Exercise Sheet 5

Theory of neural Dynamics and application to ML based on RC

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Problem: 1

(a) Simulate the stochastic HH neuron model in Eq. (S1) and plot the time series of the membrane potential V and the corresponding phase portrait in the (V-n) plane for a duration of T = 200 ms, the input current is $I_0 = 5.2\mu A/cm^2$, the initial conditions are V(0) = 0, n(0) = 0.35, m(0) = 0.06, h(0) = 0.6, and when the noise amplitude is:

i. $\sigma = 0.0$

ii. $\sigma = 0.2$

iii. $\sigma = 2.5$

iv. $\sigma = 5.0$

(b) Repeat part 1 above when the input current is $I_0 = 6.6\mu A/cm^2$.

(c) Compare and describe the results obtained in part 1 and part 2 above in a maximum of 10 sentences description.

(d) For a time duration of T = 300 ms, an input current of $I_0 = 6.6\mu A/cm^2$, initial conditions as mentioned above, and the noise amplitudes in the interval $\sigma \in [0, 4]$, calculate the mean firing rate (also known as mean spiking rate) N for different values of $\sigma \in [0, 4]$ with a noise step size of $d\sigma = 0.1$. A spike is counted when $V(t) \geq 50mV$. The "mean" in the *mean firing rate* refers to the average over 25 different trials of each value of the noise amplitude in the interval [0,4]. Plot the graph of the mean firing rate N (on the vertical axis) against the noise amplitude $\sigma \in [0, 4]$ on the horizontal axis.

(e) In not more than 10 sentences, describe what you observe in the graph obtained in part 4 above.

Solution.

(a) Time series and Phase plots for $\sigma = 0.0$, 0.2, 2.5 and 5.0 and $I_0 = 5.2\ \mu A/cm^2$:

(i.) $\sigma = 0.0$

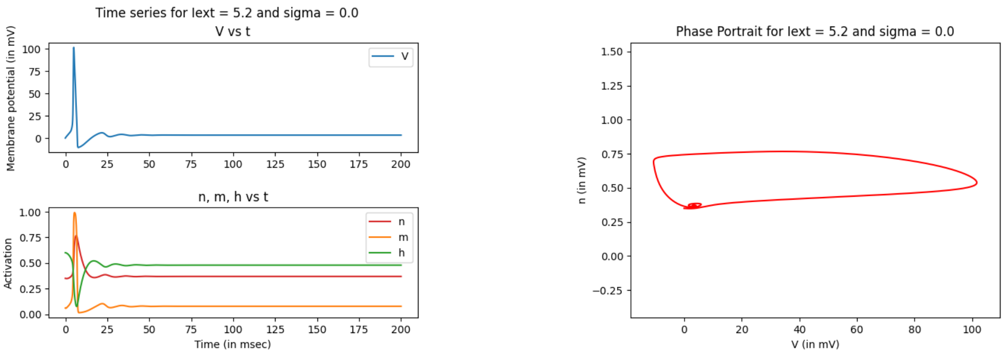

Figure 1: Time Series of membrane potential V and activations n,m and h and Phase Portrait in (V-n) plane of the stochatic HH model for $\sigma = 0.0$ and $I_0 = 5.2\ \mu A/cm^2$

(ii.) $\sigma = 0.2$

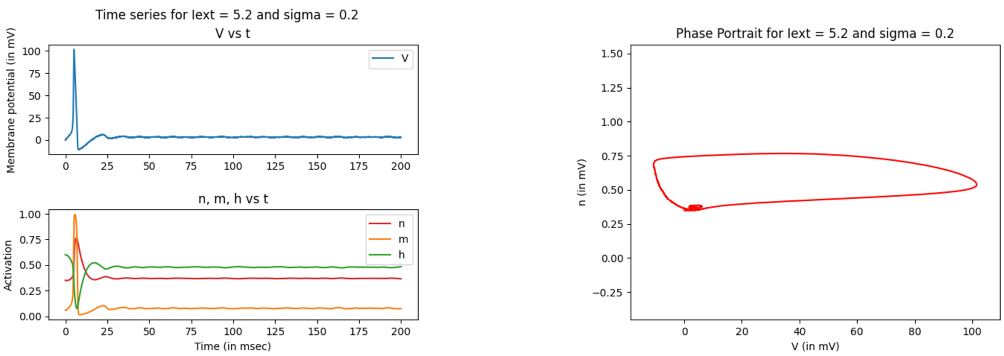

Figure 2: Time Series of membrane potential V and activations n,m and h and Phase Portrait in (V-n) plane of the stochatic HH model for $\sigma = 0.2$ and $I_0 = 5.2\ \mu A/cm^2$

