Motivation

$\mathcal{N} = (2,2)$ LG model with ADE sing. $\xrightarrow{\text{flows}}$ $\mathcal{N} = (2,2)$ minimal model

(Levde)– Vafa–Warner
Martinec

$$\frac{SU(2)_K \times U(1)_2}{U(1)_{K+2}}$$

A_{n-1}	$X^n + Y^2$	$\sum |\chi_\ell|^2$
D_{n+1}	$X^n + X Y^2$	$\sum |\chi_{2\ell}|^2 + \sum \chi_{2\ell+1} \overline{\chi}_{K-2\ell-1}$
E_6	$X^4 + Y^3$	$|\chi_0 + \chi_6|^2 + |\chi_3 + \chi_7|^2 + |\chi_4 + \chi_{10}|^2$
E_7	$X^3 Y + Y^3$	$|\chi_0 + \chi_{16}|^2 + |\chi_4 + \chi_{12}|^2 + |\chi_6 + \chi_{10}|^2$
		$\quad + (\chi_2 + \chi_{14}) \overline{\chi}_8 + \chi_8 (\overline{\chi}_2 + \overline{\chi}_{14}) + |\chi_8|^2$
E_8	$X^5 + Y^3$	$|\chi_0 + \chi_{10} + \chi_{18} + \chi_{28}|^2 + |\chi_6 + \chi_{12} + \chi_{16} + \chi_{22}|^2$

$\mathcal{N} = (0,2)$ LG model $w/$ $c = \overline{c} < 3$ Gholson–Melnikov

"small LG model"

		Gadde–Putrov
$a_{m,n}$	$\Psi_1 (\phi_1^m + \phi_2^n) + \Psi_2 \phi_1 \phi_2$	$\Rightarrow \left(\dfrac{SU(2)_{mn}}{U(1)_{mn}} \times U(1)_{\frac{mn(mn+2)}{p^2}} \right) \otimes \overline{\left(\dfrac{SU(2)_{mn} \times U(1)_2}{U(1)_{mn+2}} \right)}$
b_K	$\Psi_1 (\phi_1^k + \phi_2^2) + \Psi_2 \phi_1^2 \phi_2$	$p = G.C.D. (m,n)$
c	$\Psi_1 (\phi_1^3 + \phi_2^3) + \Psi_2 \phi_1^3 \phi_2$	
d	$\Psi_1 (\phi_1^3 + \phi_2^2) + \Psi_2 \phi_1^2 \phi_2^2$	ADE classificath ?!
e	$\Psi_1 (\phi_1^3 + \phi_2^2) + \Psi_2 \phi_1 \phi_2^2$	We could solve only b_{K-4}.

$a_{n,2} = A_n^{(2,2)}$

$a_{n-1,2} = D_{n+1}^{(2,2)}$ $E_6 = A_2 \oplus A_1$
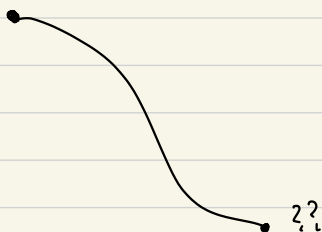
$b_2 = a_{3,3}$ $E_8 = A_3 \oplus A_1$

$b_3 = E_7$

Modular invariance beyond the ADE classification   in 2d CFT

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Problem

$\mathcal{N} = (0,2)$ LG model   with $W = Y_1 (\phi_1^4 + \phi_2^2) + Y_2 \phi_1^2 \phi_2$

(E-Term is zero)

$??$

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# of chiral $=$ # of Fermi.

gravitational anomaly

$\text{Tr} \, \gamma_3 = c - \bar{c} = 0$

C-extremization $c = \bar{c} = \dfrac{75}{27}$

	ϕ_1	ϕ_2	Y_1	Y_2
$U(1)_R$	$\frac{5}{27}$	$\frac{10}{27}$	$\frac{7}{27}$	$\frac{7}{27}$
$U(1)_\ell$	1	2	-4	-4

$\searrow$ 't Hooft anomaly $= 27.$

Candidate of IR CFT.

$$\left(\frac{SU(2)_{25}}{U(1)_{25}} \times U(1)_{\frac{27}{2}} \right) \otimes \overline{\left(\frac{SU(2)_{25} \times U(1)_2}{U(1)_{27.}} \right)}$$

$??$

parafermion PF_{25}
$c = \dfrac{3 \cdot 25}{25+2} - 1 = \dfrac{16}{9}.$

minimal model. $MM_{25.}$

Elliptic genus   in NS·NS

$$EG(\tau, z) = q^{-\frac{25}{216}} \frac{\theta(y^{-4} q^{\frac{17}{27}})^2}{\theta(y \, q^{\frac{5}{54}.}) \, \theta(y^2 \, q^{\frac{5}{27}})}$$

