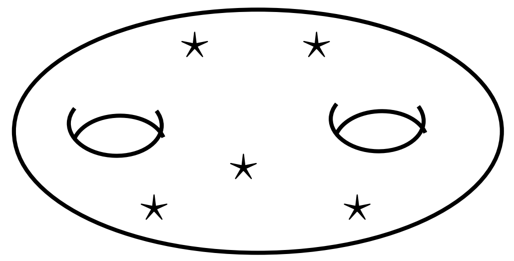
Class S on S^2

Joint with Yiwen Pan and Jiahao Zhang [arXiv:2310.07965]

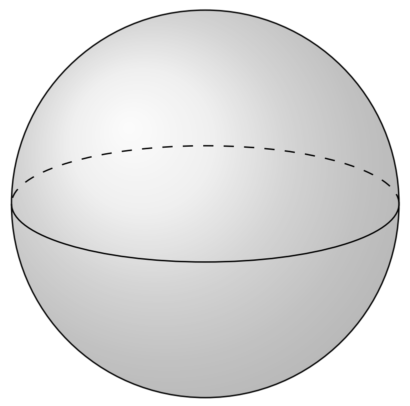
Satoshi Nawata (Fudan University)

Summary

6d N=(2,0) SCFT



4d N=2 SCFT of class S $SU(2)_R \times U(1)_r$



Top'l twist with $U(1)_R \times U(1)_r$

Top'l twist with $U(1)_r$

2d N=(0,2) theory

2d N=(0,4) theory

Only limited to Lagrangian

Landau-Ginzburg Dual

Relation to chiral algebra

incorporate punctures of all types

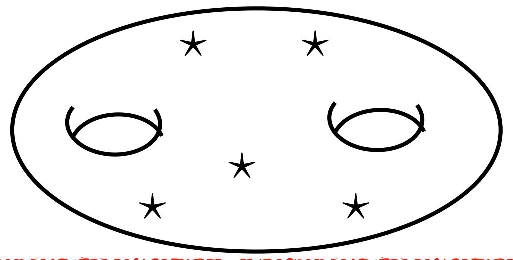
Remarkably simple elliptic genus

TQFT structures

Relation to Vafa-Witten?

Summary

6d N=(2,0) SCFT



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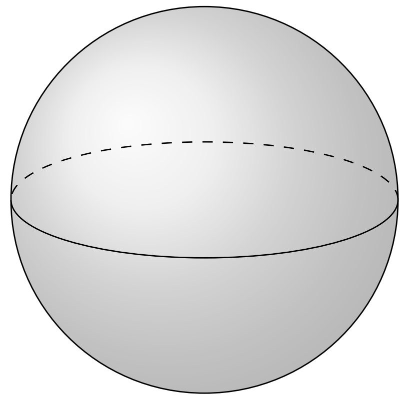

Top'l twist with $U(1)_R \times U(1)_r$

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2d N=(0,2) theory

- Only limited to Lagrangian
- Landau-Ginzburg Dual
- Relation to chiral algebra

2d N=(0,4) theory

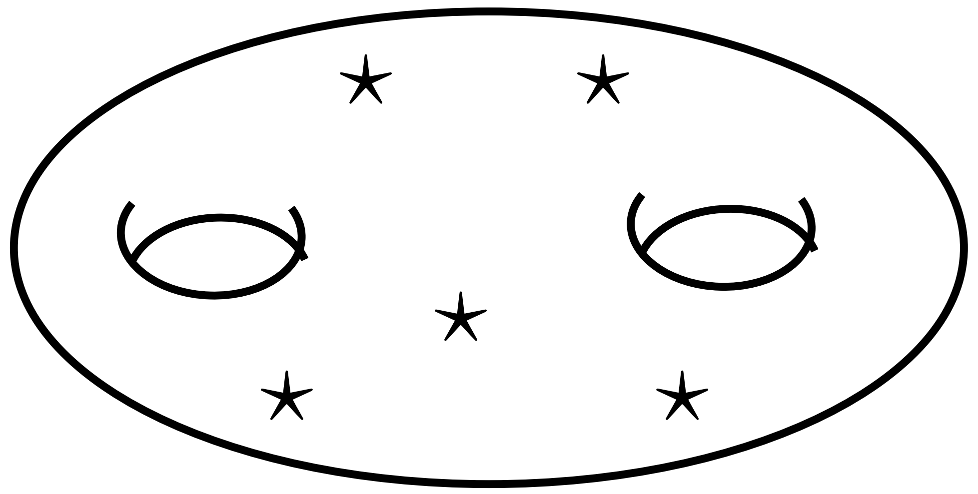
- incorporate punctures of all types
- Remarkably simple elliptic genus
- TQFT structures
- Relation to Vafa-Witten?

Review of Class S

Gaiotto '09

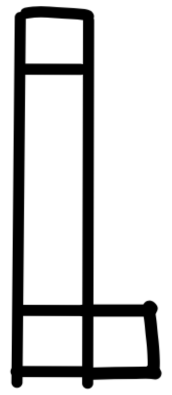
Low-energy effective theory of N M5-branes $\longleftrightarrow$ 6d N=(2,0) SCFT of type A_{N-1}

M5-branes wrap on punctured Riemann surface $C_{g,n}$

$C_{g,n}$ 

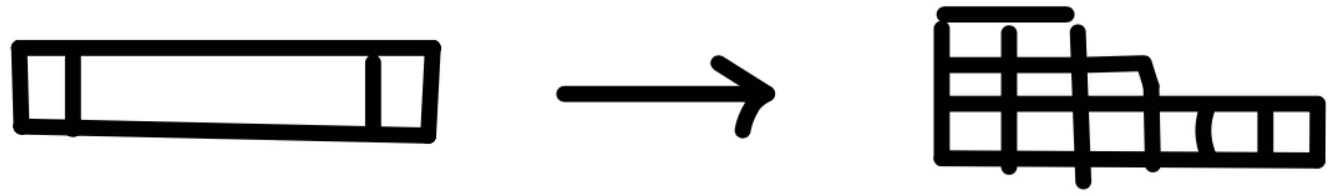
punctures are labelled by partitions of N

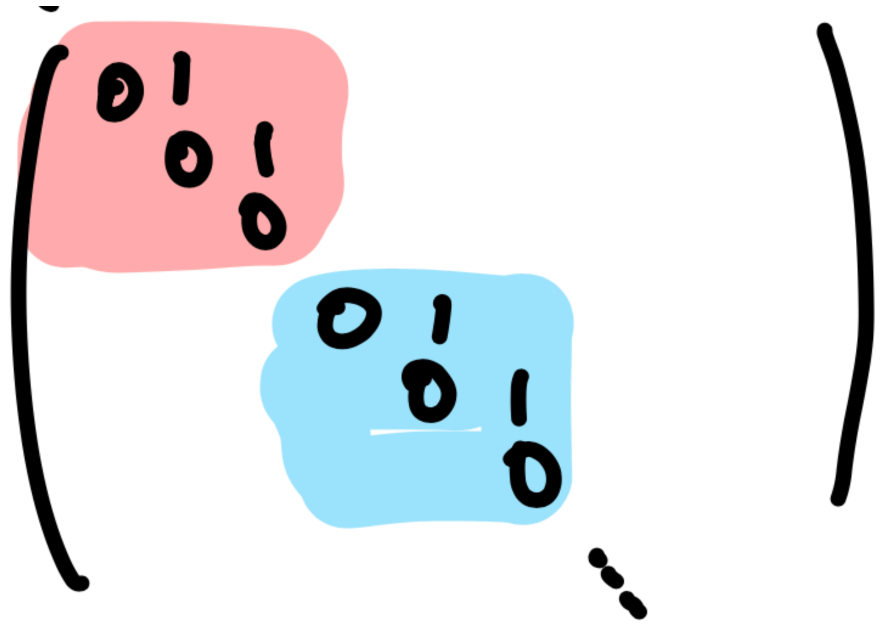
 full

 simple

An N=2 theory $T[C_{g,n}]$ by called class S: In general, no Lagrangian description

From full puncture, one can obtain punctures of any types by Nilpotent Higgsing



$\mathcal{O} \to \langle \mathcal{O} \rangle =$ 

4d N=2 Superconformal index

Gadde-Razamat-Rastelli-Yan '11

The partition function of $S^1 \times S^3$: $\mathcal{I}^{\rm 4d}(p,q,t) = \mathrm{Tr}(-1)^F\, e^{-\beta\,\tilde{\delta}_{1\dot{-}}}\, p^{j_1-j_2-r}\, q^{j_1+j_2-r}\, t^{R+r} \prod_a z_a^{f_a}$

