Upper and lower bounds on cdf of generalized non-central chi-squared

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1 Introduction

Let $Z \sim N(0, I_p)$, and let $\mu \in \mathbb{R}^p$ and Σ a positive semidefinite matrix. Define the generalized noncentral chi-squared distribution with noncentrality μ and shape Σ as the distribution of

$$Y = (Z + \mu)^T \Sigma (Z + \mu)$$

Let $V \Lambda V^T = \Sigma$ be the eigendecomposition of Σ, and let $\eta = V^T \Sigma$. Then

$$Y \overset{d}{=} (Z + \eta)^T \Lambda (Z + \eta) = \sim_{i=1}^p \lambda_i W_i$$

where $W_i \sim \chi_1^2(\eta_i^2)$. Recall that the mgf of the noncentral chi-sqaured with one df is given by

$$\mathbf{E}[e^{tW_i}] = \frac{\exp[\frac{\eta_i^2 t}{1-2t}]}{\sqrt{1-2t}}$$

It follows that the moment-generating function of Y is given by

$$\mathbf{E}[e^{tY}] = \prod_{i=1}^p \mathbf{E}[e^{\lambda_i t W_i}] = \prod_{i=1}^p \frac{\exp[\frac{\eta_i^2 \lambda_i t}{1-2t\lambda_i}]}{\sqrt{1-2t\lambda_i}}$$

2 Bound

We wish to bound the probability $\Pr[Y < x]$. We have

$$\begin{aligned}
\log \Pr[Y < x] &= \log \Pr[e^{tY} > e^{tx}] \text{ for } t < 0 \\
&\leq \log\left(\frac{\mathbf{E}[e^{tY}]}{e^{tx}}\right) \\
&= \log(\mathbf{E}[e^{tY}]) - tx \\
&= \left(\frac{-1}{2}\sum_{i=1}^{p}\log(1-2t\lambda_i)\right) + \left(\sum_{i=1}^{p}\frac{\eta_i^2\lambda_i t}{1-2t\lambda_i}\right) - tx
\end{aligned}$$

Now consider minimizing the bound over t. The derivative of the bound wrt t is

$$-x + \sum_{i=1}^{p}\frac{\lambda_i(1+\eta_i)^2}{1-2\lambda_i t} + \frac{2\eta_i^2\lambda_i^2 t}{(1-2\lambda_i t)^2}$$

which, to a first-order approximation in $1/t$, is

$$-\frac{p}{2t} - x$$

Hence this implies that for small x, the optimal value of t is approximately

$$t^* \approx -\frac{p}{2x}.$$

Plugging in this value of t yields the bound

$$\log \Pr[Y < x] < -\frac{1}{2}\left(\sum_{i=1}^{p}\log\left(1+\frac{\lambda_i p}{x}\right)\right) + \frac{p}{2}\left(1 - \frac{1}{x}\sum_{i=1}^{p}\frac{\eta_i^2\lambda_i}{1+\frac{\lambda_i p}{x}}\right)$$