A gentle introduction to the 3d/3d correspondence.

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Many thanks to people @ Yau Center!

Brane setting in M-theory

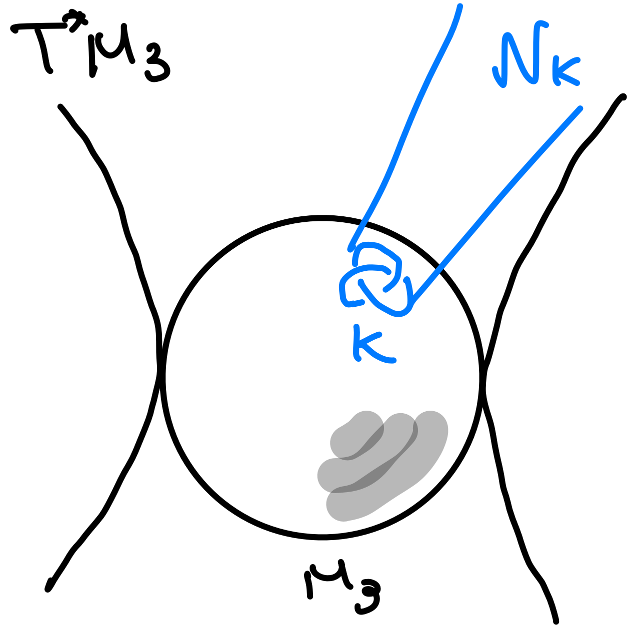
Witten

Gopakumar-Vafa.

space-time $\mathbb{R} \times \mathbb{C}_q \times \mathbb{C}_t \times T^*M_3$

N M5 $\mathbb{R} \times \mathbb{C}_q$ M_3

M5 $\mathbb{R} \times \mathbb{C}_q$ N_K



3d $\mathcal{N}=2$ thy $T[M_3]$ / $T[M_3, K]$ $\longleftrightarrow$ $\mathbb{C}$ CS thy

3d/3d Corresp.

$t = q = e^{\frac{2\pi i}{k - is}}$

$t \neq q$ ok for Seifert M_3 refined CS.

$$S = \frac{k-is}{4\pi}\int_{M_3} \mathrm{Tr}\left[A \wedge dA + \frac{2}{3}A^3\right] + \frac{k+is}{4\pi}\int_{M_3} \mathrm{Tr}\left[\bar{A} \wedge d\bar{A} + \frac{2}{3}\bar{A}^3\right]$$

the action of $\mathbb{C}$ CS.

Compactification of 6d $\mathcal{N}=(2,0)$ thy

6d (2,0) thy.: $SO(6)_E \times Sp(2)_R \cong SO(6)_E \times SO(5)_R$

M-thy on $\mathbb{R} \times \mathbb{C}_q \times M_3$.

$$SO(3) \times \underbrace{SO(3) \times SO(3)}_{\text{Topol. Twist.}} \times SO(2)$$

3d $\mathcal{N}=2$ Euclidean   3d $\mathcal{N}=2$ R-sym

Yamazaki-Terashima

DGG GPV. etc..

6d $\mathcal{N}=(2,0)$ has $SO(5)$ R-sym

$\mathbb{C}$ gauge field.

$\rightsquigarrow \mathcal{A} = A_\mu + i\phi_\mu$.

$$SO(5)_R \circlearrowright \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_5 \end{pmatrix} \longrightarrow \begin{matrix} SO(3) \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} \\ \times \\ SO(2) \begin{pmatrix} \phi_4 \\ \phi_5 \end{pmatrix} \end{matrix}$$

1-form after Top. Tw.

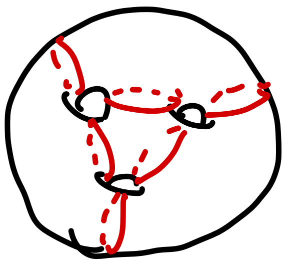
3d $\mathcal{N}=2$ R-sym.

Some Examples of $T[M_3]$

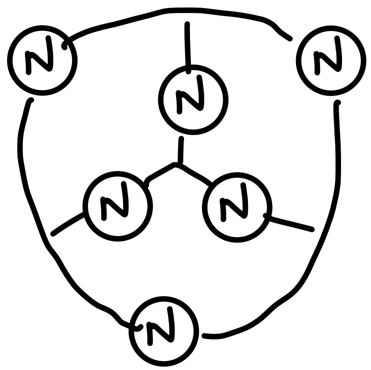
- Simplest Example. $M_3 = S^1 \times \Sigma_g$

$T[S^1 \times \Sigma_g]$ is S^1-reduction of class S thy $T[\Sigma_g]$

3d $\mathcal{N}=4$.



pants decomp of Σ_g.



