Homework 2: Due at class on Oct 18

1 Lorentz group

1.1

Show that the generators

$$J^{\mu\nu} = i\left(x^{\mu}\partial^{\nu} - x^{\nu}\partial^{\mu}\right)$$

obey the Lorentz algebra

$$\left[J^{\mu\nu}, J^{\rho\sigma}\right] = i\left(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho}\right) \ . \tag{1}$$

Find the commutation relations of $J^{\mu\nu}$ with the generators of translations $P^{\rho} = -i\partial^{\rho}$. These define the Lie algebra of the Poincare group (Lorentz + translations).

1.2

A 4-vector V is transformed as $V \to \exp\left(-\frac{i\omega_{\mu\nu}J^{\mu\nu}}{2}\right)V$ under a Lorentz transformation. Show that an infinitesimal Lorentz transformation can be written as

$$-\frac{i\omega_{\mu\nu}J^{\mu\nu}}{2} = -\sum_{a} i\left(\theta_a L_a + \beta_a K_a\right) = \begin{pmatrix} 0 & -\beta_1 & -\beta_2 & -\beta_3 \\ -\beta_1 & 0 & -\theta_3 & \theta_2 \\ -\beta_2 & \theta_3 & 0 & -\theta_1 \\ -\beta_3 & -\theta_2 & \theta_1 & 0 \end{pmatrix}$$

where K_a and L_a generate Lorentz boosts and space-rotations, respectively. Rewrite the Lorentz algebra in terms of K_a and L_a.

1.3

Show that the matrices

$$J_a^{+} \equiv \frac{1}{2}\left(L_a + iK_a\right), \quad J_a^{-} \equiv \frac{1}{2}\left(L_a - iK_a\right)$$

satisfy the commutation relations

$$\left[J_a^{+}, J_b^{+}\right] = i\epsilon_{abc}J_c^{+} \ , \quad \left[J_a^{-}, J_b^{-}\right] = i\epsilon_{abc}J_c^{-} \ , \quad \left[J_a^{+}, J_b^{-}\right] = 0 \ .$$

Therefore, the Lorentz algebra can be understood as two copies of the $\mathfrak{su}(2)$ algebra

$$\left[s^a, s^b\right] = i\epsilon^{abc}s^c$$

where $s^a = \sigma^a/2$, $(a = 1, 2, 3)$ are a half of the Pauli matrices.

2 Clifford algebra

2.1

The gamma matrices form the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \,. \tag{2}$$

Show that the generators of Lorentz transformations in the Dirac spinor representation

$$S^{\mu\nu} = \frac{i}{4}\left[\gamma^\mu, \gamma^\nu\right]$$

satisfy the Lorentz algebra (1). Compute the commutation relation $[S^{\mu\nu}, \gamma^\rho]$, and compare it with Problem 1.1.

2.2

Use the Clifford algebra (2) to show that $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ obeys

$$\left\{\gamma^5, \gamma^\mu\right\} = 0, \quad \left(\gamma^5\right)^2 = 1 \,.$$

Show that the chirality projection operators

$$P_L = \frac{1-\gamma^5}{2}, \quad P_R = \frac{1+\gamma^5}{2}$$

satisfy $P_L^2 = P_L, \quad P_R^2 = P_R, \quad P_L P_R = P_R P_L = 0.$

2.3

Show that when $m = 0$, $\psi \to e^{i\theta\gamma^5}\psi$ is a symmetry of the Dirac Lagrangian, which is called the **axial symmetry**. Find the corresponding Noether current.

2.4

Check that the basis of the gamma matrices chosen in Peskin-Schroeder

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \tag{3}$$

obey the Clifford algebra (2). Express γ^5 in this basis.

3 Magnetic moment from Dirac equation

The Dirac equation in the presence of the electromagnetic field is

$$(i\not{D} - m)\psi = 0$$

where $D_\mu = \partial_\mu + ieA_\mu$. Using the convention (3) of the gamma matrices in Peskin-Schroeder, show that non-relativistic limit of the equation for the plane wave

$$\psi(x) = \begin{pmatrix} u_L(p) \\ u_R(p) \end{pmatrix} e^{-ip\cdot x}$$

is

$$H_{\rm nr} = \frac{1}{2m}\boldsymbol{\sigma}\cdot(\mathbf{p} - e\mathbf{A})\boldsymbol{\sigma}\cdot(\mathbf{p} - e\mathbf{A}) + eA^0$$

Rewrite this Hamiltonian in the following form

$$H_{\rm nr} = \frac{1}{2m}(\mathbf{p} - e\mathbf{A})^2 - g\frac{e}{2m}\mathbf{s}\cdot\mathbf{B} + eA^0$$

and read off the g-factor. Here $\mathbf{s} = \boldsymbol{\sigma}/2$.

4 Spinor identities

4.1

In the lecture, we see that the plane wave solutions to the Dirac equation are given by

$$\psi = u_s(p)e^{-ip\cdot x}\ , \quad \psi = v_s(p)e^{ip\cdot x}\ ,$$

where

$$u_s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi_s \\ \sqrt{p\cdot\bar{\sigma}}\xi_s \end{pmatrix}\ , \quad v_s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi_s \\ -\sqrt{p\cdot\bar{\sigma}}\xi_s \end{pmatrix}\ .$$

Taking the Weyl spinor basis as $\xi_s^\dagger\xi_{s'} = \delta_{ss'}$, show that if we sum over polarization states of the spinor, we have

$$\sum_s u_s(p)\overline{u}_s(p) = \not{p} + m\ , \qquad \sum_s v_s(p)\overline{v}_s(p) = \not{p} - m\ .$$

Note that the formulas implicitly involves the spinor indices, which can be explict as

$$\sum_s u_{sa}(p)\overline{u}_{sb}(p) = \gamma^\mu_{ab}p_\mu + m\delta_{ab}\ .$$

4.2

Show $\overline{u}_s(p)\gamma^\mu u_{s'}(p) = 2p^\mu\delta_{ss'}$. This is a simple version of the **Gordon identity**.