Supervised Evaluation of Representations

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What is a representation?

- It is often hypothesized that *complex objects* (images, sounds, etc.) can be mapped into a *low-dimensional* space
- A *representation* is a nonlinear, dimensionality-reducing mapping

Why do we care?

- Application 1: Neuroscience. Different regions of the brain have different representations of the same sensory stimuli.
- Application 2: Machine learning. A good representation leads to more accurate prediction.

Example: Facial recognition

- Used to tag images in software, security

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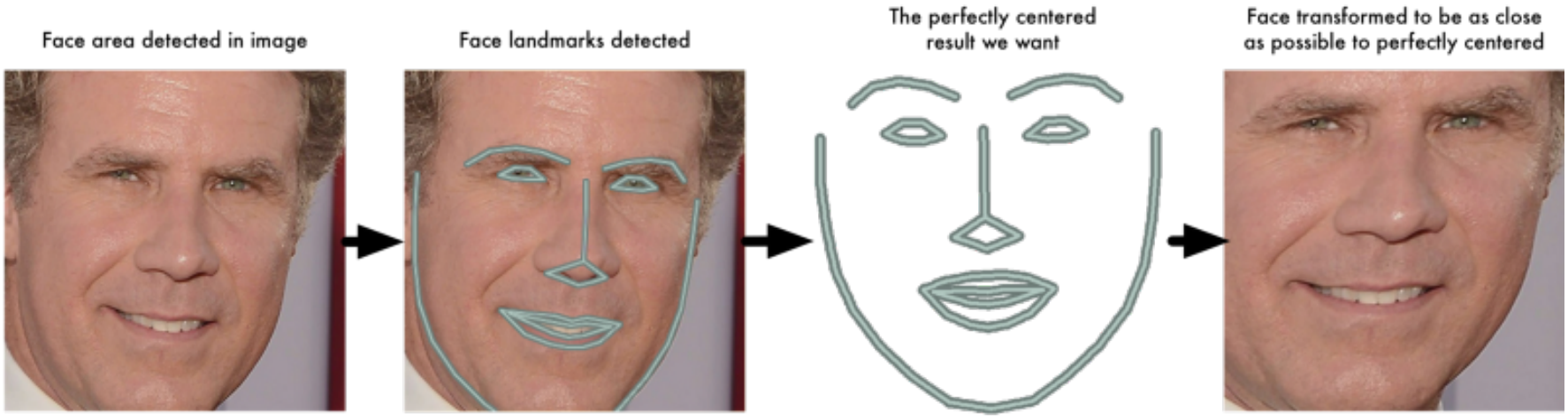

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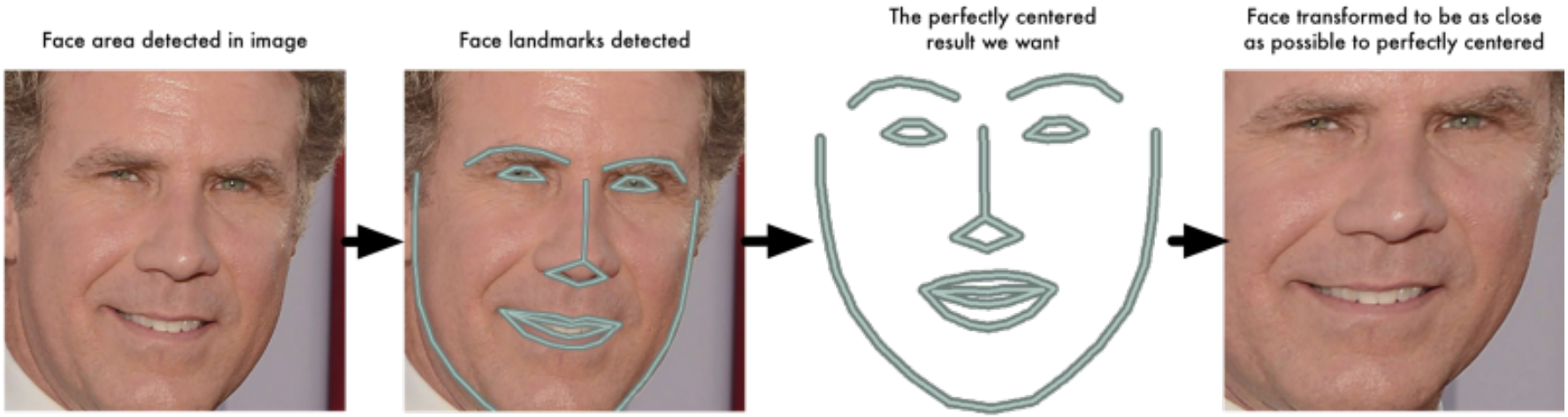
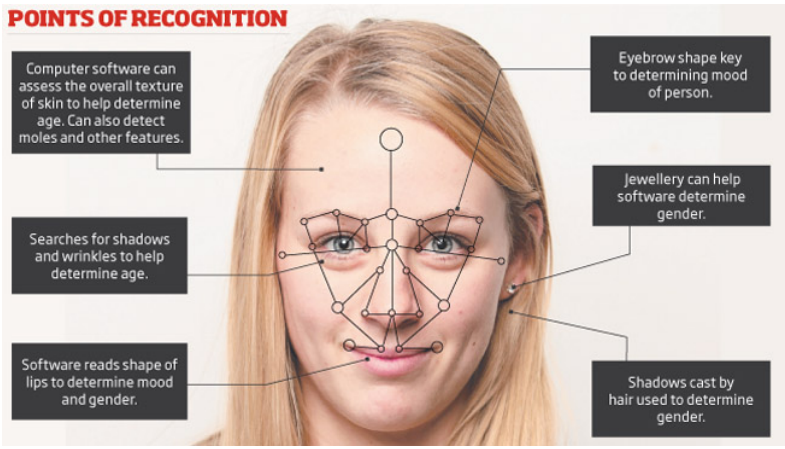


- Feature extraction

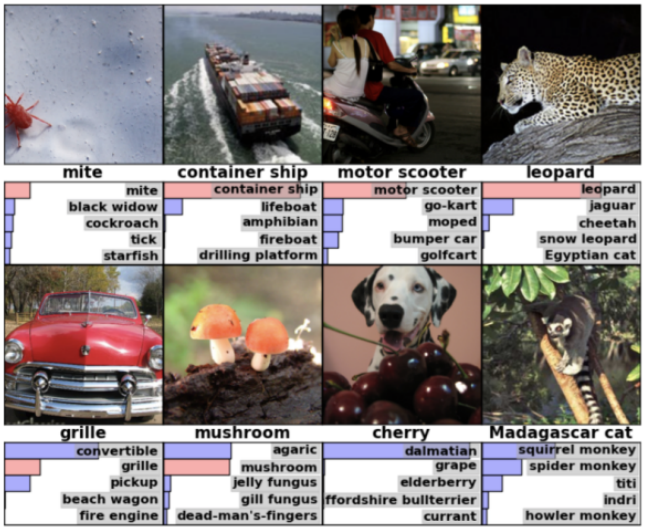


What defines a good representation?

Section 2

Randomized classification and extrapolation

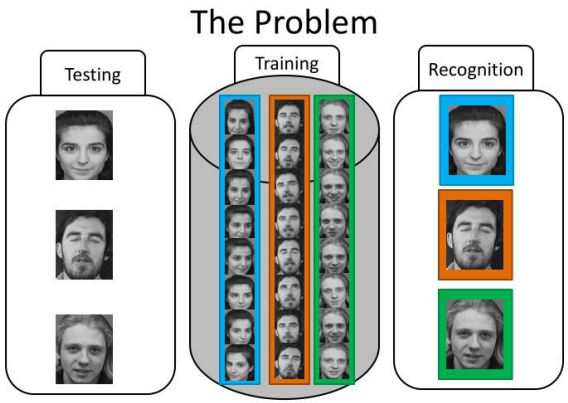
Multi-class classification



- MNIST digit recognition: 10 categories
- Human motion database: 51 categories
- ImageNet: 22,000 categories
- Wikipedia: 325,000 categories

from Krizhevsky et al. 2012

Accuracy vs. number of classes



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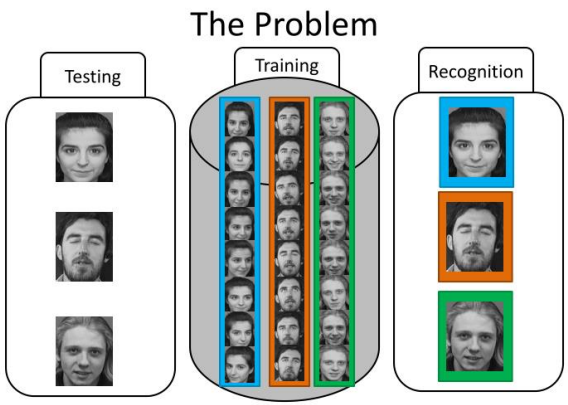

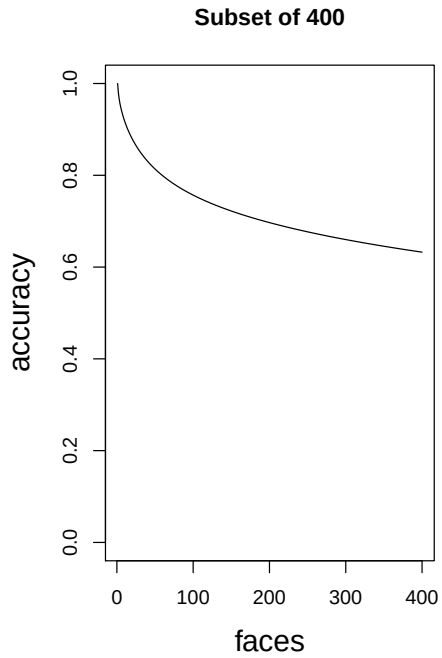

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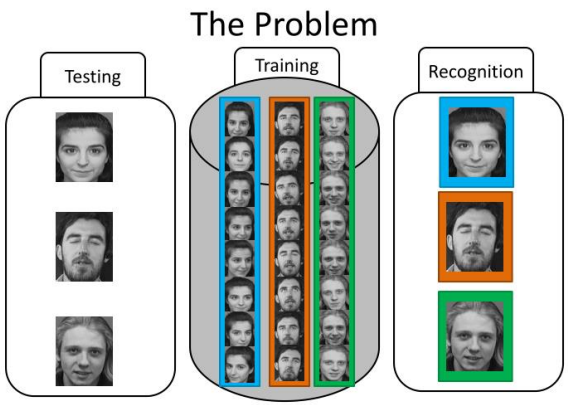
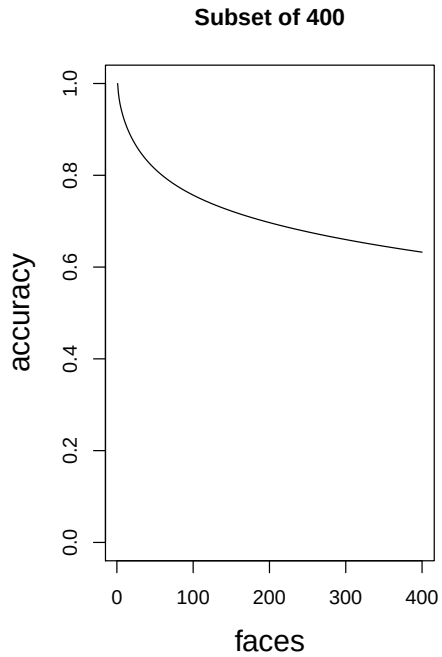
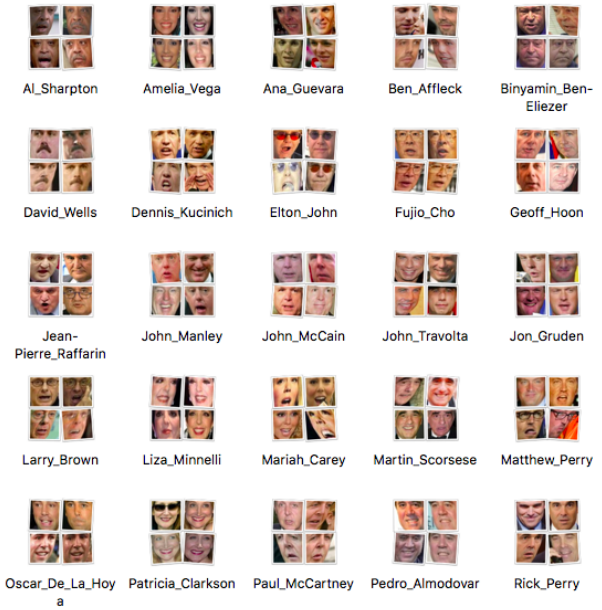




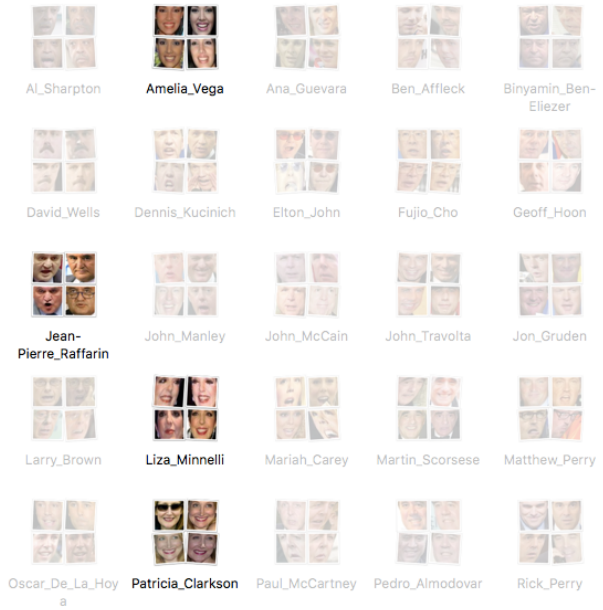
How does the accuracy scale with the number of classes (faces)?