$q = e^{2\pi i \tau}$

$y = e^{2\pi i z}$

$\theta(x) = \prod_{i=1}^{\infty} (1 - x \, q^i)(1 - x^{-1} q^{i+1})$

Among chiral primaries ($\bar{L}_0 = \frac{\bar{J}_0}{2}$), states ($L_0 = \begin{cases} r_\phi/2 & \text{chiral} \\ (r_\psi - 1)/2 & \text{Fermi} \end{cases}$)

form Topological Heterotic ring (Quantum sheaf cohomology)

$$\mathcal{H}_{\text{Top}} = \mathbb{C}[\phi, \phi_2] \Big/ (\phi_1^4 + \phi_2^2, \phi_1 \phi_2)$$

$$= \text{Span}\left[\phi_1^i\right]_{i=0}^{5} \oplus \text{Span}\left[\phi_2, \phi, \phi_2\right]$$

One can check the OPE with stress-energy Tensor

$$T = \sum_{a=1}^{2}\left[\left(1 - \frac{r_{\phi a}}{2}\right) \partial\phi_a \, \partial\bar{\phi}_a - \frac{r_{\phi a}}{2} \phi_a \partial^2\bar{\phi}_a\right]$$

$$+ \sum_{a=1}^{2} \frac{i}{2}(1 + r_{\psi a}) \psi_a \partial\bar{\psi}_a - \frac{i}{2}(1 - r_{\psi a}) \partial\psi_a \, \bar{\psi}_a \, \Big]$$

Modular invariant partition function.

$$\mathcal{Z} = \text{Tr}_{\text{NSNS}} \, q^{L_0 - \frac{c}{24}} \, y^{J_0} \, \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}} \, \bar{x}^{\bar{J}_0}$$

$$= \sum_{\text{wts}} N_{\ell\bar{\ell}}^{SU(2)} \, N_{\mu \times \bar{\mu}} \, \chi_{\ell,\nu}^{PF_{25}} \, \chi_{\lambda}^{U(1)_{27}} \, \chi_{\bar{\ell},\bar{\rho}}^{MM_{25}}$$

consistent with $EG(\tau, \bar{z})$ on $\bar{L}_0 = \frac{\bar{J}_0}{2}$.

U(1) part can be fixed by. rational Transformation

$$\begin{pmatrix} \frac{27}{2} & \\ & 27 \end{pmatrix} = R^T \begin{pmatrix} 25 & \\ & 2 \end{pmatrix} R \qquad\qquad R = \frac{1}{10}\begin{pmatrix} 2 & 10 \\ 25 & -10 \end{pmatrix}$$

$$\Rightarrow \mathcal{Z} = \sum_{\text{wts}} N_{\ell\bar{\ell}}^{SU(2)} \, \chi_{\ell,5m}^{PF_{25}} \, \chi_{\frac{27m - 5n}{2}}^{U(1)_{27}} \, \chi_{\bar{\ell}, n}^{MM_{25}}$$

Gadde - Putrov
Gannon

$$N_{\ell\bar{\ell}}^{nd} = \left(\delta_{2.\ell} - \delta_{14.\ell} + \delta_{20.\ell}\right)\left(\delta_{2.\bar{\ell}} - \delta_{14.\bar{\ell}} + \delta_{20.\bar{\ell}}\right)$$

$$+ \left(\delta_{5.\ell} - \delta_{11.\ell} + \delta_{23.\ell}\right)\left(\delta_{5.\bar{\ell}} - \delta_{11.\bar{\ell}} + \delta_{23.\bar{\ell}}\right)$$

$$= \begin{pmatrix} & \overset{2}{1} & & \overset{14}{-1} & \overset{20}{1} \\ & & & & \\ & -1 & & 1 & -1 \\ & 1 & & -1 & 1 \end{pmatrix} + \begin{pmatrix} \overset{5}{1} & & \overset{11}{-1} & & \overset{23}{1} \\ & & & & \\ -1 & & 1 & & -1 \\ 1 & & -1 & & 1 \end{pmatrix}$$

Commutes with   S. & T   for $SU(2)_{25}$.

$$N_{\ell\bar{\ell}}^{SU(2)} = \delta_{\ell\bar{\ell}} - \frac{1}{3} N_{\ell\bar{\ell}}^{nd} \qquad\qquad \text{consistent with } EG(T.Z).$$

Hilbert space

There is an identity of PF_{25} character

$$3 = \sum_{m=0}^{4} \chi_{2.10m}^{PF_{25}} - \chi_{14.10m}^{PF_{25}} + \chi_{20.10m}^{PF_{25}}$$

$$= \sum_{m=0}^{4} \chi_{5.10m+5}^{PF_{25}} - \chi_{11.10m+5}^{PF_{25}} + \chi_{23.10m+5}^{PF_{25}}$$

that   counts   primaries

$$|\ell.m\rangle_{PF_{25}} = |2.0\rangle \quad |20.20\rangle \quad |20.30\rangle$$

$$|\ell.m\rangle_{PF_{25}} = |23.25\rangle , |5.5\rangle \quad |5.45\rangle$$

$\Longrightarrow$ $N_{\ell\bar{\ell}}^{nd}$ add   or   eliminate   a   certain   linear   combination.   of   these   states.

