Homework 6 (Due at class on April 21)

1 Wold-sheet supersymmetry

1.1 Supersymmetric transformation

Show that the action

$$S^{\mathrm{m}} = \frac{1}{4\pi}\int d^2z \left(\frac{2}{\alpha'}\partial X\bar{\partial}X + \psi^\mu\bar{\partial}\psi_\mu + \overline{\psi}^\mu\partial\overline{\psi}_\mu\right) \tag{1.1}$$

is invariant under the supersymmetric transformation

$$\delta_{\epsilon,\bar{\epsilon}}X^\mu = -\sqrt{\frac{\alpha'}{2}}\left(\epsilon\psi^\mu + \bar{\epsilon}\overline{\psi}^\mu\right), \qquad \delta_\epsilon\psi^\mu = \sqrt{\frac{2}{\alpha'}}\,\epsilon\partial X^\mu, \qquad \delta_{\bar{\epsilon}}\overline{\psi}^\mu = \sqrt{\frac{2}{\alpha'}}\,\bar{\epsilon}\bar{\partial}X^\mu .$$

Also, show that the combination of supersymmetric transformations provides derivatives

$$[\delta_{\epsilon_1,\bar{\epsilon}_1}, \delta_{\epsilon_2,\bar{\epsilon}_2}]\mathcal{O} = 2\epsilon_1\epsilon_2\partial\mathcal{O} + 2\bar{\epsilon}_1\bar{\epsilon}_2\bar{\partial}\mathcal{O} .$$

1.2 Superspace formalism

This theory can be formulated in terms of superspace $(z,\bar{z},\theta,\bar{\theta})$ where $(\theta,\bar{\theta})$ are anti-commuting Grassmann coordinates. We can introduce the superfield

$$Y^\mu(z,\bar{z},\theta,\bar{\theta}) = X^\mu(z,\bar{z}) + i\theta\psi^\mu(z,\bar{z}) + i\bar{\theta}\overline{\psi}^\mu(z,\bar{z}) + \frac{1}{2}\bar{\theta}\theta F^\mu(z,\bar{z}) ,$$

as well as the superderivative

$$D = \frac{\partial}{\partial\theta} + \theta\partial_z , \quad \overline{D} = \frac{\partial}{\partial\bar{\theta}} + \bar{\theta}\bar{\partial}_{\bar{z}} .$$

The field F^μ is called an **auxiliary field**. Then, show that the action (1.1) at $\alpha' = 2$ is equivalent to

$$S = \frac{1}{4\pi}\int d^2zd^2\theta\; \overline{D}Y^\mu DY_\mu ,$$

where the superspace integral is defined as $d^2\theta = d\theta d\bar{\theta}$, and

$$\int d\theta d\bar{\theta}\;\bar{\theta}\theta = 1 .$$

1.3 Supersymmetric ghost

Let us define the ghost superfields as

$$B = \beta + \theta b , \qquad C = c + \theta\gamma .$$

Show that the supersymmetric ghost action can be written as

$$S^{\mathrm{gh}} = \frac{1}{2\pi}\int d^2zd^2\theta\; B\overline{D}C .$$

2 $\mathcal{N} = 1$ superconformal algebra

In general, a 2d superconformal field theory has the following OPEs for the stress-energy tensor $T(z)$ and the supercurrent $G(z)$

$$\begin{aligned} T(z)T(w) &\sim \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial_w T(w) \\ T(z)G(w) &\sim \frac{3/2}{(z-w)^2}G(w) + \frac{1}{z-w}\partial_w G(w) \\ G(z)G(w) &\sim \frac{2c/3}{(z-w)^3} + \frac{2}{z-w}T(w)\ . \end{aligned} \tag{2.1}$$

As in the lecture, we carry out the mode expansion

$$T_B(z) = \sum_{m\in\mathbb{Z}} \frac{L_m}{z^{m+2}}\ , \qquad G(z) = \sum_{r\in\mathbb{Z}+\nu} \frac{G_r}{z^{r+3/2}}$$

where $\nu = 0$ and $\nu = \frac{1}{2}$ correspond to Ramond sector and Neveu-Schwarz sector, respectively. From the OPEs (2.1), derive $\mathcal{N} = 1$ superconformal algebra

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0} \\ [L_m, G_r] &= \left(\frac{1}{2}m - r\right)G_{r+m} \\ \{G_r, G_s\} &= 2L_{r+s} + \frac{c}{3}(r^2 - \frac{1}{4})\delta_{r+s,0}\ . \end{aligned}$$