Introduction to string theory: Fall-2017

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Abstract: These are lecture notes of the course on string theory at Fudan in Fall 2017. Each section is written for a lecture for about 2.5 hours. Comments are welcome. If you find typos, please let us know.

Contents

1 Lecture 1

1.1 Motivation and Overview

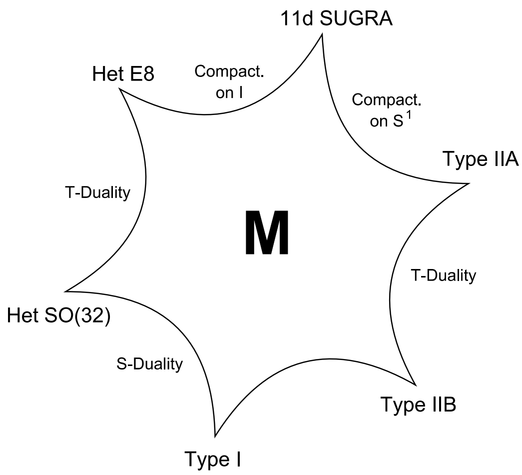
1.1.1 Why do we study string theory?

• For quantum gravity

One of the major problems in theoretical physics is to provide unified description of all the forces in Nature. The standard model has unified electro-magnetic, weak and strong force based on quantum field theories whereas general relativity for gravity is formulated within the framework of classical physics. A quantum theory of gravity is needed in order to reconcile general relativity with the principles of quantum mechanics. However, it is known that the renormalization procedure does not cure ultraviolet divergence for gravity. Therefore, naive quantization of general relativity does not give a consistent theory.

String theory not only yields the first quantization of gravity but also resolves the renormalization problem of gravity by replacing point particles by vibrating strings. To my knowledge, string theory is currently the only theory which has this property. Once we discover a candidate to unify gravity and quantum mechanics of all the forces, it is inevitable to try to understand it as well as we can although there is never any guarantee we can achieve.

Indeed, string theory has given deep insights about quantum gravity. The Beckenstein-Hawking formula [Bek73, BCH73, Haw75] was microscopically derived by counting D-brane states with fixed mass and charge for certain (called extremal) black holes in string theory [SV96]. The AdS/CFT correspondence [Mal99] conjectures that non-perturbative definition of a string theory on AdS background is described by a conformal field theory, and it partially resolves Hawking information paradox of black hole [Haw76]. Interestingly, a generalization of the Beckenstein-Hawking formula, called the Ryu-Takayanagi entanglement entropy formula [RT06], connects quantum gravity and quantum information theory, which is actively studied in recent years.[1] These developments certainly pose questions to basic concepts of space-time.



• **Rich arena for physics theories**

String theory has shed a lot of light on the behavior of an established physical theory. These range from proof of positive energy, quark confinement, heavy iron collisions, quantum critical behavior, quantum black holes, quantum information, etc. Moreover it sometimes sets up a new framework to study established physical theory. In my opinion, there are two reasons for it. One reason is that string theory can generates vast families (in fact, infinitely many) of quantum field theories in various dimensions. There are five types of string theories, Type I, IIA, IIB, Heterotic SO(32), $E_8 \times E_8$ and M-theory is a limit of large dilaton expectation value in Type IIA string theory. In addition, Type II theories can contain Dp-branes, and M-theory can have M2 and M5-branes. Although there are finitely many string theories, they can "engineer" infinitely many quantum field theories depending on manifolds string theory lives and brane configurations. The other reason is duality, an equivalence between two different descriptions at quantum level. In fact, string theories of five types are related by a web of dualities (see figure), and the AdS/CFT correspondence is another kind of duality. More dualities have been (and hopefully will be) discovered in string theories and a duality plays a crucial role to show drastically different viewpoint for established physical theory.

Through recent developments in string theory, especially in the study of M5-branes, we have learned that there are vast families of QFTs which do not admit Lagrangian description. They are intrinsically strongly-coupled so that techniques in QFT we have developed for decades cannot be used. Currently, we are trying to understand QFTs without Lagrangian description case by case by using dualities, dimensional reduction on some manifolds, RG flows or perturbing theories. However, there is no consistent unified description for these theories, and we therefore need new framework to describe QFTs without Lagrangian description.

• **For mathematical structure**

There is a certain group of people who are interested in string theory because

[1] Ling-Yan Hung at Fudan has been working on this area.

it has brought about a lot of new insights in mathematics, especially in geometry.[2] String theory has repeatedly provided new ways to look at geometry, which arise as inevitable consequences of trying to understand how physical theory should be. Mirror symmetry [CDLOGP91], Seiberg-Witten invariants [Wit94], AGT relation [AGT10], etc are salient examples. Moreover, string dualities have led highly non-trivial conjectures that connect seemingly different subjects of mathematics. String theory provides many insights in geometry partly because we do not understand the theory. In contrast to the fact that Einstein theory of gravity has been constructed based on Riemannian geometry, we do not understand geometric foundation for string theory yet. However, physicists can come up with new insights in geometry because physicists stumble upon very deep theory we do not understand well. It is certain that there are more mathematical secrets in string theory.

• **Because we do not know what it is**

As mentioned, we do not understand the core new concept string theory is based on, but we know it is remarkably rich. One of the reason that string theory is an exciting topic to work on for students is precisely that so much things are not understood yet. It sometimes framed criticisms that string theorists do not understand the theory. That's true. But if we understood it, there would be ground-breaking insights both in physics and mathematics. The fact that so little is understood and such relatively small pieces are actually such big discoveries in its own right makes us exciting. Of course, there are still a lot more to do!

1.1.2 Very very brief history

String theory was invented originally by accident in trying to solve a different problem, subsequently developed by a long and fortunate process of tinkering. Therefore, the history of string theory itself is interesting. For detailed history of string theory, we refer to [Gre99, Sch00, Sch11, Oog15, Pol17].[3]

String theory was first developed by aiming at describing hadron physics. People came up an idea that a meson is a little string with charges at its end and the meson resonances are vibrational states of the string. Although this physical picture, which first emerged from the Veneziano amplitude [Ven68], is now believed to be qualitatively correct at a description of strongly interacting particles, QCD became far more successful to describe details of strong interactions soon later.

However, a small group of physicists has continued to study string theory in 70s and it was found that string theory includes massless spin-two particles, suggesting that string theory can be a theory of gravity. Furthermore, a tachyon which is present

[2] Indeed I belong to this group.

[3] Yuji Tachikawa has recommended us a good book [CCCDV12] that describes the early history of string theory more in detail. Thanks to Yuji!

in bosonic string theory turned out to be absent by incorporating supersymmetry, and Type IIA and IIB as well as Type I theories have been constructed.

In 1984, Green and Schwarz showed [GS84] that anomaly pointed out by Alvarez-Gaumé and Witten is cancelled if the gauge group of Type I string theory is SO(32), which led to the first string revolution. A number of people has started working in the field of string theory. During the first revolution, Heterotic string theory has been constructed.

In 1995, Witten has proposed [Wit95] that five types of string theories are related by dualities and they are moreover five different limits of one bigger theory called M-theory. This has led to the second string revolution during which D-branes are proposed and non-perturbative aspects of string theories have been extensively studied. These results bore fruits as discoveries of a number of dualities, understanding of quantum nature of black holes and the AdS/CFT correspondence.

The second revolution has driven more researchers to study D-brane worldvolume theories and their relations to quantum gravity. However, dynamics on M2-branes and M5-branes that are fundamental degrees of freedoms in M-theory have remained elusive. Triggered by the work of Bagger-Lambert [BL07], the worldvolume theory for M2-branes was found to be a highly supersymmetric exquisite mixture of Chern-Simons gauge theory with some scalars and spinors [ABJM08]. On the other hand, the worldvolume theory on M5-branes is difficult to analyze because it is known that it does not admit Lagrangian description. However, the paper of Gaiotto [Gai12] has ignited the study of properties of M5-branes by wrapping them on a certain manifold, which still continues to be active so far.

The following picture shows the timeline of string theory and when the standard textbooks were written.[4] Obviously, the textbooks do not cover developments after their publications. We will mainly use Polchinski's textbook [Pol98], but I do not recommend to stick only on one book when you study a subject. In a book, some parts are explained very well, but some parts are sometimes hard to grasp for some people. Therefore, it is always a good idea to look for books, notes and papers that suit you best.

1.1.3 Very short highlights

• The basic idea in string theory is different elementary particles are all vibrational modes of a single type of string. There are two types of strings, open and closed, and a trajectory of a string is called the **string worldsheet** which is parametrized by (σ, τ). If the typical size ℓ_s of a string is smaller than the resolution that an accelerator can provide, we cannot see this in the experiment involving elementary particles.

$$\text{(Planck scale) } 10^{-33} \text{ cm} \le \ell_s \le 10^{-17} \text{ cm (TeV scale)}$$

[4] Of course, there are many other books [Zwi06, Din07, IU12, BLT13] and notes [Ura05, Ton09, Wra11, Hos15, Wei15].

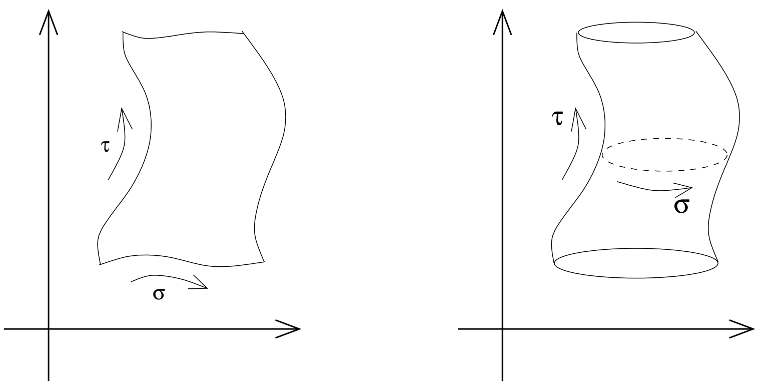
2010∼	M5-brane dynamics	
2007∼	M2-brane dynamics	Becker-Becker-Schwarz [BBS06]
		Polchinski [Pol98]
1997	AdS/CFT correspondence [Mal99]	
1995	Second string revolution [Wit95]	
		Green-Schwarz-Witten [GSW87]
1984	First string revolution [GS84]	
1974	Graviton in string theory	
∼1970	Birth of string theory	

Figure 1: History of string theory and textbooks

We often use the parameter

$$\alpha' = \ell_s^2$$

which is the only free parameter in string theory. As we will see, the couplings in string theory are expectation values of dynamical fields (so-called moduli) which take their value dynamically.



• Application of the rules of quantization provides us with the Fock space of string excitations. In bosonic string, the massless modes include among others

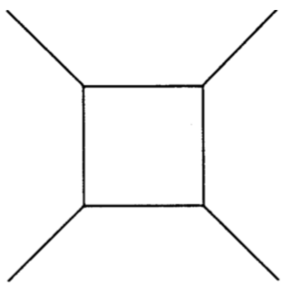
$$\begin{array}{rccc} \text{open string :} & A_\mu & \text{spin 1} & \text{W-boson} \\ \text{closed string :} & g_{\mu\nu} & \text{spin 2} & \text{graviton} \end{array}$$

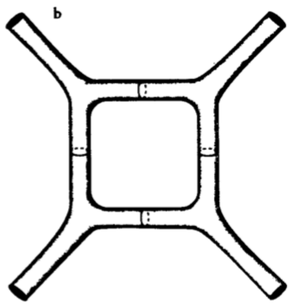
In addition, one finds a tower of massive string excitations of mass

$$\begin{aligned} \text{open bosonic :} \quad M^2 &= \frac{1}{\alpha'}(N-1) \\ \text{closed bosonic :} \quad M^2 &= \frac{4}{\alpha'}(N-1) \quad N = 0, 1, 2, \ldots \end{aligned}$$

Note that the lowest lying state has dimension of negative [mass]^2 ($N = 0$), which is called **tachyon**. The bosonic theory is consistent only when the number of spacetime dimensions is 26.

• In the point-particle picture, the divergence comes when all four vertices come close to each other. On the other hans, the string word-sheet has no vertices. Thus, when we sum over all surfaces, we do not encounter configuration analogous to collapsed vertices. String theory amplitudes have no ultraviolet (shot distance) divergence. As a result, string theory provides a finite quantum theory of gravity. Moreover, this is even better than renormalizable quantum field theories since here there is no divergence in the first place.





• Strings besides vibrating in usual spacetime can also have some internal degree of freedom. These internal degrees of freedom are fermionic. This can be quantized in a similar manner to that for bosonic strings. Then, a superstring can be considered as a string that has symmetry between bosonic states and fermionic states. In superstring theories, the good feature (e.g. emergence of gravity and UV finiteness) are preserved, and one can incorporate fermions. Moreover, as mentioned, the tachyonic mode is absent.

1. the theory is consistent in 10 spacetime dimensions.

2. there are five fully consistent string theories in $d = 10$.

   Type IIA, Type IIB, Type I (open+closed string), $E_8 \times E_8$ Heterotic, SO(32) Heterotic

3. Type II theories can have D-branes that accommodate open strings. Type IIA (resp. IIB) has Dp-branes where p is even (resp. odd), and Type I does D9, D5, D1-branes.
4. Furthermore all the five apparently different string theories have been found to be different limits of the same underlying theory called M-theory
5. In the low energy scale $\ll 1/\ell_s$, a theory is described by supergravity which combines supersymmetry and general relativity.

• How do we reduce the number of dimensions from 10 to 4? The answer is Kaluza-Klein mechanism/compactification. If we take the 10-dimensional space of the form $\mathbb{R}^{1,3} \times M$ where M is a 6-dimensional compact space and its size is much smaller than resolution of most powerful accelerator $\sim 10^{-17}$cm, then a (9+1)-dimensional theory looks like (3+1)-dimensional. In fact, in order for a theory to be consistent, M has to be a Calabi-Yau manifold (which admits Ricci-flat metric). More importantly, this extra dimensions give "room" to derive the complexity of the real world from a simple setting.

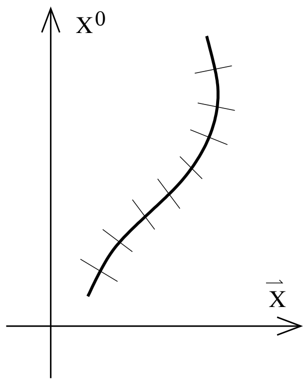
1.2 Bosonic string theory

Notation

In anticipation of string theory, we consider D-dimensional Minkowski space $\mathbb{R}^{1,D-1}$. Throughout these notes, we work with signature

$$\eta_{\mu\nu} = \mathrm{diag}(-1, +1, +1, \ldots, +1)$$

Nambu-Goto action



To begin with, let us recall the case of relativistic point particle

$$\gamma : \tau \to x^\mu(\tau) \in \mathbb{R}^{1,D-1} \ .$$

The action of relativistic point particle with mass m is

$$S = m\int_a^b ds = m\int_a^b \Big(-\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\Big)^{\frac{1}{2}} d\tau \ .$$

In a similar fashion, we consider a string

$$\Sigma : (\tau,\sigma) \to X^\mu(\tau,\sigma) \in \mathbb{R}^{1,D-1} \ .$$

The analogous action, called the **Nambu-Goto action**, for the string is given by the area of a string worldsheet Σ swept by a string

$$S_{\rm NG} = T\int_\Sigma \text{Area} = T\int_\Sigma d^2\sigma\sqrt{-\det(\gamma_{ab})} \ , \qquad \gamma_{ab} = \eta_{\mu\nu}\partial_a X^\mu \partial_b X^\nu$$

where $(\sigma^0,\sigma^1) = (\tau,\sigma)$, and $T := 1/(2\pi\alpha')$ is the **string tension**. The tension is usually defined as mass per unit length. However, we don't know how to quantize the Nambu-Goto action. Instead, for quantization of string worldsheet, we reformulate the action as follows.

String sigma-model action

Alternatively, we remove the square root and consider the following action

$$S_\sigma = \frac{T}{2}\int d^2\sigma\sqrt{-\det h}\; h^{ab}\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}$$

which is called the **string sigma-model action**.[5] Classically the string sigma-model action is equivalent to the Nambu-Goto action.

Symmetry of S_σ

1. **D-dim spacetime Poincare invariance**

   The action is certainly invariant under the Poincare transformation

   $$X^\mu(\sigma) \to \Lambda^\mu{}_\nu X^\nu(\sigma) + V^\mu \qquad \Lambda \in \mathrm{SO}(1,D-1)$$

   where Λ is a Lorentz transformation and V is the translation.

2. **Worldsheet diffeomorphism invariance**

   Under the re-parametrization

   $$\widetilde{X}^\mu(\widetilde{\sigma}) = X^\mu(\sigma) \quad \text{with} \quad \widetilde{\sigma} = \widetilde{\sigma}(\sigma) \ ,$$
   $$\widetilde{h}^{ab} = h^{cd}\frac{\partial\widetilde{\sigma}^a}{\partial\sigma^c}\frac{\partial\widetilde{\sigma}^b}{\partial\sigma^d} \ ,$$

   the action is invariant $\widetilde{S}(X^\mu, h_{ab}) = \widetilde{S}(\widetilde{X}^\mu, \widetilde{h}_{ab})$. We have two parameter family of gauge symmetry.

[5]This action is called the Polyakov action in [Pol98]. However, according to [BBS06], it was discovered by Brink, Di Vecchia and Howe and by Deser and Zumino several years before Polyakov skillfully used it for path-integral quantization of the string. So let's follow the notation of the progenitor, J.H. Schwarz.

3. **Weyl invariance**

   The action is also invariant under the metric $\widetilde{h}_{ab} = e^{\phi(\sigma)} h_{ab}$ by an arbitrary function $\phi(\sigma)$. This symmetry is called the **Weyl symmetry**.

Stress-energy tensor

The stress-energy tensor is defined by

$$\begin{aligned} T_{ab} &= -\frac{4\pi}{\sqrt{-h}} \frac{\partial S}{\partial h^{ab}} \\ &= -\frac{1}{\alpha'} \left[\partial_a X \partial_b X - \frac{1}{2} h_{ab} \, \partial_c X \partial^c X \right] \end{aligned} \tag{1.1}$$

which satisfies

$$\begin{aligned} \text{traceless :} \qquad & T^a{}_a = 0 \\ \text{conservation :} \qquad & \nabla_a T^{ab} = 0 \end{aligned} \tag{1.2}$$

Note that the traceless condition indeed follows from the Weyl invariance (exercise), and the conservation can be shown by using the equation of motion for X^μ in the following.

$$\begin{aligned} \frac{\partial S}{\partial h^{ab}} &= 0 \ \rightarrow \ T_{ab} = 0 \\ \frac{\partial S}{\partial X^\mu} &= 0 \ \rightarrow \ \Delta X_\mu = -\frac{1}{\sqrt{-h}} \partial_a (\sqrt{-h} h^{ab} \partial_b X^\mu) = 0 \end{aligned} \tag{1.3}$$

Gauge fixing

The string sigma-model action S_σ is equipped with the symmetries above and we need to fix it for quantization. Physics does not depend on a choice of gauge fixing, but if we choose a clever gauge fixing, our life becomes much easier. Actually, gauge fixing in [BBS06] makes our life clearer than that in [Pol98], so we use the gauge fixing in [BBS06] as follows.

The metric h_{ab} is a 2×2 symmetric matrix and the woldsheet diffeo invariance tells us that there is a coordinate σ^a such that the metric is a diagonal $h_{ab} = e^\omega \eta_{ab}$. Then, the Weyl transformation brings it to the 2d Minkowski metric η_{ab}. Therefore, one can use $h_{ab} = \eta_{ab}$. If we use the light-cone coordinates on the worldsheet

$$\sigma^\pm = \tau \pm \sigma \ ,$$

then the metric is written as

$$ds^2 = -d\sigma^+ d\sigma^- \ .$$

However, there are residual transformations that leaves the 2d Minkowski metric η_{ab} invariant, which is called **conformal symmetry**:

$$\sigma^+ \rightarrow f(\sigma^+) \ , \qquad \sigma^- \rightarrow g(\sigma^-)$$

with a Weyl transformation simultaneously. To fix the conformal symmetry, we introduce the spacetime light-cone coordinates,

$$X^{\pm} = \sqrt{\frac{1}{2}}(X^0 \pm X^{D-1}) \ .$$

Then we impose

$$X^{+} = x^{+} + \alpha' p^{+} \tau \ . \ , \tag{1.4}$$

which is called **light-cone gauge**.

Mode Expansions

After the gauge fixing, the the equations of motion simply read

$$\partial_+ \partial_- X^\mu = 0 \quad \text{where} \quad \partial_\pm = \frac{1}{2}(\partial_\tau \pm \partial_\sigma) \ .$$

The most general solution is a factorization of left-moving and right-moving waves

$$X^\mu(\sigma, \tau) = X^\mu_L(\sigma^+) + X^\mu_R(\sigma^-)$$

for arbitrary functions X^μ_L and X^μ_R. For a closed string, we impose a periodic condition as

$$X^\mu(\sigma, \tau) = X^\mu(\sigma + 2\pi, \tau) \ . \tag{1.5}$$

so that the most general can be expanded in Fourier modes

$$\begin{aligned} X^\mu_L(\sigma^+) &= \frac{1}{2}x^\mu + \frac{1}{2}\alpha' p^\mu \, \sigma^+ + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \, \widetilde{\alpha}^\mu_n \, e^{-in\sigma^+} \ , \\ X^\mu_R(\sigma^-) &= \frac{1}{2}x^\mu + \frac{1}{2}\alpha' p^\mu \, \sigma^- + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \, \alpha^\mu_n \, e^{-in\sigma^-} \ , \end{aligned} \tag{1.6}$$

where the reality of X^μ requires that the coefficients of the Fourier modes obey

$$\alpha^\mu_n = (\alpha^\mu_{-n})^* \quad , \quad \widetilde{\alpha}^\mu_n = (\widetilde{\alpha}^\mu_{-n})^* \ .$$

This mode expansion will be very important when we come to the quantum theory. We have to keep in mind that we have to impose the equation of motion

$$T_{--} = (\partial_- X)^2 = 0 \qquad T_{++} = (\partial_+ X)^2 = 0 \ . \tag{1.7}$$

We will see the implication of these constraints in the quantum theory.

Quantizations

The momentum conjugate to X^μ is defined in this gauge

$$\Pi_\mu = \frac{\delta S}{\delta(\partial_\tau X^\mu)} = T\,\partial_\tau X^\mu\ .$$

The canonical quantization promotes X^μ and Π_μ to operators that is subject to

$$\begin{aligned}
&[X^\mu(\sigma,\tau),\Pi_\nu(\sigma',\tau)] = i\delta(\sigma-\sigma')\,\delta^\mu{}_\nu && ,\\
&[X^\mu(\sigma,\tau),X^\nu(\sigma',\tau)] = [\Pi_\mu(\sigma,\tau),\Pi_\nu(\sigma',\tau)] = 0 && .
\end{aligned}$$

We translate these into commutation relations for the Fourier modes x^μ, p^μ, α^μ_n and $\widetilde{\alpha}^\mu_n$. Using the mode expansion, we find (exercise)

$$[x^\mu,p_\nu] = i\delta^\mu{}_\nu \quad \text{and} \quad [\alpha^\mu_n,\alpha^\nu_m] = [\widetilde{\alpha}^\mu_n,\widetilde{\alpha}^\nu_m] = n\,\eta^{\mu\nu}\delta_{n+m,0}\ , \tag{1.8}$$

Therefore, like harmonic oscillators, the creation operators are $\alpha^\mu_{-n},\widetilde{\alpha}^\mu_{-n}$ and the annihilation operators are $\alpha^\mu_n,\widetilde{\alpha}^\mu_n$ for $n\in\mathbb{N}$ so that the Hilbert space is spanned by

$$\alpha^{\mu_1}_{-n_1}\cdots\alpha^{\mu_k}_{-n_k}\widetilde{\alpha}^{\mu_1}_{-n_1}\cdots\widetilde{\alpha}^{\mu_k}_{-n_k}|k\rangle \qquad \text{where} \qquad p^\mu|k\rangle = k^\mu|k\rangle\ .$$

Let us consider the implication of the constraints (1.7). Using the mode expansions (1.6), the stress-energy tensor can be expressed as

$$T_{--} = \alpha'\sum_n L_n e^{-in\sigma^-} \qquad T_{++} = \alpha'\sum_n \widetilde{L}_n e^{-in\sigma^+}$$

where

$$L_m = \frac{1}{2}\sum_{n\neq 0}\eta_{\mu\nu} : \alpha^\mu_{m-n}\alpha^\nu_n : \qquad \widetilde{L}_m = \frac{1}{2}\sum_{n\neq 0}\eta_{\mu\nu} : \widetilde{\alpha}^\mu_{m-n}\widetilde{\alpha}^\nu_n : \tag{1.9}$$

with $\alpha^\mu_0 = \widetilde{\alpha}^\mu_0 = \sqrt{\frac{\alpha'}{2}}p^\mu$. We will learn L_m are generators of **Virasoro algebra**. According to the normal-ordering : : prescription, the lowering operators always appear to the right of the raising operators. However, it is easy to see that the normal ordering matters only in L_0 and $\widetilde{L}_0$. Thus, in the quantum theory, the constraint (1.7) on the stress-energy tensor can be interpreted by

$$L_n|\text{phys}\rangle = 0 \qquad \widetilde{L}_n|\text{phys}\rangle = 0 \qquad \text{for} \quad n>0$$

so that

$$\langle\text{phys}'|L_n|\text{phys}\rangle = 0 \qquad \langle\text{phys}'|\widetilde{L}_n|\text{phys}\rangle = 0 \qquad \text{for} \quad n>0\ .$$

Light-cone quantization

To see the effect of normal ordering in L_0, let us consider the meaning of the light-cone gauge (1.4) in the quantum theory. It is easy to see that (1.4) implies

$$\alpha_n^+ = 0 \quad \text{for} \quad n \neq 0 \ .$$

Then, the equation of motion

$$2\partial_+ X^- \partial_+ X^+ = \sum_{i=1}^{D-2} \partial_+ X^i \partial_+ X^i$$

tells us

$$\alpha_n^- = \sqrt{\frac{1}{2\alpha'}} \frac{1}{p^+} \sum_{m=-\infty}^{+\infty} \sum_{i=1}^{D-2} \alpha_{n-m}^i \alpha_m^i \ .$$

In particular, for $n = 0$, we have

$$M^2 = 2p^+p^- - \sum_{i=1}^{D-2} p^i p^i = \frac{4}{\alpha'} \left(\sum_{n>0} \sum_{i=1}^{D-2} \alpha_{-n}^i \alpha_n^i + \frac{D-2}{2} \sum_{n>0} n \right)$$

The final term clearly diverges. Fortunately, we have nice regularization of this divergence

$$\begin{aligned} \sum_{n=1}^{\infty} n \longrightarrow \sum_{n=1}^{\infty} n e^{-\epsilon n} &= -\frac{\partial}{\partial \epsilon} \sum_{n=1}^{\infty} e^{-\epsilon n} \\ &= -\frac{\partial}{\partial \epsilon} (1 - e^{-\epsilon})^{-1} \\ &= \frac{1}{\epsilon^2} - \frac{1}{12} + \mathcal{O}(\epsilon) \end{aligned}$$

Obviously the $1/\epsilon$ piece diverges as $\epsilon \to 0$. This term should be renormalized away so that we have

$$\sum_{n=1}^{\infty} n = -\frac{1}{12} \ .$$

Hence, we obtain

$$M^2 = \frac{4}{\alpha'} \left(N - \frac{D-2}{24} \right) = \frac{4}{\alpha'} \left(\widetilde{N} - \frac{D-2}{24} \right) \ , \tag{1.10}$$

where the number operators are defined by

$$N = \sum_{i=1}^{D-2} \sum_{n>0} \alpha_{-n}^i \alpha_n^i \quad , \quad \tilde{N} = \sum_{i=1}^{D-2} \sum_{n>0} \tilde{\alpha}_{-n}^i \tilde{\alpha}_n^i \ .$$

It is easy to see from (9.3)

$$N = \widetilde{N} \tag{1.11}$$

which is called the **level-matching condition**. Moreover, if $D > 2$, we have the state with negative $[\text{mass}]^2$ for $N = 0 = \widetilde{N}$ which is called the **tachyon**.

The First Excited States

Let us look at the first excited states. The level-matching condition (1.11) requires us to act both right- α^j_{-1} and left-moving $\overline{\alpha}^i_{-1}$ creation operator on $|0;k\rangle$ simultaneously. Thus, there are $(D-2)^2$ states at $N = 1 = \overline{N}$

$$\alpha^i_{-1}\overline{\alpha}^j_{-1}\,|0;k\rangle\ , \tag{1.12}$$

whose mass

$$M^2 = \frac{4}{\alpha'}\left(1 - \frac{D-2}{24}\right)\ .$$

It is easy to see that the state is under a representation of the little group $\mathrm{SO}(D-2) \subset \mathrm{SO}(1, D-1)$ and therefore it should be massless. Interestingly, this is only the case if the dimension of spacetime is

$$D = 26\ . \tag{1.13}$$

This is the critical dimension of the bosonic string. In the following sections, we will see from many different viewpoints that bosonic string theory is consistent only in this critical dimension.

Then, the states (1.12) transform in the $\mathbf{24} \otimes \mathbf{24}$ representation of SO(24), which decomposes into three irreducible representations:

$$\begin{aligned}
&\text{Graviton } G^{\mu\nu} \quad \alpha^\mu_{-1}\overline{\alpha}^\nu_{-1}|0;k\rangle \quad (\text{symmetric in } \mu \text{ and } \nu, \text{ and traceless})\ ,\\
&B\text{-field } B^{\mu\nu} \quad \alpha^\mu_{-1}\overline{\alpha}^\nu_{-1}|0;k\rangle \quad (\text{anti-symmetric in } \mu \text{ and } \nu)\ ,\\
&\text{Dilaton } \Phi \qquad \alpha^\mu_{-1}\overline{\alpha}_{-1,\mu}|0;k\rangle\ .
\end{aligned} \tag{1.14}$$

Each irreducible representation gives rise to a massless field in spacetime. As above, the traceless symmetric, anti-symmetric, and the trace part correspond to the graviton $G_{\mu\nu}$, (Kalb-Ramond) B-field $B_{\mu\nu}$, and the dilaton Φ, respectively. Therefore, graviton arises naturally from the quantization of closed strings! These massless fields play a pivotal role throughout the lecture notes.

2 Lecture 2

Note that in this lecture we use α' rather than string tension $T = \frac{1}{2\pi\alpha'}$. World-sheet(WS) index here is denoted by a, b rather than α, β. h_{ab} is WS metric; do not confuse with the induced metric $h_{\alpha\beta} = \eta_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu$ used in the last lecture. Convention in this note basically follows Polchinski's one, though we adopt slightly different normalization for Noether current in §2.2.4.

2.1 Towards string interaction

What we have learnt is

- Mass spectrum of a string by light-cone quantization in flat space-time (ST).

What we have NOT learnt is

- string interactions,
- extension to curved background (curved ST).

Especially because of the first reason(string interaction) we want to express our analysis in terms of *path integral*, which is more natural to illustrate string interaction. (If you are not familiar with path integral, consult the appendix of Polchinski Vol.1. for derivation.) See Fig. 2, which is a naive but appropriate extension from quantum field theory (QFT) interaction to string interaction. Each end of the cylinders in

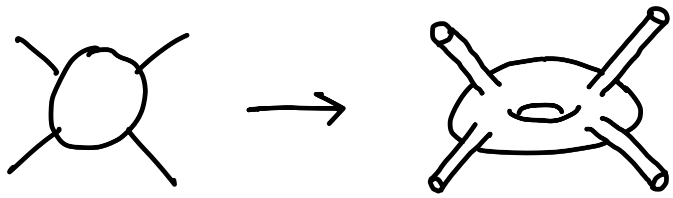


Figure 2: A Feynman diagram of QFT (left) and a string interaction diagram (right). Each end of the cylinders connecting to the torus(Riemann surface) in the string interaction diagram corresponds to an initial/final state.

the Figure corresponds to an initial/final state of string. It is quite cumbersome to use *state* expression. Instead, we will use vertex operator $\widehat{V}$ to express those states, which is explained in the following subsection. Given that we adopt the operator expression the string interaction is written as in Fig. 3 as well as

$$A_n = \sum_g \int (\mathcal{D}h_{ab})_{g,n} \int \mathcal{D}X^\mu e^{-S_\sigma[X^\mu, h_{ab}]} \widehat{V}_1 \cdots \widehat{V}_n \ ,$$

where n is the number of in-coming/out-going strings, and g is the number of holes(genus) of the Riemann surface (genus is 1 in the figures).

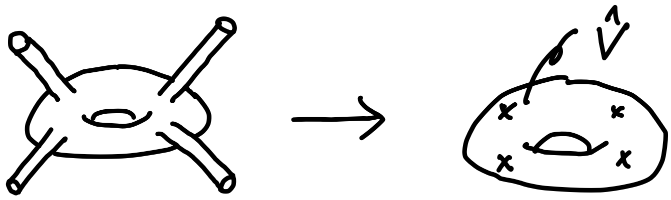


Figure 3: String interaction in state expression (left) and operator expression (right).

2.1.1 Path integral method

In path integral method expectation values(partition function, 2pt func etc.) are given in the following form

$$\langle \mathcal{O}[X] \rangle = \int \mathcal{D}X e^{iS[X]} \mathcal{O}[X] \ , \qquad S = \frac{1}{4\pi\alpha'} \int d\sigma d\tau \left\{ (\partial_\tau X)^2 - (\partial_\sigma X)^2 \right\} \ ,$$

where $\mathcal{O}$ is an arbitrary operator (eg. for partition function $\mathcal{O} = 1$, for 2pt func $\mathcal{O} = X^\mu X^\nu$). Note that we often omit the (super-)sub-script μ if it is not crucial. We usually **Wick rotate** ($\tau \to -it$) the theory so that it converges.

$$\langle \mathcal{O}[X] \rangle = \int \mathcal{D}X e^{-S_E[X]} \mathcal{O}[X] \ , \qquad S_E = \frac{1}{4\pi\alpha'} \int d\sigma dt \left\{ (\partial_t X)^2 + (\partial_\sigma X)^2 \right\} \ .$$

The subscript E will be omitted hereafter, and we will always work in the WS Euclidean signature.

(Even if you do not know path integral well) What we need to understand (at least for today) is that we can manipulate it as like normal integral:

- total derivative vanishes (assumption)

$$0 = \int \mathcal{D}X \frac{\delta}{\delta X} \left(e^{-S_E[X]} \mathcal{O}[X] \right) \ .$$

Note that with subscripts the functional derivative gives

$$\frac{\delta X^\mu(w)}{\delta X^\nu(z)} = \delta^\mu_\nu \delta^2(z - w) \ .$$

2.1.2 State-operator correspondence

As we have seen the string sigma model is Weyl invariant:

$$S_\sigma \left[X^\mu, h_{ab} \right] = S_\sigma \left[X^\mu, e^{\phi(\sigma)} h_{ab} \right] \ .$$

This (and together with WS diff) can be regarded as a property of the matter theory as we saw in the previous lecture, and we say that matter theory of X is conformally invariant or the matter theory is a conformal field theory (CFT). We can use this freedom to set WS to be $\mathbb{R}^2$ (recall that the topology of (closed) string is cylinder).

$$\begin{aligned} \mathbb{R}^2 : \quad ds^2 &= dx^2 + dy^2 = dr^2 + r^2 d\theta^2 \\ &= r^2 \left[d(\log r)^2 + d\theta^2 \right] \ , \text{Cylinder} : \quad ds^2 &= dt^2 + d\sigma^2 \ , \end{aligned}$$

where the overall factor r^2 in $\mathbb{R}^2$ is identified with the Weyl factor $e^{\phi(\sigma)}$, and can be removed. So we can identify:

$$t = \log r \ , \qquad \sigma = -\theta \ .$$

The reason for the minus sign will be clear later. This identification tells us that there is a correspondence between

- CFT on a semi-infinite cylinder $(\sigma, t;\ t \leq 0)$, and
- CFT on a disk $(x^2 + y^2 \leq 1)$ with the origin removed ,

provided that we choose the boundary conditions to be the same. Especially, the initial *state* of the string $(t = -\infty)$ corresponds to a *local operator* inserted at the origin, which is called vertex operator (see Fig. 4). This is the state-operator

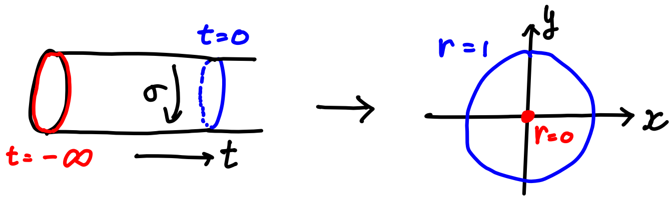


Figure 4: State-operator correspondence. Initial state(red circle) is mapped to a local operator(red dot) at the origin.

correspondence and we will express string states by the vertex operators (see again Fig. 3).

Now you may wonder if there is any way to express commutation relations in terms of operators so that we can do parallel procedures of the canonical quantization in terms of operators. This is what we do in the next section.

2.2 Conformal field theory in 2d

We will learn operator analysis of 2d conformal field theory (CFT), including OPE, Ward-Takahashi identity etc., and we will see Virasoro algebra, which is associated to the conformal symmetry.

2.2.1 Massless scalar theory in 2d

We study massless scalar fields as the easiest example. We adopt Euclidean metric $\eta_{ab} = \mathrm{diag}(+,+)$, though the procedure is parallel for Lorentz metric. The action of massless scalar fields X^μ $(\mu = 1, 2, \ldots, D)$ is given by

$$S = \frac{1}{4\pi\alpha'} \int d^2x \ \left(\partial_1 X \cdot \partial_1 X + \partial_2 X \cdot \partial_2 X\right) \ ,$$

where $d^2x = dx_1 dx_2$, $x_1 = x$, $x_2 = y$, and $\partial_i = \frac{\partial}{\partial x_i}$. (The subscript E will be omitted hereafter.)

It is convenient to introduce complex coordinates and accordingly vectors, metric etc.

- Coordinates

$$z = x + iy \ , \qquad \bar{z} = x - iy \ .$$

- Derivatives

$$\partial \equiv \partial_z = \frac{\partial}{\partial z} = \frac{1}{2}(\partial_1 - i\partial_2) \ , \quad \overline{\partial} \equiv \partial_{\overline{z}} = \frac{\partial}{\partial \overline{z}} = \frac{1}{2}(\partial_1 + i\partial_2) \ ,$$

so that $\partial z = 1$, $\partial\overline{z} = 0$, $\overline{\partial}z = 0$, $\overline{\partial}\overline{z} = 1$.

- Vectors

$$v^z = v^1 + iv^2 \ , \qquad v^{\overline{z}} = v^1 - iv^2 \ .$$

- Metric

$$h_{z\overline{z}} = h_{\overline{z}z} = \frac{1}{2} \ , \quad h_{zz} = h_{\overline{z}\overline{z}} = 0 \ , \quad h^{z\overline{z}} = h^{\overline{z}z} = 2 \ , \quad h^{zz} = h^{\overline{z}\overline{z}} = 0 \ .$$

So lower-index vectors are $v_z = h_{z\overline{z}}v^{\overline{z}} = \frac{1}{2}(v^1 - iv^2)$ etc.

- Integration measure

$$d^2z = dzd\overline{z} = 2dxdy = 2d^2x \ ,$$

where the factor 2 comes from the Jacobian. One can also write

$$\sqrt{h}d^2z = \sqrt{|\det h_{ab}|}d^2z = d^2x \ .$$

- Delta function

$$\delta^2(z) = \delta(z)\delta(\overline{z}) = \frac{1}{2}\delta(x)\delta(y) = \frac{1}{2}\delta^2(x) \ ,$$

so that

$$1 = \int d^2z \ \delta^2(z) = \int d^2x \ \delta^2(x) \ .$$

Note that although z and $\overline{z}$ are complex conjugate of each other, we will treat them as independent complex numbers.

Stokes' theorem (it is called Green's theorem in 2d) will be used frequently.

$$\int_D d^2z \left(\partial\overline{V} + \overline{\partial}V\right) = \frac{1}{i}\oint_{\partial D}\left(Vdz - \overline{V}d\overline{z}\right) \ ,$$

where D is an arbitrary region (typically small disk) and ∂D is its boundary. V and $\overline{V}$ are independent factor. Using the Stokes' theorem one can prove following equation.

$$\partial\overline{\partial}\log|z|^2 = 2\pi\delta^2(z) \ . \tag{2.1}$$

(Keep this in mind. We will use this later.)

With the complex coordinate convention, the action becomes

$$S = \frac{1}{2\pi\alpha'} \int d^2z \; \partial X \cdot \overline{\partial} X \qquad \left(= \frac{1}{4\pi\alpha'} \int \sqrt{h} d^2z \; h^{ab} \partial_a X \cdot \partial_b X \right) \; .$$

The equation of motion (E.O.M) is given by

$$\begin{aligned} \delta S &= \frac{1}{2\pi\alpha'} \int d^2z \left(\partial X \cdot \overline{\partial}(\delta X) + \partial(\delta X) \cdot \overline{\partial} X \right) \\ &= -\frac{1}{\pi\alpha'} \int d^2z \; \delta X \cdot \partial\overline{\partial} X = 0 \; , \\ \therefore \quad & \partial\overline{\partial} X^\mu = 0 \; . \end{aligned}$$

General solution to the E.O.M is

$$\begin{aligned} X^\mu(z, \overline{z}) &= X^\mu_R(z) + X^\mu_L(\overline{z}), \\ X^\mu_R(z) &= \frac{1}{2} x^\mu - i\frac{\alpha'}{2} p^\mu \log z + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \frac{\alpha^\mu_n}{z^n} \; , \\ X^\mu_L(\overline{z}) &= X^\mu_R(\overline{z})\big|_{\alpha_n \to \widetilde{\alpha}_n} \; . \end{aligned}$$

Notice that

$$\begin{aligned} z &= re^{i\theta} = e^{\log r + i\theta} = e^{i\tau - i\theta} \\ &= e^{i\sigma^-} \; , \\ \overline{z} &= e^{i\sigma^+} \; , \end{aligned}$$

and compare to

$$X^\mu_R(\sigma^-) = \frac{1}{2} x^\mu + \frac{1}{2} \alpha' p^\mu \, \sigma^- + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \, \alpha^\mu_n \, e^{-in\sigma^-} \; .$$

2.2.2 Operator analysis (Path integral quantization)

Let us consider quantum version of the E.O.M by using a path integral formulation. As like a normal integral we assume that "total derivative" vanishes in path integral. For example, in the massless free scalar case we have

$$0 = \int \mathcal{D}X \; \frac{\delta}{\delta X} e^{-S[X]} = \int \mathcal{D}X \; e^{-S[X]} \frac{1}{\pi\alpha'} \partial\overline{\partial} X = \frac{1}{\pi\alpha'} \langle \partial\overline{\partial} X \rangle \; ,$$

which is nothing but Ehrenfest's theorem. The E.O.M is satisfied as an operator equation if there is no other operator nearby.

Furthermore, the same procedure can be done with operator insertions. The easiest example is

$$0 = \int \mathcal{D}X \; \frac{\delta}{\delta X_\mu(z, \overline{z})} \left(e^{-S[X]} X^\nu(w, \overline{w}) \right) = \langle \frac{1}{\pi\alpha'} \partial_z \partial_{\overline{z}} X^\mu(z, \overline{z}) X^\nu(w, \overline{w}) + \eta^{\mu\nu} \delta^2(z - w) \rangle \; .$$

Therefore,

$$\partial\overline{\partial}X^\mu(z,\overline{z})X^\nu(w,\overline{w}) = -\pi\alpha'\eta^{\mu\nu}\delta^2(z-w) \ ,$$

where note that this is operator equation (not classical equation) and we simply omitted $\langle\rangle$ symbols. The right hand side is understood due to a quantum effect.

Recalling Eq. (2.1) we have

$$X(z,\overline{z})^\mu X^\nu(w,\overline{w}) = -\frac{\alpha'}{2}\eta^{\mu\nu}\log|z-w|^2 + :X(z,\overline{z})X(w,\overline{w}): \ ,$$

where we introduced normal ordering $:\mathcal{O}:$, meaning that the divergence at $z \to w$ has been subtracted. We also write it as follows.

$$:X^\mu(z,\overline{z})X^\nu(w,\overline{w}):= X^\mu(z,\overline{z})X^\nu(w,\overline{w}) - \eta^{\mu\nu}G(z,w) \ ,$$

where $G(z,w) = -\frac{\alpha'}{2}\log|z-w|^2$. Notice that

$$\partial_z\partial_{\overline{z}} :X(z,\overline{z})X(w,\overline{w}):= 0 \ . \tag{2.2}$$

As we will see later the divergent term is important and meaningful. This is why it is convenient to introduce the normal ordering so that we can separate the divergent terms from the other.

For an arbitrary functional of fields the OPE is expressed as follows.

$$:f[X]:= \exp\left[-\frac{1}{2}\int d^2zd^2w \ G(z,w)\frac{\delta}{\delta X^\mu(z,\overline{z})}\frac{\delta}{\delta X_\mu(w,\overline{w})}\right]f[X] \ .$$

For example,

$$\begin{aligned} :X_1X_2X_3X_4X_5:=&X_1X_2X_3X_4X_5 - (G_{12}X_3X_4X_5+\cdots)_{10\ \text{terms}} \\ &+ (G_{12}G_{34}X_5+\cdots)_{15\ \text{terms}} \ , \end{aligned}$$

where $X_i = X(z_i)$ and $G_{ij} = G(z_i,z_j)$.

Multiplication of normal ordered product is given as follows.

$$:f[X]::g[X]:= \exp\left[\int d^2zd^2w \ G(z,w) \ \frac{\delta}{\delta X^\mu(z,\overline{z})}\bigg|_f \frac{\delta}{\delta X_\mu(w,\overline{w})}\bigg|_g\right] :fg[X]: \ .$$

Example 1:

$$:\partial X(z,\overline{z})::X(w,\overline{w}):=:\partial X(z,\overline{z})X(w,\overline{w}): +\partial_z G(z,w) =:\partial X(z,\overline{z})X(w,\overline{w}): -\frac{\alpha'}{2}\frac{1}{z-w} \ .$$

Example 2:

$$\begin{aligned} :e^{ik\cdot X(z,\overline{z})}::e^{ik'\cdot X(w,\overline{w})}: &= \exp\left[(ik)\cdot(ik')\left(-\frac{\alpha'}{2}\log|z-w|^2\right)\right] :e^{ik\cdot X(z,\overline{z})+ik'\cdot X(w,\overline{w})}: \\ &= |z-w|^{\alpha' k\cdot k'} \ :e^{ik\cdot X(z,\overline{z})+ik'\cdot X(w,\overline{w})}: \ . \end{aligned}$$

2.2.3 Operator product expansion (OPE)

In general field theory, product of a pair of fields can be expanded by a single operator

$$\Phi^i(z)\Phi^j(w) = \sum_k C_k^{ij}(z-w)\Phi^k(w)\ .$$

Note that in 2d CFT language any local operator is regarded as an independent field (e.g. $\Phi = X, \partial X, :XX:$ etc.).

Typically, for a massless free scalar field theory we have

$$\begin{aligned} X^\mu(z,\overline{z})X^\nu(w,\overline{w}) &= -\frac{\alpha'}{2}\eta^{\mu\nu}\log|z-w|^2 + :X^\mu X^\nu(w,\overline{w}): \\ &+ \sum_{k=1}^{\infty}\frac{1}{k!}\left\{(z-w)^k :\partial^k X^\mu X^\nu(w,\overline{w}): + (\overline{z}-\overline{w})^k :\overline{\partial}^k X^\mu X^\nu(w,\overline{w}):\right\}\ . \end{aligned}$$

The second and the third terms of RHS can be understood as Taylor expansion of $:X(z)X(w):$ in terms of z. Mixing terms $(:\partial^m\overline{\partial}^n XX:)$ vanish due to "E.O.M", Eq. (2.2).

As we will see, only the divergent term is important. We will introduce $\sim$ that is define as "equal up to non-divergent term" so that we can forget about non-divergent terms. For example

$$X^\mu(z,\overline{z})X^\nu(w,\overline{w}) \sim -\frac{\alpha'}{2}\eta^{\mu\nu}\log|z-w|^2\ .$$

2.2.4 Noether's theorem and Ward-Takahashi identities

When an action is invariant under a certain transformation δ (namely, $\delta S = 0$) we say the theory has a (classical) symmetry. Furthermore, the measure of the path integral is also invariant under the transformation we say the theory has a symmetry at quantum level. On the other hand, if the measure is not invariant, then, we say the theory has anomaly and/or the symmetry is anomalous. It is non-trivial to see if the theory has anomaly or not.

If the theory has a symmetry there is a corresponding conserved current j^a and the space integral of its time component is the conserved charge $Q = \int_{\text{space}} j^0$, which generate the symmetry $\delta X = [Q, X]$. The current can be derived by Noether's procedure. Even if you are not familiar with the Noether's theorem we will see alternative method.

Let δ be a symmetry $\delta S = 0$ and assume it acts on a field as follows: $\delta X(z) = \epsilon(\cdots)$, where ϵ is a small parameter. If we promote the parameter to be WS coordinate dependent (i.e. $\widehat{\delta}X = \epsilon(z,\overline{z})(\cdots)$), then $\widehat{\delta}S$ is no longer zero but has to take the following form.

$$\widehat{\delta}S = \int \frac{d^2x}{2\pi}\epsilon(x,y)\partial_a j^a = \int \frac{d^2z}{2\pi}\epsilon(z,\overline{z})\left(\partial\overline{j} + \overline{\partial}j\right)\ ,$$

where we introduced $j = j_z$ and $\overline{j} = j_{\overline{z}}$. If the parameter is constant $\delta S =$ (total derivative) $= 0$. j^a **is Noether current**. Noether current is conserved, which means that $\partial_a j^a = 0$ under assumption of E.O.M. This implies that $j(\overline{j})$ is a holomorphic(anti-holomorphic) function of z. Indeed, this is true for all the example we will see.

Ward-Takahashi identity

Now let us study the transformation of a point operator $\mathcal{O}[X](w, \overline{w})$ under the symmetry δ. For this purpose, we set

$$\epsilon(z, \overline{z}) = \begin{cases} \epsilon \quad (\text{const.}) & \text{for } z \in D_w \ , \\ 0 & \text{for } z \notin D_w \ , \end{cases}$$

where D_w is a disk containing w. Then, the variation of the path integral

$$0 = \int \mathcal{D}X \ \widehat{\delta} \left[e^{-S} \mathcal{O}(w, \overline{w}) \right] = \int \mathcal{D}X e^{-S} \ \left[\delta\mathcal{O}(w, \overline{w}) - \widehat{\delta}S \cdot \mathcal{O}(w, \overline{w}) \right] .$$

Therefore, we have

$$\begin{aligned} \delta\mathcal{O}(w, \overline{w}) &= \int \frac{d^2 z}{2\pi} \epsilon(z, \overline{z}) \left(\partial\overline{j}(\overline{z}) + \overline{\partial} j(z) \right) \mathcal{O}(w, \overline{w}) \\ &= \frac{\epsilon}{2\pi i} \oint_{\partial D_w} \Big(dz \ j(z) - d\overline{z} \ \overline{j}(\overline{z}) \Big) \mathcal{O}(w, \overline{w}) \ , \end{aligned}$$

This is called the **Ward-Takahashi identity**.

