

Homework 3: Due at class on Oct 26

1 More practice with gamma matrices

Show the following identities without using explicit representation of the matrices:

$$\text{Tr}(\gamma^\mu \gamma^\nu) = 4g^{\mu\nu}$$

$$\text{Tr}(\gamma^\mu \gamma^\nu \gamma^\kappa \gamma^\lambda) = 4(g^{\mu\nu} g^{\kappa\lambda} - g^{\mu\kappa} g^{\nu\lambda} + g^{\mu\lambda} g^{\nu\kappa})$$

$$\text{Tr}(\gamma^5 \gamma^\mu \gamma^\nu \gamma^\kappa \gamma^\lambda) = -4i\epsilon^{\mu\nu\kappa\lambda}$$

$$\text{Tr}(\gamma^{\mu_1} \gamma^{\mu_2} \dots \gamma^{\mu_{2n+1}}) = 0$$

2 Feymann diagrams for simple integrals (Alex Maloney's homework)

In this problem we will consider a toy version of version of scalar ϕ^4 theory, where instead of computing a full path integral we compute a single integral:

$$Z(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\phi \exp \left\{ -\frac{1}{2}\phi^2 - \frac{\lambda}{4!}\phi^4 \right\} \quad (2.1)$$

where ϕ is just a real number, rather than a field. This is the path integral for “0-dimensional scalar ϕ^4 field theory”.

2.1

Compute $Z(0)$, $Z(0.01)$, $Z(0.1)$, $Z(0.4)$, $Z(1)$, $Z(5)$ numerically by using your favourite numerical computing program. Ask or collaborate with your friends if you don't have one.

2.2

Replace $\exp(-\lambda\phi^4/24)$ with its series expansion in λ . Find the complete series expansion in λ in closed form:

$$Z(\lambda) = \sum_{n=0}^{\infty} c_n \lambda^n \quad (2.2)$$

and find an explicit expression for c_n . (Do this by expanding the exponent, exchanging orders of summation and integration, and doing the integral for each term in the series.)

2.3

Let us now develop the Feynman diagram expansion for $Z(\lambda)$. Explain how each c_n could be computed by using (appropriately modified) Feynman rules for our simple theory. Describe the Feynman rules for this theory and draw the appropriate “Feynman Diagrams” for the first few terms c_n . Check that this matches the answer you got in Problem 2.3 for the first few c_n . Note that we are computing the analog of “vacuum” diagrams in QFT.

2.4

Evaluate the order λ^0 and λ^1 terms in this series numerically for the values of λ given in Problem 2.1. For which values of λ does the order λ^1 term help improve the accuracy of our perturbative expansion for $Z(\lambda)$?

2.5

What happens to the integral $Z(\lambda)$ when λ is negative? Conclude from your answer that the radius of convergence (in λ) of the series you developed in Problem 2.3 must be zero.

2.6

Using Stirling’s approximation, find the asymptotic form of the c_n for large n .

Now consider the individual terms $\lambda^n c_n$ in the series for $Z(\lambda)$. Show that, for fixed λ , these terms will decrease as function of n until they reach a minimum at some critical value of n (call it $n_0(\lambda)$), after which they start increasing until they diverge at $n \rightarrow \infty$. Use your approximate form for c_n to estimate $n_0(\lambda)$ for small λ . This shows explicitly that the radius of convergence of this series is zero.

Now, show that the smallest term in the series is

$$\lambda^{n_0} c_{n_0} \sim e^{f(\lambda)}$$

where $f(\lambda)$ is a function you should determine. You need only determine the leading behaviour of $f(\lambda)$ at small λ .

2.7

Based on this, you might conclude that the series in Problem 2.3 is useless. But that is not the case at all; it still contains lots of useful information about $Z(\lambda)$! The expansion

$$e^{-x} = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} x^m = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \dots$$

has the following property for $x > 0$: the partial sums

$$f_n(x) \equiv \sum_{m=0}^n \frac{(-1)^m}{m!} x^m$$

are alternately strict over-estimates and strict under-estimates of the actual function; that is, for $x > 0$, $f_0(x) = 1 > e^{-x}$, $f_1(x) = (1 - x) < e^{-x}$, $f_2(x) = (1 - x + x^2/2) > e^{-x}$ and so forth with the $<, >$ alternating.

Use this property to show that the partial sums found above (i.e. (2.2) with n cut off at $0, 1, 2, 3, \dots$), are alternately over-estimates and under-estimates of $Z(\lambda)$. Therefore, the true answer $Z(\lambda)$ always lies between neighbouring terms in the series of partial sums. Use this property to find a bound for $Z(\lambda)$ at $\lambda = 1$, by evaluating alternating terms until they start to diverge. How tight is the bound? Repeat for $Z(0.4)$ and $Z(0.1)$. How does this compare with your numerical answer for Z ?

Argue that, as λ gets smaller and smaller, one can use the Feynman diagram expansion to place tighter and tighter bounds on $Z(\lambda)$. Estimate for how tight the bound will be (i.e. how big the error terms will be) as a function of λ , when λ is small. You may find the function $f(\lambda)$ in Problem 2.6 useful.

We conclude that, while the series does not converge, it gives us very good information about the value of $Z(\lambda)$. A series with this property - zero radius of convergence but the ability to give good information near the origin - is called an *asymptotic series*. Feynman diagram expansions in QFT typically only give asymptotic series, rather than convergent series.

2.8

Show that the Borel transform

$$\mathcal{B}\mathcal{Z}(\lambda) = \sum_{\ell=0}^{\infty} \frac{c_{\ell}\lambda^{\ell}}{\ell!}$$

converges provided $|\lambda| < \frac{3}{2}$ and that in this case

$$\mathcal{Z}(\lambda) = \int_0^{\infty} dz e^{-z} \mathcal{B}\mathcal{Z}(z\lambda)$$

so that $\mathcal{Z}(\lambda)$ may be recovered from its Borel transform.

2.9

By expanding $e^{-\frac{1}{2}\phi^2}$ in the integral (2.1) obtain the strong coupling expansion

$$\mathcal{Z}(\lambda) = \frac{1}{2\sqrt{\pi}} \sum_{L=0}^{\infty} \frac{(-1)^L}{L!} \Gamma\left(\frac{L}{2} + \frac{1}{4}\right) \left(\frac{6}{\lambda}\right)^{\frac{L}{2} + \frac{1}{4}}$$

for $\mathcal{Z}(\lambda)$ as a series in $1/\sqrt{\lambda}$. For $\lambda = 0.1$, how many terms does one need to obtain the value at which the weak coupling expansion appeared to converge?