Fuzzy Ranking of Human Development: A Proposal

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Abstract: In this paper we propose to measure human development as a fuzzy concept and characterize a fuzzy relation for its ranking. We stress that there exists a great deal of vagueness in measuring a country's level of human development; one such source of vagueness we investigate is the weighting embedded in UNDP's human development index. We propose to evaluate the resulting fuzziness in human development ranking with a truth value function. A truth value is simply a function of the probability that a randomly drawn bundle of weights will rank one country as having a higher level of human development than another country. We derive general closed-form and easily computable formulae for calculating these truth values. The method derived is equally applicable to fuzzy rankings with other composite indices.

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1. Introduction

In an effort to ensure that "the primary objective of development is to benefit people," the United Nations Development Programme (UNDP) issued its first Human Development Report in 1990. The Report promises to be of "seminal nature" and makes "a significant contribution to the development dialogue." In the ensuing years, the Report has indeed kept its promise and the release of the Report has now become a much anticipated annual event. The part of the Report that has largely been responsible for this publicity is the country ranking of human development. Over the years, countries move up and down in ranks and sensation often arises when countries switch positions, especially at the very top of the list.

The force that shapes the ranking is the Human Development Index (HDI). For a country, its HDI is a simple arithmetic (and lately, geometric) average of three components: life expectancy, literacy, and real GDP per capita. The inventors of HDI pointed out that the objective of human development is "to create an enabling environment for people to enjoy long, healthy and creative lives" and national income accounts as "means" do not adequately reflect how well the objective has been achieved. Instead, a measure directly based on the "ends" such as health and literacy should be used to measure the progress of human development. Although there are multitude of "ends" in human development, for the sake of data availability and the desire to have a focused and sharp measure, the designers of HDI decided to include only the three components in the calculation. In the HDI computation, each component is normalized to have a maximum value of 1 so an HDI would lie between 0 and 1.

Not surprisingly, in the years since the debut of HDI, researchers have questioned and debated about every aspect of the HDI construction; see McGillivray and Noorbakhsh (2004) for a survey of the early literature. A particular focus of the research has been on the linear structure of HDI. That is, the HDI is an equally weighted sum of the three components. One implication of the additive structure is that the three components are perfectly substitutable to each other. In an early assessment of HDI, Desai (1991) points out that perfect substitution "can hardly be appropriate" and suggests to use "a log additive form" - a suggestion that has apparently been heeded and adopted by UNDP in recent years. Much of the recent research, however, has been on the robustness of using a given set of weights. That is, this literature is concerned with whether the ranking obtained using one set of weights will not be altered if a different set of weights are plugged in. For example, McGillivray and Noorbakhsh (2007) consider different weights in computing HDI and examine the correlation among the resulting country rankings. Foster et al. (2009, 2013) develop a Lorenz-type ε-robustness dominance condition to check the robustness of a given HDI ranking when the initial weights are allowed to deviate in a certain manner (ε-contamination). Permanyer (2011) follows a similar line of robustness-checking and considers different mechanisms of weighting. Cherchye et al. (2008) derive conditions for robust ranking when the weights, the normalization and the aggregation method are allowed to change simultaneously.

It is important to point out that the literature so far has treated human development as a well-defined concept and assumes that all countries can be rank ordered by the level of human development. There are strong reasons, however, to believe that the notion of human development is inexact and there are situations where two nations' human development may not be unequivocally ranked. In this sense, the notion of human development is similar to other socioeconomic phenomena such as poverty and inequality where the imprecise nature has been well recognized and reflected in the measurement. For example, in measuring income inequality, Sen (1973) "argues persuasively, the notion of inequality may be inherently imprecise and to do away with the imprecision by brute force may not be the best approach" (Basu, 1987) and, following Sen's insight, Basu (1987) and Ok (1995) made path-breaking steps in taking the imprecision into account in inequality measurement. In poverty measurement, Sen (1981, p. 13) also pointed out that the notion of poverty is imprecise since "while the concept of a nutritional requirement is a rather loose one, there is no reason to suppose that the concept of poverty is clear cut and sharp ... a certain amount of vagueness is implicit in both the concepts." Over the last two decades, researchers have utilized the tools developed in the fuzzy set theory to the measurement of poverty. Contributions include Cerioli and Zani (1990), Cheli and Lemmi (1995), Shorrocks and Subramanian (1994), Chakravarty (2006), and Zheng (2013). In light of these developments, it is natural also to apply the fuzzy set theory to measuring human development.

Human development is generally defined as "a process of enlarging people's choices" (UNDP, 1990). Clearly, the definition itself does not give rise to a formula for its measurement. Even with UNDP's laborious elaborations in the last three decades, the notion of human development remains vague. Conceptually, there are two main sources that contribute to the vagueness in the measurement. The first source is what to be included in the measurement. Although living a long healthy life, receiving good education and enjoying a decent material consumption are justified to be the key components of HDI, what constitutes human development is certainly a debatable topic (again see McGillivray and Noorbakhsh (2004) for a survey) and will likely never get settled. The second source of vagueness is how to aggregate the different components in the measurement. Even if one accepts the choice of components in HDI, there are ambiguities surrounding the functional form to aggregate the three components and particularly, the level of importance that each component is assigned to. The equal-weighting employed in HDI is meant to indicate the equal importance of the three components in evaluating the level of human development (UNDP, 1991, p. 88). But this practice is "almost certainly incorrect, as it implies that each component is equally important, in terms of well-being achievement, at all points of time and levels of achievement, and in all regions, countries, cultures, levels of development, and so on" (McGillivray and Noorbakhsh, 2004, p. 11). Streeten (1994, p. 235) summarizes well the vagueness in measuring human development - "not only are the weights of the three components arbitrary, but also what is excluded and what is included."

