復旦大學

本科毕业论文

论文题目：Modular Tensor Category from Argyres-Douglas Theory

阿盖尔斯-道格拉斯理论的模张量范畴

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摘 要

本文首先在第一部分对共形场论与重整化群流、超对称理论、以及李代数的邓金图与嘉当分类做一简略介绍，为后文对阿盖尔斯-道格拉斯理论等的介绍做出准备。但对于再前置的量子场论、庞加莱 (Poinc'are) 对称与李群李代数基本知识不再赘述。

共形场论 (conformal theory, CFT) 也就是有共形对称的场论；在传统意义下，也即可以理解为保角变换——仅考虑局部的意义下，一个夹角经过变换并不改变角度；用度规张量 (metric tensor) 来理解，就是度规张量在这里为不变量。共形变换形成的共形群共可有四类元素生成：移动 (translation)、放缩 (dilation)、旋转 (rotation)，还有一个特殊共形变换 (SCT)。

在共形场论中，重整化群流 (renormalization group flow; RG flow) 可以视为一条能量沿着它逐渐降低，最后走到系统红外 (infrared, IR，即低能) 极限的路径。它从紫外 (ultraviolet, UV，即高能) 一边出发，通过对系统施加的一定的畸形 (deformation)，而达到红外不动点 (fixed point)，在本文中这样的畸形是真空期待值 (vacuum expectation value, vev)。不动点即是一个对于重整化群流稳定的点；对于非共形的一个理论，可对它作重整化群流，直到它撞到不动点上，之后再受重整化群流作用，也一直待在不动点，也就是成为了共形场论。

超对称 (Supersymmetry, SUSY) 在理论物理如今被寄予厚望。超对称理论当中的超荷，一共有 $4\mathcal{N}$ 种；其中 $\mathcal{N}$ 为自旋荷 (spinor charge) 数。当 $\mathcal{N}$ 为 1 时，为普通超对称；大于等于 2 时，为扩展超对称 (extended supersymmetry)。文中简单介绍了超对称理论中的代数，以及从超对称理论的拉格朗日量中提取出来的超势 (superpotential)。

对半单 (semisimple) 李代数 (Lie algebra) 作邓金图 (Dynkin diagram)，可以推出对于半单李代数的卡坦分类 (Cartan classification)。每个半单李代数对应一个邓金图，并且，通过对对合法邓肯图的尝试，我们会发现能将所有邓金图（也就是相当于李代数）划分为四个系列（A_n、B_n、C_n、D_n）与五个额外李代数（$E_{6,7,8}$、F_4、G_2）当中的一个。其中，A、D、E 类称为单线（simply laced），因为其邓金图不含重叠二线与三项，只有单线。

阿盖尔斯-道格拉斯 (Argyres-Douglas, AD) 理论[1] 是一种 $\mathcal{N}=2$ 超共形场论 (SCFT)。一般地，由于其中的互相作用，以及库伦分支 (Coulomb branch) 中具有的

特殊轨迹 (loci) 等原因，阿盖尔斯-道格拉斯理论不具有拉格朗日量 (Lagrangian) 的描述。不过，它可以作为一个简单最小化模型（李-杨模型 Lee-Yang Model）的红外不动点；在诸 $\mathcal{N}=2$ 超共形场论中，中心荷 c 在阿盖尔斯-道格拉斯理论这儿取为最小。这表示阿盖尔斯-道格拉斯理论就是一种 $\mathcal{N}=2$ 最小化超共形场论，于是，它可以作为某些重整化群流的终点。我们可以通过重整化群流，从带有拉格朗日量描述的理论出发，指向阿盖尔斯-道格拉斯理论，并保证所涉量在重整化群流之下不变，我们就可以说得到了阿盖尔斯-道格拉斯理论的拉格朗日量描述。

阿盖尔斯-道格拉斯理论可以使用给对应李代数卡坦分类的方法。以及，被[2]加以推广的阿盖尔斯-道格拉斯理论可以包含 (A_1,G), 其中 G 是单线的半单李群。在[3] 还介绍了 H_0 H_1 H_2 系列阿盖尔斯-道格拉斯理论与流向其的部分理论。

在本文中，我们从 $\mathcal{N}=1$ 的拉格朗日量理论出发，向带有非阿贝尔 (non-Abelian) 味对称 (flavor symmetry) 的红外不动点做重整化群流。然后，加入一个 $\mathcal{N}=1$ 手性多重态，变形为这个味对称的伴随表示，通过超势的组合，给出一个真空期待值；这样，就可以定义出一个红外不动点的理论。这样就有了一个从 $\mathcal{N}=1$ 的拉格朗日量理论到 $\mathcal{N}=2$ 阿盖尔斯-道格拉斯理论的流。

[3, 4] 通过 a 中心荷最小化 (a-maximization) 等重要性质，推算出了红外不动点上阿盖尔斯-道格拉斯理论的一些数值 (matter content)，包括各种在理论中会展现到的元素及其在各种对称性之下展现的荷。这两篇文章对于 (A_1,A_{2N-1})、(A_1,A_{2N})、(A_1,D_{2N}) 和 (A_1,D_{2N+1}) 这四个 (A_1,G) 型阿盖尔斯-道格拉斯理论系列已经列出了结果，利用这些表格，我们可以去研究阿盖尔斯-道格拉斯理论的一些性质。

[5] 介绍的 A-模型，以及它具有的两种势，解对应的贝特拟设方程 (Bethe ansatz equation)，可以解出贝特真空 (Bethe vacua)；并可以求出手柄胶合算子 (handle-gluing operator) 与纤维化算子 (fibering operator)。

模张量范畴 (modular tensor category) 大致说来，是一种将有理二维共形场论的拓扑结构编码的范畴。通过这种范畴，可以在某物上定义 SL(2,$\mathbb{Z}$) 群向它的投影。在 SL(2,$\mathbb{Z}$) 中，S 和 T 矩阵有其代数性质；它们可以附在 $\mathcal{N}=2$ 阿盖尔斯-道格拉斯理论上，在模张量范畴中，S 和 T 矩阵是模数群的生成元，与 SL(2,$\mathbb{Z}$) 的 S，T 矩阵相对应；它们分别可以被解释为手柄胶合算子与纤维化算子。它们仍然满足类似于在 SL(2,$\mathbb{Z}$) 中然而退化了的性质。它们不一定还是模为 1(unitary) 的；比如李-杨模型 (Lee-Yang model) 就是一种非模一的模张量范畴。

在本文中，对于阿盖尔斯-道格拉斯理论，我们设所在四维世界有一维为 S^1 环，其上可以作用模的对称性，可以用来形成一个绕着环的和乐群 (holonomy group)；这个理论还在剩余的三维空间内发生拓扑缠绕，体现其 R-对称性。

当和乐群与总体对称性 Z_N 互质时，这三维拓扑量子场论为半单；同时它与被和乐群与阿盖尔斯-道格拉斯理论决定的模张量范畴互相联系。这一模张量范畴可以由这样的几何性质或者其他一些性质都可决定。

考虑环绕拓扑划分函数 (twisted topological partition function)，这可以用来给相关的模张量范畴以编码，而这又装配了 S 与 T 矩阵，那么使用这两个矩阵，我们就可以方便地写出这四维流形上的拓扑划分方程。

在本文中我们尝试计算出前几个阿盖尔斯-道格拉斯理论：(A_1, A_2), (A_1, A_3), (A_1, D_3), (A_1, A_4), 与 (A_1, D_4) 的模张量范畴。方法是，通过已经由重整化群流被推出的理论的各荷数据，使用求贝特真空的方式与和乐鞍 (holonomy saddle)，算出它们的手柄胶合与纤维化算子。

计算的过程主要在[5]：我们通过和乐鞍方法[6]，分别用相互之间过程几乎相同，但是在重要参数 x 的赋值上已经不同的两种方式来计算；首先，对 x 解出贝特拟设方程；然后分别算出手柄胶合算子与纤维化算子，并将参数 x 代入贝特拟设方程的解，即贝特真空；计算过程中一直大量使用着通过向阿盖尔斯-道格拉斯理论的重整化群流所得到的数据成果。而计算时所用的参数所能有的存在取值的规定，同时有的参数但凡不取某个值，整个计算就很快无法继续下去，通过尝试，我们可以将这些具体的参数值代入。最后，将和乐鞍方法之下分别计算的两部分结果组合起来，得到手柄胶合算子以及纤维化算子的若干组解。

计算的结果是：

对 (A_1, A_2) 理论，手柄胶合算子 $\mathscr{H}_1 = \frac{5+\sqrt{5}}{2}$；$\mathscr{H}_2 = \frac{5-\sqrt{5}}{2}$，（对应的）纤维化算子 $\mathscr{F}_1 = e^{-\frac{i\pi}{30}}$；$\mathscr{F}_1 = e^{\frac{11i\pi}{30}}$。

对 (A_1, A_3) 理论，手柄胶合算子 $\mathscr{H}_1 = \mathscr{H}_2 = \mathscr{H}_3 = 3$，纤维化算子 $\mathscr{F}_1 = e^{\frac{i\pi}{2}}$，$\mathscr{F}_2 = \mathscr{F}_3 = e^{-\frac{i\pi}{6}}$。

