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硕 士 学 位 论 文

（学术学位）

二维球面上的 $\mathcal{S}$ 类理论

Class $\mathcal{S}$ on S^2

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中文摘要

近些年来对于量子场论 (QFT) 和弦论的学习研究持续揭示了一系列深刻内联的关系网络。6 维 $\mathcal{N}=(2,0)$ 超对称理论 (SCFT) 处于这一网络的核心，它描述了 M-理论中 M5 膜世界体上的低能动力学。因其在持有超对称共形代数的同时有着最高的超对称性和最大的时空维度，6 维 $\mathcal{N}=(0,2)$ 超对称共形场论显得格外重要。由于缺乏拉氏量描述，直接处理 6 维 $\mathcal{N}=(0,2)$ 超对称共形场论的动力学十分困难。但是以它为中心枢纽，通过在不同流形上紧致化可以衍生出众多低维度的量子场论。当 M5 膜用合适的拓扑扭曲 (topological twist) 缠绕在特定的一个流形 M 时，6 维 $\mathcal{N}=(0,2)$ 超对称共形场论将有效的成为一个有关流形 M 的低维量子场论 $\mathcal{T}[M]$，从而导致几何和量子场论间的丰富联系。从 Gaiotto 的构造[1]开始，随后沿着这个方向的发展阐明了不同的量子场论如何可以从 M5 膜中几何的构造出来，揭示了承载着量子场论的深刻几何结构。

一大族的 4 维 $\mathcal{N}=2$ 超对称共形场论 $\mathcal{T}[C]$ 通过考虑 M5 膜卷绕在带孔黎曼面 C 上而构造出来[1]，合称$\mathcal{S}$ 类理论。这些曲面的复模空间编码了这些超对称共形场论的规范耦合常数，这些曲面上的孔则确定了 M5 膜的边界条件，从而决定了这些超对称共形场论的味对称性和算符谱。$\mathcal{S}$ 类构造提供了一个具体的、低维的视角观察 M5 膜动力学，同时也提供了 4 维 $\mathcal{N}=2$ 理论的几何观点，从而加深了我们对 M5 膜的理解。

精确超对称配分函数在加深上述几何理解中发挥着实质性的作用。特别是枚数 BPS 态的超对称共形指标 (supercomformal index) 以简单却强大而闻名，这里 BPS 态是保持部分超对称的态。由于在理论的微小变化下保持不变，超对称共形指标甚至蕴涵了在强相互作用区间的希尔伯特空间的信息。通过将超对称共形指标与黎曼曲面 C 上的关联函数对应，一系列杰出的工作[2–6]揭示了 $\mathcal{T}[C]$ 理论的超对称共形指标潜在的拓扑量子场论 (TQFT) 结构。这一拓扑量子场论描述将许多 4 维理论 $\mathcal{T}[C]$ 上物理操作，如规范化，Higgs 化和非局域算符的插入，映射到对应黎曼曲面 C 上的几何操作或对象。从这一几何观点下，$\mathcal{S}$ 类理论和它理论的一系列指标可以通过粘合基本的几何对象得到，因而广义 S 对偶变得显然起来了。

本文中，通过进一步在 S^2 上使用拓扑扭曲紧致化 $\mathcal{S}$ 类理论，我们继续推进了这一研究方向。4 维 $\mathcal{N}=2$ 超对称共形场论一般有 $\mathrm{SU}(2)_R\times\mathrm{U}(1)_r$ 的 R 对称性。在球面 S^2 上使用 $U(1)_R\times U(1)_r$ 拓扑扭曲，$\mathcal{S}$ 类理论将转变成 2 维 $\mathcal{N}=(0,2)$ 理论[7–8]。如果使用 $\mathrm{U}(1)_r$ 进行拓扑扭曲，$\mathcal{S}$ 类理论将会变成 2 维 $\mathcal{N}=(0,4)$ 的理论[9]。在本文中，我们给出了 A 类且亏格 $g>0$ 的 $(0,2)$ 和 $(0,4)$ 理论的椭圆亏格 (elliptic genus) 的表达式，这些表达式十分简单且紧凑，类似于有拉格朗日描述 $\mathcal{S}$ 类理论的结果。概括的说，全部类型的椭圆亏格有以下的形式

$$\mathcal{I}^{2d}_{g,n}=\mathcal{H}^{g-1}\prod_{i=1}^{n}\mathcal{I}_{\lambda_i}(b_i). \tag{1}$$

这里，$\mathcal{H}$ 表示黎曼曲面 $C_{g,n}$ 一个把手的贡献，$\mathcal{I}_{\lambda_i}$ 表示第 i 类型孔的贡献。惊讶的是，$\mathcal{H}$ 和

$\mathcal{I}_{\lambda_i}$ 的贡献都是 theta 函数的乘积。这种结构揭示了这类二维理论椭圆亏格的拓扑量子场论构造。

另外，在红外处这些 $(0,2)$ 理论的左移分支与紧致化前的理论 $\mathcal{T}$ 的手征代数 $\chi(\mathcal{T})$ 有一定的联系。具体来说，我们发现这些 $(0,2)$ 椭圆亏格是手征代数 $\chi(\mathcal{T})$ 特征标的线性组合，通过含味的模微分方程 (flavored modular differential equation) 可以验证这个事实。这一关系说明 $(0,2)$ 理论拥有手征代数 $\chi(\mathcal{T})$ 的红外对称性。因此，用以研究手征代数的技术可应用于获取红外处 2 维 $\mathcal{N}=(0,2)$ 希尔伯特空间和关联函数的理解。

本篇论文的内容分为 4 个主要部分：§2 背景知识回顾，§3 $\mathcal{N}=(0,2)$ 理论和 §4 $\mathcal{N}=(0,4)$ 的计算分析，以及 §5 主要结论。背景部分的意图是为不熟悉相关背景的读者提供必要的前置知识，以便这些读者能够顺利地阅读后续的内容。但由于篇幅的限制、内容相关程度的取舍以及本人知识的储备，这一部分是无法尽善尽美的，许多细节仍会被省略甚至有些内容可能未被提到，感兴趣的读者可以进一步阅读引用的参考文献或者自己寻找其他相关材料。除了后续计算中使用的特殊函数被放到了论文末尾的附录A，正文中的大多数使用的记号都会在第一次出现时加以解释。本篇论文难免出现疏漏，还请读者不吝指出，本人在此表示感谢。

在 §2.1中，我们回顾二维 $\mathcal{N}=(0,2)$ 和 $\mathcal{N}=(0,4)$ 超对称场论的构成以及后续相关的概念——手征代数/顶点算符代数 (VOA)。对于 $\mathcal{N}=(0,2)$ 理论，基本的超多重态有三种：矢量多重态 V、手征多重态 Φ 和费米多重态 Ψ。使用这些基本的多重态，我们可以构造出每一多重态对应的作用量分支。此外，我们还可以为 (0,2) 理论构造超势能 $\mathcal{W}$，

$$\mathcal{W}=\Psi_a J^a(\Phi_i)\ , \tag{2}$$

这里 $J^a(\Phi_i)$ 是费米多重态的全纯函数。超势能对应的作用量分支由下面的表达式给出

$$S_J=\int d^2x d\theta^+\ \mathcal{W}+h.c.\ . \tag{3}$$

对于 $\mathcal{N}=(0,4)$ 理论，基本的多重态可以分解为 (0,2) 理论多重态的组合。其中 (0,4) 矢量多重态由一个 (0,2) 矢量多重态 V 和一个 (0,2) 费米多重态 Θ 构成。(0,4) 费米多重态由一对 (0,2) 费米多重态 Ψ、$\tilde{\Psi}$ 组成。需要小心的是 (0,4) 理论中有两种不同的高多重态 (hypermultiplet)，它们均由两个 (0,2) 手征多重态构成，但它们在 R-对称性的变换下表现不同。类似于 (0,2) 理论，由这些多重态出发，作用量的各个分支均可以被构造出来，包括超势能对应的分支。在 §2.1.3中，我们引入了手征代数这一概念，它是二维共形场论中所有半纯算符的洛朗展开系数对易子的集合。特别的是，如果考虑二维共形场论的能动张量分量 $T(z)$ 的洛朗展开，我们将得到著名的 Virosara 代数，

$$[L_n,L_m]=(n-m)L_{n+m}+\frac{c}{12}(n^3-n)\delta_{n+m,0}\ , \tag{4}$$

其中 $T(z)=\sum_n z^{-n-2}L_n$。

椭圆亏格是研究二维超对称场论的重要工具，它可以帮助我们发现或者验证对偶理论。在 §2.2中，我们给出了 (0,2)Ramond 和 NS 分支椭圆亏格的表达式，

$$\begin{aligned}\mathcal{I}^{(0,2),R}(z,q)&=\mathrm{Tr}_R(-1)^F q^{H_L}\bar{q}^{H_R}\prod_i z_i^{f_i}\ ,\\ \mathcal{I}^{(0,2),NS}(z,q)&=\mathrm{Tr}_{NS}(-1)^F q^{H_L}\bar{q}^{H_R-\frac{1}{2}J_R}\prod_i z_i^{f_i}\ ,\end{aligned} \tag{5}$$

其中 H_L 和 H_R 分别是左移哈密顿量和右移哈密顿量，J_R 是 $\mathrm{U}(1)_R$ 对称群的生成元，f_i 是味对称群的嘉当 (极大子环群) 生成元。(0,2) 理论各种多重态的椭圆亏格可以通过特定的计算方式得到，同时得益于 Jeffrey-Kirwan 留数的帮助，我们也能够计算更一般的规范理论的椭圆亏格，具体的 JK 留数的计算流程可参见 §2.2.3。对于 (0,4) 理论，我们总是将其看作 (0,2) 理论来计算其椭圆亏格，需要提醒的是，定义 (0,4) 椭圆亏格时我们将多余的 R 对称群当作了味对称群。

在 §2.3中，我们回顾了关于超共形指标的背景知识。在 §2.3.1中，我们列出了一般四维超共形代数 $\mathfrak{sl}(2,2|\mathcal{N})$ 的对易关系，这些对易关系引导我们定义超共形指标和理论的维度约化。在 §2.3.3中，我们定义了四维 $\mathcal{N}=2$ 理论的超共形指标，

$$\mathcal{I}^{4\mathrm{d}}(p,q,t) = \mathrm{Tr}(-1)^F e^{-\beta\tilde{\delta}_{2\dot{-}}} p^{j_1+j_2-r} q^{j_2-j_1-r} t^{R+r} \prod_a z_a^{f_a} \ , \tag{6}$$

其中 j_1 和 j_2 是洛伦兹代数 $\mathfrak{su}(2)_1 \oplus \mathfrak{su}(2)_2$ 的嘉当生成元，R 和 r 是 R 对称性 $\mathfrak{su}(2)_R \oplus \mathfrak{u}(1)_r$ 的嘉当生成元，f_a 是为味对称性的嘉当生成元，这一指标可以看作更精细的 Witten 指标。当我们取 $t \to q$ 极限时，超共形指标将约化为记录舒尔算符/态的舒尔指标。大体上讲，通过舒尔算符，我们可以从四维超共形场论中恢复二维手征代数/顶点算符代数 (VOA)，而舒尔指标就是手征代数的特征标。四维超共形场论和二维手征代数的对应关系给出了四维和二维中心荷、舒尔指标和二维环面配分函数的联系，我们简要的在 §2.3.4中列出了这些结论。

在 §2.4中，我们回忆了有关四维 $\mathcal{S}$ 类理论的基本构成，这类理论是从 6d $\mathcal{N}=(0,2)$ 理论在一个黎曼曲面 C 上紧致化而来。对于一个 $\mathcal{S}$ 类理论来说，它包含以下基本信息：

- 承载其紧致化之前的 6d $\mathcal{N}=(2,0)$ 理论的李代数 $\mathfrak{g}$，李代数 $\mathfrak{g}$ 的种类只可以是节点间单线连接的 Dynkin 图对应的 A、D、E 三类李代数。

- 黎曼曲面 $C_{g,n}$，又称为 UV 曲线 (复数意义下为曲线)，这里 g 是亏格的数量，n 是曲面 $C_{g,n}$ 上孔的数量。

- 一个嵌入映射 $\rho_i : \mathfrak{su}(2) \to \mathfrak{g}$，它决定黎曼曲面 $C_{g,n}$ 孔对应的缺陷的对称性。

一般的 $\mathcal{S}$ 类理论，即 UV 曲线为某个特定 $C_{g,n}$ 的理论，可以由一些更基本的理论组装成，这些理论的 UV 曲线是带有三个孔的二维球面 $C_{0,3}$，我们称之为"trinion" 理论。反之，一般的 $\mathcal{S}$ 类理论也可以拆解为这些基本的单元。这一良好的特性在随后讨论的二维理论中被验证仍然是正确的。

在 §3中，我们描述了如何使用不同选择的拓扑扭曲将四维 $\mathcal{S}$ 类理论约化到二维 $\mathcal{N}=(0,2)$ 或者 $\mathcal{N}=(0,4)$，然后从二维 $\mathcal{S}$ 类理论的基本单元 $U_N^{(0,2)}$ 计算了一系列有拉氏量描述理论的椭圆亏格。从这些计算结果中，我们发现它们具有拓扑量子场论的结构。此外，这些椭圆亏格的计算结果显示出与朗道-金兹堡模型对偶的可能性，并与约化前四维理论的手征代数有关。通过计算结果，我们进一步确定了可能与之对偶的朗道-金兹堡理论，同时也确定了这些椭圆亏格是相应四维超共形场论手征代数特征标的线性组合（可以利用含味模微分方程 (flavored modular differential equation) 确认这一事实）。在 §3.2中，我们主要讨论的是规范群为 SU(2) 和 U(1) 以及它们有限直积的二维超共形场论。我们首先利用论文[9]中

的恒等式，

$$\frac{\eta(q)^5\vartheta_1(c_1^2c_2^2)}{\vartheta_1(c_1^2)\vartheta_1(c_2^2)\vartheta_1(c_1c_2b_1^{\pm}b_2^{\pm})} = \int_{\text{JK}} \frac{da}{2\pi i a}\frac{\eta(q)^8\vartheta_1(a^{\pm 2})}{2\vartheta_4(c_1b_1^{\pm}a^{\pm})\vartheta_4(c_2b_2^{\pm}a^{\pm})} \ , \tag{7}$$

递归地从 $SU(2)$ SQCD 理论的椭圆亏格得到了一般的线性箭图对应理论的椭圆亏格，这一结果也同样被论文[10]得到。随后，在 §3.2.3我们计算了 UV 曲线为 $C_{1,1}$ 的 SU(2) 规范理论的椭圆亏格 $\mathcal{I}_{1,1}^{(0,2),2}$，

$$\mathcal{I}_{1,1}^{(0,2),2} = \frac{\eta(q)}{\vartheta_4(c_1)}\cdot\frac{\vartheta_4(c_1)}{\vartheta_1(c_1^2)} \ , \tag{8}$$

其中，c_1 是记录味对称性的逸度，因子 $\frac{\eta(q)}{\vartheta_4(c_1)}$ 来源于自由手征多重态，而第二个因子 $\frac{\vartheta_4(c_1)}{\vartheta_1(c_1 2)}$ 是带一个伴随表示的手征多重态的 SU(2) 规范理论的椭圆亏格，是相应理论 VOA 的特征标的线性组合。更加一般的圈状箭图对应的 SU(2) 规范理论的椭圆亏格为

$$\mathcal{I}_{1,n}^{(0,2),2} = \prod_{i=1}^{n}\frac{\eta(q)}{\vartheta_1(c_i^2)} \ . \tag{9}$$

之后，在 §3.2.6我们继续计算了 UV 曲线为 $C_{0,2}$ 的两种 SU(2) × SU(2) × U(1) 规范理论的椭圆亏格，两个理论的计算结果均为

$$\mathcal{I}_{2,0}^{(0,2),2} = \frac{2\vartheta_4(d^2)^2}{\eta(q)\vartheta_1(d^4)} \ , \tag{10}$$

其中，逸度 d 是 U(1) 规范群的对角嘉当元。我们计算的一系列理论都被验证具有拓扑量子场论性质，一般的 UV 曲线 $C_{g>0,n}$ 对应的规范理论给出的椭圆亏格 (§3.2.7)，

$$\mathcal{I}_{g>0,n}^{(0,2),2}(c_1,\dots,c_n) = \prod_{j=1}^{g-1}\frac{2\vartheta_4(d_j^2)^2}{\eta(q)\vartheta_1(d_j^4)}\prod_{i=1}^{n}\frac{\eta(q)}{\vartheta_1(c_i^2)} \ . \tag{11}$$

类似的事情在更高阶的规范群 SU(N) 上同样成立，详见 §3.3。对于 SU(N) 线性箭图对应的理论，我们得到了如下的结果，

$$\mathcal{I}_{0,n,2}^{(0,2),N} = \frac{\eta(q)^{N^2+n-1}\vartheta_\alpha(\prod_{i=1}^{n}c_i^N)}{\prod_{i=1}^{n}\vartheta_\beta\left(c_i^N\right)\cdot\prod_{A,B=1}^{N}\vartheta_\gamma(b_{0,A}\,b_{n,B}^{-1}\prod_{i=1}^{n}c_i) \ ,} \tag{12}$$

这里

$$\alpha = \begin{cases} 1 & n\cdot N \text{ even} \\ 4 & n\cdot N \text{ odd}\end{cases} \ , \qquad \beta = \begin{cases} 1 & N \text{ even} \\ 4 & N \text{ odd}\end{cases} \ , \qquad \gamma = \begin{cases} 1 & n \text{ even} \\ 4 & n \text{ odd}\end{cases} \ . \tag{13}$$

对于一般的 UV 曲线为 $C_{g>0,n}$ 的规范理论，椭圆亏格的计算结果为

$$\mathcal{I}_{g>0,n}^{(0,2),N} = \prod_{j=1}^{g-1}\frac{(-1)^\beta N\vartheta_\beta(d_j^N)^2}{\eta(q)\vartheta_1(d_j^{2N})}\prod_{i=1}^{n}\frac{\eta(q)}{\vartheta_\alpha(c_i^N)} \ , \tag{14}$$

这里当 N 为偶数时 $\alpha=1,\beta=4$，当 N 为奇数时 $\alpha=4,\beta=1$。以上这些椭圆亏格的计算结果显示出这类 (0,2) 理论拓扑场论的结构：这类理论的椭圆亏格完全由其对应的 UV 曲线的亏格数量和孔洞数量决定，且亏格和孔洞对椭圆亏格的贡献是相互独立的。在最后一节 §3.4中，我们记录了利用对于 (0,2) 理论中无拉氏量描述的椭圆亏格计算的尝试。

在 §4中，我们计算了一系列从四维 $\mathcal{S}$ 类约化而来的二维 $\mathcal{N}=(0,4)$ 理论的椭圆亏格。与 $\mathcal{N}=(0,2)$ 相同的是我们依旧从相应的 (0,4) 理论基本单元 $U_N^{(0,4)}$ 出发，得到一般的黎曼曲面 $C_{g,n}$ 对应的理论的椭圆亏格。不同于 (0,2) 的是，我们利用(7)得到了一些非拉氏量描述的理论的椭圆亏格。具体而言，在 §4.1我们通过 JK-留数得到了任意黎曼曲面 $C_{g,n}$ 对应的 SU(2) 规范节点的椭圆亏格，

$$\mathcal{I}_{g,n}^{(0,4),2}(c_1,\dots,c_n)=\left(\frac{\vartheta_1(v^2)\vartheta_1(v^4)}{\eta(q)^2}\right)^{g-1}\prod_{i=1}^{n}\frac{\eta(q)^2\vartheta_1(v^4)}{\vartheta_1(v^2)\vartheta_1(v^2c_i^{\pm 2})}\ , \tag{15}$$

其中任意 c_i 是味对称子群 SU(2) 的嘉当生成元。由于规范节点均为 SU(2)，所以对应的理都是有拉氏量描述的。而对于更高阶的规范群 SU(N)，非拉氏量描述的理论是不可避免的。我们通过 Argyres-Seiberg-Gaiotto 对偶，从简单孔的黎曼曲面的椭圆亏格 $\mathcal{I}_{N-1,1}^{(0,4),N}$ 得到了最大孔的黎曼曲面的椭圆亏格 $\mathcal{I}_{[1^N]}^{(0,4),N}$，进一步根据 $\mathcal{S}$ 类理论的特性，我们可以通过 Higgsing 得到任意类型孔的黎曼曲面的椭圆亏格。于是，我们找到了这类具有拓扑量子场论结构椭圆亏格的一般形式，

$$\mathcal{I}_{g,n}^{(0,4),N}=(\mathcal{H}_N)^{g-1}\prod_{i=1}^{n}\mathcal{I}_{\lambda_i}^{(0,4),N}(b_i)\ . \tag{16}$$

其中来自把手的贡献为，

$$\mathcal{H}_N=\frac{\prod_{M=1}^{N}\vartheta_1(v^{2M})^2}{\eta(q)^{2(N-1)}\vartheta_1(v^2)\vartheta_1(v^{2N})}\ , \tag{17}$$

来自不同类型孔 λ_i 的贡献为 $\mathcal{I}_{\lambda_i}$，最为关键的最大孔的椭圆亏格 $\mathcal{I}_{[1^N]}^{(0,4),N}$ 为

$$\mathcal{I}_{[1^N]}^{(0,4),N}(b)=\frac{\eta(q)^{N^2-N}\prod_{M=1}^{N}\vartheta_1(v^{2M})}{\prod_{A,B}^{N}\vartheta_1(v^2b_A/b_B)}\ . \tag{18}$$

由此可见，这类 (0,4) 理论被 UV 曲线的孔洞类型和亏格和孔洞数量分类，具有同一椭圆亏格的理论间是潜在对偶的。

对于 §4这一部分，我们有一些尚不完全理解的问题，一是这些被我们研究的 (0,4) 理论都对应于亏格 g 大于 0 的 UV 曲线，$g=0$ 的 UV 曲线对应的椭圆亏格并没有一个紧致的形式且规范群阶数越高越复杂，二是这些亏格 g 大于 0 的 (0,4) 理论从椭圆亏格的结果上看有明显对偶于朗道-金兹堡理论的迹象，但构造这些朗道-金兹堡理论的超势却不是显而易见的。另外，与 $\mathcal{N}=(0,2)$ 理论不同的是，我们尚并不清楚这些 $\mathcal{N}=(0,4)$ 理论的椭圆亏格与舒尔指标或者说 (VOA) 之间的具体关系。以上这些问题仍有待进一步的探索。

关键词：二维超共形场论；椭圆亏格；$\mathcal{S}$ 类理论；手征代数；舒尔指标；特征标
中图分类号：O413.1

Abstract

Starting from the compactification of four-dimensional class $\mathcal{S}$ theories with a chosen topological twist, we investigate various quiver gauge theories derived from the trinion theories on S^2 with supersymmetry $\mathcal{N} = (0, 2)$ and $\mathcal{N} = (0, 4)$, respectively. Utilizing the JK-residue method, we compute several examples in both (0,2) and (0,4) cases. The closed form of elliptic genera for these theories suggests the associated Landau-Ginzburg duals. We also unveil the TQFT structure of elliptic genera corresponding to the n-punctured Riemann surfaces $C_{g,n}$. Specifically, we further relate the $\mathcal{N} = (0, 2)$ elliptic genera to the chiral algebra/VOAs of the original four-dimensional theories through the observation of the Schur index. The elliptic genera can explicitly represent as linear combinations of characters of the VOAs.

Keywords: SCFT; Elliptic genus; Class $\mathcal{S}$; Chiral algebra/VOAs; Schur index; Character
CLC code: O413.1

Chapter 1: Introduction

The study of quantum field theory (QFT) and string theory over the years has continually revealed deeper structures and interconnected webs of relationships. At the heart of this intricate web lies the 6d $\mathcal{N} = (2,0)$ superconformal field theory (SCFT), which describes the low-energy dynamics on the worldvolume of M5-branes in M-theory. The 6d $\mathcal{N} = (2,0)$ SCFT stands out due to its maximal supersymmetry and the highest spacetime dimension that hosts a superconformal algebra. While directly handling the dynamics of 6d $\mathcal{N} = (2,0)$ SCFT is challenging due to the lack of a Lagrangian description, it serves as a central hub from which a plethora of lower-dimensional theories can be derived through compactifications on various manifolds. When M5-branes wrap a certain manifold M with a suitable topological twist, it effectively gives rise to a lower-dimensional QFT $\mathcal{T}[M]$ associated to the manifold, leading to a rich interplay between geometry and QFT. Starting from Gaiotto's construction[1], subsequent development along this direction elucidated how various QFTs can be geometrically engineered from M5-branes, revealing a profound geometric structure underlying the space of QFTs.

A large family of 4d $\mathcal{N} = 2$ SCFTs $\mathcal{T}[C]$ is constructed in paper[1] by considering M5-branes wrapping Riemann surfaces C with punctures. These theories are collectively known as *theories of class* $\mathcal{S}$. The complex moduli of the surfaces encode the gauge couplings of the SCFTs, and the punctures on the Riemann surface prescribing boundary conditions for the M5-brane determine the flavor symmetry and operator spectrum in the SCFTs. The class $\mathcal{S}$ construction furthers our understanding of M5-branes by offering a concrete, lower-dimensional perspective on the dynamics of M5-branes, and at the same time offers a geometric viewpoint on the resulting 4d $\mathcal{N} = 2$ theories.

Exact supersymmetric partition functions play an essential role in enhancing this geometric viewpoint. In particular, superconformal indices stand out as simple, yet powerful observables that count BPS states (states that preserve a portion of supersymmetry). Since they are invariant under exactly marginal deformations, they encode crucial information about the Hilbert space even in the strong coupling regime. A series of outstanding works[2–6] unveiled a topological quantum field theory (TQFT) structure underlying the superconformal indices of $\mathcal{T}[C]$ by identifying the indices with correlation functions on the Riemann surfaces C. The TQFT description maps various physical manipulations on the 4d theory $\mathcal{T}[C]$, such as gauging, Higgsing and insertion of non-local operators, to geometrical operations and objects on the corresponding Riemann surface C. In this geometric viewpoint, theories of class $\mathcal{S}$ and its indices can be built by gluing simple building blocks, and generalized S-duality becomes apparent.

A remarkable development in the study of 4d $\mathcal{N} = 2$ SCFTs is the deep connection to 2d chiral algebra/vertex operator algebra (VOA)[11–14], referred to as an *SCFT/VOA correspondence*. In paper[11], it is shown that any 4d $\mathcal{N} = 2$ SCFT $\mathcal{T}^{4\mathrm{d}}$ contains a protected subsector consisting of so-called Schur operators restricted on a two-dimensional plane, which furnishes a 2d chiral algebra $\chi(\mathcal{T}^{4\mathrm{d}})$. Two notable examples of Schur operators are the $\mathrm{SU}(2)_R$ current and the flavor moment map operators, which represent the stress-energy tensor and Kac-Moody generators in $\chi(\mathcal{T}^{4\mathrm{d}})$. Through the correspondence, rich 4d physics is reincarnated in 2d context, inspiring deeper understanding and new construction of chiral algebras. In turn, by leveraging the rigidity of chiral algebras, we gain novel perspectives on 4d SCFTs[15–24]. In particular, the Schur index, which is a special limit of 4d $\mathcal{N} = 2$

superconformal index, gets mapped to the vacuum character of $\chi(\mathcal{T}^{\text{4d}})$. As discussed in paper[19], the fact that the stress-energy tensor in $\chi(\mathcal{T}^{\text{4d}})$ is not a Higgs branch operator implies the existence of a certain null state (or descendant of null) in the Verma module of $\chi(\mathcal{T}^{\text{4d}})$. Through Zhu's recursion formula[25–26], such a state translates to a modular differential equation that the Schur index should solve. In fact, additional null states may exist which lead to a set of flavored modular differential equations that all module characters of $\chi(\mathcal{T}^{\text{4d}})$ must satisfy[27–28], putting stringent constraints on both the chiral algebra and the 4d physics.

In this thesis, we push forward this research direction by further compactifying class $\mathcal{S}$ theories on S^2 with a topological twist. The 4d $\mathcal{N}=2$ SCFTs have an inherent $\mathrm{SU}(2)_R \times \mathrm{U}(1)_r$ R-symmetry. A topological twist on S^2 using $\mathrm{U}(1)_R \times \mathrm{U}(1)_r$ results in 2d $\mathcal{N}=(0,2)$ theories[7–8]. On the other hand, a twist with $\mathrm{U}(1)_r$ gives rise to 2d $\mathcal{N}=(0,4)$ theories[9]. In this work, we provide remarkably simple closed-form expressions of the (0,2) theories analogous to Lagrangian class $\mathcal{S}$ theories, and those of the (0,4) theories of all class $\mathcal{S}$ theories of type A with genus $g>0$. Schematically, the elliptic genera for both classes take the form

$$\mathcal{I}^{\text{2d}}_{g,n} = \mathcal{H}^{g-1}\prod_{i=1}^{n}\mathcal{I}_{\lambda_i}(b_i)\ . \tag{1.1}$$

Here, $\mathcal{H}$ denotes the contribution from a handle on a Riemann surface $C_{g,n}$, and $\mathcal{I}_{\lambda_i}$ represents the contribution from the i-th puncture. Surprisingly, each of these contributions, both $\mathcal{H}$ and $\mathcal{I}_{\lambda_i}$, can be expressed as a product of theta functions. This structure, as we will demonstrate, naturally unveils the explicit TQFT construction of elliptic genera on Riemann surfaces $C_{g,n}$ for both classes of 2d theories.

Moreover, in the infrared, the left-moving sector of a (0,2) theory from compactifying $\mathcal{T}^{\text{4d}}$ exhibits a connection to the associated chiral algebra $\chi(\mathcal{T}^{\text{4d}})$. Concretely, we propose that the (0,2) elliptic genus is a linear combination of characters of the chiral algebra $\chi(\mathcal{T}^{\text{4d}})$, which can be verified using flavored modular linear differential equations. This relation suggests that the $(0,2)$ theory is endowed with the VOA $\chi(\mathcal{T}^{\text{4d}})$ as the infrared symmetry. Techniques for studying VOAs can be applied to gain insights into the Hilbert space and correlation functions of 2d $\mathcal{N}=(0,2)$ theories at the infrared fixed point.

This thesis is structured as follows. In §2, we explore the essential background knowledge underpinning this thesis. Given that the 2d theories we later discuss trace back to the 4d class $\mathcal{S}$, this chapter navigates through both $2d$ and $4d$ realms. In §2.1, we examines the supermultiplets and Lagrangian formulations for $\mathcal{N}=(0,2)$ and $\mathcal{N}=(0,4)$ theories, culminating in the introduction of the pivotal concept of chiral algebra/Vertex Operator Algebra (VOA). Moving on to §2.2, our attention converges on the potent tool of elliptic genera within 2d superconformal theories. Additionally, we explain the JK-residue method for computing the elliptic genus. In §2.3, we explain the superconformal index for $4d$ $\mathcal{N}=2$ superconformal field theories(SCFTs), along with its consequential limit, the Schur index, which is closely related to chiral algebra/VOA. In §2.4, a concise review of class $\mathcal{S}$ theories is provided, including the class $\mathcal{S}$ data and the building blocks:trinion theories. While this chapter aims to offer comprehensive insights, readers are encouraged to further read related literature for a more nuanced understanding, considering the inherent limitations of the author's perspective.

In §3, we explore 2d $N=(0,2)$ quiver gauge theory analogous to Lagrangian class S theories. Our primary objective is to unveil the duality between these theories and Landau-Ginzburg (LG) models while also exploring their connection to VOAs. In §3.1, the focus is on establishing the relation between 2d $(0,2)$ theories and 4d $\mathcal{N}=2$ SCFTs. We consider a twisted compactification of 4d $\mathcal{N}=2$ SCFTs on S^2, which leads to 2d (0,2) quiver gauge theories. We further discuss the connection between the 4d $\mathcal{N}=2$ Schur index and the (0,2) elliptic genera. Additionally, we put forth a conjecture that the (0,2) elliptic genus can be expressed as a linear combination of characters of the corresponding chiral algebra. In §3.2, we examine 2D (0,2) $\mathrm{SU}(2)\times\mathrm{U}(1)$ quiver gauge theories. Here, we investigate our proposal on a case-by-case basis, demonstrating that these theories have LG duals.

In specific cases, we identify the elliptic genus as a linear combination of characters of the associated VOA. When explicit characters are not available, we check that the elliptic genus solves the modular linear differential equations that constrain the VOA characters. In the end, we demonstrate that the (0,2) elliptic genera exhibit a TQFT structure on Riemann surfaces with a minimal number of U(1) gauge groups. In §3.3, we extend the computation to $\mathrm{SU}(N) \times \mathrm{U}(1)$ gauge theories. Lastly, §3.4 collects a few remarks on non-Lagrangian theories, providing a perspective on this particular area of study.

In §4, we study 2d $\mathcal{N} = (0,4)$ theories from another twisted compactification of A-type class $\mathcal{S}$ theories on S^2. Since theories in this class generally lack Lagrangian descriptions, we make use of the elliptic inversion formula to compute their elliptic genera in §4.2 and §4.3. Given that such a $(0,4)$ theory is characterized by a Riemann surface with punctures decorated by embedding $\mathrm{SU}(2) \hookrightarrow \mathrm{SU}(N)$, in §4.4, we propose a Higgsing procedure to derive the contributions to the (0,4) elliptic genus from the puncture data. We will show that the elliptic genera of all these theories can be reorganized as simple products of theta functions, and they exhibit a TQFT structure under the cut-and-join operations on Riemann surfaces. We end this section by commenting on future directions and open problems.

Appendix A consolidates the notations and conventions, and introduces definitions of special functions and modular forms used throughout this thesis. Detailed computation for JK residue can be found in Appendix B and C.

Chapter 2: Review

2.1 2d $\mathcal{N} = (0,2)$ and $\mathcal{N} = (0,4)$ theories

In this section, we review the basic facts related to $\mathcal{N} = (0,2)$ and $\mathcal{N} = (0,4)$ theory in $d = 1+1$, mainly taking from these papers[29–31]. Additionally, we delve into 2d chiral algebra in §2.1.3 for the convenience of discussing superconformal field theories.

2.1.1 $\mathcal{N} = (0,2)$

(0,2) Superalgebra

Chiral (0,2) supersymmetry is generated by two supercharges, Q_+ and $\bar{Q}_+ = Q_+^\dagger$, the bosonic generators H, P and M of translations and rotations, and the generator J_R of $\mathrm{U}(1)_R$ R-symmetry. The (0,2) superalgebra is given by

$$\begin{aligned} Q_+^2 = \bar{Q}_+^2 = 0 \ , \qquad & \{Q_+, \bar{Q}_+\} = 2(H-P) \ , \\ [M, Q_+] = -Q_+ \ , \qquad & [M, \bar{Q}_+] = -\bar{Q}_+ \ , \\ [J_R, Q_+] = -Q_+ \ , \qquad & [J_R, \bar{Q}_+] = Q_+ \ . \end{aligned}$$

Superspace

2d $\mathcal{N} = (0,2)$ superspace consists of the spacetime coordinates $x_\pm = x^0 \pm x^1$, and Grassmann variables θ^+ and $\bar{\theta}^+$. Theories with $\mathcal{N} = (0,2)$ supersymmetry have U(1) R-symmetry under which θ^+ has charge $+1$. The supersymmetry generators are

$$Q_+ = \frac{\partial}{\partial\theta^+} + i\bar{\theta}^+(\partial_0 + \partial_1) \ , \qquad \bar{Q}_+ = -\frac{\partial}{\partial\bar{\theta}^+} - i\theta^+(\partial_0 + \partial_1) \ . \tag{2.1}$$

And the superderivatives are

$$D_+ = \frac{\partial}{\partial\theta^+} - i\bar{\theta}^+(\partial_0 + \partial_1) \ , \qquad \overline{D}_+ = -\frac{\partial}{\partial\bar{\theta}^+} + i\bar{\theta}^+(\partial_0 + \partial_1) \ . \tag{2.2}$$

They satisfy the following relations

$$\{D_+, D_+\} = \{\bar{D}_+, \bar{D}_+\} = 0 \ , \qquad \{\bar{D}_+, D_+\} = 2i\partial_+. \tag{2.3}$$

Vector Multiplet

The vector multiplet V lives in a real superfield. The component expansion of V is

$$V = (A_0 - A_1) - i\theta^+\bar{\zeta}_- - i\bar{\theta}^+\zeta_- + \theta^+\bar{\theta}D \ . \tag{2.4}$$

where A is a gauge field, ζ_- is an adjoint-valued left-handed fermion, and D is a real auxiliary field.

To construct gauge theories, a standard method is to define a new multiplet which contains the field strength. So, we should introduce the gauge covariant derivatives first,

$$\mathcal{D}_+ = \frac{\partial}{\partial\theta^+} - i\bar{\theta}^+(\mathcal{D}_0 + \mathcal{D}_1) \ , \qquad \bar{\mathcal{D}}_+ = -\frac{\partial}{\partial\bar{\theta}^+} + i\theta^+(\mathcal{D}_0 + \mathcal{D}_1) \ , \qquad \mathcal{D}_- = \mathcal{D}_0 - \mathcal{D}_1 \ , \tag{2.5}$$

where $\mathcal{D}_0 + \mathcal{D}_1 = \partial_0 + \partial_1 - i(A_0 + A_1)$ and $\mathcal{D}_0 - \mathcal{D}_1 = \partial_0 - \partial_1 - iV$.

The field strength, $F_{01} = \partial_0 A_1 - \partial_1 A_0 - i[A_0, A_1]$ contains in a Fermi multiplet,

$$\Upsilon = [\bar{\mathcal{D}}_+, \mathcal{D}_-] = -\zeta_- - i\theta^+(D - iF_{01}) - i\theta^+\bar{\theta}^+(\mathcal{D}_0 + \mathcal{D}_1)\zeta_- \ . \tag{2.6}$$

The standard action for vector multiplet is given by

$$\begin{aligned} S_\Upsilon &= \frac{1}{8g^2}\mathrm{Tr}\int d^2x d^2\theta \ \Upsilon^\dagger\Upsilon \\ &= \frac{1}{g^2}\mathrm{Tr}\int d^2x \left(\frac{1}{2}F_{01}^2 + i\bar{\zeta}_-(\mathcal{D}_0 + \mathcal{D}_1)\zeta_- + D^2\right) \ . \end{aligned} \tag{2.7}$$

Chiral Multiplet

An $\mathcal{N} = (0,2)$ Chiral multiplet is a complex-valued bosonic superfield Φ which satisfies the equation

$$\bar{\mathcal{D}}_+\Phi = 0 \ . \tag{2.8}$$

The component expansion is

$$\Phi = \phi + \theta^+\psi_+ - i\theta^+\bar{\theta}^+(D_0 + D_1)\phi \ , \tag{2.9}$$

where ψ_+ is a right-moving fermion and ϕ is a complex scalar, each transforming in the same representation of gauge group.

The action for the chiral multiplet is written as

$$\begin{aligned} S_\Phi &= \int d^2x d^2\theta \ \bar{\Phi}(\mathcal{D}_0 - \mathcal{D}_1)\Phi \\ &= \int d^2x(-|D_\mu\phi|^2 + i\bar{\psi}_+(D_0 - D_1)\psi_+ - i\bar{\phi}\zeta_-\psi_+ + i\bar{\psi}_+\bar{\zeta}_-\phi + \bar{\phi}D\phi) \ . \end{aligned} \tag{2.10}$$

Fermi Multiplet

For $\mathcal{N} = (0,2)$ theories, left-moving fermions do not necessarily have bosonic superpartners. A fermion ψ_- of this kind in a Fermion multiplet Ψ obeys the condition

$$\bar{\mathcal{D}}_+\Psi = E(\Phi_i) \ , \tag{2.11}$$

where $E(\Phi_i)$ is a holomorphic function of the chiral multiplet Φ_i. The condition (2.11) requires $E(\Phi_i)$ transforms in the same manner as Ψ under any symmetries. The choice of E determines the interaction of the theory and also appears in the component expansion,

$$\begin{aligned} \Psi =& \psi_- - \theta^+G - i\theta^+\bar{\theta}^+(D_0 + D_1)\psi_- - \bar{\theta}^+E \\ =& \psi_- - \theta^+G - i\theta^+\bar{\theta}^+(D_0 + D_1)\psi_- - \bar{\theta}^+E(\phi_i) + \theta^+\bar{\theta}^+\frac{\partial E}{\partial\phi^i}\psi_{+,i} \ , \end{aligned} \tag{2.12}$$

here G is a complex auxiliary field.

