BPS states, Torus knots and Twisted wild character varieties

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Prologue

There are two stories...

The one story

For C genus g curve and fix group GL_n,

$$\mathcal{M}_{\mathrm{B}}^{n,d} := \{A_1, B_1, \ldots, A_g, B_g \in \mathrm{GL}_n | \prod_{i=1}^{\mathrm{g}} A_i^{-1} B_i^{-1} A_i B_i = e^{\frac{2\pi i d}{n}} \mathrm{Id}\} / \mathrm{PGL}_n$$

$$\mathcal{M}_{\mathrm{Dol}}^{n,d} := \left\{ \begin{array}{c} \text{moduli space of semistable rank } n \\ \text{degree } d \text{ Higgs bundles } (E, \phi) \\ \phi \in H^0(C, End(E) \otimes K) \text{ Higgs field} \end{array} \right\}$$

Non-Abelian Hodge correspondence: Hyper-Kähler rotation $\mathcal{M}_{\mathrm{B}}^{n,d} \cong \mathcal{M}_{\mathrm{Dol}}^{n,d}$

Conjecture (de Cataldo, Hausel, Migliorini)

$$Gr_{2k}^W H^j \left(\mathcal{M}_B^{n,d}\right) = Gr_k^P H^j \left(\mathcal{M}_{Dol}^{n,d}\right)$$

- weight filtration $W_0 \subset \cdots \subset W_i \subset \cdots \subset W_{2k} = H_c^k(\mathcal{M}_B^{n,d}; \mathbb{Q})$
- perverse Leray filtration $P_0 \subset \cdots \subset P_i \subset \cdots \subset P_k = H_c^k(\mathcal{M}_{Dol}^{n,d}; \mathbb{Q})$

The one story

weighted Poincaré polynomials

$$WP(\mathcal{M}_{\mathrm{B}}^{n,d}; u, v) = \sum_{i,j} u^{i/2} v^j \dim Gr_i^W H_c^j(\mathcal{M}_{\mathrm{B}}^{n,d})$$

Conjecture (Hausel,Rodriguez-Villegas)

$$\sum_{\lambda} \prod_{\lambda} \frac{\left(z^{2l+1}-w^{2a+1}\right)^{2g}}{\left(z^{2l+2}-w^{2a}\right)\left(z^{2l}-w^{2a+2}\right)} x^{|\lambda|} = \exp\left(\sum_{n,k} \frac{WP\left(\mathcal{M}_B^{n,d}; w^{2k}, -(zw)^{-2k}\right)}{\left(z^{2k}-1\right)\left(1-w^{2k}\right)(zw)^{-d_n}} \frac{x^{nk}}{k} \right)$$

- C with tame ramification [Hausel,Letellier,Rodriguez-Villegas]
- C with wild ramification [Hausel,Mereb,Wong]

The other story

Suppose $\Sigma \subset \mathbb{C}^2$ is a reduced rational plane curve with a single singular point ν and $\pi : \mathbb{C}^2 \to \mathbb{C}^1$ projection onto one of the coordinate axes

Then $\Sigma \simeq$ affine part of a spectral curve for a meromorphic Hitchin system on $\mathbb{P}^1$ with a pole at ∞

The wild non-abelian Hodge correspondence identifies the (symplectic leaf containing) the compactified Jacobian of the projective completion of Σ with a wild character variety $\mathcal{S}_\Sigma$ on $\mathbb{P}^1$ parametrizing Stokes data at ∞

The other story

- $L \subset S^3$ - the link of the singular point $\nu \in \Sigma$
- $P_L(a,q) \in \mathbb{Q}\left(q^{1/2}\right)\left[a^{\pm}\right]$ - the HOMFLY polynomial of L
- $P_L^{(0)}(q)$ - the a^0 term in $\left(aq^{-1/2}\right)^{m(\nu)} P_L(a,q)$, where $m(\nu)$ is the Milnor number of $\nu \in \Sigma$

Conjecture (Shende, Treumann, Zaslow)

$$P_L^{(0)}(q) = (1-q)^{b(\nu)} WP\left(\mathcal{S}_\Sigma; q, -1\right)$$

where $b(\nu)$ = number of branches of Σ at ν.

