Homework 13: Due at Dec 25

Prob. 1 DBI action

DBI action for Dp-brane is given by

$$S_{\mathrm{DBI}} = \int d^{p+1}\sigma \mathcal{L}_{\mathrm{DBI}} \ , \tag{1}$$

$$\mathcal{L}_{\mathrm{DBI}} = T_{\mathrm{D}p} - T_{\mathrm{D}p}\sqrt{-\det\left(G_{ab} + kF_{ab} - B_{ab}\right)} \ , \tag{2}$$

where we assumed the world-sheet metric h_{ab} is flat (i.e. $h_{ab} = \eta_{ab} = \mathrm{diag}(-1, +1, \ldots, +1)$), the first constant term is added so that the vacuum energy becomes zero, k is $2\pi\alpha' = 2\pi l_s^2$, $T_{\mathrm{D}p} = \frac{2\pi}{g_s(2\pi l_s)^{p+1}}$, and

$$G_{ab} = \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} \ , \tag{3}$$

namely, we assume the space-time is flat. In the following we always assume $B_{ab} = 0$.

Static gauge is defined by $X^i = \sigma^i$ for $(i = 0, 1, \cdots, p)$, and rewrite $X^s = \phi^s(\sigma^i)$ for $(s = p+1, \cdots, 9)$. Displacement field is defined by $\boldsymbol{D} = \frac{\partial \mathcal{L}}{\partial \boldsymbol{E}}$, where $\boldsymbol{E} = F_{i0}$ $(i = 1, \cdots, p)$.

Prob. 1.1 Gauge theory on D-brane

Consider D3-brane and take the static gauge.

- Expand the Lagrangian (2) up to a quadratic order of fields and write down a canonical Lagrangian.
- Compare the coupling constant derived from the expansion and the canonical one g:

$$\mathcal{L}_{\mathrm{gauge}} = -\frac{1}{4g^2} F_{ab}F^{ab} \ . \tag{4}$$

Prob. 1.2 Features of the DBI action 1: Schwinger effect

Consider Dp-brane, take the static gauge, set $\phi^s = 0$, and also set the magnetic field to zero; $\boldsymbol{B} = \boldsymbol{0}$.

- Calculate the displacement field $\boldsymbol{D} = \frac{\partial \mathcal{L}_{\mathrm{DBI}}}{\partial \boldsymbol{E}}$.
- Assume only $F_{10} = E$ has non-zero value and plot E in terms of D that is a corresponding non-zero component of the displacement field. Explain physically why there are bounds for E. (This is called **Schwinger effect**.)

Prob. 1.3 Features of the DBI action 2: Self-energy

Consider the same setting as Prob. 1.2 with $p = 3$.

- Calculate the self-energy U_Q of a charged particle:

$$U_Q = \int d^3\sigma \mathcal{H}(\boldsymbol{D}) \ , \qquad \mathcal{H}(\boldsymbol{D}) = \boldsymbol{E}\cdot\boldsymbol{D} - \mathcal{L}_{\mathrm{DBI}} \ , \tag{5}$$

where $d^3\sigma$ is a spacial(not time) volume form. The charged particle is characterized by

$$\boldsymbol{\nabla} \cdot \boldsymbol{D} = Q\delta^3(\boldsymbol{\sigma}) \ , \tag{6}$$

where $\boldsymbol{\sigma}$ is a world-sheet space(not time) vector. You can use the following constant:

$$c \equiv \int_0^\infty dx \left(\sqrt{1+x^4} - x^2\right) \simeq 1.236 \ . \tag{7}$$

- Compare the self-energy with that of a normal electromagnetism:

$$\mathcal{L}_{\rm EM} = \frac{\varepsilon}{2}\boldsymbol{E}^2 \ . \tag{8}$$

Discuss also $l_s \to 0$ limit in the self-energy.

Prob. 1.4 Brane bending

Consider Dp-brane, take the static gauge, set $\boldsymbol{B} = \boldsymbol{0}$. Assume only one of ϕ^s has non-zero value, say $\phi^9 \equiv X$. Then, the Lagrangian becomes

$$\widetilde{\mathcal{L}}_{\rm DBI} = T_{{\rm D}p} - T_{{\rm D}p}\sqrt{\left(1-\boldsymbol{\mathcal{E}}^2\right)(1+(\boldsymbol{\nabla}X)^2) + \left(\boldsymbol{\mathcal{E}}\cdot\boldsymbol{\nabla}X\right)^2 - \dot{X}^2} \ , \tag{9}$$

where $\boldsymbol{\mathcal{E}} = k\boldsymbol{E}$, $\dot{X} = \frac{\partial X}{\partial \sigma^0}$.

a) Show $\widetilde{\mathcal{L}}_{\rm DBI} = \mathcal{L}_{\rm DBI}$ in the case that only one of components of $\boldsymbol{E}$ is non-zero, say E_1, and X only depends on σ^0, σ^1, and σ^2. (This special case implies full identity $\widetilde{\mathcal{L}}_{\rm DBI} = \mathcal{L}_{\rm DBI}$ thanks to rotation symmetry.)