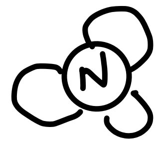
no Lagrangian

Benini Benvenuti Tachikawa

$\longleftrightarrow$

Mirror sym

g adj hypers



mirror has Lag description

- Slight generalization $\mathcal{O}(p) \to \Sigma_g$.

$$T[\mathcal{O}(p) \to \Sigma_g] = 3d\ \mathcal{N}=2 \begin{cases} U(N) \text{ vect. multi} \\ (2g+1) \text{ adj chiral. multi} \\ \text{level } p. \text{ CS term} \end{cases}$$

Equivariant Verlinde formula & Coulomb branch index

Space-Time $\quad T^*L(p,1) \times S^1 \times T^*\Sigma_g$.

N M5-branes $\quad \underbrace{L(p,1) \times S^1}_{\text{4d } \mathcal{N}=2 \ T[\Sigma_g]} \times \Sigma_g$.

$S^1 \times \Sigma_g$: 3d $\mathcal{N}=2$ thy $T[L(p,1)]$

$$L(p,1) = \begin{array}{c} \mathcal{O}(p) \\ \downarrow \\ S^2 \end{array}$$

$T[L(p,1)] = U(N) \begin{cases} \text{rect multi} \\ \text{adj chiral} \\ \text{level } p \text{ CS} \end{cases}$

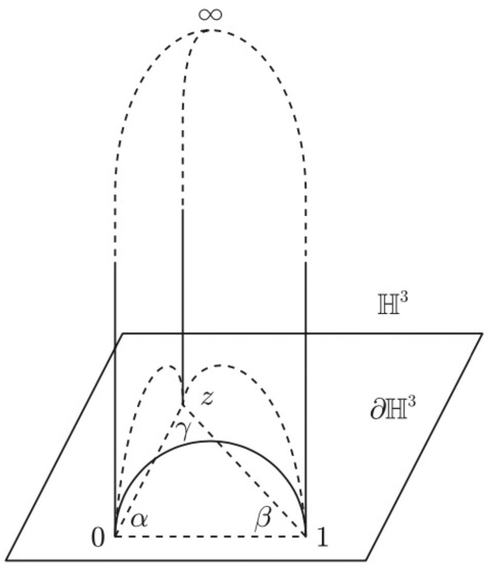
$$I^{4d\ \mathcal{N}=2}_{\text{Coulomb}}\left[T[\Sigma_g]\right] \cong \sum_{x:\,\text{BAE}} x^p \cdot \mathcal{H}(x)^{1-g}$$

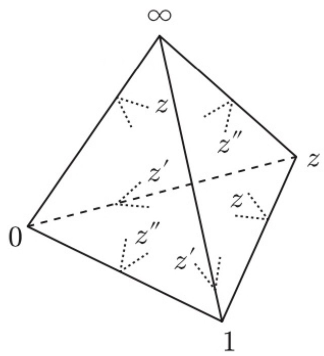
$$\cong \dim_t H^0(\mathcal{M}_H(\Sigma_g), \mathcal{L}^p)$$

Gukov-Pei

Gukov-Pei-Yan-Ye

Ideal Tetrahedra Triangulation & 3d $\mathcal{N}=2$ thy





a hyperbolic M_3 admits ideal tetrahedra triangulation

$$z, \quad z' = \frac{1}{1-z}, \quad z'' = 1 - z^{-1}$$

T_Δ = free chiral w/ $U(1)$ flavor sym $-\frac{1}{2}$ CS level

e.g. squashed S^3_b partition function

quantum dilog.

$$Z_{T_\Delta} = e_b\left(\frac{iQ}{2} + \frac{z}{2\pi b}\right)$$

$$= s_b\left(\frac{iQ}{2} + \frac{z}{2\pi b}\right) e^{\frac{i\pi}{2}\left(\frac{iQ}{2} + \frac{z}{2\pi b}\right)^2 + \frac{i\pi}{24}(2-Q^2)}$$

$$e_b(x) := \frac{\left(e^{2\pi\left(x+\frac{iQ}{2}\right)b}; q^2\right)_\infty}{\left(e^{2\pi\left(x-\frac{iQ}{2}\right)b^{-1}}; \tilde{q}^2\right)_\infty} = \frac{\left(-qe^{2\pi b x}; q^2\right)_\infty}{\left(-\tilde{q}e^{2\pi b^{-1} x}; \tilde{q}^2\right)_\infty}$$

