Homework 6: Due at class on Dec 14

1 Bremsstrahlung

In the lecture, the IR divergence of the electron vertex is given by

$$
\begin{aligned}
\delta F_1^{\text{ren}}(q^2) &= \delta F_1(q^2) - \delta F_1(0) = \frac{\alpha}{2\pi} f_{\text{IR}}(q^2) \times \log\left(\frac{-q^2}{\mu^2}\right) + \text{ regular terms} \\
f_{\text{IR}}(q^2) &= -1 + \int_0^1 d\xi \left(\frac{m^2 - q^2/2}{m^2 - q^2\xi(1-\xi)}\right) .
\end{aligned}
\tag{1.1}
$$

This cancels with the infrared singularity in the $\mathcal{O}(\alpha)$ cross-section for a single photon emission,

$$
d\sigma\left[e^-(p) \to e^-\left(p'\right) + \gamma\right] = d\sigma_0\left[e^-(p) \to e^-\left(p'\right)\right] \times \frac{\alpha}{\pi} I(\beta, \beta') \times \frac{dE_\gamma}{E_\gamma}
$$

where

$$
I(\beta, \beta') = \int \frac{d\Omega_{\boldsymbol{k}}}{4\pi} |\boldsymbol{k}|^2 \left[\frac{2p \cdot p'}{(k \cdot p)\,(k \cdot p')} - \frac{m^2}{(k \cdot p)^2} - \frac{m^2}{(k \cdot p')^2}\right] .
$$

Show that $2 f_{\text{IR}}(q^2) = I(\beta, \beta')$.

2 1-loop correction in scalar QED

The Lagrangian of the scalar QED is given by

$$
\mathcal{L}_{\text{scalar}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_\mu \phi)^* D^\mu \phi - m^2 \phi^* \phi \tag{2.1}
$$

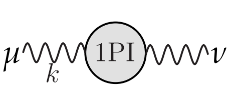
where the covariant derivative is expressed by $D_\mu = \partial_\mu + ieA_\mu$. Find the diagrams which contribute to the 1PI diagrams of the scalar propagator in the order $\mathcal{O}(e^2)$ and write down the integral expression for $\Pi_\phi^{\text{1-loop}}(k)$

$i\Pi_\phi(k) \equiv$  $= e^2 \Pi_\phi^{\text{1-loop}}(k) + \mathcal{O}(e^4)$. (2.2)

Perform momentum integration by using the dimensional regularization.

3 Vacuum polarization in scalar QED

In the scalar QED, find the diagrams which contribute to the 1PI diagrams of the photon propagator in the order $\mathcal{O}(e^2)$ and write down the integral expression for $\Pi^{\mu\nu}_{\text{1-loop}}(k)$

$i\Pi^{\mu\nu}(k) \equiv \mu$  $\nu = e^2 \Pi^{\mu\nu}_{\text{1-loop}}(k) + \mathcal{O}(e^4)$, (3.1) and perform momentum integration by using the dimensional regularization. Show that

$$i\Pi^{\mu\nu}(k) \propto \left(k^2 g^{\mu\nu} - k^\mu k^\nu\right) \tag{3.2}$$

so that $k_\mu \delta\Pi^{\mu\nu} = 0$. (Hint: there are 2 diagrams, and you have add them together to show the Ward identity.)