復旦大學

本科毕业论文

论文题目：An Introduction to 3d $\mathcal{N} = 2$ Supersymmetric Gauge Theory

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Contents

摘　要

二十世纪初提出的狭义相对论和量子力学构成了现代物理的两大基石。为了将两种理论统一在一个理论框架内讨论，我们引入了量子场论这一理论。量子场论已经展示出了它惊人的解释性与可靠性，例如量子电动力学对电磁相互作用相关散射等行为的准确预测，而最广为人知的量子场论模型之一就是粒子的标准模型。标准模型包含了 61 种粒子，并且在很高的能标下与实验数据的匹配依然精准，例如在 CERN 的 LHC（大型强子对撞机）上进行的各种实验都没有离开标准模型的范围。然而量子场论有它自身的困难，其中之一就是对引力场的量子化问题。如果一个量子场论包含引力，那么传递引力相互作用的引力子将会是一种自旋为 2 的粒子。然而对于自旋大于等于 2 的粒子我们无法对它做重整化，这就意味着许多量子场论的方法不能使用。同时我们知道由于对称性自发破缺这一现象的存在，可能大自然本身秉有更加深层次的对称性。这就启发研究者们思考是否可以通过引入一些高能标下可能得到恢复的对称性使得引力可以被统一到量子场论的框架之中。当然，超出标准模型的研究还有很多种可能，比如引入新的粒子、引入额外的时空维度、引入更多对称等等。超对称理论就是候选理论之一。

超对称首先在弦理论中对二维世界面理论的研究中出现。一开始它仅仅被当做是一种纯粹的数学工具，但是不久之后研究者们便意识到它本身拥有极为深远的意义。超对称之所以被这样称呼，是因为它意味着一种玻色子和费米子之间的对称。引入超对称之后，标准模型的每一个粒子都会拥有它相应的超对称伴子。这也就让超对称同时成为解释暗物质的候选模型之一，因为标准模型粒子与它们的超对称伴子之间几乎只有引力的相互作用。使研究者们更感兴趣的地方在于，超对称理论的微扰论不存在一圈以上的高圈图，这也就意味着理论的路径积分可以被精确计算。由于粒子和它的超对称伴子质量相同，它们应当有相同的质量重整化。实际上每一级圈图中玻色子贡献的部分都会被相对应的费米子的贡献所抵消。这一点可以通过一个简单的模型进行理解：考虑我们在量子力学中讨论的简谐振子模型。当我们选择对易关系的时候，实际上我们考虑的是一个玻色简谐振子，这时我们得到系统的基态能量是 $\frac{1}{2}\hbar\omega$。而假如我们引入超对称，利用反对易关系推导费米简谐振子的基态，会得到基态能量为 $-\frac{1}{2}\hbar\omega$。可以看到，这时两种简谐振子对系统能量的贡献刚好抵消。事实上类似地，在超对称模型当中，所有的高阶微扰都会恰好相互抵消，使我们能够精确计算路径积分的性质。

超对称的生成元是旋量场，因此当它作用在玻色态时会生成一个费米态，反

之，当作用在费米态时会生成玻色态。当然，如果用超对称变换连续作用一个态两次会回到原来的态，但是会产生一个位移。超对称扩展了李代数的概念，将反对易关系引入代数结构当中，称为分级代数或直接称为超代数。我们可以这样考虑构建超对称代数：Q 作为旋量场因此代数应当是反对称的；狭义相对论要求 Q 作为算符必须满足洛伦兹变换；同时旋量还在 $SL(2,\mathbb{C})$ 下变换，这使我们可以另外加入一个指标。另外，由于我们还可以引入超过一个的超对称生成元，这使我们可以进一步给不同的超对称生成元之间加上一个指标进行区分。不同种类的生成元数量记作 $\mathcal{N}$。显然最简单的情况就是 $\mathcal{N}=1$ 的情况。本论文讨论的是三维 $\mathcal{N}=2$ 的情况。在文中我们会看到三维 $\mathcal{N}=2$ 的代数可以通过做降维从四维 $\mathcal{N}=1$ 的代数中得到。可以进行降维这个操作，是由于那些维数 d 和超对称生成元数量 $\mathcal{N}$ 之和相等的理论拥有相同的粒子谱。也就是说，对应于各种自旋的粒子数量是相同的。超对称从被构建起就是与弦论紧密相连的。超对称与弦论的结合就是大名鼎鼎的超弦理论。直至今日也是最热门的统一理论候选人之一。经常用于支持超对称的一个“证据”是，在大爆炸之后的约 10^{-39} 秒，宇宙温度将会接近 $10^{28}K$，这时我们应当有某些概念将三种基本相互作用（强相互作用、弱相互作用与电磁相互作用）统一到一个高度对称的理论中去。而一般情况下强相互作用和弱相互作用强度相差 10^{13} 倍（虽然不能直接比较）。如果将超对称引入图像当中，利用重整化的效应可以使得三种力在如此超高能标下达到统一。这个“证据”的缺点在于，我们目前在粒子加速器上可达到的能标和预期能标之间相差将近 10^{15}，这使得我们不能轻率的假设在更高的能标下不会出现新的实验证据。目前还可以通过观测高能宇宙射线等方式寻找证明超对称存在的证据。同时超对称可以构建超引力理论，在黑洞物理领域也有用武之地。

我们有很多对三维的超对称规范理论产生兴趣的原因。例如三维理论可以看作是在一个圈上研究四维理论，因此它能够反映一些四维理论的性质。同时三维 $\mathcal{N}=2$ 理论也有它自己的特点，比如在物质场的数量上没有渐近自由边界；可以存在一些新的耦合常数，比如陈–西蒙斯项，实质量等；在 $U(1)$ 理论上我们可以检验法耶特–伊利欧珀洛斯项的有效性；出现拓扑相和 BPS 粒子（例如斯格明子）等等。同时三维 $\mathcal{N}=2$ 超对称规范理论还可以由在 $\mathcal{M}_3$ 上定义的六维 $(2,0)$ 超对称规范理论做紧致化来得到。这可以引出一个当代规范理论的一个重要问题：3d-3d 对应。这是一种规范/引力对应，但在本论文中不做涉及。

从量子力学中我们知道一个双电子系统会组成单态和三态。而在超对称当中，我们对于不同的 $\mathcal{N}$ 会有不同的超对称多重态。此时有着不同自旋的粒子会结合为超对称多重态。一般来说，在 $\mathcal{N}=2$ 的理论中我们有向量多重态和超多重态两种。当我们研究超对称时它们代表着不同的粒子类型。本篇论文将不会着重讨论有关超场与超空间的内容，因为这两个概念一般在四维 $\mathcal{N}=1$ 理论的

相关介绍中引入。有兴趣的读者可以参考 Wess et al.[1]，Müller-Kirsten et al.[2]，Krippendorf et al.[3]，Bilal[4]，Terning[5]，Weinberg[6] 进行进一步的学习。

本篇论文的最后部分我们引入了三维扁球以及建立在三维扁球上的配分函数。最早对三维球面上的配分函数进行研究是 2009 年卡普斯汀等人在 Kapustin et al.[7] 中完成的。其中使用了局域化的方法，这种方法由佩斯顿在 2007 年的 Pestun[8] 中引入。“局域化” 这个术语指的是一种将无穷维路径积分转化为有限维路径积分的技巧。它通过超对称算符作用在作用量函数上为零这个性质论证只有鞍点处的点才会对路径积分有贡献将无穷维积分转化为有限维积分。局域化是我们在超对称规范理论中精确地计算超对称路径积分的极有力的工具。而鞍点处的边界条件取决于背景流形，因为超对称变换会根据背景流形的变化而变化，这样便意味这将场建立在流形上。一般来说局域化可以用来检验三维理论之间的对偶性以及规范/引力对应，在文章的最后我们会看到两个例子在取特殊极限时会变为相同的形式。本文讨论的两种具有 $\mathcal{N}=2$ 超对称和 $U(1)_R$ R 对称性的量子场论是由在紫外（UV）的自由拉格朗日量定义的，而它们会流向强关联的红外（IR）三维 $\mathcal{N}=2$ 超对称共形场论（SCFT）。其中一个重要的例子就是 M-理论中 M2-膜的低能超对称共形场论。

就如斯蒂芬 · 温伯格在他关于超对称的量子场论论著中所说：“如果（像我预计的一样）超对称确实被证明与自然的性质相关联，它将展示出一种纯粹理论性洞察力的巨大成功”[6]。

关键字：超对称；规范理论；扁球模型；路径积分；局域化

Abstract

In this thesis, we first reviewed what we have known so far about fundamental physics. After that we discussed the properties about Poincare group and spinors. We introduced the supersymmetric algebra in 4d first and use dimensional reduction to obtain the algebra in 3d. We introduced the squashed 3-sphere partition function after discussing about 3d $\mathcal{N} = 2$ supermultiplets and supersymmetric Lagrangian in flat space. At last, we showed some examples about $Z_{S_b^3}$.

