Supervised Evaluation of Representations

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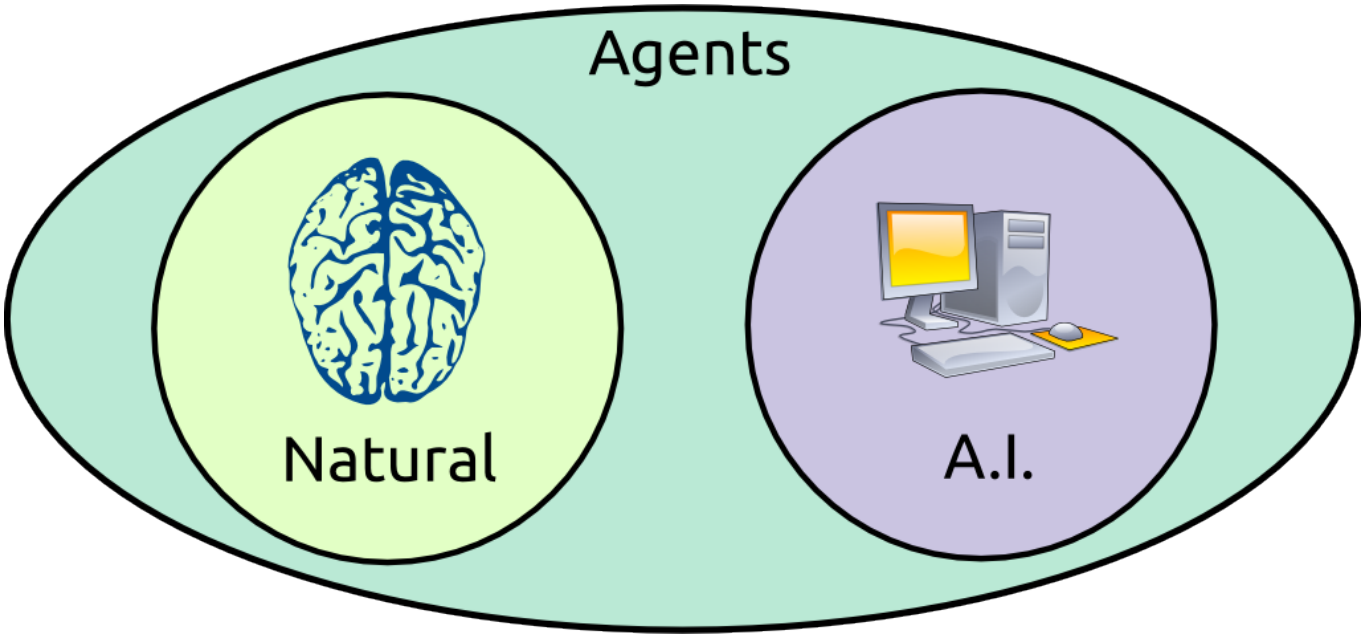
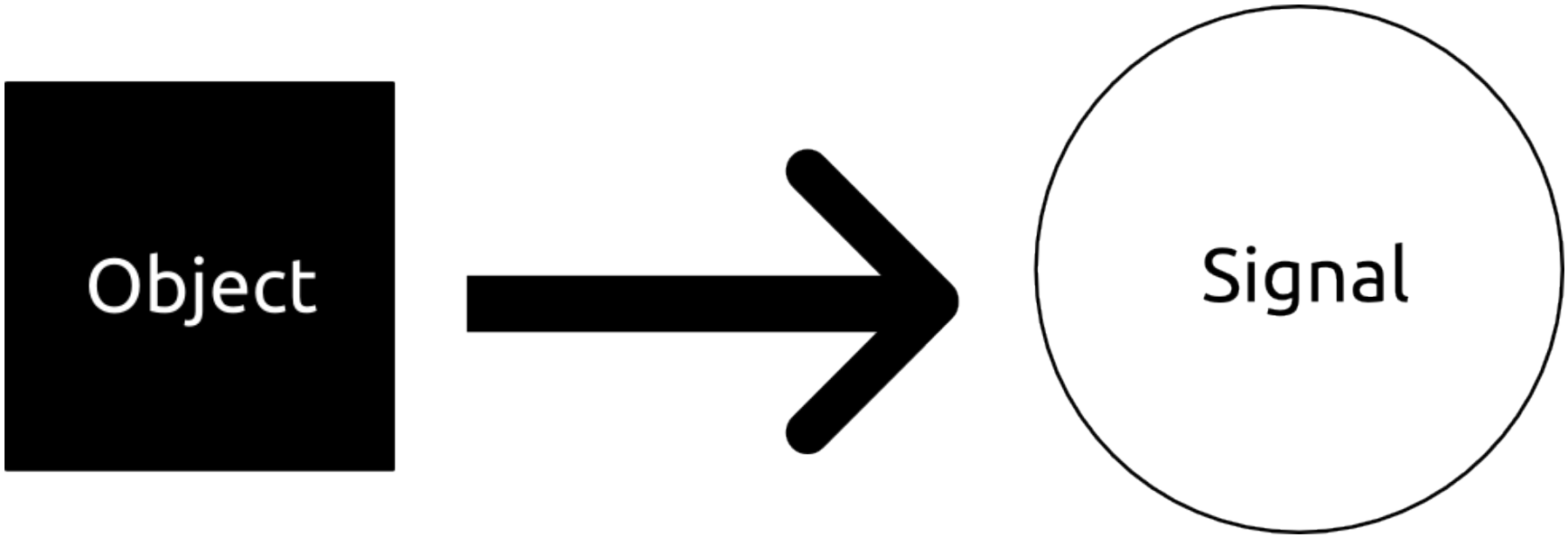
Overview



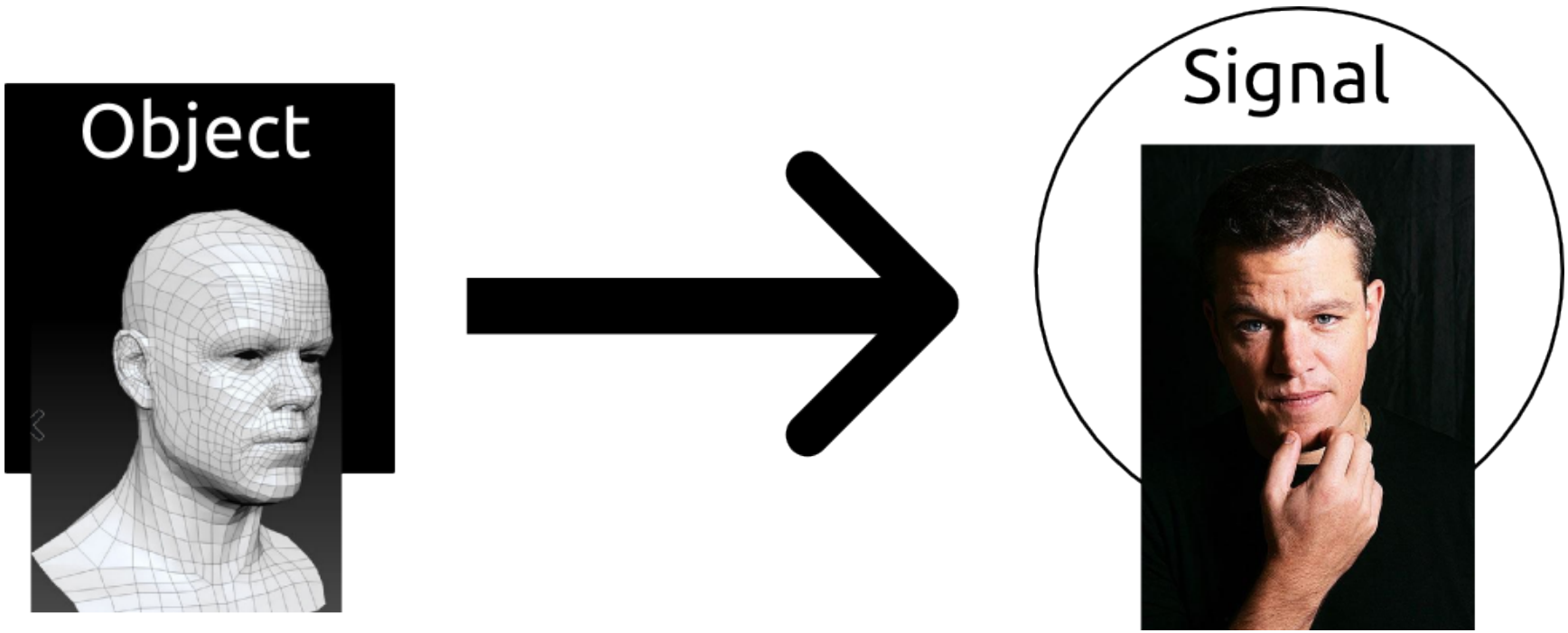
Human brains and machine learning algorithms tackle similar types of problems.

Perception

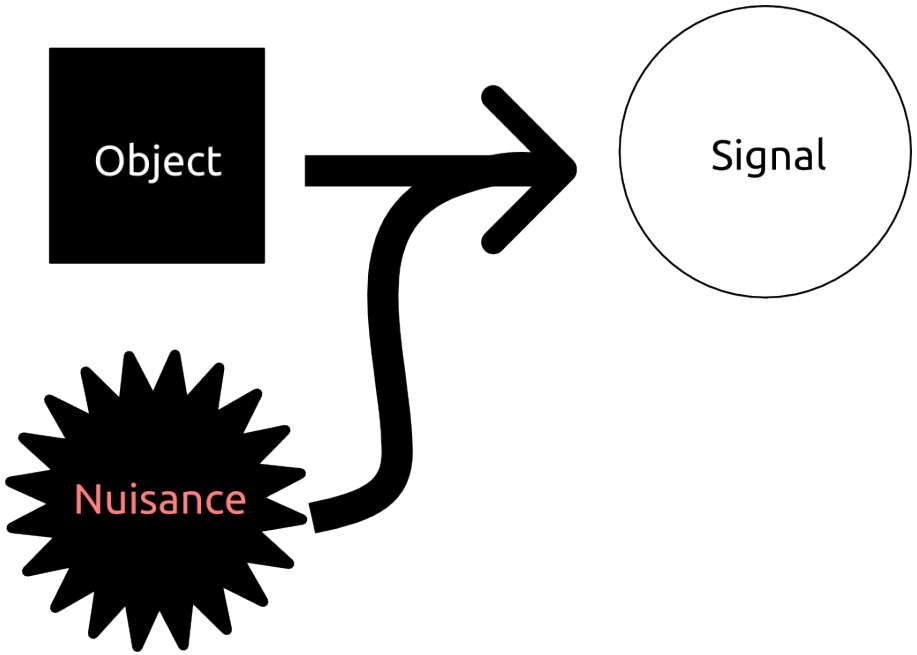


Perception: the problem of inferring *objects* in the environment given observed *signals*.

Example: face recognition



Perception



The problem is complicated because there exist some *nuisance parameters*, so the mapping from object to signal is not one-to-one.

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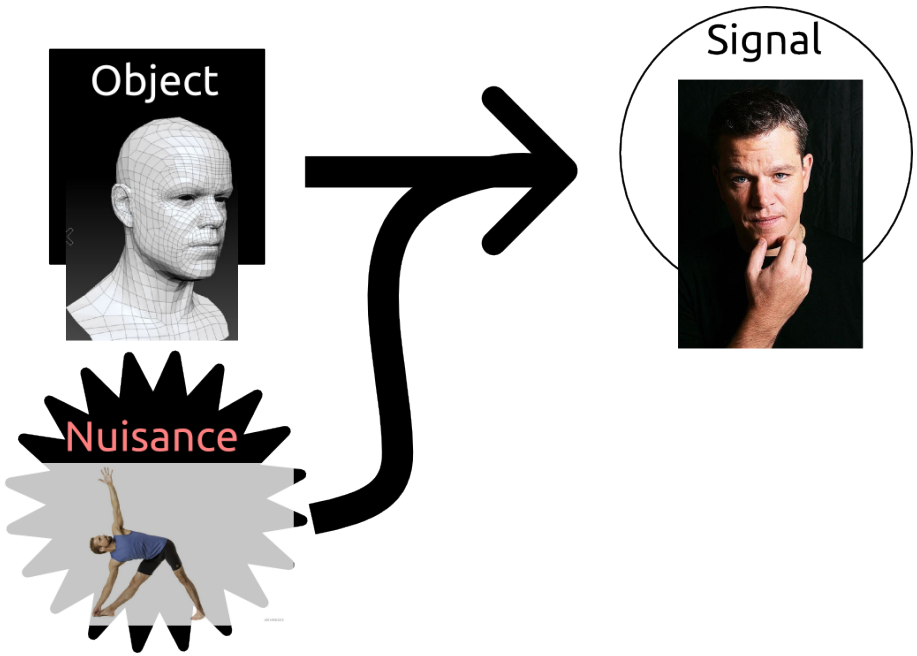
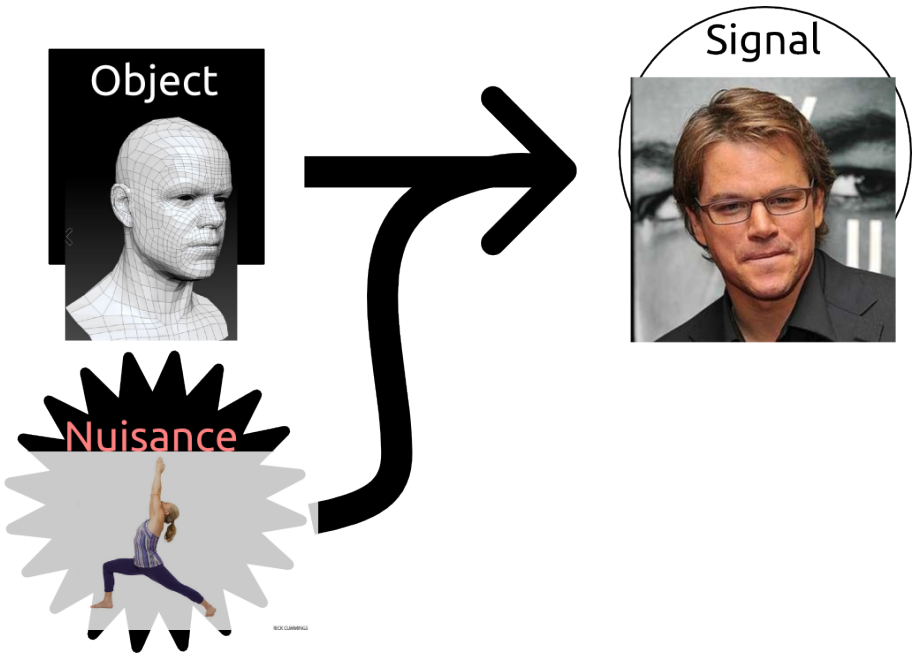


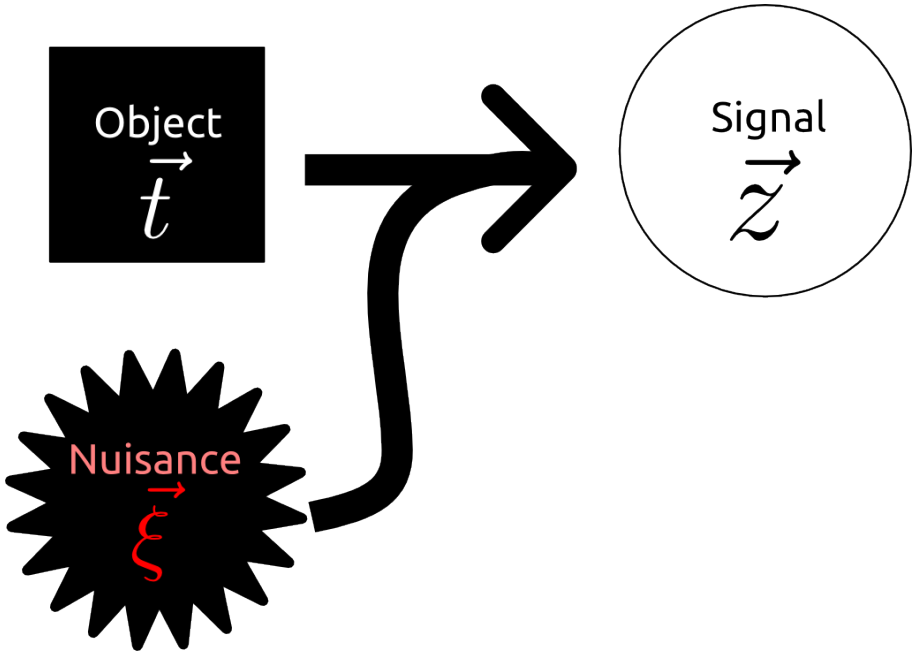
In face recognition, the *pose* (including hairstyle) and *lighting* are nuisance parameters.

