1 Bosonic string on a torus

1.1

The torus partition function on a circle S^1 of Radius R is given by

$$Z^{25} = \mathrm{Tr} q^{L_0 - \frac{1}{24}} \overline{q}^{\overline{L}_0 - \frac{1}{24}} \tag{1.1}$$

L_0 and $\overline{L}_0$ are zero mode virasoro generators

$$L_0 = \frac{\alpha' {p_L}^2}{4} + \sum_{n=1}^{\infty} \hat{\alpha}_{-n} \hat{\alpha}_n \tag{1.2}$$

$$\overline{L}_0 = \frac{\alpha' {p_R}^2}{4} + \sum_{n=1}^{\infty} \hat{\tilde{\alpha}}_{-n} \hat{\tilde{\alpha}}_n \tag{1.3}$$

with right and left momentum

$$p_L = \frac{n}{R} + \frac{wR}{\alpha'} \tag{1.4}$$

$$p_R = \frac{n}{R} - \frac{wR}{\alpha'} \tag{1.5}$$

Then we can express Z^{25} in terms of Dedekind η function

$$\begin{aligned} Z^{25} &= (q\overline{q})^{-\frac{1}{24}} \mathrm{Tr} q^{\frac{\alpha' {p_L}^2}{4} + \sum_{n=1}^{+\infty} \hat{\alpha}_{-n}\hat{\alpha}_n} \overline{q}^{\frac{\alpha' {p_R}^2}{4} + \sum_{n=1}^{+\infty} \hat{\tilde{\alpha}}_{-n}\hat{\tilde{\alpha}}_n} \\ &= (q\overline{q})^{-\frac{1}{24}} \sum_{n,w=-\infty}^{+\infty} q^{\frac{\alpha' {p_L}^2}{4}} \overline{q}^{\frac{\alpha' {p_R}^2}{4}} \sum_{N_n, \widetilde{N}_n = 0}^{+\infty} q^{nN_n} \overline{q}^{n\widetilde{N}_n} \\ &= (q\overline{q})^{-\frac{1}{24}} \sum_{n,w=-\infty}^{+\infty} q^{\frac{\alpha' {p_L}^2}{4}} \overline{q}^{\frac{\alpha' {p_R}^2}{4}} \prod_{n=1}^{+\infty} |1 - q^n|^{-2} \\ &= |\eta(q)|^{-2} \sum_{n,w=-\infty}^{+\infty} q^{\frac{\alpha' {p_L}^2}{4}} \overline{q}^{\frac{\alpha' {p_R}^2}{4}} \end{aligned} \tag{1.6}$$

The partition function for the non-compact space $\mathbb{R}^{1,24}$

$$Z^{1,24} = \text{const} \times |\eta(q)|^{-46} \tag{1.7}$$

The matter part contributes $|\eta(q)|^{-50}$(25 dimension), and the ghost part contribute $|\eta(q)|^4$ as usual.

To recover each mass formula term. We need expand the whole partion function.

$$
\begin{aligned}
Z^{1,24}Z^{25} &= |\eta(q)|^{-46}|\eta(q)|^{-2}\sum_{n,w=-\infty}^{+\infty} q^{\frac{\alpha' {p_L}^2}{4}}\overline{q}^{\frac{\alpha' {p_R}^2}{4}} \\
&= \left[(q\overline{q})^{-\frac{1}{24}}\prod_{n=1}^{+\infty}(1-q^n)^{-1}(1-\overline{q}^n)^{-1}\right]^{24}\sum_{n,w=-\infty}^{+\infty} q^{\frac{\alpha' {p_L}^2}{4}}\overline{q}^{\frac{\alpha' {p_R}^2}{4}} \\
&= \left[(q\overline{q})^{-\frac{1}{24}}\prod_{n=1}^{+\infty}\sum_{N_n=0}^{+\infty} q^{nN_n}\sum_{\widetilde{N}_n=0}^{+\infty}\overline{q}^{n\widetilde{N}_n}\right]^{24}\sum_{n,w=-\infty}^{+\infty} q^{\frac{\alpha' {p_L}^2}{4}}\overline{q}^{\frac{\alpha' {p_R}^2}{4}} \\
&= \sum_{N_n=0}^{+\infty}\sum_{\tilde{N}_n=0}^{+\infty}\sum_{n,w=-\infty}^{+\infty} q^{\frac{\alpha' {p_L}^2}{4}-1+\sum_{n=1}^{\infty}24nN_n}\overline{q}^{\frac{\alpha' {p_R}^2}{4}-1+\sum_{n=1}^{\infty}24n\widetilde{N}_n}
\end{aligned}
\tag{1.8}
$$

