Knot homologies and Super-A-polynomials

based on
joint work with Ramadevi and Zodinmawia [arXiv:1209.1409]
& on-going work with Gukov and Stosic

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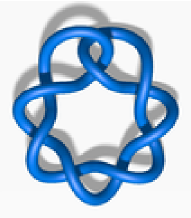


Chern-Simons theory and knot invariants

- Chern-Simons theory with gauge group G on a three-manifold M is given in terms of the path integral

$$Z_{CS}(M) = \int \mathcal{D}A e^{\frac{ik}{4\pi} S_{CS}(M;A)}$$

where $k \in \mathbb{Z}$ is the level and $S_{CS}(A)$ is

$$S_{CS}(M;A) = \int_M \mathrm{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A)$$

- The theory it is topological since the action is independent of the metric on M.
- The natural observable in the theory is the Wilson loop

$$W_R(K) = \mathrm{Tr}_R P \exp \oint_K A$$

associated to a knot K in M taken in various representations R of the gauge group G

- Expectation values of Wilson loops in CS theory become knot invariants [Witten '88]

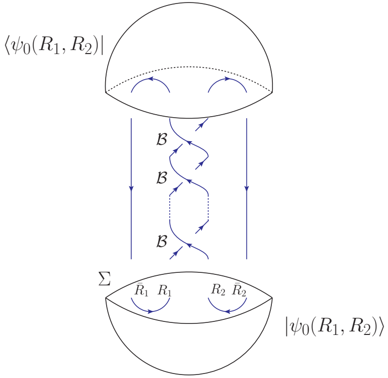
$$\langle W_R(K) \rangle = \int \mathcal{D}A \, e^{\frac{ik}{4\pi} S_{CS}(A)} \, W_R(K)$$

Chern-Simons theory and knot invariants

- Colored Jones polynomial $J_R(K; q)$ is obtained from Chern-Simons theory with $SU(2)$ gauge group where $q = \exp(\frac{2\pi i}{k+2})$
- Expectation value of Wilson loop with $G = SU(N)$ computes the HOMFLY polynomial $P(K; a, q)$ where $a = q^N$ and $q = \exp(\frac{2\pi i}{k+N})$

  - Practically, evaluations of Wilson loop are done by using braid operations on WZW conformal blocks

$$
\begin{aligned}
& \langle W_R(T^{2,2p+1}) \rangle \\
= \; & \langle \psi_0(R) | \mathcal{B}^{2p+1} | \psi_0(R) \rangle \\
= \; & \sum_{Q \in R \otimes R} \lambda_Q(R,R)^{2p+1} N_{RR}^{Q} \dim_q Q
\end{aligned}
$$



Skein relations for polynomial knot invariants

- Alexander polynomials $\Delta(K;q)$

$\Delta\Big($  $\Big) - \Delta\Big($  $\Big) = \left(q^{1/2} - q^{-1/2}\right)\Delta\Big($  $\Big)$

For trefoil, $\Delta\Big($  $\Big) = q - 1 + q^{-1}$

- Jones polynomials $J(K;q)$

$qJ\Big($  $\Big) - q^{-1}J\Big($  $\Big) = \left(q^{1/2} - q^{-1/2}\right)J\Big($  $\Big)$

For trefoil, $J\Big($  $\Big) = q + q^3 - q^4$

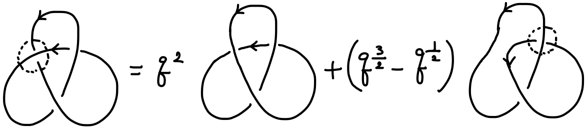
- HOMFLY polynomials $P(K;a,q)$

$a^{1/2}P\Big($  $\Big) - a^{-1/2}\Delta\Big($  $\Big) = \left(q^{1/2} - q^{-1/2}\right)P\Big($  $\Big)$

For trefoil, $P\Big($  $\Big) = aq^{-1} + aq - a^2$

Skein relations for polynomial knot invariants

- Jone polynomial $J($  $; q)$ of trefoil as example for computation.



$$= q^2 + (q^{\frac{3}{2}} - q^{\frac{1}{2}})\left(q^2 \bigcirc\bigcirc + (q^{\frac{3}{2}} - q^{\frac{1}{2}}) \bigcirc\right)$$

$$= q + q^3 - q^4$$

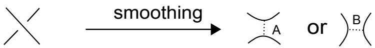
- polynomial knot invariants are all Laurent polynomials with integer coefficients
  $\longrightarrow$ homological understanding of knot invariants

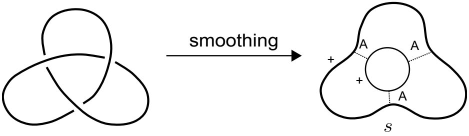
Khovanov Homolgy

- Jone polynomial can be realized as q-graded Euler characteristics of bi-graded homology group [Khovanov '99]

$$J(K;q,t) = \sum_{n,j} (-1)^n q^j \dim(H_{nj}).$$

- Construction of bi-graded chain complex





$n_B(s) = 0$

$j(s) = n_B(s) + \text{no of } (+) - \text{no of } (-) \qquad j(s) = 0 + 2 = 2$

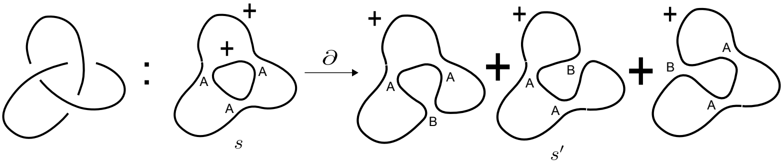
- C_{nj} is the vector space with basis as states labeled by $n(s) = n$ and $j(s) = j$.
- $\{C_{n,j}\}$ form chain complex from with differential

$$\partial : C_{n,j} \longrightarrow C_{n+1,j}, \quad \partial^2 = 0$$

Khovanov Homolgy

- Khovanov homology as bi-graded homology group

$$C_{*j}: \quad C_{0j} \xrightarrow{\partial} C_{1j} \xrightarrow{\partial} C_{2j} \xrightarrow{\partial} \ldots$$



$$n_B(s) = 0 \qquad j(s) = 0 + 2 = 2$$

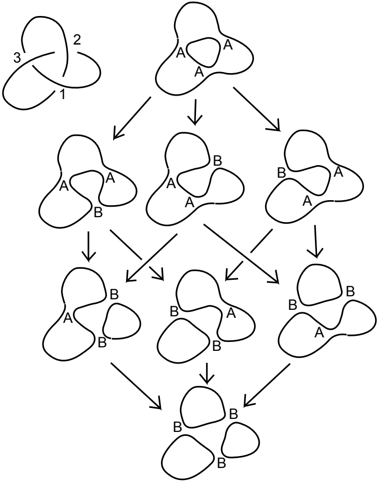
$$n_B(s') = 1 \qquad j(s') = 1 + 1 = 2$$

