

Homework 12 (Due at class on Dec 15)

Prob. 1 (S-duality)

In the lecture, the low-energy effective action of Type I string is given by

$$\begin{aligned} S_I &= S_{\text{grav}} + S_{\text{YM}} \\ S_{\text{grav}} &= \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-G} \left[e^{-2\Phi} (R + 4\partial_\mu \Phi \partial^\mu \Phi) - \frac{1}{2} |\tilde{G}_3|^2 \right] \\ S_{\text{YM}} &= -\frac{1}{2g_{10}^2} \int d^{10}x \sqrt{-G} e^{-\Phi} \text{Tr}_V |F_2|^2 . \end{aligned} \quad (1)$$

Also, the low-energy effective action of Heterotic SO(32) is given by

$$\begin{aligned} S_{\text{Het}} &= S_{\text{grav}} + S_{\text{YM}} \\ S_{\text{grav}} &= \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-G} e^{-2\Phi} \left[R + 4\partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2} |\tilde{H}_3|^2 \right] \\ S_{\text{YM}} &= -\frac{1}{2g_{10}^2} \int d^{10}x \sqrt{-G} e^{-2\Phi} \text{Tr}_V |F_2|^2 \end{aligned} \quad (2)$$

1.1 Explain why the Yang-Mills action S_{YM} in Type I (1) has $e^{-\Phi}$ whereas S_{YM} in Heterotic SO(32) (2) has $e^{-2\Phi}$.

1.2 Show that Type I (1) and Heterotic SO(32) (2) actions are related by the following field definitions

$$\begin{aligned} G_{\mu\nu}^I &= e^{-\Phi^H} G_{\mu\nu}^H , & \Phi^I &= -\Phi^H \\ \tilde{G}_3^I &= \tilde{H}_3^H , & A^I &= A^H . \end{aligned}$$

Prob. 2 (Heterotic M-theory)

2.1 Show that the length of a line interval S^1/\mathbb{Z}_2 in Heterotic M-theory is $R = g_{\text{Het}}^{\frac{2}{3}} \ell_p$ by using the following duality chain in the lecture note:

$$\text{HE} \xrightarrow{T} \text{HO} \xrightarrow{S} \text{Type I} \xrightarrow{T} \text{Type I}' \xrightarrow{\text{strong coupling}} \text{M-theory}$$

Note that T-duality on a circle S^1 relate radii and string coupling constants as

$$\tilde{R} = \ell_s^2 / R , \quad \tilde{g}_s = \ell_s g_s / R ,$$

whereas S-duality on a circle S^1 relate them as

$$\tilde{R} = R / \sqrt{g_s} , \quad \tilde{g}_s = 1 / g_s ,$$

where $\ell_s = \sqrt{\alpha'}$ and the definition of ℓ_p is as in Lecture note 12. Note that tilde denotes parameters in the dual theory.

2.2 Give an explanation how Heterotic strings and fivebranes are related to M2 and M5-branes in Heterotic M-theory up on the compactification on a segment S^1/\mathbb{Z}_2 . Namely, argue how Heterotic strings and fivebranes become M2 and M5-branes in the strong coupling regime and vice versa.

Prob. 3 (Type I with D1-brane)

Let us consider the massless spectrum of string excitations in the D1-D1 and the D1-D9 sector of a D1-brane along directions X^0, X^1 in Type I string theory and compare this to the massless fields on the worldsheet of the $SO(32)$ heterotic string.

3.1 Let us first consider the D1-D1 strings. Here X^I, ψ_{\pm}^I , $I = 2, \dots, 9$ have DD boundary conditions while $X^{\mu}, \psi_{\pm}^{\mu}$, $\mu = 0, 1$ have NN boundary conditions.

As in the bosonic open string, the NN boundary condition identifies the right and left fermion modes $\psi_n^{\mu} = \tilde{\psi}_n^{\mu}$ so that

$$\psi^{\mu}(\tau, \sigma) = \sum_{n \in \mathbb{Z} + \nu} \psi_n^{\mu} (e^{-in(\tau-\sigma)} + e^{-in(\tau+\sigma)}) ,$$

where ν takes the values 0 (R) or $\frac{1}{2}$ (NS). On the other hand, the DD boundary condition identifies the right and left fermion modes by sign $\psi_n^I = -\tilde{\psi}_n^I$ so that

$$\psi^I(\tau, \sigma) = \sum_{n \in \mathbb{Z} + \nu} \psi_n^I (e^{-in(\tau-\sigma)} - e^{-in(\tau+\sigma)}) .$$

Show that worldsheet parity $\Omega : \sigma \rightarrow \pi - \sigma$ acts on the modes as

$$\Omega \psi_n \Omega^{-1} = \pm e^{i\pi n} \psi_n , \quad +/- : \text{NN/DD} .$$

The actions of the orientifold Ω on the vacua of the D1-string given by

$$\Omega |0\rangle_{NS} = -i |0\rangle_{NS} , \quad \Omega |s_0 = \frac{1}{2}, \mathbf{s}\rangle_R = -e^{i\pi(s_1+s_2+s_3+s_4)} |s_0 = \frac{1}{2}, \mathbf{s}\rangle_R$$

with s_0 the spin in directions X^0, X^1 and s_1, \dots, s_4 the spin in the normal directions. Then, show that the orientifold projection $(1 + \Omega)/2$ keeps $\psi_{-\frac{1}{2}}^I |0\rangle$ for the normal directions and removes $\psi_{-\frac{1}{2}}^{\mu} |0\rangle$ for the tangent directions in the NS sector. In the R sector, show that the ground states **16** spanned by $|s_0 = \frac{1}{2}, \mathbf{s}\rangle$ are projected onto **8_c** by the orientifold action.

3.2 Next, let us consider D1-D9 string. Here X^I, ψ_{\pm}^I , $I = 2, \dots, 9$ have DN boundary conditions while $X^{\mu}, \psi_{\pm}^{\mu}$, $\mu = 0, 1$ have NN boundary conditions as before. In Homework 7 Problem 1, we have seen that the bosonic open string with DN boundary condition admits the following mode expansion

$$X = c + i\sqrt{\frac{\alpha'}{2}} \sum_{n \in \mathbb{Z} + 1/2} \frac{\alpha_n}{n} (e^{-in\sigma^-} - e^{-in\sigma^+}) .$$

In fact, the supersymmetry requires the periodicity of fermion $\psi(\tau, \sigma)$ (or mode ψ_n) in the R sector to be the same as for $X(\tau, \sigma)$ (or mode α_n). In the NS sector, it is the opposite (modes differ by $\frac{1}{2}$). Using this fact, compute the zero point energy in the NS and R sector. In addition, find the massless spectrum in the D1-D9 string after the GSO projection. Note that since Chan-Paton factors are allowed in the free string end, there are 32 of them.