Refined and colored extensions have been studied in [Oblomkov,Rassmusen,Shende] and [Diaconescu,Hua,Soibelman]

Introduction

Topology of character varieties with complex group is very interesting. Many mathematicians have studied it over decades since the original work of Hitchin. However, it is still difficult to analyze it, especially with wild ramifications.

On the other hand, knot invariants are very handy because it can be boiled down to simple combinatorics. Even I can handle them to some extent.

The Goal: Motivated by the two stories above, I will provide conjectural connection of cohomology groups of twisted wild character varieties to torus knot invariants.

The broad map

Twisted wild character variety $\mathcal{S}_{n,\ell,r}$ — Weighted Poincaré polynomials

NAH, and $P = W$

Moduli space $\mathcal{H}_{n,\ell,r}$ of twisted irregular Higgs bundles — Perverse Poincaré polynomials

Spectral correspondence

Moduli space of pure sheaves F of dimension one on spectral surface S

Gopakumar-Vafa expansion

Refined stable pairs $\mathcal{O}_Y \xrightarrow{s} F$ of CY3 — Refined PT invariants

Large N duality

Refined Chern-Simons theory of torus knot — Refined Chern-Simons invariants

The broad map

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Moduli space of twisted irregular flat connections

Main object of study: Moduli space of flat connections with twisted irregular singularity at p on a curve C

Data:

- (C, p) smooth projective curve with a marked point.
- For any triple (n, ℓ, r) we study $\mathcal{C}_{n,\ell,r}$ the moduli space of pairs (V, A)
- V is a rank $N = r\ell$, degree 0 vector bundle on C
- A a meromorphic $\mathfrak{gl}(N, \mathbb{C})$-connection on V subject to the following conditions:

Moduli space of twisted irregular flat connections

Flat bundle (V, A) with twisted irregular singularity at p on a curve C consists of the following data

- Any proper nontrivial sub-bundle $V' \subset V$ preserved by A has degree $\deg(V') < 0$. (flatness condition) [Narasimhan,Seshadri]
- A has a simple pole at ∞ with residue $\mathbf{1}_N/2i\pi N$. (deg=1 in Higgs side)
- A has a pole of order n at the marked point p and there exists a local trivialization of V at p which identifies it to $A_{n,\ell,r}$ up to holomorphic terms where

$$A_{n,\ell,r} = -2dQ_{n,\ell,r} - \ell^{-1} R \frac{dz}{z}$$

is a $\mathfrak{gl}(N, \mathbb{C})$-valued one-form with

$$Q_{n,\ell,r}(z) = \frac{z^{1-n}}{1-n} J_{\ell,r} + \frac{z^{2-n}}{2-n} E_{\ell,r}$$

Moduli space of twisted irregular flat connections

the word, "twisted", refers to the fact that the fixed Laurent tail at p does not have coefficients in a Cartan subalgebra of $\mathfrak{gl}(N,\mathbb{C})$

$$J_{\ell,r} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ \mathbf{1}_r & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \mathbf{1}_r & 0 \end{pmatrix}, \qquad E_{\ell,r} = \begin{pmatrix} 0 & 0 & \cdots & 0 & \mathbf{1}_r \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

The residue R is the diagonal block-matrix

$$R = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & \mathbf{1}_r & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & (\ell-1)\mathbf{1}_r \end{pmatrix}.$$

Twisted wild character varieties

Betti version of $\mathcal{C}_{n,\ell,r}$ is constructed by [Boalch,Yamakawa] as variety of twisted Stokes data

- let $\rho : \widetilde{\Delta} \to \Delta$ be the $\ell : 1$ cover $z = w^\ell$
- $\rho^* dQ_{n,\ell,r}$ can be diagonalizable so that Stokes group can be studied via this covering map
- there are two types anti-Stokes directions for the twisted wild ramification
  - complete half-period: Stokes group is a unipotent subgroup
  - incomplete half-period: Stokes group is a smaller unipotent subgroup