(iii.) $\sigma = 2.5$

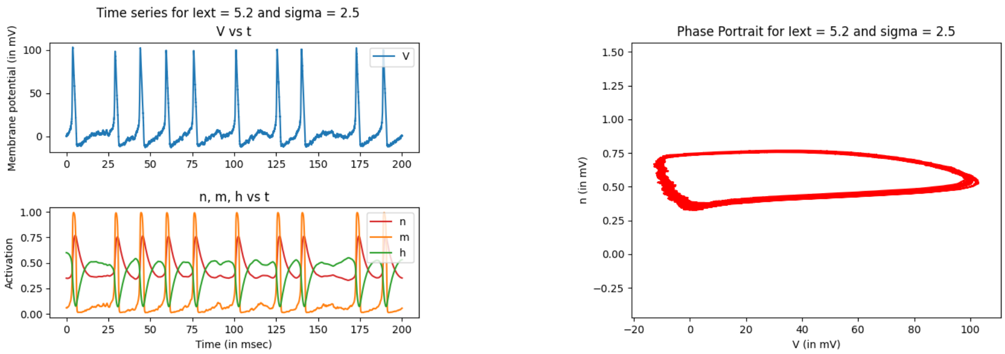

Figure 3: Time Series of membrane potential V and activations n,m and h and Phase Portrait in (V-n) plane of the stochatic HH model for $\sigma = 2.5$ and $I_0 = 5.2\ \mu A/cm^2$

(iv.) $\sigma = 5.0$

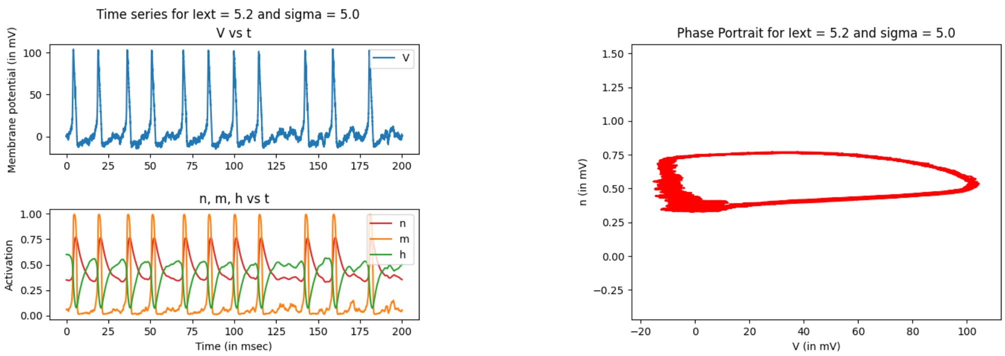

Figure 4: Time Series of membrane potential V and activations n,m and h and Phase Portrait in (V-n) plane of the stochatic HH model for $\sigma = 5.0$ and $I_0 = 5.2\ \mu A/cm^2$

(b) Time series and Phase plots for $\sigma = 0.0$, 0.2, 2.5 and 5.0 and $I_0 = 6.6\ \mu A/cm^2$:

(i.) $\sigma = 0.0$

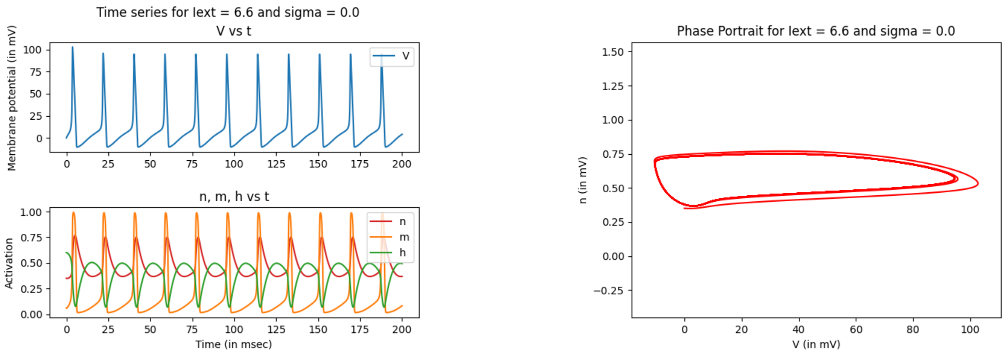

Figure 5: Time Series of membrane potential V and activations n,m and h and Phase Portrait in (V-n) plane of the stochatic HH model for $\sigma = 0.0$ and $I_0 = 6.6\ \mu A/cm^2$

