

復旦大學

本科毕业论文



论文题目：塞伯格-威腾理论与瞬子配分函数

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完成日期：2022 年 9 月 8 日

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摘要

量子场论起源于狄拉克等人试图统一量子力学和狭义相对论的尝试。这一框架现在已经成为了现代物理学的基石之一，而经由此框架发展出的粒子物理的标准模型则经受住了加速器实验的考验，在已有的能标下，理论计算和实验结果均相符。我们当然希望最终存在一种统一的量子场论描述一切粒子物理现象，然而，现有的电弱统一理论与量子色动力学（QCD）在高能标下重整化群流不一定流向同一个不动点，因而不能精确地统一。

此问题的解决方案之一即超对称量子场论。超对称是一种超越时空本身最大对称性限制（Coleman-Mandula 止步定理）的对称性，其核心在于对理论中每一个玻色（费米）粒子，引入一个费米（玻色）的超伴侣粒子。超伴侣粒子的存在使得高阶的量子修正相互消除，因而其微扰论是一圈精确（1-loop exact）的，相应的路径积分可以进行精确计算。引入超对称后，电弱统一理论与量子色动力学的重整化群流即流向同一个不动点。

为了实现超对称，我们可以在量子场论中引入不同数量的超荷（supercharge），即超对称变换生成元。超荷的数目越多，超对称对理论的限制就越大。不同种类的生成元的个数记为 \mathcal{N} ，四维时空中 $\mathcal{N} > 4$ 的理论包含引力子，因而不再是量子场论而是超引力理论； $\mathcal{N} = 4$ 时超对称对理论的限制极强，理论是自对偶（self-dual）且是唯一的。而 $\mathcal{N} = 1$ 的理论限制太弱，仍受到量子修正影响，因而只能分析理论的相（phase）。

而四维 $\mathcal{N} = 2$ 的理论称为塞伯格-威腾理论（Seiberg-Witten theory），此理论受限较小但依然具有一圈精确的特点。一般而言，从紫外（UV）拉氏量出发获得该理论的低能有效理论（LEEA）需要做严格的费曼路径积分积掉高能自由度，但是塞伯格和威腾提出，在此理论中无需做路径积分即可求得低能有效理论。通过全纯性分析他们发现该低能有效理论由一个称为“初势（prepotential）”的全纯函数的二阶导数完全决定，而除了一圈图的贡献之外全部非微扰贡献来自于瞬子（instanton）。

受四维 $\mathcal{N} = 4$ 理论中的 Olive-Montonen 猜想启发，塞伯格和威腾发现该理论存在“电磁对偶性”，原理论的耦合常数 g 在此对偶下变为 $1/g$ ，因此原理论的红外强耦合区域在对偶理论中变为红外自由的，可以微扰求解，因此可以很容易地确定红外的真空结构

随后他们利用真空模空间上的绕异（monodromy）方法得到瞬子的贡献，即间接确定了该理论的初势，从而完全解出了该低能有效理论。

涅克拉索夫则希望通过直接计算，即“瞬子计数”的方法得到初势，他通过使用超对称局域化方法将初势的计算转化为特定背景中的超对称量子力学，最终化为杨图（Young tableaux）的求和，藉此获得所谓的瞬子配分函数。对瞬子配分函数求对数即可得到初势，继而解出此理论。

对于现实世界中的量子色动力学（QCD），由于其是渐近自由的，对于其低能有效理论以及现象（如夸克禁闭）我们仍知之甚少。在塞伯格-威滕理论中，磁单极在某些特定真空态中是无质量自由度，因此会产生所谓“单极子凝聚”。类比于超导体中的“库珀对”，单极子凝聚也会产生类似于 Meissner 效应的现象，即超导体完全排除磁场；而对于 QCD，“磁场”变成“色磁场”，即夸克禁闭。同时这种单极子的凝聚也导致了手征对称性破缺。在分析非超对称的 QCD 时，'t Hooft 等人曾提出单极子凝聚导致夸克禁闭的假说，但苦于无法进行实际计算，而塞伯格-威滕理论则给出了一个绝佳的例证，无疑是对禁闭问题更深刻的理解。

从上述案例可以看出，分析超对称的 QCD 因其具有更强的对称性而相对容易，由此我们可以初窥量子场论中神秘的非微扰现象，进而对非超对称的 QCD 有更深刻的理解。

同时，塞伯格-威滕理论也为数学的发展提供了全新的视角。

数学上，四维流形的拓扑学在低维拓扑中有着最为丰富的内涵，四维流形的不变量则是刻画其拓扑结构的重要特征。在此之前，唐纳森（Donaldson）提出了所谓唐纳森不变量，即，考虑四维流形上的自对偶规范场方程的瞬子解，这些瞬子构成一个模空间，而唐纳森不变量则是这个瞬子模空间上的特定积分。这一方法开辟了这方面的研究的新途径，并逐渐成为四维流形拓扑的基础，但可惜的是，唐纳森理论的计算量极为庞大。在 Atiyah 的启发下，威滕发现这一不变量实际上对应于一个拓扑量子场论的关联函数，而该理论可以通过修改塞伯格-威滕理论中的超对称生成元（称为“扭变”）来获得。

这一扭变的塞伯格-威滕理论的红外极限给出拓扑场论中的磁单极方程，威滕随后证明唐纳森不变量即该方程经典解的个数，极大简化了唐纳森理论的计算。从物理上来看这一理论并无特别的难以理解之处，但从数学上来看这一结论则极为重要。

塞伯格-威滕理论自提出后一直是高能理论物理研究的重要方向。塞伯格-威滕在原始文献（仅含纯规范场的情形）发表后立即对包含物质的 SQCD 进行了研究，随后很快被推广到规范群为 $SO(N)$ 及 $Sp(N)$ 的情形。同时，塞伯格-威滕理论的提出还启发了弦论中的对偶的发现，其本身与弦论以及膜的动力学的关系时至今日依然是研究的热点。四维中包含八个超对称生成元的共形场论（Argyres-Douglas 理论）和 Gaiotto 对偶的发现也是基于对塞伯格-威滕理论的深入研究。同时，从六维 $\mathcal{N} = (0, 2)$ 理论如何约化得到塞伯格-威滕理论，以及它

所对应的神秘的 $M5$ -膜动力学仍有待我们进一步探索。

本文将按如下顺序介绍塞伯格-威滕理论与瞬子配分函数：

在第一章中，我们将简单介绍塞伯格-威滕理论的大致方法以及在塞伯格-威滕之后对于四维 $\mathcal{N} = 2$ 超对称量子场论的研究进展，并列出相关文献。

在第二章中，我们将介绍如何利用四维 $\mathcal{N} = 1$ 超对称规范理论的方法构造 $\mathcal{N} = 2$ 理论的拉氏量表述，以及如何构造矢量多重态和超多重态（hypermultiplets），同时简述了超对称规范理论的 NSVZ β 函数和其“精确性（exactness）”的来源。

在第三章中，我们将引入 $\mathcal{N} = 2$ 超场形式，改写相应的拉氏量，并讨论对应的真空模空间的性质与分类，并分析对本文最为重要的 $SU(2)$ 模空间的性质。

在第四章中，我们将引入初势，给出低能有效理论的表达式，计算一圈图以及瞬子对初势的贡献。同时我们还将复现塞伯格-威滕理论中的重要一步，即实现该理论的电磁对偶性（electro-magnetic duality）并讨论其对偶群 $SL(2, \mathbb{Z})$ 以及模空间度规的不变性。

在第五章中，我们通过拓扑方法分析真空模空间，并得出模空间上存在奇点。我们通过研究奇点附近的绕异矩阵，得出了此和乐不平庸的模空间参数化了一族椭圆曲线，称为 Seiberg-Witten 曲线。

在第六章中，我们对 Seiberg-Witten 曲线以及其上相应的闭链积分作了详细的分析。同时我们简要介绍了塞伯格-威滕理论的可积性以及其与可积模型周期 Toda 链的关系，导出 Seiberg-Witten 曲线与周期 Toda 链的谱曲线的对应关系，并利用谱曲线算出了瞬子对初势的修正系数。

在第七章中我们介绍了涅克拉索夫的瞬子计数方法，首先引入了 Ω 背景上的五维超对称规范理论，简单介绍了在其上如何利用超对称局域化将瞬子贡献约化到一个超对称量子力学理论。作为例子，我们计算了 $k = 0$ 和 $k = 1$ 的情况，发现其与第六章中的结果一致。

第八章中，我们还考虑了更高秩（rank）的规范群的情况，并作了相应计算。

关键字： 超对称量子场论；非微扰效应；超弦理论

Abstract

Quantum field theory arose out of attempts by Dirac to unify quantum mechanics and special relativity. This framework has now become one of the cornerstones of modern physics, and the standard model of particle physics developed through this framework has withstood the test of accelerator experiments and found an exact match. We certainly hope that there will eventually exist a unified quantum field theory to describe all phenomena in particle physics. However, the existing electroweak theory and quantum chromodynamics (QCD) does not necessarily flow to the same fixed point at high energy scales, and thus cannot be precisely unified.

One of the solutions to this problem is supersymmetric quantum field theory. Supersymmetry is a kind of symmetry beyond the maximum symmetry limit of space-time itself (Coleman-Mandula no-go theorem). The core of this is to introduce a Fermionic (Bosonic) super-partner for every Boson (Fermion) particle. The existence of the super-partner particles cancels the high-order quantum corrections, so its perturbation theory is exact, and the corresponding path integral can be calculated exactly. In the presence of supersymmetry, the renormalization group flows of electroweak unified theory and QCD meet at the same fixed point.

To realize supersymmetry, we can introduce different numbers of supercharges, i.e., supersymmetric transformation generators, into quantum field theory. The larger the number of supercharges, the more constraining the supersymmetry is on the theory. We denote the number of different kinds of generators as \mathcal{N} , the theory of $\mathcal{N} > 4$ in four-dimensional spacetime contains graviton, so it is no longer a quantum field theory but supergravity; When $\mathcal{N} = 4$, supersymmetry constraints the theory very strongly, and the theory is self-dual and unique. The constraint of $\mathcal{N} = 1$ is too weak, and it is still affected by quantum correction, so only the phase of the theory can be analyzed.

The four-dimensional $\mathcal{N} = 2$ theory is called the Seiberg-Witten theory, which is less constrained but still has the characteristics of 1-loop exactness. In general, obtaining the low-energy effective action (LEEA) of the theory from the ultraviolet (UV) Lagrangian requires performing the Feynman path integral to integrate out the high-energy massive degrees of freedom, but Seiberg and Witten proposed that in this theory, LEEA can be obtained without doing path integral. Through holomorphic analysis, they found that the low-energy efficient theory is completely determined by the second derivative

of a holomorphic function called the “prepotential”, while all non-perturbative contributions except for the one-loop diagram come from the instantons.

