Homework 3: Due at class on March 22

1

Show that an open ball $B_n = \{x \in \mathbb{R}^n |\ |x| < 1\}$ is diffeomorphic to $\mathbb{R}^n$ by constructing a smooth bijection map.

2

For elements $x = (x^0, \cdots, x^n)$ and $y = (y^0, \cdots, y^n)$ of $\mathbb{C}^{n+1} \backslash \{0\}$, we define an equivalence relation $x \sim y$ by

$$x = \alpha y \qquad \alpha \in \mathbb{C} \ .$$

Let us define $\mathbb{C}P^n$ by $(\mathbb{C}^{n+1} \backslash \{0\})/\sim$. Show that $\mathbb{C}P^n$ is a manifold and $\mathbb{C}P^1$ is diffeomorphic to S^2. The space is called a complex projective space.

3

For a positive integer d, we define a map $f^{(d)} : \mathbb{C} \to \mathbb{C};\ z \mapsto z^d$. Let $z = x + iy$ $(x, y \in \mathbb{R})$, and consider $f^{(d)}$ as a function of x and y. Compute the Jacobian matrix of $f^{(d)}$.

4

Show that the tangent bundle TS^1 of a circle S^1 is diffeomorphic to $S^1 \times \mathbb{R}$.

5

Are there zeros of the vector fields in Example 3.10 of the lecture note?

6

Construct a smooth vector field on S^2 which vanishes only at one point explicitly in terms of local coordinates.

7

Show that the Lie bracket satsfies the Jacobi identity. Show that, for $X_1, X_2 \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$,

$$[fX_1, X_2] = f[X_1, X_2] - X_2(f)X_1 \ , \qquad [X_1, fX_2] = f[X_1, X_2] + X_1(f)X_2 \ .$$

8

Construct an example of unorientable manifolds except the Möbius strip and even-dimensional real projective space.