Homework 4: Due at class on March 31

1. Let ω be the n-form on the space $\mathbb{R}^{n+1}\backslash\{0\}$ defined by

$$\omega = \frac{1}{|x|^{n+1}} \sum_{i=1}^{n+1} (-1)^{i-1} x_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_{n+1} \ ,$$

where $|x| = (x_1^2 + \cdots x_{n+1}^2)^{1/2}$. Let $\iota : S^n \hookrightarrow \mathbb{R}^{n+1}$ be the unit sphere in $\mathbb{R}^{n+1}$. Show that $\iota^*\omega$ is the generator of n-the de Rham cohomology $H^n_{dR}(S^n)$ of the n-sphere. In fact, the de Rham cohomology of S^n is

$$H^k_{dR}(S^n) = \begin{cases} \mathbb{R} & k = 0,\ n \\ 0 & \text{otherwise} \end{cases} .$$

2. Let $\iota : S^2 \hookrightarrow \mathbb{R}^3$ be the inclusion of the 2-sphere with unit radius. Let $g : ds^2 = \sum_{i=1}^3 dx^i \otimes dx^i$ be the standard metric of $\mathbb{R}^3$. Find the induced metric $\iota^* g$ on S^2 in terms of the polar coordinate of $\mathbb{R}^3$.

$$\begin{aligned} x_1 &= r \sin\theta \cos\phi \\ x_2 &= r \sin\theta \sin\phi \\ x_3 &= r \cos\theta \end{aligned}$$

3. Let $ds^2 = -dt^2 + dx^2 + dy^2$ be the Minkowski metric on $\mathbb{R}^3$ and $-t^2 + x^2 + y^2 = -1$ for $t > 0$ be the space-like surface (hyperboloid $\mathbf{S}$). (See Figure 1.) Find the induced metric on the hyperboloid S in terms of the polar coordinate

$$\begin{aligned} t &= r \cosh\rho \\ x &= r \sinh\rho \cos\phi \\ y &= r \sinh\rho \sin\phi \end{aligned}$$

4. Let us consider the unit disk $\mathbf{D}$ on the x-y plane. Let $\pi(P)$ be the intersection point of the unit disk and the line between the point $(0, 0, -1)$ and $P \in \mathbf{S}$. By assigning $\pi(P)$ to P, there is one-to-one map from the unit disk D and the hyperboloid $\mathbf{S}$. Show that the map $\pi^{-1} : \mathbf{D} \to \mathbf{S}$ is determined by

$$(u, v) \mapsto \left(\frac{2u}{1 - u^2 - v^2}, \frac{2v}{1 - u^2 - v^2}, \frac{1 + u^2 + v^2}{1 - u^2 - v^2} \right) .$$

Find the metric on the unit disk pull-backed by this map. The unit disk $\mathbf{D}$ with this induced metric is called the Poincaré disk.

The red curve on the hyperboloid is an intersection with a plane (pink in Figure 2) that goes through the origin. We put a disk (called Klein's disk) on the bottom of the hyperboloid, which allows us to get the corresponding straight line (green) on Klein's disk. This curve is mapped by $\pi : \mathbf{S} \to \mathbf{D}$ to an (red) arc in the Poincare disk. If the green line is represented by $x = a$ (Figure 3), find the equation for the red curve.

5. Let $\mathbf{H} = \{(x, y)|y > 0\}$ be the upper half plane (Yellow area in Figure 4). By reversing in terms of the circle with radius $\sqrt{2}$ around the origin $(0, -1)$ (Figure 4), we have the map $J : \mathbf{H} \to \mathbf{D}$

$$(x, y) \mapsto \left(\frac{2x}{x^2 + (y+1)^2}, 1 - \frac{2(y+1)}{x^2 + (y+1)^2} \right) .$$

Find the induced metric on the upper half plane by this map. Find the area of the triangle with angles (α, β, γ) bounded by half-circles with respect to the metric (Figure 5). Here, we can use the fact that the area of the triangle in the left of Figure 5 is the same as that of the triangle in the right of Figure 5. Compare with the area of a triangle on the 2-sphere (Homework 1).

6. Let (M, g) be an oriented Riemannian manifold, compact and without boundary. We define a linear operator $\delta = (-1)^{n(k+1)+1} * d*$. Show that

$$(d\omega, \eta) = (\omega, \delta\eta)$$

with respect to the metric on $\Omega^k(M)$ induced by the metric g.

7. The Maxwell equations are written as

$$\begin{aligned} \nabla \cdot \mathbf{E} &= \rho \ , \qquad & \nabla \times \mathbf{B} &= \mathbf{J} + \frac{\partial \mathbf{E}}{\partial t} \ , \\ \nabla \cdot \mathbf{B} &= 0 \ , \qquad & \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \ . \end{aligned}$$

Let us write the gauge potential

$$A = A_\mu dx^\mu = \phi dt + A_1 dx^1 + A_2 dx^2 + A_3 dx^3$$

and the current

$$J = J_\mu dx^\mu = \rho dt + J_1 dx^1 + J_2 dx^2 + J_3 dx^3 \ .$$

Then, the field strength can be written as $F = dA$. Show that the Maxwell equations are equivalent to the following equations

$$dF = 0 \ , \qquad \delta F = -j \ .$$

Find the equation of motion for the following action

$$S = -\frac{1}{4} \int F \wedge *F - \int A \wedge *J \ .$$

Discuss this for both a positive definite metric and a Lorentzian signature metrics.

