How many faces can be recognized? Performance extrapolation for multi-class classification

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Abstract

The difficulty of multi-class classification generally increases with the number of classes. *Recognition systems* employ such classifiers in order to recognize people, spoken words, or chemicals: it is often of interest to know how many species the system can be trained to recognize before dropping below a minimum accuracy threshold. However, before such systems are deployed, typically only a small number of species are available for testing the system. Can we predict how well the recognition system will scale with an increased number of classes? We distinguish between two types of multi-class classifiers: *separable* classifiers, which include k-nearest neighbors, multinomial regression, one-vs-one and one-vs-all classifiers, *non-separable* classifiers, such as deep neural networks and decision trees. For recognition systems based on separable classifiers, the problem of predicting scalability reduces to the problem of estimating the higher-order moments of a *conditional accuracy distribution*, which in turn can be estimated from data.

1 Introduction

Object recognition, face recognition (or more generally person recognition) and language are a few of the cognitive building blocks which are fundamental to human cognition, and which can be understood as examples of generalized classification tasks. Machine classification can be employed to mimic this power of recognition. A robot equipped with a camera can algorithmically segment its input image into objects, and to learn to recognize unique objects and people which regularly appear in its environment. A general approach to implement such a recognition ability starts by employing some parametric featurization of the object to be identified. For example, for the task of face recognition, one might define features such as the proportions between the eyes and the relative position and size of the nose. The full domain of the recognition task is a collection of *instances* (e.g. photographs of faces) which is divided into *species.* The recognition system can be implemented by training a multi-class classifier to assign instances to their corresponding species. While the system is in deployment, new species may be added to the system: when this happens, the classifier is retrained (or updated) using training data from the species to be added.

A limitation to such recognition systems, whether they be natural or artificial, is that the performance of the system (in terms of correct classification) can degrade if there are too many species. A face

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recognition algorithm can have very high success rate if it only needs to distinguish between 100 different faces, but its identifications may be less reliable when it needs to distinguish between 10000 different faces. The consequences of such errors may be severe: Cole (2005) lists 22 cases of fingerprint misindentification in criminal trials.

Therefore, in the case of engineering recognition systems, it is of much practical interest to be able to evaluate the reliability of such systems before they are deployed. Yet, during the development phase, typically data from only a fraction of the species in the domain are available. From the empirical performance of the classifier on this initial subset of species, can we predict the performance of the system on a larger subset of species (or the entire domain)?

A related problem arises in studying *naturally occurring* recognition systems: for instance, human memory. Neuroscientists may be directly interested in the number of different cues which can be recalled by a subject. A similar dilemma arises, where data from only small number of species can be obtained, due to experimental constraints. From this data, can we predict the number of species which can be distinguished by the recognition system, above a minimum accuracy threshold?

We adress this problem of *performance extrapolation* under the assumption that the initial subset of species is an i.i.d. sample from a larger population of species. For a restricted family of classifiers– *non-pooling* classifiers, we show that the problem of performance extrapolation reduces to a problem of nonparametric moment estimation. But this is still a difficult problem, and in section 4 we find limitations to traditional maximum likelihood and Bayesian inference techniques. We propose two novel estimators which have reasonable empirical performance: a likelihood-based estimator with constrained moments, and an estimator based on the mutual information estimator of Anon 2016.

1.1 Multi-class classification

We provide a very brief overview of existing methods for multi-class classification. We neglect a number of important aspects of multiclass classification, including categorical predictors, missing data, model selection, and regularization: for a fuller treatment of the subject, see (CITE) (CITE) or (CITE).

Let z be a label from $\{1, \ldots, k\}$, and let $\mathcal{Y} \subset \mathbb{R}^p$ be a space of feature vectors. Suppose that (z, y) is drawn from a joint distribution F. We observe independent pairs (z_i, y_i) for $i = 1, \ldots, n$. The *multi-class classification problem* is to learn a *classification rule* for predicting the label z as a function of y: in other words, to construct a function $f : \mathcal{Y} \to \{1, \ldots, k\}$ such that the *risk*

$$\mathbf{E}\mathcal{L}(z, f(y))$$

is minimized. Common examples of loss functions are the 0-1 loss

$$\mathcal{L}(z, \hat{z}) = I(z = \hat{z})$$

and the weighted loss

$$\mathcal{L}(z, \hat{z}) = C_{z,\hat{z}}$$

where C is a $k \times k$ cost matrix. The weighted loss is often used in multi-class classification problems, because the cost of confusing similar classes (cats and dogs) is less severe than confusing very different classes (dogs and airplanes.) In this paper we specialize to the 0-1 loss, as it provides the simplest setting for developing theory.