Setup

1. Population of categories $\pi(y)$

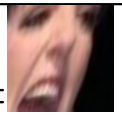
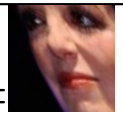
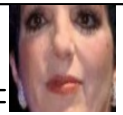


2. Subsample k labels, $y_1, \ldots, y_k$



Setup

3. Collect training and test data $x_j^{(i)}$ (faces) for labels (people) $\{y_1, \ldots, y_k\}$.

Label	Training			Test
y_1=Amelia	$x_1^{(1)} =$ 	$x_1^{(2)} =$ 	$x_1^{(3)} =$ 	$x_1^* =$ 
y_2=Jean-Pierre	$x_2^{(1)} =$ 	$x_2^{(2)} =$ 	$x_2^{(3)} =$ 	$x_2^* =$ 
y_3=Liza	$x_3^{(1)} =$ 	$x_3^{(2)} =$ 	$x_3^{(3)} =$ 	$x_3^* =$ 
y_4=Patricia	$x_4^{(1)} =$ 	$x_4^{(2)} =$ 	$x_4^{(3)} =$ 	$x_4^* =$ 

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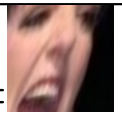
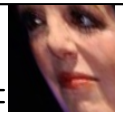
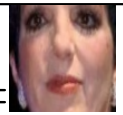
y_3=Liza	$x_3^{(1)} =$ 	$x_3^{(2)} =$ 	$x_3^{(3)} =$ 	$x_3^* =$ 
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Can we analyze how error depends on k?

Key assumption: marginal classifier

- The classifier is *marginal* if it learns a model *independently* for each class.

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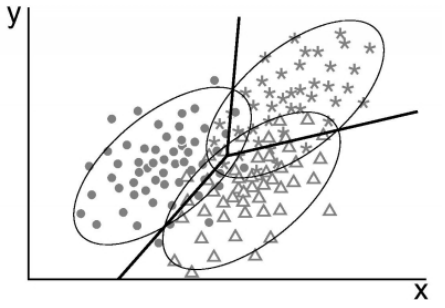
- Examples: LDA/QDA 

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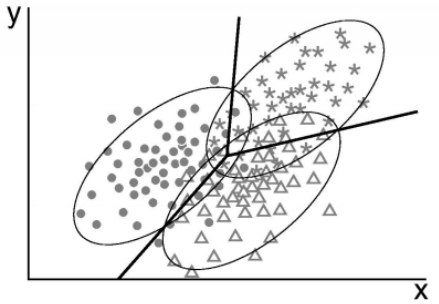
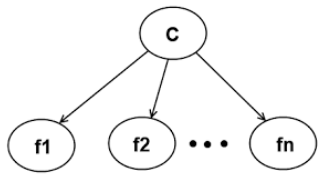
- Examples: LDA/QDA 

, naïve Bayes 

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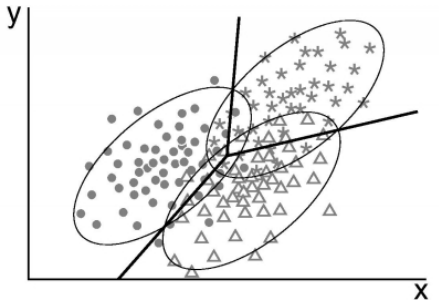
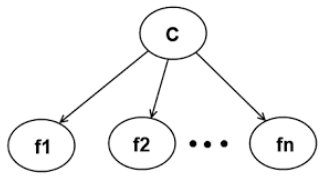
- Examples: LDA/QDA 

, naïve Bayes 

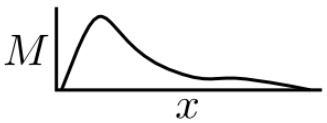
- Non-marginal classifiers: Multinomial logistic, multilayer neural networks, k-nearest neighbors

Definitions

$\hat{F}_{y^{(i)}}$ is the empirical distribution obtained from the training data for label $y^{(i)}$.

Classification Rule

$$M_{y^{(1)}}(x) = \mathcal{M}(\hat{F}_{y^{(1)}})(x)$$



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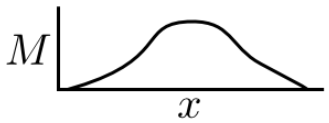

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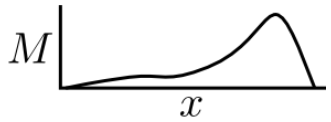
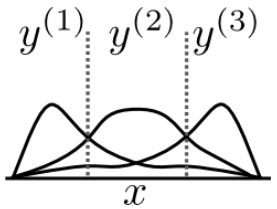
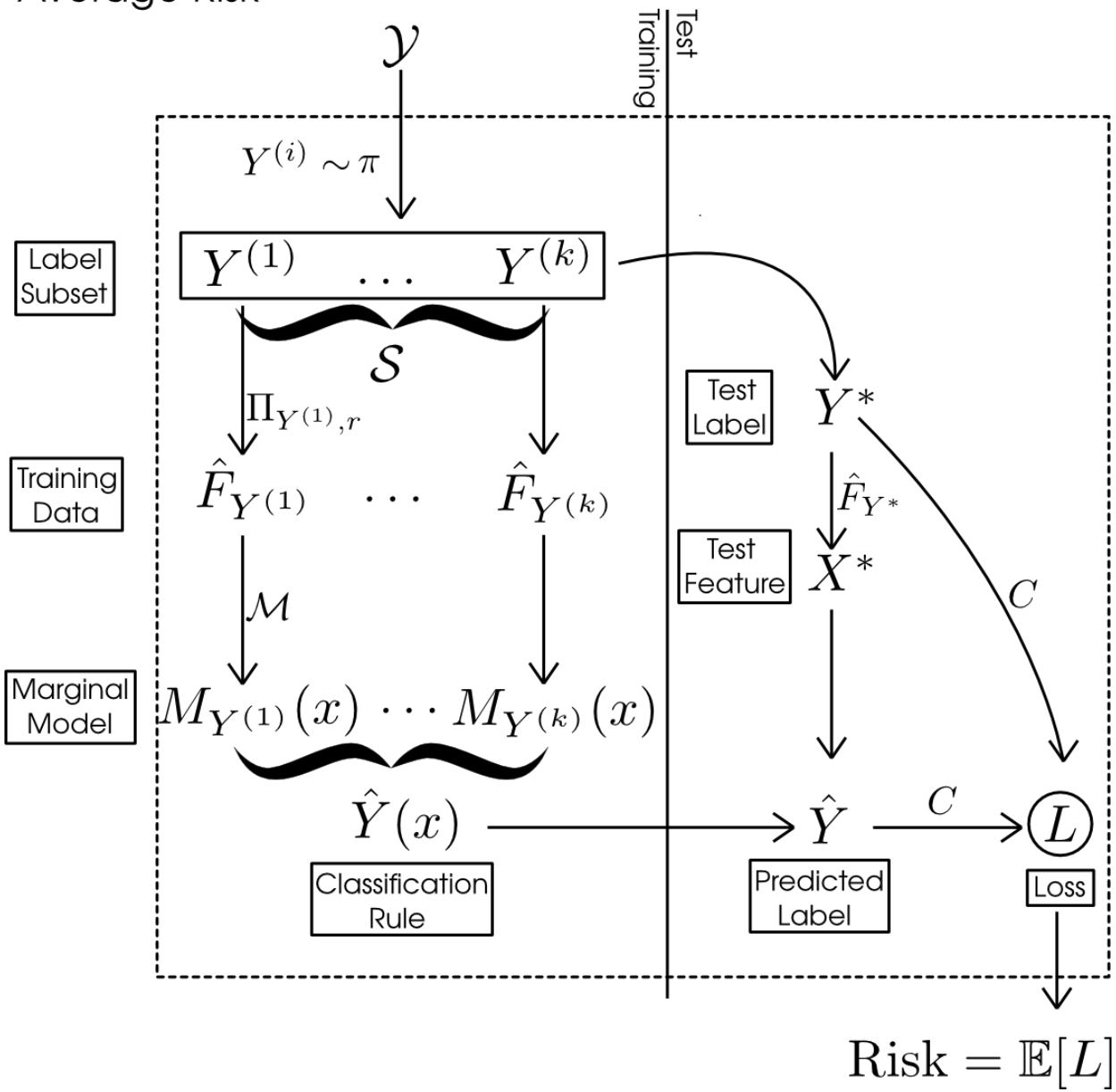


$$\hat{Y}(x) = \text{argmax}_{y \in \mathcal{S}} M_y(x)$$



Average Risk



Theoretical Result

Theorem. (Z., Achanta, Benjamini.) Suppose π, $\{F_y\}_{y \in \mathcal{Y}}$ and marginal classifier $\mathcal{F}$ satisfy *(some regularity condition)*. Then, there exists some function $\bar{D}(u)$ on $[0,1] \to [0,1]$ such that the k-class average risk is given by

$$\text{AvRisk}_k = (k-1) \int \bar{D}(u) u^{k-2} du.$$

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What is this $\bar{D}(u)$ function? We will explain in the following toy example...

Toy example

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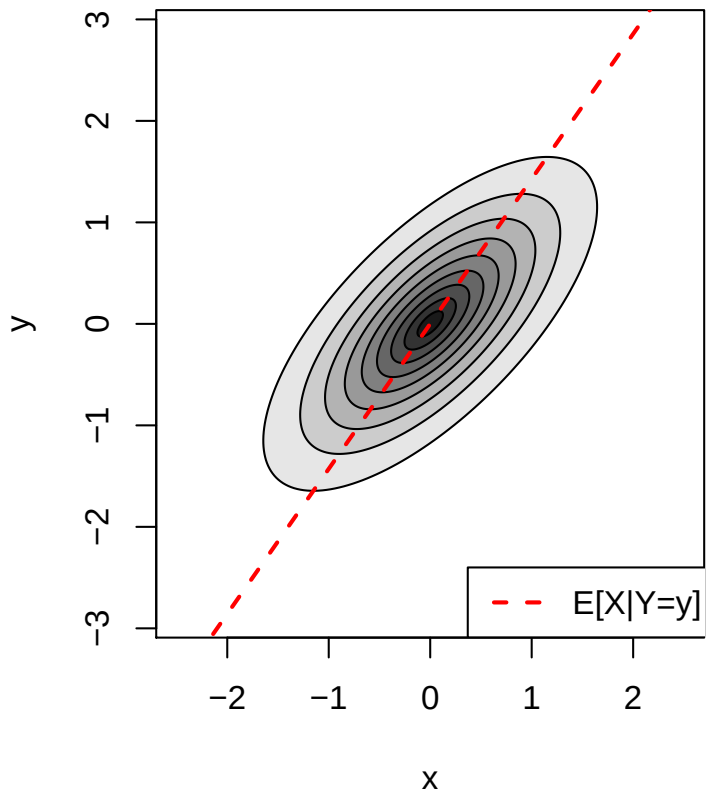
$X|Y \sim N(\rho Y, 1-\rho^2)$ i.e. $(Y, X) \sim N(0, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix})$.



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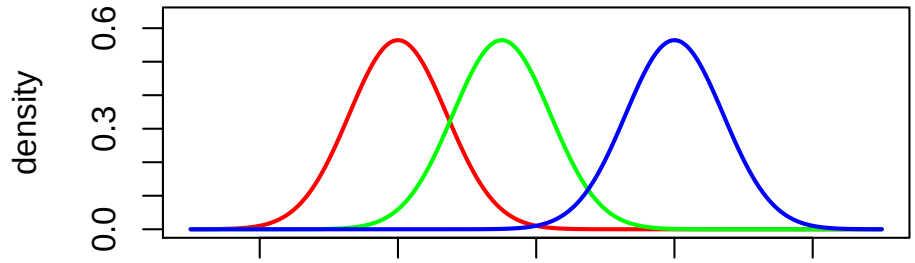
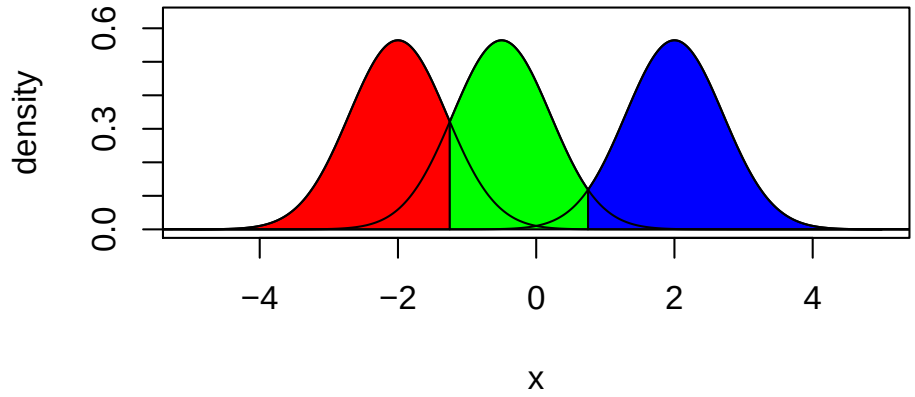


- Suppose $k = 3$, and we draw Y_1, Y_2, Y_3.
- The *Bayes rule* is the optimal classifier and depends on knowing the true densities:

$$\hat{y}(x) = \text{argmax}_{y_i} p(x|y_i)$$

- The *Bayes Risk*, which is the misclassification rate of the optimal classifier.