对 (A_1, D_3) 理论，手柄胶合算子 $\mathscr{H}_1 = \mathscr{H}_2 = 3$，纤维化算子 $\mathscr{F}_1 = e^{-\frac{i\pi}{18}}$，$\mathscr{F}_2 = e^{\frac{4i\pi}{9}}$。

对 (A_1, D_4) 理论，手柄胶合算子 $\mathscr{H}_1 = 4$，纤维化算子 $\mathscr{F} = e^{\frac{2i\pi}{3}}$。

对于[7] 已经讨论的 (A_1, A_2) 与 (A_1, A_3) 理论，我们的计算与之相符。在这篇论文中，还完成了导出 S 与 T 矩阵的过程，并讨论了它们的特别性质：例如，这是最简单的的非模一的模张量范畴；并在最小 $\mathbb{Z}_5$ 和乐群下，能与 (2, 5) 维拉索罗最小化模型（即李-杨模型）联系。

对于 (A_1, D_3), (A_1, D_4), 计算出来的结果有些存疑，有可能并不正确，有可能是尚未发现或注意到其背后的结构与细节。

(A_1, A_4) 在计算时与前述几个阿盖尔斯-道格拉斯理论在各方面更加复杂；计算由同组一位学长完成，他同时计算出了其与 (2, 7) 维拉索罗最小化模型的联系。

我们发现，随着 (A_1,G) 型阿盖尔斯-道格拉斯理论中李代数 G 秩的升高，对

其手柄胶合与纤维化算子以及模张量范畴的计算越来越复杂，估计本文中通过重整化群流提供的数据的计算方法无法支撑更进一步对整个 (A_1,G) 型系列的计算，如何找到一个通用算法对整个 (A_1,G) 型适用也留下了一个问题。

关键字：理论物理；超对称；重整化群流；阿盖尔斯-道格拉斯理论；模张量范畴

中图分类号：

Abstract

Using the matter contents derived by RG flow from $\mathcal{N} = 1$ theories with Lagrangian description, and the procedure of bethe ansatz equation with holonomy saddle, we have done calculations of the modular tensor category, $\mathrm{SL}(2, \mathbb{Z})$ matrices from supersymmetric localization for several exact Argyres-Douglas theories: (A_1, A_2), (A_1, A_3), (A_1, D_3), (A_1, A_4), and (A_1, D_4),

Keywords: theoretical physics; supersymmetry; renormalization group flow; Argyres-Douglas Theory; modular tensor category

CLC number:

List of Symbols

a	A central charge
A_n	A series of Lie algebras (Cartan classification)
β	Callan-Symanzik beta function;
	Parameter associated with q
b	Charge of $U(1)_B$
C	Central element
c	A central charge
D_n	A series of Lie algebras (Cartan classification)
E_4	Euler density
η	Dedekind eta function
$\mathscr{F}$	Fibering operator
$g_{\mu\nu}$	Matric tensor
Γ_0	(A specialized) Elliptic gamma-function
H	Hessian of the twisted superpotential
$H_{0,1,2}$	A series of AD theories
$\mathscr{H}$	Handle-gluing operator
$\mathscr{N}$	Number of spinor charges
ν	Parameter associated with y
Ω	Dilaton
Π	Flux operator
$Q^A_\alpha, \overline{Q}_{\dot{\alpha}A}$	Supercharges
q	A fugacity
R	R-symmetry
S	S matrix (MTC)
$\square$	Single box
T	T matrix (MTC)
$\mathscr{T}_{\mathrm{IR}}$	Infrared fixed point theory
τ	Parameter associated with q
θ	Theta function
θ_0	Reduced theta function

u	Parameter associated with x
v	Parameter associated with b
$\mathscr{W}$	Superpotential;
	Weyl tensor
x	A fugacity
y	A fugacity
Z	Central charges;
	Topological partition function

Chapter 1
Preliminaries

In this part we briefly introduce some preliminary knowledge of conformal field theory (CFT) and renormalization group (RG) flow, supersymmetry (SUSY), and Cartan classification of Lie algebras. Readers are assumed to have learnt basic QFT (quantum field theory), poincáre symmetry, Lie group and Lie algebra.

1.1 Conformal field theory

Conformal field theory (CFT) means a field theory with conformal symmetry, i.e., the angle between two arbitrary curves crossing each other is preserved during the (so-called conformal) transformation. A conformal transformation is locally equivalent to a rotation and a dilation.[8, 9]

By mathematical definition, a conformal transformation of the coordinates is an invertible mapping $\boldsymbol{x} \to \boldsymbol{x}'$, leaving the metric tensor $g_{\mu\nu}$ invariant up to a scale:

$$g'_{\mu\nu}(\boldsymbol{x}') = \Lambda(\boldsymbol{x}) g_{\mu\nu}(\boldsymbol{x}) \tag{1.1}$$

The set of conformal transformations manifestly forms a group, and the Poincáre group is a subgroup of it. There are 4 kinds of generators of the conformal group:

Translation

$$P_\mu = -i\partial_\mu, \tag{1.2}$$

Dilation

$$D = -ix^\mu \partial_\mu, \tag{1.3}$$

Rotation

$$L_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu), \tag{1.4}$$

and the so-called special conformal transformation (SCT)

$$K_\mu = -i\left(2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu\right). \tag{1.5}$$

1.1.1 RG flow; Fixed point

Renormalization group (RG) flow can be considered as a trajectory taking the low-energy (infrared, IR) limit of a system, by lowering the renormalization scale μ. It may start from the ultraviolet (UV; or high-energy) side, and we deform it by some relevant deformation Δ, which turns to be the vacuum expectation value (vev) in our case.[8, 10, 11]

Fixed point is somewhere that is stationary under renormalization-group flow. For a non-conformal theory, one can do the RG flow until it hit the fixed point, then stay on the fixed point, and therefore become conformal.

A quantum field theory does not make sense without a regularization prescription that introduces a scale in the theory. This scale breaks the conformal symmetry, except at particular values of the parameters, which constitute a renormalization-group fixed point.

The Callan-Symanzik β-function of a theory is

$$\beta = \mu \frac{\partial \lambda}{\partial \mu}, \tag{1.6}$$

where λ is the rescaling factor appearing in the Lagrangian.

CFT describes physics at a fixed point of a (quantum) gauge theory. A fixed point u_j^* of the renormalization group is a point in parameter space that is unaffected by the renormalization procedure. In other words, it is characterized by a vanishing beta function:

$$\beta_i \left(u_j^* \right) = 0 \tag{1.7}$$

The theory is conformal if and only if β vanishes. (For instance, since in QED, β is calculated to be

$$\beta(e) = \mu \frac{\partial e}{\partial \mu} = \frac{e^3}{12\pi^2} \neq 0, \tag{1.8}$$

QED is not a conformal theory.[12])

1.2 Supersymmetry

Supersymmetry (SUSY) is now expected to play a fundamental role in particle physics, as the supersymmetry algebra is the only non-trivial extension of Poincáre symmetry and the only graded Lie algebra of symmetries of the S-matrix consistent with relativistic quantum field theory.

In supersymmetry, there are $4\mathcal{N}$ supercharges, Q_α^A and $\overline{Q}_{\dot{\alpha}A}$, where $A = 1, 2, ..., \mathcal{N}$ and $\alpha = 1, 2$. Here $\mathcal{N} \geq 1$ is the number of spinor charges. The algebra with $\mathcal{N} =$

1 is called the supersymmetry algebra, while those with $\mathcal{N} > 1$ are called extended supersymmetry algebras.

The supercharges obey the SUSY algebra:

$$\left\{ Q_\alpha^A, \overline{Q}_{\dot\beta B} \right\} = 2\left(\sigma^\mu\right)_{\alpha\dot\beta} P_\mu \delta_B^A \tag{1.9}$$

$$\left\{ Q_\alpha^A, Q_\beta^B \right\} = \epsilon_{\alpha\beta} Z^{AB} \tag{1.10}$$

$$\{\overline{Q}_{\dot\alpha A}, \overline{Q}_{\dot\beta B}\} = \epsilon_{\alpha\beta} Z^*_{AB} \tag{1.11}$$

where the central charges Z^{AB} are antisymmetric, i.e.

$$Z^{AB} = -Z^{BA}, \tag{1.12}$$

and commute with all the generators.

In $\mathcal{N} = 2$ there is one and only central charge, while in $\mathcal{N} = 1$ there is no one.

1.2.1 Superpotential

In SUSY theory we also have a function called superpotential. Given a superpotential, two ”partner potentials” are derived that can each serve as a potential (in the Schrödinger equation), and they have the same spectrum (characteristic energy).

Consider the Lagrangian (density) with the form

$$\mathscr{L} = \Phi^\dagger \Phi|_{\theta\theta\overline{\theta\theta}} + \left[\left(g\Phi + \frac{m\Phi^2}{2} + \frac{\lambda\Phi^3}{3} \right) |_{\overline{\theta\theta}} + \text{h.c.} \right] \tag{1.13}$$

(Here θ and $\overline{\theta}$ are a pair of anticommuting Grassmann variables: Weyl spinors.)

