

Estimating mutual information for high-dimensional sparse relationships

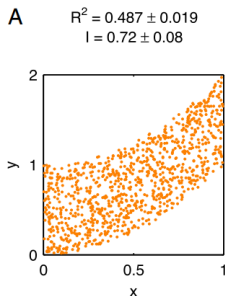
Charles Zheng

Stanford University

January 16, 2017

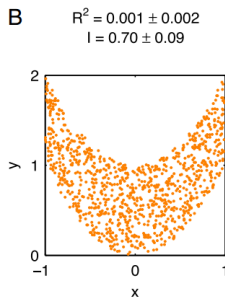
(Joint work with Yuval Benjamini, Hebrew University.)

Overview

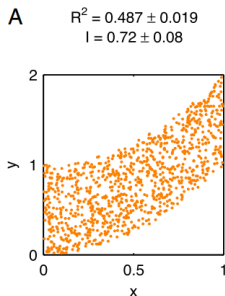


Mutual information $I(\vec{X}; \vec{Y})$

- measures dependence between two random vectors, \vec{X} and \vec{Y}

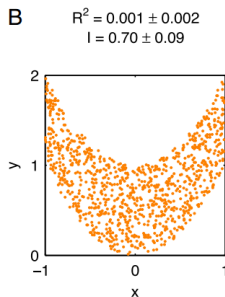


Overview

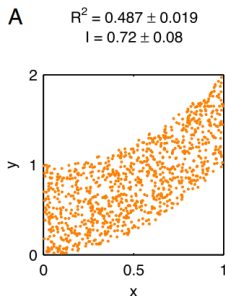


Mutual information $I(\vec{X}; \vec{Y})$

- measures dependence between two random vectors, \vec{X} and \vec{Y}
- applies to nonlinear and multidimensional relationships (unlike correlation)

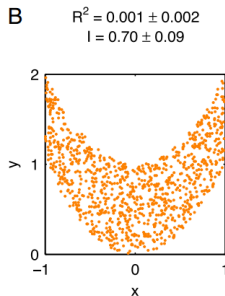


Overview

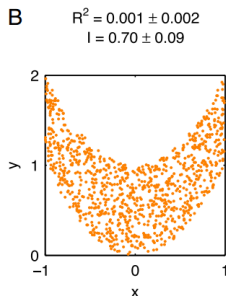
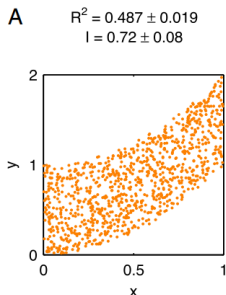


Mutual information $I(\vec{X}; \vec{Y})$

- measures dependence between two random vectors, \vec{X} and \vec{Y}
- applies to nonlinear and multidimensional relationships (unlike correlation)
- is *difficult to estimate* in high dimensions



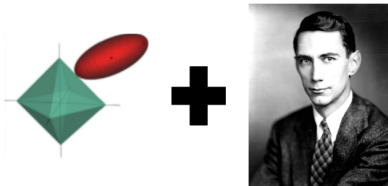
Overview



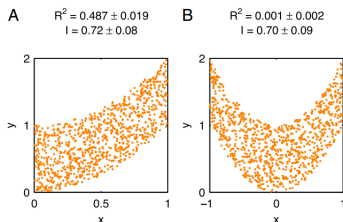
Mutual information $I(\vec{X}; \vec{Y})$

- measures dependence between two random vectors, \vec{X} and \vec{Y}
- applies to nonlinear and multidimensional relationships (unlike correlation)
- is *difficult to estimate* in high dimensions

We combine *machine learning* (sparse estimation) with *information theory* to obtain better estimates of $I(\vec{X}; \vec{Y})$



Mutual information $I(X; Y)$



Introduced in Shannon's 1948 paper, "A mathematical theory of communication"

$$I(X; Y) = \int \log \left(\frac{p(x, y)}{p(x)p(y)} \right) p(x, y) dx dy$$

Image credit Kinney et al. 2014.

Applications of $I(X; Y)$

Mutual information has since been applied to many areas outside of information theory

Applications of $I(X; Y)$

Mutual information has since been applied to many areas outside of information theory

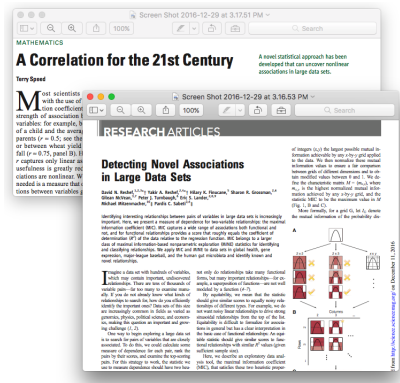
Applications [\[edit \]](#)

In many applications, one wants to maximize mutual information (thus

- In [search engine technology](#), mutual information between phrases
- In [telecommunications](#), the [channel capacity](#) is equal to the mutual information
- [Discriminative training](#) procedures for [hidden Markov models](#) have
- [RNA secondary structure](#) prediction from a [multiple sequence alignment](#)
- [Phylogenetic profiling](#) prediction from pairwise presence and absence
- Mutual information has been used as a criterion for [feature selection](#) the [minimum redundancy feature selection](#).
- Mutual information is used in determining the similarity of two documents
- Mutual information of words is often used as a significance function for word pairs; rather, one counts instances where 2 words occur adjacent to each other, goes up with N.
- Mutual information is used in [medical imaging](#) for [image registration](#) reference image, this image is deformed until the mutual information is maximized
- Detection of [phase synchronization](#) in [time series](#) analysis
- In the [infomax](#) method for neural-net and other machine learning,

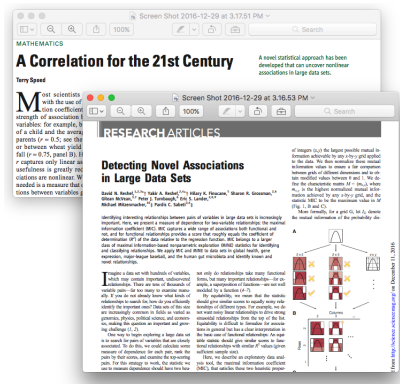
Engineering, biology, computer science, physics, medicine

Comparing $I(X; Y)$ with Pearson correlation



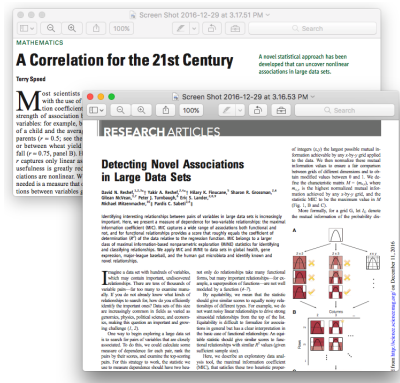
- In many applications scientists are interested in *dependence*, not *correlation* (Reshef et al. 2011, Speed 2011).

