Homework 8: Due at class on April 30

1. Let us define $S^3 = \{(x^0, x^1, x^2, x^3) \in \mathbb{R}^4 | \sum_{i=0}^3 (x^i)^2 = 1\}$ and $S^1 = \{(x^0, x^1, x^2, x^3) \in \mathbb{R}^4 | (x^0)^2 + (x^1)^2 = 1\}$. Then, show that $S^3 \backslash S^1$ is homotopic to S^1.

2. Let us identify $S^2 = \mathbb{C} \cup \{\infty\}$. Then, a holomorphic map $g : \mathbb{C} \to \mathbb{C}; z \mapsto z^n$ ($n \in \mathbb{Z}$) can be extended to $g : S^2 \to S^2$. Find the mapping degree $\deg g$ of g.

3. **Fundamental theorem of algebra**

   We define $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = z^n + a_1 z^{n-1} + \cdots + a_n$ for $n \geq 1$. In addition, by writing $z = x + iy$, we define one-form

$$\omega = \mathrm{Im}\frac{dz}{z} = \frac{-ydx}{x^2 + y^2} + \frac{xdy}{x^2 + y^2} \ .$$

Then, show that

$$\frac{1}{2\pi} \int_{C_R} f^* \omega = n \ ,$$

where C_R is the circle with sufficiently large radius R. (Hint: construct homotopy between f and g above.) If there were no zero points $f(z) = 0$ inside C_R, show that

$$\frac{1}{2\pi} \int_{C_R} f^* \omega = 0$$

by using the Stokes theorem.

4. Find Poincaré dual pairs (non-trivial intersection pairings) in the real-valued homology group $H_\ell (\Sigma_g; \mathbb{R})$ of a Riemann surface Σ_g of genus g.