Then, the action for the fermi multiplet is given by

$$\begin{aligned} S_\Psi &= \int d^2x \ d^2\theta\bar{\Psi}\Psi \\ &= \int d^2x \ i\bar{\psi}_-(D_0 + D_1)\psi_- + |G|^2 - |E(\phi_i)|^2 - \bar{\psi}_-\frac{\partial E}{\partial\phi_i}\psi_{+,i} + \bar{\psi}_{+,i}\frac{\partial\bar{E}}{\partial\bar{\phi}_i}\psi_- \ . \end{aligned} \tag{2.13}$$

Superpotential

There are two ways to construct potential terms in theories with $\mathcal{N} = (0,2)$ supersymmetry. The first one is introduced in (2.13). Another way is considered by introducing the holomorphic function $J^a(\Phi_i)$ for Fermi multiplet Ψ_a. This is usually referred to as superpotential in $\mathcal{N} = (0,2)$ theories, which is given by

$$\mathcal{W} = \Psi_a J^a(\Phi_i) \ . \tag{2.14}$$

The action for the superpotential $\mathcal{W}$ is written as

$$\begin{aligned} S_J &= \int d^2x d\theta^+ \ \sum_a \Psi_a J^a(\Phi_i) + \text{h.c.} \\ &= \sum_a \int d^2x G_a J^a(\phi_i) + \sum_i \psi_{-,a} \frac{\partial J^a}{\partial \phi_i} \psi_{+,i} + \text{h.c.} \ . \end{aligned} \tag{2.15}$$

After integrating out the auxiliary field G_a, the action S_J is proportional to $|J^a(\phi_i)|^2$. Note that S_J is preserved by supersymmetry if and only if $\bar{D}_+(\Psi_a J^a) = 0$, which leads to the condition

$$E \cdot J := \sum_a E_a J^a = 0 \ . \tag{2.16}$$

't Hooft Anomaly and Central Charge

Since (0,2) theories are chiral theories, one must pay attention to anomalies. Let us consider a (0,2) theory with a global symmetry F described by a simple Lie algebra. The 't Hooft anomaly coefficient k_F associated with this symmetry can be determined by

$$\text{Tr}\ \gamma^3 f^a f^b = k_F \delta^{ab} \ . \tag{2.17}$$

Here, f^a represents the generators of F, γ^3 denotes the gamma matrix that quantifies chirality, and the trace is taken over Weyl fermions in the theory. For a global anomaly, the computation involves evaluating the difference between the sums over the sets of (0,2) chiral and Fermi multiplets:

$$k_F = \sum_{\Phi \in (0,2)\ \text{chiral}} T(R^{\Phi_F}) - \sum_{\Psi \in (0,2)\ \text{Fermi}} T(R^{\Psi_F}) \ , \tag{2.18}$$

where $T(R_F)$ represents the index of the representation R_F of F. Note that the (0,2) supermultiplet of the gauge invariant field strength can be treated as a Fermi multiplet, and the gaugino contributes to the anomaly if it is charged under F. In this paper, we focus solely on SU(N) groups as instances of non-Abelian symmetries. For these groups, the indices are given by $T(\square) = \frac{1}{2}$ and $T(\mathbf{adj}) = N$. In cases where the theory possesses two U(1) symmetries, $\mathrm{U}(1)_1$ and $\mathrm{U}(1)_2$, with charges f_1 and f_2 respectively, a mixed 't Hooft anomaly can emerge:

$$k_{12} = \text{Tr}(\gamma^3 f_1 f_2) \ . \tag{2.19}$$

Certainly, any gauge anomaly must vanish for the theory to be well-defined.

In particular, the anomaly associated with the $\mathrm{U}(1)_R$-symmetry is related to the right-moving central charge c_R as

$$c_R = 3\text{Tr}(\gamma^3 R^2) \ . \tag{2.20}$$

To determine $\mathrm{U}(1)_R$-charges of various fields, the c-extremization is performed[32–33] if the theory meets the following two assumptions:

1. the theory is bounded, and the energy spectrum is bounded from below,

2. the classical vacuum moduli space is normalizable.

Once c_R is determined, the left-moving central charge can be obtained from the gravitational anomaly, which is the difference between the number of chiral and Fermi multiplets

$$c_R - c_L = \mathrm{Tr}(\gamma^3) \ . \tag{2.21}$$

In the end, let's summarize the (0,2) supermultiplets briefly:

Multiplets	Superfield	Componets
Vector	V	(A, ζ_-)
Chiral	Φ	(ϕ, ψ_+)
Fermi	Ψ	(ψ_-)

Table 2-1 (0,2) supermultiplets

2.1.2 $\mathcal{N} = (0,4)$

Theories with $\mathcal{N} = (0,4)$ supersymmetry have four real right-moving supercharges. The R-Symmetry is $\mathfrak{so}(4)_R \cong \mathfrak{su}(2)^-_R \times \mathfrak{su}(2)^+_R$. The supercharges transform in the $(\mathbf{2},\mathbf{2})_+$ representation, the subscript denoting the chirality under the $\mathrm{SO}(1,1)$ Lorentz group.

Now, we construct (0,4) multiplets from (0,2) multiplets.

Vector Multiplet

The $\mathcal{N} = (0,4)$ vector multiplet Λ consists of an $\mathcal{N} = (0,2)$ vector multiplet V and an Fermi multiplet Θ in adjoint representation. The Fermi multiplet Θ satisfies the equation $\bar{\mathcal{D}}_+\Theta = E_\Theta$ as (2.11). The pair of left moving complex fermions transforms as $(\mathbf{2},\mathbf{2})_-$ under the R-symmetry and a triplet of auxiliary fields transforms as $(\mathbf{3},\mathbf{1})$. The action of $\mathcal{N} = (0,4)$ vector multiplet is given by the sum of (2.7) and (2.13).

Hypermultiplet

There are two different types of hypermultiplets, hyper and twisted-hyper, distinguished by how they couple to a vector multiplet. They are also referred to as J-type hypermultiplets and E-type hypermultiplets.

An $\mathcal{N} = (0,4)$ hypermultiplet consists of two (0,2) chiral multiplets, Φ and $\tilde{\Phi}$. The representations of them are conjugate to each other under gauge transformation. The pair of complex scalars transforms as $(\mathbf{2},\mathbf{1})$ under the R-symmetry, the pair of fermions transforms as $(\mathbf{1},\mathbf{2})_+$.

The kinetic terms for $\mathcal{N} = (0,4)$ hypermultiplet comes from its components Φ and $\tilde{\Phi}$ as (2.10) . Additionally, there is a coupling to the Fermi multiplet Θ in the $\mathcal{N} = (0,4)$ vector multiplet, which is given by the (0,2) superpotential

$$J^\Theta = \Phi\tilde{\Phi} \qquad \Rightarrow \qquad \mathcal{W}_\Theta = \tilde{\Phi}\Theta\Phi \ . \tag{2.22}$$

An $\mathcal{N} = (0,4)$ twisted hypermultiplet also contains two (0,2) chiral multiplets, Φ' and $\tilde{\Phi}'$. The difference comes from how R-symmetry acts on the twisted hypermultiplet. Precisely, the pair of scalars transform as $(\mathbf{1},\mathbf{2})$, and the pair of right-moving fermions transform as $(\mathbf{2},\mathbf{1})_+$. The R-symmetry for twisted hypermultiplet imposes a condition on Θ in (0,4) vector multiplet,

$$E_\Theta = \Phi'\tilde{\Phi}' \ . \tag{2.23}$$

The combination $\Phi'\tilde{\Phi}'$ then transforms in the adjoint representation of the gauge group.

To preserve $\mathcal{N}=(0,2)$ supersymmetry, the condition (2.16), $E\cdot J=0$, is necessary. If there is only a single vector multiplet Λ which contributes to a (0,2) Fermi multiplet Θ, this condition reduces to $E_\Theta J^\Theta=0$, which cannot be naively satisfied if we try to couple hypermultiplets or twisted hypermultiplets to the vector multiplet Λ. This leads to the need for (0,4) Fermi multiplets below.

Fermi Multiplet

An $\mathcal{N}=(0,4)$ an Fermi multiplet consists of a pair of $\mathcal{N}=(0,2)$ Fermi multiplets[①], Ψ and $\tilde{\Psi}$, transforming in the conjugate representations of the gauge group. The left handed fermions transforms as $(\mathbf{1},\mathbf{1})_-$ under the R-symmetry.

The action for the (0,4) Fermi multiplet is given by the sum of two (0,2) Fermi multiplet actions as (2.13); no further coupling between Ψ, $\tilde{\Psi}$ and Θ is needed.

The $\mathcal{N}=(0,4)$ supermultiplets are summarized here for convenience,

Multiplets	(0,2) Superfields	Componets	$\mathfrak{su}(2)_R^-\times\mathfrak{su}(2)_R^+$
Vector	Vector + Fermi	$(A_\mu,\zeta_{-,a})$	$(\mathbf{1},\mathbf{1})$,$(\mathbf{2},\mathbf{2})_-$
Hypermultiplet	Chiral + Chiral	$(\phi^a,\psi_{+,b})$	$(\mathbf{2},\mathbf{1})$,$(\mathbf{1},\mathbf{2})_+$
Twisted-Hyper	Chiral + Chiral	(ϕ'_a,ψ'^{b}_{+})	$(\mathbf{1},\mathbf{2})$,$(\mathbf{2},\mathbf{1})_+$
Fermi	Fermi + Fermi	(ψ_-^a)	$(\mathbf{1},\mathbf{1})_-$

Table 2-2 (0,4) supermultiplets

where $a,b,\mu=0,1$.

2.1.3 Chiral symmetry in two dimension

The global conformal symmetry in two dimensions is $\mathfrak{sl}(2,\mathbb{C})$. The complexification of the Lie algebra is generated by the following holomorphic and anti-holomorphic operators,

$$\begin{aligned} L_{-1}&=-\partial_z\ , \quad L_0=-z\partial_z,\ L_1=-z^2\partial_z\ ,\\ \bar{L}_{-1}&=-\bar{\partial}_z\ , \quad \bar{L}_0=-\bar{z}\bar{\partial}_z,\ \bar{L}_1=-\bar{z}^2\bar{\partial}_z\ . \end{aligned} \tag{2.24}$$

These operators obey the $\mathfrak{sl}(2,\mathbb{C})\times\mathfrak{sl}(2,\mathbb{C})$ commutation relations,

$$\begin{aligned} [L_1,L_{-1}]&=2L_0\ ,\quad [L_0,L_\pm]=\mp L_\pm\ ,\\ [\bar{L}_1,\bar{L}_{-1}]&=2\bar{L}_0\ ,\quad [\bar{L}_0,\bar{L}_\pm]=\mp\bar{L}_\pm\ . \end{aligned} \tag{2.25}$$

For simplicity, we will assume that the space of local operators decomposes into a direct sum of irreducible highest weight representations of the global conformal group. Such representations are labelled by eigenvalues of L_0 and $\bar{L}_0$ of the highest weight state as follows,

$$L_0|h,\bar{h}\rangle=h|h,\bar{h}\rangle,\quad \bar{L}_0|h,\bar{h}\rangle=\bar{h}|h,\bar{h}\rangle\ . \tag{2.26}$$

The infinite chiral symmetry arises from the meromorphic operators in 2d. A local operator $\mathcal{O}(z)$ which obeys the meromorphicity condition $\partial_{\bar{z}}\mathcal{O}(z)=0$ will transform as a trivial representation under the anti-holomorphic part of the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$. By locality, such an operator $\mathcal{O}(z)$ has the following mode expansion,

$$\mathcal{O}(z)=\sum_n z^{-n-h}\mathcal{O}_n\ ,\quad \mathcal{O}_n:=\oint\frac{dz}{2\pi i}z^{n+h-1}\mathcal{O}(z)\ . \tag{2.27}$$

① In principle, the (0,4) Fermi multiplet doesn't need to be a pair of (0,2) Fermi multiplets.

The operator product expansion(OPE) of two meromorphic operators $\mathcal{O}(z)$ and $\mathcal{O}'(z)$ only contains meromorphic operators, and the singular terms will determine the commutation relations among $\mathcal{O}_n$ and $\mathcal{O}'_m$. Here, the commutation relation is defined under radial quantization by

$$[A,B] := \oint_0 dw \oint_w dz\ a(z)b(w)\ , \tag{2.28}$$

where $A := \oint a(z)dz$ and $B := \oint b(w)dw$.

A canonical example is the energy-momentum tensor $T_{\mu\nu}$ in a 2d CFT. The conservation of $T_{\mu\nu}$ leads to two independent equations:

$$\partial_{\bar{z}} T_{zz} = 0 \ \Rightarrow T_{zz} := T(z)\ , \quad \partial_z T_{\bar{z}\bar{z}} = 0 \ \Rightarrow T_{\bar{z}\bar{z}} := \bar{T}(\bar{z})\ . \tag{2.29}$$

Hence, the energy-momentum $T(z)$ is a meromorphic operator. The mode expansion of $T(z)$ is defined as (2.27),

$$L_n := \oint \frac{dz}{2\pi i} z^{n+1} T(z)\ . \tag{2.30}$$

The associated algebra of L_n is known as *Virasoro algebra*,

$$\begin{aligned}[L_n, L_m] &= \oint_0 dw\ w^{n+1} \oint_w dw\ z^{m+1} T(z)T(w) \\ &\overset{(2.32)}{=} (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0}\ .\end{aligned} \tag{2.31}$$

Here, we replace the $T(z)T(w)$ with the following OPE,

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{(z-w)}\ . \tag{2.32}$$

Similarly, global symmetries can give rise to conserved holomorphic currents $J^A(z)$ with $(h,\bar{h}) = (1,0)$. The OPE of $J^A(z)J^B(w)$ will take the generic form

$$J^A(z)J^B(w) \sim \frac{\delta^{AB}k}{(z-w)^2} + \sum_C i f^{ABC} \frac{J^C(w)}{z-w}\ , \tag{2.33}$$

where f^{ABC} is the structure constant of the Lie algebra of the corresponding global symmetry. The commutation relations of the modes form an affine Lie algebra at level k,

$$J_n^A := \oint z^n J^A(z), \quad [J_n^A, J_m^B] = \sum_C i f^{ABC} J^C_{m+n} + mk\delta^{AB}\delta_{n+m,0}\ . \tag{2.34}$$

The algebra of all meromorphic operators constitutes the *chiral algebra* or *vertex operator algebra*(VOA) of a 2d CFT. Hence, the Virasoro algebra should be contained in the corresponding chiral algebra.

2.2 Elliptic genera

In the analysis of 2d supersymmetric theories, a fundamental tool is the elliptic genus[34], which counts BPS states protected under RG flow. Conceptually, the elliptic genus can be understood as a partition function defined on a torus with a complex structure parameter τ, where fermions exhibit periodic boundary conditions along the temporal circle. Moreover, the spatial circle allows for two distinct types of boundary conditions, Ramond and Neveu-Schwarz (NS), both applicable to the left- and right-moving sectors. For the sake of simplicity, we will focus on the Ramond-Ramond and NS-NS sectors, referring to them as the Ramond and NS sectors, respectively. Here, we review some basic facts about the elliptic genus, mainly following the papers[9,35].

2.2.1 $\mathcal{N}=(0,2)$ Elliptic genera

For a 2d $\mathcal{N}=(0,2)$ supersymmetric field theory with a flavor symmetry group $\mathcal{F}$, the elliptic genus in the Ramond and NS sectors is defined respectively as

$$\mathcal{I}^{(0,2),R}(z,q)=\mathrm{Tr}_R(-1)^F q^{H_L}\overline{q}^{H_R}\prod_i z_i^{f_i}\ , \tag{2.35}$$

$$\mathcal{I}^{(0,2),NS}(z,q)=\mathrm{Tr}_{NS}(-1)^F q^{H_L}\overline{q}^{H_R-\frac{1}{2}J_R}\prod_i z_i^{f_i}\ , \tag{2.36}$$

where the left- and right-moving Hamiltonians are $2H_L=H+iP$ and $2H_R=H-iP$ respectively in the Euclidean signature, J_R represents the $\mathrm{U}(1)_R$ charge operator, and f_i are the Cartan generators of the flavor symmetry. In a superconformal theory, these operators correspond to the zero-mode generators L_0, $\overline{L}_0$, and $\overline{J}_0$ of the superconformal algebra.

Due to supersymmetry, only right-moving ground states ($H_R=0$) contribute to the elliptic genus in the Ramond sector, while right-moving chiral primary states ($H_R=\frac{R}{2}$) in the right-moving sector contribute to the elliptic genus in the NS sector. Consequently, the elliptic genera in both sectors are holomorphic functions of q.

For $\mathcal{N}=(0,2)$ theories described by a Lagrangian, the computation of the elliptic genus depends on the specific details of the gauge theory and its matter content, as outlined in the papers[35–36]. Let us consider the contributions from different types of multiplets:

Chiral Multiplet: The contribution of an $\mathcal{N}=(0,2)$ chiral multiplet Φ in a representation λ of the gauge and flavor group is

$$\mathcal{I}_{\Phi}^{(0,2)_{\mathrm{R}}}(z,q)=\prod_{w\in\lambda} i\frac{\eta(q)}{\vartheta_1(z^w)}\ ,\qquad \mathcal{I}_{\Phi}^{(0,2)_{\mathrm{NS}}}(z,q)=\prod_{w\in\lambda}\frac{\eta(q)}{\vartheta_4(q^{\frac{r-1}{2}}z^w)}. \tag{2.37}$$

Fermi Multiplet: The contribution of an $\mathcal{N}=(0,2)$ Fermi multiplet Ψ in a representation λ of the gauge and flavor group is given by

$$\mathcal{I}_{\Psi}^{(0,2)_{\mathrm{R}}}(z,q)=\prod_{w\in\lambda} i\frac{\vartheta_1(z^w)}{\eta(q)}\ ,\qquad \mathcal{I}_{\Psi}^{(0,2)_{\mathrm{NS}}}(z,q)=\prod_{w\in\lambda}\frac{\vartheta_4(q^{\frac{r}{2}}z^w)}{\eta(q)}. \tag{2.38}$$

Vector Multiplet: The contribution of an $\mathcal{N}=(0,2)$ vector multiplet V with gauge group G is

$$\mathcal{I}_{V,G}^{(0,2)_{\mathrm{R|NS}}}(q,z)=\frac{\eta(q)^{2\,\mathrm{rk}G}}{|W_G|}\prod_{\alpha\in\Delta} i\frac{\vartheta_1(z^\alpha)}{\eta(q)}\ , \tag{2.39}$$

where ω runs over the weights of representation λ and α runs over the roots of gauge group G. Note that the elliptic genera in the NS sector for chiral and Fermi multiplet depend on the R-charge r of the multiplet.

Then, the elliptic genus of a quiver gauge theory can be schematically expressed as the Jeffrey-Kirwan (JK) residue integral[36–39]

$$\mathcal{I}^{(0,2)_{\mathrm{R|NS}}}=\int_{\mathrm{JK}}\prod_{\mathrm{gauge}}\frac{dz}{2\pi iz}\mathcal{I}_{\Lambda,G}^{(0,2)_{\mathrm{R|NS}}}(q,z)\prod_{\mathrm{matter}}\mathcal{I}_{\Phi}^{(0,2)_{\mathrm{R|NS}}}(q,z)\mathcal{I}_{\Psi}^{(0,2)_{\mathrm{R|NS}}}(q,z)\ . \tag{2.40}$$

’t Hooft Anomaly

’t Hooft anomalies described in (2.17) can be observed directly at the level of $(0,2)$ elliptic genera. For an $\mathrm{SU}(N)$ global symmetry whose fugacities are denoted by $b_{1,\ldots,N}$ with $\prod_{i=1}^N b_i=1$ in an elliptic genus, the corresponding ’t Hooft anomaly can be seen in the shift of the elliptic genus as

$$\mathcal{I}^{(0,2)}(b)\to\left(qb_i/b_j\right)^{2k_F}\mathcal{I}^{(0,2)}(b)\qquad\text{as}\qquad b_i\to qb_i, b_j\to b_j/q\ . \tag{2.41}$$

For a U(1) 't Hooft anomaly, the elliptic genus behaves as

$$\mathcal{I}^{(0,2)}(c) \to (-q^{1/2}c)^{k_F}\mathcal{I}^{(0,2)}(c) \qquad \text{as} \qquad c \to qc \ , \tag{2.42}$$

where c is the corresponding U(1) fugacity. In particular, for a theory to be gauge anomaly-free, the integrand of its elliptic genus (2.40) must be elliptic with respect to gauge fugacities $z_i \to qz_i$ (i.e invariant under shifts of the gauge fugacities $z_i \to qz_i$).

2.2.2 $\mathcal{N} = (0,4)$ Elliptic genera

Since a 2d (0,4) theory is also a (0,2) theory, one can first decompose a (0,4) supersymmetry algebra into its (0,2) subalgebra and compute its (0,2) elliptic genus, which gives us the (0,4) elliptic genus. The R-symmetry of (0,4) theory is $\mathrm{SU}(2)^1_R \times \mathrm{SU}(2)^+_R$, we may choose (0,2) R-charge as $J_R = (1-\gamma)R^- + (1+\gamma)R^+$, then the rest combination $R_v = 2(R^- - R^+)$ will become a global symmetry in (0,2) algebra[9].

The different multiplets contribute to a $(0,4)$ elliptic genus as follows:

Hypermultiplet: The contribution of an $\mathcal{N} = (0,4)$ hypermultiplet (Φ, Φ') in a representation λ of the gauge and flavor group is

$$\mathcal{I}^{(0,4)_\mathrm{R}}_{(\Phi,\Phi')}(z,q) = \prod_{w\in\lambda} i\frac{\eta(q)}{\vartheta_1(vz^w)} \ , \qquad \mathcal{I}^{(0,4)_\mathrm{NS}}_{(\Phi,\Phi')}(z,q) = \prod_{w\in\lambda} \frac{\eta(q)}{\vartheta_4(q^{-\frac{1}{2}}vz^w)} \ . \tag{2.43}$$

Twisted Hypermultiplet: The contribution of an $\mathcal{N} = (0,4)$ twisted hypermultiplet $(\tilde{\Phi}, \tilde{\Phi}')$ in a representation λ of the gauge and flavor group is

$$\mathcal{I}^{(0,4)_\mathrm{R}}_{(\tilde{\Phi},\tilde{\Phi}')}(z,q) = \prod_{w\in\lambda} i\frac{\eta(q)}{\vartheta_1(v^{-1}z^w)} \ , \qquad \mathcal{I}^{(0,4)_\mathrm{NS}}_{(\tilde{\Phi},\tilde{\Phi}')}(z,q) = \prod_{w\in\lambda} \frac{\eta(q)}{\vartheta_4(v^{-1}z^w)} \ . \tag{2.44}$$

Fermi Multiplet: The contribution of an $\mathcal{N} = (0,4)$ Fermi multiplet (Ψ, Ψ') in a representation λ of the gauge and flavor group is given by

$$\mathcal{I}^{(0,4)_\mathrm{R}}_{(\Psi,\Psi')}(z,q) = \prod_{w\in\lambda} i\frac{\vartheta_1(z^w)}{\eta(q)} \ , \qquad \mathcal{I}^{(0,4)_\mathrm{NS}}_{(\Psi,\Psi')}(z,q) = \prod_{w\in\lambda} \frac{\vartheta_4(z^w)}{\eta(q)} \ . \tag{2.45}$$

Vector Multiplet: The contribution of an $\mathcal{N} = (0,4)$ vector multiplet Λ with gauge group G is

$$\mathcal{I}^{(0,4)_{\mathrm{R|NS}}}_{\Lambda,G}(q,z) = \frac{\Big(\eta(q)\vartheta_1(v^2)\Big)^{2\,\mathrm{rk}G}}{|W_G|} \prod_{\alpha\in\Delta} \frac{\vartheta_1(v^2z^\alpha)\vartheta_1(z^\alpha)}{\eta(q)^2} \ . \tag{2.46}$$

Here, we choose γ to be 1, and the fugacity v labels the anti-diagonal Cartan generator of $\mathrm{SU}(2)^-_R \times \mathrm{SU}(2)^+_R$ mentioned above. Also, we will choose the R sector for $(0,4)$ computation in chapter 4.

2.2.3 Jeffrey-Kirwan residue integrals

Since the elliptic genera receive contributions from both non-degenerate and degenerate poles in general, a thorough review of the JK residue integral definition[36–38] is beneficial for the thesis to be self-contained.

In the context of a rank-r gauge theory, the elliptic genus computed through the JK-residue technique integrates an r-form over specific cycles, and is conventionally represented as

$$\mathcal{I}^{\mathrm{2d}} = \int_{\mathrm{JK}} \prod_{i=1}^{r} \frac{da_i}{2\pi i a_i}\mathcal{Z}(a) = \oint_{\text{special cycles}} \mathcal{Z}(a)da_1 \wedge ... \wedge da_r \ . \tag{2.47}$$

As stated in the main text, $a_i = e^{2\pi i \mathsf{a}_i}$, and similarly for other variables except for $q = e^{2\pi i \tau}$. For our purpose, the integrand $\mathcal{Z}$, as a function of a_i, is separately elliptic in each a_i, namely,

$$\mathcal{Z}(\dots, \mathsf{a}_i + \tau, \dots) = \mathcal{Z}(\dots, \mathsf{a}_i, \dots), \quad \mathcal{Z}(\dots, \mathsf{a}_i + 1, \dots) = \mathcal{Z}(\dots, \mathsf{a}_i, \dots) \ . \tag{2.48}$$

More concretely, $\mathcal{Z}$ takes the form of certain ratios of the Jacobi theta functions ϑ_1, and poles come from the zeros of ϑ_1 in the denominator. Each pole is given as a solution to a set of pole equations

$$\sum_{i=1}^{r} Q_a^i \mathsf{a}_i + \mathsf{b}_a = m_a + n_a \tau \ , \qquad Q_a^i \in \mathbb{Z} \ , m_a, n_a \in \mathbb{N} \ , a = 1, 2, \dots, r \ , \tag{2.49}$$

coming from some factors $\vartheta_1(Q_a^i \mathsf{a}_i + \mathsf{b}_a)^{N_a}$ in the denominator of $\mathcal{Z}$. Note that m_a, n_a only take values in a finite range in $\mathbb{N}$ that will be determined by the charge vectors $Q_a = (Q_a^1, Q_a^2, \dots, Q_r^r)$. A few remarks follow.

- Zeros from numerators may arise in certain solutions of the pole equations, reducing the pole's order. If the total order of the pole is below r, it is not included in the JK residue.
- At some poles a_*, there may be $n > r$ factors of ϑ_1^N's simultaneously made zero by a_*, associated with n different charge vectors $Q_1, \dots, Q_n$. This is referred to as a degenerate pole.
- A pole associated to precisely r different ϑ_1^N factors and therefore r different charge vectors $Q_1, \dots, Q_r$ is referred to as a non-degenerate pole.
- The range of m_a, n_a is not unique. We start by rearranging the Q_a terms such that $Q_i^i \neq 0$. Then m_a, n_a are defined by methods like the Hermite or Smith normal form decomposition of the (reordered) integral square matrix $(Q)_{ai} := Q_a^i$. In the Smith decomposition,

  $$UQV = D, \qquad \begin{array}{l} U, V \text{ are integral and } |\det U| = |\det V| = 1 \\ D \text{ is diagonal.} \end{array}$$

  Then we fix the range $m_a, n_a, = 0, 1, \dots, D_{aa} - 1$. Alternatively, in Hermite decomposition, $UQ = T$ with a unimodular integral U, and T is an upper triangular integral matrix. In this case, $m_a, n_a = 0, 1, \dots, T_{aa} - 1$. Note that although $T_{aa} \neq D_{aa}$ in general, the final result of the JK-residue computation will be the same.

For any pole a_* satisfying (2.49), we need to compute the corresponding JK residue. We follow the constructive definition of JK residue[36]. To begin, one picks a generic reference vector $\eta \in \mathfrak{h}^*$ of the gauge group. If a_* is a non-degenerate pole with charge vectors $Q_1, \dots, Q_r$, its contribution is given by

$$\underset{\mathsf{a}_*}{\text{JK-Res}}(\eta)\mathcal{Z} = \delta(Q, \eta) \frac{1}{|\det Q|} \operatorname{Res}_{\epsilon_r = 0} \cdots \operatorname{Res}_{\epsilon_1 = 0} \mathcal{Z} \bigg|_{Q_a \mathsf{a} + \mathsf{b}_a = m_a + n_a \tau + \epsilon_a} \ , \tag{2.50}$$

where $\delta(Q, \eta)$ equals one when η is inside the cone spanned by $Q_1, \dots, Q_r$, and zero otherwise. The residues are calculated in sequence.

For a degenerate pole, we identify an associated set of charge vectors, $Q_* = \{Q_1, \dots, Q_n\}$, with $n > r$. From the set Q_*, a collection of geometric objects can be defined.

- Given any r-sequence of linearly independent charge vectors $(Q_{a_1}, \dots, Q_{a_r})$ from Q_*, we can construct a flag, F. This flag is essentially a series of nested subspaces of $\mathbb{R}^r$:

  $$\{0\} \subset F_1 \subset \dots \subset F_r = \mathbb{R}^r, \quad F_\ell = \operatorname{span}\{Q_{a_1}, \dots, Q_{a_\ell}\} \ . \tag{2.51}$$

  Note that different sequences may give rise to the same flag. When this happens, we only consider one of them. The sequence $(Q_{a_1}, \dots, Q_{a_r})$ is often called a basis $\mathcal{B}(F, Q_*)$ of F in Q_*. Given an F, the basis in Q_* is generally not unique, but we pick an arbitrary one.

- From each flag F and its basis $\mathcal{B}(F, Q_*)$, one constructs a sequence of vectors

$$\kappa(F, Q_*) := (\kappa_1, \dots, \kappa_r), \quad \kappa_a = \sum_{\substack{Q \in Q_* \\ Q \in F_a}} Q \ . \tag{2.52}$$

  One further defines $\operatorname{sign} F := \operatorname{sign} \det \kappa(F, Q_*)$.

- For each $\kappa(F)$, one constructs a closed cone $\mathsf{c}(F, Q_*)$ spanned by $\kappa(F, Q_*)$.

With these objects defined, the JK residue of the given degenerate pole a_* is given by

$$\underset{\mathsf{a}_*}{\text{JK-Res}}(\eta)\mathcal{Z} = \sum_F \delta(F, \eta) \frac{\operatorname{sign} F}{\det \mathcal{B}(F, Q_*)} \underset{\epsilon_r = 0}{\text{Res}} \cdots \underset{\epsilon_1 = 0}{\text{Res}} \mathcal{Z} \Big|_{\substack{Q_{a_1}\mathsf{a} + \mathsf{b}_{a_1} = m_{a_1} + n_{a_1}\tau + \epsilon_1 \\ \dots \\ Q_{a_r}\mathsf{a} + \mathsf{b}_{a_r} = m_{a_r} + n_{a_r}\tau + \epsilon_r}} , \tag{2.53}$$

where the sum is over all flags constructed out of Q_* associated to a_*. Again, $\delta(F, \eta)$ equals one if the closed-cone $\mathsf{c}(F, Q_*)$ contains η, and zero otherwise. This definition of JK-Res naturally extends to non-degenerate poles, where there are precisely r vectors in Q_*, and

$$\kappa(F, Q_*) = \mathcal{B}(F, Q_*), \qquad \frac{\operatorname{sign} F}{\det \mathcal{B}(F, Q_*)} = \frac{1}{|\det \mathcal{B}(F, Q_*)|} \ . \tag{2.54}$$

The result clearly reduces to the previous definition of JK-residue for the non-degenerate case. Finally, given a generic η,

$$\int_{\text{JK}} \prod_{i=1}^r \frac{da_i}{2\pi i a_i} \mathcal{Z}(a) = \sum_{\mathsf{a}_*} \underset{\mathsf{a}_*}{\text{JK-Res}}(\eta)\mathcal{Z}(a) \ . \tag{2.55}$$

Although the structure of poles and the results of individual JK residues often differ drastically when η varies across chambers, the overall result is independent of the choice of η.

2.3 4d $\mathcal{N} = 2$ SCFTs

In this section, we'd like to review some basic knowledge about $\mathcal{N} = 2$ superconformal field theories(SCFTs). This section is primarily based on the papers[11,40–42]. Let's begin with the generic superconformal algebra in 4d.

2.3.1 4d Superconformal algebra

It's well known that the conformal symmetries for $d \geq 3$ spacetime consist of 4 parts: Lorentz symmetry, translation symmetry, special conformal translations(SCT) , and dilation. Thus, the dimension of 4d conformal group is given by

$$\underbrace{\frac{1}{2}d(d-1)}_{\text{Lorentz}} + \underbrace{d}_{\text{translation}} + \underbrace{d}_{\text{SCT}} + \underbrace{1}_{\text{dilation}} = \frac{1}{2}(d+2)(d+1) \ . \tag{2.56}$$

The corresponding Lie algebra is $\mathfrak{so}(6, \mathbb{C})$ after complexification, which obeys the following commutation relations:

$$\begin{aligned}
&[M_\alpha{}^\beta, M_\gamma{}^\delta] = \delta_\gamma{}^\beta M_\alpha{}^\delta - \delta_\alpha{}^\delta M_\gamma{}^\beta \ , && [M^{\dot\alpha}{}_{\dot\beta}, M^{\dot\gamma}{}_{\dot\delta}] = \delta^{\dot\alpha}{}_{\dot\delta} M^{\dot\gamma}{}_{\dot\beta} - \delta^{\dot\gamma}{}_{\dot\beta} M^{\dot\alpha}{}_{\dot\delta} \ , \\
&[M_\alpha{}^\beta, P_{\gamma\dot\gamma}] = \delta_\gamma{}^\beta P_{\alpha\dot\gamma} - \frac{1}{2}\delta_\alpha{}^\beta P_{\gamma\dot\gamma} \ , && [M^{\dot\alpha}{}_{\dot\beta}, P_{\gamma\dot\gamma}] = \delta^{\dot\alpha}{}_{\dot\gamma} P_{\gamma\dot\beta} - \frac{1}{2}\delta^{\dot\alpha}{}_{\dot\beta} P_{\gamma\dot\gamma} \ , \\
&[M_\alpha{}^\beta, K^{\dot\gamma\gamma}] = -\delta_\alpha{}^\gamma K^{\dot\gamma\beta} + \frac{1}{2}\delta_\alpha{}^\beta K^{\dot\gamma\gamma} \ , && [M^{\dot\alpha}{}_{\dot\beta}, K_{\dot\gamma\gamma}] = -\delta^{\dot\gamma}{}_{\dot\beta} K^{\dot\alpha\gamma} + \frac{1}{2}\delta_\alpha{}^\beta K^{\dot\gamma\gamma} \ , \\
&[H, P_{\alpha\dot\alpha}] = P_{\alpha\dot\alpha}, && [H, K^{\dot\alpha\alpha}] = -K^{\dot\alpha\alpha} \\
&[K^{\alpha\dot\alpha}, P_{\beta\dot\beta}] = \delta_\beta{}^\alpha \delta^{\dot\alpha}{}_{\dot\beta} H + \delta_\beta{}^\alpha M^{\dot\alpha}{}_{\dot\beta} + \delta^{\dot\alpha}{}_{\dot\beta} M_\beta{}^\alpha \ .
\end{aligned} \tag{2.57}$$

where M, P, K, H are the generators of Lorentz symmetry, translation symmetry, SCT and dilation, respectively. Here, we introduce the spinor notion

$$\sigma^{\mu}_{\alpha\dot{\alpha}} = (\mathbb{1}, \vec{\sigma}) \ , \quad \bar{\sigma}^{\mu}_{\alpha\dot{\alpha}} = (-\mathbb{1}, \vec{\sigma}) \ , \quad X^a = \epsilon^{ab} X_b \ , \quad X_a = \epsilon_{ab} X^b \ , \tag{2.58}$$

for the conformal algebra , where $\vec{\sigma}$ are the vectors of Pauli matrices and $\epsilon_{-+} = -\epsilon_{+-} = 1$, $\epsilon^{\dot{+}\dot{-}} = -\epsilon^{\dot{-}\dot{+}} = 1$. And we also rewrite the generators of the conformal group as follows:

$$\begin{aligned} &\mathrm{P}_{\alpha\dot{\alpha}} = (\sigma^{\mu})_{\alpha\dot{\alpha}} P_{\mu} \ , && \mathrm{K}^{\alpha\dot{\alpha}} = (\bar{\sigma}^{\mu})^{\alpha\dot{\alpha}} K_{\mu} \ , \\ &M_{\alpha}{}^{\beta} = -\frac{1}{4} i (\sigma^{\mu}\bar{\sigma}^{\nu})_{\alpha}{}^{\beta} M_{\mu\nu} \ , && M^{\dot{\alpha}}{}_{\dot{\beta}} = -\frac{1}{4} i (\bar{\sigma}^{\mu}\sigma^{\nu})^{\dot{\alpha}}{}_{\dot{\beta}} M_{\mu\nu} \ . \end{aligned} \tag{2.59}$$

The superconformal algebra is realized as $\mathfrak{su}(2,2|\mathcal{N})$ or $\mathfrak{sl}(4|\mathcal{N})$ after complexification[40]①, in the case of four dimensions, by including supercharges $Q^I_{\alpha}, \tilde{Q}_{I\dot{\alpha}}$ as well as superconformal supercharges $S_I^{\alpha}, \tilde{S}^{I\dot{\alpha}}$, where $I = 1, \dots, \mathcal{N}$.

These supercharges satisfy the following anti-commutation relations:

$$\begin{aligned} \{Q^I_{\alpha}, \tilde{Q}_{J\dot{\alpha}}\} = \delta^I{}_J \mathrm{P}_{\alpha\dot{\alpha}} \ , \quad \{\tilde{S}^{I\dot{\alpha}}, \tilde{S}_J{}^{\alpha}\} = \delta^I{}_J \mathrm{K}^{\dot{\alpha}\alpha} \ , \{Q^I_{\alpha}, S_J{}^{\beta}\} &= \frac{1}{2}\delta^I{}_J \delta_{\alpha}{}^{\beta} H + \delta^I{}_J M_{\alpha}{}^{\beta} - \delta_{\alpha}{}^{\beta} R^I{}_J \ , \\ \{\tilde{S}^{I\dot{\alpha}}, \tilde{Q}_{J\dot{\beta}}\} &= \frac{1}{2}\delta^I{}_J \delta^{\dot{\alpha}}{}_{\dot{\beta}} H + \delta^I{}_J M^{\dot{\alpha}}{}_{\dot{\beta}} + \delta^{\dot{\alpha}}{}_{\dot{\beta}} R^I{}_J \ , \end{aligned} \tag{2.60}$$

where R^I_J are the generators of $\mathfrak{u}(\mathcal{N})$ R-symmetry ($\mathfrak{u}(\mathcal{N}) \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{gl}(\mathcal{N}, \mathbb{C})$), for which the Lie algebra is given by

$$[R^I{}_J, R^K{}_L] = \delta^K{}_J R^I{}_L - \delta^I{}_L R^K{}_J \ . \tag{2.61}$$

The R-symmetry acting on the supercharges obeys the following commutation relations:

$$\begin{aligned} &[R^I{}_J, Q^K_{\alpha}] = \delta^K{}_J Q^I_{\alpha} - \frac{1}{4}\delta^I{}_J Q^K_{\alpha} \ , && [R^I{}_J, \tilde{Q}_{K\dot{\alpha}}] = -\delta^I{}_K \tilde{Q}_{J\dot{\alpha}} + \frac{1}{4}\delta^I{}_J \tilde{Q}^K_{\dot{\alpha}} \ , \\ &[R^I{}_J, S_K{}^{\alpha}] = -\delta^I{}_K S_J{}^{\alpha} + \frac{1}{4}\delta^I{}_J S_K{}^{\alpha} \ , && [R^I{}_J, \tilde{S}^{K\dot{\alpha}}] = \delta^K{}_J \tilde{S}^{I\dot{\alpha}} - \frac{1}{4}\delta^I{}_J \tilde{S}^{K\dot{\alpha}} \ . \end{aligned} \tag{2.62}$$

When $\mathcal{N} = 2$, we also express the generators of R-symmetry in terms of generators of $\mathfrak{su}(2)_R \times \mathfrak{u}(1)_r$,

$$R^1{}_2 = R^+ \ , \quad R^2{}_1 = R^- \ , \quad R^1{}_1 = \frac{1}{2} r + R, \quad R^2{}_2 = \frac{1}{2} r - R \ , \tag{2.63}$$

where R is the Cartan generator of $\mathfrak{su}(2)_R$ and r is the generator of $\mathfrak{u}(1)_r$.

The remaining commutation relations involve the supercharges with the conformal symmetry, and they are given by:

$$\begin{aligned} &[M_{\alpha}{}^{\beta}, Q^I_{\gamma}] = \delta_{\gamma}{}^{\beta} Q^I_{\alpha} - \frac{1}{2}\delta_{\alpha}{}^{\beta} Q^I_{\gamma} \ , && [M^{\dot{\alpha}}{}_{\dot{\beta}}, \tilde{Q}_{I\dot{\delta}}] = \delta^{\dot{\alpha}}{}_{\dot{\delta}} \tilde{Q}_{I\dot{\beta}} - \frac{1}{2}\delta^{\dot{\alpha}}{}_{\dot{\beta}} \tilde{Q}_{I\dot{\delta}} \\ &[M_{\alpha}{}^{\beta}, S_I{}^{\gamma}] = -\delta_{\alpha}{}^{\gamma} S_I{}^{\beta} + \frac{1}{2}\delta_{\alpha}{}^{\beta} S_I{}^{\gamma} \ , && [M^{\dot{\alpha}}{}_{\dot{\beta}}, \tilde{S}^{I\dot{\gamma}}] = -\delta^{\dot{\gamma}}{}_{\dot{\beta}} \tilde{S}^{I\dot{\alpha}} + \frac{1}{2}\delta^{\dot{\alpha}}{}_{\dot{\beta}} \tilde{S}^{I\dot{\gamma}} \ , \\ &[H, Q^I_{\alpha}] = \frac{1}{2} Q^I_{\alpha} \ , && [H, \tilde{Q}_{I\dot{\alpha}}] = \frac{1}{2}\tilde{Q}_{I\dot{\alpha}} \ , \\ &[H, S_I{}^{\alpha}] = -\frac{1}{2} S_I{}^{\alpha} \ , && [H, \tilde{S}^{I\dot{\alpha}}] = -\frac{1}{2}\tilde{S}^{I\dot{\alpha}} \ , \\ &[K^{\dot{\alpha}\alpha}, Q^I_{\beta}] = \delta_{\beta}{}^{\alpha} \tilde{S}^{I\dot{\alpha}} \ , && [K^{\dot{\alpha}\alpha}, \tilde{Q}_{I\dot{\beta}}] = \delta^{\dot{\alpha}}{}_{\dot{\beta}} S_I{}^{\alpha} \ , \\ &[P_{\alpha\dot{\alpha}}, S_I{}^{\beta}] = -\delta_{\alpha}{}^{\beta} \tilde{Q}_{I\dot{\alpha}} \ , && [P_{\alpha\dot{\alpha}}, \tilde{S}^{I\dot{\beta}}] = -\delta^{\dot{\beta}}{}_{\dot{\alpha}} Q^I_{\alpha} \ . \end{aligned} \tag{2.64}$$

The superconformal representations are constructed by acting the supercharges Q, $\tilde{Q}$ and $\mathfrak{su}(2)_R$ generators on superconformal primary states which are annihilated by the SCT

① To be precise, the physically relevant superconformal symmetry is $\mathfrak{psu}(2,2|4)$ for $\mathcal{N}=4$, since $\sum_I R^I{}_I$ is central.

generators K and the conformal supercharges S, $\tilde{S}$. Usually, this construction forms the *long multiplet* which is a $2^{4\mathcal{N}}$ times as large as a conformal multiplet. However, when the quantum numbers satisfy specific shortening conditions, the representations of superconformal algebra will reduce to *short multiplets*. For $\mathcal{N}=2$ and $\mathcal{N}=4$, the short multiplets are classified in the paper[40].

