Upper and lower bounds on cdf of generalized non-central chi-squared

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November 13, 2015

1 Introduction

Let $Z \sim N(0, I_p)$, and let $\mu \in \mathbb{R}^p$ and Σ a positive semidefinite matrix. Define the generalized noncentral chi-squared distribution with noncentrality μ and shape Σ as the distribution of

$$Y = (Z + \mu)^T \Sigma (Z + \mu)$$

Let $V \Lambda V^T = \Sigma$ be the eigendecomposition of Σ, and let $\eta = V^T \Sigma$. Then

$$Y \stackrel{d}{=} (Z + \eta)^T \Lambda (Z + \eta) = \sim_{i=1}^p \lambda_i W_i$$

where $W_i \sim \chi_1^2(\eta_i^2)$. Recall that the mgf of the noncentral chi-sqaured with one df is given by

$$\mathbf{E}[e^{tW_i}] = \frac{\exp[\frac{\eta_i^2 t}{1-2t}]}{\sqrt{1-2t}}$$

It follows that the moment-generating function of Y is given by

$$\mathbf{E}[e^{tY}] = \prod_{i=1}^p \mathbf{E}[e^{\lambda_i t W_i}] = \prod_{i=1}^p \frac{\exp[\frac{\eta_i^2 \lambda_i t}{1-2t\lambda_i}]}{\sqrt{1-2t\lambda_i}}$$

2 Upper bound

We wish to bound the probability $\Pr[Y < x]$. We have

$$\begin{aligned}
\log \Pr[Y < x] &= \log \Pr[e^{tY} > e^{tx}] \text{ for } t < 0 \\
&\leq \log\left(\frac{\mathbf{E}[e^{tY}]}{e^{tx}}\right) \\
&= \log(\mathbf{E}[e^{tY}]) - tx \\
&= \left(\frac{-1}{2}\sum_{i=1}^{p}\log(1-2t\lambda_i)\right) + \left(\sum_{i=1}^{p}\frac{\eta_i^2\lambda_i t}{1-2t\lambda_i}\right) - tx
\end{aligned}$$

Now consider minimizing the bound over t. The derivative of the bound wrt t is

$$-x + \sum_{i=1}^{p}\frac{\lambda_i(1+\eta_i)^2}{1-2\lambda_i t} + \frac{2\eta_i^2\lambda_i^2 t}{(1-2\lambda_i t)^2}$$

which, to a first-order approximation in $1/t$, is

$$-\frac{p}{2t} - x$$

Hence this implies that for small x, the optimal value of t is approximately

$$t^* \approx -\frac{p}{2x}.$$

Plugging in this value of t yields the bound

$$\log \Pr[Y < x] < -\frac{1}{2}\left(\sum_{i=1}^{p}\log\left(1+\frac{\lambda_i p}{x}\right)\right) + \frac{p}{2}\left(1 - \frac{1}{x}\sum_{i=1}^{p}\frac{\eta_i^2\lambda_i}{1+\frac{\lambda_i p}{x}}\right)$$

3 Lower bound

From the previous notation, the probability is written as an integral over an ellipse $\mathcal{E}$:

$$\Pr[Y < x] = \int_{\mathcal{E}}\phi(\nu - z)dz$$

where

$$\phi(\nu - z) = \left(\frac{1}{2\pi}\right)^{p/2}\exp^{-||\nu-z||^2/2}$$

and

$$\mathcal{E} = \{z \in \mathbb{R}^p : z^T \Lambda z < x\}.$$

Define the height of the ellipse as

$$h = \max_{z \in \mathcal{E}} \left(\frac{\eta}{||\eta||} \right)^T z = \sqrt{x\rho},$$

where

$$\rho = \frac{\eta^T \Sigma^{-1} \eta}{||\eta||^2},$$

and let z^* be the point in $\mathcal{E}$ which attains $z^T \eta = h||\eta||$, whence

$$z^* = \sqrt{\frac{x}{\rho}} \Sigma^{-1} \frac{\eta}{||\eta||}.$$

Define the fractional cap with fraction u by

$$\mathcal{C}_u = \{z \in \mathcal{E} : \eta^T z < uh||\eta||\}.$$

Intuitively, this cap contains points $z \in \mathcal{E}$ with a large density $\phi(z - \nu)$. The volume of the cap is given by

$$x^{p/2} \mathrm{Vol}(B_1)/\sqrt{|\Lambda|} J(u)$$

where

$$J_p(u) = \frac{\int_{1-2u}^{1} (1 - x^2)^{\frac{p-1}{2}} dx}{\int_{-1}^{1} (1 - x^2)^{\frac{p-1}{2}} dx} = \frac{\Pr[\mathrm{Beta}(\frac{1}{2}, \frac{p+1}{2}) > (1 - 2u)^2]}{2} \text{ for } u < \frac{1}{2}.$$

Let us proceed to lower bound the density in $\mathcal{C}_u$. This amounts to upper bounding the distance

$$\max_{z \in \mathcal{C}_u} ||z - \eta||^2.$$

Let us define

$$m(u) = (||\eta|| - h(1 - 2u))^2 + \left(||\eta/||\eta|| - z^*|| + \sqrt{\frac{x}{\lambda_{min}} (1 - (1 - 2u)^2)} \right)$$

where $\lambda_{min} = \min_{i=1}^p \lambda_i$. We claim that

$$\max_{z \in \mathcal{C}_u} ||z - \eta||^2 \leq m(u).$$

To prove the claim, for any $z \in \mathcal{C}_u$, decompose the difference into parallel and orthogonal components,

$$\eta - z = \alpha \frac{\eta}{||\eta||} + \beta \eta^{\perp}$$

where $\eta^{\perp}$ is orthogonal to η. Hence

$$||\eta - z||^2 = \alpha^2 + \beta^2.$$

It suffices to show that

$$\alpha \leq ||\eta|| - h(1 - 2u)$$

and

$$\beta \leq ||\eta/||\eta|| - z^*|| + \sqrt{\frac{x}{\lambda_{min}}(1 - (1 - 2u)^2)}$$

for every $z \in \mathcal{C}_u$.

Hence,

$$\log \Pr[Y < x] \leq \log(\phi(\sqrt{m(u)})) + \log \mathrm{Vol}(B_{\sqrt{x}}) - \frac{1}{2} \log |\lambda| + \log J_p(u)$$