Comparing $I(X; Y)$ with Pearson correlation



- In many applications scientists are interested in *dependence*, not *correlation* (Reshef et al. 2011, Speed 2011).
- Only mutual information (and derived quantities) measures dependence directly

Comparing $I(X; Y)$ with Pearson correlation



- In many applications scientists are interested in *dependence*, not *correlation* (Reshef et al. 2011, Speed 2011).
- Only mutual information (and derived quantities) measures dependence directly

Problems with mutual information

- Hard to interpret (compared to R^2)
- Hard to estimate (compared to R^2)

Can we make $I(X; Y)$ easier to interpret?

- Define the “informational correlation” (Linfoot 1957)

$$\text{Cor}_{\text{Info}}(X, Y) = \sqrt{1 - e^{-2I(X; Y)}}$$

Can we make $I(X; Y)$ easier to interpret?

- Define the “informational correlation” (Linfoot 1957)

$$\text{Cor}_{\text{Info}}(X, Y) = \sqrt{1 - e^{-2I(X; Y)}}$$

- Then $\text{Cor}_{\text{Info}}(X, Y) \in [0, 1]$.
- For (X, Y) bivariate normal,

$$|\text{Cor}_{\text{Pearson}}(X, Y)| = \text{Cor}_{\text{Info}}(X, Y)$$

Can we make $I(X; Y)$ easier to interpret?

- Define the “informational correlation” (Linfoot 1957)

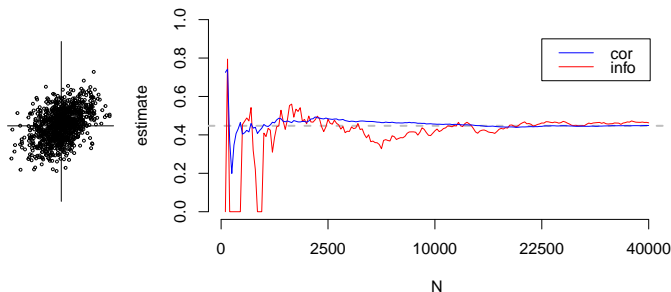
$$\text{Cor}_{\text{Info}}(X, Y) = \sqrt{1 - e^{-2I(X; Y)}}$$

- Then $\text{Cor}_{\text{Info}}(X, Y) \in [0, 1]$.
- For (X, Y) bivariate normal,

$$|\text{Cor}_{\text{Pearson}}(X, Y)| = \text{Cor}_{\text{Info}}(X, Y)$$

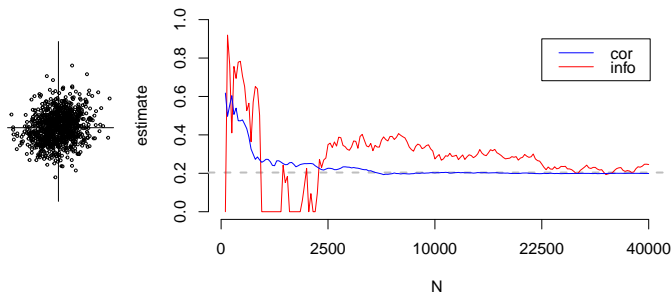
Difficulty of estimating $I(X; Y)$

Example with $\text{Cor}_{\text{Pearson}}(X, Y) = \text{Cor}_{\text{Info}}(X, Y) = 0.44$.



Difficulty of estimating $I(X; Y)$

Example with $\text{Cor}_{\text{Pearson}}(X, Y) = \text{Cor}_{\text{Info}}(X, Y) = 0.2$.



How to estimate $I(X; Y)$

Suppose we observe pairs $(X_i, Y_i)_{i=1}^n$ iid from density $p(x, y)$

- Definition of mutual information:

$$I(X; Y) = \int \log \left(\frac{p(x, y)}{p(x)p(y)} \right) p(x, y) dx dy$$

How to estimate $I(X; Y)$

Suppose we observe pairs $(X_i, Y_i)_{i=1}^n$ iid from density $p(x, y)$

- Definition of mutual information:

$$I(X; Y) = \int \log \left(\frac{p(x, y)}{p(x)p(y)} \right) p(x, y) dx dy$$

- Kernel density estimate approaches estimate $p(x, y)$ (Beirlant et al. 2001, Ivanov and Rozhkova 1981)
- Nearest neighbor estimators rely on distance-based computations (Mnatsakanov et al. 2008, Gorja et al. 2005, Singh et al. 2003)

How to estimate $I(X; Y)$

Suppose we observe pairs $(X_i, Y_i)_{i=1}^n$ iid from density $p(x, y)$

- **Plug-in estimate:**

$$\hat{I}(X; Y) = \int \log \left(\frac{\hat{p}(x, y)}{\hat{p}(x)\hat{p}(y)} \right) \hat{p}(x, y) dx dy$$

- Kernel density estimate approaches estimate $p(x, y)$ (Beirlant et al. 2001, Ivanov and Rozhkova 1981)
- Nearest neighbor estimators rely on distance-based computations (Mnatsakanov et al. 2008, Goria et al. 2005, Singh et al. 2003)

Problems in high dimensions

- Density estimation is known to have *exponential complexity* with respect to dimensionality.
 - E.g. to get the same precision, you need 10 observations for univariate X, Y but 1000 for trivariate \vec{X}, \vec{Y} .

