Homework 8: Due at class on April 28

1. Let us identify $S^2 = \mathbb{C} \cup \{\infty\}$. Then, a holomorphic map $g : \mathbb{C} \to \mathbb{C}; z \mapsto z^n$ can be extended to $g : S^2 \to S^2$. Find the mapping degree $\deg g$ of g.

2. **Fundamental theorem of algebra**
We define $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = z^n + a_1 z^{n-1} + \cdots + a_n$ for $n \geq 1$. In addition, by writing $z = x + iy$, we define one-form

$$\omega = \mathrm{Im}\frac{dz}{z} = \frac{-ydx}{x^2+y^2} + \frac{xdy}{x^2+y^2} \ .$$

Then, show that

$$\frac{1}{2\pi}\int_{C_R} f^*\omega = n \ ,$$

where C_R is the circle with sufficiently large radius R. (Hint: construct homotopy between f and g above.) If there were no zero points $f(z) = 0$ inside C_R, show that

$$\frac{1}{2\pi}\int_{C_R} f^*\omega = 0$$

by using the Stokes theorem.

3. Show that Euler characteristics of a compact Lie group is zero.

4. Find real dimensions of the following Lie groups

- Complex general linear group: $\mathrm{GL}(n,\mathbb{C}) = \{A \in \mathrm{M}_n(\mathbb{C}) |\det A \neq 0\}$
- Complex special linear group: $\mathrm{SL}(n,\mathbb{C}) = \{A \in \mathrm{GL}(n,\mathbb{C}) |\det A = 1\}$
- Unitary group $\mathrm{U}(n) = \{A \in \mathrm{GL}(n,\mathbb{C}) | AA^\dagger = I\}$
- Special unitary group $\mathrm{SU}(n) = \{A \in \mathrm{U}(n) |\det A = 1\}$
- Real general linear group: $\mathrm{GL}(n,\mathbb{R}) = \{A \in \mathrm{M}_n(\mathbb{R}) |\det A \neq 0\}$
- Real special linear group: $\mathrm{SL}(n,\mathbb{R}) = \{A \in \mathrm{GL}(n,\mathbb{R}) |\det A = 1\}$
- Orthogonal group $\mathrm{O}(n) = \{A \in \mathrm{GL}(n,\mathbb{R}) | AA^T = I\}$
- Special orthogonal group $\mathrm{SO}(n) = \{A \in \mathrm{O}(n) |\det A = 1\}$

5. Write down the definitions of the following Lie algebras: $\mathfrak{gl}(n,\mathbb{C})$, $\mathfrak{sl}(n,\mathbb{C})$, $\mathfrak{su}(n)$ and $\mathfrak{so}(n)$.

6. Show that the group of Lorentz transformations can be expressed $\mathrm{SL}(2,\mathbb{C})/\pm\mathrm{Id}$. Hint: If we define

$$A := \begin{pmatrix} t+x & z+yi \\ z-yi & t-x \end{pmatrix}$$

where $(t, x, y, z) \in \mathbb{R}^4$, then we have $t^2 - x^2 - y^2 - z^2 = \det A$.

7. Show that $\mathrm{SO}(4) \cong \{\mathrm{SU}(2)\times\mathrm{SU}(2)\}/\{\pm\mathrm{Id}\}$, where $\mathrm{Id} \hookrightarrow \mathrm{SU}(2)\times\mathrm{SU}(2)$ is the diagonal embedding. The hint is given as follows.

Let $\mathbb{H}$ be the quaternion in which an element $x \in \mathbb{H}$ can be expressed as

$$x = x_1 + x_2 i + x_3 j + x_4 k$$

where $x_a \in \mathbb{R}$ $(a = 1, \cdots, 4)$ and

$$i^2 = j^2 = k^2 = -1 \ , \quad ij = -ji = k \ , \quad jk = -kj = i \ , \quad ki = -ik = j \ .$$

We define the imaginary part of x as

$$\mathrm{Im}\ x = x_2 i + x_3 j + x_4 k$$

so that the conjugate $\overline{x}$ is written as

$$\overline{x} = x_1 - x_2 i - x_3 j - x_4 k$$

Therefore, the multiplication becomes

$$\overline{xy} = \overline{y} \cdot \overline{x}$$

The norm of x is

$$|x|^2 = x\overline{x} = \overline{x}x = x_1^2 + x_2^2 + x_3^2 + x_4^2$$

From this viepoint, $\mathrm{SU}(2)$ can be considered as a group of unit quaternions $\mathrm{SU}(2) = \{x \in \mathbb{H} |\ |x| = 1\}$. Then $\mathrm{SU}(2) \times \mathrm{SU}(2)$ acts on $\mathbb{H}$ by rotations in the following way:

$$x \mapsto q_1 x q_2^{-1}$$

is a rotation of $\mathbb{R}^4 = \mathbb{H}$ for q_1, $q_2 \in \mathrm{SU}(2)$. Then $(-q_1, -q_2)$ represents the same rotation as (q_1, q_2). Show that these represent all the rotations of $\mathbb{R}^4 = \mathbb{H}$ so that it is isomorphic to $\mathrm{SO}(4)$.