Written in terms of elliptic gamma function: $\Gamma(z;p,q) = \prod_{m,n=0}^{\infty} \frac{1-p^{m+1}q^{n+1}/z}{1-p^m q^n z}$

Various limits: Relevant here is Schur index $(t=q)$

TQFT structure is manifest $\mathcal{I}^{\rm 4d} = \sum_\mu H_\mu^{2g-2+n} \prod_i \psi_\mu(b_i)$

SCFT/VOA correspondence: 4d N=2 SCFT → VOA

Beem et al '13

$c^\chi = -12c^{4d}$; non-unitary

Schur index is the vacuum character of VOA

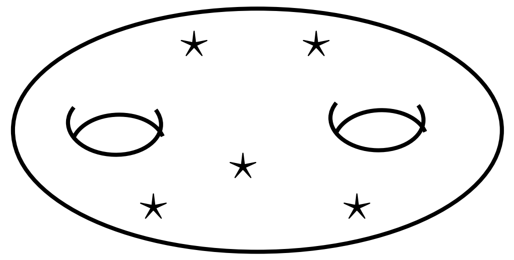
For Lagrangian: VOA = BRST cohomology

Hyper multi $\to$ $\beta\gamma$-system

Vector multi $\to$ bc-system

Twisted compactification of 4d N=2 theory on S^2

6d N=(2,0) SCFT



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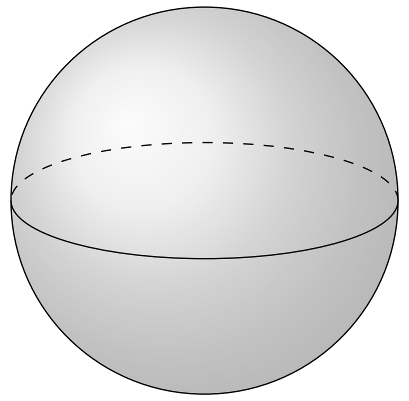

Top'l twist with $U(1)_r$

2d N=(0,2) theory

2d N=(0,4) theory

$U(1)_R$-charges must be integral for compactification to be well-defined

$R < 1$: chirals

$R > 1$: Fermi's

(0,2) chiral	$\longleftarrow$	4d hyper	$\longrightarrow$	(0,4) hyper
(0,2) vector		4d vector		(0,4) vector

Symmetries of 4d N=2 supercharges and fields

For (0,2), we need further twist by $U(1)_f$ flavor symmetry

		Spacetime		R-sym		Flavor				
		$\mathrm{SU}(2)_1$	$\mathrm{SU}(2)_2$	$\mathrm{SU}(2)_R$	$\mathrm{U}(1)_r$	$\mathrm{U}(1)_f$	$\mathrm{U}(1)_{T^2}$	$\mathrm{U}(1)_{S^2}$	$\mathrm{U}(1)^{(0,2)}$	$\mathrm{U}(1)^{(0,4)}$
SUSY	Q^1_-	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	0	-1	-1	0	0
	Q^1_+	$\frac{1}{2}$	0	$\frac{1}{2}$	1	0	1	1	2	2
	Q^2_-	$-\frac{1}{2}$	0	$-\frac{1}{2}$	1	0	-1	-1	-1	0
	Q^2_+	$\frac{1}{2}$	0	$-\frac{1}{2}$	1	0	1	1	1	2
	$\tilde{Q}^1_{\dot{-}}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	-1	0	-1	1	1	0
	$\tilde{Q}^1_{\dot{+}}$	0	$\frac{1}{2}$	$\frac{1}{2}$	-1	0	1	-1	-1	-2
	$\tilde{Q}^2_{\dot{-}}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	-1	0	-1	1	0	0
	$\tilde{Q}^2_{\dot{+}}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	-1	0	1	-1	-2	-2
1/2-Hypers	q	0	0	$\frac{1}{2}$	0	1	0	0	0	0
	$\tilde{q}$	0	0	$\frac{1}{2}$	0	-1	0	0	1	0
Adj	Φ	0	0	0	2	0	0	0	1	2

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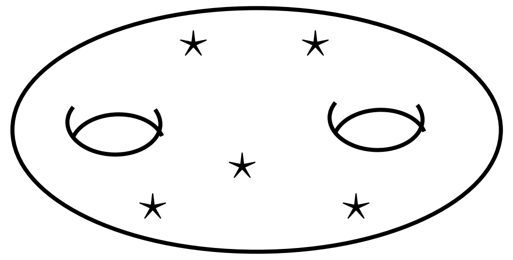

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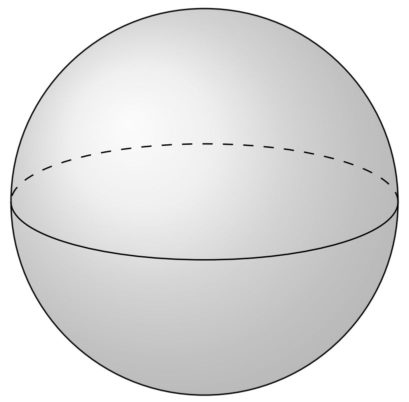

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	$\longleftarrow$		$\longrightarrow$	
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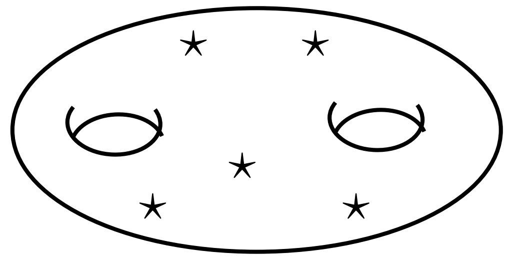

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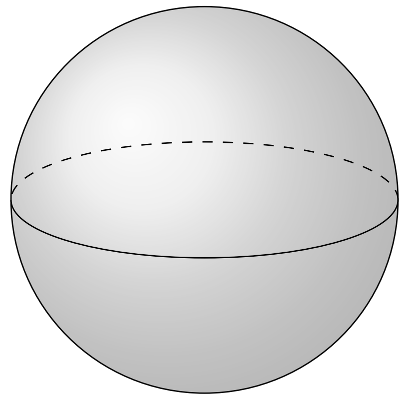

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Review of 2d N=(0,2) Theory

Elliptic genus

Chiral

$$\mathcal{I}^{(0,2)\mathrm{NS}}_{\mathrm{chi}}(\tau,u)=\prod_{w\in\lambda}\frac{\eta(q)}{\vartheta_4(q^{\frac{r-1}{2}}z^w)}$$

Fermi

$$\mathcal{I}^{(0,2)\mathrm{NS}}_{\mathrm{fer}}(\tau,u)=\prod_{w\in\lambda}\frac{\vartheta_4(q^{\frac{r}{2}}z^w)}{\eta(q)}.$$

Vector

$$\mathcal{I}^{(0,2)\mathrm{R|NS}}_{\mathrm{vec}}(q,z)=\frac{\eta(q)^{2\,\mathrm{rk}G}}{|W_G|}\prod_{\alpha\in\Delta}i\frac{\vartheta_1(z^\alpha)}{\eta(q)}.$$

$$\mathcal{I}^{(0,2)\mathrm{R|NS}}=\int_{\mathrm{JK}}\prod_{\mathrm{gauge}}\frac{dz}{2\pi iz}\mathcal{I}^{(0,2)\mathrm{R|NS}}_{\mathrm{vec}}(q,z)\prod_{\mathrm{matter}}\mathcal{I}^{(0,2)\mathrm{R|NS}}_{\mathrm{chi}}(q,z)\mathcal{I}^{(0,2)\mathrm{R|NS}}_{\mathrm{fer}}(q,z)$$