Problems in high dimensions

- Density estimation is known to have *exponential complexity* with respect to dimensionality.
 - E.g. to get the same precision, you need 10 observations for univariate X, Y but 1000 for trivariate \vec{X}, \vec{Y} .
- Many applications with high-dimensional X, Y .
 - Gene expression time series
 - Functional magnetic resonance imaging

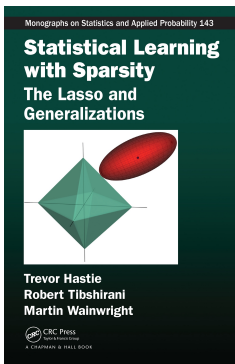
Problems in high dimensions

- Density estimation is known to have *exponential complexity* with respect to dimensionality.
 - E.g. to get the same precision, you need 10 observations for univariate X, Y but 1000 for trivariate \vec{X}, \vec{Y} .
- Many applications with high-dimensional X, Y .
 - Gene expression time series
 - Functional magnetic resonance imaging
- One approach is to assume joint multivariate normality of X, Y , but this reduces mutual information to a linear statistic.

Problems in high dimensions

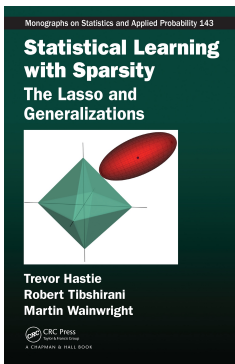
- Density estimation is known to have *exponential complexity* with respect to dimensionality.
 - E.g. to get the same precision, you need 10 observations for univariate X, Y but 1000 for trivariate \vec{X}, \vec{Y} .
- Many applications with high-dimensional X, Y .
 - Gene expression time series
 - Functional magnetic resonance imaging
- One approach is to assume joint multivariate normality of X, Y , but this reduces mutual information to a linear statistic.
- Other approaches: binning (Bialek et al. 1991, Paninski 2003), confusion matrix of a classifier (Treves 1997, Quiroga et al. 2009)

First idea: Use sparsity!



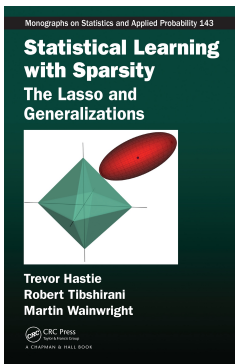
- *Sparsity* refers to existence of low-dimensional structure hidden in high-dimensional data.

First idea: Use sparsity!



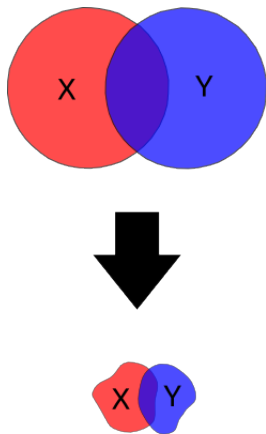
- *Sparsity* refers to existence of low-dimensional structure hidden in high-dimensional data.
- E.g. suppose X is 100-dimensional but Y is only a function of (X_5, X_9) .

First idea: Use sparsity!



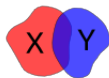
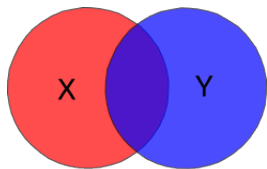
- *Sparsity* refers to existence of low-dimensional structure hidden in high-dimensional data.
- E.g. suppose X is 100-dimensional but Y is only a function of (X_5, X_9) .
- Can we exploit sparsity to obtain a good estimate of $I(X; Y)$ even under low sample sizes?

Dimension reduction vs. sparsity?

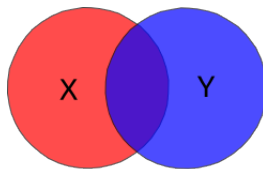


Unsupervised dimension reduction

Dimension reduction vs. sparsity?



Unsupervised dimension reduction



Sparsity = supervised dim. reduction

Second idea: link prediction accuracy to mutual information

- If $I(X; Y) > 0$, then X carries information about Y and vice-versa.

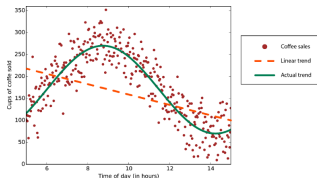
Second idea: link prediction accuracy to mutual information

- If $I(X; Y) > 0$, then X carries information about Y and vice-versa.
- Therefore, we can *predict* Y from X (or X from Y)

Second idea: link prediction accuracy to mutual information

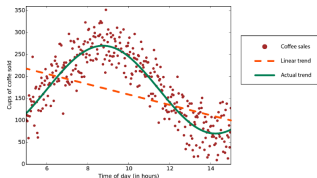
- If $I(X; Y) > 0$, then X carries information about Y and vice-versa.
- Therefore, we can *predict* Y from X (or X from Y)
- We know that often *prediction accuracy* implies a lower bound for *mutual information* (e.g. Fano 1952)

Background: Regression



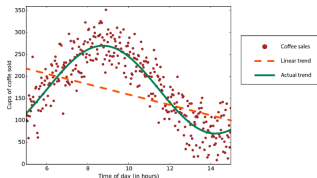
- Suppose you observe $(\vec{X}^{(i)}, Y^{(i)})_{i=1}^n$ where $Y^{(i)} = f(\vec{X}^{(i)}) + \epsilon$, where f is an unknown function and ϵ is noise. (Also, assume $\mathbf{E}[\epsilon] = 0$.)

Background: Regression



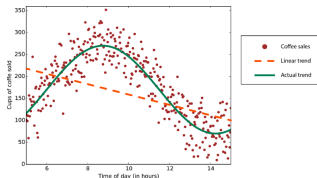
- Suppose you observe $(\vec{X}^{(i)}, Y^{(i)})_{i=1}^n$ where $Y^{(i)} = f(\vec{X}^{(i)}) + \epsilon$, where f is an unknown function and ϵ is noise. (Also, assume $\mathbf{E}[\epsilon] = 0$.)
- The goal in regression is to recover the unknown function f .