In this paper, we focus on this last source of vagueness in human development measurement and propose a method to fuzzificate human development ranking. In doing so, we accept that human development consists of only the three components and the aggregation operator is a linear function. Thus the HDI is a weighted average of the three components. Different from other aforementioned research, we do not assume an initial set of weights and thus do not give particular preference to any specific way of aggregation. Instead, we consider all possible weights - this reflects the reality that "it is probably impossible to achieve agreement on what the weights should be" (UNDP 1993). Consequently, although we may still have situations where one country unambiguously has a higher level of human development than another country, we often face situations where we fail to draw definite conclusions due to the disagreement on the weights. In the absence of a definite conclusion, we provide a "truth value" or a level of confidence for one country to have a higher level of human development than another country. We believe that the fuzzy ranking approach we develop in this paper is informative, it complements the official ranking of HDIs with "truth values" of the rankings. For example, in 2004, Canada (ranked 6) is ranked above the US (ranked 8) but only about 51% of all possible weights support the ranking. That is, nearly half of all possible weights would rank the two countries the other way around!

In a sense, the approach we propose in this paper can be seen as a dual to the robustness-testing approach that Foster et al. and other researchers have advanced. While the robustness-testing approach starts from a given weighting (usually equal weights) - thus with a known ranking - and checks the scope to which the ranking will remain valid, our approach does not have a known ranking to begin with but rather identifies the extent to which the ranking in either direction can be true.

The rest of the paper is organized as follows. In the next section we characterize a fuzzy relation that one country has a higher level of human development than another country. We deduce that the membership function, or the "truth value" function is a function of the probability that a randomly drawn set of weights will show one country's HDI to be greater than another country's HDI. We then derive closed-form and easily computable formulae to calculate the "greater-than" probability of the HDI weights. Finally we illustrate our approach with the HDI rankings of six countries in 2004. Section 3 concludes.

2. Fuzzy Human Development Relation: the Proposal

The approach to be developed is not restricted to measuring human development, it is applicable to all composite indices. For ease of reference, however, we refer to the composite index considered in this paper as the "human development index" which indicates the levels of human development for a country. The index is denoted h.

Suppose h is determined by $n \geq 2$ human development (HD) factors or components which are quantified as continuous variables x_1, x_2, ..., and x_n. We further assume that x_i's are all positive. Although the functional form of h in terms of x_1, x_2, ..., and x_n is imprecise, we do assume that, *ceteris paribus*, each component x_i contributes positively to the level of h.

2.1. Fuzzification of human development rankings

Now consider two vectors of components $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$ for two countries and the corresponding levels of human development are $h(x)$ and $h(y)$, respectively. Here we also assume that the functional form h, though vague and inexact, is the same for both countries. The issue is the comparison of the level of human development between the two countries. An important point we wish to reiterate is that often the comparison is inconclusive and neither country can be judged to have a unambiguously higher level of human development due to the imprecise nature of the h-function.

Nevertheless, we do have a situation where we can declare unequivocally that one country has a higher level of h than another country. This occurs when one country's HD components are uniformly greater than or no less than the other country's corresponding HD components. For ease of presentation, we denote

$$\begin{aligned} x &\succsim y \text{ if } x_i \geq y_i \text{ for all } i\text{; } x \succ y \text{ if additionally } x \neq y \\ x &\precsim y \text{ if } x_i \leq y_i \text{ for all } i\text{; } x \prec y \text{ if additionally } x \neq y\text{; and} \\ x, y &\in \Lambda \text{ if neither } x \succsim y \text{ nor } x \precsim y. \end{aligned}$$

Thus, $h(x) \geq h(y)$ if $x \succsim y$ and $h(x) \leq h(y)$ if $x \precsim y$; strict inequalities ($>$ and $<$) follow if additionally $x \neq y$. If $x, y \in \Lambda$, then a clear verdict on whether $h(x) > h(y)$ or $h(x) < h(y)$ requires that the h-function be more completely specified. But as we argued in the Introduction, it is improbable to have such a fully specified function for human development measurement. Thus, if $x, y \in \Lambda$, then neither $h(x) > h(y)$ nor $h(x) < h(y)$ can be completely true; it is true only to a certain degree or true with a certain "truth value".

If we denote $\mu(x, y)$ as the "truth value" that country x has a higher level of human development than country y, then we can express the comparison between $h(x)$ and $h(y)$ using a membership function from the fuzzy set theory (see Klir and Yuan, 1995, for a detailed introduction of the theory). Let $\mu(x, y)$ be a continuous function which takes values between 0 and 1: $\mu(x, y) = 0$ indicates that country x definitely has a lower level than y and $\mu(x, y) = 1$ indicates that country x definitely has a higher level than y. That is,

$$\mu(x, y) = \begin{cases} 0 & \text{if } x \prec y \\ f(x, y) & \text{if } x, y \in \Lambda \\ 1 & \text{if } x \succ y \end{cases} \tag{2.1}$$

for some continuous function $f(x, y)$. Note that we do not consider the trivial case of $x = y$.

To fully characterize the membership function $\mu(x, y)$, we need to specify function $f(x, y)$. From the above discussions, we know that $f(x, y)$ must be increasing in each x_i and decreasing in each y_i. This condition and the continuity of $f(x, y)$ imply that $f(x, y) = 1$ if $x \succ y$ and $f(x, y) = 0$ if $x \prec y$. But for $x, y \in \Lambda$, $0 < f(x, y) < 1$ and the value of $f(x, y)$ depends on the way that human development is measured and depends on the source of vagueness. In the literature, the level of human development is often measured as a weighted average of the HD components:

$$h(x) = \sum_{i=1}^{n} w_i x_i \equiv wx \tag{2.2}$$

where w_i's are weights that are drawn from the space $\Psi = \{w = (w_1, w_2, ..., w_n) | w_i \geq 0 \text{ for all } i \text{ and } \sum_{i=1}^{n} w_i = 1\}$.[1] With this specification, the vagueness/fuzziness in measuring h arises from choosing the weights w_i's. As argued in the Introduction, the use of any specific set of weights is deemed to be arbitrary since there is no "right" procedure to determine such a set of weights.

