Homework 3: Due at class on Oct 9

1 Derivations

1.1 Energy-momentum tensor in complex coordinate

Show (3.46) by performing the coordinate transformation.

1.2 Two-point function

Obtain (3.69) from (3.68).

1.3 Virasoro algebra

Derive the Virasoro algebra (3.110) from (3.109) by performing the contour integral.

2 Vertex operator and OPE

Show that $:e^{ik\varphi}:$ is a primary field in the free boson theory and find its conformal dimension. In addition, show that $\partial^n\varphi$ ($n \geq 2$) is not a primary field.

3 Schwarzian derivatives

In the lecture note (3.87), we have learned that, under the conformal transformation $z \to w(z)$, the energy-momentum tensor transforms

$$T(w) = \left(\frac{dw}{dz}\right)^{-2}\left[T(z) - \frac{c}{12}\{w;z\}\right],$$

where $\{w;z\}$ is the additional term called the Schwarzian derivative:

$$\{w;z\} = \frac{w'''}{w'} - \frac{3}{2}\left(\frac{w''}{w'}\right)^2.$$

3.1

Derive the infinitesimal transformation (3.86) from the finite version (3.87).

3.2 Under $\mathrm{SL}(2,\mathbb{C})$

For an element of $\mathrm{SL}(2,\mathbb{C})$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C})$$

show that

$$\{w;z\} = 0 \quad \text{for } w = \frac{az+b}{cz+d} ,$$

and

$$\left\{\frac{aw+b}{cw+d};z\right\} = \{w;z\} .$$

Show that the energy-momentum tensor $T(z)$ is a quasi-primary but not primary.

3.3 Free boson (Bonus problem: 2pt)

The energy-momentum tensor of the free boson is

$$T(z) = -\frac{1}{2} : \partial_z\varphi\partial_z\varphi :$$

where the normal ordering can be defined as

$$: \partial_z\varphi\partial_z\varphi := \lim_{w\to z}\left(\partial_z\varphi(z)\partial_w\varphi(w) + \frac{1}{(z-w)^2}\right) .$$

Since $\partial_z\varphi$ is the primary field of conformal dimension one, it transforms as

$$\partial_z\varphi(z)\partial_w\varphi(w) = f'(z)f'(w)\partial_{\widetilde{z}}\varphi(\widetilde{z})\partial_{\widetilde{w}}\varphi(\widetilde{w})$$

under the conformal transformation $z \to \widetilde{z} = f(z)$. Hence we have

$$: \partial_z\varphi(z)\partial_w\varphi(w) : -\frac{1}{(z-w)^2} = f'(z)f'(w)\left[: \partial_{\widetilde{z}}\varphi(\widetilde{z})\partial_{\widetilde{w}}\varphi(\widetilde{w}) : -\frac{1}{(\widetilde{z}-\widetilde{w})^2}\right]$$

Taking limit $z \to w$, show that

$$\lim_{z\to w}\left[\frac{f'(z)f'(w)}{(f(z)-f(w))^2} - \frac{1}{(z-w)^2}\right] = \frac{1}{6}\{f(w);w\} .$$

What is the central charge of the free boson?