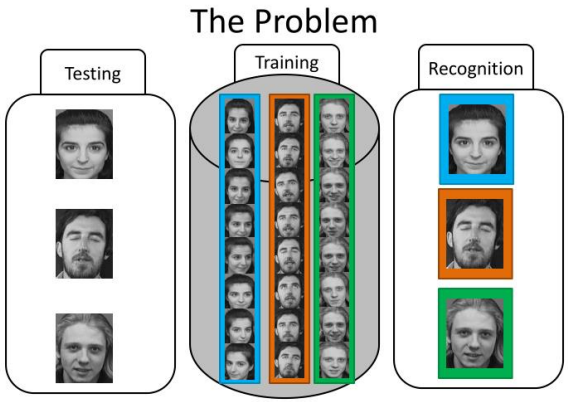

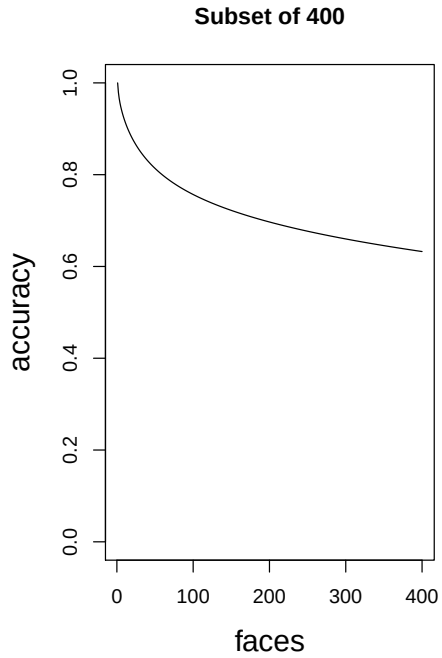

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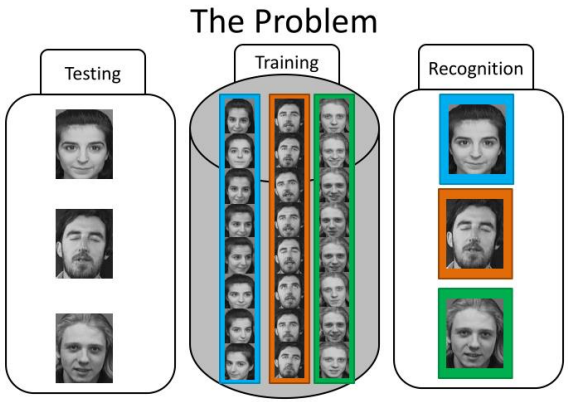
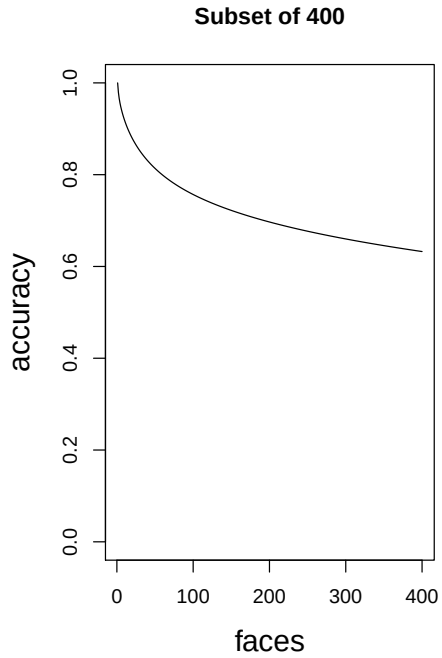
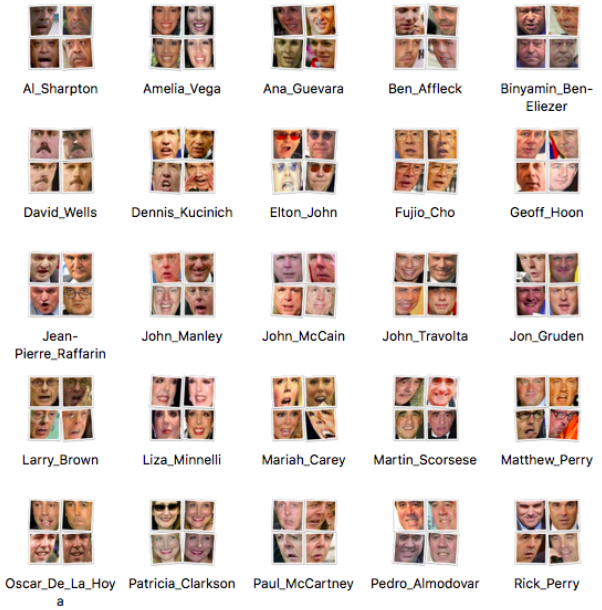




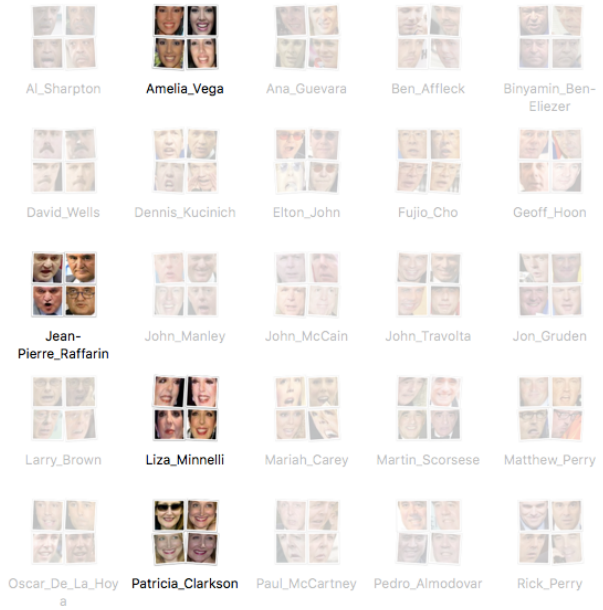
How does the accuracy scale with the number of classes (faces)?

Setup

1. Population of categories $\pi(y)$

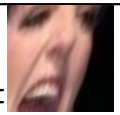
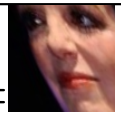
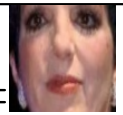


2. Subsample k labels, $y_1, \ldots, y_k$



Setup

3. Collect training and test data $x_j^{(i)}$ (faces) for labels (people) $\{y_1, \ldots, y_k\}$.

Label	Training			Test
y_1=Amelia	$x_1^{(1)} =$ 	$x_1^{(2)} =$ 	$x_1^{(3)} =$ 	$x_1^* =$ 
y_2=Jean-Pierre	$x_2^{(1)} =$ 	$x_2^{(2)} =$ 	$x_2^{(3)} =$ 	$x_2^* =$ 
y_3=Liza	$x_3^{(1)} =$ 	$x_3^{(2)} =$ 	$x_3^{(3)} =$ 	$x_3^* =$ 
y_4=Patricia	$x_4^{(1)} =$ 	$x_4^{(2)} =$ 	$x_4^{(3)} =$ 	$x_4^* =$ 

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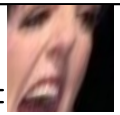
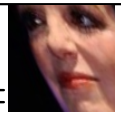
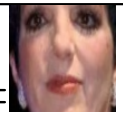
y_3=Liza	$x_3^{(1)} =$ 	$x_3^{(2)} =$ 	$x_3^{(3)} =$ 	$x_3^* =$ 
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Can we analyze how error depends on k?

Key assumption: marginal classifier

- The classifier is *marginal* if it learns a model *independently* for each class.

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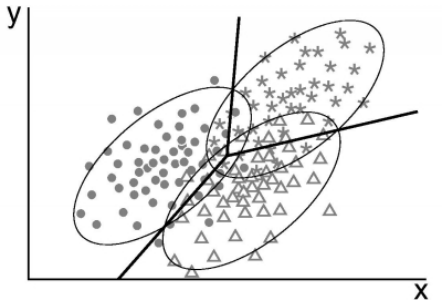
- Examples: LDA/QDA 

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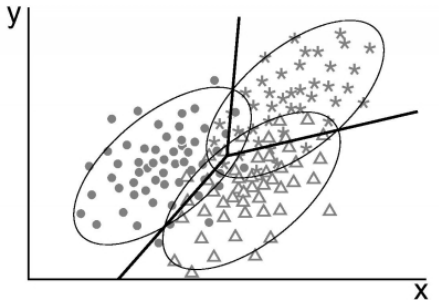
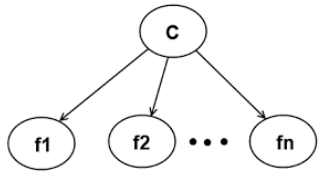
- Examples: LDA/QDA 

, naïve Bayes 

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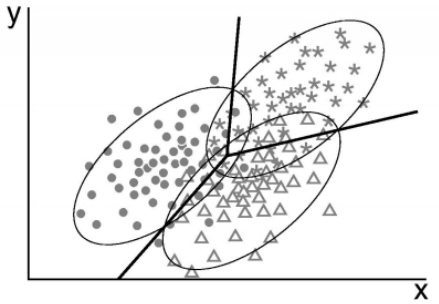
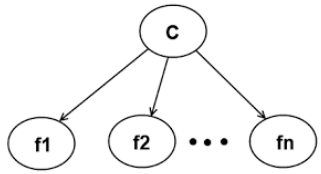
- Examples: LDA/QDA 

, naïve Bayes 

- Non-marginal classifiers: Multinomial logistic, multilayer neural networks, k-nearest neighbors

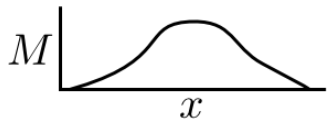
Definitions

$\hat{F}_{y^{(i)}}$ is the empirical distribution obtained from the training data for label $y^{(i)}$.

Classification Rule

$$M_{y^{(1)}}(x) = \mathcal{M}(\hat{F}_{y^{(1)}})(x)$$

$$M_{y^{(2)}}(x) = \mathcal{M}(\hat{F}_{y^{(2)}})(x)$$



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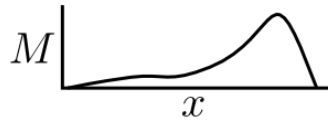
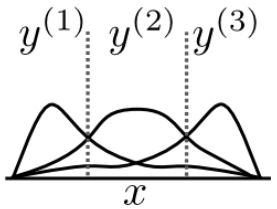
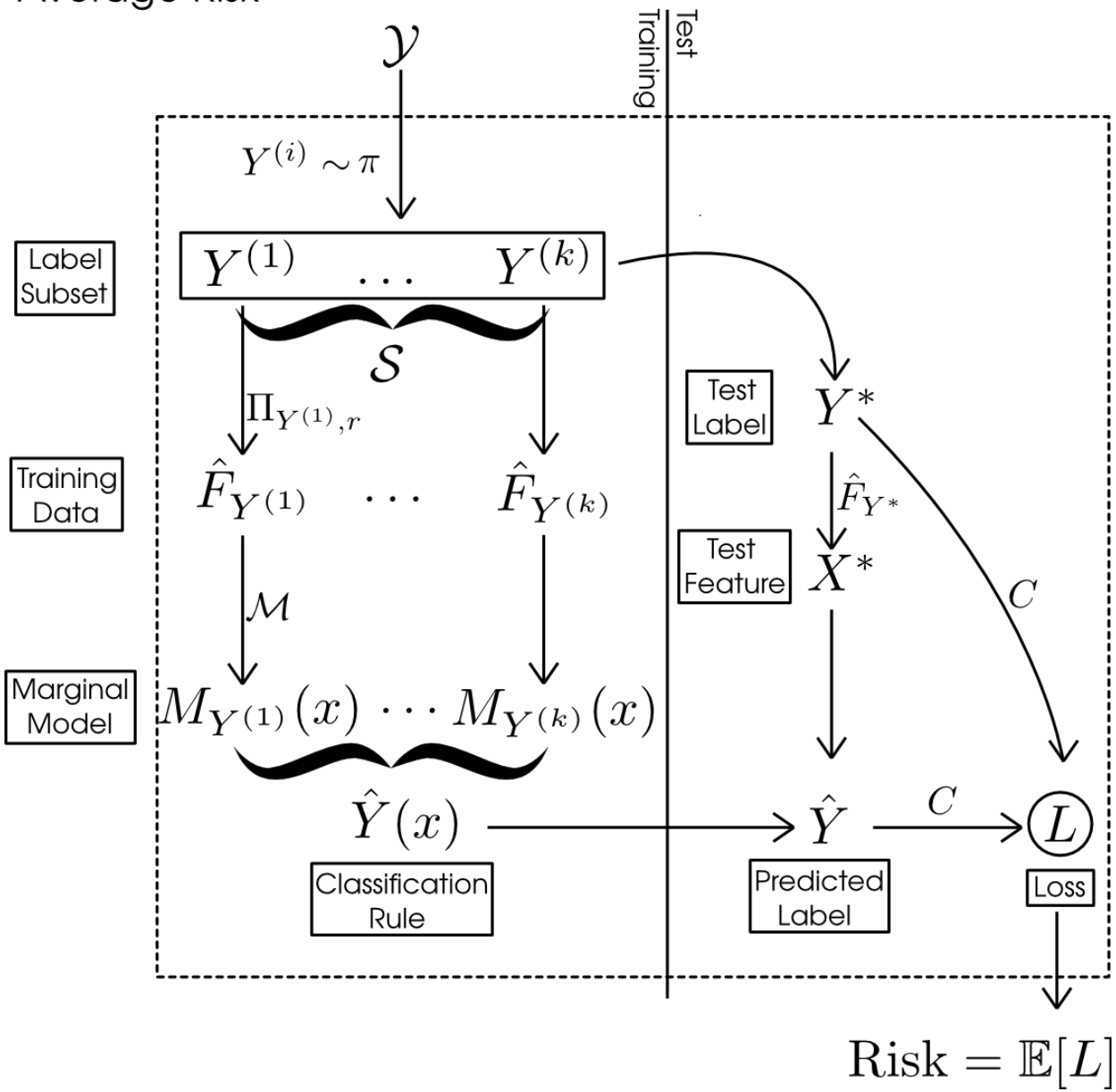


$$\hat{Y}(x) = \text{argmax}_{y \in \mathcal{S}} M_y(x)$$



Average Risk



Theoretical Result

Theorem. (Z., Achanta, Benjamini.) Suppose π, $\{F_y\}_{y \in \mathcal{Y}}$ and marginal classifier $\mathcal{F}$ satisfy *(some regularity condition)*. Then, there exists some function $\bar{D}(u)$ on $[0,1] \to [0,1]$ such that the k-class average risk is given by

$$\text{AvRisk}_k = (k-1) \int \bar{D}(u) u^{k-2} du.$$

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What is this $\bar{D}(u)$ function? We will explain in the following toy example...

Toy example

$Y_1, \ldots, Y_k \overset{iid}{\sim} N(0, 1)$;

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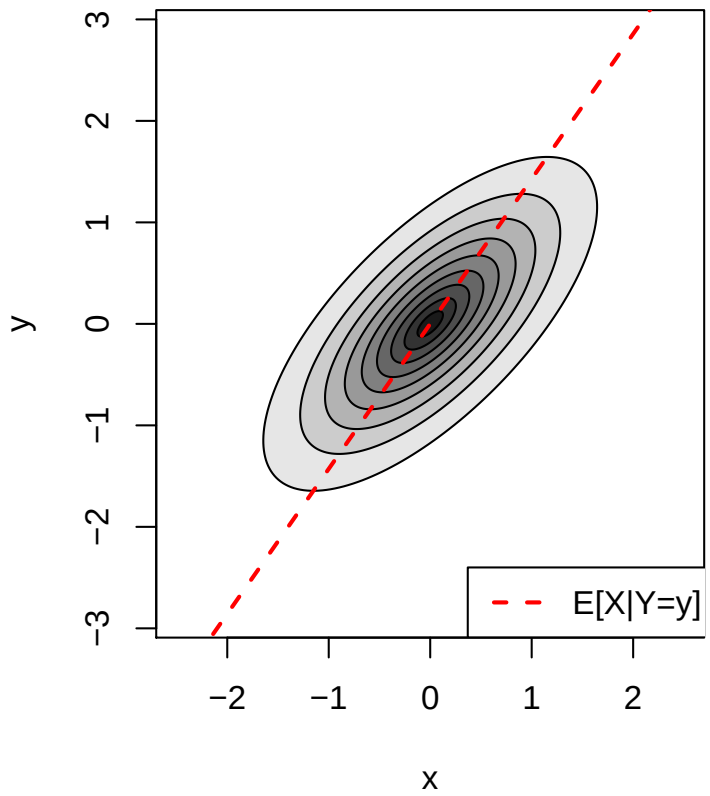
$X|Y \sim N(\rho Y, 1-\rho^2)$ i.e. $(Y, X) \sim N(0, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix})$.



