Lecture 5

1 Quantization via path integral

We have been studying bosonic string theory, but there are several caveats.

• In the light-cone quantization, Lorentz invariance is not manifest. Can we quantize strings in a way that is manifestly Lorentz invariant?

• We have seen that the Weyl symmetry of the string sigma model is anomalous on a general curved background. How can the bosonic string be anomaly-free?

• Although we learnt that string amplitude is expressed via Feynman path integral

$$A_n = \sum_g \int [\mathcal{D}h_{ab}]_{g,n} \int \mathcal{D}X^\mu e^{-S_\sigma[X^\mu, h_{ab}]}\, \widehat{V}_1 \cdots \widehat{V}_n \ , \tag{1.1}$$

we do not know how to perform this path integral. In particular, the path integral is endowed with huge gauge symmetries, WS diffeomorphism and Weyl symmetry. How can we treat integration measure and fix gauge in the path integral?

To answer to these question, we will study quantization procedure via path integral, which is often called **modern covariant quantization**. This method uses the analogue of the Faddeev-Popov method of gauge theories. Furthermore, the physical state condition is implemented via the BRST symmetry.

After gauge-fixing the reparametrization and Weyl symmetry, the integral over h_{ab} turns into a path-integral over ghost CFT, which has $c = -26$. Therefore, in order for the theory to be anomaly-free, the original theory has to be a CFT with $c = 26$.

Faddeev-Popov gauge fixing

The integration measure $[\mathcal{D}h_{ab}]_{g,n}$ is over all the metric s on 2d surfaces with genus g and n marked points. However, this integral has Weyl symmetry and wold-sheet diffeomorphisms under which the world-sheet metric is transformed as

$$h^\zeta_{ab}(\widetilde{\sigma}) = e^{2\omega(\sigma)} \frac{\partial \sigma^c}{\partial \widetilde{\sigma}^a} \frac{\partial \sigma^d}{\partial \widetilde{\sigma}^b} h_{cd}(\sigma) \ . \tag{1.2}$$

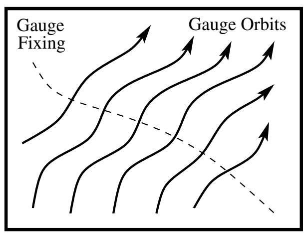

Figure 1

These symmetries are redundant and integrating along these directions just gives rise to the volume of the symmetry group. (In Figure 1, the solid arrows schematically draws gauge redundancy and the dotted line shows physically distinct configurations.) Thus, we need to carry out gauge-fixing.

Thankfully, there is a standard method to fix gauge, introduced by Faddeev and Popov. A basic idea is to insert the identity of the following form in the path integral:

$$1 = \int \mathcal{D}\zeta \; \delta(h - \hat{h}^{\zeta}) \det\left(\frac{\delta \hat{h}^{\zeta}}{\delta\zeta}\right) \tag{1.3}$$

where the Jacobian factor $\det\left(\frac{\delta \hat{h}^{\zeta}}{\delta\zeta}\right)$ is called **Faddeev-Popov determinant** and we denote it by $\Delta_{FP}[\hat{h}]$. The insertion of this identity into the path integral fixes the metric as $\hat{h}$ because of the delta function. In addition, because $\int \mathcal{D}\zeta$ integral only contributes an infinite multiplicative factor, we can discard this integral. Therefore, after Faddeev-Popov gauge fixing, the form of the path integral can be schematically written as

$$Z[\hat{h}] = \int \mathcal{D}X \; \Delta_{FP}[\hat{h}]\, e^{-S_{\sigma}[X,\hat{h}]} \tag{1.4}$$

We should make several remarks about this procedure.

• First, the conformal Killing transformations are residual gauge symmetries not fixed above. We have to throw away these residual gauge symmetries in the path integral in order to avoid over-counting. Indeed we will be careful to fix this extra residual gauge freedom when computing string amplitudes.

• Second, there are caveats related to global properties of the world-sheet Riemann surface $\Sigma_{g,n}$. In fact, metrics on a Riemann surface encodes the information of "shape" of the Riemann surface $\Sigma_{g,n}$ called **complex moduli**, which is not accounted for by local gauge transformations ζ. The space which parametrizes 'shape" of the Riemann surface is called **moduli space of Riemann surface** $\Sigma_{g,n}$ which is $(6g-6+2n)$-$\dim_{\mathbb{R}}$:

$$\mathcal{M}_{g,n} := \frac{[\mathcal{D}h_{ab}]_{g,n}}{\text{Diff} \times \text{Weyl}}$$

Therefore the path integral actually involves integral over $\mathcal{M}_{g,n}$ as well. At this moment, we postpone both the issues and will come back to them in the next lecture on string amplitudes.

