Homework 1: Due at class on March 5

1. Euler Characteristic

1. Find the Euler characteristic of a surface of genus g (Figure 1).

2. Find the Euler characteristic of a Klein bottle (Figure 2).

3. Find the area of the following triangle (Figure 4) bounded by great circles on a 2-sphere with radius one.

4. Generalize it to the area of n-gon on a 2-sphere with radius one. (An example of a pentagon is drawn in Figure 5.)

5. Prove the Euler characteristic of a 2-sphere is equal to two by using a cell decomposition (Figure 3).
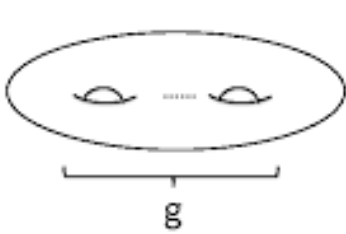

Figure 1: A surface of genus g
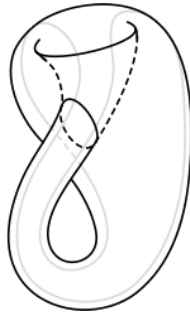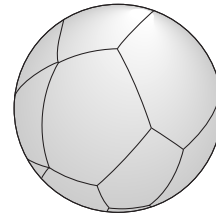

Figure 2: A Klein bottle
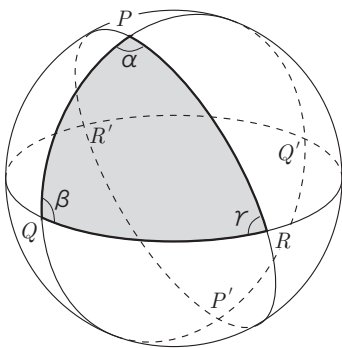

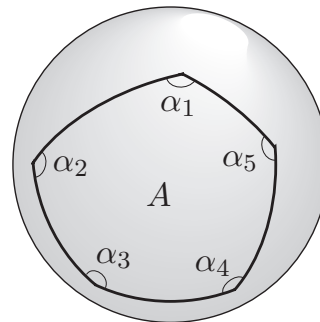Figure 3: cell decomposition of a 2-sphere

Figure 4: A triangle on a 2-sphere

Figure 5: A pentagon on a 2-sphere

2. For elements $x = (x^0, \cdots, x^n)$ and $y = (y^0, \cdots, y^n)$ of $\mathbb{R}^{n+1} \backslash \{0\}$, we define an equivalence relation $x \sim y$ by

$$x = \alpha y \qquad \forall \alpha \in \mathbb{R} .$$

Let us define $\mathbb{R}P^n$ by $(\mathbb{R}^{n+1}\backslash\{0\})/\sim$. Show that $\mathbb{R}P^n$ is a manifold and $\mathbb{R}P^1$ is diffeomorphic to S^1. The space is called a real projective space.

3. For elements $x = (x^0, \cdots, x^n)$ and $y = (y^0, \cdots, y^n)$ of $\mathbb{C}^{n+1}\backslash\{0\}$, we define an equivalence relation $x \sim y$ by

$$x = \alpha y \qquad \forall \alpha \in \mathbb{C}.$$

Let us define $\mathbb{C}P^n$ by $(\mathbb{C}^{n+1}\backslash\{0\})/\sim$. Show that $\mathbb{C}P^n$ is a manifold and $\mathbb{C}P^1$ is diffeomorphic to S^2. The space is called a complex projective space.