Let us see an example of the free scalar field.

$$\begin{aligned} & S = \frac{1}{2\pi\alpha'} \int d^2 z \ \partial X \cdot \overline{\partial} X \ , \qquad \delta X = \epsilon g(z) \quad (\text{holomorphic}) \\ \rightarrow \quad & \widehat{\delta}S = \frac{-2}{2\pi\alpha'} \int d^2 z \ \epsilon(z, \overline{z}) \overline{\partial} \left(g \partial X \right) \\ & j = -\frac{2}{\alpha'} g \partial X \ , \qquad \overline{j} = 0. \end{aligned}$$

Check that $\delta X(w) = \epsilon \oint_{\partial D} \frac{dz}{2\pi i} \ j(z) X(w)$ is consistent.

3 Lecture 3

Conformal field theories (CFTs) play distinctive role in quantum field theories, string theory, statistical mechanics, and condensed matter physics. Moreover, 2d CFTs are particularly rich because they have infinite dimensional symmetry [BPZ84]. In the previous lecture, we have studied an important property of conformal field theories, the state-operator correspondence. In this lecture, we will learn significant feature of 2d conformal field theories more in detail. However, we glimpse only a tip of iceberg and the subject would actually deserve the entire one semester. If you are interested in this fertile subject, we refer to the standard references [Gin88, FMS12, BP09].

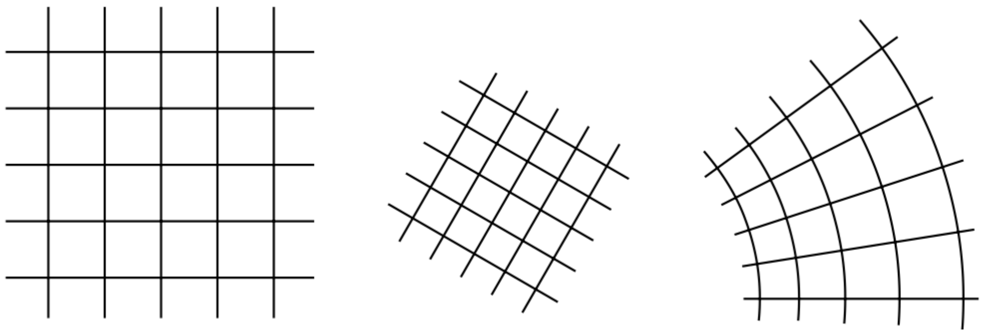
3.1 Conformal transformations

A CFT in any dimension is a quantum field theory which is invariant under conformal transformations at quantum level. A **conformal transformation** is a change of coordinates $x^a \to \widetilde{x}^a(x)$ such that the metric changes by

$$g_{ab}(x) \to \Omega^2(\sigma) g_{ab}(x) \ . \tag{3.1}$$

This means that the theory behaves the same at all length scales.

In the string sigma model action, the metric is dynamical and the conformal transformations (3.1) consist of a subgroup of diffeomorphisms so that it can be canceled by a Weyl symmetry of the metric. However, in the following, we fix a 2d Euclidian flat metric and the transformation (3.1) should be thought of as physical symmetry, taking the point x^α to point $\widetilde{x}^\alpha$.



2d flat space

In complex coordinate, the 2d flat metric can be written as $ds^2 = dzd\overline{z}$. Under a holomorphic map,

$$z \to f(z) \ ,$$

the metric is transformed as

$$ds^2 = dzd\overline{z} \quad \to \quad ds^2 = \frac{\partial f}{\partial z}\frac{\partial \overline{f}}{\partial \overline{z}} dzd\overline{z} \ .$$

Therefore, all the holomorphic maps are conformal transformations where $\left|\frac{\partial f}{\partial z}\right|^2$ is a conformal factor (corresponding to $\Omega^2(\sigma)$ in (3.1)). Moreover, it is easy to show that all 2d conformal transformations are indeed holomorphic functions (Exercise). This set is infinite-dimensional, corresponding to the the coefficients of the Laurent series of holomorphic functions in some neighborhood. This infinity is what makes conformal symmetry so powerful in two dimensions.

3.2 Conformal Ward-Takahashi identity

Recall that in a field theory, continuous symmetries correspond to conserved currents. Hence, let us consider the Noether current for conformal symmetry.

For an infinitesimal conformal transformation $x^\mu \to x^\mu + \epsilon^\mu(x)$, the metric is transformed as

$$\eta_{\mu\nu} \to \eta_{\mu\nu} + \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu \ .$$

Since this is conformal transformation, it is proportional to $\eta_{\mu\nu}$ so that we have

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = (\partial_\rho \epsilon^\rho)\eta_{\mu\nu} \ .$$

The current for the conformal transformation can be written as

$$j_\mu = T_{\mu\nu}\epsilon^\nu \ ,$$

where the straightforward calculation provides the stress-energy tensor

$$T_{\mu\nu} = -2\pi\Big[\frac{\partial L}{\partial(\partial^\mu\phi)}\partial_\nu\phi - \delta_{\mu\nu}L\Big] \ . \tag{3.2}$$

If we assume ϵ is constant, it is easy to see that the the conservation of the current implies the conservation of the stress-energy tensor:

$$\partial_\mu j^\mu = 0 \quad \to \quad \partial^\mu T_{\mu\nu} = 0 \ . \tag{3.3}$$

For general $\epsilon^\mu(x)$, the conservation of the current gives the traceless condition of $T_{\mu\nu}$:

$$0 = \partial^\mu j_\mu = \frac{1}{2}T_{\mu\nu}(\partial^\mu\epsilon^\nu + \partial^\nu\epsilon^\mu) = \frac{1}{2}T^\mu{}_\mu(\partial_\rho\epsilon^\rho) \quad \to \quad T^\mu{}_\mu = 0 \ . \tag{3.4}$$

In the complex coordinate $z = x^1 + ix^2$, the traceless condition can be written as

$$T_{z\overline{z}} = T_{\overline{z}z} = 0$$

and the conservation of the stress-energy tensor can be written as

$$\partial_{\overline{z}} T_{zz} = 0 \ , \qquad \partial_z T_{\overline{z}\overline{z}} = 0 \ .$$

Thus, the non-vanishing components of the stress-energy tensor factorize to a chiral and anti-chiral field,

$$T(z) := T_{zz} \quad \text{and} \quad \overline{T}(\overline{z}) := T_{\overline{z}\overline{z}} \ .$$

As a result, the Noether currents for conformal transformations $z \to z + \epsilon(z)$ and $\overline{z} \to \overline{z} + \overline{\epsilon}(\overline{z})$ are

$$j(z) = \epsilon(z)T(z) \ , \quad \overline{j}(\overline{z}) = \overline{\epsilon}(\overline{z})\overline{T}(\overline{z}) \ .$$

The application to the Ward-Takahashi identity leads to **conformal Ward-Takahashi identity**

$$\delta_{\epsilon.\overline{\epsilon}}\mathcal{O}(w,\overline{w}) = \frac{1}{2\pi i}\oint_{C_w} dz\ \epsilon(z)T(z)\mathcal{O}(w,\overline{w}) + \frac{1}{2\pi i}\oint_{C_{\overline{w}}} d\overline{z}\ \overline{\epsilon}(\overline{z})\overline{T}(\overline{z})\mathcal{O}(w,\overline{w}) \ , \tag{3.5}$$

where the contour integral is taken as a counter-clockwise circle both in z and in $\overline{z}$ (thereby explaining the sign difference of the second term).

3.3 Primary fields

Definition 3.1. *Fields depending only on z, i.e. $\phi(z)$, are called* ***chiral or holomorphic fields*** *and fields $\overline{\phi}(\overline{z})$ only depending on $\overline{z}$ are called* ***anti-chiral or anti-holomorphic fields****.*

Definition 3.2. *If a field ϕ that transforms under the scaling $z \to \lambda z$ as*

$$\phi(z,\overline{z}) \to \phi'(\lambda z, \overline{\lambda}\overline{z}) = \lambda^{-h}\overline{\lambda}^{-\overline{h}}\phi(z,\overline{z}) \ , \tag{3.6}$$

it has ***weight*** *$(h,\overline{h})$. Using these quantities (rather than the scaling dimension), we can define quasi-primary fields.*

Definition 3.3. *Under a conformal transformation $z \to f(z)$, a field transforming according to the rule*

$$\phi(z,\overline{z}) \to \phi'(f(z),\overline{f}(\overline{z})) = \left(\frac{\partial f}{\partial z}\right)^{-h}\left(\frac{\partial \overline{f}}{\partial \overline{z}}\right)^{-\overline{h}}\phi(z,\overline{z}) \tag{3.7}$$

it is called a ***primary field*** *of weight $(h,\overline{h})$.*

How do primary fields transform infinitesimally? Under the infinitesimal conformal transformation $z \to f(z) = z - \epsilon(z)$, we know that

$$\begin{aligned} \left(\frac{\partial f}{\partial z}\right)^{-h} &= 1 + h\partial_z\epsilon(z) + O(\epsilon^2) \ , \\ \phi(z-\epsilon(z),\overline{z}) &= \phi(z) - \epsilon(z)\partial_z\phi(z,\overline{z}) + O(\epsilon^2) \ . \end{aligned} \tag{3.8}$$

Hence, under an infinitesimal conformal transformation, the variation of a primary field is given by

$$\delta_\epsilon\phi(z,\overline{z}) = \left(h\partial_z\epsilon + \epsilon\partial_z + \overline{h}\partial_{\overline{z}}\overline{\epsilon} + \overline{\epsilon}\partial_{\overline{z}}\right)\phi(z,\overline{z}). \tag{3.9}$$

Consequently, using simple complex analysis

$$\begin{aligned} (\partial_w\epsilon(w))\phi(w,\overline{w}) &= \frac{1}{2\pi i}\oint_{C_w} dz\ \frac{\epsilon(z)\phi(w,\overline{w})}{(z-w)^2} \\ \epsilon(w)(\partial_w\phi(w,\overline{w})) &= \frac{1}{2\pi i}\oint_{C_w} dz\ \frac{\epsilon(z)\partial_w\phi(w,\overline{w})}{z-w} \ , \end{aligned} \tag{3.10}$$

one can read off the OPE of a primary operator ϕ of weight $(h,\tilde{h})$ with the stress-energy tensor T (anti-chiral part $\overline{T}$ can be obtained by complex conjugate)

$$T(z)\,\phi(w,\overline{w}) = h\frac{\phi(w,\overline{w})}{(z-w)^2} + \frac{\partial_w\phi(w,\overline{w})}{z-w} + \text{regular terms}\cdots$$

In general, the OPE of an operator $\mathcal{O}$ of weight $(h, \tilde{h})$ with the stress-energy tensor T and $\overline{T}$ takes the form

$$T(z)\,\mathcal{O}(w,\overline{w}) = \quad \cdots + h\frac{\mathcal{O}(w,\overline{w})}{(z-w)^2} + \frac{\partial\mathcal{O}(w,\overline{w})}{z-w} + \cdots$$

One of the main interest in a CFT is to calculate correlation functions of primary fields. Indeed, the conformal Ward-Takahashi identity can be applied to a correlation function of primary fields

$$\langle T(z)\phi_1(w_1,\overline{w}_1)\cdots\phi_n(w_n,\overline{w}_n)\rangle = \sum_{i=1}^{n}\left(\frac{h_i}{(z-w_i)^2}+\frac{\partial_{w_i}}{z-w_i}\right)\langle\phi_1(w_1,\overline{w}_1)\cdots\phi_n(w_n,\overline{w}_n)\rangle\ .$$

Moreover, conformal symmetry is so powerful that it determines the forms of two-point and three-point functions of primary fields (Exercise).

- **2-point function**

  For chiral primary operators ϕ_i with weight h_i $(i = 1, 2)$, their 2-point function is of form

$$\langle\phi_1(z_1)\phi_2(z_2)\rangle = \delta_{h_1h_2}\frac{d_{12}}{(z_1-z_2)^{2h_1}} \tag{3.11}$$

If d_{12} is non-degenerate, the fields can be normalized such that $d_{12} = \delta_{12}$.

- **3-point function**

  A 3-point function is also completely fixed up to the appearance of a **structure constant** C_{ijk},

$$\langle\phi_1(z_1)\phi_2(z_2)\phi_2(z_3)\rangle = \frac{C_{123}}{(z_1-z_2)^{h_1+h_2-h_3}(z_2-z_3)^{h_2+h_3-h_1}(z_3-z_1)^{h_3+h_1-h_2}}$$

The structure constant depends on a CFT and, in general, it is not easy to determine it.

- **Multi-point function**

  The computation of multi-point functions involves **conformal blocks** with the 3-point function. The details are explained in [FMS12, BP09].

3.4 Free scalar field

Now let us study conformal Ward-Takahashi identity in the simplest example, the free scalar field:

$$S = \frac{1}{2\pi\alpha'}\int d^2z\ (\partial X\overline{\partial}X)\ .$$

Let us recall that the stress-energy tensor in 2d free scalar theory is

$$T_{ab} = -\frac{1}{\alpha'}\left(\partial_a X\partial_b X - \frac{1}{2}\,g_{ab}(\partial X)^2\right)\ , \tag{3.12}$$

In addition, since the equation of motion for X is $\partial_z\overline{\partial}_{\overline{z}}X = 0$, the general classical solution decomposes as $X(z,\overline{z}) = X(z) + \overline{X}(\overline{z})$. From the equation of motion, we find the conserved chiral and anti- chiral worldsheet currents $j(z) := i\partial X(z)$ and $\overline{j}(\overline{z}) := i\overline{\partial X}(\overline{z})$. Moreover, the stress-energy tensor looks much simpler in complex coordinates. It is simple to check that $T_{z\overline{z}} = 0$ while

$$T(z) = -\frac{1}{\alpha'}\,\partial_z X(z)\partial_z X(z)\ , \quad \overline{T}(\overline{z}) = -\frac{1}{\alpha'}\,\overline{\partial}_{\overline{z}}\overline{X}(z)\overline{\partial}_{\overline{z}}\overline{X}(\overline{z})\ .$$

From the definition (3.7), one can see that $X(z,\overline{z})$ is a primary field of weight (0,0). However, since the weight is of (0,0), the two-point function does not exactly take the form (3.11). Indeed, the OPE XX tells us that the propagator takes the form

$$\langle X(z,\overline{z})X(w,\overline{w})\rangle = -\frac{\alpha'}{2}\log|z-w|^2\ .$$

Also, the currents $\partial X(z)$, $\overline{\partial X}(\overline{z})$ are primary fields of weight (1,0) and (0,1) respectively. An immediate check is their correlation function

$$\langle \partial X(z)\partial X(w)\rangle = -\frac{\alpha'}{2}\frac{1}{(z-w)^2}\ ,$$

which takes the form (3.11). To convince ourselves completely, we need to compute the OPE with the stress-energy tensor by Wick's theorem

$$\begin{aligned} T(z)\,\partial X(w) &= -\frac{1}{\alpha'}\,:\partial X(z)\partial X(z):\partial X(w) \\ &= -\frac{1}{\alpha'}\Big[\,:\partial X(z)\partial X(z)\partial X(w): + \partial X(z)\langle\partial X(z)\partial X(w)\rangle\Big] \\ &= \frac{\partial X(w)}{(z-w)^2} + \frac{\partial^2 X(w)}{z-w} + \text{regular terms}\cdots \end{aligned} \tag{3.13}$$

This is indeed the OPE for a primary operator of weight $h = 1$.

Finally, lets check to see the OPE of the stress-energy tensor TT. This is again just an exercise in Wick contractions.

$$\begin{aligned} T(z)\,T(w) &= \frac{1}{\alpha'^2}\ :\partial X(z)\,\partial X(z)::\partial X(w)\,\partial X(w): \\ &= \frac{2}{\alpha'^2}\left(-\frac{\alpha'}{2}\frac{1}{(z-w)^2}\right)^2 - \frac{4}{\alpha'^2}\frac{\alpha'}{2}\frac{:\partial X(z)\,\partial X(w):}{(z-w)^2} + \dots \end{aligned} \tag{3.14}$$

The factor of 2 in front of the first term comes from the two ways of performing two contractions; the factor of 4 in the second term comes from the number of ways of performing a single contraction. Continuing,

$$\begin{aligned} T(z)\,T(w) &= \frac{1/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} - \frac{2}{\alpha'}\frac{\partial^2 X(w)\,\partial X(w)}{z-w} + \dots \\ &= \frac{1/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \dots \end{aligned} \tag{3.15}$$

We learn that T is **not** a primary operator in the theory of a single free scalar field. It is an operator of weight $(h, \overline{h}) = (2, 0)$, but it fails the primary test on account of the $(z-w)^{-4}$ term. In fact, this property of the stress-energy tensor is general in all CFTs which we now explore in more detail.

3.5 OPE of stress-energy tensor

For the free scalar field, we have already seen that T has weight $(h, \tilde{h}) = (2, 0)$. This remains true in any CFT. The reason for this is simple: T_{ab} has dimension $\Delta = 2$ because we obtain the energy by integrating over space. It has spin $s = 2$ because it is a symmetric 2-tensor. But these two pieces of information are equivalent to the statement that T is an operator of weight $(2, 0)$. This means that the TT OPE takes the form,

$$T(z)\,T(w) = \ldots + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \ldots$$

and similar for $\overline{TT}$. What other terms could we have in this expansion? Since each term has dimension $\Delta = 4$, the unitarity indeed tells us that the singular part of the OPE takes

$$T(z)\,T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \ldots$$

Ⓡ

From the OPE of the stress-energy tensor, one can see its variation under an infinitesimal conformal transformation $z \to z - \epsilon(z)$

$$\begin{aligned} \delta_\epsilon T(w) &= \frac{1}{2\pi i}\oint_{C_w} dz\ \epsilon(z) T(z) T(w) \\ &= \epsilon(w)\partial T(w) + 2\epsilon'(w) T(w) + \frac{c}{12}\epsilon'''(w) \end{aligned} \tag{3.16}$$

One can verify by a straightforward computation that this is the infinitesimal version of the following transformation under finite transformation $z \to w(z)$:

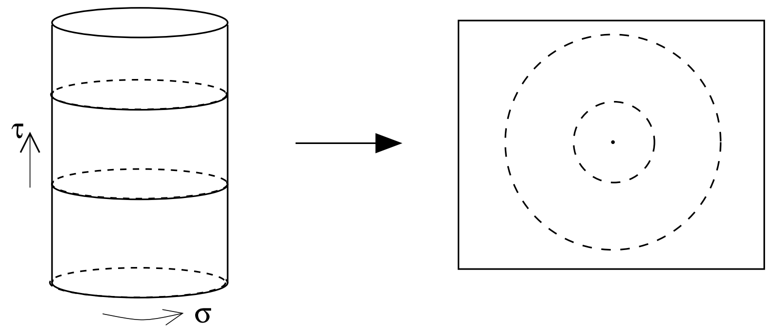
$$T'(w) = \left(\frac{\partial w}{\partial z}\right)^{-2}\left[T(z) - \frac{c}{12} S\left(w, z\right)\right], \tag{3.17}$$

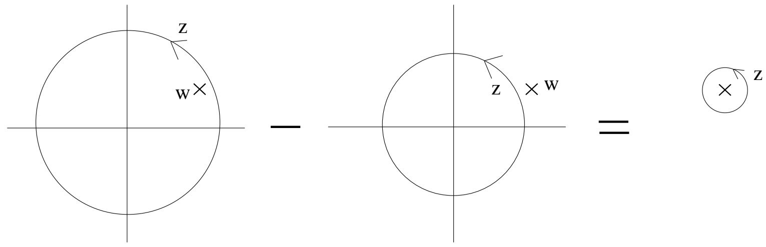
where the **Schwarzian derivative** S is defined as

$$S(w, z) := \frac{1}{(\partial_z w)^2}\left((\partial_z w)(\partial_z^3 w) - \frac{3}{2}(\partial_z^2 w)^2\right). \tag{3.18}$$

Using this conformal transformation law, one can see that the mapping from the plane to the cylinder $z = e^{-iw}$ $(w = \sigma + it)$ leads to

$$T_{cyl}(w) = -z^2 T(z) + \frac{c}{24}. \tag{3.19}$$





The Laurent mode expansion of the stress-energy tensor on the cylinder is therefore

$$T_{cyl}(w) = -\sum_{n\in\mathbb{Z}} \left(L_n - \frac{c}{24}\delta_{n,0}\right) e^{inw}. \tag{3.20}$$

We want to look at the Hamiltonian, which is defined by

$$H \equiv \int d\sigma\ T_{\tau\tau} = -\int d\sigma\,(T_{ww} + \overline{T}_{\overline{ww}})$$

The conformal transformation then tells us that the ground state energy on the cylinder is

$$E = -\frac{c+\overline{c}}{24}$$

This is indeed the (negative) Casimir energy on a cylinder. For a free scalar field, we have $c = \overline{c} = 1$ and the energy density $E = -1/12$. This is what we have seen in the quantization of bosonic string theory!

Virasoro algebra

The mode expansion of the stress-energy tensor is expressed as

$$T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$$

It is natural to find the commutation relation of the generators L_m. The commutator can be computed by

$$[L_m, L_n] = \left(\oint \frac{dz}{2\pi i}\oint \frac{dw}{2\pi i}\ -\ \oint \frac{dw}{2\pi i}\oint \frac{dz}{2\pi i}\right)\ z^{m+1}w^{n+1}\,T(z)\,T(w)$$

A clever manipulation of the contour makes life easier

$$
\begin{aligned}
[L_m, L_n] &= \oint \frac{dw}{2\pi i} \oint_w \frac{dz}{2\pi i}\ z^{m+1} w^{n+1}\, T(z)\, T(w) \\
&= \oint \frac{dw}{2\pi i} \operatorname{Res}\left[z^{m+1} w^{n+1} \left(\frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \dots \right)\right]
\end{aligned}
\tag{3.21}
$$

A simple computation (Exercise) leads to the **Virasoro algebra**

$$
[L_m, L_n] = (m-n) L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m+n,0}
$$

4 Lecture 4

Since we learnt conformal field theories and techniques in the theories, we would like to connect them to string theory, especially by considering vertex operators.

In conformal field theory the conformal symmetry is essential. However, classical conformal symmetry, which is related to WS diff and Weyl sym, breaks at quantum level. We regard this anomaly Weyl anomaly. We will look into important examples of the anomaly.

4.1 Vertex operators

Let us consider the vertex operators, which has been postponed to do from the second lecture. We learnt the state-operator correspondence, which tells us a state corresponds to a certain local operator (vertex operator). We also learnt that a closed string spectrum includes tachyon, graviton, etc. Why not consider the corresponding local operators of the states ?

Recall a few spectra in a closed string:

$$
\begin{aligned}
&\text{Tachyon } \phi && |0;k\rangle \ , \\
&\text{Graviton } G^{\mu\nu} && \alpha^\mu_{-1}\widetilde{\alpha}^\nu_{-1}|0;k\rangle \quad \text{(symmetric in } \mu \text{ and } \nu\text{, and traceless)} \ , \\
&\text{B-field } B^{\mu\nu} && \alpha^\mu_{-1}\widetilde{\alpha}^\nu_{-1}|0;k\rangle \quad \text{(anti-symmetric in } \mu \text{ and } \nu\text{)} \ , \\
&\text{Dilaton } \Phi && \alpha^\mu_{-1}\widetilde{\alpha}_{-1,\mu}|0;k\rangle \ .
\end{aligned}
$$

Note that in order to extract physical states we need to contract the polarization tensor $\zeta_{\mu\nu}$, which satisfies $k^\mu \zeta_{\mu\nu} = 0$ ($\zeta^G_{\mu\nu} = \zeta^G_{\nu\mu}$ and $\zeta^{G,\mu}_\mu = 0$ etc.). Let us first focus on the tachyon state, which is nothing but a vacuum state with a certain momentum k^μ. It was defined by $p|0;k\rangle = k|0;k\rangle$, namely,

$$
|0;k\rangle = e^{ik\cdot x}|0;0\rangle \quad \to \quad e^{ik\cdot X(0,0)} \ .
$$

The final replacement is by the state-operator correspondence.

Next we consider the first excited states. Remind ourselves that the closed string mode expansion is given by

$$X^\mu(z,\bar{z}) = X^\mu_R(z) + X^\mu_L(\bar{z}),$$
$$X^\mu_R(z) = \frac{1}{2}x^\mu - i\frac{\alpha'}{2}p^\mu \log z + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\frac{\alpha^\mu_n}{z^n}\ .$$

Therefore, each mode can be expressed as follows.

$$\alpha^\mu_{-m} = i\sqrt{\frac{2}{\alpha'}}\oint\frac{dz}{2\pi i}z^{-m}\partial X^\mu(z) \quad \to \quad i\sqrt{\frac{2}{\alpha'}}\frac{1}{(m-1)!}\partial^m X(0).$$

At the last replacement we forgot the "classical" mode expansion and regarded $\partial X(z)$ as a local operator. Now we have the correspondence of tachyon, graviton etc. as follows.

$$\begin{aligned} |0;k\rangle \quad &\to \quad e^{ikX} \\ \zeta_{\mu\nu}\alpha^\mu_{-1}\tilde{\alpha}^\nu_{-1}|0;k\rangle \quad &\to \quad \zeta_{\mu\nu}\partial X^\mu\bar{\partial}X^\nu e^{ikX} \end{aligned} \tag{4.1}$$

The string amplitude is

$$A_n = \sum_g\int\left(\mathcal{D}h_{ab}\right)_{g,n}\int\mathcal{D}X^\mu e^{-S_\sigma[X^\mu,h_{ab}]}\widehat{V}_1\cdots\widehat{V}_n\ .$$

Note that, naively say,

$$\left(\mathcal{D}h_{ab}\right)_{g,n} = \left(\mathcal{D}h_{ab}\right)_{g,0}d^2z_1\cdots d^2z_n$$

(naively say, this is because Weyl rescaling can move points on the Riemann surface to anywhere), so we can re-write the amplitude as

$$A_n = \sum_g\int\left(\mathcal{D}h_{ab}\right)_{g,n}\int\mathcal{D}X^\mu e^{-S_\sigma[X^\mu,h_{ab}]}\prod_{i=1}^n\int d^2z\sqrt{h}V_i\ .$$

Here $\int d^2z\sqrt{h}V_i$ is an operator of the CFT, and hence, it must be Weyl & WS diff $\sim$ conformal invariant.

4.1.1 Mass from vertex operator

Now, let us consider a constant scaling $z\to\lambda z$ and $\bar{z}\to\bar{\lambda}\bar{z}$. Under the scaling, a field transforms as $\phi(z,\bar{z})\to\lambda^{-h}\bar{\lambda}^{-\bar{h}}\phi(z,\bar{z})$, which should compensate the scaling of the measure $dzd\bar{z}\to\lambda\bar{\lambda}dzd\bar{z}$. Namely, $h=\bar{h}=1$. Conformal dimension of the vertex operators are

Name	$\mathcal{O}$	$(h,\bar{h})$
Tachyon	$e^{ik\cdot X}$	$(\frac{\alpha'k^2}{4},\frac{\alpha'k^2}{4})$
1st excited states	$\zeta_{\mu\nu}\partial X^\mu\bar{\partial}X^\nu e^{ikX}$	$(1+\frac{\alpha'k^2}{4},1+\frac{\alpha'k^2}{4})$

The consistency condition for the tachyon leads that

$$M^2 = -k_{\text{Tachyon}}^2 = -\frac{4}{\alpha'} \ .$$

Similarly, that for the first excited states leads

$$M^2 = -k_{\text{1st}}^2 = 0 \ .$$

Both results are consistent with the analysis from the first (and the second) lecture.

4.2 Weyl anomaly

As we have seen the Weyl symmetry ($T_a^a = 0$) is crucial. However, in general matter theories, the symmetry has anomaly. An important example is a string sigma model that is extended to curved space-time(ST):

$$S = \frac{1}{4\pi\alpha'} \int \sqrt{h} d^2z \ h^{ab} \partial_a X^\mu \partial_b X^\nu g_{\mu\nu}(X) \ .$$

This is called (string) non-linear sigma model(NLSM). The anomaly is characterized by β-functions.

Even if the matter theory has no anomaly on flat world-sheet(WS), $\mathbb{R}^2$, the same theory has anomaly on curved WS. As we will see it is characterized by the central charge $T_a^a = -\frac{c}{12} R^{(2)}$. Therefore, the string sigma model has anomaly. However, the path integral for the metric h_{ab} turns into that of "ghost CFT" due to gauge-fixing, and it leads $c = -26$. This means that the matter theory has to have $c = 26$!

In total we have two types of anomalies:

1. Anomaly from curved WS $T_a^a = -\frac{c}{12} R^{(2)}$,
2. Anomaly from curved ST $\beta[G_{\mu\nu}] \neq 0$.

We will learn these anomalies in this order (ghost CFT will be covered in the next lecture).

4.2.1 Weyl anomaly from curved WS

As is stated $T_a^a = A \neq 0$. The form of A is highly restricted by symmetries: A should be

- WS diff invariant ,
- zero on $\mathbb{R}^2$,
- WS mass dimension two .

These restrictions lead to

$$T^a_a = aR^{(2)} \ ,$$

where $R^{(2)}$ is a WS Ricci scalar.

Let us derive $T^a_a = -\frac{c}{12}R^{(2)}$. Take the WS to be conformally flat (which is always possible using WS diff)

$$ds^2 = e^{2\Omega(x,y)}(dxdx + dydy) = e^{2\Omega(z,\overline{z})}dzd\overline{z} \ .$$

For the metric we have

$$R^{(2)} = -2\partial^a\partial_a\Omega \ ,$$

therefore, the diagonal element of the EM tensor becomes

$$T_{z\overline{z}} = \frac{1}{4}e^{2\Omega}\ T^a_a = -2a\partial\overline{\partial}\Omega \ .$$

The other components can be derived from $\nabla_a T^a_b = 0$.

$$\begin{aligned} 0 &= \nabla^z T_{zz} + \nabla^{\overline{z}} T_{\overline{z}z} = \nabla_{\overline{z}} T_{zz} + \nabla_z T_{\overline{z}z} = \overline{\partial}\left(T_{zz} - 2a\left(\partial\partial\Omega - \partial\Omega\partial\Omega\right)\right) \\ \therefore \quad & T_{zz} = 2a\left(\partial\partial\Omega - \partial\Omega\partial\Omega\right) + T(z) \qquad (T(z):\ \text{holomorphic}) \end{aligned}$$

Let us recall that the conformal Ward-Takahashi identity *in flat WS*:

$$\begin{aligned} \delta_{\epsilon,\overline{\epsilon}}\mathcal{O}(w,\overline{w}) &= \frac{1}{2\pi i}\oint_{\partial M}\left\{dz\ \epsilon(z)T(z) - d\overline{z}\ \overline{\epsilon}(\overline{z})\overline{T}(\overline{z})\right\}\mathcal{O}(w,\overline{w}) \\ &= \int_M \frac{d^2z}{2\pi}\left\{\overline{\partial}\left(\epsilon(z)T(z)\right) + \partial\left(\overline{\epsilon}(\overline{z})\overline{T}(\overline{z})\right)\right\}\mathcal{O}(w,\overline{w}) \ . \end{aligned}$$

(Notice that the convention is different from the last lecture.) Note that EM tensor is conserved, $\partial^a T_{ab} = 0$, equivalently, $\overline{\partial}T(z) = 0 = \partial\overline{T}(\overline{z})$ with $T(z) = T_{zz}$, $\overline{T}(\overline{z}) = T_{\overline{z}\overline{z}}$, and $T_{z\overline{z}} = 0$. On the other hand, *in the curved WS*, the current conservation is expressed with a covariant derivative $\nabla^a T_{ab} = 0$. For example,

$$\begin{aligned} &\nabla^a T_{az} = \nabla^z T_{zz} + \nabla^{\overline{z}} T_{\overline{z}z} = \overline{\partial}T(z) = 0 \\ &\text{with} \quad T(z) = T_{zz} - 2a\left(\partial\partial\Omega - \partial\Omega\partial\Omega\right) \ . \end{aligned} \tag{4.2}$$

Similar for $\overline{T}(\overline{z})$. Hence, we have the same form of conformal Ward-Takahashi identity. Especially,

$$\delta_\epsilon T(z) = \epsilon(z)\partial T(z) + 2\partial\epsilon(z)T(z) + \frac{c}{12}\partial^3\epsilon(z) \ .$$

On the other hand, from the expression (4.2) we have

$$\delta_\epsilon T(z) = \delta_\epsilon T_{zz}(z) - 2a\left(\partial\partial\delta_\epsilon\Omega - 2\partial\Omega\partial\delta_\epsilon\Omega\right) \ . \tag{4.3}$$

Using (finite) transformations:

$$
\begin{aligned}
&z \to \widetilde{z} = z - \epsilon(z) \ , \\
&T_{zz}(z) \to \widetilde{T}_{\widetilde{z}\widetilde{z}}(\widetilde{z}) = (\partial_z \widetilde{z})^{-2} \, T_{zz}(z) \ , \\
&\Omega(z) \to \widetilde{\Omega}(\widetilde{z}) = \Omega(z) - \frac{1}{2} \log |\partial_z \widetilde{z}|^2 \ ,
\end{aligned}
$$

the transformation of (4.3) becomes

$$
\delta_\epsilon T(z) = \epsilon(z) \partial T(z) + 2\partial\epsilon(z) T(z) - a\partial^3 \epsilon(z) \ .
$$

Therefore,

$$
T^a_a = aR^{(2)} = -\frac{1}{12} c R^{(2)}.
$$

4.2.2 Non-linear sigma model (Graviton included)

So far we only consider a flat space-time(ST). However, if we expect that the string theory describe general relativity the action should contain general metric $G_{\mu\nu}(X)$:

$$
S[X^\mu, h_{ab}] = \frac{1}{4\pi\alpha'} \int d^2x \left(\sqrt{h} h^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X) \right) \ .
$$

An action of this shape is called non-linear sigma model(NLSM). Now let us consider an almost flat metric:

$$
G_{\mu\nu}(X) = \eta_{\mu\nu} + f_{\mu\nu}(X) \ .
$$

Then, the partition function becomes

$$
\begin{aligned}
Z &= \int \mathcal{D}h_{ab} \mathcal{D}X^\mu e^{-S} \\
&= \int \mathcal{D}h_{ab} \mathcal{D}X^\mu e^{-S_0} \left(1 + \frac{1}{4\pi\alpha'} \int d^2x \left(\sqrt{h} h^{ab} \partial_a X^\mu \partial_b X^\nu f_{\mu\nu}(X) \right) + \cdots \right) \ .
\end{aligned}
$$

Notice that the perturbative part is nothing but the graviton operator with wave function $f_{\mu\nu}(X) = \zeta_{\mu\nu} e^{ik\cdot X}$.

Let us again consider general case $G_{\mu\nu}(X)$. As a 2d field theory we can consider the vacuum expectation value(vev) for X, which we set to X_0:

$$
\hat{X}(\sigma, \tau) = X_0 + X(\sigma, \tau) \ .
$$

On the other hand, X_0 is a certain point in ST and we will expand the metric around this point. If you choose the coordinate nicely (Riemann normal coordinate) the expansion of the metric can be written as follows.

$$
G_{\mu\nu}(X) = G_{\mu\nu} - \frac{1}{3} R_{\mu\lambda\nu\rho} X^\lambda X^\rho + \mathcal{O}(X^3) \ ,
$$

where $G_{\mu\nu}$ and $R_{\mu\lambda\nu\rho}$ are a metric and a Riemann tensor at X_0, respectively. In a field theory sense those are coupling constants:

$$\begin{aligned} S &= \frac{1}{4\pi\alpha'} \int d^2x\, \partial^a X^\mu \partial_a X^\nu \left(G_{\mu\nu} - \frac{1}{3} R_{\mu\lambda\nu\rho} X^\lambda X^\rho + \cdots \right) \\ &= \frac{1}{2} \int d^2x\, \partial^a X^\mu \partial_a X^\nu \left(G_{\mu\nu} - \frac{2\pi\alpha'}{3} R_{\mu\lambda\nu\rho} X^\lambda X^\rho + \mathcal{O}(\alpha'^2) \right) \ . \end{aligned}$$

We rescaled the field $X \to \sqrt{2\pi\alpha'}X$ so that the expansion looks like "stringy expansion".

4.2.3 Perturbation theory for NLSM (Weyl anomaly from curved ST)

Note that discussion here is very naive and for the details one should consult [CT89].

We want to check if the theory (NLSM) has Weyl anomaly. As we briefly saw it is an interacting theory, and in general, interacting theories have non-trivial β-functions:

$$\beta[\lambda] \equiv E\frac{\partial}{\partial E}\lambda(E) = \frac{\partial}{\partial(\log E)}\lambda(E) \ ,$$

where λ is a coupling constant and E is a characteristic energy scale. When we consider a global scaling of coordinate:$z \to \widetilde{z} = sz = (1-\epsilon)z = e^{-\epsilon}z$, enegy scales oppositely: $E \to \widetilde{E} = \frac{1}{s}E = e^{\epsilon}E$. So the β-function can be written as

$$\beta[\lambda] = \frac{\partial}{\partial\epsilon}\lambda(\epsilon) \ .$$

The variation of the action is expressed in two ways:

$$\delta_\epsilon S = \begin{cases} \int \frac{d^2x}{2\pi}\sqrt{h}\delta_\epsilon h^{ab} T_{ba} = -\epsilon \int \frac{d^2x}{2\pi} T^a_{\ a} \ , \\ \frac{1}{4\pi\alpha'} \int d^2x\, \partial^a X^\mu \partial_a X^\nu \left(\epsilon \frac{\partial}{\partial\epsilon} G_{\mu\nu}(\epsilon) + \cdots \right) \ , \end{cases}$$

where the first variation is a formal transformation of the theory, and in quantum regime, it should be proportional to the trace part of the EM tensor. On the other hand, the second variation is the actual theory with an assumption that ϵ dependence of the theory is only in the coupling constants. Identifying them we have

$$\therefore \qquad T^a_{\ a} = -\frac{1}{2\alpha'}\beta[G_{\mu\nu}]\partial^b X^\mu \partial_b X^\nu + \cdots \ .$$

This shows that the anomaly is parametrized by β-functions.

Let us consider perturbation theory, namely loop corrections to two-point function etc, so that we can see if the theory is anomalous.

$$\begin{aligned} &\langle X^\mu(x_1) X^\nu(x_2) \rangle \\ &\quad = \int \frac{d^2k}{(2\pi)^2}\frac{2\pi\alpha'}{k^2} e^{ik\cdot(x_1-x_2)} \left\{ G^{\mu\nu} + \frac{2\pi\alpha'}{3} R^{\mu\nu} \left(\int \frac{d^2p}{(2\pi)^2}\frac{1}{p^2} + \frac{1}{k^2}\int \frac{d^2p}{(2\pi)^2} \right) + \cdots \right\} \ . \end{aligned}$$

We further focus on the logarithmic divergence and introduce regularization parameters:

$$\int_E^\Lambda \frac{d^2p}{(2\pi)^2}\frac{1}{p^2} = \frac{1}{2\pi}\log\left(\frac{\Lambda}{E}\right) ,$$

where Λ is an ultra-violet(UV) energy scale supposed to be ∞, and E is an infra-red(IR) energy scale supposed to be our life energy scale, which is very law (~ 0).

The divergence can be subtracted by counter terms as follows. Define $\hat{S} = S + S_{\rm ct}$, which is called bare action:

$$\hat{S} = \frac{1}{4\pi\alpha'}\int d^2x\ \partial^a\hat{X}^\mu\partial_a\hat{X}^\nu\left(\hat{G}_{\mu\nu} - \frac{1}{3}\hat{R}_{\mu\lambda\nu\rho}\hat{X}^\lambda\hat{X}^\rho + \cdots\right)$$

with

$$\hat{X}^\mu = Z^\mu_\nu X^\nu \ , \quad Z^\mu_\nu = \delta^\mu_\nu + \sum_{n=1}^\infty \alpha'^n Z^\mu_{(n),\nu}(\Lambda/E) \ ,$$
$$\hat{G}_{\mu\nu} = G_{\mu\nu} + \sum_{n=1}^\infty \alpha'^n G^{(n)}_{\mu\nu}(\Lambda/E) \ ,\text{etc.}$$

Physical action is S, which describes IR physics of energy scale E, on the other hand, $\hat{S}$ is called bare action, which describes UV physics of energy scale Λ. Note that the bare action only depends on Λ (not on E), and hence, the bare coupling constants ($\hat{G}_{\mu\nu}$ etc) only depends on high energy Λ. The counter terms leads other contributions

$$\begin{aligned}&\langle X^\mu(x_1)X^\nu(x_2)\rangle \\ &\sim \left\{G^{\mu\nu} + \frac{\alpha'}{3}R^{\mu\nu}\log\left(\frac{\Lambda}{E}\right) - \alpha'\left(G^{\mu\nu}_{(1)} + Z^{\mu\nu}_{(1)} + Z^{\nu\mu}_{(1)}\right) + \cdots\right\} .\end{aligned} \tag{4.4}$$

Unfortunately, the equation above cannot fix the ration between $G_{(1)}$ and $Z_{(1)}$. We need further information like 4-pt function etc. to determine the ratio. We simply list the result:

$$G^{(1)}_{\mu\nu} = R_{\mu\nu}\log\left(\frac{\Lambda}{E}\right) \ , \quad Z^{(1)}_{\mu\nu} = -\frac{1}{3}R_{\mu\nu}\log\left(\frac{\Lambda}{E}\right) \ ,$$

which does cancel the divergent term in (4.4). From the result we can derive the β-function:

$$G_{\mu\nu}(E,\Lambda) = \hat{G}_{\mu\nu}(\Lambda) - \alpha' R_{\mu\nu}\log\left(\frac{\Lambda}{E}\right) ,$$
$$\beta[G_{\mu\nu}] = \frac{\partial}{\partial(\log E)}G_{\mu\nu}(E,\Lambda) = \alpha' R_{\mu\nu} .$$

Therefore, in order for the theory to be anomaly free we need the space-time to be Ricci flat ($R_{\mu\nu} = 0$). This is equivalent for Hilbert-Einstein action to satisfy its E.O.M.

4.2.4 NLSM (general)

So far only the graviton has been included but not B-field or dilaton. Here, we simply give the action and their β-functions.

$$S = \frac{1}{4\pi\alpha'}\int d^2x\left(\sqrt{h}h^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X) + i\varepsilon^{ab}\partial_a X^\mu \partial_b X^\nu B_{\mu\nu}(X) + \alpha'\sqrt{h}R^{(2)}\Phi(X)\right) .$$

Weyl anomaly:

$$T^a_a = -\frac{1}{2\alpha'}\beta[G_{\mu\nu}]\partial_a X^\mu \partial_b X^\nu - \frac{i}{2\alpha'}\beta[B_{\mu\nu}]\varepsilon^{ab}\partial_a X^\mu \partial_b X^\nu - \frac{1}{2}\beta[\Phi]R^{(2)}$$

β-functions:

$$\begin{aligned}
\beta[G_{\mu\nu}] &= \alpha' R_{\mu\nu} + 2\alpha'\nabla_\mu\nabla_\nu\Phi - \frac{\alpha'}{4}H_{\mu\lambda\rho}H_\nu{}^{\lambda\rho} + \mathcal{O}(\alpha'^2) , \\
\beta[B_{\mu\nu}] &= -\frac{\alpha'}{2}\nabla^\lambda H_{\lambda\mu\nu} + \alpha'\nabla^\lambda\Phi H_{\lambda\mu\nu} + \mathcal{O}(\alpha'^2) , \\
\beta[\Phi] &= \frac{D-26}{6} - \frac{\alpha'}{2}\nabla^2\Phi + \alpha'\nabla^\lambda\Phi\nabla_\lambda\Phi - \frac{\alpha'}{24}H_{\mu\lambda\rho}H^{\mu\lambda\rho} + \mathcal{O}(\alpha'^2) .
\end{aligned}$$

There are a few remarks on them:

- String theory to be consistent requires all the β-functions to be zero.
- Note that 2d gravity has no physical D.O.F, and the Einstein-Hilbert action gives a topological number(Euler characteristic). So if the dilaton is constant we have

$$\begin{aligned}
S_{\text{dilaton}} &= \frac{1}{4\pi\alpha'}\int d^2x\left(\alpha'\sqrt{h}R^{(2)}\Phi(X)\right) \\
&\to \frac{1}{4\pi}\int d^2x\left(\sqrt{h}R^{(2)}\Phi\right) = \Phi(2-2g)
\end{aligned}$$

  Define $g_{\text{str}} = e^\Phi$ and the action leads

$$e^{-S_{\text{dilaton}}} = g_{\text{str}}^{2g-2} .$$

  g_{str} can be understood as a string coupling as follows. See Fig. 5. n-point string tree amplitude can be understood $n-2$ cylinders attaching to a cylinder. So the amplitude should proportional to g_{str}^{n-2}. Higher loop (higher genus) amplitude can be derived by attaching g cylinders to the tree amplitude, and the amplitude should be $\hat{A}_{n,g} \propto g_{\text{str}}^{n-2+2g}$. Usually, vertex operators are re-normalized so that g_{str}^n is included in the definition of $V_1 \cdots V_n$. Therefore, the re-normalized amplitude should be

$$A_{n,g} \propto g_{\text{str}}^{2g-2} ,$$

  which coincide with the dilaton action.

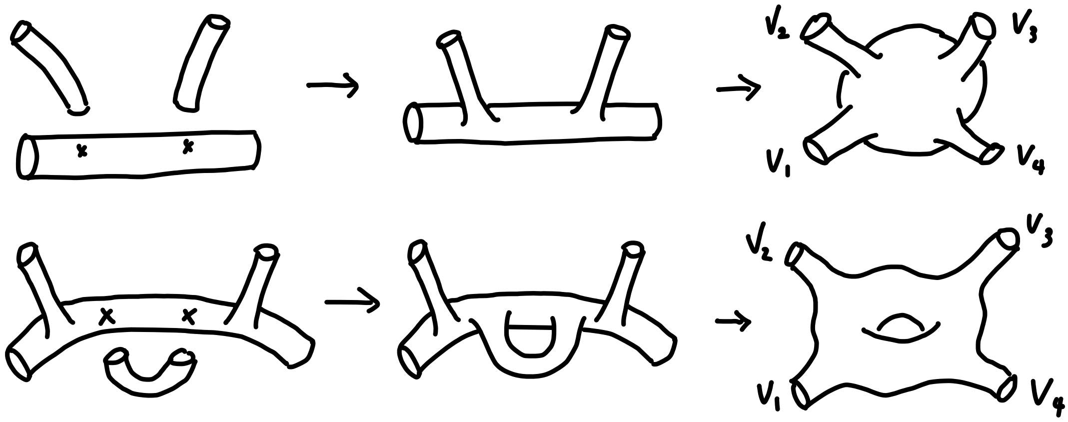


Figure 5: 4-pt amplitude example. The upper one is a construction of 4pt tree amplitude from cylinders. The lower one is a construction of 4-pt 1-loop amplitude from the tree amplitude.

- Trivial β-function ($\beta = 0$) is equivalent to the E.O.M of the following ST action.

$$S_{\text{eff}} = \frac{1}{2\kappa_D^2}\int d^DX\sqrt{-G}e^{-2\Phi}\left[\frac{2(D-26)}{3\alpha'} + R - \frac{1}{12}H_{\mu\lambda\rho}H^{\mu\lambda\rho} + 4\nabla^\lambda\Phi\nabla_\lambda\Phi + \mathcal{O}(\alpha'^2)\right] .$$

- B-field is a higher dimensional analogy of gauge fields. It has gauge transformation

$$\delta B_{\mu\nu} = \partial_\mu\Lambda_\nu - \partial_\nu\Lambda_\mu ,$$

and field strength $H_{\mu\nu\lambda}$:

$$H_{\mu\nu\lambda} = \partial_\mu B_{\nu\lambda} + \partial_\nu B_{\lambda\mu} + \partial_\lambda B_{\mu\nu} .$$

B-field plays an important role with open string.

5 Lecture 5

5.1 Quantization via path integral

We have been studying bosonic string theory, but there are several caveats.

• In the light-cone quantization, Lorentz invariance is not manifest. Can we quantize strings in a way that is manifestly Lorentz invariant?

• We have seen that the Weyl symmetry of the string sigma model is anomalous on a general curved background. How can the bosonic string be anomaly-free?

• Although we learnt that string amplitude is expressed via Feynman path integral

$$A_n = \sum_g \int [\mathcal{D}h_{ab}]_{g,n}\int \mathcal{D}X^\mu e^{-S_\sigma[X^\mu,h_{ab}]}\,\widehat{V}_1\cdots\widehat{V}_n , \tag{5.1}$$

we do not know how to perform this path integral. In particular, the path integral is endowed with huge gauge symmetries, WS diffeomorphism and Weyl symmetry. How can we treat integration measure and fix gauge in the path integral?

To answer to these question, we will study quantization procedure via path integral, which is often called **modern covariant quantization**. This method uses the analogue of the Faddeev-Popov method of gauge theories. Furthermore, the physical state condition is implemented via the BRST symmetry.

After gauge-fixing the reparametrization and Weyl symmetry, the integral over h_{ab} turns into a path-integral over ghost CFT, which has $c = -26$. Therefore, in order for the theory to be anomaly-free, the original theory has to be a CFT with $c = 26$.

Faddeev-Popov gauge fixing

The integration measure $[\mathcal{D}h_{ab}]_{g,n}$ is over all the metric s on 2d surfaces with genus g and n marked points. However, this integral has Weyl symmetry and wold-sheet diffeomorphisms under which the world-sheet metric is transformed as

$$h^{\zeta}_{ab}(\widetilde{\sigma}) = e^{2\omega(\sigma)} \frac{\partial \sigma^c}{\partial \widetilde{\sigma}^a} \frac{\partial \sigma^d}{\partial \widetilde{\sigma}^b} h_{cd}(\sigma) \ . \tag{5.2}$$

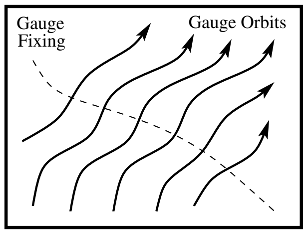


Figure 6

These symmetries are redundant and integrating along these directions just gives rise to the volume of the symmetry group. (In Figure 6, the solid arrows schematically draws gauge redundancy and the dotted line shows physically distinct configurations.) Thus, we need to carry out gauge-fixing. Thankfully, there is a standard method to fix gauge, introduced by Faddeev and Popov. A basic idea is to insert the identity of the following form in the path integral:

$$1 = \int \mathcal{D}\zeta \ \delta(h - \hat{h}^{\zeta}) \det \left(\frac{\delta \hat{h}^{\zeta}}{\delta \zeta} \right) \tag{5.3}$$

where the Jacobian factor $\det \left(\frac{\delta \hat{h}^{\zeta}}{\delta \zeta} \right)$ is called **Faddeev-Popov determinant** and we denote it by $\Delta_{FP}[\hat{h}]$. The insertion of this identity into the path integral fixes the metric as $\hat{h}$ because of the delta function. In addition, because $\int \mathcal{D}\zeta$ integral only contributes an infinite multiplicative factor, we can discard this integral. Therefore, after Faddeev-Popov gauge fixing, the form of the path integral can be schematically written as

$$Z[\hat{h}] = \int \mathcal{D}X \ \Delta_{FP}[\hat{h}] \, e^{-S_\sigma[X,\hat{h}]} \tag{5.4}$$

We should make several remarks about this procedure.

