

Homework 1: Due at class on Sep 21

1 Lagrangian and symmetry

1.1

Let us consider the Lagrangian of the electromagnetic field

$$\mathcal{L}_{\text{EM}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - A_\mu J^\mu$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and J_μ is a 4-vector. Derive the Euler-Lagrange equations for $A_\mu(x)$. Write the equations in standard form of the Maxwell's equations by identifying $E^i = -F^{0i}$ and $\epsilon^{ijk}B^k = -F^{ij}$.

1.2

Show that the Lagrangian of a complex scalar field

$$\mathcal{L}_{\text{cpx scalar}} = \partial_\mu \phi^*(x) \partial^\mu \phi(x) - m^2 \phi^*(x) \phi(x)$$

is invariant under the global U(1) symmetry

$$\phi(x) \rightarrow e^{i\alpha} \phi(x), \quad \phi^*(x) \rightarrow e^{-i\alpha} \phi^*(x), \quad \alpha \in \mathbb{R}$$

Find the corresponding Noether current.

1.3

Let us consider the Lagrangian of a complex scalar field coupled to electromagnetic field

$$\mathcal{L} = (D_\mu \phi)^* D^\mu \phi - m^2 \phi^* \phi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

where $D_\mu = \partial_\mu + ieA_\mu$. Find the energy momentum $T^{\mu\nu}$.

Show that \mathcal{L} is invariant under the U(1) gauge (sometimes called **local**) symmetry

$$\begin{aligned} \phi(x) &\rightarrow e^{i\alpha(x)} \phi(x) \\ A_\mu(x) &\rightarrow A_\mu(x) - \frac{1}{e} \partial_\mu \alpha(x) \end{aligned}$$

Note that α for the **global** U(1) symmetry in Problem 1.2 does not depend on x .

2 Complex scalar field

Let us consider quantum theory of a complex scalar field in Problem 1.2.

2.1

The mode expansion of the complex scalar field $\phi(x)$ is

$$\begin{aligned}\phi(x) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_p e^{ip \cdot x} + b_p^\dagger e^{-ip \cdot x} \right) \\ \phi^*(x) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(b_p e^{ip \cdot x} + a_p^\dagger e^{-ip \cdot x} \right).\end{aligned}$$

Imposing the canonical commutation relations

$$[\phi(x), \pi(y)] = i\delta^{(3)}(x - y) = [\phi^*(x), \pi^*(y)],$$

and the others vanish, derive the commutation relations of the creation and annihilation operators $a_p, a_p^\dagger, b_p, b_p^\dagger$.

2.2

Show that the Noether charge Q for the global $U(1)$ symmetry in Problem 1.2 can be expressed as follows in terms of the modes of ϕ :

$$Q = - \int \frac{d^3p}{(2\pi)^3} \left(a_p^\dagger a_p - b_p^\dagger b_p \right)$$

after dropping a constant term. Show that the state $a_p^\dagger|0\rangle$ has the $U(1)$ charge -1 and the state $b_p^\dagger|0\rangle$ has the $U(1)$ charge $+1$.

2.3

Let us consider the operators in the Heisenberg picture

$$\begin{aligned}\phi(x) &= \phi(x, t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_p e^{-ip \cdot x} + b_p^\dagger e^{ip \cdot x} \right) \\ \phi^*(x) &= \phi^*(x, t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(b_p e^{-ipx} + a_p^\dagger e^{ip \cdot x} \right)\end{aligned}$$

Using the commutation relations of the creation and annihilation operators to calculate $\langle 0|T[\phi(x)\phi(y)]|0\rangle$, $\langle 0|T\phi^*(x)\phi^*(y)|0\rangle$, and $\langle 0|T\phi(x)\phi^*(y)|0\rangle$. Compare with the Feynman propagator the real scalar field.

3 Propagator

The Feynman propagator of a real scalar field in momentum space is given by

$$D_F(x - y) = \langle 0|T\phi(x)\phi(y)|0\rangle = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip \cdot (x-y)}$$

where the $i\epsilon$ is there to ensure that the correct poles are included in the p_0 integral. Compute D_F explicitly and write it as a function of the Lorentz-invariant length $s^2 = (x - y)_\mu (x - y)^\mu$ in terms of the modified Hankel and Bessel functions in the limit $\epsilon \rightarrow 0$:

$$H_n^{(1)}(z) = \frac{-2i}{\sqrt{\pi} \left(n - \frac{1}{2}\right)!} \left(-\frac{1}{2}z\right)^n \int_1^{\infty - i\epsilon} e^{-izx} (x^2 - 1)^{n-1/2} dx$$

$$K_n(z) = \frac{\sqrt{\pi}}{\left(n - \frac{1}{2}\right)!} \left(\frac{1}{2}z\right)^n \int_1^\infty e^{-zx} (x^2 - 1)^{n-1/2} dx$$

where $K_n(z) = i^{n+1}(\pi/2)H_n^{(1)}(iz)$.

Discuss the behaviors of the propagator in the following regimes by using the properties of the modified Hankel and Bessel functions. (See [Wikipedia](#).)

- $s \rightarrow 0$
- s^2 is large and positive (timelike separation)
- s^2 is large and negative (spacelike separation)
- $m^2 \rightarrow 0$