The number of state: Pochinski (1.3.37)

$$
N = \sum_{2}^{D-1}\sum_{n=1}^{+\infty} nN_{im} \tag{1.9}
$$

Where D is dimension. Rewrite the partition function

$$
Z^{1,24}Z^{25} = \sum_{N_n=0}^{+\infty}\sum_{\widetilde{N}_n=0}^{+\infty}\sum_{n,w=-\infty}^{+\infty} q^{\frac{\alpha' {p_L}^2}{4}-1+N}\overline{q}^{\frac{\alpha' {p_R}^2}{4}-1+\widetilde{N}} \tag{1.10}
$$

For each term, which means for each fixed N_n,$\widetilde{N}_n$,n and w, it should be real which gives the level matching conditon

$$
\begin{aligned}
0 &= \frac{\alpha' p_L^2}{4} + N - \frac{\alpha' p_R^2}{4} - \tilde{N} \\
&= \frac{\alpha'}{4}\times\frac{4nw}{\alpha'} + N - \widetilde{N} \\
&= nw + N - \widetilde{N}
\end{aligned}
\tag{1.11}
$$

Which is the desired level matching condition. The partition function now writes

$$
\begin{aligned}
Z^{1,24}Z^{25} &= \sum_{N_n=0}^{+\infty}\sum_{\widetilde{N}_n=0}^{+\infty}\sum_{n,w=-\infty}^{+\infty} |q|^{\frac{\alpha' {p_L}^2}{4}-1+N+\frac{\alpha' {p_R}^2}{4}-1+\widetilde{N}} \\
&= \sum_{N_n=0}^{+\infty}\sum_{\widetilde{N}_n=0}^{+\infty}\sum_{n,w=-\infty}^{+\infty} |q|^{\frac{\alpha'}{2}\left[\frac{n^2}{R^2}+\frac{\omega^2R^2}{\alpha'^2}+\frac{2}{\alpha'}(N+\widetilde{N}-2)\right]}
\end{aligned}
\tag{1.12}
$$

So each term means the right hand sides of the mass formula $M^2 = \frac{n^2}{R^2} + \frac{\omega^2 R^2}{\alpha'^2} + \frac{2}{\alpha'}(N + \widetilde{N} - 2)$.

1.2

Write the more explicit form of partition funciton Z^{25}

$$\begin{aligned}
Z^{25} =& |\eta(q)|^{-2}\sum_{n,w=-\infty}^{+\infty} q^{\frac{\alpha' {p_L}^2}{4}} \bar{q}^{\frac{\alpha' {p_R}^2}{4}} \\
=& |\eta(q)|^{-2}\sum_{n,w=-\infty}^{+\infty} \exp 2\pi i(\tau_1 + i\tau_2) \times \frac{\alpha'}{4}\big(\frac{n}{R} + \frac{wR}{\alpha'}\big)^2 \times \qquad (1.13)\\
&\times \exp -2\pi i(\tau_1 - i\tau_2) \times \frac{\alpha'}{4}\big(\frac{n}{R} - \frac{wR}{\alpha'}\big)^2 \qquad (1.14)
\end{aligned}$$

Calculate the factor in exponential

$$\begin{aligned}
&2\pi i(\tau_1 + i\tau_2) \times \frac{\alpha'}{4}\big(\frac{n}{R} + \frac{wR}{\alpha'}\big)^2 - 2\pi i(\tau_1 - i\tau_2) \times \frac{\alpha'}{4}\big(\frac{n}{R} - \frac{wR}{\alpha'}\big)^2 \\
=& 2\pi i\tau_1 \times \frac{\alpha'}{4} \times \frac{4nw}{\alpha'} - 2\pi\tau_2 \times \frac{\alpha'}{4} \times 2\big(\frac{n^2}{R^2} + \frac{w^2R^2}{\alpha'^2}\big) \\
=& 2\pi i\tau_1 nw - \pi\tau_2\big(\frac{\alpha' n^2}{R^2} + \frac{w^2R^2}{\alpha'}\big) \qquad (1.15)
\end{aligned}$$

The partition function now is

$$Z^{25} = |\eta(q)|^{-2}\sum_{n,w=-\infty}^{+\infty} \exp 2\pi i\tau_1 nw - \pi\tau_2\big(\frac{\alpha' n^2}{R^2} + \frac{w^2R^2}{\alpha'}\big) \qquad (1.16)$$

Using the Poisson resummation formula.

$$\sum_{n=-\infty}^{\infty} \exp(-\pi a n^2 + 2\pi i b n) = a^{-1/2}\sum_{m=-\infty}^{\infty} \exp[-\frac{\pi(m-b)^2}{a}] \qquad (1.17)$$

