Lecture 2

Note that in this lecture we use α' rather than string tension $T = \frac{1}{2\pi\alpha'}$. World-sheet(WS) index here is denoted by a, b rather than α, β. h_{ab} is WS metric; do not confuse with the induced metric $h_{\alpha\beta} = \eta_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu$ used in the last lecture. Convention in this note basically follows Polchinski's one, though we adopt slightly different normalization for Noether current in Sec. 2.4.

1 Towards string interaction

What we have learnt is

- Mass spectrum of a string by light-cone quantization in flat space-time (ST).

What we have NOT learnt is

- string interactions,
- extension to curved background (curved ST).

Especially because of the first reason(string interaction) we want to express our analysis in terms of *path integral*, which is more natural to illustrate string interaction. (If you are not familiar with path integral, consult the appendix of Polchinski Vol.1. for derivation.) See Fig. 1, which is a naive but appropriate extension from quantum field theory (QFT) interaction to string interaction. Each end of the cylinders in

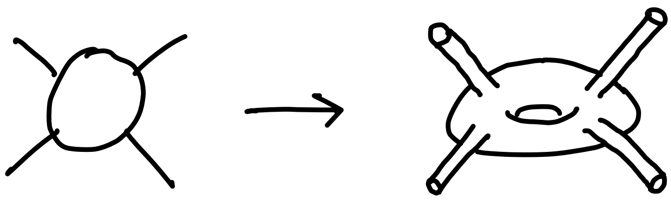

Figure 1: A Feynman diagram of QFT (left) and a string interaction diagram (right). Each end of the cylinders connecting to the torus(Riemann surface) in the string interaction diagram corresponds to an initial/final state.

the Figure corresponds to an initial/final state of string. It is quite cumbersome to use *state* expression. Instead, we will use vertex operator $\widehat{V}$ to express those states, which is explained in the following subsection. Given that we adopt the operator expression the string interaction is written as in Fig. 2 as well as

$$A_n = \sum_g \int (\mathcal{D}h_{ab})_{g,n} \int \mathcal{D}X^\mu e^{-S_\sigma[X^\mu, h_{ab}]} \widehat{V}_1 \cdots \widehat{V}_n \ ,$$

where n is the number of in-coming/out-going strings, and g is the number of holes(genus) of the Riemann surface (genus is 1 in the figures).

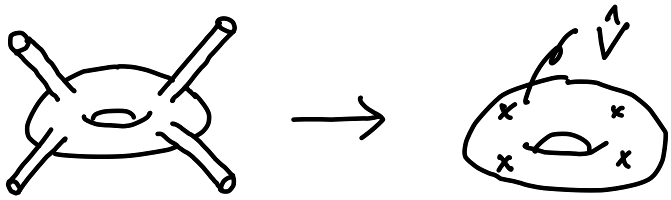

Figure 2: String interaction in state expression (left) and operator expression (right).

1.1 Path integral method

In path integral method expectation values(partition function, 2pt func etc.) are given in the following form

$$\langle \mathcal{O}[X] \rangle = \int \mathcal{D}X e^{iS[X]} \mathcal{O}[X] \ , \qquad S = \frac{1}{4\pi\alpha'} \int d\sigma d\tau \left\{ (\partial_\tau X)^2 - (\partial_\sigma X)^2 \right\} \ ,$$

where $\mathcal{O}$ is an arbitrary operator (eg. for partition function $\mathcal{O} = 1$, for 2pt func $\mathcal{O} = X^\mu X^\nu$). Note that we often omit the (super-)sub-script μ if it is not crucial. We usually **Wick rotate** ($\tau \to -it$) the theory so that it converges.

$$\langle \mathcal{O}[X] \rangle = \int \mathcal{D}X e^{-S_E[X]} \mathcal{O}[X] \ , \qquad S_E = \frac{1}{4\pi\alpha'} \int d\sigma dt \left\{ (\partial_t X)^2 + (\partial_\sigma X)^2 \right\} \ .$$

The subscript E will be omitted hereafter, and we will always work in the WS Euclidean signature.