(ii.) $\sigma = 0.2$

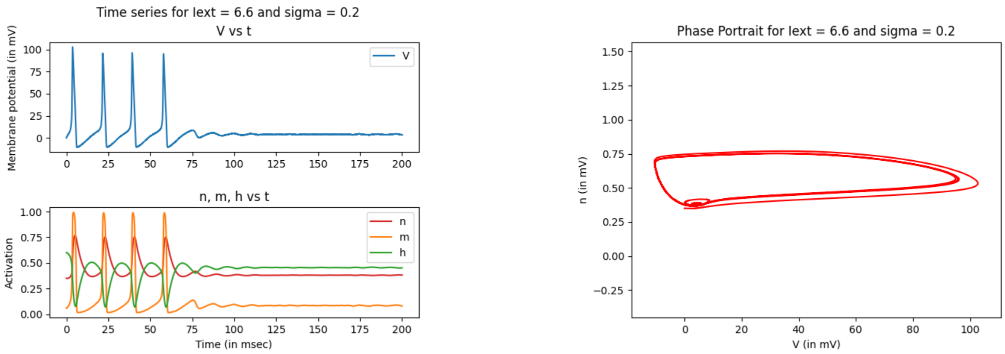

Figure 6: Time Series of membrane potential V and activations n,m and h and Phase Portrait in (V-n) plane of the stochatic HH model for $\sigma = 0.2$ and $I_0 = 6.6\ \mu A/cm^2$

(iii.) $\sigma = 2.5$

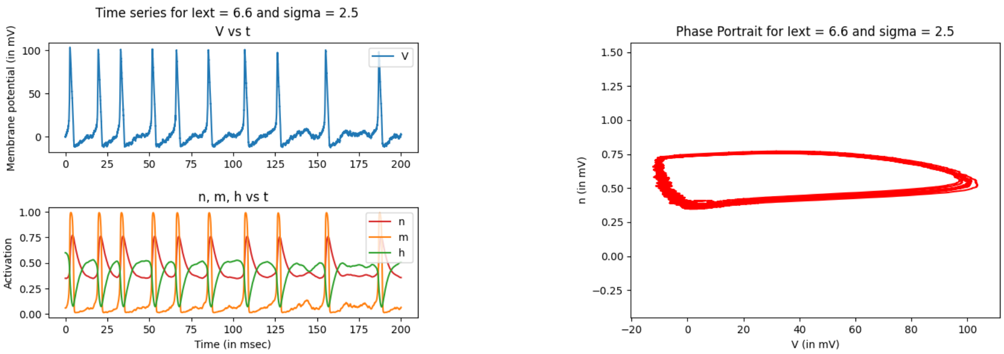

Figure 7: Time Series of membrane potential V and activations n,m and h and Phase Portrait in (V-n) plane of the stochatic HH model for $\sigma = 2.5$ and $I_0 = 6.6\ \mu A/cm^2$