We can denote the superpotential $\mathscr{W}[\Phi]$ by

$$\mathscr{W} \ := g\Phi + \frac{m\Phi^2}{2} + \frac{\lambda\Phi^3}{3} \tag{1.14}$$

For more details or introduction to SUSY theory, one may refer to[13-15].

1.3 Cartan classification of Lie algebras

A Lie algebra is called to be semisimple if it is a direct sum of simple Lie algebras (non-abelian Lie algebras without any non-zero proper ideals).

A type of graph, called Dynkin diagram, is used to represent semisimple algebras via the configuration of root system. For each Lie algebra there is only one corresponding Dynkin diagram. What’s more, by analyzing the rules one Dynkin diagram should obey as long as it represents a semisimple Lie algebra, we can list all kinds of the Dynkin diagrams (i.e., all the Lie algebras) exhaustively, as shown in Fig.1-1.

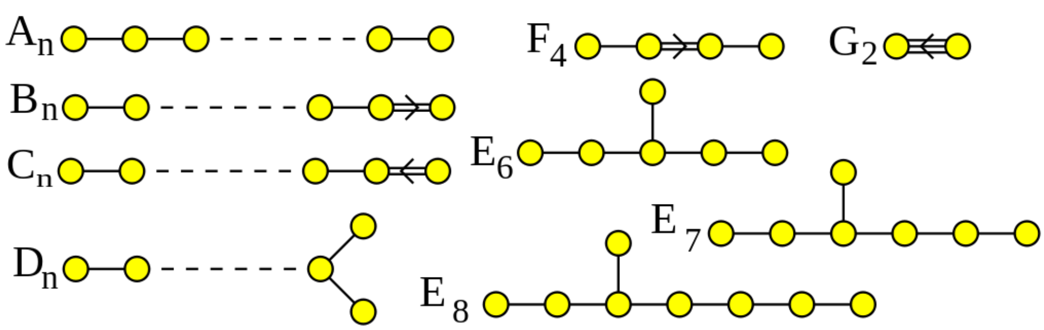

Figure 1-1 The Dynkin diagrams[16]

They have the following correspondence for the Lie algebras associated to classical groups over the complex numbers:

A_n : $\mathfrak{sl}(n+1)$, the special linear Lie algebra ($n \geq 1$).

B_n : $\mathfrak{so}(2n+1)$, the odd-dimensional special orthogonal Lie algebra ($n \geq 2$).

C_n : $\mathfrak{sp}(2n)$, the symplectic Lie algebra ($n \geq 2$).

D_n : $\mathfrak{so}(2n)$, the even-dimensional special orthogonal Lie algebra ($n \geq 4$).

Besides the above 4 "classical" series, there are and only are 5 more (therefore called exceptional) semisimple algebras: $E_{6,7,8}$, F_4 and G_2.

By this process, all the semisimple Lie algebras as well as Lie groups are classified into 7 kinds, 4 of which are infinite series. This is called Cartan classification.

A Dynkin diagram with no multiple edges (while in other diagrams some edges are doubled or tripled) is called simply laced, as are the corresponding Lie algebra and Lie group. We can see from Fig.1-1 that the A_n, D_n, $E_{6,7,8}$ series are simply laced.

For more details or fundamental Lie algebra and group theory, one may refer to[17-24].

Chapter 2
Introduction

2.1 Argyres-Douglas theory

The Argyres-Douglas (AD) theory, introduced in[1] and generalized in[2], is a kind of $\mathcal{N} = 2$ superconformal field theory (SCFT), which usually have fractional scaling dimensions for the Coulomb branch operators and dimensional coupling constants.

Any $\mathcal{N} = 2$ SCFTs have a protected sector described by the two-dimensional chiral algebra. For the AD theory and its generalizations, the corresponding chiral algebras are non-unitary minimal models or given by a simple coset. In particular, the Argyres-Douglas theory that we find as the IR fixed point has the chiral algebra given by the simplest minimal model, namely, the Yang-Lee model (firstly introduced by[25, 26]). Moreover, the central charge c takes the minimal value among the interacting unitary four-dimensional $\mathcal{N} = 2$ SCFTs. These facts indicate that the AD theory is just the $\mathcal{N} = 2$ minimal SCFT.[27]

For a general Argyres-Douglas theory $\mathcal{T}$, which is an interacting SCFT (an AD theory is an intrinsically strongly coupled theory, and includes special loci in the Coulomb branches), one can not find a Lagrangian description.

However, we might reach its proper description by RG flow starting from a theory with Lagrangian, and aiming at the theory itself as an IR fixed point. When the SCFT is obtained as an IR fixed point of a Lagrangian theory, one can easily compute the index from the matter content in the UV.

2.1.1 Generalized AD theory

The A-D-E classification of AD theories is introduced in[28, 29]. Here the Lie algebras are in Cartan classification and are simple-laced, i.e., they are A_n, D_n or $E_{6,7,8}$.

Engineered using type IIB string theory construction, theories of a pair of Dynkin diagrams (G, G') are introduced in[30]. This notion means the BPS quiver has the shape of the product of G and G' Dynkin diagram.

Based on A type M5 brane construction of AD theory in[31], the generalizes AD theory is introduced in[2]; the class of theories (A_1, G) is introduced, where G denotes one Lie algebra in A_n, D_n, $E_{6,7,8}$ series.

The AD theory from pure SU$(n+1)$ gauge theory can be labeled as $(A_1,\ A_n)$ theory, and those from SO$(2n)$ gauge theory correspond to $(A_1,\ D_n)$, and finally those derived from E_n gauge theory are labeled as $(A_1,\ E_n)$. These labels denote the shape of the BPS quiver of the corresponding SCFTs.[32, 33]

The $H_{0,1,2}$ series also appears in[3]:

$H_0 = (A_1, A_2)$;

$H_1 = (A_1, A_3) = (A_1, D_3)$;

$H_2 = (A_1, D_4)$.

2.1.2 Central charges a, c

Any conformal theory, supersymmetric or not, in four dimensions has two important numbers associated to it: these are referred to as the a and the c conformal anomalies. The conformal anomalies measure, among other things, the failure of the expectation value of the trace of the stress-energy tensor to vanish when the theory is put on a curved background metric $g_{\mu\nu}$,

$$\langle T^{\mu}{}_{\mu} \rangle = \frac{c}{16\pi^2} W^2 - \frac{a}{16\pi^2} E_4 \tag{2.1}$$

where W is the Weyl tensor and E_4 is the Euler density, both built from certain combinations of the metric $g_{\mu\nu}$ and its derivatives.

$$W^2 = R^2_{\mu\nu\rho\sigma} - 2R^2_{\mu\nu} + \frac{1}{3}R^2 \tag{2.2}$$

$$E_4 = R^2_{\mu\nu\rho\sigma} - 4R^2_{\mu\nu} + R^2 \tag{2.3}$$

The central charge a is given in terms of the 't Hooft anomaly coefficients of the IR superconformal R-symmetry as:[10, 27]

$$a = \frac{3}{32}\left(3\mathrm{Tr}R^3 - \mathrm{Tr}R\right) \tag{2.4}$$

$$c = \frac{1}{32}\left(9\mathrm{Tr}R^3 - 5\mathrm{Tr}R\right) \tag{2.5}$$

where R is the R-symmetry in the superconformal group.

The central charge c takes the minimal value among the interacting unitary four-dimensional $\mathcal{N} = 2$ SCFTs. By[34] we also learn that the superconformal R-symmetry of any 4d SCFT is exactly and uniquely determined by the a-maximization principle: it is the R-symmetry, among all possibilities, which (locally) maximizes the combination of a.

The central charge formulae for exact (A_1, G) series are summarized in[2]:

Theories	a	c
(A_1, A_{2n-1})	$\frac{-5-5n+12n^2}{24(1+n)}$	$\frac{3n^2-n-1}{6(n+1)}$
(A_1, A_{2n-2})	$\frac{(n-1)(24n-5)}{24(2n+1)}$	$\frac{(n-1)(6n-1)}{6(2n+1)}$
(A_1, D_{2n+2})	$\frac{6n+1}{12}$	$\frac{3n+1}{6}$
(A_1, D_{2n+1})	$\frac{n(8n+3)}{8(2n+1)}$	$\frac{n}{2}$

Table 2-1 The central charges of AD theory from A_1 theory

2.1.3 RG flow to AD theory

In this thesis we are interested in the RG flow from an $\mathcal{N}=1$ Lagrangian theory to the IR fixed point governed by an $\mathcal{N}=2$ AD theory with non-Abelian flavor symmetry F. Then we add an $\mathcal{N}=1$ chiral multiplet M transforming in the adjoint representation of F, via the superpotential coupling with the moment map operator μ of F given by

$$\mathcal{W} = \mathrm{Tr} M \mu \tag{2.6}$$

Then to the chiral multiplet M, we give a nilpotent vacuum expectation value (vev)

$$\langle M \rangle = \rho\left(\sigma^{+}\right), \tag{2.7}$$

where the embedding ρ is from $\mathfrak{su}(2)$ to $\mathfrak{f}$, the Lie algebra of F. This procedure defines the theory $\mathcal{T}_{\mathrm{IR}}[\mathcal{T}, \rho]$ in the IR fixed point.

