Homework 6: Due at class on Nov 2

1 Fusion rule

Following (5.36) of the lecture note, find the explicit fusion rule of the product

$$\phi_{1,3} \times \phi_{2,2}$$

2 Descendant of the identity operator

The vacuum $|0\rangle$ corresponds to the identity operator $\mathbf{1}(w)$ whose OPE with the energy-momentum tensor is

$$T(z)\mathbf{1}(w) = \frac{0}{(z-w)^2}\mathbf{1}(w) + \frac{1}{z-w}\frac{\partial}{\partial w}\mathbf{1}(w) + L_{-2}\mathbf{1}(w) + \ldots$$

where the conformal dimension of $\mathbf{1}(w)$ is zero and $\partial_w \mathbf{1}(w) = 0$. Therefore, by the state-operator correspondence identifies

$$L_{-2}|0\rangle \ \leftrightarrow \ T(w) \ ,$$

so that the energy-momentum tensor can be understood as the descendant of the identity operator $\mathbf{1}(w)$. Furthermore, we have the correspondence

$$L_{-n-2}|0\rangle \ \leftrightarrow \ \frac{1}{n!}\left(\frac{\partial}{\partial w}\right)^n T(w) \quad \text{for } n \geq 0 \ .$$

Zamolodchikov has introduced the following descendant

$$\Lambda(w) \equiv :T(w)T(w): -\frac{3}{10}\frac{\partial^2}{\partial w^2}T(w) \ .$$

Compute the OPE $T(z)\Lambda(w)$ by using generalized Wick theorem in Appendix 6.B of [FMS97]. Find the value of the central charge when $\Lambda(w)$ becomes a primary field. From the value of the central charge, find which minimal model is related to $\Lambda(w)$.

3 Feigin-Fuchs representation

We shall derive the Kac determinant (5.18) of the lecture note by following Feigin-Fuchs.

3.1 Kac determinant

Show that the parameters in the Kac determinant (5.18) obey

$$\alpha_+ + \alpha_- = 2\sqrt{-h_0} \ , \qquad \alpha_+\alpha_- = -1 \ .$$

3.2 Screening charge

Let us consider the linear dilaton CFT in Problem 3 of Homework 5 where the background charge is set to

$$Q=\sqrt{2h_0}=i\sqrt{\frac{1-c}{12}}\ .$$

Recalling that a vertex operator $V_k(z)=:e^{ik\varphi(z)}:$ in the linear dilaton CFT has conformal dimension $h=k(k+2iQ)/2$, derive that the conformal dimension of the vertex operator $V_{\sqrt{2}\alpha_\pm}(w)$ is equal to one. Hence, $\mathbf{S}_\pm=\oint\frac{dw}{2\pi i}V_{\sqrt{2}\alpha_\pm}(w)$ are the screening charge. Show that the state

$$\mathbf{S}_\pm^r|k\rangle \qquad \text{for } r\in\mathbb{Z}_{>0}$$

is a singular vector where $|k\rangle=\lim_{z\to0}V_k(z)|0\rangle$ if it is not zero.

3.3 Integral representation

It admits the integral representation as

$$\begin{aligned}\mathbf{S}_+^r|k\rangle&=\frac{1}{(2\pi i)^r}\int dz_1\cdots dz_r V_{\sqrt{2}\alpha_+}(z_1)\cdots V_{\sqrt{2}\alpha_+}(z_r)\,|k\rangle\\ &=\frac{1}{(2\pi i)^r}\int dz_1\cdots dz_r\prod_{1\le i<j\le r}(z_i-z_j)^{2\alpha_+^2}:V_{\sqrt{2}\alpha_+}(z_1)\cdots V_{\sqrt{2}\alpha_+}(z_r):|k\rangle\ .\end{aligned}$$

Let us recall that

$$a_0|k\rangle=k|k\rangle\ ,\quad e^{ip\varphi_0}|k\rangle=|k+p\rangle\ ,\quad a_n|k\rangle=0\ ,\ (n>0)\ .$$

In addition, the normal-ordering can be understood as

$$:V_{\sqrt{2}\alpha_+}(z_1)\cdots V_{\sqrt{2}\alpha_+}(z_r):=V^+_{\sqrt{2}\alpha_+}(z_1)\cdots V^+_{\sqrt{2}\alpha_+}(z_r)\,V^-_{\sqrt{2}\alpha_+}(z_1)\cdots V^-_{\sqrt{2}\alpha_+}(z_r)\ .$$

Here we write the creation operator $V_k^+(z)$

$$V^+_{\sqrt{2}\alpha_+}(z)=\exp\left(\sqrt{2}\alpha_+\sum_{n=0}^\infty t_n(z)\right)=\left(\sum_{n=0}^\infty p_m z^m\right)e^{i\sqrt{2}\alpha_+\varphi_0}\ .$$

Then, show that the integral expression can be written as

$$\mathbf{S}_+^r|k\rangle=\frac{1}{(2\pi i)^r}\int dz_1\cdots dz_r\prod_{1\le i<j\le r}(z_i-z_j)^{2\alpha_+^2}\sum_{m_i\ge0}\prod_{i=1}^r p_{m_i}z_i^{m_i+k\sqrt{2}\alpha_+}|k+r\sqrt{2}\alpha_+\rangle\ .$$

In order for this state to be non-zero, the integral should be invariant under the scaling $z_i\to\lambda z_i$, which requires

$$r+\frac{r(r-1)}{2}2\alpha_+^2+\sum_{i=1}^r m_i+rk\sqrt{2}\alpha_+=0\ .$$

If we assume that the level $\sum_{i=1}^r m_i$ of the singular vector $\mathbf{S}_+^r|k\rangle$ is equal to rs, show that k has to be

$$k=\frac{1-r}{2}\sqrt{2}\alpha_++\frac{1+s}{2}\sqrt{2}\alpha_-\ . \tag{1}$$

This implies that $\mathbf{S}_+^r|k\rangle$ is a non-zero singular vector at level rs if k is subject to (1). Compute the conformal dimension of $\mathbf{S}_+^r|k\rangle$ and show that it is equal to $h_{r,s}$ in the Kac determinant (5.18) of the lecture note.

References

[FMS97] P. Francesco, P. Mathieu, and D. Sénéchal, *Conformal field theory*, Springer Science & Business Media, 1997.