Let us look for a static solution, namely, all the fields are independent of σ^0. Then, we can express $\boldsymbol{E}$ by $\boldsymbol{E} = \boldsymbol{\nabla}A_0$.

b) Derive equations of motion of X and A_0. Confirm that both equations are satisfied if

$$k\boldsymbol{E} = k\boldsymbol{\nabla}A_0 = \pm\boldsymbol{\nabla}X \ , \qquad k\boldsymbol{\nabla}^2 A_0 = \boldsymbol{\nabla}^2 X = 0 \ . \tag{10}$$

c) Show that when (10) holds we have

$$\boldsymbol{D} = k^2 T_{{\rm D}p}\boldsymbol{E} \ , \tag{11}$$

and the energy

$$U = \int d^p\sigma \left(\boldsymbol{E}\cdot\boldsymbol{D} - \widetilde{\mathcal{L}}_{\rm DBI}\right) = k^2 T_{{\rm D}p}\int d^p\sigma \boldsymbol{E}^2 \ . \tag{12}$$

Consider a string is ending on the Dp-brane (stretching to the 9th direction), which is a source of the electric (displacement) field

$$\boldsymbol{\nabla}\cdot\boldsymbol{D} = \delta^p(\boldsymbol{\sigma}) \ . \tag{13}$$

d) Derive the solution for $\boldsymbol{D}$ as well as X and show that the energy U is diverging.

Let us regularize the energy by a cut off δ:

$$U(\delta) = k^2 T_{{\rm D}p}\int_{r>\delta} d^p\sigma \boldsymbol{E}^2 \ . \tag{14}$$

e) Show that $U(\delta) = T_{\rm str}|X(\delta)| = \frac{1}{k}|X(\delta)|$. Draw the shape of the Dp-brane, and discuss the meaning of $U(\delta)$.

Prob. 2 Bound states and supersymmetry

In type II theories we have left 16 supercharges Q_L and right 16 supercharges Q_R. Existence of string or branes break some of supersymmetry. Conditions are summarized in Table 1.

Table 1: Supersymmetry conditions for branes.

IIA branes	Conditions
F1-String(01)	$\Gamma^{01}Q_L = Q_L \ , \quad \Gamma^{01}Q_R = -Q_R$
NS5-brane(012345)	$\Gamma^{012345}Q_L = Q_L \ , \quad \Gamma^{012345}Q_R = Q_R$
Dp-brane(p: even)	$\Gamma^{01\cdots p}Q_R = Q_L$

IIB branes	Conditions
F1-String(01)	$\Gamma^{01}Q_L = Q_L \ , \quad \Gamma^{01}Q_R = -Q_R$
NS5-brane(012345)	$\Gamma^{012345}Q_L = Q_L \ , \quad \Gamma^{012345}Q_R = -Q_R$
Dp-brane(p: odd)	$\Gamma^{01\cdots p}Q_R = Q_L$

- How many supersymmetry remain if there exist Dp-brane ? (We call this Dp system and similar for multi branes.)
- How many supersymmetry remain for D0-D2 system ? D0 lies on (0) direction and D2 lies on (012) directions.
- How many supersymmetry remain for F1-D1 system ? Both string lie on (01) directions.
- How many supersymmetry remain for D0-D4 system ? D0 lies on (0) direction and D4 lies on (01234) directions.
- Show that D4-brane carries D0-brane RR-charge when there exist non-trivial instanton configuration ($n \neq 0$):
$$\frac{1}{8\pi^2}\int F_{(2)} \wedge F_{(2)} = n \in \mathbb{Z} \ . \tag{15}$$

Prob. 3 Branes ending on a brane in IIB superstring

Start from equations of motions of IIB SUGRA:

$$dG_{\text{odd}} = H_{(3)}G_{\text{odd}} \ , \tag{16}$$
$$dH_{(3)} = 0 \ , \tag{17}$$
$$dH_{(7)} = -\frac{1}{2}\left[(\mathcal{T}G_{\text{odd}})G_{\text{odd}}\right]_{(8)} \ , \tag{18}$$

introduce brane currents and construct current conservation equations. From the equations make a table of branes, which can end on some branes. Is the table compatible with S-duality ?