Example: face recognition



The same object can map to multiple signals.

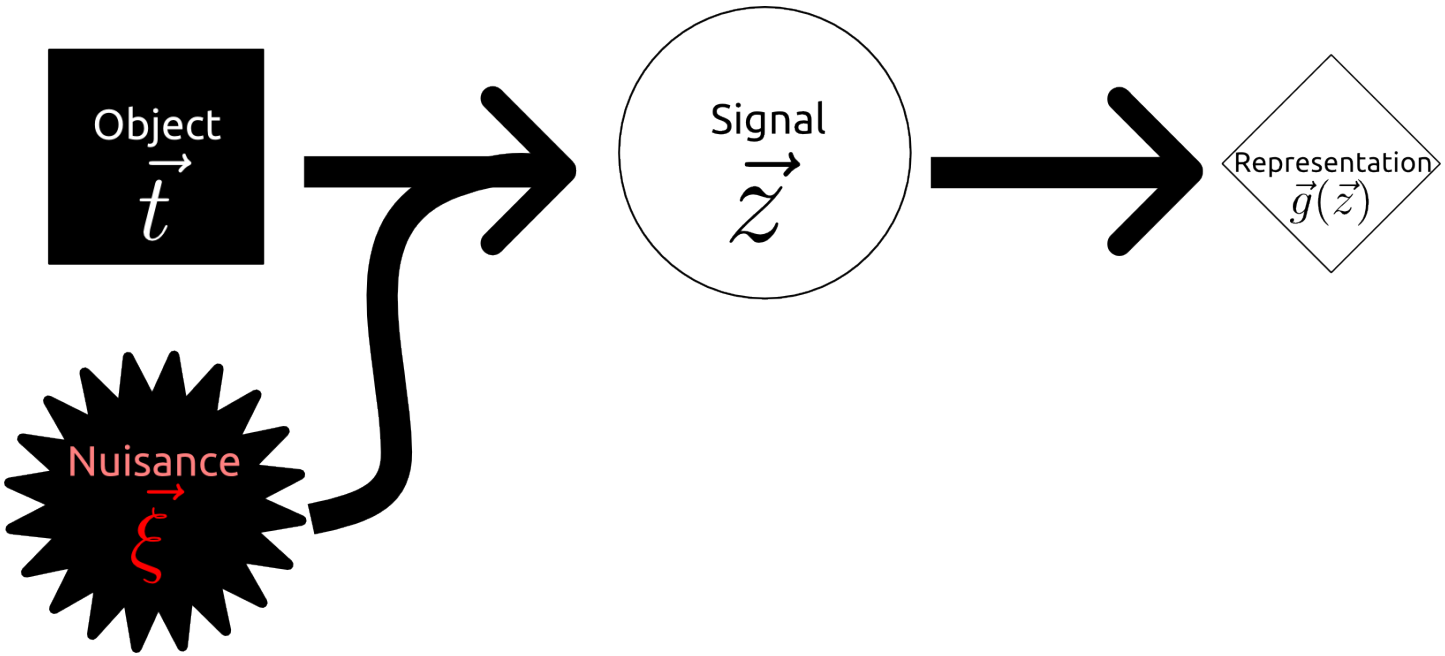
Perception



Assume there exists a function ψ that maps objects and nuisance parameters to signals:

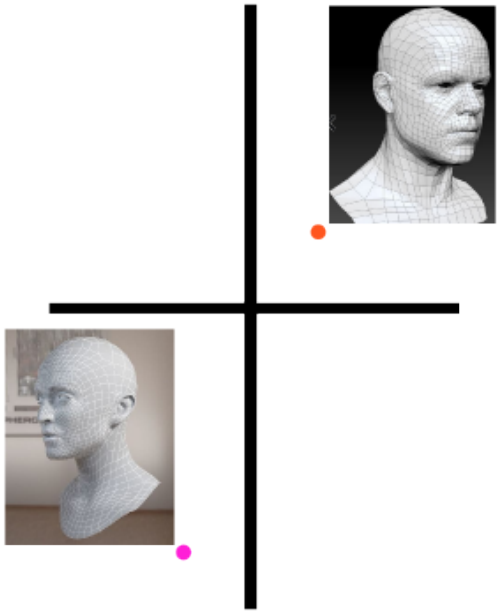
$$\vec{z} = \psi(\vec{t}, \vec{\xi}).$$

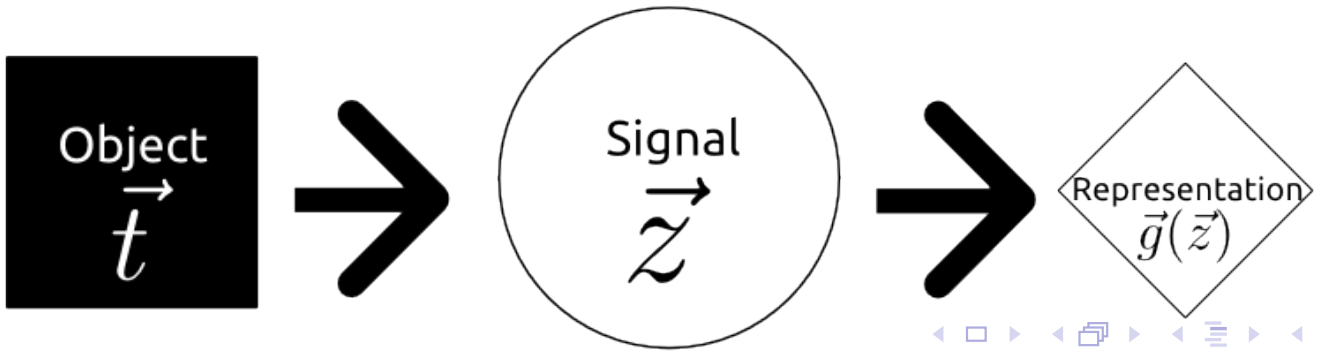
What is a representation?



A dimensionality-reducing mapping $\vec{g}$ of the signal.

A good representation...





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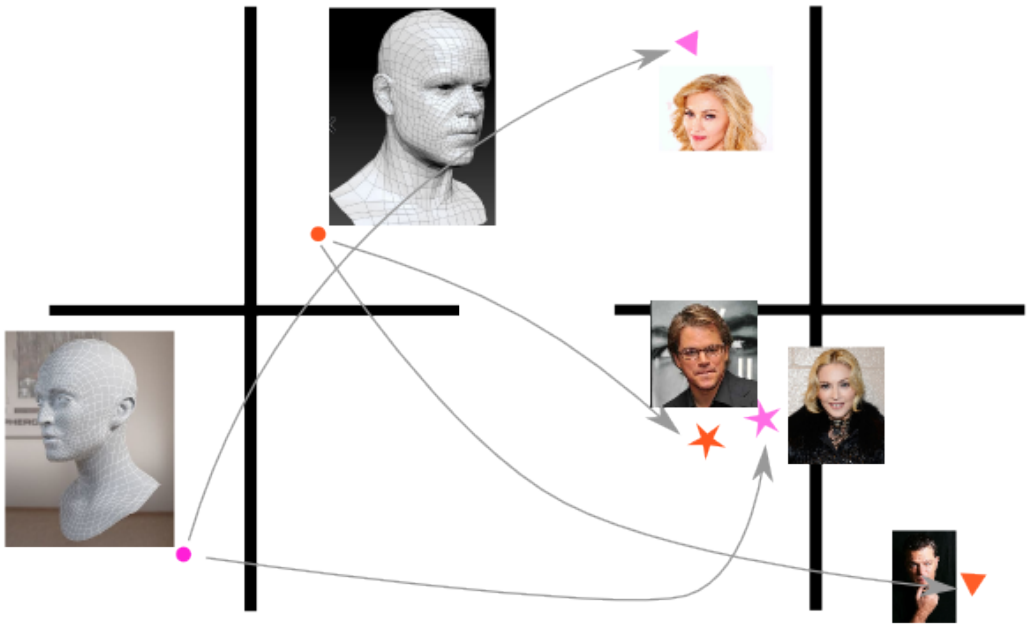

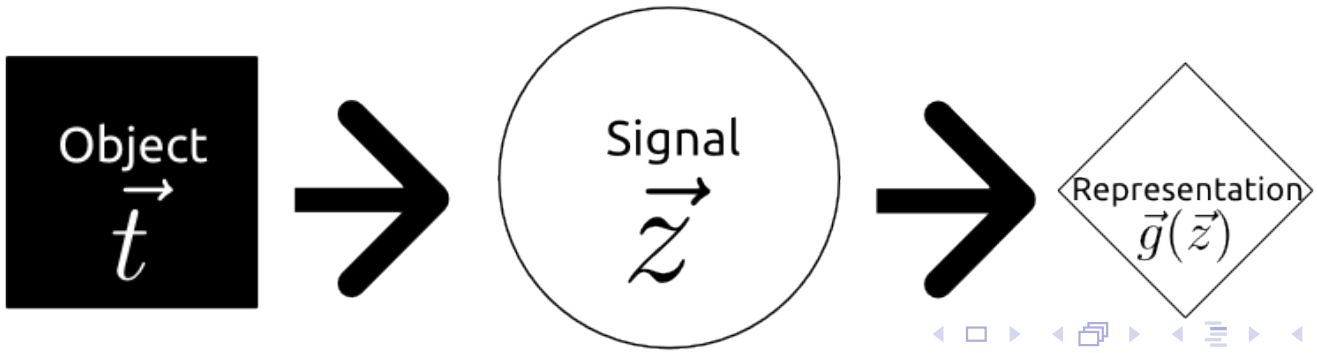

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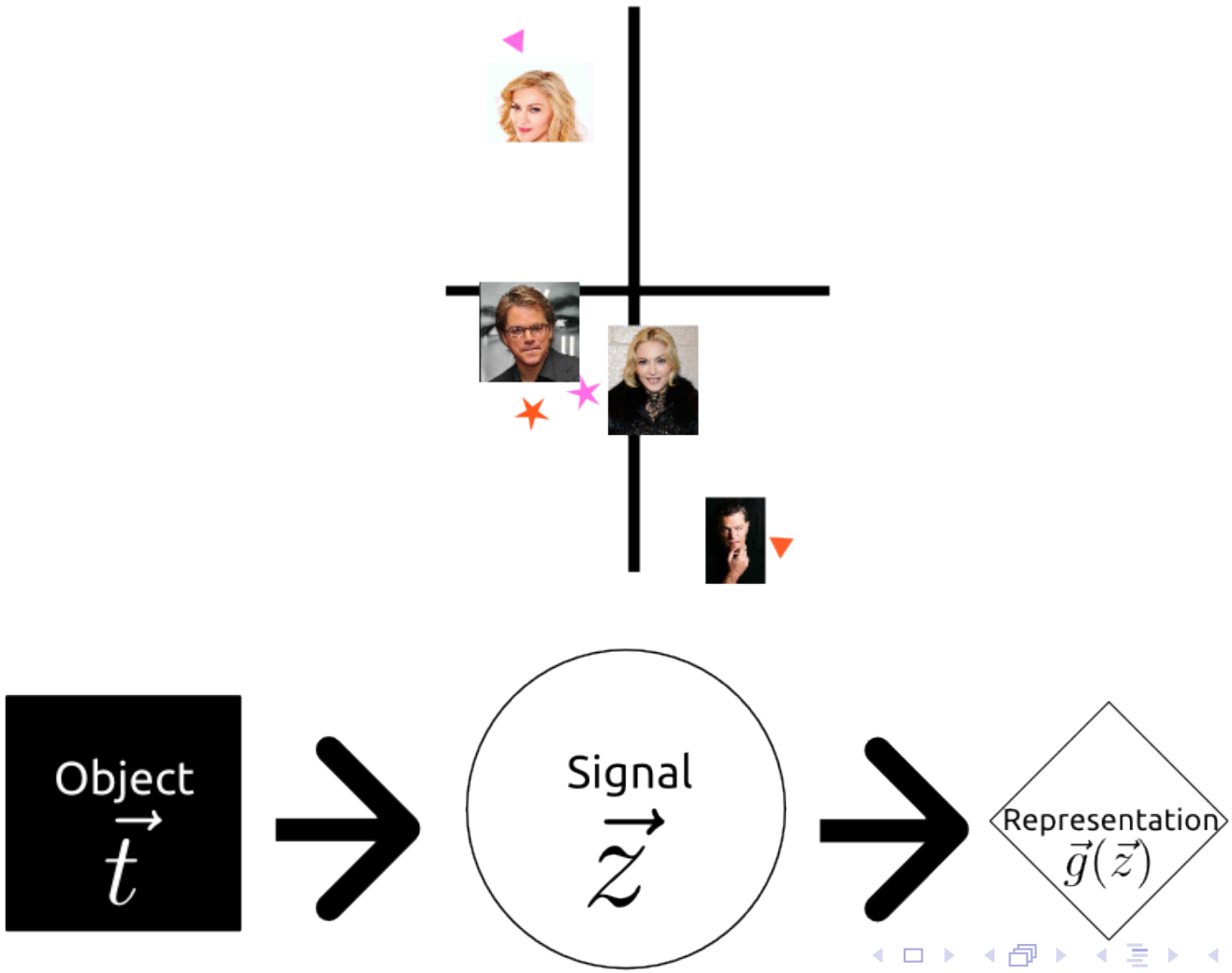
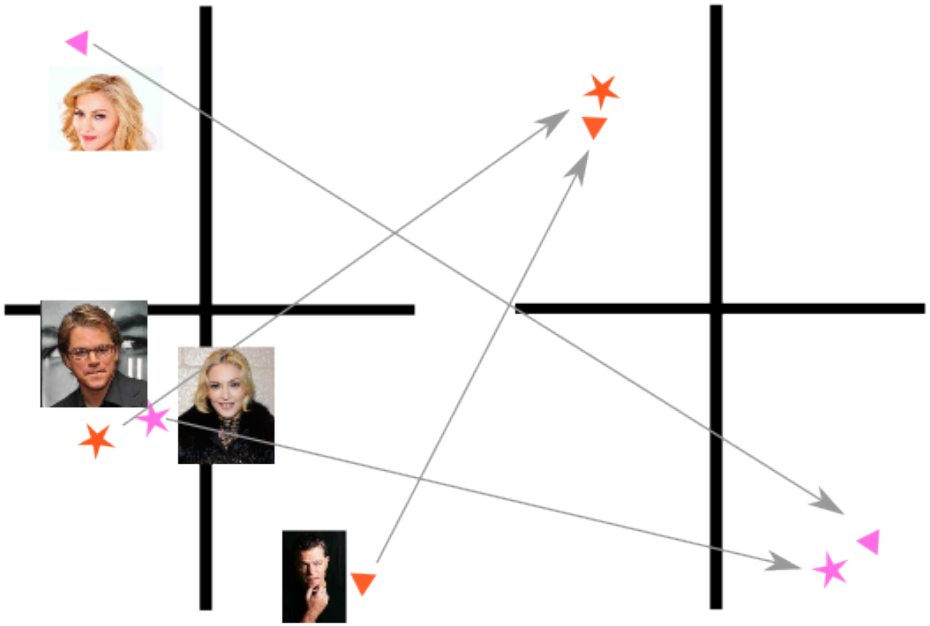
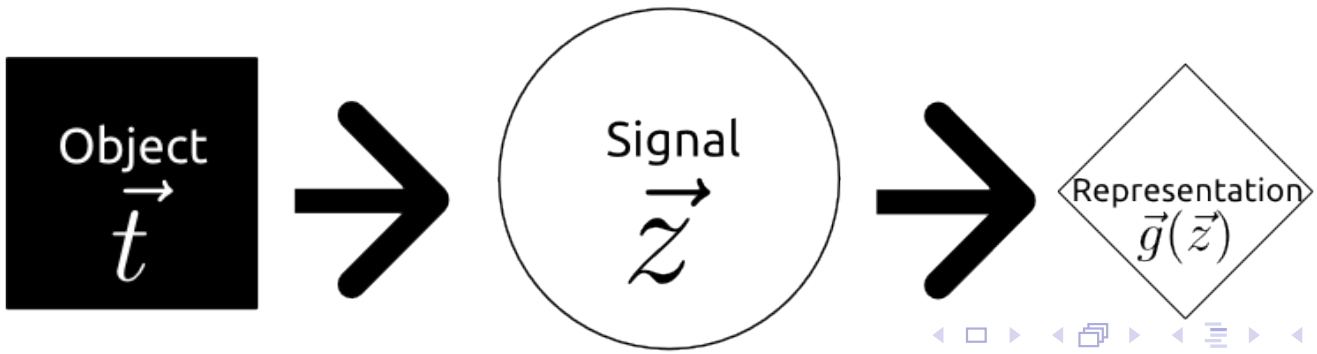


...captures the object space geometry





Why do we care?

