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硕 士 学 位 论 文

（学术学位）

杨图计算带物质的瞬子配分函数

Instantons partition function with matter on Young diagrams

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中文摘要

配分函数在物理中扮演这至关重要的角色，无论是在统计力学亦或是在量子场论中。有了配分函数，我们就能知道真空的构成，从而计算出各种我们感兴趣的物理量，譬如关联函数，可观测量的平均值等。而在量子场论中，想要精确计算配分函数以及它的非微扰贡献意味着我们需要在给定的底流形上计算无穷维的路径积分，这一般来讲是不可能的。但是幸运的是，近几十年理论物理的发展，特别是超对称理论的发展，让计算某一些情况下配分函数的精确值成为了可能。并且通过这些精确解，我们也能测试更多量子场论的性质和预测。

在量子场论中，我们知道对于渐近自由的杨米尔斯理论，在低能有效的极限下作用量会变得越来越复杂，耦合常数变得越来越大，因此我们在计算过程中必须考虑微扰和非微扰的贡献，也就是需要 Wilsonian 有效作用量。这其中跟量子有效作用量不同的是，Wilsonian 依赖于系统的红外截止能量。

在三十年前，Seiberg 和 Witten 在四维 $\mathcal{N}=2$ 超对称杨米尔斯理论具有启发性的研究工作中[1–2]告诉我们该理论的低能有效拉格朗日量（Wilsonian）可以由在奇点附近的全纯性来大体决定。例如在 $SU(N)$ 中，经典势能是在希格斯场 $\boldsymbol{H}$ 属于 $SU(N)$ 的 Cartan 子群时为零，通过这个条件，我们能够得出低能有效作用量为

$$S_{\text{eff}}=\frac{1}{4\pi}\,\text{Im}\left\{\frac{1}{2\pi i}\int d^4x\,d^4\theta\,\mathcal{F}(\Phi)\right\}$$

而这里的 $\mathcal{F}$ 就是前势（prepotential）。并且在积掉所有有质量的场后，在低能有效近似下这个前势可以写成[3]

$$\mathcal{F}=\mathcal{F}_{\text{class}}+\mathcal{F}_{\text{1-loop}}+\mathcal{F}_{\text{inst}}$$

而其中瞬子贡献 $\mathcal{F}_{\text{inst}}$ 是非微扰的，它可以写成不同数目瞬子的前势的和

$$\mathcal{F}_{\text{inst}}(a,\Lambda)=\sum_{k=1}^{\infty}\mathcal{F}_k(a)\Lambda^{k\beta}$$

这里 Λ 是系统产生的能标，$a=(a_1,a_2,...)$ 是向量多重态。

但是当 $k>2$ 时，$\mathcal{F}_k$ 便非常难以计算。而这一问题在二十年前被 Nekrasov 结合了一些局域化技巧解决了[4]。Nekrasov 使用了等变上同调积分以及 Ω 背景，

得出了配分函数可以写成

$$Z(a,m,\Lambda,\epsilon)=\exp\left[\frac{1}{\epsilon_1\epsilon_2}\mathscr{F}_{\text{inst}}(a,m,\Lambda,\epsilon)\right]$$

其中 $\epsilon_{1,2}$ 时 Ω 背景中的参数，可以看成两个复平面绕原点旋转的角度。而 $\mathscr{F}_{\text{inst}}$ 在 $\epsilon_{1,2}\to 0$ 时是解析的。并且通过等变作用的不变点，Nekrasov 给出了四维 $\mathcal{N}=2$ 的 A 型规范群的精确积分形式，并且观察到这些不变点可以被归类为杨图的形式。例如在在非精细极限 $\epsilon_1=-\epsilon_2=\hbar$ 下，$SU(N)$ 的杨图求和形式是

$$Z=\sum_{\vec{\lambda}}\mathfrak{q}^{|\vec{\lambda}|}\prod_{(l,i)\neq(n,j)}\frac{a_{ln}+\hbar(\lambda_{l,i}-\lambda_{n,j}+j-i)}{a_{ln}+\hbar(j-i)}$$

其中 $a_{li}=a_l-a_i$ 是库仑支参数，而 $\vec{\lambda}=(\lambda_1,...,\lambda_N)$ 为 N 个杨图，每一个杨图遵循 $\lambda_{l,1}\geq\lambda_{l,2}...$，他们的总格子数需要为瞬子数 k，也就是 $|\vec{\lambda}|=\sum_{l,i}\lambda_{l,i}=k$。我们至今都不是很明朗为什么不变点可以被杨图归类。这一研究对物理和数学影响深远，例如，我们可以用杨图归类求和来连接关于拓扑顶点的计算[5–6]，在 AGT 关系中也是极为重要的[7]。

直到近几年，人们才开始研究关于 A 型规范群以外的理论，并且开始研究如何用杨图分类的方式来写下它们的精确形式。譬如两年前，Nawata-Zhu 找到了五维下 BCD 型规范群纯超杨米尔斯理论的配分函数的精确形式[8]。我们需要运用 ADHM 构造，Haar 测度等等数学工具来构造 BCD 型瞬子模空间和局域化积分[9–10]。

在去年，有关研究被进一步拓展，带有一个物质的配分函数的杨图求和形式被找到[11]。有意思的是，这个物质的质量 m 在某些情况下和非精细化的参数 $\hbar$ 有这相似的定位，这使得本来可以用普通杨图归类的极点现在需要多两个维度来归类，也就是说，我们需要引入四维杨图的概念。例如在五维的 $SO(n)$ $\mathcal{N}=1^*$ 理论中，hypermultiplet 在反对称表示下的杨图求和形式为

$$\begin{aligned}&\widetilde{Z}_k(\vec{A};q)\\&=\sum_{\vec{\lambda}}C^{\text{4d}}_{\vec{\lambda},\vec{A}}\left(\frac{\text{sh}(m\pm\hbar)\ \text{sh}^2(0)}{\text{sh}^2(\hbar)\ \text{sh}^2(m)}\right)^k\\&\prod_{s=1}^{\|\vec{\lambda}\|}\prod_{x\in\lambda^{(s)}}\frac{\text{sh}^4(2\phi_s(x))\ \text{sh}^2(2\phi_s(x)+m-\hbar)\ \text{sh}^2(2\phi_s(x)-m+\hbar)\prod_{t=1}^{N}\text{sh}(\phi_s(x)\pm a_t\pm m)}{\prod_{t=1}^{\|\vec{\lambda}\|}\text{sh}^2(\phi_s(x)\pm a_t)}\\&\prod_{s\leq t}^{\|\vec{\lambda}\|}\prod_{\substack{x\in\lambda^{(s)},\,y\in\lambda^{(t)}\\x<y}}\frac{\text{sh}^4(\phi_s(x)\pm\phi_t(y))\ \text{sh}(\phi_s(x)\pm\phi_t(y)\pm m\pm\hbar)}{\text{sh}^2(\phi_s(x)\pm\phi_t(y)\pm\hbar)\ \text{sh}^2(\phi_s(x)\pm\phi_t(y)\pm m)}\end{aligned}$$

其中 $C^{\text{4d}}_{\vec{\lambda},\vec{A}}$ 是一些常数，它们对应每个不一样的杨图有不一样的值，但是它们的通式我们还不能找到。而 $\phi_s(x)=a_s+(x_1-x_2)\hbar+(x_3-x_4)m$，$x=(x_1,x_2,x_3,x_4)$ 是每个杨图每一格的坐标。我们也能通过这些精确形式检验许多由模理论构造的以及由李代数表示推导出来的关系式，例如对于 $SO(8)$，我们能够用我们的公

式作适当的变量代换后（譬如 $a_1 \to (a_1 + a_2 + a_3 + a_4)/2$），检验它的三重对称性，也就是 $SO(8)$ 的基本表示，旋量表示，共轭旋量表示是同构的。

这篇论文主要介绍了用杨图分类极点的方法计算五维 $\mathcal{N} = 1$ 超杨米尔斯理论的瞬子配分函数。主要分为两个部分：第一部分是第二到第六章，介绍了相关配分函数的理论基础和构造方法；第二部分是第七章，汇总了我们对瞬子配分函数在非精细化下的最新计算进展[11]。

以下是本篇论文的详细构成：

第一章简要介绍了顺子配分函数计算的发展历程和本文章的构成。

第二章介绍了四维 $\mathcal{N} = 2$ 的超杨米尔斯规范场的构成。我们首先简要复习了一下超对称代数的定义，其中最重要的是对超荷的定义。多少个超荷直接决定了内禀对称性的李群是什么。譬如在四维情况下 $\mathcal{N} = 2$ 超对称的八个超荷带来了 $SU(2)$ 的内禀对称，这也为接下来和拓扑扭曲提供了可能。接着我们介绍了超对称的表示，也就是各种超场，包括 $\mathcal{N} = 1$ 的手性超场，向量超场以及 $\mathcal{N} = 2$ 的手性超场，超多重态。然后我们用这些场构造了超杨米尔斯作用量，用 $\mathcal{N} = 2$ 的手性超场 Ψ 可以写成一个简单的形式

$$S_{\text{YM}} = \frac{1}{4\pi h^\vee} \operatorname{Im}\left(\int d^4x d^4\theta \frac{\tau}{2} \operatorname{Tr} \Psi^2\right)$$

而且在其中我们也看到了一个只关乎于底流形拓扑性质的项，我们把它记作 S_{top}，这一项在积分后会得出一个重要的拓扑不变量，我们称之为瞬子数 k

$$k = -\frac{1}{32\pi^2 h^\vee} \int d^4x \operatorname{Tr} F_{\mu\nu} \star F^{\mu\nu}$$

最后我们介绍了拓扑扭曲，也就是把左手自旋指标和内禀对称性混合在一起当成类似洛伦兹变换的指标。这样做的好处是超杨米尔斯作用量可以写成 BRST 恰当形式，从而让积分局限在瞬子模空间 $\mathscr{M}^G$ 上。

第三章我们简要介绍了如何用 ADHM 构造不同规范群的瞬子模空间。首先对于不同的规范群 G 我们可以定义相对应的向量空间 $\mathscr{V}$, 例如对于 A 型李群 $U(N)$, 我们可以定义 k 维的向量空间以及它的对偶群 $U(k)$ 和对偶向量空间 $\mathscr{W} = \mathbb{R}^N$。接着，我们定义向量空间之间的四个映射 (I, J, B_1, B_2)，只要它们满足 ADHM 方程组，那它们模掉对偶群，就是我们所要的瞬子模空间 $\mathscr{M}^G$。例如在 $U(N)$ 的情况下，ADHM 方程组是

$$\mu_{\mathbb{R}} = -\mu^3 = II^\dagger - J^\dagger J + [B_1, B_1^\dagger] + [B_2, B_2^\dagger] = 0$$
$$\mu_{\mathbb{C}} = \frac{1}{2}(\mu^1 - i\mu^3) = \mu^- = IJ + [B_1, B_2] = 0$$

则 $\mathscr{M}_k^G = \mu^{-1}(0)/G_D$ 就是我们想要的瞬子模空间，其中 G_D 是 G 的对偶群，k 是瞬子数。

在第四章中，我们简要介绍了从超杨米尔斯作用量到 Nekrasov 瞬子配分函数所需的数学工具以及简要推导。首先我们介绍了等变积分的概念，从而可以把路径积分从原本在 $\mathscr{M}_k^G$ 流形变回对所有映射坐标 (I, J, B_1, B_2) 的积分。然后我们介绍了等变欧拉类和 Duistermaat-Heckman 公式，有了它们，我们就可以把流形上无穷维的积分局限在有限维的留数积分上面。其中 Duistermaat-Heckman 公式是

$$\int_M \frac{\omega^n}{n!} e^{-\langle \mu, \xi \rangle} = \sum_{x_f} \frac{e^{-\langle \mu(x_f), \xi \rangle}}{\prod_\alpha \omega_\alpha}$$

这里 $\omega_\alpha \equiv \langle \omega_\alpha(x_f), \xi \rangle$，而 x_f 是相应的群 G 作用在流形上的不变点。而等变欧拉类是

$$\mathrm{Eu}_g(E) = \frac{1}{(2\pi)^{\dim F/2}} \int_{\Pi F} \mathscr{D}\chi\, e^{\frac{1}{4}\chi R_{\mu\nu}\psi^\mu\psi^\nu\chi + \frac{1}{2}\chi\phi^a T_a^o \chi}$$

紧接着，我们用 Ω 背景，将原本在非紧致流形上的积分紧致化，从而使得这个积分有限。然后我们结合上述的数学工具，给出对于 k 瞬子的 Nekrasov 瞬子配分函数为

$$Z_k(a, m, \epsilon) = \int_{\mathscr{M}_k} \mathrm{Eu}_g(\mathscr{D}_k)$$

而作为补充，我们也介绍了万有丛和等变陈示性类定义，以及等变指标的计算，从而让我们能够计算带有物质的超杨米尔斯理论。

第五章我们运用前面章节的铺垫，构造 ABCD 型李群作为度规群的配分函数的有限维留数积分形式。首先对于给定的规范群 G，总群环面 $\mathbb{T}$ 是由味对称群环面 $\mathbb{T}_F$，度规群环面 $\mathbb{T}_G$，对偶群环面 $\mathbb{T}_D$ 和 Ω 背景产生的旋转 $\mathbb{T}_{\epsilon_1,\epsilon_2}$ 直积而来的。而我们可以清楚看到这个总环面 $\mathbb{T}$ 在流形 $\mathscr{X} = (I, J, B_1, B_2)$ 上的固定点只有原点。也就是说，我们可以先算出这个群的等变指标，再用 Duistermaat-Heckman 公式，将流形坐标在原点上的对应权重写出来，便可以得到瞬子配分函数。例如纯 A 型李群 $SU(N)$ 中，他的对应总群权重是

$$\begin{aligned} B_{s,ij} &: \quad \phi_i - \phi_j + \epsilon_s \\ I_{i\ell} &: \quad \phi_i - a_\ell - \epsilon_+ \\ J_{\ell i} &: \quad a_\ell - \phi_i - \epsilon_+ \\ \chi_{\mathbb{C},ij} &: \quad \phi_i - \phi_j - 2\epsilon_+ \end{aligned}$$

则最后它的有限维积分形式为

$$z_k^{\mathrm{adj,gauge}}(a, \phi; \epsilon) = \frac{1}{k!} \prod_{s=1}^{N} \prod_{i=1}^{k} \frac{1}{\left(\epsilon_+ \pm (\phi_i - a_s)\right)} \prod_{i \neq j}^{k} (\phi_i - \phi_j) \prod_{i,j}^{k} \frac{2\epsilon_+ - \phi_i + \phi_j}{\epsilon_{1,2} - \phi_i + \phi_j}$$

对于有物质的理论，我们可以运用等变指标计算。

第六章中，我们简要介绍了五膜网的基础定义，包括 D5 膜，NS5 膜，O5 平面等。也介绍了我们在第七章中所需的五维 $\mathcal{N}=1$ 带物质的超杨米尔斯理论的构造方法。从这些构造方法中我们可以看出各种理论之间的对偶关系，这些对偶关系也是我们在第七章中计算的检验标准方法之一。

最后第七章中，我们介绍了我们在瞬子配分函数计算的新进展。我们顺着 2021 年 Nawata-Zhu 的工作方向[8]，找到了五维 $\mathcal{N}=1$ 的 ABCD 型李群为度规群并且加上一个物质的瞬子配分函数的积分形式，以及他们的杨图求和形式。例如对于 $SU(N)$ 带上一个对称表示下 hypermultiplet 的有限维积分形式是

$$Z^{\mathrm{rep}}_{SU(N),k,\kappa}=\frac{1}{k!2^k}\oint_{\mathrm{JK}}\prod_{I=1}^{k}\frac{d\phi_I}{2\pi i}\cdot e^{\kappa\sum_{I=1}^{k}\phi_I}z^{\mathrm{vec}}_{SU(N),k}z^{\mathrm{fund}}_{SU(N),k}$$

其中

$$z^{\mathrm{vec}}_{SU(N),k}=\frac{\prod_{I\neq J}\mathrm{sh}(\phi_I-\phi_J)\cdot\prod_{I,J}\mathrm{sh}(2\epsilon_+-\phi_I+\phi_J)}{\prod_{I,J}\mathrm{sh}(\epsilon_{1,2}+\phi_I-\phi_J)\prod_{I=1}^{k}\prod_{s=1}^{N}\mathrm{sh}(\epsilon_+\pm(\phi_I-a_s))}$$

$$z^{\mathrm{sym}}_{SU(N),k}(m)=\prod_{I=1}^{k}\mathrm{sh}\left(2\phi_I+m\pm\epsilon_-\right)\prod_{s=1}^{N}\mathrm{sh}\left(\phi_I+a_s+m\right)\prod_{I<J}^{k}\frac{\mathrm{sh}\left(\phi_I+\phi_J+m\pm\epsilon_-\right)}{\mathrm{sh}\left(-\epsilon_+\pm(\phi_I+\phi_J+m)\right)}$$

相应的对杨图求和的形式为

$$\begin{aligned}&Z^{\mathrm{sym}}_{SU(N),k,\kappa}\\=&(-1)^{kN+\lceil\frac{k}{2}\rceil}\sum_{\vec{\lambda}}\prod_{s=1}^{N}\prod_{x\in\lambda^{(s)}}e^{\kappa\phi_s(x)}\frac{\mathrm{sh}(2\phi_s(x)+m\pm\hbar)\cdot\prod_{t=1}^{N}\mathrm{sh}(\phi_s(x)+a_t+m)}{\prod_{t=1}^{N}\mathrm{sh}^2(N_{s,t}(x))}\\&\times\prod_{s\le t}^{N}\prod_{\substack{x\in\lambda^{(s)},y\in\lambda^{(t)}\\x<y}}\frac{\mathrm{sh}(\phi_s(x)+\phi_t(y)+m\pm\hbar)}{\mathrm{sh}^2(\phi_s(x)+\phi_t(y)+m)}\end{aligned}$$

其中 $\vec{\lambda}=(\lambda^{(1)},\ldots,\lambda^{(N)})$ 代表 N 个杨图组成的杨图列，而每一个杨图列的格子总数是 k，也就是 $\sum_{s=1}^{N}|\lambda^{(s)}|=k$，然后我们需要把所有可能的杨图列求和。在这一章里，我们还会首次描述运用四维杨图来计算配分函数，例如在带有一个反对称表示的 htpermultiplet 的 $Sp(N)$ 规范理论，在非精细极限下就需要四维杨图求和，其中两个维度是来自 Ω 背景，另外两个来自物质。

关键字：瞬子配分函数；ADHM 构造；五维 N=1 规范理论；四维杨图；Jeffrey-Kirwan 留数；等变积分

中图分类号：O413.1

Abstract

This thesis introduces the Nekrasov partition function and computes instanton contributions. We first present the four-dimensional Super Yang-Mills theory, with and without matter, by reviewing supersymmetry algebra, superfields, and topological twists in $\mathcal{N} = 2$ supersymmetry theory. We then apply topological twists to localize the action on instanton moduli space $\mathscr{M}^G$ with gauge group G and construct instanton moduli spaces for ABCD-type gauge groups using ADHM construction.

Next, we employ equivariant integration, G-equivariant cohomology, Ω-background, and the Duistermaat-Heckman formula to localize the instanton partition function on the finite-dimensional path integral over $\mathscr{M}^G$. This reduces it to finite complex integrals that can theoretically be computed using Jeffrey-Kirwan residue (JK-residue), which is formally defined in this thesis.

We then review the general construction of the instanton partition function using the universal bundle $\mathscr{E}$ and briefly discuss Brane construction in $5d$ $\mathcal{N} = 1$ Super Yang-Mills gauge theory with $O5$, $O7$-planes, and D5, D7-branes.

Lastly, we extend the theory to five dimensions. We present instanton partition functions for various $5d$ gauge theories with matter beyond the fundamental representation as sums over $2d$ or $4d$ Young diagrams under the unrefined limit, relating to opposite parameters ϵ_1 and ϵ_2 in the Ω-background. Young diagrams classify poles in corresponding instanton partition functions, and by using explicit expressions, we verify identities predicted by Higgsing procedures of fivebrane web diagrams and representation theory.

Keywords: instanton partition function; ADHM construction; 5d N=1 gauge theory; 4d Young diagrams; Jeffrey-Kirwan residue; equivariant integral

CLC code: O413.1

Chapter 1
Introduction

Partition functions play a crucial role in physics, whether in statistical mechanics or quantum field theory. With the partition function, we can know the structure of the vacuum, and thus calculate various physical quantities that we are interested in, such as the correlation function, the average value of observables, etc. In quantum field theory, the exact calculation of the partition function and its non-perturbative contribution means that we need to calculate the infinite-dimensional path integral on the given base manifold, which is generally impossible. Fortunately, the development of theoretical physics in recent decades, especially the development of supersymmetry theory, has made it possible to calculate the exact value of the partition function in some cases. And with these exact solutions, we can also test many more properties and predictions of quantum field theory.

In quantum field theory, we know that for asymptotically free Yang-Mills theory, the action will become more and more dressed in the low-energy effective limit, and the coupling constant will become larger and larger, so we must consider the perturbative and non-perturbative contributions, that is, the Wilsonian effective action is required. Unlike the quantum effective action, the Wilsonian effective action depends on the infrared cutoff energy of the system.

Thirty years ago, Seiberg and Witten's seminal work on the four-dimensional $\mathcal{N} = 2$ supersymmetric Yang-Mills theory[1–2] told us that the theory's low energy effective Lagrangian (Wilsonian) can be roughly determined by the holomorphism near the singularity. For example, the classical potential energy is zero in $SU(N)$ when the Higgs field H belongs to the Cartan subgroup of $SU(N)$. Through this condition, we can obtain the low-energy effective action as

$$S_{\text{eff}} = \frac{1}{4\pi} \operatorname{Im} \left\{ \frac{1}{2\pi i} \int d^4x \, d^4\theta \, \mathscr{F}(\Phi) \right\} \tag{1.1}$$

And the $\mathscr{F}$ here is the prepotential. And after integrating all massive fields, the prepotential can be written[3] under the low-energy efficient approximation as

$$\mathscr{F} = \mathscr{F}_{\text{class}} + \mathscr{F}_{\text{1-loop}} + \mathscr{F}_{\text{inst}} \tag{1.2}$$

where the instanton contribution $\mathscr{F}_{\text{inst}}$ is non-perturbative, it can be written as the sum of the prepotentials with different numbers of instantons

$$\mathscr{F}_{\text{inst}}(a, \Lambda) = \sum_{k=1}^{\infty} \mathscr{F}_k(a) \Lambda^{k\beta} \tag{1.3}$$

Here Λ is the energy scale generated by the system, and $a = (a_1, a_2, ...)$ is the vector multiplet.

But when $k > 2$, $\mathscr{F}_k$ is very difficult to calculate. This problem was solved by Nekrasov twenty years ago with some localization techniques[4]. Nekrasov used the equivariant cohomology integral and the Ω-background and concluded that the partition function can be written as

$$Z(a, m, \Lambda, \epsilon) = \exp\left[\frac{1}{\epsilon_1\epsilon_2}\mathscr{F}_{\text{inst}}(a, m, \Lambda, \epsilon)\right] \tag{1.4}$$

where $\epsilon_{1,2}$ is the parameter in the Ω-background, which can be regarded as the rotation angle of two complex planes around the origin. And $\mathscr{F}_{\text{inst}}$ is analytic at $\epsilon_{1,2} \to 0$. And through the fixed points of the equivariant action, Nekrasov gave the exact integral form of the gauge group of type A in four dimensions $\mathscr{N} = 2$ and observed that these fixed points can be classified into the form of a Young diagram. For example, in the unrefined limit $\epsilon_1 = -\epsilon_2 = \hbar$, the summation form of $SU(N)$ is

$$Z = \sum_{\vec{\lambda}} \mathfrak{q}^{|\vec{\lambda}|} \prod_{(l,i)\neq(n,j)} \frac{a_{ln} + \hbar(\lambda_{l,i} - \lambda_{n,j} + j - i)}{a_{ln} + \hbar(j - i)} \tag{1.5}$$

Where $a_{li} = a_l - a_i$ is the Coulomb branch parameter, and $\vec{\lambda} = (\lambda_1, ..., \lambda_N)$ represents N Young diagrams, and each Young diagram follows $\lambda_{l,1} \geq \lambda_{l,2}...$, their total box number needs to be the instanton number k, that is, $|\vec{\lambda}| = \sum_{l,i} \lambda_{l,i} = k$. We are still not very clear why all the fixed points can be classified by the Young diagram. This research has far-reaching impacts on physics and mathematics. For example, we can use the Young diagram classification summation to connect the calculation of topological vertices[5–6], which is also extremely important in the AGT relationship[7].

In recent years, people have begun to study theories about gauge groups other than type A, and how to write down their exact forms in terms of Yang diagram classifications. For example, two years ago, Nawata-Zhu found the exact form[8] of the partition function of the pure superYang-Mills theory of the BCD-type gauge group in five dimensions. We need to use ADHM structure, Haar measure, and other mathematical tools to construct BCD-type instanton module space and localized integral[9–10].

In the last year, the relevant research was further extended, and the partition function with one matter in terms of summation of the Young diagram was found[11]. Interestingly, the mass m of this substance has a similar orientation to the unrefined parameter $\hbar$ in some cases, which makes the poles that can be classified by ordinary Yang diagram now need two more dimensions to classify, that is to say, we need to introduce the concept of four-dimensional Yang diagram. For example, in the five-dimensional $SO(n)$ $\mathscr{N} = 1^*$ theory, the Yang diagram summation form of the hypermultiplet under the antisymmetric representation is

$$\begin{aligned}
&\widetilde{Z}_k(\vec{A}; q) \\
&= \sum_{\vec{\lambda}} C^{4\text{d}}_{\vec{\lambda},\vec{A}} \left(\frac{\text{sh}(m \pm \hbar)\ \text{sh}^2(0)}{\text{sh}^2(\hbar)\ \text{sh}^2(m)}\right)^k \\
&\prod_{s=1}^{\|\vec{\lambda}\|} \prod_{x\in\lambda^{(s)}} \frac{\text{sh}^4(2\phi_s(x))\ \text{sh}^2(2\phi_s(x) + m - \hbar)\ \text{sh}^2(2\phi_s(x) - m + \hbar) \prod_{t=1}^{N} \text{sh}(\phi_s(x) \pm a_t \pm m)}{\prod_{t=1}^{\|\vec{\lambda}\|} \text{sh}^2(\phi_s(x) \pm a_t)} \\
&\prod_{s\leq t}^{\|\vec{\lambda}\|} \prod_{\substack{x\in\lambda^{(s)},\, y\in\lambda^{(t)} \\ x<y}} \frac{\text{sh}^4(\phi_s(x) \pm \phi_t(y))\ \text{sh}(\phi_s(x) \pm \phi_t(y) \pm m \pm \hbar)}{\text{sh}^2(\phi_s(x) \pm \phi_t(y) \pm \hbar)\ \text{sh}^2(\phi_s(x) \pm \phi_t(y) \pm m)}
\end{aligned} \tag{1.6}$$

where $C^{4\text{d}}_{\vec{\lambda},\vec{A}}$ are some constants, they have different values corresponding to different Young diagrams, but we do not know their general form. And $\phi_s(x) = a_s + (x_1 - x_2)\hbar +$

$(x_3 - x_4)m$, $x = (x_1, x_2, x_3, x_4)$ is the coordinate of each box of each Young diagram. We can also use these precise forms to test many relational expressions constructed by brane webs and derived from Lie algebra representations. For example, for $SO(8)$, we can use our formulas by changing the variables(such as $a_1 \to (a_1 + a_2 + a_3 + a_4)/2$) to check its triality, that is, the fundamental, the spinor, and the conjugate spinor representation of $SO(8)$ are isomorphic.

The thesis is organized as follows. In chapter 2, we recall some basic definitions of $4d$ $\mathcal{N} = 2$ and $\mathcal{N} = 1$ supersymmetry and introduce super Yang-Mills theory and the topological twist. In chapter 3, we introduce the ADHM construction to construct the instanton moduli space that we localize to after applying the topological twist on the super Yang-Mills action. In chapter 4, we review the equivariant integral and Duistermaat-Heckman formula, introduce the Ω-background, and construct the Nekrasov partition function in the form of Euler class. In chapter 6, we introduce the brane webs configuration of different gauge theories. Lastly, in chapter 7, we write down several instanton partition function expressions in terms of summing over Young diagrams and check many dualities in the brane webs approach or Lie algebra approach.

Chapter 2

4d $\mathscr{N} = 2$ Super Yang-Mills theory

In this chapter, we will introduce the basic concepts in supersymmetry and super Yang-Mills theory in $4d$ $\mathscr{N} = 2$ SUSY. We mainly follow the thesis of Shadchin[12].

2.1 Supersymmetry algebra

Supersymmetry (SUSY) is a symmetry between Boson and Fermion that is heavily used in mathematical physics. Recall the generators of *Poincaré* algebra are P_μ and $J_{\mu\nu}$ and the commutation relations between them are:

$$\begin{aligned} [P_\mu, P_\nu] &= 0 \\ [J_{\mu\nu}, P_\rho] &= i g_{\rho[\nu} P_{\mu]} \\ [J_{\mu\nu}, J_{\rho\sigma}] &= i g_{[\nu_{[\rho}} J_{\mu]_{\sigma]}} \end{aligned} \tag{2.1}$$

Where the bracket [] means anti-symmetrize the labels. To generalize the *Poincaré* algebra to Super *Poincaré* algebra, we need some extra generators to relate the additional space coordinates that correspond to Fermions, which are called supercharges $\mathcal{Q}_\alpha^A$ and their hermitian conjugate partners $\bar{\mathcal{Q}}_{A,\dot{\alpha}} = (\mathcal{Q}_\alpha^A)^\dagger$. The index A runs from 1 to $\mathscr{N}$. Then the super *Poincaré* algebra now is enlarged from *Poincaré* algebra by additional relations:

$$\begin{aligned} [P_\mu, \mathcal{Q}_\alpha^A] &= 0 \\ [P_\mu, \bar{\mathcal{Q}}_{A,\dot{\alpha}}] &= 0 \\ [J_{\mu\nu}, \mathcal{Q}_\alpha^A] &= i\sigma_{\mu\nu,\alpha}{}^{\beta} \mathcal{Q}_\beta^A \\ [J_{\mu\nu}, \bar{\mathcal{Q}}_A^{\dot{\alpha}}] &= i\bar{\sigma}_{\mu\nu}{}^{\dot{\alpha}}{}_{\dot{\beta}} \mathcal{Q}_A^{\dot{\beta}} \\ \{\mathcal{Q}_\alpha^A, \bar{\mathcal{Q}}_{B,\dot{\beta}}\} &= 2\sigma^\mu_{\alpha\dot{\beta}} P_\mu \sigma_B^A \\ \{\mathcal{Q}_\alpha^A, \mathcal{Q}_\beta^B\} &= \epsilon_{\alpha\beta} Z^{AB} \mathscr{Z} \\ \{\bar{\mathcal{Q}}_{A,\dot{\alpha}}, \bar{\mathcal{Q}}_{B,\dot{\beta}}\} &= \epsilon_{\dot{\alpha}\dot{\beta}} Z^*_{AB} \mathscr{Z} \end{aligned} \tag{2.2}$$

Where Z^{AB} is an anti-symmetric matrix with respect to so-called R-symmetry. And $\mathscr{Z}$ is the operator that relates to the central extension of SUSY algebra called central charge.

The superspace of the SUSY contains extra dimensions relate to Fermions and we denote them as $\mathscr{N}$ left-handed spinor coordinates θ_A^α and $\mathscr{N}$ right-handed spinor coordinates $\bar{\theta}^{A,\dot{\alpha}}$. With the additional central charge coordinate we denote z, the complete coordinates of superspace become:

$$z^a = (x^\mu, \theta_A^\alpha, \bar{\theta}^{A,\dot{\alpha}}, z) \tag{2.3}$$

With the coordinates 2.3, we can write down the supercharges explicitly as:

$$\mathcal{Q}^A_\alpha = \frac{\partial}{\partial\theta^\alpha_A} + \sigma^\mu_{\alpha\dot\beta}\bar\theta^{A,\dot\beta}P_\mu + \frac{1}{2}\epsilon_{\alpha\beta}Z^{AB}\theta^\beta_B\mathcal{Z}$$
$$\bar{\mathcal{Q}}_{A,\dot\alpha} = \frac{\partial}{\partial\bar\theta^{\dot\alpha,A}} + \theta^\beta_A\sigma^\mu_{\beta\dot\alpha}P_\mu + \frac{1}{2}\epsilon_{\dot\alpha\dot\beta}Z^*_{AB}\bar\theta^{B,\dot\beta}\mathcal{Z} \tag{2.4}$$

Where the momentum operator is in the familiar form: $P_\mu = i\partial_\mu$, and the central charge operator is just $\mathcal{Z} = i\partial_z$.

So with all these generators and coordinates, all the group elements of the super *Poincaré* group can be obtained by exponential all the generators. And they take the form:

$$g(z^a,\omega^{\mu\nu}) = \exp\Big[-ix^\mu P_\mu + \theta^\alpha_A\mathcal{Q}^A_\alpha + \bar\theta^{B,\dot\beta}\bar{\mathcal{Q}}_{B,\dot\beta} - iz\mathcal{Z}\Big]\exp\Big[-\frac{i}{2}\omega^{\mu\nu}J_{\mu\nu}\Big] \tag{2.5}$$

Let $\tilde g(z^a) = g(z^a,0)$, and we can obtain the vielbein ${e_a}^b$ and the spinor connection $\omega^{\mu\nu}_a$ by computing the following identity:

$$\tilde g^{-1}(z^a)d\tilde g(z^a) = dz^a e^\mu_a P_\mu + dz^a {e_a}^\alpha_A\mathcal{Q}^A_\alpha + dz^a {e_a}^{B,\dot\beta}\bar{\mathcal{Q}}_{B,\dot\beta} + dz^a {e_a}^z\mathcal{Z} + \frac{1}{2}dz^a\omega^{\mu\nu}_a J_{\mu\nu} \tag{2.6}$$

Then all the covariant derivatives which are defined as $\mathcal{D}_b = {(e^{-1})_b}^a\left(\partial_a + \frac{1}{2}\omega^{\mu\nu}_a S_{\mu\nu}\right)$ can be obtained. And they are:

$$\mathcal{D}_\mu = i\partial_\mu$$
$$\mathcal{D}^A_\alpha = \frac{\partial}{\partial\theta^\alpha_A} - i\sigma^\mu_{\alpha\dot\beta}\bar\theta^{A,\dot\beta}\partial_\mu - \frac{i}{2}\epsilon_{\alpha\beta}Z^{AB}\theta^\beta_B\frac{\partial}{\partial z}$$
$$\bar{\mathcal{D}}_{A,\dot\alpha} = \frac{\partial}{\partial\bar\theta^{A,\dot\alpha}} - i\theta^\beta_A\sigma^\mu_{\beta\dot\alpha}\partial_\mu - \frac{i}{2}\epsilon_{\dot\alpha\dot\beta}Z^*_{AB}\bar\theta^{B,\dot\beta}\frac{\partial}{\partial z}$$
$$\mathcal{D}_z = i\partial_z \tag{2.7}$$

The commutation relations between these covariant derivatives are:

$$\{\mathcal{D}^A_\alpha, \bar{\mathcal{D}}_{B,\dot\beta}\} = -2i\sigma^\mu_{\alpha\dot\beta}\partial_\mu\delta^A_B$$
$$\{\mathcal{D}^A_\alpha, \mathcal{D}^B_\beta\} = -i\epsilon_{\alpha\beta}Z^{AB}\partial_z$$
$$\{\bar{\mathcal{D}}_{A,\dot\alpha}, \bar{\mathcal{D}}_{B,\dot\beta}\} = -i\epsilon_{\dot\alpha\dot\beta}Z^*_{AB}\partial_z \tag{2.8}$$

It can be shown that the covariant constants in the $\bar\theta$ and z direction take the form:

$$y^\mu = x^\mu - i\theta_A\sigma^\mu\bar\theta^A \tag{2.9}$$

Which means it satisfies $\bar{\mathcal{D}}_{A,\dot\alpha}y^\mu = \mathcal{D}_z y^\mu = 0$.

