Homework 4: Due on Nov 23

Caution: When solving homework problems, it is important to show your derivation at each step. Nowadays, many online tools make it easy to find answers, but the primary goal of these assignments is to deepen your understanding through hands-on problem-solving. By working through the calculations yourself, you engage more deeply with the material, making the learning process more meaningful, rather than simply copying answers from external sources.

Equation numbers below are as in the lecture notes.

1 Landau Parameters and Fermi Surface Stability

Consider a three-dimensional Fermi liquid with short-range two-body interactions. The effective interaction potential is given by

$$V(\boldsymbol{x}-\boldsymbol{x}')$$

and the Landau parameters F_ℓ characterize the interaction between quasiparticles on the Fermi surface.

1. For a contact interaction
$$V(\boldsymbol{x}-\boldsymbol{x}') = \lambda_1\, \delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}'),$$
compute the Landau parameters F_ℓ to leading order in λ_1.

2. For a short-range derivative interaction
$$V(\boldsymbol{x}-\boldsymbol{x}') = -\lambda_2\, \nabla^2 \delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}'),$$
show that its Fourier transform is $V(\boldsymbol{q}) = \lambda_2 q^2$, and determine the corresponding Landau parameters F_ℓ to leading order in λ_2.

3. The stability condition of a Fermi liquid is given by
$$1 + \frac{F_\ell}{2\ell+1} > 0 \quad \text{for all } \ell.$$
Using the results of 1 and 2, analyze qualitatively the stability of the Fermi surface. Sketch the regions in the (λ_1, λ_2) phase diagram where the system becomes unstable.

2 Quasi-particles in BCS states

Show that the operators defined in Eq. (8.37) satisfy the anticommutation relations

$$\left\{A_{\boldsymbol{k}}, A^\dagger_{\boldsymbol{k}'}\right\} = \left\{B_{\boldsymbol{k}}, B^\dagger_{\boldsymbol{k}'}\right\} = \delta_{\boldsymbol{k},\boldsymbol{k}'},$$

and verify that

$$\{A_{\boldsymbol{k}}, B_{-\boldsymbol{k}}\} = 0.$$

Furthermore, show that for the BCS wave function given in Eq. (8.22),

$$A_{\boldsymbol{k}}|\Psi_{\mathrm{BCS}}\rangle = B_{-k}|\Psi_{\mathrm{BCS}}\rangle = 0.$$

This means that $|\Psi_{\mathrm{BCS}}\rangle$ is the "vacuum" of the quasiparticles represented by $A_{\boldsymbol{k}}$ and B_{-k}. In addition, show that the following state is proportional to $|\Psi_{\mathrm{BCS}}\rangle$:

$$\prod_{\boldsymbol{k}} A_{\boldsymbol{k}} B_{-\boldsymbol{k}} |\mathbf{vac}\rangle$$

As a result, the quasiparticle excited states can be constructed as

$$A_{\boldsymbol{k}}^{\dagger}|\Psi_{\mathrm{BCS}}\rangle, \qquad B_{-k}^{\dagger}|\Psi_{\mathrm{BCS}}\rangle.$$