Inspired by the Olive-Montonen conjecture in the four-dimensional $\mathcal{N} = 4$ theory, Seiberg and Witten found that there is an “electromagnetic duality” in the theory, and the coupling constant g of the original theory becomes $1/g$ under this duality, so the infrared strong coupling region of the original theory becomes infrared-free in the dual theory, which can be solved by perturbation theory, so the infrared vacuum structure can be easily determined.

Then they obtained the contribution of the instanton by using the monodromy method in the vacuum moduli space. And the prepotential of the theory was determined, which means the low-energy effective theory was completely solved.

Nekrasov hopes to obtain the prepotential through direct calculation, that is, the method of “instanton counting”. By using the supersymmetric localization method, he transforms the calculation of the prepotential into supersymmetric quantum mechanics in a specific background and finally becomes summation over Young tableaux to obtain the so-called instanton partition function. The prepotential can be obtained by taking the logarithm of the instanton partition function, and then the theory can be solved.

For real-world quantum chromodynamics (QCD), because it is asymptotically free, little is known about its low-energy efficient theory and phenomena such as quark confinement. In the Seiberg-Witten theory, magnetic monopoles are massless degrees of freedom in certain vacuum states, thus producing so-called “monopole condensations”. Analogous to the “Cooper pair” in superconductors, monopole condensation also produces a phenomenon similar to the Meissner effect, that is, the superconductor completely excludes the magnetic field; while for QCD, the “magnetic field” becomes a “color magnetic field”, which is quark confinement. At the same time, the condensation of this monopole also leads to the breaking of chiral symmetry. When analyzing non-supersymmetric QCDs, 't Hooft proposed the hypothesis that monopole condensation leads to quark confinement, but they were unable to perform actual calculations, and the Seiberg-Witten theory gave an excellent example. This is undoubtedly a deeper understanding of the confinement problem.

As can be seen from the above cases, it is relatively easy to analyze supersymmetric QCDs because of their stronger symmetry. From this, we can get a first glimpse of the mysterious non-perturbative phenomena in quantum field theory, and then have more insights into non-supersymmetric QCDs.

In this thesis, we will give a pedagogical review of the Seiberg-Witten theory on

4d $\mathcal{N} = 2$ supersymmetric gauge theory and briefly introduce Nekrasov's instanton counting method that recovers the Seiberg-Witten prepotential.

Keywords: Supersymmetric quantum field theory; non-perturbative effect; super-string theory

第 1 章 Introduction

In 1994, The seminal work of Seiberg and Witten^[1-2] revolutionizes supersymmetric QFTs and also string theory. They considered the 4d $\mathcal{N} = 2$ $SU(2)$ pure Yang-Mills theory, and found the low energy effective action from this particular UV action without performing the Feynman path integral. In Seiberg-Witten theory, the low energy effective action is determined completely by a holomorphic function $\mathcal{F}(\Phi)$ called *prepotential*. The second derivative of the prepotential is $1/g$ where g is the gauge coupling. So after quantum correction, it becomes the effective coupling constant, which should contain loop corrections and nonperturbative effects. Seiberg and Witten discovered that the β -function of this theory is one-loop exact, and the nonperturbative effects come only from instantons.

Inspired by the Olive-Montonen conjecture in 4d $\mathcal{N} = 4$ theory, they found that the original theory is dual to an IR-free “magnetic” theory via electro-magnetic duality. Since the coupling constant in the magnetic theory is $1/g$ in terms of the original coupling constant g , the behavior of the original electric theory in the strong-coupling region can be described by the magnetic theory perturbatively. And the vacuum structure can be obtained accordingly.

Then by using monodromy methods on the vacuum moduli space, they finally succeeded in deriving the instanton contributions and thus the prepotential.

Soon, Nekrasov proposed a new method called instanton counting that can recover the Seiberg-Witten prepotential. He reformulated the problem of calculating instanton coefficients into supersymmetric quantum mechanics on the Ω -background by using supersymmetric localization. And by summing over all Young tables, we can get the so-called *instanton partition function* which becomes prepotential after taking the logarithm.

There are many excellent books and reviews on this elegant subject, some user-friendly ones are^[3-5]. The relation between Seiberg-Witten theory and string theory, see^[6]. A concise handbook with useful results is^[7]. For further topics, the book by Tachikawa^[8] will be of great help. A new review concentrating on the mathematical structure behind Seiberg-Witten theory is^[9]. In this review, we will mostly follow the author’s note of the lectures given by Satoshi Nawata.

第 2 章 Lagrangian for 4d $\mathcal{N} = 2$ theories

In this section, we will construct the Lagrangian for 4d $\mathcal{N} = 2$ pure Yang-Mills theory in the language of $\mathcal{N} = 1$ theories. We will see that the superfield formalism of 4d $\mathcal{N} = 1$ theories only works partially in 4d $\mathcal{N} = 2$ theories. This part takes the notation of^[9].

2.1 Supercharges

$\mathcal{N} = 2$ means we need to introduce 8 supercharges \mathcal{Q} in the theory. These supercharges are Weyl spinors that obey the following commutation relation

$$\{\mathcal{Q}_\alpha^I, \bar{\mathcal{Q}}_{\dot{\alpha}J}\} = 2\delta_J^I \sigma_{\alpha\dot{\alpha}}^\mu P_\mu, \quad \{\mathcal{Q}_\alpha^I, \mathcal{Q}_\beta^J\} = \epsilon_{\alpha\beta} \epsilon^{IJ} Z, \quad [\mathcal{Q}_\alpha^I, P_\mu] = 0. \quad (2.1)$$

In which $I, J = 1, 2$ labels different supercharges. α and $\dot{\alpha}$ s are spinor indices, P_μ the translation generator and Z is called the central charge.

This theory admits an $\mathfrak{su}(2)_R \times U(1)_r$ R-symmetry if we treat the supercharge \mathcal{Q} (or $\bar{\mathcal{Q}}$) as a left (right) hand Weyl spinor. The $\mathfrak{su}(2)_R$ act on the spin-1/2 doublet of supercharges as

$$SU(2)_R \curvearrowright \begin{pmatrix} \mathcal{Q}_\alpha^1 \\ \mathcal{Q}_\alpha^2 \end{pmatrix} \quad (2.2)$$

In general, the $\mathfrak{u}(1)_r$ is anomalous, but it is non-anomalous in an SCFT.

2.2 $\mathcal{N} = 2$ supermultiplets

Now we consider the supersymmetric multiplets in $\mathcal{N} = 2$ theories with gauge group \mathfrak{g} . In analogy with the $\mathcal{N} = 1$ case, pure Yang-Mills theories can have only vector multiplets, and the hypermultiplets somehow stand for matters.

2.2.1 Vector multiplet

Unlike the $\mathcal{N} = 1$ case, the $\mathcal{N} = 2$ vector multiplet is composed of four fields (if we omit auxiliary ones), which can be regarded as a $\mathcal{N} = 1$ chiral multiplet and a $\mathcal{N} = 1$ vector multiplet, all of them are in adjoint representation.

$$\mathcal{N} = 2 \text{ Vector multiplet } V : \begin{cases} A_\mu^A \\ \eta_\alpha^A & \lambda_\alpha^A \\ & \phi^A \end{cases} \quad (2.3)$$

here ϕ is the scalar field, A_μ the gauge field and η, λ the gaugino. Since they are all in adjoint representation, the latin indices $A = 1, \dots, \dim \mathfrak{g}$. Fields on the same row have the same R -charge and the same spin in the same column.

The $\mathcal{N} = 2$ vector multiplet $V^{\mathcal{N}=2}$ can be written in the language of $\mathcal{N} = 1$ theory as $(\Phi, V^{\mathcal{N}=1})$, where Φ consists of ϕ and η , and $V^{\mathcal{N}=1}$ contains λ and A_μ . Inside the $\mathcal{N} = 1$ multiplets ϕ and η , λ and A_μ are related by supercharge \mathcal{Q}^1 , while ϕ and λ , η and A_μ are related by \mathcal{Q}^2 .

2.2.2 Hypermultiplet

$\mathcal{N} = 2$ hypermultiplet can be decomposed into a $\mathcal{N} = 1$ chiral multiplet and a $\mathcal{N} = 1$ anti-chiral multiplet.

$$\mathcal{N} = 2 \text{ Hypermultiplet } \Phi : \begin{cases} \tilde{\psi}_\alpha^{\dagger i} \\ q^i & \tilde{q}^{\dagger i} \\ \psi_\alpha^i \end{cases} \quad (2.4)$$

The index $i = 1, \dots, N_f$ marks the flavor. q and ψ_α forms a chiral multiplet Q and so is \tilde{q}^\dagger and ψ_α^\dagger in anti-chiral multiplet \tilde{Q}^\dagger .

q and \tilde{q}^\dagger forms a $SU(2)_R$ doublet

$$\begin{pmatrix} q \\ \tilde{q}^\dagger \end{pmatrix} \circ SU(2)_R \quad (2.5)$$

which means $SU(2)_R$ can rotate $\mathcal{N} = 1$ chiral multiplet Q and anti-chiral multiplet \tilde{Q}^\dagger .

2.3 Lagrangians

Now we can write down the Lagrangian for $\mathcal{N} = 2$ theories, for pure Yang-Mills

$$\mathcal{L}_{\text{vect.}} = \frac{\tau}{16\pi i} \int d^2\theta \left(\frac{1}{2} \text{tr} W^2 + \text{h.c.} \right) + \frac{\text{Im}\tau}{8\pi} \int d^2\theta d^2\bar{\theta} \text{tr} (\Phi^\dagger e^V \Phi). \quad (2.6)$$

Here W is defined as the supersymmetric field strength by $W_\alpha = -\frac{1}{4} \bar{D}^2 D_\alpha V$ in $\mathcal{N} = 1$ theories.

For $\mathcal{N} = 2$ SQCDs with N_f flavors, we introduce the matter part and interaction part as

$$\mathcal{L}_{\text{SQCD}} = \mathcal{L}_{\text{vect.}} + \mathcal{L}_{\text{hyp.}} + \mathcal{L}_{\text{int.}}. \quad (2.7)$$

the matter part with gauge coupling can be written as

$$\mathcal{L}_{\text{hyp.}} = \int d^2\theta d^2\bar{\theta} (Q_i^\dagger e^V Q_i + \tilde{Q}_i^\dagger e^V \tilde{Q}_i). \quad (2.8)$$

where the index i runs from 1 to N_f . And also the interaction part

$$\mathcal{L}_{\text{int.}} = \int d^2\theta (\tilde{Q}_i \Phi Q_i + \text{h.c.}). \quad (2.9)$$

Note that this interaction is between s-quarks Q , \tilde{Q} and the chiral multiplet Φ , and similar cubic terms always appear in $\mathcal{N} = 2$ theories. Also, if that matter is massive, we can add the mass term which reads

$$\mathcal{L}_{\text{mass}} = \int d^2\theta (m\tilde{Q}Q_i + \text{h.c.}). \quad (2.10)$$

2.4 Renormalizability

In 4d, the exact β -function of a $SU(N_c)$ supersymmetric gauge theory is the Novikov-Shifman-Vainshtein-Zakharov (NSVZ) β -function, which is derived by using instanton methods in^[10–11].

