復旦大學

本科毕业论文(设计)

论 文 题 目: 利用 Alday-Gaiotto-Tachikawa 对偶研究 Argyres-Douglas 理论

姓　　名: 臧亦驰　　学　　号: 21300190008

院　　系: 物理学系

专　　业: 物理学（强基计划）

指 导 教 师: 绳田聪　　职　　称: 副研究员

单　　位: 复旦大学物理学系

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摘 要

本文主要讨论了如何通过 Alday-Gaiotto-Tachikawa 对偶实现 Argyres-Douglas 理论相关物理量的计算。这篇文章的内容以综述为主，较为系统地介绍了理解 AGT 对偶和 AD 理论所需要的背景知识、4 维超对称共形场论和 2 维 Liouville 共形场论的基础、以及关于 AD 理论中 Nekrasov 瞬子配分函数计算所需要使用的 Gaiotto 不规则态，在关于一类具有半整数阶的不规则态的计算过程中，我们提出了一种新的参数化方法，可以使计算变得更加清晰并且具有更明确的物理内涵。超对称性和共形对称性是两种超越了 Coleman-Mandula 不可行定理的物理对称性。它们在现代理论物理，特别是超弦理论中扮演了重要的作用。本文首先对这两种对称性和它们对应的场论进行了介绍。简单来说，超对称性描述了一种在费米子场和玻色子场之间相互转化的费米对称性，它需要用到分阶李代数来描述。而共形对称性描述了所谓的共性变换或者说保角变换对应的对称性。特别的是，在 2 维时空中，共形对称性被提升为了一个无穷维的对称性，它的量子版本由 Virasoro 代数描述。这一特性使得 2 维共形场论获得了很多优秀的性质，对它们的求解通常可以比一般的量子场论精确得多。

随后，我们讨论了 4 维超对称共形场论的构造。4 维超对称共形场论一般来说可以被看作 4 维的超对称量子场论在重整化群流下的低能量有效理论，因此我们首先介绍了一般的 4 维超对称量子场论。4 维超对称量子场论的真空模空间由超对称拉格朗日量密度中的实势能项的极小值给出，根据 4 维 $\mathcal{N}=2$ 超对称的矢量多重态和超多重态中的标量场是否获得非零的真空期望值，可以将理论的真空模空间分为 Coulomb 分支和 Higgs 分支。在有些情况下还会出现混合分支。这其中，Higgs 分支通常不受量子修正的影响，而 Coulomb 分支则会在重整化群流的作用下跑动。其中，Coulomb 分支中的某些区域就会导致低能量区有效理论出现共形对称性。Argyres-Douglas 理论是一类 4 维的超对称共形场论，其中超对称的数量是 $\mathcal{N}=2$，也就是说，其具有 8 个超对称荷。这类理论来自 Coulomb 分支中的一些很特殊的点，它们通常具有非常复杂的性质。举例来说，它的 Coulomb 分支多重态中的标量场可能具有分数阶的共形维度，并且这类理论中的 BPS 粒子态有可能携带有相互非局域的电荷和磁荷。这意味着这类理论很可能是无法用拉格朗日量密度描述的。另一种理解方式是，这一理论天然具有强耦合理论的性质，因而不像传统的有拉格朗日量密度的场论可以获得自由理论极限。这些性质导致 AD 理论的配分函数计算变得非常困难。通常，4 维超对称共形场论的配分函数可以被分解为经典阶、一圈图阶和瞬子阶的部分，而瞬子阶的贡献，也就是著名的 Nekrasov 瞬子配分函数，是这之中最难计算的部分。这些计算即使对于一般的超对称共形场论来说，也需要涉及对复杂的瞬子模空间的积分。对于 AD 理论，这样的计算更加困难。

AGT 对偶被认为是一种可以有效求解 AD 理论的方式。为了理解这一理论，我们在文中讨论了 4 维超对称共形场论的另一种构造，称为 $\mathcal{S}$ 类理论构造。简单来说，我们将 6 维的 $\mathcal{N}=(2,0)$ 超对称共形场论紧化在一个 2 维的黎曼面 $\mathcal{C}$ 上，由此得到一个 4 维的有效理论。从更高维的 M 理论角度来看，这相当于是将一张或多张 M5 膜缠绕在黎曼面 $\mathcal{C}$ 上。由于一种被称为拓扑扭转的技巧，M5 膜在其中两个垂直方向上的运动从低能量红外理论的角度看来可以被解读为 $\mathcal{C}$ 上的一个 1-形式场，称为 Higgs 场。通过这一 Higgs 场，我们可以写出紧化后的

4 维理论的 Seiberg-Witten 曲线，这有助于我们理解紧化后得到的是什么理论。一般而言，紧化选取的黎曼面 $\mathcal{C}$ 上可以具有多个穿孔，这些穿孔对应于 Higgs 场的奇点。一般来说，这样的奇点都是一阶的，我们称这样的奇点为规则奇点，此时紧化得到的 4 维超对称共形场论通常也是规则的。为了得到 AD 理论，需要在黎曼面上引入不规则的穿孔，此时 Higgs 场会具有高于一阶的奇点。由 R 对称性的要求，这样的黎曼面只能选取为黎曼球面，而且只能支持一个这样的不规则奇点，和至多一个规则奇点。

我们指出，在 $\mathcal{S}$ 类构造中，一个自然的想法是，既然可以将 6 维超对称共形场论紧化在 2 维的黎曼面上来得到 4 维超对称共形场论，是否也可以将 6 维的理论紧化在某个 4 维流形上来得到一个 2 维理论。一个通常的选择是一个压扁的超球。这样的紧化结果是黎曼面 $\mathcal{C}$ 上的 2 维 Toda 共形场论。由于此时 4 维和 2 维的理论都是共形不变的，我们理应可以将其中任意一个缩小至可以忽略不计，而保持理论不变。因此，自然的推测是，这样紧化得到的 4 维超对称共形场论和 2 维共形场论是对应的。这就是 AGT 对偶的基本图像。

通过 AGT 对偶，我们可以将 4 维 AD 理论的计算转化为 2 维共形场论中的计算，而正如我们提到的，2 维共形场论具有良好的性质，无穷维 Virasoro 代数的存在使得这类理论中的很多计算变得可能。具体来说，我们考虑了规范代数为 A_1 的 AD 理论，此时理论的 2 维共形场论变为 A_1 型 Toda 理论，也就是 Liouville 共形场论。由于 AD 理论在 $\mathcal{S}$ 类理论的构造下对应于具有不规则穿孔的黎曼球面，我们所讨论的 Liouville 共形场论也应该具有不规则的穿孔，而根据共形场论中的态-算符对应关系，这样的不规则穿孔就对应了 Liouville 理论中的不规则态。我们指出，取决于具体的黎曼球上的奇点分布，AD 理论的 Nekrasov 瞬子配分函数在 AGT 框架下对应于 Liouville 理论中的不规则态与真空态或初级态的内积，因此，求解 4 维超对称共形场论中 Nekrasov 瞬子配分函数的问题就被转化为了求解 2 维 Liouville 场论中的不规则态问题。

在本文中，我们依次讨论了整数阶和半整数阶的不规则态在 Liouville 理论中的构造和求解的问题。我们采用了 D. Gaiotto 提出的自由场表示的方法，通过 Liouville 标量场的模式展开得到的产生湮灭算符构造了整数阶的不规则态，并讨论了它的求解方法。随后我们指出，目前对于半整数阶不规则态，尚不存在这样的自由场表示。而我们通过创新性地考察具有 $\mathbb{Z}_2$ 扭转的 2 维共形场论中的标量场模式展开，模仿 D. Gaiotto 的方法给出了半整数阶不规则态的自由场表示构造，随后，我们通过具体例子的计算，证明了我们给出的表示确实可以用于求解不规则态，进而求解对应 AD 理论的 Nekrasov 瞬子配分函数。我们计算的结果也和现有的一致结论吻合。最后，作为结尾，我们还给出了一些可行的未来研究方向，这主要包括将现有的方法应用在具有更高阶规范代数 A_n 的 AD 理论上。这类理论将对偶于一般的 Toda 理论，其具有更高自旋的对称守恒流，组成的对称代数称为 W-代数。另一种可能的方向是考虑具有面缺陷的 AD 理论和其他 4 维超对称共形场论，这将对应于具有仿射 $\mathfrak{su}(2)$ 对称代数的 2 维共形场论。

简单来说，我们综述了现有的利用 AGT 理论求解 A_1 型 AD 理论 Nekrasov 瞬子配分函数的方法，并其中一类理论的求解方式做出了改进，提出了一种更简洁且具有物理内涵的半整数阶不规则态的表示方法。这种方法具有一定的可拓展性，可能可以应用到 W-代数或仿射 $\mathfrak{su}(2)$ 代数的情形中。

关键词：共形场论；超对称共形场论；S 类理论；AGT 对偶

Abstract

In this thesis, we discuss the behavior of Argyres-Douglas theories. This is a family of 4-dimensional $\mathcal{N} = 2$ superconformal field theories that exhibit unusual features such as fractional conformal dimensions or mutually non-local BPS particles. These theories generally do not admit a Lagrangian description. To describe them, we use the class $\mathcal{S}$ construction, which is a top down approach from the compactification of 6-dimensional $\mathcal{N} = (2,0)$ SCFTs. In this way, a nice duality between the 4-dimensional theories and the 2-dimensional theory living on the compactification surface $\mathcal{C}$ emerges. This is known as the Alday-Gaiotto-Tachikawa duality. Within this duality paradigm, we are able to use the powerful tool of 2-dimensional CFT to analyze the otherwise intractable Argyres-Douglas theory. In particular, we consider the simplest A_1 type theories, which are dual to the Liouville theory. We demonstrate the computation of irregular Gaiotto states in Liouville theory, and explain how they are related to the Nekrasov instanton partition function of Argyres-Douglas theories. We also propose a new way to parameterize the irregular states of half-integer ranks.

Keywords: Conformal Field Theory; Super-conformal Field Theory; Class S Theory; AGT Duality

Chapter 1

Introduction

Symmetry has been at the center of theoretical physics ever since the discovery of Noether's theorem. In the long history of the development of the modern quantum field theory (QFT), the analysis of symmetry has led to all kinds of remarkable successes and achievements. In particular, there are two astonishing discoveries of symmetry types, known as conformal symmetry and supersymmetry, that have significantly enriched our techniques to study QFT in various dimensions and geometries. It is inevitable to encounter field theories with conformal symmetry and supersymmetry in the course of string theory. With the constraint of symmetry we are able to delve into the vast world of super-string / M theory and study the rich structure behind. Although the idea of QFT has been proposed for nearly one century, its true nature has not been fully understood yet. One especially difficult question is about the behavior of quantum fields in the strongly-coupled region, where the traditional perturbation method fails. In the context of conformal field theory (CFT) or supersymmetric quantum field theory (SQFT) when extra symmetry constraints are employed, the new tools shed light on the strong coupling dynamics. It helps reveal the mysterious nature of QFTs. In 4-dimensional $\mathcal{N} = 2$ superconformal field theories (SCFTs), where the amount of symmetry constraints is balanced so that we do not lose too much diversity while keeping track of the field behaviors, we are able to take a glimpse at the exotic behavior of certain non-traditional QFT.

The Argyres-Douglas (AD) theories are a special type of 4-dimensional $\mathcal{N} = 2$ SCFT first proposed by P. Argyres and M. Douglas in 1995[1]. Since the first discovery, there has been generalizations in different ways, eventually making it a large family of SCFTs[2–6]. These theories are notoriously difficult to tackle, as they generally do not admit a Lagrangian description. Although they can be obtained by an RG flow from the UV $\mathcal{N} = 2$ SQFT, it is hard to read off its physical information from the UV description. With traditional approaches, it is hard to compute the Nekrasov partition function of those theories[7]. Only partial results have been reported recently[8].

One major breakthrough is made by F. Alday, D. Gaiotto and Y. Tachikawa[9], using the class $\mathcal{S}$ construction, which is a top-down approach to construct 4-dimensional $\mathcal{N} = 2$ SCFTs from the compactification of 6-dimensional $\mathcal{N} = (2, 0)$ SCFT, the maximally symmetric theory in 6-dimension, on a Riemann surface $\mathcal{C}$[10–12]. They find that there exists a duality between the 4-dimensional $\mathcal{N} = 2$ SCFTs and a 2-dimensional Liouville CFT living on the Riemann surface $\mathcal{C}$. By explicit computation of partition function and conformal blocks, they were able to confirm this duality in regular theories. It is conjectured that this duality, known as the Alday-Gaiotto-Tachikawa (AGT) duality holds beyond regular cases, and is valid even for AD theories. This duality provides a powerful tool to analyze the mysterious AD theories. 2-dimensional CFTs are famous for their infinite-dimensional symmetry algebra, and hence have strong constraints on the behavior of fields[13]. In particular, Liouville theory is an exactly solvable CFT[14–17]. Therefore, computation from the 2-dimensional side can be much simpler.

From the class $\mathcal{S}$ picture, AD theories correspond to the Riemann surfaces $\mathcal{C}$ with an irregular puncture[3–4]. Therefore, general computations involve irregular states in Liouville theory[11]. Several computations have been done to evaluate Nekrasov instanton partition functions for type (A_1, A_2), (A_1, A_3) and (A_1, D_4) AD theories[18–19]. There is also discussion on the generalized AGT duality for AD theories with A_2 or higher rank gauge algebra, in which case the 2-dimensional theory becomes the general A_n-Toda theory[20–22]. These theories enjoy a higher spin symmetry algebra, known as the W-algebra[23]. Some analysis has been made for these types of theories[24–25].

The thesis is organized as follows. In chapter 2, we provide the preliminaries about CFTs and SCFTs. We briefly introduce the conformal and superconformal algebras as well as their representations. In chapter 3, we introduce the 4-dimensional $\mathcal{N} = 2$ SCFTs following two paths. The first is through low-energy effective theories in the Coulomb branch of 4-dimensional $\mathcal{N} = 2$ SQFTs, which is also the path along which AD theories were first discovered. The second is the class $\mathcal{S}$ construction, which naturally leads to the AGT duality. In chapter 4, we begin with an introduction to the 2-dimensional side theory, the Liouville theory. We then illustrate the explicit way to compute the Nekrasov instanton partition function from the 2-dimensional side. In particular, in section 4.3, we propose a new way to realize the free field representation of half-integer rank irregular states, which can help better understand the behavior of AGT duality for certain types of AD theories and simplify the computation.

Chapter 2

Conformal and superconformal symmetries

The two important theories involved in our discussion are 2-dimensional CFTs and 4-dimensional SCFTs. CFTs are quantum field theories that respect conformal symmetry, and SCFTs are quantum field theories that respect superconformal symmetry, which is a combination of supersymmetry and conformal symmetry. In this chapter, we explain the basic aspects of the symmetry algebras of these theories, i.e., the conformal algebra and superconformal algebra.

2.1 Conformal symmetries

Conformal symmetry arises from the natural notion of scaling invariance. This is one characteristic behavior of quantum field theory at the renormalization group fix points. Conformal symmetry is a larger symmetry that contains scale invariance as well as other symmetries. Nevertheless, it can be proved that under certain conditions, (for example, in 2-dimension), scale invariance leads to conformal symmetry[26–28]. Therefore, the study of conformal symmetry and conformal field theories is an important subject of field theory.[13,29]

2.1.1 Conformal transformations

A conformal transformation of the space-time is a transformation that preserves the metric up to a factor. Mathematically, it can be defined as a diffeomorphism φ between two Riemannian manifolds (M, g) and (M', g') of dimension d, such that its pullback on the metric satisfies

$$\varphi^*(g') = \Lambda g, \tag{2.1}$$

where Λ is some scalar function on M. Using local coordinates $x' = \varphi(x)$, the conformal condition can be written as

$$g'_{\alpha\beta}(x') \frac{\partial x'^\alpha}{\partial x^\mu} \frac{\partial x'^\beta}{\partial x^\nu} = \Lambda(x) g_{\mu\nu}(x). \tag{2.2}$$

A characteristic feature of conformal transformation is that it preserves the angle of tangent vectors.