The multi-class classification problem is a generalization of the more common *binary classification* problem, in which $k = 2$. Existing approaches for multi-class classification fall into two categories: methods which directly learn a multi-class mapping $f : \mathcal{Y} \to \{1, \ldots, k\}$, and 'divide-and-conquer' methods which convert the multi-class problem into a combination of multiple binary classification problems. Methods in the first category include k-nearest neighbors, multinomial logistic regression, linear discriminant analysis (LDA), quadratic discriminant analysis (QDA), decision trees, and random forests (CITE). Divide-and-conquer approaches include one-vs-all and one-vs-one (CITE).

2 The recognition problem

In the recognition problem, there exists a large or infinite population of *species*, e.g. individuals, and an infinite population of *measurements*, and each measurement is associated with a single species. A

recognition system is an algorithm which contains a database of measurements from a number of species: each measurement is represented within the recognition system by a real vector y which lies in the space $\mathcal{Y}$, and labeled with an integer-valued ID key. The ID key is the recognition system's internal representation of the species associated to the measurements: we assume that the database is accurate in the sense that two measurements in the database share the same ID key if and only if they are associated with the same species. Given a query measurement, the system answers with an ID number: this answer is correct if and only if the query measurement and the database measurements labeled with the same ID are associated to the same species. New data can be added to the recognition system: we assume that the integrity of the database remains intact, so that an added measurement from species already existing in the database is labelled with the same ID, and that an added measurement from a new species is labelled with a new ID key.

Let us assume that each species can be *uniquely* parameterized by a parameter vector x, and let $\mathcal{X}$ denote the space of parameter vectors. Note that the parameter x may or may not observed by the recognition system: in the latter case, the system only has access to arbitrary ID keys. Let $p(x)$ be the population distribution of species, and assume that a conditional feature vector distribution $p(y|x)$ be defined for every $x \in \mathcal{X}$. We require that $p(x)$ be a density with respect to Lesbegue measure, but $p(y|x)$ is allowed to include Dirac delta components.

A natural approach for implementing a recognition system is to train a multi-class classifier on its database, treating each species as a distinct class. Whenever new data is added, the classifier is retrained. Depending on which multi-class classifier is used, we can categorize the recognition system as either a *separable* system, or a *non-separable* system. The database of the classifier therefore consists of pairs $\{(x^{(i)}), y^{(i),j})\}$ where $x^{(i)}$ is the parameter of the ith species in the database, and $y^{(i),j}$ is the jth measurement associated with that species. Without loss of generality, let i be the ID key of the species $x^{(i)}$. Let k denote the total number of species in the database, and let n_i be the number of measurements associated with the ith species. The recognition system maps queries y to an ID key; let us refer to this mapping as $f : \mathcal{Y} \to \{1, \ldots, k\}$. We may study the recognition system as more data is added over time: let $t = 1, 2, \ldots$ refer to discrete time stages, and let k_t denote the number of classes in the database at time t, and f_t denote the mapping from queries to answers at time t; also define $n_{i,t}$ analogously.

