Homework 1: Due at class on March 9

1 Path integrals

1.1 free particle

The Lagrangian of a free particle is given by

$$L(\dot{x}, x) := \frac{m}{2}\dot{x}^2 \,. \tag{1}$$

Show that the propagator is given by

$$\left\langle x_f \left| e^{-iHT} \right| x_i \right\rangle = \sqrt{\frac{m}{2\pi \mathrm{i} T}} \exp\left\{ \frac{\mathrm{i}m}{2T} \left(x_f - x_i\right)^2 \right\} \,,$$

and they obey the covolution

$$\int_{-\infty}^{\infty} \mathrm{d}x\, K\left(x_f, x, t_f - t\right) K\left(x, x_i, t - t_i\right) = K\left(x_f, x_i, t_f - t_i\right) \,.$$

1.2 harmonic oscillator

The Lagrangian of a harmonic oscillator with mass $m > 0$ and frequency $\omega > 0$ is given by

$$L(\dot{x}, x) := \frac{m}{2}\dot{x}^2 - \frac{m}{2}\omega^2 x^2 \,. \tag{2}$$

Show that the propagator is give by

$$\begin{aligned} \left\langle x_f \left| e^{-iHT} \right| x_i \right\rangle &= \sqrt{\frac{m\omega}{2\pi i \sin(\omega T)}}\, e^{iS_{\mathrm{cl}}} \\ S_{\mathrm{cl}} &= \frac{m\omega}{2} \frac{1}{\sin(\omega T)} \left[\cos(\omega T) \left(x_f^2 + x_i^2\right) - 2x_i x_f \right] \end{aligned} \tag{3}$$

Note that the eigenfunctions of the harmonic oscillator is expressed by the Hermite polynomials $H_n(x)$, and they satisfy

$$\frac{1}{\sqrt{1-\rho^2}} \exp\left\{ -\frac{\rho^2\left(x^2+y^2\right) - 2\rho x y}{1-\rho^2} \right\} = \sum_{n=0}^{\infty} \frac{(\rho/2)^n}{n!} H_n(x) H_n(y) \,.$$

2 Peskin-Schroeder: Problem 9.2

3 Generating functional for QED

The generating functional for QED is given by

$$Z\left(J_\mu, \eta, \overline{\eta}\right) = \int \mathcal{D}A_\mu \mathcal{D}\psi \mathcal{D}\overline{\psi} e^{i\int d^4x \left(\mathcal{L}_{QED} + \mathcal{L}_{GF} + J^\mu A_\mu + \overline{\eta}\psi + \overline{\psi}\eta\right)}$$

where

$$\mathcal{L}_{QED} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}(i\not{D} - m)\psi \qquad \mathcal{L}_{GF} = -\frac{1}{2\xi}(\partial \cdot A)^2 \,.$$

3.1

Determine $Z_0\left[J^\mu,\eta,\overline{\eta}\right]$ so that

$$Z\left[J^\mu,\eta,\overline{\eta}\right]=\exp\left\{(-e)\int d^4x\frac{\delta}{\delta\eta_\alpha(x)}\left(\gamma^\mu\right)_{\alpha\beta}\frac{\delta}{\delta\overline{\eta}_\beta(x)}\frac{\delta}{\delta J_\mu(x)}\right\}Z_0\left[J^\mu,\eta,\overline{\eta}\right]$$

3.2

We perform the perturbative expansion

$$Z=Z_0\left[1+(-ie)Z_1+(-ie)^2Z_2+\cdots\right]$$

where we have subtracted the vacuum-vacuum amplitudes in Z_i, namely $Z[0]=1$. Express Z_1 and Z_2 by using Feynman Diagrams.