diagonal spectrum has 10 states w/ $L_0 = \frac{\widehat{J_0}}{2} = \overline{L_0}$

$$|5s, 5s\rangle_{PF} \otimes |-s\rangle_{U(1)_{\frac{25}{2}}} \otimes |5s, -5s\rangle_{MM_{25}}$$

$$|5s, 50-5s\rangle_{PF} \otimes |-s\rangle_{U(1)_{\frac{25}{2}}} \otimes |5s, -5s\rangle_{MM_{25}}$$

$$1 \longleftrightarrow |0.0\rangle_{PF}$$

$$\phi_1 \longleftrightarrow |5.5\rangle_{PF} + |5.45\rangle_{PF} \qquad\qquad \color{red}{|5.5\rangle_{PF} - |5.45\rangle_{PF}}$$

$$\phi_1^2 \longleftrightarrow |10,10\rangle_{PF} + |10.40\rangle_{PF} \qquad \color{red}{\phi_2 \longleftrightarrow |10.10\rangle_{PF} - |10.35\rangle}$$

$$\phi_1^3 \longleftrightarrow |15.15\rangle_{PF} + |15.35\rangle_{PF} \qquad \color{red}{\phi_1 \phi_2 \longleftrightarrow |15.15\rangle_{PF} - |15.30\rangle_{PF}}$$

$$\phi_1^4 \longleftrightarrow |20.20\rangle_{PF} + |20.30\rangle_{PF}$$

$$\color{red}{\phi_1^2 \phi_2 \longleftrightarrow |20.20\rangle_{PF} - |20.30\rangle_{PF}}$$

$$\phi_1^5 \longleftrightarrow |25.25\rangle_{PF}$$

In fact, $\phi_1^2 \partial \phi_2$ is primary.

$$(\chi^{PF_{25}}_{|20.20\rangle - |20.30\rangle} - 1)\, \chi^{U(1)_{\frac{25}{2}}}_{-4} = (\chi^{PF_{25}}_{20.20} - 1)\, \chi^{U(1)_{\frac{25}{2}}}_{-4}$$

$$= (q + 3q^2 + 6q^3 + \cdots) \chi^{U_{\frac{25}{2}}}_{-4}$$

$\color{blue}{\Rightarrow \text{parafermionic sym}}$
$\color{blue}{\text{is broken } \frac{SU(2)_{25}}{U(1)_{25}}}$

$$\downarrow$$

$$2(\phi_1^2 \partial \phi_2 + 2\phi_1 \phi_2 \partial \phi_1) = 4\phi_1 \partial \phi_1 \partial \phi_2 + \phi_1^2 \partial^2 \phi_2 + 2\phi_2(\partial \phi_1)^2 + 2\phi_1 \phi_2 \partial^2 \phi_2$$

$$H_5^{\widehat{PF}_{25}} \cong \bigoplus V^{PF}_{5, \text{comps}} \big/ \mathbb{C}\, (|5.5\rangle - |5.45\rangle)$$

$$H_{20.}^{\widehat{PF}_{25}} \cong \bigoplus V^{PF}_{20, 10\text{um}} \big/ \mathbb{C}\, (|20.20\rangle - |20.30\rangle)$$

$$\color{red}{\text{isom. graded by } L_0}$$

$$\mathcal{H}_2^{\widetilde{PF_{25}}} \cong \oplus V_{2.10m}/\mathbb{C}\,|2.0\rangle$$

$$\mathcal{H}_{23}^{\widetilde{PF_{25}}} \cong \oplus V_{23.10m+5}/\mathbb{C}\,|23.25\rangle$$

graded vector sp.

$$\mathcal{H}_{14}^{\widetilde{PF_{25}}} \cong \mathbb{C}(|20.20\rangle + |20.30\rangle) \oplus V_{14.10m}$$

$$\mathcal{H}_{11}^{\widetilde{PF_{25}}} \cong \mathbb{C}(|5.57\rangle + |5.45\rangle) \oplus V_{11.10m+5}$$

$$\mathcal{H}_{\ell}^{\widetilde{PF}} \cong \oplus V_{\ell.10m+5(\ell \bmod 2)}$$

$$\Rightarrow \quad \text{Hilbert space.} \qquad \mathcal{H} = \bigoplus_{\ell.n} \mathcal{H}_{\ell}^{\widetilde{PF_{25}}} \otimes V_{\frac{27\ell - 5n}{2}}^{U(1)_{\frac{27}{2}}} \otimes \overline{V}_{\ell.n}^{MM_{25}}$$

$$\text{LG} \quad \underset{RG \; flow}{\Longrightarrow} \quad \left(\widetilde{PF_{25}} \times U(1)_{\frac{27}{2}} \right) \otimes \left(\overline{\frac{SU(2)_{25} \times U(1)_2}{U(1)_{27}}} \right)$$