Background: Regression



- Suppose you observe $(\vec{X}^{(i)}, Y^{(i)})_{i=1}^n$ where $Y^{(i)} = f(\vec{X}^{(i)}) + \epsilon$, where f is an unknown function and ϵ is noise. (Also, assume $\mathbf{E}[\epsilon] = 0$.)
- The goal in regression is to recover the unknown function f .
- In *linear regression*, we assume f is linear.

Background: Regression



- Suppose you observe $(\vec{X}^{(i)}, Y^{(i)})_{i=1}^n$ where $Y^{(i)} = f(\vec{X}^{(i)}) + \epsilon$, where f is an unknown function and ϵ is noise. (Also, assume $\mathbf{E}[\epsilon] = 0$.)
- The goal in regression is to recover the unknown function f .
- In *linear regression*, we assume f is linear.
- if we do not assume a particular form for f , we can use *nonparametric regression*.

Background: Sparse regression

- When \vec{X} is high dimensional, classical regression techniques perform poorly.

Background: Sparse regression

- When \vec{X} is high dimensional, classical regression techniques perform poorly.
- If the true function f only depends on a small number of components in \vec{X} , we can still do well if we use *sparse* regression methods.

Background: Sparse regression

- When \vec{X} is high dimensional, classical regression techniques perform poorly.
- If the true function f only depends on a small number of components in \vec{X} , we can still do well if we use *sparse* regression methods.

	<i>Classical</i>	<i>Sparse</i>
<i>Linear</i>	Ordinary Least-Squares (Legendre 1805)	Elastic net (Zou 2008)
<i>Nonpar.</i>	LOWESS (Cleveland 1979)	Random forests (Breiman 2001)

Our proposal

Suppose we observe pairs $(X_i, Y_i)_{i=1}^n$ iid from density $p(x, y)$.

- 1 Estimate a (sparse) regression model for $\mathbf{E}[y|x]$.
- 2 Assess the *prediction accuracy* of the model using *identification risk*
- 3 Use the identification risk to obtain a lower bound for the mutual information $I(X; Y)$

Multiple-response regression

- Pairs $(x_i, y_i)_{i=1}^n$, where X is p -dimensional and Y is q -dimensional.
- Data matrices $\mathbf{X}_{n \times p}$, $\mathbf{Y}_{n \times q}$.
- For each column of Y , fit sparse model $Y^{(i)} \approx X^T \beta^{(i)} + \epsilon$, e.g. by using elastic net (Zou 2008),

$$\hat{\beta}^{(i)} = \operatorname{argmin}_{\beta} \|\mathbf{X}^T \beta^{(i)} - Y^{(i)}\|^2 + \lambda_2 \|\beta^{(i)}\|_2^2 + \lambda_1 \|\beta^{(i)}\|_1$$

- Or, fit a *random forest* model for each column of Y (Breiman 2001)

Regression vs Identification loss

- Independent *test set* $(x_i^*, y_i^*)_{i=1}^k$.
- Use model to predict $\hat{y}_i^* = (x_i^*)^T \hat{B}$ for $i = 1, \dots, k$.

Two ways to evaluate the predictive accuracy of the regression model:

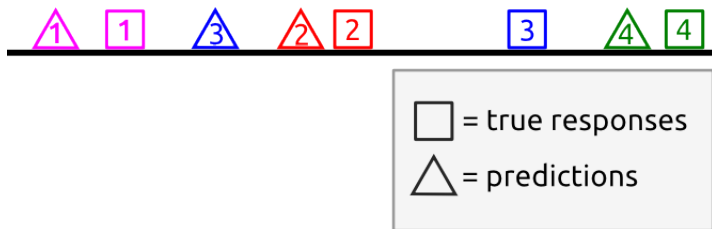
- Regression (mean squared-error) loss:

$$\text{MSE} = \frac{1}{k} \sum_{i=1}^k \|y_i^* - \hat{y}_i^*\|^2.$$

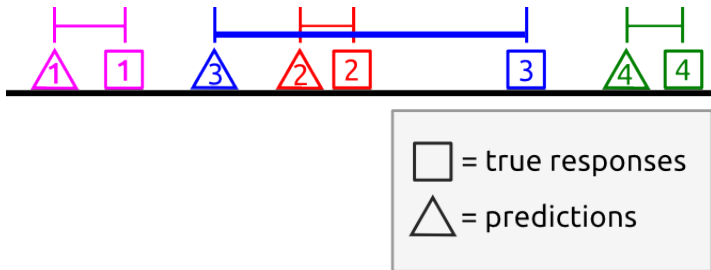
- Identification loss (Kay 2008):

$$\text{IdLoss}_k = \frac{1}{k} \sum_{i=1}^k (1 - I\{\hat{y}_i^* \text{ is nearest neighbor of } y_i^*\}).$$