2.4.1 Condition for asymptotic freedom

For sake of simplicity, we just show the final form of the NSVZ β -function here which reads

$$\beta = \mu \frac{d}{d\mu} g^2(\mu) = -\frac{2N_c - N_f}{8\pi} g^4(\mu) \quad (2.11)$$

If we require the theory to be an asymptotic-free theory so $\beta \leq 0$, this is equivalent to $N_f \leq 2N_c$, and if $N_f > 2N_c$ the coupling constant of this theory diverges at UV.

2.4.2 Origin of exactness

In $\mathcal{N} = 1$ theory, the Lagrangian consists of three parts. The superpotential part \mathcal{W} is non renormalizable by holomorphy, and the vector superfield (gauge field) term $\text{Tr}(W_\alpha W^\alpha)$ is one-loop exact. But the Kähler potential part \mathcal{K} receives higher-order quantum correction, and thus the $\mathcal{N} = 1$ theory will receive infinite loop corrections.

But in $\mathcal{N} = 2$ theory, the Kähler potential is related to the vector superfield term, which is one-loop exact, by supersymmetry. So when they are combined, we can easily conclude that the full Lagrangian is one-loop exact and receives no higher-order quantum corrections.

第 3 章 $\mathcal{N} = 2$ superfield formalism

To proceed, we need to introduce the $\mathcal{N} = 2$ superfield formalism instead of using $\mathcal{N} = 1$ superspace. But unlike the $\mathcal{N} = 1$ theories not everything can be written concisely in $\mathcal{N} = 2$ superspace.

3.1 $\mathcal{N} = 2$ superspace

In $\mathcal{N} = 1$ case, we introduced two superspace coordinate bases θ_α and $\bar{\theta}^{\dot{\alpha}}$. We need to introduce two more superspace coordinates $\tilde{\theta}_\alpha$ and $\tilde{\bar{\theta}}^{\dot{\alpha}}$ in $\mathcal{N} = 2$ case.

Now we can rewrite the supermultiplets in terms of $\mathcal{N} = 2$ superspace. The $\mathcal{N} = 2$ vector multiplet can be interpreted as the $\mathcal{N} = 2$ *chiral superfield*

$$\Psi = \Phi(\tilde{y}, \theta) + \sqrt{2}\tilde{\theta}^\alpha W_\alpha(\tilde{y}, \theta) + \tilde{\theta}^\alpha \tilde{\bar{\theta}}_\alpha G(\tilde{y}, \theta) \quad (3.1)$$

In which Φ is the chiral multiplet, $\tilde{\mathcal{D}}_\alpha \Phi = 0$. W_α is the field strength and G is the auxiliary field which reads

$$G(\tilde{y}, \theta) = \Phi^\dagger(\tilde{y} - i\theta\sigma\bar{\theta}, \theta, \bar{\theta}) \exp[-2gV(\tilde{y} - i\theta\sigma\bar{\theta}, \theta, \bar{\theta})]|_{\tilde{\theta}\bar{\theta}} \quad (3.2)$$

The coordinate \tilde{y}^μ is defined as $\tilde{y}^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta} + i\tilde{\theta}\sigma^\mu\tilde{\bar{\theta}} = y^\mu + i\tilde{\theta}\sigma^\mu\tilde{\bar{\theta}}$, while y^μ is the $\mathcal{N} = 1$ superspace coordinate satisfying $\tilde{\mathcal{D}}_\alpha y^\mu = 0$

Field (3.1) is called *chiral* because Ψ satisfies the constraint $\tilde{\mathcal{D}}_\alpha \Psi = 0$ and $\tilde{\bar{\mathcal{D}}}_{\dot{\alpha}} \Psi = 0$. Expanding all terms in (3.1) resembles the same field components as the $\mathcal{N} = 2$ vector multiplet.

But it is very unfortunate that we cannot perform the same trick on $\mathcal{N} = 2$ hypermultiplet i.e. $\mathcal{N} = 2$ hypermultiplet only admits $\mathcal{N} = 1$ description.

For pure Yang-Mills (YM) theory the Lagrangian can be written in a compact form

$$\mathcal{L}_{\text{vect.}} = \text{Im} \left[\frac{\tau}{16\pi} \int d^2\theta d^2\tilde{\theta} \frac{1}{2} \text{Tr} \Psi^2 \right], \quad (3.3)$$

with τ the complexified YM coupling

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g_{\text{YM}}^2} \quad (3.4)$$

Expand and perform integral in (3.3) we will get

$$\begin{aligned} \mathcal{L}_{\text{vect.}} = & \frac{1}{g^2} \text{Tr} \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - i\lambda\sigma^\mu \mathcal{D}_\mu \bar{\lambda} + \frac{1}{2} D^2 \right] + \frac{\theta}{32\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu} \\ & - \text{Tr} \left[|\mathcal{D}_\mu \phi|^2 - i\bar{\eta}\sigma^\mu \mathcal{D}_\mu \eta + F^\dagger F - g\phi^\dagger [D, \phi] + \sqrt{2}ig (\bar{\eta}[\bar{\lambda}, \phi] - \phi^\dagger \{\lambda, \eta\}) \right]. \end{aligned} \quad (3.5)$$

We are interested in the auxiliary field part in the Lagrangian, i.e.

$$\begin{aligned}\mathcal{L}_{\text{aux.}} &= \frac{1}{g^2} \text{Tr} \left[\frac{1}{2} D^2 - g \phi^\dagger [D, \phi] + F^\dagger F \right] \\ &= -\frac{1}{2} \text{Tr} ([\phi^\dagger, \phi]^2)\end{aligned}\tag{3.6}$$

in the second step we substitute the equation of motion of D and F^*

$$F = 0,\tag{3.7}$$

$$D = g[\phi^\dagger, \phi]\tag{3.8}$$

into the auxiliary Lagrangian. The second line of (3.6) is called the *bosonic potential* $V(\phi)$ for $\mathcal{N} = 2$ pure Yang-Mills, which is non-negative and will give constraint on possible supersymmetric vacuum.

$$V(\phi) = \frac{1}{2} \text{Tr} ([\phi^\dagger, \phi]^2) \geq 0.\tag{3.9}$$

Consider the vacuum expectation value (VEV) of ϕ , $\langle \phi \rangle = \phi_0$. If $[\phi_0^\dagger, \phi_0] > 0$, this means we have a non-zero scalar potential and the $\mathcal{N} = 2$ supersymmetry is broken, while $[\phi_0^\dagger, \phi_0] = 0$ indicates the unbroken supersymmetric vacua i.e. the state $|\text{vac}\rangle$ satisfying $\mathcal{Q}|\text{vac}\rangle = 0$ for supersymmetry to preserve.

From this condition we conclude that ϕ should be the *Cartan subalgebra* of the gauge group G . Take $G = SU(N)$, the Cartan subalgebra is the traceless diagonal matrices

$$\phi_0 = \text{diag}(\phi_1, \phi_2, \dots, \phi_N), \quad \sum_i \phi_i = 0.\tag{3.10}$$

These matrices formed the *Coulomb branch* of the *moduli space of vacua* \mathcal{M}_C

$$\mathcal{M}_C = \left\{ \phi_0 = \text{diag}(\phi_1, \phi_2, \dots, \phi_N) \mid \sum_i \phi_i = 0 \right\}.\tag{3.11}$$

This moduli space has a complex dimension $\dim_{\mathbb{C}} \mathcal{M}_C = N - 1$, since it is parameterized by $N - 1$ Coulomb branch parameters

$$\text{Tr} \phi^k = \phi_1^k + \phi_2^k + \dots + \phi_N^k, \quad k = 2, \dots, N\tag{3.12}$$

which are gauge invariant.

Now, if the VEV of ϕ is non-zero, the $SU(N)$ gauge symmetry is spontaneously broken to $U(1)^{N-1}$. Since $\dim_{\mathbb{C}} SU(N) = N^2 - 1$, there are $N^2 - 1$ massless gauge bosons in the $SU(N)$ gauge theory and $N - 1$ massless photons (since they are $U(1)$ gauge bosons, i.e. abelian) after spontaneous symmetry breaking.

By this Higgsing procedure, $N(N - 1)$ gauge bosons become massive. So in the low energy effective action (LEEA), these massive bosons whose mass is $m \sim \phi_0$ have to be integrated out by performing the Feynman path integral. But Seiberg and Witten found in^[1] that we can determine the LEEA without actually doing this path integral, as we will introduce in the following chapters.

3.2 Classification of the moduli space of vacua

If we construct the $\mathcal{N} = 2$ SQCD by adding hypermultiplets like in 2.3, the moduli space of vacua will change. The classical moduli space is characterized by the constraint from the auxiliary field part of the Lagrangian. In pure Yang-Mills theory we have already seen the D -term and F -term equations (3.7) and (3.8). The D -term equation only involves a commutator and the F -term equation is trivial. But in $\mathcal{N} = 2$ SQCD, the D -term equation is more complicated, and since there are extra terms like $F^\dagger F + (\dots)F$, the F -term equation becomes nontrivial.

The classical moduli space of vacua takes the form of

$$\mathcal{M}_{cl} = \{(\phi, q, \tilde{q}) | D = 0, F = 0\} / (\text{gauge symmetry}). \quad (3.13)$$

Due to different constraint, this moduli space can be divided into several branches:

- Coulomb branch $\mathcal{M}_C = \mathcal{M}_{cl}|_{q=\tilde{q}=0}$;
- Higgs branch $\mathcal{M}_H = \mathcal{M}_{cl}|_{\phi=0}$;
- Mixed branch $\mathcal{M}_{\text{mix}} = \mathcal{M}_{cl}|_{q,\tilde{q},\phi \neq 0}$.

For Coulomb branch, if the gauge group of the SQCD is $G = SU(N_c)$ and has N_f flavors, its real dimension

$$\dim_{\mathbb{R}} \mathcal{M}_C = 4N_c N_f - 2(N_c^2 - 1) - (N_c^2 - 1) - (N_c^2 - 1) = 4N_c(N_f - N_c) + 4. \quad (3.14)$$

The constraint $2(N_c^2 - 1)$ comes from F -term, and two $(N_c^2 - 1)$ come from the D -term and gauge symmetry respectively. Also, we observe that the real dimension of \mathcal{M}_C is an integer multiple of 4, which means it is a hyperkähler manifold. Also note that if $N_f < N_c$ there is no Higgs branch, and when $N_f > N_c$ the Higgs branch appears and receives quantum correction.