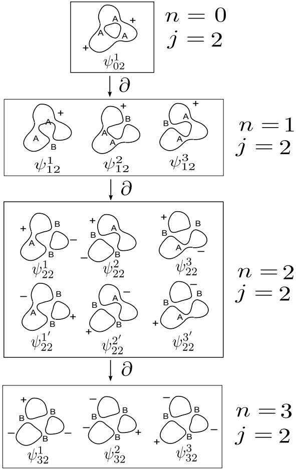
$$H_n(C_{*j}) = \frac{\ker(\partial : C_{n,j} \longrightarrow C_{n+1}, j)}{\mathrm{Image}(\partial : C_{n-1,j} \longrightarrow C_{n,j})}$$

- The Poincare polynomial of Khovanov homology is more powerful than Jones polynomial

$$Kh(K; q, t) = \sum_{n,j} t^n q^j \dim(H_{nj}).$$

Khovanov Homolgy





Categoifications

- the Poincaré polynomial of the colored $\mathfrak{sl}_2$ homology $\mathcal{H}^{\mathfrak{sl}_2,R}_{i,j}$ [Webster '10, Cooper, Krushkal '10, Frenkel, Stroppel, Sussan '10]

$$\mathcal{P}^{\mathfrak{sl}_2}_R(K;q,t)=\sum_{i,j}t^j q^i \dim\mathcal{H}^{\mathfrak{sl}_2,R}_{i,j}(K)\ ,$$

- q-graded Euler characteristic gives colored Jones polynomial:

$$J_R(K;q)=\mathcal{P}^{\mathfrak{sl}_2}_R(q,t=-1)=\sum_{i,j}(-1)^j q^i \dim\mathcal{H}^{\mathfrak{sl}_2,R}_{i,j}(K)\ .$$

- The Poincaré polynomial of the colored $\mathfrak{sl}_N$ homology [Khovanov-Rozansky '04]

$$\mathcal{P}^{\mathfrak{sl}_N}_R(K;q,t)=\sum_{i,j}t^i q^j \dim\mathcal{H}^{\mathfrak{sl}_N,R}_{i,j}(K)\ .$$

is related to the colored HOMFLY polynomial via

$$\mathcal{P}^{\mathfrak{sl}_N}_R(K;q,t=-1)=P_R(K;a=q^N,q)$$

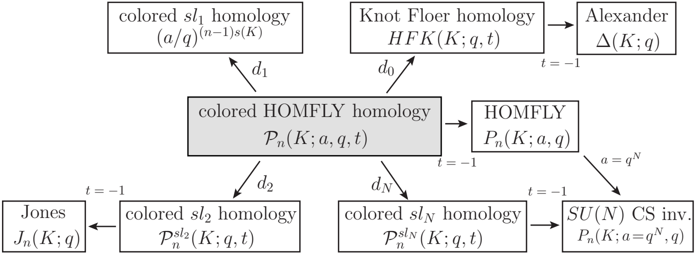
Colored superpolynomials

- the Poincaré polynomial of triply-graded homology $\mathcal{H}^R_{i,j,k}$ [Dunfield-Gukov-Rasmussen '05]

$$\mathcal{P}_R(K;a,q,t)=\sum_{i,j,k}a^iq^jt^k\dim\mathcal{H}^R_{i,j,k}(K)\ .$$

- The (a,q)-graded Euler characteristic of the triply-graded homology theory is equivalent to the colored HOMFLY polynomial

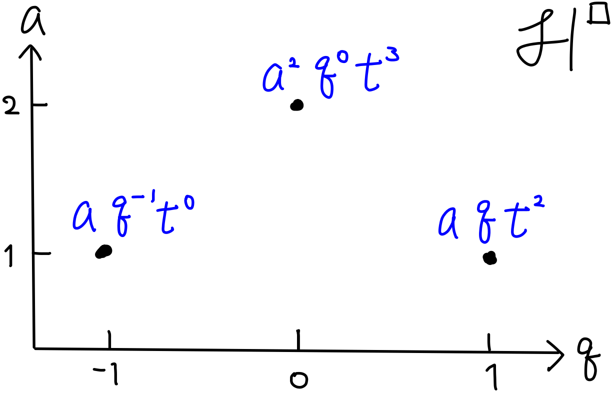
$$P_R(K;a,q)=\sum_{i,j,k}(-1)^ka^iq^j\dim\mathcal{H}^R_{i,j,k}(K)\ .$$



Differentials

- Uncolored Superpolynomial of Trefoil

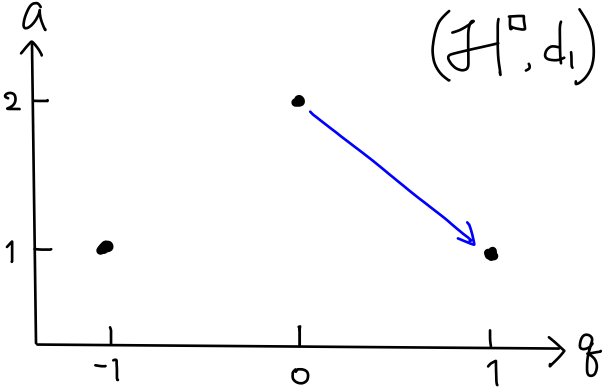
$$\mathcal{P}_{S^1}(\; ; a, q, t) = aq^{-1} + a^2t^3 + aqt^2 \ .$$



Differentials

- d_1-differential acts on uncolored HOMFLY homology for trefoil

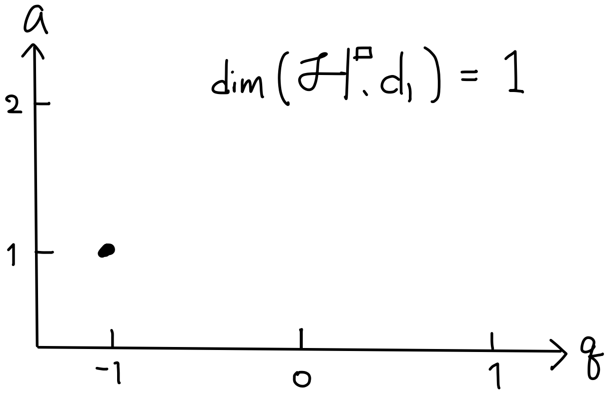
$$\mathcal{P}_{S^1}(\text{trefoil}; a, q, t) = aq^{-1} + (1 + a^{-1}qt^{-1})a^2t^3 \ .$$