Keywords: Supersymmetry; Gauge Theories; Squashed Sphere; Path Integral; Localization

Chapter 1 Introduction

This introduction targets at those undergraduate students who have learned quantum field theory and would like to know more about 3d $\mathcal{N} = 2$ supersymmetric gauge theory. Let's start from what we have known so far.

1.1 Basic Theories

Quantum mechanics and special relativity are the two basic theories we have. The framework to make these two theories consistent is called quantum field theory (QFT). In QFT, we regard elementary particles as the excitations of the quantum fields, and the particles can be divided into bosons and fermions. The spin of bosons are integer ($s = n, n \in \mathbb{Z}$) and the spin of fermions are half-integer ($s = n + \frac{1}{2}, n \in \mathbb{Z}$). The main difference between bosons and fermions is that fermions satisfy the "Pauli exclusion principle", which means two identical fermions cannot occupy the same quantum state. QFT tells us how particles interact.

The most famous and successful quantum field theory is called the Standard Model (SM). The Standard Model describes three kinds of interactions:

- Strong interaction and electroweak interaction by mediated $s = 1$ particles (photons, gluons, $W^{\pm}$, Z)
- Matter fields by $s = \frac{1}{2}$ particles (quarks, leptons)
- Higgs mechanism by $s = 0$ Higgs boson

The Standard Model includes 61 kinds of particles and its effectiveness has been proved at very high energy scale, like on the LHC at CERN. However, quantum field theory has its own difficulties. One of these difficulties is that if we want to quantize the gravitational field, the graviton is spin-2 particle, which means it cannot be renormalized.

1.2 Basic Principles

A symmetry is the invariance under transformation. Basically we can define two general kinds of symmetries for elementary particles: spacetime symmetries and internal symmetries. Spacetime symmetries correspond to transformations acting on spacetime coordinates and internal symmetries correspond to transformations of different fields. The Standard Model has its internal symmetry as

$$G_{\mathrm{SM}} = SU(3)_{\mathrm{C}} \times SU(2)_{\mathrm{L}} \times U(1) \tag{1.1}$$

Due to a kind of process called Spontaneous Symmetry Breaking, we realize that symmetry can hide. The existence of hidden symmetries indicates that there could be huge number of fundamental symmetries of nature despite that we have already observed a very limited number of symmetries because of the energy scale we can reach. If we want to go beyond the Standard Model, we may add new particles, set extra dimensions, bring in supersymmetry, etc. Supersymmetry is one of the candidates.

Supersymmetry (SUSY) first appeared in string theory where it was the symmetry of the two-dimensional world-sheet theory. It was considered as a pure theoretical tool at first. Later physicists realized that supersymmetry could be a symmetry for four-dimensional QFT and is directly relevant to elementary particle physics. One of the reasons why elementary particle physicists want to consider is that supersymmetry can cancel out many loop corrections. We call it supersymmetry because it is a symmetry between bosons and fermions. Now every particle in the Standard Model has its super-partner particle. That makes supersymmetry also a candidate for explaining dark matter. Supersymmetry has also found applications in condensed matter systems, in particular the disordered systems. In the last part of this thesis, we will meet a supersymmetric partition function called the squashed-sphere partition function and some of its applications.

We are interested in 3d supersymmetric gauge theories for various reasons. 3d theories can be obtained from studying 4d theories on a circle, thus it reflects some properties of 4d theories. 3d $\mathcal{N} = 2$ theories also have its own special properties: there is no asymptotic freedom bound on the number of matter fields; there could be new supersymmetry coupling constants like the Chern–Simons term, real masses; from $U(1)$ theories we can examine the Fayet–Iliopoulos terms; topological phases and BPS-particles (e.g. Skyrmions, vortices) occur, etc[9, 10].

From quantum mechanic we know that for a two electron system, they can form a singlet state and a triplet state. In supersymmetry, we can have supermultiplets for different $\mathcal{N}$. Particles with different spins unite into symmetry multiplets. Usually we have vector multiplet and hyper multiplet in $\mathcal{N} = 2$ theories. They represent different particles when we talk about supersymmetry. The concepts about superfield and superspace are not detailed discussed in this thesis because they are usually first introduced and detailed discussed when talking about 4d $\mathcal{N} = 1$ theories. Here are some introductory books if readers are interested: Wess et al.[1], Müller-Kirsten et al.[2], Krippendorf et al.[3], Bilal[4], Terning[5], Weinberg[6]

Although there is no direct experimental evidence and only a little indirect evidence that indicate supersymmetry connects to the real world, it's still a beautiful theory mathematically and has profound implications for fundamental physics. As S.Weinberg said in his book about supersymmetry: "If (as I except) supersymmetry does turn out to be relevant to nature, it will represent a striking success of purely theoretical insight"[6].

Chapter 2 Poincare Symmetry and Spinors

Let's start from some mathematical basics.

2.1 Properties of Poincare and Lorentz Groups

The Poincare group contains Lorentz transformations and translations. It acts on the spacetime coordinates x^μ as

$$x'^\mu = \Lambda^\mu_\nu x^\nu + a^\mu \tag{2.1}$$

Here Λ^μ_ν is the Lorentz transformation and a^μ is the translation. Lorentz transformation leaves the Minkowski spacetime metric tensor $g^{\mu\nu} = (1, -1, -1, -1)$ invariant:

$$\Lambda^T g \Lambda = g \tag{2.2}$$

In the Poincare group, the generators of the translations are denoted as P^μ and the generators of the Lorentz transformations are denoted as $M^{\mu\nu}$. The corresponding Poincare algebra is

$$[P^\mu, P^\nu] = 0 \tag{2.3}$$

$$[M^{\mu\nu}, P^\gamma] = i(P^\mu g^{\nu\gamma} - P^\nu g^{\mu\gamma}) \tag{2.4}$$

$$[M^{\mu\nu}, M^{\rho\sigma}] = i(M^{\mu\sigma} g^{\nu\rho} + M^{\nu\rho} g^{\mu\sigma} - M^{\mu\rho} g^{\nu\sigma} - M^{\nu\sigma} g^{\mu\rho}) \tag{2.5}$$

A 4-dimensional matrix representation of $M^{\mu\nu}$ is $(M^{\rho\sigma})^\mu_\nu = i(g^{\sigma\mu}\delta^\rho_\nu - g^{\rho\mu}\delta^\sigma_\nu)$.