Now let us take an infinitesimal version of (1.2) where a Weyl transformation is parameterized by $\omega(\sigma)$ and an infinitesimal diffeomorphism by $\delta\sigma^{\alpha} = \epsilon^{\alpha}(\sigma)$. Subsequently, the change of the metric is read off

$$\delta\hat{h}_{ab} = 2\omega\hat{h}_{ab} + \nabla_a\epsilon_b + \nabla_b\epsilon_a := 2\widetilde{\omega}\hat{h}_{ab} + (P\cdot\epsilon)_{ab}$$

where we decompose it into

$$\begin{aligned}(P\cdot\epsilon)_{ab} &= \nabla_a\epsilon_b + \nabla_b\epsilon_a - h_{ab}(\nabla\cdot\epsilon)\\ \widetilde{\omega} &= \omega + \frac{1}{2}(\nabla\cdot\epsilon)\ .\end{aligned} \tag{1.5}$$

Indeed, the operator P maps vectors ϵ_a to symmetric traceless 2-tensors $(P \cdot \epsilon)_{ab}$. Thus, the Faddeev-Popov determinant can be written as

$$\Delta_{FP}[\hat{h}] = \det \frac{\delta(P \cdot \epsilon, \widetilde{\omega})}{\delta(\epsilon, \omega)} = \det \begin{vmatrix} P & 0 \\ * & 1 \end{vmatrix} = \det P \ .$$

To compute $\det P$, we use **Faddeev-Popov ghosts**, which can be understood as an infinite-dimensional version of the following integral. Given a matrix M_{ij}, its determinant can be expressed as a Grassmann integral

$$\int \prod_{i=1}^{n} d\psi_i d\theta_i \exp(\theta_i M_{ij} \psi_j) = \det M \ .$$

where θ, ψ are Grassmann variables. Accordingly, we introduce anti-commuting fermionic fields, c^a (ghosts) and b_{ab} (anti-ghost) where b_{ab} transforms as a symmetric traceless tensor and c^a as a vector. Then, we can express

$$\Delta_{FP}[\hat{h}] = \int \mathcal{D}b\mathcal{D}c \exp\left(\frac{i}{2\pi} \int d^2\sigma \sqrt{\hat{h}}\, b^{ab} (P \cdot c)_{ab}\right) := \int \mathcal{D}b\mathcal{D}c \ \exp[iS_{\text{gh}}] \ ,$$

where the ghost action can be written as

$$S_{\text{gh}} = \frac{1}{2\pi} \int d^2\sigma \sqrt{-\hat{h}} \ b^{ab} \nabla_a c_b \ . \tag{1.6}$$

Something lovely has happened. Although the ghost fields were introduced as some auxiliary constructs, they now appear on the same footing as the dynamical fields X. Consequently the Faddeev-Popov gauge fixing results in a fermionic 2d CFT, usually called bc **ghost CFT**.

Let us make some remarks about the equation of motion of S_{gh}

- The equation of motion for ca is given by $P \cdot c = 0$. Therefore the solutions for c are in one-to-one correspondence with the **conformal Killing vectors**, which are the generators of the residual symmetry.

- The equation of motion for b_{ab} is $\nabla_a b^{ab} = 0$. We will understand the geometric meaning of these equations when discussing the moduli space of Riemann surfaces.

To understand the properties of bc ghost CFT, it is convenient to use Euclidean signature so that we will perform Wick rotation in what follows. Then, the factor of i in the action disappears. The expression for the full partition function (1.4) is

$$Z[\hat{h}] = \int \mathcal{D}X\mathcal{D}b\mathcal{D}c \ \exp\left(-S_\sigma[X, \hat{h}] - S_{\text{gh}}[b, c, \hat{h}]\right) \ . \tag{1.7}$$

bc ghost CFT

Now let us study the bc ghost CFT more in detail. For this purpose, we pick the conformal gauge $\hat{h}_{ab} = e^{2\omega}\delta_{ab}$. In this metric, the non-trivial Christofell connections are $\Gamma^z_{zz} = 2\partial\omega$, $\Gamma^{\bar{z}}_{\bar{z}\bar{z}} = 2\bar{\partial}\omega$ so that the ghost action can be written as

$$\begin{aligned} S_{\text{ghost}} &= \frac{1}{2\pi}\int d^2z \ \left(b_{zz}\nabla_{\bar{z}}c^z + b_{\bar{z}\bar{z}}\nabla_z c^{\bar{z}}\right) \\ &= \frac{1}{2\pi}\int d^2z \ b_{zz}\,\partial_{\bar{z}}c^z + b_{\bar{z}\bar{z}}\,\partial_z c^{\bar{z}} \end{aligned} \tag{1.8}$$