- 1) Neuroscience. The brain is hypothesized to use representations for cognitive purposes

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- 1) Neuroscience. The brain is hypothesized to use representations for cognitive purposes
- 2) Machine learning. Representations turn out to be useful for many Machine Learning tasks!

How can we tell if a representation is good?

- Method 1: *Ground truth*. If we happen to know the object parameters $\vec{t}$ (e.g. we simulated the data).

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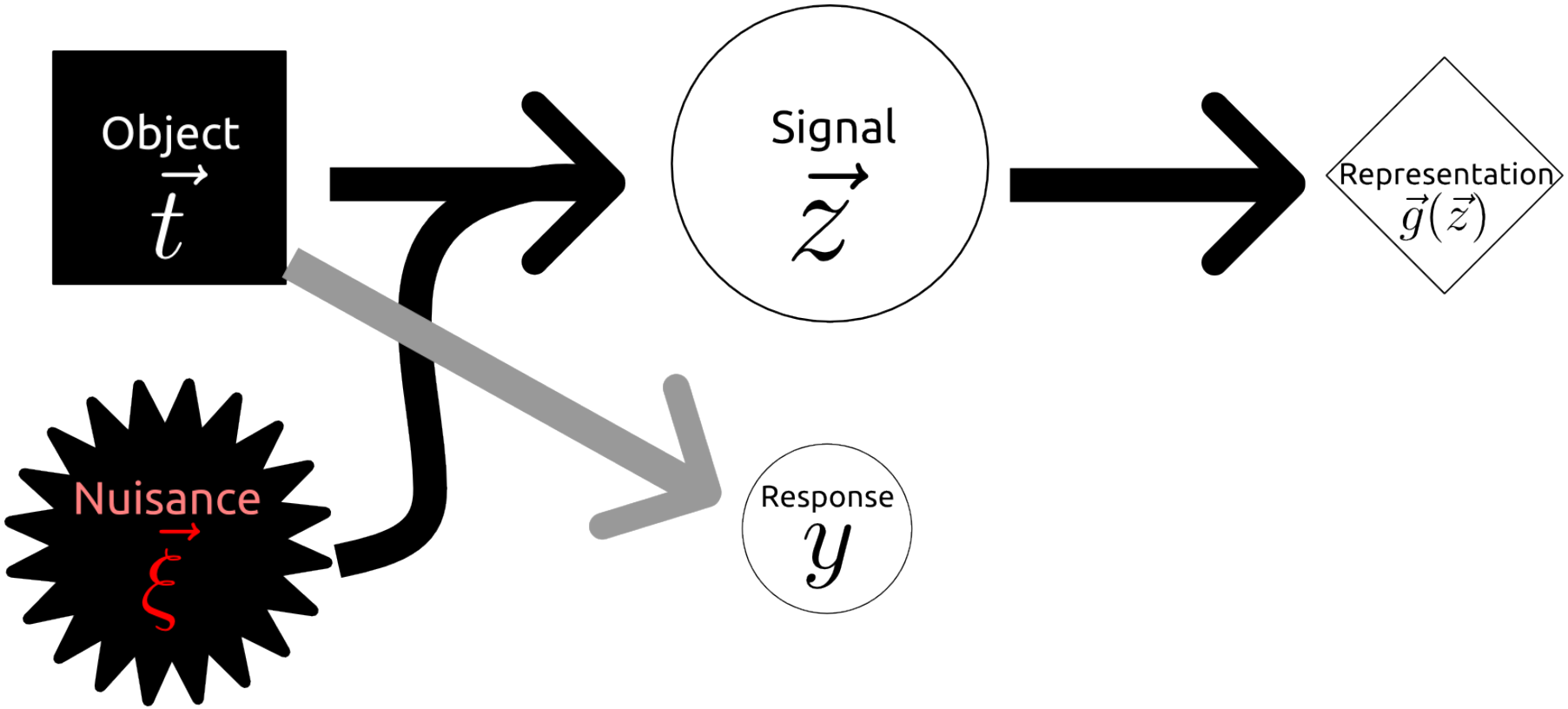
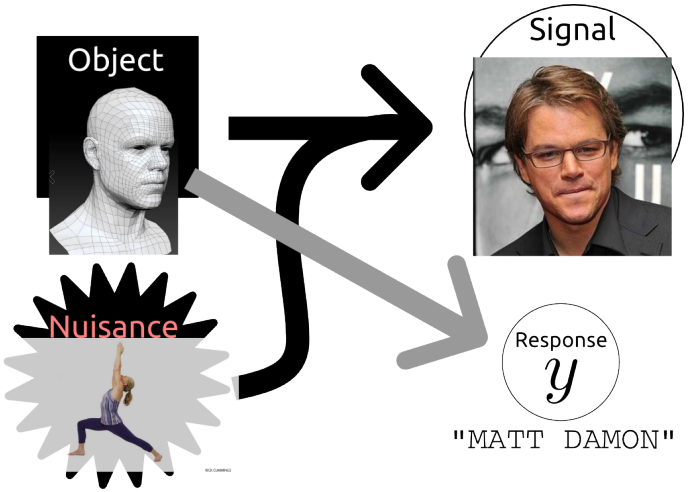
- Method 1: *Ground truth*. If we happen to know the object parameters $\vec{t}$ (e.g. we simulated the data).
- Method 2: *End result*. By the performance of the representation on a machine learning task.
- Method 3: *Supervised*. If we have a *response variable* Y which can be used to infer distances in $\vec{t}$.

Supervised evaluation of representations



Example: face recognition



- The ID of the individual is an appropriate *response* variable...

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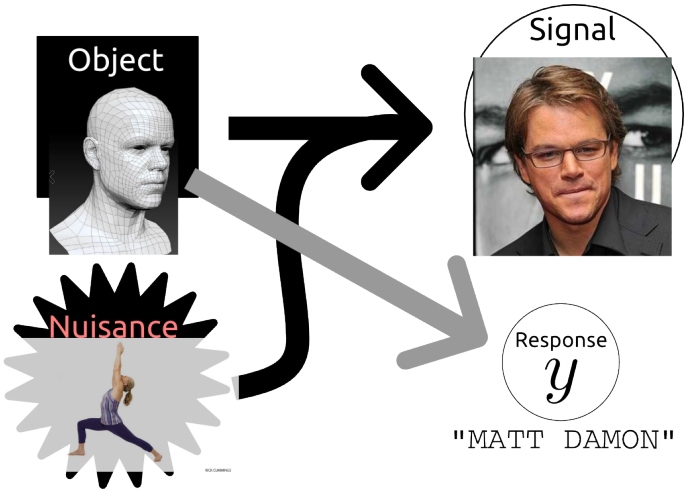

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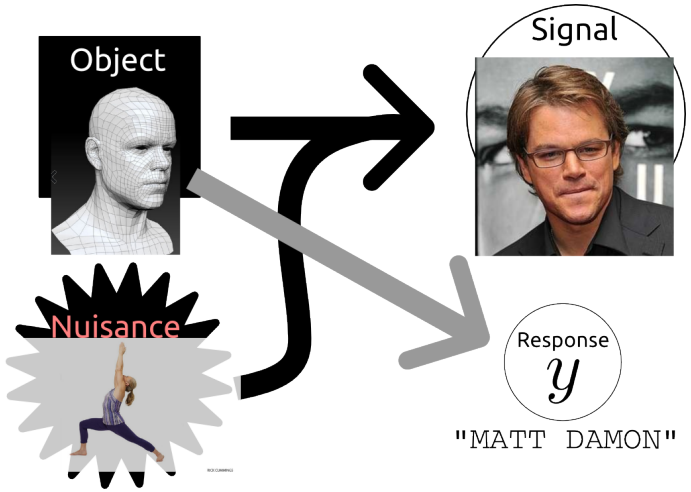


- The ID of the individual is an appropriate *response* variable...
- ...because two photos labeled with the same ID must belong to the same object $\vec{t}$
- That is, for $d(y, y')$ being the zero-one distance,

$$d(y, y') = 0 \Leftrightarrow d(\vec{t}, \vec{t}') = 0.$$

Outline

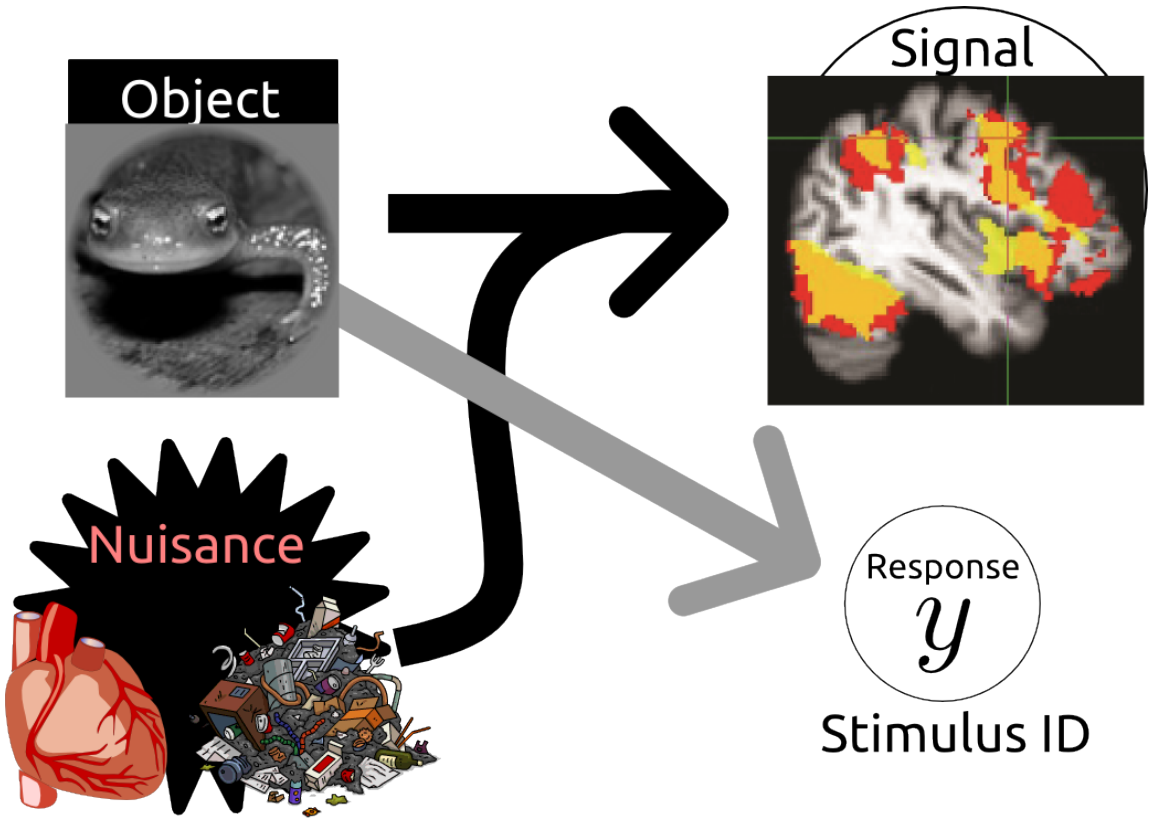
- *Previous work*. Identification accuracy, a method for evaluating representations
- *Contribution 1*. Extrapolation of identification accuracy.
- *Contribution 2*. Link between identification accuracy and mutual information.

Section 2

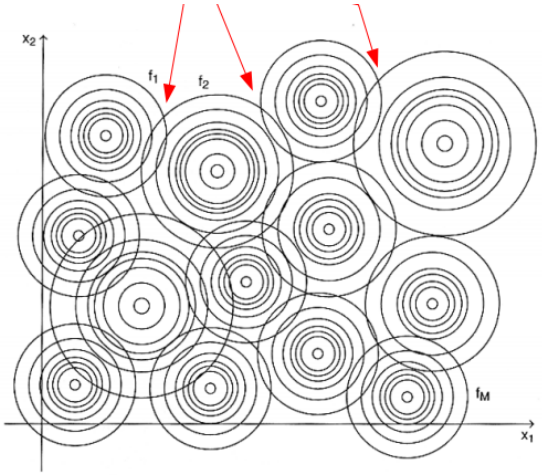
Identification

Identifying natural images from fMRI data

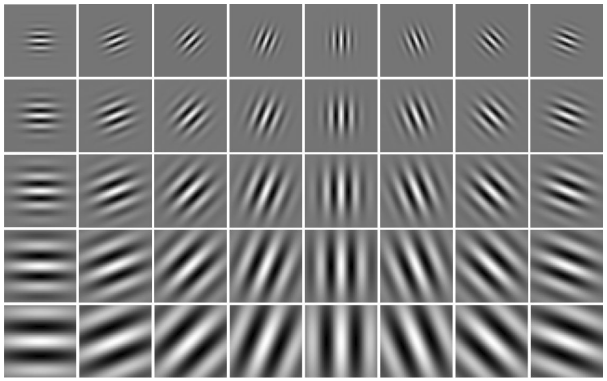
Kay, Naselaris, Prenger and Gallant (2008), *Nature*.



Comparing two different natural image bases



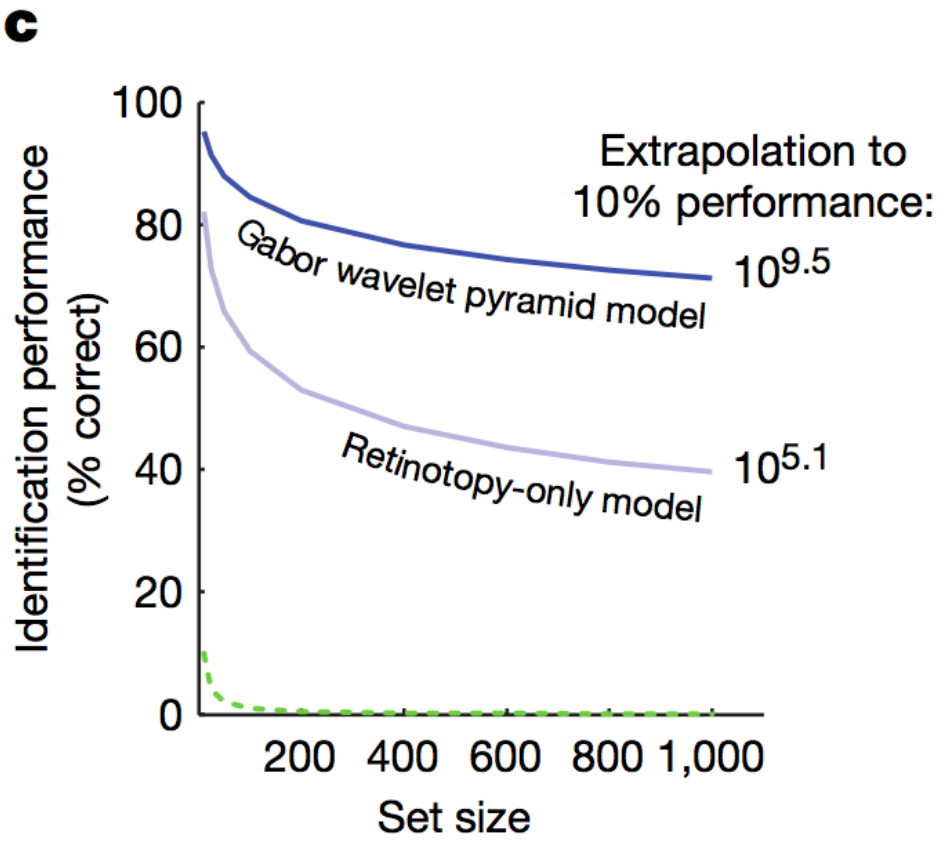
Retinotopic (Gaussian) basis



Gabor filter basis

Identifying natural images from fMRI data

Kay et al. used *identification accuracy* as a metric for the quality of the representation



Multiple-response regression

- Pairs $(x_i, y_i)_{i=1}^n$, where X is p-dimensional and Y is q-dimensional.
- Data matrices $\boldsymbol{X}_{n \times p}$, $\boldsymbol{Y}_{n \times q}$.
- For each column of Y, fit sparse model $Y^{(i)} \approx X^T \beta^{(i)} + \epsilon$, e.g. by using elastic net (Zou 2008),

$$\hat{\beta}^{(i)} = \text{argmin}_\beta ||\boldsymbol{X}^T \beta^{(i)} - Y^{(i)}||^2 + \lambda_2 ||\beta^{(i)}||_2^2 + \lambda_1 ||\beta^{(i)}||_1$$

Regression vs Identification accuracy

- Independent *test set* $(x_i^*, y_i^*)_{i=1}^k$.
- Use model to predict $\hat{y}_i^* = (x_i^*)^T \hat{B}$ for $i = 1, \ldots, k$.

