Knot invariants, A-polynomials, BPS states

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collaboration with Rama, Zodin, Gukov, Stosic, Sulkowski…

Chern-Simons theory

Witten

M compact 3-manifold

A gauge field with semi-simple gauge group

- action is independent of metric of M

$$S_{CS}(M,k) = \frac{k}{4\pi}\int_M \mathrm{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A)$$

$k \in \mathbb{Z}$ is inverse of gauge coupling (CS level)

- The partition function is itself a 3-manifold invariant

Witten-Reshetikhin-Turaev invariant

$$Z_k(M) = \int \mathcal{D}A \; e^{iS_{CS}(M,k)}$$

Invariants can be evaluated algebraically by using surgery

Chern-Simons theory

Witten

- Wilson loop along a knot colored by a representation R

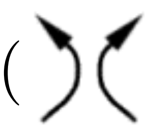
$$W_R(K) = \mathrm{Tr}_R P \exp(\oint_K A)$$

Expectation value is a knot invariant

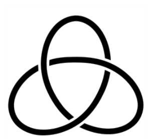
$$\overline{J}_{G,R}(M,K) = \frac{1}{Z_k(M)} \int \mathcal{D}A \; W_R(K) e^{iS_{CS}(M,k)}$$

This is indeed a Laurent polynomial of $q = \exp\left(\frac{2\pi i}{k+h^\vee}\right)$

- $M = S^3,\ G = SU(2),\ R = \square$, then it is Jones polynomial

$q^2 \overline{J}($  $) - q^{-2}\overline{J}($  $) = (q - q^{-1})\overline{J}($  $)$

For example, Jones poly of the trefoil is

$\overline{J}($  $) = q + q^3 + q^5 - q^9$

Chern-Simons theory

Witten

- Wilson loop along a knot colored by a representation R

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Expectation value is a knot invariant

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- Two variable invariants

$M = S^3,\ G = SU(N)$

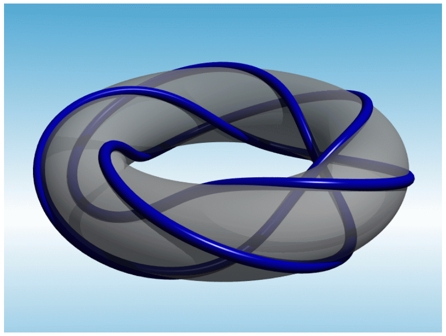
colored HOMFLY polynomials: $\overline{P}(K;a,q)$ where $a = q^N$

$M = S^3,\ G = SO(N)/Sp(N)$

colored Kauffman polynomials: $\overline{F}(K;a,q)$ where $a = q^{N-1}$

Torus knots

- Torus knots can be evaluated by modular matrices



$$T_{\lambda\mu} = q^{C_2(\lambda)}\delta_{\lambda\mu}$$

$$S_{\lambda\mu} = s_\lambda(q^\rho)s_\mu(q^{\rho+\lambda})$$

- Moreover, using the Adams operation **Rosso-Jones**

$$(S_{0\lambda})^m = \sum_\mu C_\lambda^\mu(m) S_{0\mu}$$

In principle, one can determine the invariants for any rep.

$$\overline{J}_\lambda(T_{m,n}) = \sum_\mu C_\lambda^\mu(m) q^{-\frac{m}{n}C_2(\mu)} S_{0\mu}$$

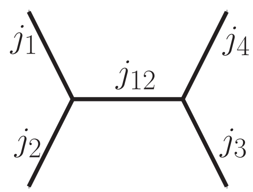
- HOMFLY polynomial of torus knots can be expressed

$$P(T_{m,n}; a, q) \propto {}_2\varphi_1(q^{1-m}, q^{1-n}; q^{1-m-n}; q, a)$$

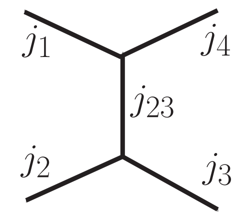
Non-torus knots

- Non-torus knots can be evaluated
  by braiding operations on WZW conformal blocks

$\longrightarrow$ we need quantum 6j-symbols



$$= \sum_{j_{23}} \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{23} \end{Bmatrix}$$



- expression for SU(N) gauge group with symmetric rep

S.N. Ramadevi, Zodinmawia

$$\begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_{12} \\ \lambda_3 & \lambda_4 & \lambda_{23} \end{Bmatrix} = \Delta(\lambda_1,\lambda_2,\lambda_{12})\Delta(\lambda_3,\lambda_4,\lambda_{12})\Delta(\lambda_1,\lambda_4,\lambda_{23})\Delta(\lambda_2,\lambda_3,\lambda_{23})[N-1]!$$
$$\sum_{z\in\mathbb{Z}_{\geq 0}} (-)^z [z+N-1]! C_z(\{\lambda_i\},\lambda_{12},\lambda_{23}) \Big\{ \left[\tfrac{1}{2}\langle \lambda_1+\lambda_2+\lambda_3+\lambda_4, \alpha_1^\vee + \alpha_{N-1}^\vee\rangle - z\right]!$$
$$\times \left[\tfrac{1}{2}\langle \lambda_1+\lambda_3+\lambda_{12}+\lambda_{23}, \alpha_1^\vee + \alpha_{N-1}^\vee\rangle - z\right]! \left[\tfrac{1}{2}\langle \lambda_2+\lambda_4+\lambda_{12}+\lambda_{23}, \alpha_1^\vee + \alpha_{N-1}^\vee\rangle - z\right]!$$
$$\times \left[z - \tfrac{1}{2}\langle \lambda_1+\lambda_2+\lambda_{12}, \alpha_1^\vee + \alpha_{N-1}^\vee\rangle\right]! \left[z - \tfrac{1}{2}\langle \lambda_3+\lambda_4+\lambda_{12}, \alpha_1^\vee + \alpha_{N-1}^\vee\rangle\right]!$$
$$\times \left[z - \tfrac{1}{2}\langle \lambda_1+\lambda_4+\lambda_{23}, \alpha_1^\vee + \alpha_{N-1}^\vee\rangle\right]! \left[z - \tfrac{1}{2}\langle \lambda_2+\lambda_3+\lambda_{23}, \alpha_1^\vee + \alpha_{N-1}^\vee\rangle\right]! \Big\}^{-1}$$