3.3 The case of $SU(2)$

We solve the D -term equation for $SU(2)$ pure Yang-Mills theory by setting $q = \tilde{q} = 0$, then the lowest rank scalar of the vector multiplet can be written as

$$\phi_{\text{CB}} = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \quad (3.15)$$

in which CB denotes the Coulomb Branch. Since we only limit our discussions within the Seiberg-Witten analysis of $SU(2)$ pure Yang-Mills theory in the following sections, we can temporarily omit this subscript. Now, the VEV of ϕ breaks the $SU(2)$ gauge symmetry to $U(1)$, this means on a generic point of the Coulomb branch where supersymmetry is preserved there is an $\mathcal{N} = 2$ $U(1)$ (abelian) theory. For this simple case with hypermultiplets, the superpotential terms (2.8) give a mass to these charged matters $m \sim |a|$. So in the low energy regime, these matters also decouple.

The VEV (3.15) does not fix gauge redundancy completely. Note that ϕ and $-\phi$ are gauge equivalent so there is an extra \mathbb{Z}_2 gauge symmetry, thus a is not gauge-invariant. Instead $u = \text{tr}\phi^2 = a^2$ is indeed gauge-invariant, as we have claimed in (3.12) which is truly the parameter on the Coulomb branch.

Another interesting feature of u is that if we roughly regard $a = \sqrt{u}$, then this function is multi-valued. This means a is not globally defined but u is. This property will play an important role in finding the duality of Seiberg-Witten theory.

第 4 章 Effective action and electro-magnetic duality

Finally, we can start reproducing the Seiberg-Witten analysis for $\mathcal{N} = 2$ pure Yang-Mills. We will first introduce the LEEA and prepotential and then manifest the electro-magnetic duality in this theory.

4.1 Low energy effective action

By using the $\mathcal{N} = 2$ superspace, the most general Lagrangian for a pure Yang-Mills theory with arbitrary holomorphic function $\mathcal{F}(\Psi)$ takes the form

$$\begin{aligned}\mathcal{L}_{\text{eff.}} &= \frac{1}{4\pi} \text{ImTr} \int d^2\theta d^2\bar{\theta} \mathcal{F}(\Psi) \\ &= \frac{1}{8\pi} \text{Im} \left(\int d^2\theta \mathcal{F}_{ab}''(\Phi) W^{a\alpha} W_{\alpha}^b + 2 \int d^2\theta d^2\bar{\theta} (\Phi^\dagger e^{2gV})^a \mathcal{F}'_a(\Phi) \right)\end{aligned}\quad (4.1)$$

in which we use the short-hand notation $\mathcal{F}_a(\Phi) = \partial\mathcal{F}/\partial\Phi^a$, $\mathcal{F}_{ab}(\Phi) = \partial^2\mathcal{F}/\partial\Phi^a\partial\Phi^b$. The \mathcal{F} is the so-called *prepotential* which we will determine later. And we note that the second derivative of \mathcal{F} is the effective coupling as well as the metric on the moduli space of vacua

$$ds^2 = g_{ij} d\phi^i d\bar{\phi}^j = \text{Im} \frac{\partial^2 \mathcal{F}}{\partial\phi_i \partial\phi_j} d\phi^i d\bar{\phi}^j \quad (4.2)$$

where ϕ (and $\bar{\phi}$) is the scalar component of the $\mathcal{N} = 1$ chiral superfield Φ . This means that as a metric, \mathcal{F}'' is positive definite.

For this theory to be renormalizable, $\mathcal{F}(\Psi)$ should be at least quadratic in Ψ . Now we start with the quadratic term and try to find the quantum corrections in the low-energy regime. Recall that there is a $U(1)_{\mathcal{R}}$ symmetry broken by the chiral anomaly, the anomalous current is

$$\partial_\mu J_5^\mu = -\frac{N_c}{8\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu} \quad (4.3)$$

So under this $U(1)_{\mathcal{R}}$ transformation $\Phi \mapsto \Phi e^{2i\alpha}$, the action will not stay invariant but change by a anomalous term

$$\delta\mathcal{L}_{\text{eff.}} = -\frac{\alpha N_c}{8\pi^2} F \tilde{F} \quad (4.4)$$

Since this term only comes from terms involving $F\tilde{F}$ in the Lagrangian, we can write

$$\frac{1}{16\pi} \text{Im} [\mathcal{F}''(e^{2i\alpha}\Phi) (-FF + iF\tilde{F})] = \frac{1}{16\pi} \text{Im} [\mathcal{F}''(\Phi) (-FF + iF\tilde{F})] - \frac{\alpha N_c}{8\pi^2} F\tilde{F}. \quad (4.5)$$

which implies that the prepotential is restricted by

$$\mathcal{F}''(e^{2i\alpha}\Phi) = \mathcal{F}''(\Phi) - \frac{2\alpha N_c}{\pi}. \quad (4.6)$$

Taking α as an infinitesimal parameter,

$$\frac{\partial^3 \mathcal{F}}{\partial \Phi^3} = \frac{N_c}{\pi} \frac{i}{\Phi}. \quad (4.7)$$

Integrate with respect to Φ , we get

$$\mathcal{F}_{1\text{-loop}}(\Phi) = \frac{i}{2\pi} \Phi^2 \ln \frac{\Phi^2}{\Lambda^2}. \quad (4.8)$$

This is the one-loop correction to the effective coupling in low-energy. The Λ in (4.8) is a fixed energy scale. As we have stated before, the β -function of $\mathcal{N} = 2$ super Yang-Mills theory (SYM) is one-loop exact. As a result, the effective coupling receives perturbative quantum correction up to one-loop too, thus is exact perturbatively.

But Seiberg and Witten^[1] argued that this LEEA does receive non-perturbative effects from instantons. This part can be derived roughly by the following. The correction to \mathcal{F} from configuration of k instantons is proportional to a factor $\exp(-8\pi^2 k/g^2)$ and by holomorphy it does not receive corrections from anti-instantons. Using the one-loop β -function

$$\beta(g) = -g^3/4\pi^2 \quad (4.9)$$

integrate the above and we find

$$e^{-8\pi^2 k/g^2} = \left(\frac{\Lambda}{\Phi}\right)^{4k} \quad (4.10)$$

by proper normalization. To balance the $U(1)_{\mathcal{R}}$ symmetry, the k -instanton correction should be proportional to Φ^2 . Altogether, the correction to the prepotential is

$$\delta \mathcal{F} = \frac{i}{2\pi} \Phi^2 \ln \frac{\Phi^2}{\Lambda^2} + \sum_{k=1}^{\infty} \mathcal{F}_k \left(\frac{\Lambda}{\Phi}\right)^{4k} \Phi^2, \quad (4.11)$$

and finally including the classical coupling $\tau_{\text{cl.}} \Phi^2/2$, the full prepotential is

$$\mathcal{F} = \frac{\tau_{\text{cl.}}}{2} \Phi^2 + \frac{i}{2\pi} \Phi^2 \ln \frac{\Phi^2}{\Lambda^2} + \sum_{k=1}^{\infty} \mathcal{F}_k \left(\frac{\Lambda}{\Phi}\right)^{4k} \Phi^2, \quad (4.12)$$

with

$$\tau_{\text{cl.}} = \frac{\theta}{2\pi} + \frac{4\pi i}{g_{\text{YM}}^2} \quad (4.13)$$

the complexified YM coupling.

The second derivative of prepotential (which is the physical coupling) reads

$$\tau(\mu) = \tau_{\text{cl.}} + \frac{i}{2\pi} \log \frac{\mu}{\Lambda} + \sum_n c_n \left(\frac{\Lambda}{\mu}\right)^{4n} \quad (4.14)$$

From UV to IR theory, the classical coupling $\tau_{\text{cl.}}$ is replaced by \mathcal{F}'' , which becomes the effective coupling at low energy.

4.2 Duality transformation

As we have seen the metric of the moduli space of vacua (4.2)

$$ds^2 = \text{Im } \tau(\phi) d\phi d\bar{\phi}. \quad (4.15)$$

with $\tau = \partial^2 \mathcal{F} / \partial \phi^2$. The metric τ should be positive-definite as defined. So it cannot be defined globally since the harmonic function cannot have a minimum. To explore the possible change of variable keeping the form of the metric (4.2), as^[1] did, we introduce a new “coordinate” so that the new metric is a pull back of the previous coordinates. Define

$$\phi_D = \mathcal{F}'(\phi) \quad (4.16)$$

So the new prepotential differs with the original one by a Legendre transformation

$$\mathcal{F}_D(\phi_D) = \mathcal{F}(\phi) - \phi \phi_D. \quad (4.17)$$

And the metric becomes

$$ds^2 = \text{Im } d\phi_D d\bar{\phi} = -\frac{i}{2} (d\phi_D d\bar{\phi} - d\phi d\bar{\phi}_D). \quad (4.18)$$

So if we use ϕ_D as the local parameter, we will get a dual theory preserving the metric (4.2) on the moduli space of vacua, with some different coupling constant.

Now consider the gauge theory terms in the Lagrangian (4.1). In $\mathcal{N} = 1$ language, the relevant terms are

$$\frac{1}{8\pi} \text{Im} \int d^2 \theta \tau(\phi) W^2 \quad (4.19)$$

To preserving the Bianchi identity $\mathcal{D}W = 0$, we introduce a dual “vector” field V_D as Lagrange multiplier by adding the following term to the Lagrangian

$$\frac{1}{4\pi} \text{Im} \int d^4 x d^4 \theta V_D \mathcal{D}W = \frac{1}{4\pi} \text{Re} \int d^4 x d^4 \theta i V_D W = -\frac{1}{4\pi} \text{Im} \int d^4 x d^2 \theta W_D W \quad (4.20)$$

Then we can integrate out the gauge field strength $F_{\mu\nu}$. Physically this term can be regarded as coupling V_D to a magnetic monopole

$$\frac{1}{8\pi} \int V_{D\mu} \epsilon^{\mu\nu\rho\sigma} \partial_\nu F_{\rho\sigma} \quad (4.21)$$

if we don't use the superspace formalism.

So we rewrite the interaction part of the Lagrangian as follows to see the invariance after duality transformation

$$\begin{aligned}
 \mathcal{L}_{\text{int.}} &= \text{Im} \int d^4\theta \Phi^\dagger e^{-V} \mathcal{F}'(\Phi) \\
 &= \text{Im} \int d^4\theta (\mathcal{F}'_D(\Phi_D))^\dagger e^{-V} \Phi_D \\
 &= \text{Im} \int d^4\theta \Phi_D^\dagger e^{-V} (\mathcal{F}'_D(\Phi_D)).
 \end{aligned} \tag{4.22}$$

By integrating out W with Gaussian integral the effective action for gauge field term with Lagrange multiplier becomes

$$\begin{aligned}
 \mathcal{S} &= \int \mathcal{D}W \mathcal{D}V_D \exp\left(\frac{i}{32\pi} \text{Im} \int d^4x d^2\theta \mathcal{F}''(\Phi) W^\alpha W_\alpha + \mathcal{D}_\alpha W^\alpha V_D\right) \\
 &= \int \mathcal{D}V_D \exp\left(\frac{i}{32\pi} \text{Im} \int d^4x d^2\theta \frac{-1}{\mathcal{F}''(\Phi)} W_D^\alpha W_\alpha^D\right).
 \end{aligned} \tag{4.23}$$

We can easily read off the coupling constant for the dual theory, namely

$$\mathcal{F}_D''(\Phi_D) = \frac{-1}{\mathcal{F}''(\Phi)} \tag{4.24}$$

So the coupling constant actually transforms as

$$\tau \mapsto \tau_D = -\frac{1}{\tau} \tag{4.25}$$

by duality, which means it is an *electro-magnetic duality* or *S-duality*.