• First, the conformal Killing transformations are residual gauge symmetries not fixed above. We have to throw away these residual gauge symmetries in the path integral in order to avoid over-counting. Indeed we will be careful to fix this extra residual gauge freedom when computing string amplitudes.

• Second, there are caveats related to global properties of the world-sheet Riemann surface $\Sigma_{g,n}$. In fact, metrics on a Riemann surface encodes the information of "shape" of the Riemann surface $\Sigma_{g,n}$ called **complex moduli**, which is not accounted for by local gauge transformations ζ. The space which parametrizes 'shape" of the Riemann surface is called **moduli space of Riemann surface** $\Sigma_{g,n}$ which is $(6g-6+2n)$-$\dim_{\mathbb{R}}$:

$$\mathcal{M}_{g,n} := \frac{[\mathcal{D}h_{ab}]_{g,n}}{\text{Diff} \times \text{Weyl}}$$

Therefore the path integral actually involves integral over $\mathcal{M}_{g,n}$ as well. At this moment, we postpone both the issues and will come back to them in the next lecture on string amplitudes.

Now let us take an infinitesimal version of (5.2) where a Weyl transformation is parameterized by $\omega(\sigma)$ and an infinitesimal diffeomorphism by $\delta\sigma^\alpha = \epsilon^\alpha(\sigma)$. Subsequently, the change of the metric is read off

$$\delta\hat{h}_{ab} = 2\omega\hat{h}_{ab} + \nabla_a\epsilon_b + \nabla_b\epsilon_a := 2\widetilde{\omega}\hat{h}_{ab} + (P\cdot\epsilon)_{ab}$$

where we decompose it into

$$\begin{aligned}(P\cdot\epsilon)_{ab} &= \nabla_a\epsilon_b + \nabla_b\epsilon_a - h_{ab}(\nabla\cdot\epsilon)\\ \widetilde{\omega} &= \omega + \frac{1}{2}(\nabla\cdot\epsilon)\ .\end{aligned} \tag{5.5}$$

Indeed, the operator P maps vectors ϵ_a to symmetric traceless 2-tensors $(P\cdot\epsilon)_{ab}$. Thus, the Faddeev-Popov determinant can be written as

$$\Delta_{FP}[\hat{h}] = \det\frac{\delta(P\cdot\epsilon,\widetilde{\omega})}{\delta(\epsilon,\omega)} = \det\begin{vmatrix} P & 0\\ * & 1\end{vmatrix} = \det P\ .$$

To compute $\det P$, we use **Faddeev-Popov ghosts**, which can be understood as an infinite-dimensional version of the following integral. Given a matrix M_{ij}, its determinant can be expressed as a Grassmann integral

$$\int\prod_{i=1}^n d\psi_i d\theta_i \exp(\theta_i M_{ij}\psi_j) = \det M\ .$$

where θ,ψ are Grassmann variables. Accordingly, we introduce anti-commuting fermionic fields, c^a (ghosts) and b_{ab} (anti-ghost) where b_{ab} transforms as a symmetric traceless tensor and c^a as a vector. Then, we can express

$$\Delta_{FP}[\hat{h}] = \int\mathcal{D}b\mathcal{D}c\ \exp\left(\frac{i}{2\pi}\int d^2\sigma\sqrt{\hat{h}}\,b^{ab}(P\cdot c)_{ab}\right) := \int\mathcal{D}b\mathcal{D}c\ \exp[iS_{\text{gh}}]\ ,$$

where the ghost action can be written as

$$S_{\rm gh} = \frac{1}{2\pi}\int d^2\sigma\sqrt{-\hat{h}}\; b^{ab}\nabla_a c_b\ . \tag{5.6}$$

Something lovely has happened. Although the ghost fields were introduced as some auxiliary constructs, they now appear on the same footing as the dynamical fields X. Consequently the Faddeev-Popov gauge fixing results in a fermionic 2d CFT, usually called bc **ghost CFT**.

Let us make some remarks about the equation of motion of $S_{\rm gh}$

• The equation of motion for ca is given by $P\cdot c = 0$. Therefore the solutions for c are in one-to-one correspondence with the **conformal Killing vectors**, which are the generators of the residual symmetry.

• The equation of motion for b_{ab} is $\nabla_a b^{ab} = 0$. We will understand the geometric meaning of these equations when discussing the moduli space of Riemann surfaces.

To understand the properties of bc ghost CFT, it is convenient to use Euclidean signature so that we will perform Wick rotation in what follows. Then, the factor of i in the action disappears. The expression for the full partition function (5.4) is

$$Z[\hat{h}] = \int \mathcal{D}X\mathcal{D}b\mathcal{D}c\ \exp\left(-S_\sigma[X,\hat{h}] - S_{\rm gh}[b,c,\hat{h}]\right)\ . \tag{5.7}$$

bc **ghost CFT**

Now let us study the bc ghost CFT more in detail. For this purpose, we pick the conformal gauge $\hat{h}_{ab} = e^{2\omega}\delta_{ab}$. In this metric, the non-trivial Christofell connections are $\Gamma^z_{zz} = 2\partial\omega$, $\Gamma^{\bar z}_{\bar z\bar z} = 2\bar\partial\omega$ so that the ghost action can be written as

$$\begin{aligned} S_{\rm ghost} &= \frac{1}{2\pi}\int d^2z\ \left(b_{zz}\nabla_{\bar z}c^z + b_{\bar z\bar z}\nabla_z c^{\bar z}\right) \\ &= \frac{1}{2\pi}\int d^2z\ b_{zz}\,\partial_{\bar z}c^z + b_{\bar z\bar z}\,\partial_z c^{\bar z} \end{aligned} \tag{5.8}$$

For the sake of simplicity, let us define

$$b = b_{zz}\ ,\qquad \bar b = b_{\bar z\bar z}\ ,\qquad c = c^z\ ,\qquad \bar c = c^{\bar z}\ .$$

Then, the action simplifies to

$$S_{\rm gh} = \frac{1}{2\pi}\int d^2z\ \left(b\,\bar\partial c + \bar b\,\partial\bar c\right)$$

Which gives the equations of motion

$$\bar\partial b = \partial\bar b = \bar\partial c = \partial\bar c = 0$$

So we see that b and c are holomorphic fields, while $\bar{b}$ and $\bar{c}$ are anti-holomorphic.

We can compute the OPEs of these fields using the standard path integral techniques that we learnt before. In what follows, we will just focus on the holomorphic piece of the CFT. We have, for example,

$$0 = \int \mathcal{D}b\mathcal{D}c \, \frac{\delta}{\delta b(z)} \, \left[e^{-S_{\rm gh}}\, b(w)\right] = \int \mathcal{D}b\mathcal{D}c \; s^{-S_{\rm gh}} \left[-\frac{1}{2\pi}\,\bar{\partial}c(z)\, b(w) + \delta(z-w)\right]$$

which tells us that

$$\bar{\partial}c(z)\, b(w) = 2\pi\, \delta(z-w)$$

Similarly, looking at $\delta/\delta c(z)$ gives

$$\bar{\partial}b(z)\, c(w) = 2\pi\, \delta(z-w)$$

We can integrate both of these equations using our favorite formula $\bar{\partial}(1/z) = 2\pi\delta(z,\bar{z})$. We learn that the OPEs between fields are given by

$$\begin{aligned} b(z)\, c(w) &= \frac{1}{z-w} + \ldots \\ c(w)\, b(z) &= \frac{1}{w-z} + \ldots \end{aligned}$$

In fact the second equation follows from the first equation and Fermi statistics. The OPEs of $b(z)\, b(w)$ and $c(z)\, c(w)$ have no singular parts because they vanish as $z \to w$.

In any CFT, it is of most importance to find the form of the stress energy tensor. The stress energy tensor is obtained via Noether's theorem with respect to world sheet transformations $\delta z = \epsilon(z)$, under which

$$\delta b = (\epsilon\partial + 2(\partial\epsilon))b \ , \qquad \delta c = (\epsilon\partial - (\partial\epsilon))c \ .$$

Indeed both b and c are primary fields with weights $h = 2$ and $h = -1$, respectively, that can be easily seen from their index structure b_{zz} and c^z. From these rules, one can deduce the form of the stress-energy tensor

$$T^{\rm gh}(z) = -2 : b(z)\partial c(z) : + : c(z)\partial b(z) :$$

In fact, this form can be obtained from the first principle, namely the variation of the action under the metric (Exercise).

The OPEs of b and c with the stress tensor are

$$\begin{aligned} T^{\rm gh}(z)\, c(w) &= -\frac{c(w)}{(z-w)^2} + \frac{\partial c(w)}{z-w} + \ldots \\ T^{\rm gh}(z)\, b(w) &= \frac{2b(w)}{(z-w)^2} + \frac{\partial b(w)}{z-w} + \ldots \end{aligned}$$

confirming that b and c have weight 2 and -1, Finally, we can compute the TT OPE to determine can be written as (Exercise)

$$T^{\text{gh}}(z)\,T^{\text{gh}}(w) = \frac{-13}{(z-w)^4} + \frac{2T^{\text{gh}}(w)}{(z-w)^2} + \frac{\partial T^{\text{gh}}(w)}{z-w} + \ldots$$

Now one can read off the central charge of the bc ghost system which is

$$c^{\text{gh}} = 2(-13) = -26 \ .$$

We learnt that the Weyl symmetry is anomalous unless $c = 0$. Since the Weyl symmetry is a gauge symmetry, the theory must be Weyl anomaly-free. Since the total central charge of the string sigma model and ghost theory (5.7) is given by $c = c^X + c^{\text{gh}}$, the dimension of the target space must be

$$D = 26.$$

Again, we obtain the critical dimension of the bosonic string theory!

5.2 BRST quantization

In 4d non-Abelian QFT, the Lagrangian with Faddeev-Popov ghosts has the continuous symmetry, called **BRST symmetry** (Becchi-Rouet-Stora-Tyupin). The BRST symmetry is generated by a nilpotent charge Q_B with $Q_B^2 = 0$ that commutes with the Hamiltonian. The nilpotency of the BRST charge has strong consequences. Since all physical states must be BRST-invariant, we require a physical state is annihilated by Q_B

$$Q_B|\text{phys}\rangle = 0 \ .$$

However, one can always add a state of the form $Q_B|\chi\rangle$ since this state will be annihilated by Q_B because of the nilpotency. However, this state is orthogonal to all physical states and therefore it is a **null state**. Thus, two states related by

$$|\psi'\rangle = |\psi\rangle + Q_B|\chi\rangle$$

have the same inner products and are indistinguishable. This is the remnant in the gauge-fixed version of the original gauge symmetry. As a result, the Hilbert space of physical states is isomorphic to the Q_B-cohomology, i.e.

$$\mathcal{H}^{\text{phys}} \cong \frac{\mathcal{H}^{Q_B\text{-closed}}}{\mathcal{H}^{Q_B\text{-exact}}}$$

This elegant method to determine the physical Hilbert space in a gauge fixed action with ghosts is known as **BRST quantization**.

We are now ready to apply this formalism to the bosonic string. Expressed in the world-sheet light-cone coordinates, we obtain the following BRST transformations

(Exercise):

$$\delta_B X^\mu = i\epsilon(c\partial + \overline{c}\overline{\partial})X^\mu\,,$$
$$\delta_B c = i\epsilon c\partial c \qquad \delta_B \overline{c} = i\epsilon\overline{c}\overline{\partial}\overline{c}\,,$$
$$\delta_B b = i\epsilon(T^X + T^{\rm gh}) \qquad \delta_B \overline{b} = i\epsilon(\overline{T}^X + \overline{T}^{\rm gh})\,.$$

Actually, this can also be shown using the BRST formalism [Pol98, §4]. The Noether theorem tells us the holomorphic part of the BRST current takes the form (Exercise)

$$\begin{aligned} j_B &= c(z)T^X(z) + \frac{1}{2} : c(z)T^{\rm gh}(z) : + \frac{3}{2} : \partial^2 c(z) : \\ &= c(z)T^X(z) + : b(z)c(z)\partial c(z) : + \frac{3}{2} : \partial^2 c(z) : \ . \end{aligned} \tag{5.9}$$

and the BRST charge is defined by

$$Q_B = \oint \frac{dz}{2\pi i}\, j_B\,.$$

Indeed the anomaly now shows up in Q_B^2 so that the BRST charge is nilpotent if and only if $D = 26$. Furthermore, using the mode expansions of b and c

$$b(z) = \sum_{b\in\mathbb{Z}} \frac{b_m}{z^{m+2}} \qquad c(z) = \sum_{c\in\mathbb{Z}} \frac{c_m}{z^{m-1}}\ ,$$

we can express the BRST charge in terms of the X^μ Virasoro operators and the ghost oscillators as

$$Q_B = \sum_n c_n(L^X_{-n} - \delta_{n,0}) + \sum_{m,n} \frac{m-n}{2} : c_m c_n b_{-m-n} : \ . \tag{5.10}$$

In the case of closed strings, there is of course the anti-holomorphic part $\overline{Q}_B$, and the total BRST charge is $Q_B + \overline{Q}_B$.

We will find the physical spectrum in the BRST context. According to our previous discussion, the physical states will have to be annihilated by the BRST charge, and not be of the form $Q_B|\ \rangle$.

First we have to describe our extended Hilbert space that includes the ghosts. As far as the X^μ oscillators are concerned the situation is the same as in the previous sections, so we need only be concerned with the ghost Hilbert space. The full Hilbert space will be a tensor product of the two. Since the ghosts generate a two-state spin system $|\uparrow\rangle, |\downarrow\rangle$, we can write all the states can be generated from $|k,\uparrow\rangle, |k,\downarrow\rangle$ by acting creation operators where $|k\rangle = |0;k\rangle$ denotes the vacuum of the matter theory.

For the ghost system, we impose the positive ghost oscillator modes annihilate the states

$$b_{n>0}|\uparrow\rangle = b_{n>0}|\downarrow\rangle = c_{n>0}|\uparrow\rangle = c_{n>0}|\downarrow\rangle = 0$$

However, there is a subtlety because of the presence of the zero modes b_0 and c_0 which satisfy $b_0^2 = c_0^2 = 0$ and $\{b_0, c_0\} = 1$ (Exercise).

From the light-cone quantization, we know that there is only one vacuum called Tachyon. So we have to pick the ghost vacuum among the two spin states. For this purpose, we further impose one more condition, namely

$$b_0|\text{phys}\rangle = 0\,. \tag{5.11}$$

This is sometimes called the **Siegel gauge** [GSW87, §3.2]. Thus, under this condition, we have

$$\begin{aligned} b_0|\downarrow\rangle &= 0\,, \qquad b_0|\uparrow\rangle = |\downarrow\rangle\,, \\ c_0|\uparrow\rangle &= 0\,, \qquad c_0|\downarrow\rangle = |\uparrow\rangle\,. \end{aligned}$$

Imposing also (5.11) implies that the correct ghost vacuum is $|\downarrow\rangle$. We can now create states from this vacuum by acting with the negative modes of the ghosts b_m, c_n. We cannot act with c_0 since the new state does not satisfy the Siegel condition (5.11). Now, we are ready to describe the physical states in the open string. Note that since Q_B in (5.10) has "level" zero, we can impose BRST invariance on physical states level by level.

At level zero, there is only one state that is the total vacuum $|k,\downarrow\rangle$

$$0 = Q_B|k,\downarrow\rangle = (L_0^X - 1)c_0|k,\downarrow\rangle = (L_0^X - 1)|k,\uparrow\rangle\,.$$

Because we have the mode expansion of the 0th Virasoro generator

$$\begin{aligned} L_0^X &= \alpha' p^2 + \alpha_{-1}\cdot\alpha_1 + \cdots\ , \\ L_0^{\text{gh}} &= b_{-1}c_1 - c_{-1}b_1 + \cdots\ , \end{aligned} \tag{5.12}$$

the only non-trivial condition at level zero is $m^2 = -1/\alpha'$, which is the tachyon. Note that the Virasoro generators of the bc ghost is

$$L_m^{\text{gh}} = \sum_{m\in\mathbb{Z}} (2m-n) : b_n c_{m-n} : - \,\delta_{m,0}\ .$$

At the first level, the possible operators that can act on the vacuum $|k,\downarrow\rangle$ are α_{-1}^μ, b_{-1} and c_{-1}. The most general state of this form is then

$$|\psi\rangle = (\zeta\cdot\alpha_{-1} + \beta b_{-1} + \gamma c_{-1})|k,\downarrow\rangle\,, \tag{5.13}$$

which has 28 parameters: a 26-dimensional vector ζ_μ and two more constants β, γ. From the definition of j_B, one can derive that

$$\{Q_B, b_0\} = L_0^X + L_0^{\text{gh}} - 2.$$

From $b_0|\psi\rangle = 0$, we deduce $0 = \{Q_B, b_0\}|\psi\rangle = (L_0^X + L_0^{\text{gh}} - 2)|\psi\rangle$. This yields the mass shell condition $k^2 = 0$. In addition, the BRST condition demands

$$0 = Q_B|\psi\rangle = 2((k\cdot\zeta)c_{-1} + \beta k\cdot\alpha_{-1})|k,\downarrow\rangle\ .$$

This only holds if $k\cdot\zeta = 0$ and $\beta = 0$. Therefore the BRST condition removes the unphysical anti-ghost excitations as well as all polarizations that are not orthogonal to the momentum, thereby eliminating 2 out the $26+2$ original states. So there are only 26 parameters left.

Next we have to make sure that this state is not Q_B-exact: a general state $|\chi\rangle$ is of the same form as (5.13), but with parameters ζ'^{μ}, β' and γ'. So the most general Q_B-exact state at this level with $k^2 = 0$ will be

$$Q_B|\chi\rangle = 2(k\cdot\zeta' c_{-1} + \beta' k\cdot\alpha_{-1})|k,\downarrow\rangle\ .$$

This means that the c_{-1} part in (5.13) is BRST-exact and that the polarization has the equivalence relation $\zeta_\mu \sim \zeta_\mu + 2\beta' k_\mu$. This leaves us with the 24 physical degrees of freedom we expect for a massless vector particle in 26 dimensions. In sum, the physical state at level 1 is

$$\{|\zeta;k\rangle\ ;\quad k\cdot\zeta = 0\}/\ \zeta_\mu \sim \zeta_\mu + 2\beta' k_\mu\ .$$

The same procedure can be followed for the higher levels. In the case of the closed string we have to include the barred operators, and of course we have to use $Q_B + \overline{Q}_B$.

6 Lecture 6

Finally, we will discuss string amplitude. The amplitude tell us about the feature of the string theory.

6.1 Global part of the gauge fixing

As we naive showed the string amplitude is written as follows.

$$A_n = \sum_g \int \frac{[\mathcal{D}h_{ab}]_{g,0}}{(\text{Weyl}\times\text{diff})} \int \mathcal{D}X^\mu e^{-S_\sigma[X^\mu,h_{ab}]} \prod_{i=1}^{n} \int d^2z\sqrt{h}V_i\ .$$

In this section we will learn the degrees of the metric integration in the string amplitude after the (Weyl $\times$ diff) gauge fixing. Namely,

$$\frac{[\mathcal{D}h_{ab}]_{g,n}}{(\text{Weyl}\times\text{diff})} = 6g - 6 + 2n\ .$$

6.1.1 Naive explanation

The $2n$ is simply coming from the integration of the vertex operators, which is needed for the vertex operator to be Weyl & diff-invariant. The $6g$ can be understood in a naive way as follows. Even after gauge fixing (say fixing metric locally) there is a freedom to change the "shape" of world-sheet(WS), which is called metric moduli (we simply denote moduli). Let us consider to increase the number of genus g. As we did in the lecture 4 we can do that by attaching a cylinder to a certain Riemann surface Σ_g. When this is done we need to specify where the end points of the cylinder are and its "shape" of the cylinder. The end points are denoted by two complex points p_1, p_2. Around the points we introduce complex coordinate z_1, z_2 and impose one condition $z_1 z_2 = c$, where c is a constant whose phase specifies twist and magnitude specifies length (see Fig. 7). In total there are another 6 parameters to describe the shape of WS.

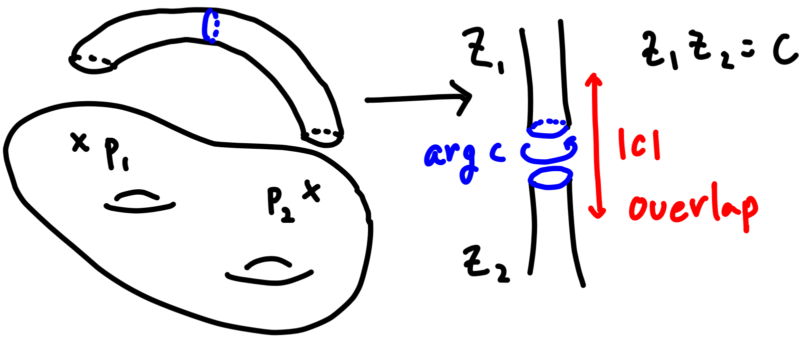


Figure 7: Attaching a pair of cylinders to Σ_g, and patching the pair of cylinders by $z_1 z_2 = c$.

Let us see some concrete examples. As we saw in the Problem 5 sphere has 6 symmetries(Conformal Killing Group(CKG)) so that we can move three points on sphere to arbitrary locations. Therefore, sphere amplitude has $2n - 6$ integrations. For torus there are 2 moduli and 2 CKG, thus torus amplitude has $2n$ integrations. From the discussion above for genus g Riemann surface its amplitude has $6g - 6 + 2n$ integrations.

6.1.2 Conformal Killing Group & Metric moduli of space

From the previous lecture we know the variation of the metric:

$$\delta h_{ab} = (P \cdot \epsilon)_{ab} + 2\widetilde{\omega} h_{ab} ,$$

where

$$\begin{aligned}
(P \cdot \epsilon)_{ab} &= \nabla_a \epsilon_b + \nabla_b \epsilon_a - h_{ab}(\nabla \cdot \epsilon) , \\
\widetilde{\omega} &= \omega + \frac{1}{2}(\nabla \cdot \epsilon) .
\end{aligned}$$

Conformal Killing Vectors are transformations that do not change metric, namely, $\delta h_{ab} = 0$. ω is totally fix by $\nabla \cdot \epsilon$. Therefore, they are the solutions of

$$(P \cdot \epsilon)_{ab} = 0 \ .$$

On the other hand, moduli is given by all the transformations perpendicular to the transformation δh_{ab}, which we will denote $\delta^{\perp} h_{ab}$. It can be derived as follows.

$$\begin{aligned} 0 &= \int \sqrt{h} d^2\sigma \ \delta^{\perp} h_{ab} \left[(P \cdot \epsilon)^{ab} + 2\widetilde{\omega} h^{ab} \right] \\ &= \int \sqrt{h} d^2\sigma \ \left[\left(P^T \cdot \delta^{\perp} h \right)_a \epsilon^a + 2\widetilde{\omega} h^{ab} \delta^{\perp} h_{ab} \right] \ . \end{aligned}$$

To satisfy the equation for arbitrary ϵ and ω it is required that

$$\begin{aligned} & h^{ab} \delta^{\perp} h_{ab} = 0 \ , \\ & \left(P^T \cdot \delta^{\perp} h \right)_a = \nabla^b \delta^{\perp} h_{ba} = 0 \ . \end{aligned}$$

CK-eq and moduli eq becomes simpler in conformal gauge in complex coordinates:

$$\begin{aligned} & \partial \overline{\epsilon} = \overline{\partial} \epsilon = 0 \ , \\ & \partial \delta^{\perp} h_{\overline{zz}} = \overline{\partial} \delta^{\perp} h_{zz} = 0 \ . \end{aligned}$$

Examples

For sphere we have

$$\begin{aligned} & \delta^{\perp} h_{zz} = \delta^{\perp} h_{\overline{zz}} = 0 \ , \\ & \epsilon = a_0 + a_1 z + a_2 z^2 \ , \\ & \overline{\epsilon} = a_0^* + a_1^* z + a_2^* z^2 \ . \end{aligned}$$

Therefore, there are 6 CKVs and no modulus.

For torus we have

$$\begin{aligned} & \delta^{\perp} h_{zz} = a \ , \\ & \epsilon = b \ . \end{aligned}$$

Therefore, there are 2 CKVs and 2 moduli.

Relation to ghost zero modes

Action for ghost is

$$S_{\text{gh}} = \frac{1}{2\pi} \int \sqrt{h} d^2 z \ b_{ab} \nabla^a c^b \ .$$

As we saw in the previous lecture the nonzero ghost modes cancel non-physical modes in X^μ. On the other hand, its zero mode

$$\begin{aligned} &P\cdot c = \nabla_a c_b + \nabla_b c_a - h_{ab}(\nabla\cdot c) = 0 \ , \\ &\nabla^a b_{ab} = 0 \ . \end{aligned}$$

corresponds to CKG and moduli. Since these modes are absent in the action we need to insert appropriate zero modes to derive a non-trivial amplitude (because $\int d\theta\cdot 1 = 0$).

Table 1: The number of zero modes of b and c.

	$g=0$	$g=1$	$g\geq 2$
c	6	2	0
b	0	2	$6g-6$

6.1.3 Ghost number anomaly

Ghost fields have so called ghost number $[c]=1$ and $[b]=-1$, and other fields have zero ghost number. Correspondingly, there is a ghost number current $j=cb$ for the transformation $\delta c = \epsilon c$ and $\delta b = -\epsilon b$. The current satisfies the conservation law $\bar\partial j = 0$. However, it has anomaly in curved WS:

$$\nabla^z j_z = \kappa\cdot R^{(2)} \ .$$

OPE with EM tensor leads

$$\begin{aligned} T(z)j(w) &= -\ {:}(2b\partial c + \partial b c)(z){:}{:}cb(w){:} \\ &\sim \frac{-3}{(z-w)^3} + \frac{j(w)}{(z-w)^2} + \frac{\partial j(w)}{z-w} \ . \end{aligned}$$

From this the infinitesimal transformations can be derived as follows.

$$\delta j = \epsilon\partial j + \partial\epsilon j - \frac{3}{2}\partial^2\epsilon \ .$$

On the other hand, for the conformal gauge $ds^2 = e^{2\omega}dzd\bar z$, the WS curvature is $R = -8e^{-2\omega}\partial\bar\partial\omega$. Therefore, we can derive

$$j_z = -4\kappa\partial\omega + j(z) \ ,$$

where $j(z)$ is the holomorphic current. As we did before (for EM tensor anomaly) we deduce that

$$\begin{aligned} &\delta j(z) = \epsilon\partial j(z) + \partial\epsilon j(z) - 2\kappa\partial^2\epsilon \ , \\ &\therefore \qquad \kappa = -\frac{3}{4} \ . \end{aligned}$$

This anomaly puts constraint on non-vanishing ghost correlators. Let us first consider classical action of δ:

$$\frac{1}{\epsilon}\delta\left\langle\prod_{i=1}^{m}c(z_i)\prod_{j=1}^{n}b(z_j)\right\rangle=(m-n)\left\langle\prod_{i=1}^{m}c(z_i)\prod_{j=1}^{n}b(z_j)\right\rangle\ .$$

On the other hand, using holomorphic current equation

$$0=\overline{\partial}J(z)=\nabla^z j_z-\kappa\cdot R^{(2)}\ ,$$

the Ward-Takahashi indentity becomes

$$\begin{aligned}\frac{1}{\epsilon}\widehat{\delta}\left\langle\prod_{i=1}^{m}c(z_i)\prod_{j=1}^{n}b(z_j)\right\rangle&=\left\langle\left(\int\frac{d^2x}{2\pi}\sqrt{h}\overline{\partial}J(z)\right)\prod_{i=1}^{m}c(z_i)\prod_{j=1}^{n}b(z_j)\right\rangle\\&=\left\langle\left(\int\frac{d^2x}{2\pi}\sqrt{h}\left(\nabla^z j_z+\frac{3}{4}R^{(2)}\right)\right)\prod_{i=1}^{m}c(z_i)\prod_{j=1}^{n}b(z_j)\right\rangle\\&=\left\langle\left(\int\frac{3d^2x}{8\pi}\sqrt{h}R^{(2)}\right)\prod_{i=1}^{m}c(z_i)\prod_{j=1}^{n}b(z_j)\right\rangle=(3-3g)\left\langle\prod_{i=1}^{m}c(z_i)\prod_{j=1}^{n}b(z_j)\right\rangle\ .\end{aligned}$$

Therefore, we can conclude that $\#c-\#b=3-3g$, and similarly, $\#\overline{c}-\#\overline{b}=3-3g$.

6.1.4 Ghost action and the zero modes

Keeping the zero modes in mind we redo the BRST quantization.

$$\begin{aligned}S_{\rm gh}&=Q_B\int\frac{d^2x}{4\pi}\sqrt{\widehat{h}}\left(h^{ab}-\widehat{h}^{ab}(t)\right)\\&\simeq\int\frac{d^2x}{4\pi}\left\{B_{ab}\left(h^{ab}-\widehat{h}^{ab}(t)\right)-b_{ab}\left((P\cdot c)^{ab}-\frac{\partial}{\partial t_k}\widehat{h}^{ab}(t)\xi_k\right)+4\pi\eta_{ai}c^a(\widehat{\sigma}_i)+\cdots\right\}\ ,\end{aligned}$$

where $\cdots$ is an abbreviation of vanishing terms when $h_{ab}=\widehat{h}_{ab}(t)$, i runs for $\#c+\#\overline{c}\equiv\mu$ and k runs for $\#b+\#\overline{b}\equiv\nu$. With this gauge fixing the string amplitude is written as follows.

$$\begin{aligned}A_n=&\sum_g\int d^\mu t\int\mathcal{D}b\mathcal{D}c\mathcal{D}X e^{-S_\sigma[X,h]-S_{\rm gh}[b,c]-\lambda\chi}\\&\times\prod_{l=\mu/2+1}^{n}\int d^2z_l\prod_{i=1}^{\mu/2}c\overline{c}(\sigma_i)\prod_{k=1}^{\nu}(b,\partial_k h)\prod_{j=1}^{n}g_{\rm st}\sqrt{h}V_j\ .\end{aligned}$$

where χ is the Euler number, λ is a vacuum expectation value of a dilaton, $g_{\rm st}=e^\lambda$, and

$$(b,\partial_k h)=\int\frac{d^2x}{4\pi}\sqrt{\widehat{h}}\ b_{ab}\frac{\partial}{\partial t_k}\widehat{h}^{ab}(t)\ .$$

This is **the most general amplitude formula** we will consider in the remaining part.

6.2 Tree amplitude

Let us consider $g=0$ case, namely, sphere case. In this case we have $\mu=6$ and $\nu=0$. Then, the amplitude formula becomes

$$
\begin{aligned}
A_{g=0,n} &= g_{\rm st}^{-2}\int \mathcal{D}b\mathcal{D}c\mathcal{D}X e^{-S_\sigma[X,h]-S_{\rm gh}[b,c]}\prod_{l=4}^{n}\int d^2z_l\prod_{i=1}^{3}c\bar{c}(\sigma_i)\prod_{j=1}^{n}g_{\rm st}\sqrt{h}V_j\\
&= g_{\rm st}^{n-2}\prod_{l=4}^{n}\int d^2z_l\left\langle\prod_{j=1}^{n}\sqrt{h}V_j\right\rangle_X \langle c\bar{c}(z_1)c\bar{c}(z_2)c\bar{c}(z_3)\rangle_{bc}\ ,
\end{aligned}
$$

where we divided the amplitude to the matter sector and the ghost sector. Note that there is no easy definition for less than 3pt amplitude.

Though we can consider any vertex operators we will focus on the simplest example, which is tachyon vertex operators:

$$
V_j = :e^{ik_j\cdot X(z_j,\bar{z}_j)}:\ .
$$

6.2.1 Matter sector

In the following argument we will focus on the important part. To be precise we need subtle argument for metric and zero modes. However, all of the subtleties vanish in the end. (For the interested students see [Pol98, §6.2])

Let us consider a generating functional

$$
Z[J] = \left\langle \exp\left(i\int\sqrt{h}d^2zJ(z,\bar{z})'\cdot X(z,\bar{z})\right)\right\rangle\ .
$$

Expand X^μ in a complete set X_I:

$$
\begin{aligned}
&X^\mu = \sum_I x_I^\mu X_I\ ,\\
&\Delta X_I = -\omega_I^2 X_I\ ,\\
&\int\sqrt{h}d^2zX_IX_{I'} = \delta_{II'}\ .
\end{aligned}
$$

Then,

$$
Z[J] = \prod_{I,\mu}\int dx_I^\mu\ \exp\left(-\frac{\omega_I^2X_I\cdot X_I}{4\pi\alpha'}+ix_I\cdot J_I\right)\ ,
$$

where

$$
J_I^\mu = \int\sqrt{h}d^2zJ^\mu X_I\ .
$$

Note that for the zero mode $\omega_I = 0$ and hence the integral leads delta function. So the result is

$$
\begin{aligned}
Z[J] &= C(2\pi)^D \delta^D(J_0) \prod_{I\neq 0} \exp\left(-\frac{\pi\alpha' J_I \cdot J_I}{\omega_I}\right) \\
&= C(2\pi)^D \delta^D(J_0) \prod_{I\neq 0} \exp\left(-\frac{1}{2}\int h d^2 z d^2 w J(z)\cdot J(w) G(z,w)\right) \ ,
\end{aligned}
$$

where $G(z,w) = -\frac{\alpha'}{2}\log|z-w|^2$.

When we consider tachyon vertex operators it corresponds to

$$
J(z) = \sum_{i=1}^{n} k_i \delta^2(z - z_i) \ .
$$

Hence, the amplitude is

$$
\left\langle \prod_{j=1}^{n} \sqrt{h} V_j \right\rangle_X = C_X (2\pi)^D \delta^D(\sum k_i) \prod_{i<j}^{n} |z_{ij}|^{\alpha' k_i \cdot k_j} \ .
$$

Note that we ignored the divergent part that is coming from $i = j$ because the vertex operators are normal ordered ones.

6.2.2 Ghost sector

For the ghost sector

$$
\langle c\bar{c}(z_1) c\bar{c}(z_2) c\bar{c}(z_3) \rangle_{bc} \ ,
$$

we only need to consider the zero modes, which is

$$
c(z) = c_0 + c_1 z + c_2 z^2 \ .
$$

Therefore,

$$
\langle c\bar{c}(z_1) c\bar{c}(z_2) c\bar{c}(z_3) \rangle_{bc} = C_{bc} \int \prod_{i=0}^{2} d\bar{c}_i dc_i \ c\bar{c}(z_1) c\bar{c}(z_2) c\bar{c}(z_3) = C_{bc} |z_{12}|^2 |z_{23}|^2 |z_{31}|^2 \ .
$$

6.2.3 Shapiro-Virasoro amplitude

Let us see 4pt amplitude of tachyons.

$$
A_{0,4} = g_{\rm st}^2 C_{\rm 4pt} (2\pi)^D \delta^D(\sum k_i) \int d^2 z_4 \prod_{i<j}^{4} |z_{ij}|^{\alpha' k_i \cdot k_j} \prod_{i<j}^{3} |z_{ij}|^2 \ .
$$

Since we have 6 CKG we can set (z_1, z_2, z_3) to $(0, 1, \infty)$, then the expression reduces to

$$
A_{0,4} = g_{\rm st}^2 C_{\rm 4pt} (2\pi)^D \delta^D(\sum k_i) B\left(-\frac{\alpha' s}{4} - 1, -\frac{\alpha' t}{4} - 1, -\frac{\alpha' u}{4} - 1\right) \ ,
$$

where

$$
\begin{aligned}
B(a,b,c) &= \int d^2z |z|^{2a-2}|1-z|^{2b-2} = \pi\frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(a+b)\Gamma(b+c)\Gamma(c+a)} \ , \\
s &= -k_{1+2}^2 = -k_{3+4}^2 = -2k_1\cdot k_2 - \frac{8}{\alpha'} \ , \\
t &= -k_{1+3}^2 = -k_{2+4}^2 = -2k_1\cdot k_3 - \frac{8}{\alpha'} \ , \\
s &= -k_{1+4}^2 = -k_{2+3}^2 = -2k_1\cdot k_4 - \frac{8}{\alpha'} \ , \qquad (s+t+u = -\frac{16}{\alpha'}) \ .
\end{aligned}
$$

This is called Shapiro-Virasoro amplitude. For the derivation of the B function see [GSW87, §3.2] Vol.1 pp.386 and pp.373.

Let us discuss the poles in the amplitude so that we can see what kind of states propagate. The s-channel poles (see Fig. 8) are

$$
-\frac{\alpha' s}{4} - 1 \in \mathbb{Z}_{\leq 0}, \qquad s = -k_{1+2}^2 = m^2 = \frac{4}{\alpha'}(n-1) \quad (n \in \mathbb{Z}_{\geq 0}) \ ,
$$

where m is a mass of intermediate states. As it shows the intermediate states could be tachyon, graviton etc.

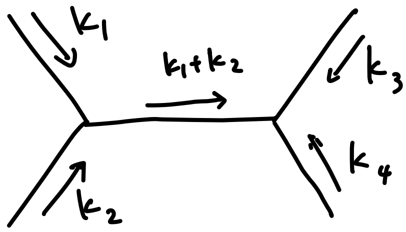


Figure 8: s-channel.

We can fix the overall constant from unitarity (see Fig. 9). The 4pt amplitude

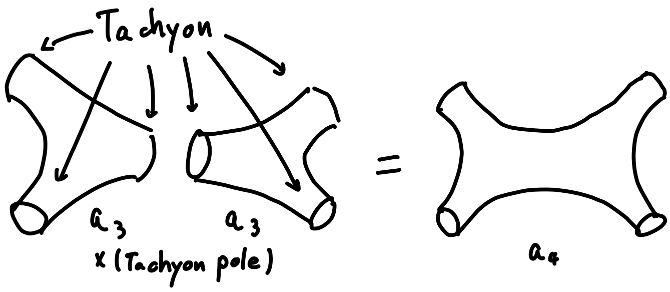


Figure 9: Unitarity requirement.

can be divided into two 3pt amplitude as in Fig. At s-channel tachyon pole it becomes

$$
\begin{aligned}
A_{0,3} &= a_3(2\pi)^D\delta^D(\sum k_i) \ , \qquad a_3 \simeq C g_{\text{st}} \ , \\
A_{0,4} &= a_4(2\pi)^D\delta^D(\sum k_i) \ , \qquad a_4 \simeq -\frac{4\pi}{\alpha'}\frac{1}{s+\frac{4}{\alpha'}} C g_{\text{st}}^2 \ .
\end{aligned}
$$

Unitarity requires

$$a_4 = \frac{(a_3)^2}{s + \frac{4}{\alpha'}} , \qquad \therefore \quad C = -\frac{4\pi}{\alpha'} .$$

7 Lecture 7

We have seen so far that bosonic strings suffer from two major problems:

• Their spectrum always contains a tachyon. In that respect their vacuum is unstable.

• They do not contain spacetime fermions. This lack of fermionic states is in contrast to observations and makes the bosonic string unrealistic.

Both of these challenges are remedied in superstring theory. Supersymmetry is a symmetry that exchanges bosons and fermions. The worldsheet superstring theory consists of a bosonic and a fermionic sector. The bosonic sector is identical to the worldsheet theory of the bosonic string. We can therefore view our efforts up to now as a preliminary study of one half of the superstring theory. In fact, we will see in this lecture that the presence of fermions resolves the problem of Tachyon. Moreover, we will learn that the critical dimension of superstring theory is $D = 10$.

There are five superstring theories as follows and we will study them in this order.

Type IIA & IIB

Closed oriented strings (if there is no D-brane). IIA: Ramond ground states with opposite chirality. IIB: Ramond ground states with same chirality.

Type I

Open and closed unoriented strings, including Yang-Mills degrees of freedom with SO(32) gauge group.

Heterotic SO(32) **&** $E_8 \times E_8$

Type II right-movers & bosonic left-movers, including Yang-Mills degrees of freedom with either SO(32) or $E_8 \times E_8$ gauge group.

There exist two major formulations of superstring theory. Both formulations enjoy supersymmetry on the worldsheet and in spacetime, but they differ in the following respect:

• In the **Ramond-Neveu-Schwarz (RNS) formulation**, supersymmetry is manifest on the worldsheet, but not in spacetime.

• In the **Green-Schwarz (GS) formulation** [GSW87, §5] [BBS06, §5], supersymmetry is manifest in spacetime, but not on the worldsheet .

More recently, the pure-spinor [Ber04] has been developed as yet another approach to the superstring. In this course, we will only discuss the RNS formalism.

7.1 RNS formulation

7.1.1 Superconformal field theories

With the complex coordinate convention, the action becomes

$$S^{\mathrm{m}} = \frac{1}{4\pi}\int d^2z\left(\frac{2}{\alpha'}\partial X^\mu\overline{\partial}X_\mu + \psi^\mu\overline{\partial}\psi_\mu + \widetilde{\psi}^\mu\partial\widetilde{\psi}_\mu\right) \tag{7.1}$$

where the equations of motion tell us $\psi^\mu(z)$ (resp. $\widetilde{\psi}^\mu(\overline{z})$) is chiral (resp. anti-chiral). The action is invariant under **supersymmetric transformation** (Exercise)

$$\delta X^\mu = -\sqrt{\frac{\alpha'}{2}}\left(\epsilon\psi^\mu + \overline{\epsilon}\widetilde{\psi}^\mu\right) , \qquad \delta\psi^\mu = \sqrt{\frac{2}{\alpha'}}\,\epsilon\partial X^\mu , \qquad \delta\widetilde{\psi}^\mu = \sqrt{\frac{2}{\alpha'}}\,\overline{\epsilon}\overline{\partial}X^\mu . \tag{7.2}$$

The Noether theorem implies that there are currents for the supersymmetry

$$T_F(z) = i\sqrt{\frac{2}{\alpha'}}\psi^\mu\partial X^\mu , \qquad \widetilde{T}_F(z) = i\sqrt{\frac{2}{\alpha'}}\widetilde{\psi}^\mu\overline{\partial}X^\mu . \tag{7.3}$$

which is called **supercurrents**. Indeed, X^μ, ψ^μ and $\widetilde{\psi}^\mu$ are primary fields of weight (0,0), $(\frac{1}{2},0)$ $(0,\frac{1}{2})$, respectively, and therefore their OPEs are

$$X^\mu(z,\overline{z})X^\nu(0,0) \sim -\sqrt{\frac{\alpha'}{2}}\eta^{\mu\nu}\ln|z|^2 , \quad \psi^\mu(z)\psi^\nu(0) \sim \frac{\eta^{\mu\nu}}{z} , \quad \widetilde{\psi}^\mu(\overline{z})\widetilde{\psi}^\nu(0) \sim \frac{\eta^{\mu\nu}}{\overline{z}} . \tag{7.4}$$

Using the OPEs, one can show the supersymmetric transformation (7.2) (exercise).

The stress-energy tensor of the action (15.1) is

$$T_B(z) = -\frac{1}{\alpha'}\,\partial_z X^\mu\partial X_\mu - \frac{1}{2}\psi^\mu\partial\psi_\mu \tag{7.5}$$

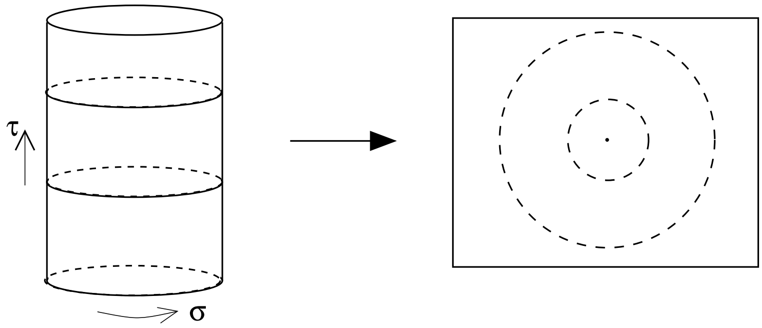
along with their complex conjugates $\widetilde{T}_B$, $\widetilde{T}_F$. Their OPE's can be computed by using (7.4)

$$\begin{aligned}
T_B(z)T_B(w) &\sim \frac{3D}{4(z-w)^4} + \frac{2T_B(w)}{(z-w)^2} + \frac{\partial_w T_B(w)}{z-w} \\
T_B(z)T_F(w) &\sim \frac{3T_F(w)}{2(z-w)^2} + \frac{\partial_w T_F(w)}{z-w} \\
T_F(z)T_F(w) &\sim \frac{D}{(z-w)^3} + \frac{2T_B(w)}{z-w} .
\end{aligned} \tag{7.6}$$

and similarly for the anti-chiral part. The central charge of the theory is

$$c^{\mathrm{m}} = \frac{3}{2}D \tag{7.7}$$

where each scalar and fermion contributes 1 and 1/2, respectively.



Ramond vs Neveu-Schwarz

In superstring theory, the fermionic fields on the closed string may be either periodic or anti-periodic on the circle around the string, corresponding to two different spinor bundles. It is conventional to denote these spin structures by **Ramond (R)** and **Neveu-Schwarz (NS)**, defined as follows.

$$\begin{aligned} &\psi^\mu(t,\sigma+2\pi) = +\psi^\mu(t,\sigma) && \text{R: periodic on cylinder} \\ &\psi^\mu(t,\sigma+2\pi) = -\psi^\mu(t,\sigma) && \text{NS: anti-periodic on cylinder} \end{aligned} \tag{7.8}$$

As we have seen before, the mapping $z = e^{iw}$ from the cylinder $w = -it - \sigma$ to the 2-plane $z \in \mathbb{C}$ is a conformal map (Figure above). Under the conformal map, the primary field ψ^μ with weight $(\frac{1}{2}, 0)$ is transformed as

$$\psi^\mu(z) = \left(\frac{dz}{dw}\right)^{-\frac{1}{2}} \psi^\mu(w) = \text{const} \times e^{-i\frac{w}{2}} \psi^\mu(w) \ .$$

Hence the (anti-)periodicity assignments are reversed between the cylinders and the plane:

$$\begin{aligned} &\psi^\mu(e^{2\pi i} z) = -\psi^\mu(z) && \text{R: anti-periodic on plane} \\ &\psi^\mu(e^{2\pi i} z) = +\psi^\mu(z) && \text{NS: periodic on plane} \end{aligned} \tag{7.9}$$

The boundary conditions for anti-chiral fields $\widetilde{\psi}^\mu$ are defined in a similar fashion.

As in the bosonic string, one can decompose ψ^μ and $\widetilde{\psi}^\mu$ in modes

$$\psi^\mu(z) = \sum_{n\in\mathbb{Z}+\nu} \frac{\psi^\mu_n}{z^{n+1/2}} \ , \qquad \widetilde{\psi}^\mu(\overline{z}) = \sum_{n\in\mathbb{Z}+\nu} \frac{\widetilde{\psi}^\mu_n}{\overline{z}^{n+1/2}}$$

where ν takes the values 0 (R) and $\frac{1}{2}$ (NS). The canonical quantization leads to the algebra

$$\{\psi^\mu_m, \psi^\nu_n\} = \eta^{\mu\nu}\delta_{m+n,0} \qquad \{\widetilde{\psi}^\mu_m, \widetilde{\psi}^\nu_n\} = \eta^{\mu\nu}\delta_{m+n,0} \ .$$

The mode expansion must be carried out with care here, since we must distinguish between Ramond and Neveu-Schwarz sectors.

$$T_B(z) = \sum_{m\in\mathbb{Z}} \frac{L_m}{z^{m+2}} \ , \qquad T_F(z) = \sum_{r\in\mathbb{Z}+\nu} \frac{G_r}{z^{r+3/2}} \ , \tag{7.10}$$

where the generators can be written in terms of the modes (exercise)

$$L^{\mathrm{m}}_m = \frac{1}{2}\sum_{n\in\mathbb{Z}} : \alpha_{m-n}\alpha_n : + \frac{1}{4}\sum_{r\in\mathbb{Z}+\nu}(2r-m) : \psi_{m-r}\psi_r : + a^{\mathrm{m}}\delta_{m,0}$$
$$G^{\mathrm{m}}_r = \sum_{n\in\mathbb{Z}} \alpha_n\psi_{r-n} \ . \tag{7.11}$$

The normal ordering constant a can be determined like in the bosonic string theory. Each periodic boson contributes $-\frac{1}{24}$. The fermionic contributions are

$$-\tfrac{1}{2}\sum_{r=0}^{\infty} r = \tfrac{1}{24} \qquad \text{R-sector}$$
$$-\tfrac{1}{2}\sum_{r=0}^{\infty}(r+\tfrac{1}{2}) = -\tfrac{1}{48} \qquad \text{NS-sector} \ .$$

Including the shift $\frac{1}{24}c = \frac{1}{16}D$ gives

$$a^{\mathrm{m}} = \tfrac{1}{24}c^{\mathrm{m}} + \Big(-\tfrac{1}{24}+\tfrac{1}{24}\Big)D = \tfrac{1}{16}D \qquad \text{R-sector}$$
$$a^{\mathrm{m}} = \tfrac{1}{24}c^{\mathrm{m}} + \Big(-\tfrac{1}{24}-\tfrac{1}{48}\Big)D = 0 \qquad \text{NS-sector} \ . \tag{7.12}$$

In fact, the generators L_m and G_r form the algebra called the $\mathcal{N}=1$ **superconformal algebra** with central charge (7.7) (exercise).

