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本科毕业论文(设计)

论 文 题 目: Gauged Linear Sigma Models, B-branes in GLSMs, and Super Calabi-Yau Manifolds

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摘　要

在本论文中，我们简要介绍了一类被称为“规范线性 σ-模型”（Gauged Linear Sigma Model，简称 GLSM）的二维 $\mathcal{N}=(2,2)$ 超对称规范理论。该理论由 Edward Witten 于三十余年前首次提出，作为研究弦传播与弦紧化的重要工具，GLSM 提供了弦世界面 (worldsheet) 上的紫外理论。通过重整化群流，紫外线性 σ-模型会根据其的 Fayet-Iliopoulos 参数（FI 参数）演化为不同的低能理论——以文中介绍的五次 Calabi-Yau 模型为例，其低能理论分别为 Calabi-Yau 流形上的非线性 σ-模型和 Landau-Ginzburg 模型。除以上的经典分析外，线性 σ-模型的量子行为同样引人入胜：我们特别考察了库仑分支（Coulomb branch）上的量子修正，这些修正将经典上同调环转化为几何相的靶空间的量子上同调环。原则上，对于规范线性 σ-模型，我们可以自由分配物质多重态规范荷，而这其中最令我们关注的是满足 Calabi-Yau 条件的 GLSM。数学上，该条件要求几何靶空间的第一陈类 c_1 为 0，结合 $\mathcal{N}=(2,2)$ 超对称性可推知靶空间必为 Calabi-Yau 流形；物理上则表现为所有的物质场总和满足电中性条件 $\sum_i Q_i = 0$，这样的电中性条件保持轴矢 R-对称性（axial R-symmetry）免受反常破坏。我们之所以聚焦 Calabi-Yau 情形，正因其为弦紧化提供了理想场域。进一步研究发现，这类 Calabi-Yau GLSM 具有标度不变性，且通常被认为具备共形不变性，这使我们能在超共形场论框架下展开研究。

镜像对称是我们在本文中介绍的 $\mathcal{N}=(2,2)$ 理论的另一精妙之处，尤其体现在 Calabi-Yau GLSM 中。该对称性表现为超对称代数中存在一个 $\mathbb{Z}_2$ 自同构，可交换 A 型（A-type）与 B 型（B-type）超对称生成元，并同时交换轴矢与矢量 R 对称性（axial and vector R-symmetry）。这种对称性在物理上可通过 T-对偶（T-dual）实现，即交换动量模式与缠绕模式（winding modes）。对于非线性 σ-模型，其镜像理论可以由上述的 T-对偶给出，通常由一个 Landau-Ginzburg 模型描述。更深刻的是，对于靶空间为 Calabi-Yau 的非线性 σ-模型，镜像对称（假说）存在一种几何诠释：对任意 Calabi-Yau 流形 M，存在另一 Calabi-Yau 流形 $\widetilde{M}$，使得 M 上的 A/B 型超对称非线性 σ-模型等价于 $\widetilde{M}$ 上的 B/A 型超对称非线性 σ-模型。在本文的第三章，我们通过物理理论的手征环（chiral rings）与几何形变（deformation）的双重论证，发现互为镜像对称的两个 Calabi-Yau 流形交换了 Kähler 模空间与复模空间。然而存在一类刚性 Calabi-Yau 流形（rigid Calabi-Yau manifolds）不允许任何复形变，导致其复模空间平庸，进而使得若其镜像的 Calabi-Yau 空间存在，则 Kähler 模空间也必须平庸——这显然构成矛盾，因为 Calabi-Yau 空间是 Kähler，其上存在 Kähler 形式，而对任意一个 Kähler 形式我们都可以对其进行放缩得到新的 Kähler 形式，因此模空间的维数至少为 1。为了解决这一问题，赋予刚性 Calabi-Yau 流形以镜像对偶，Sethi 创造性的提出刚性 Calabi-Yau 的镜像可解释为具有费米坐标的 Calabi-Yau 超流形；Schwartz 则进一步提出了超流形与纯玻色流形中超曲面（hypersurfaces）的 A-扭曲模型（A- twisted model）之间的等价性。

我们关注这种连接费米坐标流形与纯玻色流形中超曲面完全交的等价关系。现有的证据来自多重角度：Aganagic 和 Vafa 提出了对于含有超坐标的规范线性 σ-模型的 T-对偶操作步骤，通过在其对偶的 Landau-Ginzburg 模型中加入一对鬼场（ghost fields），可以得到其镜像对称的理论为超 Landau-Ginzburg 模型。

进一步研究其对偶超 Landau-Ginzburg 理论的周期积分（period integral），我们可以得到等价性的证据。同时，研究 A-扭曲模型的扭曲手征环（twisted chiral rings）、椭圆亏格（Elliptic Genus）以及球面配分函数等，也可以进一步验证两种模型的等价关系。这篇文章里，我们试图通过计算半球配分函数验证这一等价性。

然而，具有边界的世界面超对称理论构建颇具挑战性，一个重要的原因在于边界处超荷的守恒性。考虑到边界处的部分平移对称性会破缺，因此我们只能选择超对称荷的组合，使其反对易关系中不出现破缺的平移生成元。在本文中，我们主要考虑保持 B-型超对称生成元的边界条件，称之为 B-膜。在弦论里，带边界的规范线性 σ-模型可作为开弦世界面理论的紫外描述，弦的端点则对应了世界面的边界。因此，对于这样一类带边规范线性 σ-模型，我们除了需要考虑超对称的边界条件外，还需谨慎处理开弦端点处由一个 $\mathbb{Z}_2$ 分次（$\mathbb{Z}_2$-graded）Chan-Paton 向量空间标记的自由度。具体来说，我们需要考虑规范线性 σ-模型中规范场和 R-对称性在 Chan-Paton 向量空间上的表示。而后，规范玻色子（gauge boson）与快子场（tachyon）的信息可在分次 Chan-Paton 空间上表述为规范联络与奇自同态（odd endomorphism），而其描述的 B-膜则可表示为 R-荷分次的向量丛复形（complexes of vector bundles）。向量丛复形的 quasi-同构（quasi-isomorphism）等价于物理膜的 D-同构（D-isomorphism），这里的 D-同构指两个 B-膜在低能有效理论中的膜相同，这种性质可用于粘合多个膜而保持低能性质不变。同时，我们注意到规范线性 σ-模型的低能有效理论的相取决于 FI 参量，即取决于处于理论模空间的位置，因此我们可以考虑将某个 B-膜沿模空间穿越不同相。然而，在相边界的窗口（window）上，配分函数的收敛性或边界势能的有（下）界性会对 Chan-Paton 向量空间上的规范群表示施加限制，此即分次限制定则（the grade restriction rule）。同时我们介绍了半球配分函数的定义与计算，作为带边规范线性 σ-模型的一个具体例子。具体的配分函数的计算让我们对体系的性质有非常详尽的了解

在论文最后部分，我们引入 B-膜的半球配分函数来检验前述的等价性。由于费米坐标的存在，必须修正快子场参数 Q 及对应 Chan-Paton 向量空间。我们通过研究发现：为保持快子场 Q 的费米性，不同于利用常见的 Clifford 代数反对易关系的构造，现需将费米坐标对应的鬼场与满足对易关系而非反对易关系的 $[a, \overline{a}] = 1$ 的玻色型产生湮灭算符结合，用于构造表示空间。构造玻色坐标的普通场对应的表示空间费米塔会因反对易关系截断，但基于玻色生成元构建的玻色塔则会向上无限延伸，导致计算半球配分函数时膜因子出现奇点。令人惊喜的是，新出现的极点恰好补偿了因费米物质单圈行列式而损失的奇点，我们从而在 B-膜半球层面验证了等价性。

综上所述，我们发展了一种构建超 B 膜快子场与 Chan-Paton 向量空间的新方法，借此验证了射影空间中超曲面与一类超加权射影空间之间的等价关系。

关键词：规范线性西格玛模型；B-膜；半球配分函数，超卡拉比-丘流形
中图分类号：O413.1

Abstract

In this thesis, we have briefly introduced several concepts in Gauged Linear Sigma Models, which is a class of two-dimensional $(2,2)$ supersymmetric gauge theories. Moreover, we are especially interested in GLSMs that satisfy the Calabi-Yau condition, and flow to Non-linear Sigma Models with Calabi-Yau targets at the low-energy. Mirror symmetry is an extremely intriguing area, which predicts that each Calabi-Yau has a mirror Calabi-Yau. However, the existence of rigid Calabi-Yau's breaks such a geometrical correspondence, and could be cured by enlarging our playground to supermanifolds. Introducing supermanifolds into physics leads to a proposed equivalence that we are going to check, which relates hypersurfaces in ordinary manifolds and supermanifolds. Our goal of this paper is to check such an equivalence at the level of the hemisphere, and thus we also made a rough introduction on GLSMs with boundaries, among which we are most interested in B-branes. B-branes can be formulated on a hemisphere, and the partition function was already exactly computed. Thus, with the help of the hemisphere partition function, we employed a new method in constructing the tachyon profile and the Chan-Paton vector space for a class of super B-branes, which allows us to check the equivalence between hypersurfaces in projective space and a class of super weighted projective spaces.

Keywords: Gauged Linear Sigma Models; B-branes; Hemisphere Partition Function; Super Calabi-Yau Manifolds

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Chapter 1

Introduction

Gauged linear sigma models (GLSMs) were first introduced by Edward Witten over thirty years ago[1]. They are a class of two-dimensional $\mathcal{N} = (2, 2)$ supersymmetric gauge theories, with matter fields in some representations of the gauge group G. GLSMs are vital tools in studying string propagation and string compactification. Moreover, they flow to different phases, which are different low-energy theories through renormalization group flow, e.g., the geometric phase and Landau-Ginzburg phase for the quintic Calabi-Yau model.

In some phases of a GLSM, the low-energy dynamics are supersymmetric nonlinear sigma models (NLSMs) with Kähler target space, which form an interesting class of nontrivial field theories in two spacetime dimensions[2]. Due to the nonrenormalization theorems[3] from $\mathcal{N} = 2$ supersymmetry, a subset of the correlators does not receive any perturbative correction from their classical values in any order. Thus, the classical value directly reflects the geometric properties of the target space.

While the FI parameter of the GLSM lies in a certain regime where the IR theory corresponds to an NLSM, we can compute some RG invariant quantities of the NLSM via the linear model with gauge fields, e.g., elliptic genera, correlators in topological field theories. Correlators in topological twisted theories encode geometric information for target spaces called Gromov-Witten invariants, which can be used to construct quantum cohomology by promoting the ordinary cohomology cup product to the (small) quantum product.

So far, we are considering closed strings, where the linear sigma models provide an UV description of the worldsheet theories. It's natural for us to extend the LSM description for open strings, or say, study D-branes wrapped in the target space, which plays a central role in string theory and related fields. In[4], the authors' main goal is to construct a UV description of the worldsheet theory with boundary in the entire moduli space, using linear sigma models. Besides specifying strict boundary conditions for all multiplets, they tend to introduce boundary interactions to cancel boundary terms given by supersymmetry transformations.

We mainly consider D-branes preserving $\mathcal{N} = 2_B$ supersymmetry, namely B-branes. Since there are extra degrees of freedom at the end points of the open strings[5], labeled by the Chan-Paton vector spaces, a supersymmetric Wilson line on the worldsheet boundary can also be added to the action for the gauge theory, which completes the entire (category of) boundary conditions of the UV theory. These Wilson line branes specified by Chan-Panton vector spaces can be sent to branes supporting the line bundles in the low-energy theory via RG flow. Starting from some phases of the UV theory, the low-energy D-branes obtained above are objects in the derived category $D(X_r)$ of the Higgs branch of the target space X_r.

To understand a theory clearly, we most naturally think of exactly extracting its partition function, and in supersymmetric quantum field theory, this is most often done by using the localization technique. Localization has been a powerful tool in obtaining exact results, especially the partition function in supersymmetric theories[6]. As an application of supersymmetry localization for $\mathcal{N} = (2,2)$ gauge theories, the torus partition function (Elliptic Genera) was first computed in[7–8], and also the two-sphere partition was obtained in[9–10]. These models are all formulated on closed worldsheets, and we are also interested in theories with boundaries. This was done by K. Hori and M. Romo via applying localization techniques to compute the partition function on the hemisphere of a class of (2,2) supersymmetric gauge theories in dimension two[11]. The hemisphere partition function can also be used to derive the Grade Restriction Rule for B-brane transport, which is first obtained in[4] via boundary potential analysis.

In this paper, we focus on the Gauged Linear Sigma Models with super Calabi-Yau target spaces, formulate them on worldsheets with boundaries, eg, hemisphere, and compute the hemisphere partition function. Such theories can be constructed by adding ghost multiplets, where the fields have opposite statistics, i.e., Grassman odd scalars and Grassman even spinors. The two-sphere partition function as well as the Elliptic Genera for super GLSMs was already obtained in[12] by applying the localization procedure for the ghost multiplets, and was used to check the equivalence between GLSMs on super manifolds and on hypersurfaces. To further confirm such equivalence in the level of hemisphere, we should take care of the boundary conditions and the brane factors contributing to the one-loop determinant, which plays a crucial role in the hemisphere partition function.

Chapter 2

Gauged Linear Sigma Models

2.1 $\mathcal{N} = (2,2)$ Supersymmetry

To formulate supersymmetry in various dimensions, we first review spinors as irreducible representations of the Lorentz group $SO^+(a,b)$, where the spacetime dimension $d = a+b$. We are most interested in the case where $d = 2$ for both Minkowski space, but for completeness, we also pay attention to $d = 4$ as well as Euclidean spaces. We first note that the dimension of the irreducible spinor representation only depends on the spacetime dimension d.

$$\mathrm{Dim} = 2^{\lceil \frac{d-2}{2} \rceil}. \tag{2.1}$$

Furthermore, the type of the irreducible spinor of $SO^+(a,b)$ only depends on the difference $a - b \mod 8$.

$a - b \mod 8$	Irreducible Spinor Representations
0	Two real spinors, inequivalent.
± 2	Two complex spinors C and its conjugate $\bar{C}$, inequivalent.
± 3	One pseudoreal spinor.
4	Two different pseudoreal spinors, inequivalent.

Table 2-1 Table of Irreducible Spinor Representations

In $d = 1+1$ dimension, $\mathcal{N} = (2,2)$ supersymmetry has two copies of right-moving Majorana-Weyl supercharges Q_+^1, Q_+^2 and two copies of left-moving Majorana-Weyl supercharges Q_-^1, Q_-^2. These are real spinors, and we can rewrite them into two complex Weyl spinors with different chiralities.

$$\begin{aligned} Q_+ &\equiv Q_+^1 + \mathrm{i}Q_+^2, \quad \overline{Q}_+ \equiv Q_+^1 - \mathrm{i}Q_+^2, \\ Q_- &\equiv Q_-^1 + \mathrm{i}Q_-^2, \quad \overline{Q}_- \equiv Q_-^1 - \mathrm{i}Q_-^2. \end{aligned} \tag{2.2}$$

For theories supported on a Minkowski-like worldsheet, the supercharges are complex conjugates to each other, i.e., $(Q)^\dagger = \overline{Q}$, since Q^1 and Q^2 are Majorana-Weyl supercharges. However, for Euclidean QFTs, as we argued before, $(Q)^\dagger \neq \overline{Q}$.

In the quantum theory, the supersymmetry transformation of an operator $\mathcal{O}$ is generated by the conserved charges $Q_\pm$ and $\overline{Q}_\pm$:

$$\delta\mathcal{O} = \left[\hat{\delta}, \mathcal{O}\right], \tag{2.3}$$

where

$$\hat{\delta} := i\varepsilon_+ Q_- - i\varepsilon_- Q_+ - i\bar{\varepsilon}_+ \overline{Q}_- + i\bar{\varepsilon}_- \overline{Q}_+. \tag{2.4}$$

Keep in mind that we are dealing with a Poincaré invariant quantum field theory, where there are other conserved charges: Hamiltonian, momentum, and angular momentum.

$$\{H, P, M\}. \tag{2.5}$$

The (anti-)commutation relations of the symmetry transformations imply the following (anti-)commutation relation of the generators[13].

$$Q_+^2 = Q_-^2 = \overline{Q}_+^2 = \overline{Q}_-^2 = 0, \tag{2.6}$$

$$\{Q_\pm, \overline{Q}_\pm\} = H \pm P, \tag{2.7}$$

$$\{\overline{Q}_+, \overline{Q}_-\} = \{Q_+, Q_-\} = 0, \tag{2.8}$$

$$\{Q_-, \overline{Q}_+\} = \{Q_+, \overline{Q}_-\} = 0, \tag{2.9}$$

$$[iM, Q_\pm] = \mp Q_\pm, \quad [iM, \overline{Q}_\pm] = \mp \overline{Q}_\pm. \tag{2.10}$$

Furthermore, if the theory preserves both vector and axial R-rotations, there are also corresponding vector and axial R-charges.

$$F_V,\ F_A. \tag{2.11}$$

These vector and axial R-charges are rotations on the supercharges. The vector-like rotation acts on the supercharges with the same charge for different chiralities, while the axial-like rotation has different charges with respect to the chirality is $+$ or $-$.

$$[iF_V, Q_\pm] = -iQ_\pm, \quad [iF_V, \overline{Q}_\pm] = i\overline{Q}_\pm, \tag{2.12}$$

$$[iF_A, Q_\pm] = \pm iQ_\pm, \quad [iF_A, \overline{Q}_\pm] = \pm i\overline{Q}_\pm. \tag{2.13}$$

We can slightly relax the condition of anti-commutators of supercharges with opposite chiralities.

$$\{\overline{Q}_+, \overline{Q}_-\} = Z, \quad \{Q_+, Q_-\} = Z^*, \tag{2.14}$$

$$\{Q_-, \overline{Q}_+\} = \widetilde{Z}, \quad \{Q_+, \overline{Q}_-\} = \widetilde{Z}^*. \tag{2.15}$$

Here, Z and $\widetilde{Z}$ are called central charges and must commute with other symmetry generators.

These anti-commutators are products of supercharges with different chiralities, and thus are automatically M invariant. The requirement for H and P invariance forces central charges to be world sheet scalars. However, from the R-symmetry action we can see that F_V is conserved only if $Z = 0$ and F_A is conserved only if $\widetilde{Z} = 0$. In the following chapters, we only consider theories with vanishing Z and $\widetilde{Z}$ unless otherwise stated.

2.2 Field Theory Background

2.2.1 Superspace and Superfields

The $2d$ $\mathcal{N} = (2, 2)$ superspace is parametrized by two Grassman even coordinates x^0, x^1 (sometimes denoted as $x^{\pm\pm} := x^0 \pm x^1$), and four Grassman odd coordinates $\theta^\pm, \overline{\theta}^\pm$. The derivatives with respect to these coordinates are written in a shorthand notation as ∂_0, ∂_1 ($\partial_{\pm\pm}$), $\partial_\pm$, $\overline{\partial}_\pm$.

Superspace can be constructed as a coset space of the super-Poincaré group modulo the Lorentz group, following the standard coset construction approach. The coset construction can be used together with the (anti-)commutation relations to deduce the action of the supercharges on the superfields, represented by derivatives on the superspace. Explicitly, the supercharge generators written in differential operators are

$$\mathcal{Q}_\pm = \frac{\partial}{\partial\theta^\pm} + \frac{\mathrm{i}}{2}\overline{\theta}^\pm\left(\frac{\partial}{\partial x^0} \pm \frac{\partial}{\partial x^1}\right) = \partial_\pm + \mathrm{i}\overline{\theta}^\pm\partial_{\pm\pm}, \tag{2.16}$$

$$\overline{\mathcal{Q}}_\pm = -\frac{\partial}{\partial\overline{\theta}^\pm} - \frac{\mathrm{i}}{2}\theta^\pm\left(\frac{\partial}{\partial x^0} \pm \frac{\partial}{\partial x^1}\right) = \overline{\partial}_\pm - \mathrm{i}\theta^\pm\partial_{\pm\pm}. \tag{2.17}$$

As a consistency check, these differential operators satisfy the anti-commutation relations

$$\left\{\mathcal{Q}_\pm, \overline{\mathcal{Q}}_\pm\right\} = -2\mathrm{i}\partial_{\pm\pm}, \tag{2.18}$$

with other anti-commutators vanishing. We can also introduce a set of super-derivative operators

$$D_\pm = \partial_\pm - \mathrm{i}\overline{\theta}^\pm\partial_{\pm\pm}, \quad \overline{D}_\pm = -\overline{\partial}_\pm + \mathrm{i}\theta^\pm\partial_{\pm\pm}\ . \tag{2.19}$$

These differential operators satisfy similar anti-commutation relations

$$\left\{D_\pm, \overline{D}_\pm\right\} = 2\mathrm{i}\partial_{\pm\pm}. \tag{2.20}$$

Moreover, they anti-commute with the supercharges, i.e.,

$$\{D, \mathcal{Q}\} = \left\{\overline{D}, \mathcal{Q}\right\} = \left\{D, \overline{\mathcal{Q}}\right\} = \left\{\overline{D}, \overline{\mathcal{Q}}\right\} = 0. \tag{2.21}$$

We can use such properties to introduce supersymmetry invariant constraints on superfields, which can eliminate degrees of freedom and can be used to construct various types of supersymmetric actions.

2.2.2 $\mathcal{N} = (2, 2)$ Multiplets

Chiral Superfield

A *chiral superfield* is a superfield Φ that is annihilated by the anti-holomorphic super-derivatives.

$$\overline{D}_\pm\Phi = 0. \tag{2.22}$$

We can expand a chiral superfield in the y coordinates defined by

$$y^{\pm\pm} := x^{\pm\pm} - \mathrm{i}\theta^\pm\overline{\theta}^\pm, \tag{2.23}$$

where

$$\overline{D}_\pm y^{\pm\pm} = 0. \tag{2.24}$$

Thus, the dependence of $x^{\pm\pm}$ and $\overline{\theta}^\pm$ in $\Phi(x^{\pm\pm}, \theta^\pm, \overline{\theta}^\pm)$ must be combined into $y^{\pm\pm}$'s, and the chiral multiplet can be expanded as

$$\Phi(x^{\pm\pm}, \theta^\pm, \overline{\theta}^\pm) = \Phi(y^{\pm\pm}, \theta^\pm) = \phi(y) + \theta^\pm\psi_\pm(y) + \theta^+\theta^- F(y) \tag{2.25}$$

If Φ_1 and Φ_2 are both chiral superfields, then the product $\Phi_1\Phi_2$ is also a chiral superfield. Thus, we can define something called chiral rings.

An *anti-chiral superfield* $\overline{\Phi}$ can be defined similarly.

$$D_{\pm}\overline{\Phi} = 0. \tag{2.26}$$

Twisted chiral multiplet

Instead of the case in $4d$, constraints involving super-derivatives with different chiralities are independent in $2d$, and thus we can "twist" the condition of the chiral superfield, which defines the *twisted chiral superfields*.

$$\overline{D}_+U = D\_U = 0. \tag{2.27}$$

Similarly, we can define the twisted $\tilde{y}$ coordinates by

$$\tilde{y}^{\pm\pm} := x^{\pm\pm} \mp \mathrm{i}\theta^{\pm}\overline{\theta}^{\pm}, \tag{2.28}$$

satisfying

$$\overline{D}_+\tilde{y}^{++} = 0, \quad D\_\tilde{y}^{--} = 0. \tag{2.29}$$

Then the twisted chiral multiplet can be expanded as

$$\begin{aligned} U(x^{\pm\pm}, \theta^{\pm}, \overline{\theta}^{\pm}) &= U(\tilde{y}^{\pm\pm}, \theta^{+}, \overline{\theta}^{-}) \\ &= v(\tilde{y}^{\pm\pm}) + \theta^{+}\overline{\chi}_{+}(\tilde{y}^{\pm\pm}) + \overline{\theta}^{-}\chi_{-}(\tilde{y}^{\pm\pm}) + \theta^{+}\overline{\theta}^{-}E(\tilde{y}^{\pm\pm}) \end{aligned} \tag{2.30}$$

It comes to our attention that if we exchange the role of $\{Q_+, Q_-\}$ and $\{\overline{Q}_+, Q_-\}$ as well as their complex conjugate, the chirals and the twisted chirals are exchanged. Indeed, that's an operation related to mirror symmetry, which turns chiral multiplets into twisted chiral multiplets. Sigma models containing both chiral and twisted chiral superfields are quite lovely[1].

Similarly, *twisted anti-chiral superfield* $\overline{U}$, which is the complex conjugate of the twisted chiral U, is defined by

$$D_+\overline{U} = \overline{D}\_\overline{U} = 0. \tag{2.31}$$

Vector multiplet

Let's consider a Lagrangian of a chiral superfield Φ

$$\mathcal{L} = \int \mathrm{d}^4\theta\, \overline{\Phi}\Phi. \tag{2.32}$$

Such an expression is invariant under the global rotation $\Phi \to \mathrm{e}^{i\alpha}\Phi$.

However, if we lift the constant α to be a chiral superfield $A = A(x, \theta, \overline{\theta})$, the Lagrangian is no longer invariant under such a gauge transformation, and we should compensate for it by introducing a real superfield $V = V(x^{\pm\pm}, \theta^{\pm}, \overline{\theta}^{\pm}) = V^{\dagger}$, which transforms as

$$V \to V + \mathrm{i}(\overline{A} - A) + \cdots. \tag{2.33}$$

Then, the Lagrangian

$$\mathcal{L} = \int \mathrm{d}^4\theta\, \overline{\Phi}\mathrm{e}^{V}\Phi \tag{2.34}$$

is invariant under gauge transformations. Such a real scalar superfield V transforms as (2.33) is called a *vector superfield*.

With the help of the gauge transformations, we can eliminate lower components and expand V for $2d$ $\mathcal{N}=(2,2)$ theories as[13]

$$\begin{aligned} V = &\theta^-\bar{\theta}^-(v_0 - v_1) + \theta^+\bar{\theta}^+(v_0 + v_1) - \theta^-\bar{\theta}^+\sigma - \theta^+\bar{\theta}^-\bar{\sigma} \\ &+ i\theta^-\theta^+(\bar{\theta}^-\lambda_- + \bar{\theta}^+\lambda_+) + i\bar{\theta}^-\bar{\theta}^+(\theta^-\lambda_- + \theta^+\lambda_+) + \theta^-\theta^+\bar{\theta}^-\bar{\theta}^+ D. \end{aligned} \tag{2.35}$$

Here σ is a complex scalar that will play a crucial role in the Columb branch analysis of GLSMs.

Such an expression can also be obtained from a $4d$ $\mathcal{N}=1$ theory[14], and make reductions of the x^1, x^2 coordinates[1]. As a result of the reduction of the Lorentz group $SO(3,1)$ to $SO(1,1)$, the vector fields v_1 and v_2 are invariant under $SO(1,1)$, and we combine these two into a complex scalar.

$$\sigma = \frac{v_1 - iv_2}{\sqrt{2}}, \quad \bar{\sigma} = \frac{v_1 + iv_2}{\sqrt{2}}. \tag{2.36}$$

This is just the complex scalar appearing in the expansion of V in dimension two.

With the presence of the vector superfield, we can define the covariant superderivatives as

$$\mathcal{D}_\pm = \mathrm{e}^V D_\pm \mathrm{e}^{-V}, \quad \overline{\mathcal{D}}_\pm = \mathrm{e}^{-V}\overline{D}_\pm \mathrm{e}^V. \tag{2.37}$$

One can then define the field-strength superfield of the superspace gauge field, which is itself a twisted chiral superfield.

$$\Sigma = \frac{1}{2\sqrt{2}}\left\{\overline{\mathcal{D}}_+, \mathcal{D}_-\right\} \tag{2.38}$$

As we argued before, such a twisted chiral can be expanded as

$$\Sigma = \sigma(\tilde{y}) + \mathrm{i}\theta^+\overline{\lambda}_+(\tilde{y}) - \mathrm{i}\overline{\theta}^-\lambda_-(\tilde{y}) + \theta^+\overline{\theta}^-[D(\tilde{y}) - \mathrm{i}v_{01}(\tilde{y})], \tag{2.39}$$

where $v_{01} := \partial_0 v_1 - \partial_1 v_0$ is the field strength of v_μ.

2.2.3 Lagrangians

There are some natural ways to write down the supersymmetric Lagrangians in superfield language, which are respectively *D-terms* and *F-terms*.

D-term Lagrangians are constructed by integrating over the whole superspace, i.e., $\left(\theta^\pm, \overline{\theta}^\pm\right)$.

$$\mathcal{L}_{\text{D-term}} = \int \mathrm{d}^4\theta\, K = \int \mathrm{d}^2\theta^\pm\, \mathrm{d}^2\overline{\theta}^\pm\, K. \tag{2.40}$$

Here K could be any $(2,2)$ superfield, usually a series of compositing superfields of the theory. We also want it to be hermitian to get real actions. Such Lagrangians can result in supersymmetric invariant actions, since the top component of superfields transforms as a total derivative under a supersymmetry transformation. Thus, if we believe fields asymptotically drop off suitably quickly, such D-terms are supersymmetric. Note that if there are boundaries on the worldsheet, things will get complicated, and we will take care of these subtleties later in this paper.

Similarly, we can construct F-terms for chiral superfields, which is integrated over the half superspace $\theta^{\pm}$, and the dependence of $\overline{\theta}^{\pm}$ is suppressed in the y coordinates and then it was manually set to 0.

$$\mathcal{L}_{\text{F-term}} = \int \mathrm{d}^2\theta^{\pm} \; W(\Phi)\big|_{\overline{\theta}^{\pm}=0}\,, \tag{2.41}$$

where $W(\Phi)$ is a (polynomial) function of chiral superfields Φ's and thus a chiral superfield.