Non-torus knots

- 6j-symbols with multiplicity structure are intrinsically different from those with multiplicity-free

Gu-Jockers

    - Complex numbers
    - Sign (2j- and 3j-phases)
    - Not even Hermitian

- Mutant knots can be detected by HOMFLY polynomials colored by hook-like rep

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Open problem

- Find expressions of 6j-symbols with multiplicity beyond [2,1]-rep.
- Find properties of HOMFLY polynomials colored by hook-like rep

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De Tour: quantum 3-manifold invariants

- Any compact oriented closed 3-mfd is obtained by Dehn surgery on a framed link in S^3

Lickorish

- WRT invariants can be obtained

$$\tau_N(M) = \sum_{\lambda} S_{0\lambda_1} \cdots S_{0\lambda_n} \overline{J}_{\lambda_1 \cdots \lambda_n}(L)$$

Lawrence-Zagier

- Connection of SU(2) WRT invariants to modular form

$M = \Sigma(2,3,5)$: Poincare homology sphere

$\tau_N(M)$: Limiting values $\tau \to 1/N$ of Eichler integral of weight-3/2 vector-valued modular form $\Phi(\tau)$

Open problem

Higher rank WRT invariants and their relation to modular forms

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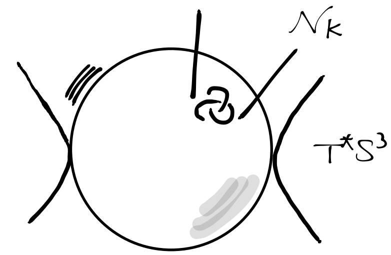
String theory

- U(N) Chern-Simons theory is realized in topological A-model
- A-model can be embedded in M-theory **Witten**

spacetime	$\mathbb{R} \times TN_4 \times T^*S^3$
N M5	$\mathbb{R} \times D \times S^3$
M5	$\mathbb{R} \times D \times N_K$

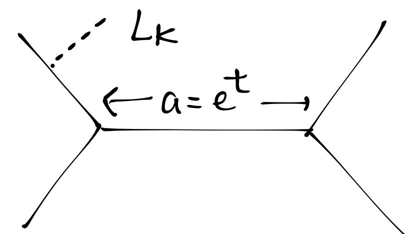


Gopakumar-Vafa

- At large N, there is geometric transition from deformed to resolved conifold **Ooguri-Vafa**

$$T^*S^3 \longrightarrow X = \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{CP}^1$$

spacetime	$\mathbb{R} \times TN_4 \times X$
M5	$\mathbb{R} \times D \times L_K$



- BPS states associated to M2-M5 bound states can be identified with knot homology $\mathcal{H}_{\text{knot}} \cong \mathcal{H}_{\text{BPS}}$ **Gukov-Schwarz-Vafa**

String theory

- The large N duality can be seen at the level of partition functions

$$\sum_{\lambda} \overline{P}_{\lambda}(K;a,q)s_{\lambda}(x) = \exp\left(\sum_{d=1}\sum_{\mu}\frac{1}{d}f_{\mu}(K;a^d,q^d)x^{d\mu}\right)$$

where $f_{\mu}(K,a,q)$ contains information of BPS degeneracies

$$f_{\mu}(K,a,q) = \sum_{Q,s}\frac{N_{\mu,Q,s}(K)a^Q q^s}{q-q^{-1}}$$

the parameters are identified by $q=e^{-g_s},\ a=e^{-t}$

- Since $N_{\mu,Q,s}(K)$ counts the number of BPS degeneracies, they should be integers.

Labastida-Marino

- This can be written in infinite product formula

$$\sum_{\lambda} \overline{P}_{\lambda}(K;a,q)s_{\lambda}(x) = \prod_{\mu}\prod_{i,j}\prod_{k=0}^{\infty}(1-a^i q^{j+k}x^{\mu})^{N_{\mu,i,j}(K)}$$

String theory

Witten

- The M2-brane realization of knot in topological strings

spacetime	$\mathbb{R} \times TN_4 \times T^*S^3$
N M5	$\mathbb{R} \times \; D \; \times \; S^3$
M2	$\mathbb{R} \times \; \{0\} \; \times \; \Sigma_K$

- After geometric transition, we are left with

spacetime	$\mathbb{R} \times TN_4 \times X$
M2	$\mathbb{R} \times \; \{0\} \; \times \Sigma_K$

Open problem

- M2-brane (D2-brane) configurations of representation R
- More precise formulation in this setting and understand differences between M5-branes and M2-branes

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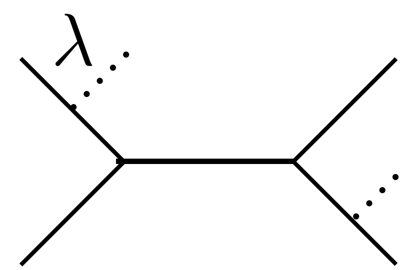
Refined Chern-Simons theory

