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本 科 毕 业 论 文

二维有理共形场论的特征标

Characters of 2d rational CFT

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摘　要

共形场论 (CFTs)，简单来说就是引入共形对称性的量子场论，其为现今唯一较为成熟的非微扰量子场论，在近几十年一直备受研究者的重视，并且在弦论、凝聚态等物理领域中均有重要应用。共形场论可根据时空维数分为两类：二维/三维以上共形场论。其中三维以上共形群生成元仅有平移、旋转、缩放和特殊共性变换四种并且个数有限，理论的约束条件较少，与之对应的是其初级场即表示个数无限。而二维中每个全纯函数均对应一个共形变换，具有无限个共形对称生成元，对称性比三维以上共形场论更高，在某些情况下理论仅含有限个初级场，其希尔伯特空间由有限个不可约表示张成，此时理论的中心荷和初级场共形维数均为有理数，故称为二维有理共形场论 (RCFTs)。

对二维共形对称性的现代研究始于 Belavin，Polyakov 和 Zamolodchikov 在 1984 年撰写的论文[6]，他们将刚刚由 Kac、Feigin 和 Fuchs 等人发展起来的 Virasoro 代数的表示理论与局部算符代数的想法相结合，提出了构造完全可解二维共形场论，亦即所谓极小模型的方法。随后在此工作的基础之上，人们发现极小模型可和多种二维统计系统在临界温度处相等同。而在共形对称性的基础上引入额外的对称性以构造共形场论的过程中，更多的可解模型诸如 WZW 模型、陪集模型以及超对称共形场论等逐渐被人们发现。

对于一般的量子场论而言，其配分函数可视为平移算符在 Hilbert 空间的迹，而二维有理共形场论的特殊之处在于其平移算符可分为完全独立的全纯和共轭全纯两部分。在特定表示空间对平移算符的全纯或共轭全纯部分求迹即得此表示下的所谓特征标。若对场施加双周期边界条件，等效于将场局限在环面上，此时配分函数具有模不变性。由于配分函数可写为一系列表示下的全纯特征标和共轭全纯特征标乘积之和，环面配分函数的模不变性意味着特征标在模群作用下具有协变性质，此性质使得特征标成为构造环面配分函数以及对二维有理共形场论进行分类的一个有力工具。虽然二维有理共形场论的分类问题至今仍未完全解决，但中心荷小于等于 1 的酉极小模型和 $\hat{\mathfrak{su}}(2)_k$、$\hat{\mathfrak{su}}(3)_k$ WZW 模型等等特殊情况下的分类已为人们所熟知，并且研究者们仍在不断取得更多的二维有理共形场论分类进展[7–9]。

本文的研究对象为二维有理共形场论的特征标，文中的记号与公式主要参考[10]。本文的结构如下：

第一章为对二维有理共形场论的简介，给出其定义和主要性质，引出下文中的极小模型、WZW 模型和陪集模型等三种 RCFTs。

第二章为极小模型，其唯一对称性为共形对称性，共形变换的生成元即能动张量满足 Virasoro 代数。根据 RCFTs 仅具有有限多个初级场的条件可以确定其中心荷与初级场共形维数应该满足的条件，满足这些条件的理论即称为极小模型，酉极小模型的中心荷恒为小于等于 1 的正有理数。由平移算符的全纯或反全纯部分的迹可以定义 Virasoro 特征标，根据极小模型的不可约表示的模结构可得特征标的显式表达式。极小模型最简单的例子是单个玻色子和费米子，其作用量满足共形对称性，分别属于中心荷 $c = 1$ 和 $c = 1/2$ 的 RCFTs。通过研究玻色子和费米子的 Fock 空间发现其特征标可由 Dedekind 函数和 Jacobi's theta 函数表示，这些函数在模群下都具有特殊的变换性质，这对之后构造具有模不变性的环面配分函数和确定特征标形式起到了重要作用。本章还以二维 Ising 模型为例说明了二维格点统计模型在临界温度处与极小模型的等价性。

第三章为 WZW 模型，其在共形对称性的基础之上具有额外的李群对称性，其作用量变分得到的守恒流称为仿射流，仿射流的洛朗展开系数满足仿射李代数。通过仿射流可以构造满足 Virasoro 代数的共形变换守恒流，这意味着在对称李群下协变的初级场也满足 Virasoro 初级场的条件。由于固定级数的仿射李代数表示个数有限，可知 WZW 初级场的个数也有限，因此 WZW 模型属于 RCFTs。为了区分共轭表示，WZW 特征标在 Virasoro 特征标的基础之上额外引入了 Cartan 子代数生成元的迹，计算表明其与仿射李代数可积权表示的特征标等价。本章讨论了一维紧化玻色子、顶点算符表示以及多费米子表示的特征标，其均对应不同的 WZW 模型。

第四章为陪集模型，陪集模型可通过两个 WZW 模型的商得到，其中一个 WZW 模型的对称李群为另一个的子群。陪集模型的生成元由两个 WZW 模型共形生成元之差定义，由于陪集关系存在，这两个 WZW 模型的生成元相互独立，因此陪集模型的生成元满足 Virasoro 代数，另外由 WZW 模型表示个数有限可知陪集模型也属于 RCFTs。本章通过陪集构造了极小模型和仲费米子模型，同时介绍了 $\mathscr{N} = 1, 2$ 的超对称共形场论的陪集构造。陪集的特征标由分支规则确定，一般无法直接显式写出，但对某些特殊情况陪集特征标具有较为简单的表达式。

本文第五章主要为对前四章特征标结果的应用，此章重新推导了文章[11]中 $\mathscr{T}_{222}$ 理论的配分函数，补充了一些文章中省略的推导过程。该文章给出了二维 $\mathscr{N} = (0, 2)$ 超对称共形场论 (SCFT) 的精确解，对研究非微扰超对称量子场论具有重要意义。二维 $\mathscr{N} = (0, 2)$ 超对称量子色动力学 (SQCD) 在重整化群流作用下流向低能标共形不动点，得到 $\mathscr{N} = (0, 2)$ SCFT。通过计算理论中场的迹异常可知此 SCFT 的 Hilbert 空间由全纯 WZW 模型表示空间和共轭全纯 $\mathscr{N} = 2$ 超

对称共形场论表示空间的直积构成，直积系数由级-秩对偶矩阵确定。其中全纯 WZW 模型的特征标可直接写出，而共轭全纯部分特征标可等价于中心荷为 1 的 $\mathcal{N}=2$ 超对称极小模型特征标，由此得到所需的 SCFT 配分函数。根据配分函数全纯特征标的仿射 E_6 对称性我们讨论了所谓的三重对称性，即对 SQCD 中的手性多重态和费米多重态的味对称群进行轮换可得到对偶理论，此三重对称性也可局部作用于一般的二维 $\mathcal{N}=2$ 的 SQCD 箭袋图的单个规范群节点并得到其对偶理论。最后本文通过计算三种不同二维 $\mathcal{N}=2$ 的 SQCD 的椭圆亏格验证了三重对称性。

本文的工作为介绍和梳理三种有理共形场论的特征标，并在第五章中利用已知结果重新推导了 $\mathcal{N}=(0,2)$ SCFT 的配分函数，分析配分函数对称性的来源。主要创新点在于利用三重对称性构造了 SQCD 箭图的五边形关系，并且通过计算椭圆亏格的 q 级数展开验证了一些具体箭图例子中三重对称性的正确性。本文的主要目的在于总结归纳共形场论历史上发展成果和了解近几年新的进展与突破，熟悉近现代数学物理工具和研究方法，并由此接触与之相关的超对称量子场论和弦论等领域的研究内容，为今后继续深入研究数学物理打下基础。

关键词：有理共形场论; 特征标; 仿射李代数; 超对称

Abstract

Conformal field theory (CFTs), simply referred to as the quantum field theory with conformal symmetry, is the only relatively mature non-perturbative quantum field theory at present. It has been paid attention to by researchers in recent decades, and has important applications in the various fields of physics such as string theory and condensed matter theory. CFTs can be divided into two categories according to the space-time dimension: CFTs in 2d or above 3d. There are only four kinds of generators of conformal group above 3d: translation, rotation, scaling and special conformal transformation. The numbers of 3d generators are finite, means there are few constrains, corresponding to an infinitely many primary fields. While in 2d, Each holomorphic function corresponds to a conformal transformation, hence there are infinitely many conformal symmetric generators. In some cases, the theory contains only finitely many primary fields, and its Hilbert space is spanned by finitely many irreducible representations. In this case, the central charge and conformal dimensions of primary fields are rational numbers, hence it is called two-dimensional rational conformal field theory (RCFTs).

Modern research on 2d conformal symmetry began with the 1984 paper[6] by Belavin, Polyakov, and Zamolodchikov. They combined the representation theory of Virasoro algebras, which had just been developed by Kac, Feigin, and Fuchs, with the idea of local operator algebras, to propose a method for constructing fully solvable 2d conformal field theories, the so-called minimal models. On the basis of this work, it is found that the minimal models can be equivalent to many 2d statistical systems at the critical temperature. In the process of constructing conformal field theory by introducing additional symmetries on the basis of conformal symmetry, more solvable models such as WZW model, coset model and supersymmetric conformal field theory are gradually discovered.

For a general quantum field theory, the partition function can be regarded as the trace of the translation operator in Hilbert space, while the RCFTs are special in that its translation operators can be divided into completely independent holomorphic and conjugate holomorphic parts. Tracing the holomorphic or conjugate holomorphic part of a translation operator in a particular representation space gives the so-called characters. If a double periodic boundary condition is applied to the field, then the partition function becomes modular invariant. This means the characters transform covariantly under the modular group which makes them a powerful tool for constructing torus partition functions and classifying 2d RCFTs. Although the classification of 2d RCFTs has not yet been completely solved, for some special cases such as the unitary minimal models with central charges less than or equal to 1, the $\hat{\mathfrak{su}}(2)_k$ and $\hat{\mathfrak{su}}(3)_k$ WZW models, the classifications are already well known. And researchers are still making more progress in classifying 2d RCFTs.

In this thesis we mainly focus on the characters of three types of RCFTs: the minimal models, the WZW models and the coset models, and we will apply the results to the calculation of the torus partition function of a SCFT. In the last chapter we constructed the pentagon relation of the quiver diagrams of 2d $\mathcal{N} = (0, 2)$ SQCD by triality, and we verified the triality by computing the elliptic genus of some theories.

Keywords: RCFTs;Characters;Affine Lie algebra;Supersymmetry

第 1 章　Introduction to 2d RCFTs

Roughly speaking, the rational conformal theories (RCFTs) are CFTs that satisfy the following conditions:

1. It is endowed with a chiral algebra $\mathscr{A} \otimes \bar{\mathscr{A}}$ that contains Virasoro algebra as a subalgebra.

2. It is unitary with a unique vacuum.

3. There are finitely many primary fields of $\mathscr{A}$ and $\bar{\mathscr{A}}$.

4. The Hilbert space $\mathscr{H}$ is spanned by finitely many irreducible representations of $\mathscr{A} \otimes \bar{\mathscr{A}}$

$$\mathscr{H} = \bigoplus_{i,j} \mathscr{N}_{i,j} \mathscr{V}_j \otimes \mathscr{V}_j \tag{1.1}$$

where the irreducible representation $\mathscr{V}_i$ arises from primary fields ϕ_i.

5. the operator algebra is closed.

$$\phi_i(z)\phi_j(w) \sim \frac{C_{ijk}}{(z-w)^{h_i+h_j-h_k}} \phi_k(w) + \cdots \tag{1.2}$$

These theories are called rational because the central charge and conformal dimensions of primary fields are both rational numbers. Since CFTs with space-time dimensions that are equal to or greater than three always have infinitely many primary fields[12], RCFTs can only occur in two dimensions.

This article will study three types of 2d RCFTs and focus on their characters: the minimal models, the WZW models and the coset models. Finally, we will apply the characters into the calculation of the torus partition function of a $\mathscr{N} = (0,2)$ SCFT. Most of the content of this article is based on the classical CFT textbook[10]; we will use the notations and formulas in it without citations below.

第 2 章　Minimal models

2.1　Verma modules

The commutation relation of 2d conformal transformation generators is the well-known Virasoro algebra

$$[L_m, L_n] = mL_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0} \tag{2.1}$$

The representations of Virasoro algebra are called Verma modules, which consist of the Hilbert space

$$\mathscr{H} = \bigoplus_{h,\bar{h}} V(c,h) \otimes \bar{V}(c,\bar{h}) \tag{2.2}$$

With fixed c, each Verma module is characterized by the conformal dimension h of the primary. The states in a Verma module can be obtained by acting raising operators on the highest-weight state $|h\rangle$

$$|\psi\rangle = L_{-n_1}L_{-n_2}\cdots L_{-n_i}|h\rangle \tag{2.3}$$

with conformal dimension $h' = h + n_1 + n_2 + \cdots + n_i = h + n$, where n is called the level of the state. For generic c, the number of Verma modules is usually infinity, while there indeed exist theories that contain only finite conformal families, with the central charge and conformal dimensions of primaries given by[6]

$$\begin{aligned} c &= 1 - 6\frac{(p-p')^2}{pp'} \qquad && p, p', r, s \in \mathbb{Z}_+, \quad \gcd(p,p') = 1 \\ h_{r,s} &= \frac{(pr-p's)^2-(p-p')^2}{4pp'} && 1 \le r \le p'-1, \quad 1 \le s \le p-1 \end{aligned} \tag{2.4}$$

We can see the number of primaries is $(p-1)(p'-1)/2$, which is finite. These theories are called the minimal models, denoted by $\mathscr{M}(p,p')$. Especially, $\mathscr{M}(p+1,p)$ are unitary, its lowest dimension operator is the identity with conformal dimension 0.

The Verma module of minimal models is actually reducible[13], which means there exists at least one singular vector $|\chi\rangle$ whose norm $\langle\chi|\chi\rangle$ equals to zero. This singular vector and its descendants form a submodule V_χ which is orthogonal to the whole Verma module $V(c,h)$, and the irreducible module $M(c,h)$ can be obtained by quotienting out all the singular submodules of $V(c,h)$

The submodules of $V_{rs} \equiv V(c, h_{r,s})$ can be determined by the following property of $h_{r,s}$

$$\begin{aligned} h_{r,s} + rs &= h_{p'+r,p-s} \\ h_{r,s} + (p'-r)(p-s) &= h_{r,2p-s} \end{aligned} \tag{2.5}$$

This means we need to quotient out two submodules $V_{p'+r,p-s}, V_{r,2p-s}$ of $V_{r,s}$ to get the the irreducible module $M_{r,s}$

$$M_{r,s} = V_{r,s}/(V_{p'+r,p-s} \oplus V_{r,2p-s}) \tag{2.6}$$

But $V_{p'+r,p-s}, V_{r,2p-s}$ share two submodules $V_{2p'+r,s}, V_{r,2p+s}$, so the direct sum is the union quotienting out these two submodules

$$V_{p'+r,p-s} \oplus V_{r,2p-s} = V_{p'+r,p-s} \cup V_{r,2p-s}/(V_{2p'+r,s} \oplus V_{r,2p+s}) \tag{2.7}$$

Continue this process and we get the following module structure of $V_{r,s}$

$$\begin{tikzcd}
 & (p'+r, p-s) \arrow[r] \arrow[ddr] & (2p'+r, p-s) \cdots \arrow[r] \arrow[ddr] & (kp'+r, (-1)^k s + [1-(-1)^k]p/2) \cdots \\
(r,s) \arrow[ur] \arrow[dr] & & & \\
 & (r, 2p-s) \arrow[r] \arrow[uur] & (r, 2p+s) \cdots \arrow[r] \arrow[uur] & (r, kp + (-1)^k s + [1-(-1)^k]p/2) \cdots
\end{tikzcd} \tag{2.8}$$

2.2 Virasoro characters

The Virasoro characters of the Verma module $V(c, h)$ is defined as

$$\chi_{(c,h)}(\tau) = \mathrm{Tr}_V\, q^{L_0 - c/24} \qquad q \equiv e^{2\pi i \tau} \tag{2.9}$$

According to 2.3, the number of independent level-n states in the Verma module is just $p(n)$, the number of partitions of n, and its generating function is

$$\frac{1}{\varphi(q)} = \prod_{n=1}^{\infty} \frac{1}{1-q^n} = \sum_{n=0}^{\infty} p(n) q^n \tag{2.10}$$

Hence if there are no singular vectors, the Virasoro character can be written as

$$\chi_{(c,h)}(\tau) = \frac{q^{h+(1-c)/24}}{\eta(\tau)} \tag{2.11}$$

where $\eta(\tau)$ is the Dedekind function

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1-q^n) \tag{2.12}$$

However, the Verma module actually contains singular vectors, so we should subtract their contribution from them. From the module structure of $V_{r,s}$ in 2.8, we can get the character of the irreducible module $M_{r,s}$

$$\chi_{(r,s)}(q) = \frac{q^{-c/24}}{\varphi(q)} \left[q^{h_{r,s}} + \sum_{k=1}^{\infty} (-1)^k \{ q^{h_{r+kp',(-1)^k s+[1-(-1)^k]p/2}} + q^{h_{r,kp+(-1)^k s+[1-(-1)^k]p/2}} \} \right] \tag{2.13}$$

The irreducible character can be expressed more compactly as

$$\chi_{(r,s)}(q) = K^{(p,p')}_{r,s}(q) - K^{(p,p')}_{r,-s}(q) \tag{2.14}$$

where we introduce the function

$$K^{(p,p')}_{r,s}(q) = \frac{q^{-c/24}}{\varphi(q)} \sum_{n\in\mathbb{Z}} q^{(2pp'n+pr-p's)^2/4pp'} \tag{2.15}$$

2.3 2d Ising model

For 2d lattice statistic theories with finite size L, the correlation function has the following form at the critical temperature T_c

$$\left\langle s_i s_{i+n} \right\rangle \sim \frac{1}{|n|^a} \tag{2.16}$$

where the critical index a is a fractional constant. This is similar to the OPE of primaries in 2d CFT

$$\phi(z,\bar{z})\phi(w,\bar{w}) \sim \frac{1}{|z-w|^a} \tag{2.17}$$

where the conformal dimensions of ϕ satisfy $2h + 2\bar{h} = a$. So we can identify the statistical models at the critical point with the minimal models. Consider the 2d Ising model, its correlation function at the critical point behaviors as

$$\left\langle \sigma_i \sigma_{i+n} \right\rangle = \frac{1}{|n|^{1/4}} \qquad \left\langle \varepsilon_i \varepsilon_{n+i} \right\rangle = \frac{1}{|n|^2} \tag{2.18}$$

Suppose the scaling fields have no spin, which means $h = \bar{h}$. Then, the conformal dimensions are

$$(h,\bar{h})_\sigma = (\frac{1}{16}, \frac{1}{16}) \qquad (h,\bar{h})_\varepsilon = (\frac{1}{2}, \frac{1}{2}) \tag{2.19}$$

Plus the vacuum $h = \bar{h} = 0$, this is indeed the $\mathscr{M}(4,3)$ minimal model with central charge $c = 1/2$, we list its characters expansion up to $O(q^6)$ together with other three statistic minimal models below

$(p,p'),c$	$h_{r,s}$	$q^{-h_{r,s}+c/24}\chi_{r,s}(q)$
$(5,2),-22/5$	$h_{1,1}=0$	$1+q^2+q^3+q^4+q^5+2q^6$
Yang-Lee	$h_{1,2}=-2/5$	$1+q+q^2+q^3+2q^4+2q^5+3q^6$
$(4,3),1/2$	$h_{1,1}=0$	$1+q^2+q^3+2q^4+2q^5+3q^6$
Ising	$h_{2,1}=1/16$	$1+q+q^2+q^3+2q^4+2q^5+3q^6$
	$h_{1,2}=1/2$	$1+q+q^2+2q^3+2q^4+3q^5+4q^6$
$(5,4),7/10$	$h_{1,1}=0$	$1+q^2+q^3+2q^4+2q^5+4q^6$
Tricrit Ising	$h_{2,1}=7/16$	$1+q+q^2+2q^3+3q^4+4q^5+6q^6$
	$h_{1,2}=1/10$	$1+q+q^2+2q^3+3q^4+4q^5+6q^6$
	$h_{1,3}=3/5$	$1+q+2q^2+2q^3+4q^4+5q^5+7q^6$
	$h_{2,2}=3/80$	$1+q+2q^2+3q^3+4q^4+6q^5+8q^6$
	$h_{3,1}=3/2$	$1+q+q^2+2q^3+3q^4+4q^5+6q^6$
$(5,4),4/5$	$h_{1,1}=0$	$1+q^2+q^3+2q^4+2q^5+4q^6$
3-state Potts	$h_{2,1}=2/5$	$1+q+q^2+2q^3+3q^4+4q^5+6q^6$
	$h_{3,1}=7/5$	$1+q+3q^2+3q^3+4q^4+5q^5+8q^6$
	$h_{1,3}=2/3$	$1+q+2q^2+2q^3+4q^4+5q^5+8q^6$
	$h_{4,1}=3$	$1+q+2q^2+3q^3+4q^4+5q^5+8q^6$
	$h_{2,3}=1/15$	$1+q+2q^2+3q^3+5q^4+7q^5+10q^6$

表 2-1　Characters of statistic minimal models

2.4　Torus partition functions

The torus partition function has double periods (ω_1,ω_2), and it can be expressed as the trace of the translation operator

$$Z(\omega_1,\omega_2)=\mathrm{Tr}\exp\big(-H\ \mathrm{Im}\,\omega_2+iP\ \mathrm{Re}\,\omega_2\big) \tag{2.20}$$

Regard the torus as a periodic cylinder with circumference $L=\omega_1\in\mathbb{R}$, then the Hamiltonian and momentum operators are

$$H=\frac{2\pi}{L}(L_0+\bar{L}_0-c/12) \qquad P=\frac{2\pi i}{L}(L_0-\bar{L}_0) \tag{2.21}$$

Introducing the period ratio $\tau=\omega_2/\omega_1$, now the torus partition function can be expressed as

$$Z(\tau)=\mathrm{Tr}\left(q^{L_0-c/24}\bar{q}^{\bar{L}_0-c/24}\right) \tag{2.22}$$

where $q=\exp(2\pi i\tau)$ and $\bar{q}=\exp(-2\pi i\bar{\tau})$. We start the discussion of the partition functions of minimal models with bosons and fermions.

