4d $\mathcal{N} = 1$ Supersymmetric Gauge Theories

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Abstract

In this thesis, we will first give a brief review of 4d $\mathcal{N} = 1$ supersymmetric gauge theories. Using superspace and superfield technique, we will construct the actions for supersymmetric gauge theories. As a theory flows to IR, we want to investigate the low energy effective action and IR dynamics. For this purpose, we introduce the non-renormalization theorems which would help us determine the effective action. We will explain non-renormalization theorems for chiral superfields and vector superfields, respectively. In order to discuss non-renormalization theorems for vector superfields, we also discuss anomalies in quantum field theory. Finally, we shall study Seiberg duality which shows that two different gauge theories will lead to the same nontrivial long distance physics.

Keywords Supersymmetry, Gauge Theories, Non-renormalization Theorems, Anomalies, Seiberg Duality

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Chapter 1

Introduction

In physics, naturalness means that the ratio between free parameters in the theory should take value of order one and the free parameters are not fine-tuned. In the context of quantum field theory, if we want a small renormalized value without a symmetry, the bare value has to be fine-tuned. The apparent violation of this naturalness principle in the value of the Higgs mass is known as the gauge hierarchy problem. One of the main motivation to study low energy dynamics of supersymmetric theories is that it could possibly provide a solution to this problem.[7]

Supersymmetry imposes a symmetry between fermions and bosons. The basic idea of supersymmetry originates from the Coleman-Mandula theorem [13] which states that the possible symmetries in a quantum field theory in four dimensions consist of the Poicaré group and internal symmetries. However, these symmetries take bosons into bosons and take fermions into fermions. Thus one can consider supersymmetry as a nontrivial extension by allowing both commuting and anticommuting generators. Though not proven by experiments yet, supersymmetric theories can be important models for particle physicists. Because the radiative corrections would be less important in supersymmetric theories, due to cancellation between fermion loops and boson loops. [8]

Moreover, supersymmetry gives strong constraints to the IR dynamics and supersymmetric theories provide remarkable insight to dynamics of quantum field theories as toy models. Suppose that we want to understand the IR physics, given an asymptotically free gauge theory. However it is difficult because the IR theory is usually strongly-coupled. For example, in quantum chromodynamics (QCD), the deep IR theory is the free theory of pions and glueballs. The UV theory is the theory of quarks and gluons. Around the dynamic scale of QCD, the theory is very complicated. The strategy here is that first we make guess for long distance fields and symmetries, then write down all possible effective actions (EA), which must be consistent with symmetries and selection rules. Finally, we check the original guess by using predictions from effective actions. This strategy works very well in supersymmetric theories because of holomorphy and non-renormalization theorems due to supersymmetries.

The structure of the thesis is as follows. In Chapter 2, we show how to derive effective actions in non-supersymmetric case. Then, we want to generalize it to supersymmetric gauge theories. Therefore, in Chapter 3, we introduce the superspace and superfields technique. We will give the detailed expression of chiral superfields, vector superfields and supersymmetric field strength. Also we give the supersymmetric extension of a gauge transformation. Then, in Chapter 4, we construct supersymmetric effective action which is invariant under supersymmetric transformations and gauge transformations. We construct it from both chiral superfields and vector superfields. Now we can use non-renormalization theorems to determine the supersymmetric effective actions. And before we consider non-renormalization theorems for vector superfields, we may turn to anomalies in quantum field theory which may help us to understand non-renormalization theorems better. Finally, we will give a brief review of Seiberg duality.

Writing this thesis, I refer to Argyres' lecture note [16] to a great extent. Part of my thesis is also based on my supervisor Satoshi's lecture.

Chapter 2

Effective Actions

A LEEA or Wilsonian action is of the form

$$S^{(\mu)} = \int_{\frac{1}{\mu}} d^4x \mathcal{L}_\mu(\phi). \tag{2.1}$$

It is a local action describing physics at energy scale $E < \mu$. It is obtained by averaging over (integrating out) the short distance fluctuations of the theory down to the scale μ. We assume that the effective action will be local on length scales $x \geqslant 1/\mu$ and it is unitary for processes involving energies $E < \mu$. For processes at energies near μ, classical (tree level) couplings in S_μ describe effective couplings and masses, which are not renormalized by loops since short distance degrees of freedom already integrated out. For E much less than μ, physical processes will receive quantum corrections due to fluctuations of the modes of the fields in the effective action with energies between E and μ. These corrections can be absorbed in the couplings to define a new effective action at the lower scale E. This change in the effective action is the familiar renormalization group (RG) running of the couplings.

We write the effective action as a sum of local operators $\mathcal{O}_i$ with couplings g_i

$$S^{(\mu)} = \int d^4x \sum_i g_i(\mu)\mathcal{O}_i. \tag{2.2}$$

Renormalization group equation is given by

$$\mu\frac{\partial g_i}{\partial \mu} = \beta_i(g_k, \mu). \tag{2.3}$$

When the right hand side vanishes, which is at a fixed point of the flow, such an effective theory does not change with scale. If we linearize the renormalization group flow around a fixed point, the dynamics of operators can be classified according to the sign of β-functions. Operators with negative sign are called irrelevant operators while operators with positive and zero are called relevant and marginal operators, respectively. The couplings of irrelevant operators are damped along the flow, so irrelevant operators become less important in the IR.

If the fixed point is a free theory, then by dimensional analysis we can determine the eigenvalues of the operators. The action for a free theory is given by

$$S_{\text{free}} = \int_{\frac{1}{\mu_0}} d^4x[(\partial\phi)^2 + (\bar{\psi}\partial\!\!\!/\psi) + F^{\mu\nu}F_{\mu\nu}]. \tag{2.4}$$

These fields should scale in such a way that S_{free} is independent of scale μ_0. If we scale all energies and momenta by a factor μ/μ_0 to lower the cutoff scale μ_0 to μ, then length dx scales by μ_0/μ, derivative ∂_x scales by μ/μ_0. Then, in order to keep S_{free} invariant, we have to scale scalar, spinor and vector fields by

$$\phi \to \left(\frac{\mu}{\mu_0}\right)\phi, \quad \psi \to \left(\frac{\mu}{\mu_0}\right)^{\frac{3}{2}}\psi, \quad V_\mu \to \left(\frac{\mu}{\mu_0}\right)V_\mu. \tag{2.5}$$

In general, operator $\mathcal{O}_i$ that scales as

$$\mathcal{O}_i \to \left(\frac{\mu}{\mu_0}\right)^{\Delta_i}\mathcal{O}_i, \tag{2.6}$$

has a scaling dimension Δ_i. An action with the operator $\mathcal{O}_i$ scales as

$$\int d^4x\mathcal{O}_i \to \left(\frac{\mu}{\mu_0}\right)^{\Delta_i-4}\int d^4x\mathcal{O}_i. \tag{2.7}$$

Therefore, if $\Delta_i > 4$, $\mathcal{O}_i$ is irrelevant and it is less important at IR. If $\Delta_i = 4$, $\mathcal{O}_i$ is marginal. If $\Delta_i < 4$, $\mathcal{O}_i$ is relevant and it is more important at IR. For a given local field content, there are only finitely many relevant and marginal local operators. This is why finding effective action is not a hopeless task.

Although in the marginal case the classical scaling shows that the operators do not scale at all, one must consider the quantum corrections modifying the classical scaling. To see how it works, let us consider the renormalization of a scalar theory whose effective action can be written as

$$S^{(\mu_0)} = S_{\text{free}} + \int_{\frac{1}{\mu_0}} d^4x(\mu_0)^{\Delta_i-4}\lambda_i(\mu_0)\mathcal{O}_i \tag{2.8}$$

where λ_i is a dimensionless coupling constant. Upon lowering the scale to μ we get

$$S^{(\mu)} = S_{\text{free}} + \int_{\frac{1}{\mu}} d^4x\mu^{\Delta_i-4}\lambda_i(\mu)\mathcal{O}_i \tag{2.9}$$

where the couplings $\lambda_i(\mu)$ are written as functions of the scale since they get correction coming from loops of virtual particles with energies in the range $\mu_0 > E > \mu$. In the marginal case, another coupling constant $\tilde{\lambda}_i$ is denoted by

$$\tilde{\lambda}_i = \mu^{\Delta_i-4}\lambda_i(\mu). \tag{2.10}$$

The β-function is now given by

$$\mu\frac{d\tilde{\lambda}_i}{d\mu} = (\Delta_i - 4)\tilde{\lambda}_i + \beta_i(\tilde{\lambda}_j) \tag{2.11}$$

where the first term on the right hand side is the classical scaling, while the second term is quantum correction from fluctuations $\mu_0 > E > \mu$. This β-function vanishes for vanishing $\tilde{\lambda}_j$. For example, for the marginal interaction $\mathcal{O} = \phi^4$, a one-loop calculation gives

$$\mu\frac{d\lambda}{d\mu} = +\lambda^2. \tag{2.12}$$

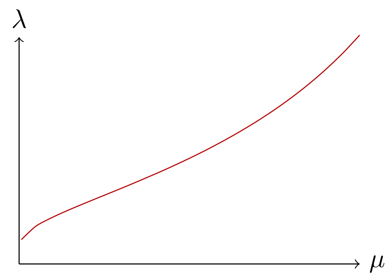


Figure 2.1: running of coupling λ

After some calculation, we can get the solution

$$\lambda(\mu) = \frac{\lambda(\mu_0)}{1 + \ln(\mu_0/\mu)}. \tag{2.13}$$

From this result, we can see that $\lambda(\mu)$ decreases as $\mu \to 0$. Thus we can see that the ϕ^4 interaction term is actually irrelevant, and that the ϕ^4 theory is IR free.

Among the interaction operators $\mathcal{O}_i$ can be terms proportional to the kinetic and mass terms of the fixed point action itself. Corrections to the mass term are relevant and hence will grow at IR. Consequently, as we flow to IR we will reach a point where the effective mass of ϕ is larger than the energy scale μ. But the response of ϕ to sources on energy scales below its mass are suppressed, and hence, decouples from the low energy physics. Essentially, its mass dominates its kinetic term and fixes ϕ to be a constant at distance greater than its inverse mass. Thus, ϕ acts like a constant in the low energy effective action. Therefore, we can"integrating out" all the ϕ degrees of freedom by dropping its kinetic term.

Corrections to the kinetic term will lead to wave function renormalization. By including the mass and kinetic term corrections, We can write the effective action as

$$S^{(\mu)} = \int d^4x \{ Z(\mu)(\partial\phi)^2 + \mu^2 m^2(\mu)\phi^2 + \lambda(\mu)\phi^4 + ... \}. \tag{2.14}$$

Notice that $m(\mu)$ is a dimensionless parameter. Z is the renormalization of the wave function and can be absorbed by redefinition of fields

$$\phi \to \phi_{C.N.} \equiv \sqrt{Z}\phi \tag{2.15}$$

where C.N. stands for canonically normalized. Therefore, we can write the action as

$$S^{(\mu)} = \int d^4x \{ (\partial\phi_{C.N.})^2 + \mu^2 m^2_{C.N.}(\mu)\phi^2_{C.N.} + \lambda_{C.N.}(\mu)\phi^4_{C.N.} + ... \}. \tag{2.16}$$

Thus, we get the renormalization of the dimensionless parameter m and coupling λ as

$$m_{C.N.}(\mu) = \frac{m(\mu)}{\sqrt{Z(\mu)}}, \quad \lambda_{C.N.}(\mu) = \frac{\lambda(\mu)}{Z^2(\mu)}. \tag{2.17}$$

The effective mass at the scale μ is $\mu m_{C.N.}(\mu)$. The physical mass should be the limit of the effective mass at arbitrarily long distances $1/\mu \to \infty$:

$$m_{\text{phys}} = \lim_{\mu \to 0} \mu m_{C.N.}(\mu) = \lim_{\mu \to 0} \frac{\mu m(\mu)}{\sqrt{Z(\mu)}}. \tag{2.18}$$

As the theory flows to IR, eventually it reaches a point where

$$m_{\text{phys}} = m_{\text{eff}} = \mu m_{C.N.}(\mu) > \mu. \tag{2.19}$$

We can conclude that mass term dominates kinetic term and fixes ϕ to be constant. Thus, for $\mu < m_{\text{eff}}$, ϕ can be treated as a constant. We can drop kinetic terms by "integrating out" ϕ. On the other hand, massless fields remain to be degrees of freedom at any μ.

2.1 Types of IR fixed points

There are three types of IR fixed points:

- Trivial theories in which all fields are massive. Therefore for $\mu < m_{\text{phys}}$, there are no propagating degrees of freedom.
- Free theories in which all massless fields are non-interacting in far IR (e.g. $\lambda\phi^4$ theory, QED).
- Interacting theories of massless degrees of freedom which are usually considered as conformal field theories.

In four dimensions, we do not have good description of interacting conformal field theories because these theories do not have Lagrangian descriptions in general. Therefore, we must limit ourselves to trivial or free theories in the IR. Fortunately, Coleman and Gross [12] found that for small enough couplings any theory of scalars, spinors and $U(1)$ vector fields in four dimensions is IR free, which gives a large class of IR free theories. Other known IR free theories are, for example, non-Abelian gauge theory with sufficiently many massless flavors.

2.2 Complex Coupling τ

We take the field content of IR effective action to be a collection of scalars ϕ^n, complex Weyl spinors ψ^a_α, and $U(1)$ vector fields A^A_μ. Here n is just an index. General form of leading (relevant) IR effective action is given by

$$\begin{aligned} S_{\text{eff}} = \int d^4x\{&\frac{1}{g}g_{nm}(\phi)(D_\mu\phi^n)^*D^\mu\phi^m + h_{ab}(\phi)\bar{\psi}^a \not{D}\psi^b + V(\phi) \\ &-\frac{1}{4(g^2)_{AB}(\phi)}F^A_{\mu\nu}F^{B\mu\nu} + \frac{\theta_{AB}(\phi)}{32\pi^2}F^A_{\mu\nu}\widetilde{F}^{B\mu\nu} + [y_{ab}(\phi)\psi^a\psi^b + c.c.]\} \end{aligned} \tag{2.20}$$

where

$$D_\mu\chi = (\partial_\mu + q_A A^A_\mu)\chi, \quad F^A_{\mu\nu} = \partial_\mu A^A_\nu - \partial_\nu A^A_\mu, \quad \widetilde{F}^A_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{A\rho\sigma}. \tag{2.21}$$

The equation (2.20) is called the gauged sigma model. Since this theory is free in the IR, no interesting dynamics involving the spinor fields occur. Thus, we can exclude the spinors from the action as

$$S_{\text{eff}} = \int d^4x\{\frac{1}{g}g_{nm}(\phi)(D_\mu\phi^n)^*D^\mu\phi^m + V(\phi) - \frac{1}{4(g^2)_{AB}(\phi)}F^A_{\mu\nu}F^{B\mu\nu} + \frac{\theta_{AB}(\phi)}{32\pi^2}F^A_{\mu\nu}\widetilde{F}^{B\mu\nu}\}. \tag{2.22}$$

We can define the complex couplings as

$$\tau_{AB}(\phi) = \frac{\theta_{AB}}{2\pi} + i\frac{4\pi}{(g^2)_{AB}}. \tag{2.23}$$

It is a complex function of ϕ^n symmetric in A and B, and whose imaginary part is positive definite. The imaginary part of τ_{AB} is a matrix of couplings and the real part is that of theta angles. Then, gauge kinetic terms can be written as

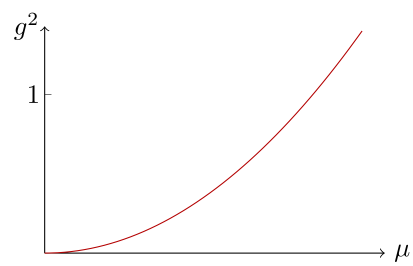
$$\frac{i\tau_{AB}}{16\pi}(\mathcal{F}^A\mathcal{F}^B) + c.c. \tag{2.24}$$

where

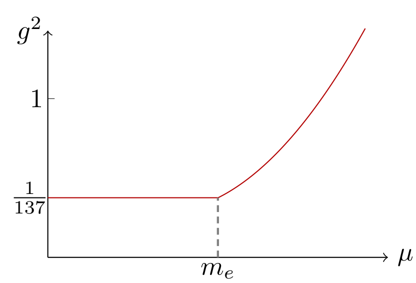
$$\mathcal{F}^{\mu\nu} = \frac{1}{2}F^{\mu\nu} - \frac{i}{2}\widetilde{F}^{\mu\nu}. \tag{2.25}$$

Since the fields are free in IR, we have to figure out what is the meaning of gauge coupling τ_{AB}. There are two kinds of vacua to consider:

- Charged field ϕ or ψ is massless. In this case the one-loop running of $U(1)$ coupling implies it is IR free theory.



- All charged fields are massive. $U(1)$ coupling stops running at energy scales below the mass of the lightest charged particle.



In this case, τ measures strength of coupling only to massive (classical) sources.

The theta angles are coefficients of the topological terms which count the instanton number of a given field configuration.

$$I_\theta = -\frac{\theta}{8\pi^2}\int \mathrm{Tr}F \wedge F \tag{2.26}$$

As long as we are on $\mathbb{R}^4$ or any four-manifold M with $H^2(M, \mathbb{Z}) = 0$, there is an elementary symmetry $\tau \to \tau + 1$, which expresses the fact that $(1/8\pi^2)\int \mathrm{Tr}F \wedge F$ is integer-valued, and that in quantum mechanics one only cares about the action modulo an integer multiple of 2π. Equivalently, θ is an angular variable, with $\theta \simeq \theta + 2\pi$.[1]

For $U(1)$ gauge group in four-dimensional space-time, there are no non-trivial instanton field configurations, and thus no physics can depend on the θ for $U(1)$ theories. However, this is not true for IR effective actions. Since the theta angles are couplings to massive sources not described by the IR effective action. Therefore, the spacetime manifold on which IR effective action is defined should be $\mathbb{R}^4$ with world lines of massive sources removed. On such manifolds there can be non-trivial $U(1)$ bundles. One of the examples is a non-Abelian gauge theory Higgsed down to $U(1)$ factors admitting magnetic monopole solutions, so there are both electrically and magnetically charged sources in the $U(1)$ IR effective action.

Finally, let us consider electric-magnetic duality. If we make field redefinition in $U(1)$ theories and the coupling τ transforms as

$$\tau \to \frac{A\tau + B}{C\tau + D} \tag{2.27}$$

where the matrix $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in SL(2, \mathbb{Z})$. This plays an important role in Seiberg-Witten theory in 4d $\mathcal{N} = 2$ theories and will not be discussed precisely in this note.

Chapter 3

Superspace and Superfields

Fields which form representations of 4d $\mathcal{N} = 1$ supersymmetry algebra are most conveniently handled in superspace language. Salam and Strathdee [2] introduced this clever notational device for working with $\mathcal{N} = 1$ supersymmetric theories that greatly simplifies manipulations of the many fields involved. All the fields in a supermultiplet are assembled in one "superfield" which is thought of as living in a "superspace" which has anticommuting coordinates in addition to the usual spacetime coordinates. In superspace notation one introduces anticommuting spinors: θ_α, $\bar{\theta}_{\dot{\alpha}}$ where α and $\dot{\alpha}$ take value in 1 or 2. Thus, except for the four spacetime coordinates, we now have four extra Grassmann coordinates. The elements of superspace are labeled by $(x, \theta, \bar{\theta})$.