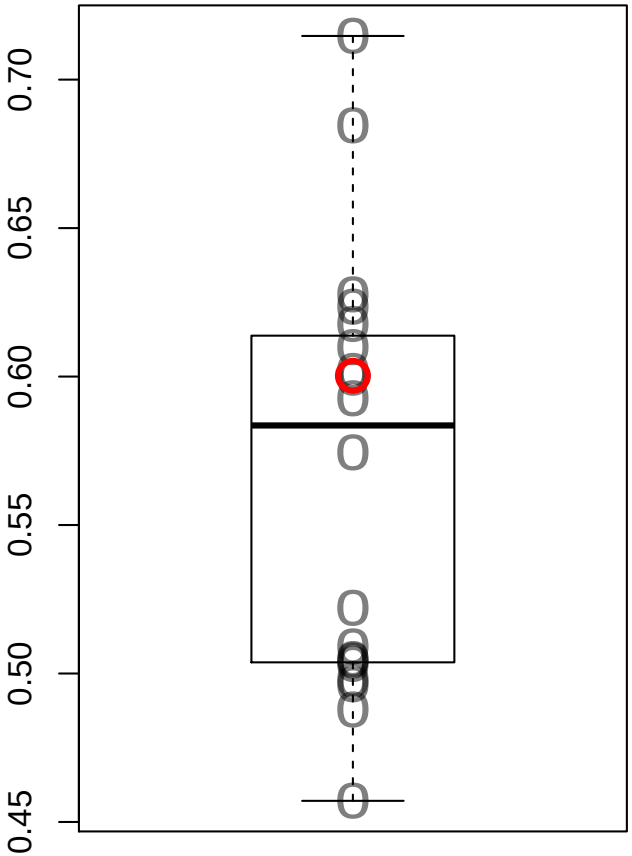
Toy example

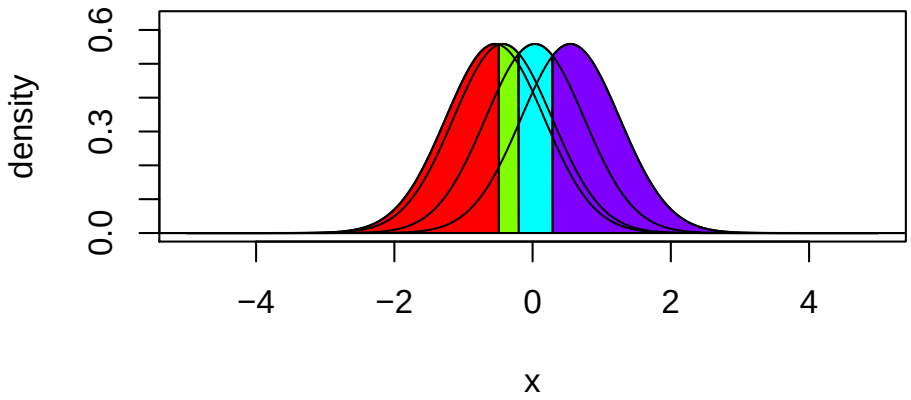


- The *Bayes Risk* is the expected test error of the Bayes rule,

$$\frac{1}{k}\sum_{i=1}^{k}\Pr[\hat{y}(x) \neq Y | Y = y_i]$$

Toy example





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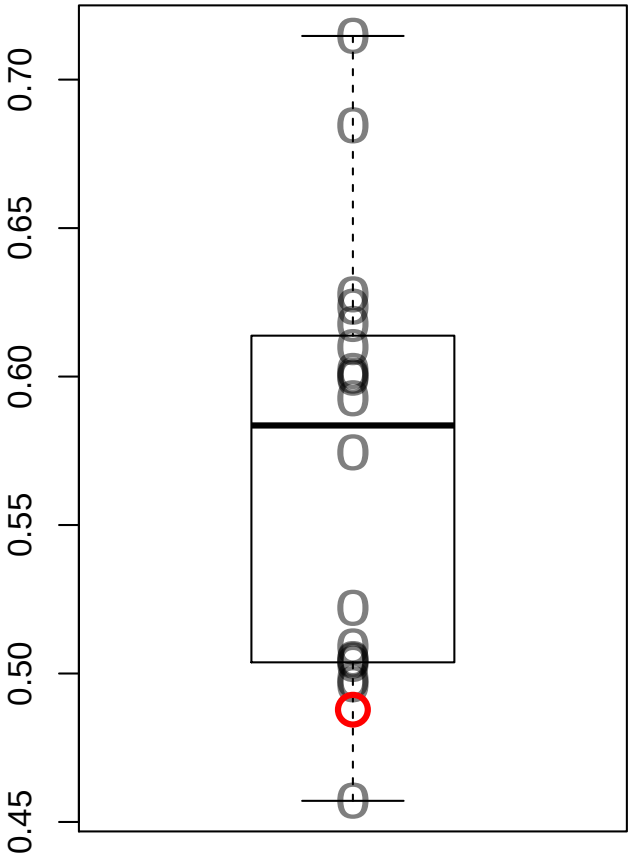

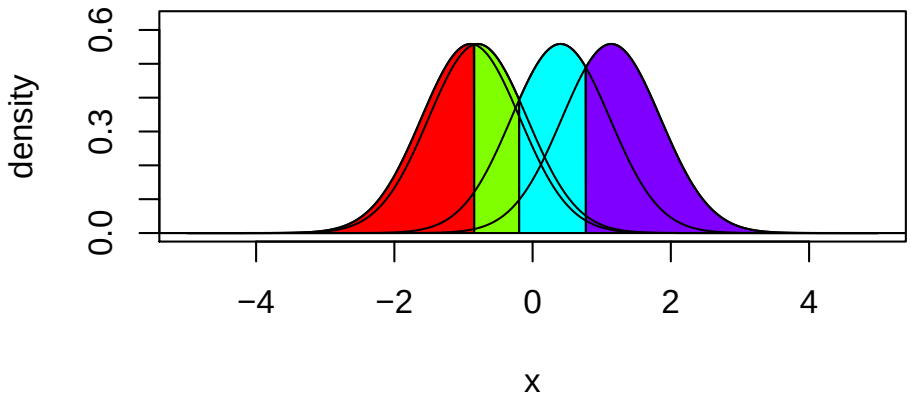

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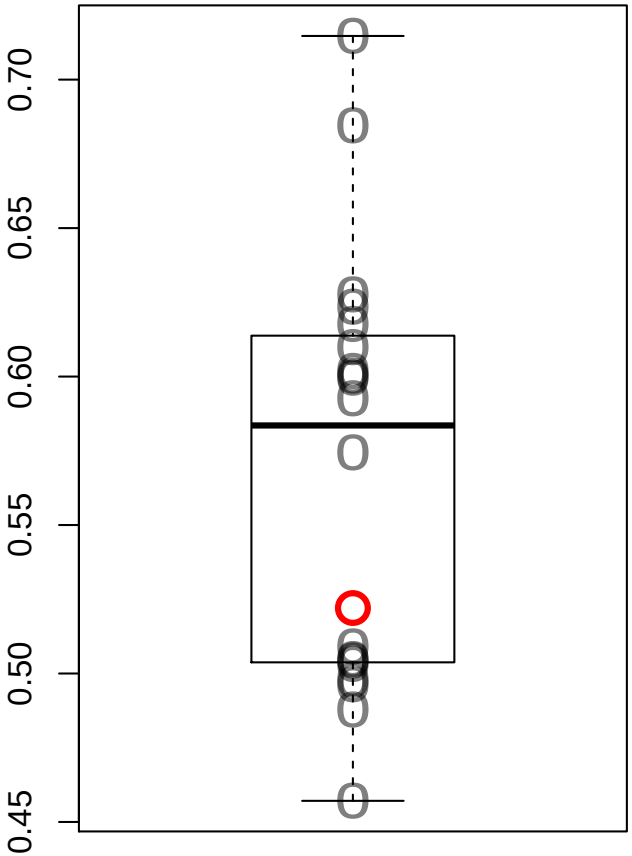

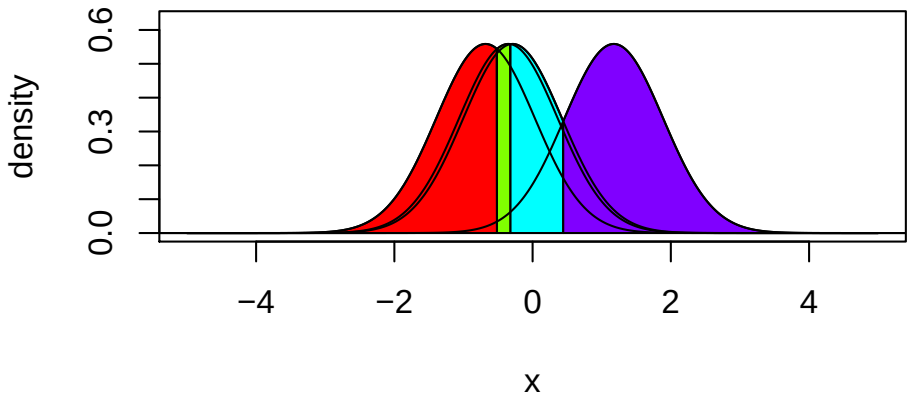

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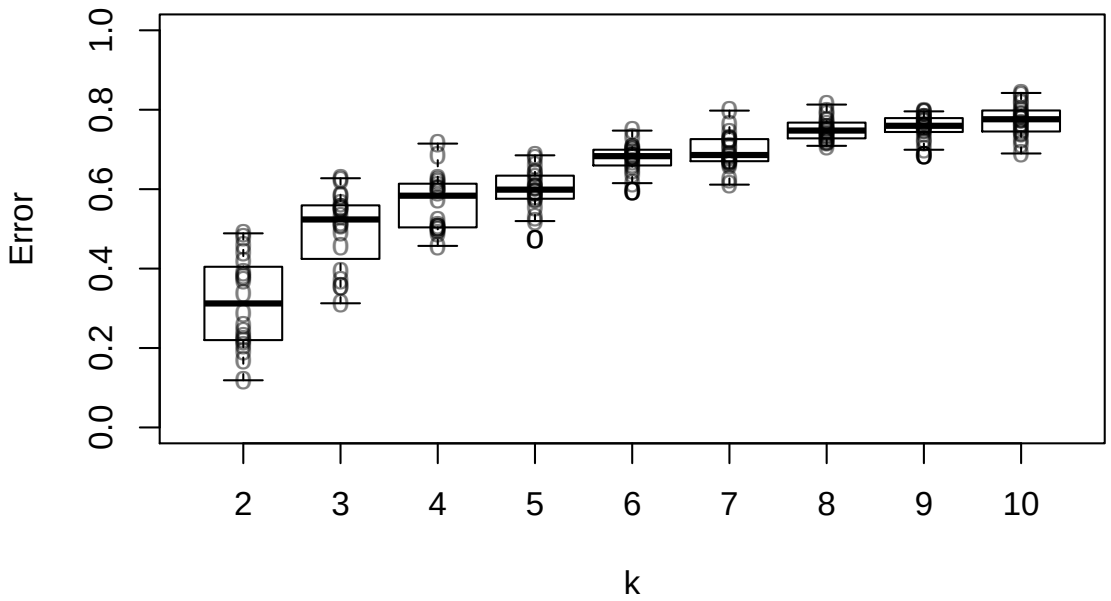

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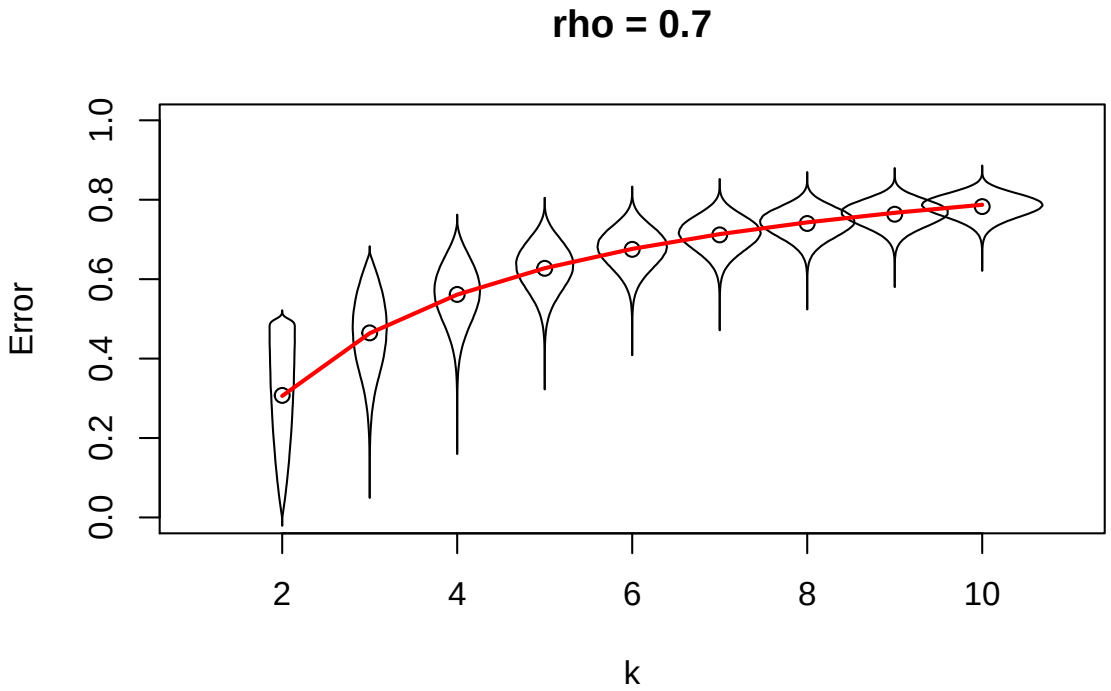
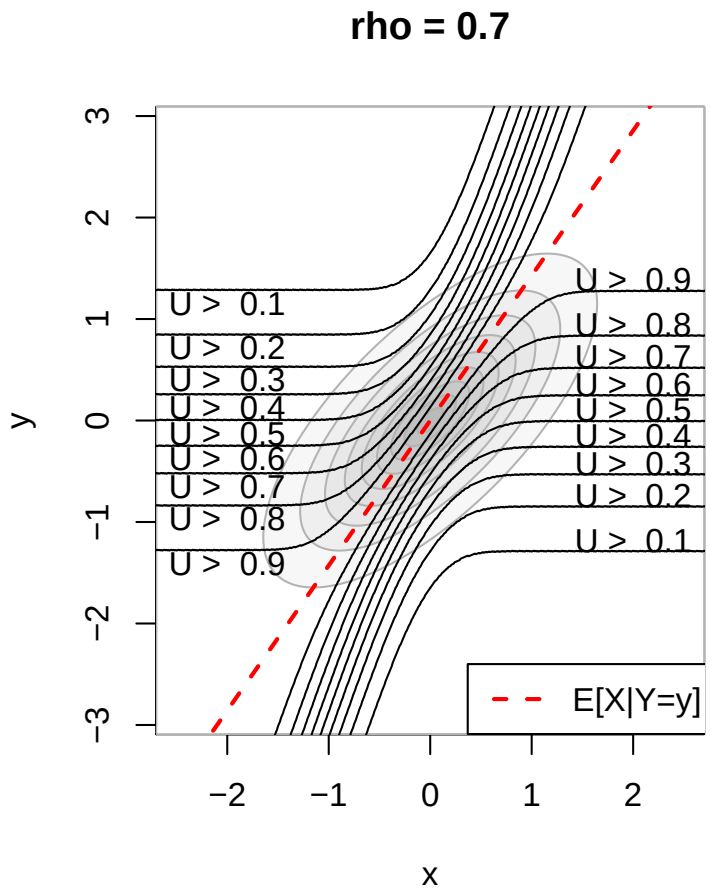