2.1 Separability

The property of separability captures the intuitive notion that *information is not shared between classes*. To formalize the notion of separability, we begin with the class of *scoring rule*-based classifier. A scoring rule $\mathcal{Q}$ is a mapping from a set of measurements $\{y^1, \ldots, y^n\}$, for $n \in \{0, 1, \ldots, \}$, to a function $q : \mathcal{Y} \to \mathbb{R}$. A scoring-based classifier constructs

$$q^{(i)} = \mathcal{Q}(y^{(i),1}, \ldots, y^{(i),n_i})$$

for each $i = 1, \ldots, k$, and then defines

$$f^{(t)}(y) = \text{argmax}_i q^{(i)}(y).$$

If we assume that the parameter vectors are observed, then the scoring rule may also take x as an argument: $q^{(i)} = \mathcal{Q}(x^{(i)}, y^{(i),1}, \ldots, y^{(i),n_i})$. For notational convenience, we assume that ties occur with probability zero: that is, $\mathcal{Q}$ satisfies the tie-breaking property

$$\Pr[\mathcal{Q}(x, S(x))(Y) = \mathcal{Q}(x, S(x))(Y')] = 0 \text{ for } Y, Y' \overset{iid}{\sim} p(y|x), \tag{1}$$

Quadratic discriminant analysis and Naive Bayes are two examples of scoring-based classifiers. For QDA, the scoring rule is given by

$$\mathcal{Q}_{QDA}(y^1, \ldots, y^n)(y) = -(y - \bar{y})^T S^{-1} (y - \bar{y})$$

where $\bar{y} = \frac{1}{n} \sum_{i=1}^n y^i$ and $S = \frac{1}{n} \sum_{i=1}^n (y^i - \bar{y})^T (y^i - \bar{y})$. In Naive Bayes, the scoring rule is

$$\mathcal{Q}_{NB}(y^1, \ldots, y^n)(y) = \sum_{i=1}^n \log \hat{f}_i(y_i)$$

where $\hat{f}_i$ is a density estimate obtained from $\{y_i^1, \ldots, y_i^n\}$.

A *separable* classifier is a classifier which can be *approximated* by a scoring-based classifier in a certain asymptotic sense.

Definition 2.1. (Separability) Suppose that for time stages $t = 1, 2, \ldots$, $k_t = O(t)$ and $n_{i,t} = O(t)$, with $x^{(1)}, x^{(2)}, \ldots$ drawn i.i.d. from $p(x)$, and $y^{(i),1}, \ldots$ drawn i.i.d. from $p(y|x^{(i)})$ for each $i = 1, 2, \ldots$. A recognition system (characterized by mappings f_t) is considered *separable* if and only there exists a scoring rule $\mathcal{Q}$ such that defining

$$\tilde{f}_t(y) = \text{argmax}_{i=1}^{k_t} \mathcal{Q}(x^{(i)}, y^{(i),1}, \ldots, y^{(i),n_t})(y)$$

we have

$$\lim_{t\to\infty} \frac{1}{k_t} \sum_{i=1}^{k_t} \Pr[f_t(y) = \tilde{f}_t(y)|y \sim p(y|x^{(i)})] \to 1.$$

We will show that certain implementations of k-nearest neighbors, LDA, one-vs-one, or one-vs-all classifiers satisfy this definition of separability.

3 Prediction Extrapolation

3.1 Problem formulation

Recall the notation from section 2.1. In order to formalize the problem of *extrapolation*, we model the data collection process as a stochastic process. Let $(\Omega, \mathcal{F}, \mathbb{P})$ define a probability space. Let $t = 1, 2, 3, \ldots$ index discrete time steps; each time step is associated with a filtration $\mathcal{F}_t$, with $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}$. At time zero, we have not observed any data. At time k, we sample a new species $x^{(k)}$ from the population distribution $p(x)$, and also observe r replicates $y^{(i),1}, \ldots, y^{(i),r}$ from the conditional distribution $p(y|x^{(i)})$. Choose some $r_1 < r$: this determines the number of training observations $S(x^{(i)}) = \{y^{(i),1}, \ldots, y^{(i),r_1}\}$ from each class. The filtration at time t is defined as the σ-algebra induced by observations from species $x^{(1)}, \ldots, x^{(k)}$, hence $\mathcal{F}_t = \sigma\{(x^{(i)}, y^{(i),j})\}_{i=1,j=1}^{t,r}$.

