

Homework 6 (Due at class on Nov 3)

1 Wold-sheet supersymmetry

1.1 Supersymmetric transformation

Show that the action

$$S^m = \frac{1}{4\pi} \int d^2z \left(\frac{2}{\alpha'} \partial X^\mu \bar{\partial} X_\mu + \psi^\mu \bar{\partial} \psi_\mu + \tilde{\psi}^\mu \partial \tilde{\psi}_\mu \right) \quad (1.1)$$

is invariant under the supersymmetric transformation

$$\delta_{\epsilon, \bar{\epsilon}} X^\mu = \sqrt{\frac{\alpha'}{2}} (\epsilon \psi^\mu + \bar{\epsilon} \tilde{\psi}^\mu) , \quad \delta_\epsilon \psi^\mu = -\sqrt{\frac{2}{\alpha'}} \epsilon \partial X^\mu , \quad \delta_{\bar{\epsilon}} \tilde{\psi}^\mu = -\sqrt{\frac{2}{\alpha'}} \bar{\epsilon} \bar{\partial} X^\mu .$$

Also, show that the combination of supersymmetric transformations provides derivatives

$$[\delta_{\epsilon_1, \bar{\epsilon}_1}, \delta_{\epsilon_2, \bar{\epsilon}_2}] \mathcal{O} = 2\epsilon_1 \epsilon_2 \partial \mathcal{O} + 2\bar{\epsilon}_1 \bar{\epsilon}_2 \bar{\partial} \mathcal{O} .$$

1.2 Superspace formalism

This theory can be formulated in terms of superspace $(z, \bar{z}, \theta, \bar{\theta})$ where $(\theta, \bar{\theta})$ are anti-commuting Grassmann coordinates. We can introduce the superfield

$$Y^\mu(z, \bar{z}, \theta, \bar{\theta}) = X^\mu(z, \bar{z}) + i\theta \psi^\mu(z, \bar{z}) + i\bar{\theta} \tilde{\psi}^\mu(z, \bar{z}) + \frac{1}{2} \bar{\theta} \theta F^\mu(z, \bar{z}) ,$$

as well as the superderivative

$$D = \frac{\partial}{\partial \theta} + \theta \partial_z , \quad \bar{D} = \frac{\partial}{\partial \bar{\theta}} + \bar{\theta} \bar{\partial}_{\bar{z}} .$$

The field F^μ is called an **auxiliary field**. Then, show that the action (1.1) at $\alpha' = 2$ is equivalent to

$$S = \frac{1}{4\pi} \int d^2z d^2\theta \bar{D} Y^\mu D Y_\mu ,$$

where the superspace integral is defined as $d^2\theta = d\theta d\bar{\theta}$, and

$$\int d\theta d\bar{\theta} \bar{\theta} \theta = 1 .$$

1.3 Supersymmetric ghost

Let us define the ghost superfields as

$$B = \beta + \theta b , \quad C = c + \theta \gamma .$$

Show that the supersymmetric ghost action can be written as

$$S^{\text{gh}} = \frac{1}{2\pi} \int d^2z d^2\theta B \bar{D} C .$$

2 $\mathcal{N} = 1$ superconformal algebra

In general, a 2d superconformal field theory has the following OPEs for the stress-energy tensor $T(z)$ and the supercurrent $G(z)$

$$\begin{aligned} T(z)T(w) &\sim \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial_w T(w) \\ T(z)G(w) &\sim \frac{3/2}{(z-w)^2}G(w) + \frac{1}{z-w}\partial_w G(w) \\ G(z)G(w) &\sim \frac{2c/3}{(z-w)^3} + \frac{2}{z-w}T(w) . \end{aligned} \tag{2.1}$$

As in the lecture, we carry out the mode expansion

$$T_B(z) = \sum_{m \in \mathbb{Z}} \frac{L_m}{z^{m+2}} , \quad G(z) = \sum_{r \in \mathbb{Z} + \nu} \frac{G_r}{z^{r+3/2}}$$

where $\nu = 0$ and $\nu = \frac{1}{2}$ correspond to Ramond sector and Neveu-Schwarz sector, respectively. From the OPEs (2.1), derive $\mathcal{N} = 1$ superconformal algebra

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m+n,0} \\ [L_m, G_r] &= \left(\frac{1}{2}m - r \right) G_{r+m} \\ \{G_r, G_s\} &= 2L_{r+s} + \frac{c}{3}(r^2 - \nu^2)\delta_{r+s,0} . \end{aligned}$$

3 First massive states

Find the number of physical degrees of freedom at the first massive level after the GSO projection both in the NS and R sectors.