By replacement $a \to \frac{\tau_2\alpha'}{R^2}$ and $b \to \tau_1 w$ the partition formula is

$$\begin{aligned}
Z^{25} &= |\eta(q)|^{-2}\big(\frac{\tau_2\alpha'}{R^2}\big)^{-1/2}\sum_{w=-\infty}^{+\infty}\exp(-\pi\tau_2\frac{w^2R^2}{\alpha'})\sum_{m=-\infty}^{\infty}\exp[-\frac{\pi R^2(m-\tau_1 w)^2}{\tau_2\alpha'}] \\
&= |\eta(q)|^{-2}\big(\frac{\tau_2\alpha'}{R^2}\big)^{-1/2}\sum_{w,m=-\infty}^{+\infty}\exp[-\frac{\pi R^2}{\tau_2\alpha'}(\tau_2^2 w^2 + m^2 - 2m\tau_1 w + \tau_1^2 w^2)] \\
&= |\eta(q)|^{-2}\big(\frac{\tau_2\alpha'}{R^2}\big)^{-1/2}\sum_{w,m=-\infty}^{+\infty}\exp[-\frac{\pi R^2}{\tau_2\alpha'}|m - w\tau|^2] \qquad (1.18)
\end{aligned}$$

Now illustrate the modular invariance of the partition function. Under the transformation $\tau \to \tau + 1$, $\tau_2 \to \tau_2$, $\eta(\tau) \to e^{i\pi/12}\eta(\tau)$. Thus the partition funciton changes

$$
\begin{aligned}
Z^{25} =&|\eta(q)|^{-2}\big(\frac{\tau_2\alpha'}{R^2}\big)^{-1/2}\sum_{w,m=-\infty}^{+\infty}\exp[-\frac{\pi R^2}{\tau_2\alpha'}|m-w\tau|^2]\\
\to&|\eta(q)|^{-2}\big(\frac{\tau_2\alpha'}{R^2}\big)^{-1/2}\sum_{w,m=-\infty}^{+\infty}\exp[-\frac{\pi R^2}{\tau_2\alpha'}|m-w(\tau+1)|^2]
\end{aligned}
\tag{1.19}
$$

is invariant if we replace the sum index m by $m+w$.

Under the transformation $\tau\to-1/\tau$, we have $\tau_2\to\tau_2/|\tau|^2$, $\eta(\tau)\to\sqrt{-i\tau}\eta(\tau)$.

$$
\begin{aligned}
Z^{25} =&|\eta(q)|^{-2}\big(\frac{\tau_2\alpha'}{R^2}\big)^{-1/2}\sum_{w,m=-\infty}^{+\infty}\exp[-\frac{\pi R^2}{\tau_2\alpha'}|m-w\tau|^2]\\
\to&|\eta(q)|^{-2}\sqrt{\tau}\big(\frac{\tau_2\alpha'}{R^2}\big)^{-1/2}\sqrt{\tau}^{-1}\sum_{w,m=-\infty}^{+\infty}\exp[-\frac{\pi R^2}{\tau_2\alpha'}|\tau|^2|m-w(-1/\tau)|^2]\\
=&|\eta(q)|^{-2}\big(\frac{\tau_2\alpha'}{R^2}\big)^{-1/2}\sum_{w,m=-\infty}^{+\infty}\exp[-\frac{\pi R^2}{\tau_2\alpha'}|m\tau+w|^2]
\end{aligned}
\tag{1.20}
$$

is invariant if we replace the sum index $m\to-w$ and $w\to m$.

1.3

In the case of self-dual $R=\sqrt{\alpha'}$. The partition function Z^{25} now is

$$
\begin{aligned}
Z^{25} =&|\eta(q)|^{-2}\sum_{n,w=-\infty}^{+\infty}q^{\frac{\alpha' {p_L}^2}{4}}\overline{q}^{\frac{\alpha' {p_R}^2}{4}}\\
=&|\eta(q)|^{-2}\sum_{n,w=-\infty}^{+\infty}q^{\frac{(n+w)^2}{4}}\overline{q}^{\frac{(n-w)^2}{4}}
\end{aligned}
\tag{1.21}
$$

Do the replacement $n\to n-w$

$$
\begin{aligned}
Z^{25} =&|\eta(q)|^{-2}\sum_{n,w=-\infty}^{+\infty}q^{\frac{n^2}{4}}\overline{q}^{\frac{(n-2w)^2}{4}}\\
=&|\eta(q)|^{-2}\sum_{w=-\infty}^{+\infty}\left(\sum_{2k+1,k\in\mathbb{Z}}q^{\frac{(2k+1)^2}{4}}\overline{q}^{\frac{(2k+1-2w)^2}{4}}+\sum_{2k,k\in\mathbb{Z}}q^{\frac{(2k)^2}{4}}\overline{q}^{\frac{(2k-2w)^2}{4}}\right)\\
=&|\eta(q)|^{-2}\sum_{w,k=-\infty}^{+\infty}q^{(k+1/2)^2}\overline{q}^{(k-w+1/2)^2}+q^{k^2}\overline{q}^{(k-w)^2}
\end{aligned}
\tag{1.22}
$$