(Even if you do not know path integral well) What we need to understand (at least for today) is that we can manipulate it as like normal integral:

- total derivative vanishes (assumption)

$$0 = \int \mathcal{D}X \frac{\delta}{\delta X} \left(e^{-S_E[X]} \mathcal{O}[X] \right) \ .$$

Note that with subscripts the functional derivative gives

$$\frac{\delta X^\mu(w)}{\delta X^\nu(z)} = \delta^\mu_\nu \delta^2(z - w) \ .$$

1.2 State-operator correspondence

As we have seen the string sigma model is Weyl invariant:

$$S_\sigma \left[X^\mu, h_{ab} \right] = S_\sigma \left[X^\mu, e^{\phi(\sigma)} h_{ab} \right] \ .$$

This (and together with WS diff) can be regarded as a property of the matter theory as we saw in the previous lecture, and we say that matter theory of X is conformally invariant or the matter theory is a conformal field theory (CFT). We can use this freedom to set WS to be $\mathbb{R}^2$ (recall that the topology of (closed) string is cylinder).

$$\begin{aligned}\mathbb{R}^2: \quad & ds^2 = dx^2 + dy^2 = dr^2 + r^2 d\theta^2 \\ & \quad = r^2 \left[d(\log r)^2 + d\theta^2 \right] , \\ \text{Cylinder}: \quad & ds^2 = dt^2 + d\sigma^2 ,\end{aligned}$$

where the overall factor r^2 in $\mathbb{R}^2$ is identified with the Weyl factor $e^{\phi(\sigma)}$, and can be removed. So we can identify:

$$t = \log r , \qquad \sigma = -\theta .$$

The reason for the minus sign will be clear later. This identification tells us that there is a correspondence between

- CFT on a semi-infinite cylinder $(\sigma, t;\ t \leq 0)$, and
- CFT on a disk $(x^2 + y^2 \leq 1)$ with the origin removed ,

provided that we choose the boundary conditions to be the same. Especially, the initial *state* of the string $(t = -\infty)$ corresponds to a *local operator* inserted at the origin, which is called vertex operator (see Fig. 3). This is the state-operator

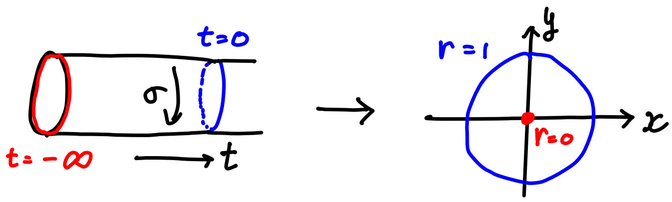

Figure 3: State-operator correspondence. Initial state(red circle) is mapped to a local operator(red dot) at the origin.

correspondence and we will express string states by the vertex operators (see again Fig. 2).

Now you may wonder if there is any way to express commutation relations in terms of operators so that we can do parallel procedures of the canonical quantization in terms of operators. This is what we do in the next section.

2 Conformal field theory in 2d

We will learn operator analysis of 2d conformal field theory (CFT), including OPE, Ward-Takahashi identity etc., and we will see Virasoro algebra, which is associated to the conformal symmetry.

2.1 Massless scalar theory in 2d

We study massless scalar fields as the easiest example. We adopt Euclidean metric $\eta_{ab} = \mathrm{diag}(+,+)$, though the procedure is parallel for Lorentz metric. The action of massless scalar fields X^μ ($\mu = 1, 2, \cdots, D$) is given by

$$S = \frac{1}{4\pi\alpha'} \int d^2x \ (\partial_1 X \cdot \partial_1 X + \partial_2 X \cdot \partial_2 X) \ ,$$

where $d^2x = dx_1 dx_2$, $x_1 = x$, $x_2 = y$, and $\partial_i = \frac{\partial}{\partial x_i}$. (The subscript E will be omitted hereafter.)

It is convenient to introduce complex coordinates and accordingly vectors, metric etc.

- Coordinates

$$z = x + iy \ , \qquad \overline{z} = x - iy \ .$$

- Derivatives

$$\partial \equiv \partial_z = \frac{\partial}{\partial z} = \frac{1}{2}(\partial_1 - i\partial_2) \ , \quad \overline{\partial} \equiv \partial_{\overline{z}} = \frac{\partial}{\partial \overline{z}} = \frac{1}{2}(\partial_1 + i\partial_2) \ ,$$

  so that $\partial z = 1$, $\partial \overline{z} = 0$, $\overline{\partial} z = 0$, $\overline{\partial}\overline{z} = 1$.

- Vectors

$$v^z = v^1 + iv^2 \ , \qquad v^{\overline{z}} = v^1 - iv^2 \ .$$

- Metric

$$h_{z\overline{z}} = h_{\overline{z}z} = \frac{1}{2} \ , \quad h_{zz} = h_{\overline{z}\overline{z}} = 0 \ , \quad h^{z\overline{z}} = h^{\overline{z}z} = 2 \ , \quad h^{zz} = h^{\overline{z}\overline{z}} = 0 \ .$$

  So lower-index vectors are $v_z = h_{z\overline{z}} v^{\overline{z}} = \frac{1}{2}(v^1 - iv^2)$ etc.