The Lorentz group has six generators. We can define $J_i \equiv \frac{1}{2}\epsilon_{ijk} M_{jk}$ as rotations and $K_i \equiv M_{0i}$ as Lorentz boosts. Then we have the Lorentz algebra:

$$[J_i, J_j] = i\epsilon_{ijk} J_k \tag{2.6}$$

$$[K_i, K_j] = -i\epsilon_{ijk} J_k \tag{2.7}$$

$$[J_i, K_j] = i\epsilon_{ijk} K_k \tag{2.8}$$

We can define $A_i = \frac{1}{2}(J_i + iK_i)$ and $B_i = \frac{1}{2}(J_i - iK_i)$ to decouple these commuting relations:

$$[A_i, A_j] = i\epsilon_{ijk} A_k \longrightarrow SU(2) \tag{2.9}$$

$$[B_i, B_j] = i\epsilon_{ijk} B_k \longrightarrow SU(2) \tag{2.10}$$

$$[A_i, B_j] = 0 \tag{2.11}$$

From the relations above, we have the relation:

$$SO(3,1) \cong SU(2) \times SU(2) \tag{2.12}$$

There is another homeomorphism

$$SO(3,1) \cong SL(2,\mathbb{C}) \tag{2.13}$$

We consider a four-vector x_μ and its corresponding 2×2 matrix representation $\tilde{x} = x_\mu \sigma^\mu$, where σ^μ is the four Pauli matrices $\sigma^\mu = (\mathbb{1}, \vec{\sigma})$ It can be easy to see that the determinant of $\tilde{x}$ is equal to $x_\mu x^\mu = x_0^2 - x_1^2 - x_2^2 - x_3^2$, which is invariant under Lorentz transformation. Meanwhile, if $N \in SL(2,\mathbb{C})$, then we have $\tilde{x} \to \tilde{x}' = N\tilde{x}N^\dagger$, and $\det \tilde{x} = \det \tilde{x}' = x_0^2 - x_1^2 - x_2^2 - x_3^2$. This establishes the map between $SO(3,1)$ and $SL(2,\mathbb{C})$.

2.2 Spinors

A spinor is a two complex-component vector

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{2.14}$$

The fundamental representation is

$$\psi'_\alpha = N_\alpha^{\ \beta} \psi_\beta; \alpha, \beta = 1, 2 \tag{2.15}$$

ψ_α here are called left-handed Weyl spinors.

And for its conjugate representation

$$\bar{\chi}'_{\dot{\alpha}} = N_{\dot{\alpha}}^{\star \dot{\beta}} \bar{\chi}_{\dot{\beta}}; \dot{\alpha}, \dot{\beta} = 1, 2 \tag{2.16}$$

$\bar{\chi}_{\dot{\alpha}}$ are called right-handed Weyl spinors.

From these relations we can also define the contravariant representations

$$\psi'^\alpha = \psi^\beta (N^{-1})_\beta^{\ \alpha} \tag{2.17}$$

$$\bar{\chi}'^{\dot{\alpha}} = \bar{\chi}^{\dot{\beta}} (N^{\star -1})_{\dot{\beta}}^{\ \dot{\alpha}} \tag{2.18}$$

Since we have defined the basic objects that we can operate on, it's time for us to consider how we can raise and lower indices. For Lorentz group, the metric tensor $g^{\mu\nu}$ is invariant under Lorentz transformation. We can find that for $SL(2,\mathbb{C})$, there is ϵ as an invariant tensor:

$$\epsilon^{\alpha\beta} = \epsilon^{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -\epsilon_{\alpha\beta} = -\epsilon_{\dot{\alpha}\dot{\beta}} \tag{2.19}$$

We can use $\epsilon^{\alpha\beta}$ to raise and lower indices:

$$\psi^{\alpha} = \epsilon^{\alpha\beta}\psi_{\beta} \tag{2.20}$$

$$\bar{\chi}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\bar{\chi}_{\dot{\beta}} \tag{2.21}$$

which means contravariant representations are not independent.

Now we try to mix the $SL(2,\mathbb{C})$ & $SO(3,1)$ indices. The transformation rule should be like

$$\sigma^{\mu}_{\alpha\dot{\alpha}} = N_{\alpha}^{\beta}\sigma^{\nu}_{\beta\dot{\gamma}}(\Lambda^{\mu}_{\nu})^{-1}N^{\star\dot{\gamma}}_{\dot{\alpha}} \tag{2.22}$$

And similar for

$$(\bar{\sigma}^{\mu})^{\dot{\alpha}\alpha} \equiv \epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}\sigma^{\mu}_{\beta\dot{\beta}} = (\mathbb{1}, -\vec{\sigma}) \tag{2.23}$$

2.3 Generators of $SL(2,\mathbb{C})$

We define

$$(\sigma^{\mu\nu})_{\alpha}^{\beta} \equiv \frac{i}{4}(\sigma^{\mu}\bar{\sigma}^{\nu} - \sigma^{\nu}\bar{\sigma}^{\mu})_{\alpha}^{\beta} \tag{2.24}$$

$$(\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}_{\dot{\beta}} \equiv \frac{i}{4}(\bar{\sigma}^{\mu}\sigma^{\nu} - \bar{\sigma}^{\nu}\sigma^{\mu})^{\dot{\alpha}}_{\dot{\beta}} \tag{2.25}$$

as the generators of $SL(2,\mathbb{C})$. This is another representation of the algebra of the Lorentz group. We can check that $\sigma^{\mu\nu}$ and $\bar{\sigma}^{\mu\nu}$ satisfy the Lorentz algebra (2.5).

And here are some useful identities:

$$\sigma^{\mu}\bar{\sigma}^{\nu} + \sigma^{\nu}\bar{\sigma}^{\mu} = 2\eta^{\mu\nu}\mathbb{1} \tag{2.26}$$

$$\operatorname{tr}\sigma^{\mu}\bar{\sigma}^{\nu} = 2\eta^{\mu\nu} \tag{2.27}$$

$$\sigma^{\mu}_{\alpha\dot{\alpha}}\bar{\sigma}_{\mu}^{\beta\dot{\beta}} = 2\delta_{\alpha}^{\beta}\delta_{\dot{\alpha}}^{\dot{\beta}} \tag{2.28}$$

$$\sigma^{\mu\nu} = \frac{1}{2i}\epsilon^{\mu\nu\rho\sigma}\sigma_{\rho\sigma} \to \text{self duality} \tag{2.29}$$

$$\bar{\sigma}^{\mu\nu} = -\frac{1}{2i}\epsilon^{\mu\nu\rho\sigma}\bar{\sigma}_{\rho\sigma} \to \text{anti-self duality} \tag{2.30}$$

These self duality and anti-self duality are important because this $\sigma^{\mu\nu}$ being antisymmetric seems to have 6 components, but actually due to the self duality it only has 3 components.

2.4 Some Properties of Spinors

We define representation without index as

$$\chi\psi \equiv \chi^{\alpha}\psi_{\alpha} = -\chi_{\alpha}\psi^{\alpha} \tag{2.31}$$

$$\bar{\chi}\bar{\psi} \equiv \bar{\chi}_{\dot{\alpha}}\bar{\psi}^{\dot{\alpha}} = -\bar{\chi}^{\dot{\alpha}}\bar{\psi}_{\dot{\alpha}} \tag{2.32}$$

Particularly, $\psi^2 = \psi\psi = \psi^\alpha\psi_\alpha = \epsilon^{\alpha\beta}\psi_\beta\psi_\alpha = \psi_2\psi_1 - \psi_1\psi_2$. Choose ψ_α to be anticommuting numbers (Grassmann numbers), we can write $\psi_\alpha\psi_\beta = \frac{1}{2}\epsilon_{\alpha\beta}(\psi\psi)$.

Here are some properties of spinors:

$$(\theta\psi)(\theta\psi) = -\frac{1}{2}(\psi\psi)(\theta\theta) \quad \text{Fierz identity} \tag{2.33}$$

$$(\psi\sigma^{\mu\nu}\chi) = -(\chi\sigma^{\mu\nu}\psi) \tag{2.34}$$

$$(\theta\psi)^\dagger = (\bar{\theta}\bar{\psi}) \tag{2.35}$$

$$(\psi\sigma^\mu\bar{\chi})^\dagger = (\chi\sigma^\mu\bar{\psi}) \tag{2.36}$$

$$(\psi_\alpha)^\dagger \equiv \bar{\psi}_{\dot{\alpha}} \tag{2.37}$$

$$\bar{\psi}^{\dot{\alpha}} = \psi^\star_\beta(\sigma^\nu)^{\beta\dot{\alpha}} \tag{2.38}$$

These properties are very useful when we do the calculations on spinors later.