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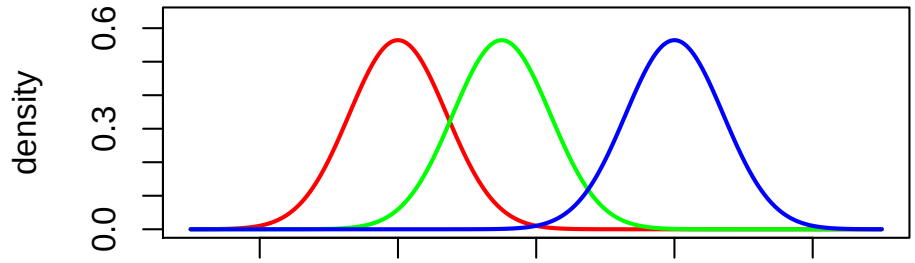
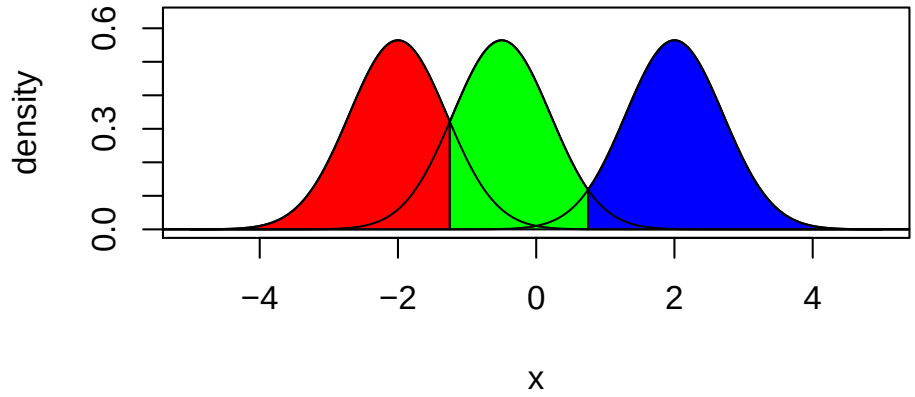


- Suppose $k = 3$, and we draw Y_1, Y_2, Y_3.
- The *Bayes rule* is the optimal classifier and depends on knowing the true densities:

$$\hat{y}(x) = \text{argmax}_{y_i} p(x|y_i)$$

- The *Bayes Risk*, which is the misclassification rate of the optimal classifier.

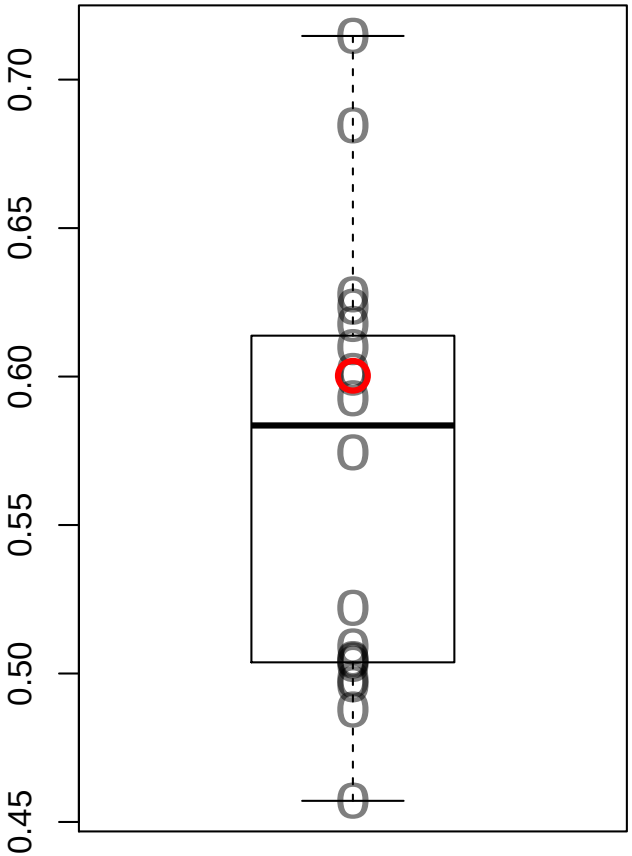
Toy example

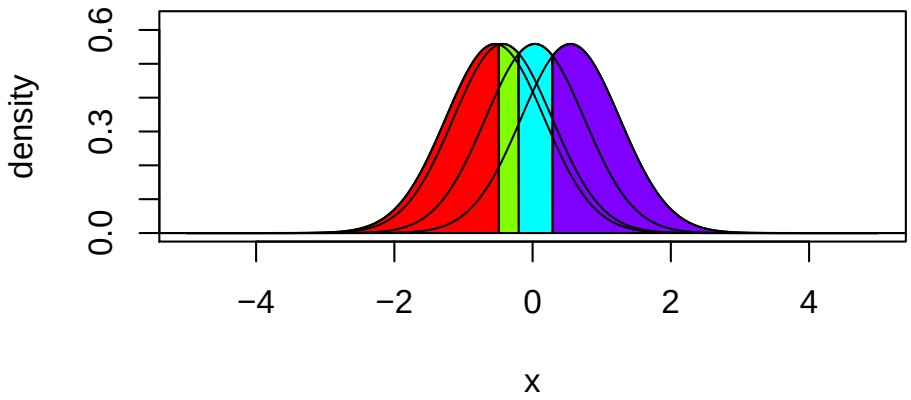


- The *Bayes Risk* is the expected test error of the Bayes rule,

$$\frac{1}{k}\sum_{i=1}^{k}\Pr[\hat{y}(x) \neq Y | Y = y_i]$$

Toy example





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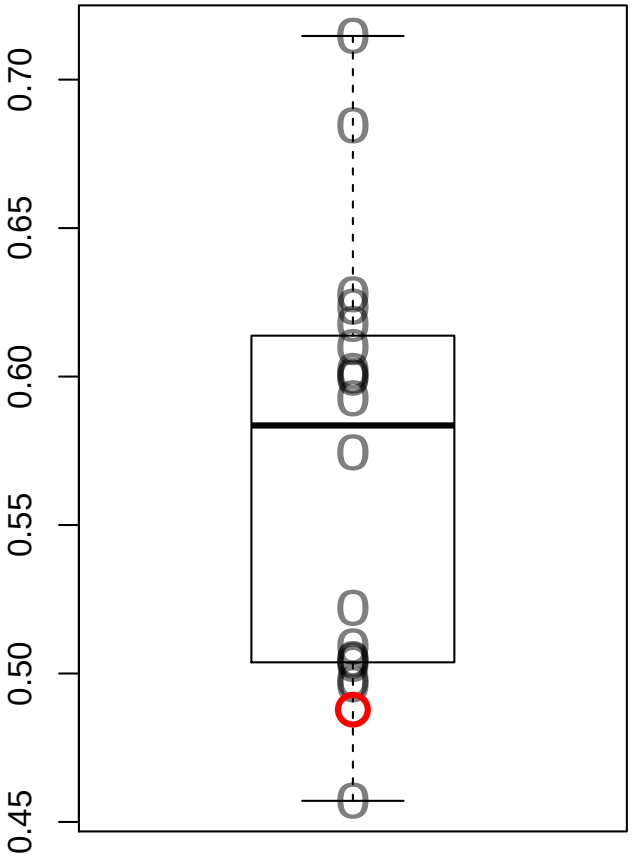

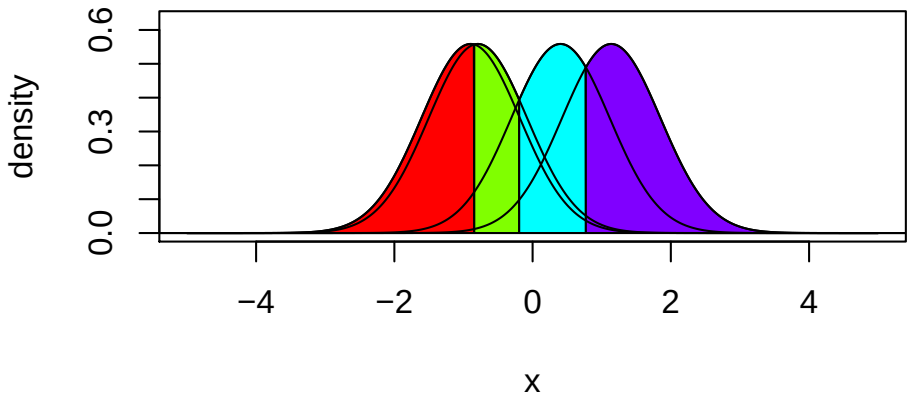

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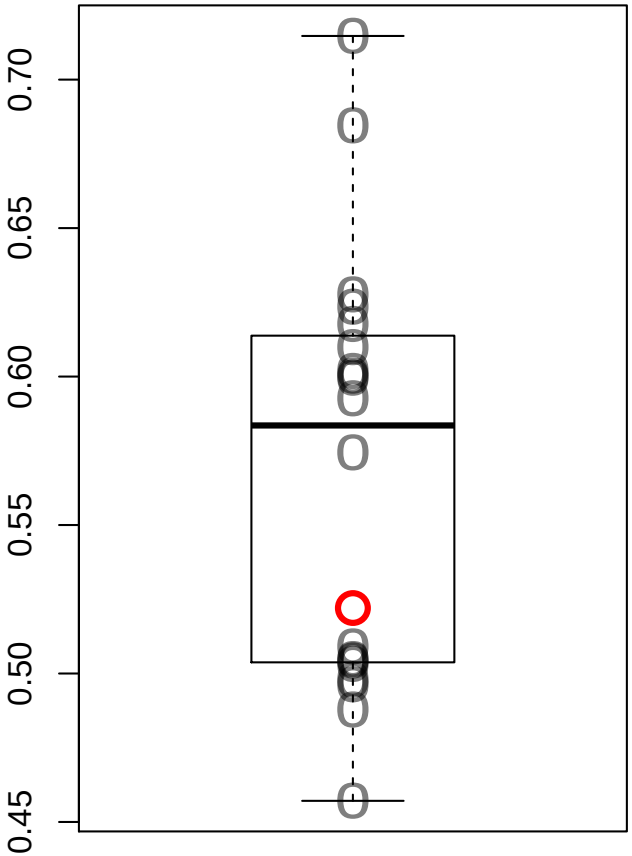

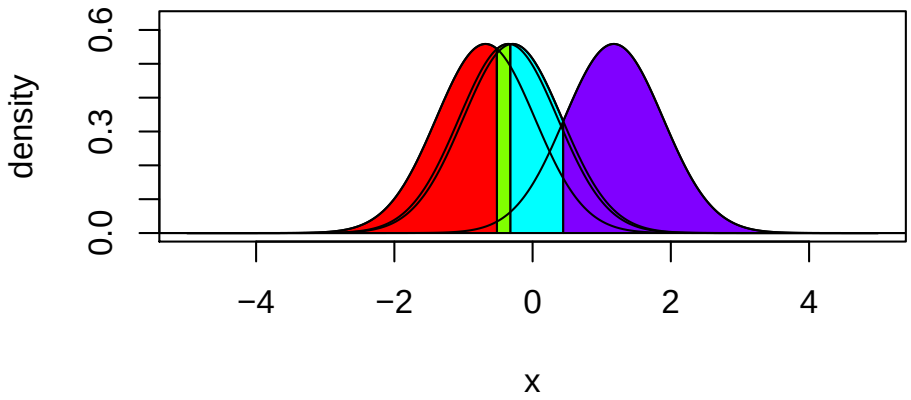

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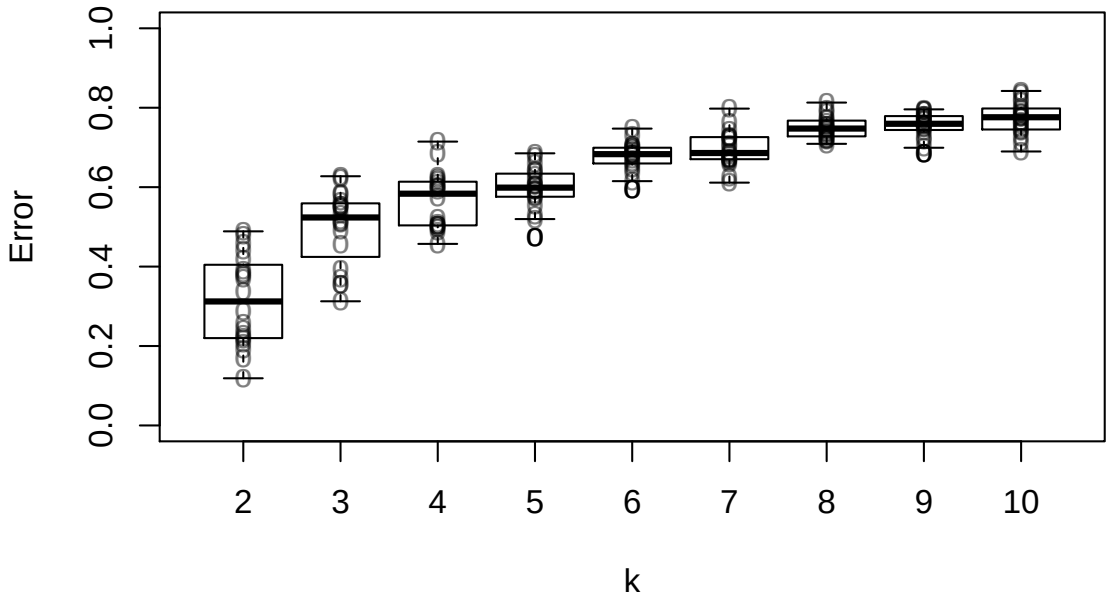

