Homework 7: Due at class on April 25

1. Let us define $S^3 = \{(x^0, x^1, x^2, x^3) \in \mathbb{R}^4 | \sum_{i=0}^{3}(x^i)^2 = 1\}$ and $S^1 = \{(x^0, x^1, x^2, x^3) \in \mathbb{R}^4 | (x^0)^2 + (x^1)^2 = 1\}$. Then, show that $S^3 \backslash S^1$ is homotopic to S^1.

2. Let us identify $S^2 = \mathbb{C} \cup \{\infty\}$. Then, a holomorphic map $g : \mathbb{C} \to \mathbb{C}; z \mapsto z^n$ $(n \in \mathbb{Z})$ can be extended to $g : S^2 \to S^2$. Find the mapping degree $\deg g$ of g.

3. **Fundamental theorem of algebra**

We define $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = z^n + a_1 z^{n-1} + \cdots + a_n$ for $n \geq 1$. In addition, by writing $z = x + iy$, we define one-form

$$\omega = \mathrm{Im}\frac{dz}{z} = \frac{-ydx}{x^2+y^2} + \frac{xdy}{x^2+y^2} \, .$$

Then, show that

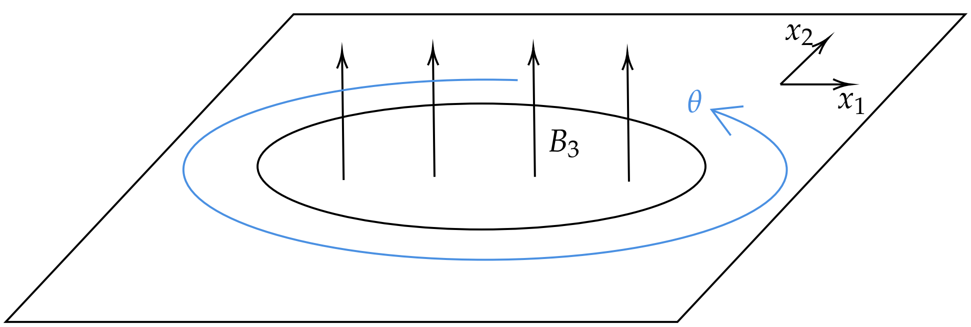
$$\frac{1}{2\pi}\int_{C_R} f^*\omega = n \, ,$$

where C_R is the circle with sufficiently large radius R. (Hint: construct homotopy between f and g above.) If there were no zero points $f(z) = 0$ inside C_R, show that

$$\frac{1}{2\pi}\int_{C_R} f^*\omega = 0$$

by using the Stokes theorem.

4. Find Poincaré dual pairs (non-trivial intersection pairings) in the real-valued homology group $H_\ell(\Sigma_g; \mathbb{R})$ of a Riemann surface Σ_g of genus g.



5. (Mapping degree and vortex) Let us consider the $3+1$-dimensional action for a scalar field interacting the electromagnetic field with the potential

$$S = \int d^4x \left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \left|D_\mu \phi\right|^2 - \lambda\left(|\phi|^2 - v^2\right)^2 \right]$$

with $D_\mu\phi = \partial_\mu\phi - iA_\mu\phi$. Let us consider the situation in which the system is invariant in the x^3 direction and $A_3 = 0$ so that all the fields depend on the (x^1, x^2) plane. We parameterise the plane by the radial coordinates $x^1 + ix^2 = re^{i\theta}$.

- Show a necessary condition for the energy to be finite is that the scalar field configuration ϕ is topologically classified by the mapping degree of $S^1_\infty \to S^1$ at the infinity of the plane. Namely, the configuration is homotomic to $\phi \to e^{in\theta}v$ as $r \to \infty$ where $n \in \mathbb{Z}$.
- Compute the Kinetic energy $\int d^2x\, |\partial_i\phi|^2$ for this configuration, which is still divergent.
- This divergence can be canceled by the gauge potential A_μ. Namely we can have $\int d^2x\, |D_i\phi|^2 < \infty$ if we choose the gauge potential appropriately. In this case, show that the magnetic flux over the plane is quantized as

$$\frac{1}{2\pi}\int d^2x B_3 = n\ .$$

6. (This is a bonus problem with extra 3 points which is NOT mandatory.)

Let us consider a non-linear sigma model $S^2 \to S^2$. In phyiscs, we usually identify $S^2 = \mathbb{R}^2 \cup \{\infty\}$ and we write the action as

$$S = \frac{1}{4\pi}\int d^2x \left(\frac{1}{2}\partial_m X^i \partial_m X^i + 2\lambda\left(X^i X^i - 1\right)^2\right)$$

where $X : \mathbb{R}^2 \to \mathbb{R}^3; x \mapsto X(x)$ and $\lambda \in \mathbb{R}_{>0}$ is the Lagrangian multiplier so that it imposes $|X|^2 - 1$. If we define

$$Q = \frac{1}{8\pi}\int d^2x \varepsilon^{ijk}\varepsilon_{mn} X^i \partial_m X^j \partial_n X^k\ ,$$

show that Q is an integer and $S \geq Q$. Find a field configuration $X : \mathbb{R}^2 \to \mathbb{R}^3$ explicitly that saturates the bound $S \geq Q$.