In the context of CFTs, we usually focus on the diffeomorphisms of (M, g) onto itself. For simplicity, we also assume that our space-time is equipped with a flat metric $\eta = \mathrm{diag}\{-1, +1, +1, \cdots\}$. Notice that the composition of two conformal transformations is also a conformal transformation. Therefore, the set of all conformal transformations on (M, η) forms a group, or more specifically, a Lie group, known as the conformal group.

2.1.2 Generators of conformal algebra

We first study the Lie algebra of this group by considering the infinitesimal transformations

$$x^\mu \mapsto x^\mu + \epsilon^\mu(x) + \mathcal{O}(\epsilon^2) \tag{2.3}$$

where $\epsilon \ll 1$. Plugging into the conformal condition (2.2), we obtain

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = (\Lambda - 1)\eta_{\mu\nu}. \tag{2.4}$$

This is known as the conformal Killing equation. When $\Lambda = 1$, this goes back to the usual Killing equation that describes the vector field preserving the metric.

The conformal Killing equation can be transformed to be independent of Λ. To do so, we take the trace of both sides to obtain

$$2\partial \cdot \epsilon = (\Lambda - 1)d \tag{2.5}$$

where d is the space-time dimension. Plugging back into (2.4), we obtain

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d}(\partial \cdot \epsilon)\eta_{\mu\nu}. \tag{2.6}$$

To solve this equation for $\epsilon^\mu(x)$, we can do the following manipulation. By acting $\partial_\rho \partial_\sigma$ on both sides of the equation and tracing over indices we obtain that for space-time dimension $d > 2$

$$\partial_\mu \partial_\nu (\partial \cdot \epsilon) = 0. \tag{2.7}$$

We can also act ∂_ρ on the equation and take permuting indices, which leads to

$$\partial_\mu \partial_\nu \epsilon_\rho = \frac{1}{d}(\eta_{\rho\mu}\partial_\nu + \eta_{\rho\nu}\partial_\mu - \eta_{\mu\nu}\partial_\rho)(\partial \cdot \epsilon). \tag{2.8}$$

From (2.7), we immediately conclude that $\partial \cdot \epsilon$ is at most linear in x^μ. That is to say, $\partial \cdot \epsilon = A + B_\mu x^\mu$. Plugging this linear form into the second equation (2.8), we can see that $\partial_\mu \partial_\nu \epsilon_\rho$ is a constant. Therefore, $\epsilon^\mu(x)$ is at most quadratic in x^μ, with its general form can be written as

$$\epsilon^\mu(x) = a^\mu + b^\mu{}_\nu x^\nu + c^\mu{}_\nu x^\nu + d^\mu{}_{\rho\sigma} x^\rho x^\sigma \quad \text{where} \quad \begin{cases} b_{\mu\nu} = b_{\nu\mu}, \\ c_{\mu\nu} = -c_{\nu\mu}, \\ d^\mu{}_{\rho\sigma} = d^\mu{}_{\rho\sigma}. \end{cases} \tag{2.9}$$

The constraints (2.6) and (2.8) further require that

$$b_{\mu\nu} = b\eta_{\mu\nu}, \quad d^\mu{}_{\rho\sigma} = \frac{1}{d}\left(\delta^\mu_\rho d^\lambda{}_{\lambda\sigma} + \delta^\mu_\sigma d^\lambda{}_{\lambda\rho} - \eta_{\rho\sigma}\eta^{\mu\nu} d^\lambda_{\lambda\nu}\right). \tag{2.10}$$

These $(a^\mu, b, c_{\mu\nu}, d^\mu{}_{\rho\sigma})$ are the infinitesimal generators of conformal transformations. We can immediately see that a^μ, b, $c_{\mu\nu}$ correspond to translations, dilation and rotations (or Lorentz boosts) respectively as finite transformations. It requires some calculation to see that $d^\mu{}_{\rho\sigma}$ corresponds to the infinitesimal transformation

$$x^\mu \mapsto x^\mu + 2x^\nu d^\lambda{}_{\lambda\nu} x^\mu - x_\rho x^\rho \eta^{\mu\nu} d^\sigma{}_{\sigma\nu} \tag{2.11}$$

and the finite transformation

$$x^\mu \mapsto \frac{x^\mu - x^2 b^\mu}{1 - 2(b\cdot x) + b^2 x^2} = \frac{x^\mu/x^2 - b^\mu}{\left(x^\mu/x^2 - b^\mu\right)^2}, \tag{2.12}$$

which is labeled by a Lorentz vector b^μ. This transformation is called the special conformal transformation (SCT). As illustrated, it can be considered as a composition of inversion, translation and another inversion. The 4 types of conformal transformations allowed in $d > 2$ and their generators in terms of differential operator representation are listed in the table 2-1.

Table 2-1 Global conformal transformations and generators

Transformation	Expression	Generator
Translation	$x^\mu \mapsto x^\mu + a^\mu$	$P_\mu = -i\partial_\mu$
Dilation	$x^\mu \mapsto \alpha x^\mu$	$D = -ix\cdot\partial$
Rotation (boost)	$x^\mu \mapsto M^\mu{}_\nu x^\nu$	$L_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu)$
SCT	$x^\mu \mapsto \frac{x^\mu - x^2 b^\mu}{1-2(b\cdot x)+b^2x^2}$	$K_\mu = -i\big(2x_\mu(x\cdot\partial) - x^2\partial_\mu\big)$

The conformal algebra generated by these generators is an extension to the Poincaré algebra spanned by P_μ and $L_{\mu\nu}$. With direct calculation, we can show that the commutators of the new generators D and K_μ are given by

$$\begin{aligned} [D, P_\mu] &= iP_\mu, \\ [D, K_\mu] &= -iK_\mu, \\ [K_\mu, P_\nu] &= 2i(\eta_{\mu\nu}D - L_{\mu\nu}), \\ [K_\mu, L_{\rho\sigma}] &= i(\eta_{\mu\rho}K_\sigma - \eta_{\mu\sigma}K_\rho). \end{aligned} \tag{2.13}$$

It can be proved using the embedding formalism that the conformal algebra of $\mathbb{R}^{1,d-1}$ is isomorphic to $\mathfrak{so}(2,d)$. Furthermore, the conformal group is isomorphic to $\mathrm{SO}(2,d)$.

2.2 Field with conformal symmetry

CFTs are field theories that respect conformal symmetry. As a special and important class of symmetry, the transformation laws of fields under conformal symmetry are discussed in this section. We introduce the crucial conservation current stress-energy tensor in CFT.

2.2.1 Noether's theorem

In a CFT, the fields also need to transform under the conformal transformation

$$\Phi(x) \mapsto \mathcal{F}[\Phi(x)]. \tag{2.14}$$

Consider again the infinitesimal transformation (2.3) and the corresponding transformation of a field

$$\Phi(x) \mapsto \Phi'(x') = \Phi(x) + \epsilon\frac{\delta\mathcal{F}}{\delta\epsilon}. \tag{2.15}$$

Generally, we assume that a field lives in a certain representation of the conformal group and conformal algebra. This representation can be denoted as

$$\delta_\epsilon \Phi(x) = \Phi'(x) - \Phi(x) = -i\epsilon G\Phi(x), \tag{2.16}$$

where G is an element of conformal algebra. Notice that in the infinitesimal transformation (2.15), the change of field not only comes from the representation of conformal algebra, but also contains the contribution from the change of coordinate. Therefore we can write the differential operator representation of conformal algebra as

$$iG\Phi(x) = \frac{\delta x^\mu}{\delta\epsilon}\partial_\mu\Phi - \frac{\delta\mathcal{F}}{\delta\epsilon}. \tag{2.17}$$

For a theory invariant under conformal transformation, the action transforms as

$$\begin{aligned} S \mapsto S' &= \int \mathrm{d}^d x' \, \mathcal{L}(\Phi', \partial'\Phi') \\ &= \int \mathrm{d}^d x \left|\frac{\partial x'}{\partial x}\right| \mathcal{L}(\mathcal{F}[\Phi], \frac{\partial x}{\partial x'}\partial\mathcal{F}[\Phi]). \end{aligned} \tag{2.18}$$

Plugging in the relation (2.17) for infinitesimal transformation of the field, we can explicitly calculate the variation of action as

$$\delta_\epsilon S = -\int \mathrm{d}^d x \, j^\mu \partial_\mu \epsilon = \int \mathrm{d}^d x \, \epsilon \partial_\mu j^\mu, \tag{2.19}$$

where j^μ is the conservation current of this transformation, given by

$$j^\mu = \left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi}\partial_\nu\Phi - \mathcal{L}\delta^\mu_\nu\right)\frac{\delta x^\nu}{\delta\epsilon} - \frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi}\frac{\delta\mathcal{F}}{\delta\epsilon}. \tag{2.20}$$

Since we want the theory to be conformal invariant, the variation of action should vanish for any ϵ. Therefore we obtain the conservation law

$$\partial_\mu j^\mu = 0 \tag{2.21}$$

as the Noether's theorem.

In particular, we can define the Noether current of coordinate transformation $x^\mu \mapsto x^\mu + \epsilon^\mu$ as the stress-energy tensor $T^{\mu\nu}$

$$\delta S = \int \mathrm{d}^d x \, T^{\mu\nu}\partial_\mu \epsilon_\nu. \tag{2.22}$$

Normally, we symmetrize the indices by adding an additional term $\partial_\rho B^{\rho\mu\nu}$. The Noether's theorem in this case implies that $\partial_\mu T^{\mu\nu} = 0$. One important property of the stress-energy tensor in CFT is that the conservation of Noether current implies that $T^{\mu\nu}$ is traceless

$$\begin{aligned} \partial_\mu j^\mu &= (\partial_\mu T^{\mu\nu})\epsilon_\nu + T^{\mu\nu}\partial_\mu\epsilon_\nu \\ &= \frac{1}{2}T^{\mu\nu}(\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu) \overset{(2.6)}{=} \frac{1}{d}T^\mu{}_\mu(\partial\cdot\epsilon) = 0. \end{aligned} \tag{2.23}$$

2.2.2 Ward identity

As a quantum version of the Noether's theorem, we want to consider the symmetry action of an operator in a correlation function. This can be done by considering the path integral expression for correlators

$$\langle \Phi(x_1)\Phi(x_2)\cdots\Phi(x_n)\rangle = \int [\mathcal{D}\Phi]\, \Phi(x_1)\Phi(x_2)\cdots\Phi(x_n)e^{-S[\Phi]}. \tag{2.24}$$

Consider the infinitesimal transformation

$$\langle \Phi(x_1)\cdots\Phi(x_n)\rangle' = \int [\mathcal{D}\Phi']\, \Phi'(x_1)\cdots\Phi'(x_n)\exp\Big(-S[\Phi] - \int \mathrm{d}^d x\, \epsilon\partial_\mu j^\mu\Big). \tag{2.25}$$

Assuming that the path integral measure being invariant under the symmetry action, we can expand the above expression to the first order of ϵ to obtain

$$\langle \delta_\epsilon\Phi(x_1)\cdots\Phi(x_n)\rangle = \int \mathrm{d}^d x\, \partial_\mu \langle j^\mu(x)\Phi(x_1)\cdots\Phi(x_n)\rangle\, \epsilon. \tag{2.26}$$

On the other hand, we can explicitly write $\delta_\epsilon\Phi(x_1)\cdots\Phi(x_n)$ using (2.16) as

$$\begin{aligned}\delta_\epsilon\Phi(x_1)\cdots\Phi(x_n) &= -i\sum_{i=1}^n \epsilon(x_i)\Phi(x_1)\cdots G\Phi(x_i)\cdots\Phi(x_n)\\ &= -i\int \mathrm{d}^d x\, \epsilon(x)\sum_{i=1}^n \Phi(x_1)\cdots G\Phi(x)\cdots\Phi(x_n)\delta(x-x_i).\end{aligned} \tag{2.27}$$

Since the above argument should hold for arbitary $\epsilon(x)$, we can conclude that

$$\frac{\partial}{\partial x^\mu}\langle j^\mu(x)\Phi(x_1)\cdots\Phi(x_n)\rangle = -i\sum_{i=1}^n \delta(x-x_i)\langle \Phi(x_1)\cdots G\Phi(x_i)\cdots\Phi(x_n)\rangle\,. \tag{2.28}$$

This is known as the Ward identity.

Recall that we can define a conservation charge

$$Q(t) = \int_{x^0=t} \mathrm{d}^{d-1}x\, j^0(x) \tag{2.29}$$

for a conservation current. By integrating over two thin time slice $t_- < x_i^0 < t_+$ sandwiching a particular operator and recall the time-ordering property of correlators, we obtain

$$[Q,\Phi] = -iG\Phi. \tag{2.30}$$

This shows that the adjoint action of conservation charge generates the corresponding symmetry action.

2.3 Conformal symmetry in 2-dimension

As we claimed above, the conformal condition in $d = 2$ is different from $d > 2$. By taking a ∂^ν on both sides of (2.6), the condition for ϵ becomes

$$\partial_\mu\epsilon^\mu = 0. \tag{2.31}$$

We can use the complex coordinate $(z, \bar{z})$ with metric

$$\mathrm{d}s^2 = (\mathrm{d}x^0)^2 + (\mathrm{d}x^1)^2 = \mathrm{d}z\,\mathrm{d}\bar{z} \tag{2.32}$$

to rewrite the condition as

$$\bar{\partial}\epsilon = 0, \quad \partial\bar{\epsilon} = 0. \tag{2.33}$$

This means that $\epsilon = \epsilon(z)$ and $\bar{\epsilon} = \bar{\epsilon}(\bar{z})$ are holomorphic and anti-holomorphic functions of the coordinates. This implies that any holomorphic function $z \mapsto w(z)$ is a conformal transformation in 2-dimension. Hence, the conformal group in 2-dimension turns out to be infinite-dimensional. This phenomenon greatly enhanced our ability to use symmetry arguments to analyze the behavior of 2-dimensional CFTs.

2.3.1 Witt algebra and global conformal transformations

Since the conformal transformation is generated by a holomorphic function, we can write the general infinitesimal transformation in terms of Laurent expansion at $z \mapsto z$

$$z \mapsto z + \epsilon(z) = z - \sum_{n\in\mathbb{Z}} \epsilon_n z^{n+1}. \tag{2.34}$$

The infinite many parameters ϵ_n corresponds to infinite many generators

$$l_n = -z^{n+1}\partial, \quad \bar{l}_n = -\bar{z}^{n+1}\bar{\partial}. \tag{2.35}$$

These generators form two copies of the so called Witt algebra

$$[l_n, l_m] = (n-m)l_{n+m}, \quad [\bar{l}_n, \bar{l}_m] = (n-m)\bar{l}_{n+m}. \tag{2.36}$$

This is the classical conformal algebra spanned by local conformal transformations in 2-dimension.

Here we emphasize the fact that these are local conformal transformations, as global transformations need to be well-defined and invertible on the Riemann sphere $\mathbb{C} \cup \{\infty\}$. By observing the behavior of l_n at 0 and ∞

$$l_n = -z^{n+1}\partial_z = w^{-n+1}\partial_w \quad \text{where} \quad w = \frac{1}{z}, \tag{2.37}$$

we can conclude that the only compatible global transformations are generated by l_{-1}, l_0 and l_1. It is easy to check that these generators indeed span a closed $\mathfrak{sl}(2,\mathbb{C})$ subalgebra of the Witt algebra.

