Introduction to string theory

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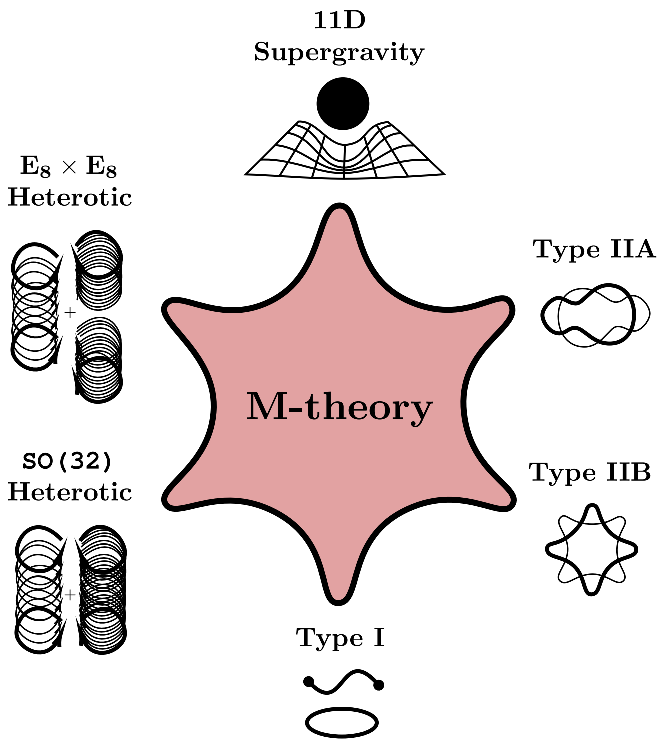
Abstract

These are lecture notes of the course on string theory at Fudan in Fall 2021. Comments are welcome. If you find typos, please let me know.



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1 Motivation and Overview

1.1 Why do we study string theory?

• For quantum gravity

One of the major problems in theoretical physics is to provide a unified description of all the forces in Nature. The standard model has unified electromagnetic, weak and strong force based on quantum field theories whereas general relativity for gravity is formulated within classical physics. A quantum theory of gravity is needed to reconcile general relativity with the principles of quantum mechanics. However, it is known that the renormalization procedure does not cure ultraviolet divergence for gravity. Therefore, naive quantization of general relativity does not give a consistent theory.

String theory not only yields the first quantization of gravity but also resolves the renormalization problem of gravity by replacing point particles by vibrating strings. To our knowledge, string theory is currently the only theory that has this property. Once we discover a candidate to unify gravity and quantum mechanics of all the forces, it is inevitable to try to understand it as well as we can although there is never any guarantee we can achieve.

Indeed, string theory has given deep insights into quantum gravity. The Beckenstein-Hawking formula [Bek73, BCH73, Haw75] was microscopically derived by counting D-brane states with fixed mass and charge for certain (called extremal) black holes in string theory [SV96]. The AdS/CFT correspondence [Mal99] conjectures that non-perturbative definition of string theory on AdS background is described by conformal field theory, and it partially resolves Hawking information paradox of black hole [Haw76]. Interestingly, a generalization of the Beckenstein-Hawking formula called the Ryu-Takayanagi entanglement entropy formula [RT06], connects quantum gravity and quantum information theory, which is actively studied in recent years. These developments certainly pose questions to basic concepts of spacetime.

• Rich arena for physics theories

String theory has shed light on the behavior of an established physical theory. These range from proof of positive energy, quark confinement, heavy-ion collisions, quantum critical behavior, quantum black holes, quantum information, etc. Moreover, it sometimes sets up a new framework to study established physical theory. In my opinion, there are two reasons for it. One reason is that string theory can generate vast families (in fact, infinitely many) of quantum field theories in various dimensions. There are five types of string theories, Type I, IIA, IIB, Heterotic SO(32), $E_8 \times E_8$ and M-theory is a limit of large dilaton expectation value in Type IIA string theory. Also, Type II theories can contain Dp-branes, and M-theory can have M2 and M5-branes. Although there are finitely many string theories, they can "engineer" infinitely many quantum field theories depending on manifolds string theory lives and brane configurations. The other reason is duality, equivalence between two different descriptions at quantum level. In fact, string theories of five types are related by a web of dualities (see the figure in the front page). The AdS/CFT correspondence is another kind of duality. More dualities have been (and surely will be) discovered in string theories, and a duality plays a crucial role in showing drastically different viewpoints for an established physical theory.

Through recent developments in string theory, especially in the study of M5-branes, we have learned that there are vast families of QFTs which do not admit Lagrangian description. They are intrinsically strongly-coupled so that techniques in QFT we have developed for decades cannot be used. Currently, we are trying to understand QFTs without Lagrangian description case by case by using dualities, dimensional reduction on some manifolds, RG flows or perturbing theories. However, there is no consistent unified description for these theories. Therefore, we need a new framework to describe QFTs without Lagrangian description.

• **For mathematical structure**

A certain group of people is interested in string theory because it has brought about a lot of new insights in mathematics, especially in geometry. String theory has repeatedly provided new ways to look at geometry, which arise as inevitable consequences of understanding how physical theory should be. Mirror symmetry [CDLOGP91], Seiberg-Witten invariants [Wit94], AGT relation [AGT10], etc are salient examples. Moreover, string dualities have led to highly non-trivial conjectures that connect seemingly different subjects of mathematics. String theory provides many insights into geometry partly because we do not understand the theory. In contrast to the fact that Einstein's theory of gravity has been constructed based on Riemannian geometry, we do not understand the geometric foundation for string theory yet. However, physicists can develop new insights into geometry because physicists stumble upon a profound theory we do not understand well. There are certainly more mathematical secrets in string theory.

• **Because we do not know what it is**

As mentioned, we do not understand the core new concept string theory is based on, but we know it is remarkably rich. One reason that string theory is an exciting topic to work on for students is that so many things are not understood yet. It sometimes framed criticisms that string theorists do not understand the theory. That is true. However, if we understood it, there would be ground-breaking insights both in physics and mathematics. The fact that so little is understood and such relatively small pieces are actually such big discoveries in their own right makes us excited. Of course, there is still a lot more to do!

1.2 Very very brief history

String theory was invented originally by accident in trying to solve a different problem, subsequently developed by a long and fortunate process of tinkering. Therefore, the history of string theory itself is interesting. For detailed history of string theory, we refer to [Gre99, Sch00, Sch11, CCCDV12, Oog15, Pol17].

String theory was first developed by aiming at describing hadron physics. People came up with the idea that a meson is a little string with charges at its end and the meson resonances are vibrational states of the string. Although this physical picture, which first emerged from the Veneziano amplitude [Ven68], is now believed to be qualitatively correct at a description of strongly interacting particles, QCD became far more successful in describing details of strong interactions soon later.

However, a small group of physicists continued to study string theory in the 70s, and it was found that string theory includes massless spin-two particles, suggesting that string theory can be a theory of gravity. Furthermore, a tachyon present in bosonic string theory turned out to be absent by incorporating supersymmetry, and Type IIA and IIB as well as Type I theories have been constructed.

In 1984, Green and Schwarz showed [GS84] that anomaly pointed out by Alvarez-Gaumé and Witten is canceled if the gauge group of Type I string theory is $\mathrm{SO}(32)$, which led to the first string revolution. A number of people have started working in the field of string theory. During the first revolution, Heterotic string theory was constructed.

In 1995, Witten proposed [Wit95b] that dualities relate five types of string theories and they are moreover five different limits of one bigger theory called M-theory. This has led to the second string revolution during which D-branes are proposed and non-perturbative aspects of string theories have been extensively studied. These results bore fruits as discoveries of a number of dualities, understanding of quantum nature of black holes and the AdS/CFT correspondence.

The second revolution has driven more researchers to study D-brane world-volume theories and their relations to quantum gravity. However, dynamics on M2-branes and M5-branes that are fundamental degrees of freedom in M-theory have remained elusive. Triggered by the work of Bagger-Lambert [BL07], the world-volume theory for M2-branes was found to be a highly supersymmetric exquisite mixture of Chern-Simons gauge theory with some scalars and spinors [ABJM08]. On the other hand, the world-volume theory on M5-branes is difficult to analyze because it is known that it does not admit Lagrangian description. However, the paper of Gaiotto [Gai12] has ignited the study of properties of M5-branes by wrapping them on a certain manifold, which continues to be active so far.

The following picture shows the timeline of string theory and when the standard textbooks were written.[1] Obviously, the textbooks do not cover developments after their publications. We will mainly use Polchinski's textbook [Pol98]. However, I do not recommend you stick to only one book when you study a subject. Some parts are explained very well in a book, but some parts are sometimes hard to grasp for some people. Therefore, it is always a good idea to look for books, notes, and papers that suit you best.

1.3 Very short highlights

• The basic idea in string theory is that different elementary particles are all vibrational modes of a single type of string. There are two types of strings, open and closed, and a trajectory of a string is called the **string world-sheet** which is parametrized by (σ, τ). If the typical size ℓ_s of a string is smaller than the resolution that an accelerator can provide, we cannot see this in the experiment involving elementary particles.

$$\text{(Planck scale) } 10^{-33} \text{ cm} \le \ell_s \le 10^{-17} \text{ cm (TeV scale)}$$

We often use the parameter

$$\alpha' = \ell_s^2$$

which is the only free parameter in string theory. As we will see, the couplings in string theory are expectation values of dynamical fields (so-called moduli) which take their value dynamically.

• Application of the quantization rules provides us with the Fock space of string excitations. In bosonic string, the massless modes include among others

$$\begin{array}{rccc} \text{open string :} & A_\mu & \text{spin 1} & \text{W-boson} \\ \text{closed string :} & g_{\mu\nu} & \text{spin 2} & \text{graviton} \end{array}$$

[1] Of course, there are many other books [Zwi06, Joh06, Din07, IU12, BLT13, Kir19, Tom22] and notes [Ura05, Ton09, Wra11, Hos15, Wei15].

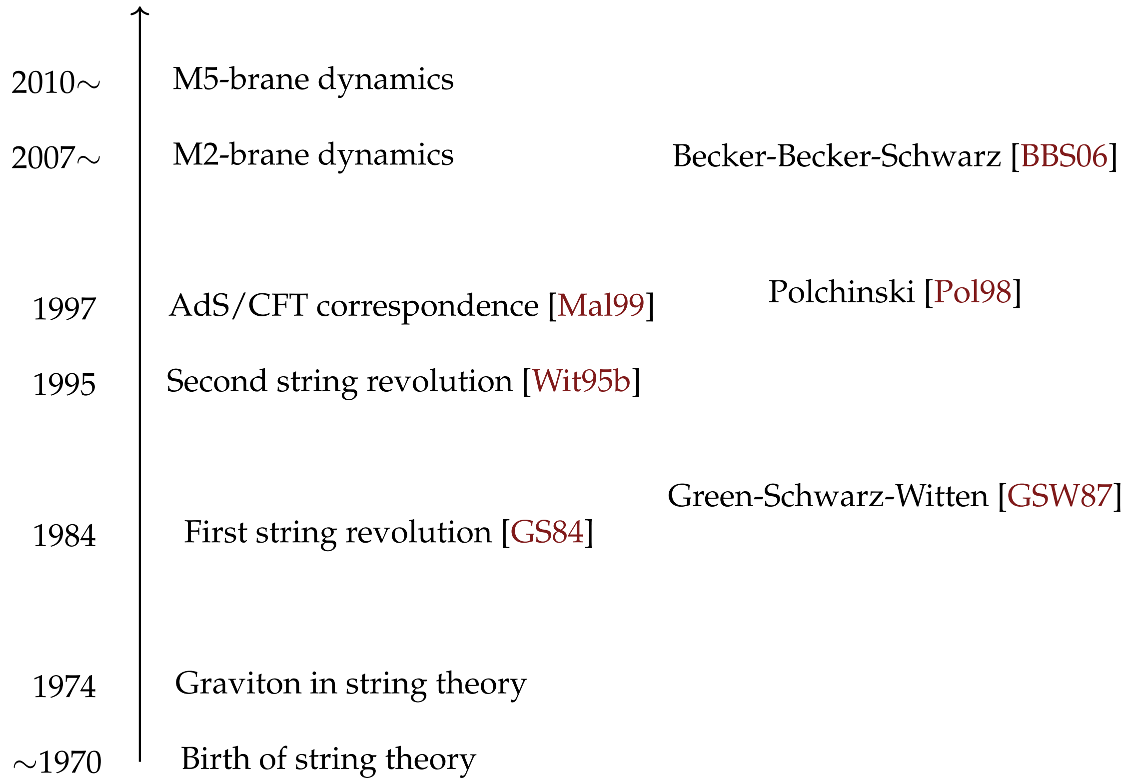


Figure 1: History of string theory and textbooks

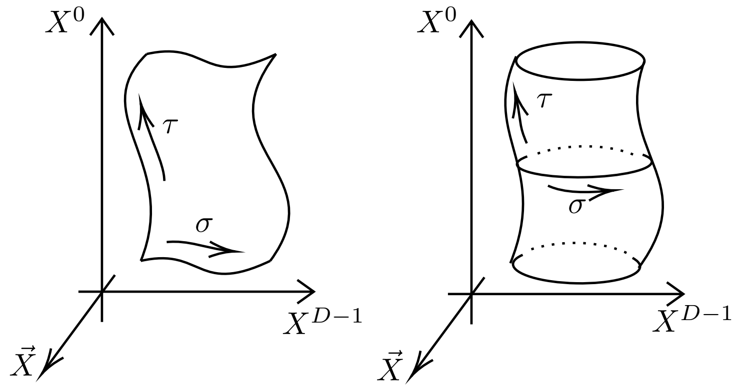
In addition, one finds a tower of massive string excitations of mass

$$\begin{aligned} \text{open bosonic :} \quad & M^2 = \frac{1}{\alpha'}(N-1) \\ \text{closed bosonic :} \quad & M^2 = \frac{4}{\alpha'}(N-1) \quad N = 0, 1, 2, \ldots \end{aligned}$$

Note that the lowest-lying state has a dimension of negative [mass]^2 ($N = 0$), which is called **tachyon**. The bosonic theory is consistent only when the number of spacetime dimensions is 26. We can also refer to [Wit18].

• In the point-particle picture, the divergence appears when all four vertices come close to each other. On the other hand, the string word-sheet has no vertices. Thus, when we sum over all surfaces, we do not encounter configuration analogous to collapsed vertices. String theory amplitudes have no ultraviolet (shot distance) divergence. As a result, string theory provides a finite quantum theory of gravity. Moreover, this is even better than renormalizable quantum field theories since there is no divergence in the first place.

• Strings, besides vibrating in usual spacetime, can also have some internal degrees of freedom. These internal degrees of freedom are fermionic. This can be quantized in a similar manner to that for bosonic strings. Then, a superstring can be considered as a string with symmetry between bosonic states and fermionic states. In superstring theories, the good features (e.g. emergence of gravity and UV finiteness) are preserved, and one can incorporate fermions. Moreover, as mentioned, the tachyonic mode is absent.



1. the theory is consistent in 10 spacetime dimensions.
2. there are five fully consistent string theories in $d = 10$.
   Type IIA, Type IIB, Type I (open+closed string), $E_8 \times E_8$ Heterotic, $\mathrm{SO}(32)$ Heterotic
3. Type II theories can have D-branes that accommodate open strings. Type IIA (resp. IIB) has Dp-branes where p is even (resp. odd), and Type I does D9, D5, D1-branes.
4. Furthermore, all the five apparently different string theories are different limits of the same underlying theory called M-theory
5. In the low energy scale $\ll 1/\ell_s$, a theory is described by supergravity which combines supersymmetry and general relativity.

• How do we reduce the number of dimensions from 10 to 4? The answer is Kaluza-Klein mechanism/compactification. If we take the 10-dimensional space of the form $\mathbb{R}^{1,3} \times M$ where M is a 6-dimensional compact space and its size is much smaller than the resolution of the most powerful accelerator $\sim 10^{-17}$cm, then a (9+1)-dimensional theory looks like (3+1)-dimensional. In fact, in order for a theory to be consistent, M has to be a Calabi-Yau manifold (which is endowed with a Ricci-flat metric). More importantly, the extra dimensions give "room" to derive the complexity of the real world from a simple setting.

1.4 Convention

The world-sheet coordinate indices are denoted by the alphabet letter $a, b, \ldots$, and the target-space coordinate indices are denoted by the Greek letter $\mu, \nu, \ldots$.

The world-sheet metric is denoted by h_{ab}.

spacetime

In anticipation of string theory, we consider D-dimensional Minkowski space $\mathbb{R}^{1,D-1}$. Throughout these notes, we work with signature

$$\eta_{\mu\nu} = \mathrm{diag}(-1, +1, +1, \ldots, +1)$$

World-sheet

We use σ^a for a local coordinate of the world-sheet and we denote the metric by h_{ab} in this local coordinate so that the measure is $\sqrt{h}d^2\sigma$ for a curved world-sheet.

We often consider the case that the world-sheet is topologically a cylinder or a plane. The Minkowski cylindrical coordinates are denoted by (τ, σ) and the Euclidean cylindrical coordinates (t, σ) where the Wick rotation is performed as $\tau = -it$. In the Euclidean coordinate, we introduce the holomorphic complex coordinate $w = it + \sigma$ for the cylinder. The conformal map brings the cylindrical coordinate to the plane coordinate by $z = e^{-iw}$.

- right-moving or holomorphic coordinate: $\sigma^- = \tau - \sigma$ (Minkowski cylinder), $w = it + \sigma$ (Euclidean cylinder), z (Euclidean plane),
- left-moving or anti-holomorphic coordinate: $\sigma^+ = \tau + \sigma$ (Minkowski cylinder), $\overline{w} = -it + \sigma$ (Euclidean cylinder), $\overline{z}$ (Euclidean plane),

In the plane coordinate, the holomorphic coordinates are given by

$$z = \sigma^1 + i\sigma^2 \,, \qquad \overline{z} = \sigma^1 - i\sigma^2$$
$$\partial \equiv \partial_z = \frac{\partial}{\partial z} = \frac{1}{2}(\partial_1 - i\partial_2) \,, \qquad \overline{\partial} \equiv \partial_{\overline{z}} = \frac{\partial}{\partial \overline{z}} = \frac{1}{2}(\partial_1 + i\partial_2) \,,$$

The metric is given by

$$d^2z = dzd\overline{z} = 2d\sigma^1 d\sigma^2 \,,$$

where the factor 2 comes from the Jacobian. One can also write

$$h_{z\overline{z}} = h_{\overline{z}z} = \frac{1}{2} \,, \quad h_{zz} = h_{\overline{z}\overline{z}} = 0 \,, \quad h^{z\overline{z}} = h^{\overline{z}z} = 2 \,, \quad h^{zz} = h^{\overline{z}\overline{z}} = 0 \,.$$

Parameters and constants

- String theory has only one parameter α', and string length and tension are written as
$$\ell_s = \sqrt{\alpha'} \,, \qquad T = \frac{1}{2\pi\alpha'} \tag{1.1}$$
- String coupling is given by an expectation value $g_s = e^{\Phi}$ of the Dilaton field
- Planck length
$$\ell_P = \sqrt{\frac{\hbar G}{c^3}} \,, \qquad \ell_P^3 = g_s \ell_s^3 \tag{1.2}$$
- Coupling constants of $D = 11$ and $D = 10$ supergravities
$$\frac{1}{2\kappa_{11}^2} = \frac{2\pi}{(2\pi\ell_p)^9} \,, \qquad \frac{1}{2\kappa_{10}^2} = \frac{2\pi}{(2\pi\ell_s)^8} \tag{1.3}$$

Acknowledgments and disclaimer

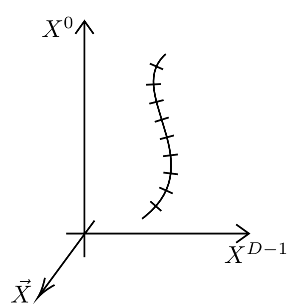
We would like to express gratitude to students at Fudan University who have participated in the lecture series in 2017 and 2021 for providing useful comments on the note. In particular, we are grateful to Jiaqi Guo and Jack Yang for helping us to draw figures. We also thank Y. Tachikawa for informing [CCCDV12] of the historical account. The literature in string theory is too vast to be cited in full, and we are not entitled to accurately acknowledge historical developments of string theory. Therefore, we would like to delegate finding original literature to the reader. As mentioned above, there are already many good books on string theory, and this note covers only a small part of string theory. Nevertheless, we would be delighted if this note becomes a concise introduction to the huge subject for students.

2 Bosonic string theory

The fist few sections are devoted to study bosonic string theory. For bosonic string theory, there is a concise and wonderful lecture note [Ton09] by D. Tong.

2.1 String sigma-model action

Nambu-Goto action



To begin with, let us recall the case of a relativistic point particle

$$\gamma : \tau \to x^\mu(\tau) \in \mathbb{R}^{1,D-1} \ .$$

The action of a relativistic point particle with mass m is

$$S = m \int_a^b ds = m \int_a^b \Big(-\eta_{\mu\nu} \frac{dX^\mu}{d\tau} \frac{dX^\nu}{d\tau} \Big)^{\frac{1}{2}} d\tau \ .$$

In a similar fashion, we consider a string moving in the target spacetime $\mathbb{R}^{1,D-1}$

$$\Sigma \ni (\tau, \sigma) \to X^\mu(\tau, \sigma) \in \mathbb{R}^{1,D-1} \ .$$

The analogous action, called the **Nambu-Goto action**, for the string is given by the area of a string world-sheet Σ swept by a string

$$S_{\rm NG} = -T \int_\Sigma \text{Area} = T \int_\Sigma d^2\sigma \sqrt{-\det(\gamma_{ab})} \ , \qquad \gamma_{ab} = \eta_{\mu\nu} \partial_a X^\mu \partial_b X^\nu \tag{2.1}$$

where $(\sigma^0, \sigma^1) = (\tau, \sigma)$, and $T := 1/(2\pi\alpha')$ is the **string tension**. The tension is usually defined as mass per unit length. However, we do not know how to quantize the Nambu-Goto action due to the square root. Instead, for quantization of string world-sheet, we reformulate the action as follows.

String sigma-model action

Alternatively, we remove the square root and consider the following action

$$S_\sigma = -\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-\det h}\, h^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} \tag{2.2}$$

which is called the **string sigma-model action**.[2] Classically, the string sigma-model action is equivalent to the Nambu-Goto action.

Symmetry of S_σ

1. **D-dim spacetime Poincaré invariance**
   The action is certainly invariant under the Poincaré transformation of the target space $\mathbb{R}^{1,D-1}$
   $$X^\mu(\sigma) \to \Lambda^\mu{}_\nu X^\nu(\sigma) + V^\mu \qquad \Lambda \in \mathrm{O}(1, D-1)$$
   where Λ is a Lorentz transformation and V is the translation.
2. **World-sheet diffeomorphism invariance**
   The string sigma-model is also invariant under a diffeomorphism $f : \widetilde{\Sigma} \to \Sigma$ of world-sheet two-dimensional manifolds. Namely, the action stays the same for (Σ, h) and $(\widetilde{\Sigma}, \widetilde{h} = f^* h)$. In physics, it is most convenient to use local coordinates to express this invariance. Namely, under a local expression of the diffeomorphism
   $$\begin{aligned} \widetilde{X}^\mu(\widetilde{\sigma}) &= X^\mu(\sigma) \quad \text{with} \quad \widetilde{\sigma} = \widetilde{\sigma}(\sigma)\,, \\ \widetilde{h}^{ab} &= h^{cd} \frac{\partial \widetilde{\sigma}^a}{\partial \sigma^c} \frac{\partial \widetilde{\sigma}^b}{\partial \sigma^d}\,, \end{aligned}$$
   the action is invariant $S_\sigma(X^\mu, h_{ab}) = S_\sigma(\widetilde{X}^\mu, \widetilde{h}_{ab})$.
3. **Weyl invariance of world-sheet metrics**
   A **Weyl transformation** $h \to e^{2\omega} h$ is a local scaling of the metric of a Riemannian manifold (Σ, h) where 2ω is an arbitrary function of Σ. The action is indeed invariant under a Weyl transformation $h_{ab} \to e^{2\omega(\sigma)} h_{ab}$ of the string world-sheet Σ.

In fact, Diff $\times$ Weyl are gauge symmetries of the theory.

Energy-momentum tensor

The energy-momentum tensor is defined by

$$T_{ab} \equiv -\frac{4\pi}{\sqrt{-h}} \frac{\delta S}{\delta h^{ab}}$$

[2] This action is called the Polyakov action in [Pol98]. However, according to [BBS06], it was discovered by Brink, Di Vecchia and Howe and by Deser and Zumino several years before Polyakov skillfully used it for path-integral quantization of the string. Therefore, let us follow the notation of the progenitor, J.H. Schwarz.

$$= -\frac{1}{\alpha'}\left[\partial_a X^\mu \partial_b X_\mu - \frac{1}{2} h_{ab}\, \partial_c X^\mu \partial^c X_\mu\right] \tag{2.3}$$

which is symmetric and subject to

$$\begin{aligned} \text{traceless}: &\qquad T^a{}_a = 0 \\ \text{conservation}: &\qquad \nabla_a T^{ab} = 0 \end{aligned} \tag{2.4}$$

Note that the traceless condition indeed follows from the Weyl invariance (exercise), and the conservation can be shown by using the equation of motion for X^μ in the following.

$$\begin{aligned} h^{ab}\frac{\delta S}{\delta h^{ab}} &= 0 \;\to\; T^a{}_a = 0 \\ \frac{\delta S}{\delta X^\mu} &= 0 \;\to\; \Delta X_\mu = -\frac{1}{\sqrt{-h}}\partial_a(\sqrt{-h}h^{ab}\partial_b X^\mu) = 0\,. \end{aligned} \tag{2.5}$$

The equation of motion with respect to the metric will be discussed in (2.11).

Gauge fixing

The string sigma-model action S_σ is equipped with the symmetries above, and we need to fix it for quantization. Physics does not depend on a choice of gauge fixing, but if we choose a clever gauge fixing, our life becomes much easier. As in [GSW87, Pol98, BBS06], we choose the light-cone gauge in the following.

The metric h_{ab} is a 2×2 symmetric matrix and the world-sheet diffeomorphism invariance tells us that there is a coordinate σ^a such that the metric is a diagonal $h_{ab} = e^\omega \eta_{ab}$. Then, the Weyl transformation brings it to the 2d Minkowski metric η_{ab}. Therefore, one can use $h_{ab} = \eta_{ab}$. If we use the light-cone coordinates on the world-sheet

$$\sigma^\pm = \tau \pm \sigma\,,$$

then the metric is written as

$$ds^2 = -d\sigma^+ d\sigma^-\,.$$

However, there are residual transformations that leave the 2d Minkowski metric η_{ab} invariant, which is called the **conformal symmetry**:

$$\sigma^+ \to f(\sigma^+)\,, \qquad \sigma^- \to g(\sigma^-)$$

with a Weyl transformation simultaneously. To fix the conformal symmetry, we introduce the spacetime light-cone coordinates,

$$X^\pm = \sqrt{\frac{1}{2}}(X^0 \pm X^{D-1})\,.$$

Then we impose

$$X^+ = x^+ + \alpha' p^+ \tau\,, \tag{2.6}$$

which is called the **light-cone gauge**.

Mode Expansions

After the gauge fixing, the equations of motion simply read

$$\partial_+\partial_- X^\mu = 0 \quad \text{where} \quad \partial_\pm = \frac{1}{2}\left(\partial_\tau \pm \partial_\sigma\right) .$$

The most general solution is a factorization of left- and right-mover

$$X^\mu(\sigma,\tau) = X_L^\mu(\sigma^+) + X_R^\mu(\sigma^-) \tag{2.7}$$

for arbitrary functions X_L^μ and X_R^μ. For a closed string, we impose a periodic condition as

$$X^\mu(\sigma,\tau) = X^\mu(\sigma+2\pi,\tau) . \tag{2.8}$$

so that the left- and right-moving fields admit the Fourier expansions

$$X_L^\mu\left(\sigma^+\right) = \frac{1}{2}x^\mu + \frac{\alpha'}{2}p^\mu\sigma^+ + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\overline{\alpha}_n^\mu e^{-in\sigma^+} , \tag{2.9}$$

$$X_R^\mu\left(\sigma^-\right) = \frac{1}{2}x^\mu + \frac{\alpha'}{2}p^\mu\sigma^- + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in\sigma^-} , \tag{2.10}$$

where the reality of X^μ requires that the coefficients of the Fourier modes obey

$$\alpha_n^\mu = \left(\alpha_{-n}^\mu\right)^* , \qquad \overline{\alpha}_n^\mu = \left(\overline{\alpha}_{-n}^\mu\right)^* . .$$

This mode expansion will be very important when we come to quantum theory. If we treat the world-sheet metric dynamically, we impose the equation of motion

$$T_{--} = -\frac{1}{\alpha'}\partial_- X^\mu \partial_- X_\mu = 0 , \qquad T_{++} = -\frac{1}{\alpha'}\partial_+ X^\mu \partial_+ X_\mu = 0 . \tag{2.11}$$

We will see the implication of these constraints in quantum theory.

2.2 Quantizations

The momentum conjugate to X^μ is defined in this gauge

$$\Pi_\mu = \frac{\delta S}{\delta\left(\partial_\tau X^\mu\right)} = \frac{1}{2\pi\alpha'}\partial_\tau X_\mu .$$

The canonical quantization promotes X^μ and Π_μ to operators that is subject to

$$\begin{aligned} &\left[X^\mu(\sigma,\tau), \Pi_\nu\left(\sigma',\tau\right)\right] = i\delta\left(\sigma-\sigma'\right)\delta_\nu^\mu \\ &\left[X^\mu(\sigma,\tau), X^\nu\left(\sigma',\tau\right)\right] = \left[\Pi_\mu(\sigma,\tau), \Pi_\nu\left(\sigma',\tau\right)\right] = 0 . \end{aligned} \tag{2.12}$$

We translate these into commutation relations for the Fourier modes x^μ, p^μ, α_n^μ and $\overline{\alpha}_n^\mu$. Using the mode expansion, we find (exercise)

$$\left[x^\mu, p_\nu\right] = i\delta_\nu^\mu \quad \text{and} \quad \left[\alpha_n^\mu, \alpha_m^\nu\right] = \left[\overline{\alpha}_n^\mu, \overline{\alpha}_m^\nu\right] = n\eta^{\mu\nu}\delta_{n+m,0} , \tag{2.13}$$

Therefore, like harmonic oscillators, the creation operators are $\alpha_{-n}^\mu, \overline{\alpha}_{-n}^\mu$ and the annihilation operators are $\alpha_n^\mu, \overline{\alpha}_n^\mu$ for $n \in \mathbb{N}$ so that the Hilbert space is spanned by

$$\alpha_{-n_1}^{\mu_1}\cdots\alpha_{-n_k}^{\mu_k}\overline{\alpha}_{-n_1}^{\mu_1}\cdots\overline{\alpha}_{-n_k}^{\mu_k}|0;k\rangle \qquad \text{where} \qquad p^\mu|0;k\rangle = k^\mu|0;k\rangle . \tag{2.14}$$

Let us consider the implication of the constraints (2.11). Using the mode expansions (2.9), the energy-momentum tensor can be expressed as

$$T_{--} = -\sum_n L_n^X e^{-in\sigma^-} \qquad T_{++} = -\sum_n \overline{L}_n^X e^{-in\sigma^+} ,$$

where

$$L_m^X = \frac{1}{2}\sum_{n\in\mathbb{Z}} \eta_{\mu\nu} : \alpha^\mu_{m-n}\alpha^\nu_n : \quad \overline{L}_m^X = \frac{1}{2}\sum_{n\in\mathbb{Z}} \eta_{\mu\nu} : \overline{\alpha}^\mu_{m-n}\overline{\alpha}^\nu_n : \tag{2.15}$$

with $\alpha_0^\mu = \overline{\alpha}_0^\mu = \sqrt{\frac{\alpha'}{2}}p^\mu$. We will learn that L_m are generators of **Virasoro algebra**. According to the normal-ordering : : prescription, the lowering operators always appear to the right of the raising operators. However, it is easy to see that the normal ordering matters only in L_0^X and $\overline{L}_0^X$. Thus, in quantum theory, the constraint (2.11) on the energy-momentum tensor can be interpreted by

$$\left(L_n^X + A\delta_{n,0}\right)|\text{phys}\rangle = 0 \qquad \left(\overline{L}_n^X + \overline{A}\delta_{n,0}\right)|\text{phys}\rangle = 0 \qquad \text{for} \quad n>0$$

where A, $\overline{A}$ are normal ordering constants so that

$$\langle \text{phys}'|L_n^X|\text{phys}\rangle = 0 \qquad \langle \text{phys}'|\overline{L}_n^X|\text{phys}\rangle = 0 \qquad \text{for} \quad n>0\,.$$

Another way to think of this constraint is as follows. A physical amplitude will not depend on the choice of gauge $h_{ab}(\sigma) + \delta h_{ab}$, i.e.

$$\delta\langle f \mid i\rangle = -\frac{1}{4\pi}\int d^2\sigma h(\sigma)^{1/2}\delta h_{ab}(\sigma)\left\langle f\left|T^{ab}(\sigma)\right|i\right\rangle = 0.$$

Light-cone quantization

To see the effect of normal ordering in L_0, let us consider the meaning of the light-cone gauge (2.6) in quantum theory. It is easy to see that (2.6) implies

$$\alpha_n^+ = 0 \quad \text{for} \quad n \neq 0\,.$$

Then, the equation of motion

$$2\partial_+X^-\partial_+X^+ = \sum_{i=1}^{D-2}\partial_+X^i\partial_+X^i$$

tells us

$$\alpha_n^- = \sqrt{\frac{1}{2\alpha'}}\frac{1}{p^+}\sum_{m=-\infty}^{\infty}\sum_{i=1}^{D-2}\alpha^i_{n-m}\alpha^i_m\,.$$

In particular, for $n=0$, we have

$$M^2 = 2p^+p^- - \sum_{i=1}^{D-2}p^ip^i = \frac{4}{\alpha'}\left(\sum_{n>0}\sum_{i=1}^{D-2}\alpha^i_{-n}\alpha^i_n + \frac{D-2}{2}\sum_{n>0}n\right)\,.$$

The final term clearly diverges. Fortunately, we have nice regularization of this divergence

$$\begin{aligned}\sum_{n=1}^{\infty} n \longrightarrow \sum_{n=1}^{\infty} n e^{-\epsilon n} &= -\frac{\partial}{\partial \epsilon} \sum_{n=1}^{\infty} e^{-\epsilon n} \\ &= -\frac{\partial}{\partial \epsilon} \left(1 - e^{-\epsilon}\right)^{-1} \\ &= \frac{1}{\epsilon^2} - \frac{1}{12} + \mathcal{O}(\epsilon) \ .\end{aligned} \tag{2.16}$$

The first term diverges as $\epsilon \to 0$ so that we renormalize this term away. Consequently, we obtain

$$\sum_{n=1}^{\infty} n = -\frac{1}{12} \ . \tag{2.17}$$

Hence, we obtain

$$M^2 = \frac{4}{\alpha'}\left(N - \frac{D-2}{24}\right) = \frac{4}{\alpha'}\left(\overline{N} - \frac{D-2}{24}\right) \ , \tag{2.18}$$

where the number operators are defined by

$$N = \sum_{i=1}^{D-2} \sum_{n>0} \alpha^i_{-n} \alpha^i_n , \quad \overline{N} = \sum_{i=1}^{D-2} \sum_{n>0} \overline{\alpha}^i_{-n} \overline{\alpha}^i_n \ . \tag{2.19}$$

It is easy to see from (2.18)

$$N = \overline{N} \tag{2.20}$$

which is called the **level-matching condition**. Moreover, if $D > 2$, we have the state $|0;k\rangle$ with negative [mass]^2

$$M^2 = -\frac{4}{\alpha'} \tag{2.21}$$

for $N = 0 = \overline{N}$ which is called the **tachyon**.

The First Excited States

Let us look at the first excited states. The level-matching condition (2.20) requires us to act both right- α^j_{-1} and left-moving $\overline{\alpha}^i_{-1}$ creation operator on $|0;k\rangle$ simultaneously. Thus, there are $(D-2)^2$ states at $N = 1 = \overline{N}$

$$\alpha^i_{-1} \overline{\alpha}^j_{-1} \left|0;k\right\rangle \ , \tag{2.22}$$

whose mass

$$M^2 = \frac{4}{\alpha'}\left(1 - \frac{D-2}{24}\right) \ .$$

It is easy to see that the state is under a representation of the little group $\mathrm{SO}(D-2) \subset \mathrm{SO}(1, D-1)$ and therefore it should be massless. Interestingly, this is only the case if the dimension of spacetime is

$$D = 26 \ . \tag{2.23}$$

This is the critical dimension of the bosonic string. In the following sections, we will see from many different viewpoints that bosonic string theory is consistent only in this critical dimension.

Then, the states (2.22) transform in the $\mathbf{24}\otimes\mathbf{24}$ representation of SO(24), which decomposes into three irreducible representations:

$$\begin{array}{lll} \text{Graviton } G^{\mu\nu} & \alpha^\mu_{-1}\overline{\alpha}^\nu_{-1}|0;k\rangle & (\text{symmetric in } \mu \text{ and } \nu, \text{ and traceless}) \ , \\ B\text{-field } B^{\mu\nu} & \alpha^\mu_{-1}\overline{\alpha}^\nu_{-1}|0;k\rangle & (\text{anti-symmetric in } \mu \text{ and } \nu) \ , \\ \text{Dilaton } \Phi & \alpha^\mu_{-1}\overline{\alpha}_{-1,\mu}|0;k\rangle & (\text{trace part}) \ . \end{array} \tag{2.24}$$

Each irreducible representation gives rise to a massless field in spacetime. As above, the traceless symmetric, anti-symmetric, and the trace part correspond to the graviton $G_{\mu\nu}$, (Kalb-Ramond) B-field $B_{\mu\nu}$, and the dilaton Φ, respectively. Therefore, graviton arises naturally from the quantization of closed strings! These massless fields play a pivotal role throughout the lecture notes.

Open strings

So far we have studied closed strings. Let us briefly summarize the light-cone quantization of an open string and the derivations of the results are left as an exercise to the reader. For an open string, a world-sheet spatial coordinate spans $\sigma \in [0,\pi]$ where the boundaries of the world-sheet are at $\sigma = 0,\pi$, and the action is given by

$$S = \frac{1}{4\pi\alpha'}\int_{\sigma=0}^{\sigma=\pi} d\tau d\sigma \left(\partial_\tau X\cdot\partial_\tau X + \partial_\sigma X\cdot\partial_\sigma X\right) \ .$$

Like the close string, the mode expansion of $X^\mu = X^\mu_L(\sigma^+) + X^\mu_R(\sigma^-)$ is given by

$$\begin{aligned} X^\mu_L(\sigma^+) &= \frac{1}{2}x^\mu + \alpha' p^\mu\,\sigma^+ + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\,\overline{\alpha}^\mu_n\, e^{-in\sigma^+} \ , \\ X^\mu_R(\sigma^-) &= \frac{1}{2}x^\mu + \alpha' p^\mu\,\sigma^- + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\,\alpha^\mu_n\, e^{-in\sigma^-} \ . \end{aligned} \tag{2.25}$$

where the momentum of an open string is defined as $\alpha^\mu_0 = \sqrt{2\alpha'}\,p^\mu$. Note that the second term differs from the closed string (2.9) by a factor of two since the spatial lengths are $l_{\text{open}} = \pi$ and $l_{\text{closed}} = 2\pi$.

The variation principle of the action leads to

$$\begin{aligned} 0 = \delta S &= \frac{1}{2\pi\alpha'}\int_{\sigma=0}^{\sigma=\pi} d\tau d\sigma \left(-\partial_\tau X\cdot\partial_\tau\delta X + \partial_\sigma X\cdot\partial_\sigma\delta X\right) \\ &= \frac{1}{2\pi\alpha'}\int_{\sigma=0}^{\sigma=\pi} d\tau d\sigma \left[-\delta X\cdot(-\partial_\tau\partial_\tau X + \partial_\sigma\partial_\sigma X) - \partial_\tau\left(\partial_\tau X\cdot\delta X\right) + \partial_\sigma\left(\partial_\sigma X\cdot\delta X\right)\right] \\ &= \frac{1}{2\pi\alpha'}\int d\tau \left[\partial_\sigma X\cdot\delta X\right]^{\sigma=\pi}_{\sigma=0} \ , \end{aligned} \tag{2.26}$$

where we use the equation of motion and $\delta X(t=\pm\infty) = 0$ at the last equality. This compels us to impose one of the following boundary conditions:

- Neumann boundary condition: $\partial_\sigma X^\mu = 0 \quad$ at $\sigma = 0,\pi$
- Dirichlet boundary condition: $X^\mu = c^\mu$ (constant) at $\sigma = 0,\pi$

which imposes the following conditions on the modes

- Neumann boundary condition: $\alpha^\mu_n = \overline{\alpha}^\mu_n$.

- Dirichlet boundary condition: $x^\mu = c^\mu, \quad p^\mu = 0, \quad \alpha_n^\mu = -\overline{\alpha}_n^\mu$.

Moreover, the light-cone gauge quantization for (2.25) leads to the open string mass spectrum

$$M^2 = 2p^+p^- - \sum_{i=1}^{D-2} p^i p^i = \frac{1}{\alpha'}\left(\sum_{n>0}\sum_{i=1}^{D-2} \alpha_{-n}^i \alpha_n^i + \frac{D-2}{2}\sum_{n>0} n\right) ,$$

where there is the difference in the normalization p^2 between closed and open string: $4p^2_{\text{open}} = p^2_{\text{closed}}$. This results in a difference by a factor of four from (2.18). Again, in $D = 26$, the open string mass spectrum becomes

$$M^2 = \frac{1}{\alpha'}(N-1) \,. \tag{2.27}$$

so that there is the tachyon vacuum at the level $N = 0$. The massless states

$$\alpha_{-1}^i|0;k\rangle \quad i = 1,\ldots,D-2 \tag{2.28}$$

at the level $N = 1$ correspond to vector (gauge) gauge bosons.

2.3 Path-integral formulations

We have obtained the bosonic string spectra, and to illustrate the interaction among string states, we will introduce **path integral**. (For more detail, see [Pol98, Appendix.A].) Figure 2 depicts an extension from quantum field theoretic (QFT) interaction to string interaction. Each end of the cylinders in the Figure corresponds to an initial/final state

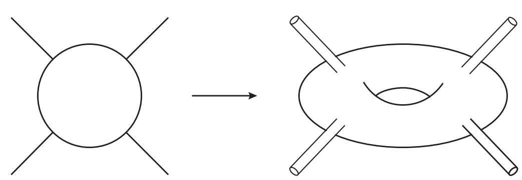


Figure 2: A Feynman diagram of QFT (left) and a string interaction diagram (right). Each end of the cylinders emanating from the torus (or a Riemann surface in general) in the string interaction diagram corresponds to an initial/final state.

of a string. The action (2.2) of the string sigma model is invariant under the world-sheet diffeomorphisms, and it is a conformal field theory as a result. As we will see in the next section §3, one can make use of the **state-operator correspondence** in a 2d CFT. Hence, instead of using the Hamiltonian formalism of string states, we will use vertex operators $\widehat{V}_i$ in the path integral formalism:

$$A_n = \sum_g \int (\mathcal{D}h_{ab})_{g,n} \int \mathcal{D}X^\mu e^{-S_\sigma[X^\mu,h_{ab}]}\widehat{V}_1\cdots\widehat{V}_n \,,$$

where n is the number of in-coming/out-going strings, and g is the number of holes(genus) of the Riemann surface (genus is 1 in Figure 3). Here we integrate over all the metric h_{ab} and field X^μ configuration up to gauge transformations, and we also sum over all the topology (genus g) of world-sheets.

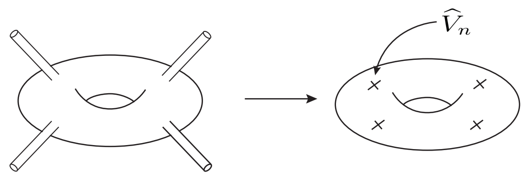


Figure 3: String interaction in state expression (left) and operator expression (right).

In the path integral method, expectation values are schematically expressed in the following form

$$\langle \mathcal{O}[X] \rangle = \int \mathcal{D}X e^{iS[X]} \mathcal{O}[X] \,, \qquad S = \frac{1}{4\pi\alpha'} \int d\sigma d\tau \left\{ (\partial_\tau X)^2 - (\partial_\sigma X)^2 \right\} \,,$$

where $\mathcal{O}$ is a gauge-invariant operator.

We usually **Wick rotate** ($\tau \to -it$) the theory so that it converges.

$$\langle \mathcal{O}[X] \rangle = \int \mathcal{D}X e^{-S_E[X]} \mathcal{O}[X] \,, \qquad S_E = \frac{1}{4\pi\alpha'} \int d\sigma dt \left\{ (\partial_t X)^2 + (\partial_\sigma X)^2 \right\} \,.$$

The subscript E will be omitted hereafter, and we will always work in the world-sheet Euclidean signature.

3 Two-dimensional conformal field theory

Conformal field theories (CFTs) play a distinctive role in quantum field theories, string theory, statistical mechanics, and condensed matter physics. Moreover, 2d CFTs are particularly rich because of infinite-dimensional symmetry [BPZ84]. We will learn operator analysis of 2d conformal field theory (CFT), including OPE, Ward-Takahashi identity, etc. Moreover, we will see Virasoro algebra, which is associated to conformal symmetry. However, we glimpse only the tip of the iceberg, and the subject would actually deserve the entire semester. If the reader is interested in this fertile subject, we refer to the standard references [Gin88, FMS12, BP09].

3.1 Conformal transformations

A CFT in any dimension is a quantum field theory invariant under conformal maps at quantum level. A conformal map between two Riemannian manifolds (Σ, h) and $(\widetilde{\Sigma}, h')$ is a diffeomorphism $f : \Sigma \to \widetilde{\Sigma}$ such that the pull-back metric f^*h' and the original metric h are related by an arbitrary function $e^{2\omega}$ of Σ

$$f^*h' = e^{2\omega} h \,.$$

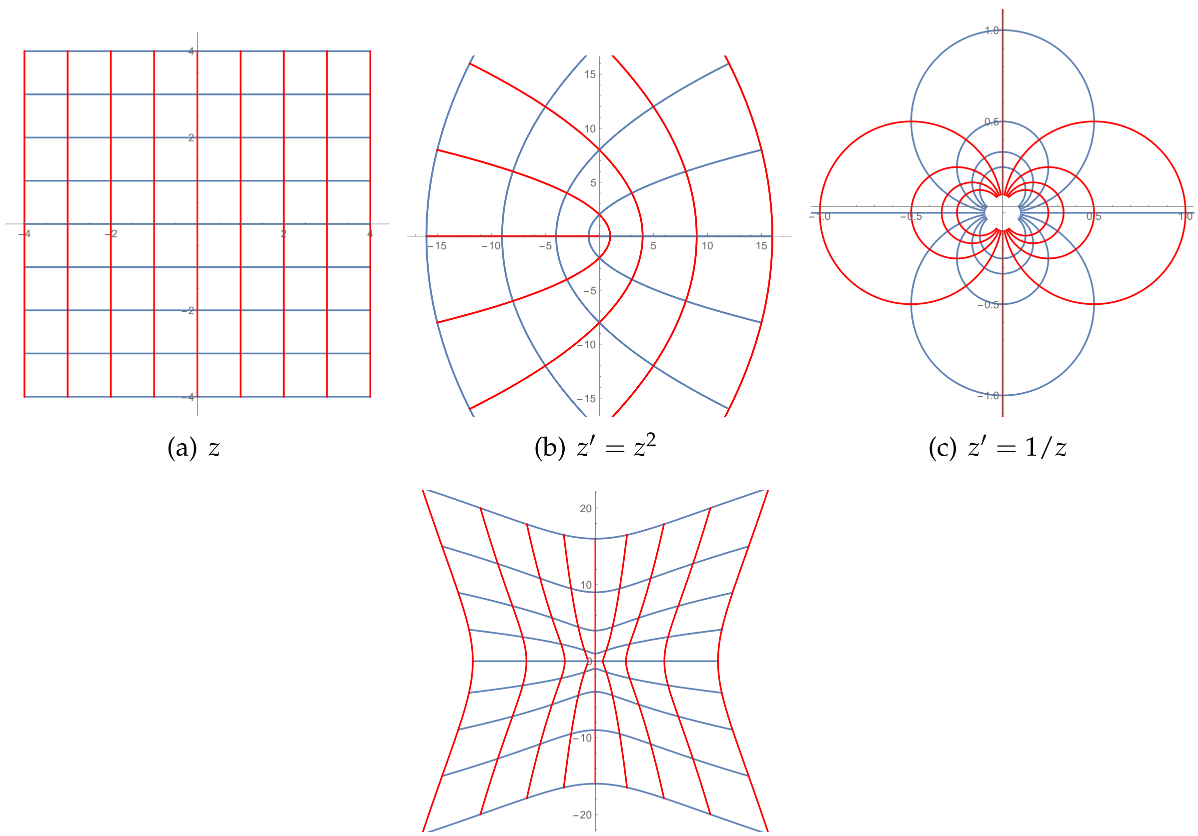
In terms of local coordinates (σ, h_{ab}) and $(\widetilde{\sigma}, h'_{ab})$, a **conformal transformation** $\sigma^a \to \widetilde{\sigma}^a(\sigma)$ relates the metrics by

$$\frac{\partial \widetilde{\sigma}^c}{\partial \sigma^a} \frac{\partial \widetilde{\sigma}^d}{\partial \sigma^b} h'_{cd}(\widetilde{\sigma}(\sigma)) = e^{2\omega(\sigma)} h_{ab}(\sigma) \,. \tag{3.1}$$

Roughly speaking, a conformal transformation is a coordinate transformation that leaves the metric invariant up to scale and thus preserves angles. This means that the theory behaves the same at all length scales.

A conformal transformation is a diffeomorphism under which the pull-back metric is related to the original metric by a Weyl transformation. Since the string sigma model (2.2) is invariant under diffeomorphisms and Weyl transformations of the world-sheet, it is conformal-invariant. Hence, we will study conformal field theory of the string world-sheet.

Note that a conformal transformation is a diffeomorphism between two Riemannian manifolds, and a Weyl transformation changes the metric keeping a manifold fixed. (Keep in mind the difference!) Therefore, if the metrics (Σ, h) and $(\Sigma, e^{2\omega}h)$ are related by a Weyl transformation, then the identity map of Σ is a conformal transformation. [3]



(a) z (b) $z' = z^2$ (c) $z' = 1/z$

(d) $z' = z|z|$

Figure 4: Coordinate transformation: The transformation from square lattice (a), onto the lattice in (b) and (c) is conformal, while the transformation onto the lattice in (d) is not.

[3]Let us clarify two terminologies at this moment. A conformal transformation is a diffeomorphism under which the metric is scaled but the infinitesimal distance ds^2 is fixed. While in the previous section, the conformal symmetry appears as a residual symmetry after gauge fixing. This is a conformal transformation along with a Weyl transformation that fixes the metric but scales the infinitesimal distance ds^2. Now the Weyl transformation is only used to undone the scaling factor due to the conformal transformation, and thus does not introduce extra degrees of freedom. These two concepts are just different ways that describe the same symmetry of our theory.

2d flat space

In complex coordinates, the 2d flat metric can be written as $ds^2 = dzd\bar{z}$. Under a holomorphic map,

$$z \to f(z) ,$$

the metric is transformed as

$$ds^2 = dzd\bar{z} \quad \to \quad ds^2 = \frac{\partial z}{\partial f}\frac{\partial \bar{z}}{\partial \bar{f}} df d\bar{f} .$$

Therefore, all the holomorphic maps are conformal transformations where $\left|\frac{\partial z}{\partial f}\right|^2$ is a conformal factor (corresponding to $e^{2\omega(\sigma)}$ in (3.1)). Moreover, it is easy to show that all 2d conformal transformations are indeed holomorphic functions (Exercise). This set is infinite-dimensional, corresponding to the coefficients of the Laurent series of holomorphic functions. Due to the infinite dimensionality, the conformal symmetry becomes so powerful in two dimensions.

3.2 State-operator correspondence

As we have seen, the action (2.2) of the string sigma model is invariant under diffeomorphisms. Therefore, it is automatically invariant under conformal transformations. We can use this freedom to set the world-sheet to be $\mathbb{R}^2$ (recall that the topology of a propagating string is a cylinder).

$$\begin{aligned}
\mathbb{R}^2 : \quad & ds^2 = d\sigma^1 d\sigma^1 + d\sigma^2 d\sigma^2 = dr^2 + r^2 d\theta^2 \\
& = r^2 \left[d\left(\log r\right)^2 + d\theta^2 \right] , \\
\text{Cylinder} : \quad & ds^2 = dt^2 + d\sigma^2 ,
\end{aligned}$$

where the overall factor r^2 in $\mathbb{R}^2$ is identified with the conformal factor $e^{2\omega(\sigma)}$. So we can identify

$$t = \log r , \qquad \sigma = -\theta ,$$

and the reason for the minus sign will be clear later. This identification tells us that there is a correspondence between

- CFT on a cylinder and
- CFT on $\mathbb{C}^\times = \mathbb{C}\backslash\{0\}$,

provided that we choose the boundary conditions to be the same. In particular, the initial **state** of the string ($t = -\infty$) corresponds to a **local operator** inserted at the origin, which is called vertex operator (Figure 5). This is the state-operator correspondence in a CFT, which plays a crucial role. We can express string states by the vertex operators (Figure 3).

Now the reader may wonder if there is any way to express commutation relations in terms of operators so that we can do parallel procedures of the canonical quantization in terms of operators.

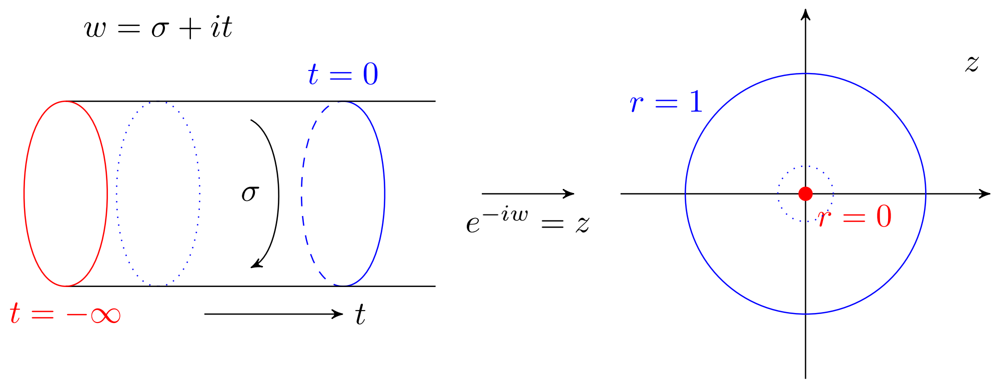


Figure 5: State-operator correspondence. The initial state (red circle) is mapped to a local operator (red dot) at the origin.

3.3 Operator product expansions in free scalar theory

We study massless free scalar fields as the easiest example. Here we adopt the Euclidean metric $\delta_{ab} = \text{diag}(+,+)$ though the procedure is parallel for the Lorentz metric. We write the action in complex coordinates

$$S_X = \frac{1}{2\pi\alpha'} \int d^2z\, \partial X^\mu \bar{\partial} X_\mu \,. \tag{3.2}$$

Then, the equation of motion is given by $\partial\bar{\partial}X^\mu = 0$, which implies that X is holomorphically factorized

$$\begin{aligned} X^\mu(z,\bar{z}) &= X^\mu(z) + \overline{X}^\mu(\bar{z}), \\ X^\mu(z) &= \frac{1}{2}x^\mu - i\frac{\alpha'}{2}p^\mu \log z + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n}\frac{\alpha_n^\mu}{z^n}\,, \\ \overline{X}^\mu(\bar{z}) &= \frac{1}{2}x^\mu - i\frac{\alpha'}{2}p^\mu \log \bar{z} + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n}\frac{\bar{\alpha}_n^\mu}{\bar{z}^n}\,, \end{aligned} \tag{3.3}$$

where

$$z = re^{i\theta} = e^{\log r + i\theta} = e^{i\tau - i\sigma} = e^{i\sigma^-}\,, \qquad \bar{z} = e^{i\sigma^+}\,.$$

This is the Euclidean version of (2.9). The (anti-)holomorphic part corresponds to the right (left) movers.

Let us consider a quantum version of the equation of motion by using a path integral formulation. Like a normal integral, we assume that the "total derivative" vanishes in the path integral. For example, in the massless free scalar case, we have

$$0 = \int \mathcal{D}X \frac{\delta}{\delta X} e^{-S[X]} = \int \mathcal{D}X\, e^{-S[X]} \frac{1}{\pi\alpha'} \partial\bar{\partial}X = \frac{1}{\pi\alpha'} \langle \partial\bar{\partial}X \rangle\,,$$

which is Ehrenfest's theorem. The equation of motion is satisfied as an operator equation if there is no other operator nearby.

Furthermore, the same procedure can be done with operator insertion:

$$0 = \int \mathcal{D}X \, \frac{\delta}{\delta X_\mu(z,\bar{z})} \left(e^{-S[X]} X^\nu(w,\overline{w}) \right) = \left\langle \frac{1}{\pi\alpha'} \partial_z \partial_{\bar{z}} X^\mu(z,\bar{z}) X^\nu(w,\overline{w}) + \eta^{\mu\nu} \delta^2(z-w) \right\rangle ,$$

yielding

$$\left\langle \partial\bar{\partial} X^\mu(z,\bar{z}) X^\nu(w,\overline{w}) \right\rangle = -\pi\alpha' \eta^{\mu\nu} \delta^2(z-w) \, . \tag{3.4}$$

Now we can use the **Stokes' theorem**

$$\int_D d^2z \left(\partial\overline{V} + \bar{\partial} V \right) = \frac{1}{i} \oint_{\partial D} \left(V dz - \overline{V} d\bar{z} \right) ,$$

where D is an arbitrary domain (typically a disk) and ∂D is its boundary. Using Stokes' theorem, we obtain

$$\partial\bar{\partial} \log |z|^2 = 2\pi \delta^2(z) \, . \tag{3.5}$$

This allows us to rewrite (3.4) into

$$X^\mu(z,\bar{z}) X^\nu(w,\overline{w}) = -\frac{\alpha'}{2} \eta^{\mu\nu} \log |z-w|^2 + :X^\mu(z,\bar{z}) X^\nu(w,\overline{w}): ,$$

where we introduced a normal ordering $:\mathcal{O}:$. This is an operator equation, and we simply omit $\langle \quad \rangle$ symbols. This expression describes the singular behavior when $X^\mu(z,\bar{z})$ approaches to $X^\nu(w,\overline{w})$.