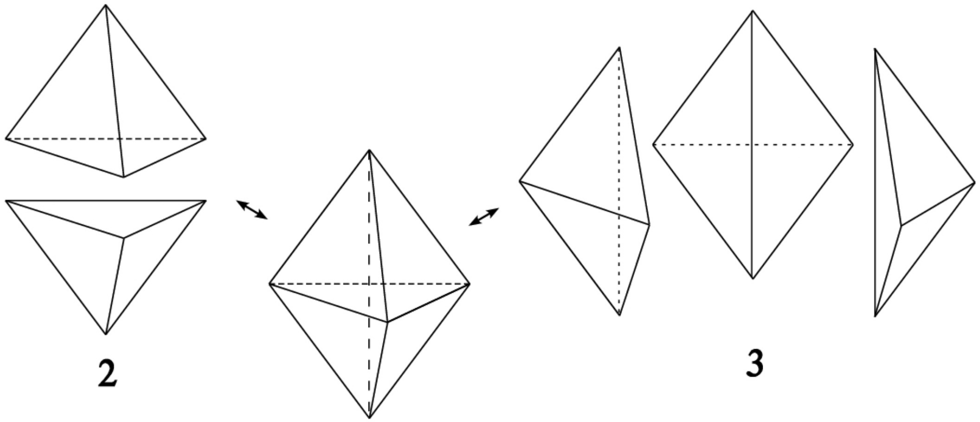
DGG.

$$\left(1 - \hat{z} - \hat{z}'^{-1}\right) Z_{T_\Delta} = 0 \qquad \hat{z}\,\psi(z) = \psi(z + 2\pi i b^2)$$

Ideal Tetrahedra Triangulation & 3d $\mathcal{N}=2$ thy

gluing of tetrahedron $\Longleftrightarrow$ introduce superpotential (+ gauging)

$$\prod_{i:\text{edge}} z_i = 1 \qquad\qquad \sum_{i:\text{edge}} r_i = 2 \quad \text{R-charge.}$$



2-3 Pachner Move

3d $\mathcal{N}=2$ SQED $N_f=1$ $\Longleftrightarrow$ XYZ model $W = XYZ$

$$\int d\sigma\, e^{2\pi i \xi \sigma}\, S_b(\sigma+r)\, S_b(\sigma-s) = e^{*}\, S_b\left(-\xi+\frac{s-r}{2}+\frac{iQ}{2}\right) S_b\left(+\xi+\frac{s-r}{2}+\frac{iQ}{2}\right) S_b\left(r-s-\frac{iQ}{2}\right)$$

quantum dilog id.

In this way, can construct $Z(T[M_3])$. for a hyperbolic M_3

But there is a caveat: pick only non-Abelian connections.

$SL(2,\mathbb{Z})$ Transformation

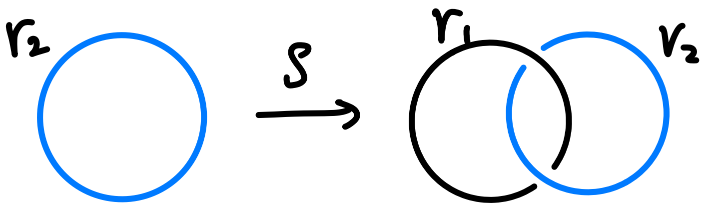
Witten

GNPSS

S-transformation

pure CS thy

3d $\mathcal{N}=2$ thy

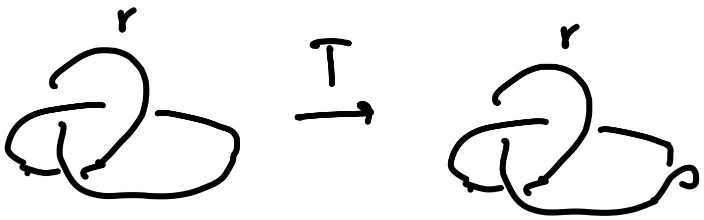


$$\frac{(a;q)_{r_2}}{(q;q)_{r_2}} \longrightarrow \sum_{i=0}^{r_2} q^{(r_1+1)i}\frac{(aq^{-1};q)_i}{(q^2;q)_i}$$

$$\mathcal{L}(A_{x_2}) \longrightarrow \mathcal{L}(A_z) + \frac{1}{2\pi}A_{x_1}\wedge dA_z$$

$$\mathcal{I}[\bigcirc](x_2,a) \longrightarrow \int \frac{dz}{z}\frac{\theta(x_1;q)\theta(z;q)}{\theta(x_1 z;q)}\mathcal{I}[\bigcirc](a,z)\ .$$