To understand the explicit transformations generated, we can examine the differential operator representation. $l_{-1} = -\partial_z$ has the same form as the translation generator and indeed generates translation $z \mapsto z + a$. $l_0 = -z\partial_z$ generates the linear multiplication $z \mapsto az$, which can be decomposed into the dilation coming from $|a|$ and the rotation from $\arg a$. The explicit generator for dilation and rotation are

$$l_0 + \bar{l}_0 = -r\partial_r, \quad i(l_0 - \bar{l}_0) = -\partial_\theta, \tag{2.38}$$

where $z = re^{i\theta}$. Finally, $l_1 = -z^2\partial_z = \partial_w$ for $w = z^{-1}$ generates a translation for the inversion of z. Therefore it is the generator for SCT. The general global transformation can be expressed in terms of

$$z \mapsto \frac{az+b}{cz+d}, \tag{2.39}$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc = 1$. We can use a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to label each transformation. In this way, the matrix group is homomorphic to the group of transformation. Notice that flipping the sign for a, b, c, d altogether results in the same transformation. Therefore, the group of global conformal transformation is the Möbius group $\mathrm{SL}(2, \mathbb{C})/\mathbb{Z}_2$.

2.3.2 Primary operators

From the algebra relation (2.36), we can observe that l_0 and $\bar{l}_0$ form the Cartan subalgebra of the conformal algebra. Therefore, we can label fields in different representation of conformal algebra by their eigenvalues under l_0 and $\bar{l}_0$ actions. Namely, we consider the fields that transforms under dilation $z \mapsto \alpha z$ as

$$\Phi(z, \bar{z}) \mapsto \alpha^{-\Delta} \bar{\alpha}^{-\bar{\Delta}} \Phi(z, \bar{z}). \tag{2.40}$$

The eigenvalues Δ and $\bar{\Delta}$ are called the conformal dimensions of the field. If a field transforms under any global transformation $z \mapsto w(z)$ as

$$\Phi(z, \bar{z}) \mapsto \left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^{-\Delta} \left(\frac{\mathrm{d}\bar{w}}{\mathrm{d}\bar{z}}\right)^{-\bar{\Delta}} \Phi(z, \bar{z}), \tag{2.41}$$

then it is called a quasi-primary field. Further more, if under any conformal transformation, either local or global, the field transforms as (2.41), then the field is called a primary field.

One important property of CFT is that many of its correlators of quasi-primary fields can be determined simply from symmetry requirement. In the following paragraphs, we demonstrate with only chiral primaries $\phi = \phi(z)$ for simplicity.

The simplest case is the one-point functions. From translation and dilation symmetries, any one-point function of quasi-primary field has to vanish

$$\langle \phi(z) \rangle = 0. \tag{2.42}$$

For two-point functions, translation symmetry requires the result to only depend on the difference of position of the operators

$$\langle \phi_1(z) \phi_2(w) \rangle = f(z - w). \tag{2.43}$$

Then, by the requirement of global transformations $z \mapsto \alpha z$ and $z \mapsto -z^{-1}$, the two-point function can be determined up to a constant

$$\langle \phi_1(z) \phi_2(w) \rangle = \frac{c_{12}}{(z-w)^{2\Delta}} \delta_{\Delta_1, \Delta_2} \quad \text{where} \quad \Delta = \Delta_1 = \Delta_2. \tag{2.44}$$

The constants c_{ij} can be fixed by the normalization of fields.

The three-point functions are more difficult. From symmetry argument, they can be determined at most up to

$$\langle\phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\rangle = \frac{C_{123}}{z_{12}^{\Delta_1+\Delta_2-\Delta_3} z_{23}^{\Delta_2+\Delta_3-\Delta_1} z_{31}^{\Delta_3+\Delta_1-\Delta_2}}, \tag{2.45}$$

where $z_{ij} = z_i - z_j$. Here the coefficients C_{ijk} can no longer be fixed by symmetry only. They generally depend on the particular theory. For a 2-dimensional CFT, the conformal dimensions of fields plus the three-point function coefficients contain all the information needed to fully determine the theory.

2.3.3 OPE and Virasoro algebra

In 2-dimension, the traceless property of stress-energy tensor (2.23) still holds. This implies $T_{z\bar{z}} = T_{\bar{z}z} = 0$. The conservation law on the remaining components becomes the (anti-)holomorphic condition

$$\bar{\partial}T_{zz} = \partial T_{\bar{z}\bar{z}} = 0. \tag{2.46}$$

We denote these two components as

$$T(z) = T_{zz}(z), \quad \bar{T}(z) = T_{\bar{z}\bar{z}}(\bar{z}). \tag{2.47}$$

Plugging the Ward identity in 2-dimensional CFT case yields

$$\delta_{\epsilon,\bar{\epsilon}}\langle\Phi\rangle = -\oint_C \frac{\mathrm{d}z}{2\pi i}\epsilon\langle T(z)\Phi\rangle + \oint_C \frac{\mathrm{d}\bar{z}}{2\pi i}\bar{\epsilon}\langle\bar{T}(\bar{z})\Phi\rangle. \tag{2.48}$$

For a primary field, we can explicitly compute $\delta_{\epsilon,\bar{\epsilon}}\Phi$. Then, from (2.48) we can obtain

$$\langle T(z)\Phi(w)\rangle = \frac{1}{z-w}\partial_w\langle\Phi(w)\rangle + \frac{\Delta}{(z-w)^2}\Phi(w) + \text{regular terms}. \tag{2.49}$$

We can conclude this in the so-called operator product expansion (OPE) form

$$T(z)\Phi(w) = \frac{\Delta\Phi(w)}{(z-w)^2} + \frac{\partial\Phi(w)}{z-w} + \cdots. \tag{2.50}$$

In CFT, products of two local operators can always be written as a sum of single operators in a power series of $(z-w)$. This is a useful tool to define the spectrum of operators in CFT. An important OPE is the TT OPE, which generally of the form

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \cdots, \tag{2.51}$$

where c is a constant called the central charge. This is a label of the theory.

Define the mode expansion of stress-energy tensor as

$$T(z) = \sum_{k\in\mathbb{Z}} L_k z^{-k-2}. \tag{2.52}$$

Equivalently

$$L_n = \oint_C \frac{\mathrm{d}z}{2\pi i} z^{n+1} T(z). \tag{2.53}$$

From the TT OPE, we can compute the commutator of these modes as

$$[L_n, L_m] = \oint_{C_0} \frac{\mathrm{d}w}{2\pi i} w^{m+1} \oint_{C_w} \frac{\mathrm{d}z}{2\pi i} z^{n+1} T(z)T(w). \tag{2.54}$$

The result is an infinite-dimensional Lie algebra called the Virasoro algebra

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0}. \tag{2.55}$$

The Virasoro algebra is the central extension of Witt algebra. It can be considered as the quantum version of the generators of 2-dimensional conformal transformations. Recall the Ward identity (2.28) with the conservation charge

$$Q = \oint_C \frac{\mathrm{d}z}{2\pi i} \epsilon(z) T(z). \tag{2.56}$$

Notice that after expansion

$$\epsilon(z) = \sum_{n\in\mathbb{Z}} \epsilon_n z^{n+1}, \tag{2.57}$$

the charge becomes

$$Q = \sum_{n\in\mathbb{Z}} \epsilon_n L_n \tag{2.58}$$

and then the Virasoro generators indeed generate the corresponding transformations

$$\delta_{\epsilon_n} \Phi = -[L_n, \Phi]. \tag{2.59}$$

2.4 Superconformal symmetry in 4-dimension

The superconformal symmetry, as the name suggests is the combination of conformal symmetry and supersymmetry. As two types of symmetries that both go beyond the limitation of Coleman-Mandula theorem[30], conformal symmetry bypasses the no-go theorem by having massless particles, while supersymmetry includes graded Lie algebra to overcome the constraint.

2.4.1 Representation of conformal algebra

In section 2.1 and 2.2, we have introduced the conformal algebra in general spacetime dimension $d > 2$. Here we briefly discuss the representation theory of this algebra in 4-dimension[31].

It is useful to decompose $\mathbb{R}^4$ into radial directions $S^3 \times \mathbb{R}$. The rotations $\mathrm{SO}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2)$ act on S^3, and the dilation acts on $\mathbb{R}$. From the algebra relation (2.13), we can observe that the Cartan subalgebra is spanned by $\{D, J_L^3, J_R^3\}$. The translations P_μ and SCTs K_μ serve as ladder operators of D. Therefore, we can label a vector in the representation by the set of eigenvalues $\{\Delta, j, \bar{\jmath}\}$.

A highest weight vector $|\Delta; j, \bar{\jmath}\rangle$ satisfies

$$D\,|\Delta; j, \bar{\jmath}\rangle = \Delta\,|\Delta; j, \bar{\jmath}\rangle\,, \quad K_\mu\,|\Delta; j, \bar{\jmath}\rangle = 0. \tag{2.60}$$

In the context of state-operator correspondence, the corresponding field $\Phi^{\Delta}_{(j,\bar{\jmath})}$ is called a primary field. In $d > 2$, there is no difference between quasi-primaries and primaries as all conformal transformations are global. A multiplet generated by this primary field is

$$\mathcal{A}^{\Delta}_{(j,\bar{\jmath})} = \mathrm{span}\left\{ \left(\prod_i P_{\mu_i} \right) \cdot \Phi^{\Delta}_{(j,\bar{\jmath})} \right\}. \tag{2.61}$$

The unitarity requires that any vector in the representation satisfies $\langle \psi | \psi \rangle \geqslant 0$. At first and second descendant level $P_\mu \Phi^{\Delta}_{(j,\bar{\jmath})}$, the unitary constraints are

$$\Delta - j - 1 - \delta_{j,0} - \bar{\jmath} - 1 - \delta_{\bar{\jmath},0} \equiv \Delta - f(j,\bar{\jmath}) \geqslant 0, \quad \Delta(\Delta - 1) \geqslant 0. \tag{2.62}$$

The second constraint demand that $\Delta = 0$ or $\Delta \geqslant 1$, which brings a gap for conformal dimensions. $\Delta = 0$ corresponds to the identity operator, and $\Delta = 1$ corresponds to the free scalar that $\partial^2 \phi = 0$.

For a primary field that saturates the BPS bound $\Delta = f(j,\bar{\jmath})$, the multiplet generated is shorter as we need to remove the null descendant states. These multiplets are called short multiplets. This condition is therefore named the shortening condition.

2.4.2 Basic supersymmetry

Supersymmetry describes a fermionic type symmetry between bosonic and fermionic fields. To achieve this, we have to extend the concept of Lie algebra to graded Lie algebra[32]. Namely, consider a vector space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with a grade function

$$\begin{aligned} \eta: \quad & \mathfrak{g}_0 \cup \mathfrak{g}_1 \to \{0,1\} \\ & v \mapsto \begin{cases} 0 & v \in \mathfrak{g}_0 \\ 1 & v \in \mathfrak{g}_1. \end{cases} \end{aligned} \tag{2.63}$$

A graded Lie algebra is a graded vector space equipped with a graded Lie bracket $[\bullet, \bullet\}$ such that

$$[x, y\} = (-1)^{\eta(x)\eta(y)+1}[y, x\}. \tag{2.64}$$

From a physical point of view, those with grade 0 are bosonic and those with grade 1 are fermionic. The graded Lie bracket is then defined as an anti-commutator between two fermionic elements and a commutator otherwise.

The supersymmetry algebra is a graded Lie algebra whose bosonic subalgebra is just the Poincaré algebra plus any internal symmetries

$$\mathfrak{g} = \mathfrak{Poin} \oplus \mathfrak{Int} \oplus \mathfrak{g}_1, \tag{2.65}$$

where the fermionic subalgebra $\mathfrak{g}_1$ is spanned by the generators of supersymmetry $\{Q^A_\alpha, \bar{Q}_{\dot{\alpha},B}\}$ called supercharges. They have a spinor index $\alpha, \dot{\alpha}$ and another index A, B that runs from 1 to $\mathcal{N}$ known as the R-symmetry index.

In the simplest case, the supersymmetry algebra takes the form[33]

$$\begin{aligned} \{Q^A_\alpha, \bar{Q}_{\dot{\alpha},B}\} &= 2\sigma^\mu_{\alpha\dot{\alpha}} P_\mu \delta^A_B, \\ \{Q^A_\alpha, Q^B_\beta\} &= \{\bar{Q}_{\dot{\alpha},A}, \bar{Q}_{\dot{\beta},B}\} = 0, \\ [P_\mu, Q^A_\alpha] &= [P_\mu, \bar{Q}_{\dot{\alpha},A}] = 0, \end{aligned} \tag{2.66}$$

with the rest of the commutators being the ordinary Poincaré algebra. The reality condition of the supercharges is

$$\left(Q_\alpha^A\right)^\dagger = \bar{Q}_{\dot{\alpha},A}. \tag{2.67}$$

We can add a central charge to the algebra as

$$\{Q_\alpha^A, Q_\beta^B\} = \epsilon_{\alpha\beta} Z^{AB}, \tag{2.68}$$

where

$$Z^{AB} = \begin{pmatrix} 0 & Z \\ -Z & 0 \end{pmatrix}, \quad Z = \mathrm{diag}\{Z_1, Z_2, \cdots\}. \tag{2.69}$$

In the case of $\mathcal{N} = 2$, the central charge is just

$$Z^{AB} = \epsilon^{AB} Z. \tag{2.70}$$

2.4.3 Representation of supersymmetry algebra

In the case $Z = 0$, we may consider the creation and annihilation operators as

$$\begin{aligned} a_\alpha^A &= \frac{1}{\sqrt{2m}} Q_\alpha^A, \\ \left(a_\alpha^A\right)^\dagger &= \frac{1}{\sqrt{2m}} \bar{Q}_{\dot{\alpha},A}, \end{aligned} \quad \text{or} \quad \begin{aligned} a^A &= \frac{1}{2\sqrt{E}} Q_1^A, \\ \left(a^A\right)^\dagger &= \frac{1}{2\sqrt{E}} \bar{Q}_{\dot{1},A}, \end{aligned} \tag{2.71}$$

depending on whether the multiplet is massive or massless. Since the supercharges are fermionic, the dimension of the multiplets are given by $2^{2\mathcal{N}}$ and $2^{\mathcal{N}}$ respectively.

In the presence of central charge Z at $\mathcal{N} = 2$, we can construct creation and annihilation operators as

$$a_\alpha = \frac{1}{\sqrt{2}}\Big(Q_\alpha^1 + \epsilon_{\alpha\beta}\big(Q_\beta^2\big)^\dagger\Big), \quad b_\alpha = \frac{1}{\sqrt{2}}\Big(Q_\alpha^1 - \epsilon_{\alpha\beta}\big(Q_\beta^2\big)^\dagger\Big) \tag{2.72}$$

as well as their conjugates $(a_\alpha)^\dagger$ and $(b_\alpha)^\dagger$. The commutation relation satisfies

$$\{a_\alpha, (a_\beta)^\dagger\} = \delta_\alpha^\beta(2m + Z), \quad \{b_\alpha, (b_\beta)^\dagger\} = \delta_\alpha^\beta(2m - Z). \tag{2.73}$$

This gives the constraint $Z \leqslant 2m$. In particular, we see that when $Z = 2m$, the states generated by b_α become null states. This is the BPS condition for $\mathcal{N} = 2$ supersymmetry.