For the sake of simplicity, let us define

$$b = b_{zz}\ , \qquad \bar{b} = b_{\bar{z}\bar{z}}\ , \qquad c = c^z\ , \qquad \bar{c} = c^{\bar{z}}\ .$$

Then, the action simplifies to

$$S_{\text{gh}} = \frac{1}{2\pi}\int d^2z \ \left(b\,\bar{\partial}c + \bar{b}\,\partial\bar{c}\right)$$

Which gives the equations of motion

$$\bar{\partial}b = \partial\bar{b} = \bar{\partial}c = \partial\bar{c} = 0$$

So we see that b and c are holomorphic fields, while $\bar{b}$ and $\bar{c}$ are anti-holomorphic.

We can compute the OPEs of these fields using the standard path integral techniques that we learnt before. In what follows, we will just focus on the holomorphic piece of the CFT. We have, for example,

$$0 = \int \mathcal{D}b\mathcal{D}c\ \frac{\delta}{\delta b(z)}\ \left[e^{-S_{\text{gh}}}\,b(w)\right] = \int \mathcal{D}b\mathcal{D}c\ s^{-S_{\text{gh}}}\left[-\frac{1}{2\pi}\,\bar{\partial}c(z)\,b(w) + \delta(z-w)\right]$$

which tells us that

$$\bar{\partial}c(z)\,b(w) = 2\pi\,\delta(z-w)$$

Similarly, looking at $\delta/\delta c(z)$ gives

$$\bar{\partial}b(z)\,c(w) = 2\pi\,\delta(z-w)$$

We can integrate both of these equations using our favorite formula $\bar{\partial}(1/z) = 2\pi\delta(z,\bar{z})$. We learn that the OPEs between fields are given by

$$\begin{aligned} b(z)\,c(w) &= \frac{1}{z-w} + \dots \\ c(w)\,b(z) &= \frac{1}{w-z} + \dots \end{aligned}$$

In fact the second equation follows from the first equation and Fermi statistics. The OPEs of $b(z)\,b(w)$ and $c(z)\,c(w)$ have no singular parts because they vanish as $z \to w$.

In any CFT, it is of most importance to find the form of the stress energy tensor. The stress energy tensor is obtained via Noether's theorem with respect to world sheet transformations $\delta z = \epsilon(z)$, under which

$$\delta b = (\epsilon\partial + 2(\partial\epsilon))b \ , \qquad \delta c = (\epsilon\partial - (\partial\epsilon))c \ .$$

Indeed both b and c are primary fields with weights $h = 2$ and $h = -1$, respectively, that can be easily seen from their index structure b_{zz} and c^z. From these rules, one can deduce the form of the stress-energy tensor

$$T^{\text{gh}}(z) = -2 : b(z)\partial c(z) : + : c(z)\partial b(z) :$$

In fact, this form can be obtained from the first principle, namely the variation of the action under the metric (Exercise).

The OPEs of b and c with the stress tensor are

$$\begin{aligned} T^{\text{gh}}(z)\, c(w) &= -\frac{c(w)}{(z-w)^2} + \frac{\partial c(w)}{z-w} + \ldots \\ T^{\text{gh}}(z)\, b(w) &= \frac{2b(w)}{(z-w)^2} + \frac{\partial b(w)}{z-w} + \ldots \end{aligned}$$

confirming that b and c have weight 2 and -1,

Finally, we can compute the TT OPE to determine can be written as (Exercise)

$$T^{\text{gh}}(z)\, T^{\text{gh}}(w) = \frac{-13}{(z-w)^4} + \frac{2T^{\text{gh}}(w)}{(z-w)^2} + \frac{\partial T^{\text{gh}}(w)}{z-w} + \ldots$$

Now one can read off the central charge of the bc ghost system which is

$$c^{\text{gh}} = 2(-13) = -26 \ .$$

We learnt that the Weyl symmetry is anomalous unless $c = 0$. Since the Weyl symmetry is a gauge symmetry, the theory must be Weyl anomaly-free. Since the total central charge of the string sigma model and ghost theory (1.7) is given by $c = c^X + c^{\text{gh}}$, the dimension of the target space must be

$$D = 26.$$

Again, we obtain the critical dimension of the bosonic string theory!

