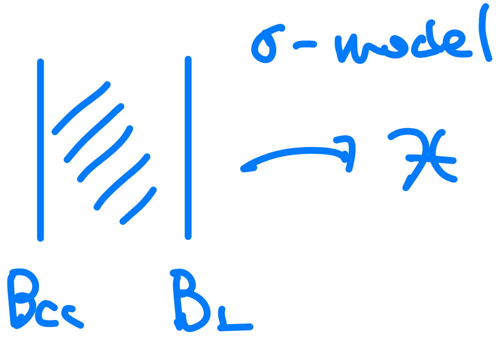
Understanding DAHA from physics

w/ Gukov, Koroteev, Pei, Saberi

Plan

- 1^{st} Talk is to understand rep thy of DAHA $\ddot{H}$

by 2d A-model on $\mathfrak{X} = \mathcal{M}_{flat}(T^2 \setminus pt, SL(2,\mathbb{C}))$



$\text{A-brane}(\mathfrak{X}, \omega_{\mathfrak{X}}) \cong \mathrm{Rep}(S\ddot{H})$

- 2^{nd} talk is to uplift the story to

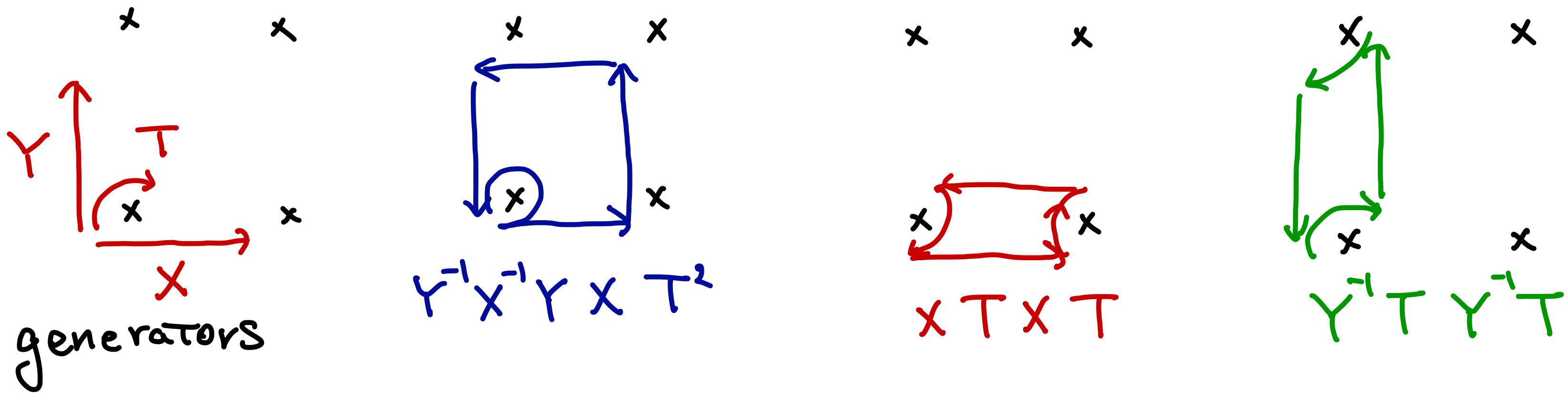
3d modularity $\quad V = K^0(\mathrm{MTC})$

4d $\mathcal{N}=2^*$ thy $\quad$ Coulomb branch, line ops

Double Affine Hecke Algebra (DAHA) Cherednik

- Orbifold fundamental group (Double affine braid group)

$$\pi_1\left((T^2 \setminus pt)/\mathbb{Z}_2\right) = \left[T^{\pm}, X^{\pm}, Y^{\pm} \;\middle|\; \begin{array}{c} Y^{-1}X^{-1}YXT^2 = 1 \\ XTXT = 1 \\ Y^{-1}TY^{-1}T = 1 \end{array} \right]$$



$q = t^c$ c: central charge.

- DAHA

$$\ddot{H} = \ddot{H}(sl_2) = \left[T^{\pm}\, X^{\pm}\, Y^{\pm} \;\middle|\; \begin{array}{c} Y^{-1}X^{-1}YXT^2 = q^{-1} \\ TXT = X^{-1} \\ TY^{-1}T = Y \end{array} \quad (T-t)(T+t^{-1}) = 0 \right]$$

Double Affine Hecke Algebra (DAHA), Cherednik

- Symmetry $PSL(2.\mathbb{Z}) \times \Xi$ (Very important!!)

$$PSL(2.\mathbb{Z}): \quad (X.Y.T) \xrightarrow{\tau_+} (X.\, q^{-\frac{1}{2}} XY.\, T)$$
$$(X.Y.T) \xrightarrow{\tau_-} (q^{\frac{1}{2}} YX.\, Y.\, T)$$
$$(X.Y.T) \xrightarrow{\sigma} (Y^{-1}.\, XT^2.\, T)$$

$$\Xi: \quad \xi_1 : X \to -X, \qquad \xi_2 : Y \to -Y.$$

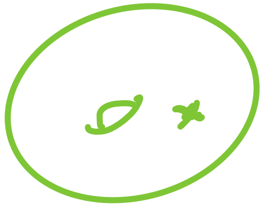
sign change.

- Spherical subalgebra. (gauge invariant part).

$$e = \frac{T + t^{-1}}{t + t^{-1}} \qquad \text{idempotent} \qquad e^2 = e.$$

$$S\ddot{H} = e\ddot{H}e = \left[\begin{array}{c|l} x = (X + X^{-1})e & [x.y]_q = (q^{-1} - q)\, z \\ y = (Y + Y^{-1})e & [y.z]_q = (q^{-1} - q)\, x \\ z = (q^{-\frac{1}{2}} Y^{-1} X + q^{\frac{1}{2}} X^{-1} Y)e & [z.x]_q = (q^{-1} - q)\, y \\ & q^{-1}x^2 + q y^2 + q^{-1} z^2 - q^{-\frac{1}{2}} xyz \\ & = (q^{-\frac{1}{2}} t - q^{\frac{1}{2}} t^{-1})^2 + (q^{\frac{1}{2}} + q^{-\frac{1}{2}})^2 \end{array}\right]$$

Spherical DAHA as $\mathcal{O}^q(\mathcal{X})$



- Classical limit $q \to 1$ of central element.

$$x^2 + y^2 + z^2 - xyz = 4 + (t - t^{-1})^2 = 2 + \operatorname{Tr} V \qquad V = \begin{pmatrix} t^2 & 0 \\ 0 & t^{-2} \end{pmatrix}$$

is moduli sp of flat $SL(2, \mathbb{C})$-connections on $T^2 \setminus \{pt\}$

Oblomkov.

$$S\ddot{H} \cong \mathcal{O}^q[\mathcal{M}_{flat}(T^2 \setminus \{pt\}, SL(2, \mathbb{C}))]$$

. deformation quantization of coord. ring w.r.t. Ω_J.

- Back to Symmetry.

Weyl.

& $i: t \to t^{-1}$

$$(x, y, z) \xrightarrow{T_+} (x, xy - z, y)$$
$$(x, y, z) \xrightarrow{T_-} (xy - z, y, x)$$
$$(x, y, z) \xrightarrow{\sigma} (y, x, xy - z)$$

$$(x, y, z) \xrightarrow{\xi_1} (-x, y, -z)$$
$$(x, y, z) \xrightarrow{\xi_2} (x, -y, -z)$$
$$(x, y, z) \xrightarrow{\xi_3} (-x, -y, z)$$

Representations of spherical DAHA.

