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本 科 毕 业 论 文

二维共形场论中边界条件对瑞丽纠缠熵的影响

Boundary condition influence Rényi entropy in 2dCFT

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摘　要

共形场论，即在共形变化下作用量不变的量子场论，由于其在统计力学 [2]，凝聚态物理 [3] 和弦论 [4] 中的广泛使用而吸引了许多研究者的关注。瑞丽纠缠熵作为一般的冯诺依曼纠缠熵的推广，由于其可提供更多关于纠缠系统的信息而在共形场论中有比较广泛的应用。但是，许多论文在讨论纠缠熵时没有指定纠缠表面的边界条件。尽管这样的做法不会影响纠缠熵的首项，但是在我们希望对纠缠熵做更精确的计算时，忽略边界条件是不合理的。关于边界条件对瑞丽纠缠熵的影响，有论文 [1] 指出在瑞丽纠缠熵中系数 n 趋于无穷的极限下，我们可以提取出一些所谓 “半普适” 的项，它们只依赖于原本选取的边界条件在该极限下变成的共形边界条件。本文主要回顾了 [1] 的结果，并以扭曲算符的方式复现了其中的推导过程。

在第一章中我们主要介绍了考虑的系统为二维欧式时空中 $t=0$ 段的一块联通的子区域 A，并将 $t=0$ 上的其他区域称为 B。在 A 的两侧各有一点于 B 相交，即 A 与 B 的纠缠点。由于在量子场论中我们一般不能直接将整个区域希尔伯特空间 $\mathscr{H}$ 分解为两个子区域上希尔伯特空间的乘积 $\mathscr{H}_A\otimes\mathscr{H}_B$。因此我们采取将两个纠缠点延长为两段很短的线段，然后将整个希尔伯特空间投影到 $\mathscr{H}\to\mathscr{H}_{A,a}\otimes\mathscr{H}_{B,a}$，其中 a 表示延长后纠缠点所形成的线段。然后我们介绍了共形场论中的一些基本知识，如共形变化，基本场，能动张量等。我们还介绍了在量子场论中如何以路径积分的形式计算瑞丽熵。其中主要概念有在 $t=0$ 的基态 $|\Omega\rangle$ 可由沿下半平面的路径积分制备，相应的 $\langle\Omega|$ 可由沿上半平面的路径积分制备。区域 A 的密度矩阵可由基态密度矩阵对区域 B 的希尔伯特空间求迹并除以整个平面的配分函数 Z_1 得到 $\rho_A=\frac{1}{Z_1}\operatorname{tr}_{\mathscr{H}_{B,a}}|\psi\rangle\langle\psi|$。其中对区域 B 求迹可以理解为将上半平面和下半平面中区域 B 所对应的点等同（可以形象地理解为将两条边粘在一起），然后对区域 B 做路径积分，Z_1 的作用是归一化常数。因此区域 A 的密度矩阵可形象地理解为在二维平面上有一条沿着区域 A 的细缝，除细缝外的区域都要做路径积分。由于瑞丽纠缠熵的定义为 $S_n=\frac{1}{1-n}\log\operatorname{tr}\rho_A^n$，我们还需要理解何为 ρ_A^n。两个 ρ_A 相乘的过程可以理解为一片二维平面上细缝的上端与另一片二维平面细缝的下端粘在一起，因此我们可以写出 $\operatorname{tr}_{\mathscr{H}_A}\rho_A^n=\frac{Z_n}{Z_1^n}$，其中 Z_n 为按照上述规则相粘的 n 篇有细缝的二维平面上的路径积分，Z_1^n 即单片平面上路径积分的 n 次方，Z_1^n 起到的仍然是归一化常数的作用。

在第二章中我们回顾了 [1] 的推导过程。此时我们要考虑延长后纠缠点上的边界条件，几何上可以想成在区域 A 的两侧我们挖去两个小洞，在这两个小洞的圆弧上我们定义了一些边界条件。在共形场论中我们可以做共形变化后再计算流形上的配分函数，配分函数只会变化一定倍数，取对数后我们可以忽略其影响。假设区域 A 的长度为 L，我们可以通过共形变化 $z \mapsto \log z - \log(L - z)$ 使得变化后的流形为一个圆柱面。由于在圆柱上的哈密顿量是已知的，我们就可以将配分函数理解为从一个边界态 $|a_1\rangle$ 经一定虚时间后传播到另一个边界态 $|a_2\rangle$。其中 Z_n 所对应的圆柱面为 n 个圆柱粘在一起形成的圆柱面，我们可以进一步做标度变化使得粘成的圆柱面周长与原本的圆柱面相等，这样我们就定义了粘成的边界态 $|a^n_{1,2}\rangle$。由于当 $n \to \infty$ 的极限时，无论我们原本定义的边界态是如何的，粘成的边界态都会流向几类共形边界态，因此 Z_n 的值是由流向的共形边界态唯一决定的。我们可以观察 Z_n 在瑞丽熵中的贡献，提取出所谓的“半普适”量。

在第三章中我们介绍了利用扭曲算符的方法计算配分函数，扭曲算符属于基本场，其在关联函数中的作用正是将不同二维平面上的细缝粘在一起。相较于共形变化的方法，其优点是即使我们考虑的区域 A 包含几段不连通的子区域，我们仍然可以很容易地写出其配分函数所对应的关联函数，而共形变化则需要考虑更为复杂的流形变化，其代价则是随着不连通子区域的增多关联函数中的扭曲算符也会增多，在计算上仍有较大难度。使用扭曲算符的方法，我们将边界条件转化为使用生成相应边界条件的算符，重新计算了 Z_n 与 Z^n_1，并成功复现了上一章的推导结果。

关键字：共形场论；瑞丽熵；边界条件，边界熵

Abstract

Rényi entanglement entropy is a generalized entanglement measure and conformal field theory is a quantum field theory which is invariant under conformal transformation. In this paper we use Rényi entanglement entropy to calculate the entanglement between a spatial region A and its complement in 2dCFT. On their entangling surface we put some boundary condition and shows the boundary condition can influence Rényi entanglement entropy at subleading order. We mainly review result in [1] that we can extract some "semi-universal" term in Rényi entropy which only depends on the conformal boundary condition that original boundary condition flows to in $n \to \infty$ limit. We also show that the derivation can be reproduced in twist operator approach.