In fact, for $(2,2)$ supersymmetry, we can also construct supersymmetric actions via so called twisted F-terms. It was obtained by integrating a twisted superfield $\widetilde{W}$ over the "twisted half" superspace θ^+ and $\overline{\theta}^-$.

$$\mathcal{L}_{\text{twisted F-term}} = \int \mathrm{d}\theta^+ \, \mathrm{d}\overline{\theta}^- \; \widetilde{W}(U)\Big|_{\theta^-=\overline{\theta}^+=0}\,. \tag{2.42}$$

For gauged linear sigma models, there are matters charged in specific representations under the gauge group, and the Lagrangian consists of the following parts.

$$L = L_{\text{kin}} + L_W + L_{\text{gauge}} + L_{\text{F-I},\theta}. \tag{2.43}$$

These terms are respectively the kinetic energy of chiral matters, the holomorphic superpotential interaction, the kinetic energy of gauge fields, and the Fayet-Iliopoulos term with theta angle term. Following[1], for $U(1)^s$ gauge theory with k chiral field of charges $Q_{i,a}$, $a=1,\cdots,s$ and $i=1,\cdots,k$, we have

$$\mathcal{L}_{\text{kin}} = \int \; \mathrm{d}^4\theta \sum_i \overline{\Phi}_i \mathrm{e}^{2\sum_a Q_{i,a}V_a}\Phi_i, \tag{2.44}$$

$$\mathcal{L}_W = -\int \; \mathrm{d}\theta^+\mathrm{d}\theta^- W\left(\Phi_i\right)\Big|_{\bar{\theta}^+=\bar{\theta}^-=0} - h.c., \tag{2.45}$$

$$\mathcal{L}_{\text{gauge}} = -\sum_a \frac{1}{4e_a^2}\int \; \mathrm{d}^4\theta \bar{\Sigma}_a \Sigma_a, \tag{2.46}$$

and

$$\begin{aligned} L_{\text{F-I},\theta} &= \int \mathrm{d}^2 y \left(-rD + \frac{\theta}{2\pi}v_{01}\right) \\ &= \frac{\mathrm{i}t}{2\sqrt{2}}\int \; \mathrm{d}^2 y \; \mathrm{d}\theta^+\mathrm{d}\bar{\theta}^-\Sigma\Big|_{\theta^-=\bar{\theta}^+=0} - \frac{\mathrm{i}\bar{t}}{2\sqrt{2}}\int \; \mathrm{d}^2 y \; \mathrm{d}\theta^-\mathrm{d}\bar{\theta}^+\bar{\Sigma}\Big|_{\theta^+=\bar{\theta}^-=0}. \end{aligned} \tag{2.47}$$

Here,

$$t = \mathrm{i}r + \frac{\theta}{2\pi} \tag{2.48}$$

is the complex coupling. The theta angle is significant in two-dimensional abelian gauge theory[15]. It induces a constant electric field in the vacuum, equal to the electric field due to a charge of strength $\theta/2\pi$.

It comes to our attention that this $L_{\text{F-I},\theta}$ could be rewritten into a similar form as the holomorphic superpotential. We call it twisted superpotential $\widetilde{W}(\Sigma)$ since the integration is over the twisted half superspace.

$$\Delta\mathcal{L} = \int \mathrm{d}\theta^+\mathrm{d}\bar{\theta}^- \; \widetilde{W}(\Sigma)\Big|_{\theta^-=\bar{\theta}^+=0} + h.c. \tag{2.49}$$

If we take a linear twisted superpotential,

$$\widetilde{W}(x) = \frac{\mathrm{i}tx}{2\sqrt{2}}, \tag{2.50}$$

it reproduce $L_{\text{F-I},\theta}$.

2.2.4 Bosonic Potential

Now we consider the simplest case where the gauge group is $G = U(1)$, and thus a single vector superfield V. The highest component of chiral supermultiplet Φ_i and vector supermultiplet V, which we denoted as F_i and D, are auxiliary fields, i.e., they don't form dynamic terms in the Lagrangian and can be eliminated by the equation of motion.

$$\begin{aligned} D &= -e^2\left(\sum_i Q_i|\phi_i|^2 - r\right), \\ F_i &= \frac{\partial W}{\partial \phi_i}. \end{aligned} \tag{2.51}$$

Here, ϕ_i is the lowest component of Φ and r is the FI parameter. And we ignore the theta angle term at this moment.

Due to Lorentz symmetry, the only possible fields with non-vanishing vacuum expectation values are the scalars. Thus, to extract the low-energy behaviors, we just write down the potential energy for the scalar fields ϕ_i and σ (the lowest component of the twisted chiral supermultiplet).

$$U(\phi_i, \sigma) = \frac{1}{2e^2}D^2 + \sum_i |F_i|^2 + 2|\sigma|^2 \sum_i Q_i^2|\phi_i|^2 \tag{2.52}$$

We will see that just as in the mean-field theory, minimizing the bosonic potential gives vacuum configurations of the system, and different values of r result in different low-energy phases, which behave similarly to tuning the mass parameter r in ϕ^4 theory.

2.2.5 Higgs Branch and GIT Quotient

We are interested in finding the vacuum where the scalars ϕ_i have a non-zero vacuum expectation value and the lowest component of gauge field strength $\sigma = 0$, which we refer to as the *Higgs branch*. From the expression of bosonic potential, the gauge-inequivalent vacua is located at the configuration where $U = 0$, modulo gauge group action.

$$U^{-1}(0, \sigma = 0)/G = V(r). \tag{2.53}$$

Thus, we obtain the Higgs branch for the bosonic potential (3.105).

$$V(r) = (\{D^{-1}(0)\} \cap \{F^{-1}(0)\})/G. \tag{2.54}$$

In general, we can construct physical realizations of GIT quotients $\mathbb{C}^n//G_{\mathbb{C}}$, via a GLSM with gauge group G and matter fields corresponding to $\mathbb{C}^n$.

For instance, we can construct $\mathbb{P}^n$ from $U(1)$ gauge theory and $Gr(k, N)$ from $U(k)$ gauge theory.

$$\begin{aligned} \mathbb{P}^n: \quad & G = U(1), \quad (n+1)\text{-chiral scalars} \quad \Phi_i, \\ Gr(k,N): \quad & G = U(k), \quad N\text{-chiral scalars in the Fund. Rep.} \quad \Phi_i^a. \end{aligned} \tag{2.55}$$

Furthermore, if we are interested in the hypersurfaces in the quotients, additional matter and a superpotential W for which the hypersurface is encoded in the critical locus should be added. This can be seen from the second term $F^{-1}(0)$ in (2.54) which determines a hypersurface in $\{D^{-1}(0)\}/G$ for a specific form of superpotential W.

2.3 Warm Up: Calabi-Yau/Landau-Ginzburg Correspondence

We now turn to illustrate the correspondence between Calabi-Yau NLSMs and Landau-Ginzburg models, and the phase structure of $\mathcal{N} = 2$ gauge theories. We take the simplest case where the gauge group is $U(1)$ with a single vector superfield V, and the matter fields are as follows.

Gauge Group	$\Phi_i, \quad i = 1, \cdots, n$	P
$U(1)$	$+1$	$-n$

Table 2-2 Gauge Charges of the $U(1)$ GLSM

Besides, we take the superpotential to be a gauge invariant function

$$W = P \cdot G(\Phi_1, \cdots, \Phi_n), \tag{2.56}$$

where G is a homogeneous polynomial of degree n. The choice of gauge charges assigned to Φ and P ensures the gauge invariance of W. Indeed, the R-charges for Φ and P are further constrained by the $+2$ (vector) R-charge of W, which is a result of the R invariance of the action.

$$S_W = -\int \mathrm{d}^2 y \, \mathrm{d}\theta^+ \mathrm{d}\theta^- W(\Phi_i)\Big|_{\bar{\theta}^+ = \bar{\theta}^- = 0} - h.c., \tag{2.57}$$

R-symmetry	θ^+	$\bar{\theta}^+$	θ^-	$\bar{\theta}^-$
$U(1)_R$	$+1$	-1	0	0
$U(1)_L$	0	0	$+1$	-1
$U(1)_A$	-1	$+1$	$+1$	-1
$U(1)_V$	-1	$+1$	-1	$+1$

Table 2-3 R-charges of $(2,2)$ Superspace Coordinates

Note that since we have

$$\int \mathrm{d}\theta \, \theta = 1, \tag{2.58}$$

the integration measure has the opposite R-charge as the coordinates θ.

The $U(1)_L$ and $U(1)_R$ are left moving and right moving R-charges, which is another basis of $U(1)_V$ and $U(1)A$ with their Lie algebra relations

$$F_A = F_R - F_L, \quad F_V = F_R + F_L. \tag{2.59}$$

The low-energy effective theory for different values of r is distinct. In this section, we focus on the Higgs branch where the scalars are massless and have non-zero vev while $\sigma = 0$. The low energy theory is the C-Y model or the L-G model depending on whether r is positive or negative.

2.3.1 Geometric (C-Y) Phase

First, we take $r >> 0$ that results in the geometric phase.

Here, we are interested in the case where G is "transverse", suggesting that the only solution of

$$\frac{\partial G}{\partial x_1} = \cdots = \frac{\partial G}{\partial x_n} = 0 \tag{2.60}$$

is at

$$x_1 = \cdots = x_n = 0. \tag{2.61}$$

Minimizing bosonic potential (3.105) indicates that

$$D = 0, \quad F_i = 0. \tag{2.62}$$

Imposing the EOM of the auxiliary fields (2.51), one can get

$$|p|^2 \left(\sum_i \left| \frac{\partial G}{\partial \phi_i} \right|^2 \right) = 0 \quad \text{and} \quad \sum_i |\phi_i|^2 - n|p|^2 - r = 0. \tag{2.63}$$

Large FI parameter r implies at least one non-vanishing ϕ_i to make D term vanish. And thus, the p field should vanish as the transversality of G.

We have

$$\{D^{-1}(0)\} = \left\{ \sum_i |\phi_i|^2 = r \right\} = S^{2n-1} \subset \mathbb{C}^n. \tag{2.64}$$

Dividing it by the gauge group $U(1)$, we obtain precisely $\mathbb{P}^{n-1}$ due to Hopf fibration.

$$U(1) \hookrightarrow S^{2n-1} \to \mathbb{P}^{n-1}. \tag{2.65}$$

Taking the intersection with $F^{-1}(0)$, the resulting gauge inequivalent vacuum is just a degree n hypersurface $G = 0$ in the projective space $\mathbb{P}^{n-1}$, which is Calabi-Yau.

$$V(r) = (\{D^{-1}(0)\} \cap \{F^{-1}(0)\})/G = \{G = 0\} \subset \mathbb{P}^{n-1}. \tag{2.66}$$

The low-energy theory for $r >> 0$ of the GLSM is an NLSM with a Calabi-Yau target space.

2.3.2 Landau-Ginzburg Phase

Now we consider $r << 0$.

For this regime, the vanishing of D implies that the p field must not vanish. Again, as a consequence of the transversality of G, from (2.51), the vanishing of F forces all ϕ's to vanish. Then, the vanishing of the D term gives

$$p = \sqrt{-\frac{r}{n}}, \quad \text{up to gauge transformation.} \tag{2.67}$$

We can expand the system around this vacuum configuration, where the ϕ_i's are massless due to the vanishing quadratic term at $n \geq 3$. And p is massive with mass proportional to r due to the D^2 term. Thus, we can integrate out the massive field p, which is equivalent to replacing it with (2.67). The resulting theory has a unique classical vacuum governed by a superpotential with a degenerate critical point, which is usually called Landau-Ginzburg theory.

In fact, the non-zero vev of p field does not totally break the gauge group, since it is charge n under the $U(1)$; rather, it breaks it down to a $\mathbb{Z}_n$ subgroup acts on ϕ_i by an n-th root of 1. This is actually an $\mathbb{Z}_n$ orbifold of a Landau-Ginzburg theory.

2.3.3 Higher Rank Examples

It's natural to consider a generalization with a higher-rank gauge group. Now, for instance, we take $G = U(1)^2$, and there are some similarities between the phase diagram of GLSMs and the a (coupled) Ising model.

Hypersurface $G_1 \cup G_2$

We take the following matter fields with superpotential $W = PG_1(\Phi) + QG_2(\Psi)$.

Gauge Group	P	Q	$\Phi_i, \quad i = 1, \cdots, n$	$\Psi_j, \quad i = 1, \cdots, m$	F-I
$U(1)_1$	$-n$	0	$+1$	0	r_1
$U(1)_2$	0	$-m$	0	$+1$	r_2

Table 2-4 GLSM Data of the Example

By analyzing bosonic energy (3.105), we want to minimize the following term

$$|G_1(\Phi)|^2 + |G_2(\Psi)|^2 \subset U(\Phi, \Psi). \tag{2.68}$$

This gives the union of two hypersurfaces $G_1(\phi) = 0$ and $G_2(\psi) = 0$ respectively in $\mathbb{P}^{n-1}$ and $\mathbb{P}^{m-1}$. There is no coupling between these two systems, and then the phase diagram is simply a double of the original system.

Hypersurface $G(\Phi, \Psi)$

We take the following matter fields with superpotential $W = PG(\Phi, \Psi)$, which can be seen as a coupling of the two $U(1)$ models.

It has come to our attention that the phase diagram no longer contains a Landau-Ginzburg phase. Here, we use the word "full coupling" with respect to coupling both Φ and Ψ into a single chiral field P.

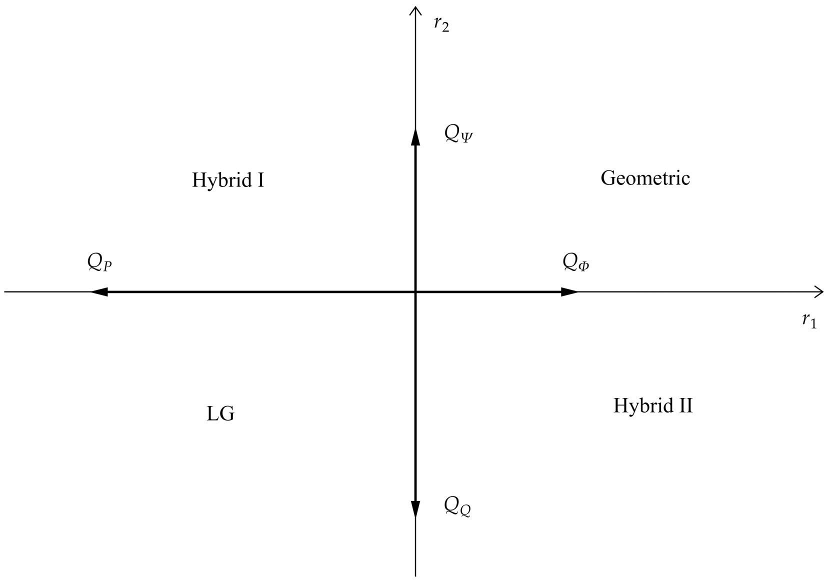


Figure 2-1 $U(1)^2$ Model without Coupling

Gauge Group	P	$\Phi_i, \quad i=1,\cdots,n$	$\Psi_j, \quad i=1,\cdots,m$	FI
$U(1)_1$	$-n$	$+1$	0	r_1
$U(1)_2$	$-m$	0	$+1$	r_2

Relation to the Phase Structure of $\mathbb{Z}_2 \times \mathbb{Z}_2$ Ising model

The phase diagram of GLSMs is very similar to the phase diagram of ϕ^4 theory, where the FI parameter r corresponds to the mass parameter. For positive and negative FI parameters, the GLSM appears similar to the paramagnetic and ferromagnetic phases in the Ising model.

We can compare the above coupled $U(1)^2$ GLSM with the coupled $\mathbb{Z}_2 \times \mathbb{Z}_2$ Ising model, where there is no fourth phase when the coupling becomes larger.

$$S_{\mathbb{Z}_2\times\mathbb{Z}_2} = \int \mathrm{d}\vec{\mathbf{x}} \left[\frac{1}{2}(\nabla\phi_1)^2 + \frac{r_1}{2}\phi_1^2 + \frac{u_1}{4}\phi_1^4 + \frac{1}{2}(\nabla\phi_2)^2 + \frac{r_2}{2}\phi_2^2 + \frac{u_2}{4}\phi_2^4 + \frac{u_{12}}{2}\phi_1^2\phi_2^2\right] \tag{2.69}$$

In such a model, $u_{12} = 0$ indicates that there is no coupling between two ϕ's, which corresponds to the GLSM with superpotential $W = PG_1(\Phi) + QG_2(\Psi)$. When the coupling becomes larger, especially

$$u_{12} > \sqrt{u_1 u_2}, \tag{2.70}$$

the phase with both $\langle\phi_1\rangle \neq 0$ and $\langle\phi_2\rangle \neq 0$ no longer exists. This corresponds to the Landau-Ginzburg phase that also vanishes when we have the (fully) coupled superpotential $W = PG(\Phi, \Psi)$.

However, at this moment, the author didn't know how to explicitly construct a GLSM with a tunable parameter as u_{12}, which can make the Landau-Ginzburg phase continuously vanish.

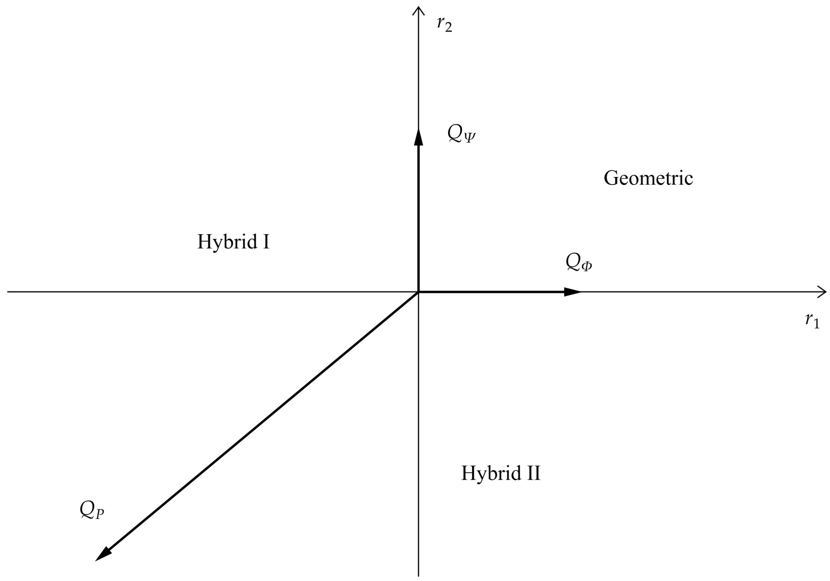
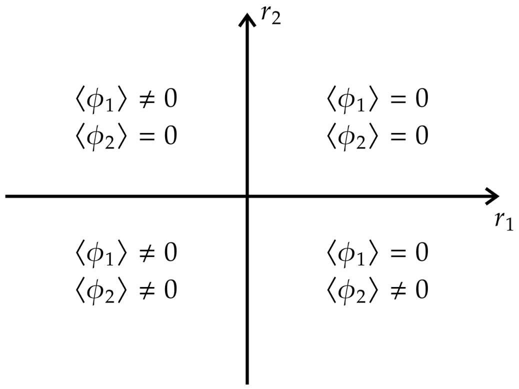


Figure 2-2 $U(1)^2$ Model with (full) $W = PG(\Phi, \Psi)$ Coupling



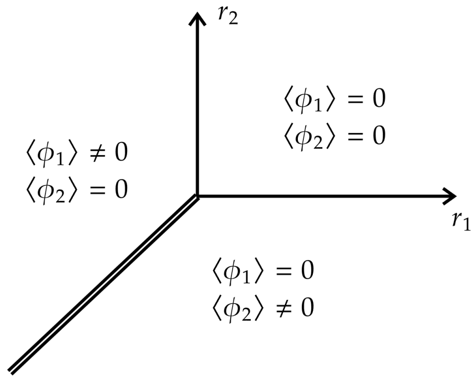


Figure 2-3 Phases of $\mathbb{Z}_2 \times \mathbb{Z}_2$ model for $u_{12} = 0$ and $u_{12} > \sqrt{u_1 u_2}$

2.4 Non-linear Sigma Models

By minimizing the bosonic potential, we are extracting IR information of a GLSM, which is actually a non-linear sigma model (NLSM) obtained by acting the renormalization group on the linear model.

An $2d$ NLSM is specified by maps from the worldsheet to the target space, which are physically realized by chiral superfields

$$\Phi: \ \Sigma \to X, \tag{2.71}$$

with Σ being a Riemann surface and X a Riemannian manifold of metric g. The moral of NLSM is that we pick local coordinates z, $\bar{z}$ on Σ and ϕ^I on X, where ϕ^I is the lowest component of the chiral multiplet Φ^I. Furthermore, the fermion component reads

$$\psi^I_\pm \in \Gamma\Big(K_\Sigma^{\mp\frac{1}{2}} \otimes \phi^* T^{(1,0)} X\Big) \tag{2.72}$$

$$\overline{\psi}^I_\pm \in \Gamma\Big(K_\Sigma^{\mp\frac{1}{2}} \otimes \phi^* T^{(0,1)} X\Big) \tag{2.73}$$

where K_Σ is the canonical bundle on Σ, and $K_\Sigma^{\mp\frac{1}{2}}$ is the spin bundle.

A key point for introducing GLSM is to compute some RG invariant quantities for the corresponding IR NLSM. This would be much more feasible than directly computing from NLSM since the UV theory is linear.

2.5 Topological Twist

The QFT can be formulated not just on the topologically trivial space, such as $\mathbb{R}^{1,1}$ and $\mathbb{R}^2$, but also on other manifolds, for instance, the two-sphere S^2 or the torus T^2. Suppose we consider a system on a two-manifold with an arbitrary metric and spin structure, as we mentioned before, the supersymmetry transformation of an operator $\mathcal{O}$ reads

$$\delta\mathcal{O} = i\epsilon_+ Q_- \mathcal{O} - i\epsilon_- Q_+ \mathcal{O} - i\bar{\epsilon}_+ \bar{Q}_- \mathcal{O} + i\bar{\epsilon}_- \bar{Q}_+ \mathcal{O}. \tag{2.74}$$

However, supersymmetry is a global symmetry, and thus we expect the action to be invariant under covariantly constant variational parameters ϵ.

While space-time has a trivial topology, a constant section of the spin bundle exists. Nevertheless, for non-contractable manifolds, it's impossible to take such a globally covariantly constant section. This is because, to obtain a global covariantly constant section, we need a flat connection on the base manifold M, provided M has a trivial topology.

This can be solved via a procedure called *topological twist*[16–17]. The basic idea is that we assume $U(1)_V$ or $U(1)_A$ is conserved, and non-spinorial fields have even charges while spinorial fields have half-integer charges. Then for a field of R-charge q with value in a vector bundle E, we replace it with $E \otimes T_\Sigma^{\otimes q/2}$ where Σ is an oriented Riemann surface on which our theory is defined[18]. A significant result is that some of the $\epsilon_\pm$ and $\bar{\epsilon}_\pm$ become world-sheet scalars because of the tensor product and thus could be chosen as a constant, which enables us to define a global supersymmetry transformation.

It is called *A-twist* by choosing $U(1)_V$ and *B-twist* by choosing $U(1)_A$. In the A-twisted theory, the scalar supercharges are $\overline{Q}_+$ and Q_-, and hence we can define $Q_A = \overline{Q}_+ + Q_-$ which is also conserved. Similarly, in the B-twisted theory, we have $Q_B = \overline{Q}_+ + \overline{Q}_-$ to be conserved.

Since we've taken advantage of the $U(1)_V$ and $U(1)_A$, it's important to make sure that they survive after quantization as a genuine symmetry of the quantum theory. In fact, the vector R-rotation remains as a symmetry for the quantum theory, and the A-twisted always exists. However, the axial R-rotation suffers from the chiral anomaly, which is proportional to the first Chern class of the target $c_1(X)$. Therefore, to perform a B-twist, we need to have Calabi-Yau manifolds as targets.

In[17], Witten argued that the correlators of Q-closed (the sum of scalar Q's) operators depend only on the Q-cohomology classes of the operators since any Q-exact term appearing in the correlation function will force it to vanish.

Furthermore, there is a natural map between Q-closed operators modulo Q-exact operators and the de Rham cohomology of X.

$$\text{BRST cohomology of the QFT} \longleftrightarrow H^\bullet_{\text{dR}}(X, \mathbb{R}). \tag{2.75}$$

For the A-twisted model, the scalar parameters are ϵ_+ and $\bar{\epsilon}_-$, and we set the other two ϵ's to be zero. The σ field, which is the lowest component of twisted chiral superfield Σ, transforms under the full $\mathcal{N} = (2, 2)$ supersymmetry as[14]

$$\delta\sigma = -i\sqrt{2}\bar{\epsilon}_+\lambda_- - i\sqrt{2}\epsilon_-\bar{\lambda}_+. \tag{2.76}$$

Thus, under the $\mathcal{N} = 2_A$ transformation, the σ field is Q-closed and can be mapped to a cohomology representative. This is the reason why we regard the 1-loop quantum corrected equation of motion of σ as the quantum cohomology ring of the target X.

$$\text{EOM of } \sigma \longleftrightarrow QH^\bullet(X). \tag{2.77}$$

2.6 Mirror Symmetry

The superpotential W is integrated over the holomorphic superspace, and thus it should have $U(1)_V$ charge $+2$ to cancel the -2 charge of the integration measure and $U(1)_A$ charge 0. Furthermore, if we recall the twisted superpotential $\widetilde{W}$ which is integrated over θ^+ and $\bar{\theta}^-$, it should be charged $+2$ under $U(1)_A$ and charge 0 under $U(1)_V$.

The $\mathcal{N} = (2, 2)$ supersymmetry algebra has an $\mathbb{Z}_2$ outer automorphism called mirror symmetry.

$$\begin{aligned} Q_- &\longleftrightarrow \bar{Q}_-, \\ U(1)_V &\longleftrightarrow U(1)_A, \\ Z &\longleftrightarrow \widetilde{Z}. \end{aligned} \tag{2.78}$$

Here, Z is the central charge of the $\mathcal{N} = 2$ supersymmetry algebra.

This indicates that under mirror symmetry, W and $\widetilde{W}$ are somehow connected. Indeed, for the A-twisted NLSM on X and the B-twisted the mirror Landau-Ginzburg model with superpotential W, we have the following chiral ring[18]

$$\mathcal{R}_A = QH^*(X) \quad \text{Quantum Cohomology Ring of } X \tag{2.79}$$

for NLSM, and

$$\mathcal{R}_B = \mathrm{Jac}(W) \quad \text{Jacobi Ring } \mathbb{C}[x_1, \cdots, x_N]/(\partial_1 W, \cdots, \partial_N W) \tag{2.80}$$

for Landau-Ginzburg model. The latter one could be used to compute the quantum cohomology ring of the geometric phase by computing the Jacobi ring of the twisted superpotential $\widetilde{W}$.

2.7 Coulomb Branch and Quantum Corrections

We've discussed the geometric phase where $r >> 0$ as well as the Landau-Ginzburg phase where $r << 0$. These two phases are both far from the origin $r = 0$, and it's natural for us to ask what happens at $r = 0$. Is there any singularity ①?

① The word "singularity" we are referring to is the non-compactness of the target space.

2.7.1 The Singularity

The classical Higgs branch where $\sigma = 0$ and ϕ_i's have non-zero vevs has been studied in Section 2.3. We are now turning to the Coulomb branch where $\phi_i = p = 0$ and σ acquires a non-zero vev. The analysis at the classical level that minimizes the bosonic potential (3.105) for both $r >> 0$ and $r << 0$ eliminates the existence of a non-trivial Coulomb branch, which means that to make the $U = 0$, the σ field must be zero.

However, things have changed when $r = 0$. Since there is no more constraint eliminating ϕ_i and p simultaneously to be zero from the D^2 term, the Coulomb branch arises. In fact, any value of σ can minimize the potential, which means the Coulomb branch is simply $\mathbb{C}$.

We then see the appearance of the singularity in the Coulomb branch, which separates the two phases.

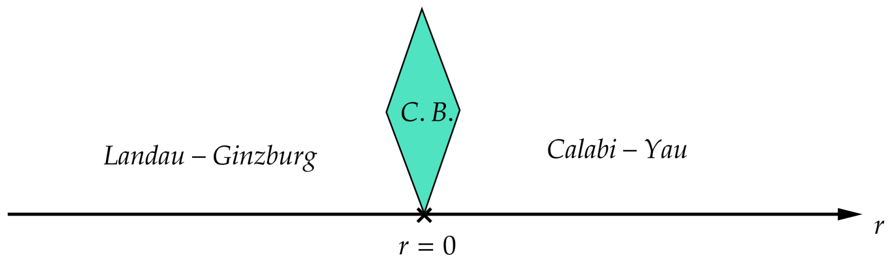


Figure 2-4 Classical Phase Diagram Separated by Columb Branch

2.7.2 One-loop Corrections

However, the above classical arguments will receive slight quantum corrections.

Indeed, for large σ regions, we can integrate out the massive chiral fields ϕ_i's, which have masses proportional to $Q_i^2|\sigma|^2$. The only contribution is the one-loop correction to the expectation value of the auxiliary field D, which comes from the one-loop ϕ^4 interactions. The result reads[2]

$$-\frac{1}{e^2}\langle D\rangle_{\text{1-loop}} = \sum_{i=1}^{n} Q_i \int \frac{\mathrm{d}^2 k}{(2\pi)^2}\frac{1}{k^2 + 2Q_i^2|\sigma|^2}. \tag{2.81}$$

Cutting off the divergent by adding a cutoff Λ, we obtain

$$-\frac{1}{e^2}\langle D\rangle_{\text{1-loop}} = \frac{1}{4\pi}\sum_{i=1}^{n} Q_i \log\left(\frac{\Lambda^2}{2Q_i^2|\sigma|^2}\right). \tag{2.82}$$

From the expression of the classical value of D in (2.51), we can regard the one-loop correction for D as a shift of r to an effective value r_{eff}. Similarly, the one-loop correction to $\langle \lambda_+ \bar{\lambda}_- \rangle$ corresponds to a shift in the theta angle θ to θ_{eff}[2]. We have the shift of the complex coupling

$$t \longrightarrow t_{eff}. \tag{2.83}$$

Recall that the F-I and theta angle term without quantum correction could be written in a unified form by using the linear twisted superpotential (3.93). For the classical Coulomb branch, we have

$$U(t) \propto \left|\frac{\partial \widetilde{W}}{\partial \sigma}\right|^2 \sim \frac{e^2}{2}\left(r^2 + \left(\frac{\theta}{2\pi}\right)^2\right). \tag{2.84}$$

The singularity is precisely at the origin, where the potential is minimized regardless of the value of σ, corresponding to a non-compact Columb branch supporting at $t = 0$.

Now, due to the quantum corrections to the complex coupling t, we have

$$\widetilde{W}(\Sigma) = \frac{1}{2\sqrt{2}}\Sigma\left(it - \frac{1}{2\pi}\sum_{i=1}^{n} Q_i \log\left(\sqrt{2}Q_i\Sigma\right)\right), \tag{2.85}$$

where, for simplicity, we've taken $\Lambda = 1$. This is just substituting the classical coupling t with the one-loop effective t_{eff}, which has a Σ dependence in general. The potential energy on C.B. reads

$$U(\sigma) = \frac{e^2}{2}\left|it - \frac{1}{2\pi}\sum_{i=1}^{n} Q_i\left(\log\left(\sqrt{2}Q_i\sigma\right) + 1\right)\right|^2. \tag{2.86}$$

For the Calabi-Yau case, where we have

$$\sum_i Q_i = 0, \tag{2.87}$$

the shift of the complex coupling t is reduced to a σ-independent form

$$t_{eff} = t + \frac{i}{2\pi}\sum_{i=1}^{n} Q_i \log\left(Q_i\right). \tag{2.88}$$

Such an effect is just a shift of a singular point. The moduli space of the complex coupling lives on a cylinder due to the periodicity of the theta angle. Thus, the singularity is complex co-dimension one, and there could be a continuous path connecting two phases, e.g., the dotted line in Fig.2-5.