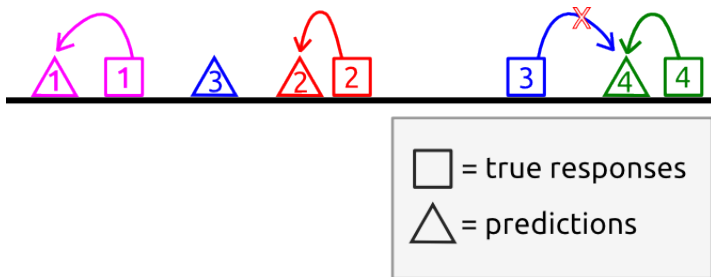
Regression vs Identification loss



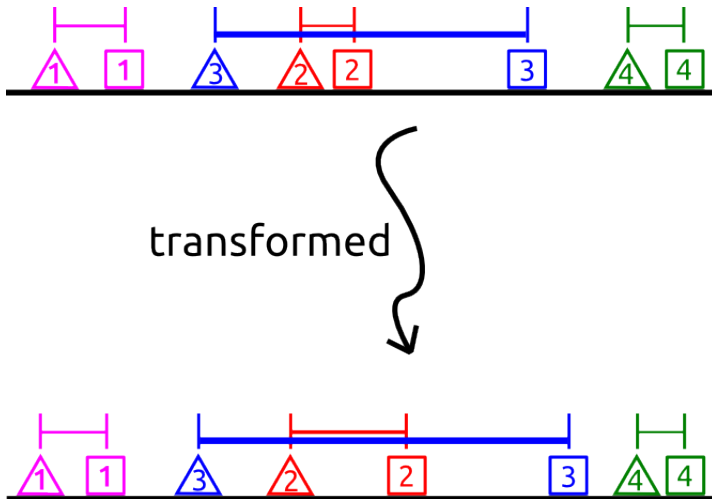
Mean-squared error



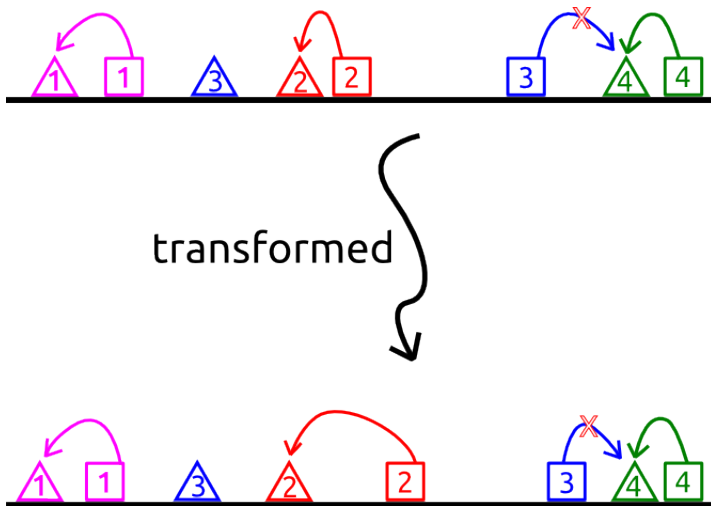
Identification loss



Mean-squared error changes under nonlinear scaling



Identification loss robust under nonlinear scaling



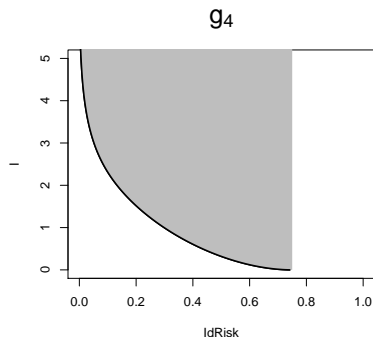
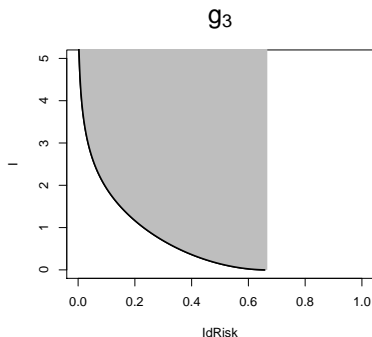
Identification loss and mutual information

- Define the identification risk as the expected identification loss

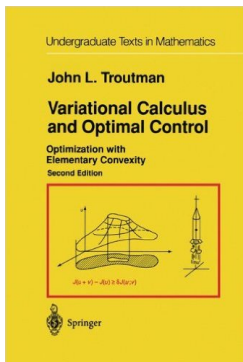
$$\text{IdRisk}_k = \mathbf{E}[\text{IdLoss}_k]$$

- Theorem.** (Z., Benjamini 2017) There exists a function g_k such that

$$I(X; Y) \geq g_k(\text{IdRisk}_k).$$



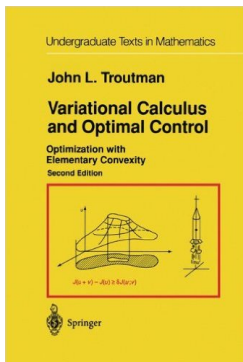
We use *calculus of variations* to obtain this result.



We use *calculus of variations* to obtain this result.

- Mutual information is a functional of $p(x, y)$.

$$I[p(x, y)] = \mathbf{E} \left[\log \frac{p(X, Y)}{p(X)p(Y)} \right].$$



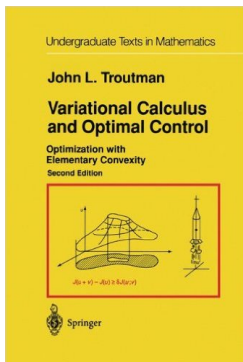
We use *calculus of variations* to obtain this result.

- Mutual information is a functional of $p(x, y)$.

$$I[p(x, y)] = \mathbf{E} \left[\log \frac{p(X, Y)}{p(X)p(Y)} \right].$$

- Identification risk is *lower-bounded* by another functional—the *Bayes risk*.

$$\text{BayesRisk}_k[p(x, y)] = 1 - \mathbf{E} \left[\max_{i=1}^k p(Y|X_i) \right].$$



We use *calculus of variations* to obtain this result.

- Mutual information is a functional of $p(x, y)$.

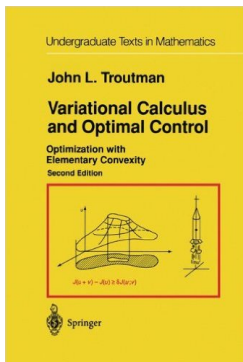
$$I[p(x, y)] = \mathbf{E} \left[\log \frac{p(X, Y)}{p(X)p(Y)} \right].$$

- Identification risk is *lower-bounded* by another functional—the *Bayes risk*.

$$\text{BayesRisk}_k[p(x, y)] = 1 - \mathbf{E}[\max_{i=1}^k p(Y|X_i)].$$

- $g_k(u) = \inf_{p(x, y)} I[p(x, y)]$

subject to $\text{BayesRisk}_k[p(x, y)] \geq u$.



Our proposal

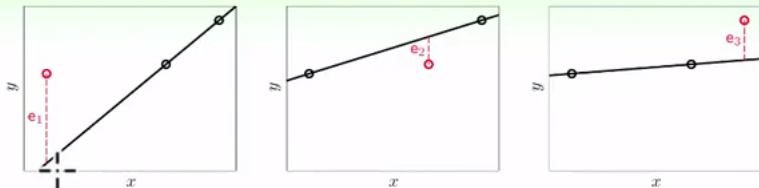
Suppose we observe pairs $(X_i, Y_i)_{i=1}^n$ iid from density $p(x, y)$.

