

Homework 2: Due at class on March 15

1. For elements $x = (x^0, \dots, x^n)$ and $y = (y^0, \dots, y^n)$ of $\mathbb{R}^{n+1} \setminus \{0\}$, we define an equivalence relation $x \sim y$ by

$$x = \alpha y \quad \alpha \in \mathbb{R}.$$

Let us define $\mathbb{R}P^n$ by $(\mathbb{R}^{n+1} \setminus \{0\}) / \sim$. Show that $\mathbb{R}P^n$ is a manifold and $\mathbb{R}P^1$ is diffeomorphic to S^1 . The space is called a real projective space.

2. For elements $x = (x^0, \dots, x^n)$ and $y = (y^0, \dots, y^n)$ of $\mathbb{C}^{n+1} \setminus \{0\}$, we define an equivalence relation $x \sim y$ by

$$x = \alpha y \quad \alpha \in \mathbb{C}.$$

Let us define $\mathbb{C}P^n$ by $(\mathbb{C}^{n+1} \setminus \{0\}) / \sim$. Show that $\mathbb{C}P^n$ is a manifold and $\mathbb{C}P^1$ is diffeomorphic to S^2 . The space is called a complex projective space.

3. Show that an open ball $B_n = \{x \in \mathbb{R}^n \mid |x| < 1\}$ is diffeomorphic to \mathbb{R}^n by constructing a smooth bijection map.

4. For a positive integer d , we define a map $f^{(d)} : \mathbb{C} \rightarrow \mathbb{C}; z \mapsto z^d$. Let $z = x + iy$ ($x, y \in \mathbb{R}$), and consider $f^{(d)}$ as a function of x and y . Compute the Jacobian matrix of $f^{(d)}$.

5. Let $M_n(\mathbb{R})$ and $M_n(\mathbb{C})$ denote the set of all $n \times n$ matrices over \mathbb{R} and \mathbb{C} , respectively. We define

$$SU(2) = \{A \in M_2(\mathbb{C}) \mid A^\dagger A = \text{Id}, \det A = 1\}$$

$$SO(3) = \{A \in M_3(\mathbb{R}) \mid A^T A = \text{Id}, \det A = 1\}.$$

5.1 Construct a double covering (2-to-1) map $SU(2) \rightarrow SO(3)$.

5.2 Show that $SU(2)$ is diffeomorphic to S^3 and $SO(3)$ is diffeomorphic to $\mathbb{R}P^3$.

6. Let e be the identity element of $SO(3)$. Show that the tangent space $T_e SO(3)$ at e is spanned by tangent vectors of curves in $SO(3)$

$$\exp(tJ_i) = 1 + tJ_i + \frac{1}{2}(tJ_i)^2 + \dots$$

at $t = 0$ where J_i ($i = x, y, z$) are defined by

$$J_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad J_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad J_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let us define the commutator by $[X, Y] = XY - YX$. Then, show that

$$[J_x, J_y] = J_z, \quad [J_y, J_z] = J_x, \quad [J_z, J_x] = J_y.$$

7. Show that the tangent space $T_e SU(2)$ is spanned by $i\sigma_x$, $i\sigma_y$ and $i\sigma_z$ (the Pauli matrices by i).