Twisted wild character varieties

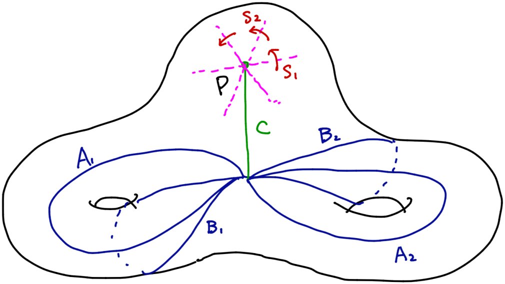
For

$$\alpha = (A_i, B_i, C, h, S_j),$$

we define a map $\mu : \mathcal{A} \to GL(N, \mathbb{C}) \times H(\overline{\partial})$ as

$$\mu(\alpha) = \left((A_1, B_1) \cdots (A_g, B_g) C^{-1} h \prod_{0 \le j \le 2n-3} S_j \, C, h^{-1} \right)$$

with $(A_i, B_i) = A_i B_i A_i^{-1} B_i^{-1}$.



Twisted wild character varieties

Then the variety of Stokes data for the flat connections parameterized by $\mathcal{C}_{n,\ell,r}$ is given by quasi-Hamiltonian reduction

$$\mathcal{S}_{n,\ell,r} = \mu^{-1}(e^{-2\pi\sqrt{-1}/N}\mathbf{1}_N, \mathcal{C}_{\mathsf{M}^{-1}})/\text{gauge transformation}$$

where M is the formal monodromy

$$\mathsf{M} = \begin{pmatrix} 0 & \mathbf{1}_r & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \mathbf{1}_r \\ \mathbf{1}_r & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Twisted wild character varieties

- We are interested in weighted Poincaré polynomial

$$WP_{n,\ell,r}(u,v) = \sum_{i,j} u^{i/2} v^j \dim Gr_i^W H^j(\mathcal{S}_{n,\ell,r})$$

  where $W_k H^j$ is the weight filtration for the mixed Hodge structure on $\mathcal{S}_{n,\ell,r}$.
- The weight filtration on cohomology is expected to satisfy the condition $W_{2k} H^j(\mathcal{S}_{n,\ell,r}) = W_{2k+1} H^j(\mathcal{S}_{n,\ell,r})$ for all k, j

The broad map

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Twisted irregular Higgs bundles

- (C, p) a curve with marked point
- $D = np \; n \geq 1$
- $M = K_C(D)$

a twisted irregular Higgs bundle (E, ϕ) of rank $N = r\ell$, $\phi : E \to E \otimes K_C(D)$, $D = np$ is subject to the condition that the restriction $\phi|_{np}$ is equivalent to $dQ_{n,\ell,r}(z)$

$$Q_{n,\ell,r}(z) = \frac{z^{1-n}}{1-n} J_{\ell,r} + \frac{z^{2-n}}{2-n} E_{\ell,r}$$

with

$$J_{\ell,r} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ \mathbf{1}_r & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \mathbf{1}_r & 0 \end{pmatrix}, \qquad E_{\ell,r} = \begin{pmatrix} 0 & 0 & \cdots & 0 & \mathbf{1}_r \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

Twisted irregular Higgs bundles

- moduli space $\mathcal{H}_{n,\ell,r}$ of semistable meromorphic Higgs bundles of rank $N = r\ell$, degree 1
- (twisted) wild non-abelian Hodge correspondence [Biquard-Boalch, Sabbah, Mochizuki] [Witten]

$$\mathcal{S}_{n,\ell,r} \cong \mathcal{H}_{n,\ell,r}$$

- $\mathcal{H}_{n,\ell,r}$ admits perverse Leray filtration $P_k H^j$ arising from the Hitchin map

Conjecture (Twisted wild version of $P = W$)

$$Gr^W_{2k} H^j(\mathcal{S}_{n,\ell,r}) = Gr^P_k H^j(\mathcal{H}_{n,\ell,r})$$

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Spectral (cover) correspondence

Translating it into different language

$$\mathcal{H}_{n,\ell,r} \cong \begin{array}{c}\text{moduli space of some sheaves} \\ \text{on a certain surface } S\end{array}$$

Baby version [Hitchin] [Beauville,Narasimhan,Ramanan]
The characteristic polynomial $\det(y\mathbf{1}-\phi)$ of the Higgs field ϕ defines a spectral over $\pi : \Sigma \to C$.

