Homework 10: Due at class on May 14

1. Derive **real** dimensions of Symplectic group $\mathrm{Sp}(n)$

2. Show that Euler characteristics of a compact Lie group is zero.

3. Show that there are matrices $A, B \in \mathfrak{gl}(n, \mathbb{C})$ such that

$$e^A e^B \neq e^{A+B} .$$

Modify this equation in such a way that the equality holds for those matrices A, B.

4. We have seen that $\mathrm{PSL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})/\{\pm\mathrm{Id}\}$ acts on the upper half plane $(\mathbf{H}, \frac{dzd\bar{z}}{(\mathrm{Im}\, z)^2})$ as an isometry group:

$$\mathrm{PSL}(2,\mathbb{R}) \times \mathbf{H} \to \mathbf{H}; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z \mapsto \frac{az+b}{cz+d} .$$

Show that this action is transitive and find the stabilizer subgroup of the point $i \in \mathbf{H}$. Find the fundamental group of $\mathrm{PSL}(2,\mathbb{R})$.

5. Show that $\mathrm{SL}(2,\mathbb{R})$ is a deformation retract to $\mathrm{SO}(2)$ in the following way.

- Write an element $A \in \mathrm{SL}(2,\mathbb{R})$ as $A = (a_1, a_2)$, where the a_i are column vectors. The Gram-Schmidt process replaces A to

$$\mathrm{SO}(2) \ni U = \left(\frac{a_1}{|a_1|}, \frac{a_2 - \frac{\langle a_1, a_2 \rangle}{|a_1|^2} a_1}{\left| a_2 - \frac{\langle a_1, a_2 \rangle}{|a_1|^2} a_1 \right|} \right) ,$$

  where $\langle a_1, a_2 \rangle$ is the standard inner product of $\mathbb{R}^2$. Construct a deformation retract that connects A to U inside $\mathrm{SL}(2,\mathbb{R})$.

Do the same exercise for $\mathrm{SL}(2,\mathbb{C})$ and which Lie group does it have a deformation retract to? Derive the homology groups and the fundamental group of $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SL}(2,\mathbb{C})$.

6. Let us define

$$\sigma_\mu \equiv (\mathbf{1}, \vec{\sigma})$$

where σ_i are the Pauli matrices. Compute that $\det X$ where

$$X := x^\mu \sigma_\mu = \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix} .$$

Give the definition of the Lorentz group $\mathrm{SO}(1,3)$. $\mathrm{SO}(1,3)$ is indeed not connected. Let us denote a subgroup of Lorentz group by $\mathrm{SO}^+(1,3)$ that satisfies the following two properties:

$$\det \Lambda^\mu{}_\nu = 1 , \qquad \Lambda^t{}_t \geq 1 \qquad \Lambda^\mu{}_\nu \in \mathrm{SO}(1,3) .$$

Show that this subgroup $\mathrm{SO}^+(1,3)$ is isomorphic to $\mathrm{SL}(2,\mathbb{C})/\{\pm\mathrm{Id}\}$. Derive the fundamental group $\pi_1(\mathrm{SO}^+(1,3))$.