$\mathcal{Q}_\alpha$, $\overline{\mathcal{Q}}_{\dot{\alpha}}$ can be expressed by differential operators in supercoordinate

$$\mathcal{Q}_\alpha = \frac{\partial}{\partial\theta^\alpha} - i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\frac{\partial}{\partial x^\mu}, \tag{3.1}$$

$$\overline{\mathcal{Q}}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\frac{\partial}{\partial x^\mu}. \tag{3.2}$$

They obey anti-commutation relations

$$\{\mathcal{Q}_\alpha, \overline{\mathcal{Q}}_{\dot{\beta}}\} = 2i\sigma^m_{\alpha\dot{\beta}}\partial_m, \tag{3.3}$$

$$\{\mathcal{Q}_\alpha, \mathcal{Q}_\beta\} = \{\overline{\mathcal{Q}}_{\dot{\alpha}}, \overline{\mathcal{Q}}_{\dot{\beta}}\} = 0. \tag{3.4}$$

We define superderivative as

$$\mathcal{D}_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\frac{\partial}{\partial x^\mu}, \tag{3.5}$$

$$\overline{\mathcal{D}}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\frac{\partial}{\partial x^\mu}, \tag{3.6}$$

which obey

$$\{\mathcal{D}_\alpha, \overline{\mathcal{D}}_{\dot{\beta}}\} = -2i\sigma^m_{\alpha\dot{\beta}}\partial_m, \tag{3.7}$$

$$\{\mathcal{D}_\alpha, \mathcal{D}_\beta\} = \{\overline{\mathcal{D}}_{\dot{\alpha}}, \overline{\mathcal{D}}_{\dot{\beta}}\} = 0. \tag{3.8}$$

Also, one can check that $\mathcal{D}$ and $\mathcal{Q}$ anticommute each other

$$\{\mathcal{D}_\alpha, \mathcal{Q}_\beta\} = \{\mathcal{D}_\alpha, \overline{\mathcal{Q}}_{\dot{\beta}}\} = 0, \tag{3.9}$$

$$\{\overline{\mathcal{D}}_{\dot{\alpha}}, \mathcal{Q}_\beta\} = \{\overline{\mathcal{D}}_{\dot{\alpha}}, \overline{\mathcal{Q}}_{\dot{\beta}}\} = 0. \tag{3.10}$$

3.1 General Superfield

A general superfield $\Phi(x, \theta, \bar{\theta})$ is a function of the supercoordinates

$$\begin{aligned}\Phi(x,\theta,\bar{\theta}) =& C(x) + \theta\xi(x) + \bar{\theta}\bar{\chi}(x) + \theta\theta M(x) + \bar{\theta}\bar{\theta}N(x) + \theta\sigma^{\mu}\bar{\theta}V_{\mu}(x) \\ &+ (\theta\theta)\bar{\theta}\bar{\lambda}(x) + (\bar{\theta}\bar{\theta})\theta\psi(x) + (\theta\theta)(\bar{\theta}\bar{\theta})D(x).\end{aligned} \tag{3.11}$$

All higher powers of θ, $\bar{\theta}$ vanish. The supersymmetric transformation law for superfields is defined as follows:

$$\delta_{\epsilon}\Phi(x,\theta,\bar{\theta}) \equiv (\epsilon\mathcal{Q} + \bar{\epsilon}\overline{\mathcal{Q}})\Phi. \tag{3.12}$$

We can see that a general superfield has many component fields and gives the reducible representation of the supersymmetry algebra. In order to get the irreducible representations, we need to impose some covariant constraints on the superfield. Different constraints generate different kinds of superfields.

3.2 Chiral Superfield

A superfield $\Phi(x, \theta, \bar{\theta})$ which satisfies $\overline{\mathcal{D}}_{\dot{\alpha}}\Phi = 0$ is called a left-handed chiral superfield, and a superfield $\Phi^{\dagger}(x, \theta, \bar{\theta})$ which satisfies $\mathcal{D}_{\alpha}\Phi^{\dagger} = 0$, is called a right-handed chiral superfield.

We first consider left-handed chiral superfields which satisfy

$$\overline{\mathcal{D}}_{\dot{\alpha}}\Phi = 0 \tag{3.13}$$

This constraint is easy to solve in terms of $y^m = x^m + i\theta\sigma^m\bar{\theta}$ and θ, for

$$\overline{\mathcal{D}}_{\dot{\alpha}}(x^m + i\theta\sigma^m\bar{\theta}) = 0, \quad \overline{\mathcal{D}}_{\dot{\alpha}}\theta = 0. \tag{3.14}$$

Any function of these variables would satisfy (3.13), therefore we can write the left-handed chiral superfield as

$$\begin{aligned}\Phi =& \phi(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y) \\ =& \phi(x + i\theta\sigma\bar{\theta}) + \sqrt{2}\theta\psi(x + i\theta\sigma\bar{\theta}) + \theta\theta F(x + i\theta\sigma\bar{\theta}) \\ =& \phi(x) + i\theta\sigma^m\bar{\theta}\partial_m\phi(x) - \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\Box\phi(x) + \sqrt{2}\theta\psi(x) - \frac{i}{\sqrt{2}}\theta\theta\partial_m\psi(x)\sigma^m\bar{\theta} + \theta\theta F(x).\end{aligned} \tag{3.15}$$

It only depends on the complex scalar fields $\phi(x)$, $F(x)$ and the left-handed Weyl spinor $\psi(x)$.

The superfield $\Phi^{\dagger}(x, \theta, \bar{\theta})$ satisfies the constraint

$$\mathcal{D}_{\alpha}\Phi^{\dagger}(x,\theta,\bar{\theta}) = 0 \tag{3.16}$$

and is called a right-handed chiral superfield. This constraint is easy to solve in terms of $z^m = x^m - i\theta\sigma^m\bar{\theta}$ and $\bar{\theta}$, for

$$\mathcal{D}_{\alpha}(x^m - i\theta\sigma^m\bar{\theta}) = 0, \quad \mathcal{D}_{\alpha}\bar{\theta} = 0 \tag{3.17}$$

Similarly, we can write the right-handed chiral superfield as

$$\Phi^{\dagger} = \phi^*(z) + \sqrt{2}\bar{\theta}\bar{\psi}(z) + (\bar{\theta}\bar{\theta})F^*(z). \tag{3.18}$$

If we also expand this result as a function of x, it can be verified that $\Phi^{\dagger}$ is the Hermitian conjugate of the left-handed chiral superfield Φ.

Also one can check that the highest component of a chiral superfield transforms into a total spacetime derivative under supersymmetric transformations.

3.3 Vector Superfield

Vector superfield satisfies the reality condition

$$V(x,\theta,\bar{\theta}) = V^{\dagger}(x,\theta,\bar{\theta}). \tag{3.19}$$

In order to satisfy the reality condition (3.19), a vector superfield should have the general expansion

$$\begin{aligned} V(x,\theta,\bar{\theta}) =& C(x) + \theta\xi(x) + \bar{\theta}\bar{\xi}(x) + (\theta\theta)M(x) + (\bar{\theta}\bar{\theta})M^*(x) + (\theta\sigma^{\mu}\bar{\theta})V_{\mu}(x) \\ &+ (\theta\theta)\bar{\theta}\bar{\lambda}(x) + (\bar{\theta}\bar{\theta})\theta\lambda(x) + (\theta\theta)(\bar{\theta}\bar{\theta})D(x) \end{aligned} \tag{3.20}$$

where $C(x)$, $V_{\mu}(x)$ and $D(x)$ are real fields. $M(x)$, $D(x)$ and $C(x)$ are scalar fields. $\lambda(x)$, $\phi(x)$ are spinor fields. $V_{\mu}(x)$ is a vector field. The vector field $V_{\mu}(x)$ lends its name to the entire multiplet $V(x,\theta,\bar{\theta})$.

A particular example of a vector superfield is the product of a right-handed chiral superfield and a left-chiral superfield, $\Phi^{\dagger}\Phi$, since in this case

$$(\Phi^{\dagger}\Phi)^{\dagger} = \Phi^{\dagger}(\Phi^{\dagger})^{\dagger} = \Phi^{\dagger}\Phi, \tag{3.21}$$

and the reality condition is satisfied. Another important example of a vector superfield is the sum of a left-handed chiral superfield and a right-handed chiral superfield, since

$$(\Phi + \Phi^{\dagger})^{\dagger} = \Phi^{\dagger} + \Phi = \Phi + \Phi^{\dagger} \tag{3.22}$$

and again the reality condition is satisfied.

Expressing $\Phi + \Phi^{\dagger}$ in terms of component fields, we have

$$\begin{aligned} \Phi + \Phi^{\dagger} =& \phi(x) + \phi^*(x) + \sqrt{2}\theta\psi(x) + \sqrt{2}\bar{\theta}\bar{\psi}(x) + (\theta\theta)F(x) + (\bar{\theta}\bar{\theta})F^*(x) \\ &+ i(\theta\sigma^{\mu}\bar{\theta})\partial_{\mu}[\phi(x) - \phi^*(x)] - \frac{i}{\sqrt{2}}(\theta\theta)\bar{\theta}\bar{\sigma}^{\mu}\partial_{\mu}\psi(x) - \frac{i}{\sqrt{2}}(\bar{\theta}\bar{\theta})\theta\sigma^{\mu}\partial_{\mu}\bar{\psi}(x) \\ &- \frac{1}{4}(\theta\theta)(\bar{\theta}\bar{\theta})\Box[\phi(x) + \phi^*(x)]. \end{aligned} \tag{3.23}$$

In this combination of chiral superfields, the coefficient of $\theta\sigma^{\mu}\bar{\theta}$ is the gradient $i\partial_{\mu}[\phi(x) - \phi^*(x)]$. This result motivates us to define the supersymmetric generalization of a gauge transformation as

$$\begin{aligned} V(x,\theta,\bar{\theta}) \to& V'(x,\theta,\bar{\theta}) \\ &= V(x,\theta,\bar{\theta}) + \Phi(x,\theta,\bar{\theta}) + \Phi^{\dagger}(x,\theta,\bar{\theta}) \\ &\equiv V(x,\theta,\bar{\theta}) + i[\Lambda(x,\theta,\bar{\theta}) - \Lambda^{\dagger}(x,\theta,\bar{\theta})] \end{aligned} \tag{3.24}$$

so that under this transformation the vector field V_{μ} of a general vector superfield transforms as

$$V_{\mu} \to V_{\mu} + i\partial_{\mu}[\phi(x) - \phi^*(x)] \tag{3.25}$$

which corresponds to an Abelian gauge transformation.

We now consider a special choice of V so that certain components of V are invariant under the supersymmetric generalization of a gauge transformation.

$$\begin{aligned} V(x,\theta,\bar{\theta}) =& C(x) + \theta\xi(x) + \bar{\theta}\bar{\xi}(x) + (\theta\theta)M(x) + (\bar{\theta}\bar{\theta})M^*(x) + (\theta\sigma^{\mu}\bar{\theta})V_{\mu}(x) \\ &+ (\theta\theta)\bar{\theta}[\bar{\lambda}(x) - \frac{i}{2}\bar{\sigma}^{\mu}\partial_{\mu}\xi(x)] + (\bar{\theta}\bar{\theta})\theta[\lambda(x) - \frac{i}{2}\sigma^{\mu}\partial_{\mu}\bar{\xi}(x)] \\ &+ (\theta\theta)(\bar{\theta}\bar{\theta})[D(x) - \frac{1}{4}\Box C(x)]. \end{aligned} \tag{3.26}$$

It can be verified that the λ and D component fields are invariant under the supersymmetric generalization of a gauge transformation, which is an important property when constructing supersymmetric Lagrangians. In this chapter, we only consider the $U(1)$ gauge group while in the next chapter we may generalize it to the non-Abelian case. Also it can be shown that the component fields C, ξ, M can be shifted independently, therefore we take a special gauge called Wess-Zumino gauge in which C, ξ, M are all zero. Although supersymmetry breaks when fixing this gauge, the vector filed still transforms as $V_\mu \to V_\mu + i\partial_\mu[\phi(x) - \phi^*(x)]$. We can easily write down the vector superfield in this gauge

$$V = (\theta\sigma^\mu\bar{\theta})V_\mu(x) + (\theta\theta)\bar{\theta}\bar{\lambda}(x) + (\bar{\theta}\bar{\theta})\theta\lambda(x) + (\theta\theta)(\bar{\theta}\bar{\theta})D(x). \tag{3.27}$$

Consequently,

$$V^2 = \frac{1}{2}(\theta\theta)(\bar{\theta}\bar{\theta})V^\mu V_\mu, \quad V^3 = 0. \tag{3.28}$$

We may consider V as the supersymmetric generalization of the Yang-Mills potential.

3.4 Supersymmetric Field Strength

Notice that λ and $\bar{\lambda}$ are the lowest-dimensional gauge invariant component fields in V. They are also the lowest-dimensional component fields in

$$\mathcal{W}_\alpha = -\frac{1}{4}(\overline{\mathcal{D}}\overline{\mathcal{D}})\mathcal{D}_\alpha V, \quad \overline{\mathcal{W}}_{\dot{\alpha}} = -\frac{1}{4}(\mathcal{D}\mathcal{D})\overline{\mathcal{D}}_{\dot{\alpha}} V. \tag{3.29}$$

They are called field strength chiral superfield since it can be shown that the field $\mathcal{W}_\alpha$ is a left-handed chiral superfield, and $\overline{\mathcal{W}}_{\dot{\alpha}}$ is a right-handed chiral superfield. Moreover, both superfields are invariant under the supersymmetric gauge transformation for

$$\mathcal{W}_\alpha \to -\frac{1}{4}(\overline{\mathcal{D}}\overline{\mathcal{D}})\mathcal{D}_\alpha(V + \Phi + \Phi^\dagger) = \mathcal{W}_\alpha - \frac{1}{4}\overline{\mathcal{D}}\{\overline{\mathcal{D}}, \mathcal{D}_\alpha\}\Phi = \mathcal{W}_\alpha \tag{3.30}$$

where we have used the property that $\overline{\mathcal{D}}\Phi = \mathcal{D}\Phi^\dagger = 0$.

After some calculation, we can get the component expansion of $\mathcal{W}_\alpha$ and $\overline{\mathcal{W}}_{\dot{\alpha}}$ in Wess-Zumino gauge:

$$\mathcal{W}_\alpha = \lambda_\alpha(y) + 2D(y)\theta_\alpha + (\sigma^{\mu\nu}\theta)_\alpha F_{\mu\nu}(y) - i(\theta\theta)\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu\bar{\lambda}^{\dot{\alpha}}(y), \tag{3.31}$$

$$\overline{\mathcal{W}}_{\dot{\alpha}} = \bar{\lambda}_{\dot{\alpha}}(z) + 2D(z)\bar{\theta}_{\dot{\alpha}} - \epsilon_{\dot{\alpha}\dot{\beta}}(\bar{\sigma}^{\mu\nu}\bar{\theta})^{\dot{\beta}}F_{\mu\nu}(z) + i(\bar{\theta}\bar{\theta})(\partial_\mu\lambda(z)\sigma^\mu)_{\dot{\alpha}}. \tag{3.32}$$

where the field strength tensor is defined as

$$F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu. \tag{3.33}$$

Like λ and D, $F_{\mu\nu}$ is also gauge invariant.

The term $\mathcal{W}^\alpha\mathcal{W}_\alpha$ is a function of y and θ and thus it is chiral. Its $\theta\theta$ component is gauge invariant as we will see it is a function of gauge invariant λ, D and $F_{\mu\nu}$. Also as the highest component of a chiral superfield, it transforms into a total spacetime derivative under supersymmetric transformations. Thus we can use these properties to construct the supersymmetric action. Also its $\theta\theta$ component is invariant under the shift of coordinate $y^\mu \to x^\mu$, so it reads

$$\int d^2\theta\mathcal{W}^\alpha\mathcal{W}_\alpha = 4D^2(x) - 2i\lambda(x)\sigma^\mu\partial_\mu\bar{\lambda}(x) - \frac{1}{2}F_{\mu\nu}(x)F^{\mu\nu}(x) + \frac{i}{2}F_{\mu\nu}(x)\widetilde{F}^{\mu\nu}(x) \tag{3.34}$$

where $\widetilde{F}$ is the dual field strength. The $d^2\theta$ here is to pick out the $\theta\theta$ component which we will discuss precisely in Grassmann integral in the next chapter.

Similarly, the $\bar{\theta}\bar{\theta}$ component of $\overline{\mathcal{W}}_{\dot{\alpha}}\overline{\mathcal{W}}^{\dot{\alpha}}$ reads

$$\int d^2\bar{\theta}\overline{\mathcal{W}}_{\dot{\alpha}}\overline{\mathcal{W}}^{\dot{\alpha}} = 4D^2(x) + 2i\partial_\mu\lambda(x)\sigma^\mu\bar{\lambda}(x) - \frac{1}{2}F_{\mu\nu}(x)F^{\mu\nu}(x) - \frac{i}{2}F_{\mu\nu}(x)\widetilde{F}^{\mu\nu}(x). \tag{3.35}$$

We can construct the supersymmetric and gauge invariant action with the couplings implicit as

$$S = \int d^4x \int d^4\theta\{\mathcal{W}^\alpha\mathcal{W}_\alpha(\bar{\theta}\bar{\theta}) + \overline{\mathcal{W}}_{\dot{\alpha}}\overline{\mathcal{W}}^{\dot{\alpha}}(\theta\theta)\} = \int d^4x\{\int d^2\theta\mathcal{W}^\alpha\mathcal{W}_\alpha + c.c.\} \tag{3.36}$$

which we will discuss precisely in the next chapter.

Chapter 4

Supersymmetric Effective Actions

With the scheme of superspace, we now generalize our discussion of effective actions in chapter 2 to the supersymmetric case. The first step is to construct the supersymmetric actions which are invariant under the supersymmetric transformations and gauge transformations. Since the super-coordinates θ, $\bar{\theta}$ are Grassmann variables, we will encounter the Grassmann integration inevitably.

4.1 Grassmann Integration

We clarify the Grassmann integration by setting

$$\int da 1 = 0, \quad \int da a = 1, \tag{4.1}$$

where a denotes a Grassmann variable. Any function of a can be written as

$$f(a) = f(0) + f^{(1)} a \tag{4.2}$$

since $a^2 = 0$. Thus,

$$\int da f(a) = f(0) \int da 1 + f^{(1)} \int da a = f^{(1)}. \tag{4.3}$$

Now we consider the Grassmann algebra G_2 generated by two elements θ_1 and θ_2. In order to define the integral

$$\int d\theta_1 d\theta_2 f(\theta_1, \theta_2),$$

first we demand that the $d\theta_A$'s also anticommute

$$\{d\theta_A, d\theta_B\} = \{d\theta_A, \theta_B\} = 0. \tag{4.4}$$

Thus we have

$$\begin{aligned} &\int d\theta_1 \int d\theta_2 1 = 0, \quad \int d\theta_1 \int d\theta_2 \theta_1 = 0, \\ &\int d\theta_1 \int d\theta_2 \theta_2 = 0, \quad \int d\theta_1 \int d\theta_2 \theta_1 \theta_2 = -1. \end{aligned} \tag{4.5}$$

The integral of an arbitrary function $f(\theta_1, \theta_2) \in G_2$ is then obtained by linearity

$$\int d\theta_1 \int d\theta_2 f(\theta_1, \theta_2) = \int d\theta_1 \int d\theta_2 [f^{(0)} + \theta_1 f^{(1)} + \theta_2 f^{(2)} + \theta_1\theta_2 f^{(3)}] = -f^{(3)}. \tag{4.6}$$

We see from this result that the integral corresponds to a projection such that the highest order component of the expansion is picked out.

It can be verified that the highest order component of a superfield transforms into a total derivative under supersymmetric transformations. A spacetime integral of such a quantity is, therefore, invariant under supersymmetric transformations. This fundamental property provides the criterion for constructing supersymmetric Lagrangian densities. A supersymmetric action integral can be described by

$$S = \int d^4x \int d^4\theta \mathcal{L} = \int d^4x \int d^2\theta d^2\bar{\theta} \mathcal{L}. \tag{4.7}$$

4.2 Construction of Lagrangians from Chiral Superfields

The most general supersymmetric and renormalizable Lagrangian constructed from chiral superfields is given by

$$\begin{aligned} \mathcal{L} =& \Phi_i^\dagger \Phi_i + (g_i \Phi_i + \frac{1}{2} m_i j \Phi_i \Phi_j + \frac{1}{3} \lambda_{ijk} \Phi_i \Phi_j \Phi_k)(\bar{\theta}\bar{\theta}) \\ &+ (g_i^* \Phi_i^\dagger + \frac{1}{2} m_{ij}^* \Phi_i^\dagger \Phi_j^\dagger + \frac{1}{3} \lambda_{ijk}^* \Phi_i^\dagger \Phi_j^\dagger \Phi_k^\dagger)(\theta\theta). \end{aligned} \tag{4.8}$$

To find a supersymmetric Lagrangian like this, we can use arguments from dimensional analysis.