The classifier also changes with time. At time t, $f^{(t)}$ is (random) function from $\mathcal{Y}$ to $\{x^{(1)}, \ldots, x^{(k)}\}$. The randomness is due to the variability of the training set, hence $f^{(t)}$ is independent of the test data $\{y^{(i),r_1+1}, \ldots, y^{(i),r}\}$ for $i = 1, \ldots, t$. Furthermore, we assume that the classifier $f^{(t)}$ is constructed in the following way. The user chooses an algorithm $\mathcal{Q}$ which constructs scoring functions $q^{(i)}$ from the training data for the ith class:

$$q^{(i)} = \mathcal{Q}(x^{(i)}, S(x^{(i)})).$$

Each $q^{(i)}$ is a function from $\mathcal{Y}$ to $\mathbb{R}$. The classifier $f^{(t)}$ is defined

$$f^{(t)}(y) = \text{argmax}_i q^{(i)}(y).$$

The generalization accuracy at time t is defined

$$\text{acc}^{(t)} = \frac{1}{k} \sum_{i=1}^{k} \Pr[f^{(t)}(y) = i|y \sim p(y|x^{(i)})].$$

The extrapolation problem is the problem of predicting $\text{acc}^{(K)}$ using only information known at time $k < K$.

3.2 Conditional accuracy

The optimal predictor of the generalization error (in mean square) is the conditional expected generalization error, $\mathbf{E}[\text{acc}^{(t)}|\mathcal{F}_k]$. However, it is easier to work with the unconditional expected generalization error $p_t \stackrel{def}{=} \mathbf{E}[\text{acc}^{(t)}]$.

Define the *conditional accuracy* function $u(x, y, S(x))$ which maps a data triple consisting of a species x, a *test* observation y, and a set of r_1 training replicates $S(x) = \{y^1, \ldots, y^{r_1}\}$ to a real number in $[0, 1]$. The conditional accuracy gives the probability that for independently drawn

$(X, S(X))$ such that $X \sim p(x)$, and $S(X) = \{Y^1, \ldots, Y^{r_1}\}$ with $Y^i \overset{iid}{\sim} p(y|X)$, that the scoring function $\mathcal{Q}(x, S(x))$ will give a higher score to y than the scoring function $\mathcal{Q}(X, S(X))$:

$$u(x, y, S(x)) = \Pr[(\mathcal{Q}(x, S(x)))(y) > (\mathcal{Q}(X, s(X)))(y)].$$

Define the *conditional accuracy* distribution μ as the law of $u(X, Y, S(X))$ when $X \sim p(x)$, $Y \sim p(y|X)$, and $S(X)$ is drawn as specified above. The significance of the conditional accuracy distribution is that the expected generalization error p_t can be written in terms of its moments.

Theorem 3.1. *Let U be defined as the random variable*

$$U = u(X, Y, S(X))$$

for X, Y drawn from $p(x, y) = p(x)p(y|x)$, and $S(X) = \{Y^1, \ldots, Y^{r_1}\}$ with $Y^i \overset{iid}{\sim} p(y|X)$ Then $p_k = \boldsymbol{E}[U^{k-1}]$.

Proof. Write $q^{(i)} = \mathcal{Q}(x^{(i)}, S(x^{(i)}))$ Note that by using conditioning and conditional independence, p_k can be written

$$\begin{aligned}
p_k &= \mathbf{E}\left[\frac{1}{k}\sum_{i=1}^{k} \Pr[q^{(i)}(Y) > \max_{j \neq i} q^{(j)}(Y)]\right] \\
&= \mathbf{E}\left[\Pr[q^{(1)}(Y) > \max_{j \neq 1} q^{(j)}(Y)]\right] \\
&= \mathbf{E}[\Pr[q^{(1)}(Y) > \max_{j \neq 1} q^{(j)}(Y) | X^{(1)}, Y, S(X^{(1)})]] \\
&= \mathbf{E}[\Pr[\cap_{j>1} q^{(1)}(Y) > q^{(j)}(Y) | X^{(1)}, Y, S(X^{(1)})]] \\
&= \mathbf{E}[\prod_{j>1} \Pr[q^{(1)}(Y) > q^{(j)}(Y) | X^{(1)}, Y, S(X^{(1)})]] \\
&= \mathbf{E}[\Pr[q^{(1)}(Y) > q^{(2)}(Y) | X^{(1)}, Y, S(X^{(1)})]^{k-1}] \\
&= \mathbf{E}[u(X^{(1)}, Y, S(X^{(1)}))^{k-1}].
\end{aligned}$$

□

Theorem 3.1 tells us that the problem of extrapolation can be approached by attempting to estimate the conditional accuracy distribution. The $(t-1)$th moment of U gives us p_t, which will in turn be a good estimate of $e^{(t)}$.