The human development index (2.2) enables us to measure the "truth value" that x indeed has a higher level of human development than y. Holding everything else the same, between x and y, an increase in any x_i should increase or at least not decrease the truth value that x has a higher h level than y. But in what sense that the truth value has been increased? Given (2.2), it is natural to think that it means for any given bundle of weights $\{w_i\}$ it becomes more likely that $h(x) > h(y)$. Or equivalently more bundles of $\{w_i\}$ will enable $h(x) > h(y)$. This interpretation suggests that we can measure the "truth value" for $h(x) > h(y)$ as a function of the "number" of weights that support the ranking. Since the weights space Ψ is continuous, the "number" of bundles is measured as the "area" or proportion of the space that renders $h(x) > h(y)$. That is,

$$f(x, y) = \hat{f}(\rho_{xy}) \tag{2.3}$$

where $\hat{f}$ is a continuous and increasing function, $\rho_{xy} = \int_{w \in \Psi} I(wx > wy) dw$ and $I(wx > wy)$ is an indicator function which is one if $wx > wy$ is true and zero otherwise. It follows that the fuzzy membership function can be further specified as

[1] In recent UNDP reports, HDI has also been calculated as a geometric mean, i.e.,

$$\hat{h}(x) = x_1^{w_1} x_2^{w_2} \cdots x_n^{w_n}.$$

Since by taking logarithm, $\ln \hat{h}(x)$ becomes a weighted arithmetic mean, the method developed in this paper can be directly applied to the comparison between $\ln \hat{h}(x)$ and $\ln \hat{h}(y)$. Thus in the paper we consider only the weighted arithmetic-mean HDI.

follows:

$$\mu(x,y)=\begin{cases}1 & \text{if } x\succ y\\ \hat{f}(\rho_{xy}) & \text{if } x,y\in\Lambda\\ 0 & \text{if } x\prec y\end{cases}. \tag{2.4}$$

2.2. Some properties of the fuzzy relation of human development

The fuzzy relation established in (2.4) is different from the crisp relation which consists of only the possibilities of $\mu(x,y)=0$ and 1. The extension to the fuzzy area $0<\mu(x,y)<1$ significantly changes the operation rules of the crisp relation. While some operation rules such as completeness ($\max\{\mu(x,y),\mu(y,x)\}>0$ for all x and y) remain largely valid in fuzzy relations, others such as reciprocality ($\mu(x,y)+\mu(y,x)=1$ for all x, y and z) no longer always hold. The most debated extension of operation from crisp ordering to fuzzy ordering is probably the property of transitivity. The max-min transitivity first introduced in Zadeh (1971) has been found to be not only "lack of intuitive appeal" but also "fruitless" in fuzzy inequality measurement (Ok, 1995, p. 119). It can be shown that the max-min transitivity is also not satisfied by the fuzzy relation defined in this paper.

Alternative transitivity operation rules have been proposed for fuzzy relations (see Ovchinnikov, 1981; and Ok, 1995 for details). One is f-transitivity that requires, for all x, y, and z, $\mu(x,y)\geq\mu(y,x)$ and $\mu(y,z)\geq\mu(z,y)$ imply $\mu(x,z)\geq\mu(z,x)$. Another is O-transitivity that, for all x, y, and z, $\mu(x,y)=1$ implies both $\mu(x,z)\geq\mu(y,z)$ and $\mu(z,y)\geq\mu(z,x)$. A third one is ω-transitivity that, for all x, y, and z, $\mu(x,y)\geq\mu(y,x)$ implies both $\mu(x,z)\geq\mu(y,z)$ and $\mu(z,y)\geq\mu(z,x)$. It is easy to see that the fuzzy relation defined in this paper satisfies O-transitivity and it can be shown that the fuzzy relation violates both f-transitivity and ω-transitivity. For this reason, we maintain O-transitivity for the fuzzy relation of human development ranking.

The reciprocality ($\mu(x,y)+\mu(y,x)=1$) is an appealing property in ranking human development: the truth values of "x has a higher level than y" and "y has a higher level than x" are summed up to 1, which makes the two events symmetric and makes it easier to interpret and compare different truth values. Ok (1995) derived the following important result for the general functional form of the truth-value function $\mu(x,y)=\hat{f}(\rho_{xy})$ that satisfies reciprocality:

$$\hat{f}(t)=\frac{1}{2}+(t-\frac{1}{2})\phi(t-t^2) \tag{2.5}$$

where ϕ is some properly specified continuous function.

Another property that is often specified for a fuzzy membership function is the S-shape of the $\hat{f}$-function. The S-shape property requires that $\hat{f}(t)$ be first convex and then concave in t; three such truth-value functions are plotted in Figure 1. Since a truth-value function is an increasing function of t, it follows that $\hat{f}(0)=0$ and

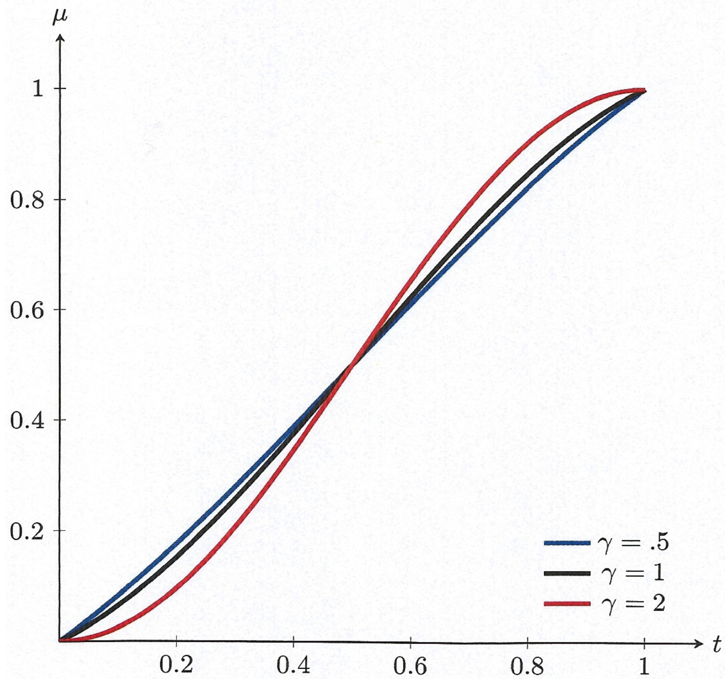

Figure 1: Truth-Value Functions

$\hat{f}(1) = 1$. Thus, we require $\phi(0) = \phi(1) = 1$. An example of the ϕ-function is the following

$$\phi(t) = (1 + t - t^2)^\gamma \tag{2.6}$$

for $\gamma \geq 0$, and thus,

$$\hat{f}(t) = \frac{1}{2} + (t - \frac{1}{2})(1 + t - t^2)^\gamma. \tag{2.7}$$

The increasingness of $\hat{f}(t)$ and the S-shape property put further restrictions on the values that γ can assume. For example, for $\gamma = 3$, $\hat{f}(t)$ would not always be increasing in t in $(0, 1)$. In Figure 1 above, we plot three "truth value" functions of (2.7) for $\gamma = 0.5, 1$ and 2. All three functions exhibit increasingness and are of the S-shape. A special case is the limiting case when $\gamma \to 0$ or $\phi(t) \to 1$, the limiting truth-value function becomes $\hat{f}(t) = t$ and $\mu(x, y) = \rho_{xy}$.

