Homework 4: Due at class on April 1

1. Let $\iota : S^2 \hookrightarrow \mathbb{R}^3$ be the inclusion of the 2-sphere with unit radius. Let $g : ds^2 = \sum_{i=0}^{2} dx^i \otimes dx^i$ be the standard metric of $\mathbb{R}^3$. Find the induced metric $\iota^* g$ on S^2 in terms of the polar coordinate of $\mathbb{R}^3$.

$$\begin{aligned} x^0 &= r \sin\theta \cos\phi \\ x^1 &= r \sin\theta \sin\phi \\ x^2 &= r \cos\theta \end{aligned}$$

Given this metric, find geodesics on S^2 and compute its Riemann, Ricci and scalar curvature.

Do parallel transport of a vector along a triangle ΔPQR on a unit sphere (Figure 1) with respect to the Levi-Civita connection of the metric and find the angle difference when it comes back. Compare it with the area of the triangle (see Homework 1).

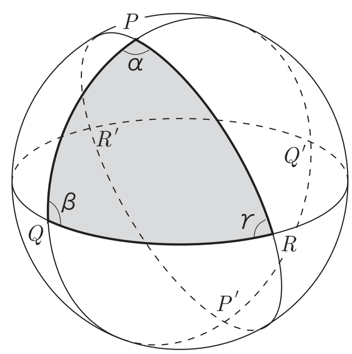

Figure 1: A triangle on a 2-sphere

2. Let $ds^2 = -dt^2 + dx^2 + dy^2$ be the Minkowski metric on $\mathbb{R}^{1,2}$ and $-t^2 + x^2 + y^2 = -1$ for $t > 0$ be the space-like surface (hyperboloid $\mathbf{S}$). (See Figure 1.) Find the induced metric on the hyperboloid $\mathbf{S}$ in terms of the polar coordinate

$$\begin{aligned} t &= r \cosh\rho \\ x &= r \sinh\rho \cos\phi \\ y &= r \sinh\rho \sin\phi \end{aligned}$$

Given this metric, find geodesics on $\mathbf{S}$ and compute its Riemann, Ricci and scalar curvature.

3. Let us consider the unit disk $\mathbf{D}$ on the x-y plane. Let $\pi(P)$ be the intersection point of the unit disk and the line between the point $(0, 0, -1)$ and $P \in \mathbf{S}$. By assigning $\pi(P)$ to P, there is one-to-one map from the unit disk D and the hyperboloid $\mathbf{S}$. Show that the map $\pi^{-1} : \mathbf{D} \to \mathbf{S}$ is determined by

$$(u, v) \mapsto \left(\frac{2u}{1 - u^2 - v^2}, \frac{2v}{1 - u^2 - v^2}, \frac{1 + u^2 + v^2}{1 - u^2 - v^2} \right) .$$

Find the metric on the unit disk pull-backed by this map. The unit disk **D** with this induced metric is called the Poincaré disk.

The red curve on the hyperboloid is an intersection with a plane (pink in Figure 2,3) that goes through the origin. We put a disk (called Klein's disk) on the bottom of the hyperboloid, which allows us to get the corresponding straight line (green) on Klein's disk. This curve is mapped by $\pi : \mathbf{S} \to \mathbf{D}$ to an (red) arc in the Poincare disk. If the green line is represented by $x = a$ (Figure 4), find the equation for the red curve.

Find geodesics on **D** and compute its Riemann, Ricci and scalar curvature.

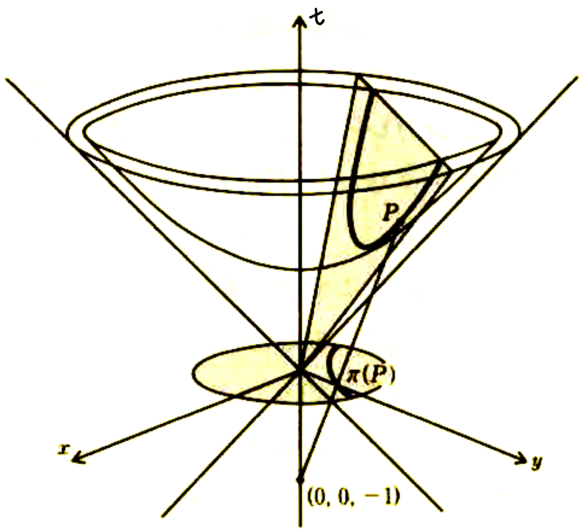

Figure 2: Hyperboloid and Poincare disk

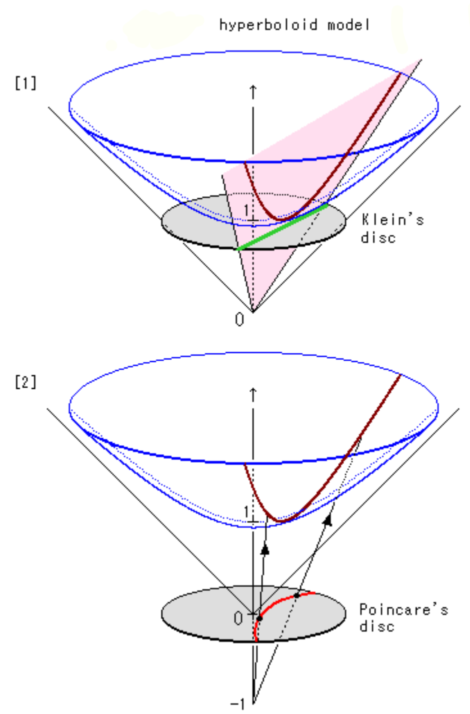

Figure 3: Hyperboloid and Poincare disk

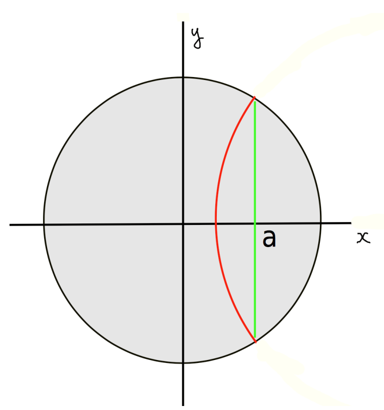

Figure 4: Poincare disk