If Σ is an irreducible spectral curve, there is a bijection between isomorphism classes of

- Pairs (E, ϕ) with spectral curve Σ
- Rank one torsion free sheaves L on Σ

Spectral correspondence

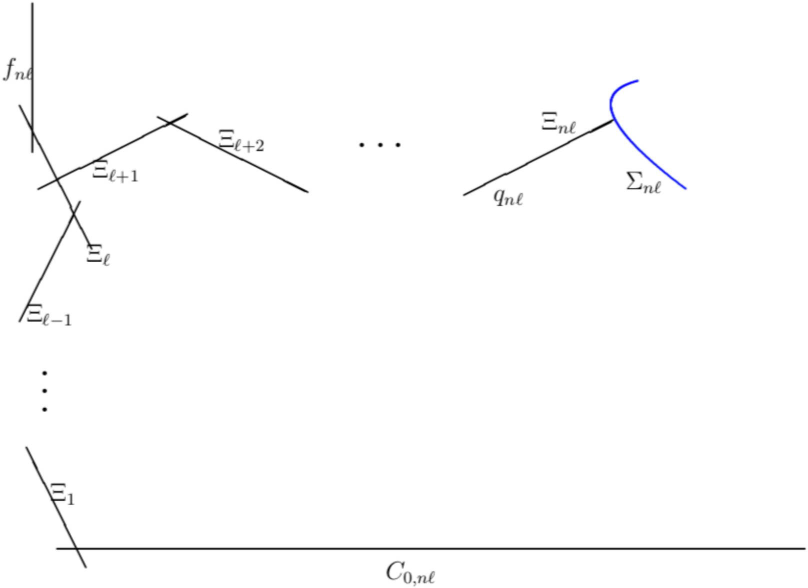
Goal: construct a holomorphic symplectic surface S in such a way that

$$\mathcal{H}_{n,\ell,r} \cong \begin{array}{c}\text{moduli space of stable}\\ \text{pure dimension one sheaves on } S\end{array}$$

[Kontsevich,Soibelman] [Diaconescu,Donagi,Pantev]

- (C, p) a curve with marked point
- $D = np$
- $M = K_C(D)$
- let $M_{n\ell}$ be $n\ell$ successive blow-ups of M, and Ξ_i be exceptional divisor
- there is a unique compact curve $\Sigma_{n\ell} \subset M_{n\ell}$

Spectral correspondence



Spectral correspondence

- $S = M_{n\ell} \backslash (\text{the support of anti-canonical divisor} = f_{n\ell} \cup \bigcup_{i=1}^{n\ell-1} \Xi_i)$
- S is a holomorphic symplectic surface
- moduli space $\mathcal{M}_{n,\ell,r}(S)$ of pure dimension one sheaves F with

$$\mathrm{ch}_1 = r[\Sigma_{n\ell}] \ , \quad \chi(F) = c = 1 - N(g-1)$$

can be endowed with stability condition called Bridgeland β-stability

Theorem (Twisted spectral correspondence)

$$\mathcal{H}_{n,\ell,r} \cong \mathcal{M}_{n,\ell,r}(S)$$

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CY3 and refined invariants

- $Y = S \times \mathbb{C}^1$
$$\mathcal{M}_{n,\ell,r}(S) \simeq \mathcal{H}_{n,\ell,r} \times \mathbb{C}^1$$
- Stable pair theory [Pandharipande,Thomas]
$PT_Y(r,c) =$ virtual number of pairs $s : \mathcal{O}_Y \to F$ generically surjective with compact support F and
$$\mathrm{ch}_1(F) = r|\Sigma_{n\ell}|,\ \chi(F) = c = 1 - N(g-1)$$
- Refined stable pair theory [Kontsevich,Soibelman]
$Z^{\mathrm{ref}}_{PT}(Y; q, y, \mathrm{x}) =$ virtual Poincaré polynomial of moduli space of such pairs.