't Hooft anomaly: $\mathrm{Tr}\,\gamma^3 f^a f^b = k_F\delta^{ab}$

central change: $c_R = 3\,\mathrm{Tr}(\gamma^3R^2)$

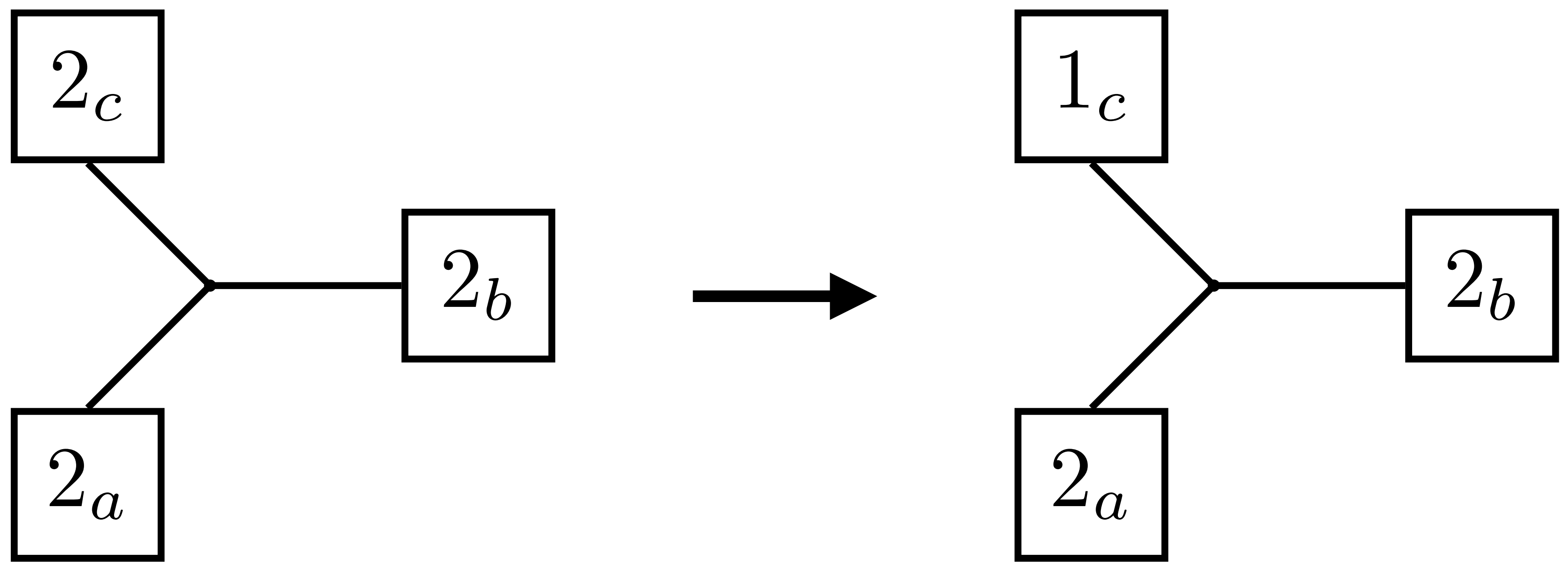
$U(1)_R$-charge: c-extremization

1. The enengy spectrum ≥ 0
2. normalizable (compact) vacuum

This assumption fails in the theory of our class

Gravitational anomaly: $c_R - c_L = \mathrm{Tr}(\gamma^3)$

4d Class S → 2d N=(0,2) theory



4d Hypermultiplet → 2d (0,2) chiral multiplet

4d vector multiplet → 2d (0,2) vector multiplet

4d building block ⟶ 2d building block

$$\mathcal{I}^{4d}_{T_2} = \Gamma(\sqrt{t}a^{\pm 1}b^{\pm 1}c^{\pm 1}) \quad \longrightarrow \quad \mathcal{I}^{(0,2)}_{U_2} = \frac{\eta(q)}{\vartheta_4(a^{\pm 1}b^{\pm 1}c)}$$

vector multiplet contribution

$$\mathcal{I}^{4d}_{\text{vec}} = \frac{1}{2}(p;p)(q;q)\Gamma\left(\frac{pq}{t}\right)\frac{\Gamma(z^{\pm 2}\frac{pq}{t})}{\Gamma\left(z^{\pm 2}\right)} \quad \longrightarrow \quad \mathcal{I}^{(0,2)}_{\text{vec}} = \frac{-\vartheta_1\left(z^{\pm 2}\right)}{2}$$

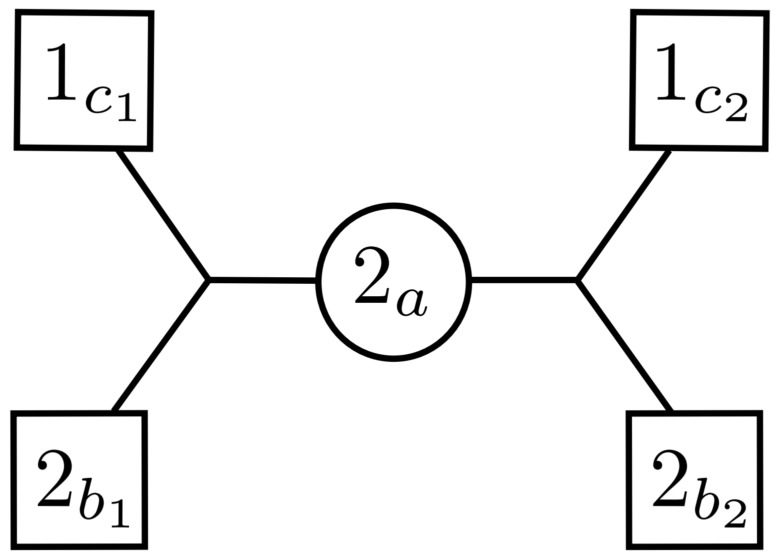
This is indeed the same as Schur limit.

Cecotti-Song-Yan-Vafa '15

The integrand for (0,2) elliptic genus and Schur index agree for theories of this class

$SU(2)$ with $N_f = 4$

Landau-Ginzburg dual



$$\mathcal{I}_{0,4}^{(0,2),2} = \int_{\mathrm{JK}} \frac{da}{2\pi i a} \mathcal{I}_{U_2}^{(0,2)}(b_1, a; c_1) \mathcal{I}_{\mathrm{vec}}^{(0,2)}(a) \mathcal{I}_{U_2}^{(0,2)}(b_2, a; c_2)$$
$$= \frac{\eta(q)^5 \vartheta_1(c_1^2 c_2^2)}{\vartheta_1(c_1^2)\vartheta_1(c_2^2)\vartheta_1(c_1 c_2 b_1^{\pm} b_2^{\pm})} \, .$$

6 chiral & 1 Fermi

$$\longleftrightarrow \qquad W = \Psi(\Phi_1 \Phi_2 + \det \widetilde{\Phi})$$

$SU(2)$ with $N_f = 4$

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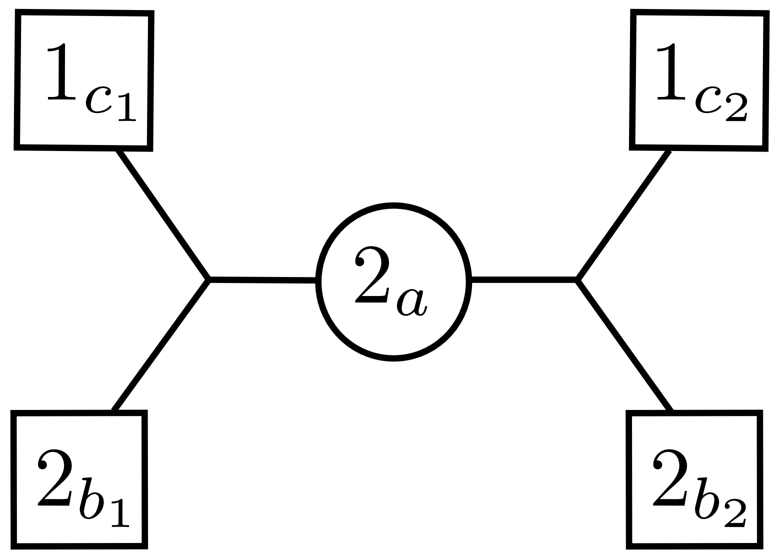


$$\mathcal{I}^{(0,2),2}_{0,4} = \int_{\mathrm{JK}} \frac{da}{2\pi i a} \mathcal{I}^{(0,2)}_{U_2}(b_1, a; c_1) \mathcal{I}^{(0,2)}_{\mathrm{vec}}(a) \mathcal{I}^{(0,2)}_{U_2}(b_2, a; c_2)$$
$$= \frac{\eta(q)^5 \vartheta_1(c_1^2 c_2^2)}{\vartheta_1(c_1^2)\vartheta_1(c_2^2)\vartheta_1(c_1 c_2 b_1^{\pm} b_2^{\pm})} \,.$$
$$= \mathrm{ch}_0^{\mathfrak{so}(8)_{-2}}(q, b, c) - \mathrm{ch}_{-2\omega_4}^{\mathfrak{so}(8)_{-2}}(q, b, c) \longleftarrow$$

