Homework 1: Due at class on March 11

1. Euler Characteristic

1. Find the Euler characteristic of a surface of genus g (Figure 1).
2. Find the Euler characteristic of a Klein bottle (Figure 2).
3. Find the area of the following triangle (Figure 4) bounded by great circles on a 2-sphere with radius one.
4. Generalize it to the area of n-gon on a 2-sphere with radius one. (An example of a pentagon is drawn in Figure 5.)
5. Prove the Euler characteristic of a 2-sphere is equal to two by using a cell decomposition (Figure 3).

Figure 1: A surface of genus g

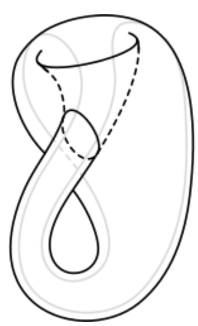

Figure 2: A Klein bottle

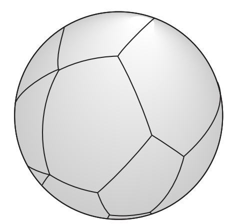

Figure 3: cell decomposition of a 2-sphere

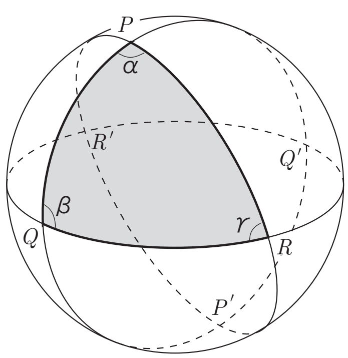

Figure 4: A triangle on a 2-sphere

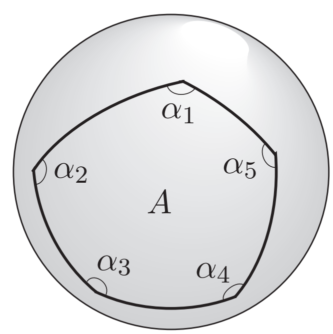

Figure 5: A pentagon on a 2-sphere

2. For elements $x = (x^0, \cdots, x^n)$ and $y = (y^0, \cdots, y^n)$ of $\mathbb{R}^{n+1} \backslash \{0\}$, we define an equivalence relation $x \sim y$ by

$$x = \alpha y \qquad \forall \alpha \in \mathbb{R} \, .$$

Let us define $\mathbb{R}P^n$ by $(\mathbb{R}^{n+1}\backslash\{0\})/\sim$. Show that $\mathbb{R}P^n$ is a manifold and $\mathbb{R}P^1$ is diffeomorphic to S^1. The space is called a real projective space.

3. For elements $x = (x^0, \cdots, x^n)$ and $y = (y^0, \cdots, y^n)$ of $\mathbb{C}^{n+1}\backslash\{0\}$, we define an equivalence relation $x \sim y$ by

$$x = \alpha y \qquad \forall \alpha \in \mathbb{C} \, .$$

Let us define $\mathbb{C}P^n$ by $(\mathbb{C}^{n+1}\backslash\{0\})/\sim$. Show that $\mathbb{C}P^n$ is a manifold and $\mathbb{C}P^1$ is diffeomorphic to S^2. The space is called a complex projective space.

4. Let $M_n(\mathbb{R})$ and $M_n(\mathbb{C})$ denote the set of all $n \times n$ matrices over $\mathbb{R}$ and $\mathbb{C}$, respectively. We define

$$\begin{aligned} SU(2) &= \{A \in M_2(\mathbb{C}) |\ A^\dagger A = \mathrm{Id},\ \det A = 1\} \\ SO(3) &= \{A \in M_3(\mathbb{R}) |\ A^T A = \mathrm{Id},\ \det A = 1\} \, . \end{aligned}$$

5.1 Construct a double covering (2-to-1) map $SU(2) \to SO(3)$.
5.2 Show that $SU(2)$ is diffeomorphic to S^3 and $SO(3)$ is diffeomorphic to $\mathbb{R}P^3$.