Ghost CFT

In bosonic string theory, we study BRST quantization with Faddeev-Popov ghost. In superstring theory, ghost fields also appear with their supersymmetric partners:

$$S^{\mathrm{gh}} = \frac{1}{2\pi}\int d^2z(b\overline{\partial}c + \beta\overline{\partial}\gamma)$$

where b, c are fermionic and β, γ are bosonic fields. Hence, the standard method tells us the $\beta\gamma$ OPEs

$$\gamma(z)\beta(w) = -\beta(z)\gamma(w) = \frac{1}{z-w} + \cdots$$

We have seen that the weights of X and ψ differ by $\frac{1}{2}$. This is the same for the ghost fields. Sine the b and c ghosts have weights 2 and -1 respectively, β and γ are primary fields of weights $(\frac{3}{2},0)$ and $(-\frac{1}{2},0)$ respectively. Hence the form of the stress energy tensor and the supercurrent are

$$T^{\mathrm{gh}}_B(z) =: (\partial b)c : -2\,\partial : bc : + : (\partial\beta)\gamma : -\frac{3}{2}\,\partial : \beta\gamma :$$
$$T^{\mathrm{gh}}_G(z) = (\partial\beta)c + \frac{3}{2}\beta\partial c - 2b\gamma \ . \tag{7.13}$$

Then, the TT OPE to determine the central charge of the ghost SCFT. The bc system contributes -26 to the central charge as we know while the $\beta\gamma$ system contributes $+11$. Hence the total central charge

$$c^{\text{tot}} = c^{\text{m}} + c^{\text{gh}} = \tfrac{3}{2}D - 26 + 11 \ .$$

Then, we happily obtain the critical dimension $D = 10$ of superstring theory if we impose the Weyl-anomaly-free condition $c^{\text{tot}} = 0$. In the following, we assume $D = 10$.

Now let us write find the Virasoro generator of the ghost ghost SCFT. The $\beta\gamma$ ghosts have the same boundary condition as the fermionic fields $\psi^\mu \widetilde{\psi}^\mu$ so that we have the mode expansions

$$\beta(z) = \sum_{r\in\mathbb{Z}+\nu} \frac{\beta_r}{z^{r+\frac{3}{2}}} \ , \quad \gamma(z) = \sum_{r\in\mathbb{Z}+\nu} \frac{\gamma_r}{z^{r-\frac{1}{2}}} \ ,$$

which satisfy the commutation relation

$$[\beta_r, \gamma_s] = \delta_{r,-s} \ .$$

Using these modes, the Virasoro generator of the ghost ghost SCFT can be expressed as

$$\begin{aligned} L_m^{\text{gh}} &= \sum_{n\in\mathbb{Z}} (m+n) : b_{m-n} c_n : + \frac{1}{2} \sum_{r\in\mathbb{Z}+\nu} (m+2r) : \beta_{m-r}\gamma_r : + a^{\text{gh}} \delta_{m,0} \\ G_r^{\text{gh}} &= \sum_{n\in\mathbb{Z}} \left[\frac{1}{2}(n+2r)\beta_{r-n} c_n + 2 b_n \gamma_{r-n} \right] \end{aligned} \tag{7.14}$$

Again, using the commutation relations of the ghost modes, one can determine the normal ordering constant

$$\begin{aligned} a^{\text{gh}} &= \tfrac{-15}{24} + \left(\tfrac{1}{12} - \tfrac{1}{12} \right) = -\tfrac{5}{8} \qquad \text{R-sector} \\ a^{\text{gh}} &= \tfrac{-15}{24} + \left(-\tfrac{1}{12} + \tfrac{1}{24} \right) = -\tfrac{1}{2} \qquad \text{NS-sector} \ . \end{aligned}$$

Combining them with (7.12) at $D = 10$, we have the total vacuum energy

$$\begin{aligned} a^{\text{tot}} &= 0 \qquad \text{R-sector} \\ a^{\text{tot}} &= -\tfrac{1}{2} \qquad \text{NS-sector} \ . \end{aligned}$$

In R sector, the vacuum energy is zero so that the Tachyon is absent. On the other hand, there is still the Tachyon in NS sector. This will be projected out by the GSO projection as we will see below.

Fermion spectrum

Before discussing about the GSO projection, let us study the fermionic spectrum generated by fermionic modes ψ^μ_r. We first consider NS spectrum since it's simpler. Since r takes half integers, we can define the ground state of NS sector as

$$\psi^\mu_r|0;k\rangle_{\rm NS}=0 \quad \text{for } r>0 \ .$$

When we include the ghost part of the vertex operator, it contributes to the total fermion number F, so that on the total matter plus ghost ground state one has the odd fermion number

$$(-1)^F|0;k\rangle_{\rm NS}=-|0;k\rangle_{\rm NS} \ . \tag{7.15}$$

Because there exists the zero modes ψ^μ_0, R-sector is more subtle. In fact, the zero modes satisfy the **Clifford algebra**

$$\{\sqrt{2}\psi^\mu_0,\sqrt{2}\psi^\nu_0\}=2\eta^{\mu\nu} \ .$$

Therefore, the ground state of R-sector becomes the spin representation of $\mathrm{SO}(1,D-1)$. In this lecture, we do not cover mathematics of Clifford algebra, spin group, spin representations. (See [Pol98, Appendix B] for physics and [Mei13, §3] for mathematics) However, we can heuristically understand the spin representation as follows. The following basis for this representation is often convenient. Form the combinations

$$\begin{aligned}\Gamma^\pm_i &= \frac{1}{\sqrt{2}}\left(\psi^{2i}_0 \pm i\psi^{2i+1}_0\right) \qquad i=1,\dots,4\\ \Gamma^\pm_0 &= \frac{1}{\sqrt{2}}\left(\psi^1_0 \mp \psi^0_0\right)\end{aligned} \tag{7.16}$$

In this basis, the Clifford algebra takes the form

$$\{\Gamma^+_i,\Gamma^-_j\}=\delta_{ij} \ , \qquad \{\Gamma^+_i,\Gamma^+_j\}=0=\{\Gamma^-_i,\Gamma^-_j\} \ . \tag{7.17}$$

The $\Gamma^\pm_i$, $i=0,\dots,4$ act as raising and lowering operators, generating the $2^5=32$ Ramond ground states:

$$|s_0,s_1,s_2,s_3,s_4;k\rangle=|\mathbf{s};k\rangle \tag{7.18}$$

where each of the s_i is $\pm\frac{1}{2}$, and where

$$\Gamma^-_i|-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2};k\rangle=0 \tag{7.19}$$

while Γ^+_i raises s_i from $-\frac{1}{2}$ to $\frac{1}{2}$. One can further define the chirality operator

$$\Gamma_{11}=(2)^5\psi^0_0\psi^1_0\psi^2_0\cdots\psi^9_0 \ ,$$

which acts on $|\mathbf{s}\rangle$ as

$$\Gamma_{11}|\mathbf{s};k\rangle=(-1)^F|\mathbf{s};k\rangle=\begin{cases}+|\mathbf{s};k\rangle & \text{even \# of } -\frac{1}{2}\\ -|\mathbf{s};k\rangle & \text{odd \# of } -\frac{1}{2}\end{cases} \ .$$

Hence, the Dirac representation $\mathbf{32}$ decomposes into a $\mathbf{16}$ with an even number of $-\frac{1}{2}$'s and $\mathbf{16'}$ with an odd number.

$$\mathbf{32}=\mathbf{16}\oplus\mathbf{16'} \ .$$

7.2 Type II string theories

Physical spectrum and the GSO Projection

Finally, let us study the physical spectrum of superstring theories. In principle, we can apply BRST quantization scheme. However, we do not need the full use of BRST quantization, indeed. To take shortcut, we can first impose to a physical state $|\psi\rangle$

$$L_n^{\mathrm{m}}|\psi\rangle = 0 \quad (n > 0) \ , \qquad G_r^{\mathrm{m}}|\psi\rangle = 0 \quad (r \geq 0) \ .$$

Since one can check (See [Pol98, (10.5.23)])

$$\{Q_B, b_n\} = L_n \ , \qquad [Q_B, \beta_r] = G_r \ ,$$

the physical states are defined modulo

$$L_n^{\mathrm{m}}|\chi\rangle \cong 0 \ , \qquad G_r^{\mathrm{m}}|\chi\rangle \cong 0 \ , \qquad \text{for} \ \ n, r < 0 \ .$$

Note that the BRST current is defined

$$j_B = cT_B^{\mathrm{m}} + \gamma T_F^{\mathrm{m}} + \frac{1}{2}\left(cT_B^{\mathrm{gh}} + \gamma T_F^{\mathrm{gh}}\right) \ .$$

Furthermore, in the RNS theory, we need to impose the **GSO (Gliozzi-Scherk-Olive) projection** in order to have an equal number of bosonic and fermionic states at each mass level.

In NS sector, the GSO projection is just to remove states with odd fermion number so that the GSO projection operator on NS sector is expressed as

$$P_{\mathrm{GSO}} = \frac{1 + (-1)^F}{2} \qquad \text{NS sector} \ .$$

At level 0, we have the Tachyon $|0; k\rangle_{\mathrm{NS}}$. However, the GSO projection removes this state because it has odd fermion number as in (7.15). At level $\frac{1}{2}$, we have massless state with vector polarization

$$|e; k\rangle_{\mathrm{NS}} = e \cdot \psi_{-\frac{1}{2}}|0; k\rangle_{\mathrm{NS}} \ ,$$

with even fermion number so we need to keep it in the spectrum. The physical state conditions are

$$\begin{aligned} 0 &= L_0|e; k\rangle_{\mathrm{NS}} = \alpha' k^2 |e; k\rangle_{\mathrm{NS}} \\ 0 &= G_{\frac{1}{2}}^{\mathrm{m}}|e; k\rangle_{\mathrm{NS}} = \sqrt{2\alpha'}\ k \cdot e|0; k\rangle_{\mathrm{NS}} \ . \end{aligned} \tag{7.20}$$

while there is a Q_B-exact condition

$$G_{-\frac{1}{2}}^{\mathrm{m}}|0; k\rangle_{\mathrm{NS}} = \sqrt{2\alpha'}\ k \cdot \psi_{-\frac{1}{2}}|0; k\rangle_{\mathrm{NS}} \ .$$

Therefore, we have

$$k^2 = 0 \ , \quad k \cdot e = 0 \ , \quad e^\mu \cong e^\mu + k^\mu \ .$$

Thus, there are degrees of freedom for 8 spacelike polarizations which form the vector representation $\mathbf{8_v}$ of SO(8).

In R sector, the GSO projection keeps one irreducible representation, and therefore the GSO projection operator can be written as

$$P^{\pm}_{\rm GSO} = \frac{1 \pm (-1)^F}{2} \qquad \text{R sector} \ .$$

Hence, there are the two choices **16** and **16′** which differ by a spacetime parity redefinition. A Ramond ground state that is massless can be expressed with spinor polarization

$$|u;k\rangle_{\rm R} = u_{\mathbf{s}}|\mathbf{s};k\rangle_{\rm R} \ .$$

The physical condition

$$0 = G^{\rm m}_0|u;k\rangle_{\rm R} = \sqrt{\alpha'}\ u_{\mathbf{s}} k \cdot \Gamma_{\mathbf{ss'}}|\mathbf{s}';k\rangle_{\rm R}$$

leads to the Dirac equation

$$u\ k \cdot \Gamma = 0 \ .$$

By choosing the momentum vector $k^\mu = (k, k, 0, \ldots, 0)$, this amounts to

$$\Gamma^+_0 u = 0 \quad \longrightarrow \quad s_0 = +\tfrac{1}{2} \ ,$$

giving a 16 degeneracy $|+, s_1, s_2, s_3, s_4\rangle$ for the physical Ramond vacuum. This is a representation of SO(8) which again decomposes into $\mathbf{8_s}$ with an even number of $-\frac{1}{2}$'s and $\mathbf{8_c}$ with an odd number.

Type II Superstrings

Now let us determine the closed string spectrum. In the closed string, there are two inequivalent choices, taking the same (IIB) or opposite (IIA) projections on the right- and left-moving spectrum. These lead to the massless sectors

$$\begin{aligned} \text{Type IIA}&: (\mathbf{8_v} \oplus \mathbf{8_s}) \otimes (\mathbf{8_v} \oplus \mathbf{8_c}) \\ \text{Type IIB}&: (\mathbf{8_v} \oplus \mathbf{8_s}) \otimes (\mathbf{8_v} \oplus \mathbf{8_s}) \end{aligned} \tag{7.21}$$

of SO(8). Although one can also choose

$$\begin{aligned} \text{Type IIA}'&: (\mathbf{8_v} \oplus \mathbf{8_c}) \otimes (\mathbf{8_v} \oplus \mathbf{8_s}) \\ \text{Type IIB}'&: (\mathbf{8_v} \oplus \mathbf{8_c}) \otimes (\mathbf{8_v} \oplus \mathbf{8_c}) \end{aligned} \tag{7.22}$$

they are equivalent after the spacetime parity redefinition.

IIA	$\mathbf{8_v}$	$\mathbf{8_c}$
$\mathbf{8_v}$	$\mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}$ $\phi \quad B_{\mu\nu} \quad G_{\mu\nu}$	$\mathbf{8_s} \oplus \mathbf{56_c}$ $\lambda^+ \quad \psi_m^-$
$\mathbf{8_s}$	$\mathbf{8_c} \oplus \mathbf{56_s}$ $\lambda^- \quad \psi_m^+$	$\mathbf{8_v} \oplus \mathbf{56_t}$ $C_n \quad C_{nmp}$

IIB	$\mathbf{8_v}$	$\mathbf{8_s}$
$\mathbf{8_v}$	$\mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}$ $\phi \quad B_{\mu\nu} \quad G_{\mu\nu}$	$\mathbf{8_c} \oplus \mathbf{56_s}$ $\lambda^+ \quad \psi_m^-$
$\mathbf{8_s}$	$\mathbf{8_c} \oplus \mathbf{56_s}$ $\lambda^+ \quad \psi_m^-$	$\mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}_+$ $C \quad C_{mn} \quad C_{mnpq}$

The various products are as follows. In the NS-NS sector, this is the same as bosonic string theory

$$\mathbf{8_v} \otimes \mathbf{8_v} = \phi \oplus B_{\mu\nu} \oplus G_{\mu\nu} = \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}. \tag{7.23}$$

In the R-R sector, the IIA and IIB spectra are respectively

$$\begin{aligned} \mathbf{8_s} \otimes \mathbf{8_c} &= [1] \oplus [3] = \mathbf{8_v} \oplus \mathbf{56_t} \\ \mathbf{8_s} \otimes \mathbf{8_s} &= [0] \oplus [2] \oplus [4]_+ = \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}_+ \ . \end{aligned} \tag{7.24}$$

Here $[n]$ denotes the n-the antisymmetric representation of SO(8), and we associate R-R n-form $C_{(n)}$ to it. Also, here $[4]_+$ means its R-R field strength $G_{(5)} = dC_{(4)}$ is self-dual

$$*G_{(5)} = G_{(5)} \ . \tag{7.25}$$

Note that the representations $[n]$ and $[8-n]$ are related by the Hodge dual so that they are related by contraction with the 8-dimensional ϵ-tensor. As we will see next, these R-R fields are associated to D-branes in Type II theories, and $[n]$ and $[8-n]$ are related by the electro-magnetic duality.

Let us first look at the massless fermionic sector. The tensor products of the NS-R and R-NS sectors are given by

$$\begin{aligned} \mathbf{8_v} \otimes \mathbf{8_c} &= \mathbf{8_s} \oplus \mathbf{56_c} \\ \mathbf{8_v} \otimes \mathbf{8_s} &= \mathbf{8_c} \oplus \mathbf{56_s} \ . \end{aligned} \tag{7.26}$$

The $\mathbf{56_{s,c}}$ correspond to gravitinos $\psi^{\pm}_{m,\alpha}$ that are superpartners of gravitons, and they have vector and one spinor indices by construction. As we will see in §**??**, they will give spacetime supersymmetry. The $\mathbf{8_{s,c}}$ are dilatino $\lambda^{\pm}_{\alpha}$ which are superpartners of the dilaton field.

8 Lecture 8

8.1 Introduction to D-branes

Physical problem of the oriented closed superstring theories (IIA, IIB) is a lack of gauge fields. On the other hand, we already know a gauge field appears from open string. Let us look into an open string further, then, we will see that open string needs an object, so called D-brane.

Note that closed superstring theories are consistent by itself. However, only open string theory is inconsistent, and it needs closed string as well (see Fig. 10).

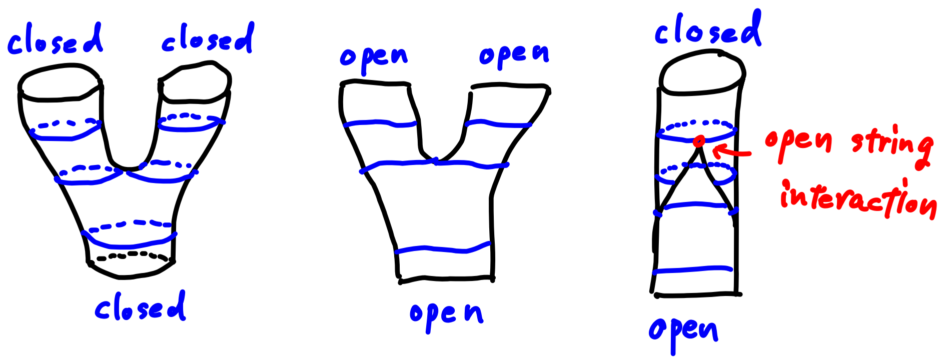


Figure 10: Open string interaction induces closed string

8.1.1 Boundary conditions & D-brane

Let us consider an open string action.

$$S_{\mathrm{E}}=\frac{1}{4\pi\alpha'}\int_{\sigma=0}^{\sigma=\pi}dtd\sigma\left(\partial_t X\cdot\partial_t X+\partial_\sigma X\cdot\partial_\sigma X\right)\ ,$$

where the subscript E shows the world-sheet(WS) is Euclidean. The variation principle leads

$$\begin{aligned}0=\delta S&=\frac{1}{2\pi\alpha'}\int_{\sigma=0}^{\sigma=\pi}dtd\sigma\left(\partial_t X\cdot\partial_t\delta X+\partial_\sigma X\cdot\partial_\sigma\delta X\right)\\&=\frac{1}{2\pi\alpha'}\int_{\sigma=0}^{\sigma=\pi}dtd\sigma\left[-\delta X\cdot\left(\partial_t\partial_t X+\partial_\sigma\partial_\sigma X\right)+\partial_t\left(\partial_t X\cdot\delta X\right)+\partial_\sigma\left(\partial_\sigma X\cdot\delta X\right)\right]\\&=\frac{1}{2\pi\alpha'}\int dt\left[\partial_\sigma X\cdot\delta X\right]_{\sigma=0}^{\sigma=\pi}\ .\end{aligned}$$

At the last equality we used E.O.M and $\delta X(t=\pm\infty)=0$. The equation compels us to impose one of either boundary conditions:

$$\begin{aligned}&\partial_\sigma X^\mu|_{\mathrm{bdy}}=0 && \text{Neumann condition}\\&\delta X^\mu|_{\mathrm{bdy}}=0\ \text{or}\ X^\mu=c^\mu\ (\text{const}) && \text{Dirichlet condition}\end{aligned}$$

From the homework 1 we saw that the solution for Dirichlet boundary condition express a momentum NON-conservation (see Fig. 11). It implies that there must be something into which the momentum goes. We call this object **D-brane**, where the D stands for Dirichlet and brane comes from membrane. D-brane is a dynamical object as it should receive the momentum, hence, it has action and interactions with string. On the other hand, in order to stay at a specific point $X^\mu=c^\mu$ D-brane must be infinitely heavy compared to string.

One can impose a certain number of Neumann condition and the rest is Dirichlet, let us set:

$$\begin{aligned}&\text{Neumann condition on } X^a && (a=0,1,\ldots,p)\\&\text{Dirichlet condition on } X^I && (I=p+1,\ldots,D-1)\end{aligned}$$

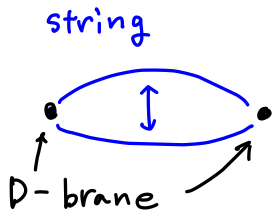


Figure 11: D-brane must exist at the ends of open string so that momentum can escape from string.

The corresponding D-brane is called Dp-brane (e.g. D3-brane). **Dp-brane** is a spatially p-dim object and space-timely $(p+1)$-dim object. Now we can visalize a configuration of D-brane and string as in Fig. 12.

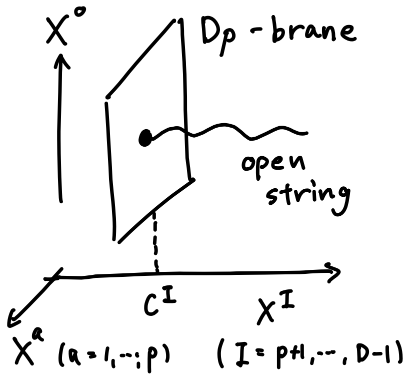


Figure 12: Visalization of Dp-brane and open string.

8.1.2 Chan-Paton factor

We can consider not only a D-brane but many D-branes, and a string now have an option on which D-brane a string ends. Let us label this option i $(i = 1, \ldots, n)$, which is called **Chan-Paton factor**. As an open string has two end points a string state has two Chan-Paton degree:

$$|N; k\rangle \quad \to \quad |N; k; ij\rangle \ .$$

Now we have n^2 massless vector states in both bosonic- or super-string theory. As usual we use $n \times n$ Hermite matrices T^a normalized to

$$\mathrm{Tr}(T^a T^b) = \delta^{ab} \ ,$$

which is a complete set for the end points:

$$|N; k; a\rangle = T^a_{ij} |N; k; ij\rangle \ .$$

You will find this is U(n) gauge fields if you compute a three point amplitude.

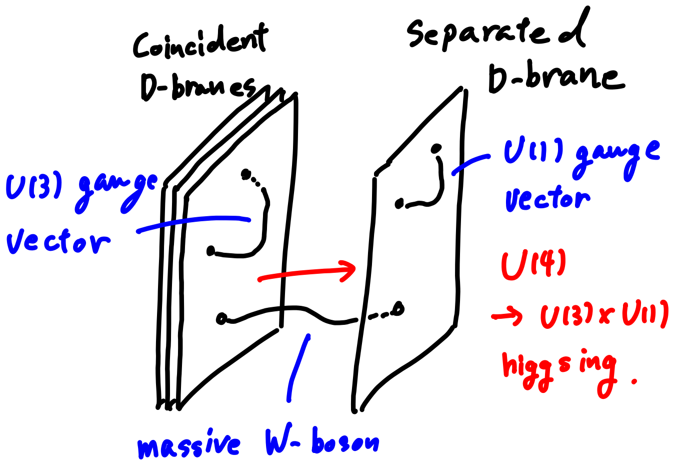


Figure 13: Many D-branes and Chan-Paton factors. U(n) gauge group and Higgs mechanism.

Note that in order to realize U(n) gauge group the D-branes must coincide at a point. Otherwise, the gauge group is broken by Higgs mechanism (see Fig. 13).

We will see, in next lectures, that other type of gauge groups (SO or Sp) can be realized in string theory.

8.1.3 D-brane in IIA/IIB superstring theory

In the previous lecture note we saw that IIA/IIB superstring theory has RR fields, which are anti-symmetric tensors and are analogy of gauge fields. Furthermore, again from an analogy of electro-magnetic dynamics, there are objects which electrically/magnetically coupled to the gauge fields like an electron/monopole. It is known that for RR fields such objects are D-branes. In order to justify the statement we need more elaborated tools. Here we simply see that which Dp-branes couple to RR fields in IIA/IIB.

Let us start with an electron/monopole in 4-dim spacetime. Electron is expressed by a source $J^\mu = (\rho, \mathbf{j}) = (q\delta^3(\mathbf{r} - \mathbf{r}(t)), \partial_t \mathbf{r}(t)\rho)$ and its coupling to gauge field is written by

$$S_J = q \int A_\mu J^\mu d^4x = q \int_C A_\mu dx^\mu = q \int_C A \ ,$$

where q is an electric charge, C is a worldline of the electron, and A is understood as 1-form. Similarly, Dp-brane ($p \leq 3$) couples to $(p+1)$-tensor RR field as follows.

$$S_{Dp} = \int_{M_{p+1}} C_{p+1} \ ,$$

where C_{p+1} is a $(p+1)$-form RR gauge potential and M_{p+1} is a world-membrane of Dp-brane. Note that an exteriror deribative of the potential gives its field strength

$G_{p+1} = dC_p$. For example of D3-brane is:

$$S_{D3} = \int_{M_4} C_4 = \frac{1}{4!}\int_{M_4} C_{\mu\nu\rho\sigma} dX^\mu \wedge dX^\nu \wedge dX^\rho \wedge dX^\sigma$$
$$= \frac{1}{4!}\int_{M_4} C_{\mu\nu\rho\sigma}\epsilon^{abcd}\frac{\partial X^\mu}{d\sigma^a}\frac{\partial X^\nu}{d\sigma^b}\frac{\partial X^\rho}{d\sigma^c}\frac{\partial X^\sigma}{d\sigma^d} d^4\sigma \ ,$$

where σ^a is a world-sheet coordinate of D3-brane and X^μ is an embeding of D3-brane into space-time. The reason why it is limited up to $p = 3$ is because we only have up to 4-tensor RR-field in IIA/IIB superstring theory.

Monopole is characterized by maginetic flux measured at surface surrounding the monopole (see Fig. 14)

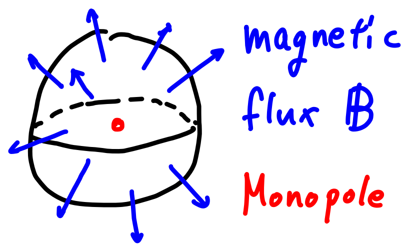


Figure 14: Monopole is measured by magnetic flux configuration.

$$q_m = \int_S B \cdot n dS = \int_{B_3} \nabla \cdot B dV \ ,$$
$$= \int_{\partial B} F_2 = \int_B dF_2 \ ,$$

where q_m is a magnetic charge, B is magnetic flux, S is a surface surrounding the monopole, B_3 is a 3-dim ball, whose boundary is S, B in the second line is B_3, and ∂B is a surface of B, which is S. Note that F_2 cannot be expressed by an exterior derivative dA globally, hence, $dF_2 \neq 0$, rather $dF_2 = \nabla \cdot B = q_m \delta^3(r)$. If we extend this idea to Dp-brane ($p \geq 3$)

$$q_m = \int_{\partial B_{D-(p+1)}} G_{D-(p+2)} = \int_{B_{D-(p+1)}} dG_{D-(p+2)} \ .$$

See Fig. 15 to understand the direction of the flux $G_{D-(p+2)}$, which leads potential $C_{D-(p+3)}$. The reason why $p \geq 3$ is because $D = 10$ and the highest tensor is [4].

If you allow to use electro-magnetic duality it is more easily understood (see Fig. 16).

String or fundamental string, denoted as F1, is electrically coupled to B-field. On the other hand, there is an object that is magnetically coupled to B-field, which is called **NS5-brane**.

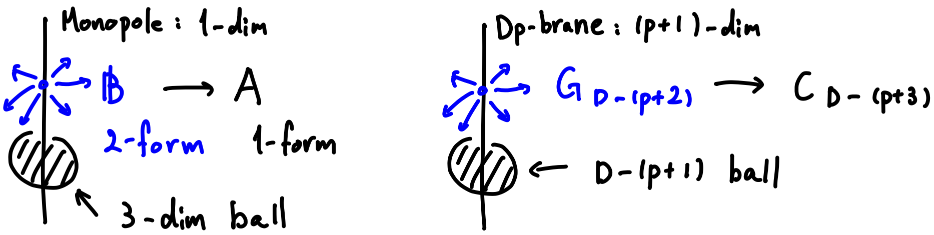


Figure 15: Higher dimensional analog of monopole: magnetically coupled Dp-brane.

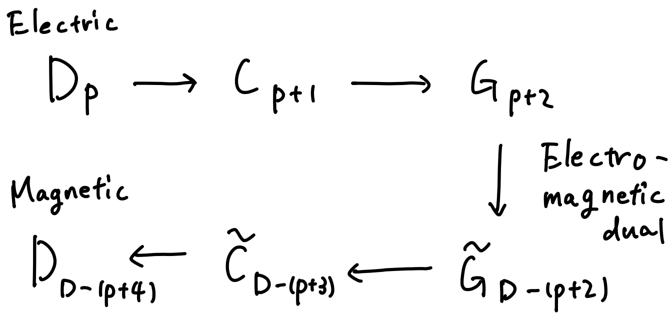


Figure 16: Electro-magnetic duality and D-brane.

Now you should notice that in IIA/IIB string theory the number p of Dp-brane is limited to a certain number. This is because the rank of RR-field (tensor) is limited. Let us summarize which D-brane can exist in which superstring theory (see Table 6).

Comments:

Table 2: Possible D-branes in IIA/IIB superstring, fundamental string, and NS5-brane.

IIA	B_2	C_1	C_3
Electric	F1	D0	D2
Magnetic	NS5	D6	D4

IIB	B_2	C_0	C_2	C_4^+
Electric	F1	D(-1)	D1	D3
Magnetic	NS5	D7	D5	D3

- Only Dp-brane for even p exist in IIA and for odd p in IIB.
- D(-1)-brane is timely localized object (like instanton).
- D8-brane can exist in IIA, which is a non-dynamical object and there is no corresponding anti-symmetric tensor.

8.2 1-loop amplitude

We continue to study amplitude, here it is 1-loop amplitude. 1-loop amplitude is important because it includes quantum information, such as anomaly.

8.2.1 Torus and its modulus & CKG

Here, we will briefly see $g = 1$ case, which is nothing but torus. Torus can be constructed from 1d flat complex space $\mathbb{C}$ by identification (see Fig. 17)

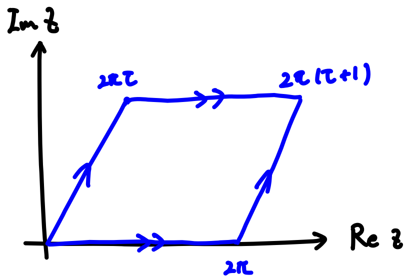


Figure 17: Torus from $\mathbb{C}$ by the identification.

$$z \sim z + 2\pi \sim z + 2\pi\tau \ ,$$

where $\tau = \tau_1 + i\tau_2$ and assume $\tau_2 > 0$. Since the metric is flat the τ is a complex modulus, which describe the "shape" of the torus. There is also a complex translation invariance, which is 2 CKG.

Torus has so called a **modular invariance**. Let us define two generators:

$$T : \tau \to \tau + 1 \ , \quad S : \tau \to -\frac{1}{\tau} \ .$$

These $T,\ S$ generates modular transformation

$$\tau \to \frac{a\tau + b}{c\tau + d} \ , \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, \mathbb{Z}) \ ,$$

The "shape" of torus is conformally invariant under the modular transformation. Therefore, the parameter space of τ is limited to the fundamental region, see Fig. 18.

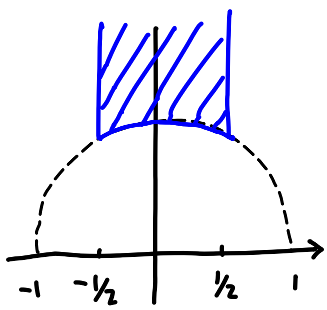


Figure 18: Fundamental region of τ.

As we did for sphere case we will encounter ghost expectation, whose contribution is only from the zero modes. In the torus case since the space is flat and finite, we can insert a WS integral as follows

$$\langle c(0)\rangle_{bc} = \langle C\rangle_{bc}\ , \quad C \equiv \int \frac{d^2z}{4\pi^2\tau_2} c(z)\ .$$

($\overline{C}$ and $B, \overline{B}$ are similarly defined.)

The string amplitude for torus case is

$$A_{1,n} = g_{\rm st}^n \frac{1}{2}\int_F d^2\tau \left\langle (b,\partial_\tau h)(b,\partial_{\overline{\tau}} h) c\overline{c}\ \sqrt{h} V_1 \prod_{j=2}^n \int d^2 z_j \sqrt{h} V_j \right\rangle\ .$$

As the same logic as before we can insert the integration even for z_1 since the amplitude should not depend on z_1:

$$A_{1,n} = g_{\rm st}^n \frac{1}{2}\frac{1}{4\pi^2\tau_2}\int_F d^2\tau \left\langle (b,\partial_\tau h)(b,\partial_{\overline{\tau}} h) C\overline{C}\ \right\rangle_{bc} \left\langle \prod_{j=1}^n \int d^2 z_j \sqrt{h} V_j \right\rangle_X\ .$$

Though seemingly the metric would not depend on τ, it does indeed depend on τ as follows. Let us consider small deformation of the metric

$$\begin{aligned} & ds^2 = dzd\overline{z} \to d(z+\epsilon\overline{z})d(\overline{z}+\overline{\epsilon}z)\ , \\ \Leftrightarrow \quad & \delta h_{zz} = \overline{\epsilon}\ , \quad \delta h_{\overline{z}\overline{z}} = \epsilon\ . \end{aligned}$$

New torus has coordinate $z+\epsilon\overline{z}$. This deformation changes the period to $(2\pi(1+\epsilon), 2\pi(\tau+\epsilon\overline{\tau}))$. Therefore the modulus becomes $\widetilde{\tau} = \frac{\tau+\epsilon\overline{\tau}}{1+\epsilon} \simeq \tau - 2i\epsilon\tau_2 \Leftrightarrow \delta\tau = -2i\epsilon\tau_2$. Now the τ derivative is

$$\partial_\tau h_{\overline{z}\overline{z}} = \lim_{\epsilon\to 0}\frac{\delta h_{\overline{z}\overline{z}}}{\delta\tau} = \frac{i}{2\tau_2}\ , \quad \Rightarrow \quad (b,\partial_\tau h) = 2\pi i B \quad \text{etc}\ ,$$

where

$$(b,\partial_\tau h) = \int \frac{d^2z}{4\pi}\sqrt{h}\ b_{ab}\frac{\partial}{\partial\tau}h^{ab}(\tau)\ .$$

8.2.2 Torus partition function for bosonic string

One may think the simplest amplitude would be tachyon amplitude, however, it is actually vacuum amplitude $n=0$, which can be calculated in torus case opposed to sphere case.

$$A_{1,0} = \frac{1}{2}\int_F \frac{d^2\tau}{\tau_2}\left\langle B\overline{B}C\overline{C}\ \right\rangle_{bc}\langle 1\rangle_X\ .$$

Matter sector

Using operator formalism

$$\langle 1\rangle_X = \mathrm{Tr}\exp\left[2\pi i\left(\tau_1 P + i\tau_2 H\right)\right] = \mathrm{Tr}\left[q^{L_0-\frac{c}{24}}\overline{q}^{\overline{L}_0-\frac{c}{24}}\right] \ ,$$

where $q = e^{2\pi i\tau}$, $P = L_0 - \overline{L}_0$, $H = L_0 + \overline{L}_0 - \frac{c}{12}$, and

$$L_0 = \frac{\alpha'}{4}p^2 + \sum_{n>0}\alpha_{-n}\cdot\alpha_n \ .$$

Note that $e^{-2\pi\tau_2 H}$ is a thermal factor or Euclideanized time translation, and $e^{2\pi i\tau_1 P}$ is a space translation. The result is (homework)

$$\begin{aligned}\langle 1\rangle_X &= \int\frac{d^Dxd^Dp}{(2\pi)^D}\exp\left(-4\pi\tau_2\frac{\alpha'}{4}p^2\right)\left|\frac{q^{-\frac{1}{24}}}{\prod_{n\geq 1}(1-q^n)}\right|^{2D} \\ &= i\frac{V_{26}}{(2\pi\ell_s)^{26}}\left(\tau_2\right)^{-13}\left|\eta(\tau)\right|^{-52} \ ,\end{aligned}$$

where $\ell_s = \sqrt{\alpha'}$, and $\eta(\tau) = q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n)$. i in the second line is from Wick-rotation of space-time momentum $p^0 \to ip_E^0$, which is needed to define Gaussian integral.

Ghost sector

Similarly,

$$\left\langle B\overline{B}C\overline{C}\ \right\rangle_{bc} = \mathrm{Tr}\left[(-1)^F b_0\overline{b}_0 c_0\overline{c}_0 q^{L_0-\frac{c}{24}}\overline{q}^{\overline{L}_0-\frac{c}{24}}\right]$$

with $c = -26$ and $L_0 = \sum_{n\in\mathbb{Z}}(n :b_{-n}c_n:) - 1$. F is a fermion number operator; $F = 1$ for fermion and $F = 0$ for boson, and hence, $(-1)^F$ anti-commutes with fermions and commutes with boson. The eigenvalue of $(-1)^F$ for a vacuum depends on the situation; usually it is $+1$, namely, $(-1)^F|0\rangle = |0\rangle$. However, there is an example gives minus sign as we saw in the previous lecture; NS sector vacuum is fermionic $(-1)^F|0\rangle_{\mathrm{NS}} = -|0\rangle_{\mathrm{NS}}$. This sign operator $(-1)^F$ ensures periodic boundary condition:

$$\mathrm{Tr}\left[(-1)^F\Psi_1\Psi_2\right] \to \mathrm{Tr}\left[(-1)^F(-1)\Psi_2\Psi_1\right] \to \mathrm{Tr}\left[(-1)^2\Psi_2(-1)^F\Psi_1\right] \to \mathrm{Tr}\left[(-1)^F\Psi_1\Psi_2\right] \ .$$

The result is (homework)

$$\left\langle B\overline{B}C\overline{C}\ \right\rangle_{bc} = q^{-1-\frac{c}{24}}\overline{q}^{-1-\frac{c}{24}}\prod_{n\geq 1}(1-q^n)^2(1-\overline{q}^n)^2 = |\eta(\tau)|^4 \ .$$

So, the final result is

$$A_{1,0} = \frac{iV}{(2\pi\ell_s)^{26}}\int_F\frac{d^2\tau}{2\tau_2}\left(\tau_2\right)^{-13}\left|\eta(\tau)\right|^{-48} \ .$$

Comments:

- There is no UV divergence (see Fig. and recall the naive argument).
- $|\eta(\tau)|^{-48} \simeq |q^{-1} + 24 + \mathcal{O}(q)|^2$, which shows correct spectrum of a closed string.
- The torus partition function is **modular invariant** (homework).

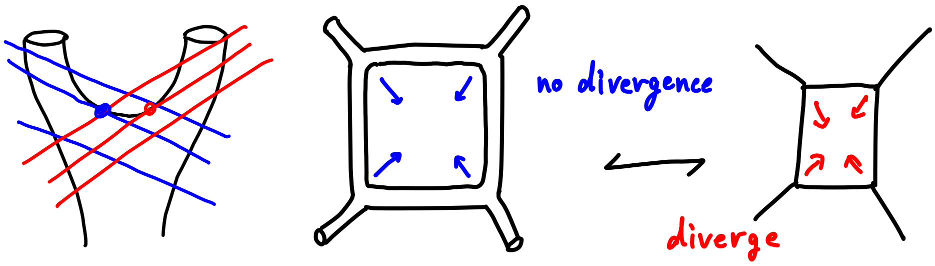


Figure 19: Naive explanation of no UV divergence in string theory.

8.2.3 Torus partition functions for superstring

As we saw in bosonic string case, 1-loop (torus) partition function indicates many informations on the theories, and requires some properties. For example, the moduar invariance is one of such requirement. Modular invariance for superstring requires GSO projection, which we naively introduced.

Torus partition function for Type II theories can be written as follows

$$Z = \frac{iV_{10}}{(2\pi\ell_s)^{10}} \int \frac{d^2\tau}{2\tau_2^2} \frac{1}{\tau_2^4 |\eta(\tau)|^{16}} Z_{\rm F}(\tau) \widetilde{Z}_{\rm F}(\tau^*) ,$$

where Z_F is ψ contribution and $\widetilde{Z}_F$ is $\widetilde{\psi}$ contribution. Since Z_F and $\widetilde{Z}_F$ is symmetric we just focus on the former. It is, in Hamilton formalism, written as follows.

$$\begin{aligned} Z_{\rm F}(\tau) &= Z_{\rm NS}^+ + Z_{\rm NS}^- + e^{i\theta} Z_{\rm R}^+ + e^{i\theta} Z_{\rm R}^- , \\ Z_{\rm S}^\pm &= {\rm Tr}\left[(\pm)^F e^{2\pi i\tau H_{\rm S}} \right] , \end{aligned}$$

where S is either NS or R, θ is a relative phase compared to NS-sector (since the NS & R are disconnected Fock spaces there may be non-trivial phase for the partition function), F is a fermion number, and the superscript is an option for periodicity of world-sheet time t direction ($-$ is periodic). The "Hamiltonian"s for fermions (in light-cone gauge) are

$$\begin{aligned} H_{\rm NS} &= L_0 - \frac{c}{24} = \sum_{r=\frac{1}{2}}^{\infty} r\psi_{-r}\cdot\psi_r - \frac{D-2}{48} , \\ H_{\rm R} &= L_0 - \frac{c}{24} = \sum_{r=1}^{\infty} r\psi_{-r}\cdot\psi_r + \frac{D-2}{24} , \end{aligned}$$

where D should be 10 for superstring, and light-cone gauge means that the space-time indices i runs from 2 to 8, e.g. $\psi \cdot \psi = \sum_{i=2}^{8} \psi^i \psi^i$ (0 and 1 are combined into $+$ and $-$).

Let us see the result. Derivation for NS sector should be straightforward, on the other hand, R sector has vacuum degeneracies

$$|\mathbf{s}; k\rangle = |s_0, s_1, s_2, s_3, s_4; k\rangle \ ,$$

where $s_i = \pm\frac{1}{2}$. Choice for s_0 is a choice of $\mathbf{16}$ or $\mathbf{16'}$, and we chose $s_0 = +\frac{1}{2}$ required by BRST quantization (Dirac equation). The other choices of s_i ($i \neq 0$) simply give degeneracies of $2^4 = 16$. Note that $(-1)^F|\mathbf{s}; k\rangle = (-1)^{\mathbf{1}\cdot(\mathbf{1}-\mathbf{2s})/\mathbf{2}}|\mathbf{s}; k\rangle$, where $\mathbf{1}\cdot(\mathbf{1}-\mathbf{2s})/\mathbf{2} = \sum_{\mathbf{i=0}}^{\mathbf{4}} \left(\frac{\mathbf{1}}{\mathbf{2}} - \mathbf{s_i}\right)$. Therefore,

$$\langle \mathbf{s}; k|(\pm 1)^F|\mathbf{s}; k\rangle = \begin{cases} 16 & \text{for } + \text{ sign} \\ +0 & \text{for } - \text{ sign, and } s_0 = \frac{1}{2} \\ -0 & \text{for } - \text{ sign, and } s_0 = -\frac{1}{2} \end{cases} ,$$

where sum over s_i ($i = 1, 2, 3, 4$) is understood. In the end, the partition functions are

$$\begin{aligned} Z_{\mathrm{NS}}^{+} &= \mathrm{Tr}\left[e^{2\pi i\tau H_{\mathrm{NS}}}\right] = q^{-\frac{1}{6}} \prod_{n=1}^{\infty}(1 + q^{n-\frac{1}{2}})^8 = \left(\frac{\vartheta_3(\tau)}{\eta(\tau)}\right)^4 , \\ Z_{\mathrm{NS}}^{-} &= \mathrm{Tr}\left[(-1)^F e^{2\pi i\tau H_{\mathrm{NS}}}\right] = \pm q^{-\frac{1}{6}} \prod_{n=1}^{\infty}(1 - q^{n-\frac{1}{2}})^8 = \pm\left(\frac{\vartheta_4(\tau)}{\eta(\tau)}\right)^4 , \\ Z_{\mathrm{R}}^{+} &= \mathrm{Tr}\left[e^{2\pi i\tau H_{\mathrm{R}}}\right] = 16 q^{\frac{1}{3}} \prod_{n=1}^{\infty}(1 + q^{n})^8 = \left(\frac{\vartheta_2(\tau)}{\eta(\tau)}\right)^4 , \\ Z_{\mathrm{R}}^{-} &= \mathrm{Tr}\left[(-1)^F e^{2\pi i\tau H_{\mathrm{R}}}\right] = \pm 0 q^{\frac{1}{3}} \prod_{n=1}^{\infty}(1 - q^{n})^8 = \pm\left(\frac{\vartheta_1(\tau)}{\eta(\tau)}\right)^4 = 0 , \end{aligned}$$

where $q = e^{2\pi i\tau}$, and the special functions are summarized in Appendix. Note that the $\pm$ in Z_{NS}^{-} is coming from the vaucuum fermion/boson statistics, namely,

$$(-1)^F|0\rangle_{\mathrm{NS}} = \pm|0\rangle_{\mathrm{NS}} \ .$$

In the previous lecture we set $(-1)^F|0\rangle_{\mathrm{NS}} = -|0\rangle_{\mathrm{NS}}$ (actually it was because of ghost vacuum) so that the tachyon state is removed from the physical states. Here it is required because of modular invariance as follows. $Z_{\mathrm{F}}(\tau)$ becomes

$$Z_{\mathrm{F}}(\tau) = \frac{1}{\eta^4(\tau)}\left\{e^{i\theta}(\vartheta_2(\tau))^4 + (\vartheta_3(\tau))^4 \pm (\vartheta_4(\tau))^4\right\}$$

Note that each term is not modular invariant (see Appendix), therefore, we need a specific combinations of them, and the correct one is

$$Z_{\mathrm{F}}(\tau) = \frac{1}{\eta^4(\tau)}\left\{-(\vartheta_2(\tau))^4 + (\vartheta_3(\tau))^4 - (\vartheta_4(\tau))^4\right\} = 0 \ .$$

This is called Jacobi identity (or Riemann identity).

Comments:

- The partition function is trivial due to supersymmetry (fermionic and bosonic contributions cancel each other).
- The partition function is modular invariant because it is independent of τ (actually it is zero).
- The combination derived above is nothing but GSO projection (see below).

The combination can be written as follows

$$\begin{aligned}
&2\mathrm{Tr}\left[\frac{1+(-1)^F}{2}e^{2\pi i\tau H_{\mathrm{NS}}}\right]-2\mathrm{Tr}\left[\frac{1\pm(-1)^F}{2}e^{2\pi i\tau H_{\mathrm{R}}}\right]\\
&=2\mathrm{Tr}\left[P^{\mathrm{NS}}_{\mathrm{GSO}}e^{2\pi i\tau H_{\mathrm{NS}}}\right]-2\mathrm{Tr}\left[P^{\mathrm{R}}_{\mathrm{GSO}}e^{2\pi i\tau H_{\mathrm{R}}}\right]\ ,
\end{aligned}$$

where GSO projections are

$$\begin{aligned}
P^{\mathrm{NS}}_{\mathrm{GSO}}&=\frac{1+(-1)^F}{2} &&\text{NS sector}\ ,\\
P^{\mathrm{R}\pm}_{\mathrm{GSO}}&=\frac{1\pm(-1)^F}{2} &&\text{R}\pm\text{ sector}\ ,
\end{aligned}$$

where $\pm$ in R-sector is an option for $\mathbf{8}_s$ or $\mathbf{8}_c$ (one can understand this as a choice of s_1). NS$-$ sector can be defined by $P^{\mathrm{NS}-}_{\mathrm{GSO}}=\frac{1-(-1)^F}{2}$ includes tachyon, hence we never consider. Note that $\pm$ in the sectors (e.g. R$\pm$ sector) is different from the superscript of the partition functions $Z^{\pm}_{\mathrm{S}}$. Also note that the definition of the GSO projection operator is different in some textbook.

8.3 Special functions

The special functions we have used in the text are summarized below. You can also consult with Polchinski [Pol98, §7.2] or Blumenhagen & Plauschinn [BP09, §4.2].

The infinite product form of them are

$$\begin{aligned}
\eta(\tau)&=q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n)\ ,\\
\vartheta_2(\tau)&=2q^{\frac{1}{8}}\prod_{n=1}^{\infty}(1-q^n)(1+q^n)^2\ ,\\
\vartheta_3(\tau)&=\prod_{n=1}^{\infty}(1-q^n)(1+q^{n-\frac{1}{2}})^2\ ,\\
\vartheta_4(\tau)&=\prod_{n=1}^{\infty}(1-q^n)(1-q^{n-\frac{1}{2}})^2\ ,
\end{aligned}$$

where $q = e^{2\pi i\tau}$. The first one is called Dedekind eta function, the others are called (Jacobi or elliptic or simply without these) theta functions.

Their T-transformations and S-transformations are

$$
\begin{aligned}
\eta(\tau+1) &= e^{i\pi/12}\eta(\tau) \ , & \eta(-1/\tau) &= \sqrt{-i\tau}\eta(\tau) \ , \\
\vartheta_2(\tau+1) &= e^{i\pi/4}\vartheta_2(\tau) \ , & \vartheta_2(-1/\tau) &= \sqrt{-i\tau}\vartheta_4(\tau) \\
\vartheta_3(\tau+1) &= \vartheta_4(\tau) \ , & \vartheta_3(-1/\tau) &= \sqrt{-i\tau}\vartheta_3(\tau) \\
\vartheta_4(\tau+1) &= \vartheta_3(\tau) \ , & \vartheta_4(-1/\tau) &= \sqrt{-i\tau}\vartheta_2(\tau) \ .
\end{aligned}
$$

There two important identities. One is already explained in the main text, called Jacobi identity:

$$
(\vartheta_3(\tau))^4 = (\vartheta_2(\tau))^4 + (\vartheta_4(\tau))^4 \ .
$$

The other is called Jacobi triple product identity:

$$
\vartheta_2(\tau)\vartheta_3(\tau)\vartheta_4(\tau) = 2\eta^3(\tau) \ ,
$$

from which you can show the modular transformation of the Dedekind eta function.

9 Lecture 9

We have seen that the critical dimensions D of bosonic and supersymmetric string theory are $D = 26$ and $D = 10$, respectively. To obtain an effective theory in lower dimensions, we can make use of **Kaluza-Klein compactifications** where the true spacetime takes the form of a direct product $M_d \times K_{D-d}$, where M_d is the d-dimensional Minkowski spacetime, and K_{D-d} is a very tiny compact manifold. As we will see, an effective theory in M_d still sees interesting "stringy" effects in this Kaluza-Klein scheme.

First we concentrate on the simple compactifications, $K = T^{D-d}$ called toroidal compactifications. Since a torus is simply a product of S^1 and are flat, the nonlinear sigma model can be described by the free two-dimensional CFT. Remarkably, this simple compactification leads to the notion of **T-duality** and **Heterotic string theories** have been constructed based on toroidal compactifications. To understand the basic properties, let us first see study the toroidal compactifications of bosonic string theory. Toroidal compactifications of superstring theories are also very interesting so that we will learn them relating to string dualities later more in detail.

The concept of D-branes was introduced in the last lecture as boundary conditions of open strings. T-duality gets particularly rich when we include D-branes so that we will study their properties more in detail.