Keywords: Conformal field theory; Rényi entropy; boundary condition; boundary entropy

Chapter 1 introduction

Conformal field theory(CFT caught eyes of many researchers due to wide usage in statistical mechanics[2], condensed matter physics[3] and string theory[4].As a generalization of common Von Neumann entropy, Rényi entanglement entropy has also been widely used in CFT because it can provide some information of entanglement systems. However, many papers discussing entanglement entropy without specifying boundary condition in the entangled surface. Although it will not influence the first term in entanglement entropy, it counts to the second term which we may be interested in some cases.Therefore, research about the influence of boundary condition on entanglement entropy can contribute to rigorousness of relevant calculation and may help to understand more physical meaning.

Consider two spatial subregion A and B in a pure state. In general cases, the Hilbert space of whole region can not be trivially separated to a tensor product of $\mathscr{H} = \mathscr{H}_A \otimes \mathscr{H}_B$, for example, it is confined by gauge invariance in lattice gauge theory[5] and contradicted to the divergent entanglement entropy in quantum field t heory. Therefore, when try to cut these two subregions, we can not ignore the boundary condition on their entangling surface. In 2d CFT, it is verified both in theory and numerical simulation[6][7] that the first term in entanglement entropy is a universal logarithm term, and the second term, which is related with boundary condition, are usually considered to be non-universal. However Ohmori,K and Tachikawa,Y [1] found for Rényi Entanglement entropy Sn, as n goes to infinite, we can extract a subleading term which is only related with the conformal boundary condition which original boundary condition flows to. Therefore, that subleading term can be described as "semi-universal" .

To define entanglement of a state $|\psi\rangle$, we need to consider a subregion A and its spatial complement B. If the total Hilbert space can be trivially separated as $\mathscr{H} = \mathscr{H}_A \otimes \mathscr{H}_B$, then we can write the reduced density matrix $\rho_A = \mathrm{tr}_{\mathscr{H}_B} |\psi\rangle\langle\psi|$. Here we introduce the Rényi entanglement entropy

$$S_n = \frac{1}{1-n} \log \mathrm{tr}\, \rho_A^n, \tag{1.1}$$

where $n \geq 1$ can be non-integer, it reproduces the common Von Neumann entropy in the limit $n \to 1$. However, as we mentioned above, tensor product separation is generally

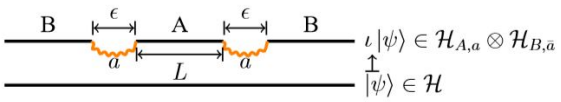

figure 1-1 cutting operation (borrowed from [1])

not allowed in quantum field theories. We need to introduce a cutting operation:

$$\iota : \mathscr{H} \to \mathscr{H}_{A,a} \otimes \mathscr{H}_{B,a}, \tag{1.2}$$

where a represent "thicken" boundary of interface between A and B, and is defined with some boundary condition. In our discussion, a is not necessarily a conformal boundary condition. After this cutting operation, the reduced density matrix should be written as:

$$\rho_A = \frac{1}{Z} tr_{\mathscr{H}_{B,a}} \iota|\psi\rangle\langle\psi|\iota^\dagger. \tag{1.3}$$

1.1 Basics in CFT

1.1.1 Conformal Transformation and Primary field

Conformal transformation is transformation which preserves the angle between local coordinates, it can be describes in transformation of metric tensor $g_{\mu\nu}$[8]:

$$\begin{aligned} x &\to x', \\ g_{\mu\nu}(x) &\to g'_{\mu\nu}(x') = \Lambda(x) g_{\mu\nu}(x). \end{aligned} \tag{1.4}$$

In this paper we focus on CFT in two dimension(2dCFT), and we always perform a wick rotation ($t \to -it$) to discuss in 2d Euclidean space.Considering the coordinates (z^0, z^1) on the plane and transformation $z^\mu \to w^\mu$, the contravariant metric tensor transforms as

$$g^{\mu\nu} \to \left(\frac{\partial w^\mu}{\partial z^\alpha}\right)\left(\frac{\partial w^\upsilon}{\partial z^\beta}\right) g^{\alpha\beta}. \tag{1.5}$$

Using $g^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, we get constrain condition:

$$\begin{aligned} \frac{\partial w^1}{\partial z^0} &= \frac{\partial w^0}{\partial z^1} \quad \text{and} \quad \frac{\partial w^0}{\partial z^0} = -\frac{\partial w^1}{\partial z^1}, \\ \frac{\partial w^1}{\partial z^0} &= -\frac{\partial w^0}{\partial z^1} \quad \text{and} \quad \frac{\partial w^0}{\partial z^0} = \frac{\partial w^1}{\partial z^1}. \end{aligned} \tag{1.6}$$

The first line is a Cauchy-Riemann equation for holomorphic function and the second line is for antiholomorphic function. Here we introduce complex coordinates $z = z^0 + iz^1$ and $\bar{z} = z^0 - iz^1$.Transformation is conformal as long as $w(z)$ is a holomorphic or antiholomorphic function. Since infinitesimal holomorphic transformation can be

expressed as Laurent expansion, there are infinite generators of field transformation in 2dCFT.

If a field transforms under any local conformal transformation as

$$\phi'(\omega,\bar{\omega}) = \left(\frac{d\omega}{dz}\right)^{-h}\left(\frac{d\bar{}}{d\bar{z}}\right)^{-\bar{h}}\phi(z,\bar{z}), \tag{1.7}$$

where h and $\bar{h}$ are confornal dimensions determined by scaling dimension and spin, then the field is a primary field. Combine this primary field transformation rule and a correlation function transformation rule$\left\langle \boldsymbol{\Phi}\left(x_1'\right)\cdots\boldsymbol{\Phi}\left(x_n'\right)\right\rangle = \left\langle \mathscr{F}\left(\boldsymbol{\Phi}\left(x_1\right)\right)\cdots\mathscr{F}\left(\boldsymbol{\Phi}\left(x_n\right)\right)\right\rangle$, we have

$$\begin{gathered}\left\langle \phi_1\left(\omega_1,\bar{\omega}_1\right)\ldots\phi_n\left(\omega_n,\bar{\omega}_n\right)\right\rangle = \\ \prod_{i=1}^{n}\left(\frac{d\omega}{dz}\right)^{-h_i}_{\omega=\omega_i}\left(\frac{d\bar{\omega}}{d\bar{z}}\right)^{-\bar{h}_i}_{\bar{\omega}=\bar{\omega}_i}\left\langle \phi_1\left(z_1,\bar{z}_1\right)\ldots\phi_n\left(z_n,\bar{z}_n\right)\right\rangle.\end{gathered} \tag{1.8}$$

In particular, we can fix the form of two point function

$$\left\langle \phi_1\left(z_1,\bar{z}_1\right)\phi_2\left(z_2,\bar{z}_2\right)\right\rangle = \frac{C_{12}}{\left(z_1-z_2\right)^{2h}\left(\bar{z}_1-\bar{z}_2\right)^{2\bar{h}}} \tag{1.9}$$

C_{12} is a non-zero constant if $h = h_1 = h_2, \bar{h} = \bar{h_1} = \bar{h_2}$, otherwise the two point function can only be zero.