Two useful multiplets in the $\mathcal{N} = 2$ supersymmetry are vector multiplet and hypermultiplet[34]. The vector multiplet $V_{\mathcal{N}=2}$ contains one complex scalar, two Weyl fermions known as gaugino and one 2-form field strength

$$V_{\mathcal{N}=2} = \{\phi, \lambda_\alpha, \bar{\lambda}_{\dot{\alpha}}, F_{\mu\nu}\}, \tag{2.74}$$

which are all in the adjoint representation of the gauge group G. The hypermultiplet $H_{\mathcal{N}=2}$ contains two complex scalars and two Weyl fermions

$$H_{\mathcal{N}=2} = \big\{q, \bar{q}, \psi_\alpha, \bar{\psi}_{\dot{\alpha}}\big\}. \tag{2.75}$$

2.4.4 Superconformal algebra

In the supersymmetry algebra (2.66), the supercharges Q may be considered as a square root of momentum P

$$Q^2 \sim P.$$

We observe that in the conformal algebra (2.13), the SCT generators K are at a similar position as P. It is natural to include another generator S as square root of K. The explicit algebra relation is given by

$$\begin{aligned} \{Q^A_\alpha, \bar{Q}_{\dot\alpha,B}\} &= 2P_\mu \sigma^\mu_{\alpha\dot\alpha}\delta^A_B, & \{S^\alpha_A, \bar{S}^{\dot\alpha,B}\} &= 2K_\mu (\bar\sigma^\mu)^{\dot\alpha\alpha}\delta^B_A, \\ \{Q^A_\alpha, Q^B_\beta\} &= 0, & \{S^\alpha_A, S^\beta_B\} &= 0. \end{aligned} \tag{2.76}$$

The anti-commutators between Q and S are

$$\begin{aligned} \{Q^A_\alpha, S^\beta_B\} &= 4\delta^A_B L_{\mu\nu}(\sigma^{\mu\nu})_\alpha{}^\beta - 2\delta^A_B \delta^\beta_\alpha D - 4\delta^\beta_\alpha R^A{}_B, \\ \{Q^A_\alpha, \bar{S}^{\dot\alpha,B}\} &= \{\bar{Q}_{\dot\alpha,A}, S^\alpha_B\} = 0, \end{aligned} \tag{2.77}$$

where $R^A{}_B$ is the R-symmetry generator. Finally, it is natural to have

$$\begin{aligned} [D, Q^A_\alpha] &= \frac{1}{2}Q^A_\alpha, & [D, S^\alpha_A] &= -\frac{1}{2}S^\alpha_A, \\ [D, \bar{Q}_{\dot\alpha,A}] &= \frac{1}{2}\bar{Q}_{\dot\alpha,A}, & [D, \bar{S}^{\dot\alpha,A}] &= -\frac{1}{2}\bar{S}^{\dot\alpha,A}. \end{aligned} \tag{2.78}$$

The R-symmetry for 4-dimensional $\mathcal{N} = 2$ superconformal symmetry is [31,34–36]

$$\mathrm{SU}(2)_R \times \mathrm{U}(1)_r. \tag{2.79}$$

Then, a general multiplet of superconformal algebra is labeled by $(\Delta, R, r, j, \bar{j})$ where R and r are charges of the corresponding R-symmetry. This long multiplet is denoted by $\mathcal{A}^\Delta_{R,r,(j,\bar{j})}$. The highest weight is annihilated by all the S. Such a field $\Phi^\Delta_{R,r,(j,\bar{j})}$ is called a superconformal primary. According to (2.76), it is also a conformal primary as it is annihilated by K.

When some combination of Qs annihilate the primary, the multiplet is shortened. This short multiplet is denoted as $\chi_{R,r,(j,\bar{j})}$. Depending on the specific shortening condition, there are $\frac{1}{2}$-BPS, $\frac{1}{4}$-BPS and $\frac{1}{8}$-BPS types, where the prefix indicates how many supercharges annihilate the highest weight.

$\mathcal{B}^A = \mathcal{B}^A_{R,r,(0,\bar{j})}$ denotes the case when $Q^A_\alpha \Phi^\Delta_{R,r,(j,\bar{j})} = 0$ for both $\alpha = \pm$. This is a $\frac{1}{4}$-BPS shortening. The condition is

$$\Delta = 2R + r, \quad j = 0. \tag{2.80}$$

Its conjugate is $\bar{\mathcal{B}}_A$ representing $\bar{Q}_{\dot\alpha,A}\Phi^\Delta_{R,r,(j,\bar{j})} = 0$. The condition becomes

$$\Delta = 2R - r, \quad \bar{j} = 0. \tag{2.81}$$

$\mathcal{C}^A = \mathcal{C}^A_{R,r,(j,\bar{j})}$ denotes the case when $\epsilon^{\alpha\beta}Q^A_\alpha\left(\Phi^\Delta_{R,r,(j,\bar{j})}\right)_\beta = 0$ or at $j = 0$ the condition being $(Q^A)^2\Phi^\Delta_{R,r,(0,\bar{j})} = 0$. This is a $\frac{1}{8}$-BPS shortening. The condition is

$$\Delta = 2 + 2j + 2R + r. \tag{2.82}$$

Combining some of these conditions gives $\frac{1}{2}$-BPS conditions. The Coulomb branch multiplet is a combination of $\mathcal{B}^1$ and $\mathcal{B}_2$. We denote it by $\mathcal{E}_r$. The condition is

$$R = j = \bar{\jmath} = 0, \quad \Delta = r. \tag{2.83}$$

The Higgs branch multiplet is a combination of $\mathcal{B}^1$ and $\bar{\mathcal{B}}_2$. We denote it by $\hat{\mathcal{B}}_R$. The condition is

$$r = j = \bar{\jmath} = 0, \quad \Delta = 2R. \tag{2.84}$$

In both cases, the superconformal primaries are scalars. Finally, we can consider a combination of $\mathcal{C}_1$ and $\bar{\mathcal{C}}_2$ denoted by $\hat{\mathcal{C}}_{(j,\bar{\jmath})}$. In the case when $j = \bar{\jmath} = 0$, the multiplet is called the stress-energy tensor multiplet. The condition is

$$R = r = 0, \quad \Delta = 2 + j + \bar{\jmath}. \tag{2.85}$$

Chapter 3

Alday-Gaiotto-Tachikawa duality

In this chapter, we introduce the concept of Alday-Gaiotto-Tachikawa duality and relevant background information. We first introduce the 4-dimensional SCFTs with $\mathcal{N}=2$ supersymmetry.

3.1 4d $\mathcal{N}=2$ SQFTs and SCFTs

3.1.1 General $\mathcal{N}=2$ Lagrangians

The supersymmetry in 4-dimension puts strong restrictions on the form of Lagrangian allowed for SQFT. Schematically, the Lagrangian should consist of[34]

$$\mathcal{L}=\mathcal{L}_{\mathrm{SYM}}+\mathcal{L}_{\mathrm{matter}}. \tag{3.1}$$

The first term contains contribution from the vector multiplet and the second term contains the hypermultiplet and its coupling to the vector multiplet.

It is convenient to employ the superfield formalism from $\mathcal{N}=1$ supersymmetric theories[32–33]. The 4-dimension $\mathcal{N}=1$ theory has a chiral superfield $\bar{D}_{\dot{\alpha}}Q=0$ with R charge $r_{\mathcal{N}=1}$ and a real vector superfield $V=V^{\dagger}$. They are given by

$$\begin{aligned}
&Q=q+i\psi\theta+F\theta\theta,\\
&V=-i\theta\sigma^{\mu}\bar{\theta}A_{\mu}+i\theta\theta\bar{\theta}\bar{\lambda}-i\bar{\theta}\bar{\theta}\theta\lambda+\frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D.
\end{aligned} \tag{3.2}$$

We can show that the $\mathcal{N}=2$ multiplets can be considered as combinations of $\mathcal{N}=1$ multiplets. In particular

$$V_{\mathcal{N}=2}\sim(V,\Phi),\quad H_{\mathcal{N}=2}\sim(Q,\bar{Q}), \tag{3.3}$$

where $r_{\mathcal{N}=1}=2/3$. We use Φ to denote the $\mathcal{N}=1$ chiral superfield in $\mathcal{N}=2$ vector multiplet. To express the supersymmetric Yang-Mills term $\mathcal{L}_{\mathrm{SYM}}$, it is convenient to write a gaugino superfield

$$W_{\alpha}=-i\lambda_{\alpha}+iF_{\mu\nu}(\sigma^{\mu\nu}\theta)_{\alpha}+D\theta_{\alpha}+(\theta\theta)\big(\sigma^{\mu}\partial_{\mu}\bar{\lambda}_{\alpha}\big). \tag{3.4}$$

It is related to V by

$$W_{\alpha}=-\frac{1}{4}\bar{D}\bar{D}D_{\alpha}V. \tag{3.5}$$

The Lagrangian can be constructed as

$$\mathcal{L}_{\mathrm{SYM}}=\frac{1}{8\pi i}\int\mathrm{d}^2\theta\,\mathrm{tr}\,(\tau W_{\alpha}W^{\alpha}+\mathrm{c.c.})+\frac{\mathrm{Im}\,\tau}{4\pi}\int\mathrm{d}^4\theta\,\mathrm{tr}\,(\Phi^{\dagger}e^{\mathrm{ad}_V}\Phi). \tag{3.6}$$

The gauge coupling τ is given by

$$\tau = \frac{\theta_{\mathrm{YM}}}{2\pi} + 4\pi i \frac{1}{g^2}. \tag{3.7}$$

For the matter field Lagrangian, there is generally kinetic terms for Q and $\bar{Q}$ with a minimal coupling with V

$$\mathcal{L}_{\mathrm{matter}} = \int \mathrm{d}^4\theta \left(Q^\dagger e^{\rho(V)} Q + \bar{Q}^\dagger e^{\bar{\rho}(V)} \bar{Q}\right) + \int \mathrm{d}^2\theta \, \bar{Q}\rho(\varPhi) Q + \text{c.c.}, \tag{3.8}$$

where ρ is some representation of gauge group G.

3.1.2 Vacuum moduli space

Before discussing the quantum behavior of the supersymmetric theory, we consider the classical vacuum moduli space. This is obtained from minimizing the scalar potential

$$V(\varPhi, q, \bar{q}) = \frac{1}{2} \operatorname{tr} D^2 + F\bar{F}, \tag{3.9}$$

where D and F are auxiliary fields in the vector and hypermultiplet respectively. We can integrate out them in the path integral to obtain a constraint on the classical value of the fields, known as the D-term and F-term equations[37]. The classical vacuum moduli space is

$$\mathcal{M}_{\mathrm{cls}} = \{\varPhi, q, \bar{q} | V(\varPhi, q, \bar{q}) = 0\} = \{\varPhi, q, \bar{q} | D = F = 0\}. \tag{3.10}$$

In the theory with gauge group $G = \mathrm{SU}(N)$ and N_f fundamental hypers $H^A_{\mathcal{N}=2}$ for $A = 1, 2, \cdots, N_f$, the D-term and F-term equations are

$$\begin{aligned} D &= \frac{1}{g^2}[\varPhi, \varPhi^\dagger] + \left(q_A (q^A)^\dagger - (\bar{q}_A)^\dagger \bar{q}^A\right)\Big|_{\text{traceless}} = 0 \\ F_\varPhi &= q_A \bar{q}^A\Big|_{\text{traceless}} = 0 \\ F_q &= \varPhi q_A = 0 \\ F_{\bar{q}} &= \bar{q}_A \varPhi = 0. \end{aligned} \tag{3.11}$$

The solution of the equations generally has three branch-like structures, categorized by how the R-symmetry is broken.

The Coulomb branch is where the hypermultiplet scalars q^A and $\bar{q}^A$ have vanishing VEVs. In this case, the only constraint is

$$[\varPhi, \varPhi^\dagger] = 0, \tag{3.12}$$

which requires the scalar of vector multiplet $\varPhi$ to be in the Cartan subalgebra of the gauge algebra $\mathfrak{g}$ up to gauge transformations. Therefore

$$\dim_{\mathbb{C}} \mathcal{M}_{\mathrm{CB}} = \operatorname{rank} \mathfrak{g}. \tag{3.13}$$

The Coulomb branch is parameterized by the VEV of Coulomb branch multiplet scalars. Since these scalars carries non-trivial $\mathrm{U}(1)_r$ charges, Coulomb branch breaks the $\mathrm{U}(1)_r$ R-symmetry and preserves the $\mathrm{SU}(2)_R$.

The Higgs branch is where the scalars of vector multiplet Φ has vanishing VEV. In this case, the remaining constraints are

$$\left(q_A(q^A)^\dagger - (\bar{q}_A)^\dagger \bar{q}^A\right)\Big|_{\text{traceless}} = q_A \bar{q}^A\Big|_{\text{traceless}} = 0 \tag{3.14}$$

modding out gauge transformations. The resulting Higgs branch is a hyper-Kähler manifold. The complex dimension is

$$\dim_{\mathbb{C}} \mathcal{M}_{\mathrm{HB}} = 2(n_H - n_V), \tag{3.15}$$

where n_H and n_V are the number of hypermultiplets and vector multiplets involved in the Higgsing. The Higgs branch is parameterized by the VEV of Higgs branch multiplet scalars. It preserves $\mathrm{U}(1)_r$ and breaks $\mathrm{SU}(2)_R$ R-symmetry. The Higgs branch generally receives no quantum correction.

There is also the mixed branch where both VEVs are non-vanishing and both R-symmetries are broken.

3.1.3 Renormalization group flow and SCFTs

According to the non-renormalization theorem in $\mathcal{N}=1$ theories[38], the superpotential is not renormalized perturbatively, while Kähler potential term can receive non-trivial quantum corrections. In $\mathcal{N}=2$ however, since superpotential and the Kähler potential are combined in (3.6), their is no independent quantum correction to Kähler potential.

This non-normalization property largely simplifies the discussion. It claims that the beta function of Yang-Mills coupling g is 1-loop exact[39–40]

$$E\frac{\mathrm{d}g}{\mathrm{d}E} = -\frac{g^3}{16\pi^2}b, \tag{3.16}$$

where b is known as the beta function coefficient[41–42]. The corresponding gauge coupling τ is

$$\tau(E) = \tau_{\mathrm{UV}} - \frac{b}{2\pi i}\log\left(\frac{E}{\Lambda_{\mathrm{UV}}}\right) + \cdots. \tag{3.17}$$

There is an important scale called the dynamic scale defined as

$$\Lambda = Ee^{2\pi i\tau(E)/b}, \tag{3.18}$$

which is an RG-invariant object. This scale marks the transition between strong and weak coupling phases of the gauge theory. Using this scale, we can write the most generic form of τ as

$$\tau(E) = -\frac{b}{2\pi i}\log\left(\frac{E}{\Lambda}\right) + \sum_{n\in\mathbb{Z}^+} a_n\left(\frac{\Lambda}{E}\right)^{bn}, \tag{3.19}$$

where each term in the infinite sum comes from the n-instanton contribution.

In the case of $b=0$, the gauge Lagrangian is already conformal invariant, resulting in an SCFT. Since b depends on the matter content, we can classify the SCFTs obtained by considering different gauge groups and the hypermultiplets where matters live[43].

When $b > 0$, the theory is asymptotically free. In this case, the IR theory can exhibit emergent superconformal symmetry. These emergent SCFTs can be found by tuning parameters that preserves $\mathcal{N} = 2$ supersymmetry, such as potential mass parameters, VEVs of fields or even the dynamic scale Λ. The basic idea is to look at regions in Coulomb branch where scale invariance emerges. This is how the first AD theory was discovered[1].

3.2 Argyres-Douglas point

Generally, the Higgs branch does not receive quantum corrections due to the non-renormalization theorem, while the Coulomb branch does. The low energy effective theories (LEET) on the Coulomb branch is helpful to study the 4-dimensional $\mathcal{N} = 2$ SCFTs.

3.2.1 Coulomb branch parameters

As we mentioned, the Coulomb branch is parameterized by the VEV of vector multiplet scalar Φ. For SU$(r+1)$ gauge group, this VEV can be diagonalized by gauge transformation

$$\langle \Phi \rangle = \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_{r+1} \end{pmatrix} \quad \text{where} \quad a_{r+1} = -\sum_{i=1}^{r} a_i. \tag{3.20}$$

These non-zero a_is lead to the massive W-bosons, hypermultiplets and monopoles, with the masses given by

$$m_W \sim |a_i - a_j|, \quad m_H \sim |a_i|, \quad m_M \sim \frac{1}{g_i^2}|a_i|, \tag{3.21}$$

where g_i is the Maxwell coupling.

Equivalently, we can use another set of coordinates for the Coulomb branch

$$u_i = \operatorname{tr} \Phi^{i+1} \tag{3.22}$$

for $i = 1, 2, \cdots, r$. This set of coordinate is gauge invariant, while $\{a_i\}$ is not. As a consequence, $\{a_i\}$ can only be defined locally. To simplify the discussion, we consider only $r = 1$ case from now on in this section.

