Homework 7: Due at class on Nov 9

1 Null states at level 3

1.1

Show that the linear combination that gives rise to null-vectors at level $N = 3$ is given by

$$|\chi_I\rangle = \left[(h_I + 1)(h_I + 2)L_{-3} - 2(h_I + 1)L_{-1}L_{-2} + L_{-1}^3\right]|\phi_I\rangle$$

where $I = (1, 3)$ or $(3,1)$.

1.2

Determine the differential equation satisfied by the correlators of the primary fields $\phi_{1,3}$ and $\phi_{3,1}$.

2 Minimal models

Consider the minimal models $\mathcal{M}_{2,2n+1}$ ($n = 1, 2, \ldots$). Compute the central charge, and determine the fusion rules of these models.
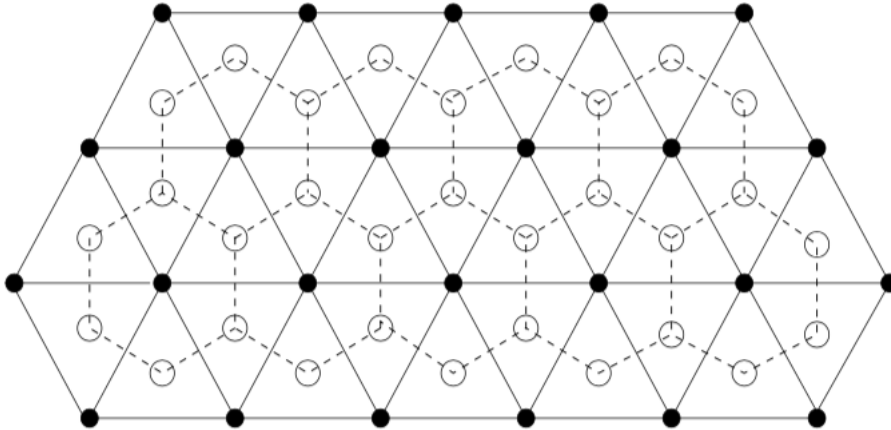
3 Star-triangle relation

Figure 1: Hexagonal and triangular lattices that are dual each other.

Let us consider the Ising model in the Hexagon lattice (white dotes) and the triangle lattice (black dotes) in Figure 1. In fact, as Figure 1 illustrates, the triangular lattice of N sites is the dual of the hexagonal lattice with $2N$ sites. By defining the coupling constant

K and L between the nearest spin in the hexagon and triangle lattice, the partition functions are written as

$$Z^{\mathrm{H}}(\mathcal{L}) = \sum_{\{\sigma\}} \exp\left[\mathcal{L}\sum \sigma_l \sigma_i\right] \ ,$$

$$Z^{\mathrm{T}}(\mathcal{K}) = \sum_{\{\sigma\}} \exp\left[\mathcal{K}\sum \sigma_l \sigma_i\right] \ ,$$

(1)

where $\mathcal{K} = -K/k_B T$ and $\mathcal{L} = -L/k_B T$. Let us show the equivalence of these partition functions

$$Z^{\mathrm{H}}(\mathcal{L}) = R^N Z^{\mathrm{T}}(\mathcal{K}) \ ,$$

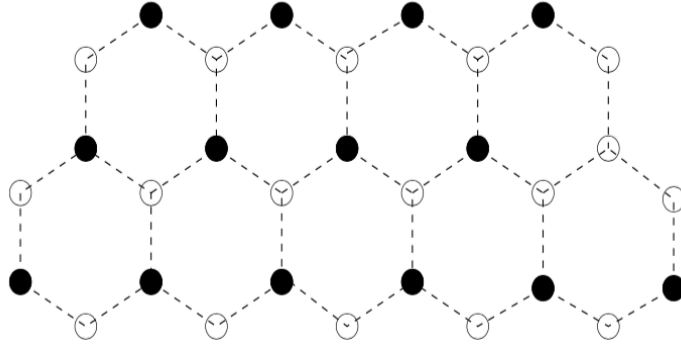which is called the **star-triangle identity**

Figure 2: Bipartition of the hexagonal lattice

First let us split sites of the hexagonal lattice into two classes as in Figure 2. Then, we introduce the Boltzmann weight

$$W\left(\sigma_b; \sigma_i, \sigma_j, \sigma_k\right) = \exp\left[\mathcal{L}\sigma_b(\sigma_i + \sigma_j + \sigma_k)\right]$$

where σ_b is a spin of a black site and $(\sigma_i, \sigma_j, \sigma_k)$ are the spins at white sites next to σ_b. Subsequently, the partition function is

$$Z^{\mathrm{H}}(\mathcal{L}) = \sum_{\sigma_a:\text{white}} \prod_{i,j,k} w^{\mathrm{H}}\left(\sigma_i, \sigma_j, \sigma_k\right)$$

where

$$w^{\mathrm{H}}\left(\sigma_i, \sigma_j, \sigma_k\right) = \sum_{\sigma_b=\pm 1} W\left(\sigma_b; \sigma_i, \sigma_j, \sigma_k\right) = 2\cosh\left(\mathcal{L}(\sigma_i + \sigma_j + \sigma_k)\right) \ .$$

Derive that

$$w^{\mathrm{H}}\left(\sigma_i, \sigma_j, \sigma_k\right) = 2\cosh^3(\mathcal{L}) + 2\cosh(\mathcal{L})\sinh^2(\mathcal{L})[\sigma_i\sigma_j + \sigma_j\sigma_k + \sigma_k\sigma_i] \ . \tag{2}$$

On the other hand, for the triangle lattice, we can just assign the Boltzmann weight to each triangle $(\sigma_i, \sigma_j, \sigma_k)$ of spins

$$\begin{aligned} w^{\mathrm{T}}\left(\sigma_i, \sigma_j, \sigma_k\right) &= R\exp\left(\mathcal{K}[\sigma_i\sigma_j + \sigma_j\sigma_k + \sigma_k\sigma_i]\right) \ , \\ &= R(\cosh^3(\mathcal{K}) + \sinh^3(\mathcal{K})) \\ &\quad + R\cosh(\mathcal{K})\sinh(\mathcal{K})(\cosh(\mathcal{K}) + \sinh(\mathcal{K}))[\sigma_i\sigma_j + \sigma_j\sigma_k + \sigma_k\sigma_i] \ . \end{aligned} \tag{3}$$

so that the partition function is

$$R^N Z^{\mathrm{T}}(\mathcal{L}) = \sum_{\sigma_a:\text{white}} \prod_{i,j,k} w^{\mathrm{T}}\left(\sigma_i, \sigma_j, \sigma_k\right)$$

It is easy to see that (2) and (3) are written in the same form. Show that w^{H} and w^{T} coincide when

$$R^2 \sinh(2\mathcal{K}) = 2\sinh^2(2\mathcal{L}) \ , \qquad (R/2)^4 = \cosh^3(\mathcal{L})\cosh(3\mathcal{L}) \ .$$