2.3.2 $\mathcal{N}=2$ Vector multiplets and hypermultiplets

For 4d $\mathcal{N}=2$ supersymmetric field theories, there are two kinds of supermultiplets: vector multiplets and hypermultiplets. They are both representations of the superconformal algebra. Therefore, it is necessary to review the basic definitions of these two multiplets.

Vector multiplet. In $\mathcal{N}=1$ language, an $\mathcal{N}=2$ vector multiplet consists of one $\mathcal{N}=1$ vector multiplet with components (λ_α, A_μ) and one $\mathcal{N}=1$ chiral multiplet $(\phi, \bar{\lambda}_{\dot{\alpha}})$, which are both in the adjoint representation of the gauge group G,

$$\begin{array}{ccc} \lambda_\alpha \longleftrightarrow A_\mu & & \mathcal{N}=1 \text{ vector multiplet} \\ \updownarrow \qquad\quad \updownarrow & & \\ \phi \longleftrightarrow \bar{\lambda}_{\dot{\alpha}} & & \mathcal{N}=1 \text{ chiral multiplet} \end{array} . \tag{2.65}$$

Vertical arrows indicate the action of the other $\mathcal{N}=1$ supersymmetry. The $\mathrm{SU}(2)_R$ R-symmetry rotates two Weyl fermions λ_α, $\bar{\lambda}_{\dot{\alpha}}$ while acts trivially on ϕ and A_μ.

Hypermultiplet. The hypermultiplet contains an $\mathcal{N}=1$ chiral (q, ψ) and an $\mathcal{N}=1$ anti-chiral $(\bar{q}^\dagger, \psi^\dagger)$

$$\begin{array}{ccc} q \longleftrightarrow \psi & & \mathcal{N}=1 \text{ chiral multiplet} \\ \updownarrow \qquad\quad \updownarrow & & \\ \bar{\psi}^\dagger \longleftrightarrow \bar{q}^\dagger & & \mathcal{N}=1 \text{ anti-chiral multiplet} \end{array} . \tag{2.66}$$

Here, the vertical arrow suggests the other $\mathcal{N}=1$ supersymmetry. The scalar q and $\bar{q}^\dagger$ transform as $\mathrm{SU}(2)_R$ doublet while the fermions ψ and $\bar{\psi}^\dagger$ transform as singlets. The $\mathcal{N}=1$ chiral and anti-chiral multiplets transform in the same representation $\mathcal{R}$ of gauge group. Therefore, q and $\bar{q}$ are in the representation $\mathcal{R}$ and its conjugate $\bar{\mathcal{R}}$.

When $\mathcal{R}$ is pseudo-real or equivalently when there is an antisymmetric invariant tensor ϵ_{ab}, we can impose the following condition on the $\mathcal{N}=1$ chiral multiplets,

$$q_a = \epsilon_{ab}\bar{q}^b \ , \tag{2.67}$$

which is compatible with $\mathcal{N}=2$ supersymmetry. The resulting multiplet is called *half hypermultiplet*.

2.3.3 Superconformal index and Schur index

The superconformal index[43] can be consider as a refined Witten index of a 4d $\mathcal{N}=1$ theory in radial quantization, which is defined by the following formula,

$$\mathcal{I}(\mu_i) = \mathrm{Tr}(-1)^F e^{-\beta\delta} \prod_i {\mu_i}^{T_i} \ , \qquad \delta := 2\{\mathcal{Q}, \mathcal{Q}^\dagger\} \ , \tag{2.68}$$

where $\mathcal{Q}$ is a chosen supercharge, and T_i are the generators of global symmetries that commute with each other and with $\mathcal{Q}$. The trace runs over the Hilbert space of the radially quantized theory on S^{d-1}, where d is the dimension of spacetime. By this setup, only the states satisfy the condition $\delta = 0$ contribute to the index $\mathcal{I}(\mu_i)$, since bosonic states and fermionic states form pairs if $\delta \neq 0$.

A 4d $\mathcal{N} = 2$ superconformal theory has the symmetry algebra $\mathfrak{su}(2,2|2)$, generated by supercharges $(Q^I_\alpha, \tilde{Q}_{I\dot\alpha})$, their superconformal partners $(S_I^\alpha, \tilde{S}^{I\dot\alpha})$ and other bosonic symmetry generators. The 4d $\mathcal{N} = 2$ superconformal index counts the 1/8-BPS states that are annihilated by one supercharge and its conformal partner, say $\tilde{Q}_{2\dot-}$ and $\tilde{S}^{2\dot-} = (\tilde{Q}_{2\dot-})^\dagger$. In other words, it provides a measure of the $\tilde{Q}_{2\dot-}$-cohomology, which consists of states that saturate the bound

$$\tilde{\delta}_{2\dot-} := 2\{\tilde{S}^{2\dot-}, \tilde{Q}_{2\dot-}\} = E - 2j_2 - 2R + r \ .$$

Here, we use the Cartan generators $\left(E := H, j_1 := -M_-{}^-, j_2 := -M^{\dot-}{}_{\dot-}, R, r\right)$ of $\mathfrak{su}(2,2|2)$. We refer to Table 3-1 for charges of supercharges under these symmetry groups. Then, the 4d $\mathcal{N} = 2$ superconformal index, denoted as $\mathcal{I}^{\text{4d}}(p,q,t)$, is defined as follows:

$$\mathcal{I}^{\text{4d}}(p,q,t) = \text{Tr}(-1)^F e^{-\beta\tilde{\delta}_{2\dot-}} p^{j_1+j_2-r} q^{j_2-j_1-r} t^{R+r} \prod_a z_a^{f_a} \ , \tag{2.69}$$

The variables z_a correspond to flavor fugacities, and f_a represent flavor charges. Evaluating the 4d $\mathcal{N} = 2$ superconformal index can be done using single-letter indices, which are given by the table below[5],

Letters	E	j_1	j_2	R	r	$\mathcal{I}(p,q,t)$
ϕ	1	0	0	0	-1	pq/t
$\lambda_{1\pm}$	$\frac{3}{2}$	$\pm\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	$-p,\ -q$
$\bar{\lambda}_{1\dot+}$	$\frac{3}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-t$
$\bar{F}_{\dot+\dot+}$	2	0	1	0	0	pq
$\partial_{-\dot+}\lambda_{1+} + \partial_{+\dot+}\lambda_{1-} = 0$	$\frac{5}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	pq
q	1	0	0	$\frac{1}{2}$	0	$\sqrt{t}$
$\bar{\psi}_{\dot+}$	$\frac{3}{2}$	0	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-pq/\sqrt{t}$
$\partial_{\pm\dot+}$	1	$\pm\frac{1}{2}$	$\frac{1}{2}$	0	0	$p,\ q$

Table 2-3 Contributions to the index from "single letters". We denote by $(\phi, \bar{\phi}, \lambda_{I,\alpha}, \bar{\lambda}_{I\dot\alpha}, F_{\alpha\beta}, \bar{F}_{\dot\alpha\dot\beta})$ the components of the adjoint $\mathcal{N} = 2$ vector multiplet, by $(q, \bar{q}, \psi_\alpha, \bar{\psi}_{\dot\alpha})$ the components of the $\mathcal{N} = 1$ chiral multiplet, and by $\partial_{\alpha\dot\alpha}$ the spacetime derivatives.

Consequently, the single letter partition functions of the $\mathcal{N} = 2$ vector and half-hyper multiplet are given by:

$$\begin{aligned} f^V &= \frac{\frac{pq}{t} - p - q - t + pq + pq}{(1-p)(1-q)} = -\frac{p}{1-p} - \frac{q}{1-q} + \frac{\frac{pq}{t} - t}{(1-p)(1-q)} \\ f^{\frac{1}{2}H} &= \frac{t^{\frac{1}{2}}(1 - \frac{pq}{t})}{(1-p)(1-q)} \end{aligned} \tag{2.70}$$

And, the contribution of the half-hypermultiplet with representation λ to the multi-particle index yields the elliptic gamma function (A.4), given by

$$\begin{aligned} \mathcal{I}^{\text{4d}}_{\frac{1}{2}H}(z;p,q,t) &= \text{PE}[f^{\frac{1}{2}H}\chi_\lambda] \\ &= \prod_{w\in\lambda}\prod_{i,j=0}^{\infty} \frac{1 - z^{-w}p^{i+1}q^{j+1}/\sqrt{t}}{1 - z^w\sqrt{t}p^iq^j} = \prod_{w\in\lambda} \Gamma(z^w\sqrt{t}) \ , \end{aligned} \tag{2.71}$$

where χ_λ is the character of representation λ and w runs over the weights of the representation λ. The 4d $\mathcal{N} = 2$ vector multiplet contributes as follows:

$$\begin{aligned} \mathcal{I}^{\text{4d}}_{\text{vec}}(z;p,q,t) &= \frac{1}{|W_G|}\text{PE}[f^V\chi_{adj}] \\ &= \frac{\kappa^{\text{rk}G}\Gamma(\frac{pq}{t})^{\text{rk}G}}{|W_G|} \prod_{\alpha\in\Delta} \frac{\Gamma(z^\alpha\frac{pq}{t})}{\Gamma(z^\alpha)} \ , \qquad \kappa = (p;p)(q;q) \end{aligned} \tag{2.72}$$

where χ_{adj} is the character of adjoint representation of gauge group G, and Δ represents the set of roots associated with the gauge group G, and $|W_G|$ is the order of the Weyl group of G. $(q;q)$ represents the function defined in (A.3). Then, the 4d $\mathcal{N}=2$ superconformal index of a quiver gauge theory can be schematically expressed as the contour integral

$$\mathcal{I}^{\text{4d}}(a;q)=\oint_{|z|=1}\prod_{\text{gauge}}\frac{dz}{2\pi i z}\mathcal{I}^{\text{4d}}_{\text{vec}}(z;p,q,t)\prod_{\text{matter}}\mathcal{I}^{\text{4d}}_{\frac{1}{2}H}(z;p,q,t)\ . \tag{2.73}$$

The 4d $\mathcal{N}=2$ superconformal index has various specializations[4–5]. Among them, the Schur index can be obtained at the specialization of $t=q$, and it counts 1/4 BPS operators consisting of Higgs branch operators annihilated by (Q^1_-,S_1^-) and $(\tilde{Q}_{2\dot{-}},\tilde{S}^{2\dot{-}})$, referred to as *Schur operators*. In the Schur limit, the hypermultiplet contribution is reduced to

$$\mathcal{I}^{\text{4d}}_H=\Gamma(z^{\pm}\sqrt{t}):=\Gamma(z\sqrt{t})\Gamma(z^{-1}\sqrt{t})\quad\xrightarrow{t\to q}\quad\mathcal{I}^{\text{Schur}}_H=\frac{\eta(q)}{\vartheta_4(z)} \tag{2.74}$$

where $\eta(q)$ is the Dedekind eta function (A.6), and the vector multiplet contribution is reduced to

$$\mathcal{I}^{\text{4d}}_{\text{vec}}=\frac{\kappa^{\text{rk}G}\Gamma(\frac{pq}{t})^{\text{rk}G}}{|W_G|}\prod_{\alpha\in\Delta}\frac{\Gamma\left(z^{\alpha}\frac{pq}{t}\right)}{\Gamma\left(z^{\alpha}\right)}\quad\xrightarrow{t\to q}\quad\mathcal{I}^{\text{Schur}}_{\text{vec}}=\frac{\eta(q)^{2\,\text{rk}G}}{|W_G|}\prod_{\alpha\in\Delta}i\frac{\vartheta_1(z^{\alpha})}{\eta(q)}\ . \tag{2.75}$$

Leveraging the state/operator correspondence, a Schur state can be obtained from a Schur operator $\mathcal{O}(0)$ applied to the vacuum. The Schur operator at the origin (anti-)commutes with the set of the four supercharges. While shifting the operator from this point generally disrupts the BPS condition, one can change the position of the operator across the $\mathbb{R}^2_{34}=\mathbb{C}_{z,\bar{z}}$ plane using the twisted translation proposed in paper[11]:

$$\mathcal{O}(z,\bar{z}):=e^{-zL_{-1}-\bar{z}\hat{L}_{-1}}\mathcal{O}(0)e^{+zL_{-1}+\bar{z}\hat{L}_{-1}}a$$

where

$$L_{-1}=P_{+\dot{+}},\quad \hat{L}_{-1}=P_{+\dot{+}}+R_1{}^2.$$

The twisted translated Schur operator $\mathcal{O}(z,\bar{z})$ is also annihilated by the two supercharges

$$\mathbb{Q}_1:=Q^1_-+\tilde{S}^{2\dot{-}}\ ,\qquad \mathbb{Q}_2:=\tilde{Q}_{2\dot{-}}-S_1^-\ . \tag{2.76}$$

Moreover, the $\bar{z}$-dependence of $\mathcal{O}(z,\bar{z})$ turns out to be $\mathbb{Q}_{1,2}$-exact. Consequently, at the level of $\mathbb{Q}_{1,2}$-cohomology, the cohomology class $\mathcal{O}(z):=[\mathcal{O}(z,\bar{z})]$ depends on the location holomorphically in z. Furthermore, the OPE coefficients of these Schur operators (as cohomology classes) are also holomorphic, forming a 2d VOA/chiral algebra on the plane $\mathbb{C}_{z,\bar{z}}$ as discussed in the paper[11].

The space of Schur operators defines the space of states of the associated VOA, and thus the Schur limit of the superconformal index, which counts the Schur operators with signs, equals the vacuum character of the associated VOA. The associated chiral algebra and the vacuum character are interesting and powerful invariants of 4d $\mathcal{N}=2$ SCFTs, which capture various aspects of four-dimensional physics. Non-perturbative dynamics in four dimensions can be probed by surface defects[44]. To study the relation of the chiral algebra, one can introduce half-BPS surface operators in $\mathcal{T}$ preserving the same nilpotent supercharges $\mathbb{Q}_{1,2}$ where the support of the surface defect transversely intersecting with the chiral algebra plane $\mathbb{C}_{z,\bar{z}}$ at the origin. From the perspective of the chiral algebra, a such surface defect introduces a non-trivial boundary condition at the origin, and the defect operators are acted on by the Schur operators in the 2d bulk through bulk-defect OPE. Therefore, it is believed that such surface defects introduce non-vacuum modules of the associated chiral algebra.

Similar to the original Schur index, one can count defect operators in the $\mathbb{Q}_{1,2}$ cohomology to obtain the defect Schur index. In general, it is difficult to compute graded dimensions of such operators from the first principle. However, a superconformal index in the presence of a surface defect can be evaluated with suitable manipulations[45–46]. A notable example of the manipulations involves vortex defects, which can be derived using the Higgsing procedure on a 4d $\mathcal{N}=2$ SCFT[45]. The vortex defect index can be computed by an appropriate residue computation on the superconformal index of the theory $\mathcal{T}^{\mathrm{UV}}$. For example, a vortex defect with vorticity k in an A_1-type class $\mathcal{S}$ theory on $C_{g,n}$ can be computed by (up to some factors q)

$$\mathcal{I}^{\mathrm{def}}_{g,n}(b_1, \ldots, b_n) = \mathrm{Res}_{b_{n+1}\to q^{\frac{k+1}{2}}} \frac{\eta(\tau)^2}{b_{n+1}} \mathcal{I}_{g,n+1}(b_1, \ldots, b_{n+1}) \ . \tag{2.77}$$

Here b_i denote the SU(2) flavor fugacities that are manifest in the class $\mathcal{S}$ construction.

In particular, the techniques to evaluate the Schur index with a surface defect have been developed to study the relation to the chiral algebra[27,47–48]. The original Schur index of $\mathcal{T}$ can be viewed as a supersymmetric partition function on $S^3_{\varphi,\chi,\theta} \times S^1_t$ (with suitable background fields turned on)①, and it localizes to a multivariate contour integral of an elliptic integrand $\mathcal{Z}(\mathsf{a}_i)$, where the integration variables a_i capture the holonomy of the dynamical gauge field along the temporal S^1[49–50]. To define a surface defect, one can also specify a BPS singular boundary condition of the gauge fields at the defect plane $\mathbb{R}^2 \subset \mathbb{R}^4$, or equivalently, at a particular T^2 in $S^3 \times S^1$. Such a singular background shifts the corresponding integration variables $\mathsf{a}_i \to \mathsf{a}_i + \lambda_i \tau$ where the λ_i reflect the singular boundary condition. As the values of λ_i vary, the shifted integration variables eventually cross the integral contour. Consequently, the Schur index with a surface defect is given by integration around a different contour, instead of the unit-circle integral like (2.73). The resulting defect Schur index is expected to be a linear combination of non-trivial characters of the associated chiral algebra.

2.3.4 The SCFT/Chiral algebra correspondence

As we mentioned above, the OPE coefficients of these Schur operators (as cohomology classes) form a chiral algebra/VOA on the plane $\mathbb{C}_{z,\bar{z}}$, which naturally gives rise to a map

$$\chi : \text{4d SCFTs} \to \text{2d chiral algebras} \ . \tag{2.78}$$

There are some important results following from this correspondence.

The first is that the global $\mathfrak{sl}(2,\mathbb{C})$ conformal symmetry enhances to a Virasoro algebra. Equivalently, for any SCFT $\mathcal{T}$, we can find a meromorphic stress tensor which is contained in $\chi[\mathcal{T}]$. The two dimensional central charge is determined by

$$c_{2d} = -12 c_{4d} \ , \tag{2.79}$$

where c_{4d} is the conformal anomaly coefficient in $4d$. This implies that when $\mathcal{T}$ is unitary $\chi[\mathcal{T}]$ cannot be unitary, since the unitary CFTs necessarily have positive central charges.

Also, global symmetries of $\mathcal{T}$ are always enhanced into affine symmetries of $\chi[\mathcal{T}]$, and the respective affine level of these flavor symmetries in $2d$ are given by

$$k_{2d} = -\frac{1}{2} k_{4d} \ , \tag{2.80}$$

where k_{4d} is the the flavor central charge in $4d$.

Finally, by the correspondence, the Schur index of $\mathcal{T}$ corresponds to the torus partition function of the chiral algebra $\chi[T]$,

$$\mathcal{I}^{\mathrm{Schur}}(q) = \mathcal{Z}_{T^2}(-1, q) \ , \tag{2.81}$$

where $\mathcal{Z}_{T^2}(x,q) = \mathrm{Tr}\ x^{j_2 - j_1} q^{L_0}$.

① Here the S^3 is viewed as a $T^2_{\varphi,\chi}$ fibering over an interval $[0,\pi/2]_\theta$. The points with $\theta = 0$ and $\theta = \pi/2$ form two special tori.

2.4 Theories of class $\mathcal{S}$ in $4d$

In this section, we review some basic knowledge of class $\mathcal{S}$ theories, mainly following the review parts in these two papers[11–12].

2.4.1 The data of class $\mathcal{S}$

Class $\mathcal{S}$ theories[1,51] are a set of theories that arise from compactification of any of $\mathcal{N}=(2,0)$ $6d$ SCFTs on a Riemann surface C, known as the UV curve, possibly with the inclusion of real codimension two defect operators at points of C.

The theories of class $\mathcal{S}$ are specified by the following data:

- A simply-laced Lie algebra $\mathfrak{g}$, which can be A_n, D_n, or $E_{6,7,8}$. This Lie algebra determines the underlying $6d$ $(2,0)$ theory.
- A (punctured) Riemann surface $C_{g,n}$, which serves as the UV curve. Here, g denotes the genus and n represents the number of punctures. In the low-energy limit, only the complex structure of $C_{g,n}$ is relevant. The complex structure moduli of the curve are identified with exactly marginal couplings in the SCFT.
- A choice of embedding $\rho_i : \mathfrak{su}(2) \to \mathfrak{g}$ (up to conjugation) for each puncture, where $i=1,\dots,n$. These choices reflect the selection of codimension two defects present at each puncture in the 6d construction. The centralizer $\mathfrak{h}_{\rho_i} \subset \mathfrak{g}$ of the embedding ρ_i is the global symmetry associated to the defect. The theory enjoys a flavor symmetry given by at least $\bigoplus_{i=1}^{n} \mathfrak{h}_{\rho_i}$ (Sometimes the flavor symmetry is larger).

Specially, if $\mathfrak{g}=\mathfrak{su}(N)$, the data at punctures can be formulated as the partition of N: $[\lambda_1^{l_1},\dots,\lambda_k^{l_k}]$ with $\sum_i l_i\lambda_i = N$ and $\lambda_i > \lambda_{i+1}$. The partition $[\lambda_1^{l_1},\dots,\lambda_k^{l_k}]$ encodes the decomposition of the fundamental representation $\mathbf{N}$ in terms of the irreducible representations of $\mathfrak{su}(2)$,

$$\mathbf{N} \to \bigoplus_{i=1}^{k} l_i V_{\frac{1}{2}(\lambda_i - 1)}, \tag{2.82}$$

where V_j denotes the spin-j representation of $\mathfrak{su}(2)$.

Another equivalent description is given by specifying a nilpotent element e in $\mathfrak{su}(2)$, which means element e satisfies the condition $(\mathrm{ad}_e)^r = 0$ for some positive integer r. The Jordan normal form of such a nilpotent element is given by

$$e = \bigoplus_{i=1}^{k} J_{\lambda_i}^{\oplus l_i}, \tag{2.83}$$

where J_m is the elementary Jordan block of size m. Every nilpotent element specify a partition of N, and vice versa. If we denote the three generators of $\mathfrak{su}(2)$ as t_0,$t_\pm$, then the embedding ρ maps t_- to e.

There are three special cases among these possible embeddings:

1. The trivial embedding is identified with the partition $[1^N]$, which leads to a defect with maximal flavor symmetry $\mathfrak{h}=\mathfrak{su}(N)$. Hence, the puncture corresponding to the partition $[1^N]$ is called maximal or full.
2. The principal embedding is labelled by the partition $[N^1]$, which leads to an empty flavor symmetry $\mathfrak{h}=\emptyset$ and an effectively null puncture.
3. The third special embedding is labelled by the partition $[N-1,1]$, and the corresponding puncture enjoys a $\mathfrak{u}(1)$ flavor symmetry if $N>2$, which is called minimal or simple. If $N=2$, the partition of maximal and minimal puncture coincide.

2.4.2 The building blocks

The building blocks for these $\mathcal{N}=2$ SCFTs of class $\mathcal{S}$ are related to three-punctured spheres, which are isolated SCFTs (having no marginal coupling) and usually don't admit a Lagrangian description. Such a theory can be denoted $T_{\mathfrak{g}}^{\rho_1,\rho_2,\rho_3}$ and we may refer it as *trinion theory*, where $\mathfrak{g}$ is the Lie algebra of the underlying 6d theory, and the ρ_i labels the defects at the three punctures. These building blocks can be assembled into more complex theories which are represented by a generalized quiver diagram.

When all three embeddings are trivial, the theory will be denoted $T_{\mathfrak{g}} := T_{\mathfrak{g}}^{[1^N],[1^N],[1^N]}$ or simply T_N if $\mathfrak{g}=A_{N-1}$. The general theory $T_N^{\rho_1,\rho_2,\rho_3}$ with non-maximal punctures can be obtained by coupling the T_N theory to a certain superconformal tail and then turning on specific Higgs branch vacuum values[45,52].(The D-type and E-type theories follow a similar procedure). Hence, the T_N theory serves as the maximal building block.

Important special cases are the theories with two maximal punctures and one minimal puncture: $T_N^{[1^N],[1^N],[N-1,1]}$. They are theories of N^2 free hypermultiplets transforming in the bifundamental representation of $\mathfrak{su}(N)\times\mathfrak{su}(N)$. Generally speaking, the theories with two maximal punctures and one minimal puncture are the only regular configuration which give rise to a Lagrangian theory for arbitrary choices of ADE type of Lie algebras. Hence, they are the building blocks for Lagrangian class $\mathcal{S}$.

The procedure to construct an arbitrary class $\mathcal{S}$ theory $\mathcal{T}[C_{g,n}]$ is parallel to the gluing of some copies of "pants": the three-punctured sphere $C_{0,3}$. Starting with $2g-2+n$ copies of the theory T_N, one can glue along a pair of maximal punctures $3g-3+n$ times by gauging the diagonal subgroup of the $\mathfrak{su}(N)\times\mathfrak{su}(N)$ flavor symmetry associated to the punctures. The gluing process will introduce an $\mathfrak{su}(N)$ gauge group in 4d SCFT and a marginal gauge coupling, leading to a theory associated with the UV curve $C_{g,n}$. The remaining maximal punctures can be reduced if desired.

Conversely, for a given UV curve $C_{g,n}$, one can decompose it into pairs of "pants", each related to the same number of weakly gauged $T_N^{\rho_1,\rho_2,\rho_3}$ theories. The equivalent pants decompositions are connected by the S-duality of these theories.

Central charges

The conformal anomalies a and c for all regular A-type theories are given by[53–54]

$$c_{4d}=\frac{2m_v+m_h}{12}\ ,\qquad a=\frac{5m_v+m_h}{24}, \tag{2.84}$$

where

$$\begin{aligned}
m_v&=\sum_i^n m_v(\rho_i)+(g-1)\Big(\frac{4}{3}h^\vee\dim\mathfrak{g}+\mathrm{rank}\mathfrak{g}\Big)\ ,\\
m_h&=\sum_i^n m_v(\rho_i)+(g-1)\Big(\frac{4}{3}h^\vee\dim\mathfrak{g}\Big)\ ,\\
m_v(\rho)&=8(w\cdot w-w\cdot\rho(t_0))+\frac{1}{2}(\mathrm{rank}\mathfrak{g}-\dim\mathfrak{g}_0)\ ,\\
m_h(\rho)&=8(w\cdot w-w\cdot\rho(t_0))+\frac{1}{2}\dim\mathfrak{g}_{\frac{1}{2}}\ .
\end{aligned} \tag{2.85}$$

In these equations, w is the Weyl vector of $\mathfrak{su}(N)$ and $h^\vee$ is the dual coxeter number which is equal to N for $\mathfrak{su}(N)$①. The Lie algebra $\mathfrak{g}$ decomposes into a direct sum of subalgebras due to the embedding generator $\rho(t_0)$:

$$\mathfrak{g}=\bigoplus_{m\in\frac{1}{2}\mathbb{Z}}\mathfrak{g}_m\ ,\qquad \mathfrak{g}_m:=\{t\in\mathfrak{g}\mid \mathrm{ad}_{\rho(t_0)}t=mt\}\ . \tag{2.86}$$

① A useful formula is $|\rho|^2=\frac{h^\vee}{12}\dim\mathfrak{g}$, which is named after Freudenthal and de Vries.

The $\mathfrak{su}(N)$ flavor symmetry associated with a full puncture has a central charge $k_{\mathfrak{su}(N)} = 2N$ ① . For a non-maximal puncture, the flavor central charge for a given simple factor $\mathfrak{h}_{\rm simp} \subset \mathfrak{h}$ is given by[54],

$$k_{\mathfrak{h}_{\rm simp}} \delta_{AB} = 2 \sum_j \mathrm{Tr}_{\mathcal{R}_j^{\rm adj}} T_A T_B \ , \tag{2.87}$$
$$\mathrm{adj}_{\mathfrak{g}} = \bigoplus_j \mathcal{R}_j^{\rm adj} \otimes V_j \ ,$$

where T_A, T_B are generators of $\mathfrak{h}_{\rm simp}$ satisfies $\mathrm{Tr}_{\mathfrak{h}_{\rm simp}} T_A T_B = h^\vee_{\mathfrak{h}_{\rm simp}} \delta_{AB}$. And $\mathcal{R}_j^{\rm adj}$ is the factor in the decomposition of $\mathfrak{su}(N)$ in the adjoint representation in terms of representations of $\mathfrak{h}_\rho \times \mathfrak{su}(2)$. For the global symmetries that extend the symmetries associated with punctures, the central charge can be deduced in terms of the embedding index.

① The general formula is $k_{\mathfrak{g}} = 2h^\vee$ if $\mathfrak{g} = A_{N-1}, D_N, E_{6,7,8}$

Chapter 3: $\mathcal{N} = (0, 2)$ gauge/Landau-Ginzburg duality and VOAs

In this chapter, we study 2d $\mathcal{N} = (0, 2)$ quiver gauge theories which is analogous to 4d $\mathcal{N} = 2$ class $\mathcal{S}$ theories[1]. We construct these (0,2) quiver gauge theories by gauging a basic building block consisting (0,2) chiral multiplets corresponding to a sphere with two maximal punctures and one minimal puncture. This family of 2d $\mathcal{N} = (0, 2)$ theories includes a class of theories obtained a particular twisted compactification on S^2, called Schur-like reductions[7], of class $\mathcal{S}$ theories with Lagrangian descriptions. For this subclass, we study the relation between the elliptic genus of a (0,2) theory and characters of the chiral algebra of the corresponding class $\mathcal{S}$ theory. Additionally, we demonstrate that, under certain conditions, the (0,2) quiver gauge theories are dual to Landau-Ginzburg models.

3.1 Relation between 2d $\mathcal{N} = (0, 2)$ theories and 4d $\mathcal{N} = 2$ SCFTs

3.1.1 Twisted compactifications of 4d $\mathcal{N} = 2$ SCFTs on S^2

A 2d $\mathcal{N} = (0, 2)$ theory can be obtained from a 4d $\mathcal{N} = 1$ gauge theory, and in particular, a 4d $\mathcal{N} = 2$ Lagrangian SCFT, by a compactification on S^2 with a certain topological twist. Such a reduction is referred to as Schur-like reduction in the paper[7]. (See also §5 in the paper[8].)

The first explicit supersymmetric localization of 4d $\mathcal{N} = 1$ theories on $T^2 \times S^2$ was carried out in paper[55]. Subsequently, the exploration of its relationship with the 2d $\mathcal{N} = (0, 2)$ elliptic genera was undertaken in the vicinity of the same period[7–9,56–57]. In this context, let us summarize the key aspects involved in the twisted compactification of 4d $\mathcal{N} = 1$ theories on S^2.

First of all, the generic holonomy group $\mathfrak{spin}(4) = \mathrm{SU}(1)_1 \times \mathrm{SU}(2)_2$ reduces to $\mathrm{U}(1)_{T^2} \times \mathrm{U}(1)_{S^2}$ on the background $T^2 \times S^2$ where $\mathrm{U}(1)_{T^2}$ (resp. $\mathrm{U}(1)_{S^2}$) is the U(1) subgroup of the diagonal (resp. anti-diagonal) subgroup of $\mathrm{SU}(2)_1 \times \mathrm{SU}(2)_2$. In order to define covariant constant supercharges on S^2, we perform a topological twist of $\mathrm{U}(1)_{S^2}$ along with a global $\mathrm{U}(1)_{\mathfrak{R}}$-symmetry. Additionally, in a Lagrangian theory, the $\mathrm{U}(1)_{\mathfrak{R}}$ charge of a 4d $\mathcal{N} = 1$ chiral multiplet must be an integer to ensure a well-defined compactification on S^2.

Under this compactification, a 4d $\mathcal{N} = 1$ chiral multiplet with $\mathrm{U}(1)_{\mathfrak{R}}$-charge $\mathfrak{r}$ decomposes into $(1 - \mathfrak{r})$ (0,2) chiral multiplets if $\mathfrak{r} < 1$, or $(\mathfrak{r} - 1)$ (0,2) Fermi multiplets if $\mathfrak{r} > 1$. When $\mathfrak{r} = 1$, the 4d chiral multiplet does not contribute to the 2d theory.

In general, the $T^2 \times S^2$ partition function involves the summation of magnetic fluxes of gauge fields. Nevertheless, if the $\mathrm{U}(1)_{\mathfrak{R}}$-charges of all chiral multiplets are non-negative, then the contributions from all non-zero flux sectors vanish, and only the zero flux contribution remains.

The R-symmetry of 4d $\mathcal{N} = 2$ superconformal theory is $\mathrm{SU}(2)_R \times \mathrm{U}(1)_r$. To perform the twisted compactification above, we treat the theory as a $\mathcal{N} = 1$ theory by selecting a $\mathrm{U}(1)_{\mathfrak{R}} \subset \mathrm{SU}(2)_R \times \mathrm{U}(1)_r$. According to the commutators (2.62) and (2.64), we can write down the corresponding charges of Q and $\tilde{Q}$. On the other hand, the charges of fields are given by the paper[7]§4.

	$\mathfrak{su}(2)_1$	$\mathfrak{su}(2)_2$	$\mathfrak{su}(2)_R$	$\mathfrak{u}(1)_r$	$\mathfrak{u}(1)_f$	$\mathfrak{u}(1)_{T^2}$	$\mathfrak{u}(1)_{S^2}$	$\mathfrak{u}(1)^{(0,2)}$	$\mathfrak{u}(1)^{(0,4)}$
Q^1_-	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	0	-1	-1	0	0
Q^1_+	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	0	1	1	2	2
Q^2_-	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	0	-1	-1	-1	0
Q^2_+	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	0	1	1	1	2
$\tilde{Q}_{1\dot{-}}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	0	-1	1	0	0
$\tilde{Q}_{1\dot{+}}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	0	1	-1	-2	-2
$\tilde{Q}_{2\dot{-}}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	0	-1	1	1	0
$\tilde{Q}_{2\dot{+}}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	0	1	-1	-1	-2
q	0	0	$\frac{1}{2}$	0	1	0	0	0	0
$\bar{q}$	0	0	$\frac{1}{2}$	0	-1	0	0	1	0
Φ	0	0	0	2	0	0	0	1	2

Table 3-1 Symmetries of 4d $\mathcal{N}=2$ supercharges and fields. The 4d $\mathcal{N}=1$ chirals $(q,\tilde{q})$ constitute an $\mathcal{N}=2$ hypermultiplets and Φ represents the $\mathcal{N}=1$ adjoint chiral in an $\mathcal{N}=2$ vector multiplet. The fifth column denotes the $\mathrm{U}(1)_f$ flavor symmetry that distinguishes q and $\tilde{q}$. The topological twist of $\mathrm{U}(1)_{S^2}$ with $\mathrm{U}(1)_{R+\frac{1}{2}(r-f)}$ results in the $\mathcal{N}=(0,2)$ supersymmetry wheareas the twist with $\mathrm{U}(1)_r$ yields the $\mathcal{N}=(0,4)$ supersymmetry.

Let us consider the choice $\mathfrak{R}=R+r$. As in Table 3-1, only the supercharges $Q^1_-, \tilde{Q}_{1\dot{-}}$ are neutral under $\mathfrak{R}+2(j_1-j_2)$ and therefore survive under this twist. These two supercharges share the same $\mathrm{U}(1)_{T^2}$ charge $2(j_1+j_2)=-1$, hence this twist leads to an $\mathcal{N}=(0,2)$ supersymmetry. The symmetry $\mathfrak{R}=R+r$ is referred to as the 2d (0,2) $\mathrm{U}(1)_R$-symmetry.

With the choice of $\mathfrak{R}$, the adjoint chiral Φ in an $\mathcal{N}=2$ vector multiplet has $\mathfrak{r}=1$, which does not contribute to the 2d theory. Consequently, an $\mathcal{N}=2$ vector multiplet simply reduces to a (0,2) vector multiplet. On the other hand, the hypermultiplet $(q,\bar{q})$ is assigned a fractional charge $\mathfrak{r}=\frac{1}{2}$. In order for their $\mathfrak{R}$-charges to be an integer, we further twist with the flavor symmetry $\mathrm{U}(1)_f$, which acts on the two chirals $(q,\bar{q})$ with opposite charges. Under the resulting $\mathfrak{R}=R+r-\frac{1}{2}f$, the scalars q and $\bar{q}$ acquire charges $\mathfrak{r}=0$ and $\mathfrak{r}=1$, respectively. Hence, an $\mathcal{N}=2$ hypermultiplet reduces to a (0,2) chiral multiplet in the representation of q. (See Table 3-1.) In this thesis, the above process of compactification on S^2 with the topological twist is referred to as the *(0,2) reduction*.

For a Lagrangian theory, an interesting observation emerges when comparing the Schur limit (2.74) and (2.75) with the contributions of (0,2) chirals (2.37) and vector multiplet (2.39) to the elliptic genus. The integrand of the elliptic genus for the 2d (0,2) theory after the reduction coincides with that of the Schur index of the original 4d $\mathcal{N}=2$ SCFT, upon suitable shifts of U(1) flavor fugacities. However, it is important to note that the computation of the (0,2) elliptic genus involves the JK residue integral whereas the Schur index is evaluated by a contour integral along the unit circles. We will provide a more detailed account of this comparison in the subsequent discussion.

3.1.2 Relation to VOAs

In the following analysis, we explore 2d (0,2) quiver gauge theories obtained by gauging the free 2d (0,2) theories associated with the three-punctured spheres. This includes, as a subclass, the theories from the (0,2) reduction of Lagrangian class $\mathcal{S}$ theories. We will demonstrate that under a certain condition, surprisingly, these quiver theories including the reduced class $\mathcal{S}$ theories are dual to Landau-Ginzburg (LG) models.

Let us discuss the central charges of the (0,2) reduction of Lagrangian class $\mathcal{S}$ theories. As illustrated below (2.20), if the two assumptions are satisfied, the central charges are determined by the c-extremization. However, the vacuum moduli space of the (0,2) reduction of a class $\mathcal{S}$ theory is non-compact, so the second assumption of the c-extremization is violated. Specifically, in such situations, a flavor current related to a non-compact direction is non-holomorphic so that it does not mix with the R-symmetry current[10,33]. Consequently, the naive application of the c-extremization to a (0,2) theory of this class can yield negative central charges. Nonetheless, the left-moving central charge turns out to coincide with that of the VOA of the parent class $\mathcal{S}$ theory[8]§5.2

$$c_L^{\text{naive}} = -12c_{4d} \ , \tag{3.1}$$

if we assign the "wrong" $\mathrm{U}(1)_R$-charges to chiral multiplets as a result of the naive application of the c-extremization

$$r_\Phi = 1 \ . \tag{3.2}$$

To get the physical central charge, we must enforce no mixing with any non-holomorphic flavor current arising from non-compact directions in the moduli space. At a practical level, we assign zero $\mathrm{U}(1)_R$-charges

$$r_\Phi = 0 \tag{3.3}$$

to the chiral multiplets which parameterize the non-compact directions. This procedure yields the correct positive central charges where the left-moving c_L is shifted from c_L^{naive} of the VOA associated to the 4d theory[8,58] by

$$c_L = c_L^{\text{naive}} + 3n_h = c_L^{\text{naive}} + 12(5c^{4d} - 4a^{4d}) \ , \tag{3.4}$$

where n_h represents the number of hypermultiplets and a^{4d}, c^{4d} are the anomaly coefficients in the corresponding class $\mathcal{S}$ theory. This suggests that the (0,2) theory flows to an SCFT fixed point despite the non-compact moduli space.

Exploring 2d (0,2) theories at the IR fixed point through the viewpoint of associated VOA is of significant importance because the BPS operators in the (0,2) IR CFT constitute a VOA. As elucidated in paper[59], the VOA associated with a (0,2) theory of this type arises from a BRST reduction of the $bc\beta\gamma$ system at zero gauge coupling, where the $\beta\gamma$ systems in the corresponding gauge representations come from the free $(0,2)$ chiral multiplets, and small bc ghosts in the adjoint of the gauge group from the free vector multiplets. The conformal weights of an involved $\beta\gamma$ system are $(h_\beta, h_\gamma) = (1-\lambda, \lambda)$ where λ is related to the correct R-charge assignment of the IR CFT. The state space of the resulting VOA is independent of the parameter λ, but the parameter affects the stress-energy tensor T of the (0,2) IR CFT. In fact, the shift (3.4) in central charge can be traced back to the difference between the stress-energy tensor T and T^χ of the VOA associated to the 4d theory by the derivative of the $U(1)$ flavor current J of the $\beta\gamma$ system, schematically[58–59]:

$$T = T^\chi + \left(\frac{1}{2} - \lambda\right) \partial J \ . \tag{3.5}$$

Therefore, the deviation from $\lambda = \frac{1}{2}$ reflects the non-compact nature of the vacuum moduli space, which invalidates the naive application of the c-extremization.

From our previous discussions, we have observed that the integrand of the (0,2) elliptic genus of the reduction of 4d $\mathcal{N}=2$ Lagrangian SCFT agrees with the integrand of the Schur index (upon appropriate redefinitions of U(1) flavor fugacities), albeit with distinct contour choices. As discussed at the end of §2.3.3, a different contour of the integrand is expected to provide the Shcur index with a surface defect, which is intrinsically tied to non-vacuum characters of the corresponding VOA $\chi(\mathcal{T}^{\text{4d}})$. Hence, we investigate the relationship between the (0,2) reduction of the class $\mathcal{S}$ theory and the corresponding VOA from this perspective.