With the replacement $w \to -w + k$, we have

$$\begin{aligned} Z^{25} =& |\eta(q)|^{-2} \sum_{w,k=-\infty}^{+\infty} q^{(k+1/2)^2} \overline{q}^{(w+1/2)^2} + q^{k^2} \overline{q}^{w^2} \\ =& \left| \eta^{-1} \sum_{n\in\mathbb{Z}} q^{n^2} \right|^2 + \left| \eta^{-1} \sum_{n\in\mathbb{Z}} q^{n^2+1/2} \right|^2 \\ =& |\chi_1(q)|^2 + |\chi_2(q)|^2 \end{aligned} \tag{1.23}$$

The χ_i are the characters of the SU(2) affine Lie algebra with level k = 1. Now the whole partition function is

$$Z^{1,24} Z^{25} = \text{const} \times |\eta(q)|^{-48} \sum_{w,k=-\infty}^{+\infty} q^{(k+1/2)^2} \overline{q}^{(w+1/2)^2} + q^{k^2} \overline{q}^{w^2} \tag{1.24}$$

According to lecture note 8, first factor $|\eta(q)|^{-48} \simeq |q^{-1} + 24 + O(q)|^2$. Its easy to see that the terms with $k = w = 0$ in the second factor contribute to usual 24 bosonic massless states. However $|q|^2$, q, $\overline{q}$ terms in the the second factor gives contributions to additional massless states, which corresponds to eight cases $k, w = \pm 1$, $k = 0, w = \pm 1$ and $k = \pm 1, w = 0$.

We now briefly conclude these cases. We denote w, n in original partition function as w_o, n_o

$$\begin{array}{llll} k = w = \pm 1 & n_o = \pm 2 & w_o = 0 & N = \widetilde{N} = 0 \\ k = -w = \pm 1 & n_o = 0 & w_o = \pm 2 & N = \widetilde{N} = 0 \\ k = \pm 1 \quad w = 0 & & n_o = w_o = \pm 1 & N = 0 \quad \widetilde{N} = 1 \\ k = 0 \quad w = \pm 1 & & n_o = -w_o = \pm 1 & N = 1 \quad \widetilde{N} = 0 \end{array}$$

The last four states are physical additional massless states with nonzero state number.

1.4

The current in the bosonic string theory is defined by

$$\begin{aligned} j^{\pm}(z) &= j^1(z) \pm i j^2(z) :- e^{\pm 2iX^{25}(z)/\sqrt{\alpha'}} \\ j^3(z) &:- i\partial X^{25}(z)/\sqrt{\alpha'} \end{aligned} \tag{1.25}$$

so we can write $j^1(z)$ and $j^2(z)$

$$j^1(z) = \frac{1}{2}(j^+(z) + j^-(z)) = \frac{1}{2}(e^{2iX^{25}(z)/\sqrt{\alpha'}} + e^{-2iX^{25}(z)/\sqrt{\alpha'}}) \tag{1.26}$$

$$j^2(z) = \frac{1}{2i}(j^+(z) + j^-(z)) = \frac{1}{2i}(e^{2iX^{25}(z)/\sqrt{\alpha'}} - e^{-2iX^{25}(z)/\sqrt{\alpha'}}) \tag{1.27}$$

Useful formula which have proved in previous notes or exercises:

$$X^{25}(z)X^{25}(0) \sim -\frac{\alpha'}{2}\ln z \tag{1.28}$$

$$:e^{ik_1X^{25}(z)} :: e^{ik_2X^{25}(0)} := z^{\frac{\alpha'}{2}k_1k_2} :e^{ik_1X^{25}(z)}e^{ik_2X^{25}(0)}: \tag{1.29}$$

$$:\partial X^{25}(z) :: e^{ikX^{25}(0)} :\sim \partial(-\frac{\alpha'}{2}\ln z)ik :e^{ikX^{25}(0)}: \tag{1.30}$$

Calculate OPE j^3j^3

$$j^3(z)j^3(0) = -\frac{1}{\alpha'}\partial X^{25}(z)\partial X^{25}(0) \sim \frac{1}{\alpha'}\partial^2(-\frac{\alpha'}{2}\ln z) = \frac{1}{2z^2} \tag{1.31}$$