Differentials

- $\mathfrak{sl}_1$ homology is one-dimensional

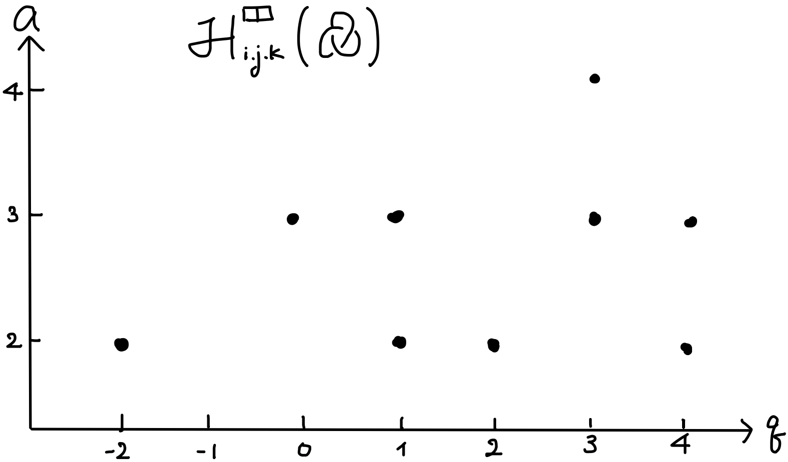
$$\mathcal{P}_{S^1}(\;;a,q,t) = aq^{-1} + (1 + a^{-1}qt^{-1})a^2t^3\ .$$



Differentials

- S^2-colored HOMFLY homology for trefoil

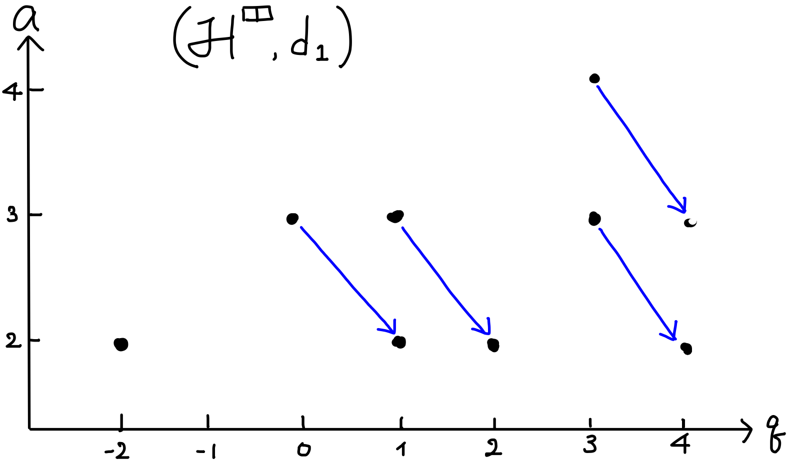
$$\mathcal{P}_{S^2}(\text{trefoil}; a, q, t) = a^2(q^{-2} + qt^2 + q^2t^2 + q^4t^4) + a^3(t^3 + qt^3 + q^3t^5 + q^4t^5) + a^4q^3t^6 \ .$$



Differentials

- d_1-differential acts on S^2-colored HOMFLY homology for trefoil

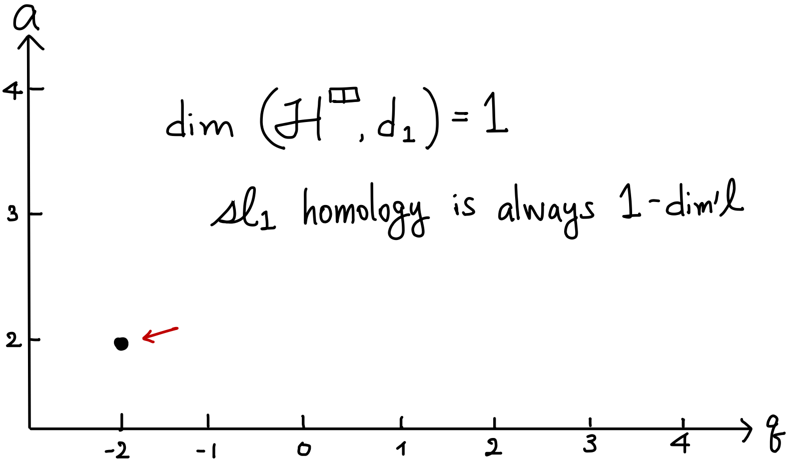
$$\mathcal{P}_{S^2}(\text{trefoil} ; a,q,t) = a^2q^{-2} + (1+a^{-1}qt^{-1})(a^3(1+q+q^3)+a^4q^4)$$



Differentials

- $\mathfrak{sl}_1$ homology is always one-dimensional

$$\mathcal{P}_{S^2}(\;\;; a, q, t) = a^2 q^{-2} + (1 + a^{-1} q t^{-1})(a^3(1 + q + q^3) + a^4 q^4) \ .$$



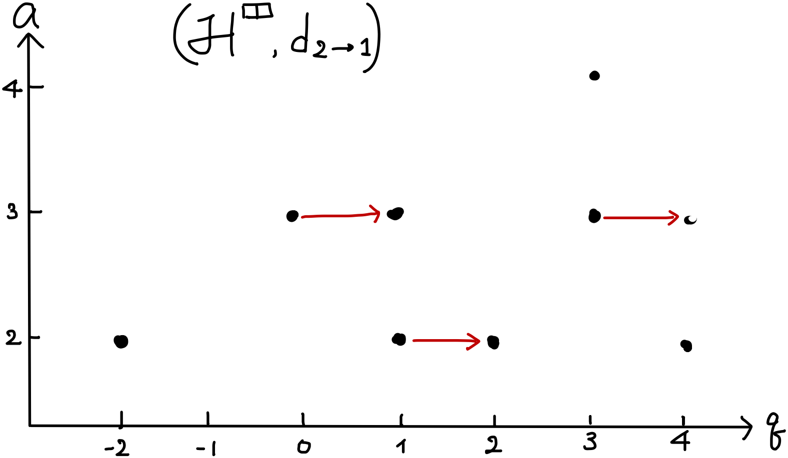
$\dim\left(\mathcal{H}^{\square}, d_1\right) = 1$

$\mathfrak{sl}_1$ homology is always 1-dim'l

Differentials

- $d2 \rightarrow$ 1-differential acts on S^2-colored HOMFLY homology for trefoi

$$\mathcal{P}_{S^2}(\ ;a,q,t) = a^2q^{-2} + a^2q^4t^4 + a^4q^3t^6 + (1+q)(a^2q^2t^2 + a^3(t^3 + q^3t^5))\ .$$