There are two primary tools for us to investigate this relation: the elliptic genus[35–36] and modular (linear) differential equations[28,60–61]. Combining these tools, we present an intriguing conjecture proposing that the NS elliptic genus of the (0,2) reduction of a

Lagrangian class $\mathcal{S}$ can be expressed as a linear combination of characters $\mathrm{ch}\chi(\mathcal{T}_N[C_{g,n}])_{\lambda_i}$ of the corresponding VOA:

$$\mathcal{I}_{g,n}^{(0,2),N}=\sum_i a_i \mathrm{ch}\chi(\mathcal{T}_N[C_{g,n}])_{\lambda_i} \tag{3.6}$$

where $a_i\in\mathbb{Q}$, and λ_i represent the highest weights of representations of $\chi(\mathcal{T}_N[C_{g,n}])$. We remark that the U(1) flavor fugacities in the elliptic genus need to be appropriately redefined to precisely align with the characters of the VOA. Moreover, since the theory is dual to an LG model, the elliptic genus can be simply expressed as a product of theta and eta functions which can be viewed as free field characters of suitable 2d $bc\beta\gamma$ systems. Consequently, it forms a Jacobi form with its index determined by the 't Hooft anomaly of the global symmetry.

3.2 SU(2)×U(1) gauge theories, LG duals and VOAs

In this section, we consider 2d $\mathcal{N}=(0,2)$ gauge theories with SU(2) and U(1) gauge groups. Analogous to the 4d $\mathcal{N}=2$ superconformal theories of class $\mathcal{S}$, the 2d theories we consider will have the building blocks depicted in Figure 3-1.

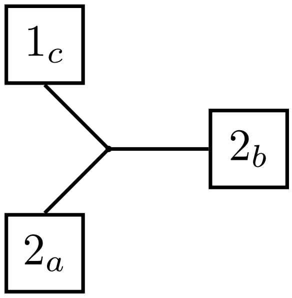

Figure 3-1 Basic building block $U_2^{(0,2)}$ for SU(2) theory

The basic building block in class $\mathcal{S}$ is the theory corresponding to a three punctured sphere $C_{0,3}$. In the case of type A_1, the 4d $\mathcal{N}=2$ theory T_2 is a free theory of 8 half-hypermultiplets with the flavor symmetry $\mathrm{SU}(2)_a\times\mathrm{SU}(2)_b\times\mathrm{SU}(2)_c$. (See the left of Figure 4-1.) We use $\mathrm{U}(1)_c\subset\mathrm{SU}(2)_c$ for the topological twist on S^2, and the (0,2) reduction of T_2 using this flavor symmetry leads to the theory of free (0,2) chiral multiplets in the representation $(2,2,1)$ of $\mathrm{SU}(2)_a\times\mathrm{SU}(2)_b\times\mathrm{U}(1)_c$ flavor symmetry. This (0,2) theory, akin to T_2, serves as the fundamental building block and is denoted as $U_2^{0,2}$. In quiver notation, we will depict this theory as a vertex with three external legs corresponding to $\mathrm{SU}(2)_a\times\mathrm{SU}(2)_b\times\mathrm{U}(1)_c$ flavor groups (see Figure 3-1).

It follows from (2.37) that the NS elliptic genus of the U_2 theory is

$$\mathcal{I}_{U_2}^{(0,2)}(a,b;c)=\frac{\eta(q)^4}{\vartheta_4(q^{-\frac{1}{2}}a^{\pm1}b^{\pm1}\tilde{c})}=\frac{\eta(q)^4}{\vartheta_4(a^{\pm1}b^{\pm1}c)}\ . \tag{3.7}$$

As emphasized in (3.3), the $\mathrm{U}(1)_R$-charge of the (0,2) chiral multiplet is zero $r=0$, but we shift the $\mathrm{U}(1)_c$ flavor fugacity by $c=q^{-\frac{1}{2}}\tilde{c}$. Note that the factors with repeated sign $\pm$ in the arguments are all multiplied. (See (A.2).) On the other hand, as seen in (2.74), the Schur limit of the superconformal index of the T_2 theory is given by[2]

$$\mathcal{I}_{T_2}^{\rm 4d}=\Gamma(\sqrt{t}a^{\pm1}b^{\pm1}c^{\pm1})\quad\xrightarrow{t\to q}\quad\mathcal{I}_{T_2}^{\rm Schur}=\frac{\eta(q)^4}{\vartheta_4(a^{\pm1}b^{\pm1}c)} \tag{3.8}$$

Therefore, the redefinition of the $\mathrm{U}(1)_c$ flavor fugacity in (3.7) ensures that the elliptic genus of the $U_2^{(0,2)}$ theory coincides with the Schur index of the T_2 theory. In this way, the elliptic genus can be decomposed into characters of the corresponding VOA.

Moreover, the contribution of a vector multiplet is the same in both the Schur index (2.75) and the elliptic genus (2.39). The SU(2) vector multiplet contribution is

$$\mathcal{I}_{\text{vec}}^{(0,2)}(a) = -\frac{\vartheta_1(a^{\pm 2})}{2} , \tag{3.9}$$

and the SU(2) gauging leads to no gauge anomaly.

As a Riemann surface $C_{g,n}$ can be constructed by gluing pants, the corresponding 4d $\mathcal{N}=2$ theory $\mathcal{T}_2[C_{g,n}]$ can be obtained by gauging the T_2 theories. However, as observed, the distinction between the 4d T_2 theory and 2d U_2 theory lies in the global symmetry. To construct (0,2) quiver gauge theories using the U_2 building block, we will outline the U(1) gauging method.

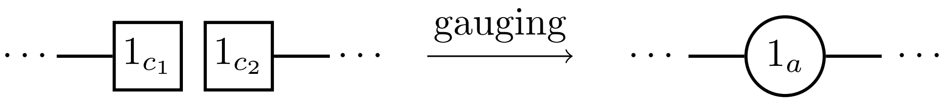

Figure 3-2 U(1) gauging

Given a $\text{U}(1)_{c_1} \times \text{U}(1)_{c_2}$ global symmetry, we gauge the anti-diagonal part $\mathsf{a} = (\mathsf{c}_1 - \mathsf{c}_2)/2$ with keeping the diagonal $\mathsf{d} = (\mathsf{c}_1 + \mathsf{c}_2)/2$ as a global symmetry. To cancel gauge anomaly during the gauging process, we include Fermi multiplets with $\text{U}(1)_a$ gauge charges ± 2. The $\text{U}(1)_R$-charge for these Fermi multiplets is taken to be $r = 0$, a value determined by the c-extremization. At the level of the elliptic genus, the gauging procedure is given by:

$$\frac{\eta(q)}{\vartheta_4(c_1\cdots)}\frac{\eta(q)}{\vartheta_4(c_2\cdots)} \quad \to \quad \eta(q)^2 \int_{\text{JK}} \frac{da}{2\pi i a}\frac{\eta(q)}{\vartheta_4(da\cdots)}\frac{\eta(q)}{\vartheta_4(da^{-1}\cdots)}\frac{\vartheta_4(a^{\pm 2})}{\eta(q)^2} \tag{3.10}$$

The details of U(1) gauging will be seen in examples in §3.2.4, §3.2.6 and below.

Through U(1) gauging, one can construct a family of (0,2) quiver gauge theories. However, when a U(1) gauge group is involved, a (0,2) theory is no longer the (0,2) reduction of a class $\mathcal{S}$ theory in general. Nonetheless, as we will demonstrate, (0,2) theories of genus $g > 0$ with $(g-1)$ U(1) gauge groups are dual to each other, making them frame-independent. We will further propose that the (0,2) reduction of the class $\mathcal{S}$ theory $\mathcal{T}_2[C_{g>0,n}]$ of type A_1 on S^2 is closely related to the corresponding (0,2) quiver theory of genus g with $(g-1)$ U(1) gauge groups. Notably, if one replaces $\vartheta_4(a^{\pm 2})$ by $\vartheta_1(a^{\pm 2})$ in the U(1) gauging, then the integrand of the elliptic genus is the same as that of the corresponding Schur index (up to a factor of 2^{g-1}), though their integration contours differ; the elliptic genus uses the JK prescription while the Schur index uses the maximal tori of the gauge groups. Additionally, the (0,2) theory of this class turns out to be dual to an LG model.

Furthermore, we explore generalized quiver gauge theories by gluing the U_2 theories, extending the (0,2) reduction of the class $\mathcal{S}$ theories. Specifically, we demonstrate that a quiver gauge theory with g loops and g U(1) gauge nodes is dual to an LG model.

3.2.1 SU(2) SQCD

First, let us consider the SU(2) gauge theory with four fundamental chiral multiplets, whose quiver diagram is presented in Figure 3-3. As described in paper[10], the naive application of the c-extremization leads to the "wrong" central charges of the theory

$$c_L^{\text{naive}} = -14 , \qquad c_R^{\text{naive}} = -9 , \tag{3.11}$$

while the enforcement of the zero R-charge to the chiral multiplets results in the "correct" positive central charges

$$c_L = 10 , \qquad c_R = 15 . \tag{3.12}$$

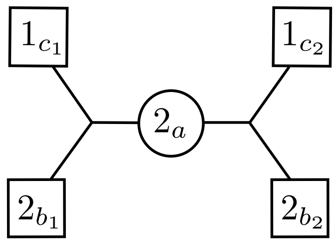

Figure 3-3 Quiver diagram for SU(2) SQCD with four fundamental chirals. In the diagram, a circle node represents a gauge group, and square nodes denote flavor groups. The number N within a node means SU(N) for $N>1$, and U(1) for $N=1$.

Nevertheless, the "wrong" left-moving central charge c_L^{naive} is equal to the central charge of the corresponding VOA $\mathfrak{so}(8)_{-2}$

This theory was first studied in the paper[9] and its elliptic genus is evaluated there as

$$\begin{aligned}\mathcal{I}_{0,4}^{(0,2),2} &= \int_{\text{JK}} \frac{da}{2\pi i a} \mathcal{I}_{U_2}^{(0,2)}(b_1,a;c_1)\mathcal{I}_{\text{vec}}^{(0,2)}(a)\mathcal{I}_{U_2}^{(0,2)}(b_2,a;c_2) \\ &= \frac{\eta(q)^5\vartheta_1(c_1^2c_2^2)}{\vartheta_1(c_1^2)\vartheta_1(c_2^2)\vartheta_1(c_1c_2b_1^{\pm}b_2^{\pm})} \ . \end{aligned} \tag{3.13}$$

As pointed out in papers[7,9–10,59], the theory exhibits an LG dual description. This dual description involves two chiral multiplets $\Phi_{1,2}$, and one chiral meson multiplet $\tilde{\Psi}_{i,j=1,2}$ with U$(1)_R$-charge of 0, as well as a Fermi multiplet Ψ with U$(1)_R$-charge of 1, and they form a J-type superpotential

$$W = \Psi(\Phi_1\Phi_2 + \det \tilde{\Psi}) \ .$$

The six chiral multiplets Φ can be regarded as the $\wedge^2 4 = 6$ representation of SU(4) so that the superpotential can be expressed as $W = \Psi\,\mathrm{Pf}(\Phi)$. Then, the expression (3.13) can be naturally understood as the elliptic genus of the LG model.

More remarkably, it has been revealed in papers[58–59] that the space of BPS states in the SU(2) SQCD has a relation to the VOA $\mathfrak{so}(8)_{-2}$, which is the VOA $\chi(\mathcal{T}_2[C_{0,4}])$ of the corresponding 4d $\mathcal{N}=2$ theory[11]. Concretely, it was found[58] that the elliptic genus (3.13) can be written as a linear combination of characters of $\mathfrak{so}(8)_{-2}$

$$\mathcal{I}_{0,4}^{(0,2),2}(q,b,c) = \text{ch}_0^{\mathfrak{so}(8)_{-2}}(q,b,c) - \text{ch}_{-2\omega_4}^{\mathfrak{so}(8)_{-2}}(q,b,c) \ . \tag{3.14}$$

where ch_0 is the vacuum character and $\text{ch}_{-2\omega_4}$ is one of the three non-vacuum characters of $\mathfrak{so}(8)_{-2}$. Therefore, the space of BPS states furnishes a non-trivial representation of the associated chiral algebra $\chi(\mathcal{T}_2[C_{0,4}])$.

The VOA $\mathfrak{so}(8)_{-2}$ contains many null states in its vacuum module. For example, the Sugawara condition $T - T_{\text{Sug}} = 0$ is a trivial null state, simply stating that the stress-energy tensor T is given by the Sugawara stress-energy tensor. The Sugawara condition is part of the so-called Joseph relations

$$(J^A J^B)|_{\mathfrak{R}=0}, \qquad (J^A J^B)_1 \sim T \ , \tag{3.15}$$

which correspond to more null states or descendants of null. Using Zhu's recursion relations[25], the null states (or their descendants) that are uncharged under the Cartan of $\mathfrak{so}(8)$ may lead to flavored modular differential equations that any $\mathfrak{so}(8)_{-2}$-character must satisfy. Concretely, there are 10 equations of weight-two, 4 equations of weight-three and 1 equation of weight-four that together constrain the characters of $\mathfrak{so}(8)_{-2}$[27–28]. In particular, the above elliptic genus $\mathcal{I}_{0,4}^{(0,2),2}(q,b,c)$ is a linear combination of characters, and therefore a solution to the set of equations. Reversing the logic, the fact that $\mathcal{I}_{0,4}^{(0,2),2}(q,b,c)$ solves the set of modular differential equations predicts that it must be some linear combination of VOA characters.

3.2.2 SU(2) linear quivers

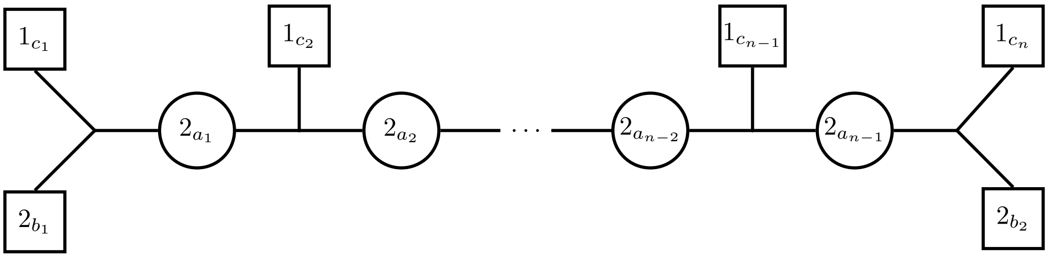

Figure 3-4 SU(2) linear quiver

Consider the SU(2) linear quiver theory with $(n-1)$ SU(2) gauge nodes as in Figure 3-4. The theory has manifest flavor symmetry $\mathrm{SU}(2)^2 \times \mathrm{U}(1)^n$. Its central charge is given by

$$c_L = 2(n+3) \ , \qquad c_R = 3(n+3) \ . \tag{3.16}$$

As explained in the paper[9]Appendix D, (3.13) can be interpreted as the elliptic inversion formula:

$$\frac{\eta(q)^5\vartheta_1(c_1^2c_2^2)}{\vartheta_1(c_1^2)\vartheta_1(c_2^2)\vartheta_1(c_1c_2b_1^\pm b_2^\pm)} = \int_{\mathrm{JK}} \frac{da}{2\pi i a} \frac{\eta(q)^8\vartheta_1(a^{\pm 2})}{2\vartheta_4(c_1b_1^\pm a^\pm)\vartheta_4(c_2b_2^\pm a^\pm)} \ . \tag{3.17}$$

Therefore, starting from a collection of U_2 theories we can repeatedly gauge the SU(2) flavor symmetries to construct a linear quiver theory, and the elliptic genus takes the following simple form:

$$\begin{aligned}
&\mathcal{I}^{(0,2),2}_{0,n+2} \\
=&\int_{\mathrm{JK}} \frac{da}{2\pi i a}\mathcal{I}^{(0,2)}_{U_2}(b_1,a_1;c_1)\mathcal{I}^{(0,2)}_{\mathrm{vec}}(a_1)\mathcal{I}^{(0,2)}_{U_2}(b_2,a_{n-1};c_n)\prod_{i=1}^{n-2}\mathcal{I}^{(0,2)}_{U_2}(a_i,a_{i+1};c_{i+1})\mathcal{I}^{(0,2)}_{\mathrm{vec}}(a_{i+1}) \\
=&\frac{\eta(q)^{n+3}\vartheta_1(\prod_{i=1}^n c_i^2)}{\vartheta_\alpha(b_1^\pm b_2^\pm\prod_{i=1}^n c_i)\prod_{i=1}^n\vartheta_1(c_i^2)} \ , \qquad \begin{cases}\alpha=1 & n \text{ even} \\ \alpha=4 & n \text{ odd}\end{cases} .
\end{aligned} \tag{3.18}$$

As also found earlier in the paper[10]§4.3, the above linear quiver theory has an LG description which consists of n chiral multiplets $\Phi_{k=1,\dots,n}$ and one chiral meson multiplet $\tilde{\Psi}_{i,j=1,2}$ with $\mathrm{U}(1)_R$-charge $r=0$, and one Fermi multiplet Ψ with $\mathrm{U}(1)_R$-charge $r=1$, forming a J-type superpotential

$$W = \Psi(\prod_{i=1}^n \Phi_i + \det\tilde{\Psi}) \ .$$

It is worth mentioning that with the shift of the $\mathrm{U}(1)_{c_i}$ flavor fugacities in (3.7), one must take this shift into account to correctly read off the $\mathrm{U}(1)_R$-charges of the superfields in the LG model from (3.18).

3.2.3 Genus one with one puncture and SU(2) with adjoint chiral

Let us consider the theory corresponding to the Riemann surface of genus one with one U(1)-puncture. Because of $2\otimes 2 = 3\oplus 1$, the theory is a product of an SU(2) gauge theory with one adjoint chiral and a free chiral. The elliptic genus can be evaluated as

$$\begin{aligned}
\mathcal{I}^{(0,2),2}_{1,1} =& \int_{\mathrm{JK}} \frac{da}{2\pi i a}\mathcal{I}^{(0,2)}_{U_2}(a,a^{-1};c_1)\mathcal{I}^{(0,2)}_{\mathrm{vec}}(a) \\
=&\frac{\eta(q)}{\vartheta_1(c_1^2)} = \frac{\eta(q)}{\vartheta_4(c_1)}\cdot\frac{\vartheta_4(c_1)}{\vartheta_1(c_1^2)} \ .
\end{aligned} \tag{3.19}$$

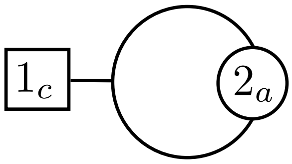

Figure 3-5 Genus one with one puncture for SU(2)

The first factor is the contribution of the free chiral, which is the vacuum character of the $\beta\gamma$-ghost, while the second factor is the elliptic genus of the SU(2) theory with one adjoint chiral. Following this computation, the SU(2) gauge theory with one adjoint chiral has an LG dual given by one chiral and one Fermi multiple. From the viewpoint of the original gauge theory, the first factor is the contributions from Tr(Φ) while the second one is from the Tr(Φ^2) and its superpartner for $\mathcal{N}=(2,2)$ supersymmetry, where Φ is the (0,2) chiral multiplet charged with $2\otimes 2$ under SU(2) gauge group.

The naive application of the c-extremization leads to the "wrong" central charges

$$c_L^{\text{naive}} = -9-1\ , \qquad c_R^{\text{naive}} = -9\ . \tag{3.20}$$

This miscalculation persists even if zero U(1)$_R$-charge of the chiral multiplet is assumed, incorrectly yielding the computed right-moving central charge to be three. To rectify this, we need to read the dimension of the moduli space. the moduli space, which is $\mathbb{C}^2$ spanned by Tr(Φ) and Tr(Φ^2), reveals that the U(1) gauge group remains unbroken in the infrared. Consequently, the precise calculation of the central charges yields

$$c_L = 5\ , \qquad c_R = 6\ . \tag{3.21}$$

We note that the elliptic genus enjoys the symmetry $c_1 \leftrightarrow c_1^{-1}$, which is the SU(2) Weyl group. Moreover, the elliptic genus admits the expansion with SU(2) characters

$$\mathcal{I}_{1,1}^{(0,2),2} = \frac{iq^{-\frac{1}{12}}}{c_1-c_1^{-1}}\left(1+\text{ch}_{\frac{1}{2}}^{\text{SU}(2)}(c_1^2)q+\text{ch}_{2}^{\text{SU}(2)}(c_1^2)q^2+\left[1+\text{ch}_{\frac{1}{2}}^{\text{SU}(2)}(c_1^2)+\text{ch}_{\frac{3}{2}}^{\text{SU}(2)}(c_1^2)\right]q^3+\cdots\right) \tag{3.22}$$

where $\text{ch}_j^{\text{SU}(2)}$ is the spin-j character of SU(2). Hence, one can observe that the U(1)$_{c_1}$ flavor symmetry gets enhanced to SU(2)$_{c_1}$ in the infrared.

The 4d $\mathcal{N}=4$ theory with SU(2) gauge group has the small $\mathcal{N}=4$ superconformal algebra as its associated VOA[11]. First of all, the central charge of the VOA is $c_{2d}=-9$ which agrees with the naive left-moving central charge of the SU(2) adjoint chiral (3.20). Moreover, the elliptic genus of the SU(2) adjoint chiral can be viewed as a special $bc\beta\gamma$ system[23]①, and is also a linear combination of the characters of the associated VOA

$$\frac{i\vartheta_4(c_1)}{\vartheta_1(c_1^2)} = \text{ch}_0^{\mathcal{N}=4}(q,c_1)+\text{ch}_1^{\mathcal{N}=4}(q,c_1)\ . \tag{3.23}$$

Here $\text{ch}_0^{\mathcal{N}=4}$ is the vacuum character and $\text{ch}_1^{\mathcal{N}=4}$ is the character of the non-vacuum irreducible module of the VOA[62], and both characters are shown to satisfy three common flavored modular differential equations from null states in the VOA[47]. Note that the flavor symmetry enhancement to SU(2) is supported from the viewpoint of the VOA since both the small $\mathcal{N}=4$ superconformal algebra and the $bc\beta\gamma$-ghost VOA are endowed with an SU(2) flavor symmetry②.

An $\mathcal{N}=(0,2)$ vector multiplet and an $\mathcal{N}=(0,2)$ adjoint chiral multiplet form an $\mathcal{N}=(2,2)$ vector multiplet. Consequently, there exists no distinction between the left- and right-moving sectors in the SU(2) adjoint chiral. This is supported by the equality of

① The conformal weights of b,c,β,γ are $3/2,-1/2,1,0$.

② The SU(2) current of this particular $bc\beta\gamma$ system is given by $J^+=\beta$, $J^0=bc+2\beta\gamma$, $J^-=\beta\gamma\gamma+\gamma bc-3/2\partial\gamma$[23].

central charges for both sectors, which are both $c_L = c_R = 3$. Moreover, the elliptic genus of this theory can be expressed using characters of the small $\mathcal{N} = 4$ superconformal algebra as in (3.23). This suggests that the infrared limit of the $\mathcal{N} = (2,2)$ SU(2) vector multiplet is equipped with the small $\mathcal{N} = 4$ superconformal algebra in both the left- and right-moving sectors, suggesting the *supersymmetry enhancement*. It deserves further investigation to determine the exact infrared Hilbert space on a torus by modular invariance as demonstrated in pepres[63–65].

3.2.4 Genus one with two punctures

Now consider the (0,2) reduction of the class $\mathcal{S}$ theory $\mathcal{T}_2[C_{1,2}]$, which is an $\mathrm{SU}(2)^2$ gauge theory coupled to two bi-fundamentals. As we saw in the previous example, we conjecture that the diagonal U(1) gauge group is unbroken in the infrared for a genus one theory. As a result, the central charges are given by

$$c_L = 7 \ , \quad c_R = 9 \ . \tag{3.24}$$

The quiver diagram is given in Figure 3-6.

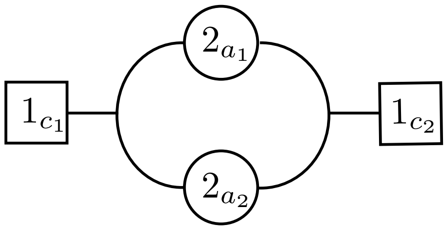

Figure 3-6 The (0,2) reduction of the class $\mathcal{S}$ theory $\mathcal{T}_2[C_{1,2}]$.

The elliptic genus of this theory is computed by the JK residue computation of the integral

$$\mathcal{I}^{(0,2),2}_{1,2}(c_1,c_2) = \int_{\mathrm{JK}} \frac{da}{2\pi i a} \mathcal{I}^{(0,2)}_{U_2}(a_1,a_2;c_1)\mathcal{I}^{(0,2)}_{U_2}(a_1^{-1},a_2^{-1};c_2)\mathcal{I}^{(0,2)}_{\mathrm{vec}}(a_1)\mathcal{I}^{(0,2)}_{\mathrm{vec}}(a_2) \tag{3.25}$$

$$= -\frac{\eta(q)^2}{\prod_{i=1}^2 \vartheta_1(c_i^2)} \ . \tag{3.26}$$

The elliptic genus implies the presence of an LG dual for the theory. This LG dual might comprise two free chiral multiplets. Potentially, there could be equal numbers of chiral and Fermi multiplets with identical charges, in addition to them. As seen in (3.22), both the $\mathrm{U}(1)_{c_i}$ flavor symmetries get enhanced to $\mathrm{SU}(2)_{c_i}$ in the infrared.

Although the module structure of the corresponding VOA $\chi(\mathcal{T}_2[C_{1,2}])$ is not explicitly known, we argue that the elliptic genus is a linear combination of the module characters. One crucial piece of evidence is that the elliptic genus is a solution to the same flavored modular differential equations[27] that the Schur index of $\mathcal{T}_2[C_{1,2}]$ satisfies. For instance, the elliptic genus satisfies a weight-two differential equation,

$$0 = \Bigg[D_q^{(1)} - \frac{1}{4}\sum_{i=1,2} D_{c_i}^2 - \frac{1}{4}\sum_{\alpha_i=\pm} E_1\begin{bmatrix} 1 \\ c_1^{\alpha_1}c_2^{\alpha_2}\end{bmatrix} \sum_{i=1,2}\alpha_i D_{c_i} - \sum_{i=1,2} E_1\begin{bmatrix} 1 \\ c_i^2 \end{bmatrix} D_{c_i} \tag{3.27}$$
$$+ 2\Bigg(E_2 + \frac{1}{2}\sum_{\alpha_i=\pm} E_2\begin{bmatrix} 1 \\ c_1^{\alpha_1}c_2^{\alpha_2}\end{bmatrix} + \sum_{i=1,2} E_2\begin{bmatrix} 1 \\ c_i^2\end{bmatrix}\Bigg)\Bigg]\mathcal{I}^{(0,2),2}_{1,2} \ .$$

The definition of the twisted Eisenstein series is given in Appendix A.2.

Furthermore, by gauging the anti-diagonal of the U(1) flavor symmetry of the two U_2 theories, one can construct two other quiver theories as SU(2) × U(1) gauge theories, corresponding to the genus-one Riemann surface with two punctures, as in Figure 3-7. Notably,

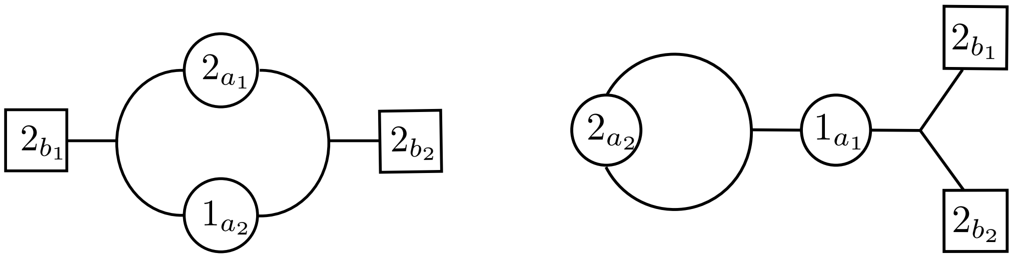

Figure 3-7 SU(2) × U(1) gauge theories corresponding to the genus-one Riemann surface with two punctures.

these theories do not originate from class $\mathcal{S}$. Nonetheless, the central charges of these theories are computed as

$$c_L = 13\ , \quad c_R = 15\ , \tag{3.28}$$

assuming that the diagonal U(1) gauge group is unbroken.

Let us first consider the left theory in Figure 3-7. The elliptic genus of this theory is

$$\mathcal{I}'_{1,2}(b_1,b_2;d_1) = \eta(q)^2 \int_{\mathrm{JK}} \frac{da}{2\pi i a} \mathcal{I}^{(0,2)}_{U_2}(a_1,b_1;d_1a_2)\mathcal{I}^{(0,2)}_{U_2}(a_1^{-1},b_2;d_1a_2^{-1})\mathcal{I}^{(0,2)}_{\mathrm{vec}}(a_1)\frac{\vartheta_4(a_2^{\pm 2})}{\eta(q)^2}\ .$$

Here, we introduce the $\mathrm{U}(1)_d$ flavor symmetry, which rotates the two chiral multiplets $U_2^{(0,2)}$ with the same phase and has a fugacity d_1. The detailed calculations of the JK residues can be found in Appendix B.1, and The final result is neatly summarized in a simplified form

$$\mathcal{I}'_{1,2}(b_1,b_2;d_1) = -2\frac{\eta(q)^2\vartheta_4(d_1^2)^2}{\vartheta_1(d_1^2b_1^{\pm}b_2^{\pm})} = 2\frac{\vartheta_4(d_1^2)^2}{\eta(q)\vartheta_1(d_1^4)}\cdot\frac{\eta(q)^3\vartheta_1(d_1^{-4})}{\vartheta_1(d_1^2b_1^{\pm}b_2^{\pm})}\ . \tag{3.29}$$

The first factor always appears when the quiver gauge theory has a U(1) gauge node. This can be understood as the contribution of two Fermi fields $\Gamma_{1,2}$ with $\mathrm{U}(1)_{d_1}$ flavor charge 2 and one chiral field Φ with $\mathrm{U}(1)_{d_1}$ flavor charge 4. The second factor of the elliptic genus can be interpreted as the contribution from one Fermi field Ψ and one chiral meson field $\tilde{\Psi}$ with $\mathrm{SU}(2)_{b_1}\times\mathrm{SU}(2)_{b_2}$ flavor symmetry. The field content and their charges are summarized as follows:

	$\Gamma_{1,2}$	Ψ	Φ	$\tilde{\Psi}$
$\mathrm{U}(1)_{d_1}$	2	-4	4	2
$\mathrm{SU}(2)_{b_i}$	$\emptyset$	$\emptyset$	$\emptyset$	2

(3.30)

Therefore, a generic J-type superpotential is

$$W = \Psi(\Phi + \det\tilde{\Psi})\ .$$

The coefficient 2 in (3.29) means that the theory is dual to two (decoupled) copies of this LG model.

The elliptic genus of the right theory in Figure 3-7 can be computed as the JK residue of the integrand

$$\mathcal{I}''_{1,2}(b_1,b_2;d_1) = \eta(q)^2 \int_{\mathrm{JK}} \frac{da}{2\pi i a} \mathcal{I}^{(0,2)}_{U_2}(a_2,a_2^{-1};da_1)\mathcal{I}^{(0,2)}_{U_2}(b_1,b_2;da_1^{-1})\mathcal{I}^{(0,2)}_{\mathrm{vec}}(a_2)\frac{\vartheta_4(a_1^{\pm 2})}{\eta(q)^2}\ .$$

The detailed computations of the JK residues are provided in Appendix B.1, while The total residue is presented in a compact form

$$\mathcal{I}''_{1,2}(b_1,b_2;d_1) = \frac{\vartheta_4(d_1^2)^2}{\eta(q)\vartheta_1(d_1^4)}\cdot\frac{\eta(q)^3\vartheta_1(d_1^8)}{\vartheta_1(d_1^{-4}b_1^{\pm 2}b_2^{\pm 2})}\cdot\prod_{i=1}^{2}\frac{\vartheta_1(b_i^4)}{\vartheta_1(b_i^{-2})}\ . \tag{3.31}$$

The elliptic genus is different from (3.29) so that the two theories in Figure 3-7 are not dual to each other. When the number of U(1) gauge nodes is equal to the genus, the theory is contingent on the quiver diagram, unlike class $\mathcal{S}$ theory.

Since (3.31) is expressed as a product of theta functions, the theory is also dual to the following LG model. The Fermi multiplets $\Gamma_{1,2}$ and Ψ and the chiral multiplets Φ and $\tilde{\Psi}_{\pm\pm}$ are similar to the aforementioned LG theory. A key distinction arises from the inclusion of two additional Fermi multiplets, $\Xi_{1,2}$, and two chiral multiplets, $\Sigma_{1,2}$. Because of these additions, the manifest flavor symmetry is $\mathrm{U}(1)_{b_1} \times \mathrm{U}(1)_{b_2}$ at UV. The charges of these fields are summarized as follows:

	$\Gamma_{1,2}$	Ψ	Ξ_j	Φ	$\tilde{\Psi}_{\epsilon_1\epsilon_2}$	Σ_j
$\mathrm{U}(1)_{d_1}$	2	8	0	4	-4	0
$\mathrm{U}(1)_{b_i}$	0	0	$4\delta_{ij}$	0	$\epsilon_i 2$	$-2\delta_{ij}$

where $\epsilon_i = \pm$. Then, a generic J-type superpotential is

$$W = \Psi \det \tilde{\Psi} + \Xi_1 \Phi \tilde{\Psi}_{-+} \Sigma_1 \Sigma_2 + \Xi_2 \Phi \tilde{\Psi}_{+-} \Sigma_1 \Sigma_2 \ .$$

The expansion of the elliptic genus (3.31) in terms of q reveals the fugacities $b_{1,2}$ arrange themselves as characters of $\mathrm{SU}(2)_{b_1} \times \mathrm{SU}(2)_{b_2}$, implying the symmetry enhancement from $\mathrm{U}(1)_{b_1} \times \mathrm{U}(1)_{b_2} \to \mathrm{SU}(2)_{b_1} \times \mathrm{SU}(2)_{b_2}$ at IR.

For the quiver theories in Figure 3-7, the corresponding VOA for the IR CFT is yet to be identified. Given their duality to the LG models, the technique in paper[66] offers a potential method for identifying the VOA. This remains an area for further study.

3.2.5 Genus one with three punctures

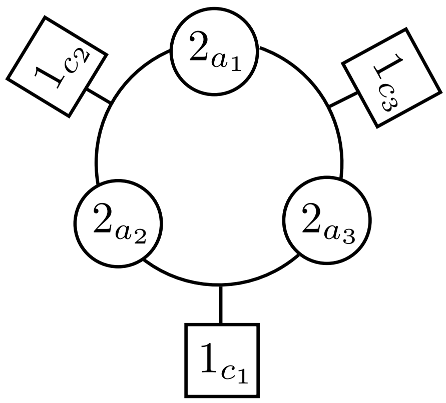

Figure 3-8 The (0,2) reduction of the class $\mathcal{S}$ theory $\mathcal{T}_2[C_{1,3}]$.

Now let us consider quiver theories of genus one with three punctures. The (0,2) reduction of the class $\mathcal{S}$ theory $\mathcal{T}_2[C_{1,3}]$ is the $\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{SU}(2)$ gauge theory in Figure 3-8. Considering that the diagonal U(1) gauge group remains unbroken, the central charges of the theory are given by

$$c_L = 9 \ , \quad c_R = 12 \ . \tag{3.32}$$

The elliptic genus is the JK residue of the integrand

$$\begin{aligned}\mathcal{I}_{1,3}^{(0,2),2} =& \int_{\mathrm{JK}} \frac{da}{2\pi i a} \mathcal{I}_{U_2}^{(0,2)}(a_1^{-1}, a_2; c_2) \mathcal{I}_{U_2}^{(0,2)}(a_2^{-1}, a_3; c_1) \mathcal{I}_{U_2}^{(0,2)}(a_3^{-1}, a_1; c_3) \prod_{i=1}^{3} \mathcal{I}_{\mathrm{vec}}^{(0,2)}(a_i) \\ =& \frac{\eta(q)^3}{\prod_{i=1}^3 \vartheta_1(c_i^2)}\end{aligned} \tag{3.33}$$

which signals the existence of an LG dual which includes three free chiral multiplets, associated to the three punctures. In addition to them, there could be equal numbers of chiral and Fermi multiplets with the same charges. All the U(1) flavor symmetries get enhanced to SU(2) in the infrared, as illustrated in (3.22).

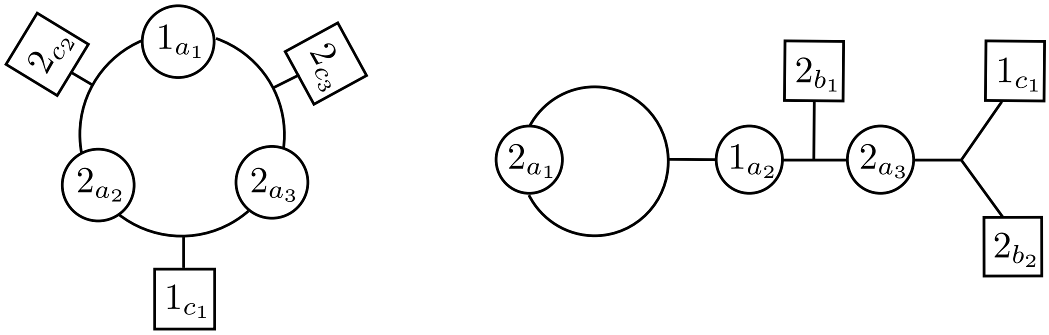

Figure 3-9 SU(2) × SU(2) × U(1) gauge theories corresponding to the genus-one Riemann surface with three punctures.

Let us now consider SU(2) × SU(2) × U(1) gauge theories for genus one with three punctures as illustrated in Figure 3-9. The central charges of the theory are given by

$$c_L = 15 \ , \quad c_R = 18 \ . \tag{3.34}$$

The elliptic genus of the left quiver theory of Figure 3-9 is

$$\mathcal{I}'_{1,3} \tag{3.35}$$

$$=\eta(q)^2 \int_{\mathrm{JK}} \frac{da}{2\pi i a} \mathcal{I}^{(0,2)}_{U_2}(a_2, c_2^{-1}; d_1 a_1) \mathcal{I}^{(0,2)}_{U_2}(a_2^{-1}, a_3; c_1) \mathcal{I}^{(0,2)}_{U_2}(a_3^{-1}, c_3; d_1 a_1^{-1}) \frac{\vartheta_4(a_1^{\pm 2})}{\eta(q)^2} \prod_{i=2}^{3} \mathcal{I}^{(0,2)}_{\mathrm{vec}}(a_i) \tag{3.36}$$

$$=2 \frac{\vartheta_4(d_1^2)^2}{\eta(q)\vartheta_1(d_1^4)} \cdot \frac{\eta(q)^4 \vartheta_1(c_1^2 d_1^4)}{\vartheta_1(c_1^2)\vartheta_4(c_1 d_1^2 b_1^{\pm} b_2^{\pm})} \ . \tag{3.37}$$

The computational details are given in Appendix B.2. The form of the elliptic genus signals an LG description. The LG model is similar to (3.30), but there is additional chiral multiplet Φ_2 and $\mathrm{U}(1)_{c_1}$ flavor symmetry:

	$\Gamma_{1,2}$	Ψ	Φ_1	Φ_2	$\tilde{\Psi}$
$\mathrm{U}(1)_{d_1}$	2	4	−4	0	−2
$\mathrm{U}(1)_{c_1}$	0	2	0	−2	−1
$\mathrm{SU}(2)_{b_i}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	2

Therefore, a generic J-type superpotential is

$$W = \Psi(\Phi_1\Phi_2 + \det \tilde{\Psi}) \ .$$

The coefficient 2 in (3.35) means that the theory is dual to two (decoupled) copies of this LG model.

There is yet another quiver gauge theory with gauge group SU(2) × SU(2) × U(1) as in the right of Figure 3-9 whose elliptic genus is

$$\mathcal{I}''_{1,3} \tag{3.38}$$

$$=\eta(q)^2 \int_{\mathrm{JK}} \frac{da}{2\pi i a} \mathcal{I}^{(0,2)}_{U_2}(a_1, a_1^{-1}; d_1 a_2) \mathcal{I}^{(0,2)}_{U_2}(b_1, a_3; d_1 a_1^{-1}) \mathcal{I}^{(0,2)}_{U_2}(a_3^{-1}, b_2; c_2) \frac{\vartheta_4(a_1^{\pm 2})}{\eta(q)^2} \prod_{i=2}^{3} \mathcal{I}^{(0,2)}_{\mathrm{vec}}(a_i) \tag{3.39}$$

$$= \frac{\vartheta_4(d_1^2)^2}{\eta(q)\vartheta_1(d_1^4)} \cdot \frac{\eta(q)^4 \vartheta_1(c_1^4 d_1^8)}{\vartheta_1(c_1^2)\vartheta_1(c_1^{-2} d_1^{-4} b_1^{\pm 2} b_2^{\pm 2})} \cdot \prod_{i=1}^{2} \frac{\vartheta_1(b_i^4)}{\vartheta_1(b_i^{-2})} \ , \tag{3.40}$$

suggesting an LG description of the theory.

