

# Estimating HRF and covariance structure

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# Estimating HRF and amplitudes

- Use one block at a time
- Code the twelve stimuli as 1-12, the “null” signal as 13, and the 6 “calibration” signals as 14-19
- Transform stimuli assignments to a matrix  $S$ , dimension of  $S$  is  $T \times K$ ,  $T$  is the time of time points,  $K$  the number of stimuli types

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 3 \\ \vdots \end{bmatrix} \rightarrow S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

# Estimating HRF and amplitudes

Transform estimated or assumed HRF  $h = (h_1, \dots, h_L)$  to matrix  $H(h)$ .  
Dimension of  $H(h)$  is  $T \times T$

$$h = \begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \\ \vdots \end{bmatrix} \rightarrow H(h) = \begin{bmatrix} h_1 & 0 & 0 & \dots \\ h_2 & h_1 & 0 & \dots \\ h_3 & h_2 & h_1 & \dots \\ h_4 & h_3 & h_2 & \dots \\ h_5 & h_4 & h_3 & \dots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

# Estimating amplitudes

Estimate the stimuli-specific amplitudes  $\alpha = (\alpha_1, \dots, \alpha_K)$  by fitting the model

$$y \sim H(h)S\alpha + \text{const} = \begin{bmatrix} h_1 & 0 & 0 & \dots \\ h_2 & h_1 & 0 & \dots \\ h_3 & h_2 & h_1 & \dots \\ h_4 & h_3 & h_2 & \dots \\ h_5 & h_4 & h_3 & \dots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \end{bmatrix} + \text{const}$$

# Estimating HRF

Suppose instead that  $\alpha$  is known and  $h$  is unknown. Then let  $t = S\alpha = (t_1, t_2, \dots)$  and define

$$A(S\alpha) = \begin{bmatrix} t_1 & 0 & 0 & \cdots \\ t_2 & t_1 & 0 & \cdots \\ t_3 & t_2 & t_1 & \cdots \\ t_4 & t_3 & t_2 & \cdots \\ t_5 & t_4 & t_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

Dimension of  $A(S\alpha)$  is  $T \times L$ , where  $L$  is duration of HRF. Then fit

$$y \sim A(S\alpha)h + \text{const}$$

Fit  $h$  and  $\alpha$  in alternating fashion until convergence.

# Regularization

Define a penalty function  $P(h)$  by

$$P(h) = (h_1 - h_2)^2 + (h_2 - h_3)^2 + \cdots + (h_{L-1} - h_L)^2$$

Now, choosing  $\lambda_h > 0$ , fit

$$h = \operatorname{argmin} \|y - A(S_\alpha)h + c\|^2 + \lambda_h P(h)$$

Similarly, fit

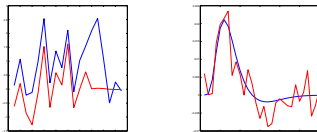
$$\alpha = \operatorname{argmin} \|y - H(h)S_\alpha + c\|^2 + \lambda_\alpha \|\alpha\|^2$$

Again, alternate until convergence.

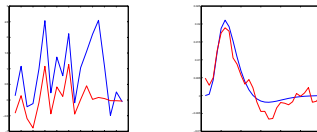
# Simulation

Compared fits with/without regularization.

- Without regularization:

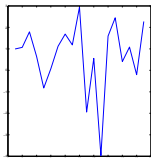


- With regularization:

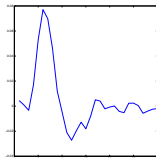


*Left: amplitudes, right: HRF, blue: truth, red: estimate*

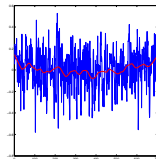
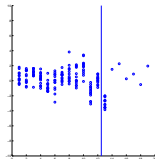
# Data: Block3



amplitudes  
events (by type)



HRF  
residuals





# Covariance estimation

- Assume Gaussian process + noise model.
- GP specified by orthonormal basis functions  $h_k$
- That is, if  $r_j$  are residuals, for  $j = 1, \dots, T$ ,

$$r_j = \epsilon_j + \sum_{k=1}^K Z_k h_{kj}$$

where  $\epsilon_j \sim N(0, \sigma_0^2)$ ,  $Z_k, W_k \sim N(0, \sigma_k^2)$  (all independent).

- Use the estimating equations

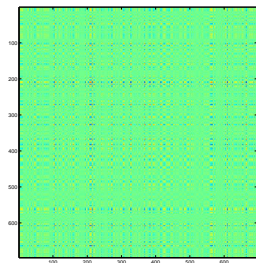
$$\mathbb{E}[r' r] = T \sigma_0^2 + 2 \sum_k \sigma_k^2$$

$$\mathbb{E}[r' h_k] = \sigma_k^2 + \sigma_0^2$$

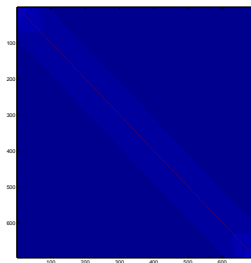
- Example: use  $h_k$  derived from  $\text{Cov}(i, j) = e^{-C(i-j)^2}$

# Covariance estimation

Estimated  $\hat{\Sigma}$



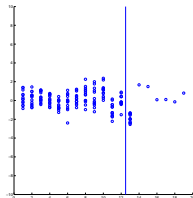
Naive estimate  $rr'$



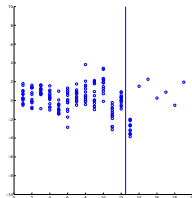
GP estimate

# Comparisons

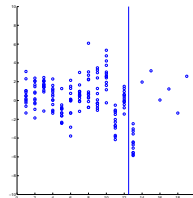
Events estimated with



SPM



fitted HRF



fitted  $h$  and GLS