The principal embedding is the one ρ such that $\mathfrak{f}$ is decomposed into rank-$\mathfrak{f}$ irreducible representations. In some cases the fixed point theory due to the principal embedding has an enhanced $\mathcal{N}=2$ supersymmetry.

Then we have a $\mathcal{N}=1$ Lagrangian theory flowing to $\mathcal{N}=2$ AD theory.

Here are some results of several AD theories $\mathcal{T}$ and their RG flows from[3, 4, 35]:

For $\mathcal{T} = H_1 = (A_1, A_3) = (A_1, D_3)$, the global symmetry to be broken by the principle embedding $F = SU(2)$. The IR fixed point theory is $\mathcal{T}_{\mathrm{IR}}[\mathcal{T}, \rho] = H_0 = (A_1, A_2)$;

For $\mathcal{T} = H_2 = (A_1, D_4)$, $F = SU(3)$, $\mathcal{T}_{\mathrm{IR}}[\mathcal{T}, \rho] = H_0$;

For $\mathcal{T} = (A_1, D_{k\geq 4})$, $F = SU(2)$, $\mathcal{T}_{\mathrm{IR}}[\mathcal{T}, \rho] = (A_1, A_{k-1})$.

This kind of flow named MS after Maruyoshi and Song is revised in[36]; this paper, however, shows not every AD theories admits UV Lagrangians via MS flow.

2.1.4 Matter content

From the "Lagrangian descriptions" by the method of RG flow,[3, 4] nave computed the full superconformal indices of the (A_1, A_n) theories and the (A_1, D_n) theories. We can see they did this by a-maximization.

The so-called "matter content"s (or field content) of charges for different chiral multiplets under different Lie groups and some other properties of the (A_1, G) series can be found in[3, 4, 35].

Beyond the tables of matter content, the decoupled operators $\mathrm{Tr}\phi^k$ should also be concerned in the following computation; $\mathrm{Tr}\phi^k$ seems to have a $U(1)_{\mathcal{F}}$ charge, which equals to k times the $U(1)_{\mathcal{F}}$ charge of ϕ. It counteract as a reciprocal of the contribution of ϕ, where it is mistakenly counted in.

(A_1, A_{2N-1}) theories

Consider the case where $\mathcal{T}$ is $\mathcal{N} = 2$ $SU(N)$ gauge theory with $2N$ fundamental hypermultiplets. This theory has $SU(2N) \times U(1)$ flavor symmetry.

The central charges are

$$a = \frac{7N^2 - 5}{24} \tag{2.8}$$
$$c = \frac{2N^2 - 1}{6} \tag{2.9}$$
$$k_{SU(N)} = 2N \tag{2.10}$$

Upon coupling the $SU(N)$ adjoint chiral multiplet M, and Higgsing via nilpotent vev, the remaining components of M are M_j , where$j = 1, ..., 2N-1$ with charge $(J_+, J_-) = (0, 2 + 2j)$.

By maximizing the trial central charge, we see various operators violate the unitary bounds: all the Coulomb branch operators $Tr\phi^i$ ($i = 2, 3, ..., N$), and M_j ($j = 1, 2, ..., N-1$). These will decouple, and we subtract the contribution of these fields. Then by re-doing the a-maximization, the field M_N has dimension 1, thus also decouples and is subtracted.

The remaining M_j with $j = N+1, ..., 2N-1$ is exactly the operator spectrum.

The matter content is

	q	$\tilde{q}$	ϕ	M_j
$SU(N)_{\text{gauge}}$	$\square$	$\overline{\square}$	adj	1
$U(1)_B$	1	-1	0	0
$U(1)_R$	1	1	0	0
$U(1)_{\mathscr{F}}$	N	N	-1	$-j-1$

Table 2-2 Matter content of (A_1, A_{2N-1}) theories

where $\square$ is called the simple box; "adj" is the adjoint representation; 1 is tribute; and the bar above means the operation of complex conjugate.

(The Young diagram or Young tableau is first introduced in[37]; fundamental introduction can be found in[23, 24]; usage in representation theory of Lie algebra can be found in[38, 39].)

The superpotential is

$$\mathscr{W} = \sum_{j=1}^{2N-1} M_j \left(\phi^{2N-1-j} q\tilde{q}\right)$$

(A_1, A_{2N}) theories

Consider the case where $\mathscr{T}$ is $\mathcal{N} = 2$ Sp(N) gauge theory with $2N+2$ fundamental hypermultiplets ($4N+4$ fundamental half-hypermultiplets). This theory has the SO($4N+4$) flavor symmetry. The central charges are

$$a = \frac{1}{24} N(14N+9) \tag{2.11}$$

$$c = \frac{1}{6} N(4N+3) \tag{2.12}$$

$$k_{\mathrm{SO}(4N+4)} = 4N \tag{2.13}$$

Couple a chiral multiplet M transforming in the adjoint representation of SO($4N+4$) and give the principal nilpotent vev to M. This will break the SO($4N+4$) flavor symmetry completely, and the remaining components of M would be M_j with$j = 1, 3, ..., 4N+1; 2N+1$ having charges $(J_+, J_-) = (0, 2j+2)$. (There are two M' s with $j = 2N+1$.)

After a-maximization of the anomalies, various operators get decoupled along the RG flow. The decoupled operators are all the Coulomb branch operators with $i = 1, 2, ..., N$ and M_j with $j = 1, 3, ..., 2N+1$ (there are two M_j' s with $j = 2N+1$); so we are left with N singlet M_j with $j = 2N+3, 2N+5, ..., 4N+1$.

The matter content is

	q	q'	ϕ	M_j	M'_{2N+1}
$Sp(N)_{\text{gauge}}$	$\square$	$\square$	adj	1	1
$U(1)_R$	1	1	0	0	0
$U(1)_{\mathscr{F}}$	$\frac{1}{2}$	$\frac{1}{2}(4N+3)$	-1	$-j-1$	$-2N-2$

Table 2-3 Matter content of (A_1, A_{2N}) theories

The superpotential is

$$\mathscr{W} = \phi q q + \sum_{i=1}^{2N+1} M_{2i-1}\left(\phi^{4N+3-2i} q' q'\right) + M'_{2N+1} q q' \tag{2.14}$$

(A_1, D_{2N}) theories

In the end result is that the operators given by $\mathrm{Tr}\phi^k$ ($2 \leq k \leq N$) and M_j ($0 \leq j \leq N-1$).

The matter content is

	q_1	$\tilde{q}_1$	q_2	$\tilde{q}_2$	ϕ	M_j
$SU(N)_{\text{gauge}}$	$\square$	$\overline{\square}$	$\square$	$\overline{\square}$	adj	1
$U(1)_1$	1	-1	1	-1	0	0
$U(1)_2$	$2N-1$	$-2N+1$	-1	1	0	0
$U(1)_R$	1	1	1	1	0	0
$U(1)_{\mathscr{F}}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{2N-1}{2}$	$\frac{2N-1}{2}$	-1	$-j-1$

Table 2-4 Matter content of (A_1, D_{2N}) theories

(A_1, D_{2N+1}) theory

In the end, the chiral operators $\mathrm{Tr}\phi^{2k}$ with $1 \leq k \leq N$ and M_j with $j = 2k+1$, $0 \leq k \leq N-1$, decouple from the interacting theory.

The matter content is

	q_1	q_2	ϕ	M_j
$Sp(N)_{\text{gauge}}$	$\square$	$\square$	adj	1
$SO(3)_{\text{flavor}}$	3	1	1	1
$U(1)_R$	1	1	0	0
$U(1)_{\mathscr{F}}$	$\frac{1}{2}$	$\frac{4N+1}{2}$	-1	$-j-1$

Table 2-5 Matter content of (A_1, D_{2N+1}) theories

2.2 Donaldson-Witten twist

Topological twist is a procedure for producing (Lagrangians for) topological quantum field theories from non-topological but supersymmetric QFTs. The so-called Donaldson-Witten twist is this kind of topological version of $\mathcal{N}=2$ SUSY theory.[40]

The topological reinterpretation of $\mathcal{N}=2$ SYM (super Yang-Mills theory) is derived by by redefining the Euclidean Lorentz group

$$\mathrm{SO}(4)_{\mathrm{spin}} = \mathrm{SU}(2)_L \otimes \mathrm{SU}(2)_R \tag{2.15}$$

in the following way:

$$\mathrm{SO}(4)'_{\mathrm{spin}} = \mathrm{SU}(2)_L \otimes \mathrm{SU}(2)'_R \tag{2.16}$$

$$\mathrm{SU}(2)'_R = \mathrm{diag}(\mathrm{SU}(2)_I \otimes \mathrm{SU}(2)_R) \tag{2.17}$$

where $\mathrm{SU}(2)_I$ is a factor of the automorphism group $\mathrm{SU}_I(2) \times \mathrm{U}_I(1)$ of the $\mathcal{N} = 2$ supersymmetry algebra.

