Homework 2: Due at class on March 18

1. Show that an open ball $B_n = \{x \in \mathbb{R}^n | \, |x| < 1\}$ is diffeomorphic to $\mathbb{R}^n$ by constructing a smooth bijection map.

2. Let (x, y) be the Cartesian coordinate of $\mathbb{R}^2$ and (r, θ) be the polar coordinate of $\mathbb{R}^2$. Write a vector field X in terms of the Cartesian coordinate that generate a flow $\varphi_t : \mathbb{R}^2 \to \mathbb{R}^2$

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} .$$

This is the rotation in $\mathbb{R}^2$. In addition, draw the schematic picture of the vector field X.

3. Write down vector fields that generate the rotation along x-, y-, z-axis in $\mathbb{R}^3$. Find the commutation relations of these vector fields. Compare the theory of angular momenta in quantum mechanics.

4. Let e be the identity element of $SO(3)$. Show that the tangent space $T_eSO(3)$ at e is spanned by tangent vectors of curves in $SO(3)$

$$\exp(tJ_i) = 1 + tJ_i + \frac{1}{2}(tJ_i)^2 + \cdots$$

at $t = 0$ where J_i $(i = x, y, z)$ are defined by

$$J_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \qquad J_x = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \qquad J_x = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} .$$

Let us define the commutator by $[X, Y] = XY - YX$. Then, show that

$$[J_x, J_y] = J_z \,, \qquad [J_y, J_z] = J_x \,, \qquad [J_z, J_x] = J_y \,.$$

5. Show that the tangent space $T_eSU(2)$ is spanned by $i\sigma_x$, $i\sigma_y$ and $i\sigma_z$ (the Pauli matrices by i).

6. Show that $\mathbb{R}P^n$ is non-orientable for even n. In addition, construct an example of unorientable manifolds except the Möbius strip and even-dimensional real projective space.