OPE j^1j^1

$$\begin{aligned}
j^1(z)j^1(0) =&\frac{1}{4} :(e^{2iX^{25}(z)/\sqrt{\alpha'}} + e^{-2iX^{25}(z)/\sqrt{\alpha'}}) :: (e^{2iX^{25}(0)/\sqrt{\alpha'}} + e^{-2iX^{25}(0)/\sqrt{\alpha'}}):\\
=&\frac{1}{4}\Bigg(z^2 :e^{2iX^{25}(z)/\sqrt{\alpha'}}e^{2iX^{25}(0)/\sqrt{\alpha'}}: + z^{-2} :e^{2iX^{25}(z)/\sqrt{\alpha'}}e^{-2iX^{25}(0)/\sqrt{\alpha'}}: +\\
&+ z^{-2} :e^{-2iX^{25}(z)/\sqrt{\alpha'}}e^{2iX^{25}(0)/\sqrt{\alpha'}}: + z^2 :e^{-2iX^{25}(z)/\sqrt{\alpha'}}e^{-2iX^{25}(0)/\sqrt{\alpha'}}:\Bigg)\\
=&\frac{1}{4}\left[z^{-2}(1+\frac{2i}{\sqrt{\alpha'}}\partial X^{25}(0)z + O(z^2)) + z^{-2}(1-\frac{2i}{\sqrt{\alpha'}}\partial X^{25}(0)z + O(z^2)) + O(z^2)\right]\\
=&\frac{1}{2z^2} + O(z^2)\\
\sim&\frac{1}{2z^2}
\end{aligned} \tag{1.32}$$

Similar for OPE j^2j^2

$$\begin{aligned}
j^2(z)j^2(0) =&-\frac{1}{4} :(e^{2iX^{25}(z)/\sqrt{\alpha'}} - e^{-2iX^{25}(z)/\sqrt{\alpha'}}) :: (e^{2iX^{25}(0)/\sqrt{\alpha'}} - e^{-2iX^{25}(0)/\sqrt{\alpha'}}):\\
=&-\frac{1}{4}\Bigg(z^2 :e^{2iX^{25}(z)/\sqrt{\alpha'}}e^{2iX^{25}(0)/\sqrt{\alpha'}}: - z^{-2} :e^{2iX^{25}(z)/\sqrt{\alpha'}}e^{-2iX^{25}(0)/\sqrt{\alpha'}}: +\\
&- z^{-2} :e^{-2iX^{25}(z)/\sqrt{\alpha'}}e^{2iX^{25}(0)/\sqrt{\alpha'}}: + z^2 :e^{-2iX^{25}(z)/\sqrt{\alpha'}}e^{-2iX^{25}(0)/\sqrt{\alpha'}}:\Bigg)\\
=&\frac{1}{4}\left[z^{-2}(1+\frac{2i}{\sqrt{\alpha'}}\partial X^{25}(0)z + O(z^2)) + z^{-2}(1-\frac{2i}{\sqrt{\alpha'}}\partial X^{25}(0)z + O(z^2)) + O(z^2)\right]\\
=&\frac{1}{2z^2} + O(z^2)\\
\sim&\frac{1}{2z^2}
\end{aligned} \tag{1.33}$$

Calculate OPE $j^3 j^1$

$$
\begin{aligned}
j^3(z)j^1(0) =& \frac{i}{2\sqrt{\alpha'}}\partial X^{25}(z) :(e^{2iX^{25}(0)/\sqrt{\alpha'}} + e^{-2iX^{25}(0)/\sqrt{\alpha'}}): \\
\sim& \frac{i}{2\sqrt{\alpha'}}\Big(- \frac{\alpha'}{2} \times \frac{2i}{\sqrt{\alpha'}} z^{-1} : e^{2iX^{25}(0)/\sqrt{\alpha'}} : - \frac{\alpha'}{2} \times \frac{-2i}{\sqrt{\alpha'}} z^{-1} : e^{-2iX^{25}(0)/\sqrt{\alpha'}} : \Big) \\
=& \frac{i}{z}\frac{1}{2i}\Big(: e^{2iX^{25}(0)/\sqrt{\alpha'}} : - : e^{-2iX^{25}(0)/\sqrt{\alpha'}} : \Big) \\
=& \frac{i}{z} j^2(0)
\end{aligned} \tag{1.34}
$$