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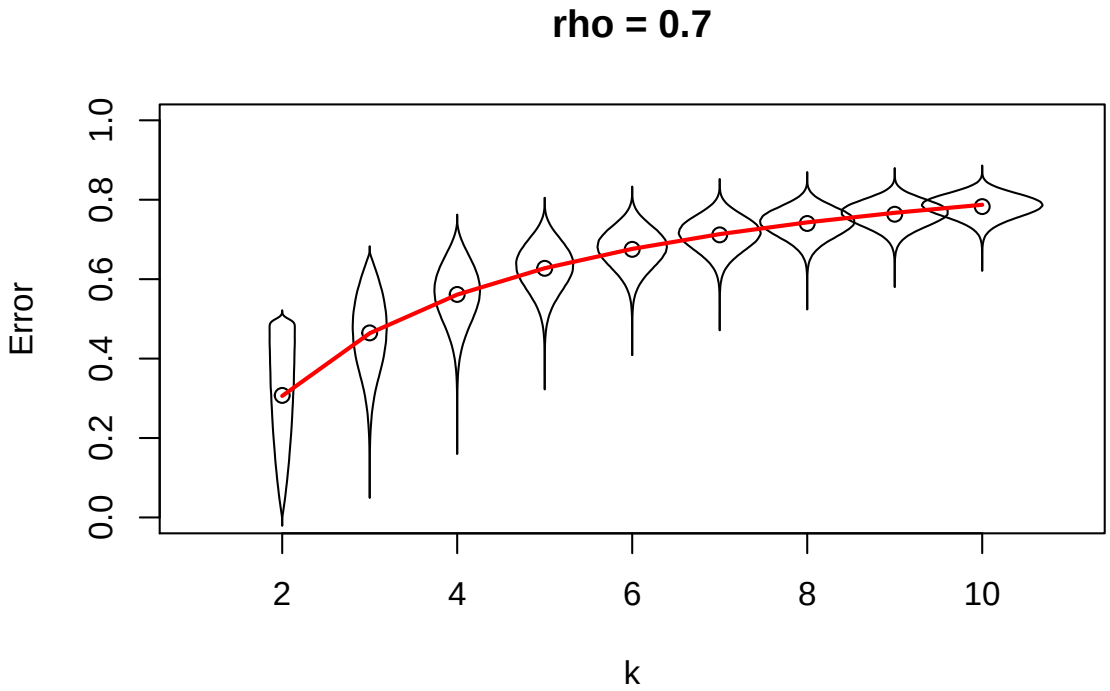
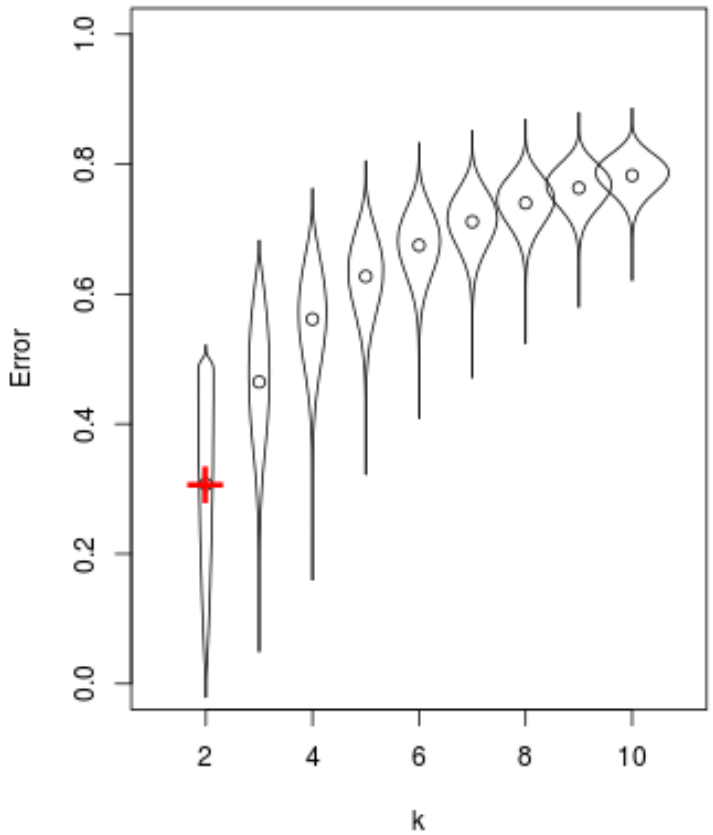
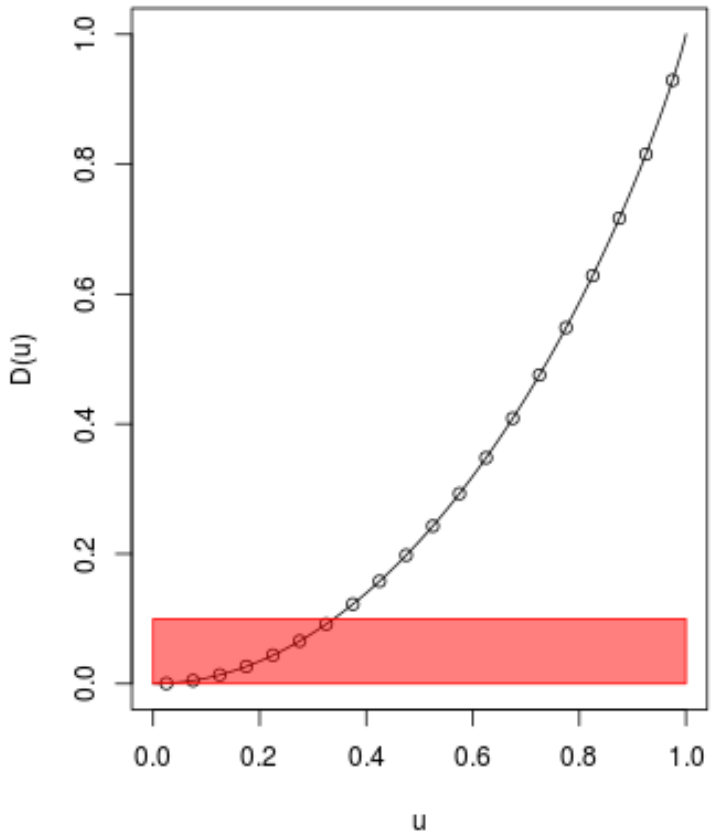


Computing average risk

$$\text{AvRisk}_k = (k-1)\int \bar{D}(u)u^{k-2}du.$$





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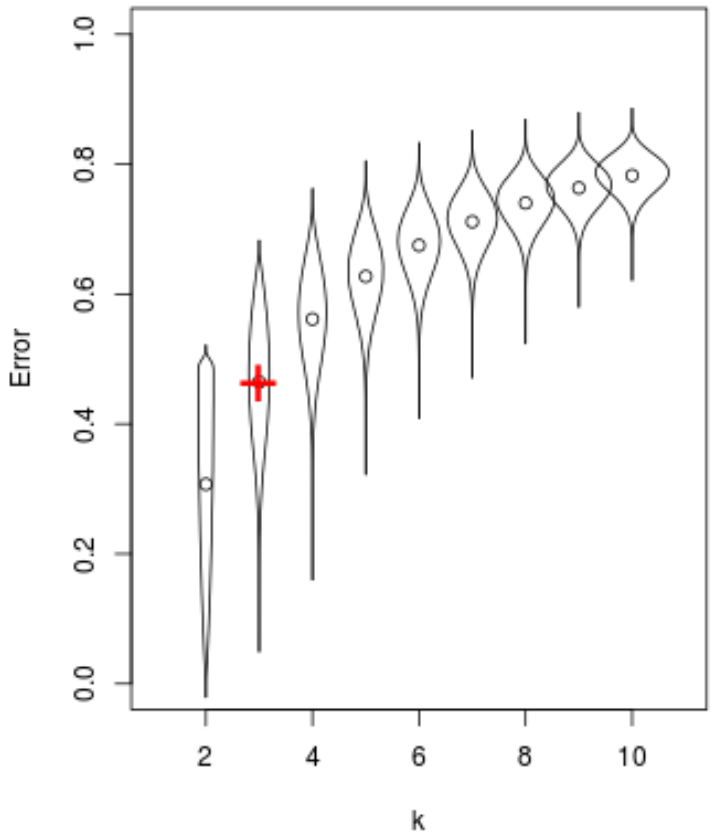

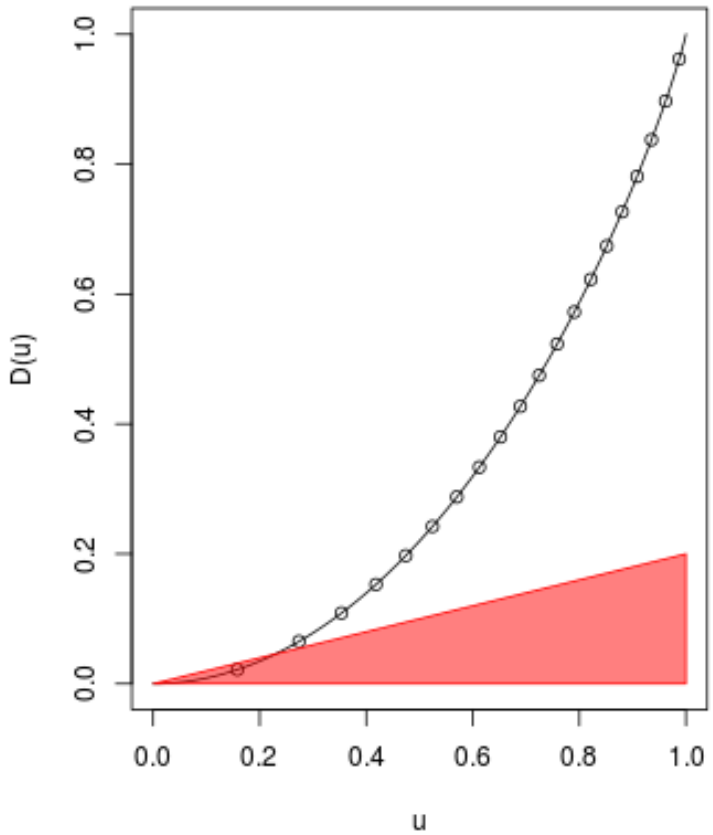

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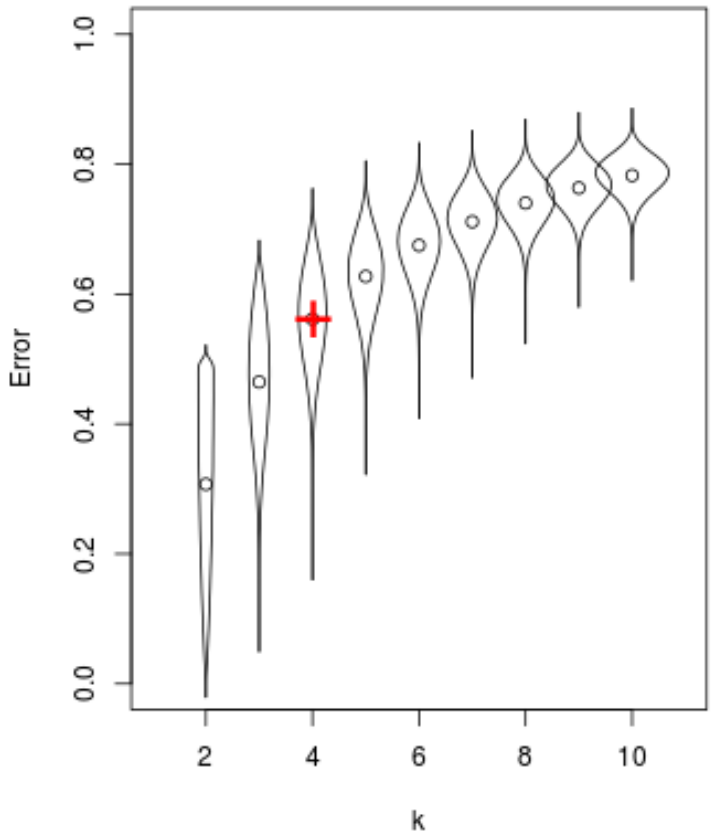

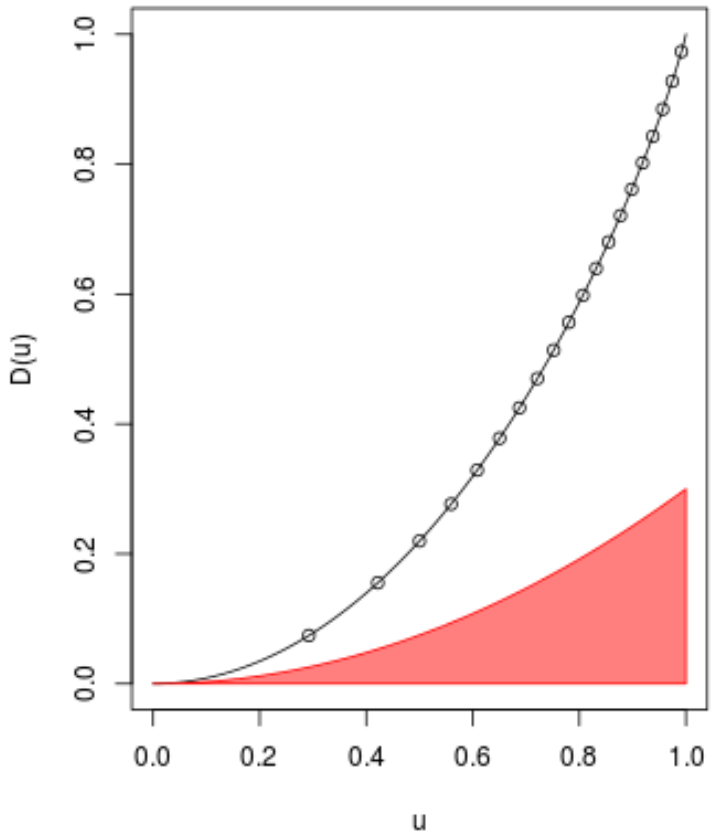

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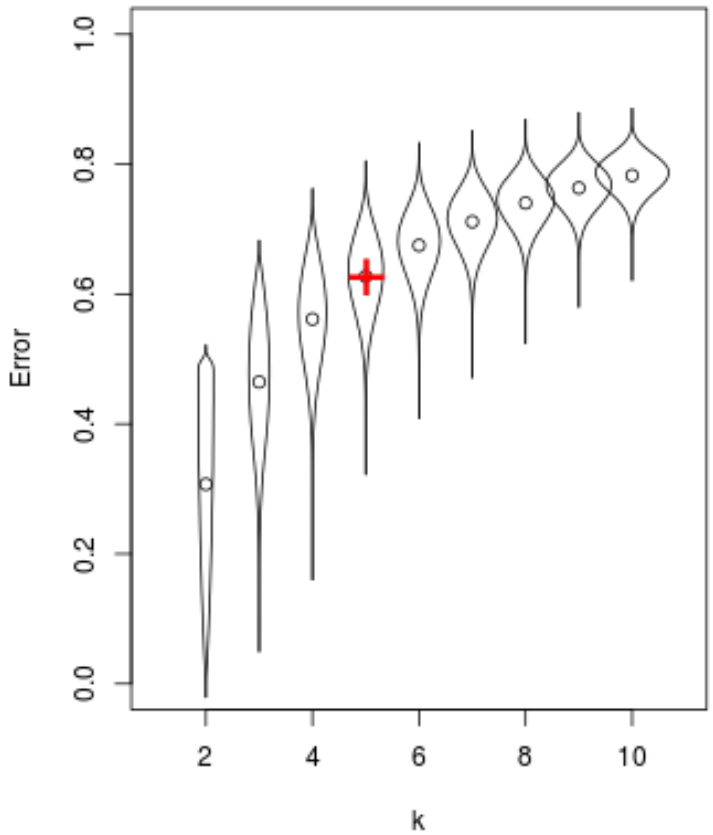