Now let's consider the connections to Dirac spinors in Minkowski spacetime. First we define

$$\gamma^\mu \equiv \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \tag{2.39}$$

Then we have the Clifford algebra for Dirac γ-matrices: $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$. And define

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -\mathbb{1}_2 & 0 \\ 0 & \mathbb{1}_2 \end{pmatrix} \tag{2.40}$$

The eigenvalues are ± 1 which indicates the chirality. If we define

$$\Sigma^{\mu\nu} \equiv \frac{i}{4}[\gamma^\mu, \gamma^\nu] = \begin{pmatrix} \sigma^{\mu\nu} & 0 \\ 0 & \bar{\sigma}^{\mu\nu} \end{pmatrix} \tag{2.41}$$

we can see that these are also the generators of Lorentz group.

The Dirac spinor is a four-component vector made from ψ_α and $\bar{\chi}^{\dot{\alpha}}$. We can express it as $\Psi_\mathrm{D} = \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix}$. Notice that $\gamma^5\Psi_\mathrm{D} = \begin{pmatrix} -\psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix}$, which means that we can define projectors

$$P_\mathrm{L} = \frac{1}{2}(1 - \gamma^5) \tag{2.42}$$

$$P_\mathrm{R} = \frac{1}{2}(1 + \gamma^5) \tag{2.43}$$

It's easy to see that $P_\mathrm{L}\Psi_\mathrm{D} = \begin{pmatrix} \psi_\alpha \\ 0 \end{pmatrix}$, $P_\mathrm{R}\Psi_\mathrm{D} = \begin{pmatrix} 0 \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix}$. We call $\begin{pmatrix} \psi_\alpha \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix}$ chiral Dirac spinors. There is a kind of special spinor called Majorana spinor, in which $\psi \equiv \chi$. Thus it looks like $\Psi_\mathrm{M} = \begin{pmatrix} \psi_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix}$.

Chapter 3 SUSY Algebra

Let's give the Poincare algebra a supersymmetric extension. First we need the concept of the graded algebras. Denote O_a as the operators of a Lie algebra. Then we have

$$O_a O_b - (-)^{\eta_a \eta_b} O_b O_a = i C^c_{ab} O_c \tag{3.1}$$

Here $\eta_a = 0$ for O_a as a bosonic generator, and $\eta_a = 1$ for O_a as a fermionic generator. This η_a is called grading.

For supersymmetry, generators are P^μ, $M^{\mu\nu}$ as Poincare generators and Q^A_α, $\bar{Q}^A_{\dot{\alpha}}$ as spinor generators. $A = 1, 2, \ldots, \mathcal{N}$. If $\mathcal{N} = 1$, the theory is called simple SUSY, and if $\mathcal{N} > 1$, the theory is called extended SUSY. In this thesis, we will mainly discuss $\mathcal{N} = 2$ SUSY in three dimensions.

3.1 SUSY Algebra in 4-dimension

Now we give a label $A, B = 1, 2, \ldots, \mathcal{N}$. The algebra for generators is

$$\{Q^A_\alpha, \bar{Q}_{\dot{\beta}B}\} = 2(\sigma^\mu)_{\alpha\dot{\beta}} P_\mu \delta^A_B \tag{3.2}$$

$$\{Q^A_\alpha, Q^B_\beta\} = \epsilon_{\alpha\beta} Z^{AB} \tag{3.3}$$

$$\{\bar{Q}^A_{\dot{\alpha}}, \bar{Q}^B_{\dot{\beta}}\} = \epsilon_{\dot{\alpha}\dot{\beta}} (Z^{AB})^\star \tag{3.4}$$

Z^{AB} are called central charges. Z^{AB} is antisymmetric ($Z^{AB} = -Z^{BA}$) and the central charges commute with all the generators, which are the origin of this name. In $\mathcal{N} = 2$, there is only one central charge $Z \equiv Z^{12}$ due to antisymmetry.

Since Q is a spinor, when it acts on a bosonic state, it produces a fermionic state. And in contrast when it acts on a fermionic state, it gives out a bosonic state. That's why we call this kind of theory "supersymmetry". It builds up a symmetry between bosons and fermions.

Denote internal symmetry generators as T_i. The commutator $[Q_\alpha, T_i]$ usually vanishes. Exceptions are $U(1)$ automorphism of the SUSY algebra, which is known as R-symmetry:

$$Q_\alpha \to e^{i\lambda} Q_\alpha, \quad \bar{Q}_{\dot{\alpha}} \to e^{-i\lambda} \bar{Q}_{\dot{\alpha}} \tag{3.5}$$

If R is the $U(1)$ generator, then we have

$$[Q_\alpha, R] = Q_\alpha \tag{3.6}$$

$$[\bar{Q}_{\dot{\alpha}}, R] = -\bar{Q}_{\dot{\alpha}} \tag{3.7}$$

The most simple case is when $\mathcal{N} = 1$, the algebra becomes

$$\{Q_\alpha, \bar{Q}_{\dot\beta}\} = 2(\sigma^\mu)_{\alpha\dot\beta} P_\mu \tag{3.8}$$

$$\{Q_\alpha, Q_\beta\} = 0 \tag{3.9}$$

$$\{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\} = 0 \tag{3.10}$$

Here comes one of the most important propositions in the supersymmetric theories: In any supersymmetric multiplet, the number of bosons and the number of fermions are always equal.

$$n_B = n_F \tag{3.11}$$

Proof: Consider a new operator $(-1)^F = (-)^F$. It counts the fermion number of a state and has properties as

$$(-)^F |B\rangle = |B\rangle, \quad |B\rangle = \text{boson} \tag{3.12}$$

$$(-)^F |F\rangle = -|F\rangle, \quad |F\rangle = \text{fermion} \tag{3.13}$$

$$(-)^F Q_\alpha = -Q_\alpha (-)^F \tag{3.14}$$

Since

$$(-)^F Q_\alpha |F\rangle = (-)^F |B\rangle = |B\rangle = Q_\alpha |F\rangle = -Q_\alpha (-)^F |F\rangle \tag{3.15}$$

we have $\{(-)^F, Q_\alpha\} = 0$.

Consider

$$\mathrm{tr}[(-)^F \{Q_\alpha, \bar{Q}_{\dot\beta}\}] = \mathrm{tr}[(-)^F Q_\alpha \bar{Q}_{\dot\beta} + (-)^F \bar{Q}_{\dot\beta} Q_\alpha] = \mathrm{tr}[-Q_\alpha (-)^F \bar{Q}_{\dot\beta} + Q_\alpha (-)^F \bar{Q}_{\dot\beta}] = 0 \tag{3.16}$$

But

$$\mathrm{tr}[(-)^F \{Q_\alpha, \bar{Q}_{\dot\beta}\}] = \mathrm{tr}[(-)^F 2\sigma^\mu_{\alpha\dot\beta} P_\mu] = 2\sigma^\mu_{\alpha\dot\beta} P_\mu \,\mathrm{tr}(-)^F \tag{3.17}$$

Because $2\sigma^\mu_{\alpha\dot\beta} P_\mu$ is arbitrary, we know that

$$\mathrm{tr}(-)^F = 0 \tag{3.18}$$

Then we have

$$\mathrm{tr}(-)^F = \sum_{\text{boson}} \langle B| (-)^F |B\rangle + \sum_{\text{fermion}} \langle F| (-)^F |F\rangle \tag{3.19}$$

$$= \sum_{\text{boson}} \langle B|B\rangle - \sum_{\text{fermion}} \langle F|F\rangle = 0 \tag{3.20}$$

Finally we have $n_B = n_F$.

3.2 Representations of the Super Poincare Algebra

Let's recall the rotation group:

$$[J_i, J_j] = i\epsilon_{ijk}J_k \tag{3.21}$$

The Casimir operator

$$J^2 = \sum_{i=1}^{3} J_i^2 \tag{3.22}$$

From quantum mechanics we know that it commutes with all the J_i and has eigenvalues $j(j+1)$. Within these representations, we can diagonalize J_3 to eigenvalues $j_3 = -j, -j+1, \ldots, j-1, j$. States are labeled like $|j, j_3\rangle$.