When calculate OPE $j^1 j^3$, the ∂ in j^3 act on "0" offer an extra minus sign, so we have

$$
j^1(z)j^3(0) \sim -\frac{i}{z} j^2(0) \tag{1.35}
$$

Similar calculation for OPE $j^3 j^2$

$$
\begin{aligned}
j^3(z)j^2(0) =& \frac{1}{2\sqrt{\alpha'}}\partial X^{25}(z) :(e^{2iX^{25}(0)/\sqrt{\alpha'}} - e^{-2iX^{25}(0)/\sqrt{\alpha'}}): \\
\sim& \frac{1}{2\sqrt{\alpha'}}\Big(- \frac{\alpha'}{2} \times \frac{2i}{\sqrt{\alpha'}} z^{-1} : e^{2iX^{25}(0)/\sqrt{\alpha'}} : + \frac{\alpha'}{2} \times \frac{-2i}{\sqrt{\alpha'}} z^{-1} : e^{-2iX^{25}(0)/\sqrt{\alpha'}} : \Big) \\
=& -\frac{i}{z}\frac{1}{2}\Big(: e^{2iX^{25}(0)/\sqrt{\alpha'}} : + : e^{-2iX^{25}(0)/\sqrt{\alpha'}} : \Big) \\
=& -\frac{i}{z} j^1(0)
\end{aligned} \tag{1.36}
$$

$$
j^2(z)j^3(0) = \frac{i}{z} j^1(0) \tag{1.37}
$$

Finally, we goes to OPE $j^1 j^2$

$$
\begin{aligned}
j^1(z)j^2(0) =& \frac{1}{4i} :(e^{2iX^{25}(z)/\sqrt{\alpha'}} + e^{-2iX^{25}(z)/\sqrt{\alpha'}})::(e^{2iX^{25}(0)/\sqrt{\alpha'}} - e^{-2iX^{25}(0)/\sqrt{\alpha'}}): \\
=& \frac{1}{4i}\Big(z^2 : e^{2iX^{25}(z)/\sqrt{\alpha'}} e^{2iX^{25}(0)/\sqrt{\alpha'}} : - z^{-2} : e^{2iX^{25}(z)/\sqrt{\alpha'}} e^{-2iX^{25}(0)/\sqrt{\alpha'}} : + \\
&+ z^{-2} : e^{-2iX^{25}(z)/\sqrt{\alpha'}} e^{2iX^{25}(0)/\sqrt{\alpha'}} : - z^2 : e^{-2iX^{25}(z)/\sqrt{\alpha'}} e^{-2iX^{25}(0)/\sqrt{\alpha'}} : \Big) \\
=& \frac{1}{4i}\Big[- z^{-2}(1 + \frac{2i}{\sqrt{\alpha'}}\partial X^{25}(0) z + O(z^2)) + z^{-2}(1 - \frac{2i}{\sqrt{\alpha'}}\partial X^{25}(0) z + O(z^2)) + O(z^2) \Big] \\
\sim& \frac{i}{z}\frac{i}{\sqrt{\alpha'}}\partial X^{25}(0) = \frac{i}{z} j^3(0)
\end{aligned} \tag{1.38}
$$

Follow the similar pattern we have

$$
j^2(z)j^1(0) \sim -\frac{i}{z}\frac{i}{\sqrt{\alpha'}}\partial X^{25}(0) = -\frac{i}{z} j^3(0) \tag{1.39}
$$

Thus the OPE can be concluded

$$j^a(z)j^b(0) \sim \frac{\delta^{ab}}{2z^2} + \frac{i\epsilon^{abc}j^c(0)}{z} \tag{1.40}$$

The oscillator modes of the currents

$$j^a(z) = \sum_{m\in\mathbb{Z}} \frac{j^a_m}{z^{m+1}} \tag{1.41}$$

So we have

$$j^a_m = \oint_c \frac{dz}{2\pi i} z^m j^a(z) \tag{1.42}$$

The commutator $\left[j^a_m, j^b_n\right]$ is

$$\begin{aligned}
\left[j^a_m, j^b_n\right] &= \left(\oint_{c_1}\oint_{c_2} - \oint_{c_2}\oint_{c_3}\right)\frac{dz}{2\pi i}\frac{dw}{2\pi i} z^m w^n j^a(z) j^b(z) \\
&= \oint_{c_2}\frac{dw}{2\pi i}\oint_{c_W}\frac{dz}{2\pi i} z^m w^n \left[\frac{\delta^{ab}}{2(z-w)^2} + \frac{i\epsilon^{abc}j^c(w)}{z-w}\right] \\
&= \oint_{c_2}\frac{dw}{2\pi i}\left[\frac{1}{2}m\omega^{n+m-1}\delta^{ab} + i\epsilon^{abc}j^c(\omega)\omega^{m+n}\right] \\
&= \delta_{n+m,0}\frac{1}{2}m\delta^{ab} + i\epsilon^{abc}j^c_{m+n}
\end{aligned} \tag{1.43}$$

2 RR field strengths and T-duality in Type II

The Clifford algebra of SO(1,9) gamma matrices defined as

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu} \qquad \mu, \nu = 0, 1 \ldots 9 \tag{2.1}$$