Two ways to evaluate the predictive accuracy of the regression model:

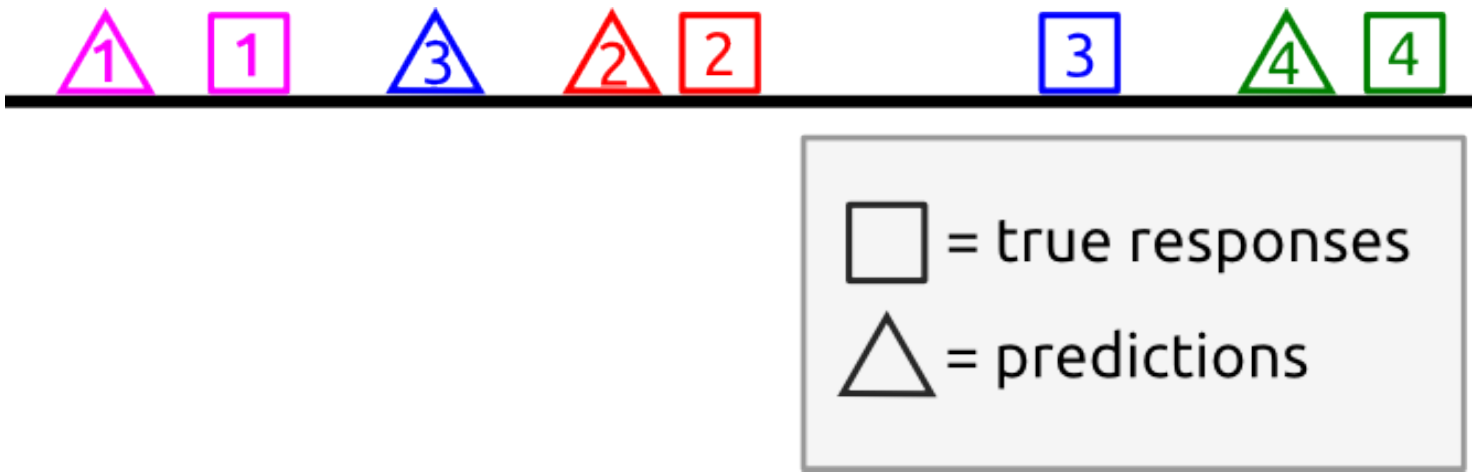
- Regression (mean squared-error) loss:

$$\text{MSE} = \frac{1}{k} \sum_{i=1}^{k} ||y_i^* - \hat{y}_i^*||^2.$$

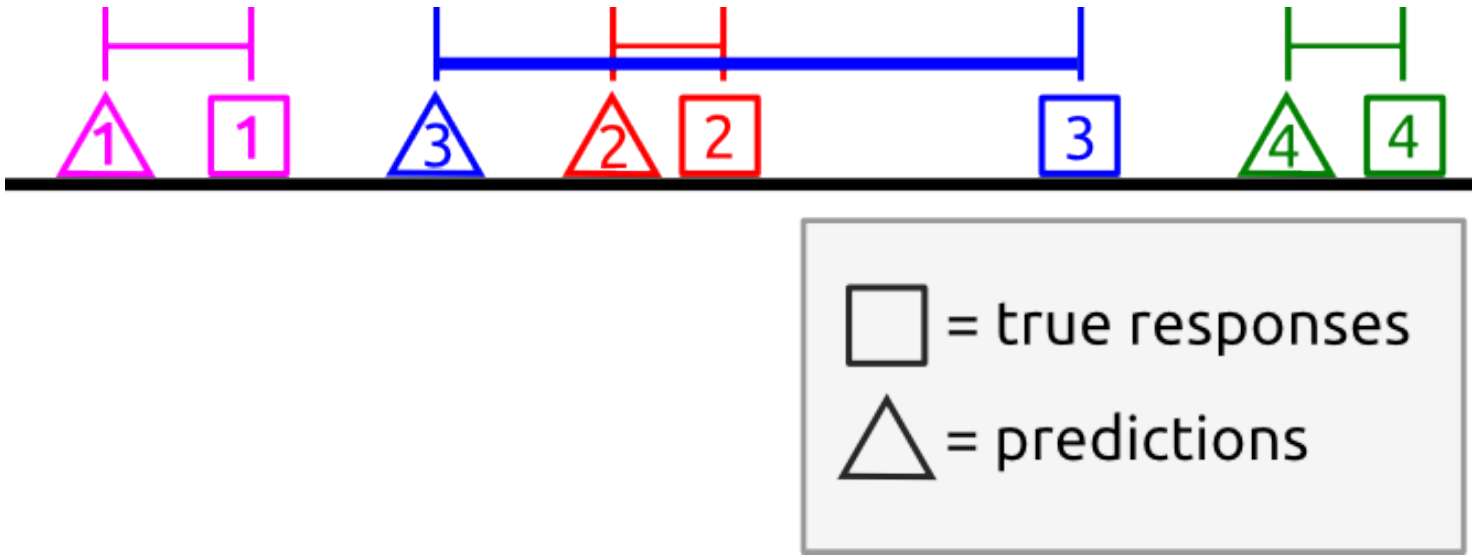
- Identification accuracy (Kay 2008):

$$\text{IdAcc}_k = \frac{1}{k} \sum_{i=1}^{k} I\{\hat{y}_i^* \text{ is nearest neighbor of } y_i^*\}.$$

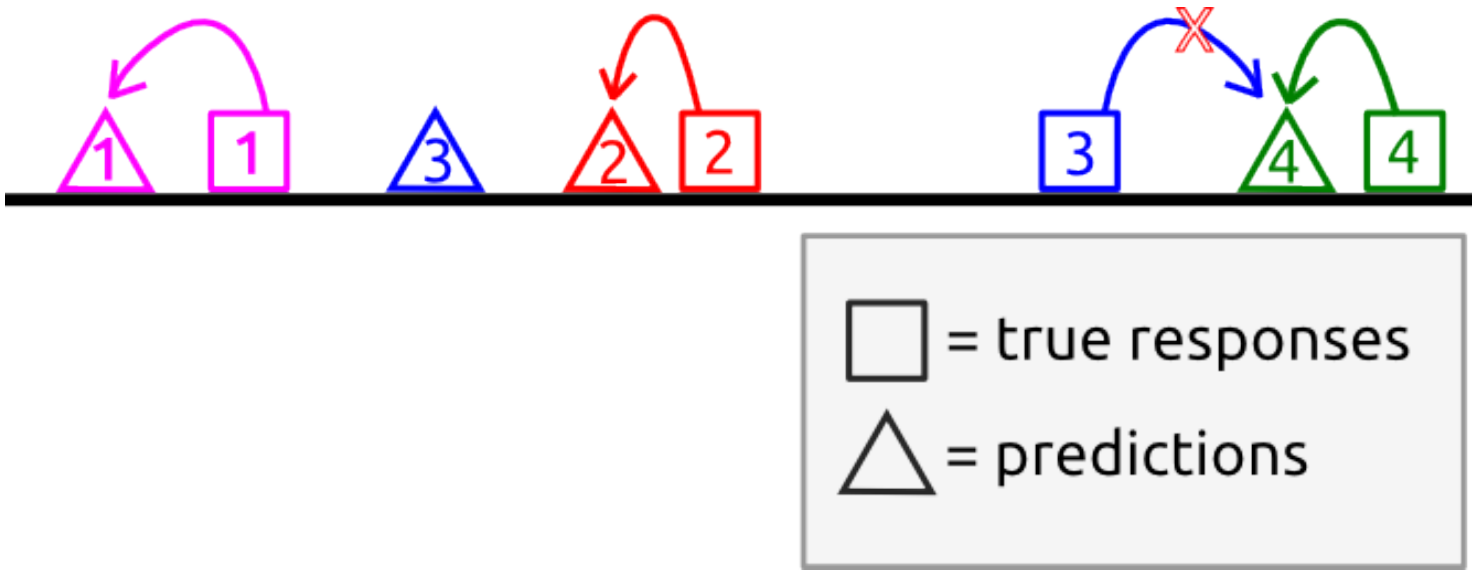
Regression vs Identification accuracy



Mean-squared error



Identification accuracy



Identification accuracy for comparing representations

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  - $\vec{g}_2$, Gabor filters

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$$\vec{Y} = B^T \vec{g}(\vec{Z}) + \epsilon$$

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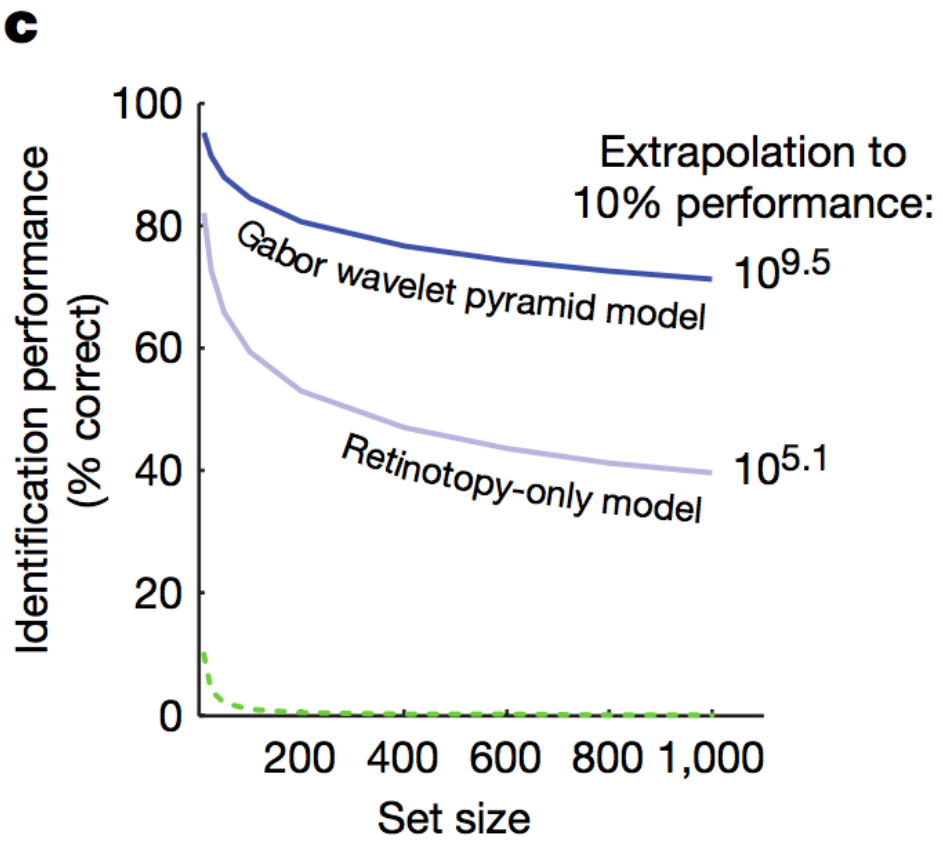
- Evaluate the identification accuracy on k test images.
- You can use a test set which is larger than k, and average the identification accuracy over k-sized subsamples

Identifying natural images from fMRI data

Gabor filters yields consistently higher accuracy than retinotopic model



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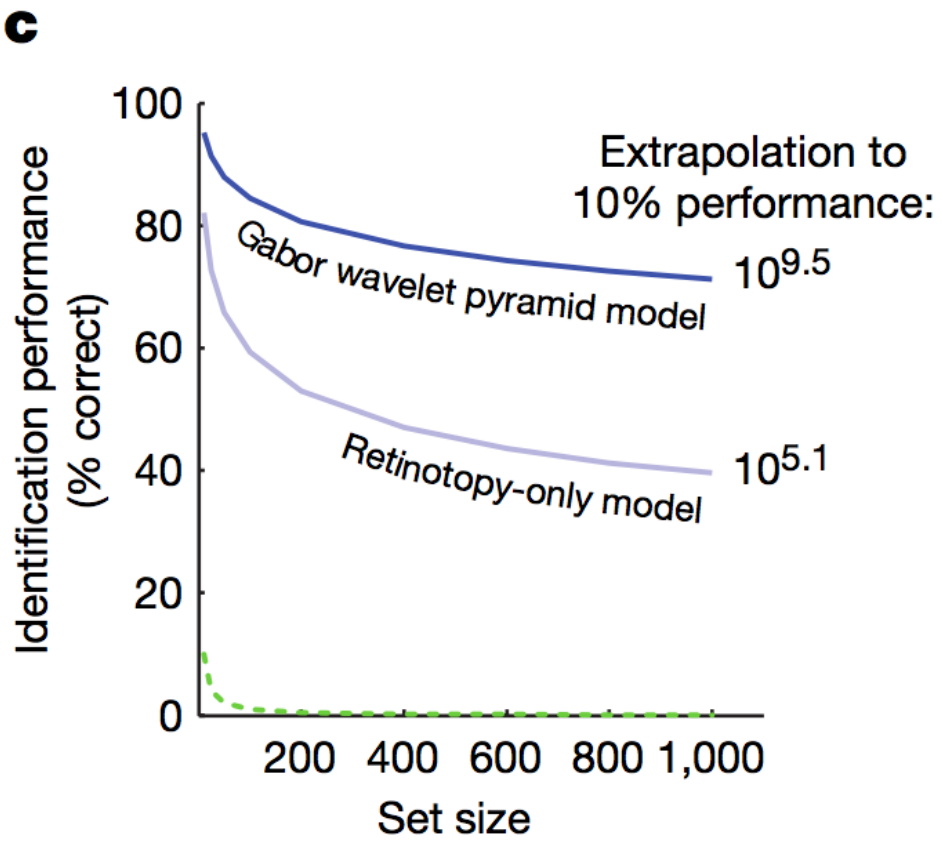


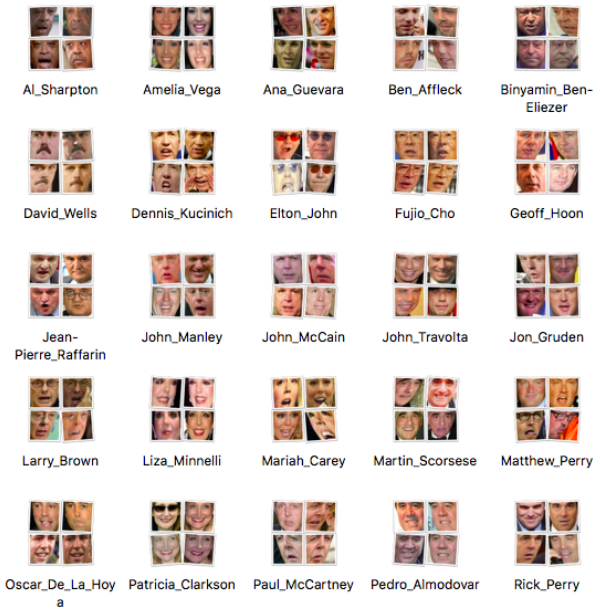
Q: Is this always the case? or could you also have intersecting curves?