(iv.) $\sigma = 5.0$

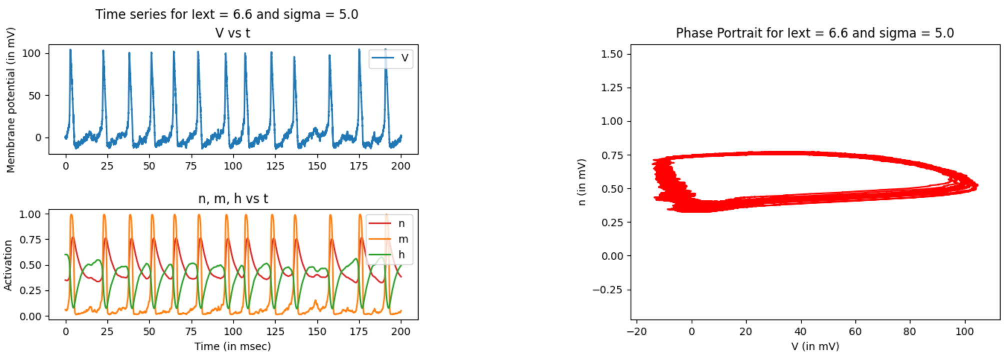

Figure 8: Time Series of membrane potential V and activations n,m and h and Phase Portrait in (V-n) plane of the stochatic HH model for $\sigma = 5.0$ and $I_0 = 6.6\ \mu A/cm^2$

(c) Given here is a stochastic Hodgkin-Huxley model which is a type II neuron model. In part (a) $I_0 = 5.2$ is clearly under the threshold for firing and when there is no or minimal noise ($\sigma = 0.0$ or 0.2), there is single spike after which the neuron returns and stays in the quiescent state. When $\sigma = 2.5$ or 4.0, the increase in noise is apparently enough to tip the neuron over the threshold and we see the familiar continuous spiking and quiescence pattern. In part (b), with $I_0 = 6.6$, we see that when there is no noise, the neuron is over the threshold and there is continuous spiking. When we add a little noise ($\sigma = 0.2$), we see a few spikes but the neuron returns to the quiescent phase, reminiscent of s neuron below the threshold. But when the noise is increased further ($\sigma = 2.5$ or 5.0), we see a return of the continuous spiking. Also, within the sections (a) and (b), compared to (i) where there no noise presence, we can see the intensification of the additive Gaussian noise as we go from (ii) to (iv) as the amplitude of the noise increases.

(d) Change in mean firing rate over $\sigma \in [0.0, 4.0]$:

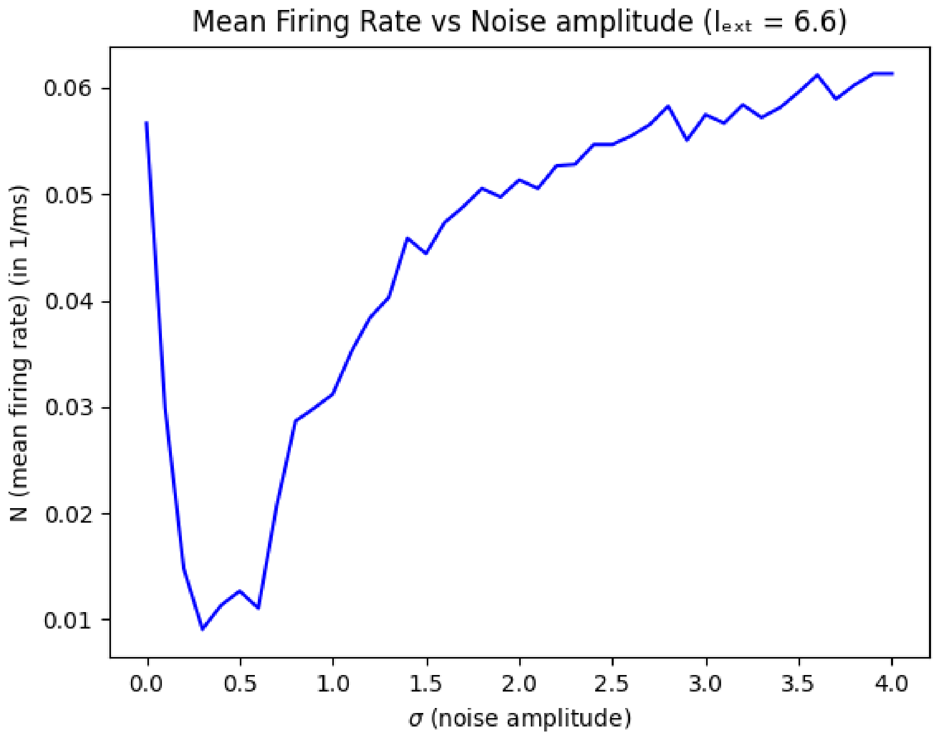

Figure 9: Plot of the mean firing rate over the amplitude of the noise, $\sigma \in [0.0, 4.0]$ with $I_0 = 6.6\ \mu A/cm^2$

(e) In the graph in part (d), we see that initial mean firing rate without the additive noise is high but immediately falls to minimum when a little noise is added. But upon increasing the amplitude of the noise, the firing rate increases rapidly, overtaking the original firing rate. At high values of σ, we see a plateauing of the firing rate. Both the increase and the decrease of the firing rate in the graph is non-linear.