2 BRST quantization

In 4d non-Abelian QFT, the Lagrangian with Faddeev-Popov ghosts has the continuous symmetry, called **BRST symmetry** (Becchi-Rouet-Stora-Tyupin). The BRST symmetry is generated by a nilpotent charge Q_B with $Q_B^2 = 0$ that commutes with the Hamiltonian. The nilpotency of the BRST charge has strong consequences. Since all physical states must be BRST-invariant, we require a physical state is annihilated by Q_B

$$Q_B|\text{phys}\rangle = 0\ .$$

However, one can always add a state of the form $Q_B|\chi\rangle$ since this state will be annihilated by Q_B because of the nilpotency. However, this state is orthogonal to all physical states and therefore it is a **null state**. Thus, two states related by

$$|\psi'\rangle = |\psi\rangle + Q_B|\chi\rangle$$

have the same inner products and are indistinguishable. This is the remnant in the gauge-fixed version of the original gauge symmetry. As a result, the Hilbert space of physical states is isomorphic to the Q_B-cohomology, i.e.

$$\mathcal{H}^{\text{phys}} \cong \frac{\mathcal{H}^{Q_B\text{-closed}}}{\mathcal{H}^{Q_B\text{-exact}}}$$

This elegant method to determine the physical Hilbert space in a gauge fixed action with ghosts is known as **BRST quantization**.

We are now ready to apply this formalism to the bosonic string. Expressed in the world-sheet light-cone coordinates, we obtain the following BRST transformations (Exercise):

$$\begin{aligned}
&\delta_B X^\mu = i\epsilon(c\partial + \overline{c}\overline{\partial})X^\mu\,,\\
&\delta_B c = i\epsilon c\partial c \qquad \delta_B\overline{c} = i\epsilon\overline{c}\overline{\partial}\overline{c}\,,\\
&\delta_B b = i\epsilon(T^X + T^{\text{gh}}) \qquad \delta_B\overline{b} = i\epsilon(\overline{T}^X + \overline{T}^{\text{gh}})\,.
\end{aligned}$$

Actually, this can also be shown using the BRST formalism [Pol98, §4]. The Noether theorem tells us the holomorphic part of the BRST current takes the form (Exercise)

$$\begin{aligned}
j_B &= c(z)T^X(z) + \frac{1}{2} :c(z)T^{\text{gh}}(z): + \frac{3}{2} :\partial^2 c(z): \\
&= c(z)T^X(z) + :b(z)c(z)\partial c(z): + \frac{3}{2} :\partial^2 c(z):\ .
\end{aligned} \tag{2.1}$$

and the BRST charge is defined by

$$Q_B = \oint \frac{dz}{2\pi i}\, j_B\,.$$

Indeed the anomaly now shows up in Q_B^2 so that the BRST charge is nilpotent if and only if $D = 26$. Furthermore, using the mode expansions of b and c

$$b(z) = \sum_{b\in\mathbb{Z}} \frac{b_m}{z^{m+2}} \qquad c(z) = \sum_{c\in\mathbb{Z}} \frac{c_m}{z^{m-1}}\ ,$$

we can express the BRST charge in terms of the X^μ Virasoro operators and the ghost oscillators as

$$Q_B = \sum_n c_n (L^X_{-n} - \delta_{n,0}) + \sum_{m,n} \frac{m-n}{2} :c_m c_n b_{-m-n}: . \tag{2.2}$$

In the case of closed strings, there is of course the anti-holomorphic part $\overline{Q}_B$, and the total BRST charge is $Q_B + \overline{Q}_B$.

We will find the physical spectrum in the BRST context. According to our previous discussion, the physical states will have to be annihilated by the BRST charge, and not be of the form $Q_B|\ \rangle$.

First we have to describe our extended Hilbert space that includes the ghosts. As far as the X^μ oscillators are concerned the situation is the same as in the previous sections, so we need only be concerned with the ghost Hilbert space. The full Hilbert space will be a tensor product of the two. Since the ghosts generate a two-state spin system $|\uparrow\rangle, |\downarrow\rangle$, we can write all the states can be generated from $|k,\uparrow\rangle, |k,\downarrow\rangle$ by acting creation operators where $|k\rangle = |0;k\rangle$ denotes the vacuum of the matter theory.

For the ghost system, we impose the positive ghost oscillator modes annihilate the states

$$b_{n>0}|\uparrow\rangle = b_{n>0}|\downarrow\rangle = c_{n>0}|\uparrow\rangle = c_{n>0}|\downarrow\rangle = 0$$

However, there is a subtlety because of the presence of the zero modes b_0 and c_0 which satisfy $b_0^2 = c_0^2 = 0$ and $\{b_0, c_0\} = 1$ (Exercise).