T-transformation



$$\overline{P}_{[r]}(K) \longrightarrow q^{\frac{r(2r+1)}{2}}\overline{P}_{[r]}(K)$$

$$\mathcal{L} \longrightarrow \mathcal{L} + \frac{1}{4\pi}A_x\wedge dA_x$$

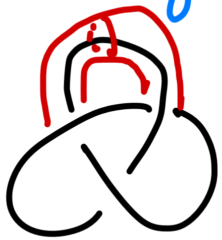
$$\mathcal{I}[K](x,a) \longrightarrow x^{\frac{1}{2}}\theta(x;q)^{-1}\mathcal{I}[K](x,a)$$

Surgery

Thm [Lickorish - Wallace]

Every closed oriented 3-mfd arises by performing an integral Dehn surgery on a link $K \subset S^3$

Dehn surgery

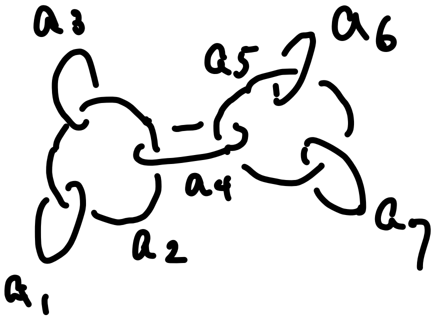


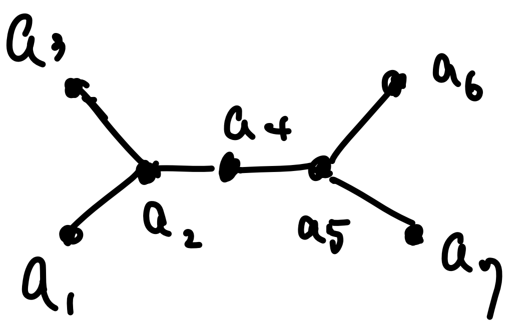
$\frac{p}{r}$ - surgery

$$S^3_{\frac{p}{r}}(K) = (S^3 \setminus K) \underset{\varphi}{\cup} (S^1 \times D^2)$$

$$\varphi = \begin{pmatrix} p & r \\ * & * \end{pmatrix} \in SL(2, \mathbb{Z})$$

Kirby diagram





$$\rightsquigarrow \quad Q_{ij} = \begin{cases} 1 & (i,j) \text{ edge} \\ a_i & i = j \\ 0 & \text{otherwise} \end{cases}$$