Problem: 2

Simulate and plot the time series (for a duration of T = 1000) of membrane potential v_i variable of the 1st, 5th, 10th, 15th, 25th FitzHugh-Nagumo (FHN) neurons in the neural network consisting of a total of N = 30 neurons and whose equation is given by Eq.(1) in the slides of Lecture 14. For this exercise, in this neural network equation, you should consider the case whereby the FHN neurons are connected via electrical synapses only and are perturbed by synaptic noise only. Consider the strength of the gap junction to 0.25, I = 0.001, $\varepsilon = 0.05$, $\beta = 0.756$, the adjacency matrix encodes a random network topology, an amplitude of 0.005 for synaptic noise, and

(a) a time delay is zero.

(b) a time delay is 5.

Please submit the Python codes (e.g., in exercise, name them like: Exercise 5 2a.py and Exercise 5 b.py and similarly in problem 1 with the HH neuron) and also the outputs of these codes showing all that is required to be shown, with labeling and legends. Jupiter Notebook also works.

Solution.

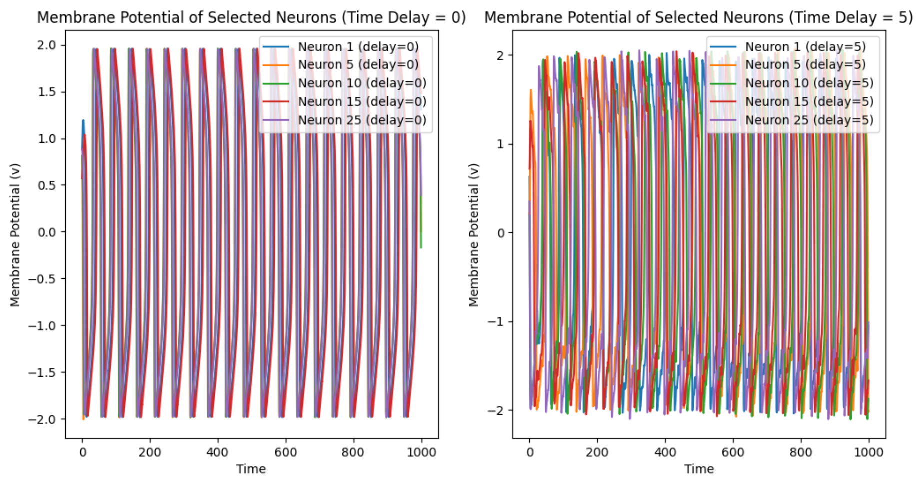

Figure 10: Time series (for a duration of T = 1000) of membrane potential v_i variable of the 1st, 5th, 10th, 15th, 25th FitzHugh-Nagumo (FHN) neurons in the neural network for parameter $\alpha = 0.1$

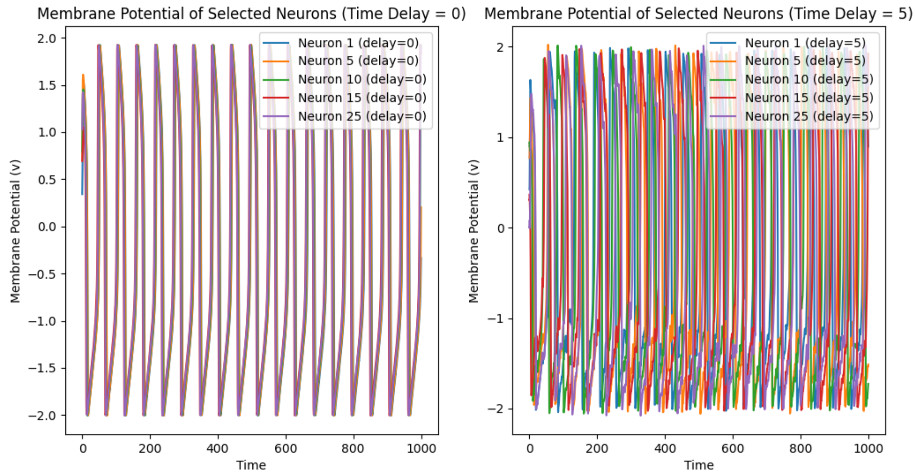

Figure 11: Time series (for a duration of T = 1000) of membrane potential v_i variable of the 1st, 5th, 10th, 15th, 25th FitzHugh-Nagumo (FHN) neurons in the neural network for parameter $\alpha = 0.3$