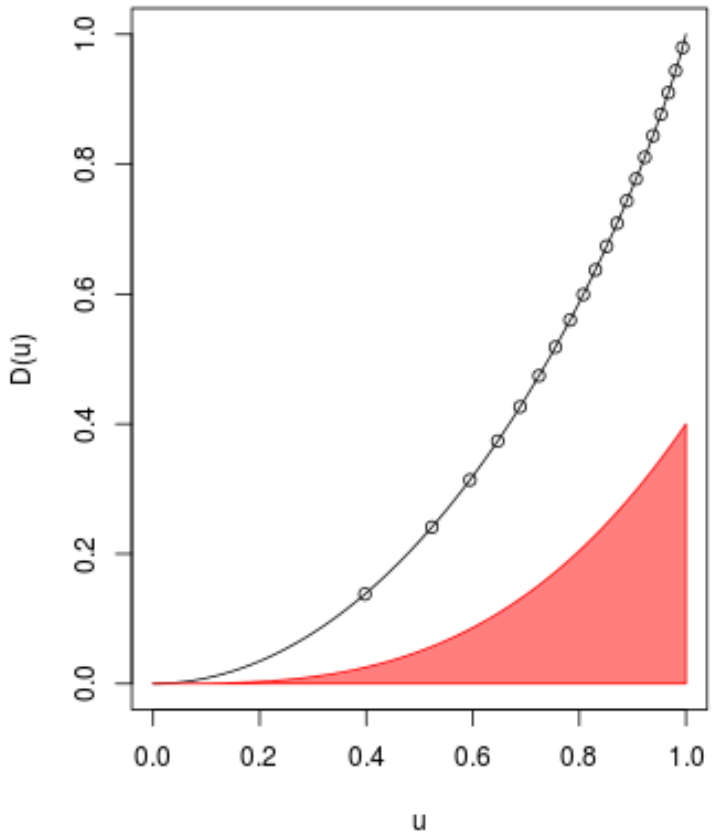

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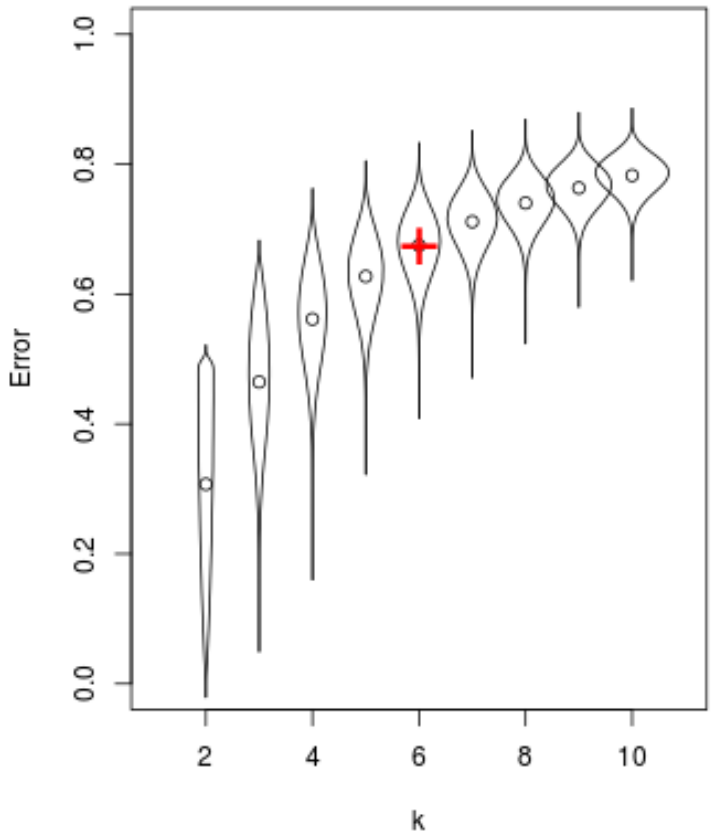
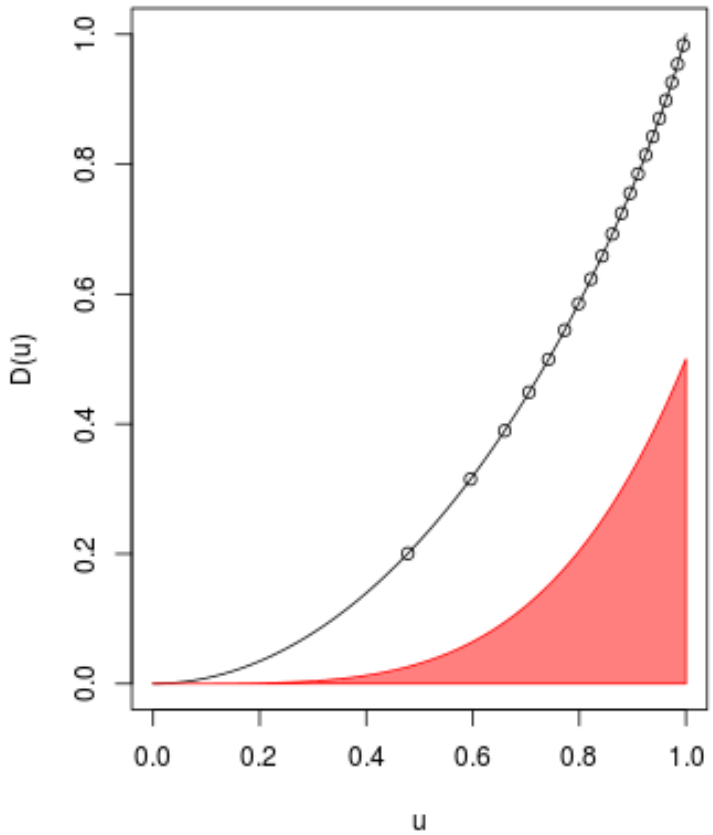
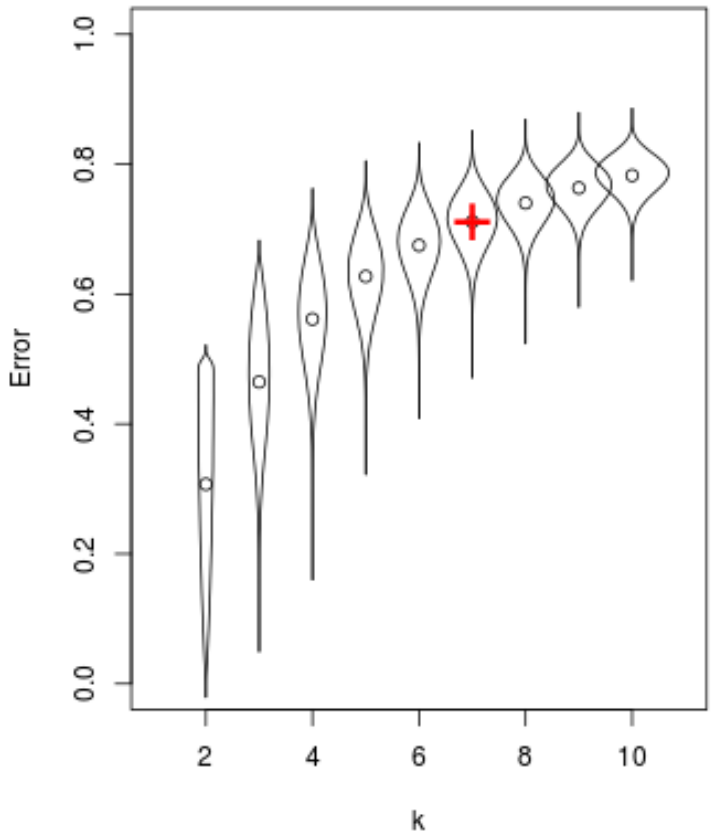