We also write it as follows:

$$:X^\mu(z,\bar{z}) X^\nu(w,\overline{w}):= X^\mu(z,\bar{z}) X^\nu(w,\overline{w}) - \eta^{\mu\nu} G(z,w) ,$$

where $G(z,w) = -\frac{\alpha'}{2} \log |z-w|^2$. Notice that

$$\partial\bar{\partial} :X(z,\bar{z}) X(w,\overline{w}):= 0 \, . \tag{3.6}$$

As we will see later, the divergent term is important and meaningful. This is the reason why it is convenient to introduce the normal ordering so that we can separate divergent terms from the other.

The normal ordering of an arbitrary functional of operators in the free scalar theory can be expressed as follows:

$$:f[X]:= \exp \left[-\frac{1}{2} \int d^2z d^2w \, G(z,w) \frac{\delta}{\delta X^\mu(z,\bar{z})} \frac{\delta}{\delta X_\mu(w,\overline{w})} \right] f[X] \, .$$

For example,

$$\begin{aligned} :X_1 X_2 X_3 X_4 X_5:= & X_1 X_2 X_3 X_4 X_5 - (G_{12} X_3 X_4 X_5 + \cdots)_{10 \text{ terms}} \\ & + (G_{12} G_{34} X_5 + \cdots)_{15 \text{ terms}} , \end{aligned}$$

where $X_i = X(z_i)$ and $G_{ij} = G(z_i, z_j)$.[4]

[4]Here the normal ordering is used to cut off UV divergence in a particular renormalization scheme. It is not necessarily equivalent to the normal ordering defined by the creation and annihilation operators in the previous section.

Moreover, a product of normal ordered operators is given as follows:

$$:f[X]::g[X]:= \exp\left[\int d^2z d^2w\, G(z,w)\, \frac{\delta}{\delta X^\mu(z,\bar{z})}\bigg|_f \frac{\delta}{\delta X_\mu(w,\bar{w})}\bigg|_g\right] :fg[X]: \ . \tag{3.7}$$

For instance, we have

$$:\partial X(z,\bar{z})::X(w,\bar{w}):=:\partial X(z,\bar{z})X(w,\bar{w}): +\partial_z G(z,w) =:\partial X(z,\bar{z})X(w,\bar{w}): -\frac{\alpha'}{2}\frac{1}{z-w} \ .$$

As another example, we can consider the OPE of vertex operators (See §3.7)

$$\begin{aligned}:e^{ik\cdot X(z,\bar{z})}::e^{ik'\cdot X(w,\bar{w})}: &= \exp\left[(ik)\cdot(ik')\left(-\frac{\alpha'}{2}\log|z-w|^2\right)\right] :e^{ik\cdot X(z,\bar{z})+ik'\cdot X(w,\bar{w})}: \\ &= |z-w|^{\alpha' k\cdot k'} \ :e^{ik\cdot X(z,\bar{z})+ik'\cdot X(w,\bar{w})}: \ .\end{aligned}$$

In a general field theory, a product of a pair of fields can be expanded by a single operator

$$\Phi^i(z)\Phi^j(w) = \sum_k C^{ij}_k(z-w)\Phi^k(w) \ .$$

This is called **operator product expansion (OPE)**, and it describes the behavior when the two operators approach each other.

Typically, for a massless free scalar field theory, we have

$$\begin{aligned}X^\mu(z,\bar{z})X^\nu(w,\bar{w}) &= -\frac{\alpha'}{2}\eta^{\mu\nu}\log|z-w|^2 + :X^\mu X^\nu(w,\bar{w}): \\ &+ \sum_{k=1}^\infty \frac{1}{k!}\left\{(z-w)^k :\partial^k X^\mu X^\nu(w,\bar{w}): + (\bar{z}-\bar{w})^k :\bar{\partial}^k X^\mu X^\nu(w,\bar{w}):\right\} \ .\end{aligned} \tag{3.8}$$

The second and the third terms of RHS can be understood as the Taylor expansions of $:X(z)X(w):$ in terms of z. Note that mixing terms ($:\partial^m\bar{\partial}^n XX:$) vanish due to the "equation of motion" (3.6).

As we will see below, divergent terms in OPE are important. Thus, we write

$$X^\mu(z,\bar{z})X^\nu(w,\bar{w}) \sim -\frac{\alpha'}{2}\eta^{\mu\nu}\log|z-w|^2 \ .$$

The symbol $\sim$ means the regular terms are ignored.

3.4 Conformal Ward-Takahashi identity

When an action is invariant under a certain transformation δ (namely, $\delta S = 0$) we say the theory has a (classical) symmetry. Furthermore, the measure of the path integral is also invariant under the transformation, the theory has the symmetry at quantum level. On the other hand, if the measure is not invariant, then the symmetry is anomalous. It is non-trivial to see if the theory has an anomaly or not.

If the theory has a symmetry, Noether's theorem states that there is the corresponding conserved current j^a. The space integral of its time component is the conserved charge $Q = \int_{\text{space}} j^0$, which generates the symmetry $\delta\phi = [Q,\phi]$. Let δ be a symmetry $\delta S = 0$ and assume it acts on a field as follows: $\delta\phi(z) = \epsilon(\cdots)$, where ϵ is a small parameter. If we promote the parameter ϵ to be world-sheet coordinate dependent (i.e. $\widehat{\delta}\phi = \epsilon(z,\bar{z})(\cdots)$), then $\widehat{\delta}S$ is no longer zero but has to take the following form:

$$\widehat{\delta}S = \int \frac{d^2\sigma}{2\pi}\epsilon(x,y)\partial_a j^a = \int \frac{d^2z}{2\pi}\epsilon(z,\bar{z})\left(\partial\bar{j} + \bar{\partial}j\right) ,$$

where we introduced $j = j_z$ and $\bar{j} = j_{\bar{z}}$. If the parameter is constant, $\delta S = $ (total derivative) $=$ 0 so that j^a is **the Noether current**. The conservation of the Noether current $\partial_a j^a = 0$ in the flat 2d implies that j ($\bar{j}$) is a holomorphic (anti-holomorphic) function of z.

Ward-Takahashi identity

Now let us study the transformation of a point operator $\mathcal{O}(w,\overline{w})$ under the symmetry δ. For this purpose, we set

$$\epsilon(z,\bar{z}) = \begin{cases} \epsilon \quad (\text{const.}) & \text{for } z \in D_w , \\ 0 & \text{for } z \notin D_w , \end{cases}$$

where D_w is a disk containing w. Then, the variation of the path integral

$$0 = \int \mathcal{D}X\, \widehat{\delta}\left[e^{-S}\mathcal{O}(w,\overline{w})\right] = \int \mathcal{D}Xe^{-S}\left[\delta\mathcal{O}(w,\overline{w}) - \widehat{\delta}S\cdot\mathcal{O}(w,\overline{w})\right].$$

Therefore, we have

$$\begin{aligned}\delta\mathcal{O}(w,\overline{w}) &= \int \frac{d^2z}{2\pi}\epsilon(z,\bar{z})\left(\partial\bar{j}(\bar{z}) + \bar{\partial}j(z)\right)\mathcal{O}(w,\overline{w}) \\ &= \frac{\epsilon}{2\pi i}\oint_{\partial D_w}\left(dz\, j(z) - d\bar{z}\,\bar{j}(\bar{z})\right)\mathcal{O}(w,\overline{w}) ,\end{aligned}$$

This is called the **Ward-Takahashi identity**.

Let us see an example of the free scalar field (3.2). It is easy to see that the action of the free scalar field is invariant under the transformation $X^\mu \to X^\mu + \epsilon^\mu$. If $\epsilon^\mu(z,\bar{z})$ has a function, then we have

$$\begin{aligned}\widehat{\delta}S &= \frac{-1}{2\pi\alpha'}\int d^2z\, \epsilon_\mu(z,\bar{z})\left[\bar{\partial}\left(\partial X^\mu\right) + \partial\left(\bar{\partial}X^\mu\right)\right] \\ j^\mu &= -\frac{1}{\alpha'}\partial X^\mu , \qquad \bar{j}^\mu = -\frac{1}{\alpha'}\bar{\partial}X^\mu .\end{aligned} \tag{3.9}$$

It is straightforward from (3.8) to check that

$$\delta X^\mu(w) = \epsilon_\nu \oint_{\partial D}\frac{dz}{2\pi i}\, j^\nu(z)X^\mu(w)$$

is consistent.

Now, let us consider the Ward-Takahashi identity for conformal symmetry. For an infinitesimal conformal transformation $\sigma^a \to \sigma^a + \epsilon^a(\sigma)$, the metric is transformed as

$$\delta_{ab} \to \delta_{ab} + \partial_a\epsilon_b + \partial_b\epsilon_a .$$

Since this is a conformal transformation, it is proportional to δ_{ab} so that we have

$$\partial_a\epsilon_b + \partial_b\epsilon_a = (\partial_\rho\epsilon^\rho)\delta_{ab} . \tag{3.10}$$

A solution to this equation is called a **conformal Killing vector**. The current for the conformal transformation can be written as

$$j_a = T_{ab}\epsilon^b ,$$

where the straightforward calculation provides the energy-momentum tensor

$$T_{ab} = -2\pi\Big[\frac{\partial L}{\partial(\partial^a\phi)}\partial_b\phi - \delta_{ab}L\Big] . \tag{3.11}$$

If we assume ϵ is constant, it is easy to see that the conservation of the current implies the conservation of the energy-momentum tensor:

$$\partial_a j^a = 0 \quad \rightarrow \quad \partial^a T_{ab} = 0 . \tag{3.12}$$

For general $\epsilon^a(\sigma)$, the conservation of the current gives the traceless condition of T_{ab}:

$$0 = \partial^a j_a = \frac{1}{2}T_{ab}(\partial^a\epsilon^b + \partial^b\epsilon^a) = \frac{1}{2}T^a{}_a(\partial_\rho\epsilon^\rho) \quad \rightarrow \quad T^a{}_a = 0 . \tag{3.13}$$

In the complex coordinate $z = x^1 + ix^2$, the traceless condition can be written as

$$T_{z\overline{z}} = T_{\overline{z}z} = 0$$

and the conservation of the energy-momentum tensor can be written as

$$\partial_{\overline{z}}T_{zz} = 0 , \qquad \partial_z T_{\overline{z}\overline{z}} = 0 .$$

Thus, the non-vanishing components of the energy-momentum tensor factorize to a chiral and anti-chiral field,

$$T(z) := T_{zz} \quad \text{and} \quad \overline{T}(\overline{z}) := T_{\overline{z}\overline{z}} .$$

As a result, the Noether currents for conformal transformations $z \to z + \epsilon(z)$ and $\overline{z} \to \overline{z} + \overline{\epsilon}(\overline{z})$ are

$$j(z) = \epsilon(z)T(z) , \quad \overline{j}(\overline{z}) = \overline{\epsilon}(\overline{z})\overline{T}(\overline{z}) .$$

The application to the Ward-Takahashi identity leads to **conformal Ward-Takahashi identity**

$$\delta_{\epsilon,\overline{\epsilon}}\mathcal{O}(w,\overline{w}) = \frac{1}{2\pi i}\oint_{C_w} dz\, \epsilon(z)T(z)\mathcal{O}(w,\overline{w}) + \frac{1}{2\pi i}\oint_{C_{\overline{w}}} d\overline{z}\, \overline{\epsilon}(\overline{z})\overline{T}(\overline{z})\mathcal{O}(w,\overline{w}) , \tag{3.14}$$

where the contour integral is taken as a counter-clockwise circle both in z and in $\overline{z}$ (thereby explaining the sign difference of the second term).

3.5 Primary fields

Let us first introduce some terminologies in 2d CFT. Fields depending only on z, i.e. $\phi(z)$, are called **chiral or holomorphic fields** and fields $\overline{\phi}(\overline{z})$ only depending on $\overline{z}$ are called **anti-chiral or anti-holomorphic fields**. If a field ϕ transforms under the scaling transformation $z \to \lambda z$ as

$$\phi(z,\overline{z}) \to \phi'(\lambda z, \overline{\lambda}\overline{z}) = \lambda^{-h}\overline{\lambda}^{-\overline{h}}\phi(z,\overline{z}) , \tag{3.15}$$

it is said to have **conformal dimension** $(h, \overline{h})$. If a field transforms under a conformal transformation $z \to f(z)$ as

$$\phi(z, \overline{z}) \to \phi'(f(z), \overline{f}(\overline{z})) = \left(\frac{\partial f}{\partial z}\right)^{-h} \left(\frac{\partial \overline{f}}{\partial \overline{z}}\right)^{-\overline{h}} \phi(z, \overline{z}) \tag{3.16}$$

it is called a **primary field** of conformal dimensions $(h, \overline{h})$. If (3.16) is true only for $f \in \mathrm{SL}(2, \mathbb{C})/\mathbb{Z}_2$, then it is called a **quasi-primary field**. Note that $\mathrm{SL}(2, \mathbb{C})/\mathbb{Z}_2$ group acts on the holomorphic coordinate as

$$z \mapsto \frac{az + b}{cz + d} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C})/\mathbb{Z}_2 \,.$$

How do primary fields transform infinitesimally? Under the infinitesimal conformal transformation $z \to f(z) = z - \epsilon(z)$, we know that

$$\begin{aligned} \left(\frac{\partial f}{\partial z}\right)^{-h} &= 1 + h\partial_z \epsilon(z) + O(\epsilon^2) \,, \\ \phi(z - \epsilon(z), \overline{z}) &= \phi(z) - \epsilon(z)\partial_z \phi(z, \overline{z}) + O(\epsilon^2) \,. \end{aligned} \tag{3.17}$$

Hence, under an infinitesimal conformal transformation, the variation of a primary field is given by

$$\delta_\epsilon \phi(z, \overline{z}) = \left(h\partial_z \epsilon + \epsilon \partial_z + \overline{h}\partial_{\overline{z}} \overline{\epsilon} + \overline{\epsilon}\partial_{\overline{z}}\right) \phi(z, \overline{z}). \tag{3.18}$$

Consequently, using simple complex analysis

$$\begin{aligned} (\partial_w \epsilon(w))\phi(w, \overline{w}) &= \frac{1}{2\pi i} \oint_{C_w} dz \, \frac{\epsilon(z)\phi(w, \overline{w})}{(z - w)^2} \\ \epsilon(w)(\partial_w \phi(w, \overline{w})) &= \frac{1}{2\pi i} \oint_{C_w} dz \, \frac{\epsilon(z)\partial_w \phi(w, \overline{w})}{z - w} \,, \end{aligned} \tag{3.19}$$

one can read off the OPE of a primary operator ϕ of conformal dimension $(h, \tilde{h})$ with the energy-momentum tensor T (anti-chiral part $\overline{T}$ can be obtained by complex conjugate)

$$T(z)\, \phi(w, \overline{w}) = h\frac{\phi(w, \overline{w})}{(z - w)^2} + \frac{\partial_w \phi(w, \overline{w})}{z - w} + \text{regular terms} \cdots \tag{3.20}$$

In general, the OPE of an operator $\mathcal{O}$ of conformal dimension $(h, \tilde{h})$ with the energy-momentum tensor T and $\overline{T}$ takes the form

$$T(z)\, \mathcal{O}(w, \overline{w}) = \quad \cdots + h\frac{\mathcal{O}(w, \overline{w})}{(z - w)^2} + \frac{\partial \mathcal{O}(w, \overline{w})}{z - w} + \cdots$$

One of the main interests in a CFT is to calculate correlation functions of primary fields. Indeed, the conformal Ward-Takahashi identity can be applied to a correlation function of primary fields

$$\langle T(z)\phi_1(w_1, \overline{w}_1) \cdots \phi_n(w_n, \overline{w}_n)\rangle = \sum_{i=1}^{n} \Big(\frac{h_i}{(z - w_i)^2} + \frac{\partial_{w_i}}{z - w_i}\Big)\langle \phi_1(w_1, \overline{w}_1) \cdots \phi_n(w_n, \overline{w}_n)\rangle \,.$$

Moreover, conformal symmetry is so powerful that it determines the forms of two-point and three-point functions of primary fields (Exercise).

• **2-point function**

For chiral primary operators ϕ_i with conformal dimension h_i ($i = 1, 2$), their 2-point function is of form

$$\langle \phi_1(z_1)\phi_2(z_2)\rangle = \delta_{h_1 h_2}\frac{d_{12}}{(z_1 - z_2)^{2h_1}} \tag{3.21}$$

If d_{12} is non-degenerate, the fields can be normalized such that $d_{12} = \delta_{12}$.

• **3-point function**

A 3-point function is also completely fixed by conformal invariance up to the appearance of a **structure constant** C_{123} (exercise) C_{ijk},

$$\langle \phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\rangle = \frac{C_{123}}{(z_1 - z_2)^{h_1+h_2-h_3}(z_2 - z_3)^{h_2+h_3-h_1}(z_3 - z_1)^{h_3+h_1-h_2}}$$

The structure constant depends on a CFT and, in general, it is not easy to determine it.

• **Multi-point function**

The computation of multi-point functions involves **conformal blocks** with the 3-point function. The details are explained in [FMS12, BP09].

Free scalar field

Now let us study conformal Ward-Takahashi identity in the simplest example, the free scalar field (3.2). Let us recall that the energy-momentum tensor in 2d free scalar theory is

$$T^X_{ab} = -\frac{1}{\alpha'}\left(\partial_a X^\mu \partial_b X_\mu - \frac{1}{2}\delta_{ab}\left(\partial_c X^\mu \partial^c X_\mu\right)\right) , \tag{3.22}$$

As in (2.7), since the equation of motion for X^μ is $\partial_z\overline{\partial}_{\overline{z}}X^\mu = 0$, the classical solution holomorphically factorizes as

$$X^\mu(z, \overline{z}) = X^\mu(z) + \overline{X}^\mu(\overline{z}) .$$

In (3.9), we find the conserved holomorphic and anti-holomorphic U(1) current

$$j^\mu(z) := -\partial X^\mu(z)/\alpha' , \quad \text{and} \quad \overline{j}^\mu(\overline{z}) := -\overline{\partial}\overline{X}^\mu(\overline{z})/\alpha' . \tag{3.23}$$

Moreover, the energy-momentum tensor becomes much simpler in complex coordinates. It is simple to check that $T^X_{z\overline{z}} = 0$ while

$$T^X(z) = -\frac{1}{\alpha'}\partial X^\mu(z)\partial X_\mu(z) , \quad \overline{T}^X(\overline{z}) = -\frac{1}{\alpha'}\overline{\partial}\overline{X}^\mu(\overline{z})\overline{\partial}\overline{X}_\mu(\overline{z}) . \tag{3.24}$$

From the definition (3.16), one can see that $X(z, \overline{z})$ is a primary field of conformal dimension $(0, 0)$. However, since the conformal dimension is of $(0, 0)$, the two-point function does not exactly take the form (3.21). Indeed, the OPE XX tells us that the propagator takes the form

$$\langle X^\mu(z, \overline{z})X^\nu(w, \overline{w})\rangle = -\frac{\alpha'}{2}\eta^{\mu\nu}\log|z - w|^2 .$$

Also, for each μ, the currents $j^\mu(z), \bar{j}^\mu(\bar{z})$ are primary fields of conformal dimension (1,0) and (0,1), respectively. Focusing only on the holomorphic part, an immediate check is their correlation function

$$\langle \partial X^\mu(z) \partial X^\nu(w) \rangle = -\frac{\alpha'}{2} \frac{\eta^{\mu\nu}}{(z-w)^2} ,$$

which takes the form (3.21). To convince ourselves completely, we need to compute the OPE with the energy-momentum tensor by Wick's theorem

$$\begin{aligned} T^X(z)\, \partial X^\mu(w) &= -\frac{1}{\alpha'} : \partial X^\nu(z) \partial X_\nu(z) : \partial X^\mu(w) \\ &= \frac{\partial X^\mu(w)}{(z-w)^2} + \frac{\partial^2 X^\mu(w)}{z-w} + \text{regular terms} \cdots \end{aligned} \tag{3.25}$$

This is indeed the OPE for a primary operator of conformal dimension $h = 1$.

Finally, let us check to see the TT OPE of the energy-momentum tensors. This can be done by applying the Wick contractions, and the result is

$$\begin{aligned} T^X(z)\, T^X(w) &= \frac{1}{\alpha'^2} : \partial X^\mu(z)\, \partial X_\mu(z) : : \partial X^\nu(w)\, \partial X_\nu(w) : \\ &= \frac{D/2}{(z-w)^4} + \frac{2T^X(w)}{(z-w)^2} + \frac{\partial T^X(w)}{z-w} + \ldots \end{aligned} \tag{3.26}$$

Therefore, the energy-momentum tensor is an operator of conformal dimension $(h, \bar{h}) = (2, 0)$. Because there is a higher singular term proportional to $(z-w)^{-4}$, the energy-momentum tensor is not a primary field. In fact, this is a general property of the TT OPE in all 2d CFTs.

3.6 Virasoro algebra

For the free scalar field, we have already seen that T has conformal dimension $(h, \tilde{h}) = (2, 0)$. This remains true in any CFT. The reason for this is simple: T_{ab} has dimension $\Delta = 2$ because we obtain the energy by integrating over space. It has spin $s = 2$ because it is a symmetric two-tensor. However, these two pieces of information are equivalent to the statement that T is an operator of conformal dimension $(2, 0)$. This means that the TT OPE takes the form,

$$T(z)\, T(w) = \ldots + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \ldots$$

and a similar one for $\overline{TT}$. What other terms could we have in this expansion? Since each term has dimension $\Delta = 4$, the unitarity indeed tells us that the singular part of the OPE takes

$$T(z)\, T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \ldots$$

From the OPE of the energy-momentum tensor, one can see its variation under an infinitesimal conformal transformation $z \to z - \epsilon(z)$

$$\delta_\epsilon T(w) = \frac{1}{2\pi i} \oint_{C_w} dz\, \epsilon(z) T(z) T(w)$$

$$= \epsilon(w)\partial T(w) + 2\epsilon'(w)T(w) + \frac{c}{12}\epsilon'''(w) \tag{3.27}$$

One can verify by a straightforward computation that this is the infinitesimal version of the following transformation under finite transformation $z \to w(z)$:

$$T'(w) = \left(\frac{\partial w}{\partial z}\right)^{-2}\left[T(z) - \frac{c}{12}S\left(w,z\right)\right], \tag{3.28}$$

where the **Schwarzian derivative** S is defined as

$$S(w,z) := \frac{1}{(\partial_z w)^2}\left((\partial_z w)(\partial_z^3 w) - \frac{3}{2}(\partial_z^2 w)^2\right). \tag{3.29}$$

Using the mapping from the plane to the cylinder $z = e^{-iw}$ $(w = \sigma + it)$ as in Figure 5, the energy-momentum tensor is transformed as

$$T_{cyl}(w) = -z^2 T(z) + \frac{c}{24}. \tag{3.30}$$

The Laurent mode expansion of the energy-momentum tensor on the cylinder is therefore

$$T_{cyl}(w) = -\sum_{n\in\mathbb{Z}}\left(L_n - \frac{c}{24}\delta_{n,0}\right)e^{inw}\,. \tag{3.31}$$

Including the contribution from the anti-holomorphic sector, the Hamiltonian is defined by

$$H \equiv \int d\sigma\; T_{\tau\tau} = -\int d\sigma\,(T_{ww} + \overline{T}_{\overline{w}\overline{w}}) = L_0 + \overline{L}_0 - \frac{c}{12}\,.$$

This tells us that the ground state energy on the cylinder is

$$E = -\frac{c}{12}\,.$$

This is indeed the (negative) Casimir energy on a cylinder. In the string sigma model, each coordinate X^μ of the target space gives rise to a free scalar theory, and it is easy to see from the computation (3.26) that each coordinate X^μ contributes to the central charge by $c = 1$. Thus, each target dimension yields the energy density $E = -1/12$ as we have seen in the quantization of bosonic string theory (2.17). Moreover, we have the central charge

$$c^X = D \tag{3.32}$$

for the string sigma model with the target space $\mathbb{R}^{1,D-1}$.

In the Euclidean flat space, the mode expansion of the energy-momentum tensor is expressed as

$$T(z) = \sum_{n\in\mathbb{Z}}\frac{L_n}{z^{n+2}}\,,$$

where the shift by two in the exponent of z is due to the conformal dimension of the energy-momentum tensor. It is natural to find the commutation relation of the generators L_m. The commutator can be computed by

$$[L_m, L_n] = \left(\oint\frac{dz}{2\pi i}\oint\frac{dw}{2\pi i} - \oint\frac{dw}{2\pi i}\oint\frac{dz}{2\pi i}\right)z^{m+1}w^{n+1}\,T(z)\,T(w)$$

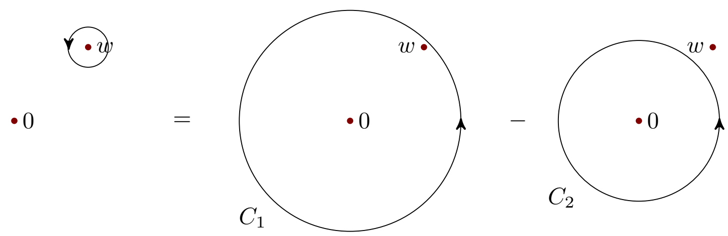


Figure 6:

Here we impose the radial ordering

$$R(A(z)B(w)) := \begin{cases} A(z)B(w) & \text{for } |z| > |w| \\ B(w)A(z) & \text{for } |w| > |z| \end{cases}$$

according to which the contour is chosen as in Figure 6. Clever manipulation of the contour makes life easier

$$\begin{aligned} [L_m, L_n] &= \oint \frac{dw}{2\pi i} \oint_w \frac{dz}{2\pi i}\, z^{m+1} w^{n+1}\, T(z)\, T(w) \\ &= \oint \frac{dw}{2\pi i} \operatorname{Res} \left[z^{m+1} w^{n+1} \left(\frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \dots \right) \right] \end{aligned} \tag{3.33}$$

A simple computation (Exercise) leads to the **Virasoro algebra**

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m+n,0}$$

For instance, the Virasoro generators of the free scalar theory are expressed in terms of the modes as in (2.15).

3.7 Vertex operators

We shall connect 2d conformal field theory to string theory, considering vertex operators. The state-operator correspondence tells us that there are corresponding local operators for the tachyon and the massless states (2.24) in the bosonic closed string theory. The tachyon state is just a vacuum state with a certain momentum k^μ. Therefore, the corresponding operator is

$$|0;k\rangle \quad \leftrightarrow \quad : e^{ikX(0,0)} : \ . \tag{3.34}$$

This can be easily verified from the operator analogue of $p^\mu|0;k\rangle = k^\mu|0;k\rangle$, namely,

$$\partial X^\mu(z) : e^{ik\cdot X(0)} : \sim \frac{-i\alpha' k^\mu}{2z} : e^{ik\cdot X(0)} : \ .$$

The first excited states are obtained by acting $\alpha^\mu_{-1}\tilde{\alpha}^\nu_{-1}$ on the vacuum $|0;k\rangle$. Each mode in (3.3) can be extracted by the Fourier transformation of $\partial X^\mu(z)$ and the corresponding operator is

$$\alpha^\mu_{-m} = i\sqrt{\frac{2}{\alpha'}} \oint \frac{dz}{2\pi i} z^{-m} \partial X^\mu(z) \quad \to \quad i\sqrt{\frac{2}{\alpha'}} \frac{1}{(m-1)!} \partial^m X(0) \ .$$

Thus, we have operators corresponding to the massless states

$$\zeta_{\mu\nu}\alpha^{\mu}_{-1}\overline{\alpha}^{\nu}_{-1}|0;k\rangle \quad \leftrightarrow \quad \zeta_{\mu\nu}\partial X^{\mu}\overline{\partial}X^{\nu} : e^{ikX} :$$

where $\zeta_{\mu\nu}$ are the polarization tensors subject to $k^{\mu}\zeta_{\mu\nu} = 0$.

The string amplitude is

$$A_n = \sum_{g} \int (\mathcal{D}h_{ab})_{g,n} \int \mathcal{D}X^{\mu} e^{-S_{\sigma}[X^{\mu},h_{ab}]} \prod_{i=1}^{n} \int d^2z\sqrt{h}V_i \, . \tag{3.35}$$

Here $\hat{V}_i = \int d^2z\sqrt{h}V_i$ is an operator corresponding to a string state. It is integrated out over the world-sheet to be Diff$\times$Weyl invariant.

Mass from vertex operator

Now, let us consider a constant scaling $z \to \lambda z$ and $\overline{z} \to \overline{\lambda}\overline{z}$. Under the scaling, a field transforms as $\phi(z,\overline{z}) \to \lambda^{-h}\overline{\lambda}^{-\overline{h}}\phi(z,\overline{z})$, which should compensate for the scaling of the measure $dzd\overline{z} \to \lambda\overline{\lambda}dzd\overline{z}$. Namely, $h = \overline{h} = 1$. Conformal dimensions of the vertex operators are

Name	$\mathcal{O}$	$(h,\overline{h})$
Tachyon	$e^{ik\cdot X}$	$(\frac{\alpha' k^2}{4}, \frac{\alpha' k^2}{4})$
1st excited states	$\zeta_{\mu\nu}\partial X^{\mu}\overline{\partial}X^{\nu}e^{ikX}$	$(1+\frac{\alpha' k^2}{4}, 1+\frac{\alpha' k^2}{4})$

The consistency condition for the tachyon leads to

$$M^2 = -k^2_{\text{Tachyon}} = -\frac{4}{\alpha'} \, .$$

Similarly, the first excited states lead to the massless condition

$$M^2 = -k^2_{\text{1st}} = 0 \, .$$

Both results are consistent with the analysis in (2.21) and (2.24). Here one may consider that one can obtain the string spectrum in arbitrary spacetime dimensions. However, as we will show in the next section, the on-shell condition only holds in $D = 26$ where the Weyl transformation is a quantum symmetry.

4 Weyl anomaly

The classical action (2.2) of the string sigma model is invariant under the Weyl symmetry, and the Weyl invariance implies the traceless condition (2.4) of the energy-momentum tensor. However, we will see that there can be an anomaly in the Weyl symmetry if the string world-sheet is curved. As we will see, it is characterized by the central charge $T^{a}_{\ a} = -\frac{c}{12}R^{(2)}$ where $R^{(2)}$ is the world-sheet Ricci scalar curvature.

Moreover, in string theory, a target space can also be a non-trivial curved spacetime where the action becomes

$$S = \frac{1}{4\pi\alpha'} \int \sqrt{h}d^2z \, h^{ab}\partial_a X^{\mu}\partial_{\,b}X^{\nu}G_{\mu\nu}(X) + \cdots \, .$$

This is no longer a free theory, and it describes an interacting non-linear theory, called the **(string) non-linear sigma model**. In an interacting quantum field theory, the breakdown of scale invariance is described in terms of a β function. In the non-linear sigma model, the anomaly of the Weyl invariance due to the curved target spacetime is also characterized by β-functions.

4.1 Weyl anomaly from curved world-sheet

Although the string sigma model is classically Weyl-invariant, in quantum theory, the trace of the energy-momentum tensor can be nonzero due to anomaly. Nonetheless, it is

- world-sheet diff invariant,
- zero on a flat world-sheet $\mathbb{R}^2$,
- world-sheet mass dimension two.

Hence, they constrain the form of the trace of the energy-momentum tensor as

$$T^a_{\ a} = kR^{(2)} \ ,$$

where $R^{(2)}$ is the world-sheet Ricci scalar curvature. In this subsection, we shall determine the coefficient k.

Given a curved world-sheet Riemann surface (Σ, h_{ab}) with non-trivial metric, we can always take a local coordinate (called **conformal gauge**) of the world-sheet Riemann surface in which the metric is a conformally flat

$$ds^2 = e^{2\omega(\sigma^1,\sigma^1)}(d\sigma^1 d\sigma^1 + d\sigma^2 d\sigma^2) = e^{2\omega(z,\overline{z})} dz d\overline{z} \ . \tag{4.1}$$

This local coordinate is called an **isothermal coordinate**. In the isothermal coordinate, the scalar curvature and non-trivial Christoffel symbols are expressed as

$$R^{(2)} = -8e^{-2\omega}\partial\overline{\partial}\omega \ , \qquad \Gamma^z_{zz} = 2\partial\omega, \quad \Gamma^{\overline{z}}_{\overline{z}\overline{z}} = 2\overline{\partial}\omega \tag{4.2}$$

and the other Christoffel symbols are zero $\Gamma^a_{bc} = 0$. Therefore, the diagonal element of the energy-momentum tensor becomes

$$T_{z\overline{z}} = \frac{1}{4}e^{2\omega} \ T^a_{\ a} = -2k\partial\overline{\partial}\omega \ .$$

Also, the conservation $\nabla_a T^a_{\ b} = 0$ of the energy-momentum tensor now becomes

$$\begin{aligned} 0 =& \nabla^z T_{zz} + \nabla^{\overline{z}} T_{\overline{z}z} = \nabla_{\overline{z}} T_{zz} + \nabla_z T_{\overline{z}z} \\ =& \overline{\partial} T_{zz} + (\partial - 2\partial\omega)(-2k\partial\overline{\partial}\omega) \\ =& \overline{\partial}\left(T_{zz} - 2k\left(\partial\partial\omega - \partial\omega\partial\omega\right)\right) \end{aligned}$$

This implies that T_{zz} deviates from a holomorphic operator $T(z)$

$$T_{zz} = 2k\left(\partial\partial\omega - \partial\omega\partial\omega\right) + T(z) \ . \tag{4.3}$$

On a curved world-sheet, the conformal Ward-Takahashi identity (3.14) becomes

$$\delta_{\epsilon,\overline{\epsilon}}\mathcal{O}(w,\overline{w}) = \oint_M \frac{d^2z}{2\pi}\left\{\left(\nabla_{\overline{z}}\epsilon(z)T_{zz} + \nabla_z\epsilon(z)T_{\overline{z}z}\right) + \left(\nabla_z\overline{\epsilon}(\overline{z})T_{\overline{z}\overline{z}} + \nabla_{\overline{z}}\overline{\epsilon}(\overline{z})T_{z\overline{z}}\right)\right\}\mathcal{O}(w,\overline{w})$$

$$= \int_M \frac{d^2z}{2\pi} \left\{ \overline{\partial} \left(\epsilon(z) T(z) \right) + \partial \left(\overline{\epsilon}(\overline{z}) \overline{T}(\overline{z}) \right) \right\} \mathcal{O}(w, \overline{w}) ,$$

where we use the current conservation

$$T_{z\overline{z}}(\nabla^z \epsilon^{\overline{z}} + \nabla^{\overline{z}} \epsilon^z) = 0$$

from the first and second line. This gives the transform property of $T(z)$ as in the flat space (3.27)

$$\delta_\epsilon T(z) = \epsilon(z) \partial T(z) + 2\partial\epsilon(z) T(z) + \frac{c}{12} \partial^3 \epsilon(z) .$$

On the other hand, (4.3) yields

$$\delta_\epsilon T(z) = \delta_\epsilon T_{zz} - 2k \left(\partial\partial\delta_\epsilon \omega - 2\partial\omega\partial\delta_\epsilon\omega \right) . \tag{4.4}$$

Using (finite) transformations,

$$\begin{aligned} & z \to \widetilde{z} = z - \epsilon(z) , \\ & T_{zz} \to \widetilde{T}_{\widetilde{z}\widetilde{z}} = \left(\partial_z \widetilde{z} \right)^{-2} T_{zz} , \\ & \omega(z) \to \widetilde{\omega}(\widetilde{z}) = \omega(z) - \frac{1}{2} \log \left| \partial_z \widetilde{z} \right|^2 , \end{aligned}$$

(4.4) can be expressed as

$$\delta_\epsilon T(z) = \epsilon(z) \partial T(z) + 2\partial\epsilon(z) T(z) - k\partial^3 \epsilon(z) .$$

Finally, we see that the Weyl anomaly is proportional to the central charge

$$T^a_{\ a} = kR^{(2)} = -\frac{c}{12} R^{(2)} . \tag{4.5}$$

We have seen in (3.32) that the central charge of the string sigma model is equal to the dimension D of the target space. Hence, naively looking, the bosonic string theory would suffer from the Weyl anomaly. However, as we will see in §5.1, we introduce "ghost CFT" to fix gauge in the world-sheet path integral, and the ghost CFT has $c^{\mathrm{gh}} = -26$. As a result, the bosonic string theory is Weyl invariant at quantum level only if $D = 26$.

4.2 Non-linear sigma model

In §2, we start with a flat D-dimensional target space, and the quantization of bosonic string naturally leads to the graviton (2.24). Therefore, in general, we can consider that the target spacetime is curved with a non-trivial metric so that the action is

$$S[X^\mu, h_{ab}] = \frac{1}{4\pi\alpha'} \int d^2\sigma \left(\sqrt{h} h^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X) \right) .$$

This is called the non-linear sigma model. If we consider a perturbation from the flat-metric in this action

$$G_{\mu\nu}(X) = \eta_{\mu\nu} + f_{\mu\nu}(X) ,$$

then the partition function becomes

$$Z = \int \mathcal{D}h_{ab}\mathcal{D}X^\mu e^{-S}$$
$$= \int \mathcal{D}h_{ab}\mathcal{D}X^\mu e^{-S_0}\left(1 + \frac{1}{4\pi\alpha'}\int d^2\sigma\left(\sqrt{h}h^{ab}\partial_a X^\mu \partial_b X^\nu f_{\mu\nu}(X)\right) + \cdots\right) .$$

Notice that the perturbative part is the graviton operator with wave function $f_{\mu\nu}(X) = \zeta_{\mu\nu}e^{ik\cdot X}$.

In a curved target, the theory is no longer free, so that we need to take into account various quantum effects. We will see that β-functions encode the anomaly of scaling invariance at quantum level. Note that the discussion here is brief, and the reader is referred to [CT89] for the details. As a 2d field theory, we can consider the vacuum expectation value (VEV) for X, which we set to X_0:

$$\widehat{X}(\sigma,\tau) = X_0 + X(\sigma,\tau) .$$

On the other hand, X_0 is a certain point in spacetime and we will expand the metric around this point:

$$G_{\mu\nu}(X) = G_{\mu\nu} - \frac{1}{3}R_{\mu\lambda\nu\rho}X^\lambda X^\rho + \mathcal{O}(X^3) ,$$

where $G_{\mu\nu}$ and $R_{\mu\lambda\nu\rho}$ are a metric and a Riemann curvature tensor of the target spacetime at X_0, respectively. In a field-theoretic sense, these can be understood as coupling constants

$$S = \frac{1}{4\pi\alpha'}\int d^2\sigma\, \partial^a X^\mu \partial_a X^\nu\left(G_{\mu\nu} - \frac{1}{3}R_{\mu\lambda\nu\rho}X^\lambda X^\rho + \cdots\right)$$
$$\to \frac{1}{2}\int d^2\sigma\, \partial^a X^\mu \partial_a X^\nu\left(G_{\mu\nu} - \frac{2\pi\alpha'}{3}R_{\mu\lambda\nu\rho}X^\lambda X^\rho + \mathcal{O}(\alpha'^2)\right) .$$

Here we rescale the field $X \to \sqrt{2\pi\alpha'}X$ so that the expansion looks like a "stringy expansion".

Perturbation theory for non-linear sigma model

We want to check if the theory (non-linear sigma model) has Weyl anomaly. As we briefly saw, it is an interacting theory, and interacting theories have generally non-trivial β-functions:

$$\beta[g] \equiv E\frac{\partial}{\partial E}g(E) = \frac{\partial}{\partial(\log E)}g(E) ,$$

where g is a coupling constant and E is a characteristic energy scale. When we consider a global scaling of coordinate:$z \to \widetilde{z} = \lambda z = (1-\epsilon)z = e^{-\epsilon}z$, energy scales oppositely: $E \to \widetilde{E} = \frac{1}{\lambda}E = e^\epsilon E$. So the β-function can be written as

$$\beta[g] = \frac{\partial}{\partial\epsilon}g(\epsilon) .$$

The variation of the action is expressed in two ways:

$$\delta_\epsilon S = \begin{cases} \int \frac{d^2\sigma}{2\pi}\sqrt{h}\delta_\epsilon h^{ab}T_{ba} = -\epsilon \int \frac{d^2\sigma}{2\pi} T^a_{\ a} \,, \\ \frac{1}{4\pi\alpha'}\int d^2\sigma\, \partial^a X^\mu \partial_a X^\nu \left(\epsilon \frac{\partial}{\partial\epsilon} G_{\mu\nu}(\epsilon) + \cdots\right) \,, \end{cases}$$

where the first variation is a formal transformation of the theory, and in quantum regime, it should be proportional to the trace part of the energy-momentum tensor. On the other hand, the second variation is the actual theory with an assumption that ϵ dependence of the theory is only in the coupling constants. Identifying them, we have

$$T^a_{\ a} = -\frac{1}{2\alpha'}\beta[G_{\mu\nu}]\partial^b X^\mu \partial_b X^\nu + \cdots \,.$$

This shows that the anomaly is parametrized by β-functions.

Let us consider perturbation theory, namely loop corrections to two-point function etc, so that we can see if the theory is anomalous.

$$\begin{aligned}&\langle X^\mu(\sigma_1) X^\nu(\sigma_2)\rangle \\ &\quad = \int \frac{d^2k}{(2\pi)^2}\frac{2\pi\alpha'}{k^2} e^{ik\cdot(\sigma_1-\sigma_2)} \left\{ G^{\mu\nu} + \frac{2\pi\alpha'}{3}R^{\mu\nu}\left(\int \frac{d^2p}{(2\pi)^2}\frac{1}{p^2} + \frac{1}{k^2}\int\frac{d^2p}{(2\pi)^2}\right) + \cdots \right\} \,.\end{aligned}$$

Among terms proportional to $R^{\mu\nu}$ in the integrand, the first integral provides a logarithmic and the second does quadratic divergence. Since the quadratic divergence is discarded by dimensional regularization, we focus on the logarithmic divergence and introduce regularization parameters:

$$\int_E^\Lambda \frac{d^2p}{(2\pi)^2}\frac{1}{p^2} = \frac{1}{2\pi}\log\left(\frac{\Lambda}{E}\right) \,,$$

where Λ is an ultra-violet (UV) energy scale supposed to be ∞, and E is an infra-red (IR) energy scale supposed to be our life energy scale, which is very low (~ 0).

The divergence can be subtracted by counter terms as follows. The bare action $\widehat{S}, = S + S_{\mathrm{ct}}$ describes UV physics of energy scale Λ whereas the physical action is S describes IR physics of energy scale E. They are related by renormalization

$$\widehat{S} = \frac{1}{4\pi\alpha'}\int d^2\sigma\, \partial^a \widehat{X}^\mu \partial_a \widehat{X}^\nu \left(\widehat{G}_{\mu\nu} - \frac{1}{3}\widehat{R}_{\mu\lambda\nu\rho}\widehat{X}^\lambda \widehat{X}^\rho + \cdots\right)$$

with

$$\begin{aligned}&\widehat{X}^\mu = Z^\mu_\nu X^\nu \,, \quad Z^\mu_\nu = \delta^\mu_\nu + \sum_{n=1}^\infty \alpha'^n Z^\mu_{\nu,(n)}(\Lambda/E) \,, \\ &\widehat{G}^{\mu\nu} = G^{\mu\nu} + \sum_{n=1}^\infty \alpha'^n G^{\mu\nu}_{(n)}(\Lambda/E) \,, \text{etc.}\end{aligned}$$

Note that the bare action only depends on Λ (not on E), and hence, the bare coupling constants ($\widehat{G}_{\mu\nu}$ etc) only depend on high energy Λ. The counter terms lead to other contributions

$$\langle X^\mu(\sigma_1) X^\nu(\sigma_2)\rangle \sim \left\{ G^{\mu\nu} + \frac{\alpha'}{3}R^{\mu\nu}\log\left(\frac{\Lambda}{E}\right) - \alpha'\left(G^{\mu\nu}_{(1)} + Z^{\mu\nu}_{(1)} + Z^{\nu\mu}_{(1)}\right) + \cdots \right\} \,. \tag{4.6}$$

Unfortunately, the equation above cannot fix the ratio between $G_{(1)}$ and $Z_{(1)}$. We need further information like a 4-point function to determine the ratio. We simply list the result:

$$G^{\mu\nu}_{(1)} = R^{\mu\nu} \log\left(\frac{\Lambda}{E}\right) , \quad Z^{\mu\nu}_{(1)} = -\frac{1}{3} R^{\mu\nu} \log\left(\frac{\Lambda}{E}\right) ,$$

which does cancel the divergent term in (4.6). From the result we can derive the β-function:

$$\begin{aligned} G_{\mu\nu}(E,\Lambda) &= \widehat{G}_{\mu\nu}(\Lambda) - \alpha' R_{\mu\nu} \log\left(\frac{\Lambda}{E}\right) , \\ \beta[G_{\mu\nu}] &= \frac{\partial}{\partial(\log E)} G_{\mu\nu}(E,\Lambda) = \alpha' R_{\mu\nu} . \end{aligned}$$

Therefore, for the theory to be anomaly-free, the spacetime is required to be Ricci-flat ($R_{\mu\nu} = 0$).

Non-linear sigma model (general)

If we incorporate all the massless states (2.24), we can write the most general form of the non-linear sigma model for a closed string as

$$S_{\text{closed}} = \frac{1}{4\pi\alpha'} \int d^2\sigma \left(\sqrt{h} h^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X) + i\varepsilon^{ab} \partial_a X^\mu \partial_b X^\nu B_{\mu\nu}(X) + \alpha' \sqrt{h} R^{(2)} \Phi(X) \right) . \tag{4.7}$$

Note that the B-field is a higher dimensional analogue of gauge fields (mathematically called “gerbe”). Its gauge transformation is given by

$$\delta B_{\mu\nu} = \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu , \tag{4.8}$$

and the field strength $H_{\mu\nu\lambda}$ is defined as

$$H_{\mu\nu\lambda} = \partial_\mu B_{\nu\lambda} + \partial_\nu B_{\lambda\mu} + \partial_\lambda B_{\mu\nu} .$$

The B-field often is the source of “stringy” effects, and it plays an important role on many occasions.

Again, the Weyl anomaly is characterized by β-functions for the massless fields

$$T^a_{\ a} = -\frac{1}{2\alpha'} \beta[G_{\mu\nu}] \partial_a X^\mu \partial^a X^\nu - \frac{i}{2\alpha'} \beta[B_{\mu\nu}] \varepsilon^{ab} \partial_a X^\mu \partial_b X^\nu - \frac{1}{2} \beta[\Phi] R^{(2)}$$

where β-functions are expressed as

$$\begin{aligned} \beta[G_{\mu\nu}] &= \alpha' R_{\mu\nu} + 2\alpha' \nabla_\mu \nabla_\nu \Phi - \frac{\alpha'}{4} H_{\mu\lambda\rho} H_\nu^{\ \lambda\rho} + \mathcal{O}(\alpha'^2) , \\ \beta[B_{\mu\nu}] &= -\frac{\alpha'}{2} \nabla^\lambda H_{\lambda\mu\nu} + \alpha' \nabla^\lambda \Phi H_{\lambda\mu\nu} + \mathcal{O}(\alpha'^2) , \\ \beta[\Phi] &= \frac{D-26}{6} - \frac{\alpha'}{2} \nabla^2 \Phi + \alpha' \nabla^\lambda \Phi \nabla_\lambda \Phi - \frac{\alpha'}{24} H_{\mu\lambda\rho} H^{\mu\lambda\rho} + \mathcal{O}(\alpha'^2) . \end{aligned}$$

In order for the theory to be non-anomalous, all the β-functions must be zero, and the vanishing of $\beta[\Phi]$ again requires $D = 26$. The perturbative quantum field theory on the world-sheet can describe the classical gravity from the vanishing of β function.

1. The equation $\beta[G_{\mu\nu}]=0$ is analogous to Einstein's equation with source terms from the B-field and the dilaton.
2. The equation $\beta[B_{\mu\nu}]=0$ is the generalization of the Maxwell's equation.

The condition that β-functions vanish ($\beta=0$) is equivalent to the equation of motion for the following spacetime effective action [5]

$$S_{\text{eff}}=\frac{1}{2\kappa_D^2}\int d^DX\sqrt{-G}e^{-2\Phi}\left[\frac{2(D-26)}{3\alpha'}+R-\frac{1}{12}H_{\mu\lambda\rho}H^{\mu\lambda\rho}+4\nabla^\lambda\Phi\nabla_\lambda\Phi+\mathcal{O}(\alpha'^2)\right]. \tag{4.9}$$

The action (4.7) tells us another important fact in string theory. The dilaton field is coupled to the world-sheet Ricci scalar. If the dilaton takes an expectation value $\langle\Phi(X)\rangle=\Phi$, the dilaton action gives the Euler characteristic of the world-sheet Riemann surface

$$\begin{aligned}S_{\text{dilaton}}&=\frac{1}{4\pi\alpha'}\int d^2\sigma\left(\alpha'\sqrt{h}R^{(2)}\Phi(X)\right)\\&\to\frac{1}{4\pi}\int d^2\sigma\left(\sqrt{h}R^{(2)}\Phi\right)=\Phi(2-2g)\end{aligned}$$

In other words, the 2d gravity is topological with no physical degrees of freedom. Defining $g_s=e^\Phi$, and the dilaton part of the action becomes

$$e^{-S_{\text{dilaton}}}=g_s^{2g-2}\,.$$

This implies that g_s can be understood as a string coupling. An n-point string tree amplitude can be understood as $n-2$ cylinders attaching to a cylinder. Hence, the amplitude should be proportional to g_s^{n-2}. A higher loop (higher genus) amplitude can be derived by attaching g cylinders to the tree amplitude, and the amplitude should be $\widehat{A}_{n,g}\propto g_s^{n-2+2g}$. (See Figure 7.) Usually, vertex operators are re-normalized so that g_s^n is included in the definition of $V_1\cdots V_n$. Therefore, the re-normalized amplitude should be

$$A_{n,g}\propto g_s^{2g-2}\,,$$

which coincide with the dilaton action. In conclusion, the string coupling g_s is just the expectation value of the dilaton field.

5 BRST quantization

5.1 Quantization via path integral

We have been studying bosonic string theory, but there are several caveats.

• In the light-cone quantization (2.6), Lorentz invariance is not manifest. Can we quantize strings in a way that is manifestly Lorentz invariant?

[5]We are in the regime where the perturbation theory works well. Namely, the length scale of the target space is long compared to the string scale. We thus can ignore the internal structure of the string and write the following low energy effective theory for quantum gravity.

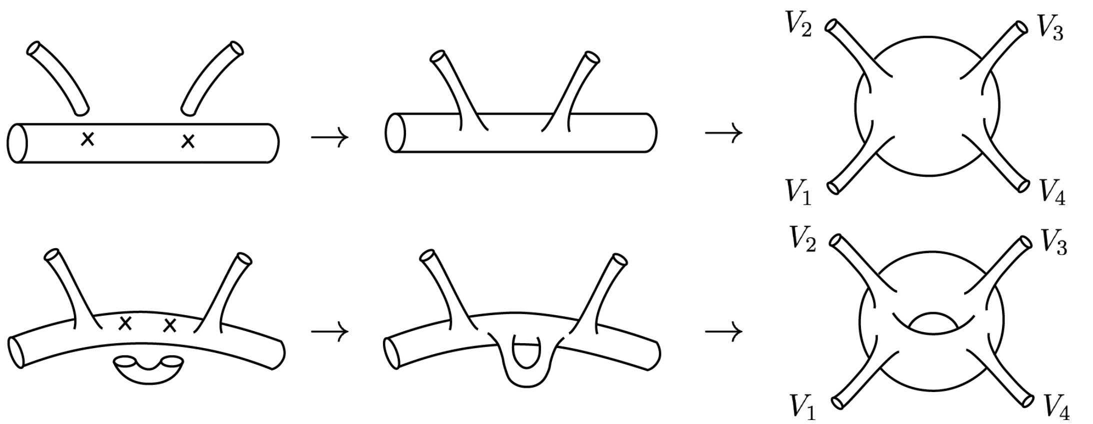


Figure 7: 4-point amplitude example. The upper one is a construction of 4pt tree amplitude from cylinders. The lower one is a construction of 4-point 1-loop amplitude from the tree amplitude.

• We have seen that the Weyl symmetry of the string sigma model is anomalous (4.5) on a curved world-sheet. How can the bosonic string be anomaly-free?

• Although we learned that string amplitude is expressed via Feynman path integral (3.35), we do not know how to perform this path integral. In particular, the path integral is endowed with huge gauge symmetries, world-sheet diffeomorphism and Weyl symmetry. How can we treat integration measure and fix gauge in the path integral?

To answer these questions, we will study the quantization procedure via path integral, which is often called **modern covariant quantization**. This method uses the analog of the Faddeev-Popov method of gauge theories. Furthermore, the physical state condition is imposed via the BRST symmetry.

After gauge-fixing the reparametrization and Weyl symmetry, the integral over h_{ab} turns into a path-integral over ghost CFT, which has $c^g = -26$. Therefore, in order for the theory to be Weyl anomaly-free, the matter part of the theory has to be a CFT with $c^X = 26$.

Faddeev-Popov gauge fixing

The integration measure can be written as $[\mathcal{D}X_\mu][\mathcal{D}h_{ab}]_{g,n}$ where the scalar fields and the metric on 2d surfaces with genus g and n marked points are integrated out. However, if there is no anomaly, this integral has the world-sheet diffeomorphisms and Weyl symmetry under which the field configurations are transformed as

$$\begin{aligned} X^{\zeta}_{\mu}(\widetilde{\sigma}) &= X^{(}_{\mu}\sigma) \\ h^{\zeta}_{ab}(\widetilde{\sigma}) &= e^{2\omega(\sigma)}\frac{\partial\sigma^c}{\partial\widetilde{\sigma}^a}\frac{\partial\sigma^d}{\partial\widetilde{\sigma}^b}h_{cd}(\sigma)\ , \end{aligned} \tag{5.1}$$

where ζ indicates Diff$\times$Weyl. These symmetries are redundant and integrating along these directions just gives rise to the volume of the symmetry group. (In Figure 8, the solid arrows schematically draw gauge redundancy and the dotted line shows physically distinct configurations.) Thus, we need to carry out gauge-fixing. Thankfully, there is a

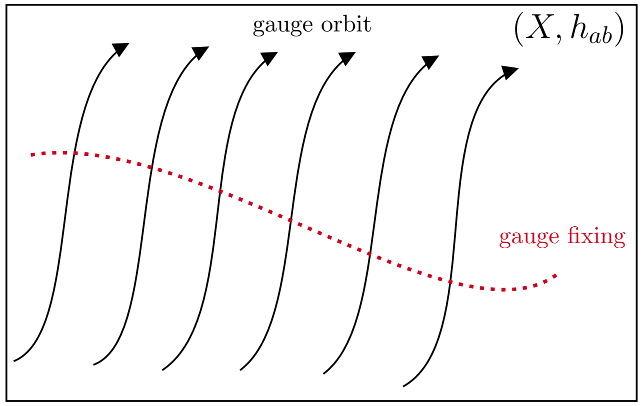


Figure 8: We need to fix a gauge in the field configuration space. Physically inequivalent configurations are depicted by the dotted line.

standard method to fix gauge, introduced by Faddeev and Popov. A basic idea is to insert the identity of the following form in the path integral:

$$1 = \int \mathcal{D}\zeta\, \delta(h - \hat{h}^{\zeta}) \det\left(\frac{\delta \hat{h}^{\zeta}}{\delta\zeta}\right) \tag{5.2}$$

where the Jacobian factor $\det\left(\frac{\delta \hat{h}^{\zeta}}{\delta\zeta}\right)$ is called **Faddeev-Popov determinant** and we denote it by $\Delta_{FP}[\hat{h}]$. Since $\mathcal{D}\zeta$ is a gauge-invariant measure, $\Delta_{FP}[\hat{h}]$ is independent of ζ so that it can be factored out of the integral above. The insertion of this identity into the path integral fixes the metric as $\hat{h}$ because of the delta functional. Because $\int \mathcal{D}\zeta$ integral only contributes an infinite multiplicative factor, we can discard this integral. Therefore, after Faddeev-Popov gauge fixing, the form of the path integral can be schematically written as

$$Z[\hat{h}] = \int \mathcal{D}X\, \Delta_{FP}[\hat{h}]\, e^{-S_\sigma[X,\hat{h}]} \tag{5.3}$$

We should make several remarks about this procedure.

• First, the conformal transformations are residual gauge symmetries not fixed above. We have to throw away these residual gauge symmetries in the path integral in order to avoid over-counting. Indeed we will be careful to fix this extra residual gauge freedom when computing string amplitudes.

• Second, there are caveats related to global properties of the world-sheet Riemann surface $\Sigma_{g,n}$. In fact, metrics on a Riemann surface encode the information of "shape" of the Riemann surface $\Sigma_{g,n}$ called **complex moduli**, which is not accounted for by local gauge transformations ζ. The space which parametrizes "shape" of the Riemann surface is called **moduli space of Riemann surface** $\Sigma_{g,n}$ which is $(6g-6+2n)$-$\dim_{\mathbb{R}}$:

$$\mathcal{M}_{g,n} := \frac{[\mathcal{D}h_{ab}]_{g,n}}{\text{Diff} \times \text{Weyl}}$$

Therefore, the path integral involves integrals over $\mathcal{M}_{g,n}$ as well. At this moment, we postpone both the issues and will come back to them in §6 on string amplitudes.

