Estimating HRF and covariance structure

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Estimating amplitudes

Estimate the stimuli-specific amplitudes $\alpha = (\alpha_1, \ldots, \alpha_K)$ by fitting the model

$$y \sim H(h)S\alpha + \text{const} = \begin{bmatrix} h_1 & 0 & 0 & \cdots \\ h_2 & h_1 & 0 & \cdots \\ h_3 & h_2 & h_1 & \cdots \\ h_4 & h_3 & h_2 & \cdots \\ h_5 & h_4 & h_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \end{bmatrix} + \text{const}$$

Estimating HRF

Suppose instead that α is known and h is unknown. Then let $t = S\alpha = (t_1, t_2, \ldots)$ and define

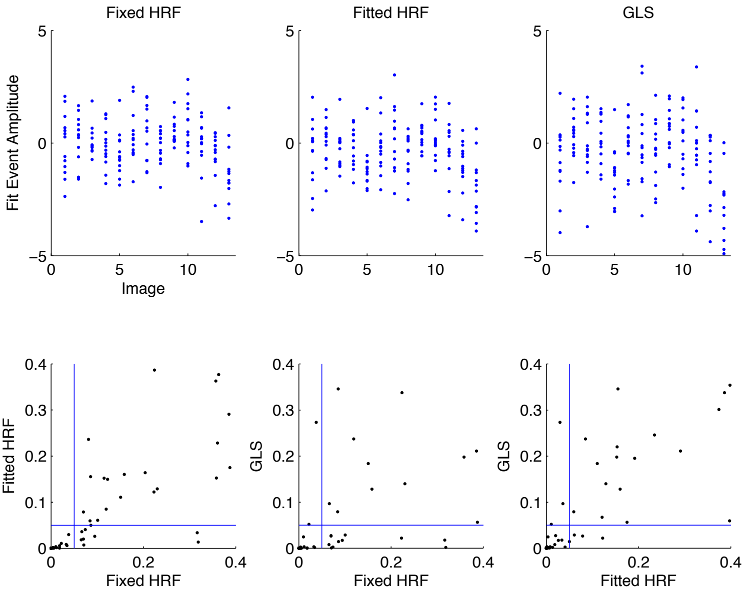
$$A(S\alpha) = \begin{bmatrix} t_1 & 0 & 0 & \cdots \\ t_2 & t_1 & 0 & \cdots \\ t_3 & t_2 & t_1 & \cdots \\ t_4 & t_3 & t_2 & \cdots \\ t_5 & t_4 & t_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

Dimension of $A(S\alpha)$ is $T \times L$, where L is duration of HRF. Then fit

$$y \sim A(S\alpha)h + \text{const}$$

Fit h and α in alternating fashion until convergence.

Comparisons



Estimating HRF and amplitudes

- Code the twelve stimuli as 1-12 in one block, the "null" signal as 13, and the 6 "calibration" signals as 14-19
- Transform stimuli assignments to a matrix S, dimension of S is $T \times K$, T is the time of time points, K the number of stimuli types

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 3 \\ \vdots \end{bmatrix} \rightarrow S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

Estimating HRF and amplitudes

Transform estimated or assumed HRF $h = (h_1, \ldots, h_L)$ to matrix $H(h)$. Dimension of $H(h)$ is $T \times T$

$$h = \begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \\ \vdots \end{bmatrix} \to H(h) = \begin{bmatrix} h_1 & 0 & 0 & \cdots \\ h_2 & h_1 & 0 & \cdots \\ h_3 & h_2 & h_1 & \cdots \\ h_4 & h_3 & h_2 & \cdots \\ h_5 & h_4 & h_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

Regularization

Define a penalty function $P(h)$ by

$$P(h) = (h_1 - h_2)^2 + (h_2 - h_3)^2 + \cdots + (h_{L-1} - h_L)^2$$

Now, choosing $\lambda_h > 0$, fit

$$h = \text{argmin}||y - A(S\alpha)h + c||^2 + \lambda_h P(h)$$

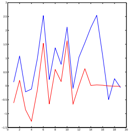
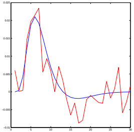
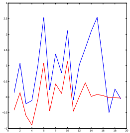
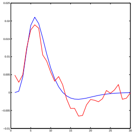
Similarly, fit

$$\alpha = \text{argmin}||y - H(h)S\alpha + c||^2 + \lambda_\alpha ||\alpha||^2$$

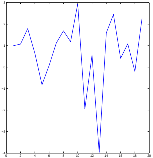
Again, alternate until convergence.

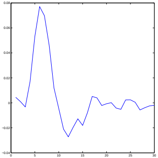
Simulation

Compared fits with/without regularization.

- Without regularization:  
- With regularization:  

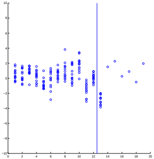
Left: amplitudes, right: HRF, blue: truth, red: estimate

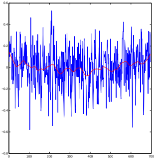




amplitudes HRF

events (by type) residuals





Covariance estimation

- Assume Gaussian process + noise model.
- GP specified by orthonormal basis functions h_k
- That is, if r_j are residuals, for $j = 1, \ldots, T$,

$$r_j = \epsilon_j + \sum_{k=1}^{K} Z_k h_{kj}$$

where $\epsilon_j \sim N(0, \sigma_0^2)$, $Z_k, W_k \sim N(0, \sigma_k^2)$ (all independent).

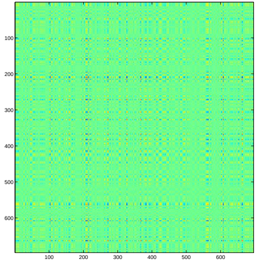
- Use the estimating equations

$$\mathbb{E}[r' r] = T\sigma_0^2 + 2\Sigma_k \sigma_k^2$$

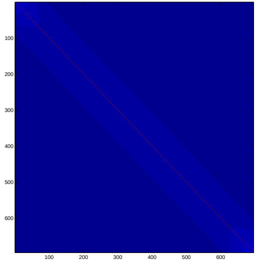
$$\mathbb{E}[r' h_k] = \sigma_k^2 + \sigma_0^2$$

- Example: use h_k derived from $Cov(i,j) = e^{-C(i-j^2)}$

Estimated $\hat{\Sigma}$



Naive estimate rr'



GP estimate