

# Homework 6: Due at class on May 25

## 1 Higgs mechanism in electroweak theory

The Glashow-Weinberg-Salam theory is the part of the Standard Model (SM) of particle physics which describes the electroweak interactions by a non-Abelian gauge theory with the gauge group  $SU(2)_L \times U(1)_Y$ . In a one-family approximation, the SM has the following particle content:

	$E_L = \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L$	$e_R$	$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$	$T^a W_\mu^a$	$B_\mu$
Hypercharge $Y$	$-\frac{1}{2}$	$-1$	$+\frac{1}{2}$	$0$	$0$
$SU(2)_L$ rep.	<b>2</b>	<b>1</b>	<b>2</b>	<b>3</b>	<b>1</b>
Lorentz rep.	$(\frac{1}{2}, 0)$	$(0, \frac{1}{2})$	$(0, 0)$	$(\frac{1}{2}, \frac{1}{2})$	$(\frac{1}{2}, \frac{1}{2})$

where  $E_L, e_R$  contain Dirac spinors and the superscripts in the Higgs doublet denote electromagnetic charges. The corresponding Lagrangian is given by

$$\begin{aligned} \mathcal{L} = & \bar{E}_L (i \not{D}) E_L + \bar{e}_R (i \not{D}) e_R - \frac{1}{4} W_{\mu\nu}^a W^{\mu\nu a} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} \\ & + (D_\mu \phi)^\dagger (D^\mu \phi) - \mu^2 \phi^\dagger \phi - \lambda (\phi^\dagger \phi)^2 - (\bar{E}_L \phi e_R + \bar{e}_R \phi^\dagger E_L) \end{aligned} \quad (1)$$

with

$$\begin{aligned} D_\mu &= \partial_\mu - ig W_\mu^a T^a - ig' Y B_\mu \\ B_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu, \quad W_{\mu\nu}^a = \partial_\mu W_\nu^a - \partial_\nu W_\mu^a + g \epsilon^{abc} W_\mu^b W_\nu^c \end{aligned} \quad (2)$$

### 1.1

Write down how the covariant derivative (2) acts on the left- and right-handed leptons doublets and on the Higgs-doublet.

### 1.2

Show that the Lagrangian (1) is Lorentz invariant and gauge invariant as well.

### 1.3

For the Higgs mechanism to work we need  $\mu^2 < 0$ . For which value of  $|\phi|$  does the Higgs potential obtain a minimum? By an  $SU(2)_L$  rotation we can choose the vacuum expectation value (VEV) of the Higgs field to be of the form  $\langle \phi \rangle = \frac{1}{\sqrt{2}}(0, v)^T$ . This leads to a redefinition of the excitation modes of the Higgs fields,

$$\phi(x) = \exp \left\{ \frac{i}{v} \xi^a(x) T^a \right\} \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}}(v + \eta(x)) \end{pmatrix}$$

with  $\zeta^a(x)$  and  $\eta(x)$  being real fields and  $T^a$  the generators of  $SU(2)_L$ . Now we apply an  $SU(2)_L$  gauge transformation such that the angular excitations  $\zeta^a(x)$  vanish. This gauge transformation is called *unitary gauge*. Show that the Higgs potential in the unitary gauge is given by

$$V(\phi) = -\mu^2\eta^2(x) + \lambda v\eta^3(x) + \frac{\lambda}{4}\eta^4(x)$$

What is the mass of the  $\eta$  field? Compare the degrees of freedom (DOF) in the Higgs sector to the situation before symmetry breakdown.

## 1.4

Consider the kinetic energy terms of the Higgs field in (1). Show that

$$(D_\mu\phi)^\dagger(D^\mu\phi) = \frac{1}{2}\partial_\mu\eta\partial^\mu\eta + \frac{1}{4}g^2(v+\eta)^2W_\mu^-W^{+\mu} + \frac{1}{8}(v+\eta)^2(W_\mu^3 \quad B_\mu) \begin{pmatrix} g^2 & -g'g \\ -g'g & g'^2 \end{pmatrix} \begin{pmatrix} W^{3\mu} \\ B^\mu \end{pmatrix} \quad (3)$$

