

Homework 6: Due at class on May 25

1 Higgs mechanism in electroweak theory

The Glashow-Weinberg-Salam theory is the part of the Standard Model (SM) of particle physics which describes the electroweak interactions by a non-Abelian gauge theory with the gauge group $SU(2)_L \times U(1)_Y$. In a one-family approximation, the SM has the following particle content:

	$E_L = \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L$	e_R	$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$	$T^a W_\mu^a$	B_μ
Hypercharge Y	$-\frac{1}{2}$	-1	$+\frac{1}{2}$	0	0
$SU(2)_L$ rep.	$\mathbf{2}$	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{3}$	$\mathbf{1}$
Lorentz rep.	$(\frac{1}{2}, 0)$	$(0, \frac{1}{2})$	$(0, 0)$	$(\frac{1}{2}, \frac{1}{2})$	$(\frac{1}{2}, \frac{1}{2})$

where E_L, e_R contain Dirac spinors and the superscripts in the Higgs doublet denote electromagnetic charges. The corresponding Lagrangian is given by

$$\mathcal{L} = \bar{E}_L (iD^\mu) E_L + \bar{e}_R (iD^\mu) e_R - \frac{1}{4} W_{\mu\nu}^a W^{\mu\nu a} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} + (D_\mu \phi)^\dagger (D^\mu \phi) - \mu^2 \phi^\dagger \phi - \lambda (\phi^\dagger \phi)^2 - (\bar{E}_L \phi e_R + \bar{e}_R \phi^\dagger E_L) \quad (1)$$

with

$$D_\mu = \partial_\mu - ig W_\mu^a T^a - ig' Y B_\mu \quad (2)$$

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu, \quad W_{\mu\nu}^a = \partial_\mu W_\nu^a - \partial_\nu W_\mu^a + g \epsilon^{abc} W_\mu^b W_\nu^c$$

1.1

Write down how the covariant derivative (2) acts on the left- and right-handed leptons doublets and on the Higgs-doublet.

1.2

Show that the Lagrangian (1) is Lorentz invariant and gauge invariant as well.

1.3

For the Higgs mechanism to work we need $\mu^2 < 0$. For which value of $|\phi|$ does the Higgs potential obtain a minimum? By an $SU(2)_L$ rotation we can choose the vacuum expectation value (VEV) of the Higgs field to be of the form $\langle \phi \rangle = \frac{1}{\sqrt{2}}(0, v)^T$. This leads to a redefinition of the excitation modes of the Higgs fields,

$$\phi(x) = \exp \left\{ \frac{i}{v} \xi^a(x) T^a \right\} \left(\begin{array}{c} 0 \\ \frac{1}{\sqrt{2}}(v + \eta(x)) \end{array} \right)$$

with $\xi^a(x)$ and $\eta(x)$ being real fields and T^a the generators of $SU(2)_L$. Now we apply an $SU(2)_L$ gauge transformation such that the angular excitations $\xi^a(x)$ vanish. This gauge transformation is called *unitary gauge*. Show that the Higgs potential in the unitary gauge is given by

$$V(\phi) = -\mu^2\eta^2(x) + \lambda v\eta^3(x) + \frac{\lambda}{4}\eta^4(x)$$

What is the mass of the η field? Compare the degrees of freedom (DOF) in the Higgs sector to the situation before symmetry breakdown.

1.4

Consider the kinetic energy terms of the Higgs field in (1). Show that

$$\begin{aligned} (D_\mu\phi)^\dagger(D^\mu\phi) &= \frac{1}{2}\partial_\mu\eta\partial^\mu\eta + \frac{1}{4}g^2(v+\eta)^2W_\mu^-W^{+\mu} \\ &\quad + \frac{1}{8}(v+\eta)^2(W_\mu^3 \quad B_\mu) \begin{pmatrix} g^2 & -g'g \\ -g'g & g'^2 \end{pmatrix} \begin{pmatrix} W^{3\mu} \\ B^\mu \end{pmatrix} \end{aligned} \quad (3)$$