1.1.2 energy-momentum tensor and conformal ward identity

The canonical energy-momentum tensor is a conserved current associated with invariance under translation $x'^{\mu} \to x^{\mu} + \epsilon^{\mu}$

$$T_{\mathrm{c}}^{\mu\nu} = -\eta^{\mu\nu}\mathscr{L} + \frac{\partial\mathscr{L}}{\partial\left(\partial_{\mu}\boldsymbol{\Phi}\right)}\partial^{\nu}\boldsymbol{\Phi}. \tag{1.10}$$

We have freedom to modify canonical energy-momentum without changing $\partial_{\mu}T^{\mu\nu} = 0$

$$T_{\mathrm{B}}^{\mu\nu} = T_{\mathrm{c}}^{\mu\nu} + \partial_{\rho}B^{\rho\mu\nu}, \quad B^{\rho\mu\nu} = -B^{\mu\rho\nu}. \tag{1.11}$$

In CFT theory we always have that action is invariant under rotation and scale transformation. Rotation invariance implies we can make the energy-momentum tensor symmetric $T^{\mu\nu} = T^{\nu\mu}$, and scale invariance implies we can make energy-momentum tensor traceless $T^{\mu}_{\mu} = 0$. An infinitesimal transformation of field can be written as

$$\boldsymbol{\Phi}'(x) = \boldsymbol{\Phi}(x) - i\omega_a G_a \boldsymbol{\Phi}(x). \tag{1.12}$$

Let us denote X as a string $\boldsymbol{\Phi}\left(x_1\right)\cdots\boldsymbol{\Phi}\left(x_n\right)$ and consider its correlation function $\langle X\rangle = \frac{1}{Z}\int[d\boldsymbol{\Phi}]\,X\exp -S[\boldsymbol{\Phi}]$. We can change the field $\boldsymbol{\Phi}(x) \to \boldsymbol{\Phi}'(x)$ without changing the

integration result. Assuming the functional integration measure is invariant ($\left[d\boldsymbol{\Phi}'\right] = [d\boldsymbol{\Phi}]$), we have

$$\langle X\rangle = \frac{1}{Z}\int [d\boldsymbol{\Phi}]\,(X+\delta X)\exp -\left\{S[\boldsymbol{\Phi}] + \int dx\partial_\mu j_a^\mu.\omega_a(x)\right\} \tag{1.13}$$

Expand the result at first order of $\omega_a(x)$, we get the general Ward identity

$$\begin{aligned}&\tfrac{\partial}{\partial x^\mu}\left\langle j_a^\mu(x)\boldsymbol{\Phi}\left(x_1\right)\cdots\boldsymbol{\Phi}\left(x_n\right)\right\rangle \\ &\quad = -i\textstyle\sum_{i=1}^n \delta\left(x-x_i\right)\left\langle\boldsymbol{\Phi}\left(x_1\right)\cdots G_a\boldsymbol{\Phi}\left(x_i\right)\cdots\boldsymbol{\Phi}\left(x_n\right)\right\rangle,\end{aligned} \tag{1.14}$$

where $j_a^\mu(x)$ is the Noether current associated with transformation invariance. In CFT we can find $j_a^\mu(x)\omega_a = T^{\mu\nu}\epsilon_v$, intergrate over the whole domain D we have the conformal Ward identity

$$\delta_\epsilon\langle X\rangle = \int_D \mathrm{d}^2x\partial_\mu\left\langle T^{\mu\nu}(x)\epsilon_v(x)X\right\rangle. \tag{1.15}$$

If we use complex coordinates $(z,\bar{z})$ and introduce a renormalized energy-momnetum tensor

$$T = -2\pi T_{zz},\quad \bar{T} = -2\pi T_{\bar{z}\bar{z}}. \tag{1.16}$$

Using Gauss's theorem, the conformal Ward identity can be written as

$$\delta_{\epsilon,\bar{\epsilon}}\langle X\rangle = -\frac{1}{2\pi i}\oint_{\mathrm{C}}\mathrm{d}z\epsilon(z)\langle T(z)X\rangle + \frac{1}{2\pi i}\oint_{\mathrm{C}}\mathrm{d}\bar{z}\bar{\epsilon}(\bar{z})\langle\bar{T}(\bar{z})X\rangle, \tag{1.17}$$

where $\epsilon = \epsilon^z, \bar{\epsilon} = \epsilon^{\bar{z}}$ and $C = \partial D$ is the contour. Variation of primary field under an infinitesimal holomorphic conformal transformation is

$$\delta_\epsilon\phi = -\epsilon\partial\phi - h\phi\partial\epsilon. \tag{1.18}$$

Replacing X with a primary field $\phi(\omega,\bar{\omega})$, in operator product expansion(OPE) we have

$$T(z)\phi(w,\bar{w}) \sim \frac{h}{(z-w)^2}\phi(w,\bar{w}) + \frac{1}{z-w}\partial_w\phi(w,\bar{w}). \tag{1.19}$$

For antiholomorphic part we can similarly get

$$\bar{T}(\bar{z})\phi(w,\bar{w}) \sim \frac{\bar{h}}{(\bar{z}-\bar{w})^2}\phi(w,\bar{w}) + \frac{1}{\bar{z}-\bar{w}}\partial_{\bar{w}}\phi(w,\bar{w}). \tag{1.20}$$

This two equation are the equavilant definition of primary field.

1.1.3 Radial Quantization and Virasoro Algebra

We first define our theory on an infinite space-time cylinder, with time t going from $-\infty$ to $+\infty$ along the length direction, and space going from 0 to L along the circumference direction, the points $(0,t)$ and (L,t) are identified. The cylinder can be

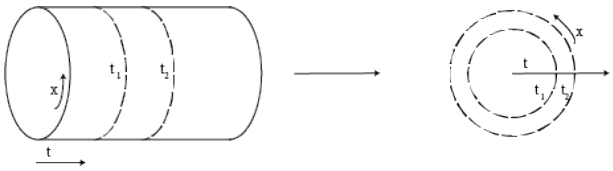

figure 1-2 radial quantization

described by a complex coordinate $\omega = t + ix$. We can map the cylinder to a complex plane through $z = e^{\frac{2\pi\omega}{L}}$ Now the original time direction is along the radial direction and space direction is along the angle direction. Infinite past($t \to -\infty$) is mapped to the origin $z = 0$ and the infinite future($t \to +\infty$) now lies on the infinite point on the Riemann sphere. After this radial quantization, state is defined at the circle where radial to the origin is the same. Since Euclidian time $\tau = it$ should be reversed while space is invariant under hermitian conjugate, $\bar{z}$ in radial quantization should be defined as $\bar{z} = \frac{1}{z^*}$