Defining the U-function

Define $U_x(y)$ as follows:

- Suppose we have test instance (face) x whose true label (person) is y.
- Let Y' be a random *incorrect* label (person).
- Use the classifier to guess whether x belongs to y or Y'.
- Define $U_x(y)$ as the probabilility of success (randomizing over training data).

Toy example



$$U_y(x) = \Pr[d(x, \rho Y') > d(x, \rho y)], \text{ for } Y' \sim N(0, 1).$$

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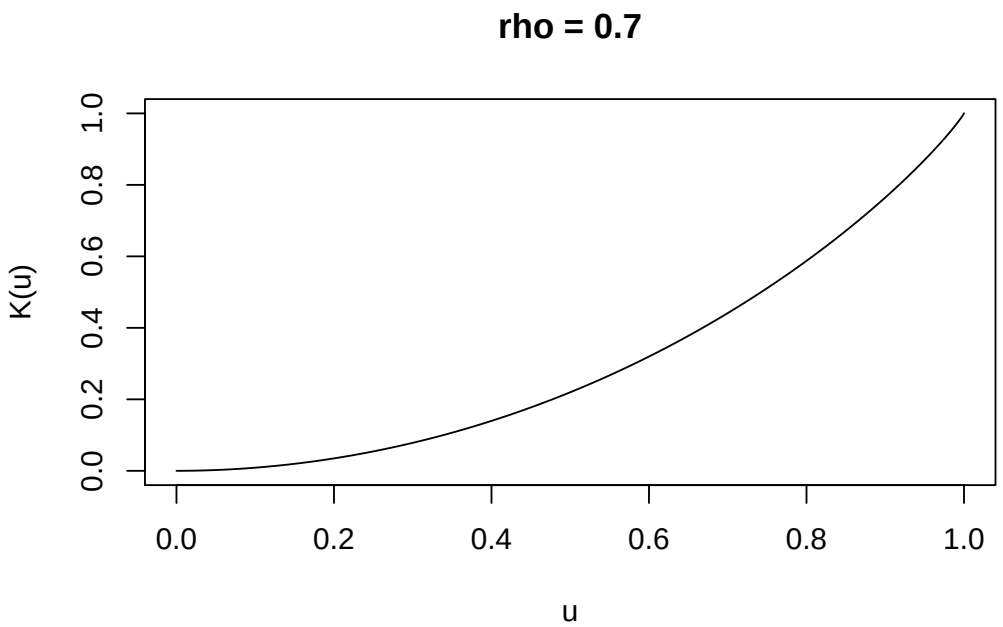
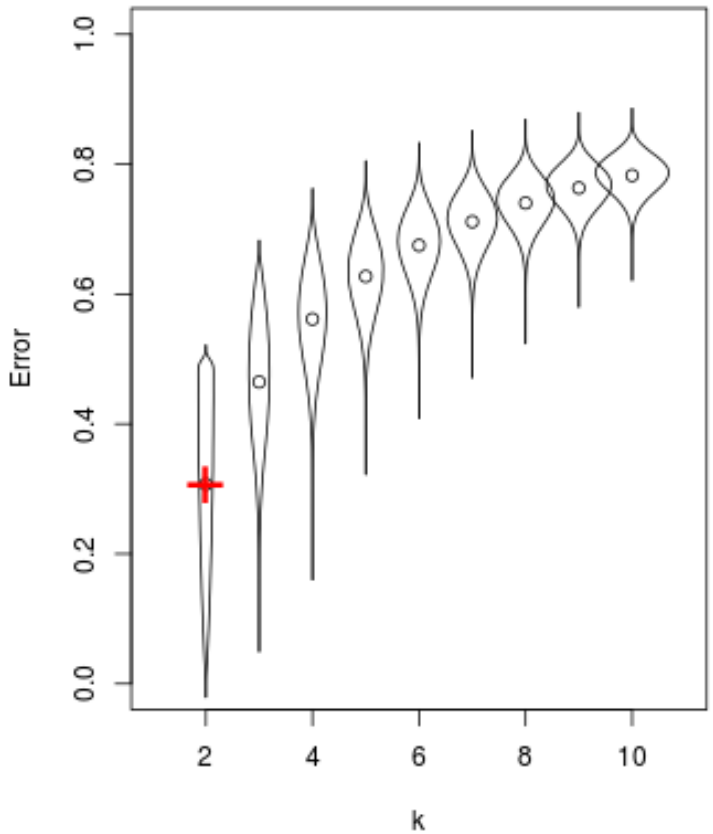
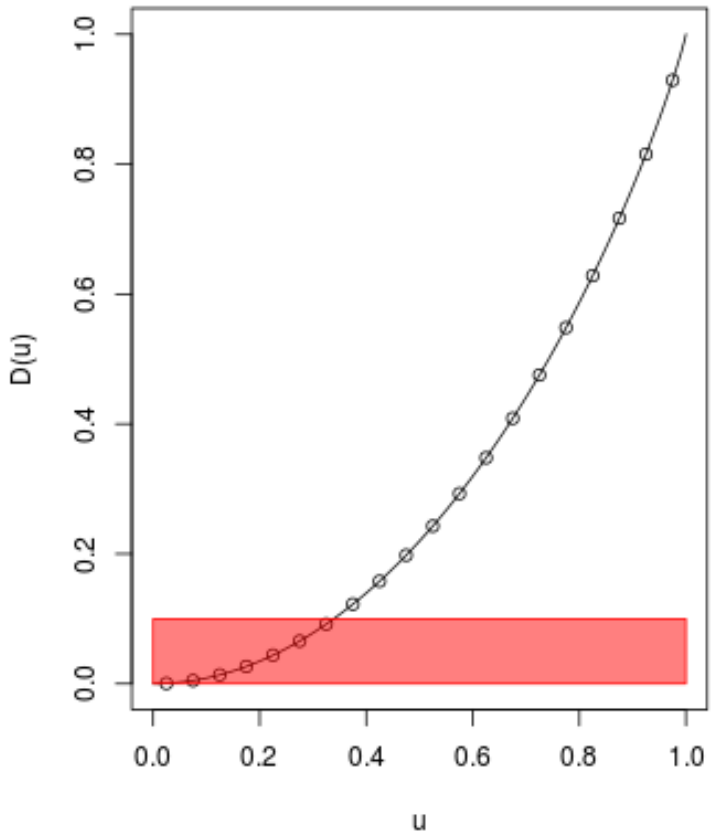