From the light-cone quantization, we know that there is only one vacuum called Tachyon. So we have to pick the ghost vacuum among the two spin states. For this purpose, we further impose one more condition, namely

$$b_0|\mathrm{phys}\rangle = 0 . \tag{2.3}$$

This is sometimes called the **Siegel gauge** [GSW87, §3.2]. Thus, under this condition, we have

$$\begin{aligned} b_0|\downarrow\rangle &= 0 , \qquad & b_0|\uparrow\rangle &= |\downarrow\rangle , \\ c_0|\uparrow\rangle &= 0 , \qquad & c_0|\downarrow\rangle &= |\uparrow\rangle . \end{aligned}$$

Imposing also (2.3) implies that the correct ghost vacuum is $|\downarrow\rangle$. We can now create states from this vacuum by acting with the negative modes of the ghosts b_m, c_n. We cannot act with c_0 since the new state does not satisfy the Siegel condition (2.3). Now, we are ready to describe the physical states in the open string. Note that since Q_B in (2.2) has "level" zero, we can impose BRST invariance on physical states level by level.

At level zero, there is only one state that is the total vacuum $|k,\downarrow\rangle$

$$0 = Q_B|k,\downarrow\rangle = (L_0^X - 1)c_0|k,\downarrow\rangle = (L_0^X - 1)|k,\uparrow\rangle .$$

Because we have the mode expansion of the 0th Virasoro generator

$$L_0^X = \alpha' p^2 + \alpha_{-1} \cdot \alpha_1 + \cdots \ ,$$
$$L_0^{\rm gh} = b_{-1} c_1 - c_{-1} b_1 + \cdots \ , \tag{2.4}$$

the only non-trivial condition at level zero is $m^2 = -1/\alpha'$, which is the tachyon. Note that the Virasoro generators of the bc ghost is

$$L_m^{\rm gh} = \sum_{m\in\mathbb{Z}} (2m-n) : b_n c_{m-n} : - \ \delta_{m,0} \ .$$

At the first level, the possible operators that can act on the vacuum $|k,\downarrow\rangle$ are α^μ_{-1}, b_{-1} and c_{-1}. The most general state of this form is then

$$|\psi\rangle = (\zeta \cdot \alpha_{-1} + \beta b_{-1} + \gamma c_{-1})|k,\downarrow\rangle \ , \tag{2.5}$$

which has 28 parameters: a 26-dimensional vector ζ_μ and two more constants β, γ. From the definition of j_B, one can derive that

$$\{Q_B, b_0\} = L_0^X + L_0^{\rm gh} - 2.$$

From $b_0|\psi\rangle = 0$, we deduce $0 = \{Q_B, b_0\}|\psi\rangle = (L_0^X + L_0^{\rm gh} - 2)|\psi\rangle$. This yields the mass shell condition $k^2 = 0$. In addition, the BRST condition demands

$$0 = Q_B|\psi\rangle = 2((k\cdot\zeta)c_{-1} + \beta k \cdot \alpha_{-1})|k,\downarrow\rangle \ .$$

This only holds if $k \cdot \zeta = 0$ and $\beta = 0$. Therefore the BRST condition removes the unphysical anti-ghost excitations as well as all polarizations that are not orthogonal to the momentum, thereby eliminating 2 out the $26 + 2$ original states. So there are only 26 parameters left.

Next we have to make sure that this state is not Q_B-exact: a general state $|\chi\rangle$ is of the same form as (2.5), but with parameters ζ'^μ, β' and γ'. So the most general Q_B-exact state at this level with $k^2 = 0$ will be

$$Q_B|\chi\rangle = 2(k\cdot\zeta' c_{-1} + \beta' k \cdot \alpha_{-1})|k,\downarrow\rangle \ .$$

This means that the c_{-1} part in (2.5) is BRST-exact and that the polarization has the equivalence relation $\zeta_\mu \sim \zeta_\mu + 2\beta' k_\mu$. This leaves us with the 24 physical degrees of freedom we expect for a massless vector particle in 26 dimensions. In sum, the physical state at level 1 is

$$\{|\zeta; k\rangle \ ; \quad k\cdot\zeta = 0\} / \ \zeta_\mu \sim \zeta_\mu + 2\beta' k_\mu \ .$$

The same procedure can be followed for the higher levels. In the case of the closed string we have to include the barred operators, and of course we have to use $Q_B + \overline{Q}_B$.

References

[GSW87] M. B. Green, J. H. Schwarz, and E. Witten. *Superstring Theory Vol. 1,2*. Cambridge University Press, 1987.

[Pol98] J. Polchinski. *String theory. Vol. 1, 2*. Cambridge University Press, 1998.