We can expand the energy-momentum tensor as

$$\begin{aligned} T(z) &= \textstyle\sum_{n\in\mathbf{Z}} z^{-n-2} L_n \quad L_n = \frac{1}{2\pi i}\oint dz z^{n+1} T(z) \\ \bar{T}(\bar{z}) &= \textstyle\sum_{n\in\mathbf{Z}} \bar{z}^{-n-2} \bar{L}_n \quad \bar{L}_n = \frac{1}{2\pi i}\oint d\bar{z} \bar{z}^{n+1} \bar{T}(\bar{z}) \end{aligned} \tag{1.21}$$

The mode operators L_n and $\bar{L}_n$ are called Virasoro algebra, they are generators of local conformal transformation on Hilbert space. In particular, the operator $L_0 + \bar{L}_0$ generates dilation along the radial direction, which is exactly time translation. Therefore, $L_0 + \bar{L}_0$ is proportional to the Hamiltonian up to a constant term.

Virasoro algebra obeys the following relation

$$\begin{aligned} \left[L_n, L_m\right] &= (n-m)L_{n+m} + \tfrac{c}{12} n\left(n^2-1\right)\delta_{n+m,0} \\ \left[L_n, \bar{L}_m\right] &= 0 \\ \left[\bar{L}_n, \bar{L}_m\right] &= (n-m)\bar{L}_{n+m} + \tfrac{c}{12} n\left(n^2-1\right)\delta_{n+m,0} \end{aligned} \tag{1.22}$$

where c called central charge is a quantity related with trace anomaly.

1.2 Calculation of Rényi entropy in 2d

Consider a single interval A between $[\mu,\nu]$ at $\tau = 0$ slice in 2d Euclidean space,and we define its spatial complement as B. Wave function of the ground state $|\Psi\rangle$ at t=0 can be prepared by path integral over the whole plane at t<0

$$\Psi\left(\phi_0(x)\right) = \int_{t_E=-\infty}^{\phi\left(t_E=0,x\right)=\phi_0(x)} D\phi e^{-\mathcal{S}(\phi)}. \tag{1.23}$$

Similarly wave function of$\langle\Psi|$ can be prepared by path integral over the upper half plane. The partition function of $t = 0$ region can be written as

$$Z_1 = \int \left[\mathscr{D}\phi_0(t=0,\vec{x})\right] \langle\Psi|\phi_0\rangle \langle\phi_0|\Psi\rangle, \tag{1.24}$$

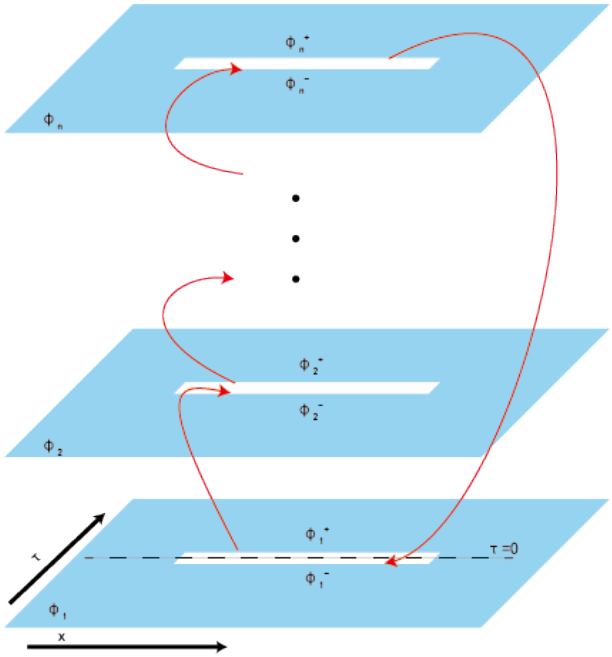

figure 1-3 "glued" manifold. Edges connected by red arrow are identified.

which is just the path integral over the whole plane.

Density matrix of region A is defined as $\rho_A = \frac{1}{Z_1} \mathrm{tr}_{\mathscr{H}_{B,a}} |\psi\rangle\langle\psi|$, in quantum field theory it can be written as

$$\rho_A = \frac{1}{Z_1} \int \left[\mathscr{D}\phi^B(t=0, \vec{x} \in B)\right] \left\langle \phi^B | \Psi \right\rangle \left\langle \Psi | \phi^B \right\rangle \tag{1.25}$$

Now we need to be careful about the field slightly upper or lower than region A. These fields have time $t = 0^+, 0^-$, which are labelled as ϕ^+ and ϕ^-. In order to get ρ_A^n, we just need to consider n copies of ρ_A

$$\left[\rho_A\right]_{\phi_1^+\phi_1^-} \left[\rho_A\right]_{\phi_2^+\phi_2^-} \cdots \left[\rho_A\right]_{\phi_n^+\phi_n^-}. \tag{1.26}$$

The density matrix product between different copies can be realized by identify $\phi_i^+(x) = \phi_{i+1}^-(x)$[9].As required by trace over $\mathscr{H}_A$, we should also identify $\phi_n^+(x)$ and $\phi_1^-(x)$. Therefore, $\mathrm{tr}_{\mathscr{H}_A} \rho_A^n = \frac{Z_n}{Z_1^n}$, where Z_n is defined as path integral over n copies of 2d plane with such "gluing" condition.

So far we have described the common path integral formalism in 2dQFT. The interesting point for Rényi entropy in 2dCFT is we can perform conformal transformation to make the partion function Z_n over "glued" manifold easier to calculate. We can also take the so-called "twist operator" approach and the twist operator is a primary field in 2dCFT. These two approaches will be described in more details in following chapters.

Chapter 2 Rényi entropy in 2dCFT with boundaries

In this chapter we mainly review the result from [1].We still consider a single inter-val A between $[\mu, \nu]$ at $t = 0$ slice in 2d CFT, the difference is now we put two different boundary condition a_1, a_2 at the "thicken" entangling surface between A and its spatial complement B.