Now let us take an infinitesimal version of (5.1) where a Weyl transformation is parameterized by $\omega(\sigma)$ and an infinitesimal diffeomorphism by $\delta\sigma^\alpha = \epsilon^\alpha(\sigma)$. Subsequently, the change of the metric under a gauge transformation is read off

$$\delta\hat{h}_{ab} = 2\omega\hat{h}_{ab} + \nabla_a\epsilon_b + \nabla_b\epsilon_a := 2\widetilde{\omega}\hat{h}_{ab} + (P\cdot\epsilon)_{ab}\ , \tag{5.4}$$

where we decompose it into

$$\begin{aligned}(P\cdot\epsilon)_{ab} &= \nabla_a\epsilon_b + \nabla_b\epsilon_a - h_{ab}(\nabla\cdot\epsilon)\\ \widetilde{\omega} &= \omega + \frac{1}{2}(\nabla\cdot\epsilon)\ .\end{aligned} \tag{5.5}$$

Indeed, the operator P maps vectors ϵ_a to symmetric traceless two-tensors $(P\cdot\epsilon)_{ab}$. Thus, the Faddeev-Popov determinant can be written as

$$\Delta_{FP}[\hat{h}] = \det\frac{\delta(P\cdot\epsilon,\widetilde{\omega})}{\delta(\epsilon,\omega)} = \det\begin{vmatrix} P & 0\\ * & 1\end{vmatrix} = \det P\ .$$

To compute $\det P$, we use **Faddeev-Popov ghosts**, which can be understood as an infinite-dimensional version of the following integral. Given a matrix M_{ij}, its determinant can be expressed as a Grassmann integral

$$\int\prod_{i=1}^{n} d\psi_i d\theta_i \exp\left(\theta_i M_{ij}\psi_j\right) = \det M\ .$$

where θ,ψ are Grassmann variables. Accordingly, we introduce anti-commuting fermionic fields, c^a (ghosts) and b_{ab} (anti-ghost) where b_{ab} transforms as a symmetric traceless tensor and c^a as a vector. Then, we can express

$$\Delta_{FP}[\hat{h}] = \int\mathcal{D}c\mathcal{D}b\ \exp\left(\frac{i}{2\pi}\int d^2\sigma\sqrt{-\hat{h}}\, b^{ab}(P\cdot c)_{ab}\right) := \int\mathcal{D}c\mathcal{D}b\ \exp\left[iS_g\right]\ ,$$

where the ghost action can be written as

$$S_g = \frac{1}{2\pi}\int d^2\sigma\sqrt{-\hat{h}}b^{ab}\nabla_a c_b\ . \tag{5.6}$$

Even though the bc ghost fields were introduced to fix a gauge, they look like dynamical fields with the action above. Consequently, the Faddeev-Popov gauge fixing results in a fermionic 2d CFT, usually called bc **ghost CFT**.

Let us make some remarks about the equation of motion of S_g

• The equation of motion for c_a is given by

$$P\cdot c = \nabla_a c_b + \nabla_b c_a - h_{ab}(\nabla\cdot c) = 0\ . \tag{5.7}$$

Therefore the solutions for c are in one-to-one correspondence with the **conformal Killing vectors**, which are the generators of the residual symmetry.

• The equation of motion for b_{ab} is

$$\nabla_a b^{ab} = 0\ . \tag{5.8}$$

We will understand the geometric meaning of these equations when discussing the moduli space of Riemann surfaces.

To understand the properties of the bc ghost CFT, it is convenient to use the Euclidean signature so that we will perform Wick rotation in what follows. Then, the factor of i in the action disappears. The expression for the full partition function (5.3) is

$$Z[\hat{h}] = \int \mathcal{D}X\mathcal{D}c\mathcal{D}b \, \exp\left(-S_\sigma[X,\hat{h}] - S_g[b,c,\hat{h}]\right) \,. \tag{5.9}$$

bc ghost CFT

Now let us study the bc ghost CFT more in detail. For this purpose, we pick the conformal gauge (4.1) for a world-sheet metric. Using (4.2), the ghost action can be written as

$$\begin{aligned} S_g &= \frac{1}{2\pi}\int d^2z \, \left(b_{zz}\nabla_{\bar{z}}c^z + b_{\bar{z}\bar{z}}\nabla_z c^{\bar{z}}\right) \\ &= \frac{1}{2\pi}\int d^2z \, b_{zz}\,\partial_{\bar{z}}c^z + b_{\bar{z}\bar{z}}\,\partial_z c^{\bar{z}} \end{aligned} \tag{5.10}$$

For the sake of simplicity, let us define

$$b = b_{zz}\,, \qquad \bar{b} = b_{\bar{z}\bar{z}}\,, \qquad c = c^z\,, \qquad \bar{c} = c^{\bar{z}}\,.$$

Then, the action simplifies to

$$S_g = \frac{1}{2\pi}\int d^2z \, \left(b\,\bar{\partial}c + \bar{b}\,\partial\bar{c}\right)\,, \tag{5.11}$$

which gives the equations of motion

$$\bar{\partial}b = \partial\bar{b} = \bar{\partial}c = \partial\bar{c} = 0 \tag{5.12}$$

Thus, we see that b and c are holomorphic fields, while $\bar{b}$ and $\bar{c}$ are anti-holomorphic.

To compute the OPEs of the bc ghost fields, we use the path integral techniques in §3.3:

$$0 = \int \mathcal{D}c\mathcal{D}b\frac{\delta}{\delta b(z)}\left[e^{-s_g}b(w)\right] = \int \mathcal{D}c\mathcal{D}bs^{-s_g}\left[-\frac{1}{2\pi}\bar{\partial}c(z)b(w) + \delta(z-w)\right]\,,$$

which tells us that

$$\bar{\partial}c(z)\,b(w) = 2\pi\,\delta(z-w)\,.$$

We can perform a similar computation for $c(z)$, which yields

$$\bar{\partial}b(z)\,c(w) = 2\pi\,\delta(z-w)\,.$$

We can integrate both of these equations using (3.5). Then, we obtain the bc OPE

$$\begin{aligned} b(z)\,c(w) &= \frac{1}{z-w} + \ldots \\ c(w)\,b(z) &= \frac{1}{w-z} + \ldots \end{aligned}$$

In fact, the second equation follows from the first equation and Fermi statistics. Hence, the OPEs of $b(z)\,b(w)$ and $c(z)\,c(w)$ are trivial for the obvious reason.

In any CFT, it is of most importance to find the form of the energy-momentum tensor. The energy-momentum tensor is obtained via Noether's theorem with respect to world-sheet transformations $\delta z = \epsilon(z)$, under which

$$\delta b = (\epsilon\partial + 2(\partial\epsilon))b \ , \qquad \delta c = (\epsilon\partial - (\partial\epsilon))c \ .$$

Indeed both b and c are primary fields with conformal dimensions $h = 2$ and $h = -1$, respectively, which can be easily seen from their index structure b_{zz} and c^z. From these rules, one can deduce the form of the energy-momentum tensor

$$T^g(z) = -2 : b(z)\partial c(z) : + : c(z)\partial b(z) : \ . \tag{5.13}$$

In fact, this form can be obtained from the first principle, namely the variation of the action under the metric (Exercise).

The OPEs of b and c with the stress tensor are

$$\begin{aligned} T^g(z)c(w) &= -\frac{c(w)}{(z-w)^2} + \frac{\partial c(w)}{z-w} + \ldots \\ T^g(z)b(w) &= \frac{2b(w)}{(z-w)^2} + \frac{\partial b(w)}{z-w} + \ldots \end{aligned}$$

so that b and c are primary fields of conformal dimension 2 and -1, respectively. Consequently, they admit the mode expansions

$$b(z) = \sum_{m\in\mathbb{Z}} \frac{b_m}{z^{m+2}} \qquad c(z) = \sum_{m\in\mathbb{Z}} \frac{c_m}{z^{m-1}} \ .$$

The ghost OPEs give the commutation relations

$$\{b_m, c_n\} = \delta_{m+n,0} \ , \quad \{c_m, c_n\} = \{b_m, b_n\} = 0 \ . \tag{5.14}$$

Also, the Virasoro generators of the bc ghost are expressed as

$$L^g_m = \sum_{n\in\mathbb{Z}} (2m - n) : b_n c_{m-n} : + a^g \delta_{m,0} \ . \tag{5.15}$$

The constant a^g in the zero mode L^g_0 is determined by the Casimir energy $\frac{c^g}{24}$ in (3.31) and the commutation relation of the bc modes $\frac{1}{12} = -\sum_{n>0} n$

$$a^g = \frac{-26}{24} + \frac{1}{12} = -1 \ .$$

This is consistent with the commutation relation $L_0 = [L_1, L_{-1}]$ ([Pol98, (2.7.20)]).

Finally, we can compute the TT OPE (Exercise)

$$T^g(z)\, T^g(w) = \frac{-13}{(z-w)^4} + \frac{2T^g(w)}{(z-w)^2} + \frac{\partial T^g(w)}{z-w} + \ldots$$

Now one can read off the central charge of the bc ghost system, which is

$$c^g = 2(-13) = -26 \ .$$

We learned that the Weyl symmetry is anomalous unless $c = 0$. Since the Weyl symmetry is a gauge symmetry, the theory must be Weyl anomaly-free. Since the total central charge of the string sigma model and ghost theory (5.9) is given by $c = c^X + c^g$, the dimension of the target space must be $D = 26$. Again, we obtain the critical dimension of the bosonic string theory!

5.2 BRST quantization

In 4d Yang-Mills theory, the Lagrangian with Faddeev-Popov ghosts has the continuous symmetry, called **BRST symmetry** (Becchi-Rouet-Stora-Tyupin). The BRST symmetry is generated by a nilpotent charge Q_B ($Q_B^2 = 0$) that commutes with the Hamiltonian. The nilpotency of the BRST charge has substantial consequences. The BRST transformations come from the gauge symmetry, all physical states must be BRST-invariant. Hence, we require a physical state to be annihilated by Q_B

$$Q_B|\text{phys}\rangle = 0\,.$$

However, one can always add a state of the form $Q_B|\chi\rangle$ since this state will be annihilated by Q_B because of the nilpotent BRST charge. However, this state is orthogonal to all physical states, and therefore it is a **null state**. Thus, two states related by

$$|\psi'\rangle = |\psi\rangle + Q_B|\chi\rangle$$

have the same inner products with all the physical states and they are thus indistinguishable. This is the remnant in the gauge-fixed version of the original gauge symmetry. As a result, the Hilbert space of physical states is isomorphic to the Q_B-cohomology, i.e.

$$\mathcal{H}^{\text{phys}} \cong \frac{\mathcal{H}^{Q_B\text{-closed}}}{\mathcal{H}^{Q_B\text{-exact}}}\,. \tag{5.16}$$

This convariant way of determining the physical Hilbert space with ghosts is known as **BRST quantization**.

We are now ready to apply this formalism to the bosonic string [Pol98, §4]. Combining (3.2) and (5.11), the action we consider is

$$S_X + S_g = \frac{1}{2\pi}\int d^2z\,\left(\frac{1}{\alpha'}\partial X^\mu\overline{\partial}X_\mu + b\,\overline{\partial}c + \overline{b}\,\partial\overline{c}\right)\,.$$

This is invariant under the following BRST transformations:

$$\begin{aligned}
&\delta_B X^\mu = i\epsilon(c\partial + \overline{c}\overline{\partial})X^\mu\,,\\
&\delta_B c = i\epsilon c\partial c \qquad \delta_B\overline{c} = i\epsilon\overline{c}\overline{\partial}\overline{c}\,,\\
&\delta_B b = i\epsilon(T^X + T^g) \qquad \delta_B\overline{b} = i\epsilon(\overline{T}^X + \overline{T}^g)\,,
\end{aligned} \tag{5.17}$$

where the explicit forms of the energy-momentum tensors can be found in (3.24) and (5.13). Note that we impose (5.12) here. This exhibits typical features of the BRST transformation: a bosonic field is transformed into a fermionic field and vice versa, and the ghost field b is transformed into the energy-momentum tensor. Noether's theorem tells us that there is a classical current associated to the BRST symmetry, and the holomorphic part of the BRST current takes the form (Exercise)

$$\begin{aligned}
j_B &= c(z)T^X(z) + \frac{1}{2} : c(z)T^g(z) : + \frac{3}{2} : \partial^2 c(z) : \\
&= c(z)T^X(z) + : b(z)c(z)\partial c(z) : + \frac{3}{2} : \partial^2 c(z) :\ .
\end{aligned} \tag{5.18}$$

and the BRST charge is defined by

$$Q_B = \oint \frac{dz}{2\pi i}\, j_B\,.$$

It follows from (5.17) that

$$\{Q_B, b(z)\} = (T^X + T^g) \to \{Q_B, b_m\} = L^X_m + L^g_m \,. \tag{5.19}$$

Computing $j_B j_B$ OPE, one can convince oneself that the BRST charge is nilpotent $Q_B^2 = 0$ if and only if $D = 26$ (Exercise). Therefore, j_B is a primary field with $h = 1$, and Q_B is a conserved charge at quantum level only when $D = 26$. Furthermore, we can express the BRST charge in terms of the X^μ Virasoro operators and the ghost oscillators as

$$Q_B = \sum_n c_n (L^X_{-n} - \delta_{n,0}) + \sum_{m,n} \frac{m-n}{2} : c_m c_n b_{-m-n} : , \tag{5.20}$$

where the normal ordering constant comes from $\{Q_B, b_0\} = L^X_0 + L^g_0$. In the case of closed strings, there is the anti-holomorphic part $\overline{Q}_B$, and the total BRST charge is $Q_B + \overline{Q}_B$.

We will find the physical open string states in the BRST context. According to our previous discussion, the physical states will have to be annihilated by the BRST charge, and not be of the form $Q_B|\ \ \rangle$.

First, we have to describe our extended Hilbert space which includes the ghosts. As far as the X^μ oscillators are concerned, the situation is the same as in (2.14), so we need to consider only the ghost Hilbert space. Indeed the ghost commutation relations (5.14) generate a two-state spin system $|\uparrow\rangle, |\downarrow\rangle$ where

$$\begin{aligned} &b_0|\downarrow\rangle = 0\,, \qquad b_0|\uparrow\rangle = |\downarrow\rangle\,, \\ &c_0|\uparrow\rangle = 0\,, \qquad c_0|\downarrow\rangle = |\uparrow\rangle\,, \\ &b_{n>0}|\uparrow\rangle = b_{n>0}|\downarrow\rangle = c_{n>0}|\uparrow\rangle = c_{n>0}|\downarrow\rangle = 0\,. \end{aligned} \tag{5.21}$$

The full Hilbert space will be a tensor product of the two $|k,\uparrow\rangle, |k,\downarrow\rangle$ by acting creation operators where $|k\rangle = |0;k\rangle$ denotes the vacuum of the matter theory. From the light-cone quantization, we know that there is only one vacuum called Tachyon. Therefore, we have to pick the ghost vacuum among the two spin states. For this purpose, we further impose one more condition, namely

$$b_0|\text{phys}\rangle = 0\,. \tag{5.22}$$

This is sometimes called the **Siegel gauge** [GSW87, §3.2]. Under this condition, (5.21) tells us that the correct ghost vacuum is $|\downarrow\rangle$. We can now create states from this vacuum by acting with the negative modes of the ghosts b_m, c_n. Note that $c_0|\downarrow\rangle = |\uparrow\rangle$ does not satisfy the Siegel condition (5.22). Also, (5.22) yields the condition

$$0 = \{Q_B, b_0\}|\text{phys}\rangle = (L^X_0 + L^g_0)|\text{phys}\rangle\,. \tag{5.23}$$

Therefore, if a physical state is at level N with momentum k, $|\text{phys}\rangle = |N;k\rangle$, then we have

$$k^2 = \frac{1-N}{\alpha'} \tag{5.24}$$

which is consistent with the light-cone gauge.

Now, let us impose the physical condition (5.16) for open-string states level by level. To this end, let us explicitly write the open-string mode expansions of the zero modes of the Virasoro generators

$$L^X_0 = \alpha' p^2 + \sum_{n=1}^{\infty} \alpha_{-n}\cdot\alpha_n = \alpha' p^2 + \alpha_{-1}\cdot\alpha_1 + \cdots\,,$$

$$L_0^g = -1 \sum_{n \in \mathbb{Z}} n : b_{-n} c_n := -1 + b_{-1} c_1 + c_{-1} b_1 + \cdots \,, \tag{5.25}$$

(Compare with (2.15) and (5.15).) At level zero, there is only the vacuum state and the BRST quantization leads to $|k, \downarrow\rangle$

$$Q_B |k, \downarrow\rangle = (L_0^X - 1) c_0 |k, \downarrow\rangle = (L_0^X - 1) |k, \uparrow\rangle = 0 \,.$$

and it is not Q_B-exact.

At the first level, the possible operators that can act on the vacuum $|k, \downarrow\rangle$ are α_{-1}^μ, b_{-1} and c_{-1}. The most general state of this form is then

$$|\psi\rangle = (\zeta \cdot \alpha_{-1} + \beta b_{-1} + \gamma c_{-1}) |k, \downarrow\rangle \,, \tag{5.26}$$

which has 28 parameters: a 26-dimensional vector ζ_μ and two more constants β, γ. First we note that (5.24) yields the massless condition $k^2 = 0$. In addition, the BRST condition demands

$$0 = Q_B |\psi\rangle = 2((k \cdot \zeta) c_{-1} + \beta k \cdot \alpha_{-1}) |k, \downarrow\rangle \,.$$

This only holds if $k \cdot \zeta = 0$ and $\beta = 0$. Therefore the BRST condition removes the unphysical anti-ghost excitations as well as all polarizations that are not orthogonal to the momentum, thereby eliminating two out of the $26 + 2$ original states. Hence, there are only 26 parameters left.

Next, we have to make sure that this state is not Q_B-exact: a general state $|\chi\rangle$ is of the same form as (5.26), but with parameters ζ'^μ, β' and γ'. Thus, the most general Q_B-exact state at this level with $k^2 = 0$ will be

$$Q_B |\chi\rangle = 2(k \cdot \zeta' c_{-1} + \beta' k \cdot \alpha_{-1}) |k, \downarrow\rangle \,.$$

This means that the c_{-1} part in (5.26) is BRST-exact and that the polarization has the equivalence relation $\zeta_\mu \sim \zeta_\mu + 2\beta' k_\mu$. This leaves us with the 24 physical degrees of freedom we expect for a massless vector particle in 26 dimensions. In sum, the physical state at level one is

$$\{|\zeta; k\rangle \,; \quad k \cdot \zeta = 0\} / \, \zeta_\mu \sim \zeta_\mu + 2\beta' k_\mu \,.$$

The same procedure can be followed for the higher levels of open string states. It can be proved that $\mathcal{H}_{\text{light-cone}}$ is isomorphic to the Q_B-cohomology, and the inner product is positive-definite. In the case of the closed string, we have to use $Q_B + \overline{Q}_B$ for the BRST quantization.

6 Bosonic string amplitudes

Finally, we will discuss string amplitude. The amplitude tells us about features of string theory. As we naively showed, the string amplitude is written as follows.

$$A_n = \sum_g \int \frac{[\mathcal{D} h_{ab}]_{g,0}}{(\mathrm{Diff} \times \mathrm{Weyl})} \int \mathcal{D} X^\mu e^{-S_\sigma[X^\mu, h_{ab}]} \prod_{i=1}^n \int d^2\sigma \sqrt{h} V_i \,.$$

In this section, we will learn the degrees of the metric integration in the string amplitude after the (Diff $\times$ Weyl) gauge fixing. Namely,

$$\dim \frac{[\mathcal{D} h_{ab}]_{g,n}}{(\mathrm{Diff} \times \mathrm{Weyl})} = 6g - 6 + 2n \,.$$

The $2n$ is simply coming from the integration of the vertex operators, which is needed for the vertex operator to be Diff $\times$ Weyl-invariant. The $6g$ can be understood in a naive way as follows. Even after gauge fixing (say fixing metric locally) there is a freedom to change the "shape" of the world-sheet, which is called metric moduli (we simply denote moduli). Let us increase the number of genus g. As in §4, we can do that by attaching a cylinder to a Riemann surface Σ_g. When this is done, we need to specify where the endpoints of the cylinder are and its "shape" of the cylinder. The endpoints are denoted by two complex points p_1, p_2. Around the points, we introduce complex coordinates z_1, z_2 and impose one condition $z_1 z_2 = c$, where c is a constant whose phase specifies twist and magnitude specifies length (see Figure 9). In total, there are another six parameters to describe the shape of the world-sheet.

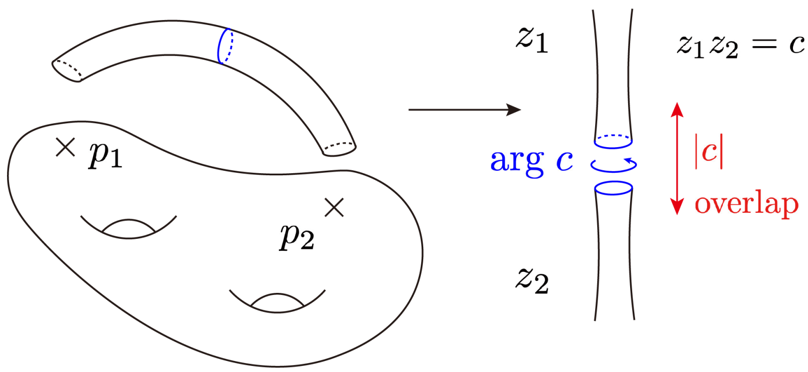


Figure 9: Attaching a pair of cylinders to Σ_g, and patching the pair of cylinders by $z_1 z_2 = c$.

6.1 Teichmüller space and moduli space of Riemann surfaces

Let us briefly introduce Teichmüller space $\mathcal{T}_g$ and moduli space $\mathcal{M}_g$ of Riemann surfaces of genus g. For the sake of brevity, we focus on an oriented closed Riemann surface without punctures. It is known that every Riemann surface admits a Riemannian metric of constant curvature:

positive A two-sphere S^2 ($g = 0$) with a fixed radius in $\mathbb{R}^3$ has positive constant curvature with respect to the induced metric from the standard metric of $\mathbb{R}^3$

zero A two-torus T^2 ($g = 1$) constructed as a quotient space $T^2 = \mathbb{C}/\Gamma$ has zero curvature where Γ is a two-dimensional lattice.

negative A Riemann surface of genus $g > 1$ can be constructed from a $4g$-polygon with geodesic arcs and each angle $2\pi/4g$ in the Poincare disc $\mathbb{D}$ where the edges are identified via words

$$a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} .$$

(See Figure 10.) Hence, it admits a negative constant curvature.

Let us first consider the case of genus one. As mentioned, a torus is constructed from a quotient space $\mathbb{C}_w/\Gamma_\tau$. As far as "shape" is concerned, we can set one edge to $1 \in \mathbb{C}$ without loss of generality. Then, an identification is given by

$$w \cong w + 2\pi(m + n\tau) , \qquad (m, n) \in \mathbb{Z} \times \mathbb{Z} , \tag{6.1}$$

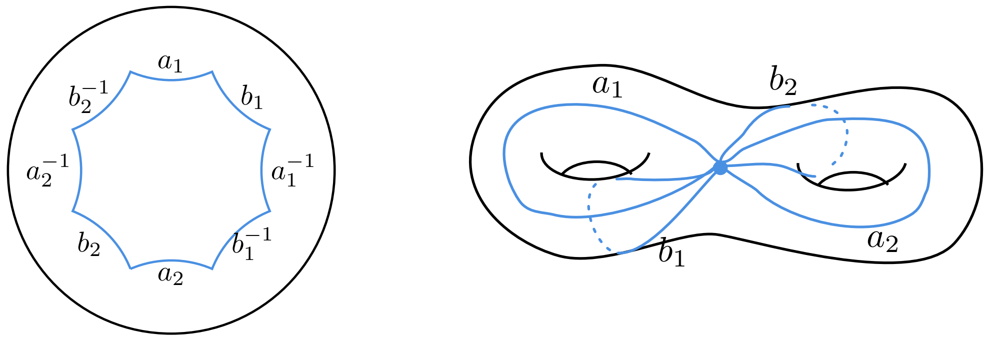


Figure 10: A $4g$-polygon with geodesic arcs and each angle $2\pi/4g$ in the Poincare disc $\mathbb{D}$ ($g = 2$)

where $\tau = \tau_1 + i\tau_2$ with $\tau_2 > 0$. In this coordinate, the metric is flat $ds^2 = dw d\overline{w}$ so that w is an isothermal coordinate. Another way to see "shape" of a torus is

$$(\sigma^1, \sigma^2) \cong (\sigma^1, \sigma^2) + 2\pi(m, n)$$

with a Riemannian metric

$$ds^2 = \left| d\sigma^1 + \tau d\sigma^2 \right|^2 . \tag{6.2}$$

In fact, the holomorphic coordinate w and (σ^1, σ^2) are related by

$$w = \sigma^1 + \tau\sigma^2 , \tag{6.3}$$

so that τ encodes the information about holomorphicity. Thus, the parameter τ is called a complex structure that describes the "shape" of the torus, roughly speaking. The space of complex structure τ is therefore the upper half place $\mathbb{H}$, which can be identified with the Teichmüller space $\mathcal{T}_{g=1}$ of a torus.

There are global transformations that leave "shape" as it is. The lattice Γ_τ is invariant under transformations

$$T : \tau \to \tau + 1 , \quad S : \tau \to -\frac{1}{\tau} , \tag{6.4}$$

up to a scale. These transformations $T,\ S$ generate the **modular transformation**

$$\tau \to \frac{a\tau + b}{c\tau + d} , \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2, \mathbb{Z}) , \tag{6.5}$$

Hence, the modular transformations do not change the "shape" of a torus, but it acts on the Teichmüller space $\mathbb{H}$ as in 11. As a result, the moduli space of complex structures of a torus is

$$\mathcal{M}_{g=1} = \mathcal{T}_{g=1}/\mathrm{PSL}(2, \mathbb{Z}) , \tag{6.6}$$

which is the shaded region in Figure 11, called the **fundamental region**.

Now let us consider a Riemann surface of genus $g > 1$. The Poincare disk with the hyperbolic metric

$$\left(\mathbb{D}, ds^2 = \frac{4(dx^2 + dy^2)}{(1 - x^2 - y^2)^2}\right)$$

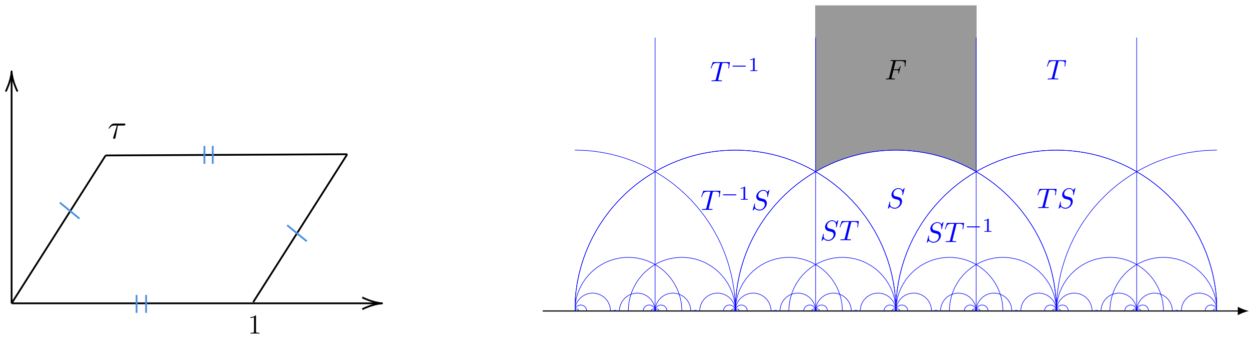


Figure 11: The Teichmüller space of a torus is the upper half-plane, and the mapping class group $\mathrm{PSL}(2,\mathbb{Z})$ acts on it. The moduli space is the fundamental region F (the shaded region).

has negative constant curvature, and geodesics are portions of circles that intersect the disk boundary at right angles. As briefly mentioned above, a Riemann surface of genus $g > 1$ can be constructed by a $4g$-polygon with geodesic arcs and each angle $2\pi/4g$ in the Poincare disk. However, not all Riemann surfaces of genus $g > 1$ with negative constant curvature (hyperbolic Riemann surfaces) are constructed in this way. The **moduli space** of Riemann surfaces indeed parametrizes hyperbolic Riemann surfaces that are not related by an isometry.

A hyperbolic Riemann surface of genus $g > 1$ admits a **pants decomposition** by cutting along $3g-3$ simple closed curves into $2g-2$ pants. In fact, one can take geodesic arcs for $3g-3$ simple closed curves and measure their lengths with respect to the hyperbolic metric. The "shape" of each pants is uniquely determined by the lengths of three edges. Thus, when we construct a hyperbolic Riemann surface by gluing pants, the "shape" will be determined by twist angles at gluing along the simple closed curves. These data will determine how to construct a hyperbolic Riemann surface from $2g-2$ pants, and the collection of the length coordinates l_i and the twist coordinates θ_i

$$\left(l_1, \cdots, l_{3g-3}; \theta_1, \cdots, \theta_{3g-3}\right) \tag{6.7}$$

are referred to as **Fenchel-Nielsen coordinates**. The Fenchel-Nielsen coordinates indeed parametrize the space of "complex structure" called **Teichmüller space** $\mathcal{T}_g$, and we have the bijection

$$\mathcal{T}_g \cong \mathbb{R}_+^{3g-3} \times \mathbb{R}^{3g-3}$$

In other words, $\mathcal{T}_g$ can be identified with an open disk in $\mathbb{R}^{6g-6}$. Moreover, the Teichmüller space is endowed with a complex structure and Kähler form

$$\omega = \sum_{i=1}^{3g-3} dl_i \wedge d\theta_i\ , \tag{6.8}$$

referred to as the **Weil-Petersson form** so that it can be considered as a Kähler manifold.

The relation between the Teichmüller space $\mathcal{T}_g$ and the moduli space $\mathcal{M}_g$ is given by the mapping class group MCG_g. The Teichmüller space classifies hyperbolic Riemann surface by isometries isotopic to the identity map. However, there are isometries that are not isotopic to the identity map. (In physics, they are sometimes called **large gauge transformations**.) The mapping class group appears as the quotient of orientation-preserving

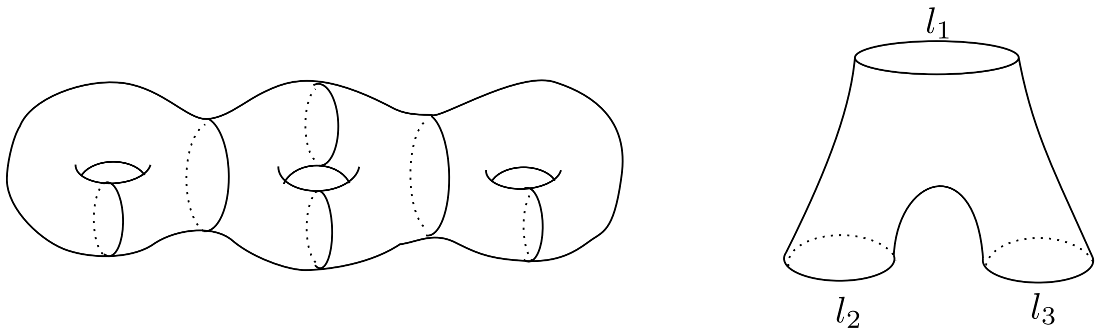


Figure 12: A pants decomposition of Riemann surface

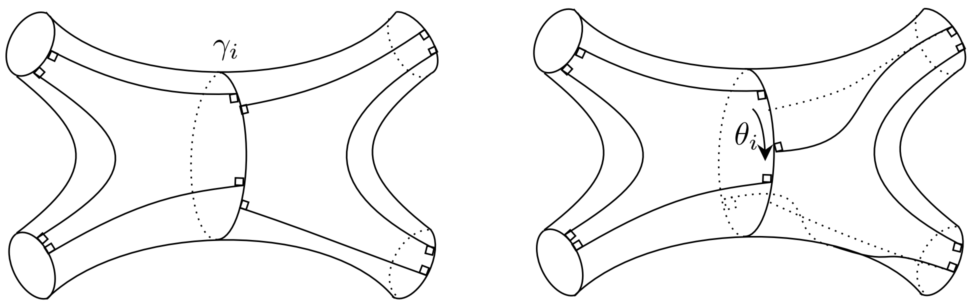


Figure 13: twist coordinates

diffeomorphisms by diffeomorphisms isotopic to the identity map

$$\mathrm{MCG}_g = \frac{\mathrm{Diff}_+(\Sigma_g)}{\mathrm{Diff}_0(\Sigma_g)} \ .$$

Although this definition looks horrendous, the theorem of Lickorish states that the mapping class group MCG_g of Riemann surfaces of genus g is a discrete group generated by Dehn twists along $3g-1$ simple closed curves shown in Figure 15. A Dehn twist is an isometry that generates a 2π-twist along a simple closed curve C of a hyperbolic Riemann surface Σ_g. Consequently, the moduli space of Riemann surfaces of genus g can be identified as the quotient space

$$\mathcal{M}_g = \mathcal{T}_g / \mathrm{MCG}_g \ .$$

In the case of a torus, $\mathcal{T}_{g=1}$ is the upper half-plane and MCG_g is $\mathrm{PSL}(2,\mathbb{Z})$, so that $\mathcal{M}_{g=1}$ is the fundamental region.

We can also discuss $\mathcal{T}_g$ and $\mathcal{M}_g$ from the viewpoint of complex analytics very briefly. More elaborate explanations can be found in [Nel87] for string theory. A Riemann surface Σ is defined by a local coordinate system $\left\{ \left(U_j, z_j\right) \right\}_{j\in J}$ that are patched by biholomorphic mappings

$$z_k \circ z_j^{-1} : z_j \left(U_j \cap U_k\right) \to z_k \left(U_j \cap U_k\right) \ .$$

A local coordinate system defines a complex structure on Σ. Given Riemann surfaces R and S, a biholomorphic map $f : R \to S$ is a holomorphic map $f : R \to S$ which has the holomorphic inverse mapping $f^{-1} : S \to R$. The Teichmüller space is the space

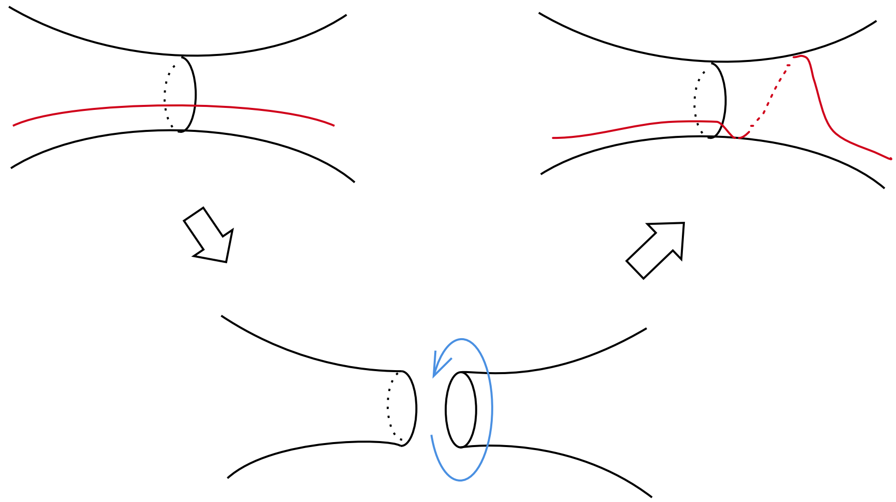


Figure 14: The Dehn twist along a simple closed curve

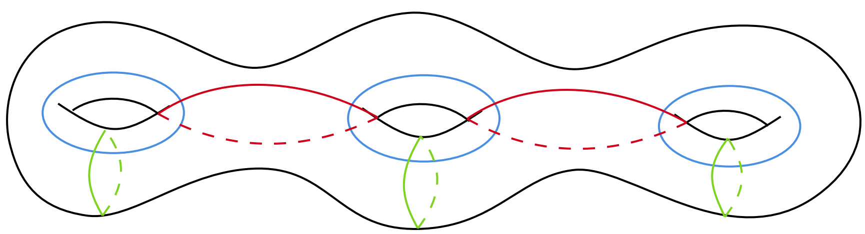


Figure 15: Choice of $3g-1$ cycles for Dehn twists. ($g=3$ here)

of complex structures modulo biholomorphic equivalence. One can reconstruct a complex structure from a real two-dimensional oriented closed manifold with a Riemannian metric, which can be expressed on a local coordinate (U, σ^a) as

$$ds^2 = h_{ab} d\sigma^a d\sigma^b .$$

Writing $z = \sigma^1 + i\sigma^2$, the metric takes the form

$$ds^2 = \lambda |dz + \mu d\bar{z}|^2 . \tag{6.9}$$

where

$$\lambda = \frac{1}{4}\left(h_{11} + h_{22} + 2\sqrt{h_{11}h_{22} - h_{12}^2}\right) , \qquad \mu = \frac{h_{11} - h_{22} + 2ih_{12}}{h_{11} + h_{22} + 2\sqrt{h_{11}h_{22} - h_{12}^2}} .$$

Moreover, we can choose an isothermal coordinate w such that

$$\rho dw d\overline{w} = \rho \left|w_z\right|^2 \left|dz + \frac{w_{\bar{z}}}{w_z} d\bar{z}\right|^2 .$$

Comparing (6.9), the existence of an isothermal coordinate is indeed equivalent to the existence of a solution to

$$\mu = \frac{w_{\bar{z}}}{w_z} = \frac{\frac{\partial w}{\partial \bar{z}}}{\frac{\partial w}{\partial z}} . \tag{6.10}$$

This is referred to as the **Beltrami equation**, and there is always a solution w. Moreover, $\{(U_j, w_j)\}_{j\in J}$ defines a complex structure, and we write the resulting Riemann surface Σ. For example, in the case of a torus, we define a complex structure (6.3) from a Riemannian metric (6.2). Even if we perform a Weyl transformation on the metric, the resulting Riemann surface Σ' is biholomorphically equivalent to the Riemann surface Σ constructed from the original metric. Therefore, the Teichmüller space can be identified with

$$\mathcal{T}_g = \frac{\mathrm{Met}(\Sigma_g)}{\mathrm{Weyl} \times \mathrm{Diff}_+} .$$

In fact, $\mu = \mu_{\bar{z}}{}^z$ in (6.10) can be considered as a $(-1,1)$-form, called **Beltrami differential**, which encodes the space of metrics up to Weyl transformations. However, there are gauge equivalent configurations due to diffeomorphisms. To distinguish non-trivial variation in the space $L^\infty_{(-1,1)}(\Sigma)$ of Beltrami differential, Teichmüller has introduced a paring $(\cdot,\cdot)$ with a **holomorphic quadratic differential** (or $(2,0)$-form) $\varphi_{zz}(z)dz \otimes dz$:

$$(\mu, \varphi) \equiv \int_\Sigma \mu_{\bar{z}}{}^z \varphi_{zz}\, \mathrm{d}z \wedge \mathrm{d}\bar{z} . \tag{6.11}$$

Then, Teichmüller's Lemma states that the subspace of Beltrami differentials for a trivial variation is given by

$$\mathcal{N} = \left\{ \mu \mid (\mu, \varphi) = 0 \quad \text{for } \forall \varphi \in H^0(\Sigma, K^{\otimes 2}) \right\} .$$

Hence, the holomorphic tangent space of the Teichmüller space at Σ can be identified with

$$T^{(1,0)}_\Sigma \mathcal{T}_g \cong L^\infty_{(-1,1)}(\Sigma)/\mathcal{N} , \tag{6.12}$$

and the holomorphic cotangent space with the space of holomorphic quadratic differentials

$$T^{*(1,0)}_\Sigma \mathcal{T}_g \cong H^0(\Sigma, K^{\otimes 2}) \tag{6.13}$$

Note that K is the cotangent bundle on Σ, and it has degree $2g-2$ so that the Riemann-Roch formula tells us

$$\dim H^0(\Sigma, K^{\otimes 2}) = \deg(K^{\otimes 2}) - g + 1 = 3g - 3 .$$

The space of anti-holomorphic quadratic differentials has the same dimension.

6.2 Gauge fixing and string amplitudes

Conformal Killing vectors & Metric moduli of space

The gauge transformation of a world-sheet metric under diffeomorphism and Weyl transformation is given in (5.4). Moreover, conformal Killing vectors ϵ

$$P \cdot \epsilon = 0 . \tag{6.14}$$

does not change the metric infinitesimally. The existence of conformal killing vectors means that there are zero modes of the ghost c-field as pointed out in (5.7). Thus, we need extra care even after introducing the bc ghost. In addition, the physical inequivalent change of a metric is perpendicular to (5.4) as in Figure 8. Writing the change $\delta^{\perp}h_{ab}$, we have

$$\begin{aligned} 0 &= \int d^2\sigma\sqrt{h}\,\delta^{\perp}h_{ab}\left[(P\cdot\epsilon)^{ab}+2\widetilde{\omega}h^{ab}\right] \\ &= \int d^2\sigma\sqrt{h}\,\left[\left(P^T\cdot\delta^{\perp}h\right)_a\epsilon^a+2\widetilde{\omega}h^{ab}\delta^{\perp}h_{ab}\right] . \end{aligned}$$

To satisfy the orthogonality for arbitrary ϵ and ω, it is required that

$$h^{ab}\delta^{\perp}h_{ab}=0\,,\qquad \left(P^T\cdot\delta^{\perp}h\right)_a=\nabla^b\delta^{\perp}h_{ba}=0\,.$$

Therefore, solutions to this equation are equivalent to zero modes of the ghost b-field (5.8). Since these zero modes are absent in the action, we need to insert appropriate zero modes to derive a non-trivial amplitude (because $\int db\cdot 1=0=\int dc\cdot 1$). In a conformal gauge, equations for conformal Killing vectors and metric moduli become

$$\begin{aligned} &\partial\overline{\epsilon}=\overline{\partial}\epsilon=0\,, \\ &\partial\delta^{\perp}h_{\overline{zz}}=\overline{\partial}\delta^{\perp}h_{zz}=0\,. \end{aligned}$$

Hence, variations of the metric moduli correspond to holomorphic quadratic differentials, which can be identified with the holomorphic cotangent space (6.13) of the Teichmüller space at Σ.

Let us look at some examples. For a two-sphere, we have the stereographic projections, and the two local coordinates are related by $w=1/z$, yielding

$$\begin{aligned} \delta w &= \frac{\partial w}{\partial z}\delta z=-z^{-2}\delta z \\ \delta h_{ww} &= \left(\frac{\partial w}{\partial z}\right)^{-2}\delta h_{zz}=z^4\delta h_{zz} \end{aligned}$$

CKV δz, is holomorphic at $w=0$ if it grows no more rapidly than z^2 as $z\to\infty$. On the other hand, there is no moduli for metric

$$\begin{aligned} &\delta^{\perp}h_{zz}=\delta^{\perp}h_{\overline{zz}}=0\,, \\ &\epsilon=a_0+a_1z+a_2z^2\,, \\ &\overline{\epsilon}=a_0^*+a_1^*\overline{z}+a_2^*\overline{z}^2\,. \end{aligned}$$

Therefore, there are 6 CKVs and no modulus.

On the torus, the only holomorphic doubly periodic functions are the constants, so there are two real moduli and two real CKVs.

$$\delta^{\perp}h_{zz}=a\,,\qquad \epsilon=b\,.$$

As we have seen, the dimension of the metric moduli is $6g-6$ for a Riemann surface of genus $g>1$. However, there is no CKV. Thus, if we write the dimension of the conformal Killing group by μ and the dimension of the metric moduli by ν, we summarize the results in Table 1.

	$g=0$	$g=1$	$g\geq 2$
μ	6	2	0
ν	0	2	$6g-6$

Table 1: The number of zero modes of b and c.

Ghost number anomaly

We can derive the conclusion that we need to insert the ghost fields in correlation functions to have non-trivial results by considering the ghost number anomaly. The ghost action in (5.11) is invariant under the transformation

$$\delta_g c = \epsilon c\ , \qquad \delta_g b = -\epsilon b\ .$$

The corresponding Noether current is a ghost number current $j = cb$ under which we can define the ghost number $[c] = 1$ and $[b] = -1$. At a flat world-sheet, the current satisfies the conservation law $\bar{\partial} j = 0$. However, it has an anomaly in a curved world-sheet proportional to the scalar curvature like in (4.5)

$$\nabla^z j_z = \kappa \cdot R^{(2)}\ . \tag{6.15}$$

Let us determine the coefficient κ. The OPE of the energy-momentum tensor and the ghost number current leads to

$$\begin{aligned} T(z)j(w) &= -:(2b\partial c + \partial b c)(z)::cb(w): \\ &\sim \frac{-3}{(z-w)^3} + \frac{j(w)}{(z-w)^2} + \frac{\partial j(w)}{z-w}\ . \end{aligned}$$

Its infinitesimal version is

$$\delta j = \epsilon \partial j + \partial \epsilon j - \frac{3}{2}\partial^2 \epsilon\ . \tag{6.16}$$

On the other hand, in the conformal gauge (4.1), (6.15) can be expressed as

$$j_z = -4\kappa \partial \omega + j(z)\ ,$$

where $j(z)$ is the holomorphic current. Using this expression, we find a transformation of $j(z)$ as

$$j(z) = \frac{\partial \tilde{z}}{\partial z}\tilde{j}(\tilde{z}) + 2\kappa \frac{\partial}{\partial z} \ln \frac{\partial \tilde{z}}{\partial z}\ ,$$

which yields

$$\delta j(z) = \epsilon \partial j(z) + \partial \epsilon j(z) + 2\kappa \partial^2 \epsilon\ .$$

Comparing this with (6.16), we have $\kappa = -\frac{3}{4}$.

This anomaly puts a constraint on non-vanishing ghost correlators. Let us consider the variation of the following correlation function under the ghost number symmetry δ_g

$$\delta_g \Big\langle \prod_{j=1}^{n} b(z_j) \prod_{i=1}^{m} c(z_i) \Big\rangle = (m-n) \Big\langle \prod_{j=1}^{n} b(z_j) \prod_{i=1}^{m} c(z_i) \Big\rangle\ .$$

On the other hand, using holomorphic current equation

$$0=\bar{\partial}j(z)=\nabla^z j_z+\frac{3}{4}\cdot R^{(2)}\ ,$$

the Ward-Takahashi identity becomes

$$\begin{aligned}\delta_g\Big\langle\prod_{j=1}^n b(z_j)\prod_{i=1}^m c(z_i)\Big\rangle=&\Big\langle\Big(\int\frac{d^2z}{2\pi}\sqrt{h}\bar{\partial}j(z)\Big)\prod_{j=1}^n b(z_j)\prod_{i=1}^m c(z_i)\Big\rangle\\ =&\Big\langle\Big(\int\frac{d^2z}{2\pi}\sqrt{h}\big(\nabla^z j_z+\frac{3}{4}R^{(2)}\big)\Big)\prod_{j=1}^n b(z_j)\prod_{i=1}^m c(z_i)\Big\rangle\\ =&\Big\langle\Big(\int\frac{3d^2z}{8\pi}\sqrt{h}R^{(2)}\Big)\prod_{j=1}^n b(z_j)\prod_{i=1}^m c(z_i)\Big\rangle\\ =&(3-3g)\Big\langle\prod_{j=1}^n b(z_j)\prod_{i=1}^m c(z_i)\Big\rangle\ .\end{aligned}$$

Therefore, we can conclude that $\#c-\#b=3-3g$, and similarly we have $\#\bar{c}-\#\bar{b}=3-3g$. This is consistent with Table 1.

General closed-string amplitude

Now let us heuristically derive the closed string amplitude. If the conformal Killing group is present, we insert the ghost zero modes $\#c+\#\bar{c}\equiv\mu$ at specified positions to fix the gauge. If enough numbers of vertex operators are present, the gauge fixing can be done by pairing $\mu/2$ vertex operators with the ghost zero modes at specific positions

$$c\bar{c}V(\sigma_j)\qquad(j=1,\ldots,\mu/2)\ .$$

Moreover, we need to integrate over the physically inequivalent world-sheet metric, which is the moduli space of the world-sheet Riemann surface $\mathcal{M}_g$. This can be done by using the non-degenerate pairing with the cotangent and tangent bundle of the moduli space introduced in (6.11). Since the b and $\bar{b}$ zero modes are sections of (anti-)holomorphic cotangent bundle of the moduli space, they are paired with (anti-)holomorphic tangent vectors as

$$(b,\partial_k h)=\int\frac{d^2\sigma}{4\pi}\sqrt{\hat{h}}\,b_{ab}\frac{\partial}{\partial t_k}\hat{h}^{ab}(t)\ ,\qquad(\bar{b},\bar{\partial}_k h)=\int\frac{d^2\sigma}{4\pi}\sqrt{\hat{h}}\,b_{ab}\frac{\partial}{\partial\bar{t}_k}\hat{h}^{ab}(t)\ .$$

With these insertions, the general closed-string amplitude in the bosonic string theory is written as

$$\begin{aligned}A_{g,n}=&\int_{\mathcal{M}_g}d^\nu t\int\mathcal{D}[c,\bar{c},b,\bar{b},X]\ e^{-S_\sigma[X,h]-S_{\rm gh}[b,c]-\Phi\chi(\Sigma_g)}\\ &\prod_{k=1}^{\nu/2}(b,\partial_k h)(\bar{b},\bar{\partial}_k h)\prod_{i=1}^{\mu/2}g_s\sqrt{h}c\bar{c}V_i(\sigma_i)\prod_{l=\mu/2+1}^{n}\int d^2\sigma_l\sqrt{h}g_sV_l(\sigma_l)\ ,\end{aligned}\tag{6.17}$$

where $\chi(\Sigma_g)$ is the Euler characteristics $(2-2g)$, and Φ is a vacuum expectation value of a dilaton that gives the string coupling $g_s=e^\Phi$. Let us now consider explicit examples and carry out computations.

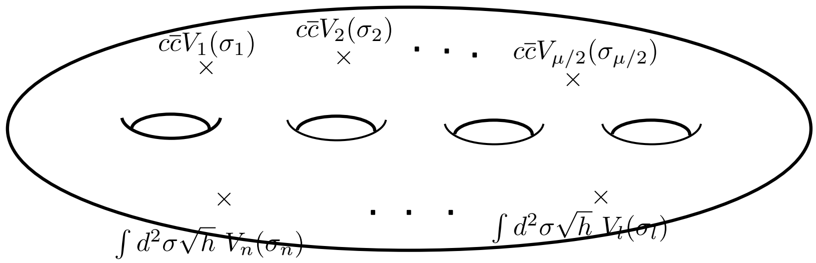


Figure 16: $\mu/2$ vertex operators are paired with $c\bar{c}$ at fixed positions and the others are integrated over the Riemann surface.

6.3 Tree amplitude

The first example is the tree level amplitude in string theory: a sphere with $n \geq 3$ punctures. Since we have $\mu = 6$ and $\nu = 0$ for a sphere, we need to insert at least three vertex operators to fix conformal Killing vectors. Considering a sphere $S^2 \cong \mathbb{C} \cup \{\infty\}$, we give the flat metric locally. Then, the amplitude formula becomes

$$\begin{aligned}
A_{g=0,n} &= g_s^{-2} \int \mathcal{D}[c,\bar{c},b,\bar{b},X]\; e^{-S_\sigma[X,h]-S_{\text{gh}}[b,c]} \prod_{i=1}^{3} c\bar{c}(z_i,\bar{z}_i) g_s V_i(z_i,\bar{z}_i) \prod_{l=4}^{n} \int d^2 z_l g_s V_l(z_l,\bar{z}_l) \\
&= g_s^{n-2} \prod_{l=4}^{n} \int d^2 z_l \left\langle \prod_{j=1}^{n} V_j(z_j,\bar{z}_j) \right\rangle_X \left\langle c\bar{c}(z_1,\bar{z}_1) c\bar{c}(z_2,\bar{z}_2) c\bar{c}(z_3,\bar{z}_3) \right\rangle_{bc} ,
\end{aligned}$$

where we fix the positions of three operators $c\bar{c}(z_i,\bar{z}_i) g_s V(z_i,\bar{z}_i)$ for $i = 1,2,3$. As a result, the amplitude factorizes into the matter sector and the ghost sector.

Let us compute when all the vertex operators are Tachyons (3.34):

$$V_j(z_j,\bar{z}_j) = :e^{ik_j\cdot X(z_j,\bar{z}_j)}: \ .$$

Then, the OPE of the Tachyon vertex operators can be computed from (3.7), which yields

$$\left\langle \prod_{j=1}^{n} V_j(z_j,\bar{z}_j) \right\rangle_X = C_X (2\pi)^D \delta^D\left(\sum k_i\right) \prod_{i<j}^{n} |z_{ij}|^{\alpha' k_i\cdot k_j} .$$

To compute the ghost sector, we perform an integral over the zero modes

$$\begin{aligned}
c(z) &= c_0 + c_1 z + c_2 z^2 , \\
\bar{c}(\bar{z}) &= \bar{c}_0 + \bar{c}_1 \bar{z} + \bar{c}_2 \bar{z}^2 .
\end{aligned}$$

Then, the integral is indeed very simple

$$\begin{aligned}
\left\langle c\bar{c}(z_1,\bar{z}_1) c\bar{c}(z_2,\bar{z}_2) c\bar{c}(z_3,\bar{z}_3) \right\rangle_{bc} =& C_{bc} \int \prod_{i=0}^{2} d\bar{c}_i dc_i \; c(z_1)\bar{c}(\bar{z}_1) c(z_2)\bar{c}(\bar{z}_2) c(z_3)\bar{c}(\bar{z}_3) \\
=& C_{bc} \det \begin{vmatrix} 1 & 1 & 1 \\ z_1 & z_2 & z_3 \\ z_1^2 & z_2^2 & z_3^2 \end{vmatrix} \det \begin{vmatrix} 1 & 1 & 1 \\ \bar{z}_4 & \bar{z}_5 & \bar{z}_6 \\ \bar{z}_4^2 & \bar{z}_5^2 & \bar{z}_6^2 \end{vmatrix} \\
=& C_{bc} |z_{12}|^2 |z_{23}|^2 |z_{31}|^2 .
\end{aligned}$$

Shapiro-Virasoro amplitude

Let us explicitly perform the four-point amplitude

$$A_{0,4} = g_s^2 C_{4\text{pt}} (2\pi)^D \delta^D(\sum k_i) \int d^2 z_4 \prod_{i<j}^{4} |z_{ij}|^{\alpha' k_i \cdot k_j} \prod_{i<j}^{3} |z_{ij}|^2 .$$

To fix the conformal Killing vectors, we can set (z_1, z_2, z_3) to $(0, 1, \infty)$. Then the expression reduces to

$$A_{0,4} = g_s^2 C_{4\text{pt}} (2\pi)^D \delta^D(\sum k_i) B\left(-\frac{\alpha' s}{4} - 1, -\frac{\alpha' t}{4} - 1, -\frac{\alpha' u}{4} - 1\right) , \tag{6.18}$$

where

$$\begin{aligned}
B(a,b,c) &= \int d^2 z |z|^{2a-2} |1-z|^{2b-2} = \pi \frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(a+b)\Gamma(b+c)\Gamma(c+a)} , \\
s &= -k_{1+2}^2 = -k_{3+4}^2 = -2k_1 \cdot k_2 - \frac{8}{\alpha'} , \\
t &= -k_{1+3}^2 = -k_{2+4}^2 = -2k_1 \cdot k_3 - \frac{8}{\alpha'} , \\
u &= -k_{1+4}^2 = -k_{2+3}^2 = -2k_1 \cdot k_4 - \frac{8}{\alpha'} , \qquad (s+t+u = -\frac{16}{\alpha'}) .
\end{aligned}$$

This is called the **Shapiro-Virasoro amplitude**. It is easy to see that Shapiro-Virasoro amplitude has a permutation symmetry among s, t, u. For the derivation of the B function, the reader can refer to [GSW87, §7(Vol.1 pp.386 and pp.373)].

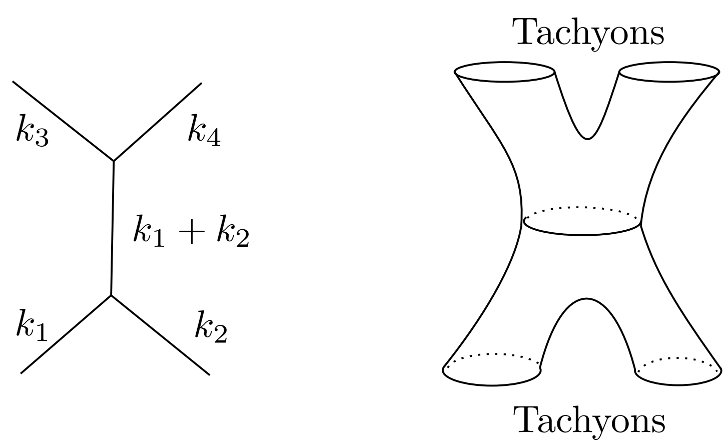


Figure 17: The s-channel of the Shapiro-Virasoro amplitude.