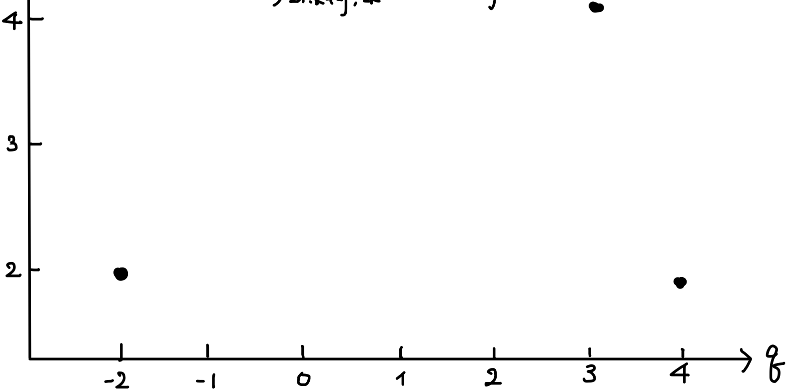


Differentials

- $d2 \to 1$-differential acts on S^2-colored HOMFLY homology for trefoi

$\mathcal{P}_{S^2}($  $; a, q, t) = a^2 q^{-2} + a^2 q^4 + a^4 q^3 + (1+q)(a^2 q^2 t^2 + a^3(t^3 + q^3 t^5))\ .$

$$(\mathcal{H}^{\square\square}, d_{2\to 1})_{2i,\, k+2j,\, 2k} \cong \mathcal{H}^{\square}_{i,j,k}$$



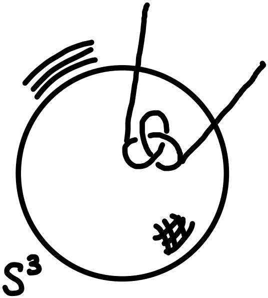
Knot homologies as BPS states

- the setup by the five-brane configuration in M-theory [Witten '92 '11]

$$
\begin{array}{rcl}
\text{space-time} & : & \mathbb{R} \times T^* S^3 \times M_4 \\
N \text{ M5-branes} & : & \mathbb{R} \times S^3 \times D \\
\text{M5-brane} & : & \mathbb{R} \times L_K \times D
\end{array}
$$

- $D \cong \mathbb{R}^2$ is the cigar in the Taub-NUT space $M_4 \cong \mathbb{R}^4$

$$L_K = \{(x, p) \in T^* S^3 \mid x = x(s),\ p \perp \dot{x}(s)\}$$



Knot homologies as BPS states

- Via geometric transition, the setup can be transformed [Ooguri Vafa '01, Gukov Schwarz Vafa '04]

$$\begin{aligned} \text{space-time} \quad &: \quad \mathbb{R} \times X \times M_4 \\ \text{M5-brane} \quad &: \quad \mathbb{R} \times L_K \times D \end{aligned}$$

  where X is the resolved conifold, *i.e.* $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ bundle over $\mathbb{P}^1$

- BPS states associated to M2-M5 configuration are candidates for the sl(N) knot homology

$$\mathcal{H}_{\rm BPS} \cong \mathcal{H}_{\rm knot}$$

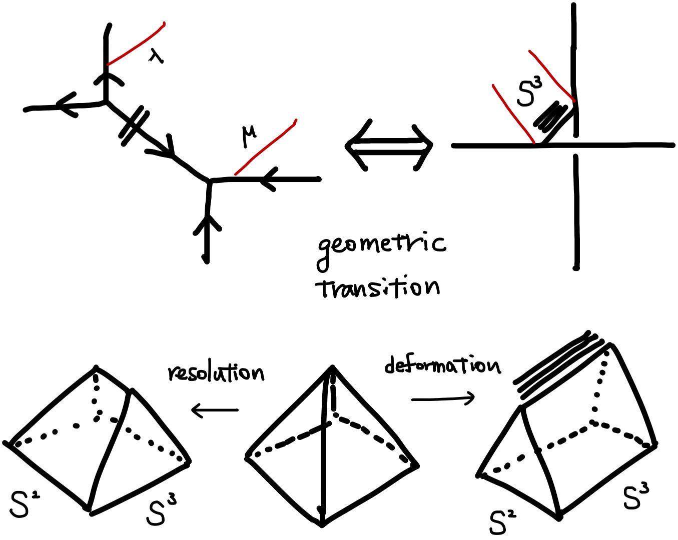
- $U(1)_P \times U(1)_F$ arises as the rotation symmetry of the tangent and normal bundle of the cigar $D \cong \mathbb{R}^2$ in the Taub-NUT space $M_4 \cong \mathbb{R}^4$
- quantum q-grading and the homological t-grading are fugacities for the equivariant action of $U(1)_P \times U(1)_F$

$$z_1 \longrightarrow q z_1 \qquad z_2 \longrightarrow t z_2$$

  where (z_1, z_2) is the local coordinate of M_4

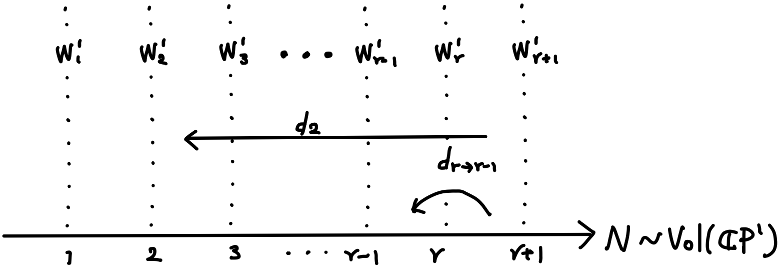
- a-grading is the Kahler parameter of $\mathbb{P}^1$

Geometric Transition

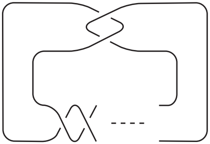


Differentials as Wall-crossing

- As you vary Kahler parameter $N \sim \log a$ (size of $\mathbb{P}^1$), the BPS spectrum jumps at wall of marginal stability
- In this case, wall of marginal stability is zero-dimensional
- Colored differentials allow us to move from one chamber to others with smaller representations



Twist knot K_p



p full twist

p	-4	-3	-2	-1	0	1	2	3	4
knots	$\mathbf{10_1}$	$\mathbf{8_1}$	$\mathbf{6_1}$	$\mathbf{4_1}$	$\mathbf{0_1}$	$\mathbf{3_1}$	$\mathbf{5_2}$	$\mathbf{7_2}$	$\mathbf{9_2}$

The correspondence between the twist number and the knots in Rolfsen's table

- Colored Jones polynomials and A-polynomials are known
- Quantum A-polynomials were already computed for $p = -14 \cdots 15$.

