Homework 12: Due at class on June 3

1 Chern classes of $\mathbb{C}P^n$

An arbitrary point $\ell = [z_0; \cdots ; z_n] \in \mathbb{C}P^n$ is a complex line $\mathbb{C}$ through the origin of $\mathbb{C}^{n+1}$. So we can construct a complex line bundle L over the complex projective space $\mathbb{C}P^n$ as

$$L = \{(\ell, w) \in \mathbb{C}P^n \times \mathbb{C}^{n+1} \,|\, w \in \ell\},$$
$$= \{([z_0; \cdots ; z_n], w) \in \mathbb{C}P^n \times \mathbb{C}^{n+1} \,|\, w = \lambda \cdot (z_0, \cdots, z_n), \quad \lambda \in \mathbb{C}\}. \tag{1.1}$$

(The notation is the same as in Problem 4 of Homework 10.) This is called the **Hopf line bundle**. If we consider points with unit length in each fiber of the Hopf line bundle, then we obtain a sphere

$$S^{2n+1} = \{(\ell, z) \in \mathbb{C}P^n \times \mathbb{C}^{n+1} \,|\, z \in \ell, \, |z| = 1\}.$$

Therefore, the Hope line bundle is the associated line bundle of the Hopf S^1-bundle.

In fact, the Whiteney sum of the tangent bundle $T\mathbb{C}P^n$ and the trivial bundle ϵ is the Whiteney sum of $(n+1)$ copies of L^*

$$T\mathbb{C}P^n \oplus \epsilon \cong L^* \oplus \cdots \oplus L^*.$$

where L^* is the dual line bundle of L. (See Morita Proposition 5.13 for the derivation.)

Let us first consider the case of $n = 1$. We know that cohomology $H^2(\mathbb{C}P^1; \mathbb{Z}) \cong \mathbb{Z}$ and its generator x is the volume form. Show that $c_1(L) = -x$ and $c_1(T\mathbb{C}P^1) = 2x$. In addition, find the curvature of the Ehresmann connection

$$A = i \,\mathrm{Im}(\bar{z}_0 dz_0 + \bar{z}_1 dz_1)$$

given in Problem 4 of Homework 10, and compare it with the volume form. Actually, show that over $\mathbb{C}P^1$ one can find an infinite family of non-equivalent complex line bundles.

Find the total Chern class of $T\mathbb{C}P^n$. In fact, the $2i$-th de Rham cohomology group of $\mathbb{C}P^n$ is isomorphic to $\mathbb{R}$ generated by $x^i = x \wedge \cdots \wedge x$:

$$H^{2i}_{dR}(\mathbb{C}P^n) \cong \mathbb{R}\langle x^i \rangle \quad \text{for} \quad i \leq n.$$

Moreover, the de Rham cohomology groups of $\mathbb{C}P^n$ has a ring structure where the multiplication is given by the wedge product:

$$H^*_{dR}(\mathbb{C}P^n) \cong \mathbb{R}[x]/(x^{n+1}).$$

2 Chern-Simons and Wess-Zumino-Witten

Let us define the Chern-Simons action over a 3-manifold M

$$S_{CS} = \frac{k}{4\pi} \int_M \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right).$$

Under the gauge transformation

$$A \to g^{-1}dg + g^{-1}Ag \,,$$

show that the action is transformed as

$$S_{CS} \to S_{CS} + \frac{k}{4\pi} \int_M d \ \text{Tr}\left(g^{-1}Ag \wedge g^{-1}dg\right) - \frac{k}{12\pi} \int_M \text{Tr}\left((g^{-1}dg)^3\right) \,.$$

Let us consider the case $M = S^3$ and $G = \text{SU}(2)$. Then, the second term vanishes because S^3 is closed. Show that the third term is

$$\frac{k}{12\pi} \int_M \text{Tr}\left((g^{-1}dg)^3\right) = 2\pi k \deg(g) \,,$$

where $\deg(g)$ is the mapping degree of

$$g : S^3 \to \text{SU}(2) \cong S^3 \,.$$

(Hint: $\mu = g^{-1}dg$ is the Maurer-Cartan form and $\frac{1}{12} \text{Tr}(\mu \wedge \mu \wedge \mu)$ is the left-invariant volume form of $\text{SU}(2)$.) Argue that the level k must be an integer in order for the partition function to be well-defined.

3 Chern-Simons from 4-dimension

Show the following identity

$$\text{Tr}(F \wedge F) = d \ \text{Tr}\left(A \wedge dA + \frac{2}{3}A \wedge A \wedge A\right) \,.$$

Therefore, if there exists a four-manifold B whose boundary is $\partial B = M$, then we can indeed define Chern-Simons action by

$$S_{CS} = \frac{k}{4\pi} \int_B \text{Tr}(F \wedge F) \,,$$

when G is compact and semi-simple.