Poincare group involves the Pauli–Lubanski vector

$$W_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}P^\nu M^{\rho\sigma} \tag{3.23}$$

where $\epsilon_{\mu\nu\rho\sigma}$ is the Levi-Civita symbol.

Now we gives the Poincare Casimirs

$$C_1 \equiv P^\mu P_\mu, \quad C_2 \equiv W^\mu W_\mu \tag{3.24}$$

and they commute with all the generators. For finite dimensional representations, $SO(2)$ is a subgroup and W_1, W_2 equal to zero. In that case, $W^\mu = \lambda P^\mu$ and we label the states as $|p^\mu, \lambda\rangle$, where λ is called helicity. $|p^\mu, \lambda\rangle \to |p^\mu, -\lambda\rangle$ under CPT transformation.

To satisfy

$$e^{2\pi i\lambda}|p^\mu, \lambda\rangle = \pm|p^\mu, \lambda\rangle \tag{3.25}$$

λ has to be integer or half-integer, e.g $\lambda = 0, \pm\frac{1}{2}, \pm1, \ldots$. For example,

$$\lambda = 0 \to \text{Higgs} \tag{3.26}$$

$$\lambda = \pm\frac{1}{2} \to \text{quarks} + \text{leptons} \tag{3.27}$$

$$\lambda = \pm1 \to \text{photons}, \ W^\pm, \ Z, \ \text{gluons} \tag{3.28}$$

$$\lambda = \pm2 \to \text{graviton} \tag{3.29}$$

3.3 Massless Supermultiplet

We choose a reference frame for massless particles where $P_\mu = (E, 0, 0, E)$. Consider the algebra

$$\{Q_\alpha^A, \bar{Q}_{\dot\beta B}\} = 2(\sigma^\mu)_{\alpha\dot\beta}P_\mu\delta_B^A = 2E(\sigma^0+\sigma^3)_{\alpha\dot\beta}\delta_B^A = 4E\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}_{\alpha\dot\beta}\delta_B^A \tag{3.30}$$

which implies Q_2^A is zero in the representation:

$$\{Q_2^A, \bar{Q}_{\dot{2}}^A\} = 0 \tag{3.31}$$

If we define $a^A \equiv \frac{Q_1^A}{2\sqrt{E}}$, $a^{\dagger A} \equiv \frac{\bar{Q}_{\dot{1}}^A}{2\sqrt{E}}$, we have $\{a^A, a_B^\dagger\} = \delta_B^A$.

We can start from choosing a vacuum state $|\Omega\rangle$, which will be annihilated by all the a^A. For $a^{\dagger A}|\Omega\rangle$, helicity is $\lambda_{\min} + \frac{1}{2}$, and there are $\mathcal{N}$ states (here $A, B = 1, \ldots, \mathcal{N}$). For $a^{\dagger A} a^{\dagger B}|\Omega\rangle$, helicity is $\lambda_{\min} + 1$, and $\binom{\mathcal{N}}{2} = \frac{\mathcal{N}(\mathcal{N}-1)}{2}$ states. The rest can be done in the manner till that $a^{\dagger \mathcal{N}} a^{\dagger(\mathcal{N}-1)} \ldots a^{\dagger 1}|\Omega\rangle$, as the helicity becomes $\lambda_{\min} + \frac{\mathcal{N}}{2}$, and only 1 state ($\binom{\mathcal{N}}{\mathcal{N}} = 1$). The total number of states will be $2^{\mathcal{N}}$.

For $\mathcal{N} = 2$ case,

- If $\lambda_{\min} = 0$, we have $\mathcal{N} = 2$ vector multiplet

$$\begin{array}{ccc} & \lambda = 0 & \\ \lambda = \frac{1}{2} & & \lambda = \frac{1}{2} \\ & \lambda = 1 & \end{array} \tag{3.32}$$

  We can see that it can be decomposed to two $\mathcal{N} = 1$ multiplets: one chiral multiplet and one vector multiplet. Here are chiral multiplets with $\lambda = 0, \frac{1}{2}$ and vector or gauge multiplets with $\lambda = \frac{1}{2}, 1$.

- If $\lambda_{\min} = -\frac{1}{2}$, we have $\mathcal{N} = 2$ hypermultiplet

$$\begin{array}{ccc} & \lambda = -\frac{1}{2} & \\ \lambda = 0 & & \lambda = 0 \\ & \lambda = \frac{1}{2} & \end{array} \tag{3.33}$$

From this we can see that $\mathcal{N} = 2$ hypermultiplet can be decomposed to two $\mathcal{N} = 1$ chiral multiplets.

3.4 Massive Supermultiplet

Let's consider the rest frame $P_\mu = (m, 0, 0, 0)$. We have

$$\{Q_\alpha^A, \bar{Q}_{\dot{\beta}B}\} = 2m \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \delta_B^A \tag{3.34}$$

$$\{Q_\alpha^A, Q_\beta^B\} = \epsilon_{\alpha\beta} Z^{AB} \tag{3.35}$$

In the massive case, the central charges can be non-zero. Therefore we need to distinguish two cases:

- $Z^{AB} = 0$

  There would be $2\mathcal{N}$ creation and annihilation operators defined by

$$a_\alpha^A \equiv \frac{Q_\alpha^A}{\sqrt{2m}}, \quad a_{\dot{\alpha}}^{\dagger A} \equiv \frac{\bar{Q}_{\dot{\alpha}}^A}{\sqrt{2m}} \tag{3.36}$$

  giving $2^{2\mathcal{N}}$ states.

  In the $\mathcal{N} = 2$ case, we have

$$|\Omega\rangle \quad \text{1 spin-0 state} \tag{3.37}$$

$$a_{\dot{\alpha}}^{\dagger A} |\Omega\rangle \quad \text{4 spin-}\frac{1}{2}\text{ states} \tag{3.38}$$

$$a_{\dot{\alpha}}^{\dagger A} a_{\dot{\beta}}^{\dagger B} |\Omega\rangle \quad \text{3 spin-0 states \& 3 spin-1 states} \tag{3.39}$$

$$a_{\dot{\alpha}}^{\dagger A} a_{\dot{\beta}}^{\dagger B} a_{\dot{\gamma}}^{\dagger C} |\Omega\rangle \quad \text{4 spin-}\frac{1}{2}\text{ states} \tag{3.40}$$

$$a_{\dot{\alpha}}^{\dagger A} a_{\dot{\beta}}^{\dagger B} a_{\dot{\gamma}}^{\dagger C} a_{\dot{\delta}}^{\dagger D} |\Omega\rangle \quad \text{1 spin-0 state} \tag{3.41}$$

  16 states in total, which is exactly what we have predicted.

- $Z^{AB} \neq 0$

  Define a scalar $\mathscr{H}$ [3]

$$\mathscr{H} \equiv (\bar{\sigma}^0)^{\dot{\beta}\alpha} \{Q_\alpha^A - \Gamma_\alpha^A, \bar{Q}_{\dot{\beta}A} - \bar{\Gamma}_{\dot{\beta}A}\} \geqslant 0 \tag{3.42}$$

  in which

$$\Gamma_\alpha^A \equiv \epsilon_{\alpha\beta} U^{AB} \bar{Q}_{\dot{\gamma}} (\bar{\sigma}^0)^{\dot{\gamma}\beta} \tag{3.43}$$

  U is an unitary matrix. From $\{Q_\alpha^A, \bar{Q}_{\dot{\beta}}^B\}$ we can see

$$\mathscr{H} = 8m\mathcal{N} - 2\,\mathrm{tr}\{ZU^\dagger + UZ^\dagger\} \geqslant 0 \tag{3.44}$$

  Due to polar decomposition theorem, Z can be express as $Z = HV$. Here H is a hermitian and $V = (V^\dagger)^{-1}$. Choose $U = V$, we have

$$\mathscr{H} = 8m\mathcal{N} - 4\,\mathrm{tr}\,H = 8m\mathcal{N} - 4\,\mathrm{tr}(\sqrt{Z^\dagger Z}) \geqslant 0 \tag{3.45}$$

  This condition gives us

$$m \geqslant \frac{1}{\mathcal{N}}\,\mathrm{tr}\{\sqrt{Z^\dagger Z}\} \tag{3.46}$$

This is called the Bogomolnyi–Prasad–Sommerfield (BPS)-bound for mass. States that $m = \frac{1}{\mathcal{N}}\,\mathrm{tr}\{\sqrt{Z^\dagger Z}\}$ are called BPS-states. The multiplet of BPS-states are shorter, having only $2^{\mathcal{N}}$ states.