Since (3.38) is expressed as a product of theta functions, the theory is also dual to the following LG model. The Fermi multiplets $\Gamma_{1,2}$ and Ψ and the chiral multiplets $\Phi_{1,2}$ and $\tilde{\Psi}_{\pm\pm}$ are similar to the aforementioned LG theory. A key distinction arises from the inclusion of two additional Fermi multiplets, $\Xi_{1,2}$, and two chiral multiplets, $\Sigma_{1,2}$. Because of these additions, the manifest flavor symmetry is $\mathrm{U}(1)_{b_1} \times \mathrm{U}(1)_{b_2}$ at UV. The charges of these fields are summarized as follows:

	$\Gamma_{1,2}$	Ψ	Ξ_j	Φ_1	Φ_2	$\tilde{\Psi}_{\epsilon_1\epsilon_2}$	Σ_j
$\mathrm{U}(1)_{d_1}$	2	8	0	4	0	-4	0
$\mathrm{U}(1)_{c_1}$	0	4	0	0	2	-2	0
$\mathrm{U}(1)_{b_i}$	0	0	$4\delta_{ij}$	0	0	$\epsilon_i 2$	$-2\delta_{ij}$

where $\epsilon_i = \pm$. Then, a generic J-type superpotential is

$$W = \Psi \det \tilde{\Psi} + \Xi_1\Phi_1\Phi_2\tilde{\Psi}_{-+}\Sigma_1\Sigma_2 + \Xi_2\Phi_1\Phi_2\tilde{\Psi}_{+-}\Sigma_1\Sigma_2 \ .$$

Like (3.31), the expansion of the elliptic genus (3.38) shows that a flavor symmetry is enhanced to $\mathrm{SU}(2)_{b_1} \times \mathrm{SU}(2)_{b_2}$ at IR.

3.2.6 Genus two

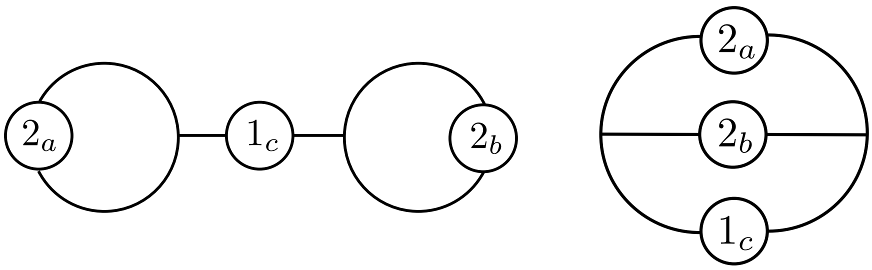

Figure 3-10 Genus two for SU(2)

For the genus-two case (with no puncture), we have two different quiver descriptions as depicted in Figure 3-10. As seen in §3.2.3, when a diagonal SU(2) subgroup of $\mathrm{SU}(2)^2$ in the U_2 theory is gauged, the Cartan subgroup U(1) remains unbroken. Therefore, for the theory of genus two, the $\mathrm{U}(1)^2$ gauge group is unbroken. Considering this fact, the central charges of this theory are computed as

$$c_L = 10 \ , \quad c_R = 9 \ . \tag{3.41}$$

It is straightforward to show that the two different descriptions are dual to each other. Certainly, the Lagrangian descriptions provide the different expressions of the elliptic genera

$$\mathcal{I}^{(0,2),2}_{\bigcirc\!\!-\!\!\bigcirc} = \int_{\mathrm{JK}} \frac{da}{2\pi i a}\frac{db}{2\pi i b}\frac{dc}{2\pi i c}\mathcal{I}^{(0,2)}_{U_2}(a,a^{-1};dc)\mathcal{I}^{(0,2)}_{U_2}(b,b^{-1};dc^{-1})\mathcal{I}^{(0,2)}_{\mathrm{vec}}(a)\mathcal{I}^{(0,2)}_{\mathrm{vec}}(b)\vartheta_4(c^{\pm 2}) \tag{3.42}$$

$$\mathcal{I}^{(0,2),2}_{\ominus} = \int_{\mathrm{JK}} \frac{da}{2\pi i a}\frac{db}{2\pi i b}\frac{dc}{2\pi i c}\mathcal{I}^{(0,2)}_{U_2}(a,b;dc)\mathcal{I}^{(0,2)}_{U_2}(a^{-1},b^{-1};dc^{-1})\mathcal{I}^{(0,2)}_{\mathrm{vec}}(a)\mathcal{I}^{(0,2)}_{\mathrm{vec}}(b)\vartheta_4(c^{\pm 2}) \tag{3.43}$$

where the last factors $\vartheta_4(c^{\pm 2})$ correspond to the contributions from the U(1) vector multiplet and the two Fermi multiplets. Both JK residues can be straightforwardly evaluated, and the results agree, suggesting that the two theories are dual to each other.

Alternatively, the two elliptic genera can be computed from other building blocks. The former theory can be obtained by gluing two copies of the theory of genus one with one puncture whose elliptic genus is given in (3.19). Similarly, the latter can be obtained by gluing the two punctures in the theory of genus one with two punctures whose elliptic genus is given in (3.25). In this approach, it becomes more evident that they are identical, and moreover the JK residue provides a remarkably simple result

$$\mathcal{I}_{2,0}^{(0,2),2}=\eta(q)^2\int_{\text{JK}}\frac{dc}{2\pi ic}\frac{\vartheta_4(c^{\pm 2})}{\vartheta_1(d^2c^{\pm 2})}=\frac{2\vartheta_4(d^2)^2}{\eta(q)\vartheta_1(d^4)}\ . \tag{3.44}$$

This can be compared to the S-duality in class $\mathcal{S}$ theory. While the process of U(1) gauging may not naturally fit in the context of the (0,2) reduction of the class $\mathcal{S}$ theory as explained below (3.10), the gluing procedure explained above is analogous to the class $\mathcal{S}$ construction. In fact, since the flavor symmetries get enhanced to SU(2) for genus one theories, we can gauge the anti-diagonal SU(2) to obtain the (0,2) reduction of the class $\mathcal{S}$ theory $\mathcal{T}_2[C_{2,0}]$ of genus two whose elliptic genus is

$$\begin{aligned}&-\int_{\text{JK}}\frac{dc}{2\pi ic}\mathcal{I}_{1,1}^{(0,2),2}(cd)\frac{\vartheta_1(c^{\pm 2})}{2}\mathcal{I}_{1,1}^{(0,2),2}(c^{-1}d)=-\int_{\text{JK}}\frac{dc}{2\pi ic}\mathcal{I}_{1,2}^{(0,2),2}(cd,c^{-1}d)\frac{\vartheta_1(c^{\pm 2})}{2}\\&=-\frac{\eta(q)^2}{2}\int_{\text{JK}}\frac{dc}{2\pi ic}\frac{\vartheta_1(c^{\pm 2})}{\vartheta_1(d^2c^{\pm 2})}=\frac{\vartheta_1(d^2)^2}{\eta(q)\vartheta_1(d^4)}\ .\end{aligned} \tag{3.45}$$

Recalling that the Jacobi theta functions ϑ_1 and ϑ_4 are related by (A.8), the result differs from (3.44) merely by the shift $d\to q^{1/4}d$ of the $\mathrm{U}(1)_d$ flavor fugacity, up to a factor. Indeed, comparing the SU(2) vector multiplet contribution (3.9) and U(1) gauging (3.10) at the level of elliptic genera, the difference appears only in $\vartheta_1(a^{\pm 2})$ and $\vartheta_4(a^{\pm 2})$, up to a factor of 2 which is the order of the Weyl group of SU(2). Hence, the duality between the two (0,2) theories in Figure 3-10 is analogous to the S-duality in the class $\mathcal{S}$ theory $\mathcal{T}_2[C_{2,0}]$.

As a result, one can expect the relation between the elliptic genera (3.44,3.45) and characters of the VOA $\chi(\mathcal{T}_2[C_{2,0}])$. In 4d, the associated VOA of the genus-two A_1-theory $\mathcal{T}_2[C_{2,0}]$ of class $\mathcal{S}$ is studied in papers[67–68]. In particular, null states are present at level-four and six, which are expected to give rise to a weight-four and a weight-six flavor differential equation[27]. The above elliptic genus (3.45) is indeed a solution to these two equations once the U(1) fugacity d is rescaled $d\to d^{1/2}$. Therefore, it is natural to argue that the elliptic genus is some linear combination of module characters of $\chi(\mathcal{T}_2[C_{2,0}])$

$$\frac{\vartheta_1(d)^2}{\eta(q)\vartheta_1(d^2)}=\sum_i a_i\mathrm{ch}_{\lambda_i}^{\chi(\mathcal{T}_2[C_{2,0}])}(d)\ ,\qquad a_i\in\mathbb{Q}. \tag{3.46}$$

Likewise, we can argue that, upon rescaling $d\to d^{1/2}$, the elliptic genus (3.44) of the genus two theory constructed from the U_2 theories can be written in a similar way as

$$\frac{2\vartheta_4(d)^2}{\eta(q)\vartheta_1(d^2)}=2q^{1/4}d\sum_i a_i\mathrm{ch}_{\lambda_i}^{\chi(\mathcal{T}_2[C_{2,0}])}(q^{1/2}d)\ . \tag{3.47}$$

3.2.7 General Riemann surfaces and TQFT structure

For a (0,2) quiver theory of genus $g>0$ constructed from the U_2 theory, the minimal number of U(1) gauge groups is $g-1$. Hence, based on the previous results, we can consider a (0,2) quiver theory analogous to the class $\mathcal{S}$ theory $\mathcal{T}_2[C_{g>0,n}]$ where the numbers of SU(2) and U(1) gauge groups are $2(g-1)+n$ and $g-1$, respectively. At a generic point of the moduli space of chiral multiplet, we conjecture that $\mathrm{U}(1)^g$ gauge group is unbroken. Consequently, the central charges of the theory are

$$c_L=7g-4+2n\ ,\qquad c_R=3(2g-1+n)\ . \tag{3.48}$$

Regardless of their quiver descriptions (or frames), these theories all flow to the same infrared (IR) theory. By introducing U(1) flavor fugacities c_i for the external punctures and d_i for the U(1) gauging at UV, the elliptic genus of the theory can be expressed in a simple form

$$\mathcal{I}^{(0,2),2}_{g>0,n}(c_1,\dots,c_n)=\prod_{j=1}^{g-1}\frac{2\vartheta_4(d_j^2)^2}{\eta(q)\vartheta_1(d_j^4)}\prod_{i=1}^{n}\frac{\eta(q)}{\vartheta_1(c_i^2)}\ . \tag{3.49}$$

Therefore, the flavor symmetry associated to each puncture gets enhanced to SU(2) at IR as seen in (3.22). Remarkably, the integral formula (3.44) guarantees that the above form of the (0,2) elliptic genera is consistent with the TQFT structure as in Figure 3-11

$$\mathcal{I}^{(0,2),2}_{g=g_1+g_2,n_1+n_2-2}=\int_{\mathrm{JK}}\frac{da}{2\pi i a}\mathcal{I}^{(0,2),2}_{g_1,n_1}(\dots,d_{g-1}a)\mathcal{I}^{(0,2),2}_{g_2,n_2}(d_{g-1}a^{-1},\dots)\vartheta_1(a^{\pm2})\ , \tag{3.50}$$

$$\mathcal{I}^{(0,2),2}_{g+1,n-2}=\int_{\mathrm{JK}}\frac{da}{2\pi i a}\mathcal{I}^{(0,2),2}_{g,n}(\dots,d_g a,d_g a^{-1})\vartheta_1(a^{\pm2})\ . \tag{3.51}$$

As we recall, the elliptic genus of the (0,2) reduction of the class $\mathcal{S}$ theory of genus one with n punctures is

$$\mathcal{I}^{(0,2),2}_{1,n}(c_1,\dots,c_n)=\prod_{i=1}^{n}\frac{\eta(q)}{\vartheta_1(c_i^2)}\ , \tag{3.52}$$

which exhibit the enhancement to SU(2) for each puncture in the infrared. To construct the (0,2) reduction of the class $\mathcal{S}$ theory $\mathcal{T}_2[C_{g>0,n}]$ of type A_1, we can repeatedly gauge anti-diagonal SU(2) of genus one theories with multiple punctures. Upon the rescaling $d\to q^{1/4}d$, the resulting elliptic genus differs from (3.49) by merely a factor of $(2q^{1/2}d^2)^{g-1}$. In this sense, the TQFT structure of the elliptic genus can be attributed to the class $\mathcal{S}$ construction.

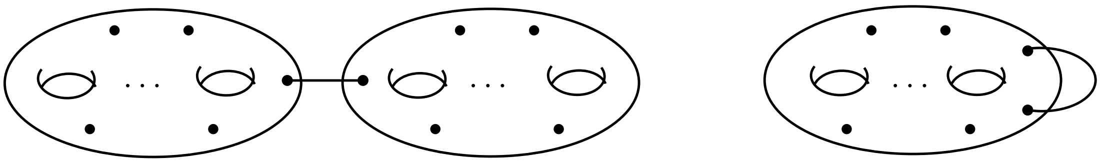

Figure 3-11 The gluing minimal punctures leads to a new Riemann surface, and (0,2) elliptic genera are consistent with the cut-and-join procedure on Riemann surfaces $C_{g,n}$.

From the perspective of the chiral algebra, we observe a connection to class $\mathcal{S}$ theory. The VOA corresponding to any Lagrangian $\mathcal{N}=2$ SCFT can be constructed using the gauging method described in paper[11]. Yet, putting this method into practice is intricate. Detailing the VOA structure and its representation theory is notably challenging, even for class $\mathcal{S}$ theories of type A_1. Nevertheless, if we change $\vartheta_4(a^{\pm2})$ to $\vartheta_1(a^{\pm2})$ for the U(1) gauging (3.10), the JK integrand of the elliptic genus, when derived from the UV Lagrangian, coincides with that of the Schur index of $\mathcal{T}_2[C_{g>0,n}]$ up to a factor, and the JK residue gives

$$\prod_{j=1}^{g-1}\frac{2\vartheta_1(d_j^2)^2}{\eta(q)\vartheta_1(d_j^4)}\prod_{i=1}^{n}\frac{\eta(q)}{\vartheta_1(c_i^2)}\ . \tag{3.53}$$

Building upon the results in papers[27,47–48], certain residues of the integrand correspond to the Schur index with surface defects of Gukov-Witten type①. Given its role as a surface defect index, one expects (3.53) to satisfy all the flavored modular differential equations associated to the VOA $\chi(\mathcal{T}_2[C_{g,n}])$. In light of these observations, we propose that the elliptic genus (3.49) can be written in terms of a linear combination of characters of the VOA $\chi(\mathcal{T}_2[C_{g>0,n}])$ upon an appropriate redefinition of the U(1) fugacities and a certain overall factor.

When the number of U(1) gauge groups is g for a theory of genus g, it admits an LG dual, but it depends on the quiver description as we saw in the examples of §3.2.4 and §3.2.5. It would be interesting to study the VOA structures for theories of this type.

① Up to some prefactors of q to account for the different stress-energy tensors involved[69].

3.3 SU(N)×U(1) gauge theories, LG duals and VOAs

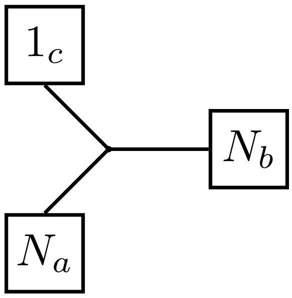

Figure 3-12 A building block $U_N^{(0,2)}$, representing N^2 (0,2) chiral multiplets with $\mathrm{SU}(N)_a\times\mathrm{SU}(N)_b\times\mathrm{U}(1)_c$ flavor symmetry.

Now let us move on to cases of higher rank. For class $\mathcal{S}$ of type A_{N-1}, punctures are classified by a partition of N, and theories do not admit a Lagrangian description in general. As seen in §3.1.1, we perform the (0,2) reduction for 4d $\mathcal{N}=2$ Lagrangian theories of class $\mathcal{S}$. Consequently, the basic building block is a sphere with two maximal punctures and one minimal puncture, corresponding to N^2 hypermultiplets. (See the right of Figure 4-1.) Its (0,2) reduction yields (0,2) N^2 chiral multiplets, with the flavor symmetry represented as $\mathrm{U}(N^2)$ and includes the subgroup $\mathrm{SU}(N)_a\times\mathrm{SU}(N)_b\times\mathrm{U}(1)_x$. This particular (0,2) theory, labeled as U_N, serves as our fundamental building block, with its quiver illustrated in Figure 3-12. As highlighted in (3.3), the c-extremization is invalid for theories of this class, and the $\mathrm{U}(1)_R$-charge of the (0,2) chiral multiplet is $r=0$. The NS elliptic genus of U_N is given by

$$\mathcal{I}_{U_N}^{(0,2)}(a,b,c)=\prod_{i,j=1}^{N}\frac{\eta(q)}{\vartheta_4(q^{-\frac{1}{2}}\tilde{c}a_ib_j)}=\prod_{i,j=1}^{N}\frac{\eta(q)}{\vartheta_4(ca_ib_j)} \tag{3.54}$$

where we redefine the $\mathrm{U}(1)_c$ flavor fugacity by $c=q^{-\frac{1}{2}}\tilde{c}$. Note that we impose the condition $\prod_{i=1}^{N}a_i=1=\prod_{j=1}^{N}b_j$ on the SU(N) fugacities. On a related note, as seen in (2.74), the Schur limit of the superconformal index for a sphere with two maximal punctures and one minimal puncture is given by

$$\mathcal{I}^{\mathrm{4d}}=\prod_{i,j=1}^{N}\Gamma(\sqrt{t}(ca_ib_j)^{\pm1})\quad\xrightarrow{t\to q}\quad\mathcal{I}^{\mathrm{Schur}}=\prod_{i,j=1}^{N}\frac{\eta(q)}{\vartheta_4(ca_ib_j)}\ . \tag{3.55}$$

Consequently, by redefining the $\mathrm{U}(1)_c$ flavor fugacity as in (3.54), the elliptic genus of the U_N theory agrees with the Schur index above.

The gauging procedure of the U_N theories is as usual. The contribution of a vector multiplet is the same in both the Schur index (2.75) and the elliptic genus (2.39). The SU(N) vector multiplet contribution is

$$\mathcal{I}_{\mathrm{vec}}^{(0,2)}(a)=\frac{\eta(q)^{2N}}{N!}\prod_{A\neq B}i\frac{\vartheta_1(a_A/a_B)}{\eta(q)}\ . \tag{3.56}$$

and the SU(N) gauging leads to no gauge anomaly. In this way, the integrand of the superconformal index and the (0,2) elliptic genus agree for a class $\mathcal{S}$ Lagrangian theory.

However, in 2d (0,2) theories, one can also gauge the U(1) symmetry of the U_N theory. This U(1) gauging is similar to the U_2 case, but the U(1) gauge charges of the two Fermi multiplets must be $\pm N$ to avoid gauge anomaly. Following the c-extremization, the $\mathrm{U}(1)_R$-charge for these Fermi multiplets is assigned to be $r=0$. Consequently, the gauging procedure is then applied to the elliptic genus as described:

$$\frac{\eta(q)}{\vartheta_4(q,c_1\cdots)}\frac{\eta(q)}{\vartheta_4(q,c_2\cdots)} \quad\rightarrow\quad \eta(q)^2\int_{\mathrm{JK}}\frac{da}{2\pi i a}\frac{\eta(q)}{\vartheta_4(da\cdots)}\frac{\eta(q)}{\vartheta_4(da^{-1}\cdots)}\frac{\vartheta_4(a^{\pm N})}{\eta(q)^2} \tag{3.57}$$

Once a (0,2) quiver theory involves a U(1) gauging of the U_N theories, the theory is no longer the (0,2) reduction of a class $\mathcal{S}$ theory. Nonetheless, one can consider the elliptic genus of the theory in the infrared.

3.3.1 Gauge/LG duality for linear quivers

The (0,2) reduction of class $\mathcal{S}$ theory of higher rank is quite restricted because the class $\mathcal{S}$ theory should have a Lagrangian description. Consequently, a theory sought to be built upon the SU(N) gauging of the U_N theories. One such example is a linear quiver as in Figure 3-13. The central charges are given by

$$c_L = 2(N^2+n-1)\ , \qquad c_R = 3(N^2+n-1)\ . \tag{3.58}$$

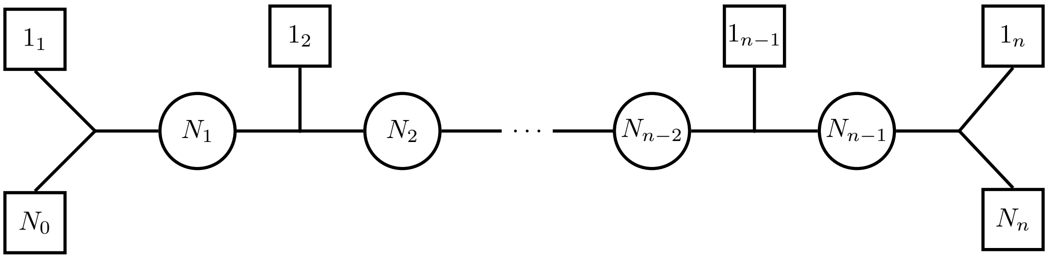

Figure 3-13 An SU(N) linear quiver where the subscripts are added solely for node numbering purposes.

As the simplest example, we can consider the (0,2) SU(N) SQCD with N fundamentals and N anti-fundamentals. As found in paper[9]Eqn.(4.7), the computation of its elliptic genus is equivalent to the higher rank rendition of the elliptic inversion formula (3.17)

$$\frac{\eta(q)^{N^2+1}\vartheta_1(c_1^N c_2^N)}{\vartheta_\alpha(c_1^N)\vartheta_\alpha(c_2^N)\prod_{A,B=1}^N\vartheta_1(c_1c_2b_{0,A}b_{2,B}^{-1})} = \frac{\eta(q)^{2N^2}}{N!}\int_{\mathrm{JK}}\frac{da}{2\pi i a}\prod_{A,B,i=1}^{N}\frac{\prod_{j\neq i}\vartheta_1(a_i/a_j)}{\vartheta_4(c_1b_{0,A}a_i^{-1})\vartheta_4(c_2b_{2,B}^{-1}a_i)}\ , \tag{3.59}$$

where $\alpha=1$ for even N and $\alpha=4$ for odd N. To evaluate the elliptic genus of a linear quiver, we repeatedly apply the elliptic inversion formula (3.59), and it therefore takes a simple form

$$\mathcal{I}_{0,n,2}^{(0,2),N} = \frac{\eta(q)^{N^2+n-1}\vartheta_\alpha(\prod_{i=1}^n c_i^N)}{\prod_{i=1}^n\vartheta_\beta\left(c_i^N\right)\cdot\prod_{A,B=1}^N\vartheta_\gamma(b_{0,A}\,b_{n,B}^{-1}\prod_{i=1}^n c_i)} \tag{3.60}$$

where

$$\alpha=\begin{cases}1 & n\cdot N \text{ even}\\ 4 & n\cdot N\text{ odd}\end{cases}\ ,\qquad \beta=\begin{cases}1 & N\text{ even}\\ 4 & N\text{ odd}\end{cases}\ ,\qquad \gamma=\begin{cases}1 & n\text{ even}\\ 4 & n\text{ odd}\end{cases}\ . \tag{3.61}$$

Taking into account the shift of the $\mathrm{U}(1)_{c_i}$ fugacities in (3.7), the form of the elliptic genus tells us that the theory is dual to an LG model with one Fermi multiplet Ψ, n chiral multiplets Φ and one chiral meson multiplet $\tilde{\Phi}_{i,j=1,\ldots,n}$, forming a J-type superpotential

$$W = \Psi\Big(\prod_{i=1}^n\Phi_i + \det\tilde{\Psi}\Big)\ .$$

3.3.2 Gauge/LG duality for circular quivers

The other class of the (0,2) reduction of class $\mathcal{S}$ theory is a circular quiver as in Figure 3-14. Again, for a genus-one theory, the diagonal Cartan subgroup $\mathrm{U}(1)^{N-1}$ is unbroken. As a result, the central charges for genus 1 with n punctures are given by

$$c_L = 2n + 3(N-1) \ , \qquad c_R = 3n + 3(N-1) \ . \tag{3.62}$$

Some of the explicit JK residue computations of elliptic genera can be found in Appendix B.5 and B.6.

As the simplest example, let us consider the theory at genus-one with one puncture. In other words, it is an SU(N) gauge theory with one adjoint chiral, which is the (0,2) reduction of the 4d $\mathcal{N}=4$ SU(N) theory, and an additional free chiral multiplet. The evaluation of its elliptic genus can be understood as an elliptic inversion formula of another kind

$$\begin{aligned} \mathcal{I}_{1,1}^{(0,2),N} &= \frac{\eta(q)^N}{N!\vartheta_4(c)^N}\int_{\mathrm{JK}}\frac{da}{2\pi i a}\prod_{j\neq i}\frac{\vartheta_1(a_i/a_j)}{\vartheta_4(ca_i/a_j)} \\ &= \frac{\eta(q)}{\vartheta_\alpha(c^N)} \end{aligned} \tag{3.63}$$

where $\alpha=1$ for even N and $\alpha=4$ for odd N. As a simple check of the central charge, the elliptic genus can be written as

$$\mathcal{I}_{1,1}^{(0,2),N} = \frac{\eta(q)}{\vartheta_4(c)}\cdot\frac{\vartheta_4(c)}{\vartheta_\alpha(c^2)}\cdot\frac{\vartheta_\alpha(c^2)}{\vartheta_\alpha(c^3)}\cdots\frac{\vartheta_\alpha(c^{N-1})}{\vartheta_\alpha(c^N)} \tag{3.64}$$

which are contributions from $\mathrm{Tr}(\Phi), \mathrm{Tr}(\Phi^2), \ldots \mathrm{Tr}(\Phi^N)$ as well as their superpartner.

We can remove a free hypermultiplet factor $\eta(\tau)/\vartheta_4(c)$ from the above expression and obtain

$$\mathcal{I}_{\mathcal{N}=4}^{(2,2),N} = \frac{\vartheta_4(c)}{\vartheta_\alpha(c^N)} \ . \tag{3.65}$$

We note that this expression is precisely the vacuum character of $N-1$ copies of $bc\beta\gamma$ systems labeled by $i=1,\ldots,N-1$, with the following conformal weights h and U(1) charges m.

	h	m
b_i	$\frac{1}{2}(d_i+1)$	$\frac{1}{2}(d_i-1)$
c_i	$-\frac{1}{2}(d_i-1)$	$-\frac{1}{2}(d_i-1)$
β_i	$\frac{1}{2}d_i$	$\frac{1}{2}d_i$
γ_i	$1-\frac{1}{2}d_i$	$-\frac{1}{2}d_i$

Here $d_i = i+1$ denotes the degree of the i-th invariant of SU(N). The vacuum character reads (up to a factor of i)

$$q^{\frac{1}{8}(N^2-1)}\prod_{i=1}^{N-1}\frac{(c^{d_i-1}q^{\frac{d_i+1}{2}};q)(c^{-d_i+1}q^{\frac{1-d_i}{2}};q)}{(c^{d_i}q^{\frac{d_i}{2}};q)(c^{-d_i}q^{1-\frac{d_i}{2}};q)} = \frac{\vartheta_4(c)}{\vartheta_\alpha(c^N)} \ . \tag{3.66}$$

The $bc\beta\gamma$ system serves as a free field realization of the chiral algebra $\chi^{\mathcal{N}=4,N}$ of the 4d $\mathcal{N}=4$ SU(N) SYM[23], and therefore is a reducible module of $\chi^{\mathcal{N}=4,N}$. Hence, the above vacuum character is naturally a reducible module character of $\chi^{\mathcal{N}=4,N}$.

As discussed at the end of §3.2.3, the combination of a (0,2) vector multiplet and an adjoint chiral multiplet forms a (2,2) vector multiplet. Consequently, there is no distinction between the left- and right-moving sectors. The analysis above suggests that the infrared theory is described by the chiral algebra $\chi^{\mathcal{N}=4,N}$.

As a generalization of this case, the circular quiver in Figure 3-14 can be obtained by SU(N) gauging of the ends of the linear quiver in Figure 3-13. The elliptic genus is given by

$$\mathcal{I}_{1,n}^{(0,2),N} = \prod_{i=1}^{n} \frac{\eta(q)}{\vartheta_\alpha(c_i^N)} \ . \tag{3.67}$$

where again $\alpha = 1$ for even N and $\alpha = 4$ for odd N. Extrapolating from the $\mathcal{N} = 4$ discussion, we conjecture that the elliptic genus continues to be a module character of the chiral algebra of the 4d $\mathcal{N} = 2$ circular quiver theory.

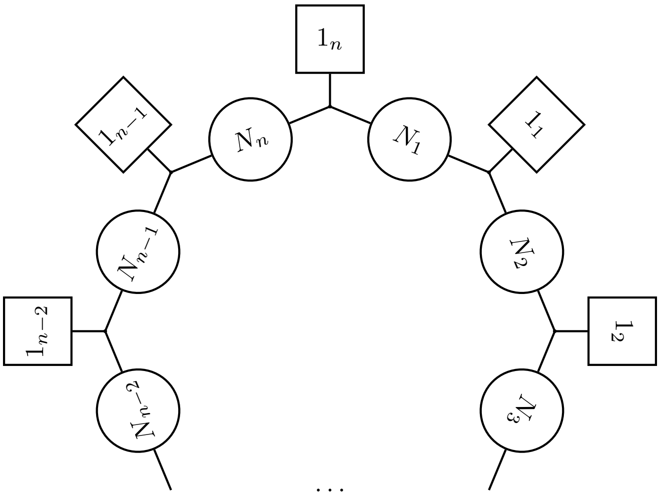

Figure 3-14 An SU(N) circular quiver where the subscripts are added solely for node numbering purposes.

3.3.3 General Riemann surfaces

Other quiver theories cannot be obtained from the (0,2) reduction of class $\mathcal{S}$ theory because they involve a U(1) gauge group. Nonetheless, one can consider SU(N) × U(1) quiver gauge theory of genus $g > 0$ with n punctures where the numbers of SU(N) and U(1) gauge groups are $2(g-1)+n$ and $g-1$, respectively. We assume that $\mathrm{U}(1)^{g(N-1)}$ gauge group is unbroken so that the complex dimension of the moduli space of chiral multiples is $(Ng - 1 + n)$. Thus, the central charges are given by

$$c_L = (3N + 1)g - 4 + 2n \ , \qquad c_R = 3(Ng - 1 + n) \ , \tag{3.68}$$

due to $\mathrm{U}(1)^{(N-1)g}$ gauge group is unbroken.

By introducing U(1) flavor fugacities c_i for the external punctures and d_i for the U(1) gauging, the elliptic genus of the theory can be expressed in a simple form (up to sign)

$$\mathcal{I}_{g>0,n}^{(0,2),N} = \prod_{j=1}^{g-1} \frac{(-1)^\beta N \vartheta_\beta(d_j^N)^2}{\eta(q)\vartheta_1(d_j^{2N})} \prod_{i=1}^{n} \frac{\eta(q)}{\vartheta_\alpha(c_i^N)} \ , \tag{3.69}$$

where $\alpha = 1, \beta = 4$ for even N and $\alpha = 4, \beta = 1$ for odd N. This form is independent of quiver descriptions. Therefore, regardless of the quiver descriptions, we claim these theories all flow to the same infrared (IR) theory. Applying the formula

$$\eta(q)^2 \int_{\mathrm{JK}} \frac{dc}{2\pi i c} \frac{\vartheta_4(c^{\pm N})}{\vartheta_\alpha(d^N c^{\pm N})} = \frac{(-1)^\beta N \vartheta_\beta(d^N)^2}{\eta(q)\vartheta_1(d^{2N})} \ , \tag{3.70}$$

one can convince oneself that (3.69) is consistent with the TQFT structure as in Figure 3-11.

The distinctions between SU(2) and SU(N) become evident in theories of genus g that have g U(1) gauge nodes. As demonstrated in Appendix B.6, the elliptic genus evaluation for the left quiver theory depicted in Figure 3-15 reveals that it is dual to an LG model. Contrarily, the explicit evaluation shows that the elliptic genus of the right quiver theory in Figure 3-15 does not factorize into theta functions. This observation implies that the right quiver theory does not possess an LG dual description.

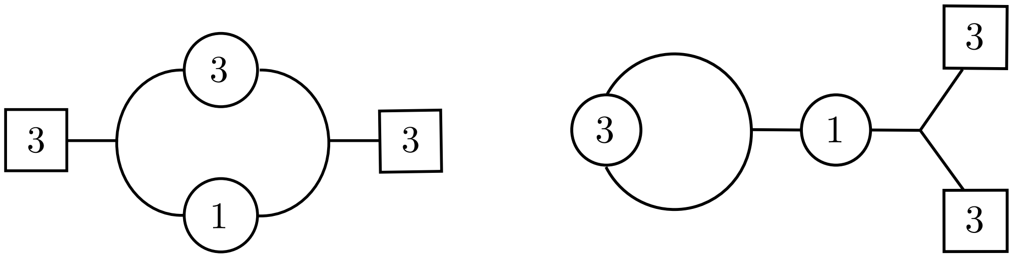

Figure 3-15 $\mathrm{SU}(3) \times \mathrm{U}(1)$ gauge theories corresponding to the genus-one Riemann surface with two punctures.

3.4 Comments on non-Lagrangian cases

The (0,2) reduction described in §3.1.1 can potentially be applied to non-Lagrangian class $\mathcal{S}$ theories, including the trinion theories T_N and the Argyres-Douglas theories. However, to perform a consistent reduction, the $(0,2)$ U(1) $\mathfrak{R}$-charges, represented as $\mathfrak{R} = R + \frac{r-f}{2}$, must take integer values. This imposes stringent conditions on which class $\mathcal{S}$ theories can undergo consistent reduction. Upon examining the Higgs and Coulomb branch operators, the reduction seems possible for simple theories such as the T_3 theory and the (A_1, D_4) Argyres-Douglas theory. On the other hand, the (A_1, D_{2n+1}) theories do not admit consistent reduction since the dimensions of their Coulomb branch operators are fractional, resulting in a non-integral value for $R + \frac{r-f}{2}$. In the following, we will discuss potential candidates for the $(0, 2)$ elliptic genus in these non-Lagrangian cases.

Recall that from §2.2, (0,2) elliptic genus in the Ramond sector① is expected to be a Jacobi form of weight-0 with a non-zero index which captures the 't Hooft anomaly of the flavor symmetry. Additionally, we further conjecture in (3.6) that the (0,2) elliptic genus for a class $\mathcal{S}$ theory $\mathcal{T}[C_{g,n}]$ should be some linear combination of the module characters of the associated VOA $\chi(\mathcal{T}[C_{g,n}])$. Indeed, both of the expected properties place strong constraints on the form of an elliptic genus.

The T_3 trinion theory of type A_2 is endowed with E_6 flavor symmetry[70–71]. As uncovered in paper[1]Figure 19, the SU(2) gauging of the T_3 trinion theory leads to the infinite coupling limit of SU(3) $N_f = 6$ superconformal theory. The corresponding VOA is the affine Lie algebra $(\hat{\mathfrak{e}}_6)_{-3}$ with level -3[11]. The algebra $\left(\hat{\mathfrak{e}}_6\right)_{-3}$ is endowed with irreducible representations with the following highest weights[72]:

$$0, \quad -3\omega_1, \quad -3\omega_6, \quad \omega_1 - 2\omega_3, \quad \omega_6 - 2\omega_5, \quad -2\omega_2, \quad -\omega_4\ . \tag{3.71}$$

Using the pure spinor formalism[58,73], the following combination of the $\left(\hat{\mathfrak{e}}_6\right)_{-3}$ characters is considered

$$\mathcal{I}^{\mathfrak{e}_6}(m, q) = \mathrm{ch}_0^{(\hat{\mathfrak{e}}_6)_{-3}}(m, q) - \mathrm{ch}_{-3\omega_1}^{(\hat{\mathfrak{e}}_6)_{-3}}(m, q). \tag{3.72}$$

This partition function can be expressed by a combination of theta functions as follows:

$$\mathcal{I}^{\mathfrak{e}_6}(m, q) = \frac{\eta(q)^{10}\left(\Theta^{\mathfrak{e}_6}_{\omega_1}(\widetilde{m}, q) - \Theta^{\mathfrak{e}_6}_{\omega_6}(\widetilde{m}, q)\right)}{\prod_{w \in \mathbf{S}} \vartheta_1(m^w_{\mathfrak{d}_5})}\ , \tag{3.73}$$

① In the previous subsections, we consider (0,2) elliptic genus in the NS sector. Nonetheless, it is straightforward to transform it to the Ramond sector simply by replacing ϑ_4 by ϑ_1.

where the two theta functions $\Theta^{\mathfrak{e}_6}_{\omega_1,\omega_6}$ are defined[74–75] as

$$\Theta^{\mathfrak{e}_6}_{\omega_1}(m,q) = \frac{q^{1/6}}{2}\sum_{k=1}^{4}\sigma_k m_0 \vartheta_k(m_0^3 q)\prod_{j=1}^{5}\vartheta_k(m_j) \ ,$$
$$\Theta^{\mathfrak{e}_6}_{\omega_6}(m,q) = \frac{q^{1/6}}{2}\sum_{k=1}^{4}\sigma_k m_0^{-1} \vartheta_k(m_0^3 q^{-1})\prod_{j=1}^{5}\vartheta_k(m_j) \ ,$$

with $-\sigma_1 = \sigma_2 = \sigma_3 = -\sigma_4 = 1$. Here, $m = (m_0, m_{\mathfrak{d}_5}) = (m_0, m_1, \dots, m_5)$ are the fugacities for $\mathfrak{e}_6$ ($m_{i>0}$ are also fugacities for the subalgebra $\mathfrak{d}_5$) in the orthogonal basis.① In the numerator of (3.73), we use $\widetilde{m} = (m_0^2, m_1, \dots, m_5)$, and in the denominator, $\mathbf{S} = [0,0,0,0,1]$ is the spin representation of $\mathfrak{d}_5$.

Using the branching rules, one can establish the relationships between the $\mathfrak{e}_6$ fugacities m and the $\mathfrak{a}_3 \oplus \mathfrak{a}_3 \oplus \mathfrak{a}_3$ fugacities x_i, y_j, z_k in the fundamental weight basis (or the omega basis in paper[76–77]):

$$m_0 = x_1^{\frac{1}{2}} z_2^{-\frac{1}{2}}, \qquad m_3 = x_1^{\frac{1}{2}} y_1^{-1} z_2^{\frac{1}{2}}, \tag{3.74}$$
$$m_1 = x_1^{-\frac{1}{2}} x_2^{-1} z_1 z_2^{\frac{1}{2}}, \qquad m_4 = x_1^{\frac{1}{2}} y_1 y_2^{-1}, \tag{3.75}$$
$$m_2 = x_1^{\frac{1}{2}} x_2 z_1 z_2^{\frac{1}{2}}, \qquad m_5 = x_1^{\frac{1}{2}} y_2 z_2^{\frac{1}{2}} \ . \tag{3.76}$$

Recall from (2.42) that the 't Hooft anomaly can be read off from the shift property of (3.73). In particular, we focus on the three simple $SU(3)$ flavor subgroups with the fugacities x_i, y_j, z_k:

$$\begin{aligned}
&x_1 \to x_1 q\ ,\ x_2 \to x_2/q, && \mathcal{I}^{\mathfrak{e}_6} \to \frac{q^2 x_1 z_2^2}{x_2^3}\,\mathcal{I}^{\mathfrak{e}_6}\\
&y_1 \to y_1 q\ ,\ y_2 \to y_2/q, && \mathcal{I}^{\mathfrak{e}_6} \to (q y_1/y_2)^9\,\mathcal{I}^{\mathfrak{e}_6}\\
&z_1 \to z_1 q\ ,\ z_2 \to z_2/q, && \mathcal{I}^{\mathfrak{e}_6} \to \frac{q^2 z_1^3}{x_1 z_2}\,\mathcal{I}^{\mathfrak{e}_6}\ .
\end{aligned} \tag{3.77}$$

In four dimensions, theThe E_6 theory is relatedd to the $SU(3)$ SQCD through the Argyres-Seiberg duality, where two of the $SU(3)$ flavor symmetries of the former are identified with the two $SU(3)$ flavor symmetriesy of the latter. If the above $\mathcal{I}^{\mathfrak{e}_6}$ truly represents the elliptic genus of the reduced E_6 theory, then its $SU(3)^2$ 't Hooft anomaly should match with that of the reduced $SU(3)$ SQCD. The elliptic genus $\mathcal{I}^{(0,2),3}_{0,2,2}$ of 2d (0,2) SU(3) SQCD with 3 fundamentals and 3 anti-fundamentals is the $N = 3$ specialization of (3.59), and it has the shift properties

$$\begin{aligned}
&c_i \to c_i q, && \mathcal{I}^{(0,2),3}_{0,2,2} \to q^{\frac{9}{2}} c_i^9\, \mathcal{I}^{(0,2),3}_{0,2,2}\\
&b_{i,1} \to b_{i,1} q,\ b_{i,2} \to b_{i,2}/q, && \mathcal{I}^{(0,2),3}_{0,2,2} \to (q b_{i,1}/b_{i,2})^9\, \mathcal{I}^{(0,2),3}_{0,2,2}
\end{aligned}$$

where c_i are the two U(1) fugacities, and $b_{i,j}$ are the two SU(3) fugacities. Hence, when we perform SU(2) gauging on the expression given in (3.73), it appears that we do not arrive at the (0,2) elliptic genus for SU(3) SQCD with $N_f = 6$. It could be worthwhile to explore alternative combinations of $(\hat{\mathfrak{e}}_6)_{-3}$ characters, distinct from (3.72), in order to compare with the SU(3) SQCD with $N_f = 6$. Indeed, by the logic of[58], a linear combination of the vacuum character and the character with the highest weight $-\omega_4$ is the most promising starting point.

Argyres-Douglas theories[78–79] constitute another interesting class of non-Lagrangian theories, whose construction involves a higher-order pole of the Higgs field in the Hitchin system[80].

① In contrary, the fugacities in paper[58] are expressed in the alpha basis[76–77].