Computing average risk

$$\text{AvRisk}_k = (k-1)\int \bar{D}(u)u^{k-2}du.$$

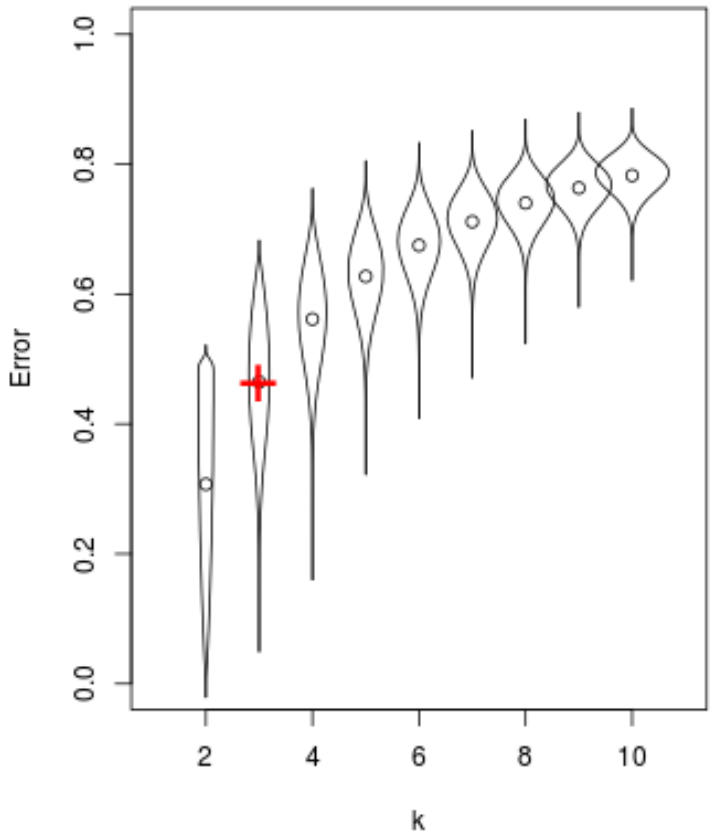


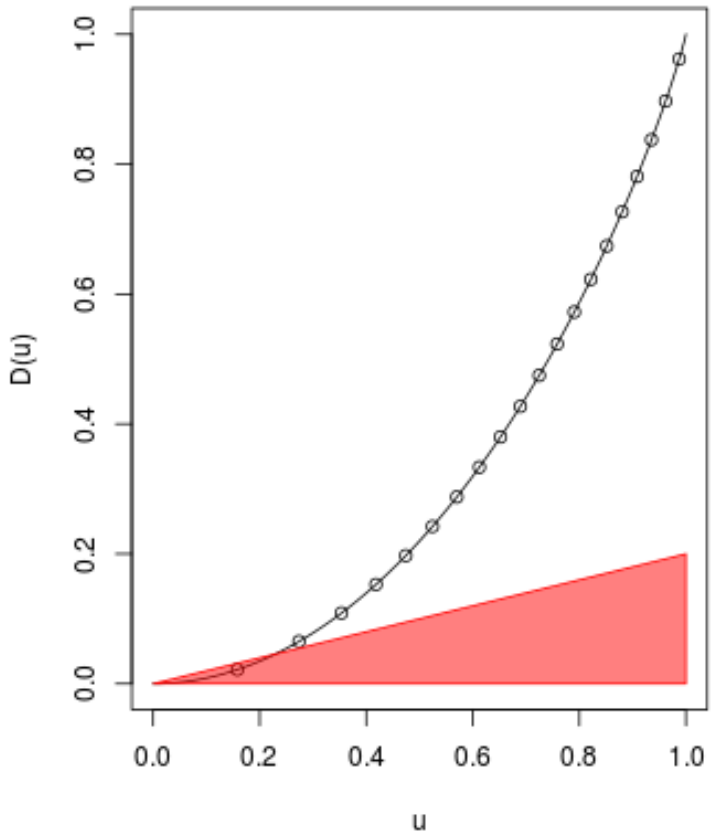


$(k = 2)$

Computing average risk

$$\text{AvRisk}_k = (k-1)\int \bar{D}(u)u^{k-2}du.$$

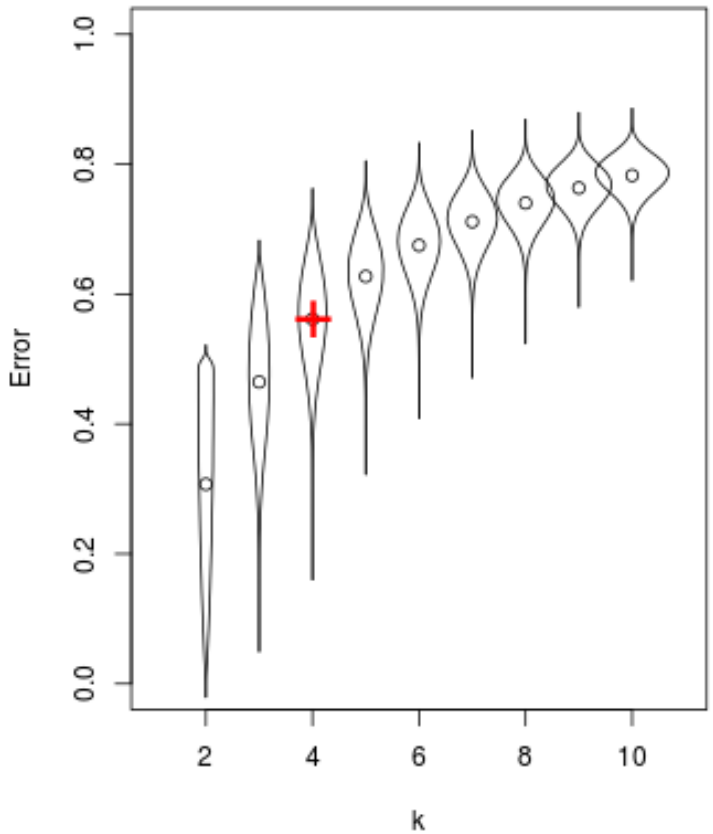


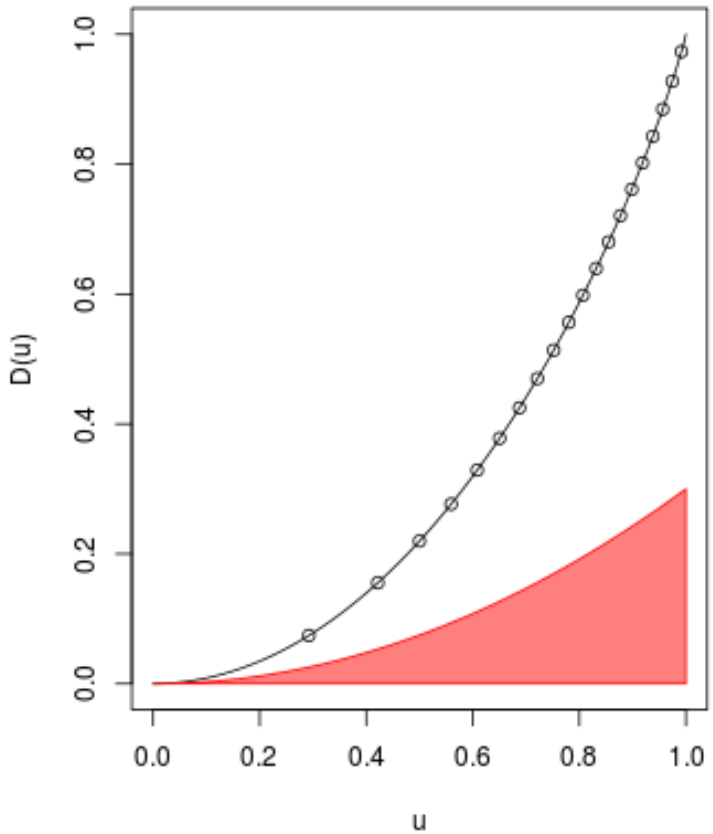


$(k = 3)$

Computing average risk

$$\text{AvRisk}_k = (k-1)\int \bar{D}(u)u^{k-2}du.$$

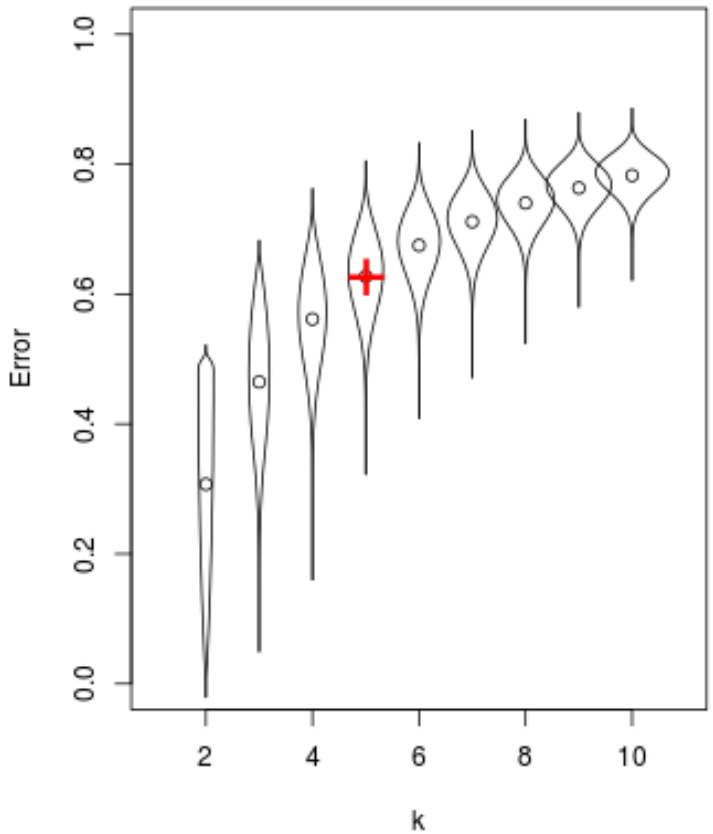


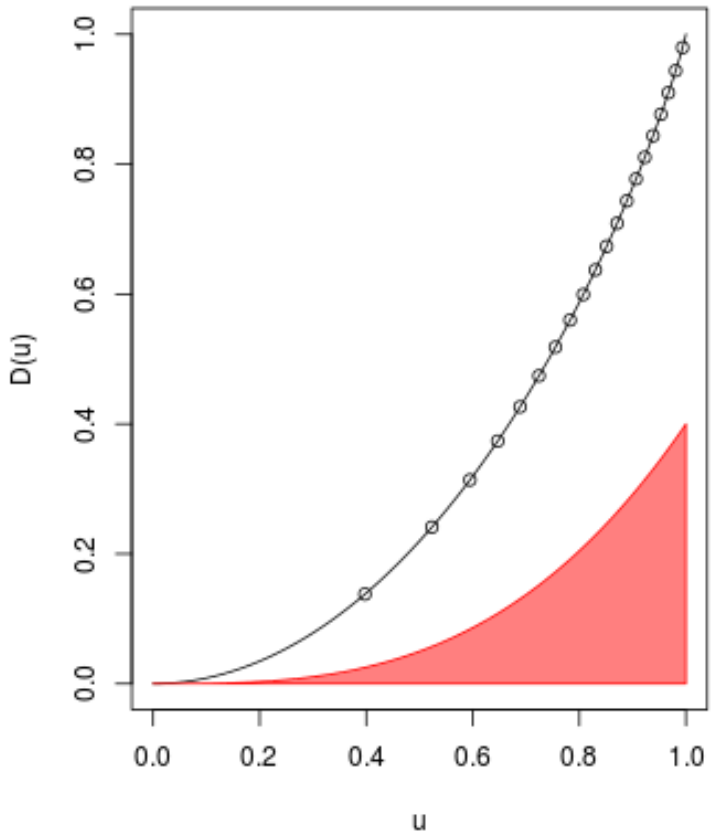


$(k = 4)$

Computing average risk

$$\text{AvRisk}_k = (k-1)\int \bar{D}(u)u^{k-2}du.$$

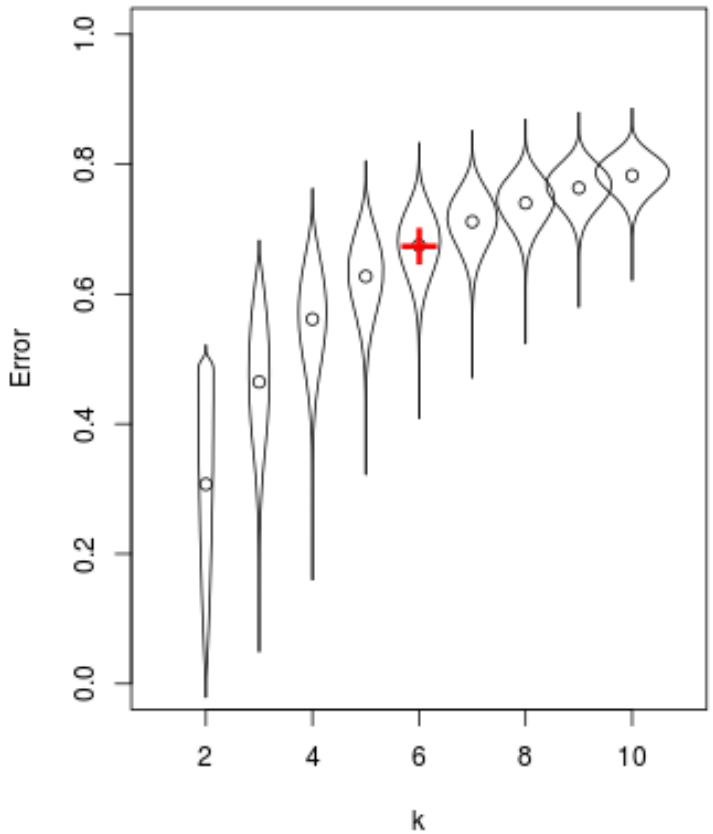


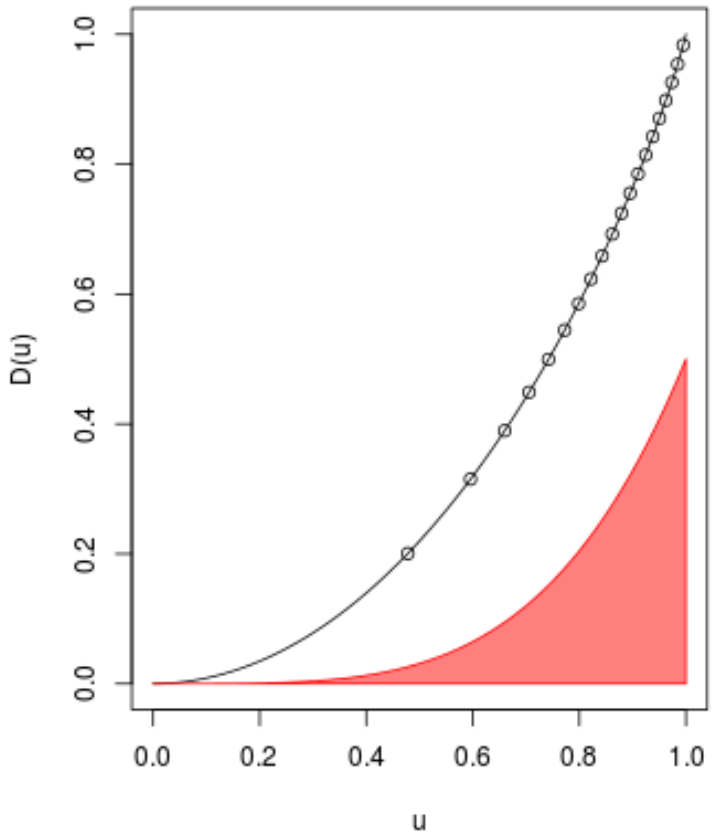


$(k = 5)$

Computing average risk

$$\text{AvRisk}_k = (k-1)\int \bar{D}(u)u^{k-2}du.$$

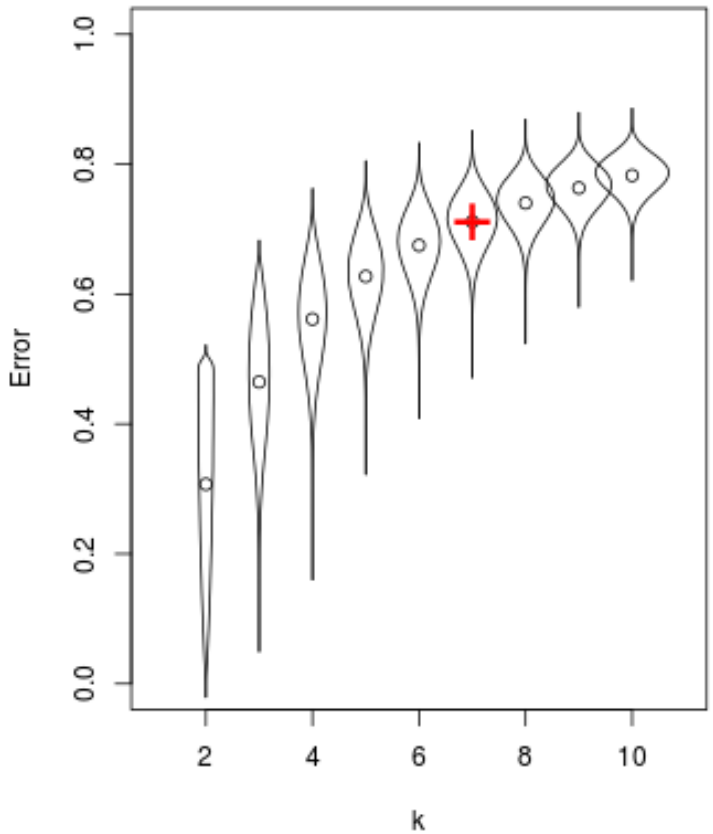


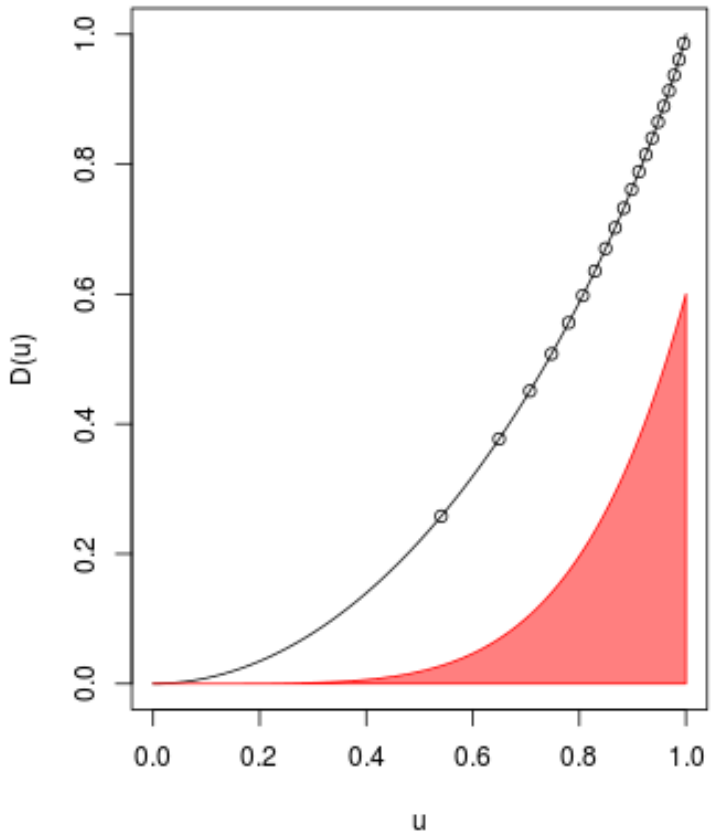


$(k = 6)$

Computing average risk

$$\text{AvRisk}_k = (k-1)\int \bar{D}(u)u^{k-2}du.$$

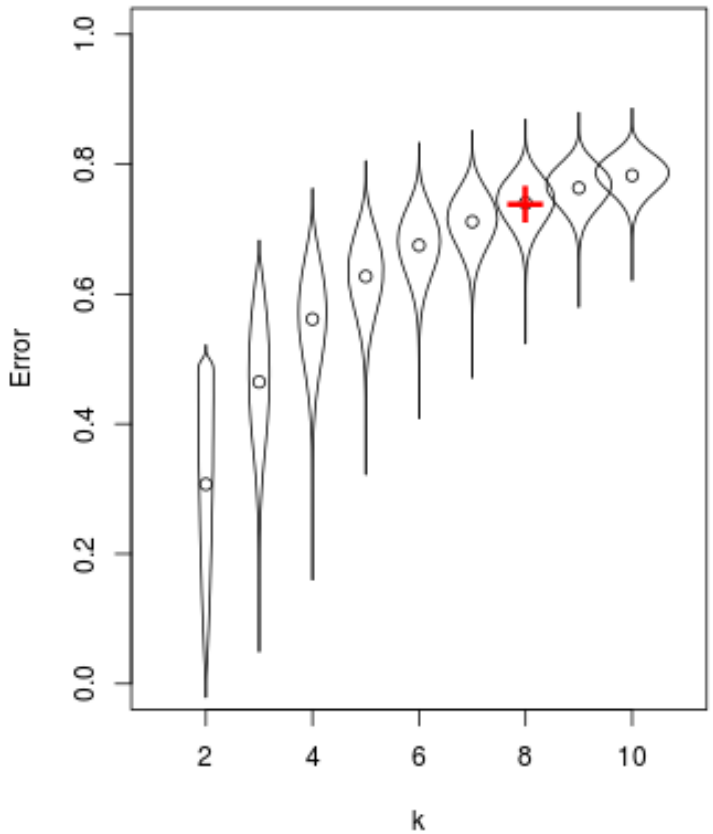


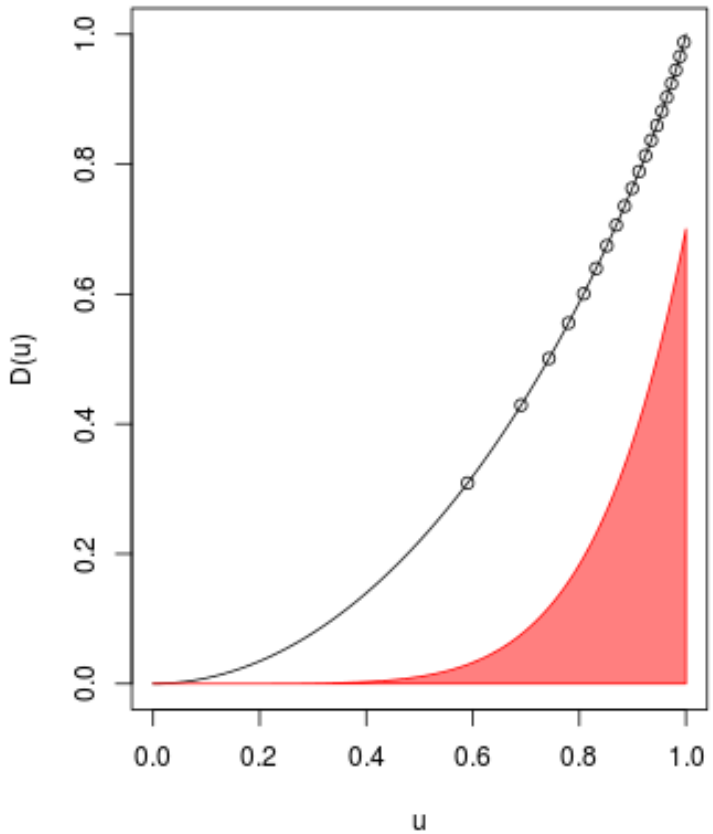


$(k = 7)$

Computing average risk

$$\text{AvRisk}_k = (k-1)\int \bar{D}(u)u^{k-2}du.$$

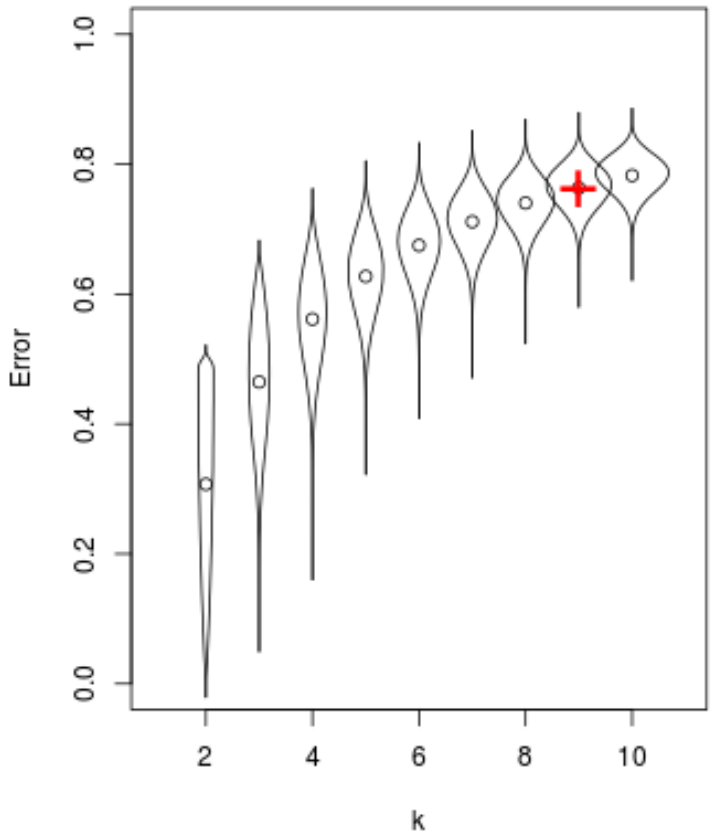


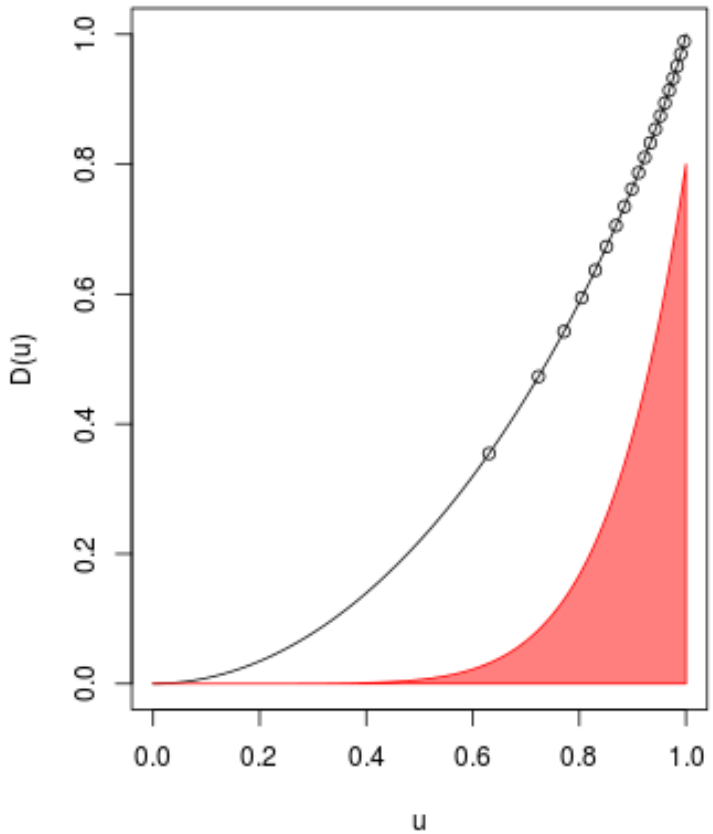


$(k = 8)$

Computing average risk

$$\text{AvRisk}_k = (k-1)\int \bar{D}(u)u^{k-2}du.$$

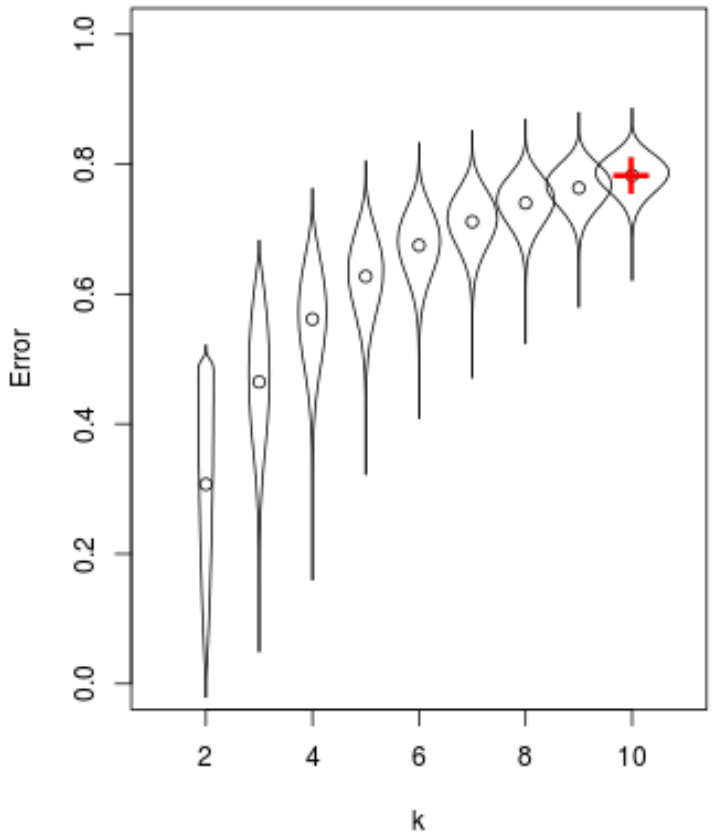


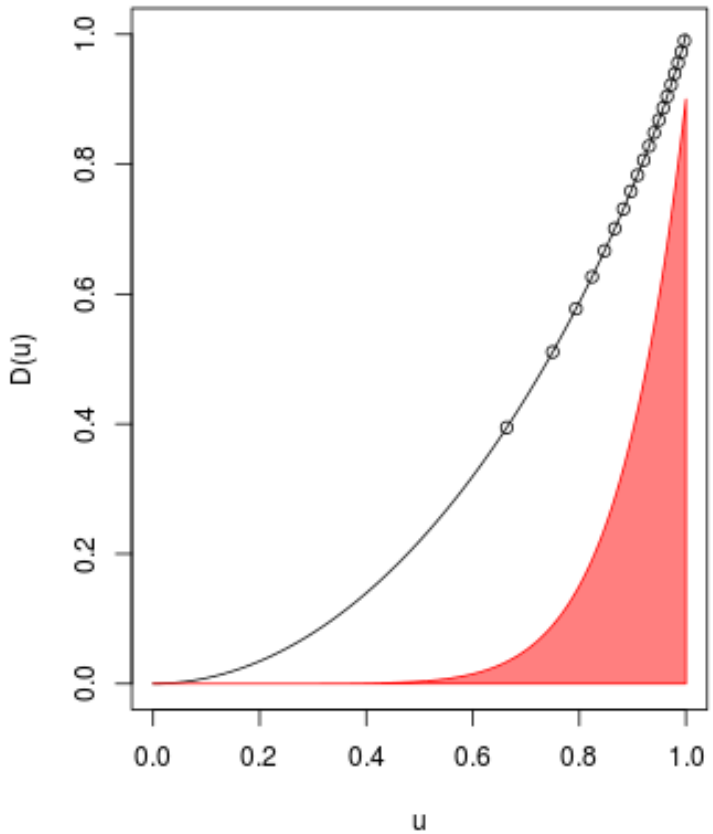