As we know, the pure Maxwell theory enjoys the electric-magnetic (EM) duality. Naturally, in the $\mathcal{N} = 2$ supersymmetric gauge theory, we want to consider a dual object for a, who lives in the same multiplet as the gauge field A_μ. This can be defined from

$$a_D = \frac{\partial \mathcal{F}}{\partial a}, \tag{3.23}$$

where $\mathcal{F}$ is the holomorphic prepotential that basically determines the low energy effective action. a and a_D are generally holomorphic functions of u. The metric of $\mathcal{M}_{\text{CB}}$ can be locally written as

$$\mathrm{d}s^2 = \operatorname{Im}\, \mathrm{d}a_D\, \mathrm{d}\bar{a}\,. \tag{3.24}$$

This pair of coordinate receives the $\mathrm{SL}(2,\mathbb{Z})$ duality action from the pure Maxwell theory

$$\begin{pmatrix} a_D \\ a \end{pmatrix} \mapsto \begin{pmatrix} f & g \\ h & l \end{pmatrix} \begin{pmatrix} a_D \\ a \end{pmatrix}, \tag{3.25}$$

and so does the IR coupling

$$\tau_{\mathrm{IR}} = \frac{\partial a_D}{\partial a} \quad \mapsto \quad \frac{f\tau_{\mathrm{IR}} + g}{h\tau_{\mathrm{IR}} + l}. \tag{3.26}$$

3.2.2 Seiberg-Witten geometry

Generally, (a_D, a) can be multi-valued functions of u and show certain monodromy behavior. From a geometric construction, there is way to precisely describe their behavior[44–45].

Inspired by the $\mathrm{SL}(2,\mathbb{Z})$ invariance, we interpret the complex effective coupling τ_{IR} as a torus T^2 associated to each point of the u-plane. It can be described by an elliptic curve Σ_u

$$y^2 = x^3 + f(u,m)x + g(u,m), \tag{3.27}$$

which is a hypersurface in $\mathbb{C}P^2$. m is a possible mass deformation. This curve is known as the Seiberg-Witten curve. There are naturally two 1-cycles on the curve Σ_u that forms the basis of $H^1(\Sigma_u, \mathbb{Z})$, named A-cycle and B-cycle. The Coulomb branch parameters can then be expressed as

$$a = \oint_A \lambda, \quad a_D = \oint_B \lambda, \tag{3.28}$$

where λ is a differential 1-form on Σ_u known as the Seiberg-Witten differential. This differential satisfies the special Kähler constraint

$$\frac{\partial \lambda}{\partial u} = \frac{\mathrm{d}x}{y} + \mathrm{d}\phi\,, \tag{3.29}$$

where $\mathrm{d}\phi$ is some arbitrary exact 1-form. In this way, the effective gauge coupling

$$\tau_{\mathrm{IR}} = \frac{\partial a_D}{\partial a} = \frac{\displaystyle\oint_A \mathrm{d}x\,/y}{\displaystyle\oint_B \mathrm{d}x\,/y} \tag{3.30}$$

is identified with the complex structure of the torus Σ_u.

3.2.3 Argyres-Douglas point

At certain isolated points of the Coulomb branch, the theory turns into an interacting SCFT with bizarre characteristics. These theories are known as the Argyres-Douglas (AD) theories. These points in $\mathcal{M}_{\mathrm{CB}}$ are called Argyres-Douglas points. Historically, the first AD theory was found in the Coulomb branch of SU(3) pure gauge theory. The method for locating such theories is soon generalized to theory broader gauge groups and various flavors[2,5–6].

One characteristic feature of AD theories is that the Coulomb branch operators often have fractional scaling dimensions. As an example, consider an SU(2) theory with Seiberg-Witten curve

$$y^2 = x^3 + f(u)x + g(u). \tag{3.31}$$

We analyze the scaling property of the curve Σ_u to realize conformal invariance explicitly. Since u is the VEV of the Coulomb branch multiplet scalar, it is naturally charged under $\mathrm{U}(1)_r$ R-symmetry, and therefore the scaling action

$$u \mapsto \xi^r u, \tag{3.32}$$

where $\xi \in \mathbb{C}^*$ and r is the R-charge of u as well as the conformal dimension of the Coulomb branch chiral primary. To find the exact conformal dimension, we use the fact that the integral of Seiberg-Witten differential λ on the 1-cycle gives the mass of the BPS particles. Therefore the scaling dimension should be 1

$$\Delta[\lambda] = 1. \tag{3.33}$$

Then, from the relation of special Kähler constraint (3.29) and the requirement that each term in the Seiberg-Witten curve should have the same scaling dimension, we can obtain the scaling dimension of the Coulomb branch chiral primary as

$$\Delta[u] = r = \frac{6}{6-n} \quad \text{or} \quad \frac{4}{4-n} \geqslant 1, \tag{3.34}$$

where n is a positive integer. We see that in most cases, the scaling dimension is fractional.

Another exotic property of AD theories is the existence of massless BPS particles that are mutually non-local with respect to their electric and magnetic charges. As a consequence, AD theories are typically strongly coupled, and do not admit a Lagrangian description. This makes the theories difficult to tackle with.

3.3 Class $\mathcal{S}$ construction

The $\mathcal{S}$ in the name class $\mathcal{S}$ theory stands for six, indicating that the construction is related to 6-dimensional theories. Generally speaking, the idea is to construct the 4-dimensional SCFT from the compactification of 6-dimensional SCFT on a Riemann surface $\mathcal{C}$[10–11].

To start with, we consider the 6-dimensional SCFT of $\mathcal{N} = (2, 0)$, which is the maximally symmetric theory possible in this dimension. Denote such a theory with gauge algebra $\mathfrak{g}$ as $\mathcal{T}_{\mathfrak{g}}$. An interaction theory can be constructed from the M theory point of view. In particular by considering the dynamics of stacks of M5-branes after remove the decoupled center of mass degrees of freedom. As an example, an N stack of M5-branes corresponds to the $\mathcal{T}_{\mathfrak{su}(N)}$ SCFT.

3.3.1 Constructing 4d SCFT from 6d

To construct the 4-dimensional theory from 6-dimension, we first build the map between 4-dimensional and 6-dimensional supersymmetries. The bosonic part of the 6-dimensional symmetries are

$$\mathfrak{so}(2, 6) \oplus \mathfrak{so}(5)_R. \tag{3.35}$$

Since the Riemann surface $\mathcal{C}$ that we compactify on may not preserve the supersymmetries we want, it is necessary to perform a topological twist[34]. First, we consider the following subalgebra

$$\left(\mathfrak{so}(1,3)_{4\mathrm{d}} \oplus \mathfrak{so}(2)_{\mathcal{C}}\right) \oplus \left(\mathfrak{so}(2)_r \oplus \mathfrak{so}(3)_R\right) \subset \mathfrak{so}(2,6) \oplus \mathfrak{so}(5)_R. \tag{3.36}$$

We then define a twisted rotation $\mathfrak{so}(2)_{\text{twist}}$ on $\mathcal{C}$ by considering the diagonal part of $\mathfrak{so}(2)_r \oplus \mathfrak{so}(2)_{\mathcal{C}}$. Then, the remaining bosonic symmetries are

$$\mathfrak{so}(1,3)_{4\mathrm{d}} \oplus \mathfrak{so}(2)_{\text{twist}} \oplus \mathfrak{so}(3)_R \cong \mathfrak{so}(1,3)_{4\mathrm{d}} \oplus \mathfrak{so}(2)_{\text{twist}} \oplus \mathfrak{su}(2)_R. \tag{3.37}$$

In this way, after compactification we can obtain the $\mathfrak{su}(2)_R$ R-symmetry as well as two Weyl supercharges required to build the 4-dimensional $\mathcal{N}=2$ supersymmetry algebra.

In the 6-dimensional theory, the scalars Φ_I with an $\mathfrak{so}(5)_R$ R-symmetry index in the Cartan subalgebra of $\mathfrak{g}$ parameterize the vacua with their VEVs. After compactification, three of them $\{\Phi_3, \Phi_4, \Phi_5\}$ still rotate under the remaining $\mathfrak{so}(3)_R \cong \mathfrak{su}(2)_R$ R-symmetry, which parameterize the Higgs branch. The other two $\{\Phi_1, \Phi_2\}$ rotate under the $\mathfrak{so}(2)_r \cong \mathfrak{u}(1)_r$ R-symmetry, hence parameterizing the Coulomb branch. Note that after topological twist, $\mathfrak{so}(2)_r$ is associated with rotation on $\mathcal{C}$. Therefore, these two fields are no longer scalars. We can combine them into a 1-form field on $\mathbb{C}$

$$\Phi(z)\,\mathrm{d}z = (\Phi_1 + i\Phi_2)\,\mathrm{d}z \tag{3.38}$$

known as the Higgs field or Hitchin field[46]. Using this field, the Seiberg-Witten curve $\Sigma \subset T^*\mathcal{C}$ is given by[11]

$$\langle \det(x - \Phi(z)) \rangle = 0, \tag{3.39}$$

where $x \in \mathbb{C}$ is coordinate on the fiber. This curve turns out to be an n-cover of $\mathcal{C}$, which can be considered as the low energy behavior of the n-wrapping M5-branes on $\mathcal{C}$.

3.3.2 Punctures on $\mathcal{C}$

In general, the Riemann surface $\mathcal{C}$ can contain different types of punctures, which greatly enriched the types of theories we arrive at. Consider the gauge group $\mathrm{SU}(N)$. Since $\Phi(z)$ is in the adjoint representation, given the Seiberg-Witten curve expression (3.39), we can explicitly write the curve as

$$x^N = \sum_{i=2}^{N} \phi_i(z) x^{N-i}, \tag{3.40}$$

where ϕ_i are polynomials of gauge invariant combinations of the Higgs field like $\operatorname{tr}\Phi^i(z)$. A puncture on $\mathcal{C}$ can then be translated into a pole of the Hitchin field, where we can do the expansion locally

$$\Phi(z) = \frac{A}{z} + \mathcal{O}(1), \tag{3.41}$$

where $A \in \mathfrak{su}(N)$ specifies the nature of the puncture. Since $\Phi(z)$ and therefore A is not gauge invariant, we can classify the defect by the conjugacy class of

A in $\mathfrak{su}(N)$. There are generally two types: the semisimple and the nilpotent ones[47]. In the semisimple case, eigenvalues of A encodes the mass parameter of BPS particles in LEET. This is referred to as the mass parameter of the puncture, which breaks conformal symmetry. In the nilpotent case, there is no such intrinsic scale and conformal invariance is manifestly preserved.

As an example, consider the 6-dimensional $\mathcal{N} = (2,0)$ SCFT with $\mathfrak{su}(2)$ gauge algebra compactified on a four-puncture sphere. To each puncture we can associate a mass parameter m_i, and the Higgs field is locally

$$\Phi(z) \sim \frac{1}{z}\begin{pmatrix} m_i & 0 \\ 0 & -m_i \end{pmatrix} + \mathcal{O}(1). \tag{3.42}$$

Assuming the punctures are located at $(0, 1, q, \infty)$, and the Seiberg-Witten curve is then

$$\begin{aligned} x^2 = \phi_2(z) &= \frac{1}{2}\operatorname{tr}\Phi^2(z) \\ &= \frac{1}{z(z-1)(z-q)}\left(\frac{q}{z}m_1^2 + \frac{q(q-1)}{z-1}m_2^2 + \frac{z-q}{z-1}m_3^2 + zm_4^2 + u\right), \end{aligned} \tag{3.43}$$

which is just the Seiberg-Witten curve of the conformal SU(2) SQCD with the usual Coulomb branch parameter $u \in \mathbb{C}$.

3.3.3 Irregular puncture and AD theories

Apart from the regular punctures where Higgs field has a single pole, we can generalize the notion to include a more singular behavior of[48]

$$\Phi(z) \sim \frac{A}{z^{2+k/b}} + \mathcal{O}\big(z^{-1-k/b}\big), \tag{3.44}$$

where $k, b \in \mathbb{Z}$. Since the order of pole is not necessarily integer, for Φ to be well defined on $\mathcal{C}$, it is required that Φ undergoes a gauge transformation as we go around the pole

$$\Phi(e^{2\pi i}z) = g\Phi(z)g^{-1} \quad \text{s.t.} \quad gAg^{-1} = e^{2\pi i\frac{k}{b}}A. \tag{3.45}$$

Generally, when the theory contains only regular punctures, there is no restriction on the position of the puncture or the topology of $\mathcal{C}$. However, recall that there exists a twisted $\mathrm{U}(1)_{\text{twist}} \cong \mathrm{SO}(2)_{\text{twist}}$ R-symmetry that acts simultaneously on the field $\Phi(z)$ and $\mathcal{C}$. In order to keep leading order A untouched, the $\mathrm{U}(1)_{\text{twist}}$ action has to take the form

$$\mathrm{U}(1)_{\text{twist}}: \quad \Phi \mapsto e^{i\alpha}\Phi, \quad z \mapsto e^{i\alpha\frac{k}{k+b}}z. \tag{3.46}$$

This is only possible for $\mathcal{C} = \mathbb{C}P^1 = S^2$ with exactly one irregular puncture located at $z = 0$. We can at most have an additional regular puncture at $z = \infty$.

In the A_1 gauge theory, as an example, there are two types of irregular punctures, namely

$$\begin{aligned} \text{type I}: \quad \Phi(z) &= \frac{\lambda}{z^n}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{A_{n-1}}{z^{n-1}} + \cdots, \\ \text{type II}: \quad \Phi(z) &= \frac{\lambda}{z^{n+\frac{1}{2}}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{A_{n-1}}{z^{n-\frac{1}{2}}} + \cdots. \end{aligned} \tag{3.47}$$

Depending on whether there exists a regular puncture, the 4-dimensional AD theory corresponding to the singularity $\sim z^{-n-1}$ is of type (A_1, D_{2n}) or (A_1, A_{2n-3}) according to the classification mentioned in section 3.4.

Notice that for type II, there is a branch cut, which means a gauge transformation is needed. Also, there is no $\sim z^{-1}$ term. Therefore, there is no mass parameter encoded in the singularity.

The irregular punctures are important in the class $\mathcal{S}$ construction of AD theory. It is impossible to obtain the characteristic fractional scaling dimension of AD theory from regular punctures only. Sometimes it is necessary to also include twisted punctures.

3.4 Type IIB string construction and $(\mathfrak{g}_1, \mathfrak{g}_2)$ singularities

In this section, we give a schematic introduction to the type IIB string theory construction of 4-dimensional $\mathcal{N} = 2$ SCFT. From the M theory / IIB duality point of view, it is natural to seek for such a construction.

The idea is to compactify the type IIB theory on a non-compact Calabi-Yau (CY) manifold. Also, we want to add singularities to the CY manifold to have some interesting interaction theories as the output. Consider the following CY manifold W that is defined by a polynomial F

$$W = \{F(x_1, x_2, x_3, x_4) = 0 | (x_i) \in \mathbb{C}^4\}. \tag{3.48}$$

The manifold has an isolated canonical 3-fold hypersurface singularity (IHS) at the origin $x_i = 0$, where $F = \mathrm{d}F = 0$ has the unique solution[49].

The effective theory of type IIB compactified on W is then typically 4-dimensional $\mathcal{N} = 2$ gauge theories. To make it conformal, the polynomial F needs to be quasi-homogeneous[50–51]

$$F(\xi^{q_i} x_i) = \xi F(x_i), \quad \xi \in \mathbb{C}^*, \tag{3.49}$$

required by the $\mathrm{U}(1)_r$ R-symmetry, and the weights q_i has to satisfy

$$\sum_{i=1}^{4} q_i > 1. \tag{3.50}$$

In particular, we can focus on a class of IHS of the form[52]

$$F = F_{\mathfrak{g}_1}(x_1, x_2) + F_{\mathfrak{g}_2}(x_3, x_4), \tag{3.51}$$

where

$$\begin{aligned} &F_{A_N}(x, y) = x^2 + y^{N+1}, \quad F_{D_N}(x, y) = x^2 y + y^{N-1}, \\ &F_{E_6}(x, y) = x^3 + y^4, \qquad F_{E_7}(x, y) = x^3 + x y^3, \qquad F_{E_8}(x, y) = x^3 + y^5. \end{aligned} \tag{3.52}$$

This class of singularities turns out to represent a large family of AD theories, denoted as $(\mathfrak{g}_1, \mathfrak{g}_2)$ type theory. Notice that $(\mathfrak{g}_1, \mathfrak{g}_2)$ and $(\mathfrak{g}_2, \mathfrak{g}_1)$ clearly represent the same theory, as well as some other equivalences[4,52]

$$(A_1, D_4) \sim (A_2, A_2), \quad (A_1, E_6) \sim (A_3, A_2), \quad (D_4, A_3) \sim (E_6, A_2), \cdots. \tag{3.53}$$

From the class $\mathcal{S}$ construction, type $(A_n, \mathfrak{g})$ theories are compactified from the $\mathcal{N} = (2, 0)$ A_n SCFT.