with $W^{\pm\mu} := \frac{1}{\sqrt{2}}(W^{1\mu} \mp iW^{2\mu})$

1.5

The masses of the gauge bosons are given by the terms that are quadratic in the fields, e.g. $\frac{1}{4}g^2v^2W_\mu^-W^{+\mu} = m_W^2W_\mu^-W^{+\mu}$, where $m_W = \frac{1}{2}vg$. However, to see the masses of W_μ^3 and B_μ one has to diagonalize the matrix in (3):

$$\frac{1}{8}(W_\mu^3 \quad B_\mu)\mathcal{O}^T\mathcal{O} \begin{pmatrix} g^2 & -g'g \\ -g'g & g'^2 \end{pmatrix} \mathcal{O}^T\mathcal{O} \begin{pmatrix} W^{3\mu} \\ B^\mu \end{pmatrix} = \frac{1}{2}(Z_\mu \quad A_\mu) \begin{pmatrix} m_Z^2 & 0 \\ 0 & m_A^2 \end{pmatrix} \begin{pmatrix} Z^\mu \\ A^\mu \end{pmatrix}$$

Determine this orthogonal matrix \mathcal{O} by computing the corresponding eigenvalues and eigenvectors. What are the masses of the Z_μ and A_μ fields? Compare the DOF in the gauge sector to the situation before the symmetry breakdown. What can you say about the total amount of DOF?

1.6

As you know, an orthogonal 2×2 matrix can be written as

$$\mathcal{O} = \begin{pmatrix} \cos\theta_W & -\sin\theta_W \\ \sin\theta_W & \cos\theta_W \end{pmatrix}$$

Write $\cos\theta_W$ in terms of g' and g . Show for the ratio of the W - and Z -boson masses

$$\frac{m_W}{m_Z} = \cos\theta_W$$

The angle θ_W is sometimes called Weinberg angle or weak mixing angle.

1.7

Finally, consider the covariant derivative (2). Substitute the fields B_μ and W_μ^a by W_μ^\pm, Z_μ and A_μ and show

$$D_\mu = \partial_\mu - i \frac{g}{\sqrt{2}} \begin{pmatrix} 0 & W_\mu^+ \\ W_\mu^- & 0 \end{pmatrix} - i \frac{1}{\sqrt{g^2 + g'^2}} Z_\mu \left(g^2 T^3 - g'^2 Y \right) - ie Q A_\mu$$

where we have defined the electric charge $e = \frac{g'g}{\sqrt{g'^2 + g^2}}$ and $Q := T^3 + Y$

1.8

Consider the following terms of the Lagrangian:

$$\mathcal{L} \supset \bar{E}_L (iD) E_L + \bar{e}_R (iD) e_R$$

Find the interaction terms of the fermions with the gauge bosons. For the weak interaction, analyze its V-A structure $\frac{1}{2} (c_V + c_A \gamma^5)$. Draw the corresponding Feynman diagrams (Note: use $i\mathcal{L}$, drop all fields and you get the vertex factor).

1.9

Using the unitary gauge, insert the shifted Higgs field

$$\phi(x) = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}}(v + \eta(x)) \end{pmatrix}$$

into the Yukawa couplings of the Lagrangian:

$$\mathcal{L} \supset -G_e \left(\bar{E}_L \phi e_R + \bar{e}_R \phi^\dagger E_L \right)$$

Show that the mass of the electron is $m_e = \frac{G_e v}{\sqrt{2}}$.

2 Anomaly cancellation in Standard Model

A condition for a QFT to be well-defined is the cancellation of anomalies associated with local symmetries. Show that the SM is anomaly free and

$$\mathcal{A}^{abc} = \text{tr} \left[t^a \{ t^b, t^c \} \right] = 0$$

2.1

List the different triangle diagrams that exist in terms of group structure. Using the fact that $SU(3)_C$ is a vector-like interaction and properties of $SU(2)_L$, argue that the only nontrivial contributions are those with an odd number of hypercharge insertions. Show that some relations are equivalent to requiring cancellation of electric charge inside each generation.

2.2

Are baryon number (B) and lepton number (L) currents anomaly free?