When $\mathcal{N} = 2$, we can represent the central charge Z^{AB} as

$$Z^{AB} = \begin{pmatrix} 0 & q_1 \\ -q_1 & 0 \end{pmatrix} \to m \geqslant \frac{q_1}{2} \tag{3.47}$$

Chapter 4 3-dimensional $\mathcal{N} = 2$ Supermultiplets and SUSY Lagrangians in Flat Space

There is a common but highly non-trivial statement that for a given supersymmetric observable, the saddle point approximation will give an exact answer from the view point of path integral. That's one of the reasons why we are interested in supersymmetric theories. Here we will focus on supersymmetric partition functions on closed 3-manifolds $\mathcal{M}_3$. These theories will have $\mathcal{N} = 2$ supersymmetry and $U(1)_R$ R-symmetry.

The partition functions are defined as path integrals

$$Z_{\mathcal{M}_3}(y) = \int [\mathcal{D}\phi]_{\mathcal{M}_3} e^{-S[\phi;y]} \tag{4.1}$$

y is a continuous complex parameter coupling to the conserved flavor symmetry currents of 3d $\mathcal{N} = 2$ QFT.

To compute these partition functions, we need two generalized Killing spinors ϵ and $\bar{\epsilon}$ with opposite R-charges. The supersymmetric background on $\mathcal{M}_3$ is called "half-BPS geometry". The Killing spinor here is a spinor field $\phi \in \Gamma(s)$ with Killing number $\beta \in \mathbb{C}$ which satisfying the differential equation $\nabla_X \phi = \beta X \cdot \phi$ for all vector fields X on $\mathcal{M}$.

Let's consider 3d $\mathcal{N} = 2$ theories on $\mathbb{R}^3$ first. In this 3d theory, the Lorentz group $SO(3, 1)$ in Minkowski spacetime becomes the 3-dimensional Lorentz group $SO(2, 1)$. And we have the homeomorphism

$$SO(2, 1) \cong SL(2, \mathbb{R}) \tag{4.2}$$

which means the only central charge Z for $\mathcal{N} = 2$ theories is real and we now have four real supercharges. The Euclidean signature can be obtained through Wick rotation.

We can denote these four real supercharges as Q_α and $\bar{Q}_\alpha$, where $\alpha = 1, 2$. Here we have the supersymmetric algebra for 3d $\mathcal{N} = 2$ theory:

$$\{Q_\alpha, Q_\beta\} = \{\bar{Q}_\alpha, \bar{Q}_\beta\} = 0 \tag{4.3}$$

$$\{Q_\alpha, \bar{Q}_\beta\} = 2(\sigma^\mu)_{\alpha\beta} P_\mu + 2i\epsilon_{\alpha\beta} Z \tag{4.4}$$

Since the spectrum (the number of scalar, fermion, vector, …) is the same for n-dimensional $\mathcal{N} = N$ theories where $n + N = n' + N'$, we know that this algebra can be derived from dimensional reduction to 4d $\mathcal{N} = 1$ SUSY algebra.

The $U(1)_R$ symmetry can rotate Q and $\bar{Q}$ in opposite phase, or we say that we choose the convention that Q has charge -1 and $\bar{Q}$ has charge $+1$ under $U(1)_R$ symmetry. And this property indicates that in (4.4) the left hand side of the equation should be supercharges from different indices because the right hand side has no $U(1)_R$ charge. There are four σ^μ matrices in 4d and three σ^μ matrices in 3d. Here is a little difference between the ways we choose σ^μ matrices. In 3d case, we usually choose

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma^3_{\alpha\beta} = i\epsilon_{\alpha\beta} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \tag{4.5}$$

and this gives the factor of the Z term in the right hand side of (4.4).

There is a special symmetry in abelian gauge fields in 3d. For every $U(1)$ gauge field A_μ, we can define a current:

$$j^\mu_T = \frac{1}{4\pi}\epsilon^{\mu\nu\rho}F_{\nu\rho} \tag{4.6}$$

It is conserved due to Bianchi identity. The corresponding global symmetry is usually called “topological symmetry”. The states that have $U(1)_T$ symmetry carry non-trivial $U(1)$ magnetic charge. The corresponding local operators are called “monopole operators”.

4.1 Supermultiplets

In 3d $\mathcal{N} = 2$ theories, there would be two kinds of supermultiplets: vector multiplet and chiral multiplet. Now let’s give out the basic properties of these supermultiplets. The spinor representation of SUSY operators is

$$\mathcal{Q}_\alpha = \frac{\partial}{\partial\theta^\alpha} - i\sigma^\mu_{\alpha\beta}\bar{\theta}^\beta\partial_\mu \tag{4.7}$$

$$\bar{\mathcal{Q}}_\alpha = \frac{\partial}{\partial\bar{\theta}^\alpha} - i\theta^\beta\sigma^\mu_{\beta\alpha}\partial_\mu \tag{4.8}$$

The SUSY transformations can be observed from the variation of superfields:

$$\delta S = i[S, \epsilon Q + \bar{\epsilon}\bar{Q}] = i(\epsilon\mathcal{Q} + \bar{\epsilon}\bar{\mathcal{Q}}) \tag{4.9}$$

4.1.1 Vector Multiplet

We can choose “Wess–Zumino gauge” to reduce the vector multiplet in the form:

$$\mathcal{V} = \mathcal{V}(\sigma, A_\mu, \lambda, \bar{\lambda}, D) \tag{4.10}$$

Here σ is a real scalar; A_μ is a 3-dimensional gauge field; λ and $\bar{\lambda}$ are gaugino (supersymmetric partner of corresponding gauge boson); D is an auxiliary field.

Choose covariant gauge-derivative as $D_\mu = \partial_\mu - iA_\mu$ and the field strength $F_{\mu\nu}$ as non-abelian

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu] \tag{4.11}$$

We can obtain the SUSY transformations for these fields[11]:

$$\delta\sigma = -\epsilon\bar{\lambda} + \bar{\epsilon}\lambda \tag{4.12}$$

$$\delta A_\mu = -i(\epsilon\gamma_\mu\bar{\lambda} + \bar{\epsilon}\gamma_\mu\lambda) \tag{4.13}$$

$$\delta\lambda = i\epsilon D - i\gamma^\mu\epsilon(D_\mu\sigma + \frac{1}{2}\epsilon_{\mu\nu\rho}F^{\nu\rho}) \tag{4.14}$$

$$\delta\bar{\lambda} = -i\bar{\epsilon}D + i\gamma^\mu\bar{\epsilon}(D_\mu\sigma - \frac{1}{2}\epsilon_{\mu\nu\rho}F^{\nu\rho}) \tag{4.15}$$

$$\delta D = \epsilon\gamma^\mu D_\mu\bar{\lambda} - \epsilon[\sigma, \bar{\lambda}] - \bar{\epsilon}\gamma^\mu D_\mu\lambda - \bar{\epsilon}[\sigma, \lambda] \tag{4.16}$$

Here ϵ and $\bar{\epsilon}$ are a parameter spinors in $\mathbb{R}^3$.

4.1.2 Chiral Multiplet

Now we write the chiral multiplet as

$$\boldsymbol{\Phi} = \boldsymbol{\Phi}(\phi, \psi, F) \tag{4.17}$$

to include matter fields.