- Integration measure

$$d^2z = dz d\overline{z} = 2dxdy = 2d^2x \ ,$$

  where the factor 2 comes from the Jacobian. One can also write

$$\sqrt{h} d^2z = \sqrt{|\det h_{ab}|} d^2z = d^2x \ .$$

- Delta function

$$\delta^2(z) = \delta(z)\delta(\overline{z}) = \frac{1}{2}\delta(x)\delta(y) = \frac{1}{2}\delta^2(x) \ ,$$

  so that

$$1 = \int d^2z \ \delta^2(z) = \int d^2x \ \delta^2(x) \ .$$

Note that although z and $\overline{z}$ are complex conjugate of each other, we will treat them as independent complex numbers.

Stokes' theorem (it is called Green's theorem in 2d) will be used frequently.

$$\int_D d^2z\left(\partial\overline{V}+\overline{\partial}V\right)=\frac{1}{i}\oint_{\partial D}\left(V dz-\overline{V}d\overline{z}\right)\ ,$$

where D is an arbitrary region (typically small disk) and ∂D is its boundary. V and $\overline{V}$ are independent factor. Using the Stokes' theorem one can prove following equation.

$$\partial\overline{\partial}\log|z|^2=2\pi\delta^2(z)\ . \tag{2.1}$$

(Keep this in mind. We will use this later.)

With the complex coordinate convention, the action becomes

$$S=\frac{1}{2\pi\alpha'}\int d^2z\ \partial X\cdot\overline{\partial}X\quad\left(=\frac{1}{4\pi\alpha'}\int\sqrt{h}d^2z\ h^{ab}\partial_a X\cdot\partial_b X\right)\ .$$

The equation of motion (E.O.M) is given by

$$\begin{aligned}\delta S&=\frac{1}{2\pi\alpha'}\int d^2z\left(\partial X\cdot\overline{\partial}(\delta X)+\partial(\delta X)\cdot\overline{\partial}X\right)\\&=-\frac{1}{\pi\alpha'}\int d^2z\ \delta X\cdot\partial\overline{\partial}X=0\ ,\\ \therefore\quad &\partial\overline{\partial}X^\mu=0\ .\end{aligned}$$

General solution to the E.O.M is

$$\begin{aligned}X^\mu(z,\overline{z})&=X^\mu_R(z)+X^\mu_L(\overline{z}),\\ X^\mu_R(z)&=\frac{1}{2}x^\mu-i\frac{\alpha'}{2}p^\mu\log z+i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq0}\frac{1}{n}\frac{\alpha^\mu_n}{z^n}\ ,\\ X^\mu_L(\overline{z})&=X^\mu_R(\overline{z})\big|_{\alpha_n\to\tilde{\alpha}_n}\ .\end{aligned}$$

Notice that

$$\begin{aligned}z&=re^{i\theta}=e^{\log r+i\theta}=e^{i\tau-i\theta}\\&=e^{i\sigma^-}\ ,\\ \overline{z}&=e^{i\sigma^+}\ ,\end{aligned}$$

and compare to

$$X^\mu_R(\sigma^-)=\frac{1}{2}x^\mu+\frac{1}{2}\alpha'p^\mu\,\sigma^-+i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq0}\frac{1}{n}\,\alpha^\mu_n\,e^{-in\sigma^-}\ .$$

2.2 Operator analysis (Path integral quantization)

Let us consider quantum version of the E.O.M by using a path integral formulation. As like a normal integral we assume that "total derivative" vanishes in path integral. For example, in the massless free scalar case we have

$$0 = \int \mathcal{D}X \ \frac{\delta}{\delta X} e^{-S[X]} = \int \mathcal{D}X \ e^{-S[X]} \frac{1}{\pi\alpha'} \partial\overline{\partial} X = \frac{1}{\pi\alpha'} \langle \partial\overline{\partial} X \rangle \ ,$$

which is nothing but Ehrenfest's theorem. The E.O.M is satisfied as an operator equation if there is no other operator nearby.

Furthermore, the same procedure can be done with operator insertions. The easiest example is

$$0 = \int \mathcal{D}X \ \frac{\delta}{\delta X_\mu(z,\overline{z})} \left(e^{-S[X]} X^\nu(w,\overline{w}) \right) = \langle \frac{1}{\pi\alpha'} \partial_z \partial_{\overline{z}} X^\mu(z,\bar{z}) X^\nu(w,\bar{w}) + \eta^{\mu\nu} \delta^2(z-w) \rangle \ .$$

Therefore,

$$\partial\overline{\partial} X^\mu(z,\bar{z}) X^\nu(w,\bar{w}) = -\pi\alpha' \eta^{\mu\nu} \delta^2(z-w) \ ,$$

where note that this is operator equation (not classical equation) and we simply omitted $\langle\rangle$ symbols. The right hand side is understood due to a quantum effect.