The factor $\mathrm{U}_I(1)$ is called R-symmetry group, and it enters in a suitable redefinition of the $\mathrm{U}_g(1)$ ghost number group:

$$\mathrm{U}_g(1) \to \mathrm{U}'_g(1) = \mathrm{diag}(\mathrm{U}_g(1) \times \mathrm{U}_I(1)) \tag{2.18}$$

For physical fields ($g = 0$), the R-symmetry charge r coincides with the ghost number of the topologically twisted theory. This procedure is generally referred to as topological twisting. This twisting procedure applied to the rigid $\mathcal{N} = 2$ supersymmetric Yang–Mills theory yields topological Yang–Mills theory.[41]

And the partition function Z in this theory is a topological invariant (actually the simplest one.)[42]

2.3 A-model and Bethe vacua

The A-model is first introduced in[43]; Here we study it in[5]. Define the four-dimensional A-model of some gauge theory as the low-energy effective theory on the Coulomb branch in two dimensions, subjected to the topological A-twist. It has isolated vacua.

The four-dimensional A-model is fully determined in terms of two potentials, $\mathcal{W}(u, v; \tau)$ and $\Omega(u, v; \tau)$. The effective twisted superpotential $\mathcal{W}$ governs the dynamics of the low energy effective theory on $\mathbb{R}^2 \times T^2$; and the effective dilaton Ω governs the coupling to curved space. The fields u_a are Coulomb branch coordinates in the $N = (2, 2)$ theory. The parameters v_α for the flavor group can be similarly defined. Their higher-dimensional origin manifests itself by the periodic identifications under $U(1)_a$ and $U(1)_\alpha$ large gauge transformations on T^2 respectively.

The A-model vacua correspond to the solutions of the Bethe equations

$$e^{2\pi i \frac{\partial \mathcal{W}(u,v;\tau)}{\partial u_a}} = 1, \qquad a = 1, ..., \mathrm{rk}(\mathfrak{g}) \tag{2.19}$$

The solutions are called Bethe vacua, and are two dimensional vacua of the theory compactified on T^2.

One can build a number of "canonical" A-model operators from $\mathscr{W}$ and Ω, such as the flavor flux operator

$$\Pi_\alpha = e^{2\pi i \frac{\partial \mathscr{W}}{\partial \nu_\alpha}} \tag{2.20}$$

the handle-gluing operator

$$\mathscr{H} = e^{2\pi i \Omega} \det_{ab} \left(\frac{\partial^2 \mathscr{W}(u, \nu; \tau)}{\partial u_a \partial u_b} \right)$$

and the fibering operator $\mathscr{F}$

$$\mathscr{F} = e^{2\pi i \frac{\partial \mathscr{W}}{\partial \tau}} \tag{2.21}$$

In the thesis we consider a four-dimensional $\mathcal{N} = 1$ supersymmetric gauge theory on $\mathbb{R}^2 \times T^2$, with a modular parameter τ for the torus.

Consider an $\mathcal{N} = 1$ gauge theory for $\mathbf{G}$ semi-simple, with chiral multiplets Φ_i in representations $\mathfrak{R}_i$ of $\mathfrak{g}$. We also turn on generic background parameters $\nu\alpha$ for any flavor symmetry $\mathbf{G}_F$, with $\nu_i = \omega_i(\nu)$ and ω_i the flavor weight as defined above. We simply obtain:

$$\mathscr{W}(u, \nu; \tau) = \sum_i \sum_{\rho \in \mathfrak{R}_i} \mathscr{W}_\Phi(\rho_i(u) + \nu_i; \tau) \tag{2.22}$$

Here the sum goes over all weights $\rho_i(u)$ of the representations of $\mathfrak{g}$.

Given the twisted superpotential, we may define several A-model operators. The flux operator $\Pi_\mathbf{a}$ is a local operator that inserts one unit of $\mathrm{U}(1)_a$ flux on Σ_g, given by:

$$\Pi_\mathbf{a} = e^{2\pi i \frac{\partial \mathscr{W}}{\partial \mathbf{u_a}}} \tag{2.23}$$

where the index $\mathbf{a} = (a, \alpha)$ run over both the gauge and flavor group. This can be considered as a combination of (2.20) and the LHS of (2.19).

2.4 Modular tensor category

A modular tensor category (MTC) is roughly a category that encodes the topological structure underlying a rational 2-dimensional conformal field theory. It is a CFT analogue of the classical notion of tensor category for representations of (modules for) a group or a Lie algebra. In such a category one can define a projective action of the group $\mathrm{SL}(2, \mathbb{Z})$ on an appropriate object.[44-46]

2.4.1 S and T matrices

In the group

$$\mathrm{SL}(2, \mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = 1;\ a, b, c, d \in \mathbb{Z} \right\} \tag{2.24}$$

one can find two generators, the so-called S and T matrices, to be

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{2.25}$$

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \tag{2.26}$$

satisfying

$$S^2 = (ST)^3 = C = -I, \tag{2.27}$$

where C is the central element satisfying

$$C^2 = I. \tag{2.28}$$

The group $\mathrm{SL}(2, \mathbb{Z})$ plays an important role in the type IIB superstrings.[9, 47]

Since the four-dimensional A-model is defined by compactification on a torus, one expects that is behaves naturally under modular transformations. The MTC associated to a $\mathcal{N} = 2$ Argyres-Douglas theory also comes equipped with S and T matrices. Now in the MTC, S and T matrices are generators of the modular group, corresponding to the $\mathrm{SL}(2, \mathbb{Z})$ matrices interpreted as handle gluing operator and fibering operator respectively.

For a single chiral multiplet, the S and T generators acting on the superpotential are:

$$S: \quad \mathscr{W}_\Phi\left(\frac{u}{\tau}; -\frac{1}{\tau}\right) = \frac{\mathscr{W}_\Phi(u, \tau)}{\tau} + \frac{u^3}{6\tau^2} + \frac{u}{4\tau} \tag{2.29}$$

$$T: \quad \mathscr{W}_\Phi(u; \tau + 1) = \mathscr{W}_\Phi(u, \tau) + \frac{u^3}{6\tau(\tau + 1)} - \frac{u}{12} \tag{2.30}$$

which satisfy the $\mathrm{SL}(2, \mathbb{Z})$ relations

$$S^2 = C \tag{2.31}$$

$$(ST)^3 = C \tag{2.32}$$

where the center C acts as charge conjugation[5, 7]

However we will see not every MTC is to be ”unitary”. In the (A_1, A_2) theory, the S and T are computed to be the minimal ”Lee-Yang theory”, which is non-unitary.

2.4.2 Twisted topological partition function and MTC

For our (generalized) Argyres-Douglas theory $\mathscr{T}$, we construct a 3d TQFT by choosing the 4d space-time to be

$$M_4 = S^1 \times M_3. \tag{2.33}$$

A general feature of an AD theory is that it possesses a $\mathbb{Z}_F$ global symmetry, which can be used to turn on a non-trivial holonomy $\gamma \in \mathbb{Z}_F$ along the S^1 in the above geometry. Further, the theory is topologically twisted along M_3 by identifying the Riemannian holonomy group Spin(3) with $SU(2)_R \subset SU(2)_R \times U(1)_r$ of the R-symmetry of the 4d $\mathcal{N} = 2$ Argyres-Douglas theory, and by a standard argument the partition function is expected to be a topological invariant of M_3.

When the holonomy is co-prime to F, i.e. $\gamma \in \mathbb{Z}_F^\times$, the 3d TQFT is expected to be semisimple, and this 3d TQFT is associated to a MTC determined by $\mathscr{T}$ and γ. This MTC, in turn, can be determined either from the geometry of the wild Hitchin moduli space[48] or from the chiral algebra[49].

The values of F and numbers of fixed points for each of the four (A_1, G) series listed above are listed in[50]:

Theories	F	Fixed pts
(A_1, A_{2n+1})	$n+2$	$\frac{(n+1)(n+2)}{2}$
(A_1, D_{2n+1})	$2n+1$	$2n+1$
(A_1, D_{2n})	n	n^2
(A_1, A_{2n})	$2n+3$	$n+1$

Table 2-6 Value of F and number of fixed points for each of the four (A_1, G) series

We consider the twisted topological partition function $Z_{\mathscr{T}}(M_4)$, which is noted in[51] to be conveniently encoded in the data of an MTC associated to that theory, which comes equipped with S and T matrices. Using these S and T matrices, we can conveniently write the topological partition function on the four-manifold M_4, where M_3 is defined by an arbitrary plumbing graph, as

$$Z_{AD}\left(S^1 \times M_3\right) = \sum_{\lambda_v} \prod_{\text{vertices}} S_{0\lambda_v}^{2-\deg(v)} T_{\lambda_v \lambda_v}^{a_v} \prod_{\text{edges}} S_{\lambda_v \lambda_v'} \tag{2.34}$$

Denote the total space of a degree-p circle bundle over a genus-g Riemann surface as $L_g(p)$. Using localization to compute,

$$Z_{AD}\left(T^2 \times \Sigma_g\right) = \sum_{\lambda} (S_{0\lambda})^{2-2g} T^{p}_{\lambda\lambda} \tag{2.35}$$

here λ depends on the size of the S and T matrices.