- From perspective of refined topological string, the refined modular S, T matrices are defined **Aganagic-Shakirov**

$$T_{\lambda\mu} = \frac{q_1^{\frac{1}{2}||\lambda||^2}}{q_2^{\frac{1}{2}||\lambda^t||^2}} \frac{q_2^{\frac{N}{2}|\lambda|}}{q_1^{\frac{1}{2N}|\lambda|^2}} \cdot \delta_{\lambda\mu} \,,$$

$$S_{\lambda\mu} = M_\lambda(q_1^\rho) M_\mu(q_1^\rho q_2^\lambda)$$



- Using refined modular matrices, one can evaluate torus knot invariants that

- This is equivalent to Poincare polynomial of HOMFLY homology for torus knots/links

Geometric engineering

Gorsky-Negut

- By geometric engineering, this is equivalent to 5d U(1) Nekrasov partition function with surface operator associated to a knot K

$$\overline{\mathcal{P}}(T_{n,m}:a,q,t)$$
$$=\sum_{\mu\vdash n}\frac{\widetilde{\gamma}^n}{\widetilde{g}_\mu}\sum_{\text{of shape }\mu}^{\text{SYT}}\frac{\prod_{i=1}^n \chi_i^{S_{m/n}(i)}(1-a\chi_i)(q\chi_i-t)}{\left(1-\frac{q\chi_2}{t\chi_1}\right)\dots\left(1-\frac{q\chi_n}{t\chi_{n-1}}\right)}\prod_{1\le i<j\le n}\frac{(\chi_j-q\chi_i)(t\chi_j-\chi_i)}{(\chi_j-\chi_i)(t\chi_j-q\chi_i)}$$

where χ_i denotes the q,t-content of the box labeled by i in the tableau, and

$$S_{m/n}(i)=\left\lfloor\frac{im}{n}\right\rfloor-\left\lfloor\frac{(i-1)m}{n}\right\rfloor, \qquad \widetilde{\gamma}=\frac{(t-1)(q-1)}{q(t-q)}$$
$$\widetilde{g}_\mu=\prod_{\square\in\lambda}(1-q^{a(\square)}t^{l(\square)+1})\prod_{\square\in\lambda}(1-q^{-a(\square)-1}t^{-l(\square)})$$

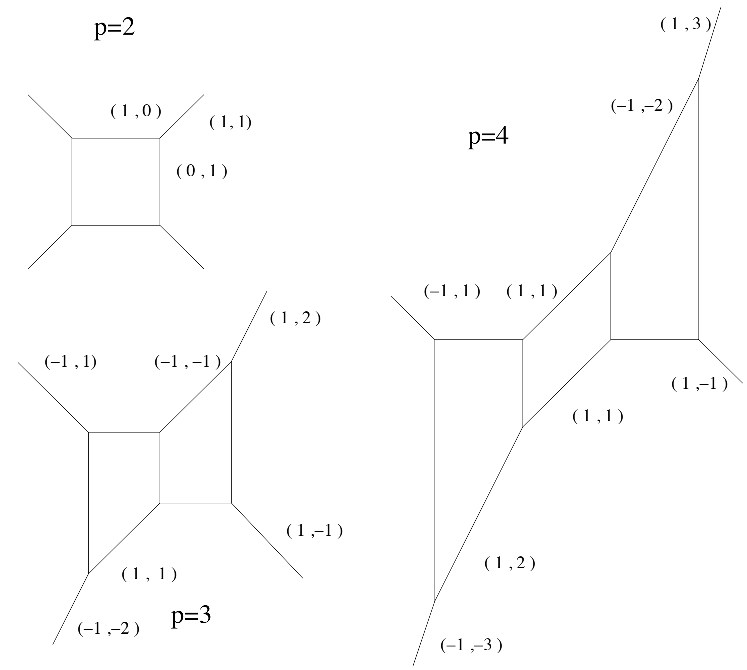
- the expression admits the following integral representation

$$\overline{\mathcal{P}}(T_{n,m}:a,q,t)=\int\frac{\prod_{i=1}^n z_i^{S_{m/n}(i)}\cdot\frac{1-az_i}{z_i-1}}{\left(1-qt\frac{z_2}{z_1}\right)\dots\left(1-qt\frac{z_n}{z_{n-1}}\right)}\prod_{1\le i<j\le n}\omega\left(\frac{z_i}{z_j}\right)\frac{dz_1}{2\pi iz_1}\dots\frac{dz_n}{2\pi iz_n}$$

geometric transition

Chern-Simon theory on
Lens space $S^3/\mathbb{Z}_p$

$\longleftrightarrow$ topological string
on toric CY



Open problem

- Evaluate (refined) invariants of an unknot in Lens space by using (refined) topological vertex
- Perform SL(2,Z) transformation and obtain invariants of torus knots

knot homology

- Categorifications of quantum knot invariants lead to doubly-graded homology **Khovanov, Khovanov-Rozansky, Webster, ...**

$$J^{\mathfrak{g}}_{\lambda}(K;q) = \sum_{i,j} (-1)^j q^i \ \dim \mathcal{H}^{\mathfrak{g}}_{\lambda}(K)_{i,j}$$

- Poincare polynomials are more powerful than quantum invariants. Even uncolored sl(2) homology distinguish some mutant pair

$$\mathcal{P}^{\mathfrak{g}}_{\lambda}(K;q,t) = \sum_{i,j} q^i t^j \ \dim \mathcal{H}^{\mathfrak{g}}_{\lambda}(K)_{i,j}$$