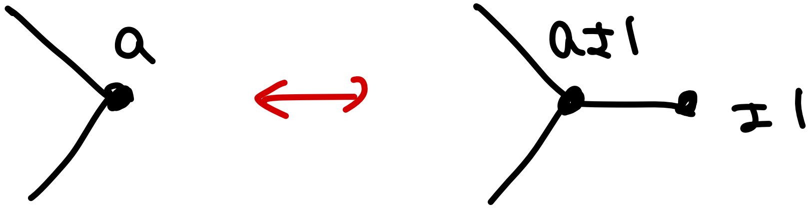
adjacent matrix

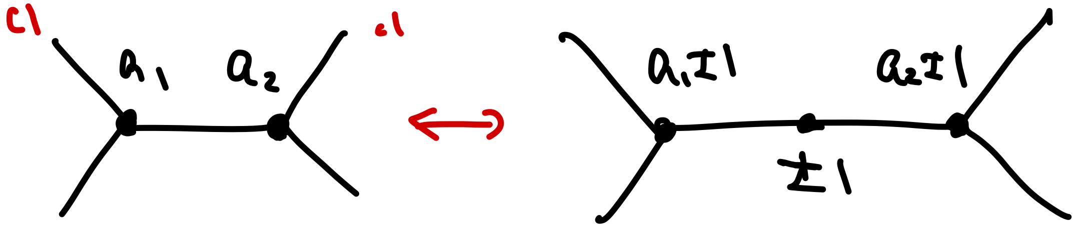
linking

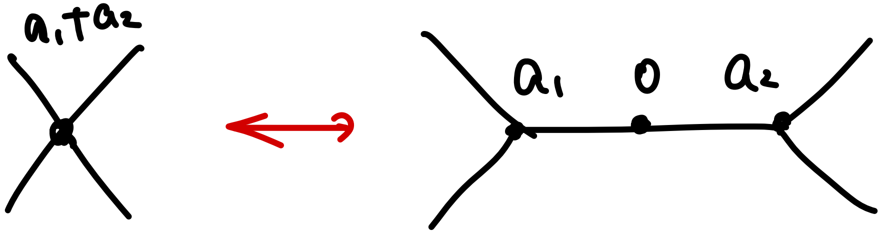
Topological invariance

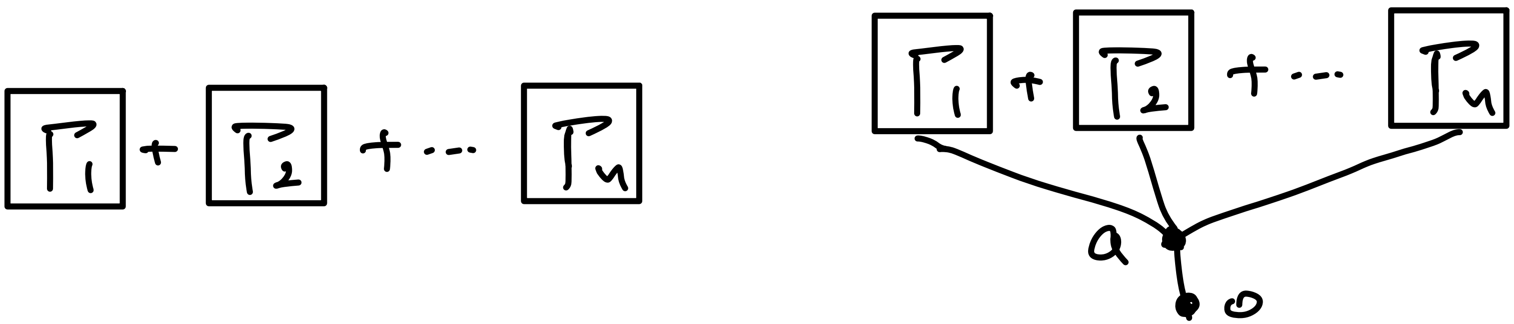
Kirby-Neumann move.

Surgeries are not unique!

1. 

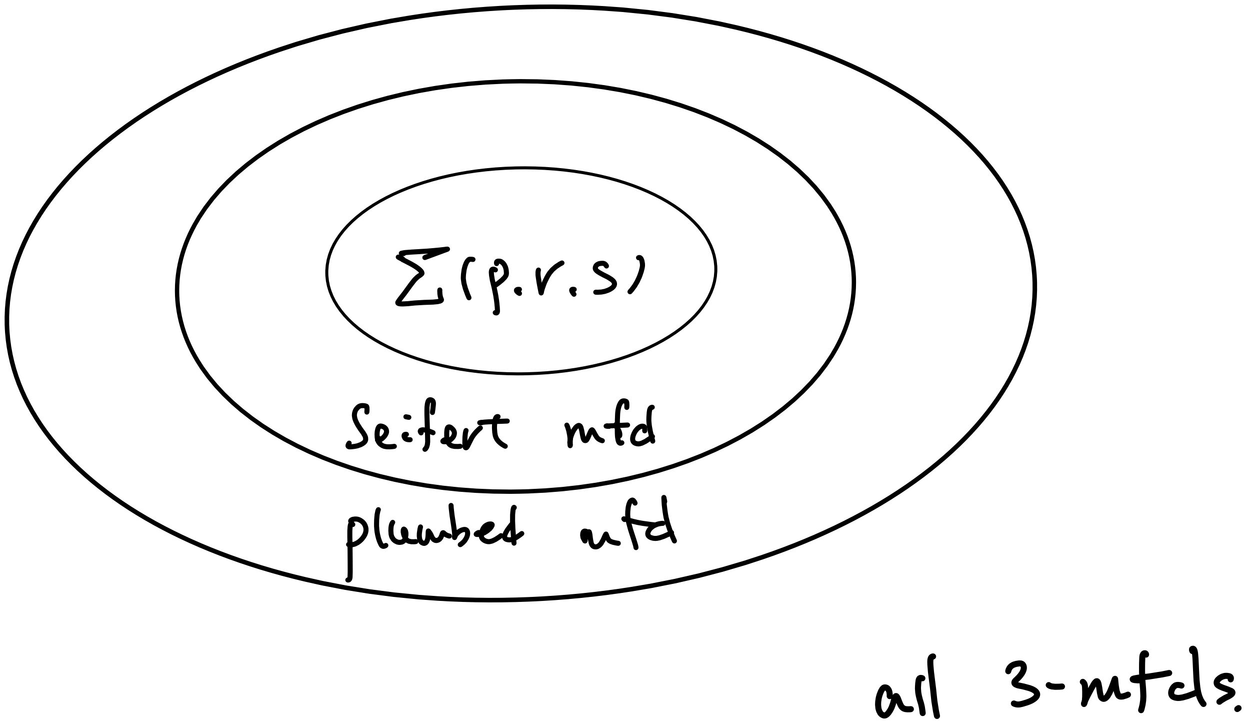
2. 