- 1 Estimate a (sparse) regression model for $\mathbf{E}[y|x]$.
- 2 Compute *identification loss*, IdLoss_k , using *leave-k-out*.
- 3 Estimate mutual information using

$$\hat{I}_{\text{IdLoss}}(X; Y) = g_k(\text{IdLoss}_k).$$

What is leave-k-out cross-validation?

Illustration of Leave-One-Out



- Randomly hold out a subset of size k .
- Use remaining data to predict the held-out data.
- Obtain the average prediction error.

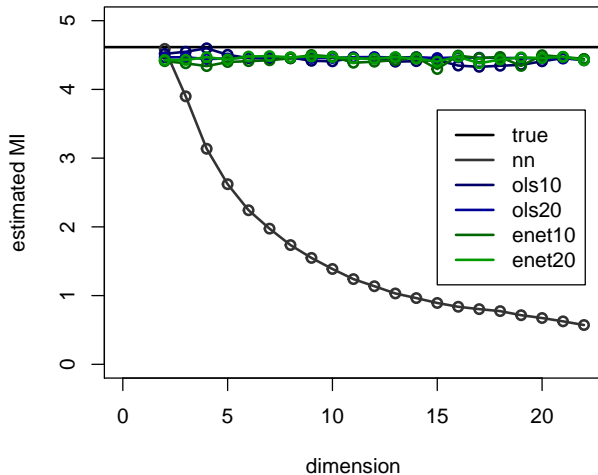
Image credit Hsuan-Tien Lin

Section 2

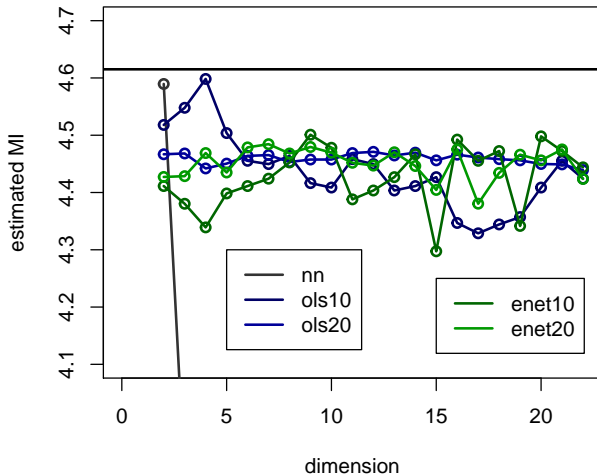
Applications

- Generate data: $(Y_1, Y_2) = (X_1, X_2)^T B + \epsilon$ where B is a randomly generated coefficient matrix.
- Add extra noise dimensions X_3, X_4, \dots
- $n = 1000$.
- Compare Nearest-Neighbor estimator (Mnatsakov et al, 2008, implemented in FNN) with our method using OLS and elastic net (sparse).

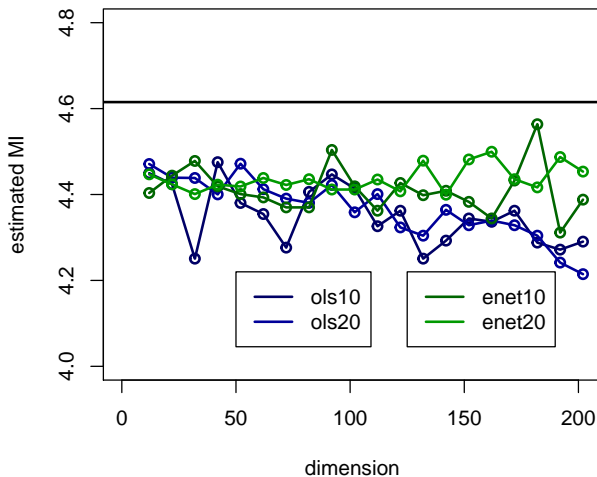
Simulation Results - I. low dimension



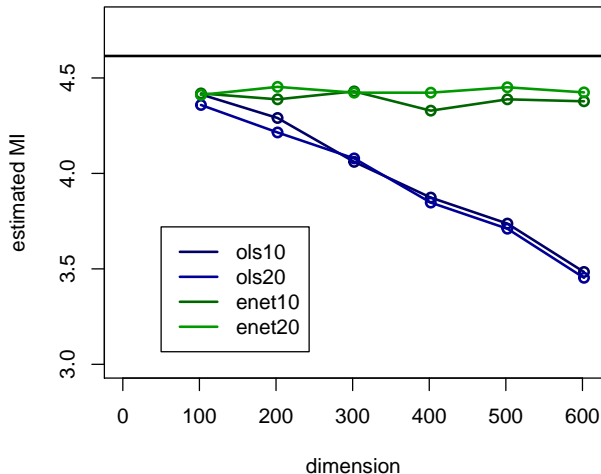
Simulation Results - I. low dimension



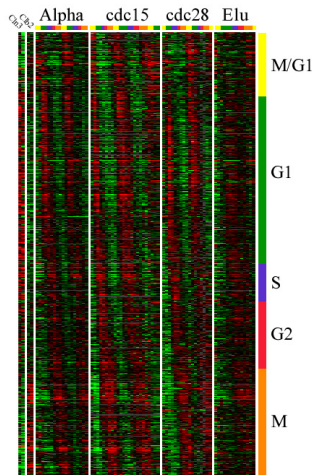
Simulation Results - II. medium dimension



Simulation Results - III. high dimension



Application to gene expression time series



- Data from Spellman et al. 1998
- Expression levels of 6178 yeast genes during cell cycle
- Total 73 time points per gene

Groups of genes

Group	No. genes
unknown	396
cell cycle	27
DNA replication	27
transport	19
cytoskeleton	17
chromatin structure	16

Total 145 different categories (only top 6 shown).

Canonical correlations between time series

Top canonical correlation (Hotelling 1936)

	CC	DR	Tr	Cy	CS
CC		1	1	1	1
DR			1	0.99	0.99
Tr				0.99	0.98
Cy					0.98
CS					

CC = cell cycle, DR = DNA replication, Tr = transport,
Cy = cytoskeleton, CS = chromatin structure

Sparse canonical correlations between time series

Using sparse CCA* (Witten and Tibshirani 2009).