- There is a triply-graded homology behind HOMFLY polynomials **Khovanov-Rozansky**

$$P_{\lambda}(K;a,q) = \sum_{i,j,k} (-1)^k a^i q^j \ \dim \mathcal{H}_{\lambda}(K)_{i,j,k}$$

knot homology

HOMFLY homology

$$\mathcal{P}_\lambda(K;a,q,t) = \sum_{i,j,k} a^i q^j t^k \ \dim \mathcal{H}_\lambda(K)_{i,j,k}$$

$t=-1$

HOMFLY polynomial

$$P_\lambda(K;a,q)$$

sl(N) homology

$$J_\lambda^{\mathfrak{sl}(N)}(K;q,t) = \sum_{i,j,k} q^i t^j \ \dim \mathcal{H}_\lambda^{\mathfrak{sl}(N)}(K)_{i,j}$$

$a=q^N$

$t=-1$

sl(N) quantum invariant

$$J_\lambda^{\mathfrak{sl}(N)}(K;q)$$

knot homology

HOMFLY homology

$$\mathcal{P}_\lambda(K;a,q,t)=\sum_{i,j,k} a^i q^j t^k \ \dim \mathcal{H}_\lambda(K)_{i,j,k}$$

$t=-1$

d_N differential

HOMFLY polynomial

$$P_\lambda(K;a,q)$$

sl(N) homology

$$J_\lambda^{\mathfrak{sl}(N)}(K;q,t)=\sum_{i,j,k} q^i t^j \ \dim \mathcal{H}_\lambda^{\mathfrak{sl}(N)}(K)_{i,j}$$

$a=q^N$

$t=-1$

sl(N) quantum invariant

$$J_\lambda^{\mathfrak{sl}(N)}(K;q)$$

knot homology

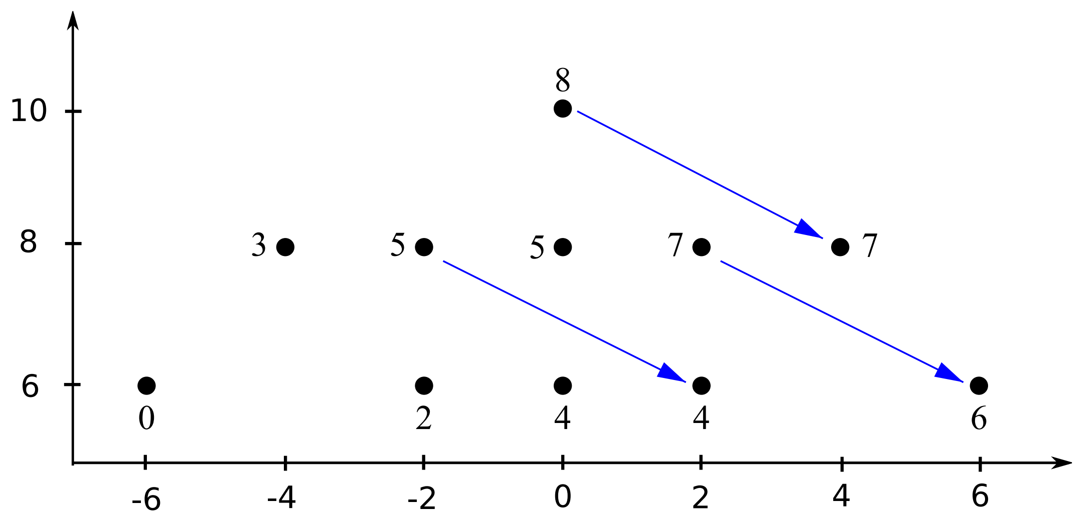
- homology w.r.t. d_N differentail is isomorphic to sl(N) homology

$$\deg\, d_N = (-2, 2N, -1)$$

For example,

Dunfield-Gukov-Rassmusen

$$\begin{aligned}\mathcal{P}(T_{3,4}) =& a^{10}t^8 + a^8(q^{-4}t^3 + q^{-2}t^5 + t^5 + q^2t^7 + q^4t^7) \\ &+ a^6(q^{-6}t^0 + q^{-2}t^2 + t^4 + q^2t^4 + q^6t^6)\end{aligned}$$



- taking homology with respect to d_2 kills six generators

$$\mathcal{P}^{\mathfrak{sl}(2)}(T_{3,4}) = q^6t^0 + q^{10}t^2 + q^{12}t^3 + q^{12}t^4 + q^{16}t^5$$