The Γ-function has a pole at non-positive integers. This shows that the amplitude in (6.18) has poles in the s-channel:

$$-\frac{\alpha' s}{4} - 1 \in \mathbb{Z}_{\leq 0}, \qquad s = -k_{1+2}^2 = M^2 = \frac{4}{\alpha'}(n-1) \quad (n \in \mathbb{Z}_{\geq 0}) ,$$

where m is the mass of a propagating state as in Figure 17. Since the mass spectra are the same as in (2.18), this implies that the propagating states are tachyon, graviton, B-field, etc. The four-point amplitude can be divided into two three-point amplitudes as in Figure 17. At the s-channel tachyon pole, it becomes

$$\begin{aligned} A_{0,3} &= a_3(2\pi)^D\delta^D(\sum k_i)\,, \qquad a_3 \simeq Cg_s\,, \\ A_{0,4} &= a_4(2\pi)^D\delta^D(\sum k_i)\,, \qquad a_4 \simeq -\frac{4\pi}{\alpha'}\frac{1}{s+\frac{4}{\alpha'}}Cg_s^2\,. \end{aligned}$$

We can fix the overall constant from unitarity

$$a_4 = \frac{(a_3)^2}{s+\frac{4}{\alpha'}}\,, \qquad \therefore \quad C = -\frac{4\pi}{\alpha'}\,.$$

Although we have considered a closed-string tree-level amplitude here, the result for open-string amplitude, so-called the **Veneziano amplitude** [Ven68], has historically proceeded. The discovery of the Veneziano amplitude is widely acknowledged as the starting point for the developments leading to string theory. The very first examples of string amplitudes by Veneziano [Ven68], Virasoro [Vir69], Shapiro [Sha70], as well as Koba and Nielsen [KN69a, KN69b] appeared long before the formulation of string theory. The early history of string theory before the superstring revolution in 1984 is recounted in [CCCDV12].

6.4 One-loop amplitude

Let us consider a torus amplitude which is analogous to one-loop amplitude so that it encodes quantum corrections. First, we will use the cylindrical holomorphic coordinate $w = t + i\sigma$ in Figure 5 with periodicity (6.1).

As seen in §6.2, there are two constant conformal Killing vectors and metric moduli. Note that an integral (with an appropriate normalization) over a torus automatically picks up zero modes

$$c(0) = \int \frac{d^2w}{4\pi^2\tau_2}c(w)$$

(The other zero modes $\overline{c}(0), b(0), \overline{b}(0)$ are similarly obtained.)

Let us consider an infinitesimal deformation of the flat metric

$$ds^2 = dwd\overline{w} \to d(w+\epsilon\overline{w})d(\overline{w}+\overline{\epsilon}w)\,,$$

where $\delta h_{ww} = \overline{\epsilon}$ and $h_{\overline{w}\overline{w}} = \epsilon$. One can set a new isothermal coordinate is $\widetilde{w} = w + \epsilon\overline{w}$. This changes the period of a torus to

$$\widetilde{w} = \widetilde{w} + (2\pi(1+\epsilon), 2\pi(\tau+\epsilon\overline{\tau}))\,.$$

Consequently, the complex modulus becomes

$$\widetilde{\tau} = \frac{\tau+\epsilon\overline{\tau}}{1+\epsilon} \simeq \tau - 2i\epsilon\tau_2$$

so that $\delta\tau = -2i\epsilon\tau_2$. Thus, a tangent vector of the metric moduli is

$$\partial_\tau h_{\overline{w}\overline{w}} = \frac{\delta h_{\overline{w}\overline{w}}}{\delta\tau} = \frac{i}{2\tau_2}\,,$$

so that we have

$$(b, \partial_\tau h) = \int \frac{d^2 w}{4\pi} \sqrt{h}\, b_{ww}(w) \frac{\partial}{\partial \tau} h_{\overline{w}\overline{w}}(\tau) = 2\pi i b_{ww}(0) \ .$$

The torus amplitude is expressed as an integral over the fundamental region F in Figure 11 with bc zero modes insertion

$$A_{1,n} = g_s^n \frac{1}{2} \int_F d^2\tau \Big\langle (b, \partial_\tau h)(\overline{b}, \partial_{\overline{\tau}} h) c(0)\overline{c}(0) \sqrt{h} V_1(0) \prod_{j=2}^{n} \int d^2 w_j \sqrt{h} V_j(w_j, \overline{w}_j) \Big\rangle \ .$$

where $\frac{1}{2}$ is due to the symmetry $w \to -w$. Since the zero mode on a torus can be obtained by integrating an operator out, the expression can be manipulated into

$$A_{1,n} = g_s^n \frac{1}{2} \int_F \frac{d^2\tau}{\tau_2} \langle b(0)\overline{b}(0)c(0)\overline{c}(0) \rangle_{bc} \Big\langle \prod_{j=1}^{n} \int d^2 w_j \sqrt{h} V_j(w_j, \overline{w}_j) \Big\rangle_X \ .$$

Torus partition function for bosonic string

Let us explicitly compute the amplitude without vertex operators ($n = 0$):

$$A_{1,0} = \frac{1}{2} \int_F \frac{d^2\tau}{\tau_2} \langle b(0)\overline{b}(0)c(0)\overline{c}(0) \rangle_{bc} \langle 1 \rangle_X \ .$$

The Feynman path integral over a torus with complex structure τ can be written in terms of the Hamiltonian formalism

$$\langle 1 \rangle_X = \mathrm{Tr} \exp\left[2\pi i \left(\tau_1 P + i\tau_2 H\right)\right] = \mathrm{Tr}\left[q^{L_0^X - \frac{c^X}{24}} \overline{q}^{\overline{L}_0^X - \frac{c^X}{24}} \right] \ ,$$

where $q = e^{2\pi i \tau}$. Note that the space and time translation are generated by the zero mode of the Virasoro generator (2.15)

$$P = L_0^X - \overline{L}_0^X \ , \qquad H = L_0^X + \overline{L}_0^X - \frac{c^X}{12} \ .$$

The trace is taken over the Hilbert space (2.14). Writing the holomorphic Fock space with a fixed target coordinate as

$$|n_1, n_2, n_3, \cdots ; k\rangle = \alpha_{-1}^{n_1} \alpha_{-2}^{n_2} \alpha_{-3}^{n_3} \cdots |0; k\rangle, \quad \text{with} \quad n_i \in \mathbb{Z}_{\geq 0} \ , \tag{6.19}$$

its generating function can be written as

$$\begin{aligned} \mathrm{Tr}_{\mathcal{H}^X}(q^{L_0 - \frac{c}{24}}) &= q^{-\frac{1}{24}} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \Big\langle k; n_1, n_2, \cdots \Big| q^{L_0^X} \Big| n_1, n_2, \cdots ; k \Big\rangle \\ &= q^{\frac{\alpha'}{4}k^2 - \frac{1}{24}} \prod_{n=1}^{\infty} \frac{1}{1 - q^n} = q^{\frac{\alpha'}{4}k^2} \frac{1}{\eta(\tau)} \ , \end{aligned}$$

where $\eta(\tau)$ is the **Dedekind eta-function**. Hence, we have the bosonic torus partition function

$$\langle 1 \rangle_X = \int \frac{d^D x d^D p}{(2\pi)^D} \exp\left(-4\pi\tau_2 \frac{\alpha'}{4} p^2\right) \left| \frac{q^{-\frac{1}{24}}}{\prod_{n\geq 1}(1 - q^n)} \right|^{2D}$$

$$= i\frac{V_D}{(2\pi\ell_s)^D}(\tau_2)^{-D/2}|\eta(\tau)|^{-2D} ,$$

where i in the second line is from Wick-rotation of spacetime momentum $p^0 \to ip_E^0$.

Similarly, we can write the ghost partition function over a torus with complex structure τ as

$$\left\langle b(0)\overline{b}(0)c(0)\overline{c}(0)\right\rangle_{bc} = \mathrm{Tr}\left[(-1)^F b_0\overline{b}_0 c_0\overline{c}_0 q^{L_0^g-\frac{c^g}{24}}\overline{q}^{\overline{L}_0^g-\frac{c^g}{24}}\right]$$

with $c^g = -26$, and the zero mode of the Virasoro generator (5.15) is

$$L_0^g = -\sum_{n\in\mathbb{Z}} n : b_n c_{-n} : -1 . \tag{6.20}$$

Here F is the fermion number operator; $F = 1$ for fermion and $F = 0$ for boson. The insertion of $(-1)^F$ means that the periodic boundary condition is imposed on the ghost fields along the time circle. The trace is taken over the Hilbert space spanned by physical states that are constructed creation operators b_{-n}, c_{-m} for $n, m \in \mathbb{Z}_{>0}$ (as well as anti-holomorphic operators) on the bc ghost vacuum $|\downarrow\downarrow\rangle$

$$\mathrm{Tr}_{\mathcal{H}^g}[(-1)^F b_0 c_0 q^{L_0^g-\frac{c^g}{24}}] = q^{-1-\frac{c^g}{24}}\prod_{n\geq 1}(1-q^n)^2$$

Thus, we have

$$\left\langle b(0)\overline{b}(0)c(0)\overline{c}(0)\right\rangle_{bc} = q^{-1-\frac{c^g}{24}}\overline{q}^{-1-\frac{c^g}{24}}\prod_{n\geq 1}(1-q^n)^2(1-\overline{q}^n)^2 = |\eta(\tau)|^4 .$$

In conclusion, the torus amplitude is

$$A_{1,0} = \frac{iV_D}{(2\pi\ell_s)^D}\int_F \frac{d^2\tau}{2\tau_2}(\tau_2)^{-D/2}|\eta(\tau)|^{-2(D-2)} . \tag{6.21}$$

At the critical dimension $D = 26$, the expansion of the Dedekind eta-function

$$|\eta(\tau)|^{-48} \simeq \left|q^{-1} + 24 + \mathcal{O}(q)\right|^2 ,$$

gives the spectrum of the bosonic string theory. Furthermore, it is invariant under $\mathrm{PSL}(2,\mathbb{Z})$. See §7.4.

Note that the limit $\tau_2 \to 0$ describes the ultraviolet (UV) regime since the Euclidean time becomes small. The integral over the fundamental domain F avoids this UV region. Consequently, there is no UV divergence in one-loop amplitude of the bosonic string theory thanks to the modular invariance of a torus.

7 Superstring theories

We have seen so far that bosonic strings suffer from two major problems:

- Their spectrum always contains a tachyon. In that respect, their vacuum is unstable.

- They do not contain spacetime fermions. This lack of fermionic states is in contrast to observations and makes the bosonic string unrealistic.

Both of these challenges are remedied in superstring theory. Supersymmetry is a symmetry that exchanges bosons and fermions. The world-sheet superstring theory consists of a bosonic and a fermionic sector. The bosonic sector is identical to the world-sheet theory of the bosonic string. Therefore, we can view our efforts up to now as a preliminary study of one-half of the superstring theory. In fact, we will see that the presence of fermions resolves the problem of Tachyon. Moreover, we will learn that the critical dimension of superstring theory is $D = 10$.

There are five superstring theories as follows, and we will study them in this order.

Type IIA & IIB

Oriented string theories that can incorporate open strings if there are D-branes. IIA: Ramond ground states with opposite chirality, and Dp-branes (p even). IIB: Ramond ground states with the same chirality, and Dp-branes (p odd).

Type I

An open and closed unoriented string theory, including Yang-Mills degrees of freedom with SO(32) gauge group, that can incorporate D1, D5, D9-branes.

Heterotic SO(32) **&** $E_8 \times E_8$

Type II right-movers & bosonic left-movers, including Yang-Mills degrees of freedom with either SO(32) or $E_8 \times E_8$ gauge group.

There exist two major formulations of superstring theory. Both formulations enjoy supersymmetry on the world-sheet and in spacetime, but they differ in the following respect:

• In the **Ramond-Neveu-Schwarz (RNS) formulation**, supersymmetry is manifest on the world-sheet, but not in spacetime.

• In the **Green-Schwarz (GS) formulation** [GSW87, §5] [BBS06, §5], supersymmetry is manifest in spacetime, but not on the world-sheet .

More recently, the pure-spinor formalism [Ber04] has been developed as yet another approach to the superstring. In this course, we will only discuss RNS formalism.

7.1 RNS formulation

With the complex coordinate convention, the action becomes

$$S^{\mathrm{m}} = \frac{1}{4\pi}\int d^2z \left(\frac{2}{\alpha'}\partial X^\mu \overline{\partial} X_\mu + \psi^\mu \overline{\partial}\psi_\mu + \overline{\psi}^\mu \partial \overline{\psi}_\mu\right) \tag{7.1}$$

where the equations of motion tell us $\psi^\mu(z)$ (resp. $\overline{\psi}^\mu(\overline{z})$) is chiral (resp. anti-chiral). The action is invariant under **supersymmetric transformation** (Exercise)

$$\delta X^\mu = -\sqrt{\frac{\alpha'}{2}}\left(\epsilon\psi^\mu + \overline{\epsilon}\overline{\psi}^\mu\right) , \qquad \delta\psi^\mu = \sqrt{\frac{2}{\alpha'}}\,\epsilon\partial X^\mu , \qquad \delta\overline{\psi}^\mu = \sqrt{\frac{2}{\alpha'}}\,\overline{\epsilon}\overline{\partial} X^\mu . \tag{7.2}$$

The Noether theorem implies that there are currents for the supersymmetry

$$T_F(z) = i\sqrt{\frac{2}{\alpha'}}\psi^\mu \partial X^\mu , \qquad \overline{T}_F(z) = i\sqrt{\frac{2}{\alpha'}}\overline{\psi}^\mu \overline{\partial} X^\mu . \tag{7.3}$$

which is called **supercurrents**. Indeed, X^μ, ψ^μ and $\overline{\psi}^\mu$ are primary fields of conformal dimension (0,0), $(\frac{1}{2},0)$ $(0,\frac{1}{2})$, respectively, and therefore their OPEs are

$$X^\mu(z,\overline{z})X^\nu(0,0) \sim -\sqrt{\frac{\alpha'}{2}}\eta^{\mu\nu}\ln|z|^2\,, \quad \psi^\mu(z)\psi^\nu(0) \sim \frac{\eta^{\mu\nu}}{z}\,, \quad \overline{\psi}^\mu(\overline{z})\overline{\psi}^\nu(0) \sim \frac{\eta^{\mu\nu}}{\overline{z}}\,. \tag{7.4}$$

Using the OPEs, one can show the supersymmetric transformation (7.2) (exercise).

The energy-momentum tensor of the action (7.1) is

$$T_B(z) = -\frac{1}{\alpha'}\partial_z X^\mu \partial X_\mu - \frac{1}{2}\psi^\mu\partial\psi_\mu \tag{7.5}$$

along with their complex conjugates $\overline{T}_B$, $\overline{T}_F$. Their OPEs can be computed by using (7.4)

$$\begin{aligned}
T_B(z)T_B(w) &\sim \frac{3D}{4(z-w)^4} + \frac{2T_B(w)}{(z-w)^2} + \frac{\partial_w T_B(w)}{z-w} \\
T_B(z)T_F(w) &\sim \frac{3T_F(w)}{2(z-w)^2} + \frac{\partial_w T_F(w)}{z-w} \\
T_F(z)T_F(w) &\sim \frac{D}{(z-w)^3} + \frac{2T_B(w)}{z-w}\,.
\end{aligned} \tag{7.6}$$

and similarly for the anti-chiral part. The central charge of the theory is

$$c^{\mathrm{m}} = \frac{3}{2}D \tag{7.7}$$

where each scalar and fermion contributes 1 and 1/2, respectively.

Ramond vs Neveu-Schwarz

In superstring theory, the fermionic fields on the closed string may be either periodic or anti-periodic on the circle around the string, corresponding to two different spinor bundles. It is conventional to denote these spin structures by **Ramond (R)** and **Neveu-Schwarz (NS)**, defined as follows.

$$\begin{aligned}
&\psi^\mu(t,\sigma+2\pi) = +\psi^\mu(t,\sigma) && \text{R: periodic on cylinder} \\
&\psi^\mu(t,\sigma+2\pi) = -\psi^\mu(t,\sigma) && \text{NS: anti-periodic on cylinder}
\end{aligned} \tag{7.8}$$

As in Figure 5, the mapping $z = e^{-iw}$ from the cylinder $w = \sigma + it$ to the 2-plane $z \in \mathbb{C}$ is a conformal map (Figure above). Under the conformal map, the primary field ψ^μ with conformal dimension $(\frac{1}{2},0)$ is transformed as

$$\psi^\mu(z) = \left(\frac{dz}{dw}\right)^{-\frac{1}{2}}\psi^\mu(w) = \text{const} \times e^{-i\frac{w}{2}}\psi^\mu(w)\,.$$

Hence the (anti-)periodicity assignments are reversed between the cylinders and the plane:

$$\begin{aligned}
&\psi^\mu(e^{2\pi i}z) = -\psi^\mu(z) && \text{R: anti-periodic on plane} \\
&\psi^\mu(e^{2\pi i}z) = +\psi^\mu(z) && \text{NS: periodic on plane}
\end{aligned} \tag{7.9}$$

The boundary conditions for anti-chiral fields $\overline{\psi}^\mu$ are defined in a similar fashion.

As in the bosonic string, one can decompose ψ^μ and $\overline{\psi}^\mu$ in modes

$$\psi^\mu(z) = \sum_{n\in\mathbb{Z}+\nu} \frac{\psi_n^\mu}{z^{n+1/2}}\,, \qquad \overline{\psi}^\mu(\overline{z}) = \sum_{n\in\mathbb{Z}+\nu} \frac{\overline{\psi}_n^\mu}{\overline{z}^{n+1/2}}$$

where ν takes the values 0 (R) and $\frac{1}{2}$ (NS). The canonical quantization leads to the algebra

$$\{\psi_m^\mu, \psi_n^\nu\} = \eta^{\mu\nu}\delta_{m+n,0} \qquad \{\overline{\psi}_m^\mu, \overline{\psi}_n^\nu\} = \eta^{\mu\nu}\delta_{m+n,0}\,. \tag{7.10}$$

The mode expansion must be carried out with care here since we must distinguish between Ramond and Neveu-Schwarz sectors.

$$T_B(z) = \sum_{m\in\mathbb{Z}} \frac{L_m}{z^{m+2}}\,, \qquad T_F(z) = \sum_{r\in\mathbb{Z}+\nu} \frac{G_r}{z^{r+3/2}}\,, \tag{7.11}$$

where the generators can be written in terms of the modes (exercise)

$$\begin{aligned} L_m^{\mathrm{m}} &= \frac{1}{2}\sum_{n\in\mathbb{Z}} :\alpha_{m-n}\alpha_n: + \frac{1}{4}\sum_{r\in\mathbb{Z}+\nu}(2r-m):\psi_{m-r}\psi_r: + a^{\mathrm{m}}\delta_{m,0} \\ G_r^{\mathrm{m}} &= \sum_{n\in\mathbb{Z}} \alpha_n\psi_{r-n}\,. \end{aligned} \tag{7.12}$$

The normal ordering constant a can be determined like in the bosonic string theory. Each periodic boson contributes $-\frac{1}{24}$. The fermionic contributions are

$$\begin{aligned} -\tfrac{1}{2}\sum_{r=0}^{\infty} r &= \tfrac{1}{24} &\qquad& \text{R-sector} \\ -\tfrac{1}{2}\sum_{r=0}^{\infty}(r+\tfrac{1}{2}) &= -\tfrac{1}{48} && \text{NS-sector}\,. \end{aligned} \tag{7.13}$$

Including the shift $\frac{1}{24}c = \frac{1}{16}D$ gives

$$\begin{aligned} a^{\mathrm{m}} &= \tfrac{1}{24}c^{\mathrm{m}} + \Big(-\tfrac{1}{24} + \tfrac{1}{24}\Big)D = \tfrac{1}{16}D &\qquad& \text{R-sector} \\ a^{\mathrm{m}} &= \tfrac{1}{24}c^{\mathrm{m}} + \Big(-\tfrac{1}{24} - \tfrac{1}{48}\Big)D = 0 && \text{NS-sector}\,. \end{aligned} \tag{7.14}$$

In fact, the generators L_m and G_r form the algebra called the $\mathcal{N}=1$ **superconformal algebra** with central charge (7.7) (exercise).

Ghost CFT

In the bosonic string theory, we study the BRST quantization with the Faddeev-Popov bc ghost. In superstring theory, ghost fields also appear with their supersymmetric partners:

$$S^{\mathrm{gh}} = \frac{1}{2\pi}\int d^2z(b\overline{\partial}c + \beta\overline{\partial}\gamma)$$

where b, c are fermionic and β, γ are bosonic fields. Hence, the standard method tells us the $\beta\gamma$ OPEs

$$\gamma(z)\beta(w) = -\beta(z)\gamma(w) \sim \frac{1}{z-w}$$

We have seen that the conformal dimensions of X and ψ differ by $\frac{1}{2}$. This is the same for the ghost fields. Since the b and c ghosts have conformal dimensions 2 and -1 respectively, β and γ are primary fields of conformal dimensions $(\frac{3}{2}, 0)$ and $(-\frac{1}{2}, 0)$ respectively. Hence the form of the energy-momentum tensor and the supercurrent are

$$
\begin{aligned}
T_B^{\mathrm{gh}}(z) &=: (\partial b)c : -2\,\partial : bc : + : (\partial\beta)\gamma : -\frac{3}{2}\,\partial : \beta\gamma : \\
T_G^{\mathrm{gh}}(z) &= (\partial\beta)c + \frac{3}{2}\beta\partial c - 2b\gamma \,.
\end{aligned}
\tag{7.15}
$$

Then, the TT OPE determines the central charge of the ghost SCFT. The bc system contributes -26 to the central charge as we know, while the $\beta\gamma$ system contributes $+11$. Hence the total central charge

$$
c^{\mathrm{tot}} = c^{\mathrm{m}} + c^{\mathrm{gh}} = \tfrac{3}{2}D - 26 + 11 \,.
$$

Then, we happily obtain the critical dimension $D = 10$ of superstring theory if we impose the Weyl-anomaly-free condition $c^{\mathrm{tot}} = 0$. In the following, we assume $D = 10$.

Now let us write the Virasoro generators of the ghost SCFT. The $\beta\gamma$ ghosts have the same boundary condition as the fermionic fields $\psi^\mu \overline{\psi}^\mu$ so that we have the mode expansions

$$
\beta(z) = \sum_{r\in\mathbb{Z}+\nu} \frac{\beta_r}{z^{r+\frac{3}{2}}} \,, \quad \gamma(z) = \sum_{r\in\mathbb{Z}+\nu} \frac{\gamma_r}{z^{r-\frac{1}{2}}} \,,
$$

which satisfy the commutation relation

$$
[\beta_r, \gamma_s] = \delta_{r,-s} \,.
$$

Using these modes, we express the Virasoro generator of the ghost SCFT as

$$
\begin{aligned}
L_m^{\mathrm{gh}} &= \sum_{n\in\mathbb{Z}} (2m - n) : b_n c_{m-n} : + \frac{1}{2} \sum_{r\in\mathbb{Z}+\nu} (m + 2r) : \beta_{m-r}\gamma_r : + a^{\mathrm{gh}}\delta_{m,0} \\
G_r^{\mathrm{gh}} &= \sum_{n\in\mathbb{Z}} \left[\frac{1}{2}(n + 2r)\beta_{r-n}c_n + 2b_n\gamma_{r-n} \right]
\end{aligned}
\tag{7.16}
$$

Again, using the commutation relations of the ghost modes, one can determine the normal ordering constant

$$
\begin{aligned}
a^{\mathrm{gh}} &= \tfrac{-15}{24} + \left(\tfrac{1}{12} - \tfrac{1}{12} \right) = -\tfrac{5}{8} \qquad \text{R-sector} \\
a^{\mathrm{gh}} &= \tfrac{-15}{24} + \left(\tfrac{1}{12} + \tfrac{1}{24} \right) = -\tfrac{1}{2} \qquad \text{NS-sector} \,.
\end{aligned}
$$

Combining them with (7.14) at $D = 10$, we have the total vacuum energy

$$
\begin{aligned}
a^{\mathrm{tot}} &= 0 \qquad \text{R-sector} \\
a^{\mathrm{tot}} &= -\tfrac{1}{2} \qquad \text{NS-sector} \,.
\end{aligned}
$$

In the R sector, the vacuum energy is zero so that the Tachyon is absent. On the other hand, there is still the Tachyon in the NS sector. This will be projected out by the GSO projection, as we will see below.

7.2 Physical spectrum and the GSO Projection

Before discussing the GSO projection, let us study the fermionic spectrum generated by fermionic modes ψ^μ_r. We first consider the NS spectrum since it's simpler. Since r takes half integers, we can define the ground state of the NS sector as

$$\psi^\mu_r|0;k\rangle_{\rm NS} = 0 \quad \text{for } r > 0 \, .$$

When we include the ghost part of the vertex operator, it contributes to the total fermion number F, so that the matter plus ghost ground state has the odd fermion number

$$(-1)^F|0;k\rangle_{\rm NS} = -|0;k\rangle_{\rm NS} \, . \tag{7.17}$$

Because there exist the zero modes ψ^μ_0, R-sector is more subtle. In fact, the canonical commutation relation (7.10) of the zero modes satisfies the **Clifford algebra**

$$\{\sqrt{2}\psi^\mu_0, \sqrt{2}\psi^\nu_0\} = 2\eta^{\mu\nu} \, .$$

Therefore, we can identify them with Gamma matrices $\Gamma^\mu = \sqrt{2}\psi^\mu_0$, and the ground state of R-sector becomes the spin representation of $\mathrm{SO}(1, D-1)$. For mathematics of Clifford algebra, spin group, and spin representations, we refer to [Pol98, Appendix B]. However, we can heuristically understand the spin representation as follows. The following basis for this representation is often convenient:

$$\begin{aligned} \Gamma^\pm_i &= \frac{1}{\sqrt{2}}\left(\psi^{2i}_0 \pm i\psi^{2i+1}_0\right) \qquad i = 1, \ldots, 4 \\ \Gamma^\pm_0 &= \frac{1}{\sqrt{2}}\left(\psi^1_0 \pm \psi^0_0\right) \end{aligned} \tag{7.18}$$

In this basis, the Clifford algebra takes the form

$$\{\Gamma^+_i, \Gamma^-_j\} = \delta_{ij} \, , \qquad \{\Gamma^+_i, \Gamma^+_j\} = 0 = \{\Gamma^-_i, \Gamma^-_j\} \, . \tag{7.19}$$

The $\Gamma^\pm_i$, $i = 0, \ldots, 4$ act as raising and lowering operators, generating the $2^5 = 32$ Ramond ground states:

$$|s_0, s_1, s_2, s_3, s_4; k\rangle = |\mathbf{s}; k\rangle \tag{7.20}$$

where each of the s_i is $\pm\frac{1}{2}$, and where

$$\Gamma^-_i| -\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}; k\rangle = 0 \tag{7.21}$$

while Γ^+_i raises s_i from $-\frac{1}{2}$ to $\frac{1}{2}$. One can further define the chirality operator

$$\Gamma_{11} = (2)^5 \psi^0_0 \psi^1_0 \psi^2_0 \cdots \psi^9_0 \, , \tag{7.22}$$

which acts on $|\mathbf{s}\rangle$ as

$$\Gamma_{11}|\mathbf{s}; k\rangle = (-1)^F|\mathbf{s}; k\rangle = \left\{ \begin{array}{ll} +|\mathbf{s}; k\rangle & \quad \text{even \# of } -\frac{1}{2} \\ -|\mathbf{s}; k\rangle & \quad \text{odd \# of } -\frac{1}{2} \end{array} \right. \, .$$

Hence, the Dirac representation **32** decomposes into a $\mathbf{16_s}$ with an even number of $-\frac{1}{2}$'s and $\mathbf{16_c}$ with an odd number.

$$\mathbf{32} = \mathbf{16_s} \oplus \mathbf{16_c} \, . \tag{7.23}$$

Now let us study the physical spectrum of superstring theories including the GSO projection. In principle, we can apply the BRST quantization scheme. However, we do not need the full use of BRST quantization, indeed. To take a shortcut, we can first impose to a physical state $|\psi\rangle$

$$L_n^{\rm m}|\psi\rangle = 0 \quad (n>0)\,, \qquad G_r^{\rm m}|\psi\rangle = 0 \quad (r\ge 0)\,. \tag{7.24}$$

Since one can check (See [Pol98, (10.5.23)])

$$\{Q_B, b_n\} = L_n\,, \qquad [Q_B,\beta_r] = G_r\,,$$

the physical states are defined modulo

$$L_n^{\rm m}|\chi\rangle \cong 0\,, \qquad G_r^{\rm m}|\chi\rangle \cong 0\,, \qquad \text{for } \; n,r<0\,. \tag{7.25}$$

Note that the BRST current is defined

$$j_B = cT_B^{\rm m} + \gamma T_F^{\rm m} + \frac{1}{2}\left(cT_B^{\rm gh} + \gamma T_F^{\rm gh}\right)\,.$$

Furthermore, in the RNS theory, we need to impose the **GSO (Gliozzi-Scherk-Olive) projection** in order to have an equal number of bosonic and fermionic states at each mass level.

In the NS sector, the GSO projection is just to remove states with odd fermion number so that the GSO projection operator on the NS sector is expressed as

$$P_{\rm GSO}^{\rm NS} = \frac{1+(-1)^F}{2} \qquad \text{NS sector}\,. \tag{7.26}$$

At level 0, we have the Tachyon $|0;k\rangle_{\rm NS}$. However, the GSO projection removes this state because it has an odd fermion number as in (7.17). At level $\frac{1}{2}$, we have a massless state with vector polarization

$$|\zeta;k\rangle_{\rm NS} = \zeta\cdot\psi_{-\frac{1}{2}}|0;k\rangle_{\rm NS}\,,$$

with even fermion number so we need to keep it in the spectrum. The physical state conditions (7.24) are

$$\begin{aligned} 0 &= L_0|\zeta;k\rangle_{\rm NS} = \alpha' k^2|\zeta;k\rangle_{\rm NS} \\ 0 &= G_{\frac{1}{2}}^{\rm m}|\zeta;k\rangle_{\rm NS} = \sqrt{2\alpha'}\,k\cdot\zeta|0;k\rangle_{\rm NS}\,. \end{aligned} \tag{7.27}$$

while there is a Q_B-exact condition (7.25)

$$G_{-\frac{1}{2}}^{\rm m}|0;k\rangle_{\rm NS} = \sqrt{2\alpha'}\,k\cdot\psi_{-\frac{1}{2}}|0;k\rangle_{\rm NS}\,.$$

Therefore, we have

$$k^2 = 0\,, \quad k\cdot\zeta = 0\,, \quad \zeta^\mu \cong \zeta^\mu + k^\mu\,.$$

Thus, there are degrees of freedom for 8 spacelike polarizations which form the vector representation $\mathbf{8_v}$ of SO(8).

A Ramond ground state that is massless can be expressed with spinor polarization

$$|u;k\rangle_{\rm R} = u_{\mathbf{s}}|\mathbf{s};k\rangle_{\rm R}\,.$$

The physical state conditions (7.24)

$$0 = G_0^{\mathrm{m}}|u;k\rangle_{\mathrm{R}} = \sqrt{\alpha'}\ u_{\mathbf{s}}k\cdot\Gamma_{\mathbf{ss'}}|\mathbf{s}';k\rangle_{\mathrm{R}}$$

leads to the Dirac equation

$$u\,k\cdot\Gamma = 0\ .$$

By choosing the momentum vector $k^\mu = (k,k,0,\ldots,0)$, this amounts to

$$\Gamma_0^+ u = 0 \quad\longrightarrow\quad s_0 = +\tfrac{1}{2}\ , \tag{7.28}$$

giving 16 degeneracies $|+,s_1,s_2,s_3,s_4\rangle$ for the physical Ramond vacuum. This is a representation **16** of SO(8) which again decomposes into $\mathbf{8_s}$ with an even number of $-\frac{1}{2}$'s and $\mathbf{8_c}$ with an odd number:

$$\mathbf{16} = \mathbf{8_s}\oplus\mathbf{8_c}\ . \tag{7.29}$$

In the R sector, the GSO projections will pick one of these two irreducible representations, and therefore the GSO projection operators can be written as

$$P^{\mathrm{R}\pm}_{\mathrm{GSO}} = \frac{1\pm(-1)^F}{2} \qquad \text{R sector}\ . \tag{7.30}$$

Indeed, the two choices $\mathbf{8_s}$ and $\mathbf{8_c}$ differ by the spacetime parity redefinition.

7.3 Torus partition functions for superstring theory

Let us see the role of the GSO projections in the torus (one-loop) partition function of superstring theory. Although the computation is very similar to that in §6.4, fermionic torus partition functions are more interesting due to boundary conditions along circles.

Since the critical dimension of superstring theory is $D = 10$, the bosonic contribution can be read off from (6.21). Consequently, the torus partition function for superstring theory can be written as follows

$$Z = \frac{iV_{10}}{(2\pi\ell_s)^{10}}\int\frac{d^2\tau}{2\tau_2^2}\,\frac{1}{\tau_2^4|\eta(\tau)|^{16}}Z_{\mathrm{F}}(\tau)\overline{Z}_{\mathrm{F}}(\overline{\tau})\ ,$$

where Z_F is ψ contribution and $\overline{Z}_F$ is $\overline{\psi}$ contribution.

As usual, we will focus on the holomorphic sector Z_F and we sum over all the possible boundary conditions for fermions. Since the NS & R boundary conditions give rise to disconnected Fock spaces, we can introduce a relative phase $e^{i\theta}$ between them. Hence, the partition function in the Hamilton formalism is written as follows.

$$\begin{aligned} Z_{\mathrm{F}}(\tau) &= Z^+_{\mathrm{NS}} + Z^-_{\mathrm{NS}} + e^{i\theta}Z^+_{\mathrm{R}} + e^{i\theta}Z^-_{\mathrm{R}}\ ,\\ Z^\pm_{\mathrm{S}} &= \mathrm{Tr}\left[(\pm)^F e^{2\pi i\tau H_{\mathrm{S}}}\right]\ ,\end{aligned}$$

where S is either NS or R, and the superscript is an option for periodicity of world-sheet time t direction ($-$ is periodic). In the light-cone gauge, the "Hamiltonian"s for fermions are

$$H_{\mathrm{NS}} = L_0 - \frac{c}{24} = \sum_{r=\frac{1}{2}}^{\infty} r\psi_{-r}\cdot\psi_r - \frac{D-2}{48}\ ,$$

$$H_{\rm R} = L_0 - \frac{c}{24} = \sum_{r=1}^{\infty} r\psi_{-r}\cdot\psi_r + \frac{D-2}{24}\,, \tag{7.31}$$

where D should be 10 for superstring, and the light-cone gauge means that the spacetime indices i run from 2 to 9, namely

$$\psi\cdot\psi = \sum_{i=2}^{9}\psi^i\psi^i.$$

Although the ground state of the NS sector is unique, the R sector has vacuum degeneracies as in (7.20) before imposing any physical condition. However, the physical state condition requires $s_0 = +\frac{1}{2}$ (7.28) so that there are $2^4 = 16$ degeneracies on which the operator $(-1)^F$ acts as

$$(-1)^F|\mathbf{s};k\rangle = (-1)^{\sum_{i=0}^{4}\left(\frac{1}{2}+s_i\right)}|\mathbf{s};k\rangle$$

Therefore, we have

$$\sum_{\mathbf{s}}\langle\mathbf{s};k|(\pm 1)^F|\mathbf{s};k\rangle = \begin{cases}16 & \text{for + sign}\\ 0 & \text{for − sign,}\end{cases},$$

where sum over s_i $(i = 1,2,3,4)$ is understood. In the presence of the fermion zero modes, the index $\mathrm{Tr}(-1)^F$ vanishes because $s_i = \pm\frac{1}{2}$ gives the opposite sign. (This is the generic feature of the index.)

Counting the eigenvalues of (7.31) in the fermionic Fock spaces with a given boundary condition, the torus partition functions are therefore written as

$$\begin{aligned}
Z_{\rm NS}^{+} &= \mathrm{Tr}\left[e^{2\pi i\tau H_{\rm NS}}\right] = q^{-\frac{1}{6}}\prod_{n=1}^{\infty}(1+q^{n-\frac{1}{2}})^8 = \left(\frac{\vartheta_3(\tau)}{\eta(\tau)}\right)^4,\\
Z_{\rm NS}^{-} &= \mathrm{Tr}\left[(-1)^F e^{2\pi i\tau H_{\rm NS}}\right] = -q^{-\frac{1}{6}}\prod_{n=1}^{\infty}(1-q^{n-\frac{1}{2}})^8 = -\left(\frac{\vartheta_4(\tau)}{\eta(\tau)}\right)^4,\\
Z_{\rm R}^{+} &= \mathrm{Tr}\left[e^{2\pi i\tau H_{\rm R}}\right] = 16q^{\frac{1}{3}}\prod_{n=1}^{\infty}(1+q^{n})^8 = \left(\frac{\vartheta_2(\tau)}{\eta(\tau)}\right)^4,\\
Z_{\rm R}^{-} &= \mathrm{Tr}\left[(-1)^F e^{2\pi i\tau H_{\rm R}}\right] = 0q^{\frac{1}{3}}\prod_{n=1}^{\infty}(1-q^{n})^8 = \left(\frac{\vartheta_1(\tau)}{\eta(\tau)}\right)^4 = 0\,,
\end{aligned}$$

where $q = e^{2\pi i\tau}$, and modular functions are summarized in §7.4. Remarkably, we obtain the Jacobi theta functions $\vartheta_i(\tau)$ $(i = 1,\ldots,4)$ depending on boundary conditions. Moreover, the $\mathrm{SL}(2,\mathbb{Z})$ action on the boundary condition over a torus easily tells us their modular properties as in Figure 18. Note that the minus sign in $Z_{\rm NS}^{-}$ comes from the fact (7.17) that the NS vacuum is fermionic.

Therefore, the torus partition function for fermions is given by

$$Z_{\rm F}(\tau) = \frac{1}{\eta^4(\tau)}\left\{\vartheta_3(\tau))^4 - (\vartheta_4(\tau))^4 + e^{i\theta}(\vartheta_2(\tau))^4\right\}$$

In order for $Z_{\rm F}(\tau)$ to be modular invariant, the relative phase factor is uniquely determined as $e^{i\theta} = -1$ so that

$$Z_{\rm F}(\tau) = \frac{1}{\eta^4(\tau)}\left\{\vartheta_3(\tau))^4 - (\vartheta_4(\tau))^4 - (\vartheta_2(\tau))^4\right\} = 0\,.$$

In fact, the partition function is zero due to supersymmetry and this is called the Jacobi-Riemann identity (7.34). The partition function is zero so that it is trivially modular invariant.

The combination derived above is indeed expressed by using the GSO projection operators (7.26) (7.30):

$$\begin{aligned} Z_{\mathrm{F}}(\tau) =& 2\,\mathrm{Tr}\left[\frac{1+(-1)^F}{2}e^{2\pi i\tau H_{\mathrm{NS}}}\right] - 2\,\mathrm{Tr}\left[\frac{1\pm(-1)^F}{2}e^{2\pi i\tau H_{\mathrm{R}}}\right] \\ =& 2\,\mathrm{Tr}\left[P^{\mathrm{NS}}_{\mathrm{GSO}}e^{2\pi i\tau H_{\mathrm{NS}}}\right] - 2\,\mathrm{Tr}\left[P^{\mathrm{R},\pm}_{\mathrm{GSO}}e^{2\pi i\tau H_{\mathrm{R}}}\right] . \end{aligned}$$

and the minus sign in the R sector indeed comes from spacetime spin-statistics. Note that a choice of the GSO projections $\pm$ in the R sector does not affect the result due to the fermion zero modes. In other words, the GSO projection is compatible with the modular-invariance of the torus partition function.

7.4 Modular functions

It is quite amusing to see that the modular functions such as the Dedekind eta function and Jacobi theta functions defined in the 19th century naturally arise in string theory. Here we summarize the definition and the basic properties of the modular functions describing torus partition functions. The reader may also refer to [Pol98, §7.2] and [BP09, §4.2].

The infinite product form of them are

$$\begin{aligned} \eta(\tau) &= q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n) , \\ \vartheta_2(\tau) &= 2q^{\frac{1}{8}}\prod_{n=1}^{\infty}(1-q^n)(1+q^n)^2 , \\ \vartheta_3(\tau) &= \prod_{n=1}^{\infty}(1-q^n)(1+q^{n-\frac{1}{2}})^2 , \\ \vartheta_4(\tau) &= \prod_{n=1}^{\infty}(1-q^n)(1-q^{n-\frac{1}{2}})^2 , \end{aligned} \tag{7.32}$$

where $q = e^{2\pi i\tau}$. The first one is called the **Dedekind eta function**, and the others are the **Jacobi theta functions**.

Their T- and S-transformations are read off

$$\begin{aligned} \eta(\tau+1) &= e^{i\pi/12}\eta(\tau) , & \eta(-1/\tau) &= \sqrt{-i\tau}\eta(\tau) , \\ \vartheta_2(\tau+1) &= e^{i\pi/4}\vartheta_2(\tau) , & \vartheta_2(-1/\tau) &= \sqrt{-i\tau}\vartheta_4(\tau) \\ \vartheta_3(\tau+1) &= \vartheta_4(\tau) , & \vartheta_3(-1/\tau) &= \sqrt{-i\tau}\vartheta_3(\tau) \\ \vartheta_4(\tau+1) &= \vartheta_3(\tau) , & \vartheta_4(-1/\tau) &= \sqrt{-i\tau}\vartheta_2(\tau) . \end{aligned} \tag{7.33}$$

There are two important identities. One is called the **Jacobi-Riemann identity**:

$$(\vartheta_3(\tau))^4 = (\vartheta_2(\tau))^4 + (\vartheta_4(\tau))^4 . \tag{7.34}$$

The other is **Jacobi triple product identity**:

$$\vartheta_2(\tau)\vartheta_3(\tau)\vartheta_4(\tau) = 2\eta^3(\tau) , \tag{7.35}$$

from which the modular property of the Dedekind eta function follows.

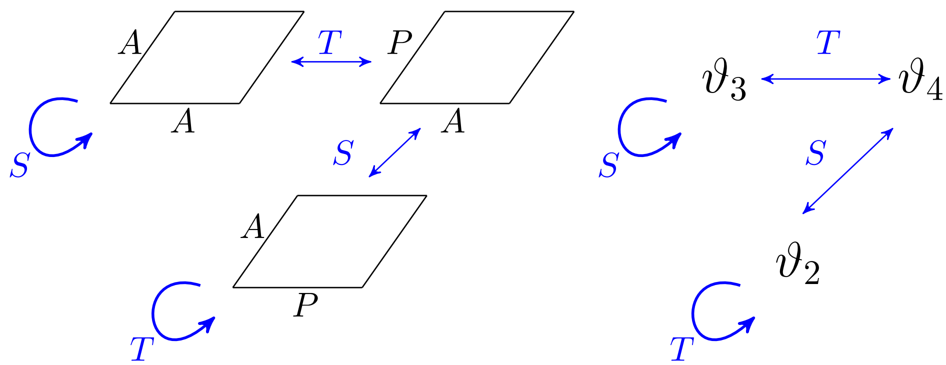


Figure 18: the T and S transformations of ϑ-functions. A and P represent anti-periodic and periodic boundary conditions of fermions.

8 Type II superstring theories

We have seen that the triality $\mathbf{8_v}$, $\mathbf{8_s}$, $\mathbf{8_c}$ of the eight-dimensional irreducible representations of $\mathrm{SO}(8)$ appear as the massless spectrum of the RNS superstring theory after the GSO projections. One way to explain the critical dimension $D = 10$ of superstring theory is that the little group $\mathrm{SO}(8)$ of the Lorentz group $\mathrm{SO}(1,9)$ of the spacetime has these special eight-dimensional irreducible representations. As explained, the GSO projections (7.30) in the R sector pick one of the irreducible spinor representations $\mathbf{8_s}$, $\mathbf{8_c}$ of $\mathrm{SO}(8)$. In this section, we will obtain superstring theories of two different types, depending on the sign in the GSO projection operators. The analysis of massless fields in Type II theories predicts extended objects, called **D-branes** [Pol95].

8.1 Type II superstrings

Like bosonic closed string theory, massless spectrum even in closed superstring can be determined by taking tensor products of massless states in the left- and right-moving sector. However, we now have a choice between $\mathbf{8_s}$ and $\mathbf{8_c}$ in the R sector. Hence, there are two inequivalent ways to construct superstring theory: the same (IIB) or opposite (IIA) choices on the right- and left-moving spectrum. These lead to the massless sectors

$$\begin{aligned}\text{Type IIA}: &\ (\mathbf{8_v}\oplus\mathbf{8_s})\otimes(\mathbf{8_v}\oplus\mathbf{8_c})\\ \text{Type IIB}: &\ (\mathbf{8_v}\oplus\mathbf{8_s})\otimes(\mathbf{8_v}\oplus\mathbf{8_s})\end{aligned} \tag{8.1}$$

of $\mathrm{SO}(8)$. Although one can also choose

$$\begin{aligned}\text{Type IIA}': &\ (\mathbf{8_v}\oplus\mathbf{8_c})\otimes(\mathbf{8_v}\oplus\mathbf{8_s})\\ \text{Type IIB}': &\ (\mathbf{8_v}\oplus\mathbf{8_c})\otimes(\mathbf{8_v}\oplus\mathbf{8_c})\end{aligned} \tag{8.2}$$

they are equivalent after the spacetime parity redefinition. By the construction, Type IIB theory is chiral whereas IIA theory is not chiral. Also, the tensor products of the NS-NS and R-R sectors provide massless bosonic fields whereas those of the NS-R and R-NS sectors give massless fermionic fields.

IIA	$\mathbf{8_v}$	$\mathbf{8_c}$
$\mathbf{8_v}$	$\mathbf{1}\oplus\mathbf{28}\oplus\mathbf{35}$ $\phi\ \ B_{\mu\nu}\ \ G_{\mu\nu}$	$\mathbf{8_s}\oplus\mathbf{56_c}$ $\lambda^+\ \ \psi_m^-$
$\mathbf{8_s}$	$\mathbf{8_c}\oplus\mathbf{56_s}$ $\lambda^-\ \ \psi_m^+$	$\mathbf{8_v}\oplus\mathbf{56_t}$ $C_n\ \ C_{nmp}$

IIB	$\mathbf{8_v}$	$\mathbf{8_s}$
$\mathbf{8_v}$	$\mathbf{1}\oplus\mathbf{28}\oplus\mathbf{35}$ $\phi\ \ B_{\mu\nu}\ \ G_{\mu\nu}$	$\mathbf{8_c}\oplus\mathbf{56_s}$ $\lambda^+\ \ \psi_m^-$
$\mathbf{8_s}$	$\mathbf{8_c}\oplus\mathbf{56_s}$ $\lambda^+\ \ \psi_m^-$	$\mathbf{1}\oplus\mathbf{28}\oplus\mathbf{35}_+$ $C\ \ C_{mn}\ \ C_{mnpq}$

Table 2: Massless fields in Type IIA and IIB theory.

Let us first look at the massless bosonic sector. In the NS-NS sector, this is the same as bosonic string theory

$$\mathbf{8_v}\otimes\mathbf{8_v}=\phi\oplus B_{\mu\nu}\oplus G_{\mu\nu}=\mathbf{1}\oplus\mathbf{28}\oplus\mathbf{35}. \tag{8.3}$$

Thus, the new ingredients come from the R-R sector, and the IIA and IIB spectra are respectively

$$\begin{aligned}\mathbf{8_s}\otimes\mathbf{8_c}&=[1]\oplus[3]=\mathbf{8_v}\oplus\mathbf{56_t}\\ \mathbf{8_s}\otimes\mathbf{8_s}&=[0]\oplus[2]\oplus[4]_+=\mathbf{1}\oplus\mathbf{28}\oplus\mathbf{35}_+\ .\end{aligned} \tag{8.4}$$

Here $[n]$ denotes the n-the antisymmetric representation of $\mathrm{SO}(8)$, and we associate R-R n-form $C_{(n)}$ to it. Also, here $[4]_+$ means its R-R field strength $G_{(5)}=dC_{(4)}$ is self-dual

$$*\,G_{(5)}=G_{(5)}\ . \tag{8.5}$$

Note that the representations $[n]$ and $[8-n]$ are related by the Hodge dual so that they are related by contraction with the 8-dimensional ϵ-tensor. As we will see next, these R-R fields are associated to D-branes in Type II theories, and $[n]$ and $[8-n]$ are related by the electro-magnetic duality.

Let us first look at the massless fermionic sector. The tensor products of the NS-R and R-NS sectors are given by

$$\begin{aligned}\mathbf{8_v}\otimes\mathbf{8_c}&=\mathbf{8_s}\oplus\mathbf{56_c}\\ \mathbf{8_v}\otimes\mathbf{8_s}&=\mathbf{8_c}\oplus\mathbf{56_s}\ .\end{aligned} \tag{8.6}$$

The $\mathbf{56_{s,c}}$ correspond to gravitinos $\psi^{\pm}_{m,\alpha}$ that are superpartners of gravitons, and they have vector and one spinor indices by construction. As we will see in §12, they will give spacetime supersymmetry. The $\mathbf{8_{s,c}}$ are dilatino $\lambda^{\pm}_{\alpha}$ which are superpartners of the dilaton field.

8.2 Introduction to D-branes

The massless spectrum from closed strings analyzed above does not incorporate gauge fields because gauge fields arise from an open string. To incorporate open strings in Type II theories, we need to introduce D-branes. Note that closed superstring theories are consistent in themselves as we see in Heterotic string theory §11. However, only open string theory is inconsistent, and it requires closed strings as well (see Figure 19).

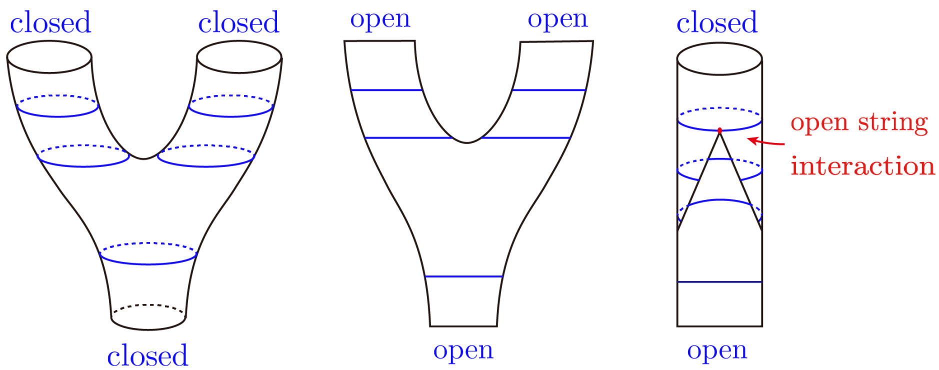


Figure 19: Open string interaction induces closed string

Boundary conditions & D-brane

As seen in (2.26), we have imposed the two types of boundary conditions for open strings. In fact, the Dirichlet boundary condition (Figure 20) does not conserve the momenta of open strings (exercise). It implies that there must be an object into which the momentum goes. We call this object a D-brane where the D stands for Dirichlet and brane comes from the membrane. Indeed, we can impose the Neumann boundary condition to some coordinates and the Dirichlet to the other coordinates:

$$\begin{aligned}&\text{Neumann condition on } X^a \quad (a = 0, 1, \ldots, p)\\&\text{Dirichlet condition on } X^I \quad (I = p+1, \ldots, D-1)\end{aligned}$$

The corresponding D-brane is called a **Dp-brane**, which extends to a p-dimensional subspace or a $(p+1)$-dimensional spacetime. Now we can visualize a configuration of D-brane and string as in Figure 20. A D-brane is a dynamical object as it should receive the momentum so that it has action and interactions with string. (See §14.) On the other hand, in order to give the Dirichlet boundary condition $X^I = c^I$, a D-brane must be infinitely heavier than a string.

Chan-Paton factor

We can consider not only a single D-brane but also a stack of D-branes, and a string now has a choice of D-branes to which a string ends. Let us label this option i ($i = 1, \ldots, n$), which is called **Chan-Paton factor**. As an open string has two endpoints, Chan-Paton degree of freedom is specified by

$$|N; k\rangle \quad \to \quad |N; k; ij\rangle \ . \tag{8.7}$$

Now we have n^2 massless vector states in both bosonic string and superstring theory. As usual, we use $n \times n$ Hermitian matrices T^a normalized to

$$\mathrm{Tr}\left(T^a T^b\right) = \delta^{ab} \ ,$$

which consist of a complete set for data of the open string endpoints:

$$|N = 1; k; a\rangle = T^a_{ij} |N = 1; k; ij\rangle \ . \tag{8.8}$$

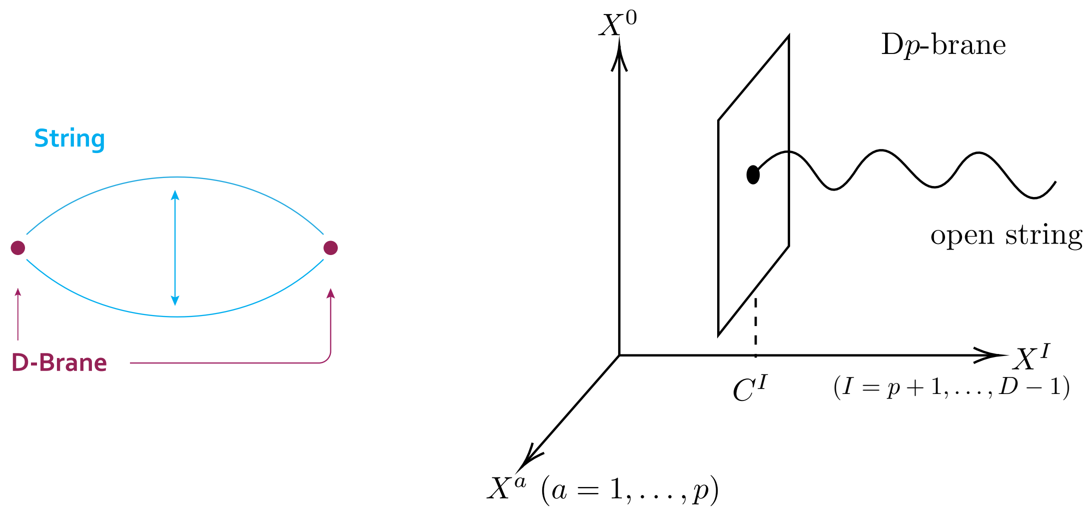


Figure 20: D-brane must exist at the ends of the open string so that momentum can escape from the string.

These states correspond to a $\mathrm{U}(n)$ gauge field. (You can check it by a three-point amplitude.)

Note that in order to realize $\mathrm{U}(n)$ gauge group, the D-branes must coincide at a point. Otherwise, the gauge group is broken by Higgs mechanism (see Figure 21).

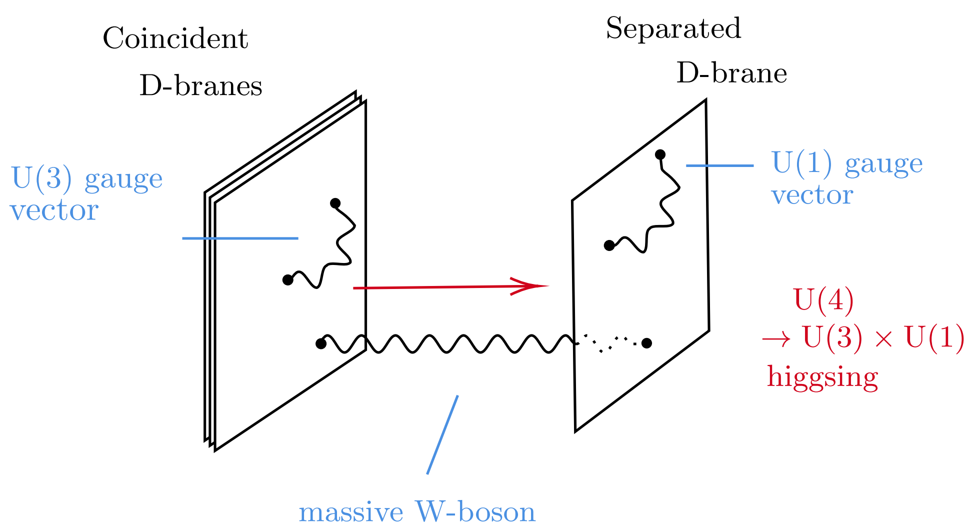


Figure 21: Many D-branes and Chan-Paton factors. $\mathrm{U}(n)$ gauge group and Higgs mechanism.

We will see in §10.1 that other types of gauge groups (SO or Sp) can be realized in string theory with orientifold planes.

D-brane in IIA/IIB superstring theory

D-branes arise as boundary conditions of open strings, and they carry gauge fields. Since D-branes are intrinsic to string theory, it reveals many interesting facets so that we will further investigate their properties. In §8.1, we saw that Type II superstring theories are endowed with R-R fields, which are anti-symmetric tensors analogous to gauge fields. Like an electron/monopole is coupled to the gauge fields, D-branes are coupled to the R-R fields in Type II theories.

To use the analogy of electromagnetism, let us quickly review an electron/monopole interacting with the electromagnetic fields in the $(3+1)$-dim spacetime. An electron is expressed by a source

$$J^{\mu} = (\rho, \mathbf{j}) = (q\delta^3(\mathbf{r} - \mathbf{r}(t)), \partial_t \mathbf{r}(t)\rho) ,$$

and its coupling to the gauge field A is given by

$$S_J = q_e \int A_\mu J^\mu d^4x = q_e \int_\gamma A_\mu dx^\mu = q_e \int_\gamma A ,$$

where q_e is an electric charge, and γ is the world-line of the electron. Since the Maxwell equation is

$$* d * F_{(2)} = J_e \tag{8.9}$$

, we can obtain the electric charge by

$$q_e = \int_{S^2} *F_{(2)} .$$

On the other hand, the electromagnetic duality

$$dF_{(2)} = J_m \tag{8.10}$$

tells us that a magnetic monopole is a source $J_m = q_m \delta^3(r)$ of a magnetic flux. Therefore, the magnetic charge can be measured as

$$q_m = \int_S F_{(2)} = \int_B dF_{(2)} ,$$

where a surface S around the monopole. It is well-known that the charges of the electron and monopole obey the Dirac quantization condition (for instance, see [Pol98, §13.3]):

$$q_e q_m \in 2\pi\mathbb{Z} . \tag{8.11}$$

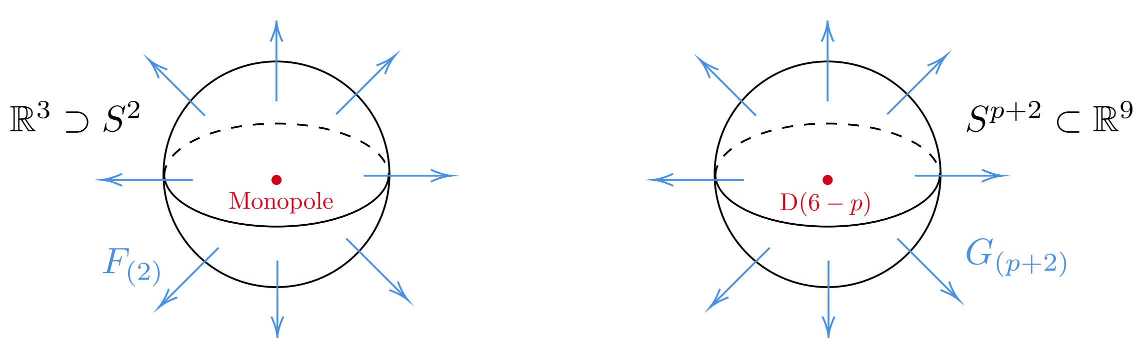


Figure 22: Higher dimensional analog of monopole: magnetic flux of D$(6-p)$-brane.