	CC	DR	Tr	Cy	CS
CC		0.96	0.87	0.92	0.94
DR			0.83	0.88	0.95
Tr				0.83	0.78
Cy					0.90
CS					

CC = cell cycle, DR = DNA replication, Tr = transport,
Cy = cytoskeleton, CS = chromatin structure

*: using CCApermute in R package PMA

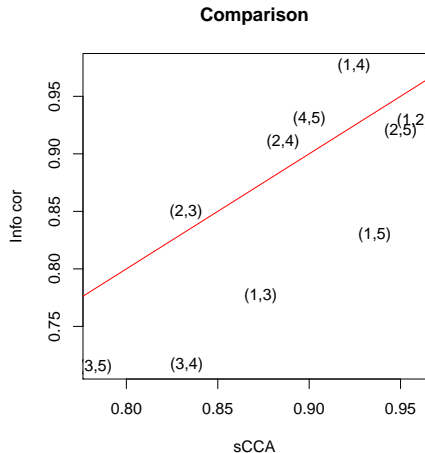
Information correlations between time series

Taking the max of $\hat{I}(X; Y)$ and $\hat{I}(Y; X)$.

	CC	DR	Tr	Cy	CS
CC		0.93	0.78	0.98	0.83
DR			0.85	0.91	0.92
Tr				0.72	0.71
Cy					0.93
CS					

CC = cell cycle, DR = DNA replication, Tr = transport,
Cy = cytoskeleton, CS = chromatin structure

Comparing sparse CCA and Cor_{Info}



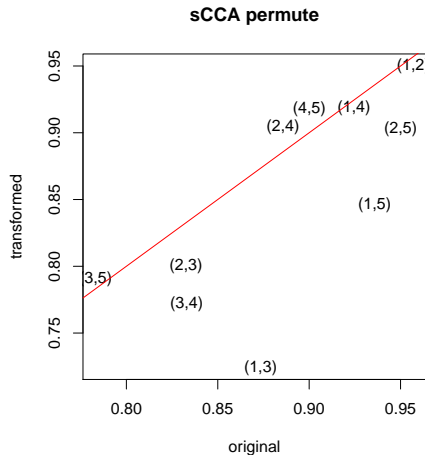
(1) cell cycle, (2) DNA replication, (3) transport,
(4) cytoskeleton, (5) chromatin structure

Invariance properties

Transform data from each group with random rotation...

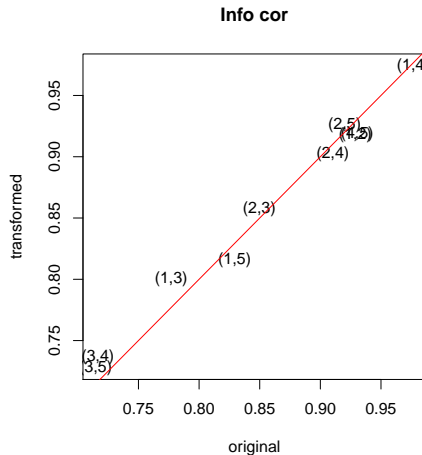
Invariance properties

Transform data from each group with random rotation...



Invariance properties

Transform data from each group with random rotation...



Conclusions

- Mutual information, and derived Cor_{Info} are useful measures of correlation, but hard to estimate.

Conclusions

- Mutual information, and derived Cor_{Info} are useful measures of correlation, but hard to estimate.
- Our method targets high-dimensional data with sparsity.

- Mutual information, and derived Cor_{Info} are useful measures of correlation, but hard to estimate.
- Our method targets high-dimensional data with sparsity.
- How to use: choose a regression model suited to the model assumptions. Our method allows you to convert the prediction accuracy of the model, IdLoss_k into an estimate of $I(\vec{X}; \vec{Y})$.

- Mutual information, and derived Cor_{Info} are useful measures of correlation, but hard to estimate.
- Our method targets high-dimensional data with sparsity.
- How to use: choose a regression model suited to the model assumptions. Our method allows you to convert the prediction accuracy of the model, IdLoss_k into an estimate of $I(\vec{X}; \vec{Y})$.
- Example application: measure of joint information between two tables which is robust to transformations.

Related work and future directions

- What if data is high-dimensional, but not sparse? We have another method based on high-dimensional asymptotics (ZB 2016).

Related work and future directions

- What if data is high-dimensional, but not sparse? We have another method based on high-dimensional asymptotics (ZB 2016).
- Estimating quantities related to mutual information, such as *transfer information*, *stimulus-specific information* and *redundancy* (Borst and Theunissen 1999)

Related work and future directions

- What if data is high-dimensional, but not sparse? We have another method based on high-dimensional asymptotics (ZB 2016).
- Estimating quantities related to mutual information, such as *transfer information*, *stimulus-specific information* and *redundancy* (Borst and Theunissen 1999)
- Inferring resting-state brain networks.

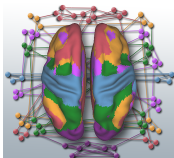


Image credit Simons Foundation

Section 3

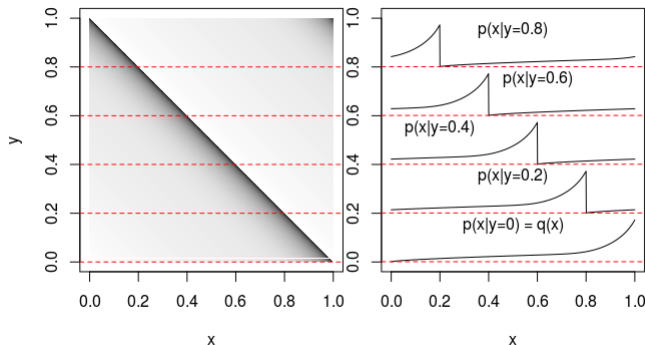
The End

References

- Reshef et al, 2011. "Detecting Novel Associations in Large Datasets." *Science*.
- Speed, 2011. "A correlation for the 21st century." *Science*.
- Linfoot, 1957. "An informational measure of correlation." *Information and Control*.
- Kay, 2008. "Identifying natural images from human brain activity." *Nature*.
- Mnatsakanov, et al, (2008). "K-nearest neighbor estimators of entropy." *Mathematical Methods of Statistics*
- Spellman et al., (1998). "Comprehensive Identification of Cell Cycle-regulated Genes of the Yeast *Saccharomyces cerevisiae* by Microarray Hybridization." *Molecular Biology of the Cell*.
- Hotelling, H. (1936). "Relations Between Two Sets of Variates". *Biometrika*.
- Witten, Daniela M., and Robert J. Tibshirani. (2009). "Extensions of sparse canonical correlation analysis with applications to genomic data." *Statistical applications in genetics and molecular biology*