Colored Jones polynomials of twist knots

- The double-sum expressions [Habiro '03][Masbaum '03]

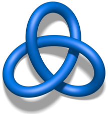
$$J_n(K_p;q) = \sum_{k=0}^{\infty}\sum_{\ell=0}^{k} q^k (q^{1-n};q)_k (q^{n+1};q)_k \times (-1)^\ell q^{\ell(\ell+1)p+\ell(\ell-1)/2}(1-q^{2\ell+1})\frac{(q;q)_k}{(q;q)_{k+\ell+1}(q;q)_{k-\ell}} .$$

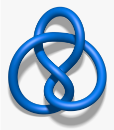
- The multi-sum expressions

$$J_n(K_{p>0};q) = \sum_{s_p\geq\cdots\geq s_1\geq 0}^{\infty} q^{s_p}\left(q^{1-n};q\right)_{s_p}\left(q^{1+n};q\right)_{s_p}\prod_{i=1}^{p-1} q^{s_i(s_i+1)}\begin{bmatrix} s_{i+1} \\ s_i \end{bmatrix}_q$$

$$J_n(K_{p<0};q) = \sum_{s_{|p|}\geq\cdots\geq s_1\geq 0}^{\infty} (-1)^{s_{|p|}} q^{-\frac{s_{|p|}(s_{|p|}+1)}{2}}\left(q^{1-n};q\right)_{s_{|p|}}\left(q^{1+n};q\right)_{s_{|p|}} \times\prod_{i=1}^{|p|-1} q^{-s_i(s_{i+1}+1)}\begin{bmatrix} s_{i+1} \\ s_i \end{bmatrix}_q$$

Colored superpolynomials of trefoil and figure-8





- Colored superpolynomials of trefoil and figure-8 are known [Fuji Gukov Sulkowski '12][Itoyama, Mironov, Morozov^2 '12]

$$
\begin{aligned}
\mathcal{P}_n(\mathbf{3_1}; a, q, t) &= (-t)^{-n+1} \sum_{k=0}^{\infty} q^k \frac{(-atq^{-1}; q)_k}{(q; q)_k} (q^{1-n}; q)_k (-at^3 q^{n-1}; q)_k \ , \\
\mathcal{P}_n(\mathbf{4_1}; a, q, t) &= \sum_{k=0}^{\infty} (-1)^k a^{-k} t^{-2k} q^{-k(k-3)/2} \frac{(-atq^{-1}; q)_k}{(q; q)_k} (q^{1-n}; q)_k (-at^3 q^{n-1}; q)_k \ .
\end{aligned}
$$

- Colored superpolynomials $\mathcal{P}_n(a, q, t)$ for $\mathbf{5_2}$ and $\mathbf{6_1}$ are also known up to $n = 3$ [Gukov Stosic '11]

- One can do educated guess on colored superpolynomials of twist knots

Colored superpolynomials of twist knots

- The double-sum expressions

$$\mathcal{P}_n(K_p; a, q, t) = \sum_{k=0}^{\infty}\sum_{\ell=0}^{k} q^k \frac{(-atq^{-1};q)_k}{(q;q)_k}(q^{1-n};q)_k(-at^3q^{n-1};q)_k \times (-1)^\ell a^{p\ell} t^{2p\ell} q^{(p+1/2)\ell(\ell-1)} \frac{1-at^2q^{2\ell-1}}{(at^2q^{\ell-1};q)_{k+1}} \begin{bmatrix} k \\ \ell \end{bmatrix}_q .$$

- The multi-sum expressions

$$\mathcal{P}_n(K_{p>0}; a, q, t) = (-t)^{-n+1} \sum_{s_p \geq \cdots \geq s_1 \geq 0}^{\infty} q^{s_p} \frac{(-atq^{-1};q)_{s_p}}{(q;q)_{s_p}}(q^{1-n};q)_{s_p}(-at^3q^{n-1};q)_{s_p} \times \prod_{i=1}^{p-1}(at^2)^{s_i} q^{s_i(s_i-1)} \begin{bmatrix} s_{i+1} \\ s_i \end{bmatrix}_q$$

$$\mathcal{P}_n(K_{p<0}; a, q, t) = \sum_{s_{|p|} \geq \cdots \geq s_1 \geq 0}^{\infty} (-1)^k a^{-s_{|p|}} t^{-2s_{|p|}} q^{-s_{|p|}(s_{|p|}-3)/2} \frac{(-atq^{-1};q)_{s_{|p|}}}{(q;q)_{s_{|p|}}}(q^{1-n};q)_{s_{|p|}}(-at^3q^{n-1};q)_{s_{|p|}} \times \prod_{i=1}^{|p|-1}(at^2)^{-s_i} q^{-s_i(s_{i+1}-1)} \begin{bmatrix} s_{i+1} \\ s_i \end{bmatrix}_q .$$

Colored Jones polynomials of twist knots

- The double-sum expressions [Habiro '03][Masbaum '03]

$$J_n(K_p;q) = \sum_{k=0}^{\infty}\sum_{\ell=0}^{k} q^k (q^{1-n};q)_k (q^{n+1};q)_k \times (-1)^\ell q^{\ell(\ell+1)p+\ell(\ell-1)/2}(1-q^{2\ell+1})\frac{(q;q)_k}{(q;q)_{k+\ell+1}(q;q)_{k-\ell}} .$$

- The multi-sum expressions

$$J_n(K_{p>0};q) = \sum_{s_p\geq\cdots\geq s_1\geq 0}^{\infty} q^{s_p}\left(q^{1-n};q\right)_{s_p}\left(q^{1+n};q\right)_{s_p}\prod_{i=1}^{p-1} q^{s_i(s_i+1)}\begin{bmatrix} s_{i+1} \\ s_i \end{bmatrix}_q$$

$$J_n(K_{p<0};q) = \sum_{s_{|p|}\geq\cdots\geq s_1\geq 0}^{\infty} (-1)^{s_{|p|}} q^{-\frac{s_{|p|}(s_{|p|}+1)}{2}}\left(q^{1-n};q\right)_{s_{|p|}}\left(q^{1+n};q\right)_{s_{|p|}} \times \prod_{i=1}^{|p|-1} q^{-s_i(s_{i+1}+1)}\begin{bmatrix} s_{i+1} \\ s_i \end{bmatrix}_q$$

Colored superpolynomials of twist knots

- The double-sum expressions

$$\mathcal{P}_n(K_p;a,q,t) = \sum_{k=0}^{\infty}\sum_{\ell=0}^{k} q^k \frac{(-atq^{-1};q)_k}{(q;q)_k}(q^{1-n};q)_k(-at^3q^{n-1};q)_k \times(-1)^\ell a^{p\ell}t^{2p\ell}q^{(p+1/2)\ell(\ell-1)}\frac{1-at^2q^{2\ell-1}}{(at^2q^{\ell-1};q)_{k+1}}\begin{bmatrix} k \\ \ell \end{bmatrix}_q .$$