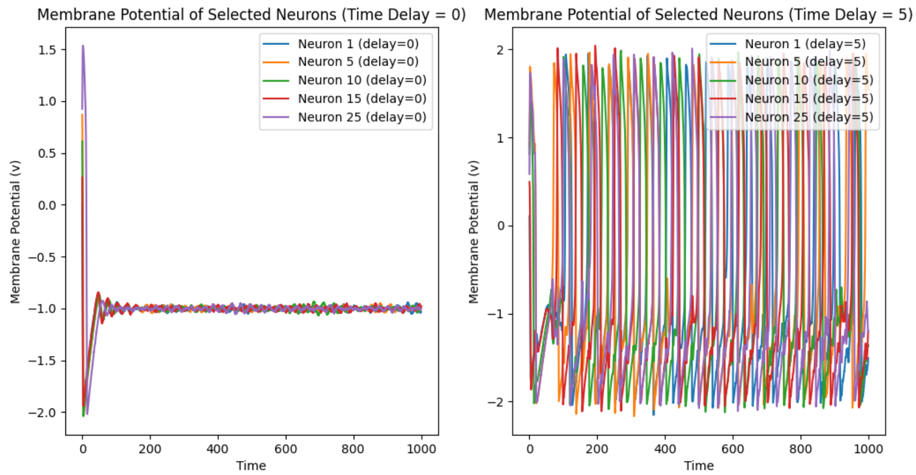

Figure 12: Time series (for a duration of T = 1000) of membrane potential v_i variable of the 1st, 5th, 10th, 15th, 25th FitzHugh-Nagumo (FHN) neurons in the neural network for parameter $\alpha = 0.5$

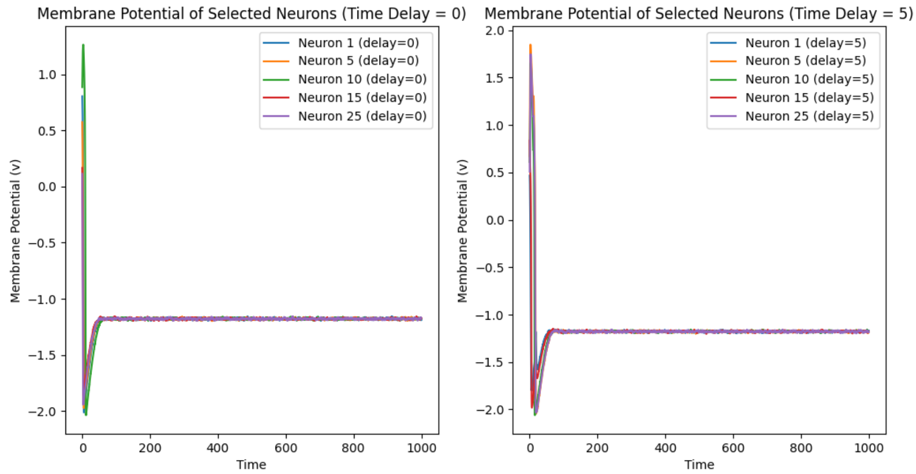

Figure 13: Time series (for a duration of T = 1000) of membrane potential v_i variable of the 1st, 5th, 10th, 15th, 25th FitzHugh-Nagumo (FHN) neurons in the neural network for parameter $\alpha = 0.7$

Problem: 3

From the slides of Lecture 15, show all the steps that are involved in obtaining the synchronization error dynamical system given in Eq.(6), using the coupled neuron model given in Eq.(2) and the definition of the errors between the two neurons given in Eq.(5).