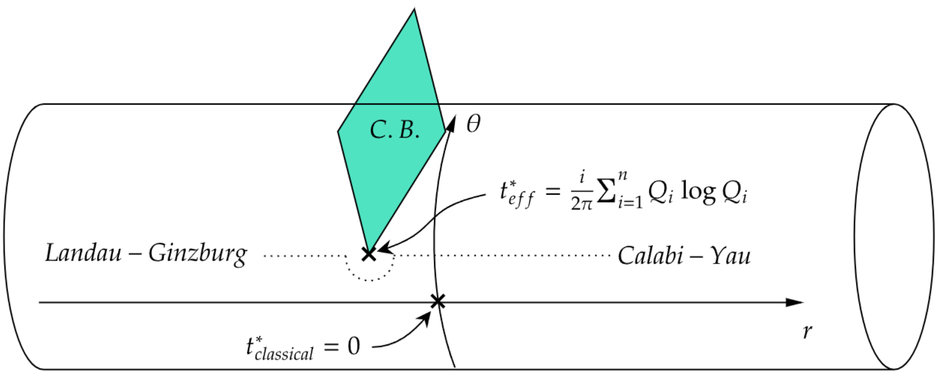


Figure 2-5 Quantum Corrected Phase Diagram with Theta Angle

2.7.3 Quantum Cohomology

In (2.77) and (2.80), we mentioned that the equation of motion of the σ field, or equivalently,

$$\exp\left(\frac{\partial\widetilde{W}(\sigma)}{\partial\sigma}\right) = 1 \tag{2.89}$$

gives the quantum cohomology ring of X. The exponential was taken due to the periodicity of the theta angle.

However, as we've seen above, for the Calabi-Yau case, the equation of motion won't produce any information for the ring. Instead, the EOM gives the position of the Coulomb branch, but the Coulomb branch itself has no shape (becomes non-compact) at that point. Thus, we turn to the simplest Fano case where we drop the P field as well as the superpotential W. In the geometric phase, we have

$$X = \mathbb{P}^{n-1}. \tag{2.90}$$

By taking the critical locus of $\widetilde{W}$ in (2.85), we obtain

$$\sigma^n = \left(\frac{\Lambda}{e}\right)^n e^{2\pi i t}. \tag{2.91}$$

Absorbing the constants into the definition of the quantum parameter q, we get the ring relation for the σ field.

$$\sigma^n = q. \tag{2.92}$$

This is the quantum cohomology ring of $\mathbb{P}^{n-1}$, where the σ is regarded as the first Chern class by the argument of R-charge, the generator of the classical cohomology ring of $\mathbb{P}^{n-1}$.

$$\sigma \sim c_1(X) \in H^{1,1}[X]. \tag{2.93}$$

The classical cohomology ring of $\mathbb{P}^{n-1}$ is generated by a single generator c_1 with a truncated relation.

$$H^*[\mathbb{P}^{n-1}] = H^*[\text{pt.}][\![x]\!]/x^n \tag{2.94}$$

Furthermore, the quantum cohomology ring of $\mathbb{P}^{n-1}$ is a deformation of the classical ring, with a quantum parameter q.

$$QH^*[\mathbb{P}^{n-1}] = QH^*[\text{pt.}][\![x]\!]/(x^n - q) \tag{2.95}$$

This is precisely the ring relation from the EOM of the σ field (2.92).

In mathematics, quantum cohomology is defined by modifying the cup product $\cup$ of classical cohomology by the quantum product $\star$ containing the information of Gromov-Witten invariants, which encodes the data of $4d$ Yukawa couplings in string compactifications.

Another comment for the projective space model is that the Euler characteristic number of vacua of the two phases is the same. For the geometric phase, we have

$$\chi(\mathbb{P}^{n-1}) = n. \tag{2.96}$$

In fact, there is no Landau-Ginzburg phase for the $\mathbb{P}^{n-1}$ model. However, the equation of motion (2.92) has n-th power in σ and gives n distinct vacua on the Coulomb branch. Thus, the Euler characteristic number is an invariant when transporting among different phases.

2.8 Bethe/Gauge Correspondence

In[19], the authors proposed a correspondence between (effective) $2d$ supersymmetric gauge theories and quantum integrable systems.

The Heisenberg spin chain, also known as the $SU(2)$ XXX model, is defined on a one-dimensional length L lattice where each site has spin-$\frac{1}{2}$. The Bethe ansatz equation (BAE) of such a spin chain coincides with the (equivariant) quantum cohomology ring of an NLSM with twisted masses.

$$\text{BAE with Twisted Masses} \longleftrightarrow \text{Equivariant Quantum Cohomology} \tag{2.97}$$

The twisted masses are some quadratic terms for fundamental chirals, which serve as a mass and contribute to the vacuum equation by modifying σ since the vacuum expectation values of σ's are the masses of the chirals in the low-energy limit.

$$L_{\widetilde{m}} = \int \mathrm{d}^4\theta \ \mathrm{Tr}_{\mathcal{R}} \Phi^\dagger \left(\mathrm{e}^{\widetilde{V}} \otimes \mathrm{Id}_R\right)\Phi. \tag{2.98}$$

Here, $\mathcal{R}$ is the representation of the theory

$$\mathcal{R} = M \otimes R. \tag{2.99}$$

R is the representation of the gauge group, and M is the representation of the flavor symmetry group. The twisted masses are added with respect to the representation space M of flavor symmetries.

In mathematics, twisted masses correspond to equivariant parameters. For example, the NLSM with $T^*Gr(k,N)$ target space has the following equivariant quantum cohomology rings,

$$\prod_{a=1}^{N} \frac{\sigma_i + m_a^{\mathrm{f}}}{\sigma_i - m_a^{\bar{\mathrm{f}}}} = -e^{2\pi i \hat{t}} \prod_{j=1}^{k} \frac{\sigma_i - \sigma_j - m^{\mathrm{adj}}}{\sigma_i - \sigma_j + m^{\mathrm{adj}}}, \tag{2.100}$$

which coincides with the BAE of the XXX-model with appropriate boundary conditions.

Chapter 3

Sigma Models on Calabi-Yau's and Super Calabi-Yau's

3.1 Moduli Space of Calabi-Yau Manifolds

In this section, we first have a brief review of complex geometry following[13,20], which is an important background for the analysis of complex deformation and Kähler deformation of a Calabi-Yau manifold.

3.1.1 Complex Structure

We know that a complex manifold to be a topological space with local charts that are isomorphic to open sets in $\mathbb{C}^n$ with holomorphic transition functions. It's obvious that the second condition is much more strict, since every even-dimensional real manifold locally looks like $\mathbb{C}^n$. Thus, given a $2n$-dimensional real manifold, one should ask what the condition is for it to be a complex manifold, i.e., when it can be endowed with holomorphic transition functions while identifying each local patch to $\mathbb{C}^n$.

We first focus on the (co-)tangent bundle. The holomorphic requirement of the transition matrix L in coordinates

$$z^a = x^a + \mathrm{i}y^a, \quad \overline{z}^{\overline{a}} = x^a - \mathrm{i}y^a \tag{3.1}$$

can be expressed in the well-known Cauchy-Riemann condition

$$L = -JLJ. \tag{3.2}$$

Here, J is a $(1,1)$-type tensor field satisfies $J^2 = -\mathbb{1}_{2n\times 2n}$, with a matrix representation with respect to the (x^a, y^a) coordinates.

$$J = \begin{pmatrix} 0 & -\mathbb{1}_{n\times n} \\ \mathbb{1}_{n\times n} & 0 \end{pmatrix}. \tag{3.3}$$

Note that since J is a $(1,1)$-type tensor, it's endomorphisms on both tangent and cotangent bundles.

$$J \in \mathrm{Hom}(\mathfrak{X}, \mathfrak{X}) \cong \mathrm{Hom}(\varLambda^1, \varLambda^1). \tag{3.4}$$

We can also seek a different expression for J since it squares to $-\mathbb{1}$: A set of coordinates where J is diagonalized and the eigenvalues are $\pm$i's. In fact, the z^a's did the job, and we can check it by direct computation.

$$J\mathrm{d}z = J\mathrm{d}x + \mathrm{i}J\mathrm{d}y = -\mathrm{d}y + \mathrm{i}\mathrm{d}y = \mathrm{i}\mathrm{d}z, \tag{3.5}$$

$$J\mathrm{d}\overline{z} = J\mathrm{d}x - \mathrm{i}J\mathrm{d}y = -\mathrm{d}y - \mathrm{i}\mathrm{d}y = -\mathrm{i}\mathrm{d}z. \tag{3.6}$$

We've secretly decided to work with the complexified (co-)tangent bundle $TM \otimes \mathbb{C}$ in (3.1) by taking linear combinations of (x, y) with coefficients in $\mathbb{C}$. The complexified tangent bundle $TM \otimes \mathbb{C}$ is decomposed to, with eigenvalues $+\mathrm{i}$ and $-\mathrm{i}$ respectively, the holomorphic tangent bundle TM (by abuse of notation) and the anti-holomorphic tangent bundle $\overline{T}M$, after specifying the complex structure J.

$$TM \otimes \mathbb{C} = TM \oplus \overline{T}M. \tag{3.7}$$

Furthermore, there are projection operators from $TM \otimes \mathbb{C}$ onto the holomorphic and anti-holomorphic tangent bundles.

$$P = \frac{1}{2}(1 - \mathrm{i}J), \quad \overline{P} = \frac{1}{2}(1 + \mathrm{i}J). \tag{3.8}$$

Finally, the condition for globally well-defined and compatible transition functions indicates tha the Lie bracket of two holomorphic vectors should always be a holomorphic vector, or equivalently, the Nijenhuis tensor must vanish.

$$\overline{P}[PX, PY] = 0 \tag{3.9}$$

$$\Updownarrow \tag{3.10}$$

$$N(X, Y) := [JX, JY] - J[X, JY] - J[JX, Y] - [X, Y] = 0, \tag{3.11}$$

for all X and Y.

3.1.2 Symplectic Structure, Hermitian Metrics and Kählar Manifolds

A non-degenerate and closed (skew-symmetric) differential 2-form ω on a smooth manifold M is called the symplectic structure. It is called almost symplectic if it's only non-degenerate, and is called presymplectic if it's only closed.

Let's stop for a moment and recall some known structures on a smooth manifold M. We have Riemannian manifolds (M, g) equipped with a *Riemann metric*, which is a non-degenerate symmetric bilinear $(0, 2)$-tensor g, almost symplectic manifolds (M, ω) with an *almost symplectic form*, which is a non-degenerate skew-symmetric $(0, 2)$-tensor ω, and almost complex manifolds (M, J) with an *almost complex structure* J, which is a $(1, 1)$-tensor satisfying $J^2 = -\mathbb{1}$. What are the relations between these structures?

We first consider an almost symplectic manifold (M, ω) with a given almost complex structure J. We say that ω and J are compatible if

$$\omega(JX, JY) = \omega(X, Y), \quad \forall X, Y \in \mathfrak{X}(M), \tag{3.12}$$

$$\omega(\bullet, J\bullet) \text{ is non-degenerate.} \tag{3.13}$$

Due to the second property, we can define a non-degenerate symmetric bilinear $(0, 2)$-tensor by

$$g(X, Y) := \omega(X, JY) = \omega(Y, JX) = g(Y, X), \tag{3.14}$$

which is just the Riemann metric structure compatible with the almost complex structure J.

$$g(JX, JY) = g(X, Y). \tag{3.15}$$

On a manifold (M, ω, J) with a compatible almost symplectic structure and almost complex structure, an almost Hermitian metric could be defined as a positive-definite inner product at every point on M.

$$TM \otimes \overline{T}M \to \mathbb{C}. \tag{3.16}$$

In local coordinates z^i, we can express it by $h_{i\bar{j}}\,\mathrm{d}z^i\mathrm{d}\overline{z}^{\bar{j}}$, where the matrix $h_{i\bar{j}}$ is Hermitian. In fact, an almost Hermitian structure could be explicitly constructed via g and ω, whose real part and imaginary part are respectively the Riemann metric and the almost symplectic form.

$$h(X,Y) := g(X,Y) - \mathrm{i}\omega(X,Y). \tag{3.17}$$

A manifold equipped with an almost Hermitian metric is called an almost Hermitian manifold. If we lift the almost complex structure to a complex structure, i.e., J is torsion free, then the almost Hermitian structure is lifted to a Hermitian structure and (M, h) is a Hermitian manifold.

As we just saw, the above construction is started from ω and J. In fact, we could also assume the existence of h and J or other combinations.

From the above construction, we see that a Hermitian metric allows us to define a related $(1,1)$-form

$$\omega = \frac{\mathrm{i}}{2} h_{i\bar{j}}\,\mathrm{d}z^i \wedge \mathrm{d}\overline{z}^{\bar{j}}, \tag{3.18}$$

which is real $\omega = \overline{\omega}$ and usually called the Kähler form if it is a symplectic form, not only an almost symplectic form, i.e., $\mathrm{d}\omega = 0$.

From the integrable condition $\mathrm{d}\omega = 0$, we can express the metric locally as

$$h_{i\bar{j}} = \frac{\partial^2 K(z,\overline{z})}{\partial z^i \partial \overline{z}^{\bar{j}}}, \tag{3.19}$$

where $K(z,\overline{z})$ is called the Kähler potential, and $h_{i\bar{j}}$ is invariant under local transformations

$$K(z,\overline{z}) \to K(z,\overline{z}) + F(z) + G(\overline{z}). \tag{3.20}$$

3.1.3 Calabi-Yau Manifolds

In 1957, Calabi conjectured the existence of a Kähler metric with vanishing Ricci tensor for any compact Kähler manifold with vanishing first Chern class and proved the uniqueness. In 1977, Yau proved the existence[21–22].

Equivalently, a Calabi–Yau manifold can be defined as a complex n-manifold M whose bundle of $(n,0)$-forms is trivial. More algebraic-geometrically speaking, Calabi-Yau manifolds have a trivial canonical bundle K_M. Triviality indicates that we have a globally well-defined, nowhere vanishing section of a holomorphic top form $\varOmega$. And every global $(n,0)$-form could be written as $f\varOmega$, for f some function on M.

For compact Calabi-Yau n-folds, holomorphy forces f to be constant function and thus we have $h^{n,0}(M) = 1$. If M is simply connected, then $h^{1,0}(M) = h^{0,1}(M) = 0$. After computation, for a compact and simply connected Calabi-Yau three-fold, the undetermined elements in the Hodge diamond are just $h^{1,1}$ and $h^{2,1}$.

3.1.4 Kähler Moduli Space and Complex Moduli Space

As we've seen before, the Kähler forms are closed $(1,1)$-forms, and thus the undetermined $h^{1,1}(M)$ is the number of possible Kähler forms of a Calabi-Yau manifold M. We say that $h^{1,1}$ is the dimension of the *Kähler moduli space*.

We are also interested in complex deformation. For general Calabi-Yau's, it's very hard to globally grasp the information of the complex moduli space, and we can locally look at the infinitesimal deformation ϵ with respect to the complex structure J.

One should bare in mind that the deformation must preserve the integrability of J, i.e., subject to the constraint $N[J+\epsilon]=0$. Furthermore, there is redundancy from the redefinition of the coordinates, which means J and $\widetilde{J}$ should be identified as the same point in the complex moduli space if they can be related via coordinate transformations.

Thus, we conclude that the complex moduli is the space of ϵ satisfying $(J+\epsilon)^2=-\mathbb{1}$ and $N[J+\epsilon]=0$, modulo coordinate redefinitions.

From the condition $(J+\epsilon)^2=-\mathbb{1}$, we can get $\epsilon J+J\epsilon=0$, which indicates that the pure indices ($\partial_a\otimes \mathrm{d}z^b$ and $\overline{\partial}_{\overline{a}}\otimes \mathrm{d}\overline{z}^{\overline{b}}$) components of ϵ vanish. Thus, we can write ϵ as a $(0,1)$-form with values in the holomorphic tangent bundle, i.e., $\epsilon_{\text{hol}}=\left(\epsilon^a_{\overline{b}}\partial_a\right)\mathrm{d}\overline{z}^{\overline{b}}$, and as well the anti-holomorphic counterpart. For the equation $N=0$, one can linearize it and obtain $\overline{\partial}\epsilon=0$.

We further consider the change of J by a (infinitesimal) coordinate transformation generated by a vector field $v_i\partial_i$. The Jacobian matrix is just $M^i_j=\delta^i_j+\partial_j v^i$.

$$J'=M^{-1}JM=\left(J^i_j-v^i_{,k}J^k_j+J^i_k v^k_{,j}\right)\partial_i\otimes \mathrm{d}x^j. \tag{3.21}$$

In complex coordinates, such an expression reads

$$\begin{aligned} J' &= \left(J^i_j-v^i_{,k}J^k_j+J^i_k v^k_{,j}\right)\partial_i\otimes \mathrm{d}x^j && (3.22)\\ &= J-\mathrm{i}\partial v+\mathrm{i}\overline{\partial}v+\mathrm{i}\partial v-\mathrm{i}\partial\overline{v} && (3.23)\\ &= J+\mathrm{i}\overline{\partial}v-\mathrm{i}\partial\overline{v}:=J+\overline{\partial}v_{\text{hol}}+\partial v_{\text{anti-hol}}, && (3.24)\end{aligned}$$

where $v_{\text{anti-hol}}=\overline{v}_{\text{hol}}$.

Finally, we find that the in-equivalent complex deformation is labeled by a closed (with respect to $\overline{\partial}$) $(0,1)$-form with values in the holomorphic tangent bundle, modulo $\overline{\partial}v_{\text{hol}}$, which is an exact $(0,1)$-form with values also in the holomorphic tangent bundle. Then, one can conclude that the complex moduli space is classified by the cohomology group

$$H^1_{\overline{\partial}}(M,TM). \tag{3.25}$$

We further consider the case where M is a Calabi-Yau manifold, i.e., the canonical bundle K_M is trivial. The wedge pairing (with TM referring to the holomorphic tangent bundle)

$$\wedge:\ TM\otimes\wedge^{n-1}TM\to\wedge^n TM=K_M=1 \tag{3.26}$$

induces an isomorphism

$$TM\cong\left(\wedge^{n-1}TM\right)^*=\wedge^{n-1}T^*M. \tag{3.27}$$

Finally, we see that the *complex moduli space* of a Calabi-Yau manifold is characterized by

$$H^1_{\bar{\partial}}(M, \wedge^{n-1}T^*M) = H^{n-1,1}(M). \tag{3.28}$$

3.2 Mirror Symmetry for Calabi-Yau Manifolds

Recall that in Sec.2.6 we mentioned a $\mathbb{Z}_2$ automorphism inside the $2d$ $\mathcal{N} = 2$ supersymmetry algebra. By performing $Q_- \leftrightarrow \overline{Q}_-$ and $F_V \leftrightarrow F_A$, the role of two types of supercharges

$$Q_A = \overline{Q}_+ + Q_- \quad \text{and} \quad Q_B = \overline{Q}_+ + \overline{Q}_- \tag{3.29}$$

got exchanged. In this section, we revisit mirror symmetry and especially focus on the SCFT case, where the target space is Calabi-Yau manifolds.

3.2.1 $\mathcal{N} = (2, 2)$ SCFTs

For the case of supersymmetric sigma models with Calabi–Yau targets, as we have seen, the Kähler class does not flow and the theory has zero one-loop running since $c_1 = 0$. We believe that we will get a Super Conformal Field Theory in the IR.

In $2d$ CFTs, locally we have a symmetry algebra given by

$$\mathrm{Vir} \oplus \overline{\mathrm{Vir}}, \tag{3.30}$$

where

$$\mathrm{Vir} = \mathrm{Span}\{L_n : n \in \mathbb{Z}\} \tag{3.31}$$

is the well-known Virasoro algebra, a central extension of the Witt algebra. Note that this is infinite-dimensional, since a finite-dimensional Lie algebra does not admit a nontrivial central extension.

Moreover, we can perform a $(2, 2)$ extension of Vir, which is given by adding the following set of generators.

$$G^\pm_s \quad \text{odd generators, } s \in \mathbb{Z} + a,\ a \in \mathbb{R}, \tag{3.32}$$

$$J_m \quad \text{even generators, } m \in \mathbb{Z}, \tag{3.33}$$

satisfying the graded commutation relations if we adopt the notation in[23–24]

$$\begin{aligned}
&[J_m, J_n] = \frac{c}{3} m\delta_{m+n,0},\\
&[L_m, J_n] = -nJ_{m+n},\\
&\{G^\pm_r, G^\pm_s\} = 0,\\
&\{G^+_r, G^-_s\} = 2L_{r+s} + (r-s)J_{r=s} + \frac{c}{3}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0},\\
&[L_m, G^\pm_r] = \left(\frac{1}{2}m - r\right) G^\pm_{m+r},\\
&[J_m, G^\pm_r] = \pm G^\pm_{m+r}.
\end{aligned} \tag{3.34}$$

And from unitarity, we define adjoint operators by

$$(J_n)^\dagger = J_{-n}, \quad (L_n)^\dagger = L_{-n}, \quad (G_r^\pm)^\dagger = G_{-r}^\mp. \tag{3.35}$$

We can denote the $(2,2)$ Super Conformal algebra as

$$(2,2)\ \text{SCA} := S(2,2) \oplus \overline{S}(2,2) \tag{3.36}$$

with $\left\{S, \overline{S}\right\}_{\mathbb{Z}_2} = 0$ and Vir $\subset S(2,2)$. Moreover, just as the case for Virasoro algebra, these generators can be regarded as the mode coefficients of (super) currents, i.e., we can construct the superconformal algebra from the expansion of the energy-momentum tensor T, two supercurrents G^+ and G^-, and a $U(1)$ current J which forms a closed algebra with them:

$$T(z) = \sum_{n\in\mathbb{Z}} \frac{L_n}{z^{n+2}}, \tag{3.37}$$

$$G^\pm(z) = \sum_{m\in\mathbb{Z}+\nu} \frac{G_m}{z^{m+3/2}}, \tag{3.38}$$

$$J(z) = \sum_{n\in\mathbb{Z}} \frac{L_n}{z^{n+1}}, \tag{3.39}$$

in which $\nu = 0$ for the Ramond sector and $\nu = 1/2$ for the Neveu-Schwarz sector.

And the graded commutators among the generators of $S(2,2)$ are encoded in the OPEs.

$$T(z)T(0) \sim \frac{c/2}{z^4} + \frac{2T(0)}{z^2} + \frac{\partial T(0)}{z}, \tag{3.40}$$

$$J(z)J(0) \sim \frac{c/3}{z^2}, \tag{3.41}$$

$$T(z)J(0) \sim \frac{J(0)}{z^2} + \frac{\partial J(0)}{z}, \tag{3.42}$$

$$G^\pm(z)G^\pm(0) \sim 0, \tag{3.43}$$

$$G^+(z)G^-(0) \sim \frac{2c/3}{z^3} + \frac{2J(0)}{z^2} + \frac{2T(0) + \partial J(0)}{z}, \tag{3.44}$$

$$T(z)G^\pm(0) \sim \frac{3G^\pm(0)/2}{z^2} + \frac{\partial G^\pm(0)}{z}, \tag{3.45}$$

$$J(z)G^\pm(0) \sim \frac{\pm G^\pm(0)}{z}. \tag{3.46}$$

3.2.2 Chiral Rings

Let's take a moment off and have a brief introduction to chiral rings in $(2,2)$ theories, not only for conformal theories. Recall that there are two kinds of supercharge combinations: A-type and B-type. We can verify that either $Q = Q_A$ or $Q = Q_B$, it is nilpotent, i.e.,

$$Q^2 = 0, \tag{3.47}$$

provided the central charges vanish $Z = \widetilde{Z} = 0$.

Naturally, due to the nilpotency, we can think of defining the Q-cohomology

$$H_Q := \frac{\operatorname{Ker} Q}{\operatorname{Im} Q}, \quad Q = Q_A \,\text{or}\, Q_B. \tag{3.48}$$

One can define the chiral operator $\mathcal{O}_{\text{chiral}}$ to be Q_B closed operators

$$[Q_B, \mathcal{O}_{\text{chiral}}] = 0, \tag{3.49}$$

and the twisted chiral operator $\mathcal{O}_{\text{twisted-chiral}}$ to be Q_A closed operators

$$[Q_A, \mathcal{O}_{\text{twisted-chiral}}] = 0. \tag{3.50}$$

For instance, the lowest component ϕ of a chiral superfield $\varPhi$ is annihilated by $\overline{Q}_\pm$ and thus a chiral operator. Similarly, the lowest component σ of a twisted chiral superfield $\varSigma$ is a twisted chiral operator.

We then immediately find that there is a ring structure of Q-cohomology class operators, since for Q-closed operators $\mathcal{O}_1$ and $\mathcal{O}_2$, their product $\mathcal{O}_1\mathcal{O}_2$ remains Q-closed. Such a ring is called the chiral ring for $Q = Q_B$ and the twisted chiral ring for $Q = Q_A$.

So far, we are considering flat worldsheets, where a constant section of the spin bundle always exists, and we can use it to construct the supersymmetry transformation parameters $\epsilon_\pm$ and $\overline{\epsilon}_\pm$. Now, we relax the worldsheet to a curved Riemann surface with general topologies, which forces us to perform a topological twist.

We just briefly summarize the impact of the two twists on chiral superfields with trivial R-charges, especially for the spinor component, since the scalars are not affected under twisting.

$$\varPhi = \phi + \theta^+\psi_+ + \theta^-\psi_- + \cdots. \tag{3.51}$$

For A-twist theories, the fermions that become scalars under the modified Lorentz group are ψ_- and $\overline{\psi}_+$.

And for B-twisted theories, they are $\overline{\psi}_-$ and $\overline{\psi}_+$.

It comes to our attention that the above fermions are scalars with Grassmann odd values. Thus, the philosophy in relating physical operators and cohomology classes is regarding the products of these fermions as wedge products of differential forms or tangent vectors. However, we should examine the supersymmetry transformation of all of the operators.

We first focus on the A-twist of Non-linear Sigma Models. Part of the supersymmetry variation for the generator $\widehat{\delta} = \overline{\epsilon}_-\overline{Q}_+ + \epsilon_+ Q_-$ reads

$$\delta\phi^i = \epsilon_+\psi_-^i, \quad \delta\psi_-^i = 0, \tag{3.52}$$

$$\delta\overline{\phi}^i = \overline{\epsilon}_-\overline{\psi}_+^i, \quad \delta\overline{\psi}_+^i = 0. \tag{3.53}$$

Thus, one can perform such an identification

$$\psi_-^i \leftrightarrow \mathrm{d}z^i \quad \text{and} \quad \overline{\psi}_+^{\bar{i}} \leftrightarrow \mathrm{d}\overline{z}^{\bar{i}}, \tag{3.54}$$

with the differential operators identified as

$$Q_- \leftrightarrow \partial \quad \text{and} \quad \overline{Q}_+ \leftrightarrow \overline{\partial}. \tag{3.55}$$

We can check that

$$\left\{Q_-, \omega_{i,\bar{j}}(\phi,\overline{\phi})\,\psi_-^i\overline{\psi}_+^{\bar{j}}\right\} = \frac{\partial\omega_{i,\bar{j}}}{\partial\phi_k}\psi_-^i\overline{\psi}_+^{\bar{j}}\psi_-^k. \tag{3.56}$$

and its $\overline{Q}_+$ counterpart. Such an expression precisely aligns with the action of ∂ on a differential form

$$\partial\Big(\omega_{i,\bar{j}}(z,\overline{z})\mathrm{d}z^i\mathrm{d}\overline{z}^{\bar{j}}\Big) = \frac{\partial\omega_{i,\bar{j}}(z,\overline{z})}{\partial z^k}\mathrm{d}z^i\mathrm{d}\overline{z}^{\bar{j}}\mathrm{d}z^k. \tag{3.57}$$

Thus, for A-twist NLSMs with target space X, there is a one-to-one correspondence,

$$\text{Physical Operators in } H_{Q_A} \cong H^*_{dR}(X). \tag{3.58}$$

Now, we turn to the B-twisted NLSMs with a Calabi-Yau manifold M as their target. As mentioned before, performing a B-type topological twist requires the Calabi-Yau condition.

It will be convenient for later analysis to define

$$\overline{\psi}_-^{\bar{i}} + \overline{\psi}_+^{\bar{i}} = -\eta^{\bar{i}}, \quad \overline{\psi}_-^{\bar{i}} - \overline{\psi}_+^{\bar{i}} = g^{\bar{i}j}\theta_j, \tag{3.59}$$

where $g^{\bar{i}j}$ is the Kähler metric associated with the NLSM. Now, part of the supersymmetry variation for the generator $\widehat{\delta} = \overline{\epsilon}Q_B$ reads

$$\delta\phi^i = 0, \quad \delta\theta_i = 0, \tag{3.60}$$

$$\delta\overline{\phi}^i = \overline{\epsilon}\eta^{\bar{i}}, \quad \delta\eta^{\bar{i}} = 0. \tag{3.61}$$

The space of physical operators can be identified with

$$\eta^{\bar{i}} \leftrightarrow \mathrm{d}\overline{z}^{\bar{i}} \quad \text{and} \quad \theta_i \leftrightarrow \frac{\partial}{\partial z^i}, \tag{3.62}$$

while the supercharge Q_B is identified to the anti-holomorphic differential $\overline{\partial}$.

We can also make a consistency check.

$$\left\{Q_B, \omega_{\bar{i}}^{j}(\phi,\overline{\phi})\,\eta^{\bar{i}}\theta_j\right\} = \frac{\partial\omega_{i,\bar{j}}}{\partial\overline{\phi}_{\bar{k}}}\eta^{\bar{i}}\theta_j\eta^{\bar{k}}, \tag{3.63}$$

which precisely aligns with the action of $\overline{\partial}$ on a poly-vector field $\Omega^{0,p}(M,\wedge^q TM)$.

$$\overline{\partial}\Big(\omega_{\bar{i}}^{j}(z,\overline{z})\mathrm{d}z^i\mathrm{d}\overline{z}^{\bar{j}}\Big) = \frac{\partial\omega_{\bar{i}}^{j}(z,\overline{z})}{\partial\overline{z}_{\bar{k}}}\mathrm{d}\overline{z}^{\bar{i}}\mathrm{d}\overline{z}^{\bar{k}}\frac{\partial}{\partial z^j}. \tag{3.64}$$

Thus, for a B-twist NLSM with its target space a Calabi-Yau manifold M, there is also a one-to-one correspondence between the Q_B cohomology and the Dolbeault cohomology groups.

$$\text{Physical Operators in } H_{Q_B} \cong \bigoplus_{p,q=0}^{\dim_{\mathbb{C}} M} H^{0,p}(M,\wedge^q TM). \tag{3.65}$$

3.2.3 Chiral Primaries

We would also like to introduce a subset of fields in $2d$ $(2,2)$ SCFTs called the chiral primary fields[25].

