Homework 8: Due at class on April 26

1. Let us define $S^3 = \{(x^0, x^1, x^2, x^3) \in \mathbb{R}^4 | \sum_{i=0}^{3} (x^i)^2 = 1\}$ and $S^1 = \{(x^0, x^1, x^2, x^3) \in \mathbb{R}^4 | (x^0)^2 + (x^1)^2 = 1\}$. Then, show that $S^3 \backslash S^1$ is homotopic to S^1.

2. Let us identify $S^2 = \mathbb{C} \cup \{\infty\}$. Then, a holomorphic map $g : \mathbb{C} \to \mathbb{C}; z \mapsto z^n$ $(n \in \mathbb{Z})$ can be extended to $g : S^2 \to S^2$. Find the mapping degree $\deg g$ of g.

3. **Fundamental theorem of algebra**

We define $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = z^n + a_1 z^{n-1} + \cdots + a_n$ for $n \geq 1$. In addition, by writing $z = x + iy$, we define one-form

$$\omega = \mathrm{Im}\frac{dz}{z} = \frac{-ydx}{x^2+y^2} + \frac{xdy}{x^2+y^2} \ .$$

Then, show that

$$\frac{1}{2\pi} \int_{C_R} f^* \omega = n \ ,$$

where C_R is the circle with sufficiently large radius R. (Hint: construct homotopy between f and g above.) If there were no zero points $f(z) = 0$ inside C_R, show that

$$\frac{1}{2\pi} \int_{C_R} f^* \omega = 0$$

by using the Stokes theorem.

4. **This is a bonus problem with extra 5 points which is NOT mandatory.**

4.1 Show that the following two embeddings $\mathbb{C}P^k \hookrightarrow \mathbb{C}P^n$ are homotopic

$$\begin{aligned} \mathbb{C}P^k &= \left\{ \left[z^0, z^1, \cdots, z^k, 0, \cdots, 0\right] \in \mathbb{C}P^n \right\} \subset \mathbb{C}P^n \\ \widetilde{\mathbb{C}P}^k &= \left\{ \left[0, \cdots, 0, z^{n-k}, \cdots, z^n\right] \in \mathbb{C}P^n \right\} \subset \mathbb{C}P^n \end{aligned}$$

4.2 Derive the cohomology ring $(H^*(\mathbb{C}P^n; \mathbb{R}), \cup)$.

4.3 Find the intersection number $\mathbb{C}P^k \cdot \widetilde{\mathbb{C}P}^{n-k}$ in $\mathbb{C}P^n$ for all k.

4.4 If $n > m$, show that there is no smooth map $f : \mathbb{C}P^n \to \mathbb{C}P^m$ which induces a non-trivial map $f^* : H^2(\mathbb{C}P^m; \mathbb{R}) \to H^2(\mathbb{C}P^n; \mathbb{R})$.