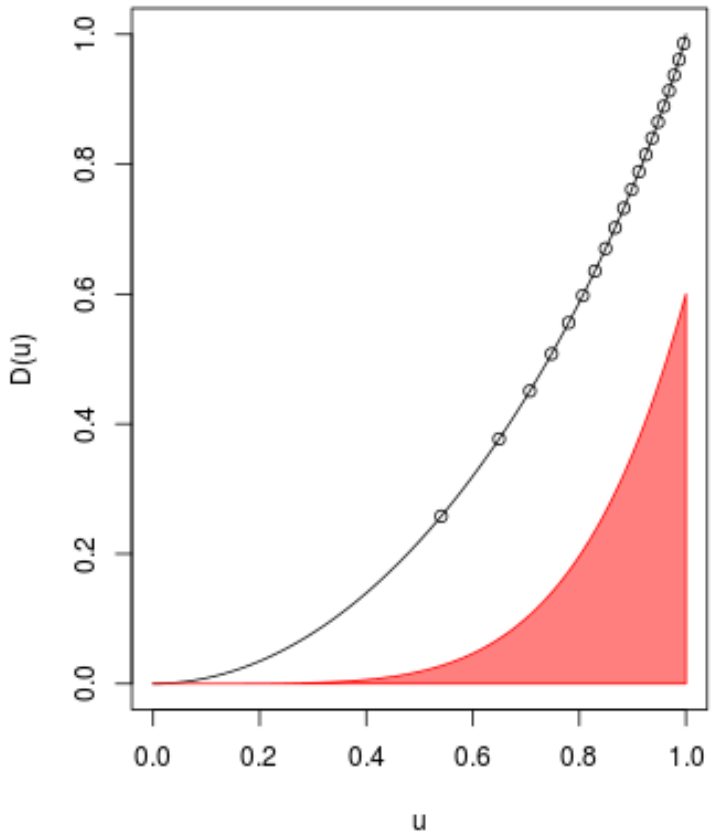


$(k = 6)$

Computing average risk

$$\text{AvRisk}_k = (k-1)\int \bar{D}(u)u^{k-2}du.$$

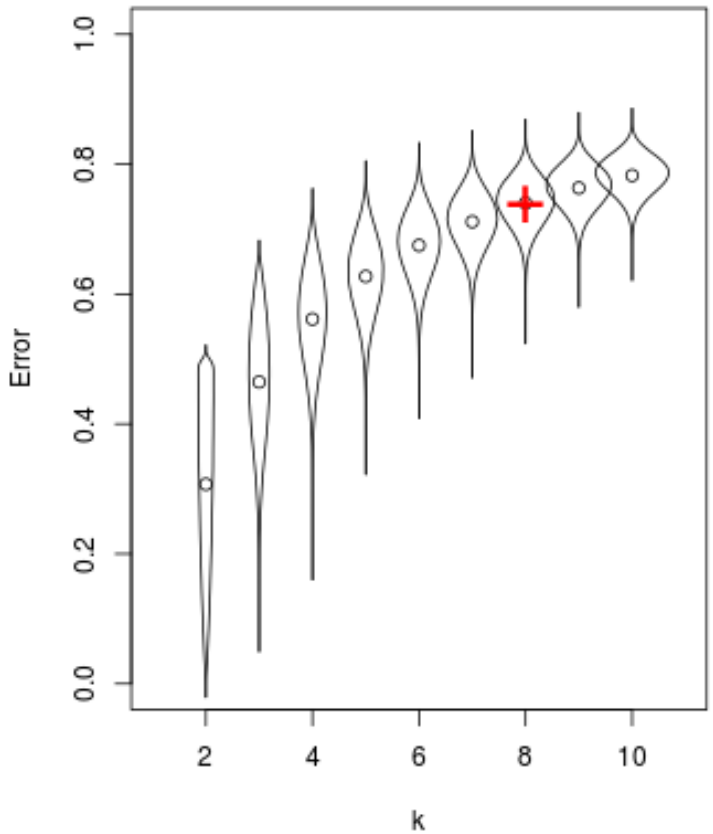


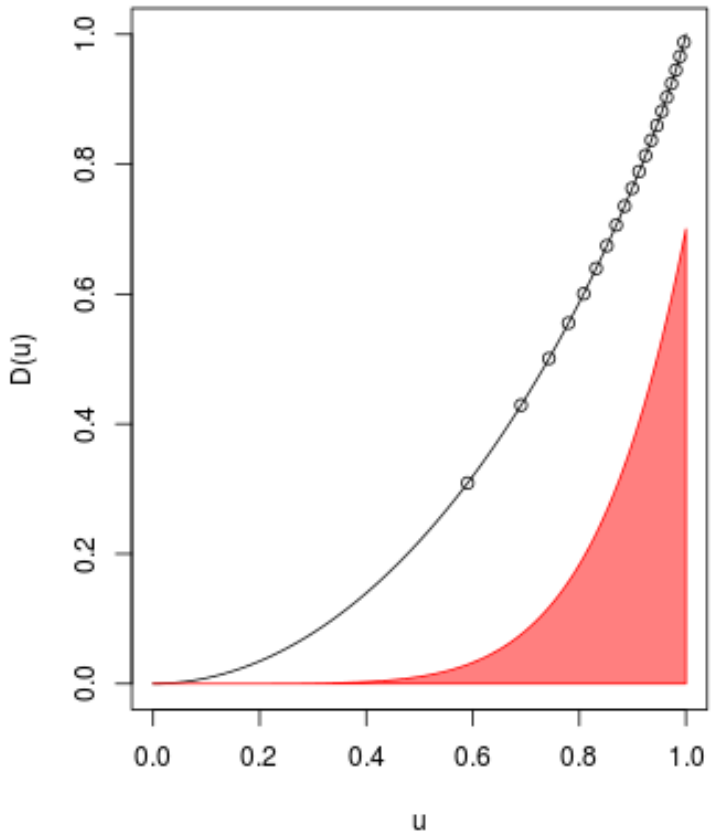


$(k = 7)$

Computing average risk

$$\text{AvRisk}_k = (k-1)\int \bar{D}(u)u^{k-2}du.$$

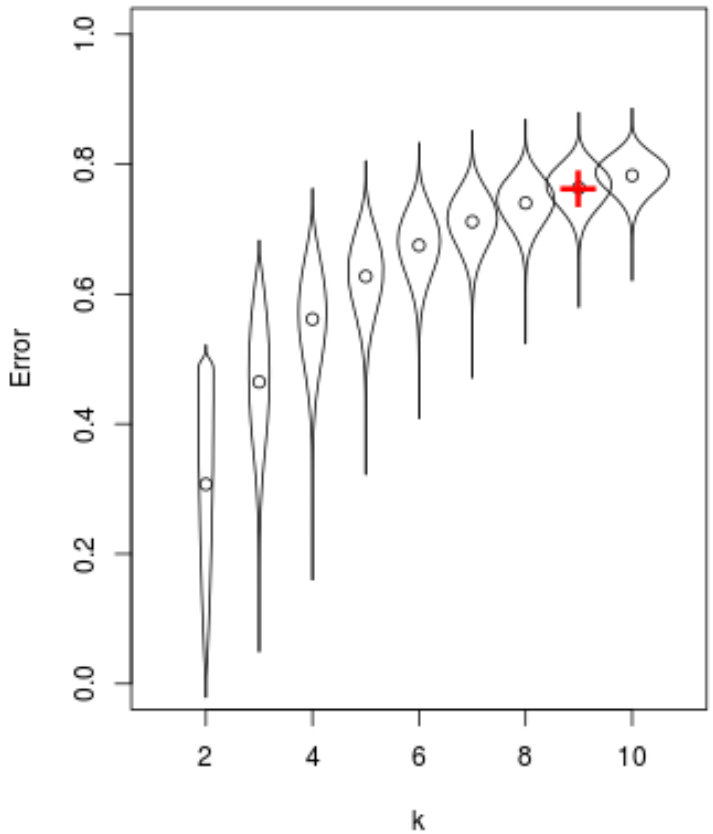


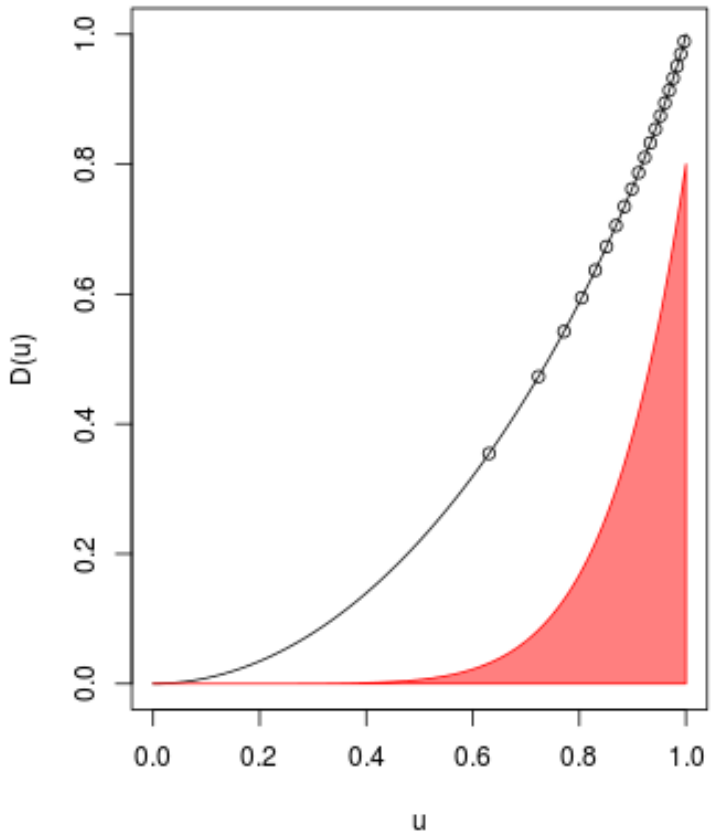


$(k = 8)$

Computing average risk

$$\text{AvRisk}_k = (k-1)\int \bar{D}(u)u^{k-2}du.$$

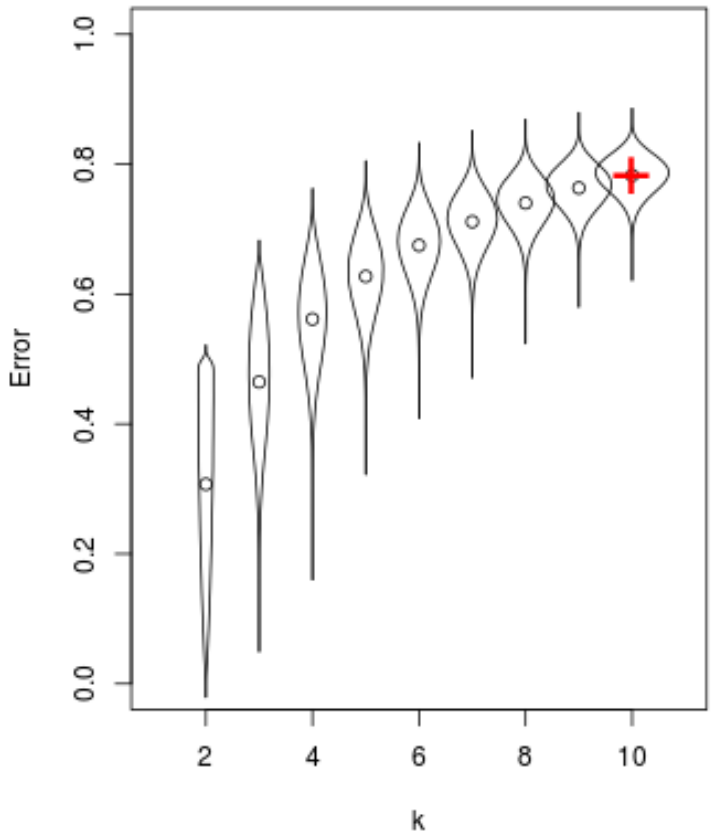


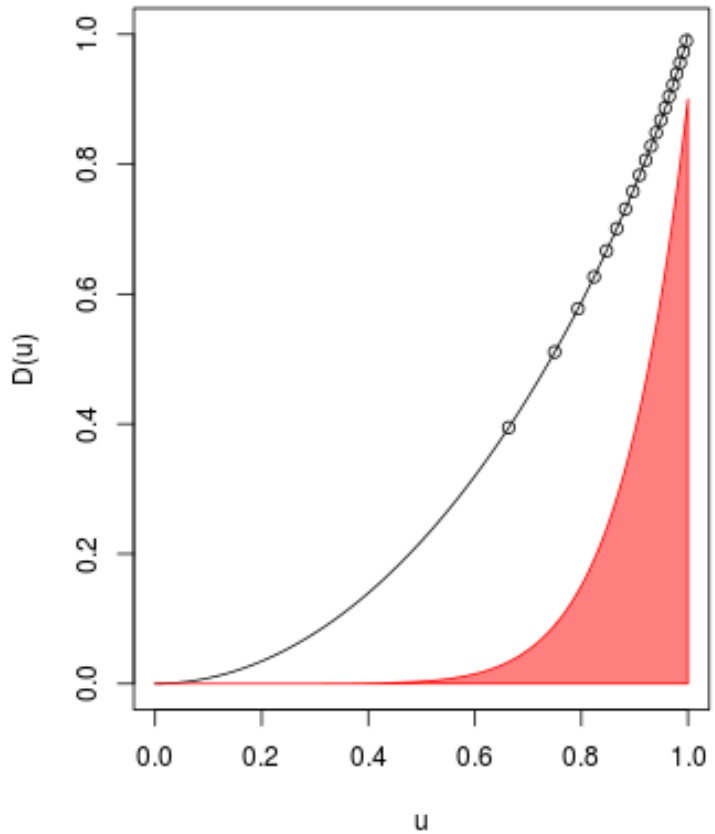


$(k = 9)$

Computing average risk

$$\text{AvRisk}_k = (k-1)\int \bar{D}(u)u^{k-2}du.$$





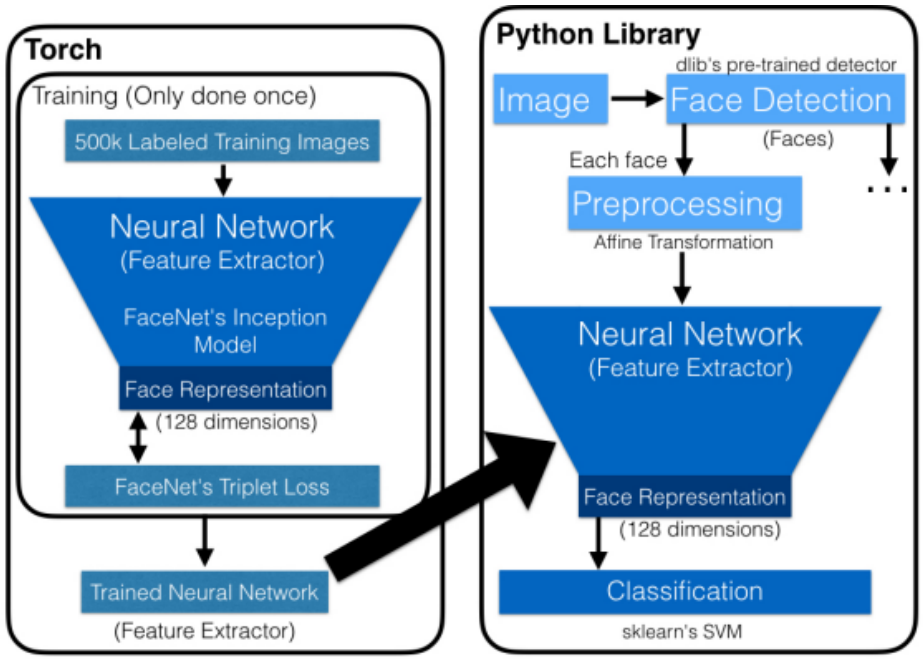
$(k = 10)$

Implication: estimate $\bar{D}(u)$ to predict risk

- Theoretical result links k-class average risk to $\bar{D}(u)$ function
- In real data, we do not know $\bar{D}(u)$ since it depends on the unknown joint distribution
- However, given a model, we can estimate $\bar{D}(u)$

Facial recognition example

- Data: faces from "Labeled Faces in the Wild."
- 1672 people with at least 2 photos
- Featurization: trained neural network from OpenFace



Facial recognition example

- Let us first subsample 400 faces (out of 1672)
- Randomly choose 1 face as training and 1 as test for each person
- Use 1-nearest neighbor.
  - NOTE: 1-NN with 1 example/class is equivalent to LDA with $\Sigma = I$: this fits marginal classifier assumption!

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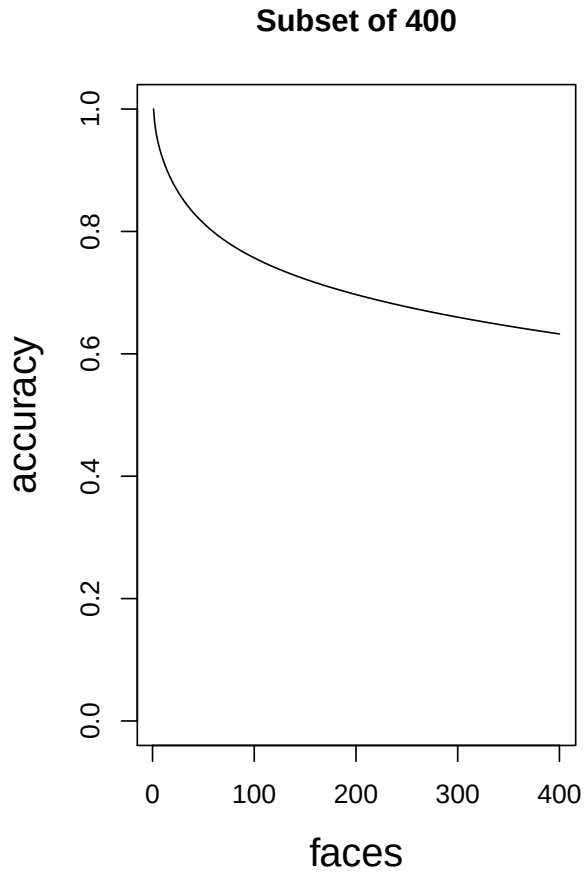

Facial recognition example

- Let us first subsample 400 faces (out of 1672)
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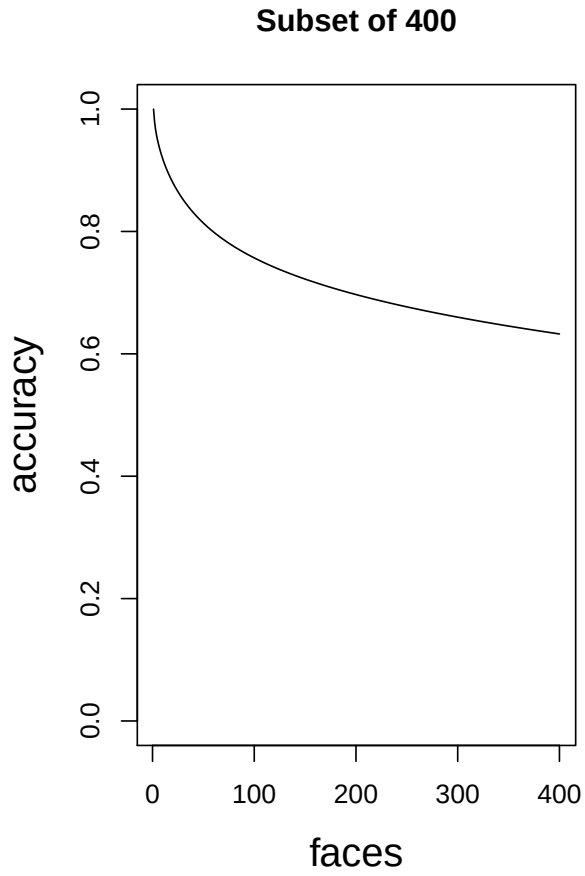
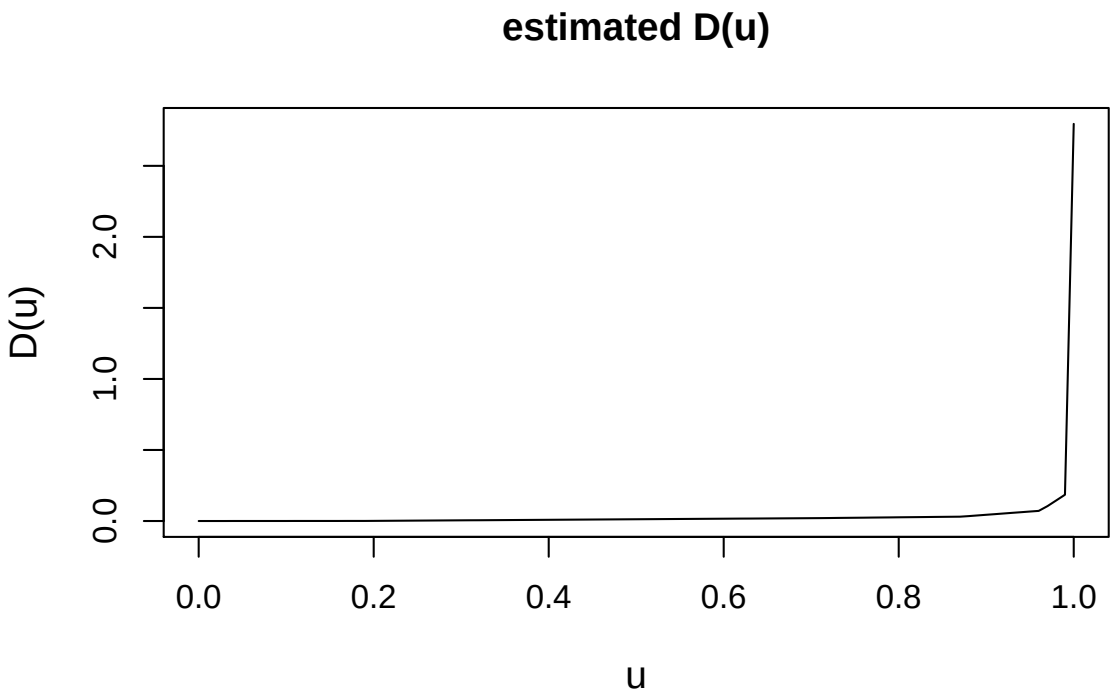


Can we predict the accuracy on the full set of 1672?

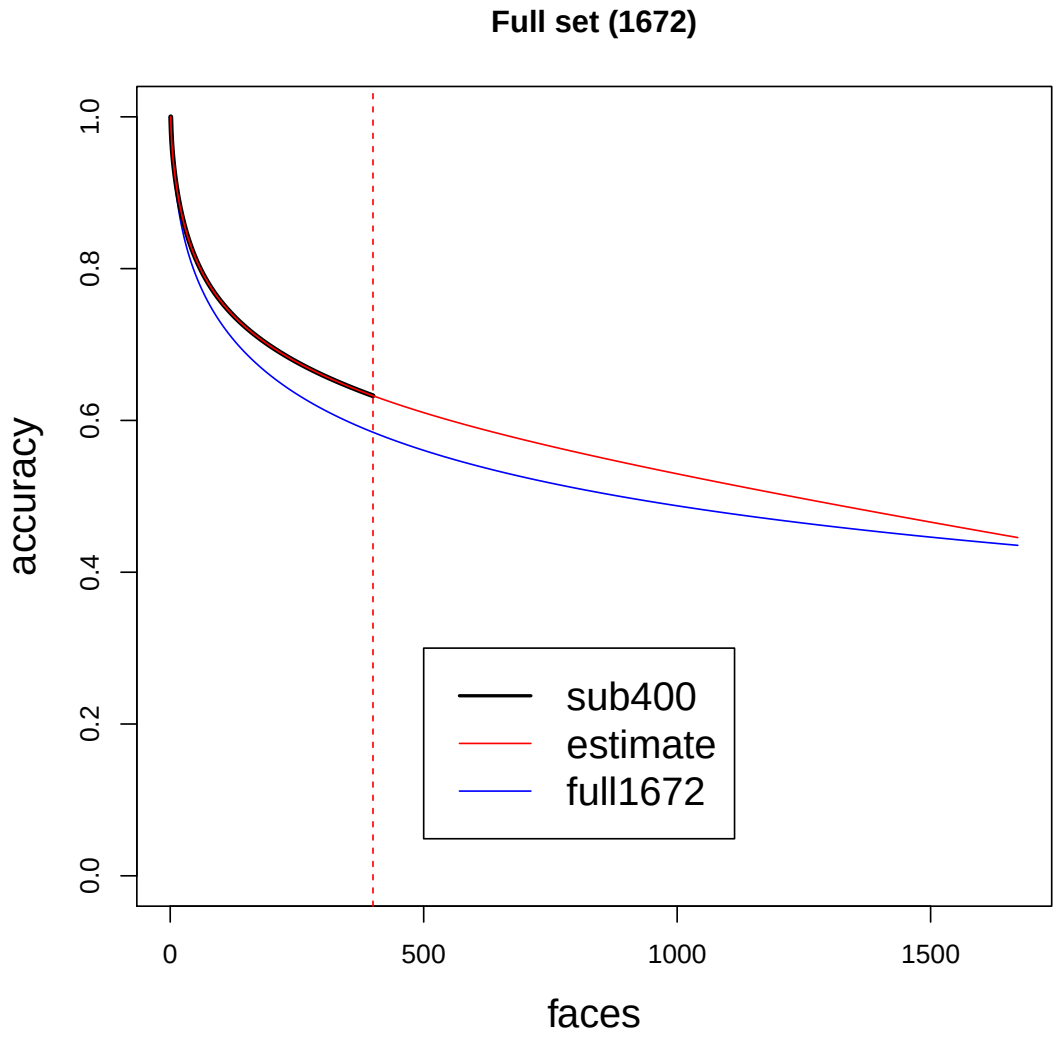
Estimated $\bar{D}(u)$

Using linear spline basis ($p = 10000$) and nonnegativity constraint.

