Homework 5: Due at class on April 2

1. Let $ds^2 = -dt^2 + dx^2 + dy^2$ be the Minkowski metric on $\mathbb{R}^{1,2}$ and $-t^2 + x^2 + y^2 = -1$ for $t > 0$ be the space-like surface (hyperboloid **S**). (See Figure 1.) Find the induced metric on the hyperboloid **S** in terms of the polar coordinate

$$\begin{aligned} t &= r \cosh \rho \\ x &= r \sinh \rho \cos \phi \\ y &= r \sinh \rho \sin \phi \end{aligned}$$

Given this metric, find geodesics on **S** and compute its Riemann, Ricci and scalar curvature.

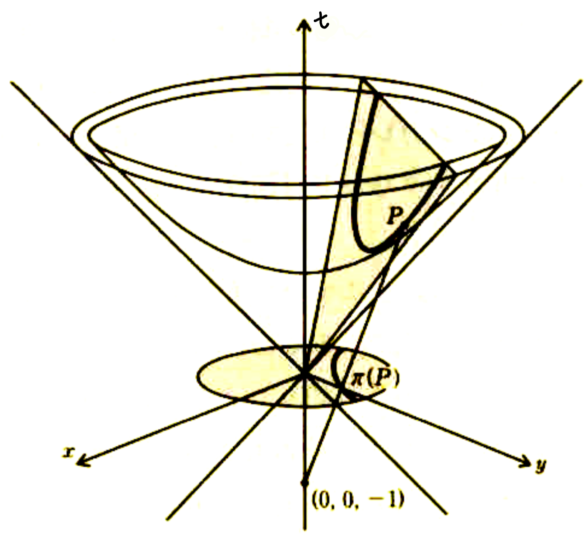

Figure 1: Hyperboloid and Poincare disk

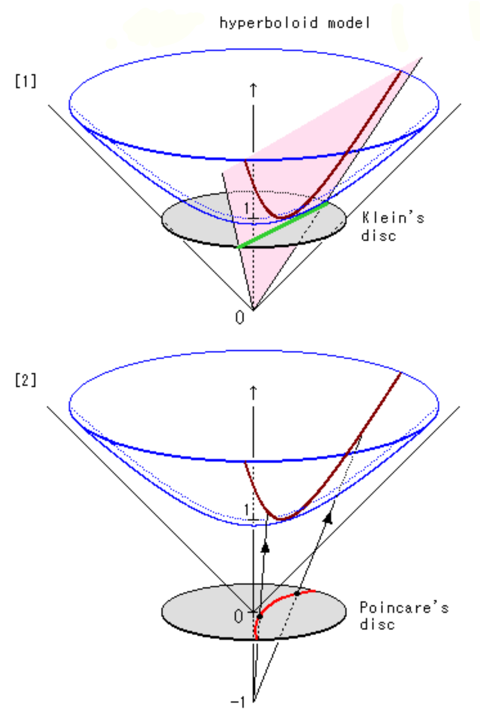

Figure 2: Hyperboloid and Poincare disk

2. Let us consider the unit disk **D** on the x-y plane. Let $\pi(P)$ be the intersection point of the unit disk and the line between the point $(0, 0, -1)$ and $P \in \mathbf{S}$. By assigning $\pi(P)$ to P, there is one-to-one map from the unit disk D and the hyperboloid **S**. Show that the map $\pi^{-1} : \mathbf{D} \to \mathbf{S}$ is determined by

$$(u, v) \mapsto \left(\frac{2u}{1 - u^2 - v^2}, \frac{2v}{1 - u^2 - v^2}, \frac{1 + u^2 + v^2}{1 - u^2 - v^2} \right) .$$

Find the metric on the unit disk pull-backed by this map. The unit disk **D** with this induced metric is called the Poincaré disk.

The red curve on the hyperboloid is an intersection with a plane (pink in Figure 1,) that goes through the origin. We put a disk (called Klein's disk) on the bottom of the hyperboloid,

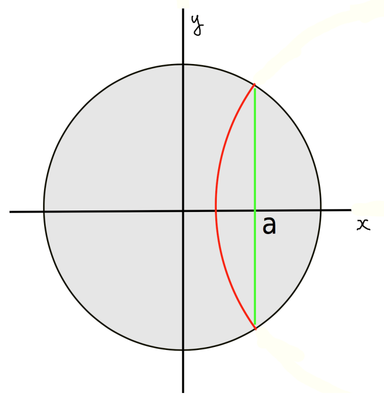

Figure 3: Poincare disk

which allows us to get the corresponding straight line (green) on Klein's disk. This curve is mapped by $\pi : \mathbf{S} \to \mathbf{D}$ to an (red) arc in the Poincare disk. If the green line is represented by $x = a$ (Figure 3), find the equation for the red curve.