The classical scaling of the superfields can be determined by dimension counting. If we assign scaling dimension -1 to x^μ, according to supersymmetry algebra, we must assign dimensions to superspace quantities as follows:

	dimension
x^μ, dx^μ	-1
$\partial/\partial x^\mu$	$+1$
θ	$-\frac{1}{2}$
$d\theta, \partial/\partial\theta$	$+\frac{1}{2}$

Thus, we can get that the classical scaling dimension of a chiral superfield Φ is $+1$. As a result, its propagating components, ϕ and ψ, have their usual dimensions of $+1$ and $+3/2$ respectively. This, therefore, leads to a unique choice for a free (quadratic) massless action

$$S_{\text{kin}} = \int d^4x d^4\theta \Phi_i^\dagger \Phi_i. \tag{4.9}$$

Also, mass terms can be written as

$$S_{\text{mass}} = \frac{1}{2} \int d^4x d^2\theta m_{ij} \Phi_i \Phi_j \tag{4.10}$$

where the dimension of m_{ij} is $+1$. Interaction terms can be written as

$$S_{\text{int}} = \frac{1}{3} \int d^4x d^2\theta \lambda_{ijk} \Phi_i \Phi_j \Phi_k \tag{4.11}$$

where the coupling λ_{ijk} is dimensionless.

4.3 Kähler Potential and Superpotential

As we have discussed in the last section, the most general supersymmetric and renormalizable Lagrangian density which involves only chiral superfields has a form

$$\mathcal{L} = \mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{pot}}, \tag{4.12}$$

where the kinetic term is

$$\mathcal{L}_{\text{kin}} = \Phi_i^\dagger \Phi_i, \tag{4.13}$$

and

$$\mathcal{L}_{\text{pot}} = (g_i\Phi_i + \frac{1}{2}m_{ij}\Phi_i\Phi_j + \frac{1}{3}\lambda_{ijk}\Phi_i\Phi_j\Phi_k)(\bar{\theta}\bar{\theta}) + h.c. \tag{4.14}$$

is the potential term.

The kinetic term $\Phi_i^\dagger\Phi_i$ is called Kähler potential. In general, when constructing Lagrangian from chiral superfields, there exists such Kähler potential, which is a real function of left-chiral superfields and corresponding right-chiral superfields with no derivatives. Next, we shall discuss the superpotential.

First, we consider the case of a theory with a single chiral superfield. Then the Lagrangian has the form

$$\mathcal{L} = \Phi^\dagger\Phi + [(g\Phi + \frac{1}{2}m\Phi^2 + \frac{1}{3}\lambda\Phi^3)(\bar{\theta}\bar{\theta}) + h.c.]. \tag{4.15}$$

Defining $W(\Phi)$ as

$$W(\Phi) \equiv g\Phi + \frac{1}{2}m\Phi^2 + \frac{1}{3}\lambda\Phi^3, \tag{4.16}$$

we can write the action in the form

$$S = \int d^4x \int d^4\theta\{\Phi^\dagger\Phi + [W(\Phi)(\bar{\theta}\bar{\theta}) + h.c.]\} = \int d^4x\{\int d^4\theta\Phi^\dagger\Phi + [\int d^2\theta W(\Phi) + c.c.]\}. \tag{4.17}$$

The functional $W(\Phi)$, which is a polynomial of the superfield Φ is called superpotential. The superpotential contains mass terms and interaction terms.

If we write the chiral superfield as

$$\Phi(y,\theta) = \phi(y) + \sqrt{2}\theta\psi(y) + (\theta\theta)F(y), \tag{4.18}$$

then, the Grassman integral can be rewritten as

$$\int d^4\theta W(\Phi)(\bar{\theta}\bar{\theta}) = \int d^2\theta W(\Phi) = \frac{\partial W}{\partial\phi}F - \frac{1}{2}\frac{\partial^2 W}{\partial\phi^2}\psi\psi \tag{4.19}$$

where

$$\frac{\partial W}{\partial\phi} \equiv \frac{\partial W(\phi)}{\partial\phi(x)}. \tag{4.20}$$

Here, $W(\phi)$ is the superpotential (4.16) with the superfield Φ replaced by its lowest component $\phi(x)$.

In general, if we have a bunch of chiral superfields Φ_i, $i = 1, 2, \cdots, n$. Then a superpotential is a function of these n superfields

$$W(\Phi) = W(\Phi_1, \Phi_2, \cdots, \Phi_n). \tag{4.21}$$

Then,

$$\int d^4\theta W(\Phi_1, \Phi_2, \cdots, \Phi_n)(\bar{\theta}\bar{\theta}) = \int d^2\theta W(\Phi_1, \Phi_2, \cdots, \Phi_n) = \sum_i W_i F_i - \frac{1}{2}\sum_{i,j} W_{ij}\psi_i\psi_j, \quad (4.22)$$

where

$$W_i = \frac{\partial W(\phi_1, \phi_2, \cdots, \phi_n)}{\partial \phi_i}, \quad (4.23)$$

$$W_{ij} = \frac{\partial^2 W(\phi_1, \phi_2, \cdots, \phi_n)}{\partial \phi_i \partial \phi_j}. \quad (4.24)$$

After some calculation and using equation of motion to eliminate the auxiliary fields F, the action integral becomes

$$\begin{aligned} S &= \int d^4x\{\int d^4\theta \Phi^\dagger \Phi + \int d^2\theta W(\Phi_1, \Phi_2, \cdots, \Phi_n) + \int d^2\bar{\theta}\overline{W}(\Phi_1^\dagger, \Phi_2^\dagger, \cdots, \Phi_n^\dagger)\} \\ &= \int d^4x\{i\partial_\mu \bar{\psi}_i(x)\sigma^\mu \psi_i(x) - \phi_i^*(x)\Box\phi_i(x) - \frac{1}{2}W_{ij}\psi_i(x)\psi_j(x) - \frac{1}{2}\overline{W}_{ij}\bar{\psi}_i(x)\bar{\psi}_j(x) \\ &\quad - |W_i|^2\}. \end{aligned} \quad (4.25)$$

A scalar potential term in the Lagrangian density, which does not contain any fermionic field, is related to the superpotential by

$$V(\phi, \phi^*) = |W_i|^2 = \sum_{i=1}^{n} \left|\frac{\partial W}{\partial \phi_i}\right|^2. \quad (4.26)$$

From the expression above, we observe that the scalar potential is always greater than or equal to zero. The energy of the ground state is minimized if the scalar potential is zero. Supersymmetry is unbroken if and only if the scalar potential is zero.

4.4 Construction of Lagrangians from Vector Superfields

As we have discussed before, we can construct the supersymmetric and gauge invariant action of a pure Abelian gauge theory as

$$S = \int d^4x \int d^4\theta\{\mathcal{W}^\alpha \mathcal{W}_\alpha(\bar{\theta}\bar{\theta}) + \overline{\mathcal{W}}_{\dot{\alpha}}\overline{\mathcal{W}}^{\dot{\alpha}}(\theta\theta)\}. \quad (4.27)$$

In terms of its component fields this integral becomes

$$S = \int d^4x[8D^2(x) - F_{\mu\nu}(x)F^{\mu\nu}(x) - 4i\lambda(x)\sigma^\mu\partial_\mu\bar{\lambda}(x)]. \quad (4.28)$$

This is an action for a supersymmetric Abelian gauge theory. It has the following properties:

- $D(x)$ is an auxiliary field and we can eliminated it with the help of equation of motion.
- Supersymmetry requires a massless fermionic partner $\lambda(x)$ of the massless gauge boson $V_\mu(x)$ which is called gaugino.

- The invariance of (4.28) under supersymmetric transformations is manifest since the integrand is the $\theta\theta$ or correspondingly $\bar{\theta}\bar{\theta}$ component of a chiral superfield which transforms into a spacetime derivative under supersymmetric transformations.

- The invariance of (4.28) under gauge transformations is manifest since the action is a functional of the fields $\lambda(x)$, $D(x)$, $F_{\mu\nu}(x)$ which are gauge invariant.

4.5 Gauge Invariant Interactions

The action we constructed from vector superfields is gauge invariant manifestly while the action we constructed from chiral superfields is not. But we can modify it a bit to make it gauge invariant. In this section, we are going to consider the gauge invariant interactions of chiral and vector superfields. We will first discuss the $U(1)$ case and construct the supersymmetric quantum electrodynamics (SQED). Then we will generalize the results to non-Abelian gauge groups and construct the supersymmetric quantum chromodynamics (SQCD).

4.5.1 Abelian Gauge Group

In a $U(1)$ gauge theory, chiral superfields Φ_l, $\Phi_l^\dagger$ transform by a phase:

$$\Phi_l \to e^{-it_l\Lambda(x)}\Phi_l, \quad \Phi_l^\dagger \to e^{it_l\Lambda^\dagger(x)}\Phi_l^\dagger, \tag{4.29}$$

where Λ is a left-handed chiral superfield thus under the transformation Φ_l remain left-handed chiral superfields. Correspondingly, $\Lambda^\dagger$ is a right-handed chiral superfield.

However, the Kähler potential $\Phi_l^\dagger\Phi_l$ as we constructed in the previous sections is no longer invariant under such local transformations for

$$\Phi_l^\dagger\Phi_l \to \Phi_l^\dagger\Phi_l e^{it_l(\Lambda^\dagger-\Lambda)}. \tag{4.30}$$

It is easy to see that $\Phi_l^\dagger\Phi_l$ may be rendered invariant by introducing the vector superfield V with its transformation law

$$V \to V + i(\Lambda - \Lambda^\dagger). \tag{4.31}$$

Then $\Phi_l^\dagger e^{t_l V}\Phi_l$ is gauge invariant. Then the full gauge invariant Lagrangian can be written as

$$\mathcal{L} = \Phi_l^\dagger e^{t_l V}\Phi_l + [\mathcal{W}^\alpha\mathcal{W}_\alpha(\bar{\theta}\bar{\theta}) + W(\Phi)(\bar{\theta}\bar{\theta}) + h.c.]. \tag{4.32}$$

The supersymmetric extension of electrodynamics is constructed in terms of two chiral superfields Φ_+ and Φ_-. They transform under the local $U(1)$ as

$$\Phi_+ \to e^{-ie\Lambda}\Phi_+, \quad \Phi_- \to e^{ie\Lambda}\Phi_-. \tag{4.33}$$

The Lagrangian of SQED is given by

$$\mathcal{L} = \Phi_+^\dagger e^{eV}\Phi_+ + \Phi_-^\dagger e^{-eV}\Phi_- + \mathcal{W}^\alpha\mathcal{W}_\alpha(\bar{\theta}\bar{\theta}) + \overline{\mathcal{W}}_{\dot{\alpha}}\overline{\mathcal{W}}^{\dot{\alpha}}(\theta\theta) + m(\Phi_+\Phi_-(\bar{\theta}\bar{\theta}) + \Phi_+^\dagger\Phi_-^\dagger(\theta\theta)). \tag{4.34}$$

4.5.2 Non-Abelian Gauge Group

Now we consider the case when the gauge group is Non-Abelian. The generators of the gauge group are T^a. The transformation law of non-Abelian compact groups is given by

$$\Phi \to e^{-i\Lambda}\Phi, \quad \Phi^\dagger \to \Phi^\dagger e^{i\Lambda^\dagger}, \tag{4.35}$$

where Λ is a matrix

$$\Lambda_{ij} = T^a_{ij}\Lambda_a. \tag{4.36}$$

The Lagrangian (4.32) is still invariant under non-Abelian gauge transformations in linear order if we enhance the gauge transformation law $V \to V + i(\Lambda - \Lambda^\dagger)$ to

$$e^V \to e^{-i\Lambda^\dagger} e^V e^{i\Lambda} \tag{4.37}$$

and also put trace. Here, V, like Λ, is also a matrix

$$V_{ij} = T^a_{ij} V_a. \tag{4.38}$$

We also need to generalize the field strength chiral superfield to the non-Abelian case:

$$\mathcal{W}_\alpha = -\frac{1}{4}\overline{\mathcal{D}\mathcal{D}}e^{-V}\mathcal{D}_\alpha e^V, \quad \overline{\mathcal{W}}_{\dot\alpha} = \frac{1}{4}\mathcal{D}\mathcal{D}e^V\overline{\mathcal{D}}_{\dot\alpha}e^{-V} \tag{4.39}$$

with transformation law

$$\mathcal{W}_\alpha \to e^{-i\Lambda}\mathcal{W}_\alpha e^{i\Lambda}, \quad \overline{\mathcal{W}}_{\dot\alpha} \to e^{i\Lambda^\dagger}\overline{\mathcal{W}}_{\dot\alpha}e^{-i\Lambda^\dagger}. \tag{4.40}$$

In Wess-Zumino gauge, we can obtain the component expression of $\mathcal{W}_\alpha$:

$$\mathcal{W}_\alpha = \lambda_\alpha(y) + 2D(y)\theta_\alpha + (\sigma^{\mu\nu}\theta)_\alpha F_{\mu\nu}(y) - i(\theta\theta)(\sigma^\mu D_\mu\bar\lambda(y))_\alpha \tag{4.41}$$

where now the field strength

$$F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu - \frac{i}{2}[V_\mu, V_\nu] \tag{4.42}$$

and the covariant derivative

$$D_\mu\bar\lambda = \partial_\mu\bar\lambda - \frac{i}{2}[V_\mu, \bar\lambda]. \tag{4.43}$$

Therefore, (3.34) remains gauge invariant if we replace $\partial_\mu\bar\lambda$ by $D_\mu\bar\lambda$ and $F_{\mu\nu}$ is defined as above and put trace. We, again, introduce the complex coupling constant

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2} \tag{4.44}$$

Then, we can construct the action as

$$\begin{aligned} S_{\text{gauge}} &= \int d^4x d^2\theta \frac{i\tau}{16\pi}\mathrm{Tr}\mathcal{W}^\alpha W_\alpha + c.c. \\ &= \frac{1}{g^2}\mathrm{Tr}(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - i\lambda\sigma^\mu D_\mu\bar\lambda + 2D^2) + \frac{\theta}{32\pi^2}\mathrm{Tr}F_{\mu\nu}\widetilde{F}^{\mu\nu}. \end{aligned} \tag{4.45}$$

We see it produces the gauge kinetic term:

$$-\frac{1}{4g^2}\mathrm{Tr}F_{\mu\nu}F^{\mu\nu} + \frac{\theta}{32\pi^2}\mathrm{Tr}F_{\mu\nu}\widetilde{F}^{\mu\nu} = \frac{i\tau}{16\pi}\mathrm{Tr}\mathcal{F}^2 + c.c. \tag{4.46}$$

We can write down the most general action as a combination of the Kähler potential, superpotential and the supersymmetric field strength term:

$$S = \int d^4x\{\int d^4\theta\Phi^\dagger e^V\Phi + [\int d^2\theta W(\Phi) + c.c.]\} + S_{\text{gauge}}. \tag{4.47}$$

4.6 Supersymmetric QCD

We now construct the action for supersymmetric QCD (SQCD). The gauge group is $SU(N_c)$. The flavor symmetry group is $SU(N_f)$. There are N_f quarks transforming in the fundamental representation ($\mathbf{N}_c$) of the gauge group as well as antiquarks in the anti-fundamental representation ($\overline{\mathbf{N}}_c$). They are expressed by chiral superfields:

$$Q_L^i = q_L^i + \sqrt{2}\theta\psi_L^i - \theta\theta f_L^i, \quad \widetilde{Q}_{iL} = \tilde{q}_{iL} + \sqrt{2}\theta\tilde{\psi}_{iL} - \theta\theta\tilde{f}_{iL}. \tag{4.48}$$

where $i = 1, 2, \cdots, N_c$ and $L = 1, 2, \cdots, N_f$. Then, a general action for SQCD is given by

$$S = \int d^4x\{\frac{1}{2}[Q^\dagger e^V Q + \widetilde{Q}^\dagger e^{-V}\widetilde{Q}]_D + [\frac{i\tau}{16\pi}\mathrm{Tr}\mathcal{W}^2 + W(Q, \widetilde{Q}) + c.c.]_F\}, \tag{4.49}$$

where we introduced the D-term and F-term. As we have demonstrated, the action and the Lagrangian is related by $S = \int d^4x \int d^4\theta\mathcal{L}$. When we construct Lagrangian from chiral superfields, the highest component is the $(\theta\theta)$ component which is called an F-term. Similarly, when we construct Lagrangian from general superfields, a D-term refers to the $(\theta\theta)(\bar{\theta}\bar{\theta})$ component. $[\cdots]_F$ and $[\cdots]_D$ simply extract the F-term and D-term, respectively.

For massless SQCD, the superpotential term $W(Q, \widetilde{Q})$ vanishes. The symmetries in SQCD is of great importance as we will see in later chapters, like in . Thus, we will explain these symmetries in more details. Since all terms in the Lagrangian are diagonal in the flavor indices of the quarks and separately in the flavor indices of the antiquarks, there is an $SU(N_f)_L \times SU(N_f)_R$ global symmetry. In addition, there is an $U(1)_B$ denoting the "baryon number" under which the quarks and antiquarks transform as

$$Q \to e^{ib}Q, \quad \widetilde{Q} \to e^{-ib}\widetilde{Q}, \tag{4.50}$$

as well as an $U(1)_A$ denoting the axial symmetry under which the quarks and antiquarks transform as

$$Q \to e^{ia}Q, \quad \widetilde{Q} \to e^{ia}\widetilde{Q}. \tag{4.51}$$

We also have the global $U(1)_R$ symmetry acting as

$$Q(x,\theta) \to e^{-iq}Q(x, e^{iq}\theta), \quad \widetilde{Q}(x,\theta) \to e^{-iq}\widetilde{Q}(x, e^{iq}\theta), \quad V(x,\theta,\bar{\theta}) \to V(x, e^{iq}\theta, e^{-iq}\bar{\theta}). \tag{4.52}$$

Thus, the global symmetry group in the massless case is $SU(N_f)_L \times SU(N_f)_R \times U(1)_A \times U(1)_B \times U(1)_R$.

