Homework 10 (Due at class on May 19)

1 Poisson Resummation Formula

The Poisson resummation formula allows us to express sums over a lattice as sums over its dual lattice, a powerful tool in understanding the transformation properties of partition functions under S-modular transformations. This problem set explores the proof of this formula and its applications to the behavior of the Dedekind η-function under modular transformations.

(i) Given a function $f(x)$ which approaches zero suitably fast for $x \to \pm\infty$, i.e. to be precise we consider $f(x)$ to be a Schwartz function. Then the Fourier transformed function $\hat{f}(p)$ is given by

$$\hat{f}(p) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x p} dx.$$

Prove the Poisson resummation formula for a Schwartz function f

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n).$$

Hint: What are the Fourier coefficients of the periodic function $F(x) = \sum_{n\in\mathbb{Z}} f(x+n)$?

(ii) Use the Poisson resummation formula to derive a summation identity for the function

$$f(x) = e^{-ax^2+bx}, \quad \mathrm{Re}(a) > 0.$$

(iii) Use the Jacobi triple product identity

$$\prod_{n=1}^{\infty} (1-q^n)\left(1+q^{n-1/2}y\right)\left(1+q^{n-1/2}y^{-1}\right) = \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}n^2} y^n, \quad |q| < 1 y \neq 0,$$

to derive the infinite sum formula

$$\eta(q) = \sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{3}{2}\left(n-\frac{1}{6}\right)^2}$$

for the Dedekind eta function. Apply this result to determine the behavior of the Dedekind eta function $\eta(\tau)$ with respect to modular transformations.

Hint: In order to find the modular formula $\eta(-1/\tau) = \sqrt{-i\tau}\,\eta(\tau)$, split the infinite sum into three suitable infinite sums after Poisson resummation.

(iv) Show that $\tau_2^{1/2}|\eta(q)|^2$ is invariant under modular transformation.

(v) Extend the Poisson resummation formula to a function f on $\mathbb{R}^n$ and a lattice Λ in $\mathbb{R}^n$:

$$\sum_{\lambda\in\Lambda} f(\lambda) = \sum_{\lambda^*\in\Lambda^*} \hat{f}(\lambda^*),$$

where Λ^* is the dual lattice. Define $\hat{f}$ in this context.

2 Even, Unimodular Lattices and T-duality

In the context of toroidal compactifications of string theory and the heterotic string, a significant subsector is represented by left-moving and right-moving bosons with a non-trivial zero-mode sector. To facilitate a deeper understanding of such conformal field theories (CFTs), we explore a CFT comprising N left-moving and M right-moving bosons. The zero modes (p_L, p_R) of these bosons are valued in a lattice $\Gamma_{N,M} \subset \mathbb{R}^{N+M}$. This lattice is equipped with a Lorentzian metric defined as $\|(p_L, p_R)\|^2 = p_L^2 - p_R^2$ and is integral, meaning that all scalar products between lattice points yield integers. We define the rank of the lattice as $N + M$ and its signature as $N - M$. A lattice is termed even (type II) if $\|x\|^2$ is even for all $x \in \Gamma_{N,M}$, and odd (type I) otherwise. A unimodular, or self-dual, lattice has a unit cell volume of 1. We assume that $\Gamma_{N,M}$ is both even and unimodular.

(i) Calculate the partition function $Z(\tau, \overline{\tau})$ of this CFT.

(ii) Demonstrate that $Z(\tau, \overline{\tau})$ is modular invariant. *Hint: Use Poisson resummation.*

(iii) Construct an example of an even, unimodular Lorentzian lattice $\Gamma_{1,1}$. Discuss whether a Euclidean lattice $\Gamma_{2,0}$ of this type exists and justify your answer.

(iv) Consider the positive, even, unimodular root-lattice of E_8, generated by basis vectors $e_i \in \mathbb{R}^8, i = 1, 2, \ldots, 8$, with the Cartan matrix $C_{ij} = e_i \cdot e_j$,

$$C_{ij} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

Verify that this lattice is even and unimodular and identify a set of vectors e_i that realize the above Cartan matrix.

(v) The orthogonal group $O(N, M; \mathbb{R})$ acts on lattices of rank $N + M$ and signature $N - M$. Verify that the properties of being even and unimodular are invariant under this transformation, and argue that the partition function $Z(\tau, \overline{\tau})$ remains unchanged when replacing the lattice $\Gamma_{N,M}$ with $\Lambda \cdot \Gamma_{N,M}$, where $\Lambda \in O(N, M; \mathbb{Z})$ is a Lorentz transformation and a lattice automorphism. Such transformations are known as T-duality.

(vi) The Hasse-Minkowski classification of unimodular, indefinite lattices states that, up to isomorphism, there is exactly one lattice for a given rank, signature and type (see for example J.-P. Serre, A course in arithmetics, theorem V.2.2.5 and theorem V.2.2.6). Use this result to argue why the moduli space $\mathcal{M}_{N,M}$ of physically inequivalent lattices is given by $(N, M > 0)$

$$\mathcal{M}_{N,M} = \frac{\mathcal{M}^0_{N,M}}{O(N, M; \mathbb{Z})}, \quad \mathcal{M}^0_{N,M} = \frac{O(N, M; \mathbb{R})}{(O(N; \mathbb{R}) \times O(M; \mathbb{R}))}.$$

Find its dimension. Further, describe the moduli spaces $\mathcal{M}^0_{1,1}$ and $\mathcal{M}_{1,1}$.