Homework 11: Due at class on May 23

1 Instanton

Let (M, g) be an oriented closed 4-dimensional Riemannian manifold. We denote an orthonormal basis of one-forms $\Omega^1(M)$ by $e^1, \cdots, e^4$. The Hodge star operator

$$* : \Omega^2(M) \to \Omega^2(M)$$

is subject to $* \cdot * = 1$. Therefore, the eigenvalues of the Hodge star operator $*$ on $\Omega^2(M)$ are ± 1 so that we denote the decomposition of $\Omega^2(M)$ by its eigenvalues

$$\Omega^2(M) = \Omega^2_+ \oplus \Omega^2_- \ . \tag{1.1}$$

In fact, we can write a basis of Ω^2_+ as

$$e^1 \wedge e^2 + e^3 \wedge e^4 \ , \quad e^1 \wedge e^3 + e^4 \wedge e^2 \ , \quad e^1 \wedge e^4 + e^2 \wedge e^3 \ ,$$

whereas a basis of Ω^2_- is spanned by

$$e^1 \wedge e^2 - e^3 \wedge e^4 \ , \quad e^1 \wedge e^3 - e^4 \wedge e^2 \ , \quad e^1 \wedge e^4 - e^2 \wedge e^3 \ .$$

Let P be a principal G-bundle over M. In addition, let A be a connection and F be its curvature. According to (1.1), we can decompose

$$F = F_+ + F_- \ .$$

If $F_- = 0$, namely $*F = F$, then the connection A is called **self-dual connection or instanton**. On the other hand, if $F_+ = 0$, the connection A is called **anti-self-dual connection or anti-instanton**. Show that both self-dual and anti-self-dual connections satisfy the Yang-Mills equation $*d_A * F = 0$.

By setting the Yang-Mills coupling $g_{YM} = 1$, the Yang-Mills action is

$$\mathscr{YM}(A) = \frac{1}{2} \int_M \mathrm{Tr}(F \wedge *F) \ .$$

In addition, the integral of the first Pontryagin class $4\pi^2 p_1(P)$ is called the first Pontryagin number

$$p(P) = \frac{1}{2} \int_M \mathrm{Tr}(F \wedge F) \ .$$

Show the following equality

$$\mathscr{YM}(A) \geq |p(P)|$$

where the equality holds when A is either self-dual or anti-self-dual connection.

2 Belavin-Polyakov-Schwarz-Tyupkin instanton

Let $\mathbb{H}$ be the quaternion where an element $x \in \mathbb{H}$ can be expressed as

$$x = x_1 + x_2 i + x_3 j + x_4 k$$

where $x_a \in \mathbb{R}$ $(1 \leq a \leq 4)$ and

$$i^2 = j^2 = k^2 = -1\,, \quad ij = -ji = k\,, \quad jk = -kj = i\,, \quad ki = -ik = j\,.$$

We define the imaginary part of x as

$$\mathrm{Im}\; x = x_2 i + x_3 j + x_4 k$$

so that the conjugate $\overline{x}$ is written as

$$\overline{x} = x_1 - x_2 i - x_3 j - x_4 k$$

Therefore, the multiplication becomes

$$\overline{xy} = \overline{y} \cdot \overline{x}$$

The norm of x is

$$|x|^2 = x\overline{x} = \overline{x}x = x_1^2 + x_2^2 + x_3^2 + x_4^2$$

The symplectic group $\mathrm{Sp}(1) = \{x \in \mathbb{H} |\; |x| = 1\}$ is a Lie group, which is isomorphic to $\mathrm{SU}(2)$. Its Lie algebra $\mathfrak{sp}(1)$ can be written

$$\mathfrak{sp}(1) = \mathrm{Im}\; \mathbb{H} = \{x_2 i + x_3 j + x_4 k |\; x_2, x_3, x_4 \in \mathbb{R}\} \quad \text{with the Lie bracket} \quad [x, y] = \mathrm{Im}\; xy\,.$$

The connection A of $\mathrm{SU}(2)$-principal bundle over $\mathbb{R}^4$ can be understood as one-form taking value on $\mathrm{Im}\mathbb{H}$

$$A = \sum_{a=1}^{4} A_a(x) dx^a\,, \qquad \text{where} \quad A_a(x) \in \mathrm{Im}\; \mathbb{H}\,.$$

Let us define the connection by

$$A = \mathrm{Im} \left\{ \frac{\overline{x} dx}{1 + |x|^2} \right\}$$

Compute its curvature F and show that it is anti-self-dual. This configuration is named as **Belavin-Polyakov-Schwarz-Tyupkin anti-instanton**.

Given a scale transformation $\lambda : \mathbb{H} \to \mathbb{H}; x \mapsto \frac{x}{\lambda}$ $(\lambda > 0)$, the pull-back of the connection by this map

$$(\lambda^* A)(x) = \mathrm{Im} \left\{ \frac{\overline{x} dx}{\lambda^2 + |x|^2} \right\}$$

is anti-self-dual. Furthermore, one can change the position x_0 of instanton by

$$A_{\lambda, x_0}(x) = \mathrm{Im} \left\{ \frac{\overline{(x - x_0)} dx}{\lambda^2 + |x - x_0|^2} \right\}\,.$$

The 5 parameters (λ, x_0) (size and position) are called **moduli of instanton**.

3 Prasad-Sommerfield monopole

Let $P = \mathbb{R}^3 \times G$ be the trivial principal G-bundle over $\mathbb{R}^3$ and A be the connection of P. In addition, let Φ be a section of the adjoint bundle $P \times_{Ad} \mathfrak{g}$. Namely, the field Φ takes its value on the Lie algebra $\mathfrak{g}$ and D_A acts on it as $D_A\Phi = d\Phi + [A, \Phi]$. We define the action

$$S = \frac{1}{2}\int \mathrm{Tr}(F \wedge *F) + \frac{1}{2}\int \mathrm{Tr}(D_A\Phi \wedge *D_A\Phi) ,$$

called Yang-Mills-Higgs action. Show that the equations of motion for this action becomes

$$\begin{aligned} * d_A * F + [\Phi, D_A\Phi] = 0 \\ * d_A * D_A\Phi = 0 . \end{aligned}$$

If field configurations (A, Φ) satisfy

$$*F = \pm D_A\Phi ,$$

which is called **Bogomol'nyi equation or non-Abelian monopole equation**, then show that (A, Φ) obeys the equations of motion.

For $G = \mathrm{SU}(2)$, show that the following configuration is the solution of Bogomol'nyi equation

$$\begin{aligned} A(x) &= \frac{i}{2}\left(\frac{1}{\sinh r} - \frac{1}{r}\right)(\vec{r} \times \vec{\sigma}) \cdot d\vec{x} , \\ \Phi(x) &= \mp\frac{i}{2}\left(\frac{1}{\tanh r} - \frac{1}{r}\right)(\vec{r} \cdot \vec{\sigma}) , \end{aligned}$$

where $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are Pauli matrices and $\vec{r} = (x_1, x_2, x_3)/r$ is the unit vector in $\mathbb{R}^3$ with $r = |x|$. This is called **Prasad-Sommerfield monopole**. Find the value of the action for this solution.