Mass terms in SQCD would break flavor symmetries. We can add a gauge invariant mass term

$$W(Q, \widetilde{Q}) = m_{LM}Q_L^i\widetilde{Q}_{i,M}. \tag{4.53}$$

This is a quark mass term and m_{LM} is the $N_f \times N_f$ mass matrix. Using the global symmetry of the other terms in the Lagrangian one can diagonalize this term so that it becomes

$$W(Q, \widetilde{Q}) = \sum_L m_L Q_L^i\widetilde{Q}_{i,L}. \tag{4.54}$$

4.7 Construction of Supersymmetric Effective Action

Generally, we write the supersymmetric actions as

$$S = \int d^4x\{[\widetilde{K}]_D + [\widetilde{W} + h.c.]_F\} \tag{4.55}$$

In this action, $\widetilde{K}$ is a general superfield and $\widetilde{W}$ is a chiral superfield. The general superfield $\widetilde{K}$ can be expanded as

$$\widetilde{K} = K(\Phi^\dagger e^V, \Phi) + \widehat{K}(\Phi^\dagger e^V, \Phi, \mathcal{D}, \overline{\mathcal{D}}, \partial_\mu) \tag{4.56}$$

where K is Kähler potential and $\widehat{K}$ is a function which contains the derivatives of the chiral superfields. We can also expand the chiral superfield $\widetilde{W}$ as

$$\widetilde{W} = W(\Phi) + \tau(\Phi)\mathrm{Tr}(\mathcal{W}_\alpha \mathcal{W}^\alpha) + \mathcal{O}((\mathcal{W}_\alpha \mathcal{W}^\alpha)^2) + \widehat{W}(\Phi, \partial_\mu) + ... \tag{4.57}$$

When we scale all energies and momenta by a factor μ/μ_0 to lower the cutoff scale fromμ_0 to μ, then, the supercoordinates scale as

$$x \to \left(\frac{\mu_0}{\mu}\right) x, \quad dx \to \left(\frac{\mu_0}{\mu}\right) dx, \quad \partial_\mu \to \left(\frac{\mu}{\mu_0}\right) \partial_\mu \tag{4.58}$$

$$\theta \to \left(\frac{\mu_0}{\mu}\right)^{\frac{1}{2}} \theta, \quad d\theta \to \left(\frac{\mu}{\mu_0}\right)^{\frac{1}{2}} d\theta, \quad \partial_\theta \to \left(\frac{\mu}{\mu_0}\right)^{\frac{1}{2}} \partial_\theta \tag{4.59}$$

In order to keep the action invariant, we have to rescale the superfields as

$$\Phi \to \left(\frac{\mu}{\mu_0}\right) \Phi, \quad \mathcal{W}^\alpha \to \left(\frac{\mu}{\mu_0}\right)^{\frac{3}{2}} \mathcal{W}^\alpha \tag{4.60}$$

Now we would like to determine whether an operator is relevant or not. They key is to find the scaling dimension of the marginal operator. We will find that the scaling dimension for marginal chiral superfields and marginal vector superfields are different. If a chiral superfield $\mathcal{O}_i$ scales as

$$\mathcal{O}_i \to \left(\frac{\mu}{\mu_0}\right)^{\Delta_i} \mathcal{O}_i, \tag{4.61}$$

then, the action due to the operator scales as

$$\int d^4x \int d^2\theta \mathcal{O}_i \to \left(\frac{\mu}{\mu_0}\right)^{\Delta_i - 3} \int d^4x d^2\theta \mathcal{O}_i. \tag{4.62}$$

When $\Delta_i = 3$, the chiral superfield $\mathcal{O}_i$ is marginal. If a general superfield η_i scales as

$$\eta_i \to \left(\frac{\mu}{\mu_0}\right)^{\Delta_i} \eta_i, \tag{4.63}$$

then the action due to the operator scales as

$$\int d^4x d^4\theta \eta_i \to \left(\frac{\mu}{\mu_0}\right)^{\Delta_i - 2} \int d^4x d^4\theta \eta_i. \tag{4.64}$$

When $\Delta_i = 2$, the general superfield η_i is marginal.

Now We write down the relevant terms of the supersymmetric effective action as follows.

$$S^{(\mu)} = \int d^4x \{ \int d^2\theta Z(\mu) \Phi^\dagger e^V \Phi + \int d^2\theta [\tau(\mu) \mathrm{Tr} \mathcal{W}^2 + W(\Phi)] + c.c. \}, \tag{4.65}$$

where the Kähler potential term receives the wave function renormalization. Also the scaling dimension of the superpotential $W(\Phi)$ is no greater than three so that it is marginal or relevant. For U(1) gauge group, we may also add a Fayet-Iliopoulos term

$$\int d^4x \, \xi \int d\theta_\alpha \mathcal{W}^\alpha. \tag{4.66}$$

Chapter 5

Non-renormalization Theorems for Chiral Superfields

5.1 Symmetries and Holomorphy

The basic approach will be to consider the low energy effective action for the light fields, by integrating out degrees of freedom above some scale. Assuming that we are working above the scale of possible supersymmetry breaking, the effective action will have a linearly realized supersymmetry which can be made manifest by working in terms of superfields.

We will focus on a particular contribution to the effective Lagrangian: the superpotential term

$$\int d^2\theta W_{\rm eff}(\Phi). \tag{5.1}$$

The key fact is that supersymmetry requires $W_{\rm eff}$ to be holomorphic in the chiral superfields Φ, and independent of $\Phi^\dagger$. We will think of all the coupling constant g in the tree level superpotential $W_{\rm tree}$ and the scale Λ as background fields. Then, the quantum effective superpotential is constrained by

- Symmetries and selection rules: By assigning transformation laws both to the fields and to the coupling constants, the theory has a large symmetry. The effective Lagrangian should be invariant under it.
- Holomorphy: $W_{\rm eff}$ is independent of $g^\dagger$. This is the key property. Just as the superpotential is holomorphic in the fields, it is also holomorphic in the coupling constants (the background fields). This is unlike the effective Lagrangian in non-supersymmetric theories, which is not subject to any holomorphy restrictions.
- Various limits: $W_{\rm eff}$ can be analyzed approximately at weak coupling. The singularities have physical meaning and can be controlled.

Often these conditions completely determine $W_{\rm eff}$. The point is that a holomorphic function is determined by its asymptotic behavior and singularities. The results can be highly non-trivial, revealing interesting non-perturbative dynamics.[6]

When there is a Coulomb phase, the kinetic terms for the gauge fields are also constrained by the above considerations. The relevant term in the effective Lagrangian is

$$\int d^2\theta \mathrm{Im}[\tau_{\rm eff}(\Phi, g, \Lambda)\mathcal{W}_\alpha^2], \tag{5.2}$$

where $\mathcal{W}_\alpha^2$ gives the supersymmetric completion of $F^2 + iF\widetilde{F}$, therefore,

$$\tau_{\text{eff}} \sim \frac{\theta_{\text{eff}}}{2\pi} + i\frac{4\pi}{g_{\text{eff}}^2}, \tag{5.3}$$

is the effective gauge coupling constant. $\tau_{\text{eff}}(\Phi, g, \Lambda)$ is holomorphic in its arguments and can often be exactly determined.

To explain this scheme more specifically, consider an IR effective action

$$S_{\text{eff}} = \int d^4x\{\int d^4\theta Z\widetilde{\Phi}^\dagger e^V\widetilde{\Phi} + \int d^2\theta[W_{\text{eff}}(\widetilde{\Phi}) + \tau_{\text{eff}}(\widetilde{\Phi})\text{Tr}\widetilde{\mathcal{W}^\alpha}\widetilde{\mathcal{W}_\alpha}] + c.c.\}. \tag{5.4}$$

The action of the UV theory which is asymptotically free is given by

$$S_{\text{UV}} = \int d^4x\{\int d^4\theta\Phi^\dagger e^V\Phi + \int d^2\theta[W_{\text{UV}}(\Phi) + \tau\text{Tr}(\mathcal{W}^\alpha\mathcal{W}_\alpha)] + c.c.\}. \tag{5.5}$$

Now we consider a specific superpotential

$$W_{UV}(\Phi) = \lambda_2\Phi^2 + \lambda_3\Phi^3. \tag{5.6}$$

Here the superfields Φ, $\mathcal{W}_\alpha$ in the UV are not necessarily the same as $\widetilde{\Phi}$, $\widetilde{W_\alpha}$ in the IR. We would like to figure out how W_{eff}, τ_{eff} and Z depend on the parameters λ_i and τ.

We treat λ_i and τ as background fields. That is to say we can consider them as left-handed chiral superfields, and therefore, they are functions of y and θ: $\lambda_i \to \lambda_i(y, \theta)$, $\tau \to \tau(y, \theta)$, where $y = x + i\bar{\theta}\sigma\theta$. Then, their value in the action can be understood as vacuum expectation value (VEV) of these functions

$$\lambda_i = \langle\lambda_i(y, \theta)\rangle, \quad \tau = \langle\tau(y, \theta)\rangle. \tag{5.7}$$

Since left-handed chiral superfields can only enter holomorphically in W_{eff} and τ_{eff}, the background fields λ_i and τ can only enter them holomorphically as well.

Let us turn off vector multiplet. Since this is just a theory of scalars and spinors, which is IR free for weak enough coupling. We assume that the IR field content is the same as UV field content. Now we can see the power of holomorphy. Suppose that the effective superpotential has the form:

$$W_0 = \lambda\Phi. \tag{5.8}$$

Then, we can say that there is a U(1) global symmetry $\Phi \to e^{i\epsilon}\Phi$, broken by λ. Thus, thinking of λ as a superfield and transforms as $\lambda \to e^{-i\epsilon}\lambda$, we can restore this symmetry. Without holomorphy, we could write infinitely many possible contributions to effective superpotential

$$W_{\text{eff}} = \cdots + \frac{\bar{\lambda}^2}{\Phi^2} + \frac{\bar{\lambda}}{\Phi} + \lambda\Phi + \lambda^2\bar{\lambda}\Phi + \cdots + \lambda e^{-\frac{1}{|\lambda|^2}}\Phi + \cdots. \tag{5.9}$$

But with holomorphy, the only terms allowed are of the form

$$W_{\text{eff}} = \sum_\alpha(\lambda\Phi)^\alpha, \quad \alpha > 0. \tag{5.10}$$

This already rules out non-perturbative contributions. Also, $\alpha > 0$ since the theory is free as λ goes to zero. Note that Kähler potential is not holomorphic, therefore it does not get these restrictions.

Seiberg summarized this argument prescriptively, and constrained the effective superpotential by holomorphy in UV couplings, global symmetries broken by the coupling and smoothness of the physics in weak coupling limits.

5.2 Non-renormalization Theorem of Superpotential

We start with the Wess-Zumino model of one single left-chiral superfield Φ with superpotential at scale μ_0

$$W^{(\mu_0)} = \frac{1}{2}\mu_0\lambda_2\Phi^2 + \frac{1}{3}\lambda_3\Phi^3. \tag{5.11}$$

The UV effective action is given by

$$S^{(\mu_0)} = \int d^4x\{\frac{1}{2}[\Phi^\dagger\Phi]_D + [\frac{1}{2}\mu_0\lambda_2\Phi^2 + \frac{1}{3}\lambda_3\Phi^3 + c.c.]_F\}. \tag{5.12}$$

By holomorphy, the effective superpotential at scale $\mu < \mu_0$ is

$$W^{(\mu)} = W^{(\mu)}(\Phi, \lambda_2, \lambda_3; \mu, \mu_0), \tag{5.13}$$

which means it is only a function of Φ, λ_2 and λ_3. The microscopic superpotential is invariant under a global $U(1) \times U(1)_R$ symmetry if we assign charges:

	$U(1)$	$\times$	$U(1)_R$
Φ	1		1
λ_2	-2		0
λ_3	-3		-1

This implies selection rules for these symmetries constraining the effective superpotential to be neutral under $U(1)$ symmetry and have charge $+2$ under the $U(1)_R$ symmetry. Thus it must have the form

$$W^{(\mu)} = \mu\lambda_2\Phi^2 g(\frac{\lambda_3\Phi}{\mu\lambda_2}) \tag{5.14}$$

where g is an arbitrary holomorphic function. In the $\lambda_3 \to 0$ limit, keeping λ_2 fixed, the theory is free, so if we expand g, only terms with non-negative integers powers of λ_3 can appear:

$$W^{(\mu)} = \sum_{n\geqslant 0} g_n(\mu\lambda_2)^{1-n}\lambda_3^n\Phi^{n+2} \tag{5.15}$$

Similarly, in the $\lambda_2 \to 0$ limit, terms with $n > 1$ is not allowed. So the superpotential at scale μ has the form

$$W^{(\mu)} = g_0\mu\lambda_2\Phi^2 + g_1\lambda_3\Phi^3 \tag{5.16}$$

Then, the effective action can be written as

$$S^{(\mu)} = \int d^4x\{\frac{1}{2}Z[\Phi^\dagger\Phi + \cdots]_D + [g_0\mu\lambda_2\Phi^2 + g_1\lambda_3\Phi^3 + c.c]_F\}, \tag{5.17}$$

where we have included a wave function renormalization Z of the Kähler term. The Kähler term may also receive other corrections which will be discussed later.

Now we have to determine the coupling constants g_0 and g_1. In the $\lambda_3 \to 0$ limit, the theory is free with a mass $2g_0\mu\lambda_2/Z$. Equating this to the mass in the UV action, we get

$$g_0 = \frac{1}{2}\frac{\mu_0}{\mu}Z. \tag{5.18}$$

We can choose

$$Z = \frac{\mu}{\mu_0} \quad (\text{when } \lambda_3 = 0) \tag{5.19}$$

so that $g_0 = 1/2$. When $\lambda_3 \neq 0$, Z will receive corrections shifting it from the value above.

The constant g_1 can be determined by matching the results of perturbation theory in λ_3. Since the Φ^3 vertex appears in both proportional to the same coupling λ_3, they must match at tree level (classically), which gives $g_1 = 1/3$. Thus, the effective superpotential is

$$W^{(\mu)} = \frac{1}{2}\mu\lambda_2\Phi^2 + \frac{1}{3}\lambda_3\Phi^3. \tag{5.20}$$

We have shown that couplings are not renormalized. The low energy superpotential receives no quantum corrections and it only differs from the UV superpotential by the classical scalings.

To generalize non-renormalization result a bit further, let us consider a more complicated example. The superpotential at scale μ_0 is given by

$$W^{(\mu_0)} = \mu_0^2\lambda_1\Phi + \mu_0\lambda_2\Phi^2 + \cdots + \mu_0^{3-r}\lambda_r\Phi^r + \cdots \tag{5.21}$$

which has the global symmetries

	$U(1)$	$\times$	$U(1)_R$
Φ	$+1$		$+1$
λ_r	$-r$		$2-r$

Thus selection rules for these symmetries constrain the effective superpotential at scale μ to have the form

$$W^{(\mu)} = \mu^2\lambda_1\Phi g(\frac{\lambda_2}{\lambda_1}\frac{\Phi}{\mu}, \frac{\lambda_3}{\lambda_1}\frac{\Phi^2}{\mu^2}, \cdots, \frac{\lambda_r}{\lambda_1}\frac{\Phi^{r-1}}{\mu^{r-1}}). \tag{5.22}$$

Similarly, we may get that the effective superpotential receives no quantum corrections. Thus, the effective action has the form

$$S^{(\mu)} = \int d^4x\{\frac{1}{2}[Z\Phi^\dagger\Phi]_D + [\sum_r \mu^{3-r}\lambda_r\Phi^r + c.c.]_F\}. \tag{5.23}$$

More generally, if we have a bunch of chiral superfields. The UV superpotential at scale μ_0 can be written as

$$W^{(\mu_0)} = \sum_r \mu_0^{3-\Delta_r}\lambda_r\mathcal{O}_r, \tag{5.24}$$

where

$$\mathcal{O}_r = \prod_n \Phi_n^{r_n}, \tag{5.25}$$

is a chiral superfield with scaling dimension $\Delta_r = \sum_n r_n$. Then the effective superpotential has the form

$$W^{(\mu)} = \sum_r \mu^{3-\Delta_r}\lambda_r\mathcal{O}_r. \tag{5.26}$$

5.3 Kähler Term Renormalization

Since Kähler term is a function of both left-handed and right-handed chiral superfields. It is not protected by this holomorphy argument. It will get wave function renormalization or even higher derivatives. Thus, in order to see how the canonical couplings Z are renormalized, we define canonical chiral superfield by

$$\Phi_n \to \Phi_n^{C.N.} = \sqrt{Z_n(\mu)}\Phi_n. \tag{5.27}$$

From (5.26), we can write the scale dependent scalings as

$$\lambda_r(\mu) = \mu^{3-\Delta_r}\lambda_r. \tag{5.28}$$

Since the chiral superfields Φ_n get canonically renormalized, we see that the couplings $\lambda_r(\mu)$ also have to be renormalized as

$$\lambda_r^{C.N.}(\mu) = \mu^{3-\Delta_r}(\prod_n Z_n^{-\frac{r_n}{2}}(\mu))\lambda_r. \tag{5.29}$$

This implies that the exact RG equation for the physical superpotential couplings

$$\mu\frac{d\lambda_r^{C.N.}(\mu)}{d\mu} = \lambda_r^{C.N.}(\mu)(3 - \Delta_r - \frac{1}{2}\sum_n r_n\gamma_n(\mu)). \tag{5.30}$$

where

$$\gamma_n(\mu) = \frac{d\ln Z_n(\mu)}{d\ln\mu} \tag{5.31}$$

is called the anomalous dimension of Φ_n. This β-function is an exact RG equation, but we do not know how to compute $\gamma_n(\mu)$ exactly.

An interesting example arises in a theory with two left-handed chiral superfields Φ_1, Φ_2 and superpotential

$$W = \lambda\Phi_1\Phi_2^2. \tag{5.32}$$

To get susy vacua, we take

$$\frac{\partial W}{\partial\phi_1} = 0, \quad \frac{\partial W}{\partial\phi_2} = 0, \tag{5.33}$$

and we get

$$\phi_2 = 0, \quad \phi_1 = \text{arbitrary}, \tag{5.34}$$

implying a whole moduli space $\mathcal{M}$ is degenerate with inequivalent classical ground states. By the non-renormalization theorem of the superpotential, this conclusion does not change once quantum effects are taken into account.

However, quantum effects make the Kähler potential renormalized and thus change the metric on $\mathcal{M}$ from its classical value. Since

$$K = \phi_1^*\phi_1 + \phi_2^*\phi_2, \tag{5.35}$$

the metric induced on $\mathcal{M}$ is

$$ds^2 = d\phi_1 d\phi_1^*. \tag{5.36}$$

Spectrum at any point on $\mathcal{M}$ is one massless chiral superfield Φ_1 and one massive chiral superfield Φ_2 with mass propotional to $|\lambda\langle\Phi_1\rangle|$. At scales above the mass of Φ_2, one loop diagrams of Φ_2 contribute to Φ_1 propagators, and the Kähler term can be rewritten as

$$K = \phi_1^*\phi_1 - \#\phi_1^*\phi_1 |\lambda|^2 \ln\left|\frac{\phi_1}{\mu_0}\right|^2 + \cdots . \tag{5.37}$$

The first term comes from the tree level. We are assuming that the scale of the IR effective action μ is smaller than the mass of Φ_2. The second term comes from the massive Φ_2 at one loop which contributes the usual logarithm of its mass over the cutoff mass μ_0. The sign of the one loop term results in a growing K as $\phi_1 \to 0$, which in turn implies that the canonically renormalized couplings are going to zero in this limit. Thus the one loop perturbative result becomes exact as we approach the origin of $\mathcal{M}$.

Since the Kähler metric is given by $ds^2 = g_{1\bar{1}} d\phi_1 d\phi_1^*$ with

$$g_{1\bar{1}} = \partial_1\partial_{\bar{1}}K \simeq -|\lambda|^2 \ln\phi_1\phi_1^* + \text{const.} \tag{5.38}$$

We see that the metric is singular at $\phi_1 = 0$. The physical meaning behind the singularity at $\phi_1 = 0$ is that when $\phi_1 = 0$ a particle multiplet Φ_2 is becoming massless. The assumption, the scale μ is smaller than the mass of Φ_2, under which we computed IR effective action breaks down. Generally speaking, a singularity in moduli space is a sign of new massless degrees of freedom. To get a smooth description of physics, we need to include them.

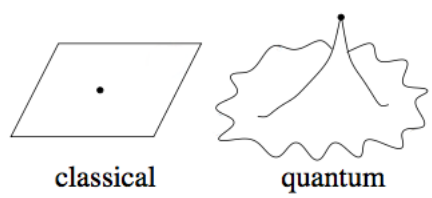


Figure 5.1: classical and quantum moduli space

5.4 Moduli Space of Vacua

As we have seen, the supersymmetric vacua corresponds to the zero point of scalar potential. Given a general supersymmetric Lagrangian

$$\mathcal{L} = \sum_f \int d^4\theta Q_f^\dagger e^{T_f V} Q_f + \int d^2\theta\tau \mathrm{Tr}\mathcal{W}_\alpha\mathcal{W}^\alpha + h.c., \tag{5.39}$$

the scalar potential is given by

$$V_D = D^a \sum_f Q_f^\dagger T_f^a Q_f - \frac{1}{g^2}\sum_a (D^a)^2, \tag{5.40}$$

where $\sum_a D^a T^a$ is the D-term. To eliminate D^a, we use the equation of motion of D^a and have

$$D^a = \frac{g^2}{2}\sum_f Q_f^\dagger T_f^a Q_f. \tag{5.41}$$

Plugging this back into the scalar potential, we get

$$V_D = \frac{1}{g^2}\sum_a (D^a)^2. \tag{5.42}$$

Set $V_D = 0$, we get $D^a = 0$, $a = 1, \cdots, |G|$. Thus the classical moduli space of vacua is

$$\mathcal{M}_{\rm cl} = \{\langle Q_f \rangle | D^a = 0\}. \tag{5.43}$$

In this description, we have modded out the gauge transformation of Q_f.