In a similar fashion, a Dp-brane ($p \leq 3$) supported on a world-membrane M_{p+1} is electrically coupled to the R-R $(p+1)$-form $C_{(p+1)}$ as

$$S_p = \mu_p \int_{M_{p+1}} C_{(p+1)} \, . \tag{8.12}$$

where μ_p is the charge of the Dp-brane. An exterior derivative of the R-R potential gives its R-R field strength $G_{(p+2)} = dC_{(p+1)}$, and its electromagnetic dual is given by its Hodge dual

$$\widetilde{G}_{(8-p)} = *G_{(p+2)}$$

From Gauss's law, the charge is given by the flux of $\widetilde{G}_{(8-p)}$ over a sphere S^{8-p} around the Dp-brane

$$\mu_p = 2\kappa_{10}^2 \int_{S^{8-p}} \widetilde{G}_{(8-p)} \, ,$$

where $2\kappa_{10}^2 = (2\pi)^7 \alpha'^4$. The magnetic dual of the Dp-brane is a D$(6-p)$-brane, and its magnetic charge is accordingly given by

$$\mu_{6-p} = 2\kappa_{10}^2 \int_{S^{p+2}} G_{(p+2)} \, .$$

They must obey the Dirac quantization condition

$$2\kappa_{10}^2 \mu_p \mu_{6-p} \in 2\pi\mathbb{Z} \, . \tag{8.13}$$

Writing the flux $\widetilde{G}_{(8-p)} = d\widetilde{C}_{(7-p)}$, the D$(6-p)$-brane is magnetically coupled to $\widetilde{C}_{(7-p)}$. Now, reading off R-R fields from Table 2, we see that there are Dp-branes for even p in Type IIA and for odd p in Type IIB theory.

$$\begin{array}{llcccl}
\text{Electric} & \mathrm{D}p & \longleftrightarrow & C_{(p+1)} \xrightarrow{\ d\ } & G_{(p+2)} & \\
 & & & & \Big\updownarrow & \begin{array}{l}\text{EM dual}\\ \text{Hodge dual}\end{array} \\
\text{Magnetic} & \mathrm{D}(6-p) & \longleftrightarrow & \widetilde{C}_{(7-p)} \xrightarrow{\ d\ } & \widetilde{G}_{(8-p)} &
\end{array}$$

Figure 23: Electro-magnetic duality and D-brane.

Even from the NS-NS sector, we can predict the presence of an extended object in string theory. A string or a fundamental string, denoted as F1, is electrically coupled to the B-field. On the other hand, there is an object magnetically coupled to the B-field, which is called **NS5-brane**. There are always fundamental strings and NS5-branes in string theory of five types.

In conclusion, we summarize extended objects in Type II superstring theory in Table 8.2.

Note that a D(-1)-brane is a timely localized object (called an instanton). There are D8-branes in Type IIA theory, which are non-dynamical so that there is no corresponding R-R anti-symmetric tensor. Moreover, these branes can be understood as a decay of the space-filling D9-brane and anti-D9-brane pair D9-$\overline{\mathrm{D9}}$ [Sen98b], which admits a beautiful mathematical interpretation [Wit98c] by K-theory.

IIA	$B_{(2)}$	$C_{(1)}$	$C_{(3)}$
Electric	F1	D0	D2
Magnetic	NS5	D6	D4

IIB	$B_{(2)}$	$C_{(0)}$	$C_{(2)}$	$C^{+}_{(4)}$
Electric	F1	D(−1)	D1	D3
Magnetic	NS5	D7	D5	D3

Table 3: Extended objects in Type II superstring theory

9 T-duality

We have seen that the critical dimensions D of bosonic and supersymmetric string theory are $D = 26$ and $D = 10$, respectively. To obtain an effective theory in lower dimensions, we can make use of **Kaluza-Klein compactifications** where the true spacetime takes the form of a direct product $M_d \times K_{D-d}$, where M_d is the d-dimensional Minkowski spacetime, and K_{D-d} is a very tiny compact manifold. As we will see, an effective theory in M_d still sees interesting "stringy" effects in this Kaluza-Klein scheme.

First, we concentrate on the simple compactifications, $K = T^{D-d}$ called toroidal compactifications. Since a torus is simply a product of S^1 and it is flat, the nonlinear sigma model can be described by the free two-dimensional CFT. Remarkably, this simple compactification leads to the notion of **T-duality** and **Heterotic string theories** have been constructed based on toroidal compactifications as we will see in §11. To understand the basic properties, let us first see the toroidal compactifications of bosonic string theory. Toroidal compactifications of superstring theories are also very interesting so that we will learn about them relating to string dualities later more in detail.

The concept of D-branes was introduced as boundary conditions of open strings. T-duality gets particularly rich when we include D-branes so that we will study their properties more in detail. The explanation of T-duality in Polchinski's lecture notes [Pol96b] is so elegant that we just simply follow it in this section.

9.1 S^1 compactification in closed bosonic string

To begin with, let us first study the simplest case of the spacetime $\mathbb{R}^{1,24} \times S^1$ where we compactify 25-th direction on a circle S^1 of radius R. For closed strings, we have the familiar mode expansion (3.3). Now let us take a close look at the zero modes which can be written as

$$X^\mu(z,\bar{z}) = x^\mu + \bar{x}^\mu - i\sqrt{\frac{\alpha'}{2}}\left(\alpha_0^\mu + \bar{\alpha}_0^\mu\right) t + \sqrt{\frac{\alpha'}{2}}\left(\bar{\alpha}_0^\mu - \alpha_0^\mu\right)\sigma + \text{ oscillators.}$$

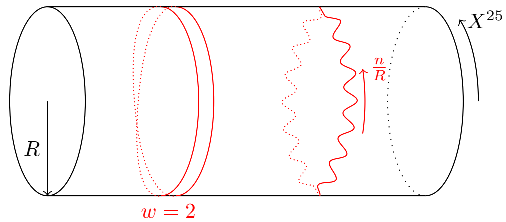
where the spacetime momentum of the string is

$$p^\mu = \frac{1}{\sqrt{2\alpha'}}(\alpha_0^\mu + \bar{\alpha}_0^\mu) \ .$$

Under $\sigma \to \sigma + 2\pi$, the oscillator term is periodic and $X^\mu(z,\bar{z})$ changes by $2\pi\sqrt{(\alpha'/2)}(\bar{\alpha}_0^\mu - \alpha_0^\mu)$. For a non-compact spatial direction $\mathbb{R}^{1,24}$, X^μ is single-valued $X^\mu(t,\sigma) = X^\mu(t,\sigma + 2\pi)$, which requires

$$\alpha_0^\mu = \bar{\alpha}_0^\mu = \sqrt{\frac{\alpha'}{2}}p^\mu \ , \qquad \mu = 0,1,\ldots,24 \ .$$

On the other hand, since the 25-th direction is put on the circle S^1 of radius R, it has a period $X^{25} \sim X^{25} + 2\pi R$. Hence, the momentum p^{25} can take the values n/R for



$n \in \mathbb{Z}$ where n is called **Kaluza-Klein momentum**. Also, under $\sigma \sim \sigma + 2\pi$, $X^{25}(z, \bar{z})$ can change by $2\pi w R$ where w is called the **winding number**. Thus, we have

$$\alpha_0^{25} + \overline{\alpha}_0^{25} = \frac{2n}{R}\sqrt{\frac{\alpha'}{2}}, \qquad \overline{\alpha}_0^{25} - \alpha_0^{25} = \sqrt{\frac{2}{\alpha'}} w R$$

implying

$$\alpha_0^{25} = \left(\frac{n}{R} - \frac{wR}{\alpha'}\right)\sqrt{\frac{\alpha'}{2}}, \qquad \overline{\alpha}_0^{25} = \left(\frac{n}{R} + \frac{wR}{\alpha'}\right)\sqrt{\frac{\alpha'}{2}}. \tag{9.1}$$

Now let us study their mass spectrum. The mass formula for the string with one dimension compactified on a circle can be interpreted from a 25-dimensional viewpoint in which one regards each of the Kaluza-Klein momenta, which are given by n, as distinct particles. Thus, the mass formula is given by

$$M^2 = -\sum_{\mu=0}^{24} p^\mu p_\mu$$

where μ runs only over the non-compact dimensions. Hence, we can write the mass formula as

$$M^2 = \frac{n^2}{R^2} + \frac{w^2 R^2}{\alpha'^2} + \frac{2}{\alpha'}(N + \overline{N} - 2)\,.$$

where N and $\overline{N}$ are the right and left number operators (2.19). Using (9.1), we can express the difference between the two expressions

$$N - \overline{N} = nw\,, \tag{9.2}$$

so that the level matching condition is modified due to the S^1 compactification. In a similar fashion, the mass can be expressed in terms of the Kaluza-Klein momentum and the winding number

$$M^2 = \frac{n^2}{R^2} + \frac{w^2 R^2}{\alpha'^2} + \frac{2}{\alpha'}(N + \overline{N} - 2) \tag{9.3}$$

The mass spectra (9.3) of the theories at radius R and α'/R are identical when the winding and Kaluza-Klein modes are interchanged $n \leftrightarrow w$. This symmetry of the bosonic string theory is called **T-duality**. From the viewpoint of strings, a circle of radius R is equivalent to that of radius α'/R, and this is the main reason why we can avoid UV divergence in string theory. This shows that strings see geometry in an unprecedented and intriguing way, and this is one of the remarkable phenomena called "stringy geometry".

It is easy to see from (9.1) that this interchange amounts to

$$\alpha_0^{25} \to -\alpha_0^{25}, \quad \overline{\alpha}_0^{25} \to \overline{\alpha}_0^{25}. \tag{9.4}$$

In fact, it is not just the zero mode, but the entire right-moving part of the compact coordinate that flips the sign under the T-duality transformation

$$X'^{25}(z,\bar{z}) = -X^{25}(z) + \overline{X}^{25}(\bar{z}) \ . \tag{9.5}$$

Remarkably, the energy-momentum tensor, OPEs and therefore all of the correlation functions are invariant under this rewriting. In other words, T-duality, relating the two theories with radius R and α'/R, is an exact symmetry of perturbative closed string theory.

Because of the T-duality, a theory with compactification radius R is equivalent to the theory with compactification radius α'/R. Thus, this implies that there is a "minimal radius" $R = \sqrt{\alpha'}$ in string theory which is called **self-dual radius**. At self-dual radius, the duality $R \to \alpha'/R$ maps R back to its original value where we can expect something interesting to occur. In the next section, we will study physics at the self-dual radius.

Self-dual radius: $R = \sqrt{\alpha'}$

As we know, the massless spectra of bosonic string theory include gravitons. Hence, let us see the effect of the S^1 compactification on the gravitons. In the Kaluza-Klein mechanism $M_{25} \times S^1$, the metric is decomposed into compact and non-compact spacetime direction

$$ds^2 = G_{MN}dx^M dx^N = G_{\mu\nu}dx^\mu dx^\nu + G_{25,25}(dx^{25} + A_\mu dx^\mu)^2 \ . \tag{9.6}$$

where the fields $G_{\mu\nu}$, $G_{25,25}$, and A_μ are allowed to depend only on the non-compact coordinates x^μ ($\mu = 0, 1, \ldots 24$). Under a coordinate transformation

$$x'^{25} = x^{25} + \lambda(x^\mu)$$

the part $G_{\mu,25} = G_{25,\mu}$ of the metric transforms as

$$A'_\mu = A_\mu - \partial_\mu \lambda \ .$$

Thus, it behaves as U(1) gauge field, and gauge transformations arise as part of the higher-dimensional coordinate transformation. On the other hand, the part $G_{25,25}$ of the metric behaves as a scalar field. Indeed, writing $G_{25,25} = e^\sigma$, the Ricci scalar for the metric (9.6) can be written as

$$R_{26} = R_{25} - 2e^{-\sigma}\nabla^2 e^\sigma - \frac{1}{4}e^{2\sigma}F_{\mu\nu}F^{\mu\nu} \ .$$

Actually, it is straightforward to see the corresponding vertex operators at generic radius R:

$$\begin{aligned}
\partial X^\mu \overline{\partial} X^\nu e^{ik\cdot X} \quad &\longleftrightarrow \quad G_{\mu\nu}, B_{\mu\nu}, \phi \\
\partial X^\mu \overline{\partial} X^{25} e^{ik\cdot X},\ \partial X^{25} \overline{\partial} X^\mu e^{ik\cdot X} \quad &\longleftrightarrow \quad A^\mu \ , B_{\mu,25} \\
\partial X^{25} \overline{\partial} X^{25} e^{ik\cdot X} \quad &\longleftrightarrow \quad \sigma
\end{aligned} \tag{9.7}$$

where $\nu = 0, \ldots, 24$ runs the coordinate indices for M_{25}. In fact, the middle line indicates that the theory has $\mathrm{U}(1)_\ell \times \mathrm{U}(1)_r$ gauge symmetry at generic radius R.

	n	w	$\overline{N}$	N
A	± 1	± 1	0	1
B	± 1	∓ 1	1	0
C	± 2	0	0	0
D	0	± 2	0	0

However, at the self-dual radius $R = \sqrt{\alpha'}$, the mass formula (9.3) becomes

$$M^2 = \frac{1}{\alpha'}(n^2 + w^2 + 2(N + \overline{N} - 2)) ,$$

so that the massless spectra actually get enlarged. In addition to the generic solution $n = w = 0$, $N = \overline{N} = 1$, there are now also
Hence, the states corresponding to A and B contain four new gauge bosons with vertex operators

$$\overline{\partial} X^\mu e^{\pm 2iX^{25}(z)/\sqrt{\alpha'}} e^{ik\cdot X} , \qquad \partial X^\mu e^{\pm 2i\overline{X}^{25}(\overline{z})/\sqrt{\alpha'}} e^{ik\cdot X} .$$

Indeed, C and D also give rise to new gauge bosons (exercise). It is expected that the new gauge bosons must combine with the old into a non-Abelian theory. In fact, if one can define the current

$$j^\pm(z) = j^1(z) \pm i\, j^2(z) := e^{\pm 2iX^{25}(z)/\sqrt{\alpha'}} \qquad j^3(z) := i\,\partial X^{25}(z)/\sqrt{\alpha'} ,$$

they satisfy the OPEs (Exercise)

$$j^a(z) j^b(0) \sim \frac{k\delta^{ab}}{2z^2} + \frac{i\epsilon^{abc} j^c(0)}{z} .$$

with $k = 1$. Here ϵ^{abc} is the structure constant of SU(2). This is precisely the definition of SU(2) affine Lie algebra with level $k = 1$. The same story is repeated for the left movers. Hence we see that we have an enhancement of gauge symmetry from $\mathrm{U}(1)_\ell \times \mathrm{U}(1)_r$ to $\mathrm{SU}(2)_\ell \times \mathrm{SU}(2)_r$ at $R = \sqrt{\alpha'}$.

In fact, when the theory moves away from the self-dual radius $R = \sqrt{\alpha'}$, the $\mathrm{SU}(2)_\ell \times \mathrm{SU}(2)_r$ gauge symmetry is Higgsed. The world-sheet action is deformed by turning on the marginal operator

$$V_{a\overline{a}} := j_a \overline{j}_{\overline{a}} e^{ik\cdot X} ,$$

which is equivalent to giving the VEV of the (3, 3)-component of the Higgs field. As a result, when the theory is away from the self-dual radius, the $\mathrm{SU}(2)_\ell \times \mathrm{SU}(2)_r$ gauge symmetry is spontaneously broken down to a $\mathrm{U}(1)_\ell \times \mathrm{U}(1)_r$.

9.2 T-duality of open strings

Now let us consider T-duality on the open string spectrum in the S^1 compactification. At the end of §2.2, we briefly study open string spectra. There, we learned that Neumann boundary condition on the X^μ-direction is achieved by imposing $\alpha^\mu_n = \overline{\alpha}^\mu_n$ while Dirichlet boundary condition is by $\alpha^\mu_n = -\overline{\alpha}^\mu_n$. Since T-duality along the 25-th direction transforms the modes as in (9.4), it therefore exchanges Neumann to Dirichlet condition [DLP89, Lei89]:

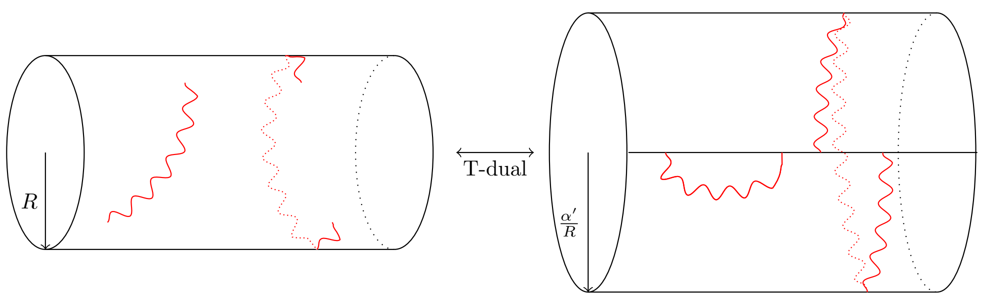
$$\partial_\sigma X^{25}\big|_{\sigma=0,\pi} = 0 \quad \to \quad \partial_t X'^{25}\big|_{\sigma=0,\pi} = 0 .$$

Suppose that an open string with Neumann boundary condition has KK momentum n along the 25-th direction, and we perform T-duality on that circle. Then, a simple calculation

$$X'^{25}(\pi) - X'^{25}(0) = \int_0^\pi d\sigma \partial_\sigma X'^{25} = i \int_0^\pi d\sigma \partial_t X^{25} \tag{9.8}$$

$$= 2\pi\alpha' p^{25} = \frac{2\pi\alpha' n}{R} = 2\pi n R' . \tag{9.9}$$

tells us that the X^{25} coordinate of the open string endpoints is fixed after T-duality. T-duality transforms the KK momentum n to winding number n as in the figure below. This can be interpreted as follows. Open strings can freely move on a space-filling D25-brane. After T-duality, the space-filling D25-brane becomes a D24-brane with its X^{25} coordinate fixed. As a result, all the endpoints of open strings are constrained to the fixed X^{25}-direction whereas they are free to move in the other 24 spatial dimensions. More generally, the boundary condition of open strings can be imposed at any dimension. Since T-duality interchanges Neumann and Dirichlet boundary conditions, T-duality tangent to a Dp-brane brings it to a D$(p-1)$-brane. On the other hand, T-duality orthogonal to a Dp-brane turns it into a D$(p+1)$-brane.



We can ask a question which position of the X^{25} coordinate a D24-brane is located in the figure above. This question is related to Chan-Paton factors we encountered in §8.2. Suppose that there are K space-filling D25-branes, which give rise to U(K) spacetime gauge fields. In the current setting, X^{25} direction is compactified on the circle S^1 so that we can include a Wilson loop on S^1 as

$$A_{25} = \mathrm{diag}\{\theta_1, \theta_2, \ldots, \theta_K\}/2\pi R \tag{9.10}$$

The insertion of the Wilson loop breaks the gauge group as U$(K) \to$ U$(1)^K$, and the broken gauge group is abelian so that we can write it as

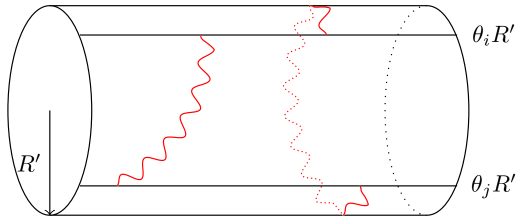
$$A_{25} = -i\Lambda^{-1}\partial_{25}\Lambda, \quad \Lambda = \mathrm{diag}\{e^{iX^{25}\theta_1/2\pi R}, e^{iX^{25}\theta_2/2\pi R}, \ldots, e^{iX^{25}\theta_N/2\pi R}\} .$$

Then, under the translation $X^{25} \to X^{25} + 2\pi R$, an open string state $|N=1;n;ij\rangle$ is shifted by a phase

$$e^{i(\theta_j - \theta_i)} , \tag{9.11}$$

which means its momentum is shifted by $(\theta_j - \theta_i)/2\pi$ from an integer n. Under T-duality, the momentum is transformed into the winding number. Therefore, θ_i can be understood as the X^{25} coordinate of the i-th D24-brane after T-duality. Namely, the open string state $|N=1;n;ij\rangle$ is mapped to an open string of length

$$X'^{25}(\pi) - X'^{25}(0) = \left(2\pi n + \theta_j - \theta_i\right) R' .$$



There are in general K D24-branes at different positions as schematically depicted in the following figure.

Then, the $(D-1) = 24$-dimensional mass of this open string is

$$\begin{aligned} M^2 &= \left(p^{25}\right)^2 + \frac{1}{\alpha'}(N-1) \\ &= \left(\frac{\left[2\pi n + \left(\theta_j - \theta_i\right)\right] R'}{\pi\alpha'}\right)^2 + \frac{1}{\alpha'}(N-1)\,. \end{aligned} \tag{9.12}$$

Hence, massless gauge bosons arise only from open strings $|N=1; n=0; ii\rangle$ whose endpoints are on the same D-brane. Due to the string tension, open strings that stretch between different D-branes become massive (gauge field). Therefore, this can be understood as the Higgs mechanism in which X^{25} expectation values (coordinates) of K D24-branes take different values, and the gauge group is broken as $\mathrm{U}(K) \to \mathrm{U}(1)^K$. This situation is called **Coulomb phase**, where the Goldstone bosons are "eaten" by gauge fields $|N=1; n=0; ij\rangle$ which become massive.

Let us make an important remark. So far, we have treated D-branes just as rigid boundary conditions. However, D-branes are dynamical so that they can fluctuate in shape and position. As we will see in §14, §15, §16, the gravitational and gauge dynamics on a stack of D-branes makes string theory very intriguing.

9.3 T-duality of Type II superstrings

Let us briefly discuss how the T-duality acts on Type II superstring compactified on $M_9 \times S^1$. We have seen that the T-duality acts as the parity transformation on the right-moving sector

$$X^9(z) \quad \longleftrightarrow \quad -X'^9(z)$$

The superconformal invariance requires

$$\psi^9 \leftrightarrow -\psi'^9\,.$$

However, this implies that the chirality of the right-moving R sector ground state is reversed: the raising and lowering operators $\psi^8 \pm i\psi'^9$ are interchanged. (See (7.18).) In other words, T-duality is a spacetime parity operation on just one side of the worldsheet, so it reverses the relative chiralities of the right- and left-moving ground states. As a result, Type IIA theory with compactification radius R is T-dualized to Type IIB theory with radius α'/R.

Chiral spinors are now defined by using (7.22)

$$\Gamma_{11}\psi_\pm = \pm\psi_\pm$$

and we define $\overline{\psi}_\pm = \psi_\pm^\dagger \Gamma^0$. Then, the vertex operators of the R-R field strengths $G_{\mu_1\cdots\mu_{p+2}}$ are expressed as spinor bilinears

$$\text{IIA}: \overline{\psi}_-^L \Gamma^{\mu_1\cdots\mu_{p+2}} \psi_+^R, \quad \text{IIB}: \overline{\psi}_+^L \Gamma^{\mu_1\cdots\mu_{p+2}} \psi_+^R$$

where ψ^R (ψ^L) comes from the right (left) movers and

$$\Gamma^{\mu_1\cdots\mu_{p+2}} = \Gamma^{[\mu_1} \dots \Gamma^{\mu_{p+2}]}$$

is the antisymmetric product of $(p+2)$ gamma matrices.

Since the IIA and IIB theories have different R-R fields, T-duality in the 9th direction will transform one into the other. This can be seen from the action of T-duality on the spin fields as

$$\psi_\alpha^R(z) \to \beta_9 \psi_\alpha^R(z), \qquad \overline{\psi}_\alpha^L(\bar{z}) \to \overline{\psi}_\alpha^L(\bar{z})$$

for $\beta_9 = \Gamma^9\Gamma^{11}$, the parity transformation (9-reflection) on the spinors. Hence, the R-R fields are transformed as

$$\begin{aligned} C_9 &\to C \\ C_\mu, C_{\mu\nu 9} &\to C_{\mu 9}, C_{\mu\nu} \\ C_{\mu\nu\lambda} &\to C_{\mu\nu\lambda 9}\,. \end{aligned} \tag{9.13}$$

Figure 24 illustrates the behavior of D-branes under the T-duality, and Table 4 shows different D-brane configurations in Type II theories that are dual to each other.

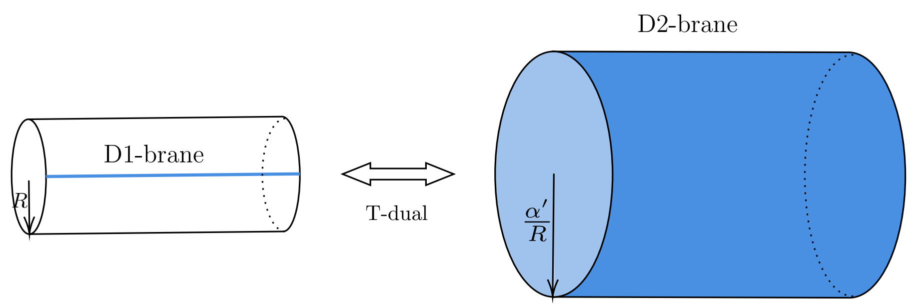


Figure 24: D1-brane and its T-dual D2-brane on $\mathbb{R} \times S^1$.

10 Type I superstring theory

So far we have considered oriented world-sheet Riemann surfaces. However, it is quite natural to consider processes in Figure 25, which yields an unoriented world-sheet. Therefore, we will consider the so-called **unoriented string**, which leads to Type I superstring theory. As we will see below, an unoriented string theory needs to incorporate open strings.

Open strings can end on D-branes in Type II theories. However, without D-branes, Type II theories cannot incorporate open strings due to supersymmetry as follows. In fact, $D = 10$ $\mathcal{N} = 2$ supersymmetry requires $\mathbf{8}_v$ to have two superpartners as like for a

		0	1	2	3	4	5	6	7	8	9
IIB	NS5	×	×	×	×	×	×				
	D1	×	×								
	D5	×	×					×	×	×	×

$\updownarrow$ T-dual on X^6

		0	1	2	3	4	5	6	7	8	9
IIA	NS5	×	×	×	×	×	×				
	D2	×	×					×			
	D4	×	×						×	×	×

$\updownarrow$ T-dual on X^2

		0	1	2	3	4	5	6	7	8	9
IIB	NS5	×	×	×	×	×	×				
	D3	×	×	×				×			
	D5	×	×	×					×	×	×

$\updownarrow$ S-dual

		0	1	2	3	4	5	6	7	8	9
IIB	D5	×	×	×	×	×	×				
	D3	×	×	×				×			
	NS5	×	×						×	×	×

Table 4: D-branes and dualities in Type II theories. The last two configurations will appear in §14.3.

graviton to have two gravitinos in Table 2. On the other hand, after the GSO projection, open strings can have one of the following massless states:

$$\begin{aligned} P_{\text{GSO}}: &\quad \text{NS, R+} = \mathbf{8}_v + \mathbf{8}_s\ , \\ \widetilde{P}_{\text{GSO}}: &\quad \text{NS, R}- = \mathbf{8}_v + \mathbf{8}_c\ , \end{aligned}$$

where $\mathbf{8}_v$ represents a gauge field and $\mathbf{8}_{s,c}$ are gauginos. Hence, the gauge field has only one superpartner so that it cannot exist in Type II theories without D-branes.

In fact, the existence of (parallel) D-branes breaks a half of supersymmetries so that open strings can end on D-branes. To consider unoriented string theory, we will introduce an **orientifold plane** or **O-plane** that breaks a half of supersymmetry. With an O-plane, it is natural to consider unoriented open string theory. Although the argument below mainly focuses on the bosonic string for simplicity, it is straightforward to apply for superstrings.

10.1 Orientifold

As in Figure 25, an orientation flip exchanges left- and right-movers for closed strings and a reversal of the spatial direction for open strings.

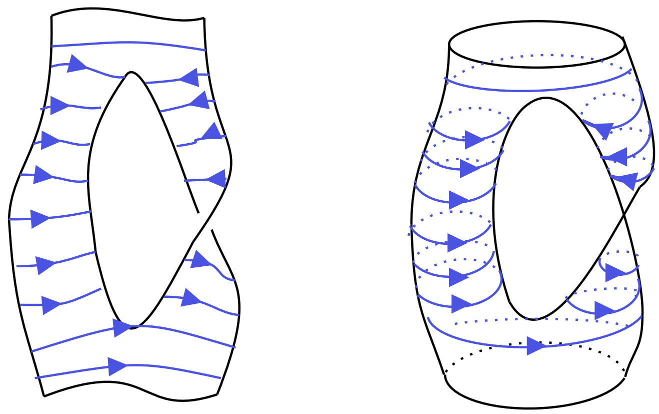


Figure 25: Unoriented processes. The left is open string one, and the right is closed string one.

Let us first take a look at an orientation flip operator Ω on a bosonic closed string. It exchanges the left and right-moving modes which ends up with a reversal of the spatial direction up to the overall sign:

$$\Omega : \begin{cases} X(t,\sigma) & \leftrightarrow & X(t,-\sigma) \\ b(t,\sigma) & \leftrightarrow & \bar{b}(t,-\sigma) \\ c(t,\sigma) & \leftrightarrow & -\bar{c}(t,-\sigma) \end{cases} , \qquad \Omega : \begin{cases} \alpha_n^\mu & \leftrightarrow & \bar{\alpha}_n^\mu \\ b_n & \leftrightarrow & \bar{b}_n \\ c_n & \leftrightarrow & \bar{c}_n \end{cases} , \tag{10.1}$$

where t is the Euclidean world-sheet time, not τ. Thus, the projection by the orientation flip, called an orientifold, keeps only Ω invariant states such as the tachyon vacuum, gravitons and dilaton, but it projects out the B-field. Since a closed string is electrically coupled to the B-field, this implies that an unoriented closed string is not stable and it should decay.

Now, let us introduce a space-filling D25-brane and consider an open string with the Neumann boundary condition. As in (2.26), the Neumann boundary condition identifies the left and right-moving modes. Then, the orientation flip operator acts as

$$\Omega : \begin{cases} X(t,\sigma) & \leftrightarrow & X(t,\pi-\sigma) \\ b(t,\sigma) & \leftrightarrow & \bar{b}(t,\pi-\sigma) \\ c(t,\sigma) & \leftrightarrow & -\bar{c}(t,\pi-\sigma) \end{cases} , \qquad \Omega : \begin{cases} \alpha_n^\mu & \leftrightarrow & (-1)^n\alpha_n^\mu \\ b_n & \leftrightarrow & (-1)^n b_n \\ c_n & \leftrightarrow & (-1)^n c_n \end{cases} , \tag{10.2}$$

Now recall that we can incorporate Chan-Paton degrees of freedom (8.7) for open string states. The orientation flip Ω exchanges the Chan-Paton indices $i \leftrightarrow j$. Hence, writing an open string state as in (8.8)

$$|N,k;T\rangle \equiv |N,k;ij\rangle T_{ij} ,$$

Ω acts on the state as

$$\Omega|N,k;T\rangle = (-1)^N|N,k;T^t\rangle .$$

Therefore, it imposes the condition

$$\begin{aligned} T^t &= T \quad \text{for } N \text{ even} \\ T^t &= -T \quad \text{for } N \text{ odd} . \end{aligned} \tag{10.3}$$

In particular, a gauge boson arises at level one (2.28) so that we need to impose the anti-symmetric condition $T^t = -T$ on the Chan-Paton factor. Consequently, it gives rise to a $\mathrm{SO}(n)$ gauge field.

In general, when orientation is flipped, we can also transform the Chan-Paton index by a matrix P at the same time.

$$\Omega|N;k;ij\rangle = (-1)^N P_{jj'} \left|N;k;j'i'\right\rangle P^{-1}_{i'i}$$

The projection operator is subject to $\Omega^2 = 1$, which gives the constraint

$$\Omega^2|N;k;ij\rangle = [(P^T)^{-1}P]_{ii'} \left|N;k;i'j'\right\rangle (P^{-1}P^T)_{j'j} \,, \quad \Rightarrow \quad P^t P^{-1} = \pm 1$$

Furthermore, there is a $\mathrm{U}(n)$ gauge equivalence relation $T \sim \widetilde{T} = UTU^{-1}$ ($U \in \mathrm{U}(n)$), which amounts to $\widetilde{P} \sim UPU^t$, Taking the gauge equivalence into account, $P^t P^{-1} = 1$ leads to $P = 1$, in which the gauge Lie algebra becomes $\mathrm{SO}(n)$ as above. On the other hand, for $P^t P^{-1} = -1$, the rank needs to be even $n = 2k$, and we can take a basis such that

$$P = i \begin{pmatrix} 0 & -\mathbf{1}_{k\times k} \\ \mathbf{1}_{k\times k} & 0 \end{pmatrix}$$

In this situation, the gauge Lie algebra becomes $\mathrm{Sp}(k)$ so that $T^t = -PTP$. We denote the orientifold flip projections that give rise to an SO and Sp gauge group by an O^-- and O^+-plane, respectively.

Unoriented superstring spectrum

Now, let us consider the orientation flip operation Ω in superstring theories. Since Ω exchanges the left- and right-moving modes, both must have the same spectra. Hence, it can only act on Type IIB theory but not on Type IIA theory. (See (8.1).) The orientation flip projection of Type IIB theory is called **Type I superstring theory**. In Table 2, the flip projects out the B-field [2] in the NS-NS sector, as well as a half of the NS-R R-NS sector $\mathbf{8}_c + \mathbf{56}_s$ (only the diagonal part survives). Supersymmetry requires that the number of bosons and fermions are the same, which implies that [0] and $[4]_+$ are eliminated and only the second rank anti-symmetric field [2] survives in the R-R sector. This implies that Ω projects out D(−1)-, D3-, D7-branes. Then, D1- and D5-branes are electrically and magnetically coupled to the R-R two-form $C_{(2)}$, respectively. The remaining states are

$$\begin{aligned}[0] \oplus [2] \oplus (2) \oplus \mathbf{8}_c \oplus \mathbf{56} =& \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35} \oplus \mathbf{8}_c \oplus \mathbf{56}_s \\ & \Phi \quad C_{\mu\nu} \;\; G_{\mu\nu} \;\; \lambda^- \quad \psi^+_\mu\end{aligned}$$

Furthermore, to incorporate open-strings $\mathbf{8}_v + \mathbf{8}_s$, Type I theory incorporates space-filling non-dynamical D9-Branes. The orientation flip operation Ω can be understood as the effect of an O9-plane.

In conclusion, Type I is an unoriented, open plus closed superstring theory in which the massless spectrum is

$$[0] \oplus [2] \oplus (2) \oplus \mathbf{8}_c \oplus \mathbf{56} \oplus (\mathbf{8}_v \oplus \mathbf{8}_s)_{\mathrm{SO}(n) \text{ or } \mathrm{Sp}(n)} \,.$$

However, there is one important quantum effect we need to take into account. An O9-plane has R-R charge that has to cancel with the charge of D9-branes. From the world-sheet perspective, this corresponds to the **tadpole cancellation**. In what follows, we will investigate the tadpole cancellation in bosonic string theory, and will see that the gauge group needs to be $\mathrm{SO}(2^{13})$. A similar analysis tells us that the gauge group must be $\mathrm{SO}(32)$ in Type I superstring theory.

10.2 Amplitude of Type I theory

As above, unoriented open strings give rise to either $\mathrm{SO}(n)$ or $\mathrm{Sp}(n)$ massless gauge fields. However, unless the tadpole cancellation is satisfied, the theory is still anomalous. We will see it in bosonic string theory.

Cylinder

First, let us evaluate an "oriented" open string one-loop amplitude where a worldsheet is a cylinder (or annulus) as in the left of Figure 26. We consider that there are n D25-branes (or D9-branes for superstring) so that all of the boundary conditions are of Neumann type. The evaluation is parallel to §6. It is clear from Figure 26 that we notice

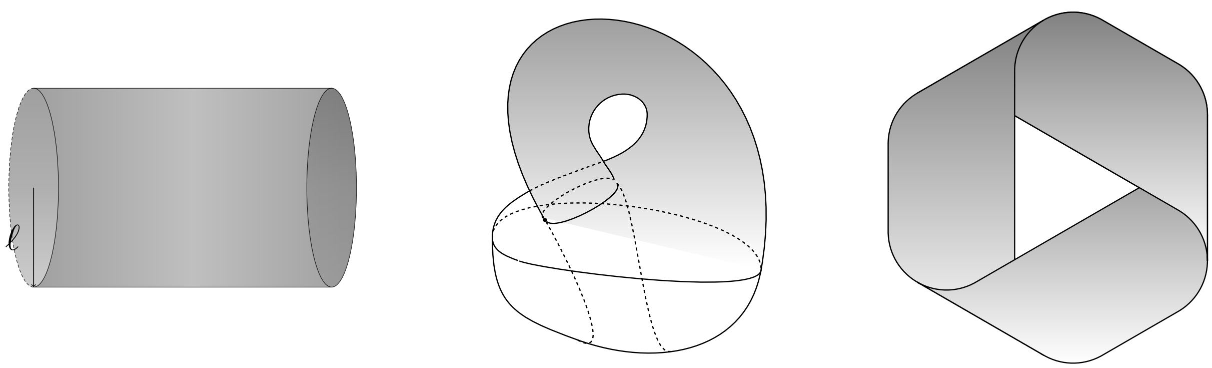


Figure 26: Cylinder, Klein bottle and Möbius strip.

the following facts:

- The range is $0 \leq \mathrm{Re}\, w \leq \pi$, the period is $w \sim w + 2\pi i l$.
- There is a real modulus l so that the amplitude needs one b zero mode insertion.
- There is a translational isometry, a shift of $\mathrm{Im}\, w$ so that the amplitude needs one c zero mode insertion.

Consequently, the cylinder partition function is written as

$$A_{0,C} = \int \frac{dl}{2l} \langle b_0 c_0 \rangle_{\mathrm{gh}} \langle 1 \rangle_{\mathrm{mat}} ,$$

where the ghost zero modes in open strings are read off

$$\begin{aligned} b_0 &= \frac{1}{2\pi} \int_0^{\pi} \left[dw\, b(w) + d\overline{w}\, \overline{b}(\overline{w}) \right] , \\ c_0 &= \frac{i}{2\pi} \int_0^{\pi} \left[dw\, c(w) - d\overline{w}\, \overline{c}(\overline{w}) \right] . \end{aligned}$$

Using operator formalism, we can derive each contribution as

$$\langle 1 \rangle_{\mathrm{mat}} = n^2 \ \mathrm{Tr} \left[q^{L_0^X - \frac{c}{24}} \right] = n^2 \cdot \frac{i V_{26}}{(2\pi)^{26}} (2l\alpha')^{-13} \cdot \eta(il)^{-26} ,$$

$$\langle b_0 c_0 \rangle_{\text{gh}} = \text{Tr}\left[(-1)^F b_0 c_0 q^{L_0^g - \frac{c}{24}}\right] = \eta(il)^2 \,,$$

where $q = e^{-2\pi l}$ and the Virasoro generators are in (5.25) for open strings. Note that n^2 comes from the trace over the Chan-Paton degrees of freedom. Therefore, the cylinder partition function is

$$A_{0,C} = n^2 \cdot \frac{iV_{26}}{(2\pi\ell_s)^{26}} \int_0^\infty \frac{dl}{2l} \frac{1}{(2l)^{13}\eta(il)^{24}} \,, \tag{10.4}$$

where $\ell_s = \sqrt{\alpha'}$ is the string length.

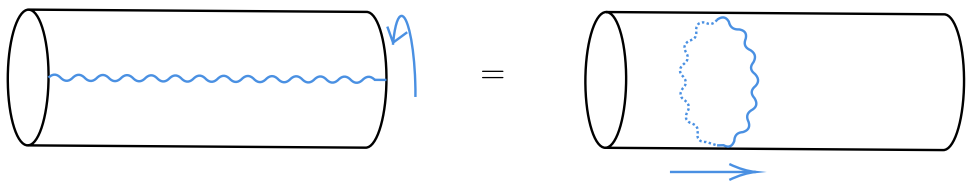


Figure 27: Pictorial description of the equivalence between open string one-loop and closed string propagation.

Let us look into its physical implication. When $l \to 0$, the cylinder becomes a thin tube, and it can be regarded as a propagation of a closed string as in Figure 27. This is called the **open-closed duality**. This is justified by rewriting the amplitude. Using the modular property (7.33) of the Dedekind η-function, we can rewrite it in terms of $l = s^{-1}$ as

$$A_{0,C} = \frac{n^2}{2^{13}} \cdot \frac{iV_{26}}{(2\pi\ell_s)^{26}} \int_0^\infty \frac{ds}{2} \, \eta(is)^{-24} \,, \tag{10.5}$$

where

$$\eta(is)^{-24} = q^{-1} + 24 + \cdots \equiv \sum_{N=0}^{\infty} \mathcal{N}_N q^{N-1} \quad (q = e^{-2\pi s}) \,.$$

In (10.4), there is an UV divergence from $l \to 0$, as opposed to the closed string case. The UV divergence $l \to 0$ in an open string is replaced by the IR divergence $s \to \infty$ of a closed string propagation, which can be understood as particle propagations (sum of lines) as follows (see also Figure 28).

$$\int_0^\infty ds \sum_{N=0}^{\infty} \mathcal{N}_N e^{-2\pi s(N-1)} \sim \sum_i \int_0^\infty ds \, e^{-s(k^2+m_i^2)} = \sum_i \frac{1}{k^2+m_i^2}\bigg|_{k=0} \,.$$

We can see that the IR divergence is from massless particle propagation (graviton etc.), which is absorbed or emitted from D25-branes. This point will be revisited at the end of §14.1. In conclusion, the divergence is due to the existence of the D25-branes, which have definite tension (this is why they emit graviton/dilaton). To cure this divergence, we will consider unoriented string amplitudes when world-sheets are Klein bottle and Möbius strip as in Figure 26. The reason will become clear later.

Klein bottle amplitude

Next, let us evaluate an unoriented closed string one-loop amplitude when a world-sheet is a Klein bottle. Since the Klein bottle can be realized by identifying the top and

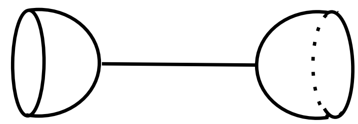


Figure 28: Intermediate propagation is replaced by particles (lines).

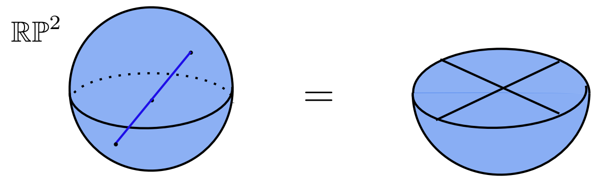


Figure 29: $\mathbb{R}P^2$ is obtained by identifying the antipodal points on S^2. This is equivalent to the southern hemisphere and the equators

the bottom string by the orientation flip operator as in Figure 30, the amplitude can be written as

$$A_{0,K} = \int \frac{dl}{2l} \mathrm{Tr}\left[\Omega(-1)^F \frac{1}{2}(b_0 + \bar{b}_0)\frac{1}{2}(c_0 + \bar{c}_0) q^{L_0 + \bar{L}_0 - \frac{c}{12}}\right] \qquad (q = e^{-2\pi l}) .$$

Using (10.1), we can rewrite it as

$$A_{0,K} = \frac{1}{2}\int \frac{dl}{2l} \mathrm{Tr}\left[(-1)^F b_0 c_0 q^{2L_0^{\mathrm{tot}} - \frac{c^{\mathrm{tot}}}{12}}\right] ,$$

where L_0 and c are the Virasoro generators and the central charge for closed strings. Thus, the result is

$$A_{0,K} = \frac{iV_{26}}{(2\pi\ell_s)^{26}} \int \frac{dl}{2l} \frac{1}{l^{13}\eta(2il)^{24}} = 2^{13}\frac{iV_{26}}{(2\pi\ell_s)^{26}} \int \frac{ds}{2}\eta(is)^{-24} . \tag{10.6}$$

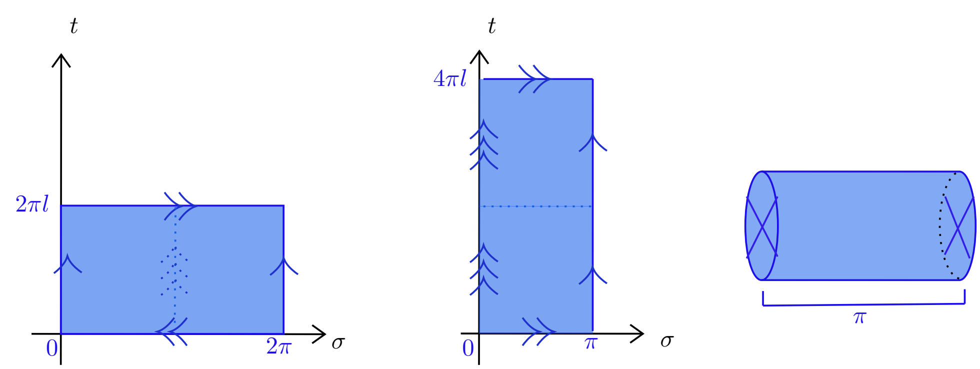


Figure 30: Klein bottle. It can be described by a cylinder with cross cap boundary on both ends.

Möbius strip amplitude

Finally, let us evaluate an unoriented open string one-loop amplitude when a worldsheet is a Möbius strip. (See Figure 31.) It has

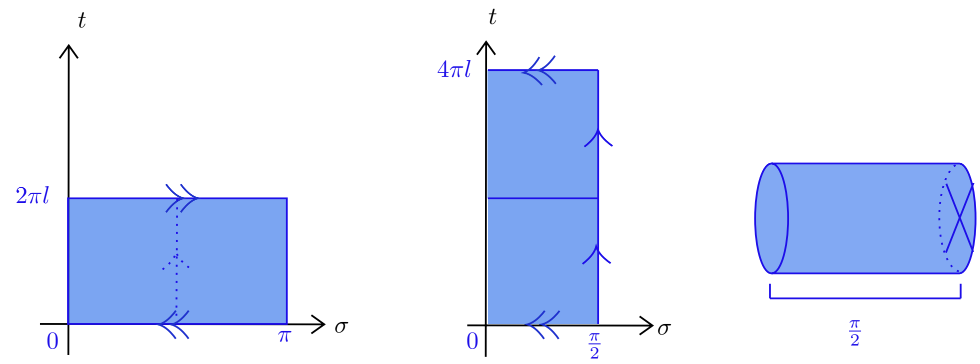


Figure 31: Möbius strip.

- The range is $0 \leq \sigma \leq \pi$, the period is $2\pi l$, coming together with the orientation flip: $(t, \sigma) \sim (t + 2\pi l, \pi - \sigma)$.
- There is a real modulus l; the amplitude needs one b zero mode insertion.
- There is a real isometry, which is the time translation; the amplitude needs one c zero mode insertion.

Thus, the Möbius strip amplitude is given by

$$A_{0,M} = \int \frac{dl}{2l} \operatorname{Tr} \left[\Omega (-1)^F b_0 c_0 q^{L_0 - \frac{c}{12}} \right] \qquad (q = e^{-2\pi l}) \, .$$

The trace is over Hilbert space of the matter and ghost sectors on the strip as well as the Chan-Paton factors, which give in total n^2 degeneracy for each state. We can divide the effect of Ω into two parts as follows.

$$\Omega |\Phi; T\rangle = |\Omega\Phi; PT^t P\rangle \equiv \Omega_\Phi \cdot \Omega_T |\Phi; T\rangle \, .$$

Let us see the Ω_T, which is defined as

$$\Omega_T = \frac{PT^t P}{T} \, .$$

In the case of $\mathrm{SO}(n)$, which means $P = 1$,

$$\Omega_{T,\mathrm{SO}} = \frac{T^t}{T} = \begin{cases} +1 & \text{(for symmetric } T) \\ -1 & \text{(for anti-symmetric } T) \end{cases} \, ,$$

Therefore,

$$\operatorname{Tr}_{T,\mathrm{SO}} [\Omega] = \frac{n(n+1)}{2} - \frac{n(n-1)}{2} = n \, .$$

In the case of $\mathrm{Sp}(n)$ (exercise), we have

$$\operatorname{Tr}_{T,\mathrm{Sp}} [\Omega] = -n \, .$$

The contribution from the matter and ghost part can be evaluated as

$$\operatorname{Tr}_\Phi \left[\Omega (-1)^F b_0 c_0 q^{L_0 - \frac{c}{12}} \right] = \frac{iV_{26}}{(2\pi \ell_s)^{26}} \frac{1}{(2l)^{13}} \cdot q^{-1} \prod_{N=1}^{\infty} \frac{(1-(-q)^N)^2}{(1-(-q)^N)^{26}}$$

$$= \frac{iV_{26}}{(2\pi\ell_s)^{26}} \frac{1}{(2l)^{13}} \cdot \frac{-1}{\eta(il+\frac{1}{2})^{24}} .$$

Therefore, the Möbius string amplitude is

$$\begin{aligned} A_{0,M} &= \pm n \cdot \frac{iV_{26}}{(2\pi\ell_s)^{26}} \int \frac{dl}{2l} \frac{1}{(2l)^{13}} \cdot \frac{-1}{\eta(il+\frac{1}{2})^{24}} , \\ &= \mp n \cdot \frac{iV_{26}}{(2\pi\ell_s)^{26}} \int \frac{ds}{2} \eta(is+\frac{1}{2})^{-24} , \end{aligned}$$

where we use the modular property (7.33); $\sqrt{2l}\eta(il+\frac{1}{2}) = \eta(\frac{i}{4l}+\frac{1}{2})$.

To sum up, three amplitudes are (introduced additional $\frac{1}{2}$ factor for Cylinder as an unoriented amplitude)

$$\begin{aligned} A_{0,C} &= \frac{n^2}{2^{13}} \cdot \frac{iV_{26}}{(2\pi\ell_s)^{26}} \int_0^\infty \frac{ds}{2} \eta(is)^{-24} , \\ A_{0,K} &= 2^{13} \cdot \frac{iV_{26}}{(2\pi\ell_s)^{26}} \int \frac{ds}{2} \eta(is)^{-24} , \\ A_{0,M} &= \mp 2n \cdot \frac{iV_{26}}{(2\pi\ell_s)^{26}} \int \frac{ds}{2} \eta(is+\frac{1}{2})^{-24} , \end{aligned}$$

where

$$\begin{aligned} \eta(is)^{-24} &= q^{-1} + 24 + \mathcal{O}(q) \qquad (q = e^{-2\pi s}) , \\ \eta(is+\frac{1}{2})^{-24} &= -q^{-1} + 24 + \mathcal{O}(q) . \end{aligned}$$

As we saw in the oriented string case, the massless states lead to IR singularity. On the other hand, in the unoriented open string case, we have

$$\frac{1}{2^{13}} \left[n \mp 2^{13} \right]^2 \cdot \frac{iV_{26}}{(2\pi\ell_s)^{26}} \int_0^\infty \frac{ds}{2} \cdot 24 ,$$

which vanish only for $SO(2^{13}) = SO(8192)$. This cancellation can be illustrated as like Figure 32. The cross cap shows another object (other than D-brane) that absorbs and

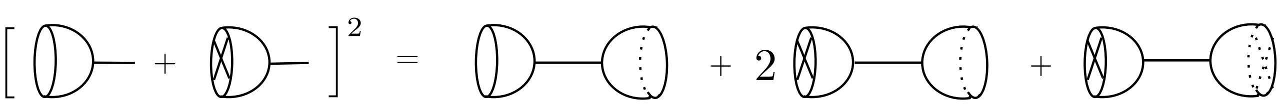


Figure 32: Pictorial expression for the unoriented open string amplitude.

emits gravitons etc., which is called O-plane. In this situation, it should be space-filling. Hence, it is O25$^\pm$-plane, and a single O25$^-$-plane cancels the tension of 2^{13} D25-branes.

Although our discussion was in the bosonic string, parallel argument perfectly works for superstring, which implies **the IR divergence vanishes for** $SO(2^5) = SO(32)$ [PC88]. This is another way to show the Green-Schwarz anomaly cancellation [GS84]. This means that

$$\text{Type I} = \text{Type IIB} + 32 \text{ D9-branes} + \text{O9}^-\text{-plane} .$$

Note that in the superstring case, D-branes and O-plane have RR-charge in addition to tension, which has relations

$$T_{\text{O9}^\pm} = \pm 32 \cdot T_{\text{D9}} \quad (\text{tension}) , \quad \mu_{\text{O9}^\pm} = \pm 32 \cdot \mu_{\text{D9}} \quad (\text{RR-charge}) .$$

10.3 T-duality of Type I theory

Let us recall T-duality. Consider X^i is S^1 compactified and T-duality acts as follows:

$$T_i: \quad X^i(z,\bar{z}) = X^i(z) + \overline{X}^i(\bar{z}) \quad \to \quad X'^i(z,\bar{z}) = X^i(z) - \overline{X}^i(\bar{z})\ .$$

On the other hand, the orientation flip acts as follows:

$$\Omega: \quad X^i(z,\bar{z}) = X^i(z) + \overline{X}^i(\bar{z}) \quad \to \quad X^i(\bar{z},z) = \overline{X}^i(z) + X^i(\bar{z})\ .$$

Therefore, in the T-dual coordinate X' the orientation flip acts as

$$\Omega: \quad X'^i(z,\bar{z}) \quad \to \quad -X'^i(\bar{z},z)\ .$$

This is understood as spacetime **oribfold** as well as world-sheet orientation flip (see Figure 33), which is called **orientifold**. Therefore, the dual space is not S^1 but $S^1/\mathbb{Z}_2$ with

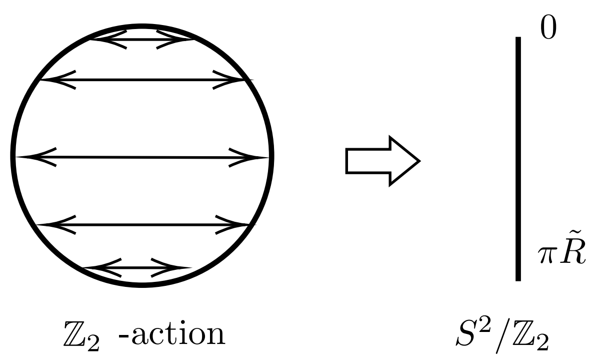


Figure 33: $\mathbb{Z}_2$ orbifold of S^1.

radius $\widetilde{R} = \frac{\alpha'}{R}$. Note that there are two fixed points where O-planes sit and induce the spacetime reversal and the orientation flip.

Let us consider Type I superstring theory with X^9 compactified on S^1 and take T-duality along the S^1. With a proper Wilson line

$$A_9 = i\begin{pmatrix} & -a_1 & & & \\ a_1 & & & & \\ & & & -a_2 & \\ & & a_2 & & \\ & & & & \ddots \end{pmatrix}$$

D8-branes sit at different points in $\mathbb{Z}_2$ symmetric way (see Figure 34). Note that an O9$^-$-plane splits into two O8$^-$-planes. Accordingly, tension and RR-charge reduce by 2. In the end, T-dual of Type I on S^1 is

$$\text{Type I}' \;=\; \text{Type IIA on } S^1/\mathbb{Z}_2 \quad \text{with} \quad 2\,\text{O8}^-\text{-plane} + (16+16)\ \text{D8-branes}\ . \tag{10.7}$$

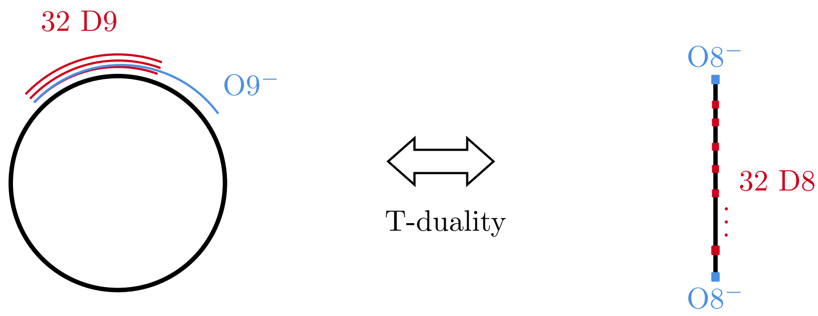


Figure 34: T-dual of Type I superstring theory.

