Homework 3: Due at class on April 6

1 SU(3) Lie algebra

The standard basis for the generators of SU(3) in the fundamental representation is given by the so-called Gell-Mann matrices, which are the analogues of the Pauli matrices for SU(2) :

$$\begin{aligned}
t^1 &= \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad t^2 = \frac{1}{2}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad t^3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
t^4 &= \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad t^5 = \frac{1}{2}\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad t^6 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\
t^7 &= \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad t^8 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}
\end{aligned} \tag{1}$$

This is usually referred to as the fundamental representation “**3** of SU(3)”.

1.1

Explain why there are exactly eight of these matrices and why they form a basis for Hermitian, traceless 3×3 matrices.

1.2

Evaluate all of the commutators of the Gell-Mann matrices to compute the structure constants, and show that they are antisymmetric. (This is a kind of tedious exercise, so you can compute only a representative sample if you don't feel like computing them all.)

1.3

Check that these are normalized so that

$$\mathrm{Tr}\left(t^a t^b\right) = C(\mathbf{3})\delta^{ab}$$

for a constant $C(\mathbf{3})$ which you should compute.

1.4

Compute the quadratic Casimir C_2 directly from its definition

$$t^a t^a = C_2(\mathbf{3}) \cdot \mathbf{1}$$

where $\mathbf{1}$ is the identity matrix. And verify the identity (Peskin&Schroeder (15.94)) for this representation:

$$d(r)C_2(r) = d(G)C(r)$$

1.5

Note that t^1, t^2, t^3 form a natural sub-algebra which acts on rows one and two of a three component column vector. Find two other natural SU(2) sub-algebras, which act on rows one and three and rows two and three, respectively. Of course, these three SU(2) sub-algebras do not commute with one another.

1.6

Consider the $\mathrm{SU}(2) \subset \mathrm{SU}(3)$ generated by t^1, t^2, t^3. How does the **3** of SU(3) transform under this SU(2) subgroup?

1.7

Since all orthogonal matrices are also unitary, there is a natural $\mathrm{SO}(3) \subset \mathrm{SU}(3)$. Which Gell-Mann matrices generate this SO(3)? How does the **3** of SU(3) transform under this SO(3)?

2 Quadratic Casimir operator for $\mathrm{SU}(N)$

For each representation r of $\mathrm{SU}(N)$ the generators t_r^a are normalized such, that

$$\mathrm{Tr}\left[t_r^a t_r^b\right] = C(r)\delta^{ab}$$

where $C(r)$ is some constant depending on the representation under consideration. Using the fact that I and the generators t_f^a in the fundamental representation provide a basis for $N \times N$ Hermitian matrices, where $a = 1 \dots N^2 - 1$

2.1

derive the Fierz identity:

$$\sum_a \left(t_f^a\right)_{ij} \left(t_f^a\right)_{kl} = C(f)\left(\delta_{il}\delta_{kj} - \frac{1}{N}\delta_{ij}\delta_{kl}\right)$$

The quadratic Casimir operator $C_2(r)$ is defined by

$$C_2(r)\mathbf{1}_{d(r)} = \sum_a t_r^a t_r^a$$

where $\mathbf{1}_{d(r)}$ is the $d(r) \times d(r)$ identity matrix.

2.2

Using the Fierz identity, derive the quadratic Casimir of the fundamental representation, $C_2(f)$

2.3

Derive the more general relation

$$C_2(r) = C(r)\frac{d(\mathrm{SU}(N))}{d(r)} = C(r)\frac{N^2-1}{d(r)}$$

and with this find $C_2(f)$ (this should be the same as in the previous question) and $C_2(\mathbf{adj})$, the quadratic Casimir of the adjoint representation. Express $C_2(\mathbf{adj})$ in terms of N and $C(f)$.

3 BRST transformations

Let us consider the gauge-fixed Yang-Mills Lagrangian with fermions

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{4}\left(F^a_{\mu\nu}\right)^2 + \bar{\psi}(i\not{D} - m)\psi + \mathcal{L}_{\mathrm{GF}} \\
\mathcal{L}_{\mathrm{GF}} &= \frac{\xi}{2}\left(B^a_\mu\right)^2 + B^a\partial_\mu A^a_\mu + \partial_\mu\bar{c}^a\left(D_\mu c\right)^a \ .
\end{aligned}$$

Under the BRST transformations

$$\begin{aligned}
\delta A^a_\mu &= \epsilon\left(QA_\mu\right)^a = \epsilon D^{ab}_\mu c^b \\
\delta\psi &= \epsilon(Q\psi) = ig\epsilon c^a t^a \psi \\
\delta c^a &= \epsilon(Qc)^a = -\frac{1}{2}g\epsilon f^{abc}c^b c^c \\
\delta\bar{c}^a &= \epsilon(Q\bar{c})^a = \epsilon B^a \\
\delta B^a &= \epsilon(QB)^a = 0\,,
\end{aligned}$$

show $Q(Q\phi)$ vanishes for all the fields ϕ. In addition, show that $\mathcal{L}_{\mathrm{GF}}$ can be written as a BRST-exact

$$\mathcal{L}_{\mathrm{GF}} = Q\left(\bar{c}^a\partial_\mu A^a_\mu + \frac{\xi}{2}\bar{c}^a B^a\right) \ .$$