- polynomial representations.

$$\mathrm{pol} : S\ddot{H} \longrightarrow \mathrm{End}(\mathbb{C}_{q,t}[X, X^{-1}]^{\mathbb{Z}_2})$$

$$x \longmapsto X + X^{-1}$$

$$y \longmapsto \frac{tX - t^{-1}X^{-1}}{X - X^{-1}} q^{\partial_x} + \frac{t^{-1}X - tX^{-1}}{X - X^{-1}} q^{-\partial_x}$$ Macdonald difference op.

$$z \longmapsto X^{-1}\frac{tX - t^{-1}X^{-1}}{X - X^{-1}} q^{\partial_x} + X\frac{t^{-1}X - tX^{-1}}{X - X^{-1}} q^{-\partial_x}$$

- basis of $\mathbb{C}_{q,t}[X, X^{-1}]^{\mathbb{Z}_2}$ is spanned by Macdonald poly $\{P_\ell(x;q,t)\}$

$$y \cdot P_j(x;q,t) = (q^j t + q^{-j} t^{-1}) P_j(x;q,t)$$

- raising and lowering operators

$$R_j = x - q^{j-\frac{1}{2}} t z \qquad R_j \cdot P_j = (1 - q^{-2j} t^{-2}) P_{j+1}$$

$$L_j = x - q^{-j-\frac{1}{2}} t^{-1} z \qquad L_j \cdot P_j = -\frac{(1 - q^{2(j-1)} t^4)(1 - q^{2j})}{q^{2j} t^2 (1 - q^{2(j-1)} t^2)} P_{j-1}$$

Finite-dimensional Rep

$$1 \underset{L}{\overset{R}{\rightleftarrows}} P_1 \underset{L}{\overset{R}{\rightleftarrows}} P_2 \underset{R}{\overset{R}{\rightleftarrows}} \cdots \qquad P_{n-1} \underset{L}{\overset{R}{\rightleftarrows}} P_n \rightleftarrows \cdots$$

null vector

$\rightsquigarrow \quad \mathrm{S}\ddot{\mathrm{H}} \circlearrowright V = \mathbb{C}_{q,t}[X^{\pm}]^{\mathbb{Z}_2} / (P_n)$ finite-dim rep.

shortening condition

$$\frac{(1-q^{2j})(1-q^{j-1}t^2)(1+q^{j-1}t^2)}{q^{2j}t^2(1-q^{2(j-1)}t^2)} = 0$$

① $q^{2n} = 1$

② $t^2 = -q^{-k}$ $\rightsquigarrow$ different finite-dim reps

③ $t^2 = q^{-(2\ell-1)}$

we will see geometrically.

Geometry of Hitchin moduli space

- $$\mathcal{M}_{flat}(C, SL(2,\mathbb{C})) \cong \mathcal{M}_H(C, SU(2))$$

$$\left\{ \begin{matrix} \text{flatness of } SL(2,\mathbb{C})\text{-conn} \\ A+\varphi+\varphi^* \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} (E,\Phi) \\ \varphi \in \mathrm{End}(E)\otimes K_C \end{matrix} \;\middle|\; \begin{matrix} F_A + [\phi,\phi^*]=0 \\ \bar{D}_A \phi = 0 \end{matrix} \right\}$$

non-Abelian Hodge.

Hyperkähler manifold

from C

$\downarrow$

$(I \;\; J, K)$

$\uparrow$

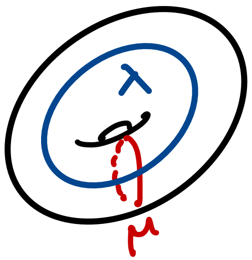
from character variety.

$$\Omega_J = \frac{dx\wedge dy}{2z - xy}.$$

(x,y,z). holom. in J.

- $$x^2+y^2+z^2-xyz = 2+\mathrm{Tr}\,V.$$

no holonomy $V = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$



$$\mu \sim \begin{pmatrix} X & * \\ 0 & X^{-1} \end{pmatrix}$$

$$\lambda \sim \begin{pmatrix} Y & * \\ 0 & Y^{-1} \end{pmatrix}$$

$$\begin{matrix} \mathcal{M}_{flat}(T^2, SL(2,\mathbb{C})) & = & \dfrac{\mathbb{C}^\times \times \mathbb{C}^\times}{\mathbb{Z}_2} \\ \cup & & \cup \\ \mathcal{M}_{flat}(T^2, SU(2)) & = & \dfrac{S^1\times S^1}{\mathbb{Z}_2} \end{matrix}$$

Ramification

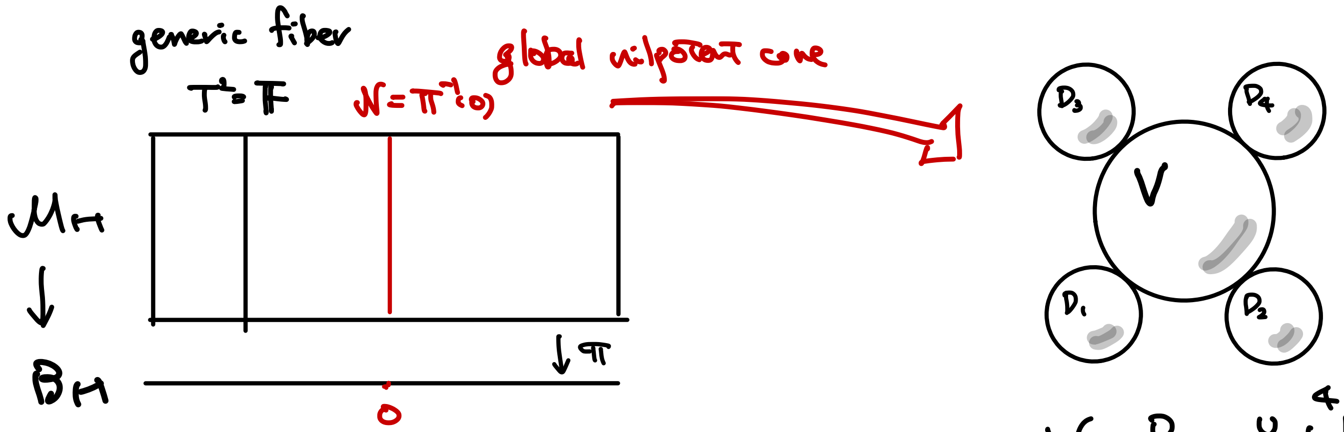
$$A = i\alpha\, d\theta$$

$$\varphi = \frac{1}{2}(\beta + i\gamma)\frac{dz}{z}$$

$t = \exp(-\pi(\gamma + i\alpha))$ complex str on J.

When $\alpha \neq 0$. $\beta = 0 = \gamma$

- Hitchin fibration $\pi : \mathcal{M}_H \longrightarrow B_H = H^0(C, K_C^{\otimes 2}) = \{\mathrm{Tr}\,\phi^2\}$



affine $\hat{D}_4$ singularity

Kodaira I_0^* type.

$$\mathcal{N} = \mathrm{Bun}_G \cup \bigcup_{i=1}^{4} D_i$$

- 2^{nd} homology of $\mathcal{M}_H$ has relation

$$[F] = 2[V] + \sum_{i=1}^{4}[D_i]$$

- Exceptional fibers $D_1 \cdots D_4$ show up when we turn on α.