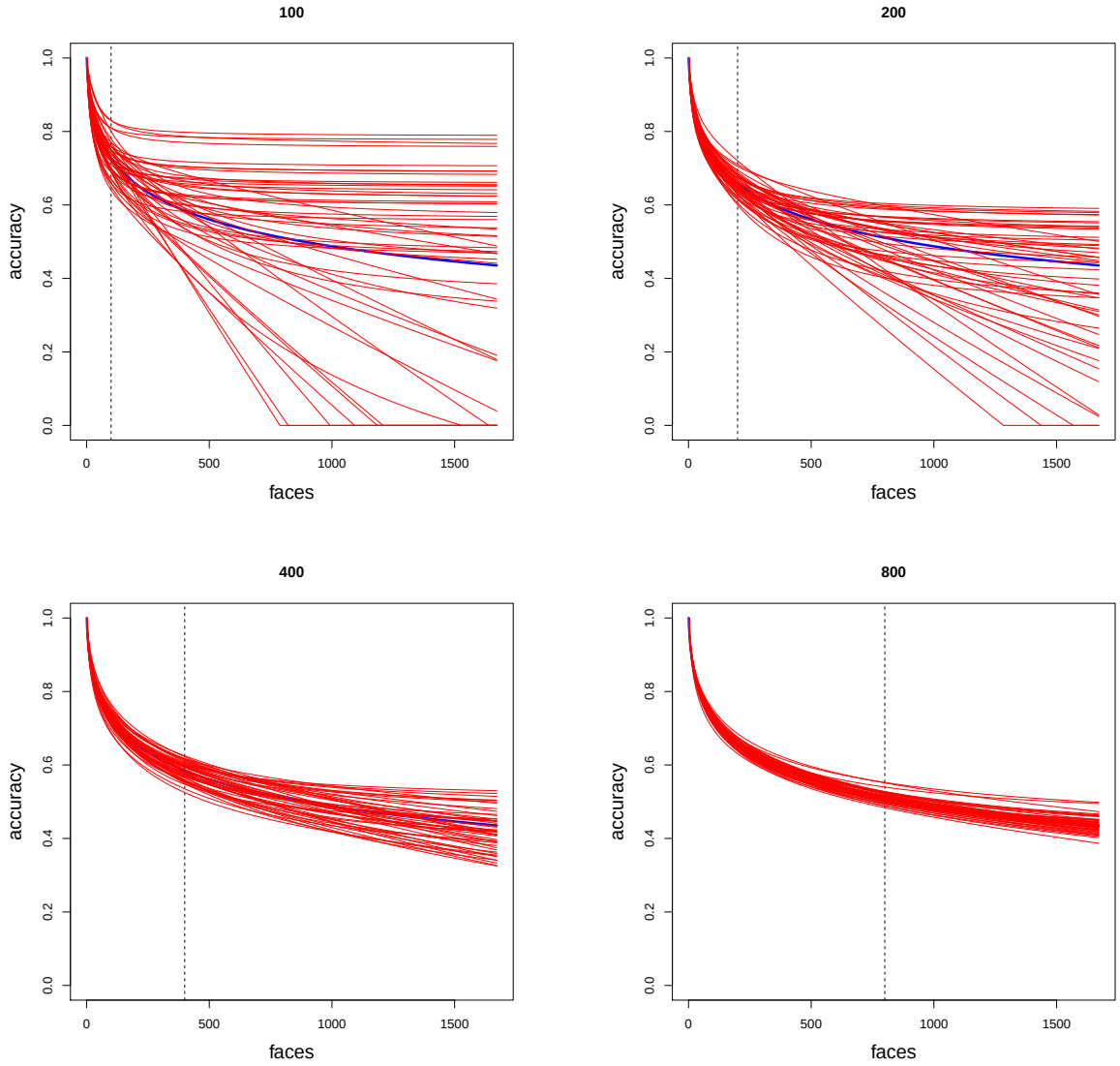


Estimated risk

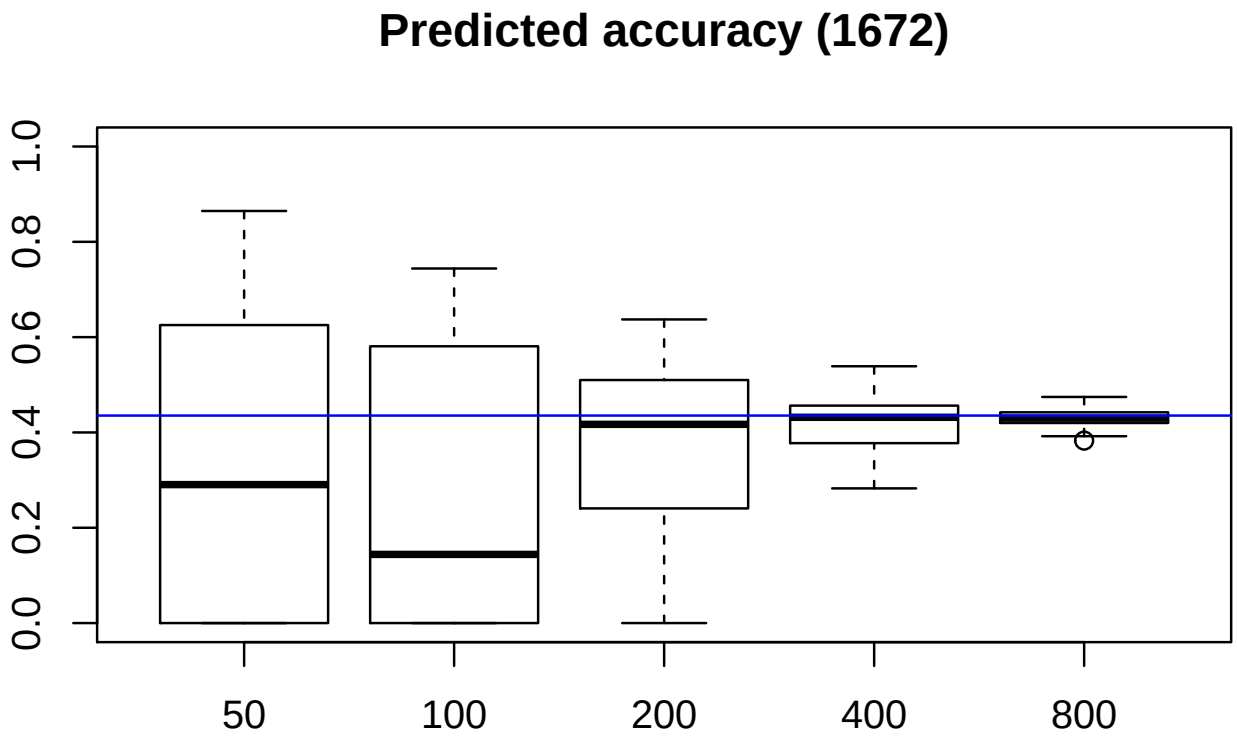
Compare to test risk at $K = 1672$



Estimated risk: more experiments



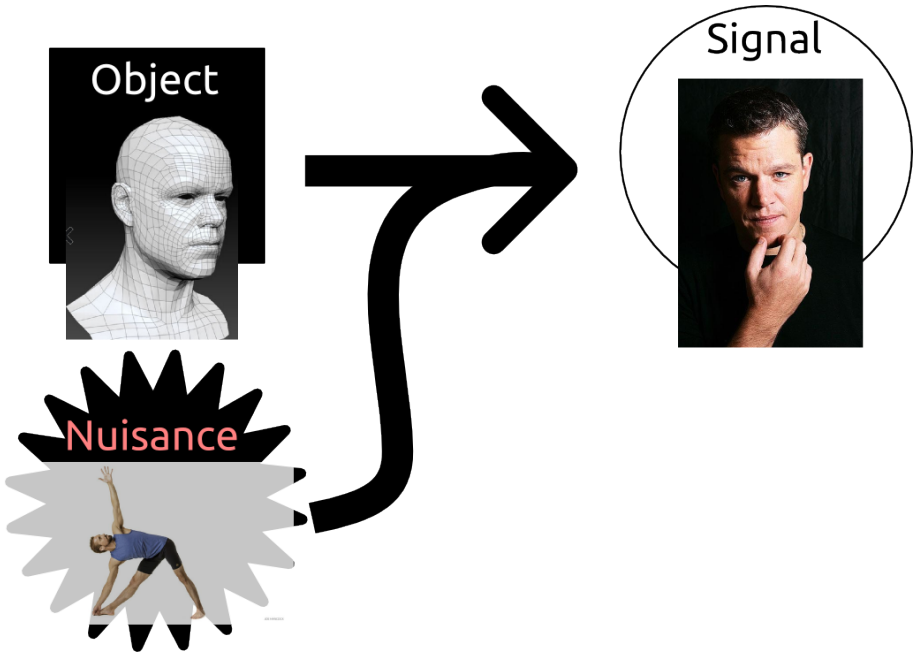
Estimated risk: more experiments



Section 2

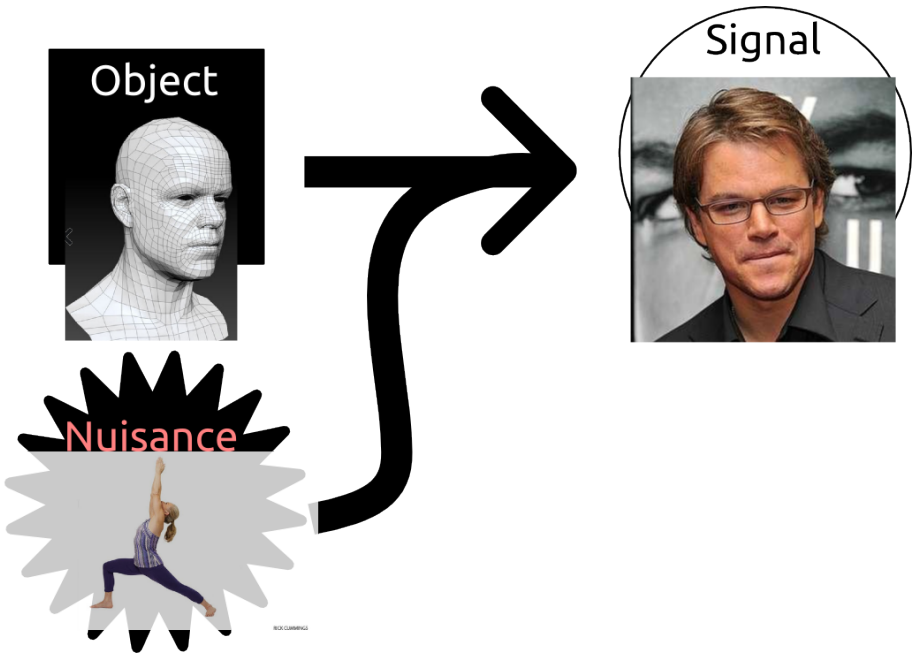
Geometry of representations

Example: face recognition



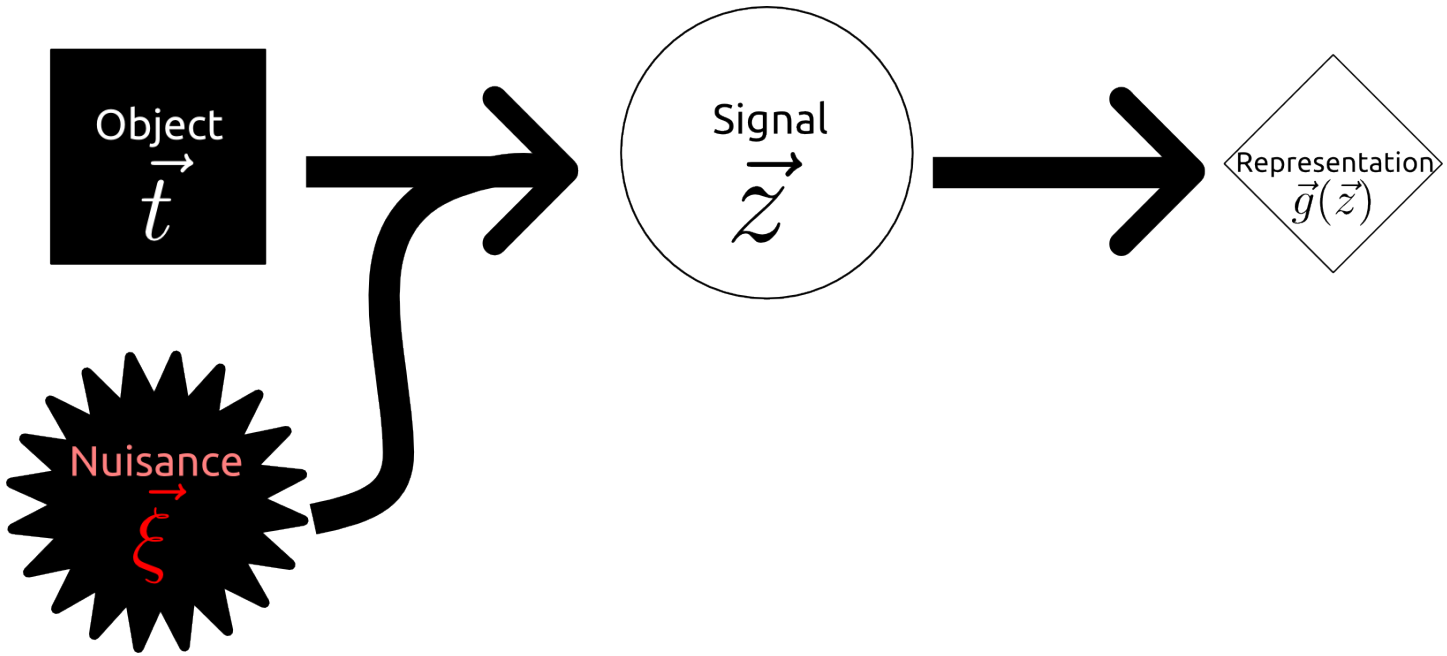
In face recognition, the *pose* (including hairstyle) and *lighting* are nuisance parameters.

Example: face recognition



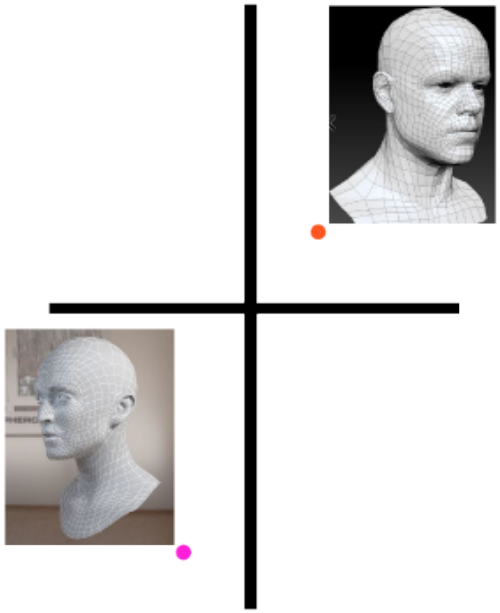
The same object can map to multiple signals.