2.4.1 Boson and Fermion

Free Boson ($c = 1$)

The holomorphic Fock space of a free boson consists of states

$$\left|n_1, n_2, \cdots\right\rangle = a_{-1}^{n_1} a_{-2}^{n_2} \cdots |0\rangle \qquad n_i \in \mathbb{Z}_{\geq 0} \tag{2.23}$$

and the operator form of L_0 is

$$L_0 = \frac{1}{2}a_0^2 + \sum_{k=1}^{\infty} a_{-k}a_k \tag{2.24}$$

From the commutation relation between L_0 and $a_{-k}, k > 0$, we can get

$$[L_0, a_{-k}] = ka_{-k} \quad \Rightarrow \quad q^{L_0}a_{-k} = a_{-k}q^{L_0+k} \tag{2.25}$$

The central charge of a free boson is $c = 1$

$$\begin{aligned} \mathrm{Tr}\left(q^{L_0-c/24}\right) &= q^{-1/24} \sum_{n_i \geq 0} \langle n_1, n_2, \cdots | \, q^{L_0} \left|n_1, n_2, \cdots\right\rangle \\ &= q^{-1/24} \sum_{n_i \geq 0} \prod_{i=1}^{\infty} q^{in_i} \\ &= \frac{1}{\eta(\tau)} \end{aligned} \tag{2.26}$$

So the partition function should have the form $Z(\tau) \propto 1/|\eta(\tau)|^2$, the coefficient can be determined by the integral of a_0

$$\int \mathrm{d}a_0 (q\bar{q})^{a_0^2/2} = \int \mathrm{d}a_0 e^{-2\pi\tau_2 a_0^2} = \frac{1}{\sqrt{2\tau_2}} \tag{2.27}$$

Combining with this coefficient, the torus partition function is indeed modular invariant.

$$Z(\tau) = \frac{1}{\sqrt{2\tau_2}|\eta(\tau)|^2} \tag{2.28}$$

Compactified Boson ($c = 1$)

Compactified boson can be obtained by restricting the the value of the boson field on a circle with radius R

$$\varphi(z, \bar{z}) \sim \varphi(z, \bar{z}) + 2\pi Rm \qquad m \in \mathbb{Z} \tag{2.29}$$

The mode expansion of φ is

$$\varphi(z, \bar{z}) = \varphi_0 - i\frac{2\pi}{l}[(a_0 + \bar{a}_0)t - i(a_0 - \bar{a}_0)x] + i \sum_{n \neq 0} \frac{1}{n}\left(a_n e^{2\pi n z/l} + \bar{a}_n^{2\pi n \bar{z}/l}\right) \tag{2.30}$$

Since the boson filed is defined on the torus, we can impose periodic condition $\varphi(t, x + l) = \varphi(t, x) + 2\pi R w \quad w \in \mathbb{Z}$. Moreover, the momentum $p = (a_0 + \bar{a}_0)/2$ is quantized as $p = n/R$ by the compactification 2.32 , where $n \in \mathbb{Z}$ is called the Kaluza-Klein momentum. These gives the following restriction of $a_0, \bar{a}_0$

$$\begin{aligned} a_0 - \bar{a}_0 &= Rw \\ a_0 + \bar{a}_0 &= 2n/R \end{aligned} \tag{2.31}$$

Now the partition function becomes

$$\begin{aligned} Z_R(\tau) &= \frac{1}{|\eta(\tau)|^2} \sum_{a_0, \bar{a}_0} q^{a_0^2/2} \bar{q}^{\bar{a}_0^2/2} \\ &= \frac{1}{|\eta(\tau)|^2} \sum_{n, w \in \mathbb{Z}} q^{(n/R + Rw/2)^2/2} \bar{q}^{(n/R - Rw/2)^2/2} \end{aligned} \tag{2.32}$$

It has the T-duality $R \leftrightarrow 2/R$. Using the Poisson resummation formula

$$\sum_{n \in \mathbb{Z}} \exp\left(-\pi a n^2 + b n\right) = \frac{1}{\sqrt{a}} \sum_{n \in \mathbb{Z}} \exp\left[-\frac{\pi}{a}(n + b/2\pi i)^2\right] \tag{2.33}$$

and taking $a = R^2/2\tau_2, b = \pi w R^2 \tau_1/\tau_2$, we can manifest the modular invariance of the partition function

$$Z_R(\tau) = Z(\tau) R \sum_{n, w} \exp\left(-\frac{\pi R^2 |w\tau - n|^2}{2\tau_2}\right) \tag{2.34}$$

Free Fermion ($c = 1/2$)

We denote the fermion space-time torus boundary condition by (u, v), where u, v take the value 0 for R boundary and 1/2 for NS boundary. The holomorphic Fock space for a free fermion consists of states

$$\left|n_{1-\nu}, n_{2-\nu}, \cdots\right\rangle = b_{-1+\nu}^{n_{1-\nu}} b_{-2+\nu}^{n_{2-\nu}} \cdots |0\rangle \qquad n_{i-\nu} \in \mathbb{Z}_{\geq 0} \tag{2.35}$$

and the operator form of L_0 with respect to space boundary conditions are

$$\begin{aligned} L_0^{\mathrm{NS}} &= \sum_{k>0} k b_{-k} b_k \qquad (\mathrm{NS} : k \in \mathbb{Z} + \frac{1}{2}) \\ L_0^{\mathrm{R}} &= \sum_{k>0} k b_{-k} b_k + \frac{1}{16} \qquad (\mathrm{R} : k \in \mathbb{Z}) \end{aligned} \tag{2.36}$$

From the commutation relation between L_0 and $b_{-k}, k > 0$, we can get

$$[L_0, b_{-k}] = -k b_{-k} \quad \Rightarrow \quad q^{L_0} b_{-k} = b_{-k} q^{L_0 - k} \tag{2.37}$$

Consider $\langle \psi(z) X \rangle$, the correlation function of fermions, it is nonzero only when X is a product of odd number fermion fields. When z crosses a time period, $\psi(z)$ exchanges with X, there should generate a minus sign. Hence we should insert a fermion number operator

$$(-1)^F \qquad \text{where} \quad F = \sum_{k\geq 0} b_{-k} b_k \tag{2.38}$$

in the trace for the time R boundary. Calculating the trace with respect to the four boundary conditions, they can be expressed in terms of the Jacobi's theta functions and the Dedekind function

$$\begin{aligned}
d_{0,0} &= \mathrm{Tr}\left[(-1)^F q^{L_0^{\mathrm{R}}-1/48}\right] = q^{1/24}\prod_{n=0}^{\infty}(1-q^n) = 0 \\
d_{0,1/2} &= \mathrm{Tr}\left[q^{L_0^{\mathrm{R}}-1/48}\right] = q^{1/24}\prod_{n=0}^{\infty}(1+q^n) = \sqrt{\frac{2\theta_2(\tau)}{\eta(\tau)}} \\
d_{1/2,0} &= \mathrm{Tr}\left[(-1)^F q^{L_0^{\mathrm{NS}}-1/48}\right] = q^{-1/48}\prod_{r=1/2}^{\infty}(1-q^r) = \sqrt{\frac{\theta_4(\tau)}{\eta(\tau)}} \\
d_{1/2,1/2} &= \mathrm{Tr}\left[q^{L_0^{\mathrm{NS}}-1/48}\right] = q^{-1/48}\prod_{r=1/2}^{\infty}(1+q^r) = \sqrt{\frac{\theta_3(\tau)}{\eta(\tau)}}
\end{aligned} \tag{2.39}$$

But the trace is not the character yet, since the central charge $c = 1/2$, which corresponds to the minimal model $\mathscr{M}(4,3)$, the allowed conformal dimensions of primaries are thus $h_{1,1} = 0, h_{2,1} = 1/2, h_{1,2} = 1/16$. The three Verma modules have descendants with conformal dimensions $\mathbb{Z} \geq \mathbb{0}, \mathbb{Z}_{\geq 0} + 1/2, \mathbb{Z}_{\geq 0} + 1/16$ respectively. The characters are thus

$$\begin{aligned}
\chi_{1,1} &= \frac{1}{2}(d_{1/2,1/2} + d_{1/2,0}) = \frac{1}{2}\left(\sqrt{\frac{\theta_3(\tau)}{\eta(\tau)}} + \sqrt{\frac{\theta_4(\tau)}{\eta(\tau)}}\right) \\
\chi_{2,1} &= \frac{1}{2}(d_{1/2,1/2} - d_{1/2,0}) = \frac{1}{2}\left(\sqrt{\frac{\theta_3(\tau)}{\eta(\tau)}} - \sqrt{\frac{\theta_4(\tau)}{\eta(\tau)}}\right) \\
\chi_{1,2} &= \frac{1}{2}d_{0,1/2} = \sqrt{\frac{\theta_2(\tau)}{2\eta(\tau)}}
\end{aligned} \tag{2.40}$$

So the torus partition function of a free fermion is

$$\begin{aligned}
Z &= |\chi_{1,1}|^2 + |\chi_{2,1}|^2 + |\chi_{1,2}|^2 \\
&= \frac{1}{2}\left(\left|\frac{\theta_2}{\eta}\right| + \left|\frac{\theta_3}{\eta}\right| + \left|\frac{\theta_4}{\eta}\right|\right)
\end{aligned} \tag{2.41}$$

This is the same as the partition function of the 2d Ising model and it is indeed modular invariant.

2.4.2 ADE classification

Now we start constructing general torus partition functions of the minimal models, where the modular property of Virasoro characters plays an important role. Here we just give the results. The irreducible character transforms under the two generators $\mathscr{T}: \tau \to \tau+1$ and $\mathscr{S}: \tau \to -1/\tau$ of the modular group $\mathrm{SL}(2,\mathbb{Z})$ as

$$\begin{aligned} \chi_{r,s}(\tau+1) &= \sum_{(\rho,\sigma)\in E_{p,p'}} \mathscr{T}_{rs;\rho\sigma}\chi_{\rho,\sigma}(\tau) \\ \chi_{r,s}(-1/\tau) &= \sum_{(\rho,\sigma)\in E_{p,p'}} \mathscr{S}_{rs;\rho\sigma}\chi_{\rho,\sigma}(\tau) \end{aligned} \tag{2.42}$$

The explicit forms of the modular matrices above are

$$\begin{aligned} \mathscr{T}_{rs;\rho\sigma} &= \delta_{r,\rho}\delta_{s,\sigma}e^{2\pi i(h_{r,s}-c/24)} \\ \mathscr{S}_{rs;\rho\sigma} &= 2\sqrt{\frac{2}{pp'}}(-1)^{1+s\rho+r\sigma}\sin\left(\pi\frac{p}{p'}r\rho\right)\sin\left(\pi\frac{p'}{p}s\sigma\right) \end{aligned} \tag{2.43}$$

where $\mathscr{T}$ and $\mathscr{S}$ are both unitary, and $E_{p,p'}$ is the the set of integer pairs (ρ,σ) which satisfy

$$\begin{aligned} &1 \le \rho \le p'-1 \\ &1 \le \sigma \le p-1 \\ &p'\rho \le p\sigma \end{aligned} \tag{2.44}$$

Since the Hilbert space of a minimal model with central charge c is

$$\mathscr{H} = \bigoplus_{h,\bar{h}} M(c,h)\otimes M(c,\bar{h}) \tag{2.45}$$

the partition function is expected to has the following form

$$Z(\tau) = \sum_{(r,s),(t,u)\in E_{p,p'}} \mathscr{M}_{rs;tu}\chi_{r,s}(\tau)\bar{\chi}_{t,u}(\bar{\tau}) \tag{2.46}$$

where $\mathscr{M}_{rs;tu}$ is the multiplicities of the irreducible module tensor product $M_{rs}\otimes M_{tu}$. Specially, the identity operator is non-degenerate, which means $\mathscr{M}_{11;11} = 1$. Using the modular property 2.42 of the characters, the modular invariance of the partition function is equivalent to the following property of the multiplicities matrix

$$\mathscr{M}\mathscr{T} = \mathscr{T}\mathscr{M} \qquad \mathscr{M}\mathscr{S} = \mathscr{S}\mathscr{M} \tag{2.47}$$

The first condition $\mathscr{M}\mathscr{T} = \mathscr{T}\mathscr{M}$ simply means the left and right conformal dimensions are equal modulo 1, that is $h = \bar{h} \mod 1$. The simplest case is $h = \bar{h}$ and the partition function becomes diagonal

$$Z_{A_{p'-1},A_{p-1}} = \sum_{(r,s)\in E_{p,p'}} |\chi_{r,s}|^2 \tag{2.48}$$

As we label the partition function by the Lie algebra pair $(A_{p'-1}, A_{p-1})$, it has been proven in[14] that all torus partition functions of unitary minimal models $\mathscr{M}(p+1, p)$ with $c < 1$ can be labeled by simply laced Lie algebra pairs of A, D, E type, this is called the ADE classification. We just list the results below.

p	$Z_{\mathscr{M}(p+1,p)}$	Type
≥ 3	$\frac{1}{2}\sum_{r=1}^{p}\sum_{s=1}^{p-1}\lvert\chi_{r,s}\rvert^2$	(A_p, A_{p-1})
$4l+1$	$\frac{1}{2}\sum_{(r,s)\in E_{p,p'}}^{r\text{ odd}}\lvert\chi_{r,s}+\chi_{p'-r,s}\rvert^2$	(D_{2l+2}, A_{4l})
$4l+2$	$\frac{1}{2}\sum_{(r,s)\in E_{p,p'}}^{s\text{ odd}}\lvert\chi_{r,s}+\chi_{r,p-s}\rvert^2$	(A_{4l+2}, D_{2l+2})
$4l-1$	$\frac{1}{2}\sum_{s=1}^{4l-2}\left[\sum_{r\text{ odd}}\lvert\chi_{r,s}\rvert^2+\lvert\chi_{2l,s}\rvert^2+\sum_{r\text{ even}}\chi_{r,s}\bar{\chi}_{p'-r,s}\right]$	(D_{2l+1}, A_{p-1})
$4l$	$\frac{1}{2}\sum_{r=1}^{4l}\left[\sum_{s\text{ odd}}\lvert\chi_{r,s}\rvert^2+\lvert\chi_{r,2l}\rvert^2+\sum_{s\text{ even}}\chi_{r,s}\bar{\chi}_{r,p-s}\right]$	(A_p, D_{2l+1})
11	$\frac{1}{2}\sum_{s=1}^{10}\left[\lvert\chi_{1,s}+\chi_{7,s}\rvert^2+\lvert\chi_{4,s}+\chi_{8,s}\rvert^2+\lvert\chi_{5,s}+\chi_{11,s}\rvert^2\right]$	(E_6, A_{10})
12	$\frac{1}{2}\sum_{r=1}^{12}\left[\lvert\chi_{r,1}+\chi_{r,7}\rvert^2+\lvert\chi_{r,4}+\chi_{r,8}\rvert^2+\lvert\chi_{r,5}+\chi_{r,11}\rvert^2\right]$	(A_{12}, E_6)
17	$\frac{1}{2}\sum_{s=1}^{16}[\lvert\chi_{1,s}+\chi_{17,s}\rvert^2+\lvert\chi_{5,s}+\chi_{13,s}\rvert^2+\lvert\chi_{7,s}+\chi_{11,s}\rvert^2$ $+\lvert\chi_{9,s}\rvert^2+\chi_{9,s}(\bar{\chi}_{3,s}+\bar{\chi}_{15,s})+(\chi_{3,s}+\chi_{15,s})\bar{\chi}_{9,s}]$	(E_7, A_{16})
18	$\frac{1}{2}\sum_{r=1}^{18}[\lvert\chi_{r,1}+\chi_{r,17}\rvert^2+\lvert\chi_{r,5}+\chi_{r,13}\rvert^2+\lvert\chi_{r,7}+\chi_{r,11}\rvert^2$ $+\lvert\chi_{r,9}\rvert^2+\chi_{r,9}(\bar{\chi}_{r,3}+\bar{\chi}_{r,15})+(\chi_{r,3}+\chi_{r,15})\bar{\chi}_{r,9}]$	(A_{18}, E_7)
29	$\frac{1}{2}\sum_{s=1}^{28}[\lvert\chi_{1,s}+\chi_{11,s}+\chi_{19,s}+\chi_{29,s}\rvert^2$ $+\lvert\chi_{7,s}+\chi_{13,s}+\chi_{17,s}+\chi_{23,s}\rvert^2]$	(E_8, A_{28})
30	$\frac{1}{2}\sum_{r=1}^{30}[\lvert\chi_{r,1}+\chi_{r,11}+\chi_{r,19}+\chi_{r,29}\rvert^2$ $+\lvert\chi_{r,7}+\chi_{r,13}+\chi_{r,17}+\chi_{r,23}\rvert^2]$	(A_{30}, E_8)

表 2-2　Torus partition function of unitary minimal models

第 3 章　WZW models

This chapter introduces the Wess-Zumino-Witten (WZW) models[15], which are RCFTs with affine Lie algebraic symmetry. In searching for conformal theories with additional Lie-algebraic symmetry, it is natural to consider the following nonlinear sigma model

$$S_0 = \frac{1}{4a^2} \int \mathrm{d}^2 x \, \mathrm{Tr}'(\partial^\mu g^{-1} \partial_\mu g) \tag{3.1}$$

where $g(x)$ is a matrix bosonic field and transforms under some unitary representation of the Lie group G. Clearly the action is conformally invariant classically and has a global $G \times G$ invariance.