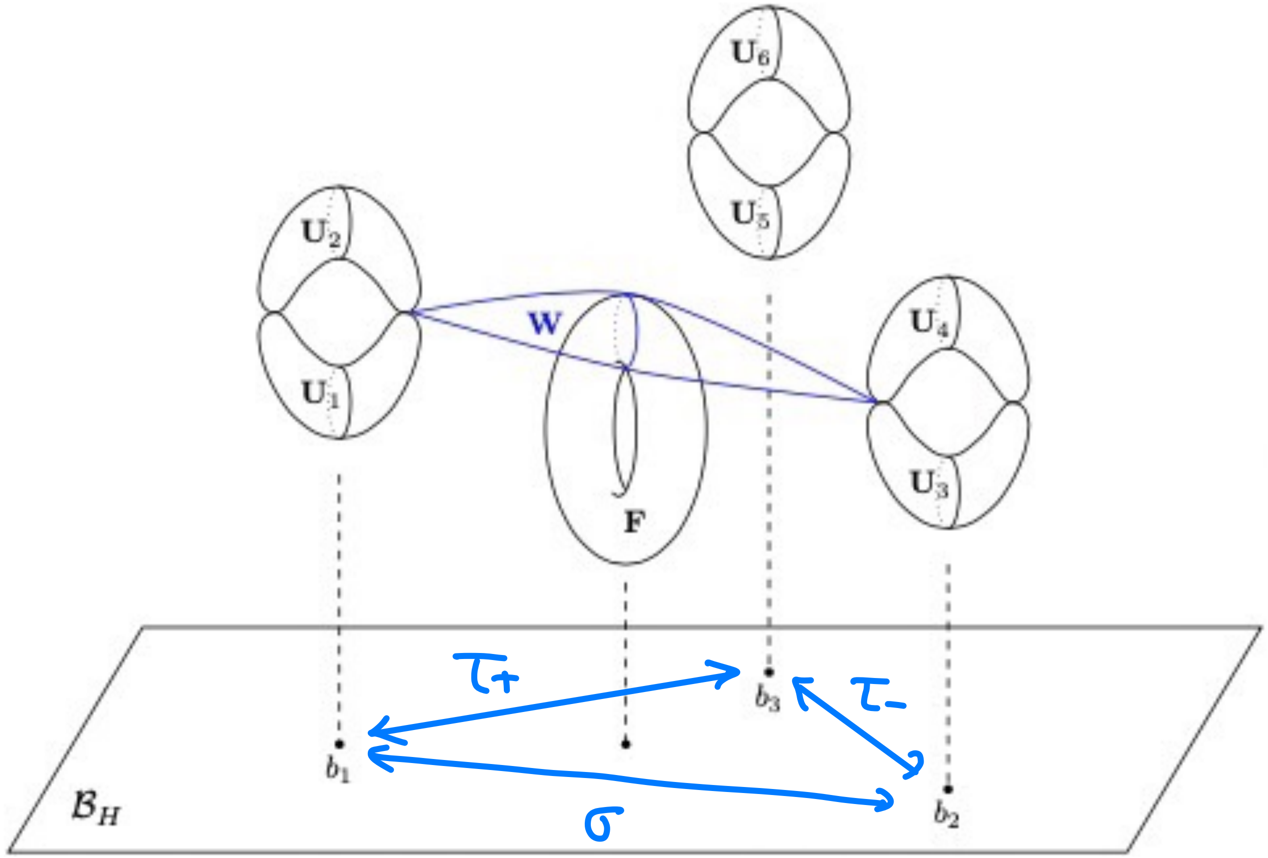
Geometry of Hitchin moduli space

At generic ramification $\beta, \gamma \neq 0$.

$$\alpha = \int_{D_i} \frac{\omega_I}{\pi} \qquad \beta = \int_{D_i} \frac{\omega_J}{\pi} \qquad \gamma = \int_{D_i} \frac{\omega_K}{\pi}$$

$$\int_F \frac{\omega_I}{2\pi} = 1$$

$$\int_F \frac{\omega_J}{2\pi} = 0 = \int_F \frac{\omega_K}{2\pi}.$$



$$U_1 = V + D_1 + D_2 \qquad U_3 = V + D_1 + D_3 \qquad U_5 = V + D_1 + D_4$$
$$U_2 = V + D_3 + D_4 \qquad U_4 = V + D_2 + D_4 \qquad U_6 = V + D_2 + D_3$$

as. H_2 homology class

A-model on $\mathcal{X} = \mathcal{M}_{flat}(T^2 \setminus pt, SL(2,\mathbb{C}))$

- 2d A-model on $\Sigma \to (\mathcal{X}, \omega_{\mathcal{X}})$
- Σ can have boundary. Boundary condition is described by A-branes.
- Usual A-branes are "flat unitary bundle E over Lagrangian L" $\mathcal{B}_L = \begin{pmatrix} E \\ \downarrow \\ L \end{pmatrix}$
- Kapustin-Orlov some A-branes are supported on coisotropic submfd. Σ
  (i.e. submfd locally defined by Poisson commuting fns)
- One distingushed object "canonical coisotropic brane $\mathcal{B}_{cc}$"

$$\mathcal{B}_{cc} = \begin{pmatrix} \mathcal{L} \\ \downarrow \\ \mathcal{M}_H \end{pmatrix} \qquad c_1(\mathcal{L}) = F$$

$$F + B + i\omega_{\mathcal{X}} = \frac{\Omega_J}{i\hbar} \rightsquigarrow q = e^{2\pi i \hbar}$$

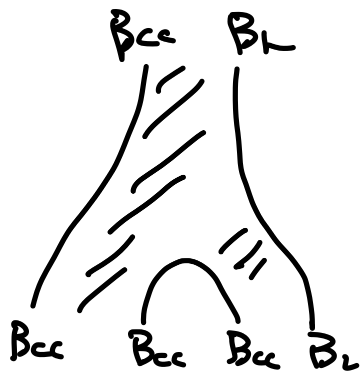
$$\tau = \exp(-\pi(\gamma + i\alpha))$$

Brane quantization Gukov - Witten.

- (B_{cc}, B_{cc})-string : deformation quantization of holom. fns w.r.t. J over M_H.

- joining of (B_{cc}, B_{cc})-string and (B_{cc}, B_L)-string gives rise to another (B_{cc}, B_L)-string.

$$\begin{array}{ccc} \mathrm{Hom}(B_{cc}, B_{cc}) & \longleftrightarrow & S\ddot{H} \\ \circlearrowleft & & \circlearrowleft \\ \mathrm{Hom}(B_{cc}, B_L) & \longleftrightarrow & V_L \\ \cap & & \cap \\ \text{A-brane}(\mathfrak{X}, \omega_{\mathfrak{X}}) & \longleftrightarrow & \mathrm{Rep}(S\ddot{H}) \end{array}$$



- Given A-brane, there is the corresponding module V_L

compact A-brane B_L $\longleftrightarrow$ finite-dimensional module V_L

$$\dim V = \int_L ch(B_{cc} \otimes B_L^{-1})\, Td(L) \quad \text{Hirzebruch-Riemann-Roch}$$

Brane with compact support & finite-dim reps

- generic fiber F

$$\text{HRR} \to \frac{1}{\hbar} = 2n \longrightarrow \text{shortening cond} \quad ① \; q^{2n} = 1$$

encode holonomy position

$$\mathrm{Hom}(B_{cc}, B_F) = \mathbb{C}_{q,t}[X^{\pm}]^{\mathbb{Z}_2} \Big/ (X^{2n} + X^{-2n} - a - a^{-1})$$