9.1 S^1 compactification in closed bosonic string

To begin with, let us first study the simplest case of the spacetime $\mathbb{R}^{1,24} \times S^1$ where we compactify 25-th direction on a circle S^1 of radius R. For closed strings, we have the familiar mode expansion $X^\mu(z,\overline{z}) = X^\mu(z) + \overline{X}^\mu(\overline{z})$ with

$$X^\mu(z) = x^\mu + i\sqrt{\frac{\alpha'}{2}}\left(-\alpha_0^\mu \ln z + \sum_{m\neq 0} \frac{\alpha_m^\mu}{m z^m}\right),$$

$$\overline{X}^\mu(\overline{z}) = \widetilde{x}^\mu + i\sqrt{\frac{\alpha'}{2}}\left(-\widetilde{\alpha}_0^\mu \ln \overline{z} + \sum_{m\neq 0} \frac{\widetilde{\alpha}_m^\mu}{m \overline{z}^m}\right).$$

Now let us take a close look at the zero modes which can be written as

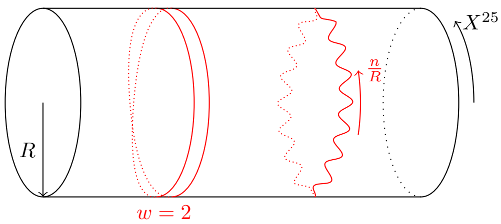
$$X^\mu(z,\overline{z}) = x^\mu + \widetilde{x}^\mu - i\sqrt{\frac{\alpha'}{2}}(\alpha_0^\mu + \widetilde{\alpha}_0^\mu)t + \sqrt{\frac{\alpha'}{2}}(\widetilde{\alpha}_0^\mu - \alpha_0^\mu)\sigma + \text{oscillators}.$$

where the spacetime momentum of the string is

$$p^\mu = \frac{1}{\sqrt{2\alpha'}}(\alpha_0^\mu + \widetilde{\alpha}_0^\mu)\ .$$

Under $\sigma \to \sigma + 2\pi$, the oscillator term are periodic and $X^\mu(z,\overline{z})$ changes by $2\pi\sqrt{(\alpha'/2)}(\widetilde{\alpha}_0^\mu - \alpha_0^\mu)$. For a non-compact spatial direction $\mathbb{R}^{1,24}$, X^μ is single-valued $X^\mu(t,\sigma) = X^\mu(t,\sigma+2\pi)$, which requires

$$\alpha_0^\mu = \widetilde{\alpha}_0^\mu = \sqrt{\frac{\alpha'}{2}}p^\mu\ , \qquad \mu = 0, 1, \ldots, 24\ .$$



On the other hand, since the 25-th direction is put on the circle S^1 of radius R, it has period $X^{25} \sim X^{25} + 2\pi R$. Hence, the momentum p^{25} can take the values n/R for $n \in \mathbb{Z}$ where n is called **Kaluza-Klein momentum**. Also, under $\sigma \sim \sigma + 2\pi$, $X^{25}(z,\overline{z})$ can change by $2\pi w R$ where w is called the **winding number**. Thus, we have

$$\alpha_0^{25} + \widetilde{\alpha}_0^{25} = \frac{2n}{R}\sqrt{\frac{\alpha'}{2}}\ , \qquad \widetilde{\alpha}_0^{25} - \alpha_0^{25} = \sqrt{\frac{2}{\alpha'}}wR\ ,$$

implying

$$\alpha_0^{25} = \left(\frac{n}{R} - \frac{wR}{\alpha'}\right)\sqrt{\frac{\alpha'}{2}}\ , \qquad \widetilde{\alpha}_0^{25} = \left(\frac{n}{R} + \frac{wR}{\alpha'}\right)\sqrt{\frac{\alpha'}{2}}\ . \tag{9.1}$$

Now let us study their mass spectrum. The mass formula for the string with one dimension compactified on a circle can be interpreted from a 25-dimensional viewpoint in which one regards each of the Kaluza-Klein momenta, which are given by n, as distinct particles. Thus, the mass formula is given by

$$M^2 = -\sum_{\mu=0}^{24} p^\mu p_\mu$$

where μ runs only over the non-compact dimensions. Hence, we can write the mass formula as

$$M^2 = \frac{2}{\alpha'}(\alpha_0^{25})^2 + \frac{4}{\alpha'}(N-1) = \frac{2}{\alpha'}(\widetilde{\alpha}_0^{25})^2 + \frac{4}{\alpha'}(\widetilde{N}-1).$$

where N and $\widetilde{N}$ are the right and left numbering operator (See lecture note 1). Using (9.1), we can express the difference of the two expression

$$N - \widetilde{N} = nw\ , \tag{9.2}$$

so that the level matching condition is modified due to the S^1 compactification. In a similar fashion, the mass can be expressed in terms of the Kaluza-Klein momentum and the winding number

$$M^2 = \frac{n^2}{R^2} + \frac{w^2R^2}{\alpha'^2} + \frac{2}{\alpha'}(N + \widetilde{N} - 2) \tag{9.3}$$

The mass spectra (9.3) of the theories at radius R and α'/R are identical when the winding and Kaluza-Klein modes are interchanged $n \leftrightarrow w$. This symmetry of the bosonic string theory is called **T-duality**. This is the first and still most striking indication that strings see spacetime geometry differently from the way we are used to. Indeed, many other examples of 'stringy geometry' or 'quantum geometry' are closely related to this.

It is easy to see from (9.1) that this interchange amounts to

$$\alpha_0^{25} \to -\alpha_0^{25}, \quad \widetilde{\alpha}_0^{25} \to \widetilde{\alpha}_0^{25}.$$

In fact, it is not just the zero mode, but the entire right-moving part of the compact coordinate that flips sign under the T-duality transformation

$$X'^{25}(z,\overline{z}) = -X^{25}(z) + \overline{X}^{25}(\overline{z})\ . \tag{9.4}$$

Remarkably, the energy-momentum tensor, OPEs and therefore all of the correlation functions are invariant under this rewriting. In other words, T-duality, relating the two theories with radius R and α'/R, is an exact symmetry of perturbative closed string theory.

Because of the T-duality, a theory with compactification radius R is equivalent to the theory with compactification radius α'/R. Thus, this implies that there is a "minimal radius" $R = \sqrt{\alpha'}$ in string theory which is called **self-dual radius**. At self-dual radius, the duality $R \to \alpha'/R$ maps R back to its original value where we can expect something interesting to occur. In the next section, we will study the physics at the self-dual radius.

9.1.1 Self-dual radius: $R = \sqrt{\alpha'}$

As we know, the massless spectra of bosonic string theory includes graviton. Hence, let us see the effect of the S^1 compactification on the graviton. In the Kaluza-Klein mechanism $M^{25} \times S^1$, the metric is decomposed into compact and non-compact spacetime direction

$$ds^2 = G_{MN}dx^M dx^N = G_{\mu\nu}dx^\mu dx^\nu + G_{25,25}(dx^{25} + A_\mu dx^\mu)^2 \ . \tag{9.5}$$

where the fields $G_{\mu\nu}$, $G_{25,25}$, and A_μ are allowed to depend only on the non-compact coordinates x^μ ($\mu = 0, 1, \ldots 24$). Under a coordinate transformation

$$x'^{25} = x^{25} + \lambda(x^\mu)$$

the part $G_{\mu,25} = G_{25,\mu}$ of the metric transforms as

$$A'_\mu = A_\mu - \partial_\mu \lambda \ .$$

Thus, it behaves as U(1) gauge field, and gauge transformations arise as part of the higher-dimensional coordinate transformation. On the other hand, the part $G_{25,25}$ of the metric behaves as a scalar field. Indeed, writing $G_{25,25} = e^\sigma$, the Ricci scalar for the metric (9.5) can be written as

$$R_{26} = R_{25} - 2e^{-\sigma}\nabla^2 e^\sigma - \frac{1}{4}e^{2\sigma}F_{\mu\nu}F^{\mu\nu} \ .$$

Actually, it is straightforward to see the corresponding vertex operators at generic radius R:

$$\begin{aligned}
\partial X^\mu \overline{\partial} X^\nu e^{ik\cdot X} \quad &\longleftrightarrow \quad G_{\mu\nu}, B_{\mu\nu}, \phi \\
\partial X^\mu \overline{\partial} X^{25} e^{ik\cdot X}, \ \partial X^{25} \overline{\partial} X^\mu e^{ik\cdot X} \quad &\longleftrightarrow \quad A^\mu \, , B_{\mu,25} \\
\partial X^{25} \overline{\partial} X^{25} e^{ik\cdot X} \quad &\longleftrightarrow \quad \sigma
\end{aligned} \tag{9.6}$$

where $\nu = 0, \ldots, 24$ runs the coordinate indices for M_{25}. In fact, the middle line indicates that the theory has $\mathrm{U}(1)_\ell \times \mathrm{U}(1)_r$ gauge symmetry at generic radius R.

	n	w	$\widetilde{N}$	N
A	± 1	± 1	0	1
B	± 1	∓ 1	1	0
C	± 2	0	0	0
D	0	± 2	0	0

However, at the self-dual radius $R = \sqrt{\alpha'}$, the mass formula (9.3) becomes

$$M^2 = \frac{1}{\alpha'}(n^2 + w^2 + 2(N + \widetilde{N} - 2)) ,$$

so that the massless spectra actually gets enlarged. In addition to the generic solution $n = w = 0$, $N = \widetilde{N} = 1$, there are now also
Hence, the states corresponding to A and B contain four new gauge bosons (A,B,C,D contain new massless scalars too that will appear in Homework 8), with vertex operators

$$\overline{\partial} X^\mu e^{\pm 2iX^{25}(z)/\sqrt{\alpha'}} e^{ik\cdot X} , \qquad \partial X^\mu e^{\pm 2i\overline{X}^{25}(\bar{z})/\sqrt{\alpha'}} e^{ik\cdot X} .$$

It is expected that the new gauge bosons must combine with the old into a non-Abelian theory. In fact, if one can define the current

$$j^\pm(z) = j^1(z) \pm i\, j^2(z) := e^{\pm 2iX^{25}(z)/\sqrt{\alpha'}} \qquad j^3(z) := i\, \partial X^{25}(z)/\sqrt{\alpha'} ,$$

they satisfy the OPEs (Exercise)

$$j^a(z) j^b(0) \sim \frac{k\delta^{ab}}{2z^2} + \frac{i\epsilon^{abc} j^c(0)}{z} .$$

with $k = 1$. Here ϵ^{abc} is the structure constant of SU(2). This is precisely the definition of SU(2) affine Lie algebra with level $k = 1$. The same story is repeated for the left movers. Hence we see that we have an enhancement of gauge symmetry from $\mathrm{U}(1)_\ell \times \mathrm{U}(1)_r$ to $\mathrm{SU}(2)_\ell \times \mathrm{SU}(2)_r$ at $R = \sqrt{\alpha'}$.

In fact, when the theory moves away from the self-dual radius $R = \sqrt{\alpha'}$, the $\mathrm{SU}(2)_\ell \times \mathrm{SU}(2)_r$ gauge symmetry is Higgsed. The world sheet action is deformed by turning on the marginal operator

$$V_{a\tilde{a}} := j_a \bar{j}_{\tilde{a}} e^{ik\cdot X} ,$$

which is equivalent to giving the VEV of the $(3,3)$-component of Higgs filed. As a result, when the theory is away from the self-dual radius, the $\mathrm{SU}(2)_\ell \times \mathrm{SU}(2)_r$ gauge symmetry is spontaneously broken down to a $\mathrm{U}(1)_\ell \times \mathrm{U}(1)_r$.

9.2 T-duality of Open Strings

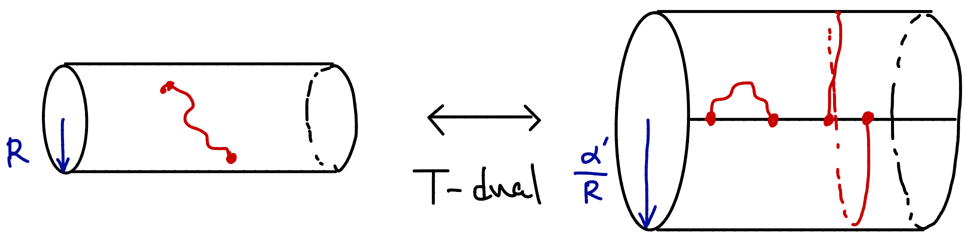
Now let us consider T-duality on the open string spectrum in the S^1 compactification. At the end of §??, we briefly study open string spectra. There, we learned that Neumann boundary condition on the X^μ-direction is achieved by imposing $\alpha^\mu_n = \overline{\alpha}^\mu_n$ while Dirichlet boundary condition is by $\alpha^\mu_n = -\overline{\alpha}^\mu_n$. Since T-duality along the 25-th direction transforms the modes as in (9.4), it therefore exchanges Neumann to Dirichlet condition [? ?]:

$$\partial_\sigma X^{25}\big|_{\sigma=0,\pi} = 0 \quad \to \quad \partial_t X'^{25}\big|_{\sigma=0,\pi} = 0 \ .$$

Suppose that an open string with Neumann boundary condition has KK momentum n along the 25-th direction, and we perform T-duality on that circle. Then, a simple calculation

$$\begin{aligned} X'^{25}(\pi) - X'^{25}(0) &= \int_0^\pi d\sigma \partial_\sigma X'^{25} = i\int_0^\pi d\sigma \partial_t X^{25} && (9.7) \\ &= 2\pi\alpha' p^{25} = \frac{2\pi\alpha' n}{R} = 2\pi n R'. && (9.8) \end{aligned}$$

tells us that the X^{25} coordinate of the open string endpoints is fixed after T-duality. T-duality transforms the KK momentum n to winding number n as in the figure below. This can be interpreted as follows. Open strings can freely move on a space-filling D25-brane. After T-duality, the space-filling D25-brane becomes a D24-brane with its X^{25} coordinate fixed. As a result, all the endpoints of open strings are constrained to the fixed X^{25}-direction whereas they are free to move in the other 24 spatial dimensions. More generally, the boundary condition of open strings can be imposed at any dimension. Since T-duality interchanges Neumann and Dirichlet boundary conditions, T-duality tangent to a Dp-brane brings it to a D$(p-1)$-brane. On the other hand, T-duality orthogonal to a Dp-brane turns it into a D$(p+1)$-brane.



We can ask a question which position of the X^{25} coordinate a D24-brane is located in the figure above. This question is related to Chan-Paton factors we encountered in §??. Suppose that there are K space-filling D25-branes, which give rise to U(K) spacetime gauge fields. In the current setting, X^{25} direction is compactified on the circle S^1 so that we can include a Wilson loop on S^1 as

$$A_{25} = \text{diag}\{\theta_1, \theta_2, \ldots, \theta_K\}/2\pi R \tag{9.9}$$

The insertion of the Wilson loop breaks the gauge group as $\mathrm{U}(K) \to \mathrm{U}(1)^K$, and the broken gauge group is abelian so that we can write it as

$$A_{25} = -i\Lambda^{-1}\partial_{25}\Lambda, \quad \Lambda = \mathrm{diag}\{e^{iX^{25}\theta_1/2\pi R}, e^{iX^{25}\theta_2/2\pi R}, \ldots, e^{iX^{25}\theta_N/2\pi R}\} .$$

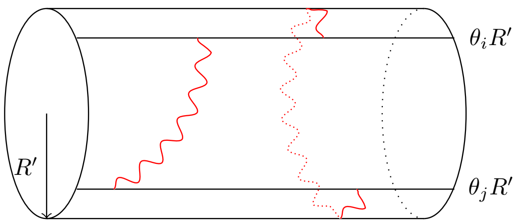
Then, under the translation $X^{25} \to X^{25} + 2\pi R$, an open string state $|N=1;n;ij\rangle$ is shifted by a phase

$$e^{i(\theta_j - \theta_i)} , \tag{9.10}$$

which means its momentum is shifted by $(\theta_j - \theta_i)/2\pi$ from an integer n. Under T-duality, the momentum is transformed into the winding number. Therefore, θ_i can be understood as the X^{25} coordinate of the i-th D24-brane after T-duality. Namely, the open string state $|N=1;n;ij\rangle$ is mapped to an open string of length

$$X'^{25}(\pi) - X'^{25}(0) = (2\pi n + \theta_j - \theta_i)\, R' .$$

There are in general K D24-branes at different positions as schematically depicted in the following figure.



Then, the $(D-1) = 24$-dimensional mass of this open string is

$$\begin{aligned} M^2 &= \left(p^{25}\right)^2 + \frac{1}{\alpha'}(N-1) \\ &= \left(\frac{[2\pi n + (\theta_j - \theta_i)]\, R'}{\pi\alpha'}\right)^2 + \frac{1}{\alpha'}(N-1) . \end{aligned} \tag{9.11}$$

Hence, massless gauge bosons arise only from open strings $|N=1;n=0;ii\rangle$ whose endpoints are on the same D-brane. Due to the string tension, open strings that stretch between different D-branes become massive (gauge field). Therefore, this can be understood as the Higgs mechanism in which X^{25} expectation values (coordinates) of K D24-branes take different values, and the gauge group is broken as $\mathrm{U}(K) \to \mathrm{U}(1)^K$. This situation is called **Coulomb phase**, where the Goldstone bosons are "eaten" by gauge fields $|N=1;n=0;ij\rangle$ which become massive.

In addition, let us make an important remark. So far, we have treat D-branes just as rigid boundary conditions. However, D-branes are dynamical so that they can fluctuate in shape and position. As we will see in the subsequent lectures (hopefully!), the gravitational and gauge dynamics on a stack of D-branes makes string theory very intriguing.

9.3 T-duality of Type II Superstrings

To end this lecture, let us briefly discuss how the T-duality acts on Type II superstring compactified on $M^9 \times S^1$. We have seen that the T-duality acts as the parity transformation on the right-moving sector

$$X^9(z) \quad \longleftrightarrow \quad -X'^9(z)$$

The superconformal invariance requires

$$\psi^9 \leftrightarrow -\psi'^9 \ .$$

However, this implies that the chirality of the right-moving R sector ground state is reversed: the raising and lowering operators $\psi'^8 \pm i\psi'^9$ are interchanged. In other words, T-duality is a spacetime parity operation on just one side of the world-sheet, and so reverses the relative chiralities of the right- and left-moving ground states. As a result, Type IIA theory with compactification radius R is T-dualized to Type IIB theory with radius α'/R.

Since the IIA and IIB theories have different R-R fields, T-duality in the 9th direction must transform one set into the other. The action of duality on the spin fields is of the form

$$\psi^R_\alpha(z) \to \beta_9 \psi^R_\alpha(z), \qquad \overline{\psi}^L_\alpha(\overline{z}) \to \overline{\psi}^L_\alpha(\overline{z})$$

for $\beta_9 = \Gamma^9\Gamma^{11}$, the parity transformation (9-reflection) on the spinors. Since the RR vertex operators are written as

$$\overline{\psi}^L \Gamma^{\mu_1 \cdots \mu_p} \psi^R \ ,$$

Hence, the RR fields are transformed as

$$\begin{aligned} C_9 \quad &\to \quad C \\ C_\mu, C_{\mu\nu 9} \quad &\to \quad C_{\mu 9}, C_{\mu\nu} \\ C_{\mu\nu\lambda} \quad &\to \quad C_{\mu\nu\lambda 9} \ . \end{aligned}$$

10 Lecture 10

10.1 Type I superstring theory

What we will learn:

- Type I superstring theory.
- In Type I theory, we only have D1-, D5-, and D9-, branes.
- $O9^-$-plane is needed for Type I to be consistent theory.

- T-duality of Type I theory.

So far, we have learnt two kinds of superstring theory, which are Type IIA and IIB. Both of them are closed superstring theory. Note that we also learnt about D-branes in Type II theories, which leads open string. However, note that we are considering a situation in which there exist D-branes, so it is not a theory.

Actually, an open string cannot exist in Type II theories (without D-branes), due to supersymmetry as follows. After GSO projection an open string can have one of the following massless states:

$$\begin{aligned} P_{\text{GSO}}: &\quad \text{NS+},\ \text{R+} = \mathbf{8}_v + \mathbf{8}_s\ , \\ \widetilde{P}_{\text{GSO}}: &\quad \text{NS+},\ \text{R}- = \mathbf{8}_v + \mathbf{8}_c\ , \end{aligned}$$

where $\mathbf{8}_v$ is a space-time vector field and $\mathbf{8}_{s,c}$ are gauginos. $\mathcal{N} = 2$ supersymmetry requires for $\mathbf{8}_v$ to have two superpartners as like for a graviton to have two gravitinos. However, as you see, the gauge field has only one superpertner, and hence, cannot exist in Type II theories.

The existence of (parallel) D-branes, as you might notice, breaks the half of supersymmetries, and hence, an open string can exist in such a situation.

The world-sheets(WS) we have considered are oriented ones. However, it is quite natural to consider processes in Fig. 20. Therefore, we will consider so called

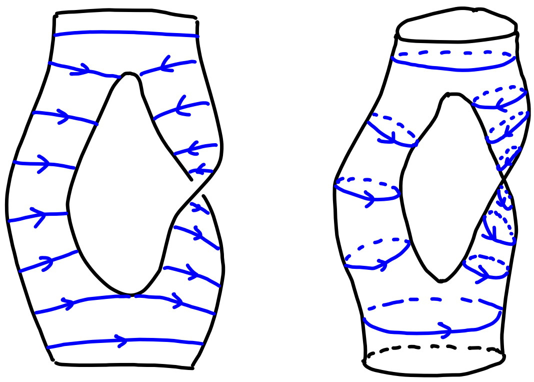


Figure 20: Unoriented processes. The left is open string one, and the right is closed string one.

unoriented string in this lecture, which leads Type I superstring theory. Argument below is almost in bosonic string, for simplicity, but it is straightfoward to apply for superstring.

10.1.1 Orientation flip operation

As the name indicates the orientation flip is nothing but an exchange of left-/right-mover for closed string and a reversal of the direction for open string.

Closed string

Let us define an orientation flip operator Ω for bosonic string, which flips the orientation of an closed string:

$$\Omega : \begin{cases} X(t,\sigma) & \leftrightarrow & X(t,-\sigma) \\ b(t,\sigma) & \leftrightarrow & \overline{b}(t,-\sigma) \\ c(t,\sigma) & \leftrightarrow & -\overline{c}(t,-\sigma) \end{cases} ,$$

where t is the Eculideanized WS time, not τ (though there is no special meaning on it at this point). Or, the operator acts on the modes as

$$\Omega : \begin{cases} \alpha_n^\mu & \leftrightarrow & \widetilde{\alpha}_n^\mu \\ b_n & \leftrightarrow & \overline{b}_n \\ c_n & \leftrightarrow & \overline{c}_n \end{cases} ,$$

where the modes are

$$\begin{aligned} X(t,\sigma) &= x^\mu - i\alpha' p^\mu t + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n} \left(\alpha_n^\mu e^{n(-t+i\sigma)} + \widetilde{\alpha}_n^\mu e^{n(-t-i\sigma)} \right) , \\ b(t,\sigma) &= -\sum_{n\in\mathbb{Z}} b_n e^{n(-t+i\sigma)} , \quad \overline{b}(t,\sigma) = -\sum_{n\in\mathbb{Z}} \overline{b}_n e^{n(-t-i\sigma)} , \\ c(t,\sigma) &= i\sum_{n\in\mathbb{Z}} c_n e^{n(-t+i\sigma)} , \quad \overline{c}(t,\sigma) = -i\sum_{n\in\mathbb{Z}} \overline{c}_n e^{n(-t-i\sigma)} , \end{aligned}$$

where we can also use $w = -it - \sigma \leftarrow \tau - \sigma$ ($z = e^{iw}$).

We keep only Ω invariant states:

OK	Tachyon	$\vert k\rangle$
OK	Dilaton	$\alpha_{-1}\cdot\widetilde{\alpha}_{-1}\vert k\rangle$
NG	B-field	$\alpha_{-1}^{[\mu}\widetilde{\alpha}_{-1}^{\nu]}\vert k\rangle$
OK	Graviton	$\left(\alpha_{-1}^{(\mu}\widetilde{\alpha}_{-1}^{\nu)} - \frac{\delta^{\mu\nu}}{D}\alpha_{-1}\cdot\widetilde{\alpha}_{-1}\right)\vert k\rangle$
	$\vdots$	

No B-field means that a closed string, which is electrically coupled to B-fields, is not a stable object and should decay.

Open string

Let us put a D25-brane and consider Neumann boundary conditions. Then similarly, we define the orientation flip operator for open string:

$$\Omega : \begin{cases} X(t,\sigma) & \leftrightarrow & X(t,\pi-\sigma) \\ b(t,\sigma) & \leftrightarrow & \overline{b}(t,\pi-\sigma) \\ c(t,\sigma) & \leftrightarrow & -\overline{c}(t,\pi-\sigma) \end{cases} ,$$

or, for the modes expansion

$$\Omega : \begin{cases} \alpha_n^\mu & \leftrightarrow & (-1)^n \alpha_n^\mu \\ b_n & \leftrightarrow & (-1)^n b_n \\ c_n & \leftrightarrow & (-1)^n c_n \end{cases} ,$$

where the modes are

$$\begin{aligned}
X(t,\sigma) &= x^\mu - 2i\alpha' p^\mu t + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{\alpha_n^\mu}{n} \left(e^{n(-t+i\sigma)} + e^{n(-t-i\sigma)} \right) , \\
b(t,\sigma) &= -\sum_{n\in\mathbb{Z}} b_n e^{n(-t+i\sigma)} , \quad \bar{b}(t,\sigma) = -\sum_{n\in\mathbb{Z}} b_n e^{n(-t-i\sigma)} , \\
c(t,\sigma) &= i\sum_{n\in\mathbb{Z}} c_n e^{n(-t+i\sigma)} , \quad \bar{c}(t,\sigma) = -i\sum_{n\in\mathbb{Z}} c_n e^{n(-t-i\sigma)} .
\end{aligned}$$

Naively, the massless vector field state is Ω variant. However, we can keep the state by using Chan-Paton factor:

$$|\Phi;\Lambda\rangle \equiv |\Phi;ij\rangle \Lambda_{ij} .$$

When the orientation flips, the Chan-Paton indices are exchanged, so

$$\Omega|\Phi;\Lambda\rangle = |\Omega\Phi;\Lambda^T\rangle .$$

Therefore, Ω invariant states are

$$\begin{array}{lll} \text{Tachyon} & |k;\Lambda\rangle & \Lambda^T = \Lambda \\ \text{Vector} & \alpha_{-1}^\mu |k;\Lambda\rangle & \Lambda^T = -\Lambda \\ \vdots & & \end{array}$$

The $n \times n$ hermitian, anti-symmetric matrix forms a SO(n) algebra. Therefore, the vector field is identified with a SO(n) gauge field.

When we flip the orientation there could be a shuffle of the Chan-Paton index because the D-branes are coincident. Let us denote this as follows.

$$\begin{aligned}
\Omega|\Phi;ij\rangle &= |\Omega\Phi;kl\rangle P_{kj} P_{il}^{-1} \quad (P \in \mathrm{U}(n)) , \\
\Omega|\Phi;\Lambda\rangle &= |\Omega\Phi; P\Lambda^T P^{-1}\rangle .
\end{aligned}$$

There is a natural constraint $\Omega^2 = 1$, and equivalence relation: $\Lambda \sim \widetilde{\Lambda} = U\Lambda U^{-1}$ ($U \in \mathrm{U}(n)$). The constraint leads

$$\begin{aligned}
&\Omega^2|\Phi;\Lambda\rangle = \Omega|\Omega\Phi;P\Lambda^T P^{-1}\rangle = |\Phi; P(P\Lambda^T P^{-1})^T P^{-1}\rangle , \\
&\therefore \qquad \Lambda = PP^{-T}\Lambda(PP^{-T})^{-1} .
\end{aligned}$$

For general Λ to satisfy the relation above we need $PP^{-T} = 1$. Since the order of the index is artificial, physics is invariant under the re-labelling, $\Lambda \sim \widetilde{\Lambda} = U\Lambda U^{-1}$, and this leads the equivalence class for P as well:

$$P\Lambda^T P^{-1} = P(U^{-1}\widetilde{\Lambda}U)^T P^{-1} = PU^T\widetilde{\Lambda}^T U^{-T}P^{-1} = U^{-1}(UPU^T)\widetilde{\Lambda}^T(UPU^T)^{-1}U \ .$$

Thus, $\widetilde{P} \sim UPU^T$. Subject to the constraint and the equivalence class, we have two physically inequivalent choice for P:

- $P = 1 \qquad \Lambda^T = -\Lambda \qquad \cdots$ SO(n) gauge symmetry.
- $P = i\begin{pmatrix} 0 & -\mathbf{1}_{k\times k} \\ \mathbf{1}_{k\times k} & 0 \end{pmatrix} \qquad (n = 2k)$
  $\Lambda^T = \begin{pmatrix} 0 & -\mathbf{1}_{k\times k} \\ \mathbf{1}_{k\times k} & 0 \end{pmatrix}\Lambda\begin{pmatrix} 0 & -\mathbf{1}_{k\times k} \\ \mathbf{1}_{k\times k} & 0 \end{pmatrix} \qquad \cdots$ Sp(n) gauge symmetry.

Unoriented superstring spectrum

The orientation flip is a swap of left- and right-mover, therefore, IIB theory has the world-sheet parity Ω because it is a non-chiral theory. The flip projection eliminates B-field [2] in NS-NS sector, as well as half of NS-R R-NS sector $\mathbf{8}_c + \mathbf{56}_s$ (only diagonal part survive). Supersymmetry requires that the number of bosons and fermions are the same, which implies that [0] and $[4]_+$ are eliminated and only the second rank anti-symmetric field [2] survives in R-R sector. The remaining states are

$$\begin{aligned}[0] + [2] + (2) + \mathbf{8}_c + \mathbf{56} &= \mathbf{1} + \mathbf{28} + \mathbf{35} + \mathbf{8}_c + \mathbf{56}_s \\ &= \Phi + C_{\mu\nu} + G_{\mu\nu} + \lambda^- + \psi^+_\mu \ .\end{aligned}$$

Absence of the C and $C^+_{\mu\nu\rho\sigma}$ means that there is no D(-1)-, D3-, D7-branes in Type I theory. **Only D1-, D5-, D9-branes exist in Type I theory**, as the first two D-branes are electrically and magnetically coupled to $C_{\mu\nu}$, respectively. D9 is space-filling and non-dynamical, and is necessary to have an open string. Actually, this Type I closed superstring theory is inconsistent as we will see, and it necessarily includes open string $\mathbf{8}_v + \mathbf{8}_s$ (choice of the GSO projection follows from IIB). Therefore, the spectrum of Type I theory (unoriented, open plus closed supersting) is as follows.

$$[0] + [2] + (2) + \mathbf{8}_c + \mathbf{56} + (\mathbf{8}_v + \mathbf{8}_s)_{SO(n) \text{ or } \mathrm{Sp}(n)} \ .$$

Anomaly or tadpole cancellation argument, which we will see in the following subsection, shows that **only** SO(32) **is consistent**.

10.1.2 Amplitude of Type I theory

As we saw in the previous subsection unoriented open string has a SO(n) or Sp(n) massless gauge field. However, all of them are anomalous, which means that the

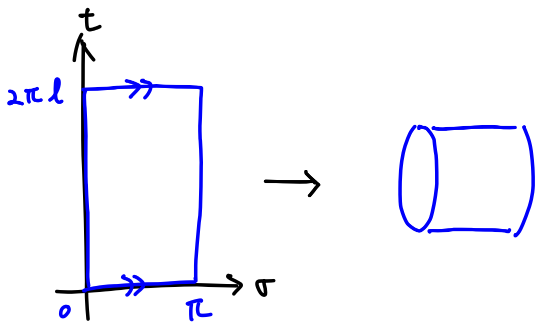


Figure 21: Cylinder.

theories are inconsistent, except SO(32) as we will see below. There are a few ways to see the anomaly. We utilize vacuum amplitudes.

First, let us consider "oriented" open string amplitude, whose WS is cylinder(=annulus), see Fig. 21. It is clear from the picture that

- The range is $0 \le \mathrm{Re}w \le \pi$, the period is $w \sim w + 2\pi i l$.
- There is a real modulus l; the amplitude needs a b zero mode insertion.
- There is a real isometry, shift of $\mathrm{Im}w$; the amplitude needs a c zero mode insertion.

The cylinder partition function is

$$A_{0,C} = \int \frac{dl}{2l} \langle b_0 c_0 \rangle_{\mathrm{gh}} \langle 1 \rangle_{\mathrm{mat}} \ ,$$

where

$$\begin{aligned} b_0 &= \frac{1}{2\pi} \int_0^\pi \left[dw\ b(w) + d\overline{w}\ \overline{b}(\overline{w}) \right] \ , \\ c_0 &= \frac{i}{2\pi} \int_0^\pi \left[dw\ c(w) - d\overline{w}\ \overline{c}(\overline{w}) \right] \ . \end{aligned}$$

Assume that there are n D25- (or D9- for superstring) branes so that all of the boundary condition is Neumann. Using operator formalism we can derive each contributions as follows.

$$\begin{aligned} \langle 1 \rangle_{\mathrm{mat}} &= n^2\ \mathrm{Tr}\left[q^{L_0 - \frac{c}{24}} \right] \qquad (q = e^{-2\pi l} \ , \ L_0 = \alpha' p^2 + \sum_{n=1}^\infty \alpha_{-n} \cdot \alpha_n) \\ &= n^2 \cdot \frac{i V_{26}}{(2\pi)^{26}} (2 l \alpha')^{-13} \cdot \eta(il)^{-26} \ . \\ \langle b_0 c_0 \rangle_{\mathrm{gh}} &= \mathrm{Tr}\left[(-1)^F b_0 c_0 q^{L_0 - \frac{c}{24}} \right] = \eta(il)^2 \qquad (L_0 = \sum_{n \in \mathbb{Z}} n :b_{-n} c_n: -1) \ . \end{aligned}$$

Therefore, the amplitude is

$$A_{0,C} = n^2 \cdot \frac{iV_{26}}{(2\pi\ell_s)^{26}} \int_0^\infty \frac{dl}{2l} \frac{1}{(2l)^{13}\eta(il)^{24}} ,$$

where $\ell_s = \sqrt{\alpha'}$ is the string length. Let us look into the physical information that can be read off from the amplitude.

- UV divergence: there is a UV divergence from $l \to 0$, as opposed to the closed string case.
- Open string short 1-loop = closed string long propagation (see Fig. 22).
  This is justified by rewriting the amplitude. Using $\eta(il) = l^{-\frac{1}{2}}\eta(il^{-1})$ we have

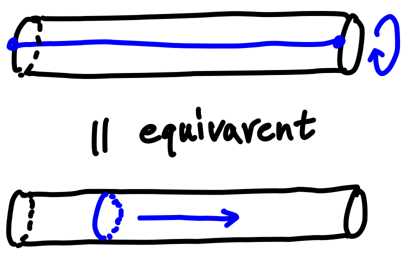


Figure 22: Pictorical "proof" of the equivalence between open string 1-loop and closed string propagation.

$$A_{0,C} = \frac{n^2}{2^{14}} \cdot \frac{iV_{26}}{(2\pi\ell_s)^{26}} \int_0^\infty ds\ \eta(is)^{-24} ,$$

$$\text{where} \quad \eta(is)^{-24} = q^{-1} + 24 + \cdots \equiv \sum_{N=0}^{\infty} \mathcal{N}_N q^{N-1} \quad (q = e^{-2\pi s}) .$$

  Compare with the torus partition function (exercise).
- The UV divergence $l \to 0$ is replaced by IR divergence $s \to \infty$ of a closed string propagation, which can be understood particle propagations (sum of lines) as follows (see also Fig. 23).

Figure 23: Intermediate propagation is replaced by particles(lines).

$$\int_0^\infty ds \sum_{N=0}^{\infty} \mathcal{N}_N e^{-2\pi s(N-1)} \sim \sum_i \int_0^\infty ds\ e^{-s(k^2+m_i^2)}\bigg|_{k=0} = \sum_i \frac{1}{k^2+m_i^2}\bigg|_{k=0} .$$

  We can see that the IR divergence is from massless particle propagation (graviton etc.), which is absorbed or emitted from D25-branes.

- In conclusion, the divergence is due to the existence of the D25-branes, which has definite tension (this is why they emit graviton/dilaton). Can this not be eliminated ?? $\rightarrow$ unoriented string.

Klein bottle amplitude

Next, let us consider a Klein bottle for WS and compute the amplitude on it. As the Klein bottle can be realized by orientation flip operator as in Fig. 24 it should be

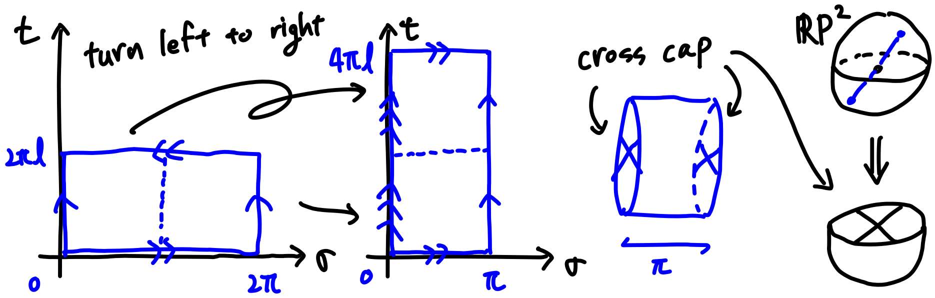


Figure 24: Klein bottle. It can be described by a cylinder with cross cap boundary on both ends.

$$A_{0,K} = \frac{1}{2}\int \frac{dl}{2l}\mathrm{Tr}\left[\Omega(-1)^F \frac{1}{2}(b_0+\bar{b}_0)\frac{1}{2}(c_0+\bar{c}_0)q^{L_0+\bar{L}_0-\frac{c}{12}}\right] \qquad (q=e^{-2\pi l}) \ ,$$

where the factor of $\frac{1}{2}$ is from the projection operator $\frac{1+\Omega}{2}$, or you can also regard it as an additional gauge redundancy $w \rightarrow \overline{w}$.

One can rewrite the amplitude as follows.

$$A_{0,K} = \frac{1}{2}\int \frac{dl}{2l}\mathrm{Tr}\left[(-1)^F b_0 c_0 q^{2L_0-\frac{c}{12}}\right] \ ,$$

where $c = c_{\text{mat}} + c_{\text{gs}} = 0$, and

$$L_0 = \frac{\alpha'}{4}p^2 + \sum_{n=1}^{\infty}(\alpha_{-n}\cdot\alpha_n + nb_{-n}c_n + nc_{-n}b_n) - 1 \ .$$

Thus, the result is

$$\begin{aligned} A_{0,K} &= \frac{iV_{26}}{(2\pi\ell_s)^{26}}\int \frac{dl}{4l}\frac{1}{l^{13}\eta(2il)^{24}} \ , \\ &= 2^{13}\frac{iV_{26}}{(2\pi\ell_s)^{26}}\int \frac{ds}{4}\eta(is)^{-24} \ . \end{aligned}$$

Möbius strip amplitude

Finally, let us consider a Möbius strip for WS (see Fig. 25). It has

- The range is $0 \le \sigma \le \pi$, the period is $2\pi l$, coming together with the orientation flip: $(t,\sigma) \sim (t+2\pi l, \pi - \sigma)$.

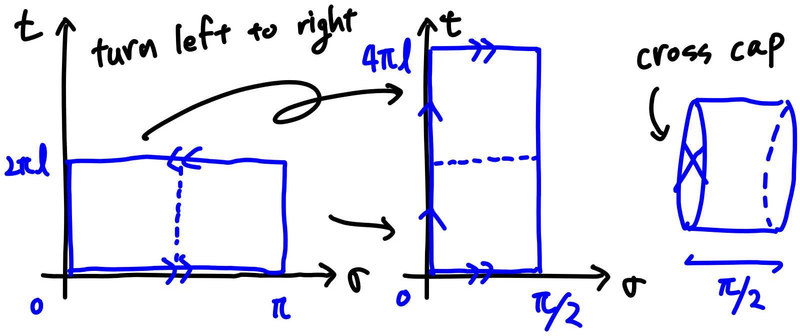


Figure 25: Möbius strip.

- There is a real modulus l; the amplitude needs a b zero mode insertion.
- There is a real isometry, shift of t; the amplitude needs a c zero mode insertion.

The Möbius strip amplitude is

$$A_{0,M} = \frac{1}{2}\int \frac{dl}{2l} \mathrm{Tr}\left[\Omega(-1)^F b_0 c_0 q^{L_0 - \frac{c}{12}}\right] \qquad (q = e^{-2\pi l}) \ .$$

The trace is over Hilbert space of the matter and ghost sectors on the strip as well as the Chan-Paton index, which gives in total n^2 degeneracy for each state. We can divide the effect of Ω into two parts as follows.

$$\Omega|\Phi;\Lambda\rangle = |\Omega\Phi; P\Lambda^T P\rangle \equiv \Omega_\Phi \cdot \Omega_\Lambda |\Phi;\Lambda\rangle \ .$$

Let us see the Ω_Λ, which is defined as

$$\Omega_\Lambda = \frac{P\Lambda^T P}{\Lambda} \ .$$

In the case of SO(n), which means $P = 1$,

$$\Omega_{\Lambda,SO} = \frac{\Lambda^T}{\Lambda} = \begin{cases} +1 & \text{(for symmetric } \Lambda) \\ -1 & \text{(for anti-symmetric } \Lambda) \end{cases} \ ,$$

Therefore,

$$\mathrm{Tr}_{\Lambda,SO}\left[\Omega\right] = \frac{n(n+1)}{2} - \frac{n(n-1)}{2} = n \ .$$

In the case of Sp(n) (exercise)

$$\mathrm{Tr}_{\Lambda,Sp}\left[\Omega\right] = -n \ .$$

The matter and ghost part is

$$\begin{aligned} \mathrm{Tr}_\Phi\left[\Omega(-1)^F b_0 c_0 q^{L_0-\frac{c}{12}}\right] &= \frac{iV_{26}}{(2\pi\ell_s)^{26}} \frac{1}{(2l)^{13}} \cdot q^{-1} \prod_{n=1}^{\infty} \frac{(1-(-q)^n)^2}{(1-(-q)^n)^{26}} \\ &= \frac{iV_{26}}{(2\pi\ell_s)^{26}} \frac{1}{(2l)^{13}} \cdot \frac{-1}{\eta(il+\frac{1}{2})^{24}} \ . \end{aligned}$$

Therefore, the Möbius strim amplitude is

$$A_{0,M} = \pm n \cdot \frac{iV_{26}}{(2\pi\ell_s)^{26}} \int \frac{dl}{4l} \frac{1}{(2l)^{13}} \cdot \frac{-1}{\eta(il+\frac{1}{2})^{24}} \ ,$$
$$= \mp n \cdot \frac{iV_{26}}{(2\pi\ell_s)^{26}} \int \frac{ds}{2} \eta(is+\frac{1}{2})^{-24} \ ,$$

where we used $\sqrt{2l}\eta(il+\frac{1}{2}) = \eta(\frac{i}{4l}+\frac{1}{2})$.

To sum up, three amplitudes are (introduced additional $\frac{1}{2}$ factor for Cylinder as an unoriented amplitude)

$$A_{0,C} = \frac{n^2}{2^{13}} \cdot \frac{iV_{26}}{(2\pi\ell_s)^{26}} \int_0^\infty \frac{ds}{4} \eta(is)^{-24} \ ,$$
$$A_{0,K} = 2^{13} \cdot \frac{iV_{26}}{(2\pi\ell_s)^{26}} \int \frac{ds}{4} \eta(is)^{-24} \ ,$$
$$A_{0,M} = \mp 2n \cdot \frac{iV_{26}}{(2\pi\ell_s)^{26}} \int \frac{ds}{4} \eta(is+\frac{1}{2})^{-24} \ ,$$

where

$$\eta(is)^{-24} = q^{-1} + 24 + \mathcal{O}(q) \qquad (q = e^{-2\pi s}) \ ,$$
$$\eta(is+\frac{1}{2})^{-24} = -q^{-1} + 24 + \mathcal{O}(q) \ .$$

As we saw in oriented string case the massless states lead IR singularity. On the other hand, in the unoriented open string case we have

$$\frac{1}{2^{13}} \left[n \mp 2^{13} \right]^2 \cdot \frac{iV_{26}}{(2\pi\ell_s)^{26}} \int_0^\infty \frac{ds}{4} \cdot 24 \ ,$$

which vanish for SO($2^{13} = 8192$). This cancellation can be illustrated as like Fig. 26. The cross cap shows another object (other than D-brane) that absorb and emit

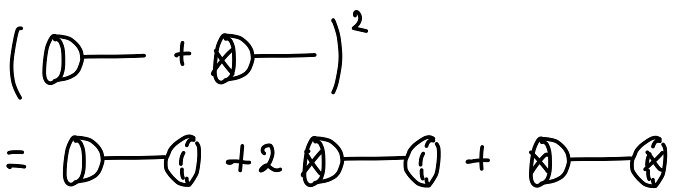


Figure 26: Pictorical expression for the unoriented open string amplitude.

gravitons etc., which is called O-plane. In this situation it should be space-filling, hence, it is O25$^{\pm}$-plane (+ for Sp and − for SO). O25$^{\pm}$-plane has $\pm 2^{13}$ times that of D25-brane, and so single O25^{-}-plane cancel tension of 2^{13} D25-branes.

Although our discussion was in bosonic string, parallel argument perfectly works for superstring and it leads that **the IR divergence vanishes for** SO(32). This means that

$$\text{Type I} = \text{Type IIB} + 32 \text{ D9-branes} + \text{O9}^{-}\text{-plane} \ .$$

Note that in the superstring case D-branes and O-plane have RR-charge in addition to tension, which has relations

$$T_{\text{O9}^{\pm}} = \pm 32 \cdot T_{\text{D9}} \quad (\text{tension}) \ , \qquad Q_{\text{O9}^{\pm}} = \pm 32 \cdot Q_{\text{D9}} \quad (\text{RR-charge}) \ .$$

10.1.3 T-duality of Type I theory

Let us recall T-duality. Consider X^i is S^1 compactified and T-duality acts as follows:

$$T_i : \quad X^i(z,\overline{z}) = X^i(z) + \overline{X}^i(\overline{z}) \quad \to \quad X'^i(z,\overline{z}) = X^i(z) - \overline{X}^i(\overline{z}) \ .$$

On the other hand, the orientation flip acts as follows:

$$\Omega : \quad X^i(z,\overline{z}) = X^i(z) + \overline{X}^i(\overline{z}) \quad \to \quad X^i(\overline{z},z) = \overline{X}^i(z) + X^i(\overline{z}) \ .$$

Therefore, in the T-dual coordinate X' the orientation flip acts as

$$\Omega : \quad X'^i(z,\overline{z}) \quad \to \quad -X'^i(\overline{z},z) \ .$$

This is understood as space-time **oribfold** as well as world-sheet orientation flip (see Fig. 27), which is called **orientifold**. Therefore, the dual space is not S^1 but $S^1/\mathbb{Z}_2$

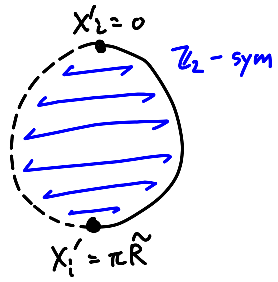


Figure 27: $\mathbb{Z}_2$ orbifold of S^1.

with radius $\widetilde{R} = \frac{\alpha'}{R}$. Note that there are two fixed points, where O-planes sit and causes the ST reversal and the orientation flip.

Let us consider Type I superstring theory with X^9 compactified on S^1 and take T-duality along the S^1. With a proper Wilson line

$$A_9 = i \begin{pmatrix} & -a_1 & & & \\ a_1 & & & & \\ & & & -a_2 & \\ & & a_2 & & \\ & & & & \ddots \end{pmatrix}$$

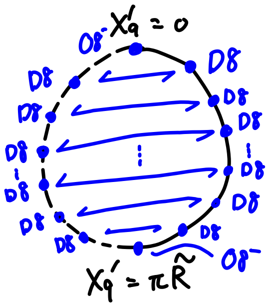


Figure 28: T-dual of Type I superstring theory.

D8-branes sit at different points in $\mathbb{Z}_2$ symmetric way (see Fig. 28). Note that an O9$^-$-plane splits into two O8$^-$-plane. Accordingly, tension and RR-charge reduce by 2.

In the end, T-dual of Type I on S^1 is

$$\text{Type IIA on } S^1/\mathbb{Z}_2 \quad \text{with} \quad 2 \text{ O8}^-\text{-plane } + 32 \text{ D8-branes} .$$

Of course, one can consider further T-dualities along other directions. Each T-duality doubles the number of O-planes, and hence, reduces the tension and the RR-charges. Namely, we have following relations:

$$T_{\mathrm{O}p^\pm} = \pm 2^{p-4} \cdot T_{\mathrm{D}p} \quad (\text{tension}) , \quad Q_{\mathrm{O}p^\pm} = \pm 2^{p-4} \cdot Q_{\mathrm{D}p} \quad (\text{RR-charge}) .$$

11 Lecture 11

We are ready to learn **Heterotic string theories** [GHMR85b, GHMR85a, GHMR86]. Heterotic string is a hybrid construction of the left-moving sector of the 26-dimensional bosonic string and the right-moving sector of 10-dimensional superstring. The 16 extra bosons of the left-movers are compactified on particular 16-dimensional tori, leading to SO(32) or $E_8 \times E_8$. Since 16-dimensional tori have very special properties, we can also describe the left-movers in terms of 32 free fermions whose current algebra is associated to either SO(32) or $E_8 \times E_8$ with level $k = 1$. This hybridization of two different kinds of modes has been referred to as **heterosis**.