CY3 and refined invariants

Generating functions

$$Z_{PT}^{\mathrm{ref}}(Y; q, y, \mathrm{x}) = 1 + \sum_{r \geq 1,\, c \in \mathbb{Z}} PT_{Y}^{\mathrm{ref}}(r, c\,; y) q^{c} \mathrm{x}^{r}$$

Goal: compute the generating function.

- Specialize $C = \mathbb{P}^1$ with one marked point $\sigma_{n\ell} \in C$
- Torus action $\mathbb{C}^{\times} \times S \to S$
- Refined virtual localization [Nekrasov,Okounokov][Maulik]
- Stable pair theory localizes on a single rational curve $\Sigma_{n\ell}$ in S

Stable pairs conjecture

Conjecture (Spectral correspondence and Gopakumar-Vafa expansion)

$$\ln Z^{\mathrm{ref}}_{PT}(Y; q, y, \mathrm{x}) = -\sum_{s\geq 1}\sum_{r\geq 1}\frac{\mathrm{x}^{sr}}{s}\frac{y^{sr\#}(qy^{-1})^{s\#}P_{n,\ell,r}(q^{-s}y^{-s}, y^s)}{(1-q^{-s}y^{-s})(1-q^s y^{-s})}$$

where $P_{n,\ell,r}$ are perverse Poincaré polynomials

$$P_{n,\ell,r}(u,v) = \sum_{j,k} u^i v^j \dim Gr^P_i H^j(\mathcal{H}_{n,\ell,r}) \ .$$

Direct localization computations of $PT^{\mathrm{ref}}_Y(r, c\,; y)$ are possible in principle but are hard

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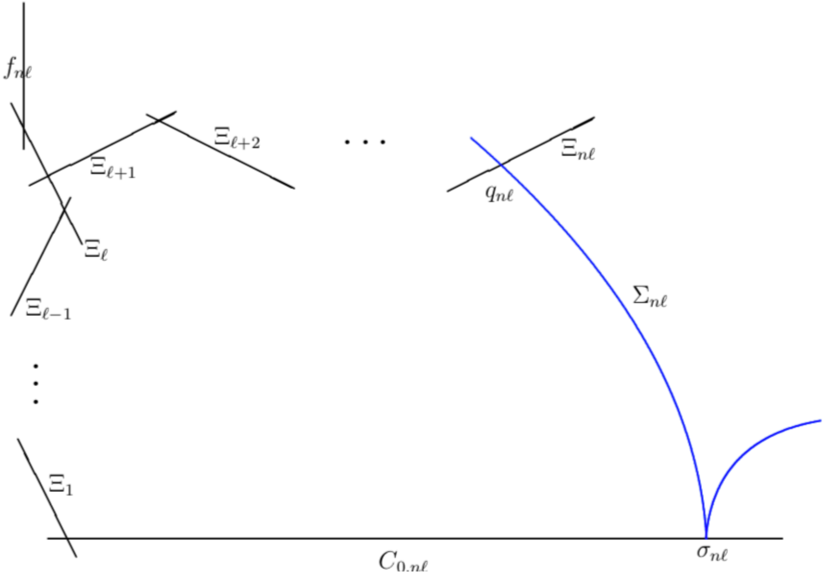
Compact curve $\Sigma_{n\ell}$ on S



Refined invariants and refined Chern-Simons theory

- Local equation of the torus invariant curve $\Sigma_{n\ell}$ at $\sigma_{n\ell}$. The curve has a unique singular point of the form

$$v^\ell = w^{(n-2)\ell-1}.$$

- We can use large N duality [Oblomkov,Shende,Rasmussen] [Diaconescu,Hua,Soibelman]