3.5 AGT duality

Given the class $\mathcal{S}$ construction of the 4-dimensional SCFT by compactification of 6-dimensional $\mathcal{N} = (2,0)$ $\mathfrak{g}$ SCFT on the 2-dimensional $\mathcal{C}$, it is natural to consider instead compactifying the 4-dimensional part to obtain a 2-dimensional theory. It turns out the result is a Toda CFT with gauge algebra $\mathfrak{g}$ on $\mathcal{C}$. Since both the 4-dimensional and 2-dimensional side theories are conformal, we may scale either of them to zero, thereby claiming both sides are equivalent. This is the rough picture of AGT duality[20].

To start with, consider the 4-dimensional theory on a squashed sphere $S_b^4 \subset \mathbb{R}^5$

$$b(x_1^2 + x_2^2) + \frac{1}{b}(x_3^2 + x_4^2) + x_5^2 = R^2, \tag{3.54}$$

where b is the squashing parameter. For the 4-dimensional SCFT that has Lagrangian description, the S_b^4 partition function can be computed from supersymmetry localization[53–54]

$$Z_{S_b^4} = \int \mathrm{d}a\, Z_{\text{cls}}(a, q, \bar{q}) Z_{\text{1-loop}}(a) Z_{\text{inst}}(a, q) Z_{\text{inst}}(a, \bar{q}), \tag{3.55}$$

where q is the gauge coupling and a is the constant value of a vector multiplet scalar, which can be put into the Cartan subalgebra of $\mathfrak{g}$. In the partition function, classical contributions are

$$Z_{\text{cls}}(a, q, \bar{q}) = Z_{\text{cls}}(a, q) Z_{\text{cls}}(a, \bar{q}) = |q|^{R^2|a|^2}, \tag{3.56}$$

the 1-loop contributions can be computed from the Lagrangian and the remaining part is the Nekrasov (anti-)instanton partition function[7,55]

$$Z_{\text{inst}}(a, q) = \sum_{k=0}^{\infty} q^k Z_{\text{inst},k}(a) = 1 + \mathcal{O}(q) \tag{3.57}$$

coming from the (anti-)instanton contribution at the north (and south) poles $x_5 = \pm R$ of S_b^4. It depends on the Ω-background parameters

$$\epsilon_1 = \frac{b}{R}, \quad \epsilon_2 = \frac{1}{bR}. \tag{3.58}$$

The k-instanton contribution $Z_{\text{inst},k}$ is hard to compute as it requires integrating over the k-instanton moduli space, which is extremely singular.

In the simplest case of $\mathfrak{g} = A_1$, the $\mathfrak{g}$ Toda theory becomes the Liouville theory. Computations from theories with Lagrangian description hint a connection between the 4-dimensional SCFT and the Liouville theory as[9]

$$Z_{S_b^4} = \langle V_{\alpha_1}(z_1) V_{\alpha_2}(z_2) \cdots V_{\alpha_n}(z_n) \rangle_{\text{Liouville}}, \tag{3.59}$$

where V_α are vertex operators which we introduce in the section 4.1. α_is, the Liouville momenta are given by $\frac{Q}{2} + Rm_i$ where m_is are the mass parameters. As we know, in CFT, the n-point function can be decomposed into conformal blocks as

$$\langle V_{\alpha_1}(z_1) V_{\alpha_2}(z_2) \cdots V_{\alpha_n}(z_n) \rangle = \int \mathrm{d}^{n-1}\beta\, C(\boldsymbol{\alpha}, \boldsymbol{\beta}) \mathcal{F}(\boldsymbol{\alpha}, \boldsymbol{\beta}, q) \mathcal{F}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \bar{q}). \tag{3.60}$$

Comparing this to the S_b^4 partition function (3.55), it is natural to see the correspondence

$$Z_{\text{1-loop}}(a) = C(\boldsymbol{\alpha}, \boldsymbol{\beta}), \quad Z_{\text{cls}}(a,q) Z_{\text{inst}}(a,q) = \mathcal{F}(\boldsymbol{\alpha}, \boldsymbol{\beta}, q). \tag{3.61}$$

There is an additional connection between ϕ_2 and the stress-energy tensor $T(z)$ that

$$\phi_2(z) = \frac{\langle \Delta_{\alpha_0} | T(z) V_{\alpha_1} V_{\alpha_2} \cdots | \Delta_{\alpha_n} \rangle}{\langle \Delta_{\alpha_0} | V_{\alpha_1} V_{\alpha_2} \cdots | \Delta_{\alpha_n} \rangle}. \tag{3.62}$$

We can then compute the Seiberg-Witten curve out of it.

AGT duality is conjectured to be held by even non-Lagrangian 4-dimensional $\mathcal{N} = 2$ SCFTs, or theories with higher gauge symmetry $\mathfrak{g}$. In these cases the corresponding objects in 2-dimension involve the computation of irregular punctures, which we introduce in the next section. This duality greatly simplifies the computation of Nekrasov partition function in 4-dimension, as we are able to use the powerful tool of infinite dimensional symmetry algebra in 2-dimension for the analysis.

Table 3-1 Dictionary of AGT duality in A_1 case

Objects in 4d $\mathcal{N} = 2$ A_1 SCFTs	Objects in 2d Liouville theory
Deformation parameters $\epsilon_{1,2}$	Liouville charge $Q = b + b^{-1} = \epsilon_1 + \epsilon_2$
Mass parameters m	Insertion of Liouville vertex operators V_m
Coulomb branch parameter a	Insertion of V_α with $\alpha = Q/2 + a$
$Z_{\text{1-loop}}$	Product of OPE coefficients given by DOZZ formula (4.13)
Z_{inst}	Liouville conformal blocks
$Z_{S_b^4}$	Liouville correlation functions

Chapter 4

Irregular representation of Virasoro algebra

As discussed in the last chapter, we are interested to employ the techniques from 2-dimensional CFT to compute objects in 4-dimensional SCFTs, in particular AD theories that were difficult to obtain from traditional approaches. In this chapter, we develop the argument especially for A_1 type AD theories as the simplest example.

4.1 Liouville theory

The 2-dimensional corresponding theory of A_1 type AD theory is Liouville theory, an irrational exact-solvable 2-dimensional CFT. This theory gets its name from the Liouville equation

$$\partial\bar{\partial}\varphi = \lambda e^{\varphi},$$

which is the classical equation of motion for Liouville theory. Liouville theory comes as a toy model for 2-dimensional quantum gravity, and there is a nice geometrical interpretation for this equation. Recall that in 2-dimension, the Riemannian curvature is completely determined by the scalar curvature R and that the metric $g_{\mu\nu}$ has only 3 independent components. As a consequence, we can characterize a physical metric with only one scalar field (after subtracting 2 gauge degrees of freedom from reparameterization) as

$$g_{\mu\nu}(x) = e^{\varphi(x)} h_{\mu\nu},$$

where $h_{\mu\nu}$ is a fixed metric known as the reference metric. The curvature is then given by

$$R = -e^{-\varphi}\partial\bar{\partial}\varphi.$$

Then, in this context Liouville equation describes a 2-dimensional geometry with a constant curvature.

In this section, we sketch the key features of Liouville theory, especially the stress-energy tensor, primary operators and the DOZZ formula for conformal blocks[14–16,56–58].

4.1.1 Action of Liouville theory

The action of Liouville theory is given by

$$S_{\text{Liouville}} = \frac{1}{4\pi}\int_{\Sigma} \mathrm{d}^2 z\, \sqrt{h}\Big(\frac{1}{2}h^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi + QR_h\varphi + 4\pi\mu e^{2b\varphi}\Big), \tag{4.1}$$

where h is the reference metric, related to the physical metric by

$$g_{\alpha\beta} = e^{2b\varphi} h_{\alpha\beta}, \tag{4.2}$$

and R_h is the curvature related to the reference metric. μ is the cosmological constant. Q is called the Liouville charge, and is related to b as

$$Q = b + \frac{1}{b}. \tag{4.3}$$

The stress-energy tensor of Liouville theory can be explicitly computed as

$$\begin{aligned} &T_{z\bar{z}} = 0, \\ &T_{zz} = T(z) = -{:}\partial\varphi\partial\varphi{:} + Q\partial^2\varphi, \\ &T_{\bar{z}\bar{z}} = \bar{T}(z) = -{:}\bar{\partial}\varphi\bar{\partial}\varphi{:} + Q\bar{\partial}^2\varphi. \end{aligned} \tag{4.4}$$

It is then straightforward to give the TT OPE as

$$T(z)T(w) = \left(\frac{1}{2} + 3Q^2\right)\frac{1}{(z-w)^4} + \cdots \tag{4.5}$$

and the central charge of Liouville theory is then given by

$$c = 1 + 6Q^2. \tag{4.6}$$

4.1.2 Primary operators

We should observe that given the transformation of the metric under conformal transformation

$$g_{\mu\nu} \mapsto g_{\alpha\beta}\frac{\partial x'^{\alpha}}{\partial x^{\mu}}\frac{\partial x'^{\beta}}{\partial x^{\nu}}, \tag{4.7}$$

the transformations of the Liouville field φ and $\partial\varphi$ under $z \mapsto w(z)$ are then

$$\begin{aligned} \varphi(z) &\mapsto \varphi(z) - \frac{Q}{2}\ln\left|\frac{\partial w}{\partial z}\right|^2 \\ \partial\varphi(z) &\mapsto \frac{\partial w}{\partial z}\partial\varphi(z) - \frac{Q}{2}\frac{\partial}{\partial z}\ln\left|\frac{\partial w}{\partial z}\right|. \end{aligned} \tag{4.8}$$

This shows that both φ and $\partial\varphi$ are not conformal primary operators. This can also be demonstrated in the OPE

$$T(z)\partial_w\varphi(w) = \frac{Q/2}{(z-w)^2} + \frac{\partial_w\varphi(w)}{z-w} + \cdots. \tag{4.9}$$

Instead, the conformal primaries in Liouville theory are given by the vertex operators

$$V_\alpha(z) = {:}e^{2\alpha\varphi}{:}, \tag{4.10}$$

which transforms as

$$V_\alpha(z) \mapsto \left(\frac{\partial w}{\partial z}\right)^{-\alpha Q}\left(\frac{\partial \bar{w}}{\partial \bar{z}}\right)^{-\alpha Q} V_\alpha(z). \tag{4.11}$$

The classical conformal weights of these operators are $\Delta_{\text{cls}} = \bar{\Delta}_{\text{cls}} = \alpha Q$. They receive a modification at quantum level, and the quantum conformal weights are

$$\Delta_\alpha = \bar{\Delta}_\alpha = \alpha(Q-\alpha). \tag{4.12}$$

This can also be seen from the TV_α OPE. The parameter α is called the Liouville momentum of the vertex operator.

4.1.3 DOZZ formula

A nice property of Liouville theory is that it is exactly-solvable. In particular, its three-point functions (2.45) can be solved by the famous DOZZ formula[59]

$$C_{123} = \left(\pi\mu\gamma(b^2)b^{2-2b^2}\right)^{(Q-\sum_i \alpha_i)/b} \times \frac{\Upsilon_0 \Upsilon(2\alpha_1)\Upsilon(2\alpha_2)\Upsilon(2\alpha_3)}{\Upsilon(\sum_i \alpha_i - Q)\Upsilon(\alpha_1+\alpha_2-\alpha_3)\Upsilon(\alpha_2+\alpha_3-\alpha_1)\Upsilon(\alpha_3+\alpha_1-\alpha_2)}, \tag{4.13}$$

where the special function $\Upsilon(x) \equiv \Upsilon_b(x)$ can be expressed in terms of an integral

$$\log \Upsilon(x) = \int_0^\infty \frac{\mathrm{d}t}{t}\left[\left(\frac{Q}{2}-x\right)^2 e^{-t} - \frac{\sinh^2\left(\frac{Q}{2}-x\right)\frac{t}{2}}{\sinh\frac{bt}{2}\sinh\frac{t}{2b}}\right] \tag{4.14}$$

and the γ function is just standard notation

$$\gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)}. \tag{4.15}$$

Some nice properties of this Υ function include

$$\begin{aligned} \Upsilon(x+b) &= \gamma(bx)b^{1-2bx}\Upsilon(x) \\ \Upsilon_b(x) &= \Upsilon_{b^{-1}}(x) \\ \Upsilon(x) &= \Upsilon(Q-x). \end{aligned} \tag{4.16}$$

4.2 Irregular Gaiotto state in Liouville theory

As we know, a primary state given by

$$|\Delta_\alpha\rangle = V_\alpha(0)\,|0\rangle \tag{4.17}$$

satisfies the Virasoro action

$$L_0\,|\Delta_\alpha\rangle = \Delta_\alpha\,|\Delta_\alpha\rangle\,, \quad L_n\,|\Delta_\alpha\rangle = 0 \quad \text{for} \quad n>0. \tag{4.18}$$

This can be considered as the state corresponding to a regular puncture on the Riemann sphere.

Generally, we can require a state to be a common eigenstate of several Virasoro generators without breaking the commutation relation (2.55). In particular, we are interested in the generators $L_{n\geqslant 0}$ that forms a subalgebra[60]

$$[L_n, L_m] = (n-m)L_{n+m}, \quad n,m \geqslant 0. \tag{4.19}$$

From this commutation relation, we can observe that a maximum of simultaneous diagonalization can take place for the set $\{L_n, L_{n+1}, \cdots, L_{2n}\}$. We denote such a state as $|I^{(n)}\rangle$, called the irregular Gaiotto state. It satisfies

$$L_k\,|I^{(n)}\rangle = \begin{cases} 0 & k>2n, \\ \Lambda_k\,|I^{(n)}\rangle & n\leqslant k\leqslant 2n, \end{cases} \tag{4.20}$$

where Λ_ks are parameters of the state. We can further complete the action of Virasoro generators L_k for $0 \leqslant k < n$ by including differential operators with respect to the parameters Λ_k

$$L_k \left|I^{(n)}\right\rangle = \begin{cases} 0 & k > 2n, \\ \Lambda_k \left|I^{(n)}\right\rangle & n \leqslant k \leqslant 2n, \\ \left(f_k(\Lambda) + \sum_{\ell=n}^{2n-k} (\ell - k)\Lambda_{\ell+k} \frac{\partial}{\partial \Lambda_\ell} \right) \left|I^{(n)}\right\rangle & 0 \leqslant k < n, \end{cases} \tag{4.21}$$

where $f_k(\Lambda)$ are certain functions of the parameters. It is straightforward to check that (4.21) is compatible with the commutation relations (4.19).

Sometimes, we write (4.21) in a more compact way

$$T_>(y) \left|I^{(n)}\right\rangle = \left(\sum_{k=n}^{2n} \frac{\Lambda_k}{y^{k+2}} + \sum_{k=0}^{n-1} \frac{\mathcal{L}_k}{y^{k+2}} + \frac{L_{-1}}{y} \right) \left|I^{(n)}\right\rangle, \tag{4.22}$$

where $T_>(z)$ is a part of the stress-tensor that annihilates the vacuum

$$T_>(z) = \sum_{k \geqslant -1} \frac{L_k}{z^{k+2}}, \tag{4.23}$$

and $\mathcal{L}_k$ denotes the action of L_k on the irregular state that contains differential operators.

