Hw 6.

1.1

$$\delta S = \frac{1}{4\pi}\int d^2z \left[\frac{2}{\alpha'}\partial\delta X\bar\partial X + \frac{2}{\alpha'}\partial X\bar\partial\delta X + \delta\psi^\mu\bar\partial\psi_\mu - \bar\partial\delta\psi_\mu\psi^\mu + \delta\tilde\psi^\mu\partial\tilde\psi_\mu - \partial\delta\tilde\psi_\mu\tilde\psi^\mu \right]$$

$$= \frac{1}{4\pi}\int d^2z\left[-\sqrt{\frac{2}{\alpha'}}\left(\partial\{\underline{\epsilon\psi^\mu} + \underline{\bar\epsilon\tilde\psi^\mu}\}\bar\partial X_\mu + \partial X^\mu\bar\partial\{\underline{\epsilon\psi_\mu} + \underline{\bar\epsilon\tilde\psi_\mu}\}\right) + \sqrt{\frac{2}{\alpha'}}\left(\underline{\epsilon\partial X^\mu\bar\partial\psi_\mu} - \underline{\bar\partial\epsilon\partial X_\mu\psi^\mu} + \underline{\bar\epsilon\bar\partial X^\mu\partial\tilde\psi_\mu} - \underline{\partial\bar\epsilon\bar\partial X_\mu\tilde\psi^\mu}\right)\right]$$

note that for term $\int d^2z\, \bar\partial\epsilon\partial X_\mu\psi^\mu$

partial int. $= -\int d^2z\, \epsilon\,\partial X_\mu\bar\partial\psi^\mu$

hence terms have same color underline cancelled each other.

which leads to $\delta S = 0$

We can show that, for any field

$\mathcal{O} \sim \rho X^\mu + \sigma\psi^\mu$, if we only consider holomorphic part

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}]\mathcal{O} = \delta_{\epsilon_1}\left(-\sqrt{\frac{\alpha'}{2}}\rho\,\epsilon_2\psi^\mu + \sqrt{\frac{2}{\alpha'}}\sigma\epsilon_2\partial X^\mu\right) - \delta_{\epsilon_2}\left(-\sqrt{\frac{\alpha'}{2}}\epsilon_1\psi^\mu + \sqrt{\frac{2}{\alpha'}}\epsilon_1\partial X^\mu\right)$$

$$= -\sqrt{\frac{\alpha'}{2}}\rho\,\epsilon_2\sqrt{\frac{2}{\alpha'}}\epsilon_1\partial X^\mu - \sqrt{\frac{2}{\alpha'}}\sigma\epsilon_2\sqrt{\frac{\alpha'}{2}}\epsilon_1\partial\psi^\mu + \sqrt{\frac{\alpha'}{2}}\epsilon_1\sqrt{\frac{2}{\alpha'}}\epsilon_2\partial X^\mu + \sqrt{\frac{2}{\alpha'}}\epsilon_1\sqrt{\frac{\alpha'}{2}}\epsilon_2\partial X^\mu$$

$$= 2\epsilon_1\epsilon_2\partial(\rho X^\mu + \sigma\psi^\mu) = 2\epsilon_1\epsilon_2\partial\mathcal{O}$$

the same for antiholomorphic part. hence

$$[\delta_{\epsilon_1,\bar\epsilon_1}, \delta_{\epsilon_2,\bar\epsilon_2}]\mathcal{O} = 2\epsilon_1\epsilon_2\partial\mathcal{O} + 2\bar\epsilon_1\bar\epsilon_2\bar\partial\mathcal{O}$$

1.2

$$\bar D Y^\mu = (\partial_{\bar\theta} + \bar\theta\bar\partial)(X^\mu + i\theta\psi^\mu + i\bar\theta\tilde\psi^\mu + \tfrac{1}{2}\bar\theta\theta F^\mu)$$

$$= \bar\theta\bar\partial X^\mu + i\bar\theta\theta\bar\partial\psi^\mu + i\tilde\psi^\mu + \tfrac{1}{2}\theta F^\mu$$

$$D Y^\mu = (\partial_\theta + \theta\partial)(X^\mu + i\theta\psi^\mu + i\bar\theta\tilde\psi^\mu + \tfrac{1}{2}\bar\theta\theta F^\mu)$$

$$= \theta\partial X_\mu + i\psi_\mu + i\theta\bar\theta\partial\tilde\psi_\mu - \tfrac{1}{2}\bar\theta F_\mu$$

note that $\int d\theta(a + b\theta) = b$, hence only $\bar\theta\theta$ term survive the integral $\int d^2\theta$