As a simple example, consider $U(1)$ gauge theory with a matter superfield Q of charge 1 and $\widetilde{Q}$ of charge -1. The scalar potential is

$$V_D = \frac{g^2}{4}(|Q|^2 - \left|\widetilde{Q}\right|^2)^2 \tag{5.44}$$

Thus there is a continuum of degenerate vacua labeled by $\langle Q \rangle = \langle \widetilde{Q} \rangle = a$, for any complex a. The moduli space is, then, given by

$$\mathcal{M}_{\rm cl} = \{\langle Q \rangle \langle \widetilde{Q} \rangle | V_D = 0\} \equiv \{\langle X \rangle\}, \tag{5.45}$$

where we have also modded out the gauge transformation.

In vacua with $a \neq 0$, the gauge group is broken by a supersymmetric version of Higgs mechanism. The gauge field gets mass $|a|$ by "eating" one chiral superfield degree of freedom from the matter fields. Since we started with two superfields, one superfield degree of freedom remains massless. The remaining massless superfield can be given a gauge invariant description as $X = Q\widetilde{Q}$. In the vacuum labeled as above by a, $\langle X \rangle = a^2$. Because a is arbitrary, there is no potential for X, classically X is a "modulus" field whose expectation value labels a classical moduli space of degenerate vacua. [6]

Another example is a SQED $U(1)$ theory with N_f flavors. The charge of Q_f is $+1$ while the charge of $\widetilde{Q}_f$ is -1. The scalar potential is given by

$$V_D = \frac{g^2}{4}[\sum_f (|Q_f|^2 - \left|\widetilde{Q}_f\right|^2)]^2. \tag{5.46}$$

The moduli space is given by

$$\mathcal{M}_{cl} = \{\langle Q_f \rangle \langle \widetilde{Q}_f \rangle | V_D = 0\} = \{\langle M_{f\tilde{g}} = Q_f \widetilde{Q}_g \rangle\} \tag{5.47}$$

We see that there is singularity at $M_{f\tilde{g}} = 0$. It reflects the presence of additional massless degrees of freedom. Since SQED is IR free, this also holds in quantum theory. We will see in SQCD other mechanisms, depending on the number of flavors N_f and the number of colors N_c, which will be discussed later.

Chapter 6

Anomalies

From classical symmetries, using Noether theorem, one can derive the corresponding conserved currents. However, when we consider quantum corrections like one-loop correction to the theory, the currents are not conserved any more and the symmetries are broken. Thus, we say the symmetries are anomalous. We will show a class of anomalies. Trace anomalies are associated to breaking of scale invariance symmetry. Another important anomaly, is an anomaly for chiral symmetry. First, we will discuss the anomalies in non-supersymmetric case, then we will generalize it to the supersymmetric case.

6.1 Trace Anomalies

One of the most important anomaly is the scale invariance anomaly. Recall the dilatation current D^μ which satisfies

$$\partial^\mu D_\mu = T^\mu_{\ \mu}. \tag{6.1}$$

Even when $T^\mu_{\ \mu}$ vanishes classically, it is a composite operator, which means it is a product of fields at the same point. Therefore, it must be regulated quantum mechanically.

For example, let us consider a QCD theory

$$\mathcal{L} = -\frac{1}{4g^2}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) + i\bar{\psi}\not{D}\psi. \tag{6.2}$$

Using dimensional regularization we obtain

$$T^\mu_{\ \mu} = \frac{d-4}{4}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) + (1-d)i\bar{\psi}\not{D}\psi. \tag{6.3}$$

Classically, $d = 4$ and from the equation of motion of ψ, we can get $T^\mu_{\ \mu} = 0$. Quantum mechanically, we compute in the background field gauge

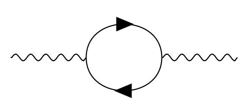


Figure 6.1: one loop correction

$$T^{\mu}_{\ \mu} = \frac{b}{32\pi^2}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) + \mathcal{O}(g^2). \tag{6.4}$$

On the other hand, under scaling $x \to e^{\epsilon}x$, the stress-energy tensor is given by

$$T^{\mu}_{\ \mu} = \left(\frac{\partial \mathcal{L}}{\partial \epsilon} - 4\mathcal{L}\right)\Bigg|_{\epsilon=0} = \frac{\partial(-\frac{1}{4g^2})}{\partial \epsilon}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}). \tag{6.5}$$

Comparing this result with (6.4) we get

$$\frac{\partial g^{-2}}{\partial \epsilon} = -\frac{b}{8\pi^2} + \mathcal{O}(g^2). \tag{6.6}$$

Under this scaling, the energy scales as $\mu_0 \to e^{-\epsilon}\mu_0 \equiv \mu$. We can get $\epsilon = \ln(\mu_0/\mu)$. Then we can rewrite (6.6) as

$$\frac{1}{g^2(\epsilon)} - \frac{1}{g^2(0)} = -\frac{b\epsilon}{8\pi^2} \tag{6.7}$$

which is equivalent to

$$\frac{1}{g^2(\mu)} - \frac{1}{g^2(\mu_0)} = \frac{b}{8\pi^2}\ln(\frac{\mu}{\mu_0}). \tag{6.8}$$

Absorbing the constant on the left hand side, we can write

$$\frac{1}{g^2(\mu)} = -\frac{b}{8\pi^2}\ln\left(\frac{|\Lambda|}{\mu}\right) \tag{6.9}$$

where

$$|\Lambda| \equiv \mu_0 \exp\left(-\frac{8\pi^2}{bg^2(\mu_0)}\right) \tag{6.10}$$

is the dynamical scale of the QCD and it is independent of the choice of μ_0. When scale of the effective theory approaches to $|\Lambda|$, effective coupling diverge, thus we need to consider higher loop and non-perturbative effects. We can think of $|\Lambda|$ as the approximate scale at which effective coupling become strong. The trading of information between $\{g^2(\mu_0), \mu_0\}$ and $|\Lambda|$ is known as "dimensional transmutation". In a theory with many gauge groups: $G_1 \times G_2 \times \cdots \times G_n$, there are correspondingly many gauge scales: $\{|\Lambda_1|, |\Lambda_2|, \cdots, |\Lambda_n|\}$.

The behavior of the effective coupling has different behaviors in the IR depending on the sign of b. For $b > 0$, the coupling is weak in the UV and runs to strong coupling in the IR. If $b < 0$, then the theory is weakly coupled in the IR and runs to strong coupling in the UV.

Computation of b coefficient gives

$$b = \frac{11}{6}T(\mathbf{adj}) - \frac{1}{3}\sum_j T(\mathbf{R}_j) - \frac{1}{6}\sum_a T(\mathbf{R}_a), \tag{6.11}$$

where the sum on j is over Weyl fermions with the j-th fermion in the $\mathbf{R}_j$ representation of the gauge group, and the sum on a is over complex bosons in representations $\mathbf{R}_a$. Recall that $T(\mathbf{R})$ is called the index of the representation $\mathbf{R}$. It has the form

$$T(\mathbf{R}) = \frac{C(\mathbf{R})}{C(\mathbf{fund})} \tag{6.12}$$

with the quadratic Casimir $C(\mathbf{R})$ given by

$$C(\mathbf{R})\delta^{ab} = \mathrm{Tr}_{\mathbf{R}}(T^aT^b). \tag{6.13}$$

For classical groups:

Dynkin	G	**fund**	C(**fund**)	C(**adj**)	dimG	rankG
A_{N-1}	$SU(N)$	N	$\frac{1}{2}$	N	N^2-1	$N-1$
C_N	$Sp(2N)$	$2N$	$\frac{1}{2}$	$N+1$	$N(2N+1)$	N
$B_{\frac{N-1}{2}} D_{\frac{N}{2}}$	$SO(N)$	N	1	$N-2$	$\frac{N(N-1)}{2}$	$[\frac{N}{2}]$

Thus, for $SU(N_c)$ QCD with N_f flavors in fundamental representation ($\mathbf{N}_c$), there are $2N_f$ Weyl fermions and zero bosons transforming in fundamental representation. From the table above, $T(\mathbf{adj}) = 2N_c$, $T(\mathbf{fund}) = 1$. We obtain

$$b = \frac{11}{6} 2N_c - \frac{2}{3} N_f = \frac{11N_c - 2N_f}{3}. \tag{6.14}$$

In summary, quantum effects in a classically scale invariant Yang-Mills theory make the gauge coupling g run with scale. Moreover, other dimensionless couplings can also run, for example, coupling λ of $\lambda\phi^4$ and Yukawa coupling g of $g\phi\psi^2$. Recall from the renormalization of the Kähler potential, $Z(\mu)\Phi^\dagger\Phi$, the wave function renormalization Z runs as

$$\frac{d\ln Z}{d\ln\mu} = \gamma, \tag{6.15}$$

where γ is the anomalous dimension.

Another gauge coupling is the theta angle. The instanton term is given by

$$\frac{\theta}{16\pi^2}\mathrm{Tr}(F_{\mu\nu}\widetilde{F}^{\mu\nu}), \tag{6.16}$$

where the instanton number

$$\frac{1}{16\pi^2}\int d^4x \mathrm{Tr}(F_{\mu\nu}\widetilde{F}^{\mu\nu}) \tag{6.17}$$

is an integer. Therefore, the coupling θ possesses periodicity:

$$\theta \sim \theta + 2\pi. \tag{6.18}$$

Since the term $\mathrm{Tr}(F\widetilde{F})$ is a total derivative, θ does not run perturbatively. Recall what we have defined as a complex coupling

$$\tau(\mu) = \frac{\theta}{2\pi} + i\frac{4\pi}{g^2(\mu)}. \tag{6.19}$$

Then we can define complex RG invariant scale

$$\Lambda = |\Lambda|\, e^{i\frac{\theta}{b}} = \mu e^{2\pi i \frac{\tau(\mu)}{b}} \tag{6.20}$$

or

$$\tau(\mu) = \frac{b}{2\pi i}\ln(\frac{\Lambda}{\mu}) \tag{6.21}$$

Now "dimensional transmutation" means the trading of information between $\tau(\mu)$ and Λ. At one loop, we can write gauge kinetic term as

$$\int d^2\theta \frac{1}{32\pi^2}\ln(\frac{\Lambda^b}{\mu^b})\mathrm{Tr}(\mathcal{W}^\alpha\mathcal{W}_\alpha) + c.c. \tag{6.22}$$

6.2 Chiral Anomalies and Gauge Anomalies

Before the discussion of chiral anomalies, we need to know what chiral symmetries are. Chiral symmetries are those in which left-handed fermions (ψ_L) and right-handed fermions (ψ_R) transform differently. Let us consider that the fermions are coupled to gauge fields with gauge symmetry group G. T^a's are generators of the gauge group. If left-handed fermions transform in the $\mathbf{R}$ representation

$$\psi_L \to (1 + i\alpha^a T_a^{(\mathbf{R})})\psi_L, \tag{6.23}$$

taking complex conjugate

$$\psi_R \to (1 - i\alpha^a (T_a^{(\mathbf{R})})^T)\psi_R = (1 + i\alpha^a T_a^{(\overline{\mathbf{R}})})\psi_R, \tag{6.24}$$

we find that right-handed fermions transform in the $\overline{\mathbf{R}}$ representation.

In a free field theory of a Majorana fermion, the conserved currents for the symmetry group G is

$$J^{\mu a} = \bar{\psi}\gamma^\mu (\frac{1-\gamma^5}{2}) T^a \psi. \tag{6.25}$$

We couple this current to gauge boson fields and compute the gauge invariance at one loop. Consider the triangle diagram as shown in figure 6.2.

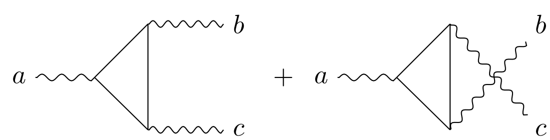


Figure 6.2: triangle diagram

It is proportional to

$$\mathrm{Tr}_{\mathbf{R}}(T^a\{T^b, T^c\})\epsilon_{\mu\nu\rho\sigma} k_2^\mu k_3^\nu \xi_2^\rho \xi_3^\sigma, \tag{6.26}$$

where k's are the momenta and ξ's are the polarizations of the external fields b, c. We symmetrize the generators T^b, T^c in the trace because we impose Bose symmetry between the external gauge fields. Differentiating this result, we obtain the anomaly

$$\partial_\mu J^{\mu a} \propto \mathrm{Tr}_R(T^a\{T^b, T^c\}) F_b^{\mu\nu} \widetilde{F}_{c\mu\nu}. \tag{6.27}$$

Generally, if we have many fermions in the representation, the anomaly would be proportional to $\sum_i \mathrm{Tr}_{\mathbf{R}_i}(T^a\{T^b, T^c\})$. We see that if $\sum_i \mathrm{Tr}_{\mathbf{R}_i}(T^a\{T^b, T^c\}) = 0$, the theory is anomaly free. Real or pseudoreal representations will lead to no anomalies. Because for these representations, $T^a = -(T^a)^T$. Thus,

$$\mathrm{Tr}(T^a\{T^b, T^c\}) = -\mathrm{Tr}((T^a)^T\{(T^b)^T, (T^c)^T\}) = -\mathrm{Tr}(T^a\{T^b, T^c\}), \tag{6.28}$$

the anomaly vanishes. From this result, we see only massless fermions contribute to the anomaly for the fermion mass term is in a real representation. However, if $\sum_i \mathrm{Tr}_{\mathbf{R}_i}(T^a\{T^b, T^c\}) \neq 0$, the chiral symmetry is anomalous which we will discuss precisely later. The discussion above gives constraint to the gauge group that it should have complex representations. For example, $U(1)$ representations are complex. The complex conjugate of a charge q is a charge $-q$.

Now the global symmetry group is denoted by H. And the gauge symmetry group is denoted by G. Therefore, we have symmetries $H \times G$ and all the generators T^a are split into T^a_H and T^a_G. But note that the global symmetries do not have associated field strength, indicating that gauging was a fake.

6.2.1 Gauge Anomalies

If all three generators in the trace are generators of gauge group G. Then $\partial_\mu J^\mu_G \neq 0$ will lead to the breaking of gauge invariance. Thus, the theory is inconsistent.

If the anomaly does not vanish, then, the trace term is not zero. As we have explained above, the gauge group must have complex representation. Also there must be a gauge invariant symmetric tensor d^{ABC}. The only groups which have complex representations and a symmetric three-index tensor are the $SU(N)$ groups for $N \geqslant 3$. [16]

6.2.2 Chiral Anomalies

We now turn to the physics of anomalous global symmetries. T^a is a generator of H and T^b, T^c are generators of G. From the triangle diagram, the anomaly is

$$\partial_\mu J^\mu_H = \frac{1}{32\pi^2} \sum_i \mathrm{Tr}_{\mathbf{R}_i}(T_H\{T^a_G, T^b_G\}) F^{\mu\nu}_a \widetilde{F}_{b\mu\nu} \tag{6.29}$$

Evaluate the right hand side component:

$$\mathrm{Tr}_{\mathbf{R}_H \otimes \mathbf{R}_G}(T_H\{T^a_G, T^b_G\}) = 2\mathrm{Tr}_{\mathbf{R}_H}(T_H)\mathrm{Tr}_{\mathbf{R}_G}(T^a_G T^b_G) = 2\mathrm{Tr}_{\mathbf{R}_H}(T_H)\delta^{ab} C(\mathbf{R}_G). \tag{6.30}$$

If H is semi-simple, $\mathrm{Tr}_{\mathbf{R}_H}(T_H) = 0$, thus, the anomaly vanishes. There are only anomalies in global $U(1)$ symmetries. We restrict ourselves to the case where the global symmetry group $H = U(1)$ and the fermion ψ_i has global charge q_i and transform in $\mathbf{R}_G$ representation of the gauge group G. Then, the anomaly is

$$\partial^\mu J_\mu = \frac{1}{16\pi^2} \sum_i q_i T(\mathbf{R}_i) \mathrm{Tr}_{\mathbf{fund}}(F^{\mu\nu} \widetilde{F}_{\mu\nu}). \tag{6.31}$$

Here $\mathbf{R}_i$ corresponds to the representation of gauge group G and q_i is the global $U(1)$ charge. If there are two or three global currents, the anomaly vanishes. But we will see it has consequence as low energy selection rule.

Since anomaly is proportional to $\mathrm{Tr}_{\mathbf{fund}}(F\widetilde{F})$, which is the θ term in action. The effects of the anomaly are equivalent to treating θ parameter as field and assigning it transformation properties under global $U(1)$ symmetries according to

$$\psi^i \to e^{iq_i\epsilon}\psi^i \tag{6.32}$$

$$\theta \to \theta + \epsilon \sum_i q_i T(\mathbf{R}_i). \tag{6.33}$$

Then the anomaly can be given by

$$\partial^\mu J_\mu = \frac{\delta \mathcal{L}}{\delta \epsilon} = \frac{\delta}{\delta \epsilon}\left(\frac{\theta}{16\pi^2} \mathrm{Tr}_{\mathbf{fund}}(F^{\mu\nu}\widetilde{F}_{\mu\nu})\right) \tag{6.34}$$

In this way, chiral anomaly is now seen in classical action as explicit breaking. Since anomaly appear only through θ terms, there is at most one anomalous global $U(1)$ symmetry per gauge factor.

Another implication of the chiral anomaly is the anomalous $U(1)$ charge violation:

$$\Delta Q = \int dt\partial_0 Q = \int dtd^3x\partial_0 J_0 = \int d^4x \frac{\sum_i q_i T(\mathbf{R}_i)}{16\pi^2}\mathrm{Tr}_{\mathbf{fund}}(F\widetilde{F}) = [\sum_i q_i T(\mathbf{R}_i)]n, \tag{6.35}$$

where we dropped a total derivative of the current and n is the instanton number. However, this processes changing the instanton number is suppressed at weak coupling by $\exp(-\frac{8\pi^2}{g^2}|n|)$. Therefore, though $U(1)$ is broken, at weak gauge coupling the magnitude of breaking effects is tiny. If there exists a massless Weyl chiral fermion, there will be chiral symmetry broken by anomaly. One can use this property and shift θ to 0, and thus, θ angle has no effect in this theory.

To summarize, scale and chiral anomalies in gauge theories can be written in classical action by

$$S = \int d^4x[\frac{i\tau}{16\pi}\mathrm{Tr}(\mathcal{F})^2 + c.c.] \tag{6.36}$$

with

$$\tau(\mu) = \frac{\theta}{2\pi} + i\frac{4\pi}{g^2(\mu)} = \frac{1}{2\pi i}\ln(\frac{\Lambda^b}{\mu^b}), \tag{6.37}$$

where

$$\Lambda^b = \mu^b e^{2\pi i\tau}. \tag{6.38}$$

From the transformation property of θ parameter

$$\theta \to \theta + \epsilon(\sum_i q_i T(\mathbf{R}_i)), \tag{6.39}$$

the strong coupling scale transforms as

$$\Lambda^b \to e^{i\epsilon(\sum_i q_i T(\mathbf{R}_i))}\Lambda^b. \tag{6.40}$$

Therefore, the classical action can be rewritten as

$$S = \int d^4x[\frac{1}{32\pi^2}\ln(\frac{\Lambda}{\mu})^b\mathrm{Tr}(\mathcal{F}^2) + c.c.] \tag{6.41}$$

with

$$b = \frac{11}{6}T(\mathbf{adj}) - \frac{1}{3}\sum_i T(\mathbf{R}_i) - \frac{1}{6}\sum_a T(\mathbf{R}_a), \tag{6.42}$$

and Λ^b has charge $\sum_i q_i T(\mathbf{R}_i)$ under the $U(1)$ chiral symmetry.

