Homework 11: Due at class on May 19

1. Pontryagin classes of sphere

For the tangent bundle TS^n of a sphere, show that its total Pontryagin class is trivial $p(TS^n) = 1$.

2. Chern classes of $\mathbb{C}P^n$

An arbitrary point $\ell = [z_0; \cdots ; z_n] \in \mathbb{C}P^n$ is a complex line $\mathbb{C}$ through the origin of $\mathbb{C}^{n+1}$. So we can construct a complex line bundle L over the complex projective space $\mathbb{C}P^n$ as

$$\begin{aligned} L &= \{(\ell, w) \in \mathbb{C}P^n \times \mathbb{C}^{n+1} |\ w \in \ell\} \ , \\ &= \{([z_0; \cdots ; z_n], w) \in \mathbb{C}P^n \times \mathbb{C}^{n+1} |\ w = \lambda \cdot (z_0, \cdots , z_n) \ , \quad \lambda \in \mathbb{C}\} \ . \end{aligned} \tag{1}$$

(The notation is the same as in Problem 4 of Homework 9.) This is called the **Hopf line bundle**. If we consider points with unit length in each fiber of the Hopf line bundle, then we obtain a sphere

$$S^{2n+1} = \{(\ell, z) \in \mathbb{C}P^n \times \mathbb{C}^{n+1} |\ z \in \ell \ , \ |z| = 1\} \ .$$

Therefore, the Hope line bundle is the associated line bundle of the Hopf S^1-bundle.

In fact, the Whiteney sum of the tangent bundle $T\mathbb{C}P^n$ and the trivial bundle ϵ is the Whiteney sum of $(n + 1)$ copies of L^*

$$T\mathbb{C}P^n \oplus \epsilon \cong L^* \oplus \cdots \oplus L^* \ .$$

where L^* is the dual line bundle of L. (See Morita Proposition 5.13 for the derivation.)

Let us first consider the case of $n = 1$. We know that cohomology $H^2(\mathbb{C}P^1; \mathbb{Z}) \cong \mathbb{Z}$ and its generator x is the volume form. Show that $c_1(L) = -x$ and $c_1(T\mathbb{C}P^1) = 2x$. In addition, find the curvature of the Ehresmann connection

$$A = i \ \mathrm{Im}(\overline{z}_0 dz_0 + \overline{z}_1 dz_1)$$

given in Problem 4 of Homework 9, and compare it with the volume form. Actually, show that over $\mathbb{C}P^1$ one can find an infinite family of non-equivalent complex line bundles.

Find the total Chern class of $T\mathbb{C}P^n$. In fact, the $2i$-th de Rham cohomology group of $\mathbb{C}P^n$ is isomorphic to $\mathbb{R}$ generated by $x^i = x \wedge \cdots \wedge x$:

$$H^{2i}_{dR}(\mathbb{C}P^n) \cong \mathbb{R}\langle x^i \rangle \quad \text{for} \quad i \leq n \ .$$

Moreover, the de Rham cohomology groups of $\mathbb{C}P^n$ has a ring structure where the multiplication is given by the wedge product:

$$H^*_{dR}(\mathbb{C}P^n) \cong \mathbb{R}[x]/(x^{n+1}) \ .$$

3. Chern-Simons and Wess-Zumino-Witten

Let us define the Chern-Simons action over a 3-manifold M

$$S_{CS} = \frac{k}{4\pi}\int_M \mathrm{Tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right) .$$

Under the gauge transformation

$$A \to g^{-1}dg + g^{-1}Ag ,$$

show that the action is transformed as

$$S_{CS} \to S_{CS} + \frac{k}{4\pi}\int_M d\ \mathrm{Tr}(g^{-1}Ag\wedge g^{-1}dg) - \frac{k}{12\pi}\int_M \mathrm{Tr}((g^{-1}dg)^3) .$$

Let us consider the case $M = S^3$ and $G = \mathrm{SU}(2)$. Then, the second term vanishes because S^3 is closed. Show that the third term is

$$\frac{k}{12\pi}\int_M \mathrm{Tr}((g^{-1}dg)^3) = 2\pi k \deg(g) ,$$

where $\deg(g)$ is the mapping degree of

$$g : S^3 \to \mathrm{SU}(2) \cong S^3 .$$

(Hint: $\mu = g^{-1}dg$ is the Maurer-Cartan form and $\frac{1}{6}\mathrm{Tr}(\mu\wedge\mu\wedge\mu)$ is the left-invariant volume form of $\mathrm{SU}(2)$.) Argue that the level k must be an integer in order for the partition function to be well-defined.

4. Chern-Simons from 4-dimension

Show the following identity

$$\mathrm{Tr}(F\wedge F) = d\ \mathrm{Tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right) .$$

Therefore, if there exists a four-manifold B whose boundary is $\partial B = M$, then we can indeed define Chern-Simons action by

$$S_{CS} = \frac{k}{4\pi}\int_B \mathrm{Tr}(F\wedge F) ,$$

when G is compact and semi-simple.