Solution.
We have,

$$\begin{cases} e_v = v_2 - v_1 \\ e_w = w_2 - w_1 \end{cases} \tag{5}$$

$$\begin{cases} \frac{dv_1}{dt} = v_1(a - v_1)(v_1 - 1) - w_1 + I^{ext} + G^e(v_2 - v_1) \\ \frac{dw_1}{dt} = \varepsilon(bv_1 - cw_1) \\ \frac{dv_2}{dt} = v_2(a - v_2)(v_2 - 1) - w_2 + I^{ext} - G^e(v_2 - v_1) \\ \frac{dw_2}{dt} = \varepsilon(bv_2 - cw_2) \end{cases} \tag{2}$$

Differentiating Eq.(5), we have,

$$\begin{cases} \frac{de_v}{dt} = \frac{dv_2}{dt} - \frac{dv_1}{dt} \\ \frac{de_w}{dt} = \frac{dw_2}{dt} - \frac{dw_1}{dt} \end{cases}$$

So,

$$\frac{de_w}{dt} = \frac{dw_2}{dt} - \frac{dw_1}{dt}$$

$$\implies \frac{de_w}{dt} = [\varepsilon(bv_2 - cw_2)] - [\varepsilon(bv_1 - cw_1)]$$

$$\implies \frac{de_w}{dt} = \varepsilon[(b(v_2 - v_1) - c(w_2 - w_1)]$$

$$\implies \frac{de_w}{dt} = \varepsilon(be_v - ce_w) \tag{A}$$

On the other hand,

$$\frac{de_v}{dt} = \frac{dv_2}{dt} - \frac{dv_1}{dt}$$

$$\implies \frac{de_v}{dt} = [v_2(a-v_2)(v_2-1)-w_2+I^{ext}-G^e(v_2-v_1)]-[v_1(a-v_1)(v_1-1)-w_1+I^{ext}+G^e(v_2-v_1)]$$

$$\implies \frac{de_v}{dt} = [v_2(a - v_2)(v_2 - 1) - v_1(a - v_1)(v_1 - 1) + w_1 - w_2 - 2G^e(v_2 - v_1)$$

$$\implies \frac{de_v}{dt} = [v_2(a - v_2)(v_2 - 1) - v_1(a - v_1)(v_1 - 1) - e_w - 2G^e e_v$$

$$\implies \frac{de_v}{dt} = v_1^3 - v_2^3 + av_2^2 - av_1^2 - av_2 + av_1 + v_2^2 - v_1^2 - e_w - 2G^e e_v$$

$$\implies \frac{de_v}{dt} = (v_1 - v_2)^3 + 3v_1v_2(v_1 - v_2) + av_2^2 - av_1^2 - av_2 + av_1 + v_2^2 - v_1^2 - e_w - 2G^e e_v$$

$$\implies \frac{de_v}{dt} = -e_v^3 + 3v_1^2v_2 - 3v_1v_2^2 + av_2^2 - av_1^2 - av_2 + av_1 + v_2^2 - v_1^2 - e_w - 2G^e e_v$$

$$\implies \frac{de_v}{dt} = a(v_2^2 - v_1^2) + v_2^2 - v_1^2 + 3v_1^2v_2 - 3v_1v_2^2 - av_2 + av_1 - e_v^3 - e_w - 2G^e e_v$$

$$\implies \frac{de_v}{dt} = a[(v_2-v_1)^2+2v_1v_2-2v_1^2]+[(v_2-v_1)^2+2v_1v_2-2v_1^2]+3v_1^2v_2-3v_1v_2^2-av_2+av_1-e_v^3-e_w-2G^e e_v$$

$$\implies \frac{de_v}{dt} = ae_v^2 + e_v^2 + 2av_1v_2 - 2av_1^2 + 2v_1v_2 - 2v_1^2 + 3v_1^2v_2 - 3v_1v_2^2 - av_2 + av_1 - e_v^3 - e_w - 2G^e e_v$$

$$\implies \frac{de_v}{dt} = 2av_1v_2 - 2av_1^2 + 2v_1v_2 - 2v_1^2 + 3v_1^2v_2 - 3v_1v_2^2 - av_2 + av_1 + (a+1)e_v^2 - e_v^3 - e_w - 2G^e e_v$$

$$\implies \frac{de_v}{dt} = 3v_1^2v_2 - 3v_1v_2^2 + 2av_1v_2 - 2av_1^2 + 2v_1v_2 - 2v_1^2 - av_2 + av_1 + (a+1)e_v^2 - e_v^3 - e_w - 2G^e e_v$$

