Homework 2: Due at class on Oct 12

1 Lorentz group

1.1

Show that the generators
$$J^{\mu\nu} = i\left(x^\mu \partial^\nu - x^\nu \partial^\mu\right)$$
obey the Lorentz algebra
$$\left[J^{\mu\nu}, J^{\rho\sigma}\right] = i\left(g^{\nu\rho} J^{\mu\sigma} - g^{\mu\rho} J^{\nu\sigma} - g^{\nu\sigma} J^{\mu\rho} + g^{\mu\sigma} J^{\nu\rho}\right) . \tag{1.1}$$

Find the commutation relations of $J^{\mu\nu}$ with the generators of translations $P^\rho = -i\partial^\rho$. These define the Lie algebra of the Poincare group (Lorentz + translations).

1.2

A 4-vector V is transformed as $V \to \exp\left(-\frac{i\omega_{\mu\nu}J^{\mu\nu}}{2}\right)V$ under a Lorentz transformation. Show that an infinitesimal Lorentz transformation can be written as
$$-\frac{i\omega_{\mu\nu}J^{\mu\nu}}{2} = -\sum_a i\left(\theta_a L_a + \beta_a K_a\right) = \begin{pmatrix} 0 & -\beta_1 & -\beta_2 & -\beta_3 \\ -\beta_1 & 0 & -\theta_3 & \theta_2 \\ -\beta_2 & \theta_3 & 0 & -\theta_1 \\ -\beta_3 & -\theta_2 & \theta_1 & 0 \end{pmatrix}$$
where K_a and L_a generate Lorentz boosts and space-rotations, respectively. Rewrite the Lorentz algebra in terms of K_a and L_a.

1.3

Show that the matrices
$$J_a^+ \equiv \frac{1}{2}\left(L_a + iK_a\right), \quad J_a^- \equiv \frac{1}{2}\left(L_a - iK_a\right)$$
satisfy the commutation relations
$$\left[J_a^+, J_b^+\right] = i\epsilon_{abc} J_c^+ , \quad \left[J_a^-, J_b^-\right] = i\epsilon_{abc} J_c^- , \quad \left[J_a^+, J_b^-\right] = 0 .$$
Therefore, the Lorentz algebra can be understood as two copies of the $\mathfrak{su}(2)$ algebra
$$\left[s^a, s^b\right] = i\epsilon^{abc} s^c$$
where $s^a = \sigma^a/2$, $(a = 1, 2, 3)$ are a half of the Pauli matrices.

2 Derivations

2.1

Derive the canonical anticommutation relations

$$\begin{aligned}\left\{\psi_a(\boldsymbol{x}), \psi_b^\dagger(\boldsymbol{y})\right\} &= \delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})\delta_{ab} \\ \{\psi_a(\boldsymbol{x}), \psi_b(\boldsymbol{y})\} = \left\{\psi_a^\dagger(\boldsymbol{x}), \psi_b^\dagger(\boldsymbol{y})\right\} &= 0\end{aligned} \tag{2.1}$$

from the following relations

$$\left\{a_{\boldsymbol{p}}^r, a_{\boldsymbol{q}}^{s\dagger}\right\} = \left\{b_{\boldsymbol{p}}^r, b_{\boldsymbol{q}}^{s\dagger}\right\} = (2\pi)^3\delta^{(3)}(\boldsymbol{p}-\boldsymbol{q})\delta^{rs}, \tag{2.2}$$

and others anti-commutes. Or the other way around, namely derive from ψ-$\psi^\dagger$ anticommutation relations to a-b anticommutation relations.