The chirality matrix is $\Gamma^{11} = \Gamma^0\Gamma^1\cdots\Gamma^9$, calculate $\Gamma^{11}\Gamma^{11}$:

$$\Gamma^{11}\Gamma^{11} = \Gamma^0\Gamma^1\cdots\Gamma^9\Gamma^0\Gamma^1\cdots\Gamma^9 = (-1)^{9+8+\cdots+1}(\Gamma^0)^2(\Gamma^1)^2\cdots(\Gamma^9)^2 = (-1)^{46} = 1 \tag{2.2}$$

The anticommutator of Γ^{11} and Γ^μ vanishes

$$\{\Gamma^{11}, \Gamma^\mu\} = \Gamma^{11}\Gamma^\mu + \Gamma^\mu\Gamma^{11} = \Gamma^{11}\Gamma^\mu + (-1)^9\Gamma^{11}\Gamma^\mu = 0 \tag{2.3}$$

Chiral spinors are now defined by $\Gamma^{11}\psi_\pm = \pm\psi_\pm$, then we have

$$\overline{\psi}_\pm\Gamma^{11} = \psi^\dagger_\pm\Gamma^0\Gamma^{11} = (\Gamma^{11\dagger}\Gamma^{0\dagger}\psi_\pm)^\dagger \tag{2.4}$$

Notice that

$$\Gamma^{11\dagger} = \Gamma^{9\dagger}\cdots\Gamma^{0\dagger} = \Gamma^0\Gamma^9\Gamma^8\cdots\Gamma^0(\Gamma^0)^{-1} = (-1)^{45}\Gamma^0\Gamma^0\Gamma^1\cdots\Gamma^9(\Gamma^0)^{-1} = -\Gamma^0\Gamma^{11}(\Gamma^0)^{-1} \tag{2.5}$$

So, we have

$$\overline{\psi}_{\pm}\Gamma^{11} = (\Gamma^0\Gamma^{11}\Gamma^0(\Gamma^0)^{-1}\psi_{\pm})^{\dagger} = \pm(\Gamma^0\psi_{\pm})^{\dagger} = \pm\psi_{\pm}^{\dagger}(-\Gamma^0) = \mp\overline{\psi}_{\pm} \tag{2.6}$$

The R R field strengths $G_{\mu_1\cdots\mu_{p+1}}$ as bilinears

$$\text{II A}: \quad \overline{\psi}_{-}^{L}\Gamma^{\mu_1\cdots\mu_{p+1}}\psi_{+}^{R} \qquad \text{II B}: \quad \overline{\psi}_{+}^{L}\Gamma^{\mu_1\cdots\mu_{p+1}}\psi_{+}^{R} \tag{2.7}$$

Since the spacetime indexes are antisymmetric $p+1 \leq 10$. Consider Type II A

$$\overline{\psi}_{-}^{L}\Gamma^{\mu_1\cdots\mu_{p+1}}\psi_{+}^{R} = \overline{\psi}_{-}^{L}\Gamma^{11}\Gamma^{\mu_1\cdots\mu_{p+1}}\Gamma^{11}\psi_{+}^{R} = \overline{\psi}_{-}^{L}(\Gamma^{11})^2(-1)^{p+1}\Gamma^{\mu_1\cdots\mu_{p+1}}\psi_{+}^{R} = (-1)^{p+1}\overline{\psi}_{-}^{L}\Gamma^{\mu_1\cdots\mu_{p+1}}\psi_{+}^{R} \tag{2.8}$$

So, for type II A, the spinor bilinears will vanish unless p is odd. p can be $1, 3, 5, 7, 9$. Similar for Type II B

$$\overline{\psi}_{+}^{L}\Gamma^{\mu_1\cdots\mu_{p+1}}\psi_{+}^{R} = -\overline{\psi}_{+}^{L}\Gamma^{11}\Gamma^{\mu_1\cdots\mu_{p+1}}\Gamma^{11}\psi_{+}^{R} = (-1)^{p+2}\overline{\psi}_{+}^{L}\Gamma^{\mu_1\cdots\mu_{p+1}}\psi_{+}^{R} \tag{2.9}$$

For type II B, the spinor bilinears will vanish unless p is even. So p can be $0, 2, 4, 6, 8$

Since $\beta_9 = \Gamma^9\Gamma^{11}$, we have

$$\{\beta_9, \Gamma^9\} = \{\Gamma^9\Gamma^{11}, \Gamma^9\} = \Gamma^9\{\Gamma^{11}, \Gamma^9\} - [\Gamma^9, \Gamma^9]\Gamma^{11} = 0 \tag{2.10}$$

For $\mu \neq 9$

$$[\beta_9, \Gamma^{\mu}] = [\Gamma^9\Gamma^{11}, \Gamma^{\mu}] = \Gamma^9\{\Gamma^{11}, \Gamma^{\mu}\} - \{\Gamma^9, \Gamma^{\mu}\}\Gamma^{11} = 0 \tag{2.11}$$