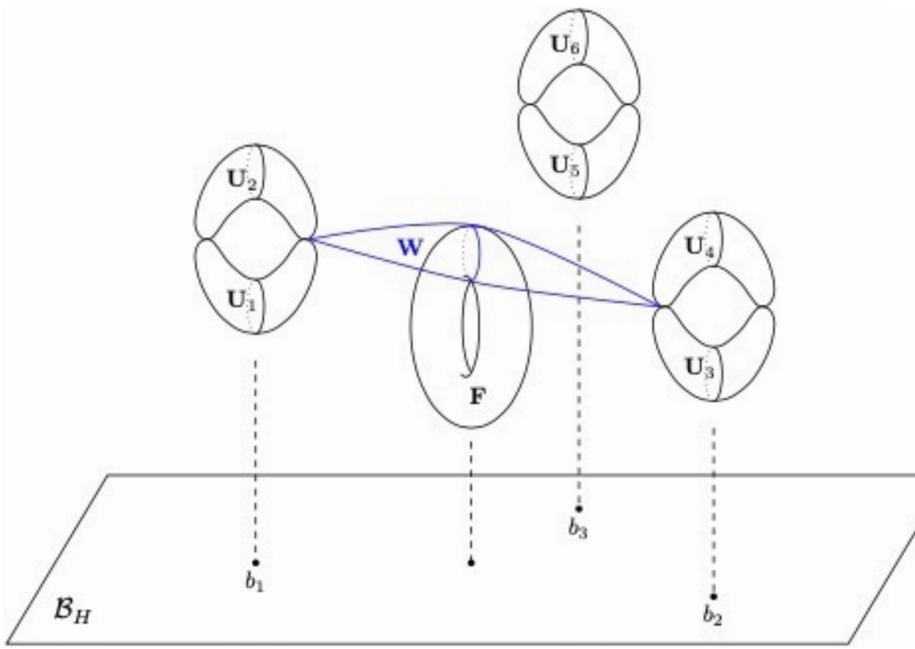
- irreducible comp U_1

$$\mathrm{Hom}(B_{cc}, B_{U_1}) = \mathbb{C}_{q,t}[X^{\pm}]^{\mathbb{Z}_2} \Big/ (P_n)$$

$$0 \to U_2 \to \overset{a=-1}{F} \to U_1 \to 0$$

$$0 \to U_2 \to \zeta_2 \cdot i(\overset{a=-1}{F}) \to U_1 \to 0$$

generates $\mathrm{Hom}^1(U_1, U_2)$



Brane with compact support & finite-dim reps

- moduli space of G-bundle V

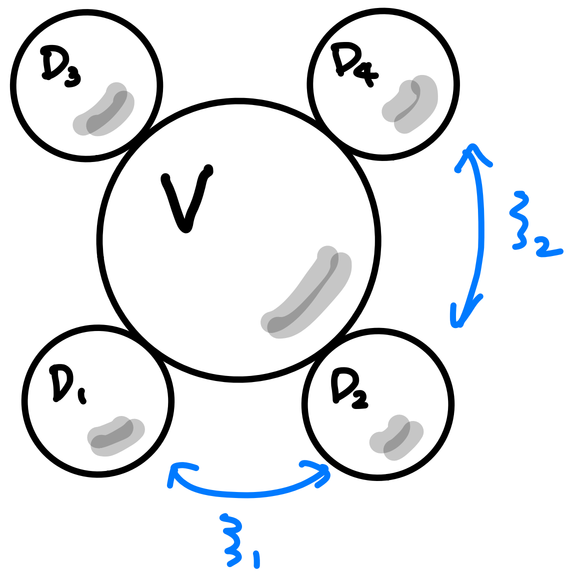
HRR $\rightsquigarrow$ shortening cond ② $t^2 = -q^{-k}$

$$\mathrm{Hom}(B_{cc}, B_V) = \mathbb{C}_{q,t}[X^{\pm}]^{\mathbb{Z}_2} / (P_k)$$

- Exceptional divisor D_i

HRR $\rightsquigarrow$ shortening cond ③ $t^2 = q^{-2\ell+1}$

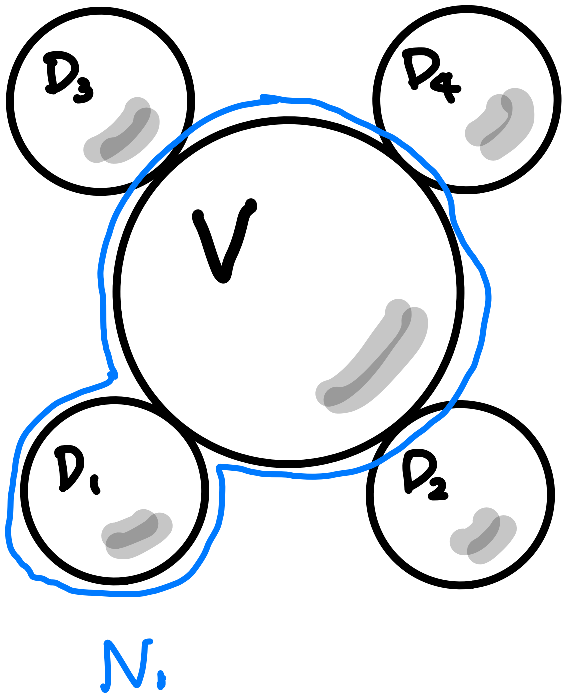
$$\begin{matrix} \mathrm{Hom}(B_{cc}, B_{D_1}) \\ \oplus \\ \mathrm{Hom}(B_{cc}, B_{D_2}) \end{matrix} = \mathbb{C}_{q,t}[X^{\pm}]^{\mathbb{Z}_2} / (P_{2\ell})$$



Bound states of branes & Morphism structure

$$0 \to V \to N_1 \to D_1 \to 0$$

generates $\mathrm{Hom}^1(D_1, V)$



This gives evidence

$$\text{A-brane}(\mathcal{X}, \omega_{\mathcal{X}}) \cong \mathrm{Rep}(\ddot{\mathfrak{sl}})$$

Advantages.

- new reps
- geometric understanding.

Fukaya.

- New phenomena in A-brane
- lift to higher-dim physics

Intermission

PSL(2.$\mathbb{Z}$) action

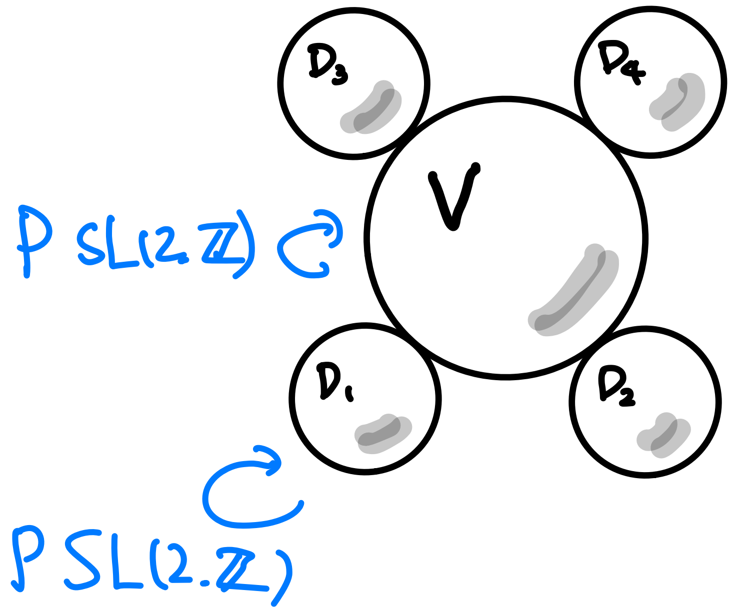
Sït & target space $\mathcal{X}$ enjoy PSL(2.$\mathbb{Z}$) action

Compact Lagrangian V & D_1 are invariant under PSL(2.$\mathbb{Z}$)

PSL(2.$\mathbb{Z}$) $\circlearrowright$ Hom(B_{cc}.B_V)

$\rightsquigarrow$

PSL(2.$\mathbb{Z}$) $\circlearrowright$ Hom(B_{cc}.B_{D_1})



question

Which modular rep?

answer from 3d physics

3d / 3d Correspondence

$$N\ M5 \quad \begin{matrix} S^1 \times \mathbb{R}^2_q \times \mathbb{R}^2_t \times T^* M_3 \\ \underbrace{S^1 \times \mathbb{R}^2_q}_{3d\ \mathcal{N}=2\ T[M_3]} \times pt \times \underbrace{M_3}_{\mathbb{C}\ CS.} \end{matrix}$$

M_3: Seifert mfd.