$(k = 9)$

Computing average risk

$$\text{AvRisk}_k = (k-1)\int \bar{D}(u)u^{k-2}du.$$





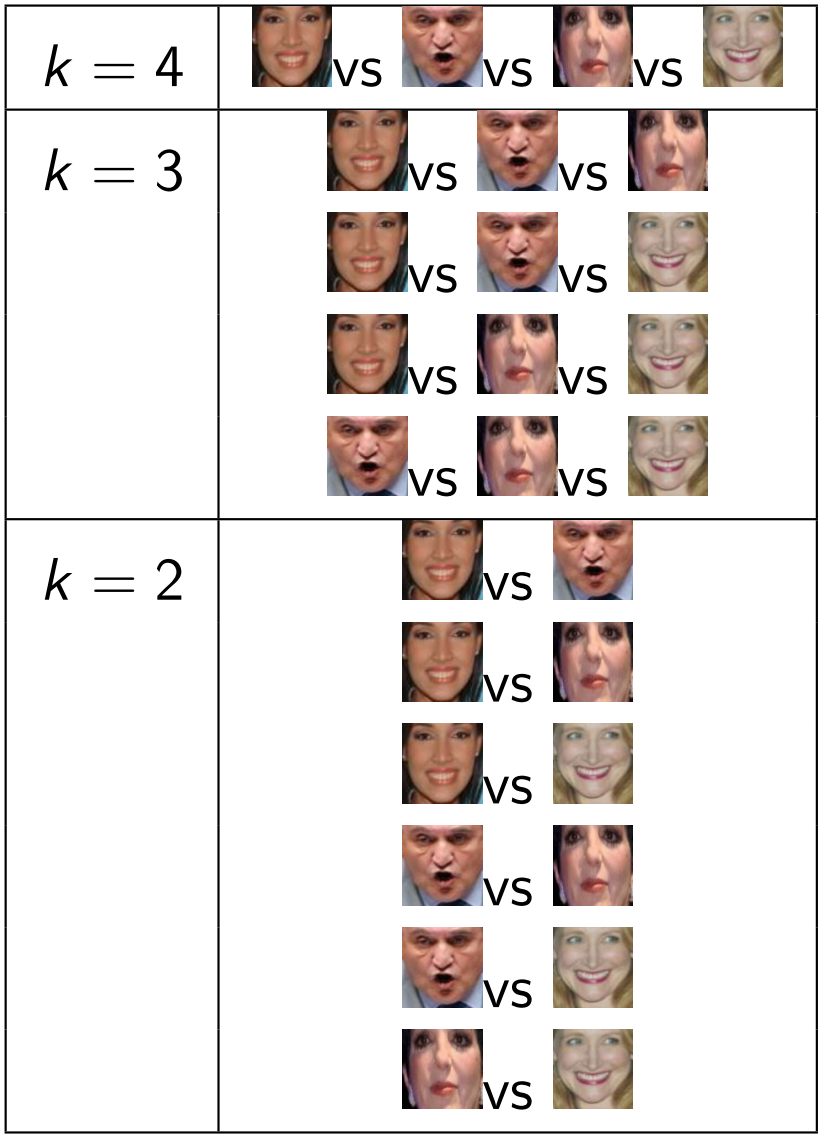
$(k = 10)$

Implication: estimate $\bar{D}(u)$ to predict risk

- Theoretical result links k-class average risk to $\bar{D}(u)$ function
- In real data, we do not know $\bar{D}(u)$ since it depends on the unknown joint distribution
- However, given a model, we can estimate $\bar{D}(u)$

Subsampled risk estimates

- Suppose we have data for k classes (subsampled from π)
- The test error TestErr_k is an unbiased estimate for AvRisk_k
- For any $\ell < k$, we can estimate AvRisk_ℓ by *subsampling* the k classes, and taking the average test risk, AvTestErr_ℓ.



Parametric modelling approach

Assume that for set of basis functions $h_1, \ldots, h_\ell$, we have

$$\bar{D}(u) = \sum_{\ell=1}^{m} \beta_\ell h_\ell(u).$$

Then

$$\text{AvRisk}_k = \sum_{\ell=1}^{m} \beta_\ell H_{\ell,k} = \beta^T \vec{H}_k$$

where

$$H_{\ell,k} = (k-2) \int_0^1 h_\ell(u) u^{k-2} du.$$

and $\vec{H}_k = (H_{1,k}, \ldots, H_{\ell,k})$

Prediction extrapolation as regression

1. Choose basis $h_1, \ldots, h_\ell$

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Prediction extrapolation as regression

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3. Fit regression model

$$\hat{\beta} = \text{argmin}_\beta \sum_{i=2}^{k} \left(\text{AvTestErr}_i - \vec{H}_i^T \beta_\ell \right)^2$$