- The multi-sum expressions

$$\mathcal{P}_n(K_{p>0};a,q,t) = (-t)^{-n+1}\sum_{s_p\geq\cdots\geq s_1\geq 0}^{\infty} q^{s_p}\frac{(-atq^{-1};q)_{s_p}}{(q;q)_{s_p}}(q^{1-n};q)_{s_p}(-at^3q^{n-1};q)_{s_p} \times\prod_{i=1}^{p-1}(at^2)^{s_i}q^{s_i(s_i-1)}\begin{bmatrix} s_{i+1} \\ s_i \end{bmatrix}_q$$

$$\mathcal{P}_n(K_{p<0};a,q,t) = \sum_{s_{|p|}\geq\cdots\geq s_1\geq 0}^{\infty}(-1)^k a^{-s_{|p|}}t^{-2s_{|p|}}q^{-s_{|p|}(s_{|p|}-3)/2}\frac{(-atq^{-1};q)_{s_{|p|}}}{(q;q)_{s_{|p|}}}(q^{1-n};q)_{s_{|p|}}(-at^3q^{n-1};q)_{s_{|p|}} \times\prod_{i=1}^{|p|-1}(at^2)^{-s_i}q^{-s_i(s_{i+1}-1)}\begin{bmatrix} s_{i+1} \\ s_i \end{bmatrix}_q .$$

Cancelling differentials and Rasmussen s-invarinats

- the action of the differential d_1 on the colored superpolynomials

$$\begin{aligned}\mathcal{P}_{n+1}(K_{p>0};a,q,t) &= a^n q^{-n} + (1+a^{-1}qt^{-1})Q^{\mathfrak{sl}_1}_{n+1}(K_{p>0};a,q,t)\ ,\\ \mathcal{P}_{n+1}(K_{p<0};a,q,t) &= 1 + (1+a^{-1}qt^{-1})Q^{\mathfrak{sl}_1}_{n+1}(K_{p<0};a,q,t)\ ,\end{aligned}$$

- the action of the differential d_{-n} on the colored superpolynomials

$$\begin{aligned}\mathcal{P}_{n+1}(K_{p>0};a,q,t) &= a^n q^{n^2} t^{2n} + (1+a^{-1}q^{-n}t^{-3})Q_{n+1}(K_{p>0};a,q,t)\ ,\\ \mathcal{P}_{n+1}(K_{p<0};a,q,t) &= 1 + (1+a^{-1}q^{-n}t^{-3})Q_{n+1}(K_{p<0};a,q,t)\ .\end{aligned}$$

- The exponents of the remaining monomials are consistent with the Rassmusen's s-invariant of the twist knots, $s(K_{p>0})=1$ and $s(K_{p<0})=0$ [Rasmussen '04]

$$\begin{aligned}\deg\left(\mathcal{H}^{S^n}_{*,*,*}(K),d_1\right) &= \Big(n\,s(K)\,,\,-n\,s(K)\,,\,0\Big)\ ,\\ \deg\left(\mathcal{H}^{S^n}_{*,*,*}(K),d_{-n}\right) &= \left(n\,s(K)\,,\,n^2\,s(K)\,,\,2n\,s(K)\right)\ ,\end{aligned}$$

Volume conjecture and A-polynomials

- The volume conjecture relates "quantum invariants" of knots to "classical" 3d topology [Kashaev '99][Murakami^2 '00]

$$\lim_{n\to\infty} \frac{2\pi}{n} \log \left| J_n(K; q = e^{\frac{2\pi i}{n}}) \right| = \mathrm{Vol}(S^3 \backslash K) \ .$$

- The relation b/w volume conjecture and A-polynomial [Gukov '03]

$$\log y = -x \frac{d}{dx} \lim_{\substack{n,k\to\infty \\ e^{i\pi n/k}=x}} \frac{1}{k} \log J_n(K; q = e^{\frac{2\pi i}{k}}) \ ,$$

gives the zero locus of the A-polynomial $A(K; x, y)$ of the knot K.

- A-polynomial $A(K; x, y)$ is a character variety of $SL(2, \mathbb{C})$-representation of the fundamental group of the knot complement

$$\mathcal{M}_{\mathrm{flat}}(SL(2,\mathbb{C}), T^2) \overset{\mathrm{Lag.sub.}}{\supset} \mathcal{M}_{\mathrm{flat}}(SL(2,\mathbb{C}), S^3 \backslash K) = \{(x,y) \in \mathbb{C}^\times \times \mathbb{C}^\times | A(K; x, y) = 0\}$$

Qauntum volume conjecture

- Quantum version of the volume conjecture $\rightarrow$ AJ conjecture [Gukov '03][Garoufalidis '03]

$$\widehat{A}(K;\hat{x},\hat{y},q)J_n(K;q)=0$$

where action of $\hat{x}$ and $\hat{y}$ on the set of colored Jones polynomials as

$$\hat{x}J_n(K;q^n)=q^nJ_n(K;q^n)\ , \quad \hat{y}J_n(K;q)=J_{n+1}(K;q)\ .$$

- Find difference equations of colored Jones polynomials

$$a_kJ_{n+k}(K;q)+...+a_1J_{n+1}(K;q)+a_0J_n(K;q)=0$$

where $a_k=a_k(K;\widehat{x},q)$ and

$$\widehat{A}(K;\widehat{x},\widehat{y};q)=\sum a_i(K;\widehat{x},q)\widehat{y}^i$$

- Taking the classical limit $q=e^{\hbar}\rightarrow 1$, quantum (non-commutative) A-polynomials reduces to ordinary A-polynomials

$$\widehat{A}(K;\widehat{x},\widehat{y};q)\rightarrow A(K;x,y) \quad \text{as} \quad q\rightarrow 1$$

Super-A-polynomials

- Refinement of quantum and classical A-polynomials [Fuji Gukov Sulkowski '12]

Quantum operator	provides recursion for	classical limit
$\widehat{A}^{\rm super}(\widehat{x},\widehat{y};a,q,t)$	colored superpolynomial	$A^{\rm super}(x,y;a,t)$
$\widehat{A}^{\rm ref}(\widehat{x},\widehat{y};q,t)$	colored $\mathfrak{sl}_N$ homology	$A^{\rm ref}(x,y,t)$
$\widehat{A}^{\rm Q-def}(\widehat{x},\widehat{y};a,q)$	colored HOMFLY	$A^{\rm Q-def}(x,y;a)$
$\widehat{A}(\widehat{x},\widehat{y};q)$	colored Jones	$A(x,y)$

$$\begin{tikzcd}
 & \widehat{A}^{\rm super}(\widehat{x},\widehat{y};a,q,t) \arrow[dl, "a=q^2"'] \arrow[dr, "t=-1"] & \\
\widehat{A}^{\rm ref}(\widehat{x},\widehat{y};q,t) \arrow[dr, "t=-1"'] & & \widehat{A}^{\rm Q-def}(\widehat{x},\widehat{y};a,q) \arrow[dl, "a=q^2"] \\
 & \widehat{A}(\widehat{x},\widehat{y};q) &
\end{tikzcd}
\quad \xrightarrow{q=1} \quad
\begin{tikzcd}
 & A^{\rm super}(x,y;a,t) \arrow[dl, "a=1"'] \arrow[dr, "t=-1"] & \\
A^{\rm ref}(x,y,t) \arrow[dr, "t=-1"'] & & A^{\rm Q-def}(x,y;a) \arrow[dl, "a=1"] \\
 & A(x,y) &
\end{tikzcd}$$