Properties of HOMFLY homology

Gukov-Stosic

Gorsky-Gukov-Stosic

- sl(N) differential

$$H_*(\mathcal{H}_\lambda, d_N) \cong \mathcal{H}^{\mathfrak{sl}(N)}$$

- colored differential $d_{[r]\to[k]}$

$$H_*(\mathcal{H}_{[r]}, d_{[r]\to[k]}) \cong \mathcal{H}_{[k]}$$

- mirror symmetry

$$(\mathcal{H}_\lambda)_{i,j,k} \cong (\mathcal{H}_{\lambda^T})_{i,-j,*}$$

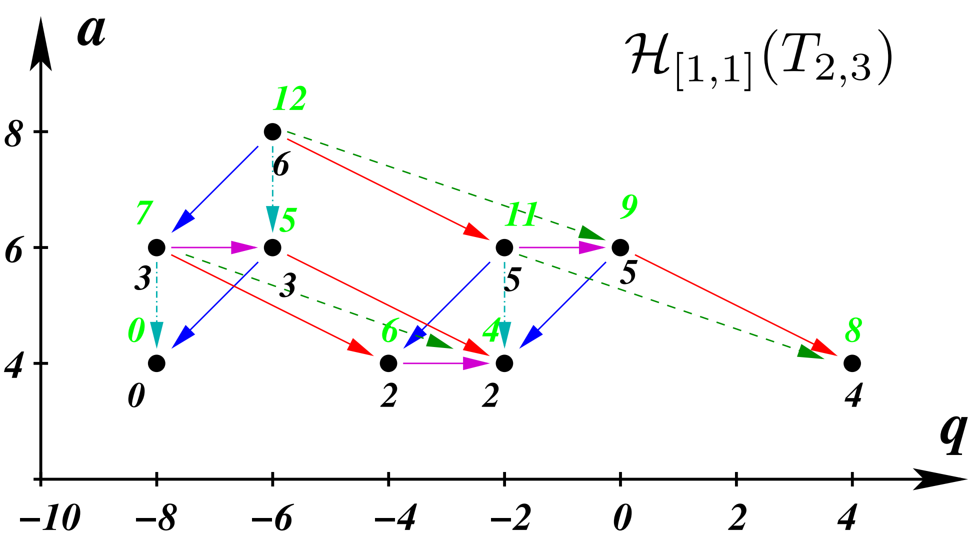
- exponential growth property

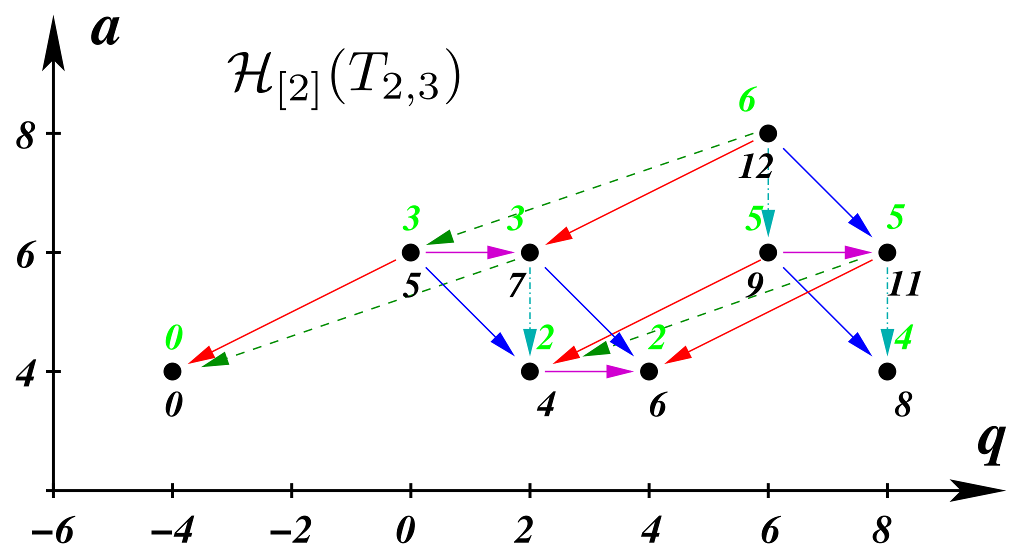
$$\dim \mathcal{H}_{[r]} = [\dim \mathcal{H}_{[1]}]^r$$

