Homework 8: Due at class on May 2

1. Find the fundamental group $\pi_1(\Sigma_g)$ of the Riemann surface of genus g.

2. Show that Euler characteristics of a compact Lie group is zero.

3. Show that $\mathrm{SO}(4) \cong \{\mathrm{SU}(2) \times \mathrm{SU}(2)\}/\{\pm \mathrm{Id}\}$, where $\mathrm{Id} \hookrightarrow \mathrm{SU}(2) \times \mathrm{SU}(2)$ is the diagonal embedding. The hint is given as follows.

Let $\mathbb{H}$ be the quaternion in which an element $x \in \mathbb{H}$ can be expressed as

$$x = x_1 + x_2 i + x_3 j + x_4 k$$

where $x_a \in \mathbb{R}$ $(a = 1, \cdots, 4)$ and

$$i^2 = j^2 = k^2 = -1\,, \quad ij = -ji = k\,, \quad jk = -kj = i\,, \quad ki = -ik = j\,.$$

We define the imaginary part of x as

$$\mathrm{Im}\, x = x_2 i + x_3 j + x_4 k$$

so that the conjugate $\overline{x}$ is written as

$$\overline{x} = x_1 - x_2 i - x_3 j - x_4 k$$

Therefore, the multiplication becomes

$$\overline{xy} = \overline{y} \cdot \overline{x}$$

The norm of x is

$$|x|^2 = x\overline{x} = \overline{x}x = x_1^2 + x_2^2 + x_3^2 + x_4^2$$

From this viepoint, $\mathrm{SU}(2)$ can be considered as a group of unit quaternions $\mathrm{SU}(2) = \{x \in \mathbb{H} |\, |x| = 1\}$. Then $\mathrm{SU}(2) \times \mathrm{SU}(2)$ acts on $\mathbb{H}$ by rotations in the following way:

$$x \mapsto q_1 x q_2^{-1}$$

is a rotation of $\mathbb{R}^4 = \mathbb{H}$ for q_1, $q_2 \in \mathrm{SU}(2)$. Then $(-q_1, -q_2)$ represents the same rotation as (q_1, q_2). Show that these represent all the rotations of $\mathbb{R}^4 = \mathbb{H}$ so that it is isomorphic to $\mathrm{SO}(4)$.

4. Derive the fundamental groups of $\mathrm{SO}(2)$, $\mathrm{SO}(3)$ and $\mathrm{SO}(4)$.

5. Show that $\mathrm{SL}(2, \mathbb{R})$ is a deformation retract to $\mathrm{SO}(2)$ in the following way.

- Write an element $A \in \mathrm{SL}(2, \mathbb{R})$ as $A = (a_1, a_2)$, where the a_i are column vectors. The Gram-Schmidt process replaces A to

$$\mathrm{SO}(2) \ni U = \left(\frac{a_1}{|a_1|}, \frac{a_2 - \frac{\langle a_1, a_2 \rangle}{|a_1|^2} a_1}{|a_2 - \frac{\langle a_1, a_2 \rangle}{|a_1|^2} a_1|} \right) ,$$

  where $\langle a_1, a_2 \rangle$ is the standard inner product of $\mathbb{R}^2$. Construct a deformation retract that connects A to U inside $\mathrm{SL}(2, \mathbb{R})$.

Do the same exercise for $\mathrm{SL}(2, \mathbb{C})$ and which Lie group does it have a deformation retract to? Derive the homology groups and the fundamental group of $\mathrm{SL}(2, \mathbb{R})$ and $\mathrm{SL}(2, \mathbb{C})$.