10.4 Op-plane

Of course, one can consider further T-dualities along other directions. In particular, an orientifold p-plane (Op) in Type II string theory is defined [DP96, Dab97] as the $\mathbb{Z}_2$ involution

$$\mathrm{O}p: \quad \mathbb{R}^{1,p} \times \mathbb{R}^{9-p}/I_{9-p}\Omega \cdot \begin{cases} 1 & p = 0, 1 \bmod 4 \\ (-1)^{F_L} & p = 2, 3 \bmod 4, \end{cases} \tag{10.8}$$

where I_{9-p} is the involution of all coordinates in the transverse space $\mathbb{R}^{9-p}$ and F_L is the left-moving spacetime fermion number operator. The presence of $(-1)^{F_L}$ for $p = 2, 3 \bmod 4$ can be understood from the requirement that the generator square to one on fermions. Due to the spacetime involution I_{9-p} transverse to the Op-plane, the number of Dp-branes parallel to the Op-plane is effectively doubled. Therefore, a stack of n Dp-branes along the Op^--plane leads to an SO$(2n)$ gauge theory (Dynkin type D_n) in Type II theory. To realize an SO$(2n+1)$ gauge group (Dynkin type B_{n+1}), we need a combination of an Op^--plane and $\frac{1}{2}$ Dp-brane, which is denoted by $\widetilde{\mathrm{O}p}^-$. Consequently, n Dp-branes with $\widetilde{\mathrm{O}p}^-$-plane give rise to an SO$(2n+1)$ gauge theory as a world-volume theory. As we will see in §13.1, it is natural to incorporate an $\widetilde{\mathrm{O}p}^+$-plane in Type II theory, which also gives rise to an Sp(k) gauge theory like an Op^+-plane. They behave differently under the S-duality of Type IIB theory.

Each T-duality doubles the number of O-planes, and hence, reduces the tension and the R-R charges. Namely, we have the following relations:

$$T_{\mathrm{O}p^\pm} = \pm 2^{p-5} \cdot T_{\mathrm{D}p} \quad (\text{tension})\,, \quad \mu_{\mathrm{O}p^\pm} = \pm 2^{p-5} \cdot \mu_{\mathrm{D}p} \quad (\text{R-R charge})\,.$$

Since an $\widetilde{\mathrm{O}p}^-$-plane is a combination of an Op^--plane and $\frac{1}{2}$ Dp-brane, there is $\frac{1}{2}$-shift in R-R charge as summarized in Table 10.4. Op-planes bring richness to string theory such as quiver gauge theories, the topology of $\mathbb{R}\mathbf{P}^p$, real K-theory, Dirac quantization conditions, etc, which are beyond the scope of this lecture. We refer to [Wit98b, HK00, HZ99, TY19] and references therein.

11 Heterotic string theories

We are ready to learn **Heterotic string theories** [GHMR85b, GHMR85a, GHMR86]. Heterotic string is a hybrid construction of the left-moving sector of the 26-dimensional bosonic string and the right-moving sector of 10-dimensional superstring. The 16 extra

O-plane	gauge group	RR-charge
$\mathrm{O}p^-$	$\mathrm{SO}(2n)$	-2^{p-5}
$\mathrm{O}p^+$	$\mathrm{Sp}(2n)$	$+2^{p-5}$
$\widetilde{\mathrm{O}p}^-$	$\mathrm{SO}(2n+1)$	$-2^{p-5}+\frac{1}{2}$
$\widetilde{\mathrm{O}p}^+$	$\mathrm{Sp}(2n)$	$+2^{p-5}$

bosons of the left-movers are compactified on particular 16-dimensional tori, leading to $\mathrm{SO}(32)$ or $E_8 \times E_8$. Since 16-dimensional tori have very special properties, we can also describe the left-movers in terms of 32 free fermions whose current algebra is associated to either $\mathrm{SO}(32)$ or $E_8 \times E_8$ with level $k = 1$. This hybridization of two different kinds of modes has been referred to as **heterosis**.

11.1 Bosonic construction

Toroidal compactifications

We have learned the S^1 compactification in §9 so that we now generalize our analysis to the compactification of bosonic string theory on a D-dimensional torus T^D. The resulting theory is effectively $(26-D)$-dimensional. The torus is defined by identifying points in the D-dimensional internal space as follows (compact dimensions are labeled with capital letters):

$$X^I \sim X^I + 2\pi e_i^I w^i = X^I + 2\pi W^I\,, \qquad \text{for} \quad w^i \in \mathbb{Z}\,. \tag{11.1}$$

The $\mathbf{e}_i = \{e_i^I\}$ $(i = 1\cdots D)$ are D linear independent vectors called **vielbein** which generate a D-dimensional lattice Λ. In addition, the vielbein brings the metric into the standard Euclidean form:

$$G_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j = e_i^I e_j^J \delta_{IJ}, \qquad X^I \equiv e_i^I X^i \tag{11.2}$$

The torus on which we compactify is obtained by dividing $\mathbb{R}^D$ by Λ:

$$T^D = \frac{\mathbb{R}^D}{2\pi\Lambda}\,.$$

The momentum p^I conjugate to the coordinates X^I on the torus is quantized as $\mathbf{p}\cdot\mathbf{W} \in \mathbb{Z}$. Therefore, the momentum p takes its value on the dual lattice Λ^*

$$\Lambda^* \equiv \{e^{*Ii} n_i; \quad n_i \in \mathbb{Z}\}, \qquad G^{ij} = \mathbf{e}^{*i}\cdot\mathbf{e}^{*j} = e_I^{*i} e_J^{*j} \delta^{IJ}.$$

The condition which a closed string in the compact directions $X^I(z,\bar{z}) = X_R^I(z) + \overline{X}_L^I(\bar{z})$ obey is

$$X^I(\sigma + 2\pi, \tau) = X^I(\sigma,\tau) + 2\pi W^I$$

so that W^I are analogues of winding numbers. We express the mode expansion for the compact direction as follows:

$$X_R^I(z) = x^I - i\sqrt{\frac{\alpha'}{2}}\, p_R^I \ln z + i\sqrt{\frac{\alpha'}{2}} \sum_{m\neq 0} \frac{\alpha_m^I}{m z^m},$$

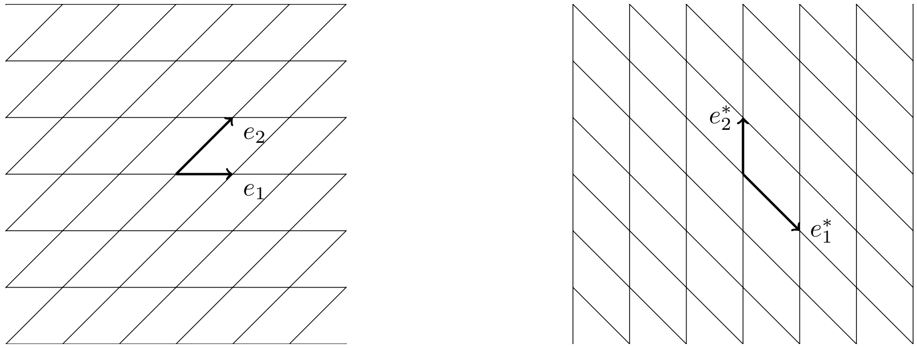


Figure 35: Lattice and its dual lattice

$$\overline{X}_L^I(\overline{z}) = \overline{x}^I - i\sqrt{\frac{\alpha'}{2}}p_L^I \ln \overline{z} + i\sqrt{\frac{\alpha'}{2}}\sum_{m\neq 0}\frac{\overline{\alpha}_m^I}{m\overline{z}^m}.$$

where the zero modes are

$$\begin{aligned}
\mathbf{p}_L := p_L^I &= \frac{1}{\sqrt{2}}\left[\sqrt{\alpha'}p^I + \frac{W^I}{\sqrt{\alpha'}}\right] = \frac{1}{\sqrt{2}}\left[\sqrt{\alpha'}e^{*Ii}n_i + \frac{e_i^I}{\sqrt{\alpha'}}w^i\right] , \\
\mathbf{p}_R := p_R^I &= \frac{1}{\sqrt{2}}\left[\sqrt{\alpha'}p^I - \frac{W^I}{\sqrt{\alpha'}}\right] = \frac{1}{\sqrt{2}}\left[\sqrt{\alpha'}e^{*Ii}n_i - \frac{e_i^I}{\sqrt{\alpha'}}w^i\right] .
\end{aligned} \tag{11.3}$$

The mass formula and the level matching condition are now

$$\begin{aligned}
\alpha' M^2 &= 2(N + \overline{N} - 2) + (\alpha' p_I p^I + \frac{1}{\alpha'}W_I W^I) \\
&= 2(N + \overline{N} - 2) + (\alpha' n_i n_j G^{ij} + \frac{1}{\alpha'}w^i w^j G_{ij}) \\
N - \overline{N} &= p_I W^I = n_i w^i
\end{aligned} \tag{11.4}$$

As we have seen before, the expressions for p_L and p_R suggest **T-duality** between the winding number W^I and the momentum p^I. In fact, **T-duality** is equivalence between a pair of compactification lattices $\mathbf{e}_i$ and $\mathbf{e}_i'$ that are related as $\sqrt{\alpha'}\mathbf{e}_i' = \frac{\mathbf{e}^{*i}}{\sqrt{\alpha'}}$. These two compactifications give the same spectrum since their allowed values of the momenta are related as

$$\mathbf{p}_L \leftrightarrow \mathbf{p}_L'; \qquad \mathbf{p}_R \leftrightarrow -\mathbf{p}_R' \tag{11.5}$$

by interchanging the labels n_i and w^i.

Now let us combine the zero modes into the $(D+D)$-dimensional vectors $\mathbf{P} = (\mathbf{p}_L, \mathbf{p}_R)$. This construction treats Λ and Λ^* on equal footing as

$$\mathbf{P} = \mathbf{E}^{*i}n_i + \mathbf{E}_j w^j,$$

where

$$\mathbf{E}_j = \frac{1}{\sqrt{\alpha'}}\left(\mathbf{e}_j, -\mathbf{e}_j\right) , \quad \mathbf{E}^{*i} = \sqrt{\alpha'}\left(\mathbf{e}^{*i}, \mathbf{e}^{*i}\right).$$

Note that the length of the lattice is normalized by the string length $\sqrt{\alpha'} = \ell_s$. Hence $\mathbf{P}$ takes value in a $(D+D)$-dimensional lattice $\Gamma_{D,D}$ spanned by $\{\mathbf{E}^{*i}\}$ and $\{\mathbf{E}_j\}$ that satisfies the following properties:

- **Lorentzian** if the signature of the metric G is $((+1)^D, (-1)^D)$,
- **integral** if $v \cdot w \in \mathbb{Z}$ for all $v, w \in \Gamma_{D,D}$,
- **even** if $\Gamma_{D,D}$ is integral and v^2 is even for all $v \in \Gamma_{D,D}$,
- **self-dual** if $\Gamma_{D,D} = (\Gamma_{D,D})^*$,
- **unimodular** if $\mathrm{Vol}(\Gamma_{D,D}) = |\det G| = 1$.

In fact, the metric of this lattice is defined by

$$\mathbf{P} \cdot \mathbf{P}' = (\mathbf{p}_L \cdot \mathbf{p}'_L - \mathbf{p}_R \cdot \mathbf{p}'_R) = n_i w'^i + n'_i w^i$$

so that it is Lorentzian. Because of $\mathbf{P} \cdot \mathbf{P} \in 2\mathbb{Z}$, it is even. The self-dual property will be shown in Homework. The unimodular property $\mathrm{Vol}(\Gamma_{D,D}) = \mathrm{Vol}(\Gamma_{D,D}) = 1$ immediately follows from the self-dual property. The lattice $\Gamma_{D,D}$ in the torus compactification of the string is called **Narain lattice**.

The partition function of the bosonic string compactified on a torus T^D is easy to write down:

$$Z^{\mathrm{bos}}_{\Gamma_{D,D}} = \frac{1}{\tau_2^{(24-D)/2} |\eta(q)|^{48}} \sum_{(\mathbf{p}_R, \mathbf{p}_L) \in \Gamma_{D,D}} q^{\frac{1}{2}\mathbf{p}_R^2} \overline{q}^{\frac{1}{2}\mathbf{p}_L^2}$$

where $|\eta(q)|^{48}$ is the bosonic oscillator contribution and $\tau_2^{(24-D)/2}$ comes from the integral of non-compact momenta. This is easy to generalize to Type II string compactified on T^D

$$Z^{\text{Type II}}_{\Gamma_{D,D}} = \frac{1}{\tau_2^{(8-D)/2} |\eta(q)|^{24}} \frac{1}{4} \Big| - \vartheta_2^4(\tau) + \vartheta_3^4(\tau) - \vartheta_4^4(\tau) \Big|^2 \sum_{(\mathbf{p}_R, \mathbf{p}_L) \in \Gamma_{D,D}} q^{\frac{1}{2}\mathbf{p}_R^2} \overline{q}^{\frac{1}{2}\mathbf{p}_L^2}$$

which vanishes by virtue of the Jacobi-Riemann identity.

Heterotic strings

After we learn about toroidal compactifications, we are ready to introduce the $D = 10$ Heterotic string. As mentioned at the beginning, Heterotic string is a combination of the left-moving sector of the 26-dimensional bosonic string and the right-moving sector of the 10-dimensional superstring. The left-moving bosonic string is compactified on a 16-dimensional torus so that the momenta of the additional chiral bosons $X^I(\overline{z})$ takes value on 16-dimensional lattice Γ_{16}, **i.e** $\mathbf{p}_L \in \Gamma_{16}$. Hence, the partition function of Heterotic string can be written as

$$Z^{\mathrm{het}}(\tau) = \frac{1}{\tau_2^4 \eta(q)^{12} \eta(\overline{q})^{24}} \Big(- \vartheta_2^4(\tau) + \vartheta_3^4(\tau) - \vartheta_4^4(\tau) \Big) \sum_{\mathbf{p}_L \in \Gamma_{16}} \overline{q}^{\frac{1}{2}\mathbf{p}_L^2} \tag{11.6}$$

Here $\eta(q)^8 \eta(\overline{q})^{24}$ is the bosonic oscillator contribution, the τ_2^4 factor arises from the zero modes of the noncompact transverse coordinates and $\vartheta_i^4/\eta(q)^4$ comes from the world-sheet fermions. The most interesting part of this partition function is the lattice sum

$$P(\tau) := \sum_{\mathbf{p}_L \in \Gamma_{16}} \overline{q}^{\frac{1}{2}\mathbf{p}_L^2}$$

Since the partition function (11.6) should be invariant under the modular transformation $\mathrm{SL}(2,\mathbb{Z})$, the modular transformation (7.33) of η and ϑ_i tell us that

$$T : P(\tau + 1) = P(\tau)\,, \qquad S : P(-1/\tau) = \tau^8 P(\tau)\,.$$

The invariance under T-transformation clearly demands that $\mathbf{p}_L^2 \in 2\mathbb{Z}$ so that Γ_{16} must be **even**. For the S-transformation, we make use of the Poisson resummation formula

$$\sum_{\mathbf{p}\in\Lambda} e^{-\pi\alpha(\mathbf{p}+\mathbf{x})^2+2\pi i\mathbf{y}\cdot(\mathbf{p}+\mathbf{x})} = \frac{1}{\mathrm{Vol}(\Lambda)\alpha^{\dim\Lambda/2}} \sum_{\mathbf{q}\in\Lambda^*} e^{-2\pi i\mathbf{q}\cdot\mathbf{x}-\frac{\pi}{\alpha}(\mathbf{y}+\mathbf{q})^2}$$

which amounts to

$$P(-1/\tau) = \frac{\tau^8}{\mathrm{Vol}(\Lambda)} \sum_{\mathbf{p}_L\in(\Gamma_{16})^*} \overline{q}^{\frac{1}{2}\mathbf{p}_L^2} .$$

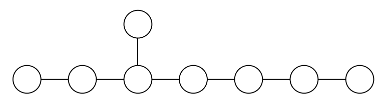
This requires that the lattice Γ_{16} is **self-dual**, *i.e.* $(\Gamma_{16})^* = \Gamma_{16}$ so that $\mathrm{Vol}(\Lambda) = 1$.

It turns out that there are only two even self-dual Euclidean lattices in 16 dimensions

- the root lattice of $E_8 \times E_8$
- the weight lattice of $\mathrm{Spin}(32)/\mathbb{Z}_2$

The metric G_{ij} of the root lattice of E_8 is the Cartan matrix of E_8 [6]:

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix} .$$



Let us read off massless fields in Heterotic string theory more carefully. As usual, there is the tachyonic vacuum of the bosonic string. At the massless level, we have oscillator excitations $\overline{\alpha}^{\mu}_{-1}|0\rangle$, $\overline{\alpha}^{I}_{-1}|0\rangle$ in the left-moving sector. The former transform like spacetime vectors while the internal oscillator excitations correspond to the left-moving part of the Abelian $\mathrm{U}(1)^{16}$ gauge boson. They form the **Cartan subalgebra** of $E_8 \times E_8$ or $\mathrm{SO}(32)$. Both the root lattice of $E_8 \times E_8$ and the weight lattice of $\mathrm{Spin}(32)/\mathbb{Z}_2$ contain 480 vectors of $(\text{length})^2 = 2$ and generate the 496-dimensional non-Abelian gauge bosons of these groups. Remarkably, although Heterotic strings are closed strings, gauge fields show up thanks to the extra 16-dimensional tours! This can be also understood as a novel **stringy effect** and gauge groups are restricted only to either $E_8 \times E_8$ or $\mathrm{SO}(32)$ in order for the theory to be consistent. Moreover, we have seen that $\mathrm{SO}(32)$ gauge group appears in Type I string theory. As we will see in §13, this is not coincidental because Type I and Heterotic $\mathrm{SO}(32)$ are related by **S-duality**.

As a result, the massless spectra of Heterotic string are as follows

• Gravitons, B-fields, dilaton in $D = 10$

$$\psi^{\mu}_{-\frac{1}{2}}|0\rangle_{\mathrm{NS}} \otimes \overline{\alpha}^{\nu}_{-1}|0\rangle$$

• their supersymmetric partners, gravitino and dilatino

$$|\mathbf{s}\rangle_{\mathrm{R}} \otimes \overline{\alpha}^{\nu}_{-1}|0\rangle$$

• 496 gauge bosons of $E_8 \times E_8$ or $\mathrm{SO}(32)$

$$\psi^{\mu}_{-\frac{1}{2}}|0\rangle_{\mathrm{NS}} \otimes \overline{\alpha}^{I}_{-1}|0\rangle , \qquad \psi^{\mu}_{-\frac{1}{2}}|0\rangle_{\mathrm{NS}} \otimes |\mathbf{p}_L^2 = 2\rangle$$

[6] Unfortunately, we do not have time to talk about exceptional Lie algebras or classification of semi-simple Lie algebras [Kir08]. In particular, if you want to get some intuition of weight and root lattices, see [Kir08, Fig 7.3, Fig 8.1, Fig 8.2] for A_2. If you want to understand the structure E_8 related to string theory, we refer to [GSW87, §6].

• 496 supersymmetric partners, gaugini

$$|\mathbf{s}\rangle_{\rm R} \otimes \overline{\alpha}^I_{-1}|0\rangle\ , \qquad |\mathbf{s}\rangle_{\rm R} \otimes |\mathbf{p}_L^2 = 2\rangle$$

Indeed, Heterotic string theory is $D = 10\ \mathcal{N} = 1$ supergravity coupled to $D = 10\ \mathcal{N} = 1$ $E_8 \times E_8$ or $\mathrm{SO}(32)$ super-Yang-Mills theory so that it has 16 real supersymmetric charges.

11.2 Fermionic construction

The 16 bosonic fields compactified on the self-dual lattice can be described by fermionic fields, which is called **fermionization**. Therefore, we will describe fermionic construction of Heterotic string theory next.

The world-sheet action of Heterotic string theory is given by

$$\begin{aligned} S^{\rm m} &= \frac{1}{4\pi}\int d^2z\, \left(\frac{2}{\alpha'}\partial X^\mu \overline{\partial} X_\mu + \psi^\mu \overline{\partial}\psi_\mu + \overline{\lambda}^A \partial \overline{\lambda}_A\right) \\ S^{\rm gh} &= \frac{1}{2\pi}\int d^2z\, (b\overline{\partial}c + \overline{b}\partial\overline{c} + \beta\overline{\partial}\gamma) \end{aligned} \tag{11.7}$$

where μ are 10-dimensional indices and the right-moving sector is supersymmetric. In order for the theory to be Weyl-anomaly free, the central charge

$$c^{\rm tot} = c^X + c^\psi + c^{bc} + c^{\beta\gamma} + c^\lambda = 10 + \frac{5}{2} - 26 + \frac{11}{2} + c^\lambda = c^\lambda - 8$$

should vanish. Since each Majorana-Weyl anti-chiral fermion $\overline{\lambda}^A$ contributes $\frac{1}{4}$ to the central charge, we need 32 left-moving fermions $\overline{\lambda}^A$ in the action.

It turns out that there are two possible boundary conditions on the left-moving fermions $\overline{\lambda}_A$ which give rise to fully consistent string theories. If we impose the same boundary condition to all, it leads to $\mathrm{SO}(32)$ gauge group. On the other hand, if we impose one boundary condition to a half and the other boundary condition to the other half, we obtain $E_8 \times E_8$ gauge group.

Heterotic $\mathrm{SO}(32)$ (HO)

For Heterotic $\mathrm{SO}(32)$, we impose the same boundary condition to all the left-moving fermions as

$$\begin{aligned} \overline{\lambda}^A(t,\sigma+2\pi) &= +\overline{\lambda}^A(t,\sigma) && \text{R: periodic on cylinder} \\ \overline{\lambda}^A(t,\sigma+2\pi) &= -\overline{\lambda}^A(t,\sigma) && \text{NS: anti-periodic on cylinder} \end{aligned} \tag{11.8}$$

so that there is a global symmetry $\mathrm{SO}(32)$ that rotates $\overline{\lambda}^A$ ($A = 1,\ldots,32$). In order for the theory to be consistent, we have to impose GSO projection on the left-moving sector. In HO theory, we pick only states with odd fermionic numbers in NS sector and those with even fermionic number

$$P^{\rm HO}_{\rm NS} := \frac{1-(-1)^F}{2}$$

whereas we keep only the states with even fermion number

$$P^{\rm HO}_{\rm R} := \frac{1+(-1)^F}{2}\ .$$

In addition, we have to impose the level matching condition

$$N - a = \overline{N} - \overline{a} \tag{11.9}$$

where the normal ordering constants in the left-moving sector are

$$\overline{a}_{\rm NS} = \frac{8}{24} + \frac{32}{48} = 1\,, \qquad \overline{a}_{\rm R} = \frac{8}{24} - \frac{32}{24} = -1\,.$$

Here the first term comes from the left-moving bosonic field $\overline{X}^i$ whereas the second term depends on the boundary condition (7.13) of $\overline{\lambda}^A$. Hence, the R sector contains only massive states. Contrary to the supersymmetric right-mover, the Tachyon state $|0\rangle_{\rm NS}$ in the NS sector is preserved under the GSO projection. However, there is no corresponding state in the right-moving sector so that it does not obey the level matching condition (11.9). As a result, the left-moving Tachyon is not included in the spectrum. Then, the first excited states after the GSO projection in the NS sector are

$$\begin{aligned} \overline{\alpha}^i_{-1}|0\rangle_{\rm NS}\,, \qquad & (\mathbf{8_v}, \mathbf{1}) \\ \overline{\lambda}^A_{-1/2}\overline{\lambda}^B_{-1/2}|0\rangle_{\rm NS}\,, \qquad & (\mathbf{1}, \mathbf{adj}) \end{aligned}$$

where the bold letters are the representations of SO(8) $\times$ SO(32). The adjoint representation **adj** of SO(32) is the antisymmetric tensor with dimension $32 \times 31/2 = \mathbf{496}$. The following table shows the massless spectrum of HO where the first row represents the $D = 10$ $\mathcal{N} = 1$ supergravity multiplet whereas the second row shows $\mathcal{N} = 1$ gauge multiplet in the adjoint of SO(32) as we have seen in the bosonic construction.

Left\Right	$\mathbf{8_v}$	$\mathbf{8_c}$
$(\mathbf{8_v}, \mathbf{1})$	$\mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}$ $\phi\ \ B_{\mu\nu}\ \ G_{\mu\nu}$	$\mathbf{8_s} \oplus \mathbf{56_c}$ $\lambda^+\ \ \psi^-_m$
$(\mathbf{1}, \mathbf{496})$	SO(32) gauge boson $A^\mu_{[A,B]}$	SO(32) gaugini $\eta_{[A,B]}$

Heterotic $E_8 \times E_8$ (HE)

The second Heterotic string theory is obtained by dividing the $\overline{\lambda}^A$ into two sets of 16 with independent boundary conditions,

$$\overline{\lambda}^A(t, \sigma + 2\pi) = \begin{cases} \epsilon_1 \overline{\lambda}^A(t,\sigma) & A = 1, \ldots, 16 \\ \epsilon_2 \overline{\lambda}^A(t,\sigma) & A = 17, \ldots, 32 \end{cases}$$

where $\epsilon_i = \pm 1$. Therefore, in the left-moving sector, we need to take the following boundary conditions into account

$$(\mathrm{NS}_1, \mathrm{NS}_2)\,, \quad (\mathrm{R}_1, \mathrm{NS}_2)\,, \quad (\mathrm{NS}_1, \mathrm{R}_2)\,, \quad (\mathrm{R}_1, \mathrm{R}_2)\,.$$

Consequently, the global symmetry is broken to $\mathrm{SO}(16)_1 \times \mathrm{SO}(16)_2$. The GSO projection is imposed to the two sets of left-movers independently:

$$P^{\rm HE}_{\mathrm{NS}_i} := \frac{1 - (-1)^F}{2} \qquad P^{\rm HE}_{\mathrm{R}_i} := \frac{1 + (-1)^F}{2}\,.$$

We also apply for the level-matching condition (11.9). The normal ordering constant in each boundary condition is

$$\bar{a}_{\mathrm{NS}_1,\mathrm{NS}_2} = 1\,, \qquad \bar{a}_{\mathrm{R}_1,\mathrm{NS}_2} = \bar{a}_{\mathrm{NS}_1,\mathrm{R}_2} = \frac{8}{24} + \frac{16}{48} - \frac{16}{24} = 0\,, \qquad \bar{a}_{\mathrm{R}_1,\mathrm{R}_2} = -1\,.$$

Again, $(\mathrm{R}_1, \mathrm{R}_2)$ boundary condition has only massive states. Although the Tachyon state $|0\rangle_{\mathrm{NS}_1,\mathrm{NS}_2}$ in the NS sector is preserved under the GSO projection, it does not obey the level-matching condition (11.9) so that it is not present in the spectrum. Then, the massless states are

$$\begin{aligned} \bar{\alpha}^i_{-1}|0\rangle_{\mathrm{NS}_1,\mathrm{NS}_2}\,, &\qquad (\mathbf{8_v},\mathbf{1},\mathbf{1}) \\ \bar{\lambda}^A_{-1/2}\bar{\lambda}^B_{-1/2}|0\rangle_{\mathrm{NS}_1,\mathrm{NS}_2}\,, &\qquad (\mathbf{1},\mathbf{adj},\mathbf{1}) \text{ or } (\mathbf{1},\mathbf{1},\mathbf{adj}) \end{aligned} \tag{11.10}$$

where the bold letters are the representations of $\mathrm{SO}(8) \times \mathrm{SO}(16)_1 \times \mathrm{SO}(16)_2$. Note that the GSO projection requires either $1 \le A, B \le 16$ or $17 \le A, B \le 32$ in (11.10). The adjoint representation of SO(16) is of $16 \times 15/2 = \mathbf{120}$ dimensions.

In $(\mathrm{R}_1, \mathrm{NS}_2)$ and $(\mathrm{NS}_1, \mathrm{R}_2)$, the ground states are massless since the normal ordering constant is zero. Since the 16 $\bar{\lambda}^A_0$ zero modes form 8 raising and 8 lowering operators

$$\bar{\lambda}^{K\pm}_0 = 2^{-1/2}(\bar{\lambda}^{2K-1}_0 \pm i\bar{\lambda}^{2K}_0)\,, \qquad K = 1, \ldots, 8 \text{ or } K = 9, \ldots, 16\,,$$

the $2^8 = \mathbf{256}$-dimensional spinor representation of SO(16) becomes massless. However, the GSO projection picks positive chirality $\mathbf{128}$ out of $\mathbf{256} = \mathbf{128} + \mathbf{128}'$ in the Ramond sector. Hence, the ground states are $(\mathbf{1},\mathbf{128},\mathbf{1})$ and $(\mathbf{1},\mathbf{1},\mathbf{128})$ under $\mathrm{SO}(8) \times \mathrm{SO}(16)_1 \times \mathrm{SO}(16)_2$ in $(\mathrm{R}_1, \mathrm{NS}_2)$ and $(\mathrm{NS}_1, \mathrm{R}_2)$, respectively .

All in all, the left-moving massless states form the representations of $\mathrm{SO}(8) \times \mathrm{SO}(16)_1 \times \mathrm{SO}(16)_2$

$$(\mathbf{8_v},\mathbf{1},\mathbf{1}) + (\mathbf{1},\mathbf{120},\mathbf{1}) + (\mathbf{1},\mathbf{1},\mathbf{120}) + (\mathbf{1},\mathbf{128},\mathbf{1}) + (\mathbf{1},\mathbf{1},\mathbf{128})$$

This spectrum strongly suggests that gauge symmetry is enhanced $\mathrm{SO}(16) \to E_8$ because E_8 has dimension $\mathbf{120} + \mathbf{128} = \mathbf{248}$ which is also the dimension of the adjoint representation E_8. In fact, E_8 has an SO(16) subgroup under which the E_8 adjoint $\mathbf{248}$ transforms as $\mathbf{120} + \mathbf{128}$. Hence, the massless spectrum is the $D = 10$ $\mathcal{N} = 1$ supergravity multiplet plus an $\mathcal{N} = 1$ $E_8 \times E_8$ gauge multiplet. Even in fermionic construction, we reproduce the 496-dimensional adjoint representations of both SO(32) and $E_8 \times E_8$ gauge groups.

No D-branes in Heterotic strings

We have seen that D-branes are charged to R-R fields in Type II theories. However, there is no R-R field in Heterotic string theories because there is only world-sheet supersymmetry in the right-moving sector. In other words, although the R-R $(p+2)$-form field strength G in Type II theories can be expressed as

$$G = \bar{\psi}^L \Gamma^{\mu_1 \cdots \mu_{p+2}} \psi^R\,,$$

there is no ψ^L in Heterotic string theories. Hence, there is no D-brane in Heterotic string theories. Consequently, Heterotic string theories are the theories of closed strings[7]. However, apart from the fundamental strings, there are extended objects, **NS5-branes or Heterotic fivebranes**, in Heterotic string theories and they are magnetically charged under the B-field.

[7]However, Polchinski argues in [Pol06] that there exist open Heterotic strings.

12 Supergravity

String theory includes massless states as well as massive states. However, in the energy regions much lower than $1/\ell_s$, we do not see the stringy massive states because the mass $M^2 \sim \frac{1}{\alpha'} \sim \frac{1}{\ell_s^2}$ is assumed to be very heavy. Therefore, in the low-energy region, we can describe the theory by an **effective theory**, which only contains the massless particles (lightest states). The effective theory, of course, does not contain all the information of the original theory, however, it does give us some information about the original theory.

For the bosonic string theory, the low-energy effective action for the massless fields is given in (4.7). Below we deal with its supersymmetric versions, type **IIA/IIB supergravity**, that are low-energy effective theories of IIA/IIB superstring theory. In §13.4, we will deal with the low-energy effective actions of Type I and Heterotic string theory. The theory of supergravity is very broad, and we refer to [FVP12] for more detail.

12.1 Local supersymmetry

Before going to Type IIA/IIB supergravity, let us see the basic idea of supergravity.

Roughly speaking, a global supersymmetry algebra takes the form

$$\{\epsilon_2 Q, \overline{\epsilon}_1 \overline{Q}\} = i\overline{\epsilon}_1 \Gamma^\mu \epsilon_2 P_\mu \ . \tag{12.1}$$

(For instance, see (16.10).) The key idea in supergravity is that supersymmetry holds locally so that the spinor parameters ϵ are arbitrary functions of the spacetime coordinates $\epsilon \to \epsilon(x)$. The supersymmetry algebra will then involve local translation parameters $\overline{\epsilon}_1 \gamma^\mu \epsilon_2$ which must be viewed as diffeomorphisms. Thus local supersymmetry requires gravity. Note that we use x^μ for local coordinates of the spacetime instead of X^μ, which has been adopted in the previous sections.

First of all, in order to define spinors in a curved space, we need to introduce vielbeins $e^a_\mu(x^\rho)$. The metric can be the orthonormal one at one point by coordinate transformations, and vielbeins transforms a local coordinate x^μ into the orthonormal coordinate at the point:

$$x^a = e^a_\mu x^\mu \ , \quad x^\mu = e^\mu_a x^a \ ,$$

which is defined through spacetime metric G_{MN} by

$$G_{\mu\nu}(x^\rho) = \eta_{ab} e^a_\mu(x^\rho) e^b_\nu(x^\rho) \ .$$

Now we can use spinor representations and gamma matrices thanks to the vielbeins:

$$\{\Gamma^a, \Gamma^b\} = 2\eta^{ab} \ .$$

If we include gravity in a theory, the translational invariance is promoted to invariance under general coordinate transformations. Similarly, global Lorentz symmetry is promoted to local Lorentz symmetry (this is done by the vielbeins):

$$\Lambda^a{}_b e^\mu_a(x^\rho) e^b_\nu(x^\rho) \equiv \Lambda^\mu{}_\mu(x^\rho) \ .$$

Gravity can be interpreted as the gauge field for general coordinate transformations. Since the gauge transformation of a spin-1 gauge field can be generally written as

$$\delta A_\mu = D_\mu \lambda \ ,$$

the gauge transformation of the gravity field will take the form

$$\delta e^a_\mu = D_\mu \lambda^a \ , \tag{12.2}$$

where λ^a is a general coordinate transformation, and D_μ is a covariant derivative. Similarly, we need to introduce a gauge field for local supersymmetry in supergravity, and the corresponding gauge field is a **gravitino**. (Recall that gravitino appears as a massless field in superstring theory.) Then, the supersymmetry transformations for the gravitino and vielbeins are given by

$$\delta\psi^\alpha_\mu = D_\mu \epsilon^\alpha \ , \qquad \delta_\epsilon e^a_\mu = i\bar{\epsilon}\Gamma^a\psi_\mu \ . \tag{12.3}$$

Then, the square of the supersymmetry transformation is as we want:

$$\begin{aligned}[\delta_1, \delta_2]\, e^a_\mu =& iD_\mu(\bar{\epsilon}_1\Gamma^a\epsilon_2) \\ [\delta_1, \delta_2]\, \psi_\mu =& i\bar{\epsilon}_1\Gamma^\rho\epsilon_2(D_\rho\psi_\mu - D_\mu\psi_\rho) \ .\end{aligned} \tag{12.4}$$

Supergravity theory always includes the two gauge fields, e^a_μ and ψ^α_μ, and the action is given by

$$S = \frac{1}{2\kappa_D^2}\int d^D x\, e\, \left[R - 2i\psi_\mu\Gamma^{MNP}D_\nu\psi_P\right] \ ,$$

where $e = \det e^a_\mu$. It is straightforward to check that the action is invariant under local supersymmetry transformation (12.3). This action can be regarded as the supersymmetric version of the Einstein-Hilbert action.

12.2 $D = 11$ supergravity

Supersymmetry puts a strong constraint on the spacetime dimension. If we limit ourselves to consider fields up to spin-2, then it is known that the highest dimension is $D = 11$ [Nah78]. Roughly speaking, this is because the degrees of freedom for fermions grow exponentially as $2^{[D/2]}$ whereas those of bosons grow by the power law of D (for instance, $\frac{(D-1)(D-2)}{2} - 1$ for gravitons). Therefore, to balance fermions and bosons with spin less than two, we cannot go arbitrarily higher.

The $D = 11$ supergravity [CJS78] consists of three fields;

- one is the graviton G_{MN} (44 states)
- three-form field $M_{(3)}$ (84 states)
- gravitino ψ_M (128 states)

We can see that the numbers of fermions and bosons are balanced. Although the existence of fermions is crucial, we write the bosonic part of the action of the $D = 11$ supergravity:

$$2\kappa_{11}^2 S_{11} = \int d^{11}x\sqrt{-G}\left[R - \frac{1}{2}K_{(4)}^2\right] - \frac{1}{6}\int d^{11}x\, M_{(3)} \wedge K_{(4)} \wedge K_{(4)} \ , \tag{12.5}$$

where $K_{(4)} = dM_{(3)}$ is the field strength, and $K_{(4)}^2 = K_{(4)} \wedge *K_{(4)}$. The fermionic part of the action follows from supersymmetry in principle. Note that there is only one parameter κ_{11} defined as

$$\frac{1}{2\kappa_{11}^2} = \frac{2\pi}{(2\pi\ell_p)^9} \ , \tag{12.6}$$

which is written in terms of Planck length

$$\ell_{\rm P} = \sqrt{\frac{\hbar G}{c^3}} \, .$$

The critical dimension of superstring theory is $D = 10$. Nonetheless, Type IIA $D = 10$ supergravity can be obtained from the dimensional reduction of the $D = 11$ supergravity on S^1, as we see below. Furthermore, the $D = 11$ supergravity gives an important clue to the strong coupling behavior of superstring theory.

12.3 $D = 10$ IIA supergravity

Now we compactify the $D = 11$ supergravity on S^1 of radius R. In the following, $C_{(p+1)}$ is the R-R $(p+1)$-form and $B_{(2)}$ is the B-field. Also, $G_{(p+2)}$ and $H_{(3)}$ are their field strengths. On the reduction, the three-form field $M_{(3)}$ decomposes into the massless fields of Type IIA string theory in Table 2:

$$M_{(3)} = C_{(3)} + B_{(2)} \wedge d\theta \, , \quad K_{(4)} = \widehat{G}_{(4)} + H_{(3)} \wedge (d\theta + C_{(1)}) \, , \tag{12.7}$$

where

$$\widehat{G}_{(4)} = G_{(4)} - C_{(1)} \wedge H_{(3)} \, . \tag{12.8}$$

Like (9.6), the metric takes the form

$$ds_{11}^2 = G_{MN} dx^M dx^N = G_{\mu\nu} dx^\mu dx^\nu + R^2 (d\theta + C_{(1)})^2 \, ,$$

where the compactified direction is denoted as θ.

Now we rewrite the action (12.5) in terms of the massless fields of Type IIA theory. However, we want to obtain the action of the form (4.9) in the NS-NS sector. For this purpose, we consider the radius of the circle depends on the spacetime coordinate and it can be written as the Dilaton fields as

$$R = \ell_p e^{\frac{2}{3}\Phi} \, . \tag{12.9}$$

Then, the action (12.5) is written as

$$\begin{aligned}
S_{\rm A,NS} &= \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-G} e^{-2\Phi} \left[R + 4\partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2} H_{(3)}^2 \right] \, , \\
S_{\rm A,R} &= -\frac{1}{4\kappa_{10}^2} \int d^{10}x \sqrt{-G} \left[G_{(2)}^2 + \widehat{G}_{(4)}^2 \right] \, , \\
S_{\rm A,CS} &= -\frac{1}{4\kappa_{10}^2} \int B_{(2)} \wedge G_{(4)} \wedge G_{(4)} \, ,
\end{aligned} \tag{12.10}$$

where the coupling constants are related by

$$\frac{2\pi R}{2\kappa_{11}^2} = \frac{e^{-2\Phi}}{2\kappa_{10}^2} \, . \tag{12.11}$$

Since it is the low-energy effective action of Type IIA string theory, the dimensional analysis of the coupling constant yields

$$\frac{1}{2\kappa_{10}^2} = \frac{2\pi}{(2\pi\ell_s)^8} \, . \tag{12.12}$$

Recalling that an expectation value of the dilaton is the string coupling $g_s = e^{\Phi}$, (12.9) gives

$$\left(\frac{R}{\ell_p}\right)^3 = g_s^2 \,. \tag{12.13}$$

Also, (12.11) provides the relation

$$\frac{R}{\ell_p^3} = \frac{1}{\ell_s^2} \tag{12.14}$$

Combining these two relations, we have

$$R = g_s \ell_s \,. \tag{12.15}$$

12.4 $D = 10$ IIB supergravity

As seen in §9, Type IIA and IIB are T-dual to each other so that the action of Type IIB supergravity is consistent with the T-duality. In fact, the action takes a very similar form

$$\begin{aligned}
S_{\mathrm{B,NS}} &= \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-G}e^{-2\Phi}\left[R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}H_{(3)}^2\right] \,, \\
S_{\mathrm{B,R}} &= -\frac{1}{4\kappa_{10}^2}\int d^{10}x\sqrt{-G}\left[G_{(1)}^2 + \widehat{G}_{(3)}^2 + \frac{1}{2}\widehat{G}_{(5)}^2\right] \,, \\
S_{\mathrm{B,CS}} &= -\frac{1}{4\kappa_{10}^2}\int C_{(4)} \wedge H_{(3)} \wedge G_{(3)} \,,
\end{aligned} \tag{12.16}$$

where

$$\begin{aligned}
\widehat{G}_{(3)} &= G_{(3)} - C_{(0)}H_{(3)} \,, \\
\widehat{G}_{(5)} &= G_{(5)} - \frac{1}{2}C_{(2)} \wedge H_{(3)} + \frac{1}{2}B_{(2)} \wedge G_{(3)} \,.
\end{aligned} \tag{12.17}$$

Note that the action for the NS sector is the same as that of Type IIA.

As in (8.5), Type IIB supergravity has the self-dual 5-form $G_{(5)}$. In the notation here, the self-dual condition is

$$*\widehat{G}_{(5)} = \widehat{G}_{(5)} \,,$$

which does not follow from the action. In fact, if the field satisfies the self-dual condition, its kinetic action becomes trivial

$$\int \widehat{G}_{(5)} \wedge *\widehat{G}_{(5)} = 0 \,.$$

Therefore, the self-duality condition needs to be imposed by hand.

13 String dualities

We have introduced superstring theories of five types: Type IIA, IIB, I, and Heterotic SO(32) and $E_8 \times E_8$. In fact, the seminal paper [Wit95b] reveals that these string theories are related by dualities, which led to the second string revolution. Indeed, we have already learned

- Type IIA and IIB are T-dual to each other

• Type I is the orientifold projection of Type IIB

In this section, we will learn string dualities extensively studied in the second string revolution after [Wit95b].

• Type IIB has $\mathrm{SL}(2,\mathbb{Z})$ symmetry so that it is self-dual under S-duality

• The strong coupling regime of Type IIA is described by M-theory on S^1

• Heterotic SO(32) and $E_8 \times E_8$ are T-dual to each other

• Heterotic SO(32) is S-dual to Type I

• The strong coupling regime of Heterotic $E_8 \times E_8$ is described by M-theory on $S^1/\mathbb{Z}_2$

• Heterotic string on T^4 is dual to Type IIA on K3

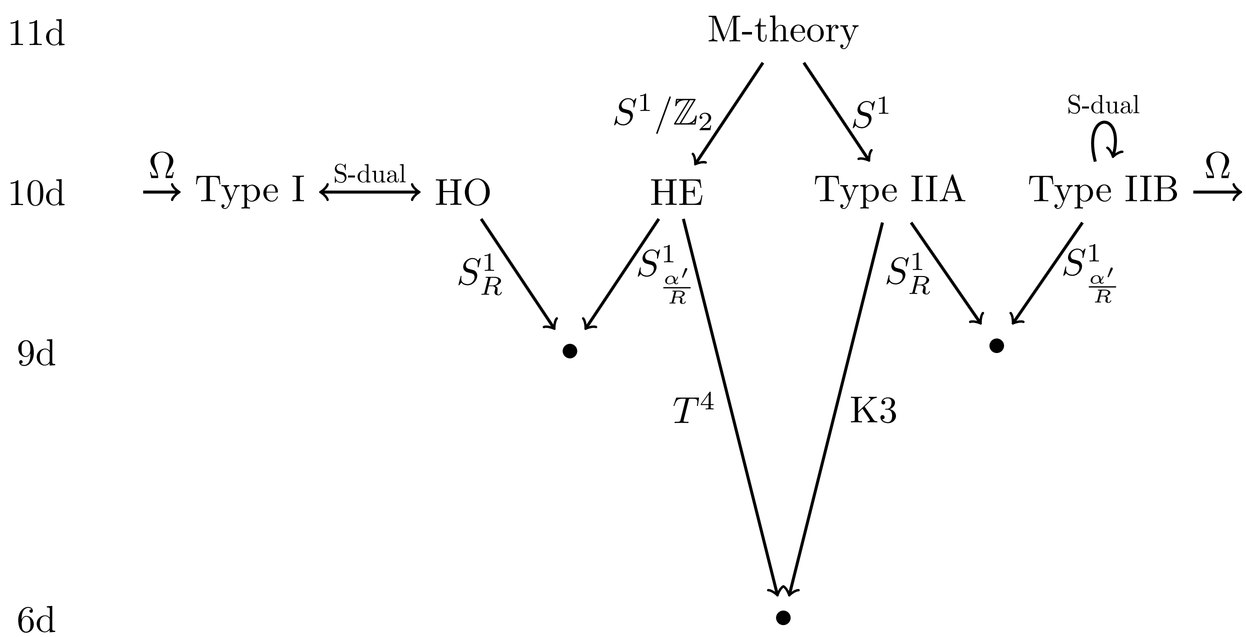


Figure 36: Duality web of string theory

Even for these dualities, we can cover only key points in this section. More details can be found in [Pol98, BBS06]. Moreover, we just see the tip of the iceberg, and there are much more string dualities. Thus, we refer to good reviews [Asp96, FL98, OY96, Pol96a, Pol96b, Tow96, Sch97, Sen97, Dij97, Vaf97, Sen98a, Joh00] written during the second string revolution for this rich subject. All in all, these dualities tell us that quantum strings somehow see geometry from drastically different viewpoints. I hope you will get some feeling of it in this section.

13.1 S-duality of Type IIB supergravity

The electromagnetic duality seen in (8.9) and (8.10) is a basic example of duality. The generalization to the Yang-Mills theory is proposed by Goddard-Nuyts-Olive [GNO77] and Montonen-Olive [MO77]. The Montonen-Olive duality exchanges the Yang-Mills coupling by

$$g_{\mathrm{YM}} \leftrightarrow 1/g_{\mathrm{YM}} \,, \tag{13.1}$$

and it also exchanges a fundamental particle into a soliton. Thus, this duality is also called the **strong-weak duality** or **S-duality**. Although standard techniques in quantum field theory cannot be applied to the strong coupling regime, the S-duality provides new insights into non-perturbative dynamics in the strong-coupling regime. One of the reasons why string theory sheds new light on physical theories is that the strong-weak dualities show up in string theory [Sen94, Duf95], which often have geometric origins.

In fact, Type IIB string theory enjoys the S-duality. To see it, let us rewrite the action by rescaling the metric $G_{E,\mu\nu} = e^{-\Phi/2}G_{\mu\nu}$, and by introducing

$$\tau = C_{(0)} + ie^{-\Phi} \,, \tag{13.2}$$

and

$$\mathbb{M} = \frac{1}{\mathrm{Im}\tau}\begin{pmatrix} |\tau|^2 & -\mathrm{Re}\tau \\ -\mathrm{Re}\tau & 1 \end{pmatrix} \,, \qquad \mathbb{F}_{(3)} = \begin{pmatrix} H_{(3)} \\ G_{(3)} \end{pmatrix} \,,$$

Then, the action is rewritten as

$$\begin{aligned} S_{\mathrm{B}} =& \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-G_E}\left[R_E - \frac{\partial_\mu\tau\partial^\mu\overline{\tau}}{2(\mathrm{Im}\tau)^2} - \frac{1}{2}\mathbb{F}_{(3)}\cdot\mathbb{M}\cdot\mathbb{F}_{(3)} - \frac{1}{4}\widetilde{G}_{(5)}^2\right] \\ & - \frac{1}{4\kappa_{10}^2}\int C_{(4)}\wedge\mathbb{F}_{(3)}^T\wedge\epsilon\mathbb{F}_{(3)} \,, \end{aligned}$$

where $\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Remarkably, this action is invariant under the $\mathrm{SL}(2,\mathbb{R})$ transformation:

$$\tau' = \frac{a\tau + b}{c\tau + d} \,, \quad \mathbb{M}' = (\Lambda^{-1})^T\mathbb{M}\Lambda^{-1} \,, \quad \mathbb{F}'_{(3)} = \Lambda\mathbb{F}_{(3)} \,, \quad \Lambda = \begin{pmatrix} d & c \\ b & a \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}) \,.$$

On the other hand, the metric G_E and $C_{(4)}$ are invariant under the transformation.

Let us consider its physical implications. Since $C_{(4)}$ is invariant, a D3-brane is invariant. On the other hand, since the 2-form fields $B_{(2)}$ and $C_{(2)}$ are transformed by $\mathrm{SL}(2,\mathbb{R})$, the extended objects coupled to these fields are transformed accordingly. Namely, F1 and D1-branes are electrically coupled to the 2-form fields whereas NS5 and D5-branes are magnetically coupled to them, respectively. Thus, they are transformed under $\mathrm{SL}(2,\mathbb{R})$ as

$$\begin{pmatrix} \mathrm{F1}' & \mathrm{D1}' \end{pmatrix} = \begin{pmatrix} \mathrm{F1} & \mathrm{D1} \end{pmatrix}\Lambda^{-1} \,, \qquad \begin{pmatrix} \mathrm{NS5}' \\ \mathrm{D5}' \end{pmatrix} = \Lambda\begin{pmatrix} \mathrm{NS5} \\ \mathrm{D5} \end{pmatrix} \,, \qquad \text{as} \qquad \begin{pmatrix} B'_{(2)} \\ C'_{(2)} \end{pmatrix} = \Lambda\begin{pmatrix} B_{(2)} \\ C_{(2)} \end{pmatrix}$$

Due to the Dirac quantization condition, the electric and the magnetic charges of D-branes must be integers so that the true symmetry in Type IIB string theory is indeed $\mathrm{SL}(2,\mathbb{Z})$ [HT95].

In particular, if we choose

$$\Lambda = S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \,,$$

the fields transform as $\tau \leftrightarrow -1/\tau$, and the extended objects are exchanged as

$$\text{F1} \leftrightarrow \text{D1}\,, \qquad \text{NS5} \leftrightarrow \text{D5}\,.$$

Assuming that $\langle C_0 \rangle = 0$, this results in the S-duality of the string coupling constant, $g_s \leftrightarrow 1/g_s$. Other elements of $SL(2,\mathbb{Z})$ lead to infinitely many bound states of F1 and D1: (p,q)-string, and those of NS5 and D5: (p,q) 5-branes.

The combination (13.2) of the fields behaves as a complex structure of a torus and the theory enjoys the modular transformations (6.4) and (6.5) of a torus. Hence, a $D = 12$ dimensional theory, called **F-theory**, is proposed in [Vaf96, MV96a, MV96b] by considering a torus fibration over ten-dimensional spacetime of Type IIB theory, which can be interpreted as the geometric origin of the field combination (13.2).

13.2 M-theory

We would also like to understand the strong coupling region of Type IIA theory. As we obtain Type IIA supergravity by compactification of the $D = 11$ supergravity on S^1, the radius is proportional to the string coupling constant (12.15) in the unit of the string length. This suggests that in the strong coupling region $g_s \gg 1$ of Type IIA string theory, another spacetime direction emerges (dimensional decompactification). Although the strong coupling behavior may look rather bizarre, it turns out that a number of phenomena in string theory can be explained "more naturally" from the $D = 11$ strongly-coupled Type IIA "string theory". Witten calls the $D = 11$ "string theory" **M-theory** where M stands for Magic, Mystery, or Membrane according to him.

Since the $D = 11$ supergravity is endowed with the 3-form fields $M_{(3)}$, there must be corresponding objects coupled to the field electrically and magnetically. They are called **M2-brane** and **M5-brane**, which are $(1+2)$- and $(1+5)$-dimensional objects, respectively. Let us assume that they have the following tensions and charges

$$T_{M2} = \mu_{M2} = \frac{2\pi}{(2\pi\ell_p)^3}\,, \qquad T_{M5} = \mu_{M5} = \frac{2\pi}{(2\pi\ell_p)^6}\,.$$

The compactification of M-theory on a circle S^1 leads to the IIA superstring theory. Since it is compactified on S^1, the momenta of a particle along S^1 are quantized as n/R ($n \in \mathbb{Z}$), and it can be identified with D0-branes. An M2-brane on S^1 becomes a fundamental string whereas an M2-brane in $D = 10$ reduces to a D2-brane. Similarly, an M5-brane on S^1 becomes a D4-brane whereas an M5-brane in $D = 10$ reduces to an NS5-brane. A D6-brane arises geometrically as a Kaluza-Klein monopole, which we do not deal with in this lecture note. (See [Joh02, §15.2] for instance.) Let us see this by comparing the branes and their tensions in Table 5. One can convince oneself that the tensions of extended objects perfectly agree. Note that the tensions and charges of D-branes in Type IIA theory are related by the string coupling constant

$$g_s T^{\text{eff}}_{\text{D}p} = \mu_p\,. \tag{13.3}$$

In fact, charges of a Dp and D$(6-p)$-brane can be read off from Table 5, and it is easy to check that they are subject to the Dirac quantization condition (8.13). We will come back to this relation in §14.1.

Note that M-theory is *not* even defined in a sense that we do not know how to quantize the M-branes. For instance, the world-volume theory on M5-branes is endowed with $OSp(2,6|2)$ superconformal symmetry, called 6d $\mathcal{N} = (2,0)$ superconformal field theory [Wit95a]. The field contents are as follows

- 2-form tensor field B with self-dual field strength $dB = H = - * H$
- spinors $\psi_{\alpha,a}$ with $\psi_{\alpha,a} = J_{\alpha\beta}J_{ab}\overline{\psi}^{\beta,b}$ $(\alpha, a = 1, 2, 3, 4)$
- 5 scalars ϕ_i $(i = 1, \ldots, 5)$

The source for the tensor field is an M2-brane ending on the M5-brane, called self-dual string, $*dH = J$. However, we do not know how to write down Lagrangian of the theory for nonabelian cases because of the self-dual 3-form H:

$$\frac{1}{g^2}\int d^6x\, H \wedge *H = -\frac{1}{g^2}\int d^6x\, H \wedge H = 0\,. \tag{13.4}$$

It is believed that 6d $\mathcal{N} = (2, 0)$ SCFTs are classified by $A_n, D_n, E_{6,7,8}$ root system.

Nevertheless, the existence of such theories tells us a lot. Especially, even though no effective description of M5-branes in flat space is known, M5-branes wrapped on manifolds give rise to surprising dualities between a d-dim topological theory and $(6 - d)$-dim supersymmetric gauge theories. A salient example is the AGT relation [AGT10] where M5-branes are wrapped on Riemann surface. Also, the highest dimension of SCFTs is $D = 6$ [Nah78], and string theory is indispensable for the study of 6d SCFTs. The reader is referred to [HR19] for this subject.

Dimension	0	1	2	4	5	6
M on S^1	KK-mom.	M2/S^1	M2	M5/S^1	M5	KK-mono.
	$\frac{1}{R}$	$\frac{2\pi\cdot 2\pi R}{(2\pi\ell_p)^3}$	$\frac{2\pi}{(2\pi\ell_p)^3}$	$\frac{2\pi\cdot 2\pi R}{(2\pi\ell_p)^6}$	$\frac{2\pi}{(2\pi\ell_p)^6}$	$\frac{2\pi(2\pi R)^2}{(2\pi\ell_p)^9}$
IIA	D0	F1	D2	D4	NS5	D6
	$\frac{2\pi}{g_s(2\pi\ell_s)}$	$\frac{2\pi}{(2\pi\ell_s)^2}$	$\frac{2\pi}{g_s(2\pi\ell_s)^3}$	$\frac{2\pi}{g_s(2\pi\ell_s)^5}$	$\frac{2\pi}{g_s^2(2\pi\ell_s)^6}$	$\frac{2\pi}{g_s(2\pi\ell_s)^7}$

Table 5: Wrapped/unwrapped M-branes and the corresponding extended objects in Type IIA theory with their effective tensions.

13.3 Heterotic T-duality

Let us consider the T-duality in Heterotic strings on a circle S^1 in [Nar86, NSW87, Gin87]. In the bosonic construction, the bosonic left-moving sector is compactified on an even self-dual Euclidean lattice of 16-dimensions. There are only two such lattices: the weight lattice $\Gamma_{\mathrm{SO}(32)}$ of SO(32) and the root lattice $\Gamma_{E_8} \oplus \Gamma_{E_8}$ of $E_8 \times E_8$ as we have seen in §11.1.

One may also describe the compactification on a circle S^1 in terms of lattices. As we have seen, the left-moving and right-moving momenta compactified boson takes the value on the lattice $\Gamma^{1,1}$ of the Lorentzian signature. Hence, the circle compactification results in adding ($\oplus$) the lattice $\Gamma^{1,1}$ to the original lattice.

It is a useful mathematical fact that for Lorentzian lattices, there is only unique even unimodular Lorentzian lattice for each rank. Therefore, the theorem implies

$$\Gamma_{\mathrm{SO}(32)} \oplus \Gamma^{1,1} \cong \Gamma^{1,17} \cong \Gamma_{E_8} \oplus \Gamma_{E_8} \oplus \Gamma^{1,1}\,.$$

Together with the metric G and the B-field, they parameterize the moduli space

$$\mathcal{M} = \frac{\mathrm{O}(1,17)}{\mathrm{O}(1)\times\mathrm{O}(17)} \Big/ \mathrm{O}(1,17;\mathbb{Z}), \tag{13.5}$$

where $\mathrm{O}(1,17;\mathbb{Z})$ is the T-duality group. It is called **Narain moduli space**. Different points in the moduli space correspond to physically distinct compactifications, e.g. the gauge groups can be different, although always of rank 18. At generic points, it is $\mathrm{U}(1)^{18}$ that corresponds to the fact that Wilson loops generically break the 10d gauge group to $\mathrm{U}(1)^{18}$.