When M_3 is the lens space $L(p,1)$, supersymmetry can be preserved for generic values of the fugacity $\mathfrak{t}$, and the partition function is expected to compute the equivalent index of a line bundle on the Coulomb branch. For generalized Argyres-Douglas theories, such quantities are sometimes referred to as the "wild Hitchin characters."

Chapter 3
Computation Procedure

3.1 Aim of the thesis

In this thesis, I plan to use the data in the matter contents, given directly by[7] or derived from the general series in[3, 4], to calculate the modular tensor category properties of the first few Argyres-Douglas theories: (A_1, A_2), (A_1, A_3), (A_1, D_3), (A_1, D_4) and (A_1, A_4).

Using the procedure of solving the Bethe vacua of A-model in[5] and the method of holonomy saddle in[6], we can obtain the handle-gluing and the fibering operators of the AD theories, which is associated with the S and T matrices by the twisted partition function.

The result will be compared with what is done in[7], and the process of the computation will also be discussed.

3.2 Definition of some parameters

For convenience, we introduce parameters ν, τ etc., define them to be related to the fugacities y, q etc. by

$$x = e^{2\pi i u}, \tag{3.1}$$

$$y = e^{2\pi i \nu}, \tag{3.2}$$

$$q = e^{2\pi i \tau}, \tag{3.3}$$

$$q = e^{-2\pi\beta} \tag{3.4}$$

From now on we consider the pairs of x and u, etc., are intrinsically related, i.e., for a function $f(x)$, we can have u including in f as $u = \dfrac{\log x}{2\pi i}$, and $f(x)$ and $f(u)$ are intrinsically equal using this translation.

3.3 Theta function and Holonomy saddle

The theta function is defined as

$$\theta(u;\tau) = x^{-\frac{1}{2}} q^{\frac{1}{12}} \prod_{k=0}^{\infty} \left(1 - x q^{k}\right)\left(1 - x^{-1} q^{k+1}\right) \tag{3.5}$$

The reduced theta function is its product term:

$$\theta_0(u;\tau) = \prod_{k=0}^{\infty} \left(1 - xq^k\right)\left(1 - x^{-1}q^{k+1}\right) \tag{3.6}$$

which is an infinite product.

On θ_0 we do a method called holonomy saddle introduced in[6], that is, for every AD theory we use two kinds (cases) of θ_0 to compute:

- ”unshifted θ_0”

  Let $q \to 0$ directly, and we immediately have

  $$\theta_0 = 1 - x \tag{3.7}$$

  (as x and u are related, one can also rewrite it in u.)

  (In practice, sometimes we have to set $q \to 0$ later to avoid some indeterminate during the process.)

- ”shifted θ_0”

  In this case there are 3 steps:

  1. Keep 3 terms remained,

     $$\theta_0 = (1 - x)(1 - xq)\left(1 - x^{-1}q\right) \tag{3.8}$$

     instead of only 1 term as in the previous case;
  2. Shift $x \to xq^{\frac{1}{2}}$;
  3. Then at last let $q \to 0$.

3.4 Bethe ansatz equation

Here in the following our method is mainly given in[5], and we just continue from the end of Section 2.3 which refers to the same paper, and enter the concrete computation for AD theories using the data of the matter contents in[3, 4, 7].

For a single chiral multiplet with twisted superpotential, we have the contribution

$$\Pi^{\Phi}(u;\tau) = \frac{e^{2\pi i\left(-\frac{u^2}{2\tau}+\frac{u}{2}-\frac{\tau}{12}\right)}}{\theta_0(u;\tau)} \tag{3.9}$$

Thus

$$\Pi_a(u, v; \tau) = e^{2\pi i \frac{\partial \mathscr{W}}{\partial u_a}} = \prod_i \prod_{\rho_i \in \mathfrak{R}_i} \Pi^\Phi \left(\sum_a \rho_i^a u_a + v_i \right)^{\rho_i^a} \tag{3.10}$$

$$\Pi_\alpha(u, v; \tau) = e^{2\pi i \frac{\partial \mathscr{W}}{\partial v_\alpha}} = \prod_i \prod_{\rho_i \in \mathfrak{R}_i} \Pi^\Phi \left(\sum_a \rho_i^a u_a + v_i \right)^{\omega_i^\alpha} \tag{3.11}$$

for the gauge and flavor flux operators, respectively.

Here, for the outer product, i runs over all of the q's, ϕ, and the M_j, i.e., every column in the matter content table.

For the inner product, it goes over all weights $\rho_i(u)$ of the representations $\mathfrak{R}_i$ (i.e., content in the row of the gauge group in the matter content table) of $\mathfrak{g}$. One can use `WeightSystem` in the Mathematica application LieART[52] to look up them. For SU(2), the weights of adj, $\square$ and 1 are {2, 0, -2}, {1, -1} and {0} respectively.

The parameter v_i is just the $U(1)_{\mathscr{F}}$ charge in the matter content for each i.

We solve the bethe ansatz equation (BAE) of x:

$$\Pi_a(x) = 1 \tag{3.12}$$

The solution is the so-called Bethe vacua, and they will be substitute into the result of the following process.

3.5 Handle-gluing operator

We have dilaton

$$\Omega = \Omega_{\text{mat}}(u, v; \tau) + \Omega_{\text{vec}}(u, v; \tau) \tag{3.13}$$

where

$$e^{2\pi i \Omega_{\text{mat}}} = \prod_i \prod_{\rho_i \in \mathfrak{R}_i} \Pi^\Phi (\sum_a \rho_i^a u_a + v_i; \tau)^{r_i - 1} =: \texttt{Mat} \tag{3.14}$$

$$e^{2\pi i \Omega_{\text{vec}}} = \eta(\tau)^{-2\text{rk}(\mathfrak{g})} \prod_{\alpha \in \mathfrak{g}} \Pi^\Phi(\alpha(u); \tau) =: \texttt{Vec} \tag{3.15}$$

The parameter r_i in (3.14) is just the charge of $U(1)_R$ in the matter content. We can see both and only ϕ and M_j count in this equation, and we have to additionally subtract the term(s) of decoupling.

The parameter α denotes the roots in $\mathfrak{g}$. One can use `RootSystem` in LieART[52] to look up them. For SU(2) the roots are 2, 0, -2.

The Dedekind η-function in (3.15) is

$$\eta = q^{\frac{1}{24}} \prod_{k=1}^{\infty} \left(1 - q^k\right) \tag{3.16}$$

For convenience we can just compute the LHSs of (3.14) and (3.15), denoted as `Mat` and `Vec` respectively, and let `Dil` be their product

$$\texttt{Dil} = \texttt{Mat} \times \texttt{Vec} = e^{2\pi i \Omega} \tag{3.17}$$

Hessian of the twisted superpotential

$$H = \det_{ab} \frac{\partial^2 \mathscr{W}(u, v; \tau)}{\partial u_a \partial u_b} = \det_{ab} \left(\frac{1}{2\pi i} \frac{\partial \log \Pi_a}{\partial u_b} \right) \tag{3.18}$$

and with a little trick using (3.1)(3.12),

$$H = x \frac{\partial \Pi_a}{\partial x} \tag{3.19}$$

The handle-gluing operator is:

$$\mathscr{H} = e^{2\pi i \Omega} H = H \times \texttt{Dil} \tag{3.20}$$

3.6 Fibering operator

Define

$$\Gamma_0(u; \tau) = \prod_{n=0}^{\infty} \left(\frac{1 - x^{-1} q^{n+1}}{1 - x q^{n+1}} \right)^{n+1} \tag{3.21}$$

then the fibering operator contribution of a single chiral multiplet of unit charge

$$\mathscr{F}^{\Phi}(u; \tau) = e^{2\pi i \left(\frac{u^3}{6\tau^2} - \frac{u}{12} \right)} \Gamma_0 \tag{3.22}$$

the fibering operator

$$\mathscr{F}(\mathbf{u}; \tau) = \prod_I \mathscr{F}_I^{\Phi}(Q_I(\mathbf{u}); \tau) = \prod_i \prod_{\rho_i \in \mathfrak{R}_i} \mathscr{F}^{\Phi}(\sum_a \rho_i^a u_a + v_i; \tau) \tag{3.23}$$

Here we should consider the decoupling again.

Then, we should put the Bethe vacua, i.e., solution of the Bethe ansatz equation into the handle-gluing and fibering operators. The other parameters should be replaced as some exact numbers as their physics ask it to be. For some parameters the two operators just can not be solved (get meaningful values) unless them come to be some appropriate values.

At last we combine the two parts of solutions from both "unshifted" and "shifted" computation.