2.2 Supermultiplets

In this section, we introduce the supermultiplets we need.

In the field theory perspective, all the elementary particles are irreducible representations of the *Poincaré* group, while they are not an irreducible representation of the super *Poincaré* group. We call the latter supermultiplets due to the fact that they contain more than one particle (at least one fermion and one boson).

- $\mathcal{N}=1$ chiral multiplet:

  We can consider the simplest case: a scalar function $\Phi(x,\theta,\bar{\theta})$. Note that it is easy to show that this field is a reducible representation of the super *Poincaré* group if we impose the condition $\bar{\mathcal{D}}_{\dot{\alpha}}\Phi(x,\theta,\bar{\theta})=0$ since the covariant derivatives are commute with SUSY transformation. And we can solve this issue by replacing the coordinate as $y^{\mu}=x^{\mu}-i\theta_{A}\sigma^{\mu}\bar{\theta}^{A}$. Then we can do the expansion of the chiral multiplet as:

$$\begin{aligned}\Phi(y,\theta)=&H(x)+i\theta\sigma^{m}u\bar{\theta}\partial_{\mu}H(x)-\frac{1}{4}(\theta\theta)(\bar{\theta}\bar{\theta})\partial_{\mu}\partial^{\mu}H(x)+\sqrt{2}\theta\psi(x)\\&-\frac{i}{\sqrt{2}}\theta\theta(\partial_{\mu}\psi(x)\sigma^{\mu}\bar{\theta})+\theta\theta f(x)\end{aligned}\tag{2.10}$$

  Where $H(x)$ is a scalar field, $\psi^{a}(x)$ is a Weyl spinor, and $f(x)$ has no derivative terms, equivalently it has no dynamical property, thus it is called an auxiliary field.

- $\mathcal{N}=1$ vector multiplet:

  A general scalar function on $\mathcal{N}=1$ superspace should satisfy the floowing condition:

$$V(x,\theta,\bar{\theta})=V^{\dagger}(x,\theta,\bar{\theta})\tag{2.11}$$

  We can expand it and see what properties its component fields have from this relation:

$$\begin{aligned}V(x,\theta,\bar{\theta})=&\varphi(x)+\sqrt{2}\theta\chi(x)+\sqrt{2}\bar{\theta}\bar{\chi}(x)+\theta\theta g(x)+\bar{\theta}\bar{\theta}g^{\dagger}(x)+\theta\sigma^{\mu}\bar{\theta}A_{\mu}(x)\\&-i(\bar{\theta}\bar{\theta})\theta\left(\lambda(x)+\frac{1}{\sqrt{2}}\sigma^{\mu}\partial_{\mu}\bar{\chi}(x)\right)+i(\theta\theta)\bar{\theta}\left(\bar{\lambda}(x)+\frac{1}{\sqrt{2}}\bar{\sigma}^{\mu}\partial_{\mu}\chi(x)\right)\\&+\frac{1}{2}(\theta\theta)(\bar{\theta}\bar{\theta})\left(D(x)-\frac{1}{2}\partial_{\mu}\partial^{\mu}\varphi(x)\right)\end{aligned}\tag{2.12}$$

  And it yields $\varphi^{\dagger}(x)=\varphi(x)$, $D^{\dagger}(x)=D(x)$ and $A_{\mu}^{\dagger}(x)=A_{\mu}(x)$ is the gauge boson of gauge theory.

  But this is not an irreducible representation! To fix it we need to fix a gauge. Here we can choose the Wess-Zumino gauge which does not fix all the degrees of freedom but can have great simplification as:

$$V_{WZ}(x,\theta,\bar{\theta})=\theta\sigma^{\mu}\bar{\theta}A_{\mu}(x)-i(\bar{\theta}\bar{\theta})(\theta\lambda(x))+i(\theta\theta)(\bar{\theta}\bar{\lambda}(x))+\frac{1}{2}(\theta\theta)(\bar{\theta}\bar{\theta})D(x)\tag{2.13}$$

- SUSY field strength defining the field strength as an invariant expression under *Wess − Zumino* gauge transformation as:

$$W_{\alpha}(x,\theta,\bar{\theta})=-\frac{1}{8}\bar{\mathcal{D}}_{\dot{\alpha}}\bar{\mathcal{D}}^{\dot{\alpha}}e^{-2V(x,\theta,\bar{\theta})}\mathcal{D}_{\alpha}e^{2V(x,\theta,\bar{\theta})}\tag{2.14}$$

And expand it in y coordinates then it becomes chiral ($\bar{\mathscr{D}}_{\dot{\alpha}} W_\alpha = 0$):

$$W_\alpha(y,\theta) = -i\lambda_\alpha(y) + \theta_\alpha D(y) - i\sigma^{\mu\nu}{}_\alpha{}^\beta \theta_\beta F_{\mu\nu}(y) - \theta\theta\sigma^\mu \nabla_\mu \bar{\lambda}^{\dot{\beta}}(y) \tag{2.15}$$

Where $F_{\mu\nu}$ is the good old field strength:

$$F_{\mu\nu} = \partial_{[\mu} A_{\nu]} - i[A_\mu, A_\nu] \tag{2.16}$$

And it also satisfies the reality condition:

$$\bar{\mathscr{D}}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}} = \mathscr{D}^\alpha W_\alpha \tag{2.17}$$

Then we consider the (irreducible) representation of $\mathcal{N}=2$ SUSY[13].

- $\mathcal{N}=2$ chiral multiplet:

  Chirality means the superfield satisfies $\bar{\mathscr{D}}_{A,\dot{\alpha}} \Psi = 0$. We expand it with the covariant constant coordinate y (again, A is the r-symmetry index that can take $1,2$):

$$\begin{aligned}\Psi(y,\theta) =& H(y) + \sqrt{2}\theta_A \psi^A(y) + \frac{1}{\sqrt{2}}\theta_A \sigma^{\mu\nu}\theta^A F_{\mu\nu}(y) + \frac{1}{\sqrt{2}}\theta_A L^A{}_B(y)\theta^B \\ & - \frac{2i\sqrt{2}}{3}(\theta^A\theta^B)\left(\theta_A\left\{\sigma^\mu\nabla_\mu\bar{\psi}_B(y) + \frac{1}{\sqrt{2}}[H^\dagger(x), \psi_B(y)]\right\}\right) \\ & - \frac{1}{3}(\theta^A\theta_B)(\theta_A\theta_B)\left(\nabla^2 H^\dagger(y) - [H^\dagger(y), D(y)] - \frac{i}{\sqrt{2}}\{\bar{\psi}^C(y), \bar{\psi}_C(y)\}\right)\end{aligned} \tag{2.18}$$

  Where $L^A{}_B$ is a matrix with auxiliary fields as its components.

- $\mathcal{N}=2$ hypermultiplet, which relate to matter in the $\mathcal{N}=2$ SUSY

  Consider a doublet of the internal R-symmetry $SU(2)_R$ that is denoted as $Q^A(x,\theta,\bar{\theta},z)$. In order to let $\mathscr{D}^A_\alpha Q^B$ and $\bar{\mathscr{D}}^A_{\dot{\alpha}} Q^B$ to be the fundamental representation of $1 \oplus 3$, which is the 1, we need to impose the condition:

$$\begin{aligned}\mathscr{D}^A_\alpha Q^B &= \frac{1}{2}\epsilon^{AB}\mathscr{D}_{C,\alpha}Q^C \\ \bar{\mathscr{D}}^A_{\dot{\alpha}} Q^B &= \frac{1}{2}\epsilon^{AB}\bar{\mathscr{D}}_{C,\dot{\alpha}}Q^C\end{aligned} \tag{2.19}$$

  After expansion, we can show the hypermultiplet consists of several fields: q^A, which are two complex scalar fields; χ_α and $\tilde{\chi}_\alpha$ are two spinors and X^A are two auxiliary fields.

2.3 $\mathcal{N}=2$ Super Yang-Mills action

In this section, we will briefly review the Super Yang-Mills theory in $\mathcal{N}=2$ SUSY.

It is described by the chiral superfields in $\mathcal{N}=2$ as:

$$\begin{gathered} A_\mu(x) \\ \psi^1_\alpha(x) = \psi_\alpha(x) \qquad \psi^2_\alpha(x) = \lambda_\alpha(x) \\ H(x) \end{gathered} \tag{2.20}$$

Where A_μ is a vector field that describes a gauge boson, $\psi_\alpha^{1,2}$ represent two gluinos (the SUSY partner of gluons) and $H(x)$ is a complex scalar field that describes Higgs field. Since the gauge boson takes the adjoint representation of the gauge group G, thus all the other contents in the chiral multiplet H and $\psi_\alpha^{1,2}$ also take the value of adjoint representation of G.

And we also need to include hypermultiplet when the matter comes into play:

$$\begin{gathered} \chi_\alpha(x) \\ q^1(x)=q(x) \qquad q^2(x)=\tilde{q}^\dagger(x) \\ \bar{\tilde{\chi}}_\alpha(x) \end{gathered} \tag{2.21}$$

Where similarly, χ_α and $\bar{\tilde{\chi}}_\alpha$ are two Weyl spinors and $q^{1,2}$ are complex bosons. And unlike the chiral multiplet, the hypermultiplet takes the value in the representation we choose. The top and the bottom lines of 2.20 and 2.21 are singlets for $SU(2)_R$ symmetry, while the middle lines are representing two doublets.

With the above contents, the super Yang-Mills action is given in the aspect of contains no higher than two derivatives terms and renormalizable uniquely in terms of $\mathcal{N}=1$ superfields as (integrate out all the auxiliary fields):

$$S_{\mathrm{YM}}=\frac{1}{8\pi h^\vee}\,\mathrm{Im}\left[\tau_0\,\mathrm{Tr}\left(\int d^4x\,d^2\theta\,W^\alpha W_\alpha+\int d^4x\,d^2\theta\,d^2\bar{\theta}\,\Psi^\dagger e^{2V}\Psi\right)\right] \tag{2.22}$$

Where $\tau_0=\frac{4\pi i}{g_0^2}+\frac{\Theta_0}{2\pi}$ with g_0^2 being the coupling constant in Yang-mills theory and the Θ_0 being instanton angle. Θ_0 contributes as a topological term with the instanton number $k\in\mathbb{Z}$ as $\Theta_0 k$:

$$k=-\frac{1}{32\pi^2h^\vee}\int d^4x\,\mathrm{Tr}\,F_{\mu\nu}\star F^{\mu\nu} \tag{2.23}$$

$h^\vee$ is the dual Coxeter number that depends on the gauge group.

Using $\mathcal{N}=2$ chiral superfield, we can write the action as:

$$S_{\mathrm{YM}}=\frac{1}{4\pi h^\vee}\,\mathrm{Im}\left(\int d^4x\,d^4\theta\,\frac{\tau}{2}\,\mathrm{Tr}\,\Psi^2\right) \tag{2.24}$$

This theory is asymptotically free. So when we go to low energy, we need to consider the perturbative and non-perturbative corrections which lead to the Wilsonian effective action:

$$\begin{aligned} S_{\mathrm{eff}}=\frac{1}{8\pi}\,\mathrm{Im}\Bigg\{&\frac{1}{2\pi i}\int d^4x\,d^4\theta\,\mathcal{F}^{lm}(\Phi)W_l^\alpha W_{m,\alpha} \\ &+\frac{1}{2\pi i}\int d^2\theta\,d^2\bar{\theta}\,[\Phi^\dagger e^{2V}]_l\mathcal{F}^l(\Phi)\Bigg\} \end{aligned} \tag{2.25}$$

Where we have set $\frac{1}{2}[H,H^\dagger]^2=0$, which means $[H,H^\dagger]=0$ to preserve $\mathcal{N}=2$ SUSY. This means H is an element of the Cartan subgroup of G. (same as gauge field $A_{l,\mu}(x)$, $l=1,...,r$, $r=\mathrm{rank}\,G$). And we have introduced the holomorphic function $\mathcal{F}(a)$ on Coulombs branch parameters a_l, $l=1,...,r$, which is called prepotential:

$$\mathcal{F}^l(a)=\frac{\partial\mathcal{F}(a)}{\partial a_l},\quad \mathcal{F}^{lm}=\frac{\partial^2\mathcal{F}}{\partial a_l\partial a_m} \tag{2.26}$$

Thus the effective action becomes the following form:

$$S_{\text{eff}} = \frac{1}{4\pi} \operatorname{Im} \left\{ \frac{1}{2\pi i} \int d^4x \, d^4\theta \, \mathcal{F}(\Psi) \right\} \tag{2.27}$$

The classical contribution of prepotential can be read from 2.24, which simply traces all the diagonal matrix (Cartan subgroup of G):

$$\mathcal{F}_{\text{class}}(a) = \pi i \tau_0 \sum_{l=1}^{r} a_l^2 \equiv \pi i \tau_0 \langle a, a \rangle \tag{2.28}$$

Thanks to the work of Seiberg[3], we can obtain all the perturbation corrections to the prepotential as:

$$\begin{aligned} \mathcal{F}_{\text{pert}} = & -\sum_{\alpha \in \Delta^+} \langle \alpha, a \rangle^2 \left(\ln \left| \frac{\langle \alpha, a \rangle}{\Lambda} \right| - \frac{3}{2} \right) \\ & + \frac{1}{2} \sum_{\varrho \in \text{reps}} \sum_{\lambda \in \boldsymbol{\omega}_\varrho} (\langle a, \lambda \rangle + m_\varrho)^2 \left(\ln \left| \frac{\langle a, \lambda \rangle + m_\varrho}{\Lambda} \right| - \frac{3}{2} \right) \end{aligned} \tag{2.29}$$

Where ϱ is a representation of G, Δ^+ are the positive roots of the Lie algebra $\mathfrak{g}$, Λ is the dynamically generated scale and finally, m_ϱ is the mass of the matter multiplet.

And now come to the instanton part of the prepotential. The most general form it can take is:

$$\mathcal{F}_{\text{inst}}(a, \Lambda) = \sum_{k=1}^{\infty} \mathcal{F}_k(a) \Lambda^{k\beta} \tag{2.30}$$

Where β is the leading coefficient of the Beta function. This expansion is all come from the non-perturbative contributions caused by different vacua, which can be shown by looking at the RG flow of the complex coupling constant τ that can be obtained from 2.29:

$$\tau(\Lambda_1) = \tau(\Lambda_2) + \frac{\beta}{2\pi i} \ln \frac{\Lambda_1}{\Lambda_2} \tag{2.31}$$

We can choose $\tau(\Lambda) = \tau_0 + \frac{\beta}{2\pi i} \ln \Lambda$ and define the instanton counting parameter $\mathfrak{q}$ which is important in later computation:

$$\mathfrak{q} \equiv e^{2\pi i \tau} = e^{-\frac{8\pi^2}{g^2}} e^{i\Theta} = e^{2\pi i \tau_0} \Lambda^{\beta} \tag{2.32}$$

2.4 Topological twist and BRST-exact form

A very important property of $\mathcal{N} = 2$ super Yang-Mills theory is that it can do the topological twist. Since the theory consists with four dimensional Lorentz symmetry and $\mathcal{N} = 2$ SUSY, the Lie algebra is: $\mathfrak{su}(2)_L \times \mathfrak{su}(2)_R \times \mathfrak{su}(2)_I$. Here we donate the internal symmetry as I, and the left and right spinor indices are denoted L and R. So we can twist the SUSY indices with the right spinor indices to redefine the Lorentz group in the theory. First we can define a fermionic operator $\bar{\mathcal{Q}}$ that also called BRST operator as:

$$\bar{\mathcal{Q}} \equiv \epsilon^{A\dot{\alpha}} \bar{\mathcal{Q}}_{A,\dot{\alpha}} \tag{2.33}$$

It turns out that besides the topological terms that are proportional to Tr $F_{\mu\nu} \star F^{\mu\nu}$, all the terms are $\bar{\mathcal{Q}}$ exact in the action 2.24 Which we will see later.

With the twisting, we can redefine all the fields:

$$\psi_{A,\alpha} = \frac{1}{2}\sigma^{\mu}_{\alpha A}\psi_{\mu}, \qquad \bar{\psi}^{A,\dot{\alpha}} = \frac{1}{2}\epsilon^{A\dot{\alpha}}\bar{\psi} + \frac{1}{2}\bar{\sigma}_{\mu\nu}{}^{A\dot{\alpha}}\bar{\psi}^{\mu\nu} \tag{2.34}$$

Where this field $\bar{\psi}^{\mu\nu}$ is by definition anti-self-dual: $\bar{\psi}^{\mu\nu} = -i \star \bar{\psi}^{\mu\nu}$. All the other twisted supercharges can be redefined as follow:

$$\mathcal{Q}_{\mu} = \bar{\sigma}^{A\alpha}_{\mu}\mathcal{Q}_{A,\alpha}, \qquad \bar{\mathcal{Q}}_{\mu\nu} = \bar{\sigma}_{\mu\nu}{}^{A\dot{\alpha}}\bar{\mathcal{Q}}_{A,\dot{\alpha}} \tag{2.35}$$

Thus the Yang-Mills action can be the $\bar{\mathcal{Q}}$ exact form[14] after expansion into fields as (more detail in[12]):

$$S_{\text{YM}} = \text{Im}\left[\bar{\mathcal{Q}}\left\{\frac{\tau_0}{16\pi h^{\vee}}\int d^4x\, \text{Tr}\left((F_{\mu\nu})^{-}\bar{\psi}^{\mu\nu} - i\sqrt{2}\psi^{\mu}\nabla_{\mu}H^{\dagger} + i\bar{\psi}[H, H^{\dagger}]\right)\right\}\right] \tag{2.36}$$

Where $(F_{\mu\nu})^{\pm} = \frac{1}{2}(F_{\mu\nu} \pm i \star F_{\mu\nu})$ is the self or anti-self dual anti-symmetric tensor. And we already integrate out all the auxiliary fields $f(x)$, $f^{\dagger}(x)$, and $D(x)$. And we already used the equation of motion for $\bar{\psi}_{\mu\nu}$:

$$(\nabla_{[\mu}\psi_{\nu]})^{-} = \sqrt{2}[H, \bar{\psi}_{\mu\nu}] \tag{2.37}$$

And we can show that the topological term in the action is $\bar{\mathcal{Q}}$ closed since $F^{\mu\nu}$ satisfies the Bianchi identity:

$$\bar{\mathcal{Q}}\int d^4x\, F_{\mu\nu} \star F^{\mu\nu} = 2i\int d^4x\, (\nabla_{[\mu}\psi_{\nu]}) \star F^{\mu\nu} = -4i\int d^4x\, \psi_{\nu}\nabla_{\mu} \star F^{\mu\nu} = 0 \tag{2.38}$$

And this topological part we denote S_{top}:

$$S_{\text{top}} = \frac{\Theta_0}{32\pi^2 h^{\vee}}\int d^4x\, \text{Tr}\left[F_{\mu\nu} \star F^{\mu\nu}\right] \tag{2.39}$$

S_{top} is incariant under:

$$A_{\mu} \mapsto A_{\mu} - \nabla_{\mu}\alpha + \alpha_{\mu} \tag{2.40}$$

Where $\alpha(x)$ and $\alpha_{\mu}(x)$ are both gauge $\mathfrak{g}$ valued while $\alpha_{\mu}(x) + A_{\mu}(x)$ need to be in the same gauge class.

To further fix the gauge, we need the help of ghost: ϕ, Lagrange multiplier: b, $H_{\mu\nu}$ and η and antighosts: $\bar{c}$, $\chi_{\mu\nu}$ and λ. And the ghost number of these supplementary fields is shown in table:2-1.

Fields	A_μ	c	ψ_μ	ϕ	b	$H_{\mu\nu}$	η	$\bar{c}$	$\chi_{\mu\nu}$	λ
Ghost number	0	+1	+1	+2	0	0	-1	-1	-1	-2
Statistics	B	F	F	B	B	B	F	F	F	B

Table 2-1 Properties of fields

We can compute all the algebraic properties of these new fields which we neglect in this thesis. And we can obtain the gauge fixed action in a very neat form by this gauge fixing ($\nabla_\mu A^\mu = (F_{\mu\nu})^- = \nabla^\mu \psi_\mu = 0$):

$$
\begin{aligned}
S_{\text{fixed}} =& S_{\text{top}} + \bar{\mathcal{Q}} V_{\text{YM}} \\
V_{\text{YM}} =& \frac{1}{h^\vee g_0^2} \int d^4x \\
& \operatorname{Tr}\left[\frac{1}{2}\chi^{\mu\nu}\Big((F_{\mu\nu})^- + \frac{1}{4}H_{\mu\nu}\Big) + \frac{i}{8}\lambda\nabla_\mu\psi^\mu + \bar{c}(\nabla_\mu A^\mu + b)\right]
\end{aligned} \tag{2.41}
$$

So the total action 2.36 can be recover by adding another $\bar{\mathcal{Q}}$ exact term: $\bar{\mathcal{Q}}V'$, where:

$$
V' = \frac{-i}{128 h^\vee g_0^2} \int d^4x\, Tr\Big[\eta[\phi,\lambda]\Big] \tag{2.42}
$$

Then:

$$
S_{\text{YM}} = S_{\text{top}} + \bar{\mathcal{Q}}(V_{\text{YM}} + V') \tag{2.43}
$$

Where we have redefined the fields notation in order to simplify the formula a bit more as below:

$$
\phi = -2\sqrt{2}H, \quad \chi_{\mu\nu} = \bar{\psi}_{\mu\nu}, \quad \lambda = -2\sqrt{2}H^\dagger, \quad \eta = -4\bar{\psi} \tag{2.44}
$$

Of course, we can add matter to the action. Luckily, the matter sector of the action is also can be written as $\bar{\mathcal{Q}}$ exact form. And one can check indeed S_{matter} can be obtained by acting the BRST operators on the sum of the following expressions:

$$
\begin{aligned}
V_{\text{mat}} =& \frac{1}{h^\vee g_0^2}\int d^4x\ \operatorname{Tr}\Bigg[-\frac{i}{2}\chi_{\mu\nu}q^\dagger_{\dot\alpha}\bar\sigma^{\mu\nu,\dot\alpha}{}_{\dot\beta}q^{\dot\beta} - \frac{1}{4}\Big(\bar\mu_{\dot\alpha}\lambda q^{\dot\alpha} - q^\dagger_{\dot\alpha}\lambda\mu^{\dot\alpha}\Big) \\
& + 2\bar\nu^\alpha(\sigma^\mu_{\alpha\dot\alpha}\nabla_\mu q^{\dot\alpha} - h_\alpha) - 2\Big(\nabla_\mu q^\dagger_{\dot\alpha}\bar\sigma^{\mu,\dot\alpha\alpha} - \bar h^\alpha\Big)\nu_\alpha\Bigg]
\end{aligned} \tag{2.45}
$$

$$
V_{\text{mass}} = \frac{1}{h^\vee g_0^2}\int d^4x\ \operatorname{Tr}\left[-\frac{1}{4}m(q^\dagger_{\dot\alpha}\mu^{\dot\alpha} + \bar\mu_{\dot\alpha}q^{\dot\alpha})\right] \tag{2.46}
$$

Where we have done the topological twist $q^A \to q^{\dot\alpha}$ and, again for simplicity, defined some new fields:

$$
\mu^{\dot\alpha} = \sqrt{2}\bar{\tilde\chi}^{\dot\alpha}, \quad \nu_\alpha = \frac{\sqrt{2}}{2}\chi_\alpha, \quad \bar\mu_{\dot\alpha} = \sqrt{2}\bar\chi_{\dot\alpha}, \quad \bar\nu^\alpha = \frac{\sqrt{2}}{2}\tilde\chi^\alpha \tag{2.47}
$$

Before we have the BRST operator as $\bar{\mathcal{Q}}^2 = \mathcal{G}(\phi)$, where $\mathcal{G}(\phi)$ is the gauge transformation depends on the parameter ϕ (all the transformations produced by $\bar{\mathcal{Q}}$ are omitted in this paper, the detail we refer to Shadchin's thesis[12]). After we adding the hypermultiplet, the BRST operator now needs to be modified as:

$$\bar{\mathcal{Q}}^2 = \mathcal{G}(\phi) + \mathrm{F}(m) \tag{2.48}$$

Where $\mathrm{F}(m)$ is a gauge independence operator that can multiply all the hypermultiplets with $\pm m$. For an infinitesimal version, one can show:

$$Q \mapsto Q' = e^m Q, \quad \tilde{Q} \mapsto \tilde{Q}' = e^{-m}\tilde{Q} \tag{2.49}$$

Finally, we have obtained the full action:

$$S = S_{\mathrm{top}} + \bar{\mathcal{Q}}(V_{\mathrm{YM}} + V' + V_{\mathrm{mat}} + V_{\mathrm{mass}}) \tag{2.50}$$

Which is very useful in later content.

Chapter 3
ADHM construction

In this chapter, we will introduce the ADHM construction[15] to obtain the moduli space of the solutions of the self-dual equation which is called the instanton moduli space. First, we describe the localization of the action, and then we will go to ADHM constructions of type ABCD groups. Again, we mainly follow Shadchin's thesis[12].

3.1 Localization

From above, we conclude that the action of super Yang-Mills theory can be presented by $\bar{\mathcal{Q}}$ exact form and topological part as 2.50. Here, we first consider the vacuum expectation value (VEV) of $\bar{\mathcal{Q}}$ closed observable $\mathcal{O}$ (i.e $\bar{\mathcal{Q}}\mathcal{O} = 0$) for pure Yang-mills theory: $S = S_{\text{top}} + \bar{\mathcal{Q}}(V_{\text{YM}} + V')$ as:

$$\langle \mathcal{O} \rangle = \int \mathscr{D}X\, \mathcal{O} e^{S_{\text{top}} + \bar{\mathcal{Q}}(V_{\text{YM}} + V')} \tag{3.1}$$

Where $\mathscr{D}X$ represents the measure of path integral $\mathscr{D}X = \mathscr{D}A\mathscr{D}\psi\mathscr{D}\eta\mathscr{D}\chi\mathscr{D}H\mathscr{D}\lambda$. With the action taking as this form, we can directly see that the VEV of $\mathcal{O}$ is unchanged if we add any BRST exact term $\bar{\mathcal{Q}}\delta V$ to the action:

$$\begin{aligned}\langle \mathcal{O} \rangle' &= \int \mathscr{D}X\, \mathcal{O} e^{S + \bar{\mathcal{Q}}\delta V} \\ &= \langle \mathcal{O} \rangle + \int \mathscr{D}X\, \mathcal{O} e^{S} \bar{\mathcal{Q}}\delta V \\ &= \langle \mathcal{O} \rangle + \int \mathscr{D}X\, \bar{\mathcal{Q}}(\mathcal{O} e^{S} \delta V) \\ &= \langle \mathcal{O} \rangle \end{aligned} \tag{3.2}$$

Where we have used the Leibniz rule for $\bar{\mathcal{Q}}$ and we already know $\bar{\mathcal{Q}}S = \bar{\mathcal{Q}}\mathcal{O} = 0$. So we can modify the action into a more convenient form without affecting the VEV of BRST closed observable as $S_{\text{top}} + \bar{\mathcal{Q}}\tilde{V}$, where:

$$\tilde{V} = \int d^4x\ \text{Tr}\left[- \chi^{\mu\nu}\Big(t(F_{\mu\nu})^- - \frac{1}{4}H_{\mu\nu} \Big) + i\lambda \nabla_\mu \psi^\mu \right] \tag{3.3}$$

Here we introduce an arbitrary parameter t that will not lead to any new singularity of the Lagrangian. So we can take t to infinity later on. Then we integrate out the Lagrange multiplier $H_{\mu\nu}$ and obtain:

$$\bar{\mathcal{Q}}\tilde{V} = \int d^4x\ \text{Tr}\left[- t^2 (F_{\mu\nu})^-(F^{\mu\nu})^- + t\chi^{\mu\nu}(\nabla_{[\mu}\psi_{\nu]})^- + i\eta\nabla^\mu\psi_\mu + i\lambda\nabla^\mu\nabla_\mu\phi \right] \tag{3.4}$$

If we take $t \to \infty$, we can see the integral localizes on the space that only contains self-dual fields. Note that although the second term in the expression 3.4 is only at the order of $O(t^1)$, we need to keep it to balance the Faddeev-Popov determinant.

After all, this behavior that the whole integral only locates at the space of self-dual fields leads us to find a way to construct the instanton moduli space, and that is ADHM construction.

3.2 ADHM construction for $SU(N)$

To construct the self-dual connection with the gauge group $SU(N)$, first, we need a $(N + 2k) \times 2k$ complex matrix denoted $\Delta_{\dot{\alpha}}$ carrying a left spinor index that satisfies the following equation:

$$\Delta_{\dot{\alpha}} = \mathscr{A}_{\dot{\alpha}} + \mathscr{B}^{\alpha} x_{\alpha\dot{\alpha}} \tag{3.5}$$

Where $x_{\alpha\dot{\alpha}} = \sigma^{\mu}_{\alpha\dot{\alpha}} x_{\mu}$ is the coordinate of $\mathbb{R}^4$. Assuming $\Delta_{\dot{\alpha}}$ has the maximal rank $2k$. And we define a annihilator of $\Delta^{\dagger\dot{\alpha}}$ as $v(x)$ (i.e $\Delta^{\dagger\dot{\alpha}} v(x) = 0$). And $v(x)$ is a $(2k+N)\times N$ matrix that is normalized as:

$$v^{\dagger}(x) v(x) = \mathbb{1}_N \tag{3.6}$$

With this data, we can already construct the $U(N)$ connection in the fundamental representation as $A_{\mu}(x)$ with the property $A_{\mu}^{\dagger} = A_{\mu}$:

$$A_{\mu}(x) \equiv i v^{\dagger}(x) \partial_{\mu} v(x) \tag{3.7}$$

In order to let the curvature $F_{\mu\nu}$ induced by this connection A_{μ} satisfies the self-dual condition, we need to impose on $\Delta_{\dot{\alpha}}$ the factorization condition:

$$\Delta^{\dagger\dot{\alpha}} \Delta_{\dot{\beta}} = \delta^{\dot{\alpha}}_{\dot{\beta}} \mathscr{R}^{-1} \tag{3.8}$$

From this definition, we know $\mathscr{R}$ is an invertible $k \times k$ hermitian matrix. With this factorization condition, we can obtain:

$$v v^{\dagger} = \mathbb{1}_{2k+N} - \Delta_{\dot{\alpha}} \mathscr{R} \Delta^{\dagger\dot{\alpha}} \tag{3.9}$$

And we can also see that the curvature is indeed self-dual:

$$F_{\mu\nu} = \partial_{[\mu} A_{\nu]} - i[A_{\mu}, A_{\nu}] = -2i\eta^{i}_{\mu\nu} v^{\dagger} \mathscr{B}^{\alpha} \mathscr{R} \mathscr{B}^{\dagger}_{\beta} v \tau_{i,\alpha}{}^{\beta} \tag{3.10}$$

Where $\eta^{i}_{\mu\nu}$ is so called self-dual $t'Hooft$ symbols ($\bar{\eta}^{i}_{\mu\nu}$ is anti-self-dual $t'Hooft$ symbols) that consist with three matrices:

$$\eta^1_{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad \eta^2_{\mu\nu} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \eta^3_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$
$$\bar{\eta}^1_{\mu\nu} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad \bar{\eta}^2_{\mu\nu} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \bar{\eta}^3_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{3.11}$$

And $\tau_{i,\alpha}{}^{\beta}$ are the standard pauli matrices:

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.12}$$

Furthermore, since the curvature must obey the Yang-Mills equation, therefore it must satisfy the condition: $\nabla_\mu F^{\mu\nu} = 0$. Taking the trace with respect to gauge group is also zero $\partial_\mu \operatorname{Tr} F^{\mu\nu} = 0$. So using integral by parts and the cyclic property of trace we can show:

$$\begin{aligned} 0 = \int d^4x \operatorname{Tr}\left[A_\nu \partial_\mu \operatorname{Tr} F^{\mu\nu}\right] &= -\frac{N}{2}\int d^4x \operatorname{Tr} F_{\mu\nu} \operatorname{Tr} F^{\mu\nu} \\ \implies \operatorname{Tr} F_{\mu\nu} = 0 \quad &\implies \operatorname{Tr} A_\mu = \partial_\mu \alpha \end{aligned} \tag{3.13}$$

This implies $F_{\mu\nu}$ is traceless in terms of gauge group value and α is just some scalar in gauge group value. And then we can say that $A_\mu(x)$ is the anti-self dual connection of $SU(N)$ in fundamental representation.