As for chiral multiplet, the SUSY transformation goes like[11]

$$\delta\phi = \sqrt{2}\epsilon\psi \tag{4.18}$$

$$\delta\psi = \sqrt{2}\epsilon F + \sqrt{2}i\sigma\bar{\epsilon}\phi - \sqrt{2}i\gamma^\mu\bar{\epsilon}D_\mu\phi \tag{4.19}$$

$$\delta F = -\sqrt{2}i\sigma\bar{\epsilon}\psi + 2i\phi\bar{\epsilon}\bar{\lambda} - \sqrt{2}i\bar{\epsilon}\gamma^\mu D_\mu\psi \tag{4.20}$$

We can directly write out the SUSY transformation of the CPT conjugate of $\boldsymbol{\Phi}$ field $\bar{\boldsymbol{\Phi}} = \bar{\boldsymbol{\Phi}}(\bar{\phi}, \bar{\psi}, \bar{F})$

$$\delta\bar{\phi} = -\sqrt{2}\bar{\epsilon}\bar{\psi} \tag{4.21}$$

$$\delta\bar{\psi} = \sqrt{2}\bar{\epsilon}\bar{F} - \sqrt{2}i\sigma\epsilon\bar{\phi} + \sqrt{2}i\gamma^\mu\epsilon D_\mu\bar{\phi} \tag{4.22}$$

$$\delta\bar{F} = -\sqrt{2}i\sigma\epsilon\bar{\psi} + 2i\bar{\phi}\epsilon\psi - \sqrt{2}i\epsilon\gamma^\mu D_\mu\bar{\psi} \tag{4.23}$$

We should note that the real central charge $Z = -\sigma$ in Wess–Zumino gauge (σ is the real scalar), depending on the field.

4.2 Supersymmetric Lagrangians

We can write the supersymmetric Yang–Mills–Chern–Simons Matter Lagrangian as

$$\mathscr{L} = \mathscr{L}_{\text{YM}} + \mathscr{L}_{\text{CS}} + \underbrace{\mathscr{L}_{\bar{\Phi}\Phi} + \mathscr{L}_{W} + \mathscr{L}_{\bar{W}}}_{\text{Matter}} \tag{4.24}$$

Their contributions are as follows:

The supersymmetric Yang–Mills term is[11]

$$\mathscr{L}_{\text{YM}} = \frac{1}{|e_0^2|}\operatorname{tr}\Big(\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D_\mu\sigma D^\mu\sigma - i\bar{\lambda}\gamma^\mu D_\mu\lambda - i\bar{\lambda}[\sigma,\lambda] - \frac{1}{2}D^2\Big) \tag{4.25}$$

where the dimension of the squared Yang–Mills gauge coupling constant is mass ($[e_0^2] = 1$).

The non-supersymmetric Chern–Simons term is

$$\mathscr{L}_{\text{CS}} = \frac{k}{4\pi}\operatorname{tr}\Big(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\Big) \tag{4.26}$$

and in the supersymmetric non-abelian case we have

$$\mathscr{L}_{\text{CS}} = \frac{k}{4\pi}\operatorname{tr}\Big(i\epsilon^{\mu\nu\rho}(A_\mu\partial_\nu A_\rho - \frac{2i}{3}A_\mu A_\nu A_\rho) - 2D\sigma + 2i\bar{\lambda}\lambda\Big) \tag{4.27}$$

This $(-2D\sigma)$ term plays an important role in supersymmetric localization.

An new extra term can be added into the Lagrangian and it is known as the Fayet–Iliopoulos term (FI term).

$$\mathscr{L}_{\text{FI}} = -\xi D, \qquad \xi = \frac{1}{2\pi}m_T \tag{4.28}$$

ξ is a constant parameter. It is gauge invariant for $U(1)$ gauge theories because the corresponding field is not charged, thus it can only occur in abelian theories. This expression indicates that the 3d FI term is the real mass of the topological symmetry.

Here comes the Lagrangian of matter. If we couple a chiral multiplet $\boldsymbol{\Phi}$ to a vector multiplet $\mathscr{V}$, the kinetic terms reads

$$\mathscr{L}_{\bar{\Phi}\Phi} = D_\mu\bar{\phi}D^\mu\phi - i\bar{\psi}\gamma^\mu D_\mu\psi - \bar{F}F + \bar{\phi}D\phi + \underbrace{\bar{\phi}\sigma^2\phi - i\bar{\psi}\sigma\psi}_{\text{mass term}} + i\sqrt{2}(\bar{\phi}\lambda\psi + \bar{\psi}\bar{\lambda}\phi) \tag{4.29}$$

At last the interaction term in superpotential $W(\phi)$ gives as

$$\mathscr{L}_W = F^i\frac{\partial W}{\partial\phi^i} - \frac{1}{2}\psi^i\psi^j\frac{\partial W}{\partial\phi^i\partial\phi^j} \tag{4.30}$$

the conjugate superpotential term $\mathscr{L}_{\bar{W}}$ can be similarly obtained.

Chapter 5 The Squashed-sphere Partition Function: An Example

The study of partition function on squashed 3-sphere was first introduced in Kapustin et al.[7]. It's one of the most well-studied models in 3d SUSY theories. One of the reasons why we prefer to study these theories on a compact manifold is that the partition function of theories on it is finite, well-defined observable.

For a partition function of a field, we usually write it as

$$Z[\Phi] = \int [\mathscr{D}\Phi] e^{i\mathscr{L}[\Phi]} \tag{5.1}$$

It's an infinite-dimensional path integral. Usually, we cannot get the exact result easily except for the free field, and sometimes divergence occurs thus we need to regularize it. Luckily, by a method called localization[8], we can turn this partition function into a finite-dimensional integral

$$Z_{\mathscr{M}} = \int d\sigma Z(\sigma) \tag{5.2}$$

on the background manifold $\mathscr{M}$. Here σ is the real scalar in vector multiplet.

In Euclidean signature, the path integral partition function becomes

$$Z[\Phi] = \int [\mathscr{D}\Phi] e^{-\mathscr{L}[\Phi]} \tag{5.3}$$

We assume $\mathscr{V}$ an functional satisfying $\mathscr{Q}^2\mathscr{V} = 0$. $\mathscr{Q}\mathscr{V}$ is a $\mathscr{Q}$-exact term, thus its expectation value is 0. We can deform $\mathscr{L} \to \mathscr{L} + t\mathscr{Q}\mathscr{V}$, where t is an arbitrary parameter. Now Z becomes $Z(t) = \int [\mathscr{D}\Phi] e^{-\mathscr{L}[\Phi] - t\mathscr{Q}\mathscr{V}}$. We can find that once we take the derivative of Z with respect to t

$$\frac{\mathrm{d}Z}{\mathrm{d}t} = \int [\mathscr{D}\Phi](\mathscr{Q}\mathscr{V}) e^{-\mathscr{L}[\Phi] - t\mathscr{Q}\mathscr{V}} \tag{5.4}$$

Because $\mathscr{L}$ is invariant under $\mathscr{Q}$ transformation ($\mathscr{Q}\mathscr{L} = 0$) , and $\mathscr{Q}^2\mathscr{V} = 0$, we can write the integrand in a $\mathscr{Q}$-exact form, which is exactly 0

$$\frac{\mathrm{d}Z}{\mathrm{d}t} = \int [\mathscr{D}\Phi](\mathscr{Q}\mathscr{V}) e^{-\mathscr{L}[\Phi] - t\mathscr{Q}\mathscr{V}} = \int [\mathscr{D}\Phi]\mathscr{Q}(\mathscr{V} e^{-\mathscr{L}[\Phi] - t\mathscr{Q}\mathscr{V}}) = 0 \tag{5.5}$$

Now we know that Z is independent of t. Take t to the large limit ($t \to \infty$), and we can see that only points with $\mathscr{Q}\mathscr{V} = 0$ can contribute to the partition function.