3.3 Properties of the conditional accuracy distribution

The conditional error distribution μ is determined by $p(x, y)$ and $\mathcal{Q}$. What can we say about the the conditional accuracy distribution without making any assumptions on either $p(x, y)$ or $\mathcal{Q}$? The answer is: not much–for an arbitrary probability measure ν on $[0, 1]$, one can construct $p(x, y)$ and $\mathcal{Q}$ such that $\mu = \nu$.

Theorem 3.2. *Let U be defined as in Theorem 2.1, and let μ denote the law of U. Then, for any probability distribution ν on $[0, 1]$, one can construct a joint distribution $p(x, y)$ and a scoring rule $\mathcal{Q}$ such that $\mu = \nu$.*

Proof. Let X and Y have the degenerate joint distribution $X = Y \sim Unif[0, 1]$. Let G be the cdf of ν, $G(x) = \int_0^x d\nu(x)$, and let $H(u) = \sup_x\{G(x) \leq u\}/$. Define $\mathcal{Q}$ by

$$(\mathcal{Q}(x, S(x)))(y) = \begin{cases} 0 & \text{if } x > y + H(y) \\ 0 & \text{if } y + H(y) > 1 \text{ and } x \in [H(y) - y, y] \\ 1 + x - y & \text{if } x \in [y, y + H(y)] \\ 1 + y + x & \text{if } y + H(y) > 1 \text{ and } x \in [0, H(y) - y]. \end{cases}$$

One can verify that $u(x, x, S(x)) = H(u)$. Therefore, the cdf of U is equal to G, as needed. □

In practice, however, the scoring rule $\mathcal{Q}$ must approximate a monotonic function of the conditional density $p(y|x)$ in order to yield an effective classifier. It is therefore notable that in the case that

(X, Y) have a density with respect to Lesbegue measure, taking an *optimal* scoring rule, with the property that $\mathcal{Q}(x, y, S(x)) = g(p(y|x))$ for monotonic g, the distribution of U has a monotonically increasing density.

Theorem 3.1. *Let U be defined as in Theorem 2.2, and let μ denote the law of U. Suppose (X, Y) has a density $p(x, y)$ with respect to Lebesgue measure on $\mathcal{X} \times \mathcal{Y}$, and $\mathcal{Q}(x, S(x))$ satisfies the property of monotonicity*

$$p(y|x) > p(y'|x) \text{ implies } \mathcal{Q}(x, S(x))(y) > \mathcal{Q}(x, S(x))(y')$$

and the property of tie-breaking (1)*, then μ has a density $\eta(u)$ on $[0, 1]$ which is monotonic in u.*

Proof. Choose $0 < u < v < 1$ and $0 < \delta < \min(u, 1 - v, v - u)$. For $x \in \mathcal{X}$, define the set

$$\underline{J}_x = \{y \in \mathcal{Y} : \int_{\mathcal{Y}} I(p(y|x) > p(w|x))p(w|x)dw \in [u - \delta, u + \delta]\}$$

and

$$\bar{J}_x = \{y \in \mathcal{Y} : \int_{\mathcal{Y}} I(p(y|x) > p(w|x))p(w|x)dw \in [v - \delta, v + \delta]\}$$

One can verify that for all $x \in \mathcal{X}$,

$$\int_{\underline{J}_x} p(y|x)dy \leq \int_{\bar{J}_x} p(y|x)dy.$$

Yet, since

$$\Pr[U \in [u - \delta, u + \delta]] = \Pr[\cup_{\mathcal{X}} x \times \underline{J}_x]$$
$$\Pr[U \in [v - \delta, v + \delta]] = \Pr[\cup_{\mathcal{X}} x \times \underline{J}_x].$$

we obtain

$$\Pr[U \in [u - \delta, u + \delta]] \leq \Pr[U \in [v - \delta, v + \delta]].$$

Taking $\delta \to 0$, we conclude the theorem. □

4 Nonparametric Estimation

Let us assume that U has a density $\eta(u)$. While $U = u(x, y)$ cannot be directly observed, we can estimate $u(x^{(i)}, y^{(i),r_1+j}, S(x^{(i)}))$ for any $1 \leq i \leq k$, $1 \leq j \leq r_2$, where $r_2 = r - r_1$ is the number of testing repeats.