Following the notation in[24] and working in the Neveu-Schwartz sector, we first define a primary state $|\Psi\rangle$ with respect to the $(2,2)$ superconformal algebra as

$$G_r^{\pm}|\Psi\rangle = 0, \quad r \geq \frac{1}{2}, \tag{3.66}$$
$$J_n|\Psi\rangle = L_n|\Psi\rangle = 0, \quad n \geq 1. \tag{3.67}$$

In fact, the second equation can be deduced from the first condition by unitarity and (anti-)commutation relations. As we did in the $2d$ CFTs, we can diagonalize the primary states in L_0 and J_0, and label them by their eigenvalues, conformal weight h and $U(1)$ charge q.

$$L_0|\Psi\rangle = h|\Psi\rangle, \quad J_0|\Psi\rangle = q|\Psi\rangle. \tag{3.68}$$

Then, a chiral primary state $|\Phi\rangle$ is defined to be annihilated by one more supercharge mode

$$G^{+}_{-1/2}|\Phi\rangle = 0. \tag{3.69}$$

In the operator product language, it is written as

$$G^{+}(z)\Phi(w) \sim \text{regular}. \tag{3.70}$$

Similarly, we can define anti-chiral primaries by

$$G^{-}_{-1/2}|\Phi\rangle = 0. \tag{3.71}$$

Moreover, we should also take into account the anti-holomorphic counterpart of the above arguments, i.e., we can define the chiral primaries and anti-chiral primaries in the sense of anti-holomorphic sectors.

$$\overline{G}^{+}_{-1/2}|\Phi\rangle = 0, \tag{3.72}$$
$$\overline{G}^{-}_{-1/2}|\Phi\rangle = 0. \tag{3.73}$$

Now we focus on the holomorphic sector, from the anti-commutation relations, as well as the unitary condition, we have

$$0 \leq \langle\Psi|\left\{G^{\pm}_{1/2}, G^{\mp}_{-1/2}\right\}|\Psi\rangle = \langle\Psi|2L_0 \pm J_0|\Psi\rangle. \tag{3.74}$$

For $|\Psi\rangle$ a chiral primary, the left hand side vanishes by definition, and hence we obtain that

$$h = \frac{1}{2}q \tag{3.75}$$

for a chiral primary field, and

$$h = -\frac{1}{2}q \tag{3.76}$$

for anti-chiral primaries.

The chiral primaries also form a ring called the chiral primary ring. In fact, there are four rings, (c, c), (a, c), (c, a), and (a, a). One should note that there are only two independent rings, since other rings can be obtained by conjugation. Let's take (c, c) and (a, c) as independent rings and explain why we call them like that. First, c stands for chiral and a stands for anti-chiral. Recall that there are holomorphic sectors and anti-holomorphic sectors, given by $G^{\pm}$ and $\overline{G}^{\pm}$. We can respectively define chirals and anti-chirals in these two sectors, and write the holomorphic one in the first place and the anti-holomorphic one in the second.

We have to mention that the notation in this subsection does not precisely align with previous chapters. For instance, the chiral and anti-chiral are distinguished by G^+ and G^-, while in the $(2, 2)$ language, they are characterized by Q and $\overline{Q}$. Specifically, we have

$$\begin{aligned} Q_+ &\leftrightarrow \overline{G}^-_{-1/2}, \quad Q_- \leftrightarrow G^-_{-1/2}, \\ \overline{Q}_+ &\leftrightarrow \overline{G}^+_{-1/2}, \quad \overline{Q}_- \leftrightarrow G^+_{-1/2}. \end{aligned} \tag{3.77}$$

With such an identification, we can see that the chiral rings $\mathcal{R}_A$ and $\mathcal{R}_B$ in the $(2, 2)$ language correspond to the chiral rings (a, c) and (c, c) in the SCA language. For example, we consider the $\mathcal{R}_B$ ring, where the state $|\phi\rangle$ is annihilated by Q_B and also has a definite R-charge. By assuming the F_A is conserved, we have

$$\mathrm{e}^{-\mathrm{i}\alpha F_A} Q_B \mathrm{e}^{\mathrm{i}\alpha F_A} |\phi\rangle = 0, \tag{3.78}$$

which results in

$$G^+_{-1/2} |\phi\rangle = \overline{G}^+_{-1/2} |\phi\rangle = 0, \tag{3.79}$$

the chiral condition in the SCA. For simplicity, now we only consider the holomorphic part, and there is a statement that in an SCFT, a chiral field can be decomposed as

$$|\phi\rangle = |\phi_0\rangle + G^+_{-1/2} |\phi_1\rangle, \tag{3.80}$$

where $|\phi_0\rangle$ is a chiral primary and $|\phi_1\rangle$ is an arbitrary field. Thus, there is an isomorphism between R_B and (c, c).

3.2.4 Deformations

For a given scale-invariant QFT, we can deform it by adding local operators, and we call such a deformation irrelevant, marginal, or relevant if the dimension of the added operator Δ is greater than, equal to, or less than the spacetime dimension d. A relevant deformation grows along the RG flow, while an irrelevant deformation decays. For a marginal deformation, it depends on the specific operator, and we can define it to be exactly marginal if it remains invariant under the RG flow. The moduli space of conformal field theories is coordinatized by exactly marginal deformation parameters.

In $2d$ CFTs, the scaling dimension of an operator of conformal weight $(h, \overline{h})$ is $\Delta = h + \overline{h}$, while the spin is $s = h - \overline{h}$. Thus, a marginal operator should satisfy $h + \overline{h} = 2$. We further restrict on spinless operators, where they have conformal weights $(h, \overline{h}) = (1, 1)$. Then we claim that these operators can deform a given conformal field theory, and the deformation generates a family of CFTs continuously related to each other on different points of the moduli space.

However, as we mentioned above, not every marginal operator deforms the CFT in a manner we expect, i.e., they are marginal relevant or marginal irrelevant. For instance, some marginal operators in the original theory won't keep marginal after deformation. We need to specify the set of exact marginal operators. By superconformal Ward Identity arguments[25–26], one can show that two kinds of operators fall into the exact marginal region.

First, we start from an element ϕ in the (c, c) ring, with conformal weight $(h, \overline{h}) = (1/2, 1/2)$, and $U(1)$ charges $(q, \overline{q}) = (1, 1)$. We can construct an intermediate operator $\hat{\phi}$ by

$$\hat{\phi}(w, \overline{w}) = \oint \mathrm{d}z G^{-}(z)\phi(w, \overline{w}). \tag{3.81}$$

And then, we construct

$$\varPhi_{(1,1)}(w, \overline{w}) = \oint \mathrm{d}\overline{z}\overline{G}^{-}(\overline{z})\hat{\phi}(w, \overline{w}). \tag{3.82}$$

One can verify that such an operator $\varPhi_{(1,1)}$ satisfies $(h, \overline{h}) = (1, 1)$ and $(q, \overline{q}) = (0, 0)$ by using commutation relations.

The second exact marginal operator can be constructed from an element of the (a, c) ring, with $(h, \overline{h}) = (1/2, 1/2)$ and $(q, \overline{q}) = (-1, 1)$. Then, we similarly first construct

$$\hat{\phi}(w, \overline{w}) = \oint \mathrm{d}\overline{z}\overline{G}^{-}(\overline{z})\phi(w, \overline{w}), \tag{3.83}$$

and

$$\varPhi_{(-1,1)}(w, \overline{w}) = \oint \mathrm{d}z G^{+}(z)\hat{\phi}(w, \overline{w}). \tag{3.84}$$

By the above construction, we can see that $\varPhi_{(1,1)}$ has $(h, \overline{h}) = (1, 1)$ and $(q, \overline{q}) = (0, 0)$.

We thus conclude that the deformation of $2d$ $(2, 2)$ SCFTs is generated either by $(q, \overline{q}) = (1, 1)$ elements in the (c, c) ring, or by $(q, \overline{q}) = (-1, 1)$ elements in the (a, c) ring.

One can also consider deformation operators in a $2d$ $(2, 2)$ QFTs, not necessarily conformal theories. From the constraints of supersymmetry invariance, there are three types of deformations[18].

$$\varDelta_D \mathcal{L} = Q_+ Q_- \overline{Q}_- \overline{Q}_+ \mathcal{K}, \tag{3.85}$$

$$\varDelta_A \mathcal{L} = Q_+ \overline{Q}_- \mathcal{O}_A \text{ and its adjoint}, \tag{3.86}$$

$$\varDelta_B \mathcal{L} = Q_+ Q_- \mathcal{O}_B \text{ and its adjoint}. \tag{3.87}$$

Here $\mathcal{K}$ is an arbitrary superfield, while $\mathcal{O}_A$ is twisted-chiral and $\mathcal{O}_B$ is chiral. They are called D-term deformations, twisted F-term deformations, and F-term deformations. From a cohomological point of view, we can see that twisted F-terms modulo D-terms are $\mathcal{R}_A$, and F-terms modulo D-terms are $\mathcal{R}_B$.

One can compare (3.85) with the above SCFT constructions. Especially,

$$\varPhi_{(1,1)} \leftrightarrow \varDelta_B \mathcal{L} = Q_+ Q_- \mathcal{O}_B, \tag{3.88}$$

$$\varPhi_{(-1,1)} \leftrightarrow \varDelta_A \mathcal{L} = Q_+ \overline{Q}_- \mathcal{O}_A. \tag{3.89}$$

They come from the same construction in the $(2,2)$ supersymmetry language. We expect that the charge constraint on ϕ's can also be derived from the R-invariance of the action.

There are two kinds of exactly marginal deformations for a $(2,2)$ SCFT, and it's natural to ask how these two operators parametrize the moduli space. In fact, there is a natural metric on the moduli space G called the Zamolodchikov metric[27]. And we can block-diagonalize the tangent space of the moduli space for $\Phi_{(1,1)}$ and $\Phi_{(-1,1)}$ operators. As a consequence, at least locally, we can regard the SCFT moduli space as a product of two sectors, where we can independently perform either type of deformation.

We further consider a concrete example of $2d$ $\mathcal{N}=(2,2)$ SCFTs with a geometric background: A superconformal non-linear sigma model with a Calabi-Yau manifold M as its target.

As mentioned above, there should be two types of deformation of the Calabi-Yau NLSM, corresponding to $\Phi_{(1,1)}$ and $\Phi_{(-1,1)}$, which are generated from a $(q,\overline{q})=(1,1)$ element of the (c,c) ring and a $(q,\overline{q})=(-1,1)$ element of the (a,c) ring. Since we are working in the framework of Calabi-Yau NLSMs, it's better to seek a geometric interpretation of these two types of exact marginal operators.

We claim that the (c,c) ring and the (a,c) ring defined in the last subsection align with the chiral rings $\mathcal{R}_B$ and twisted chiral rings $\mathcal{R}_A$ we mentioned before, since they are Q_B-closed by chiral, Q_A closed by anti-chiral, and can't be Q-exact by primary. Recall that there is a map between chiral/twisted-chiral rings and physical operators, where we map the $\mathcal{R}_A$ ring to the differential forms and the $\mathcal{R}_B$ ring to the poly vector fields. One can further grade the map by the R-charges, by

$$\text{For } (c,c) \text{ ring} \quad (q,\overline{q}) \leftrightarrow \Omega^{0,\overline{q}}(M,\wedge^q TM), \tag{3.90}$$

$$\text{For } (a,c) \text{ ring} \quad (q,\overline{q}) \leftrightarrow \Omega^{-q,\overline{q}}(M). \tag{3.91}$$

Surprisingly, we find that for a $(1,1)$ deformation generated from the (c,c) ring, there is a corresponding element in $H^{0,1}_{\overline{\partial}}(M,TM)$, and a $(-1,1)$ deformation generated from the (a,c) ring, there is a corresponding element in $H^{1,1}_d(M)$. By contracting with the Calabi-Yau form Ω, we have that two types of exact marginal deformation, $\Phi_{(1,1)}$ and $\Phi_{(-1,1)}$, correspond to

$$H^{n-1,1}(M) \quad \text{and} \quad H^{1,1}(M). \tag{3.92}$$

These are exactly the moduli space of complex deformation and Kähler deformation of a Calabi-Yau manifold M! Such constructions establish that the geometric meaning of two types of conformal field theory operators corresponds to exact marginal deformations.

Intuitively, we can draw the moduli space of both $(2,2)$ SCFTs and the corresponding ones for Calabi-Yau manifolds.

In[23], there is a very concise example that illustrates the effects of complex and Kähler deformation, in a case where we consider a one-dimensional Calabi-Yau manifold $T^2=\mathbb{C}/(\mathbb{Z}\oplus\tau\mathbb{Z})$. A Kähler deformation only changes the volume and leaves the shape intact, while a complex deformation changes the shape and leaves the volume fixed.

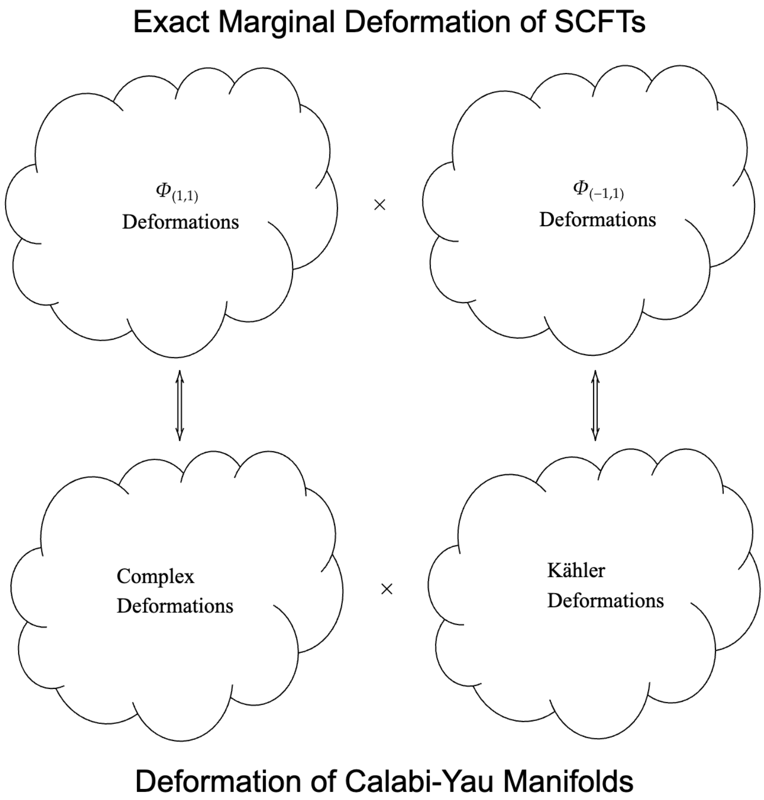


Figure 3-1 Moduli Space of SCFTs and Calabi-Yau's

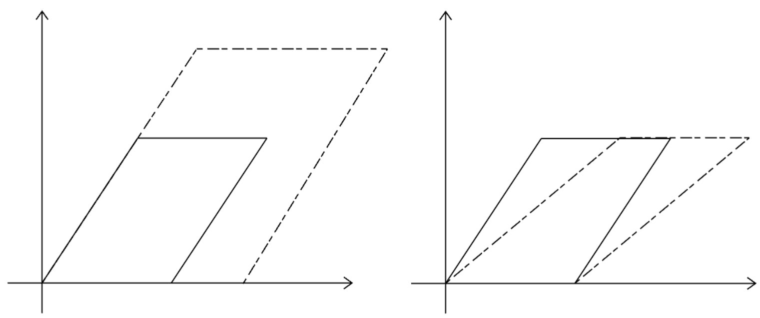


Figure 3-2 Kähler Deformation and Complex Deformation of T^2

One can recall the computation for the quintic Calabi-Yau GLSM in Chapter. 2, where there is a twisted superpotential term given by

$$\widetilde{W}(x) = \frac{\mathrm{i}tx}{2\sqrt{2}}. \tag{3.93}$$

It's a Q_A closed operator and thus can be regarded as a twisted F-term deformation, which results in the target Calabi-Yau as a Kähler deformation, and the complex coupling t corresponds to the deformation parameter. One can further check that the coupling $t = \mathrm{i}r + \frac{\theta}{2\pi}$ went into the D-term equation, or more specifically, the FI parameter r determined the "radius" of the sphere that becomes the projective space via Hopf fibration.

$$D = -e^2\left(\sum_i Q_i|\phi_i|^2 - r\right). \tag{3.94}$$

3.2.5 Mirror Symmetry Revisit

We are interested in worldsheet theories with $\mathcal{N} = (2,2)$ superconformal symmetry, as they provide the playground in superstring compactification, especially for the heterotic strings, as the resulting theory has space-time supersymmetry.

We can consider a $(2,2)$ non-linear sigma model with a Calabi-Yau target, which gives a class of string vacua. However, there is an ambiguity that for a given physical theory, we can't uniquely construct the corresponding target manifold, i.e., there are pairs of Calabi-Yau's that lead to the physically equivalent theory.

In particular, we should examine how geometric rings (cohomology) on the target manifold get mapped to rings of physical operators. The cohomology ring of the manifold is associated with a ring of primary operators in the field theory in the large radius limit [28]. Nevertheless, we've seen that there are two kinds of rings that could be our candidates: The (c,c) ring and the (a,c) ring. Note that these two rings are exchanged by exchanging the supercharges $Q_A \leftrightarrow Q_B$, and consequently, the $U(1)_V \leftrightarrow U(1)_A$.

Then the mirror symmetry (conjecture) states that if the cohomology ring of the target Calabi-Yau M is identified with the (c,c) ring, then there is a manifold $\widetilde{M}$ called the mirror of M, whose cohomology ring can be identified with the (a,c) ring. Furthermore, we consider physical operators with $U(1)_L \times U(1)_R$ charges (p,q) in the (c,c) ring. After exchanging $U(1)_A \leftrightarrow U(1)_V$ and the supercharges, they become elements in the (a,c) ring with new $U(1)_L \times U(1)_R$ charges $(-p,q)$. As argued before, elements in the (c,c) ring with charges (p,q) correspond to

$$H^{0,q}_{\bar\partial}(M, \wedge^p TM) \cong H^{d-p,q}(M), \quad \text{with}\, d = \dim_{\mathbb{C}} M \tag{3.95}$$

while elements in the (a,c) ring with charges $(-p,q)$ correspond to

$$H^{p,q}(\widetilde{M}). \tag{3.96}$$

Since they can be regarded as the same ring in the physical theory, up to $Q_A \leftrightarrow Q_B$ and $U(1)_V \leftrightarrow U(1)_A$, we conclude that for M and its mirror $\widetilde{M}$, the Hodge numbers are related.

$$\tilde{h}^{p,q} = h^{d-p,q}. \tag{3.97}$$

For Calabi-Yau manifolds, there are two kinds of important deformations that correspond to the Kähler structure and the complex structure, with their moduli space of dimension $h^{1,1}$ and $h^{d-1,1}$. We immediately discover that these two moduli spaces are exchanged under the mirror map, i.e., the Kähler moduli space of M becomes its mirror's complex moduli space and vice versa.

3.3 Super Manifolds as Mirrors

The argument of mirror symmetry in the previous section is a powerful tool for us to establish relations between different and unexpected manifolds. Specifically, for a Calabi-Yau manifold M, we can seek its mirror $\widetilde{M}$ whose Hodge diamond is flipped $\widetilde{h}^{p,q} = h^{d-p,q}$.

However, there exists a long-standing obstruction preventing us from generalizing the concept of mirror symmetry to all Calabi-Yau manifolds. One can consider a family of Calabi-Yau's that is rigid, i.e., Calabi-Yau manifolds that do not admit any complex deformation and thus without any complex structure moduli. Mirror symmetry, as we pointed out before, exchanges the Kähler moduli and the complex moduli. Then, one can see that the mirror of a rigid Calabi-Yau does not possess any Kähler moduli space. We can see that any Kähler manifold should admit at least one deformation to the Kähler structure, which is simply scaling it by a factor. Thus, the mirror manifold can not be Kähler in a conventional sense. In fact, such a mirror of a rigid manifold can usually be described as a Landau-Ginzburg model, but such an LG model does not admit a geometric interpretation via sigma models.

To overcome this issue, Sethi proposed that one could enlarge the space of Calabi-Yau manifolds and include Calabi-Yau supermanifolds[28], i.e., strings that propagate on a rigid manifold can be equivalently described via mirror symmetry by strings propagating on a supermanifold. Thus, we can associate a sigma model interpretation of the mirror Landau-Ginzburg model, which means we can find a UV GLSM, whose low-energy theory contains both the LG phase and a sigma model phase.

3.3.1 Examples: Superprojective Spaces and One-loop Running

In this part, we generalize the concept of projective spaces into supermanifolds, which are manifolds with Grassmann odd coordinates. Similar to the definition of the projective spaces, we first write down the homogeneous coordinates for the superprojective space $\mathbb{CP}^{n|m}$ by

$$(z^1, \cdots, z^{n+1} | \theta^1, \cdots, \theta^m), \tag{3.98}$$

with the quotient given by the equivalence relations of the $\mathbb{C}^*$ action λ

$$z^i \sim \lambda z^i, \quad \theta^i \sim \lambda \theta^i. \tag{3.99}$$

Similarly, we can define the weighted superprojective space $\mathbb{WP}^{n|m}_{[Q_1,\cdots,Q_{n+1}|\widetilde{Q}_1,\cdots,\widetilde{Q}_m]}$ by

$$z^i \sim \lambda^{Q_i} z^i, \quad \theta^i \sim \lambda^{\widetilde{Q}_i} \theta^i. \tag{3.100}$$

Note that such a superprojective space has a different geometry than its bosonic base projective space, as the bosonic coordinates could all vanish, while some of the fermionic coordinates remain non-zero.

We would like to construct a $G = U(1)$ GLSM corresponding to these superweighted projective spaces and investigate the one-loop running pattern of such super sigma models. For instance, we consider GLSMs for $\mathbb{WP}^{n|m}_{[Q_1,\cdots,Q_n|\widetilde{Q}_1,\cdots,\widetilde{Q}_m]}$ [12,29]. There are two kinds of chiral multiplets:

- $n+1$ Grassmann even chirals Φ_i, $i = 1, \cdots, n+1$, with $U(1)$ gauge charges Q_i and $U(1)_V$ R-charges R_i.

- m Grassmann odd chirals $\widetilde{\Phi}_\mu$, $\mu = 1, \cdots, m$, with $U(1)$ gauge charges $\widetilde{Q}_\mu$ and $U(1)_V$ R-charges $\widetilde{R}_\mu$.

We refer to the Grassmann odd chirals as chiral superfields with values in Grassmann numbers. In the multiplet $(\tilde{\phi}_\mu, \widetilde{\psi}^\pm_\mu)$, the scalar $\tilde{\phi}_\mu$ and spinor $\widetilde{\psi}^\pm_\mu$ have opposite spin-statistics to the ordinary case, and thus we sometimes call it a ghost multiplet.

By adding Grassmann odd multiplets, only the Lagrangian for the chiral superfields is modified, and now consists of two parts

$$\mathcal{L}_{\text{kin}} = \mathcal{L}^{\text{even}}_{\text{kin}} + \mathcal{L}^{\text{odd}}_{\text{kin}} \tag{3.101}$$

$$= \int \mathrm{d}^4\theta \sum_i \overline{\Phi}_i \mathrm{e}^{2Q_i V} \Phi_i + \int \mathrm{d}^4\theta \sum_\mu \overline{\widetilde{\Phi}}_\mu \mathrm{e}^{2\widetilde{Q}_\mu V} \widetilde{\Phi}_\mu, \tag{3.102}$$

while leaving

$$\mathcal{L}_{\text{gauge}} \quad \text{and} \quad \mathcal{L}_{\text{F-I},\theta} \tag{3.103}$$

intact.

Now we examine the changes for the D term.

$$D = -e^2 \left(\sum_i Q_i |\phi_i|^2 + \sum_\mu \widetilde{Q}_\mu \overline{\tilde{\phi}}_\mu \tilde{\phi}_\mu - r \right). \tag{3.104}$$

Here, ϕ_i is the lowest component of Φ_i, and $\tilde{\phi}_\mu$ is the lowest component of $\widetilde{\Phi}_\mu$. At this moment, we only consider the case where $W = 0$, and thus the corresponding F term equation is trivial.

To extract the low-energy behaviors of super GLSMs, we also like to write down the potential energy for the scalar fields ϕ_i's, ghost fields $\tilde{\phi}_\mu$'s and σ field.

$$U(\phi_i, \sigma) = \frac{1}{2e^2} D^2 + 2|\sigma|^2 \left(\sum_i Q_i^2 |\phi_i|^2 + \sum_\mu \widetilde{Q}_\mu^2 \overline{\tilde{\phi}}_\mu \tilde{\phi}_\mu \right). \tag{3.105}$$

By classical analysis $U = 0$, we can similarly extract the low-energy behavior of the system. One can first work in the $r >> 0$ phase and consider the Higgs branch, where $\sigma = 0$ and thus we have the D-term equation.

$$\sum_i Q_i |\phi_i|^2 + \sum_\mu \widetilde{Q}_\mu \overline{\tilde{\phi}}_\mu \tilde{\phi}_\mu - r = 0. \tag{3.106}$$

Finally, we divide the D-term equation, which is an equation of a super sphere, by the $(Q_1, \cdots, Q_{n+1} | \widetilde{Q}_1, \cdots, \widetilde{Q}_m)$ representation of the gauge group $U(1)$, and the resulting target space X is thus given by the super toric variety, which is actually the super weighted projective space $\mathbb{WP}^{n|m}_{[Q_1, \cdots, Q_{n+1} | \widetilde{Q}_1, \cdots, \widetilde{Q}_m]}$.

$$X \simeq \frac{\mathbb{C}^{n+1|m} - \Delta}{\mathbb{C}^*} = \mathbb{WP}^{n|m}_{[Q_1, \cdots, Q_{n+1} | \widetilde{Q}_1, \cdots, \widetilde{Q}_m]}. \tag{3.107}$$

We are especially interested in the one-loop running behavior of the model, and looking for the condition of conformal invariance.

Similarly, the contribution of the one-loop running of the D term, or equivalently, the one-loop running of two-point functions of $|\phi_i|^2$ and $\overline{\widetilde{\phi}}_\mu\widetilde{\phi}_\mu$, comes from the $Q^2|\phi|^4$ interaction term in the Lagrangian. For ϕ_i fields, we've already computed the renormalization

$$-\frac{1}{e^2}\langle D\rangle^{\text{even}}_{\text{1-loop}} = \frac{1}{4\pi}\sum_i Q_i \log\left(\frac{\Lambda^2}{2Q_i^2|\sigma|^2}\right). \tag{3.108}$$

For the ghost field, as we've learned from Feynman rules, a fermionic loop will result in an overall -1 factor for the diagram. Consequently, we have the one-loop running

$$-\frac{1}{e^2}\langle D\rangle^{\text{odd}}_{\text{1-loop}} = -\frac{1}{4\pi}\sum_\mu \widetilde{Q}_\mu \log\left(\frac{\Lambda^2}{2\widetilde{Q}_\mu^2|\sigma|^2}\right). \tag{3.109}$$

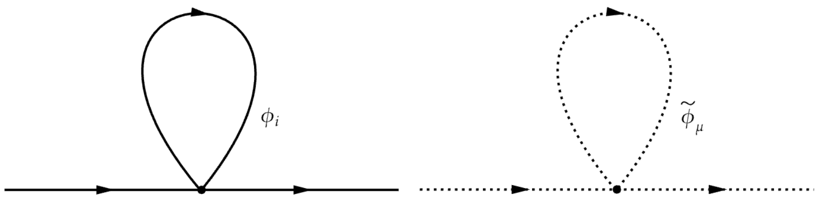


Figure 3-3 One-loop ϕ^4 Renormalization of the D-term from ϕ_i and $\tilde{\phi}_\mu$

We can absorb the shift of the D-term into the effective FI parameter r_{eff}, which is given by

$$r_{\text{eff}} = r - \frac{1}{4\pi}\left(\sum_i Q_i \log\left(\frac{\Lambda^2}{2Q_i^2|\sigma|^2}\right) - \sum_\mu \widetilde{Q}_\mu \log\left(\frac{\Lambda^2}{2\widetilde{Q}_\mu^2|\sigma|^2}\right)\right). \tag{3.110}$$

With the corresponding shift in θ, the holomorphic complex coupling t is shifted to

$$t_{\text{eff}} = t + \frac{\mathrm{i}}{2\pi}\left(\sum_i Q_i \log\left(\frac{\sqrt{2}Q_i\sigma}{\Lambda}\right) - \sum_\mu \widetilde{Q}_\mu \log\left(\frac{\sqrt{2}\widetilde{Q}_\mu\sigma}{\Lambda}\right)\right). \tag{3.111}$$

By choosing a proper physical scale μ, one can compute the beta function of the one-loop renormalization.

$$\beta = \frac{\partial r_{\text{eff}}}{\partial \log\mu} \propto \sum_i Q_i - \sum_\mu \widetilde{Q}_\mu. \tag{3.112}$$

We then observe that for theories with charges satisfying

$$\sum_i Q_i - \sum_\mu \widetilde{Q}_\mu = 0, \tag{3.113}$$

the theory has zero beta function and also the r_{eff} is independent of σ and Λ. Such a scale-invariant theory is believed to be conformal invariant for the bosonic case and the metric will flow to a non-trivial Ricci flat metric under the RG due to Yau's theorem.

The analysis above shows that a superchiral field $\widetilde{\Phi}_\mu$ with gauge charge $\widetilde{Q}_\mu$ behaves like a chiral field with a negative charge $-Q_\mu$. Indeed, in[30] the author checked that a topological A-model on an $(n|m)$ supermanifold aligns with an A-model with an $(n-m)$-dimensional manifold as its target space.

3.3.2 Calabi-Yau Supermanifolds

As we mentioned before, Shing-Tung Yau proved[21–22] the Calabi conjecture for ordinary (bosonic) manifolds, that is, the vanishing first Chern class is necessary and sufficient for the existence of a Ricci flat metric.

For the supermanifold case, there is no such theorem, and we have to be careful about the analysis. We follow[28,31] to generalize the spirit of Calabi-Yau to supermanifolds.