6.2.3 ’t Hooft Anomaly Matching

There is one other property of anomalies that is of importance. It concerns the triangle diagrams with three global currents, i.e. T^a, T^b, T^c are generators of H. This will not lead to any anomalous symmetry breaking. The argument of ’t Hooft [3] shows that they compute scale independence information about the theory. That is to say the anomalies are constant along RG flow. To prove this, we can use a spectator scheme.

Consider a theory described by a Lagrangian $\mathcal{L}$ at some scale μ, with global symmetries generated by currents J^μ_A. Gauge these symmetries by adding in new gauge fields V^A_μ, thus giving the new theory

$$\mathcal{L}' = \mathcal{L} + \int d^4x[\frac{1}{g^2}\mathrm{Tr}\mathcal{F}^2 + J_A V^A]. \tag{6.43}$$

This may not be a consistent theory. Because the newly gauged currents J_A would lead to non-vanishing triangle diagrams. In that case, to exactly cancel the anomalies, we can add in a set of new free fermion fields ψ_S which are called spectator fields. Denoting the currents of the fermions ψ_S by j^S_A, we then have the enlarged and anomaly-free theory

$$\mathcal{L}'' = \mathcal{L} + \int d^4x[\frac{1}{g^2}\mathrm{Tr}\mathcal{F}^2 + \bar{\psi}_S \partial\!\!\!/ \psi_S + (J^S_A + J_A)V^A]. \tag{6.44}$$

When the theory is weakly coupled, the anomalies are constant along RG flow so that the total anomaly remains zero for all RG scale μ.

6.3 Anomalies in Supersymmetric Gauge Theories

Now we specialize our previous description to supersymmetric gauge theories. The action for a supersymmetric gauge theory is given by

$$S = \int d^4x\{[\int d^2\theta \frac{1}{32\pi^2}\ln(\frac{\Lambda^b}{\mu^b})\mathrm{Tr}(\mathcal{W}^2)] + c.c. + \cdots\}, \tag{6.45}$$

where we have omitted the superpotential and Kähler potential. As we have demonstrated before, the anomalies is equivalent to treating Λ^b as fields. We would like to evaluate b and the charge of Λ^b.

The chiral superfield Φ_i would include a Weyl fermion and a complex boson both transforming in $\mathbf{R}_i$ representation of the gauge group. Also the vector superfield includes a Weyl fermion (the gaugino) and a gauge boson transforming in the adjoint representation. Then,

$$\begin{aligned} b &= \frac{11}{6}T(\mathbf{adj}) - \frac{1}{3}(T(\mathbf{adj}) + \sum_i T(\mathbf{R}_i)) - \frac{1}{6}(\sum_i T(\mathbf{R}_i)) \\ &= \frac{3}{2}T(\mathbf{adj}) - \frac{1}{2}\sum_i T(\mathbf{R}_i). \end{aligned} \tag{6.46}$$

The Λ^b charge $\sum_i q_i T(\mathbf{R}_i)$ depends on q_i. That is to say it depends on the definition of chiral $U(1)$. Since there are many $U(1)$'s in free theory, there are many choices. First we consider an axial flavor $U(1)_i$ symmetry. Axial flavor $U(1)_i$'s are symmetries under which only one chiral superfield rotates:

$$U(1)_i \equiv \begin{cases} \Phi_i & \to e^{i\epsilon}\Phi_i \qquad \psi_i \text{ has charge } 1 \\ \Phi_j & \to \Phi_j \qquad \psi_j \text{ has charge } 0 \quad (j \neq i) \\ \mathcal{W}_\alpha & \to \mathcal{W}_\alpha \qquad \lambda_\alpha \text{ has charge } 0 \end{cases} \tag{6.47}$$

We can get the charge of Λ^b

$$\sum_i q_i T(\mathbf{R}_i) = T(\mathbf{R}_i), \tag{6.48}$$

where the summation is over all the fermions. Another choice is the $U(1)_R$ symmetry. Under $U(1)_R$, θ transforms as

$$\theta \to e^{i\epsilon}\theta \tag{6.49}$$

which means the R-charge of θ is one (i.e. $R[\theta] = +1$).Also, we can get $R[\tau \mathrm{Tr}\mathcal{W}^2] = +2$. An especially convenient such R-symmetry is:

	$U(1)_R$
Φ_i	$2/3$
$\mathcal{W}_\alpha$	1

We can get the R-charge of the component fields:

$$R[\psi_i] = -\frac{1}{3}, \quad R[\phi_i] = \frac{2}{3}, \quad R[\lambda_\alpha] = 1, \quad R[F_{\mu\nu}] = 0. \tag{6.50}$$

This is classical symmetry because $R[\mathrm{Tr}(\mathcal{W}^2)] = +2$, $R[\Phi^\dagger\Phi] = 0$ and Kähler terms are invariant. We can get

$$\sum q_i T(\mathbf{R}_i) = T(\mathbf{adj}) - \frac{1}{3}\sum_i T(\mathbf{R}_i) = \frac{2}{3}b, \tag{6.51}$$

where b is given by (6.46). Thus, Λ^b has $U(1)_R$ charge $2b/3$ or Λ has $U(1)_R$ charge $2/3$. There are many other R symmetries possible, for example, $U(1)_R + \sum_i \alpha_i U(1)_i$. But we will call this one above *the* $U(1)_R$.

6.4 Superspace Formalization of Anomalies

We now write the anomaly in superspace language. The conservation of current is obtained from equation of motion and we will enhance the equation of motion to superspace equation of motion.

6.4.1 Konishi Anomaly

Consider the axial flavor symmetry $U(1)_i$ under which only one chiral superfield transforms as

$$\Phi_i \to e^{i\epsilon}\Phi_i. \tag{6.52}$$

The chiral anomaly then reads

$$\partial_\mu J_i^\mu = \frac{T(\mathbf{R}_i)}{16\pi^2}\mathrm{Tr}(F_{\mu\nu}\widetilde{F}^{\mu\nu}). \tag{6.53}$$

Note that classically the conservation of the current follows from equations of motion

$$\partial_\mu(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}) - \frac{\partial\mathcal{L}}{\partial\phi} = 0. \tag{6.54}$$

Similarly, given a supersymmetric action

$$\begin{aligned} S &= \int d^4x[\int d^4\theta\Phi^\dagger e^V\Phi + \int d^2\theta(W(\Phi) + \tau\mathrm{Tr}\mathcal{W}^2) + c.c.] \\ &= \int d^4y d^2\theta[\overline{\mathcal{D}}^2(\Phi^\dagger e^V\Phi) + W(\Phi) + \tau\mathrm{Tr}\mathcal{W}^2] + \int d^4x d^2\bar\theta[\overline{W}(\Phi^\dagger) + \bar\tau\mathrm{Tr}\overline{\mathcal{W}}^2], \end{aligned} \tag{6.55}$$

treating Φ and $\Phi^\dagger$ independently, superspace equation of motion can be obtained by

$$0=\frac{\delta S}{\delta \Phi_i(y,\theta)}=\overline{\mathcal{D}}^2(\Phi_i^\dagger e^V)+\frac{\partial W}{\partial \Phi_i}. \tag{6.56}$$

Then,

$$-\overline{\mathcal{D}}^2(\Phi_i^\dagger e^V\Phi_i)=+\Phi_i\frac{\partial W}{\partial \Phi_i}. \tag{6.57}$$

However, quantum mechanically, as we can see in (6.53), we should add anomaly on right hand side

$$-\overline{\mathcal{D}}^2(\Phi_i^\dagger e^V\Phi_i)=+\Phi_i\frac{\partial W}{\partial \Phi_i}+\frac{T(\mathbf{R}_i)}{32\pi^2}\mathrm{Tr}(\mathcal{W}^2). \tag{6.58}$$

This is called the Konishi anomaly.

6.4.2 $U(1)_R$ Anomaly

Now we want to compute the $U(1)_R$ anomaly in superspace language. However, We cannot use the same trick as above, because $U(1)_R$ does not commute with supersymmetry generators. Recall that under the $U(1)_R$ symmetry, chiral superfields and supersymmetric field strength transform as

$$\Phi_i\to e^{i\frac{2}{3}\epsilon}\Phi_i,\quad \mathcal{W}_\alpha\to e^{i\epsilon}\mathcal{W}_\alpha. \tag{6.59}$$

We can get the anomaly

$$\partial_\mu J_R^\mu=\frac{\frac{2}{3}b}{32\pi^2}\mathrm{Tr}(F_{\mu\nu}\widetilde{F}^{\mu\nu}). \tag{6.60}$$

But if there is a scale invariance symmetry under which the fields transform as follows

$$\Phi_i\to e^{\delta}\Phi_i,\quad \mathcal{W}_\alpha\to e^{\frac{3}{2}\delta}\mathcal{W}_\alpha. \tag{6.61}$$

We obtain the trace anomaly

$$T_\mu{}^\mu=\partial_\mu D^\mu=\frac{b}{32\pi^2}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}). \tag{6.62}$$

We can get

$$\frac{2}{3}T_\mu{}^\mu+i\partial_\mu J_R^\mu=\frac{2}{3}\frac{b}{32\pi^2}\mathrm{Tr}(\mathcal{F}^2). \tag{6.63}$$

If we supersymmetrize this result, we get the anomaly

$$\frac{2}{3}\frac{b}{32\pi^2}\mathrm{Tr}(\mathcal{W}^2)+3W-\sum_i\Phi_i\frac{\partial W}{\partial \Phi_i}. \tag{6.64}$$

This indicates that scale and $U(1)_R$ currents should appear together in the same superfield as $\frac{2}{3}D^\mu+iJ_R^\mu$. It turns out that we can write a vector superfield to include this term as

$$V_\mu=(\frac{2}{3}D_\mu+iJ_\mu^R)+\theta^\alpha S_{\mu\alpha}+c.c.+\bar\theta\sigma^\nu\theta\{\frac{2}{3}T_{\mu\nu}+\partial_{[\mu}J_{\nu]}^R\}+\cdots. \tag{6.65}$$

We define $V_{\alpha\dot{\alpha}} = \sigma^{\mu}_{\alpha\dot{\alpha}} V_\mu$, then,

$$\overline{\mathcal{D}}^{\dot{\alpha}} V_{\alpha\dot{\alpha}} = \frac{2}{3}\mathcal{D}_\alpha \left(3W - \sum_i \Phi_i \frac{\partial W}{\partial \Phi_i} + \frac{b}{32\pi^2}\mathrm{Tr}(\mathcal{W}^2) - \frac{1}{2}\overline{\mathcal{D}}^2 \left[\sum_i \gamma_i \Phi_i^\dagger e^V \Phi_i \right] \right). \tag{6.66}$$

Plugging Konishi anomaly into this expansion gives

$$\overline{\mathcal{D}}^{\dot{\alpha}} V_{\alpha\dot{\alpha}} = \frac{2}{3}\mathcal{D}_\alpha \left(\left[3W - \sum_i (1 - \frac{1}{2}\gamma_i)\Phi_i \frac{\partial W}{\partial \Phi_i} \right] + \left[b + \sum_i \frac{1}{2}\gamma_i T(\mathbf{R}_i) \right] \frac{1}{32\pi^2}\mathrm{Tr}\mathcal{W}^2 \right). \tag{6.67}$$

If the superpotential has the form $W = \sum_r \lambda_r \mathcal{O}_r$, where $\mathcal{O}_r = \prod_i \Phi_i^{n_{ir}}$ with scaling dimension $\Delta_r = \sum_i n_{ir}$. Then we obtain

$$\overline{\mathcal{D}}^{\dot{\alpha}} V_{\alpha\dot{\alpha}} = \frac{2}{3}\mathcal{D}_\alpha \left(\sum_r \lambda_r \left[3 - \Delta_r + \frac{1}{2}\sum_i n_{ir}\gamma_i \right] \mathcal{O}_r + \left[b + \sum_i \frac{1}{2}\gamma_i T(\mathbf{R}_i) \right] \frac{1}{32\pi^2}\mathrm{Tr}\mathcal{W}^2 \right). \tag{6.68}$$

Therefore conditions for scale invariance are

$$\Delta_r - 3 - \frac{1}{2}\sum_i n_{ir}\gamma_i = 0, \tag{6.69}$$

$$b + \frac{1}{2}\sum_i \gamma_i T(\mathbf{R}_i) = \frac{3}{2}T(\mathbf{adj}) - \frac{1}{2}\sum_i (1 - \gamma_i) T(\mathbf{R}_i) = 0. \tag{6.70}$$

Note that the expansion for $U(1)_R$ anomaly is exact. But trace anomaly was computed only at one loop. Since supersymmetry relates them, it must be that both β-functions computed above are exact. These scale invariance conditions have a lot of the same information as non-renormalization theorems. We will see how to recover them from non-renormalization argument for vector superfields.

Chapter 7

Non-renormalization Theorem for Vector Superfields

In Chapter 5, we have introduced the non-renormalization theorem for chiral superfields. Now we turn to the vector superfield case. The UV action at energy scale μ_0 can be given by

$$S^{(\mu_0)} = \int d^4x \{ \int d^4\theta \Phi_i^\dagger e^V \Phi_i + \int d^2\theta \widetilde{W}^{(\mu_0)} + c.c. \} \tag{7.1}$$

with general superpotential

$$\widetilde{W}^{(\mu_0)} = \frac{i\tau_0}{16\pi} \mathrm{Tr}(\mathcal{W}^2) + W^{(\mu_0)}(\Phi_i, \lambda_0) \tag{7.2}$$

We want to find dependence of $\widetilde{W}^{(\mu)}$ on τ_0, λ_0 in effective action at scales $\mu < \mu_0$. Let us consider the interacting case where $S^{(\mu_0)}$ describes an asymptotically free theory, i.e. one that is weakly coupled when μ_0 is large enough. In general, we do not know what the relation is between the (free) UV field content, and the effective IR field content. However, we can do something more modest: let us try to derive non-renormalization theorems for the effective action at a scale $\mu < \mu_0$, but only slightly less: $\frac{\mu_0 - \mu}{\mu_0} \ll 1$. Then, if μ_0 is large enough, the theory is weakly coupled, as we can be confident that if we run only slightly down in scale, the theory will remain weakly coupled, and the same field content will be good. Therefore, the effective generalized superpotential is written as

$$\widetilde{W}^{(\mu)} = \frac{i}{16\pi} \tau_\mu(\Phi_i, \lambda_0, \tau_0) \mathrm{Tr}(\mathcal{W}^2) + W^{(\mu)}(\Phi_i, \lambda_0, \tau_0). \tag{7.3}$$

7.1 Non-renormalization of the complex coupling

Recall what we has defined as a complex coupling, $\tau_0 \equiv \frac{b}{2\pi i} \ln(\frac{\Lambda}{\mu_0})$. The rotation of the θ angle

$$\theta \to \theta + 2\pi \tag{7.4}$$

would lead to

$$\Lambda^b \to e^{2\pi i} \Lambda^b, \quad \tau_0 \to \tau_0 + 1. \tag{7.5}$$

The periodicity of the θ angle should remain true as we flow down to μ, since it only depends on the topological quantization of the $\mathrm{Tr}(F\widetilde{F})$ term. Thus, we can write down the selection rule for $\tau_\mu(\Phi_i, \lambda_0, \Lambda^b)$ as

$$\tau_\mu(\Phi_i, \lambda_0, e^{2\pi i}\Lambda^b) = \tau_\mu(\Phi_i, \lambda_0, \Lambda^b) + 1. \tag{7.6}$$

Since τ_μ is a holomorphic function of its arguments, it must have the form

$$\tau_\mu = \frac{b}{2\pi i} \ln(\frac{\Lambda}{\mu}) + h(\Lambda^b, \Phi_i, \lambda_0), \tag{7.7}$$

where h is an arbitrary single-valued holomorphic function of its arguments. Since we are dealing with an asymptotically free theory, the weak coupling limit is $\Lambda \to 0$. For this limit to have smooth physics, only positive powers of Λ^b should appear. Thus,

$$\tau_\mu = \frac{b}{2\pi i} \ln(\frac{\Lambda}{\mu}) + \sum_{n=1}^{\infty} \Lambda^{nb} h_n(\Phi_i, \lambda_0). \tag{7.8}$$

Recall that $\ln \Lambda$ is proportional to $1/g^2$, we can get that Λ^{nb} is proportional to $\exp(-\frac{8\pi^2 n}{g^2})$. So we learn that the holomorphic coupling does not get any corrections in perturbative theory beyond one loop, though it may get non-perturbative corrections.

7.2 Superpotential in the Presence of Vector Superfield

The superpotential satisfies a similar "non-renormalization" theorem. If we turned off the gauge coupling ($\Lambda \to 0$), then we would just have our previous superpotential non-renormalization theorem: $W^{(\mu)} = W^{(\mu_0)}(\Phi_i, \lambda_0, \mu)$. Turning on Λ (the gauge interactions) can only add new terms holomorphic in Λ^b and vanishing as $\Lambda \to 0$, therefore, we can write the effective superpotential as

$$W^{(\mu)} = W^{(\mu_0)} + \sum_{n=1}^{\infty} \Lambda^{nb} g_n(\Phi_i, \lambda_0). \tag{7.9}$$

Again, it implies that the superpotential W gets no perturbative corrections, but may get non-perturbative ones. We can tighten these results considerably using selection rules from other global symmetries. In Chapter 6, we see that chiral anomalies are equivalent to treat θ angles as fields. (Please be careful here and do not confuse θ angles with the supercoordinate θ.) Thus, it generates new global symmetry. Now suppose that superpotential is of the form

$$W^{(\mu_0)} = \sum_r \lambda_r \mathcal{O}_r, \quad \mathcal{O}_r = \prod_i \Phi_i^{n_{ir}}, \tag{7.10}$$

where Φ_i is gauge invariant and the scaling dimension of the operator $\mathcal{O}_r$ is $\Delta_r = \sum_i n_{ir}$.