Classical super-A-polynomials

- Categorification of generalized volume conjecture [Fuji Gukov Sulkowski '12]

$$\log y = -x \frac{d}{dx} \lim_{\substack{n,k\to\infty \\ e^{i\pi n/k}=x}} \frac{1}{k} \log \mathcal{P}_n(K; a, q = e^{\frac{2\pi i}{k}}, t) \ ,$$

gives the zero locus of the super-A-polynomial $A(K; x, y; a, q, t)$ of the knot K.

Knot	$A^{\rm super}(K; x, y; a, t)$
$\mathbf{5_2}$	$\left(1+at^3x\right)^3 y^4$ $-a(1+at^3x)^2(2-x+tx-2t^2x+3t^2x^2+at^2x^2+4at^3x^2-2at^3x^3+2at^4x^3+2at^5x^3-at^5x^4+2a^2t^5x^4+2a^2t^6x^4-a^2t^6x^5+a^2t^7x^5+a^3t^8x^6)y^3$ $-a^2(x-1)\left(1+at^3x\right)(1+tx-2t^2x+2t^2x^2-2t^3x^2+4at^3x^2+t^4x^2-3t^4x^3+at^4x^3-2at^5x^3+4at^5x^4-4at^6x^4+6a^2t^6x^4-4at^7x^4+3at^7x^5-a^2t^7x^5+2a^2t^8x^5+2a^2t^8x^6-2a^2t^9x^6+4a^3t^9x^6+a^2t^{10}x^6-a^3t^{10}x^7+2a^3t^{11}x^7+a^4t^{12}x^8)y^2$ $+a^3t^3x^2(x-1)^2(1+tx-t^2x-t^3x^2+2at^3x^2+2at^4x^2+2at^4x^3-2at^5x^3-2at^6x^3+3at^6x^4+a^2t^6x^4+4a^2t^7x^4+a^2t^7x^5-a^2t^8x^5+2a^2t^9x^5+2a^3t^{10}x^6)y$ $-a^5t^{11}x^7(x-1)^3$

Quantum super-A-polynomials

- Categorification of quantum volume conjecture

$$\widehat{A}^{\rm super}(K;\hat{x},\hat{y};a,q,t)\mathcal{P}_n(K;a,q,t)=0$$

where the operators $\hat{x}$ and $\hat{y}$ acts on the set of colored superpolynomials as

$$\hat{x}\mathcal{P}_n(K;a,q,t)=q^n\mathcal{P}_n(K;a,q,t)\ ,\quad \hat{y}\mathcal{P}_n(K;q)=\mathcal{P}_{n+1}(K;a,q,t)\ .$$

- Find difference equations of colored superpolynomials

$$a_k\mathcal{P}_{n+k}(K;q)+\ldots+a_1\mathcal{P}_{n+1}(K;q)+a_0\mathcal{P}_n(K;q)=0$$

where $a_k=a_k(K;\widehat{x};a,q,t)$ and

$$\widehat{A}^{\rm super}(K;\widehat{x},\widehat{y};a,q,t)=\sum a_i(K;\widehat{x};a,q,t)\widehat{y}^i$$

- Taking the classical limit $q=e^{\hbar}\to 1$, quantum (non-commutative) super-A-polynomials reduces to classical super-A-polynomials

$$\widehat{A}^{\rm super}(K;\widehat{x},\widehat{y};a,q,t)\to A^{\rm super}(K;x,y;a,q,t)\quad\text{as}\quad q\to 1$$