However, the coupling constant a^2 acquires a scale dependence at the quantum level, which breaks the scale invariance. Actually, the conserved current of S_0 is

$$J_\mu = g^{-1} \partial_\mu g \tag{3.2}$$

If we write the current in terms of complex variables, the conservation law becomes

$$\partial_z \tilde{J}_{\bar{z}} + \partial_{\bar{z}} \tilde{J}_z = 0 \quad \text{where} \quad \tilde{J}_z = g^{-1} \partial_z g, \ \tilde{J}_{\bar{z}} = g^{-1} \partial_{\bar{z}} g \tag{3.3}$$

The two currents $\tilde{J}_z$ and $\tilde{J}_{\bar{z}}$ are inseparable which violates the property of 2d CFT. In order to enhance the symmetry and get separable conserved currents, so-called Wess-Zumino term must be added :

$$\Gamma = \frac{-i}{24\pi} \int_B \mathrm{d}^3 y \, \epsilon_{\alpha\beta\gamma} \, \mathrm{Tr}'(\tilde{g}^{-1} \partial^\alpha \tilde{g} \tilde{g}^{-1} \partial^\beta \tilde{g} \tilde{g}^{-1} \partial^\gamma \tilde{g}) \tag{3.4}$$

where B is a 3d manifold whose boundary is the compactification of the original 2d space, and $\tilde{g}$ is the extension of g to B. However, there exist two choices of B since a 2d space divides the compactified 3d space into two parts. Taking orientation into consideration, the action difference $\Delta\Gamma$ whose integration range extends to the whole compact 3d space is actually an integer multiple of $2\pi i$. This means in order to eliminate the ambiguity between the two choices of B, the coupling constant in front of Γ must be an integer. Now, the complete action can be written as

$$S = S_0 + k\Gamma \qquad k \in \mathbb{Z} \tag{3.5}$$

The equation of motion with respect to this action is

$$\left(1+\frac{a^2k}{4\pi}\right)\partial_z(g^{-1}\partial_{\bar{z}}g)+\left(1-\frac{a^2k}{4\pi}\right)\partial_{\bar{z}}(g^{-1}\partial_z g)=0 \tag{3.6}$$

For $a^2 = 4\pi/k$, there are indeed two separable conserved currents

$$J_z = \partial_z g g^{-1} \qquad J_{\bar{z}} = g^{-1}\partial_{\bar{z}} g \tag{3.7}$$

Additionally, S is invariant under $g(z,\bar{z}) \to \Omega(z)g(z,\bar{z})\bar{\Omega}^{-1}(\bar{z})$ when $a^2 = 4\pi/k$, which means the global $G\times G$ symmetry of the sigma model is extended to a local $G(z)\times G(\bar{z})$ symmetry. Now this model is called the $\hat{\mathfrak{g}}_k$ Wess-Zumino-Witten (WZW) model; we can write its action as

$$S^{\mathrm{WZW}} = \frac{k}{16\pi}\int \mathrm{d}^2x\,\mathrm{Tr}'(\partial^\mu g^{-1}\partial_\mu g) + k\Gamma \tag{3.8}$$

3.1 Sugawara construction

Since the currents of the WZW model are separately conserved, we can define

$$J(z) \equiv -\frac{k}{\sqrt{2}}J_z(z), \quad \bar{J}(\bar{z}) \equiv \frac{k}{\sqrt{2}}J_{\bar{z}}(\bar{z}) \tag{3.9}$$

Using the conformal Ward-Takahashi identity to compute JJ OPE, we get the so-called affine current algebra

$$J^a(z)J^b(w) \sim \frac{k\delta_{ab}/2}{(z-w)^2} + \sum_c if_{abc}\frac{J^c(w)}{(z-w)} \tag{3.10}$$

Taking the Laurent expansion $J^a(z) = \sum_{n\in\mathbb{Z}} z^{-n-1}J^a_n$, the commutation relations of modes J^a_n are indeed the same as the $\hat{\mathfrak{g}}_k$ affine Lie algebra

$$[J^a_n, J^b_m] = \sum_c if_{abc}J^c_{n+m} + \frac{k}{2}n\delta_{ab}\delta_{n+m,0} \tag{3.11}$$

while the anti-holomorphic current $\bar{J}$ provide another independent copy of 3.11

The WZW action gives the classical form of the energy-momentum tensor: $T(z) = (1/k)\sum_a J^aJ^a$, while its quantum version needs to impose normal ordering and renormalize the coefficient to make $T(z)J(w) \supset \partial J(w)/(z-w)$, the result is

$$T(z) = \frac{1}{k+h^\vee_{\mathfrak{g}}}\sum_a (J^aJ^a)(z) \tag{3.12}$$

Calculating TT OPE, we get the Sugawara central charge

$$c(\hat{\mathfrak{g}}_k) = \frac{k\dim\mathfrak{g}}{k+h^\vee_{\mathfrak{g}}} \tag{3.13}$$

which is fractional and always greater than or equal to 1. Take mode expansion $T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2}L_n$, 3.12 becomes

$$L_n = \frac{1}{k + h^\vee_\mathfrak{g}} \sum_a \sum_m : J^a_m J^a_{n-m} : \tag{3.14}$$

Hence, the complete affine Lie and Virasoro algebra is

$$\begin{aligned}
[L_n, L_m] &= (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0} \\
[L_n, J^a_m] &= -mJ^a_{n+m} \\
[J^a_n, J^b_m] &= \sum_c i f_{abc} J^c_{n+m} + \frac{k}{2} n\delta_{ab}\delta_{n+m,0}
\end{aligned} \tag{3.15}$$

3.2 WZW primary fields and characters

While the Virasoro primaries transform covariantly under scaling, the WZW primaries are defined as fields that transform covariantly under the $G(z) \times G(\bar{z})$ action, which implies the OPE between the currents and primaries should be

$$\begin{aligned}
J^a(z)\phi_{\lambda,\mu}(w,\bar{w}) &\sim \frac{-t^a_\lambda \phi_{\lambda,\mu}(w,\bar{w})}{z-w} \\
\bar{J}^a(\bar{z})\phi_{\lambda,\mu}(w,\bar{w}) &\sim \frac{\phi_{\lambda,\mu}(w,\bar{w})t^a_\mu}{\bar{z}-\bar{w}}
\end{aligned} \tag{3.16}$$

where t^a_λ is a matrix in the λ representation. Focus on the holomorphic sector, taking mode expansion of the current $J^a(z) = \sum_n (z-w)^{-n-1} J^a_n(w)$ and define the $\phi_\lambda(0)\left|0\right\rangle = \left|\phi_\lambda\right\rangle$, the condition of WZW primary fields above becomes

$$\begin{aligned}
J^a_0 \left|\phi_\lambda\right\rangle &= -t^a_\lambda \left|\phi_\lambda\right\rangle \\
J^a_n \left|\phi_\lambda\right\rangle &= 0 \qquad \text{for } n > 0
\end{aligned} \tag{3.17}$$

It can be seen from 3.14 that the WZW primaries are also Virasoro primaries with conformal dimension h_λ, but the inverse is not true.

$$L_0 \left|\phi_\lambda\right\rangle = h_\lambda \left|\phi_\lambda\right\rangle \qquad h_\lambda = \frac{\sum_a t^a_\lambda t^a_\lambda}{k + h^\vee_\mathfrak{g}} = \frac{(\lambda, \lambda + 2\rho)}{2(k + h^\vee_\mathfrak{g})} \tag{3.18}$$

where $\mathcal{Q} \equiv \sum_a t^a_\lambda t^a_\lambda$ is the quadratic Casimir operator, it has the same eigenvalue $(\lambda, \lambda + 2\rho)/2$ for all states in representation $\hat{\lambda}$. Thus for certain $\hat{\mathfrak{g}}_k$, there exists a one-to-one correspondence between WZW primary fields $\left|\phi_\lambda\right\rangle$ and representations $\hat{\lambda} \in P^k_+$. Since the number of representations of fixed level is finite, there are indeed finite WZW primary fields, which means the WZW models indeed belong to RCFTs. Descendants can be obtained by acting a series of J^a_{-n} on the primary

$$J^a_{-n_1} J^b_{-n_2} \cdots \left|\phi_\lambda\right\rangle \qquad n_i \in \mathbb{Z}_{\geq 0} \tag{3.19}$$

We define the characters of $\hat{\mathfrak{g}}_k$ WZW models in analogy with the Virasoro characters 2.9 as

$$\begin{aligned}\chi_{\hat{\lambda}}(\tau) &\equiv \mathrm{Tr}_{\hat{\lambda}}\, e^{2\pi i\tau(L_0-c/24)} \\ &= e^{2\pi i\tau(h_\lambda-c/24)}\sum_n d(n)e^{2\pi i n\tau}\end{aligned} \tag{3.20}$$

where $d(n)$ is the number of level n states. Using the Freudenthal-de Vries strange formula

$$12|\rho|^2 = h^\vee_{\mathfrak{g}}\dim\mathfrak{g} \tag{3.21}$$

The conformal dimension $h_{\hat{\lambda}}$ relates to the modular anomaly $m_{\hat{\lambda}}$ as

$$h_{\hat{\lambda}} - c/24 = m_{\hat{\lambda}} \tag{3.22}$$

Hence, the $\hat{\mathfrak{g}}_k$ WZW characters are the same as the specialized characters of the $\hat{\mathfrak{g}}_k$ affine Lie algebra defined in Appendix B. In order to distinguish the conjugate representations, a $\zeta = \sum_{i=1}^r z_i\alpha_i^\vee$ dependence is usually introduced

$$\chi_{\hat{\lambda}}(z;\tau) = \mathrm{Tr}_{\hat{\lambda}}\, e^{2\pi i\tau(L_0-c/24)}e^{-2\pi i\sum_j z_j h_j} \tag{3.23}$$

where h_j is the Chevalley generator of Lie algebra $\mathfrak{g}$

3.3 Boson

3.3.1 Compactified boson: $\hat{\mathfrak{u}}(1)_k$

Consider the free boson filed $\varphi(z)$ compactified on a circle with radius R, from the OPE

$$\partial\varphi(z)\partial\varphi(w) \sim -\frac{1}{(z-w)^2} \tag{3.24}$$

we see that $i\partial\varphi(z)/\sqrt{2}$ is a $\hat{\mathfrak{u}}(1)_1$ current and the level can be changed by scaling. Recall the partition function 2.32, there seems to be an infinite number of $\hat{\mathfrak{u}}(1)$ characters due to the range of n and w

$$\chi_{n,w}(q) = \frac{1}{\eta(q)}q^{(n/R+wR/2)^2/2} \tag{3.25}$$

However, if we choose R^2 to be fractional, that is $R^2 = 2p'/p$, the characters can be reorganized into a finite set

$$\chi_l^{(pp')}(q) = \frac{1}{\eta(q)}\sum_{u\in\mathbb{Z}} q^{pp'(u+l/2pp')^2} \tag{3.26}$$

where integer l take the value $-pp'+1 \le l \le pp'$. This is indeed the affine Lie character of $\hat{\mathfrak{u}}(1)_k$ defined in B.33 with $k = 2pp'$. The conformal dimension of the corresponding primary field $|\phi_l\rangle$ is $l^2/2k$, and the Sugawara central charge is 1 since there is only one boson field.

3.3.2 Vertex representation: $\hat{\mathfrak{su}}(2)_1$

Besides the $\hat{\mathfrak{u}}(1)_1$ current $H(z) \equiv i\partial\varphi(z)/\sqrt{2}$, we can construct two vertex operators $E^{\pm}(z) \equiv e^{\pm i\sqrt{2}\varphi(z)}$ with conformal dimension 1 from the free boson field $\varphi(z)$, they form the $\hat{\mathfrak{su}}(2)_1$ current algebra together. In fact, they satisfy the OPEs of the Cartan-Weyl basis of $\hat{\mathfrak{su}}(2)_1$

$$\begin{aligned} E^+(z)E^-(w) &\sim \frac{1}{(z-w)^2} + \frac{2H(w)}{z-w} \\ H(z)E^{\pm}(w) &\sim \frac{\pm E^{\pm}(w)}{z-w} \\ H(z)H(w) &\sim \frac{1/2}{(z-w)^2} \end{aligned} \tag{3.27}$$

The $\hat{\mathfrak{su}}(2)_1$ algebra has two integrable modules $L_{[1,0]}$ and $L_{[0,1]}$ with highest weight conformal dimension 0 and 1/4 respectively. We consider $L_{[1,0]}$ first; its highest weight state is just the vacuum, hence the Fock space consists of the following states

$$\left|p;\{n_i\}\right\rangle = N_{\{n_i\}} a_{-1}^{n_1} a_{-2}^{n_2} \cdots \left|p;\{0\}\right\rangle \tag{3.28}$$

where $N_{\{n_i\}}$ is some constant and $|p;\{0\}\rangle$ is obtained by acting the vertex operators on the vacuum

$$|p;\{0\}\rangle = (E^{\pm}(0))^m\,|0;\{0\}\rangle = \left|\pm\sqrt{2}m;\{0\}\right\rangle \tag{3.29}$$

$L_0 = a_0^2/2 + \sum_{n>0} a_{-n}a_n$ determines the conformal dimension of $\left|p;\{n_i\}\right\rangle$

$$L_0\left|p;\{n_i\}\right\rangle = \left[m^2 + \sum_{k=1}^{\infty} k n_k\right]\left|\pm\sqrt{2}m;\{n_i\}\right\rangle \tag{3.30}$$

While for $L_{[0,1]}$ with highest weight state $e^{i\varphi(0)/\sqrt{2}}\,|0;\{0\}\rangle$, we just need to replace m with $m+1/2$. Using the result of the compactified boson, the characters are

$$\chi^{\hat{\mathfrak{su}}(2)_1}_{[1-\lambda_1,\lambda_1]}(z;\tau) = \frac{1}{\eta(q)} \sum_{m\in\mathbb{Z}+\lambda_1/2} q^{m^2} x^m \qquad (x = e^{-2\pi i z}) \tag{3.31}$$

3.4 Fermion

3.4.1 Real free-fermion : $\hat{\mathfrak{so}}(N)_1$

In this section, we show that the $\hat{\mathfrak{so}}(N)_1$ current algebra can be recovered by N real independent fermions. Consider the following fermions OPEs, which transform in the vector representation (ω_1) of $\mathfrak{so}(N)$

$$\psi_i(z)\psi_j(w) \sim \frac{\delta_{ij}}{z-w} \qquad 1 \le i,j \le N \tag{3.32}$$

Suppose the transformation matrices are t^a_{ij}, where $a = (r, s)$, $1 \le r < s \le N$ denotes the index of the algebra generators. An explicit realization of t^a_{ij} is

$$\begin{aligned}
&t^a_{ij} \equiv t^{(r,s)}_{ij} = \frac{i}{\sqrt{2}}(\delta^r_i \delta^s_j - \delta^r_j \delta^s_i) \\
&\mathrm{Tr}\left(t^a t^b\right) = \delta_{ab} \\
&\sum_a t^a_{ij} t^a_{kl} = -\frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \\
&[t^a, t^b] = \sum_c i f_{abc} t^c \qquad h^\vee_{\mathfrak{so}(N)} = f_{abc} f^{abc} = N - 2
\end{aligned} \tag{3.33}$$

With the fermions and transformation matrices, the bosonic dimension 1 current can be constructed as

$$J^a(z) = \beta \sum_{i,j} (\psi_i t^a_{ij} \psi_j)(z) \tag{3.34}$$

where β is some constant. Calculate the current OPE, we get

$$J^a(z) J^b(w) \sim 2\beta \frac{i f_{abc} J^c(w)}{z - w} + 2\beta^2 \frac{\mathrm{Tr}\left(t^a t^b\right)}{(z - w)^2} \tag{3.35}$$

This is just the form of affine current algebra, and we can determine $\beta = 1/2$ from the $(z - w)^{-1}$ term. Hence the level of this current algebra is $k = 4\beta^2 \,\mathrm{Tr}\left(t^a t^b\right) = 1$, we get the desired $\hat{\mathfrak{so}}(N)_1$ current algebra. The Sugawara central charge is

$$c(\hat{\mathfrak{so}}(N)_1) = \frac{\dim \mathfrak{so}(N)}{1 + h^\vee_{\mathfrak{so}(N)}} = \frac{N(N-1)/2}{1 + N - 2} = \frac{N}{2} \tag{3.36}$$

which is expected since each fermion contributes 1/2 to the central charge

$\hat{\mathfrak{so}}(N)_1$ characters

Recall the integrable condition of a representation $\hat{\lambda}$

$$\lambda_0 = k - (\lambda, \theta) \ge 0 \tag{3.37}$$

It turns out that the only integrable representation of $\hat{\mathfrak{so}}(N)_1$ are $\hat{\omega}_0, \hat{\omega}_1, \hat{\omega}_r$ for $N = 2r + 1$ and $\hat{\omega}_0, \hat{\omega}_1, \hat{\omega}_{r-1}, \hat{\omega}_r$ for $N = 2r$, using 3.18 and we can find the conformal dimensions of corresponding primaries. The vacuum state $\hat{\omega}_0$ has dimension 0 while the other primaries have dimension

$$\begin{aligned}
&N = 2r + 1: \quad h_{\hat{\omega}_1} = \frac{1}{2} \quad h_{\hat{\omega}_r} = \frac{2r + 1}{16} \\
&N = 2r: \quad h_{\hat{\omega}_1} = \frac{1}{2} \quad h_{\hat{\omega}_{r-1}} = h_{\hat{\omega}_r} = \frac{r}{8}
\end{aligned} \tag{3.38}$$

Since $\hat{\mathfrak{so}}(N)_1$ can be represented by N fermions, we can easily write its partition functions using that of the free fermions

$$Z = \sum_{\hat{\lambda}\in P_+^1} |\chi_{\hat{\lambda}}|^2 \tag{3.39}$$

where P_+^1 denotes the set of integrable weights in level 1 and the characters are found by modular covariance

$$\begin{aligned}
&\text{for } N = 2r\ :\\
&\chi_{\hat{\omega}_0} = \frac{1}{2}\left(\frac{\theta_3^r + \theta_4^r}{\eta^r}\right) \quad \chi_{\hat{\omega}_1} = \frac{1}{2}\left(\frac{\theta_3^r - \theta_4^r}{\eta^r}\right) \quad \chi_{\hat{\omega}_{r-1}} = \chi_{\hat{\omega}_r} = \frac{1}{2}\frac{\theta_2^r}{\eta^r}\\
&\text{for } N = 2r+1\ :\\
&\chi_{\hat{\omega}_0} = \frac{1}{2}\left(\frac{\theta_3^{r+\frac{1}{2}} + \theta_4^{r+\frac{1}{2}}}{\eta^{r+\frac{1}{2}}}\right) \quad \chi_{\hat{\omega}_1} = \frac{1}{2}\left(\frac{\theta_3^{r+\frac{1}{2}} - \theta_4^{r+\frac{1}{2}}}{\eta^{r+\frac{1}{2}}}\right) \quad \chi_{\hat{\omega}_r} = \frac{1}{\sqrt{2}}\frac{\theta_2^{r+\frac{1}{2}}}{\eta^{r+\frac{1}{2}}}
\end{aligned} \tag{3.40}$$