Recalling Eq. (2.1) we have

$$X(z,\bar{z})^\mu X^\nu(w,\bar{w}) = -\frac{\alpha'}{2} \eta^{\mu\nu} \log|z-w|^2 + \ :X(z,\bar{z}) X(w,\bar{w}): \ ,$$

where we introduced normal ordering $:\mathcal{O}:$, meaning that the divergence at $z \to w$ has been subtracted. We also write it as follows.

$$:X^\mu(z,\bar{z}) X^\nu(w,\bar{w}):= X^\mu(z,\bar{z}) X^\nu(w,\bar{w}) - \eta^{\mu\nu} G(z,w) \ ,$$

where $G(z,w) = -\frac{\alpha'}{2} \log|z-w|^2$. Notice that

$$\partial_z \partial_{\overline{z}} \ :X(z,\bar{z}) X(w,\bar{w}):= 0 \ . \tag{2.2}$$

As we will see later the divergent term is important and meaningful. This is why it is convenient to introduce the normal ordering so that we can separate the divergent terms from the other.

For an arbitrary functional of fields the OPE is expressed as follows.

$$:f[X]:= \exp\left[-\frac{1}{2} \int d^2z d^2w \ G(z,w) \frac{\delta}{\delta X^\mu(z,\overline{z})} \frac{\delta}{\delta X_\mu(w,\overline{w})} \right] f[X] \ .$$

For example,

$$\begin{aligned} :X_1 X_2 X_3 X_4 X_5: &= X_1 X_2 X_3 X_4 X_5 - (G_{12} X_3 X_4 X_5 + \cdots)_{10 \text{ terms}} \\ &+ (G_{12} G_{34} X_5 + \cdots)_{15 \text{ terms}} \ , \end{aligned}$$

where $X_i = X(z_i)$ and $G_{ij} = G(z_i, z_j)$.

Multiplication of normal ordered product is given as follows.

$$:f[X]::g[X]:= \exp\left[\int d^2z d^2w\ G(z,w)\ \frac{\delta}{\delta X^\mu(z,\bar{z})}\bigg|_f \frac{\delta}{\delta X_\mu(w,\bar{w})}\bigg|_g\right] :fg[X]: \ .$$

Example 1:

$$:\partial X(z,\bar{z})::X(w,\bar{w}):=:\partial X(z,\bar{z})X(w,\bar{w}): +\partial_z G(z,w) =:\partial X(z,\bar{z})X(w,\bar{w}): -\frac{\alpha'}{2}\frac{1}{z-w} \ .$$

Example 2:

$$\begin{aligned}:e^{ik\cdot X(z,\bar{z})}::e^{ik'\cdot X(w,\bar{w})}: &= \exp\left[(ik)\cdot(ik')\left(-\frac{\alpha'}{2}\log|z-w|^2\right)\right] :e^{ik\cdot X(z,\bar{z})+ik'\cdot X(w,\bar{w})}: \\ &= |z-w|^{\alpha' k\cdot k'}\ :e^{ik\cdot X(z,\bar{z})+ik'\cdot X(w,\bar{w})}: \ .\end{aligned}$$

2.3 Operator product expansion (OPE)

In general field theory, product of a pair of fields can be expanded by a single operator

$$\Phi^i(z)\Phi^j(w) = \sum_k C_k^{ij}(z-w)\Phi^k(w) \ .$$

Note that in 2d CFT language any local operator is regarded as an independent field (e.g. $\Phi = X, \partial X, :XX:$ etc.).

Typically, for a massless free scalar field theory we have

$$\begin{aligned}X^\mu(z,\overline{z})X^\nu(w,\overline{w}) &= -\frac{\alpha'}{2}\eta^{\mu\nu}\log|z-w|^2 + :X^\mu X^\nu(w,\overline{w}): \\ &+ \sum_{k=1}^{\infty}\frac{1}{k!}\left\{(z-w)^k\ :\partial^k X^\mu X^\nu(w,\overline{w}): + (\overline{z}-\overline{w})^k\ :\overline{\partial}^k X^\mu X^\nu(w,\overline{w}):\right\} \ .\end{aligned}$$

The second and the third terms of RHS can be understood as Taylor expansion of $:X(z)X(w):$ in terms of z. Mixing terms ($:\partial^m\overline{\partial}^n XX:$) vanish due to "E.O.M", Eq. (2.2).