2.1 Map to cylinder

Assuming A's length is $L = |\mu - \nu|$, by conformal transformation $z \mapsto \log z - \log(L-z)$, original manifold is mapped to a cylinder with circumference 2π. The shape of boundaries on cylinder may not be flat because it depends on the explict shape of "thicken" boundary, but we can always flatten it by performing some small conformal transformation.We can understand length of "thicken" boundary ϵ as cut off, so length of cylinder is $l = \log(\frac{L}{\epsilon})^2 + O(\frac{\epsilon}{L})$. Now the "glued" manifold is just a cylinder with circumference $2n\pi$ and length l, we can also perform a scale transformation to make the circumference 2π and length $\frac{l}{n}$. We know on a cylinder with circumference 2π the Hamiltonian operator is $H = L_0 + \bar{L}_0 - \frac{c+\bar{c}}{24}$ [10], notice L_0, L_0 are dilation operators on complex plane. If the boundary condition is free, we can write down $Z_n = \mathrm{tr}\exp(-\frac{l}{n}(L_0+\bar{L}_0-\frac{c+\bar{c}}{24}))$. In presence of boundary condition $a_{1,2}$, the partition function should be understood as boundary state $|a_1\rangle$ propagate for imaginary time $\delta = \frac{l}{n}$ to another boundary state $|a_2\rangle$. We can write the partition as

$$Z_n \propto \langle a_1^n| \exp(-\frac{l}{n}(L_0 + \bar{L}_0 - \frac{c+\bar{c}}{24}))|a_2^n\rangle. \tag{2.1}$$

where $|a_{1,2}^n\rangle$ means "glued" boundary state on that scaled cylinder. Let $|0\rangle$ be the state with lowest conformal dimension that couples to both $|a_{1,2}\rangle$.In 2d unitary CFTs with discrete spectrum $|0\rangle$is just the vacuum state $|\Omega\rangle$.Inserting two complete basis $\sum_{i=0}^{\infty}|i\rangle\langle i|$ on each side in Z_n's bracket:

$$Z_n \propto \sum_i \sum_j \langle a_1^n|i\rangle\langle i| \exp(-\frac{l}{n}(L_0 + \bar{L}_0 - \frac{c+\bar{c}}{24}))|j\rangle\langle j|a_2^n\rangle. \tag{2.2}$$

2.2 Leading term

In the limit $\frac{L}{\epsilon} > 1$, the leading term is nothing but the lowest dimension state term:

$$Z_n \sim \left\langle a_1^n|0\right\rangle \exp\left(\frac{\ell}{n}\left(\frac{c}{12} - 2h_0\right)\right)\left\langle 0|a_2^n\right\rangle . \tag{2.3}$$

Similarly, Z_1^n can be written as

$$Z_1^n \propto \langle a_1| \exp(-l(L_0 + \bar{L}_0 - \frac{c+\bar{c}}{24}))|a_2\rangle^n. \tag{2.4}$$

We can also insert two complete basis and the leading term is

$$Z_1^n \sim \langle a_1|0\rangle \exp(-l(L_0 + \bar{L}_0 - \frac{c+\bar{c}}{24}))\langle 0|a_2\rangle^n. \tag{2.5}$$

Using $l \sim \log(\frac{L}{\epsilon})^2$, we can recover the famous logrithm result in Rényi entropy $S_n = \frac{1}{1-n}\log\frac{Z_n}{Z_1^n}$:

$$S_n \sim \left(1 + \frac{1}{n}\right)\frac{c_{\text{eff}}}{6}\log\frac{L}{\epsilon}, \tag{2.6}$$

where $c_{\text{eff}} = c - 24h_0$ is the effective central charge[11]. For vacuum state we have $h_0 = 0$, so $c_{\text{eff}} = c$. Notice that boundary condition also contributes to the entropy, more explictily we can write

$$\begin{aligned} S_n = &\left(1 + \frac{1}{n}\right)\frac{c}{6}\left(\log\frac{L}{\epsilon}\right) \\ &+ \frac{1}{1-n}\left(s\left(a_1^n\right) - ns\left(a_1^1\right) + s\left(a_2^n\right) - ns\left(a_2^1\right)\right) + O(e^{-\frac{l}{n}min(\Delta_1)}), \end{aligned} \tag{2.7}$$

where $s(a) = \log\langle a|0\rangle$ is the Affleck-Ludwig boundary entropy[12], i.e. the leading order of boundary entropy. h_1 corresponds to the second-lowest conformal dimension of state that couples to both $|a_{1,2}\rangle$. We know $s(a^n)$ term comes from Z_n while $ns(a)$ term comes from Z_1^n. The interesting point here is as $n \to \infty$, the "glued" boundary condition $a_{1,2}^{\infty}$ becomes conformal boundary condition. Therefore, even if we put different non-conformal boundary condition, that boundary entropy term could be the same in $n \to \infty$ limit.

2.3 Subleading term

If we look into subleading terms in Z_n and Z_1^n, we can find a interesting rule:

$$\begin{aligned} Z_n &\sim \left[\left\langle a_1^n|0\right\rangle \exp\left(\frac{l}{n}\left(\frac{c}{12} - 2h_0\right)\right)\left\langle 0|a_2^n\right\rangle\right](1 + O((\frac{L}{\epsilon})^{-4\frac{h_1}{n}}) + O((\frac{L}{\epsilon})^{-4\frac{h_2}{n}}) + ...) \\ Z_1^n &\sim \left[\langle a_1|0\rangle \exp(l(\frac{c}{12} - 2h_0))\langle 0|a_2\rangle\right]^n (1 + O((\frac{L}{\epsilon})^{-4h_1}) + O((\frac{L}{\epsilon})^{-4h_2}) + ...)^n \end{aligned} \tag{2.8}$$

which means when we expand subleading terms in $\log Z$.$\log Z_n$ will produce some terms at order $O((\frac{l}{\epsilon})^{-m\frac{h_1}{n}}))$, where power of $\frac{L}{\epsilon}$ is proptional to $\frac{1}{n}$.While $\log Z_1^n$ will only produce terms at order $O((\frac{l}{\epsilon})^{-mh_1}))$. If we can gather all the terms in S_n with power proptional to $\frac{1}{n}$ of $\frac{L}{\epsilon}$, then we can get all subleading contribution from $\log Z_n$. We can define the fractional power part as S_n^{frac}. In the limit $n \to \infty$, we know boundary state in Z_n flows to a conformal boundary state. We can write

$$\lim_{n\to\infty} Z_n\Big|_{\frac{L}{\epsilon}=q^{-n/2}} = Z^{\text{conf}}\left(q; a_1^{(\infty)}, a_2^{(\infty)}\right) \tag{2.9}$$

where $Z^{\text{conf}}\left(q; a_1^{(\infty)}, a_2^{(\infty)}\right) = \left\langle a_1^{(\infty)} \left| q^{L_0+L_0-c/12} \right| a_2^{(\infty)} \right\rangle$ is the cylinder partition function with conformal boundary. In this limit we can extract the fractional power part in Rényi entropy:

$$\lim_{n\to\infty}(1-n)S_n^{\text{frac}}\Big|_{L/\epsilon=q^{-n/2}} = \log\left(q^{\frac{c_{\text{eff}}}{12}} Z^{\text{conf}}\left(q; a_1^{(\infty)}, a_2^{(\infty)}\right)\right) - s\left(a_1^{(\infty)}\right) - s\left(a_2^{(\infty)}\right) \tag{2.10}$$

where $\log q^{\frac{c_{\text{eff}}}{12}}$ can seperate the leading order $\frac{c_{\text{eff}}}{6}\log\frac{L}{\epsilon}$ from $\log Z^{\text{conf}}\left(q; a_1^{(\infty)}, a_2^{(\infty)}\right)$. The result only depends on which conformal boundary $a_{1,2}$ flows to in $n \to \infty$ limit, so we can call it "semi-universal".

