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Suppose $X_1, \ldots, X_n$ are uniform on the simplex, so $X_1 + \cdots + X_n = 1$. We wish to compute $\Pr[a_1 X_1 + \cdots + a_n X_n > 0]$.

Fact: one can write $X_i = E_i / (\sum_i E_i)$ where $E_1, \ldots, E_n$ are iid exponential. Hence

$$\Pr[a_1 X_1 + \cdots + a_n X_n > 0] = \Pr[a_1 E_1 + \cdots + a_n E_n > 0]$$

WLOG take $a_1 \geq \cdots \geq a_n$. The general case is easy to write for $a_1 > \cdots > a_n$, but it is also straightforward to work out what happens if $a_i = a_j$ for some $i \neq j$.

For $a > b > 0$, the distribution of $aE_1 + bE_2$ is

$$\begin{aligned}
f_{a,b}(z) &= \int_0^z \frac{1}{a} e^{-x/a} \frac{1}{b} e^{-(z-x)/b} dx \\
&= \frac{1}{ab} \int_0^z e^{-z/b} e^{-x(\frac{1}{a} - \frac{1}{b})} dx \\
&= \frac{1}{ab} e^{-z/b} \left[\frac{1}{\frac{1}{a} - \frac{1}{b}} e^{-x(\frac{1}{a} - \frac{1}{b})} \right]_0^z \\
&= \frac{1}{a-b} e^{-z/b} [e^{-z(a^{-1} - b^{-1})} - 1] \\
&= \frac{1}{a-b} [e^{-z/a} - e^{-z/b}] = \frac{a^{-1} e^{-z/a}}{a^{-1}(a-b)} + \frac{b^{-1} e^{-z/b}}{b^{-1}(b-a)}
\end{aligned}$$

It is clear from the above form that the distribution of $aE_1 + bE_2 + cE_3$ for $a > b > c > 0$ is

$$\begin{aligned}
f_{a,b,c}(z) &= \frac{\frac{1}{a-c} e^{-z/a} + \frac{1}{c-a} e^{-z/c}}{a^{-1}(a-b)} + \frac{\frac{1}{b-c} e^{-z/b} + \frac{1}{c-b} e^{-z/c}}{b^{-1}(b-a)} \\
&= \frac{\frac{a}{a-c} e^{-z/a} + \frac{a}{c-a} e^{-z/c}}{(a-b)} + \frac{\frac{b}{b-c} e^{-z/b} + \frac{b}{c-b} e^{-z/c}}{(b-a)} \\
&= \frac{a}{(a-c)(a-b)} e^{-z/a} + \frac{b}{(b-c)(b-a)} e^{-z/b} + \frac{c}{(c-a)(c-b)} e^{-z/b}
\end{aligned}$$

Now it is easy to see what will happen for general $a_1 > \ldots > a_m > 0$ since if for $f_{a_1,\ldots,a_i}$ the coefficient of the e^{-z/a_j} term is $C_{j,i}$, the coefficient of the

e^{-z/a_j} term for $f_{a_1,\ldots,a_{i+1}}$ will be $\frac{a_1}{a_1-a_{i+1}}C_i$. Hence the general form is

$$f_{a_1,\ldots,a_m}(z) = \sum_{j=1}^{m} \frac{a_j^{m-1}}{\prod_{k\neq j}(a_j - a_k)} e^{-z/a_j}$$

Now suppose that $a_1 > \cdots > a_m > 0 > a_{m+1} > \cdots > a_n$. We need to compute

$$\begin{aligned}
\Pr[a_1E_1 + \cdots + a_nE_n > 0] &= \Pr[a_1E_1 + \cdots + a_mE_m > (-a_{m+1})E_{m+1} + \cdots + (-a_n)E_n] \\
&= \int_0^\infty \int_0^x f_{a_1,\ldots,a_m}(x) f_{-a_{m+1},\ldots,-a_n}(y)dydx \\
&= \int_0^\infty \int_0^x \left[\sum_{j=1}^{m} \frac{a_j^{m-1}}{\prod_{m\geq k\neq j}(a_j - a_k)} e^{-x/a_j}\right] \left[\sum_{j=m+1}^{n} \frac{(-a_j)^{n-m-1}}{\prod_{m<k\neq j}(-a_j + a_k)} e^{-y/(-a_j)}\right] \\
&= \sum_{j=1}^{m} \sum_{\ell=m+1}^{n} \int_0^\infty \int_0^x \frac{a_j^{m-1}}{\prod_{m\geq k\neq j}(a_j - a_k)} e^{-x/a_j} \frac{(-a_\ell)^{n-m-1}}{\prod_{m<k\neq \ell}(-a_\ell + a_k)} e^{-y/(-a_\ell)}dydx \\
&= \sum_{j=1}^{m} \sum_{\ell=m+1}^{n} C_jC_\ell \int_0^\infty \int_0^x e^{-x/a_j} e^{-y/(-a_\ell)}dydx \\
&= \sum_{j=1}^{m} \sum_{\ell=m+1}^{n} C_jC_\ell a_j(-a_\ell) \Pr[\mathit{Exponential}(a_j) > \mathit{Exponential}(-a_\ell)] \\
&= \sum_{j=1}^{m} \sum_{\ell=m+1}^{n} C_jC_\ell a_j(-a_\ell) \frac{a_j}{a_j - a_\ell} = \sum_{j=1}^{m} \sum_{\ell=m+1}^{n} C_jC_\ell \frac{a_j^2(-a_\ell)}{a_j - a_\ell}
\end{aligned}$$

where $C_j = \frac{a_j^{m-1}}{\prod_{m\geq k\neq j}(a_j-a_k)}$ and $C_\ell = \frac{(-a_\ell)^{n-m-1}}{\prod_{m<k\neq \ell}(-a_\ell+a_k)}$.