Prediction extrapolation as regression

1. Choose basis $h_1, \ldots, h_\ell$

$$\bar{D}(u) = \sum_{\ell=1}^{m} \beta_\ell h_\ell(u).$$

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4. For $K > k$, predict AvRisk_K as

$$\widehat{\text{AvRisk}}_K = \vec{H}_K^T \beta_\ell.$$

Examples of basis functions

- Polynomials, $1, x, x^2, \ldots$
- Cubic splines
- Linear splines, $[x - t_\ell]_+$

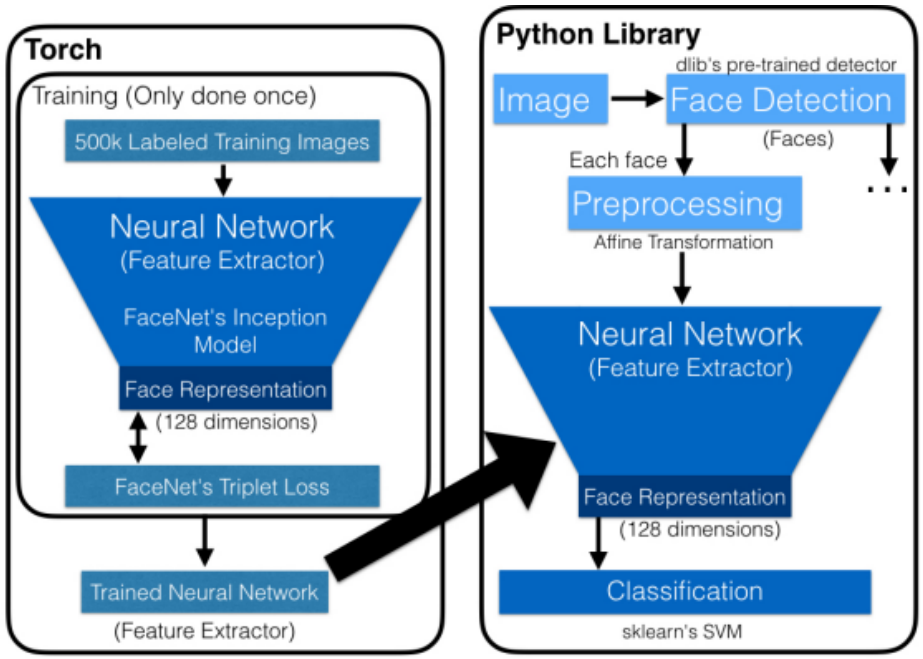
Optional constraint: assume β is *non-negative*. In the case of linear splines, this results in a *convex* fit.

Section 3

Applications

Facial recognition example

- Data: faces from "Labeled Faces in the Wild."
- 1672 people with at least 2 photos
- Featurization: trained neural network from OpenFace



Facial recognition example

- Let us first subsample 400 faces (out of 1672)
- Randomly choose 1 face as training and 1 as test for each person
- Use 1-nearest neighbor.
  - NOTE: 1-NN with 1 example/class is equivalent to LDA with $\Sigma = I$: this fits marginal classifier assumption!

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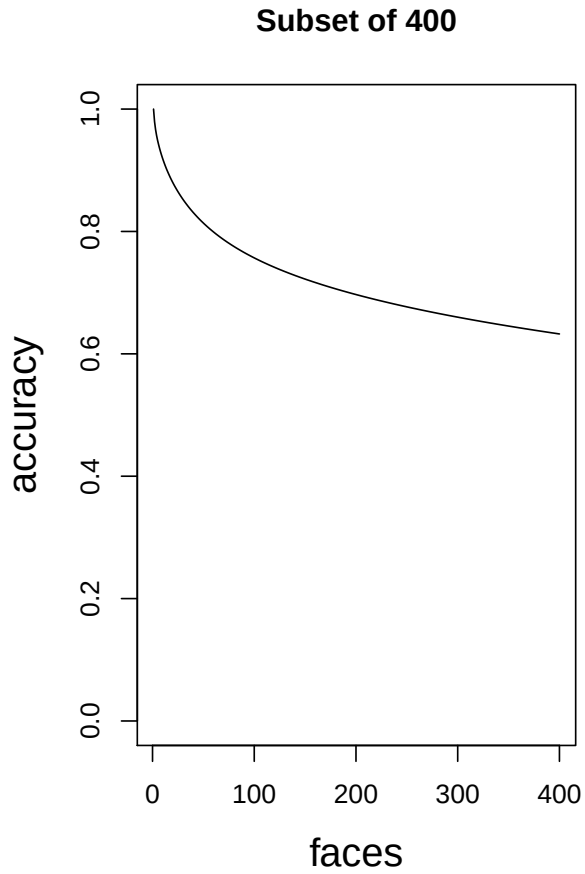

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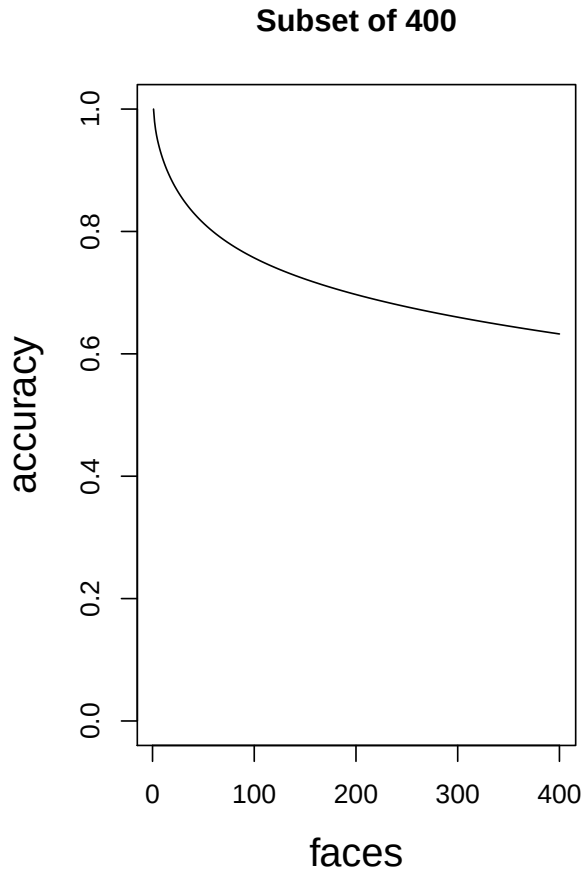
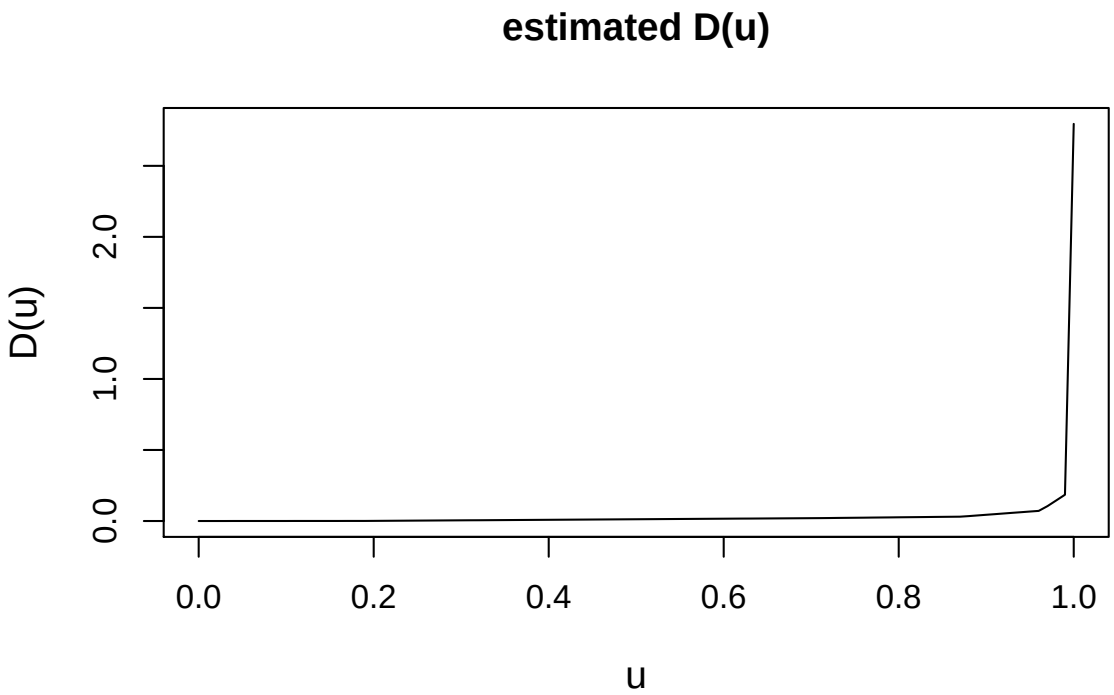


Can we predict the accuracy on the full set of 1672?