We can extend the above analysis to higher level by considering fermions transforming in representation λ, and the level of the current algebra equals to the Dynkin index x_λ, where

$$\mathrm{Tr}\left(t_\lambda^a t_\lambda^a\right) = x_\lambda \delta_{ab} \tag{3.41}$$

3.4.2 Complex free-fermion : $\hat{\mathfrak{u}}(N)_1 \equiv \hat{\mathfrak{su}}(N)_1 \oplus \hat{\mathfrak{u}}(1)$

Consider N complex free fermions, their OPEs are

$$\begin{aligned}
&\psi_i(z)\psi_j^\dagger(w) \sim \psi_i^\dagger(z)\psi_j(w) \sim \frac{\delta_{ij}}{z-w}\\
&\psi_i(z)\psi_j(w) \sim \psi_i^\dagger(z)\psi_j^\dagger(w) \sim 0
\end{aligned} \tag{3.42}$$

Similar with the real free-fermion case, if the complex fields are under the ω_1 representation of $\mathfrak{su}(N)$, then the transformation matrices t_{ij}^a satisfy

$$\begin{aligned}
&\mathrm{Tr}\left(t^a t^b\right) = \frac{1}{2}\delta_{ab}\\
&\sum_a t_{ij}^a t_{kl}^a = \frac{1}{2}\left(\delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{kl}\right)\\
&[t^a, t^b] = \sum_c i f_{abc} t^c \qquad h^\vee_{\mathfrak{su}(N)} = f_{abc}f^{abc} = N
\end{aligned} \tag{3.43}$$

The $\hat{\mathfrak{su}}(N)$ currents appear to be

$$J^a(z) = \sum_{i,j}(\psi_i^\dagger t_{ij}^a \psi_j)(z) \tag{3.44}$$

which has level 1, and the central charge is

$$c(\hat{\mathfrak{su}}(N)_1) = \frac{\dim \mathfrak{su}(N)}{1 + h^\vee_{\mathfrak{su}(N)}} = \frac{N^2 - 1}{1 + N} \tag{3.45}$$

In addition, we can construct a $\hat{\mathfrak{u}}(1)$ current

$$J^0(z) = \frac{\beta}{\sqrt{2}} \sum_i (\psi_i^\dagger \psi_i)(z) \tag{3.46}$$

which has undetermined level $\beta^2 N$, and the central charge is $c(\hat{\mathfrak{u}}(1)) = 1$. The $\hat{\mathfrak{u}}(1)$ current and $\hat{\mathfrak{su}}(N)$ currents are actually independent

$$J^0(z) J^a(w) \sim 0 \tag{3.47}$$

Hence we get a representation of $\hat{\mathfrak{u}}(N)_1$ with expected Sugawara central charge

$$c(\hat{\mathfrak{u}}(N)_1) = c(\hat{\mathfrak{su}}(N)_1) + c(\hat{\mathfrak{u}}(1)) = N \tag{3.48}$$

The discussion above can also be extended to higher levels by considering other representations of $\mathfrak{su}(N)$ rather than ω_1

第 4 章　Cosets

4.1　GKO construction and coset characters

Consider two WZW models with affine Lie algebra $\hat{\mathfrak{g}}$ and $\hat{\mathfrak{p}}$, where $\hat{\mathfrak{p}}$ is a subalgebra of $\hat{\mathfrak{g}}$ with embedding index x_e, which means if the level of $\hat{\mathfrak{g}}$ is k, then the level of $\hat{\mathfrak{p}}$ is $x_e k$. We denote their conformal generators by L^g_m, L^p_m, affine Lie algebra generators by $J^a_n, \tilde{J}^{a'}_n$ and energy-momentum tensors by T_g, T_p respectively . Since the generators of $\hat{\mathfrak{p}}$ are linear combinations of the generators of $\hat{\mathfrak{g}}$

$$\tilde{J}^{a'}_n = \sum_a m_{a'a} J^a_n \tag{4.1}$$

the following commutators are equal

$$[L^g_m, \tilde{J}^{a'}_n] = [L^p_m, \tilde{J}^{a'}_n] = -n\tilde{J}^{a'}_n \tag{4.2}$$

This shows $L^{(g/p)}_m \equiv L^g_m - L^p_m$ commutes with $\tilde{J}^{a'}_n$ and L^p_m, hence

$$\begin{aligned} T_{(g/p)}(z)\tilde{J}^a(w) &\sim 0 \\ T_{(g/p)}(z)T_p(w) &\sim 0 \end{aligned} \tag{4.3}$$

where $T_{(g/p)}(z) \equiv T_g(z) - T_p(z)$. So $L^{(g/p)}_m$ also forms the Virasoro algebra

$$[L^{(g/p)}_m, L^{(g/p)}_n] = (m-n)L^{(g/p)}_{m+n} + \frac{c\left(\hat{\mathfrak{g}}_k/\hat{\mathfrak{p}}_{x_e k}\right)}{12}(m^3-m)\delta_{m+n,0} \tag{4.4}$$

where the central charge is

$$c\left(\hat{\mathfrak{g}}_k/\hat{\mathfrak{p}}_{x_e k}\right) = c\left(\hat{\mathfrak{g}}_k\right) - c\left(\hat{\mathfrak{p}}_{x_e k}\right) = \frac{k \dim \mathfrak{g}}{k + h^\vee_{\mathfrak{g}}} - \frac{x_e k \dim \mathfrak{p}}{x_e k + h^\vee_{\mathfrak{p}}} \tag{4.5}$$

This is called the Goddard-Kent-Olive (GKO) construction[16]; it gives new RCFTs by quotient two WZW models; these theories are referred to as the costs.

Now we define the characters of the coset. Suppose $\hat{\lambda}, \hat{\mu}$ are representations of $\hat{\mathfrak{g}}, \hat{\mathfrak{p}}$ respectively, if the branching rule of representations is

$$\hat{\lambda} \mapsto \bigoplus_{\hat{\mu}} b_{\hat{\lambda}\hat{\mu}}\hat{\mu} \tag{4.6}$$

then the coset characters $\chi_{\{\hat\lambda;\hat\mu\}}$ are defined by

$$\chi_{\mathscr{P}\hat\lambda}(\zeta;\tau;t) = \sum_{\hat\mu \in P_+^{kx_e}} \chi_{\{\hat\lambda;\hat\mu\}}(\tau)\chi_{\hat\mu}(\zeta;\tau;t) \tag{4.7}$$

where $\mathscr{P}$ is the projection matrix of the embedding $\mathfrak{p} \subset \mathfrak{g}$ and

$$\chi_{\{\hat\lambda;\hat\mu\}}(\tau) = q^{m_{\hat\lambda} - m_{\hat\mu}} b_{\hat\lambda\hat\mu} \tag{4.8}$$

The modular property of coset characters is simple. They transform covariantly under the following matrices, which are productions of the modular transformation matrices of the affine Lie algebra characters

$$\begin{aligned} \mathscr{T}_{\{\hat\lambda;\hat\mu\},\{\hat\lambda';\hat\mu'\}} &= \mathscr{T}^{(k)}_{\hat\lambda\hat\lambda'}\mathscr{T}^{(kx_e)}_{\hat\mu\hat\mu'} \\ \mathscr{S}_{\{\hat\lambda;\hat\mu\},\{\hat\lambda';\hat\mu'\}} &= \mathscr{S}^{(k)}_{\hat\lambda\hat\lambda'}\mathscr{S}^{(kx_e)}_{\hat\mu\hat\mu'} \end{aligned} \tag{4.9}$$

4.1.1 Minimal models

Consider the $\hat{\mathfrak{su}}(2)$ diagonal coset

$$\frac{\hat{\mathfrak{su}}(2)_k \oplus \hat{\mathfrak{su}}(2)_1}{\hat{\mathfrak{su}}(2)_{k+1}} \qquad k \in \mathbb{Z}_{\geq 1} \tag{4.10}$$

It has a central charge

$$\begin{aligned} c &= c(\hat{\mathfrak{su}}(2)_k) + c(\hat{\mathfrak{su}}(2)_1) - c(\hat{\mathfrak{su}}(2)_{k+1}) \\ &= 1 - \frac{6}{(k+2)(k+3)} \end{aligned} \tag{4.11}$$

This is exactly the central charge of the unitary minimal models. The unitarity is obvious since the integrable highest weight representations with positive integer levels of affine Lie algebras are all unitary.

As for non-unitary minimal model $\mathscr{M}_{p,p'}$ with $|p-p'| \neq 1$, the level has to be fractional

$$c = 1 - \frac{6(p-p')^2}{pp'} \quad \Rightarrow \quad k = \frac{3p'-2p}{p-p'} \tag{4.12}$$

The integer level representation does not work, so we need to introduce the so-called admissible representations of $\hat{\mathfrak{su}}(2)_k$. Suppose the fractional level $k = t/u$, where t and u are coprime positive integers, we decompose the highest weight $\hat\lambda$ of $\hat{\mathfrak{su}}(2)_k$ into integer and fractional parts

$$\hat\lambda = \hat\lambda^I - (k+2)\hat\lambda^F \tag{4.13}$$

where $\hat\lambda^I$ and $\hat\lambda^F$ are two integrable weights with level

$$\begin{aligned} k^I &= u(k+2) - 2 \geq 0 \\ k^F &= u - 1 \geq 0 \end{aligned} \tag{4.14}$$

Since the number of integrable weights of $\hat{\mathfrak{su}}(2)_k$ at integer level is $k+1$, there are $(k^I+1)(k^F+1)$ admissible representations $\hat{\lambda}$ at fractional level. Using the Weyl-Kac character formula and specializing at $\hat{\xi} = -2\pi i(z\omega_1;\tau;0)$, the characters of admissible representations of $\hat{\mathfrak{su}}(2)_k$ becomes

$$\chi_{\hat{\lambda}}(z;\tau) = \frac{\Theta^{(d)}_{b_+}(z/u;\tau) - \Theta^{(d)}_{b_-}(z/u;\tau)}{\Theta^{(2)}_{1}(z;\tau) - \Theta^{(2)}_{-1}(z;\tau)} \tag{4.15}$$

where $d = u^2(k+2), b_\pm = u[\pm(\lambda^I_1+1) - (k+2)\lambda^F_1]$ and

$$\Theta^{(d)}_b = \sum_{l\in\mathbb{Z}+b/2d} q^{dl^2} x^{dl} \qquad (x = e^{-2\pi i z}) \tag{4.16}$$

If we choose $p = u(k+2)+u$ and $p' = u(k+2)$, then the non-unitary minimal models are recovered.

4.1.2 Parafermions: $\hat{\mathfrak{su}}(2)_k/\hat{\mathfrak{u}}(1)_k$

In analogy with the vertex representation $\hat{\mathfrak{su}}(2)_1$, we introduce three currents to represent $\hat{\mathfrak{su}}(2)_k$

$$\begin{aligned} J^+(z) &= \sqrt{k}\psi_{\text{par}}(z)e^{i\sqrt{2/k}\varphi(z)} \\ J^-(z) &= \sqrt{k}\psi^\dagger_{\text{par}}(z)e^{-i\sqrt{2/k}\varphi(z)} \\ J^0(z) &= i\sqrt{k/2}\partial\varphi(z) \end{aligned} \tag{4.17}$$

where $\psi_{\text{par}}, \psi^\dagger_{\text{par}}$ are called the parafermions. In order to make $J^\pm$ have conformal dimension 1, the conformal dimension of parafermions should be $(k-1)/k$, which means the parafermions are absent or become free fermions when $k=1$ or $k=2$ respectively.

Now we consider the characters of $\hat{\mathfrak{su}}(2)_k$ in terms of string functions in B.34. Since the string functions are invariant under translation $t_{\alpha^\vee}$, we can translate all the $\hat{\mathfrak{su}}(2)_k$ weights $\hat{\mu} \in \Omega^{\max}_{\hat{\lambda}}$ into weights $\hat{\nu}$ such that $-k+1 \le \nu_1 \le k$, and the expression of characters becomes

$$\text{ch}_{\hat{\lambda}} = \sum_{\hat{\mu}\in\Omega^{\max}_{\hat{\lambda}}} \sigma^{(\hat{\lambda})}_{\hat{\mu}}(e^{-\delta})e^{\hat{\mu}} = \sum_{-k+1\le\nu_1\le k} \sigma^{(\hat{\lambda})}_{\hat{\nu}}(e^{-\delta}) \sum_{\alpha^\vee\in Q^\vee} e^{t_{\alpha^\vee}\hat{\nu}} \tag{4.18}$$

Letting $\lambda_1 \equiv l, \nu_1 = m$ and Specializing at $\hat{\xi} = -2\pi i(0;\tau;0)$, it can be further written as the combination of $\hat{\mathfrak{u}}(1)_k$ characters

$$\chi^{\hat{\mathfrak{su}}(2)_k}_l(q) = \sum_{m=-k+1}^{k} \eta(q)c^l_m(q)\chi^{\hat{\mathfrak{u}}(1)_k}_l(q) \tag{4.19}$$

where $c^l_m(q)$ is the normalized string functions defined as

$$c^l_m(q) = q^{h_l - h_m - 3k/(k+2)} \sigma^l_m(q) \tag{4.20}$$

According to the branching rule 4.19, the parafermions characters are just a product of the Dedekind function and the normalized string functions

$$\chi^{\hat{\mathfrak{su}}(2)_k/\hat{\mathfrak{u}}(1)_k}_{\{l;m\}} = \eta(q) c^l_m(q) \tag{4.21}$$

4.2 Supercoset

This section follows Yoichi Kazama and Hisao Suzuki's articles[1,17], which give coset construction of $\mathcal{N} = 1, 2$ superconformal algebra and the parts of minimal models are from[18–19].

4.2.1 $\mathcal{N} = 1$

In order to extend the WZW model to the super affine Lie algebra case, which means constructing the superconformal current $T(Z) = G(z)/2 + \theta T(z)$ by the super affine Lie algebra current $J^a(Z) = j^a(z) + \theta J^a(z)$, it is convenient to introduce the modified affine current

$$\begin{aligned} \hat{J}^a(z) &\equiv J^a(z) - J^a_f(z) \\ J^a_f(z) &\equiv -\frac{i}{k} f_{abc} (j^b j^c)(z) \end{aligned} \tag{4.22}$$

They generate two independent affine Lie algebras

$$\begin{aligned} \hat{J}^a(z)\hat{J}^b(w) &\sim \frac{\hat{k}\delta_{ab}/2}{(z-w)^2} + \frac{i f_{abc} \hat{J}^c(w)}{z-w} \\ J^a_f(z) J^b_f(w) &\sim \frac{h^\vee_{\mathfrak{g}} \delta_{ab}/2}{(z-w)^2} + \frac{i f_{abc} J^c_f(w)}{z-w} \\ \hat{J}^a(z) j^b(w) &\sim \hat{J}^a(z) J^b_f(w) \sim 0 \end{aligned} \tag{4.23}$$

where $\hat{k} = k - h^\vee_{\mathfrak{g}}$ is called the bosonic level. Then the superconformal currents can be expressed in terms of $\hat{J}^a$ and j^a as

$$\begin{aligned} T_g(z) &= \frac{1}{k} \sum_a [(\hat{J}^a \hat{J}^a)(z) - (j^a \partial j^a)(z)] \\ G_g(z) &= \frac{2}{k} \left(\sum_a (j^a \hat{J}^a)(z) - \frac{i}{3k} f_{abc} (j^a j^b j^c)(z) \right) \end{aligned} \tag{4.24}$$

The central charge is

$$c_g = \frac{1}{2} \dim \mathfrak{g} + \frac{\hat{k} \dim \mathfrak{g}}{\hat{k} + h^\vee_{\mathfrak{g}}} \tag{4.25}$$

Similarly, we can construct the super WZW model corresponding to the subalgebra $\hat{\mathfrak{p}}$. Setting the embedding index $x_e = 1$ for convenience, we consider $\{J^{a'}(z)\}$ the currents of $\hat{\mathfrak{p}}$ as a subset of $\{J^a(z)\}$, and define

$$\begin{aligned} \tilde{J}^{a'}(z) &\equiv J^{a'}(z) - J_f^{a'}(z) = \hat{J}^{a'}(z) - \frac{i}{k} f_{a'\bar{b}\bar{c}}(j^{\bar{b}} j^{\bar{c}})(z) \\ J_f^{a'}(z) &\equiv -\frac{i}{k} f_{a'b'c'}(j^{b'} j^{c'})(z) \end{aligned} \tag{4.26}$$

where $\bar{a}, \bar{b}, \cdots$ stand for the index of the coset $\mathfrak{g}/\mathfrak{p}$. This decomposition also generates two independent affine Lie algebras with level $k - h_{\mathfrak{p}}^{\vee}$ and $h_{\mathfrak{p}}^{\vee}$. The corresponding superconformal currents are

$$\begin{aligned} T_p(z) &= \frac{1}{k} \sum_{a'} \left((\tilde{J}^{a'} \tilde{J}^{a'})(z) - (j^{a'} \partial j^{a'})(z) \right) \\ G_p(z) &= \frac{2}{k} \left(\sum_{a'} (j^{a'} \tilde{J}^{a'})(z) - \frac{i}{3k} f_{a'b'c'}(j^{a'} j^{b'} j^{c'})(z) \right) \end{aligned} \tag{4.27}$$

The superconformal currents of the coset are constructed by $T_{(g/p)}(Z) \equiv T_g(Z) - T_p(Z)$, the result is

$$\begin{aligned} T_{(g/p)} &= \frac{1}{k} \left(\hat{J}^{\bar{a}} \hat{J}^{\bar{a}} - \frac{\hat{k}}{k} j^{\bar{a}} \partial j^{\bar{a}} + \frac{2i}{k} f_{\bar{a}\bar{b}\bar{c}} \hat{J}^{\bar{a}} j^{\bar{b}} j^{\bar{c}} - \frac{2}{k} f_{\bar{a}\bar{c}\bar{d}} f_{\bar{b}\bar{c}\bar{d}} j^{\bar{a}} \partial j^{\bar{b}} - \frac{1}{k^2} f_{\bar{a}\bar{b}\bar{c}} f_{\bar{a}\bar{d}\bar{e}} j^{\bar{b}} j^{\bar{c}} j^{\bar{d}} j^{\bar{e}} \right) \\ G_{(g/p)} &= \frac{2}{k} \left(j^{\bar{a}} \hat{J}^{\bar{a}} - \frac{i}{3k} f_{\bar{a}\bar{b}\bar{c}} j^{\bar{a}} j^{\bar{b}} j^{\bar{c}} \right) \end{aligned} \tag{4.28}$$

where we omit the normal ordering and summation with respect to all index pairs. The central charge of the coset can be obtained by the GG OPE

$$c_{(g/p)} = \frac{3\hat{k}}{2k} \dim(\mathfrak{g}/\mathfrak{p}) + \frac{1}{2k} f_{\bar{a}\bar{b}\bar{c}} f_{\bar{a}\bar{b}\bar{c}} \tag{4.29}$$