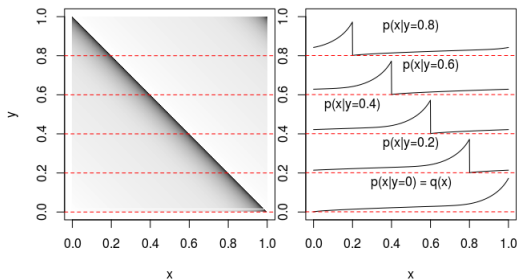
Reduced Problem

Rather than show the whole proof, we consider a simplified problem to illustrate the methods.



Actually, the simplified problem is equivalent to the full problem and we get the same answer (but this is non-trivial).

Reduced Problem



- $p(x, y)$ on unit square with uniform marginals.
- The conditional distributions $p(x|y)$ are just “shifted” copies of a common density, $q(x)$, on $[0, 1]$

$$p(x|y) = q(x - y + I\{x < y\})$$

- Furthermore, $q(x)$ is increasing in x .

The information and average Bayes error can be written in terms of $q(x)$.

$$I[p(x, y)] = \int_0^1 q(x) \log q(x) dx$$

$$\text{BayesAcc}_k[p(x, y)] = \int_{[0,1]^k} \max_{i=1}^k q(x_i) dx_1 \cdots dx_k$$

Overload the notation and “redefine” information and average Bayes error as functionals of $q(x)$.

$$I[q(x)] \stackrel{\text{def}}{=} \int_0^1 q(x) \log q(x) dx$$

$$\text{BayesAcc}_k[q(x)] \stackrel{\text{def}}{=} \frac{1}{k} \int_{[0,1]^k} \max_{i=1}^k q(x_i) dx_1 \cdots dx_k$$

Optimization problem

We now pose the question: how do we find $q(x)$ which maximizes $\text{BayesAcc}_k[q(x)]$ subject to $I[q(x)] \leq \iota$?

- *Domain of the optimization:* Recall that $q(x)$ satisfies $q(x) \geq 0$, $\int_0^1 q(x)dx = 1$, and is increasing in x . Let \mathcal{Q} denote the space of functions on $[0, 1] \rightarrow [0, \infty)$ which are increasing in x .
- *Constraints:* We have two remaining constraints, $I[q(x)] \leq \iota$ and $\int_0^1 q(x)dx = 1$.

Hence the problem is

maximize $_{q(x) \in \mathcal{Q}}$ $\text{BayesAcc}_k[q(x)]$ subject to $\int_0^1 q(x)dx = 1$ and $I[q(x)] \leq \iota$.

Optimization problem

maximize $_{q(x) \in \mathcal{Q}}$ BayesAcc $_k[q(x)]$ subject to $\int_0^1 q(x)dx = 1$ and $I[q(x)] \leq \iota$.

- Does a solution exist? Yes, because the space of measures with density $q(x)$ satisfying $I[q(x)] \leq \iota$ is tight, and both the constraints and objective are continuous wrt to the topology of weak convergence.
- Given a solution $q^*(x)$ exists, there exist Lagrange multipliers $\lambda \in \mathbb{R}$ and $\nu > 0$ such that q^* minimizes

$$\begin{aligned}\mathcal{L}[q(x)] &= -\text{BayesAcc}_k[q(x)] + \lambda \int_0^1 q(x)dx + \nu I[q(x)] \\ &= \int_0^1 (-t^{k-1} + \lambda + \nu \log q(x))q(x)dx.\end{aligned}$$

Functional derivatives

- Taylor expansions are a useful trick for computing functional derivatives
- We can compute the functional derivative of $\mathcal{L}[q(x)]$ by writing

$$\begin{aligned}\mathcal{L}[q(x) + \epsilon \xi(x)] &= \int_0^1 (-t^{k-1} + \lambda + \nu \log(q(x) + \epsilon \xi(x)))(q(x) + \epsilon \xi(x)) dx. \\ &\approx \int (q(x) + \epsilon \xi(x))(-t^{k-1} + \lambda + \nu \{\log q(x) + \frac{\epsilon \xi(x)}{q(x)}\}) dx \\ &\approx \mathcal{L}[q(x)] + \int_0^1 (-t^{k-1} + \lambda + \nu(1 + \log q(x))) \epsilon \xi(x) dx.\end{aligned}$$

- Hence

$$\nabla \mathcal{L}[q](x) = -t^{k-1} + \lambda + \nu(1 + \log q(x))$$

Variational magic!

Suppose we set the functional derivative to 0,

$$0 = \nabla \mathcal{L}[q](t) = -t^{k-1} + \lambda + \nu + \nu \log q(t).$$

Then we conclude that the optimal $q^*(t)$ takes the form

$$q^*(t) = \alpha e^{\beta t^{k-1}}$$

for some $\alpha > 0$, $\beta > 0$.

From the constraint $\int q(t) dt = 1$, we get

$$q_\beta(t) = \frac{e^{\beta t^{k-1}}}{\int e^{\beta t^{k-1}} dt}.$$

Theorem. For any $\iota > 0$, there exists $\beta_\iota \geq 0$ such that defining

$$q_\beta(t) = \frac{\exp[\beta t^{k-1}]}{\int_0^1 \exp[\beta t^{k-1}]},$$

we have

$$\int_0^1 q_{\beta_\iota}(t) \log q_{\beta_\iota}(t) dt = \iota.$$

Then,

$$\sup_{I(X;Y)=\iota} \text{BayesAcc}_k = \int_0^1 q_{\beta_\iota}(t) t^{k-1} dt = g_k^{-1}(\iota).$$