moreover, we have

$$\mathcal{P}_{[r]}(K; a, q=1, t) = [\mathcal{P}_{[1]}(K; a, q=1, t)]^r$$

Properties of HOMFLY homology

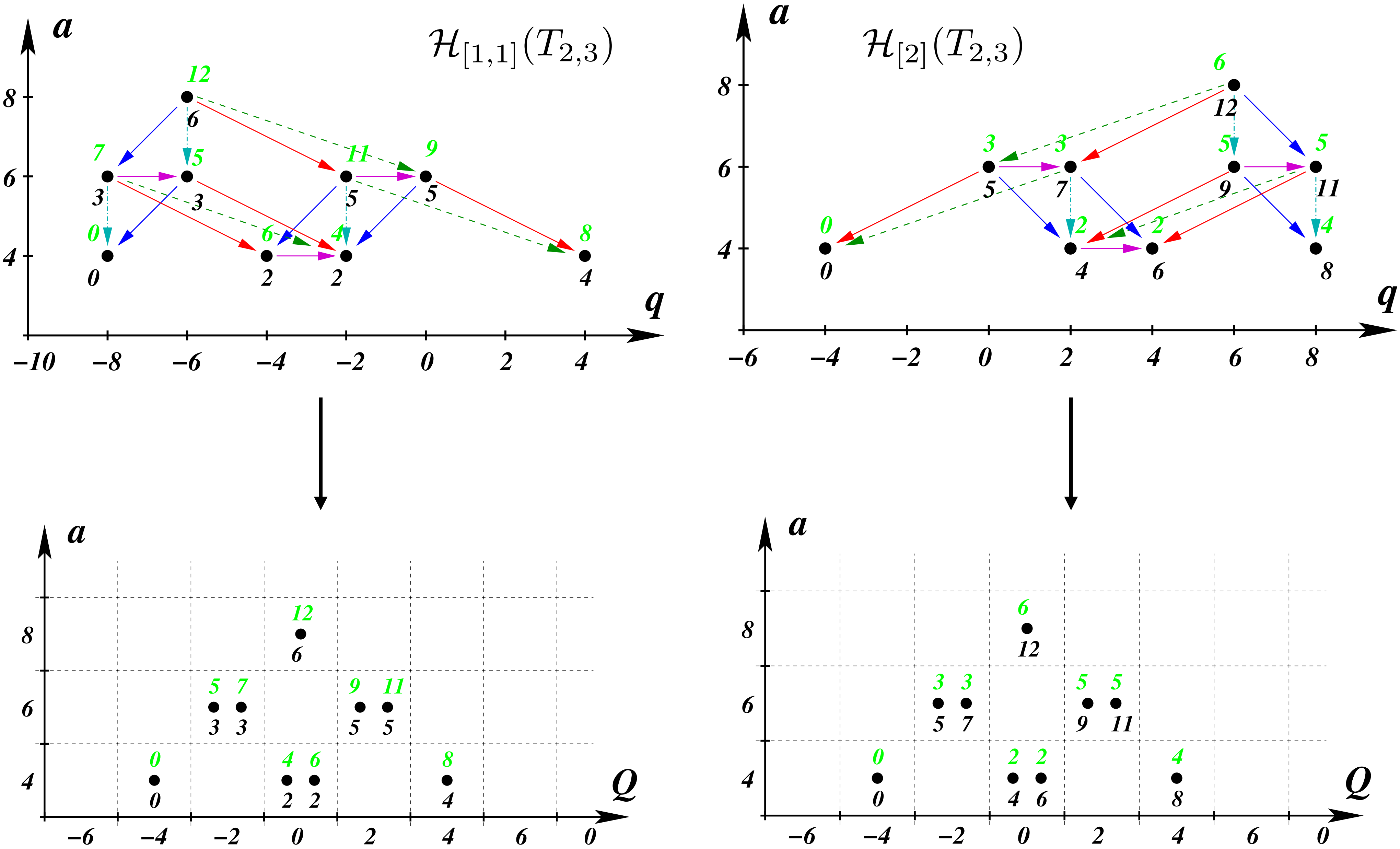




- There are two-ways to assign homological gradings
- Using these two gradings, we can introduce auxiliary Q-grading

$$Q = q + t_r - t_c$$

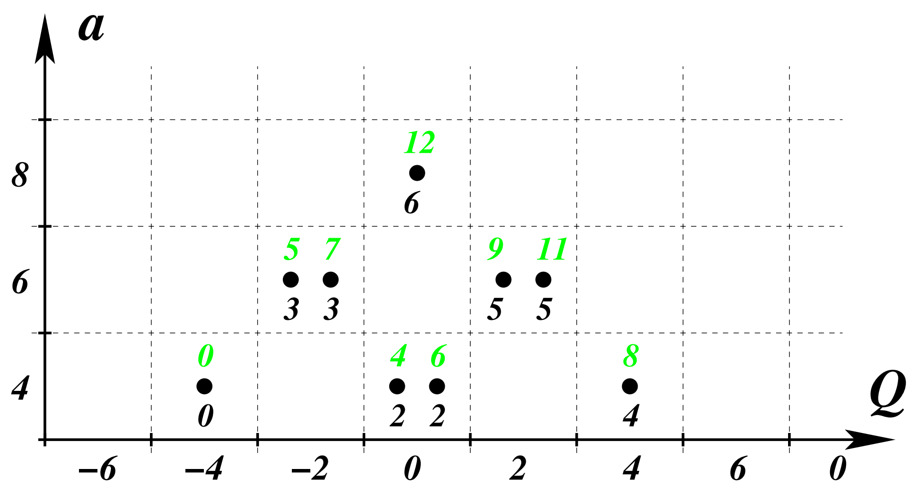
Properties of HOMFLY homology

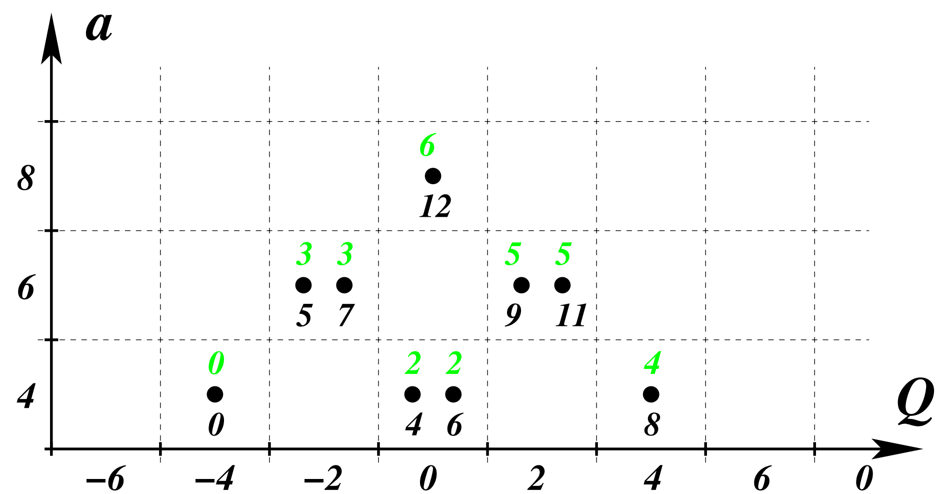


Properties of HOMFLY homology

- With Q-grading, the properties of HOMFLY homology become more transparent
  - Self-symmetry $(\widetilde{\mathcal{H}}_{[r]})_{i,j,k,\ell} \cong (\widetilde{\mathcal{H}}_{[r]})_{i,-j,k-j,\ell-rj}$
  - refined exp growth prop

$$\widetilde{\mathcal{P}}_{[r]}(K;a,Q,t_r,t_c=1) = [\widetilde{\mathcal{P}}_{[1]}(K;a,Q,t_r,t_c=1)]^r$$