3. 

4. 

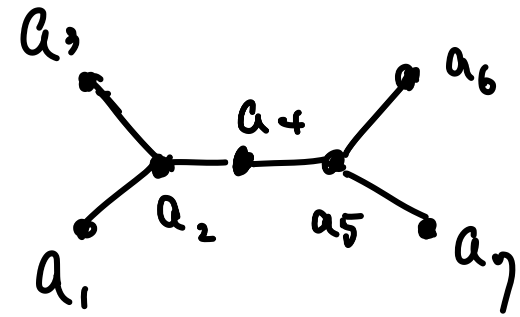
indep of a!

Space of 3-manifolds.



$\hat{Z}$-invariant

$G_{\mathbb{C}} = SL(2,\mathbb{C})$ GPV, GPPV



$$\hat{Z}_b(\text{graph}; q) = \int_{|x_i|} \prod_{j:\text{vert}} \frac{dx_j}{2\pi i x_j}\, q^{-\frac{a_j+3}{4}} \left(x_j - \frac{1}{x_j}\right)^2$$

$$\times \prod_{(i-j)\text{ edges}} \left(x_i - \frac{1}{x_i}\right)^{-1} \left(x_j - \frac{1}{x_j}\right)^{-1} \Theta^Q_b(x)$$

$$\Theta^Q_b(x) = \sum_{n \in Q\mathbb{Z}^{|\text{vert}|} + b} q^{-(n \cdot Q^{-1} n)}\, x_i^{n_i}$$

$\cong H^1(M_3;\mathbb{Z})/\mathbb{Z}_2$

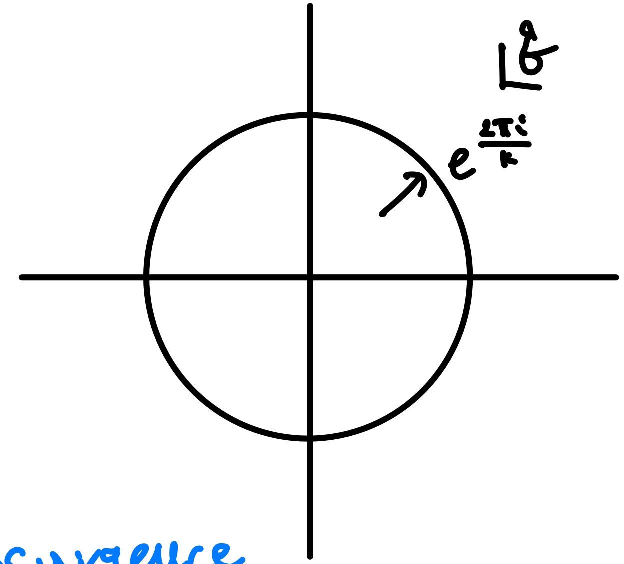
$b \in \mathrm{Coker}\, Q \cong \mathrm{Spin}^c(M_3)$

- $\hat{Z}_b = q^{\Delta}(c_0 + c_1 q + c_2 q^2 + \cdots) \in q^{\Delta_b}\mathbb{Z}[[q]]$

- $\hat{Z}_b$ converges in $|q| < 1$

- invariant under Kirby-Neumann moves.