First, let's review the bosonic case, where we focus on an ordinary Kähler manifold with its Kähler potential K, and the Ricci tensor reads

$$R_{i\bar{j}} = (\ln\det(g))_{,i\bar{j}}, \tag{3.114}$$

where

$$\det(g) = \det\left(\overline{\partial}\partial K\right) \tag{3.115}$$

obeys the Monge-Ampere equation $\det\left(\overline{\partial}\partial K\right) = 1$ for Calabi-Yau manifolds. However, for supermanifolds, due to the existence of the fermionic coordinates, the crossing components

$$\overline{\partial}^b\partial^f K \quad \text{and} \quad \overline{\partial}^f\partial^b K \tag{3.116}$$

are fermionic, i.e., the supermetric

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{3.117}$$

has fermionic blocks B and C. We adopt a new definition of the super determinant of g as

$$\mathrm{sdet}(g) := \frac{\det(A)}{\det(D - CA^{-1}B)} = \frac{\det(A - BD^{-1}C)}{\det(D)}, \tag{3.118}$$

with the super trace given by

$$\mathrm{str}(g) := \mathrm{tr}(A) - \mathrm{tr}(D). \tag{3.119}$$

Now we can define the potential of the super Ricci tensor for a supermanifold by

$$\ln\mathrm{sdet}(g) = \ln\det\left(K_{i\bar{j}} - K_{i\bar{\mu}}K^{\bar{\mu}\nu}K_{\nu\bar{j}}\right) - \ln\det\left(K_{\mu\bar{\nu}}\right). \tag{3.120}$$

Here, the index takes its value as $\alpha = i$ for bosonic coordinates and $\alpha = \mu$ for fermionic coordinates, and we use the notation that $K_{\alpha\bar{\beta}} = g_{\alpha\bar{\beta}} = \partial_\alpha\overline{\partial}_{\bar{\beta}}K$. The fermionic part $K_{\mu\bar{\nu}}$ should be invertible due to metric non-degeneracy, which physically ensures the propagators of the ghost fields are well-defined.

One can perform some concrete computations for the probably one of the easiest supermanifolds, $\mathbb{CP}^{n|m}$, with its Kähler potential K given by a natural extension of the Fubini-Study Kähler potential of an ordinary projective space.

$$K = \ln(1 + z^\alpha z_\alpha) \tag{3.121}$$

$$= \ln(1 + z^i z_i) + \sum_p \frac{(-)^{p+1}(\theta^\mu\theta_\mu)^p}{p(1 + z^i z_i)^p}, \tag{3.122}$$

where

$$z_\alpha = \delta_{\alpha\overline{\alpha}}\overline{z}^{\overline{\alpha}}. \tag{3.123}$$

The computation in[28] shows that for $\mathbb{CP}^{n|m}$, the super Ricci tensor is proportional to the Kähler potential K.

$$K_R = \ln \mathrm{sdet} K_{\alpha\overline{\beta}} = -(n-m+1)K. \tag{3.124}$$

Such a result implies that fermionic coordinates contribute to the super-first Chern class with a negative sign. Moreover, the superprojective space $\mathbb{CP}^{n|n+1}$ admits a super Ricci flat metric, induced from a natural extension of the Fubini-Study Kähler potential. It is thus a Calabi-Yau supermanifold in any sense, both with vanishing first Chern class and admitting super Ricci flat metric.

The fermionic part of the Kähler potential for a super weighted projective space $\mathbb{WP}^{n|m}_{[1,\cdots,1|\widetilde{Q}_1,\cdots,\widetilde{Q}_m]}$ is generalized as[28]

$$K^f = \sum_\mu \frac{(\theta^\mu\theta_\mu)}{(1+z^i z_i)^{\widetilde{Q}_\mu}} + O(\theta^i\theta_i)^2. \tag{3.125}$$

For instance, the super weighted projective space $\mathbb{WP}^{2|1}_{[1,1|2]}$ has a potential

$$K = \ln(1+z\overline{z}) + \frac{\theta\overline{\theta}}{(1+z\overline{z})^2}. \tag{3.126}$$

The super first Chern class of $\mathbb{WP}^{n|m}_{[Q_1,\cdots,Q_n|\widetilde{Q}_1,\cdots,\widetilde{Q}_m]}$ vanishes if we have $\sum_\mu \widetilde{Q}_\mu = \sum_i Q_i$, i.e., a fermion coordinate with charge $\widetilde{Q}_\mu$ has contributions $\propto -\widetilde{Q}_\mu$. However, we can compute that for a supermanifold with vanishing super first Chern class, e.g., the weighted projective space $\mathbb{WP}^{2|1}_{[1,1|2]}$, its super Ricci potential K_R corresponding to (3.126) has a fermionic term proportional to $\frac{2\theta\overline{\theta}}{(1+z\overline{z})^2}$, which violates the super Ricci flatness. We anticipate drawing some inspiration from the Ricci flow equation.

$$\frac{\partial}{\partial\Lambda}\left(\overline{\partial}\partial K\right) = \frac{\partial g_{\alpha\overline{\beta}}}{\partial\Lambda} \propto -R_{\alpha\overline{\beta}} = -\overline{\partial}\partial K_R. \tag{3.127}$$

We claim that the super metric does get renormalized from the non-zero super Ricci tensor, and one can expect that such an effect might lead to a super Ricci flat but degenerate metric. Equivalently, the physical theory does get renormalized and flows to a conformal model with a degenerate metric[28].

It comes to our attention that for the supermanifold, Yau's theorem[21–22] does not hold. In fact, vanishing super first Chern class is only sufficient to ensure the trivialness of the bosonic part contribution of the super Ricci potential. Moreover, it was proved in[31] that for a Kähler supermanifold with only one complex fermionic dimension, a super Ricci-flat (super)metric exists if and only if the bosonic part of the metric has vanishing scalar curvature. As a consequence, there does not exist a super Ricci-flat metric on $\mathbb{WP}^{n|1}_{[1,\cdots,1|\widetilde{Q}]}$.

In the latter part of this paper, we refer to super Calabi-Yau's as Kähler supermanifolds with vanishing super first Chern class, which is a much weaker condition than the existence of a super Ricci-flat metric.

3.4 LG Mirrors of Supermanifolds

As we argued before, a rigid Calabi-Yau model has a Landau-Ginzburg mirror, and can further be geometrically explained as a sigma model with a super Calabi-Yau target space. One could also ask for a given GLSM with a super Calabi-Yau target, can we derive its mirror as a super Landau-Ginzburg model? Such a procedure might provide an alternative method for deriving mirror symmetry for compact Calabi-Yau and super Calabi-Yau's.

For instance, in[30], Schwarz established a correspondence between A-models on Calabi-Yau hypersurfaces and certain A-models on super Calabi-Yau's, that are, super weighted projective spaces. Thus, if we can derive the super Landau-Ginzburg mirror of a given super GLSM, comparing the Landau-Ginzburg integration (LG periods) of the dual LG theories leads to strong evidence for the equivalence between GLSMs on supermanifolds and GLSMs on certain hypersurfaces, e.g., see in[32].

In this section, we mainly follow the procedure developed by Aganagic and Vafa in[33], which is a fermionic generalization of the T-duality in[34].

3.4.1 T-duality for Bosonic Fields

We first review the concept of T-duality for the bosonic case, where there is a charge Q bosonic chiral superfield $\varPhi$ coupled to an $\mathcal{N} = (2,2)$ vector multiplet in a $U(1)$ gauge theory. Recall that a theory with both chiral and twisted chiral fields is quite lovely, since mirror symmetry exchanges the role of them. We thus add a twisted chiral superfield $\varSigma$ in the $U(1)$ vector multiplet. As we've seen, the chiral multiplet has mass $Q\varSigma$ from coupling.

We dualize the chiral field $\varPhi$ by replacing it with a twisted chiral superfield Y , which is periodic in the imaginary direction. It is defined as follows.

$$\mathrm{Re}\ Y = |\varPhi|^2, \tag{3.128}$$

and the phase angle of $\varPhi = |\varPhi|\mathrm{e}^{\mathrm{i}\theta}$ is T-dual to the imaginary shift ρ of $Y + \mathrm{i}\rho$. Such a duality exchanges the momentum modes of the phase of $\varPhi$ and the winding modes of the Y, and thus is a T-duality.

One can ask what happens to the twisted superpotential $\widetilde{W}(\varSigma)$ after such a duality procedure, as the newly introduced field Y is twisted chiral and could appear in $\widetilde{W}$. In fact, the form of

$$\widetilde{W}(\varSigma, Y) \tag{3.129}$$

could be fixed by symmetry analysis. As Y is periodic in the imaginary direction, the twisted superpotential must depend on Y by $\exp(-Y)$. However, if the top component $\varSigma$ is quantized, i.e., its integral over the worldsheet takes values in integers, a term proportional to $\varSigma Y$ with integer coefficients is allowed since a shift in Y by $2\pi\mathrm{i}$ (more precisely, a shift of the bottom component of Y) leaves the action invariant. In fact, such a term is from the action after duality, and the coefficient is the $U(1)$ charge Q. The twisted superpotential can thus be written as

$$\widetilde{W}(\varSigma, Y) = Q\varSigma Y + f(\exp(-Y)). \tag{3.130}$$

After further matching the BPS masses, the form of $f(\exp(-Y))$ is uniquely fixed by $f(\exp(-Y)) = \exp(-Y)$.

Recall that for linear sigma models, there is an additional term in the twisted superpotential (with a rescaled t comparing with our notation before)

$$\widetilde{W}_{\mathrm{LSM}} = -t\Sigma. \tag{3.131}$$

We now slightly generalize the model by introducing multiple Φ_i's with charges Q_i in the original theory, and similarly, T-dualize them by Y_i's. The Σ dependence in the resulting twisted superpotential reads

$$\Sigma\left(\sum_i Q_i Y_i - t\right) \subset \widetilde{W}(\Sigma, Y). \tag{3.132}$$

By integrating out Σ, there comes a constraint

$$\sum_i Q_i Y_i - t = 0, \tag{3.133}$$

whose real part is exactly the D-term equation in the original theory

$$\sum_i Q_i |\Phi|_i^2 - r = 0. \tag{3.134}$$

Finally, the superpotential takes the form

$$W(Y_i) = \sum_i \exp(-Y_i), \tag{3.135}$$

with the Y_i's constrained by (3.133).

The Landau-Ginzburg periods of the T-dualized LG theory are evaluated by

$$\Pi_\alpha = \int_\alpha \prod_i \mathrm{d}Y_i \exp(-W(Y_i)) \tag{3.136}$$

$$= \int_\alpha \prod_i \mathrm{d}Y_i\, \delta\left(\sum_i Q_i Y_i - t\right) \exp\left(-\sum_i \exp(-Y_i)\right). \tag{3.137}$$

The above period integral is performed on a middle-dimensional cycle α with fields replaced by their constant modes. We will explain these concepts later in this section.

Duality between different theories can be verified by their mirror Landau-Ginzburg period integral. However, the exact computation of an LG period is usually very difficult, and thus, we sometimes adopt alternative strategies. For instance, we can see that the only parameter in the Landau-Ginzburg periods is t, and it's natural to consider differential equations with respect to it, i.e., the Picard-Fuchs equations. It's quite strong evidence for duality between the two theories if their Landau-Ginzburg periods satisfy the same Picard-Fuchs equation.

3.4.2 T-duality for Fermionic Fields

Now we turn to the case where the chiral superfield $\varPhi$ has opposite spin-statistics, i.e., we introduce a fermionic chiral superfield $\varTheta$ with its gauge charge Q.

Regardless of its statistics, we can similarly rotate it by

$$\varTheta \to \varTheta \exp(\mathrm{i}\omega), \tag{3.138}$$

where ω is bosonic and just the phase that we want to dualize. Thus, we again introduce a (bosonic) twisted chiral superfield X whose imaginary part is dual to ω. We also demand that

$$\varTheta\varTheta^* = -\mathrm{Re}\ X, \tag{3.139}$$

which is a natural extension of the bosonic case. Similarly, by matching the BPS spectrum to the original theory, the twisted superpotential is given by

$$\widetilde{W}(\varSigma, X) = -Q\varSigma X + \exp(-X). \tag{3.140}$$

Nonetheless, introducing a single bosonic field X can not match the central charge on two sides, as a ghost field contributes -1 while X is bosonic and thus contributes $+1$. To compensate, we should add two more ghost fields η and χ. By adding extra fields, the spectrum might change. The best thing we could expect is that one boson and one fermion cancel in pairs in the partition function. Thus, we need η and χ to have mass $Q\varSigma$. The twisted superpotential reads

$$\widetilde{W}(\varSigma, X, \eta, \chi) = -Q\varSigma(X - \eta\chi) + \exp(-X). \tag{3.141}$$

After shifting $X \to X + \eta\chi$, we finally get

$$\widetilde{W}(\varSigma, X, \eta, \chi) = -Q\varSigma X + \exp(-X)(1 + \eta\chi). \tag{3.142}$$

We start with a $-Q\varSigma X$ instead of $Q\varSigma X$. This is because we can either start from integrating out $\varTheta$ in the original theory, or integrating out X, η, and χ in the dual theory, and the effective twisted superpotential $\widetilde{W}_{\text{eff}}(\varSigma)$ should be the same. However, it's different from the case where we integrate out a bosonic field $\varPhi$ with charge Q, and there is an extra negative sign coming from the fermionic nature of $\varTheta$. Thus, we need to flip the sign in $-Q\varSigma X$ to recover a correct $\widetilde{W}_{\text{eff}}(\varSigma)$.

Combining the above results, for multiple bosonic chiral superfields $\varPhi_i$ with charges Q_i and fermionic chiral superfields $\varTheta_\mu$ with charges $\widetilde{Q}_\mu$, the superpotential is given by

$$W = \sum_i \exp(-Y_i) + \sum_\mu \exp\big(-X_\mu\big)(1 + \eta_\mu\chi_\mu), \tag{3.143}$$

where the bosonic fields Y_i and X_μ subjecting to the constraint

$$\sum_i Q_i Y_i - \sum_\mu \widetilde{Q}_\mu X_\mu - t = 0. \tag{3.144}$$

The super Landau-Ginzburg periods are given by

$$\begin{aligned}\Pi_\alpha &= \int_\alpha \left(\prod_i \mathrm{d}Y_i\right)\left(\prod_\mu \mathrm{d}X_\mu \mathrm{d}\eta_\mu \mathrm{d}\chi_\mu\right)\exp\left(-W\right)\\ &= \int_\alpha \left(\prod_i \mathrm{d}Y_i\right)\left(\prod_\mu \mathrm{d}X_\mu \mathrm{d}\eta_\mu \mathrm{d}\chi_\mu\right)\delta\left(\sum_i Q_i Y_i - \sum_\mu \widetilde{Q}_\mu X_\mu - t\right)\\ &\qquad\qquad \times \exp\left(-\sum_i \exp\left(-Y_i\right) - \sum_\mu \exp\big(-X_\mu\big)(1+\eta_\mu\chi_\mu)\right).\end{aligned} \tag{3.145}$$

3.4.3 The Hodge Bundle and Periods

As promised before, we now give a conceptual introduction to period integrals and their relation to the complex structure.

To begin with, we first recall that a complex structure determines the decompositions of the (co)tangent bundle into holomorphic and anti-holomorphic parts. For instance, we consider a Calabi-Yau three-fold M, and thus we have the decomposition

$$H^3(M) = \bigoplus_{r+s=3} H^{r,s}(M). \tag{3.146}$$

Furthermore, variation with respect to the complex structure will not change H^3, as the exterior derivative d itself does not depend on the complex structure, although its decomposition $d = \overline{\partial} + \partial$ does.

We learn that the cohomology group H^3 forms a bundle over the complex moduli space $\mathcal{M}_M$, whose decomposition depends on the base point. We call such a bundle a Hodge bundle, and obviously, the Calabi–Yau form $\Omega \in H^{(3,0)}(M)$ is a section of this bundle. Moreover, the position of the Calabi–Yau form Ω as a line in the fiber H^3 can be used to describe the complex structure.

An illustrative diagram for the Hodge bundle of a Calabi-Yau three-fold M can be shown as follows.

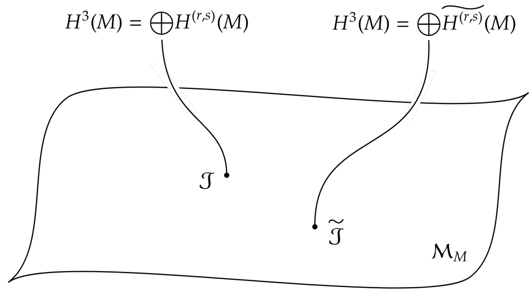


Figure 3-4 The Hodge Bundle over Complex Moduli Space

Given M a Calabi-Yau three-fold and its Hodge bundle over the complex moduli space $\mathcal{M}_M$ with fibers $H^3(M,\mathbb{C})$, one can define a natural hermitian metric on the fiber: Let θ, $\eta \in H^3(M,\mathbb{C})$, then the hermitian product is defined as

$$(\theta, \eta) := \mathrm{i} \int \theta \wedge \overline{\eta}. \tag{3.147}$$

Equipped with a hermitian metric, one can further find[13] a symplectic basis of real integer three-forms α_a, β^b, $a, b = 1, \cdots, h^3/2$ such that

$$(\alpha_a, \alpha_b) = (\beta^a, \beta^b) = 0, \quad \text{and} \quad (\alpha_a, \beta^b) = i\delta_a^b. \tag{3.148}$$

There are also basis homology cycles (A^a, B_b) for $H_3(M, \mathbb{Z})$ that dual to α_a, β^b, i.e., the three-forms are delta functions supported on these cycles.

$$\int_{A_a} \alpha_b = \delta_b^a \quad \text{and} \quad \int_{B_a} \beta^b = \delta_a^b. \tag{3.149}$$

With the help of the basis α_a, β^b, we can expand the Calabi-Yau form Ω by

$$\Omega = z^a \alpha_a + w_b \beta^b. \tag{3.150}$$

Based on our arguments before, the coefficients z^a and w_b will vary as we move in the complex moduli space. Furthermore, these coefficients, which determine the Calabi-Yau form, can describe the base point of the fiber, or say, overdetermine the complex structure by freedom counting. We can express them in terms of the period integrals.

$$z^a = \int_{A^a} \Omega \quad \text{and} \quad w_b = \int_{B_b} \Omega. \tag{3.151}$$

In a word, we can describe the complex structure of a Calabi-Yau (redundantly) by the period integrals.

Chapter 4

GLSMs with Boundaries

At this point, we have discussed the main essence of Linear Sigma Models formulated on a worldsheet without boundary, which provide an Ultra-Violet description of closed string worldsheet theories.

A global picture of the theory on the moduli space has been established for a class of simple but important models, e.g., the quintic Calabi-Yau model, whose low-energy theory is either a Calabi-Yau non-linear sigma model or a Landau-Ginzburg model, depending on the FI parameter r , which actually determines the point on the complex moduli space where the theory is located.

As we mentioned before, D-branes wrapped in Calabi-Yau manifolds play an essential role in string theory, especially those with open strings. And it's natural to generalize the methods we introduced in Linear Sigma Models to study D-branes, which means we have to construct an Ultra-Violet worldsheet theory with boundaries in the LSM language. We would also expect to have a global understanding of the theory in different regions of the moduli space, where the theory may behave in various patterns.

In this paper, we mainly follow[4,11,18,34] to construct GLSMs with boundaries and illustrate some important results, for instance, the grade restriction rule and hemisphere partition function.

4.1 $\mathcal{N} = (2,2)$ Theories with Boundaries

In this section, we formulate $\mathcal{N} = (2,2)$ theories on a worldsheet Σ with a non-trivial boundary $\partial\Sigma$, i.e., we study D-branes in $(2,2)$ supersymmetric theories. More precisely speaking, as we will see, due to the existence of the boundary, there is only a diagonal part of the $\mathcal{N} = (2,2)$ supersymmetry can be preserved.

4.1.1 Warm Up: Boundary Conditions in $(2,2)$ Theories

To extract supersymmetric boundary conditions, we first review the $(2,2)$ supersymmetry algebra, where there are four supercharges $Q_\pm$ and $\overline{Q}_\pm$, subject to the anti-commutation relation

$$\{Q_\pm, \overline{Q}_\pm\} = H \pm P, \tag{4.1}$$

where the Hamiltonian H and the momentum P correspond to translations in the time and space directions.

However, if we formulate the theory on a Σ where $\partial\Sigma \neq 0$, the boundary will break a subset of translation symmetries $\{H, P\}$. For instance, we take the worldsheet as a strip $\mathbb{S} = [0,\pi] \times \mathbb{R}$, where the spatial direction is restricted to an interval $[0,\pi]$. We lose the momentum P as our symmetry generator, while with an appropriate time-independent boundary condition, the Hamiltonian H survives. As a consequence, we should specify a subset of the $(2,2)$ supersymmetry algebra that is closed and does not involve P, i.e.,

$$Q, Q^\dagger \in \widetilde{\mathrm{Sym}}(2,2) \quad \text{and} \quad \{Q, Q^\dagger\} \propto H. \tag{4.2}$$

As one can check, there are two classes of supercharge combinations that satisfy the above condition, which are

$$Q_A = Q_- + \overline{Q}_- \quad Q_B = \overline{Q}_- + \overline{Q}_+. \tag{4.3}$$

In general, we can also consider combinations of supercharges that are rotated by the corresponding R-charges, which also meet the constraint.

$$\mathrm{e}^{\mathrm{i}\alpha F_V} Q_A \mathrm{e}^{-\mathrm{i}\alpha F_V} = \mathrm{e}^{\mathrm{i}\alpha}\overline{Q}_+ + \mathrm{e}^{-\mathrm{i}\alpha} Q_-, \tag{4.4}$$

$$\mathrm{e}^{\mathrm{i}\beta F_A} Q_B \mathrm{e}^{-\mathrm{i}\beta F_B} = \mathrm{e}^{\mathrm{i}\beta}\overline{Q}_+ + \mathrm{e}^{-\mathrm{i}\beta} \overline{Q}_-. \tag{4.5}$$

For simplicity, we omit such R-rotations and only consider the boundary conditions that preserve the bare Q_A and Q_B.

Conservation of Q and $Q^\dagger$ forces that the corresponding supercurrent must vanish at the boundary, for either $Q = Q_A$ or $Q = Q_B$. Recall that we have supercurrents $G^\mu_\pm$ and $\overline{G}^\mu_\pm$ with respect to supercharges $Q_\pm$ and $\overline{Q}_\pm$. For $Q = Q_A$, we have

$$\overline{G}^1_+ + G^1_- = G^1_+ + \overline{G}^1_- = 0, \tag{4.6}$$

and for $Q = Q_B$,

$$\overline{G}^1_+ + \overline{G}^1_- = G^1_+ + G^1_- = 0. \tag{4.7}$$

D-branes preserving Q_A and Q_B type supersymmetry are called A-branes and B-branes.

To obtain the supersymmetric boundary conditions, we can first consider a $\mathcal{N} = 1$ subalgebra of the $(2,2)$ supersymmetry, which is given by

$$Q^1_\pm := Q_\pm + \overline{Q}_\pm. \tag{4.8}$$

Its diagonal part,

$$Q^1 := Q^1_+ + Q^1_-, \tag{4.9}$$

is conserved regardless we impose A-type or B-type boundary conditions, as we can see that

$$\begin{aligned} Q^1 &= Q_+ + \overline{Q}_+ + Q_- + \overline{Q}_- \\ &= Q_A + \overline{Q}_A = Q_B + \overline{Q}_B. \end{aligned} \tag{4.10}$$

Thus, for either A-brane or B-brane boundary conditions, one can start from such a $\mathcal{N} = 1$ diagonal subalgebra Q^1.

Let's start from a supersymmetric non-linear sigma model

$$\phi : \ \Sigma \to X \tag{4.11}$$

with a Kähler target X. If the worldsheet Σ has a boundary, the ordinary argument of supersymmetry invariance of the action by total derivatives

$$\int \delta_{\mathrm{SUSY}}(\mathcal{L}) = \int \mathrm{d}(**) \tag{4.12}$$

does not hold due to the presence of the boundary.

For simplicity, we formulate the theory on the left half-plane $\Sigma = (-\infty, 0] \times \mathbb{R}$, which has a boundary $\partial\Sigma = 0 \times \mathbb{R}$ mapped to a submanifold γ of the target X.

$$\phi:\ \partial\Sigma \to \gamma \subset X. \tag{4.13}$$

We say that the boundary ends on a D-brane wrapped on γ.

One can first perform the variation $\delta\phi^I$, which gives the classical equation of motion that forces $\partial_1\phi^I$ to be normal to γ. Moreover, the invariance under the diagonal $\mathcal{N}=1$ supersymmetry gives constraints on the boundary both for bosons and fermions. It's convenient to define

$$T_b := \partial_0\phi^I, \quad N_b := \partial_1\phi^I, \tag{4.14}$$
$$T_f := \psi_-^I + \psi_+^I, \quad N_f := \psi_-^I - \psi_+^I. \tag{4.15}$$

And the $\mathcal{N}=1$ invariance indicates that $T_\bullet$ is tangent to γ and $N_\bullet$ is normal to γ. We further have a constraint on γ itself, which forces $\mathrm{Im}(W)$ to be constant on γ.

At this moment, we do not consider B fields or any coupling to a gauge field.

Keep in mind that the above conditions, either from the classical equation of motions, or from the diagonal part of the $\mathcal{N}=1$ supersymmetry, are common constraints for both $\mathcal{N}=2_A$ and $\mathcal{N}=2_B$ supersymemtric boundary conditions.

Now, we turn to specifying supersymemtric boundary conditions corresponding to A-branes.

The supercurrents computed in the bulk theory can be computed via Noether's theorem by

$$G_\pm^0 = g_{i\bar{j}}\left(\partial_0 \pm \partial_1\right)\overline{\phi}^j \psi_\pm^i \mp \frac{\mathrm{i}}{2}\overline{\psi_\mp^i}\partial_{\bar{i}}\overline{W}, \tag{4.16}$$

$$G_\pm^1 = \mp g_{i\bar{j}}\left(\partial_0 \pm \partial_1\right)\overline{\phi}^j \psi_\pm^i - \frac{\mathrm{i}}{2}\overline{\psi_\mp^i}\partial_{\bar{i}}\overline{W}, \tag{4.17}$$

$$\overline{G}_\pm^0 = g_{i\bar{j}}\overline{\psi}_\pm^j\left(\partial_0 \pm \partial_1\right)\phi^i \pm \frac{\mathrm{i}}{2}\psi_\mp^i\partial_i W, \tag{4.18}$$

$$\overline{G}_\pm^1 = \mp g_{i\bar{j}}\overline{\psi}_\pm^{\bar{j}}\left(\partial_0 \pm \partial_1\right)\phi^i + \frac{\mathrm{i}}{2}\psi_\mp^i\partial_i W. \tag{4.19}$$

For A-branes, we want the spacial component of the A-type supercurrents vanish, i.e.,

$$\overline{G}_+^1 + G_-^1 = 0 \quad \text{at the boundary.} \tag{4.20}$$

By using the condition from $\mathcal{N}=1$ supersymmetry that $\mathrm{Im}(W)$ should be a constant on γ, and also by adopting the notation in (4.14), we can write the supercurrent as

$$\overline{G}_+^1 + G_-^1 = \frac{\mathrm{i}}{2}\omega(N_b, N_f) - \frac{\mathrm{i}}{2}\omega(T_b, T_f) + \frac{\mathrm{i}}{2}N_f^I\partial_I\mathrm{Re}(W). \tag{4.21}$$

These terms should independently vanish, and for the first two terms we have

$$\omega|_{T\gamma} = 0 \quad \text{and} \quad \omega^{-1}|_{(T\gamma)^{\perp}} = 0. \tag{4.22}$$

Such constraints indicate that γ is a middle-dimensional Lagrangian submanifold of the target X. Here we omit the analysis for the real part of the superpotential W.

As we can see, an A-brane is a middle dimensional Lagrangian submanifold of the target X with respect to the symplectic form ω given by the Kähler structure on X, and the imaginary part of W should be constant along the brane.

We now consider B-branes, where the spacial component of the B-type supercurrents should vanish, i.e.,

$$\overline{G}^1_+ + \overline{G}^1_- = 0 \quad \text{at the boundary.} \tag{4.23}$$

Similarly, we can write the expression in terms of $T_\bullet$'s and $N_\bullet$'s.

$$\overline{G}^1_+ + \overline{G}^1_- = -\frac{\mathrm{i}}{2}\omega(T_b, N_f) + \frac{\mathrm{i}}{2}\omega(N_b, T_f) + \frac{\mathrm{i}}{2}T^I_f \partial_I \mathrm{Re}(W). \tag{4.24}$$

For the B-type supercharges to be conserved, we have the condition

$$\omega(T, N) = 0 \tag{4.25}$$

for the first two terms. Thus, we have

$$\omega(T, N) = (JT, gN) = 0, \tag{4.26}$$

where J is the complex structure. Since we know that

$$(T, gN) = g(T, N) = 0, \tag{4.27}$$

the above condition is just

$$JT \subset T, \tag{4.28}$$

which indicates that $T\gamma$ is a complex subspace of (X, J). Furthermore, vanishing of the third term, along with the condition from the $\mathcal{N} = 1$ supersymmetry, forces W to be locally constant on γ.

Finally, we conclude that a B-brane wraps on a complex submanifold γ of (X, J) with a locally constant W on it.

Note that we should also check if the A-brane and B-brane boundary conditions are invariant under the corresponding supersymmetry transformations, since a symmetry should preserve both the action and the allowed field configuration space.

4.1.2 Adding $\mathcal{N} = 1$ Boundary Interactions

We have seen that for a $\mathcal{N} = (2, 2)$ supersymmetric theory formulated on a worldsheet with boundaries, there is only a subset of supersymmetry transformations that can be preserved, namely, A-type and B-type on flat worldsheets. Moreover, we have to assign boundary conditions for those fields in the theory to make the supercurrent at the boundary vanish.

In fact, we can consider taking another strategy by adding a boundary term for the theory, whose supersymmetry transformation can exactly cancel the extra term from the supersymmetry variation in the bulk. Thus, with the additional boundary term added, the theory formulated on a worldsheet with boundaries is supersymmetric, regardless of boundary conditions.

