

# Homework 10 (Due at class on Dec 1)

## 1 Fermionization

### 1.1

At the special radius  $R = \sqrt{\frac{\alpha'}{2}}$  of the circle compactification, one can redefine a bosonic field  $H(z) := \sqrt{\frac{2}{\alpha'}} X(z)$  so that the OPE are given by

$$H(z)H(0) \sim -\ln z .$$

Let us also consider two Majorana-Weyl fermions  $\psi^1, \psi^2$  with OPE

$$\psi^i(z)\psi^j(0) \sim \frac{\delta^{ij}}{z} .$$

We can define the complex fermion

$$\psi(z) = 2^{-1/2}(\psi^1(z) + i\psi^2(z)) , \quad \bar{\psi}(z) = 2^{-1/2}(\psi^1(z) - i\psi^2(z)) .$$

Show the equivalence of operators in boson and fermion

$$:e^{iH}: \cong \psi , \quad :e^{-iH}: \cong \bar{\psi} , \quad i\partial H \cong \psi\bar{\psi} , \quad T_H \cong T_\psi ,$$

by calculating the OPEs of operators in both theories and comparing.

### 1.2

At the special radius  $R = \sqrt{\frac{\alpha'}{2}}$ , show that the torus partition function can be written as

$$Z^{25} = \frac{1}{2|\eta(q)|^2} \left[ \left| \sum_n q^{n^2/2} \right|^2 + \left| \sum_n (-1)^n q^{n^2/2} \right|^2 + \left| \sum_n q^{\frac{1}{2}(n+1/2)^2} \right|^2 \right] .$$

Verify that this is the torus partition function of a free complex fermion with all the possible boundary conditions (AA, AP, PA, PP). This is called the **diagonal modular invariant** partition function.

## 2 Check the central charge

The world-sheet action of Heterotic string theory is given by

$$S^m = \frac{1}{4\pi} \int d^2z \left( \frac{2}{\alpha'} \partial X^\mu \bar{\partial} X_\mu + \psi^\mu \bar{\partial} \psi_\mu + \tilde{\lambda}^A \partial \tilde{\lambda}_A \right)$$

$$S^{gh} = \frac{1}{2\pi} \int d^2z (b \bar{\partial} c + \bar{b} \partial \bar{c} + \beta \bar{\partial} \gamma)$$

where  $\mu$  are 10-dimensional indices and the right-moving sector is supersymmetric. In the lecture, I have explained that if there are 32 left-moving fermions  $\tilde{\lambda}^A$ , there is no Weyl-anomaly  $c^{\text{tot}} = 0$ . Explain in detail the value of each contribution  $c^X, c^\psi, c^{bc}, c^{\beta\gamma}, c^\lambda$  in the lecture note.

### 3 Toroidal compactifications

Let us consider the toroidal compactification in the presence of  $B$ -field

$$S = -\frac{1}{4\pi\alpha'} \int d^2\sigma (\delta_{IJ}\eta^{\alpha\beta} + B_{IJ}\epsilon^{\alpha\beta}) \partial_\alpha X^I \partial_\beta X^J ,$$

where  $\epsilon^{01} = -1$  and  $\eta^{\alpha\beta} = \text{diag}(-1, 1)$ . The bosonic field is quantized as in the lecture note

$$\begin{aligned} X^I(z) &= x^I - i\sqrt{\frac{\alpha'}{2}} p_R^I \ln z + i\sqrt{\frac{\alpha'}{2}} \sum_{m \neq 0} \frac{\alpha_m^I}{m z^m}, \\ \bar{X}^I(\bar{z}) &= \tilde{x}^I - i\sqrt{\frac{\alpha'}{2}} p_L^I \ln \bar{z} + i\sqrt{\frac{\alpha'}{2}} \sum_{m \neq 0} \frac{\tilde{\alpha}_m^I}{m \bar{z}^m}. \end{aligned}$$

where the bosonic field is periodically identified on the lattice  $W^I \in \Lambda$

$$X^I(\sigma + 2\pi, \tau) = X^I(\sigma, \tau) + 2\pi W^I . \quad (3.1)$$

#### 3.1

Show that the center of mass momentum

$$\pi_I := \int_0^{2\pi} d\sigma \Pi_I , \quad \Pi_I = \frac{\delta S}{\delta \dot{X}^I} ,$$

together with (3.1) imply that

$$(p_I)_{L,R} := \delta_{IJ} p_{L,R}^J = \sqrt{\frac{\alpha'}{2}} \left( \pi_I \pm \frac{1}{\alpha'} (\delta_{IJ} \mp B_{IJ}) W^J \right) .$$

Here left-modes get the upper sign and right-modes get the lower sign whenever you find the notation  $\pm$  and  $\mp$ .