11.1 Bosonic construction

11.1.1 Toroidal compactifications

We have learnt the S^1 compactification so that we now generalize our analysis to the case of D-dimensions compactified on a torus T^D. The resulting theory is effectively $(26 - D)$-dimensional. The torus is defined by identifying points in the D-dimensional internal space as follows (compact dimensions are labeled with capital letters):

$$X^I \sim X^I + 2\pi e_i^I n^i = X^I + 2\pi W^I , \qquad \text{for} \quad n^i \in \mathbb{Z} . \tag{11.1}$$

The $\mathbf{e}_i = \{e_i^I\}$ $(i = 1 \cdots D)$ are D linear independent vectors called **vielbein** which generate a D-dimensional lattice Λ. In addition, the vielbein brings the metric into the standard Euclidean form:

$$G_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j = e_i^I e_j^J \delta_{IJ}, \qquad X^I \equiv e_i^I X^i \tag{11.2}$$

The torus on which we compactify is obtained by dividing $\mathbb{R}^D$ by Λ:

$$T^D = \frac{\mathbb{R}^D}{2\pi\Lambda} \ .$$

The momentum p^I conjugate to the coordinates X^I on the torus is quantized as $\mathbf{p} \cdot \mathbf{W} \in \mathbb{Z}$. Therefore, the momentum p takes its value on the dual lattice Λ^*

$$\Lambda^* \equiv \{e^{*Ii} m_i; \quad m_i \in \mathbb{Z}\}, \qquad G^{ij} = \mathbf{e}^{*i} \cdot \mathbf{e}^{*j} = e_I^{*i} e_J^{*j} \delta^{IJ}.$$

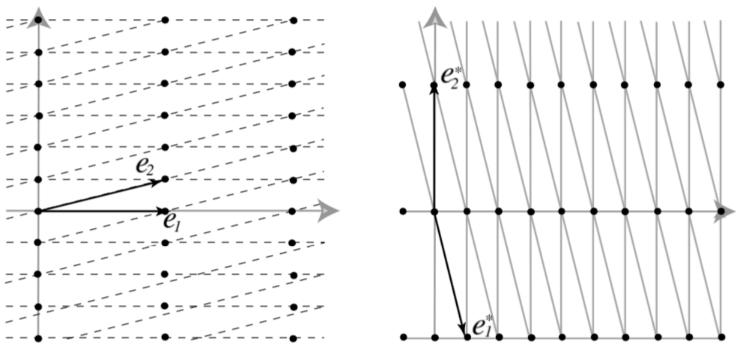


Figure 29: Lattice and its dual lattice

The condition which a closed string in the compact directions has to satisfy is

$$X^I(\sigma + 2\pi, \tau) = X^I(\sigma, \tau) + 2\pi W^I$$

so that W^I are analogues of winding number. We express the mode expansion for the compact direction as follows:

$$X^I(z) = x^I - i\sqrt{\frac{\alpha'}{2}} p_R^I \ln z + i\sqrt{\frac{\alpha'}{2}} \sum_{m \neq 0} \frac{\alpha_m^I}{m z^m},$$
$$\overline{X}^I(\overline{z}) = \widetilde{x}^I - i\sqrt{\frac{\alpha'}{2}} p_L^I \ln \overline{z} + i\sqrt{\frac{\alpha'}{2}} \sum_{m \neq 0} \frac{\widetilde{\alpha}_m^I}{m \overline{z}^m}.$$

where the zero modes are

$$\begin{aligned}
\mathbf{p}_L := p_L^I &= \frac{1}{\sqrt{2}}\left[\sqrt{\alpha'} p^I + \frac{W^I}{\sqrt{\alpha'}}\right] = \frac{1}{\sqrt{2}}\left[\sqrt{\alpha'} e^{*Ii} m_i + \frac{e_i^I}{\sqrt{\alpha'}} n^i\right] \ , \\
\mathbf{p}_R := p_R^I &= \frac{1}{\sqrt{2}}\left[\sqrt{\alpha'} p^I - \frac{W^I}{\sqrt{\alpha'}}\right] = \frac{1}{\sqrt{2}}\left[\sqrt{\alpha'} e^{*Ii} m_i - \frac{e_i^I}{\sqrt{\alpha'}} n^i\right] \ .
\end{aligned} \tag{11.3}$$

The mass formula and the level matching condition are now

$$\begin{aligned}\alpha' M^2 &= 2(N+\widetilde{N}-2)+(\alpha' p_I p^I+\frac{1}{\alpha'}W_I W^I)\\ &= 2(N+\widetilde{N}-2)+(\alpha' m_i m_j G^{ij}+\frac{1}{\alpha'}n^i n^j G_{ij})\\ N-\widetilde{N} &= p_I W^I = m_i n^i \end{aligned} \tag{11.4}$$

As we have seen before, the expressions for p_L and p_R suggest **T-duality** between the winding number W^I and the momentum p^I. In fact, **T-duality** is equivalence between a pair of compactification lattices $\mathbf{e}_i$ and $\mathbf{e}'_i$ that are related as $\sqrt{\alpha''}\mathbf{e}'_i = \frac{\mathbf{e}^{*i}}{\sqrt{\alpha''}}$. These two compactifications give the same spectrum since their allowed values of the momenta are related as

$$\mathbf{p}_L \leftrightarrow \mathbf{p}'_L; \qquad \mathbf{p}_R \leftrightarrow -\mathbf{p}'_R \tag{11.5}$$

by interchanging the labels m_i and n^i.

Now let us combine the zero modes into the $(D+D)$-dimensional vectors $\mathbf{P} = (\mathbf{p}_L, \mathbf{p}_R)$. This construction treats Λ and Λ^* on equal footing as

$$\mathbf{P} = \mathbf{E}^{*i} m_i + \mathbf{E}_j n^j ,$$

where

$$\mathbf{E}_j = \frac{1}{\sqrt{\alpha'}}(\mathbf{e}_j, -\mathbf{e}_j) \ , \quad \mathbf{E}^{*i} = \sqrt{\alpha'}\left(\mathbf{e}^{*i}, \mathbf{e}^{*i}\right).$$

Note that the length of the lattice is normalized by the string length $\sqrt{\alpha'} = \ell_s$. Hence $\mathbf{P}$ takes value in a $(D+D)$-dimensional lattice $\Gamma_{D,D}$ spanned by $\{\mathbf{E}^{*i}\}$ and $\{\mathbf{E}_j\}$ that satisfies the following properties:

- **Lorentzian** if the signature of the metric G is $((+1)^D, (-1)^D)$,
- **integral** if $v \cdot w \in \mathbb{Z}$ for all $v, w \in \Gamma_{D,D}$,
- **even** if $\Gamma_{D,D}$ is integral and v^2 is even for all $v \in \Gamma_{D,D}$,
- **self-dual** if $\Gamma_{D,D} = (\Gamma_{D,D})^*$,
- **unimodular** if $\mathrm{Vol}(\Gamma_{D,D}) = |\det G| = 1$.

In fact, the metric of this lattice is defined by

$$\mathbf{P}\cdot\mathbf{P}' = (\mathbf{p}_L\cdot\mathbf{p}_L - \mathbf{p}_R\cdot\mathbf{p}_R) = m_i n'^i + m'_i n^i$$

so that it is Lorentzian. Because of $\mathbf{P}\cdot\mathbf{P} \in 2\mathbb{Z}$, it is even. The self-dual property will be shown in Homework. The unimodular property $\mathrm{Vol}(\Gamma_{D,D}) = \mathrm{Vol}(\Gamma_{D,D}) = 1$ immediately follows from the self-dual property. The lattice $\Gamma_{D,D}$ in the torus compactification of the string is called **Narain lattice**.

The partition function of the bosonic string compactified on a torus T^D is easy to write down:

$$Z^{\mathrm{bos}}_{\Gamma_{D,D}} = \frac{1}{\tau_2^{(24-D)/2}|\eta(q)|^{48}} \sum_{(\mathbf{p}_R,\mathbf{p}_L)\in\Gamma_{D,D}} q^{\frac{1}{2}\mathbf{p}_R^2}\overline{q}^{\frac{1}{2}\mathbf{p}_L^2}$$

where $|\eta(q)|^{48}$ is the bosonic oscillator contribution and $\tau_2^{(24-D)/2}$ comes from the integral of non-compact momenta. This is easy to generalize to Type II string compactified on T^D

$$Z^{\text{Type II}}_{\Gamma_{D,D}} = \frac{1}{\tau_2^{(8-D)/2}|\eta(q)|^{24}}\frac{1}{4}\Big| -\vartheta_2^4(\tau)+\vartheta_3^4(\tau)-\vartheta_4^4(\tau)\Big|^2 \sum_{(\mathbf{p}_R,\mathbf{p}_L)\in\Gamma_{D,D}} q^{\frac{1}{2}\mathbf{p}_R^2}\overline{q}^{\frac{1}{2}\mathbf{p}_L^2}$$

which vanishes by virtue of the Jacobi-Riemann identity.

11.1.2 Heterotic strings

After our discussion of toroidal compactifications, we are now prepared to introduce the ten-dimensional Heterotic string. As mentioned in the beginning, Heterotic string is a combination of the left-moving sector of the 26-dimensional bosonic string combined with the right-moving sector of the 10-dimensional superstring. The left-moving bosonic string is compactified on a 16-dimensional torus so that the momenta of the additional chiral bosons $X^I(\overline{z})$ takes value on 16-dimensional lattice Γ_{16}, *i.e* $\mathbf{p}_L \in \Gamma_{16}$. Hence, the partition function of Heterotic string can be written as

$$Z^{\text{het}}(\tau) = \frac{1}{\tau_2^4\eta(q)^{12}\eta(\overline{q})^{24}}\Big(-\vartheta_2^4(\tau)+\vartheta_3^4(\tau)-\vartheta_4^4(\tau)\Big)\sum_{\mathbf{p}_L\in\Gamma_{16}} \overline{q}^{\frac{1}{2}\mathbf{p}_L^2} \tag{11.6}$$

Here $\eta(q)^8\eta(\overline{q})^{24}$ is the bosonic oscillator contribution, the τ_2^4 factor arises from the zero modes of the uncompactified transverse coordinates and $\vartheta_i^4/\eta(q)^4$ comes from the world-sheet fermions. The most interesting part of this partition function is the lattice sum

$$P(\tau) := \sum_{\mathbf{p}_L\in\Gamma_{16}} \overline{q}^{\frac{1}{2}\mathbf{p}_L^2}$$

Since the partition function (11.6) should be invariant under the modular transformation $\mathrm{SL}(2,\mathbb{Z})$, the modular transformation of η and ϑ_i tell us that

$$T: P(\tau+1) = P(\tau)\ , \qquad S: P(-1/\tau) = \tau^8 P(\tau)\ .$$

The invariance under T-transformation clearly demands that $\mathbf{p}_L^2 \in 2\mathbb{Z}$ so that Γ_{16} must be **even**. For the S-transformation, we make use of the Poisson resummation formula

$$\sum_{\mathbf{p}\in\Lambda} e^{-\pi\alpha(\mathbf{p}+\mathbf{x})^2+2\pi i\mathbf{y}\cdot(\mathbf{p}+\mathbf{x})} = \frac{1}{\mathrm{Vol}(\Lambda)\alpha^{\dim\Lambda/2}}\sum_{\mathbf{q}\in\Lambda^*} e^{-2\pi i\mathbf{q}\cdot\mathbf{x}-\frac{\pi}{\alpha}(\mathbf{y}+\mathbf{q})^2}$$

which amounts to

$$P(-1/\tau) = \frac{\tau^8}{\mathrm{Vol}(\Lambda)}\sum_{\mathbf{p}_L\in(\Gamma_{16})^*} \overline{q}^{\frac{1}{2}\mathbf{p}_L^2}\ .$$

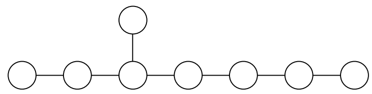
This requires that the lattice Γ_{16} is **self-dual**, *i.e.* $(\Gamma_{16})^* = \Gamma_{16}$ so that $\mathrm{Vol}(\Lambda)=1$.

It turns out that there are only two even self-dual Euclidean lattices in 16 dimensions

- the root lattice of $E_8 \times E_8$
- the weight lattice of $\mathrm{Spin}(32)/\mathbb{Z}_2$

The metric G_{ij} of the root lattice of E_8 is the Cartan matrix of E_8 [6]:

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix} .$$



Let us investigate the on-shell spectrum of Heterotic string more carefully. As usual, there is the tachyonic vacuum of the bosonic string. At the massless level we have oscillator excitations $\widetilde{\alpha}^{\mu}_{-1}|0\rangle$, $\widetilde{\alpha}^{I}_{-1}|0\rangle$ in left-moving sector. The former transform like space-time vectors whereas the internal oscillator excitations correspond to the left-moving part of the Abelian $\mathrm{U}(1)^{16}$ gauge boson. They build the **Cartan subalgebra** of $E_8 \times E_8$ or SO(32). Both the root lattice of $E_8 \times E_8$ and the weight lattice of $\mathrm{Spin}(32)/\mathbb{Z}_2$ contain 480 vectors of $(\text{length})^2 = 2$ and generate the 496-dimensional non-Abelian gauge bosons of these groups. Remarkably, although Heterotic strings are closed strings, gauge fields show up thanks to the extra 16-dimensional tori! This can be also understood as novel **stringy effect** and gauge groups are restricted only to either $E_8 \times E_8$ or SO(32) in order for the theory to be consistent. Moreover, we have seen that SO(32) gauge group appears in Type I string theory. As we will see in the subsequent lecture, this is not coincident because Type I and Heterotic SO(32) are related by **S-duality**.

As a result, the massless spectra of Heterotic string are as follows

- Gravitons, B-fields, dilaton in 10D
$$\psi^{\mu}_{-\frac{1}{2}}|0\rangle_{\mathrm{NS}} \otimes \widetilde{\alpha}^{\nu}_{-1}|0\rangle$$
- their supersymmetric partners, gravitino and dilatino
$$|\mathbf{s}\rangle_{\mathrm{R}} \otimes \widetilde{\alpha}^{\nu}_{-1}|0\rangle$$
- 496 gauge bosons of $E_8 \times E_8$ or SO(32)
$$\psi^{\mu}_{-\frac{1}{2}}|0\rangle_{\mathrm{NS}} \otimes \widetilde{\alpha}^{I}_{-1}|0\rangle \ , \qquad \psi^{\mu}_{-\frac{1}{2}}|0\rangle_{\mathrm{NS}} \otimes |\mathbf{p}_L^2 = 2\rangle$$
- 496 supersymmetric partners, gaugini
$$|\mathbf{s}\rangle_{\mathrm{R}} \otimes \widetilde{\alpha}^{I}_{-1}|0\rangle \ , \qquad |\mathbf{s}\rangle_{\mathrm{R}} \otimes |\mathbf{p}_L^2 = 2\rangle$$

Indeed, Heterotic string theory is 10D $\mathcal{N} = 1$ supergravity coupled to 10D $\mathcal{N} = 1$ $E_8 \times E_8$ or SO(32) super-Yang-Mills theory so that it has 16 real supersymmetric charges.

[6]Unfortunately, we do not have time to talk about exceptional Lie algebra or a classification of semi-simple Lie algebras [Kir08]. In particular, if you want to get some intuition of weight and root lattices, see [Kir08, Fig 7.3, Fig 8.1, Fig 8.2] for A_2. If you want to understand the structure E_8 related to string theory, we refer to [GSW87, §6].

11.2 Fermionic construction

The 16 bosonic fields compactified on the self-dual lattice can be described by fermionic fields, which is called **fermionization**. Therefore, we will describe fermionic construction of Heterotic string theory next.

The world-sheet action of Heterotic string theory is given by

$$\begin{aligned} S^{\mathrm{m}} &= \frac{1}{4\pi}\int d^2z \left(\frac{2}{\alpha'}\partial X^\mu \overline{\partial} X_\mu + \psi^\mu \overline{\partial}\psi_\mu + \widetilde{\lambda}^A \partial \widetilde{\lambda}_A\right) \\ S^{\mathrm{gh}} &= \frac{1}{2\pi}\int d^2z \; (b\overline{\partial}c + \overline{b}\partial\overline{c} + \beta\overline{\partial}\gamma) \end{aligned} \tag{11.7}$$

where μ are 10-dimensional indices and the right-moving sector is supersymmetric. In order for the theory to be Weyl-anomaly free, the central charge

$$c^{\mathrm{tot}} = c^X + c^\psi + c^{bc} + c^{\beta\gamma} + c^\lambda = 10 + \frac{5}{2} - 26 + \frac{11}{2} + c^\lambda = c^\lambda - 8$$

should vanish. Since each Majorana-Weyl anti-chiral fermion $\widetilde{\lambda}^A$ contributes $\frac{1}{4}$ to the central charge, we need 32 left-moving fermions $\widetilde{\lambda}^A$ in the action.

It turns out that there are two possible boundary conditions on the left-moving fermions $\widetilde{\lambda}_A$ which give rise to fully consistent string theories. If we impose the same boundary condition to all, it leads to SO(32) gauge group. On the other hand, if we impose one boundary condition to a half and the other boundary condition to the other half, we obtain $E_8 \times E_8$ gauge group.

11.2.1 Heterotic SO(32) (HO)

For Heterotic SO(32), we impose the same boundary condition to all the left-moving fermions as

$$\begin{aligned} \widetilde{\lambda}^A(t,\sigma+2\pi) &= +\widetilde{\lambda}^A(t,\sigma) && \text{R: periodic on cylinder} \\ \widetilde{\lambda}^A(t,\sigma+2\pi) &= -\widetilde{\lambda}^A(t,\sigma) && \text{NS: anti-periodic on cylinder} \end{aligned} \tag{11.8}$$

so that there is a global symmetry SO(32) that rotates $\widetilde{\lambda}^A$ $(A = 1, \ldots, 32)$. In order for the theory to be consistent, we have to impose GSO projection on the left-moving sector. In HO theory, we pick only states with odd fermionic number in NS sector and those with even fermionic number

$$P^{\mathrm{HO}}_{\mathrm{NS}} := \frac{1-(-1)^F}{2}$$

whereas we keep only the states with even fermion number

$$P^{\mathrm{HO}}_{\mathrm{R}} := \frac{1+(-1)^F}{2} \ .$$

In addition, we have to impose the level matching condition

$$N - a = \widetilde{N} - \widetilde{a} \tag{11.9}$$

where the normal ordering constants in the left-moving sector are

$$\widetilde{a}_{\mathrm{NS}} = \frac{8}{24} + \frac{32}{48} = 1 \ , \qquad \widetilde{a}_{\mathrm{R}} = \frac{8}{24} - \frac{32}{24} = -1 \ .$$

Here the first term comes from the left-moving bosonic field $\overline{X}^i$ and the second term comes from $\widetilde{\lambda}^A$. Hence, the R sector contains only massive states. Contrary to the supersymmetric right-mover, the Tachyon state $|0\rangle_{\mathrm{NS}}$ in the NS sector is preserved under the GSO projection. However, there is no corresponding state in the right-moving sector so that it does not obey the level matching condition (11.9). As a result, the left-moving Tachyon is not included in the spectrum. Then, the first excited states after the GSO projection in the NS sector are

$$\begin{aligned} \alpha^i_{-1}|0\rangle_{\mathrm{NS}} \ , \qquad & (\mathbf{8_v}, \mathbf{1}) \\ \widetilde{\lambda}^A_{-1/2}\widetilde{\lambda}^B_{-1/2}|0\rangle_{\mathrm{NS}} \ , \qquad & (\mathbf{1}, \mathbf{adj}) \end{aligned}$$

where the bold letters are the representations of SO(8) $\times$ SO(32). The adjoint representation **adj** of SO(32) is the antisymmetric tensor with dimension $32 \times 31/2 =$ **496**. The following table shows the massless spectrum of HO where the first row represent the 10d $\mathcal{N} = 1$ supergravity multiplet whereas the second row shows $\mathcal{N} = 1$ gauge multiplet in the adjoint of SO(32) as we have seen in the bosonic construction.

Left\Right	$\mathbf{8_v}$	$\mathbf{8_c}$
$(\mathbf{8_v}, \mathbf{1})$	$\mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}$ $\phi \quad B_{\mu\nu} \quad G_{\mu\nu}$	$\mathbf{8_s} \oplus \mathbf{56_c}$ $\lambda^+ \quad \psi^-_m$
$(\mathbf{1}, \mathbf{496})$	SO(32) gauge boson $A^\mu_{[A,B]}$	SO(32) gaugini $\eta_{[A,B]}$

11.2.2 Heterotic $E_8 \times E_8$ (HE)

The second Heterotic string theory is obtained by dividing the $\widetilde{\lambda}^A$ into two sets of 16 with independent boundary conditions,

$$\widetilde{\lambda}^A(t, \sigma + 2\pi) = \begin{cases} \epsilon_1 \widetilde{\lambda}^A(t, \sigma) & A = 1, \ldots, 16 \\ \epsilon_2 \widetilde{\lambda}^A(t, \sigma) & A = 17, \ldots, 32 \end{cases}$$

where $\epsilon_i = \pm 1$. Therefore, in the left-moving sector, we need to take the following boundary conditions into account

$$(\mathrm{NS}_1, \mathrm{NS}_2) \ , \quad (\mathrm{R}_1, \mathrm{NS}_2) \ , \quad (\mathrm{NS}_1, \mathrm{R}_2) \ , \quad (\mathrm{R}_1, \mathrm{R}_2) \ .$$

Consequently, the global symmetry is broken to $\mathrm{SO}(16)_1 \times \mathrm{SO}(16)_2$. The GSO projection is imposed to the two sets of left-movers independently:

$$P^{\mathrm{HE}}_{\mathrm{NS}_i} := \frac{1-(-1)^F}{2} \qquad P^{\mathrm{HE}}_{\mathrm{R}_i} := \frac{1+(-1)^F}{2} \ .$$

We also apply for the level-matching condition (11.9). The normal ordering constant in each boundary condition is

$$\widetilde{a}_{\mathrm{NS}_1,\mathrm{NS}_2} = 1 \ , \qquad \widetilde{a}_{\mathrm{R}_1,\mathrm{NS}_2} = \widetilde{a}_{\mathrm{NS}_1,\mathrm{R}_2} = \frac{8}{24} + \frac{16}{48} - \frac{16}{24} = 0 \ , \qquad \widetilde{a}_{\mathrm{R}_1,\mathrm{R}_2} = -1 \ .$$

Again, $(\mathrm{R}_1, \mathrm{R}_2)$ boundary condition has only massive states. Although the Tachyon state $|0\rangle_{\mathrm{NS}_1,\mathrm{NS}_2}$ in the NS sector is preserved under the GSO projection, it does not obey the level-matching condition (11.9) so that it is not present in the spectrum. Then, the massless states are

$$\begin{aligned} &\alpha^i_{-1}|0\rangle_{\mathrm{NS}_1,\mathrm{NS}_2} \ , && (\mathbf{8_v}, \mathbf{1}) \\ &\widetilde{\lambda}^A_{-1/2}\widetilde{\lambda}^B_{-1/2}|0\rangle_{\mathrm{NS}_1,\mathrm{NS}_2} \ , && (\mathbf{1}, \mathbf{adj}, \mathbf{1}) \text{ or } (\mathbf{1}, \mathbf{1}, \mathbf{adj}) \end{aligned} \tag{11.10}$$

where the bold letters are the representations of $\mathrm{SO}(8) \times \mathrm{SO}(16)_1 \times \mathrm{SO}(16)_2$. Note that the GSO projection requires either $1 \le A, B \le 16$ or $17 \le A, B \le 32$ in (11.10). The adjoint representation of SO(16) is of $16 \times 15/2 = \mathbf{120}$ dimension.

In $(\mathrm{R}_1, \mathrm{NS}_2)$ and $(\mathrm{NS}_1, \mathrm{R}_2)$, the ground states are massless since the normal ordering constant is zero. Since the 16 $\widetilde{\lambda}^A_0$ zero modes form 8 raising and 8 lowering operators

$$\widetilde{\lambda}^{K\pm}_0 = 2^{-1/2}(\widetilde{\lambda}^{2K-1}_0 \pm i\widetilde{\lambda}^{2K}_0) \ , \qquad K = 1, \dots, 8 \text{ or } K = 9, \dots, 16 \ ,$$

the $2^8 = \mathbf{256}$-dimensional spinor representation of SO(16) becomes massless. However, the GSO projection picks positive chirality $\mathbf{128}$ out of $\mathbf{256} = \mathbf{128} + \mathbf{128}'$ in the Ramond sector. Hence, the ground states are $(\mathbf{1}, \mathbf{128}, \mathbf{1})$ and $(\mathbf{1}, \mathbf{1}, \mathbf{128})$ under $\mathrm{SO}(8) \times \mathrm{SO}(16)_1 \times \mathrm{SO}(16)_2$ in $(\mathrm{R}_1, \mathrm{NS}_2)$ and $(\mathrm{NS}_1, \mathrm{R}_2)$, respectively .

All in all, the left-moving massless states form the representations of $\mathrm{SO}(8) \times \mathrm{SO}(16)_1 \times \mathrm{SO}(16)_2$

$$(\mathbf{8_v}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{120}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{120}) + (\mathbf{1}, \mathbf{128}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{128})$$

This spectrum strongly suggests that gauge symmetry is enhanced $\mathrm{SO}(16) \to E_8$ because E_8 has dimension $\mathbf{120} + \mathbf{128} = \mathbf{248}$ which is also the dimension of the adjoint representation E_8. In fact, E_8 has an SO(16) subgroup under which the E_8 adjoint $\mathbf{248}$ transforms as $\mathbf{120} + \mathbf{128}$. Hence, the massless spectrum is the 10d $\mathcal{N} = 1$ supergravity multiplet plus an $\mathcal{N} = 1$ $E_8 \times E_8$ gauge multiplet. Even in fermionic construction, we have reproduce the 496-dimensional adjoint representations of both SO(32) and $E_8 \times E_8$ gauge groups.

11.2.3 No D-branes in Heterotic strings

We have seen that D-branes are charged to RR fields in Type II theories. However, there is no RR field in Heterotic string theories because there is only world-sheet supersymmetry in the right-moving sector. In other words, although the RR $(p+1)$-form field strength G in Type II theories can be expressed as

$$G = \overline{\psi}^L \Gamma^{\mu_1 \cdots \mu_{p+1}} \psi^R ,$$

there is no ψ^L in Heterotic string theories. Hence, there is no D-brane in Heterotic string theories. Consequently, Heteroting string theories are the theories of closed strings[7]. However, apart from the fundamental strings, there are extended objects, **NS5-branes or Heterotic fivebranes**, in Heterotic string theories and they are magnetically charged under the B-field.

12 Lecture 12

12.1 Supergravity

String theory includes massless states as well as massive states. However, in a low-energy(IR) region, we do not see the stringy massive states because the mass $M^2 \sim \frac{1}{\alpha'} \sim \frac{1}{\ell_s^2}$ is assumed to be very heavy. Therefore, in the IR limit, we can describe the theory by so called **effective theory**, which only contains the lightest particles(states). The effective theory, of course, does not contain all the information of the original theory, however, it does give us some information of the original theory.

For the bosonic string theory the lightest particles are the massless states (except non-physical tachyon), and we saw that the effective theory is (Lecture 4)

$$S_{\text{eff}} = \frac{1}{2\kappa_{26}^2} \int d^{26}X \sqrt{-G} e^{-2\Phi} \left[R - \frac{1}{12} H_{\mu\lambda\rho} H^{\mu\lambda\rho} + 4\nabla^\lambda \Phi \nabla_\lambda \Phi \right] ,$$

where $2\kappa_{26}^2$ is a gravitational coupling and it is related to the Newton constant as $2\kappa_{26}^2 = 16\pi G_N$. What we will see below is supersymmetric versions of this, that are effective theories of IIA/IIB superstring theory and called type **IIA/IIB SUGRA** (SUpersymmetric GRAvity).

12.1.1 Local SUSY

Before going to the details of the IIA/IIB SUGRA let us see general features of SUGRA.

First of all, in order to define spinors in a curved space we need a vielbein $e_M^a(x^P)$, which transform a local coordinate x^M into a tangent space coordinate (tangent vector) x^a and vice versa (see also Fig. 30):

[7] However Polchinski argues in [Pol06] that there exist open Heterotic strings.

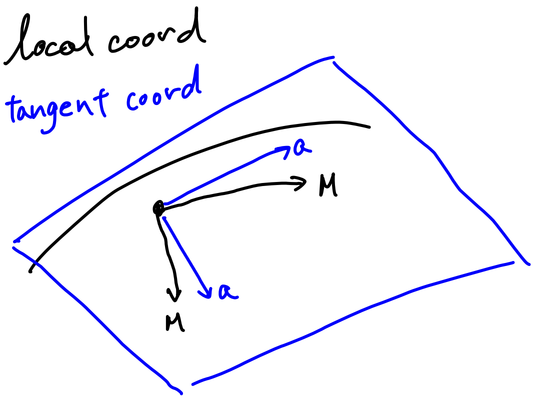


Figure 30: Local coordinate and tangent coordinate.

$$x^a = e^a_M x^M \ , \quad x^M = e^M_a x^a \ ,$$

which is defined through space-time metric G_{MN} by

$$G_{MN}(x^P) = \eta_{ab} e^a_M(x^P) e^b_N(x^P) \ .$$

Now we can use spinor representations and gamma matrices thanks to the vielbein:

$$\{\Gamma^a, \Gamma^b\} = 2\eta^{ab} \ .$$

In the case that a theory includes gravity translation invariance is promoted to general coordinate transformation invariance (the symmetry is localized). Similarly, global Lorentz symmetry is promoted to local Lorentz symmetry (this is done by the vielbein):

$$\Lambda^a{}_b e^M_a(x^P) e^b_N(x^P) \equiv \Lambda^M{}_N(x^P) \ .$$

Gravity is nothing but a gauge field for general coordinate transformation. In general, gauge field has gauge degrees of freedom

$$\delta A_M = D_M \lambda \ ,$$

and for gravity it is

$$\delta e^a_M = D_M \lambda^a \ ,$$

where λ^a is a vector gauge transformation parameter, and D_M is a covariant derivative.

Since the translation is promoted to local one supersymmetry, whose square is roughly translation, should also be promoted to local supersymmetry. Correspondingly, there must be a gauge field that transforms with a spinor gauge parameter ξ^α as follows.

$$\delta \psi^\alpha_M = D_M \xi^\alpha \ ,$$

where ψ_M^α is called Rarita-Schwinger field or **gravitino**.

SUGRA theory always includes the two gauge fields, e_M^a and ψ_M^α, and their action is given by

$$S = \frac{1}{2\kappa_D^2}\int d^D x\; e\;\left[R - 2i\psi_M \Gamma^{MNP} D_N \psi_P\right]\ ,$$

where $e = \det e_M^a$, and we omitted the spinor index α. The SUSY transformations for the fields are

$$\delta_\epsilon e_M^a = i\bar{\epsilon}\Gamma^a \psi_m\ , \qquad \delta_\epsilon \psi_M = D_M \epsilon\ .$$

12.1.2 11d SUGRA

Supersymmetry puts a strong constraint on the dimension of the theory. If we limit ourselves to consider fields up to spin-2, then, it is known that the highest dimension is 11. This is roughly because the D.O.F of fermions grows exponentially: $2^{[D/2]}$, on the other hand, that of bosons grows as a power of D: $\frac{(D-1)(D-2)}{2} - 1$ (graviton). Therefore, to balance fermions and bosons we cannot go arbitrary higher.

Although the existence of fermions is crucial, what we need, to see the connections to string theories, is the bosonic part.

Let us see the action of the 11d SUGRA.

$$2\kappa_{11}^2 S_{11} = \int d^{11}x\sqrt{-G}\left[R - \frac{1}{2}K_{(4)}^2\right] - \frac{1}{6}\int d^{11}x\ M_{(3)} \wedge K_{(4)} \wedge K_{(4)}\ ,$$

where $K_4 = dM_3$ is a field strength of a rank 3 anti-symmetric tensor M_3, and $K_{(4)}^2 = K_{(4)} \wedge *K_{(4)}$. The 11d SUGRA consists of three fields; one is the graviton G_{MN} (44 states), the other boson is rank 3 anti-symmetric tensor $M_{(3)}$ (84 states), and the gravitino ψ_M (128 states). We can see that the numbers of fermions and bosons are balanced. Note that there is only one parameter κ_{11}, which can be written in terms of Planck length ℓ_p: $\frac{1}{2\kappa_{11}^2} = \frac{2\pi}{(2\pi\ell_p)^9}$.

You may wonder why we looked into such a non-interesting theory in a sense that what we want is 10 dimension, rather than 11 dimension. One reason is that the interesting 10d SUGRA can be derived by dimensional reduction from the 11 SUGRA. Other reason, which is rather surprising, will be clear later.

12.1.3 10d Type IIA SUGRA

Let us first write down the action S_{A}, which is a sum of following three terms.

$$\begin{aligned}
S_{\mathrm{A,NS}} &= \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-G}e^{-2\Phi}\left[R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}H_{(3)}^2\right]\ ,\\
S_{\mathrm{A,R}} &= \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-G}\left[-\frac{1}{2}G_{(2)}^2 - \frac{1}{2}\widetilde{G}_{(4)}^2\right]\ ,\\
S_{\mathrm{A,CS}} &= -\frac{1}{4\kappa_{10}^2}\int B_{(2)} \wedge G_{(4)} \wedge G_{(4)}\ ,
\end{aligned}$$

where $H_{(3)} = dB_{(2)}$, $G_{(2)} = dC_{(1)}$, $\widetilde{G}_{(4)} = G_{(4)} - C_{(1)} \wedge H_{(3)}$, $G_{(4)} = dC_{(3)}$, and $2\kappa_{10}^2 = (2\pi\ell_s)^8/2\pi$.

The IIA fields are coming from reduction of 11d SUGRA fields (the compactified direction is denoted as θ):

$$\begin{aligned} &G_{MN} \Rightarrow G_{\mu\nu} \ , \quad G_{\mu\theta} \Rightarrow C_1 \ , \quad G_{\theta\theta} \Rightarrow \Phi \ , \\ &M_{\mu\nu\theta} \Rightarrow B_{\mu\nu} \ , \quad M_{\mu\nu\rho} \Rightarrow C_{(3)} \ . \end{aligned}$$

In order to see the concrete reduction let us substitute following expression:

$$\begin{aligned} &ds_{11}^2 = G_{MN}dx^M dx^N = G_{\mu\nu}dx^\mu dx^\nu + l_p^2 e^{2\sigma}(d\theta + C_{(1)})^2 \ , \\ &M_{(3)} = C_{(3)} + B_{(2)} \wedge d\theta \ , \quad K_{(4)} = \widetilde{G}_{(4)} + H_{(3)} \wedge (d\theta + C_{(1)}) \ , \end{aligned}$$

into the action, and then, the action becomes

$$\begin{aligned} S_{11} \Rightarrow &\frac{2\pi\ell_p}{2\kappa_{11}^2} \int d^{10}x\sqrt{-G}\left[e^\sigma R - \frac{1}{2}e^{3\sigma}G_{(2)}^2 - \frac{1}{2}e^{-\sigma}H_{(3)}^2 - \frac{1}{2}e^\sigma \widetilde{G}_{(4)}^2\right] \\ &- \frac{2\pi\ell_p}{4\kappa_{11}^2}\int d^{11}x \ B_{(2)} \wedge G_{(4)} \wedge G_{(4)} \ . \end{aligned} \tag{12.1}$$

Furthermore, we rescale the metric $G_{\mu\nu} = e^{-\sigma}G_{s,\mu\nu}$:

$$\Rightarrow \frac{2\pi\ell_p}{2\kappa_{11}^2}\int d^{10}x\sqrt{-G_s}\left[e^{-3\sigma}\left(R + 9\partial^\mu\sigma\partial_\mu\sigma - \frac{1}{2}H_{(3)}^2\right) - \frac{1}{2}G_{(2)}^2 - \frac{1}{2}\widetilde{G}_{(4)}^2\right] + \cdots \ .$$

Therefore, we can identify $3\sigma = 2\Phi$.

Let us define $R = \ell_p e^\sigma$, which is a radius of the compactified circle. Then, we have following relation.

$$e^{3\sigma} = e^{2\Phi} \quad \Rightarrow \quad \left(\frac{R}{\ell_p}\right)^3 = g_s^2 \ .$$

Furthermore, we compare the coupling constant of 11d SUGRA with reduction (12.1) and IIA SUGRA:

$$\frac{2\pi R}{2\kappa_{11}^2} = \frac{e^{-2\Phi}}{2\kappa_{10}^2} \quad \Rightarrow \quad \frac{2\pi(2\pi R)}{(2\pi\ell_p)^9} = \frac{2\pi}{g_s^2(2\pi\ell_p)^8} \quad \Rightarrow \quad \frac{R}{\ell_p^3} = \frac{1}{\ell_s^2} \ .$$

Combining the two relation we have $R = g_s\ell_s$.

12.1.4 10d Type IIB SUGRA

Let us first write down the action $S_{\rm B}$, which is a sum of following three terms.

$$\begin{aligned} &S_{\rm B,NS} = \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-G}e^{-2\Phi}\left[R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}H_{(3)}^2\right] \ , \\ &S_{\rm B,R} = \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-G}\left[-\frac{1}{2}G_{(1)}^2 - \frac{1}{2}\widetilde{G}_{(3)}^2 - \frac{1}{4}\widetilde{G}_{(5)}^2\right] \ , \\ &S_{\rm B,CS} = -\frac{1}{4\kappa_{10}^2}\int C_{(4)} \wedge H_{(3)} \wedge G_{(3)} \ , \end{aligned}$$

where $H_{(3)} = dB_{(2)}$, $G_{(1)} = dC_{(0)}$, $G_{(3)} = dC_{(2)}$, $G_{(5)} = dC_{(4)}$, $\widetilde{G}_{(3)} = G_{(3)} - C_{(0)}H_{(3)}$, and $\widetilde{G}_{(5)} = G_{(5)} - \frac{1}{2}C_{(2)} \wedge H_{(3)} + \frac{1}{2}B_{(2)} \wedge G_{(3)}$.

As we saw in the string theory analysis the string spectrum includes self-dual 4-form. The self-dual condition, in the language here, is $*\widetilde{G}_{(5)} = \widetilde{G}_{(5)}$, which is different from E.O.M $d * \widetilde{G}_{(5)} = 0$ or Bianchi id $d\widetilde{G}_{(5)} = 0$. Hence, the condition must be imposed by hand. This is problematic when you quantize the theory, however, it is not problem for our purpose.

12.2 String dualities

In the previous lectures we saw that there exist so called T-duality, which relates IIA, IIB, and also Type I superstring theories. Although it exchanges the space-time radius $R \leftrightarrow \widetilde{R} = \alpha'/R$ it does not affect coupling constant g_s.

Our argument is based on perturbation theory, which means that the string coupling g_s is small, otherwise we cannot discuss "1 string" states. Therefore, we do not know what happen in a strong coupling region $g_s >> 1$. Since the electromagnetic duality inverse the coupling we may wonder the same thing could happen in the string theory.

12.2.1 S-duality of Type IIB SUGRA

Let us write the action so that we can see the hidden symmetry of the action. We use $G_{E,\mu\nu} = e^{-\Phi/2}G_{\mu\nu}$, $\tau = C_{(0)} + ie^{-\Phi}$,

$$\mathbb{M} = \frac{1}{\mathrm{Im}\tau}\begin{pmatrix} |\tau|^2 & -\mathrm{Re}\tau \\ -\mathrm{Re}\tau & 1 \end{pmatrix} , \qquad \mathbb{F}_{(3)} = \begin{pmatrix} H_{(3)} \\ G_{(3)} \end{pmatrix} ,$$

and the action becomes

$$S_{\mathrm{B}} = \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-G_E}\left[R_E - \frac{\partial_\mu\tau\partial^\mu\overline{\tau}}{2(\mathrm{Im}\tau)^2} - \frac{1}{2}\mathbb{F}_{(3)}\cdot\mathbb{M}\cdot\mathbb{F}_{(3)} - \frac{1}{4}\widetilde{G}_{(5)}^2\right] - \frac{1}{4\kappa_{10}^2}\int C_{(4)} \wedge \mathbb{F}_{(3)}^T \wedge \epsilon\mathbb{F}_{(3)} ,$$

where $\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. This action is invariant under the $SL(2,\mathbb{R})$ transformation:

$$\tau' = \frac{a\tau + b}{c\tau + d} , \quad \mathbb{M}' = (\Lambda^{-1})^T\mathbb{M}\Lambda^{-1} , \quad \mathbb{F}'_{(3)} = \Lambda\mathbb{F}_{(3)} , \quad \Lambda = \begin{pmatrix} d & c \\ b & a \end{pmatrix} .$$

G_E and $C_{(4)}$ are invariant under the transformation.

Let us consider consider D-branes coupled to the RR-fields. D3-brane is invariant because $C_{(4)}$ is invariant. 5-branes (NS5 and D5) are magnetically coupled to 2-form fields ($B_{(2)}$ and $C_{(2)}$), therefore,

$$\mathbb{F}'_{(3)} = \Lambda\mathbb{F}_{(3)} \quad \Rightarrow \quad d\mathbb{F}'_{(3)} = \begin{pmatrix} J'_{\mathrm{NS5}} \\ J'_{\mathrm{D5}} \end{pmatrix} = \Lambda\begin{pmatrix} J_{\mathrm{NS5}} \\ J_{\mathrm{D5}} \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} \mathrm{NS5}' \\ \mathrm{D5}' \end{pmatrix} = \Lambda\begin{pmatrix} \mathrm{NS5} \\ \mathrm{D5} \end{pmatrix} .$$

Strings (F1 and D1) are electrically coupled to 2-form fields, hence, have a following action

$$S = \int_{\mathrm{F1}} B_{(2)} + \int_{\mathrm{D1}} C_{(2)} + \cdots \ .$$

this should be invariant, therefore,

$$\big(\mathrm{F1}\ \mathrm{D1}\big)\begin{pmatrix} B_{(2)} \\ C_{(2)} \end{pmatrix} \quad \Rightarrow \quad \big(\mathrm{F1}'\ \mathrm{D1}'\big) = \big(\mathrm{F1}\ \mathrm{D1}\big)\,\Lambda^{-1} \quad \Leftarrow \quad \begin{pmatrix} B'_{(2)} \\ C'_{(2)} \end{pmatrix} = \Lambda \begin{pmatrix} B_{(2)} \\ C_{(2)} \end{pmatrix} \ .$$

Due to the Dirac quantization condition, the electric and the magnetic charges must be integers, and hence, the true symmetry is $SL(2,\mathbb{Z})$.

In the case of $\Lambda = S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ fields transform $\tau \leftrightarrow -1/\tau$, F1 $\leftrightarrow$ D1, and NS5 $\leftrightarrow$ D5. Especially, if $\langle C_0 \rangle = 0$ $g_s \leftrightarrow 1/g_s$. Therefore, this is a strong-weak duality ! and typically called **S-duality**.

Other elements of $SL(2,\mathbb{Z})$ lead infinitely many bound states of F1 and D1: (p,q)-string, and those of NS5 and D5: (p,q) 5-branes.

The argument here implies that IIB superstring theory has also $SL(2,\mathbb{Z})$ symmetry, and there many evidences for it but there is no proof so far.

12.2.2 M-theory

Since the IIB theory has $SL(2,\mathbb{Z})$ symmetry, we may expect the similar thing happens in IIA as well. However, we immediately notice, e.g. that there is no partner for $B_{(2)}$ field, etc.

For IIA superstring theory, rather surprising phenomenon happens in strong coupling region. As we saw in the dimensional reduction of 11d SUGRA we have $R = g_s \ell_s$. This means that when we consider strong coupling region another spacetime direction emerges (de-compactification) !! So far, this deduction is purely from SUGRA analysis, however, if we assume that the 11d SUGRA is an effective theory of something like "string" theory, then, further miraculous coincidence happens.

Witten proposed such a "string"-like theory and put a name, **M-theory**. According to him the M stands for Magic, Mysterious, or Membrane. Some people also include Matrix, etc. so why not come up with your own M !

Since the 11d SUGRA has 3-form anti-symmetric fields there must be corresponding objects that coupled to the field electrically and magnetically. They are called **M2-brane** and **M5-brane**, which are $(1+2)$- and $(1+5)$-dimensional objects, respectively. Let us assume that they have following tensions and charges

$$T_{M2} = \mu_{M2} = \frac{2\pi}{(2\pi\ell_p)^3} \ , \qquad T_{M5} = \mu_{M5} = \frac{2\pi}{(2\pi\ell_p)^6} \ .$$

Table 3: Wrapped or unwrapped M-branes and corresponding string and D-branes, with their tensions.

Dimension	0	1	2	4	5	6
M on S^1	KK-mom.	M2/S^1	M2	M5/S^1	M5	KK-mono.
	$\frac{1}{R}$	$\frac{2\pi\cdot 2\pi R}{(2\pi\ell_p)^3}$	$\frac{2\pi}{(2\pi\ell_p)^3}$	$\frac{2\pi\cdot 2\pi R}{(2\pi\ell_p)^6}$	$\frac{2\pi}{(2\pi\ell_p)^6}$	$\frac{2\pi(2\pi R)^2}{(2\pi\ell_p)^9}$
IIA	D0	F1	D2	D4	NS5	D6
	$\frac{2\pi}{g_s(2\pi\ell_s)}$	$\frac{2\pi}{(2\pi\ell_s)^2}$	$\frac{2\pi}{g_s(2\pi\ell_s)^3}$	$\frac{2\pi}{g_s(2\pi\ell_s)^5}$	$\frac{2\pi}{g_s^2(2\pi\ell_s)^6}$	$\frac{2\pi}{g_s(2\pi\ell_s)^7}$

When we put this M-theory on a circle S^1, then, it should reduce to the IIA superstring theory. Let us see this by comparing the branes and their tensions (see Table 6). You should confirm that the tensions perfectly agree.

Note that M-theory is NOT even defined in a sense that we do not know how to quantize the M-branes. However, the existence of such a theory tells us a lot. Especially, even though no effective theory of M5-branes in flat space is known, M5-branes wrapped on Riemann manifolds leads amazing relation between a d-dim topological theory and $6-d$ supersymmetric gauge theories (typical one is M5 wrapped on Riemann surface and the relation is called AGT), which is an ongoing, hot research topic.

13 Lecture 13

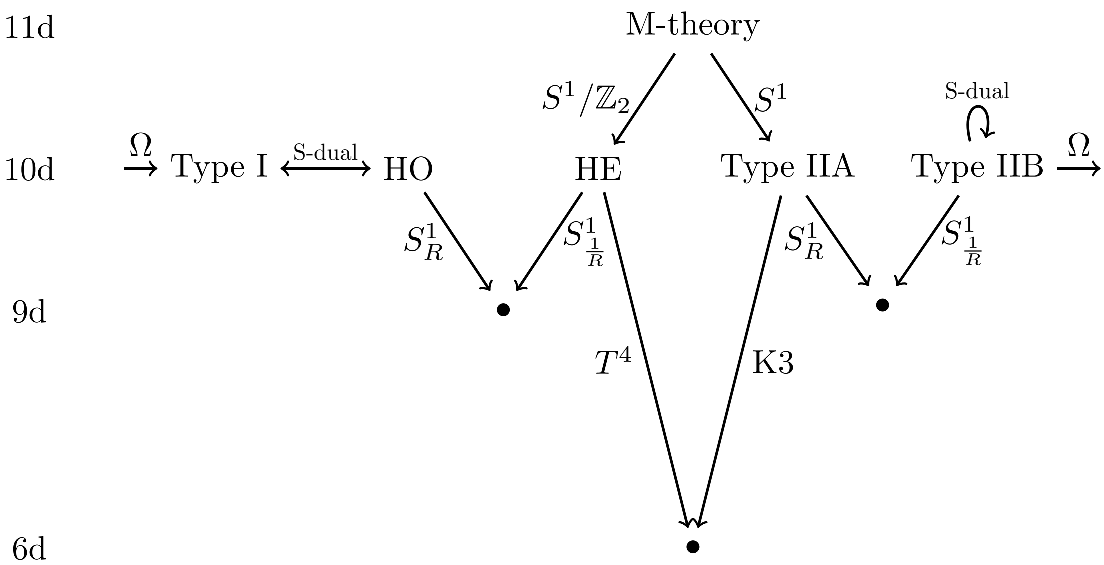
We will continue to learn string dualities extensively studied in the second string revolution after the seminal paper [Wit95]. So far, we have learned

- Type IIA and IIB are T-dual to each other
- Type I is the orientifold projection of Type IIB
- Type IIB has SL$(2,\mathbb{Z})$ symmetry so that it is self-dual under S-duality
- The strong coupling regime of Type IIA is described by M-theory on S^1

In this lecture, we will learn

- Heterotic SO(32) and $E_8 \times E_8$ are T-dual to each other
- Heterotic SO(32) is S-dual to Type I
- The strong coupling regime of Heterotic $E_8 \times E_8$ is described by M-theory on $S^1/\mathbb{Z}_2$
- Heterotic string on T^4 is dual to Type IIA on K3

Even for these dualities, we can cover only key points in this lecture. More details can be found in [Pol98, BBS06]. Moreover, we just see a tip of iceberg, and there



are much more string dualities. Thus, I refer to good reviews [Asp96, FL98, OY96, Pol96a, Pol96b, Tow96, Sch97, Sen97, Dij97, Vaf97, Sen98] written during the second string revolution for this rich subject. All in all, these dualities tell us that quantum strings somehow see geometry from drastically different viewpoints. I hope you will get some feeling of it in this lecture.

13.1 Heterotic T-duality

Let us consider the T-duality in Heterotic strings on a circle S^1 in [Nar86, NSW87, Gin87]. In the bosonic construction, the bosonic left-moving sector is compactified on an even self-dual Euclidean lattice of 16-dimensions. There are only two such lattices: the weight lattice $\Gamma_{\mathrm{SO}(32)}$ of SO(32) and the root lattice $\Gamma_{E_8} \oplus \Gamma_{E_8}$ of $E_8 \times E_8$ as we have seen in §11.1.2.

One may also describe the compactification on a circle S^1 in terms of lattices. As we have seen, the left-moving and right-moving momenta compactified boson takes the value on the lattice $\Gamma^{1,1}$ of the Lorentzian signature. Hence, the circle compactification results in adding ($\oplus$) the lattice $\Gamma^{1,1}$ to the original lattice.

It is a useful mathematical fact that for Lorentzian lattices, there is only unique even unimodular Lorentzian lattice for each rank. Therefore, the theorem implies

$$\Gamma_{\mathrm{SO}(32)} \oplus \Gamma^{1,1} \cong \Gamma^{1,17} \cong \Gamma_{E_8} \oplus \Gamma_{E_8} \oplus \Gamma^{1,1} \ .$$

Together with the metric G and the B-field, they parameterize the moduli space

$$\mathcal{M} = \frac{\mathrm{O}(1,17)}{\mathrm{O}(1) \times \mathrm{O}(17)} \Big/ \mathrm{O}(1,17;\mathbb{Z}), \tag{13.1}$$

where $\mathrm{O}(1,17;\mathbb{Z})$ is the T-duality group. It is called **Narain moduli space**. Different points in the moduli space correspond to physically distinct compactifications, e.g.

the gauge groups can be different, although always of rank 18. At generic points it is $\mathrm{U}(1)^{18}$ that corresponds to the fact that Wilson loops generically breaks 10d gauge group to $\mathrm{U}(1)^{18}$.

However, there are special subspaces of the moduli space where it is enhanced. This moduli space has exactly two asymptotic boundary points, one associated to the decomposition $\Gamma^{1,17} \cong \Gamma_{E_8} \oplus \Gamma_{E_8} \oplus \Gamma^{1,1}$, and the other to the decomposition $\Gamma^{1,17} \cong \Gamma_{\mathrm{SO}(32)} \oplus \Gamma^{1,1}$. We assign the boundary points the interpretations of types HE and HO strings, or large radius and small radius. T-duality will relate these interpretations.

In fact, starting from either Heterotic theory, there is a simple choice of Wilson line which breaks the gauge group to $\mathrm{SO}(16) \times \mathrm{SO}(16) \times \mathrm{U}(1) \times \mathrm{U}(1)$. If we leave this group unbroken, then the only remaining parameter is the radius. An analysis of the massive states shows that if we map $R \to 1/R$ while exchanging KK momenta and winding modes, then the two Heterotic theories are exchanged [Pol98, §11.6].

More generally, upon a compactification of Heterotic strings on a D-dimensional torus T^D, momenta take the values on an even self-dual Lorentzian lattice $\Gamma^{D,D+16}$. Therefore, the Narain moduli space becomes

$$\mathcal{M} = \frac{\mathrm{O}(D, D+16)}{\mathrm{O}(D) \times \mathrm{O}(D+16)} \Big/ \mathrm{O}\left(D, D+16; \mathbb{Z}\right) \ . \tag{13.2}$$

which is $D(D+16)$-dimensional.