We first use a simple example to illustrate this. Consider an $\mathcal{N}=(1,1)$ theory described by real scalar fields x^I and Majorana fermions $\psi_\pm^I$, with their Lagrangian given by

$$\mathcal{L}=\sum_{I=1}^{n}\left(\frac{1}{2}\left(\partial_t x^I\right)^2-\frac{1}{2}\left(\partial_s x^I\right)^2+\frac{i}{2}\psi_-^I\left(\partial_t+\partial_s\right)\psi_-^I+\frac{i}{2}\psi_+^I\left(\partial_t-\partial_s\right)\psi_+^I\right). \tag{4.29}$$

Such a Lagrangian describes a string propagating in $\mathbb{R}^n$, and is invariant under the $(1,1)$ supersymmetry transformation is given by

$$\delta x^I=\mathrm{i}\epsilon_-^1\psi_+^I-\mathrm{i}\epsilon_+^1\psi_-^I,\quad \delta\psi_\pm^I=\mp\epsilon_\mp^1(\partial_t\pm\partial_s)x^I, \tag{4.30}$$

up to total derivatives. Now we formulate this on a strip $\mathbb{S}=[0,L]\times\mathbb{R}$, and thus a boundary term should be added to preserve a subset of supersymmetry. By a straightforward computation, it reads

$$L=\int_0^L\mathcal{L}\mathrm{d}s+\left[-\frac{\mathrm{i}}{2}\sum_I\psi_+^I\psi_-^I\right]_0^L. \tag{4.31}$$

One can check that a diagonal part of the $(1,1)$ supersymmetry is preserved, which is given by $\epsilon_-^1=-\epsilon_+^1=\epsilon_1$.

As we have seen above, the nature of supersymmetry for a theory will be ruined by the presence of the boundary, while it can be restored by adding an appropriate boundary interaction, which makes the theory supersymmetric regardless of the boundary conditions.

It's natural to generalize such a spirit into GLSMs with boundaries. However, let's first examine the brane-antibrane system and open string tachyons between these brane pairs.

First, for the tachyon and the massless vector boson with polarization ϵ, the vertex operators are

$$V_T=k\cdot\psi\mathrm{e}^{\mathrm{i}k\cdot x},\quad k^2=1, \tag{4.32}$$

and

$$V_A^\epsilon=(\epsilon\cdot\dot{x}-(\epsilon\cdot\psi)(k\cdot\psi))\mathrm{e}^{\mathrm{i}k\cdot x},\quad k^2=k\cdot\epsilon=0. \tag{4.33}$$

To include such terms in the action, we have to find an off-shell version. For the vector boson (4.33), such a term is given by

$$\mathcal{A}_t=\dot{x}^I A_I(x)-\frac{\mathrm{i}}{4}F_{IJ}(x)\psi^I\psi^J. \tag{4.34}$$

Here F_{IJ} is the field strength of the gauge field A_I, and one can check that the $\mathcal{N}=1$ variation of $\mathcal{A}_t$ is a total time covariant derivative and hence is itself supersymmetric.

Now let's turn to the tachyon vertex operator (4.32). We should first note that such an operator is fermionic, which is a problem if we want to include it in the action. Recall that for an open string, there are extra degrees of freedom at the endpoint of the string, labeled by Chan-Paton vector space.

Here, we introduce a $\mathbb{Z}_2$-graded Chan-Paton space of the form

$$\mathcal{V} = \mathcal{V}^{\text{even}} \oplus \mathcal{V}^{\text{odd}}, \tag{4.35}$$

where $\mathcal{V}^{\text{even}}$ is the bosonic subspace, and $\mathcal{V}^{\text{odd}}$ is the fermionic subspace. One can introduce an odd endomorphism $\mathbf{T}(x)$, which maps the even subspace to odd and the odd subspace to even. We further demand that the fermionic fields $\psi^I_\pm$ as well as the fermionic parameters ϵ's anticommute with the odd maps on $\mathcal{V}$. Then, we can try such a bosonic boundary action defined by

$$\mathcal{A}_t^{(1)} = \frac{\mathrm{i}}{2}\psi^I \partial_I \mathbf{T}(x). \tag{4.36}$$

One can further verify that the $\mathcal{N} = 1$ variation is a total time derivative.

However, for a Chan-Paton vector space with rank larger than one, we have to put it into a path-ordered exponential, which reads

$$U(t_f, t_i) = \mathrm{P}\exp\left(-i\int_{t_i}^{t_f} \mathcal{A}_t \mathrm{d}t\right). \tag{4.37}$$

Thus, what we want is actually a total time covariant derivative, i.e.,

$$\delta\mathcal{A}_t = \dot{X} + \mathrm{i}[\mathcal{A}_t, X] \tag{4.38}$$

for some X. Such an issue can be solved by adding an extra term to $\mathcal{A}_t^{(1)}$, and in fact,

$$\mathcal{A}_t = \frac{\mathrm{i}}{2}\psi^I \partial_I \mathbf{T}(x) + \frac{1}{2}\mathbf{T}(x)^2 \tag{4.39}$$

is the correct boundary action we want. Combined with (4.34), we can write down an $\mathcal{N} = 1$ invariant boundary interaction that encodes the boundary degrees of freedom.

$$\mathcal{A}_t = \dot{x}^I A_I(x) - \frac{\mathrm{i}}{4}F_{IJ}(x)\psi^I\psi^J + \frac{\mathrm{i}}{2}\psi^I D_I\mathbf{T}(x) + \frac{1}{2}\mathbf{T}(x)^2. \tag{4.40}$$

Here, the D_I is just the covariant derivative with connection A_I.

$$D_I\mathbf{T} = \partial_I\mathbf{T} + \mathrm{i}[A_I, \mathbf{T}]. \tag{4.41}$$

4.1.3 Lift to $\mathcal{N} = 2$ Supersymmetry

Up to now, we have considered $\mathcal{N} = 1$ supersymmetric boundary interactions given by $(A, \mathbf{T})$, which is the superconnection of a $\mathbb{Z}_2$-graded vector bundle E. Next, let's find the condition for lifting such a boundary term to a diagonal $\mathcal{N} = 2$ supersymmetry of B-type.

We just directly provide the condition here without giving proof.

The condition for the vector boson is that the curvature of the gauge connection A is an $(1,1)$-form, i.e.,

$$F_A^{(0,2)} = F_A^{(2,0)} = 0. \tag{4.42}$$

Such a condition allows us to define a complex structure on E. And the condition on the tachyon profile $\mathbf{T}$ is that it can be decomposed into a holomorphic part and an anti-holomorphic part.

$$\mathbf{T} = \mathrm{i}Q - \mathrm{i}Q^\dagger. \tag{4.43}$$

By further computing the $\mathcal{N}=2_B$ variation of the tachyon boundary term, from the non-vanishing terms ∂Q^2 and $[Q^2, **]$ one can get a constraint on Q.

$$Q^2 = c \cdot \mathrm{id}_E, \tag{4.44}$$

where c is a numerical constant. With such a decomposition, the total boundary interaction can be written as (omitting some constant identity terms)

$$\mathcal{A}_t = A_I \dot{x}^I - \frac{\mathrm{i}}{4} F_{IJ}\psi^I\psi^J - \frac{1}{2}\psi^i D_i Q + \frac{1}{2}\overline{\psi}^{\bar{i}} D_{\bar{i}} Q^\dagger + \frac{1}{2}\left\{Q, Q^\dagger\right\}. \tag{4.45}$$

Moreover, by the standard computation of Noether's theorem, conserved charges from supersymmetry transformation, i.e., the supercharges, are found to be proportional to the tachyon profile Q by a holomorphic dependence.

Recall that the bulk theory always has a $U(1)_V$ R-symmetry, which is not affected by quantum anomaly. Now we demand that such an R-symmetry is also preserved by the boundary interaction, and then becomes a symmetry of the $\mathcal{N}=2_B$ supersymmetry algebra.

We are working with $Q_B = \overline{Q}_+ + \overline{Q}_-$ that has a vector R-charge $+1$. As we mentioned, the supercharge depends on the tachyon profile Q holomorphically and as a linear function, thus the Q should also have a vector R-charge $+1$.

Keep in mind that Q is an odd endomorphism on the Chan-Paton vector space, and thus, by demanding Q has R-charge $+1$, we are actually specifying a representation $R(\lambda)$ of $\lambda \in U(1)_V$ on the Chan-Paton vector space by

$$R(\lambda) Q(\lambda^{q_x} x) R(\lambda)^{-1} = \lambda^{+1} Q(x). \tag{4.46}$$

However, by the $\mathcal{N}=2_B$ supersymmetry, we have

$$Q^2 = c \cdot \mathrm{id}_E, \tag{4.47}$$

where c is a constant independent of the fields. Thus, the condition from the vector R-charge just fixes the c to vanish, i.e.,

$$Q^2 = 0. \tag{4.48}$$

Besides, we want the connection A_I to be invariant under $R(\lambda)$.

$$R(\lambda) A_I(\lambda^{q_x} x) R(\lambda)^{-1} = A_I(x). \tag{4.49}$$

Assigning a representation of $+1$ R-charge leads to a grading of the vector bundle E. For instance, let's consider a subbundle E^j which is the eigenbundle of R-charge j. By the complex structure, we can decompose it and only consider the holomorphic part $\mathcal{E}^j$. Acting Q on it, by the R-charge argument, will send us to $\mathcal{E}^{j+1}$. We thus have a complex of holomorphic vector bundles, with the differential given by the tachyon profile Q with $Q^2=0$.

$$\cdots \xrightarrow{Q} \mathcal{E}^{j-1} \xrightarrow{Q} \mathcal{E}^{j} \xrightarrow{Q} \mathcal{E}^{j+1} \xrightarrow{Q} \cdots. \tag{4.50}$$

4.1.4 Landau-Ginzburg Models and Matrix Factorizations

In the last part of this section, we briefly introduce the $\mathcal{N}=2_B$ boundary condition for Landau-Ginzburg models with a polynomial superpotential W.

So far, we haven't considered the superpotential W , which might affect the supersymmetry boundary condition. One can first try the LG model formulated on a worldsheet with boundaries, with the standard boundary term

$$L^{(0)}_{\mathrm{bdry}}\left[-\frac{\mathrm{i}}{2}\sum_i\left(\psi^i_+\overline{\psi}^i_- + \overline{\psi}^i_+ + \psi^i_-\right)\right]_{\partial\Sigma}, \tag{4.51}$$

which is just a complex generalization of (4.31). Thus, one can compute the $\mathcal{N}=2_B$ variation, which gives the Warner term

$$\delta\left[\int_\Sigma \mathcal{L}\mathrm{d}^2s + \int_{\partial\Sigma} L^{(0)}_{\mathrm{bdry}}\mathrm{d}t\right] = -\mathrm{Re}\int_{\partial\Sigma}\sum_i\left(\overline{\epsilon}\psi^i\frac{\partial W(x)}{\partial x_i}\right)\mathrm{d}t. \tag{4.52}$$

Such an extra term should be canceled by the boundary interaction given by $\mathcal{A}_t$, which means, the $\mathcal{N}=2_B$ variation of the vector boson and the tachyon should generate a term to compensate the Warner term, and not be self-supersymmetric anymore.

The condition for $\mathcal{A}_t$, or more specifically, for the tachyon profile Q is that

$$Q^2 = W\cdot \mathrm{id}_E. \tag{4.53}$$

One can check that, as the superpotential has vector R-charge $+2$, such a condition is compatible with the R-symmetry action on the Chan-Paton vector space, and there is no constant allowed.

Such a condition can be met by constructing factorizations of the superpotential $W(x)$, given by

$$Q(x) = \begin{pmatrix} 0 & f(x) \\ g(x) & 0\end{pmatrix}, \tag{4.54}$$

with the relation

$$f(x)g(x) = g(x)f(x) = W(x)\mathrm{id}. \tag{4.55}$$

Such a matrix $Q(x)$ is called a matrix factorization of W.

A convenient way to construct such a matrix factorization is by the Clifford algebra

$$\left\{\eta_i,\overline{\eta}_j\right\} = \delta_{i,j},\quad \left\{\eta_i,\eta_j\right\} = \left\{\overline{\eta}_i,\overline{\eta}_j\right\} = 0. \tag{4.56}$$

For instance, let's consider a polynomial superpotential given by

$$W = x_1^{d_1} + \cdots + x_N^{d_N}. \tag{4.57}$$

A class of matrix factorizations of W can be written as

$$Q_{\mathbf{L}} = \sum_{i=1}^N\left(x_i^{L_i+1}\eta_i + x_i^{d_i-L_i-1}\overline{\eta}_i\right). \tag{4.58}$$

The representation space can be constructed via a fermionic tower started from the Clifford vacuum $|0\rangle$.

4.2 $\mathcal{N}=2_B$ Linear Sigma Models

We have discussed the supersymmetric theories formulated on a worldsheet with boundaries. As one can see, essentially, there are two key parts.

To preserve supersymmetry, we have to add an appropriate boundary counterterm for the bulk theory, which makes the total action invariant under the $\mathcal{N}=2_B$ variation. Furthermore, there are extra degrees of freedom located at the boundary, which are labeled by the connection and tachyon profile on the Chan-Paton vector space and captured by a boundary line operator.

We first examine the bulk action and boundary counterterms for a $U(1)$ gauged linear sigma model. The spirit of constructing such counterterms can be explained as follows.

One can notice that if we can write the action as

$$\int \mathrm{d}^2 s\ \mathbf{Q}_B \mathbf{Q}_B^{\dagger}(**). \tag{4.59}$$

Due to the nilpotency of Q_B, the action is manifestly supersymmetric invariant.

Furthermore, from detailed computation, we can see that the action written in the form of superspace integrals can be rewritten in such **Q**-exact terms, up to total derivatives. For instance,

$$\int \mathrm{d}^4\theta L \sim \frac{1}{2}\mathbf{Q}_B \mathbf{Q}_B^{\dagger}[\overline{\mathbf{Q}}_+, \mathbf{Q}_+]k + \text{total derivatives}. \tag{4.60}$$

Here k is the lowest component of K.

Now, the key point comes: The total derivatives. The counterterms we want are just the boundary contributions of these terms. For example, let's formulate the theory on a strip $\mathbb{S}$ where the spatial direction s takes its value on an interval. Now, we only consider the total derivative terms of s in (4.60), and drop the total derivatives of time t.

A similar procedure can also be done for other terms, and in total, the action reads

$$\begin{aligned}
S_g &= \frac{1}{2\pi}\int_{\mathbb{S}} \mathrm{d}^2 s\ \mathcal{L}_g + \frac{1}{4\pi e^2}\int_{\partial\mathbb{S}} \mathrm{d}t\ \left[\frac{1}{2}\partial_1|\sigma|^2 + \mathrm{Im}(\sigma)D + \mathrm{Re}(\sigma)v_{01}\right], \\
S_{\mathrm{m}} &= \frac{1}{2\pi}\int_{\mathbb{S}} \mathrm{d}^2 s\ \mathcal{L}_m + \frac{1}{2\pi}\int_{\partial\mathbb{S}} \mathrm{d}t\ \left[\frac{\mathrm{i}}{2}\left(\bar{\psi}_-\psi_+ - \bar{\psi}_+\psi_-\right) + \mathrm{Im}(\sigma)|\phi|^2\right], \\
S_{\mathrm{FI},\theta} &= \frac{1}{2\pi}\int_{\mathbb{S}} \mathrm{d}^2 s\ \mathcal{L}_{\mathrm{FI},\theta} + \frac{1}{2\pi}\int_{\partial\mathbb{S}} \mathrm{d}t\ \mathrm{Im}(-t\sigma).
\end{aligned} \tag{4.61}$$

Adding boundary D-terms in $\mathcal{N}=2_B$, or in $(2,2)$ language, F-terms, will generally not change the low-energy behavior. And thus, we consider deforming the theory by a boundary term which is $\mathcal{N}=2_B$ but not a D-term.

$$\frac{1}{2}\int_{\partial\Sigma} \mathrm{d}\theta \mathrm{d}\overline{\theta}\ V = -\int_{\partial\Sigma}[v_0 - \mathrm{Re}(\sigma)]. \tag{4.62}$$

However, such a term is not invariant under the gauge group $U(1)$, and thus we have to form it in an exponential before adding it into the boundary action, which is the Wilson line

$$W_q\left(t_f, t_i\right) = \exp\left(-i\int_{t_i}^{t_f} q\left[v_0 - \mathrm{Re}(\sigma)\right]\mathrm{d}t\right). \tag{4.63}$$

We denote a brane supporting a charged q Wilson line as $\mathcal{W}(q)$.

Recall that the boundary degrees of freedom live in the Chan-Paton vector space, and we have already specified representations of the tachyon profile Q and the vector R-charge action $R(\lambda)$ on it. It's natural to generalize the framework for the gauge group action.

Namely, instead of considering the charge q irreducible representation of the gauge group $U(1)$, we turn to specify a (probably) reducible $(q_1, \cdots, q_n)$ representation $\rho(g)$ for the gauge group $g \in U(1)$ on the Chan-Paton vector space, i.e.,

$$\rho(g) = \begin{pmatrix} g^{q_1} & & \\ & \ddots & \\ & & g^{q_n} \end{pmatrix}. \tag{4.64}$$

As a consequence, the gauge field boundary interaction in (4.63) now becomes a matrix-valued term on the Chan-Paton vector space.

$$q\left[v_0 - \mathrm{Re}(\sigma)\right] \rightarrow \begin{pmatrix} q_1 & & \\ & \ddots & \\ & & q_n \end{pmatrix} \left[v_0 - \mathrm{Re}(\sigma)\right] = \rho^*(v_0 - \mathrm{Re}(\sigma)). \tag{4.65}$$

We denote the corresponding Wilson line brane as a direct sum

$$\mathcal{W} = \bigoplus_{i=1}^{n} \mathcal{W}(q_i). \tag{4.66}$$

Let's combine the gauge boson part and the tachyon part, which form the whole boundary interaction in the gauge theory.

$$\mathcal{A}_t = \rho^*(v_0 - \mathrm{Re}(\sigma)) + \text{Tachyon Profile}. \tag{4.67}$$

The gauge covariance of the connection from the gauge invariance of the Wilson line (4.63) reads

$$\mathrm{i}\mathcal{A}_t \xrightarrow{g} \rho(g)\mathrm{i}\mathcal{A}_t\rho(g)^{-1} + \rho(g)\partial_t\rho(g)^{-1}. \tag{4.68}$$

Such a condition indicates that the tachyon profile Q should be gauge invariant

$$\rho(g)^{-1}Q(g \cdot \phi)\rho(g) = Q(\phi). \tag{4.69}$$

And similarly, it should square to a constant times identity, and by further R-symmetry constraint, we have $Q^2 = 0$, as we don't have a superpotential here.

We would also like to demand that the R-symmetry action commutes with the gauge group action.

$$R(\lambda)\rho(g) = \rho(g)R(\lambda). \tag{4.70}$$

Such a commutativity allows us to simultaneously diagonalize these two actions on the Chan-Paton vector space $\mathcal{V}$, namely, find a basis which is a direct sum of multiple R-eigenspaces with eigenvalues $R(\lambda) = \lambda^j$

$$\mathcal{V} = \bigoplus_{j=j_{min}}^{j=j_{max}} \mathcal{V}^j, \tag{4.71}$$

and each $\mathcal{V}^j$ can be further written as $\mathcal{W}^j$, a direct sum of Wilson line branes of different gauge charges. Under such a basis, the tachyon profile Q is block-off-diagonalized.

$$Q = \begin{pmatrix} 0 & d & 0 & \cdots \\ 0 & 0 & d & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{4.72}$$

Finally, we can specify an R-graded B-brane, denoted as $\mathfrak{B}$, by the data from $(\mathcal{W}, Q, R)$, or equivalently $(\mathcal{V}, \rho, Q, R)$. The argument above can lead us to a complex that contains the information.

$$\mathcal{C}(\mathfrak{B}): \quad \cdots \xrightarrow{d^{j-1}} \mathcal{W}^{j-1} \xrightarrow{d^j} \mathcal{W}^j \xrightarrow{d^{j+1}} \mathcal{W}^{j+1} \xrightarrow{d^{j+2}} \cdots. \tag{4.73}$$

Similarly, we can also consider matrix factorizations in linear sigma models with a superpotential W. The $\mathcal{N} = 2_B$ supersymmetry will lead us to the same condition for the tachyon profile Q,

$$Q^2 = W \cdot \mathrm{id}_{\mathcal{V}}. \tag{4.74}$$

With the presence of the superpotential, the R-charges of the fields will be constrained, i.e., we have to assign proper R-charges q^ϕ to the fields ϕ involved in W.

$$W(\lambda^{q_\phi}\phi) = \lambda^2 W(\phi). \tag{4.75}$$

Moreover, we demand that such an R-rotation commutes with the gauge group action

$$\lambda^{q_{g\cdot\phi}}(g \cdot \phi) = g \cdot (\lambda^{q_\phi}\phi). \tag{4.76}$$

And consequently, we have

$$R(\lambda)Q(\lambda^{q_\phi}\phi)R(\lambda)^{-1} = \lambda Q(\phi). \tag{4.77}$$

4.3 Renormalization Group Flows

Now we have seen the construction of a UV Linear Sigma Model with gauge fields. One can ask how it will behave in the low-energy region via the renormalization group flow. Such a correspondence will lead us to some equivalences of low-energy B-type D-branes, by B-brane transportation in different phases of the LSM theory.

4.3.1 D-term deformations and brane-antibrane annihilation

In the bulk theory, the non-renormalization theorem states that the F-term does not change along the RG flow, while the D-term is sent to a unique form in the IR. Thus, we can freely deform the theory by a D-term deformation, with the low-energy behavior intact. If two branes only differ by a D-term deformation, we call they are D-isomorphic to each other.

From computation, we can obtain that for a given D-brane $(E, A, \mathbf{T})$, the deformation of the fiber metric leads to a change in the boundary interaction $\mathcal{A}_t$ by

$$\delta\mathcal{A}_t = \mathcal{D}_t(*) + \mathbf{Q}^\dagger\mathbf{Q}(**), \tag{4.78}$$

which means that the deformation of the fiber metric is a D-term deformation modulo total covariant derivative terms.

For brane-antibrane annihilation, let's first focus on the classical boundary potential given by

$$U(x) = \frac{1}{2}\mathbf{T}(x)^2, \tag{4.79}$$

where $\mathbf{T}$ can be decomposed as $\mathbf{T} = i(\mathbf{Q} - \mathbf{Q}^\dagger)$.

Classically, the vacuum should be related to the classical minimum of the tachyon potential. Quantum mechanically, this claim still holds, and we can consider the Euclidean path-integral weight

$$P\exp\left(-\int_{\partial\mathbb{S}} U(x)\sqrt{h}\mathrm{d}\tau\right). \tag{4.80}$$

As we mentioned that the deformation of the metric is irrelevant in the IR, we can freely send h to infinity. As a consequence, the partition function and thus all correlation functions vanish in the low-energy limit. The argument above indicates that we can ignore the subspace of the Chan-Paton vector space where the det($\mathbf{T}$) does not vanish.

For example, if $\det(\mathbf{T}_0) \neq 0$, we have the RG flow to the IR:

$$\begin{pmatrix} \mathbf{T}_0 & 0 \\ 0 & \mathbf{T}' \end{pmatrix} \to \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{T}' \end{pmatrix}. \tag{4.81}$$

However, things will be more complicated if there are couplings on the off-block-diag part of the Tachyon profile.

$$\begin{pmatrix} \mathbf{T}_0 & * \\ * & \mathbf{T}' \end{pmatrix} \to \begin{pmatrix} \mathbf{T}_0 & 0 \\ 0 & \mathbf{T}'' \end{pmatrix} \to \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{T}'' \end{pmatrix} \tag{4.82}$$

We used the knowledge through standard linear algebra operations.

Let's use an example in[4] to illustrate such an equivalence. We first consider a boundary interaction given by the complex

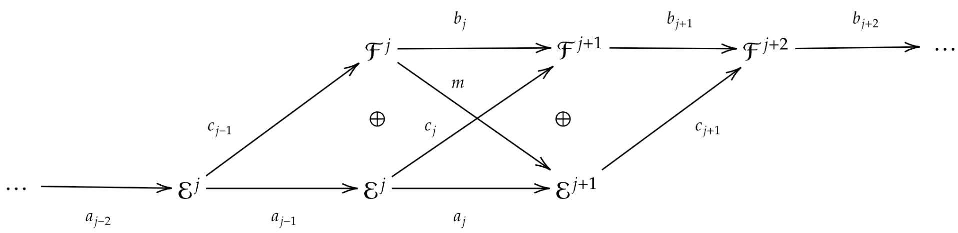


Figure 4-1 Boundary Interactions Given by a Complex

Here m is an invertible map, which gives an invertible boundary interaction $\mathbf{T}$. Now, by manipulations, we can see that such a brane is equivalent to a direct sum of two complexes. And further, the direct sum it is D-isomorphic to the bottom complex.

By computing $\mathbf{T}^2 = \{Q, Q^\dagger\}$, the positivity $\mathbf{T}^2 > 0$ is equivalent to $\mathrm{Im}Q = \mathrm{Ker}Q$, which says that the complex $\mathcal{C}(\mathcal{E}, Q)$is exact. Thus, the D-brane that corresponds to an exact complex can be ignored in the infrared limit. In the

$$
\mathcal{F}^j \xrightarrow{\quad m \quad} E^{j+1}
$$
$$
\bigoplus
$$
$$
\cdots \xrightarrow{a_{j-2}} \mathcal{E}^{j-1} \xrightarrow{a_{j-1}} \mathcal{E}^{j} \xrightarrow{c_j - b_j m^{-1} a_j} \mathcal{F}^{j+1} \xrightarrow{b_{j+1}} \mathcal{F}^{j+1} \xrightarrow{b_{j+2}} \cdots
$$

Figure 4-2 Complex Decomposition by Equivalence

language of the derived category, an exact complex is quasi-isomorphic to the zero complex.

Such an operation in (4.82) can decouple the invertible part from the theory, and as a cost, may introduce modifications on the differential of the complex, which we have already seen. The original complex is decoupled into two parts: One is D-isomorphic to the original one, and the other is generated by an invertible differential, which means it is D-isomorphic to the empty brane.

4.3.2 Quasi-isomorphisms and Cone construction

Suppose we have two D-branes $(\mathcal{E}, Q_E)$ and $(\mathcal{F}, Q_F)$, and there is a degree-zero bundle map $\varphi : \mathcal{E} \to \mathcal{F}$ with the condition $Q_F\varphi - \varphi Q_E = 0$, we can construct a new brane given by the bundles $\mathcal{E}[1] \oplus \mathcal{F}$ with the tachyon profile

$$
Q = \begin{pmatrix} -Q_E & 0 \\ \varphi & Q_F \end{pmatrix}. \tag{4.83}
$$

We claim that if and only if the φ is a quasi-isomorphism, which means it preserves the cohomology class, i.e., $\varphi^* e^{j+1} = Q_F f^j \Leftrightarrow e^{j+1} = Q_E e^j$), the resulting cone complex is exact.

It is also been proved that D-isomorphisms are equivalent to quasi-isomorphisms. For instance, for a D-brane A and B with a quasi-isomorphism $\varphi : \mathcal{C}_A \to C_B$, we can write

$$
A \simeq A + (B + \bar{B})_{\mathrm{id.}} \simeq (A + \bar{B})_{\varphi} + B \simeq B. \tag{4.84}
$$

However, the derivation of the second equivalence is much more complicated.

4.3.3 Branes as Complexes

So far, we have seen that a non-empty D-brane can only wrap on a submanifold of the target X, where the tachyon profile is strictly vanishing $\mathbf{T}(x)^2 = 0$, or equivalently, $f(x) := \{Q(x), Q(x)^\dagger\} = 0$. Thus, it's better to use the language of sheaves to describe it, i.e., the D-brane can be represented by sheaves supported on the divisor D_f. However, for simplicity, we can still define a D-brane as a complex of vector bundles on the entire space X, and such a complex is quasi-isomorphic to the complex of coherent sheaves supported on the divisor D_f.

Let's consider a simple example, the linear sigma model with target $X = \mathbb{C}^n$ and coordinates $x_1, \cdots, x_n$. As we pointed out before, it's natural to construct the representation of the tachyon profile Q via Clifford algebra generators η_i and $\overline{\eta}_j$.

Without superpotential, the tachyon profile is given by

$$
Q(x) = \sum_{i=1}^{n} x_i \eta_i \tag{4.85}
$$

which is nilpotent

$$Q(x)^2 = 0. \tag{4.86}$$

Then, starting from the Clifford vacuum $|0\rangle$ that is annihilated by all η_i's, the representation space, or equivalently, the Chan-Paton vector space where the D-brane is defined, is constructed as follows

$$\mathbb{C}\overline{\eta}_1 \cdots \overline{\eta}_l |0\rangle \xrightarrow{Q} \cdots \xrightarrow{Q} \bigoplus_{i<j} \mathbb{C}\overline{\eta}_i \overline{\eta}_j |0\rangle \xrightarrow{Q} \bigoplus_{i=1}^{n} \mathbb{C}\overline{\eta}_i |0\rangle \xrightarrow{Q} \mathbb{C}|0\rangle. \tag{4.87}$$

Such a complex corresponds to a complex of trivial vector bundles, called the Koszul complex:

$$\mathcal{K} : \mathcal{O} \xrightarrow{Q} \mathcal{O}^{\oplus n} \xrightarrow{Q} \mathcal{O}^{\oplus \binom{n}{2}} \to \cdots \to \mathcal{O}^{\oplus \binom{n}{2}} \xrightarrow{Q} \mathcal{O}^{\oplus n} \xrightarrow{Q} \mathcal{O}. \tag{4.88}$$

The boundary potential reads

$$\{Q, Q^\dagger\} = \left(\sum_{i=1}^{n} |x_i|^2 \right) \cdot \mathrm{id}_{\mathcal{V}}. \tag{4.89}$$

As we can see, such potential is vanishing only at $\{x_1 = \cdots = x_n = 0\}$. Thus, there is only a non-empty brane at the origin, which is a D0-brane. For $x \neq 0$, the corresponding brane in the IR limit is trivial, which can be used to bind to other branes for later use.

We can also consider a phase where the gauge group is not totally broken and there is an abelian finite subgroup Γ that survives. At the orbifold point $\mathfrak{p}$, the condition of gauge equivariance indicates that

$$\rho(\gamma)^{-1} \left(\sum_{i,j=1}^{n} \gamma_{ij} x_j \eta_i \right) \rho(\gamma) = \sum_{j=1}^{n} x_j \eta_j. \tag{4.90}$$

Here, we demand that the Clifford vacuum $|0\rangle$ is invariant under Γ.