Aganagic-Shakirov,
Yoshida-Sugiyama.

$$Z = \mathrm{Tr}(-1)^F q^{J_3+S} t^{R-S}$$

e.g. $M_3 = L(p,1)$

$$\underset{\substack{\uparrow \\ \mathcal{M}_{flat}(L(p,1), G_{\mathbb{C}})}}{\hat{Z}_b} = \int_{|x|=1} \frac{dx}{2\pi i x} \prod_{\alpha\in\Delta_+} \frac{(x^\alpha;q)_\infty}{(t^2 x^\alpha;q)_\infty} \Theta_{b,p}(x;q)$$

generalized Θ-fn

Macdonald

e.g. $M_3 = S^3$

$$S_{\mu\nu} = \int_{|x|=1} \frac{dx}{2\pi i x} \prod_{\alpha\in\Delta_+} \frac{(x^\alpha;q)_\infty}{(t^2 x^\alpha;q)_\infty} \Theta(x;q)\, P_\lambda(x)\, \overline{P_\mu(x)}$$

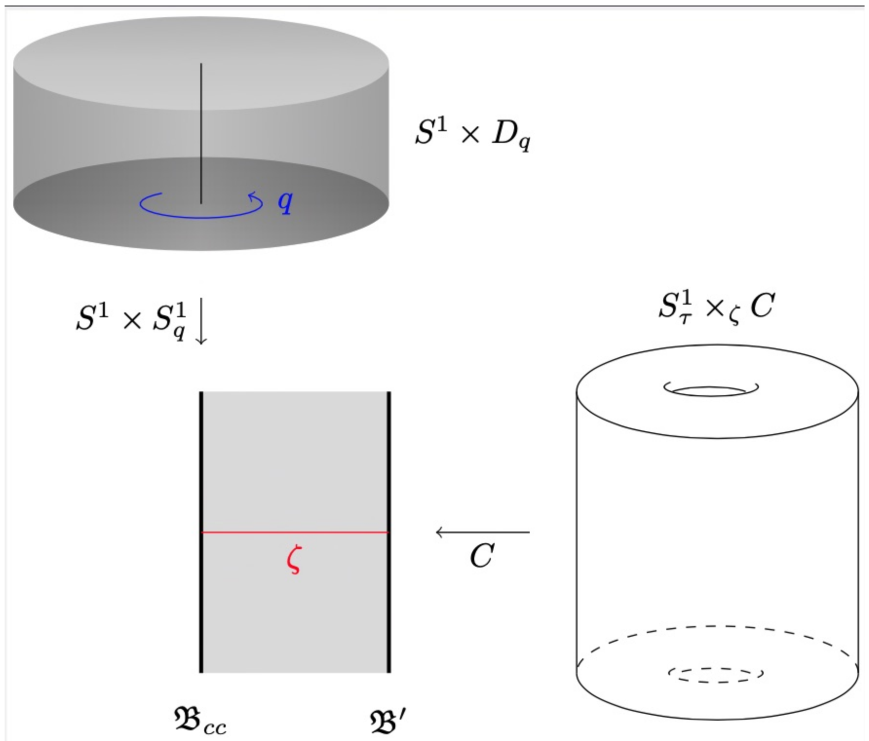
Connecting 2d σ-model to 3d Physics

$M_3 = S^1_\tau \times_\varsigma C \qquad \varsigma \in SL(2,\mathbb{Z}) \qquad$ mapping tori

6d (2,0) thy on $S^1 \times \mathbb{R}^2_q \times S^1_\tau \times_\varsigma C$

$S^1 \times S^1_q \times C \swarrow \qquad \searrow S^1 \times_\varsigma C$

2d σ-model $S^1_\tau \times I \to \mathcal{M}_H(C_p, G) \simeq$ 3d $\mathcal{N}=2$ thy $T[S^1_\tau \times_\varsigma C]$ on $S^1 \times \mathbb{R}^2_q$.



$PSL(2, \mathbb{Z})$ on $\mathrm{Hom}(\mathcal{B}_{cc}, \mathcal{B}_{V})$

shortening condition ② $t^2 = -q^{-k}$ $\rightsquigarrow$ $t = \exp\left(\frac{c\pi i}{k+2c}\right)$

$$q = \exp\left(\frac{\pi i}{k+2c}\right)$$

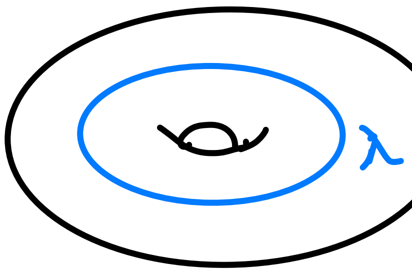
rank $(k+1)$

$$S_{ij} = P_i(t q^j ; q, t)\, P_j(t^{-1} ; q, t)\Big|_{②} \qquad \circlearrowright \mathrm{Hom}(\mathcal{B}_{cc}, \mathcal{B}_{V})$$

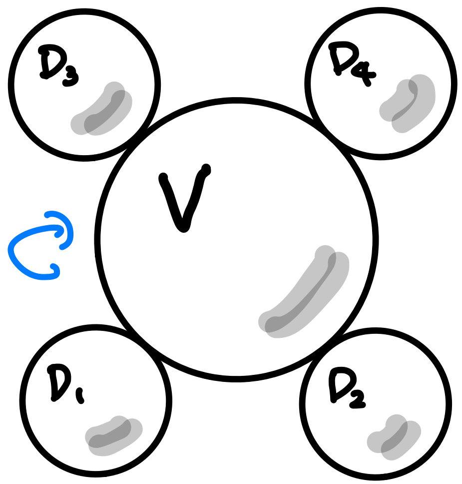
$$T_{ij} = \delta_{ij}\, q^{-\frac{j^2}{2}}\, t^{-j}\Big|_{②}$$

Refined CS.

$\xrightarrow[t=q]{}$ Verlinde module.



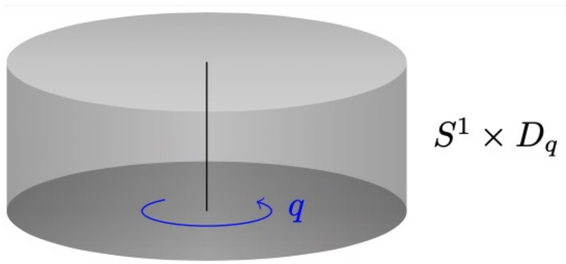
$PSL(2, \mathbb{Z}) \circlearrowright$



$PSL(2,\mathbb{Z})$ on $\mathrm{Hom}(B_{cc}, B_{p_i})$

Xie.

$(A_1, A_{2(\ell-1)})$ Argyres-Douglas thy $T[C_{\text{wild}}, SU(2)]$



$\cong$

$\varphi(z)\,dz \sim z^{\frac{2\ell-1}{2}}\sigma_3\,dz$

$$t\, q^{\frac{2\ell-1}{2}}\, \varphi(x_0+\beta, z) = \varphi(x_0, z)$$

$\longrightarrow$ shortening cond @ $T^2 q^{\frac{2\ell-1}{2}} = 1$.

(rank ℓ)

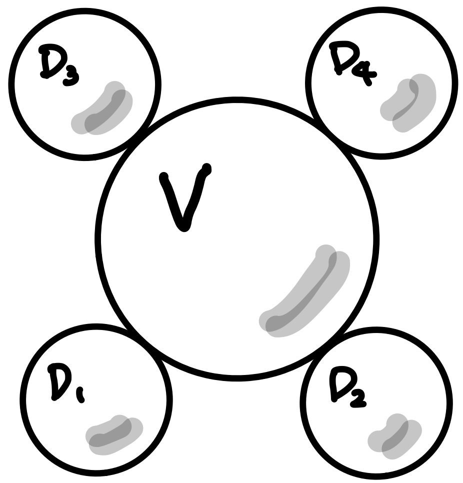
$S_{ij}|_{@}$ modular matrices

$T_{ij}|_{@}$ of Virasoro Minimal model

$\updownarrow$

chiral alg of $(A_1, A_{2\ell-2})$ thy.