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$$\widetilde{\mathcal{P}}_{[r]}(K;a,Q,t_r,t_c=1) = [\widetilde{\mathcal{P}}_{[1]}(K;a,Q,t_r,t_c=1)]^r$$

Open problem

- Understand physics of two homological gradings as well as auxiliary Q-grading
- properties of HOMFLY homology colored by hook-like Young diagrams

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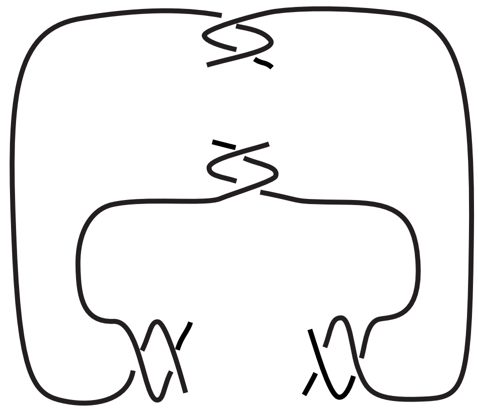
Examples



(2s-1,1,2t-1)-pretzel knot

$$\widetilde{\mathcal{P}}_{[r]}(K_{s,t};a,Q,t_r,t_c) = (-t_r t_c)^{-r(\mathrm{sgn}(s)+\mathrm{sgn}(t))/2} \sum_{i=0}^{r} \varpi_{s,i}(a,t_r,t_c)\varpi_{t,i}(a,t_r,t_c)$$
$$\times (-a^2 t_c^{2(r-1)})^{-i}(-a^2Q^{-2}t_r t_c;t_c^2)_i(-a^2Q^2t_r^3t_c^{1+2r};t_c^2)_i \begin{bmatrix} r \\ i \end{bmatrix}_{t_c^2}$$

where

$$\varpi_{p,i}(a,t_r,t_c) = \quad a^i t_c^{i(i-1)/2} \sum_{j=0}^{i} (-1)^j (at_r t_c)^{2pj} t_c^{(2p+1)j(j-1)} \frac{1-a^2t_r^2t_c^{4j}}{(a^2t_r^2t_c^{2j};t_c^2)_{i+1}} \begin{bmatrix} i \\ j \end{bmatrix}_{t_c^2}$$

Volume Conjecture

- Volume conjecture relates "quantum" knot invariants and "classical" geometry of knot complement

Kashaev

Murakami,Murakami

$$\lim_{n\to\infty}\frac{2\pi}{n}\log\left|J_n(K;q=e^{\frac{2\pi i}{n}})\right| = \mathrm{Vol}(S^3\backslash K)$$

- One can take double scaling limit

$$J_n(K;q) \overset{\substack{r\to\infty \\ \hbar\to 0}}{\sim} \exp\left(\frac{1}{\hbar}S_{-1}(x) - \frac{3}{2}\log\hbar + \sum_k \hbar^k S_k(x)\right) \qquad x = q^n : \text{fixed}$$

the leading term is given by

Gukov

Gukov-Murakami

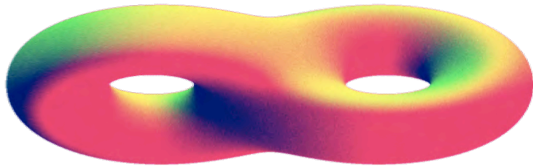
$$S_{-1}(x) = \int_{A(K;x,y(x))=0} \log y \frac{dx}{x}$$

The integral is over an algebraic curve determined by "A-polynomial" that describe moduli space of SL(2,C) flat connections

$$\mathcal{M}_{\text{flat}}(T^2, SL(2,\mathbb{C}) = \frac{\mathbb{C}^\times \times \mathbb{C}^\times}{\mathbb{Z}_2} \supset \{A(x.y)=0\} = \mathcal{M}_{\text{flat}}(S^3\backslash K, SL(2,\mathbb{C})$$

Quantum curves

Gukov
Garoufalidis



$$\xrightarrow{\text{quantization}}$$
$$\hat{x}\hat{y} - q\hat{y}\hat{x} = 0$$

$$A(K; x, y) \longrightarrow \widehat{A}(K; \hat{x}, \hat{y}, q)$$

- non-commutative deformation of Darboux coordinates leads to operator that annihilates Jones polynomial

$$\widehat{A}(K; \hat{x}, \hat{y}, q) J_n(K; q) = 0$$

$$\hat{x} J_n(K; q) = q^n J_n(K; q)$$
$$\hat{y} J_n(K; q) = J_{n+1}(K; q)$$

- Recursion relations of colored Jones polynomials amounts to quantum A-polynomial

knot homology

HOMFLY homology

$$\mathcal{P}_\lambda(K;a,q,t) = \sum_{i,j,k} a^i q^j t^k \ \dim \mathcal{H}_\lambda(K)_{i,j,k}$$

$t=-1$

d_N differential

HOMFLY polynomial

$$P_\lambda(K;a,q)$$

sl(N) homology

$$J_\lambda^{\mathfrak{sl}(N)}(K;q,t) = \sum_{i,j,k} q^i t^j \ \dim \mathcal{H}_\lambda^{\mathfrak{sl}(N)}(K)_{i,j}$$