Now we can write down the full Lagrangian for the dual theory

$$\mathcal{L}_{\text{dual}} = \frac{1}{4\pi} \int d^4\theta \Phi_D^\dagger e^{-V} (\mathcal{F}'_D(\Phi_D)) + \int d^4x d^2\theta \mathcal{F}_D''(\Phi_D) W_D^\alpha W_\alpha^D \tag{4.26}$$

which has a similar form to the original theory. We will see more important properties and results derived from this duality in the following sections.

4.3 Duality group

Not only do we have the electro-magnetic duality $\tau \mapsto -\frac{1}{\tau}$, we also observe that the action is invariant with $\tau \mapsto \tau + 1$. This reminds us of generators for the group $SL(2, \mathbb{Z})$ which is the full duality group of this theory. Recall the $SL(2, \mathbb{Z})$ (or *modular*) transform

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}, \tag{4.27}$$

with $ad - bc = 1$ and $a, b, c, d \in \mathbb{Z}$.

As is claimed in the previous chapter, we denote $\langle \phi \rangle = a$ and $\langle \phi_D \rangle = a_D$, and introduce the parameter $u = \langle \text{tr} \phi^2 \rangle$ on the moduli space of vacua. And since the only global coordinate on the moduli space is u , a and a_D should be treated as holomorphic functions of u .

We naturally consider the $SL(2, \mathbb{Z})$ action on local coordinate (a, a_D) because $\tau(a) = \partial a_D / \partial a$, which implies the metric (4.18) is invariant under this $SL(2, \mathbb{Z})$ transformation via

$$\begin{pmatrix} a_D \\ a \end{pmatrix} \rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a_D \\ a \end{pmatrix} \quad (4.28)$$

In terms of u we rewrite the metric as

$$ds^2 = \text{Im} \frac{da_D}{du} \frac{d\bar{a}}{d\bar{u}} du d\bar{u} = -\frac{i}{2} \left(\frac{da_D}{du} \frac{d\bar{a}}{d\bar{u}} - \frac{d\bar{a}_D}{d\bar{u}} \frac{da}{du} \right) du d\bar{u} \quad (4.29)$$

which is invariant if we plug in the transformation (4.28). This means the two dual theories, though very different in nature, describe the same low-energy physics. Since the magnetic theory is IR-free, we can analyze it easily using perturbative methods.

第 5 章 Structure of the vacuum

Now, return to the running coupling (4.14). Since u has a scaling dimension of 2, we can treat \sqrt{u} as a physical energy scale. Then above this energy scale, the $\mathcal{N} = 2$ theory is effectively a $SU(2)$ theory but below this energy scale, the non abelian gauge symmetry is broken into an abelian one. The $U(1)$ theory has no charged degree of freedom that contributes to the running coupling. This means the holomorphic coupling “stops” at \sqrt{u} and the perturbative β -function is zero when $E < \sqrt{u}$. So in this region, the nonperturbative effect dominates.

If the β -function is nonzero, τ runs with a nontrivial dependence on u , by its physical meaning as holomorphic coupling, the $\text{Im}\tau$ is bound non-negative. But if it is a bounded harmonic function on $\mathbb{C} \cong \mathbb{R}^2$, it must be constant over the entire Coulomb branch.

This seems to contradict the fact of running coupling, while this implies that the Coulomb branch is not \mathbb{C} but is a complex plane with singularities. These singularities have deep physical interpretations.

If taken along a closed loop, the local coordinate (a_D, a) will get transformed by an element of the monodromy group. In the presence of singularities, the monodromy can be nontrivial. In the following sections, we will analyze the effects caused by possible singularities in the moduli space of vacua and the nontrivial monodromies.

5.1 Monodromy at infinity

We first consider the easier case where $u \rightarrow \infty$. The theory is weakly coupled because of asymptotic freedom, thus the instanton part of $\mathcal{F}(a)$ can be ignored.

The prepotential without instantons is

$$\mathcal{F}(a) = \frac{1}{2}\tau_{\text{cl.}}a^2 + \frac{i}{2\pi}a^2 \ln \frac{a^2}{\Lambda}. \quad (5.1)$$

with $u = a^2/2$. And correspondingly

$$a_D = \frac{\partial \mathcal{F}}{\partial a} = \frac{i}{2\pi}a \left(\ln \frac{a^2}{\Lambda^2} + 1 \right). \quad (5.2)$$

Taking a closed loop around $u = \infty$ gives

$$u \mapsto e^{2\pi i}u. \quad (5.3)$$

then $a \mapsto e^{\pi i} a$. So the effect of this monodromy gives

$$\begin{aligned} a_D &\mapsto -a_D + 2a \\ a &\mapsto -a. \end{aligned} \tag{5.4}$$

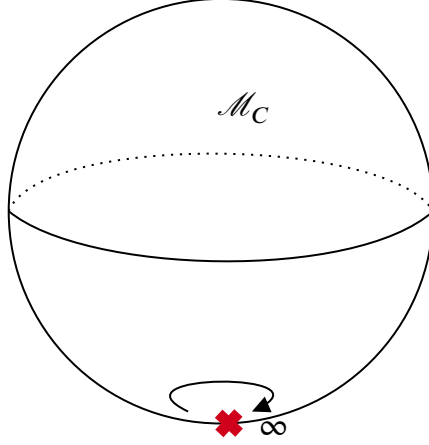


图 5.1 The monodromy at infinity.

So the transformation acts on the coordinate (a, a_D) by

$$\begin{pmatrix} a \\ a_D \end{pmatrix} \mapsto M_\infty \begin{pmatrix} a \\ a_D \end{pmatrix} \tag{5.5}$$

where we can easily write down the monodromy matrix M_∞

$$M_\infty = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix} \tag{5.6}$$

5.2 Monodromy at finite u

Now that the infinity point is a singularity, the topological triviality of the whole Coulomb branch implies that there must exist more nontrivial singularities somewhere else. And also, due to the positive-definite metric, the monodromy transformations at finite u form a subgroup of the $SL(2, \mathbb{Z})$ and are noncommutative. So except the infinity, there should be two more singularities.

Since there is a symmetry $u \leftrightarrow -u$, we can assume that the two singularities are at $u = \pm u_0$. The only dynamical scale in this theory is Λ , we can set $u = \pm \Lambda^2$ up to a constant.

At the singularity of \mathcal{M}_C , we will expect new massless degrees of freedom to appear, for example when a massive gauge boson becomes massless. Classically this happens when $u = 0$. But quantum mechanically this may shift to a non-zero u . While a

massive spin-1 multiplet becoming massless can cause inconsistency, the only possible contributions come from $\text{spin} \leq 1/2$ multiplets, namely monopoles and dyons (which have both electric and magnetic charge).

At $u = \Lambda^2$, monopoles become massless, while $a_D \rightarrow 0$. And at $u = -\Lambda^2$, the dyons become massless with $a + a_D = 0$ and $a, a_D \neq 0$.

From the electro-magnetic duality we discovered in sec.4.2, the original (or electric) theory described by Φ and W^α is effective at ∞ . So in the $a_D \rightarrow 0$, the dual magnetic coupling is $1/(\text{electric coupling})$ and thus the dual magnetic theory is weakly coupled and the description by Φ_D and $(W_D)_\alpha$ is effective. So we can work in the dual theory where we can adopt perturbative calculations.

5.2.1 The case of M_+

Using one-loop β -function, we can expand τ_D around $a_D = 0$, which reads

$$\tau_D \approx -\frac{i}{\pi} \ln a_D \quad (5.7)$$

at $u = \Lambda^2$, $a_D = 0$, a_D is a good coordinate near $u = \Lambda^2$. Then assume $a_D(u) = c(u - \Lambda^2)$ where c is a constant and integrating $\tau_D = dh_D/da_D$ over a_D , we can deduce that

$$a(u) = -h_D(u) \approx a_0 + \frac{i}{\pi} a_D \ln a_D \approx a_0 + \frac{i}{\pi} c (u - \Lambda^2) \ln (u - \Lambda^2) \quad (5.8)$$

This constant a_0 cannot be zero or all the electrically charged particles will be massless. When circles around $u = \Lambda^2$,

$$(u - \Lambda^2) \mapsto e^{2\pi i} (u - \Lambda^2) \quad (5.9)$$

a and a_D transforms as

$$\begin{aligned} a_D &\rightarrow a_D \\ a &\rightarrow a - 2a_D \end{aligned} \quad (5.10)$$

It is easy to read off the monodromy matrix

$$M_+ = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \quad (5.11)$$

5.2.2 The case of M_-

With all of the monodromies taken in the clockwise (or counter clockwise) direction as in figure 5.2, the monodromy obeys $M_\infty = M_+ M_-$. Now it is easy to find that

$$M_+ = \begin{pmatrix} -1 & -2 \\ -2 & 3 \end{pmatrix} \quad (5.12)$$

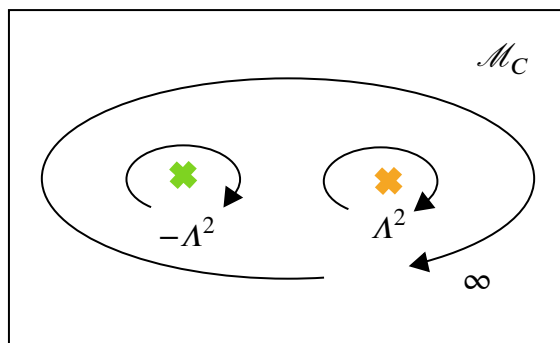


图 5.2 The monodromy in the strong coupling region.

5.3 Physical interpretation

The singularities on the moduli space of vacua indicates that some massless degrees of freedom exist. Since they cannot be gauge bosons, they are actually monopoles and dyons. At $u = \Lambda^2$ monopoles become massless and at $u = -\Lambda^2$ dyons become massless.

第 6 章 Solution of the model

Our final goal here is to determine the prepotential for $\mathcal{N} = 2$ $SU(2)$ pure SYM, so we need to perform the integral over a and a_D since $a_D = d\mathcal{F}(a)/da$.