From simple manipulations, we can find that the $\overline{\eta}_i\,|0\rangle$ transforms as the dual of the defining representation $\mathcal{R}$ of the Γ on $\mathbb{C}^n$. Thus, the Koszul complex can be constructed as follows.

$$\wedge^n \mathcal{R}^* \xrightarrow{Q} \cdots \xrightarrow{Q} \mathcal{R}^* \xrightarrow{Q} \mathbb{C}. \tag{4.91}$$

Furthermore, we can tensor such a representation with any representation ρ' of the surviving gauge group action, which we denote as $\mathcal{O}_{\mathfrak{p}}(\rho')$. In additive notation, we can write it as $\mathcal{O}_{\mathfrak{p}}(\rho_m) = \mathcal{O}_{\mathfrak{p}}(m)$.

For the Landau-Ginzburg case, the condition for a non-empty brane is the same, i.e., $\{Q(x), Q(x)^\dagger\} = 0$. However, in LG models, the Q's are matrix factorizations of the superpotential W, and thus such a condition tells us which matrix factorization is trivial. For instance,

$$Q(x) = \begin{pmatrix} 0 & \mathrm{id}_r \\ W(x)\mathrm{id}_r & 0 \end{pmatrix} \tag{4.92}$$

has a strictly positive boundary potential and thus is trivial in the low-energy.

4.4 The Grade Restriction Rule

In this section, we mainly focus on the quantum effect of gauged linear sigma models. More precisely, we consider the asymptotic behavior of the boundary interactions on the Coulomb branch, which gives the permitted region for the Coulomb branch to live on, and then, such analysis can derive the grade restriction rule, which restricts the representation of the gauge group ρ for the B-brane that can be transported in different phases.

We are now working in the Coulomb branch, which is described by a twisted chiral field σ that is governed by a twisted superpotential $\widetilde{W}(\sigma)$, as well as its supersymmetry partners. For linear sigma models, we have

$$\widetilde{W}(\Sigma) = -t\Sigma. \tag{4.93}$$

Recall that we are considering B-branes, which preserve the $\mathcal{N} = 2_B$ supersymmetry. Furthermore, we focused much more on the chirals, instead of twisted chirals. For instance, the conclusion that A-branes are Lagrangian submanifolds and B-branes are complex submanifolds is derived for the chiral fields.

Should we rederive $\mathcal{N} = 2_B$ boundary conditions for the twisted chiral field σ? In fact, we can recall that the concept of chiral and twisted chiral will be exchanged if we exchange Q_A and Q_B. Thus, for twisted chiral fields in $\mathcal{N} = 2_B$ theories, we can treat them as chiral fields concerning $\mathcal{N} = 2_A$ supersymmetry. Moreover, it should be a $\mathcal{N} = 2_A$ Landau-Ginzburg model, with its superpotential given by $\widetilde{W}(\sigma)$.

As we derived before, the A-brane condition in an LG model is a Lagrangian submanifold, with the constraint

$$\mathrm{Im}(W) = \mathrm{constant} \tag{4.94}$$

on it.

Recall that such a condition is derived from the supersymmetry invariance of the action without adding boundary terms. We can also start from an action where there is a boundary term added, in a manifestly $\mathcal{N} = 2_A$ invariant form. Thus, the supersymmetry invariance is automatically satisfied, and won't lead to any constraint on the fields.

However, this is not the end of the story. In fact, there are still constraints from the equations of motion of the fields on the boundary. Furthermore, we should choose a set of configurations for fields on the boundary that satisfy the equations of motion, and also are invariant under $\mathcal{N} = 2_A$ supersymmetry variation.

By examining the above conditions, there is only a very special class of A-branes allowed: The linear Lagrangian subspaces L, which have to go through the origin. We can also deform the Lagrangian submanifold by adding a $\mathbf{Q}_A\mathbf{Q}_A^\dagger h$ term in the action, which is a boundary D-term. Such a deformation is called a Hamiltonian deformation of the brane L.

To analyze the Coulomb branch problem, we take an alternative boundary counterterm

$$\Delta S_{\mathrm{bdry}} = \int_{\partial\mathbb{S}} \mathrm{d}t\, (-\mathrm{Im}\, W), \tag{4.95}$$

which is not automatically $\mathcal{N} = 2_A$ invariant, but the invariance can be fixed by the Lagrangian condition of the brane.

With the help of such a boundary term, the imaginary part of W does not need to be a constant, and the theory will gain a boundary potential given by

$$V_{\text{bdry}} = \text{Im } W. \tag{4.96}$$

There comes a requirement from physics, which yields the constraint that the brane L could only extend in the asymptotic direction where the boundary potential is bounded below.

We now come to consider the constraint from the boundary potential for the Coulomb branch L, which gives the admitted asymptotic direction for L. Since we are demanding the asymptotic behavior of the Coulomb branch, the σ field has large values, and thus the matter fields are heavy. We can then integrate them out, resulting in an effective boundary potential for σ

For sure that there will also be large quantum corrections for the small σ region, where the problem is hard to analyze due to the strong coupling between the Coulomb branch and the Higgs branch. Thus, the brane L is a bent line, with two asymptotic directions going out to infinity.

Moreover, there are singularities located at the imaginary line where the field satisfies $\sigma = \pm \mathrm{i}|\sigma|$. Such singular behavior of the model can be explained by the appearance of zero modes of the charged matter sector. We could also explicitly see the origin of such a singular imaginary line by the hemisphere partition functions.

For a $U(1)$ gauged linear sigma model satisfying the Calabi-Yau condition, i.e.,

$$\sum_{i}^{n} Q_i = 0, \tag{4.97}$$

the effective boundary potential from integrating out matter fields reads

$$V_{\text{bdry}}^{\text{eff}} = \frac{1}{2\pi}\left(r + \sum_{i=1}^{n} Q_i \log|Q_i|\right)\text{Im}(\sigma) - \left(\frac{\theta}{2\pi} + q\right)\text{Re}(\sigma) + \frac{1}{4}\sum_{i=1}^{n}|Q_i \text{Re}(\sigma)|\,. \tag{4.98}$$

Let's define

$$\mathcal{S} := \sum_{Q_i > 0} Q_i \tag{4.99}$$

to be the sum of the positive charges.

One can see that the allowed region for large σ, or equivalently, for the asymptotical direction of L, varies as the FI parameter changes from $+\infty$ to $-\infty$. Nonetheless, the pattern of the change of the asymptotic region concerning the variation of the FI parameter r really differs, and depends on the range where $\frac{\theta}{2\pi} + q$ located.

$$\frac{\theta}{2\pi} + q > \frac{1}{2}\mathcal{S}, \tag{4.100}$$

$$-\frac{1}{2}\mathcal{S} < \frac{\theta}{2\pi} + q < \frac{1}{2}\mathcal{S}, \tag{4.101}$$

$$\frac{\theta}{2\pi} + q < -\frac{1}{2}\mathcal{S}. \tag{4.102}$$

Keep in mind that we can deform the brane L without touching the singular imaginary line. However, for the first and the third case, while the FI parameter changes from $+\infty$ to $-\infty$, there isn't a way for us to deform L to keep it in the admitted locus without crossing over the singularity. Only for the second case, there exists such a trajectory for us to deform L.

Finally, the condition for transporting a Wilson line brane $\mathcal{W}(q)$ with gauge charge q between two phases is that the inequality

$$-\frac{1}{2}\mathcal{S} < \frac{\theta}{2\pi} + q < \frac{1}{2}\mathcal{S} \tag{4.103}$$

hold. This is called the grade restriction rule.

4.4.1 Derived Equivalence

We have already obtained the transportation rule for Wilson line branes between different phases, which is given by the restriction on the gauge charge of the Wilson line brane $\mathcal{W}(q)$. Thus, we can use such results in deriving the equivalence between different low-energy, by transporting the UV gauged model into different phases and determining the low-energy behaviors.

Let's first determine the low-energy behavior of D-branes in the UV linear sigma models in a phase where the gauge group is broken. We take the low-energy limit by sending the gauge coupling $e \to \infty$.

Under such a limit, the auxiliary field D becomes a Lagrange multiplier that imposes the D-term equation. Furthermore, we integrate out the gauge multiplet fields v_μ and σ, by setting them to the classical values from the equation of motion.

As a result, the boundary interaction of a Wilson line brane $\mathcal{W}(q)$ descends to the brane supporting the line bundle.

$$\mathcal{A}_t = q[v_0 - \mathrm{Re}(\sigma)] = \dot{x}^I A_I^{(q)}(x) - \frac{i}{4} F_{IJ}^{(q)}(x)\psi^I\psi^J. \tag{4.104}$$

Here, $\dot{x}^I A_I^{(q)}(x)$ is the pull-back connection of the holomorphic line bundle $\mathcal{O}(q)$ over X_r, the vacuum manifold. This is exactly the result we had for the non-linear sigma models, where the D-brane supports the line bundle $\mathcal{O}(q)$.

Thus, we can see that a Wilson line brane with gauge charge q can be sent to the low-energy limit by

$$\mathcal{W}(q) \to \mathcal{O}(q). \tag{4.105}$$

Or, equivalently, we have such a map between the profiles of graded D-branes

$$\mathfrak{B} = (\mathcal{V}, Q, \rho, R) \to \mathcal{B} = (E, A, Q, R). \tag{4.106}$$

If we denote by $\mathfrak{D}(\mathbb{C}^N, G)$ the set of graded D-branes in the linear sigma models with gauge group G, and $D(X_r)$ the set of graded D-branes in the low-energy theory with its vacuum manifold X_r, from the argument above, we have the map for each phase r

$$\pi_r : \mathfrak{D}(\mathbb{C}^N, G) \to D(X_r). \tag{4.107}$$

We can further transport the linear sigma model among different phases and derive the equivalence between categories $D(X_r)$.

4.4.2 An Example for B-Brane Transportation

Let's consider a $U(1)$ linear sigma model with fields P, X_1, X_2, X_3, with the gauge charges $-3, +1, +1, +1$. Two phases are $r >> 0$ and $r << 0$ respectively, where the deleted sets are

$$\Delta_+ = \{x_1 = x_2 = x_3 = 0\}, \tag{4.108}$$
$$\Delta_- = \{p = 0\}. \tag{4.109}$$

In the $r >> 0$ region, we start from a Koszul complex constructed in (4.87), which is chosen to start from a gauge charge -1 Wilson line brane.

$$\mathcal{C}(\mathfrak{B}_+) : \mathcal{W}(-1) \xrightarrow{X} W(0)^{\oplus 3} \xrightarrow{X} W(1)^{\oplus 3} \xrightarrow{X} W(2). \tag{4.110}$$

As we argued before, after deleting the origin, the boundary interaction is strictly positive, and such a Wilson line brane flows to an empty brane in the low-energy limit, which gives the map

$$\pi_+(\mathfrak{B}_+) \cong 0. \tag{4.111}$$

However, at the $r << 0$ phase, the low-energy theory is the free orbifold $X_- = \mathbb{C}^3/\mathbb{Z}_3$ due to the partially broken gauge group. The deleted set has changed to Δ_-, and the boundary potential is no longer everywhere positive. Specifically, the brane flows to a non-empty D-brane located at the origin $\mathfrak{p} = \{x_1 = x_2 = x_3 = 0\}$. We denote it as $\mathcal{O}_{\mathfrak{p}}(\bar{2})$.

Thus, we have

$$\pi_-(\mathfrak{B}_+) \cong \mathcal{O}_{\mathfrak{p}}(\bar{2}). \tag{4.112}$$

We find that the same Wilson line brane has different low-energy D-branes in different phases, which gives the empty brane in the $r >> 0$ phase while $\mathcal{O}_{\mathfrak{p}}(\bar{2})$ in the $r << 0$ phase.

Another example is $\mathfrak{B}_-$.

$$\mathcal{C}(\mathfrak{B}_-) : \mathcal{W}(q+3) \xrightarrow{p} \mathcal{W}(q). \tag{4.113}$$

We can compute that for such a brane, the boundary is proportional to $|p|^2$ and thus strictly positive, given the deleted set in the $r << 0$ phase is $\{p = 0\}$. Thus, such a brane is empty in the $r << 0$ phase, which gives the equivalence

$$\pi_-(\mathcal{W}(q+3)) \cong \pi_+(\mathcal{W}(q+3)). \tag{4.114}$$

Such a relation does not hold in the $r >> 0$ phase.

From the above analysis, we found that a Wilson line brane descends to different low-energy branes in different phases. As we mentioned, such a problem can be solved by applying the grade restriction rule, which means that if we want to transport a Wilson line brane through a window between adjacent phases, the charge of the brane complex should lie in a specific set of numbers.

For this example, the only allowed set is

$$\mathcal{T}^{\omega}_{\mathrm{I,II}} = \langle \mathcal{W}(-1), \mathcal{W}(0), \mathcal{W}(+1) \rangle \,. \tag{4.115}$$

Thus, we have to use the technique introduced before to bind the given Wilson line brane with empty branes to obtain a complex of branes whose gauge charges lie in the grade restriction range. Or equivalently, we lift $D_r(X_r)$ by a map $\omega_r^{\mathrm{I,II}}$ to the window category $\mathcal{T}^{\omega}_{\mathrm{I,II}}$. For sure, binding empty branes won't change the low-energy behavior, and thus we can freely transport the D-branes in the linear sigma model into different phases and derive equivalences among the low-energy categories.

We can illustratively sketch the above procedure for B-brane transportation via a diagram.

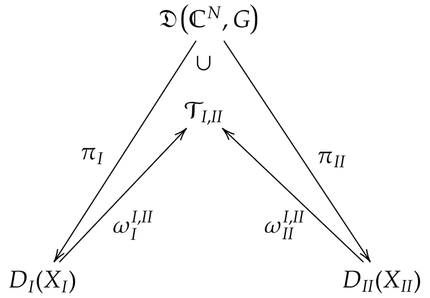


Figure 4-3 B-brane Transportation via a Window Category

Chapter 5

B-branes in Super GLSMs on the Hemisphere

Up to now, we have encountered several important concepts in two-dimensional supersymmetric gauge theories. In Chapter 3, we have introduced constructions and properties of Gauged Linear Sigma Models for both Calabi-Yau models and Super Calabi-Yau models. Moreover, we mentioned a proposed equivalence between a class of super Calabi-Yau models and hypersurface Calabi-Yau models, which implies that a ghost field contributes as an ordinary field with the same but negative charge to form a hypersurface. In Chapter 4, we have seen how to formulate supersymmetric theories on a worldsheet with boundaries. We also introduced the concept of A-branes and B-branes. Now, it's natural to consider constructing theories on the hemisphere, where the target space has some fermionic coordinates.

In this Chapter, we first introduce the hemisphere partition function for gauged linear sigma models, with $A_{(\pm)}$-type supersymmetry in the bulk and B-branes at the boundary. Then, we review the equivalence between super Calabi-Yau models and hypersurface Calabi-Yau models, through arguments on the elliptic genus, sphere partition function, etc. Finally, we formulate these two types of theories on the hemisphere and derive their partition function to check the equivalence at the level of the hemisphere.

5.1 Hemisphere Partition Function

In this section, we will give a brief review based on the work on the hemisphere partition function for B-branes, which was done by K. Hori and M. Romo[11,18]. We mainly follow these two papers in this section.

The partition function is a vital tool in analyzing the system and has a profound impact on determining the nature of a theory. For example, the convergence of the hemisphere partition function in asymptotic directions yields the grade restriction rule. Such a convergent condition can also be generalized to the non-abelian case, while the analysis we used before, based on the boundary effective potential, is insufficient anymore.

5.1.1 Supersymmetry On the Sphere and the Hemisphere

Now we first formulate supersymmetry on the sphere and also on the hemisphere. It will be convenient if we start from the superconformal transformations, which have conformal Killing spinors as variational parameters and reduce to constant parameters in flat space.

In fact, the commutator of two superconformal transformations leads to a combination of conformal transformations, axial-R, and vector-R transformations, that is

$$[\delta_1^{\mathrm{sc}}, \delta_2^{\mathrm{sc}}] = \delta_X^{\mathrm{conformal}} + \delta_{\Theta_V}^{\mathrm{vector}} + \delta_{\Theta_A}^{\mathrm{axial}}. \tag{5.1}$$

There is an ambiguity in the conformal transformation of the gauge potential v_μ, which does not resemble the standard conformal transformation and with a different parameter. All of the parameters, X, Θ_V, and Θ_A is determined by the superconformal parameter of δ_1 and δ_2.

We first consider the case for the whole two-sphere, coordinated by z-plane and w-plane with $zw = 1$, where there are four globally defined conformal Killing spinors given by

$$\mathbf{s}_{-\frac{1}{2}} = \sqrt{\frac{\partial}{\partial z}}, \quad \mathbf{s}_{\frac{1}{2}} = z\sqrt{\frac{\partial}{\partial z}}, \tag{5.2}$$

$$\tilde{\mathbf{s}}_{-\frac{1}{2}} = \sqrt{\frac{\partial}{\partial \overline{z}}}, \quad \tilde{\mathbf{s}}_{\frac{1}{2}} = \overline{z}\sqrt{\frac{\partial}{\partial \overline{z}}}. \tag{5.3}$$

These four spinors can be used to construct transformation parameters for the superconformal transformation. However, recall that we started from a superconformal theory, and we would also like to consider theories which is not conformal invariant, e.g., Fano GLSMs. Thus, we should only consider parameters that generate the isometry of the sphere, i.e., the $\mathfrak{o}(3)$ isometry that is spanned by

$$\ell_3 = -z\frac{\partial}{\partial z} + \overline{z}\frac{\partial}{\partial \overline{z}}, \quad \ell_+ = z^2\frac{\partial}{\partial z} + \frac{\partial}{\partial \overline{z}}, \quad \ell_- = -\frac{\partial}{\partial z} - \overline{z}^2\frac{\partial}{\partial \overline{z}}. \tag{5.4}$$

There are two types of combinations of the transformation parameter, A-type and B-type.

The A-type is defined as

$$Q_{(+)}^{A+} = \delta^{\mathrm{sc}}\left(0, 0, \mathbf{s}_{\frac{1}{2}}, \tilde{\mathbf{s}}_{-\frac{1}{2}}\right), \quad Q_{(+)}^{A-} = \delta^{\mathrm{sc}}\left(\mathbf{s}_{-\frac{1}{2}}, \tilde{\mathbf{s}}_{\frac{1}{2}}, 0, 0\right), \tag{5.5}$$

$$Q_{(-)}^{A+} = \delta^{\mathrm{sc}}\left(0, 0, \mathbf{s}_{-\frac{1}{2}}, -\tilde{\mathbf{s}}_{\frac{1}{2}}\right), \quad Q_{(-)}^{A-} = \delta^{\mathrm{sc}}\left(\mathbf{s}_{\frac{1}{2}}, -\tilde{\mathbf{s}}_{-\frac{1}{2}}, 0, 0\right). \tag{5.6}$$

And the B-type is

$$Q_{(+)}^{B+} = \delta^{\mathrm{sc}}\left(\mathbf{s}_{\frac{1}{2}}, 0, 0, \tilde{\mathbf{s}}_{-\frac{1}{2}}\right), \quad Q_{(+)}^{B-} = \delta^{\mathrm{sc}}\left(0, \tilde{\mathbf{s}}_{\frac{1}{2}}, \mathbf{s}_{-\frac{1}{2}}, 0\right), \tag{5.7}$$

$$Q_{(-)}^{B+} = \delta^{\mathrm{sc}}\left(\mathbf{s}_{-\frac{1}{2}}, 0, 0, -\tilde{\mathbf{s}}_{\frac{1}{2}}\right), \quad Q_{(-)}^{B-} = \delta^{\mathrm{sc}}\left(0, -\tilde{\mathbf{s}}_{-\frac{1}{2}}, \mathbf{s}_{\frac{1}{2}}, 0\right). \tag{5.8}$$

We could also consider their variants generated by the corresponding R-rotations, as we mentioned in the last chapter for the flat worldsheet case.

One can also check that, at the south pole or the north pole, a part of A-type (resp. B-type) supercharges annihilate the twisted chiral (resp. twisted chiral) fields, and thus why we use "A" and "B".

Now we turn to the hemisphere, where the isometry group is broken to $O(2)$, which is generated only by ℓ_3 rotations. Then the combination of supersymmetry transformation parameters is further constrained, and leads to a decomposition of $(+)$ and $(-)$ type. Another way to argue this is that we claim the sign in $(\pm)$ corresponds to the choice of spin structure at the boundary, and thus we have to specify one instead of both.

The $A_{(\pm)}$-type supercharges are defined as:

$$Q^{A+}_{(+)} = \delta^{\mathrm{sc}}\left(0, 0, \mathbf{s}_{\frac{1}{2}}, \tilde{\mathbf{s}}_{-\frac{1}{2}}\right), \quad Q^{A-}_{(+)} = \delta^{\mathrm{sc}}\left(\mathbf{s}_{-\frac{1}{2}}, \tilde{\mathbf{s}}_{\frac{1}{2}}, 0, 0\right), \tag{5.9}$$

for $A_{(+)}$, or

$$Q^{A+}_{(-)} = \delta^{\mathrm{sc}}\left(0, 0, \mathbf{s}_{-\frac{1}{2}}, -\tilde{\mathbf{s}}_{\frac{1}{2}}\right), \quad Q^{A-}_{(-)} = \delta^{\mathrm{sc}}\left(\mathbf{s}_{\frac{1}{2}}, -\tilde{\mathbf{s}}_{-\frac{1}{2}}, 0, 0\right) \tag{5.10}$$

for $A_{(-)}$.

Similarly, the $B_{(\pm)}$-type supersymmetry can be defined as choosing either the combination of $Q^{B+}_{(+)}$ and $Q^{B-}_{(+)}$ or the combination of $Q^{B+}_{(-)}$ and $Q^{B+}_{(-)}$.

We can check the boundary conditions at $|z| = 1$ that preserve A-type and B-type supercharges, which respectively define A-branes and B-branes. In fact, the boundary condition that preserves $A_{(\pm)}$-type supersymmetries defines the B-brane, and the boundary condition that preserves $B_{(\pm)}$-type supersymmetries defines the A-brane.

5.1.2 Supersymmetry Localization and the Result

The computation of the hemisphere partition function is extremely complicated, and here we just sketch the procedure.

First, from the standard supersymmetry localization argument, the partition function is invariant under the limit of gauge coupling $e \to 0$ and $g \to 0$, where there is a $1/g^2$ coefficient in front of matter kinetic terms. This is a consequence of the Q-exactness of the vector and chiral multiplets. Thus, we only consider the quadratic fluctuation around the configuration where the real parts of these kinetic terms are minimized.

Then, we perform mode expansions to the fields on the hemisphere. For simplicity, we first expand them on the whole sphere, and then specify the contributing modes that satisfy the boundary conditions. We view the sphere as $\mathbb{CP}^1$ and the fields as sections of line bundles $\mathcal{O}(n)$. Note that the sphere can also be regarded as a coset space $SU(2)/U(1)$. Thus, we can expand the modes by using a class of spin representations of $SU(2)$, which span the space of global sections of $\mathcal{O}(n)$ as an orthogonal basis.

After expanding the fields on the above basis, we find that there is only a subset of modes that satisfy the boundary condition. In fact, to simplify the boundary conditions, one can choose the Coulomb branch L to be the real locus $L = i\mathfrak{g} \subset \mathfrak{g}_{\mathbb{C}}$. Such a choice of L simplifies the boundary condition for the chiral multiplet, which becomes the purely Neumann boundary condition. Moreover, a real locus eliminates the boundary counterterms of the gauge and matter kinetic terms, which means we only compute the contributions from kinetic terms in the bulk.

After that, one can get the determinants and thus perform Gamma-function regularizations to cut off the infinite momentum modes. And we finally get the answer.

The hemisphere partition function of a B-brane (M, Q, γ) and $A_{(+)}$ bulk supersymmetry reads

$$
\begin{aligned}
Z_{D_2^{A(+)}}(M, Q, \gamma) = C(r\Lambda)^{c'/2} \int_{\gamma'} \mathrm{d}^\ell \sigma' \prod_{\alpha>0} \alpha(\sigma') \sinh(\pi\alpha(\sigma')) \\
\times \prod_i \Gamma\left(iQ_i(\sigma') + \frac{R_i}{2}\right) \exp\left(it_R(\sigma')\right) f_M(\sigma').
\end{aligned}
\tag{5.11}
$$

Here, we define $\sigma' := r\sigma$, where r is the radius, α's are the positive roots of the gauge group, and the factor $f_M(\sigma')$ is the brane factor, which is a trace over the Chan-Paton vector space.

$$
f_M(\sigma') := \mathrm{tr}_M \exp\left(\pi i \mathbf{r} + 2\pi\rho^*(\sigma')\right). \tag{5.12}
$$

And the t_R is defined by

$$
t_R := t - \left(\sum_i Q_i\right) \log(r\Lambda). \tag{5.13}
$$

Note that the t_R is not the effective t_{eff} we obtained before from integrating out the matter multiplets.

We will see that such a brane factor aligns with our analysis in B-brane complexes, i.e., for an empty brane, such a factor eliminates all of the contributing residues inside the integral and makes the hemisphere partition function vanish.

As we've mentioned in the derivation, the integrand was first computed on the real locus $i\mathfrak{g} \subset \mathfrak{g}_{\mathbb{C}}$. For instance, if we only consider a $U(1)$ theory, the integration locus is just the real axis. Furthermore, we can deform the contour as long as we keep it away from singular points, and also preserving the asymptotic behavior on the contour γ for large $|\sigma|$.

We could also consider the large σ limit of the hemisphere partition function. In fact, from Stirling' s formula, the asymptotic behaviour of the gamma function reads

$$
\Gamma(z) \sim \sqrt{2\pi}\ \mathrm{e}^{-z} z^{z-\frac{1}{2}}. \tag{5.14}
$$

Thus, the term of Chan-Paton weight q in the integrand behaves

$$
Z_q \sim \prod_{\alpha>0} \alpha\left(\sigma'\right) \prod_i Q_i\left(\sigma'\right)^{R_i - \frac{1}{2}} \cdot \exp\left(-2\pi \mathrm{i} r\, \widetilde{W}_{\text{eff},q}(\sigma)\right). \tag{5.15}
$$

We find that the convergence of the hemisphere partition function requires that

$$
\operatorname{Im} \widetilde{W}_{\text{eff},q}(\sigma) \tag{5.16}
$$

is bounded below, which is exactly the same condition as the boundary potential analysis.

Note that there is a sum of positive roots involved in the twisted superpotential, which is a contribution from the W-boson. Then, one could expect that the convergence of the hemisphere partition function might allow us to derive the grade restriction rule for non-abelian GLSMs.

5.1.3 Two Examples

We consider a $U(1)$ GLSM with fields $(p, x_1, \cdots, x_N)$ with gauge charges $(-d, 1, \cdots, 1)$, and subject to a superpotential $W = pf(x_1, \cdots, x_N)$. Due to the presence of the superpotential, we assign R-charges to these fields by $(2 - d\epsilon, \epsilon, \cdots, \epsilon)$.

There are two examples of B-brane data, given by $\mathfrak{B}_1 = (M_1, Q_1)$ and $\mathfrak{B}_2 = (M_2, Q_2)$, namely

$$M_1 = \mathbb{C}(0,0) \oplus \mathbb{C}(1 - d\epsilon, d), \tag{5.17}$$

$$Q_1 = \begin{pmatrix} 0 & p \\ f(x) & 0 \end{pmatrix}, \tag{5.18}$$

and

$$M_2 = \bigoplus_{j=0}^{N} \mathbb{C}(j - j\epsilon, j)^{\oplus \binom{N}{j}}, \tag{5.19}$$

$$Q_2 = \sum_{i=1}^{N} \left(x_i \overline{\eta}_i + \frac{1}{d} p \frac{\partial f(x)}{\partial x_i} \eta_i \right). \tag{5.20}$$

The boundary potential $\left\{Q, Q^\dagger\right\}$ is given by

$$\left\{Q_1, Q_1^\dagger\right\} = (|p|^2 + |f(x)|^2)\, \mathrm{id}_{M_1}, \tag{5.21}$$

$$\left\{Q_2, Q_2^\dagger\right\} = \sum_{i=1}^{N} \left(|x_i|^2 + \frac{1}{d^2} \left| p \frac{\partial f(x)}{\partial x_i} \right|^2 \right) \mathrm{id}_{M_2}\,. \tag{5.22}$$

Thus, in the geometric phase, the brane $\mathfrak{B}_2$ is empty at the low-energy, and in the Landau-Ginzburg phase, the brane $\mathfrak{B}_1$ is empty at the low-energy.

We can write the hemisphere partition function for such two B-branes. From the Gamma function factors, one can see that there are order N poles at $\sigma' = \mathrm{i}(n_x + \epsilon/2)$ and simple poles at $\sigma' = \mathrm{i}(-(n_p + 1)/2 + \epsilon/2)$, where n_x and n_p are non-negative integers labeling poles from x and p fields. Besides, we can compute the brane factors for the two B-branes. The results state that the brane factor eliminates the p-poles for $\mathfrak{B}_1$ brane and the x-poles for $\mathfrak{B}_2$ brane.

Before performing the integral, we have to specify a contour where the asymptotic behavior is convergent. We take $d = N$ for simplicity.

For the geometric phase $\zeta >> 0$, the convergence requires

$$\frac{\mathrm{Im}(\sigma')}{|\mathrm{Re}(\sigma')|} >> 0. \tag{5.23}$$

Thus, we can take the contour as

$$\gamma_+ = (\mathrm{Im}(\sigma'), \mathrm{Re}(\sigma')) = (\lambda^2, \lambda). \tag{5.24}$$

One can deform it to be closed at $+\infty$, and thus the hemisphere partition functions just count the residues of the upper half plane, which is from the x-poles. Recall that the brane factor eliminates the x-poles for the $\mathfrak{B}_2$ brane, we have

$$Z(\mathfrak{B}_2, \gamma_+) = 0, \tag{5.25}$$

which aligns with the fact that the $\mathfrak{B}_2$ brane is empty in the geometric phase.

Similarly, for the Landau-Ginzburg phase $\zeta << 0$, we have

$$\frac{\mathrm{Im}(\sigma')}{|\mathrm{Re}(\sigma')|} << 0. \tag{5.26}$$

Thus, we can take the contour as

$$\gamma_- = (\mathrm{Im}(\sigma'), \mathrm{Re}(\sigma')) = (-\lambda^2, \lambda). \tag{5.27}$$

One can also deform it to be closed at $-\infty$, and thus the hemisphere partition functions just count the residues of the p-poles. Again, recall that the brane factor eliminates the p-poles for the $\mathfrak{B}_1$ brane, and similarly, we have

$$Z(\mathfrak{B}_1, \gamma_1) = 0, \tag{5.28}$$

which again aligns with the fact that the $\mathfrak{B}_1$ brane is empty in the Landau-Ginzburg phase.

Now we turn to the singular window where $\zeta = N \log N$. We can compute that the $\mathrm{Im}(\sigma')$ dependence in the asymptotic behavior has disappeared. Thus, for any admissible contour, we can deform it either up or down. However, the existence of the convergent contour requires that

$$-\frac{N}{2} < q + \frac{\theta}{2\pi} < \frac{N}{2}, \tag{5.29}$$

which is exactly the grade restriction rule.