These are the saddle points and now we change an infinite-dimensional integral into a finite-dimensional integral

$$Z[\Phi] = \int [\mathscr{D}\Phi] e^{-\mathscr{L}[\Phi]} \to \int_{\mathscr{M}_3} d\sigma Z_{\text{1-loop}}(\sigma) e^{-\mathscr{L}} \tag{5.6}$$

5.1 The Squashed Three-sphere

When we begin to consider theories in 3d, theories on S^3 will be convenient to think about. Further more, we can add a parameter b controlling the "squashing extent" of the sphere.

The squashed 3-sphere can be described by the definition

$$S_b^3 = \{b^2(x_1^2 + x_2^2) + b^{-2}(x_3^2 + x_4^2) = r^2, \quad \{x_1, x_2, x_3, x_4 \in \mathbb{R}^4\}\} \tag{5.7}$$

we can see that when $b = 1$, S_b^3 have $SO(4)$ symmetry, and when $b \neq 1$, the symmetry breaks into $SO(2) \times SO(2)$. For simplicity, usually we take $r = 1$.

If we write it in spherical coordinates, we have the metric

$$ds^2 = (b^{-2}\cos^2\theta + b^2\sin^2\theta)d\theta^2 + b^{-2}\sin^2\theta d\phi^2 + b^2\cos^2\theta d\chi^2 \tag{5.8}$$

The range of these angles are $\theta \in [0, \frac{\pi}{2}]$, $\phi \in [0, 2\pi]$, $\chi \in [0, 2\pi]$.

In the background of S_b^3, the saddle points have relations[12]

$$\begin{aligned} &\sigma = \text{Const.} \qquad D = \frac{\sigma}{\sqrt{b^{-2}\cos^2\theta + b^2\sin^2\theta}} \\ &F_{\mu\nu} = 0 \qquad\quad\; \Phi = F = 0 \end{aligned} \tag{5.9}$$

Here we denote $\tilde{\sigma}_a = r\sigma_a$, $\tilde{m}_\alpha = rm_\alpha$, $\tilde{\sigma}_R = -\frac{i}{2}(b+b^{-1})$ to construct dimensionless parameters and r is set to 1. σ_a is the eigenvalues of σ ($\sigma = \text{diag}(\sigma_a) \in \mathbb{R}^{\text{rank}(G)}, a = 1, \dots, \text{rank}(G)$). $\tilde{m}_\alpha$ and $\tilde{\sigma}_R$ can be regarded as masses for $U(1)_\alpha$ flavor symmetry and R-symmetry.

Now we can write out the path integral for S_b^3:

$$Z_{S_b^3} = \int \prod_a d\tilde{\sigma}_a Z_{S_b^3}^{\text{CS}}(\tilde{\sigma}, \tilde{m}) Z_{S_b^3}^{\text{1-loop}}(\tilde{\sigma}, \tilde{m}) \tag{5.10}$$

Furthermore, if we consider supersymmetric Lagrangian containing Chern–Simons term, matter term and vector multiplet, the partition function becomes

$$Z = \int \frac{d^{\text{rank}(G)}\tilde{\sigma}}{|W_G|} Z_{S_b^3}^{\text{CS}} Z_{S_b^3}^{\Phi} Z_{S_b^3}^{\text{vector}} \tag{5.11}$$

$|W_G|$ is the number of generators in Weyl group.

We can then calculate the partition function for each term. The partition function for Chern–Simons term is

$$Z^{\mathrm{CS}}_{S^3_b} = e^{i\pi k \sum_{a=1}^{N} \tilde{\sigma}_a^2} \tag{5.12}$$

And for FI term,

$$Z^{\mathrm{FI}}_{S^3_b} = e^{2\pi i \xi \sum_{a=1}^{N} \tilde{\sigma}_a} \tag{5.13}$$

As for the 1-loop part, the corresponding partition function for vector multiplet is

$$Z^{\mathrm{vector}}_{S^3_b} = \left(e^{-\frac{\pi i}{12}(b^2+b^{-2}+3)}\right)^{\dim(G)} \prod_{\alpha\in\Delta^+} 4\sinh(\pi b\alpha(\tilde{\sigma}))\sinh\left(\pi b^{-1}\alpha(\tilde{\sigma})\right) \tag{5.14}$$

Here Δ^+ are gauge group's positive roots.

Particularly, for $U(N)$ gauge theories, the positive roots are $\{\sigma_i - \sigma_j\}_{1\le i<j\le N}$. So the $Z^{\mathrm{vector}}_{S^3_b}$ becomes

$$Z^{\mathrm{vector}}_{S^3_b} = e^{-\frac{\pi i N^2}{12}(b^2+b^{-2}+3)} \prod_{i,j=1;i>j}^{N} 4\sinh\left(\pi b(\tilde{\sigma}_i - \tilde{\sigma}_j)\right)\sinh\left(\pi b^{-1}(\tilde{\sigma}_i - \tilde{\sigma}_j)\right) \tag{5.15}$$

5.2 $Z_{S^3_b}$: Some Examples

Here we show some examples about 3-sphere partition functions of gauge theories.

5.2.1 Supersymmetric $U(N)_k$ Chern–Simons Theory

For $U(N)_k$ CS theory, we have

$$\begin{aligned} Z^{U(N)_k}_{S^3_b} = \frac{1}{N!}\int d^N\tilde{\sigma}\, & \underbrace{e^{2\pi i\xi\sum_{a=1}^{N}\tilde{\sigma}_a}}_{\text{FI term}}\ \underbrace{e^{i\pi k\sum_{a=1}^{N}\tilde{\sigma}_a^2}}_{\text{Chern–Simons term}} \\ & \underbrace{\cdot\, e^{-\frac{\pi i N^2}{12}(b^2+b^{-2}+3)} \prod_{i,j=1;i>j}^{N} 4\sinh\left(\pi b(\tilde{\sigma}_i-\tilde{\sigma}_j)\right)\sinh\left(\pi b^{-1}(\tilde{\sigma}_i-\tilde{\sigma}_j)\right)}_{\text{Vector multiplet term}} \end{aligned} \tag{5.16}$$

A special case is the $U(1)_k$ theory. In this case we have

$$Z^{U(1)_k}_{S^3_b} = e^{-\frac{\pi i}{12}(b^2+b^{-2}+3)}\int d\tilde{\sigma} e^{2\pi i\xi\tilde{\sigma}} e^{i\pi k\tilde{\sigma}^2} \tag{5.17}$$

Now we can do the Gaussian integral to get

$$Z^{U(1)_k}_{S^3_b} = H(k) e^{-\frac{\pi i}{12}(b^2+b^{-2})} \frac{e^{-\frac{\pi i \xi^2}{k}}}{\sqrt{|k|}} \tag{5.18}$$

Here $H(k)=1$ if $k>0$ and $H(k)=-i$ if $k<0$.

And $U(1)_1$ case is

$$Z^{U(1)_1}_{S^3_b} = e^{-\frac{\pi i}{12}(b^2+b^{-2})} e^{-\pi i\xi^2} \tag{5.19}$$

5.2.2 $U(1)_{\frac{1}{2}} + \Phi$ Supersymmetric Gauge Theory

Similarly, in this case the partition function is

$$Z_{S_b^3}^{U(1)_{\frac{1}{2}}+\Phi} = e^{-\frac{\pi i}{12}(b^2+b^{-2}+3)} \int d\tilde{\sigma} e^{2\pi i \xi \tilde{\sigma}} e^{i\pi\tilde{\sigma}^2} \tilde{\Phi}_b(\tilde{\sigma} + \tilde{\sigma}_R(r-1)) \tag{5.20}$$

r here is an arbitrary R-charge in $\mathbb{R}$ for chiral multiplet.

It can be obtained that when take the $\tilde{\sigma} \to -\infty$ limit, $\tilde{\Phi} \to e^{-i\pi\tilde{\sigma}^2} e^{-\frac{\pi i}{12}(b^2+b^{-2})}$, and when $\tilde{\sigma} \to \infty$, $\tilde{\Phi} \to 1$. Moreover, if we take the limit $\xi \to -\infty$ and $r = 1$, the expression (5.20) recovers the result in (5.19).

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