Homework 3: Due at class on March 25

1. Let $\varphi_t : M \to M$ be the flow generated by a vector field $X \in \mathfrak{X}(M)$. Then, show that for another vector field $Y \in \mathfrak{X}(M)$

$$[X, Y] = \lim_{t\to 0} \frac{(\varphi_{-t})_* Y - Y}{t}$$

2. Given a vector field $X \in \mathfrak{X}(M)$, the Lie derivative

$$L_X : \Omega^k(M) \to \Omega^k(M)$$

can be defined by

$$L_X(\omega) = \lim_{t\to 0} \frac{\varphi_t^* \omega - \omega}{t}$$

where $\varphi_t : M \to M$ be the flow as above. Show that it satisfies

$$L_X\omega(X_1, \cdots, X_k) = X\omega(X_1, \cdots, X_k) - \sum_{I=1}^{k} \omega(X_1, \cdots, [X, X_i], \cdots, X_k)$$

for $X_1, \cdots, X_k \in \mathfrak{X}(M)$.

3. Define an n-form ω on the space $\mathbb{R}^{n+1}\backslash\{0\}$, obtained from $\mathbb{R}^{n+1}$ by removing the origin, by

$$\omega = \frac{1}{|x|^{n+1}} \sum_{i=0}^{n} (-1)^i x^i dx^0 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n \ .$$

Prove that $d\omega = 0$. Let $\iota : S^n \hookrightarrow \mathbb{R}^{n+1}$ be the unit sphere in $\mathbb{R}^{n+1}$. Show that $\iota^*\omega$ is the generator of n-the de Rham cohomology $H^n_{dR}(S^n)$ of the n-sphere. In fact, the de Rham cohomology of S^n is

$$H^k_{dR}(S^n) = \begin{cases} \mathbb{R} & k = 0,\ n \\ 0 & \text{otherwise} \end{cases} .$$

In the case of $n = 2$, evaluate the integral

$$\int_{S^2} \iota^* \omega \ .$$

4. Let $T^2 = S^1 \times S^1$ be the torus. Using the formula for de Rham cohomology of a product space,

$$H^k_{dR}(M \times N) \cong \bigoplus_{k=p+q} H^p_{dR}(M) \otimes H^q_{dR}(N) \ ,$$

find de Rham cohomology $H^*_{dR}(T^2)$. Find the Poincare dual of each generator of $H^*_{dR}(T^2)$.

5. The Maxwell equations are written as

$$\nabla \cdot \mathbf{E} = \rho \ , \qquad \nabla \times \mathbf{B} = \mathbf{J} + \frac{\partial \mathbf{E}}{\partial t} \ ,$$

$$\nabla \cdot \mathbf{B} = 0\,, \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}\,.$$

Let us write the gauge potential

$$A = A_\mu dx^\mu = \phi dt + A_1 dx^1 + A_2 dx^2 + A_3 dx^3$$

and the current

$$J = J_\mu dx^\mu = \rho dt + J_1 dx^1 + J_2 dx^2 + J_3 dx^3\,.$$

Then, the field strength can be written as $F = dA$. Show that the Maxwell equations are equivalent to the following equations

$$dF = 0\,, \qquad \delta F = -j\,.$$

Find the equation of motion for the following action

$$S = -\frac{1}{2}\int F \wedge *F - \int A \wedge *J\,.$$

Discuss this for both a positive definite metric and a Lorentzian signature metrics.

6. (This is a bonus problem with extra 3 points which is NOT mandatory.) Find the preimage of a smooth map $f : \mathrm{SO}(3) \to \mathrm{SO}(3); g \mapsto g^2$ and the rank of f_* at every point of SO(3). [Hint: An element $g \neq 1 \in \mathrm{SO}(3)$ is an rotation along a certain axis in $\mathbb{R}^3$.]