Find geodesics on $\mathbf{D}$ and compute its Riemann, Ricci and scalar curvature.

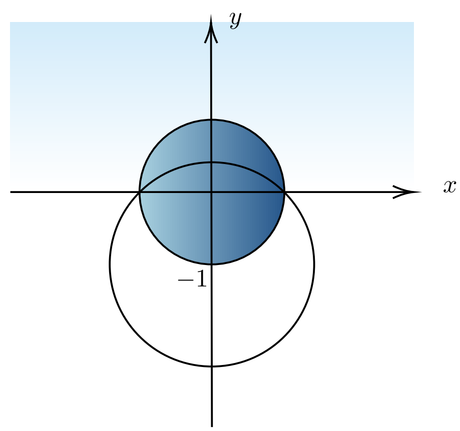

Figure 4: Hyperboloid and Poincare disk

3. Let $\mathbf{H} = \{(x, y) | y > 0\}$ be the upper half plane (light blue area in Figure 4). We invert the upper half plane $\mathbf{H}$ to $\mathbf{D}$ in terms of the circle with radius $\sqrt{2}$ around the center $(0, -1)$, and take the reflection with respect to x-axis (Figure 4). This gives a map $J : \mathbf{H} \to \mathbf{D}; (x, y) \mapsto (u, v)$

$$u = \frac{2x}{x^2 + (y+1)^2}\,, \qquad v = 1 - \frac{2(y+1)}{x^2 + (y+1)^2}\,.$$

1. Find the induced metric on the upper half plane by this map.

2. Find geodesics on $\mathbf{H}$ and compute its Riemann, Ricci and scalar curvature.

3. Find the area of the triangle with angles (α, β, γ) bounded by half-circles with respect to the metric (Figure 5). Here, we can use the fact that the area of the triangle in the left

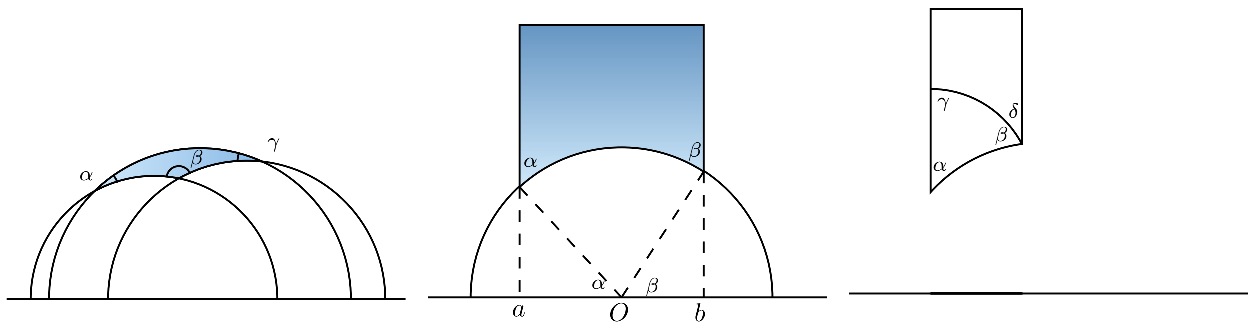

Figure 5: triangle in the upper half plane

of Figure 5 is the same as that of the triangle in the right of Figure 5. Compare with the area of a triangle on the 2-sphere (Homework 1).

4. Do parallel transport of a vector along the triangle with respect to the Levi-Civita connection of the metric and find the angle difference when it comes back. Describe the difference between the sphere and the upper half plane.

5. The **Möbius transformation** of the upper half plane $\mathbf{H} = \{z = x + iy \mid y > 0\}$ is a rational function of the form
$$f(z) = \frac{az+b}{cz+d},$$
where $ad - bc = 1$ with $a, b, c, d \in \mathbb{R}$. If f_1 and f_2 are Möbius transformations, prove that $f_1 \circ f_2$ is also a Möbius transformation. Show that this is an isometry group for the metric.