6 chiral & 1 Fermi

$$\longleftrightarrow \quad W = \Psi(\Phi_1 \Phi_2 + \det \widetilde{\Phi})$$

Eager-Lockhart-Sharpe '19

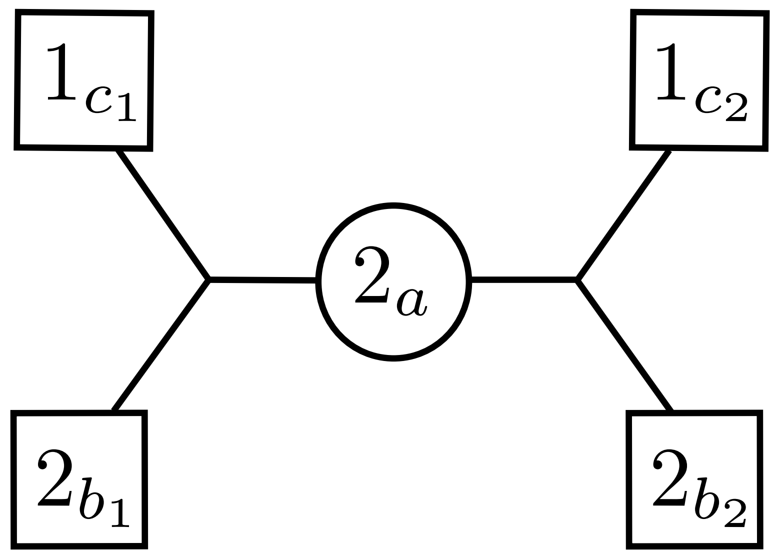
$\mathfrak{so}(8)_{-2}$: VOA of $SU(2)$ with $N_f = 4$

Naive c-extremization: $c_R = 3[8(r_\varphi - 1)^2 - 0 \cdot r_\psi^2 - \dim G] \to -9 \quad c_L = -15$

$$\longrightarrow r_\varphi = 1,\ r_\psi = 0,$$

$SU(2)$ with $N_f = 4$

Landau-Ginzburg dual



$$\begin{aligned}\mathcal{I}^{(0,2),2}_{0,4} &= \int_{\text{JK}} \frac{da}{2\pi i a} \mathcal{I}^{(0,2)}_{U_2}(b_1, a; c_1)\mathcal{I}^{(0,2)}_{\text{vec}}(a)\mathcal{I}^{(0,2)}_{U_2}(b_2, a; c_2) \\ &= \frac{\eta(q)^5 \vartheta_1(c_1^2 c_2^2)}{\vartheta_1(c_1^2)\vartheta_1(c_2^2)\vartheta_1(c_1 c_2 b_1^{\pm} b_2^{\pm})} \,. \\ &= \mathrm{ch}_0^{\mathfrak{so}(8)_{-2}}(q,b,c) - \mathrm{ch}_{-2\omega_4}^{\mathfrak{so}(8)_{-2}}(q,b,c)\end{aligned}$$

6 chiral & 1 Fermi

$$\longleftrightarrow \quad W = \Psi(\Phi_1\Phi_2 + \det\widetilde{\Phi})$$

$\longleftarrow$ $\mathfrak{so}(8)_{-2}$: VOA of $SU(2)$ with $N_f = 4$

~~Naive c-extremization~~: $c_R = 3[8(r_\varphi - 1)^2 - 0 \cdot r_\psi^2 - \dim G] \to -9 \quad c_L = -15$

$$\longrightarrow r_\varphi = 1,\ r_\psi = 0,$$

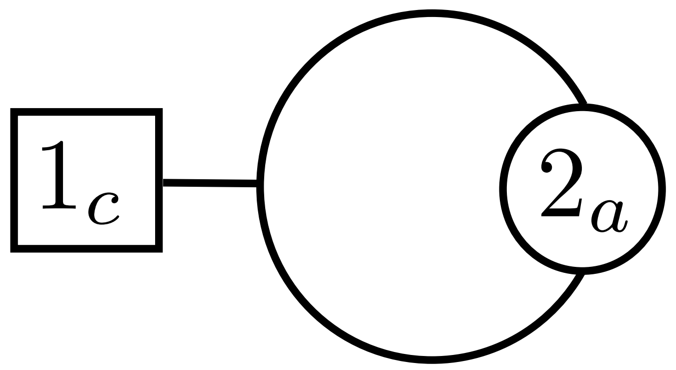
The central charges are negative, and non-unitary?

The vacuum moduli space $\Phi_1\Phi_2 + \det\widetilde{\Phi} = 0$ is non-compact, so the assumption fails.

$$c_R = 3[8 \cdot 1 - \dim G] = 15 \quad c_L = 10$$

The right-moving central charge is 3 x dim of the vacuum moduli space

Free chiral + $SU(2)$ with adjoint



$$
\begin{aligned}
\mathcal{I}^{(0,2),2}_{1,1} &= \int_{\text{JK}} \frac{da}{2\pi i a} \mathcal{I}^{(0,2)}_{U_2}(a, a^{-1}; c_1) \mathcal{I}^{(0,2)}_{\text{vec}}(a) \\
&= \frac{\eta(q)}{\vartheta_1(c_1^2)} = \frac{\eta(q)}{\vartheta_4(c_1)} \cdot \frac{\vartheta_4(c_1)}{\vartheta_1(c_1^2)} \,.
\end{aligned}
$$

Free chiral

SU(2)+adj

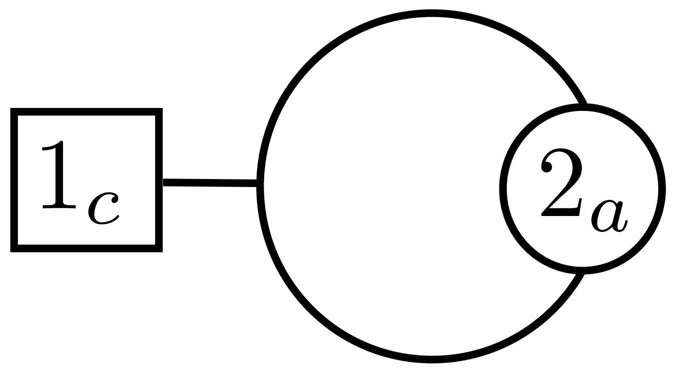
$$
\frac{i\vartheta_4(c_1)}{\vartheta_1(c_1^2)} = \mathrm{ch}_0^{\mathcal{N}=4}(q, c_1) + \mathrm{ch}_1^{\mathcal{N}=4}(q, c_1)
$$

Small N=4 SCA: VOA of $SU(2)$ with adjoint

Naive c-extremization: $c_R = 3[4(r_\varphi - 1)^2 - 0 \cdot r_\psi^2 - \dim G] \to -9 \qquad c_L = -1 - 9$

$$
\longrightarrow r_\varphi = 1,\ r_\psi = 0,
$$

Free chiral + $SU(2)$ with adjoint



$$\mathcal{I}^{(0,2),2}_{1,1} = \int_{\mathrm{JK}} \frac{da}{2\pi i a} \mathcal{I}^{(0,2)}_{U_2}(a, a^{-1}; c_1)\mathcal{I}^{(0,2)}_{\mathrm{vec}}(a)$$
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Free chiral

SU(2)+adj

$$\frac{i\vartheta_4(c_1)}{\vartheta_1(c_1^2)} = \mathrm{ch}_0^{\mathcal{N}=4}(q, c_1) + \mathrm{ch}_1^{\mathcal{N}=4}(q, c_1)$$

Small N=4 SCA: VOA of $SU(2)$ with adjoint

~~Naive c-extremization~~: $c_R = 3[4(r_\varphi - 1)^2 - 0 \cdot r_\psi^2 - \dim G] \to -9 \quad c_L = -1 - 9$

$$\longrightarrow r_\varphi = 1, \ r_\psi = 0,$$

It is because of non-compact vacuum : $c_R = 6 \quad c_L = -1 + 6$

N=(0,2) SU(2) with adjoint chiral $\quad = \quad$ N=(2,2) vector multiplet

In the IR, VOA at least contains small N=4 SCA as sub algebra: supersymmetric enhancement?