$PSL(2,\mathbb{Z}) \circlearrowright$

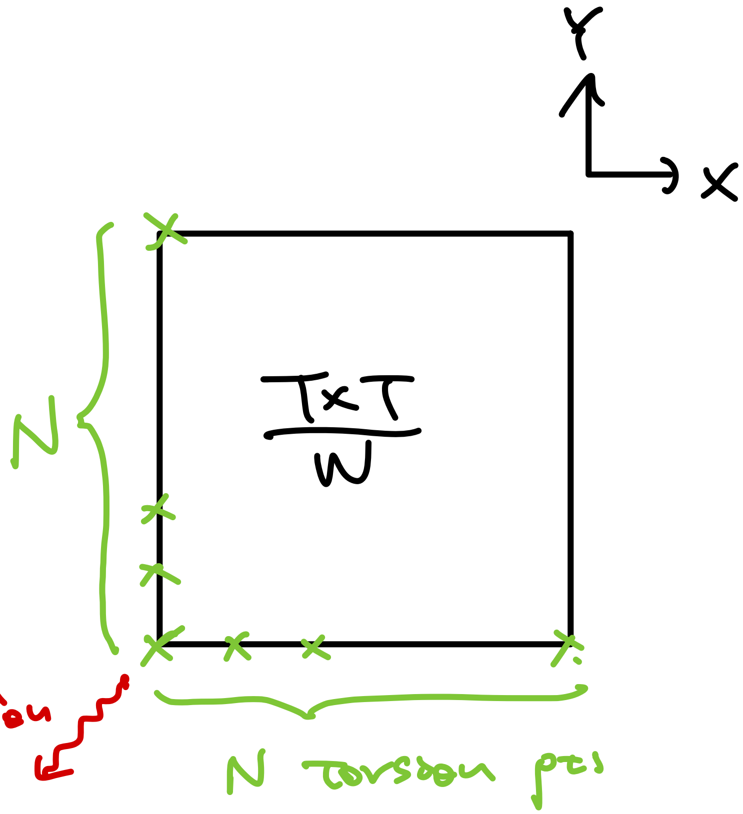


Higher rank generalization

$$\mathcal{M}_{flat}(T^2, SL(N,\mathbb{C})) = \frac{T_{\mathbb{C}} \times T_{\mathbb{C}}}{W}$$

$$\cup \qquad\qquad \cup$$

$$\mathcal{M}_{flat}(T^2, SU(N)) = \frac{T \times T}{W}$$



after ramification

Shortening cond $t^N q^M = 1$.

$PSL(2,\mathbb{Z}) \circlearrowright$  analogous to $\mathbb{P}_1$

$$K_0(MTC(A_{N-1}, A_{M-1})) \cong Hom(B_{cc}, B_{\mathbb{P}_1^{(N)}}) \cong \mathcal{H}_{bottom}(T_{N,M})$$

Gorsky Oblomkov

Rasmussen - Shende.

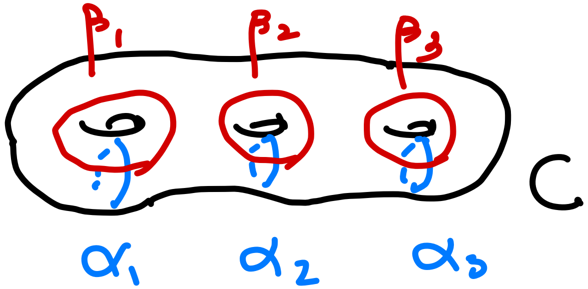
4d $\mathcal{N}=2$ of class S & Coulomb branch

Thy. $T[C, G, L]$

Gaiotto

Gaiotto-Moore-Neitzke

Tachikawa.



Hitchin System

$$\begin{array}{ccc} \Sigma & \hookrightarrow & T^*C \\ & & \downarrow \\ & & C \end{array}$$

symplectic basis

$(\alpha_1 \cdots \alpha_g \quad \beta_1 \cdots \beta_g) \in H_1(C)$

$\alpha_i \cdot \beta_j = \delta_{ij} \qquad \alpha_i \cdot \alpha_j = 0 = \beta_i \cdot \beta_j$

$\downarrow$ maximal isotropic sublattic.

Coulomb branch

$\mathcal{M}_C(C, G, L) \cong \mathcal{M}_H(C, G)/L^\vee$

$L \subset (H^1(C, Z(G)), \omega)$

$\pi : \mathcal{M}_C(C, G, L) \to B_u.$

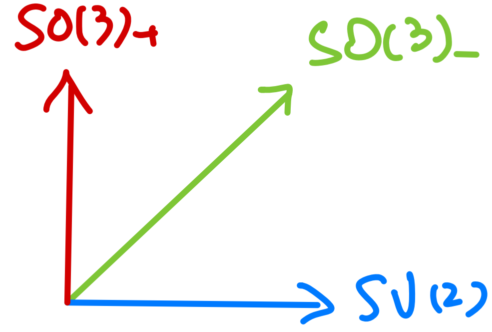
Completely integrable system.

4d $\mathcal{N}=2^*$ thy of Type A_1

$C_p = T^2 \setminus pt.$

$G = SU(2)$

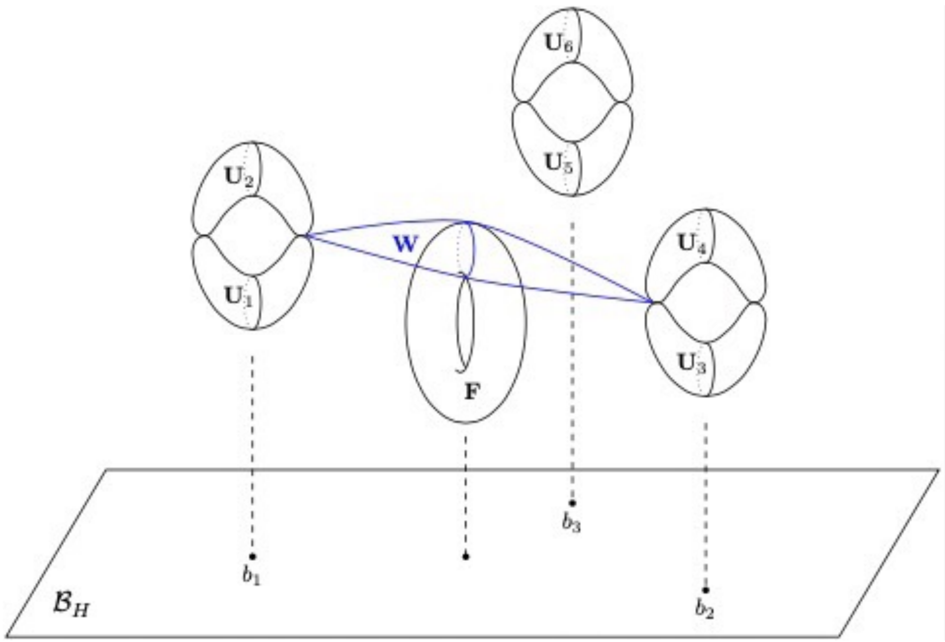
$H^1(C, \mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \supset$ $(1,0)$, $(0,1)$, M.I.L $(1,1)$