3.7 The S, T matrices

The derivation from the handle-gluing and fibering operators to the S, T matrices for (A_1, A_2) and (A_1, A_3) theories are introduced in[7] individually in accordance with the result of these two operators. For these two theories we show the same result as this paper, and for the other theories we just stop at the handle-gluing and fibering operators.

Chapter 4
Results

4.1 (A_1, A_2) Theory

This is the so-called H_0 theory.

The superpotential is:

$$\mathscr{W} = q\phi q + uq'\phi q \tag{4.1}$$

The matter content:

(From now on we multiply all the previous $U(1)_{\mathscr{F}}$ charge by a factor $\frac{2}{F}$. Here $F = 5$.)

	q	q'	ϕ	M_5
$SU(2)_{\text{gauge}}$	$\square$	$\square$	adj	1
$U(1)_R$	1	1	0	0
$U(1)_{\mathscr{F}}$	$\frac{1}{5}$	$\frac{7}{5}$	$-\frac{2}{5}$	$-\frac{12}{5}$

Table 4-1 Matter content of (A_1, A_2) Theory

There are 2 fixed points and a decoupled operator $\mathrm{Tr}\phi^2$.

For $M_3 = L_g(p)$, the partition function is given by

$$Z_{(A_1,A_2)}\left(S^1 \times L_g(p)\right) = \sum_i (\mathscr{H}_i)^{g-1} (\mathscr{F}_i)^p \tag{4.2}$$

We can compare it with (2.35) and obtain the value of S and T matrices.

As an example of some formulae above in chapter 3, we write down them with matter content plugged in:

$$
\begin{aligned}
\Pi_a =&\Pi^{\Phi}\left(u+\frac{1}{5}\nu\right)\Pi^{\Phi}\left(-u+\frac{1}{5}\nu\right)^{-1}\\
&\Pi^{\Phi}\left(u+\frac{7}{5}\nu\right)\Pi^{\Phi}\left(-u+\frac{7}{5}\nu\right)^{-1}\\
&\Pi^{\Phi}\left(2u-\frac{2}{5}\nu\right)^{2}\Pi^{\Phi}\left(-2u-\frac{2}{5}\nu\right)^{-2}
\end{aligned}
\tag{4.3}
$$

$$
\begin{aligned}
\texttt{Mat} =&\Pi^{\Phi}\left(2u-\frac{2}{5}\nu\right)^{-1}\Pi^{\Phi}\left(-\frac{2}{5}\nu\right)^{-1}\Pi^{\Phi}\left(-2u-\frac{2}{5}\nu\right)^{-1}\\
&\Pi^{\Phi}\left(-\frac{12}{5}\nu\right)^{-1}\Pi^{\Phi}\left(-\frac{4}{5}\nu\right)
\end{aligned}
\tag{4.4}
$$

$$
\texttt{Vec} = \Pi^{\Phi}(2u)\Pi^{\Phi}(0)\Pi^{\Phi}(-2u)
\tag{4.5}
$$

where $\Pi^{\Phi}\left(-\frac{4}{5}\nu\right)$ is included as the decoupling $\mathrm{Tr}\phi^2$, and in practise we replace $\Pi^{\Phi}(0)$ by $e^{2\pi i\left(\frac{-\tau}{12}\right)}$ beforehand.

$$
\begin{aligned}
\mathscr{F} =&\mathscr{F}^{\Phi}\left(u+\frac{1}{5}\nu\right)\mathscr{F}^{\Phi}\left(-u+\frac{1}{5}\nu\right)\\
&\mathscr{F}^{\Phi}\left(u+\frac{7}{5}\nu\right)\mathscr{F}^{\Phi}\left(-u+\frac{7}{5}\nu\right)\\
&\mathscr{F}^{\Phi}\left(2u-\frac{2}{5}\nu\right)\mathscr{F}^{\Phi}\left(-\frac{2}{5}\nu\right)\mathscr{F}^{\Phi}\left(-2u-\frac{2}{5}\nu\right)\\
&\mathscr{F}^{\Phi}\left(-\frac{12}{5}\nu\right)\mathscr{F}^{\Phi}\left(-\frac{4}{5}\nu\right)^{-1}
\end{aligned}
\tag{4.6}
$$

where $\mathscr{F}^{\Phi}\left(-\frac{4}{5}\nu\right)^{-1}$ is the decoupling.

- For the "unshifted" θ_0:

  We put exact value $y = \pm 1$ into the result of Π_a (since it represents the chiral algebra).

  We can see for $y = 1$, it appears to be $\Pi_a = 1$, which is just tribute. (which makes the Hessian H vanishes for $y = 1$.) Then we just consider the case of $y = -1$.

  There are 4 nondegenerated solutions for the BAE. List the result:

BAE Solution x	Handle-gluing $\mathscr{H}$	Fibering $\mathscr{F}$
-1	ComplexInfinity	$e^{-\frac{i\pi}{30}}$
1	ComplexInfinity	$e^{-\frac{i\pi}{30}}$
$e^{\frac{2i\pi}{3}}$	$\frac{5+\sqrt{5}}{2}$	$e^{-\frac{i\pi}{30}}$
$e^{-\frac{2i\pi}{3}}$	$\frac{5+\sqrt{5}}{2}$	$e^{-\frac{i\pi}{30}}$

Table 4-2 The result with $y = -1$, unshifted θ_0 in (A_1, A_2) Theory

Two of them give (complex)infinite value for the handle-gluing operator, while the other two gives the same handle-gluing operator

$$\mathscr{H} = -y + y^{\frac{1}{5}} + y^{-\frac{1}{5}} - y^{-1} = \frac{5+\sqrt{5}}{2} \tag{4.7}$$

The result of the fibering operator is independent to x (All the x terms vanish with $q \to 0$), being

$$\mathscr{F} = e^{-\frac{i\pi}{30}} \tag{4.8}$$

- For "shifted" θ_0:

  We again have tribute case when $y = 1$, and we put $y = -1$ in, the solution of BAE is $x = 1, i, -1, -i$.

  List the result of the handle-gluing and fibering operators:

BAE Solution x	Handle-gluing $\mathscr{H}$	Fibering $\mathscr{F}$
-1	ComplexInfinity	$e^{-\frac{19i\pi}{30}}$
$-i$	$\frac{5-\sqrt{5}}{2}$	$e^{\frac{11i\pi}{30}}$
i	$\frac{5-\sqrt{5}}{2}$	$e^{\frac{11i\pi}{30}}$
1	ComplexInfinity	$e^{-\frac{19i\pi}{30}}$

Table 4-3 The result with $y = -1$, shifted θ_0 in (A_1, A_2) Theory

We can see only $y = \pm i$ gives both meaningful handle-gluing and fibering operators, where the handle-gluing operator is

$$\mathscr{H} = -y + y^{\frac{7}{5}} + y^{-\frac{7}{5}} - y^{-1} = \frac{5-\sqrt{5}}{2} \tag{4.9}$$

And the corresponding fibering operator is

$$\mathscr{F} = e^{\frac{11i\pi}{30}} \tag{4.10}$$

Our result agrees with[7], where the result is written in $\mathfrak{t} = e^{2\pi i}$.

Then according to the localization method in[7], we can obtain the S and T matrices by

$$S = \begin{pmatrix} -\left(\frac{5-\sqrt{5}}{2}\right)^{-\frac{1}{2}} & \left(\frac{5+\sqrt{5}}{2}\right)^{-\frac{1}{2}} \\ \left(\frac{5+\sqrt{5}}{2}\right)^{-\frac{1}{2}} & \left(\frac{5-\sqrt{5}}{2}\right)^{-\frac{1}{2}} \end{pmatrix} = \frac{2}{\sqrt{5}} \begin{pmatrix} -\sin\frac{2\pi}{5} & \sin\frac{\pi}{5} \\ \sin\frac{\pi}{5} & \sin\frac{2\pi}{5} \end{pmatrix} \tag{4.11}$$

$$T = \begin{pmatrix} e^{\frac{11i\pi}{30}} & 0 \\ 0 & e^{-\frac{i\pi}{30}} \end{pmatrix} \tag{4.12}$$

4.2 (A_1, A_3) Theory

This is the H_1 theory.

The superpotential is

$$\mathscr{W} = uq\tilde{q} \tag{4.13}$$

Field content: $(F = 3)$

	q	$\tilde{q}$	ϕ	M_3
$SU(2)_{\text{gauge}}$	$\square$	$\square$	adj	1
$U(1)_B$	1	-1	0	0
$U(1)_R$	1	1	0	0
$U(1)_{\mathscr{F}}$	$\frac{4}{3}$	$\frac{4}{3}$	$-\frac{2}{3}$	$-\frac{8}{3}$

Table 4-4 Matter content of (A_1, A_3) Theory

For the $U(1)_B$ we use b to denote the content, and define υ (Greek upsilon) as

$$b = e^{2\pi i \upsilon} \tag{4.14}$$

the charge b is put into the computation as well as y in every term it appears.

- "Unshifted" θ_0

  The $y = 1$ case is tribute.