6.1 Seiberg-Witten curve

The three monodromy matrices M_{\pm} and M_{∞} do not generate the whole $SL(2, \mathbb{Z})$ group. Instead, they generate a congruence group $\Gamma(2)$ consisting of matrices congruent to 1 modulo 2, i.e.

$$\Gamma(2) = \{M \in SL(2, \mathbb{Z}) | M_{ij} = \delta_{ij}(\text{mod } 2)\} \quad (6.1)$$

Moreover, the u -plane punctured at 1, -1 , and ∞ can be regarded as the quotient of the upper half-plane H by $\Gamma(2)$.

The family of curves parametrized by $H/\Gamma(2)$ is given by

$$y^3 = (x - \Lambda)(x + \Lambda)(x - u). \quad (6.2)$$

This is the famous *Seiberg-Witten curve*, denoted as E_u . Here u serves as a complex structure that parametrized these curves i.e. the shape of the curve varies with u , see figure 6.1 for a rough picture.

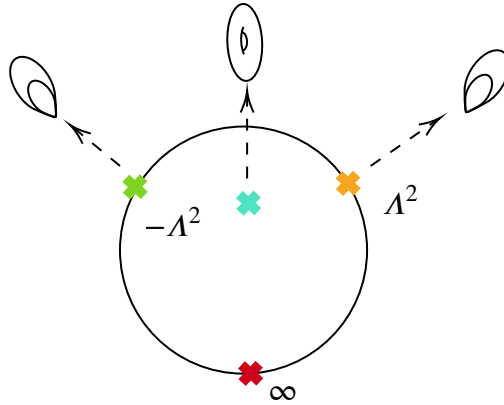


图 6.1 The Seiberg-Witten curves.

This curve E_u forms a double cover of the u plane with four branch points Λ , $-\Lambda$, u and ∞ . Consider the following way to fix the branch cut: one from Λ to Λ , the other from u to ∞ . We can easily see that the loop on the x -plane that goes around one of the two cuts corresponds to the a-cycle of the torus. On the other hand, a loop that intersects both cuts corresponds to the b-cycle on the torus, see figure 6.2.

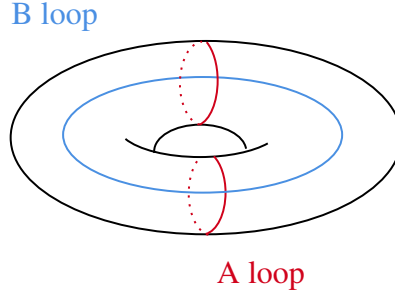


图 6.2 Schematic illustration of the A-loop and B-loop.

To find out a and a_D on the genus one Riemann surface E_u , we pick up two independent one-cycles A and B . These one-cycles form a local basis for the first homology group $H^1(E_u, \mathbb{C})$. Also, a one-cycle can be paired with elements λ from the first cohomology group

$$A, B \rightarrow \oint_{A,B} \lambda \quad (6.3)$$

We can also choose a basis for these one-forms λ on E_u . Take

$$\lambda_A = \frac{dx}{y}, \quad \lambda_B = \frac{x dx}{y}. \quad (6.4)$$

also define

$$b_{A,B} = \oint_{A,B} \lambda_{A,B} \quad (6.5)$$

Thus

$$\tau_u = \frac{b_B}{b_A} \quad (6.6)$$

serves as the modular parameter of the torus which is invariant under $SL(2, \mathbb{Z})$ action.

Pick an arbitrary section

$$\lambda = a_A(u) \lambda_A + a_B(u) \lambda_B \quad (6.7)$$

and claim that

$$a_D = \oint_{\gamma_B} \lambda, \quad a = \oint_{\gamma_A} \lambda \quad (6.8)$$

If λ is a form with vanishing residue then, on circling a singularity, a_D and a transform in the right way under $\Gamma(2)$. The above identification of a_D and a implies that

$$\frac{da_D}{du} = \oint_A \frac{d\lambda}{du}, \quad \frac{da}{du} = \oint_B \frac{d\lambda}{du}. \quad (6.9)$$

Since

$$\tau = \frac{da_D}{da} = \frac{da_D/du}{da/du} \quad (6.10)$$

we make the following identification

$$\tau = \frac{b_B}{b_A} = \tau_u. \quad (6.11)$$

The function $f(u)$ is fixed by the asymptotic behavior of the theory near the singularities on the u plane and is given by $f(u) = -\sqrt{2}/(4\pi)$. Then we get the explicit form of λ (also called *Seiberg-Witten differential*)

$$\lambda = \frac{\sqrt{2}}{2\pi} \frac{\sqrt{x-u}}{\sqrt{x^2-1}} dx \quad (6.12)$$

Now that all the ingredients are ready, recall $\tau = 1/\mathcal{F}''$, we can now get the exact form of prepotential \mathcal{F} by integrations!

6.2 Prepotential from the curve

We have the explicit form of (6.12) now but the evaluation involves using a lot of hypergeometric functions. Instead, we use a much simpler method by using *integrable systems*.

It is originally observed in^[12] that the Seiberg-Witten solution to $\mathcal{N} = 2$ SYM has nontrivial relation with complexified periodic Toda chain. This idea is further developed in^[13–14]. For a recent review, see^[15].

The spectral curve of a N -body Toda chain exactly coincides with the Riemann surface introduced in the $SU(N)$ gauge theory. In our $SU(2)$ case, the spectral curve reads

$$\Lambda^2(z + \frac{1}{z}) = (x^2 - u) \quad (6.13)$$

in which $u = p^2 - \cosh q$ with p, q generalized momentum and coordinate of the periodic Toda system. The action variable

$$a = \oint p \, dq = \oint \sqrt{u - \cosh q} \, dq = \oint x \frac{dz}{z}, \quad \lambda = x \frac{dz}{z} \quad (6.14)$$

with $z = e^q$. Around different loops this action variable will coincide with a and a_D .

$$a = \frac{1}{2\pi i} \oint_A \lambda, \quad a_D = \frac{1}{2\pi i} \oint_B \lambda. \quad (6.15)$$

Since $a_D = \partial \mathcal{F}(a)/\partial a$, integrating twice and we get the expansion of \mathcal{F} as

$$2\pi i \mathcal{F}(a) = -4a^2 \log \frac{a}{\Lambda} + \sum_{k=1}^{\infty} c_k \frac{\Lambda^{4k}}{a^{4k-2}} \quad (6.16)$$

Now we follow^[8,16] to determine the numerical factor c_k .

We adopt the renormalization group relation in^[17], which is

$$2\pi i \Lambda \frac{\partial}{\partial \Lambda} \mathcal{F}(a, \Lambda) = 4u \quad (6.17)$$

Set $u = \tilde{a}^2$ with \tilde{a} the renormalized VEV, we can perform the integration as follows

$$\begin{aligned} a(\tilde{a}) &= \frac{1}{2\pi i} \oint_A \lambda = \frac{1}{2\pi i} \oint_A \left(\sum_{l=0}^{\infty} \left(z + \frac{1}{z} \right)^l \frac{(-1)^l (2l)!}{(1-2l)(l!)^2 2^{2l}} \frac{\Lambda^{2l}}{a^{2l-1}} \right) \frac{dz}{z} \\ &= \tilde{a} \sum_{k=0}^{\infty} \frac{\Lambda^{4k}}{\tilde{a}^{4k}} \frac{1}{(1-4k)2^{4k}} \frac{(4k)!}{(2k)!k!k!}. \end{aligned} \quad (6.18)$$

A self-consistent calculation is to substitute (6.16) into (6.17), then we have

$$a^2 \left(1 + \sum_{k=1}^{\infty} k c_k \frac{\Lambda^{4k}}{a^{4k}} \right) = \tilde{a}^2 \quad (6.19)$$

Then by comparing coefficients, we can easily see the first few terms of the instanton corrections

$$c_1 = \frac{1}{2}, c_2 = \frac{5}{64}, c_3 = \frac{3}{64}, c_4 = \frac{1469}{32768}, \dots \quad (6.20)$$

第 7 章 Instanton counting

In the seminal paper^[18], Nekrasov purposed a new way to derive the prepotential of $\mathcal{N} = 2$ $SU(2)$ pure SYM directly from the so-called *instanton counting*. By considering a closely related 5d gauge theory and making use of equivariant localization, the prepotential can be precisely recovered. We will review the $SU(2)$ SYM case following^[8] and then extend it to the $SU(N)$ SYM.

7.1 From 5d to 4d

To perform instanton counting for a 4d gauge theory with gauge group G , we first consider a 5d gauge theory of the same gauge group. Then, when we compactify this theory on S^1 with radius β and then take the zero-radius limit $\beta \rightarrow 0$, we can recover the 4d theory.

Now we introduce the Ω -background. Since the 5d theory live on \mathbb{R}^5 , regard \mathbb{R}^5 as $\mathbb{C}^2 \times \mathbb{R}$ with complex coordinate (z_1, z_2) and real coordinate x_5 . Then we make the following identification

$$(z_1, z_2, x_5) \sim (z_1 e^{i\beta\epsilon_1}, z_2 e^{i\beta\epsilon_2}, x_5 + \beta) \quad (7.1)$$

To preserve supersymmetry, we need to introduce a corresponding $SU(2)_R$ rotation

$$\text{diag} \left(e^{i\beta(\epsilon_1+\epsilon_2)/2}, e^{-i\beta(\epsilon_1+\epsilon_2)/2} \right) \in SU(2)_R \quad (7.2)$$

and a gauge rotation $e^{i\beta\vec{a}} \in G$, where \vec{a} an element of Lie algebra of G .

Regarding the x_5 direction as the time direction, then the system on \mathbb{R}^4 is forced to rotate by $\beta\epsilon_{1,2}$. So any excitations far from the origin are suppressed. The effective scale of the system is $1/\epsilon_1\epsilon_2$ and thus the partition function of the system can be written as

$$\log Z(\epsilon_{1,2}, \tau_{UV}; \vec{a}) = \frac{1}{\epsilon_1\epsilon_2} F(\tau_{UV}; \vec{a}) + \text{terms less singular in } \epsilon_{1,2}. \quad (7.3)$$

in which F stands for the density of free energy, τ_{UV} is the 5d coupling constant constructed from the 't Hooft coupling g_{UV5} . The gauge theory part of the 5d Lagrangian can be written as

$$\int dx_5 \int d^4x \frac{1}{2g_{UV5}^2} \text{tr} F_{\mu\nu} F^{\mu\nu}. \quad (7.4)$$

By integrating out x_5 we can get the coupling

$$\tau_{UV} = \beta \frac{4\pi i}{g_{UV5}^2} \quad (7.5)$$

We here claim that this free energy \mathcal{F} equals the prepotential:

$$2\pi i \mathcal{F}(\tau_{UV}; \vec{a}) = F(\tau_{UV}; \vec{a}) = \lim_{\epsilon_{1,2} \rightarrow 0} \epsilon_1 \epsilon_2 \log Z(\tau_{UV}; \vec{a}) \quad (7.6)$$

So after taking the limit $\beta \rightarrow 0$, we can then get the prepotential. For proof of this claim, see^[18].