Let us first consider the (A_1, D_4) theory. The rank-one theory contains a Coulomb branch operator with conformal dimension $\Delta = -3/2$, and therefore an integral r-charge $r = 2\Delta = 3$, suggesting a possible S^2 reduction and a corresponding (0,2) elliptic genus. The associated VOA is given by the Kac-Moody algebra $\widehat{\mathfrak{su}}(3)_{-3/2}$[6,81]. The level $k = -3/2$ with respect to the SU(3) flavor symmetry is called *boundary admissible* in Mathematics literature. There are four irreducible admissible highest weight modules with affine weights

$$-\frac{3}{2}\hat{\omega}_0, \quad -\frac{3}{2}\hat{\omega}_1, \quad -\frac{3}{2}\hat{\omega}_2, \quad \hat{\rho} = -\frac{1}{2}\sum_{i=0}^{2}\hat{\omega}_i \ , \tag{3.78}$$

where $\hat{\rho}$ is the affine Weyl vector. The characters are given by

$$\mathrm{ch}_{-\frac{3}{2}\hat{\omega}_0} = \frac{\eta(\tau)\vartheta_1(\mathrm{b}_1 - 2\mathrm{b}_2|2\tau)\vartheta_1(-\mathrm{b}_1 - \mathrm{b}_2|2\tau)\vartheta_1(-2\mathrm{b}_1 + \mathrm{b}_2|2\tau)}{\eta(2\tau)\vartheta_1(\mathrm{b}_1 - 2\mathrm{b}_2|\tau)\vartheta_1(-\mathrm{b}_1 - \mathrm{b}_2|\tau)\vartheta_1(-2\mathrm{b}_1 + \mathrm{b}_2|\tau)} \ , \tag{3.79}$$

$$\mathrm{ch}_{-\frac{3}{2}\hat{\omega}_1} = -\frac{\eta(\tau)\vartheta_4(\mathrm{b}_1 - 2\mathrm{b}_2|2\tau)\vartheta_4(-\mathrm{b}_1 - \mathrm{b}_2|2\tau)\vartheta_1(-2\mathrm{b}_1 + \mathrm{b}_2|2\tau)}{\eta(2\tau)\vartheta_1(\mathrm{b}_1 - 2\mathrm{b}_2|\tau)\vartheta_1(-\mathrm{b}_1 - \mathrm{b}_2|\tau)\vartheta_1(-2\mathrm{b}_1 + \mathrm{b}_2|\tau)} \ , \tag{3.80}$$

$$\mathrm{ch}_{-\frac{3}{2}\hat{\omega}_2} = -\frac{\eta(\tau)\vartheta_1(\mathrm{b}_1 - 2\mathrm{b}_2|2\tau)\vartheta_4(-\mathrm{b}_1 - \mathrm{b}_2|2\tau)\vartheta_4(-2\mathrm{b}_1 + \mathrm{b}_2|2\tau)}{\eta(2\tau)\vartheta_1(\mathrm{b}_1 - 2\mathrm{b}_2|\tau)\vartheta_1(-\mathrm{b}_1 - \mathrm{b}_2|\tau)\vartheta_1(-2\mathrm{b}_1 + \mathrm{b}_2|\tau)} \ , \tag{3.81}$$

$$\mathrm{ch}_{-\frac{1}{2}\hat{\rho}} = -\frac{\eta(\tau)\vartheta_4(\mathrm{b}_1 - 2\mathrm{b}_2|2\tau)\vartheta_1(-\mathrm{b}_1 - \mathrm{b}_2|2\tau)\vartheta_4(-2\mathrm{b}_1 + \mathrm{b}_2|2\tau)}{\eta(2\tau)\vartheta_1(\mathrm{b}_1 - 2\mathrm{b}_2|\tau)\vartheta_1(-\mathrm{b}_1 - \mathrm{b}_2|\tau)\vartheta_1(-2\mathrm{b}_1 + \mathrm{b}_2|\tau)} \ ,$$

where the first one is the vacuum character of $\widehat{\mathfrak{su}}(3)_{-3/2}$ as well as the Schur index of the (A_1, D_4) theory. The modular S-matrix is given by

$$S = -\frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix} \ . \tag{3.82}$$

There are four eigenvectors of S, with eigenvalues respectively $(-1, -1, -1, 1)$,

$$\mathrm{ch}_{-\frac{3}{2}\hat{\omega}_0} - \mathrm{ch}_{-\frac{3}{2}\hat{\rho}}, \quad \mathrm{ch}_{-\frac{3}{2}\hat{\omega}_0} + \mathrm{ch}_{-\frac{3}{2}\hat{\omega}_1}, \quad \mathrm{ch}_{-\frac{3}{2}\hat{\omega}_0} + \mathrm{ch}_{-\frac{3}{2}\hat{\omega}_2}, \tag{3.83}$$

and finally

$$\mathrm{ch}_{-\frac{3}{2}\hat{\omega}_0} - \mathrm{ch}_{-\frac{3}{2}\hat{\omega}_1} - \mathrm{ch}_{-\frac{3}{2}\hat{\omega}_2} + \mathrm{ch}_{-\frac{3}{2}\hat{\rho}} \ . \tag{3.84}$$

Unfortunately, neither the $+1$ eigenvector nor any linear combination of the -1 eigenvectors behaves consistently under the shift of both SU(3) flavor fugacities b_1 and b_2. Consequently, no linear combination of the $\widehat{\mathfrak{su}}(3)_{-3/2}$ characters satisfies the expected properties of a (0,2) elliptic genus.

Let us also consider the (A_1, D_{2n+1}) Argyres-Douglas theories which enjoy SU(2) flavor symmetry. Since these theories contain Coulomb branch operators with fractional r-charge, a valid (0,2) reduction is *not* anticipated. Below, we contend that from the VOA perspective, a (0,2) elliptic genus does not exist. The associated VOAs are given by $\widehat{\mathfrak{su}}(2)_{k=-\frac{4n}{2n+1}}$[6,81]. In this case, the level $k = -\frac{4n}{2n+1} = -2 + \frac{2}{2n+1}$ is also boundary admissible, and the VOA has admissible affine weights given by (where $u := 2n + 1$)[82–83]

$$\hat{\lambda}_{k,j} = (k + \frac{2j}{u})\hat{\omega}_0 - \frac{2j}{u}\hat{\omega}_1 \ , \qquad j = 0, 1, 2, \ldots, u - 1 = 2n \ . \tag{3.85}$$

They are highest weights of irreducible highest weight modules $L(\hat{\lambda}_{k,j})$ of $\widehat{\mathfrak{su}}(2)_{k=-\frac{4n}{2n+1}}$, whose characters are given by a simple formula

$$\mathrm{ch}\, L(\hat{\lambda}_{k,j}) = z^{-\frac{2j}{u}} q^{\frac{j^2}{2u}} \frac{\vartheta_1(2\mathrm{z} - j\tau|u\tau)}{\vartheta_1(2\mathrm{z}|\tau)} \ . \tag{3.86}$$

Here, z is the flavor SU(2) fugacity. The modular S-matrix is given by

$$S_{jj'} = \sqrt{\frac{2}{u^2(k+2)}} e^{\pi i(j+j')} e^{\pi i(jj'(k+2))} \sin\left(\frac{\pi}{k+2}\right) \ . \tag{3.87}$$

However, none of the eigenvectors of the S-matrix transforms itself (up to a factor) under the shift $\mathsf{z} \to \mathsf{z}+\tau$ since each character transforms in the following manner (the subscript follows a cyclic rule such that $j \sim j+2n+1$):

$$\mathrm{ch}(\hat{\lambda}_{k,j}) \xrightarrow{\mathsf{z}\to\mathsf{z}+\tau} (b^2 q)^{-k}\,\mathrm{ch}(\hat{\lambda}_{k,j-2}) \ . \tag{3.88}$$

This implies that a Jacobi form from $\mathrm{ch}(\hat{\lambda}_{k,j})$ must take the form

$$\mathrm{const}\cdot\sum_{j=0}^{2n}\mathrm{ch}(\hat{\lambda}_{k,j}) \ , \tag{3.89}$$

which is never an eigenvector of the S-matrix (3.87). Therefore, we conclude that no linear combination of the $\widehat{\mathfrak{su}}(2)_{k=-\frac{4n}{2n+1}}$ characters satisfies the expected properties of a (0,2) elliptic genus.

Chapter 4: $\mathcal{N} = (0,4)$ elliptic genera for class $\mathcal{S}$ theories on S^2

In this chapter, we study $\mathcal{N} = (0,4)$ theories obtained by a distinct twisted compactification of class $\mathcal{S}$ theories of type A on S^2. In these theories, we perform a topological twist on $\mathrm{U}(1)_{S^2}$ with 4d $\mathcal{N} = 2$ superconformal R-symmetry $\mathrm{U}(1)_r \subset \mathrm{SU}(2)_R \times \mathrm{U}(1)_r$ as discussed in papers[9,84]. Referencing Table 3-1, the four supercharges $Q^I_{-}, \tilde{Q}_{I\dot{-}}$ $(I = 1,2)$ survive under this twist, and they possess the identical $\mathrm{U}(1)_{T^2}$ charge. Therefore, this twist preserves 2d $N = (0,4)$ supersymmetry and thus, we refer to this twisted compactification as *the (0,4) reduction* of class $\mathcal{S}$ theories. For 4d $\mathcal{N} = 2$ SCFT, $\mathrm{U}(1)_r$-charges of operators are integral, eliminating the need for an additional twist by a flavor symmetry. The 2d $N = (0,4)$ supersymmetry has $\mathrm{SO}(4)_R \cong \mathrm{SU}(2)^-_R \times \mathrm{SU}(2)^+_R$ as the UV R-symmetry where 4d $\mathrm{SU}(2)_R$ is identified with 2d $\mathrm{SU}(2)^-_R \subset \mathrm{SO}(4)_R$. This subgroup subsequently evolves into the affine $\widehat{\mathfrak{su}(2)}$ Lie algebra within the small $\mathcal{N} = 4$ superconformal algebra in the infrared. Given the (0,4) reduction of a class $\mathcal{S}$ theory, we consider its IR SCFT on the Higgs branch where $\mathrm{SU}(2)^+_R$ becomes the small $\mathcal{N} = 4$ superconformal R-symmetry in the right-moving sector. For a detailed analysis of the symmetries within this context, readers are directed to[9].

In the (0,4) reduction, a 4d $\mathcal{N} = 2$ hypermultiplet reduces to a 2d $\mathcal{N} = (0,4)$ hypermultiplet (two (0,2) chirals with opposite charges). Likewise, a 4d $\mathcal{N} = 2$ vector multiplet reduces to a 2d $\mathcal{N} = (0,4)$ vector multiplet ((0,2) vector + (0,2) adjoint Fermi). Consequently, for a Lagrangian theory, the basic building blocks in 2d are as follows:

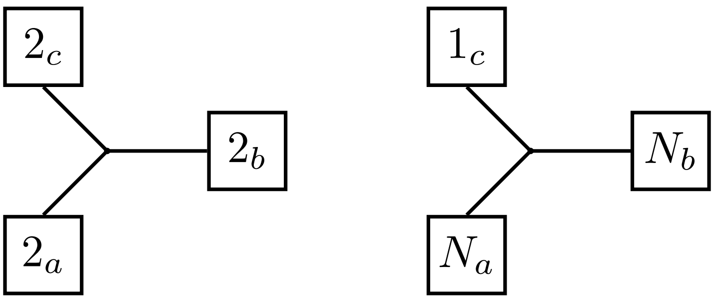

Figure 4-1 Left: A building block $U_2^{(0,4)}$ for type A_1, representing eight (0,2) chiral multiplets with $\mathrm{SU}(2)^3$ flavor symmetry. Right: A building block for Lagrangian theories $U_N^{(0,4)}$ of type A_{N-1}, representing a (0,4) free hypermultiplet with $\mathrm{SU}(N)_a \times \mathrm{SU}(N)_b \times \mathrm{U}(1)_c$ flavor symmetry.

For type A_1, it corresponds to a sphere with three punctures. For type A_{N-1}, a sphere with one minimal puncture and two maximal punctures gives rise to this building block. For simplicity in notation (without distinguishing types of punctures), we denote its contribution to a (0,4) elliptic genus as $\mathcal{I}^{(0,4)}_{0,3}$. The explicit contributions of this and the vector multiplet are provided as follows:

$$\mathcal{I}_{0,3}^{(0,4)}(a,b,c)=\prod_{i,j=1}^{N}\frac{\eta(q)^2}{\vartheta_1(v(ca_ib_j)^{\pm})}\ ,$$
$$\mathcal{I}_{\text{vec}}^{(0,4)}(a)=\frac{\left(\vartheta_1(v^2)\eta(q)\right)^{N-1}}{N!}\prod_{\substack{A,B=1\\A\neq B}}^{N}\frac{\vartheta_1(v^2a_A/a_B)\vartheta_1(a_A/a_B)}{\eta(q)^2}\ , \tag{4.1}$$

where the SU(N) fugacities condition is implicitly imposed

$$\prod_{i=1}^{N}a_i=1=\prod_{i=1}^{N}b_i\ . \tag{4.2}$$

The fugacity v is for the Cartan subgroup of the anti-diagonal of $\mathrm{SU}(2)_R^-\times\mathrm{SU}(2)_R^+$ R-symmetry that commutes with the supercharges. In this physical setup, it is argued in papers[9,85] that the (0,4) elliptic genus is expected to be the Vafa-Witten partition function[86] on $C_{g,n}\times S^2$. In paper[9], the S-duality of (0,4) theories with genus zero is confirmed by evaluating the elliptic genera. Notably, using the elliptic inversion formula (3.17), the elliptic genus of the (0,4) reduction of the non-Lagrangian T_3 trinion theory was obtained there. The primary focus of this thesis is to explore the (0,4) theories with a genus greater than zero, using elliptic genera.

In theories with a genus of zero, one can determine the right-moving central charge using the $\mathrm{SU}(2)_R^+$ anomaly

$$c_R=6(2k_R)=6(n_h-n_v)\ . \tag{4.3}$$

Here, k_R denotes the $\mathrm{SU}(2)_R^+$ anomaly coefficient, which can be computed as in (2.17) at UV, and $2k_R$ represents the level of the affine $\mathrm{SU}(2)_R^+$-symmetry. For a class $\mathcal{S}$ theory $\mathcal{T}_N[C_{g=0,n}]$ lacking Lagrangian description, n_h and n_v can be evaluated from partitions of N assigned to punctures. The explicit treatment can be found in paper[9,54], and we omit the details here. The left-moving central charge c_L can be derived from the gravitational anomaly (2.21). Note that $2k_R$ represents the quaternionic dimension of the Higgs branch. Moreover, the $q\to 0$ limit of the elliptic genus agrees with the Hilbert series of the Higgs branch[87] (the computational techniques are developed in paper[88–90]).

On the other hand, the situation drastically changes for theories with a genus greater than zero. In a theory with genus $g>0$, the $\mathrm{U}(1)^{(N-1)g}$ gauge symmetry is unbroken at a generic point of the Higgs branch[87,91].① Since a theory is expected to flow to a CFT on its Higgs branch, the right-moving central charge is six times the quaternionic dimension of the Higgs branch. A general formula for the quaternionic dimension of the Higgs branch of an A_{N-1} type class $\mathcal{S}$ is given in[91]. The gravitational anomaly is given by $c_R-c_L=2(n_h-n_v)$ where n_h,n_v are the effective number of hypermultiplets and vector multiplets, respectively. Therefore, the central charges for the (0,4) theory of genus $g>0$ are given by

$$\begin{aligned}c_R&=6(n_h-n_v+g(N-1))=6\Big[N-1+\frac{1}{2}\sum_{i=1}^{n}(N^2-N-\dim_{\mathbb{C}}\mathcal{O}_i)\Big]\ ,\\ c_L&=4(n_h-n_v)+6g(N-1)=(4+2g)(N-1)+2\sum_{i=1}^{n}(N^2-N-\dim_{\mathbb{C}}\mathcal{O}_i)\end{aligned} \tag{4.4}$$

where $\mathcal{O}_i$ represents the nilpotent orbit corresponding to the i-th puncture[54]. A nilpotent orbit is labelled by a partition λ of N whose complex dimension is calculatedgiven asby

$$\dim_{\mathbb{C}}O_\lambda=N^2-\sum_i l_i^2$$

① The term "Higgs branch" refers to the moduli space of hypermultiplets where the gauge symmetry is "maximally Higgsed". Namely, it is the configuration in which the number of massless Abelian vector multiplets is minimized. This particular branch was previously referred to as the "Kibble branch" in[87].

where $\lambda^t = [l_1, l_2 \cdots]$ is the transpose of the Young diagram transpose to λ. As we will see below, the $q \to 0$ limit of the elliptic genus is no longer equal to the Hilbert series of the Higgs branch. This is very similar to the relation between the Hall-Littlewood index and the Higgs branch Hilbert series[5] in which the agreement can be seen for theories with genus zero but not higher.

In the following, we present closed-form expressions for the (0,4) elliptic genera of theories where the genus $g > 0$. If a theory has a Lagrangian description with a gauge group of adequately low total rank, one can straightforwardly compute the elliptic genus through the JK-residue method. To determine the elliptic genus of non-Lagrangian theories at higher genus, we exploit the inversion formula in paper[9,92–93], namely performing additional gauging in Lagrangian theories. For detailed calculations, readers can refer to Appendix C, which provides explicit JK residue computations of (0,4) elliptic genera. The resulting closed-form expressions are remarkably simple, aligning well with the TQFT structure on punctured Riemann surfaces $C_{g,n}$.

4.1 Type A_1

Class $\mathcal{S}$ theories of type A_1 all have Lagrangian descriptions and are completely specified by the genus g and the number of (regular) punctures n. We shall focus on theories at genus $g \geq 1$ with an arbitrary number of punctures. To compute elliptic genera, we can gauge the basic building block illustrated in the left of Figure 4-1 by using (4.1). For $g = 1, n = 1$, the elliptic genus is computed by JK-residue where one only encounters non-degenerate poles,

$$\begin{aligned} \mathcal{I}_{1,1}^{(0,4),2}(c) &= \int_{\mathrm{JK}} \frac{da}{2\pi i a} \mathcal{I}_{0,3}^{(0,4)}(a, a^{-1}, c) \mathcal{I}_{\mathrm{vec}}^{(0,4)}(a) \\ &= \frac{\eta(q)^2 \vartheta_1(v^4)}{\vartheta_1(v^2)\vartheta_1(v^2 c^{\pm 2})} \ . \end{aligned} \tag{4.5}$$

The expression is a simple ratio of the theta functions, and the LG dual theory is described in paper[9]§2.2.3. A similar computation can be performed for $g = 1, n \geq 1$, which yields

$$\mathcal{I}_{1,n}^{(0,4),2}(c_1, \dots, c_n) = \prod_{i=1}^{n} \frac{\eta(\tau)^2 \vartheta_1(v^4)}{\vartheta_1(v^2)\vartheta_1(v^2 c_i^{\pm 2})} \ . \tag{4.6}$$

While the Kibble branch Hilbert series was computed in paper[87]§4.2.2 for $n = 2$, the relation between the elliptic genus and the Hilbert series is unclear. Consequently, although the form of the elliptic genus suggests the existence of an LG dual theory, its precise description remains unknown to us.

It is straightforward to obtain the elliptic genus for higher genera. For example, the theory of genus two

$$\begin{aligned} \mathcal{I}_{2,0}^{(0,4),2} &= \int_{\mathrm{JK}} \frac{da}{2\pi i a} \mathcal{I}_{1,2}^{(0,4),2}(a, a^{-1}) \mathcal{I}_{\mathrm{vec}}^{(0,4)}(a) = \int_{\mathrm{JK}} \frac{da}{2\pi i a} (\mathcal{I}_{1,1}^{(0,4),2}(a))^2 \mathcal{I}_{\mathrm{vec}}^{(0,4)}(a) \\ &= \frac{\vartheta_1(v^2)\vartheta_1(v^4)}{\eta(q)^2} \end{aligned} \tag{4.7}$$

Moreover, we can increase the genus g by gluing together any number of pairs of punctures, and the final result takes a simple form

$$\mathcal{I}_{g,n}^{(0,4),2}(c_1, \dots, c_n) = \left(\frac{\vartheta_1(v^2)\vartheta_1(v^4)}{\eta(q)^2} \right)^{g-1} \prod_{i=1}^{n} \frac{\eta(q)^2 \vartheta_1(v^4)}{\vartheta_1(v^2)\vartheta_1(v^2 c_i^{\pm 2})} \ . \tag{4.8}$$

This result is consistent with the cut-and-join TQFT structure on $C_{g,n}$ so that

$$\mathcal{I}_{g+1,n}^{(0,4),2}(c_1, \dots, c_n) = \int_{\mathrm{JK}} \frac{da}{2\pi i a} \mathcal{I}_{g,n+2}^{(0,4),2}(c_1, \cdots, c_n, a, a^{-1}) \mathcal{I}_{\mathrm{vec}}^{(0,4)}(a) \ ,$$

$$\mathcal{I}_{g_1+g_2,n_1+n_2}^{(0,4),2}(c_1, \dots, c_{n_1+n_2}) = \int_{\mathrm{JK}} \frac{da}{2\pi i a} \mathcal{I}_{g_1,n_1+1}^{(0,4),2}(c_1, \cdots, a) \mathcal{I}_{\mathrm{vec}}^{(0,4)}(a) \mathcal{I}_{g_2,n_2+1}^{(0,4),2}(c_{n_1+1}, \cdots, a^{-1}) \ .$$

4.2 Type A_2

For class $\mathcal{S}$ theories of type A_2, there are two types of regular punctures the minimal punctures with flavor symmetry U(1) and the maximal punctures with flavor symmetry SU(3). We denote the number of these punctures as n_1 and n_3, respectively. To begin with, we can gauge the basic building block illustrated in the right of Figure 4-1 by using (4.1) to compute the elliptic genus for $g = 1$ with several minimal punctures. This can be swiftly computed using the JK residue from which we can postulate a general formula

$$\mathcal{I}^{(0,4),3}_{g=1;n_1,0}(c_1, \dots, c_{n_1}) = \prod_{i=1}^{n_1} \frac{\eta(q)^2 \vartheta_1(v^6)}{\vartheta_1(v^2)\vartheta_1(v^3 c_i^{\pm 3})} . \tag{4.9}$$

To access the elliptic genus in the presence of maximal punctures, we apply the ellip-

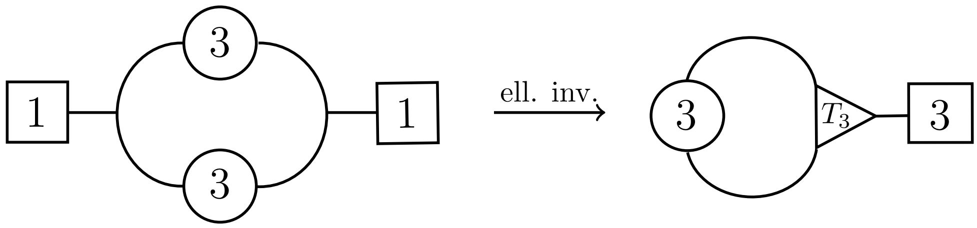

Figure 4-2 Application of the elliptic inversion formula leads to the elliptic genus for a maximal puncture.

tic inversion formula in paper[9] that computes the (0,4) elliptic genus of the T_3 theory. Specifically, starting from $g = 1, n_1 = 2$, we use the inversion formula (3.17) to obtain $g = 1, n_3 = 1$ (Figure 4-2)

$$\begin{aligned}\mathcal{I}^{(0,4),3}_{g=1;0,n_3=1}(b) =& \frac{\eta(q)^5}{2\vartheta_1(v^2 z^{\pm 2})} \int_{\text{JK}} \frac{ds}{2\pi i s} \frac{\vartheta_1(s^{\pm 2})\vartheta_1(v^{-2})}{\vartheta_1(v^{-1} s^{\pm 1} z^{\pm 1})} \mathcal{I}^{(0,4),3}_{g=1;n_1,0}(s^{\frac{1}{3}}/r, s^{-\frac{1}{3}}/r) \\ =& \frac{\eta(q)^6 \vartheta_1(v^2)\vartheta_1(v^4)\vartheta_1(v^6)}{\prod_{A,B=1}^{3} \vartheta_1(v^2 b_A/b_B)}\end{aligned} \tag{4.10}$$

where the SU(3) fugacities for the maximal puncture is identified by $(b_1, b_2, b_3) = (rz, r/z, r^{-2})$. The detailed computations of the elliptic genus for type A_2 theories are collected in Appendix C.2. In summary, the elliptic genus for the (0,4) reduction of class $\mathcal{S}$ theory $\mathcal{T}_3[C_{g;n_1,n_3}]$ is given by the following simple form

$$\mathcal{I}^{(0,4),3}_{g;n_1,n_3} = \left(\frac{\vartheta_1(v^2)\vartheta_1(v^4)^2\vartheta_1(v^6)}{\eta(q)^4} \right)^{g-1} \mathcal{I}^{(0,4),3}_{1;n_1,0} \mathcal{I}^{(0,4),3}_{1;0,n_3} , \tag{4.11}$$

where

$$\mathcal{I}^{(0,4),3}_{g=1;0,n_3} = \prod_{i=1}^{n_3} \frac{\eta(q)^6 \vartheta_1(v^2)\vartheta_1(v^4)\vartheta_1(v^6)}{\prod_{A,B=1}^{3} \vartheta_1(v^2 b_{iA}/b_{iB})} . \tag{4.12}$$

4.3 Type A_3

For type A_3 theories, there are four types of regular punctures whose partitions and flavor symmetries are given as follows:

- $[3, 1]$ U(1)
- $[2, 1, 1]$ SU(2) × U(1)

- [2, 2] SU(2)
- [1, 1, 1, 1] SU(4)

We use the notations n_1, n_2, n_3, n_4 to denote the numbers of these punctures, respectively. For $g = 1$ only with minimal punctures ($n_2 = n_3 = n_4 = 0$), it is straightforward to compute the elliptic genus

$$\mathcal{I}^{(0,4),4}_{1;n_1,0,0,0} = \prod_{i=1}^{n_1} \frac{\eta(q)^2\vartheta_1(v^8)}{\vartheta_1(v^2)\vartheta_1(v^4c_i^{\pm4})} \ . \tag{4.13}$$

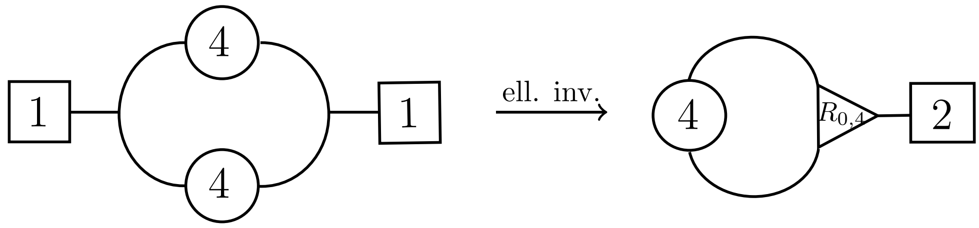

Figure 4-3 Application of the elliptic inversion formula leads to the elliptic genus for a [2,1,1] puncture.

Applying the elliptic inversion formula in paper[93], we can calculate the elliptic genus of genus-one theories that have [2,1,1], and [2,2] punctures. As a specific case, when there is solely one [2,1,1] puncture, the elliptic genus is as follows:

$$\begin{aligned}\mathcal{I}^{(0,4),4}_{g=1;0,1,0,0} =& \frac{\eta(q)^5}{2\vartheta_1(v^2z^{\pm2})}\int_{\mathrm{JK}}\frac{ds}{2\pi i s}\frac{\vartheta_1(s^{\pm2})\vartheta_1(v^{-2})}{\vartheta_1(v^{-1}s^{\pm1}z^{\pm1})}\mathcal{I}^{(0,4),3}_{g=1;2,0,0,0}(s^{\frac14}/r, s^{-\frac14}/r)\\ =&\frac{\eta(q)^6\vartheta_1(v^6)\vartheta_1(v^8)}{\vartheta_1(v^2)^2\vartheta_1(v^2z^{\pm2})\vartheta_1(v^3z^{\pm}r^{\pm4})} \ ,\end{aligned} \tag{4.14}$$

where z, r denote the SU(2) and U(1) fugacities respectively. (See Figure 4-3.) With only one $[2, 2]$ puncture, the elliptic genus

$$\begin{aligned}\mathcal{I}^{(0,4),4}_{g=1;0,0,1,0} =& \frac{\eta(q)^5\vartheta_1(v^2)\vartheta_1(v^{-2})}{2\vartheta_1(v^4)}\int_{\mathrm{JK}}\frac{ds}{2\pi i s}\frac{\vartheta_1(s^{\pm2})}{\vartheta_1(s^{\pm1})\vartheta_1(v^{-2}s^{\pm1})}\mathcal{I}^{(0,4),3}_{g=1;2,0,0,0}(s^{\frac14}/w, s^{-\frac14}/w)\\ =& \frac{\eta(q)^4\vartheta_1(v^6)\vartheta_1(v^8)}{\vartheta_1(v^2)\vartheta_1(v^4)\vartheta_1(v^4w^{\pm4})\vartheta_1(v^2w^{\pm4})} \ .\end{aligned} \tag{4.15}$$

While the derivation of the elliptic genus for the theory of genus one with a maximal puncture remains unknown, extrapolation from the results in (4.5) and (4.10) allows us to propose the following expression

$$\mathcal{I}^{(0,4),4}_{g=1;0,0,0,1} = \frac{\eta(q)^{12}\vartheta_1(v^2)\vartheta_1(v^4)\vartheta_1(v^6)\vartheta_1(v^8)}{\prod_{A,B=1}^4\vartheta_1(v^2b_A/b_B)} \ . \tag{4.16}$$

The validity of our proposed formula can be tested by examining the S-duality in Figure 4-4. Gauging this theory as in the right figure leads to the theory of genus one with three minimal punctures so that we can compare the result with (4.13). The detailed computations of the elliptic genera are collected in Appendix C.3.

This allows us to further increase the genus g, and finally, the elliptic genus of a general A_3 theory is given by

$$\mathcal{I}^{(0,4),4}_{g;n_1,n_2,n_3,n_4} = \left(\frac{\vartheta_1(v^2)\vartheta_1(v^4)^2\vartheta_1(v^6)^2\vartheta_1(v^8)}{\eta(q)^6}\right)^{g-1}\mathcal{I}^{(0,4),4}_{1;n_1,0,0,0}\mathcal{I}^{(0,4),4}_{1;0,n_2,0,0}\mathcal{I}^{(0,4),4}_{1;0,0,n_3,0}\mathcal{I}^{(0,4),4}_{1;0,0,0,n_4} \ ,$$

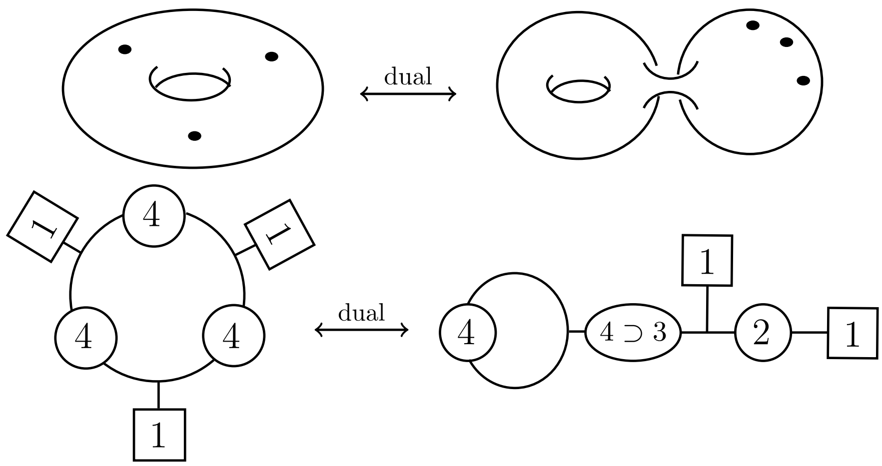

Figure 4-4 The S-duality in the class $\mathcal{S}$ theory of type A_3. The right theory involves gauging the A_3 trinion theory.

where

$$\mathcal{I}^{(0,4),4}_{1;n_1,0,0,0} = \prod_{i=1}^{n_1} \frac{\eta(q)^2\vartheta_1(v^8)}{\vartheta_1(v^2)\vartheta_1(v^4 c_i^{\pm 4})} \ , \tag{4.17}$$

$$\mathcal{I}^{(0,4),4}_{1;0,n_2,0,0} = \prod_{i=1}^{n_3} \frac{\eta(q)^6\vartheta_1(v^6)\vartheta_1(v^8)}{\vartheta_1(v^2)^2\vartheta_1(v^2 z_i^{\pm 2})\vartheta_1(v^3 z_i^{\pm} r_i^{\pm 4})} \ , \tag{4.18}$$

$$\mathcal{I}^{(0,4),4}_{1;0,0,n_3,0} = \prod_{i=1}^{n_2} \frac{\eta(q)^4\vartheta_1(v^6)\vartheta_1(v^8)}{\vartheta_1(v^2)\vartheta_1(v^4)\vartheta_1(v^2 w_i^{\pm 4})\vartheta_1(v^4 w_i^{\pm 4})} \ , \tag{4.19}$$

$$\mathcal{I}^{(0,4),4}_{1;0,0,0,n_4} = \prod_{i=1}^{n_4} \frac{\eta(q)^{12}\vartheta_1(v^2)\vartheta_1(v^4)\vartheta_1(v^6)\vartheta_1(v^8)}{\prod_{A,B=1}^{4}\vartheta_1(v^2 b_{iA}/b_{iB})} \ . \tag{4.20}$$

4.4 Type A_{N-1} and TQFT structure

From the above results, we can observe a simple TQFT structure in the $\mathcal{N} = (0,4)$ elliptic genus of the class $\mathcal{S}$ theory at genus $g \geq 1$. This structure suggests that the elliptic genus can be expressed as a straightforward product of contributions from individual punctures and handles. Therefore, the $\mathcal{N} = (0,4)$ elliptic genus corresponding to type A_{N-1} is expected to have the following form:

$$\mathcal{I}^{(0,4),N}_{g,n} = (\mathcal{H}_N)^{g-1} \prod_{i=1}^{n} \mathcal{I}^{(0,4),N}_{\lambda_i}(b_i) \ , \tag{4.21}$$

where g is the genus of the associated Riemann surface, and n collectively denotes the number of punctures, with their internal data represented by partitions (Young diagrams). The function $\mathcal{I}_{\lambda_i}$ captures the contribution from the i-th puncture labelled by a partition λ_i, and $\mathcal{H}_N$ encapsulates the contribution originating from a handle. Loosely speaking, this expression resembles the TQFT expression of the 4d $\mathcal{N} = 2$ superconformal index[4–5], which involves an infinite sum over representations of SU(N) schematically as

$$\mathcal{I}^{\text{4d}} = \sum_{\mu} H_{\mu}^{2g-2+n} \prod_{i} \psi_{\mu}^{(\lambda_i)}(b_i) \ . \tag{4.22}$$

We expect the elliptic genus $\mathcal{I}_{g,n}$ to obey a TQFT structure under cutting and gluing. Let us consider the maximal puncture corresponding to the integer partition $[1^N]$, which contributes

$$\mathcal{I}^{(0,4),N}_{[1^N]}(b) = \frac{\eta(q)^{N^2-N}\prod_{M=1}^{N}\vartheta_1(v^{2M})}{\prod_{A,B}^{N}\vartheta_1(v^2 b_A/b_B)} , \tag{4.23}$$

where b_A denotes the SU(N) flavor fugacities with constraint $b_1 \cdots b_N = 1$.

Consider two Riemann surfaces, labeled as C_{g_1,n_1} and C_{g_2,n_2}, each with a maximal puncture. By SU(N) gauging, these two maximal punctures can be joined together, which results in a new Riemann surface $C_{g_1+g_2,n_1+n_2-2}$. Similarly, if a Riemann surface $C_{g,n}$ possesses more than two maximal punctures, by gauging the diagonal of the SU$(N)^2$ flavor symmetry originating from these two maximal punctures, we can transform this surface into a new Riemann surface $C_{g+1,n-2}$. These processes can be visualized in Figure 4-5. For the form of (0,4) elliptic genus (4.21) to be compatible with these procedures, the handle contribution must be

$$\begin{aligned}\mathcal{H}_N &= \int_{\mathrm{JK}} \frac{da}{2\pi i a}\mathcal{I}^{(0,4),N}_{[1^N]}(a)\mathcal{I}^{(0,4),N}_{[1^N]}(a^{-1})\mathcal{I}^{(0,4)}_{\mathrm{vec}}(a) \\ &= \frac{\prod_{M=1}^{N}\vartheta_1(v^{2M})^2}{N!\eta(q)^{N-1}\vartheta_1(v^2)^{N+1}}\int_{\mathrm{JK}} \frac{da}{2\pi i a}\prod_{A\neq B}\frac{\vartheta_1(a_A/a_B)}{\vartheta_1(v^2 a_A/a_B)} , \\ &= \frac{\prod_{M=1}^{N}\vartheta_1(v^{2M})^2}{\eta(q)^{2(N-1)}\vartheta_1(v^2)\vartheta_1(v^{2N})} .\end{aligned} \tag{4.24}$$

The JK integral is analogous to the one in (3.63). These results (4.23) and (4.24) reduce to those in the previous examples when $N = 2, 3, 4$. Furthermore, given that the (0,4) elliptic genus form in (4.21) receives only local contributions, verifying the following properties is straightforward

$$\begin{aligned}\mathcal{I}^{(0,4),N}_{g_1+g_2,n_1+n_2-2}(b,c) &= \int_{\mathrm{JK}} \frac{da}{2\pi i a}\mathcal{I}^{(0,4),N}_{g_1,n_1}(b,a)\mathcal{I}^{(0,4),N}_{g_2,n_2}(c,a^{-1})\mathcal{I}^{(0,4)}_{\mathrm{vec}}(a) , \\ \mathcal{I}^{(0,4),N}_{g+1,n-2}(b) &= \int_{\mathrm{JK}} \frac{da}{2\pi i a}\mathcal{I}^{(0,4),N}_{g,n}(b,a,a^{-1})\mathcal{I}^{(0,4)}_{\mathrm{vec}}(a) .\end{aligned} \tag{4.25}$$

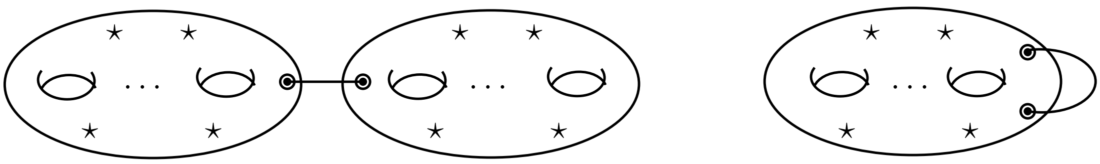

Figure 4-5 The gluing maximal punctures leads to a new Riemann surface, and (0,4) elliptic genera are consistent with the cut-and-join procedure on Riemann surfaces $C_{g,n}$.

The contribution from other punctures can be derived from the maximal one using nilpotent Higgsing. When transitioning from the maximal one to another type, an operator $\mathcal{O}$, which is charged under the SU(N) flavor symmetry, acquires a nilpotent VEV $\langle\mathcal{O}\rangle$ with Jordan blocks of sizes 1 and $N-1$, which specifies an embedding SU(2) $\hookrightarrow$ SU(N)[54]. Following[93], we propose that this nilpotent Higgsing procedure can be implemented at the level of the elliptic genus as follows. The contribution from a puncture defined by an integer partition λ of N is given by

$$\mathcal{I}_\lambda(c) = \lim_{b\to c}\left[\frac{K_\lambda(c)}{K_{[1^N]}(b)}\right]_{\Gamma(t^\alpha z)\to\frac{\eta(q)}{\vartheta_1(v^{2\alpha}z)}} \mathcal{I}_{[1^N]}(b) . \tag{4.26}$$

Here, b denotes the flavor fugacities associated to the puncture, and the function K is defined using the plethystic exponential (A.5) as

$$K_\lambda(c) := \mathrm{PE}\left[\sum_j \frac{t^{j+1}-pqt^j}{(1-p)(1-q)}\mathrm{ch}^f_{\mu_j}(c)\right] . \tag{4.27}$$

The ratio $K_\lambda/K_{[1^N]}$ in (4.26) can always be expressed by elliptic Gamma functions, which will be shown at the end of this section.

The replacement $b \to c$ and the K_λ should be understood in the following way[94]. Recall that the integer partition λ captures an embedding of SU(2) into SU(N). The adjoint representation of SU(N) decomposes with respect to this embedding

$$\mathbf{adj} = \oplus_j \mu_j \otimes \sigma_j \ , \tag{4.28}$$

where σ_j denotes the spin-j representation of the embedded SU(2), and the μ_j denotes a representation of the commutant, namely, the flavor symmetry f of the puncture. With the decomposition, one writes down an equality of characters

$$\mathrm{ch}_{\mathbf{adj}}(b) = \sum_j \mathrm{ch}^f_{\mu_j}(c)\mathrm{ch}^{\mathrm{SU}(2)}_{\sigma_j}(t^{1/2}) \ , \quad \mathrm{ch}^{\mathrm{SU}(2)}_{\sigma_j}(t^{1/2}) = \sum_{m=-j}^{j} t^m \ . \tag{4.29}$$

determining the substitution $b \to c$ (up to Weyl transformation)①.