Section 3

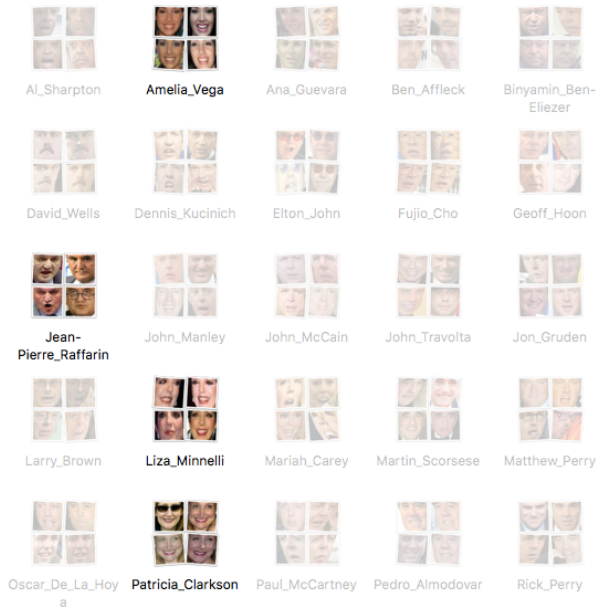
Extrapolation

Randomized multi-class classification

1. Population of categories $\pi(y)$

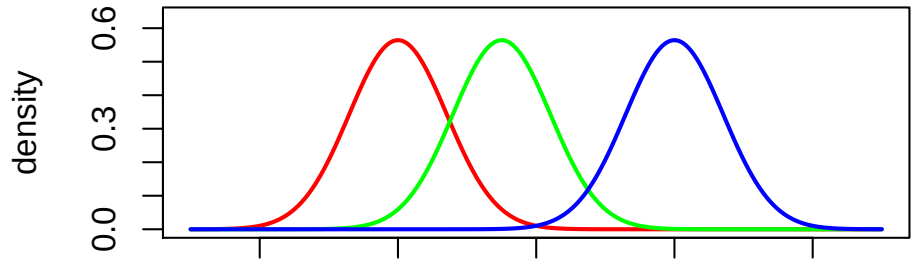


2. Subsample k labels, $y_1, \ldots, y_k$



Identification is a special case of a *randomized classification* task.

Toy example

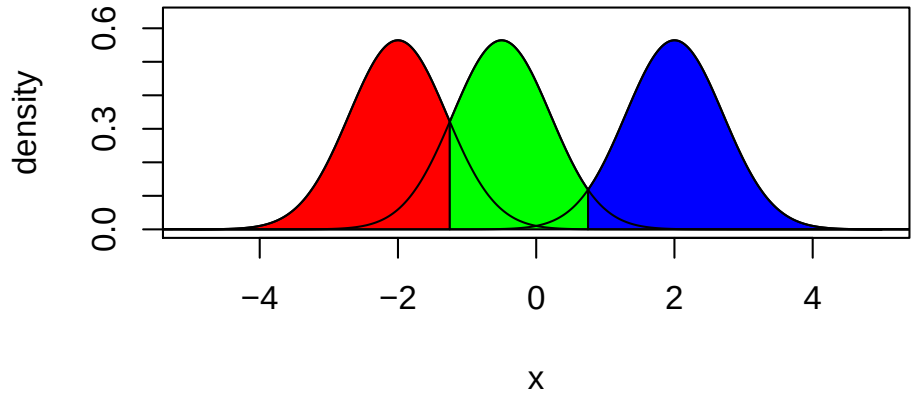


- Suppose $k = 3$, and we draw Y_1, Y_2, Y_3.
- The *Bayes rule* is the optimal classifier and depends on knowing the true densities:

$$\hat{y}(x) = \text{argmax}_{y_i} p(x|y_i)$$

- The *Bayes Risk*, which is the misclassification rate of the optimal classifier.

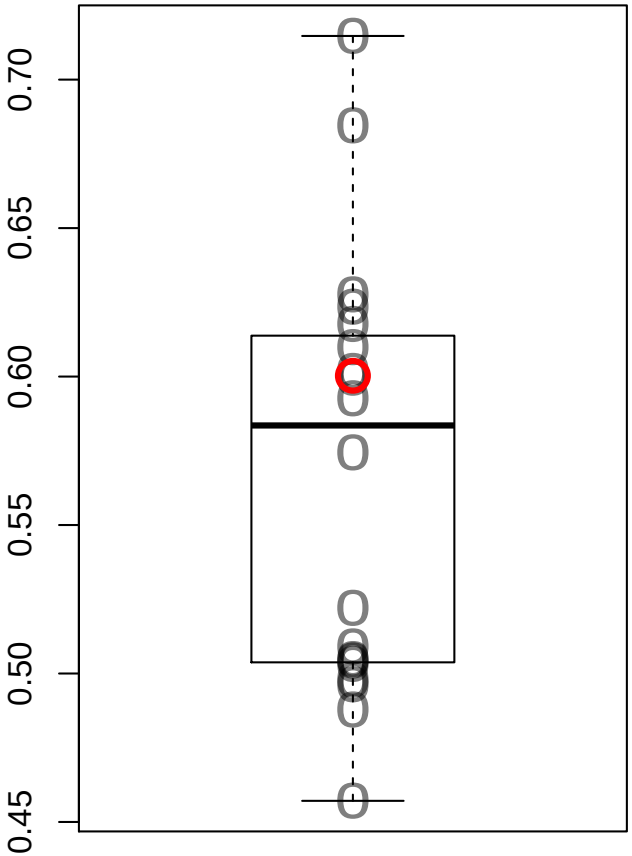
Toy example

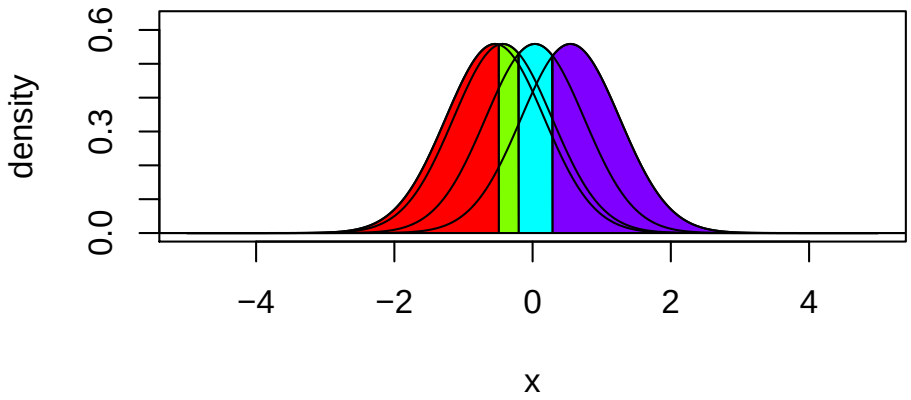


- The *Bayes Risk* is the expected test error of the Bayes rule,

$$\frac{1}{k}\sum_{i=1}^{k}\Pr[\hat{y}(x) \neq Y | Y = y_i]$$

Toy example





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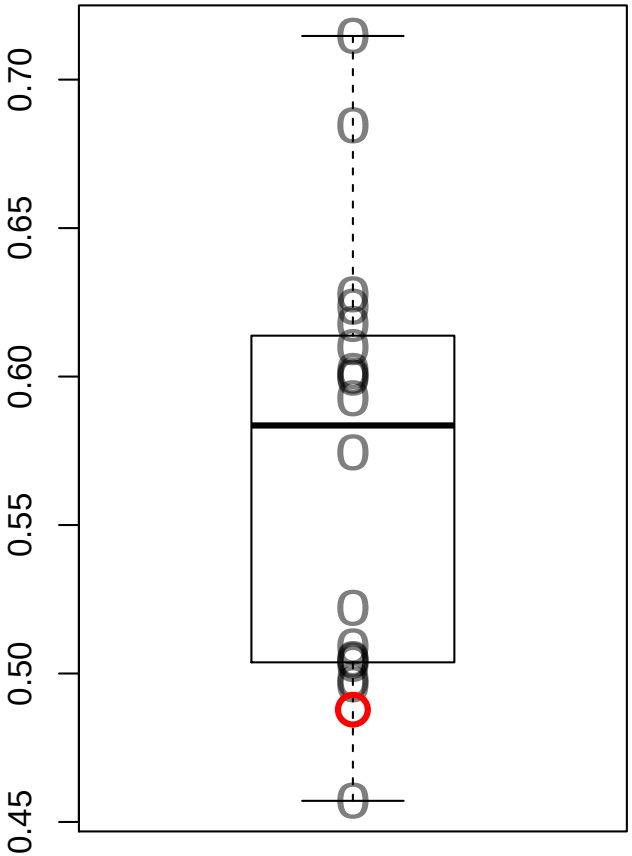

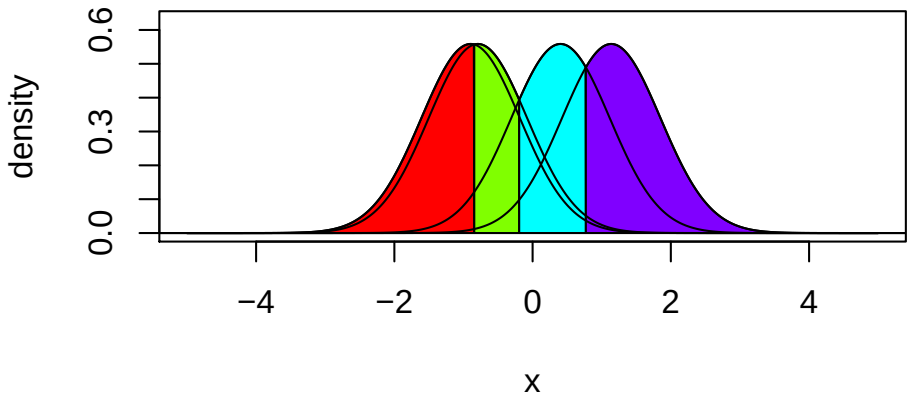

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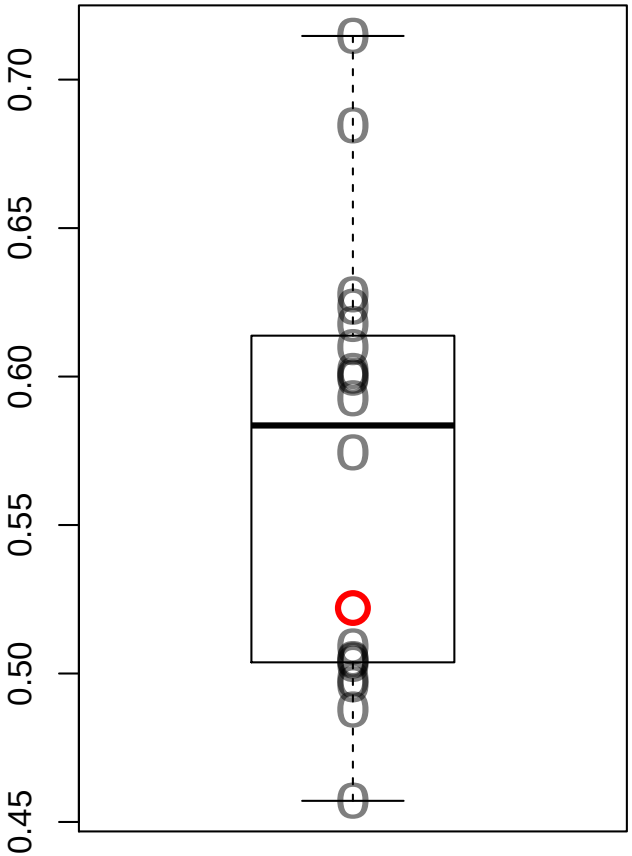

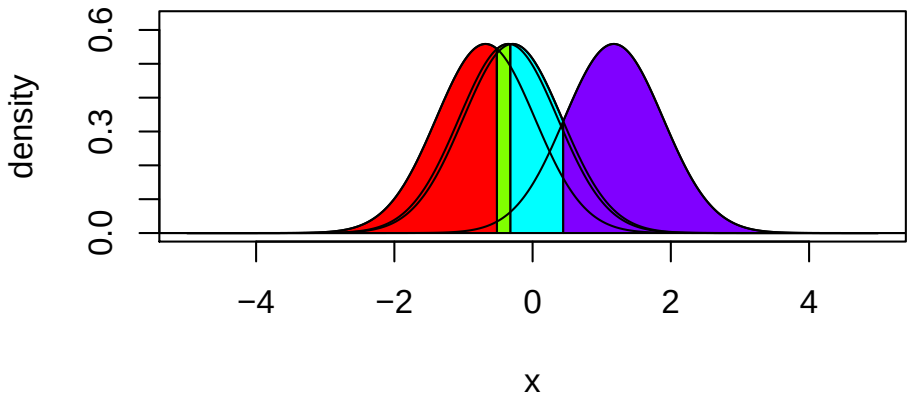

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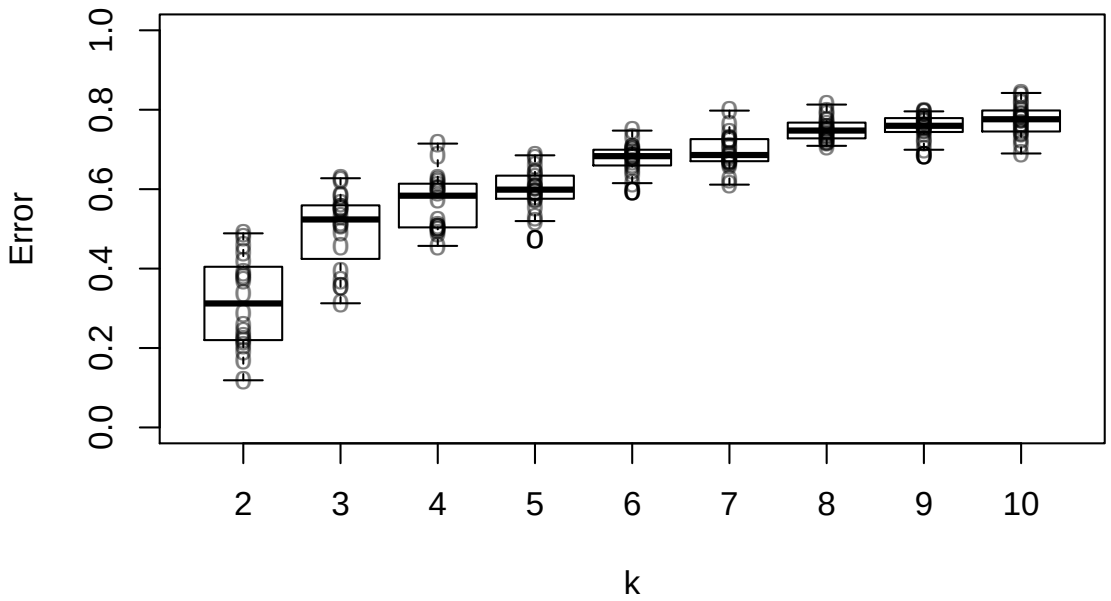

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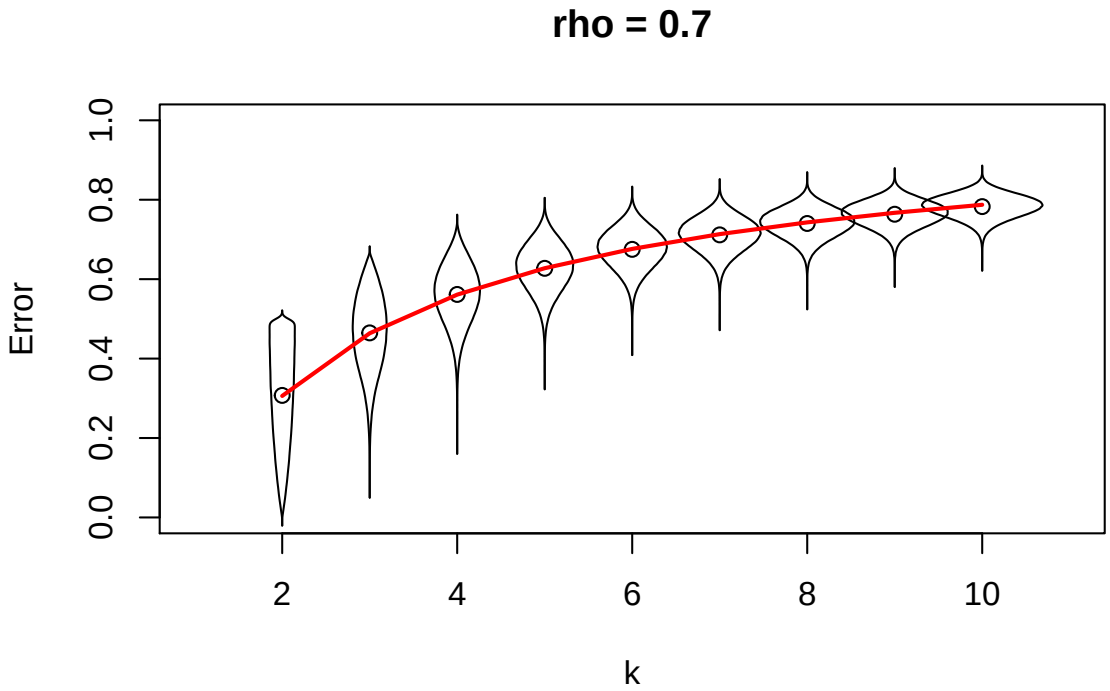