With the truth values, we can also define a "closest crisp relation" (Ok, 1994) that is associated with the fuzzy ranking. For example, following Ok (1995), we can define that x (strictly) fuzzy dominates y if $\mu(x, y)(>) \geq \mu(y, x)$ or using the characteristic function

$$\chi_\mu(x, y) = \begin{cases} 1 & \text{if } \mu(x, y) \geq \mu(y, x) \\ 0 & \text{otherwise} \end{cases} .$$

With reciprocality, it implies that $\chi_\mu(x, y) = 1$ if and only if $\mu(x, y) \geq 0.5$. If we treat a truth value as a level of confidence, then we can define a generalized crisp

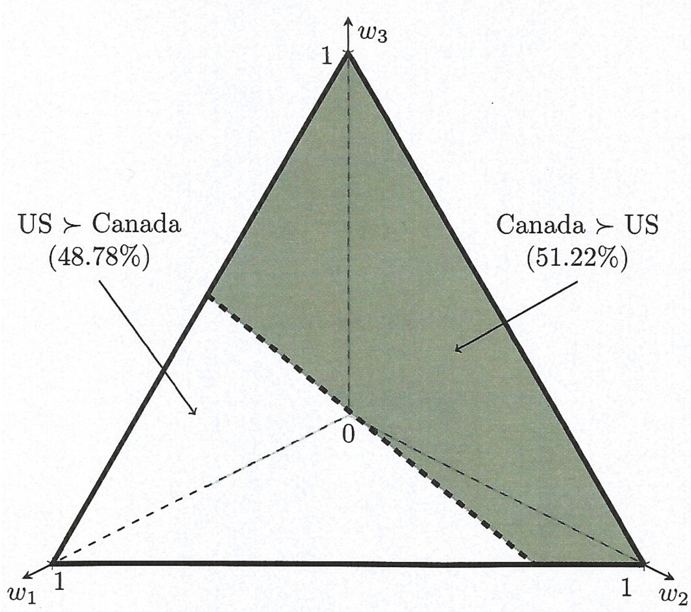

Figure 2: Human Development: US vs. Canada

relation with a different threshold of truth value. For example, we could define that $\chi_\mu(x, y) = 1$ if and only if $\mu(x, y) \geq 0.9$.

2.3. The computation of ρ_{xy}: the general case for any n

Thus, once we have specified the functional form for $\hat{f}(t)$, to obtain the truth value, all we need is to compute ρ_{xy}. For the cases of $n = 2$ and 3, ρ_{xy} can be computed as a ratio of two lengths and a ratio of two areas, respectively, and it can also be easily depicted graphically ($n = 3$ is depicted in Figure 2 above). But for the general case with $n > 3$, ρ_{xy} cannot be depicted graphically and the method of computation for $n = 2$ and 3 is no longer workable. Below, we follow the interpretation of ρ_{xy} as a probability and derive general formulae of it for any $n \geq 2$.

Recall that for two vectors of HD factors $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$, ρ_{xy} is defined as $\rho_{xy} = \int_{w \in \Psi} I(wx > wy)dw$ where $I(wx > wy)$ is an indicator function that is one if $wx > wy$ is true and zero otherwise. Since Ψ is a simplex with $\sum_{i=1}^n w_i = 1$, ρ_{xy} can be interpreted as the probability that a randomly drawn vector of weights $w = (w_1, w_2, ..., w_n)$ from the simplex (weights space) such that $h(x) = wx > wy = h(y)$. That is,

$$\rho_{xy} = P\{wx > wy | w \in \Psi\}.$$

Denoting $a = x - y$ and expanding the vector, we have

$$\rho_{xy} = P\{a_1 w_1 + a_2 w_2 + \cdots + a_n w_n > 0 | \sum_{i=1}^{n} w_i = 1; w_i \geq 0\}. \tag{2.8}$$

A first step, and a necessary one, in computing the probability is to derive the probability distribution of $aw = a_1w_1 + a_2w_2 + \cdots + a_nw_n$. Since within the simplex, each weight w_i can assume any value between 0 and 1 equally likely, we assume that each w_i, $i = 1, 2, .., n$, follows a uniform distribution on $[0, 1]$.

Since w_i and w_j for $i \neq j$ are not independently distributed, a key step is to transform w into a new vector such that the transformed variables are independent. This is accomplished via a vector of iid exponential random variables. Let $z_1, z_2, ..., z_n$ be n iid random variables with each variable following an exponential distribution with density function:

$$f(z) = \begin{cases} \lambda e^{-\lambda z} & z \geq 0 \\ 0 & z < 0 \end{cases} \tag{2.9}$$

for some $\lambda > 0$. The instrument for the transformation is to set

$$w_i = \frac{z_i}{\sum_{j=1}^{n} z_j}, \tag{2.10}$$

then

$$P\{a_1w_1 + a_2w_2 + \cdots + a_nw_n > 0\} = P\{a_1z_1 + a_2z_2 + \cdots + a_nz_n > 0\} \tag{2.11}$$

since $\sum_{j=1}^{n} z_j > 0$. Now $z_1, z_2, ...,$ and z_n are independently and identically distributed and we need to derive the distribution of $a_1z_1 + a_2z_2 + \cdots + a_nz_n$. For exponential distributions z_i's, the sum $z_1 + z_2 + \cdots + z_n$ has been known as a reliability issue in engineering - the failure rate of the remaining $n - i$ components when i components have failed - which has been studied by engineers and mathematicians (for example, Cox, 1967; Scheuer, 1988; and Amari and Misra, 1997). For completeness, here we derive the distribution of $a_1z_1 + a_2z_2 + \cdots + a_nz_n$ as a lemma.