As we will see, only the divergent term is important. We will introduce $\sim$ that is define as "equal up to non-divergent term" so that we can forget about non-divergent terms. For example

$$X^\mu(z,\overline{z})X^\nu(w,\overline{w}) \sim -\frac{\alpha'}{2}\eta^{\mu\nu}\log|z-w|^2 \ .$$

2.4 Noether's theorem and Ward-Takahashi identities

When an action is invariant under a certain transformation δ (namely, $\delta S = 0$) we say the theory has a (classical) symmetry. Furthermore, the measure of the path integral is also invariant under the transformation we say the theory has a symmetry at quantum level. On the other hand, if the measure is not invariant, then, we say the theory has anomaly and/or the symmetry is anomalous. It is non-trivial to see if the theory has anomaly or not.

If the theory has a symmetry there is a corresponding conserved current j^a and the space integral of its time component is the conserved charge $Q = \int_{\text{space}} j^0$, which generate the symmetry $\delta X = [Q, X]$. The current can be derived by Noether's procedure. Even if you are not familiar with the Noether's theorem we will see alternative method.

Let δ be a symmetry $\delta S = 0$ and assume it acts on a field as follows: $\delta X(z) = \epsilon(\cdots)$, where ϵ is a small parameter. If we promote the parameter to be WS coordinate dependent (i.e. $\widehat{\delta} X = \epsilon(z, \overline{z})(\cdots)$), then $\widehat{\delta} S$ is no longer zero but has to take the following form.

$$\widehat{\delta} S = \int \frac{d^2x}{2\pi} \epsilon(x, y) \partial_a j^a = \int \frac{d^2z}{2\pi} \epsilon(z, \overline{z}) \left(\partial \overline{j} + \overline{\partial} j \right) \ ,$$

where we introduced $j = j_z$ and $\overline{j} = j_{\overline{z}}$. If the parameter is constant $\delta S =$ (total derivative) $= 0$. j^a **is Noether current**. Noether current is conserved, which means that $\partial_a j^a = 0$ under assumption of E.O.M. This implies that $j(\overline{j})$ is a holomorphic(anti-holomorphic) function of z. Indeed, this is true for all the example we will see.

Ward-Takahashi identity

Now let us study the transformation of a point operator $\mathcal{O}[X](w, \overline{w})$ under the symmetry δ. For this purpose, we set

$$\epsilon(z, \overline{z}) = \begin{cases} \epsilon \quad \text{(const.)} & \text{for } z \in D_w \ , \\ 0 & \text{for } z \notin D_w \ , \end{cases}$$

where D_w is a disk containing w. Then, the variation of the path integral

$$0 = \int \mathcal{D}X \ \widehat{\delta} \left[e^{-S} \mathcal{O}(w, \overline{w}) \right] = \int \mathcal{D}X e^{-S} \ \Big[\delta \mathcal{O}(w, \overline{w}) - \widehat{\delta} S \cdot \mathcal{O}(w, \overline{w}) \Big] \, .$$

Therefore, we have

$$\begin{aligned} \delta \mathcal{O}(w, \overline{w}) &= \int \frac{d^2z}{2\pi} \epsilon(z, \overline{z}) \left(\partial \overline{j}(\overline{z}) + \overline{\partial} j(z) \right) \mathcal{O}(w, \overline{w}) \\ &= \frac{\epsilon}{2\pi i} \oint_{\partial D_w} \Big(dz \ j(z) - d\overline{z} \ \overline{j}(\overline{z}) \Big) \mathcal{O}(w, \overline{w}) \ , \end{aligned}$$

This is called the **Ward-Takahashi identity**.

Let us see an example of the free scalar field.

$$
\begin{aligned}
&S = \frac{1}{2\pi\alpha'} \int d^2z \; \partial X \cdot \overline{\partial} X \;, \qquad \delta X = \epsilon g(z) \quad \text{(holomorphic)} \\
\rightarrow \quad &\widehat{\delta} S = \frac{-2}{2\pi\alpha'} \int d^2z \; \epsilon(z, \overline{z}) \overline{\partial} \left(g \partial X \right) \\
&j = -\frac{2}{\alpha'} g \partial X \;, \qquad \overline{j} = 0.
\end{aligned}
$$

Check that $\delta X(w) = \epsilon \oint_{\partial D} \frac{dz}{2\pi i} \; j(z) X(w)$ is consistent.