

Homework 9: Due at class on May 5

1. Suppose that we have a connection ∇ on a rank- r vector bundle $\pi : E \rightarrow M$. Suppose that we have local trivializations of E over two open sets U_α and U_β in M . On these open sets, the connection can be written in terms of one-form A_α and A_β taking value on $\mathfrak{gl}(r, \mathbb{R})$. Given a transition function $g_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow \mathrm{GL}(r, \mathbb{R})$, show that the connections are related by

$$A_\beta = g_{\alpha\beta}^{-1} A_\alpha g_{\alpha\beta} + g_{\alpha\beta}^{-1} d(g_{\alpha\beta}) ,$$

and the curvature forms are related by

$$F_\beta = g_{\alpha\beta}^{-1} F_\alpha g_{\alpha\beta}$$

on $U_\alpha \cap U_\beta$.

2. Prove that the Bianchi identity

$$dF + A \wedge F - F \wedge A = 0 .$$

3. Show that there is a line (rank one vector) bundle L over S^2 such that $TS^2 \oplus L$ is trivial where TS^2 is the tangent bundle of S^2 . Hint: The tangent bundle of the flat space is trivial $T\mathbb{R}^n \cong \mathbb{R}^{2n}$.

4. Hopf fibration and Dirac monopole

We define $S^{2n+1} = \{(z_0, \dots, z_n) \in \mathbb{C}^{n+1} \mid \sum_{i=0}^n |z_i|^2 = 1\}$ and let the complex projective space $\mathbb{C}P^n$ be the quotient space S^{2n+1} / \sim where we define an equivalent relation $(z_0, \dots, z_n) \sim \lambda(z_0, \dots, z_n)$ with $\lambda \in \mathrm{U}(1)$. We denote a point of $\mathbb{C}P^n$ by $[z_0; \dots; z_n]$.

When $n = 1$, we can consider that S^3 is the principal $\mathrm{U}(1)$ -bundle over $\mathbb{C}P^1$

$$\begin{array}{ccc} S^1 & \longrightarrow & S^3 \\ & & \downarrow \pi \\ & & \mathbb{C}P^1 \end{array} .$$

First of all, show that $\mathbb{C}P^1$ can be identified with $S^2 \subset \mathbb{R}^3$ by a map

$$[z_0; z_1] \mapsto (z_0 \bar{z}_1 + \bar{z}_0 z_1, -i(z_0 \bar{z}_1 - \bar{z}_0 z_1), |z_0|^2 - |z_1|^2) .$$

Let us define trivializations of this bundle over $U_N = S^2 \setminus \{\text{north pole}\}$ and $U_S = S^2 \setminus \{\text{south pole}\}$ as

$$\begin{aligned} \pi^{-1}(U_N) &\rightarrow U_N \times S^1; (z_0; z_1) \mapsto \left([z_0; z_1], \frac{z_0}{|z_0|} \right) \\ \pi^{-1}(U_S) &\rightarrow U_S \times S^1; (z_0; z_1) \mapsto \left([z_0; z_1], \frac{z_1}{|z_1|} \right) . \end{aligned}$$

Find the transition function between these trivializations. Let us consider embedding

$$\begin{aligned} f_N : U_N &\hookrightarrow \pi^{-1}(U_N) : (\theta, \phi) \mapsto \left(\cos \frac{\theta}{2}, \sin \frac{\theta}{2} e^{-i\phi} \right) \\ f_S : U_S &\hookrightarrow \pi^{-1}(U_S) : (\theta, \phi) \mapsto \left(\cos \frac{\theta}{2} e^{i\phi}, \sin \frac{\theta}{2} \right) , \end{aligned}$$

where (θ, ϕ) is the standard polar coordinate of S^2 . Defining a connection

$$A = i \operatorname{Im}(\bar{z}_0 dz_0 + \bar{z}_1 dz_1)$$

on $S^3 \subset \mathbb{C}^2$, show that

$$\begin{aligned} f_N^* A &= -\frac{i}{2}(1 - \cos \theta)d\phi \quad \text{on } U_N \\ f_S^* A &= \frac{i}{2}(1 + \cos \theta)d\phi \quad \text{on } U_S . \end{aligned}$$

Show that they are related by gauge transformations and its curvature F is well-defined on S^2 . This connection provides **Dirac magnetic monopole**

$$\frac{i}{2\pi} \int_{S^2} F = 1 .$$

5. Let us consider the case that a Lie group G acts on a manifold M

$$G \times M \rightarrow M; (g, p) \mapsto gp .$$

We call the action is **transitive** if any two points of M can be transformed by the action of G . Suppose that G acts on M transitively, and we pick an arbitrary point $p \in M$. We call $H = \{g \in G | gp = p\}$ a **stabilizer subgroup** of G . Then, we have diffeomorphism $M \cong G/H$ and it is called a **homogeneous space**. Moreover, G can be interpreted as a principle H -bundle over M . Let us see this construction in the following examples.

Let us consider the $\operatorname{SO}(2)$ subgroup of $\operatorname{SO}(3)$ as

$$\operatorname{SO}(3) \supset \operatorname{SO}(2) = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

Show that $S^2 \cong \operatorname{SO}(3)/\operatorname{SO}(2)$ and $\operatorname{SO}(3)$ can be considered as S^1 bundle over S^2 . Construct local trivializations of this bundle over $U_N = S^2 \setminus \{\text{north pole}\}$ and $U_S = S^2 \setminus \{\text{south pole}\}$, and find the corresponding transition function at the equator. Construct a $\operatorname{U}(1)$ -connection on this bundle and find its curvature.

6. We have seen that $\operatorname{SL}(2, \mathbb{R})$ acts on the upper half plane \mathbf{H} :

$$\operatorname{SL}(2, \mathbb{R}) \times \mathbf{H} \rightarrow \mathbf{H}; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z \mapsto \frac{az + b}{cz + d} .$$

Show that this action is transitive and find the stabilizer subgroup of the point $i \in \mathbf{H}$. Find the fundamental group of $\operatorname{SL}(2, \mathbb{R})$.