However, there are special subspaces of the moduli space where it is enhanced. This moduli space has exactly two asymptotic boundary points, one associated to the decomposition $\Gamma^{1,17} \cong \Gamma_{E_8} \oplus \Gamma_{E_8} \oplus \Gamma^{1,1}$, and the other to the decomposition $\Gamma^{1,17} \cong \Gamma_{\mathrm{SO}(32)} \oplus \Gamma^{1,1}$. Therefore, these boundary points are associated to Type HE and HO string, or large and small radii. T-duality will relate these boundary points.

In fact, starting from either Heterotic theory, there is a simple choice of Wilson line which breaks the gauge group to $\mathrm{SO}(16) \times \mathrm{SO}(16) \times \mathrm{U}(1) \times \mathrm{U}(1)$. If we leave this group unbroken, then the only remaining parameter is the radius. An analysis of the massive states shows that if we map $R \to 1/R$ while exchanging KK momenta and winding modes, then the two Heterotic theories are exchanged [Pol98, §11.6].

More generally, upon a compactification of Heterotic strings on a D-dimensional torus T^D, momenta take the values on an even self-dual Lorentzian lattice $\Gamma^{D,D+16}$. Therefore, the Narain moduli space becomes

$$\mathcal{M} = \frac{\mathrm{O}(D,D+16)}{\mathrm{O}(D) \times \mathrm{O}(D+16)} \Big/ \mathrm{O}\left(D, D+16; \mathbb{Z}\right) . \tag{13.6}$$

which is $D(D+16)$-dimensional.

13.4 S–duality between Type I and Heterotic $\mathrm{SO}(32)$

Now let us see S-duality between Type I theory and Heterotic $\mathrm{SO}(32)$ theory [Wit95b, Dab95, Hul95, PW96] from low-energy effective actions.

One can obtain Type I supergravity action from Type IIB by setting to zero the IIB fields C_0, $B_{(2)}$, and C_4 that are removed by the Ω projection. In addition, we include $\mathrm{SO}(32)$ gauge fields with appropriate dilaton dependence

$$\begin{aligned} S_{\mathrm{I}} &= S_{\mathrm{grav}} + S_{\mathrm{YM}} \\ S_{\mathrm{grav}} &= \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-G} \left[e^{-2\Phi} (R + 4\partial_\mu \Phi \partial^\mu \Phi) - \frac{1}{2} |\widetilde{G}_{(3)}|^2 \right] \\ S_{\mathrm{YM}} &= -\frac{1}{2g_{10}^2} \int d^{10}x \sqrt{-G}\, e^{-\Phi}\, \mathrm{Tr}\, |F_{(2)}|^2 \end{aligned} \tag{13.7}$$

where $F_{(2)}$ is the $\mathrm{SO}(32)$ field strength and the trace is in the adjoint representation. Here $G_{(3)}$ is the field strength of the R-R 2-form $C_{(2)}$

$$\widetilde{G}_{(3)} = dC_{(2)} - \frac{\kappa_{10}^2}{g_{10}^2} \omega_3$$

with the Chern-Simons 3-form

$$\omega_3 = \mathrm{Tr} \left(A dA - \frac{2i}{3} A^3 \right) .$$

The gauge coupling constant g_{10} and the gravitational constant κ_{10} are related by $\kappa_{10}^2 / g_{10}^2 = \alpha'/4$, which is determined by anomaly cancellation. Under the gauge transformation

$\delta A = d\lambda - i[A, \lambda]$, the Chern-Simons term transforms as

$$\delta\omega_3 = d\,\mathrm{Tr}(\lambda A)$$

Hence, it comes with

$$\delta C_{(2)} = \frac{\kappa_{10}^2}{g_{10}^2}\,\mathrm{Tr}(\lambda dA)\;.$$

Heterotic strings have the same supersymmetry as Type I string and so we expect the same action. However, in the absence of open strings or R-R fields the dilaton dependence should be $e^{-2\Phi}$ throughout:

$$\begin{aligned}
S_{\mathrm{Het}} &= S_{\mathrm{grav}} + S_{\mathrm{YM}} \\
S_{\mathrm{grav}} &= \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-G}\,e^{-2\Phi}\left[R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}|\widetilde{H}_{(3)}|^2\right] \\
S_{\mathrm{YM}} &= -\frac{1}{2g_{10}^2}\int d^{10}x\sqrt{-G}\,e^{-2\Phi}\,\mathrm{Tr}\,|F_{(2)}|^2
\end{aligned} \tag{13.8}$$

where the 3-form $\widetilde{H}_3$ is the field strength of the B-field equipped with Chern-Simons form

$$\widetilde{H}_{(3)} = dB_{(2)} - \frac{\kappa_{10}^2}{g_{10}^2}\omega_3\;.$$

Indeed the low-energy effective actions of Type I (13.7) and Heterotic SO(32) (13.8) are related by the following field definitions (Homework)

$$\begin{aligned}
G^I_{\mu\nu} &= e^{-\Phi^H}G^H_{\mu\nu}\,, & \Phi^I &= -\Phi^H \\
\widetilde{G}^I_{(3)} &= \widetilde{H}^H_{(3)}\,, & A^I &= A^H\;.
\end{aligned} \tag{13.9}$$

Recalling that the vacuum expectation value of the dilaton is the string coupling $g_s = e^{\Phi}$, we see that the strong coupling limit of one theory is related to the weak coupling limit of the other theory and vice versa.

In Type I theory, D1-branes and D5-branes are electrically and magnetically charged under $C_{(2)}$, respectively. In Heterotic SO(32) theory, fundamental strings and NS5-branes are electrically and magnetically charged under $B_{(2)}$, respectively. The S-duality maps them as [PW96]

Type I	$\leftrightarrow$	Heterotic SO(32)
D1-branes	$\leftrightarrow$	F-strings
D5-branes	$\leftrightarrow$	NS5-branes

One can provide another evidence of this duality by looking at the massless spectrum. We have seen that Heterotic SO(32) has massless fields:

1. $\mathbf{8_v}$ of SO(8): bosonic right-moving $X^i(z)$
2. $\mathbf{8_c}$ of SO(8): fermionic right-moving $\psi^i(z)$
3. $\mathbf{32}$ of SO(32): left-moving Majorana-Weyl fermion $\widetilde{\lambda}^a(\bar{z})$

Correspondingly, one can see the massless BPS excitations from D1-strings stretched in the x_1-direction in Type I theory (Homework):

1. $\mathbf{8_v}$ of SO(8): normal bosonic excitations of D1-D1 strings

2. $\mathbf{8_c}$ of SO(8): right-moving fermionic excitations of D1-D1 strings
3. **32** of SO(32): left-moving fermionic excitations of D1-D9 strings

Further evidence of this duality has been assembled by comparing tensions, $F^4_{(2)}$ interactions, and so on [Pol98, §14.3].

13.5 Heterotic $E_8 \times E_8$ string from M-theory

Now we shall consider the strong-coupling behavior of Heterotic $E_8 \times E_8$ theory. Taking T-duality and S-duality, Heterotic $E_8 \times E_8$ theory is dual to Type I theory. As seen in §10.3, the T-dual to Type I theory is the Type I′ theory, which is Type IIA theory on a line segment $S^1/\mathbb{Z}_2$ where O8$^-$-planes sit at the two ends, and $(16+16)$ D8-branes are distributed on $S^1/\mathbb{Z}_2$. In the strong coupling regime, the M-theory circle will emerge, and it is described as M-theory on $S^1 \times S^1/\mathbb{Z}_2$.

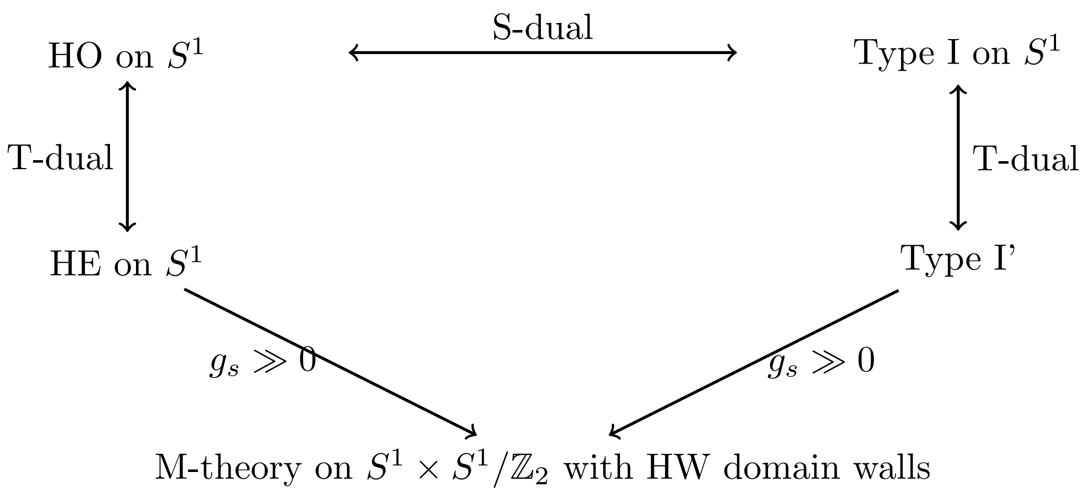


Figure 37: Duality web for Heterotic M-theory

Interestingly enough, the relative position of O8$^-$-planes and D8-branes in Type I′ string theory may be adjusted. This freedom goes away in the M-theory limit; the D8-branes have to be stuck at the O8$^-$-planes, and they become the domain walls of M-theory, which are called **Hořava-Witten domain wall** or **M9-branes** [HW96b, HW96a].

Its low-energy effective description is given by $D = 11$ supergravity on $S^1/\mathbb{Z}_2$ which gives rise to gravitational anomaly [AGW84]. In order to cancel such anomaly, non-Abelian gauge fields have to be present at the boundaries in order to employ a Green-Schwarz mechanism [GS84]. (Homework) This mechanism that bulk anomaly cancels with boundary anomaly is called **anomaly inflow**. Indeed the low-energy effective theory at the Hořava-Witten domain wall is described by 10d $\mathcal{N} = 1$ SYM with E_8 gauge group that cancels anomaly.

As in Type IIA, the distance between the two boundaries is related to Heterotic coupling $R = g_{\text{het}}^{\frac{3}{2}} \ell_p$. Hence, the line segment $S^1/\mathbb{Z}_2$ shrinks at the weak coupling regime, leading to Heterotic $E_8 \times E_8$ string theory. The reason to pick $E_8 \times E_8$ is that the anomalies must be canceled on both boundaries, and there is no way to distribute SO(32) between two boundaries (it's a simple group with no factors). Using the previous terminology Heterotic $E_8 \times E_8$ string theory can be viewed as M-theory compactified on $S_1/\mathbb{Z}_2$. This setup is called **Hořava-Witten M-theory** or **Heterotic M-theory** [HW96b, HW96a].

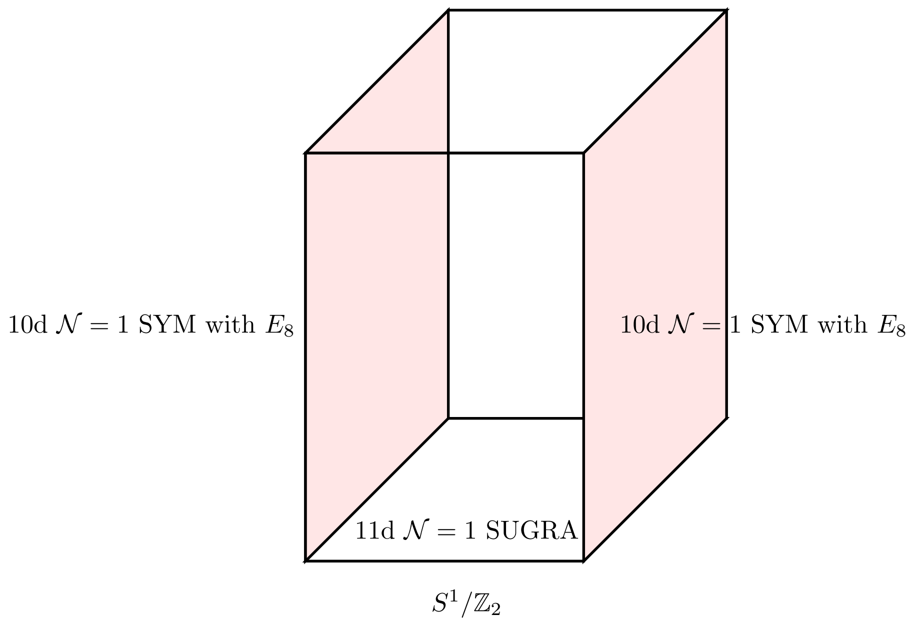


Figure 38: Low-energy effective description of Heterotic M-theory. Hořava-Witten domain walls at the two boundaries give rise to $\mathcal{N}=1$ SYM with E_8 gauge group and they cancel bulk anomaly.

13.6 Duality between Heterotic on T^4 and Type IIA on K3

Let us now see one more non-trivial duality. Although we have studied only toroidal compactifications, we have seen a rich web of dualities. In string theory, a theory is consistent if we compactify it on a Calabi-Yau manifold. Since there are wide varieties of Calabi-Yau manifolds, string dualities involving them are much richer. It has still been an active research area both in physics and mathematics. Here, we deal with the next simplest Calabi-Yau manifold called **K3 surface**.

K3 surface

A K3 surface is a resolution of $T^4/\mathbb{Z}_2$. We write a 4-torus as

$$T^4 = \mathbb{R}^4/\mathbb{Z}^4 = \{\mathbf{x} = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 | x_i \sim x_i + 1\}$$

and the $\mathbb{Z}_2$ action is a reflection $x_i \to -x_i$. Note that this action has $2^4 = 16$ fixed points given by choice of midpoints or the origin in any of the four x_i. Thus, the resulting space $T^4/\mathbb{Z}_2$ is singular at any of these 16 fixed points. The neighborhood of a singular point is indeed a cone of $\mathbb{R}P^3$. To make it smooth, let us consider the set of vectors of length ≤ 1 in the tangent bundle of TS^2

$$V = \{(v_1, v_2) \in S^2 \times T_{v_1}S^2 |\, |v_2| \leq 1\} .$$

Then the boundary of V is $\partial V = \mathbb{R}P^3$ so that you can replace the neighborhood of each singular point by V. Since V is a smooth manifold, the resulting space is smooth, and it is a K3 surface. This smoothing procedure is called **resolution** or **blow-up**.

Although the construction of a K3 surface is rather simple, its geometry is surprisingly fertile. First of all, it is a Calabi-Yau manifold, namely a Ricci-flat Kähler manifold. In real four dimensions, there are only two topologically equivalent compact closed Calabi-Yau manifolds, T^4 and K3. Moreover, it is a hyper-Kähler manifold. (Let us not go into detail about hyper-Kähler manifolds.)

Let us now briefly look at the topological property of K3 surfaces. The resolution of the 16 singular points provides 16 elements of $H^2(K3, \mathbb{Z})$ in addition to $6 = {}_4C_2$ tori in T^4. Therefore, we have $H^2(K3; \mathbb{Z}) \cong \mathbb{Z}^{22}$. Moreover, the Hodge diamond as a complex manifold turns out to be

$$\begin{array}{ccccc} & & h^{0,0} & & \\ & h^{1,0} & & h^{0,1} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{2,1} & & h^{1,2} & \\ & & h^{2,2} & & \end{array} = \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array} .$$

Since it is a real 4-dimensional manifold, one can consider the intersection matrix of rank 22

$$Q(\alpha_i, \alpha_j) = \alpha_i \cap \alpha_j \qquad \alpha_i \in H_2(K3; \mathbb{Z})$$

In fact, in a certain nice basis, the intersection matrix can be written as follows

$$Q(\alpha_i, \alpha_j) \sim 2(-E_8) \oplus 3 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

where $-E_8$ denotes the 8×8 matrix given by minus the Cartan matrix of the Lie algebra E_8. Hence, we may decompose

$$H^2(K3, \mathbb{R}) = H^+ \oplus H^-,$$

where $H^\pm$ represents the cohomology of the space of (anti-)self-dual 2-forms. We then see that

$$\dim H^+ = 3, \quad \dim H^- = 19 .$$

The moduli space of non-trivial metric deformations on a K3 is 58-dimensional and given by the coset space [Asp96]

$$\mathcal{M}_{\mathrm{K3}} = \mathbb{R}^+ \times \frac{\mathrm{O}(3,19)}{\mathrm{O}(3) \times \mathrm{O}(19)} \Big/ \mathrm{O}(3,19,\mathbb{Z}) ,$$

where the second factor is the Teichmüller space for Ricci-flat metrics of volume one on a K3 surface and the first factor is associated with the size of the K3.

This is not the end of the story if we consider string propagation on $K3$. For each element of $H_2(K3, \mathbb{Z})$, we can turn on the B-field. Because of $H_2(K3, \mathbb{Z}) = \mathbb{Z}^{22}$, we have 22 additional real parameters and that makes the total dimension of moduli space $58 + 22 = 80$. It turns out that this moduli space is isomorphic to

$$\mathcal{M}_{\mathrm{K3}}^{\mathrm{stringy}} = \frac{\mathrm{O}(4,20)}{\mathrm{O}(4) \times \mathrm{O}(20)} \Big/ \mathrm{O}(4,20,\mathbb{Z}) . \tag{13.10}$$

Substituting $D = 4$ into (13.6), one can see that this is exactly the same as the Narain moduli space for Heterotic string on T^4!

Heterotic on T^4/Type IIA on K3

The relations between Heterotic on T^4 and Type IIA on K3 can be seen by comparing the effective actions in $D=6$. On the Heterotic side, for generic Wilson lines the $E_8 \times E_8$ or $\mathrm{SO}(32)$ gauge symmetry is broken to $\mathrm{U}(1)^{16}$. Including the KK gauge bosons from T^4 compactification, the gauge group becomes $\mathrm{U}(1)^{24}$ and the effective 6d supergravity action of Heterotic string is

$$S_{\mathrm{Het}} = \frac{1}{2\kappa_6^2}\int d^6x\sqrt{-G}\,e^{-2\Phi}\left[R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}|\widetilde{H}_{(3)}|^2 - \frac{\kappa_6^2}{2g_6^2}\sum_{I=1}^{24}|F^I_{(2)}|^2\right] .$$

Type IIA superstring theory compactified on K3 breaks a half of supersymmetries so that there are 16 supercharges as in Heterotic string. It also gives rise to $\mathrm{U}(1)^{24}$ gauge fields which, via KK-reduction, all arise from the R-R sector. One comes from the one-form $C_{(1)}$ with indices in the 6d non-compact spacetime, and another one from the three-form $C_{(3)}$, which is Hodge-dual to a massless vector in 6d. Because of $H_2(K3,\mathbb{Z}) = \mathbb{Z}^{22}$, the three-form $C_{(3)}$ on two-cycles of K3 gives 22 vectors. (Thus, we have $1+1+22=24$ $\mathrm{U}(1)$ gauge fields.) As a result, the effective Type IIA action compactified on K3 is

$$S_{\mathrm{IIA}} = \frac{1}{2\kappa_6^2}\int d^6x\sqrt{-G}\left[e^{-2\Phi}\left(R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}|\widetilde{H}_{(3)}|^2\right) - \frac{\kappa_6^2}{2g_6^2}\sum_{I=1}^{24}|F^I_{(2)}|^2\right] .$$

It is straightforward to show that the two actions are equivalent via the following field redefinition

$$\begin{aligned} \Phi^{\mathrm{H}} &= -\Phi^{\mathrm{IIA}}\,, & G^{\mathrm{H}} &= e^{-2\Phi^{\mathrm{IIA}}}G^{\mathrm{IIA}} \\ A^{\mathrm{H}} &= A^{\mathrm{IIA}}\,, & \widetilde{H}^{\mathrm{H}} &= e^{-2\Phi^{\mathrm{IIA}}} * \widetilde{H}^{\mathrm{IIA}}\,. \end{aligned}$$

More dualities

Of course, what we have glimpsed are merely a few representative examples of string dualities. For instance, compactifications of M-theory on tori with orbifold actions, one can find many dualities. These dualities have been discovered at by using similar arguments above. Some examples of such conjectured dualities are given below [DM96a, Wit96, Sen96]:

M-theory on		
K3	$\leftrightarrow$	Heterotic/Type I on T^3
$T^5/\mathbb{Z}_2$	$\leftrightarrow$	IIB on K3
$T^8/\mathbb{Z}_2$	$\leftrightarrow$	Type I/Heterotic on T^7
$T^9/\mathbb{Z}_2$	$\leftrightarrow$	Type IIB on $T^8/\mathbb{Z}_2$

In each case, $\mathbb{Z}_2$ acts as reversing the sign of all the coordinates of T^D. These dualities have been checked by taking Type IIA limit of M-theory.

14 D-brane dynamics

So far, D-branes are treated as static rigid objects that are associated to boundary conditions of an open string. Actually, it has dynamics like a fundamental string, and this section introduces the dynamics of D-branes. After we derive the action for the world-volume theory of Dp-branes in the first half, we discuss the dynamics of D-branes. Branes control non-perturbative dynamics in string theory, which reveals the richness and depth of string theory. Therefore, the study of D-branes is remarkably broad and we cover a tiny part of it. The reader is referred to [Joh06] for more detail.

14.1 D-brane action

When we quantize a world-sheet in §2, we use the string sigma action (2.2) instead of the Nambu-Goto action (2.1) to avoid the complexity of the square root. There, the dimension of a world-sheet to be two is crucial for quantization, and we cannot simply follow the same procedure for the fundamental string to quantize a Dp-brane, in general. Thus, we will first learn the effective action of a D-brane in this section.

Let us summarize the properties the D-brane action should be endowed with:

1. it contains scalars as the spacetime coordinates of the world-volume of a D-brane
2. it includes gauge fields living on a D-brane, which arises from an open string massless spectrum
3. it involves B-field because open stings can end on a D-brane
4. it couples to the R-R-field $C_{(p)}$ via (8.12)
5. it can possess supersymmetry (though we focus only on the bosonic part in this section)

The first point will be addressed by the Nambu-Goto action (2.1) for a D-brane, namely the volume of the D-brane. We write the world-volume coordinates of Dp-branes by $\sigma^a (a = 0, 1, \ldots, p)$, and $X^\mu(\sigma^a)$ maps from the world-sheet to the spacetime. Then, the Nambu-Goto action (2.1) for a D-brane can be written as

$$S_{\mathrm{D}p} = -T^{\mathrm{eff}}_{\mathrm{D}p} \int d^{p+1}\sigma \sqrt{-\det\left(G_{\mu\nu}\frac{\partial X^\mu}{\partial\sigma^a}\frac{\partial X^\nu}{\partial\sigma^b}\right)}$$

where $T^{\mathrm{eff}}_{\mathrm{D}p}$ is an effective Dp-brane tension which we will discuss shortly. Starting the Nambu-Goto action with the R-R coupling (8.12), we will generalize the action by using the T-duality and gauge invariance. In this way, we will incorporate the second and third point.

Let us consider the simple setup as in Figure 24 where D1 and D2-brane are related by the T-duality. Here we are interested in a part $\mathbb{R}_t \times \mathbb{R} \times S^1$ of the spacetime where the D-brane is located, and we assume that its metric is flat $ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu$ (i.e. $G_{\mu\nu} = \eta_{\mu\nu}$). As explained in §9.2, the position X_2 ($\sim X_2 + 2\pi R$) of the D1-brane, and the gauge field A_2 ($\sim A_2 + 1/R$) on the D2-brane are related under the T-duality by

$$X_2 = 2\pi\alpha' A_2 \ .$$

Now let us consider a situation in which the D1-brane has dynamics, namely it vibrates as $X^2 = X^2(X^1)$ (see Figure 39). Then, the vibration of the D1-brane is translated as non-trivial field strength $F_{12} = \partial_1 A_2(X^1) \neq 0$ on the D2-brane. Parametrizing the world-volume coordinates as $X^0 = \sigma^0$, $X^1 = \sigma^1$, the Nambu-Goto action of the D1-brane becomes

$$S_{\mathrm{D}1} = -T^{\mathrm{eff}}_{\mathrm{D}1} \int d^2\sigma \sqrt{-\det\left(G_{\mu\nu}\frac{\partial X^\mu}{\partial\sigma^a}\frac{\partial X^\nu}{\partial\sigma^b}\right)} \quad \text{with ,}$$

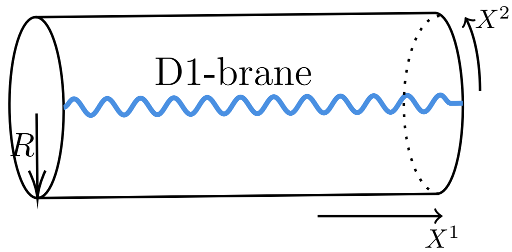


Figure 39: Vibrating D1-brane.

$$= -T^{\text{eff}}_{\text{D1}} \int d\sigma^0 d\sigma^1 \sqrt{1 + \left(\frac{\partial X^2}{\partial \sigma^1}\right)^2} \; .$$

In order for the D-brane action to be invariant under the T-duality, we incorporate the field strength of the gauge field in such a way that

$$\begin{aligned} S_{\text{D2}} &= -T^{\text{eff}}_{\text{D2}} \int d^3\sigma \sqrt{-\det\left(G_{\mu\nu}\frac{\partial X^\mu}{\partial \sigma^a}\frac{\partial X^\nu}{\partial \sigma^b} + 2\pi\alpha' F_{ab}\right)} \\ &= -T^{\text{eff}}_{\text{D2}} \cdot 2\pi\widetilde{R} \cdot \int d\sigma^0 d\sigma^1 \sqrt{1 + (2\pi\alpha' F_{12})^2} \; . \end{aligned}$$

D-brane tension

The dimension analysis tells us that a D-brane tension should be proportional to

$$T_{\text{D}p} \sim \frac{\text{mass}}{p\text{-dim vol}} \quad \Rightarrow \quad T_{\text{D}p} \sim \frac{1}{\ell_s^{p+1}} \; .$$

From the argument above in order for the two actions to coincide we need $T^{\text{eff}}_{\text{D1}} = 2\pi\widetilde{R}T^{\text{eff}}_{\text{D2}}$. On the other hand, $T_{\text{D}p}$ should be independent of the spacetime geometry including the radius R. Note that D-brane effective theory is supposed to reproduce open string amplitude, whose leading contribution is the disk amplitude $\sim e^{-\langle\Phi\rangle}$. Thus, we reach the following form

$$S_{\text{D}p} = -T_{\text{D}p} \int d^{p+1}\sigma \, e^{-\Phi(X)} \sqrt{-\det\left(G_{\mu\nu}\partial_a X^\mu \partial_b X^\nu + 2\pi\alpha' F_{ab}\right)} \; . \tag{14.1}$$

Then, the ratio of the tensions of D1 and D2-brane is

$$2\pi\widetilde{R} = \frac{T^{\text{eff}}_{\text{D1}}}{T^{\text{eff}}_{\text{D2}}} = \frac{T_{\text{D1}}e^{-\Phi}}{T_{\text{D2}}e^{-\widetilde{\Phi}}} = \frac{T_{\text{D1}}}{T_{\text{D2}}} \cdot \frac{\widetilde{R}}{\ell_s} \quad \Rightarrow \quad T_{\text{D1}} = 2\pi\ell_s \cdot T_{\text{D2}} \; .$$

Note that the dilation field transforms under the T-duality as $e^{-\widetilde{\Phi}} = e^{-\Phi}\frac{\widetilde{R}}{\ell_s}$ (exercise). For a D-brane in superstring theory, the correct normalization is

$$T_{\text{D}p} = \frac{2\pi}{(2\pi\ell_s)^{p+1}} \; . \tag{14.2}$$

The B-field

In (4.7), we see the action of the bosonic closed string. For an open string, we need to add a boundary term in order for the action to be invariant under gauge transformations:

$$\begin{aligned}S_{\text{open}} =& \frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\left(\sqrt{h}h^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X) + i\varepsilon^{ab}\partial_a X^\mu \partial_b X^\nu B_{\mu\nu}(X) + \alpha'\sqrt{h}R^{(2)}\Phi(X)\right) \\ & + i\int_{\partial\Sigma} d\sigma^0 \partial_0 X^\mu A_\mu \\ & = \cdots + \frac{1}{2\pi\alpha'}\int_\Sigma B_{(2)} + \int_{\partial\Sigma} A_{(1)} \,.\end{aligned} \tag{14.3}$$

Here the gauge transformation (4.8) of the B-field $\delta_B B_{(2)} = d\Lambda_{(1)}$ is compensated by that of the gauge field $A_{(1)}$, which is

$$\delta_B A_{(1)} = -\frac{\Lambda_{(1)}}{2\pi\alpha'} \,.$$

Therefore, (14.1) can be generalized by incorporating the B-field as

$$S_{\text{D}p} = -T_{\text{D}p}\int d^{p+1}\sigma\, e^{-\Phi(X)}\sqrt{-\det\left(G_{ab} + 2\pi\alpha' F_{ab} + B_{ab}\right)} \,. \tag{14.4}$$

This is called the **Dirac-Born-Infeld (DBI) action**.

Generalization of R-R coupling and the DBI action

In addition, a Dp-brane couples to the R-R field $C_{(p+1)}$ via the action (8.12), which can be written in terms of local coordinates as

$$S_{\text{R-R D}p} = \mu_p \cdot \int_{\text{D}p} C_{(p+1)} = \mu_p \cdot \int d^{p+1}\sigma\, C_{\mu_1\cdots\mu_{p+1}}(X)\frac{\partial X^{\mu_1}}{\partial\sigma^1}\cdots\frac{\partial X^{\mu_{p+1}}}{\partial\sigma^{p+1}} \,.$$

As above, let us consider the T-duality for the vibrating D1-brane in the R-R coupling. Recalling that R-R fields are transformed under the T-duality by (9.13), the T-duality connects the following two expressions

$$\begin{aligned}S_{\text{R-R D1}} &= \mu_1 \cdot \int d\sigma^0 d\sigma^1 \left(C_{01} + C_{02}\frac{\partial X^2}{\partial\sigma^1}\right) \,, \\ S_{\text{R-R D2}} &= \mu_2 \cdot \int d\sigma^0 d\sigma^1 d\sigma^2 \left(\widetilde{C}_{012} + \widetilde{C}_0 \cdot 2\pi\alpha' F_{12}\right) \,,\end{aligned}$$

where $C_{01} \leftrightarrow \widetilde{C}_{012}$, $C_{02} \leftrightarrow \widetilde{C}_0$, and $X^2 \leftrightarrow 2\pi\alpha' A_2$. This can be understood as follows. The vibrating D1-brane consists of a straight D1-brane along X^1 and local vibration along X^2. After T-duality along X^2, the vibration part gives non-trivial flux $F_{12} \neq 0$ or equivalently gives D0-branes. (See left Figure 41.) Therefore, the generalization of the R-R coupling (9.13) is

$$S_{\text{D}p} = \mu_p \cdot \int C_{\text{RR}} \wedge \exp\left(2\pi\alpha' F_{(2)} + B_{(2)}\right) \,, \tag{14.5}$$

where $C_{RR} = \sum_n C_{(n)}$. Note that the B-field appears with $F_{(2)}$ due to the gauge invariance.

Finally, the fully general form of Dp-brane action is given as follows:

$$\begin{aligned} S_{\mathrm{D}p} = & -T_{\mathrm{D}p} \int d^{p+1}\sigma\, e^{-\Phi(X)} \sqrt{-\det\left(G_{ab} + 2\pi\alpha' F_{ab} - B_{ab}\right)} \\ & + \mu_p \cdot \int C_{RR} \wedge \exp\Big(2\pi\alpha' F_{(2)} + B_{(2)}\Big) , \end{aligned} \tag{14.6}$$

where $G_{ab} = G_{\mu\nu}\partial_a X^\mu \partial_b X^\nu$ and $B_{ab} = B_{\mu\nu}\partial_a X^\mu \partial_b X^\nu$.

Though we focused on the bosonic part so far, there is a fermionic part so that they form spacetime supersymmetry. Here we only write down the leading fluctuation:

$$-i \int d^{p+1}\sigma\ \mathrm{Tr}\left(\overline{\psi}\Gamma^a D_a \psi\right) .$$

For the full nonlinear supersymmetric form, one should consult with, for example, [Tse99].

So far, we consider the world-volume effective action of a single D-brane, and non-Abelian generalization for multiple D-branes remains an open problem although there is a proposal [Tse97, Mye99].

D-brane tensions and charges

As seen in (8.13), D-brane charges satisfy the Dirac quantization condition. It is easy to see that the D-brane tension (14.2) is also subject to the Dirac quantization condition. Hence, we can set $\mu_p = \frac{2\pi}{(2\pi\ell_s)^{p+1}} = T_{\mathrm{D}p}$. The tension induces gravitational force (graviton & dilaton) between Dp-branes, which is attractive whereas the R-R charge induces repulsive force. Thus, the equality of the charge and the tension is crucial for multiple Dp-branes to exist stably.

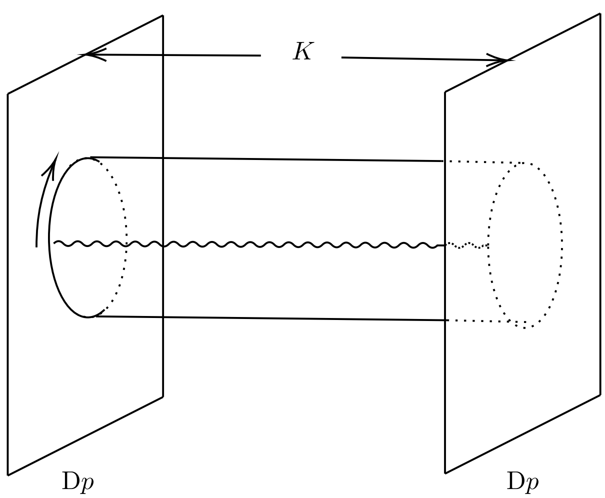


Figure 40:

Let us evaluate the gravitational force between two Dp-branes. As Figure 27 illustrates, a closed string amplitude between the Dp-branes can be interpreted as the one-loop amplitude of an open string between the Dp-branes. The contribution of a particle with mass m at one-loop to the free energy can be evaluated as

$$
\begin{aligned}
F &= -V_{p+1}\frac{1}{2}\int \frac{d^{p+1}k}{(2\pi)^{p+1}}\log\left(k^2+m^2\right) \\
&= -V_{p+1}\int\frac{dt}{2t}\int\frac{d^{p+1}k}{(2\pi)^{p+1}}e^{-2\pi\alpha'\left(k^2+m^2\right)t} \\
&= -iV_{p+1}\int\frac{dt}{2t}(8\pi^2\ell_s^2 t)^{-\frac{p+1}{2}}e^{-2\pi\alpha' m^2 t}
\end{aligned}
\tag{14.7}
$$

Since the mass of an open string mode between the branes at distant K is given by

$$
\alpha' m^2 = L_0 + \frac{K^2}{\ell_s^2}\,, \tag{14.8}
$$

the free energy takes the form

$$
F = -iV_{p+1}\int\frac{dt}{2t}(8\pi^2\ell_s^2 t)^{-\frac{p+1}{2}}q^{\frac{K^2}{\ell_s^2}}\operatorname{Tr}\left(q^{L_0}\right)\,. \tag{14.9}
$$

where

$$
\operatorname{Tr}\left(q^{L_0}\right) = q^{-\frac{1}{2}}\prod_{m=1}^{\infty}\frac{(1-q^{m-1/2})^8}{(1-q^m)^8} = \frac{\eta(it/2)^8}{\eta(it)^{16}}\,. \tag{14.10}
$$

Note that the denominator is the bosonic and the numerator is the fermionic contribution. As we learn in (10.5), the leading contribution from a closed string can be read off in the limit $t\to 0$ when the cylinder becomes a thin tube. Therefore, we make the modular transformation (7.33) and we take the leading order

$$
\begin{aligned}
F &= -iV_{p+1}\frac{1}{(2\pi\ell_s)^{p+1}}\int\frac{dt}{2t}(2t)^{\frac{7-p}{2}}e^{-2\pi\frac{K^2}{\ell_s^2}t} \\
&= -iV_{p+1}\frac{1}{(2\pi\ell_s)^{p+1}}\left(\frac{\ell_s^2}{\pi K^2}\right)^{\frac{7-p}{2}}\int\frac{dx}{x}x^{\frac{7-p}{2}}e^{-x} \\
&= -iV_{p+1}\frac{1}{(2\pi\ell_s)^{p+1}}\left(\frac{\ell_s^2}{\pi K^2}\right)^{\frac{7-p}{2}}\Gamma\left(\frac{7-p}{2}\right) \\
&= \text{const}\cdot\ell_s^{6-2p}G_{9-p}(K^2)
\end{aligned}
\tag{14.11}
$$

where $G_d(K)$ is the massless scalar Green's function in d dimensions, the inverse of $-\nabla^2$. In the $D=10$ supergravity §12, the Newton constant is given by $2\kappa_{10}^2=(2\pi\ell_s)^8/2\pi$ so that the Newton's potential is given by

$$
V_{\mathrm{G}} = -\text{const}\cdot\kappa_{10}^2T_{\mathrm{D}p}^2G_{9-p}(r)\,. \tag{14.12}
$$

Therefore, we can conclude that $T_{\mathrm{D}p}\propto\ell_s^{-(p+1)}$, which justifies (14.2).

For the B-field, the normalization of the quantization condition is different from (8.13)

$$
T_{\mathrm{F1}}\cdot T_{\mathrm{NS5}}\cdot 2\kappa_{10}^2g_s^2\in 2\pi\mathbb{Z}\,.
$$

Since $T_{\mathrm{F1}}=\frac{2\pi}{(2\pi\ell_s)^2}$, we have

$$
T_{\mathrm{NS5}} = \frac{2\pi}{T_{\mathrm{F1}}\cdot 2\kappa_{10}^2g_s^2} = \frac{2\pi}{(2\pi\ell_s)^6g_s^2}\,.
$$

See also Table 5.

14.2 Branes and strings ending on branes

Now we consider D-brane systems in Type IIA theory. Let us recall that Maxwell equations lead to the charge conservation law:

$$\begin{array}{lcl} d *F_{(2)} = J_e & \Rightarrow & dJ_e = 0 \\ dF_{(2)} = J_m & & dJ_m = 0 \end{array} .$$

Because of the charge conservation, a world-line of a charged particle cannot have an endpoint and therefore it must be either a closed path or an infinitely long line. If we apply this logic to branes, we may find the same result for branes. However, the charge conservation in supergravity is quite non-trivial due to the non-linearity of the equation of motions. In this subsection, we will study the physics of D-branes from the equation of motions in (massive) Type IIA supergravity.

Massive IIA supergravity

When Type IIA supergravity action is introduced in (12.10), we do not include the R-R field corresponding to D8-brane. As explained in §8.2, D8-branes are non-dynamical, and its R-R field is a 9-form so that its field strength is 10-form $G_{(10)}$. Consequently, $d *G_{(10)} = 0$ leads to $*G_{(10)} = G_{(0)} \equiv m$), which clearly shows that D8-brane is non-dynamical. However, it actually has a constant but non-trivial contribution to the action, called **massive IIA supergravity**. Since it is a constant, it contributes as a mass term, called **Romans mass** to the action [Rom86].

Let us write the massive IIA supergravity action (we omit the wedge product $\wedge$ and the Hodge star $*$ here):

$$\begin{aligned} S_{\mathrm{A,NS}} &= \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-G}e^{-2\Phi}\left[R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}H_{(3)}^2\right] , \\ S_{\mathrm{A,R}} &= \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-G}\left[-\frac{1}{2}m^2 - \frac{1}{2}G_{(2)}^2 - \frac{1}{2}G_{(4)}^2\right] , \\ S_{\mathrm{A,CS}} &= \frac{1}{2\kappa_{10}^2}\int \left[-\frac{1}{2}B_{(2)}G_{(4)}G_{(4)} + \frac{1}{2}B_{(2)}^2G_{(2)}G_{(4)} - \frac{1}{6}B_{(2)}^3G_{(2)}^2 \right. \\ &\qquad\qquad \left. -\frac{m}{6}B_{(2)}^3G_{(4)} + \frac{m}{8}B_{(2)}^4G_{(2)} - \frac{m^2}{40}B_{(2)}^5\right] , \end{aligned}$$

where

$$\begin{aligned} H_{(3)} &= dB_{(2)} , \\ G_{(2)} &= dC_{(1)} + mB_{(2)} , \\ G_{(4)} &= dC_{(3)} + dC_{(1)}B_{(2)} + \tfrac{1}{2}mB_{(2)}^2 . \end{aligned} \tag{14.13}$$

From this action, we obtain the following equations of motions for $C_{(1)}$ and $C_{(3)}$

$$-d *G_{(2)} = H_{(3)} *G_{(4)} , \tag{14.14}$$

$$d *G_{(4)} = H_{(3)}G_{(4)} . \tag{14.15}$$

Since we know the relation between the field strengths and their gauge fields (14.13), we obtain the following Bianchi identities

$$dG_{(2)} = mH_{(3)} , \tag{14.16}$$

$$dG_{(4)} = H_{(3)}G_{(2)} \,. \tag{14.17}$$

If we relabel m by $G_{(0)}$ and define the dual field strengths:

$$G_{(10)} = *G_{(0)} \,, \qquad G_{(8)} = - *G_{(2)} \,, \qquad G_{(6)} = *G_{(4)} \,,$$

then the equations of motions (14.14) and the Bianchi identities (14.16) can be repackaged into

$$dG_{\text{even}} = H_{(3)}G_{\text{even}} \,.$$

where

$$\begin{aligned} C_{\text{odd}} =& C_{(1)} + C_{(3)} + C_{(5)} + C_{(7)} + C_{(9)} \,, \\ G_{\text{even}} =& G_{(0)} + G_{(2)} + G_{(4)} + G_{(6)} + G_{(8)} + G_{(10)} \,. \end{aligned} \tag{14.18}$$

This equation can be solved as follows:

$$G_{\text{even}} = e^{B_{(2)}} \left(m + dC_{\text{odd}} \right) \,.$$

The equation of motion for the B-field is given as follows

$$d\left(e^{-2\Phi} *H_{(3)}\right) = m *G_{(2)} + *G_{(4)}G_{(2)} - \frac{1}{2}G_{(4)}^2 \,.$$

If we define the dual field strength $H_{(7)} = e^{-2\Phi} *H_{(3)}$, then, the equation of motion becomes

$$dH_{(7)} = \frac{1}{2}\left[(*G_{\text{even}})G_{\text{even}}\right]_{(8)} \tag{14.19}$$

The Bianchi identity is trivial $dH_{(3)} = 0$. Note that those field strengths are invariant under the gauge transformations of B-field as well as R-R fields:

$$\begin{aligned} &\delta_B B_{(2)} = d\lambda_{(1)} \,, \qquad \delta_B C_{\text{odd}} = -\lambda_{(1)} \left(m + dC_{\text{odd}} \right) \,, \\ &\delta_C B_{(2)} = 0 \,, \qquad \delta_C C_{\text{odd}} = d\lambda_{\text{even}} \,, \end{aligned}$$

where we introduced a formal sum of gauge parameters

$$\lambda_{\text{even}} = \lambda_{(0)} + \lambda_{(2)} + \lambda_{(4)} + \lambda_{(6)} + \lambda_{(8)} \,.$$

Now let us introduce brane currents $J^{\text{F1}}_{(8)}$, $J^{\text{NS5}}_{(4)}$, and

$$J_{\text{odd}} = J^{\text{D8}}_{(1)} + J^{\text{D6}}_{(3)} + J^{\text{D4}}_{(5)} + J^{\text{D2}}_{(7)} + J^{\text{D0}}_{(9)} \,. \tag{14.20}$$

Then, we can include them to the equations of motions:

$$\begin{aligned} dH_{(3)} &= J^{\text{NS5}}_{(4)} \,, \\ dH_{(7)} &= J^{\text{F1}}_{(8)} + \frac{1}{2}(* G_{\text{even}})G_{\text{even}} \,, \\ dG_{\text{even}} &= J_{\text{odd}} + H_{(3)}G_{\text{even}} \,. \end{aligned}$$

Now we can derive the following "conservation" laws:

$$\begin{aligned} dJ^{\text{NS5}}_{(4)} &= 0 \,, \\ dJ^{\text{F1}}_{(8)} &= - \left[J_{\text{odd}}(*G_{\text{even}})\right]_{(9)} \,, \\ dJ_{\text{odd}} &= -J^{\text{NS5}}_{(4)}G_{\text{even}} - J_{\text{odd}}H_{(3)} \,. \end{aligned}$$

From these equations, we can deduce several facts:

- NS5-brane cannot have boundaries,
- F1 string can end on any D-branes,
- Dp-brane can end on NS5-brane up to $p = 6$ (D8 cannot),
- Dp-brane can end on D$(p+2)$-brane.

A similar analysis can be performed in Type IIB supergravity, or we can take the T-duality suitably. The results are summarized in Table 14.2.

Brane	Branes end on
F1	nothing
NS5-brane	D0, D2, D4, D6
D0-brane	F1
D2-brane	F1, D0
D4-brane	F1, D2
D6-brane	F1, D4
D8-brane	F1, D6

Brane	Branes end on
F1	nothing
NS5-brane	D1, D3, D5
D1-brane	F1
D3-brane	F1, D1
D5-brane	F1, D3
D7-brane	F1, D5

Table 6: Which branes can end on a brane in Type IIA (left) and IIB (right).

D0-D2 bound states and Myers effect

The general R-R coupling (14.5) includes all possible R-R fields C_{RR} in the theory. This implies that Dp-branes can include lower-dimensional D$(p-2n)$-branes ($n \in \mathbb{Z}_{\geq 0}$), forming a bound state.

Let us consider a concrete example of a D2-brane supported on Σ where the R-R coupling at $B_{(2)} = 0$ is

$$S \sim \frac{1}{2\pi} \int_{\mathbb{R}_t \times \Sigma} \left(C_{(3)} + F_{(2)} C_{(1)} \right) ,$$

Note that the $F_{(2)}$ flux needs to be quantized

$$\frac{1}{2\pi} \int_{\Sigma} F_{(2)} = n \in \mathbb{Z} .$$

If $C_{(3)}$ is zero, the D2-brane tries to shrink to a point due to its tension. On the other hand, $C_{(1)}$ part remains finite because the flux $F_{(2)}$ is quantized as

$$S \sim \frac{1}{2\pi} \int_{\mathbb{R}_t \times \Sigma} \left(F_{(2)} C_{(1)} \right) \to n \int_{\mathbb{R}_t} C_{(1)} .$$

This is equivalent to n D0-branes. Namely, when the D2-brane with n flux shrinks to a point, n D0-branes remain. This can be understood as a bound state of D2- and D0-branes (see Figure 41).

Let us consider the opposite process of the previous argument. When there are n D0-branes in the non-trivial $C_{(3)}$ background flux, they can become a D2-brane with the $F_{(2)}$ flux. Due to the non-trivial $C_{(3)}$ flux, being a D2-brane is energetically more preferable than being the D0-branes (see Figure 41). This is similar to the polarization phenomenon in electromagnetism, and in this case, it is called **Myers effect** [Mye99].

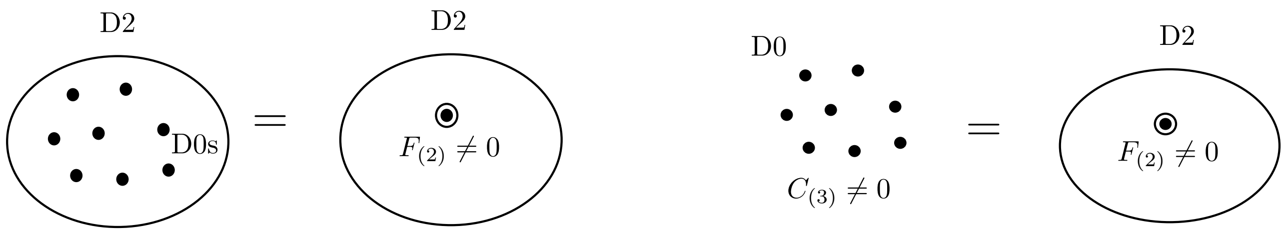


Figure 41: Myers effect: transition from D0-branes to D0-D2 .

14.3 Brane creations/annihilations

In fact, the last two Type IIB D-brane configurations in Table 4 preserve a quarter of supersymmetries, and they give rise to 3d $\mathcal{N} = 4$ theories in X^{012}. Let us first consider the third configuration of Table 4. As in Figure 42, N D3-branes suspended by NS5-branes give rise to $\mathrm{U}(N)$ gauge group (or vector multiplet), and F1-strings between D3- and D5-branes give rise to matters (hypermultiplet) in the fundamental representation. Also, bifundamental matter (hypermultiplet) arises from F1-strings between two adjacent D3-branes. The resulting theory is often represented by a quiver diagram. 3d $\mathcal{N} = 4$ supersymmetry is endowed with $\mathrm{SU}(2)_C \times \mathrm{SU}(2)_H$ R-symmetry where $\mathrm{SU}(2)_C$ acts on the X^{345} direction and $\mathrm{SU}(2)_H$ acts on the X^{789} direction in the Type IIB setup. Also, 3d $\mathcal{N} = 4$ supersymmetric theories have moduli spaces of vacua, called Coulomb and Higgs branch. Both are hyper-Kähler manifolds. In the brane realization, the Coulomb branch corresponds to the motion of D3-branes along X^{345} and the Higgs branch corresponds to the motion of D3-branes along X^{789}. Therefore, a large class of 3d $\mathcal{N} = 4$ theories can be studied by using Type IIB brane configurations in the third configuration of Table 4.

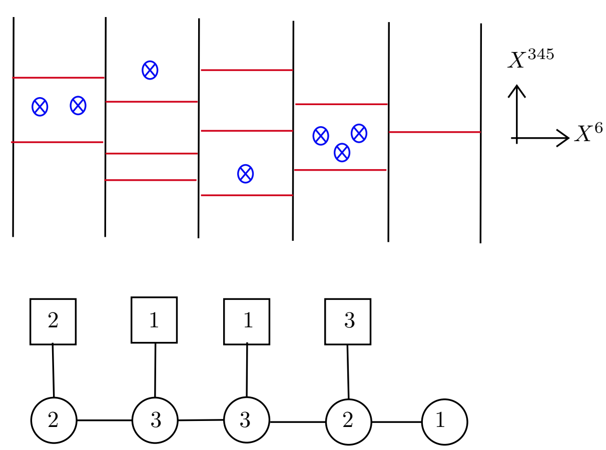


Figure 42: Type IIB brane system where NS5/D5/D3-branes are colored by black/blue/red, and the corresponding 3d $\mathcal{N} = 4$ quiver theory where circles/squares represent unitary gauge/flavor groups.

What makes this system very intriguing is a brane creation/annihilation process called **Hanany-Witten transition** [HW97]. As a D5-brane crosses an NS5-brane, a D3-brane is created or annihilated. See Figure 43. In the Hanany-Witten transition, the s-rule constrains that at most one D3-brane can be suspended between an NS5 and a D5-brane.

As in Table 4, Type IIB theory enjoys the S-duality which exchanges NS5 and D5-

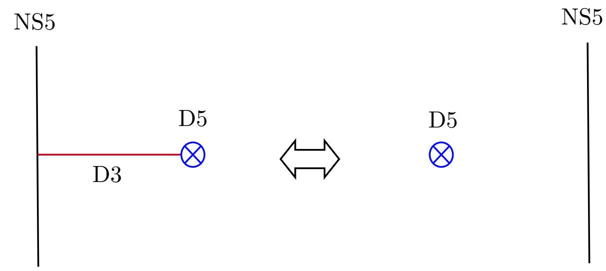


Figure 43: The crossing of NS5 and D5-brane leads to D3-brane creation/annihilation.

branes. The combination of the S-duality and the Hanany-Witten transition predicts non-trivial duality in 3d $\mathcal{N} = 4$ theories, called the 3d $\mathcal{N} = 4$ **mirror symmetry** [IS96]. One example of mirror symmetry is illustrated in Figure 44. Since the S-duality exchanges NS5 and D5-branes, so does the roles of the Coulomb and Higgs branch (namely the roles of X^{345} and X^{789}). Therefore, 3d $\mathcal{N} = 4$ supersymmetric theories can be studied from many perspectives such as branes, dualities, geometry of moduli spaces, and quantum field theories. They also admit mathematically very deep interpretations in representation theory, the geometry of hyper-Kahler manifolds, and equivalences of categories.

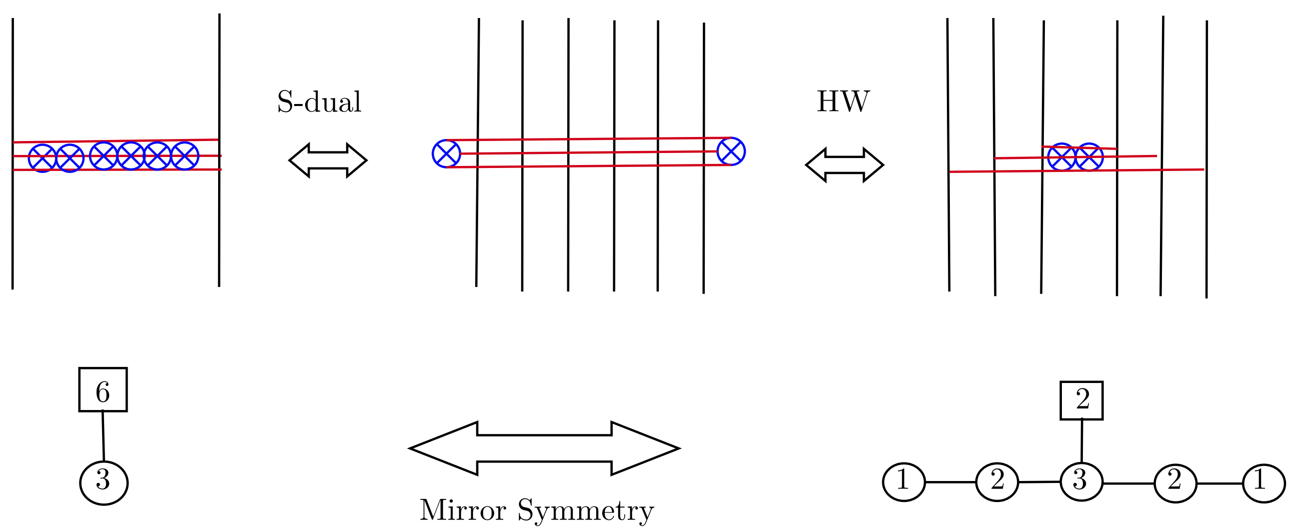


Figure 44: The S-duality maps from the left to middle brane configuration. Moving the two D5-branes into the middle involves the Hanany-Witten transition, and D3-branes are annihilated. The 3d $\mathcal{N} = 4$ theories from the left and the right brane configuration are dual to each other.

The study of supersymmetric theories by D-branes is very fruitful and successful. Brane dynamics provides profound insights into supersymmetric theories. Since the subject is very rich and broad, the reader is referred to, for instance, [HW97, DM96b, Wit97, AHK98, GK99, Joh00] as a starting point.

15 Black holes in string theory

In the previous section, we have seen that D-branes are dynamical objects and D-branes can end on others forming bound states. Moreover, they were ideally suited for studying black holes.

A large number of D-branes is heavy enough to produce a black hole by wrapping a cycle in a compact manifold. There is a large degeneracy due to open strings attaching to D-branes, which gives a statistical interpretation of the thermodynamic entropy. This leads to a precise microscopic accounting for the Bekenstein-Hawking entropy of the supersymmetric black holes, as shown by Strominger-Vafa [SV96].

The study of black holes in string theory by using D-branes has led to the celebrated AdS/CFT correspondence [Mal99]. (See Maldacena's Ph.D. thesis [Mal96] for instance.)

First let us briefly summarize the basics of black holes in general relativity and the laws of black hole thermodynamics studied in the early 70s [Bek72, Bek73, BCH73, Haw75]. For more detail, I refer to a wonderful lecture note [Tow97]. Also I refer the reader to [Wei21] for a historical account of black hole entropy.

15.1 Black holes

To begin with, we consider the Einstein-Maxwell action

$$\frac{1}{16}\pi \int d^4x\sqrt{g}\left(\frac{1}{G}R - F_{\mu\nu}F^{\mu\nu}\right), \tag{15.1}$$

where G is Newton's constant. In this subsection, we shall review black hole solutions to the action (15.1) and see that they are characterized by mass M, charge Q and angular momentum J.

Schwarzschild metric

If there is no electromagnetic fields $F = 0$ in the action (15.1), the equation of motion is $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} = 0$, which has a spherically symmetric, static solution

$$ds^2 \equiv g_{\mu\nu}dx^\mu dx^\nu = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1}dr^2 + r^2 d\Omega^2,$$

where t is the time, r is the radial coordinate, and $d\Omega$ is the canonical metric of a 2-sphere. This metric describes the spacetime outside a gravitationally collapsed non-rotating star with zero electric charge, called **Schwarzschild metric**. It is well-known that the **event horizon** appears at

$$g^{rr} = 0\,,$$

and the sphere $r = 2GM$ is indeed the event horizon of the Schwarzschild black hole with mass M.

It turns out that much of the interesting physics having to do with the quantum properties of black holes comes from the region near the event horizon. To examine the region *near* $r = 2GM$, we analytically continued to the Euclidean metric $t = -it_E$, and we set

$$r - 2GM = \frac{x^2}{8GM}\,.$$

Then, the metric near the event horizon $r = 2GM$

$$ds_{\mathrm{E}}^2 \approx (\kappa x)^2 dt_{\mathrm{E}}^2 + dx^2 + \frac{1}{4\kappa^2} d\Omega^2 \,,$$

where $\kappa = \frac{1}{4GM}$ is called the **surface gravity** because it is indeed the acceleration of a static particle near the horizon as measured at spatial infinity. Note that the surface gravity is defined by using Killing vector at the horizon, precisely speaking [Tow97]. The first part of the metric is just $\mathbb{R}^2$ with polar coordinates if we make the periodic identification

$$t_E \sim t_E + \frac{2\pi}{\kappa} \,.$$

Using the relation between Euclidean periodicity and temperature, we can deduce **Hawking temperature** of the Schwarzschild black hole

$$k_B T_H = \frac{\hbar\kappa}{2\pi c} = \frac{\hbar c^3}{8\pi G M} \,. \tag{15.2}$$

Here we restore the Boltzman constant k_B, and the speed of light c. This is a very heuristic way to introduce the Hawking temperature which was not originally found in this way.

