Homework 9: Due at class on May 10

1. Find the fundamental group $\pi_1(\Sigma_g)$ of the Riemann surface of genus g.

2. Find the fundamental group $\pi_1(\mathbb{R}P^n)$ of the n-dimensional real projective space $\mathbb{R}P^n$.

3. Find the fundamental group $\pi_1(K)$ of the 3-dimensional complex K in Problem 5 of Homwork 7.

4. Find **real** dimensions of the following Lie groups

- Complex general linear group: $\mathrm{GL}(n,\mathbb{C}) = \{A \in \mathrm{M}_n(\mathbb{C}) | \det A \neq 0\}$
- Complex special linear group: $\mathrm{SL}(n,\mathbb{C}) = \{A \in \mathrm{GL}(n,\mathbb{C}) | \det A = 1\}$
- Unitary group $\mathrm{U}(n) = \{A \in \mathrm{GL}(n,\mathbb{C}) | A^\dagger A = I\}$
- Special unitary group $\mathrm{SU}(n) = \{A \in \mathrm{U}(n) | \det A = 1\}$
- Real general linear group: $\mathrm{GL}(n,\mathbb{R}) = \{A \in \mathrm{M}_n(\mathbb{R}) | \det A \neq 0\}$
- Real special linear group: $\mathrm{SL}(n,\mathbb{R}) = \{A \in \mathrm{GL}(n,\mathbb{R}) | \det A = 1\}$
- Symplectic group $\mathrm{Sp}(n,\mathbb{R}) = \{A \in \mathrm{GL}(2n,\mathbb{R}) | A^{\mathrm{T}} J A = J$ where $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}\}$
- Orthogonal group $\mathrm{O}(n,\mathbb{R}) = \{A \in \mathrm{GL}(n,\mathbb{R}) | A^T A = I\}$
- Special orthogonal group $\mathrm{SO}(n,\mathbb{R}) = \{A \in \mathrm{O}(n,\mathbb{R}) | \det A = 1\}$

5. Show that the group of Lorentz transformations is isomorphic to $\mathrm{SL}(2,\mathbb{C})/\pm\mathrm{Id}$. Hint: If we define

$$A := \begin{pmatrix} t+x & z+yi \\ z-yi & t-x \end{pmatrix}$$

where $(t,x,y,z) \in \mathbb{R}^4$, then we have $t^2 - x^2 - y^2 - z^2 = \det A$.

6. Show that $\mathrm{SO}(4) \cong \{\mathrm{SU}(2) \times \mathrm{SU}(2)\}/\{\pm \mathrm{Id}\}$, where $\mathrm{Id} \hookrightarrow \mathrm{SU}(2) \times \mathrm{SU}(2)$ is the diagonal embedding. The hint is given as follows.

Let $\mathbb{H}$ be the quaternion in which an element $x \in \mathbb{H}$ can be expressed as

$$x = x_1 + x_2 i + x_3 j + x_4 k$$

where $x_a \in \mathbb{R}$ $(a = 1, \cdots, 4)$ and

$$i^2 = j^2 = k^2 = -1 \ , \quad ij = -ji = k \ , \quad jk = -kj = i \ , \quad ki = -ik = j \ .$$

We define the imaginary part of x as

$$\mathrm{Im}\ x = x_2 i + x_3 j + x_4 k$$

so that the conjugate $\overline{x}$ is written as

$$\overline{x} = x_1 - x_2 i - x_3 j - x_4 k$$

Therefore, the multiplication becomes

$$\overline{xy} = \overline{y} \cdot \overline{x}$$

The norm of x is

$$|x|^2 = x\overline{x} = \overline{x}x = x_1^2 + x_2^2 + x_3^2 + x_4^2$$

From this viepoint, SU(2) can be considered as a group of unit quaternions $\mathrm{SU}(2) = \{x \in \mathbb{H} | \ |x| = 1\}$. Then $\mathrm{SU}(2) \times \mathrm{SU}(2)$ acts on $\mathbb{H}$ by rotations in the following way:

$$x \mapsto q_1 x q_2^{-1}$$

is a rotation of $\mathbb{R}^4 = \mathbb{H}$ for $q_1,\ q_2 \in \mathrm{SU}(2)$. Then $(-q_1, -q_2)$ represents the same rotation as (q_1, q_2). Show that these represent all the rotations of $\mathbb{R}^4 = \mathbb{H}$ so that it is isomorphic to SO(4).

7. **This is a bonus problem with extra 3 points which is NOT mandatory.**

Theorem 0.1 (Borsuk-Ulam theorem) *For every continuous map $f : S^2 \to \mathbb{R}^2$, there exists a pair of anti-podal points x and $-x$ in S^2 with $f(x) = f(-x)$.*

Let us give a proof by contradiction. Suppose to the contrary that $f : S^2 \to \mathbb{R}^2$ is continuous, but $f(x) \neq f(-x)$ for $^\forall x \in S^2$. Then we define a continuous map $g : S^2 \to S^1$ by

$$g : x \mapsto \frac{f(x) - f(-x)}{|f(x) - f(-x)|}$$

which satisfies $g(-x) = -g(x)$. Without loss of generality, we set $g(1, 0, 0) = 1$. The equator of S^2 can understood as a loop

$$\beta : [0, 1] \to S^2; s \mapsto (\cos 2\pi s, \sin 2\pi s, 0)\ .$$

Show that $\alpha = g \circ \beta : [0, 1] \to S^1$ is contractible, which means that it is homotopic to the constant map $\alpha \sim 1$.

On the other hand, for $s \in [0, \frac{1}{2}]$, we have $\beta(s) = -\beta(s + \frac{1}{2})$ so that $\alpha(s) = -\alpha(s + \frac{1}{2})$. Let $\tilde{\alpha} : [0, 1] \to \mathbb{R}$ be the lift of α to the universal cover. Then, show that $\tilde{\alpha}(0) = \tilde{\alpha}(1) + n$ where n is an odd integer. Therefore, it contradicts with $\alpha \sim 1$.

Here is a "corollary". At every instant, there must be a pair of antipodal points on the earth having the same temperature and the same barometric pressure.

Using the Borsuk-Ulam theorem, give a proof of the following theorem.

Theorem 0.2 (Ham-Sandwich theorem) *Given any three sets in space, there exists a plane which bisects all three sets, in the sense that the part of each set which lies on one side of the plane has the same volume as the part of the same set which lies on the other side of the plane.*