$$S = \frac{1}{4\pi}\int d^2z\, d^2\theta\, \bar D Y^\mu D Y_\mu = \frac{1}{4\pi}\int d^2z\, d^2\theta\left[\bar\theta\theta\bar\partial X^\mu\partial X_\mu - \bar\theta\theta\bar\partial\psi^\mu\psi_\mu - \theta\bar\theta\tilde\psi_\mu\partial\tilde\psi^\mu + \tfrac{1}{4}\bar\theta\theta F^2\right]$$

$$= \frac{1}{4\pi}\int d^2z\left(\partial X^\mu\bar\partial X_\mu + \psi_\mu\bar\partial\psi^\mu + \tilde\psi_\mu\partial\tilde\psi^\mu + \tfrac{1}{4}F\right)$$

Since F is auxiliary, it yeilds it's EOM: $F = 0$.

hence $S = \frac{1}{4\pi}\int d^2z\left(\partial X\bar\partial X + \psi_\mu\bar\partial\psi^\mu + \tilde\psi_\mu\partial\tilde\psi^\mu\right)$

1.3. $\bar D C = (\partial_{\bar\theta} + \bar\theta\bar\partial)(c + \theta\gamma) = \bar\theta\bar\partial c + \bar\theta\theta\bar\partial\gamma$

$S^{gh} = \frac{1}{2\pi}\int d^2z\, d^2\theta\,(\beta + \theta b)(\bar\theta\bar\partial c + \bar\theta\theta\bar\partial\gamma)$, keep only $\bar\theta\theta$ term

$$= \frac{1}{2\pi}\int d^2z\, d^2\theta\left(\bar\theta\theta\beta\bar\partial\gamma + \underline{\theta b\bar\theta\bar\partial c}\right) \quad = -\theta\bar\theta b\bar\partial c = \bar\theta\theta b\bar\partial c$$

$$= \frac{1}{2\pi}\int d^2z\left(\beta\bar\partial\gamma + b\bar\partial c\right)$$

2. we can have $L_m = \oint \frac{dz}{2\pi i} T_B z^{m+1}$, $G_r = \oint \frac{dz}{2\pi i} G z^{r+\frac{1}{2}}$

$$[L_m, L_n] = \oint \frac{dz}{2\pi i} \oint_{w\to z} \frac{dw}{2\pi i} T_B(z) T_B(w) z^{m+1} w^{n+1}$$

$$= \oint \frac{dz}{2\pi i} \oint_{w\to z} \frac{dw}{2\pi i} \left[\frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2} \sum_l \frac{L_l}{w^{l+2}} + \sum_l \frac{(-l-2)}{z-w} \frac{L_l}{w^{l+3}} \right] z^{m+1} w^{n+1}$$

$$= \oint \frac{dz}{2\pi i} \oint_{w\to z} \frac{dw}{2\pi i} \left[-\frac{(n+1)n(n-1)}{(z-w)} \frac{c}{12} z^{m+1} w^{n-2} - \sum_l \frac{2L_l}{z-w} (n-l-1) w^{n-l-2} z^{m+1} + \sum_l \frac{(-l-2)}{z-w} L_l z^{m+1} w^{n-l-2} \right]$$

$$= \oint \frac{dz}{2\pi i} \left[-\frac{c}{12} n(n^2-1) z^{m+n-1} - \sum_l 2L_l (n-l-1) z^{m+n-l-1} + \sum_l (-l-2) L_l z^{m+n-l-1} \right]$$

$$= \frac{c}{12} m(m^2-1)\delta_{m+n,0} + L_{m+n}(2m+2-m-n-2)$$

$$= (m-n) L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m+n,0}$$

$$[L_m, G_r] = \oint \frac{dz}{2\pi i} \oint_{w\to z} \frac{dw}{2\pi i} T_B(z) G(w) z^{m+1} w^{r+\frac{1}{2}}$$

$$= \oint \frac{dz}{2\pi i} \oint_{w\to z} \frac{dw}{2\pi i} \left[\frac{3/2}{(z-w)^2} \sum_s \frac{G_s}{w^{s+3/2}} + \frac{1}{z-w} \sum_s (s+\tfrac{3}{2}) \frac{G_s}{w^{s+5/2}} \right] z^{m+1} w^{r+\frac{1}{2}}$$

$$= \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} \left[-\frac{3/2}{z-w} \sum_s (r-s-1) G_s z^{m+1} w^{r-s-2} - \frac{1}{z-w} \sum_s (s+\tfrac{3}{2}) G_s z^{m+1} w^{r-s-2} \right]$$

$$= \oint \frac{dz}{2\pi i} \sum_s \frac{3}{2}(s+1-r) G_s z^{m+r-s-1} - \sum_s (s+\tfrac{3}{2}) G_s z^{m+r-s-1} \Big]$$

$$= \frac{3}{2}(m+r+1-r) G_{m+r} - (m+r+\tfrac{3}{2}) G_{m+r}$$ since r and s take the same b.c. hence $r \pm s$ are integer.

