Homework 5: Due at class on April 11

1. Derive all the curvature identities in either (6.8) or (6.9) of the lecture note.

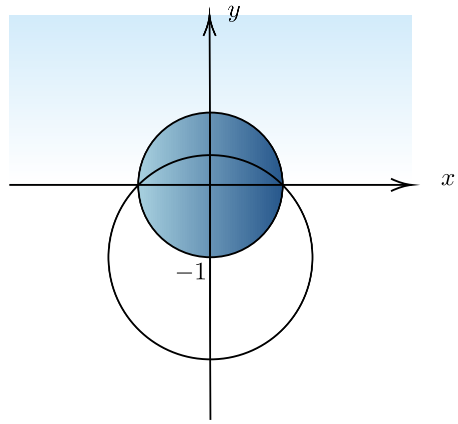

Figure 1: Hyperboloid and Poincare disk

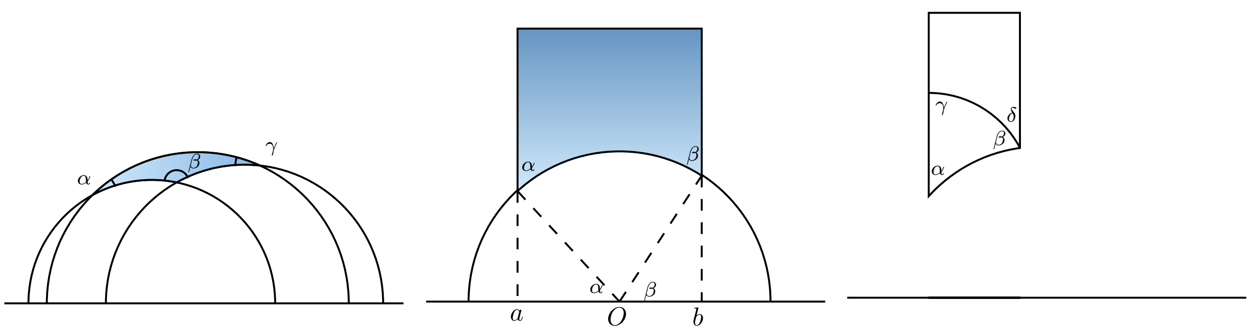

Figure 2: triangle in the upper half plane

2. Let $\mathbf{H} = \{(x,y)|y > 0\}$ be the upper half plane (Yellow area in Figure 1). We invert the upper half plane $\mathbf{H}$ to $\mathbf{D}$ in terms of the circle with radius $\sqrt{2}$ around the center $(0,-1)$, and take the reflection with respect to x-axis (Figure 1). This gives a map $J : \mathbf{H} \to \mathbf{D}; (x,y) \mapsto (u,v)$

$$u = \frac{2x}{x^2 + (y+1)^2}\,, \qquad v = 1 - \frac{2(y+1)}{x^2 + (y+1)^2}\,.$$

1. Find the induced metric on the upper half plane by this map.
2. Find geodesics on $\mathbf{H}$ and compute its Riemann, Ricci and scalar curvature.
3. Find the area of the triangle with angles (α, β, γ) bounded by half-circles with respect to the metric (Figure 2). Here, we can use the fact that the area of the triangle in the left of Figure 2 is the same as that of the triangle in the right of Figure 2. Compare with the area of a triangle on the 2-sphere (Homework 1).

4. Do parallel transport of a vector along the triangle with respect to the Levi-Civita connection of the metric and find the angle difference when it comes back. Describe the difference between the sphere and the upper half plane.

5. The **Möbius transformation** of the upper half plane $\mathbf{H} = \{z = x + iy \mid y > 0\}$ is a rational function of the form
$$f(z) = \frac{az+b}{cz+d} ,$$
where $ad - bc = 1$ with $a, b, c, d \in \mathbb{R}$. If f_1 and f_2 are Möbius transformations, prove that $f_1 \circ f_2$ is also a Möbius transformation. Show that this is an isometry group for the metric.

3. (Kepler's two-body problem)

Let us consider one of the first examples of integrable systems solved by the Liouville theorem: The Kepler two-body problem of planetary motion. Taking the center-of-mass frame, the potential $V(r)$ of the system depends only on the radius, and the Hamiltonian is given by
$$H = \frac{1}{2}\sum_{i=1}^{3} p_i^2 + V(r) .$$

1. Show that the angular momentum
$$\vec{J} = (J_1, J_2, J_3) , \quad J_{ij} = x_i p_j - x_j p_i = \epsilon_{ijk} J_k$$
is conserved.

2. Given the standard symplectic form $\omega = \sum_{i=1}^{3} dp_i \wedge dx_i$, compute the Poisson brackets
$$\{J_i, J_j\} = -\epsilon_{ijk} J_k .$$
Show that the following three physical quantities commute under the Poisson bracket
$$H, \quad J_3, \quad J^2 = J_1^2 + J_2^2 + J_3^2$$

3. Rewrite the Liouville 1-form
$$\alpha = \sum_i p_i dx_i = p_r dr + p_\theta d\theta + p_\phi d\phi \tag{0.1}$$
in terms of the polar coordinates
$$x_1 = r\sin\theta\cos\phi, \quad x_2 = r\sin\theta\sin\phi, \quad x_3 = r\cos\theta .$$
Rewrite the conserved quantities H, J_3, J^2 in terms of the polar coordinates and (p_r, p_θ, p_ϕ).

4. Without loss of generality, we can rotate our coordinate system such that in a new system $\vec{J}$ has only the third component: $\vec{J} = (0, 0, J_3)$. This can be simply done by setting $\theta = \frac{\pi}{2}$. Kepler's 2nd law states that the areal (sectorial) velocity is constant, and in this situation, it is nothing but the conservation of J_3 because the areal velocity is

$$\frac{dA}{dt} = \frac{1}{2} r^2 \dot{\phi} = \frac{1}{2} J_3 \,.$$

Under this situation, show that an integral of the Liouville 1-form (0.1) becomes

$$S = \int \alpha = \pm \int^r dr \sqrt{2(H - V) - \frac{J^2}{r^2}} + \int^\phi J_3 d\phi \tag{0.2}$$

where the sign $\pm$ is chosen in such a way that it is consistent with p_r. Derive the equations of motion for the angle variables

$$\psi_H = \frac{\partial S}{\partial H}, \quad \psi_J = \frac{\partial S}{\partial J} \,.$$

Discuss their physical consequence. In particular, under which condition is an orbit of the motion closed?

5. Let us assume that the potential takes the form

$$V(r) = -\frac{k}{r} \,.$$

Show the Kepler's 1st law: a planet describes an ellipse with the Sun at one focus. Let T be the revolution period of a planet and a be the major semi-axes of ellipse. Show the Kepler's 3rd law:

$$T = \frac{2\pi}{\sqrt{k}} a^{\frac{3}{2}} \,.$$

Refer to Wikipedia page for the terminolgy.