Usually, the functions f_k take complicated forms and are hard to compute. We can simplify the expression by a different parameterization with parameters $\{c_0, c_1, \cdots c_n\}$. The connection between these two sets of parameters are given by

$$\Lambda_k = -\sum_{\ell=k-n}^{n} c_{k-\ell} c_\ell + (n+1)Q c_n \delta_{kn}. \tag{4.24}$$

Using this new set of parameters, the action of Virasoro generators can be expressed explicitly as

$$L_k \left|I^{(n)}\right\rangle = \begin{cases} 0 & k > 2n, \\ -\sum_{\ell=k-n}^{n} c_{k-\ell} c_\ell \left|I^{(n)}\right\rangle & n < k \leqslant 2n, \\ \left(-\sum_{\ell=0}^{n} c_{n-\ell} c_\ell + (n+1)Q c_n \right) \left|I^{(n)}\right\rangle & k = n, \\ \left(-\sum_{\ell=0}^{k} c_{k-\ell} c_\ell + (k+1)Q c_k + \sum_{\ell=1}^{n-k} \ell c_{\ell+k} \frac{\partial}{\partial c_\ell} \right) \left|I^{(n)}\right\rangle & 0 \leqslant k < n. \end{cases} \tag{4.25}$$

This explicit expression can be naturally derived from a method called the free field representation, which we introduce in section 4.2.2.

According to the state-operator correspondence, an irregular state corresponds to an irregular puncture on the sphere.

4.2.1 Collision limit for irregular state

One important feature of the irregular state is that it can be constructed as a collision limit of primary fields or regular states in this context. It corresponds to an irregular puncture on the sphere. To illustrate this idea, consider the state before collision

$$|R^{(n)}(z)\rangle = \prod_{i=1}^{n} V_{\alpha_i}(z_i)\,|\Delta_{\alpha_0}\rangle\,. \tag{4.26}$$

Then, given the OPE of stress-energy tensor with primary fields (2.50), we can write the action of $T_>(y)$ on this state as

$$T_>(y)\,|R^{(n)}\rangle = \sum_{i=0}^{n}\left(\frac{\Delta_{\alpha_i}}{(y-z_i)^2} + \frac{1}{y-z_i}\frac{\partial}{\partial z_i}\right)|R^{(n)}\rangle\,, \tag{4.27}$$

where $z_0 = 0$. Recall that

$$L_{-1}\,|R^{(n)}(z)\rangle = \sum_{i=0}^{n}\frac{\partial}{\partial z_i}\,|R^{(n)}(z)\rangle\,, \tag{4.28}$$

we can rewrite (4.27) as

$$\begin{aligned} T_>(y)\,|R^{(n)}\rangle &= \left(\sum_{i=0}^{n}\frac{\Delta_{\alpha_i}}{(y-z_i)^2} + \sum_{i=1}^{n}\frac{z_i}{y(y-z_i)}\frac{\partial}{\partial z_i} + \frac{L_{-1}}{y}\right)|R^{(n)}\rangle \\ &= \left(T_{\text{sing}}(y) + \sum_{(i\neq j)=1}^{n}\frac{\alpha_i\alpha_j}{(y-z_i)(y-z_j)}\right. \\ &\qquad\qquad \left. + \sum_{i=1}^{n}\frac{z_i}{y(y-z_i)}\frac{\partial}{\partial z_i} + \frac{L_{-1}}{y}\right)|R^{(n)}\rangle\,, \end{aligned} \tag{4.29}$$

where

$$T_{\text{sing}}(y) = -(\partial\phi_{\text{sing}})^2 + Q\partial^2\phi_{\text{sing}}, \quad \partial\phi_{\text{sing}} = -\sum_{i=0}^{n}\frac{\alpha_i}{y-z_i}. \tag{4.30}$$

Consider a normalization on the state

$$|R^{(n)}\rangle \mapsto \prod_{(i\neq j)=0}^{n}(z_i-z_j)^{\alpha_i\alpha_j}\,|R^{(n)}\rangle\,. \tag{4.31}$$

Then $T_>(y)$ action becomes

$$T_>(y)\,|R^{(n)}\rangle = \left(T_{\text{sing}}(y) + \sum_{i=1}^{n}\frac{z_i}{y(y-z_i)}\frac{\partial}{\partial z_i} + \frac{L_{-1}}{y}\right)|R^{(n)}\rangle\,. \tag{4.32}$$

In the collision process, we take the limit $z_i \to 0$ so that the primary fields all collide onto the origin. In order to have a finite result, it turns out that we have to simultaneously take the limit $\alpha_i \to \infty$ as well. To avoid any divergent behavior, certain combinations of z_i and α_i are kept finite. In particular

$$c_k = \sum_{i=0}^{n}\alpha_i \sum_{(j_1<j_2<\cdots<j_k)\neq i}\prod_{r=1}^{k}\left(-z_{j_r}\right) \tag{4.33}$$

should be fixed at finite values. We can express ϕ_{sing} and consequently T_{sing} in terms of these c_ks

$$\partial\phi_{\text{sing}} = -\sum_{k=0}^{n} \frac{c_k y^{n-k}}{\prod_{j=0}^{n}(y-z_j)}. \tag{4.34}$$

The differential operators become

$$z_i \frac{\partial}{\partial z_i} = z_i \sum_{k=1}^{n} \frac{\partial c_k}{\partial z_i} \frac{\partial}{\partial c_k}. \tag{4.35}$$

Then, it is straightforward to plug these back into (4.32) and take the proper limit of $z_i \to 0$ and $\alpha_i \to \infty$. The result exactly reproduces (4.22). Therefore, the collision

$$\lim_{\substack{z_i\to 0\\ \alpha_i\to\infty}} |R^{(n)}(z)\rangle$$

is indeed a realization of the irregular state satisfying (4.25).

Notice that in the construction (4.33), we have

$$c_0 = \sum_{i=0}^{n} \alpha_i, \tag{4.36}$$

representing the sum of Liouville momentum of the vertex operators. For this reason, c_0 is sometimes denoted as α.

4.2.2 Free field representation for integer rank

The exact form (4.25) of the irregular representation can be derived naturally from the free field representation. To start with, consider a chiral Liouville field with mode expansion

$$\phi(z) = q - \alpha_p \ln z + \sum_{k\neq 0} \frac{i}{k} a_k z^{-k} \quad \text{where} \quad \alpha_p = ip + \frac{Q}{2}. \tag{4.37}$$

The modes satisfy the following commutation relations and reality conditions

$$[q,p] = \frac{i}{2}, \quad [a_k, a_\ell] = \frac{k}{2}\delta_{k+\ell,0}, \quad p^\dagger = p, \quad q^\dagger = q, \quad a_k^\dagger = a_{-k}. \tag{4.38}$$

For the simplicity of notation, we can also denote

$$a_0 = -i\alpha_p = p - \frac{iQ}{2}. \tag{4.39}$$

As the zero mode, it indeed commutes with all other modes and satisfies $a_0^\dagger = a_0$. In this way, it is convenient to write the mode expansion for

$$\partial\phi(z) = -i\sum_{k\in\mathbb{Z}} a_k z^{-k-1}. \tag{4.40}$$

Plugging this mode expansion into the stress-energy tensor of Liouville theory (4.4)

$$\begin{aligned} T(z) &= -{:}\partial\phi(z)\partial\phi(z){:} + Q\partial^2\phi(z) \\ &= \sum_{k,\ell} {:}a_k a_\ell{:} z^{-k-\ell-2} + iQ\sum_k (k+1) a_k z^{-k-2} = \sum_k L_k z^{-k-2}, \end{aligned} \tag{4.41}$$

we can obtain the expression of the Virasoro generators from these modes

$$L_k = \sum_{\ell \in \mathbb{Z}} :a_{k-\ell} a_\ell: + i(k+1)Q a_k. \tag{4.42}$$

We can check using (4.38) that this indeed satisfies the Virasoro algebra (2.55) with Liouville central charge (4.6).

To construct the irregular representation, consider a coherent state such that

$$a_k \left|\boldsymbol{c}^{(n)}\right\rangle = \begin{cases} 0 & k > n, \\ -ic_k \left|\boldsymbol{c}^{(n)}\right\rangle & 0 \leqslant k \leqslant n, \\ \dfrac{i}{2}(-k)\dfrac{\partial}{\partial c_{-k}} \left|\boldsymbol{c}^{(n)}\right\rangle & -n \leqslant k < 0. \end{cases} \tag{4.43}$$

It is easy to check that this is compatible with the commutation relation (4.38) and the state $\left|\boldsymbol{c}^{(n)}\right\rangle$ is a realization of irregular state $\left|I^{(n)}\right\rangle$ satisfying (4.25).

4.2.3 Recursive solution of irregular state

To construct a general regular state that satisfies (4.27), it is useful to consider the screened vertex operator $V^{\alpha}_{\beta_i,\beta_j}$ of conformal dimension Δ_α that maps from two Verma modules

$$V^{\alpha}_{\beta_i,\beta_j}: \quad \mathcal{M}_{\Delta_{\beta_i}} \to \mathcal{M}_{\Delta_{\beta_j}}. \tag{4.44}$$

The general regular state can be then constructed as

$$\left|R^{(n)}(\boldsymbol{z};\boldsymbol{\beta})\right\rangle = \prod_{i=1}^{n} V^{\alpha_i}_{\beta_{i-1},\beta_i}(z_i) \left|\Delta_{\alpha_{n+1}}\right\rangle \quad \text{where} \quad \begin{array}{l} \beta_0 \equiv \alpha_0, \\ \beta_n \equiv \alpha_{n+1}. \end{array} \tag{4.45}$$

There are $n-1$ intermediate Liouville momenta β_i to label the state. The natural idea is that these parameters should also appear for irregular state $\left|I^{(n)}(c,\alpha;\beta)\right\rangle$. This implies that from the equations (4.25) alone, we cannot fully determine the irregular state.

To fully solve the state, we need to impose boundary conditions for the irregular states as $c_n \to 0$ one by one. Namely, the ansatz is proposed as

$$\left|I^{(n)}(\boldsymbol{c},\alpha;\boldsymbol{\beta})\right\rangle = \left(\prod_{i=1}^{n} c_i^{\nu_i}\right) e^{S(\boldsymbol{c};\alpha,\beta_{n-1})} \sum_{k=0}^{\infty} c_n^k \left|I_k^{(n-1)}(\tilde{\boldsymbol{c}},\beta_{n-1};\tilde{\boldsymbol{\beta}})\right\rangle, \tag{4.46}$$

where $\tilde{\boldsymbol{c}}$ and $\tilde{\boldsymbol{\beta}}$ denotes the tuple without c_n or β_{n-1}. $\left|I_k^{(n-1)}\right\rangle$ is a general descendant state of rank $n-1$

$$\left|I_k^{(n-1)}\right\rangle = \sum_{\text{level}=nk} x_{\text{des}} c_{i_1}^{-p_1} c_{i_2}^{-p_2} \cdots \left(\frac{\partial}{\partial c_{j_1}}\right)^{q_1} \left(\frac{\partial}{\partial c_{j_2}}\right)^{q_2} \cdots L_{-\ell_1}^{r_1} L_{-\ell_2}^{r_2} \cdots \left|I^{(n-1)}\right\rangle, \tag{4.47}$$

where x_{des}s are coefficients to be determined, and the level of the descendant is given by

$$\text{level} = \sum_s p_s i_s + \sum_t q_t j_t + \sum_u r_u \ell_u. \tag{4.48}$$

4.3 Half-integer rank irregular states

As discussed in section 3.3.3, there are two types of irregular singularities in A_1 type theories (3.47). In the above section, we only discussed the state $|I^{(n)}\rangle$ where $n \in \mathbb{Z}$, corresponding to the type I puncture. To include type II singularities, we have to consider an irregular state $|I^{(n-\frac{1}{2})}\rangle$ of half-integer rank.

Although the integer rank states have been well understood, the general form of half-integer rank states is still not fully understood. In the recent studies, it is proposed that a half-integer irregular state can be parameterized by

$$L_k\,|I^{(n-\frac{1}{2})}\rangle = \begin{cases} 0 & k \geqslant 2n, \\ \Lambda_k\,|I^{(n-\frac{1}{2})}\rangle & n \leqslant k < 2n, \\ \left(f_k(\Lambda) + \displaystyle\sum_{\ell=n}^{2n-k-1} (\ell - k)\Lambda_{\ell+k}\frac{\partial}{\partial \Lambda_\ell}\right)|I^{(n-\frac{1}{2})}\rangle & 0 \leqslant k < n, \end{cases} \tag{4.49}$$

where f_k is some function defined by a recursion relation[61].

4.3.1 Free field representation of half-integer rank

Similar to the case of integer rank, the function f_k is extremely complicated. This greatly increases the difficulty of solving a general irregular state of half-integer. The parameters in (4.49) are also hard to interpret physically. Thus, it is necessary to construct a free field representation in this case.

Naive extension of the method used in section 4.2.2 fail as the dimensions of parameters do not match. Inspired by the branch cutting behavior of type II singularities in (3.47), we construct an analogue of (4.25) with a $\mathbb{Z}_2$-twisted field[62].

Consider an $\mathbb{Z}_2$-twist line defect from $z = 0$ to $z = \infty$, with two twist operators placed at endpoints $\sigma(0)$ and $\sigma(\infty)$. A free chiral scalar field in the twisted sector exhibits the monodromy behavior

$$\phi(z)\sigma(0) \sim z^{1/2}\tau(0). \tag{4.50}$$

The mode expansion of $\partial\phi(z)$ is given by

$$\partial\phi(z) = -i \sum_{r\in\mathbb{Z}+\frac{1}{2}} \tilde{a}_r z^{-r-1}, \tag{4.51}$$

with commutation relation

$$[\tilde{a}_r, \tilde{a}_s] = \frac{r}{2}\delta_{r+s,0}. \tag{4.52}$$

There is an issue that stress-energy tensor in (4.4) does not admit the $\mathbb{Z}_2$ symmetry. Hence we need to work with a new stress-energy tensor

$$T(z) = -{:}\partial\phi(z)\partial\phi(z){:}, \tag{4.53}$$

which is the $Q \to 0$ limit of (4.4). In this case, the Virasoro generators are given by

$$L_k = \sum_{r\in\mathbb{Z}+\frac{1}{2}} {:}\tilde{a}_{k-r}\tilde{a}_r{:}. \tag{4.54}$$

To construct the analogue of free field representation, we can consider a coherent state that

$$\tilde{a}_r \left|\boldsymbol{c}^{(n-\frac{1}{2})}\right\rangle = \begin{cases} 0 & r \geqslant n+\dfrac{1}{2}, \\ -ib_r \left|\boldsymbol{c}^{(n-\frac{1}{2})}\right\rangle & \dfrac{1}{2} \leqslant r \leqslant n-\dfrac{1}{2}, \\ \dfrac{i}{2}(-r)\dfrac{\partial}{\partial b_{-r}} \left|\boldsymbol{c}^{(n-\frac{1}{2})}\right\rangle & -n+\dfrac{1}{2} \leqslant r \leqslant -\dfrac{1}{2}. \end{cases} \tag{4.55}$$

Such a state is indeed an irregular state of half-integer rank $\left|I^{(n-\frac{1}{2})}\right\rangle$. The explicit form of Virasoro generator actions is

$$L_k \left|I^{(n-\frac{1}{2})}\right\rangle = \begin{cases} 0 & k \geqslant 2n, \\ -\displaystyle\sum_{r=k-n+\frac{1}{2}}^{n-\frac{1}{2}} b_{k-r} b_r \left|I^{(n-\frac{1}{2})}\right\rangle & n \leqslant k < 2n, \\ \left(-\displaystyle\sum_{r=\frac{1}{2}}^{k-\frac{1}{2}} b_{k-r} b_r + \sum_{r=\frac{1}{2}}^{n-k-\frac{1}{2}} r b_{r+k} \frac{\partial}{\partial b_r}\right) \left|I^{(n-\frac{1}{2})}\right\rangle & 0 \leqslant k < n. \end{cases} \tag{4.56}$$

In this construction, the L_0 action only contains differential operator. This is aligned with the result given in the recent research where they also find $f_0 = 0$ in (4.49)[19,61].