The derivation utilizes the Laplace transform technique but needs to distinguishes between two situations: (i) all a_i's are distinct, and (ii) some of a_i's are identical. In both cases, all a_i's are nonzero; if some a_i's are zero, then reset n to the number of non-zero a_i's and all results remain valid.

Lemma 1. Suppose each random variable z_i, $i = 1, 2, ..., n$, follows an exponential distribution as specified in (2.9), then (i) if all a_i's are distinct and non-zero then the probability distribution function for $v = a_1z_1 + a_2z_2 + \cdots + a_nz_n$ is

$$P\{v < z\} = \sum_{i=1}^{n} \left[\frac{a_i^{n-1}}{\Pi_{j=1, j\neq i}^{n} (a_i - a_j)} \right] \frac{\lambda}{a_i} e^{-\frac{\lambda}{a_i} z}; \tag{2.12}$$

(ii) if some of a_i's are equal, say,

$$\begin{aligned} a_1 &= \cdots = a_{r_1} = b_1 \\ a_{r_1+1} &= \cdots = a_{r_1+r_2} = b_2 \\ &\cdots \\ a_{r_{m-1}+1} &= \cdots = a_{r_{m-1}+r_m} = b_m, \end{aligned} \tag{2.13}$$

and all b_i's are distinct and non-zero. The probability distribution function for $v = b_1\tilde{z}_1 + b_2\tilde{z}_2 + \cdots + b_m\tilde{z}_m$ is

$$P\{v < z\} = \Pi_{j=1}^{m}\tilde{\lambda}_j^{r_j}\sum_{k=1}^{m}\sum_{l=1}^{r_k}\frac{\Phi_{kl}(-\tilde{\lambda}_k)}{(r_k-l)!(l-1)!}z^{r_k-l}e^{-\tilde{\lambda}_k z} \tag{2.14}$$

with $\tilde{\lambda}_i = \lambda/b_i$ and

$$\Phi_{kl}(z) = \frac{d^{l-1}}{dz^{l-1}}\{\Pi_{j=1,j\neq i}^{m}(\tilde{\lambda}_j + z)^{-r_j}\}. \tag{2.15}$$

Proof. Since

$$E(e^{-sa_i z_i}) = \int_0^\infty \lambda e^{-sa_i z}e^{-\lambda z}dz = \int_0^\infty \lambda e^{-(\lambda+sa_i)z}dz = \frac{\lambda}{\lambda + a_i s}, \tag{2.16}$$

the Laplace transform of the probability distribution function (p.d.f.) of $v = a_1z_1 + a_2z_2 + \cdots + a_nz_n$ is (Cox, 1967, p. 10):

$$\begin{aligned} k^*(s) &= E[\exp\{-s(a_1z_1 + a_2z_2 + ... + a_nz_n)\}] \\ &= E(e^{-sa_1z_1}\cdots e^{-sa_nz_n}) \\ &= E(e^{-sa_1z_1})\cdots E(e^{-sa_nz_n}) \\ &= \left(\frac{\lambda}{\lambda + a_1 s}\right)\cdots\left(\frac{\lambda}{\lambda + a_n s}\right). \end{aligned} \tag{2.17}$$

The p.d.f. of $v = a_1z_1 + a_2z_2 + \cdots + a_nz_n$ is the inverse of the Laplace transform

$$k(z) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{sz}k^*(s)ds$$

but often is available from the tables of Laplace transforms such as in Erdélyi (1954).

If we rewrite $k^*(s)$ as

$$\begin{aligned} k^*(s) &= \frac{\lambda^n}{\Pi_i a_i}\frac{1}{(\lambda_1' + s)\cdots(\lambda_n' + s)} \\ &= \frac{\lambda^n}{\Pi_i a_i}\frac{1}{\Pi_i(s + \lambda_i')} = \frac{\Pi_i\lambda_i'}{\Pi_i(s + \lambda_i')} \end{aligned} \tag{2.18}$$

where $\lambda_i' = \lambda/a_i$. Since a_i's are distinct and non-zero, λ_i''s are also distinct and non-zero. This type of $k^*(s)$ matches formula 20 on page 232 in the tables of transforms edited by Erdélyi (1954). Making proper notational substitution in formula 20 yields the p.d.f. of $a_1z_1 + a_2z_2 + \cdots + a_nz_n$ as given in (2.12).

To prove case (ii), we first note that $a_1z_1 + a_2z_2 + \cdots + a_nz_n$ can now be written as

$$v = b_1\tilde{z}_1 + b_2\tilde{z}_2 + \cdots + b_m\tilde{z}_m \tag{2.19}$$

with $\tilde{z}_i = \sum_{j=r_{i-1}+1}^{r_{i-1}+r_i} z_j$, $r_0 \equiv 1$ and $r_m \equiv n$. The Laplace transform of $b_i\tilde{z}_i$ is

$$E(e^{-sb_i\tilde{z}_i}) = \left(\frac{\lambda}{\lambda + b_i s}\right)^{r_i}. \tag{2.20}$$

Thus, the Laplace transform of $b_1\tilde{z}_1 + b_2\tilde{z}_2 + \cdots + b_m\tilde{z}_m$ is

$$h^*(s) = \Pi_j \left(\frac{\lambda}{\lambda + b_j s}\right)^{r_j} = \Pi_j \left(\frac{\tilde{\lambda}_j}{\tilde{\lambda}_j + s}\right)^{r_j} \tag{2.21}$$

where $\tilde{\lambda}_j = \lambda/b_j$.

The p.d.f. of $v = b_1\tilde{z}_1 + b_2\tilde{z}_2 + \cdots + b_m\tilde{z}_m$ again can be obtained from Erdélyi's tables of transforms (p. 232, equation 21) with proper substitutions and it leads to (2.14). $\square$

One may naturally wonder, since case (i) is a special case of (ii), i.e., $r_1 = r_2 = \ldots = r_m$ and $m = n$, why there is the need to derive them separately. The reason is that in case (ii) the results are substantially more complicated than case (i) even though in theory setting all r_i's to one should reduce (ii) to (i).