Consider an $U(1)_i$ symmetry under which chiral superfields transform as $\Phi_i \to e^{i\epsilon}\Phi_i$, $\mathcal{W}_\alpha \to \mathcal{W}_\alpha$. Now we sum the $U(1)_i$ over all fermions and denote this symmetry $U(1)_{\text{tot}}$. Then, under this symmetry, $\mathcal{O}_r$ and λ_r transform as

$$\mathcal{O}_r \to e^{i\sum_i n_{ir}\epsilon}\mathcal{O}_r = e^{i\Delta_r \epsilon}\mathcal{O}_r, \tag{7.11}$$

$$\lambda_r \to e^{-i\sum_i n_{ir}\epsilon}\lambda_r = e^{-i\Delta_r \epsilon}\lambda_r. \tag{7.12}$$

And from the anomalies, we can get

$$\theta \to \theta + \epsilon T(\mathbf{R}_i), \quad \Lambda^b \to e^{i\epsilon T(\mathbf{R}_i)}\Lambda^b. \tag{7.13}$$

Similarly, for the $U(1)_R$ symmetry, the charge of each component field is assigned:

$$\Phi_i \to \Phi_i, \quad \psi_i \to e^{-i\epsilon}\psi_i, \tag{7.14}$$
$$\mathcal{W}_\alpha \to e^{i\epsilon}\mathcal{W}_\alpha, \quad \lambda_\alpha \to e^{i\epsilon}\lambda_\alpha. \tag{7.15}$$

Since the superpotential term has R-charge +2, we have:

$$W^{(\mu_0)} \to e^{2i\epsilon}W^{(\mu_0)}. \tag{7.16}$$

One way is to assign R-cahrge as follows

$$\mathcal{O}_r \to \mathcal{O}_r, \quad \lambda_r \to e^{2i\epsilon}\lambda_r. \tag{7.17}$$

Then, from the anomaly we obtain the transformation law of θ angle and strong gauge coupling:

$$\theta \to \theta + \epsilon[T(\mathbf{adj}) - \sum_i T(\mathbf{R}_i)], \tag{7.18}$$
$$\Lambda^b \to e^{i\epsilon[T(\mathbf{adj}) - \sum_i T(\mathbf{R}_i)]}\Lambda^b. \tag{7.19}$$

We can summarize the charges using a table below:

	$U(1)_{\text{tot}}$	$\times$	$U(1)_R$
Φ_i	1		0
$\mathcal{W}_\alpha$	0		1
λ_r	$-\Delta_r$		2
Λ^b	$\sum_j T(\mathbf{R}_j)$		$T(\mathbf{adj}) - \sum_i T(\mathbf{R}_i)$

Before we discuss the non-perturbative contribution to the superpotential, now we are interested in a non-perturbative contribution to $\tau_\mu \mathrm{Tr}(\mathcal{W}^2)$. The contribution is proportional to

$$\Lambda^{b\alpha}\prod_r \lambda_r^{\alpha_r}\prod_i \Phi_i^{\beta_i}\mathrm{Tr}(\mathcal{W}^2). \tag{7.20}$$

From weak coupling limit $\Lambda, \lambda_r \to 0$, we see there are no negative powers of them, $\alpha, \alpha_r \geqslant 0$. Also, from the fact that τ_μ is single valued in this limit, we can get that both α and α_r are integers. Then, according to the selection rule from anomalous symmetries, the $U(1)_{\text{tot}}$ charge of this term should be zero and the $U(1)_R$ charge should be +2:

$$\alpha\sum_i T(\mathbf{R}_i) - \sum_r \alpha_r\Delta_r + \sum_i \beta_i = 0, \tag{7.21}$$
$$\alpha[T(\mathbf{adj}) - \sum_i T(\mathbf{R}_i)] + 2\sum_r \alpha_r + 2 = 2. \tag{7.22}$$

From these equations, we can get

$$\alpha[\frac{3}{2}T(\mathbf{adj}) - \frac{1}{2}\sum_i T(\mathbf{R}_i)] + \sum_r (3-\Delta_r)\alpha_r + \sum_i \beta_i = 0, \tag{7.23}$$

or

$$\alpha b + \sum_r (3-\Delta_r)\alpha_r + \sum_i \beta_i = 0. \tag{7.24}$$

We were interested in asymptotically free theories, that is to say all microscopic couplings are relevant or marginal, namely,

$$b \geqslant 0, \quad 3-\Delta_r \geqslant 0. \tag{7.25}$$

Since $\alpha, \alpha_r \geqslant 0$, we can get $\sum_i \beta_i \leqslant 0$. Therefore, the corrections only involve Φ_i's to negative powers. Furthermore, non-perturbative corrections to gauge coupling is proportional to $\Lambda^{b\alpha}\prod_r \lambda_r^{\alpha_r}\mathrm{Tr}(\mathcal{W}^2)$, we can get that $\beta_i = 0$ and $\alpha b + \sum_r (3-\Delta_r)\alpha_r = 0$. There is no solution to this equation. Therefore, the (holomorphic) gauge couplings are one loop exact.

We can analyze contributions to superpotential similarly, though it is a bit more complicated. The non-perturbative contribution to the superpotential $\lambda_s\mathcal{O}_s$ gives

$$\Lambda^{b\alpha}\prod_r \lambda_r^{\alpha_r}\mathcal{O}_s. \tag{7.26}$$

The selection rule from the symmetries indicates that it has $U(1)_{\rm tot}$ charge 0 and $U(1)_R$ charge +2:

$$\alpha\sum_i T(\mathbf{R}_i) - \sum_r \alpha_r\Delta_r + \Delta_s = 0, \tag{7.27}$$

$$\alpha[T(\mathbf{adj}) - \sum_i T(\mathbf{R}_i)] + 2\sum_r \alpha_r = 2. \tag{7.28}$$

From these two equations we can get

$$\alpha b + \sum_r (3-\Delta_r)\alpha_r = 3-\Delta_s. \tag{7.29}$$

Again, for asymptotically free theory, $b \geqslant 0$, $3-\Delta_r \geqslant 0$ and $\alpha, \alpha_r \geqslant 0$. We can get that $\Delta_s \leqslant 3$. The only solution is

$$\alpha = 0, \quad \alpha_s = 1, \quad \alpha_r = 0 \ (r \neq s). \tag{7.30}$$

Then, the correction reads $\lambda_s\mathcal{O}_s$. Thus, tree-level superpotential is not renormalized. But non-perturbative corrections with inverse powers of Φ_i are allowed.

7.3 Exact β-functions

Let us interpret these non-renormalization theorems. They applied to couplings in the holomorphic generalized superpotential only. The Kähler term is not protected. So there will be wave function renormalizations as well:

$$S^{(\mu)} = \int d^4x[\int d^4\theta Z_i\Phi_i^\dagger e^V\Phi_i + \int d^2\theta\{\frac{1}{32\pi^2}\ln(\frac{\Lambda}{\mu})^b\mathrm{Tr}\mathcal{W}^2 + \sum_r \mu^{3-\Delta_r}\lambda_r\mathcal{O}_r\} + c.c.]. \tag{7.31}$$

We are interested in the $\mathrm{Tr}\mathcal{W}^2$ here, we may write it as

$$-\frac{1}{4}\int d^4x \int d^2\theta(\frac{1}{g^2}-\frac{i\theta}{8\pi^2})\mathrm{Tr}\mathcal{W}^2 \tag{7.32}$$

We need to rescale fields to canonically normalize. We want the coefficient of $\Phi_i^\dagger e^V \Phi_i$ is equal to one and the real part of the coefficient of $\mathrm{Tr}\mathcal{W}^2$ is also one, therefore, we put $V = g_{C.N.}V_{C.N.}$, and the Kähler term becomes $\Phi_i^{\dagger C.N.} e^{g_{C.N.}V_{C.N.}}\Phi_i^{C.N.}$.

The wave function renormalization is given by

$$\Phi_i \to \Phi_i = \frac{1}{\sqrt{Z_i}}\Phi_i^{C.N.}. \tag{7.33}$$

However, this change of variables rescales fermion as

$$\psi_i \to \frac{1}{\sqrt{Z_i}}\psi_i^{C.N.}. \tag{7.34}$$

In our supersymmetric scheme, this rescaling may have an anomaly (equivalently, it is a non-trivial Jacobian in path integral). We can see as follows. If we think of $1/\sqrt{Z_i}$ as chiral superfield, then it is complex and includes phase rotation as well as scaling. The anomaly in the phase rotation then must also apply to rescaling:

$$\psi_i \to e^{\epsilon_i}\psi_i = e^{i(-i\epsilon_i)}\psi_i, \tag{7.35}$$

$$\theta \to \theta - \sum_i (i\epsilon_i)T(\mathbf{R}_i), \tag{7.36}$$

where

$$\epsilon_i = \ln(1/\sqrt{Z_i}) = -\frac{1}{2}\ln Z_i. \tag{7.37}$$

Therefore,

$$\theta \to \theta + \frac{i}{2}\sum_i(\ln Z_i T(\mathbf{R}_i)). \tag{7.38}$$

Similarly, the field strength chiral superfield scales as

$$\mathcal{W}_\alpha \to \mathcal{W}_\alpha = g_{C.N.}\mathcal{W}_\alpha^{C.N.} = e^{-\frac{1}{2}\ln\left(\frac{1}{g^2_{C.N.}}\right)}\mathcal{W}_\alpha^{C.N.}, \tag{7.39}$$

so that θ angle transforms as

$$\theta \to \theta + \frac{i}{2}\ln(\frac{1}{g^2_{C.N.}})T(\mathbf{adj}). \tag{7.40}$$

Therefore, the net result of the renormalization of both chiral superfield Φ_i and field strength chiral superfield $\mathcal{W}_\alpha$ will lead to θ angle transforming as

$$\theta \to \hat{\theta} = \theta + \frac{i}{2}[T(\mathbf{adj})\ln(\frac{1}{g^2_{C.N.}}) + \sum_i \ln Z_i T(\mathbf{R}_i)]. \tag{7.41}$$

The Lagrangian is given by

$$\mathcal{L}_\mu \to \mathcal{L}_\mu^{C.N.} = \int d^4\theta \Phi_i^{\dagger C.N.} e^{g_{C.N.}V_{C.N.}}\Phi_i^{C.N.} - \frac{1}{4}\int d^2\theta(\frac{g^2_{C.N.}}{g^2(\mu)} - \frac{i\hat{\theta}g^2_{C.N.}}{8\pi^2})\mathrm{Tr}\mathcal{W}^2_{C.N.} + c.c. \tag{7.42}$$

In order to get the canonical renormalization of the coupling g, we require that $g^2_{C.N.}$ satisfies that the coefficient in front of $\mathrm{Tr}\mathcal{W}^2_{C.N.}$ is equal to $(1-\frac{i\theta g^2_{C.N.}}{8\pi^2})$ so that we get

$$\frac{g^2_{C.N.}}{g^2(\mu)}+\frac{g^2_{C.N.}}{16\pi^2}[T(\mathbf{adj})\ln(\frac{1}{g^2_{C.N.}})+\sum_i \ln Z_i T(\mathbf{R}_i)]=1. \tag{7.43}$$

After some simplification, we obtain

$$\frac{1}{g^2_{C.N.}}=\frac{1}{g^2(\mu)}+\sum_i\frac{\ln Z_i T(\mathbf{R}_i)}{16\pi^2}+\frac{T(\mathbf{adj})}{16\pi^2}\ln(\frac{1}{g^2_{C.N.}}). \tag{7.44}$$

Taking $\dfrac{d}{d\ln\mu}$ on both sides, we obtain

$$\frac{d(1/g^2_{C.N.})}{d\ln\mu}=\frac{b+\dfrac{1}{2}\sum_i T(\mathbf{R}_i)\gamma_i}{8\pi^2-\dfrac{1}{2}T(\mathbf{adj})g^2_{C.N.}}, \tag{7.45}$$

where we use $\frac{d}{d\ln\mu}(\frac{1}{g^2(\mu)})=\frac{b}{8\pi^2}$, $\frac{d\ln Z_i}{d\ln\mu}=\gamma_i$. This is called the NSVZ β-functions. [14] In general, we cannot compute the anomalous dimension γ_i non-perturbatively. However, we can consider supersymmetric pure Yang-Mills theory without chiral superfields. Then we can set $\gamma_i=0$, yeilding

$$\frac{d(1/g^2_{C.N.})}{d\ln\mu}=\frac{\frac{3}{2}T(\mathbf{adj})}{8\pi^2-\frac{1}{2}T(\mathbf{adj})g^2_{C.N.}}. \tag{7.46}$$

It is easy to solve this differential equation and we get

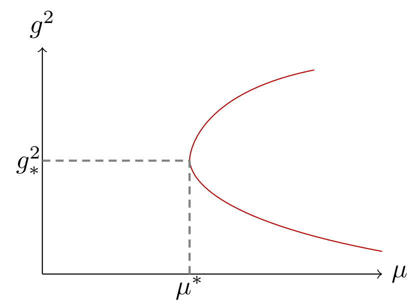


Figure 7.1: RG flow of gauge coupling in pure SYM (a)

From the figure above we see that since we can only flow down in μ, there are two branches, $g^2>g^2_*$ and $g^2<g^2_*$. If the theory is asymptotically free, only $g^2<g^2_*$ branch is physical. Then we just ignore the $g^2>g^2_*$ branch and the figure becomes

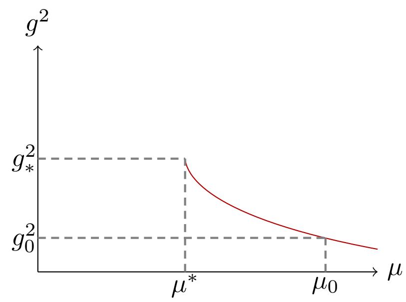


Figure 7.2: RG flow of gauge coupling in pure SYM (b)

We see as μ flows down, there is a stop at μ^*. Therefore it is not a fixed point since it does not go to $\mu = 0$. The glue ball operator $\mathrm{Tr}\mathcal{W}^2$ gets mass $\sim \mu^*$, and one can integrate it out at IR.

Non-renormalization theorems, by themselves, do not solve strong coupling problem: they can't tell you what the new effective low energy degrees of freedom will be.

Chapter 8

Seiberg Duality

Before we introduce the Seiberg duality, we first show what SQCD is. We just consider $SU(N_c)$ super Yang-Mills with N_f flavors:

	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_B$	$U(1)_R$
Q^i_a	$\square$		1	$\frac{N_f-N_c}{N_f}$
$\widetilde{Q}^b_j$		$\bar{\square}$	-1	$\frac{N_f-N_c}{N_f}$
λ			0	1

where Q^i_a, $\widetilde{Q}^b_j$ denote chiral and anti-chiral superfields transforming in fundamental ($\square$) and anti-fundamental ($\bar{\square}$) representations of $SU(N_c)$, λ denotes a gaugino transforming in the adjoint representation of $SU(N_c)$. We are interested in the IR behavior of these theories, which differ qualitatively for different values of N_f.

8.1 SQCD for $N_f = 0$

In this section, we mainly follow Yuji Tachikawa's lecture note [17]. When $N_f = 0$, we restore to the pure supersymmetric Yang-Mills. The gauge group is $SU(N_c)$. Under the $U(1)_R$ symmetry, if we shift θ angle by 2π, there is a discrete rotation of the gaugino λ_α:

$$\lambda_\alpha \to e^{2\pi i/(2N_c)}\lambda_\alpha. \tag{8.1}$$

This is a symmetry generating $\mathbb{Z}_{2N_c}$. Also, Λ rotates as

$$\Lambda \to e^{2\pi i/(3N_c)}\Lambda. \tag{8.2}$$

Now we consider the gaugino condensate $\langle\lambda_\alpha\lambda^\alpha\rangle$, which should be of mass dimension 3 and of R-charge 2. We can write

$$\langle\lambda_\alpha\lambda^\alpha\rangle = c\Lambda^3. \tag{8.3}$$

From the symmetry of Λ, we see that if we multiply by $e^{2\pi i N_c}$ on both sides of (8.3), the $\mathbb{Z}_{2N_c}$ is spontaneously broken to $\mathbb{Z}_2$, yielding N_c solutions

$$\langle\lambda_\alpha\lambda^\alpha\rangle = ce^{2\pi i l/N_c}\Lambda^3. \tag{8.4}$$

Here, $l = 0, 1, \cdots, N_c - 1$. There are N_c vacua with supersymmetry unbroken. We write $\langle S \rangle \equiv \langle \lambda_\alpha \lambda^\alpha \rangle = (\Lambda^{3N_c})^{\frac{1}{N_c}}$. The vacua is represented by $\langle W \rangle$ satisfying

$$\langle S \rangle = \frac{\partial \langle W \rangle}{\partial \ln \Lambda^{3N_c}}. \tag{8.5}$$

Solving this, we obtain

$$\langle W \rangle = N_c (\Lambda^{3N_c})^{\frac{1}{N_c}}. \tag{8.6}$$

Taking (inverse) Legendre transformation with respect to the source $\ln \Lambda^{3N_c}$, we obtain

$$W_{\text{eff}} = N_c S (1 + \ln \frac{\Lambda^3}{S}), \tag{8.7}$$

which is called a Veneziano-Yankielowicz superpotential. [18]

8.2 SQCD for $0 < N_f < N_c$

Recall that full perturbative running of holomorphic gauge coupling τ is given by

$$\tau = \frac{\theta_{YM}}{2\pi} + i \frac{4\pi}{g^2(\mu)} = \frac{b_0}{2\pi i} \ln \frac{\Lambda}{\mu} \tag{8.8}$$

where $b_0 = 3N_c - N_f$. If we add flavors to pure supersymmetric Yang-Mills theory, classically our moduli space is parametrized by VEV's of meson fields

$$M^i{}_j = Q^i_a \widetilde{Q}^a_j. \tag{8.9}$$

Next we would discuss the Affleck-Dine-Seiberg (ADS) [11] superpotential which is of importance in SQCD for $N_f < N_c$. We now define the $U(1)_A$ symmetry as

$$Q \to e^{i\alpha} Q, \quad \widetilde{Q} \to e^{i\alpha} \widetilde{Q}, \quad \Lambda^b \to e^{i2N_f\alpha} \Lambda^b. \tag{8.10}$$

Treating θ_{YM} as a VEV of a certain field. This will help us construct the non-perturbative superpotential. We, then, have the following superfields to our disposal: $\mathcal{W}^a$, Λ, M with charges

	$U(1)_A$	$U(1)_R$
$\mathrm{Tr}\mathcal{W}^2$	0	2
Λ^b	$2N_f$	0
$\det M$	$2N_f$	$2(N_f - N_c)$

We can see from the table that only powers of Λ^b are allowed and only $SU(N_f)_L \times SU(N_f)_R$ invariance is possible. It follows that we can have terms like

$$\Lambda^{bn} (\mathcal{W}^a \mathcal{W}^a)^m (\det M)^p, \tag{8.11}$$

in the effective superpotential W_{eff}. Since W_{eff} must be neutral under $U(1)_A$ and have R-charge $+2$ under $U(1)_R$, we obtain:

$$n \cdot 2N_f + p \cdot 2N_f = 0, \tag{8.12}$$

$$2m + p \cdot 2(N_f - N_c) = 2. \tag{8.13}$$

From these two equations, we obtain

$$n = -p = \frac{1-m}{N_c - N_f}. \tag{8.14}$$

The weak coupling limit requires $n \geqslant 0$. From the relation above, we can get that $p \leqslant 0$. Also, in the case $N_f < N_c$, it is easy to see that m should satisfy $m \leqslant 1$. On the other hand, to have meaningful expansion in derivatives, m should satisfy $m \geqslant 0$. Therefore, there are only two possible terms: $m = 1$ and $m = 0$. The former refers to the tree-level term while the latter refers to the ADS superpotential:

$$W_{ADS} = C_{N_c,N_f}\left(\frac{\Lambda^{3N_c-N_f}}{\det M}\right)^{\frac{1}{N_c-N_f}}. \tag{8.15}$$

Note that there is no vacuum. Instead, W_{ADS} is minimized for infinite $\langle M\rangle$ which is just pure supersymmetric Yang-Mills. We will now do two consistency checks to determine C_{N_c,N_f}. One is by moduli and the other is by mass perturbation.

Let us consider the case when one of the flavors acquires a large VEV. Thus, we break $SU(N_c)$ to $SU(N_c-1)$ and are left with $N_f - 1$ flavors in low energy effective theory. The IR theory has β-function coefficient

$$b_L = 3(N_c-1) - (N_f-1) = 3N_c - N_f - 2, \tag{8.16}$$

and its holomorphic scale Λ_L must match at scale $V \equiv$ "mass of heavy gauge boson" as

$$\left(\frac{\Lambda}{V}\right)^{b} = \left(\frac{\Lambda_L}{V}\right)^{b_L}. \tag{8.17}$$

After some simplification, we can get that

$$\frac{\Lambda^{3N_c-N_f}}{V^2} = \Lambda_{N_c-1,N_f-1}^{3N_c-N_f-2}, \tag{8.18}$$

where $\Lambda_{N_c-1,N_f-1} \equiv \Lambda_L$. Plug this back into (8.15), we obtain

$$W_{ADS}(N_c,N_f) = C_{N_c,N_f}\left(\frac{\Lambda_{N_c,N_f}^{3N_c-N_f}}{\det M}\right)^{\frac{1}{N_c-N_f}} = C_{N_c,N_f}\left(\frac{V^2\Lambda_{N_c-1,N_f-1}^{3N_c-N_f-2}}{V^2\det \widetilde{M}}\right)^{\frac{1}{N_c-N_f}}, \tag{8.19}$$

where $\widetilde{M}$ is the remaining $(N_f-1)\times(N_f-1)$ meson matrix satisfying $\det M = V^2\det\widetilde{M} + \cdots$. The ellipsis here denotes the decoupled singlets. Hence, we recover $W_{ADS}(N_c-1,N_f-1)$ provided that $C_{N_c,N_f} = C_{N_c-1,N_f-1}$. Repeating this result N_f times and notice that $N_f < N_c$ in this case, we obtain $C_{N_c,N_f} = C_{N_c-N_f,0}$.