Minimal models

Consider the following diagonal coset

$$\frac{[\hat{\mathfrak{su}}(2)]_{k+2} \oplus [\hat{\mathfrak{su}}(2)]_4}{[\hat{\mathfrak{su}}(2)]_{k+4} \oplus [\hat{\mathfrak{su}}(2)]_2} = \frac{\hat{\mathfrak{su}}(2)_k \oplus \hat{\mathfrak{su}}(2)_2}{\hat{\mathfrak{su}}(2)_{k+2}} \qquad k \in \mathbb{Z}_{\geq 1} \tag{4.30}$$

where $[\hat{\mathfrak{g}}]$ stands for the superalgebra associated with $\hat{\mathfrak{g}}$. Its central charge and conformal dimension of primaries are given by

$$\begin{aligned} c &= \frac{3}{2} \left(1 - \frac{8}{(k+2)(k+4)} \right) \\ h_{r,s} &= \frac{[(k+4)r - (k+2)s]^2 - 4}{8(k+2)(k+4)} + \nu \end{aligned} \tag{4.31}$$

where $\nu = 0$ in the NS sector with $r-s$ even and $\nu = 1/16$ in the R sector with $r-s$ odd. This is called the $\mathcal{N}=1$ minimal model. The branching rule of characters is

$$\chi_l^{\hat{\mathfrak{su}}(2)_k}(\tau)\chi_s^{\hat{\mathfrak{su}}(2)_2}(\tau) = \sum_{m=0}^{k+2} \chi_{l,s}^m(\tau)\chi_m^{\hat{\mathfrak{su}}(2)_{k+2}}(\tau) \tag{4.32}$$

4.2.2 $\mathcal{N}=2$

Considering the $\mathcal{N}=2$ SCA, the OPE about fermionic currents $G^{\pm}(z)$ is

$$\begin{aligned} G^+(z)G^-(w) &\sim \frac{2c/3}{(z-w)^3} + \frac{2J(w)}{(z-w)^2} + \frac{2T(w)+\partial J(w)}{(z-w)} \\ G^+(z)G^+(w) &\sim G^-(z)G^-(w) \sim 0 \end{aligned} \tag{4.33}$$

For convenience, we define

$$\begin{aligned} G^0(z) &= \frac{1}{\sqrt{2}}(G^+(z)+G^-(z)) \\ G^1(z) &= \frac{1}{\sqrt{2}i}(G^+(z)-G^-(z)) \\ J^{ij} &= iJ(z)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{aligned} \tag{4.34}$$

The OPE 4.33 becomes

$$G^i(z)G^j(w) \sim \frac{2c\delta^{ij}/3}{(z-w)^3} + \frac{2J^{ij}(w)}{(z-w)^2} + \frac{2T(w)\delta^{ij}+\partial J^{ij}(w)}{(z-w)} \tag{4.35}$$

When $i,j=0$, this is indeed the $\mathcal{N}=1$ fermionic current OPE. We want to construct G^i in terms of $j^{\bar{a}}, \hat{J}^{\bar{a}}$ to recover the above algebra. The most general form of fermionic operators with dimension 3/2 is

$$G^i = \frac{2}{k}\left(h^i_{\bar{a}\bar{b}}j^{\bar{a}}\hat{J}^{\bar{b}} - \frac{i}{3k}S^i_{\bar{a}\bar{b}\bar{c}}j^{\bar{a}}j^{\bar{b}}j^{\bar{c}}\right) \tag{4.36}$$

In order to identify G^0 with the $\mathcal{N}=1$ current, the coefficients should satisfy

$$h^0_{\bar{a}\bar{b}} = \delta_{\bar{a}\bar{b}} \qquad S^0_{\bar{a}\bar{b}\bar{c}} = f_{\bar{a}\bar{b}\bar{c}} \tag{4.37}$$

Moreover, by comparing the OPE of G^i with 4.35, we get the following constraints

$$\begin{aligned} & h^1_{\bar{a}\bar{b}} + h^1_{\bar{b}\bar{a}} = 0 \qquad h^1_{\bar{a}\bar{p}}h^1_{\bar{p}\bar{b}} = -\delta_{\bar{a}\bar{b}} \\ & h^1_{\bar{a}\bar{d}}f_{\bar{d}\bar{b}\bar{e}} = f_{\bar{a}\bar{d}\bar{e}}h^1_{\bar{d}\bar{b}} \qquad S^1_{\bar{a}\bar{b}\bar{c}} = h^1_{\bar{a}\bar{p}}h^1_{\bar{b}\bar{q}}h^1_{\bar{c}\bar{r}}f_{\bar{p}\bar{q}\bar{r}} \\ & f_{\bar{a}\bar{b}\bar{c}} = h^1_{\bar{a}\bar{p}}h^1_{\bar{b}\bar{q}}f_{\bar{p}\bar{q}\bar{c}} + h^1_{\bar{b}\bar{p}}h^1_{\bar{b}\bar{q}}f_{\bar{p}\bar{q}\bar{a}} + h^1_{\bar{c}\bar{p}}h^1_{\bar{a}\bar{q}}f_{\bar{p}\bar{q}\bar{b}} \end{aligned} \tag{4.38}$$

This shows that $h^1_{\bar{a}\bar{b}}$ is a almost complex structure. In fact, these constraints are equivalent to the statement: The supercoset $[\hat{\mathfrak{g}}]/[\hat{\mathfrak{p}}]$ is $\mathcal{N} = 2$ supersymmetric if and only if

$$\begin{aligned} &(a) \quad \mathrm{rank}(\mathfrak{g}) - \mathrm{rank}(\mathfrak{p}) = 2n \quad n \in \mathbb{Z}_{\geq 0} \\ &(b) \quad \frac{\mathfrak{g}}{\mathfrak{p} \oplus 2n\ \mathfrak{u}(1)} \quad \text{is Kählerian} \end{aligned} \tag{4.39}$$

Instead of finding all the Kählerian space, we just consider the following simple solution here

$$S^1_{\bar{a}\bar{b}\bar{c}} = f^1_{\bar{a}\bar{b}\bar{c}} = 0 \tag{4.40}$$

Coset spaces that satisfy 4.40 are called Hermitian symmetric spaces (HSSs), they have no extra $\hat{\mathfrak{u}}(1)$ symmetry, that is $\mathrm{rank}(\mathfrak{g}) = \mathrm{rank}(\mathfrak{p})$. HSSs have been classified completely as follows

$\mathfrak{g}/\mathfrak{p}$	$c([\hat{\mathfrak{g}}]_k/[\hat{\mathfrak{p}}]_k)$
$\mathfrak{su}(m+n)/\mathfrak{su}(m) \oplus \mathfrak{su}(n) \oplus \mathfrak{u}(1)$	$3\hat{k}mn/(\hat{k}+m+n)$
$\mathfrak{so}(n+2)/\mathfrak{so}(n) \oplus \mathfrak{so}(2)$	$3\hat{k}n/(\hat{k}+n) \quad n \geq 2$
$\mathfrak{so}(3)/\mathfrak{so}(2)$	$3\hat{k}/(\hat{k}+2)$
$\mathfrak{so}(2n)/\mathfrak{su}(n) \oplus \mathfrak{u}(1)$	$3\hat{k}n(n-1)/2(\hat{k}+2n-2)$
$\mathfrak{sp}(n)/\mathfrak{su}(n) \oplus \mathfrak{u}(1)$	$3\hat{k}n(n+1)/2(\hat{k}+n+1)$
$E_6/\mathfrak{so}(10) \oplus \mathfrak{u}(1)$	$48\hat{k}/(\hat{k}+12)$
$E_7/E_6 \oplus \mathfrak{u}(1)$	$81\hat{k}/(\hat{k}+18)$

表 4-1 Hermitian symmetric spaces

An important property of the HSSs is that we can define a projection operator from the complex structure

$$P_{\pm\bar{a}\bar{b}} = \frac{1}{2}(\delta_{\bar{a}\bar{b}} \mp i h^1_{\bar{a}\bar{b}}) \tag{4.41}$$

It projects $\hat{J}^{\bar{a}}$ into two commutative sets

$$\begin{aligned} &\hat{J}^{\bar{a}}_{\pm}(z) = P_{\pm\bar{a}\bar{b}}\hat{J}^{\bar{a}}(z) \\ &J^{\bar{a}}_{+}(z)J^{\bar{b}}_{+}(w) \sim J^{\bar{a}}_{-}(z)J^{\bar{b}}_{-}(w) \sim 0 \end{aligned} \tag{4.42}$$

Characters

The $\mathcal{N}=2$ SCA can be reformulated in the Cartan-Weyl basis of $\hat{\mathfrak{g}}$, which has the following OPEs

$$
\begin{aligned}
E^{\alpha}(z)E^{-\beta}(w) &\sim \frac{N_{\alpha,-\beta}E^{\alpha-\beta}(w)}{z-w} \\
E^{\alpha}(z)E^{-\alpha}(w) &\sim \frac{k}{(z-w)^2}+\frac{\alpha\cdot H(w)}{z-w} \\
H^{i}(z)H^{j}(z) &\sim \frac{k\delta_{ij}}{(z-w)^2}
\end{aligned}
\tag{4.43}
$$

For the HSSs, the rank of coset is zero, means the generators of the coset are all of type $E^{\bar{\alpha}}$. Moreover, we can divide these generators into two commutative sets:

$$
\bar{\Delta}=\bar{\Delta}_{+}+\bar{\Delta}_{-} \qquad E^{\bar{\alpha}}(z)E^{\bar{\beta}}(w)\sim 0 \quad \text{for } \alpha,\beta\in\bar{\Delta}_{+} \text{ or } \alpha,\beta\in\bar{\Delta}_{-}
\tag{4.44}
$$

In addition, we need $D/2$ complex fermions or D real fermions to realize the supersymmetry, where $D=\dim(\mathfrak{g}/\mathfrak{p})$

$$
\psi_{\bar{\alpha}}(z)\psi_{-\bar{\beta}}=\frac{\delta_{\bar{\alpha},\bar{\beta}}}{z-w}
\tag{4.45}
$$

Now the $\mathcal{N}=2$ SCA currents can be expressed as

$$
\begin{aligned}
G_{\pm}(z) &= \sqrt{\frac{2}{k+h_{\mathfrak{g}}^{\vee}}}\sum_{\bar{\alpha}\in\bar{\Delta}_{+}}\psi_{\pm\bar{\alpha}}(z)E^{\mp\bar{\alpha}}(z) \\
T(z) &= \frac{1}{k+h_{\mathfrak{g}}^{\vee}}J^{a}(z)J^{a}(z)-\frac{1}{2}\sum_{\bar{\alpha}\in\bar{\Delta}_{+}}\left[\psi_{\bar{\alpha}}(z)\partial\psi_{-\bar{\alpha}}(z)+\psi_{-\bar{\alpha}}(z)\partial\psi_{\bar{\alpha}}(z)\right] \\
&\quad -\frac{1}{2(k+h_{\mathfrak{g}}^{\vee})}\left[H(z)+\sum_{\bar{\alpha}\in\bar{\Delta}_{+}}\bar{\alpha}\psi_{\bar{\alpha}}(z)\psi_{-\bar{\alpha}}(z)\right]^{2} \\
&\quad -\frac{1}{k+h_{\mathfrak{g}}^{\vee}}\sum_{\bar{\alpha}\in\bar{\Delta}_{+}}\left[E^{-\bar{\alpha}}(z)+\sum_{\bar{\beta}-\bar{\gamma}=-\bar{\alpha}}\psi_{\bar{\beta}}(z)\psi_{-\bar{\gamma}}(z)\right]\left[E^{\bar{\alpha}}(z)+\sum_{\bar{\beta}-\bar{\gamma}=\bar{\alpha}}\psi_{\bar{\beta}}(z)\psi_{-\bar{\gamma}}(z)\right] \\
J(z) &= \frac{1}{k+h_{\mathfrak{g}}^{\vee}}\sum_{\bar{\alpha}\in\bar{\Delta}_{+}}\left[\psi_{\bar{\alpha}}(z)\psi_{-\bar{\alpha}}(z)-\bar{\alpha}\cdot H(z)\right]
\end{aligned}
\tag{4.46}
$$

Since the D real fermions can be represented by $\hat{\mathfrak{so}}(D)_1$, We can actually realize the $\mathcal{N}=2$ SCA by the ordinary GKO coset

$$
\frac{\hat{\mathfrak{g}}_{\text{bos}}\oplus\hat{\mathfrak{so}}(D)_{1}}{\hat{\mathfrak{p}}_{\text{bos}}}
\tag{4.47}
$$

Here we define the affine characters slightly different from the Appendix B, that means we ignore the modular anomaly and use the Cartan-Weyl basis instead of the Chevalley basis

$$\chi_{\Lambda}^{\hat{\mathfrak{g}}_k}(\tau, z_i) = \mathrm{Tr}_{\mathscr{H}_\Lambda}\, q^{L_0 - z\cdot H_0} \tag{4.48}$$

Thus the $\mathcal{N} = 2$ super coset characters $\chi_{\Lambda,v}^{\lambda}$ can be obtained by the following branching rule according to 4.46 and 4.47

$$\chi_{\Lambda}^{\hat{\mathfrak{g}}_k}(\tau, 0)\chi_{v}^{\hat{\mathfrak{so}}(D)_1}(\tau, z_l = z) = \sum_{\lambda} \chi_{\Lambda,v}^{\lambda}(\tau, z)\chi_{\lambda}^{\hat{\mathfrak{p}}_{k+h^\vee_{\mathfrak{g}}-h^\vee_{\mathfrak{p}}}}\left(\tau, z_i = \frac{\sum_{\bar{\alpha}\in\bar{\Delta}_+}\bar{\alpha}^i}{k+h^\vee_{\mathfrak{g}}} z\right) \tag{4.49}$$

Minimal models

Consider the following KS supercoset, it represents the $\mathcal{N} = 2$ minimal models which are all unitary

$$\frac{[\hat{\mathfrak{su}}(2)]_{k+2}}{[\hat{\mathfrak{u}}(1)]_{k+2}} = \frac{\hat{\mathfrak{su}}(2)_k \oplus \hat{\mathfrak{so}}(2)_1}{\hat{\mathfrak{u}}(1)_{k+2}} = \frac{\hat{\mathfrak{su}}(2)_k \oplus \hat{\mathfrak{u}}(1)_2}{\hat{\mathfrak{u}}(1)_{k+2}} \qquad k \in \mathbb{Z}_{\geq 1} \tag{4.50}$$

This supercoset has central charge

$$c = \frac{3k}{k+2} \tag{4.51}$$

with the conformal dimension and $\hat{\mathfrak{u}}(1)$ charge of primaries given by

$$\begin{aligned} (h^l_{m,s}, Q_{m,s}) &= \left(\frac{l(l+2)-m^2}{4(k+2)} + \frac{s^2}{8}, q_{m,s} = -\frac{m}{k+2} + \frac{s}{2}\right) \\ \text{for} \qquad & 0 \leq l \leq k, \quad 0 \leq |m-s| \leq l \qquad l, m, s \in \mathbb{Z} \end{aligned} \tag{4.52}$$

where m is defined modulo $2(k+2)$, $s = 0, 2$ in the NS sector and $s = \pm 1$ in the R sector.

The branching rule of characters of this coset read

$$\chi_l^{\hat{\mathfrak{su}}(2)_k}(\tau)\chi_s^{\hat{\mathfrak{u}}(1)_2}(\tau) = \sum_{m=-k-1}^{k+2} \chi^l_{m,s}(\tau)\chi_m^{\hat{\mathfrak{u}}(1)_{k+2}}(\tau) \tag{4.53}$$

We already know the form of $\hat{\mathfrak{u}}(1)_k$ and $\hat{\mathfrak{su}}(2)_k$ characters, expressing them in terms of the Theta functions and normalized string functions, the above equation becomes

$$\sum_{\substack{m=-k+1 \\ l+m=0 \mod 2}}^{k} c^{(k)}_{l,m}(\tau)\Theta^{(k)}_m(\tau)\frac{\Theta^{(2)}_s(\tau)}{\eta(\tau)} = \sum_{m=-k-1}^{k+2} \chi^l_{m,s}(\tau)\frac{\Theta^{(k+2)}_m(\tau)}{\eta(\tau)} \tag{4.54}$$

Using the following identity of the Theta functions

$$\Theta^{(k)}_l \Theta^{(k')}_{l'} = \sum_{j=1}^{k+k'} \Theta^{(k+k')}_{2k'j+l+l'}\Theta^{(kk'(k+k'))}_{2kk'j-lk'+l'k} \tag{4.55}$$

we can write down the coset characters $\chi^l_{m,s}$ explicitly

$$\chi^l_{m,s}(q) = \sum_{j=1}^{k+2} c^{(k)}_{l,m-4j-s}(\tau)\Theta^{(2k(k+2))}_{-2m+(4j+s)(k+2)} \tag{4.56}$$

with restriction that $l + m + s \in 2\mathbb{Z}$.

第 5 章　Exact solution of 2d $\mathcal{N} = (0,2)$ SCFT

This chapter reproduces the results in the third chapter of[11] which gives the exact solution of 2d $\mathcal{N} = (0,2)$ SQCD in low energy, other references are[20–23].

5.1　2d $\mathcal{N} = (0,2)$ SQCD

The 2d $\mathcal{N} = (0,2)$ supersymmetry admits three types of representations: the chiral multiplet Φ, the Fermi multiplet Ψ and the vector multiplet V:

$$
\begin{aligned}
\Phi &= \phi + \sqrt{2}\theta^+\psi_+ - i\theta^+\bar{\theta}^+\partial_+\phi \\
\Psi &= \psi_- - \sqrt{2}\theta^+ G - i\theta^+\bar{\theta}^+\partial_+\psi_- - \sqrt{2}\bar{\theta}^+ E \\
V &= v - 2i\theta^+\bar{\lambda}_- - 2i\bar{\theta}^+\lambda_- + 2\theta^+\bar{\theta}^+ D
\end{aligned}
\tag{5.1}
$$

Consider a 2d $\mathcal{N} = (0,2)$ SQCD with $\mathrm{U}(N_c)$ gauge group which has N_b fundamental chiral multiplets Φ with charge $+1$ and N_f fundamental Fermi multiplets Ψ with charge -1. The theory should be gauge anomaly free to avoid unphysical longitudinal polarizations. The chiral multiplets, Fermi multiplets and vector multiplet contributes $N_b/2, -N_f/2$ and $-N_c$ to the gauge anomaly, respectively. In order to cancel the $\mathrm{SU}(N_c)$ gauge anomaly, $2N_c - N_b + N_f$ chiral-Fermi pairs (P^a_α, Γ^s_a) should be added, where P^a are anti-fundamental with charge -1 while Γ_a is neutral. Additionally, the $\mathrm{U}(1)$ anomaly can be canceled by introducing two extra Fermi multiplets $\Omega_{1,2}$ in the determinant representation. We denote the theory by $\mathcal{T}_{N_1 N_2 N_3}$ where

$$
N_1 = 2N_c + N_f - N_b \qquad N_2 = N_b \qquad N_3 = N_f \tag{5.2}
$$

And it is convenient to define

$$
N = \frac{N_1 + N_2 + N_3}{2} \qquad n_i = N - N_i \tag{5.3}
$$

The information of fields is summarized below

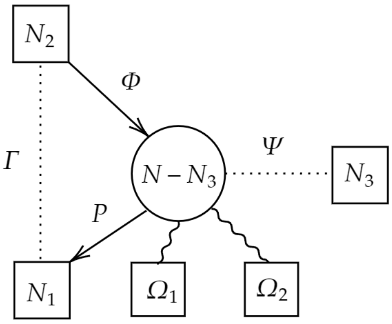

图 5-1　quiver diagram

	Φ	Ψ	P	Γ	$\Omega_{1,2}$
$\mathrm{U}(N - N_3)$	$\square$	$\overline{\square}$	$\overline{\square}$	1	det
$\mathrm{SU}(N_1)$	1	1	$\square$	$\overline{\square}$	1
$\mathrm{SU}(N_2)$	$\overline{\square}$	1	1	$\square$	1
$\mathrm{SU}(N_3)$	1	$\square$	1	1	1
SU(2)	1	1	1	1	$\square$

表 5-1: 2d $\mathcal{N} = (0, 2)$ SQCD

	Φ	Ψ	P	Γ	Ω_1	Ω_2
$\mathrm{U}(1)_1$	0	0	1	-1	$-N_1$	0
$\mathrm{U}(1)_2$	-1	0	0	1	$-N_2$	0
$\mathrm{U}(1)_3$	0	1	0	0	0	N_3
$\mathrm{U}(1)_4$	1	1	-1	0	n_3	$-n_3$

表 5-2: charges of U(1)

Table 5-2 shows that the classical $\mathrm{U}(1)_4$ symmetry is broken at the quantum level while the other three U(1) symmetries are preserved. Computing the trace anomaly $k_F = \mathrm{Tr}\,\gamma^3 J_F \cdot J_F$ of flavor symmetry F gives the level of representations

$$k_{\mathrm{SU}(N_i)} = -\frac{n_i}{2}, \quad k_{\mathrm{U}(1)_i} = -\frac{N N_i}{2} \qquad \text{for } i = 1, 2, 3 \tag{5.4}$$

Moreover, the left-moving and right-moving central charge can be computed by the R-symmetry anomaly

$$c_R = 3\,\mathrm{Tr}\,\gamma^3 R^2 = \frac{3 n_1 n_2 n_3}{N} \qquad c_L = c_R - \mathrm{Tr}\,\gamma^3 = \sum_{i=1}^{3} \left(\frac{n_i (N_i^2 - 1)}{N} + 1 \right) \tag{5.5}$$

5.1.1　2d $\mathcal{N} = (0, 2)$ Triality

From 5.4 and 5.5 we see that the central charges and trace anomalies are invariant under the cyclic permutation D with respect to N_1, N_2, N_3, say $N_1 \to N_2 \to N_3 \to N_1$. This means we get the same theories under D, which is called the triality[20]. This triality is illustrated in the figure 5-2.

