Homework 6: Due at class on Dec 13

1 Dimensional regularization

Using dimensional regularization $d = 4 - \epsilon$, and Wick rotation ($p^0 \to ip^0_E$), show

$$\int \frac{d^dk}{(2\pi)^d} \frac{k^{2a}}{[k^2 - \Delta + i\tilde{\epsilon}]^b} = \frac{i}{16\pi^2}(-\Delta)^{a-b+2}\left(\frac{4\pi}{\Delta}\right)^{\frac{\epsilon}{2}} \frac{\Gamma(2 + a - \frac{\epsilon}{2})\Gamma(b - a - 2 + \frac{\epsilon}{2})}{\Gamma(b)\Gamma(2 - \frac{\epsilon}{2})}$$

The integral over the solid angle in d-dimensional euclidean space is given by

$$\int d\Omega_d = \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})} .$$

2 Optical theorem in massless QED

Let us consider the optical theorem in the massless QED. Show that the unpolarized cross-section σ of the scattering$A(k_1)B(k_2) \to C(p_1)D(p_2)$ in CM-frame namely

$$\sigma = \frac{1}{s}\,\mathrm{Im}(\mathcal{M})$$

where $i\mathcal{M}$ the unpolarized (average) Feynman amplitude of the process $A(k_1)B(k_2) \to A(k_1)B(k_2)$.

In the lecture, we have seen that the vacuum polarization takes the form

$$\Pi^{\mu\nu}(q) = \left(q^2 g^{\mu\nu} - q^\mu q^\nu\right)\Pi(q^2)$$

Also, the dimensional regularization $d = 4 - \epsilon$ provides

$$\Pi(q^2) = -\frac{2\alpha}{\pi}\int_0^1 dx x(1-x)\left(\frac{2}{\epsilon} - \gamma + \log(4\pi) - \log\frac{-x(1-x)q^2 - i\tilde{\epsilon}}{\mu^2}\right)$$

where we replace the electric charge e by $e\mu^\epsilon$.

Show that in the case of $e^-e^+ \to \mu^-\mu^+$ scattering in CM-frame ($m_e = m_\mu = 0$) the unpolarized cross-section (order $\mathcal{O}(\alpha^2)$) is given by

$$\sigma = \frac{4\pi\alpha^2}{3s}$$

using the optical theorem and the expression for $\mathrm{Im}(\Pi(q^2))$.

3 Vacuum polarization in scalar QED

In the scalar QED, find the diagrams which contribute to the 1PI diagrams of the photon propagator in the order $\mathcal{O}(e^2)$ and write down the integral expression for $\Pi^{\mu\nu}_{\text{1-loop}}(k)$

$i\Pi^{\mu\nu}(k) \equiv \mu$  $\nu = e^2\Pi^{\mu\nu}_{\text{1-loop}}(k) + \mathcal{O}(e^4)$,

and perform momentum integration by using the dimensional regularization.