Homework 5: Due at class on April 7

1. Let ω be the n-form on the space $\mathbb{R}^{n+1}\setminus\{0\}$ defined by

$$\omega = \frac{1}{|x|^{n+1}} \sum_{i=0}^{n} (-1)^i x^i dx^0 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n .$$

Let $\iota : S^n \hookrightarrow \mathbb{R}^{n+1}$ be the unit sphere in $\mathbb{R}^{n+1}$. Show that $\iota^*\omega$ is the generator of n-the de Rham cohomology $H^n_{dR}(S^n)$ of the n-sphere. In fact, the de Rham cohomology of S^n is

$$H^k_{dR}(S^n) = \begin{cases} \mathbb{R} & k = 0,\ n \\ 0 & \text{otherwise} \end{cases} .$$

2. Let $\iota : S^2 \hookrightarrow \mathbb{R}^3$ be the inclusion of the 2-sphere with unit radius. Let $g : ds^2 = \sum_{i=0}^{2} dx^i \otimes dx^i$ be the standard metric of $\mathbb{R}^3$. Find the induced metric ι^*g on S^2 in terms of the polar coordinate of $\mathbb{R}^3$.

$$x^0 = r\sin\theta\cos\phi$$
$$x^1 = r\sin\theta\sin\phi$$
$$x^2 = r\cos\theta$$

3. Let $ds^2 = -dt^2 + dx^2 + dy^2$ be the Minkowski metric on $\mathbb{R}^3$ and $-t^2 + x^2 + y^2 = -1$ for $t > 0$ be the space-like surface (hyperboloid $\mathbf{S}$). (See Figure 1.) Find the induced metric on the hyperboloid $\mathbf{S}$ in terms of the polar coordinate

$$t = r\cosh\rho$$
$$x = r\sinh\rho\cos\phi$$
$$y = r\sinh\rho\sin\phi$$

4. Let us consider the unit disk $\mathbf{D}$ on the x-y plane. Let $\pi(P)$ be the intersection point of the unit disk and the line between the point $(0, 0, -1)$ and $P \in \mathbf{S}$. By assigning $\pi(P)$ to P, there is one-to-one map from the unit disk D and the hyperboloid $\mathbf{S}$. Show that the map $\pi^{-1} : \mathbf{D} \to \mathbf{S}$ is determined by

$$(u, v) \mapsto \left(\frac{2u}{1 - u^2 - v^2}, \frac{2v}{1 - u^2 - v^2}, \frac{1 + u^2 + v^2}{1 - u^2 - v^2} \right) .$$

Find the metric on the unit disk pull-backed by this map. The unit disk $\mathbf{D}$ with this induced metric is called the Poincaré disk.

The red curve on the hyperboloid is an intersection with a plane (pink in Figure 2) that goes through the origin. We put a disk (called Klein's disk) on the bottom of the hyperboloid, which allows us to get the corresponding straight line (green) on Klein's disk. This curve is mapped by $\pi : \mathbf{S} \to \mathbf{D}$ to an (red) arc in the Poincare disk. If the green line is represented by $x = a$ (Figure 3), find the equation for the red curve.

5. Let $\mathbf{H} = \{(x, y)|y > 0\}$ be the upper half plane (Yellow area in Figure 4). By reversing in terms of the circle with radius $\sqrt{2}$ around the origin $(0, -1)$ (Figure 4), we have the map $J : \mathbf{H} \to \mathbf{D}$

$$(x, y) \mapsto \left(\frac{2x}{x^2 + (y+1)^2}, 1 - \frac{2(y+1)}{x^2 + (y+1)^2} \right) \ .$$

Find the induced metric on the upper half plane by this map. Find the area of the triangle with angles (α, β, γ) bounded by half-circles with respect to the metric (Figure 5). Here, we can use the fact that the area of the triangle in the left of Figure 5 is the same as that of the triangle in the right of Figure 5. Compare with the area of a triangle on the 2-sphere (Homework 1).

6. We consider the unite 2-sphere $S^2 \subset \mathbb{R}^3$ with metric induced from $\mathbb{R}^3$ as in problem 2. Let us define a smooth map $f : S^2 \to \mathbb{R}$ by $f(x^0, x^1, x^2) = x^2$. Then, explicitly write the gradient vector of f and its flow φ_t.

7. The Maxwell equations are written as

$$\nabla \cdot \mathbf{E} = \rho \ , \qquad \nabla \times \mathbf{B} = \mathbf{J} + \frac{\partial \mathbf{E}}{\partial t} \ ,$$

$$\nabla \cdot \mathbf{B} = 0 \ , \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \ .$$