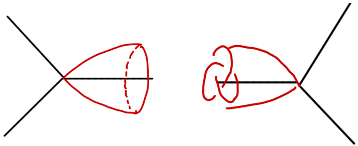
$$\left(\begin{array}{c} \text{Refined PT invariants of} \\ \text{the singular plane curves } \Sigma_{n\ell} \end{array} \right) \leftrightarrow \left(\begin{array}{c} \text{Refined CS invariants of} \\ T_{\ell,(n-2)\ell-1} \text{ torus knots} \end{array} \right)$$

- In refined Chern-Simons theory [Aganagic,Shakirov], torus knot invariants can be evaluated by modular matrices

$$S_{\lambda\mu} = P_\lambda(t^\rho s^\mu) P_\mu(t^\rho) \ , \qquad T_{\lambda\mu} = \delta_{\lambda\mu} s^{\frac{1}{2}\sum_i \lambda_i(\lambda_i - 1)} t^{\sum_i \lambda_i (i-1)}$$

Refined invariants and refined Chern-Simons theory

Topological vertex formalism tells us that the contribution is localized at $\Sigma_{n\ell}$ and ∞



Conjecture (Large N duality)

The generating function of refined stable pairs in Y is expressed as

$$Z^{\text{ref}}_{PT}(Y;q,y,\mathrm{x}) = 1 + \sum_{|\lambda|>0} T^{\#}_{\lambda\lambda} W^{\text{ref}}_{\lambda}(\bigcirc) W^{\text{ref}}_{\lambda}(T_{\ell,(n-2)\ell-1})\mathrm{x}^{|\lambda|}\Big|_{s=qy,\ t=qy^{-1}}.$$

Epilogue

Definition: the HMW partition function is the function

$$Z_{HMW}(z,w) = 1 + \sum_{|\lambda|>0} \Omega_\lambda^{g,n}(z,w)\widetilde{H}_\lambda(\mathrm{x}; z^2, w^2)$$

- the sum over all Young diagrams λ
- for each such λ

$$\Omega_\lambda^{g,n} = \prod_{\square\in\lambda} \frac{(-z^{2a(\square)}w^{2l(\square)})^{-mn-2g+2}(z^{2a(\square)+1} - w^{2l(\square)+1})^{2g}}{(z^{2a(\square)+2} - w^{2l(\square)})(z^{2a(\square)} - w^{2l(\square)+2})},$$

- the infinite set of variables $\mathrm{x} = (x_1, x_2, \ldots)$.
- $\widetilde{H}_\lambda(\mathrm{x}; z^2, w^2)$ is the modified Macdonald polynomial

Epilogue

Next let $\mathbb{H}_{g,m,n}(z,w;\mathrm{x}_{(1)},\ldots,\mathrm{x}_{(m)})$ be defined by

$$\ln Z_{HMW}(z,w)=\sum_{k\geq 1}\frac{(-1)^{(mn-1)|\mu|}\mathbb{H}_{g,m,n}(z^k,w^k;\mathrm{x}^k))}{(1-z^{2k})(w^{2k}-1)}.$$

For each Young diagram ρ, let $h_\rho(\mathrm{x})$ and $p_\rho(\mathrm{x})$ be the complete and power sum symmetric functions respectively.

Conjecture

$$WP_{g,m,n,\ell}(u,-1)$$
$$=u^{\#}\left\langle \mathbb{H}_{g,m,n}(u^{\frac{1}{2}},u^{-\frac{1}{2}};\mathrm{x}_{(1)},\ldots,\mathrm{x}_{(m)}),\bigotimes_{a=1}^{m}h_{(\ell-1,1)}(\mathrm{x}_{(2a-1)})\otimes p_{(\ell)}(\mathrm{x}_{(2a)})\right\rangle$$

where $\langle\ ,\ \rangle$ is the natural pairing on symmetric functions.