We can further apply this triality to a general quiver diagram with multiple gauge nodes, now the triality acts on each individual gauge node ⓘ. If we denote the nodes

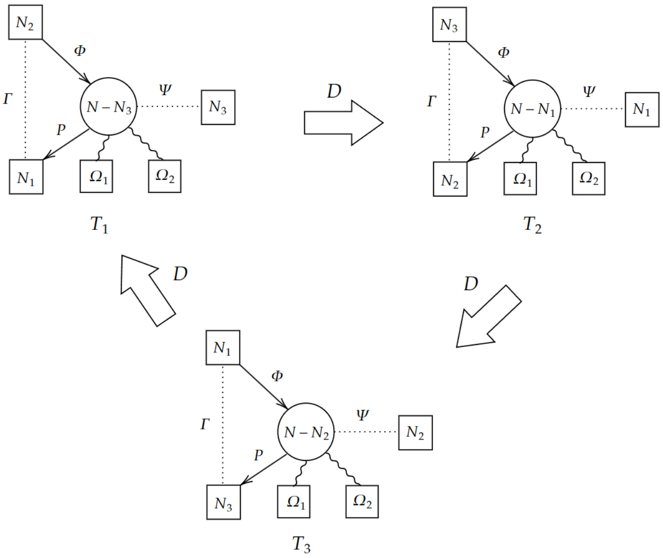

图 5-2 2d $\mathcal{N} = (0,2)$ triality. The theories are the same in all three frames T_1, T_2, T_3

as $X_i \equiv \{ⓙ : ⓙ \to ⓘ\}$, $Y_i \equiv \{ⓙ : ⓙ \leftarrow ⓘ\}$, $Z_i \equiv \{ⓙ : ⓙ - - ⓘ\}$, then the triality rules on an individual gauge node ⓘ are:

1. Draw the same type of arrows from ⓚ $\subset Y_i \cup Z_i$ to all ⓙ $\subset X_i$ that connects ⓚ to ⓘ.

2.Change the arrows that connect ⓚ so that X_i, Y_i, Z_i becomes Z_i', X_i', Y_i', respectively.

3.the rank of the new gauge group becomes $(N_i' = \sum_{ⓙ \in X_i} N_j) - N_i$.

4.cancel all the fermi-bose pairs.

We can understand the rules by the figure 5-3. We also constructed a pentagon relation in the figure 5-4 by the triality.

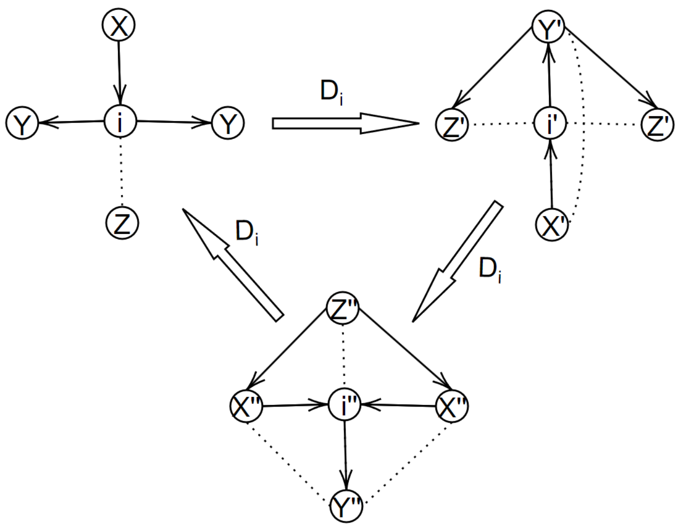

图 5-3 triality rules

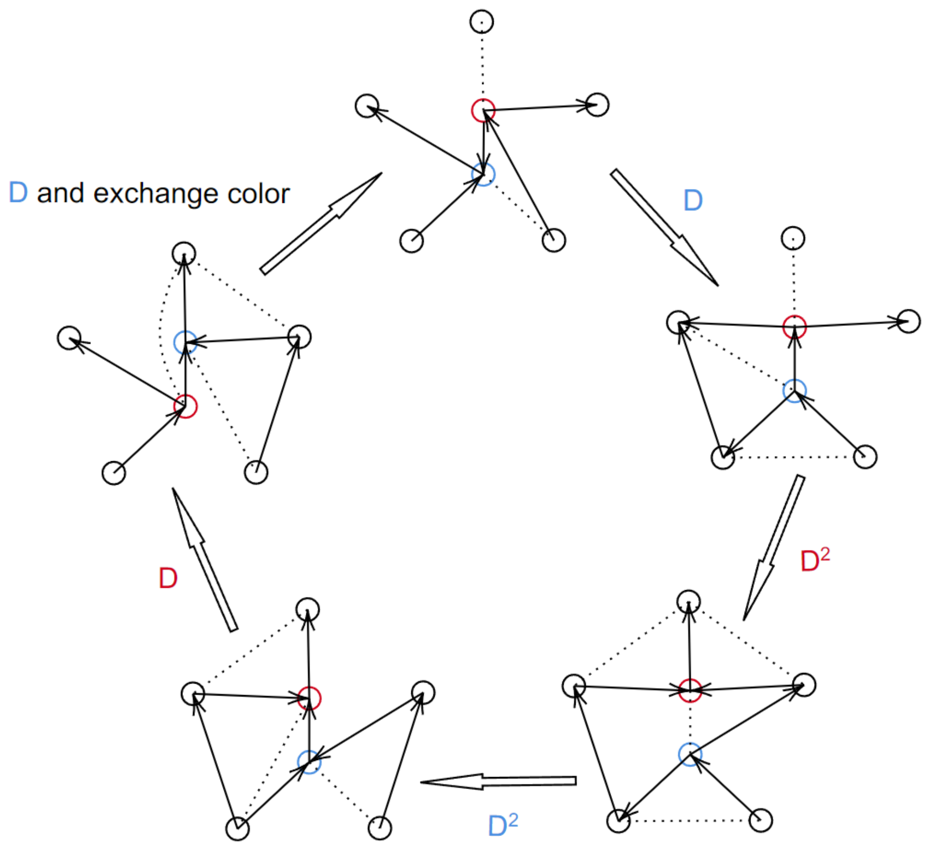

图 5-4 Pentagon relation

5.1.2 Elliptic genus

The flavored elliptic genus[24–25] is another powerful quantity to verify the triality, which contains all the information of anomalies and central charges of the theory. For this 2d $\mathcal{N} = (0,2)$ SQCD in frame T_1, the elliptic genus is defined as

$$\begin{aligned} I_{N_1,N_2,N_3} =& (q;q)^{2N_c} \oint \prod_{i=1}^{N_c} \frac{\mathrm{d}\zeta_\alpha}{2\pi i \zeta_\alpha} \prod_{\alpha \neq \beta} \theta(\zeta_\alpha/\zeta_\beta) \times \\ & \frac{\prod_{a,s} \theta(q^{\frac{1+R_\Gamma}{2}} x_s/y_a) \prod_{\alpha,i} \theta(q^{\frac{1}{2}} z_i/\zeta_\alpha) \prod_{\pm} \theta(q^{\frac{1}{2}} w^{\pm} \prod_\alpha \zeta_\alpha)}{\prod_{\alpha,s} \theta(q^{\frac{R_\Phi}{2}} \zeta_\alpha/x_s) \prod_{\alpha,a} \theta(q^{\frac{R_P}{2}} y_a/\zeta_\alpha)} \end{aligned} \tag{5.6}$$

where $\{y_a : a = 1, \cdots, N_1\}, \{x_s : s = 1, \cdots, N_2\}, \{z_i : i = 1, \cdots, N_3\}, \{\zeta_\alpha : \alpha = 1, \cdots, N_c\}, \{w^\pm\}$ are the fugacities of symmetry group $\mathrm{SU}(N_1)$, $\mathrm{SU}(N_2)$, $\mathrm{SU}(N_3)$, $\mathrm{SU}(N_c)$ and $\mathrm{SU}(2)_w$, respectively. The function $\theta(a,q)$ above is defined as

$$\theta(a,q) = \frac{(a;q)}{(q/a;q)} \qquad (a;q) = \prod_{i=0}^{\infty} (1 - aq^i) \tag{5.7}$$

The integral can be computed by taking residue at poles $\zeta_\alpha = q^{-\frac{R_\Phi}{2}} \tilde{x}_\alpha$, summing all the residues, the expression 5.6 becomes

$$\begin{aligned} I_{N_1,N_2,N_3} =& \sum_{\{\tilde{x}_\alpha \subset x_s\}} \prod_{\alpha \neq \beta} \theta(\tilde{x}_\alpha/\tilde{x}_\beta) \times \\ & \frac{\prod_{a,s} \theta(q^{\frac{1+R_\Gamma}{2}} x_s/y_a) \prod_{\alpha,i} \theta(q^{\frac{1+R_\Phi}{2}} z_i/\tilde{x}_\alpha) \prod_{\pm} \theta(q^{\frac{1-N_c R_\Phi}{2}} w^{\pm} \prod_\alpha \tilde{x}_\alpha)}{\prod_{\tilde{x}_\alpha \neq x_s} \theta(\tilde{x}_\alpha/x_s) \prod_{\alpha,a} \theta(q^{-\frac{1+R_\Gamma}{2}} y_a/\tilde{x}_\alpha)} \end{aligned} \tag{5.8}$$

Ignoring the R charges R_Φ, R_Γ and R_P, we computed the leading order of the q-expansion of the elliptic genus $I_{2,3,5}$, $I_{2,4,6}$, $I_{3,3,6}$ and their cyclic permutation with respect to N_1, N_2, N_3. The results below indeed satisfy the triality up to a minus sign.

$$
\begin{aligned}
I_{235} &= \frac{q^{2/3}(w-1)^2(x_1-y_1)(x_1x_2-y_1)(y_1-x_2)(x_1y_1-1)(x_2y_1-1)(x_1x_2y_1-1)}{wx_1^2x_2^2y_1^3} + O(q^{5/3}) \\
I_{352} &= \frac{q^{2/3}(w-1)^2(y_1-z_1)(y_2-z_1)(y_1y_2-z_1)(y_1z_1-1)(y_2z_1-1)(y_1y_2z_1-1)}{wy_1^2y_2^2z_1^3} + O(q^{5/3}) \\
I_{523} &= \frac{q^{2/3}(w-1)^2(x_1-z_1)(x_1z_1-1)(x_1z_2-1)(x_1-z_2)(x_1-z_1z_2)(z_1z_2x_1-1)}{wz_1^2z_2^2x_1^3} + O(q^{5/3}) \\
I_{246} &= \frac{q^{5/6}(w-1)^2(x_1-y_1)(x_2-y_1)(x_3-y_1)(x_1x_2x_3-y_1)}{wx_1^2x_2^2x_3^2y_1^4} \times \\
&\quad (x_1y_1-1)(x_2y_1-1)(x_3y_1-1)(x_1x_2x_3y_1-1) + O(q^{25/24}) \\
I_{462} &= \frac{q^{5/6}(w-1)^2(y_1-z_1)(y_2-z_1)(y_3-z_1)(y_1y_2y_3-z_1)}{wy_1^2y_2^2y_3^2z_1^4} \times \\
&\quad (y_1z_1-1)(y_2z_1-1)(y_3z_1-1)(y_1y_2y_3z_1-1) + O(q^{25/24}) \\
I_{624} &= \frac{q^{5/6}(w-1)^2(z_1-x_1)(z_2-x_1)(z_3-x_1)(z_1z_2z_3-x_1)}{wz_1^2z_2^2z_3^2x_1^4} \times \\
&\quad (z_1x_1-1)(z_2x_1-1)(z_3x_1-1)(z_1z_2z_3x_1-1) + O(q^{25/24}) \\
I_{336} &= \frac{iq^{11/12}(w-1)^2(x_1-y_1)(x_2-y_1)(x_1x_2y_1-1)(x_1-y_2)(x_2-y_2)(x_1x_2y_2-1)}{wx_1^3x_2^3y_1^3y_2^3} \times \\
&\quad (x_1x_2-y_1y_2)(x_1y_1y_2-1)(x_2y_1y_2-1) + O(q^{25/24}) \\
I_{363} &= \frac{iq^{11/12}(w-1)^2(y_1-z_1)(z_1-y_2)(y_1y_2z_1-1)(y_1-z_2)(y_2-z_2)(y_1y_2z_2-1)}{wy_1^3y_2^3z_1^3z_2^3} \times \\
&\quad (y_1y_2-z_1z_2)(y_1z_1z_2-1)(y_2z_1z_2-1) + O(q^{25/24}) \\
I_{633} &= \frac{iq^{11/12}(w-1)^2(z_1-x_1)(x_1-z_2)(z_1z_2x_1-1)(z_1-x_2)(z_2-x_2)(z_1z_2x_2-1)}{wz_1^3z_2^3x_1^3x_2^3} \times \\
&\quad (z_1z_2-x_1x_2)(z_1x_1x_2-1)(z_2x_1x_2-1) + O(q^{25/24})
\end{aligned}
\tag{5.9}
$$

5.2 2d $\mathcal{N} = (0,2)$ SCFT

The SQCD flows to a conformal fixed point in infra-red, leading to a 2d $\mathcal{N} = (0,2)$ SCFT. From 5.4, we see the trace anomalies are all negative, which means the affine Lie symmetries are all left-moving. Moreover, the double of the absolute value of a trace anomaly is the level of corresponding affine Lie symmetry. Hence the left-moving affine current algebra $\mathfrak{H}$ and its level-rank dual $\mathfrak{H}^t$ are

$$\mathfrak{H}=\bigoplus_{i=1}^{3}\left(\hat{\mathfrak{su}}(N_i)_{n_i}\oplus\hat{\mathfrak{u}}(1)_{NN_i}\right)\qquad \mathfrak{H}^t=\bigoplus_{i=1}^{3}\left(\hat{\mathfrak{su}}(n_i)_{N_i}\oplus\hat{\mathfrak{u}}(1)_{Nn_i}\right) \tag{5.10}$$

The Sugawara charge of $\mathfrak{H}$ is

$$c_{\mathfrak{H}}=\frac{k\dim\mathfrak{g}}{k+h^\vee_{\mathfrak{g}}}=\sum_{i=1}^{3}\left(\frac{n_i(N_i^2-1)}{n_i+N_i}+1\right) \tag{5.11}$$

which is equal to the left-moving central charge c_L of the $(0,2)$ SQCD. This means the left-moving Hilbert space of the SCFT is the same as the module of the $\mathfrak{H}$ WZW model, while the right-moving one forms a representation of the $\mathcal{N}=2$ superconformal algebra.

$$\mathscr{H}=\bigoplus_{\lambda}\mathscr{H}^{\lambda}_{\mathrm{LWZW}}\otimes\mathscr{H}^{\lambda}_{R} \tag{5.12}$$

So, the torus partition function of this SCFT has the form

$$Z(\tau)=\sum_{\lambda}\chi_\lambda(\tau,\xi)K_\lambda(\bar{\tau},\bar{\eta}) \tag{5.13}$$

where χ_λ and K_λ is the left-moving $\mathfrak{H}$ character and right-moving $\mathcal{N}=2$ character respectively. The modular invariance of the torus partition function requires K_λ to transform as the $\mathfrak{H}^t$ character under the modular $\mathcal{S}$ matrix. Moreover, K_λ should transform as a singlet under all affine symmetries. The candidate for K_λ appears to be the Kazama-Suzuki (KS) supercoset $[\mathfrak{G}]/[\mathfrak{H}^t]$, by comparing the central charge of the supercoset with c_R, we can get

$$[\mathfrak{H}^t]=\bigoplus_{i=1}^{3}[\hat{\mathfrak{u}}(n_i)]_N\qquad [\mathfrak{G}]=[\hat{\mathfrak{u}}(N)]_N=[\hat{\mathfrak{u}}(1)]_{N^2}\oplus[\hat{\mathfrak{su}}(N)]_N \tag{5.14}$$

It can be seen that $\mathfrak{G}/\mathfrak{H}^t$ is Kähler and the supersymmetry is indeed enhanced to $\mathcal{N}=2$. Moreover, the bosonic level of $\hat{\mathfrak{su}}(N)_N$ is $\hat{k}=N-h^\vee_{\mathfrak{su}(N)}=0$, it only admits trivial representation, hence the bosonic part of $\mathfrak{G}$ is simply $\hat{\mathfrak{u}}(1)_{N^2}$, and the fermionic part is $\hat{\mathfrak{so}}(D)_1$ where $D=\dim\mathfrak{G}/\mathfrak{H}^t$. So the characters $C^{\lambda^t}_{\Lambda,v}$ of this supercoset are defined by

$$\chi_{\Lambda}^{\hat{\mathfrak{u}}(1)_{N^2}}(\bar{\tau},0)\chi_{v}^{\hat{\mathfrak{so}}(D)_1}(\bar{\tau},\bar{\eta})=\sum_{\lambda^t}C^{\lambda^t}_{\Lambda,v}(\bar{\tau},\bar{\eta})\chi^{\mathfrak{H}^t}_{\lambda^t}(\bar{\tau},\frac{\sum_{\alpha\in\Delta_+}\alpha}{h^\vee_{\mathfrak{G}}}\bar{\eta}) \tag{5.15}$$

$C^{\lambda^t}_{\Lambda,v}$ are invariant under the modular group up to a modular anomaly factor when $\Lambda=\Lambda_0$ and $v=v_0$

$$\Lambda_0=\bigoplus_{r=1}^{N}rN\qquad v_0=\hat{\omega}_0\oplus\hat{\omega}_1 \tag{5.16}$$

In fact, the left of 5.15 are the characters of free fermions in the NS sector

$$\chi_{\Lambda_0}^{\hat{\mathfrak{u}}(1)_{N^2}}(\bar{\tau},0)=\frac{\theta_3(0|\bar{\tau})}{\eta(\bar{\tau})}\qquad \chi_{v_0}^{\hat{\mathfrak{so}}(D)_1}(\bar{\tau},\bar{\eta})=\left(\frac{\theta_3(\bar{\eta}|\bar{\tau})}{\eta(\bar{\tau})}\right)^{D/2} \tag{5.17}$$

Now $C^{\lambda^t}_{\Lambda_0,v_0}$ satisfy all the properties of K_λ, hence K_λ can be expressed as

$$K_\lambda(\bar{\tau},\bar{\eta})=\sum_{[\lambda^t]}L_{\lambda\lambda^t}C^{\lambda^t}_{\Lambda,v} \tag{5.18}$$

where $L_{\lambda\lambda^t}$ is the level-rank duality matrix and $[\lambda^t]$ denotes the class of λ^t. The Hilbert space of $\mathcal{N}=(0,2)$ SCFT becomes

$$\mathcal{H}=\bigoplus_{\lambda[\lambda^t]}L_{\lambda\lambda^t}\mathcal{H}^{\lambda}_{\text{LWZW}}\otimes\mathcal{H}^{\lambda^t}_{RKZ} \tag{5.19}$$

where $\mathcal{H}^{\lambda}_{\text{LWZW}}$ is the module of the left-moving $\mathfrak{H}$ WZW model and $\mathcal{H}^{\lambda^t}_{RKZ}$ is the module of the right-moving KS supercoset $[\mathfrak{G}]/[\mathfrak{H}^t]$.