7.2 Reduction to supersymmetric quantum mechanics

The perturbative part of the partition corresponds to the one-loop part in the prepotential (4.12). What we are interested in is the nonperturbative contributions from instantons. The energy of a gauge field can be written as

$$\int d^4x \frac{1}{2g_{UV5}^2} \text{tr} F_{\mu\nu} F^{\mu\nu} \quad (7.7)$$

which is bound from below by $8\pi^2|k| = g_{UV5}^2$, and k the instanton number.

The instanton configurations can be described by the k instanton moduli space M_k . Given a point $p \in M_k$, we have an instanton solution $F_{\mu\nu}(x_{1,2,3,4}; p)$ for $\mu, \nu = 1, 2, 3, 4$. As the time $t = x_5$ changes, the shape of the instanton varies, forming a path $p(t)$ in M_k .

Then we can rewrite the 5d action for gauge theory as follows

$$\int dt \int d^4x \frac{1}{2g_{UV5}^2} \text{tr} F_{\mu\nu} F^{\mu\nu} = \int dt \left(\frac{8\pi^2}{2g_{UV5}^2} + G_{IJ}(p) \partial_t p^I(t) \partial_t p^J(t) \right). \quad (7.8)$$

The first term on the right-hand side comes from the part with x_5 , while the second term only consists of $x_{1,2,3,4}$. The index I, J runs from 1 to $\dim M_k$, and G_{IJ} serves as the metric on M_k .

Since anti-instantons only contribute to the anti-holomorphic part of the partition function, we can safely write

$$Z_{\text{non-pert.}} = \sum_{k=0}^{\infty} Z_k, \quad Z_k = e^{2\pi i \tau_{UV} k} \tilde{Z}_k. \quad (7.9)$$

where the constant energy shift comes from the terms with x_5 dependence, and \tilde{Z}_k is the partition function of the supersymmetric quantum mechanics on M_k with the zero energy at the ground state.

7.3 Instanton counting of Seiberg-Witten theory

Now we start to compute the instanton partition function (or Nekrasov partition function) for $SU(2)$ pure SYM.

7.3.1 $k = 0$ case

In $k = 0$ case the moduli space M_0 is a single point. It is a 1d Hilbert space with zero Hamiltonian, so we have

$$Z_0 = 1. \quad (7.10)$$

7.3.2 $k = 1$ case

The $k = 1$ moduli space is a little bit more complicated. We need to first specify its center-of-mass position in \mathbb{C}^2 (or \mathbb{R}^4). Also, An $SU(2)$ instanton has a size (radius) ρ , which is a non-negative real number, and a gauge direction in the group manifold. Since the gauge direction takes values in the $SO(3)$ group manifold, which is $\mathbb{R}P^2 = S^3/\mathbb{Z}_2$. Combining with ρ we get $\mathbb{C}^2/\mathbb{Z}_2$.

Then, the coordinate on the moduli space is (z_1, z_2, u, v) with $(u, v) \sim -(u, v)$. Under the rotation by $\beta\epsilon_{1,2}$, the coordinate transforms by

$$(z_1, z_2, u, v) \mapsto (e^{i\beta\epsilon_1} z_1, e^{i\beta\epsilon_2} z_2, e^{i\beta(\epsilon_1+\epsilon_2)/2} u, e^{i\beta(\epsilon_1+\epsilon_2)/2} v). \quad (7.11)$$

Under gauge rotation

$$\text{diag}(e^{i\beta a}, e^{-i\beta a}) \in SU(2) \quad (7.12)$$

z_1, z_2 remain unchanged, but u, v will transform as follows

$$(u, v) \mapsto (e^{i\beta a} u, e^{-i\beta a} v). \quad (7.13)$$

Since the supersymmetric wavefunctions are holomorphic functions of z_1 , The state whose wavefunction is z_1^n gets multiplied by $e^{-i\beta\epsilon_1}$ by the spatial rotation. The partition function of the degrees of freedom on the motion described by z_1 is by tracing over the corresponding Hilbert space. Therefore, this part contributes

$$\sum_{n=0}^{\infty} e^{in\beta\epsilon_1} = \frac{1}{1 - e^{i\beta\epsilon_1}} \quad (7.14)$$

By similar analysis we can also write down the contributions of z_2 and (u, v) , but notice that there is a constraint $(u, v) \sim -(u, v)$ so that for wave functions of the form $u^n v^m$,

only even $n + m$ survives. Therefore the complete contributions are

$$\begin{aligned} & \sum_{n \geq 0, m \geq 0, n+m: \text{ even}} e^{in\beta((\epsilon_1+\epsilon_2)/2+a)} e^{im\beta((\epsilon_1+\epsilon_2)/2-a)} \\ &= \frac{1 + e^{i\beta(\epsilon_1+\epsilon_2)}}{\left(1 - e^{i\beta(\epsilon_1+\epsilon_2+2a)}\right) \left(1 - e^{i\beta(\epsilon_1+\epsilon_2-2a)}\right)}. \end{aligned} \quad (7.15)$$

Finally, combining the constant energy shift in (7.9), the one-instanton contribution is

$$Z_1 = e^{2\pi i \tau_{UV}} \frac{1}{1 - e^{i\beta\epsilon_1}} \frac{1}{1 - e^{i\beta\epsilon_2}} \frac{1 + e^{i\beta(\epsilon_1+\epsilon_2)}}{\left(1 - e^{i\beta(\epsilon_1+\epsilon_2+2a)}\right) \left(1 - e^{i\beta(\epsilon_1+\epsilon_2-2a)}\right)}. \quad (7.16)$$

Next step would be taking the $\beta \rightarrow 0$ limit. By one-loop running for τ_{UV} , we can expect that $e^{2\pi i} \sim \beta^4$ by identifying $\Lambda \sim \beta^{-1}$ which makes this limit reasonable.

Let us take the $\beta \rightarrow 0$ limit fixing

$$\Lambda^4 := \beta^{-4} e^{2\pi i \tau_{UV}} \quad (7.17)$$

The limit is then

$$Z_1 \rightarrow \Lambda^4 \frac{1}{2} \frac{1}{\epsilon_1} \frac{1}{\epsilon_2} \frac{1}{(\epsilon_1 + \epsilon_2)/2 + a} \frac{1}{(\epsilon_1 + \epsilon_2)/2 - a} \quad (7.18)$$

Plugging into (7.6), we have

$$2\pi i \mathcal{F}_{\text{non-pert.}} = \lim_{\epsilon_{1,2} \rightarrow 0} \epsilon_1 \epsilon_2 \log \left(1 + e^{2\pi i \tau_{UV}} Z_1 + \dots \right) = \frac{1}{2} \frac{\Lambda^4}{a^2} + \dots. \quad (7.19)$$

which coincide with the corresponding term in (6.20).

Similar computation can be done for any k , and the result can be summarized in a combinatorial formula

$$\begin{aligned} Z_k &= \sum_{Y_1, Y_2} \prod_{n, m=1}^2 \prod_{s \in Y_n} \frac{1}{1 - e^{i\beta \left(-L_{Y_m}(s)\epsilon_1 + (A_{Y_n}(s)+1)\epsilon_2 + a_m - a_n \right)}} \\ &\quad \prod_{t \in Y_m} \frac{1}{1 - e^{i\beta \left((L_{Y_n}(t)+1)\epsilon_1 - A_{Y_m}(s)\epsilon_2 + a_m - a_n \right)}}. \end{aligned} \quad (7.20)$$

where the sum runs over pairs (Y_1, Y_2) of Young diagrams with the number of total boxes being k , $s \in Y$ denotes that s is a box in a Young diagram Y , and finally the functions $A_Y(s)$ and $L_Y(s)$ are the arm length and the leg length of the box s in a Young diagram Y .

For a mathematical proof that (7.20) recovers the Seiberg-Witten curve, see Okounkov-Nekrasov^[19] for a combinatorial proof, Nakajima-Yoshioka^[20] by blow-up method on instanton moduli space and Braverman-Etingof^[21] using affine Lie algebra.

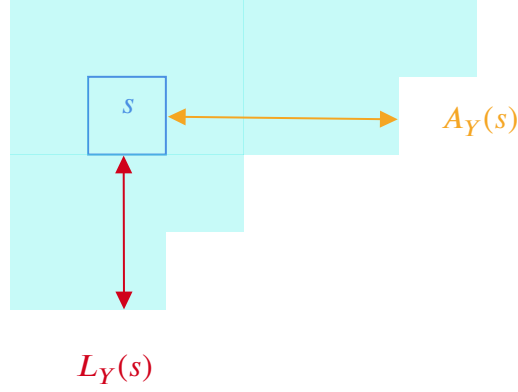


图 7.1 The definition of the arm length and the leg length in a Young diagram.

By summing over the contributing pairs of diagrams $(\square, 0)$ and $(0, \square)$ we can easily recover the result in (7.19).

For $k = 2$ case the contributing diagrams are $(\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}, 0)$, $(\square\square, 0)$, (\square, \square) , $(0, \square\square)$, $(0, \begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix})$, the computation gives that

$$Z_2 \rightarrow \Lambda^8 \frac{\left(8(\epsilon_1 + \epsilon_2)^2 + \epsilon_1 \epsilon_2 - 8a^2\right)}{\epsilon_1^2 \epsilon_2^2 \left((\epsilon_1 + \epsilon_2)^2 - 4a^2\right) \left((2\epsilon_1 + \epsilon_2)^2 - 4a^2\right) \left((\epsilon_1 + 2\epsilon_2)^2 - 4a^2\right)} \quad (7.21)$$

in the limit $\beta \rightarrow 0$.

Plugging into (7.6) we can conclude that

$$\begin{aligned} 2\pi i \mathcal{F}_{\text{non-pert}} &= \lim_{\epsilon_{1,2} \rightarrow 0} \epsilon_1 \epsilon_2 \log(1 + Z_1 + Z_2 + \dots) \\ &= \lim_{\epsilon_{1,2} \rightarrow 0} \epsilon_1 \epsilon_2 \left(Z_1 + \left(Z_2 - \frac{1}{2} Z_1^2 \right) + \dots \right) \\ &= \frac{1}{2} \frac{\Lambda^4}{a^2} + \frac{5}{64} \frac{\Lambda^8}{a^6} + \dots \end{aligned} \quad (7.22)$$

which matches with the first two terms in (6.20).

第 8 章 Extending to higher rank

Now we extend the discussion to higher rank i.e. the $SU(N)$ case. The Seiberg-Witten-like analysis is considered in^[22] and^[23]. Some parts of this chapter are modified from Yuji Tachikawa's master thesis.