Theoretical Result

Theorem. (Z., Achanta, Benjamini.) Suppose π, $\{F_y\}_{y \in \mathcal{Y}}$ and marginal classifier $\mathcal{F}$ satisfy *(some regularity condition)*. Then, there exists some function $\bar{D}(u)$ on $[0,1] \to [0,1]$ such that the k-class average risk is given by

$$\text{AvRisk}_k = (k-1) \int \bar{D}(u) u^{k-2} du.$$

Theoretical Result

Theorem. (Z., Achanta, Benjamini.) Suppose π, $\{F_y\}_{y \in \mathcal{Y}}$ and marginal classifier $\mathcal{F}$ satisfy *(some regularity condition)*. Then, there exists some function $\bar{D}(u)$ on $[0,1] \to [0,1]$ such that the k-class average risk is given by

$$\text{AvRisk}_k = (k-1) \int \bar{D}(u) u^{k-2} du.$$

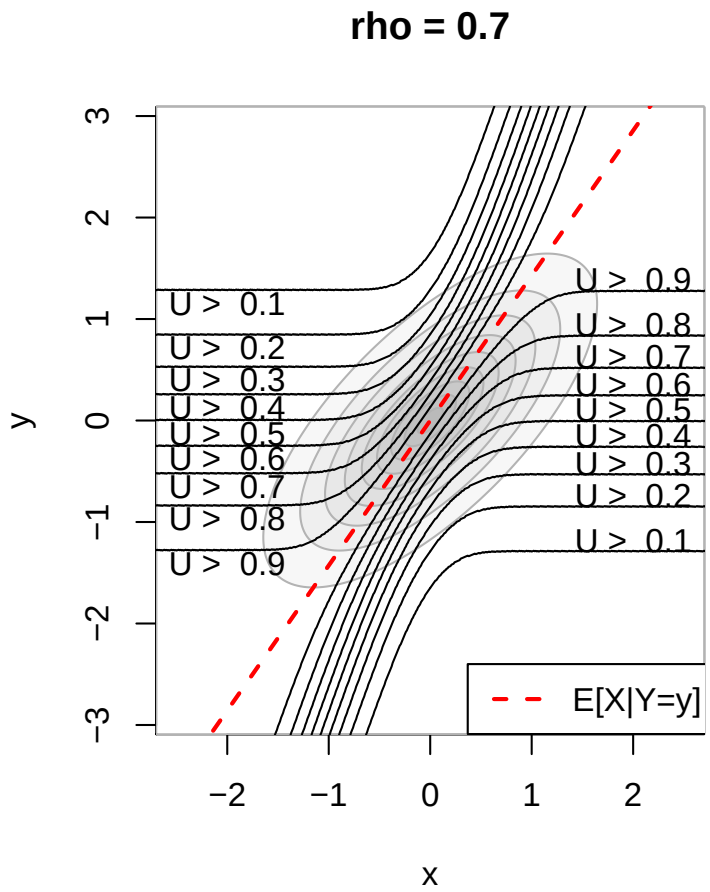
Remark. This result also applies to the *Bayes identification risk*.

Defining the U-function

Define $U_x(y)$ as follows:

- Suppose we have test instance (face) x whose true label (person) is y.
- Let Y' be a random *incorrect* label (person).
- Use the classifier to guess whether x belongs to y or Y'.
- Define $U_x(y)$ as the probabilility of success (randomizing over training data).

Toy example



$$U_y(x) = \Pr[d(x, \rho Y') > d(x, \rho y)], \text{ for } Y' \sim N(0, 1).$$

Defining $\bar{D}(u)$

- Define random variable as $U_Y(X)$ for (Y, X) drawn from the joint distribution.

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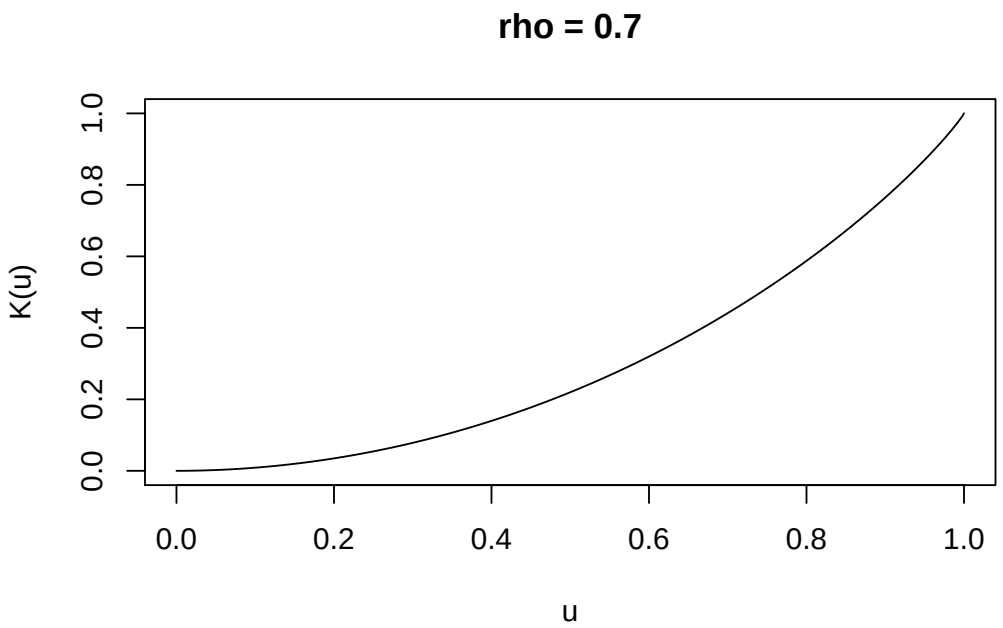
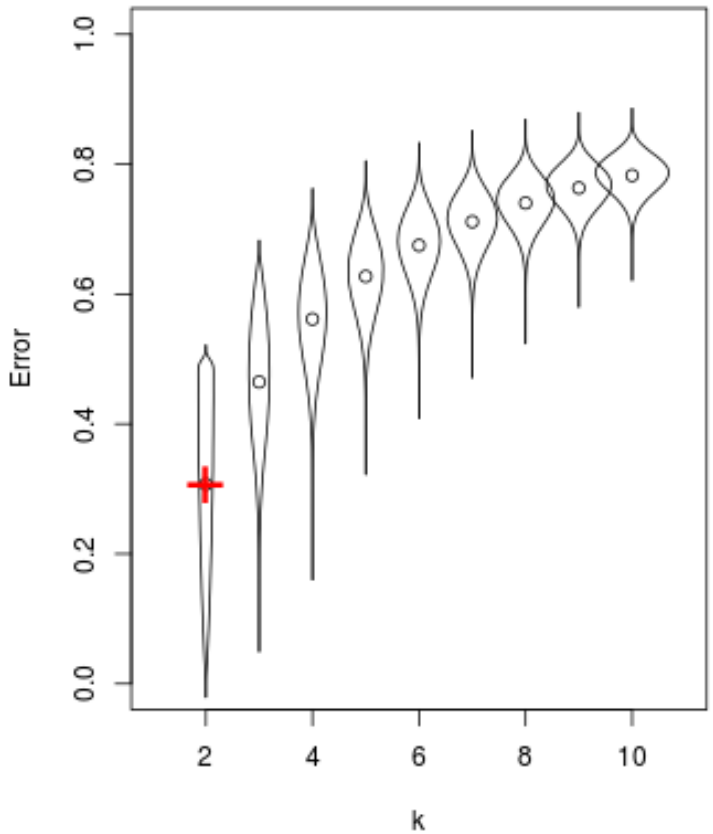
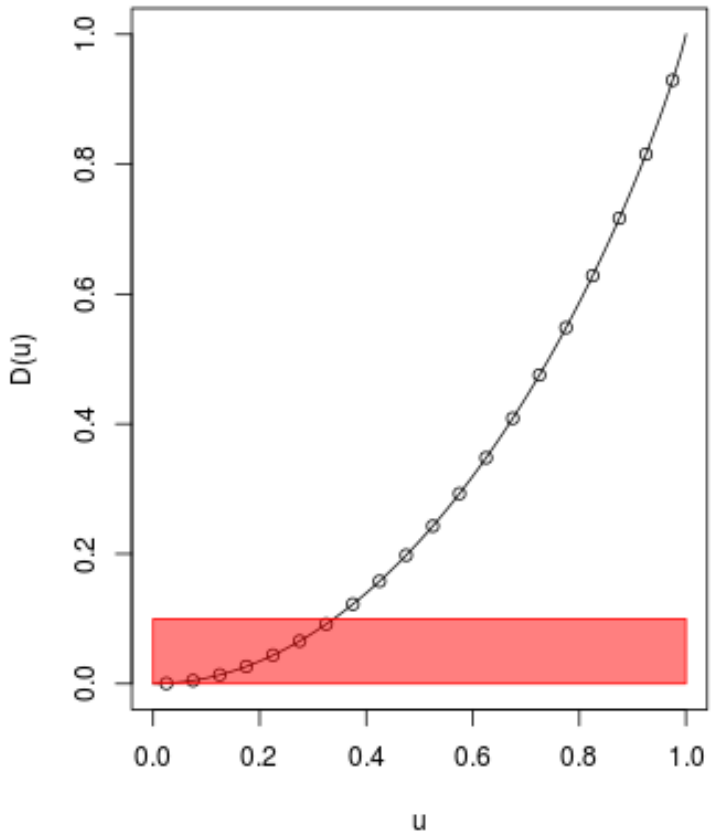


