Homework 4: Due at class on Oct 19

1 Modular invariant

Show that (4.98) is modular S-invariant by using the Poisson resummation formula (4.99).

2 Modular transformation

The modular group is defined by

$$\mathrm{SL}(2, \mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \quad a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1 \right\} .$$

Show that

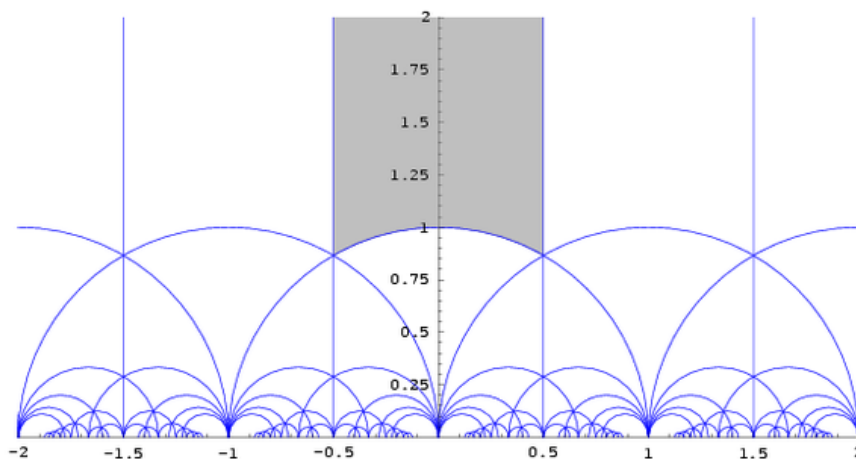$$\mathrm{Im}(\gamma \cdot \tau) = \frac{\mathrm{Im}\,\tau}{|c\tau + d|^2} ,$$

where

$$\gamma : \tau \mapsto \gamma \cdot \tau = \frac{a\tau + b}{c\tau + d} .$$

Therefore, if $\mathrm{Im}\,\tau > 0$, then $\mathrm{Im}(\gamma \cdot \tau) > 0$ so that the upper half-plane

$$\mathbb{H} = \{x + iy \mid y > 0; x, y \in \mathbb{R}\}$$

receives the action of the modular group. Express elements of modular group that maps the gray region to blue, red and green region in terms of modular S and T transformations

$$T : \tau \to \tau + 1 , \qquad S : \tau \to -1/\tau .$$

3 Boson-Fermion correspondence

We have learnt that the OPE of the free boson is

$$\varphi(z)\varphi(0) \sim -\ln z \ .$$

Let us also consider two Majorana-Weyl fermions ψ^1, ψ^2 with OPE

$$\psi^i(z)\psi^j(0) \sim \frac{\delta^{ij}}{z} \ .$$

We can define the complex fermion

$$\psi(z) = 2^{-1/2}(\psi^1(z) + i\psi^2(z)), \qquad \overline{\psi}(z) = 2^{-1/2}(\psi^1(z) - i\psi^2(z)) \ .$$

Show the equivalence of operators in boson and fermion

$$: e^{i\varphi} : \cong \psi, \quad : e^{-i\varphi} : \cong \overline{\psi}, \quad i\partial\varphi \cong \psi\overline{\psi}, \quad T_\varphi \cong T_\psi \ ,$$

by calculating the OPEs of operators in both theories and comparing. Note that the energy-momentum tensor of the complex fermion is

$$T_\psi := -\frac{1}{2} : \psi\partial\overline{\psi} : -\frac{1}{2} : \overline{\psi}\partial\psi : \ .$$