8.1 The curve

For this case, the gauge group is completely broken into $U(1)^{N-1}$, and the metric $\text{Im } \tau_{ij}$ should be positive definite. The electro-magnetic duality group extends to $Sp(2N - 2, \mathbb{Z})$ accordingly.

A natural extension of the original Seiberg-Witten curve is

$$y^2 = P(x)^2 - \Lambda^{2N} \quad (8.1)$$

with

$$P(x) = \langle \det(x - \phi) \rangle = \sum_p u_p x^{n-p}. \quad (8.2)$$

so a_k and $a_{D,k}$ takes the form

$$a_k = \frac{1}{2\pi i} \oint_{A_k} \lambda, \quad a_{D,k} = \frac{1}{2\pi i} \oint_{B_k} \lambda. \quad (8.3)$$

And the Seiberg-Witten differential is now

$$\lambda = \frac{x dP(x)}{y}. \quad (8.4)$$

Next, we shall determine the prepotential by actually doing this integral. One way is to solve the Picard-Fuchs equations that a and a_D satisfy. But a simpler method is to determine the coefficients recursively, following^[16,24].

8.1.1 Classical moduli

Now we first show without proof the form of a_i in terms of the classical moduli \tilde{a}_i appearing in the definition of the curves

$$P(x) = \prod_i (x - \tilde{a}_i) \quad (8.5)$$

The quantum correction is of the form

$$a_k = \tilde{a}_k + \sum_{m=1}^{\infty} \frac{\Lambda^{2m}}{2^{2m}(m!)^2} \left(\frac{\partial}{\partial \tilde{a}_k} \right)^{2m-1} S_k(\tilde{a}_k)^m, \quad (8.6)$$

where

$$S_k(x) = \frac{\Lambda^{2(N-1)}}{\prod_{i \neq k} (x - \tilde{a}_i)^2}. \quad (8.7)$$

8.1.2 Linear recursion relation

By using the renormalization group equation derived in ^[17] for $SU(2)$ and in ^[25–26] for $SU(N)$

$$\Lambda \frac{\partial \mathcal{F}}{\partial \Lambda} = \frac{N}{\pi i} \sum_k \tilde{a}_k^2 \quad (8.8)$$

First, we calculate $\Lambda(\partial \mathcal{F} / \partial \Lambda)$ by substitute the full form of prepotential

$$\mathcal{F} = \frac{N}{\pi i} \sum_k a_k^2 + \frac{i}{4\pi} \sum_{i < j} (a_i - a_j)^2 \log \frac{(a_i - a_j)^2}{\Lambda^2} + \sum_{m=1}^{\infty} \frac{\Lambda^{2Nm}}{2m\pi i} c_m(a) \quad (8.9)$$

into (8.8), we get

$$\sum_k \tilde{a}_k^2 = \sum_k a_k^2 + \sum_{m=1}^{\infty} \Lambda^{2Nm} c_m(a) \quad (8.10)$$

Then, substituting a_k by \tilde{a}_k

$$0 = \sum_k \left(\sum_{m=0}^{\infty} \Lambda^{2Nm} \Delta_k^{(m)}(\tilde{a}) \right)^2 - \sum_k \left(\Delta_k^{(0)}(\tilde{a}) \right)^2 + \sum_{m=1}^{\infty} \Lambda^{2Nm} c_m \left(\sum_{n=0}^{\infty} \Lambda^{2Nn} \Delta_k^{(n)}(\tilde{a}) \right) \quad (8.11)$$

where we define

$$\Delta_k^{(m)}(x) = \frac{1}{2^{2m}(m!)^2} \left(\frac{\partial}{\partial x} \right)^{2m-1} S_k(x)^m, \quad \Delta_k^{(0)}(x) = \tilde{a}_k. \quad (8.12)$$

With this, we can compute to any order the instanton correction c_m recursively, by substituting \tilde{a}_k s with a_k s.

The first several contributions are

$$\begin{aligned} -c_1 &= \sum_k 2\Delta_k^{(0)}\Delta_k^{(1)} \\ -c_2 &= \sum_k \left(2\Delta_k^{(0)}\Delta_k^{(2)} + \left(\Delta_k^{(1)} \right)^2 \right) + \sum_k \Delta_k^{(1)} \frac{\partial}{\partial a_k} c_1 \\ -c_3 &= \sum_k \left(2\Delta_k^{(0)}\Delta_k^{(3)} + 2\Delta_k^{(1)}\Delta_k^{(2)} \right) \\ &\quad + \sum_k \left(\Delta_k^{(1)} \frac{\partial}{\partial a_k} c_2 + \Delta_k^{(2)} \frac{\partial}{\partial a_k} c_1 \right) + \frac{1}{2} \sum_{k,l} \Delta_k^{(1)} \Delta_l^{(1)} \frac{\partial^2}{\partial a_k \partial a_l} c_1 \end{aligned} \quad (8.13)$$

8.2 Nekrasov's Formula

The instanton counting for $SU(N)$ is similar to the $SU(2)$ case. Note that we now have N Young tables and for a k -instanton contribution we require the total number of

boxes in all Young tables is k . Here we present the formula without derivation, for a detailed calculation, see^[18,27]. By setting $\epsilon_1 = -\epsilon_2 = \epsilon$, the 5d partition function reads

$$\begin{aligned}
 Z &= \sum_k q^k \sum_{(Y_1, \dots, Y_N), \Sigma \# Y_i = k} \sum_{i,j} \sum_{s \in Y_i \cup Y_j} \frac{1}{\sinh \frac{\beta}{2} \left(a_i - a_j + \epsilon \left(-L_{Y_j}(s) - A_{Y_i}(s) - 1 \right) \right)} \frac{1}{\sinh \frac{\beta}{2} \left(a_i - a_j + \epsilon \left(L_{Y_i}(s) + A_{Y_j}(s) + 1 \right) \right)} \\
 &= \sum_k q^k \sum_{(Y_1, \dots, Y_N), \Sigma \# Y_i = k} \sum_{(i,m) \neq (j,n)} \frac{\sinh \frac{\beta}{2} (a_i - a_j + \epsilon (y_{i,n} - y_{j,m} + m - n))}{\sinh \frac{\beta}{2} (a_i - a_j + \epsilon(m - n))}
 \end{aligned} \tag{8.14}$$

in which $y_{i,m}$ is the total number of boxes in the m -th row of the i -th Young table Y_i .

Taking $\beta \rightarrow 0$ limit with Coulomb branch parameters a_i fixed, we get the 4d partition function

$$Z = \sum_k \Lambda^{2Nk} \sum_{(Y_1, \dots, Y_N), \Sigma \# Y_i = k} \sum_{(i,m) \neq (j,n)} \frac{(a_i - a_j + \epsilon (y_{i,n} - y_{j,m} + m - n))}{(a_i - a_j + \epsilon(m - n))}. \tag{8.15}$$

Recall the prepotential is related to the instanton partition function via

$$\mathcal{F} = \lim_{\epsilon \rightarrow 0} \epsilon^2 \log Z \tag{8.16}$$

Up to two instantons, the results are

$$\begin{aligned}
 c_1 &= \frac{1}{2} \sum_l \prod_{k \neq l} \frac{1}{(a_k - a_l)^2}, \\
 c_2 &= \frac{1}{4} \sum_l \sum_{k \neq l, m \neq l} \frac{1}{a_k - a_l} \frac{1}{a_m - a_l} \prod_{k \neq l} \frac{1}{(a_k - a_l)^2} + \frac{3}{8} \sum_l \sum_{k \neq l} \frac{1}{(a_k - a_l)^2} \prod_{k \neq l} \frac{1}{(a_k - a_l)^2} \\
 &\quad + \frac{1}{4} \sum_{l \neq m} \frac{1}{(a_l - a_m)^2} \prod_{k \neq l} \frac{1}{(a_k - a_l)^2} \prod_{k \neq m} \frac{1}{(a_k - a_m)^2}.
 \end{aligned} \tag{8.17}$$

This exactly agrees with the result in (8.13). For larger k , writing down the explicit expression is relatively hard. But we can verify this relation to arbitrarily high order using numerical methods. In the file attached, we numerically verified the result of the Seiberg-Witten curve with instanton partition function to $k = 4$ case.

第 9 章 Concluding remarks

In this thesis, we presented a concise review of Seiberg-Witten theory and computed explicitly the instanton contributions in the prepotential \mathcal{F} . We also presented Nekrasov's instanton counting method and derived the instanton partition functions for $SU(2)$ and $SU(3)$.

Though the discussions in this thesis are limited to pure SYM cases. The SQCDs with N_f flavor (hypermultiplets) can also be calculated via instanton counting, for details, see^[27] and related papers.

Also, we only considered the case of $SU(N)$ (or A -type), for BCD -type instanton counting, we can use a different method by taking the *Jeffery-Kirwan* residues for a supersymmetric quantum mechanics, for mathematical formulation see^[28–29]. For the BCD -type instanton counting method, see^[30–33].

For readers interested in the mathematical structure behind instanton counting, please see^[34].

Finally, there is much new progress regarding 4d $\mathcal{N} = 2$ supersymmetric gauge theories, especially the discoveries in the 6d $\mathcal{N} = (0, 2)$ theory, relations with topological string theory, the Argyres-Douglas CFTs, and the Gaiotto dualities and so on. Due to the limited knowledge of the author, we had to stop here. The author will try to explore these fascinating topics in the near future.

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致 谢

我首先要感谢我的指导老师绳田聪研究员，他耐心地为我讲解超对称的基础知识，使我能入门这一领域。我也要感谢万义顿教授，吴咏时教授和虞跃教授对我的教导，尤其是周洋研究员，我从他们那里获益良多。

同时，感谢郭家祺、王思源、吴晋渊、于鑫阳、舒畅、王逸舟、李添翼、朱津纬、菊池谦（Ken Kikuchi）、王昊、江加群和陈艳艳等友人在科研上给予我的帮助。

感谢周韞斐同学给予我的温暖，让我在困窘的日子里有所慰藉。

最后我要感谢我的家人对我的一贯支持。

附 录

Using the `Mathematica` file

In this `Mathematica` file attached we provide the computer program used to calculate instanton coefficients. In the Chapter 7 part, we checked the instanton coefficients of pure $SU(2)$ SYM using the Nekrasov partition function. We checked this up to 4 instantons.

In Chapter 8 part, we first use the linear recursion in Chan-D'Hoker^[16] to derive the instanton coefficients of pure $SU(2)$ SYM again. We checked this up to 4 instantons.

Then we calculated pure $SU(3)$ SYM by both linear recursion and instanton counting, and found exact agreement up to 4 instantons.

Due to the limit of computational power, we cannot do more. Also, the file is created with `Mathematica` version 13.0.1.

指导教师对论文学术规范的审查意见：

☐ 本人经过尽职审查，未发现毕业论文有学术不端行为。

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日期： 20 年 月 日

指导教师评语：

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20 年 月 日

签名：

20 年 月 日

学分

成绩

备注：

教务处制