5.3 $\mathcal{T}_{222}$

Now we study the theory $\mathcal{T}_{222}$ in low energy with all $N_i=2$, the left moving symmetry and its level-rank dual are

$$\mathfrak{H}=3\left(\hat{\mathfrak{su}}(2)_1\oplus\hat{\mathfrak{u}}(1)_6\right)\qquad \mathfrak{H}^t=3\left(\hat{\mathfrak{u}}(1)_3\right) \tag{5.20}$$

Denote the highest-weight representation of the i-th copy of $\hat{\mathfrak{su}}(2)_1\oplus\hat{\mathfrak{u}}(1)_6$ by $\lambda_i\in\{.,\square\}\otimes\{-2,-1,0,1,2,3\}$, and $\lambda_i^t\in\{-1,0,1\}$ with respect to $\hat{\mathfrak{u}}(1)_3$. The elements of the level-rank duality matrix $L_{\lambda\lambda^t}$ are listed in the table below, where the entries not shown are zero.

	$(.,-2)$	$(.,0)$	$(.,2)$	$(\square,1)$	$(\square,3)$	$(\square,-1)$
-1	1			1		
0		1			1	
1			1			1

表 5-3　$L_{\lambda\lambda^t}$

While the corresponding KS supercoset is

$$\frac{[\mathfrak{G}]}{[\mathfrak{H}^t]}=\frac{[\hat{\mathfrak{u}}(3)]_3}{3[\hat{\mathfrak{u}}(1)]_3}=\frac{[\hat{\mathfrak{su}}(3)]_3}{[\hat{\mathfrak{su}}(2)]_3\oplus[\hat{\mathfrak{u}}(1)]_3}\oplus\frac{[\hat{\mathfrak{su}}(2)]_3}{[\hat{\mathfrak{u}}(1)]_3}\oplus\frac{[\hat{\mathfrak{u}}(1)]_9}{[\hat{\mathfrak{u}}(1)]_3} \tag{5.21}$$

This is indeed the $\mathcal{N}=2$ minimal model with central charge $c=1$, it has three primaries whose conformal dimensions are $(h,Q)\in\{(0,0),(1/6,1/3),(1/6,-1/3)\}$ respectively, and their characters are given by

$$\chi^{\mathcal{N}=2}_{(h,Q)}(\bar{q},\bar{y})=\frac{\sum\limits_{n\in\mathbb{Z}}\bar{y}^{n+Q}\bar{q}^{\frac{3}{2}(n+Q)^2}}{\eta(\bar{\tau})} \tag{5.22}$$

Since dim $\hat{\mathfrak{u}}(3)/3\hat{\mathfrak{u}}(1)=6$, the branching rule 5.15 becomes

$$\frac{\theta_3(0|\bar{\tau})}{\eta(\bar{\tau})}\left(\frac{\theta_3(\bar{\eta}|\bar{\tau})}{\eta(\bar{\tau})}\right)^3=\sum_{\lambda^t}C^{\lambda_1^t,\lambda_2^t,\lambda_3^t}_{\Lambda_0,\upsilon_0}(\bar{q},\bar{y})\chi^{\hat{\mathfrak{u}}(1)_3}_{\lambda_1^t}(\bar{q},\bar{y}^{2/3})\chi^{\hat{\mathfrak{u}}(1)_3}_{\lambda_2^t}(\bar{q},1)\chi^{\hat{\mathfrak{u}}(1)_3}_{\lambda_3^t}(\bar{q},\bar{y}^{-2/3}) \tag{5.23}$$

Solving this equation, the only non-zero characters are

$(\lambda_1^t,\lambda_2^t,\lambda_3^t)$	$C^{\lambda_1^t,\lambda_2^t,\lambda_3^t}_{\Lambda_0,\upsilon_0}$
$(0,0,0),(1,1,1),(-1,-1,-1)$	$\chi^{\mathcal{N}=2}_{(0,0)}$
$(1,0,-1),(-1,1,0),(0,-1,1)$	$\chi^{\mathcal{N}=2}_{(1/6,1/3)}$
$(-1,0,1),(1,-1,0),(0,1,-1)$	$\chi^{\mathcal{N}=2}_{(1/6,-1/3)}$

表 5-4 $\frac{[\hat{\mathfrak{u}}(3)]_3}{3[\hat{\mathfrak{u}}(1)]_3}$ characters

Recalling the Hilbert spaces of $\mathcal{N}=(0,2)$ SCFT 5.19, Now we can write down the torus partition function of $\mathscr{T}_{222}$ in low energy explicitly

$$\begin{aligned}Z_{\mathscr{T}_{222}}&=\sum_{\lambda[\lambda^t]}L_{\lambda_1\lambda_1^t}L_{\lambda_2\lambda_2^t}L_{\lambda_3\lambda_3^t}\chi^{\mathfrak{H}}_{\lambda_1,\lambda_2,\lambda_3}(\tau,\xi_1,\xi_2,\xi_3)C^{\lambda_1^t,\lambda_2^t,\lambda_3^t}_{\Lambda_0,\upsilon_0}(\bar{\tau},\bar{\eta})\\&=\chi^{\mathcal{N}=2}_{(0,0)}(\bar{\tau},\bar{\eta})(\Xi_{0,0,0}(\tau)+\Xi_{1,1,1}(\tau)+\Xi_{-1,-1,-1}(\tau))\\&+\chi^{\mathcal{N}=2}_{(1/6,1/3)}(\bar{\tau},\bar{\eta})(\Xi_{1,0,-1}(\tau)+\Xi_{0,-1,1}(\tau)+\Xi_{-1,1,0}(\tau))\\&+\chi^{\mathcal{N}=2}_{(1/6,-1/3)}(\bar{\tau},\bar{\eta})(\Xi_{-1,0,1}(\tau)+\Xi_{0,1,-1}(\tau)+\Xi_{1,-1,0}(\tau))\end{aligned} \tag{5.24}$$

where $\Xi_{a,b,c}(\tau)\equiv\Xi_{a,b,c}(\tau,\xi_1,\xi_2,\xi_3)$ is

$$\begin{aligned}\Xi_{a,b,c}(\tau,\xi_1,\xi_2,\xi_3)&=\Xi_a(\tau,\xi_1)\Xi_b(\tau,\xi_2)\Xi_c(\tau,\xi_3)\\\Xi_{-1}(\tau,\xi)&=\chi^{\hat{\mathfrak{su}}(2)_1\oplus\hat{\mathfrak{u}}(1)_6}_{(\cdot,-2)}(\tau,\xi)+\chi^{\hat{\mathfrak{su}}(2)_1\oplus\hat{\mathfrak{u}}(1)_6}_{(\square,1)}(\tau,\xi)\\\Xi_0(\tau,\xi)&=\chi^{\hat{\mathfrak{su}}(2)_1\oplus\hat{\mathfrak{u}}(1)_6}_{(\cdot,0)}(\tau,\xi)+\chi^{\hat{\mathfrak{su}}(2)_1\oplus\hat{\mathfrak{u}}(1)_6}_{(\square,3)}(\tau,\xi)\\\Xi_1(\tau,\xi)&=\chi^{\hat{\mathfrak{su}}(2)_1\oplus\hat{\mathfrak{u}}(1)_6}_{(\cdot,2)}(\tau,\xi)+\chi^{\hat{\mathfrak{su}}(2)_1\oplus\hat{\mathfrak{u}}(1)_6}_{(\square,-1)}(\tau,\xi)\end{aligned} \tag{5.25}$$

It can be written more compactly in terms of the affine E_6 characters

$$Z_{\mathscr{T}_{222}}=\chi^{\mathcal{N}=2}_{(0,0)}(\bar{\tau},\bar{\eta})\chi^{(E_6)_1}_{\bullet}(\tau,\xi_i)+\chi^{\mathcal{N}=2}_{(\frac{1}{3},\frac{1}{6})}(\bar{\tau},\bar{\eta})\chi^{(E_6)_1}_{\square}(\tau,\xi_i)+\chi^{\mathcal{N}=2}_{(\frac{1}{3},-\frac{1}{6})}(\bar{\tau},\bar{\eta})\chi^{(E_6)_1}_{\overline{\square}}(\tau,\xi_i) \tag{5.26}$$

We can explain the E_6 symmetry of this partition function by the triality. Consider $\mathcal{T}_{222}$, the gauge group becomes U(1) in all three frames. In frame T_1, since the fundamental representation of U(1) is the same as the determinant representation, the flavor symmetry SU(2) of $\Omega_{1,2}$ combines with SU($N_3 = 2$) of Ψ to form SU(4). The other two frames also manifest a SU(4) symmetry, and these three SU(4) symmetries combine together to form an affine E_6 symmetry.

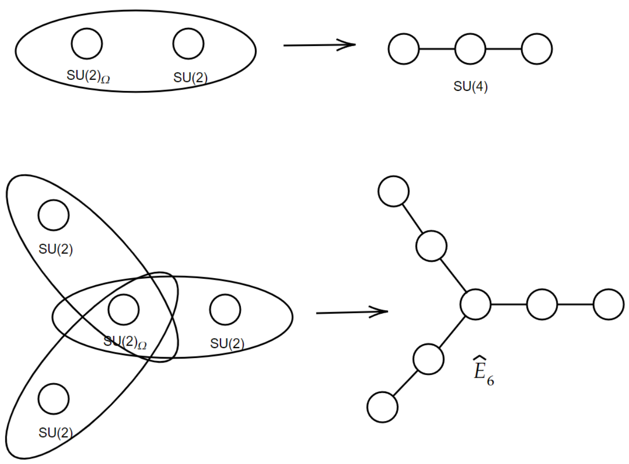

图 5-5 $\hat{E}_6$ symmetry enhancement

附录 A　Theta function

Defining $q = \exp(2\pi i\tau)$ and $y = \exp(2\pi i z)$, Jacobi's theta functions are

$$\begin{aligned}
\theta_1(z|\tau) &= -i \sum_{r\in\mathbb{Z}+1/2} (-1)^{r-1/2} y^r q^{r^2/2} \\
\theta_2(z|\tau) &= \sum_{r\in\mathbb{Z}+1/2} y^r q^{r^2/2} \\
\theta_3(z|\tau) &= \sum_{n\in\mathbb{Z}} y^n q^{n^2/2} \\
\theta_4(z|\tau) &= \sum_{n\in\mathbb{Z}} (-1)^n y^n q^{n^2/2}
\end{aligned} \tag{A.1}$$

Using Jacobi's triple product formula

$$\prod_{n=1}^{\infty}(1-q^n)(1+q^{n-1/2}t)(1+q^{n-1/2}/t) = \sum_{n\in\mathbb{Z}} q^{n^2/2} t^n \tag{A.2}$$

The theta functions become

$$\begin{aligned}
\theta_1(z|\tau) &= -iy^{1/2}q^{1/8}\prod_{n=1}^{\infty}(1-q^n)\prod_{n=0}^{\infty}(1-yq^{n+1})(1-y^{-1}q^n) \\
\theta_2(z|\tau) &= y^{1/2}q^{1/8}\prod_{n=1}^{\infty}(1-q^n)\prod_{n=0}^{\infty}(1+yq^{n+1})(1+y^{-1}q^n) \\
\theta_3(z|\tau) &= \prod_{n=1}^{\infty}(1-q^n)\prod_{r\in\mathbb{Z}_+ +1/2}^{\infty}(1+yq^r)(1+y^{-1}q^r) \\
\theta_4(z|\tau) &= \prod_{n=1}^{\infty}(1-q^n)\prod_{r\in\mathbb{Z}_+ +1/2}^{\infty}(1-yq^r)(1-y^{-1}q^r)
\end{aligned} \tag{A.3}$$

They relate to the Dedekind function by

$$\eta^3(\tau) = \frac{1}{2}\theta_2(\tau)\theta_3(\tau)\theta_4(\tau) \tag{A.4}$$

and the modular properties are

$$\begin{aligned}
&\theta_2(\tau+1) = e^{i\pi/4}\theta_2(\tau) && \theta_2(-1/\tau) = \sqrt{-i\tau}\theta_4(\tau) \\
&\theta_3(\tau+1) = \theta_4(\tau) && \theta_3(-1/\tau) = \sqrt{-i\tau}\theta_3(\tau) \\
&\theta_4(\tau+1) = \theta_3(\tau) && \theta_4(-1/\tau) = \sqrt{-i\tau}\theta_2(\tau) \\
&\eta(\tau+1) = e^{i\pi/12}\eta(\tau) && \eta(-1/\tau) = \sqrt{-i\tau}\eta(\tau)
\end{aligned} \tag{A.5}$$

附录 B　Affine Lie algebra

B.1　Algebra structure

Affine Lie algebra, also known as the Kac-Moody algebra, can be obtained from finite Lie algebra $\mathfrak{g}$ by the following few steps. First,constructing algebra $\tilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$,the elements in $\tilde{\mathfrak{g}}$ are also Laurent polynomials of t. Then extend $\tilde{\mathfrak{g}}$ with central charge $\hat{k}$ which commutes with all the generators of $\mathfrak{g}$. In additional,it is necessary to add a generator $L_0 = -\frac{\mathrm{d}}{\mathrm{d}t}$ which tracks the power of t in the generators. Now the structure of affine Lie algebra becomes:

$$\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\hat{k} \oplus \mathbb{C}L_0 \tag{B.1}$$

Denote the generators of $\hat{\mathfrak{g}}$ that associated with t by $J_n^a \equiv J^a \otimes t^n$ where $\{J^a\}$ are the generators of $\mathfrak{g}$. If we require $\{J_n^a\}$ to be orthonormal with respect to the Killing form, that is $K(J_n^a, J_m^b) = \delta^{ab}\delta_{n+m,0}$, then there is actually only one possible central extension which satisfies the Jacobi identities, the correspondent commutation relations are:

$$[J_n^a, J_m^b] = \sum_c i f^{ab}{}_c J_{n+m}^c + \hat{k} n \delta^{ab} \delta_{n+m,0} \tag{B.2}$$

We can also write down the Killing form about $\hat{k}$ and L_0, and it is convenient to redefine $L_0 \to L_0 + a\hat{k}$ to make $K(L_0, L_0) = 0$.

$$\begin{aligned} &K(J_n^a, J_m^b) = \delta^{ab}\delta_{n+m,0} \qquad && K(J_n^a, \hat{k}) = 0 \\ &K(\hat{k}, \hat{k}) = 0 && K(J_n^a, L_0) = 0 \\ &K(L_0, \hat{k}) = -1 && K(L_0, L_0) = 0 \end{aligned} \tag{B.3}$$

B.2　Affine weights and roots

Now the Cartan generators $\{H^i\}$ of the finite algebra $\mathfrak{g}$ together with $\hat{k}$ and L_0 forms the maximal Cartan subalgebra of $\hat{\mathfrak{g}}$, while the other generators E_n^α are all ladder operators. The eigenvalues given by affine Cartan generators $\{H^1, \cdots, H^r; \hat{k}; -L_0\}$ acting on a state forms the affine weight

$$\hat{\lambda} = (\lambda; k_\lambda; n_\lambda) \tag{B.4}$$

where k_λ and n_λ are called level and grade respectively. The Killing form induces the scalar product rule of affine weights

$$(\hat{\lambda}, \hat{\mu}) = (\lambda, \mu) + k_\lambda n_\mu + k_\mu n_\lambda \tag{B.5}$$

In the adjoint representation, weights are called roots. Since $\hat{k}$ commutes with all generators, the level of a affine root $\hat{\alpha}$ is always zero

$$\hat{\alpha} = (\alpha; 0; n) \tag{B.6}$$

The scalar product of affine roots is the same as the finite roots product.

$$(\hat{\alpha}, \hat{\beta}) = (\alpha, \beta) \tag{B.7}$$

We introduce the imaginary root δ, and it is obvious that $(\delta, \delta) = 0$

$$\delta = (0; 0; 1) \tag{B.8}$$

With δ, the full affine roots set can be expressed as

$$\hat{\Delta} = \{\alpha + n\delta | n \in \mathbb{Z}, \alpha \in \Delta\} \cup \{n\delta | n \in \mathbb{Z}, n \neq 0\} \tag{B.9}$$

where Δ is the set of finite roots. We can further determine the set of positive roots $\hat{\Delta}_+$ and corresponding simple roots.

$$\hat{\Delta}_+ = \{\alpha + n\delta | n > 0, \alpha \in \Delta\} \cup \{\alpha | \alpha \in \Delta_+\} \tag{B.10}$$

We can check that any element in $\hat{\Delta}_+$ can be written as a sum of the affine simple roots $\{\alpha_i\}, i = 0, 1, \cdots, r$, where α_i are just the finite roots for $i \neq 0$, and

$$\alpha_0 \equiv (-\theta; 0; 1) \tag{B.11}$$

where θ is the highest root of $\mathfrak{g}$, and its length is normalized to $\sqrt{2}$. The affine simple coroots are given by

$$\hat{\alpha}^\vee = \frac{2}{|\hat{\alpha}|^2}(\alpha; 0; n) = (\alpha^\vee; 0; \frac{2}{|\alpha|^2} n) \tag{B.12}$$

The scalar product of simple roots and coroots gives the extended Cartan matrix which encodes all the information about the affine Lie algebra $\hat{\mathfrak{g}}$.

$$\hat{A}_{ij} = (\alpha_i, \alpha_j^\vee) \qquad 0 \leq i, j \leq r \tag{B.13}$$

We can continue to define the affine fundamental weights which duals to the affine coroots

$$\hat{\omega}_0 = (0; 1; 0), \qquad \hat{\omega}_i = (\omega_i; a_i^\vee; 0) \quad \text{for} \quad i \neq 0 \tag{B.14}$$

where $a_i^\vee$ are the comarks defined by the expansion of the highest root $\theta = \sum_{i=1}^r a_i^\vee \alpha_i^\vee$. Now we can expand the affine weight with the fundamental weights basis, where the expand coefficients λ_i are called the Dynkin labels.

$$\hat{\lambda} = \sum_{i=0}^{r} \lambda_i \hat{\omega}_i + l\delta \qquad l \in \mathbb{R} \tag{B.15}$$

We ignore the imaginary root and denote it as $\hat{\lambda} = [\lambda_0, \lambda_1, \cdots, \lambda_r]$. Specially, the affine Weyl vector $\hat{\rho}$ is defined as a weight with all Dynkin labels equal to 1

$$\hat{\rho} = [1, 1, \cdots, 1] \tag{B.16}$$

B.3 Affine Weyl group

The affine Weyl reflection $s_{\hat{\alpha}}$ of an affine weight $\hat{\lambda}$ with respect to a real affine root $\hat{\alpha}$ is defined as

$$s_{\hat{\alpha}}\hat{\lambda} = \hat{\lambda} - (\hat{\lambda}, \hat{\alpha}^\vee)\hat{\alpha} \tag{B.17}$$

One important property of the affine Weyl reflection is that it preserve the scalar product of affine weights. All Weyl reflections generate the affine Weyl group, denoted as $\hat{W}$. In fact, the affine Weyl reflection $s_{\hat{\alpha}}$ with $\hat{\alpha} = \alpha + m\delta$ can be decomposed into the product of finite Weyl reflection s_α and translation $t_{\alpha^\vee}$

$$s_{\hat{\alpha}}\hat{\lambda} = s_\alpha (t_{\alpha^\vee})^m \hat{\lambda} \tag{B.18}$$

where the translation $t_{\alpha^\vee}$ is defined as

$$t_{\alpha^\vee} \equiv s_{-\alpha+\delta} s_\alpha = s_\alpha s_{\alpha+\delta} \qquad (t_{\alpha^\vee})(t_{\beta^\vee}) = t_{\alpha^\vee + \beta^\vee} \tag{B.19}$$

All translations generate the coroot lattice $Q^\vee = \bigoplus_{i=1}^r \mathbb{Z}_i \alpha_i^\vee$. Since the finite Weyl group W and $Q^\vee$ have no common elements except the identity, the affine Weyl group $\hat{W}$ is indeed a semidirect product of W and $Q^\vee$.