Let us write the gauge potential

$$A = A_\mu dx^\mu = \phi dt + A_1 dx^1 + A_2 dx^2 + A_3 dx^3$$

and the current

$$J = J_\mu dx^\mu = \rho dt + J_1 dx^1 + J_2 dx^2 + J_3 dx^3 \ .$$

Then, the field strength can be written as $F = dA$. Show that the Maxwell equations are equivalent to the following equations

$$dF = 0 \ , \qquad \delta F = -j \ .$$

Find the equation of motion for the following action

$$S = -\frac{1}{4} \int F \wedge *F - \int A \wedge *J \ .$$

Discuss this for both a positive definite metric and a Lorentzian signature metrics.
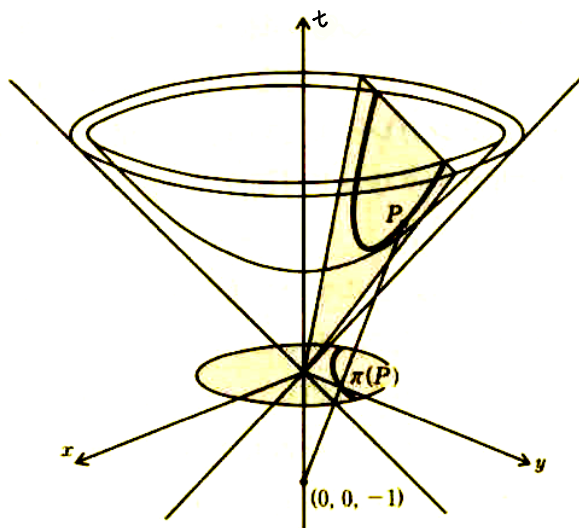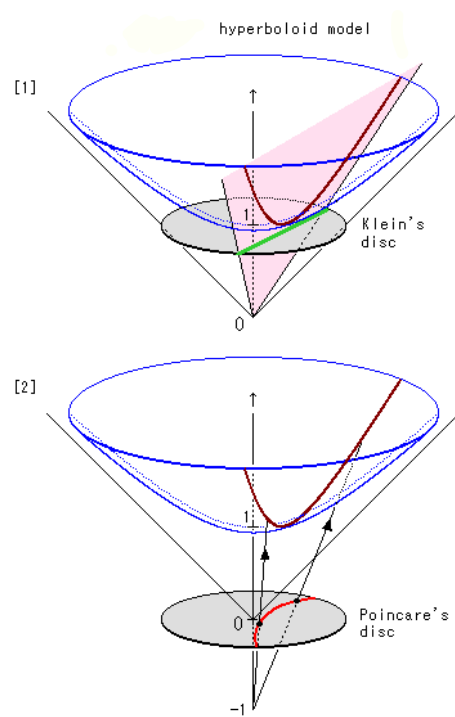
Figure 1: Hyperboloid and Poincare disk
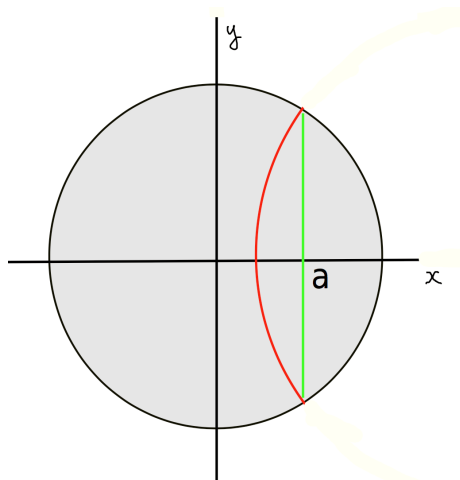

Figure 2: Hyperboloid and Poincare disk

Figure 3: Poincare disk

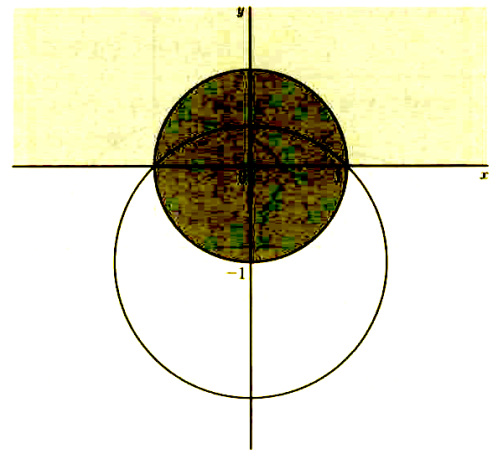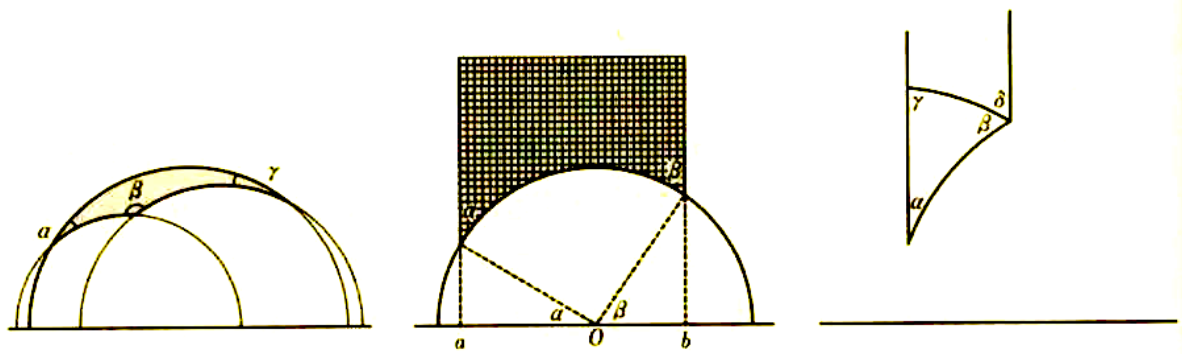Figure 4: Hyperboloid and Poincare disk

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Figure 5: triangle in the upper half plane