Computing average risk

$$\text{AvRisk}_k = (k-1)\int \bar{D}(u)u^{k-2}du.$$

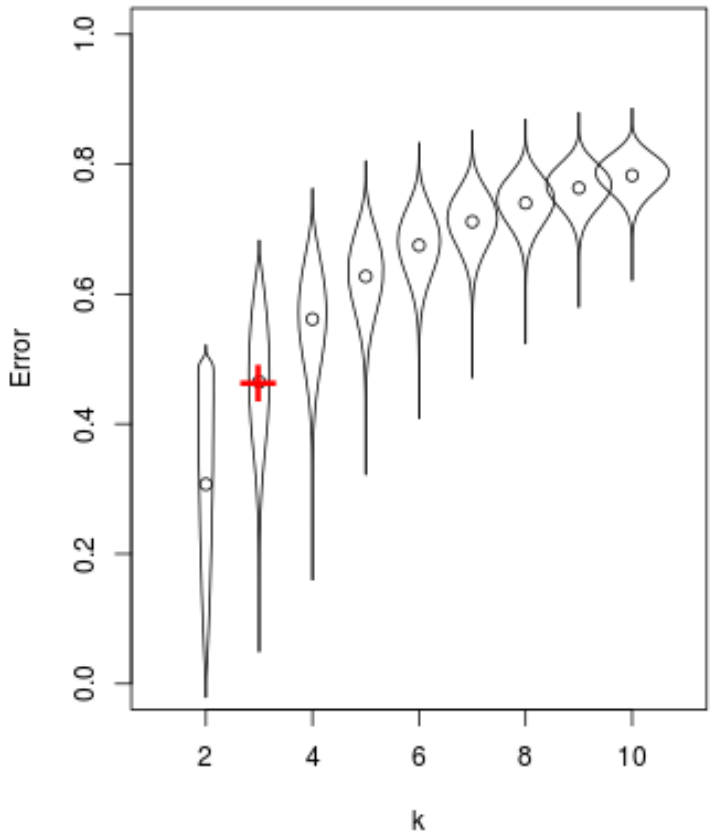


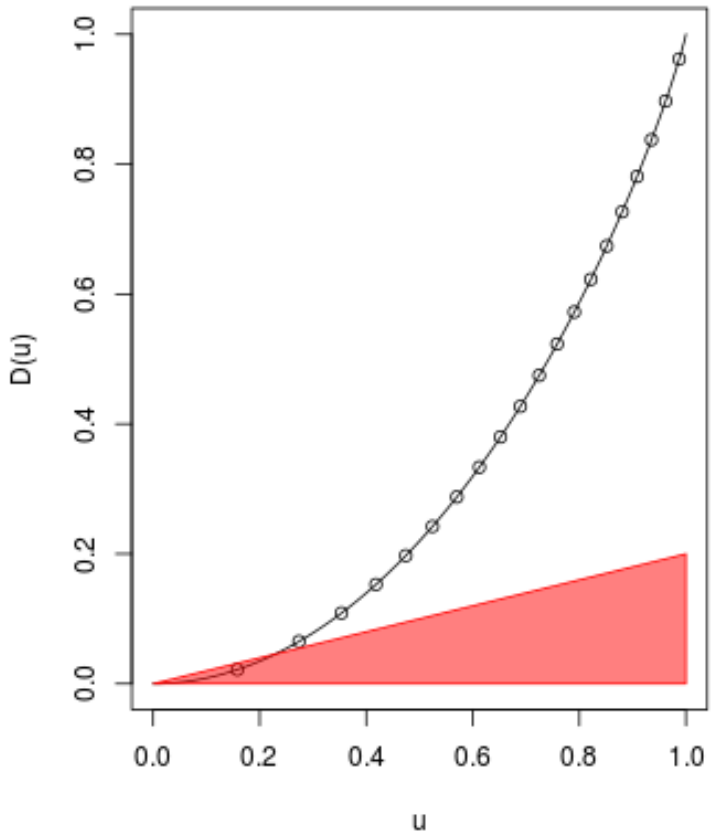


$(k = 2)$

Computing average risk

$$\text{AvRisk}_k = (k-1)\int \bar{D}(u)u^{k-2}du.$$

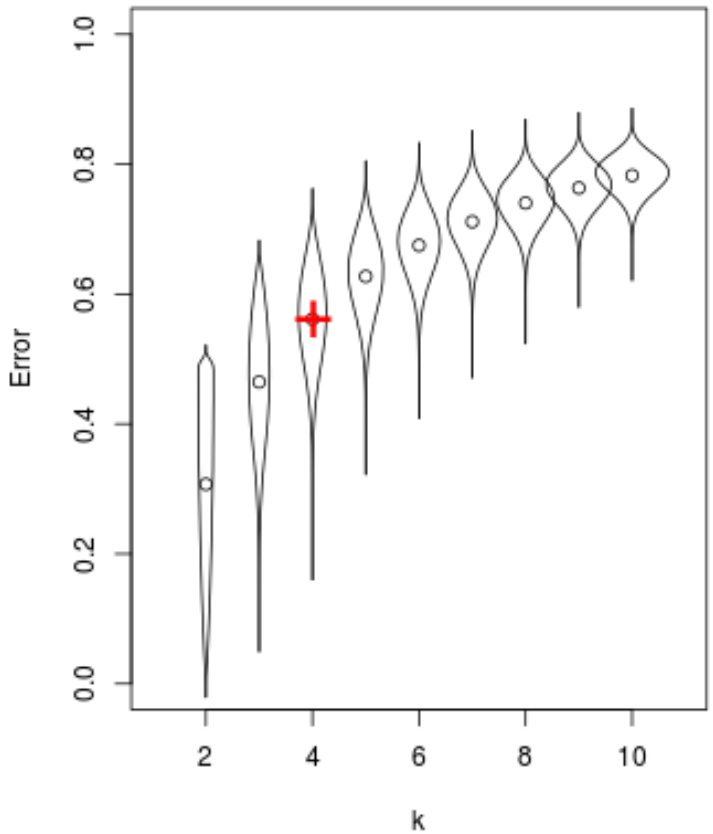


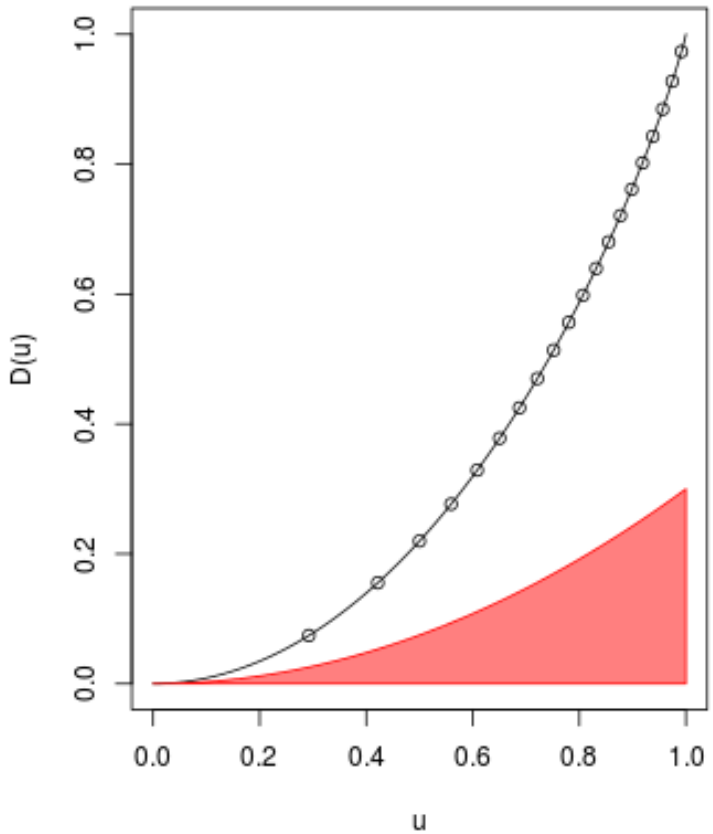


$(k = 3)$

Computing average risk

$$\text{AvRisk}_k = (k-1)\int \bar{D}(u)u^{k-2}du.$$

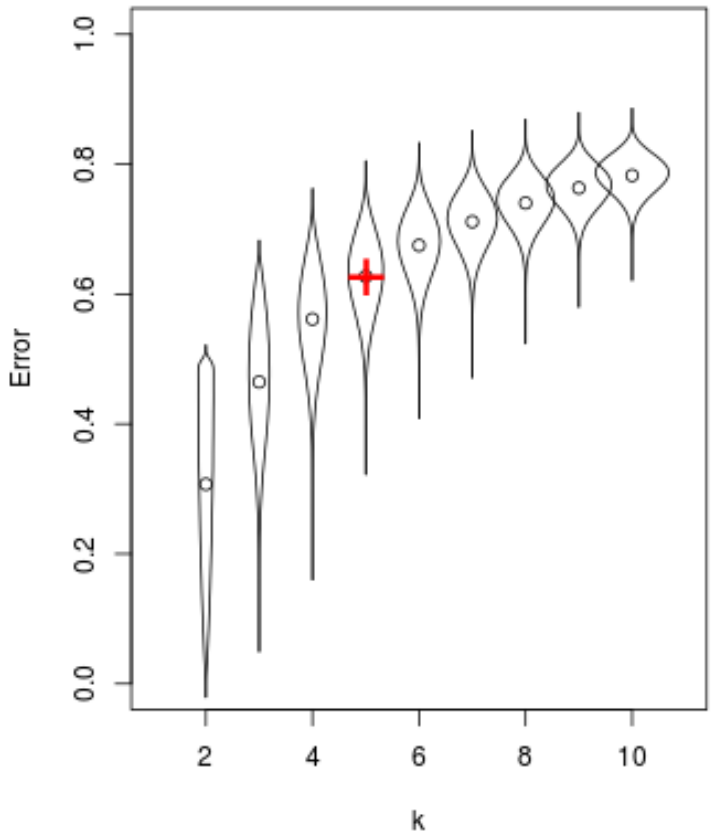


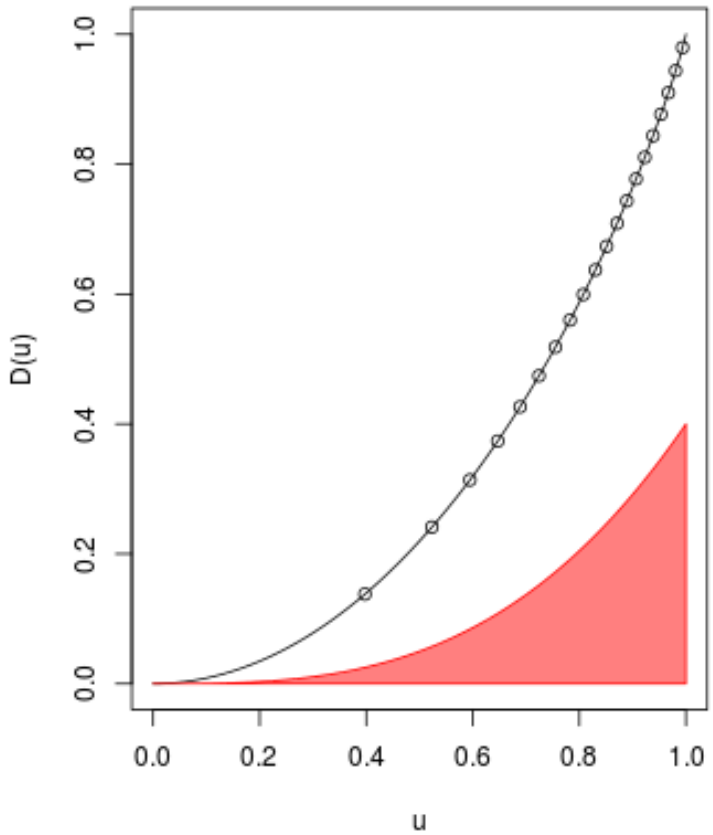


$(k = 4)$

Computing average risk

$$\text{AvRisk}_k = (k-1)\int \bar{D}(u)u^{k-2}du.$$

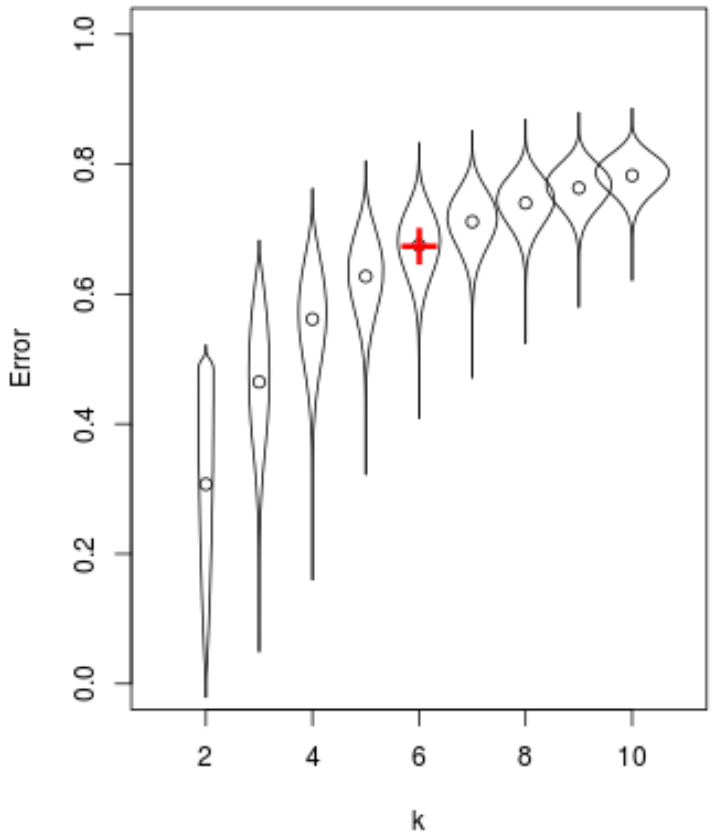


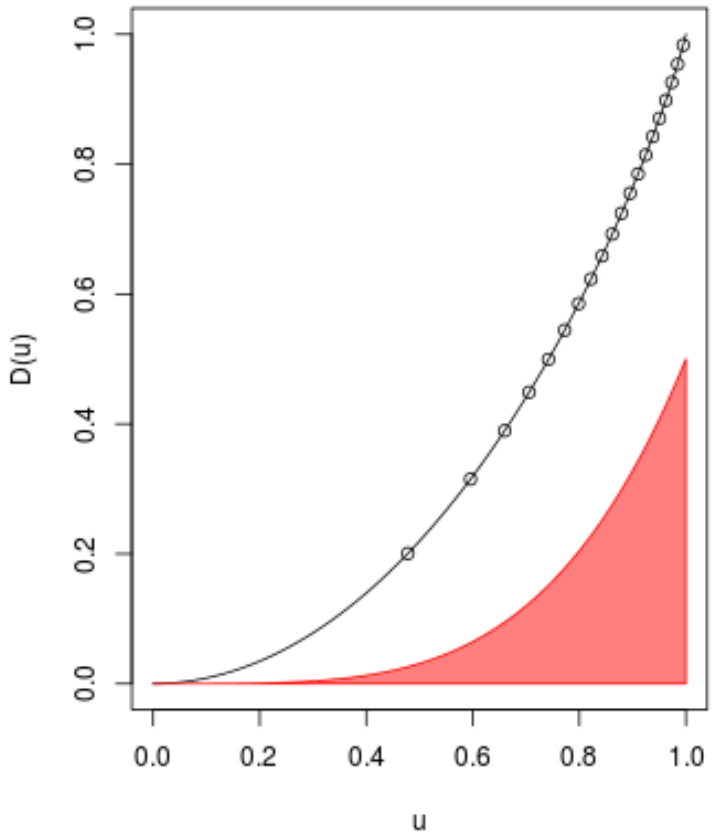


$(k = 5)$

Computing average risk

$$\text{AvRisk}_k = (k-1)\int \bar{D}(u)u^{k-2}du.$$

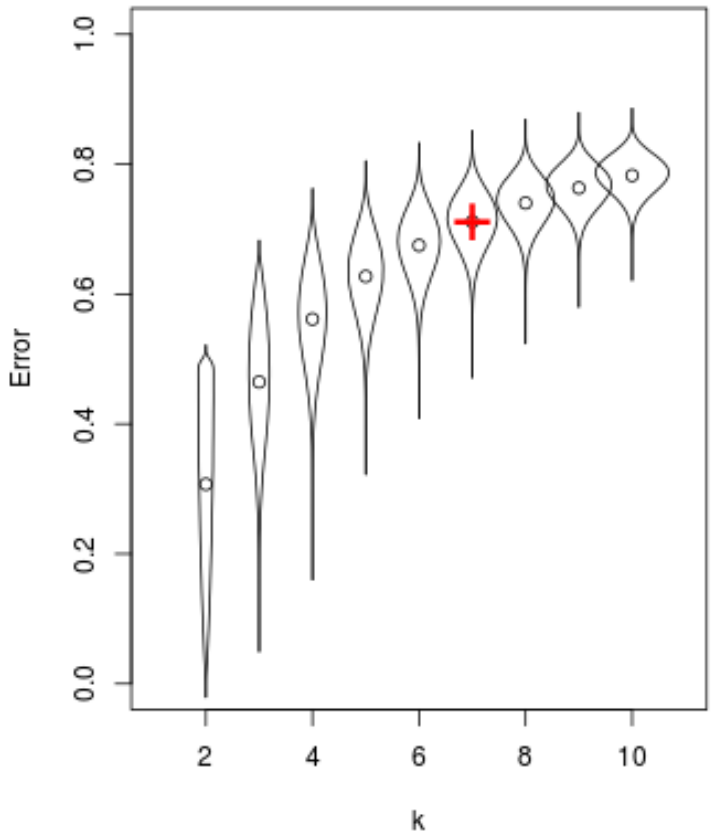


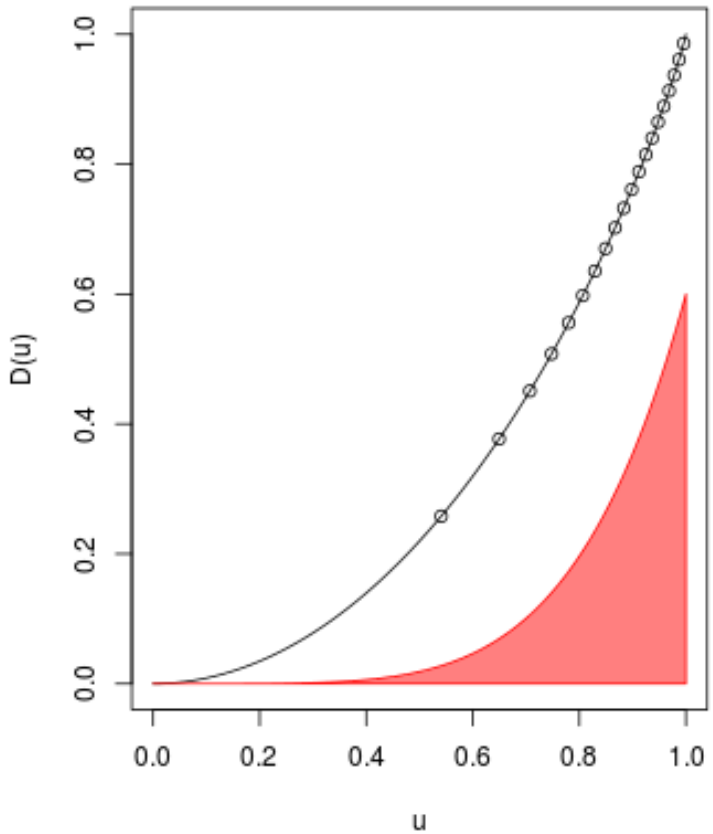


$(k = 6)$

Computing average risk

$$\text{AvRisk}_k = (k-1)\int \bar{D}(u)u^{k-2}du.$$

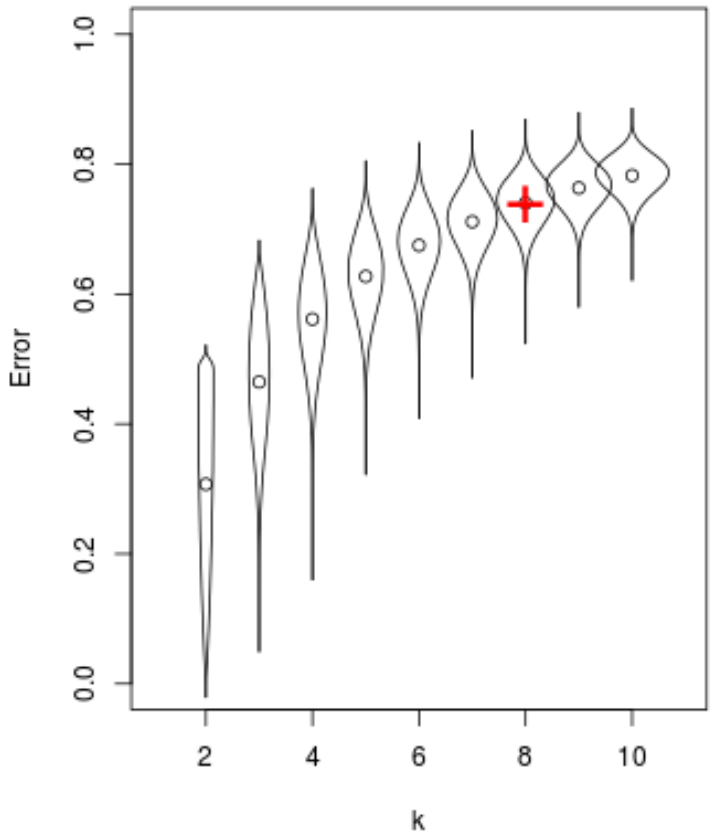


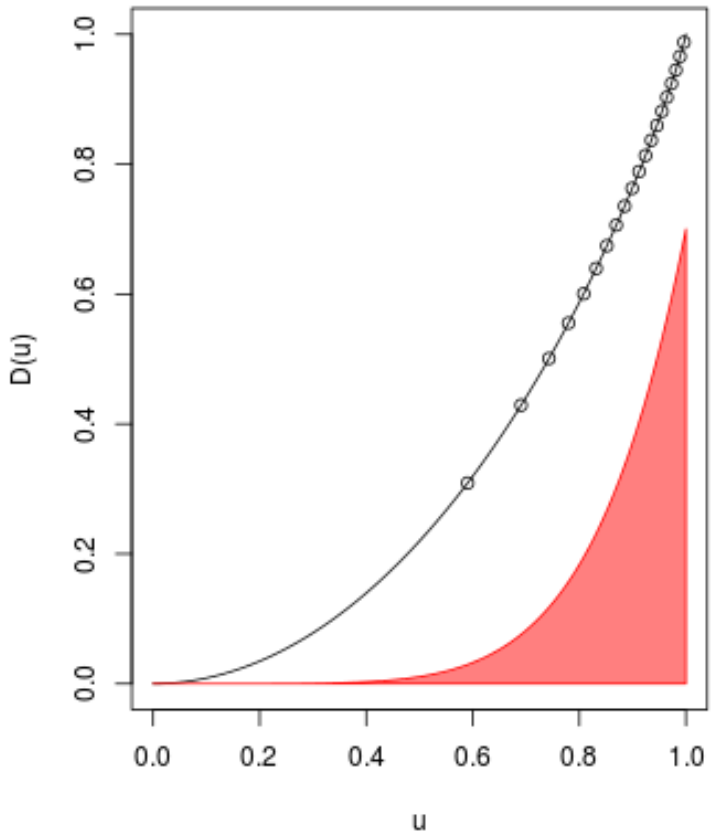


$(k = 7)$

Computing average risk

$$\text{AvRisk}_k = (k-1)\int \bar{D}(u)u^{k-2}du.$$

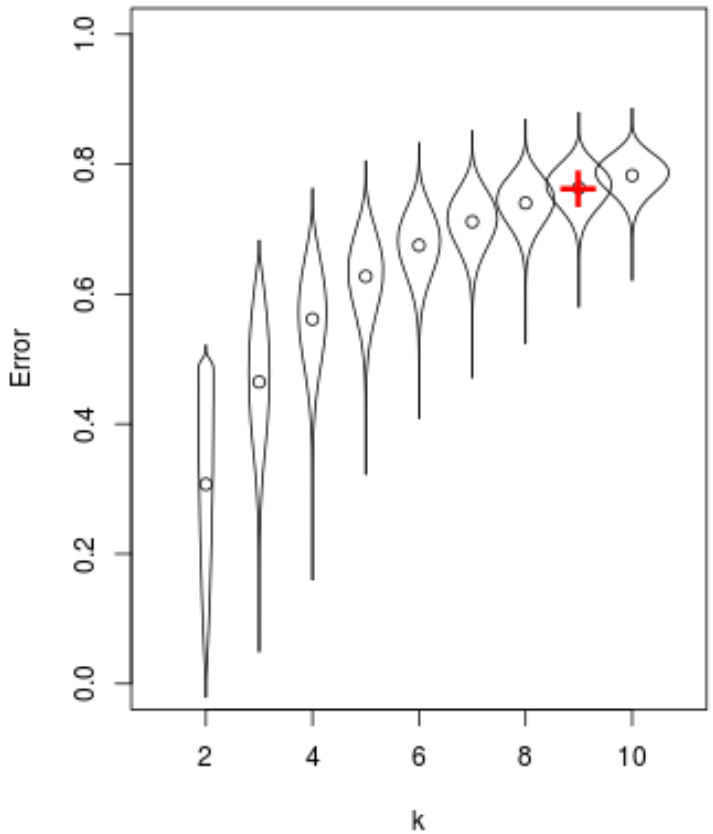


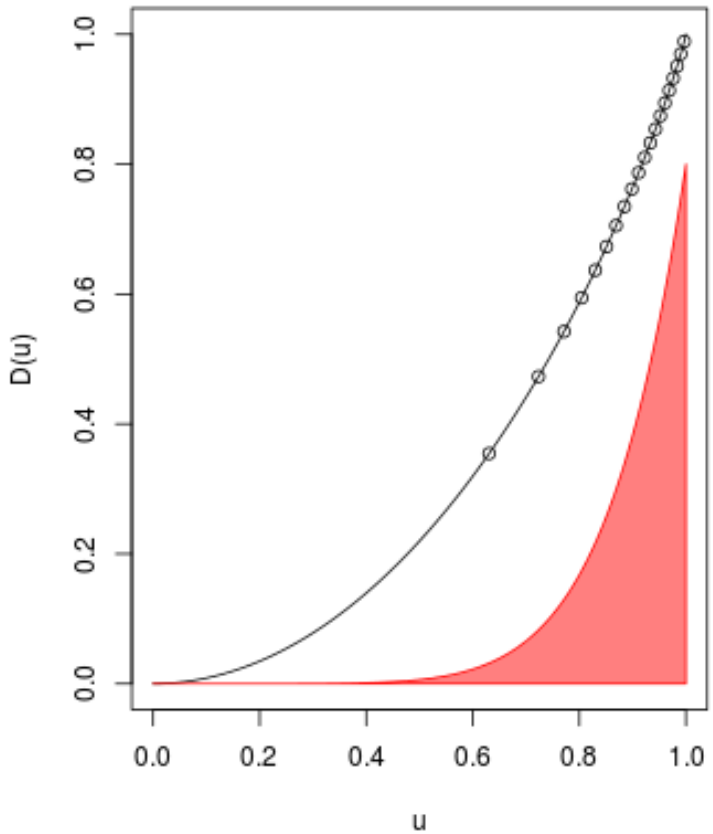


$(k = 8)$

Computing average risk

$$\text{AvRisk}_k = (k-1)\int \bar{D}(u)u^{k-2}du.$$

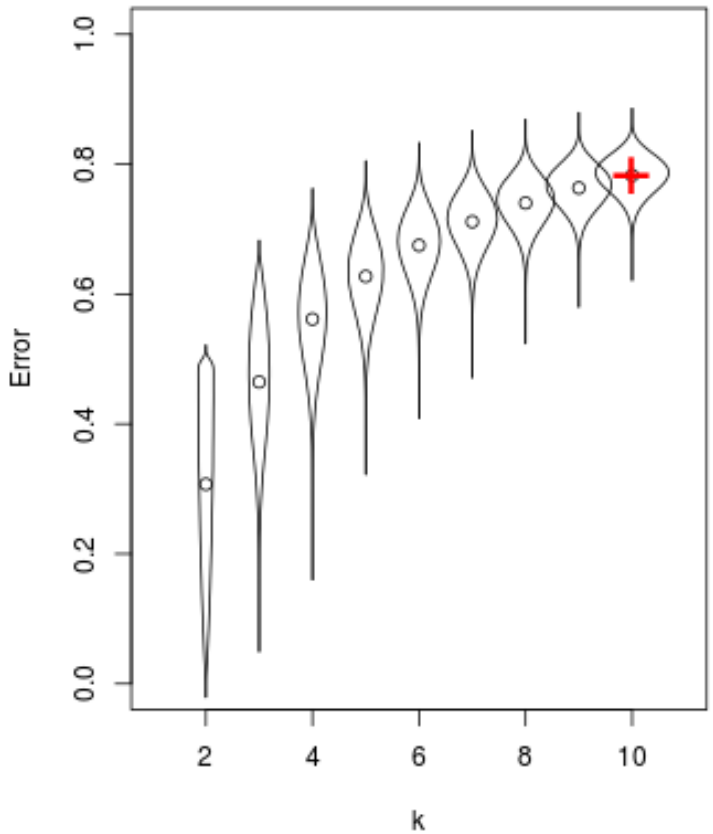


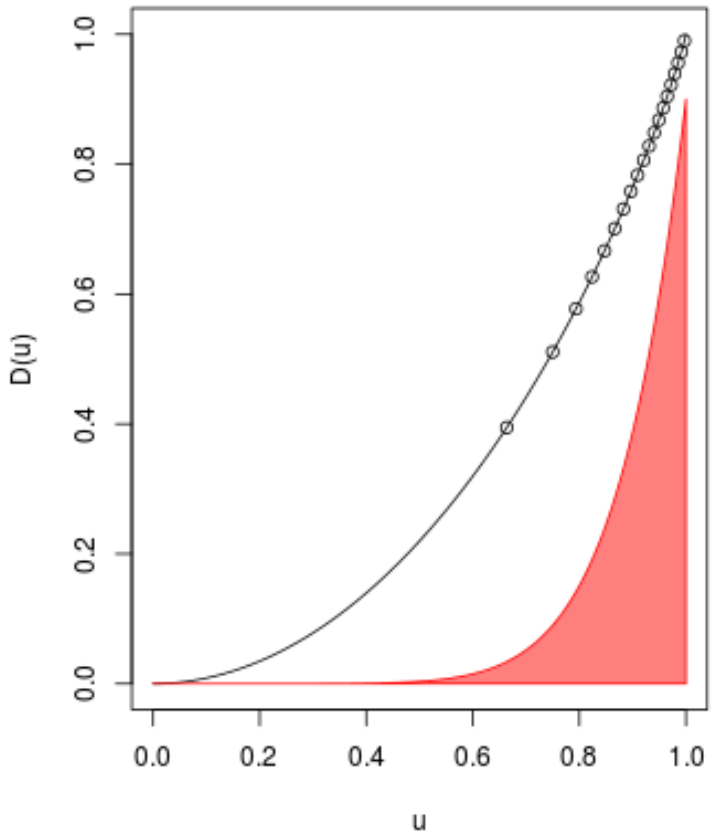


$(k = 9)$

Computing average risk

$$\text{AvRisk}_k = (k-1)\int \bar{D}(u)u^{k-2}du.$$





$(k = 10)$

Implication: estimate $\bar{D}(u)$ to predict risk

- Theoretical result links k-class average risk to $\bar{D}(u)$ function
- In real data, we do not know $\bar{D}(u)$ since it depends on the unknown joint distribution
- However, given a model, we can estimate $\bar{D}(u)$

So... can accuracy curves intersect?

- In general, the answer is *yes*.