4.3.2 Solution of half-integer states

Similar to the case of integer rank irregular states, (4.56) alone cannot fully determine the state. We need an ansatz similar to (4.46) to recursively solve the state. However, we may observe that there is no dimensionless parameters in $\{b_r\}$ that can be considered as the Liouville momentum of the state. The direct assumption

$$\left|I^{(n-\frac{1}{2})}\right\rangle = f(\boldsymbol{b}) \sum_{k=0}^{\infty} b_{n-\frac{1}{2}}^k \left|I_k^{(n-\frac{3}{2})}\right\rangle \tag{4.57}$$

cannot serve as the ansatz as we miss the set of parameters $\{\beta_i\}$.

Hence, we need to take a different approach. The idea is that we can use integer rank irregular states, to which we already know the solutions, as a handle. That is to construct a relation like

$$\left|I^{(n-\frac{1}{2})}(\boldsymbol{b})\right\rangle = f(\boldsymbol{b}) \sum_{k=0}^{\infty} b_{n-\frac{1}{2}}^k \left|I_k^{(n-1)}\right\rangle. \tag{4.58}$$

We use an example to better illustrate the recursive solution. Consider the rank 3/2 irregular state

$$\begin{aligned}
L_3 \,|I^{(3/2)}\rangle &= -b_{3/2}^2 \,|I^{(3/2)}\rangle \\
L_2 \,|I^{(3/2)}\rangle &= -2b_{1/2}b_{3/2} \,|I^{(3/2)}\rangle \\
L_1 \,|I^{(3/2)}\rangle &= \left(-b_{1/2}^2 + \frac{1}{2}b_{3/2}\frac{\partial}{\partial b_{1/2}}\right) |I^{(3/2)}\rangle \\
L_0 \,|I^{(3/2)}\rangle &= \left(\frac{1}{2}b_{1/2}\frac{\partial}{\partial b_{1/2}} + \frac{3}{2}b_{3/2}\frac{\partial}{\partial b_{3/2}}\right) |I^{(3/2)}\rangle \,.
\end{aligned} \tag{4.59}$$

In the limit to $|I^{(1)}\rangle$, we want to send the L_3 action $b_{3/2}^2$ to zero while keeping $b_{1/2}b_{3/2}$ finite. Since the integer rank state $|I^{(1)}\rangle$ labeled by the parameter c_1 has L_2 action c_1^2, it is natural to consider the following change of variable

$$c_1 = \sqrt{2}b_{1/2}^{1/2}b_{3/2}^{1/2}, \quad \Lambda_3 = -b_{3/2}^2. \tag{4.60}$$

Then, the Virasoro generators act as

$$\begin{aligned}
L_3 \,|I^{(3/2)}\rangle &= \Lambda_3 \,|I^{(3/2)}\rangle \\
L_2 \,|I^{(3/2)}\rangle &= -c_1^2 \,|I^{(3/2)}\rangle \\
L_1 \,|I^{(3/2)}\rangle &= \left(\frac{c_1^4}{4\Lambda_3} - \frac{1}{2}\frac{\Lambda_3}{c_1}\frac{\partial}{\partial c_1}\right) |I^{(3/2)}\rangle \\
L_0 \,|I^{(3/2)}\rangle &= \left(c_1\frac{\partial}{\partial c_1} + 3\Lambda_3\frac{\partial}{\partial \Lambda_3}\right) |I^{(3/2)}\rangle \,.
\end{aligned} \tag{4.61}$$

With a proper ansatz

$$\begin{aligned}
|I^{(n-\frac{1}{2})}\rangle &= c_1^{\gamma_1}\Lambda_3^{\gamma_3} e^{-\frac{4}{3}(Q-\beta)\frac{c_1^3}{\Lambda_3} + \frac{c_1^6}{12\Lambda_3^2}} \sum_{k=0}^{\infty} \Lambda_3^k \,|I_k^{(1)}\rangle \\
&= b_{1/2}^{\gamma_1/2} b_{3/2}^{\gamma_1/2+2\gamma_3} \exp\left(\frac{8\sqrt{2}}{3}(Q-\beta)\frac{b_{1/2}^{3/2}}{b_{3/2}^{1/2}} + \frac{2b_{1/2}^3}{3b_{3/2}}\right) \sum_{k=0}^{\infty} (-b_{3/2}^2)^k \,|I_k^{(1)}\rangle \,,
\end{aligned} \tag{4.62}$$

where

$$\gamma_1 + 3\gamma_3 = \beta(Q-\beta) = \Delta_\beta, \tag{4.63}$$

we can derive the following constraints for the general descendants

$$\begin{aligned}
L_3 \,|I_k^{(1)}\rangle &= |I_{k-1}^{(1)}\rangle \\
(L_2 + c_1^2)\,|I_k^{(1)}\rangle &= 0 \\
(L_1 - 2c_1(Q-\beta))\,|I_k^{(1)}\rangle &= \left(-\frac{\gamma_1}{2c_1^2} - \frac{1}{2c_1}\frac{\partial}{\partial c_1}\right) |I_{k-1}^{(1)}\rangle \\
L_0 \,|I_k^{(1)}\rangle &= \left(\Delta_\beta + 3k + c_1\frac{\partial}{\partial c_1}\right) |I_k^{(1)}\rangle \,.
\end{aligned} \tag{4.64}$$

A general descendant is in the form of

$$|I_k^{(1)}\rangle = \sum_{i+j+|Y|=3k} x_{i,j,Y}\, c_1^{-i} \left(\frac{\partial}{\partial c_1}\right)^j L_{-Y}\,|I^{(1)}\rangle \,, \tag{4.65}$$

where $Y = (y_1, y_2, \cdots)$ and $|Y| = \sum_i y_i$. By solving the coefficients $x_{i,j,Y}$ from (4.64) order by order, we can then solve the state $|I^{(3/2)}\rangle$. The result is equivalent to the one using (4.49) parameterization[61].

4.4 Computation of instanton partition function

In the generalized AGT correspondence, where we introduce irregular singularities in the 4-dimensional theory, the Nekrasov instanton partition function should have a 2-dimensional counterpart that also includes irregular punctures, or irregular states as we introduced. In particular, the type (A_1, A_{2n-3}) and (A_1, D_{2n}) AD theories correspond to the sphere with one irregular puncture and zero or one regular puncture. Then, by considering an analogue of (3.59) and (3.62), the corresponding instanton partition function can be expressed as

$$Z_{\text{inst}}^{(A_1, A_{2n-3})} = \langle 0 | I^{(n)} \rangle \,, \quad Z_{\text{inst}}^{(A_1, D_{2n})} = \langle \Delta | I^{(n)} \rangle \,, \tag{4.66}$$

and the Seiberg-Witten curves are

$$x^2 = \phi_2(z) = \frac{\langle 0 | T(z) | I^{(n)} \rangle}{\langle 0 | I^{(n)} \rangle} \quad \text{or} \quad \frac{\langle \Delta | T(z) | I^{(n)} \rangle}{\langle \Delta | I^{(n)} \rangle}. \tag{4.67}$$

Using the ansatz (4.46) or (4.58), we can then recursively solve the Nekrasov instanton partition functions.

As an example, we consider (A_1, A_3) type AD theory[18]. The Seiberg-Witten curve is given by

$$\begin{aligned} x^2 = \frac{\langle 0 | T(z) | I^{(3)} \rangle}{\langle 0 | I^{(3)} \rangle} &= \frac{u}{z^4} - \frac{\Lambda_3}{z^5} - \frac{\Lambda_4}{z^6} - \frac{\Lambda_5}{z^7} - \frac{\Lambda_6}{z^8} \\ &= \frac{u}{z^4} - \frac{2c_1 c_2 + 2c_3(\alpha - 2Q)}{z^5} - \frac{c_2^2 + 2c_1 c_3}{z^6} - \frac{2c_2 c_3}{z^7} - \frac{c_3^2}{z^8}, \end{aligned} \tag{4.68}$$

where

$$u = c_3 \frac{\partial}{\partial c_1} \ln \langle 0 | I^{(3)} \rangle - c_2(2\alpha - 3Q) - c_1^2. \tag{4.69}$$

We can employ a change of variable

$$z \mapsto -\frac{i c_3^{1/3} z}{1 + i c_2 z / c_3^{2/3}}, \quad T(z) \mapsto -\frac{1 + i c_2 z / c_3^{2/3}}{c_3^{2/3}} T(z), \tag{4.70}$$

and the curve then becomes

$$x^2 = \frac{\langle 0 | T(z) | I^{(3)} \rangle}{\langle 0 | I^{(3)} \rangle} = \frac{\tilde{u}}{z^4} + \frac{2i(2Q - \alpha)}{z^5} + \frac{\frac{c_2^2}{2c_3^{4/3}} - \frac{2c_1}{c_3^{1/3}}}{z^6} + \frac{1}{z^8}. \tag{4.71}$$

Through comparing the curve with the one computed from (3.39), we find that $\tilde{u}$ corresponds to the VEV of Coulomb branch operator, $2i(2Q - \alpha)$ corresponds to the mass parameter and $\frac{c_2^2}{2c_3^{4/3}} - \frac{2c_1}{c_3^{1/3}}$ is the relevant coupling. An observation is that there are more parameters than needed in the Seiberg-Witten curve. Therefore, we can set $c_1 = 0$ to eliminate a redundant one.

The corresponding irregular state can be solved from the ansatz

$$|I^{(3)}(\boldsymbol{c},\alpha;\boldsymbol{\beta})\rangle = c_2^{\rho_2} c_3^{\rho_3} \exp\left[(\alpha-\beta)\left(\frac{2c_1c_2}{c_3} - \frac{c_2^3}{2c_3^2} - \frac{c_1^2}{c_2}\right)\right] \sum_{k=0}^{\infty} c_3^k \, |I_k^{(2)}(\tilde{\boldsymbol{c}},\beta_2;\tilde{\boldsymbol{\beta}})\rangle \,. \tag{4.72}$$

To compute the Nekrasov instanton partition function, we need to work out the inner products

$$D_k \equiv \langle 0|I_k^{(2)}\rangle \,. \tag{4.73}$$

Recall the definition for general descendants $|I_k^{(2)}\rangle$s (4.47), the inner products should then consist of

$$c_1, c_2, \frac{\partial}{\partial c_1}\langle 0|I^{(2)}\rangle \,, \frac{\partial}{\partial c_2}\langle 0|I^{(2)}\rangle \,, \langle 0|L_{-i}|I^{(2)}\rangle \,. \tag{4.74}$$

c_1 is already set to zero, and L_{-i} annihilates $\langle 0|$. The derivatives can be determined from

$$\begin{aligned} \langle 0|L_0|I^{(2)}\rangle &= \left(\varDelta_\beta + c_1\frac{\partial}{\partial c_1} + 2c_2\frac{\partial}{\partial c_2}\right)\langle 0|I^{(2)}\rangle = 0 \\ \langle 0|L_1|I^{(2)}\rangle &= \left(2c_1(Q-\beta) + c_2\frac{\partial}{\partial c_1}\right)\langle 0|I^{(2)}\rangle = 0. \end{aligned} \tag{4.75}$$

As $c_1 \to 0$, $\partial/\partial c_1$ acting on $\langle 0|I^{(2)}\rangle$ is also vanishing. Then, only even order D_{2m} produces non-zero results. In fact, we can completely solve $\langle 0|I^{(2)}\rangle$ up to an irrelevant constant.

Therefore, the Nekrasov instanton partition function can be expressed in a series expansion

$$Z_{\text{inst}}^{(A_1,A_3)} = \left\langle 0\middle|I^{(3)}\right\rangle = c_2^{\rho_2} c_3^{\rho_3} e^{(\alpha-\beta)S_3(\boldsymbol{c})} \left\langle 0\middle|I^{(2)}\right\rangle \sum_{k=0}^{\infty}\left(\frac{2c_3^2}{c_2^3}\right)^k D_{2k}. \tag{4.76}$$

Chapter 5

Conclusion and outlooks

In this thesis, we review the construction of 4-dimensional $\mathcal{N}=2$ SCFTs from the class $\mathcal{S}$ theory. It provides a top-down understanding of this family of theories. In particular, we provide an introduction to the AD theories. These theories are known for their exotic behaviors, as they generally do not admit Lagrangian descriptions. Hence, it is hard to use traditional methods to compute relevant quantities. We then introduce the AGT duality that claims a duality between 4-dimensional $\mathcal{N}=2$ SCFTs and 2-dimensional CFTs. This provides a powerful tool to analyze the AD theories from the 2-dimensional side.

In particular, we analyze the situation for A_1 type AD theories, which correspond to the Liouville theory. The Nekrasov instanton partition function of AD theory is given by the inner product of irregular states in Liouville theory

$$Z_{\text{inst}}^{(A_1,A_{2n-3})} = \langle 0|I^{(n)}\rangle\,, \quad Z_{\text{inst}}^{(A_1,D_{2n})} = \langle \Delta|I^{(n)}\rangle\,. \tag{4.66}$$

By using the ansatz (4.46) and solving the equations imposed by (4.25), we are able to solve the Nekrasov instanton partition function for (A_1, A_N) and (A_1, D_N) type AD theories of integer rank.

We further propose a new way to construct the Virasoro algebra representation for half-integer rank (4.56). This is an analogue to the free field representation of integer rank[60]. We innovatively employ the mode expansion of a $\mathbb{Z}_2$-twisted chiral scalar field to construct the state. We also show that this representation leads to consistent results with existing computations.

The AGT duality leads the way to a broad world of field dynamics in lower dimensions, compactified from the M5-brane world volume. We only reveal a corner of it. Here we list some possible future research directions in this field. Some of these directions are currently under active exploration, and premature progress has already been made. However, since they are still incomplete and require further development, we did not include them in the main body of the thesis. Nevertheless, they still offer valuable insights on the subject.

- For AD theories of $A_{n\geqslant 2}$ type, the AGT dual at 2-dimensional side is the A_n-Toda theory, which admit a larger symmetry described by the $W^{(n)}$-algebra. It is possible to construct the $W^{(n)}$ generators from the scalar fields $\boldsymbol{\varphi}$ of Toda theory. For instance, the A_2-Toda theory has two independent scalar degree of freedoms ϕ_1, ϕ_2. We can then construct the analogue of the free field representation of $W^{(3)}$ currents[23]

$$\begin{aligned}
T(z) &= -\frac{1}{4}{:}\partial\phi_1(z)\partial\phi_1(z){:} - \frac{1}{4}{:}\partial\phi_2(z)\partial\phi_2(z){:} + iQ\partial^2\phi_2(z), \\
W(z) &= \frac{\sqrt{\beta}}{12i}({:}\partial\phi_1(z)\partial\phi_1(z)\partial\phi_1(z){:} - 3{:}\partial\phi_1(z)\partial\phi_2(z)\partial\phi_2(z){:}) \\
&\quad + \frac{\sqrt{\beta}}{4}Q({:}\partial\phi_1(z)\partial^2\phi_2(z){:} + 3{:}\partial\phi_2(z)\partial^2\phi_1(z){:}) + \frac{\sqrt{\beta}}{2i}Q^2\partial^3\phi_1(z).
\end{aligned} \tag{5.1}$$

Then it is possible to use a similar irregular state $|I^{(n,m)}\rangle$ to compute the corresponding Nekrasov instanton partition function.

- As a generalization of class $\mathcal{S}$ theory, it is possible to construct 4-dimensional SCFT with surface defects from M theory perspective. Instead of considering N stack of M5-branes, we can insert the branes in different directions, so that the effective theory in $\mathbb{R}^4$ only sees a $\mathbb{C} \cong \mathbb{R}^2$ surface operator. In this way, the symmetry algebra on $\mathcal{C}$ is enhanced to affine $\mathfrak{su}(2)$ algebra. A similar free field representation has been proposed in recent papers[63]. We can then apply the technique introduced in this thesis to compute partition function in the presence of defect.

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