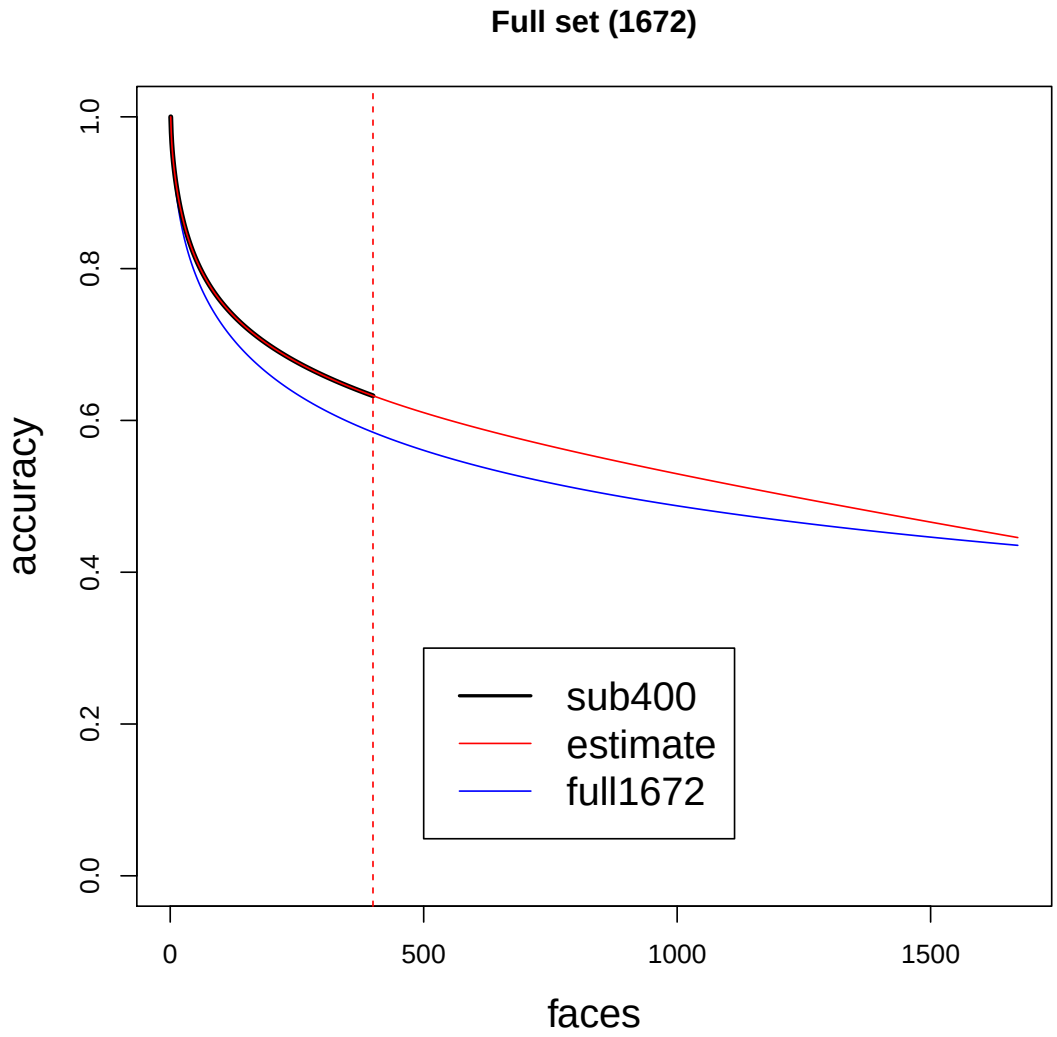
Estimated $\bar{D}(u)$

Using linear spline basis ($p = 10000$) and nonnegativity constraint.

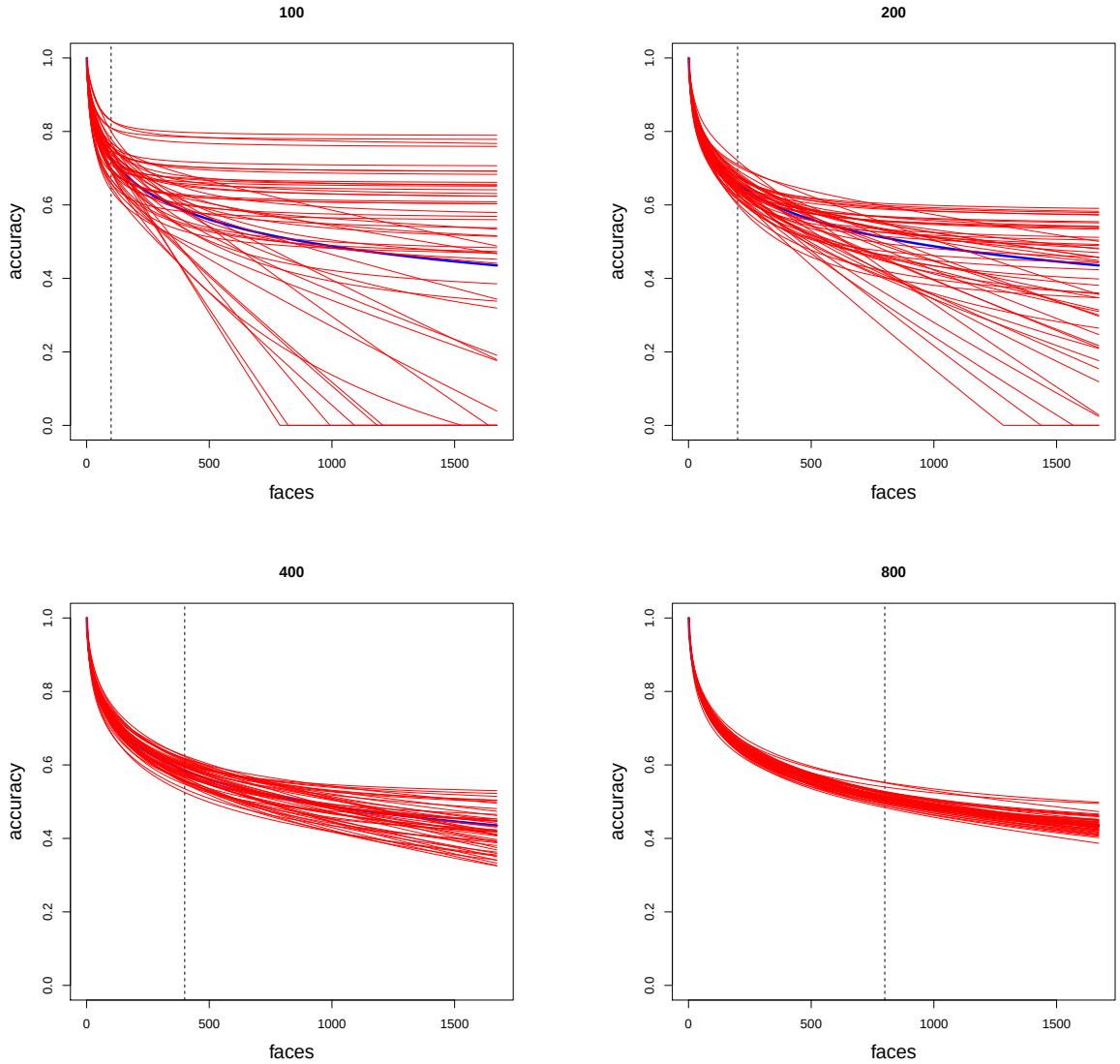


Estimated risk

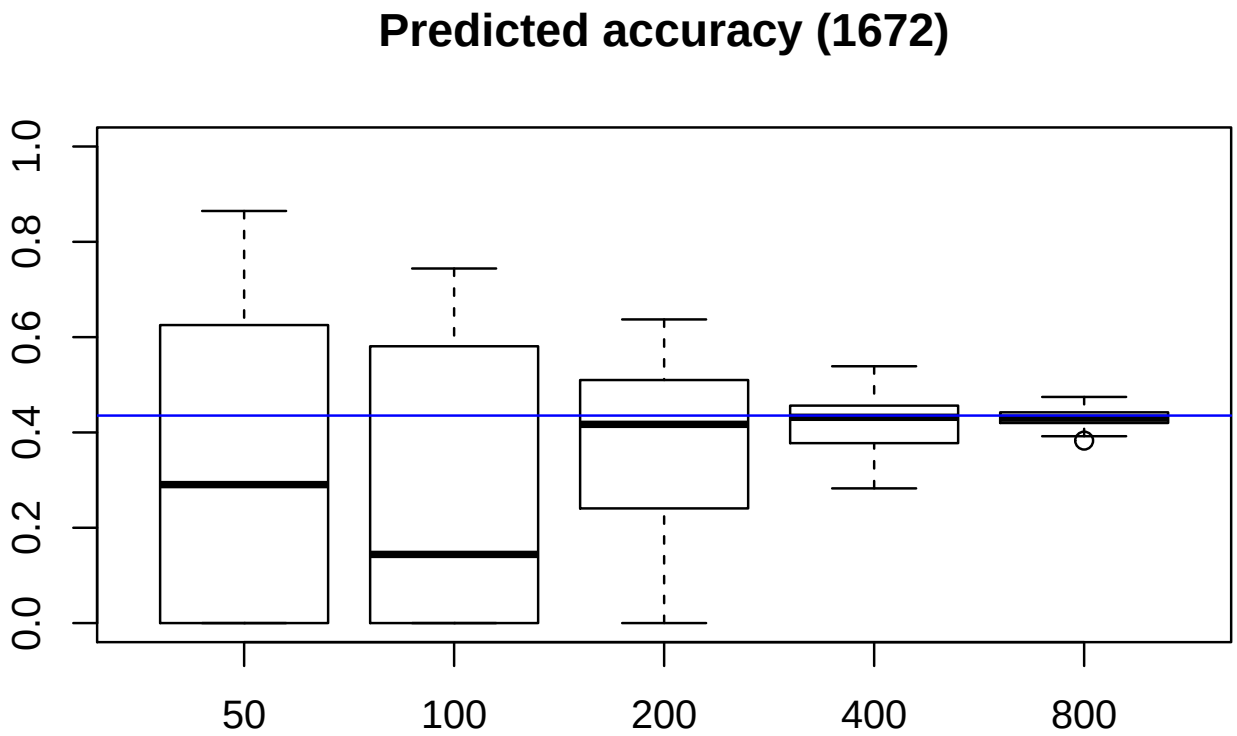
Compare to test risk at $K = 1672$



Estimated risk: more experiments

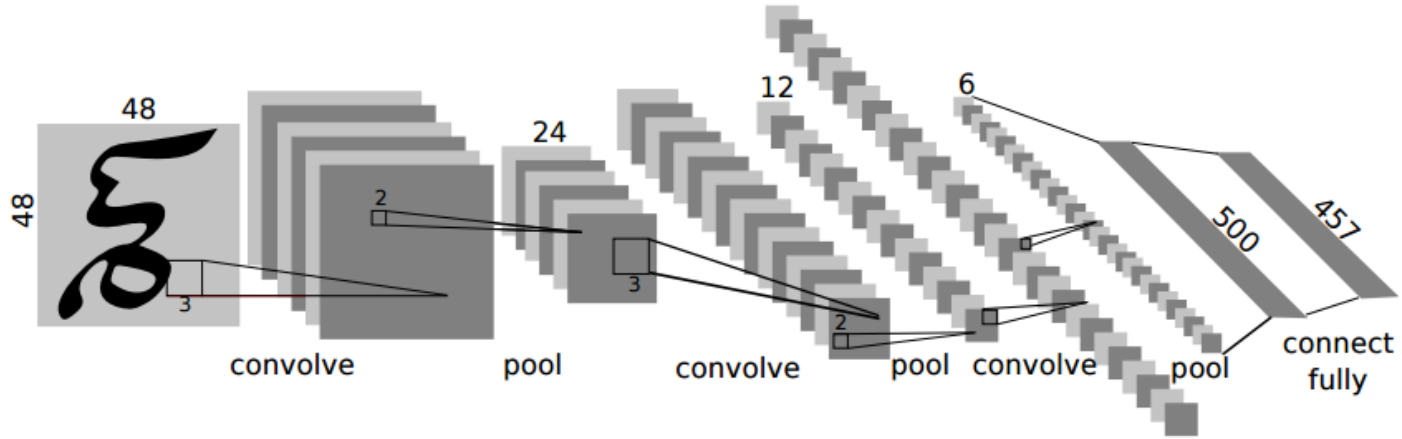


Estimated risk: more experiments



Telugu OCR example

కమ్మనిలతాంతముల కుమ్మినసివచ్చు మధు
పమ్మిల సుగీతనినదమ్మిలెగసెంజూ
తమ్మిల లసత్కిసలయమ్మిల సుగంధిముకు
ళమ్మిలను నానుచు ముదమ్మినర వాచా



Telugu OCR example

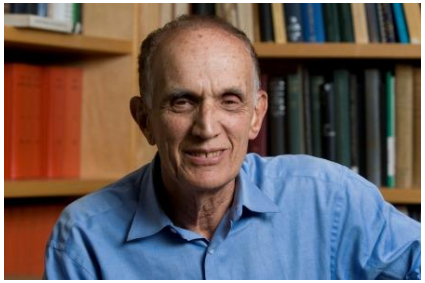
Classifier	Test err$^{(20)}$	Test err$^{(400)}$	$\hat{p}_{400}^{EXP}$	$\hat{p}_{400}^{POS}$	$\hat{p}_{400}^{(5)}$
Naive Bayes	0.049	0.399	0.108	**0.142**	0.079
Logistic	0.078	0.289	0.166	**0.188**	0.130
SVM	0.140	0.455	0.299	**0.313**	0.227
ϵ-NN	0.049	0.409	0.084	**0.590**	0.102
Deep neural net	0.005	0.014	**0.011**	0.093	0.010

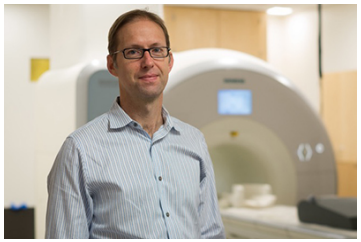
Performance extrapolation: predicting the accuracy on 400 classes using data from 20 classes on a Telugu character dataset. $\epsilon = 0.002$ for ϵ-nearest neighbors.

Section 4

Acknowledgements


CHALLENGER



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Section 5

The end