13.2 S–duality between Type I and Heterotic SO(32)

Now let us see S-duality between Type I theory and Heterotic SO(32) theory [Wit95, Dab95, Hul95, PW96] from low-energy effective actions.

One can obtain Type I supergravity action from Type IIB by setting to zero the IIB fields C_0, B_2, and C_4 that are removed by the Ω projection. In addition, we include SO(32) gauge fields with appropriate dilaton dependence

$$\begin{aligned} S_{\mathrm{I}} &= S_{\mathrm{grav}} + S_{\mathrm{YM}} \\ S_{\mathrm{grav}} &= \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-G} \left[e^{-2\Phi}(R + 4\partial_\mu \Phi \partial^\mu \Phi) - \frac{1}{2} |\widetilde{G}_3|^2 \right] \\ S_{\mathrm{YM}} &= -\frac{1}{2g_{10}^2} \int d^{10}x \sqrt{-G}\, e^{-\Phi} \mathrm{Tr}_V |F_2|^2 \end{aligned} \tag{13.3}$$

where F_2 is the SO(32) field strength and the trace is in the vector representation. Here G_3 is the field strength of the RR 2-form C_2

$$\widetilde{G}_3 = dC_2 - \frac{\kappa_{10}^2}{g_{10}^2} \omega_3$$

with the Chern-Simons 3-form

$$\omega_3 = \mathrm{Tr}_V \left(AdA - \frac{2i}{3} A^3 \right) \ .$$

The gauge coupling constant g_{10} and the gravitational constant κ_{10} are related by $\kappa_{10}^2/g_{10}^2 = \alpha'/4$, which is determined by anomaly cancelation. Under the gauge transformation $\delta A = d\lambda - i[A, \lambda]$, the Chern-Simons term transforms as

$$\delta\omega_3 = d\mathrm{Tr}_V(\lambda A)$$

Hence, it comes with

$$\delta C_2 = \frac{\kappa_{10}^2}{g_{10}^2}\mathrm{Tr}_V(\lambda dA) \ .$$

Heterotic strings have the same supersymmetry as Type I string and so we expect the same action. However, in the absence of open strings or RR fields the dilaton dependence should be $e^{-2\Phi}$ throughout:

$$\begin{aligned} S_{\mathrm{Het}} &= S_{\mathrm{grav}} + S_{\mathrm{YM}} \\ S_{\mathrm{grav}} &= \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-G}\,e^{-2\Phi}\left[R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}|\widetilde{H}_3|^2\right] \\ S_{\mathrm{YM}} &= -\frac{1}{2g_{10}^2}\int d^{10}x\sqrt{-G}\,e^{-2\Phi}\mathrm{Tr}_V|F_2|^2 \end{aligned} \tag{13.4}$$

where the 3-form $\widetilde{H}_3$ is the field strength of the B-field equipped with Chern-Simons form

$$\widetilde{H}_3 = dB_2 - \frac{\kappa_{10}^2}{g_{10}^2}\omega_3 \ .$$

Indeed the low-energy effective actions of Type I (13.3) and Heterotic SO(32) (13.4) are related by the following the field definitions (Homework)

$$\begin{aligned} G_{\mu\nu}^I = e^{-\Phi^H}G_{\mu\nu}^H \ , &\qquad \Phi^I = -\Phi^H \\ \widetilde{G}_3^I = \widetilde{H}_3^H \ , &\qquad A^I = A^H \ . \end{aligned} \tag{13.5}$$

Recalling that the vacuum expectation value of the dilaton is the string coupling $g_{st} = e^\Phi$, we see that the strong coupling limit of one theory is related to the weak coupling limit of the other theory and vice versa.

In Type I theory there are D1-branes and D5-branes that are electrically and magnetically charged under C_2, respectively. In Heterotic SO(32) theory, there are fundamental strings and NS5-branes that are electrically and magnetically charged under B_2, respectively. The S-duality maps them as [PW96]

Type I	$\leftrightarrow$	Heterotic SO(32)
D1-branes	$\leftrightarrow$	F-strings
D5-branes	$\leftrightarrow$	NS5-branes

One can provide another evidence of this duality by looking at massless spectrum. We have seen that Heterotic SO(32) has massless fields:

1. $\mathbf{8_v}$ of SO(8): bosonic right-moving $X^i(z)$
2. $\mathbf{8_c}$ of SO(8): fermionic right-moving $\psi^i(z)$
3. $\mathbf{32}$ of SO(32): left-moving Majorana-Weyl fermion $\widetilde{\lambda}^a(\overline{z})$

Correspondingly, one can see the massless BPS excitations from D1-strings stretched in the x_1-direction in Type I theory (Homework):

1. $\mathbf{8_v}$ of SO(8): normal bosonic excitations of D1-D1 strings
2. $\mathbf{8_c}$ of SO(8): right-moving fermionic excitations of D1-D1 strings
3. $\mathbf{32}$ of SO(32): left-moving fermionic excitations of D1-D9 strings

Further evidence of this duality has been assembled by comparing tensions, F_2^4 interactions, so on [Pol98, §14.3].

13.3 Heterotic $E_8 \times E_8$ string from M-theory

Now we shall consider the strong-coupling behavior of Heterotic $E_8 \times E_8$ theory. Taking T-duality and S-duality, Heterotic $E_8 \times E_8$ theory is dual to Type I theory. The T-dual to Type I theory is Type IIA theory on a line segment $S^1/\mathbb{Z}_2$ where O8$^-$-plane sit at the two ends and 16+16 D8 branes are distributed on $S^1/\mathbb{Z}_2$ as described in §10.1.3. We call this theory **Type I' theory**. In the strong coupling regime, the 11th circle will emerge and it is described M-theory on $S^1 \times S^1/\mathbb{Z}_2$.

$$\begin{tikzcd}
\text{HO on } S^1 \arrow[rr, leftrightarrow, "\text{S-dual}"] \arrow[d, leftrightarrow, "\text{T-dual}"'] & & \text{Type I on } S^1 \arrow[d, leftrightarrow, "\text{T-dual}"] \\
\text{HE on } S^1 \arrow[dr, "g_s \gg 0"'] & & \text{Type I'} \arrow[dl, "g_s \gg 0"] \\
& \text{M-theory on } S^1 \times S^1/\mathbb{Z}_2 \text{ with HW domain walls} &
\end{tikzcd}$$

Figure 31: Duality web for Heterotic M-theory

Interestingly enough, the relative position of O8$^-$-planes and D8-branes in Type I' string theory may be adjusted. This freedom goes away in the M-theory limit; the D8-branes have to be stuck at the O8$^-$-planes, and they become the domain walls of M-theory, which are called **Hořava-Witten domain wall** or **M9-branes** [HW96b, HW96a].

Its low-energy effective description is given by 11d supergravity on $S^1/\mathbb{Z}_2$ which gives rise to gravitational anomaly [AGW84]. In order to cancel such anomaly, non-Abelian gauge fields have to be present at the boundaries in order to employ a Green-Schwarz mechanism [GS84]. (Homework) This mechanism that bulk anomaly cancels with boundary anomaly is called **anomaly inflow**. Indeed the low-energy effective theory at the Hořava-Witten domain wall is indeed described by 10d $\mathcal{N}=1$ SYM with E_8 gauge group that cancel anomaly.

As in Type IIA case, the distance between the two boundaries is related to Heterotic coupling $R = g_{\text{het}}^{\frac{3}{2}}\ell_p$. Hence, the line segment $S^1/\mathbb{Z}_2$ shrinks at the weak coupling regime, leading to Heterotic $E_8 \times E_8$ string theory. The reason to pick $E_8 \times E_8$ is that the anomalies must be canceled on both boundaries, and there is no way to distribute SO(32) between two boundaries (its a simple group with no factors). Using the previous terminology Heterotic $E_8 \times E_8$ string theory can be viewed as M-theory compactified on $S_1/\mathbb{Z}_2$. This setup is called **Hořava-Witten M-theory** or **Heterotic M-theory** [HW96b, HW96a].

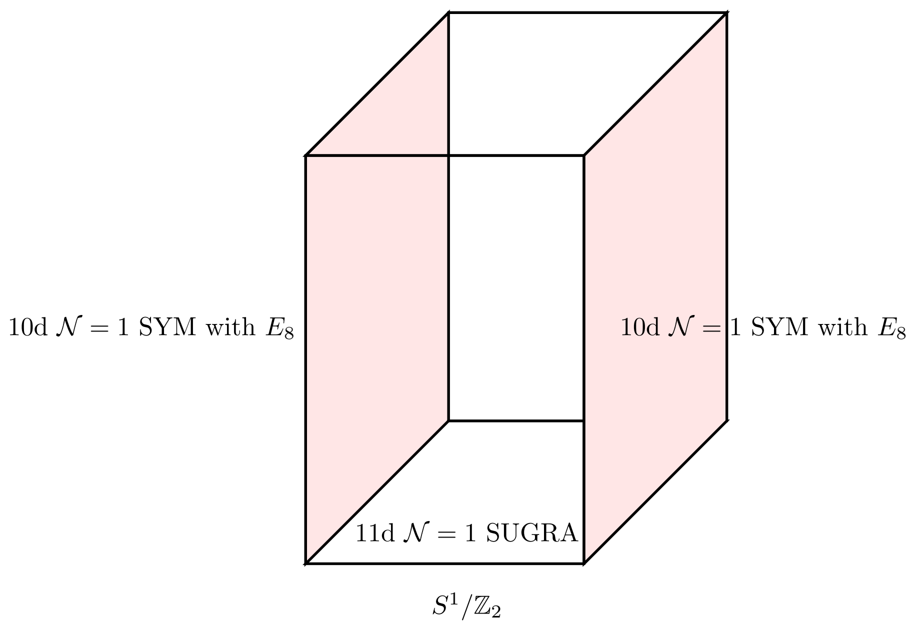


Figure 32: Low-energy effective description of Heterotic M-theory. Hořava-Witten domain walls at the two boundaries give rise to $\mathcal{N}=1$ SYM with E_8 gauge group and they cancel bulk anomaly.

13.4 Duality between Heterotic on T^4 and Type IIA on K3

Let us now see one more non-trivial duality. Although we have studied only toroidal compactifications, we have seen rich web of dualities. In string theory, a theory is consistent if we compactify it on a Calabi-Yau manifold. Since there are wide varieties of Calabi-Yau manifolds, string dualities involving them are much richer. It has still been an active research area both in physics and mathematics. In this lecture, we deal with the next simplest Calabi-Yau manifold called **K3 surface**.

13.4.1 K3 surface

A K3 surface is a resolution of $T^4/\mathbb{Z}_2$. We write a 4-torus as

$$T^4 = \mathbb{R}^4/\mathbb{Z}^4 = \{\mathbf{x} = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 | x_i \sim x_i + 1\}$$

and the $\mathbb{Z}_2$ action is a reflection $x_i \to -x_i$. Note that this action has $2^4 = 16$ fixed points given by the choice of midpoints or the origin in any of the four x_i. Thus, the resulting space $T^4/\mathbb{Z}_2$ is singular at any of these 16 fixed points. The neighborhood of a singular point is indeed a cone of $\mathbb{R}P^3$. To make it smooth, let us consider the set of vectors of length ≤ 1 in the tangent bundle of TS^2

$$V = \{(v_1, v_2) \in S^2 \times T_{v_1}S^2 | \ |v_2| \leq 1\} \ .$$

Then the boundary of V is $\partial V = \mathbb{R}P^3$ so that you can replace the neighborhood of each singular point by V. Since V is a smooth manifold, the resulting space is smooth and it is a K3 surface. This smoothing procedure is called **resolution** or **blow-up**.

Although the construction of a K3 surface is rather simple, its geometry is surprisingly fertile. First of all, it is a Calabi-Yau manifold, namely a Ricci-flat Kähler manifold. In real four dimensions, there are only two topologically equivalent compact closed Calabi-Yau manifolds, T^4 and K3. Moreover, it is a hyper-Kähler manifold. (Let's not go into detail about hyper-Kähler manifold.)

Let us now briefly look at topological property of K3 surfaces. The resolution of the 16 singular points provides 16 elements of $H^2(K3, \mathbb{Z})$ in addition to $6 = {}_4C_2$ tori in T^4. Therefore, we have $H^2(K3; \mathbb{Z}) \cong \mathbb{Z}^{22}$. Moreover, the Hodge diamond as a complex manifold turns out to be

$$\begin{array}{ccccc} & & h^{0,0} & & \\ & h^{1,0} & & h^{0,1} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{2,1} & & h^{1,2} & \\ & & h^{2,2} & & \end{array} = \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array} \ .$$

Since it is a real 4-dimensional manifold, one can consider the intersection matrix of rank 22

$$Q(\alpha_i, \alpha_j) = \alpha_i \cap \alpha_j \qquad \alpha_i \in H_2(K3; \mathbb{Z})$$

In fact, in a certain nice basis, the intersection matrix can be written as follows

$$Q(\alpha_i, \alpha_j) \sim 2(-E_8) \oplus 3 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

where $-E_8$ denotes the 8×8 matrix given by minus the Cartan matrix of the Lie algebra E_8. Hence, we may decompose

$$H^2(K3, \mathbb{R}) = H^+ \oplus H^-,$$

where $H^\pm$ represents the cohomology of the space of (anti-)self-dual 2-forms. We then see that

$$\dim H^+ = 3, \quad \dim H^- = 19 \ .$$

The moduli space of non-trivial metric deformations on a K3 is 58-dimensional and given by the coset space [Asp96]

$$\mathcal{M}_{\mathrm{K3}} \ = \ \mathbb{R}^+ \ \times \ \frac{\mathrm{O}(3,19)}{\mathrm{O}(3) \times \mathrm{O}(19)} \Big/ \ \mathrm{O}(3,19,\mathbb{Z}) \ ,$$

where the second factor is the Teichmüller space for Ricci-flat metrics of volume one on a K3 surface and the first factor is associated with the size of the K3.

This is not the end of the story if we consider string propagation on $K3$. For each element of $H_2(K3, \mathbb{Z})$, we can turn on the B-field. Because of $H_2(K3, \mathbb{Z}) = \mathbb{Z}^{22}$, we have 22 additional real parameters and that makes the total dimension of moduli space $58 + 22 = 80$. It turns out that this moduli space is isomorphic to

$$\mathcal{M}_{\mathrm{K3}}^{\mathrm{stringy}} = \frac{\mathrm{O}(4,20)}{\mathrm{O}(4) \times \mathrm{O}(20)} \Big/ \ \mathrm{O}(4,20,\mathbb{Z}) \ . \tag{13.6}$$

Substituting $D = 4$ into (13.2), one can see that this is exactly the same as the Narain moduli space for Heterotic string on T^4!

13.4.2 Heterotic on T^4/Type IIA on K3

The relations between Heterotic on T^4 and Type IIA on K3 can be seen by comparing the effective actions in $D = 6$. On Heterotic side, for generic Wilson lines the $E_8 \times E_8$ or SO(32) gauge symmetry is broken to $\mathrm{U}(1)^{16}$. Including the KK gauge bosons from T^4 compactification, the gauge group becomes $\mathrm{U}(1)^{24}$ and the effective 6d supergravity action of Heterotic string is

$$S_{\mathrm{Het}} = \frac{1}{2\kappa_6^2} \int d^6x \sqrt{-G}\, e^{-2\Phi} \left[R + 4\partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2} |\widetilde{H}_3|^2 - \frac{\kappa_6^2}{2g_6^2} \sum_{I=1}^{24} |F_2^I|^2 \right] \ .$$

Type IIA superstring theory compactified on K3 breaks a half of supersymmetries so that there are 16 supercharges as in Heterotic string. It also gives rise to $\mathrm{U}(1)^{24}$ gauge fields which, via KK-reduction, all arise from the RR sector. One comes from the one-form with indices along the six non-compact directions, i.e. C_1 , and another one from the three-form with indices C_3 , which is Hodge-dual to a massless vector in

$D = 6$. Because of $H_2(K3, \mathbb{Z}) = \mathbb{Z}^{22}$, the three-form with index structure C_3 gives 22 vectors. As a result, the effective string frame Type IIA action compactified on K3 is

$$S_{\text{IIA}} = \frac{1}{2\kappa_6^2} \int d^6x\sqrt{-G} \left[e^{-2\Phi} \Big(R + 4\partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2}|\widetilde{H}_3|^2 \Big) - \frac{\kappa_6^2}{2g_6^2} \sum_{I=1}^{24} |F_2^I|^2 \right] .$$

It is straightforward to show that the two actions are equivalent via the following field redefinition

$$\begin{aligned} \Phi^{\text{H}} &= -\Phi^{\text{IIA}} , \qquad & G^{\text{H}} &= e^{-2\Phi^{\text{IIA}}} G^{\text{IIA}} \\ A^{\text{H}} &= A^{\text{IIA}} , \qquad & \widetilde{H}^{\text{H}} &= e^{-2\Phi^{\text{IIA}}} * \widetilde{H}^{\text{IIA}} . \end{aligned}$$

13.4.3 More dualities

Of course, what we have glimpsed are merely a few representative examples of string dualities. Compactifying M-theory on various manifolds, one can find many duality relations. These duality conjectures can be arrived at by using similar arguments. Some examples of such conjectured dualities are given below [DM96, Wit96, Sen96]:

M-theory on		
K3	$\leftrightarrow$	Heterotic/Type I on T^3
$T^5/\mathbb{Z}_2$	$\leftrightarrow$	IIB on K3
$T^8/\mathbb{Z}_2$	$\leftrightarrow$	Type I/Heterotic on T^7
$T^9/\mathbb{Z}_2$	$\leftrightarrow$	Type IIB on $T^8/\mathbb{Z}_2$

In each case $\mathbb{Z}_2$ acts by reversing the sign of all the coordinates of T^n. Each of these duality conjectures satisfy the consistency condition that the theory on the right hand side, upon further compactification on a circle, is dual to Type IIA string theory compactified on the manifold on the left hand side.

14 Lecture 14

14.1 D-brane dynamics

So far we treated D-branes as a static solid object that is associated to boundary conditions. On the other hand, it actually has dynamics as like a fundamental string. We will learn the dynamics mainly through their action. Therefore, the first half is devoted to derive the action for Dp-branes. Later half is devoted for connection between D-branes, F1-string, and NS5-brane.

14.1.1 D-brane action

We assume D-branes action is something similar to Nambu-Goto action. Note that when we quantize the string action we used a string sigma action instead of Nambu-Goto action, so that we can evade complexity of the square root. There, the dimension of the world-sheet to be 2 was crucial for quantization. Therefore, we cannot follow the quantization procedure for the fundamental string to quantize Dp-brane, in general. Namely, the action we will learn here is an effective action of D-branes.

What we expect for the action is that

- it contains scalars that is a map from the world-sheet to space-time,
- it also contains vectors living on the D-branes, which arised from an open string massless spectrum,
- it involves B-field because open stinrgs end on D-branes,
- it has supersymmetry (though we only talk about bosonic part in this lecture).

The proposed effective action is called **Dirac-Born-Infeld(DBI) action**. There are several approach to the action. We assume Nambu-Goto action for D-branes and generalize it by utilizing T-duality. Hence, *the action is T-duality manifest.*

We will describe a world-sheet of Dp-branes by $\sigma^a(a = 0, 1, \ldots, p)$. $X^\mu(\sigma^a)$ are the scalars. Then, the Nambu-Goto action is

$$S_{\mathrm{D}p} = -T_{\mathrm{D}p} \int d^{p+1}\sigma \sqrt{-\det\left(G_{\mu\nu}\frac{\partial X^\mu}{\partial\sigma^a}\frac{\partial X^\nu}{\partial\sigma^b}\right)} \ ,$$

where $T_{\mathrm{D}p}$ is a Dp-brane tension, which is discussed later. x^μ, which will appear later, are used for space-time coordinate.

Let us consider a simple set up (see Fig. 33). The space-time is $\mathbb{R}_t \times \mathbb{R} \times S^1$ and

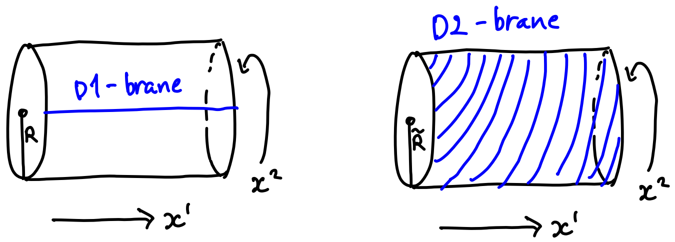


Figure 33: D1-brane and its T-dual D2-brane on $\mathbb{R} \times S^1$.

the metric is $ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu$ (i.e. $G_{\mu\nu} = \eta_{\mu\nu}$). D1-brane locates at X_2 ($\sim X_2 + 2\pi R$), and D2-brane has Wilson line A_2 ($\sim A_2 + \frac{1}{R}$). Note that here A_2 is not a 2-form but $A_{\mu=2}$. T-duality relates these quantities (see Lecture note 9):

$$X_2 = 2\pi\alpha' A_2 \ .$$

Now consider vibrating D1-brane $X^2 = X^2(X^1)$ (see Fig. 34). It maps to a field strength $F_{12} = \partial_1 X_2(X^1) \neq 0$ on D2-brane. Suppose the vibrating D1-brane is

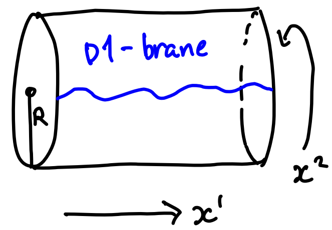


Figure 34: Vibrating D1-brane.

described by the Nambu-Goto action

$$
\begin{aligned}
S_{\mathrm{D1}} &= -T_{\mathrm{D1}} \int d^2\sigma \sqrt{-\det\left(G_{\mu\nu}\frac{\partial X^\mu}{\partial\sigma^a}\frac{\partial X^\nu}{\partial\sigma^b}\right)} \quad \text{with} \quad X^0=\sigma^0\ , \quad X^1=\sigma^1\ , \\
&= -T_{\mathrm{D1}} \int d\sigma^0 d\sigma^1 \sqrt{1+\left(\frac{\partial X^2}{\partial\sigma^1}\right)^2}\ .
\end{aligned}
$$

This expression seems to coincide with

$$
\begin{aligned}
S_{\mathrm{D2}} &= -T_{\mathrm{D2}} \int d^3\sigma \sqrt{-\det\left(G_{\mu\nu}\frac{\partial X^\mu}{\partial\sigma^a}\frac{\partial X^\nu}{\partial\sigma^b}+2\pi\alpha' F_{ab}\right)} \\
&= -T_{\mathrm{D2}}\cdot 2\pi\widetilde{R}\cdot \int d\sigma^0 d\sigma^1 \sqrt{1+(2\pi\alpha' F_{12})^2}\ .
\end{aligned}
$$

D-brane tension

From the dimension analysis D-brane tension should be the following form.

$$
T_{\mathrm{D}p} \sim \frac{\text{mass}}{p\text{-dim vol}} \quad \Rightarrow \quad T_{\mathrm{D}p} \sim \frac{1}{l_s^{p+1}}\ .
$$

From the argument above in order for the two action to coincide we need $T_{\mathrm{D1}} = 2\pi\widetilde{R}T_{\mathrm{D2}}$. On the other hand, we do not want $T_{\mathrm{D}p}$ to depend on R because $T_{\mathrm{D}p}$ should be independent of space-time geometry. Note that D-brane effective theory is supposed to reproduce open string amplitude, whose leading contribution is the disk amplitude $\sim e^{-\langle\Phi\rangle}$. Thus, we reach the following form

$$
S_{\mathrm{D}p} = -T_{\mathrm{D}p}\int d^{p+1}\sigma\, e^{-\Phi(X)}\sqrt{-\det\left(G_{\mu\nu}\partial_a X^\mu\partial_b X^\nu + 2\pi\alpha' F_{ab}\right)}\ .
$$

The ratio of effective tensions of D1 and D2 branes is

$$
\frac{T^{\mathrm{eff}}_{\mathrm{D1}}}{T^{\mathrm{eff}}_{\mathrm{D2}}} = \frac{T_{\mathrm{D1}}e^{-\Phi}}{T_{\mathrm{D2}}e^{-\widetilde{\Phi}}} = \frac{T_{\mathrm{D1}}}{T_{\mathrm{D2}}}\cdot\frac{\widetilde{R}}{l_s} = 2\pi\widetilde{R} \quad \Rightarrow \quad T_{\mathrm{D1}} = 2\pi l_s\cdot T_{\mathrm{D2}}\ .
$$

Note that the dilation field transforms under T-duality as $e^{-\widetilde{\Phi}} = e^{-\Phi}\frac{\widetilde{R}}{l_s}$ (see Homework 11 Prob. 4). For D-branes in superstring theory the correct normalization is $T_{\mathrm{D}p} = \frac{2\pi}{(2\pi l_s)^{p+1}}$.

RR charge and its normalization

We heavily used the fact that Dp-branes couples to C_{p+1} in previous lectures. Let us consider a concrete coupling. It should be the following form.

$$S_{\mathrm{D}p} = \cdots + q_{\mathrm{D}p} \cdot \int d^{p+1}\sigma \, e^{-\Phi} C_{\mu_1 \cdots \mu_{p+1}}(X) \frac{\partial X^{\mu_1}}{\partial \sigma^1} \cdots \frac{\partial X^{\mu_{p+1}}}{\partial \sigma^{p+1}} = \cdots + q_{\mathrm{D}p} \cdot \int_{\mathrm{D}p} e^{-\Phi} C_{(p+1)} \ .$$

By considering T-duality we can conclude that $q_{\mathrm{D}p} = \frac{2\pi}{(2\pi l_s)^{p+1}} = T_{\mathrm{D}p}$ (up to a constant). Note that the equality of the charge and the tension is crucial for multiple Dp-branes to co-exist statically. This is because the tension induce gravitational force (graviton & dilaton) between Dp-branes, which is attractive, on the other hand, the RR charge induce repulsive force for positively(negatively) charged objects, which are Dp-branes.

Note that we have to use proper normalization for RR-fields. The convention used above is called **string normalization**, and the one used in SUGRA is called **canonical normalization**. Let us recall the IIA SUGRA. Appropriate part is, for example,

$$S_{A,R} = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-G} \left[-\frac{1}{2} G_{(2)}^2 \right] + q_{\mathrm{D}0} \cdot \int_{\mathrm{D}0} C_{(1)} \ .$$

The same expression in the string normalization is

$$S_{A,R} = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-G} e^{-2\Phi} \left[-\frac{1}{2} G_{(2)}^2 \right] + q_{\mathrm{D}0} \cdot \int_{\mathrm{D}0} e^{-\Phi} C_{(1)} \ .$$

The Dirac quantization conditions(for the details see homework 11 Prob. 2.2) for these two expressions are the same:

$$2\kappa_{10}^2 q_e q_m \in 2\pi \mathbb{Z} \ .$$

($q_e = q_{\mathrm{D}0}$ and $q_m = q_{\mathrm{D}(6)}$ for the above case.) However, we also define $q^{\mathrm{eff}} = q e^{-\Phi} = q g_s$ in the string normalization, then, the quantization condition is

$$2\kappa_{10}^2 q_e^{\mathrm{eff}} q_m^{\mathrm{eff}} g_s^2 \in 2\pi \mathbb{Z} \ .$$

In this convention we always assume that Φ is non-dynamical. Confirm that $q_{\mathrm{D}p}^{\mathrm{eff}} = T_{\mathrm{D}p}^{\mathrm{eff}} = \frac{2\pi}{(2\pi l_s)^{p+1} g_s}$ satisfies the quantization condition. Note that some references define $q e^{-\Phi}$ as q (similarly $T_{\mathrm{D}p} e^{-\Phi}$ as $T_{\mathrm{D}p}$). The reason we used canonical normalization is simply that the expression is much simpler (especially kinetic terms).

For the B-field, the normalization is different and the quantization condition is

$$T_{\mathrm{F}1} \cdot T_{\mathrm{NS}5} \cdot 2\kappa_{10}^2 g_s^2 \in 2\pi \mathbb{Z} \ .$$

Since $T_{\mathrm{F}1} = \frac{2\pi}{(2\pi l_s)^2}$,

$$T_{\mathrm{NS}5} = \frac{2\pi}{T_{\mathrm{F}1} \cdot 2\kappa_{10}^2 g_s^2} = \frac{2\pi}{(2\pi l_s)^6 g_s^2} \ .$$

Generalization of RR coupling and the DBI action

Let us again consider vibrating D1-brane with the RR-coupling. T-duality connects the following two expression.

$$S_{\text{D1}} = \cdots + q_{\text{D1}} \cdot \int dx^0 dx^1 e^{-\Phi} \left(C_{01} + C_{02} \frac{\partial X^2}{\partial \sigma^1} \right) ,$$
$$S_{\text{D2}} = \cdots + q_{\text{D2}} \cdot \int dx^0 dx^1 dx^2 e^{-\widetilde{\Phi}} \left(\widetilde{C}_{012} + \widetilde{C}_0 \cdot 2\pi\alpha' F_{12} \right) ,$$

where $C_{01} \leftrightarrow \widetilde{C}_{012}$, $C_{02} \leftrightarrow \widetilde{C}_0$, and $X^2 \leftrightarrow 2\pi\alpha' A_2$. This can be understood as follows (see Fig. 35). The vibrating D1-brane consists of straight D1-brane along x^1 and local

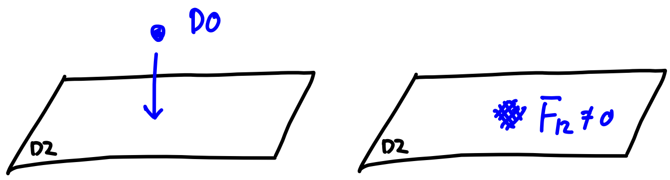


Figure 35: D0-D2 bound state.

vibration along x^2. After T-duality along x^2 the vibration part becomes D0-brane and gives $F_{12} \neq 0$. Generalization of the RR-coupling is

$$S_{\text{D}p} = \cdots + q_{\text{D}p} \cdot \int C_{\text{RR}} \wedge \exp(2\pi\alpha' F_{(2)}) ,$$

where $C_{RR} = \sum_n C_{(n)}$.

Finally, full general form of Dp-brane action is given as follows.

$$S_{\text{D}p} = -T_{\text{D}p} \int d^{p+1}\sigma\, e^{-\Phi(X)} \sqrt{-\det\left(G_{ab} + 2\pi\alpha' F_{ab} - B_{ab}\right)}$$
$$+ q_{\text{D}p} \cdot \int C_{RR} \wedge \exp(2\pi\alpha' F_{(2)} - B_{(2)}) ,$$

where $G_{ab} = G_{\mu\nu}\partial_a X^\mu \partial_b X^\nu$ and $B_{ab} = B_{\mu\nu}\partial_a X^\mu \partial_b X^\nu$. The first line is called **DBI action**. Note that we used canonical normalization for RR-fields. B-field should appear with $F_{(2)}$ due to the gauge invariance.

Let us consider a fundamental string action that is coupled to Dp-branes.

$$S = -\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-h} h^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} + \epsilon^{ab} \partial_a X^\mu \partial_b X^\nu B_{\mu\nu} + \cdots$$
$$+ \int_{\partial\Sigma} d\sigma^0 \partial_0 X^\mu A_\mu$$
$$= \cdots - \frac{1}{2\pi\alpha'} \int_\Sigma B_{(2)} + \int_{\partial\Sigma} A_{(1)} .$$

Important point here is that the gauge transformation of B-field $\delta_B B_{(2)} = d\lambda_{(1)}$ in the action is NOT invariant if there are boundaries, which is exactly the situation we consider now:

$$\delta_B \int_\Sigma B_{(2)} = \int_\Sigma d\lambda_{(1)} = \int_{\partial\Sigma} \lambda_{(1)} \neq 0 \ .$$

This is compensated if $A_{(1)}$ transform as follows:

$$\delta_B A_{(1)} = \frac{\lambda_{(1)}}{2\pi\alpha'} \ .$$

Though we focused on the bosonic part so far, there is a fermionic part so that they form space-time supersymmetry. Here we only write down the leading fluctuation:

$$-i \int d^{p+1}\sigma \,\mathrm{Tr}\left(\overline{\psi}\Gamma^a D_a \psi\right) \ .$$

For the full nonlinear supersymmetric form one should consult with, for example [Tse99].

14.1.2 Branes, Strings ending on Branes

We will look into the RR coupling further from a different view point. Maxwell equation leads charge conservation law as follows.

$$\begin{array}{lcl} d*F_{(2)} = J_e & & dJ_e = 0 \\ & \Rightarrow & \\ dF_{(2)} = J_m & & dJ_m = 0 \end{array} \ .$$

Charge conservation assure that a world-line of the charged particle does not end (closed path or infinitely long). If we apply this logic to branes we may find the same result for branes. However, charge conservation for SUGRA is quite non-trivial due to the non-linearity of the E.O.Ms. We will see the case of generalized type IIA SUGRA.

Massive IIA SUGRA

When we saw the IIA SUGRA action you may wonder why there is no RR-field corresponding to D8-brane, which is 9-form and its field strength is 10-form $G_{(10)}$. Since it is non-dynamical ($d*G_{(10)} = 0$ leads $*G_{(10)} = G_{(0)} \equiv m$) it has constant contributions to the action, called **massive IIA SUGRA**. m is called **Romans mass** because it is a constant and partly contributes as a mass term in the action.

Let us see the massive IIA SUGRA action (we omit wedge product $\wedge$ in this lecture):

$$
\begin{aligned}
S_{\mathrm{A,NS}} &= \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-G}e^{-2\Phi}\left[R+4\partial_\mu\Phi\partial^\mu\Phi-\frac{1}{2}H_{(3)}^2\right] \ , \\
S_{\mathrm{A,R}} &= \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-G}\left[-\frac{1}{2}m^2-\frac{1}{2}G_{(2)}^2-\frac{1}{2}G_{(4)}^2\right] \ , \\
S_{\mathrm{A,CS}} &= \frac{1}{2\kappa_{10}^2}\int\left[-\frac{1}{2}B_{(2)}G_{(4)}G_{(4)}+\frac{1}{2}B_{(2)}^2G_{(2)}G_{(4)}-\frac{1}{6}B_{(2)}^3G_{(2)}^2\right. \\
&\qquad\left.-\frac{m}{6}B_{(2)}^3G_{(4)}+\frac{m}{8}B_{(2)}^4G_{(2)}-\frac{m^2}{40}B_{(2)}^5\right] \ ,
\end{aligned}
$$

where

$$
\begin{aligned}
H_{(3)} &= dB_{(2)} \ , \\
G_{(2)} &= dC_{(1)}+mB_{(2)} \ , \\
G_{(4)} &= dC_{(3)}+dC_{(1)}B_{(2)}+\tfrac{1}{2}mB_{(2)}^2 \ .
\end{aligned} \tag{14.1}
$$

RR- field

From the massive IIA action we have following equation of motion for $C_{(1)}$ and $C_{(3)}$:
E.O.M

$$
\begin{aligned}
-d*G_{(2)} &= H_{(3)}*G_{(4)} \ , \\
d*G_{(4)} &= H_{(3)}G_{(4)} \ .
\end{aligned}
$$

Since we know the relation between the field strengths and their gauge fields (14.1) we have following Bianchi identities:

$$
\begin{aligned}
dG_{(2)} &= mH_{(3)} \ , \\
dG_{(4)} &= H_{(3)}G_{(2)} \ .
\end{aligned}
$$

Now we relabel m by $G_{(0)}$ and define the dual field strengths:

$$
G_{(10)} = *G_{(0)} \ , \qquad G_{(8)} = -*G_{(2)} \ , \qquad G_{(6)} = *G_{(4)} \ .
$$

Then, the E.O.M and Bianchi ids are re-written as

$$
dG_{(2n)} = G_{(2n-2)}H_{(3)} \ . \tag{14.2}
$$

Let us define following formal sum of RR-fields

$$
G_{\mathrm{even}} = G_{(0)}+G_{(2)}+G_{(4)}+G_{(6)}+G_{(8)}+G_{(10)} \ .
$$

Using the formal sum we can express (14.2) by single expression

$$
dG_{\mathrm{even}} = H_{(3)}G_{\mathrm{even}} \ .
$$

This equation can be solved as follows.

$$
\begin{aligned}
G_{\mathrm{even}} &= e^{B_{(2)}}\left(m+dC_{\mathrm{odd}}\right) \ , \\
C_{\mathrm{odd}} &= C_{(1)}+C_{(3)}+C_{(5)}+C_{(7)}+C_{(9)} \ .
\end{aligned}
$$

B-field

Field strength of the B-field is defined by

$$H_{(3)} = dB_{(2)} \ ,$$

hence, the Bianchi id is $dH_{(3)} = 0$. E.O.M is given as follows.

$$d\left(e^{-2\Phi} *H_{(3)}\right) = m *G_{(2)} + *G_{(4)}G_{(2)} - \frac{1}{2}G_{(4)}^2 \ .$$

If we define the dual field strength $H_{(7)} = e^{-2\Phi} *H_{(3)}$, then, the E.O.M becomes

$$dH_{(7)} = -\frac{1}{2}\left[(\mathcal{T}G_{\text{even}})G_{\text{even}}\right]_{(8)}$$

where

$$\mathcal{T}(dx^{i_1}\cdots dx^{i_n}) = (dx^{i_n}\cdots dx^{i_1}) \ ,$$

which is a "transpose" of differential forms.

Note that those field strengths are invariant under the gauge transformations of B-field as well as RR-fields:

$$\begin{aligned} &\delta_B B_{(2)} = d\lambda_{(1)} \ , \qquad \delta_B C_{\text{odd}} = -\lambda_{(1)}\left(m + dC_{\text{odd}}\right) \ , \\ &\delta_C B_{(2)} = 0 \ , \qquad \delta_C C_{\text{odd}} = d\lambda_{\text{even}} \ , \end{aligned}$$

where we introduced a formal sum of gauge parameters

$$\lambda_{\text{even}} = \lambda_{(0)} + \lambda_{(2)} + \lambda_{(4)} + \lambda_{(6)} + \lambda_{(8)} \ .$$

Brane currents

Let us introduce brane currents $J_{(8)}^{\text{F1}}$, $J_{(4)}^{\text{NS5}}$, and

$$J_{\text{odd}} = J_{(1)}^{\text{D8}} + J_{(3)}^{\text{D6}} + J_{(5)}^{\text{D4}} + J_{(7)}^{\text{D2}} + J_{(9)}^{\text{D0}} \ ,$$

and add to the E.O.Ms:

$$\begin{aligned} dH_{(3)} &= J_{(4)}^{\text{NS5}} \ , \\ dH_{(7)} &= J_{(8)}^{\text{F1}} - \frac{1}{2}(TG_{\text{even}})G_{\text{even}} \ , \\ dG_{\text{even}} &= J_{\text{odd}} + H_{(3)}G_{\text{even}} \ . \end{aligned}$$

From these equations we can derive following "charge conservation" law:

$$\begin{aligned} dJ_{(4)}^{\text{NS5}} &= 0 \ , \\ dJ_{(8)}^{\text{F1}} &= \left[J_{\text{odd}}(\mathcal{T}G_{\text{even}})\right]_{(9)} \ , \\ dJ_{\text{odd}} &= -J_{(4)}^{\text{NS5}}G_{\text{even}} - J_{\text{odd}}H_{(3)} \ . \end{aligned}$$

From the laws we can deduce several facts (see Table 6):

- NS5-brane cannot have boundaries,
- F1 string can end on any D-branes,
- Dp-brane can end on NS5-brane up to $p = 6$ (D8 cannot),
- Dp-brane can end on D$(p + 2)$-brane.

Table 4: Branes on which brane ends and branes that end on.

Brane	Branes end on
F1	nothing
NS5-brane	D0, D2, D4, D6
D0-brane	F1
D2-brane	F1, D0
D4-brane	F1, D2
D6-brane	F1, D4
D8-brane	F1, D6

The fact that D-brane is coupled to RR-field requires that the brane action should include $S = \int C_{\text{odd}}$. However, it is invariant under the gauge transformations. The invariant form is

$$S = \int \left(e^{2\pi\alpha' F_{(2)} - B_{(2)}} C_{\text{odd}} + m\omega \right) ,$$

where

$$\omega = \sum_n \frac{1}{(n+1)!} A_{(1)} F_{(2)}^n .$$

This is consistent with the previous analysis.

14.1.3 Bound states of D-branes

As we saw Dp-brane action has following term

$$S \sim \int e^{2\pi\alpha' F_{(2)}} C_{\text{RR}} = \int \delta_{D-p-1}(\text{D}p) e^{2\pi\alpha' F_{(2)}} C_{\text{RR}} ,$$

where we set $m = 0 = B_{(2)}$. The fact that the action includes not only $C_{(p+1)}$ but also $C_{(p+1-2n)}$ means that Dp-brane can have D$(p - 2n)$-brane charges for $n \in \mathbb{Z}_+$.

D0-D2 bound state

Let us consider a concrete example of D2-brane case. The action include following term

$$S \sim \frac{1}{2\pi} \int_{\mathbb{R}_t \times \Sigma} \left(C_{(3)} + F_{(2)} C_{(1)} \right) ,$$

where Σ is an image of world-sheet space (not time). Note that the flux is quantized;

$$\int_\Sigma F_{(2)} = 2\pi n \qquad n \in \mathbb{Z} \ .$$

Now consider a process the Σ shrinks to zero. In this process $C_{(3)}$ part becomes zero as the volume becomes zero. On the other hand, $C_{(1)}$ part remains finite because the flux $F_{(2)}$ is quantized and gives

$$S \sim \frac{1}{2\pi} \int_{\mathbb{R}_t \times \Sigma} \left(F_{(2)} C_{(1)}\right) \to n \int_{\mathbb{R}_t} C_{(1)} \ .$$

This is nothing but n D0-branes. Namely, when D2-brane with n flux shrinks to a point, n D0-branes remains. The former state can be understood that it is a bound state of D2- and D0-branes (see Fig. 36).

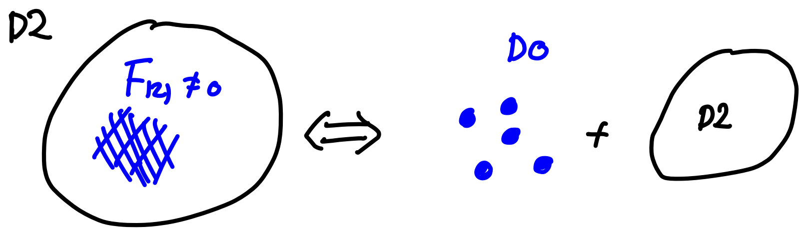


Figure 36: Transition of D0-D2 bound state and D0 + D2.

Myers effect

Let us consider the opposite process of the previous argument. When there are n D0-branes they can become D2-brane. This situation can be accelerated by inducing background $C_{(3)}$ flux. If there is $C_{(3)}$ flux, then, being D2-brane is a lower energy state than being D0-branes (see Fig. 37). This is something similar to polarization

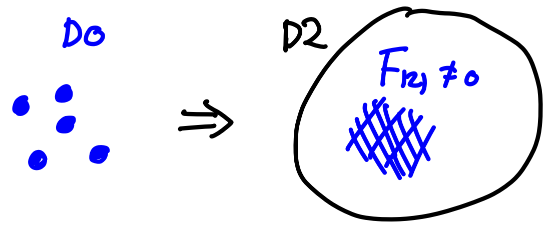


Figure 37: Myers effect: transition from D0-branes to D0-D2 bound state.

phenomenon in electro-magnetism, and in this case, it is called **Myers effect**.

Other bound states

Left it for homework.

Hanany-Witten effect

There is so called Hanany-Witten effect, which is a brane creation/annihilation process when some branes cross each other. Typical example is a cross of NS5 and D5 create/annihilate D3-brane (see Fig. 38). Another example is that Dp and Dp' for

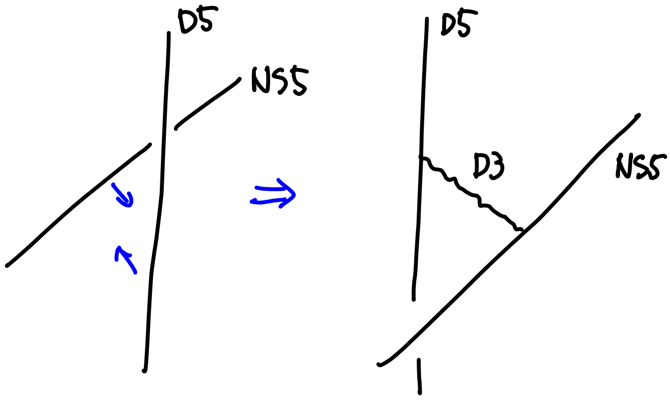


Figure 38: Hanany-Witten effect: crossing of D5- and NS5-branes.

$p + p' = 8$ create/annihilate F1-string. Crossing of two M5-branes create/annihilate M2-brane etc (look it up in the web if you are interested in).

15 Lecture 15

In the last lecture, we have seen that D-branes are dynamical objects and D-branes can end on others forming bound states. Moreover, they were ideally suited for studying black holes.

A large number of D-branes is heavy enough to produce a black hole by wrapping a cycle in a compact manifold. There is a large degeneracy due to open strings attaching to D-branes, which gives a statistical interpretation of the thermodynamic entropy. This leads to a precise microscopic accounting for the Beckenstein-Hawking entropy of the supersymmetric black holes, as shown by Strominger-Vafa [SV96].

The study of black holes in string theory by using D-branes has led to the celebrated AdS/CFT correspondence [Mal99]. (See Maldacena's Ph.D. thesis [Mal96] for instance.)

15.1 Black hole thermodynamics

First let us briefly summarize basics of black holes in general relativity and the laws of black hole thermodynamics studied in the early 70s [Bek73, BCH73, Haw75]. For more detail, I refer to a wonderful lecture note [Tow97].

15.1.1 Black holes

To begin with, we consider the Einstein-Maxwell action

$$\frac{1}{16\pi}\int d^4x\sqrt{g}(\frac{1}{G}R - F_{\mu\nu}F^{\mu\nu}) \ , \tag{15.1}$$

where G is Newton's constant. In this subsection, we shall review black hole solutions to the action (15.1) and see that they are characterized by mass M, charge Q and angular momentum J.

Schwarzschild metric

If there is no electromagnetic fields $F = 0$ in the action (15.1), the equation of motion is $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} = 0$, which has a spherically symmetric, static solution

$$ds^2 \equiv g_{\mu\nu}dx^\mu dx^\nu = -(1 - \frac{2GM}{r})dt^2 + (1 - \frac{2GM}{r})^{-1}dr^2 + r^2 d\Omega^2,$$

where t is the time, r is the radial coordinate, and $d\Omega$ is the canonical metric of a 2-sphere. This metric describes the spacetime outside a gravitationally collapsed non-rotating star with zero electric charge, called **Schwarzschild metric**. It is well-known that the **event horizon** appears at

$$g^{rr} = 0\,,$$

and the sphere $r = 2GM$ is indeed the event horizon of the Schwarzschild black hole with mass M.

It turns out that much of the interesting physics having to do with the quantum properties of black holes comes from the region near the event horizon. To examine the region *near* $r = 2GM$, we analytically continued to the Euclidean metric $t = -it_E$, and we set

$$r - 2GM = \frac{x^2}{8GM} \ .$$

Then, the metric near the event horizon $r = 2GM$

$$ds_{\mathrm{E}}^2 \approx (\kappa x)^2 dt_E^2 + dx^2 + \frac{1}{4\kappa^2}d\Omega^2 \ ,$$

where $\kappa = \frac{1}{4GM}$ is called the **surface gravity** because it is indeed the acceleration of a static particle near the horizon as measured at spatial infinity. Note that the surface gravity is defined by using Killing vector at the horizon, precisely speaking [Tow97]. The first part of the metric is just $\mathbb{R}^2$ with polar coordinates if we make the periodic identification

$$t_E \sim t_E + \frac{2\pi}{\kappa} \ .$$

Using the relation between Euclidean periodicity and temperature, we can deduce **Hawking temperature** of the Schwarzschild black hole

$$T_H = \frac{\hbar\kappa}{2\pi} = \frac{\hbar}{8\pi GM} \ . \tag{15.2}$$

This is a very heuristic way to introduce the Hawking temperature which is not originally found in this way.

Reissner-Nordström black hole

The most general static, spherically symmetric, charged solution of the Einstein-Maxwell theory (15.1) is

$$ds^2 = -\left(1 - \frac{2GM}{r} + \frac{GQ^2}{r^2}\right)dt^2 + \left(1 - \frac{2GM}{r} + \frac{GQ^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega^2, \tag{15.3}$$

with the electromagnetic field strength

$$F_{tr} = \frac{Q}{r^2} \ .$$

This solution is called the **Reissner-Nordström (RN) black hole** with mass M and charge Q. From the metric (15.3) we see that there are two event horizon for this solution where $g^{rr} = 0$ at

$$r_{\pm} = GM \pm \sqrt{(GM)^2 - GQ^2} \ .$$

Thus, r_+ defines the outer horizon of the black hole and r_- defines the inner horizon of the black hole. The area of the black hole is $4\pi r_+^2$. It turns out that the Hawking temperature of the RN black hole is

$$T_H = \frac{\sqrt{(GM)^2 - GQ^2}}{2\pi G\left(GM + \sqrt{(GM)^2 - GQ^2}\right)^2} \ .$$

For a physically sensible definition of temperature, the mass must satisfy the bound $GM^2 \geq Q^2$, and the two horizons coincide $r_+ = r_- = GM$ when this bound is saturated. In this case, the temperature of the black hole is zero and it is called **extremal black hole**.

Kerr-Newman black hole

If we relax the static condition, black holes can have angular momentum. Hence, general stationary solutions, called **Kerr-Newman black holes**, to the action (15.1) are described with three parameters. In **Boyer-Linquist coordinates**, the KN metric is

$$\begin{aligned} ds^2 = &-\frac{(\Delta - a^2 \sin^2\theta)}{\Sigma}dt^2 - 2a\sin^2\theta\frac{(r^2 + a^2 - \Delta)}{\Sigma}dt\, d\phi \\ &+ \left(\frac{(r^2 + a^2)^2 - \Delta a^2 \sin^2\theta}{\Sigma}\right)\sin^2\theta d\phi^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 \end{aligned}$$

where

$$\Sigma = r^2 + a^2 \cos^2 \theta$$
$$\Delta = r^2 - 2Mr + a^2 + e^2 \ .$$

The three parameters are M, a, and e. It can be shown that

$$a = \frac{J}{M}$$

where J is the total angular momentum, while

$$e = \sqrt{Q^2 + P^2}$$

where Q and P are the electric and magnetic (monopole) charges, respectively. The Maxwell 1-form of the KN solution is

$$A_\mu dx^\mu = \frac{Qr\left(dt - a\sin^2\theta d\phi\right) - P\cos\theta\left[adt - (r^2 + a^2)\,d\phi\right]}{\Sigma} \ .$$

15.1.2 Black hole thermodynamics

Classically, a stationary black hole is characterized by its mass M, angular momentum J, and charge Q. This is called a black hole no hair theorem. However, in [**?**], Bekenstein asks an incisive question: if we treat a black hole as a purely geometric object, by throwing a package of entropy, a cup of tea, into a black hole, the total entropy of the world outside would seem to decrease. This contradicts the second law of thermodynamics which states the total entropy never decreases. To save the second law of thermodynamics, this suggests that the black hole must have entropy. The question is how to characterize the entropy if a black hole has. The hit was hidden in the area theorem of black holes [**? ?**], stating that the total area of the black hole horizons never decreases in any process. For example, two Schwarzschild black holes with masses M_1 and M_2 can merge into a bigger black hole of mass $M = M_1 + M_2$. Since the area is proportional to the square of the mass, this is consistent with the area theorem, namely $(M_1 + M_2)^2 \geq M_1^2 + M_2^2$. On the other hand, the opposite process where a bigger black hole splits into two is never allowed by this theorem. Motivated by the area theorem, Bekenstein proposed in [**?**] that a black hole has entropy proportional to its area.