We've seen that the hemisphere partition function is indeed a powerful tool for B-brane analysis.

5.2 Proposed Equivalence: Hypersurfaces in $\mathbb{CP}^N$ v.s. $\mathbb{WP}^{N|M}$

In this section, we work on the proposed equivalence between A-twisted non-linear sigma models on hypersurfaces and A-twisted non-linear sigma models on supermanifolds. In[32], the author checked the dual super Landau-Ginzburg periods as evidence. Moreover, the author in[12] further checked the equivalence by chiral ring relations, elliptic genera, and sphere partition function.

We first review their work, and then we will check the equivalence at the level of the B-brane hemisphere partition function.

5.2.1 Equivalence by Super Landau-Ginzburg Periods

In[32], the equivalence was checked by direct computation of the super Landau-Ginzburg integration, among various types of complete intersections of hypersurfaces in super weighted projective spaces.

Recall that in[33], the authors proposed a mirror construction for supermanifolds, where they map a fermionic chiral superfield to a bosonic twisted chiral superfield X, and due to the requirement for matching the central charge, there are a pair of ghost field η and γ added into the Landau-Ginzburg side, which results in a super Landau-Ginzburg model. The pair of ghost fields gets into the period integral by a modification of the superpotential X, given by

$$\mathrm{e}^{-X}(1+\eta\gamma). \tag{5.30}$$

To simplify the problem, one can first employ the relation between periods of hypersurface $\mathit{\Pi}_{\widetilde{\mathbb{WP}}^{m-1|n}_{[s]}}$ and periods of a model without imposing such a superpotential constraint, as written in $\mathit{\Pi}_{\widetilde{\mathrm{Tot}}\left(\mathcal{O}(-s)\to\mathbb{WP}^{m-1|n}\right)}$. Such two periods can be related by a differential relation with respect to the s.

$$\begin{aligned}
&\mathit{\Pi}_{\widetilde{\mathbb{WP}}^{m-1|n}_{(Q_1,\dots,Q_m|\widetilde{Q}_1,\dots,\widetilde{Q}_n)[s]}}\\
&= -s\frac{\partial}{\partial t}\mathit{\Pi}_{\widetilde{\mathrm{Tot}}\left(\mathcal{O}(-s)\to\mathbb{WP}^{m-1|n}_{(Q_1,\dots,Q_m|\widetilde{Q}_1,\dots,\widetilde{Q}_n)}\right)}\\
&= \int\left(\prod_{i=1}^{m}dY_i\right)dY_P e^{-Y_P}\left(\prod_{a=1}^{n}dX_a d\eta_a d\gamma_a\right)\delta\left(\sum_{i=1}^{m}Q_iY_i - sY_P - \sum_{a=1}^{n}\widetilde{Q}_aX_a - t\right)\\
&\quad\times\exp\left[-\sum_{i=1}^{m}e^{-Y_i} - e^{-Y_P} - \sum_{a=1}^{n}e^{-X_a}\left(1+\eta_a\gamma_a\right)\right].
\end{aligned} \tag{5.31}$$

Moreover, we have to integrate out the twisted chiral field Y_P, which is the mirror of the P field used to generate the hypersurface constraint.

We would also like to consider $\widetilde{\mathbb{WP}}^{m-1|n}_{(Q_1,\dots,Q_m|\widetilde{Q}_1,\dots,\widetilde{Q}_n)[s_1,\cdots,s_l]}$, which is given by taking the differential of $s_1,\cdots,s_l$ for the LG period of $\widetilde{\mathrm{Tot}}\left(\bigoplus_k\mathcal{O}(-s_k)\to\mathbb{WP}^{m-1|n}\right)$.

Finally, by a manipulation of the super Landau-Ginzburg integral, we can show that there are equivalence relations given by periods.

$$\begin{aligned}
\mathit{\Pi}_{\widetilde{\mathbb{WP}}^{m-1|n}_{(Q_1,\dots,Q_m|\widetilde{Q}_1,\dots,\widetilde{Q}_n)[s_1,\cdots,s_l]}} &= \mathit{\Pi}_{\widetilde{\mathbb{WP}}^{m-1|n}_{(Q_1,\dots,Q_m|\widetilde{Q}_1,\dots,\widetilde{Q}_n,s_1,\cdots,s_{l-1})[s_l]}}\\
&= \mathit{\Pi}_{\widetilde{\mathbb{WP}}^{m-1|n}_{(Q_1,\dots,Q_m|\widetilde{Q}_1,\dots,\widetilde{Q}_n,s_1,\cdots,s_l)}}.
\end{aligned} \tag{5.32}$$

5.2.2 Equivalence by Chiral Rings and Partition Functions

In this subsection, we mainly follow[12] to check the equivalence through twisted chiral ring relations, Elliptic genera, and the sphere partition function.

To compute the twisted chiral ring relation for $\mathbb{WP}^{N|M}_{(Q_1,\dots,Q_m|\widetilde{Q}_1,\dots,\widetilde{Q}_n)}$, one could first recall that we have computed the quantum correction to the effective twisted chiral parameter t, which is given by

$$t_{\text{eff}} = t + \frac{\mathrm{i}}{2\pi}\left(\sum_i Q_i\log\left(\frac{\sqrt{2}Q_i\sigma}{\Lambda}\right) - \sum_\mu \widetilde{Q}_\mu\log\left(\frac{\sqrt{2}\widetilde{Q}_\mu\sigma}{\Lambda}\right)\right). \tag{5.33}$$

The one-loop effective twisted superpotential then reads

$$\widetilde{W}_{\text{eff}}(\sigma) = -t_{\text{eff}}\sigma. \tag{5.34}$$

The critical locus of the effective twisted superpotential gives the quantum-corrected Coulomb branch equation

$$\exp\left(\frac{\partial \widetilde{W}_{\text{eff}}(\sigma)}{\partial\sigma}\right) = 0. \tag{5.35}$$

Here, the exponential is taken to eliminate the 2π ambiguity of the theta angle. After absorbing the constants, one can obtain

$$\tilde{q} = \prod_i (Q_i\sigma)^{Q_i} \prod_\mu (\widetilde{Q}_\mu \sigma)^{-\widetilde{Q}_\mu}. \tag{5.36}$$

Compared to the twisted chiral ring relation of complete intersections of hypersurfaces given by chiral fields with charges $-\widetilde{Q}_\mu$, which is given by

$$q = \prod_i (Q_i\sigma)^{Q_i} \prod_\mu (-\widetilde{Q}_\mu \sigma)^{-\widetilde{Q}_\mu}, \tag{5.37}$$

it's easy to see that they are isomorphic rings related by

$$\tilde{q} = (-1)^{\sum_\mu \widetilde{Q}_\mu} q. \tag{5.38}$$

Such an extra factor corresponds to a shift in the theta parameter, and we will see it repeatedly in the following part.

Now we turn to a more refined check, where we formulate our theory on a torus with appropriate boundary conditions, and then compute its partition function, i.e., the Elliptic Genus.

To extract the expression of the Elliptic Genus for super GLSMs, one should first examine the supersymmetry localization procedure, which gives the one-loop determinant of the EG by their quadratic fluctuations. There will be an issue when we compute the one-loop determinant for a ghost multiplet.

One can see that a ghost multiplet $\widetilde{\Phi} = (\tilde{\phi}, \widetilde{\psi}, \cdots)$ has opposite statistics for bosons and fermions. And thus the contribution from

$$Z^{\text{1-loop}}_{\text{even}} = \prod_i \frac{\Delta(\text{Fermions})}{\Delta(\text{Bosons})} \tag{5.39}$$

is modified by

$$Z^{\text{1-loop}}_{\text{odd}} = \prod_i \frac{\Delta(\text{Bosons})}{\Delta(\text{Fermions})}. \tag{5.40}$$

After obtaining the one-loop determinants for ghost multiples, one can compute the Elliptic Genus. For instance, let's take

$$\mathbb{CP}^n_{[d]} \quad \text{and} \quad \mathbb{WP}^{n|1}_{(1,\cdots,1|d)}. \tag{5.41}$$

For a degree d hypersurface in the projective space $\mathbb{CP}^n$, we have to add a p field with gauge charge $-d$, and we would also like to assign R-charge $+2$ to it. Thus, the Elliptic Genus reads

$$Z^{\text{even}}_{T^2}(\tau, z) = -\frac{i\eta(q)^3}{\theta_1(q, y^{-1})} \oint_{u=0} \mathrm{d}u \frac{\theta_1(q, x^{-d})}{\theta_1(q, yx^{-d})} \left(\frac{\theta_1(q, y^{-1}x)}{\theta_1(q, x)}\right)^n. \tag{5.42}$$

For the supermanifold case, the p field then becomes the ghost with gauge charge $+d$. There are no constraints on R-charges from the superpotential, and then we set all of the R-charges to vanish. We note that the one-loop determinant of the ghost multiplet is the inverse of the ordinary field with the same charges. Thus, one can claim

$$\begin{aligned} Z_{T^2}^{\text{odd}}(\tau, z) &= -\frac{i\eta(q)^3}{\theta_1\left(q, y^{-1}\right)} \oint_{u=0} \mathrm{d}u \left(\frac{\theta_1\left(q, y^{-1}x^d\right)}{\theta_1\left(q, x^d\right)}\right)^{-1} \left(\frac{\theta_1\left(q, y^{-1}x\right)}{\theta_1(q, x)}\right)^n \\ &= -\frac{i\eta(q)^3}{\theta_1\left(q, y^{-1}\right)} \oint_{u=0} \mathrm{d}u \frac{\theta_1\left(q, x^d\right)}{\theta_1\left(q, y^{-1}x^d\right)} \left(\frac{\theta_1\left(q, y^{-1}x\right)}{\theta_1(q, x)}\right)^n \\ &= -\frac{i\eta(q)^3}{\theta_1\left(q, y^{-1}\right)} \oint_{u=0} \mathrm{d}u \frac{\theta_1\left(q, x^{-d}\right)}{\theta_1\left(q, yx^{-d}\right)} \left(\frac{\theta_1\left(q, y^{-1}x\right)}{\theta_1(q, x)}\right)^n \\ &= Z_{T^2}^{\text{even}}(\tau, z), \end{aligned} \tag{5.43}$$

where we employed the theta function identity

$$\theta_1(\tau, x) = -\theta_1(\tau, x^{-1}). \tag{5.44}$$

Finally, we conclude that at the level of the Elliptic Genus, these two models are equivalent. We haven't seen the theta angle shift in the EG computation, since, as we know, the twisted chiral parameter t was not captured by the Elliptic Genus.

In the last part of this section, we tend to check the partition function on S^2, whose supersymmetry localization gives

$$Z_{S^2} = \sum_m e^{-im\theta} \int \frac{\mathrm{d}\sigma}{2\pi} e^{-4\pi i\xi\sigma} Z_{\Phi}^{1-\text{ loop}}, \tag{5.45}$$

where the one-loop determinant for Grassmann even chiral superfields reads

$$Z_{\Phi}^{1-\text{loop}} = \prod_{\Phi_i} \frac{\Gamma\left(\frac{R_i}{2} - iQ_i\sigma - Q_i\frac{m}{2}\right)}{\Gamma\left(1 - \frac{R_i}{2} + iQ_i\sigma - Q_i\frac{m}{2}\right)}. \tag{5.46}$$

Similarly, for a ghost multiplet, the boson and fermion contribution change their position in the fraction, and thus we have

$$Z_{\Phi,\widetilde{\Phi}}^{1-\text{ loop}} = \prod_{\Phi_i} \frac{\Gamma\left(-iQ_i\sigma - Q_i\frac{m}{2}\right)}{\Gamma\left(1 + iQ_i\sigma - Q_i\frac{m}{2}\right)} \prod_{\widetilde{\Phi}_\mu} \frac{\Gamma\left(1 + i\widetilde{Q}_\mu\sigma - \widetilde{Q}_\mu\frac{m}{2}\right)}{\Gamma\left(-i\widetilde{Q}_\mu\sigma - \widetilde{Q}_\mu\frac{m}{2}\right)}, \tag{5.47}$$

where we have assigned R-charge zero to all fields.

For the corresponding GLSM on the hypersurface, the one-loop determinant of the chiral fields reads

$$Z_{\Phi_i,\Phi_\mu}^{1-\text{ loop}} = \prod_{\Phi_i} \frac{\Gamma\left(-iQ_i\sigma - Q_i\frac{m}{2}\right)}{\Gamma\left(1 + iQ_i\sigma - Q_i\frac{m}{2}\right)} \prod_{\Phi_\mu} \frac{\Gamma\left(1 + i\widetilde{Q}_\mu\sigma + \widetilde{Q}_\mu\frac{m}{2}\right)}{\Gamma\left(-i\widetilde{Q}_\mu\sigma + \widetilde{Q}_\mu\frac{m}{2}\right)}. \tag{5.48}$$

Here, we assign R-charge +2 to each field Φ_μ that generates the complete intersection.

Now, with the help of the Gamma function identity

$$\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)}, \tag{5.49}$$

we find that

$$Z^{1-\text{ loop}}_{\Phi_i,\Phi_\mu} = (-1)^{m\sum_\mu \widetilde{Q}_\mu} Z^{1-\text{ loop}}_{\Phi_i,\widetilde{\Phi}_\mu}. \tag{5.50}$$

Thus, we conclude that the equivalence holds at the level of the sphere partition function, modulo a shift of the theta angle, which we have already seen in the twisted chiral ring relation.

5.2.3 Equivalence by Hemisphere Partition Function

Now it comes to the main objective of this thesis, checking the equivalence between hypersurfaces in projective spaces and super weighted projective spaces, at the hemisphere level.

We first have a quick review of the hemisphere partition function

$$\begin{aligned} Z_{D_2^{A(+)}}(M,Q,\gamma) = C(r\Lambda)^{c'/2} \int_{\gamma'} \mathrm{d}^\ell\sigma' \prod_{\alpha>0} \alpha(\sigma')\sinh(\pi\alpha(\sigma')) \\ \times \prod_i \Gamma\left(iQ_i(\sigma') + \frac{R_i}{2}\right) \exp\left(it_R(\sigma')\right) f_M(\sigma'), \end{aligned} \tag{5.51}$$

where the brane factor f_M is a trace over the Chan-Paton vector space

$$f_M(\sigma') := \mathrm{tr}_M \exp\left(\pi i \mathbf{r} + 2\pi\sigma'\right), \tag{5.52}$$

and the renormalized t_R is defined as

$$t_R := t - \left(\sum_i Q_i\right)\log(r\Lambda). \tag{5.53}$$

By adding fermionic chiral superfields into the theory, we should first consider the boundary conditions. However, we claim that the A-brane and B-brane conditions are not changed, up to changing the part of the coordinates to fermionic values. Such a conclusion can be argued as follows. We examine the A-type or B-type supercurrents on the worldsheet boundary, which is expressed in a symplectic product of N_b, N_f, T_b, and T_f. Moreover, one can easily see that the supercurrent is fermionic, and the dependence of $\bullet_b$ and $\bullet_f$ is paired up. Thus, exchanging the statistics of bosons and fermions can't be seen in the supercurrent.

We then turn to the supersymmetric localization of the one-loop determinant of the fermionic chiral superfields. Similarly, we claim that for the Gamma function that appears in the hemisphere partition function, it becomes

$$\Gamma\left(iQ_i(\sigma') + \frac{R_i}{2}\right) \to \frac{1}{\Gamma\left(i\widetilde{Q}_i(\sigma') + \frac{R_i}{2}\right)}. \tag{5.54}$$

Further, the renormalized t_R should get a negative contribution from the ghost multiplet.

$$t_R := t - \left(\sum_i Q_i - \sum_\mu \widetilde{Q}_\mu\right)\log(r\Lambda). \tag{5.55}$$

Here comes the problem: Other than the Gamma function dependence in both the numerator and the denominator for the sphere partition function, we only have it in the numerator for the hemisphere. And thus, for ghost multiplets, there are only Gamma functions that appear in the denominator, which can not contribute poles since the Gamma function has no zeros.

This should be a disappointing issue since we lose poles from the p fields if we turn it into fermionic.

In fact, we proposed that such an issue can be solved by the brane factor in the hemisphere partition function, although they are usually sums of exponentials and can't contribute to any pole. For the super GLSM case, we should carefully examine the Chan-Paton vector space, which saves our theory.

We again consider the $U(1)$ GLSM we mentioned before, where the fields $(p, x_1, \cdots, x_N)$ have gauge charges $(-d, 1, \cdots, 1)$ and R-charges $(2 - d\epsilon, \epsilon, \cdots, \epsilon)$, and are subject to a superpotential $W = pf(x_1, \cdots, x_N)$. At this moment, for simplicity, we set $\epsilon = 0$. Such a GLSM reduces to the non-linear sigma model on $\mathbb{CP}^{N-1}_{[d]}$.

And we still have the same examples of B-brane data, given by $\mathfrak{B}_1 = (M_1, Q_1)$ and $\mathfrak{B}_2 = (M_2, Q_2)$, namely

$$M_1 = \mathbb{C}(0,0) \oplus \mathbb{C}(1,d), \tag{5.56}$$

$$Q_1 = \begin{pmatrix} 0 & p \\ f(x) & 0 \end{pmatrix}, \tag{5.57}$$

and

$$M_2 = \bigoplus_{j=0}^{N} \mathbb{C}(j,j)^{\oplus\binom{N}{j}}, \tag{5.58}$$

$$Q_2 = \sum_{i=1}^{N} \left(x_i \overline{\eta}_i + \frac{1}{d} p \frac{\partial f(x)}{\partial x_i} \eta_i \right). \tag{5.59}$$

Let's try to generalize the above data to the super GLSM case. As an easy example, we would like to again compare

$$\mathbb{CP}^n_{[d]} \quad \text{and} \quad \mathbb{WP}^{n|1}_{(1,\cdots,1|d)}. \tag{5.60}$$

Thus, we have to replace the charge $-d$ field p with a charge $+d$ ghost field $\tilde{p}$. Such a replacement will result in a significant change in the tachyon profile Q. We can see this by considering the constructions via Clifford algebra generators

$$\left\{\eta_i, \overline{\eta}_j\right\} = \delta_{i,j}, \quad \left\{\eta_i, \eta_j\right\} = \left\{\overline{\eta}_i, \overline{\eta}_j\right\} = 0. \tag{5.61}$$

The tachyon profile, which is fermionic on the Chan-Panton vector space, is constructed as

$$Q \sim x_i \overline{\eta}_i. \tag{5.62}$$

The fermionic behavior comes from the Clifford generators, which are also the creation and annihilation operators for fermions.

Now, by changing the statistics of the field p, to preserve the fermionic nature of Q, we have to correspondingly change the anti-commutation relations of $\overline{\eta}$'s and η's, i.e., we introduce a new set of operators $\overline{a}$ and a satisfying

$$[a, \overline{a}] = 1. \tag{5.63}$$

One can immediately realize that these are the creation and annihilation operators for bosons.

Now the key point comes: Instead of the truncation behavior of the fermionic tower, we can construct an infinite bosonic tower by repeatedly acting $\overline{a}$. Thus, the Chan-Paton vector space, then has an infinite direction given by the ghost field p. Moreover, the R-charge in this tower remains mod-two invariant due to the bosonic nature, and the gauge charge is increasing by $+d$ from p. Note that a and $\overline{a}$ are bosonic operators, and thus we demand them to be even maps on the Chan-Paton vector space, where we need a Grassmann even θ with R-charge $+1$ to compensate for the tachyon profile Q.

To illustrate, let's take the B-brane data given by $\widetilde{\mathfrak{B}}_1 = (\widetilde{M}_1, \widetilde{Q}_1)$. By changing p to a ghost field, following the argument above, we have

$$\widetilde{M}_1 = \left(\bigoplus_{k=0}^{\infty} \mathbb{C}(0, kd)\right) \oplus \left(\bigoplus_{k=0}^{\infty} \mathbb{C}(1, d+kd)\right), \tag{5.64}$$

$$\widetilde{Q}_1 = f(x)\overline{\eta} + \theta\, p\overline{a}. \tag{5.65}$$

We can draw the corresponding complex $\mathcal{C}(\widetilde{\mathfrak{B}}_1)$ as follows, where we denote the Q^{even} by the contribution in the tachyon profile from Grassmann even fields, and the Q^{odd} by the contribution from the Grassmann odd field.

$$\begin{tikzcd}
\mathbb{C}(0,0) \arrow[r, "Q^{\text{even}}"] \arrow[dr, "Q^{\text{odd}}"'] & \mathbb{C}(1,d) \arrow[dr, "Q^{\text{odd}}"] & & & \\
& \mathbb{C}(0,d) \arrow[r, "Q^{\text{even}}"] \arrow[dr, "Q^{\text{odd}}"'] & \mathbb{C}(1,2d) \arrow[dr, "Q^{\text{odd}}"] & & \\
& & \mathbb{C}(0,2d) \arrow[r, "Q^{\text{even}}"] \arrow[dr, "Q^{\text{odd}}"'] & \mathbb{C}(1,3d) \arrow[dr, "Q^{\text{odd}}"] & \\
& & & \ddots & \ddots
\end{tikzcd}$$

Figure 5-1 Complex of the Super B-brane $\widetilde{\mathfrak{B}}_1 = (\widetilde{M}_1, \widetilde{Q}_1)$

Besides, the brane factor

$$f_{\widetilde{M}_1}(\sigma') = \operatorname{tr}_{\widetilde{M}_1} \exp\left(\pi i \mathbf{r} + 2\pi\sigma'\right) \tag{5.66}$$

now involves an infinite sum. We can write it explicitly

$$f_{\widetilde{M}_1}(\sigma') = \mathrm{tr}_{\widetilde{M}_1} \exp\left(\pi i \mathbf{r} + 2\pi\sigma'\right) \tag{5.67}$$
$$= \left(\sum_k \mathrm{e}^{2\pi k d\sigma'}\right) \mathrm{tr}_{M_1} \exp\left(\pi i \mathbf{r} + 2\pi\sigma'\right) \tag{5.68}$$
$$= \left(\sum_k \mathrm{e}^{2\pi k d\sigma'}\right) f_{M_1}(\sigma'). \tag{5.69}$$

Then there comes a factor

$$\sum_k \mathrm{e}^{2\pi k d\sigma'} = \frac{1}{1-\exp(2\pi d\sigma')} = \frac{\exp(-\pi d\sigma')}{\sin(\pi \mathrm{i} d\sigma')}, \tag{5.70}$$

which tells the difference between the brane factor of the super B-brane $\widetilde{\mathfrak{B}}_1$ and the original B-brane $\mathfrak{B}_1$. Moreover, such a factor potentially introduces new poles.

Together with the Gamma function in the one-loop determinant, the $\mathfrak{B}_1$ brane and $\widetilde{\mathfrak{B}}_1$ brane respectively read

$$\mathit{\Gamma}(1-\mathrm{i}d\sigma') f_{M_1}(\sigma') \quad \text{and} \quad \frac{1}{\mathit{\Gamma}(\mathrm{i}d\sigma')} \frac{\exp(-\pi d\sigma')}{\sin(\pi \mathrm{i} d\sigma')} f_{M_1}(\sigma'). \tag{5.71}$$

Note that due to the presence of the superpotential, the p field has R-charge $+2$ in the $\mathfrak{B}_1$ brane.

Again, we use the Gamma function identity

$$\mathit{\Gamma}(1-z)\mathit{\Gamma}(z) = \frac{\pi}{\sin(\pi z)}, \tag{5.72}$$

which leads to

$$\frac{1}{\mathit{\Gamma}(\mathrm{i}d\sigma')} \frac{\exp(-\pi d\sigma')}{\sin(\pi \mathrm{i} d\sigma')} f_{M_1}(\sigma') = \frac{1}{\pi} \mathrm{e}^{-\pi d\sigma'} \mathit{\Gamma}(1-\mathrm{i}d) f_{M_1}(\sigma'). \tag{5.73}$$

We omit the constant factor $1/\pi$, and have a look at the extra phase $\exp(-\pi d\sigma')$. One can recall that there is a term

$$\exp\left(i t_R(\sigma')\right) \tag{5.74}$$

that appears in the hemisphere partition function with

$$t_R := t - \left(\sum_i Q_i\right) \log(r\mathit{\Lambda}), \quad t = \zeta - \mathrm{i}\theta \tag{5.75}$$

Thus, the extra factor can be precisely canceled by a shift of πd for the theta angle, which exactly aligns with the known result for the sphere partition function. Such a shift of the theta angle can be seen from the asymptotic behavior of the twisted effective superpotential. For ghost fields with positive $\widetilde{Q}_\mu$'s, they contribute as

$$-\sum_\mu \widetilde{Q}_\mu \log\left(\widetilde{Q}_\mu \sigma\right), \tag{5.76}$$

where the overall -1 comes from the fermionic statistics. However, for ordinary fields with negative charges, i.e., $-\widetilde{Q}_\mu$ instead, the contribution reads

$$\sum_{\mu} -\widetilde{Q}_\mu \log\left(-\widetilde{Q}_\mu \sigma\right). \tag{5.77}$$

Thus, they intrinsically differ by a factor of

$$\mathrm{i}\pi \sum_{\mu} \widetilde{Q}_\mu. \tag{5.78}$$

Finally, we conclude that

$$Z(\widetilde{\mathfrak{B}}_1, \theta + \pi d) \sim Z(\mathfrak{B}_1, \theta). \tag{5.79}$$

Thus, we checked the equivalence at the level of the hemisphere. Generalizations to multiple ghost fields are straightforward.

Chapter 6

Conclusion

In this thesis, we've had a brief introduction to a class of two-dimensional $\mathcal{N} = (2,2)$ supersymmetric gauge theories called Gauged Linear Sigma Models, which were first introduced by Edward Witten over thirty years ago. GLSMs provide the Ultraviolet theory on the worldsheet, which is a powerful tool in the study of string propagations and string compactifications. Moreover, the UV linear sigma model will descend to different low-energy theories via the renormalization group flow, depending on the FI parameter of the LSM. In the quintic Calabi-Yau model, the low-energy theories are non-linear sigma models on Calabi-Yau manifolds and the Landau-Ginzburg model, respectively. While the above analyses are classical, the quantum behavior of the linear sigma model will also be interesting. For instance, we introduced quantum corrections on the Coulomb branch, which give the worldsheet instanton corrections to the classical cohomology ring and modify it to the quantum cohomology ring of the target space of the geometric phase. In principle, we can freely assign gauge charges to the matter multiplets. Among these theories, a class of GLSMs that satisfy the Calabi-Yau condition is the one that we are most interested in. Mathematically, such a condition implies the vanishing first Chern class of the geometric target space, and with the $(2,2)$ supersymmetry, one can conclude that the geometric target space is a Calabi-Yau manifold. Physically, such a condition is given by demanding that the sum of the charges of matter multiplets is neutral, which preserves the axial R-symmetry from anomalies. We are interested in the Calabi-Yau case since it provides the playground for string compactifications. We further notice that such a Calabi-Yau GLSM is scale invariant, and was believed to be conformal invariant in general. Thus, it allows us to study these GLSMs in the framework of superconformal field theories.

Mirror symmetry is another intriguing point in $(2,2)$ theories that we've introduced in this thesis, especially in Calabi-Yau GLSMs. Mirror symmetry states that there is a $\mathbb{Z}_2$-automorphism in the supersymmetry algebra, which exchanges the A-type and B-type supersymmetry generators, as well as the axial and vector R-symmetry. Such a symmetry can be realized as a T-duality, and for non-linear sigma models, usually the mirror theory is given by a Landau-Ginzburg model. Furthermore, for Calabi-Yau NLSMs, there is a geometric interpretation, which states that for a Calabi-Yau manifold M, there exists another Calabi-Yau manifold $\widetilde{M}$, and the NLSM on M is equivalent to the NLSM on $\widetilde{M}$, after exchanging A-type and B-type supersymmetry. Moreover, as we've seen in the arguments from deformations both in physical theory and geometry, mirror Calabi-Yau's have exchanged Kählerand Complex moduli spaces. Nevertheless, there is a class of Calabi-Yau's that do not admit any complex deformation, and thus have a trivial complex moduli space. As a consequence, their mirrors have a trivial Kählermoduli space, which is impossible. Sethi proposed that the mirrors of rigid Calabi-Yau's can be explained as a class of Calabi-Yau supermanifolds, which have some fermionic coordinates. Besides, Schwartz proposed the equivalence between A-models on hypersurfaces and supermanifolds.

We are interested in such an equivalence, which connects manifolds with fermionic coordinates and complete intersections of hypersurfaces in pure bosonic manifolds. There is evidence for it, either from the dual super Landau-Ginzburg theory, the chiral rings of the A-twisted GLSMs, the Elliptic Genus, and as well as the sphere partition function. And we would like to check such an equivalence at the level of the hemisphere.

However, formulating theories on a worldsheet with boundaries is challenging due to the conservation of supercharges at the boundary. In fact, GLSMs with boundaries can be used as a UV description of open strings, and thus, other than the boundary condition for supersymmetry, we should also be careful about the extra degrees of freedom at the endpoint of an open string, which is labeled by the Chan-Paton vector space. The gauge boson and the tachyon profiles can be formulated on the graded Chan-Paton space, as connections and an odd endomorphism, respectively, and a brane can be represented by a complex of vector bundles, graded by the R-charge. Moreover, quasi-isomorphisms of the complexes are equivalent to D-isomorphisms of the physical branes, which can be used to bind branes together. While transporting a brane via the moduli space to different phases, there comes a restriction on the representation of the gauge group on the Chan-Paton vector space, which is a consequence of the convergence of the partition function.

In the last part of this paper, we introduced the hemisphere partition function for B-branes and employed it to check the proposed equivalence. Due to the presence of the fermionic coordinates, we have to modify the tachyon profile as well as the corresponding Chan-Paton vector space. We noticed that, to preserve the fermionic nature of the tachyon profile Q, we need to combine the fermionic field with bosonic creation and annihilation operators satisfying $[a, \overline{a}] = 1$ on the representation space. For the ordinary fields, the corresponding fermionic tower in the representation space will truncate due to anti-commutation relations. However, as we started from the bosonic generators, the tower is infinitely long and introduces singularities in the brane factor while computing the hemisphere partition function. Surprisingly, the newly rising poles precisely compensate for the poles which was lost due to the one-loop determinant of the fermionic matters, and thus the equivalence was checked at the level of the hemisphere.

In conclusion, we employed a new method in constructing the tachyon profile and the Chan-Paton vector space for super B-branes, which allows us to check the equivalence between hypersurfaces in projective space and a class of super weighted projective spaces.

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