Homework 11: Due at class on Dec 7

1 Modular transformations of ϑ and η-functions

Although we have been using modular transformations of the theta and eta functions, we have not proven those properties yet. Here, let us achieve this from "boson-fermion correspondence", which is confirmed in homework 4.

1. As we saw in the homework 4, a vertex operator, $V_\alpha = :e^{i\alpha\varphi}:$, plays an important role. In order for the vertex operator to be self-consistent the scalar φ should be compactified, and let us assume that the period of φ is $2\pi R$. Derive possible values of α. Using T-duality (i.e. $R \leftrightarrow \frac{2}{R}$) confirm that $R = \sqrt{2k}$ with $k = 2$ is the correct value for the correspondence.

2. Free complex fermion system has following non-trivial commutation relations:
$$\begin{aligned} &\{\psi_r, \overline{\psi}_s\} = \delta_{r+s,0}, \qquad [J_m, J_n] = m\delta_{m+n,0}, \qquad [L_m, J_n] = -nJ_{m+n}, \\ &[J_m, \psi_s] = +\psi_{m+s}, \qquad [J_m, \overline{\psi}_s] = -\overline{\psi}_{m+s}, \qquad [L_m, \psi_r] = \left(-\frac{m}{2} - r\right)\psi_{m+r}, \end{aligned} \tag{1}$$
where ψ_r, $\overline{\psi}_s$ are modes of complex fermion ψ, $\overline{\psi}$, J_m are modes of $U(1)$ current $J = \psi\overline{\psi}$, we consider NS sector and r, s are half-integers. Since L_0 and J_0 commute each other they can simultaneously have eigenvalues. Derive those eigenvalues for the eigenstates of ψ: $|n_{1/2}, n_{3/2}, \ldots\rangle = (\psi_{-\frac{1}{2}})^{n_{1/2}}(\psi_{-\frac{3}{2}})^{n_{3/2}} \cdots |0\rangle$. Calculate the holomorphic part of partition function (character):
$$\chi(\tau, z) = \mathrm{Tr}\left(q^{L_0 - \frac{1}{24}} y^{J_0}\right), \qquad q = e^{2\pi i\tau}, \quad y = e^{2\pi i z}. \tag{2}$$
The result is
$$\chi(\tau, z) = q^{-\frac{1}{24}} \prod_{n=1}^{\infty} (1 + q^{n-\frac{1}{2}} y)(1 + q^{n-\frac{1}{2}} y^{-1}). \tag{3}$$

3. Now, we compute the same character in terms of boson. Eigenstates are given by
$$|\alpha, n_1, n_2, \ldots\rangle = \lim_{z,\overline{z}\to 0} J_{-1}^{n_1} a J_{-2}^{n_2} \cdots V_\alpha(z, \overline{z})|0\rangle \tag{4}$$
where $J_n = a_n$ are the modes of the free boson in this case. Derive eigenvalues of L_0 and J_0 for the states, where the eigenvalues of V_α are given by $(L_0, J_0) = (\frac{\alpha^2}{2}, \alpha)$ (see homework 3). Calculate the character. The result is
$$\chi(\tau, z) = \frac{1}{\eta(\tau)} \sum_{n\in\mathbb{Z}} q^{\frac{n^2}{2}} y^n. \tag{5}$$

4. Using the last two results we have
$$q^{-\frac{1}{24}} \prod_{n=1}^{\infty} (1 + q^{n-\frac{1}{2}} y)(1 + q^{n-\frac{1}{2}} y^{-1}) = \frac{1}{\eta(\tau)} \sum_{n\in\mathbb{Z}} q^{\frac{n^2}{2}} y^n. \tag{6}$$

This is called the Jacobi triple product identity (triple-ness can be seen if you move the $\eta(\tau)$ to the LHS). Using this identity express theta functions ($\vartheta_2(\tau),\ \vartheta_3(\tau),\ \vartheta_4(\tau),$) in terms of sum (not product), where the definitions of the theta functions are given in Sec. 4.3 of lecture note. Derive the S- and T-transformations of the theta functions (you may use Poisson resummation formula).

