Homework 13 (Due at class on June 13)

1 Surface gravity

Section 6, Problem 4 [Wal10].

2 Cardy formula

2.1 Free boson

Let us consider the torus partition function of bosonic string theory

$$\begin{aligned} Z(\beta) &= \frac{1}{q}\prod_{s=1}^{\infty}\frac{1}{(1-q^s)^{24}} \\ &\equiv \sum_{n=-1}^{\infty} d(n)q^n \qquad q := e^{-\beta}\,. \end{aligned} \tag{2.1}$$

As explained in the lecture, $1/\beta$ can be interpreted as temperature. The degeneracy $d(N)$ can be obtained from the partition function by the inverse Laplace transform

$$d(N) = \frac{1}{2\pi i}\int d\beta e^{\beta N} Z(\beta). \tag{2.2}$$

We would like to evaluate this integral (2.2) for large N which corresponds to large world-sheet energy.

For large N, we expect that the integral receives most of its contributions from high temperature or small β region of the integrand. To evaluate the small β contribution, it is convenient to use S-dual frame

$$Z(\beta) = \left(\frac{\beta}{2\pi}\right)^{12} Z(\frac{4\pi^2}{\beta}).$$

Using this property, show that (2.2) for large N can be approximated as

$$d(N) \sim \frac{1}{2\pi i}\int \left(\frac{\beta}{2\pi}\right)^{12} e^{\beta N + \frac{4\pi^2}{\beta}} d\beta.$$

Using the saddle point analysis, show that the leading asymptotic expression for the number of states is

$$d(N) \sim \exp\left\{(4\pi\sqrt{N})\right\}.$$

2.2 D1-D5-P system

In the lecture, the torus partition function of the long string in the D1-D5-P system is given by

$$Z = \text{const}\left[\prod_{m=1}^{\infty}\frac{1+q^m}{1-q^m}\right]^4 \equiv \sum \Omega(n)q^n\,,$$

Note that the partition function can be written as

$$Z = \left[\frac{\vartheta_2(\tau)}{\eta(\tau)^3}\right]^2$$

upto some constant. Using the modular property of this partition function, show that the number of states at large D-brane charges is

$$\Omega(n = Q_1Q_5Q_P) \sim (Q_1Q_5Q_P)^{-\frac{7}{4}} \exp\left(2\pi\sqrt{Q_1Q_5Q_P}\right) .$$

References

[Wal10] R. M. Wald. *General relativity*. University of Chicago press, 2010.