Complex Chern-Simons invariant?



$$\sqrt{2k}\, Z[M_3] = \sum_{a,b\in \mathrm{Spin}^c(M_3)} e^{2\pi i k CS_a}\, S_{ab}\, \hat{Z}_b(q).$$

resurgence. [GMP]

radial limit. $\xrightarrow[q\to e^{\frac{2\pi i}{k}}]{}$ WRT$[M_3]$

$$\xrightarrow[k\to\infty]{} \sum_{a\in \pi_0(\mathcal{M}_{flat})} e^{2\pi i k CS_a}\, \frac{1}{k^{d_a+1}}\left(a_0^{(a)} + \frac{a_1^{(a)}}{k} + \frac{a_2^{(a)}}{k^2} + \dots\right)$$

Witten's asymptotic exp. conj.

e.g. $M_3 = L(5,1)$

$$S = \begin{bmatrix} 1 & 1 & 1 \\ 2 & \frac{1}{2}(\sqrt{5}-1) & \frac{1}{2}(\sqrt{5}-1) \\ 2 & \frac{1}{2}(\sqrt{5}-1) & \frac{1}{2}(\sqrt{5}-1) \end{bmatrix}$$

$\hat{Z}_0 = 1$

$\hat{Z}_1 = q^{\frac{1}{5}}$

$\hat{Z}_2 = 0$

For Knot complement ($S^3 \setminus K$)

Gukov - Manolescu

$$J_n(K;q) \xrightarrow[x = e^{n\hbar}]{\substack{\hbar \to 0 \\ n \to \infty}} \frac{1}{\Delta_K(x)} + \frac{P_1(x)}{\Delta_K(x)^3}\hbar + \frac{P_2(x)}{\Delta_K(x)^5}\frac{\hbar^2}{2!} + \cdots$$

$\longrightarrow$ Borel resum

$$F_K(x,q) = \frac{1}{2}\sum_{b\in\mathbb{Z}} \hat{Z}_b(S^3\setminus K)\, x^b$$

e.g.

$$\frac{F_{3_1}(x;q)}{x^{\frac{1}{2}} - x^{-\frac{1}{2}}} = x\sum_{k=0}^{\infty} q^k x^k (x;q^{-1})_k$$

$$\frac{F_{4_1}(x;q)}{x^{\frac{1}{2}} - x^{-\frac{1}{2}}} = \sum_{k=1}^{\infty} \frac{1}{\prod_{i=0}^{k-1}(x - x^{-1} - q^i - q^{-i})}$$

related to cyclotomic expansion!

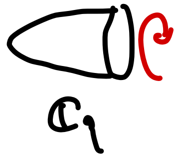
Dehn surgery = "Laplace Transf"

$$\hat{Z}(S^3_{\frac{p}{r}}(K);q) = \oint \frac{dx}{2\pi i x} \underbrace{(x^{\frac{1}{2r}} - x^{-\frac{1}{2r}})}_{\text{solid torus}} \underbrace{F_K(x;q)}_{S^3\setminus K} \underbrace{\sum_{u\in\frac{p}{r}\mathbb{Z}+\frac{k}{r}} q^{-\frac{r}{p}u^2} x^u}_{\text{gluing}}$$

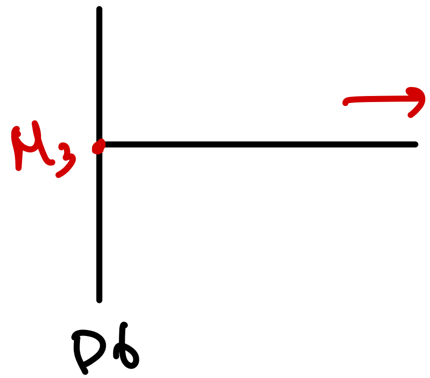
Connection to Witten's Construction

Witten "5-branes & knots."

space-time	$\mathbb{R} \times \mathbb{C}_q \times \mathbb{C}_t \times T^*M_3$	$\longrightarrow$	$\mathbb{R} \times \mathbb{R}_+ \times \mathbb{C} \times T^*M_3$
N M5	$\mathbb{R} \times \mathbb{C}_q \qquad M_3$		N D4 $\quad \mathbb{R} \times \mathbb{R}_+ \qquad M_3$
			D6 $\quad \mathbb{R} \times \mathbb{R}_+ \qquad T^*M_3$