For example, the partition $\lambda = [1^N]$ corresponding to the maximal puncture simply means a trivial embedding of SU(2), and therefore $f = \mathrm{SU}(N)$, $j = 0$ and $\mu_0 = \mathbf{adj}$. The K-function then reads

$$K_{[1^N]}(b) = \mathrm{PE}\left[\frac{t - pq}{(1-p)(1-q)}\mathrm{ch}_{\mathbf{adj}}(b)\right] , \tag{4.31}$$

where $b = (b_1, b_2, \dots, b_{N-1}, b_N)$ denotes the fugacities of the flavor SU(N). As another example, when $N = 4$, $\lambda = [2,2]$, the flavor symmetry is $f = \mathrm{SU}(2)$. The adjoint of SU(4) decomposes as $\mathbf{adj} = ([j=1] \oplus [j=0]) \otimes \sigma_{j=1} \oplus [j=1] \otimes \sigma_{j=0}$ where $[j]$ simply denotes the spin-j representation with respect to f. The replacement $b \to c$ reads $(b_1, b_2, b_3, b_4) \to (ct^{1/2}, ct^{-1/2}, c^{-1}t^{1/2}, c^{-1}t^{-1/2})$, and the K function is given by

$$K_{[2^2]}(c) = \mathrm{PE}\left[\frac{(t-pq)}{(1-q)(1-p)}(c^2 + \frac{1}{c^2} + 1) + \frac{(t^2 - pqt)}{(1-q)(1-p)}(c^2 + \frac{1}{c^2} + 2)\right] . \tag{4.32}$$

Another case of interest is the principal embedding $\lambda = [N]$ corresponding to trivial flavor symmetry. The fugacity takes the principal specialization

$$(b_1, \cdots, b_N) \to (t^{-\frac{N-1}{2}}, t^{-\frac{N-3}{2}}, \cdots, t^{\frac{N-3}{2}}, t^{-\frac{N-1}{2}}) \ . \tag{4.33}$$

The adjoint of SU(N) is decomposed into $\mathbf{adj} = \oplus_{i=1}^{N-1}\sigma_i$ and the K function is

$$K_{[N]}(c) = \mathrm{PE}\left[\sum_{i=1}^{N-1}\frac{t^{i+1} - pqt^i}{(1-p)(1-q)}\right] . \tag{4.34}$$

Note that there is no real c dependence since the corresponding flavor symmetry is trivial. We should regard the principal embedding as removing the puncture entirely. Hence,

$$\lim_{b \to c}\left[\frac{K_{[N]}(c)}{K_{[1^N]}(b)}\right]_{\Gamma(t^\alpha z) \to \frac{\eta(\tau)}{\vartheta_1(2\alpha v + z)}} \mathcal{I}_{[1^N]}(b) = 1 \ . \tag{4.35}$$

This illustrates the closure of a puncture.

① To actually obtain the decomposition (4.29) of character, one may start by finding the replacement $b \to c$ associated to the embedding. The process to find $b \to c$ effectively starts by addressing a simpler equation:

$$\mathrm{ch}_{\mathbf{fund}}(b) = \sum_j \mathrm{ch}^f_{\mu'_j}(c)\mathrm{ch}^{\mathrm{SU}(2)}_{\sigma_j}(t^{1/2}) \tag{4.30}$$

with respect to the decomposition of the SU(N) fundamental $\mathbf{fund} = \oplus_j \mu'_j \otimes \sigma_j$. Then, the replacement can then be reintroduced back to (4.29) and help determine pairs μ, σ of representations.

In this way, the contribution from a puncture with hook-type Young diagram can be read as

$$\mathcal{I}_{[N-K,1^K]} = \frac{\prod_{M=N-K+1}^{N} \vartheta_1(v^{2M})}{\eta(q)^K} \frac{\eta(q)^{2K}}{\prod_{A=1}^{K} \vartheta_1(v^{N-K+1}(r^N c_A)^{\pm})} \frac{\eta(q)^{K^2}}{\prod_{A,B=1}^{K} \vartheta_1(v^2 c_A/c_B)} , \tag{4.36}$$

where r and c_A represent the flavor fugacities.

In particular, for genus one with a simple puncture, the elliptic genus is given by

$$\begin{aligned}\mathcal{I}^{(0,4),N}_{g=1,n=1} =& \frac{\eta(q)^2 \vartheta_1(v^{2N})}{\vartheta_1(v^2)\vartheta_1(v^N c^{\pm N})} \\ =& \frac{\eta(q)^2}{\vartheta_1(vc^{\pm 1})} \prod_{i=2}^{N} \frac{\vartheta_1(v^{2i})}{\vartheta_1(v^{2(i-1)})} \frac{\vartheta_1(v^{i-1}c^{\pm(i-1)})}{\vartheta_1(v^i c^{\pm i})} .\end{aligned}$$

The first term represents the contribution from a free hypermultiplet, while the subsequent terms account for $\mathrm{Tr}(\Phi\tilde{\Phi})^i$, $\mathrm{Tr}(\Phi^i)$, and $\mathrm{Tr}(\tilde{\Phi}^i)$ $(i = 2, \dots, N)$ where $(\Phi, \tilde{\Phi})$ is the $N \otimes \overline{N}$ hypermultiplet in the UV quiver theory. Indeed, the Higgs branch of the theory[95] is

$$\frac{\mathbb{C}^N \times \mathbb{C}^N}{S_N} = \mathbb{C}^2 \times \frac{\mathfrak{t}_{\mathbb{C}} \times \mathfrak{t}_{\mathbb{C}}}{S_N} . \tag{4.37}$$

On the left-hand side, the eigenvalues of the mutually commuting $(\Phi, \tilde{\Phi})$ span $\mathbb{C}^N \times \mathbb{C}^N$ which receives the action of the Weyl group S_N. On the right-hand side, the first factor $\mathbb{C}^2$ is spanned by $\mathrm{Tr}(\Phi)$ and $\mathrm{Tr}(\tilde{\Phi})$ (the free hypermultiplet), and the second represents the Higgs branch of 4d $\mathcal{N} = 4$ SCFT[23,96] where $\mathfrak{t}_{\mathbb{C}}$ is the Cartan subalgebras of $\mathfrak{sl}(N)$. Hence, its quaternionic dimension is N, consistent with the right-moving central charge $c_R = 6N$, as derived from (4.4).

In all the above ratios of K we will take a replacement $\Gamma(t^\alpha z) \to \frac{\eta(\tau)}{\vartheta_1(2\alpha \mathsf{v}+z)}$ at the end of computation. This is possible thanks to the fact that $\lim_{b\to c} K_\lambda / K_{[1^N]}$ is always a product of elliptic Gamma functions. It can be seen by explicitly writing out the plethystic exponential

$$\lim_{b\to c} \frac{K_\lambda(c)}{K_{[1^N]}(b)} = \mathrm{PE}\left[\frac{t - pq}{(1-p)(1-q)} \sum_j \mathrm{ch}^f_{\mu_j}(c)\left(t^j - \mathrm{ch}^{\mathrm{SU}(2)}_{\sigma_j}(t^{1/2})\right)\right] . \tag{4.38}$$

For each j in the above sum, we write explicitly

$$(t - pq)(t^j - \mathrm{ch}^{\mathrm{SU}(2)}_{\sigma_j}(t^{1/2})) = -(t-pq)(t^{-j} + t^{-j+1} + \cdots + t^{j-2} + t^{j-1}) . \tag{4.39}$$

Note that the adjoint representation **adj** is real, hence we have

$$\mathrm{ch}^f_{\mu_j}(c) = \mathrm{ch}^f_{\mu_j}(c^{-1}) . \tag{4.40}$$

Therefore,

$$\begin{aligned}(t - pq)(t^j - \mathrm{ch}^{\mathrm{SU}(2)}_{\sigma_j}(t^{1/2}))\,\mathrm{ch}^f_{\mu_j}(c) =& -(t^{-j+1} + t^{-j+2} + \cdots + t^{j-1} + t^j)\,\mathrm{ch}^f_{\mu_j}(c) \\ &+ pq(t^{-j+1} + t^{-j+2} + \cdots + t^{j-1} + t^j)\,\mathrm{ch}^f_{\mu_j}(c)\Big|_{\substack{t\to t^{-1}\\ c\to c^{-1}}} .\end{aligned}$$

Here we see the structure of $x - \frac{pq}{x}$ emerges, and the ratio of K precisely forms a product of elliptic Gamma functions. At the end of the computation, the replacement $\Gamma(t^\alpha z) \to \frac{\eta(\tau)}{\vartheta_1(2\alpha \mathsf{v}+z)}$ should be performed to derive the contribution from a puncture to the $(0,4)$ elliptic genus.

Our findings give rise to various intriguing questions and potential research directions. Thus, we conclude this section by highlighting a few prospective avenues for future exploration.

Simplicity of forms: The (0,4) elliptic genus manifests in surprisingly simple forms, primarily as products of theta functions. However, such simplicity is observed only in theories where the genus is greater than zero. The underlying reasons for this remain mysterious.

Non-linear sigma model: Our investigations point toward an $\mathcal{N} = (0, 4)$ non-linear sigma model as an infrared theory. The target space of this model is the moduli space of the hypermultiplet with a nontrivial left-moving bundle. Notably, the form of the (0,4) elliptic genus strongly indicates the existence of an LG dual theory for this class of theories. An immediate challenge is to identify the superpotential of the LG model, which realizes the target space of the non-linear sigma model.

Relation with Schur indices: Prior works, such as[97–100], have brought up the relationship between (0,4) elliptic genera and Schur indices. However, the findings in these works remain observational and lack a foundational understanding. Therefore, a deeper analysis of the (0,4) elliptic genus presented in this thesis, in light of these observations, is a promising avenue.

Chapter 5: Conclusion

In the previous two chapters, we have studied the $\mathcal{N} = (0,2)$ and $\mathcal{N} = (0,4)$ theories which come from the reduction of 4d class $\mathcal{S}$ theories with two types of topological twist. In the case of (0,2) theories, we discovered the closed form of their elliptic genera which are dual to Landau-Ginzburg models. The relations to VOAs and the TQFT structure were also explored. In the case of (0,4) theories, we began with Lagrangian theories $U_N^{(0,4)}$, leveraged S-duality to arrive at elliptic genera of non-Lagrangian T_N theories. Furthermore, the elegant TQFT structure was also revealed on general Riemann surfaces $C_{g>0,n}$.

As noted in the conclusion of §4, several unresolved questions remain, warranting further contemplation and investigation.

Appendices

A Notations and conventions

In this thesis, the symbol q is defined as $q := e^{2\pi i\tau}$, where τ is a complex structure of a two-torus. Throughout the thesis, single symbols written in sans-serif type are used to represent chemical potentials. The fugacity z and the chemical potential z for either gauge or flavor symmetry are related by the equation $z = e^{2\pi i \mathsf{z}}$. Abusing notations, functions with fugacities and chemical potentials will be used interchangeably. For example, the following two notations represent the same theta function

$$\vartheta_1(\mathsf{z}) = \vartheta_1(z) \ . \tag{A.1}$$

The notation $f(a^{\pm}b^{\pm})$ is a shorthand notation used to denote the multiplication of all possible combinations of signs in the arguments. It is defined as follows:

$$\begin{aligned} f(a^{\pm}b^{\pm}) :=& f(ab)f(a^{-1}b)f(ab^{-1})f(a^{-1}b^{-1}) \ , \\ g(\pm\mathsf{a} \pm \mathsf{b}) :=& g(\mathsf{a}+\mathsf{b})g(-\mathsf{a}+\mathsf{b})g(\mathsf{a}-\mathsf{b})g(-\mathsf{a}-\mathsf{b}) \ . \end{aligned} \tag{A.2}$$

Throughout this thesis, we use the following notation for q-Pochhammer symbols

$$(z;q) := \prod_{k=0}^{\infty}(1-zq^k) \ . \tag{A.3}$$

The elliptic gamma function is defined by

$$\Gamma(z;p,q) = \prod_{m,n=0}^{\infty} \frac{1-p^{m+1}q^{n+1}/z}{1-p^m q^n z} = \mathrm{PE}\left[\frac{z-\frac{pq}{z}}{(1-q)(1-p)}\right] , \tag{A.4}$$

where PE represents the plethystic exponential

$$\mathrm{PE}[f(x,y,\cdots)] \equiv \exp\left[\sum_{d=1}^{\infty}\frac{1}{d}f\left(x^d,y^d,\cdots\right)\right] , \tag{A.5}$$

which brings the single particle index f to the multi-particle index. We often use the shorthand notation $\Gamma(z)$ for the elliptic Gamma function. The Dedekind eta function is

$$\eta(\tau) = q^{\frac{1}{24}}\prod_{n=1}^{\infty}\left(1-q^n\right) \tag{A.6}$$

where $q = e^{2\pi i\tau}$ and $\Im\tau > 0$. Often, we also use the notation $\eta(q)$. Its modular properties are

$$\eta(\tau+1) = e^{\frac{i\pi}{12}}\eta(\tau), \quad \eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\,\eta(\tau) \ . \tag{A.7}$$

A.1 Jacobi theta functions

The Jacobi theta functions are defined as a Fourier series

$$\vartheta_1(\mathsf{z}|\tau) := -i \sum_{r\in\mathbb{Z}+\frac{1}{2}} (-1)^{r-\frac{1}{2}} e^{2\pi i r z} q^{\frac{r^2}{2}}, \quad \vartheta_2(\mathsf{z}|\tau) := \sum_{r\in\mathbb{Z}+\frac{1}{2}} e^{2\pi i r z} q^{\frac{r^2}{2}},$$
$$\vartheta_3(\mathsf{z}|\tau) := \sum_{n\in\mathbb{Z}} e^{2\pi i n z} q^{\frac{n^2}{2}}, \qquad \vartheta_4(\mathsf{z}|\tau) := \sum_{n\in\mathbb{Z}} (-1)^n e^{2\pi i n z} q^{\frac{n^2}{2}}.$$

where $q = e^{2\pi i\tau}$ and $z = e^{2\pi i z}$. The Jacobi theta functions can be rewritten in the triple-product form

$$\vartheta_1(\mathsf{z}|\tau) = i q^{\frac{1}{8}} z^{-\frac{1}{2}} (q;q)(z;q)(z^{-1}q;q), \qquad \vartheta_2(\mathsf{z}|\tau) = q^{\frac{1}{8}} z^{-\frac{1}{2}} (q;q)(-z;q)(-z^{-1}q;q)\ ,$$
$$\vartheta_3(\mathsf{z}|\tau) = (q;q)(-zq^{1/2};q)(-z^{-1}q^{1/2};q)\ , \quad \vartheta_4(\mathsf{z}|\tau) = (q;q)(zq^{1/2};q)(z^{-1}q^{1/2};q)\ .$$

From the Jabobi triple products, we can easily find the relation between ϑ_1 and ϑ_4 as

$$\vartheta_4(\mathsf{z}|\tau) = -i q^{\frac{1}{8}} z^{\frac{1}{2}} \vartheta_1\left(\mathsf{z} + \frac{\tau}{2} \mid \tau\right)\ . \tag{A.8}$$

We will also use the notation $\vartheta_i(z, q)$. In either notation, the q and τ are often omitted, and we simply write $\vartheta_i(z)$ or $\vartheta_i(\mathsf{z})$.

Let us spell out some properties of the function $\vartheta_1(\mathsf{z}|\tau)$ we use in the main text. Under shifts of z we have

$$\vartheta_1(\mathsf{z} + a + b\tau|\tau) = (-1)^{a+b} e^{-2\pi i b \mathsf{z} - i\pi b^2\tau} \vartheta_1(\mathsf{z}|\tau) \tag{A.9}$$

for $a, b \in \mathbb{Z}$. Furthermore, ϑ_1 is odd with respect to z while the others are even

$$\vartheta_1(-\mathsf{z}|\tau) = -\vartheta_1(\mathsf{z}|\tau), \qquad \vartheta_{i=2,3,4}(-\mathsf{z}|\tau) = \vartheta_i(\mathsf{z}|\tau)\ .$$

The function $\vartheta_1(\mathsf{z}|\tau)$ has simple zeros in z at $\mathsf{z} = \mathbb{Z} + \tau\mathbb{Z}$, and no poles. When computing JK residues, it is notable that the derivative of z at 0 relates to $\eta(\tau)$ as follows

$$\vartheta_1'(0|\tau) = 2\pi\eta(q)^3\ .$$

From this relationship, we deduce a pole at $z = 0$ as

$$\frac{1}{\vartheta_1(\mathsf{z})} = \frac{1}{2\pi\eta(\tau)^3}\frac{1}{z} + \mathcal{O}(\mathsf{z})\ , \tag{A.10}$$

from which one easily extracts residues of ratios of Jacobi theta functions.

Under the modular transformation $\tau \xrightarrow{T} \tau+1$, $(\mathsf{z},\tau) \xrightarrow{S} (\frac{\mathsf{z}}{\tau}, -\frac{1}{\tau})$, the Jacobi theta function ϑ_1 transforms

$$\vartheta_1(\mathsf{z}|\tau+1) = e^{\frac{\pi i}{4}} \vartheta_1(\mathsf{z}|\tau), \quad \vartheta_1\left(\frac{\mathsf{z}}{\tau}| -\frac{1}{\tau}\right) = -i\sqrt{-i\tau}\, e^{\pi i \mathsf{z}^2/\tau} \vartheta_1(\mathsf{z}|\tau)\ .$$

A.2 Eisenstein series

The twisted Eisenstein series, denoted by $E_k\begin{bmatrix}\phi\\ \theta\end{bmatrix}$ with characteristics $\begin{bmatrix}\phi\\ \theta\end{bmatrix}$, are defined as a series in q,

$$E_{k\geq 1}\begin{bmatrix}\phi\\ \theta\end{bmatrix} := -\frac{B_k(\lambda)}{k!} + \frac{1}{(k-1)!}\sum_{r\geq 0}{}' \frac{(r+\lambda)^{k-1}\theta^{-1}q^{r+\lambda}}{1-\theta^{-1}q^{r+\lambda}} + \frac{(-1)^k}{(k-1)!}\sum_{r\geq 1}\frac{(r-\lambda)^{k-1}\theta q^{r-\lambda}}{1-\theta q^{r-\lambda}}.$$

Here, $\phi \equiv e^{2\pi i\lambda}$ determines $0 \leq \lambda < 1$. $B_k(x)$ represents the k-th Bernoulli polynomial. When $\phi = \theta = 1$, the prime in the sum indicates that the $r = 0$ term is omitted.

Additionally, we define

$$E_0 \begin{bmatrix} \phi \\ \theta \end{bmatrix} = -1 \ .$$

The standard, or untwisted, Eisenstein series E_{2n} is obtained from the $\theta, \phi \to 1$ limit of $E_{2n}\begin{bmatrix} \phi \\ \theta \end{bmatrix}$,

$$E_{2n}(\tau) = E_{2n} \begin{bmatrix} +1 \\ +1 \end{bmatrix} \ .$$

In contrary, taking the limit $\theta, \phi \to 1$ for odd k results in 0, with the exception of $E_1\begin{bmatrix} \phi \\ \theta \end{bmatrix}$, which is singular.

The Eisenstein series with $\phi = \pm 1$ enjoy a useful symmetry property

$$E_k \begin{bmatrix} \pm 1 \\ z^{-1} \end{bmatrix} = (-1)^k E_k \begin{bmatrix} \pm 1 \\ z \end{bmatrix} .$$

For instance, under transformations $z \to qz$ or $z \to q^{\frac{1}{2}} z$, the twisted Eisenstein series intermix with those of lower weight:

$$E_n \begin{bmatrix} \pm 1 \\ z q^{\frac{k}{2}} \end{bmatrix} = \sum_{\ell=0}^{n} \left(\frac{k}{2}\right)^{\ell} \frac{1}{\ell!} E_{n-\ell} \begin{bmatrix} (-1)^k(\pm 1) \\ z \end{bmatrix} .$$

Similarly, for the modular S-transformation, an inhomogeneous behavior is observed. For instance,

$$E_n \begin{bmatrix} +1 \\ +z \end{bmatrix} \xrightarrow{S} \left(\frac{1}{2\pi i}\right)^n \left[\left(\sum_{k \geq 0} \frac{1}{k!} (-\log z)^k y^k \right) \left(\sum_{\ell \geq 0} (\log q)^\ell y^\ell E_\ell \begin{bmatrix} +1 \\ z \end{bmatrix} \right) \right]_n , \tag{A.11}$$

B JK residues of $\mathcal{N} = (0,2)$ elliptic genera

In this appendix, we provide detailed computations of JK residue integrals for elliptic genera of 2d $\mathcal{N} = (0,2)$ quiver gauge theories, complementing the main text. To elucidate the JK residue computations, we use notations based on chemical potentials instead of fugacities here.

B.1 $g = 1, n = 2$

Given genus $g = 1$ and the number of punctures $n = 2$, one can write down different quiver gauge theories with different gauge groups.

$\mathrm{SU}(2)^2$ gauge theory

The first theory is an $\mathrm{SU}(2)^2$ gauge theory coupled to two bi-fundamentals,

	$\mathrm{SU}(2)_1 \times \mathrm{SU}(2)_2$
ϕ_1	$(2,2)$
ϕ_2	$(2,2)$

The quiver is shown in Figure 3-6. The elliptic genus of this theory is computed by the JK residue computation of the integral

$$\mathcal{I}_{1,2} = \int_{\mathrm{JK}} \frac{da_1}{2\pi i a_1} \frac{da_2}{2\pi i a_2} \frac{\eta(\tau)^8}{4} \frac{\prod_{i=1}^{2} \vartheta_1(\pm 2\mathsf{a}_i)}{\prod_{i=1}^{2} \vartheta_4(\pm \mathsf{a}_1 \pm \mathsf{a}_2 + \mathsf{c}_i)} \ . \tag{B.1}$$

Here $\mathsf{c}_{1,2}$ represent the flavor $\mathrm{U}(1) \times \mathrm{U}(1)$ fugacities. The charge covectors are drawn in Figure -1. Various reference vectors, η, can be chosen, all of which yield the same result.

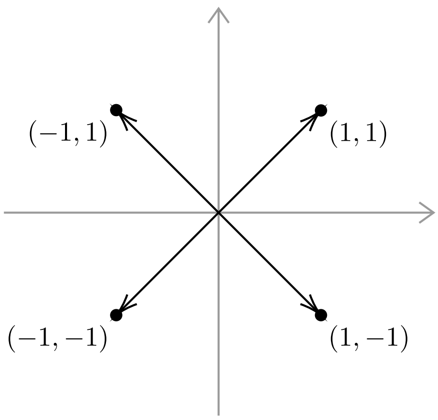

Figure -1 The charge vectors of the genus-one theory as an $SU(2)^2$ gauge theory.

For example, picking $\eta = (1,0)$ picks out one cone in $\mathbb{R}^2$ spanned by the charge vectors $(1,1)$ and $(1,-1)$. The corresponding poles are given by the set of equations

$$\mathsf{a}_1 - \mathsf{a}_2 + \mathsf{c}_1 + \frac{\tau}{2} = m_1 + n_1\tau, \quad \mathsf{a}_1 + \mathsf{a}_2 + \mathsf{c}_2 + \frac{\tau}{2} = m_1' + n_1'\tau \ , \tag{B.2}$$

and

$$\mathsf{a}_1 - \mathsf{a}_2 + \mathsf{c}_2 + \frac{\tau}{2} = m_2 + n_2\tau, \quad \mathsf{a}_1 + \mathsf{a}_2 + \mathsf{c}_1 + \frac{\tau}{2} = m_2' + n_2'\tau \ , \tag{B.3}$$

where $m_i, n_i = 0$, and $m_i, n_i = 0, 1$. There are in total 8 poles, all contribute $-\frac{1}{8}\frac{\eta(\tau)^2}{\vartheta_1(\mathsf{c}_1)\vartheta_1(\mathsf{c}_2)}$, and therefore,

$$\mathcal{I}_{1,2} = -\frac{\eta(\tau)^2}{\prod_{i=1}^2 \vartheta_1(2\mathsf{c}_i)} \ . \tag{B.4}$$

First SU(2)×U(1) gauge theory

Additionally, there are two other quiver theories as SU(2) × U(1) gauge theories that correspond to the genus-one Riemann surface with two punctures, with the quiver diagrams on the left in Figure 3-7. The first such gauge theory has an elliptic genus described by

$$\mathcal{I}_{1,2}' = \int_{\mathrm{JK}} \frac{da_1}{2\pi i a_1}\frac{da_2}{2\pi i a_2}\frac{\eta(\tau)^8}{2}\frac{\vartheta_1(\pm 2\mathsf{a}_1)\vartheta_4(\pm 2\mathsf{a}_2)}{\vartheta_4(\mathsf{a}_1 \pm \mathsf{a}_2 \pm \mathsf{b}_1 + \mathsf{d}_1)\vartheta_4(-\mathsf{a}_1 \pm \mathsf{a}_2 \pm \mathsf{b}_2 + \mathsf{d}_1)} \ .$$

Here we turn on the U(1) flavor symmetry that rotates the chiral multiplets from the two $U_2^{(0,2)}$ with the same phase with fugacity d_1. One can also check that the integrand remains separately elliptic with respect to the variables $\mathsf{a}_{1,2}$. The charge vectors are still

$$(1,1), \quad (-1,1), \quad (1,-1), \quad (-1,-1) \ . \tag{B.5}$$

One can pick any η inside the four quadrants, and the JK residue computation yields the same result,

$$\mathcal{I}_{1,2}' = \frac{2\eta(\tau)^2\vartheta_4(2\mathsf{d}_1)^2}{\vartheta_1(2\mathsf{d}_1 \pm \mathsf{b}_1 \pm \mathsf{b}_2)} \ . \tag{B.6}$$

Second SU(2)×U(1) gauge theory

A distinct SU(2) × U(1) gauge theory is depicted by the quiver diagram on the right side of Figure 3-7. The elliptic genus can be computed as the JK residue of the integrand

$$\mathcal{Z}_{1,2}'' = \frac{\eta(\tau)^{10}\vartheta_4(\pm 2\mathsf{a}_1)\vartheta_1(\pm 2\mathsf{a}_2)}{4\vartheta_4(\mathsf{a}_1 + \mathsf{d}_1)^2 \prod_{\pm} \vartheta_4(\mathsf{a}_1 \pm 2\mathsf{a}_2 + \mathsf{d}_1)\vartheta_4(-\mathsf{a}_1 \pm \mathsf{b}_1 \pm \mathsf{b}_2 + \mathsf{d}_1)} \ .$$

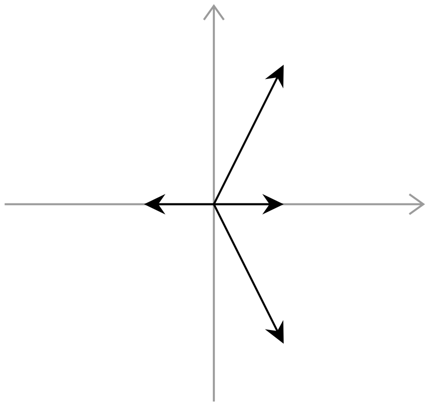

Figure -2 The charge vectors of the JK residues for the second SU(2)×U(1) gauge theory

From the denominator, we can deduce the charge vectors

$$(-1,0), \quad (1,-2), \quad (1,0), \quad (1,2)\ . \tag{B.7}$$

See Figure -2. Clearly, there are several choices for η. Let us start with $\eta=(-1,-1)$. In this case, only the cone spanned by $(-1,0)$ and $(1,-2)$ contribute, corresponding to the poles from the equation (with four different choices of the signs $\pm$)

$$\begin{aligned} -\mathsf{a}_1 \pm \mathsf{b}_1 \pm \mathsf{b}_2 + \mathsf{d}_1 + \frac{\tau}{2} &= m_1 + n_1\tau \\ \mathsf{a}_1 - 2\mathsf{a}_2 + \mathsf{d}_1 + \frac{\tau}{2} &= m_2 + n_2\tau\ , \end{aligned} \tag{B.8}$$

where $m_1, n_1 = 0$, $m_2, n_2 = 0, 1$. In total, there are 4×4 different poles. For example, poles having $-\mathsf{a}_1 - \mathsf{b}_1 - \mathsf{b}_2 + \mathsf{d}_1 + \frac{\tau}{2} = 0$ contribute

$$\frac{\eta(\tau)^4 \vartheta_4(-2\mathsf{b}_1 - 2\mathsf{b}_2 + 2\mathsf{d}_1)^2}{\vartheta_1(2\mathsf{b}_1 + 2\mathsf{b}_2)\vartheta_1(2\mathsf{b}_1 + 2\mathsf{b}_2)\vartheta_1(-2\mathsf{b}_1 - 2\mathsf{b}_2 + 4\mathsf{d}_1)\prod_{i=1}^2 \vartheta_1(2\mathsf{b}_i)}\ .$$

To summarize, the elliptic genus reads

$$\mathcal{I}''_{1,2} = \frac{\eta(\tau)^2}{\prod_{i=1}^2 \vartheta_1(2\mathsf{b}_i)} \sum_{\alpha,\beta=\pm} \frac{\alpha\beta\vartheta_4(2\alpha\mathsf{b}_1 + 2\beta\mathsf{b}_2 + 2\mathsf{d}_1)}{\vartheta_1(2\alpha\mathsf{b}_1 + 2\beta\mathsf{b}_2)\vartheta_1(2\alpha\mathsf{b}_1 + 2\beta\mathsf{b}_2 + 4\mathsf{d}_1)}\ . \tag{B.9}$$

Alternatively, one can also choose $\eta=(1,1)$. In this case, the relevant cones are spanned by the charge vectors

$$(1,2) \text{ and } (1,0), \qquad (1,2) \text{ and } (1,-2)\ . \tag{B.10}$$

The corresponding poles are from the equations

$$\mathsf{a}_1 - 2\mathsf{a}_2 + \mathsf{d}_1 + \frac{\tau}{2} = m_1 + n_1\tau, \qquad \mathsf{a}_1 + 2\mathsf{a}_2 + \mathsf{d}_1 + \frac{\tau}{2} = m_2 + n_2\tau\ , \tag{B.11}$$

with $m_1, n_1 = 0$, $m_2, n_2 = 0, 1, 2, 3$, and the equations

$$\mathsf{a}_1 + \mathsf{d}_1 + \frac{\tau}{2} = m_1 + n_1\tau, \qquad \mathsf{a}_1 + 2\mathsf{a}_2 + \mathsf{d}_1 + \frac{\tau}{2} = m_2 + n_2\tau\ , \tag{B.12}$$

with $m_1, n_1 = 0$, $m_2, n_2 = 0, 1$. The JK residue computation is, however, more subtle in this setup, due to the presence of degenerate poles

$$(\mathsf{a}_1, \mathsf{a}_2) = (-\mathsf{d}_1 - \frac{\tau}{2}, \frac{m + n\tau}{2}), \qquad m, n = 0, 1\ . \tag{B.13}$$

Note also that these degenerate poles are precisely the common solutions to (B.11), (B.12) (up to shift of a_1 by full periods $1, 1+\tau, \tau$). At these poles, the factors

$$\vartheta_4(\mathsf{a}_1 - 2\mathsf{a}_2 + \mathsf{d}_1), \vartheta_4(\mathsf{a}_1 + 2\mathsf{a}_2 + \mathsf{d}_1), \vartheta_4(\mathsf{a}_1 + \mathsf{d}_1)^2 \tag{B.14}$$

simultaneous vanish. Therefore, there are 12 non-degenerate poles and 4 degenerate poles that contribute to the elliptic genus. The former is straightforward to compute. For example,

$$\mathrm{JK}_{\substack{\mathsf{a}_1-2\mathsf{a}_2+\mathsf{d}_1+\frac{\tau}{2}=0 \\ \mathsf{a}_1+2\mathsf{a}_2+\mathsf{d}_1+\frac{\tau}{2}=\tau}} \mathcal{Z} = -\frac{\vartheta_4(2\mathsf{d}_1)^2}{16\vartheta_4(\pm\mathsf{b}_1 \pm \mathsf{b}_2 + 2\mathsf{d}_1)} \ . \tag{B.15}$$

For the degenerate poles, we follow[36]. The relevant charge vectors can be grouped into

$$Q_* = \{(1,-2),(1,2),(1,0)\} \ , \tag{B.16}$$

and that gives rise to three flags $F_{1,-2}, F_{1,2}, F_{1,0}$ led by the three vectors in Q_*. The corresponding $\kappa(F)$ and other relevant information is collected in the following table.

F	$F_{1,-2}$	$F_{1,2}$	$F_{1,0}$
$\kappa(F)$	$((1,-2),(3,0))$	$((1,2),(3,0))$	$((1,0),(3,0))$
$\operatorname{sign}\det(\kappa(F))$	1	-1	0
$\eta \in \mathsf{c}(F, Q_*)$	False	True	False

From the table, we can compute the contribution from the degenerate poles

$$= \frac{\eta(\tau)^2\vartheta_4(2\mathsf{d}_1)^2}{4\vartheta_1(\pm\mathsf{b}_1 \pm \mathsf{b}_2 + 2\mathsf{d}_1)} \ . \tag{B.17}$$

In the end, the elliptic genus computed using $\eta = (1,1)$ is

$$\mathcal{I}''_{1,2} = \frac{\eta(\tau)^4}{2}\vartheta_4(2\mathsf{d}_1)^2 \sum_{i=1}^{4} (-1)^i \frac{1}{\vartheta_i(2\mathsf{d}_1 \pm \mathsf{b}_1 \pm \mathsf{b}_2)} \ . \tag{B.18}$$

Although look different, the elliptic genus (B.9) and (B.18) are actually identical, and are equal to

$$\mathcal{I}''_{1,2} = \eta(\tau)^2\vartheta_4(2\mathsf{d}_1)^2 \frac{\vartheta_1(8\mathsf{d}_1)}{\vartheta_1(4\mathsf{d}_1)} \prod_{i=1}^{2} \frac{\vartheta_1(4\mathsf{b}_i)}{\vartheta_1(2\mathsf{b}_i)} \times \frac{1}{\vartheta_1(4\mathsf{d}_1 \pm 2\mathsf{b}_1 \pm 2\mathsf{b}_2)} \ . \tag{B.19}$$

The equivalence of the expressions is checked by the power expansion in q.

B.2 $g = 1, n = 3$

Corresponding to the genus-one Riemann surfaces $C_{1,3}$ with three punctures, there are several different 2d $\mathcal{N} = (0,2)$ theories.

$\mathrm{SU}(2)^3$ gauge theory

Let us consider first the $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$ gauge theory drawn in Figure 3-8. The elliptic genus is the JK residue of the integrand

$$\mathcal{Z}_{1,3} = -\frac{\eta(\tau)^{12}\vartheta_1(\pm 2\mathsf{a}_1)\vartheta_1(\pm 2\mathsf{a}_2)\vartheta_1(\pm 2\mathsf{a}_3)}{\prod_{A<B}\vartheta_4(\pm\mathsf{a}_A \pm \mathsf{a}_B + \mathsf{c}_1)} \ . \tag{B.20}$$

We can choose $\eta = (1, 1+\frac{1}{1000}, 1+\frac{1}{2000})$. There is no degenerate pole, and the elliptic genus is given by

$$\mathcal{I}_{1,3} = \frac{\eta(\tau)^3}{\prod_{i=1}^{3}\vartheta_1(2\mathsf{c}_i)} \ . \tag{B.21}$$

First $\mathrm{SU}(2)^2 \times \mathrm{U}(1)$ theory

Let us now consider an $\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{U}(1)$ gauge theory illustrated in the left of Figure 3-9, whose elliptic genus is the JK residue of the integrand

$$\mathcal{Z}'_{1,3} = 2\frac{\eta(\tau)^{12}\vartheta_4(\pm 2\mathsf{a}_1)\vartheta_1(\pm 2\mathsf{a}_2)\vartheta_1(\pm 2\mathsf{a}_3)}{4\vartheta_4(\pm\mathsf{a}_2\pm\mathsf{a}_3+\mathsf{c}_1)\vartheta_4(\mathsf{a}_1\pm\mathsf{a}_2\pm\mathsf{b}_2+\mathsf{d}_1)\vartheta_4(-\mathsf{a}_1\pm\mathsf{a}_3\pm\mathsf{b}_3+\mathsf{d}_1)}\ . \tag{B.22}$$

An arbitrary reference vector can be chosen, for example, $\eta=(\frac{999}{1000},\frac{1999}{2000},\frac{2999}{3000})$ which leads to 40 non-degenerate poles. The JK residue is straightforward, yielding,

$$\begin{aligned}\frac{\mathcal{I}'_{1,3}}{\eta(\tau)^3} = &-\frac{1}{2\vartheta_1(2\mathsf{c}_1)\prod_{i=2}^3\vartheta_1(2\mathsf{b}_i)}\sum_{\alpha,\beta=\pm}\frac{\alpha\beta\vartheta_1(\alpha\mathsf{b}_2+\beta\mathsf{b}_3+\mathsf{c}_1)^2}{\vartheta_4(\alpha\mathsf{b}_2+\beta\mathsf{b}_3+\mathsf{c}_1\pm 2\mathsf{d}_1)}\\ &+\frac{\vartheta_4(2\mathsf{d}_1)^2}{2\vartheta_1(2\mathsf{b}_2)\vartheta_1(2\mathsf{c}_1)}\sum_{\alpha=\pm}\frac{\alpha\vartheta_1(2\alpha\mathsf{b}_2+2\mathsf{c}_1)}{\vartheta_4(\alpha\mathsf{b}_2+\mathsf{c}_1\pm\mathsf{b}_3\pm 2\mathsf{d}_1)}\\ &+\frac{\vartheta_4(2\mathsf{d}_1)^2}{\vartheta_1(2\mathsf{b}_3)\vartheta_1(4\mathsf{d}_1)}\sum_{\alpha=\pm}\frac{\alpha\vartheta_1(2\alpha\mathsf{b}_3+4\mathsf{d}_1)}{\vartheta_4(\alpha\mathsf{b}_3+2\mathsf{d}_1\pm\mathsf{b}_2\pm\mathsf{c}_1)}\ .\end{aligned} \tag{B.23}$$

It is straightforward to check that this complicated expression is actually equal to

$$\frac{\mathcal{I}'_{1,3}}{\eta(\tau)^3} = 2\frac{\vartheta_4(2\mathsf{d}_1)^2}{\vartheta_1(4\mathsf{d}_1)}\frac{\vartheta_1(2\mathsf{c}_1+4\mathsf{d}_1)}{\vartheta_1(2\mathsf{c}_1)}\frac{1}{\vartheta_4(\mathsf{c}_1+2\mathsf{d}_1\pm\mathsf{b}_2\pm\mathsf{b}_3)}\ , \tag{B.24}$$

which signals an LG description of the quiver gauge theory.

Second $\mathrm{SU}(2)^2 \times \mathrm{U}(1)$ theory

Another quiver gauge theory with the gauge group $\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{U}(1)$ is depicted to the right in Figure 3-9. Its elliptic genus is determined by the JK residue of the integrand

$$\mathcal{Z}=\mathcal{I}^{(0,2)}_{U_2}(\mathsf{c}_1,\mathsf{b}_1,\mathsf{a}_3)\mathcal{I}^{(0,2)}_{U_2}(\mathsf{a}_2+\mathsf{d}_1,\mathsf{b}_2,-\mathsf{a}_3)\mathcal{I}^{(0,2)}_{U_2}(-\mathsf{a}_2+\mathsf{d}_1,\mathsf{a}_1,-\mathsf{a}_1)\vartheta_4(\pm\mathsf{a}_2)\prod_{i=\{1,3\}}\mathcal{I}^{(0,2)}_{\mathrm{vec}}(\mathsf{a}_i)\ .$$

The six charge vectors are given by

$$(\pm1,0,0),\quad(\pm1,0,1),\quad(0,\pm2,-1)\ . \tag{B.25}$$

There are various choices for η. For example, let us begin with

$$\eta=(\frac{999}{1000},\frac{1999}{2000},\frac{2999}{3000}). \tag{B.26}$$

With this choice, there are 64 non-degenerate poles, giving the elliptic genus

$$\begin{aligned}&\frac{\mathcal{I}''_{0,3}}{\eta(\tau)^3}\\ &=\frac{1}{\vartheta_1(2\mathsf{c}_1)\prod_{i=1}^2\vartheta_1(2\mathsf{b}_i)}\sum_{\alpha\beta=\pm}\frac{\alpha\beta\vartheta_4(2\alpha\mathsf{b}_1+2\beta\mathsf{b}_2+2\mathsf{c}_1+2\mathsf{d}_1)^2}{\vartheta_1(2\alpha\mathsf{b}_1+2\beta\mathsf{b}_2+2\mathsf{c}_1)\vartheta_1(2\alpha\mathsf{b}_1+2\beta\mathsf{b}_2+2\mathsf{c}_1+4\mathsf{d}_1)}\\ &\quad+\frac{2\vartheta_4(2\mathsf{d}_1)^2}{\vartheta_1(4\mathsf{d}_1)}\sum_{i=1}^4\frac{(-1)^i}{2\vartheta_i(\pm\mathsf{b}_1\pm\mathsf{b}_2+\mathsf{c}_1)}\ .\end{aligned} \tag{B.27}$$

For instance, another intriguing choice for η is

$$\eta=(-\frac{999}{1000},\frac{1}{1000},1)\ . \tag{B.28}$$

With this choice of η, there are 56 non-degenerate poles and 8 degenerate poles. The latter poles are given by

$$\begin{aligned} & a_1 = -\frac{m+n\tau}{2}, \quad a_2 = d_1 + \frac{\tau}{2}, \quad a_3 = +b_2 - 2d_1 - \tau, \quad m,n = 0,1 \\ \text{or,} \quad & a_1 = -\frac{m+n\tau}{2}, \quad a_2 = d_1 + \frac{\tau}{2}, \quad a_3 = -b_2 - 2d_1 - \tau, \quad m,n = 0,1 . \end{aligned} \tag{B.29}$$

The elliptic genus reads

$$\begin{aligned} & 4\frac{\vartheta_1(2b_2)\vartheta_1(4d_1)}{\eta(\tau)^3\vartheta_4(2d_1)^2}\mathcal{I}''_{0,3} \\ & = \sum_{i=1}^{4}\sum_{\alpha=\pm}\frac{(-1)^i\alpha\vartheta_1(2\alpha b_2+4d_1)}{\vartheta_i(\pm b_1+\alpha b_2\pm c_1+2d_1)} \\ & \quad + \sum_{\alpha,\beta,\gamma=\pm}\frac{2\alpha\beta\vartheta_1(4d_1)}{\vartheta_1(2b_1)\vartheta_1(2c_1)\vartheta_4(2d_1)^2}\frac{\vartheta_4(2\alpha b_1+2\beta b_2+2c_1+2\gamma d_1)^2}{\vartheta_1(2(\alpha b_1+\beta b_2+c_1+2\gamma d_1))\vartheta_1(2(\alpha b_1+\beta b_2+c_1))} . \end{aligned} \tag{B.30}$$

Although the two expressions for $\mathcal{I}''_{1,3}$ look different, they are both equal to the simple ratio

$$\frac{\mathcal{I}''_{1,3}}{\eta(\tau)^3} = \frac{\vartheta_4(2d_1)^2\vartheta_1(4c_1+8d_1)}{\vartheta_1(4d_1)\vartheta_1(2c_1)}\frac{1}{\vartheta_1(\pm 2b_1\pm 2b_2+2c_1+4d_1)}\prod_{i=1}^{2}\frac{\vartheta_1(4b_i)}{\vartheta_1(2b_i)} , \tag{B.31}$$

suggesting an LG description of the theory.