Theorem 4.1 *Assume that $\mathcal{Q}$ satisfies the tie-breaking property* (1)*. Define*

$$V_{i,j} = \sum_{i=1}^{k} I(q^{(i)}(y^{(i),j}) > q^{(j)}(y^{(i),j})).$$

Then

$$V_{i,j} \sim \textit{Binomial}(k, u(x^{(i)}, y^{(i),j}, S(x^{(i)}))).$$

□

The proof of Theorem 4.1 follows from the same conditioning argument used in Theorem 3.1.

At a high level, we have a hierarchical model where U is drawn from a density $\eta(u)$ on $[0, 1]$ and then $V_{i,j} \sim \text{Binomial}(k, U)$; therefore the marginal distribution of $V_{i,j}$ can be written

$$\Pr[V_{i,j} = \ell] = \binom{k}{\ell} \int_0^1 u^\ell (1 - u)^{k-\ell} \eta(u)du.$$

However, the observed $\{V_{i,j}\}$ do *not* comprise an i.i.d. sample.

We discuss the following three approaches for estimating $p_t = \mathbf{E}[U^{t-1}]$ based on $V_{i,j}$. The first is *unbiased estimation* based on binomial U-statistics, which is discussed in Section 4.1. The second is the *psuedolikelihood* approach. In problems where the marginal distributions are known, but the dependence structure between variables is unknown, the *psuedolikelihood* is defined as the product of the marginal distributions. For certain problems in time series analysis and spatial statistics,

the maximum psuedolikelihood estimator (MPLE) is proved to be consistent (CITE). We discuss psuedolikelihood-based approaches in Sections 4.2 and 4.3. A third approach is an adaptation of the *mutual information estimator* developed by (Anon 2016). Anon 2016 develop an asymptotic theory which relates the Bayes error to the mutual information $I(X;Y)$ and vice versa. This allows us to estimate "information" from classification error p_k, and then predict the generalization error p_t from the estimated information: details are given in Section 4.4.

4.1 Unbiased estimation

If $V \sim \text{Binomial}(k, \eta)$, then an unbiased estimator $f_t(V)$ of η^t exists if and only if $0 \leq t \leq k$.

The theory of U-statistics provides the minimal variance unbiased estimator for η^t:

$$\eta^t = \mathbf{E}\left[\frac{\binom{V}{t}}{\binom{k}{t}}\right].$$

This result can be immediately applied to yield an unbiased estimator of p_t, when $t \leq k$:

$$p_t = \mathbf{E}\left[\frac{1}{kr_2}\sum_{i=1}^{k}\sum_{j=1}^{r_2}\frac{\binom{V_{i,j}}{t}}{\binom{k}{t}}\right]. \tag{2}$$

The problem of *extrapolation* concerns the case $t > k$, in which the expression (2) is undefined. Still, the estimator (2) is worthy of study, since it has close to optimal performance for the case $t \leq k$.

4.2 Maximum pseudo-likelihood

For fixed j the quantities $\{V_{1,j}, \ldots, V_{k,j}\}$ are mutually independent, and one can write a nonparametric likelihood for the density $\eta(u)$ of U:

$$\mathcal{L}_j(\eta) = \prod_{i=1}^{k}\binom{k}{V_{i,j}}\eta(u)^{V_{i,j}}(1-\eta(u))^{k-V_{i,j}}.$$

However, it is not possible to write a likelihood for $\eta(u)$ depending on all the terms $V_{i,j}$,

4.3 Constrained pseudo-likelihood

4.4 Information-based methodology

[Mostly copy and paste, needs fixing]

We start by restating the results of ZB 2016. The asymptotic regime considered is a sequence of joint distributions $p(x, y)$ where the dimensionality of x goes to infinity. A specific example of a sequence in this regime is one where X is d-dimensional multivariate normal with covariance identity I_d, and $Y = X + E$, where E is an independent multivariate normal with covariance cdI_d, for some fixed constant $c > 0$.