And we can compute the term in Yang-Mills action that is proportional to the instanton number k by the Osborn identity[16]:

$$Tr_{\text{fund}} F_{\mu\nu} \star F^{\mu\nu} = -(\partial_\mu \partial^\mu)^2 \ln \det \mathscr{R} \tag{3.14}$$

Follow by the factorization condition 3.8, one can show the following relation (expand the $\Delta_{\dot\alpha}$ as original definition 3.5):

$$2\mathscr{R}^{-1} = \mathscr{A}^{\dagger\dot\alpha}\mathscr{A}_{\dot\alpha} + \mathscr{A}^{\dagger\dot\alpha}\mathscr{B}^{\beta}x_{\beta\dot\alpha} + x^{\dagger\dot\alpha\beta}\mathscr{B}^{\dagger}_{\beta}\mathscr{B}^{\gamma}x_{\gamma\dot\alpha} \tag{3.15}$$

We can obtain an asymptotic relation by taking $x \to \infty$:

$$\mathscr{R}^{-1} \to \frac{1}{2}x^2 \mathscr{B}^{\dagger}_{\alpha}\mathscr{B}^{\alpha} \tag{3.16}$$

where we have used the conditions derived from the factorization relation 3.8:

$$\begin{aligned} \mathscr{B}^{\dagger}_{\alpha}\mathscr{B}^{\beta} &= \frac{1}{2}\delta^{\beta}_{\alpha}\mathscr{B}^{\dagger}_{\gamma}\mathscr{B}^{\gamma} \\ \mathscr{B}^{\dagger}_{\alpha}\mathscr{A}_{\dot\alpha} &= \mathscr{A}^{\dagger}_{\dot\alpha}\mathscr{B}_{\alpha} \\ \mathscr{A}^{\dagger\dot\alpha}\mathscr{A}_{\dot\beta} &= \frac{1}{2}\delta^{\dot\alpha}_{\dot\beta}\mathscr{A}^{\dagger\dot\gamma}\mathscr{A}_{\dot\gamma} \end{aligned} \tag{3.17}$$

For $SU(N)$, we have $2N \operatorname{Tr}_{\text{fund}} F_{\mu\nu} \star F^{\mu\nu} = \operatorname{Tr} F_{\mu\nu} \star F^{\mu\nu}$. And recall the definition of instanton number 2.23, we have now:

$$\partial_\mu \partial^\mu \operatorname{Tr}_{\text{fund}} \ln \mathscr{R} \to -\frac{4k}{x^2} \quad \text{when} \quad x \to \infty \tag{3.18}$$

With the following transformation:

$$\Delta_{\dot\alpha} \to \Delta'_{\dot\alpha} = U\Delta_{\dot\alpha}M \quad \text{and} \quad v \to v' = Uv \tag{3.19}$$

where U is a $(N+2K)$ unitary matrix and M is a invertible matrix, the two initial conditions 3.8 and 3.6 are still valid. This gives us the freedom to choose the matrix $\mathscr{B}^{\alpha} = (\mathscr{B}^1, \mathscr{B}^2)$ to be a nice canonical form:

$$\mathscr{B} = \begin{pmatrix} 0 \\ \mathbb{1}_k \otimes \mathbb{1}_2 \end{pmatrix} \tag{3.20}$$

With this choice, all the information is encoded in matrix $\mathscr{A}$ and v. We can write down their components:

$$\mathscr{A}_{\dot\alpha} = (\mathscr{A}_{\dot 1}, \mathscr{A}_{\dot 2}) = \begin{pmatrix} S_{\dot 1} & S_{\dot 2} \\ X^\mu \otimes \sigma_\mu \end{pmatrix}, \quad v = \begin{pmatrix} T \\ Q_\alpha \end{pmatrix} \tag{3.21}$$

Where $S_{\dot\alpha}$ will be identified as a righthanded spinor under spacetime rotation while Q_α is lefthanded, X^μ as a vector in spacetime rotation and also a hermitian $k \times k$ matrix, and T is a scalar. Then the transformation 3.19 can be expanded as follow with the choice of the matrix $\mathscr{B}$ as canonical form:

$$S_{\dot\alpha} \to U_N S_{\dot\alpha} U_k^{-1}, \quad X^\mu \to U_k X^\mu U_k^{-1}, \quad T \to U_N T, \quad Q_\alpha \to U_k Q_\alpha \tag{3.22}$$

The factorization condition can be translated into the following form:

$$\mu^i \equiv \mathscr{A}^{\dagger\dot\alpha} \tau_{i,\dot\alpha}{}^{\dot\beta} \mathscr{A}_{\dot\beta} = 0 \tag{3.23}$$

Where we have defined three real moment maps as the key data to construct the instanton moduli space. And one can show the moment map corresponds to the i-th symplectic form and some Lie algebra element $\xi \in \mathfrak{su}(k)$ is:

$$\mu^i_\xi = -i\tau_{i,\dot\alpha}{}^{\dot\beta} \operatorname{Tr}\left[\xi \mathscr{A}^{\dagger\dot\alpha} \mathscr{A}_{\dot\beta}\right] \tag{3.24}$$

We denote all these matrices in a more traditional way:

$$J = S_{\dot 1}, \quad I = S^\dagger_{\dot 2}, \quad B_1 = X^0 - iX^3 \quad \text{and} \quad B_2 = -iX^1 + X^2 \tag{3.25}$$

Then the ADHM equations of $SU(N)$ gauge group are:

$$\begin{aligned} \mu_{\mathbb{R}} &= -\mu^3 = II^\dagger - J^\dagger J + [B_1, B_1^\dagger] + [B_2, B_2^\dagger] = 0 \\ \mu_{\mathbb{C}} &= \frac{1}{2}(\mu^1 - i\mu^2) = \mu^- = IJ + [B_1, B_2] = 0 \end{aligned} \tag{3.26}$$

To have a clear vision of these matrices, let us consider two vector spaces $\mathscr{V} = \mathbb{C}^k$ and $\mathscr{W} = \mathbb{C}^N$ and we have the following quiver representation: 3-1.

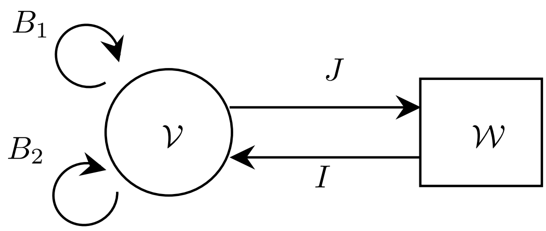


Figure 3-1 The ADHM linear data for $SU(N)$ gauge group in quiver representation. $\mathscr{V}$ is k dimensional complex vector space and $\mathscr{W}$ is a N dimensional one. $B_{1,2}$ and I, J are linear maps between them[10]

The freedom of transformations in 3.23 corresponds to the framing change of vector spaces $\mathscr{V}$ and $\mathscr{W}$. The framing change of $\mathscr{W}$ is the gauge at infinity which we also denote as G_∞, while the group corresponds to the change of frame in $\mathscr{V}$ is sometimes denoted as G_D, which means the dual group of G_∞.

The k instantons moduli space of $SU(N)$ gauge group denoted as $\mathscr{M}_k^{SU(N)}$ is then the hyper-Kähler quotient of the space of this four matrices $\mathscr{X} = (I, J, B_1, B_2)$ by its dual group $U(k)$:

$$\mathscr{M}_k^{SU(N)} = \mathscr{X}///U(k) = \mu^{-1}(0)/U(k) \tag{3.27}$$

Before the ADHM constructions for $SO(N)$ and $SP(N)$, it is worth discussing the solutions of Weyl equations and how they relate to what we have constructed.

There are k independent solutions for Weyl equations $\nabla^{\dot{\alpha}\alpha}\psi_\alpha = \bar{\sigma}^{\mu,\dot{\alpha}\alpha}\nabla_\mu\psi_\alpha$ so that they can be written in the form of $N \times k$ matrices[17]:

$$\psi^\alpha = v^\dagger \mathscr{B}^\alpha \mathscr{R} = \bar{\mathcal{Q}}^\alpha \mathscr{R} \tag{3.28}$$

And one can show the following identities in fundamental representation:

$$\begin{aligned}
&\bar{\psi}_\alpha \psi^\alpha = -\frac{1}{4}\partial_\mu \partial^\mu \mathscr{R}, \\
&\int d^4x\, \bar{\psi}_\alpha \psi^\alpha = \pi^2 \mathbb{1}_k, \\
&\int d^4x\, \bar{\psi}_\alpha \psi^\alpha x^\mu = -\pi^2 X^\mu \\
&\psi^\alpha x_{\alpha\dot{\alpha}} \to -\frac{1}{x^2} S_{\dot{\alpha}} \quad \text{when} \quad x \to \infty
\end{aligned} \tag{3.29}$$

We can also take these relations as the initial definitions of X^μ and $S_{\dot{\alpha}}$. And one can show the initial conditions that $v^\dagger v = \mathbb{1}_N$ and $\Delta^{\dagger\dot{\alpha}} v = 0$ now become:

$$\begin{aligned}
&Q_\alpha = -(X+x)^{-2}(X+x)_{\alpha\dot{\alpha}} S^{\dagger\dot{\alpha}} T \\
&T^\dagger \Big(\mathbb{1}_{\mathscr{W}} + S_{\dot{\alpha}} (X+x)^{-2} S^{\dagger\dot{\alpha}} \Big) T = \mathbb{1}_{\mathscr{W}}
\end{aligned} \tag{3.30}$$

We can identify the second equation as $T^\dagger M^\dagger M T = \mathbb{1}$, such that we identify the $M(x)T(x)$ as a group element in $U(N)$. This can be easily generated to SO and SP group as long as we replace the definition from Hermitian conjugate to transpose or add an anti-symmetric matrix $\mathbb{J}_{2N}$ respectively.

And for the solutions in adjoint representation, we can assume the following form:

$$\psi_\alpha = i v^\dagger (\mathscr{C}\mathscr{R}\mathscr{B}_\alpha^\dagger - \mathscr{B}_\alpha \mathscr{R}\mathscr{C}^\dagger) v \tag{3.31}$$

Where $\mathscr{C}$ is a $k \times (N+2k)$ constant complex matrix that satisfies $\Delta^{\dagger\dot{\alpha}}\mathscr{C} + \mathscr{C}^\dagger \Delta^{\dot{\alpha}} = 0$. Thus the $2kN$ degree of freedom of $\mathscr{C}$ exactly corresponds to $2kN$ solutions of the Weyl equation.

3.3 ADHM construction for $SO(N)$

We can construct the $SO(N)$ instanton moduli space similarly. First, we notice that the relation between the Dynkin index of adjoint representation and fundamental representation of $SO(N)$ is $\ell_{\text{adj}} = h^\vee \ell_{\text{fund}}$ while it is $\ell_{\text{adj}} = 2h^\vee \ell_{\text{fund}}$ for $SU(N)$ and $SP(N)$. So to get the trace of $F_{\mu\nu} \star F^{\mu\nu}$ in adjoint representation from fundamental representation, as we did in the previous section, we need to replace the vector space $\mathscr{V} = \mathbb{C}^k$ in the construction of $SU(N)$ to $\mathbb{C}^{2k}$.

Choosing the Darboux basis in $\mathscr{V}$ which means that split $\mathbb{C}^{2k}$ to $\mathbb{C}^2 \otimes \mathbb{C}^k$. The first index $A, B, ...$ corresponds to the $\mathbb{C}^2$ part that takes value $1, 2$. And the second index runs from 1 to k.

Thus the Weyl equation can be written as:

$$\nabla^\mu \psi_\mu = 0 \quad \text{and} \quad (\nabla_\mu \psi_\nu)^- = 0 \tag{3.32}$$

Where the index μ, ν is twisted index instead of Lorentz index, and the solutions of Weyl equations consist of four $N \times k$ matrices $\psi_{\alpha,A}$. Again, $\psi_{\alpha A} = \psi_\mu \sigma^\mu_{\alpha A}$.

So the Weyl equations implies the solutions are self-dual, also, are transitive to the gauge transformations.

Since the gauge field $-iA_\mu$ takes $SO(N)$ value, so it is a real matrix. That follows ψ_μ are real, and implies $S_{\dot\alpha}$ that defined from 3.29 can be expanded as $S_{A\dot\alpha} = S_\mu \sigma^\mu_{A\dot\alpha}$ where S_μ are real. And we can also use the definition of X^μ 3.29 to obtain:

$$\epsilon_{CA} X^{\mu,A}{}_B \epsilon^{BD} = (X^{\mu T})_C{}^D \tag{3.33}$$

Therefore, we can introduce a sympletic structure $\mathbb{J}_{2k} = \epsilon_{AB}$ and then the previous condition becomes:

$$\mathbb{J}_{2k} X^\mu \mathbb{J}^T_{2k} = X^{\mu T} \tag{3.34}$$

This condition induces the dual group $SP(k)$ which is a subgroup of $U(2k)$. We can write X^μ and $S_{\dot\alpha}$ in terms of their components

$$X^\mu = \begin{pmatrix} Y^\mu & Z^{\dagger\mu} \\ Z^\mu & Y^{\dagger\mu} \end{pmatrix}, \quad S_{\dot\alpha} = (S_{\dot 1}, S_{\dot 2}) = (J, I^\dagger) = ((K, K'), (-K'^*, K^*)) \tag{3.35}$$

Where Y^μ is a hermitian matrix and Z^μ is an anti-symmetric matrix. Then let the matrices $B_{1,2}$ be:

$$B_{1,2} = \begin{pmatrix} P_{1,2} & Q'_{1,2} \\ Q_{1,2} & P^T_{1,2} \end{pmatrix} \tag{3.36}$$

Where $Q_{1,2}$ and $Q'_{1,2}$ are anti-symmetric. With the definitions of following maps:

$$\begin{aligned}
M_{\mathbb{C}} =& [P_1, P_2] + Q'_1 Q_2 - Q'_2 Q_1 - K'^T K \\
N_{\mathbb{C}} =& Q_1 P_2 - P_2^T Q_1 + P_1^T Q_2 - Q_2 P_1 + K^T K \\
N'_{\mathbb{C}} =& Q'_1 P_2^T - P_2 Q'_1 + P_1 Q'_2 - Q'_2 P_1^T + K'^T K \\
M_{\mathbb{R}} =& \sum_{s=1}^{2} \left([P_s, P_s^\dagger] + Q_s^* Q_s - Q'_s Q'^*_s \right) - K'^T K'^* - K^\dagger K \\
N_{\mathbb{R}} =& \sum_{s=1}^{2} \left(Q_s P_s^\dagger - P_s^* Q_s + Q'^*_s P_s - P_s^T Q'^*_s \right) - K^T K'^* - K'^\dagger K \\
N'_{\mathbb{R}} =& \sum_{s=1}^{2} \left(Q'_s P_s^* - P_s^\dagger Q'_s + Q_s^* P_s^T - P_s Q_s^* \right) - K'^T K^* - K^\dagger K'
\end{aligned} \tag{3.37}$$

And with the fractorization condition 3.8, we can find the ADHM construction for $SO(N)$ with the corresponding moment maps $\mu_{\mathbb{R},\mathbb{C}}$:

$$\mu_{\mathbb{R},\mathbb{C}} = \begin{pmatrix} M_{\mathbb{R},\mathbb{C}} & N'_{\mathbb{R},\mathbb{C}} \\ N_{\mathbb{R},\mathbb{C}} & -M^T_{\mathbb{R},\mathbb{C}} \end{pmatrix} = 0 \tag{3.38}$$

And the quiver representation of ADHM construction of $SO(N)$ is shown in figure:3-2.

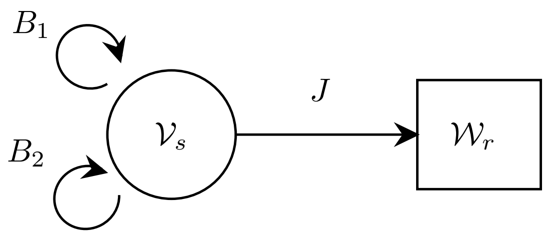


Figure 3-2 The ADHM linear data for $SO(N)$ gauge group in quiver representation. $\mathscr{V}_s$ is a $2k$ dimensional vector space with a symplectic structure which is indicated by subscript s, $\mathscr{W}_r$ is a N dimensional vector space with a real structure which is indicated by subscript r, and B_1, B_2 and J are linear maps between them.

Thus the k instantons moduli space can be given by the kernel of this moment map quotient of the dual group:

$$\mathscr{M}_k^{SO(N)} = \mu^{-1}(0)/SP(k) \tag{3.39}$$

3.4 ADHM construction for $SP(N)$

The relation of the Dynkin index between fundamental and adjoint representation is the same for the SP and SU groups. Thus we don't need to double the instanton charge. While $SP(N)$ is a subgroup of $U(2N)$, which means we can construct the ADHM equation from $SU(2N)$ with the extra condition that $SP(N)$ preserves the symplectic structure $\mathbb{J}_{2N}$.

We again choose the Darboux basis of $\mathscr{W} = \mathbb{C}^{2N}$ which means we again split the vector space into $\mathbb{C}^2 \otimes \mathbb{C}^N$ and correspondingly split the symplectic structure $\mathbb{J}_{2N} = \mathbb{J}_2 \otimes \mathbb{J}_N$. So we split the index into two, the first one A runs from 1 to 2 and the second one runs from 1 to N. Same as $SO(N)$ case, we expand the solution of Weyl equation as the form with twisted index $\psi_{\alpha A} = \psi_\mu \sigma^\mu_{\alpha A}$. The gauge field is taking the $SP(N)$ value now so it must satisfy:

$$\mathbb{J}_{2N} A^*_\mu \mathbb{J}_{2N} = -A_\mu \tag{3.40}$$

Then choosing ψ_μ as real, and following the definition of X^μ and $S_{\dot{\alpha}}$ in 3.29, we can see the matrix X^μ is hermitian, real and symmetric, which means the dual group of $SP(N)$ is $O(k) \subset U(k)$. And we also see that S_μ is real, where $S_{\dot{\alpha}} = S_\mu \sigma^\mu_{A\dot{\alpha}}$. Thus I and J are:

$$I^\dagger = \begin{pmatrix} -K'^* \\ K^* \end{pmatrix} \quad \text{and} \quad J = \begin{pmatrix} K \\ K' \end{pmatrix} \tag{3.41}$$

Thus the ADHM equations for $SP(N)$ are:

$$\begin{aligned}\mu_{\mathbb{C}} =& K^T K' - K'^T K + [B_1, B_2] \\ \mu_{\mathbb{R}} =& K^T K^* - K^\dagger K + K'^T K'^* - K'^\dagger K' + [B_1, B_1^*] + [B_2, B_2^*]\end{aligned} \tag{3.42}$$

Where the k instanton moduli space is now:

$$\mathscr{M}_k^{SP(N)} = \mu^{-1}(0)/O(k) \tag{3.43}$$

And the quiver representation of ADHM data of $SP(N)$ is shown in figure:3-3.

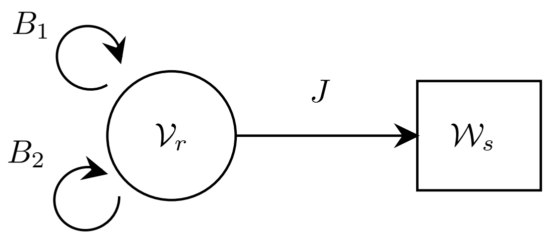


Figure 3-3 The ADHM linear data for $SP(N)$ gauge group in quiver representation. $\mathscr{V}_r$ is a k dimensional vector space with a real structure which is indicated by subscript s, $\mathscr{W}_s$ is a $2N$ dimensional vector space with a symplectic structure which is indicated by subscript r, and B_1, B_2 and J are linear maps between them.

We can summarize the information about the ADHM data in the table:3-1

G	G_D	Size of $\Delta_{\dot{\alpha}}$	Size of v	$\mathscr{V}$	$\mathscr{W}$
$U(N)$	$U(k)$	$k \times (N + 2k)$	$N \times (N + 2k)$	$\mathbb{C}^k$	$\mathbb{C}^N$
$O(N)$	$Sp(k)$	$2k \times (N + 4k)$	$N \times (N + 4k)$	$\mathbb{C}^{2k}$	$\mathbb{R}^N$
$Sp(N)$	$O(k)$	$k \times (2N + 2k)$	$2N \times (2N + 2k)$	$\mathbb{R}^k$	$\mathbb{C}^{2N}$

Table 3-1 The spaces and groups in ADHM data

Chapter 4

Nekrasov partition function

In this chapter, we will construct the Nekrasov partition function in a nutshell that relates to performing the path integral[4]. We have already seen the functional integration is localized on the instanton moduli space, which has been constructed via ADHM data in the previous chapter. However, the ADHM equations are extremely hard to solve. Thus we need to perform the integral with the help of equivariant integration[18] and Ω-background, which we will briefly review in this chapter.

4.1 Equivariant integration

What we want to explore is how to do the integral of instanton moduli space in terms of the whole coordinates space in ADHM construction. More specifically, we want to do the integral on the manifold $\mathscr{M}_k^G = \mu^{-1}(0)/G$ in terms of $\mathscr{X} = (I, J, B_1, B_2)$.

- First consider how to express the integral of $M = \mu^{-1}(0)$ as a submanifold in terms of $\mathscr{X}$. We can view $\mu \in \Gamma(E)$ as a section of a vector bundle of a fiber F: $F \hookrightarrow E \xrightarrow{\pi} \mathscr{X}$. So we can perform the integration over zero locus as follow.

  Let x^μ be coordinates of $\mathscr{X}$, and define a one form $dx^\mu = \psi^\mu$ with de Rham differential $d = \bar{\mathcal{Q}}$ on $\mathscr{X}$. Then we have $\bar{\mathcal{Q}}x^\mu = \psi$ and $\bar{\mathcal{Q}}\psi^\mu = 0$. Then we can construct a multiplet $(\chi, H) \in \Pi F^* \otimes F^*$ with respect to the fiber F, where χ is a fermion so the ΠF^* indicates a change of statistic. With the connection on E being denoted as Γ_μ, we have:

  $$\bar{\mathcal{Q}}\chi = H - \Gamma_\mu \psi^\mu \chi, \quad \bar{\mathcal{Q}}H = -\Gamma_\mu \psi^\mu H + \frac{1}{2}R_{\mu\nu}\psi^\mu\psi^\nu\chi \tag{4.1}$$

  If the bundle E is trivial, 4.1 will then be the following simple form:

  $$\bar{\mathcal{Q}}\chi = H, \quad \bar{\mathcal{Q}}H = 0 \tag{4.2}$$

  Thus the formula we want is:

  $$\int_{M=\mu^{-1}(0)} \alpha = \int_{\mathscr{X}} \mathscr{D}x\mathscr{D}\psi\mathscr{D}H\mathscr{D}\chi\, \imath^*\alpha\, e^{i\bar{\mathcal{Q}}\chi(\mu(x)-\frac{1}{2t}H)} \tag{4.3}$$

  Where $\imath$ is the inclusion map from M to $\mathscr{X}$. And from section 3.1 we know the integral is independent of t.

- Second, we try to do the integral on the $\mathscr{M}_k^G = \mu^{-1}(0)/G$ in terms of $M = \mu^{-1}(0)$ with the help of G-equivariant cohomology of M which isomorphic to de Rham cohomology of M/G:

  $$H_G^*(M) \simeq H^*(M/G) \tag{4.4}$$

Note that G acts freely on M. And the $H_G^*(M)$ can be constructed as follows.

Let Fun($\mathfrak{g}$) be the algebra of function on the lie algebra $\mathfrak{g}$ of the group G. And G can act on functions of Fun($\mathfrak{g}$) by adjoint representation, while acts on forms can be induced by left action on M. Thus we can introduce the G-equivariant complex $\Omega_G^*(M)$ from the original de Rham complex $\Omega^*(M)$ of M as:

$$\Omega_G^*(M) = (\Omega^* \otimes \mathrm{Fun}(\mathfrak{g}))^G \tag{4.5}$$

Where $(...)^G$ means the G-invariant part. Then we can define the Cartan differential $\bar{\mathcal{Q}}$ on M with $\phi \in$ Fun($\mathfrak{q}$) and a vecctor field $V(\phi) = \phi^a V_a$ on M as:

$$\bar{\mathcal{Q}} = d + i_{V(\phi)} \tag{4.6}$$

Where d is the ordinary de Rham differential and $i_{V(\phi)}$ is the interior product by the $V(\phi)$. One can easily show $\bar{\mathcal{Q}}^2 = 0$. Thus the G-equivariant cohomology of M can be defined as:

$$H_G^*(M) = \mathrm{Im}\,\bar{\mathcal{Q}} / \mathrm{Ker}\,\bar{\mathcal{Q}} \tag{4.7}$$

Then taking $\tilde{\alpha}$ in the same equivariant cohomology class as $\alpha(\phi, x, \psi)$, and with introducing the G-invariant metric $(v, w) = g_{\mu\nu} v^\mu w^\nu$ on M, we can get the integral we want:

$$\int_{M/G} \tilde{\alpha} = \int_M \frac{\mathscr{D}x\mathscr{D}\psi\mathscr{D}\phi\mathscr{D}\lambda\mathscr{D}\eta}{\mathrm{Vol}(G)} e^{i\bar{\mathcal{Q}} V_\mu(\lambda)\psi^\mu} \alpha(\phi, x, \psi) \tag{4.8}$$

where we add the projection multiplet (λ, η). Thus the differential operator acts as:

$$\bar{\mathcal{Q}} x^\mu = \psi^\mu, \quad \bar{\mathcal{Q}} \psi^\mu = V^\mu(\phi), \quad \bar{\mathcal{Q}} \lambda = \eta, \quad \bar{\mathcal{Q}} \eta = [\phi, \lambda]. \tag{4.9}$$

We can simplify 4.8 further by choosing the Haar measure on G. Let $d\phi_1 d\phi_2 ... d\phi_r$ consists with the Haar measure of G at its identity ($r = \mathrm{dim}G$), then we have:

$$\oint_M \alpha = \frac{1}{\mathrm{Vol}(G)} \int_{\mathfrak{g}} \prod_{a=1}^{r} \frac{d\phi_a}{2\pi i} \int_M \alpha(\phi) \tag{4.10}$$

Combining this, we can define the formula 4.8 as:

$$\int_{\mu^{-1}(0)/G} \tilde{\alpha} = \oint_{\mu^{-1}(0)} \int \mathscr{D}\eta\mathscr{D}\lambda\, e^{i\bar{\mathcal{Q}} V_\mu(\lambda)\psi^\mu} \alpha \tag{4.11}$$

With the formula 4.11 and 4.3 we can conclude that the formula we want has the following form:

$$\int_{\mathscr{M}_k^G = \mu^{-1}(0)/G} \tilde{\alpha} = \oint_{\mathscr{X}} \int \mathscr{D}\eta\mathscr{D}\lambda\mathscr{D}H\mathscr{D}\chi\, e^{i\bar{\mathcal{Q}}(\chi\mu + \psi_\mu V^\mu(\lambda))} \iota^* \alpha(\phi, x, \psi) \tag{4.12}$$

4.2 Euler class and the Duistermaat-Heckman formula

Before we compute the VEV of a BRST closed observable, we need to introduce two useful mathematical concepts as equivariant Euler class and the Duistermaat-Heckman localization.

- Equivariant Euler class: Consider the part from the exponential of the formula 4.3 with the choice $t = i$ (t is irrelevant):

$$i\bar{\mathcal{Q}}\chi(\mu(x) + \frac{i}{2}H) = -\frac{1}{2}(H - i\mu)^2 - \frac{1}{2}\mu^2 + i\chi\nabla_\mu \mu\psi^\mu + \frac{1}{4}\chi R_{\mu\nu}\psi^\mu\psi^\nu\chi + \frac{1}{2}\chi\phi^a T_a^\varrho \chi \tag{4.13}$$

where ∇_μ is the covariant derivative with connection Γ_μ, and $R_{\mu\nu}$ is the relevant curvature. Note that here μ is the section of the bundle while index μ is the twisted Lorentz index. We can integrate out the H field with Gaussian integral, and make a similar argument in section:3.1 about the integral 4.3 is independent of μ. Thus then we can set $\mu(x) = 0$ and define the equivariant Euler class as:

$$\int_M \alpha = \int_{\mathcal{X}} \mathcal{D}x\mathcal{D}\psi\mathcal{D}H\mathcal{D}\chi\, i^*\alpha e^{i\bar{\mathcal{Q}}\chi(\mu(x)+\frac{i}{2}H)} = \int_{\mathcal{X}} i^*\alpha\, \mathrm{Eu}_g(E) \tag{4.14}$$

Where:

$$g = e^{i\phi} \in G$$
$$\mathrm{Eu}_g(E) = \frac{1}{(2\pi)^{\dim F/2}} \int_{\Pi F} \mathcal{D}\chi\, e^{\frac{1}{4}\chi R_{\mu\nu}\psi^\mu\psi^\nu\chi + \frac{1}{2}\chi\phi^a T_a^\varrho\chi} \tag{4.15}$$

If $\bar{\mathcal{Q}} = d$ is the ordinary de Rham differential, then this becomes the ordinary Euler class:

$$\mathrm{Eu}(E) = \frac{1}{(2\pi)^{\dim F/2}} \int_{\Pi F} \mathcal{D}\chi\, e^{\frac{1}{4}\chi R_{\mu\nu}\psi^\mu\psi^\nu\chi} \tag{4.16}$$

- Duistermaat-Heckman formula: Consider a symplectic manifold M with dimension $2n$ and with symplectic form ω. And a torus $\mathbb{T}$ acts on M symplectically with Lie algebra $\mathfrak{t}$. The usual moment map can be defined as $\mu : M \to \mathfrak{t}^*$. The hamiltonian defined by $h(\xi) = \langle \mu, \xi \rangle$ with $\xi \in \mathfrak{t}$ which indicates $dh(\xi) = i_{V(\xi)}\omega$. With the fixed point x_f on M of the $\mathbb{T}$ action and finally $\omega_\alpha(x_f) \in \mathfrak{t}^*$ is a weight of the action on the tangent space $T_{x_f}M$.

  With all this definition, if we want to express the integral over M that acted on by $\mathbb{T}$ as sum of the fix points x_f, it would be the Duistermaat-Heckman formula:

$$\int_M \frac{\omega^n}{n!} e^{-\langle \mu, \xi \rangle} = \sum_{x_f} \frac{e^{-\langle \mu(x_f), \xi \rangle}}{\prod_\alpha \omega_\alpha} \tag{4.17}$$

  Where $\omega_\alpha \equiv \langle \omega_\alpha(x_f), \xi \rangle$, and the denominator of the right-hand side can be easier computed by the equivariant version of the Atiyah-Singer index theorem with treating the same quantity as the equivariant index of the Dirac operator:

$$\mathrm{Ind}_q \equiv \sum_\alpha \epsilon_\alpha e^{\omega_\alpha} \tag{4.18}$$

with $q \in \mathbb{T}$. And we will introduce the universal bundle in the latter content. Then we have the mapping:

$$\sum_{\alpha} \epsilon_\alpha e^{\omega_\alpha} \mapsto \prod_{\alpha} \omega_\alpha^{\epsilon_\alpha} \tag{4.19}$$

which can in fact be seen as proper time regularization[4,12]. Note that $\epsilon_\alpha = \pm 1$ in the Ind_q comes from the alternating sum of the cohomologies instead of fermionic and bosonic statistics, while the ϵ_α on the power of weights depends on the statistic of the coordinates.

4.3 Ω-background

To compute the prepotential of the partition function, we can use the help of Ω-background to localize all the action in zero-modes. First, $\mathcal{N} = 2$ $4d$ super Yang-Mills theory can be produced by the dimensional reduction from $\mathcal{N} = 1$ $6d$ super Yang-Mills action[12].

And we first briefly talk about this dimensional reduction. We can impose the periodic condition on the x_4 and x_5 coordinates which means taking $\mathbb{R}^6$ to $\mathbb{R}^4 \times \mathbb{T}^2$, and make x_4 and x_5 curl up to a very small circle such that all the superfields do not depend on them. So for $6d$ $\mathcal{N} = 1$ Yang-Mills action:

$$S_{\mathcal{N}=1,d=6} = \frac{1}{g^2 h^\vee} \int d^4x \ \mathrm{Tr} \left[-\frac{1}{4} F_{IJ} F^{IJ} + \frac{i}{2} \bar{\Psi}_A \Gamma^I \nabla_I \Psi^A \right] \tag{4.20}$$

With Ψ^A, $A = 1, 2$ as two $6d$ Weyl spinors, I, J runs from 0 to 5 and Γ^I is the $6d$ Gamma mattix. By the compactification, $F_{\mu 4} = \nabla_\mu A_4$ and $F_{\mu 5} = \nabla_\mu A_5$. If we let the Weyl spinors as:

$$\Psi^A = (\psi^A_\alpha,\ \chi^{A,\dot{\alpha}},\ 0,\ 0)^T \tag{4.21}$$

and define $H = \frac{A_4 + iA_5}{\sqrt{2}}$, then the $\mathcal{N} = 1$ $6d$ YM action becomes exactly $\mathcal{N} = 2$ $4d$ YM action 2.24. Then the Ω-background can re realized as the compacted torus $\mathbb{T}^2$ acts on the spacetime manifold $\mathbb{R}^4$ as a Lorentz rotations, which means that:

$$dx^\mu \mapsto dx^\mu + {\Omega_4}^\mu{}_\nu x^\nu dx^4 + {\Omega_5}^\mu{}_\nu x^\nu dx^5 \equiv dx^\mu + V^\mu_a dx^a \tag{4.22}$$

Where $a = 1, 2$. And we can write the metric as:

$$\begin{aligned} ds_6^2 =& g_{\mu\nu}(dx^\mu + V^\mu_a dx^a)(dx^\nu + V^\nu_a dx^a) - (dx^4)^2 - (dx^5)^2 \\ =& g_{\mu\nu} e^{(\mu)}_I e^{(\nu)}_J dx^I dx^J - e^{(a)}_I e^{(a)}_J dx^I dx^J \\ =& G_{IJ} dx^I dx^J \end{aligned} \tag{4.23}$$

where we have defined the general metric G_{IJ} which is flat if $[\Omega_4, \Omega_5] = 0$ and the $6d$ vielbein $e^{(J)}_I$.

With the "twisted" version of V^μ_a and ${\Omega_a}^\mu{}_\nu$:

$$V^\mu = \frac{1}{\sqrt{2}}(V^\mu_4 + iV^\mu_5) \quad \text{and} \quad \Omega^\mu{}_\nu = \frac{1}{\sqrt{2}}({\Omega_4}^\mu{}_\nu + i{\Omega_5}^\mu{}_\nu) \tag{4.24}$$

And with all this information, we can compute the new action under the Ω-background:

$$S^{\Omega}_{\mathcal{N}=1,d=6} = \frac{1}{g^2 h^\vee} \int_{\mathbb{R}^4} \mathrm{Tr}\left[-\frac{1}{4} F_{(I)(J)} F^{(I)(J)} + \frac{i}{2} \bar{\Psi}_A \Gamma^I e^J_{(I)} \nabla_J \Psi^A \right] \tag{4.25}$$

where $F_{(I)(J)}F^{(I)(J)} = \sqrt{-G} F_{IJ} F_{KL} G^{IK} G^{JL}$. And we identify the corresponding terms before and after applying the Ω-background:

$$\begin{aligned} H &\mapsto \mathscr{H} = H - iV^\mu \nabla_\mu + \frac{i}{2} \Omega^{\mu\nu} S_{\mu\nu} \\ \tau &\mapsto \tau(x,\theta) = \tau - \frac{1}{\sqrt{2}} (\bar{\Omega}_{\mu\nu})^+ \theta^\mu \theta^\nu \end{aligned} \tag{4.26}$$

where we already used the twisted coordinates $\theta^\mu = \bar{\sigma}^{\mu,A\alpha} \theta_{A,\alpha}$. Generally, we choose the Ω-background as the simple form with the Ω matrix:

$$\Omega = \frac{1}{\sqrt{2}} \begin{pmatrix} & & & -\epsilon_1 \\ & & -\epsilon_2 & \\ & \epsilon_2 & & \\ \epsilon_1 & & & \end{pmatrix} \tag{4.27}$$

This is equivalent to the coordinates transform:

$$(z_1, z_2) \mapsto (e^{i\epsilon_1} z_1, e^{i\epsilon_2} z_2) \tag{4.28}$$

where (z_1, z_2) is the coordinates $x_{\alpha\dot{\alpha}}$:

$$x_{\alpha\dot{\alpha}} = \sigma^\mu_{\alpha\dot{\alpha}} x_\mu = \begin{pmatrix} z_1 & -z_2^* \\ z_2 & z_1^* \end{pmatrix} \tag{4.29}$$

4.4 Nekrasov partition function

To get the partition function we first look at the VEV of BRST closed observables $\langle \mathcal{O} \rangle$, then the partition function is nothing but $\langle 1 \rangle$.