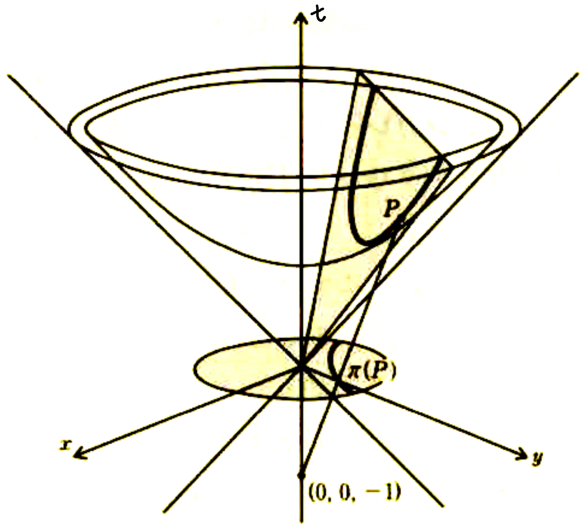

Figure 1: Hyperboloid and Poincare disk

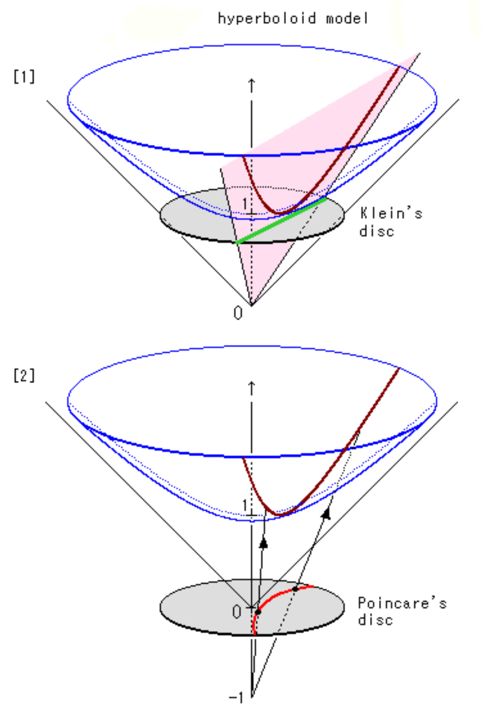

Figure 2: Hyperboloid and Poincare disk

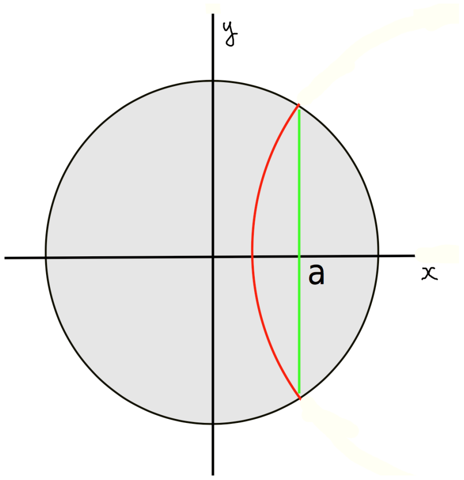

Figure 3: Poincare disk

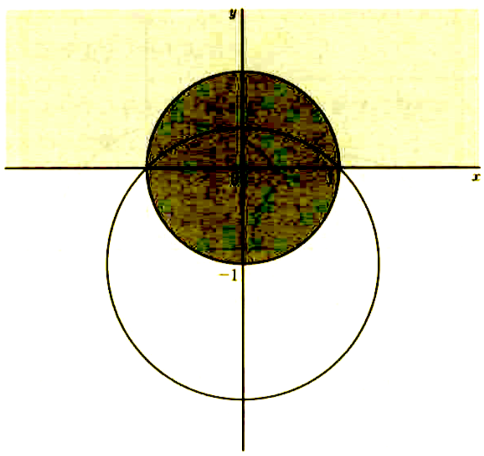

Figure 4: Hyperboloid and Poincare disk

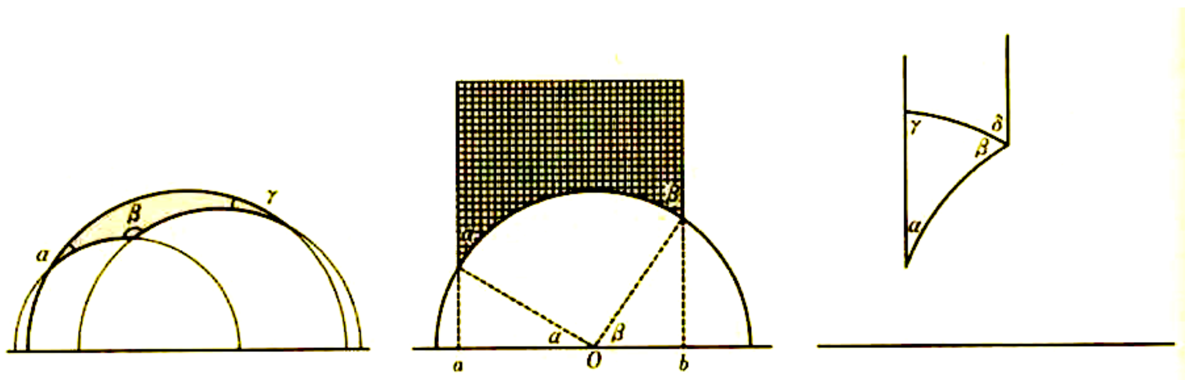

Figure 5: triangle in the upper half plane