# Homework 8 (Due at class on May 5)

## 1 Bosonic string on a circle

### 1.1

The torus partition function of the bosonic string on a circle $S^1$ of radius $R$ is given by

$$\begin{aligned} Z^{25} &= \mathrm{Tr}\ q^{L_0-1/24}\overline{q}^{\overline{L}_0-1/24}, \\ &= |\eta(q)|^{-2}\sum_{n,w} q^{\frac{\alpha'}{4}p_R^2}\overline{q}^{\frac{\alpha'}{4}p_L^2} . \end{aligned} \tag{1.1}$$

where $n$ are KK momenta and $w$ are winding numbers. If we include the non-compact space $\mathbb{R}^{1,24}$, we have to multiply the partition function of the non-compact direction

$$Z^{1,24} = \text{const} \times |\eta(q)|^{-46}$$

By expanding out the Dedekind $\eta$-functions in $Z^{1,24}Z^{25}$, show that each term means the right hand sides of the mass formula and the level matching condition:

$$\begin{aligned} M^2 &= \frac{n^2}{R^2} + \frac{w^2R^2}{\alpha'^2} + \frac{2}{\alpha'}(N+\overline{N}-2) \\ nw &= N - \overline{N} . \end{aligned}$$

### 1.2

By using the Poisson resummation formula,

$$\sum_{n\in\mathbb{Z}} \exp\Big(-\pi a n^2 + 2\pi i b n\Big) = a^{-1/2}\sum_{m\in\mathbb{Z}} \exp\left[-\frac{\pi(m-b)^2}{a}\right] ,$$

show that (1.1) is modular-invariant.

### 1.3

Show that the partition function (1.1) of the theory at the self-dual radius $R=\sqrt{\alpha'}$ can be written as

$$Z^{25} = |\chi_1(q)|^2 + |\chi_2(q)|^2 , \qquad \text{where} \quad \chi_1 = \frac{1}{\eta}\sum_n q^{n^2} \quad \chi_2 = \frac{1}{\eta}\sum_n q^{(n+1/2)^2}$$

The $\chi_i$ are the characters of the SU(2) affine Lie algebra with level $k=1$. By expanding this expression out find the massless states from above.

### 1.4

Show that the currents in the bosonic string theory defined by

$$j^{\pm}(z) = j^1(z) \pm i\, j^2(z) := e^{\pm 2iX^{25}(z)/\sqrt{\alpha'}} \qquad j^3(z) := i\,\partial X^{25}(z)/\sqrt{\alpha'}\,,$$

satisfy the OPEs

$$j^a(z) j^b(0) \sim \frac{\delta^{ab}}{2z^2} + \frac{i\epsilon^{abc} j^c(0)}{z}\,.$$

From the OPEs, show that the oscillator modes of the currents

$$j^a(z) = \sum_{m\in\mathbb{Z}} \frac{j^a_m}{z^{m+1}}\,,$$

satisfy

$$[j^a_m, j^b_n] = \frac{m}{2}\delta_{m+n,0}\delta^{ab} + i\epsilon^{abc} j^c_{m+n}\,.$$

This infinite-dimensional algebra is called the SU(2) **affine Lie algebra with level** $k = 1$. (Check that the zero modes satisfy the SU(2) Lie algebra.)

# 2 R-R field strengths and T-duality in Type II

Let

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu} \qquad \mu = 0, \cdots, 9$$

be the Clifford algebra of SO(1,9) gamma matrices. The gamma matrices have the following hermiticity property,

$$(\Gamma^\mu)^\dagger = -\Gamma^0 \Gamma^\mu (\Gamma^0)^{-1}\,.$$

By using the chirality operator $\Gamma_{11} = \Gamma^0\Gamma^1 \cdots \Gamma^9$, chiral spinors are defined by

$$\Gamma_{11}\psi_\pm = \pm\psi_\pm\,. \tag{2.1}$$

Show that

$$\overline{\psi}_\pm \Gamma_{11} = \mp\overline{\psi}_\pm$$

where $\overline{\psi}_\pm = \psi^\dagger_\pm \Gamma^0$.

We define the R-R field strengths $G^{\mu_1\cdots\mu_{p+2}}$ as spinor bilinears

$$\text{IIA}:\ \overline{\psi}^L_- \Gamma^{\mu_1\cdots\mu_{p+2}} \psi^R_+\,, \qquad\qquad \text{IIB}:\ \overline{\psi}^L_+ \Gamma^{\mu_1\cdots\mu_{p+2}} \psi^R_+\,, \tag{2.2}$$

where $\psi^R$ ($\psi^L$) comes from the right (left) movers and

$$\Gamma^{\mu_1\cdots\mu_{p+2}} = \Gamma^{[\mu_1} \ldots \Gamma^{\mu_{p+2}]}$$

is the antisymmetric product of $(p+2)$ gamma matrices. Using the chirality (2.1) of the spinors, determine for which values of $p$ the R-R field strengths (2.2) are non-zero.

In the lecture, we learn that T-duality of the 9th direction in Type II theory acts the left-moving fermion mode

$$\psi_n^9 \to -\psi_n'^9 \,, \qquad n \in \mathbb{Z} \,.$$

The action of duality on the spinor fields is of the form

$$\overline{\psi}^L \to \overline{\psi}^L \beta_9 \,, \qquad \psi^R \to \psi^R \tag{2.3}$$

where $\beta_9 = \Gamma_{11}\Gamma^9$. Show that

$$\{\beta_9, \Gamma^9\} = 0 \,, \qquad [\beta_9, \Gamma^\mu] = 0 \,, \quad \text{for} \quad \mu \neq 9 \,.$$

Using the effect of (2.3) on the R-R field strengths (2.2), show that T-duality transforms the R-R field strengths in IIA to those in IIB, and vice versa.