First, consider pure Yang-Mills theory without the hypermultiplet, the VEV of $\mathcal{O}$ is given by 3.1. We identify the section $\mu(x)$ in formula 4.12 as the anti-self-dual curvature $(F_{\mu\nu})^-$ from the definition of $\tilde{V}$ in 3.3. Then we identify the whole group that needs to be quotient out as a group that consists of two parts: the gauge group $\mathscr{G} = \{g : \mathbb{R}^4 \to G : g(\infty) = \mathbb{1}_G\}$ and the rigid gauge transformation G_∞ that corresponds to transformation at infinity.

The transformation produced by the BRST operator $\bar{\mathcal{Q}}$ depends on the gauge transformation with parameter ϕ which we have already discussed in section 2.4. Also, from section 2.3, we concluded that in order to preserve SUSY, we need to let all the fields be elements of the Cartan subgroup of the gauge group at infinity, thus $\phi(x) \to a$ with $a \in \mathfrak{g}$. Then the vacuum depends on the field configuration which leads us to denote the VEV of $\mathcal{O}$ as $\langle \mathcal{O} \rangle_a$. Also consider what we do in section 4.1, the gauge transformation, in fact, contains not only G_∞ but also the dual group G_D when we use ADHM data. Thus the square of the Cartan differential operator then consists of three parts rather with the addition of dual group $\mathscr{D}(\phi)$ compared to before 2.48:

$$\bar{\mathcal{Q}}^2 = \mathscr{G}(a) + \mathrm{F}(m) + \mathscr{D}(\phi) \tag{4.30}$$

The VEV of $\mathcal{O}$ then is the sum of all the integration over different instanton moduli space:

$$\langle \mathcal{O} \rangle_a = \sum_{k=0}^{\infty} \mathfrak{q}^k \int_{\mathcal{M}_k} \tilde{\mathcal{O}}_k \tag{4.31}$$

Where q is defined in 2.32.

Consider adding matter into the theory, we can use the G-equivariant Euler class and identify the multiplet (v_α, h_α) and $(q^{\dot{\alpha}}, \mu^{\dot{\alpha}})$ as (χ, H) and (x, ψ) in section 4.2 and redefine the section as pair $\mu = ((F_{\mu\nu})^- + iq^\dagger \bar{\sigma}_{\mu\nu} q,\ \sigma^\mu_{\alpha\dot{\alpha}} \nabla_\mu q^{\dot{\alpha}})$. Then the VEV is localized to the muduli space of Seiberg-Witten monopoles[2] which satisfies the monopole equations $\mu = 0$, i.e:

$$(F_{\mu\nu})^- + iq^\dagger_{\dot{\alpha}} \bar{\sigma}_{\mu\nu}{}^{\dot{\alpha}}{}_{\dot{\beta}} q^{\dot{\beta}} = 0 \quad \text{and} \quad \sigma^\mu_{\alpha\dot{\alpha}} \nabla_\mu q^{\dot{\alpha}} = 0 \tag{4.32}$$

And mod out the gauge transformation.

If we integral out h^α in 2.45, the remaining action is exactly the equivariant Euler class 4.15 of a bundle with fiber as solutions of Weyl equation over the instanton moduli space $\mathcal{M}_k$, which we denote as $\mathcal{D}_k$. The action on the space of this Weyl solution becomes:

$$S_{\text{mat}} = -\frac{1}{\pi^2} \int d^4x \,\text{Tr} \left[\bar{v}^\alpha (\phi + m) v_\alpha + \bar{h}^\alpha h_\alpha \right] \tag{4.33}$$

Then finally, the VEV of Yang-Mills theory with matter becomes:

$$\langle \mathcal{O} \rangle_a = \sum_{k=0}^{\infty} \mathfrak{q}^k \int_{\mathcal{M}_k} \tilde{\mathcal{O}}_k \,\text{Eu}_g(\mathcal{D}_k) \tag{4.34}$$

With the group combines with matter, dual group and gauge group: $g = (e^m, e^\phi, e^a) \in G_F \times G_D \times G_\infty$.

Now take into account the Ω-background, in which the action is still BRST exact such that we can still use the localization we have discussed to modify the action. Using the formula 2.36, we can plug in the equation of motion for ψ_μ and $\bar{\psi}_{\mu\nu}$ after begin modified under Ω-background. Then one can show the action S_{YM} is $\bar{\mathcal{Q}}_\Omega$ exact[4] with:

$$\bar{\mathcal{Q}}_\Omega = \bar{\mathcal{Q}} + \frac{1}{2\sqrt{2}} \Omega^\mu{}_\nu x^\nu \mathcal{Q}_\mu \tag{4.35}$$

And the modification of Ω-background only changes the topological term with the coupling constant being shifted:

$$\tau \to \tau(x, \theta) = \tau - \frac{1}{\sqrt{2}} (\bar{\Omega}_{\mu\nu})^+ \theta^\mu \theta^\nu + \frac{1}{4} \bar{\Omega}_{\mu\nu} \Omega^\mu{}_\rho x^\rho x^\nu \tag{4.36}$$

Also, using the SUSY algebra $\{\bar{\mathcal{Q}}, \mathcal{Q}_\mu\} = 4i\nabla_\mu$, we can obtain the square of BRST operator in this case: $(\bar{\mathcal{Q}}_\Omega)^2 = i\sqrt{2}\Omega^\mu{}_\nu x^\nu \partial_\nu$.

Note that the modified complex coupling constant can be annihilated by the operator $\mathscr{R}_\Omega$. And since we have 2.32, we can show this operator also annihilate the dynamically energy scale Λ. And the operator takes the form[4]:

$$\mathscr{R}_\Omega = \theta^\mu \partial_\mu + \frac{1}{2\sqrt{2}} \Omega^\mu{}_\nu x^\mu \frac{\partial}{\partial \theta^\mu} \tag{4.37}$$

An important observation is, using the $\bar{\mathcal{Q}}_\Omega$ exact property, we can localize the action on the zero-modes of the superfields, where the action be the form:

$$Z(a;\epsilon) = \langle 1 \rangle_a = \exp\left[\mathrm{Im}\left(\frac{1}{2\pi i} \int d^4 x d^4 \theta \, \mathscr{F}(-\frac{1}{2\sqrt{2}} a, \Lambda(x,\theta)) \right) \right] \tag{4.38}$$

We can view the $\mathscr{R}_\omega$ operator as Cartan differential: $\mathscr{R}_\omega = d + \frac{1}{2\sqrt{2}} i_V$, then apply the Duistermaat-Heckman formula with respect to the Ω-background action $\mathbb{T}_{\epsilon_1,\epsilon_2}$ with the Ω matrix is chosen as 4.27. And finally, we have the relation between the partition function and the prepotential:

$$Z(a,m,\Lambda,\epsilon) = \exp\left[\frac{1}{\epsilon_1 \epsilon_2} \mathscr{F}(a,m,\Lambda,\epsilon) \right] \tag{4.39}$$

Then we use the formula we just derived 4.34, the partition function really is:

$$Z(a,m;\epsilon) = \exp\left[\frac{1}{\epsilon_1 \epsilon_2} \mathscr{F}(a,m,\Lambda,\epsilon) \right] = \langle 1 \rangle_a = \sum_{k=0}^{\infty} \mathfrak{q}^k \int_{\mathscr{M}_k} \mathrm{Eu}_g(\mathscr{D}_k) \tag{4.40}$$

where we already consider the Ω-background, thus with the extra variables ϵ and the group now becomes: $g = (e^m, e^\phi, e^a, e^{i\epsilon}) \in \mathbb{T}_F \times \mathbb{T}_D \times \mathbb{T}_\infty \times \mathbb{T}_{\epsilon_1,\epsilon_2}$.

Lastly, we take the perturbative contribution into account, the Nekrasov partition function can be obtained as:

$$Z(a,m,\Lambda,\epsilon) = Z^{\mathrm{pert}}(a,m,\Lambda,\epsilon) \sum_{k=0}^{\infty} \mathfrak{q}^k Z_k(a,m,\epsilon) = \exp\left[\frac{1}{\epsilon_1 \epsilon_2} \mathscr{F}(a,m,\Lambda,\epsilon) \right] \tag{4.41}$$

Where the instanton partition function of given k is then:

$$Z_k(a,m,\epsilon) = \int_{\mathscr{M}_k} \mathrm{Eu}_g(\mathscr{D}_k) \tag{4.42}$$

4.5 Universal bundle and Chern character

In order to perform the Duistermaat-Heckman formula 4.17 using the Atiyah-Singer index theorem 4.18, we need to find all the representations of the solutions of the Weyl equation. However, we can directly obtain all the weights that appear in 4.17 by using the equivariant Chern character of the universal bundle[4,12,19].

For a manifold equipped with a complex structure and a hermitian metric, we can define a complex spin structure on it. And the complexified tangent bundle $TM \otimes \mathbb{C}$ of M can be viewed as homomorphisms from $S_+ \otimes L$ to $S_- \otimes L$, where $S_\pm$ is the spinor bundles with left or right chiral and L is the determinant bundle.

The dotted and undotted spinors are sections of S_+ and S_- respectively. And the Ω-backgroud action $\mathbb{T}_{\epsilon_1,\epsilon_2}$ acts on those sections as:

$$\chi_{\dot{\alpha}} \mapsto U_{+\dot{\alpha}}{}^{\dot{\beta}} \chi_{\dot{\beta}} \quad \text{and} \quad \psi_\alpha \mapsto U_{-\alpha}{}^{\beta} \psi_\beta \tag{4.43}$$

Where $U_\pm = \mathrm{diag}(e^{i\epsilon_\pm}, e^{-i\epsilon_\pm})$ with $\epsilon_\pm = \frac{1}{2}(\epsilon_1 \pm \epsilon_2)$. And the section of L transform as $s \mapsto e^{i\epsilon_+} s$.

And then we can express the ADHM data as a complex as follow:

$$\mathscr{V} \otimes \mathscr{L}^{-1} \xrightarrow{\tau} \mathscr{V} \otimes \mathscr{S}_- \oplus \mathscr{W} \xrightarrow{\sigma} \mathscr{V} \otimes \mathscr{L} \tag{4.44}$$

where $\tau \circ \sigma = 0$ to make sure it is indeed a complex, and $\mathscr{L}$, $\mathscr{S}_-$ are fibers of L, S_- respectively.

The maps τ and σ are defined as:

$$\tau = (B_1, B_2, I)^T, \quad \sigma = (B_2, -B_1, J) \tag{4.45}$$

Thus their composition $\tau \circ \sigma = 0$ is the ADHM equation.

After constructing the instanton moduli space $\mathscr{M}$ by ADHM data, we can define the coordinates on it as $\{m^I\}$, the index $I = 1, ..., \dim \mathscr{M}$. The tangent space of $\mathscr{M}$ is spanned by instantons which we can also fix a set of basis denoted as $\{a^I_\mu(x, m)\}$. The different point on $\mathscr{M}$ corresponds to different instanton gauge fields denoted as $A_\mu(x, m)$. Thus with the compensating gauge transformation α_I, we can define a connection one-form $\mathbb{A}$ on the manifold $\mathscr{M} \times \mathbb{R}^4$: $\mathbb{A} = A_\mu dx^\mu + \alpha_I dm^I$. This can be seen as the vector bundle $\mathscr{E}$ with fiber $\mathscr{W}$ on the manifold $\mathscr{M} \times \mathbb{R}^4$, which is exactly the universal bundle.

With the universal bundle $\mathscr{E}$, we can compute the equivariant index of the Dirac operator via the Atiyah-Singer theorem:

$$\mathrm{Ind}_q = \sum_\alpha \epsilon_\alpha e^{\omega_\alpha} = \int_{\mathbb{C}^2} \mathrm{Ch}_q(\mathscr{E})\, \mathrm{Td}_q(\mathbb{C}^2) \tag{4.46}$$

Where we denote the element of the torus $\mathbb{T} = \mathbb{T}_D \times \mathbb{T}_\infty \times \mathbb{T}_{\epsilon_1,\epsilon_2} \times \mathbb{T}_F$ as q. ω_α as we already seen, is weight of the $\mathbb{T}$ action on fixed point. And $\mathrm{Ch}_q(\mathscr{E})$ is the equivariant Chern character defined as:

$$\begin{aligned}
\mathrm{Ch}_q(\mathscr{E}) &= \mathrm{Tr}_{\mathscr{E}}(q) \\
&= \mathrm{Tr}_{\mathscr{W}}(q) + \mathrm{Tr}_{\mathscr{V}}(q)\Big(\mathrm{Tr}_{\mathscr{S}_-}(q) - \mathrm{Tr}_{\mathscr{L}}(q) - \mathrm{Tr}_{\mathscr{L}^{-1}}(q) \Big) \\
&= \mathrm{Tr}_{\mathscr{W}}(q) - (e^{i\epsilon_1} - 1)(e^{i\epsilon_2} - 1) e^{-i\epsilon_+} \mathrm{Tr}_{\mathscr{V}}(q)
\end{aligned} \tag{4.47}$$

And finally the factor $\mathrm{Td}_q(\mathbb{C}^2)$ is the equivariant Todd class of $\mathbb{C}^2$:

$$\mathrm{Td}_q(\mathbb{C}^2) = \frac{\epsilon_1 \epsilon_2}{(e^{i\epsilon_1} - 1)(e^{i\epsilon_2} - 1)} \tag{4.48}$$

We can perform the integral 4.46 with the help of Duistermaat-Heckman formula 4.17 with noticing the only fixed point of the $\mathbb{T}_{\epsilon_1,\epsilon_2}$ action is the origin $(z_1, z_2) = (0, 0)$ with the weights $i\epsilon_{1,2}$:

$$\mathrm{Ind}^{\mathrm{fund}}_q = \sum_\alpha \epsilon_\alpha e^{\omega_\alpha} = \frac{\mathrm{Ch}_q(\mathscr{E})|_{z_1 = z_2 = 0}}{(e^{i\epsilon_1} - 1)(e^{i\epsilon_2} - 1)} \tag{4.49}$$

And we denote the elements in $\mathbb{T}$ as:

$$\begin{aligned} q_G &= \mathrm{diag}(ia_1, ..., ia_N) \in \mathbb{T}_\infty \\ q_D &= \mathrm{diag}(i\phi_1, ..., i\phi_k) \in \mathbb{T}_D \\ q_F &= \mathrm{diag}(im_1, ..., im_{N_f}) \in \mathbb{T}_F \end{aligned} \tag{4.50}$$

where $N = \dim G$, k is the instanton number and N_f is the number of flavors. Thus we can obtain the index under the fundamental representation of $SU(N)$ as:

$$\mathrm{Ind}_q^{\mathrm{fund}} = \frac{1}{(e^{i\epsilon_1} - 1)(e^{i\epsilon_2} - 1)} \sum_{f=1}^{N_f} \sum_{\ell=1}^{N} e^{ia_\ell + im_f} - \sum_{f=1}^{N_f} \sum_{i=1}^{k} e^{i\phi_i - i\epsilon_+ + im_f} \tag{4.51}$$

We can do a similar thing to SO and SP gauge groups to obtain their index, which we omit in this paper.

Furthermore, we can use the relation between representations to compute the index in other representations. For example, the adjoint representation of $SU(N)$ is the traceless part of the tensor product of fundamental and anti-fundamental representation, thus:

$$\mathrm{Ind}_q^{\mathrm{adj}} = \int_{\mathbb{C}^2} \mathrm{Ch}_q(\mathscr{E} \otimes \mathscr{E}^*) \,\mathrm{Td}_q(\mathbb{C}^2) = \int_{\mathbb{C}^2} \mathrm{Ch}_q(\mathscr{E}) \,\mathrm{Ch}_q(\mathscr{E}^*) \,\mathrm{Td}_q(\mathbb{C}^2) \tag{4.52}$$

Chapter 5

Evaluating instanton partition function

In this chapter, we use the tools we have developed to express the closed-form expression of $SU(N)$ instanton partition functions, and list the $SO(N)$ and $SP(N)$ cases[4,20–21]. Also, we introduce Jeffrey-Kirwan residue[22–23] and use it to compute some examples.

5.1 $SU(N)$ without matter

We will first derive the $SU(N)$ instanton partition function without matter in this section.

Using ADHM data, we know that the instanton moduli space is the hyper-Kähler quotient of the set (B_1, B_2, I, J). The ADHM equations are $\mu_{\mathbb{R}} = 0$ and $\mu_{\mathbb{C}} = 0$ from 3.26. So we introduce two supplementary multiplets according to the general cases we have discussed in section 4.1:

$$(\chi_{\mathbb{R}}, H_{\mathbb{R}}), \quad (\chi_{\mathbb{C}}, H_{\mathbb{C}}) \tag{5.1}$$

The formula 4.42 can be expressed as the finite-dimensional form:

$$Z_k(a;\epsilon) = \int \frac{\mathscr{D}\phi}{\mathrm{Vol}(G)} \mathscr{D}\eta \mathscr{D}\lambda \mathscr{D}\chi \mathscr{D}H \mathscr{D}B_1 \mathscr{D}B_2 \mathscr{D}I \mathscr{D}J \mathscr{D}\psi \; e^{i\bar{\mathcal{Q}}(\chi\mu + t\chi H + \psi V(\lambda))} \tag{5.2}$$

Where:

$$\begin{aligned}
\chi\mu &= \mathrm{Tr}\left[\chi_{\mathbb{R}}\mu_{\mathbb{R}} + \frac{1}{2}\left(\chi_{\mathbb{C}}^{\dagger}\mu_{\mathbb{C}} + \chi_{\mathbb{C}}\mu_{\mathbb{C}}^{\dagger}\right)\right] \\
\chi H &= \mathrm{Tr}\left[\chi_{\mathbb{R}}H_{\mathbb{R}} + \frac{1}{2}\left(\chi_{\mathbb{C}}^{\dagger}H_{\mathbb{C}} + \chi_{\mathbb{C}}H_{\mathbb{C}}^{\dagger}\right)\right] \\
\psi V(\lambda) &= \mathrm{Tr}\left[\sum_{s=1}^{2}(\psi_s[\lambda, B_s^{\dagger}] + \bar{\psi}_s[\lambda, B_s])\right. \\
&\qquad \left. + \psi_I \lambda I - I^{\dagger}\lambda\bar{\psi}_I - J\lambda\bar{\psi}_J + \psi_J \lambda J^{\dagger}\right]
\end{aligned} \tag{5.3}$$

After integrating out the fields $\chi_{\mathbb{R}}$, H and the projection multiplet (λ, η), the superspace is now spanned by (B_1, B_2, I, J) and $\chi_{\mathbb{C}}$, since they acted by $\bar{\mathcal{Q}}$ can produce all other fields. And they acted by the torus action $\mathbb{T} = \mathbb{T}_D \times \mathbb{T}_\infty \times \mathbb{T}_{\epsilon_1,\epsilon_2}$ will transform as:

$$\begin{aligned}
B_s &\mapsto e^{\phi} B_s e^{-\phi} e^{i\epsilon_s} \\
I &\mapsto e^{\phi} I e^{-a} e^{-i\epsilon_+} \\
J &\mapsto e^{a} J e^{-\phi} e^{-i\epsilon_+} \\
\chi_{\mathbb{C}} &\mapsto e^{\phi} \chi_{\mathbb{C}} e^{-\phi} e^{-i2\epsilon}
\end{aligned} \tag{5.4}$$

The infinitesimal version would lead us to the weights of the action on different coordinates:

$$\begin{aligned} B_{s,ij} &: \quad \phi_i - \phi_j + \epsilon_s \\ I_{i\ell} &: \quad \phi_i - a_\ell - \epsilon_+ \\ J_{\ell i} &: \quad a_\ell - \phi_i - \epsilon_+ \\ \chi_{\mathbb{C},ij} &: \quad \phi_i - \phi_j - 2\epsilon_+ \end{aligned} \tag{5.5}$$

Where we denote the torus elements:

$$\begin{aligned} a &= \mathrm{diag}(ia_1, ..., ia_N) \quad \text{with} \quad N = \dim SU(N) \\ \phi &= \mathrm{diag}(i\phi_1, ..., i\phi_k) \quad \text{with} \quad k = \dim G_D \end{aligned} \tag{5.6}$$

One can observe that the fixed point of the action is only the origin. Thus we can use the Duistermaat-Hackman formula 4.17 to compute the integration:

First we integrate out the fields $\chi_{\mathbb{C}}$ and $\chi_{\mathbb{C}}^\dagger$, we can obtain the factor that contributed by them:

$$\prod_{i,j}(2\epsilon_+ - \phi_i + \phi_j) \tag{5.7}$$

Where $\epsilon_\alpha = -1$ in the formula 4.17 will bring this factor on the numerator. Similarly, for I and J, the factor is:

$$\prod_{i=1}^{k}\prod_{\ell=1}^{N}(\epsilon_+ \pm (\phi_i - a_\ell))^{-1} \tag{5.8}$$

with the $\epsilon_\alpha = 1$. And for B_s and $B_s^\dagger$ fields, we have:

$$\prod_{i,j}(\epsilon_{1,2} + \phi_i - \phi_j) \tag{5.9}$$

Where for convenience, we will use the following short-hand notation in the following content:

$$\begin{aligned} (\epsilon_{1,2} + \phi_i - \phi_j) &\equiv (\epsilon_1 + \phi_i - \phi_j)(\epsilon_1 + \phi_i - \phi_j) \\ (\epsilon_+ \pm (\phi_i - a_\ell)) &\equiv (\epsilon_+ + (\phi_i - a_\ell))(\epsilon_+ - (\phi_i - a_\ell)) \end{aligned} \tag{5.10}$$

Combine with the Haar measure of $U(k)$ dual group:

$$\frac{\mathscr{D}\phi}{\mathrm{Vol}(G)} \mapsto \frac{1}{k!}\prod_{i=1}^{k}\frac{d\phi_i}{2\pi i}\prod_{i\neq j}^{k}(\phi_i - \phi_j) \tag{5.11}$$

which can be viewed as the contribution from the weight of $\chi_{\mathbb{R}}$: $\phi_i - \phi_j$.

Thus we can then obtain the $SU(N)$ instanton partition function with gauge field taking the adjoint representation in $4d$ $\mathcal{N} = 2$ pure super Yang-Mills theory under the Ω-backgroud:

$$Z_{\mathrm{inst}}(a;\epsilon) = \sum_{k=0}^{\infty}\mathfrak{q}^k \int \prod_{i=1}^{k}\frac{d\phi_i}{2\pi i} z_k^{\mathrm{adj,gauge}}(a,\phi;\epsilon) \tag{5.12}$$

Where:

$$z_k^{\text{adj,gauge}}(a,\phi;\epsilon) = \frac{1}{k!}\prod_{s=1}^{N}\prod_{i=1}^{k}\frac{1}{\left(\epsilon_+ \pm (\phi_i - a_s)\right)}\prod_{i\neq j}^{k}(\phi_i-\phi_j)\prod_{i,j}^{k}\frac{2\epsilon_+ - \phi_i + \phi_j}{\epsilon_{1,2}-\phi_i+\phi_j} \tag{5.13}$$

5.2 $SU(N)$ with matter

We already discuss the case with matter. In fac,t it is similar to the case without matter with the addition of considering the equivariant Euler class $\mathrm{Eu}_g(\mathscr{D}_k)$. We first consider the fundamental representation.

The integral will localize on the space of solutions of Weyl equations which is in the formula 3.28. The solutions are depend on the vector $x \in \mathscr{V} = \mathbb{C}^k$, so the formula becomes:

$$\psi_x^\alpha = \bar{\mathscr{Q}}^\alpha \mathscr{R} x \tag{5.14}$$

Since the solutions are bosonic, we can introduce the fermionic solution ξ on the space $\Pi\mathscr{V}$ where, again, the symbol Π denotes changing the statistic. With the definition 3.29, the action of matter sector 4.33 can rewrite as:

$$\int \mathscr{D}x\mathscr{D}\xi\, e^{-t\bar{x}x - t\bar{\xi}(\phi+m-i\epsilon_+)\xi} \tag{5.15}$$

Where t can be arbitrary according to section 3.1.

The torus action $\mathbb{T} = \mathbb{T}_D \times \mathbb{T}_{\epsilon_1,\epsilon_2} \times \mathbb{T}_F$ act on the fermionic coordinate ξ as:

$$\xi \mapsto e^{\phi} e^{im} \xi e^{-i\epsilon_+} \tag{5.16}$$

where the torus element:

$$m = \mathrm{diag}(im_1, ...im_{N_f}) \tag{5.17}$$

The weight can be obtained as:

$$\phi_i + m_f - \epsilon_+, \quad i = 1, ..., k;\ lf = 1, ..., N_f \tag{5.18}$$

Thus the contribution now is:

$$z_k^{\text{fund}}(\phi, m;\epsilon) = \prod_{i=1}^{k}\prod_{f=1}^{N_f}(\phi_i + m_f - \epsilon_+) \tag{5.19}$$

Which is exactly the second term of 4.51 after the mapping 4.19. While the first term of 4.51 comes from the contribution of the perturbative part.

We can consider one matter in adjoint representation with mass m using the formula 4.52 which equals to sum over the fixed point:

$$\begin{aligned}\mathrm{Ind}_q^{\mathrm{adj}} =&\frac{\mathrm{Ch}_q(\mathscr{E})\,\mathrm{Ch}_q(\mathscr{E}^*)|_{z_1=z_2=0}}{(e^{i\epsilon_1}-1)(e^{i\epsilon_2}-1)}\\ =&\frac{1}{(e^{i\epsilon_1}-1)(e^{i\epsilon_2}-1)}\left[N+\sum_{\ell\neq m}^{N}e^{ia_\ell-ia_m}\right]\\ &-\sum_{i=1}^{k}\sum_{\ell=1}^{N}(e^{i\phi_i-i\epsilon_+-ia_\ell+im}+e^{-i\phi_i+ia_\ell-i\epsilon_++im})+k(1-e^{im-i\epsilon_1})(1-e^{im-i\epsilon_2})\\ &+\sum_{i\neq j}^{k}e^{i\phi_i-i\phi_j}\left(e^{im}+e^{-i2\epsilon_++im}-e^{-i\epsilon_1+im}-e^{-i\epsilon_2+im}\right)\end{aligned} \tag{5.20}$$

Thus by the mapping 4.19, we can read off the contribution from matter in adjoint representation:

$$z_k^{\mathrm{adj,matter}}=\prod_{i=1}^{k}\prod_{s=1}^{N}\Big((\phi_i-a_s)\pm(m-\epsilon_+)\Big)\prod_{i,j}^{k}\frac{m-\epsilon_{1,2}+\phi_i-\phi_j}{(m+\phi_i-\phi_j)(m-2\epsilon_++\phi_i-\phi_j)} \tag{5.21}$$

Similarly, the partition function with one symmetric or anti-symmetric representation of the matter can also be obtained in this way. First, compute the equivariant Chern character:

$$\mathrm{Ch}_q^{\mathrm{sym,anti}}(\mathscr{E})=\frac{1}{2}\left[(\mathrm{Ch}_q^{\mathrm{fund}})^2\pm\mathrm{Ch}_{q^2}^{\mathrm{fund}}\right] \tag{5.22}$$

Then the partition function that comes from matter can be read off:

$$z_k^{\mathrm{sym}}=\prod_{i=1}^{k}(2\phi_i+m\pm\epsilon_-)\prod_{s=1}^{N}(\phi_i+a_s+m)\prod_{i<j}^{k}\frac{\phi_i+\phi_j+m\pm\epsilon_-}{-\epsilon_+\pm(\phi_i+\phi_j+m)} \tag{5.23}$$

$$z_k^{\mathrm{anti}}=\prod_{i=1}^{k}\frac{\prod_{s=1}^{N}(\phi_i+a_s+m)}{-\epsilon_+\pm(2\phi_i+m)}\prod_{i<j}^{k}\frac{\phi_i+\phi_j+m\pm\epsilon_-}{-\epsilon_+\pm(\phi_i+\phi_j+m)} \tag{5.24}$$

5.3 $SO(N)$ and $SP(N)$

For the $SO(N)$ gauge group, its elements in Cartan Lie subalgebra can be represented as:

$$a=\mathrm{diag}\left\{\begin{pmatrix}0&-a_1\\a_1&0\end{pmatrix},\ldots,\begin{pmatrix}0&-a_n\\a_n&0\end{pmatrix},\diamond\right\} \tag{5.25}$$

Where $\diamond=0$ for N is odd and is absent when N is even and $n=\lfloor N/2\rfloor$. And we can diagonalize them as:

$$\tilde{a}=U^\dagger aU=\mathrm{diag}(ia_1,-ia_1,\ldots,ia_n,-ia_n,\diamond) \tag{5.26}$$

While the dual group is $SP(k)$, we can choose the diagonal elements as:

$$\phi = \begin{pmatrix} \tilde{\phi} & 0 \\ 0 & -\tilde{\phi} \end{pmatrix}, \quad \tilde{\phi} = \mathrm{diag}(i\phi_1, ..., i\phi_k) \tag{5.27}$$

With these diagonalized elements, we can follow the procedure from the $SU(N)$ case to obtain their partition function.

We omit the computation and then list some of the results:

$$z_k^{\mathrm{fund}} = \prod_{i=1}^{k}(m - \epsilon_+ \pm \phi_i) \tag{5.28}$$

$$z_k^{\mathrm{adj,gauge}} = \frac{(2\epsilon_+)^k}{\epsilon_{1,2}^k} \prod_{i<j}^{k} \frac{(\pm\phi_i \pm \phi_j)(\phi_i \pm \phi_j \pm 2\epsilon_+)}{\phi_i \pm \phi_j \pm \epsilon_{1,2}} \prod_{i=1}^{k} \frac{(\pm 2\phi_i)(2\epsilon_+ \pm 2\phi_i)}{(\phi_i \pm \epsilon_+)^\chi \prod_{s=1}^{n}(\phi_i \pm \epsilon_+ \pm a_s)} \tag{5.29}$$

$$\begin{aligned} z_k^{\mathrm{adj,matter}} =& \frac{(m - \epsilon_{1,2})^k}{m^k (m - 2\epsilon_+)^k} \prod_{i<j}^{k} \frac{\Big(\phi_i \pm \phi_j \pm (m - \epsilon_{1,2})\Big)}{(\phi_i \pm \phi_j \pm m)\Big(\phi_i \pm \phi_j \pm (m - 2\epsilon_+)\Big)} \\ & \times \prod_{i=1}^{k} \frac{\Big(\phi_i \pm (\epsilon_+ - m)\Big)^\chi \prod_{s=1}^{n} \Big(\phi_i \pm a_s \pm (\epsilon_+ - m)\Big)}{(2\phi_i \pm m)\Big(2\phi_i \pm (m - 2\epsilon_+)\Big)} \end{aligned} \tag{5.30}$$

Where $n = \lfloor N/2 \rfloor$ and $\chi \equiv N \bmod 2$.

For the case of $SP(N)$ with dual group as $SO(k)$, we also denote $n = \lfloor k/2 \rfloor$ and $\chi \equiv k \bmod 2$. And some of the results are as follows:

$$z_k^{\mathrm{fund}} = (m - \epsilon_+)^\chi \prod_{i=1}^{n} \Big((m - \epsilon_+) \pm \phi_i\Big) \tag{5.31}$$

$$\begin{aligned} z_k^{\mathrm{adj,gauge}} =& \frac{(2\epsilon_+)^n}{\epsilon_{1,2}^n} \left[\frac{1}{\epsilon_{1,2} \prod_{s=1}^{N}(\epsilon_+ \pm a_s)} \prod_{i=1}^{n} \frac{(\pm\phi_i)(\phi_i \pm 2\epsilon_+)}{(\phi_i \pm \epsilon_{1,2})}\right]^\chi \\ & \times \prod_{i=1}^{n} \frac{1}{(2\phi_i \pm \epsilon_{1,2}) \prod_{s=1}^{N}(\phi_i \pm \epsilon_+ \pm a_s)} \\ & \times \prod_{i<j}^{n} \frac{(\pm\phi_i \pm \phi_j)(\phi_i \pm \phi_j \pm 2\epsilon_+)}{\phi_i \pm \phi_j \pm \epsilon_{1,2}} \end{aligned} \tag{5.32}$$

$$
\begin{aligned}
z_k^{\text{adj,matter}} =& \frac{(m-\epsilon_{1,2})^n}{m^n\Big(m-(2\epsilon_+)^n\Big)} \prod_{i<j}^{n} \frac{\Big(\phi_i \pm \phi_j \pm (m-\epsilon_{1,2})\Big)}{(\phi_i \pm \phi_j \pm m)\Big(\phi_i \pm \phi_j \pm (m-2\epsilon_+)\Big)} \\
&\times \left[(m-\epsilon_{1,2}) \prod_{s=1}^{N} (m-\epsilon_+ \pm a_s) \prod_{i=1}^{n} \frac{\Big(\phi_i \pm (m-\epsilon_{1,2})\Big)}{(\phi_i \pm m)\Big(\phi_i \pm (m-2\epsilon_+)\Big)} \right]^{\chi} \\
&\times \prod_{i=1}^{n} \Big(2\phi_i \pm (m-\epsilon_{1,2})\Big) \prod_{s=1}^{N} \Big(\phi_i \pm a_s \pm (m-\epsilon_+)\Big)
\end{aligned} \tag{5.33}
$$

$$
\begin{aligned}
z_k^{\text{anti}} =& \frac{(m-\epsilon_{1,2})^n}{m^n\Big(m-(2\epsilon_+)^n\Big)} \prod_{i<j}^{n} \frac{\Big(\phi_i \pm \phi_j \pm (m-\epsilon_{1,2})\Big)}{(\phi_i \pm \phi_j \pm m)\Big(\phi_i \pm \phi_j \pm (m-2\epsilon_+)\Big)} \\
&\times \left[\frac{\prod_{s=1}^{N}(m-\epsilon_+ \pm a_s)}{m(m-2\epsilon_+)} \prod_{i=1}^{n} \frac{\Big(\phi_i \pm (m-\epsilon_{1,2})\Big)}{(\phi_i \pm m)\Big(\phi_i \pm (m-2\epsilon_+)\Big)} \right]^{\chi} \\
&\times \prod_{i=1}^{n} \frac{\prod_{s=1}^{N} \Big(\phi_i \pm a_s \pm (m-\epsilon_+)\Big)}{(2\phi_i \pm m)\Big(2\phi_i \pm (m-2\epsilon_+)\Big)}
\end{aligned} \tag{5.34}
$$

With all these results, we can evaluate the instanton partition function as:

$$
Z_k(a,m;\epsilon) = \frac{1}{|W_D|} \oint \prod_{i=1}^{k} \frac{d\phi_i}{2\pi i} z_k^{\text{adj,gauge}}(a,\phi;\epsilon) \prod_{\varrho\in\text{reps}} z_k^{\varrho,\text{matter}}(a,m_\varrho,\phi,\epsilon) \tag{5.35}
$$

Where $|W_D|$ is the orders of the Weyl group of the dual group. And to let the integral converge, we need to choose $(\epsilon_{1,2}, m) \mapsto (\epsilon_{1,2} + i0, m - i0)$ such that the integrated variables do not pass the poles.

