Homework 12: Due at class on Dec 14

1 Entanglement entropy

1.1 Spins of two sites

We compute the entanglement entropy in the spin systems of two sites. Let us consider a pure state

$$|\Psi\rangle = \cos\theta|0\rangle_A|1\rangle_B + \sin\theta|1\rangle_A|0\rangle_B$$

Compute the density operator restricted to the subsystem A

$$\rho_A = \mathrm{Tr}_{\mathcal{H}_B}|\Psi\rangle\langle\Psi|$$

and calculate the entanglement entropy S_A for the subsystem A. When is S_A maximized?

1.2 Entanglement entropy of harmonic oscillators

Let us consider the entanglement entropy of two harmonic oscillators A and B whose Hamiltonian is

$$H = a^\dagger a + b^\dagger b + \lambda\left(a^\dagger b^\dagger + ab\right)$$

where $\left(a, a^\dagger\right)$ and $\left(b, b^\dagger\right)$ are creation-annihilation operators with the commutation relation

$$\left[a, a^\dagger\right] = \left[b, b^\dagger\right] = 1$$

We set the coupling constant λ as

$$\lambda = \frac{2\sinh\theta\cosh\theta}{1+2\sinh^2\theta} .$$

Using new creation-annihilation operators

$$\tilde{a} = \cosh\theta\cdot a + \sinh\theta\cdot b^\dagger$$
$$\tilde{b} = \sinh\theta\cdot a^\dagger + \cosh\theta\cdot b$$

diagonalize the Hamiltonian. Show that the ground state $\tilde{a}|\Psi\rangle = \tilde{b}|\Psi\rangle = 0$ can be written as

$$|\Psi\rangle = \frac{1}{\cosh\theta}\cdot e^{-\tanh\theta a^\dagger b^\dagger}|0\rangle_A\otimes|0\rangle_B .$$

Compute the density operator ρ_A restricted to the subsystem A and the entanglement entropy S_A for the subsystem A. How does S_A behaves as a function of λ?