  We put exact value $y = -1$ into the solution, and let b be an exact even integer number. As the result in terms of b is much more complicated, we plug the first few exact values $b = 4, 6, 8$ into the result. Some results are computed numerically.

  There are 6 solutions of the Bethe equation, 4 of which lead to valued handle-gluing and fibering operators:

$$\mathscr{H} = 3 \tag{4.15}$$

$$\mathscr{F} = e^{-\frac{i\pi}{6}} \tag{4.16}$$

- "Shifted" θ_0

  The $y = 1$ case is tribute.

  The solution of BAE is $x = 1, i, -1, -i$.

  The handle-gluing operator is again

$$\mathscr{H} = 3, \tag{4.17}$$

  and the fibering operator is

$$\mathscr{F} = i = e^{\frac{i\pi}{2}}. \tag{4.18}$$

Our results agree with[7], and the partition function is

$$Z_{(A_1,A_3)}\left(S^1 \times L_g(p)\right) = \sum_i (\mathscr{H}_i)^{g-1}(\mathscr{F}_i)^p = 3^{g-1}\left(e^{\frac{i\pi p}{2}} + e^{-\frac{i\pi p}{6}} + e^{-\frac{i\pi p}{6}}\right) \tag{4.19}$$

and comparing with (2.35), we have the S and T matrices

$$S = -\frac{1}{\sqrt{3}}\begin{pmatrix} 1 & -1 & 1 \\ -1 & e^{\frac{4\pi i}{3}} & e^{-\frac{2\pi i}{3}} \\ 1 & e^{-\frac{2\pi i}{3}} & e^{\frac{4\pi i}{3}} \end{pmatrix} \tag{4.20}$$

$$T = \begin{pmatrix} e^{\frac{\pi i}{2}} & & \\ & e^{-\frac{\pi i}{6}} & \\ & & e^{-\frac{\pi i}{6}} \end{pmatrix} \tag{4.21}$$

Notice that we have one Bethe vacua corresponding to one pair of the solution of the Bethe equation, so we have both

$$\mathscr{F}_2 = \mathscr{F}_3 = e^{-\frac{\pi i}{6}} \tag{4.22}$$

since there are 4 solutions in the case of "unshifted θ_0", giving all the same handle-gluing and fibering operators.

4.3 (A_1, D_3) Theory

Although the (A_1, A_3) and (A_1, D_3) theories are conjectured to be identical in the IR, their two UV descriptions are different. Both of them are denoted as the H_1 theory.

The superpotential is

$$\mathscr{W} = q\phi q + u\tilde{q}\phi\tilde{q} \tag{4.23}$$

Field content: $(F = 3)$

	q	$\tilde{q}$	ϕ	M_3
$SU(2)_{\text{gauge}}$	$\square$	$\square$	adj	1
$SO(3)_{\text{flavor}}$	3	1	1	1
$U(1)_R$	1	1	0	0
$U(1)_{\mathscr{F}}$	$\frac{1}{3}$	$\frac{5}{3}$	$-\frac{2}{3}$	$-\frac{8}{3}$

Table 4-5 Matter content of (A_1, D_3) Theory

Here we use s to denote the $SO(3)$ flavor charge.

- ”unshifted” θ_0

  The Π_a becomes very complicated unless we lat $s = \pm 1, 0$. At the end of the day the only meaningful s value is $s = 1$. $y = 1$ still leads to tribute case. So we have $s = 1$, $y = -1$ in this case.

  The Bethe vacua are $x = \pm 1, \pm i$. Only $x = \pm i$ leads to meaningful handle-gluing operator

$$\mathscr{H}_1 = 3 \tag{4.24}$$

$$\mathscr{F}_1 = e^{-\frac{\pi i}{18}} \tag{4.25}$$

- ”shifted” θ_0

  Put $s = 1$, $y = -1$ in, Bethe vacua are $x = \pm 1, \pm i$. Only $x = \pm i$ leads to meaningful handle-gluing operator

$$\mathscr{H}_2 = 3 \tag{4.26}$$

$$\mathscr{F}_2 = e^{\frac{4\pi i}{9}} \tag{4.27}$$

4.4 (A_1, D_4) Theory

This is the H_2 theory.

Field content: $(F = 2)$

	q_1	$\tilde{q}_1$	q_2	$\tilde{q}_2$	ϕ	M_2
$SU(2)$	$\square$	$\bar{\square}$	$\square$	$\bar{\square}$	adj	1
$U(1)_1$	1	-1	1	-1	0	0
$U(1)_2$	3	-3	-1	1	0	0
$U(1)_R$	1	1	1	1	0	0
$U(1)_{\mathscr{F}}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	-1	-3

Table 4-6 Matter content of (A_1, D_4) Theory

- "Unshifted" θ_0

  None of the Bethe vacua $x = 0, \pm 1$ leads to meaningful handle-gluing operator.

- "Shifted" θ_0

  $y = 1$ is tribute; $y = -1$ leads to Bethe vacua $x = \pm 1, \pm i$, and only $x = \pm i$ leads to meaningful handle-gluing,

$$\mathscr{H} = 4 \tag{4.28}$$

$$\mathscr{F} = e^{\frac{2\pi i}{3}} \tag{4.29}$$

4.5 (A_1, A_4) Theory

Field content: $(F = 7)$

	q	$\tilde{q}$	ϕ	M_7	M_9
$Sp(2)$	$\square$	$\square$	adj	1	1
$U(1)_R$	1	1	0	0	0
$U(1)_{\mathscr{F}}$	$\frac{1}{7}$	$\frac{11}{7}$	$-\frac{2}{7}$	$-\frac{16}{7}$	$-\frac{20}{7}$

Table 4-7 Matter content of (A_1, A_4) Theory

The (A_1, A_4) theory is worth to be noticed that we have the gauge theory $Sp(2)$ different from, and unfortunately more complicated than the former ones; what's more, both decoupling of $\mathrm{Tr}\phi^2$ and $\mathrm{Tr}\phi^4$ should be considered; both M_7 and M_9 should be counted.

The calculation of this theory is done by a co-worker. It is computed numerically, and the result gives the same S and T as the so-called (2, 7) minimal model, which is mentioned in[7].

4.6 Discussions

(A_1, A_3) and (A_1, D_3) are actually the same; but we get a different result for (A_1, D_3) from (A_1, A_3) of ours and[7]. This may be an indication that we have get a wrong answer due to the mistakes during the computation or misunderstanding of the real content in the theory. However, this two theories have different start points on the RG flow, and the parameters are disparate, so they may be intrinsically different in the handle-gluing and fibering operators, and so as the S, T matrices in our computation.

Also, the computation of (A_1, D_4) ends up with giving only one solution, which is not likely to be able to derive the handle-gluing and fibering matrices. It indicates that the result may be wrong, or for this and higher theories we must consider more unseen details.

Another thing worth to be considered is that some solutions may be mistakenly extracted by putting in exact value of parameters at wrong time. On the other side, there may be some "fake" solutions rising during the procedure and seem to be legal result, while they are actually not.

We can see that the growing complexity of the process of computation is the main difficulty to obtain the information of MTC in higher theories. In our computation the (A_1, A_3) theory is computed partly numerically already, and we can imagine the higher rank theories will be much more complicated and may be eventually impossible to compute.

Although it may be harder, we are looking forward to compute higher AD theories, some of whose known properties has been summarized in[50]. (For example, the series of A_1, A_{2n+1} is of the $\mathscr{B}_{n+2}$ algebra introduced in[53]; It is reported in[7] that the (A_1, A_{2n}) theories have some properties associated with $(2, 2n+3)$ minimal model.) Further study may discover more connections between this AD theory and some other theories, which is also a verification of the computation result.

For a general equation to compute S and T matrices, etc. of the whole series of

(A_1, A_n) and (A_1, D_n) AD theories derived from the given matter content of these general series, we may need some new nice methods.

There are also two interesting observations: $y = 1$ always gives tribute case and never lead meaningful handle-gluing and fibering operators; after using the holonomy saddle to "shift" x, the Bethe vacua (solutions of the BAE) are always the 4 roots making $x^4 = 1$. To prove or disprove whether these properties work in all AD theories, we may need dig deeper in the principle of Bethe ansatz, and the holonomy saddle, and so on.

Chapter 5
Conclusions & Outlook

In the thesis, using the matter contents derived by RG flow from $\mathcal{N} = 1$ theories with Lagrangian description, and the procedure of bethe ansatz equation and holonomy saddle, we have done calculations of the modular tensor category, $\mathrm{SL}(2, \mathbb{Z})$ matrices from supersymmetric localization for several exact Argyres-Douglas theories: (A_1, A_2), $(A_1, A_3) = (A_1, D_3)$, (A_1, D_4), and (A_1, A_4), which covers the result of[7] and goes further, although the correctness is uncertain.

We are still looking forward to a general equation to compute S and T matrices, etc. of the whole series of (A_1, A_n) and (A_1, D_n) theories. Exact solutions and deep relations between some other theories are also expected.

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<tr><td colspan="4">指导教师对论文学术规范的审查意见：

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