$a=q^N$

$t=-1$

sl(N) quantum invariant

$$J_\lambda^{\mathfrak{sl}(N)}(K;q)$$

Deformation of A-polynomials

Fuji-Gukov-Sulkowski

$$
\begin{tikzcd}[column sep=small]
 & \begin{array}{c}\text{super-A-polynomial}\\ A^{\rm super}(K;x,y,a,t)\end{array} \arrow[dl, "t=-1"'] \arrow[dr, "a=1"] & \\
\begin{array}{c}\text{Q-deformed A-polynomial}\\ A^{\text{Q-def}}(K;x,y,a)\end{array} \arrow[dr, "a=1"'] & & \begin{array}{c}\text{refined A-polynomial}\\ A^{\rm ref}(K;x,y,t)\end{array} \arrow[dl, "t=-1"] \\
 & \begin{array}{c}\text{A-polynomial}\\ A(K;x,y)\end{array} &
\end{tikzcd}
$$

Deformation of A-polynomials

Q-deformed A-polynomial

$$A^{\text{Q-def}}(K; x, y, a)$$

Aganagic-Vafa

Agangic-Ekholm-Ng-Vafa

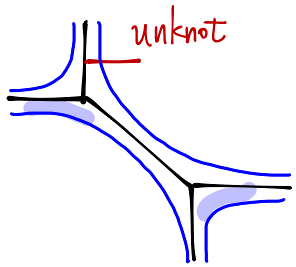
= Augmentation polynomial of knot contact homology

Mirror symmetry

- Mirror symmetry is duality b/w A-model and B-model

Kahler moduli in A-model is complex structure moduli in B-model

Aganagic Vafa



$$\xleftrightarrow{\text{mirror}} \quad wZ = 1 - xa^2 - ay + axy$$

- Given a knot K, there is a corresponding mirror geometry

$$wz = H_K(x, y, a)$$

Brini-Eynard-Marino

Mirror geometry of torus knot brane is described by SL(2,Z) transformation of unknot curve

Aganagic-Vafa

Mirror geometry for a knot K is described by Q-deformed A-polynomial

A-polynomial and topological recursion

$$J_n(K;q) \overset{r\to\infty}{\underset{\hbar\to 0}{\sim}} \exp\left(\frac{1}{\hbar}S_{-1}(x) - \frac{3}{2}\log\hbar + \sum_k \hbar^k S_k(x)\right) \qquad x = q^n : \text{fixed}$$

- Subleading terms can be obtained by applying topological recursion to character variety

- Need non-perturbative corrections with theta-functions

- Asymptotic expansion of HOMFLY polynomials colored by symmetric rep can be obtained by applying topological recursion to Q-deformed A-polynomial

Gu-Jockers-Klemm-Soroush

- Modified Kernel is constructed by Brini-Eynard-Marino curve

A-polynomial and topological recursion

- Asymptotic expansion of HOMFLY polynomials colored by symmetric rep can be obtained by applying topological recursion to Q-deformed A-polynomial

Gu-Jockers-Klemm-Soroush

- Modified kernel is constructed by Brini-Eynard-Marino curve

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Open problem

- Find a way to construct modified kernel for non-torus knot
- Matrix model construction of colored HOMFLY polynomials for non-torus knots
- Find spectral curve for torus knot in Lens space

A-polynomial and BPS states

Garoufalidis-Kucharski-Sulkowski
to appear

$$\sum_{\lambda} \overline{P}_{\lambda}(K;a,q)s_{\lambda}(x) = \prod_{\mu}\prod_{i,j}\prod_{k=0}^{\infty}(1-a^i q^{j+k}x^{\mu})^{N_{\mu,i,j}(K)}$$

- Using infinite product formula, a certain summation of BPS states can be expressed i.t.o. Moebius function

$$b_{r,\max}(T_{2,3}) \equiv \sum_{j\in\mathbb{Z}} N_{[r],\max,j}(T_{2,3}) = -\frac{1}{r^2}\sum_{k|r}\mu\left(\frac{r}{k}\right)\frac{(-1)^k}{4}\begin{pmatrix} 4k \\ k \end{pmatrix}$$

- One observation

$$\lim_{r\to\infty}\frac{b_{r+1,\max}}{b_{r,\max}} = -\frac{256}{27}$$

On the other hand,

$$\mathrm{Root}\left\{\mathrm{Discr}\left[A^{\max}(x,y),y\right]\right\} = -\frac{27}{256}$$

There is a relation b/w BPS numbers and A-polynomials

$$\lim_{r\to\infty}\frac{b_{r+1,\max}}{b_{r,\max}} = \mathrm{Root}\left\{\mathrm{Discr}\left[A^{\max}(x,y),y\right]\right\}$$

Refined BPS states

$$\sum_{\lambda} m_{\lambda}\overline{\mathcal{P}}_{\lambda}(K;a,q,t)M_{\lambda}(x;q,t)=\prod_{\mu}\prod_{i,j,k}\prod_{\ell=0}^{\infty}(1-a^{i}q^{j+\ell}t^{k}x^{\mu})^{N_{\mu,i,j,k}(K)} \qquad ???$$

Open problem

- Formulate open refined BPS states
- Define refined LMOV invariants for torus knots or any knots

Refined BPS states

$$\sum_\lambda m_\lambda \overline{\mathcal{P}}_\lambda(K;a,q,t) M_\lambda(x;q,t) = \prod_\mu \prod_{i,j,k} \prod_{\ell=0}^{\infty} (1 - a^i q^{j+\ell} t^k x^\mu)^{N_{\mu,i,j,k}(K)} \qquad ???$$

Open problem

- Formulate open refined BPS states
- Define refined LMOV invariants for torus knots or any knots

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I will be at Bonn in March and April,
so let's discuss some of open problems I listed in this talk!