B.3 $g=2, n=0$

Associated to the genus-two Riemann surface with no puncture, there are two possible quiver gauge theories with $SU(2)^2\times U(1)$ gauge groups. (See Figure 3-10.)

First $SU(2)^2\times U(1)$ theory

The theory on the left of Figure 3-10 has an elliptic genus given by the JK-residue of the integrand

$$\mathcal{Z}_{2,0} = 2\mathcal{I}^{(0,2)}_{U_2}(a_3+d_1,a_1,a_2)\mathcal{I}^{(0,2)}_{U_2}(-a_3+d_1,-a_1,-a_2)\vartheta_4(\pm 2a_3)\prod_{i=1}^{2}\mathcal{I}^{(0,2)}_{\text{vec}}(a_i) . \tag{B.32}$$

The eight charge vectors are given by $(\pm 1,\pm 1,\pm 1)$. There are many essentially equivalent choices of η. For example, we pick

$$\eta = (1, \frac{999}{1000}, \frac{4999}{5000}) . \tag{B.33}$$

With this choice of η, there are 32 non-degenerate poles. All of them contribute identically to the total JK residue. Finally, the elliptic genus reads

$$\mathcal{I}_{2,0} = -\frac{2\vartheta_4(2d_1)^2}{\eta(\tau)\vartheta_1(4d_1)} . \tag{B.34}$$

Second $SU(2)^2\times U(1)$ theory

The theory on the right of Figure 3-10 has an elliptic genus integrand

$$\mathcal{Z}'_{2,0} = 2\mathcal{I}^{(0,2)}_{U_2}(a_3+d_1,a_1,-a_1)\mathcal{I}^{(0,2)}_{U_2}(-a_3+d_1,-a_2,-a_2)\vartheta_4(\pm 2a_3)\prod_{i=1}^{2}\mathcal{I}^{(0,2)}_{\text{vec}}(a_i) . \tag{B.35}$$

The charge vectors are given by

$$(\pm 2,0,1), \quad (0,\pm 2,-1), \quad (0,0,\pm 1) . \tag{B.36}$$

Let us consider a choice of η

$$\eta = (\frac{1001}{1000}, \frac{501}{500}, \frac{1003}{1000}) \ . \tag{B.37}$$

With this choice, there are 48 non-degenerate poles and 16 degenerate poles. The latter takes the form

$$\mathsf{a}_1 = \frac{m_1 + n_1\tau}{2}, \quad \mathsf{a}_2 = -\mathsf{d}_1 - \frac{\tau}{2} + \frac{m_2 + n_2\tau}{2}, \quad \mathsf{a}_3 = -\mathsf{d}_1 - \frac{\tau}{2} \ . \tag{B.38}$$

It turns out that all 64 poles share identical contributions to the elliptic genus. In the end, we have

$$\mathcal{I}'_{2,0} = \frac{2\vartheta_4(2\mathsf{d}_1)^2}{\eta(\tau)\vartheta_1(4\mathsf{d}_1)} \ . \tag{B.39}$$

Apparently, up to a sign,

$$\mathcal{I}_{2,0} = \mathcal{I}'_{2,0} \ . \tag{B.40}$$

B.4 Genus two with n punctures

Let us briefly summarize the computation for genus-two theories with n punctures. Given n, there are essentially only two $\mathrm{SU}(2)^{n+2} \times \mathrm{U}(1)$ quiver gauge theories one can consider: if the U(1) node in the quiver diagram is removed/ungauged, one frame continues to have a connected quiver diagram, while the other frame is cut into two disconnected pieces.

The first frame is simpler. The integrand reads

$$\mathcal{Z}_{2,n} = \vartheta_4(\pm 2\mathsf{a}_{n+3}) \prod_{j=1}^{n+2} \mathcal{I}^{(0,2)}_{\text{vec}}(\mathsf{a}_j) \cdot \prod_{i=1}^{n+2} \mathcal{I}^{(0,2)}_{U_2}(\mathsf{c}_i, -\mathsf{a}_{i-1}, \mathsf{a}_i) \Big|_{\substack{\mathsf{c}_{n+1}=\mathsf{a}_{n+3}+\mathsf{d}_1 \\ \mathsf{c}_{n+2}=-\mathsf{a}_{n+3}+\mathsf{d}_1 \\ \mathsf{a}_0=\mathsf{a}_{n+2}}} \ . \tag{B.41}$$

One can pick a simple and generic $\eta = (\eta_1, \dots, \eta_{n+3})$, for example,

$$\eta_i = 1 - \frac{1}{1000000i} \ . \tag{B.42}$$

With this choice, there are only 2^{2g+n+1} non-degenerate poles, which lead to a simple elliptic genus

$$\mathcal{I}_{2,n} = 2(-1)^{n+1} \frac{\vartheta_4(2\mathsf{d}_1)^2}{\eta(\tau)\vartheta_1(4\mathsf{d}_1)} \prod_{i=1}^{n} \frac{\eta(\tau)}{\vartheta_1(2\mathsf{c}_i)} \ . \tag{B.43}$$

The second frame has an integrand

$$\mathcal{Z}'_{2,n} = \vartheta_4(\pm 2\mathsf{a}_{n+3}) \prod_{j=1}^{n+2} \mathcal{I}^{(0,2)}_{\text{vec}}(\mathsf{a}_j) \cdot \mathcal{I}^{(0,2)}_{U_2}(-\mathsf{a}_{n+3}+\mathsf{d}, \mathsf{a}_{n+2}, -\mathsf{a}_{n+2}) \prod_{i=1}^{n+1} \mathcal{I}^{(0,2)}_{U_2}(\mathsf{c}_i, -\mathsf{a}_{i-1}, \mathsf{a}_i) \Big|_{\substack{\mathsf{c}_{n+1}=\mathsf{a}_{n+3}+\mathsf{d}_1 \\ \mathsf{a}_0=\mathsf{a}_{n+1}}} \ . \tag{B.44}$$

There are different η to choose from. For example,

$$\eta = (\eta_1, \dots), \qquad \eta_i = 1 - \frac{1}{1000 \times i} \ . \tag{B.45}$$

With this choice, only non-degenerate poles contribute, yielding

$$\mathcal{I}'_{2,n} = 2(-1)^{n} \frac{\vartheta_4(2\mathsf{d}_1)^2}{\eta(\tau)\vartheta_1(4\mathsf{d}_1)} \prod_{i=1}^{n} \frac{\eta(\tau)}{\vartheta_1(2\mathsf{c}_i)} \ . \tag{B.46}$$

While other choices of η may lead to degenerate poles, the final result of the elliptic genus remains independent of η. Up to a sign, both the $\mathrm{SU}(2)^{n+3} \times \mathrm{U}(1)$ quiver gauge theory share the same elliptic genus,

$$\mathcal{I}_{2,n} = \mathcal{I}'_{2,n} \ . \tag{B.47}$$

B.5 SU(3) theory for $g = 1, n = 1$

The elliptic genus for the $\mathcal{N} = (0,2)$ SU(3) theory coupled to an adjoint chiral multiplet is given by the integral

$$\mathcal{I}_{1,1}^{(0,2),3} = \int_{\mathrm{JK}} \frac{da_1}{2\pi i a_1} \frac{da_2}{2\pi i a_2} \mathcal{I}_{U_3}^{(0,2)}(\mathsf{c}, \mathsf{a}, -\mathsf{a}) \mathcal{I}_{\mathrm{vec}}^{(0,2)}(\mathsf{a}) \ , \tag{B.48}$$

where

$$\mathcal{I}_{U_3}^{(0,2)}(\mathsf{c}, \mathsf{a}, \mathsf{b}) = \prod_{A,B=1}^{3} \frac{\eta(\tau)}{\vartheta_4(\mathsf{c} + \mathsf{a}_A + \mathsf{b}_B)} \Bigg|_{\substack{\mathsf{a}_3 = -\mathsf{a}_1 - \mathsf{a}_2 \\ \mathsf{b}_3 = -\mathsf{b}_1 - \mathsf{b}_2}} \tag{B.49}$$

$$\mathcal{I}_{\mathrm{vec}}^{(0,2)}(\mathsf{a}) = \frac{\eta(\tau)^3}{3!} \prod_{\substack{A,B=1 \\ A \neq B}}^{3} \frac{\vartheta_1(\mathsf{a}_A - \mathsf{a}_B)}{\eta(\tau)} \ . \tag{B.50}$$

Subsequent discussions involving SU(N) gauge group will continue to use these basic building blocks. One can pick some generic reference vector η, for example,

$$\eta = (1, 0), \qquad \text{or} \qquad (0, 1) \ , \tag{B.51}$$

there are always 18 non-degenerate poles, yielding

$$\mathcal{I}_{1,1}^{(0,2),3} = \frac{\eta(\tau)}{\vartheta_4(3\mathsf{c})} \ . \tag{B.52}$$

B.6 $g = 1, n = 2$ with SU(3) and/or U(1) gauge group

For the genus-one Riemann surface with two punctures, several A_2 type $\mathcal{N} = (0,2)$ theories can be defined.

SU(3)^2 gauge theory

The simplest theory is an SU(3)^3 theory, with integrand

$$\mathcal{Z} = \mathcal{I}_{U_3}^{(0,2)}(\mathsf{c}_1, \mathsf{a}_1, \mathsf{a}_2) \mathcal{I}_{U_3}^{(0,2)}(\mathsf{c}_2, -\mathsf{a}_1, -\mathsf{a}_2) \prod_{i=1}^{2} \mathcal{I}_{\mathrm{vec}}^{(0,2)}(\mathsf{a}_i) \ . \tag{B.53}$$

One can pick, for example,

$$\eta = (1, \frac{1}{1000}, -\frac{1}{19}, \frac{16}{17}) \ . \tag{B.54}$$

There are 108 non-degenerate poles, giving

$$\mathcal{I}_{1,2}^{(0,2),3} = \frac{\eta(\tau)^2}{\vartheta_4(3\mathsf{c}_1)\vartheta_4(3\mathsf{c}_2)} \ . \tag{B.55}$$

SU(3)×U(1) gauge theory

The integrand of the elliptic genus for the SU(3) × U(1) theory, depicted on the left of Figure 3-15, is

$$\mathcal{Z} = \mathcal{I}_{U_3}^{(0,2)}(\mathsf{a}_2 + \mathsf{d}_1, \mathsf{a}_1, \mathsf{b}) \mathcal{I}_{U_3}^{(0,2)}(-\mathsf{a}_2 + \mathsf{d}_1, -\mathsf{a}_1, \mathsf{b}') \mathcal{I}_{\mathrm{vec}}^{(0,2)}(\mathsf{a}_1) \vartheta_4(\pm 3\mathsf{a}_2) \ , \tag{B.56}$$

where $\vartheta_4(\pm 3\mathsf{a}_2)$ accounts for the contribution from two Fermi multiplets with ± 3 U(1) gauge charges. With a simple choice of

$$\eta = (\frac{999}{1000}, \frac{1999}{2000}, \frac{2999}{3000}) \ , \tag{B.57}$$

there are 162 non-degenerate poles, giving a fairly complicated expression as the elliptic genus,

$$\mathcal{I}'_{1,2} = -\eta(\tau)^7 \sum_{A,A'=1}^{3} \frac{\vartheta_1(-\mathsf{b}_A - \mathsf{b}'_{A'} + \mathsf{d}_1)^2}{\prod_{B\neq A}\vartheta_1(\mathsf{b}_B - \mathsf{b}_A)\prod_{B'\neq A'}\vartheta_1(\mathsf{b}'_{B'} - \mathsf{b}'_{A'})} \times \frac{1}{\vartheta_1(\mathsf{b}_A + \mathsf{b}'_{A'} + 2\mathsf{d}_1)\prod_{B\neq A, B'\neq A'}\vartheta_1(\mathsf{b}_B + \mathsf{b}'_{B'} + 2\mathsf{d}_1)} . \tag{B.58}$$

Although the expression is complicated, it can be reformulated as a simpler ratio

$$\mathcal{I}'_{1,2} = \frac{3\eta(\tau)^7\vartheta_1(3\mathsf{d}_1)^2}{\prod_{A,B=1}^{3}\vartheta_1(\mathsf{b}_A + \mathsf{b}'_B + 2\mathsf{d}_1)} . \tag{B.59}$$

C JK residues of $\mathcal{N} = (0,4)$ elliptic genera

In this Appendix, we present the detailed computation of the $\mathcal{N} = (0,4)$ elliptic genus. Recall that the basic building blocks of the elliptic genus for the Lagrangian theories of type A_{N-1}, as illustrated in Figure 4-1, are given by[9]

$$\mathcal{I}^{(0,4),N}_{0,3}(\mathsf{c},\mathsf{a},\mathsf{b}) = \prod_{A,B=1}^{N} \frac{\eta(\tau)}{\vartheta_1(\mathsf{v} \pm (\mathsf{c} + \mathsf{a}_A + \mathsf{b}_B))} , \tag{C.1}$$

$$\mathcal{I}^{(0,4),N}_{\mathrm{vec}} = \frac{1}{N!}\big(\vartheta_1(2\mathsf{v})\eta(\tau)\big)^{N-1} \prod_{\substack{A,B=1\\ A\neq B}}^{N} \frac{\vartheta_1(2\mathsf{v} + \mathsf{a}_A - \mathsf{a}_B)\vartheta_1(\mathsf{a}_A - \mathsf{a}_B)}{\eta(\tau)^2} . \tag{C.2}$$

where all the SU(N) chemical potentials satisfy the traceless condition such as $\sum_{A=1}^{N}\mathsf{a}_A = 0$. The chemical potentials $\mathsf{a},\mathsf{b},\mathsf{c}$ make manifest the flavor symmetry SU$(N)\times$SU$(N)\times$U$(1) \subset$ SU$(2N)\times$U(1). Note that our convention is slightly different from that in[9] by some constant factor.

C.1 Type A_{N-1} theories with minimal punctures at genus-one

We begin with the theory of type A_2 at genus-one with one minimal puncture associated to an U(1) flavor symmetry. The elliptic genus can be computed by the JK-residue

$$\mathcal{I}^{(0,4),3}_{1,1,0}(\mathsf{c}) = \int_{\mathrm{JK}} \prod_{A=1}^{2} \frac{da_A}{2\pi i a_A} \mathcal{I}^{(0,4),3}_{0,3}(\mathsf{c},\mathsf{a},-\mathsf{a})\mathcal{I}^{(0,4),3}_{\mathrm{vec}}(\mathsf{a}) . \tag{C.3}$$

The integration simplifies when choosing a suitable reference vector as $\eta = (1, 1-\frac{1}{1000})$, resulting in only 54 non-degenerate poles. The result of the JK-residue computation is

$$\mathcal{I}^{(0,4),3}_{1,1,0}(\mathsf{c}) = \eta(\tau)^2\Bigg[-\frac{\vartheta_1(4\mathsf{v}-2\mathsf{c})\vartheta_1(3\mathsf{v}-\mathsf{c})}{\vartheta_1(\mathsf{v}-3\mathsf{c})\vartheta_1(3\mathsf{v}-3\mathsf{c})\vartheta_1(2\mathsf{c})\vartheta_1(\mathsf{v}+\mathsf{c})} + \frac{\vartheta_1(3\mathsf{v}-\mathsf{c})\vartheta_1(3\mathsf{v}+\mathsf{c})}{\vartheta_1(\mathsf{v}-3\mathsf{c})\vartheta_1(\mathsf{v}-\mathsf{c})\vartheta_1(\mathsf{v}+\mathsf{c})\vartheta_1(\mathsf{v}+3\mathsf{c})} + \frac{\vartheta_1(3\mathsf{v}+\mathsf{c})\vartheta_1(4\mathsf{v}+2\mathsf{c})}{\vartheta_1(\mathsf{v}-\mathsf{c})\vartheta_1(2\mathsf{c})\vartheta_1(\mathsf{v}+3\mathsf{c})\vartheta_1(3\mathsf{v}+3\mathsf{c})} \Bigg] . \tag{C.4}$$

The sum of three terms can be reorganized into a simple ratio,

$$\begin{aligned}\mathcal{I}^{(0,4),3}_{1,1,0}(\mathsf{c}) &= \frac{\eta(\tau)^2\vartheta_1(6\mathsf{v})}{\vartheta_1(2\mathsf{v})\vartheta_1(3\mathsf{v}+3\mathsf{c})\vartheta_1(3\mathsf{v}-3\mathsf{c})} \\ &= q^{-1/6}(1 + 2v^2 + (c^3+c^{-3})v^3 + 3v^4 \\ &\quad + (-v^{-6} - 2v^{-4} - v^{-2} + 6 + 2(c^3+c^{-3})v + 15v^2 + \ldots)q + \ldots)\end{aligned} \tag{C.5}$$

The same computation can be done for SU(4), just more tedious. There are both non-degenerate and degenerate poles, and in the end, the elliptic genus reads

$$\begin{aligned}\mathcal{I}^{(0,4),4}_{1,1,0}(\mathsf{c}) = \eta(\tau)^2\Bigg[& -\frac{\vartheta_1(5\mathsf{v}-3\mathsf{c})\vartheta_1(4\mathsf{v}-2\mathsf{c})\vartheta_1(3\mathsf{v}-\mathsf{c})}{\vartheta_1(2\mathsf{c})\vartheta_1(2\mathsf{v}-4\mathsf{c})\vartheta_1(4\mathsf{v}-4\mathsf{c})\vartheta_1(\mathsf{v}-3\mathsf{c})\vartheta_1(\mathsf{v}+\mathsf{c})} \\ & -\frac{\vartheta_1(4\mathsf{v}-2\mathsf{c})\vartheta_1(3\mathsf{v}-\mathsf{c})\vartheta_1(3\mathsf{v}+\mathsf{c})}{\vartheta_1(2\mathsf{c})\vartheta_1(4\mathsf{c})\vartheta_1(2\mathsf{v}-4\mathsf{c})\vartheta_1(\mathsf{v}-\mathsf{c})\vartheta_1(\mathsf{v}+\mathsf{c})} \\ & +\frac{\vartheta_1(2\mathsf{v})\vartheta_1(4\mathsf{v})\vartheta_1(3\mathsf{v}-\mathsf{c})\vartheta_1(3\mathsf{v}+\mathsf{c})}{(\vartheta_1(2\mathsf{c}))^2\vartheta_1(\mathsf{v}-3\mathsf{c})\vartheta_1(2\mathsf{v}-2\mathsf{c})\vartheta_1(2\mathsf{v}+2\mathsf{c})\vartheta_1(\mathsf{v}+3\mathsf{c})} \\ & -\frac{\vartheta_1(3\mathsf{v}-\mathsf{c})\vartheta_1(3\mathsf{v}+\mathsf{c})\vartheta_1(4\mathsf{v}+2\mathsf{c})}{\vartheta_1(2\mathsf{c})\vartheta_1(4\mathsf{c})\vartheta_1(\mathsf{v}-\mathsf{c})\vartheta_1(\mathsf{v}+\mathsf{c})\vartheta_1(2\mathsf{v}+4\mathsf{c})} \\ & +\frac{\vartheta_1(3\mathsf{v}+\mathsf{c})\vartheta_1(4\mathsf{v}+2\mathsf{c})\vartheta_1(5\mathsf{v}+3\mathsf{c})}{\vartheta_1(2\mathsf{c})\vartheta_1(\mathsf{v}-\mathsf{c})\vartheta_1(\mathsf{v}+3\mathsf{c})\vartheta_1(2\mathsf{v}+4\mathsf{c})\vartheta_1(4\mathsf{v}+4\mathsf{c})}\Bigg] .\end{aligned} \tag{C.6}$$

Similar to the SU(3) case, the expression can be recast into a simple ratio,

$$\mathcal{I}^{(0,4),N}_{1,1,0}(\mathsf{c}) = \frac{\eta(\tau)^2\vartheta_1(2N\mathsf{v})}{\vartheta_1(2\mathsf{v})\vartheta_1(N\mathsf{v}\pm N\mathsf{c})} , \quad N=4 . \tag{C.7}$$

Here, the expression on the right is expected to hold for all genus-one A_{N-1} theories with one minimal puncture. Furthermore, through an even more intricate computation, we derive the elliptic genus for a circular $\mathrm{SU}(3)^2$ quiver with $\mathrm{U}(1)^2_{x_1,x_2}$ flavor symmetry, or equivalently, a theory of type A_2 at genus-one with two minimal punctures. The expression from the direct JK-residue computation is too complicated to detail here; however, it can be reorganized into a simple form

$$\mathcal{I}^{(0,4),2}_{g=1,n_1=2}(\mathsf{c}_1,\mathsf{c}_2) = \frac{\eta(\tau)^4\vartheta(6\mathsf{v})^2}{\vartheta_1(2\mathsf{v})^2\prod_{i=1}^{2}\vartheta(3\mathsf{v}\pm 3\mathsf{c}_i)} . \tag{C.8}$$

Here n denotes the number of minimal punctures. Extrapolating from the above results, it is natural to conjecture that for all the SU(N)-type theory at genus-one with n_1 minimal punctures, the elliptic genus can be written as

$$\mathcal{I}^{(0,4),N}_{g=1,n_1}(\mathsf{c}_1,\cdots,\mathsf{c}_{n_1}) = \prod_{i=1}^{n_{U(1)}}\frac{\eta(\tau)^2\vartheta_1(2N\mathsf{v})}{\vartheta_1(2\mathsf{v})\vartheta_1(N\mathsf{v}\pm N\mathsf{c}_i)} . \tag{C.9}$$

We will apply this conjecture to later computations.

C.2 Type A_2 theories at genus $g \geq 1$

The elliptic genus of $\mathcal{N}=(0,4)$ T_3 theory was computed in[9] using an elliptic inversion formula. In more detail, the elliptic genus of the $\mathcal{N}=(0,4)$ T_3 theory and the SU(3) SQCD with six fundamental flavors is related by the Argyres-Seiberg duality,

$$\mathcal{I}^{(0,4),N=3}_{\mathrm{SQCD}}(\mathsf{a},\mathsf{b},\mathsf{r},\mathsf{s}) = \int_{\mathrm{JK}}\frac{dz}{2\pi i z}\frac{\eta(\tau)}{\vartheta_1(\mathsf{v}\pm\mathsf{s}\pm\mathsf{z})}\mathcal{I}^{(0,4),2}_{\mathrm{vec}}(\mathsf{z})\mathcal{I}_{T^3}(\mathsf{a},\mathsf{b},\mathsf{c}) . \tag{C.10}$$

Within the integral, we gauge an $\mathrm{SU}(2)\subset\mathrm{SU}(3)_c$, leading to $\mathsf{c}_1=\mathsf{r}+\mathsf{z}$, $\mathsf{c}_2=\mathsf{r}-\mathsf{z}$, $\mathsf{c}_3=-2\mathsf{r}$. The integral can be inverted to compute the elliptic genus $\mathcal{I}_{T_3}$. Explicitly, it is given by a simple JK-residue computation

$$\mathcal{I}_{T_3}(\mathsf{a},\mathsf{b},\mathsf{c}) = \frac{\eta(\tau)^5}{2\vartheta_1(2\mathsf{v}\pm 2\mathsf{z})}\int_{\mathrm{JK}}\frac{ds}{2\pi i s}\frac{\vartheta_1(\pm 2\mathsf{s})\vartheta_1(\pm 2\mathsf{v})}{\vartheta_1(-\mathsf{v}\pm\mathsf{s}\pm\mathsf{z})}\mathcal{I}^{(0,4),N=3}_{\mathrm{SQCD}}(\mathsf{a},\mathsf{b},\mathsf{r},\mathsf{s}) .$$

Note that the coefficient in front of the integral has been adjusted according to our convention. On the right, $\mathsf{a},\mathsf{b},\mathsf{r},\mathsf{s}$ represent the $\mathrm{SU}(3)_a\times\mathrm{SU}(3)_b\times\mathrm{U}(1)_x\times\mathrm{U}(1)_y$ flavor chemical potentials, where

$$\mathsf{x}=\frac{\mathsf{s}}{3}+\mathsf{r}\,,\qquad \mathsf{y}=\frac{\mathsf{s}}{3}-\mathsf{r}\ . \tag{C.11}$$

After the integral, z,r combine into $\mathrm{SU}(3)$ chemical potentials

$$\mathsf{c}_1=\mathsf{r}+\mathsf{z}\,,\qquad \mathsf{c}_2=\mathsf{r}-\mathsf{z}\,,\qquad \mathsf{c}_3=-2\mathsf{r}\ . \tag{C.12}$$

The $\mathsf{a},\mathsf{b},\mathsf{c}$ denote the $\mathrm{SU}(3)^3\subset E_6$ chemical potentials for the $\mathcal{N}=(0,4)$ E_6 theory.

From the elliptic inversion formula for $\mathcal{I}_{T_3}$, it is evident that the diagonal of the two $\mathrm{SU}(3)$ flavor subgroups can be gauged, which yields the elliptic genus of the $\mathrm{SU}(3)$-type genus-one theory with one maximal puncture associated to an $\mathrm{SU}(3)_c$ flavor symmetry. Now the situation is much better: the right-hand side involves $\mathcal{I}^{(0,4),3}_{g=1,n_1=2}$ which is shown to be a simple ratio of elliptic theta functions, where n_1 denotes the number of minimal punctures. (See Figure 4-2.) When performing the JK-residue computation, we encounter only non-degenerate poles, and we obtain a simple result for the (0,4) elliptic genus with $n_3=1$ maximal puncture,

$$\mathcal{I}^{(0,4),3}_{g=1,n_1=0,n_3=1}(\mathsf{c})=\frac{\eta(\tau)^6\vartheta_1(2\mathsf{v})\vartheta_1(4\mathsf{v})\vartheta_1(6\mathsf{v})}{\prod_{A,B=1}^3\vartheta_1(2\mathsf{v}+\mathsf{b}_{iA}-\mathsf{b}_{iB})}\ . \tag{C.13}$$

Instead of directly gauging the diagonal of the two existing $\mathrm{SU}(3)$ groups, one can alternatively gauge the diagonal of the $\mathrm{SU}(3)^2$ flavor subgroup of $\mathcal{I}_{T_3}$ and a $\mathrm{SU}(3)$ linear quiver. This approach can be used to generate a genus-one theory with additional minimal punctures. In other words,

$$\mathcal{I}^{(0,4),3}_{g=1,n_1,n_3=1}=\text{elliptic inversion of }\ \mathcal{I}^{(0,4),3}_{g=1,n_1+2,n_3=0}\ . \tag{C.14}$$

Effectively, the elliptic inversion formula represents the fusion of any two minimal punctures into a single maximal one. Therefore, the inversion, or fusing two minimal punctures, can be performed successively, yielding more maximal punctures. Since the elliptic genus on the right are all simple ratios, the JK residue can be easily carried out, and at each step of the fusion, the outcome continues to be a simple ratio. In conclusion, we obtain the following general result for the theory of type A_2 for any $g\ge1$ that has n_1 minimal and n_3 maximal punctures

$$\mathcal{I}^{(0,4),3}_{g,n_1,n_3}=\left(\frac{\vartheta_1(2\mathsf{v})\vartheta_1(4\mathsf{v})^2\vartheta_1(6\mathsf{v})}{\eta(\tau)^4}\right)^{g-1}\mathcal{I}^{(0,4),3}_{g=1,n_1,0}\mathcal{I}^{(0,4),3}_{g=1,0,n_3}\ , \tag{C.15}$$

where with $N=3$,

$$\mathcal{I}^{(0,4),3}_{g=1,n_1,0}=\prod_{i=1}^{n_1}\frac{\eta(\tau)^2\vartheta_1(2N\mathsf{v})}{\vartheta_1(2\mathsf{v})\vartheta_1(N\mathsf{v}\pm N\mathsf{c}_i)}\ , \tag{C.16}$$

$$\mathcal{I}^{(0,4),3}_{g=1,0,n_3}=\prod_{i=1}^{n_3}\frac{\eta(\tau)^6\vartheta_1(2\mathsf{v})\vartheta_1(4\mathsf{v})\vartheta_1(6\mathsf{v})}{\prod_{A,B=1}^3\vartheta_1(2\mathsf{v}+\mathsf{b}_{iA}-\mathsf{b}_{iB})}\ . \tag{C.17}$$

C.3 Type A_3 theories at genus $g\ge1$

Let us now consider theories of type A_3 of class $\mathcal{S}$. There are four punctures (and the corresponding flavor symmetry) to be considered, minimal ($\mathrm{U}(1)$), $[2,1^2]$ ($\mathrm{SU}(2)\times\mathrm{U}(1)$), $[2^2]$ ($\mathrm{SU}(2)$), and maximal ($\mathrm{SU}(4)$). To begin, we have

$$\mathcal{I}^{(0,4),N=4}_{g=1;n_1,0,0,0}=\prod_{i=1}^{n_1}\frac{\eta(\tau)^2\vartheta_1(2N\mathsf{v})}{\vartheta_1(2\mathsf{v})\vartheta_1(N\mathsf{v}\pm N\mathsf{c}_i)}\ ,\qquad N=4\ , \tag{C.18}$$

where the subscript n_1, n_2, n_3, n_4 denotes the number of punctures associated with the respective flavor symmetries: U(1), SU(2) × U(1), SU(2), and SU(4).

In the context of 4d $\mathcal{N} = 2$ SCFTs, $R_{0,N}$ are non-Lagrangian theories with SU(2N) × SU(2) flavor symmetry, and they arise from a strong coupling limit of the $\mathcal{N} = 2$ SU(N) theory with 2N fundamental hypermultiplets. Concretely, as a theory of A_3-type class $\mathcal{S}$, $R_{0,4}$ corresponds to the three-punctured sphere with two maximal-puncture and a $[2, 1^2]$ puncture, where the manifest flavor subgroup is $\mathrm{SU}(4)^2 \times \mathrm{U}(1) \times \mathrm{SU}(2)$.

In 2d, the $\mathcal{N} = (0, 4)$ elliptic genus of $R_{0,4}$ theory was computed in[93] using an elliptic inversion. The Argyres-Seiberg-like duality implies an integral equality between the elliptic genus of SU(4) SQCD with eight fundamental flavors and that of the $R_{0,4}$ theory,

$$\mathcal{I}^{(0,4),N=4}_{\mathrm{SQCD}}(\mathsf{a},\mathsf{b},\mathsf{r},\mathsf{s}) = \int_{\mathrm{JK}} \frac{dz}{2\pi i z} \frac{\eta(\tau)}{\vartheta_1(\mathsf{v} \pm \mathsf{s} \pm \mathsf{z})} \mathcal{I}^{(0,4),2}_{\mathrm{vec}}(\mathsf{z}) \mathcal{I}^{(0,4)}_{R_{0,4}}(\mathsf{a},\mathsf{b},\mathsf{r},\mathsf{z}) \ , \tag{C.19}$$

which can be inverted to give

$$\mathcal{I}^{(0,4)}_{R_{0,4}}(\mathsf{a},\mathsf{b},\mathsf{r},\mathsf{z}) = \frac{\eta(\tau)^5}{2\vartheta_1(2\mathsf{v} \pm 2\mathsf{z})} \int_{\mathrm{JK}} \frac{ds}{2\pi i s} \frac{\vartheta_1(\pm 2\mathsf{s})\vartheta_1(-2\mathsf{v})}{\vartheta_1(-\mathsf{v} \pm \mathsf{s} \pm \mathsf{z})} \mathcal{I}^{(0,4),N=4}_{\mathrm{SQCD}}(\mathsf{a},\mathsf{b},\mathsf{r},\mathsf{s}) \ .$$

Here, $\mathsf{a},\mathsf{b},\mathsf{r},\mathsf{z}$ denote the chemical potentials of SU(4)×SU(4)×U(1)×SU(2) ⊂ SU(8)×SU(2) flavor symmetry of the $R_{0,4}$ theory. On the right, $\mathsf{a},\mathsf{b},\mathsf{r},\mathsf{s}$ are the chemical potentials of the $\mathrm{U}(8) = \mathrm{SU}(8)_{\mathsf{a},\mathsf{b},\mathsf{r}} \times \mathrm{U}(1)_{\mathsf{s}}$ flavor symmetry of the SU(4) gauge theory with 8 fundamental hypermultiplets. Chemical potentials r,s are related to the standard x,y (associated to the two minimal punctures of the SU(4) SQCD) by

$$\mathsf{x} = \frac{\mathsf{s}}{4} + \mathsf{r}, \qquad \mathsf{y} = \frac{\mathsf{s}}{4} - \mathsf{r} \ . \tag{C.20}$$

One can start gauging in the theory of 4^2 free hypermultiplets or handles to the punctures associated to a,b on both sides of the above equation, so that the $\mathcal{I}^{(0,4),N=4}_{\mathrm{SQCD}}$ on the right becomes $\mathcal{I}^{(0,4),N=4}_{g,n_1+2,0,0,n_4}$, while on the left, it becomes $\mathcal{I}^{(0,4),4}_{g;n_1,0,1,n_4}$. Effectively, the elliptic inversion merges two minimal punctures into a $[2, 1^2]$ puncture. (See Figure 4-3.) In the simplest case, when $g = 1$, $n_1 = 0$, $n_4 = 0$,

$$\mathcal{I}^{(0,4),4}_{g=1;0,1,0,0} = \frac{\eta(\tau)^6\vartheta_1(6\mathsf{v})\vartheta_1(8\mathsf{v})}{\vartheta_1(2\mathsf{v})^2\vartheta_1(2\mathsf{v} \pm 2\mathsf{z})\vartheta_1(3\mathsf{v} \pm \mathsf{z} \pm 4\mathsf{r})} \ . \tag{C.21}$$

The $[2, 1, 1]$ puncture can be further Higgsed to a $[2, 2]$ puncture with SU(2) flavor symmetry. The class $\mathcal{S}$ theory corresponding to a three-punctured sphere with two maximal and one $[2, 2]$ puncture has an enhanced E_7 flavor symmetry. The elliptic genus of the corresponding 2d $\mathcal{N} = (0, 4)$ theory can also be computed using an elliptic inversion formula[93]. Following the same approach as above, we perform additional gluing/gauging operations on the two maximal punctures, and we obtain the contribution from one $[2, 2]$ puncture,

$$\begin{aligned} \mathcal{I}^{(0,4),4}_{g=1;0,0,1,0}(\mathsf{w}) &= \int_{\mathrm{JK}} \frac{ds}{2\pi i s} \frac{\eta(\tau)^5\vartheta_1(\pm 2\mathsf{v})}{2\vartheta_1(4\mathsf{v})} \frac{\vartheta_1(\pm 2\mathsf{s})}{\vartheta_1(\pm \mathsf{s})\vartheta_1(-2\mathsf{v} \pm \mathsf{s})} \mathcal{I}^{(0,4),4}_{1;2,0,0,0}(\mathsf{c}_1,\mathsf{c}_2) \\ &= \frac{\eta(\tau)^6\vartheta_1(6\mathsf{v})\vartheta_1(8\mathsf{v})}{\vartheta_1(2\mathsf{v})\vartheta_1(4\mathsf{v})\vartheta_1(4\mathsf{v} \pm 4\mathsf{w})\vartheta_1(2\mathsf{v} \pm 4\mathsf{w})} \ . \end{aligned} \tag{C.22}$$

Again, w represents the SU(2) flavor chemical potential. As before, the two U(1) chemical potentials on the right are related to w by $\mathsf{c}_1 = \frac{\mathsf{s}}{4} + \mathsf{w}$ and $\mathsf{c}_2 = \frac{\mathsf{s}}{4} - \mathsf{w}$.

The contribution from the maximal puncture can be obtained by considering the generalized S-duality as shown in Figure 4-4, which requires

$$\begin{aligned}\mathcal{I}^{(0,4),4}_{1;3,0,0,0}(\mathsf{x}_{1,2,3}) = \int_{\mathrm{JK}} \frac{db_1}{2\pi i b_1}\prod_{A=1}^{2}\frac{da_A}{2\pi i a_A}\mathcal{I}^{(0,4),4}_{1;0,0,0,1}(\mathsf{c})\Big|_{\mathsf{c}_{A=1,2,3}=\mathsf{a}_A+\mathsf{r}} \\ \times \mathcal{I}^{(0,4),3}_{\mathrm{vec}}(\mathsf{a})\prod_{A=1}^{3}\prod_{i=1}^{2}\frac{\eta(\tau)^2}{\vartheta_1(\mathsf{v}\pm(-\mathsf{a}_A+\mathsf{b}_i+\mathsf{x}'_1))} \\ \times \mathcal{I}^{(0,4),2}_{\mathrm{vec}}(\mathsf{b})\prod_{i=1}^{2}\frac{\eta(\tau)^2}{\vartheta_1(\mathsf{v}\pm(-\mathsf{b}_i+\mathsf{x}'_2))}\ .\end{aligned} \tag{C.23}$$

In the above a_A and b_i are SU(3) and SU(2) fugacities with $a_3 = 1/(a_1a_2)$ and $b_1 = 1/b_2$. The U(1) fugacities on both sides are identified by

$$x'_1 = \frac{x_1^{4/3}x_3^{2/3}}{x_2^{2/3}}, \qquad x'_2 = x_2^2x_3^2, \qquad r = \left(\frac{x_3}{x_1x_2}\right)^{1/3}. \tag{C.24}$$

Explicitly, the elliptic genus of the $\mathcal{N}=(0,4)$ genus-one theory with one maximal puncture reads

$$\mathcal{I}^{(0,4),4}_{1;0,0,0,1} = \frac{\eta(\tau)^{12}\vartheta_1(2\mathsf{v})\vartheta_1(4\mathsf{v})\vartheta_1(6\mathsf{v})\vartheta_1(8\mathsf{v})}{\prod_{A,B=1}^{4}\vartheta_1(2\mathsf{v}+\mathsf{c}_A-\mathsf{c}_B)}\ . \tag{C.25}$$

Finally, by gauging the maximal punctures, we determine the elliptic genus for genus $g \geq 1$ with arbitrary punctures

$$\mathcal{I}^{(0,4),4}_{g;n_1,n_2,n_3,n_4} = \left(\frac{\vartheta_1(2\mathsf{v})\vartheta_1(4\mathsf{v})^2\vartheta_1(6\mathsf{v})^2\vartheta_1(8\mathsf{v})^2}{\eta(\tau)^6}\right)^{g-1}\mathcal{I}^{(0,4),4}_{1;n_1,0,0,0}\mathcal{I}^{(0,4),4}_{1;0,n_2,0,0}\mathcal{I}^{(0,4),4}_{1;0,0,n_3,0}\mathcal{I}^{(0,4),4}_{1;0,0,0,n_4}\ ,$$

where

$$\mathcal{I}^{(0,4),4}_{1;n_1,0,0,0} = \prod_{i=1}^{n_1}\frac{\eta(\tau)^2\vartheta_1(8\mathsf{v})}{\vartheta_1(2\mathsf{v})\vartheta_1(4\mathsf{v}\pm 4\mathsf{x}_i)} \tag{C.26}$$

$$\mathcal{I}^{(0,4),4}_{1;0,n_2,0,0} = \prod_{i=1}^{n_3}\frac{\eta(\tau)^6\vartheta_1(6\mathsf{v})\vartheta_1(8\mathsf{v})}{\vartheta_1(2\mathsf{v})^2\vartheta_1(2\mathsf{v}\pm 2\mathsf{z}_i)\vartheta_1(3\mathsf{v}\pm\mathsf{z}_i\pm 4\mathsf{r}_i)} \tag{C.27}$$

$$\mathcal{I}^{(0,4),4}_{1;0,0,n_3,0} = \prod_{i=1}^{n_2}\frac{\eta(\tau)^6\vartheta_1(6\mathsf{v})\vartheta_1(8\mathsf{v})}{\vartheta_1(2\mathsf{v})\vartheta_1(4\mathsf{v})\vartheta_1(2\mathsf{v}\pm 4\mathsf{w}_i)\vartheta_1(4\mathsf{v}\pm 4\mathsf{w}_i)}\ , \tag{C.28}$$

$$\mathcal{I}^{(0,4),4}_{1;0,0,0,n_4} = \prod_{i=1}^{n_4}\frac{\eta(\tau)^{12}\vartheta_1(2\mathsf{v})\vartheta_1(4\mathsf{v})\vartheta_1(6\mathsf{v})\vartheta_1(8\mathsf{v})}{\prod_{A,B=1}^{4}\vartheta_1(2\mathsf{v}+\mathsf{b}_{iA}-\mathsf{b}_{iB})} \tag{C.29}$$

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