Even though the formula seems like the ordinary multiple residues integrations, we still need to determine what the integration cycle is. So we still need some mathematical tools to manipulate it, which is the Jeffrey-Kirwan residue[23–25].

5.4 Jeffrey-Kirwan residue

We first follow the works of Vergne[26–27] and Nakamura[28] and briefly introduce the mathematical definition of JK-residue.

Suppose we have n integration variables denoted as $\phi = (\phi_1, ..., \phi_n) \in \mathbb{C}^n$. And $\mathbf{Q}_* \in \mathbb{R}^n$ is a finite subset of nonzero elements as a vector space. We denote the location of poles as $\phi_* = (\phi_{*1}, ..., \phi_{*n})$ and let $\hat{F}_{\phi_*}$ as the set of formal power series of $(\phi_i - \phi_{*i})_{1\le i\le n}$. Thus the integrands that appear in the Nekrasov partition function can be defined as:

$$
\hat{R}_{\mathbf{Q}_*,\phi_*} := \operatorname{span}_{\mathbb{C}} \left\{ \frac{f}{\prod_{Q\in\mathbf{Q}_*} \Big(Q\cdot(\phi-\phi_*)\Big)^{m_Q}} \middle| f \in \hat{F}_{\phi_*}, m_Q \in \mathbb{N} \right\} \tag{5.36}
$$

This is a ring over $\hat{F}_{\phi_*}$ if we invert the linear map $Q \cdot (\phi - \phi_*)$. And the part that has degree $-n$ of $(\phi - \phi_*)$ we denote as $\hat{R}_{\mathbf{Q}_*,\phi_*}[-n]$.

A subset of $\mathbf{Q}_*$ is called the basis of $\mathbf{Q}_*$ if it is a basis of $\mathbb{R}^n$. And we call a fraction f_σ basic if the $m_Q = 1$ and Q is element of basis σ of $\mathbf{Q}_*$:

$$f_\sigma := \frac{1}{\prod_{Q\in\sigma} Q \cdot (\phi - \phi_*)} \tag{5.37}$$

Then we can define a subset of $\hat{R}_{\mathbf{Q}_*,\phi_*}[-n]$ as $S_{\mathbf{Q}_*,\phi_*}$ which is span by the basic f_σ. Also, we can define $NS_{\mathbf{Q}_*,\phi_*}$ as the space spanned by the following form:

$$\frac{\psi}{\prod_{Q\in\kappa} \left(Q \cdot (\phi - \phi_*)\right)^{n_Q}} \tag{5.38}$$

Where κ is subset of $\mathbf{Q}_*$ that does not contain basis of $\mathbf{Q}_*$. Michel Brion and Michele Vergne[27] proved that there is a direct sum:

$$\hat{R}_{\mathbf{Q}_*,\phi_*}[-n] = S_{\mathbf{Q}_*,\phi_*} \oplus NS_{\mathbf{Q}_*,\phi_*} \tag{5.39}$$

And for computing the partition function, we only need to consider the projective $\mathbf{Q}_*$ which is defined as existing a vector $\delta \in \mathbb{R}^n$ such that its inner product with all elements of $\mathbf{Q}_*$ are positive.

A vector $\eta \in \mathbb{R}^n$ is generic if $\eta \notin \mathrm{span}_{\mathbb{R}}\{Q_1, ..., Q_{n-1}\}$ for any $n - 1$ elements of $\mathbf{Q}_*$.

With all this setup, the JK-residue of a basic fraction f_σ is defined as:

$$\text{JK-Res}_{\eta,\phi_*}(f_\sigma d\phi_1 \wedge ... \wedge d\phi_n) := \begin{cases} \left|\det(Q_i \cdot e_j)\right|^{-1} & \eta \in \mathrm{Cone}(Q_1, ..., Q_n) \\ 0 & \text{otherwise} \end{cases} \tag{5.40}$$

Where $\sigma = \{Q_1, ..., Q_n\}$ is a basis of $\mathbb{R}^n$ while $\{e_j\}_{1\le j\le n}$ is the ordinary canonical basis. And the cone is defined as:

$$\mathrm{Cone}(\sigma) := \left\{ \sum_{j=1}^{n} a_j Q_j \in \mathbb{R}^n \middle| a_j > 0 \right\} \tag{5.41}$$

For a projective $\mathbf{Q}_*$, JK-residue 5.40 is well defined on $S_{\mathbf{Q}_*,\phi_*}$[27].

Thus for a general JK-residue with a projective $\mathbf{Q}_*$ and a generic η at a particular pole ϕ_* is defined as:

$$\text{JK-Res}_{\eta,\phi_*} : \hat{R}_{\mathbf{Q}_*,\phi_*} \to \mathbb{C} \tag{5.42}$$

which is the composition of the maps:

$$\hat{R}_{\mathbf{Q}_*,\phi_*} \xrightarrow{\text{project}} \hat{R}_{\mathbf{Q}_*,\phi_*}[-n] \xrightarrow{5.39} S_{\mathbf{Q}_*,\phi_*} \xrightarrow{5.40} \mathbb{C} \tag{5.43}$$

Finally, we sum over all the possible poles ϕ_* that their corresponding cones contain η to obtain the JK-residue:

$$\oint_{\text{JK}} z = \sum_{\phi_*} \text{JK-Res}_{\eta,\phi_*}\, z \tag{5.44}$$

With the JK-residue, we can calculate the partition function[9]. Here we give a simple example for pure $U(1)$ with one and two instantons $k = 1, 2$.

- For $k = 1$ and $N = 1$, the formula 5.13 becomes:

$$z_1^{\text{adj,gauge}}(a, \phi; \epsilon) = \frac{2\epsilon_+}{\epsilon_{1,2}} \frac{1}{\left(\epsilon_+ \pm (\phi_1 - a_1)\right)} \tag{5.45}$$

We can write it consist with the form 5.36:

$$z_1^{\text{adj,gauge}}(a, \phi; \epsilon) = \frac{2\epsilon_+}{\epsilon_{1,2}} \frac{1}{(Q \cdot (\phi - \phi_*))(Q' \cdot (\phi - \phi'_*))} \tag{5.46}$$

Where $Q = (1)$, $Q' = (-1)$, $\phi = (\phi_1)$, $\phi_* = (a_1 - \epsilon_+)$ and $\phi'_* = (a_1 + \epsilon_+)$. They are all $n = 1$ dimensional vectors. So $\mathbf{Q}_* = \{Q, Q'\} = \{(+1), (-1)\}$. We need to choose the projective part that can be either $(+1)$ or (-1). We choose the generic vector $\eta = (1)$, then the projective $\mathbf{Q}_*$ can only be $(+1)$ such that η is inside $\text{Cone}(Q) = \{aQ \in \mathbb{R} | a > 0\}$. Thus we only need ϕ_*, not ϕ'_*. So following 5.40, the JK-residue is then:

$$\begin{aligned}
\oint_{\text{JK}} z_1^{\text{adj,gauge}} &= \frac{2\epsilon_+}{\epsilon_{1,2}} \left(\frac{1}{Q'(\phi - \phi'_*)}\right)\Bigg|_{\phi=\phi_*} \text{JK}_\text{R}\text{es}_{\eta,\phi_*} \left(\frac{1}{Q(\phi - \phi_*)}\right) \\
&= \frac{2\epsilon_+}{\epsilon_{1,2}} \left(\frac{1}{Q'(\phi_* - \phi'_*)}\right) \left|\det Q\right|^{-1} \\
&= \frac{1}{\epsilon_{1,2}}
\end{aligned} \tag{5.47}$$

- For $k = 2$, the formula 5.13 becomes:

$$z_2^{\text{adj,gauge}}(a, \phi; \epsilon) = \frac{1}{2} \frac{4\epsilon_+^2}{\epsilon_{1,2}^2} \frac{(\phi_1 - \phi_2)(\phi_2 - \phi_1)(2\epsilon_+ \pm (\phi_1 - \phi_2))}{(\epsilon_{1,2} \pm (\phi_1 - \phi_2))(\epsilon_+ \pm (\phi_{1,2} - a_1))} \tag{5.48}$$

We can read off the $\mathbf{Q}_* = \{(\pm 1, \mp 1), (\pm 1, 0), (0, \pm 1)\}$. We can choose the projective part as three different versions with $\eta = (1, 1)$:

$$\mathbf{Q}_* = \begin{cases} \{(1,0), (0,1)\} \\ \{(0,1), (1,-1)\} \\ \{(1,0), (-1,1)\} \end{cases} \tag{5.49}$$

And we have five different poles with this choice:

$$\phi_* = \begin{cases} (a_1 - \epsilon_+, a_1 - \epsilon_+) \\ (a_1 - \epsilon_+, a_1 - \epsilon_+ - \epsilon_1) \\ (a_1 - \epsilon_+, a_1 - \epsilon_+ - \epsilon_2) \\ (a_1 - \epsilon_+ - \epsilon_1, a_1 - \epsilon_+) \\ (a_1 - \epsilon_+ - \epsilon_2, a_1 - \epsilon_+) \end{cases} \tag{5.50}$$

With these locations of poles, the result of $k = 2$ is:

$$\oint_{\text{JK}} z_2^{\text{adj,gauge}} = \sum_{\phi_*} \text{JK-Res}_{\eta,\phi_*}(z_2^{\text{adj,gauge}}) = \frac{1}{2} \frac{1}{\epsilon_{1,2}^2} \tag{5.51}$$

For the pure $U(1)$ gauge group, it is quite trivial. However, things become much more difficult for general $U(n)$ with arbitrary k. For example, when $n = 2$ and $k = 2$, the result is:

$$\oint_{\mathrm{JK}} z_2^{U(2)} = \frac{-2a_1^2 + 4a_1a_2 - 2a_2^2 + q\epsilon_1^2 + 17\epsilon_1\epsilon_2 + 8\epsilon_2}{\epsilon_1^2\epsilon_2^2(2\epsilon_+ \pm (a_1 - a_2))(\epsilon_1 + 2\epsilon_+ \pm (a_1 - a_2))(\epsilon_2 + 2\epsilon_+ \pm (a_1 - a_2))} \tag{5.52}$$

When we compute higher instanton using JK-residue, even if we program nicely, it will still take a very long time since the number of poles is rising sharply. For instance, for pure $U(3)$ theory with $k = 4$, it will contain 3993 poles while $k = 3$ only has 243 poles and $k = 2$ only has 21 poles.

Thus if we can find the pattern of the location of poles, we can dramatically reduce the computation time. Fortunately, thanks to the seminal work by Nekrasov[4,29], we know that the poles of pure $U(N)$ gauge theory are classified by the Young table. And Nawata-Zhu extended the classification of poles[8] to pure BCD type of gauge group. In Chapter 7, we will show that even with one matter, the poles of ABCD gauge theory can be classified by $2d$ or $4d$ Young diagram[11].

Chapter 6

Five-branes webs

In this chapter, we will briefly review the basic concepts of the five-brane wbes construction in type IIB string theory. This will provide us one more way to check the accuracy of our calculations, or on the other hand, use our results as new evidence of Branes dynamics.

The reason we use five-branes in type IIB rather than four-branes in type IIA is later in Chapter 7 we will generalize the $4d$ super Yang-Mills theory to $5d$.

6.1 Introduction of 5-brane

It is well known that the open strings satisfy two kinds of boundary conditions as

$$\begin{aligned} \partial_\sigma X^\mu = 0 \quad &\text{at} \quad \sigma = 0, \pi \qquad \text{Neumann boundary equation} \\ \delta X^\mu = 0 \quad &\text{at} \quad \sigma = 0, \pi \qquad \text{Dirichlet boundary equation} \end{aligned} \tag{6.1}$$

where X denotes the spacetime coordinate of the string and σ means the world-sheet spatial coordinate.

The fundamental open string can end on Dirichlet p branes (Dp-branes) in type II string theory. And the Dp-brane is a dynamical object that fills the whole $(x^1, ..., x^p)$ dimensions and locates at a point in $(x^p + 1, ..., x^9)$. And the Dp-brane is defined by choosing the first $p + 1$ coordinates $(x^0, x^1, ..., x^p)$ that satisfy the Neumann boundary condition and the last $9 - p$ coordinates $(x^{p+1}, ..., x^9)$ satisfy the Dirichlet condition. The Ramond-Ramond (RR) charge of Dp-branes is equal to its tension:

$$T_p = \frac{1}{g_s l_s^{p+1}} \tag{6.2}$$

Where g_s is the fundamental string coupling and l_s is the fundamental string scale. And a fundamental string can be viewed as a D1-brane, which means that its tension is proportional to l_s^{-2}.

In the following discussion, we need to use the several branes shown in table:6-1.

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9
NS5	x	x	x	x	x	x				
D5	x	x	x	x	x		x			
D7	x	x	x	x	x			x	x	x

Table 6-1 The worldvolumes of different branes

p number of Neveu-Schwarz (NS) 5-branes and q number of D5-branes can merge in (x^5, x^6) plane to form (p, q)5-branes[30] with the corresponding string tension $T_{p,q}^{\text{string}} =$

$\sqrt{p^2+q^2}$ and the strip tension $T_{p,q}^{\text{strip}} = 1/\sqrt{p^2+q^2}$ as shown in figure 6-1. Note that p and q are co-prime pair, otherwise we can view this (p,q)5-brane as a stack of (p_0, q_0)5-branes where $(p,q) = m(p_0, q_0)$. And the charge of (p,q)5-branes should satisfy the charge conservation on each vertex: $\sum_i q_i = 0$, and $\sum_i p_i = 0$ with fixing the orientation of all 5branes in the direction towards the vertex[31]

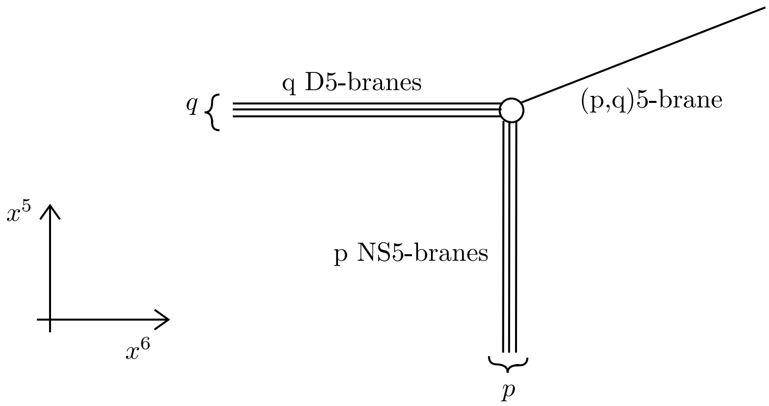


Figure 6-1 (p,q)5-branes emerge from the vertex that join q D5-branes and p NS5-branes

The presence of the Dp-branes breaks the *Poincare* symmetries of the *Minkowski* spacetime to a subspace $(1,p) \times SO(9-p)$. The excitations of the fundamental strings that attach on Dp-branes can endow the Dp-brane dynamics and will generate a massless vector field A_μ ($\mu = 0, 1, ..., p$) and $9-p$ massless scalar fields Φ^a ($a = p+1, p+2, ..., 9$).

As the fig:6-2 shown, if a string with both end attach on the same Dp-brane, there will be a global $U(1)$ symmetry arise for the strings interactions. And more generally, for N number of distinguishable Dp-branes, the global symmetry of strings interactions become $U(1)^{\times N}$. Once this N Dp-branes are coincidental, the gauge symmetry will be enhanced to $U(N)$ with all the vector fields and scalar fields taking the $U(N)$ values, which are $(A_\mu)_{ij}$ and Φ^a_{ij} respectively.

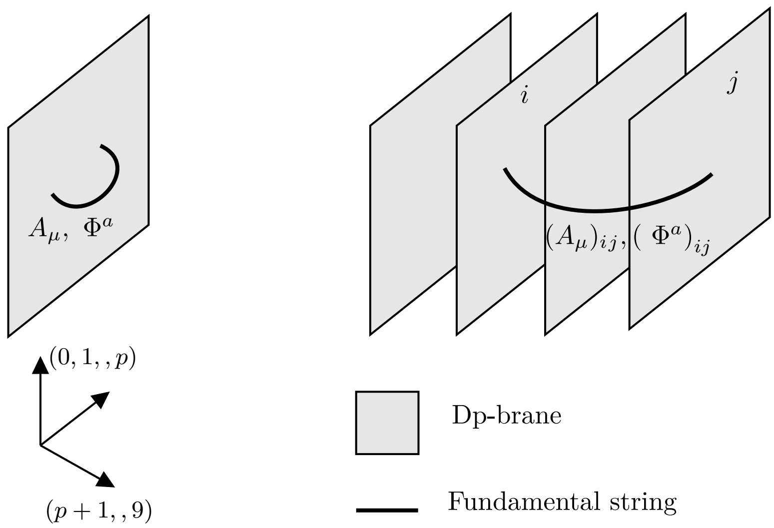


Figure 6-2 Fundamental strings with both ends on Dp-branes depict (a) a $U(1)$ gauge field and $9-p$ scalars on a single Dp-brane, or (b) a $U(N)$ gauge field and $9-p$ scalars on a stack of N coincidental Dp-branes[31]

6.2 Brane webs for ABCD type gauge group

We have discussed the $U(N)$ gauge group in a nutshell. As an example, to construct pure $SU(2)$ gauge theory, we need coincidental 2 D5-branes. And D5-branes must end on NS5-branes, so the diagram is drawn in figure 6-3 (a). Notice we need 4 extra (p,q)5-branes to complete the charge conservation on each vertex.

And to introduce quarks flavors, we need a semi-infinite D5-brane to attach to the NS5-brane for each flavor, where the number of flavors is denoted as N_f, as fig: 6-3 (b).

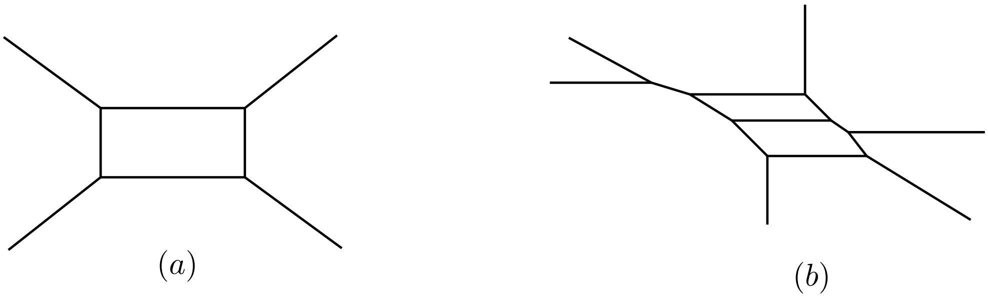


Figure 6-3 (a)Pure $SU(2)$ gauge theory. $(b)N = 3$ and $N_f = 2$.

D7-brane is a point in (x^5, x^6) plane. It can connect to five brane webs by joining the D5-branes horizontally. If we move the D7-brane that connects with a D5-brane to infinity, the brane webs then are identical to a web with a semi-infinite D5-brane, which is equivalent to a flavor. The advantage of the D7-brane will be seen soon.

To construct C and D type gauge theory, we need to flip the orientation of the string, for which we require $O5$-plane. Orientifold p-planes (Op-planes) are non-dynamical objects that can reverse the orientation of the string after reflection. And it is a generalization of the Z_2 orbifold fixed point[32]. The RR charge of Op-palne denoted as Q_{Op} is equal to 2^{p-4} times of RR charge of Dp-brane up to a sign:

$$Q_{Op} = \pm 2^{p-4} Q_{Dp} \tag{6.3}$$

There are 4 variants of $O5$-palnes as $O5^+$, $\tilde{O}5^+$, $O5^-$ and $\tilde{O}5^-$, where the main difference between them is the D5-brane charge they carry. The charge of $O5^\pm$ is ± 1, while the charge of $\tilde{O}5^+$ is $+1$ and the $\tilde{O}5^-$ is $-\frac{1}{2}$. When an NS5-brane crosses the $O5$-plane, it will partition the $O5$-plane into 2 different pieces as $O5^+$ ($\tilde{O}5^+$) changes to $O5^-$ ($\tilde{O}5^-$) in order to preserve supersymmetry. Similarly, when the O7-brane crossing $O5$-plane, it will also change from $O5^+$ ($O5^-$) into $\tilde{O}5^+$ ($\tilde{O}5^-$). And considering the quantum effect with charge conservation and so on, NS5-brane will bend after crossing $O5$-plane as figure 6-4[33]

Then the $SP(N)$ and $SO(N)$ with $N = 2N_c$ (N_c denotes number of D5-brane) can be construct by using the $O5$-plane as figure 6-5

Since the $O5$-plane works as a mirror, we can only look at the upper half of the web diagram. And then we can see in figure 6-6. By putting semi-infinite D5-branes on NS5-branes from the left and right, we can construct a theory with flavors as we desire.

Notice again all the semi-infinite D5-branes can be viewed as attached on a D7-brane on the infinite side and move the D7-brane to infinity, just like the (a) and (b) in fig: 6-7.

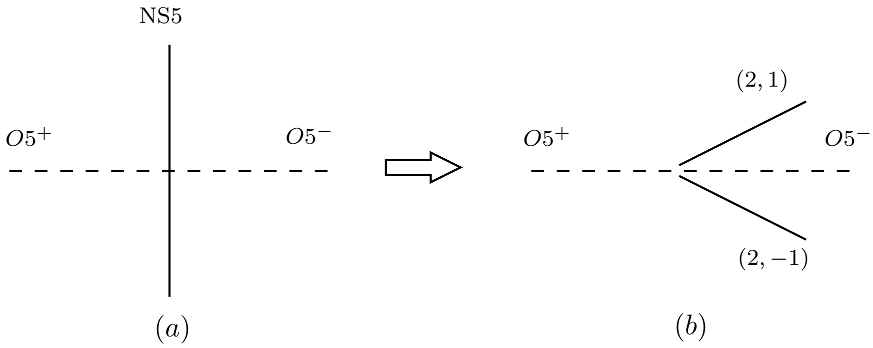


Figure 6-4 NS5-brane crossing $O5$-plane: (a) Classically; (b) Quantum mechanically

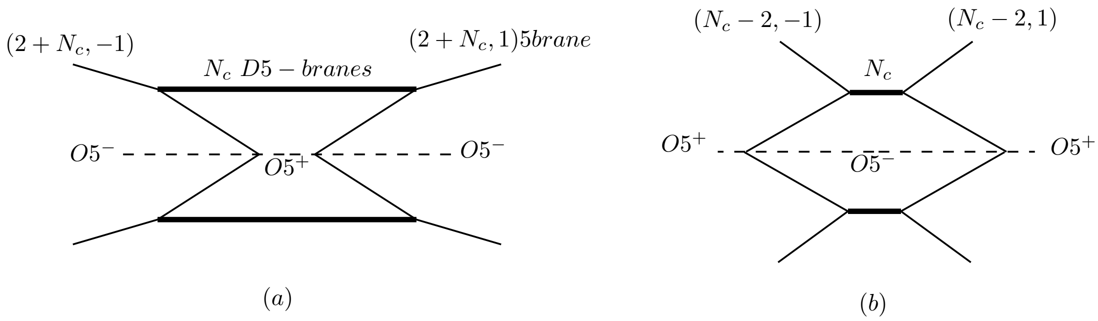


Figure 6-5 (a)$SP(2N_c)$; (b)$SO(2N_c)$

Using the D7-brane we can construct $SO(2N_c-1)$ from $SO(2N_c)+1F$. First, we can Higgsing the flavor D5-brane on the $O5$-plane as well as the bottom D5-brane like (a) of figure 6-8. Since $O5$-plane must transfer to $\tilde{O}5$-plane and vice versa, also D5-brane can not stuck on $\tilde{O}5^+$-plane. So between two D7-branes, it is forced to put an extra D5-branes between them. Finally, we can move the left D7-brane to the left infinity and transform the left $O5^+$ to $\tilde{O}5^+$ and move the right D7 brane to the left infinity with the monodromy branch to get the pure $SO(2N_c-1)$ we desire[33–34]. In figure 6-8 we show how to get $SO(7)$ from $SO(8)+1F$.

Thus the webs construction of pure $SO(2N_c+1)$ is shown in the figure:6-9.

6.3 Brane web of $SU(2N)$ with (anti)symmetric

We can construct the $SU(2N)$ with one symmetric and anti-symmetric hypermultiplet with the help of $O7$-plane in type IIB string theory[35].

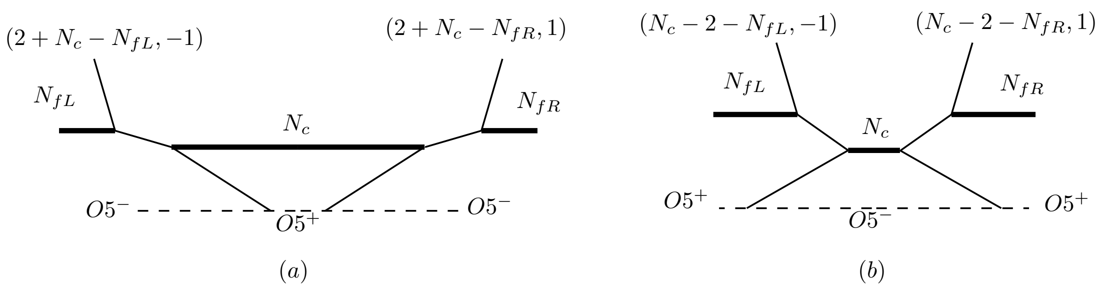


Figure 6-6 (a)$SP(2N_c)$ with $(N_{fL}+N_{fR})$ flavors; (b)$SO(2N_c)$ with $(N_{fL}+N_{fR})$ flavors

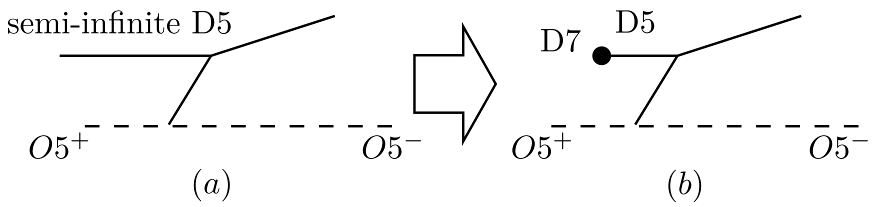


Figure 6-7 A semi-infinite D5-brane can be viewed as it attaches to a D7-brane

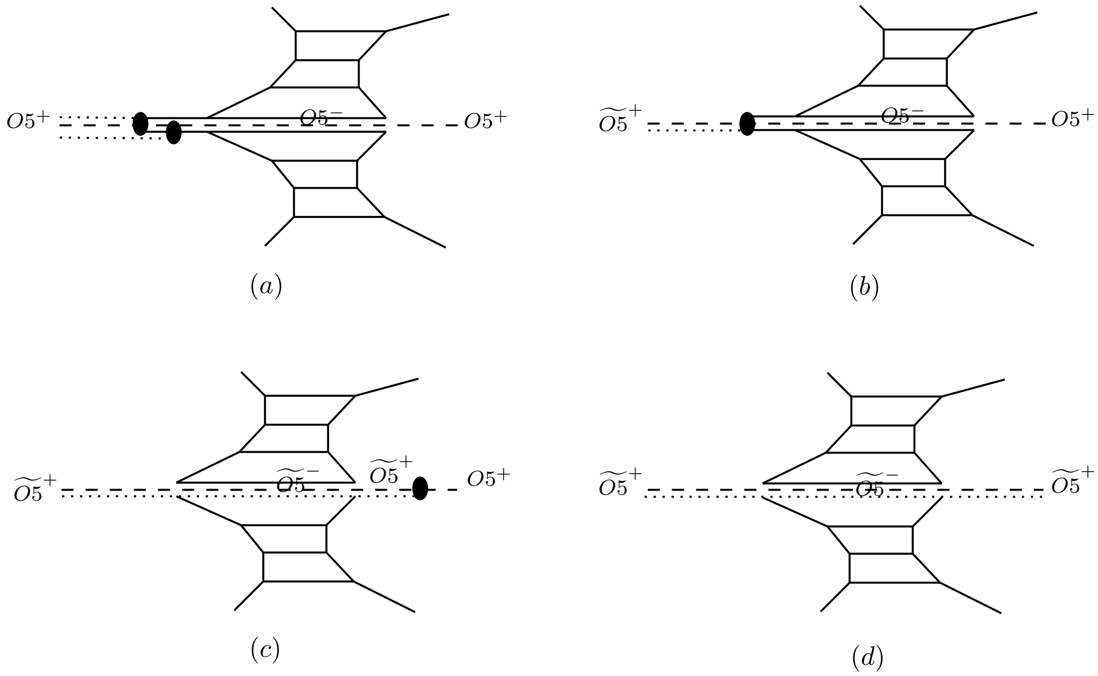


Figure 6-8 (a) Two D7-branes alone with D5-branes are brought to the $O5$-plane. The monodromy branch cut is denoted by a black dotted line, which is associated with a D7-brane. The D5-brane stuck to the $O5$-plane is denoted by a black solid line. The $O5$-plane of any kind is denoted by a black dashed line. (b) The web diagram for $SO(7)$ arising from entering the Higgs branch of $SO(8)+1F$. (c) Pull out the D7-brane to the right side. The $O5$ will transform into $\tilde{O}5$ when the D7-brane passes through it. One can check the charge conversation law at the two intersecting points. (d) Move the D7-brane to infinitely right. The resulting diagram is the whole 5-brane web diagram for pure $SO7$ gauge theories.

Firstly, to construct a 5-brane web configuration for $SU(2N)$ with an anti-symmetric hypermultiplet, we need to include one $O7^-$-plane, which is a type of orientifold 7-plane. It is parallel to D7-brane we have constructed in table:6-1 on which the external NS5-branes end. Notice the NS5-brane cannot detach from the $O7^-$ brane.

And as in the figure 6-10, we draw the whole space of $SU(2N)$ with one anti-symmetric hypermultiplet, while the physical space is only the upper plane as the previous construction using $O5$-plane. While the difference between $O7$ and $O5$ besides charge and worldvolume is that $O7$ has monodromy cut[35]. Thus two sides of the $O7$-plane are identified, which implies the figure is symmetric across the origin. The monodromy cut of $O7^-$-plane is: $M_{O7^-} = -T^{-4}$, where:

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in SL(2, \mathbb{Z}) \tag{6.4}$$

If we only consider the upper half plane of the figure:6-10, we can first do the hig-

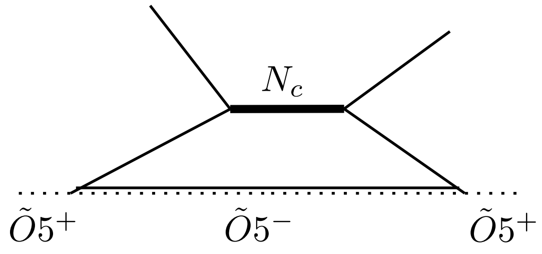


Figure 6-9 5-brane Webs construction of $SO(2N_c + 1)$

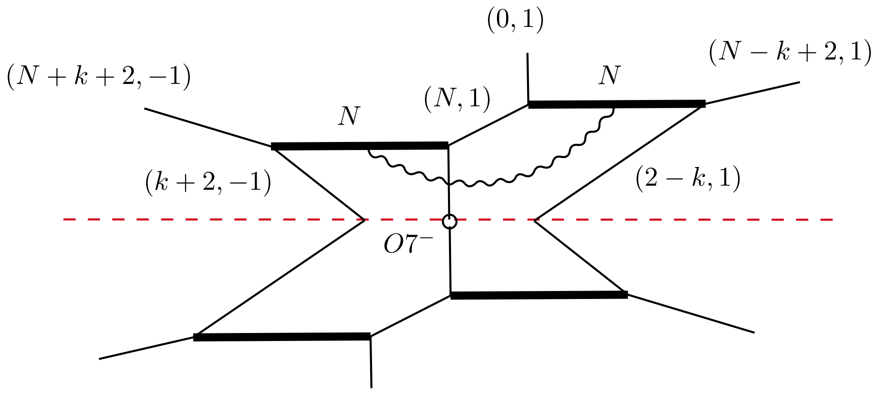


Figure 6-10 Webs configuration of $SU(2N)$ with 1 anti-symmetric hypermultiplet. Each horizontal thick line corresponds to N D5-branes. $O7^-$-plane is indicated by a circle in the origin. The red dashed line corresponds to the monodromy cut of $O7^-$-plane. The wavy curve that connects the D5-branes on either side of the NS5-brane indicates the hypermultiplet.

gsing to align all the D5-branes and identify the left and right half of the monodromy cut line by, let's say, rotating half of the diagram into the bottom. Then we can get the familiar $SP(2N)$ group with one end of the D5-branes end on half NS5-branes. As shown in figure:6-11.

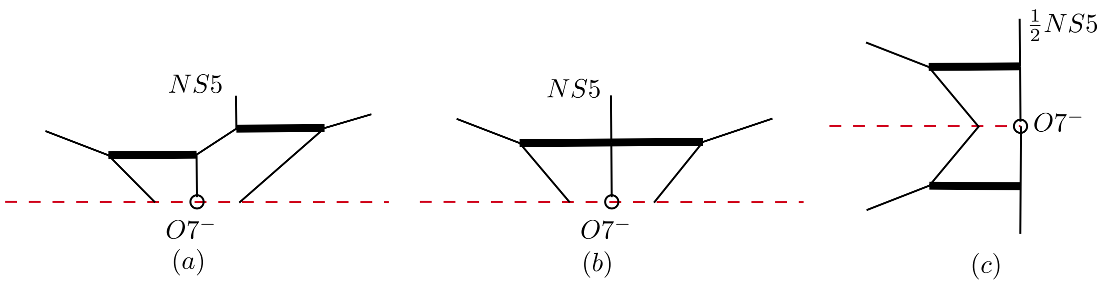


Figure 6-11 (a) the $SU(2N)$ with one anti-symmetric in figure:6-10. (b) Higgsing and aligning the D5-branes. (c), rotate the diagram to a more familiar form

To construct the $SU(2N)$ with one symmetric hypermultiplet, we simply need to replace the $O7^-$-plane to $O7^+$ plane as figure:6-12

It is worth mentioning that after we align the D5-branes like the figure:6-11(b), we can use higgsing again to remove the middle NS5-brane and the $O7^-$-plane. The rest will be exactly pure $SO(N)$ gauge theory[11], which we will check in the Chapter 7.

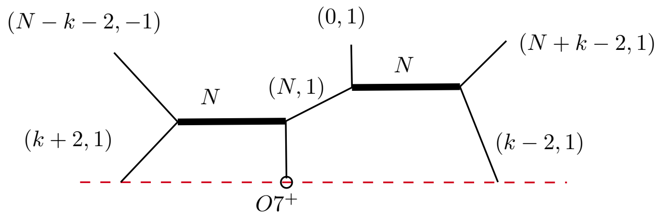


Figure 6-12 Webs diagram for $SU(2N)$ with one symmetric hypermiltiplet

Chapter 7
Young diagram computation

In this chapter, we will consider $5d$ super Yang-Mills instead of $4d$. And we will list the integral expressions at refine level and expressions in terms of summing over Young diagrams of the partition function of $SU(N)$ with symmetric and anti-symmetric hypermultiplet, $SO(n)$ with spinor, and $SP(N)$ with spinor. Then we compare our results with others using branes constructions and isomorphism between representations of Lie algebra. We refer to the work for the results of pure gauge BCD type gauge group using Young diagrams under unrefined limit[8].