Reissner-Nordström black hole

The most general static, spherically symmetric, charged solution of the Einstein-Maxwell theory (15.1) is

$$ds^2 = -\left(1 - \frac{2GM}{r} + \frac{GQ^2}{r^2}\right) dt^2 + \left(1 - \frac{2GM}{r} + \frac{GQ^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega^2, \tag{15.3}$$

with the electromagnetic field strength

$$F_{tr} = \frac{Q}{r^2} \,.$$

This solution is called the **Reissner-Nordström (RN) black hole** with mass M and charge Q. From the metric (15.3) we see that there are two event horizon for this solution where $g^{rr} = 0$ at

$$r_\pm = GM \pm \sqrt{(GM)^2 - GQ^2} \,.$$

Thus, r_+ defines the outer horizon of the black hole and r_- defines the inner horizon of the black hole. The area of the black hole is $4\pi r_+^2$. It turns out that the Hawking temperature of the RN black hole is

$$T_H = \frac{\sqrt{(GM)^2 - GQ^2}}{2\pi G\left(GM + \sqrt{(GM)^2 - GQ^2}\right)^2} \,.$$

For a physically sensible definition of temperature, the mass must satisfy the bound $GM^2 \geq Q^2$, and the two horizons coincide $r_+ = r_- = GM$ when this bound is saturated. In this case, the temperature of the black hole is zero and it is called an **extremal black hole**.

Kerr-Newman black hole

If we relax the static condition, black holes can have angular momentum. Hence, general stationary solutions, called **Kerr-Newman black holes**, to the action (15.1) are described with three parameters. In **Boyer-Linquist coordinates**, the KN metric is

$$\begin{aligned} ds^2 = & -\frac{\left(\Delta - a^2 \sin^2\theta\right)}{\Sigma} dt^2 - 2a\sin^2\theta \frac{\left(r^2+a^2-\Delta\right)}{\Sigma} dt\, d\phi \\ & + \left(\frac{\left(r^2+a^2\right)^2 - \Delta a^2 \sin^2\theta}{\Sigma}\right)\sin^2\theta d\phi^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 \end{aligned}$$

where

$$\begin{aligned} \Sigma &= r^2 + a^2\cos^2\theta \\ \Delta &= r^2 - 2Mr + a^2 + e^2 . \end{aligned}$$

The three parameters are M, a, and e. It can be shown that

$$a = \frac{J}{M}$$

where J is the total angular momentum, while

$$e = \sqrt{Q^2 + P^2}$$

where Q and P are the electric and magnetic (monopole) charges, respectively. The Maxwell 1-form of the KN solution is

$$A_\mu dx^\mu = \frac{Qr\left(dt - a\sin^2\theta d\phi\right) - P\cos\theta\left[adt - \left(r^2+a^2\right)d\phi\right]}{\Sigma} .$$

15.2 Black hole thermodynamics and Bekenstein-Hawking entropy

Classically, a stationary black hole is characterized by its mass M, angular momentum J, and charge Q. This is called a black hole no hair theorem. However, in [Bek72], Bekenstein asks an incisive question: if we treat a black hole as a purely geometric object, by throwing a package of entropy, a cup of tea, into a black hole, the total entropy of the world outside would seem to decrease. This contradicts the second law of thermodynamics which states the total entropy never decreases. To save the second law of thermodynamics, this suggests that the black hole must have entropy. The question is how to characterize the entropy if a black hole has. The hit was hidden in the area theorem of black holes [Haw71, Haw72], stating that the total area of the black hole horizons never decreases in any process. For example, two Schwarzschild black holes with masses M_1 and M_2 can merge into a bigger black hole of mass $M = M_1 + M_2$. Since the area is proportional to the square of the mass, this is consistent with the area theorem, namely $(M_1 + M_2)^2 \geq M_1^2 + M_2^2$. On the other hand, the opposite process where a bigger black hole splits into two is never allowed by this theorem. Motivated by the area theorem, Bekenstein proposed in [Bek72] that a black hole has entropy proportional to its area.

Soon after that, Bardeen, Carter and Hawking point out similarities between the laws of black hole mechanics and the laws of thermodynamics in [BCH73]. More concretely, they find the laws of corresponding to the three laws of thermodynamics.

(0) Zeroth Law: In thermodynamics, the zeroth law states that the temperature T of a thermal equilibrium object is constant throughout the body. Correspondingly, for a stationary black hole, its surface gravity $\kappa = 1/4GM$ is constant over the event horizon.

(1) First Law: The first law of thermodynamics states that energy is conserved, and the variation of energy is given by

$$dE = TdS + \mu dQ + \Omega dJ \tag{15.4}$$

where E is the energy, Q is the charge with chemical potential μ and J is the angular momentum with chemical potential Ω in the system. Correspondingly, for a black hole, the variation of its mass is given by

$$dM = \frac{\kappa}{8\pi G} dA + \mu dQ + \Omega dJ \tag{15.5}$$

where A is the area of the horizon, and κ is the surface gravity, μ is the chemical potential conjugate to Q, and Ω is the angular velocity conjugate to J.

(2) Second Law: The second law of thermodynamics states that the total entropy S never decreases, $\delta S \geq 0$. Correspondingly, for a black hole, the area theorem states that the total area of a black hole in any process never decreases, $\delta A \geq 0$.

Laws of thermodynamics	**Laws of black hole mechanics**
Temperature is constant throughout a body at equilibrium. T=constant.	Surface gravity is constant on the event horizon. κ =constant.
Energy is conserved. $dE = TdS + \mu dQ + \Omega dJ$.	Energy is conserved. $dM = \frac{\kappa}{8\pi G} dA + \mu dQ + \Omega dJ$.
Entropy never decreases. $\delta S \geq 0$.	Area never decreases. $\delta A \geq 0$.

Table 7: Laws of black hole thermodynamics

This result can be understood as one of the highlights of general relativity. Classically, a black hole is a not only geometric but also thermodynamic object. If a black hole has energy E and entropy S, then it must also have temperature T given by

$$\frac{1}{T} = \frac{\partial S}{\partial E}.$$

For example, for a Schwarzschild black hole, the area and the entropy are proportional to M^2. Hence, we can derive

$$\frac{1}{T} = \frac{\partial S}{\partial M} \sim \frac{\partial M^2}{\partial M} \sim M.$$

Therefore, black hole temperature is inversely proportional to mass M. The smaller a black hole is, the hotter it is! Moreover, if the black hole has temperature, it must thermally radiate like any hot body. The understanding of the thermal properties of black holes requires treatment beyond classical general relativity.

Hawking has applied techniques of quantum field theories on a curved background to the near-horizon region of a black hole and showed that a black hole indeed radiates [Haw75]. This can be intuitively understood as follows: in a quantum theory, particle-antiparticle creations constantly occur in the vacuum. Around the horizon, after pairs are created, antiparticles fall into a black hole due to the gravitational attraction whereas particles escape to the infinity. Although we do not deal with Hawking's calculation unfortunately (see [Tow97]), it indeed justifies this picture. Moreover, it revealed that the spectrum emitted by the black hole is precisely subject to the thermal radiation with temperature (15.2). Indeed, a black hole is not black at quantum level. Hence, we can treat a black hole as a thermal object, and the analogy of the laws in Table 7 can be understood as the natural consequence of the laws of thermodynamics. As a result, the formula for the Hawking temperature (15.2) and the first law of thermodynamics

$$c^2 dM = T_H dS = \frac{\kappa c^2}{8\pi G} dA,$$

lead to the precise relation between entropy and the area of the black hole:

$$S = \frac{k_B c^3 A}{4G\hbar} . \tag{15.6}$$

This is a universal result for any black hole, and this remarkable relation between the thermodynamic properties of a black hole and its geometric properties is called the celebrated **Bekenstein-Hawking entropy formula**. This formula involves all four fundamental constants of nature; $(G, c, k_B, \hbar)$. Also, this is the first place where the Newton constant G meets with the Planck constant $\hbar$. Thus, this formula shows a deep connection between black hole geometry, thermodynamics and quantum mechanics.

For ordinary objects, Boltzmann has given the statistical interpretation of the thermodynamic entropy of a system. We fix the macroscopic parameters (e.g. total electric charge, energy etc.) and count the number Ω of quantum states, known as microstates, each of which has the same values for the macroscopic parameters, and the entropy is expressed as

$$S = k_B \log \Omega .$$

Since the Bekenstein-Hawking entropy (15.6) behaves as the ordinary thermodynamic entropy in every aspect, it is therefore natural to ask whether the black hole entropy admits a statistical interpretation in the same way.

Furthermore, one of the most dramatic results of Hawking's work was the implication that black holes are associated with information loss. Physically speaking, we can associate information with pure states in quantum mechanics. If we throw in a pure quantum state, say, the s-wave to a black hole, then it eventually comes out as a thermal (mixed) state. Thus the net result of this process is the evolution of a pure quantum state into a mixed state, which violates the law (unitarity) of quantum mechanics. This is called the **information paradox** [Haw76]. This is because Hawking's calculation is based on the semi-classical analysis, namely, we fix the background and quantize particles. In fact, the information paradox stems from the absence of such a microscopic description of gravity.

In order to investigate the microscopic description of black hole entropy, we need quantum theory of gravity. This is precisely what string theorists have attempted to do and have been partially successful.

15.3 Black holes in string theory

In string theory on a d-dimensional compact manifold, branes can be wrapped in a cycle of the compact manifold and it looks like a point-like object in $(10-d)$-dimensional spacetime. In the regime where supergravity approximation is valid, configurations of this kind give rise to black hole solutions of the corresponding low-energy supergravity theory. Moreover, if a brane configuration preserves supersymmetry, then the corresponding solution will be an extremal supersymmetric black hole. Extremal black holes are interesting because they are stable against Hawking radiation and nevertheless have a large entropy. On the other hand, configurations without supersymmetry yield non-extremal black holes.

In general, the regime of the parameter space in which supergravity is valid is different from the regime in which the microstates counting can be performed. Thus, even if we know that a given brane configuration becomes a black hole when we go from weak to strong coupling, it is generally difficult to extract microscopic information of the black hole from the brane configuration.

For supersymmetric black holes, however, one can count the number of states at weak coupling and extrapolate the result to the black hole phase due to the BPS property. We will see that in this way, one derives the Bekenstein-Hawking entropy formula (including the precise numerical coefficient) for a 5d supersymmetric black hole [SV96]. (For more detail, I refer to [DMW02, DN12].)

D1-D5-P brane system

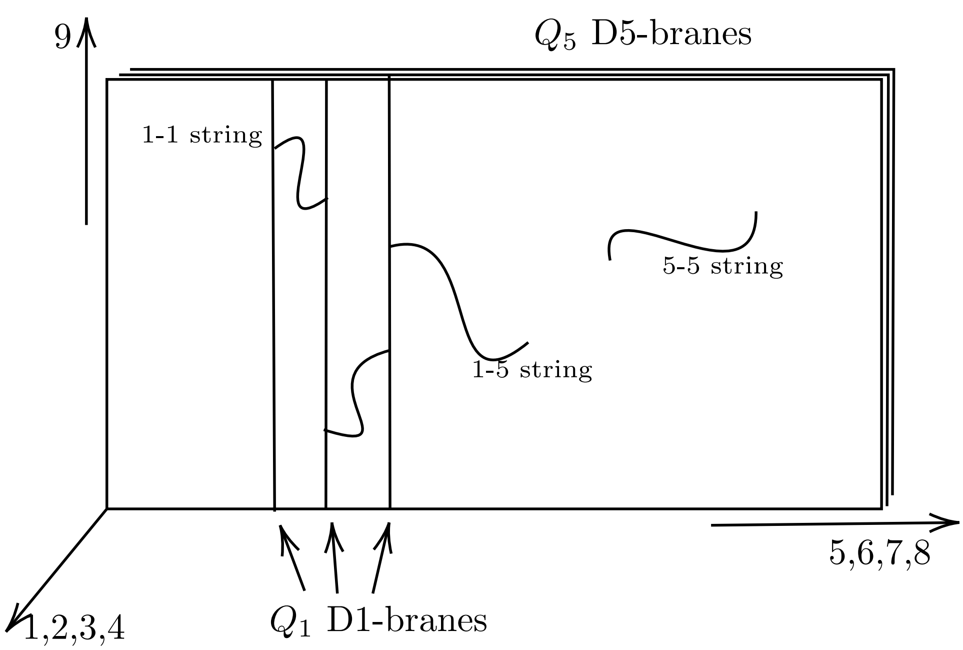


Figure 45:

Here we follow the example treated in [CM96], called the D1-D5-P brane system on $\mathbb{R}^{1,4} \times T^5$. Let us consider Type IIB compactified on a five-torus $T^5 = T^4 \times S^1$, which spans the $(x_5 \cdots x_9)$ coordinates, with Q_1 D1-branes and Q_5 D5-branes in the following

	0	1	2	3	4	5	6	7	8	9
Q_1 D1	$\times$									$\times$
Q_5 D5	$\times$					$\times$	$\times$	$\times$	$\times$	$\times$
Q_P mom										$\rightsquigarrow$

configuration. We consider that the volume of T^4 is $(2\pi)^4 V$ and the radius of S^1 is R. Here we also assume that there is an excitation by open strings carrying momenta Q_P/R in the x_9-direction. This system preserves 4 real supercharges since each constituent breaks a half of supersymmetry.

Black hole in 5d supergravity

If there are large enough D-brane charges (Q_1, Q_5, Q_P) and the five-torus is sufficiently small, the configuration produces a 5d black hole. We would like to compute the Bekenstein-Hawking entropy (15.6) of the black hole by evaluating the area of the event horizon. When 5d supergravity analysis is valid, this brane system gives rise to a 1/8-BPS black hole configuration. Ignoring the R-R field and the B-field configuration, the 5d Einstein frame metric of this solution then becomes

$$ds_5^2 = -\lambda(r)^{-\frac{2}{3}} dt^2 + \lambda(r)^{\frac{1}{3}} \left[dr^2 + r^2 d\Omega_3^2\right] ,$$

where the harmonic functions are

$$\lambda(r) = H_1(r) H_5(r) K(r) = \left(1 + \frac{r_1^2}{r^2}\right)\left(1 + \frac{r_5^2}{r^2}\right)\left(1 + \frac{r_m^2}{r^2}\right) ,$$

with

$$r_1^2 = \frac{g_s Q_1 \ell_s^6}{V} , \qquad r_5^2 = g_s Q_5 \ell_s^2 \qquad r_m^2 = \frac{g_s^2 Q_P \ell_s^8}{R^2 V} .$$

Let us briefly evaluate the validity of the supergravity analysis. In order for the α' corrections to geometry to be small, the radius parameters have to be large with respect to the string unit, $r_{1,5,m} \gg \ell_s$. Since we assume $V^{1/4}$, R are an order of the string length, this implies

$$g_s Q_1 \gg 1 , \qquad g_s Q_5 \gg 1 , \qquad g_s^2 Q_P \gg 1 . \tag{15.7}$$

To suppress string loop corrections, we need g_s to be small (but finite) so that the D-brane charges must be sufficiently large for supergravity analysis.

It turns out that the surface gravity and therefore the Hawking temperature of this black hole is zero, $T_H = 0$, as expected. The metric shows that the event horizon is located at $r = 0$ and the Bekenstein-Hawking entropy (15.6) is

$$\begin{aligned} S_{\text{macro}} &= \frac{A}{4G_5} = \frac{1}{4G_5} 2\pi^2 \left[r^2 \lambda(r)^{\frac{1}{3}}\right]^{\frac{3}{2}} \text{ at } r = 0 \\ &= 2 \frac{\pi^2}{4\left[\pi g_s^2 \ell_s^8/(4VR)\right]} (r_1 r_5 r_m)^{\frac{1}{2}} = \frac{2\pi V R}{g_s^2 \ell_s^8} \left(\frac{g_s Q_1 \ell_s^6}{V} g_s Q_5 \ell_s^2 \frac{g_s^2 Q_P \ell_s^8}{R^2 V}\right)^{\frac{1}{2}} \\ &= 2\pi \sqrt{Q_1 Q_5 Q_P} , \end{aligned} \tag{15.8}$$

where we use $G_5 = \frac{G_{10}}{(2\pi)^5 VR}$ and $16\pi G_{10} = (2\pi)^7 g_2^2 \ell_s^8$. Notice that it is also independent of R and of V while the ADM mass depends on R, V explicitly.

$$M = \frac{Q_P}{R} + \frac{Q_1 R}{g_s \ell_s^2} + \frac{Q_5 RV}{g_s \ell_s^6} .$$

Counting microstates

The next step is to identify the degeneracy of open string states of the D1-D5-P system, which can be analyzed at the limit opposite to (15.7), i.e.

$$g_s Q_1 \ll 1\,, \qquad g_s Q_5 \ll 1\,, \qquad g_s^2 Q_P \ll 1\,. \tag{15.9}$$

Further simplification can be made by taking the limit that the volume of T^4 is small as compared to the radius of the circle S^1,

$$V^{\frac{1}{4}} \ll R\,.$$

In this limit, the theory on the D-branes is an effective $2d$ theory on (x_1, x_9)-direction. Moreover, the smeared D1-branes plus D5-branes have a symmetry group $SO(1,1) \times \mathrm{SO}(4)_{\parallel} \times \mathrm{SO}(4)_{\perp}$ where $\mathrm{SO}(4)_{\parallel} \times \mathrm{SO}(4)_{\perp}$ becomes R-symmetry of the $2d$ theory which we call $\mathcal{N} = (4,4)$ 2d CFT. In the supersymmetric configuration, the left-movers are in their ground states so that we count excited right-movers.

Because the D1-branes are instantons in the D5-brane theory, the low-energy theory of interest is in fact a σ-model on the moduli space of instantons

$$\mathcal{M} = \mathrm{Sym}^{Q_1 Q_5}(T^4) = (T^4)^{Q_1 Q_5} / S_{Q_1 Q_5}\,.$$

The central charge of this 2d CFT is

$$c = n_{\text{bose}} + \frac{1}{2} n_{\text{fermi}} = 6 Q_1 Q_5\,. \tag{15.10}$$

Roughly, this central charge c can be thought of as coming from having $Q_1 Q_5$ 1-5 strings that can move in the 4 directions of T^4. Although this orbifold theory has many twisted sectors, the special point of the moduli space corresponds to a single string winding $Q_1 Q_5$ times. It turns out that counting the excitations of this **long string** is only relevant in the limit of large D-brane charges. For this long string, the level-matching condition is

$$N - \overline{N} = \frac{Q_P}{R} W\,, \quad W = Q_1 Q_5\,, \quad \to \quad N = \frac{Q_P Q_1 Q_5}{R}$$

where the left-movers are in the ground states $\overline{N} = 0$.

If N_m^i and n_m^i denote occupation numbers of the four transverse compact bosonic and fermionic oscillators, respectively, then evaluation of N gives

$$nW = \sum_{i=1}^{4} \sum_{m=1}^{\infty} m(N_m^i + n_m^i) \tag{15.11}$$

The degeneracy $\Omega(Q_1, Q_5, Q_P)$ is then given by the number of choices for N_m^i and n_m^i subject to (15.11).

The partition function of this system is the partition function for 4 bosons and an equal number of fermions

$$Z = \left[\prod_{m=1}^{\infty} \frac{1+q^m}{1-q^m}\right]^4 \equiv \sum \Omega(Q_1, Q_5, Q_P) q^N\,,$$

where $\Omega(Q_1, Q_5, Q_P)$ is the degeneracy of states at the level $N = \frac{Q_P Q_1 Q_5}{R}$. The Cardy formula [Car86] can be applied in the regime in which the KK momenta $Q_P \gg Q_1 Q_5$ much larger than the central charge (15.10) while assuming (15.9):

$$\Omega(Q_1, Q_5, Q_P) \sim \exp\left(2\pi\sqrt{Q_1 Q_5 Q_P}\right) = \exp\left(2\pi\sqrt{\frac{c}{6} Q_P}\right)$$

Therefore the microscopic D-brane statistical entropy is

$$S_{\text{micro}} = \log\left(\Omega(Q_1, Q_5, Q_P)\right) = 2\pi\sqrt{Q_1 Q_5 Q_P}\,.$$

This agrees exactly with the black hole result (15.8)!

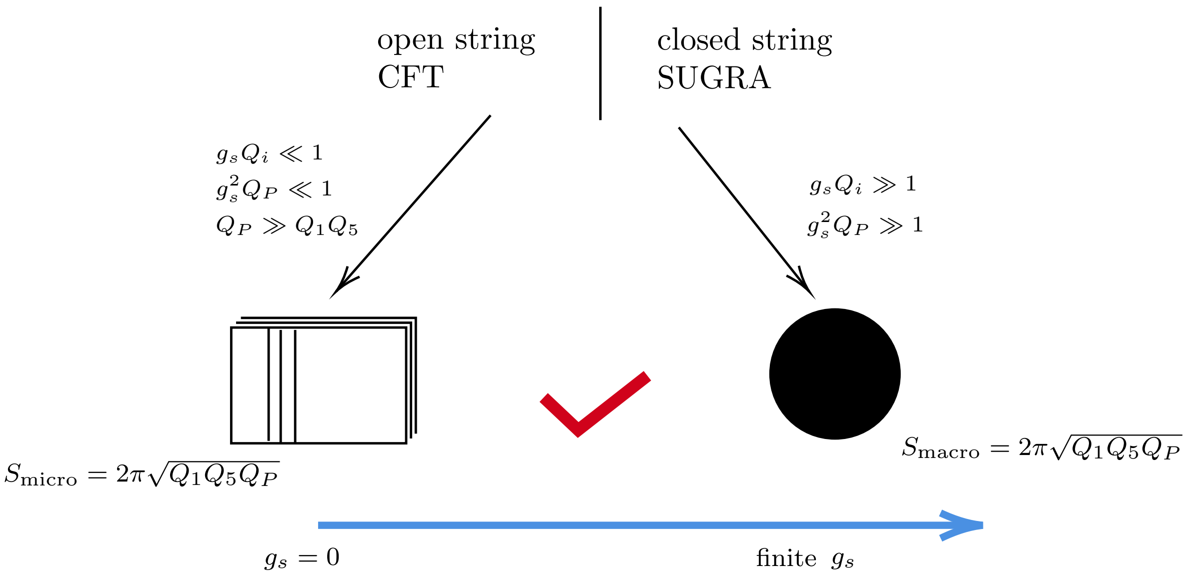


Figure 46:

More results

By coupling the low energy degrees of freedom in the D1-D5-p system to supergravity modes (therefore perturbing the extremal condition), one can also compute the rate of Hawking radiation from a black hole that agrees precisely with the Hawking calculation. Thus this provides a microscopic explanation of Hawking radiation. (See [DMW02, §8].)

In fact, vigorous research in the last decade has shown that one can show the exact match between macroscopic and microscopic calculations of black hole entropy even in finite D-brane charges.

Moreover, a generalization of Bekenstein-Hawking entropy has been proposed in [RT06] that connects quantum theory of gravity and quantum information theory. A recent study has clearly suggested that quantum entanglement must have something to do with quantum physics of spacetime.

16 Introduction to AdS/CFT correspondence

The study of black holes in string theory by using D-branes has led to the celebrated AdS/CFT correspondence [Mal99]. The AdS/CFT correspondence is the equivalence between a string theory or M-theory on an anti-de Sitter background and a conformal field theory. It has shed new light on quantum gravity as well as strongly coupled quantum field theories. Although it was proposed in the framework of string theory, it has already been studied beyond string theory, influencing other physical theories. It has attracted a large number of researchers, and it is connected to many branches of physics. For the basics of the AdS/CFT correspondence, the most famous review is [AGM$^+$00], and there are many other reviews and books, *e.g.* [DF02, Mal03, Ram15, Năs15].

We shall first study the basic properties of conformal field theories in general dimensions and geometry of anti-de Sitter space. Then, we will deal with the most famous example, Type IIB on $\mathrm{AdS}_5 \times S^5$/ 4d $\mathcal{N} = 4$ SYM.

16.1 Conformal field theory

Conformal group

A conformal field theory (CFT) is a quantum field theory that is invariant under conformal transformations. In §3, we have studied the conformal transformation for two dimensions, which is a special case. Here, we study a conformal group for arbitrary dimensions (assume $d \geq 3$).

A conformal group is defined by transformations that preserve the metric up to a local scale factor:

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \Omega^2(x) g_{\mu\nu}(x) \ .$$

In an infinitesimal form ($x'^{\mu} = x^\mu + \epsilon^\mu$) it is (compare with (3.10)

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d} \eta_{\mu\nu} \partial \cdot \epsilon \ . \tag{16.1}$$

Applying ∂^μ to (16.1) we have

$$\left(1 - \frac{2}{d}\right) \partial_\nu \partial \cdot \epsilon + \Box \epsilon_\nu = 0 \ ,$$

where $\Box = \partial \cdot \partial$. From this expression, we can see that $d = 2$ is quite special, and it leads $\partial^2 \epsilon = \partial_- \partial_+ \epsilon$, which gives infinitely many transformations. We further apply ∂^ν and reach

$$(d-1) \Box \partial \cdot \epsilon = 0 \ .$$

This expression implies ϵ_μ is up to quadratic order of x:

$$\epsilon_\mu = a_\mu + b_{\mu\nu} x^\nu + c_{\mu\nu\rho} x^\nu x^\rho \ .$$

Plugging these expressions back to the definition equations above (and its variant), we have the following constraints

$$\begin{aligned} b_{\mu\nu} &= \alpha \eta_{\mu\nu} + M_{\mu\nu} \qquad (M_{\mu\nu} = -M_{\nu\mu}) \ , \\ c_{\mu\nu\rho} &= \eta_{\mu\nu} f_\rho + \eta_{\mu\rho} f_\nu - \eta_{\nu\rho} f_\mu \qquad (f_\mu = \frac{1}{d} c^\rho{}_{\rho\mu}) \ . \end{aligned}$$

Parameters above correspond to transformations you are familiar with except f_μ, which is called special conformal transformation (SCT). See Table 16.1 for the summary.

Names	Finite transf.	Generators	Dim.
Translation	$x'^\mu = x^\mu + a^\mu$	$P_\mu = -i\partial_\mu$	$+1$
Dilat(at)ion	$x'^\mu = \alpha x^\mu$	$D = -ix \cdot \partial$	0
Lorentz/Rotation	$x'^\mu = M^\mu_{\ \nu} x^\nu$	$L_{\mu\nu} = i\left(x_\mu \partial_\nu - x_\nu \partial_\mu\right)$	0
SCT	$x'^\mu = \frac{x^\mu - (x\cdot x) f^\mu}{1 - 2f\cdot x + (f\cdot f)(x\cdot x)}$	$K_\mu = -i\left(2x_\mu x\cdot\partial - (x\cdot x)\partial_\mu\right)$	-1

Table 8: Generators of the conformal group.

The generators summarized in the table form conformal group commutation relations

$$\begin{aligned}
&[J_{ab}, J_{cd}] = i\left(\eta_{ad} J_{bc} + \eta_{bc} J_{ad} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac}\right) \ , \\
&J_{\mu\nu} = L_{\mu\nu} \ , \qquad\qquad J_{(d+1)d} = D \ , \\
&J_{\mu d} = \tfrac{1}{2}\left(K_\mu - P_\mu\right) \ , \qquad J_{\mu(d+1)} = \tfrac{1}{2}\left(K_\mu + P_\mu\right) \ ,
\end{aligned} \tag{16.2}$$

where note that $a, b, c, d = 0, 1, \ldots d+1$ and $\mu, \nu = 0, 1, \ldots, d-1$, and η_{ab} is $\mathrm{diag}(-, +, \ldots, +, -)$ for Lorentzian spacetime and $\mathrm{diag}(+, +, \ldots, +, -)$ for Euclidean space. The algebras are isomorphic to those of $\mathrm{SO}(2, d)$ and $\mathrm{SO}(1, d+1)$, respectively. Note that the SCT can be understood as an inversion ($x^\mu \to \frac{x^\mu}{x\cdot x}$) with translation $\frac{x'^\mu}{x'\cdot x'} = \frac{x^\mu}{x\cdot x} - f^\mu$. However, the inversion is not included in the algebra (since the inversion is a discrete transformation). Among others, we write down non-trivial commutation relations that involve D

$$[D, P_\mu] = -iP_\mu \ , \quad [D, K_\mu] = iK_\mu \ , \quad [P_\mu, K_\nu] = 2i\left(M_{\mu\nu} - \eta_{\mu\nu} D\right) \ , \tag{16.3}$$

which characterize representation of the conformal group. By the way, the other non-trivial ones are

$$\begin{aligned}
&[M_{\mu\nu}, P_\rho] = -i(\eta_{\mu\rho} P_\nu - \eta_{\nu\rho} P_\mu) \ , \qquad [M_{\mu\nu}, K_\rho] = -i(\eta_{\mu\rho} K_\nu - \eta_{\nu\rho} K_\mu) \ , \\
&[M_{\mu\nu}, M_{\rho\sigma}] = -i\eta_{\mu\rho} M_{\nu\sigma} \pm (\text{permutations}) \ .
\end{aligned}$$

Primary fields

In 2-dimensions, we defined the primary field by using OPE (3.20) with energy-momentum tensor (it leads to a representation of the conformal group). For general dimensions, we define it by a formal representation of the conformal group. It is known that the representation is characterized by an eigenvalue of the dilatation operator $-i\Delta$ (Δ is called the **scaling dimension** of the field, rather than **weight**), and representation of the Lorentz group. The former statement means that $\Phi(x) \to \Phi'(\lambda x) = \lambda^{-\Delta}\Phi(x)$. The commutation relations (16.3)) tell us that P_μ is the raising operator, while K_μ is the lowering operator. Therefore, there are operators annihilated by K_μ in each finite-dimensional representation of the conformal group. Such an operator is called **primary operator/field** (we use operator and field interchangeably). The action of the conformal group on the primary field is

$$\begin{aligned}
[P_\mu, \Phi(x)] &= i\partial_\mu \Phi(x) \ , \\
[M_{\mu\nu}, \Phi(x)] &= [i(x_\mu \partial_\nu - x_\nu \partial_\mu) + \Sigma_{\mu\nu}]\Phi(x) \ , \\
[D, \Phi(x)] &= i(-\Delta + x^\mu \partial_\mu)\Phi(x) \ , \\
[K_\mu, \Phi(x)] &= [i(x^2 \partial_\mu - 2x_\mu x^\nu \partial_\nu + 2x_\mu \Delta) - 2x^\nu \Sigma_{\mu\nu}]\Phi(x) \ ,
\end{aligned}$$

where $\Sigma_{\mu\nu}$ are the matrices of a finite-dimensional representation of the Lorentz group, acting on the indices of the Φ field (e.g. it is $\frac{i}{2}\gamma_{\mu\nu}$ for a spinor). There are some comments on primary fields and others:

- Fields created by acting P_μ on a primary field are called **descendant fields**.
- Fields are not, in general, by eigenfunctions of the Hamiltonian P^0, or the mass operator $-P \cdot P$, and hence, they have a continuous spectrum.
- In unitary field theories the scale dimension is bounded from below (**unitary bound**). It is $\Delta \geq (d-2)/2$ for scalars, $\Delta \geq (d-1)/2$ for spinors, and $\Delta \geq d+s-2$ for spin-s fields for $s \geq 1$. (We refer to the derivation of the bounds and other details to [Qua15, Sec. 2].)

16.2 Anti-de Sitter space

An anti-de Sitter(AdS) space is a maximally symmetric manifold with constant negative scalar curvature. It is a solution of Einstein's equations for an empty universe with negative cosmological constant. The easiest way to understand it is as follows.

A Lorentzian AdS$_{d+1}$ space can be illustrated by the hyperboloid in $(2, d)$ Minkowski space:

$$X_0^2 + X_{d+1}^2 - \sum_{i=1}^{d} X_i^2 = R^2 . \tag{16.4}$$

The metric can be naturally induced from the Minkowski space

$$ds^2 = -dX_0^2 - dX_{d+1}^2 + \sum_{i=1}^{d} dX_i^2 .$$

By construction, it has SO$(2, d)$ isometry, which is the first connection to the conformal group in d-dim.

Global coordinate

A simple solution to (16.4) is given as follows.

$$\begin{aligned} X_0^2 + X_{d+1}^2 &= R^2 \cosh^2 \rho , \\ \sum_{i=1}^{d} X_i^2 &= R^2 \sinh^2 \rho . \end{aligned}$$

Or, more concretely,

$$\begin{aligned} X_0 &= R \cosh\rho \, \cos\tau , \qquad X_{d+1} = R\cosh\rho \, \sin\tau , \\ X_i &= R \sinh\rho \, \Omega_i \quad (i = 1, \ldots, d, \text{ and } \sum_i \Omega_i^2 = 1). \end{aligned}$$

These are S^1 and S^{d-1} with radii $R\cosh\rho$ and $R\sinh\rho$, respectively. The metric is

$$ds^2 = R^2 \left(-\cosh^2\rho \, d\tau^2 + d\rho^2 + \sinh^2\rho \, d\Omega_{(d-1)}^2 \right) .$$

Note that τ is a periodic variable and if we take $0 \leq \tau < 2\pi$, the coordinate wraps the hyperboloid precisely once. This is why this coordinate is called the **global coordinate**.

The manifest sub-isometries are SO(2) and SO(d) of SO($2,d$). To obtain a causal spacetime, we simply unwrap the circle S^1, namely, we take the region $-\infty < \tau < \infty$ with no identification, which is called the **universal cover** of the hyperboloid.

In literature, another global coordinate is also used, which can be derived by redefinitions $r \equiv R \sinh \rho$ and $dt \equiv R d\tau$:

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\Omega^2_{(d-1)} \,, \qquad f(r) = 1 + \frac{r^2}{R^2} \,.$$

Poincaré coordinates

There is yet another coordinate system, called the **Poincaré coordinates**. As opposed to the global coordinate, this coordinate covers only half of the hyperboloid. It is most easily (but naively) seen in $d = 1$ case:

$$x^2 - y^2 = R^2 \,,$$

which is the hyperbolic curve. The curve consists of two isolated parts in regions $x > R$ and $x < -R$. We simply use one of them to construct the coordinate.

Let us get back to general d-dim. We define the coordinate as follows.

$$\begin{aligned}
X_0 &= \frac{1}{2u}\left(1 + u^2\left(R^2 + x_i^2 - t^2\right)\right) \,, \\
X_i &= Rux_i \qquad (i = 1, \ldots, d-1) \,, \\
X_d &= \frac{1}{2u}\left(1 - u^2\left(R^2 - x_i^2 + t^2\right)\right) \,, \\
X_{d+1} &= Rut \,,
\end{aligned}$$

where $u > 0$. As it is stated, the coordinate covers half of the hyperboloid; in the region, $X_0 > X_d$. The metric is

$$ds^2 = R^2\left(\frac{du^2}{u^2} + u^2(-dt^2 + dx_i^2)\right) = R^2\left(\frac{du^2}{u^2} + u^2 dx_\mu^2\right) \,. \tag{16.5}$$

The coordinates (u, t, x_i) are called the **Poincaré coordinates**. This metric has manifest $ISO(1, d-1)$ and SO(1,1) sub-isometries of SO($2,d$); the former is the Poincaré transformation and the latter corresponds to the dilatation

$$(u, t, x_i) \to (\lambda^{-1}u, \lambda t, \lambda x_i) \,.$$

If we further define $z = 1/u$ ($z > 0$), then,

$$ds^2 = \frac{R^2}{z^2}\left(dz^2 + dx_\mu^2\right) \,. \tag{16.6}$$

This is called **the upper(Poincaré) half-plane model**. The hypersurface given by $z = 0$ is called the **(asymptotic) boundary** of the AdS space, which corresponds to $u \sim r \sim \rho = \infty$.

16.3 Introduction to AdS$_5$/CFT$_4$ correspondence

Now let us study the most well-studied example of AdS/CFT correspondence, which is the equivalence between 4d U(N) $\mathcal{N}=4$ super-Yang-Mills (SYM) and Type IIB string theory on AdS$_5\times S^5$, which arises from the large number of D3-branes. Type IIB string theory with D3-branes contains two kinds of perturbative excitations, closed strings and open strings. If we consider the system at low energies, energies lower than the string scale $1/\ell_s$, then only the massless string states can be excited. The closed string massless states give a gravity supermultiplet in $D=10$ in Type IIB supergravity as in §12.4. The open string massless states give an $\mathcal{N}=4$ vector multiplet in $D=4$, and their low-energy effective theory is $\mathcal{N}=4$ U(N) SYM. Therefore, the duality can be also understood as **open/closed duality**.

$\mathcal{N}=4$ super-Yang-Mills theory

The low-energy effective theory of N D3-branes is 4d $\mathcal{N}=4$ U(N) SYM theory so we describe the basic properties of the $\mathcal{N}=4$ SYM. The action can be obtained by the dimensional reduction from the 10d $\mathcal{N}=1$ SCFT on $\mathbb{R}^{1,3}\times T^6$ where the 10d Lorentz group SO$(1,9)$ is decomposed to SO$(1,3)\times$SO$(6)\subset$ SO$(1,9)$:

$$\begin{aligned}S&=-\frac{1}{g_{YM}^2}\int d^{10}x\mathrm{Tr}\left[\frac{1}{4}F_{MN}^2+\frac{i}{2}\overline{\lambda}\Gamma^M D_M\lambda\right]\\&=-\frac{1}{g_{YM}^2}\int d^4x\ \mathrm{Tr}\Big[\frac{1}{4}F_{\mu\nu}^2+\frac{1}{2}(D_\mu X_m)^2+\frac{i}{2}\overline{\lambda}\Gamma^\mu D_\mu\lambda-\frac{g_{YM}}{2}\overline{\lambda}\Gamma^m[X_m,\lambda]-\frac{g_{YM}^2}{4}[X_m,X_n]^2\Big]\end{aligned}\tag{16.7}$$

where the ten-dimensional gauge fields A_M, $M=0,\ldots,9$ split the 4d gauge field A_μ, $\mu=0,\ldots,3$ and 6 scalars X_m, $m=1,\ldots,6$, and λ is a 10d Majorana-Weyl spinor dimensionally reduced to 4d. We can also add the topological term

$$S_{\mathrm{top}}=\frac{i\theta}{32\pi^2}\int d^4x\ \epsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}\big(F_{\mu\nu}F_{\rho\sigma}\big)$$

The action is invariant under the supersymmetry transformation

$$\begin{aligned}\delta X^m&=-\bar{\epsilon}\Gamma^m\lambda\\\delta A^\mu&=-\bar{\epsilon}\Gamma^\mu\lambda\\\delta\lambda&=\Big(\frac{1}{2}F_{\mu\nu}\Gamma^{\mu\nu}+D_\mu X_m\Gamma^{\mu m}+\frac{i}{2}[X_m,X_n]\Big)\epsilon\,.\end{aligned}\tag{16.8}$$

It is easy to see that 4d $\mathcal{N}=4$ SYM is classically **conformal invariant**. because the mass dimensions of the fields

$$[A_\mu]=[X^i]=1\,,\qquad[\lambda_a]=\frac{3}{2}\,,\tag{16.9}$$

so that the coupling constant is dimensionless: $[g]=[\theta]=0$. However, one has to be careful at quantum level because quantum correction generally breaks the conformal invariance. To be conformal invariant at quantum level, the beta function of the coupling constant has to vanish $\beta=0$. It turns out that 4d $\mathcal{N}=4$ SYM is the case and hence it is quantum mechanically conformal. The $\mathcal{N}=4$ supersymmetry combined with conformal symmetry forms the superconformal group SU$(2,2|4)$ which consists of the following generators

- **Conformal Symmetry** is $\mathrm{SO}(2,4) \cong \mathrm{SU}(2,2)$ in $d=4$, as we have seen in §16.1. The generators consist of translations P^μ, Lorentz transformations $L_{\mu\nu}$, dilations D and special conformal transformations K^μ with the relations (16.2);
- **R-symmetry** is $\mathrm{SO}(6)_R \cong \mathrm{SU}(4)_R$ which is manifest from the 10d viewpoint, and R-symmetry rotates the 6 scalar X^m ($m=1,\ldots,6$);
- **Poincaré supersymmetries** are generated by the supercharges $Q^I_\alpha, \overline{Q}^I_{\dot\alpha}$, $I=1,\ldots,4$ that transform under the $\mathbf{4}$ of $\mathrm{SU}(4)_R$. They can be understood as a "square root" of P_μ. Type IIB string theory has 32 supercharges and D3-branes break a half of the supersymmetries. Consequently, the 16 preserved supercharges are indeed $Q^I_\alpha, \overline{Q}^I_{\dot\alpha}$, which form $\mathcal{N}=4$ Poincaré supersymmetry;
- **Conformal supersymmetries**: are generated by the fermionic generators S^I_α and $\overline{S}^I_{\dot\alpha}$ that are superconformal partners of $Q^I_\alpha, \overline{Q}^I_{\dot\alpha}$. They can be understood as a "square root" of K_μ.

Therefore, there are 32 supercharges $Q, \overline{Q}, S, \overline{S}$ in total and they obey the anti-commutation relations

$$\begin{aligned}
&\{Q^\alpha_A, \overline{Q}^{\dot\alpha B}\} = P^{\alpha\dot\alpha}\delta^B_A, \\
&\{S^A_\alpha, \overline{S}_{\dot\alpha B}\} = K_{\alpha\dot\alpha}\delta^A_B, \\
&\{S^A_\alpha, Q^\beta_B\} = \delta^A_B M_\alpha{}^\beta + \delta^\beta_\alpha R^A_B + \delta^A_B\delta^\beta_\alpha \frac{D}{2}, \\
&\{\overline{S}_{\dot\alpha A}, \overline{Q}^{\dot\beta B}\} = \delta^B_A \overline{M}^{\dot\beta}{}_{\dot\alpha} - \delta^{\dot\beta}_{\dot\alpha} R^B_A + \delta^B_A\delta^{\dot\beta}_{\dot\alpha}\frac{D}{2}\,.
\end{aligned} \tag{16.10}$$

The $\mathcal{N}=4$ SYM enjoys **S-duality** [GNO77, MO77] that is the $\mathrm{SL}(2,\mathbb{Z})$ action on the complexified coupling constant $\tau \equiv \frac{\theta}{2\pi} + \frac{4\pi i}{g^2_{YM}}$

$$\tau \to \frac{a\tau+b}{c\tau+d} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})\,. \tag{16.11}$$

The $\mathcal{N}=4$ SYM is the world-volume theory on a stack of D3-branes, and D3-branes are invariant under the $\mathrm{SL}(2,\mathbb{Z})$ symmetry of Type IIB theory as seen in §13.1. In fact, this is the origin of the $\mathrm{SL}(2,\mathbb{Z})$ symmetry (16.11).

There is another way to realize the $\mathcal{N}=4$ SYM from string theory. Indeed, the M5-branes wrapped on a torus with complex structure τ give rise to the $\mathcal{N}=4$ SYM with the complexified coupling constant τ. Here, τ manifestly admits a geometric origin as the complex structure of a torus. Note that when $\theta=0$, the S-duality transformation amounts to $g_{YM} \to 1/g_{YM}$, thereby exchanging strong and weak coupling.[8]

Near-horizon geometry of D3-branes

Now let us study the closed string side of the system. A system of N coincident D3-branes is a classical solution of the low-energy string effective action:

$$ds^2 = H(y)^{-\frac{1}{2}}\eta_{\mu\nu}dx^\mu dx^\nu + H(y)^{\frac{1}{2}}(dy^2 + y^2 d\Omega_5^2) \tag{16.12}$$

[8]Precisely speaking, the electromagnetic duality of the $\mathcal{N}=4$ SYM depends on a choice of gauge groups, and the duality group is usually a congruence subgroup of $\mathrm{SL}(2,\mathbb{Z})$. For more detail, we refer to [AST13].

with R-R field

$$C_{(4)} = H(y)^{-1} dx^0 \wedge \cdots \wedge dx^3$$

where

$$H(y) = 1 + \frac{R^4}{y^4}\,, \qquad R^4 = 4\pi g_s N(\alpha')^2\,. \tag{16.13}$$

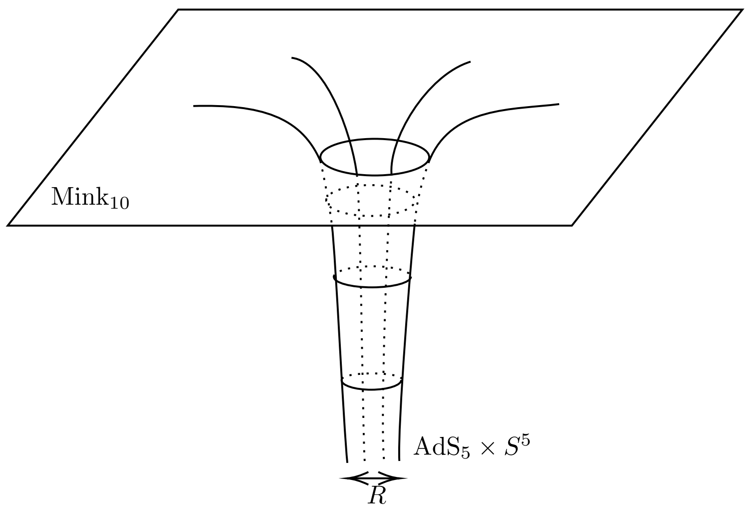


Figure 47: Minkowski region and near-horizon region

To study this geometry more closely, we consider its limit in two regimes. As $y \gg R$, we recover flat spacetime $\mathbb{R}^{1,9}$. When $y < R$, the geometry is often referred to as the **throat** and would at first appear to be singular as $y \ll R$. More precisely, the near-horizon geometry becomes apparent in the region

$$y \to 0 \qquad\qquad \alpha' \to 0 \qquad\qquad u \equiv y/R^2 \tag{16.14}$$

in which also the Regge slope is taken to zero, while u is kept fixed. In this limit, we can neglect the factor 1 in the function $H(y)$ in (16.13) and the metric in (16.12) becomes:

$$ds^2 = R^2\left[u^2 \eta_{\mu\nu} dx^\mu dx^\nu + \frac{du^2}{u^2} + d\Omega_5^2\right] \tag{16.15}$$

As we have seen in (16.5), the first part of the metric is AdS$_5$, and the other part is S^5. In conclusion, the geometry close to the brane ($y \sim 0$ or $u \sim 0$) is regular and highly symmetrical AdS$_5 \times S^5$ with the same radii

$$R^2_{\mathrm{AdS}_5} = R^2_{S^5} = \alpha' \sqrt{4\pi N g_s}\,.$$

In the limit (16.14), only the AdS region of the D3-brane geometry survives while the dynamics in the asymptotically flat region decouples from the theory. Furthermore, it turns out that the interaction between bulk and brane dynamics becomes negligible. Therefore, it is called the **decoupling limit**.

The AdS/CFT correspondence

As mentioned, the world volume theory of N coincident D3-branes is 4d $\mathcal{N}=4$ SYM with $\mathrm{U}(N)$ gauge group. On the other hand, the classical solution in (16.15) is a good approximation when the radii of AdS$_5$ and S^5 are very big:

$$\frac{R^2}{\alpha'} \gg 1 \implies N g_{YM}^2 \equiv \lambda \gg 1 \tag{16.16}$$

The fact that those two descriptions are simultaneously consistent for large values of the coupling constant λ brought Maldacena to formulate the conjecture that the strongly interacting $\mathcal{N}=4$ SYM with gauge group $\mathrm{U}(N)$ at large N is equivalent to Type IIB supergravity compactified on AdS$_5 \times S^5$. However, supergravity is not a consistent quantum theory and it is just a low-energy effective theory of string theory. Hence, the natural way to extend the equivalence at any value of λ is therefore that $\mathcal{N}=4$ SYM is equivalent to Type IIB string theory on AdS$_5 \times S^5$ [Mal99]. Namely, the following two theories are dual to each other:

- $\mathcal{N}=4$ super-Yang-Mills theory in 4-dimensions with gauge group $\mathrm{U}(N)$.
- Type IIB superstring theory on AdS$_5 \times S^5$ with the same radius R as in (16.16), where the 5-form G_5^+ has integer flux $N = \int_{S^5} G_5^+$ on S^5.

The coupling constants in the two theories are related by $g_s = g_{YM}^2$. The precise formulation of this duality will follow in the next subsection. In these two theories, we can immediately find the following correspondence as in Table 9.

4d $\mathcal{N}=4$ SYM	Type IIB on AdS$_5 \times S^5$
32 supercharges	32 supercharges
$\mathrm{SO}(2,4)$ conformal group	$\mathrm{SO}(2,4)$ isometry of AdS$_5$
$\mathrm{SU}(4)_R$ symmetry	$\mathrm{SO}(6)$ isometry of S^5
SL(2,$\mathbb{Z}$) symmetry of coupling constants	SL(2,$\mathbb{Z}$) symmetry of axio-dilaton

Table 9: Dictionary for the AdS$_5$/CFT$_4$ correspondence

This conjecture is the most general statement, which is valid at any values of coupling constant $g_s = g_{YM}^2$ and rank N. However, it is still difficult to quantize string theory at any value of g_s on a general manifold including asymptotic AdS space. Hence, it is still an open problem to prove this general statement of the conjecture. Nevertheless, taking various limits of the conjecture, we can show a variety of non-trivial evidence for the conjecture, providing new physical insight.

The 't Hooft Limit: The 't Hooft limit [tH74] is the limit in which we keep the **'t Hooft coupling** $\lambda \equiv g_{YM}^2 N = g_s N$ fixed and letting $N \to \infty$, $g_s \to 0$. As in Figure 48, the planar diagrams become dominant in this limit on the Yang-Mills side. On the AdS side, since the string coupling can be re-expressed in terms of the 't Hooft coupling as $g_s = \lambda/N$, the 't Hooft limit corresponds to the regime where weak coupling string perturbation theory is valid. Put differently, this limit of the AdS/CFT correspondence can be understood as the incarnation of the idea of 't Hooft [tH74].

Supergravity limit: While we take $N \to \infty$, in the regime that 't Hooft coupling is large $\lambda = g_s N \gg 1$, supergravity description becomes reliable. On the gauge theory side,

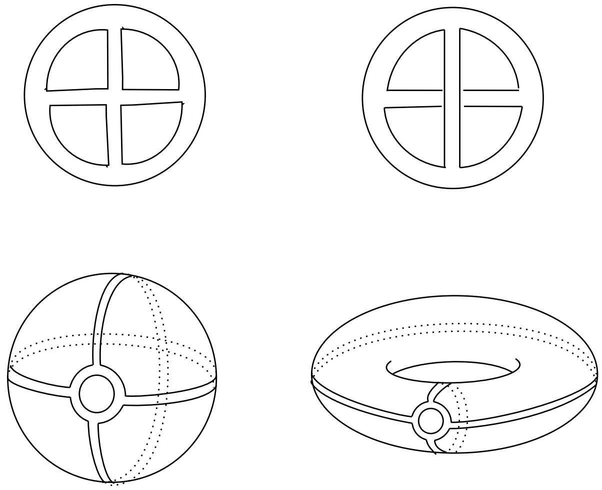


Figure 48: Planar and non-planar diagram

the theory is strongly-coupled so that perturbation techniques cannot be applied. The two theories are conjectured to be the same, but when one side is weakly coupled, the other is strongly coupled. This is a salient feature of **duality**. Thus, using the AdS/CFT correspondence, analyses of supergravity provide new insights to strongly-coupled SYM theory, such as quark confinement and mass gap.

16.4 GKPW relation

Soon after Maldacena's proposal [Mal99], a more precise formulation was given in [GKP98, Wit98a]. The gravitational partition function on asymptotically AdS space is equal to the generating function of correlation functions of the corresponding CFT:

$$Z_{\text{grav}}[\phi \to \phi_0] = \left\langle \exp\left(\int_{\partial AdS} \overline{\phi}_0 \mathcal{O} \right) \right\rangle_{\text{CFT}} \tag{16.17}$$

that is called the **GKPW** relation. For any bulk field ϕ in gravity theory on AdS, there exists the corresponding operator $\mathcal{O}$ in the CFT. The gravitational partition function can be schematically written as

$$Z_{\text{grav}}[\phi \to \phi_0] = \int_{\phi \to \phi_0} \mathcal{D}\phi \, e^{-S_{\text{string}}[\phi]} \, .$$

For instance, in the regime $\lambda \gg 1$, we can use supergravity description

$$Z_{\text{grav}}[\phi \to \phi_0] = \sum_{\text{saddle point}} e^{-S_{\text{SUGRA}}[\phi \to \phi_0]} \, .$$

Bulk field/boundary operator

The GKPW relation tells us that each field propagating in the bulk AdS space is in one-to-one correspondence with an operator in CFT. Also, the spin of the bulk field is equal to that of the CFT operator. Moreover, the mass of the bulk field fixes the scaling dimension of the CFT operator. Here are some examples:

- By definition, every gravitational theory has the graviton $g_{\mu\nu}$, a massless spin-2 particle. The dual operator must be the universal one with spin-2 in CFT. In fact, there is a natural candidate for it: the energy-momentum tensor $T_{\mu\nu}$ in CFT. The fact that the graviton is massless corresponds to the fact that the CFT stress tensor is conserved.
- If our theory of gravity has a spin-1 vector field A_μ, then the dual operator is also a spin-1 operator J_μ in CFT. In particular, if A_μ is a gauge field, then J_μ is a conserved current. In fact, the GKPW relation provides the natural coupling $e^{i\int A_\mu J^\mu}$. This implies that gauge symmetries in the bulk correspond to global symmetries in the CFT.
- A bulk scalar field is dual to a scalar operator in the CFT. The boundary value of the bulk scalar field acts as a source in the CFT.

There is a relation between the mass of the field ϕ and the scaling dimension of the corresponding operator in CFT. If we write the AdS$_{d+1}$ metric as in (16.6) the wave equation (for instance, Klein-Gordon equation for the scalar field) in the AdS$_{d+1}$ space for a field of mass m has two independent solutions, which behave like $z^{d-\Delta}$ (**non-normalizable**) and z^{Δ} (**normalizable**) for small z (close to the boundary of AdS) where

$$\Delta = \frac{d}{2} + \sqrt{\frac{d^2}{4} + R^2m^2}\,. \tag{16.18}$$

Therefore, as the massive field approaches the boundary $\epsilon \to 0$, the field on the right-hand side of (16.17) behaves as

$$\phi(\vec{x}, \epsilon) = \epsilon^{d-\Delta}\phi_0(\vec{x})\,. \tag{16.19}$$

This means that ϕ_0 has conformal dimension $d-\Delta$ so that the dual operator $\mathcal{O}$ in (16.17) has conformal dimension Δ (16.18). This is consistent with the fact that the radial direction z in (16.6) of the bulk AdS space corresponds to the scaling of the boundary CFT.

Examples

Let us now consider this correspondence in a massless scalar field in AdS$_5 \times S^5$. The Klein-Gordon equation of the massless field in AdS$_5 \times S^5$ can be written as

$$\nabla^2\phi = \nabla^2_{\mathrm{AdS}_5} + \nabla^2_{S^5}\phi = 0\,.$$

The eigenfunctions of the Laplacian $\nabla^2_{S^5}$ on a sphere are known as the spherical harmonics $Y_l(\Omega)$ (like the theory of angular momenta in quantum mechanics) so that

$$\nabla^2_{S^5} Y_l(\Omega) = -\frac{l(l+4)}{R^2} Y_l(\Omega), \quad l = 0, 1, 2, \ldots\,. \tag{16.20}$$

Writing the ten-dimensional field $\phi = \sum_l \phi_l Y_l$, the AdS$_5$ fields ϕ_l satisfy the massive Klein-Gordon equation

$$\nabla^2_{\mathrm{AdS}_5}\phi_l = m_l^2\phi_l\,, \qquad m_l^2 = \frac{l(l+4)}{R^2}\,. \tag{16.21}$$

This can be understood that the compactification on S^5 leads to a tower of the field with Kaluza-Klein (KK) masses m_l. The AdS/CFT correspondence predicts that there exist operators in the $d=4$ $\mathcal{N}=4$ SYM dual to these fields.

To see that, we read off the conformal dimension of the corresponding operator in $d = 4$ from (16.18)

$$\Delta_l = 2 + \sqrt{4 + (m_l R)^2} = 2 + \sqrt{4 + l(l+4)} = 4 + l\,. \tag{16.22}$$

First, we consider a massless scalar $l = 0$ in AdS$_5$ where the conformal dimension of the dual operator is $\Delta_0 = 4$. Also, the spherical harmonics (16.20) at $l = 0$ corresponds to the s-wave, which is transformed trivially under the SO(6) symmetry. Therefore, the dual operators are also singlet under the SU(4) R-symmetry so that it does not contain the scalars X_i of the $d = 4$ $\mathcal{N} = 4$ SYM. Consequently, the only operator with these properties is the gauge-invariant glueball operator

$$\mathcal{O} = \mathrm{Tr}\left[F_{\mu\nu}F^{\mu\nu}\right]\,.$$

Note that the conformal dimension is $\Delta = 4$ since dim$[\partial]$=dim$[A] = 1$. For higher KK modes $l > 0$, the dual operator transforms non-trivially under the SU(4) R-symmetry so that it involves the scalar X_i. The natural candidate dual to the l^{th} KK mode is

$$\mathcal{O}_{i_1,\ldots,i_l} = \mathrm{Tr}\left[X_{(i_1,\ldots,i_l)}F_{\mu\nu}F^{\mu\nu}\right]\,, \tag{16.23}$$

where $X_{(i_1,\ldots,i_l)}$ being the traceless symmetric product of l scalar fields X_i of the $\mathcal{N} = 4$ SYM. It is easy to see that the conformal dimension of this operator is $4 + l$, which is consistent with (16.22). In fact, the field/operator matching of this kind has been extended to all the fields of 10d supergravity on AdS$_5 \times S^5$.

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