Homework 1: Due at class on Sep 27

1 Lagrangian and symmetry

1.1

Let us consider the Lagrangian of the electromagnetic field

$$\mathcal{L}_{\text{EM}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - A_\mu J^\mu$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and J_μ is a 4-vector. Derive the Euler-Lagrange equations for $A_\mu(x)$. Write the equations in standard form of the Maxwell's equations by identifying $E^i = -F^{0i}$ and $\epsilon^{ijk}B^k = -F^{ij}$.

1.2

Show that the Lagrangian of a complex scalar field

$$\mathcal{L}_{\text{cpx scalar}} = \partial_\mu\phi^*(x)\partial^\mu\phi(x) - m^2\phi^*(x)\phi(x)$$

is invariant under the global U(1) symmetry

$$\phi(x) \to e^{i\alpha}\phi(x), \quad \phi^*(x) \to e^{-i\alpha}\phi^*(x), \quad \alpha \in \mathbb{R}$$

Find the corresponding Noether current.

1.3

Let us consider the Lagrangian of a complex scalar field coupled to electromagnetic field

$$\mathcal{L} = \left(D_\mu\phi\right)^* D^\mu\phi - m^2\phi^*\phi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

where $D_\mu = \partial_\mu + ieA_\mu$. Find the energy momentum $T^{\mu\nu}$.

Show that $\mathcal{L}$ is invariant under the U(1) gauge (sometimes called **local**) symmetry

$$\phi(x) \to e^{i\alpha(x)}\phi(x)$$
$$A_\mu(x) \to A_\mu(x) - \frac{1}{e}\partial_\mu\alpha(x)$$

Note that α for the **global** U(1) symmetry in Problem 1.2 does not depend on x.

2 Complex scalar field

Let us consider quantum theory of a complex scalar field in Problem 1.2.

2.1

The mode expansion of the complex scalar field $\phi(\mathbf{x})$ is

$$\begin{aligned}\phi(\mathbf{x}) &= \int \frac{d^3p}{2\pi^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left(a_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}} + b^\dagger_{\mathbf{p}}e^{-i\mathbf{p}\cdot\mathbf{x}}\right) \\ \phi^*(\mathbf{x}) &= \int \frac{d^3p}{2\pi^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left(b_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}} + a^\dagger_{\mathbf{p}}e^{-i\mathbf{p}\cdot\mathbf{x}}\right)\end{aligned} .$$

Imposing the canonical commutation relations

$$[\phi(\mathbf{x}), \pi(\mathbf{y})] = i\delta^{(3)}(\mathbf{x}-\mathbf{y}) = [\phi^*(\mathbf{x}), \pi^*(\mathbf{y})] ,$$

and the others vanish, derive the commutation relations of the creation and annihilation operators $a_{\mathbf{p}}, a^\dagger_{\mathbf{p}}, b_{\mathbf{p}}, b^\dagger_{\mathbf{p}}$.

2.2

Show that the Noether charge Q for the global U(1) symmetry in Problem 1.2 can be expressed as follows in terms of the modes of ϕ :

$$Q = -\int \frac{d^3p}{(2\pi)^3}\left(a^\dagger_{\mathbf{p}}a_{\mathbf{p}} - b^\dagger_{\mathbf{p}}b_{\mathbf{p}}\right)$$

after dropping a constant term. Show that the state $a^\dagger_{\mathbf{p}}|0\rangle$ has the U(1) charge -1 and the state $b^\dagger_{\mathbf{p}}|0\rangle$ has the U(1) charge $+1$.

2.3

Let us consider the operators in the Heisenberg picture

$$\begin{aligned}\phi(x) = \phi(\mathbf{x}, t) &= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left(a_{\mathbf{p}}e^{-ip\cdot x} + b^\dagger_{\mathbf{p}}e^{ip\cdot x}\right) \\ \phi^*(x) = \phi^*(\mathbf{x}, t) &= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left(b_{\mathbf{p}}e^{-ipx} + a^\dagger_{\mathbf{p}}e^{ip\cdot x}\right)\end{aligned}$$

Using the commutation relations of the creation and annihilation operators to calculate $\langle 0|T[\phi(x)\phi(y)]|0\rangle$, $\langle 0\,|T\phi^*(x)\phi^*(y)|\,0\rangle$, and $\langle 0\,|T\phi(x)\phi^*(y)|\,0\rangle$. Compare with the Feynman propagator the real scalar field.

3 Propagator

The Feynman propagator of a real scalar field in momentum space is given by

$$D_F(x-y) = \langle 0|T\phi(x)\phi(y)|0\rangle = \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i\epsilon}e^{-ip\cdot(x-y)}$$

where the $i\epsilon$ is there to ensure that the correct poles are included in the p_0 integral. Compute D_F explicitly and write it as a function of the Lorentz-invariant length $s^2 = (x-y)_\mu (x-y)^\mu$ in terms of the modified Hankel and Bessel functions in the limit $\epsilon \to 0$:

$$H_n^{(1)}(z) = \frac{-2i}{\sqrt{\pi}\left(n-\frac{1}{2}\right)!}\left(-\frac{1}{2}z\right)^n \int_1^{\infty - i\varepsilon} e^{-izx}\left(x^2-1\right)^{n-1/2} dx$$

$$K_n(z) = \frac{\sqrt{\pi}}{\left(n-\frac{1}{2}\right)!}\left(\frac{1}{2}z\right)^n \int_1^{\infty} e^{-zx}\left(x^2-1\right)^{n-1/2} dx$$

where $K_n(z) = i^{n+1}(\pi/2)H_n^{(1)}(iz)$.

Discuss the behaviors of the propagator in the following regimes by using the properties of the modified Hankel and Bessel functions. (See Wikipedia.)

- $s \to 0$
- s^2 is large and positive (timelike separation)
- s^2 is large and negative (spacelike separation)
- $m^2 \to 0$