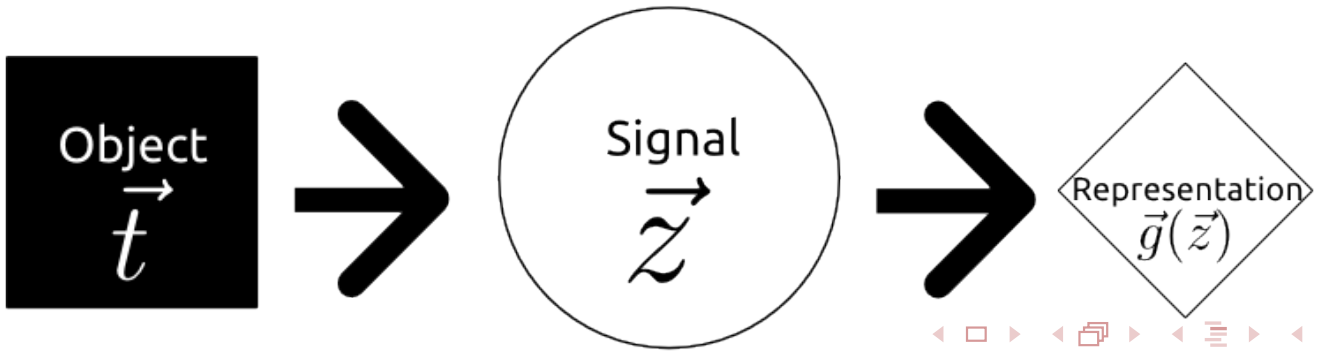
What is a representation?



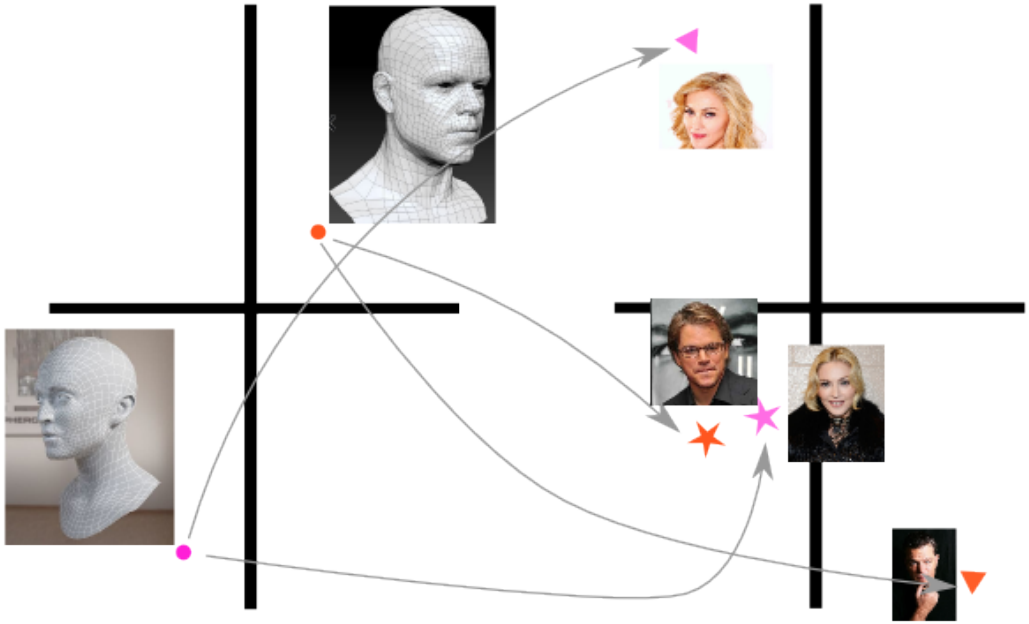
A dimensionality-reducing mapping $\vec{g}$ of the signal.

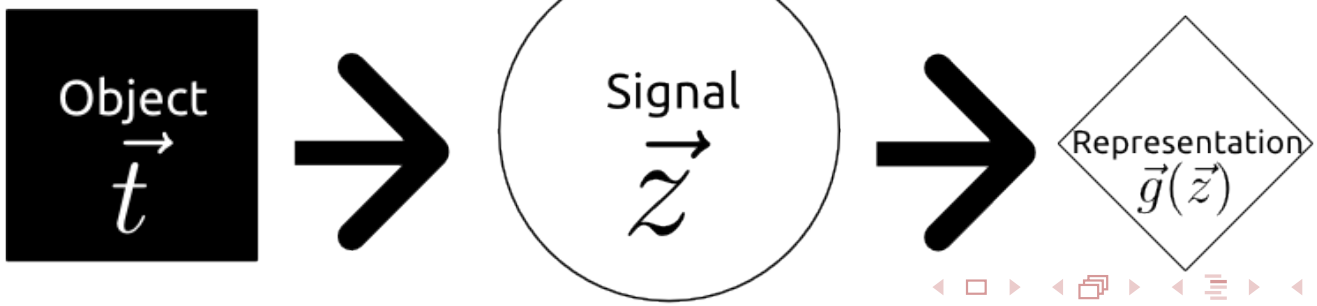
A good representation...



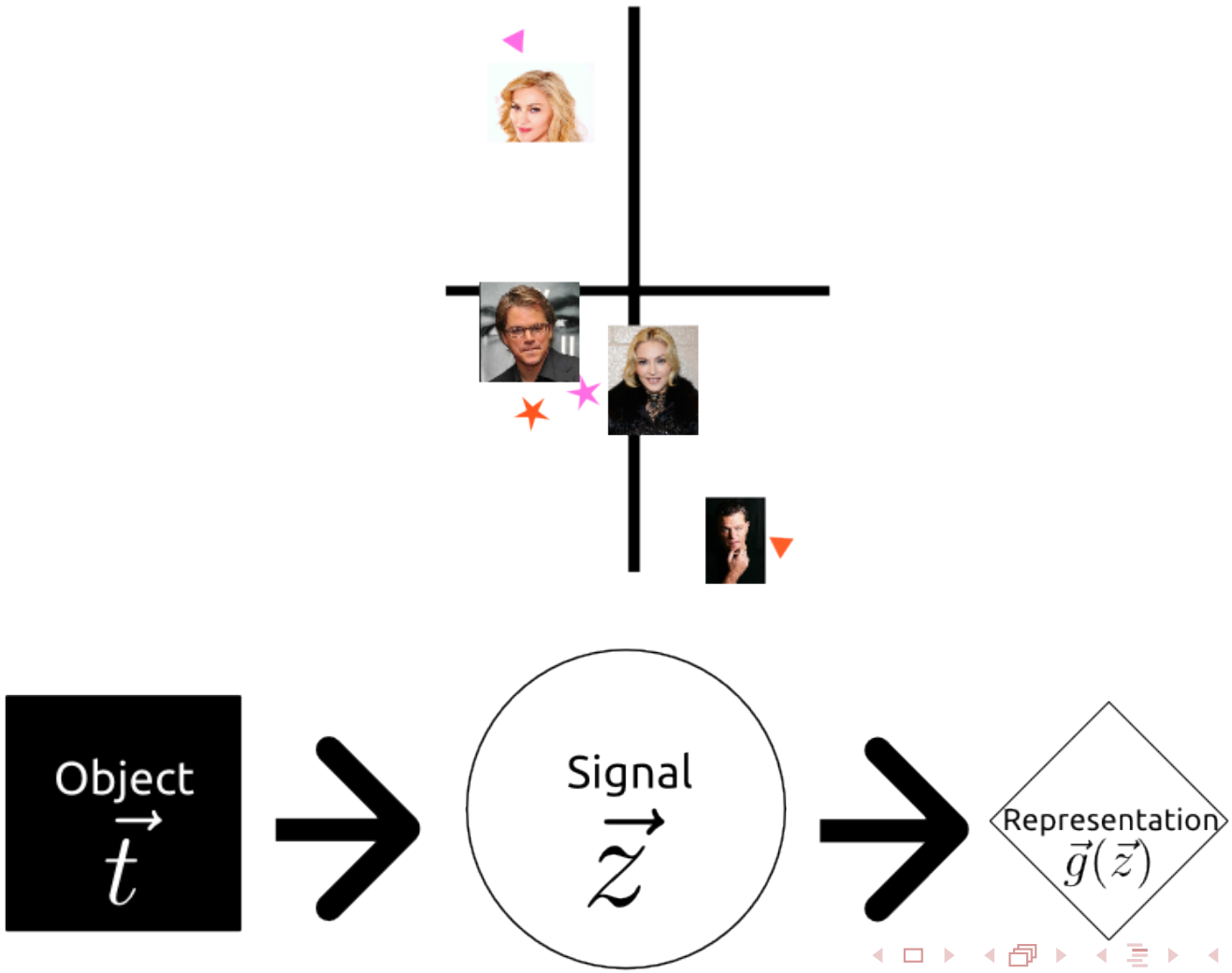


A good representation...

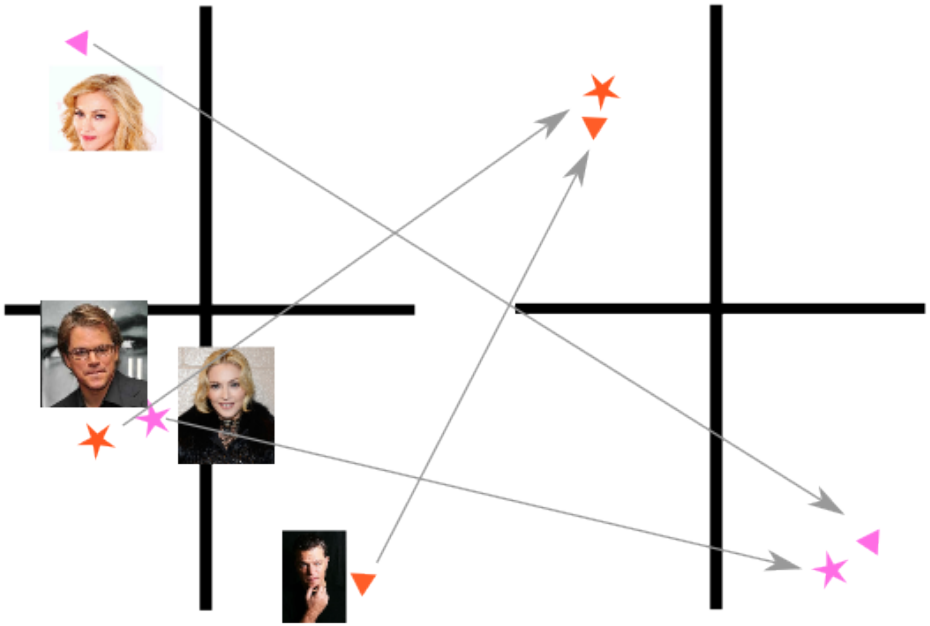


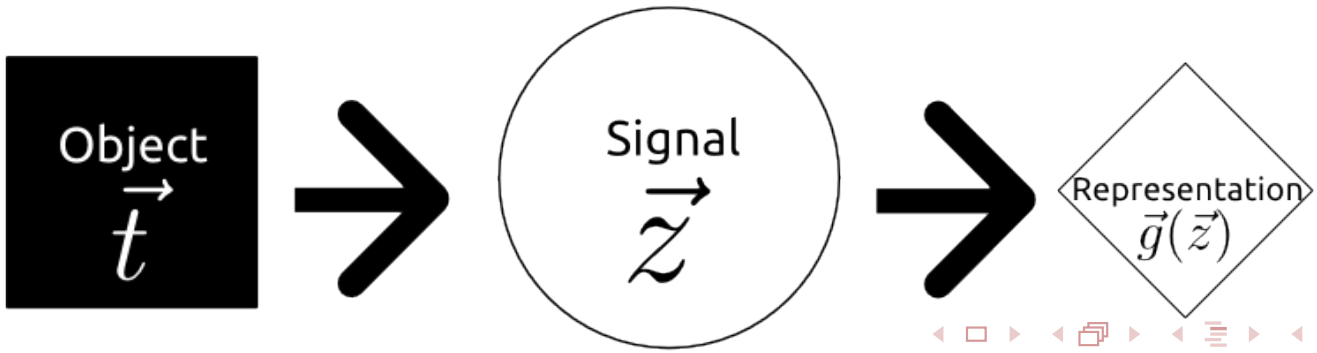


A good representation...