7.1 $5d$ $\mathcal{N}=1$ Yang-Mills theory

First, we introduce the field content in $5d$ $\mathcal{N}=1$ Yang-Mills theory by following the notation in Terashima's work[36].

For Vector multiplet, there is a $5d$ gauge boson A_m, $m=1,...,5$, a real scalar σ, a triplet of auxiliary scalars D_{IJ} and a Majorana spinor λ_I^{α}. Where $I,J=1,2$ is the internal symmetry index, and the auxiliary field satisfies:

$$(D_{IJ})^{\dagger}=D^{IJ}=\epsilon^{II'}\epsilon^{JJ'}D_{I'J'} \tag{7.1}$$

And the Yang-Mills Lagrangian takes the form:

$$\mathscr{L}_{\mathrm{YM}}=\mathrm{Tr}\left[\frac{1}{2}F_{mn}F^{mn}-D_m\sigma D^m\sigma-\frac{1}{2}D_{IJ}D^{IJ}+i\epsilon^{IJ}\lambda_I\Gamma^mD_m\lambda_J-\epsilon^{IJ}\lambda_I[\sigma,\lambda_J]\right] \tag{7.2}$$

where Γ^m is element in the $5d$ Clifford algebra $\{\Gamma^m,\Gamma^n\}=2\delta^{mn}$.

For the system with r hypermultiplets, we have scalars q_I^A, fermions ψ^A and auxiliary scalars F_I^A where $A=1,...,2r$. These fields satisfy the real conditions:

$$(q_I^A)^*=\Omega_{AB}\epsilon^{IJ}q_J^B,\quad (\phi^{A\alpha})^*=\Omega_{AB}C_{\alpha\beta}\psi^{B\beta},\quad (F_I^A)^*=\Omega_{AB}\epsilon^{IJ}F_J^B \tag{7.3}$$

where $C_{\alpha\beta}$ is the invariant tensor of $Spin(5)$ and Ω_{AB} is the invariant tensor of flavor symmetry. And then the Lagrangian is:

$$\mathscr{L}_{\mathrm{mat}}=D_m\bar{q}D^mq-\bar{q}\sigma^2q-2(i\bar{\psi}\Gamma^mD_m\psi+\bar{\psi}\sigma\psi)-i\bar{q}_ID^{IJ}q_J-4\bar{\psi}\lambda q-\bar{F}F \tag{7.4}$$

In the Ω-background, the spacetime manifold is $\mathbb{R}^5=\mathbb{C}^2\times\mathbb{R}$, and we define the coordinate transformation under Ω-background as:

$$(z_1,z_2,x_5)\mapsto(e^{i\beta\epsilon_1}z_1,e^{i\beta\epsilon_2}z_2,x_5+\beta) \tag{7.5}$$

This is equivalently forcing the original $4d$ system to rotate along the origin by angle $\beta\epsilon_{1,2}$ with the transition in the extra dimension corresponding to a different rotate angle.

If we compactify the fifth direction on a circle with the radius going to zero, i.e $x_5 \equiv x_5 + 2\pi R$ and view A_5 as a scalar field, when we compute its one-loop correction, we can interpret the instanton contribution in the $4d$ $\mathcal{N} = 2$ super Yang-Mills prepotential[4,37–38].

When it comes to finite-dimensional reduction using the equivariant Atiyah-Singer index, the mapping 4.19 should be modified as[4,39]:

$$\sum_{\alpha} \epsilon_{\alpha} e^{\omega_{\alpha}} \mapsto \prod_{\alpha} \mathrm{sh}^{\epsilon_{\alpha}}(\omega_{\alpha}) \tag{7.6}$$

Where we introduce a short-hand notation as:

$$\mathrm{sh}(x) := e^{x/2} - e^{-x/2}, \quad \mathrm{ch}(x) := e^{x/2} + e^{-x/2} \tag{7.7}$$

Notice that each factor of the $5d$ version has periodic behavior, i.e $\mathrm{sh}(x) = \mathrm{sh}(x \pm 2m\pi i)$ with $m \in \mathbb{Z}$. To avoid counting the poles repeatedly, we only consider those ϕ_* with imaginary part πi or 0[8,40].

Since in $5d$, all variables are inside of sh, which means they are all get exponentiated, we redefine the following notations to simplify things further:

$$q = e^{-\epsilon_1}, \quad t = e^{\epsilon_2}, \quad M = e^{-m}, \quad A_{\ell} = e^{-a_{\ell}} \tag{7.8}$$

7.2 $SU(N)$ with (anti-)symmetric hypermultiplet

Using 7.6 and 5.22, we can obtain the $SU(N)$ with one hypermultiplet in symmetric or anti-symmetric representation formula for $5d$ version[11]:

$$Z^{\mathrm{rep}}_{SU(N),k,\kappa} = \frac{1}{k!2^k} \oint_{\mathrm{JK}} \prod_{I=1}^{k} \frac{d\phi_I}{2\pi i} \cdot e^{\kappa \sum_{I=1}^{k} \phi_I} z^{\mathrm{vec}}_{SU(N),k} z^{\mathrm{rep}}_{SU(N),k} \tag{7.9}$$

Where we rewrite $z^{\mathrm{adj,gauge}}_{SU(N),k}$ as $z^{\mathrm{vec}}_{SU(N),k}$, and the integrands are:

$$\begin{aligned}
z^{\mathrm{vec}}_{SU(N),k} &= \frac{\prod_{I\neq J} \mathrm{sh}(\phi_I - \phi_J) \cdot \prod_{I,J} \mathrm{sh}(2\epsilon_+ - \phi_I + \phi_J)}{\prod_{I,J} \mathrm{sh}(\epsilon_{1,2} + \phi_I - \phi_J) \prod_{I=1}^{k} \prod_{s=1}^{N} \mathrm{sh}(\epsilon_+ \pm (\phi_I - a_s))} \\
z^{\mathrm{fund}}_{SU(N),k}(m) &= \prod_{I=1}^{k} \mathrm{sh}(\phi_I + m) \\
z^{\mathrm{sym}}_{SU(N),k}(m) &= \prod_{I=1}^{k} \mathrm{sh}\left(2\phi_I + m \pm \epsilon_-\right) \prod_{s=1}^{N} \mathrm{sh}\left(\phi_I + a_s + m\right) \prod_{I<J}^{k} \frac{\mathrm{sh}\left(\phi_I + \phi_J + m \pm \epsilon_-\right)}{\mathrm{sh}\left(-\epsilon_+ \pm (\phi_I + \phi_J + m)\right)} \\
z^{\mathrm{anti}}_{SU(N),k}(m) &= \prod_{I=1}^{k} \frac{\prod_{s=1}^{N} \mathrm{sh}\left(\phi_I + a_s + m\right)}{\mathrm{sh}\left(-\epsilon_+ \pm (2\phi_I + m)\right)} \prod_{I<J}^{k} \frac{\mathrm{sh}\left(\phi_I + \phi_J + m \pm \epsilon_-\right)}{\mathrm{sh}\left(-\epsilon_+ \pm (\phi_I + \phi_J + m)\right)}
\end{aligned} \tag{7.10}$$

Note that κ is the $5d$ Chern-Simons level. And from $U(N)$ to $SU(N)$ can be achieved by imposing the traceless condition to the group elements, i.e $\mathrm{Tr}\, a = \sum_{i=1}^{N} a_i = 0$. For a gauge group $SU(2N+1)$, the level κ needs to be a half-odd-integer due to the parity anomaly associated with the (anti-)symmetric matter.

It is very difficult to perform these expressions using JK-residue as we discuss in section 5.4. However, the poles can be classified by N-tuples $\vec{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(N)})$ of $2d$ Young diagrams with the total number of boxes $\sum_{s=1}^{N} |\lambda^{(s)}| = k$ in the unrefined limit $\epsilon_1 = -\epsilon_2 = \hbar$. We specify a pole location associated with a content $x = (x_1, x_2) \in \lambda^{(s)}$ as

$$\phi_s(x) = a_s + (x_1 - x_2)\hbar \tag{7.11}$$

By writing the Nekrasov factor[4]

$$N_{s,t}(x) := a_s - a_t - \hbar(a_{\lambda^{(s)}}(x) + l_{\lambda^{(t)}}(x) + 1) \tag{7.12}$$

the residue sum for a symmetric hypermultiplet is expressed as

$$\begin{aligned}
&Z^{\text{sym}}_{SU(N),k,\kappa} \\
=&(-1)^{kN+\lceil \frac{k}{2} \rceil} \sum_{\vec{\lambda}} \prod_{s=1}^{N} \prod_{x\in\lambda^{(s)}} e^{\kappa\phi_s(x)} \frac{\text{sh}(2\phi_s(x) + m \pm \hbar) \cdot \prod_{t=1}^{N} \text{sh}(\phi_s(x) + a_t + m)}{\prod_{t=1}^{N} \text{sh}^2(N_{s,t}(x))} \\
&\times \prod_{s\leq t}^{N} \prod_{\substack{x\in\lambda^{(s)}, y\in\lambda^{(t)} \\ x<y}} \frac{\text{sh}(\phi_s(x) + \phi_t(y) + m \pm \hbar)}{\text{sh}^2(\phi_s(x) + \phi_t(y) + m)}
\end{aligned} \tag{7.13}$$

Here we define a total ordering on the boxes of 2d Young diagrams:

$$\lambda^{(s)} \ni x < y \in \lambda^{(t)} \quad \text{if} \begin{cases} s < t \\ s = t, x_1 < y_1 \\ s = t, x_1 = y_1, x_2 < y_2 \end{cases} \tag{7.14}$$

For an anti-symmetric matter, the JK-poles are classified by $(N+4)$-tuples of 2d Young diagrams with the total number of boxes $\sum_{s=1}^{N+4} |\lambda^{(s)}| = k$ where the four additional effective Coulomb branch parameters are

$$a_{N+j} = -\frac{m}{2}(+\pi i)\,,\ \frac{\hbar - m}{2}(+\pi i) \tag{7.15}$$

for $j = 1, 2, 3, 4$. Then, the residue sums over Young diagrams are expressed as

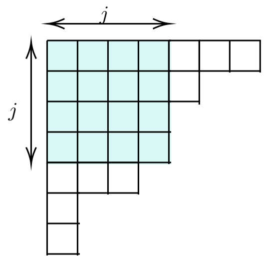
$$\begin{aligned}
&Z^{\text{anti}}_{SU(N),k,\kappa}(A_{N+1} = e^{\frac{m}{2}}, A_{N+2} = -e^{\frac{m}{2}}, A_{N+3} = e^{-\frac{\hbar-m}{2}}, A_{N+4} = -e^{-\frac{\hbar-m}{2}}) \\
=&(-1)^{kN+\lceil \frac{k}{2} \rceil} \sum_{\vec{\lambda}} C^{\text{anti}}_{\vec{\lambda},\vec{A}} \prod_{s=1}^{N+4} \prod_{x\in\lambda^{(s)}} e^{\kappa\phi_s(x)} \frac{\text{sh}^2(2\phi_s(x) + m - \hbar) \cdot \prod_{t=1}^{N} \text{sh}(\phi_s(x) + a_t + m)}{\prod_{t=1}^{N+4} \text{sh}^2(N_{s,t}(x))} \\
&\times \prod_{s\leq t}^{N+4} \prod_{\substack{x\in\lambda^{(s)}, y\in\lambda^{(t)} \\ x<y}} \frac{\text{sh}(\phi_s(x) + \phi_t(y) + m \pm \hbar)}{\text{sh}^2(\phi_s(x) + \phi_t(y) + m)}
\end{aligned} \tag{7.16}$$

Where the non-trivial multiplicity constants

$$C^{\text{anti}}_{\vec{\lambda},\vec{A}} = \prod_{s=N+1}^{N+4} C^{\text{anti}}_{\lambda^{(s)},A_s} \tag{7.17}$$

are involved due to higher-order poles coming from the additional effective Coulomb branch parameters 7.15. These multiplicity constants are in principle determined by the orders of poles and the inverse of the determinant of the Cone(Q) in 5.40. Here we conjecture these constants as follows: $C^{\text{anti}}_{\lambda^{(s)}=\varnothing,A_s} = 1$, and

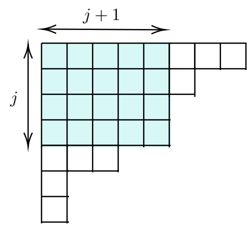
$\lambda^{(s)} =$



$$\Longrightarrow \quad C^{\text{anti}}_{\lambda^{(s)},A_s=\pm e^{-\frac{m}{2}}} = (-2)^j$$

j is the number of rows with $\lambda^{(s)}_i \geq i$ (7.18)

$\lambda^{(s)} =$



$$\Longrightarrow \quad C^{\text{anti}}_{\lambda^{(s)},A_s=\pm e^{-\frac{\hbar-m}{2}}} = (-2)^j$$

j is the number of rows with $\lambda^{(s)}_i \geq i+1$

Using these formulas we check the following identities come from the isomorphism of the representation of Lie algebra:

$$\begin{aligned} (\mathfrak{su}(3),\mathbf{AS}) \cong (\mathfrak{su}(3),\mathbf{F}) \quad &\longrightarrow \quad Z^{\text{anti}}_{SU(3),k,\kappa=1/2}(m) = Z^{\text{fund}}_{SU(3),k,\kappa=1/2}(m \to -m) \\ (\mathfrak{su}(4),\mathbf{AS}) \cong (\mathfrak{so}(6),\mathbf{V}) \quad &\longrightarrow \quad Z^{\text{anti}}_{SU(4),k,\kappa=0} = Z^{\text{vec}}_{SO(6),k} \end{aligned} \tag{7.19}$$

Where in the second equation, we change the variables on the Coulomb branch parameters from the orthogonal basis of $SU(4)$ to that of $SO(6)$:

$$A_1 \to (A_1A_2A_3)^{\frac{1}{2}}, \quad A_2 \to \left(\frac{A_1}{A_2A_3}\right)^{\frac{1}{2}}, \quad A_3 \to \left(\frac{A_2}{A_1A_3}\right)^{\frac{1}{2}}, \quad A_4 \to \left(\frac{A_3}{A_1A_2}\right)^{\frac{1}{2}} \tag{7.20}$$

From section 6.3, we construct both $SU(2N)$ symmetric and anti-symmetric brane web in figure 6-12 and figure 6-10. Thus the $SU(4)$ gauge theory with a (anti-)symmetric can be construct as figure 7-1

We can use the Higgsing procedure to remove the middle NS5-brane in $SU(N)$ with one symmetric, what's left is pure $SO(N)$ gauge theory as we construct in figure 6-5. This indicates:

$$Z^{\text{sym}}_{SU(N),2k,\kappa=\frac{1}{2}(N \bmod 2)}(a_{N-i+1} = -a_i, m = \epsilon_+) = (-1)^{k(N+1)} Z^{\text{pure}}_{SO(N),k} \tag{7.21}$$

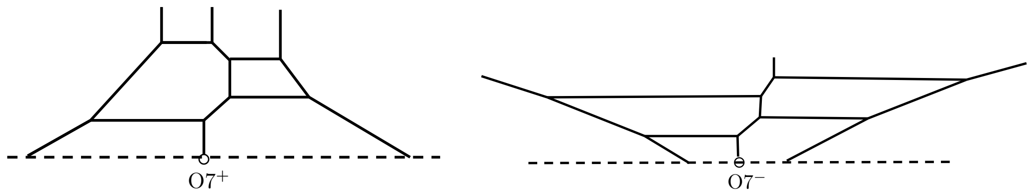


Figure 7-1 (Left-panel) $SU(4)$ gauge theory with a symmetric hypermultiplet.
(Right-panel) $SU(4)$ gauge theory with an anti-symmetric hypermultiplet.

Notice that $2k$-instanton on the left-hand side coincides with k-instanton on the right-hand side. Also, when N is odd, we set $a_{\frac{N+1}{2}} = 0$ in the left-hand side.

Likewise, Higgsing the middle NS5-brane of $SU(2N)$ with one anti-symmetric left with pure $SP(N)$ in figure 6-9. Which implies:

$$\begin{aligned} Z^{\text{anti}}_{SU(2N),k,\kappa=N \bmod 2}(a_{N-i+1} = -a_i, m = \epsilon_+) &= (-1)^{k(N+1)+\lceil\frac{k}{2}\rceil} Z^{\text{pure}}_{Sp(N),k,\theta=0} \\ Z^{\text{anti}}_{SU(2N),k,\kappa=N+1 \bmod 2}(a_{N-i+1} = -a_i, m = \epsilon_+) &= (-1)^{k(N+1)+\lceil\frac{k}{2}\rceil} Z^{\text{pure}}_{Sp(N),k,\theta=\pi} \end{aligned} \tag{7.22}$$

where the discrete θ-angle of the $Sp(N)$ theory is determined by the Chern-Simons level and the rank of the gauge group.

7.3 $SO(N)$ with spinor

7.3.1 Constructions and properties of $SO(N)$

Using $O5$-plane we can construct a $SO(n)$ gauge group with a spinor hypermultiplet[41]. To introduce two hypermultiplets in the spinor representation, a ”leg” fivebrane needs to be introduced on the right side. For instance, we draw the fivebrane diagrams for the $SO(7)$ gauge group with one spinor, and $SO(8)$ gauge theory with spinor and conjugate spinor in the figure: 7-2. As the figures show, the diagram for a hypermultiplet in the (conjugate) spinor representation can be intuitively interpreted as the “$Sp(0)$” gauge theory. In other words, roughly speaking, the theory can be regarded as an “$SO(n)$-$Sp(0)$ quiver” gauge theory.

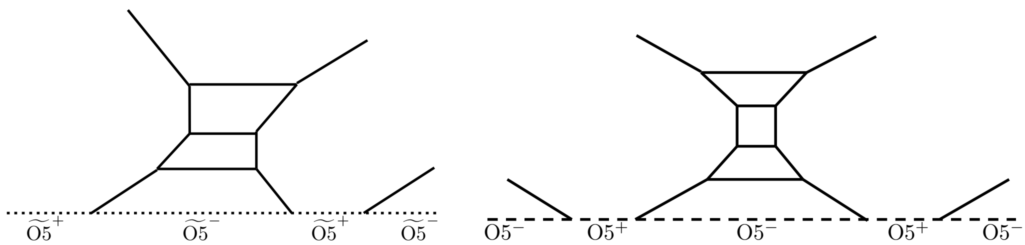


Figure 7-2 (Left-panel) $SO(7)$ gauge theory with a spinor matter.
(Right-panel) $SO(8)$ gauge theory with spinor and conjugate spinor matters.

Instantons are created by $D(p-4)$-branes in our case D1-branes that are suspended by fivebranes in the middle. On the other hand, hypermultiplet particles in the spinor representation are created by D1’-branes that are suspended by ”leg” fivebranes.

As illustrated in Figure 7-3, for $SO(n)$ gauge theory with $n \leq 6$, the ”leg” fivebranes do not intersect. For $n = 7, 8$, the two ”leg” fivebranes are parallel to each other.

In fact, from this viewpoint of quantum mechanics on D1-D1'-branes, the $SO(n)$ instanton moduli spaces with spinor matter are described using the ADHM approach[40].

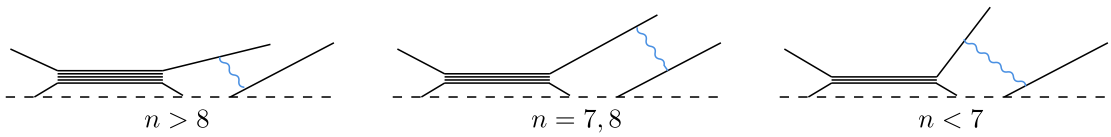


Figure 7-3 D1'-branes (blue) create hypermultiplet particles in the spinor representation.

When k D1-branes create k instantons and j D1'-branes create j hypermultiplet particles, we can view the system as $Sp(k)$-O(j) quiver gauge theory from the perspective of quantum mechanics on the D1-D1'-branes. (See figures 7-4 and 7-5.) The gauge group O(j) has two connected components, O$(j)_\pm$, which leads to generally two contributions, $Z^{SO(n)}_{k,j,\pm}$, to the partition function for a (conjugate) spinor matter. These contributions can be obtained by performing JK residue integrals[11]. However, for $n =$ odd, $Z^{SO(n)}_{k,\text{even},-}$ and $Z^{SO(n)}_{k,\text{odd},+}$ vanish due to fermionic zero modes. The full partition function for a spinor hypermultiplet can still be written as

$$Z^{\text{spinor}}_{\text{full},SO(n)} = \sum_{k,j=0}^{\infty} \mathfrak{q}^k e^{-jm} \frac{Z^{SO(n)}_{k,j,+} + Z^{SO(n)}_{k,j,-}}{2} \tag{7.23}$$

where m is the mass of the hypermultiplet. On the other hand, for a conjugate spinor hypermultiplet, the full partition function is

$$Z^{\text{conj}}_{\text{full},SO(n)} = \sum_{k,j=0}^{\infty} \mathfrak{q}^k e^{-jm} (-1)^j \frac{Z^{SO(n)}_{k,j,+} + (-1)^{\text{sign}(j)} Z^{SO(n)}_{k,j,-}}{2} \tag{7.24}$$

where $\text{sign}(j) = 0$ for $j = 0$, and $\text{sign}(j) = 1$ for $j > 0$. For $n < 9$, the full partition function Z_{full} contains not only instanton contributions but also spurious contributions.

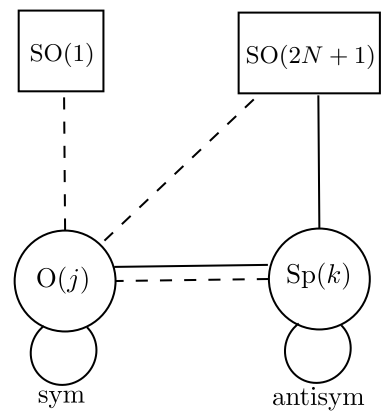


Figure 7-4 The solid line represents a hypermultiplet while the dashed line represents a Fermi multiplet.

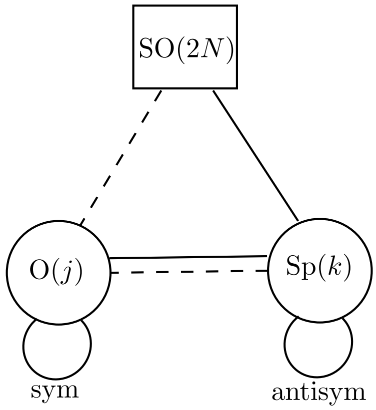


Figure 7-5 The solid line represents a hypermultiplet while the dashed line represents a Fermi multiplet.

However, we must remove spurious contributions to extract genuine instanton contributions[40]. Below, we focus only on the spinor hypermultiplet, but the story is parallel for the conjugate spinor (just insert appropriate signs as in 7.24). For $n \leq 6$, the partition function at $k = 0$ is given by

$$\sum_{j=0}^{\infty} e^{-jm} \frac{Z_{0,j,+}^{SO(n)} + Z_{0,j,-}^{SO(n)}}{2} = Z_{\text{pert}} \equiv \text{PE}\left[e^{-m} \frac{Z_{0,1,+}^{SO(n)} + Z_{0,1,-}^{SO(n)}}{2} \right] \tag{7.25}$$

where PE is the plethystic exponent that is defined as:

$$\text{PE}\left[f(x, y, \cdots) \right] \equiv \exp\left[\sum_{d=1}^{\infty} \frac{1}{d} f(x^d, y^d, \cdots) \right] \tag{7.26}$$

Consequently, the full partition function factorizes as

$$Z_{\text{full},SO(n)}^{\text{spinor}} = \widehat{Z}_{SO(n)}^{\text{spinor}}(\mathfrak{q}, \epsilon_{1,2}, m) Z_{\text{pert}}(\epsilon_{1,2}, m) \tag{7.27}$$

where we extract the genuine instanton contribution

$$\widehat{Z}_{SO(n)}^{\text{spinor}}(\mathfrak{q}, \epsilon_{1,2}, m) = \sum_{k=0}^{\infty} \mathfrak{q}^k \widehat{Z}_{SO(n),k}^{\text{spinor}}(\epsilon_{1,2}, m)\ , \quad \widehat{Z}_{SO(n),k}^{\text{spinor}}(\epsilon_{1,2}, m) = \sum_{j=0}^{\infty} e^{-jm} \widehat{Z}_{SO(n),k,j}^{\text{spinor}} \tag{7.28}$$

For the cases of $n \leq 6$, the contributions from hypermultiplet particles more than instanton numbers vanish as $\widehat{Z}_{k,j} = 0$ for $j > k$ for both the spinor and the conjugate spinor.

For $n = 7, 8$, the partition function at $k = 0$ is given by

$$\begin{aligned} \sum_{j=0}^{\infty} e^{-jm} \frac{Z_{0,j,+}^{SO(n)} + Z_{0,j,-}^{SO(n)}}{2} =& Z_{\text{pert}} Z_{\text{extra}} \\ =& \text{PE}\left[e^{-m} \frac{Z_{0,1,+}^{SO(n)} + Z_{0,1,-}^{SO(n)}}{2} \right] \text{PE}\left[-\frac{1}{2} e^{-2m} \frac{\text{ch}\, 2\epsilon_+}{\text{sh}\, \epsilon_{1,2}} \right] \end{aligned} \tag{7.29}$$

where Z_{extra} is the contribution from D1'-branes running away along the two parallel fivebranes. (See the middle of Figure 7-3.) Thus, the full partition function factorizes as

$$Z^{\text{spinor}}_{\text{full},SO(n)} = \widehat{Z}^{\text{spinor}}_{SO(n)}(\mathfrak{q},\epsilon_{1,2},m)Z_{\text{pert}}(\epsilon_{1,2},m)Z_{\text{extra}}(\epsilon_{1,2},m)\,, \tag{7.30}$$

where $\widehat{Z}^{\text{spinor}}_{SO(n)}(\mathfrak{q},\epsilon_{1,2},m)$ takes the form 7.28. In the cases of $n=7,8$, for both the spinor and the conjugate spinor, the instanton partition functions are subject to $\widehat{Z}_{k,2k-n}=\widehat{Z}_{k,n}$ so that $\widehat{Z}_{k,j}=0$ for $j>2k$.

However, for the $n\geq 9$ case, we cannot express the contribution purely from the hypermultiplet in the form of a plethystic exponent, unlike in previous examples 7.25 and 7.29. This is because the leg fivebranes meet at a certain point in this case. And we will leave a detailed investigation of this problem in future research.

For the explicit formulas, we will list them in the following.

7.3.2 Expression of $SO(2N+1)$ with one spinor

For a gauge group of B type, the contributions from $\mathrm{O}(2\ell)_-$ and $\mathrm{O}(2\ell+1)_+$ sectors vanish due to fermionic zero modes. As a result, we have a single contribution, $Z^{SO(2N+1)}_{k,j}$, for each hypermultiplet particle number j. This is another way to see that there is no distinction between the spinor representation and its conjugate for $SO(n)$ with odd n.

The brane system for $SO(2N+1)$ gauge theory with spinor hypermultiplet is schematically drawn in Figure 7-2. The instanton quantum mechanics can be derived from open strings in this brane system. $\mathcal{N}=4$ supersymmetric quantum mechanics on k D1-branes and j D1'-branes is given in Figure 7-4. The formal $SO(1)$ comes from the half D5-brane on the $\widetilde{\mathrm{O5}}^+$-plane, on the leg part of fivebranes in Figure 7-2. For 5d gauge groups of B type, the contributions from $\mathrm{O}(2\ell)_-$ and $\mathrm{O}(2\ell+1)_+$ sectors vanish due to the fermionic zero modes.

Hence, the integrand of the supersymmetric quantum mechanics for k instantons and 2ℓ hypermultiplet particles is given by

$$\begin{aligned}
&z^{SO(2N+1)}_{k,2\ell}\\
=&\frac{1}{2^k\,k!}\frac{\mathrm{sh}^k(2\epsilon_+)}{\mathrm{sh}^k(\epsilon_{1,2})}\prod_{I=1}^{k}\frac{\mathrm{sh}(2\epsilon_+\pm 2\phi_I)\,\mathrm{sh}(\pm 2\phi_I)}{\mathrm{sh}(\epsilon_+\pm\phi_I)\prod_{s=1}^N\mathrm{sh}(\epsilon_+\pm\phi_I\pm a_s)}\prod_{I>J}\frac{\mathrm{sh}(2\epsilon_+\pm\phi_I\pm\phi_J)\,\mathrm{sh}(\pm\phi_I\pm\phi_J)}{\mathrm{sh}(\epsilon_{1,2}\pm\phi_I\pm\phi_J)}\\
&\frac{1}{2^\ell\,\ell!}\frac{\mathrm{sh}^\ell(2\epsilon_+)}{\mathrm{sh}^\ell(\epsilon_{1,2})}\prod_{i=1}^{\ell}\frac{\mathrm{sh}(\pm\chi_i)\prod_{s=1}^N\mathrm{sh}(\pm\chi_i+a_s)}{\mathrm{sh}(\epsilon_{1,2}\pm 2\chi_i)}\prod_{i>j}\frac{\mathrm{sh}(2\epsilon_+\pm\chi_i\pm\chi_j)\,\mathrm{sh}(\pm\chi_i\pm\chi_j)}{\mathrm{sh}(\epsilon_{1,2}\pm\chi_i\pm\chi_j)}\\
&\prod_{I=1}^{k}\prod_{i=1}^{\ell}\frac{\mathrm{sh}(\epsilon_-\pm\phi_I\pm\chi_i)}{\mathrm{sh}(-\epsilon_+\pm\phi_I\pm\chi_i)}
\end{aligned}\tag{7.31}$$

Where ϕ_I ($I=1,\cdots,k$) are $Sp(k)$ gauge fugacities, and χ_i ($i=1,\cdots,\ell$) are $\mathrm{O}(2\ell)$ gauge fugacities so that they are integrated by the JK prescription. On the other hand, the integrand for k instantons and $(2\ell+1)$ hypermultiplet particles is given by

$$\begin{aligned}
&z^{SO(2N+1)}_{k,2\ell+1}\\
=&\frac{1}{2^k\,k!}\frac{\mathrm{sh}^k(2\epsilon_+)}{\mathrm{sh}^k(\epsilon_{1,2})}\prod_{I=1}^{k}\frac{\mathrm{sh}(2\epsilon_+\pm 2\phi_I)\ \mathrm{sh}(\pm 2\phi_I)}{\mathrm{sh}(\epsilon_+\pm\phi_I)\prod_{s=1}^{N}\mathrm{sh}(\epsilon_+\pm\phi_I\pm a_s)}\prod_{I>J}\frac{\mathrm{sh}(2\epsilon_+\pm\phi_I\pm\phi_J)\ \mathrm{sh}(\pm\phi_I\pm\phi_J)}{\mathrm{sh}(\epsilon_{1,2}\pm\phi_I\pm\phi_J)}\\
&\frac{1}{2^\ell\,\ell!}\frac{\mathrm{sh}^\ell(2\epsilon_+)}{\mathrm{sh}^{\ell+1}(\epsilon_{1,2})}\prod_{i=1}^{\ell}\frac{\mathrm{ch}(2\epsilon_+\pm\chi_i)\ \mathrm{sh}(\pm 2\chi_i)\prod_{s=1}^{N}\mathrm{sh}(\pm\chi_i+a_s)}{\mathrm{ch}(\epsilon_{1,2}\pm\chi_i)\ \mathrm{sh}(\epsilon_{1,2}\pm 2\chi_i)}\cdot\prod_{s=1}^{N}\mathrm{ch}(a_s)\\
&\prod_{i>j}\frac{\mathrm{sh}(2\epsilon_+\pm\chi_i\pm\chi_j)\ \mathrm{sh}(\pm\chi_i\pm\chi_j)}{\mathrm{sh}(\epsilon_{1,2}\pm\chi_i\pm\chi_j)}\prod_{I=1}^{k}\frac{\mathrm{ch}(\epsilon_-\pm\phi_I)}{\mathrm{ch}(-\epsilon_+\pm\phi_I)}\prod_{i=1}^{\ell}\frac{\mathrm{sh}(\epsilon_-\pm\phi_I\pm\chi_i)}{\mathrm{sh}(-\epsilon_+\pm\phi_I\pm\chi_i)}
\end{aligned}\tag{7.32}$$

For the Young diagram summation form, there are four effective Coulomb branch parameters so that even the “Sp(0)” gauge theory contributes to the unrefined partition function as we discussed in 7.16. Hence, for $Z^{SO(2N+1)}_{k,2\ell(+1)}$, the non-trivial JK poles in the unrefined limit are classified $(N+4)$-tuples $\vec{\lambda}$ of 2d Young diagrams where the numbers of boxes are subject to

$$\sum_{s=1}^{N}|\lambda^{(s)}|=k\,,\qquad \sum_{s=N+1}^{N+4}|\lambda^{(s)}|=\ell\tag{7.33}$$

The JK residue sums are expressed as

$$\begin{aligned}
&Z^{SO(2N+1)}_{k,2\ell}(A_{N+1}=q^{\frac12},A_{N+2}=-q^{\frac12},A_{N+3}=1,A_{N+4}=-1;q)=(-1)^{k+\ell}\\
&\sum_{\vec\lambda}C^{Sp}_{\vec\lambda,\vec A}\prod_{s=1}^{N}\prod_{x\in\lambda^{(s)}}\frac{\mathrm{sh}^4\,2\phi_s(x)}{\mathrm{sh}^2(\phi_s(x))\prod_{t=1}^{N}\mathrm{sh}^2(N_{s,t}(x))\ \mathrm{sh}^2(\phi_s(x)+a_t)}\prod_{s\le t}^{N}\prod_{\substack{x\in\lambda^{(s)}\\ y\in\lambda^{(t)}\\ x<y}}\frac{\mathrm{sh}^4(\phi_s(x)+\phi_t(y))}{\mathrm{sh}^2(\phi_s(x)+\phi_t(y)\pm\hbar)}\\
&\prod_{s=N+1}^{N+4}\prod_{x\in\lambda^{(s)}}\frac{\mathrm{sh}^2(\phi_s(x))\ \mathrm{sh}^4(2\phi_s(x))\prod_{t=1}^{N}\mathrm{sh}(\pm\phi_s(x)+a_t)}{\prod_{t=N+1}^{N+4}\mathrm{sh}^2(N_{s,t}(x))\ \mathrm{sh}^2(\phi_s(x)+a_t)}\prod_{N+1\le s\le t}^{N+4}\prod_{\substack{x\in\lambda^{(s)}\\ y\in\lambda^{(t)}\\ x<y}}\frac{\mathrm{sh}^4(\phi_s(x)+\phi_t(y))}{\mathrm{sh}^2(\phi_s(x)+\phi_t(y)\pm\hbar)}\\
&\prod_{s=1}^{N}\prod_{x\in\lambda^{(s)}}\prod_{t=N+1}^{N+4}\prod_{y\in\lambda^{(t)}}\frac{\mathrm{sh}(\hbar\pm\phi_s(x)\pm\phi_t(y))}{\mathrm{sh}(\pm\phi_s(x)\pm\phi_t(y))}
\end{aligned}\tag{7.34}$$

$$\begin{aligned}
&Z^{SO(2N+1)}_{k,2\ell+1}(A_{N+1}=q^{\frac12},A_{N+2}=-q^{\frac12},A_{N+3}=1,A_{N+4}=-q;q)=(-1)^{k+\ell+1}\frac{\prod_{t=1}^{3}\mathrm{ch}\,a_t}{\mathrm{sh}^2\,\hbar}\\
&\sum_{\vec\lambda}C^{Sp}_{\vec\lambda,\vec A}\prod_{s=1}^{N}\prod_{x\in\lambda^{(s)}}\frac{\mathrm{sh}^4\,2\phi_s(x)}{\mathrm{sh}^2(\phi_s(x))\prod_{t=1}^{N}\mathrm{sh}^2(N_{s,t}(x))\ \mathrm{sh}^2(\phi_s(x)+a_t)}\prod_{s\le t}^{N}\prod_{\substack{x\in\lambda^{(s)}\\ y\in\lambda^{(t)}\\ x<y}}\frac{\mathrm{sh}^4(\phi_s(x)+\phi_t(y))}{\mathrm{sh}^2(\phi_s(x)+\phi_t(y)\pm\hbar)}\\
&\prod_{s=N+1}^{N+4}\prod_{x\in\lambda^{(s)}}\frac{\mathrm{sh}^2(\phi_s(x))\ \mathrm{sh}^4(2\phi_s(x))\prod_{t=1}^{N}\mathrm{sh}(\pm\phi_s(x)+a_t)}{\prod_{t=N+1}^{N+4}\mathrm{sh}^2(N_{s,t}(x))\ \mathrm{sh}^2(\phi_s(x)+a_t)}\prod_{N+1\le s\le t}^{N+4}\prod_{\substack{x\in\lambda^{(s)}\\ y\in\lambda^{(t)}\\ x<y}}\frac{\mathrm{sh}^4(\phi_s(x)+\phi_t(y))}{\mathrm{sh}^2(\phi_s(x)+\phi_t(y)\pm\hbar)}\\
&\prod_{s=1}^{N}\prod_{x\in\lambda^{(s)}}\prod_{t=N+1}^{N+4}\prod_{y\in\lambda^{(t)}}\frac{\mathrm{sh}(\hbar\pm\phi_s(x)\pm\phi_t(y))}{\mathrm{sh}(\pm\phi_s(x)\pm\phi_t(y))}\frac{\mathrm{ch}(\hbar\pm\phi_s(x))}{\mathrm{ch}(\pm\phi_s(x))}
\end{aligned}\tag{7.35}$$

where $\phi_s(x)$ and $N_{s,t}(x)$ are given in 7.11 and 7.12, and $C^{Sp}_{\vec{\lambda},\vec{A}}$ is a constant

$$C^{Sp}_{\vec{\lambda},\vec{A}} = \prod_{s=N+1}^{N+4} C^{Sp}_{\lambda^{(s)},A_s} \,. \tag{7.36}$$

These coefficients are the same as those that appear in the pure $Sp(N)$ instanton partition function[8]: $C_{\lambda^{(s)}=\varnothing,A_s} = 1$, and

$$\begin{aligned} C^{Sp}_{\lambda^{(s)},A_s=\pm 1,\pm q^{\frac{1}{2}}} &= \frac{2^{2j-1}}{\binom{2j-1}{j-1}} \qquad \text{where } j \text{ is the number of rows with } \lambda_i^{(s)} \geq i \,, \\ C^{Sp}_{\lambda^{(s)},A_s=\pm q} &= \frac{2^{2j}}{\binom{2j+1}{j}} \qquad \text{where } j \text{ is the number of rows with } \lambda_i^{(s)} \geq i+1 \,. \end{aligned} \tag{7.37}$$

(See also 7.18 for the illustration by Young diagrams.) Note that these multiplicity coefficients are still conjectural. And we compare our result with (2.29)[40] up to 5 instantons.

