Homework 2: Due at class on Sep 29

Prob. 1 (Open string)

An open string has boundaries so that one needs to impose a boundary condition. There are two boundary conditions one can impose:

- **Neumann boundary condition**: $\partial_\sigma X^\mu = 0 \quad$ at $\sigma = 0, \pi$
- **Dirichlet boundary condition**: $X^\mu = c^\mu$ (constant) at $\sigma = 0, \pi$

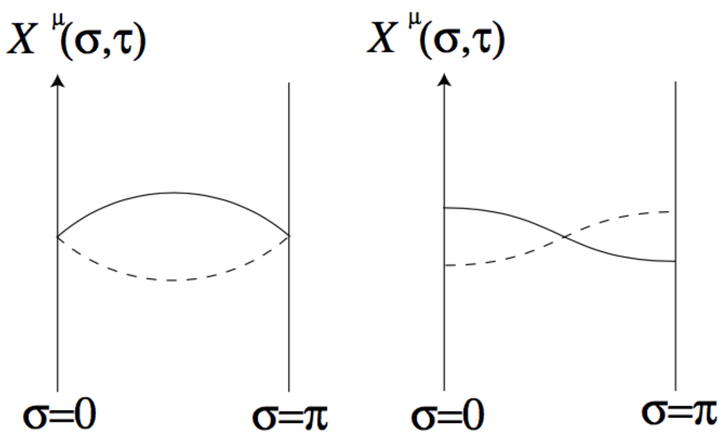

Figure 1: Dirichlet (left) and Neumann (right) boundary conditions

Like the close string, we take the mode expansion for the open string $X^\mu = X^\mu_L(\sigma^+) + X^\mu_R(\sigma^-)$ by

$$X^\mu_L(\sigma^+) = \frac{1}{2}x^\mu + \alpha' p^\mu\, \sigma^+ + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\,\widetilde{\alpha}^\mu_n\, e^{-in\sigma^+}\ ,$$
$$X^\mu_R(\sigma^-) = \frac{1}{2}x^\mu + \alpha' p^\mu\, \sigma^- + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\,\alpha^\mu_n\, e^{-in\sigma^-}\ . \tag{1}$$

Note that the second term differs from the closed string by factor of 2. Show that the boundary conditions impose the following requirements

- Neumann boundary condition requires $\alpha^a_n = \widetilde{\alpha}^a_n$.
- Dirichlet boundary condition requires $x^I = c^I, \quad p^I = 0, \quad \alpha^I_n = -\widetilde{\alpha}^I_n$.

Actually, this is an essence of Problem 3 in the homework 1.

Now let us study open string mass spectrum in the quantum theory. In the case of open strings, we can define the momentum $\alpha^\mu_0 = \sqrt{2\alpha'}\,p^\mu$. Show that the light-cone gauge quantization for (1) gives

$$2\alpha^-_n = \sqrt{\frac{1}{2\alpha'}}\frac{1}{p^+}\sum_{m=-\infty}^{\infty}\sum_{i=1}^{D-2}\alpha^i_{n-m}\alpha^i_m\ .$$

Check that $n = 0$ can be read off

$$M^2 = 2p^+p^- - \sum_{i=1}^{D-2} p^i p^i = \frac{1}{\alpha'} \left(\sum_{n>0} \sum_{i=1}^{D-2} \alpha_{-n}^i \alpha_n^i + \frac{D-2}{2} \sum_{n>0} n \right) ,$$

and explain the reason why there is the difference of factor 4 from the closed string. Again, in $D = 26$, the open string mass spectrum becomes

$$M^2 = \frac{1}{\alpha'}(N-1) .$$

so that there is the tachyon for $N = 0$. The massless states

$$\alpha_{-1}^i |0; k\rangle \qquad i = 1, \ldots, D-2$$

for $N = 1$ correspond to a vector boson.

Prob. 2

1. Start from Green's theorem in Cartesian coordinate and show that the complex version is given by the form given in the lecture.
2. Confirm that $\partial\overline{\partial} \log |z|^2 = 2\pi\delta^2(z)$.
3. Show that $\oint \frac{d\bar{z}}{2\pi i} \frac{1}{\bar{z}} = -1$.

Prob. 3

You do not have to consider space-time indices in this problem.

1. Express $:X_1 X_2 X_3 X_4:$ in terms of normal products.
2. Using the previous result express $X_1 X_2 X_3 X_4$ in terms of normal-ordered products, and confirm that it can be expressed in the form of

$$f[X] = \exp\left[\frac{1}{2}\int d^2z d^2w \; G(z,w) \frac{\delta}{\delta X(z)} \frac{\delta}{\delta X(w)}\right] :f[X]: \; .$$

Prob. 4

Let us define $T(z) = -\frac{1}{\alpha'} :\partial X^\mu \partial X_\mu:$, and $j(z) = a_\mu \partial X^\mu$. Extract divergent terms from $j(z)X^\nu(w)$, and $T(z)T(w)$.