Chapter 3 Twist Operator Approach

Although calculation in cylinder is straightforward and easy to understand, it can no longer be applied in multi-intervals(i.e.when region A contains differet intervals $[\mu_1, \nu_1], [\mu_2, \nu_2]$ ...)case because genus of manifold becomes higher. Here we introduce the twist operator approach which basically translate the constrain $\phi_i^+(x) = \phi_{i+1}^-(x)$ to some local primary field operators, which may have some advantage to be generalized in multi-intervals case.

3.1 Twist Operator

Twist fields can be defined in a QFT model when there is a global internal symmetry(invariance under an operation which does not change the position:

$$\int dxdy\mathscr{L}[\sigma\phi](x,y) = \int dxdy\mathscr{L}[\phi](x,y)$$

. In our model with n copies of plane, we can write the Largarangian as sum over Largarangian on each sheet $\mathscr{L}^{(n)} = \sum_1^n \mathscr{L}_i$. $\mathscr{L}^{(n)}$ has a symmetry under exchange of copies, we can define two kinds of twist field operators related to two permutation symmetry $\sigma : i \mapsto i+1$ and $\sigma^{-1} : i+1 \mapsto i$.We denote them as $\mathscr{T}, \tilde{\mathscr{T}}$:

$$\begin{aligned} \mathscr{T} &= \mathscr{T}_\sigma, \quad \sigma : i \mapsto i+1 \bmod n \\ \tilde{\mathscr{T}} &= \mathscr{T}_{\sigma^{-1}}, \quad \sigma^{-1} : i+1 \mapsto i \bmod n \end{aligned} \tag{3.1}$$

The function of $\mathscr{T}(a,b)$ is when we clockwise turn around (a,b), field φ_i in ith copy turns into φ_{i+1} in $(i+1)th$ copy. This can be understood as adding a constrain condition in path integral:

$$\langle \mathscr{T}(a,b)\cdots\rangle_{\mathscr{L},\mathbb{R}^2} \propto \int_{\mathscr{C}_\sigma(0,0)} [d\phi]_{\mathbb{R}^2} \exp\left[-\int_{\mathbb{R}^2} d\mathrm{x}d\mathrm{y}\mathscr{L}[\phi](\mathrm{x},\mathrm{y})\right]\cdots, \tag{3.2}$$

where the path integral condition is:

$$\mathscr{C}_\sigma(a,b) \quad : \quad \phi\left(\mathrm{x}, b^+\right) = \sigma\phi\left(\mathrm{x}, b^-\right), \quad \mathrm{x} \in [a,\infty). \tag{3.3}$$

In order to define $\tilde{\mathscr{T}}$, we just need to change the σ in path integral condition to σ^{-1}. Now we can write Z_n as two point function:$Z_n \propto \langle \mathscr{T}(\mu,0)\tilde{\mathscr{T}}(\nu,0)\rangle_{\mathscr{L}^{(n)},\mathbb{R}^2}$,it can be seen

by observing on $\in [\mu, \nu]$ we have the "gluing" condition $\phi_i(x, 0^+) = \phi_{i+1}(x, 0^-)$, while for $x \in (\nu, \infty)$ different copy is indepent because $\mathscr{T}, \tilde{\mathscr{T}}$ cancels each other. General correlation function on that "glued" manifold(labelled by $\mathscr{M}_{n,\mu,\nu}$) can be written as

$$\langle \mathcal{O}(\mathrm{x}, \mathrm{y};\ \text{sheet } i) \cdots \rangle_{\mathscr{L},\mathscr{M}_{n,\mu,\nu}} = \frac{\left\langle \mathscr{T}(\mu, 0)\, \tilde{T}(\nu, 0)\, \mathcal{O}_i(\mathrm{x}, \mathrm{y}) \cdots \right\rangle_{\mathscr{L}^{(n)},\mathbb{R}^2}}{\left\langle \mathscr{T}(\mu, 0)\, \tilde{\mathscr{T}}(\nu, 0) \right\rangle_{\mathscr{L}^{(n)},\mathbb{R}^2}}. \tag{3.4}$$

In particular, we consider the one point function of energy-momentum tensor:$\langle T(w) \rangle_{\mathscr{L},\mathscr{M}_{n,\mu,\nu}}$. It can be obtained by a conformal transformation from z in R^2 to ω in $\mathscr{M}_{n,\mu,\nu}$:

$$z = \left(\frac{w - a_1}{w - a_2} \right)^{\frac{1}{n}}. \tag{3.5}$$

Energy-momentum tensor changes as

$$\langle T(w) \rangle_{\mathscr{L},\mathscr{M}_{n,\mu,\nu}} = \left(\frac{\partial z}{\partial w} \right)^2 \langle T(z) \rangle_{\mathscr{L},\mathbb{R}^2} + \frac{c}{12} \{z, w\}, \tag{3.6}$$

where $\{z, w\} = \frac{z'''z' - (3/2)(z'')^2}{(z')^2}$ is the Schawarzian derivative. Using $\langle T(w) \rangle_{\mathscr{L},R^2} = 0$, we can get

$$\langle T(w) \rangle_{\mathscr{L},\mathscr{M}_{n,\mu,\nu}} = \frac{c\left(n^2 - 1\right)}{24n} \frac{(\mu - \nu)^2}{(w - \mu)^2 (w - \nu)^2}. \tag{3.7}$$

$\langle T(w) \rangle_{\mathscr{L},\mathscr{M}_{n,\mu,\nu}}$ can be replaced by using 3.4, now we have equation:

$$\frac{\left\langle \mathscr{T}(\mu, 0)\, \tilde{T}(\nu, 0)\, T^{(n)}(w) \right\rangle_{\mathscr{L}^{(n)},\mathbb{R}^2}}{\langle \mathscr{T}(\mu, 0)\, T(\nu, 0) \rangle_{\mathscr{L}^{(n)},\mathbb{R}^2}} = \frac{c\left(n^2 - 1\right)}{24n} \frac{(\mu - \nu)^2}{(w - \mu)^2 (w - \nu)^2} \tag{3.8}$$