With the lemma, we can now derive the probability $P\{a_1z_1 + a_2z_2 + \cdots + a_nz_n > 0\}$. To facilitate presentation, we further assume that the first q of $\{a_1, a_2, ..., a_n\}$ are positive and the remaining $n - q$ are negative, i.e., $a_i > 0$ for $i = 1, ..., q$ and $a_i < 0$ for $i = q+1, ..., n$. If a_i's are all positive or all negative, then the probability $P\{a_1z_1 + a_2z_2 + \cdots + a_nz_n > 0\}$ will be 1 and 0, respectively.

Proposition 1. Suppose each random variable z_i, $i = 1, 2, ..., n$, follows an exponential distribution specified in (2.9), then (i) if all a_i's are distinct then

$$P\{a_1z_1 + a_2z_2 + \cdots + a_nz_n > 0\} = \sum_{i=1}^{q}\sum_{\tau=q+1}^{n} C_iC_\tau \frac{a_i}{a_i - a_\tau} \tag{2.22}$$

where $C_i = \frac{a_i^{q-1}}{\Pi_{j=1,j\neq i}^{q}(a_i - a_j)}$ for $i = 1, ..., q$, and $C_\tau = \frac{a_\tau^{n-q-1}}{\Pi_{j=q+1,j\neq\tau}^{n}(a_\tau - a_j)}$ for $\tau = q+1, ..., n$; (ii) if some a_i's are identical in the way specified in Lemma 1(ii), then

$$\begin{aligned} P\{a_1z_1 + a_2z_2 + \cdots + a_nz_n &> 0\} \\ &= P\{b_1\tilde{z}_1 + b_2\tilde{z}_2 + \cdots + b_m\tilde{z}_m > 0\} \\ &= A\sum_{k=1}^{q}\sum_{l=1}^{r_k}\sum_{k'=q+1}^{m}\sum_{l'=1}^{r_{k'}} D_{kl}D_{k'l'}\left[\frac{(r_k - l + 1)!}{(-\tilde{\lambda}_{k'})^{r_{k'}-l'+1}\tilde{\lambda}_k}\right. \\ &\quad \left. + \frac{1}{\tilde{\lambda}_{k'}}\sum_{i=0}^{r_{k'}-l'} \frac{(r_{k'} - l')!}{(r_{k'} - l' - i)!}\frac{(r_{k'} - l' + r_k - l + 1 - i)!}{(-\tilde{\lambda}_{k'})^i(\tilde{\lambda}_k - \tilde{\lambda}_{k'})}\right] \end{aligned} \tag{2.23}$$

where $A = \Pi_{j=1}^{m} |\tilde{\lambda}_j|^{r_j}$, $D_{kl} = \frac{\Phi_{kl}(-\tilde{\lambda}_k)}{(r_k-l)!(l-1)!}$, $D_{k'l'} = \frac{\Phi_{k'l'}(\tilde{\lambda}_{k'})}{(r_{k'}-l')!(l'-1)!}$ and $\Phi_{kl}(\cdot)$ and $\Phi_{k'l'}(\cdot)$ are defined in (2.15).

Proof. Case (i). Since

$$P\{a_1 z_1 + ... + a_n z_n > 0\} = P\{a_1 z_1 + ... + a_q z_q > (-a_{q+1}) z_{q+1} + ... + (-a_n) z_n\}, \quad (2.24)$$

and from Lemma 1,

$$P\{a_1 z_1 + ... + a_q z_q < z\} = \sum_{i=1}^{q} \left[\frac{a_i^{q-1}}{\Pi_{j=1, j\neq i}^{q} (a_i - a_j)} \right] \frac{\lambda}{a_i} e^{-\frac{\lambda}{a_i} z}, \quad (2.25)$$

$$P\{(-a_{q+1}) z_{q+1} + ... + (-a_n) z_n < z\} = \sum_{\tau=q+1}^{n} \left[\frac{(-a_\tau)^{n-q-1}}{\Pi_{j=q+1, j\neq \tau}^{n} (-a_\tau + a_j)} \right] \frac{\lambda}{-a_\tau} e^{-\frac{\lambda}{-a_\tau} z}, \quad (2.26)$$

it follows that

$$\begin{aligned} P\{a_1 z_1 + ... + a_n z_n &> 0\} = \int_0^\infty \int_0^z \left\{ \sum_{i=1}^{q} \left[\frac{a_i^{q-1}}{\Pi_{j=1, j\neq i}^{q} (a_i - a_j)} \right] \frac{\lambda}{a_i} e^{-\frac{\lambda}{a_i} z} \right\} \\ &\left\{ \sum_{\tau=q+1}^{n} \left[\frac{(-a_\tau)^{n-q-1}}{\Pi_{j=q+1, j\neq i}^{n} (-a_\tau + a_j)} \right] \frac{\lambda}{-a_\tau} e^{-\frac{\lambda}{-a_\tau} y} \right\} dydz \\ &= \sum_{i=1}^{q} \sum_{\tau=q+1}^{n} \int_0^\infty \int_0^z \left[\frac{a_i^{q-1}}{\Pi_{j=1, j\neq i}^{q} (a_i - a_j)} \right] \left[\frac{(-a_\tau)^{n-q-1}}{\Pi_{j=q+1, j\neq \tau}^{n} (-a_\tau + a_j)} \right] \\ &\left(\frac{\lambda}{a_i} e^{-\frac{\lambda}{a_i} z} \right) \left(\frac{\lambda}{-a_\tau} e^{-\frac{\lambda}{-a_\tau} y} \right) dydz \quad (2.27) \\ &= \sum_{i=1}^{q} \sum_{\tau=q+1}^{n} C_i C_\tau \int_0^\infty \int_0^z \left(\frac{\lambda}{a_i} e^{-\frac{\lambda}{a_i} z} \right) \left(\frac{\lambda}{-a_\tau} e^{-\frac{\lambda}{-a_\tau} y} \right) dydz \end{aligned}$$

where $C_i = \frac{a_i^{q-1}}{\Pi_{j=1, j\neq i}^{q} (a_i - a_j)}$ and $C_\tau = \frac{a_\tau^{n-q-1}}{\Pi_{j=q+1, j\neq \tau}^{n} (a_\tau - a_j)}$. But it is easy to show that

$$\int_0^\infty \left(\int_0^z \frac{\lambda}{-a_\tau} e^{-\frac{\lambda}{-a_\tau} y} dy \right) \left(\frac{\lambda}{a_i} e^{-\frac{\lambda}{a_i} z} \right) dz = \frac{a_i}{a_i - a_\tau}, \quad (2.28)$$

thus

$$P\{a_1 z_1 + ... + a_n z_n > 0\} = \sum_{i=1}^{q} \sum_{\tau=q+1}^{n} C_i C_\tau \frac{a_i}{a_i - a_\tau}. \quad (2.29)$$