If we give equal VEV's to all flavors, the gauge symmetry $SU(N_c)$ is broken to $SU(N_c-N_f)$. We get a similar matching condition

$$\frac{\Lambda^{3N_c-N_f}}{V^{2N_f}} = \Lambda_{N_c-N_f,0}^{3(N_c-N_f)}, \tag{8.20}$$

yielding our pure supersymmetric Yang-Mills gaugino condensation result

$$W_{\text{eff}} = C_{N_c,N_f}\Lambda_{N_c-N_f,0}^{3}. \tag{8.21}$$

This gives us some deeper understanding of how W_{ADS} is generated if $N_c - N_f \geqslant 2$, namely, it is generated by gaugino condensation of the remaining $SU(N_c - N_f)$ gauginos.

Next let us study the case when we give a mass to one of the flavors which leads to $SU(N_c)$ with $N_f - 1$ flavors. The matching condition then reads

$$m\Lambda^{3N_c - N_f} = \Lambda_{N_c, N_f - 1}^{3N_c - N_f + 1}. \tag{8.22}$$

Before integrating out this massive flavor, we can use $M_{N_f}^{N_f}$ as a building block

$$W_{\text{exact}} = \left(\frac{\Lambda^{3N_c - N_f}}{\det M}\right)^{\frac{1}{N_c - N_f}} \cdot f(t), \tag{8.23}$$

where

$$t = mM_{N_f}^{N_f} \left(\frac{\Lambda^{3N_c - N_f}}{\det M}\right)^{-\frac{1}{N_c - N_f}}. \tag{8.24}$$

We can now take the double limit $m, \Lambda \to 0$ which leaves t arbitrary. Since we must recover our previous result plus a small mass term, therefore, the function of t should have the form:

$$f(t) = C_{N_c, N_f} + t. \tag{8.25}$$

Then, the superpotential has the form

$$W_{\text{exact}} = C_{N_c, N_f} \left(\frac{\Lambda^{3N_c - N_f}}{\det M}\right)^{\frac{1}{N_c - N_f}} + mM_{N_f}^{N_f}. \tag{8.26}$$

Integrating out $M_{N_f}^{N_f}$ and $M_{N_f}^{j}$ gives

$$M = \begin{pmatrix} \widetilde{M} & 0 \\ 0 & M_{N_f}^{N_f} \end{pmatrix}, \tag{8.27}$$

and

$$mM_{N_f}^{N_f} = \frac{C_{N_c, N_f}}{N_c - N_f} \left(\frac{\Lambda^{3N_c - N_f}}{\det M}\right)^{\frac{1}{N_c - N_f}}, \tag{8.28}$$

which after plugging into W gives

$$W_{exact} = (N_c - N_f + 1) \left(\frac{C_{N_c, N_f}}{N_c - N_f}\right)^{\frac{N_c - N_f}{N_c - N_f + 1}} \left(\frac{\Lambda_{N_c, N_f - 1}^{3N_c - N_f + 1}}{\det \widetilde{M}}\right)^{\frac{1}{N_c - N_f + 1}}. \tag{8.29}$$

For consistency,

$$C_{N_c, N_f - 1} = (N_c - N_f + 1) \left(\frac{C_{N_c, N_f}}{N_c - N_f}\right)^{\frac{N_c - N_f}{N_c - N_f + 1}}. \tag{8.30}$$

Therefore, $C_{N_c, N_f} = (N_c - N_f) C^{\frac{1}{N_c - N_f}}$, where C is some constant. An instanton calculation for $N_f = N_c - 1$ gives $C = 1$, thus, $C_{N_c, N_f} = N_c - N_f$. Note that for $N_f = N_c - 1$, W_{ADS} is generated by instantons since the gauge group is then fully broken. Finally we get the ADS superpotential of the form

$$W_{ADS} = (N_c - N_f)\left(\frac{\Lambda^{3N_c-N_f}}{\det M}\right)^{\frac{1}{N_c-N_f}}. \tag{8.31}$$

For $N_f \geqslant N_c$, no ADS superpotential can be generated since for $N_f = N_c$ the exponent $\frac{1}{N_c-N_f}$ is ill-defined and for $N_f > N_c$ there is no sensible weak coupling limit.

8.3 SQCD for $N_f = N_c$

In SQCD for $N_f = N_c$, besides the meson fields, we now have two baryons

$$B = \epsilon^{a_1\cdots a_{N_c}} Q^1_{a_1}\cdots Q^{N_c}_{a_{N_c}}, \quad \widetilde{B} = \epsilon_{a_1\cdots a_{N_c}} \widetilde{Q}^{a_1}_1 \cdots \widetilde{Q}^{a_{N_c}}_{N_c}, \tag{8.32}$$

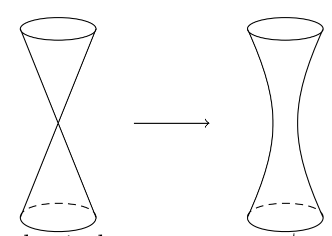
which parametrize the moduli space. By their mere definition it follows that $\det M - B\widetilde{B} = 0$ classically. However, there can be a deformation in the quantum theory that $\det M - B\widetilde{B} = a\Lambda^{2N_c}$ ($a \in \mathbb{C}$). The reason is that the right hand side goes to zero in classical limit $\Lambda \to 0$, and $2N_c$ is β-function coefficient $b = 3N_c - N_f = 2N_c$. Also it is a one-instanton correction because $e^{-S_{\text{inst}}}$ is proportional to Λ^{2N_c}.

We implement this constraint via

$$W = A(\det M - B\widetilde{B} - a\Lambda^{2N_c}). \tag{8.33}$$

Upon holomorphic decoupling, which is adding $W = mM^{N_c}_{N_c}$, and integrating out $M^{N_c}_{N_c}$, we ought to recover the ADS superpotential. Doing these steps yields $a = 1$, which leads to $\det M - B\widetilde{B} = \Lambda^{2N_c}$.

The difference of the classical and quantum moduli space is shown in the figure below. Classically, there are new massless degrees of freedom at the singularity. Quantum mechanically, the singularity is resolved and this is the expected physics of a confining vacuum. [16]



We see that the origin of field space is not part of moduli space and chiral symmetry is always broken. Moduli space is non-singular and smooth in B, $\widetilde{B}$ and M. Therefore, there is no mass gap and no additional degree of freedom. At a generic point in field space, $SU(N_c)$ is completely broken. However there are configurations where $SU(N_c)$ is not completely broken. The first is the Mesonic branch, where

$$M^i{}_j = \Lambda^2 \delta^i{}_j, \quad B = 0 = \widetilde{B}. \tag{8.34}$$

Thus the symmetry $SU(N_f)_L \times SU(N_f)_R \times U(1)_B \times U(1)_R$ is broken to $SU(N_f)_D \times U(1)_B \times U(1)_R$. The second is the Baryonic branch, where

$$M = 0, \quad B = \Lambda^{N_c} = -\widetilde{B}. \tag{8.35}$$

In this case the symmetry $SU(N_f)_L \times SU(N_f)_R \times U(1)_B \times U(1)_R$ is broken to $SU(N_f)_L \times SU(N_f)_R \times U(1)_R$. The gauge group is always broken, which leads to the Higgs phase. But also the theory is smooth in M, B, $\widetilde{B}$ and strongly coupled near the origin in field space. Thus, for small VEV's it is more appropriate to call it confining phase.

8.4 SQCD for $N_f = N_c + 1$

In SQCD for $N_f = N_c + 1$, we now have $N_c + 1$ baryons and $N_c + 1$ antibaryons:

$$B_i = \epsilon_{ij_1\cdots j_{N_c}} \epsilon^{a_1\cdots a_{N_c}} Q^{j_1}_{a_1} \cdots Q^{j_{N_c}}_{a_{N_c}}, \tag{8.36}$$

$$\widetilde{B}^i = \epsilon^{ij_1\cdots j_{N_c}} \epsilon_{a_1\cdots a_{N_c}} \widetilde{Q}^{a_1}_{j_1} \cdots \widetilde{Q}^{a_{N_c}}_{j_{N_c}}. \tag{8.37}$$

We can write the effective superpotential as

$$W_{\text{eff}} = \frac{a}{\Lambda^{2N_c-1}}(\det M - B_i M^i{}_j \widetilde{B}^j). \tag{8.38}$$

Indeed upon holomorphic decoupling one recovers previous result, provided $a = 1$. The quantum constraints, then, read

$$M\widetilde{B} = 0 = BM, \tag{8.39}$$

$$\det M \cdot (M^{-1})_i{}^j = B_i \widetilde{B}^j. \tag{8.40}$$

These are identical to classical constraints. Thus, classical moduli-space is untouched by quantum effects.

8.5 SQCD for $N_f \geqslant N_c + 2$

The previous procedure does not work any more since anomalies of the naive IR description do not match with the UV anomalies. Instead, the astonishing claim made by N. Seiberg [5] is that the IR physics for $N_f \geqslant N_c + 2$ can instead be described by a dual $SU(N_f - N_c)$ gauge theory with N_f flavors. Let us see why this is the case.

8.5.1 Conformal Window

Recall that the exact NSVZ β-function is given by

$$\beta(g) = -\frac{g^3}{16\pi^2} \frac{3N_c - N_f[1 - \gamma(g^2)]}{1 - N_c \frac{g^2}{8\pi^2}}, \tag{8.41}$$

where the anomalous dimension γ of matter-fields is

$$\gamma(g^2) = -\frac{g^2}{8\pi^2} \frac{N_c^2 - 1}{N_c} + \mathcal{O}(g^4). \tag{8.42}$$

Plug this back into the β-function and we get

$$\beta(g) = -\frac{g^3}{16\pi^2}[(3N_c - N_f) + (3N_c^2 - 2N_f N_c + \frac{N_f}{N_c})\frac{g^2}{.}8\pi^2 + \mathcal{O}(g^4)]. \tag{8.43}$$

We define a parameter ϵ as

$$\epsilon \equiv 3 - \frac{N_f}{N_c}. \tag{8.44}$$

In the limit $\epsilon \ll 1$ for large N_c, one finds an approximate IR fixed point :

$$g_*^2 = \frac{8\pi^2}{3} \frac{N_c}{N_c^2 - 1} \epsilon + \mathcal{O}(\epsilon^2), \tag{8.45}$$

which is called a Banks-Zaks fixed point. [15] We will now assume that there is an IR fixed point even if N_c is not very large, which leads to the superconformal theory below $N_f = 3N_c$. There is a non-trivial fact that in conformal theory one has $\Delta = \frac{3}{2}|R|$ for chiral fields where Δ is the mass dimension of the field and R is the R-charge of the field. Now consider the mass dimension of the meson field M:

$$\Delta(M) = \frac{3}{2}R(M) = \frac{3}{2}R(Q\widetilde{Q}) = \frac{3}{2} \cdot 2 \cdot \frac{N_f - N_c}{N_f} = 2 + \gamma_*, \tag{8.46}$$

where

$$\gamma_* = 1 - \frac{3N_c}{N_f}. \tag{8.47}$$

When $N_f = 3N_c$, we get $\gamma_* = 0$. The further fact is that, for scalar fields, the mass dimension $\Delta \geqslant 1$ because of unitarity. This gives $N_f \geqslant \frac{3}{2}N_c$. Thus, we get the conformal window

$$\frac{3}{2}N_c \leqslant N_f < 3N_c. \tag{8.48}$$

8.5.2 Seiberg duality

The natural answer to describe IR physics is in terms of mesons and baryons:

$$M^i{}_j, \quad B_{i_1 \cdots i_{N_f - N_c}}, \quad \widetilde{B}^{i_1 \cdots i_{N_f - N_c}}. \tag{8.49}$$

It is tempting to write

$$B_{i_1 \cdots i_{\overline{N}_c}} = \epsilon_{a_1 \cdots a_{\overline{N}_c}} q_{i_1}^{a_1} \cdots q_{i_{\overline{N}_c}}^{a_{\overline{N}_c}} \tag{8.50}$$

in terms of dual quarks in fundamental of $SU(\overline{N}_c)$. Here $\overline{N}_c = N_f - N_c$. More precisely, as described by Seiberg, an SQCD theory with $SU(N_c)$ gauge group, N_f flavors and no superpotential agrees at low energies with an mSQCD theory with $SU(N_f - N_c)$ gauge group, N_f flavors and superpotential $W = q_i \Phi^i{}_j \tilde{q}^j$. $\Phi^i{}_j$ is a gauge singlet transforming in (anti-)fundamental of $SU(N_f)_L$(and $SU(N_f)_R$). The fact is that mSQCD flows to an IR fixed point in the same window. As N_f becomes smaller the original theory becomes more strongly coupled. Its dual becomes more weakly coupled. This is a characteristic behavior of two dual theories. Therefore, we suggest that the two theories are dual in the sense of electric-magnetic duality. Thus, we refer to the original as electric and the dual as magnetic. That is why we write the dual theory as mSQCD. [5] Furthermore since

$$b_m = 3\overline{N}_c - N_f = 2N_f - 3N_c, \tag{8.51}$$

$N_f = \frac{3}{2}N_c$ plays the same role for mSQCD as $N_f = 3N_c$ did for SQCD. For $N_c + 2 \leqslant N_f \leqslant \frac{3}{2}N_c$, SQCD is described by mSQCD which is IR free. In conformal window, both theories are UV free and flow to IR fixed point. The duality map is

$$M \leftrightarrow \Phi: \quad {\Phi^i}_j = \frac{1}{\mu} {M^i}_j, \tag{8.52}$$

$$B \leftrightarrow b: \quad b^{j_1 \cdots j_{N_c}} = \epsilon^{i_1 \cdots i_{N_f - N_c}} B_{i_1 \cdots i_{N_f - N_c}}, \tag{8.53}$$

where b's are dual baryons and μ is the matching scale. Now we summarize the mSQCD in the table below:

	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_B$	$U(1)_R$
${q^a}_i$	$\bar{\Box}$		$\frac{N_c}{N_f - N_c}$	$\frac{N_c}{N_f}$
${\tilde{q}_b}^{\ j}$		$\Box$	$-\frac{N_f}{N_f - N_c}$	$\frac{N_c}{N_f}$
Φ	$\Box$	$\bar{\Box}$	0	$2\frac{N_f - N_c}{N_f}$
$\tilde{\lambda}$			0	1

We see that the global symmetry groups match and the mesons and baryons of SQCD are mapped to Φ, b of mSQCD. What happens with the mesons of mSQCD? The answer is that the superpotential term leads to

$$q_i \tilde{q}^j = {u_i}^j = 0, \tag{8.54}$$

by equations of motion which we will discuss precisely later.

8.5.3 Consistency Checks for the duality

To be able to believe that this duality holds one needs to check the following crucial properties. The first is the 't Hooft anomaly matching conditions which we will skip.

The second is that when acting twice with duality map on a theory, it will give back the original theory.

$$\begin{array}{ll} \text{SQCD} & SU(N_c),\ N_f,\ W = 0 \\ \downarrow & \\ \text{mSQCD} & SU(N_f - N_c),\ N_f,\ W = \frac{1}{\mu} q_i {M^i}_j \tilde{q}^j = q_i {\Phi^i}_j \tilde{q}^j \\ \downarrow & \\ \text{mmSQCD} & SU(N_c),\ N_f,\ W = \frac{1}{\mu} q_i {M^i}_j \tilde{q}^j + \frac{1}{\tilde{\mu}} d_i {u^i}_j \tilde{d}^j \end{array}$$

As shown in the flow chart, the SQCD theory is a theory with $SU(N_c)$ gauge group with N_f flavors and the superpotential $W = 0$. Acting duality map on it, we will obtain the mSQCD theory with $SU(N_f - N_c)$ gauge group with N_f flavors and the superpotential gives

$$W = \frac{1}{\mu} q_i {M^i}_j \tilde{q}^J = q_i {\Phi^i}_j \tilde{q}^j. \tag{8.55}$$

Again, acting duality map on it, we obtain the mmSQCD theory with $SU(N_c)$ gauge group with N_f flavors, the superpotential is given by

$$W = \frac{1}{\mu} q_i {M^i}_j \tilde{q}^j + \frac{1}{\tilde{\mu}} d_i {u^i}_j \tilde{d}^j, \tag{8.56}$$

where the d's are the chiral superfields of mmSQCD and $u^i{}_j$ are the mesons of mSQCD: $u = q\tilde{q}$. Choosing $\mu = -\tilde{\mu}$ yields

$$W = \frac{1}{\mu}\mathrm{Tr}(uM - dud). \tag{8.57}$$

Both M and u are massive mesons. From the equation of motion we obtain

$$u = 0, \quad M^i{}_j = d^i \tilde{d}_j. \tag{8.58}$$

Thus, the mmSQCD theory goes back to the original theory.

The third is that we can check the stability under mass perturbations $W = mM^{N_f}_{N_f}$.In dual theory this produces

$$W = \frac{1}{\mu} q_i M^i{}_j \tilde{q}^j + mM^{N_f}_{N_f}. \tag{8.59}$$

The equation of motion for M reads

$$q^a_{N_f} \tilde{q}_a^{N_f} + \mu m = 0. \tag{8.60}$$

The equation of motion for q_{N_f}, $\tilde{q}^{N_f}$ reads

$$(M\tilde{q}_a)^{N_f} = 0 = (q^a M)_{N_f}. \tag{8.61}$$

Thus we have an $SU(N_f - N_c - 1)$ dual theory with $N_f - 1$ flavors. It turns out that the diagram below is a commuting diagram.

$$\begin{tikzcd}
SU(N_c), N_f \arrow[r, leftrightarrow, "\text{dual}"] \arrow[d, "\text{mass}"] & SU(N_f - N_c), N_f \arrow[d, "\text{higgsing}"] \\
SU(N_c), N_f - 1 \arrow[r, leftrightarrow] & SU(N_f - N_c - 1), N_f - 1
\end{tikzcd}$$

Last but not least, we can reproduce SQCD for $N_f = N_c + 1$ from SQCD for $N_f = N_c + 2$. For $N_f = N_c + 2$, we have magnetic gauge group $SU(2)$. Upon holomorphic decoupling

$$W \sim q_i M^i{}_j \tilde{q}^j, \tag{8.62}$$

where q, $\tilde{q}$'s can be identified with baryons of SQCD for $N_f = N_c + 1$. Since $SU(2)$ breaks completely, we need to include instanton effects giving

$$W_{\text{eff}} \sim (B_i M^i{}_j \widetilde{B}^j - \det M). \tag{8.63}$$

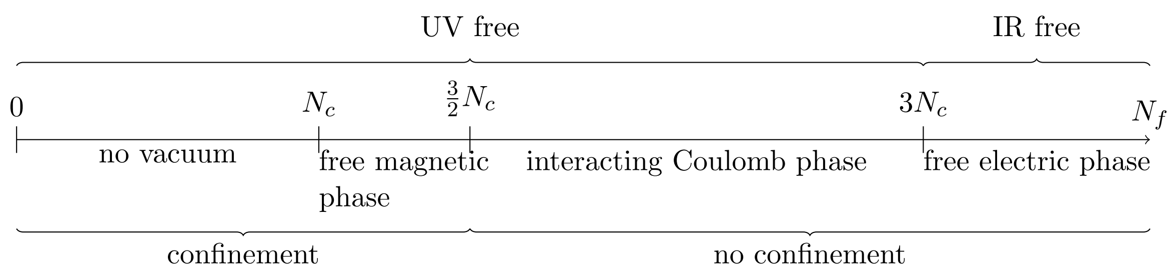
Thus, we obtain the SQCD for $N_f = N_c + 1$.

8.6 Conclusion

We summarize the conclusion as below, for each different N_f, we get

- $N_f = 0$: strict confinement, N_c distinct supersymmetric vacua,
- $0 < N_f < N_c$: runaway vacuum,
- $N_f = N_c, N_c + 1$: deformed moduli-space, confinement and Higgs phase,
- $N_c + 2 \leqslant N_f \leqslant \frac{3}{2} N_c$: magnetic free phase in IR, free quarks and gluons in UV, magnetic gauge group generated non-perturbatively,
- $\frac{3}{2} N_c < N_f < 3N_c$: superconformal QFT with IR fixed point,
- $N_f \geqslant 3N_c$: electric free phase in IR, free dual quarks and gluons in UV.

These results can be shown in the picture below.



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