In fact, we can prove $\mathscr{T}, \tilde{\mathscr{T}}$ are primary field[13] with same conformal dimension by diagonalizing twist field in a differet basis $\widetilde{\phi}_k = \frac{1}{n} \sum_{l=1}^{n} e^{2\pi i l k/n} \phi_l$. Here we just use its primary field property. We know the OPE of energy-momentum tensor with two primary fields can be written as:

$$\begin{aligned} &\left\langle \mathscr{T}(\mu, 0)\, \tilde{\mathscr{T}}(\nu, 0)\, T^{(n)}(w) \right\rangle_{\mathscr{L}^{(n)},\mathbb{R}^2} = \\ &\left(\frac{1}{w-\mu} \frac{\partial}{\partial \mu} + \frac{h_1}{(w-\mu)^2} + \frac{1}{w-\nu} \frac{\partial}{\partial \nu} + \frac{h_2}{(w-\nu)^2} \right) \left\langle \mathscr{T}(\mu, 0)\, \tilde{\mathscr{T}}(\nu, 0) \right\rangle_{\mathscr{L}^{(n)},\mathbb{R}^2}. \end{aligned} \tag{3.9}$$

Using $h_1 = h_2$ and plugging in$\left\langle \mathscr{T}(\mu, 0)\, \tilde{T}(\nu, 0) \right\rangle_{\mathscr{L}^{(n)},\mathbb{R}^2} \propto (\mu - \nu)^{-2h} (\bar{\mu} - \bar{\nu})^{(} - 2\bar{h})$, we get

$$h = \frac{c}{24} \left(n - \frac{1}{n} \right). \tag{3.10}$$

Following the same procedure for antiholomorphic part, we can also get $\bar{h} = h$.

3.2 Derivation from twist operator approach

Since twist operator is just another approach to calculate the partition function, we should be able to reproduce the result from mapping to cylinder approach. Here we treat the boundary condition as some local excitation, generated by some field $B_{a_1}(\mu - \frac{\epsilon}{2}, 0)$, $B_{a_2}(\nu + \frac{\epsilon}{2}, 0)$ acting on the vacuum state:

$$Z_n \propto \langle \prod_1^n B_{a_1,i}(\mu - \frac{\epsilon}{2}, 0) \mathcal{T}(\mu, 0) \tilde{\mathcal{T}}(\nu, 0) \prod_1^n B_{a_2,i}(\nu + \frac{\epsilon}{2}, 0) \rangle_{\mathcal{L}^{(n)}, \mathbb{R}^2}. \tag{3.11}$$

Then we can perform a 90 degree rotation to change time and space coordinate, partition function after the rotation can be written as

$$Z_n \propto \langle \prod_1^n B_{a_1,i}(0, \mu - \frac{\epsilon}{2}) \mathcal{T}(0, \mu) \tilde{\mathcal{T}}(0, \nu) \prod_1^n B_{a_2,i}(0, \nu + \frac{\epsilon}{2}) \rangle_{\mathcal{L}^{(n)}, \mathbb{R}^2}. \tag{3.12}$$

We want to insert two complete basis at differet time slice as we did in last chapter. Common choice of complete basis is $\sum_i |i\rangle\langle i|_{t=\mu,\nu}$, where $|i\rangle$ means energy eigenstates or equivalantly eigenstates of dilation operator L_0 in radial quantization.However, this choice of complete basis can not reproduce "glued" boundary state. Therefore, we choice a twisted complete basis

$$\begin{aligned} &\sum_i \mathcal{T}(0, \mu) \tilde{\mathcal{T}}(0, \mu^+) |i\rangle\langle i|_{t=\mu} \mathcal{T}(0, \mu^+) \tilde{\mathcal{T}}(0, \mu), \\ &\sum_i \mathcal{T}(0, \nu) \tilde{\mathcal{T}}(0, \nu^-) |i\rangle\langle i|_{t=\nu} \mathcal{T}(0, \nu^-) \tilde{\mathcal{T}}(0, \nu), \end{aligned} \tag{3.13}$$

where $\mu^+ = \lim_{\tau \to 0} \mu + \tau$ means time slightly upper than $(0, \mu)$, similarly for ν^-. Notice $\mathcal{T}(0, \mu) \tilde{\mathcal{T}}(0, \mu^+) \mathcal{T}(0, \mu^+) \tilde{\mathcal{T}}(0, \mu) = \mathrm{I}$, so the twisted complete basis is still a complete basis. After insertion, partition function can be written as

$$\begin{aligned} Z_n \propto \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} &\langle 0| \prod_1^n B_{a_1,i}(0, \mu) \mathcal{T}(0, \mu) \tilde{\mathcal{T}}(0, \mu^+) |k\rangle\langle k|_{t=\mu} \mathcal{T}(0, \mu^+) \tilde{\mathcal{T}}(0, \nu^-) |l\rangle \\ &\langle l|_{t=\nu} \mathcal{T}(0, \nu^-) \tilde{\mathcal{T}}(0, \nu) \prod_1^n B_{a_2,j}(0, \nu) |0\rangle_{\mathcal{L}^{(n)}, \mathbb{R}^2}. \end{aligned} \tag{3.14}$$

For state with lowest conformal dimension, which is vacuum $|0\rangle$, we have the first term

$$\begin{aligned} &\langle 0| \prod_1^n B_{a_1,i}(0, \mu) \mathcal{T}(0, \mu) \tilde{\mathcal{T}}(0, \mu^+) |0\rangle\langle 0| \mathcal{T}(0, \mu^+) \tilde{\mathcal{T}}(0, \nu^-) |0\rangle \\ &\qquad \langle 0| \mathcal{T}(0, \nu^-) \tilde{\mathcal{T}}(0, \nu) \prod_1^n B_{a_2,j}(0, \nu) |0\rangle_{\mathcal{L}^{(n)}, \mathbb{R}^2} \end{aligned} \tag{3.15}$$