To prove case (ii), similarly we have (by again assuming the first q' of b_i's are positive and the remaining $m-q'$ are negative):

$$\begin{aligned}
P\{b_1\tilde{z}_1+...+b_m\tilde{z}_m &> 0\} \qquad (2.30)\\
&= P\{b_1\tilde{z}_1+...+b_{q'}\tilde{z}_{q'} > (-b_{q'+1})\tilde{z}_{q'+1}+...+(-b_m)\tilde{z}_m\}\\
&= \int_0^\infty \int_0^z \left\{\Pi_{j=1}^{q'}\tilde{\lambda}_j^{r_j}\sum_{k=1}^{q'}\sum_{l=1}^{r_k}\frac{\Phi_{kl}(-\tilde{\lambda}_k)}{(r_k-l)!(l-1)!}z^{r_k-l}e^{-\tilde{\lambda}_k z}\right\}\\
&\quad \left\{\Pi_{j=q'+1}^{m}(-\tilde{\lambda}_j)^{r_j}\sum_{k=q'+1}^{m}\sum_{l=1}^{r_k}\frac{\Phi_{kl}(\tilde{\lambda}_k)}{(r_k-l)!(l-1)!}y^{r_k-l}e^{\tilde{\lambda}_k y}\right\}dydz\\
&= \Pi_{j=1}^{q'}\tilde{\lambda}_j^{r_j}\Pi_{j=q'+1}^{m}(-\tilde{\lambda}_j)^{r_j}\sum_{k=1}^{q'}\sum_{l=1}^{r_k}\sum_{k'=q'+1}^{m}\sum_{l'=1}^{r_{k'}}\frac{\Phi_{kl}(-\tilde{\lambda}_k)}{(r_k-l)!(l-1)!}\frac{\Phi_{k'l'}(\tilde{\lambda}_{k'})}{(r_{k'}-l')!(l'-1)!}\\
&\quad \int_0^\infty\int_0^z\left(z^{r_k-l}e^{-\tilde{\lambda}_k z}\right)\left(y^{r_{k'}-l'}e^{\tilde{\lambda}_{k'}y}\right)dydz\\
&= A\sum_{k=1}^{q'}\sum_{l=1}^{r_k}\sum_{k'=q'+1}^{m}\sum_{l'=1}^{r_{k'}}D_{kl}D_{k'l'}\int_0^\infty\left(\int_0^z y^{r_{k'}-l'}e^{\tilde{\lambda}_{k'}y}dy\right)z^{r_k-l}e^{-\tilde{\lambda}_k z}dz
\end{aligned}$$

where $D_{kl}=\frac{\Phi_{kl}(-\tilde{\lambda}_k)}{(r_k-l)!(l-1)!}$, $D_{k'l'}=\frac{\Phi_{k'l'}(\tilde{\lambda}_{k'})}{(r_{k'}-l')!(l'-1)!}$ and $A=\Pi_{j=1}^m|\tilde{\lambda}_j|^{r_j}$.

Since it can be shown that

$$\int_0^z x^n e^{-ax}dx = \frac{1}{a^{n+1}}-\frac{1}{a}\sum_{i=0}^{n}\frac{n!}{(n-i)!}\frac{z^{n-i}e^{-az}}{a^i}, \qquad (2.31)$$

thus

$$\begin{aligned}
&\int_0^\infty\left(\int_0^y x^n e^{-ax}dx\right)y^m e^{-by}dy \qquad (2.32)\\
=&\int_0^\infty\left(\frac{1}{a^{n+1}}-\frac{1}{a}\sum_{i=0}^{n}\frac{n!}{(n-i)!}\frac{z^{n-i}e^{-az}}{a^i}\right)y^m e^{-by}dy\\
=&\int_0^\infty\frac{1}{a^{n+1}}y^m e^{-by}dy-\frac{1}{a}\sum_{i=0}^{n}\frac{n!}{(n-i)!}\int_0^\infty\frac{y^{n-i}e^{-az}y^m e^{-by}}{a^i}dy\\
=&\frac{(m+1)!}{a^{n+1}b}-\frac{1}{a}\sum_{i=0}^{n}\frac{n!}{(n-i)!}\int_0^\infty\frac{y^{n+m-i}e^{-(a+b)z}}{a^i}dy\\
=&\frac{(m+1)!}{a^{n+1}b}-\frac{1}{a}\sum_{i=0}^{n}\frac{n!}{(n-i)!}\frac{(n+m+1-i)!}{a^i(a+b)},
\end{aligned}$$

here we have used $\int_0^\infty x^{n-1}e^{-x}dx=n!$.

Finally, by replacing $n=r_{k'}-l'$, $m=r_k-l$, $a=-\tilde{\lambda}_{k'}$, and $b=\tilde{\lambda}_k$ in (2.32), we have

$$
\begin{aligned}
& A\sum_{k=1}^{q'}\sum_{l=1}^{r_k}\sum_{k'=q'+1}^{m}\sum_{l'=1}^{r_{k'}} D_{kl}D_{k'l'}\int_0^\infty\left(\int_0^z y^{r_{k'}-l'}e^{\tilde{\lambda}_{k'}y}dy\right)z^{r_k-l}e^{-\tilde{\lambda}_k z}dz \\
= \; & A\sum_{k=1}^{q'}\sum_{l=1}^{r_k}\sum_{k'=q'+1}^{m}\sum_{l'=1}^{r_{k'}} D_{kl}D_{k'l'} \qquad (2.33) \\
& \left[\frac{(r_k-l+1)!}{(-\tilde{\lambda}_{k'})^{r_{k'}-l'+1}\tilde{\lambda}_k}+\frac{1}{\tilde{\lambda}_{k'}}\sum_{i=0}^{r_{k'}-l'}\frac{(r_{k'}-l')!}{(r_{k'}-l'-i)!}\frac{(r_{k'}-l'+r_k-l+1-i)!}{(-\tilde{\lambda}_{k'})^i(\tilde{\lambda}_k-\tilde{\lambda}_{k'})}\right].
\end{aligned}
$$

□

As pointed out before, the results for the case with identical a_i values are significantly more complicated than the case with distinct values. The probability for distinct a_i values can be easily calculated by computing a series of C_i values.[2] Although the probability for the case with identical a_i values can also be straightforwardly calculated, the calculation would be much simplified if we could keep sufficient number of decimal points to have unequal a_i values. Or we could treat the equal case as the limit of unequal cases, i.e., by adding or subtracting very small values from the equal values to make them unequal, then the formula for the distinct-value case can be used to obtain an approximation.