#### 3.2

It is  $\pi_I$  that generates translations and it must therefore lie on the lattice  $\Lambda_D^*$ ,  $\pi_I = e_I^{*i} m_i$  for  $m_i \in \mathbb{Z}$ . Writing  $\mathbf{g} = g_{ij} = e_i^I \delta_{IJ} e_j^J$ ,  $\mathbf{b} = b_{ij} = e_i^I B_{IJ} e_j^J$ , show that the mass formula and the level-matching condition for closed strings are modified to (a matrix notation is employed in the expression below)

$$\begin{aligned} \alpha' M^2 &= \alpha' \mathbf{m}^T \mathbf{g}^{-1} \mathbf{m} + \frac{1}{\alpha'} \mathbf{n}^T (\mathbf{g} - \mathbf{b} \mathbf{g}^{-1} \mathbf{b}) \mathbf{n} + 2 \mathbf{n}^T \mathbf{b} \mathbf{g}^{-1} \mathbf{m} + 2(N + \tilde{N} - 2) \\ N - \tilde{N} &= \boldsymbol{\pi} \cdot \mathbf{W} = \mathbf{m}^T \mathbf{n} . \end{aligned} \quad (3.2)$$

### 3.3

Use (3.2) to show that the spectrum is invariant under the map

$$\mathbf{m} \leftrightarrow \mathbf{n}, \quad \alpha' \mathbf{g}^{-1} \leftrightarrow \frac{1}{\alpha'} (\mathbf{g} - \mathbf{b} \mathbf{g}^{-1} \mathbf{b}), \quad \mathbf{b} \mathbf{g}^{-1} \leftrightarrow -\mathbf{g}^{-1} \mathbf{b}.$$

The second and the third are indeed equivalent to

$$\frac{1}{\alpha'} (\mathbf{g} + \mathbf{b}) \leftrightarrow \alpha' (\mathbf{g} + \mathbf{b})^{-1},$$

and this is the T-duality in the presence of  $B$ -field.

### 3.4

A non-trivial instructive example is the 2-torus  $T^2 = \mathbb{R}^2/\Lambda$  where  $\mathbf{e}_1, \mathbf{e}_2$  are the two generators of  $\Lambda$  and its metric is  $g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j$ . The torus has one Kähler modulus  $\sqrt{\det g_{ij}}$  and one complex structure modulus

$$\tau = \frac{|\mathbf{e}_2|}{|\mathbf{e}_1|} e^{i \text{angle}(\mathbf{e}_1, \mathbf{e}_2)} = \frac{g_{12} + i\sqrt{\det g_{ij}}}{g_{11}} = \tau_1 + i\tau_2.$$

In the presence of  $B$ -field, the Kähler modulus is complexified

$$\omega = \frac{1}{\alpha'} (B + i\sqrt{\det g_{ij}}) = \omega_1 + i\omega_2.$$

This leads to the expression

$$g_{ij} = \alpha' \frac{\omega_2}{\tau_2} \begin{pmatrix} 1 & \tau_1 \\ \tau_1 & |\tau|^2 \end{pmatrix}.$$

Using these expressions, show that the momentum contribution to the mass formula is given by

$$\begin{aligned} \mathbf{p}_L^2 &= \frac{1}{2\omega_2 \tau_2} |m_2 - \tau m_1 + \bar{\omega}(n^1 + \tau n^2)|^2 \\ \mathbf{p}_R^2 &= \frac{1}{2\omega_2 \tau_2} |m_2 - \tau m_1 + \omega(n^1 + \tau n^2)|^2. \end{aligned}$$

Show that, if we transform the momentum and winding numbers appropriately, the string spectrum is then invariant under the following two  $\text{SL}(2, \mathbb{Z})$  transformations

- T-dualities

$$\omega \rightarrow \frac{a\omega + b}{c\omega + d}$$

- Torus modular transformations

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d}$$

- **mirror symmetry** (symmetry between complexified Kähler modulus and complex structure modulus)

$$\omega \leftrightarrow \tau$$

How do the momentum and winding numbers transform?