Quantum super-A-polynomials

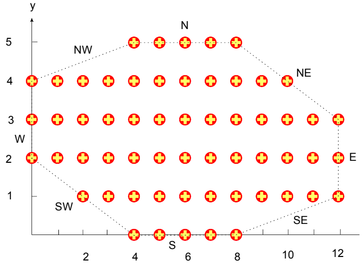
Knot	$\widehat{A}^{\text{super}}(K;\hat{x},\hat{y};a,q,t)$
$\mathbf{5_2}$	$q^6t^4\left(1+at^3\hat{x}\right)\left(1+aqt^3\hat{x}\right)\left(1+aq^2t^3\hat{x}\right)\left(1+at^3\hat{x}^2\right)\left(q+at^3\hat{x}^2\right)\left(1+aqt^3\hat{x}^2\right)\hat{y}^4$ $-aq^5t^4(1+at^3\hat{x})(1+aqt^3\hat{x})(1+at^3\hat{x}^2)(q+at^3\hat{x}^2)(1+aq^4t^3\hat{x}^2)(1+q-q^3\hat{x}+q^3t\hat{x}-q^2t^2\hat{x}-q^3t^2\hat{x}+ q^4t^2\hat{x}^2+aq^4t^2\hat{x}^2+q^5t^2\hat{x}^2+q^6t^2\hat{x}^2+aqt^3\hat{x}^2+aq^2t^3\hat{x}^2+aq^5t^3\hat{x}^2+aq^6t^3\hat{x}^2-aq^4t^3\hat{x}^3-aq^8t^3\hat{x}^3+aq^4t^4\hat{x}^3+ aq^8t^4\hat{x}^3+aq^5t^5\hat{x}^3+aq^6t^5\hat{x}^3+a^2q^5t^5\hat{x}^4-aq^8t^5\hat{x}^4+a^2q^9t^5\hat{x}^4+a^2q^6t^6\hat{x}^4+a^2q^7t^6\hat{x}^4-a^2q^9t^6\hat{x}^5+a^2q^9t^7\hat{x}^5+ a^3q^{10}t^8\hat{x}^6)\hat{y}^3$ $-a^2q^5t^4(-1+q^2\hat{x})(1+at^3\hat{x})(q+at^3\hat{x}^2)(1+aq^2t^3\hat{x}^2)(1+aq^5t^3\hat{x}^2)(1+q^2t\hat{x}-qt^2\hat{x}-q^2t^2\hat{x}+q^3t^2\hat{x}^2+q^4t^2\hat{x}^2+ at^3\hat{x}^2+aqt^3\hat{x}^2-q^3t^3\hat{x}^2+aq^3t^3\hat{x}^2-q^4t^3\hat{x}^2+aq^4t^3\hat{x}^2+q^3t^4\hat{x}^2+aq^2t^4\hat{x}^3-q^4t^4\hat{x}^3-aq^4t^4\hat{x}^3-q^5t^4\hat{x}^3- q^6t^4\hat{x}^3+aq^6t^4\hat{x}^3-aqt^5\hat{x}^3-aq^2t^5\hat{x}^3+aq^3t^5\hat{x}^3+aq^4t^5\hat{x}^3-aq^5t^5\hat{x}^3-aq^6t^5\hat{x}^3+aq^3t^5\hat{x}^4+aq^4t^5\hat{x}^4+aq^7t^5\hat{x}^4+ aq^8t^5\hat{x}^4+a^2qt^6\hat{x}^4-aq^3t^6\hat{x}^4+a^2q^3t^6\hat{x}^4-aq^4t^6\hat{x}^4+2a^2q^4t^6\hat{x}^4+a^2q^5t^6\hat{x}^4-aq^7t^6\hat{x}^4+a^2q^7t^6\hat{x}^4-aq^8t^6\hat{x}^4- aq^4t^7\hat{x}^4-2aq^5t^7\hat{x}^4-aq^6t^7\hat{x}^4-a^2q^4t^7\hat{x}^5+aq^6t^7\hat{x}^5+a^2q^6t^7\hat{x}^5+aq^7t^7\hat{x}^5+aq^8t^7\hat{x}^5-a^2q^8t^7\hat{x}^5+a^2q^3t^8\hat{x}^5+ a^2q^4t^8\hat{x}^5-a^2q^5t^8\hat{x}^5-a^2q^6t^8\hat{x}^5+a^2q^7t^8\hat{x}^5+a^2q^8t^8\hat{x}^5+a^2q^7t^8\hat{x}^6+a^2q^8t^8\hat{x}^6+a^3q^4t^9\hat{x}^6+a^3q^5t^9\hat{x}^6- a^2q^7t^9\hat{x}^6+a^3q^7t^9\hat{x}^6-a^2q^8t^9\hat{x}^6+a^3q^8t^9\hat{x}^6+a^2q^7t^{10}\hat{x}^6-a^3q^8t^{10}\hat{x}^7+a^3q^7t^{11}\hat{x}^7+a^3q^8t^{11}\hat{x}^7+a^4q^8t^{12}\hat{x}^8)\hat{y}^2$ $+a^3q^7t^7\hat{x}^2(-1+q\hat{x})(-1+q^2\hat{x})(1+at^3\hat{x}^2)(1+aq^4t^3\hat{x}^2)(1+aq^5t^3\hat{x}^2)(q+q^2t\hat{x}-q^2t^2\hat{x}+at^3\hat{x}^2-q^3t^3\hat{x}^2+ aq^4t^3\hat{x}^2+aqt^4\hat{x}^2+aq^2t^4\hat{x}^2+aqt^4\hat{x}^3+aq^5t^4\hat{x}^3-aqt^5\hat{x}^3-aq^5t^5\hat{x}^3-aq^2t^6\hat{x}^3-aq^3t^6\hat{x}^3+aq^3t^6\hat{x}^4+ a^2q^3t^6\hat{x}^4+aq^4t^6\hat{x}^4+aq^5t^6\hat{x}^4+a^2t^7\hat{x}^4+a^2qt^7\hat{x}^4+a^2q^4t^7\hat{x}^4+a^2q^5t^7\hat{x}^4+a^2q^4t^7\hat{x}^5-a^2q^4t^8\hat{x}^5+a^2q^3t^9\hat{x}^5+ a^2q^4t^9\hat{x}^5+a^3q^3t^{10}\hat{x}^6+a^3q^4t^{10}\hat{x}^6)\hat{y}$ $-a^5q^8t^{15}(-1+\hat{x})\hat{x}^7(-1+q\hat{x})(-1+q^2\hat{x})(1+aq^3t^3\hat{x}^2)(1+aq^4t^3\hat{x}^2)(1+aq^5t^3\hat{x}^2)$

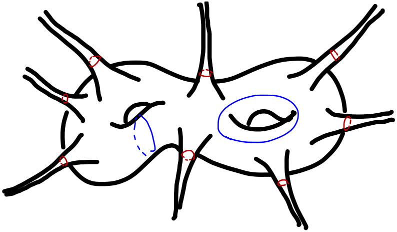
Quantizability

- In order for a classical super-A-polynomial $A^{\mathrm{super}}(K; x, y; a, t)$ to be quantizable, periods of the imaginary and real parts of along a path γ is required to obey the Bohr-Sommerfeld condition

$$\begin{aligned} \oint_\gamma \Big(\log|x| d(\arg y) - \log|y| d(\arg x) \Big) &= 0\,, \\ \frac{1}{4\pi^2}\oint_\gamma \Big(\log|x| d\log|y| + (\arg y) d(\arg x) \Big) &\in \mathbb{Q}\,. \end{aligned}$$

- This amounts to the condition that all roots of all face polynomials of its Newton polygon are roots of unity





Augmentation polynomials of knot contact homology

- Knot contact homology describes knot invariants as invariants of the Legendrian submanifolds in the contact manifold
- A knot is realized by an intersection of the cosphere bundle ST^*M of a 3-manifold $M$with the unit conormal bundle Λ_K where ST^*M admits a contact structure
- Q-deformed A-polynomial is equivalent to augmentation polynomial of knot contact homology [Ng '10][Aganagic Vafa '12]

$$
\begin{aligned}
A^{\mathrm{super}}&\left(K_{p>0}; x=-\mu, y=\frac{1+\mu}{1+U\mu}\lambda; a=U, t=-1\right)\\
&=\frac{(-1)^{p-1}(1+\mu)^{(2p-1)}}{1+U\mu}\mathrm{Aug}(K_{p>0};\mu,\lambda;U,V=1)\ ,\\
A^{\mathrm{super}}&\left(K_{p<0}; x=-\mu, y=\frac{1+\mu}{1+U\mu}\lambda; a=U, t=-1\right)\\
&=\frac{(-1)^{p}(1+\mu)^{-2p}}{1+U\mu}\mathrm{Aug}(K_{p<0};\mu,\lambda;U,V=1)\ ,
\end{aligned}
$$

Prospects and Future Directions

- Representations other than symmetric representations
- Quantum $6j$-symbolos for $U_q(\mathfrak{sl}_N)$ and their refinement
- Colored superpolynomials of other non-torus knots and links [work in progress]
- refinement of colored Kaufmann polynomials
- $SU(N)$ analogue of WRT invariants and one-parameter deformations of Mock modular forms.
- Connection to $3+3$ AGT relation. [Dimofte Gukov Hollands, DGG, Fuji Gukov Stosic Sułkowki]
  - Meaning of periods
  - Argyres-Douglas fixed points
  - Dualities
  - Integrable systems

Thank you