2.4. An illustration

To illustrate our proposal of fuzzy ranking, we apply the results of Proposition 1 to the comparison of six countries' levels of human development in 2004. These six countries are Australia, Canada, Iceland, Ireland, Sweden and the United States. The detailed component data are reported in Table 2 of Foster et al. (2009) and the table below reproduces the numbers (the components H, E and SL stand for health, education and standard of living, respectively; HDI is the equally weighted sum of the three components; the rank of each country is the rank among all 177 countries ranked). In this illustration, since all a_i values are distinct, only part (i) of Proposition 1 is applied.

Table 1. HDI Components and Ranking

Country	H	E	SL	HDI	Rank
Iceland	0.931	0.981	0.968	0.960	2
Australia	0.925	0.993	0.954	0.957	3
Ireland	0.882	0.990	0.995	0.956	4
Sweden	0.922	0.982	0.949	0.951	5
Canada	0.919	0.970	0.959	0.950	6
USA	0.875	0.971	0.999	0.948	8

[2] A simple R program has been written to compute the probability for an arbitrary n. It is available from the authors upon request.

In Table 2, we compute the truth values for the ranking of the six countries and report them in the upper triangle of the table. For the sake of simplicity, we choose the truth value function to be the limiting case of $\hat{f}(\rho_{xy}) = \rho_{xy}$, thus, the probability in the weights space that $h(x) > h(y)$ is used to measure the degree of truth that x has a higher level of human development than y. In the table, each country in the first column compares with all other countries in the first row. The "truth value" reported in each pair of comparison indicates the proportion of weights that the column country has a higher level of human development than the row country in question.

Table 2. "Truth values" of HDI rankings

Country	Iceland	Australia	Ireland	Sweden	Canada	USA
Iceland	$\approx$	0.6923	0.5447	0.9950	1	0.7306
Australia		$\approx$	0.5452	1	0.9188	0.6819
Ireland			$\approx$	0.9368	0.6710	0.9368
Sweden				$\approx$	0.6503	0.7306
Canada					$\approx$	0.5122
USA						$\approx$

As the table shows, the truth values for the rankings vary widely between 0.5122 and 1. These values provide useful information about the levels of confidence on the rankings. Without these values, the rankings between Canada and USA and between Iceland and Sweden would be regarded as equally robust. But the ranking between Canada and USA carries a truth value of 0.5122 while the truth value between Iceland and Sweden is 0.9550, hence one should have a much higher confidence in the Iceland/Sweden ranking than in the Canada/USA ranking.

It is interesting to note that none of the rankings has a truth value below 0.5. If we follow Ok (1995) and define a strictly fuzzy dominance of x over y if $\mu(x, y) > \mu(y, x)$, or $\mu(x, y) > 0.5$, then the resulting rankings yielded by Table 2 would be identical to the official HDI ranking by UNDP. Of course, here we only compare and rank six countries, we believe richer and more varied results would emerge if we apply the method to the entire set of 177 countries.

3. Conclusion

Human development, like many other socioeconomic phenomenon such as inequality and poverty, is fundamentally a fuzzy concept. Given the fact that both inequality and poverty have been examined and measured as fuzzy predicates, it is a bit surprising that human development has heretofore been regarded as a crisp quantity and an unambiguous ranking is sought between two countries' levels of human development.

In this paper, we hope we have successfully argued that ranking human development between countries is not a clean-cut exercise; there exists a great deal of vagueness in the ranking process. The vagueness comes from two main sources: one is in deciding what aspects of human development to be included in the measurement and the other is in choosing the operator to aggregate these aspects. Even if we have accepted the weighted arithmetic-mean (or weighted geometric-mean) as the "correct" structure to compute the level of human development, the assignment of weights becomes a source of vagueness and there does not exist a "right" procedure in setting the weights. The use of equal weighting adopted in the UNDP's human development index has been justified on the basis of Occam's razor - but the invoking of the razor is more an admission that "it is probably impossible to achieve agreement on what the weights should be" (McGillivray and Noorbakhsh, 2004, p. 11).

The objective of the paper has been to recognize the vagueness in weighting and explicitly take it into account in ranking human development levels. The approach we developed is the "truth value" approach. A truth value is simply a function of the probability that a randomly drawn bundle of weights will rank one country as having a higher level of human development than another country. Each truth value provides a level of confidence about a particular ranking and, in this sense, it is akin to the significance level used in econometric analysis. Certainly it is a requirement to report levels of significance in all econometric models. Thus it would be very helpful for UNDP's human development rankings to also provide the "truth values" associated with the rankings. Without these truth values, it would be impossible to tell the strong rankings from the weak ones. For example, the rankings of US versus Canada and US versus Iceland would be treated as equally valid despite the fact that the latter is much more robust than the former. Thus, our proposal to UNDP is that the truth values should be computed and reported alongside the rankings of human development.

Of course, the issue of ranking robustness has long been recognized in the literature since its inception. In fact, much of the recent research such as Foster et al. (2009) has focused on the issue. But in these previous research, an initial weighting (usually equal weights) must be specified and the aim is to test whether the resulting ranking remains valid to some perturbation in the initial weighting. In contrast, our approach does not rely on an initial weighting and thus provides an alternative and simple way to assess the robustness of the human development ranking.

Finally, it is useful to reiterate that the methodology we developed in this paper applies to all composite index rankings and "truth values" can be easily computed using the formulae we have derived and the computer program we have compiled.

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