Homework 5: Due on Dec 15

1 Smarr formula

Compute the Smarr integrals in Kerr black hole, and express them as functions of r_+ and a

$$
\begin{aligned}
M_H &= -\frac{1}{8\pi}\oint_{\mathcal{H}} \nabla^\alpha t^\beta dS_{\alpha\beta} \\
J_H &= \frac{1}{16\pi}\oint_{\mathcal{H}} \nabla^\alpha \phi^\beta dS_{\alpha\beta},
\end{aligned}
\tag{1.1}
$$

where $t^\beta = (\partial_t)^\beta$ and $\phi^\beta = (\partial_\phi)^\beta$ are Killing vectors, and $\mathcal{H}$ is the outer horizon. Show that they are subject to

$$
M_H = 2\Omega_H J_H + \frac{\kappa A}{4\pi} \tag{1.2}
$$

2 Hawking-Hartle formula

Consider a quasi-static process during which the surface area of a black hole changes. (By quasi-static we mean that dA/dv is very small.) Derive the Hawking-Hartle formula,

$$
\frac{dA}{dv} = \frac{8\pi}{\kappa}\oint_{\mathcal{H}(v)} \left(\frac{1}{8\pi}\sigma^{\alpha\beta}\sigma_{\alpha\beta} + T_{\alpha\beta}\xi^\alpha\xi^\beta\right) dS, \tag{2.1}
$$

in which ξ^α is tangent to the null generators of the event horizon and $\sigma_{\alpha\beta}$ is their shear tensor. The second term within the integral represents the effect of accreting matter on the surface area. The first term represents the effect of gravitational radiation flowing across the horizon.

3 Derivations

- Consider two sets of functions, $\{\phi_I(x)\}$ and $\{\phi'_I(x)\}$, related by the Bogoliubov transformations:

$$
\begin{aligned}
\phi'_I(x) &= \sum_I \left[\alpha_{IJ}\phi_J(x) + \beta_{IJ}\phi^*_J(x)\right], \\
\phi'^*_I(x) &= \sum_I \left[\beta^*_{IJ}\phi_J(x) + \alpha^*_{IJ}\phi^*_J(x)\right].
\end{aligned}
\tag{3.1}
$$

  Show the following conditions

$$
\begin{aligned}
\alpha\alpha^\dagger - \beta\beta^\dagger = 1, &\quad \alpha\beta^T - \beta\alpha^T = 0, \\
\alpha^\dagger\alpha - \beta^T\beta^* = 1, &\quad \alpha^\dagger\beta - \beta^T\alpha^* = 0,
\end{aligned}
\tag{3.2}
$$

- We define the (unnormalized) left- and right Rindler wedge n-particle states:

$$
|^R n, P\rangle = (b_P^{R\dagger})^n|0_{Rd}\rangle, \quad \text{and} \quad |^L n, P\rangle = (b_P^{L\dagger})^n|0_{Rd}\rangle. \tag{3.3}
$$

Show that the Minkowski vacuum is written as an entangled state

$$|0_M\rangle = \prod_P (\cosh \phi_P)^{-1} \sum_{n=0}^{\infty} e^{-n\pi\omega/\kappa} |^R n, P\rangle |^L n, P\rangle, \tag{3.4}$$

where ϕ_P is defined by $\tanh \phi_P = e^{-\pi\omega/\kappa}$.