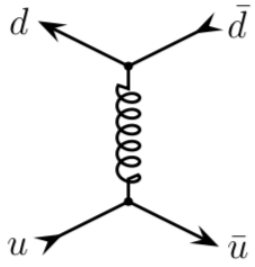
Homework 4: Due at class on April 20

1 Tree-level amplitude in QCD

Consider quark-antiquark pair production in QCD with gauge group G, specifically $u\bar{u} \to d\bar{d}$. There is only one tree diagram contributing to this process, Evaluate this dia-



gram, then sum/average the $|\mathcal{M}|^2$ over both spins and *colors* of the final/initial particles to calculate the total cross section by neglecting the quark masses. You can use the result of $e^-e^+ \to \mu^-\mu^+$.

2 Tree-level amplitude in scalar QCD

Let us consider scalar QCD with gauge group G and complex scalar fields $\Phi^i(x)$ in some representation r of G.

2.1

Write down the Lagrangian of scalar QCD and extract Feynman rules.

2.2

We consider tree-level annihilation of a scalar 'quark' Φ^i and an 'antiquark' Φ^*_j into a pair of gauge bosons with adjoint colors a and b: $\Phi + \Phi^* \to 2$ gauge bosons. At the tree level, there are four Feynman diagrams contributing to this process. Draw the diagrams and write down the tree-level annihilation amplitude.

2.3

Rewrite it in the form

$$\mathcal{M}(j + i \to a + b) = F \times \left\{t^a_r, t^b_r\right\} + iG \times \left[t^a_r, t^b_r\right] \tag{1}$$

where F and G are some functions of all the momenta and of the two vectors' polarizations. Give explicit formulae for F and G.

2.4

Next, let us sum the $|\mathcal{M}|^2$ over the gauge boson's colors and average over the scalars' colors. Show that

$$\frac{1}{d(r)^2}\sum_{ij}\sum_{ab}|\mathcal{M}|^2 = \frac{C_2(r)}{d(r)} \times \Big((4C_2(r) - C_2(\text{adj})) \times |F|^2 + C_2(\text{adj}) \times |G|^2\Big) \tag{2}$$

In particular, for scalars in the fundamental representation of the $\mathrm{SU}(N)$ gauge group,

$$\frac{1}{N^2}\sum_{ij}\sum_{ab}|\mathcal{M}|^2 = \frac{N^2-1}{2N^2}\left(\frac{N^2-2}{N} \times |F|^2 + N \times |G|^2\right) \tag{3}$$

2.5 Bonus problem

Evaluate F and G in the center of mass frame, where the gluon polarizations are purely spatial and transverse to the gluon momenta $\pm\mathbf{k}$. Combining all the results and calculate the cross-section for the annihilation process.

3 β-function in QCD

3.1

Let us consider QCD with gauge group G and n_f species of fermions in a representation r. The beta function of the gauge coupling constant is given by

$$\beta(g) = g\frac{\partial}{\partial \log \mu}\left(-\delta_1 + \delta_2 + \frac{1}{2}\delta_3\right) .$$

The derivation of δ_3 is schematically given in the lecture. Fill out the detail of the computation of δ_3, and calculate δ_1 and δ_2. Show that

$$\beta(g) = -\frac{g^3}{(4\pi)^2}\left[\frac{11}{3}C_2(G) - \frac{4}{3}n_f C(r)\right] .$$

3.2

Calculate the δ_2^c and δ_1^c counterterms for the ghosts fields. Show that the difference $\delta_1^c - \delta_2^c$ for the ghosts is exactly the same as the $\delta_1 - \delta_2$ difference for the quarks.