$$\begin{aligned}\implies \frac{de_v}{dt} = {} & -3v_1v_2^2 - 3v_1^3 + 6v_1^2v_2 + 3v_1^3 - 3v_1^2v_2 + 2av_1v_2 - 2av_1^2 + 2v_1v_2 - 2v_1^2 - av_2 + av_1 \\ & + (a+1)e_v^2 - e_v^3 - e_w - 2G^e e_v\end{aligned}$$

$$\begin{aligned}\implies \frac{de_v}{dt} = {} & -3v_1(v_2^2 + v_1^2 - 2v_1v_2) + 3v_1^3 - 3v_1^2v_2 + 2av_1v_2 - 2av_1^2 + 2v_1v_2 - 2v_1^2 - av_2 + av_1 \\ & + (a+1)e_v^2 - e_v^3 - e_w - 2G^e e_v\end{aligned}$$

$$\begin{aligned}\implies \frac{de_v}{dt} = {} & -3v_1(v_2 - v_1)^2 + 3v_1^3 - 3v_1^2v_2 + 2av_1v_2 - 2av_1^2 + 2v_1v_2 - 2v_1^2 - av_2 + av_1 \\ & + (a+1)e_v^2 - e_v^3 - e_w - 2G^e e_v\end{aligned}$$

$$\begin{aligned}\implies \frac{de_v}{dt} = {} & 3v_1^3 - 3v_1^2v_2 + 2av_1v_2 - 2av_1^2 + 2v_1v_2 - 2v_1^2 - a(v_2 - v_1) \\ & - 3v_1e_v^2 + (a+1)e_v^2 - e_v^3 - e_w - 2G^e e_v\end{aligned}$$

$$\begin{aligned}\implies \frac{de_v}{dt} = {} & 3v_1^3 - 3v_1^2v_2 + 2av_1v_2 - 2av_1^2 + 2v_1v_2 - 2v_1^2 \\ & - ae_v - 3v_1e_v^2 + (a+1)e_v^2 - e_v^3 - e_w - 2G^e e_v\end{aligned}$$

$$\begin{aligned}\implies \frac{de_v}{dt} = {} & -3v_1^2v_2 + 2v_1v_2 + 2av_1v_2 + 3v_1^3 - 2v_1^2 - 2av_1^2 \\ & - ae_v - 3v_1e_v^2 + (a+1)e_v^2 - e_v^3 - e_w - 2G^e e_v\end{aligned}$$

$$\begin{aligned}\implies \frac{de_v}{dt} = {} & -[3v_1^2v_2 - 2v_1v_2(1+a) - 3v_1^3 + 2v_1^2(1+a)] \\ & - ae_v - 3v_1e_v^2 + (a+1)e_v^2 - e_v^3 - e_w - 2G^e e_v\end{aligned}$$

$$\implies \frac{de_v}{dt} = -[3v_1^2 - 2(1+a)v_1](v_2 - v_1) - ae_v - 3v_1e_v^2 + (a+1)e_v^2 - e_v^3 - e_w - 2G^e e_v$$

$$\implies \frac{de_v}{dt} = -[3v_1^2 - 2(1+a)v_1]e_v - ae_v - 3v_1e_v^2 + (a+1)e_v^2 - e_v^3 - e_w - 2G^e e_v$$

$$\implies \frac{de_v}{dt} = -e_v^3 - 3v_1e_v^2 + (a+1)e_v^2 - [3v_1^2 - 2(1+a)v_1]e_v - ae_v - 2G^e e_v - e_w$$

$$\implies \frac{de_v}{dt} = -e_v^3 - (3v_1 - a - 1)e_v^2 - [3v_1^2 - 2(1+a)v_1 + a + 2G^e]e_v - e_w \tag{B}$$

Equations (A) and (B) together give us:

$$\begin{cases} \frac{de_v}{dt} = -e_v^3 - (3v_1 - a - 1)e_v^2 - [3v_1^2 - 2(1+a)v_1 + a + 2G^e]e_v - e_w \\ \frac{de_w}{dt} = \varepsilon(be_v - ce_w) \end{cases} \tag{6}$$

which is the error dynamical system given in Eq.(6) of the slides.