

Homework 3: Due at class on Oct 12

1 Derivations

1.1 Energy-momentum tensor in complex coordinate

Show (3.50), (3.51), (3.52) by performing the coordinate transformation.

1.2 Schwarzian derivative

Derive the infinitesimal transformation (3.92) from the finite version (3.93).

1.3 Virasoro algebra

Derive the Virasoro algebra (3.116) from (3.115) by performing the contour integral.

1.4 Commutation relations in free boson

Derive the commutation relations (4.7) and (4.8) from (4.6).

1.5 Hamiltonian in free boson

Derive the Hamiltonian (4.10) from (4.9).

1.6 Action of free fermion

Derive (4.33) from (4.30).

1.7 TT OPE in free fermion

Compute TT OPE (4.49) in the free fermion.

2 Vertex operator and OPE

Show that $:e^{ik\varphi}:$ is a primary field of weight $h = \bar{h} = k^2/2$ in the free boson theory. In addition, show that $\partial^n \varphi$ ($n \geq 2$) is not a primary field.

3 bc ghost system

The action of the bc ghost system in the Euclidian signature is given by

$$S_{\text{gh}} = \frac{1}{2\pi} \int d^2\sigma \sqrt{g} \, b^{ab} \nabla_a c_b ,$$

The energy-momentum tensor for the bc ghosts can be calculated by the definition

$$T_{ab} = -\frac{4\pi}{\sqrt{g}} \frac{\delta S}{\delta g^{ab}} ,$$

yielding

$$T(z) = -2 : b(z) \partial c(z) : + : c(z) \partial b(z) :$$

in the conformally flat metric. (If you derive this explicit form of the energy-momentum tensor, you will obtain a bonus point.) Suppose that the bc OPE is given by

$$\begin{aligned} b(z) c(w) &= \frac{1}{z-w} + \dots \\ c(w) b(z) &= \frac{1}{w-z} + \dots \end{aligned}$$

where b, c fields obey the Fermi statistics so that the second equation follows from the first equation.

Compute $T(z)b(w)$, $T(z)c(w)$, and $T(z)T(w)$ OPEs, and find the conformal dimensions of b and c as well as the central charge of the bc ghost system.

4 Schwarzian derivatives

In the lecture note (3.93), we have learned that, under the conformal transformation $z \rightarrow w(z)$, the energy-momentum tensor transforms

$$T(w) = \left(\frac{dw}{dz} \right)^{-2} \left[T(z) - \frac{c}{12} \{w; z\} \right] ,$$

where $\{w; z\}$ is the additional term called the Schwarzian derivative:

$$\{w; z\} = \frac{w'''}{w'} - \frac{3}{2} \left(\frac{w''}{w'} \right)^2 .$$

4.1 Under $\text{SL}(2, \mathbb{C})$

For an element of $\text{SL}(2, \mathbb{C})$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{C})$$

show that

$$\{w; z\} = 0 \quad \text{for } w = \frac{az + b}{cz + d} ,$$

and

$$\left\{ \frac{aw + b}{cw + d}; z \right\} = \{w; z\} .$$

4.2 Free boson

The energy-momentum tensor of the free boson is

$$T(z) = -\frac{1}{2} : \partial_z \varphi \partial_z \varphi :$$

where the normal ordering can be defined as

$$: \partial_z \varphi \partial_w \varphi := \lim_{w \rightarrow z} \left(\partial_z \varphi(z) \partial_w \varphi(w) + \frac{1}{(z-w)^2} \right) .$$

Since $\partial_z \varphi$ is the primary field of conformal dimension one, it transforms as

$$\partial_z \varphi(z) \partial_w \varphi(w) = f'(z) f'(w) \partial_{\tilde{z}} \varphi(\tilde{z}) \partial_{\tilde{w}} \varphi(\tilde{w})$$

under the conformal transformation $z \rightarrow \tilde{z} = f(z)$. Hence we have

$$: \partial_z \varphi(z) \partial_w \varphi(w) : - \frac{1}{(z-w)^2} = f'(z) f'(w) \left[: \partial_{\tilde{z}} \varphi(\tilde{z}) \partial_{\tilde{w}} \varphi(\tilde{w}) : - \frac{1}{(\tilde{z}-\tilde{w})^2} \right]$$

Taking limit $z \rightarrow w$, show that

$$\lim_{z \rightarrow w} \left[\frac{f'(z) f'(w)}{(f(z) - f(w))^2} - \frac{1}{(z-w)^2} \right] = \frac{1}{6} \{f(w); w\} .$$

What is the central charge of the free boson?