Homework 7: Due at class on Nov 10

Prob. 1 Open string spectrum

Consider an open bosonic string parametrized by $0 \le \sigma \le \pi$, stretching between two D-branes, one is D23 located at $(X^{24}, X^{23}) = (c_x, c_y)$ and the other is D24 located at $X^{24} = c$.

- Illustrate the D-branes and the string in (X^{24}, X^{23})-plane.
- State the boundary conditions for X^{22}, X^{23} and X^{24} at each end of the string.
- Derive the expression of the mode expansion of X^{22}, X^{23} and X^{24}. The result should be the form

$$X^\mu = A^\mu \sigma^+ + B^\mu \sigma^- + C^\mu + i\sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}, n\neq 0} \frac{1}{n} \alpha_n^\mu \left(f(\sigma^-) + \widetilde{f}(\sigma^+) \right) \ , \tag{1}$$

  where the normalizations of $f(\sigma^-)$ and $\widetilde{f}(\sigma^+)$ are given by $|f(0)| = \left|\widetilde{f}(0)\right| = 1$. (Derive A^μ, B^μ, C^μ, $f(\sigma^-)$, $\widetilde{f}(\sigma^+)$, and conditions for α_n^μ (if needed) for $\mu = 22,\ 23,\ 24$.)

- Derive zero modes α_0^μ for $\mu = 22,\ 23,\ 24$ from $a_0^\mu = \frac{1}{2\pi}\sqrt{\frac{2}{\alpha'}} \int_0^{2\pi} d\sigma^+ \partial_+ X^\mu$, where $\partial_+\sigma^+ = 1$ and $\partial_+\sigma^- = 0$.
- Derive the mass spectrum by light-cone quantization. You can use

$$2p^-p^+ = \frac{1}{\alpha'} \sum_{i=1}^{24} \sum_{m=1}^{\infty} \left(\alpha_{-m}^i \alpha_m^i \right) - \frac{1}{\alpha'} \frac{11}{12} + \frac{1}{2\alpha'} \sum_{i=1}^{24} \alpha_0^i \alpha_0^i. \tag{2}$$

  It would be nice if you can explain the factor of $\frac{11}{12}$ (this is not mandatory but you should be able to do).

- What is the condition for c to have a non-negative spectrum ?

Prob. 2 Partition functions

Prob. 2.1 S^1 partition function of bosons

Consider two harmonic oscillators on S^1 (time direction is compactified and its period is β) with the Hamiltonian

$$H = a^\dagger a + b^\dagger b \ . \tag{3}$$

Their commutation relations are

$$[a, a^\dagger] = 1 \ , \quad [b, b^\dagger] = 2 \ , \tag{4}$$

otherwise zero. Derive the partition function of this theory:

$$Z = \mathrm{tr}\left[e^{-\beta H} \right] \tag{5}$$

Prob. 2.2 S^1 partition function of fermions

Consider two fermions on S^1 with Hamiltonian

$$H = d^\dagger d + 2 f^\dagger f \ . \tag{6}$$

Non-trivial anti-commutation relations are

$$\{d, d^\dagger\} = 1 \ , \quad \{f, f^\dagger\} = 1 \ . \tag{7}$$

Derive the partition function of this theory with a periodic boundary condition

$$Z = \mathrm{tr}\left[(-1)^F e^{-\beta H}\right] \ , \tag{8}$$

as well as an anti-periodic boundary condition

$$Z = \mathrm{tr}\left[e^{-\beta H}\right] \ . \tag{9}$$

Note that

$$\{(-1)^F, d\} = 0, \quad \{(-1)^F, d^\dagger\} = 0, \quad \{(-1)^F, f\} = 0, \quad \{(-1)^F, f^\dagger\} = 0 \ , \tag{10}$$

and assume

$$(-1)^F |0\rangle = -|0\rangle \ . \tag{11}$$

Prob. 2.3 Torus partition function of X

Derive the torus partition function of matter sector of the bosonic string $\langle 1 \rangle_X$.

Prob. 2.4 Torus partition function of ghosts

Derive the torus partition function of holomorphic ghost sector of the bosonic string

$$\mathrm{tr}\left[(-1)^F b_0 c_0 q^{L_0 - \frac{c_{gh}}{24}}\right] \ , \tag{12}$$

where $c_{gh} = -26$, and

$$L_0 = \left(\sum_{n \in \mathbb{Z}} n \, :b_{-n} c_n:\right) - 1 \ . \tag{13}$$

The non-trivial anti-commutation relations are

$$\{c_m, b_n\} = \delta_{m+n,0} \ . \tag{14}$$

Note that c_m $(m \leq 0)$ and b_n $(n < 0)$ are creation op, and c_m $(m > 0)$ and b_n $(n \geq 0)$ are anihilation op.

Prob. 2.5 Torus partition function of the bosonic string

- Derive the number of the second excited states of the bosonic string from an expansion of $|\eta(\tau)|^{-48}$.
- The modular transformations of the τ-function are given by

$$T: \quad \eta(\tau + 1) = e^{i\pi/12} \eta(\tau) \ , \tag{15}$$

$$S: \quad \eta(-1/\tau) = \sqrt{-i\tau}\, \eta(\tau) \ . \tag{16}$$

Show the torus partition function of the bosonic string is modular invariant.