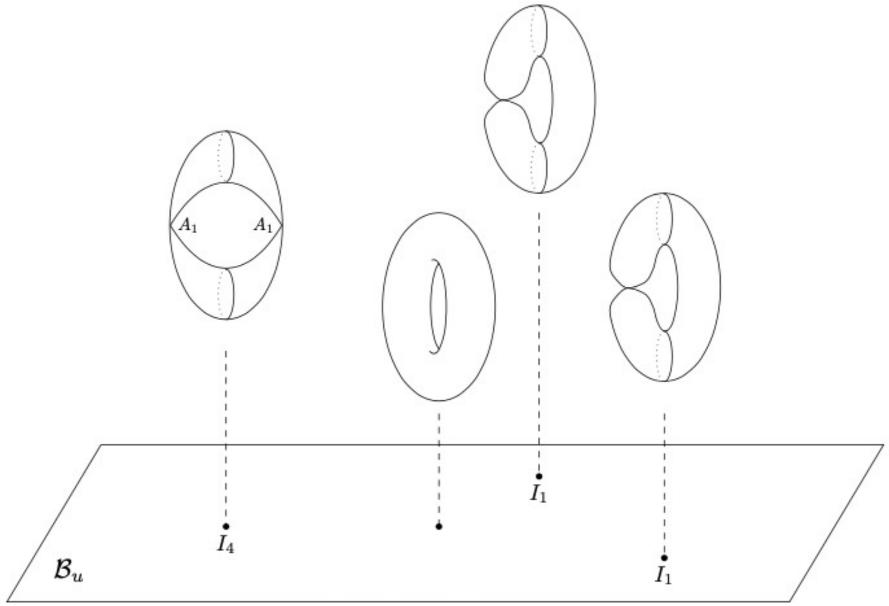


$$\begin{tikzcd}
 & SU(2) \arrow[loop above, "T"] \arrow[dl, leftrightarrow, "S"'] \arrow[dr, leftrightarrow, "TST"] & \\
SO(3)_+ \arrow[loop left, "TST"] \arrow[rr, leftrightarrow, "T"'] & & SO(3)_- \arrow[loop right, "S"]
\end{tikzcd}$$

$$\begin{tikzcd}
 & \mathcal{M}_H(C_p, SU(2)) \arrow[dl, "\xi_2"'] \arrow[d, "\xi_1"] \arrow[dr, "\xi_3"] & \\
\mathcal{M}_C(C_p, SU(2)) \arrow[dr, "\xi_1"'] & \mathcal{M}_C(C_p, SO(3)_+) \arrow[d, "\xi_2"] & \mathcal{M}_C(C_p, SO(3)_-) \arrow[dl, "\xi_1"] \\
 & \mathcal{M}_H(C_p, SO(3)) &
\end{tikzcd}$$

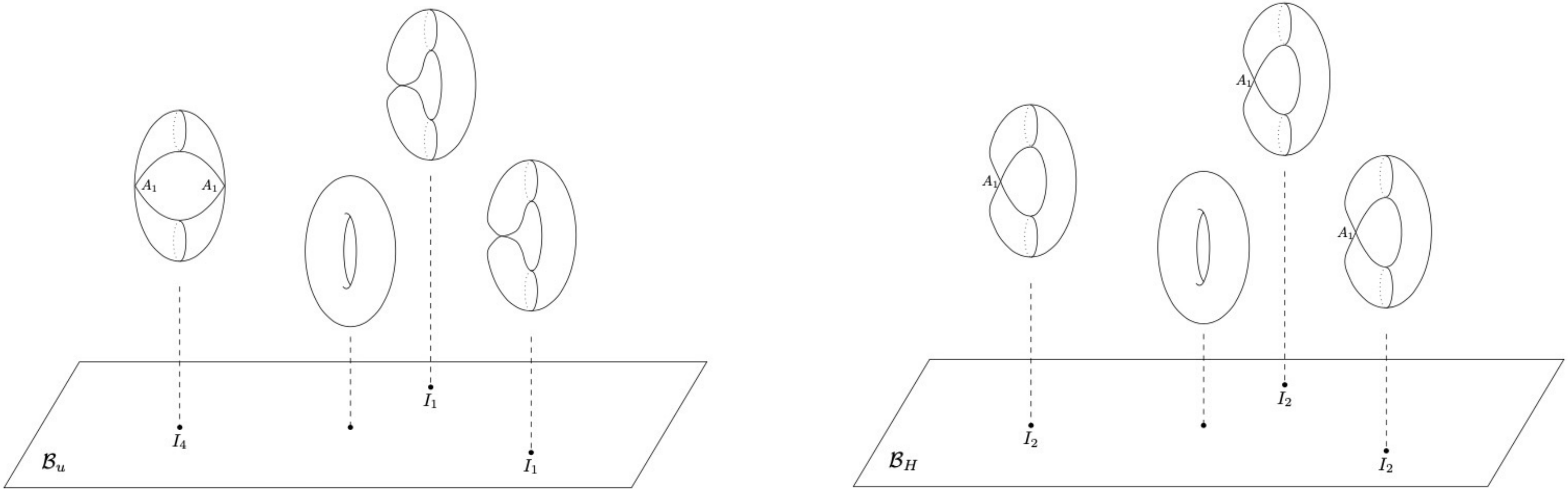
Geometry of Coulomb branch of 4d $\mathcal{N}=2^*$ of type A_1





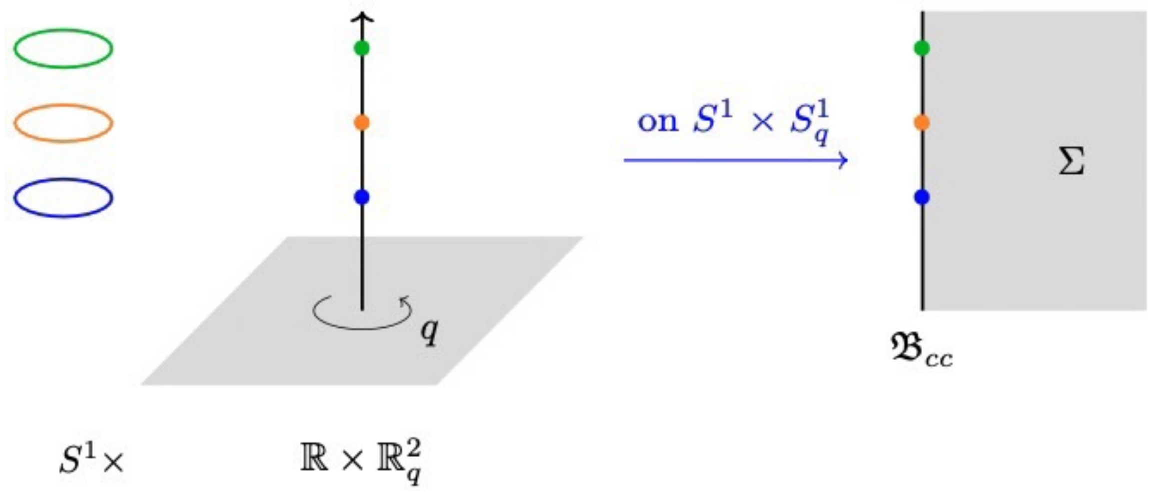
$$\mathcal{M}_H(C_p, SU_2) \xrightarrow{\mathbb{Z}_2} \mathcal{M}_C(C_p, SU_2)$$

$\mathcal{M}_H(C, SO(3))$



$$\mathcal{M}_c(C, SU(2), L) \xrightarrow{3_1} \mathcal{M}_H(C, SO(3))$$

Algebra of line operators (quantized Coulomb branch)



Ito-Okuda-Taki

Hayashi-Okuda-Yoshida

- SU(2) : $\mathcal{O}^q(\mathcal{M}_C(C_p, SU(2))) \cong S\ddot{H}^{\mathfrak{z}_2}$

$$x = (X + X^{-1})\,e \qquad y^2 - 1 = (Y^2 + 1 + Y^{-2})\,e$$

- $SO(3)_+$: $\mathcal{O}^q(\mathcal{M}_C(C_p, SO(3)_+) \cong S\ddot{H}^{\mathfrak{z}_1}$

$$x^2 - 1 = (X^2 + 1 + X^2)\,e \qquad y = (Y + Y^{-1})\,e$$

- $SO(3)_-$: $\mathcal{O}^q(\mathcal{M}_C(C_p, SO(3)_-)) \cong S\ddot{H}^{\mathfrak{z}_3}$