5. Show that

$$\sqrt{\frac{\vartheta_2(\tau)\vartheta_3(\tau)\vartheta_4(\tau)}{2\eta(\tau)^3}} = 1, \tag{7}$$

and derive the S- and T-transformations of $\eta(\tau)$.

6. Finally, confirm that the equality of a free boson on a circle with $R = 1$ or 2 is equivalent to free complex fermion by comparing those torus partition functions, namely, show that

$$\sum_{-1\le m\le 2}\left|\frac{\Theta_{m,2}(\tau)}{\eta(\tau)}\right|^2 = \frac{1}{2}\left\{\left|\frac{\vartheta_2(\tau)}{\eta(\tau)}\right|^2 + \left|\frac{\vartheta_3(\tau)}{\eta(\tau)}\right|^2 + \left|\frac{\vartheta_4(\tau)}{\eta(\tau)}\right|^2\right\}, \tag{8}$$

where

$$\Theta_{m,k}(\tau) = \sum_{n\in\mathbb{Z}} q^{k\left(n+\frac{m}{2k}\right)^2}. \tag{9}$$

2 Verlinde algebra of Ising model

There are finitely many primary fields in a rational conformal field theory like a minimal model and WZW model. The fusion rule of primary fields are closed under themselves

$$[\phi_i]\times[\phi_j] = \sum_k \mathcal{N}_{ij}^k\,[\phi_k]\ ,$$

where $\mathcal{N}_{ij}^k$ are called the fusion coefficients. As we have seen in the minimal models and WZW models, one can construct a highest weight representation associated to a primary field ϕ_i, and we write the corresponding character by $\chi_i(\tau)$. The modular transformations are

$$\chi_i(-1/\tau) = \sum_j S_{ij}\,\chi_j(\tau)\ , \qquad \chi_i(\tau+1) = \sum_j T_{ij}\,\chi_j(\tau)\ .$$

E. Verlinde has found the remarkable relationship between fusion rule and the modular S-matrices [Ver88]

$$\mathcal{N}_{ij}^k = \sum_l \frac{S_{jl}S_{il}\,(S^{-1})_{lk}}{S_{0l}}\ . \tag{10}$$

2.1 Ising model revisited

Let us recall that the Ising model is the unitary minimal model $\mathcal{M}_{p=3}$ where the primary fields are associated to

$$
\begin{aligned}
1 \quad &\Leftrightarrow \phi_1 := \phi_{1,1} \\
\epsilon \quad &\Leftrightarrow \phi_2 := \phi_{2,1} \\
\sigma \quad &\Leftrightarrow \phi_3 := \phi_{2,2}
\end{aligned}
$$

where ϵ is the energy density filed and σ is the spin field. The corresponding characters are given in (4.111) of the lecture note

$$
\begin{aligned}
\chi_1(\tau) := \chi_0(\tau) &= \frac{1}{2}\left(\sqrt{\frac{\vartheta_3(\tau)}{\eta(\tau)}} + \sqrt{\frac{\vartheta_4(\tau)}{\eta(\tau)}}\right) \\
\chi_2(\tau) := \chi_{\frac{1}{2}}(\tau) &= \frac{1}{2}\left(\sqrt{\frac{\vartheta_3(\tau)}{\eta(\tau)}} - \sqrt{\frac{\vartheta_4(\tau)}{\eta(\tau)}}\right) \\
\chi_3(\tau) := \chi_{\frac{1}{16}}(\tau) &= \frac{1}{\sqrt{2}}\sqrt{\frac{\vartheta_2(\tau)}{\eta(\tau)}}
\end{aligned}
$$

Using the properties of the ϑ and η-functions, find the 3×3 S-matrix of the Ising model. Then, compute the three 3×3 fusion matrices $\mathcal{N}_{ij}^k$ $(i = 1, 2, 3)$ by using the Verlinde formula (10). Check they reproduce the fusion rule at the end of §5.5 of the lecture note.

References

[Ver88] E. Verlinde, *Fusion rules and modular transformations in 2d conformal field theory*, Nuclear Physics B **300** (1988) 360–376.