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- However, we will see in the next section that under *high-dimension* assumptions, the Bayes accuracy curves do *not* intersect.

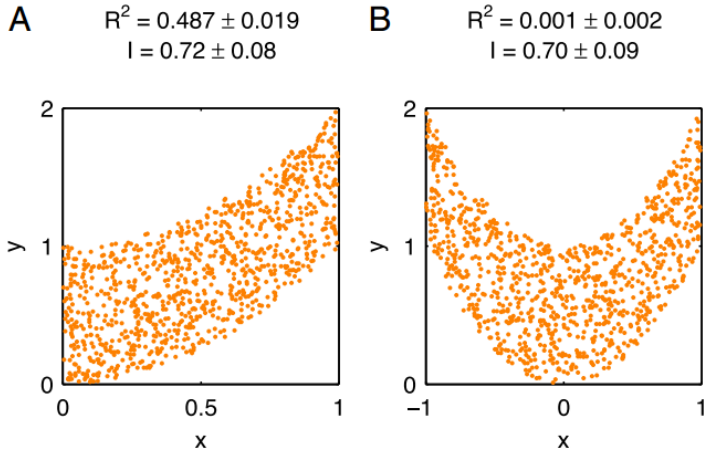
So... can accuracy curves intersect?

- In general, the answer is *yes*.
- However, we will see in the next section that under *high-dimension* assumptions, the Bayes accuracy curves do *not* intersect.
- Therefore, a single parameter, the *mutual information*, suffices to summarize the entire curve.

Section 4

Mutual Information

Mutual information $I(X; Y)$



Introduced in Shannon's 1948 paper, "A mathematical theory of communication"

$$I(X; Y) = \int \log \left(\frac{p(x, y)}{p(x)p(y)} \right) p(x, y) dxdy$$

Image credit Kinney et al. 2014.

Result 1. Lower bound for mutual information

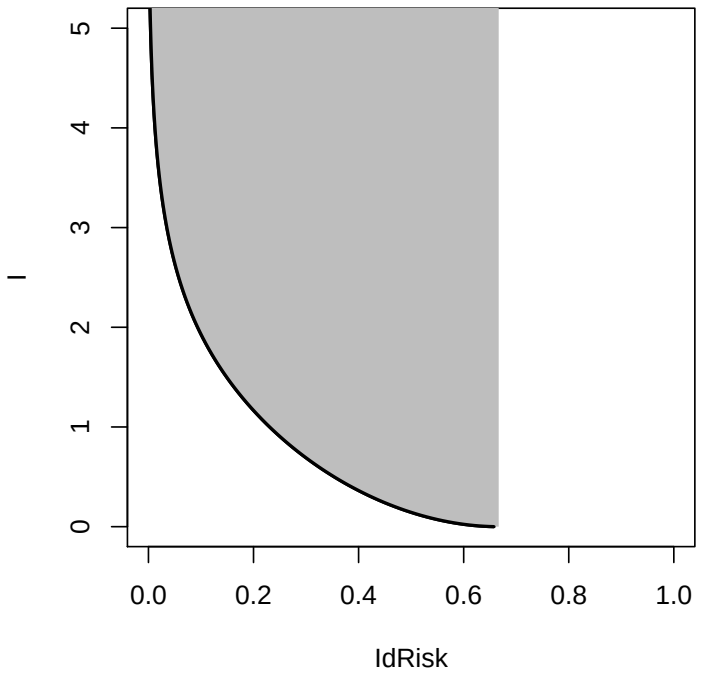
- Define the identification risk as the expected identification loss

$$\text{IdRisk}_k = \mathbf{E}[\text{IdLoss}_k]$$

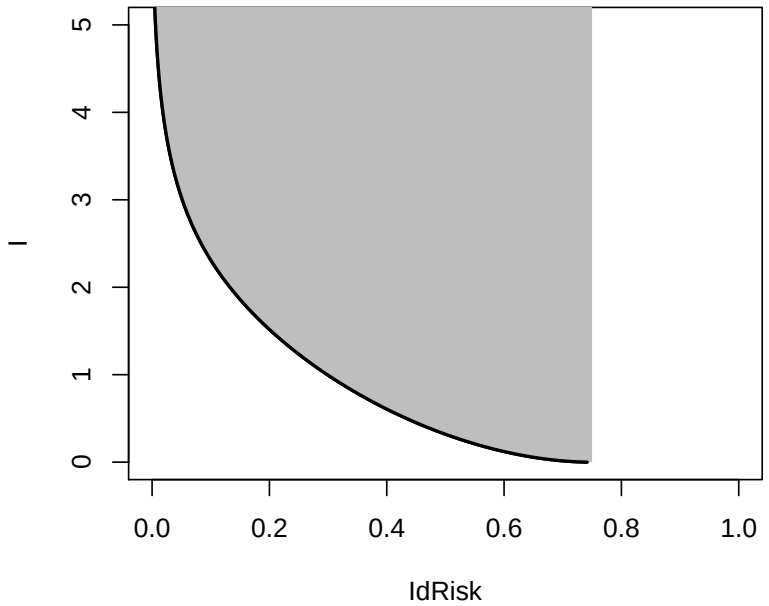
- **Theorem.** (Z., Benjamini 2017) There exists a function h_k such that

$$I(\vec{g}(\vec{Z}); \vec{Y}) \geq h_k(\text{IdRisk}_k).$$

h_3



h_4



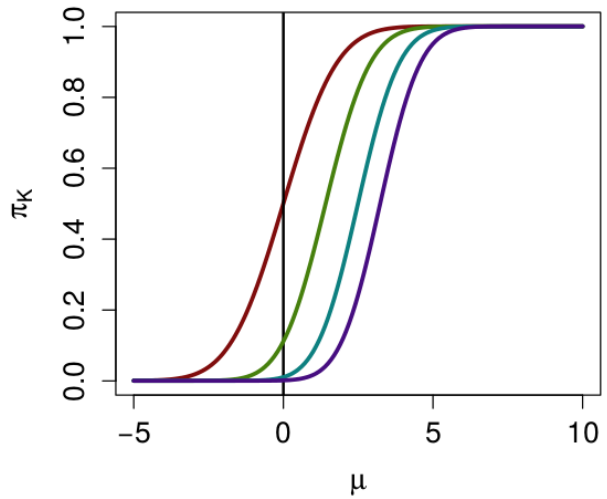
Result 2. Limiting behavior of accuracy curves

Define ABA_k as the Bayes identification accuracy (or average Bayes classification accuracy). Then under a particular high-dimensional limit,

$$\text{ABA}_k \approx \pi_k(\sqrt{2I(X;Y)}) \tag{1}$$

The function π_k is given by

$$\pi_k(c) = \int_{\mathbb{R}} \phi(z-c)\Phi(z)^{k-1} dz.$$

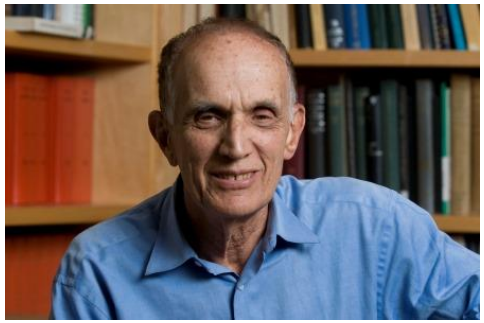


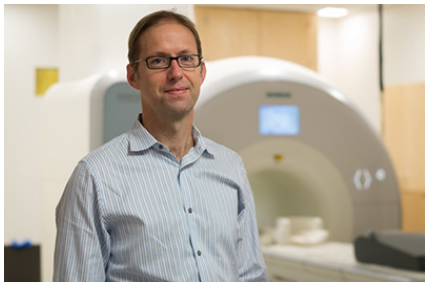
Legend: $K = \{$ 2, 9, 99, 999 $\}$

Section 5

Acknowledgements


CHALLENGER



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Year 3
Year 4
Year 5 +
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Minyong Lee
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Section 6

The end