$$= (\tfrac{1}{2}m - r) G_{r+m}.$$

$$\{G_r, G_s\} = \oint \frac{dz}{2\pi i} \oint_{w\to z} \frac{dw}{2\pi i} \left[\frac{2c/3}{(z-w)^3} + \frac{2}{z-w} \sum_m \frac{L_m}{w^{m+2}} \right] z^{r+\frac{1}{2}} w^{s+\frac{1}{2}}$$

$$= \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} \left[\frac{c/3}{z-w} (s+\tfrac{1}{2})(s-\tfrac{1}{2}) z^{r+\frac{1}{2}} w^{s-\frac{3}{2}} + \frac{2}{z-w} \sum_m L_m z^{r+\frac{1}{2}} w^{s-m-\frac{3}{2}} \right]$$

$$= \oint \frac{dz}{2\pi i} \left[\frac{c}{3} (s+\tfrac{1}{2})(s-\tfrac{1}{2}) z^{r+s-1} + 2\sum_m L_m z^{r+s-m-1} \right]$$

$$= 2L_{r+s} + \frac{c}{3}(r^2 - \tfrac{1}{4})\delta_{r+s,0}$$

note that in R sector, where $\nu = 0$, we have nontrival normal ordering shift $a^m = \frac{1}{16} D = \frac{1}{24} c$

hence $\{G_r, G_s\}_R = 2L_{r+s} + \frac{c}{12}\delta_{r+s,0} + \frac{c}{3}(r^2 - \frac{1}{4})\delta_{r+s,0} = 2L_{r+s} + \frac{c}{3} r^2 \delta_{r+s,0}$, $r, s \in \mathbb{Z}+\nu$, $\nu = 0$

while for NS sector $a^m = 0$

$$\{G_r, G_s\}_{NS} = 2L_{r+s} + \frac{c}{3}(r^2 - \tfrac{1}{4})\delta_{r+s,0} \qquad r, s \in \mathbb{Z}+\nu,\ \nu = \tfrac{1}{2}$$

In all, we can write

$$\{G_r, G_s\} = 2L_{r+s} + \frac{c}{3}(r^2 - \nu^2)\delta_{r+s,0}$$

Problem 3.

In NS sector, we should have $n = \frac{3}{2}$ to find the first massive state

$$G_{\frac{3}{2}} = \alpha_0 \psi_{\pm\frac{3}{2}} + \alpha_1 \psi_{\frac{1}{2}} + \alpha_{-1}\psi_{-\frac{1}{2}} + \cdots$$

After GSO projection only even fermion number states survive. Since vaccum have odd fermion number, we should have odd fermion operator.

hence we can have

$$\alpha_0 \psi_{-\frac{3}{2}} |0,k\rangle \text{, with } \begin{cases} \text{mass-shell condition} \\ Q_B\text{- exact condition} \end{cases} \text{for } \psi_{-\frac{3}{2}} \Rightarrow 8 \text{ DOF}$$

$$\alpha_{-1}\psi_{-\frac{1}{2}} |0,k\rangle \text{, with } \begin{cases} \text{mass-shell} \\ Q_B\text{- exact} \end{cases} \text{for both } \alpha \text{ and } \psi \Rightarrow 8 \times 8 \text{ DOF}$$

$$\psi_{-\frac{1}{2}}\psi_{-\frac{1}{2}}\psi_{-\frac{1}{2}} |0,k\rangle \text{, with } \begin{cases} \text{mass-shell} \\ Q_B\text{-exact} \\ \text{transverse to each other} \\ \text{combination of space direction} \end{cases} \text{for each } \psi \Rightarrow \frac{8\times 7\times 6}{1\times 2\times 3} \text{ DOF}$$

Therefore in NS sector we have

$8 + 64 + 56 = 128$ DOF

In R sector, we have $n = 1$

$$G_1 = \alpha_0 \psi_{\pm 1} + \alpha_1 \psi_0 + \alpha_{-1}\psi_0 + \cdots$$

we know that vaccum $|u,k\rangle$ have 8 DOF

hence $\alpha_{-1}|u,k\rangle \Rightarrow 8\times 8$ DOF

$\psi_{-1}|u,k\rangle \Rightarrow 8 \times 8$ DOF

in R sector we finall have

$64 + 64 = 128$ DOF which is same as NS sector.