$B \simeq \frac{B_0}{y}$



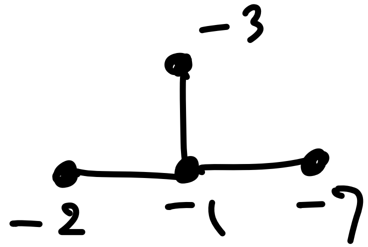
BPS eq^n

$$F^+ - \frac{1}{4} B \times B - \frac{1}{2} D_y B = 0$$

$$F_{yi} + D^j D_{ji} = 0$$

$$\hat{Z}_b(M_3) = \mathrm{Tr}_{\mathcal{H}_b} (-1)^F q^P$$

Mock, false Modular form



Example $M_3 = \Sigma(2,3,7) = S^3_1(4_1) = S_{-1}(3_1^r)$

$$\hat{Z}(q) = q^{\frac{1}{2}}(1 - q - q^5 + q^{10} - q^{11} + q^{18} + q^{30} + \cdots)$$

$$= q^{\frac{1}{2}} \sum_{n=0}^{\infty} (-1)^n \frac{q^{\frac{n(n+1)}{2}}}{(q^{n+1})_n} = F_0(q^{-1})$$

Ramanujan 7^{th} Mock theta

$$= q^{\frac{83}{168}} \tilde{\Psi}^{42,+6,14,21}(\tau)$$

false theta function

$$\hat{Z}(-\Sigma(2,3,7)) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^{n+1})_n} = F_0(q)$$

where

$$\tilde{\Psi}_1^{42+6,14,21}(\tau) = \sum_{r \in \{1,-13,-29,41\}} \tilde{\Psi}_{42,r}(\tau).$$

$$\tilde{\Psi}_{m,r}(\tau) = \Bigg[\sum_{\substack{n>0 \\ n \equiv r \bmod 2m}} - \sum_{\substack{n>0 \\ n \equiv -r \bmod 2m}} \Bigg] q^{\frac{n^2}{4m}} \xleftarrow{\text{Eichler int of wt } -\frac{3}{2}} \theta_{m,r} = \sum_{l \equiv r \bmod m} q^{\frac{l^2}{4m}} y^l$$

familiar theta-fn

Modular properties & Resurgence

GMP

Modular properties of false Θ-functions

Transseries.

$$\frac{1}{\sqrt{k}}\,\hat{\Psi}_a^{m+k}\left(\frac{1}{k}\right) \sim \sum_{n\geq 0} \frac{c_n}{k^{n+\frac{1}{2}}}\left(\frac{\pi i}{2m}\right)^n + \sum_{b\in\sigma} S_{ab}^{m+k}\,\hat{\Psi}_b^{m+k}(-k)$$

perturbative

non-perturbative

$$\xrightarrow{k\in\mathbb{Z}} \sum_{n\geq 0} \frac{c_n}{k^{n+\frac{1}{2}}}\left(\frac{\pi i}{2m}\right)^n + \sum_{b\in\sigma} S_{ab}^{m+k}\, d_b\, e^{-2\pi i k\frac{b^2}{4m}}$$

CS action of non-Abelian flat conn

$$\sim I_\alpha + \sum_\beta n_{\alpha\beta} I_\beta$$

E.g. $M_3 = \Sigma(2,3,7)$

$$= \frac{2\,\mathrm{sh}(6z)\,\mathrm{sh}(14z)}{\mathrm{ch}(21z)} + \sum_{b=1,5,11} S_{ab}\, d_b\, e^{-2\pi i k\frac{b^2}{4m}}$$

3 non-abelian flat conn.

Quantum Modular form

Zagier

QMF is a function $Q(x)$ on $\mathbb{Q}$ such that.

$$p(x) := Q(x) - Q\left(\frac{ax+b}{cx+d}\right)(cx+d)^k$$

extends to a function of $\mathbb{R}$ minus finitely many pts and analyticity or continuity property.

Quantum Modularity Conj $J_K(x) = J_n(K, q^{\frac{2\pi i m}{n}} = x)$

$$J_K\left(\frac{ax+b}{cx+d}\right) \sim J_K(x)\left(\frac{2\pi i}{\hbar}\right)^{\frac{3}{2}} e^{\frac{1}{c\hbar}[CS(K)+iVol(K)]} \, \Phi_{K,\frac{a}{c}}(\hbar)$$

formal power series of $\hbar = \frac{2\pi i}{cx+d}$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$.