with  $W^{\pm\mu} := \frac{1}{\sqrt{2}}(W^{1\mu} \mp iW^{2\mu})$

## 1.5

The masses of the gauge bosons are given by the terms that are quadratic in the fields, e.g.  $\frac{1}{4}g^2v^2W_\mu^-W^{+\mu} = m_W^2W_\mu^-W^{+\mu}$ , where  $m_W = \frac{1}{2}vg$ . However, to see the masses of  $W_\mu^3$  and  $B_\mu$  one has to diagonalize the matrix in (3):

$$\frac{1}{8}(W_\mu^3 \quad B_\mu)\mathcal{O}^T\mathcal{O} \begin{pmatrix} g^2 & -g'g \\ -g'g & g'^2 \end{pmatrix} \mathcal{O}^T\mathcal{O} \begin{pmatrix} W^{3\mu} \\ B^\mu \end{pmatrix} = \frac{1}{2} \begin{pmatrix} Z_\mu & A_\mu \end{pmatrix} \begin{pmatrix} m_Z^2 & 0 \\ 0 & m_A^2 \end{pmatrix} \begin{pmatrix} Z^\mu \\ A^\mu \end{pmatrix}$$

Determine this orthogonal matrix  $\mathcal{O}$  by computing the corresponding eigenvalues and eigenvectors. What are the masses of the  $Z_\mu$  and  $A_\mu$  fields? Compare the DOF in the gauge sector to the situation before the symmetry breakdown. What can you say about the total amount of DOF?

## 1.6

As you know, an orthogonal  $2 \times 2$  matrix can be written as

$$\mathcal{O} = \begin{pmatrix} \cos\theta_W & -\sin\theta_W \\ \sin\theta_W & \cos\theta_W \end{pmatrix}$$

Write  $\cos\theta_W$  in terms of  $g'$  and  $g$ . Show for the ratio of the  $W$  - and  $Z$  -boson masses

$$\frac{m_W}{m_Z} = \cos\theta_W$$

The angle  $\theta_W$  is sometimes called Weinberg angle or weak mixing angle.

## 1.7

Finally, consider the covariant derivative (2). Substitute the fields  $B_\mu$  and  $W_\mu^a$  by  $W_\mu^\pm, Z_\mu$  and  $A_\mu$  and show

$$D_\mu = \partial_\mu - i\frac{g}{\sqrt{2}} \begin{pmatrix} 0 & W_\mu^+ \\ W_\mu^- & 0 \end{pmatrix} - i\frac{1}{\sqrt{g^2 + g'^2}} Z_\mu (g^2 T^3 - g'^2 Y) - ieQA_\mu$$

where we have defined the electric charge  $e = \frac{g'g}{\sqrt{g'^2 + g^2}}$  and  $Q := T^3 + Y$

## 1.8

Consider the following terms of the Lagrangian:

$$\mathcal{L} \supset \bar{E}_L(i\mathcal{D})E_L + \bar{e}_R(i\mathcal{D})e_R$$

Find the interaction terms of the fermions with the gauge bosons. For the weak interaction, analyze its V-A structure  $\frac{1}{2}(c_V + c_A\gamma^5)$ . Draw the corresponding Feynman diagrams (Note: use  $i\mathcal{L}$ , drop all fields and you get the vertex factor).

## 1.9

Using the unitary gauge, insert the shifted Higgs field

$$\phi(x) = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}}(v + \eta(x)) \end{pmatrix}$$

into the Yukawa couplings of the Lagrangian:

$$\mathcal{L} \supset -G_e (\bar{E}_L \phi e_R + \bar{e}_R \phi^\dagger E_L)$$

Show that the mass of the electron is  $m_e = \frac{G_e v}{\sqrt{2}}$ .

# 2 Anomaly cancellation in Standard Model

A condition for a QFT to be well-defined is the cancellation of anomalies associated with local symmetries. Show that the SM is anomaly free and

$$\mathcal{A}^{abc} = \text{tr} [t^a \{t^b, t^c\}] = 0$$

## 2.1

List the different triangle diagrams that exist in terms of group structure. Using the fact that  $\text{SU}(3)_C$  is a vector-like interaction and properties of  $\text{SU}(2)_L$ , argue that the only nontrivial contributions are those with an odd number of hypercharge insertions. Show that some relations are equivalent to requiring cancellation of electric charge inside each generation.

## 2.2

Are baryon number ( $B$ ) and lepton number ( $L$ ) currents anomaly free?