Here we omit the time because vacum state is invariant under time translation. For state with higher confromal dimension, we define $\phi(0,t)$ as the field generating that state $|i\rangle_t$ with conformal dimension h_ϕ. We can do OPE with twist operator $\phi(0,t)\mathcal{T}(0,0)$

$$\frac{\langle\phi(0,t)\mathcal{T}(0,0)\rangle_{(\mathbb{R}^2)_n}}{\langle\mathcal{T}\rangle_{(\mathbb{R}^2)_n}} = \tilde{C}_{\phi\mathcal{T}}^{:\phi\mathcal{T}:(0,0)} t^{2\left(h_{:\phi\mathcal{T}:}-h_\phi-h_\mathcal{T}\right)} \frac{\langle:\phi\mathcal{T}:\rangle_{(\mathbb{R}^2)_n}}{\langle\mathcal{T}\rangle_{(\mathbb{R}^2)_n}}. \tag{3.16}$$

where $:\phi\mathcal{T}:(0,0)$ are primary field with lowest conformal dimension in OPE, it is the leading order when the distance between these fields are small ($t \ll 1$), we call it composite field. The composite field has conformal dimension $h_{:\mathcal{T}\phi:} = \frac{h_\phi}{n} + h_\mathcal{T}$[14]. So at the leading order we have

$$\langle l|_{t=\mu}\mathcal{T}(0,\mu^+)\tilde{\mathcal{T}}(0,\nu^-)|l\rangle_{t=\nu} \sim \langle:\phi\mathcal{T}:(0,\mu^+):\phi\mathcal{T}:(0,\nu^-)\rangle \sim \frac{1}{|\mu-\nu|^{-4h_{:\phi\mathcal{T}:}}}. \tag{3.17}$$

This equation means the two point twist operator correlation function at higher conformal dimension state $\langle l|_{t=\mu}\mathcal{T}(0,\mu^+)\tilde{\mathcal{T}}(0,\nu^-)|l\rangle$ is subleading term in $\log Z_n$ at the limit $L = |\mu-\nu| \gg 1$.

Now we can calucate the leading term in $\log Z_n$:

$$\begin{aligned}\log Z_n =& \log\langle 0|\mathcal{T}(0,\mu^+)\tilde{\mathcal{T}}(0,\nu^-)|0\rangle + \log\langle 0|\prod_1^n B_{a_1,i}(0,\mu)\mathcal{T}(0,\mu)\tilde{\mathcal{T}}(0,\mu^+)|0\rangle + \\ & \log\langle 0|\mathcal{T}(0,\nu^-)\tilde{\mathcal{T}}(0,\nu)\prod_1^n B_{a_2,j}(0,\nu)|0\rangle + ...\end{aligned} \tag{3.18}$$

After introducing the cut off ϵ, we can write

$$\log\langle 0|\mathcal{T}(0,\mu^+)\tilde{\mathcal{T}}(0,\nu^-)|0\rangle \sim -4h_\mathcal{T}\log\frac{L}{\epsilon}. \tag{3.19}$$

For $Z_1^n = \prod_1^n\langle B_{a_1,i}(0,\mu-\frac{\epsilon}{2})B_{a_2,i}(0,\nu+\frac{\epsilon}{2})\rangle_{\mathcal{L}_i,\mathbb{R}^2}$, we can also insert two complete basis field $\sum_i |i\rangle\langle i|_{t=\mu,\nu}$, the leading order in Z_1^n is still the vacuum state term.

$$\log Z_1^n = n\log\langle B_{a_1,i}(0,\mu-\frac{\epsilon}{2})\rangle + n\log\langle B_{a_2,i}(0,\nu+\frac{\epsilon}{2})\rangle + ... \tag{3.20}$$

Using $h_\mathcal{T} = \frac{c}{24}\left(n-\frac{1}{n}\right)$, we can get Rényi entropy

$$\begin{aligned}S_n =& \left(1+\frac{1}{n}\right)\frac{c}{6}\left(\log\frac{L}{\epsilon}\right) \\ &+ \frac{1}{1-n}\left(S\left(a_1^n\right) - nS\left(a_1^1\right) + S\left(a_2^n\right) - nS\left(a_2^1\right)\right) + O(e^{-\frac{l}{n}min(\Delta_i)}),\end{aligned} \tag{3.21}$$

where $S\left(a_1^1\right) = \log\langle B_{a_1}(\mu-\frac{\epsilon}{2},0)\rangle = \log\langle a_1|0\rangle$ is just the Affleck-Ludwig boundary entropy.$S\left(a_1^n\right) = \log\langle 0|\prod_1^n B_{a_1,i}(0,\mu)\mathcal{T}(0,\mu)\tilde{\mathcal{T}}(0,\mu^+)|0\rangle$ can be understood as n-copies

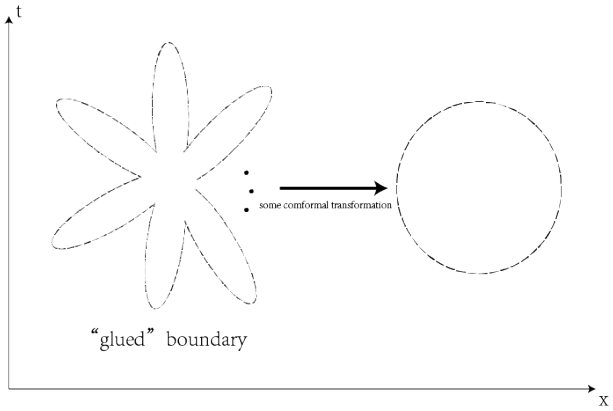

figure 3-1 "glued" boundary in R^2.

of boundary condition a_1 at a circle with radius ϵ centered at $(0, \mu - \frac{\epsilon}{2})$, and they are glued at a point $\phi_i(0-, \mu) = \phi_{i+1}(0+, \mu)$. We can perform a conformal transformation from ω in "glued" n copies of R^2 to z in R^2:

$$\omega \to z = (\omega - \mu)^{\frac{1}{n}}. \tag{3.22}$$

Picture 3-1 shows the "glued" boundary condition on R^2 can be transformed to a boundary condition on a circle, which corresponds to the state $|a_1^\infty\rangle$ in radial quantization. Therefore, $S\left(a_1^n\right)$ is again the Affleck-Ludwig boundary entropy. Now we have reproduce the same Rényi entropy result as in cylinder approach.

The subleading term in Z_n and Z_1^n is produced by composite field $:\phi\mathcal{T}:$ with higher conformal dimension. For simplicity, we denote the leading term in Z_n (Eq.3.15) as Z_n^0 and leading term in Z_1 as Z_1^0, then we can write

$$\begin{aligned} Z_n &= Z_n^0 \left[1 + O(L^{-4\frac{h_1}{n}}) + O(L^{-4\frac{h_2}{n}}) + ...\right], \\ Z_1^n &= (Z_1^0)^n \left[1 + O(L^{-4h_1}) + O(L^{-4h_2}) + ...\right]^n. \end{aligned} \tag{3.23}$$

We can see the order have the same behaviour as in cylinder approach(Eq.2.8).

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