7.3.3 Expression of $SO(2N)$ with one (conjugate) spinor

For a gauge group of D type, the two sectors from $\mathrm{O}(j)_\pm$ make non-trivial contributions. This is another way to see that the spinor representation and its conjugate are not isomorphic for $SO(n)$ for general even n.

The brane system for $SO(2N)$ gauge theory with (conjugate) spinor hypermultiplet is shown in Figure 7-2. This brane system allows us to derive the instanton quantum mechanics from open strings. In Figure 7-5, we see the $\mathcal{N} = 4$ supersymmetric quantum mechanics on k D1-branes and j D1'-branes. In this case, there are non-trivial contributions from the two connected components of the gauge group $\mathrm{O}(j)$.

To account for each contribution, we write down an integrand of the partition function for each one:

$$\begin{aligned} &z^{SO(2N)}_{k,2\ell,+} = \\ &\frac{1}{2^k\, k!}\frac{\mathrm{sh}^k(2\epsilon_+)}{\mathrm{sh}^k(\epsilon_{1,2})}\prod_{I=1}^{k}\frac{\mathrm{sh}(2\epsilon_+ \pm 2\phi_I)\ \mathrm{sh}(\pm 2\phi_I)}{\prod_{s=1}^{N}\mathrm{sh}(\epsilon_+ \pm \phi_I \pm a_s)}\prod_{I>J}\frac{\mathrm{sh}(2\epsilon_+ \pm \phi_I \pm \phi_J)\ \mathrm{sh}(\pm\phi_I \pm \phi_J)}{\mathrm{sh}(\epsilon_{1,2} \pm \phi_I \pm \phi_J)} \\ &\frac{2^{\mathrm{sign}(\ell)}}{2^\ell\, \ell!}\frac{\mathrm{sh}^\ell(2\epsilon_+)}{\mathrm{sh}^\ell(\epsilon_{1,2})}\prod_{i=1}^{\ell}\frac{\prod_{s=1}^{N}\mathrm{sh}(\pm\chi_i + a_s)}{\mathrm{sh}(\epsilon_{1,2} \pm 2\chi_i)}\prod_{i>j}\frac{\mathrm{sh}(2\epsilon_+ \pm \chi_i \pm \chi_j)\ \mathrm{sh}(\pm\chi_i \pm \chi_j)}{\mathrm{sh}(\epsilon_{1,2} \pm \chi_i \pm \chi_j)} \\ &\prod_{I=1}^{k}\prod_{i=1}^{\ell}\frac{\mathrm{sh}(\epsilon_- \pm \phi_I \pm \chi_i)}{\mathrm{sh}(-\epsilon_+ \pm \phi_I \pm \chi_i)} \end{aligned} \tag{7.38}$$

$$
\begin{aligned}
&Z^{SO(2N)}_{k,2\ell,-} = \\
&\frac{1}{2^k\,k!}\frac{\mathrm{sh}^k(2\epsilon_+)}{\mathrm{sh}^k(\epsilon_{1,2})}\prod_{I=1}^{k}\frac{\mathrm{sh}(2\epsilon_+\pm 2\phi_I)\ \mathrm{sh}(\pm 2\phi_I)}{\prod_{s=1}^{N}\mathrm{sh}(\epsilon_+\pm\phi_I\pm a_s)}\prod_{I>J}\frac{\mathrm{sh}(2\epsilon_+\pm\phi_I\pm\phi_J)\ \mathrm{sh}(\pm\phi_I\pm\phi_J)}{\mathrm{sh}(\epsilon_{1,2}\pm\phi_I\pm\phi_J)} \\
&\left[\frac{\prod_{s=1}^{N}\mathrm{sh}(2a_s)}{2^{\ell-1}\,(\ell-1)!}\frac{\mathrm{ch}(2\epsilon_+)}{\mathrm{sh}(2\epsilon_{1,2})\,\mathrm{sh}(\epsilon_{1,2})}\prod_{i=1}^{\ell-1}\frac{\mathrm{sh}(2\epsilon_+)\ \mathrm{sh}(4\epsilon_+\pm 2\chi_i)\ \mathrm{sh}(\pm 2\chi_i)\prod_{s=1}^{N}\mathrm{sh}(\pm\chi_i+a_s)}{\mathrm{sh}(\epsilon_{1,2})\ \mathrm{sh}(2\epsilon_{1,2}\pm 2\chi_i)\ \mathrm{sh}(\epsilon_{1,2}\pm 2\chi_i)}\right. \\
&\left.\prod_{i>j}\frac{\mathrm{sh}(2\epsilon_+\pm\chi_i\pm\chi_j)\ \mathrm{sh}(\pm\chi_i\pm\chi_j)}{\mathrm{sh}(\epsilon_{1,2}\pm\chi_i\pm\chi_j)}\prod_{I=1}^{k}\frac{\mathrm{sh}(2\epsilon_-\pm 2\phi_I)}{\mathrm{sh}(-2\epsilon_+\pm 2\phi_I)}\prod_{i=1}^{\ell-1}\frac{\mathrm{sh}(\epsilon_-\pm\phi_I\pm\chi_i)}{\mathrm{sh}(-\epsilon_+\pm\phi_I\pm\chi_i)}\right]^{\mathrm{sign}(\ell)}
\end{aligned} \tag{7.39}
$$

$$
\begin{aligned}
&Z^{SO(2N)}_{k,2\ell+1,+} = \\
&\frac{1}{2^k\,k!}\frac{\mathrm{sh}^k(2\epsilon_+)}{\mathrm{sh}^k(\epsilon_{1,2})}\prod_{I=1}^{k}\frac{\mathrm{sh}(2\epsilon_+\pm 2\phi_I)\ \mathrm{sh}(\pm 2\phi_I)}{\prod_{s=1}^{N}\mathrm{sh}(\epsilon_+\pm\phi_I\pm a_s)}\prod_{I>J}\frac{\mathrm{sh}(2\epsilon_+\pm\phi_I\pm\phi_J)\ \mathrm{sh}(\pm\phi_I\pm\phi_J)}{\mathrm{sh}(\epsilon_{1,2}\pm\phi_I\pm\phi_J)} \\
&\frac{\prod_{s=1}^{N}\mathrm{sh}(a_s)}{2^{\ell}\,\ell!}\frac{\mathrm{sh}^{\ell}(2\epsilon_+)}{\mathrm{sh}^{\ell+1}(\epsilon_{1,2})}\prod_{i=1}^{\ell}\frac{\mathrm{sh}(2\epsilon_+\pm\chi_i)\ \mathrm{sh}(\pm\chi_i)\prod_{s=1}^{N}\mathrm{sh}(\pm\chi_i+a_s)}{\mathrm{sh}(\epsilon_{1,2}\pm\chi_i)\ \mathrm{sh}(\epsilon_{1,2}\pm 2\chi_i)} \\
&\prod_{i>j}\frac{\mathrm{sh}(2\epsilon_+\pm\chi_i\pm\chi_j)\ \mathrm{sh}(\pm\chi_i\pm\chi_j)}{\mathrm{sh}(\epsilon_{1,2}\pm\chi_i\pm\chi_j)}\prod_{I=1}^{k}\frac{\mathrm{sh}(\epsilon_-\pm\phi_I)}{\mathrm{sh}(-\epsilon_+\pm\phi_I)}\prod_{i=1}^{\ell}\frac{\mathrm{sh}(\epsilon_-\pm\phi_I\pm\chi_i)}{\mathrm{sh}(-\epsilon_+\pm\phi_I\pm\chi_i)}
\end{aligned} \tag{7.40}
$$

$$
\begin{aligned}
&Z^{SO(2N)}_{k,2\ell+1,-} = \\
&\frac{1}{2^k\,k!}\frac{\mathrm{sh}^k(2\epsilon_+)}{\mathrm{sh}^k(\epsilon_{1,2})}\prod_{I=1}^{k}\frac{\mathrm{sh}(2\epsilon_+\pm 2\phi_I)\ \mathrm{sh}(\pm 2\phi_I)}{\prod_{s=1}^{N}\mathrm{sh}(\epsilon_+\pm\phi_I\pm a_s)}\prod_{I>J}\frac{\mathrm{sh}(2\epsilon_+\pm\phi_I\pm\phi_J)\ \mathrm{sh}(\pm\phi_I\pm\phi_J)}{\mathrm{sh}(\epsilon_{1,2}\pm\phi_I\pm\phi_J)} \\
&\frac{\prod_{s=1}^{N}\mathrm{ch}(a_s)}{2^{\ell}\,\ell!}\frac{\mathrm{sh}^{\ell}(2\epsilon_+)}{\mathrm{sh}^{\ell+1}(\epsilon_{1,2})}\prod_{i=1}^{\ell}\frac{\mathrm{ch}(2\epsilon_+\pm\chi_i)\ \mathrm{ch}(\pm\chi_i)\prod_{s=1}^{N}\mathrm{sh}(\pm\chi_i+a_s)}{\mathrm{ch}(\epsilon_{1,2}\pm\chi_i)\ \mathrm{sh}(\epsilon_{1,2}\pm 2\chi_i)} \\
&\prod_{i>j}\frac{\mathrm{sh}(2\epsilon_+\pm\chi_i\pm\chi_j)\ \mathrm{sh}(\pm\chi_i\pm\chi_j)}{\mathrm{sh}(\epsilon_{1,2}\pm\chi_i\pm\chi_j)}\prod_{I=1}^{k}\frac{\mathrm{ch}(\epsilon_-\pm\phi_I)}{\mathrm{ch}(-\epsilon_+\pm\phi_I)}\prod_{i=1}^{\ell}\frac{\mathrm{sh}(\epsilon_-\pm\phi_I\pm\chi_i)}{\mathrm{sh}(-\epsilon_+\pm\phi_I\pm\chi_i)}
\end{aligned} \tag{7.41}
$$

Then, the non-trivial JK poles in the unrefined limit are classified $(N+4)$-tuples $\vec{\lambda}$ of 2d Young diagrams where the numbers of boxes for $Z^{SO(2N)}_{k,2\ell,+}$ and $Z^{SO(2N)}_{k,2\ell+1,\pm}$ are subject to

$$
\sum_{s=1}^{N}|\lambda^{(s)}| = k\ , \qquad \sum_{s=N+1}^{N+4}|\lambda^{(s)}| = \ell \tag{7.42}
$$

whereas those for $Z^{SO(2N)}_{k,2\ell,-}$ is required to be

$$
\sum_{s=1}^{N}|\lambda^{(s)}| = k\ , \qquad \sum_{s=N+1}^{N+4}|\lambda^{(s)}| = \ell-1 \tag{7.43}
$$

Then, using the multiplicity coefficients in 7.37, the JK residue sum of each sector is expressed as

$$
\begin{aligned}
&Z^{SO(2N)}_{k,2\ell,+}(A_{N+1}=q^{\frac{1}{2}},A_{N+2}=-q^{\frac{1}{2}},A_{N+3}=1,A_{N+4}=-1;q)=2^{\mathrm{sign}(\ell)}\\
&\sum_{\vec{\lambda}} C^{Sp}_{\vec{\lambda},\vec{A}}\prod_{s=1}^{N}\prod_{x\in\lambda^{(s)}}\frac{\mathrm{sh}^4(2\phi_s(x))}{\prod_{t=1}^{N}\mathrm{sh}^2(N_{s,t}(x))\ \mathrm{sh}^2(\phi_s(x)+a_t)}\prod_{s\le t}^{N}\prod_{\substack{x\in\lambda^{(s)}\\ y\in\lambda^{(t)}\\ x<y}}\frac{\mathrm{sh}^4(\phi_s(x)+\phi_t(y))}{\mathrm{sh}^2(\phi_s(x)+\phi_t(y)\pm\hbar)}\\
&\prod_{s=N+1}^{N+4}\prod_{x\in\lambda^{(s)}}\frac{\mathrm{sh}^4(2\phi_s(x))\prod_{t=1}^{N}\mathrm{sh}(\pm\phi_s(x)+a_t)}{\prod_{t=N+1}^{N+4}\mathrm{sh}^2(N_{s,t}(x))\ \mathrm{sh}^2(\phi_s(x)+a_t)}\prod_{N+1\le s\le t}^{N+4}\prod_{\substack{x\in\lambda^{(s)}\\ y\in\lambda^{(t)}\\ x<y}}\frac{\mathrm{sh}^4(\phi_s(x)+\phi_t(y))}{\mathrm{sh}^2(\phi_s(x)+\phi_t(y)\pm\hbar)}\\
&\prod_{s=1}^{N}\prod_{x\in\lambda^{(s)}}\prod_{t=N+1}^{N+4}\prod_{y\in\lambda^{(t)}}\frac{\mathrm{sh}(\hbar\pm\phi_s(x)\pm\phi_t(y))}{\mathrm{sh}(\pm\phi_s(x)\pm\phi_t(y))}
\end{aligned}
\tag{7.44}
$$

$$
\begin{aligned}
&Z^{SO(2N)}_{k,2\ell,-}(A_{N+1}=q^{\frac{1}{2}},A_{N+2}=-q^{\frac{1}{2}},A_{N+3}=q,A_{N+4}=-q;q)=\left[(-1)^k 2\frac{\prod_{s=1}^{N}\mathrm{sh}(2a_s)}{\mathrm{sh}^2(\hbar)\,\mathrm{sh}^2(2\hbar)}\right]^{\mathrm{sign}(\ell)}\\
&\sum_{\vec{\lambda}} C^{Sp}_{\vec{\lambda},\vec{A}}\prod_{s=1}^{N}\prod_{x\in\lambda^{(s)}}\frac{\mathrm{sh}^4(2\phi_s(x))}{\prod_{t=1}^{N}\mathrm{sh}^2(N_{s,t}(x))\ \mathrm{sh}^2(\phi_s(x)+a_t)}\prod_{s\le t}^{N}\prod_{\substack{x\in\lambda^{(s)}\\ y\in\lambda^{(t)}\\ x<y}}\frac{\mathrm{sh}^4(\phi_s(x)+\phi_t(y))}{\mathrm{sh}^2(\phi_s(x)+\phi_t(y)\pm\hbar)}\\
&\Bigg[\prod_{s=N+1}^{N+4}\prod_{x\in\lambda^{(s)}}\frac{\mathrm{sh}^4(2\phi_s(x))\prod_{t=1}^{N}\mathrm{sh}(\pm\phi_s(x)+a_t)}{\prod_{t=N+1}^{N+4}\mathrm{sh}^2(N_{s,t}(x))\ \mathrm{sh}^2(\phi_s(x)+a_t)}\prod_{N+1\le s\le t}^{N+4}\prod_{\substack{x\in\lambda^{(s)}\\ y\in\lambda^{(t)}\\ x<y}}\frac{\mathrm{sh}^4(\phi_s(x)+\phi_t(y))}{\mathrm{sh}^2(\phi_s(x)+\phi_t(y)\pm\hbar)}\\
&\prod_{s=1}^{N}\prod_{x\in\lambda^{(s)}}\frac{\mathrm{sh}(2\hbar\pm2\phi_s(x))}{\mathrm{sh}^2(2\phi_s(x))}\prod_{t=N+1}^{N+4}\prod_{y\in\lambda^{(t)}}\frac{\mathrm{sh}(\hbar\pm\phi_s(x)\pm\phi_t(y))}{\mathrm{sh}(\pm\phi_s(x)\pm\phi_t(y))}\Bigg]^{\mathrm{sign}(\ell)}
\end{aligned}
\tag{7.45}
$$

$$
\begin{aligned}
&Z^{SO(2N)}_{k,2\ell+1,+}(A_{N+1}=q^{\frac{1}{2}},A_{N+2}=-q^{\frac{1}{2}},A_{N+3}=q,A_{N+4}=-1;q)=(-1)^{k+1}\frac{\prod_{s=1}^{N}\mathrm{sh}(a_s)}{\mathrm{sh}^2(\hbar)}\\
&\sum_{\vec{\lambda}} C^{Sp}_{\vec{\lambda},\vec{A}}\prod_{s=1}^{N}\prod_{x\in\lambda^{(s)}}\frac{\mathrm{sh}^4(2\phi_s(x))}{\prod_{t=1}^{N}\mathrm{sh}^2(N_{s,t}(x))\ \mathrm{sh}^2(\phi_s(x)+a_t)}\prod_{s\le t}^{N}\prod_{\substack{x\in\lambda^{(s)}\\ y\in\lambda^{(t)}\\ x<y}}\frac{\mathrm{sh}^4(\phi_s(x)+\phi_t(y))}{\mathrm{sh}^2(\phi_s(x)+\phi_t(y)\pm\hbar)}\\
&\prod_{s=N+1}^{N+4}\prod_{x\in\lambda^{(s)}}\frac{\mathrm{sh}^4(2\phi_s(x))\prod_{t=1}^{N}\mathrm{sh}(\pm\phi_s(x)+a_t)}{\prod_{t=N+1}^{N+4}\mathrm{sh}^2(N_{s,t}(x))\ \mathrm{sh}^2(\phi_s(x)+a_t)}\prod_{N+1\le s\le t}^{N+4}\prod_{\substack{x\in\lambda^{(s)}\\ y\in\lambda^{(t)}\\ x<y}}\frac{\mathrm{sh}^4(\phi_s(x)+\phi_t(y))}{\mathrm{sh}^2(\phi_s(x)+\phi_t(y)\pm\hbar)}\\
&\prod_{s=1}^{N}\prod_{x\in\lambda^{(s)}}\frac{\mathrm{sh}(\hbar\pm\phi_s(x))}{\mathrm{sh}^2(\phi_s(x))}\prod_{t=N+1}^{N+4}\prod_{y\in\lambda^{(t)}}\frac{\mathrm{sh}(\hbar\pm\phi_s(x)\pm\phi_t(y))}{\mathrm{sh}(\pm\phi_s(x)\pm\phi_t(y))}
\end{aligned}
\tag{7.46}
$$

$$
\begin{aligned}
&Z^{SO(2N)}_{k,2\ell+1,-}(A_{N+1}=q^{\frac{1}{2}},A_{N+2}=-q^{\frac{1}{2}},A_{N+3}=1,A_{N+4}=-q;q)=-\frac{\prod_{s=1}^{N}\mathrm{ch}(a_s)}{\mathrm{sh}^2(\hbar)}\\
&\sum_{\vec{\lambda}}C^{Sp}_{\vec{\lambda},\vec{A}}\prod_{s=1}^{N}\prod_{x\in\lambda^{(s)}}\frac{\mathrm{sh}^4(2\phi_s(x))}{\prod_{t=1}^{N}\mathrm{sh}^2(N_{s,t}(x))\ \mathrm{sh}^2(\phi_s(x)+a_t)}\prod_{s\le t}^{N}\prod_{\substack{x\in\lambda^{(s)}\\ y\in\lambda^{(t)}\\ x<y}}\frac{\mathrm{sh}^4(\phi_s(x)+\phi_t(y))}{\mathrm{sh}^2(\phi_s(x)+\phi_t(y)\pm\hbar)}\\
&\prod_{s=N+1}^{N+4}\prod_{x\in\lambda^{(s)}}\frac{\mathrm{sh}^4(2\phi_s(x))\prod_{t=1}^{N}\mathrm{sh}(\pm\phi_s(x)+a_t)}{\prod_{t=N+1}^{N+4}\mathrm{sh}^2(N_{s,t}(x))\ \mathrm{sh}^2(\phi_s(x)+a_t)}\prod_{N+1\le s\le t}^{N+4}\prod_{\substack{x\in\lambda^{(s)}\\ y\in\lambda^{(t)}\\ x<y}}\frac{\mathrm{sh}^4(\phi_s(x)+\phi_t(y))}{\mathrm{sh}^2(\phi_s(x)+\phi_t(y)\pm\hbar)}\\
&\prod_{s=1}^{N}\prod_{x\in\lambda^{(s)}}\frac{\mathrm{ch}(\hbar\pm\phi_s(x))}{\mathrm{ch}^2(\phi_s(x))}\prod_{t=N+1}^{N+4}\prod_{y\in\lambda^{(t)}}\frac{\mathrm{sh}(\hbar\pm\phi_s(x)\pm\phi_t(y))}{\mathrm{sh}(\pm\phi_s(x)\pm\phi_t(y))}
\end{aligned}
\tag{7.47}
$$

7.3.4 Isomorphisms of representations

Using the expressions above, we can check a range of isomorphic relations between representations of Lie algebra. For instance:

- $SO(4)$ with (conjugate) spinor representation is isomorphic to the fundamental representation of $SU(2)\times SU(2)$:

$$
\begin{aligned}
&(\mathfrak{so}(4),\mathbf{S})\cong(\mathfrak{su}(2)\oplus\mathfrak{su}(2),\mathbf{F}\oplus\varnothing):\\
&\widehat{Z}^{\text{spinor}}_{SO(4)}=Z^{\text{fund}}_{SU(2),\kappa=\frac{1}{2}}\left(A\to\sqrt{A_1A_2},\mathfrak{q}\to-\mathfrak{q}M^{-1/2}\right)\\
&\qquad\times Z^{\text{pure}}_{SU(2),\kappa=0}\left(A\to\sqrt{\frac{A_1}{A_2}}\right)Z_{U(1)};\\
&(\mathfrak{so}(4),\mathbf{C})\cong(\mathfrak{su}(2)\oplus\mathfrak{su}(2),\varnothing\oplus\mathbf{F}):\\
&\widehat{Z}^{\text{conj}}_{SO(4)}=Z^{\text{pure}}_{SU(2),\kappa=0}\left(A\to\sqrt{A_1A_2}\right)\\
&\qquad\times Z^{\text{fund}}_{SU(2),\kappa=\frac{1}{2}}\left(A\to\sqrt{\frac{A_1}{A_2}},\mathfrak{q}\to-\mathfrak{q}M^{-1/2}\right)Z_{U(1)}
\end{aligned}
\tag{7.48}
$$

- $SO(5)$ with spinor representation is isomorphic to $SP(2)$ in fundamental representation:

$$
\begin{aligned}
&(\mathfrak{so}(5),\mathbf{S})\cong(\mathfrak{sp}(2),\mathbf{F}):\\
&M^{\frac{k}{2}}\widehat{Z}^{\text{spinor}}_{SO(5),k}=Z^{\text{fund}}_{Sp(2),k}\left(A_1\to\sqrt{A_1A_2},A_2\to\sqrt{\frac{A_1}{A_2}}\right)
\end{aligned}
\tag{7.49}
$$

- $SO(6)$ with (conjugate) spinor representation is isomorphic to the fundamental representation of $SU(4)$:

$$
\begin{aligned}
&(\mathfrak{so}(6),\mathbf{S})\cong(\mathfrak{su}(4),\mathbf{F}):\\
&M^{\frac{k}{2}}\widehat{Z}^{\text{spinor}}_{SO(6),k}=(-1)^kZ^{\text{fund}}_{SU(4),k,\kappa=\frac{1}{2}}\\
&(\mathfrak{so}(6),\mathbf{C})\cong(\mathfrak{su}(4),\mathbf{F}):\\
&M^{\frac{k}{2}}\widehat{Z}^{\text{conj}}_{SO(6),k}=Z^{\text{fund}}_{SU(4),k,\kappa=-\frac{1}{2}}\left(M\to M^{-1}\right)
\end{aligned}
\tag{7.50}
$$

Note that the $U(1)$ factor is given by

$$Z_{U(1)} = \mathrm{PE}\left[-\frac{\mathfrak{q}\,\mathrm{ch}\,\epsilon_+}{2\,\mathrm{sh}\,\epsilon_{1,2}}\right] \tag{7.51}$$

Also, we need to shift the 5d Chern-Simons level by $\pm\frac{1}{2}$ for $SU(N)$ gauge theory with one fundamental because of the parity anomaly.

In particular, 7.49 implies that the top ($j=k$) and bottom ($j=0$) components of the hypermultiplet particles are equal to

$$\begin{aligned} \widehat{Z}^{SO(5)}_{k,0} &= Z^{\text{pure}}_{Sp(2),k,\theta=0}(A_1 \to \sqrt{A_1A_2}, A_2 \to \sqrt{\frac{A_1}{A_2}}) \\ \widehat{Z}^{SO(5)}_{k,k} &= Z^{\text{pure}}_{Sp(2),k,\theta=\pi}(A_1 \to \sqrt{A_1A_2}, A_2 \to \sqrt{\frac{A_1}{A_2}}) \,. \end{aligned} \tag{7.52}$$

Also, 7.50 indicate that

$$\widehat{Z}^{SO(6)}_{k,0} = (-1)^k Z^{\text{pure}}_{SU(4),k,\kappa=0}\,, \qquad \widehat{Z}^{SO(6)}_{k,k} = Z^{\text{pure}}_{SU(4),k,\kappa=1} \tag{7.53}$$

- The $SO(8)$ instanton partition functions enjoy the triality[42] as shown in figure 7-6 And the change of variables is given by:

$$\begin{tikzcd} & (so(8),\mathbf{S}) \arrow[dr, "\text{c.o.v.}"] & \\ (so(8),\mathbf{V}) \arrow[ur, "\text{c.o.v.}"] & & (so(8),\mathbf{C}) \arrow[ll, "\text{c.o.v.}"] \end{tikzcd}$$

Figure 7-6 Triality property of representations of $\mathfrak{so}(8)$ with the change of variables

$$A_1 \to \sqrt{A_1A_2A_3A_4}\,, \quad A_2 \to \sqrt{\frac{A_1A_2}{A_3A_4}}\,, \quad A_3 \to \sqrt{\frac{A_1A_3}{A_2A_4}}\,, \quad A_4 \to \sqrt{\frac{A_2A_3}{A_1A_4}} \tag{7.54}$$

We check all these relations up to 5 instantons.

7.4 $SO(n)$ in $\mathcal{N}=1^*$

The maximally supersymmetric 5d $SO(n)$ gauge theory can be realized as a worldvolume theory on D4-branes with an $O4^-$-plane in Type IIA theory. The scalar field in the adjoint (antisymmetric) hypermultiplet represents the fluctuation of a D4-brane along the transverse direction to its worldvolume. When we turn on the Ω-background transverse to the D4-branes/O4-plane, the adjoint hypermultiplet acquires a mass, giving rise to the $SO(n)$ $\mathcal{N}=1^*$ theory.

The equivariant index for the adjoint hypermultiplet is described in detail in Shadchin's thesis[12]. The integrand of the JK residue integral for the instanton partition function of the $SO(n)$ $\mathcal{N}=1^*$ theory is given by

$$\begin{aligned}
&z^{\text{adj}}_{SO(n),k}\\
&=\frac{1}{k!2^k}\prod_{I=1}^{k}\frac{\text{sh}\,(2\epsilon_+)\,\text{sh}(\pm 2\phi_I)\,\text{sh}(\pm 2\phi_I+2\epsilon_+)}{\text{sh}\,(\epsilon_{1,2})\,\text{sh}^{\chi}(\epsilon_+\pm\phi_I)\prod_{s=1}^{\lfloor\frac{n}{2}\rfloor}\text{sh}\,(\pm\phi_I\pm a_s+\epsilon_+)}\prod_{I<J}^{k}\frac{\text{sh}\,(\pm\phi_I\pm\phi_J)\,\text{sh}\,(\pm\phi_I\pm\phi_J+2\epsilon_+)}{\text{sh}\,(\pm\phi_I\pm\phi_J+\epsilon_{1,2})}\\
&\cdot\prod_{I=1}^{k}\frac{\text{sh}\,(\pm m-\epsilon_-)\,\text{sh}^{\chi}(m\pm\phi_I)\prod_{s=1}^{\lfloor\frac{n}{2}\rfloor}\text{sh}\,(\pm\phi_I\pm a_s+m)}{\text{sh}\,(\pm m-\epsilon_+)\,\text{sh}\,(\pm 2\phi_I\pm m-\epsilon_+)}\prod_{I<J}^{k}\frac{\text{sh}\,(\pm\phi_I\pm\phi_J\pm m-\epsilon_-)}{\text{sh}\,(\pm\phi_I\pm\phi_J\pm m-\epsilon_+)}
\end{aligned}\tag{7.55}$$

where $n\equiv\chi \mod 2$.