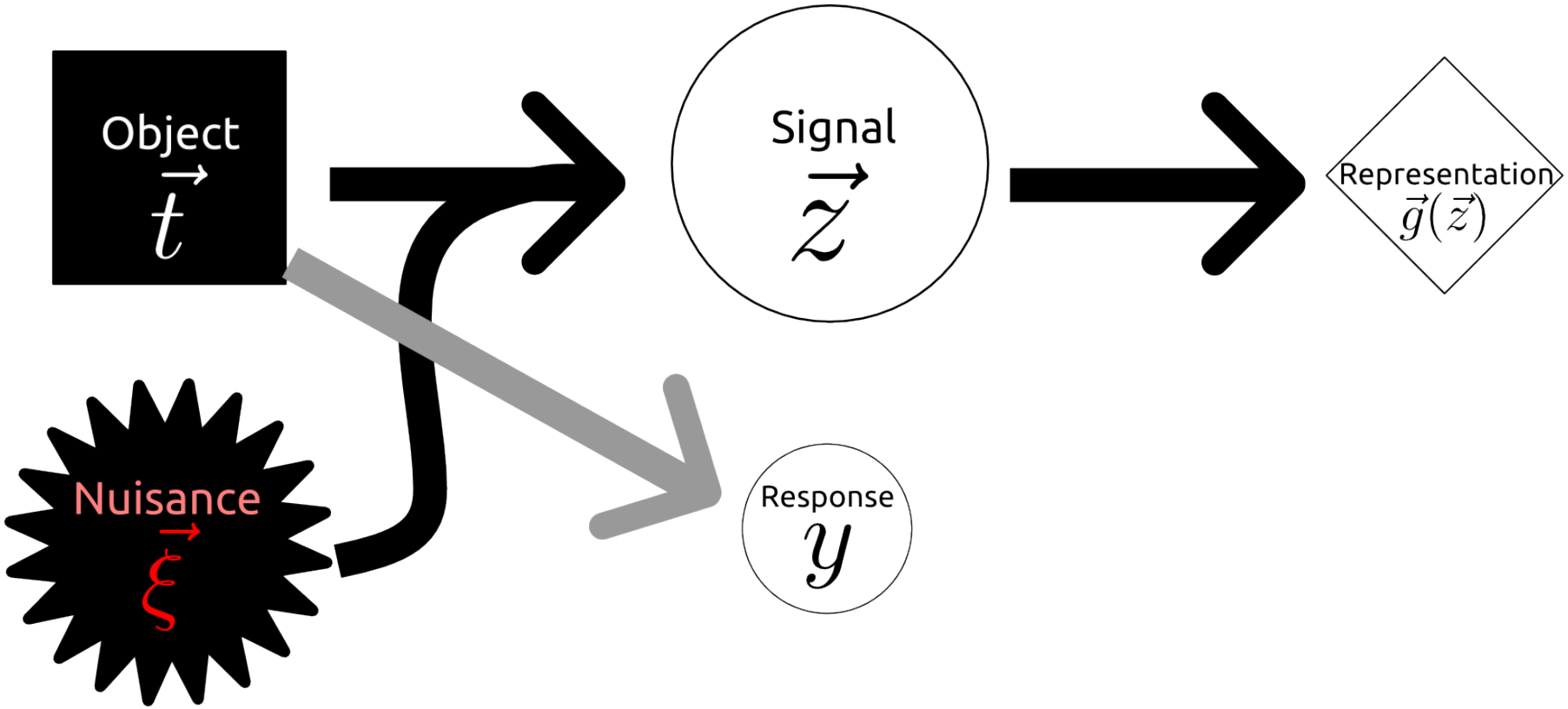


...captures the object space geometry

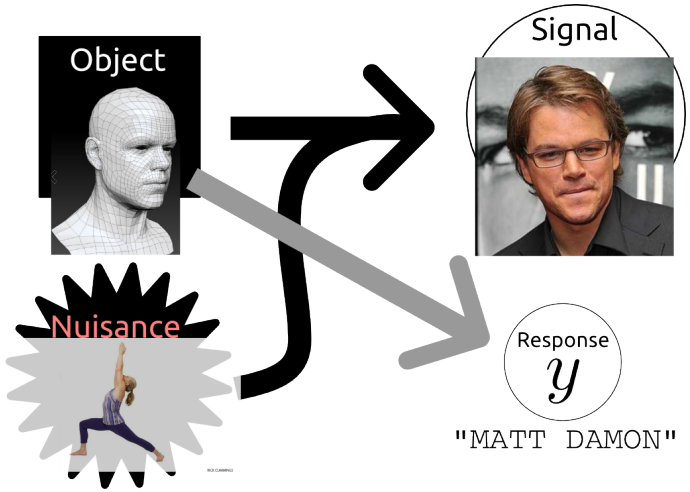




Supervised evaluation of representations

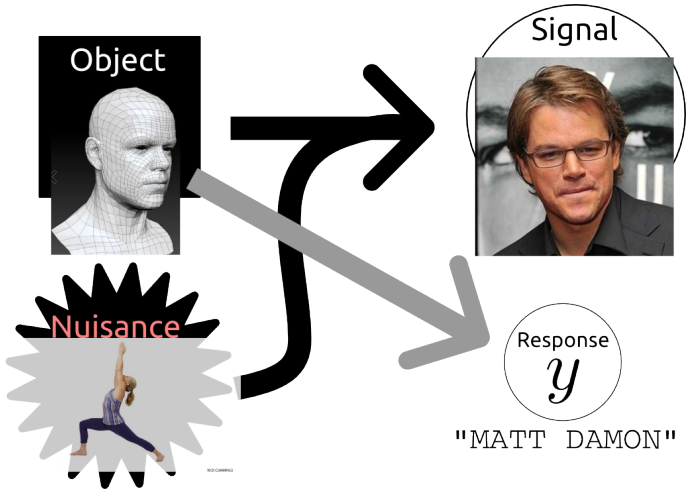


Example: face recognition



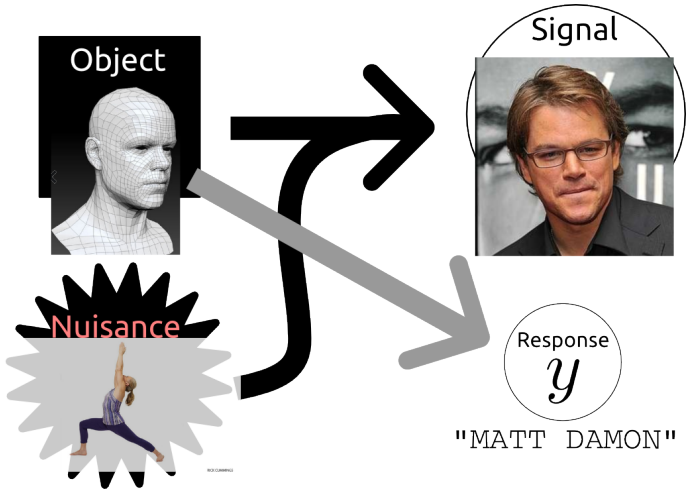
- The ID of the individual is an appropriate *response* variable...

Example: face recognition



- The ID of the individual is an appropriate *response* variable...
- ...because two photos labeled with the same ID must belong to the same object $\vec{t}$

Example: face recognition

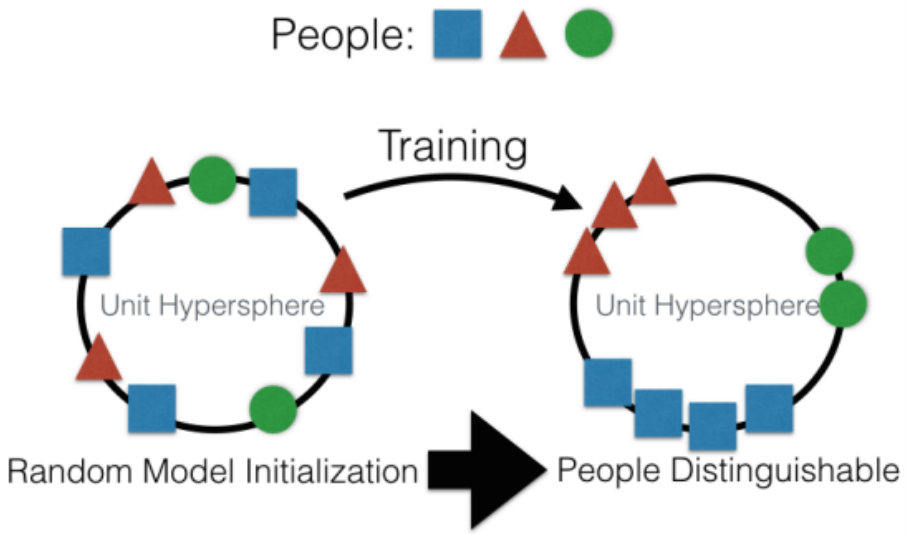


- The ID of the individual is an appropriate *response* variable...
- ...because two photos labeled with the same ID must belong to the same object $\vec{t}$
- That is, for $d(y, y')$ being the zero-one distance,

$$d(y, y') = 0 \Leftrightarrow d(\vec{t}, \vec{t}') = 0.$$

Methods for supervised evaluation of representations

- *Triplet loss* (Schroff 2015)



Methods for supervised evaluation of representations

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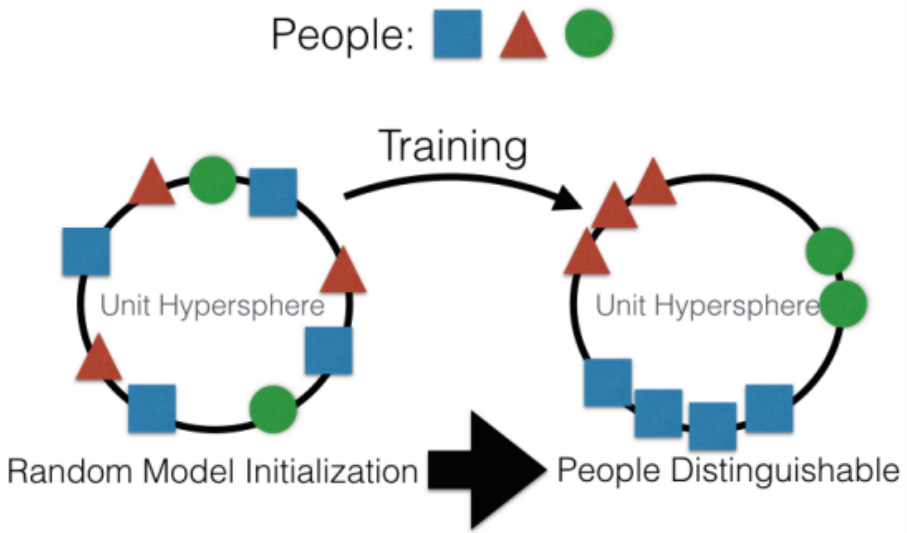
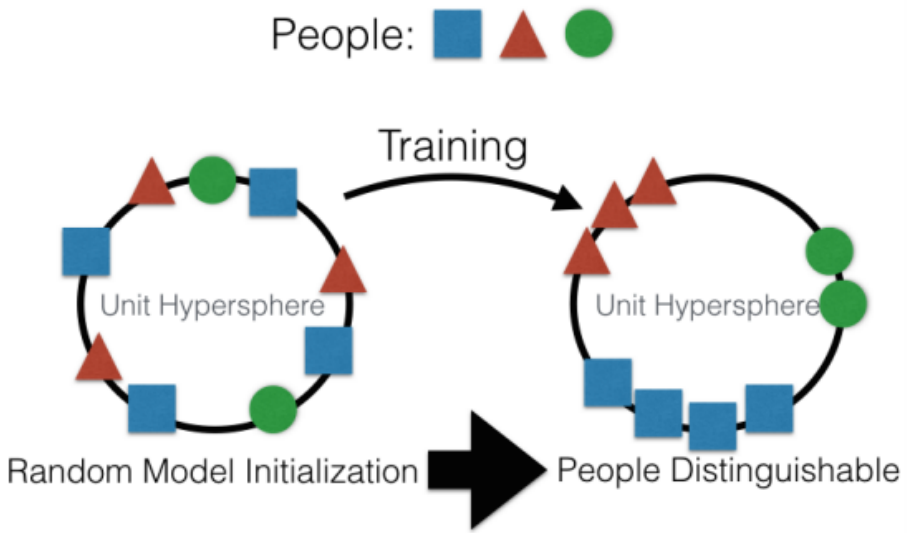


- Average risk of randomized classification (using e.g. 1-nearest neighbor)

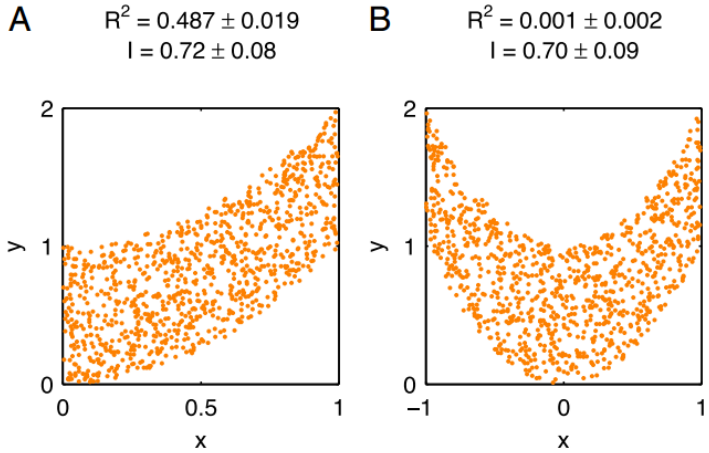
Methods for supervised evaluation of representations

- *Triplet loss* (Schroff 2015)



- Average risk of randomized classification (using e.g. 1-nearest neighbor)
- *Mutual information* $I(g(\vec{Z}), \vec{Y})$

Mutual information $I(X;Y)$



Introduced in Shannon's 1948 paper, "A mathematical theory of communication"

$$I(X;Y) = \int \log\left(\frac{p(x,y)}{p(x)p(y)}\right) p(x,y)dxdy$$

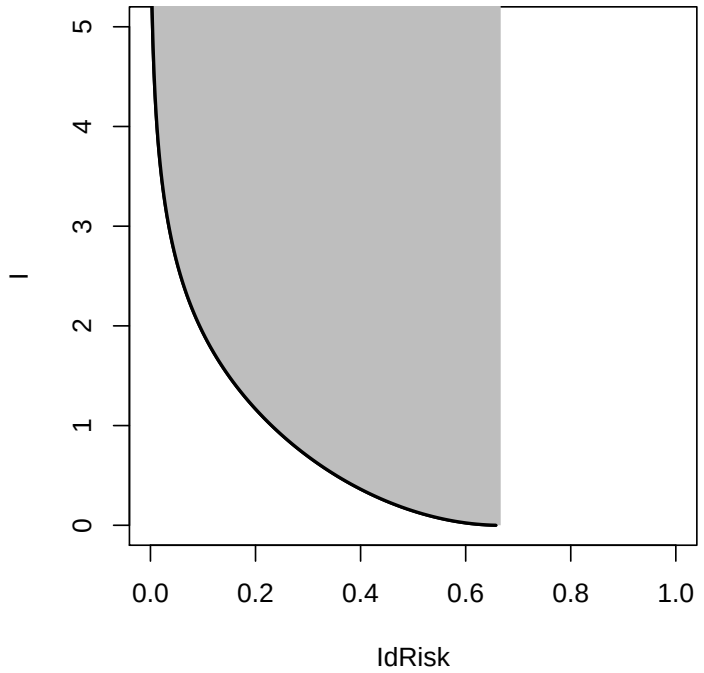
Image credit Kinney et al. 2014.

Result 1. Lower bound for mutual information

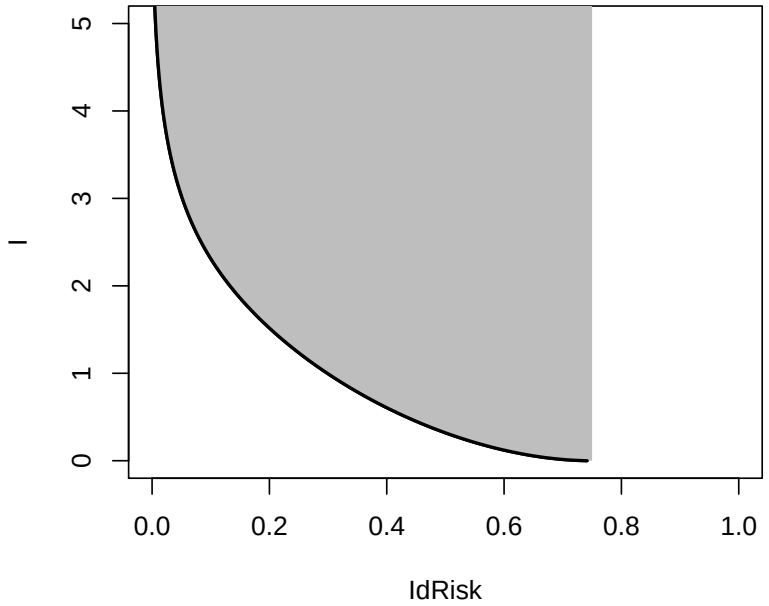
- **Theorem.** (Z., Benjamini 2017) There exists a function h_k such that

$$I(\vec{g}(\vec{Z}); \vec{Y}) \geq h_k(\text{AvRisk}_k).$$

h_3



h_4



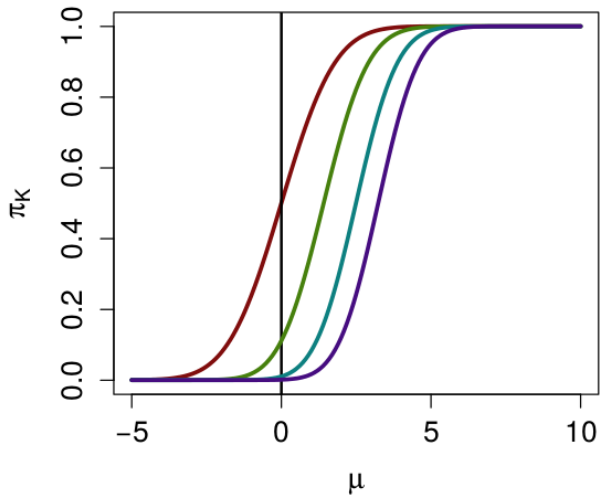
Result 2. Limiting behavior of accuracy curves

(Z., Benjamini 2016) Define ABA_k as the Bayes identification accuracy (or average Bayes classification accuracy). Then under a particular high-dimensional limit,

$$\text{ABA}_k \approx \pi_k(\sqrt{2I(X;Y)}) \tag{1}$$

The function π_k is given by

$$\pi_k(c) = \int_{\mathbb{R}} \phi(z-c)\Phi(z)^{k-1}dz.$$



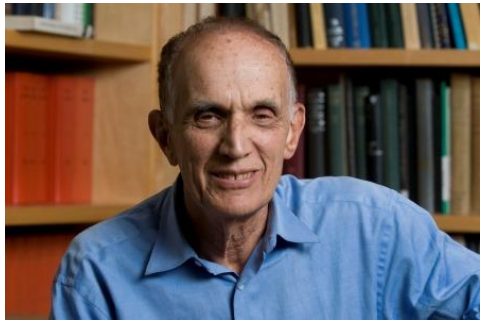
Legend: $K = \{$ 2, 9, 99, 999 $\}$

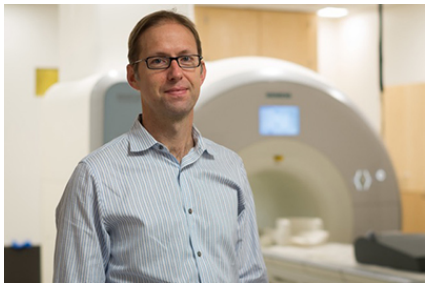
Section 3

Acknowledgements

Co-advisors

Committee





Collaborators

¡Compadres!


Year 1
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Michael Sklar
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Jelena Markovich
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Minyong Lee
Michael Celentano
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Andrew Tsao
Leying Guan
Junyang Qian
Zhou Fan
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Katelyn Gao
Bai Jiang
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Nikolaos Ignatiadis
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Eugene Katsevich
Feng Ruan
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Subhabrata Sen
Jack Gorham
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Pragya Sur
Jiyao Kou
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Murat Erdogdu
Chaojung Wang

Section 4

The end