Theorem 2. *Let $p^{[d]}(x, y)$ be a sequence of joint densities for $d = 1, 2, \ldots$ as given above. Further assume that*

- *A1.* $\lim_{d\to\infty} I(X^{[d]}; Y^{[d]}) = \iota < \infty$.
- *A2. There exists a sequence of scaling constants $a_{ij}^{[d]}$ and $b_{ij}^{[d]}$ such that the random vector $(a_{ij}\ell_{ij}^{[d]} + b_{ij}^{[d]})_{i,j=1,\ldots,K}$ converges in distribution to a multivariate normal distribution.*
- *A3. There exists a sequence of scaling constants $a^{[d]}$, $b^{[d]}$ such that*

  $$a^{[d]}u(X^{(1)}, Y^{(2)}) + b^{[d]}$$

  converges in distribution to a univariate normal distribution.

A4. For all $i \neq k$,

$$\lim_{d\to\infty} Cov[u(X^{(i)}, Y^{(j)}), u(X^{(k)}, Y^{(j)})] = 0.$$

Then for p_K as defined above, we have

$$\lim_{d\to\infty} 1 - p_K = \pi_K(\sqrt{2\iota})$$

where

$$\pi_K(c) = 1 - \int_{\mathbb{R}} \phi(z-c)\Phi(z)^{K-1}dz$$

where ϕ and Φ are the standard normal density function and cumulative distribution function, respectively.

By combining Theorems 1 and 2, we immediately compute the limiting distribution of P in the given regime **Corollary.***Let $p^{[d]}(x, y)$ be a sequence of joint densities satisfying A1-A4 as stated in Theorem 2. For any d, let $P^{[d]}$ as defined in Theorem 1. Then $P^{[d]}$ converges in distribution to P, where the cdf of P is given by*

$$\Pr[P < t] = \int_0^t \frac{\phi(\Phi^{-1}(u) - \sqrt{2\iota})}{\phi(\Phi^{-1}(u))}du.$$

Proof. By Theorem 1, the moments of $P^{[d]}$ are given by

$$\mathbf{E}[P^{[d]k-1}] = p_k^{[d]}$$

and meanwhile, Theorem 2 implies that

$$\lim_{d\to\infty} p_k^{[d]} = \int_{\mathbb{R}} \phi(z - \sqrt{2\iota})\Phi(z)^{k-1}dz.$$

Let Z be a normal $N(\sqrt{2\iota}, 1)$ variate, and define $P = \bar{\Phi}(Z)$. Then it is clear that

$$\lim_{d\to\infty} \mathbf{E}[P^{[d]k-1}] = \int_{\mathbb{R}} \phi(z - \sqrt{2\iota})\Phi(z)^{k-1}dz = \mathbf{E}[P^{k-1}]$$

for all k. Since both $P^{[d]}$ and P lie in the compact interval $[0, 1]$, the fact that the moments of $P^{[d]}$ converge to the moments of P implies that the distribution of $P^{[}d]$ converges to the distribution of P. $\square$.

The corollary identifies a parametric family of distributions $\mathcal{P} = \{P_\iota\}$ indexed by the mutual information ι. For given ι, the density of P_ι if given by

$$g_\iota(u) = \frac{\phi(\Phi^{-1}(u) - \sqrt{2\iota})}{\phi(\Phi^{-1}(u))}.$$

Note the special case $\iota = 0$, which yields $P_0 = U$, the uniform distribution on $[0, 1]$. This implies that in the special case that X is independent of Y, and hence optimal classification does no better than random guessing, $p_k = \frac{1}{k}$, which indeed matches the moments of the uniform distribution

$$\mathbf{E}[U^{k-1}] = \int_0^1 u^{k-1}du = \frac{1}{k}.$$

We see that for any given finite-dimensional joint distribution $p(x, y)$, if the distribution of P lies close to a member of the parametric family $\mathcal{P}$, the information-theoretic methodology for estimating p_N from p_k will be accurate.

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