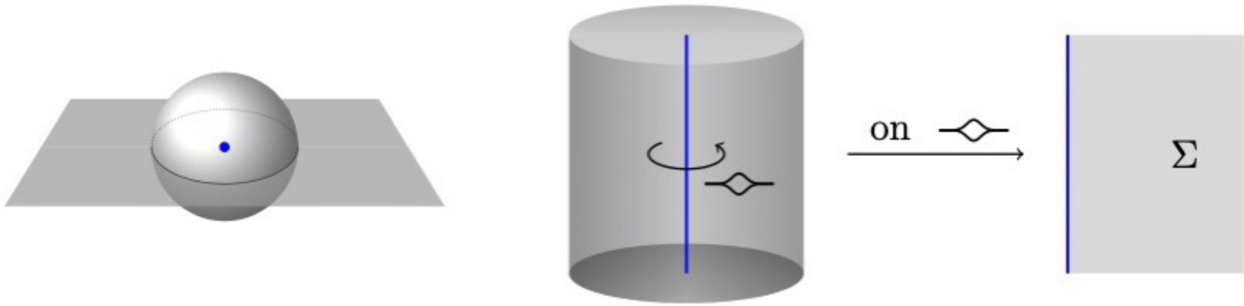
$$x^2 - 1 = (X^2 + 1 + X^2)\,e \qquad z = (q^{-\frac{1}{2}} Y^{-1} X + q^{\frac{1}{2}} X^{-1} Y)\,e$$

Category of line operators

Kapustin-Witten

BFM

BFN



(B.B.B) Wilson

(B.A.A) 't Hooft.

Compactify 4d $\mathcal{N}=4$ ($\mathcal{N}=2^*$) thy around line operators

$\downarrow$ raviolo

(B.A.A) brane in $\mathcal{M}_H(\ , G)$

$\mathcal{O} = \mathbb{C}[[z]]$

$K = \mathbb{C}((z))$

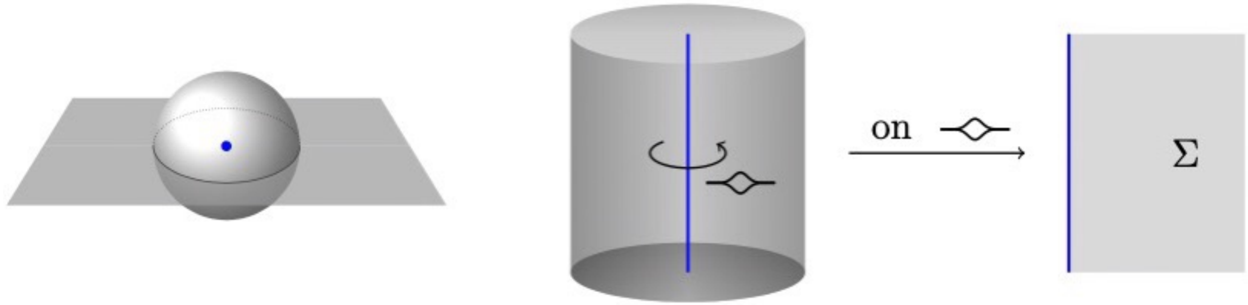
$Gr_{G_\mathbb{C}} = G_\mathbb{C}^K / G_\mathbb{C}^\mathcal{O}$

affine Grassmannian Steinberg variety.

$$R = \{ (x, [g]) \in \mathfrak{g}_\mathbb{C}^\mathcal{O} \times Gr_{G_\mathbb{C}} \mid Ad_g^{-1}(x) \in \mathfrak{g}^\mathcal{O} \}$$

$\to$ Category of line op $\cong D^b Coh^{G_\mathbb{C}^\mathcal{O}}(R)$.

Category of line operators



(B.A.A)

taking Grothendieck group

$$K^{G_{\mathbb{C}}^{\mathcal{O}}}(R) = \mathcal{O}(T_{\mathbb{C}} \times T_{\mathbb{C}}^{\vee})^{W}$$

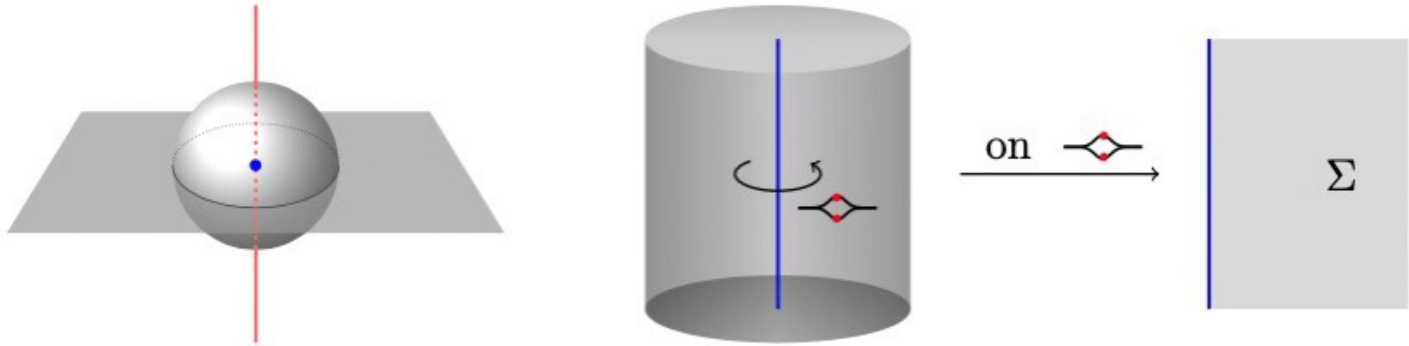
Coulomb branch of 4d $\mathcal{N}=4$ $= \dfrac{T_{\mathbb{C}} \times T_{\mathbb{C}}^{\vee}}{W} \subset \dfrac{T_{\mathbb{C}} \times T_{\mathbb{C}}}{W}$

$\rightarrow$ Turning on Ω-def (q, t)

$$K^{(G_{\mathbb{C}}^{\mathcal{O}} \times \mathbb{C}_t^{\times}) \rtimes \mathbb{C}_q^{\times}}(R) \cong \ddot{SH}(W)^{\iota^{\vee}}$$

Including surface operator

Vasserot



If we include a surface operator, raviolo has 2 ramifications.

$$R \rightsquigarrow Z = \{(x,[g]) \in \mathrm{Lie}(I) \times \mathrm{Fl}_{G_{\mathbb{C}}} \mid \mathrm{Ad}_g^{-1}(x) \in \mathrm{Lie}(I)\}.$$

affine Flag Steinberg variety

$$\mathrm{Fl}_{G_{\mathbb{C}}} = G_{\mathbb{C}}^{\mathcal{K}}/I.$$

$$\rightsquigarrow \quad K^{(I \times \mathbb{C}_t^\times) \rtimes \mathbb{C}_q^\times}(Z) \cong \ddot{H}(W)^{L^\vee}$$

$$\begin{array}{ccc} G_{\mathbb{C}}^{\mathcal{O}} & \xrightarrow{p} & G_{\mathbb{C}} \\ \cup & & \cup \\ I = p^{-1}(B) & \longrightarrow & B \end{array}$$

alg of line operators on a surface operators.

「なに，あれは眉や鼻を鑿（のみ）で作るんぢやない。あの通りの眉や鼻が木の中に埋つてゐるのを鑿と槌（つち）の力で掘り出す迄だ。丸で土の中から石を掘り出す様なものだから決して間違ふ筈はない」

私の楕円曲面論は実は私が考え出したのではなく，数学という木の中に埋まっていた楕円曲面論を私が紙と鉛筆の力で掘り出したにすぎない、と言うのが実感であった。

『ボクは算数しか出来なかった』小平邦彦: