Final exam

1 Differential forms

Let M be a C^∞ manifold and $X \in \mathfrak{X}(M)$ a vector field on M. We define a linear map, called an **interior product**,

$$i(X) : \Omega^k(M) \to \Omega^{k-1}(M)$$

by

$$\left(i(X)\omega\right)(X_1, \cdots, X_{k-1}) = k\ \omega\left(X, X_1, \cdots, X_{k-1}\right)$$

where $\omega \in \Omega^k(M)$, $X_1, \cdots, X_{k-1} \in \mathfrak{X}(M)$. Recalling that the Lie derivative $L_X : \Omega^k(M) \to \Omega^k(M)$ yields (Homework 4, Problem 2)

$$\left(L_X\omega\right)(X_1, \cdots, X_k) = X\omega\left(X_1, \cdots, X_k\right) - \sum_{i=1}^{k} \omega\left(X_1, \cdots, [X, X_i], \cdots, X_k\right)\ .$$

Show the following identities

$$\begin{aligned} L_X i(Y) - i(Y) L_X &= i([X, Y])\ , \\ L_X &= i(X)d + di(X)\ , \end{aligned}$$

for $X, Y \in \mathfrak{X}(M)$.

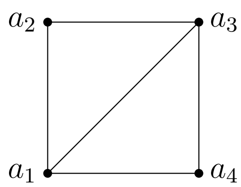
2 Homology

1. We define boundary homomorphisms

$$\partial_q : C_q(K) \to C_{q-1}(K);\ (a_0, \cdots, a_q) \mapsto \sum_{i=0}^{q} (-1)^i (a_0, \cdots, \widehat{a}_i, \cdots, a_q)\ .$$

   Show that $\partial_{q-1} \cdot \partial_q = 0$.

2. Compute the homology groups of the **one-dimensional** complex below:



3 Connection and curvature

1. Suppose that we have a connection ∇ on a rank-r vector bundle $\pi : E \to M$. Suppose that we have local trivializations of E over two open sets U_α and U_β in M. On these open sets, the connection can be written in terms of one-form A_α and A_β taking value on $\mathfrak{gl}(r, \mathbb{R})$. Given a transition function $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(r, \mathbb{R})$, show that the connections are related by

$$A_\beta = g_{\alpha\beta}^{-1} A_\alpha g_{\alpha\beta} + g_{\alpha\beta}^{-1} d(g_{\alpha\beta}) \ ,$$

   and the curvature forms are related by

$$F_\beta = g_{\alpha\beta}^{-1} F_\alpha g_{\alpha\beta}$$

   on $U_\alpha \cap U_\beta$.

2. Prove that the Bianchi identity

$$dF + A \wedge F - F \wedge A = 0 \ .$$

4 Riemannian geometry

Please follow the notation of the latest lecture note 5 for Christoffel symbols, Riemann, Ricci and scalar curvature.

A 3-dimensional anti-de Sitter space AdS_3 can be defined as follows. Let us consider a flat space $\mathbb{R}^{2,2} = (X_1, X_2, X_3, X_4)$ with a metric

$$ds^2 = -dX_1^2 - dX_2^2 + dX_3^2 + dX_4^2 \ .$$

The AdS_3 space in $\mathbb{R}^{2,2}$ is defined by the equation

$$-X_1^2 - X_2^2 + X_3^2 + X_4^2 = -1 \ . \tag{1}$$

Since the following change of variables

$$\begin{cases} X_1 = \cosh\rho \cos\tau \\ X_2 = \cosh\rho \sin\tau \\ X_3 = \sinh\rho \sin\theta \\ X_4 = \sinh\rho \cos\theta \end{cases}$$

satisfies (1), show that the induced metric from $\mathbb{R}^{2,2}$ to AdS_3 is

$$ds^2 = -\cosh^2\rho \, d\tau^2 + \, d\rho^2 + \sinh^2\rho \, d\theta^2 \ .$$

By computing Riemann curvature tensor explicitly, show that

$$R_{\mu\nu\alpha\beta} = g_{\mu\beta} g_{\nu\alpha} - g_{\mu\alpha} g_{\nu\beta} \ .$$

Then, show that it satisfies the Einstein equation with a negative consmologial constant $\Lambda = -1$

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - g_{\mu\nu} = 0 \ .$$