2.2

Derive the expression of the Hamiltonian

$$H = \int \frac{d^3p}{(2\pi)^3} E_p \sum_s \left[a_{\boldsymbol{p}}^{s\dagger}a_{\boldsymbol{p}}^s + b_{\boldsymbol{p}}^{s\dagger}b_{\boldsymbol{p}}^s - (2\pi)^3\delta(0)\right]$$

from

$$\mathcal{H} = \overline{\psi}\left(-i\gamma^i\partial_i + m\right)\psi\,. \tag{2.3}$$

2.3

Show that when $m = 0$, $\psi \to e^{i\theta\gamma^5}\psi$ is a symmetry of the Dirac Lagrangian, which is called the **axial symmetry**. Find the corresponding Noether current.

3 Magnetic moment from Dirac equation

The Dirac equation in the presence of the electromagnetic field is

$$(i\not{D} - m)\psi = 0$$

where $D_\mu = \partial_\mu + ieA_\mu$. Using the convention of the gamma matrices in Peskin-Schroeder, show that non-relativistic limit of the equation for the plane wave

$$\psi(x) = \begin{pmatrix} u_L(p) \\ u_R(p)\end{pmatrix} e^{-ip\cdot x}$$

is

$$H_{\text{nr}} = \frac{1}{2m}\boldsymbol{\sigma}\cdot(\boldsymbol{p}-e\boldsymbol{A})\boldsymbol{\sigma}\cdot(\boldsymbol{p}-e\boldsymbol{A}) + eA^0$$

Rewrite this Hamiltonian in the following form

$$H_{\text{nr}} = \frac{1}{2m}(\boldsymbol{p}-e\boldsymbol{A})^2 - g\frac{e}{2m}\boldsymbol{s}\cdot\boldsymbol{B} + eA^0$$

and read off the g-factor. Here $\boldsymbol{s} = \boldsymbol{\sigma}/2$.

4 Spinor identities

4.1

In the lecture, we see that the plane wave solutions to the Dirac equation are given by

$$\psi = u_s(p)e^{-ip\cdot x}\,, \quad \psi = v_s(p)e^{ip\cdot x}\,,$$

where

$$u_s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi_s \\ \sqrt{p\cdot\overline{\sigma}}\xi_s \end{pmatrix}\,, \quad v_s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi_s \\ -\sqrt{p\cdot\overline{\sigma}}\xi_s \end{pmatrix}\,.$$

Taking the Weyl spinor basis as $\xi_s^\dagger\xi_{s'} = \delta_{ss'}$, show that if we sum over polarization states of the spinor, we have

$$\sum_s u_s(p)\overline{u}_s(p) = \not{p} + m\,, \qquad \sum_s v_s(p)\overline{v}_s(p) = \not{p} - m\,.$$

Note that the formulas implicitly involves the spinor indices, which can be explict as

$$\sum_s u_{sa}(p)\overline{u}_{sb}(p) = \gamma^\mu_{ab}p_\mu + m\delta_{ab}\,.$$

4.2

Show $\overline{u}_s(p)\gamma^\mu u_{s'}(p) = 2p^\mu\delta_{ss'}$. This is a simple version of the **Gordon identity**.

5 Bonus problems

5.1

The gamma matrices form the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\,. \tag{5.1}$$

Show that the generators of Lorentz transformations in the Dirac spinor representation

$$S^{\mu\nu} = \frac{i}{4}\left[\gamma^\mu, \gamma^\nu\right]$$

satisfy the Lorentz algebra (1.1). Compute the commutation relation $[S^{\mu\nu}, \gamma^\rho]$, and compare it with Problem 1.1.

5.2

Use the Clifford algebra (5.1) to show that $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ obeys

$$\left\{\gamma^5, \gamma^\mu\right\} = 0, \quad \left(\gamma^5\right)^2 = 1\,.$$

Show that the chirality projection operators

$$P_- = \frac{1-\gamma^5}{2}, \quad P_+ = \frac{1+\gamma^5}{2}$$

satisfy $P_-^2 = P_-, \quad P_+^2 = P_+, \quad P_-P_+ = P_+P_- = 0.$

5.3

Check that the basis of the gamma matrices chosen in Peskin-Schroeder

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \tag{5.2}$$

obey the Clifford algebra (5.1). Express γ^5 in this basis.