Soon after that, Bardeen, Carter and Hawking point out similarities between the laws of black hole mechanics and the laws of thermodynamics in [BCH73]. More concretely, they find the laws of corresponding to the three laws of thermodynamics.

(0) Zeroth Law: In thermodynamics, the zeroth law states that the temperature T of a thermal equilibrium object is constant throughout the body. Correspondingly, for a stationary black hole, its surface gravity $\kappa = 1/4GM$ is constant over the event horizon.

(1) First Law: The first law of thermodynamics states that energy is conserved, and the variation of energy is given by

$$dE = TdS + \mu dQ + \Omega dJ \tag{15.4}$$

where E is the energy, Q is the charge with chemical potential μ and J is the angular momentum with chemical potential Ω in the system. Correspondingly, for a black hole, the variation of its mass is given by

$$dM = \frac{\kappa}{8\pi G} dA + \mu dQ + \Omega dJ \tag{15.5}$$

where A is the area of the horizon, and κ is the surface gravity, μ is the chemical potential conjugate to Q, and Ω is the angular velocity conjugate to J.

(2) Second Law: The second law of thermodynamics states that the total entropy S never decreases, $\delta S \geq 0$. Correspondingly, for a black hole, the area theorem states that the total area of a black hole in any process never decreases, $\delta A \geq 0$.

Laws of thermodynamics	**Laws of black hole mechanics**
Temperature is constant throughout a body at equilibrium. T=constant.	Surface gravity is constant on the event horizon. κ =constant.
Energy is conserved. $dE = TdS + \mu dQ + \Omega dJ$.	Energy is conserved. $dM = \frac{\kappa}{8\pi G} dA + \mu dQ + \Omega dJ$.
Entropy never decreases. $\delta S \geq 0$.	Area never decreases. $\delta A \geq 0$.

Table 5: Laws of black hole thermodynamics

This result can be understood as one of the highlights of general relativity. Classically, a black hole is a not only geometric but also thermodynamic object. If a black hole has energy E and entropy S, then it must also have temperature T given by

$$\frac{1}{T} = \frac{\partial S}{\partial E}.$$

For example, for a Schwarzschild black hole, the area and the entropy are proportional to M^2. Hence, we can derive

$$\frac{1}{T} = \frac{\partial S}{\partial M} \sim \frac{\partial M^2}{\partial M} \sim M.$$

Therefore, black hole temperature is inversely proportional to mass M. The smaller a black hole is, the hotter it is! Moreover, if the black hole has temperature, it must thermally radiate like any hot body. The understanding of the thermal properties of black holes requires treatment beyond classical general relativity.

Hawking has applied techniques of quantum field theories on a curved background to the near-horizon region of a black hole and showed that a black hole indeed radiates [Haw75]. This can be intuitively understood as follows: in a quantum theory, particle-antiparticle creations constantly occur in the vacuum. Around the horizon, after pairs are created, antiparticles fall into a black hole due to the gravitational attraction whereas particles escape to the infinity. Although we do not deal with Hawking's calculation unfortunately (see [Tow97]), it indeed justifies this picture. Moreover, it revealed that the spectrum emitted by the black hole is precisely subject to the thermal radiation with temperature (15.2). Indeed, a black hole is not black at quantum level. Hence, we can treat a black hole as a thermal object, and the analogy of the laws in Table 5 can be understood as the natural consequence of the laws of thermodynamics. As a result, the formula for the Hawking temperature (15.2) and the first law of thermodynamics

$$c^2 dM = T_H dS = \frac{\kappa c^2}{8\pi G} dA,$$

lead to the precise relation between entropy and the area of the black hole:

$$S = \frac{k_B c^3 A}{4G\hbar}\,. \tag{15.6}$$

This is a universal result for any black hole, and this remarkable relation between the thermodynamic properties of a black hole and its geometric properties is called the celebrated **Bekenstein-Hawking entropy formula**. This formula involves all four fundamental constants of nature; $(G, c, k_B, \hbar)$. Also, this is the first place where the Newton constant G meets with the Planck constant $\hbar$. Thus, this formula shows a deep connection between black hole geometry, thermodynamics and quantum mechanics.

For ordinary objects, Boltzmann has given the statistical interpretation of the thermodynamic entropy of a system. We fix the macroscopic parameters (e.g. total electric charge, energy etc.) and count the number Ω of quantum states, known as microstates, each of which has the same values for the macroscopic parameters, and the entropy is expressed as

$$S = k_B \log \Omega\ .$$

Since the Bekenstein-Hawking entropy (15.6) behaves as the ordinary thermodynamic entropy in every aspect, it is therefore natural to ask whether the black hole entropy admits a statistical interpretation in the same way.

Furthermore, one of the most dramatic results of Hawking's work was the implication that black holes are associated with information loss. Physically speaking, we can associate information with pure states in quantum mechanics. If we throw in a pure quantum state, say, the s-wave to a black hole, then it eventually comes out as a thermal (mixed) state. Thus the net result of this process is the evolution of a pure quantum state into a mixed state, which violates the law (unitarity) of quantum mechanics. This is called the **information paradox** [Haw76]. This is because Hawking's calculation is based on the semi-classical analysis, namely, we fix the background and quantize particles. In fact, the information paradox stems from the absence of such a microscopic description of gravity.

In order to investigate the microscopic description of black hole entropy, we need quantum theory of gravity. This is precisely what string theorists have attempted to do and have been partially successful.

15.2 Black holes in string theory

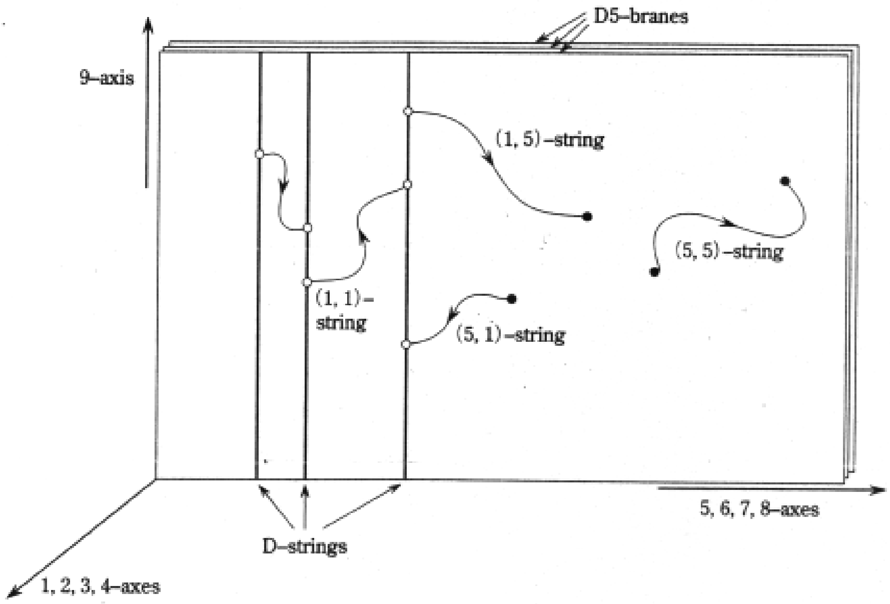
In string theory on a d-dimensional compact manifold, branes can be wrapped in a cycle of the compact manifold and it looks like a point-like object in $10-d$-dimensional space time. In the regime that supergravity approximation is valid, configurations of this kind gives rise black hole solutions of the corresponding low-energy supergravity theory. Moreover, if a brane configuration preserves supersymmetry, then the corresponding solution will be an extremal supersymmetric black hole. Extremal black holes are interesting because they are stable against Hawking radiation and nevertheless have a large entropy. On the other hand, configurations without supersymmetry yield non-extremal black holes.

In general, the regime of the parameter space in which supergravity is valid is different from the regime in which weakly coupled string theory is valid where the microstates counting can be performed. Thus, even if we know that a given brane configuration becomes a black hole when we go from a weak to a strong coupling, it is generally difficult to extract microscopic information of the black hole from the brane configuration.

For supersymmetric black holes, however, one can count the number of states at weak coupling and extrapolate the result to the black hole phase due to the BPS property. We will see that in this way, one derives the Bekenstein-Hawking entropy formula (including the precise numerical coefficient) for a 5d supersymmetric black hole [SV96]. (For more detail, I refer to [DMW02].)

15.2.1 D1-D5-P brane system

Let us consider Type IIB compactified on a five-torus $T^5 = T^4 \times S^1$, which spans the $(x_5 \cdots x_9)$ coordinates, with Q_1 D1-branes and Q_5 D5-branes in the following configuration. We consider that the volume o T^4 is $(2\pi)^4 V$ and the radius of S^1 is R. Here we also assume that there is an excitation by open strings carrying momenta Q_P/R in the x_9-direction. This system preserves 4 real supercharges since each constituent breaks a half of supersymmetry.



	0	1	2	3	4	5	6	7	8	9
Q_1 D1	$\times$									$\times$
Q_5 D5	$\times$					$\times$	$\times$	$\times$	$\times$	$\times$
Q_P mom										$\rightsquigarrow$

15.2.2 Black hole in 5d supergravity

If there are large enough D-brane charges (Q_1, Q_5, Q_P) and the five-torus is sufficiently small, the configuration produces a 5d black hole. We would like to compute the Beckenstein-Hawking entropy of the black hole by evaluating the area of the event horizon. In this regime, five-dimensional supergravity analysis can be used and it admits the corresponding 1/8-BPS solution. Ignoring RR-field and B-field configuration, the 5d Einstein frame metric of this solution then becomes

$$ds_5^2 = -\,\lambda(r)^{-2/3}dt^2 + \lambda(r)^{1/3}\left[dr^2 + r^2 d\Omega_3^2\right] ,$$

where the harmonic functions are

$$\lambda(r) = H_1(r)H_5(r)K(r) = \left(1 + \frac{r_1^2}{r^2}\right)\left(1 + \frac{r_5^2}{r^2}\right)\left(1 + \frac{r_m^2}{r^2}\right) ,$$

with

$$r_1^2 = \frac{g_{\rm s}Q_1\ell_{\rm s}^6}{V} , \qquad r_5^2 = g_{\rm s}Q_5\ell_{\rm s}^2 \qquad r_m^2 = \frac{g_{\rm s}^2 Q_P \ell_{\rm s}^8}{R^2 V} .$$

Let us briefly evaluate the validity of the supergravity analysis. In order for the α' corrections to geometry to be small, the radius parameters have to be large with respect to the string unit, $r_{1,5,m} \gg \ell_{\rm s}$. Since we assume V, R are order of the string length, this

$$g_{\rm s}Q_1 \gg 1 , \qquad g_{\rm s}Q_5 \gg 1 , \qquad g_{\rm s}^2 Q_P \gg 1 .$$

To suppress string loop corrections, we need $g_{\rm s}$ to be small so that the D-brane charges must be sufficiently large for supergravity analysis.

It turns out that the surface gravity and therefore the Hawking temperature of this black hole is zero, $T_H = 0$, as expected. The metric shows that the event horizon is located at $r = 0$ and the Bekenstein-Hawking entropy is

$$\begin{aligned} S_{\rm BH} &= \frac{A}{4G_5} = \frac{1}{4G_5} 2\pi^2 \left[r^2 \lambda(r)^{\frac{1}{3}} \right]^{\frac{3}{2}} \quad \text{at } r = 0 \\ &= \frac{2\pi^2}{4\left[\pi g_{\rm s}^2 \ell_{\rm s}^8/(4VR)\right]} (r_1 r_5 r_m)^{\frac{1}{2}} = \frac{2\pi V R}{g_{\rm s}^2 \ell_{\rm s}^8} \left(\frac{g_{\rm s} Q_1 \ell_{\rm s}^6}{V} g_{\rm s} Q_5 \ell_{\rm s}^2 \frac{g_{\rm s}^2 Q_P \ell_{\rm s}^8}{R^2 V} \right)^{\frac{1}{2}} \\ &= 2\pi \sqrt{Q_1 Q_5 Q_P}\,, \end{aligned} \tag{15.7}$$

where we use $G_5 = \frac{G_{10}}{(2\pi)^5 VR}$ and $16\pi G_{10} = (2\pi)^7 g_2^2 \ell_s^8$. Notice that it is also independent of R and of V whereas the ADM mass depends on R, V explicitly.

$$M = \frac{Q_P}{R} + \frac{Q_1 R}{g_{\rm s} \ell_{\rm s}^2} + \frac{Q_5 R V}{g_{\rm s} \ell_{\rm s}^6}\,.$$

15.2.3 Counting microstates

The next step is to identify the degeneracy of open string states of D1-D5-P system, which can be analyzed at weak coupling limit, i.e. $g_s Q_i \ll 1$. Further simplification can be made by taking the limit that the volume of T^4 is small as compared to the radius of the circle S^1,

$$V^{\frac{1}{4}} \ll R\,.$$

In this limit, the theory on the D-branes is effectively $2d$ theory on (x_1, x_9)-direction. Moreover, the smeared D1-branes plus D5-branes have a symmetry group $SO(1,1) \times \mathrm{SO}(4)_{\|} \times \mathrm{SO}(4)_{\perp}$ where $\mathrm{SO}(4)_{\perp} \cong \mathrm{SU}(2) \times \mathrm{SU}(2)$ becomes R-symmetry of the $2d$ theory which we call $\mathcal{N} = (4,4)$ 2d CFT. In the supersymmetric configuration, the right-movers are in their ground states so that we count excited left-movers.

Because the D1-branes are instantons in the D5-brane theory, the low-energy theory of interest is in fact a σ-model on the moduli space of instantons

$$\mathcal{M} = \mathrm{Sym}^{Q_1 Q_5}(T^4) = (T^4)^{Q_1 Q_5}/S_{Q_1 Q_5}\ .$$

The central charge of this 2d CFT is

$$c = n_{\rm bose} + \frac{1}{2} n_{\rm fermi} = 6 Q_1 Q_5\ .$$

Roughly, this central charge c can be thought of as coming from having $Q_1 Q_5$ 1-5 strings that can move in the 4 directions of T^4. Although this orbifold theory has many twisted sectors, the special point of the moduli space corresponds to a single string wound $Q_1 Q_5$ times. It turns out that counting the excitations of this **long**

string is only relevant in the limit of large D-brane charges. For this long string, the level-matching condition is

$$N-\widetilde{N}=\frac{Q_P}{R}W\ ,\quad W=Q_1Q_5\ ,\quad\to\quad N=\frac{Q_PQ_1Q_5}{R}$$

where the right-movers are in the ground states $\widetilde{N}=0$.

If N^i_m and n^i_m denote occupation numbers of the four transverse compact bosonic and fermionic oscillators, respectively, then evaluation of N gives

$$nW=\sum_{i=1}^{4}\sum_{m=1}^{\infty}m(N^i_m+n^i_m) \tag{15.8}$$

The degeneracy $\Omega(Q_1,Q_5,Q_P)$ is then given by the number of choices for N^i_m and n^i_m subject to (15.8).

The partition function of this system is the partition function for 4 bosons and an equal number of fermions

$$Z=\left[\prod_{m=1}^{\infty}\frac{1+q^m}{1-q^m}\right]^4\equiv\sum\Omega(Q_1,Q_5,Q_P)q^N\ ,$$

where $\Omega(Q_1,Q_5,Q_P)$ is the degeneracy of states at d=1+1 energy $N=\frac{Q_PQ_1Q_5}{R}$. At large charges, we can use the Cardy formula

$$\Omega(Q_1,Q_5,Q_P)\sim\exp\sqrt{\frac{\pi\,c\,E\,(2\pi R)}{3}}=\exp\left(2\pi\sqrt{\frac{c}{6}\,ER}\right)\ .$$

Therefore the microscopic D-brane statistical entropy is

$$S_{\text{micro}}=\log\left(\Omega(Q_1,Q_5,Q_P)\right)=2\pi\sqrt{Q_1Q_5Q_P}\,.$$

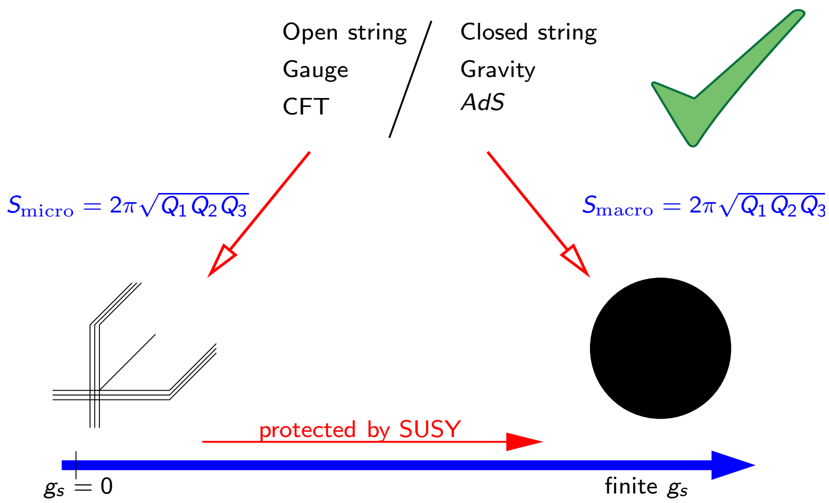
This agrees exactly with the black hole result (15.7)!

15.2.4 More results

By coupling the low energy degrees of freedom in the D1-D5-p system to supergravity modes (therefore perturbing the extremal condition), one can also compute the rate of Hawking radiation from these black that agrees precisely with the Hawking calculation. Thus this provides a microscopic explanation of Hawking radiation. (See [DMW02, §8].)

In fact, vigorous research in the last decade has shown that one can show the exact match between macroscopic and microscopic calculations of black hole entropy even in finite D-brane charges.

Moreover, a generalization of Bekenstein-Hawking entropy has been proposed in [RT06] that connects quantum theory of gravity and quantum information theory. Recent study has clearly suggested that quantum entanglement must have something to do with quantum physics of spacetime.



16 Lecture 16

The study of black holes in string theory by using D-branes has led to the celebrated AdS/CFT correspondence [Mal99]. The AdS/CFT correspondence is the equivalence between a string theory or M-theory on an anti-de Sitter background and a conformal field theory. It has shed a new light on quantum gravity as well as strongly coupled quantum field theories. Although it was proposed in the framework of string theory, it has already been studied beyond string theory, influencing other physical theories. It has attracted large number of researchers, and it is connected to many branches of physics. For the basic of the AdS/CFT correspondence, I refer to the most famous review [AGM+00].

We shall first study basic properties of conformal field theories in general dimensions and geometry of anti-de Sitter space. Then, we will deal with the most famous example, Type IIB on $\mathrm{AdS}_5 \times S^5$/ 4d $\mathcal{N} = 4$ SYM.

16.1 Conformal group

A conformal field theory (CFT) is a quantum field theory that is invariant under conformal transformations. We have studied the conformal transformation for 2-dim (in Lec 2 & 3), which is a special case. Here, we study a conformal group for arbitrary dimensions (assume $d \geq 3$).

Conformal group is defined by transformations that preserve the metric up to a local scale factor:

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \Omega^2(x) g_{\mu\nu}(x) \ .$$

In an infinitesimal form ($x'^\mu = x^\mu + \epsilon^\mu$) it is (compare with lecture note 03)

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d} \eta_{\mu\nu} \partial \cdot \epsilon \ . \tag{16.1}$$

Applying ∂^μ to Eq. (16.1) we have

$$\left(1-\frac{2}{d}\right)\partial_\nu \partial\cdot\epsilon + \Box\epsilon_\nu = 0\ ,$$

where $\Box = \partial\cdot\partial$. From this expression we can see that $d=2$ is quite special, and it leads $\partial^2\epsilon = \partial_-\partial_+\epsilon$, which gives infinitely many transformations. We further apply ∂^ν and reach

$$(d-1)\Box\partial\cdot\epsilon = 0\ .$$

This expression implies ϵ_μ is up to quadratic order of x:

$$\epsilon_\mu = a_\mu + b_{\mu\nu}x^\nu + c_{\mu\nu\rho}x^\nu x^\rho\ .$$

Plugging these expressions back to the definition equations above (and its variant), we have following constraints

$$\begin{aligned} b_{\mu\nu} &= \alpha\eta_{\mu\nu} + M_{\mu\nu} \qquad (M_{\mu\nu} = -M_{\nu\mu})\ , \\ c_{\mu\nu\rho} &= \eta_{\mu\nu}f_\rho + \eta_{\mu\rho}f_\nu - \eta_{\nu\rho}f_\mu \qquad (f_\mu = \frac{1}{d}c^\rho{}_{\rho\mu})\ . \end{aligned}$$

Parameters above correspond to transformations you are familiar with except f_μ, which is called special conformal transformation (SCT). See Table 6 for the summary. The generators summarized in the table form conformal group commutation relations

Table 6: Ingredients of the conformal group.

Names	Finite transf.	Generators	Dim.
Translation	$x'^\mu = x^\mu + a^\mu$	$P_\mu = -i\partial_\mu$	$+1$
Dilat(at)ion	$x'^\mu = \alpha x^\mu$	$D = -ix\cdot\partial$	0
Lorentz/Rotation	$x'^\mu = M^\mu{}_\nu x^\nu$	$L_{\mu\nu} = i\left(x_\mu\partial_\nu - x_\nu\partial_\mu\right)$	0
SCT	$x'^\mu = \frac{x^\mu - (x\cdot x)f^\mu}{1-2f\cdot x+(f\cdot f)(x\cdot x)}$	$K_\mu = -i\left(2x_\mu x\cdot\partial - (x\cdot x)\partial_\mu\right)$	-1

$$\begin{aligned} &[J_{ab}, J_{cd}] = i\left(\eta_{ad}J_{bc} + \eta_{bc}J_{ad} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac}\right)\ , \\ &J_{\mu\nu} = L_{\mu\nu}\ , \qquad\qquad J_{(d+1)d} = D\ , \\ &J_{\mu d} = \tfrac{1}{2}\left(K_\mu - P_\mu\right)\ ,\ J_{\mu(d+1)} = \tfrac{1}{2}\left(K_\mu + P_\mu\right)\ , \end{aligned}$$

where note that $a,b,c,d = 0,1,\ldots d+1$ and $\mu,\nu = 0,1,\ldots,d-1$, and η_{ab} is $\mathrm{diag}(-,+,\ldots,+,-)$ for Lorentzian spacetime and $\mathrm{diag}(+,+,\ldots,+,-)$ for Euclidean space. The algebras are isomorphic to those of $\mathrm{SO}(2,d)$ and $\mathrm{SO}(1,d+1)$, respectively. Note that the SCT can be understood with inversion ($x^\mu \to \frac{x^\mu}{x\cdot x}$) as follows

$\frac{x'^\mu}{x'\cdot x'} = \frac{x^\mu}{x\cdot x} - f^\mu$. However, the inversion is not included in the algebra (since the inversion is discrete transformation). Among others we write down non-trivial commutation relations that involve D

$$[D, P_\mu] = -iP_\mu \ , \quad [D, K_\mu] = iK_\mu \ , \quad [P_\mu, K_\nu] = 2i\left(M_{\mu\nu} - \eta_{\mu\nu}D\right) \ , \qquad (16.2)$$

which characterize representation of the conformal group. By the way, the other non-trivial ones are

$$\begin{aligned}
&[M_{\mu\nu}, P_\rho] = -i(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu) \ , \qquad [M_{\mu\nu}, K_\rho] = -i(\eta_{\mu\rho}K_\nu - \eta_{\nu\rho}K_\mu) \ , \\
&[M_{\mu\nu}, M_{\rho\sigma}] = -i\eta_{\mu\rho}M_{\nu\sigma} \pm (\text{permutations}) \ .
\end{aligned}$$

16.1.1 Primary fields

For 2-dim we defined the primary field by using OPE with energy-momentum tensor (it leads a representation of the conformal group). For general dimensions we define it by a formal representation of the conformal group. It is known that the representation is characterized by an eigenvalue of the dilatation operator $-i\Delta$ (Δ is called the **scaling dimension** of the field, rather than **weight**), and representation of Lorentz group. The former statement means that $\Phi(x) \to \Phi'(\lambda x) = \lambda^{-\Delta}\Phi(x)$. The commutation relations (16.2) tell us that P_μ is raising operator, while K_μ is lowering operator. Therefore, there are operators that annihilated by K_μ in each finite representation of the conformal group. Such operator is called **primary operator/field** (we use operator and field interchangeably). The action of the conformal group on the primary field is

$$\begin{aligned}
[P_\mu, \Phi(x)] &= i\partial_\mu\Phi(x) \ , \\
[M_{\mu\nu}, \Phi(x)] &= [i(x_\mu\partial_\nu - x_\nu\partial_\mu) + \Sigma_{\mu\nu}]\Phi(x) \ , \\
[D, \Phi(x)] &= i(-\Delta + x^\mu\partial_\mu)\Phi(x) \ , \\
[K_\mu, \Phi(x)] &= [i(x^2\partial_\mu - 2x_\mu x^\nu\partial_\nu + 2x_\mu\Delta) - 2x^\nu\Sigma_{\mu\nu}]\Phi(x) \ ,
\end{aligned}$$

where $\Sigma_{\mu\nu}$ are the matrices of a finite dimensional representation of the Lorentz group, acting on the indices of the Φ field (e.g. it is $\frac{i}{2}\gamma_{\mu\nu}$ for a spinor). There are some comments on primary fields and others:

- Fields created by acting P_μ on a primary field are called **descendant fields**.
- Fields are not, in general, by eigenfunctions of the Hamiltonian P^0, or the mass operator $-P\cdot P$, and hence, they have continuous spectrum.
- In unitary field theories the scale dimension is bounded from below (**unitary bound**). It is $\Delta \geq (d-2)/2$ for scalars, $\Delta \geq (d-1)/2$ for spinors, and $\Delta \geq d+s-2$ for spin-s fields for $s \geq 1$. (We refer the derivation of the bounds and other details to [Qua15, Sec. 2].)

16.2 Anti-de Sitter space

An anti-de Sitter(AdS) space is a maximally symmetric manifold with constant negative scalar curvature. It is a solution of Einstein's equations for an empty universe with negative cosmological constant. The easiest way to understand it is as follows.

A Lorentzian AdS_{d+1} space can be illustrated by the hyperboloid in $(2, d)$ Minkowski space:

$$X_0^2 + X_{d+1}^2 - \sum_{i=1}^{d} X_i^2 = R^2 \ . \tag{16.3}$$

The metric can be naturally induced from the Minkowski space

$$ds^2 = -dX_0^2 - dX_{d+1}^2 + \sum_{i=1}^{d} dX_i^2 \ .$$

By construction it has $\mathrm{SO}(2, d)$ isometry, which is a first connection to the conformal group in d-dim.

16.2.1 Global coordinate

Simple solution to (16.3) is given as follows.

$$\begin{aligned} X_0^2 + X_{d+1}^2 &= R^2 \cosh^2\rho \ , \\ \sum_{i=1}^{d} X_i^2 &= R^2 \sinh^2\rho \ . \end{aligned}$$

Or, more concretely,

$$\begin{aligned} X_0 &= R\cosh\rho\ \cos\tau \ , \qquad X_{d+1} = R\cosh\rho\ \sin\tau \ , \\ X_i &= R\sinh\rho\ \Omega_i \quad (i = 1, \ldots, d, \ \text{and} \ \sum_i \Omega_i^2 = 1). \end{aligned}$$

These are S^1 and S^{d-1} with radii $R\cosh\rho$ and $R\sinh\rho$, respectively. The metric is

$$ds^2 = R^2\left(-\cosh^2\rho\ d\tau^2 + d\rho^2 + \sinh^2\rho\ d\Omega_{(d-1)}^2\right) \ .$$

Note that τ is a periodic variable and if we take $0 \le \tau < 2\pi$ the coordinate wrap the hyperboloid precisely once. This is why this coordinate is called **global coordinate**. The manifest sub-isometries are $\mathrm{SO}(2)$ and $\mathrm{SO}(d)$ of $\mathrm{SO}(2, d)$. To obtain a causal spacetime, we simply unwrap the circle S^1, namely, we take the region $-\infty < \tau < \infty$ with no identification, which is called **universal cover** of the hyperboloid.

In literatures another global coordinate is also used, which can be derived by redefinitions $r \equiv R\sinh\rho$ and $dt \equiv Rd\tau$:

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\Omega_{(d-1)}^2 \ , \qquad f(r) = 1 + \frac{r^2}{R^2} \ .$$

16.2.2 Poincaré coordinates

There is yet another coordinates, called **Poincaré coordinates**. As opposed to the global coordinate, this coordinate covers only the half of the hyperboloid. It is most easily (but naively) seen in $d=1$ case:

$$x^2 - y^2 = R^2 \ ,$$

which is the hyperbolic curve. The curve consists of two isolated parts in regions $x > R$ and $x < -R$. We simply use one of them to construct the coordinate.

Let us get back to general d-dim. We define the coordinate as follows.

$$\begin{aligned}
X_0 &= \frac{1}{2u}\left(1 + u^2\left(R^2 + x_i^2 - t^2\right)\right) \ , \\
X_i &= Rux_i \qquad (i = 1, \ldots, d-1) \ , \\
X_d &= \frac{1}{2u}\left(1 - u^2\left(R^2 - x_i^2 + t^2\right)\right) \ , \\
X_{d+1} &= Rut \ ,
\end{aligned}$$

where $u > 0$. As it is stated the coordinate covers the half of the hyperboloid; in the region, $X_0 > X_d$. The metric is

$$ds^2 = R^2\left(\frac{du^2}{u^2} + u^2(-dt^2 + dx_i^2)\right) = R^2\left(\frac{du^2}{u^2} + u^2 dx_\mu^2\right) \ . \tag{16.4}$$

The coordinates (u, t, x_i) is called **Poincaré coordinates**. This metric has manifest $ISO(1, d-1)$ and $\mathrm{SO}(1,1)$ sub-isometries of $\mathrm{SO}(2,d)$; the former is the Poincaré transformation and the latter corresponds to the dilatation

$$(u, t, x_i) \to (\lambda^{-1}u, \lambda t, \lambda x_i) \ .$$

If we further define $z = 1/u$ ($z > 0$), then,

$$ds^2 = \frac{R^2}{z^2}\left(dz^2 + dx_\mu^2\right) \ . \tag{16.5}$$

This is called **the upper(Poincaré) half-plane model**. The hypersurface given by $z = 0$ is called **(asymptotic) boundary** of the AdS space, which corresponds to $u \sim r \sim \rho = \infty$.

16.3 Introduction to $\mathrm{AdS}_5/\mathrm{CFT}_4$ correspondence

Now let us study the most well-studied example of AdS/CFT correspondence, which is the equivalence between 4d $\mathrm{U}(N)$ $\mathcal{N} = 4$ super-Yang-Mills (SYM) and Type IIB string theory on $\mathrm{AdS}_5 \times S^5$, which arises from the large number of D3-branes. Type IIB string theory with D3-branes contains two kinds of perturbative excitations, closed strings and open strings. If we consider the system at low energies, energies lower than the string scale $1/\ell_s$, then only the massless string states can be excited. The closed string massless states give a gravity supermultiplet in 10d, and their low-energy effective theory is Type IIB supergravity. The open string massless states give an $\mathcal{N}=4$ vector supermultiplet in 4d, and their low-energy effective theory is $\mathcal{N}=4$ U(N) SYM. Therefore, it can be also understood as **open/closed duality**.

$\mathcal{N}=4$ super-Yang-Mills theory

The low-energy effective theory of N D3-branes is 4d $\mathcal{N}=4$ U(N) SYM theory so we describe the basic properties of the $\mathcal{N}=4$ SYM. The action can be obtained by the dimensional reduction from the 10d $\mathcal{N}=1$ SCFT on $\mathbb{R}^{1,3}\times T^6$ where the 10d Lorentz group SO$(1,9)$ is decomposed to SO$(1,3)\times$ SO$(6)\subset$ SO$(1,9)$:

$$\begin{aligned} S &= -\frac{1}{g_{YM}^2}\int d^{10}x \mathrm{Tr}\left[\frac{1}{4}F_{MN}^2+\frac{i}{2}\overline{\lambda}\Gamma^M D_M\lambda\right] \\ &= -\frac{1}{g_{YM}^2}\int d^4x\ \mathrm{Tr}\Big[\frac{1}{4}F_{\mu\nu}^2+\frac{1}{2}(D_\mu X_m)^2+\frac{i}{2}\overline{\lambda}\Gamma^\mu D_\mu\lambda-\frac{g_{YM}}{2}\overline{\lambda}\Gamma^m[X_m,\lambda]-\frac{g_{YM}^2}{4}[X_m,X_n]^2\Big] \end{aligned} \tag{16.6}$$

where the ten-dimensional gauge fields A_M, $M=0,\ldots,9$ split the 4d gauge field A_μ, $\mu=0,\ldots,3$ and 6 scalars X_m, $m=1,\ldots,6$, and λ is a 10d Majorana-Weyl spinor dimensionally reduced to 4d. We can also add the topological term

$$S_{\mathrm{top}}=\frac{i\theta}{32\pi^2}\int d^4x\ \epsilon^{\mu\nu\rho\sigma}\mathrm{Tr}(F_{\mu\nu}F_{\rho\sigma})$$

The action is invariant under the supersymmetry transformation

$$\begin{aligned} \delta X^m &= -\overline{\epsilon}\Gamma^m\lambda \\ \delta A^\mu &= -\overline{\epsilon}\Gamma^\mu\lambda \\ \delta\lambda &= \Big(\frac{1}{2}F_{\mu\nu}\Gamma^{\mu\nu}+D_\mu X_m\Gamma^{\mu m}+\frac{i}{2}[X_m,X_n]\Big)\epsilon\ . \end{aligned} \tag{16.7}$$

It is easy to see that 4d $\mathcal{N}=4$ SYM is classically **conformal invariant**. because the mass dimensions of the fields

$$[A_\mu]=[X^i]=1\ , \qquad [\lambda_a]=\frac{3}{2}\ , \tag{16.8}$$

so that the coupling constant is dimensionless: $[g]=[\theta]=0$. However, one has to be careful at quantum level because quantum correction generally breaks the conformal invariance. To be conformal invariant at quantum level, the beta function of the coupling constant has to vanish $\beta=0$. It turns out that 4d $\mathcal{N}=4$ SYM is the case and hence it is quantum mechanically conformal. The $\mathcal{N}=4$ supersymmetry combined with conformal symmetry forms the superconformal group SU$(2,2|4)$ which consists of the following generators

- **Conformal Symmetry** is $\mathrm{SO}(2,4) \cong \mathrm{SU}(2,2)$ in $d=4$, as we have seen in §16.1. The generators consist of translations P^μ, Lorentz transformations $L_{\mu\nu}$, dilations D and special conformal transformations K^μ with the relations (**??**);

- **R-symmetry** is $\mathrm{SO}(6)_R \cong \mathrm{SU}(4)_R$ which is manifest from the 10d viewpoint, and R-symmetry rotates the 6 scalar X^m $(m=1,\dots,6)$;

- **Poincaré supersymmetries** are generated by the supercharges $Q^I_\alpha, \overline{Q}^I_{\dot\alpha}$, $I=1,\dots,4$ that transform under the **4** of $\mathrm{SU}(4)_R$. They can be understood as a "square root" of P_μ. Type IIB string theory has 32 supercharges and D3-branes break a half of the supersymmetries. Consequently, the 16 preserved supercharges are indeed $Q^I_\alpha, \overline{Q}^I_{\dot\alpha}$, which form $\mathcal{N}=4$ Poincaré supersymmetry;

- **Conformal supersymmetries**: are generated by the fermionic generators S^I_α and $\overline{S}^I_{\dot\alpha}$ that are superconformal partners of $Q^I_\alpha, \overline{Q}^I_{\dot\alpha}$. They can be understood as a "square root" of K_μ.

Therefore, there are 32 supercharges $Q, \overline{Q}, S, \overline{S}$ in total and they obey the anti-commutation relations

$$\begin{aligned}
&\{Q^\alpha_A, \overline{Q}^{\dot\alpha B}\} = P^{\alpha\dot\alpha}\delta^B_A, \\
&\{S^A_\alpha, \overline{S}_{\dot\alpha B}\} = K_{\alpha\dot\alpha}\delta^A_B, \\
&\{S^A_\alpha, Q^\beta_B\} = \delta^A_B M_\alpha{}^\beta + \delta^\beta_\alpha R^A_B + \delta^A_B\delta^\beta_\alpha\frac{D}{2}, \\
&\{\overline{S}_{\dot\alpha A}, \overline{Q}^{\dot\beta B}\} = \delta^B_A \overline{M}^{\dot\beta}{}_{\dot\alpha} - \delta^{\dot\beta}_{\dot\alpha} R^B_A + \delta^B_A \delta^{\dot\beta}_{\dot\alpha}\frac{D}{2} \ .
\end{aligned} \tag{16.9}$$

The $\mathcal{N}=4$ SYM enjoys **S-duality** [**? ?**] that is the $\mathrm{SL}(2,\mathbb{Z})$ action on the complexified coupling constant $\tau \equiv \frac{\theta}{2\pi} + \frac{4\pi i}{g^2_{YM}}$

$$\tau \to \frac{a\tau + b}{c\tau + d} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \ . \tag{16.10}$$

The $\mathcal{N}=4$ SYM is the world-volume theory on a stack of D3-branes, and D3-branes are invariant under the $\mathrm{SL}(2,\mathbb{Z})$ symmetry of Type IIB theory as seen in §**??**. In fact, this is the origin of the $\mathrm{SL}(2,\mathbb{Z})$ symmetry (16.10).

There is another way to realize the $\mathcal{N}=4$ SYM from string theory. Indeed, the M5-branes wrapped on a torus with complex structure τ give rise to the $\mathcal{N}=4$ SYM with the complexified coupling constant τ. Here, τ manifestly admits a geometric origin as the complex structure of a torus. Note that when $\theta=0$, the S-duality transformation amounts to $g_{YM} \to 1/g_{YM}$, thereby exchanging strong and weak coupling.[8]

[8] Precisely speaking, the electromagnetic duality of the $\mathcal{N}=4$ SYM depends on a choice of gauge groups, and the duality group is usually a congruence subgroup of $\mathrm{SL}(2,\mathbb{Z})$. For more detail, we refer to [**?**].

Near-horizon geometry of D3-branes

Now let us study the closed string side of the system. A system of N coincident D3-branes is a classical solution of the low-energy string effective action:

$$ds^2 = H(y)^{-\frac{1}{2}} \eta_{\mu\nu} dx^\mu dx^\nu + H(y)^{\frac{1}{2}} (dy^2 + y^2 d\Omega_5^2) \tag{16.11}$$

with R-R field

$$C_{(4)} = H(y)^{-1} dx^0 \wedge \cdots \wedge dx^3$$

where

$$H(y) = 1 + \frac{R^4}{y^4} , \qquad R^4 = 4\pi g_s N (\alpha')^2 . \tag{16.12}$$

Figure 39: Minkowski region and near-horizon region

To study this geometry more closely, we consider its limit in two regimes. As $y \gg R$, we recover flat spacetime $\mathbb{R}^{1,9}$. When $y < R$, the geometry is often referred to as the **throat** and would at first appear to be singular as $y \ll R$. More precisely, the near-horizon geometry becomes apparent in the region

$$y \to 0 \qquad\qquad \alpha' \to 0 \qquad\qquad u \equiv y/R^2 \tag{16.13}$$

in which also the Regge slope is taken to zero, while u is kept fixed. In this limit, we can neglect the factor 1 in the function $H(y)$ in (16.12) and the metric in (16.11) becomes:

$$ds^2 = R^2 \left[u^2 \eta_{\mu\nu} dx^\mu dx^\nu + \frac{du^2}{u^2} + d\Omega_5^2 \right] \tag{16.14}$$

As we have seen in (16.4), the first part of the metric is AdS_5, and the other part is S^5. In conclusion, the geometry close to the brane ($y \sim 0$ or $u \sim 0$) is regular and highly symmetrical $\mathrm{AdS}_5 \times S^5$ with the same radii

$$R^2_{\mathrm{AdS}_5} = R^2_{S^5} = \alpha' \sqrt{4\pi N g_s} .$$

In the limit (16.13), only the AdS region of the D3-brane geometry survives while the dynamics in the asymptotically flat region decouples from the theory. Furthermore, it turns out that the interaction between bulk and brane dynamics becomes negligible. Therefore, it is called the **decoupling limit**.

The AdS/CFT correspondence

As mentioned, the world volume theory of N coincident D3-branes is 4d $\mathcal{N} = \triangle$ SYM with U(N) gauge group. On the other hand, the classical solution in (16.14) is a good approximation when the radii of AdS_5 and S^5 are very big:

$$\frac{R^2}{\alpha'} \gg 1 \implies N g_{YM}^2 \equiv \lambda \gg 1 \tag{16.15}$$

The fact that those two descriptions are simultaneously consistent for large values of the coupling constant λ brought Maldacena to formulate the conjecture that the strongly interacting $\mathcal{N}=4$ SYM with gauge group U(N) at large N is equivalent to Type IIB supergravity compactified on $\mathrm{AdS}_5 \times S^5$. However, supergravity is not a consistent quantum theory and it is just a low-energy effective theory of string theory. Hence, the natural way to extend the equivalence at any value of λ is therefore that $\mathcal{N}=4$ SYM is equivalent to Type IIB string theory on $\mathrm{AdS}_5 \times S^5$ [Mal99]. Namely, the following two theories are dual to each other:

- $\mathcal{N}=4$ super-Yang-Mills theory in 4-dimensions with gauge group U(N).
- Type IIB superstring theory on $\mathrm{AdS}_5 \times S^5$ with the same radius R as in (16.15), where the 5-form G_5^+ has integer flux $N=\int_{S^5} G_5^+$ on S^5.

The coupling constants in the two theories are related by $g_s = g_{YM}^2$. The precise formulation of this duality will follow in the next subsection. In these two theories, we can immediately find the following correspondence as in Table 7.

4d $\mathcal{N}=4$ SYM	Type IIB on $\mathrm{AdS}_5 \times S^5$
32 supercharges	32 supercharges
$SO(2,4)$ conformal group	$SO(2,4)$ isometry of AdS_5
$SU(4)_R$ symmetry	$SO(6)$ isometry of S^5
SL(2,$\mathbb{Z}$) symmetry of coupling constants	SL(2,$\mathbb{Z}$) symmetry of axio-dilaton

Table 7: Dictionary for the $\mathrm{AdS}_5/\mathrm{CFT}_4$ correspondence

This conjecture is the most general statement, which is valid at any values of coupling constant $g_s = g_{YM}^2$ and rank N. However, it is still difficult to quantize string theory at any value of g_s on a general manifold including asymptotic AdS space. Hence, it is still an open problem to prove this general statement of the conjecture. Nevertheless, taking various limits of the conjecture, we can show a variety of non-trivial evidence for the conjecture, providing new physical insight.

The 't Hooft Limit: The 't Hooft limit [tH74] is the limit in which we keep the **'t Hooft coupling** $\lambda \equiv g_{YM}^2 N = g_s N$ fixed and letting $N \to \infty$, $g_s \to 0$. As in Figure 40, the planar diagrams become dominant in this limit on the Yang-Mills side. On the AdS side, since the string coupling can be re-expressed in terms of the 't Hooft coupling as $g_s = \lambda/N$, the 't Hooft limit corresponds to the regime where weak coupling string perturbation theory is valid. Put differently, this limit of the AdS/CFT correspondence can be understood as the incarnation of the idea of 't Hooft [tH74].

Figure 40: Planar and non-planar diagram

Supergravity limit: While we take $N \to \infty$, in the regime that 't Hooft coupling is large $\lambda = g_s N \gg 1$, supergravity description becomes reliable. On the gauge theory side, the theory is strongly-coupled so that perturbation techniques cannot be applied. The two theories are conjectured to be the same, but when one side is weakly coupled, the other is strongly coupled. This is a salient feature of **duality**. Thus, using the AdS/CFT correspondence, analyses of supergravity provide new insights to strongly-coupled SYM theory, such as quark confinement and mass gap.

16.4 GKPW relation

Soon after Maldacena's proposal [Mal99], a more precise formulation was given in [GKP98, Wit98]. The gravitational partition function on asymptotically AdS space is equal to the generating function of correlation functions of the corresponding CFT:

$$Z_{\text{grav}}[\phi \to \phi_0] = \left\langle \exp\left(\int_{\partial AdS} \overline{\phi}_0 \mathcal{O} \right) \right\rangle_{\text{CFT}} \tag{16.16}$$

that is called the **GKPW** relation. For any bulk field ϕ in gravity theory on AdS, there exists the corresponding operator $\mathcal{O}$ in the CFT. The gravitational partition function can be schematically written as

$$Z_{\text{grav}}[\phi \to \phi_0] = \int_{\phi \to \phi_0} \mathcal{D}\phi \; e^{-S_{\text{string}}[\phi]} \ .$$

For instance, in the regime $\lambda \gg 1$, we can use supergravity description

$$Z_{\text{grav}}[\phi \to \phi_0] = \sum_{\text{saddle point}} e^{-S_{\text{SUGRA}}[\phi \to \phi_0]} \ .$$

Bulk field/boundary operator

Each field propagating on AdS space is in a one-to-one correspondence with an operator in the field theory. The spin of the bulk field is equal to the spin of the CFT operator; the mass of the bulk field fixes the scaling dimension of the CFT operator. Here are some examples:

- Every theory of gravity has a massless spin-2 particle, the graviton $g_{\mu\nu}$. This is dual to the stress tensor $T_{\mu\nu}$ in CFT. This makes sense since every CFT has a stress tensor. The fact that the graviton is massless corresponds to the fact that the CFT stress tensor is conserved.

- If our theory of gravity has a spin-1 vector field A_μ, then the dual CFT has a spin-1 operator J_μ. If A_μ is gauge field (massless), then J_μ is a conserved current so that gauge symmetries in the bulk correspond to global symmetries in the CFT.

- A bulk scalar field is dual to a scalar operator in the CFT. The boundary value of the bulk scalar field acts as a source in the CFT.

There is a relation between the mass of the field ϕ and the scaling dimension of the operator in the conformal field theory. Let us describe this more generally in AdS_{d+1} with metric (16.5). The wave equation in the AdS_{d+1} space in Euclidean signature for a field of mass m has two independent solutions, which behave like $z^{d-\Delta}$ (**non-normalizable**) and z^{Δ} (**normalizable**) for small z (close to the boundary of AdS), where

$$\Delta = \frac{d}{2} + \sqrt{\frac{d^2}{4} + R^2 m^2} \ . \tag{16.17}$$

Therefore, in order to get consistent behavior for a massive field, the boundary condition on the field on the right-hand side of (16.16) should in general be changed to

$$\phi(\vec{x}, \epsilon) = \epsilon^{d-\Delta}\phi_0(\vec{x}), \tag{16.18}$$

and we would eventually take the limit where $\epsilon \to 0$. Since ϕ is dimensionless, we see that ϕ_0 has dimensions of $[\text{length}]^{\Delta-d}$ which implies, through the left-hand side of (16.16), that the associated operator $\mathcal{O}$ has dimension Δ (16.17).

Examples

Let us now consider this correspondence in a massless scalar field in $\mathrm{AdS}_5 \times S^5$. The Klein-Gordon equation of the massless field in $\mathrm{AdS}_5 \times S^5$ can be written as

$$\nabla^2\phi = \nabla^2_{\mathrm{AdS}_5} + \nabla^2_{S^5}\phi = 0 \ .$$

The eigenfunctions of the Laplacian $\nabla^2_{S^5}$ on a sphere are known as the spherical harmonics $Y_l(\Omega)$ (like the theory of angular momenta in quantum mechanics) so that

$$\nabla^2_{S^5} Y_l(\Omega) = -\frac{l(l+4)}{R^2} Y_l(\Omega), \quad l = 0, 1, 2, \ldots \ . \tag{16.19}$$

Writing the ten-dimensional field $\phi = \sum_l \phi_l Y_l$, the AdS_5 fields ϕ_l satisfy the massive Klein-Gordon equation

$$\nabla^2_{\mathrm{AdS}_5}\phi_l = m_l^2\phi_l \ , \qquad m_l^2 = \frac{l(l+4)}{R^2} \ . \tag{16.20}$$

This can be understood that the compactification on S^5 leads to a tower of the field with Kaluza-Klein (KK) masses m_l. The AdS/CFT correspondence predicts that there exist operators in the $d = 4$ $\mathcal{N} = 4$ SYM dual to these fields.

To see that, we read off the conformal dimension of the corresponding operator in $d = 4$ from (16.17)

$$\Delta_l = 2 + \sqrt{4 + (m_l R)^2} = 2 + \sqrt{4 + l(l+4)} = 4 + l \ . \tag{16.21}$$

First, we consider a massless scalar $l = 0$ in AdS$_5$ where the conformal dimension of the dual operator is $\Delta_0 = 4$. Also, the spherical harmonics (16.19) at $l = 0$ corresponds to the s-wave, which is transformed trivially under the SO(6) symmetry. Therefore, the dual operators are also singlet under the SU(4) R-symmetry so that it does not contain the scalars X_i of the $d = 4$ $\mathcal{N} = 4$ SYM. Consequently, the only operator with these properties is the gauge-invariant glueball operator

$$\mathcal{O} = \mathrm{Tr}\left[F_{\mu\nu}F^{\mu\nu}\right] \ .$$

Note that the conformal dimension is $\Delta = 4$ since dim$[\partial]$=dim$[A] = 1$. For higher KK modes $l > 0$, the dual operator transforms non-trivially under the SU(4) R-symmetry so that it involves the scalar X_i. The natural candidate dual to the l^{th} KK mode is

$$\mathcal{O}_{i_1,\dots,i_l} = \mathrm{Tr}\left[X_{(i_1,\dots,i_l)}F_{\mu\nu}F^{\mu\nu}\right] \ , \tag{16.22}$$

where $X_{(i_1,\dots,i_l)}$ being the traceless symmetric product of l scalar fields X_i of the $\mathcal{N} = 4$ SYM. It is easy to see that the conformal dimension of this operator is $4 + l$, which is consistent with (16.21). In fact, the field/operator matching of this kind has been extended to all the fields of 10d supergravity on AdS$_5 \times S^5$.

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