At the unrefined level, $\hbar$ and m play a similar role in the JK residue integral. This is natural because they are the parameters for the Ω-background. Indeed, poles are classified by $(\lfloor\frac{n}{2}\rfloor+4)$-tuples of 4d young diagrams with the total number of boxes k. We specify a pole location associated with a content $x=(x_1,x_2,x_3,x_4)\in\lambda^{(s)}$ as:

$$\phi_s(x)=a_s+(x_1-x_2)\hbar+(x_3-x_4)m \tag{7.56}$$

Note that a_s $(s=1,\ldots,\lfloor\frac{n}{2}\rfloor)$ are the Coulomb branch parameters of the $SO(n)$ gauge theory. Similarly to 7.15, the other four effective Coulomb branch parameters are given by

$$a_{\lfloor\frac{n}{2}\rfloor+j}=\frac{m}{2}(+\pi i)\,,\ \frac{\hbar-m}{2}(+\pi i) \tag{7.57}$$

for $j=1,2,3,4$. For later use, we define

$$\begin{aligned}
&\widetilde{Z}_k(\vec{A};q)\\
&=\sum_{\vec{\lambda}}C^{\text{4d}}_{\vec{\lambda},\vec{A}}\left(\frac{\text{sh}(m\pm\hbar)\ \text{sh}^2(0)}{\text{sh}^2(\hbar)\ \text{sh}^2(m)}\right)^k\\
&\prod_{s=1}^{\|\vec{\lambda}\|}\prod_{x\in\lambda^{(s)}}\frac{\text{sh}^4(2\phi_s(x))\ \text{sh}^2(2\phi_s(x)+m-\hbar)\ \text{sh}^2(2\phi_s(x)-m+\hbar)\prod_{t=1}^{N}\text{sh}(\phi_s(x)\pm a_t\pm m)}{\prod_{t=1}^{\|\vec{\lambda}\|}\text{sh}^2(\phi_s(x)\pm a_t)}\\
&\prod_{s\le t}^{\|\vec{\lambda}\|}\prod_{\substack{x\in\lambda^{(s)},\,y\in\lambda^{(t)}\\ x<y}}\frac{\text{sh}^4(\phi_s(x)\pm\phi_t(y))\ \text{sh}(\phi_s(x)\pm\phi_t(y)\pm m\pm\hbar)}{\text{sh}^2(\phi_s(x)\pm\phi_t(y)\pm\hbar)\ \text{sh}^2(\phi_s(x)\pm\phi_t(y)\pm m)}
\end{aligned}\tag{7.58}$$

where $\|\vec{\lambda}\|$ is the number of 4d Young diagrams in $\vec{\lambda}$, and the total ordering on the boxes is the natural extension of 7.14. In this case, $\|\vec{\lambda}\|=\lfloor\frac{n}{2}\rfloor+4$, and the instanton partition function is

$$Z^{\text{adj}}_{SO(n),k}=\widetilde{Z}_k(\vec{A},A_{N+1}=e^{-\frac{m}{2}},A_{N+2}=-e^{-\frac{m}{2}},A_{N+3}=e^{-\frac{\hbar-m}{2}},A_{N+4}=-e^{-\frac{\hbar-m}{2}};q) \tag{7.59}$$

where $N=\lfloor\frac{n}{2}\rfloor$. As in 7.17, the non-trivial multiplicity constants are involved due to higher-order poles coming from the additional effective Coulomb branch parameters 7.57. We do *not* know how to determine these multiplicity constants for general 4d Young diagrams. However, some of the multiplicity constants $C^{\text{4d}}_{\vec{\lambda},\vec{A}}$ are listed as follow[11]:

- $C^{4d}_{\lambda, A=\pm q^{\frac{1}{2}}}$ $\quad (a = \frac{\hbar}{2}(+\pi i))$:

$\ydiagram{1} \Longrightarrow 2;\quad \ydiagram{1,1} \Longrightarrow 2;$

$\begin{ytableau} (2,1) \end{ytableau} \Longrightarrow 2;\quad \ydiagram{3} \Longrightarrow 2;\quad \ydiagram{2,1} \Longrightarrow 2;\quad \ydiagram{1,1,1} \Longrightarrow 2;$

- $C^{4d}_{\lambda, A=\pm M^{\frac{1}{2}}}$ $\quad (a = \frac{m}{2}(+\pi i))$:

$\ydiagram{1} \Longrightarrow -2;\quad \begin{ytableau} (2)^t \end{ytableau} \Longrightarrow 2;$

$\ydiagram{2,1} \Longrightarrow -2;\quad \begin{ytableau} (3)^t \end{ytableau} \Longrightarrow -2;\quad \begin{ytableau} (3) \end{ytableau} \Longrightarrow -2;\quad \begin{ytableau} (2,1) \end{ytableau} \Longrightarrow -2;$

- $C^{4d}_{\lambda, A=\pm\left(\frac{q}{M}\right)^{\frac{1}{2}}}$ $\quad (a = \frac{\hbar - m}{2}(+i\pi))$:

$\ydiagram{2} \Longrightarrow -1;\quad \begin{ytableau} (2)^t \end{ytableau} \Longrightarrow -1;$

- $C^{4d}_{\lambda, A=\pm 1}$ $\quad (a = 0(+i\pi))$:

$\ydiagram{2} \Longrightarrow 2;\quad \ydiagram{1,1} \Longrightarrow 2;\quad \begin{ytableau} (2)^t \end{ytableau} \Longrightarrow 2;\quad \begin{ytableau} (2) \end{ytableau} \Longrightarrow 2;$

$\begin{ytableau} (3)^t \end{ytableau} \Longrightarrow -2;\quad \begin{ytableau} (3) \end{ytableau} \Longrightarrow -2;\quad \ydiagram{3} \Longrightarrow 2;\quad \ydiagram{1,1,1} \Longrightarrow 2;$

$\begin{ytableau} (2)^t & \end{ytableau} \Longrightarrow -1;\quad \begin{ytableau} (2) & \end{ytableau} \Longrightarrow -1;\quad \begin{ytableau} (2)^t \\ \end{ytableau} \Longrightarrow -1;\quad \begin{ytableau} (2) \\ \end{ytableau} \Longrightarrow -1;$

- $C^{4d}_{\lambda, A=\pm q}$ $\quad (a = \hbar(+i\pi))$:

$\ydiagram{1} \Longrightarrow 1;\quad \ydiagram{1,1} \Longrightarrow 1;\quad \ydiagram{2} \Longrightarrow 2;$

$\ydiagram{1,1,1} \Longrightarrow 1;\quad \begin{ytableau} (2,1) \end{ytableau} \Longrightarrow 2;\quad \ydiagram{2,1} \Longrightarrow 2;$

- $C^{\text{4d}}_{\lambda,A=\pm M}$ $\quad (a = m(+i\pi))$:

$$\boxed{} \Longrightarrow -1; \qquad \boxed{(2)^t} \Longrightarrow 1; \qquad \boxed{(2)} \Longrightarrow -2;$$
$$\boxed{(3)^t} \Longrightarrow -1; \qquad \boxed{(2,1)} \Longrightarrow 2;$$

Where we introduce some notation for 4d Young diagrams. As in (7.56), the (x_1, x_2) direction is associated with the equivariant parameter $\hbar$, while the (x_3, x_4) direction is associated with the equivariant parameter m. We represent an (x_1, x_2) Young sub-diagram by drawing it explicitly, and we assign an (x_3, x_4) Young sub-diagram to each box of the (x_1, x_2) Young sub-diagram by monotonically decreasing positive integers $\mu = (\mu_1, \mu_2, \ldots)$. Note that we write the transposition of a 2d Young diagram μ by μ^t. For example, all the contents $x = (x_1, x_2, x_3, x_4) \in \lambda$ of the following presentation of a 4d Young diagram λ are given by

$$\begin{aligned}
\boxed{(2)}\boxed{} &\Longrightarrow \{(1,1,1,1),(1,2,1,1),(1,1,1,2)\} \\
\boxed{(2)^t}\boxed{} &\Longrightarrow \{(1,1,1,1),(1,2,1,1),(1,1,2,1)\} \\
\boxed{(2,1)} &\Longrightarrow \{(1,1,1,1),(1,1,1,2),(1,1,2,1)\}
\end{aligned} \tag{7.60}$$

And we only list the multiplicity coefficients up to Young diagrams with 3 boxes. The contributions from 4 boxes are not clear yet.

Since the adjoint representation $\mathbf{6}$ of $\mathfrak{so}(4)$ is isomorphic to the direct sum $(\mathbf{3},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3})$ of the adjoint representations of $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$, we have the following identity at the unrefined level

$$Z^{\text{adj}}_{SO(4)}(A_1, A_2, M) = Z^{\text{adj}}_{SU(2)}(A = (A_1A_2)^{\frac{1}{2}}, M) Z^{\text{adj}}_{SU(2)}(A = (A_1/A_2)^{\frac{1}{2}}, M) Z_{\text{extra}} \tag{7.61}$$

where

$$Z_{\text{extra}} = \text{PE}\left[\frac{\mathfrak{q}(2 + 3M + 2M^2)(q - M)(1 - Mq)}{(1 - \mathfrak{q})M(1 + M)^2(1 - q)^2}\right] \tag{7.62}$$

Also, the adjoint representation of $\mathfrak{so}(6)$ is isomorphic to that of $\mathfrak{su}(4)$, which leads to

$$Z^{\text{adj}}_{SO(6)}(A_1, A_2, A_3, M) = Z^{\text{adj}}_{SU(4)}(A_1, A_2, A_3, M) Z'_{\text{extra}} \tag{7.63}$$

where

$$Z'_{\text{extra}} = \text{PE}\left[\frac{\mathfrak{q}(1 + M + M^2)(q - M)(1 - Mq)}{(1 - \mathfrak{q})M(1 + M)^2(1 - q)^2}\right] . \tag{7.64}$$

We check these identities up to 5-instanton.

7.5 $Sp(N)$ with antisymmetric hypermultiplet

If we replace the $O4^-$-plane with an $O8^-$-plane in the brane configuration described in the previous section, the gauge group becomes $Sp(N)$, and the fluctuation of a $D4$-brane along the transverse directions to its worldvolume on the $O8^-$-plane describes the scalar field in the antisymmetric hypermultiplet. Hence, the D4-$O8^-$ system in Type IIA theory gives rise to the $Sp(N)$ gauge theory with the antisymmetric hypermultiplet.

The dual gauge group $O(k)$ has two components, which results in two contributions $Z^{\text{anti}}_{Sp(N),k,\pm}$ to the partition function. Then, the θ-angle can be distinguished through the following equations:

$$Z^{\text{anti}}_{Sp(N),k,\theta=0} = \frac{Z^{\text{anti}}_{Sp(N),k,+} + Z^{\text{anti}}_{Sp(N),k,-}}{2}$$
$$Z^{\text{anti}}_{Sp(N),k,\theta=\pi} = (-1)^k \frac{Z^{\text{anti}}_{Sp(N),k,+} - Z^{\text{anti}}_{Sp(N),k,-}}{2} \tag{7.65}$$

To get the integrands of This theory, let us recall the supersymmetric quantum mechanics of the k instanton moduli space of pure $Sp(N)$ gauge theory[9]. This system is described by an O(k) gauge theory with a (rank-two) symmetric hypermultiplet and $2N$ fundamental half-hypermultiplets, which have $Sp(N)$ flavor symmetry.

The introduction of a 5d antisymmetric hypermultiplet modifies the field content of the instanton quantum mechanics. Specifically, it introduces an additional symmetric hypermultiplet and $2N$ fundamental half-Fermi multiplets. (This is illustrated in Figure 7-7.) The partition function receives two contributions due to the O(k) gauge group.

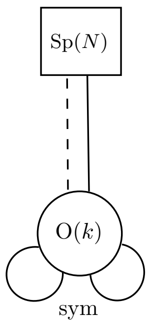


Figure 7-7 The solid line represents a hypermultiplet while the dashed line represents a Fermi multiplet.

These contributions can be calculated by performing the JK residue integrals of the following integrands:

$$\begin{aligned} z^{+}_{k=2\ell+\chi} &= \left(\frac{1}{\mathrm{sh}\,(\epsilon_{1,2}) \prod_{s=1}^{N} \mathrm{sh}\,(\pm a_s + \epsilon_+)} \cdot \prod_{I=1}^{\ell} \frac{\mathrm{sh}\,(\pm\phi_I)\,\mathrm{sh}\,(\pm\phi_I + 2\epsilon_+)}{\mathrm{sh}\,(\pm\phi_I + \epsilon_{1,2})} \right)^{\chi} \\ &\cdot \prod_{I=1}^{\ell} \frac{\mathrm{sh}\,(2\epsilon_+)}{\mathrm{sh}\,(\epsilon_{1,2})\,\mathrm{sh}(\pm 2\phi_I + \epsilon_{1,2}) \prod_{s=1}^{N} \mathrm{sh}\,(\pm\phi_I \pm a_s + \epsilon_+)} \prod_{I<J}^{\ell} \frac{\mathrm{sh}\,(\pm\phi_I \pm \phi_J)\,\mathrm{sh}\,(\pm\phi_I \pm \phi_J + 2\epsilon_+)}{\mathrm{sh}\,(\pm\phi_I \pm \phi_J + \epsilon_{1,2})} \\ &\cdot \left(\frac{\prod_{s=1}^{N} \mathrm{sh}\,(m \pm a_s)}{\mathrm{sh}\,(\pm m - \epsilon_+)} \prod_{I=1}^{\ell} \frac{\mathrm{sh}\,(\pm\phi_I \pm m - \epsilon_-)}{\mathrm{sh}\,(\pm\phi_I \pm m - \epsilon_+)} \right)^{\chi} \prod_{I=1}^{\ell} \frac{\mathrm{sh}\,(\pm m - \epsilon_-) \prod_{s=1}^{N} \mathrm{sh}\,(\pm\phi_I \pm a_s + m)}{\mathrm{sh}\,(\pm m - \epsilon_+)\,\mathrm{sh}\,(\pm 2\phi_I \pm m - \epsilon_+)} \\ &\cdot \prod_{I<J}^{\ell} \frac{\mathrm{sh}\,(\pm\phi_I \pm \phi_J \pm m - \epsilon_-)}{\mathrm{sh}\,(\pm\phi_I \pm \phi_J \pm m - \epsilon_+)} \end{aligned} \tag{7.66}$$

$$
\begin{aligned}
&z^-_{k=2\ell+1}\\
=&\frac{1}{\mathrm{sh}\,(\epsilon_{1,2})\prod_{s=1}^N \mathrm{ch}\,(\pm a_s+\epsilon_+)}\cdot\prod_{I=1}^{\ell}\frac{\mathrm{ch}\,(\pm\phi_I)\,\mathrm{ch}\,(\pm\phi_I+2\epsilon_+)}{\mathrm{ch}\,(\pm\phi_I+\epsilon_{1,2})}\\
&\cdot\prod_{I=1}^{\ell}\frac{\mathrm{sh}\,(2\epsilon_+)}{\mathrm{sh}\,(\epsilon_{1,2})\,\mathrm{sh}\,(\pm2\phi_I+\epsilon_{1,2})\prod_{s=1}^N \mathrm{sh}\,(\pm\phi_I\pm a_s+\epsilon_+)}\prod_{I<J}^{\ell}\frac{\mathrm{sh}\,(\pm\phi_I\pm\phi_J)\,\mathrm{sh}\,(\pm\phi_I\pm\phi_J+2\epsilon_+)}{\mathrm{sh}\,(\pm\phi_I\pm\phi_J+\epsilon_{1,2})}\\
&\cdot\frac{\prod_{s=1}^N \mathrm{ch}\,(m\pm a_s)}{\mathrm{sh}\,(\pm m-\epsilon_+)}\cdot\prod_{I=1}^{\ell}\frac{\mathrm{ch}\,(\pm\phi_I\pm m-\epsilon_-)}{\mathrm{ch}\,(\pm\phi_I\pm m-\epsilon_+)}\frac{\mathrm{sh}\,(\pm m-\epsilon_-)\prod_{s=1}^N \mathrm{sh}\,(\pm\phi_I\pm a_s+m)}{\mathrm{sh}\,(\pm m-\epsilon_+)\,\mathrm{sh}\,(\pm2\phi_I\pm m-\epsilon_+)}\\
&\cdot\prod_{I<J}^{\ell}\frac{\mathrm{sh}\,(\pm\phi_I\pm\phi_J\pm m-\epsilon_-)}{\mathrm{sh}\,(\pm\phi_I\pm\phi_J\pm m-\epsilon_+)}
\end{aligned}
\tag{7.67}
$$

$$
\begin{aligned}
&z^-_{k=2\ell}\\
=&\frac{\mathrm{ch}\,(\pm m-\epsilon_-)\prod_{s=1}^N \mathrm{sh}\,(2m\pm2a_s)}{\mathrm{sh}\,(\pm m-\epsilon_+)\ \mathrm{sh}\,(\pm2m-2\epsilon_+)}\\
&\cdot\frac{\mathrm{ch}\,(2\epsilon_+)}{\mathrm{sh}\,(\epsilon_{1,2})\ \mathrm{sh}\,(2\epsilon_{1,2})\prod_{s=1}^N \mathrm{sh}\,(\pm2a_s+2\epsilon_+)}\cdot\prod_{I=1}^{\ell-1}\frac{\mathrm{sh}\,(\pm2\phi_I)\,\mathrm{sh}\,(\pm2\phi_I+4\epsilon_+)}{\mathrm{sh}\,(\pm2\phi_I+2\epsilon_{1,2})}\\
&\cdot\prod_{I=1}^{\ell-1}\frac{\mathrm{sh}\,(2\epsilon_+)}{\mathrm{sh}\,(\epsilon_{1,2})\,\mathrm{sh}\,(\pm2\phi_I+\epsilon_{1,2})\prod_{s=1}^N \mathrm{sh}\,(\pm\phi_I\pm a_s+\epsilon_+)}\prod_{I<J}^{\ell-1}\frac{\mathrm{sh}\,(\pm\phi_I\pm\phi_J)\,\mathrm{sh}\,(\pm\phi_I\pm\phi_J+2\epsilon_+)}{\mathrm{sh}\,(\pm\phi_I\pm\phi_J+\epsilon_{1,2})}\\
&\cdot\prod_{I=1}^{\ell-1}\frac{\mathrm{sh}\,(\pm2\phi_I\pm2m-2\epsilon_-)}{\mathrm{sh}\,(\pm2\phi_I\pm2m-2\epsilon_+)}\frac{\mathrm{sh}\,(\pm m-\epsilon_-)\prod_{s=1}^N \mathrm{sh}\,(\pm\phi_I\pm a_s+m)}{\mathrm{sh}\,(\pm m-\epsilon_+)\,\mathrm{sh}\,(\pm2\phi_I\pm m-\epsilon_+)}\cdot\prod_{I<J}^{\ell-1}\frac{\mathrm{sh}\,(\pm\phi_I\pm\phi_J\pm m-\epsilon_-)}{\mathrm{sh}\,(\pm\phi_I\pm\phi_J\pm m-\epsilon_+)}
\end{aligned}
\tag{7.68}
$$

In the unrefined limit, the JK poles are classified using 4d Young diagrams, and a pole location at a content $x\in\lambda^{(s)}$ is as in 7.56. Additionally, all the residues can be expressed in a single universal formula, as shown in 7.58. This is similar to the relationship between the instanton partition functions of pure Yang-Mills theory for $SO(n)$ and $Sp(N)$ gauge groups[8]. This relationship still holds when including an antisymmetric hypermultiplet. However, there are more effective Coulomb branch parameters than 7.59, depending on the instanton number and plus/minus sector:

- (even,+) sector

$$
Z^{\mathrm{anti}}_{Sp(N),k=2\ell,+}=\widetilde{Z}_\ell(\vec{A};q)
\tag{7.69}
$$

where there are 8 additional effective Coulomb branch parameters ($j=1,\dots,8$)

$$
a_{N+j}=\frac{\hbar}{2}(+\pi i)\,,\ \frac{m}{2}(+\pi i)\,,\ \frac{\hbar-m}{2}(+\pi i)\,,\ 0(+\pi i)\,.
$$

- (odd,+) sector

$$
Z^{\mathrm{anti}}_{Sp(N),k=2\ell+1,+}=\frac{\prod_{s=1}^{\|\vec{\lambda}\|}\prod_{x\in\lambda^{(s)}}\mathrm{sh}(\phi_s(x)\pm m\pm\hbar)}{2\,\mathrm{sh}^2(\hbar)\,\mathrm{sh}^2(m)}\prod_{t=1}^{N}\frac{\mathrm{sh}(a_t\pm m)}{\mathrm{sh}^2(a_t)}\widetilde{Z}_\ell(\vec{A};q)
\tag{7.70}
$$

where there are 9 additional effective Coulomb branch parameters ($j=1,\dots,9$)

$$
a_{N+j}=\frac{\hbar}{2}(+\pi i)\,,\ \frac{m}{2}(+\pi i)\,,\ \frac{\hbar-m}{2}(+\pi i)\,,\ \pi i\,,\ \hbar\,,\ m\,.
$$

- (even,−) sector

$$Z^{\text{anti}}_{Sp(N),k=2\ell,+} = \frac{\text{ch}(m\pm\hbar)}{\text{sh}^2(\hbar)\,\text{sh}^2(2\hbar)\,\text{sh}^2(m)\,\text{sh}^2(2m)} \times \prod_{t=1}^{N}\frac{\text{sh}(a_t\pm m)\,\text{ch}(a_t\pm m)}{\text{sh}^2(2a_t)}\widetilde{Z}_{\ell-1}(\vec{A};q) \tag{7.71}$$

where there are 10 additional effective Coulomb branch parameters ($j=1,\ldots,10$)

$$a_{N+j}=\frac{\hbar}{2}(+\pi i)\;,\;\frac{m}{2}(+\pi i)\;,\;\frac{\hbar-m}{2}(+\pi i)\;,\;\hbar(+\pi i)\;,\;m(+\pi i)\;.$$

- (odd,−) sector

$$Z^{\text{anti}}_{Sp(N),k=2\ell+1,+} = \frac{1}{2\,\text{sh}^2(\hbar)\,\text{sh}^2(m)}\prod_{t=1}^{N}\frac{\text{ch}(a_t\pm m)}{\text{ch}^2(a_t)} \times \prod_{s=1}^{\|\vec{\lambda}\|}\prod_{x\in\lambda^{(s)}}\text{ch}(\phi_s(x)\pm m\pm\hbar)\widetilde{Z}_{\ell}(\vec{A};q) \tag{7.72}$$

where there are 9 additional effective Coulomb branch parameters ($j=1,\ldots,9$)

$$a_{N+j}=\frac{\hbar}{2}(+\pi i)\;,\;\frac{m}{2}(+\pi i)\;,\;\frac{\hbar-m}{2}(+\pi i)\;,\;0\;,\;\hbar+\pi i\;,\;m+\pi i\;.$$

The multiplicity coefficients are by far the same as $SO(n)$ $\mathcal{N}=1^*$ gauge group. So one can use the coefficients that are listed in section 7.4. Even though we do not know the general form of the coefficients.

We can verify the isomorphisms of representations by comparing the instanton partition functions of different theories. For example, the antisymmetric representation of $Sp(1)$ is trivial, so it decouples from the gauge theory and does not affect the instanton dynamics. Thus, the instanton partition function for the $Sp(1)$ theory with an antisymmetric hypermultiplet is equal to the pure $Sp(1)$ instanton partition function:

$$\begin{aligned} Z^{\text{anti}}_{Sp(1),\theta=0} &= Z^{\text{pure}}_{Sp(1),\theta=0}Z_{\text{D0}}\;, \\ Z^{\text{anti}}_{Sp(1),\theta=\pi} &= Z^{\text{pure}}_{Sp(1),\theta=\pi} \end{aligned} \tag{7.73}$$

up to the 8 translational zero modes of a single D0-brane:

$$Z_{\text{D0}} = \text{PE}\left[\frac{\mathfrak{q}}{\text{sh}(\epsilon_{1,2})\,\text{sh}(m\pm\epsilon_+)}\right] \tag{7.74}$$

Once we take T-duality from the brane system in Type IIA theory, we can construct 5d $Sp(N)$ theories with an antisymmetric hypermultiplet by fivebrane web with two O7$^-$-planes in Type IIB theory[35,43–44]. The brane configuration can be further deformed by splitting an O7$^-$-plane into two (p,q)-sevenbranes and performing Hanany-Witten transitions[45]. In the massless limit of the antisymmetric matter ($m=\epsilon_+$ at the refined level), the resulting fivebrane web takes the form of a product of pure $Sp(1)$ theories[46], as shown in Figure 7-8. This brane moves and the Higgsing procedure can also be verified at the level of instanton partition functions as:

$$\begin{aligned} Z^{\text{anti}}_{Sp(2),\theta=0}(m=\epsilon_+) &= Z^{\text{pure}}_{Sp(1),\theta=0}(A_1)Z^{\text{pure}}_{Sp(1),\theta=0}(A_2)Z_{\text{D0}} \\ Z^{\text{anti}}_{Sp(2),\theta=\pi}(m=\epsilon_+) &= Z^{\text{pure}}_{Sp(1),\theta=\pi}(A_1)Z^{\text{pure}}_{Sp(1),\theta=\pi}(A_2)\;. \end{aligned} \tag{7.75}$$

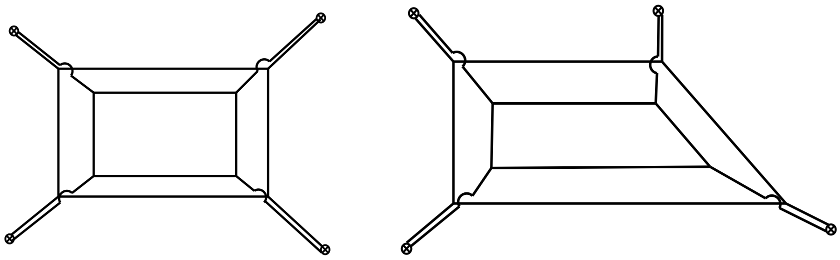


Figure 7-8 Fivebrane web configuration of $Sp(2)$ gauge theory with massless antisymmetric hypermultiplet in Type IIB theory where the left-panel represents $\theta = 0$ while the right-panel illustrates $\theta = \pi$.

Chapter 8
Conclusion

So far we have presented the unrefined partition functions of various $5d$ $\mathcal{N} = 1$ gauge theories after introducing the mathematical basics of them. And we checked multiple dualities between those theories.

In order to move forward with the research presented in this paper, we need to fully understand the meaning of the fact that poles can be classified as Young diagrams and determine the multiplicity constants in section 7.4.

Furthermore, while 7.58 currently yields non-trivial rational functions through the cancellation of sh(0) in the numerator with poles in the denominator, it would be valuable to find a closed-form expression similar to the Nekrasov factor 7.12 for 4d Young diagrams. This could provide deeper insights and simplify calculations furthermore.

Bibliography

[1] SEIBERG N, WITTEN E. Monopoles, duality and chiral symmetry breaking in n = 2 supersymmetric QCD[J/OL]. Nuclear Physics B, 1994, 431(3): 484-550. https://doi.org/10.1016%2F0550-3213%2894%2990214-3. DOI: 10.1016/0550-3213(94)90214-3.

[2] SEIBERG N, WITTEN E. Electric-magnetic duality, monopole condensation, and confinement in n=2 supersymmetric yang-mills theory[J/OL]. Nuclear Physics B, 1994, 426(1): 19-52. https://www.sciencedirect.com/science/article/pii/0550321394901244. DOI: https://doi.org/10.1016/0550-3213(94)90124-4.

[3] SEIBERG N. Supersymmetry and Nonperturbative beta Functions[J/OL]. Phys. Lett. B, 1988, 206: 75-80. DOI: 10.1016/0370-2693(88)91265-8.

[4] NEKRASOV N A. Seiberg-witten prepotential from instanton counting[M/OL]. arXiv, 2002. https://arxiv.org/abs/hep-th/0206161. DOI: 10.48550/ARXIV.HEP-TH/0206161.

[5] AGANAGIC M, KLEMM A, MARI�O M, et al. The topological vertex[J/OL]. Communications in Mathematical Physics, 2004, 254(2): 425-478. https://doi.org/10.1007%2Fs00220-004-1162-z. DOI: 10.1007/s00220-004-1162-z.

[6] AWATA H, KANNO H. Instanton counting, macdonald function and the moduli space of<i>d</i>-branes[J/OL]. Journal of High Energy Physics, 2005, 2005 (05): 039-039. https://doi.org/10.1088%2F1126-6708%2F2005%2F05%2F039. DOI: 10.1088/1126-6708/2005/05/039.

[7] ALDAY L F, GAIOTTO D, TACHIKAWA Y. Liouville correlation functions from four-dimensional gauge theories[J/OL]. Letters in Mathematical Physics, 2010, 91(2): 167-197. https://doi.org/10.1007%2Fs11005-010-0369-5. DOI: 10.1007/s11005-010-0369-5.

[8] NAWATA S, ZHU R D. Instanton counting and O-vertex[J/OL]. JHEP, 2021, 09: 190. DOI: 10.1007/JHEP09(2021)190.

[9] NEKRASOV N, SHADCHIN S. ABCD of instantons[J/OL]. Commun. Math. Phys., 2004, 252: 359-391. DOI: 10.1007/s00220-004-1189-1.

[10] BENVENUTI S, HANANY A, MEKAREEYA N. The hilbert series of the one instanton moduli space[J/OL]. Journal of High Energy Physics, 2010, 2010(6). https://doi.org/10.1007%2Fjhep06%282010%29100. DOI: 10.1007/jhep06(2010)100.

[11] CHEN Y, JIANG J, NAWATA S, et al. Instantons on young diagrams with matters [M/OL]. arXiv, 2023. https://arxiv.org/abs/2301.02342. DOI: 10.48550/ARXIV.2301.02342.

[12] SHADCHIN S. On certain aspects of string theory/gauge theory correspondence [M/OL]. arXiv, 2005. https://arxiv.org/abs/hep-th/0502180. DOI: 10.48550/ARXIV.HEP-TH/0502180.

[13] GRIMM R, SOHNIUS M, WESS J. Extended Supersymmetry and Gauge Theories[J/OL]. Nucl. Phys. B, 1978, 133: 275-284. DOI: 10.1016/0550-3213(78)90303-6.

[14] WITTEN E. Topological Quantum Field Theory[J/OL]. Commun. Math. Phys., 1988, 117: 353. DOI: 10.1007/BF01223371.

[15] ATIYAH M, HITCHIN N, DRINFELD V, et al. Construction of instantons[J/OL]. Physics Letters A, 1978, 65(3): 185-187. https://www.sciencedirect.com/science/article/pii/037596017890141X. DOI: https://doi.org/10.1016/0375-9601(78)90141-X.

[16] OSBORN H. Semiclassical Functional Integrals for Selfdual Gauge Fields[J/OL]. Annals Phys., 1981, 135: 373. DOI: 10.1016/0003-4916(81)90159-7.

[17] OSBORN H. Solutions of the dirac equation for general instanton solutions[J/OL]. Nuclear Physics B, 1978, 140(1): 45-53. https://www.sciencedirect.com/science/article/pii/0550321378903127. DOI: https://doi.org/10.1016/0550-3213(78)90312-7.

[18] CORDES S, MOORE G, RAMGOOLAM S. Lectures on 2d yang-mills theory, equivariant cohomology and topological field theories[J/OL]. Nuclear Physics B - Proceedings Supplements, 1995, 41(1-3): 184-244. https://doi.org/10.1016%2F0920-5632%2895%2900434-b. DOI: 10.1016/0920-5632(95)00434-b.

[19] LOSEV A S, MARSHAKOV A, NEKRASOV N A. Small instantons, little strings and free fermions[M/OL]. arXiv, 2003. https://arxiv.org/abs/hep-th/0302191. DOI: 10.48550/ARXIV.HEP-TH/0302191.

[20] MOORE G, NEKRASOV N, SHATASHVILI S. D -particle bound states and generalized instantons[J/OL]. Communications in Mathematical Physics, 2000, 209(1): 77-95. https://doi.org/10.1007%2Fs002200050016. DOI: 10.1007/s002200050016.

[21] MOORE G, NEKRASOV N, SHATASHVILI S. Integrating over higgs branches [J/OL]. Communications in Mathematical Physics, 2000, 209(1): 97-121. https://doi.org/10.1007%2Fpl00005525. DOI: 10.1007/pl00005525.

[22] JEFFREY L C, KIRWAN F C. Localization for nonabelian group actions[J/OL]. Topology, 1995, 34(2): 291-327. https://www.sciencedirect.com/science/article/pii/004093839400028J. DOI: https://doi.org/10.1016/0040-9383(94)00028-J.

[23] BENINI F, EAGER R, HORI K, et al. Elliptic genera of 2d $${\mathcal{n}}$$ n = 2 gauge theories[J/OL]. Communications in Mathematical Physics, 2014, 333 (3): 1241-1286. https://doi.org/10.1007%2Fs00220-014-2210-y. DOI: 10.1007/s00220-014-2210-y.

[24] HORI K, KIM H, YI P. Witten index and wall crossing[J/OL]. Journal of High Energy Physics, 2015, 2015(1). https://doi.org/10.1007%2Fjhep01%282015%29124. DOI: 10.1007/jhep01(2015)124.

[25] HWANG C, KIM J, KIM S, et al. General instanton counting and 5d scft[A]. 2015. arXiv: 1406.6793.

[26] SZENES A, VERGNE M. Toric reduction and a conjecture of batyrev and materov [J/OL]. Inventiones mathematicae, 2004, 158(3): 453-495. https://doi.org/10.1007%2Fs00222-004-0375-2. DOI: 10.1007/s00222-004-0375-2.

[27] BRION M, VERGNE M. Arrangements of hyperplanes i: Rational functions and jeffrey-kirwan residue[A]. 1999. arXiv: math/9903178.

[28] NAKAMURA S. On the jeffrey–kirwan residue of<i>BCD</i>-instantons [J/OL]. Progress of Theoretical and Experimental Physics, 2015, 2015(7): 073B02. https://doi.org/10.1093%2Fptep%2Fptv085. DOI: 10.1093/ptep/ptv085.

[29] NEKRASOV N, OKOUNKOV A. Seiberg-witten theory and random partitions [A]. 2003. arXiv: hep-th/0306238.

[30] AHARONY O, HANANY A, KOL B. Webs of (p,q) five-branes, five-dimensional field theories and grid diagrams[J/OL]. JHEP, 1998, 01: 002. DOI: 10.1088/1126-6708/1998/01/002.

[31] GIVEON A, KUTASOV D. Brane dynamics and gauge theory[J/OL]. Rev. Mod. Phys., 1999, 71: 983-1084. https://link.aps.org/doi/10.1103/RevModPhys.71.983.

[32] POLCHINSKI J, CHAUDHURI S, JOHNSON C V. Notes on D-branes[A]. 1996. arXiv: hep-th/9602052.

[33] ZAFRIR G. Brane webs and $O5$-planes[J/OL]. JHEP, 2016, 03: 109. DOI: 10.1007/JHEP03(2016)109.

[34] HAYASHI H, KIM S S, LEE K, et al. 5-brane webs for 5d $\mathcal{N} = 1$ G_2 gauge theories[J/OL]. JHEP, 2018, 03: 125. DOI: 10.1007/JHEP03(2018)125.

[35] BERGMAN O, ZAFRIR G. 5d fixed points from brane webs and o7-planes[J/OL]. Journal of High Energy Physics, 2015, 2015(12): 1-30. DOI: 10.1007/JHEP12(2015)163.

[36] TERASHIMA S. Supersymmetric gauge theories on<mml:math xmlns:mml=”http://www.w3.org/1998/Math/MathML” display=”inline”><mml:msup><mml:mi>s</mml:mi><mml:mn>4</mml:mn></mml:msup><mm [J/OL]. Physical Review D, 2014, 89(12). https://doi.org/10.1103%2Fphysrevd.89.125001. DOI: 10.1103/physrevd.89.125001.

[37] TACHIKAWA Y. Seiberg-Witten Theory and Instanton Counting[D]. 2003.

[38] NEKRASOV N. Five-dimensional gauge theories and relativistic integrable systems[J/OL]. Nuclear Physics B, 1998, 531(1-3): 323-344. https://doi.org/10.1016%2Fs0550-3213%2898%2900436-2. DOI: 10.1016/s0550-3213(98)00436-2.

[39] ATIYAH M F, SEGAL G B. The index of elliptic operators: Ii[J/OL]. Annals of Mathematics, 1968, 87(3): 531-545[2023-03-19]. http://www.jstor.org/stable/1970716.

[40] KIM H C, KIM J, KIM S, et al. 6d strings and exceptional instantons[J/OL]. Physical Review D, 2021, 103(2). https://doi.org/10.1103%2Fphysrevd.103.025012. DOI: 10.1103/physrevd.103.025012.

[41] ZAFRIR G. Brane webs and o5-planes[J/OL]. Journal of High Energy Physics, 2016, 2016(3). https://doi.org/10.1007%2Fjhep03%282016%29109. DOI: 10.1007/jhep03(2016)109.

[42] KIM J, KIM S S, LEE K H, et al. Instantons from blow-up[J/OL]. Journal of High Energy Physics, 2019, 2019(11). https://doi.org/10.1007%2Fjhep11%282019%29092. DOI: 10.1007/jhep11(2019)092.

[43] HAYASHI H, KIM S S, LEE K, et al. Dualities and 5-brane webs for 5d rank 2 SCFTs[J/OL]. Journal of High Energy Physics, 2018, 2018(12). https://doi.org/10.1007%2Fjhep12%282018%29016. DOI: 10.1007/jhep12(2018)016.

[44] HAYASHI H, KIM S S, LEE K, et al. Rank-3 antisymmetric matter on 5-brane webs[J/OL]. Journal of High Energy Physics, 2019, 2019(5). https://doi.org/10.1007%2Fjhep05%282019%29133. DOI: 10.1007/jhep05(2019)133.

[45] HANANY A, WITTEN E. Type iib superstrings, bps monopoles, and three-dimensional gauge dynamics[J/OL]. Nuclear Physics B, 1997, 492(1): 152-190. https://www.sciencedirect.com/science/article/pii/S0550321397800302. DOI: https://doi.org/10.1016/S0550-3213(97)80030-2.

[46] CHENG S, KIM S S. Refined topological vertex for a 5d $sp(n)$ gauge theories with antisymmetric matter[J/OL]. Phys. Rev. D, 2021, 104: 086004. https://link.aps.org/doi/10.1103/PhysRevD.104.086004.

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