# Homework 8: Due on Nov 14

## 1 Derivation

Derive eigenvalues and eigenvectors of the following $2N \times 2N$ matrix

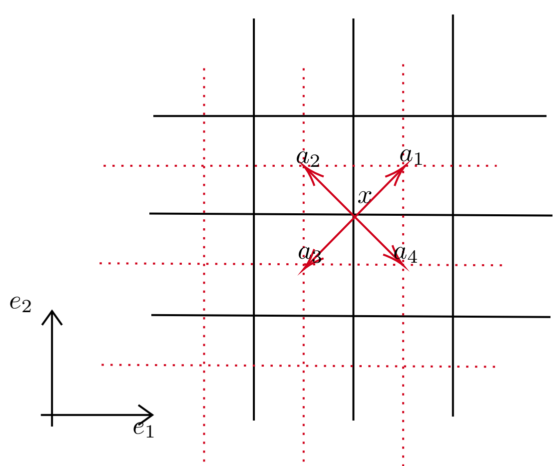
$$
-iM = -i\begin{pmatrix}
0 & B & & & & & & & & -J \\
-B & 0 & & & & & & & & \\
 & & \ddots & \ddots & & & & & & \\
 & & \ddots & 0 & B & & & & & \\
 & & & -B & 0 & J & & & & \\
 & & & & -J & 0 & B & & & \\
 & & & & & -B & 0 & J & & \\
 & & & & & & -J & 0 & \ddots & \\
 & & & & & & & \ddots & \ddots & B \\
J & & & & & & & & -B & 0
\end{pmatrix}
$$

## 2 Onsager's exact solution

Let us consider an $M \times N$ square periodic lattice. At each vertex, one can define fermionic operators

$$
\psi_i(x) = \sigma(x)\mu\left(x + a_i\right) , \qquad (i = 1, \ldots, 4)
$$

where the position is defined as below




Suppose that the branch cut from the disordered operator $\mu$ runs to the left. Then, we can write

$$
\begin{aligned}
\psi_1(x) &= \sigma(x)\mu\left(x+a_1\right) \\
&= \sigma(x)\mu\left(x+a_2\right)e^{-2K\sigma(x)\sigma(x+e_2)} \\
&= \sigma(x)\mu\left(x+a_2\right)\left(\operatorname{ch}(2K)-\operatorname{sh}(2K)\sigma(x)\sigma\left(x+e_2\right)\right) \\
&= \operatorname{ch}(2K)\psi_2(x)-\operatorname{sh}(2K)\psi_3\left(x+e_2\right)
\end{aligned}
$$

Show that

$$
\not{D}\Psi = 0, \quad \not{D} = \begin{bmatrix} -1 & \operatorname{ch}(2K) & -\operatorname{sh}(2K)D_2 & 0 \\ 0 & -1 & \operatorname{ch}(2K) & -\operatorname{sh}(2K)D_1^{-1} \\ \operatorname{sh}(2K)D_2^{-1} & 0 & -1 & \operatorname{ch}(2K) \\ -\operatorname{ch}(2K) & \operatorname{sh}(2K)D_1 & 0 & -1 \end{bmatrix}, \quad \Psi = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{bmatrix}
$$

where $D_i$ is the shift operator along $e_i$. Essentially, the partition function of fermionic model of Ising model can be understood as

$$
Z_{MN} = \int d\Psi e^{\Psi\not{D}\Psi} = (\det \not{D})^{1/2}
$$

Show that the free energy of the Ising model is

$$
\begin{aligned}
&\lim_{M,N\to\infty} \frac{1}{MN}\log Z_{MN} \\
=&\frac{1}{2}\log[2\operatorname{ch}^2(2K)] + \frac{1}{2}\int_0^{2\pi}\int_0^{2\pi}\log\left(\operatorname{ch}^2(2K)-\operatorname{sh}(2K)\left(\cos\theta_1+\cos\theta_2\right)\right)\frac{d\theta_1}{2\pi}\frac{d\theta_2}{2\pi}
\end{aligned}
$$

which is the Onsager's exact solution.

# 3 XXZ spin chain

Consider the quantum spin chain of the XXZ model with Hamiltonian

$$
H = \frac{J_1}{4}\sum_{k=1}^{N}\left(\sigma_k^x\sigma_{k+1}^x + \sigma_k^y\sigma_{k+1}^y\right) + \frac{J_2}{4}\sum_k \sigma_k^z\sigma_{k+1}^z
$$

with periodic boundary conditions. Defining the operators

$$
c_i = \left(\prod_{j<i}\sigma_j^z\right)\sigma_i{}^+, \qquad c_i^\dagger = \left(\prod_{j<i}\sigma_j^z\right)\sigma_i^-
$$

where $\sigma_i^\pm = \frac{1}{2}(\sigma_i^x \pm i\sigma_i^y)$. Show that the hamiltonian can be written as

$$
H = \frac{J_1}{2}\sum_{a=1}^{N}\left[c_a^\dagger c_{a+1} + c_{a+1}^\dagger c_a\right] + J_2\sum_{a=1}^{N}\left(c_a^\dagger c_a - \frac{1}{2}\right)\left(c_{a+1}^\dagger c_{a+1} - \frac{1}{2}\right)
$$

Setting $J_2 = 0$, diagonalize the hamiltonian.