Homework 5: Due at class on Oct 21

1 Modular invariant

Show that (4.85) is modular S-invariant by using the Poisson resummation formula (4.86).

2 Modular transformation

The modular group is defined by

$$\mathrm{SL}(2,\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \quad a,b,c,d \in \mathbb{Z}\,, \quad ad-bc=1 \right\} .$$

Show that

$$\mathrm{Im}(\gamma\cdot\tau) = \frac{\mathrm{Im}\,\tau}{|c\tau+d|^2}\,,$$

where

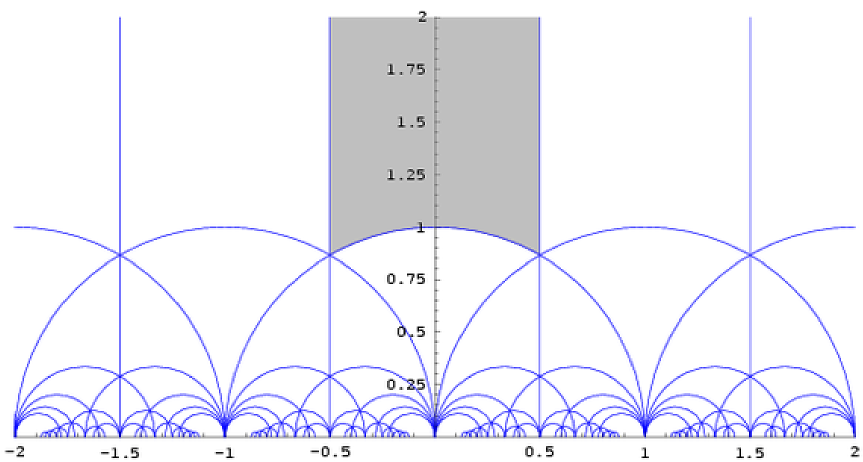
$$\gamma : \tau \mapsto \gamma\cdot\tau = \frac{a\tau+b}{c\tau+d}\,.$$

Therefore, if $\mathrm{Im}\,\tau > 0$, then $\mathrm{Im}(\gamma\cdot\tau) > 0$ so that the upper half-plane

$$\mathbb{H} = \{x+iy \mid y>0; x,y\in\mathbb{R}\}$$

receives the action of the modular group. Express elements of modular group that maps the gray region to blue, red and green region in terms of modular S and T transformations

$$T:\tau\to\tau+1\,,\qquad S:\tau\to -1/\tau\,.$$



3 Boson-Fermion correspondence

We have learnt that the OPE of the free boson is

$$\varphi(z)\varphi(0) \sim -\ln z \,.$$

Let us also consider two Majorana-Weyl fermions ψ^1, ψ^2 with OPE

$$\psi^i(z)\psi^j(0) \sim \frac{\delta^{ij}}{z} \,.$$

We can define the complex fermion

$$\psi(z) = 2^{-1/2}(\psi^1(z) + i\psi^2(z))\,, \qquad \overline{\psi}(z) = 2^{-1/2}(\psi^1(z) - i\psi^2(z))\,.$$

Show the equivalence of operators in boson and fermion

$$:e^{i\varphi}: \cong \psi\,, \quad :e^{-i\varphi}: \cong \overline{\psi}\,, \quad i\partial\varphi \cong :\psi\overline{\psi}:\,, \quad T_\varphi \cong T_\psi\,,$$

by calculating the OPEs of operators in both theories and comparing. Note that the energy-momentum tensor of the complex fermion is

$$T_\psi := -\frac{1}{2}:\psi\partial\overline{\psi}: -\frac{1}{2}:\overline{\psi}\partial\psi: \,.$$