What we learned...

$$c_L = c_L^{\text{naive}} + 3n_n = 12\left(5c^{4d} - 4a^{4d}\right)$$

Positive Negative

Cecotti-Song-Yan-Vafa ‘15
Dedushenko-Gukov ‘17

The left-moving VOA of (0,2) theory = BRST cohomology

chiral multi $\rightarrow$ $\beta\gamma$-system
Vector multi $\rightarrow$ bc-system

$$J_{BRST} = \sum_{A=1}^{\dim G} c^A \left(J_{\mathrm{m}}^A + \frac{1}{2} J_{\mathrm{gh}}^A\right)$$

$$J_{\mathrm{m}}^A = i\beta T^A \gamma, \quad J_{\mathrm{gh}}^A = -if_{BC}^A c^A b^B c^C$$

The construction is analogous to 4d N=2 VOA

The stress-energy tensor of (0,2) theory deviates from the one of VOA

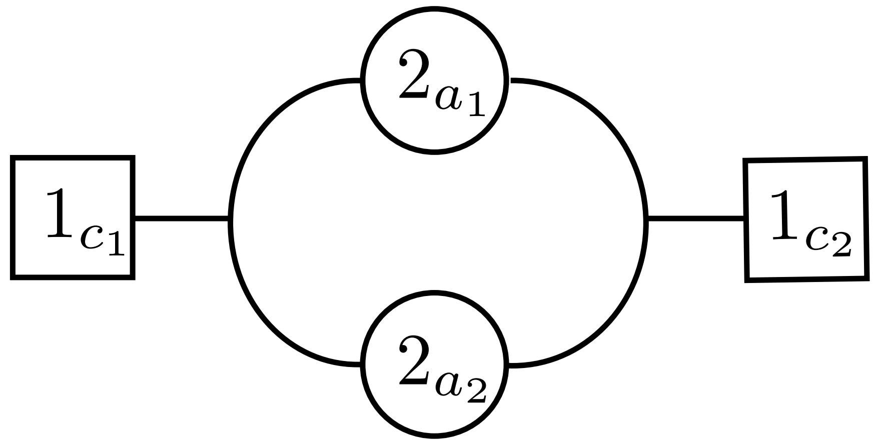
$$T^{(0,2)} = T^{\chi} + \left(\frac{1}{2} - r_*\right)\partial J$$

Current of $\beta\gamma$-system

4d N=2 VOA

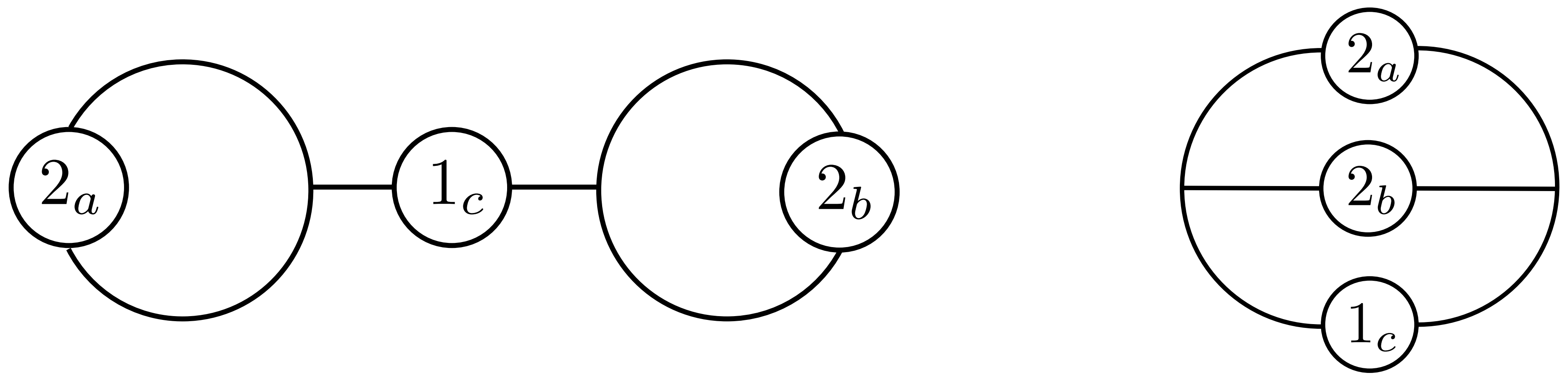
Deviation from c-ext

$g = 1, n = 2$



$$\mathscr{I}^{(0,2),2}_{1,2}(c_1, c_2) = \frac{\eta^2}{\vartheta_1(c_1^2)\,\theta_1(c_2^2)}$$

$g = 2$



$$\mathcal{I}^{(0,2),2}_{2,0} = \eta(q)^2 \int_{\mathrm{JK}} \frac{dc}{2\pi i c} \frac{\vartheta_4(c^{\pm 2})}{\vartheta_1(d^2 c^{\pm 2})} = \frac{2\vartheta_4(d^2)^2}{\eta(q)\vartheta_1(d^4)}\,.$$

Satisfy MLDE of the corresponding VOA

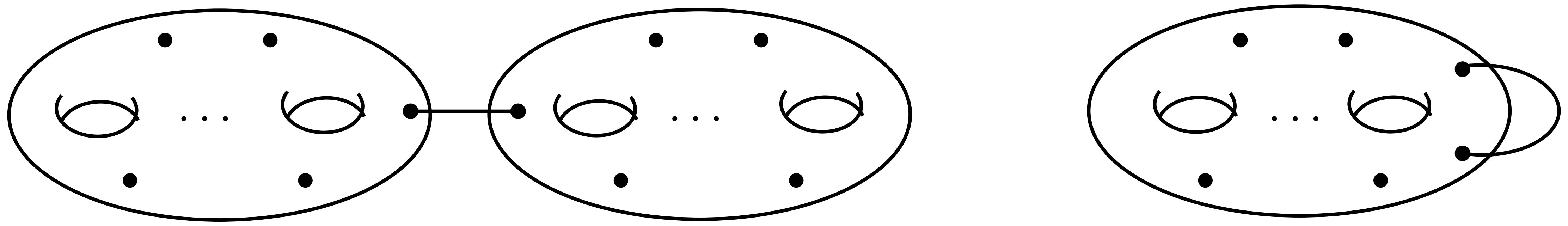
Mathur-Mukhi-Sen ‘88

genus g, n punctures with $(g-1)$ $U(1)$ gauge groups

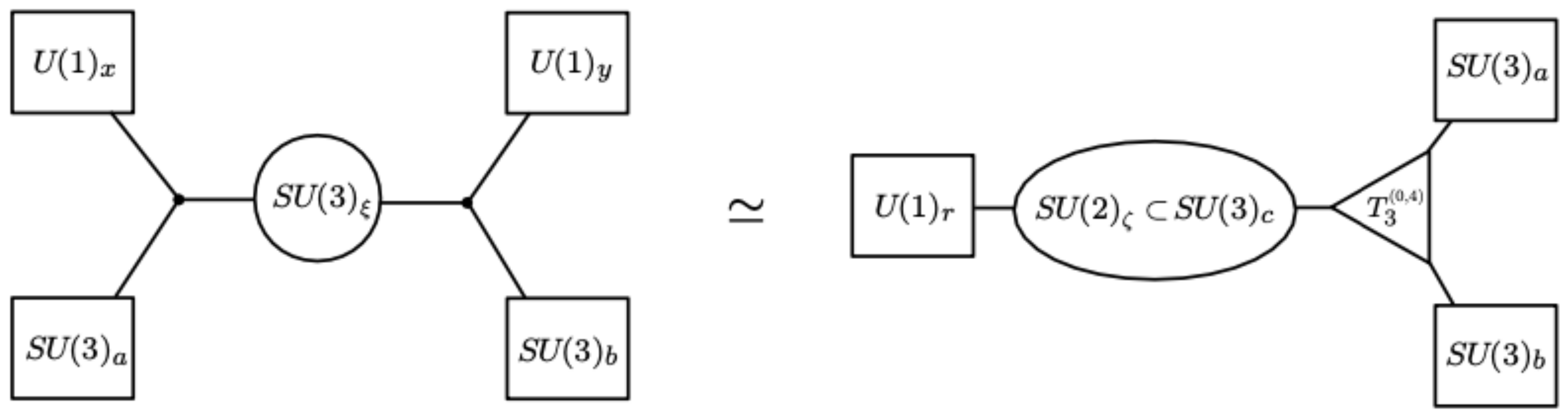
min # of $U(1)$ gauge groups for $C_{g,n}$ is $(g-1)$.

EG is independent of frames. Also. it exhibits TQFT structure

$$\mathcal{I}^{(0,2),2}_{g>0,n}(c_1, \ldots, c_n) = \prod_{j=1}^{g-1} \frac{2\vartheta_4(d_j^2)^2}{\eta(q)\vartheta_1(d_j^4)} \prod_{i=1}^{n} \frac{\eta(q)}{\vartheta_1(c_i^2)}$$



Comments



higher rank generalizations are parallel.

Non-Lagrangian

Eager-Lockhart-Sharpe '19

E_6 Minahan-Nemechansky: $I^{(E_6)_{-3}} = ch_0 - ch_{-3\omega_0}$ is known

But it doesn't lead to SU(3) N_f=6

(A_1, D_4) Argyres-Douglas theory: VOA = $\widehat{SU}(3)_{-\frac{3}{2}}$

conformal dimension of Coulomb branch are integral.

But cannot construct modular invariant combination by characters

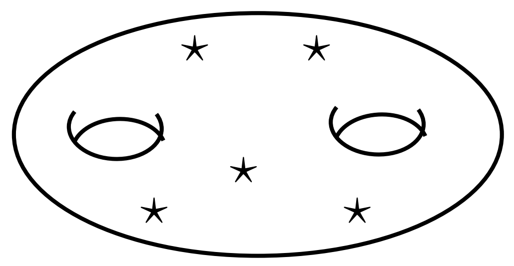
(A_1, D_{2n+1}) Argyres-Douglas theory: VOA = $\widehat{SU}(2)_{-\frac{4n}{2n+1}}$

conformal dimension of Coulomb branch are fractional.

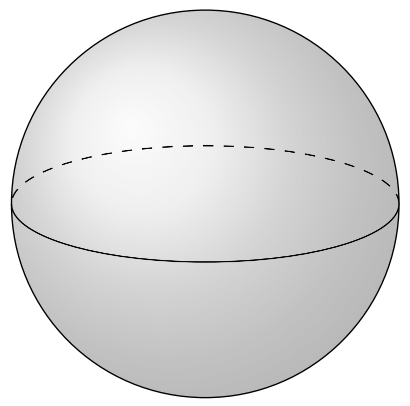
cannot construct modular invariant combination by characters

Summary

6d N=(2,0) SCFT



4d N=2 SCFT of class S $SU(2)_R \times U(1)_r$



Top'l twist with $U(1)_R \times U(1)_r$

2d N=(0,2) theory

Only limited to Lagrangian

Landau-Ginzburg Dual

Relation to chiral algebra

Top'l twist with $U(1)_r$

2d N=(0,4) theory

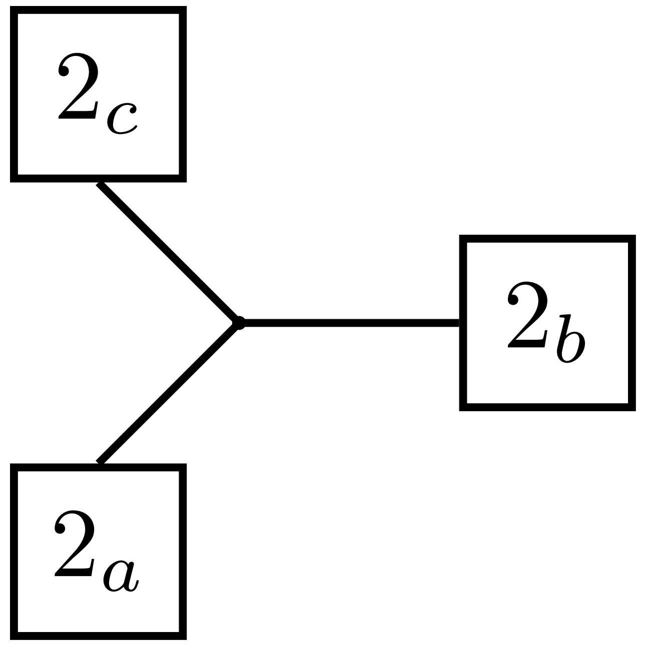
incorporate punctures of all types

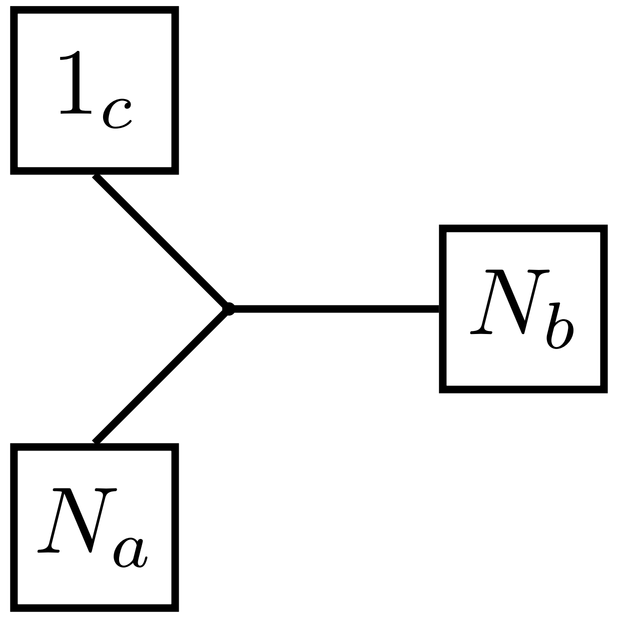
Remarkably simple elliptic genus

TQFT structures

Relation to Vafa-Witten?

4d Class S → 2d N=(0,4) theory





$$\mathcal{I}^{(0,4)}_{0,3}(a,b,c) = \prod_{i,j=1}^{N} \frac{\eta(q)^2}{\vartheta_1(v(ca_ib_j)^{\pm})} \,,$$

$$\mathcal{I}^{(0,4)}_{\text{vec}}(a) = \frac{\left(\vartheta_1(v^2)\eta(q)\right)^{N-1}}{N!} \prod_{\substack{A,B=1\\A\neq B}}^{N} \frac{\vartheta_1(v^2 a_A/a_B)\vartheta_1(a_A/a_B)}{\eta(q)^2}$$

Given a quiver description, we study SCFT on Higgs branch.

4d N=2 R-symmetry: $SU(2)_R \times U(1)_r$

affine Lie algebra of 2d small N=4 Topological twist

Elliptic genus: $\mathrm{Tr}(-1)^F q^{H_L} v^{R_+ - R_-}$

First the setting is studied by Putrov-Song-Yan: g=0

We study $g \geq 1$. Remarkably simple Elliptic genus.

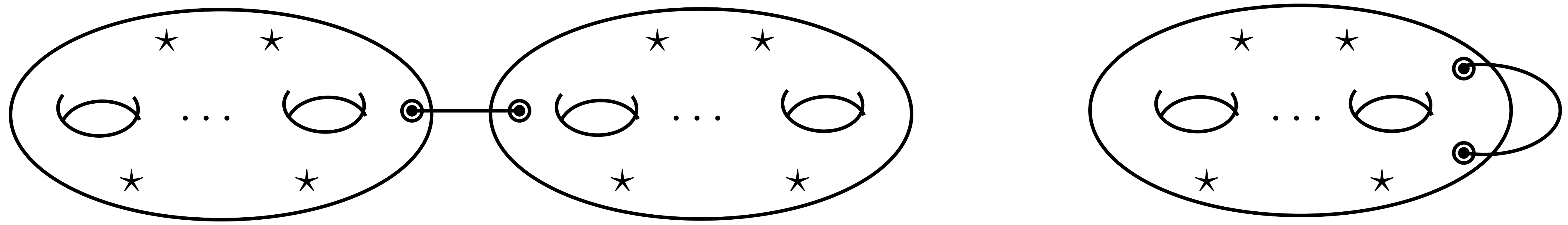
Type A_1

Straightforward to compute EG because they admit Lagrangian descriptions

$$\mathcal{I}_{g,n}^{(0,4),2}(c_1,\dots,c_n) = \left(\frac{\vartheta_1(v^2)\vartheta_1(v^4)}{\eta(q)^2}\right)^{g-1} \prod_{i=1}^{n} \frac{\eta(q)^2\vartheta_1(v^4)}{\vartheta_1(v^2)\vartheta_1(v^2 c_i^{\pm 2})}$$

For $g \geq 1$, EG is again a product of theta functions.

They exhibit TOFT structure.

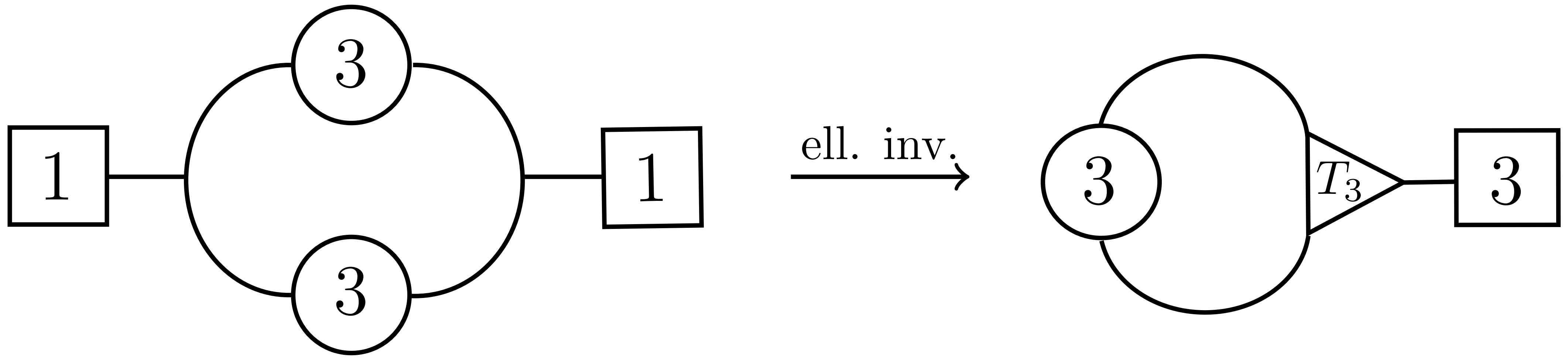


Type A_2

Easy to read off simple puncture

$$\mathcal{I}^{(0,4),3}_{g=1;n_1,0}(c_1,\ldots,c_{n_1}) = \prod_{i=1}^{n_1} \frac{\eta(q)^2\vartheta_1(v^6)}{\vartheta_1(v^2)\vartheta_1(v^3c_i^{\pm 3})}\ .$$

For maximal puncture, use elliptic inversion formula.

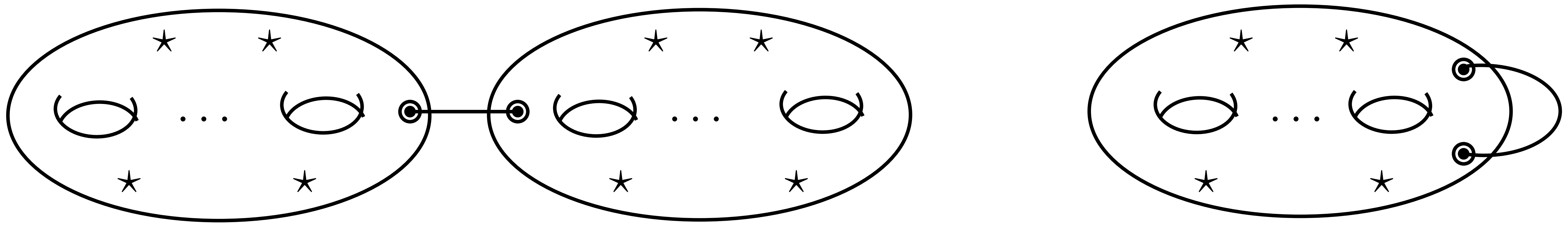


$$\mathcal{I}^{(0,4),3}_{g=1;0,n_3=1}(b) = \frac{\eta(q)^5}{2\vartheta_1(v^2z^{\pm 2})}\int_{\mathrm{JK}}\frac{ds}{2\pi i s}\frac{\vartheta_1(s^{\pm 2})\vartheta_1(v^{-2})}{\vartheta_1(v^{-1}s^{\pm 1}z^{\pm 1})}\mathcal{I}^{(0,4),3}_{g=1;n_1,0}(s^{\frac{1}{3}}/r, s^{-\frac{1}{3}}/r)$$
$$= \frac{\eta(q)^6\vartheta_1(v^2)\vartheta_1(v^4)\vartheta_1(v^6)}{\prod_{A,B=1}^{3}\vartheta_1(v^2b_A/b_B)}$$

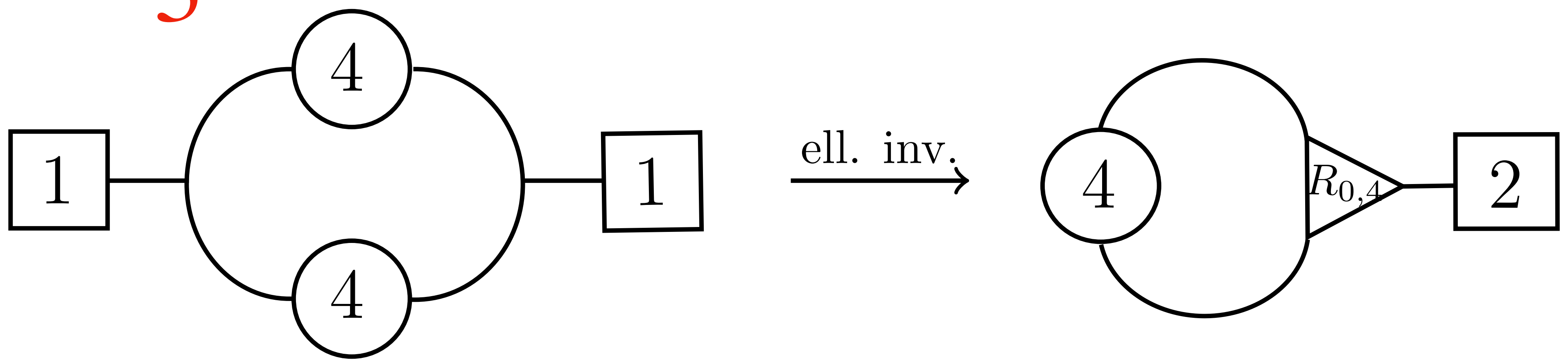
Generically,

$$\mathcal{I}^{(0,4),3}_{g;n_1,n_3} = \left(\frac{\vartheta_1(v^2)\vartheta_1(v^4)^2\vartheta_1(v^6)}{\eta(q)^4}\right)^{g-1}\mathcal{I}^{(0,4),3}_{1;n_1,0}\mathcal{I}^{(0,4),3}_{1;0,n_3}$$

They exhibit TOFT structure.



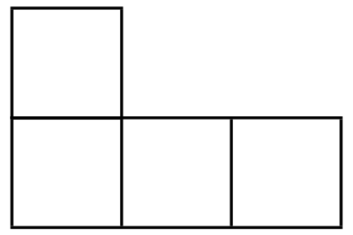
Type A_3



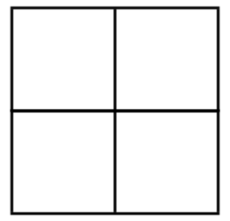
elliptic inversion formula

Agarwal-Maruyoshi-Song '18

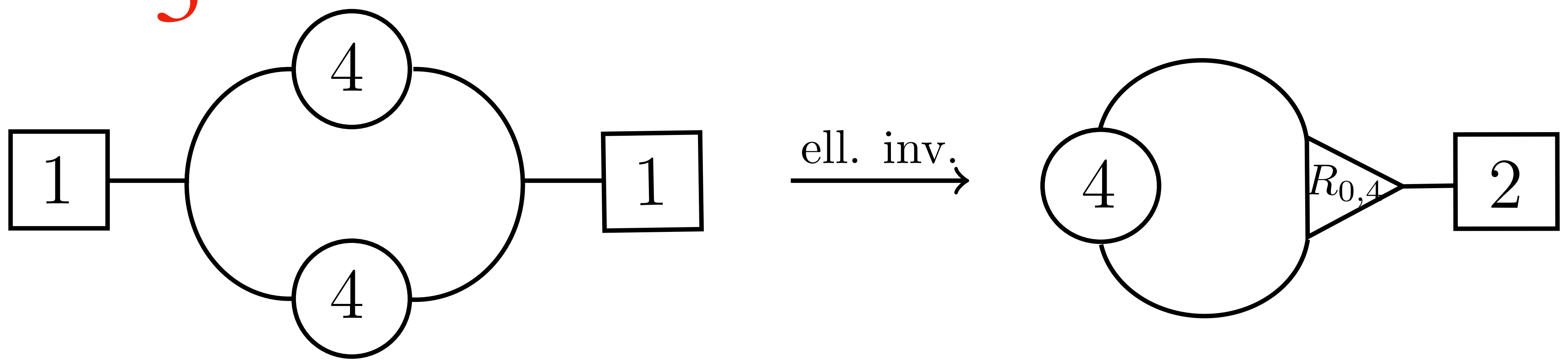
$$\mathcal{I}^{(0,4),4}_{g=1;0,1,0,0} = \frac{\eta(q)^5}{2\vartheta_1(v^2 z^{\pm 2})} \int_{\mathrm{JK}} \frac{ds}{2\pi i s} \frac{\vartheta_1(s^{\pm 2})\vartheta_1(v^{-2})}{\vartheta_1(v^{-1}s^{\pm 1}z^{\pm 1})} \mathcal{I}^{(0,4),3}_{g=1;2,0,0,0}(s^{\frac{1}{4}}/r, s^{-\frac{1}{4}}/r) \longleftrightarrow$$

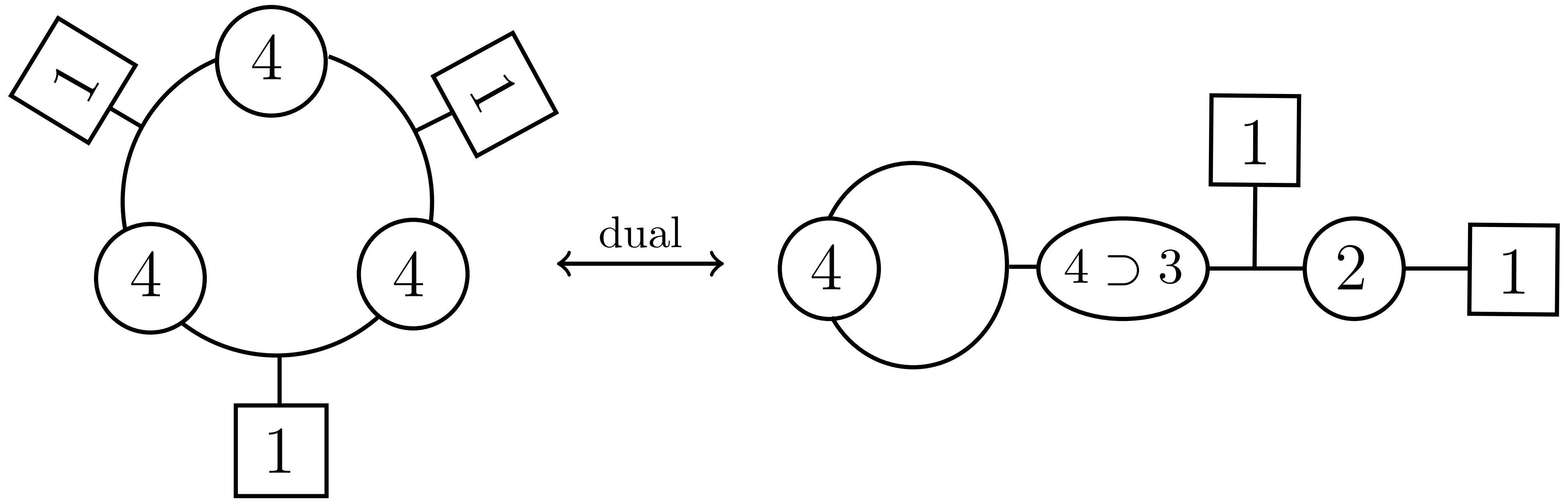


$$\mathcal{I}^{(0,4),4}_{g=1;0,0,1,0} = \frac{\eta(q)^5 \vartheta_1(v^2)\vartheta_1(v^{-2})}{2\vartheta_1(v^4)} \int_{\mathrm{JK}} \frac{ds}{2\pi i s} \frac{\vartheta_1(s^{\pm 2})}{\vartheta_1(s^{\pm 1})\vartheta_1(v^{-2}s^{\pm 1})} \mathcal{I}^{(0,4),3}_{g=1;2,0,0,0}(s^{\frac{1}{4}}/w, s^{-\frac{1}{4}}/w) \longleftrightarrow$$



Type A_3





Contributions from all the punctures are products of theta functions

$$\mathcal{I}^{(0,4),4}_{1;n_1,0,0,0} = \prod_{i=1}^{n_1} \frac{\eta(q)^2 \vartheta_1(v^8)}{\vartheta_1(v^2)\vartheta_1(v^4 c_i^{\pm 4})}$$

$$\mathcal{I}^{(0,4),4}_{1;0,n_2,0,0} = \prod_{i=1}^{n_3} \frac{\eta(q)^6 \vartheta_1(v^6)\vartheta_1(v^8)}{\vartheta_1(v^2)^2\vartheta_1(v^2 z_i^{\pm 2})\vartheta_1(v^3 z_i^{\pm} r_i^{\pm 4})}$$

$$\mathcal{I}^{(0,4),4}_{1;0,0,n_3,0} = \prod_{i=1}^{n_2} \frac{\eta(q)^4 \vartheta_1(v^6)\vartheta_1(v^8)}{\vartheta_1(v^2)\vartheta_1(v^4)\vartheta_1(v^2 w_i^{\pm 4})\vartheta_1(v^4 w_i^{\pm 4})} ,$$

$$\mathcal{I}^{(0,4),4}_{1;0,0,0,n_4} = \prod_{i=1}^{n_4} \frac{\eta(q)^{12} \vartheta_1(v^2)\vartheta_1(v^4)\vartheta_1(v^6)\vartheta_1(v^8)}{\prod_{A,B=1}^{4} \vartheta_1(v^2 b_{iA}/b_{iB})}$$

Generic formulas enjoy dualities coming from MCG of Riemann surfaces

$$\mathcal{I}^{(0,4),4}_{g;n_1,n_2,n_3,n_4} = \left(\frac{\vartheta_1(v^2)\vartheta_1(v^4)^2\vartheta_1(v^6)^2\vartheta_1(v^8)}{\eta(q)^6} \right)^{g-1} \mathcal{I}^{(0,4),4}_{1;n_1,0,0,0}\mathcal{I}^{(0,4),4}_{1;0,n_2,0,0}\mathcal{I}^{(0,4),4}_{1;0,0,n_3,0}\mathcal{I}^{(0,4),4}_{1;0,0,0,n_4} ,$$

Type A_{N-1}

EG of type A_{N-1} takes the form

$$\mathcal{I}^{(0,4),N}_{g,n} = (\mathcal{H}_N)^{g-1} \prod_{i=1}^{n} \mathcal{I}^{(0,4),N}_{\lambda_i}(b_i)$$

where the full puncture & Handle contribution

$$\mathcal{I}^{(0,4),N}_{[1^N]}(b) = \frac{\eta(q)^{N^2-N} \prod_{M=1}^{N} \vartheta_1(v^{2M})}{\prod_{A,B}^{N} \vartheta_1(v^2 b_A/b_B)}$$

$$\begin{aligned}
\mathcal{H}_N &= \int_{\mathrm{JK}} \frac{d\boldsymbol{a}}{2\pi i \boldsymbol{a}} \mathcal{I}^{(0,4),N}_{[1^N]}(\boldsymbol{a}) \mathcal{I}^{(0,4),N}_{[1^N]}(\boldsymbol{a}^{-1}) \mathcal{I}^{(0,4)}_{\mathrm{vec}}(a) \\
&= \frac{\prod_{M=1}^{N} \vartheta_1(v^{2M})^2}{N! \eta(q)^{N-1} \vartheta_1(v^2)^{N+1}} \int_{\mathrm{JK}} \frac{d\boldsymbol{a}}{2\pi i \boldsymbol{a}} \prod_{A \neq B} \frac{\vartheta_1(a_A/a_B)}{\vartheta_1(v^2 a_A/a_B)} \\
&= \frac{\prod_{M=1}^{N} \vartheta_1(v^{2M})^2}{\eta(q)^{2(N-1)} \vartheta_1(v^2) \vartheta_1(v^{2N})}.
\end{aligned}$$

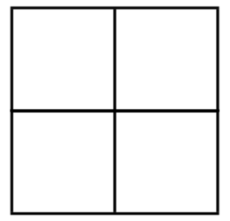
Nilpotent Higgsing $\mathscr{O} \to \langle \mathscr{O} \rangle$

$$\mathcal{I}_\lambda(c) = \lim_{b \to c} \left[\frac{K_\lambda(c)}{K_{[1^N]}(b)} \right]_{\Gamma(t^\alpha z) \to \frac{\eta(q)}{\vartheta_1(v^{2\alpha} z)}} \mathcal{I}_{[1^N]}(b)$$

$$K_\lambda(c) := \mathrm{PE}\left[\sum_j \frac{t^{j+1} - pqt^j}{(1-p)(1-q)} \mathrm{ch}^f_{\mu_j}(c) \right]$$

$SU(2) \hookrightarrow SU(N)$ determines branching

$$\mathrm{ch}_{\mathbf{adj}}(b) = \sum_j \mathrm{ch}^f_{\mu_j}(c)\, \mathrm{ch}^{\mathrm{SU}(2)}_{\sigma_j}(t^{1/2})$$

For example, puncture labelled by 

$$K_{[2^2]}(c) = \mathrm{PE}\left[\frac{(t - pq)}{(1-q)(1-p)} (c^2 + \frac{1}{c^2} + 1) + \frac{(t^2 - pqt)}{(1-q)(1-p)} (c^2 + \frac{1}{c^2} + 2) \right]$$

Future directions

Interpretation in $AdS_3 \times S^2$: Cardy limit and large N limit cf) $\omega_1 + \omega_2 - 2\varphi = 2\pi i n$

Why $g \geq 1$?

N=(0,4) non-linear sigma models. Landau-Ginzburg description

N=(2,2) twist: GLSM, Landau-Ginzburg duality, Mirror symmetry

Class S on $S^2 \rightarrow$ Class S on a Riemann surface: elliptic Bethe Ansatz equation

M5-branes on more general 4-manifolds.

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Thank you!

Many thanks to invitation and effort on organization