Under T-duality transformation $\psi^L \to \psi^L$, $\psi^R \to \beta_9\psi^R$

$$\overline{\psi}_{-}^{L}\Gamma^{\mu_1\cdots\mu_{p+1}}\psi_{+}^{R} \to \overline{\psi}_{-}^{L}\Gamma^{\mu_1\cdots\mu_{p+1}}\beta_9\psi_{+}^{R} \tag{2.12}$$

Follow the same step, we see that

$$\overline{\psi}_{-}^{L}\Gamma^{\mu_1\cdots\mu_{p+1}}\beta_9\psi_{+}^{R} = \overline{\psi}_{-}^{L}\Gamma^{11}\Gamma^{\mu_1\cdots\mu_{p+1}}\beta_9\Gamma^{11}\psi_{+}^{R} = (-1)^{p+2}\overline{\psi}_{-}^{L}\Gamma^{\mu_1\cdots\mu_{p+1}}\beta_9\psi_{+}^{R} \tag{2.13}$$

The transformed spinor bilinear vanishes unless p is even which is the case of type II B. And we find $\overline{\psi}_{-}^{L}\Gamma^{\mu_1\cdots\mu_{p+1}}\beta_9\psi_{+}^{R} \sim \overline{\psi}_{-}^{L}\beta_9\Gamma^{\mu_1\cdots\mu_{p+1}}\psi_{+}^{R}$ up to a sign factor. And

$$\overline{\psi}_{-}^{L}\beta_9\Gamma^{11} = -\overline{\psi}_{-}^{L}\Gamma^{11}\beta_9 = +\overline{\psi}_{-}^{L}\beta_9 \tag{2.14}$$

So $\overline{\psi}_{-}^{L}\beta_9 \sim \overline{\psi}_{+}^{L}$. Thus T duality transform RR field strenghths in II A to II B, and the same reason vise versa.

$$\overline{\psi}_{+}^{L}\Gamma^{\mu_1\cdots\mu_{p+1}}\psi_{+}^{R} \to \overline{\psi}_{+}^{L}\Gamma^{\mu_1\cdots\mu_{p+1}}\beta_9\psi_{+}^{R} \tag{2.15}$$

$$\overline{\psi}_{+}^{L}\Gamma^{\mu_1\cdots\mu_{p+1}}\beta_9\psi_{+}^{R} = \overline{\psi}_{+}^{L}\Gamma^{11}\Gamma^{\mu_1\cdots\mu_{p+1}}\beta_9\Gamma^{11}\psi_{+}^{R} = (-1)^{p+2}\overline{\psi}_{+}^{L}\Gamma^{\mu_1\cdots\mu_{p+1}}\beta_9\psi_{+}^{R} \tag{2.16}$$

The transformed spinor bilinear vanishes unless p is odd which is the case of type II A. And we find $\overline{\psi}_{+}^{L}\Gamma^{\mu_1\cdots\mu_{p+1}}\beta_9\psi_{+}^{R} \sim \overline{\psi}_{+}^{L}\beta_9\Gamma^{\mu_1\cdots\mu_{p+1}}\psi_{+}^{R}$ up to a sign factor. And

$$\overline{\psi}_{+}^{L}\beta_9\Gamma^{11} = -\overline{\psi}_{+}^{L}\Gamma^{11}\beta_9 = -\overline{\psi}_{+}^{L}\beta_9 \tag{2.17}$$

So $\overline{\psi}_{+}^{L}\beta_9 \sim \overline{\psi}_{-}^{L}$.

3 D-branes in Type II and T- duality

3.1

Since T-duality exchange NN to DD and ND to DN in corresponding spacetime direction, the number (# NN + # DD) and (# ND + # DN) are invariant.

3.2

Without loss of generality, we place all the spacetime dimension as small as possible. For II B theory p must be odd .Above tables list all the possible configuration, and all these configurations are dual to the D1-D5 configuration.

	0	1	2	3	4	5	6	7	8	9
D3	$\times$	$\times$	$\times$	$\times$						
D1	$\times$				$\times$					

Table 1: transform to D1-D5 by T_4, T_5

	0	1	2	3	4	5	6	7	8	9
D3	$\times$	$\times$	$\times$	$\times$						
D3	$\times$	$\times$			$\times$	$\times$				

Table 2: transform to D1-D5 by T_2, T_3

	0	1	2	3	4	5	6	7	8	9
D3	$\times$	$\times$	$\times$	$\times$						
D5	$\times$	$\times$	$\times$		$\times$	$\times$	$\times$			

Table 3: transform to D1-D5 by T_1, T_3

	0	1	2	3	4	5	6	7	8	9
D3	$\times$	$\times$	$\times$	$\times$						
D5	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$		

Table 4: transform to D1-D5 by T_1, T_2