$\hat{W}$ divides affine weights into infinite affine Weyl chambers defined as

$$\hat{C}_w = \{\hat{\lambda} | (w\hat{\lambda}, \alpha_i) \geq 0, i = 0, 1, \cdots, r\} \qquad w \in \hat{W} \tag{B.20}$$

The chamber with respect to $w = 1$ is called the fundamental chamber, in which the Dynkin labels of weights are all non-negative integers.

B.4 Highest-weight representation

From the affine weight $\hat{\lambda}$ in the fundamental chamber, we can construct the highest-weight representations. In Cartan-Weyl basis, the highest state $\left|\hat{\lambda}\right\rangle$ should annihilated by all ladder operators for positive roots

$$E_0^{\alpha}\left|\hat{\lambda}\right\rangle = E_n^{\pm\alpha}\left|\hat{\lambda}\right\rangle = H_n^i\left|\hat{\lambda}\right\rangle \qquad \text{for} \quad n, \alpha > 0 \tag{B.21}$$

As for the Cartan generators, it is more convenient to express the eigenvalues in the Chevalley basis defined as $h^i \equiv 2\alpha_i \cdot H/|\alpha_i|^2$, the eigenvalues of h^i are just the Dynkin labels

$$h^i\left|\hat{\lambda}\right\rangle = \lambda_i\left|\hat{\lambda}\right\rangle, \qquad \hat{k}\left|\hat{\lambda}\right\rangle = k\left|\hat{\lambda}\right\rangle, \qquad L_0\left|\hat{\lambda}\right\rangle = 0 \tag{B.22}$$

It should be noted that the eigenvalue of L_0 can be arbitrary by redefinition, we set it zero here just for convenience.

In order to determine the condition that a highest weight should satisfy, we project the representation $\hat{\lambda}$ onto the $\mathfrak{su}(2)$ algebra associated with a simple root $\hat{\alpha}$. We only concern about so-called integrable representations, which can be written in a direct sum of finite dimensional weight spaces. Hence the projection $\mathfrak{su}(2)$ representation is also finite dimensional, say $2j+1$. Starting from any state $\left|\hat{\lambda}'\right\rangle$ with spin $j' = (\hat{\lambda}, \hat{\alpha}^\vee)/2$, the highest spin state $|j\rangle$ can be reached by acting finite ladder operators E^α, as well as the lowest spin state $|-j\rangle$

$$j = j' + p, \qquad -j = j' - q \tag{B.23}$$

where p, q are positive integers. From the above equation, we conclude that for any root $\hat{\alpha}$ and any weight $\hat{\lambda}'$ in an integrable representation, $(\hat{\lambda}, \hat{\alpha}^\vee) = q - p$ is always an integer. If we chose the weight $\hat{\lambda}'$ to be the highest weight $\hat{\lambda}$ itself, then $p = 0$ since $\left|\hat{\lambda}\right\rangle$ is annihilated by all positive ladder operators. This gives the condition of an integrable representation $\hat{\lambda}$

$$\lambda_i \in \mathbb{Z}_{\geq 0}, \qquad i = 0, 1, \cdots, r \tag{B.24}$$

This shows $\hat{\lambda}$ locates in the fundamental affine Weyl chamber. Particularly, the zeroth Dynkin label $\lambda_0 = k - (\lambda, \theta) \in \mathbb{Z}_{\geq 0}$ indicates that the level k must be a positive integer and bounded from below by λ, θ

$$k \in \mathbb{Z}_+, \qquad k \geq (\lambda, \theta) \tag{B.25}$$

From now on, we denote the algebra $\hat{\mathfrak{g}}$ at level k as $\hat{\mathfrak{g}}_k$.

B.5 Characters

The definition of affine character of an integrable highest-weight is

$$\mathrm{ch}_{\hat\lambda} = \sum_{\hat\lambda' \in \Omega_{\hat\lambda}} \mathrm{mult}_{\hat\lambda}(\hat\lambda') e^{\hat\lambda'} \tag{B.26}$$

It can also be written as the Weyl-Kac character formula

$$\mathrm{ch}_{\hat\lambda} = \frac{\sum_{w\in\hat W} \epsilon(w) e^{w(\hat\lambda+\hat\rho)}}{\sum_{w\in\hat W} \epsilon(w) e^{w\hat\rho}} = e^{m_{\hat\lambda}\delta} \frac{\sum_{w\in W} \epsilon(w)\Theta_{w(\hat\lambda+\hat\rho)}}{\sum_{w\in W} \epsilon(w)\Theta_{w\hat\rho}} \tag{B.27}$$

where $m_{\hat\lambda}$ is called the modular anomaly

$$m_{\hat\lambda} = \frac{|\hat\lambda+\hat\rho|^2}{2(k+h^\vee_{\mathfrak g})} - \frac{|\hat\rho|}{2h^\vee_{\mathfrak g}} = \frac{|\lambda+\rho|^2}{2(k+h^\vee_{\mathfrak g})} - \frac{|\rho|^2}{2h^\vee_{\mathfrak g}} \tag{B.28}$$

and Θ is the generalized theta function

$$\begin{aligned} \Theta_{\hat\lambda} &= e^{-\frac{1}{2k}|\hat\lambda|^2\delta} \sum_{\alpha^\vee\in Q^\vee} e^{(t_{\alpha^\vee})\hat\lambda} = e^{k\hat\omega_0} \sum_{\alpha^\vee \in Q^\vee+\lambda/k} e^{k[\alpha^\vee - \frac{1}{2}|\alpha^\vee|^2\delta]} \\ \sum_{w\in\hat W} \epsilon(w) e^{w\hat\lambda} &= \sum_{w\in W} \epsilon(w) e^{\frac{1}{2k}|\lambda|^2\delta}\Theta_{w\hat\lambda} \end{aligned} \tag{B.29}$$

We denote $\mathrm{ch}_{\hat\lambda}(\hat\xi) \equiv \mathrm{ch}_{\hat\lambda}(\zeta;\tau;t)$ the characters evaluated at the point $\hat\xi = -2\pi i(\zeta;\tau;t)$, the value of the theta function at this point becomes

$$\Theta_{\hat\lambda}(\hat\xi) = e^{-2\pi i k t} \sum_{\alpha^\vee\in Q^\vee} e^{-\pi i[2k(\alpha^\vee,\zeta)+2(\lambda,\zeta)-\tau k|\alpha^\vee+\lambda/k|^2]} \tag{B.30}$$

The normalized character is introduced by dividing the modular anomaly factor, the symbol χ is understood the normalized character in the following texts. Specializing at $\hat\xi = -2\pi i(\zeta;\tau;0)$, where $\zeta = \sum_{i=1}^r z_i\alpha_i^\vee$, the character becomes

$$\chi_{\hat\lambda}(z;\tau) = \left[e^{-m_{\hat\lambda}\delta}\mathrm{ch}_{\hat\lambda}\right](\zeta;\tau;0) = q^{m_{\hat\lambda}}\,\mathrm{Tr}_{\hat\lambda}\left[q^{L_0} e^{-2\pi i\sum_j z_j h^j}\right] \tag{B.31}$$

We now consider the simplest example $\hat{\mathfrak g}_k = \hat{\mathfrak u}(1)_k$, the algebra is abelian and coroot lattice $Q^\vee = \mathbb Z$. Using the Macdonald-Weyl denominator identity

$$\sum_{w\in\hat W} \epsilon(w) e^{w\hat\rho} = e^{\hat\rho} \prod_{\hat\alpha>0} (1-e^{-\hat\alpha})^{\mathrm{mult}(\hat\alpha)} \tag{B.32}$$

for the $\hat{\mathfrak u}(1)$ case, that is $\hat\alpha = n\delta$ and $\hat\rho = \hat\omega_0$, the character becomes

$$\chi_\lambda^{\hat{\mathfrak u}(1)_k}(z;\tau) = \frac{1}{\eta(q)} \sum_{n\in\mathbb Z} e^{\pi i\tau k(n+\lambda/k)^2 - 2\pi i\zeta(kn+\lambda)} \tag{B.33}$$

Besides the Weyl-Kac character formula, there is another expression of the affine Lie characters by introducing the string functions $\sigma_{\hat{\mu}}^{(\hat{\lambda})}$

$$\mathrm{ch}_{\hat{\lambda}} = \sum_{\hat{\mu} \in \Omega_{\hat{\lambda}}^{\max}} \sigma_{\hat{\mu}}^{(\hat{\lambda})}(e^{-\delta}) e^{\hat{\mu}} \tag{B.34}$$

where $\Omega_{\hat{\lambda}}$ is the set of weights in representation $\hat{\lambda}$ and $\Omega_{\hat{\lambda}}^{\max}$ consists of weights $\hat{\mu}$ such that $\hat{\mu} + \delta \notin \Omega_{\hat{\lambda}}$, with the string functions defined by

$$\sigma_{\hat{\mu}}^{(\hat{\lambda})}(q) = \sum_{n=0}^{\infty} \mathrm{mult}_{\hat{\lambda}}(\hat{\mu} - n\delta) q^n \tag{B.35}$$

We can easily check the validity of this expression since any weight subtracting an imaginary root $n\delta$ is still a weight, with respect to acting a ladder operator H_{-n}^i. Since the Weyl reflection preserve the multiplicities of weights, we can get

$$\sigma_{w\hat{\mu}}^{(\hat{\lambda})}(q) = \sigma_{\hat{\mu}}^{(\hat{\lambda})}(q) \qquad \text{for} \quad w \in \hat{W} \tag{B.36}$$

Modular property

If we define the transformation of weight under the modular group $\tau \to (a\tau + b)/(c\tau + d)$ as

$$(\zeta; \tau; t) \to \left(\frac{\zeta}{c\tau + t}; \frac{a\tau + b}{c\tau + d}; t + \frac{c|\zeta|^2}{2(c\tau + d)} \right) \tag{B.37}$$

Then the affine characters also transform modular covariantly as

$$\begin{aligned} \chi_{\hat{\lambda}}(\zeta; \tau + 1; t) &= \sum_{\hat{\mu} \in P_+^k} \mathcal{T}_{\hat{\lambda}\hat{\mu}} \chi_{\hat{\mu}}(\zeta; \tau; t) \\ \chi_{\hat{\lambda}}(\zeta/\tau; -1/\tau + 1; t + |\zeta|^2/2\tau) &= \sum_{\hat{\mu} \in P_+^k} \mathcal{S}_{\hat{\lambda}\hat{\mu}} \chi_{\hat{\mu}}(\zeta; \tau; t) \end{aligned} \tag{B.38}$$

The explicit form of modular matrices $\mathcal{T}$ and $\mathcal{S}$ are

$$\begin{aligned} \mathcal{T}_{\hat{\lambda}\hat{\mu}} &= \delta_{\hat{\lambda}\hat{\mu}} e^{2\pi i m_{\hat{\lambda}}} \\ \mathcal{S}_{\hat{\lambda}\hat{\mu}} &= \frac{i^{|\Delta_+|} \sum_{w \in W} \epsilon(w) e^{-2\pi i (w(\lambda+\rho), \mu+\rho)/(k+h_{\mathfrak{g}}^\vee)}}{|P/Q^\vee|^{1/2} (k + h_{\mathfrak{g}}^\vee)^{\mathrm{rank}\, \mathfrak{g}/2}} \end{aligned} \tag{B.39}$$

附录 C　Supersymmetry algebra

C.1　Affine Lie superalgebra[1]

The affine Lie superalgebra can be obtained by extending the affine current $J^a(z)$ to super current $\mathcal{J}^a(Z) = j^a(z) + \theta J^a(z)$, the OPEs of the two components are given by

$$\begin{aligned} j^a(z)j^b(w) &\sim \frac{k\delta_{ab}/2}{z-w} \\ J^a(z)j^b(w) &\sim j^a(z)J^b(w) \sim \frac{if_{abc}j^c(w)}{z-w} \\ J^a(z)J^b(w) &\sim \frac{k\delta_{ab}/2}{(z-w)^2} + \frac{if_{abc}J^c(w)}{z-w} \end{aligned} \tag{C.1}$$

C.2　2d superconformal algebra

C.2.1　$\mathcal{N} = 2$[2–5]

The 2d $\mathcal{N} = 2$ superspace is described by two complex coordinates $z, \bar{z}$ and two Weyl spinors $\theta, \bar{\theta}$. In CFT, the holomorphic and anti-holomorphic parts are separable, so we can only focus on the holomorphic superspace described by $Z \equiv (z, \theta, \bar{\theta})$. The superfield in Z has the following form

$$\Phi(Z) = \phi(z) + \theta\bar{\psi}(z) + \bar{\theta}\psi(z) + \frac{i}{2}\theta\bar{\theta}g(z) \tag{C.2}$$

Suppose Φ is a primary field, which means Φ transforms as

$$\begin{aligned} \delta_E\Phi(W) = [E(Z)J(Z), \Phi(W)] &= \oint_{C(W)} \mathrm{d}Z\ E(Z)J(Z)\Phi(W) \\ &= \Delta\partial E(W)\Phi(W) + E(W)\partial\Phi(W) \\ &\quad + DE(W)\bar{D}\Phi(W) + \bar{D}E(W)D\Phi(W) - t[D, \bar{D}]E(W)\Phi(W) \end{aligned} \tag{C.3}$$

where $J(Z)$ is the supercurrent, $E(z) = \epsilon(z) - i\sqrt{2}\theta\bar{\beta}(z) + i\sqrt{2}\bar{\theta}\beta(z) - \frac{1}{2}\theta\bar{\theta}t(z)$ and $\mathrm{d}Z = \dfrac{\mathrm{d}z\ \mathrm{d}\theta\ \mathrm{d}\bar{\theta}}{2\pi i}$. The transformation forms a closed algebra given by

$$[\delta_E, \delta_F] = \delta_G \qquad G = F\partial E - E\partial F + DF\bar{D}E + \bar{D}FDE \tag{C.4}$$

This yields the current algebra

$$[J_E, J_F] = J_G + \frac{c}{6}\oint \mathrm{d}Z\ F(Z)[D,\bar{D}]\partial E(Z) \tag{C.5}$$

where the coefficient 1/6 is chosen to make c the central charge of the Virasoro algebra. Writing $J(Z)$ in terms of two bosonic fields $T(z)$, $J(z)$ and two fermionic fields $G^{\pm}(z)$

$$J(Z) = -J(z) + \frac{i}{\sqrt{2}}\theta G^{-}(z) + \frac{i}{\sqrt{2}}\bar{\theta}G^{+}(z) + \bar{\theta}\theta T(z) \tag{C.6}$$

where T is the energy-momentum tensor and J, $G^{\pm}$ are Virasoro primaries of dimension $1, 3/2, 3/2$ respectively. Carrying out the mode expansion

$$T(z) = \sum_{n\in\mathbb{Z}} \frac{L_n}{z^{n+2}}, \quad J(z) = \sum_{n\in\mathbb{Z}} \frac{J_n}{z^{n+1}}, \quad G^{\pm}(z) = \sum_{r\in\mathbb{Z}+\nu} \frac{G_r^{\pm}}{z^{r+3/2}} \tag{C.7}$$

where $\nu = 0$ and $\nu = 1/2$ for Ramond sector and Neveu-Schwarz sector respectively. The current algebra C.5 is actually equivalent to the 2d $\mathcal{N} = 2$ superconformal algebra

$$\begin{aligned}
&T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} \\
&T(z)J(w) \sim \frac{J(w)}{(z-w)^2} + \frac{\partial J(w)}{z} \qquad J(z)J(w) \sim \frac{c/3}{(z-w)^2} \\
&G^{+}(z)G^{+}(w) \sim G^{-}(z)G^{-}(w) \sim 0 \\
&G^{+}(z)G^{-}(w) \sim \frac{2c/3}{(z-w)^3} + \frac{2J(w)}{(z-w)^2} + \frac{2T(w) + \partial J(w)}{z-w} \\
&T(w)G^{\pm}(w) \sim \frac{3G^{\pm}(w)/2}{(z-w)^2} + \frac{\partial G^{\pm}(2)}{z} \qquad J(w)G^{\pm}(w) \sim \frac{\pm G^{\pm}(w)}{z-w}
\end{aligned} \tag{C.8}$$

and

$$\begin{aligned}
&[L_m, L_n] = (m-n)L_{m+n} + \frac{1}{12}c(m^3 - m)\delta_{m+n,0} \\
&[L_m, J_n] = -nJ_{m+n} \qquad [J_m, J_n] = \frac{1}{3}m\delta_{m+n,0} \\
&\{G_r^{+}, G_s^{+}\} = \{G_r^{-}, G_s^{-}\} = 0 \\
&\{G_r^{+}, G_s^{-}\} = 2L_{r+s} + 2(r-s)J_{r+s} + \frac{1}{3}c(r^2 - \frac{1}{4})\delta_{r+s,0} \\
&[L_m, G_s^{\pm}] = (\frac{1}{2}m - s)G_{m+s}^{\pm} \qquad [J_m, G_s^{\pm}] = \pm G_{m+s}^{\pm}
\end{aligned} \tag{C.9}$$

C.2.2 $\mathcal{N} = 1$

$\mathcal{N} = 1$ superalgebra admits only 1 fermionic current $G(z)$ and the energy momentum tensor $T(z)$, there is no $\hat{\mathfrak{u}}(1)$ current $J(z)$. Hence the $\mathcal{N} = 2$ superalgebra reduces to the following $\mathcal{N} = 1$ superalgebra[18]

$$\begin{aligned}
T(z)T(w) &\sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(W)}{z-w} \\
T(z)G(w) &\sim G(z)T(w) \sim \frac{3/2\ G(w)}{(z-w)^2} + \frac{\partial G(w)}{z-w} \\
G(z)G(w) &\sim \frac{2c/3}{(z-w)^3} + \frac{2T(w)}{z-w}
\end{aligned} \tag{C.10}$$

and

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{1}{12}c(m^3 - m)\delta_{m+n,0} \\
\{G_r, G_s\} &= 2L_{r+s} + \frac{1}{3}c(r^2 - \frac{1}{4})\delta_{r+s,0} \\
[L_m, G_s] &= (\frac{1}{2}m - s)G_{m+s}
\end{aligned} \tag{C.11}$$

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