Homework 9: Due at class on May 9

1. Derive **real** dimensions of the following Lie groups

- Real special linear group: $\mathrm{SL}(n,\mathbb{R}) = \{A \in \mathrm{GL}(n,\mathbb{R}) | \det A = 1\}$
- Symplectic group $\mathrm{Sp}(n,\mathbb{R}) = \{A \in \mathrm{GL}(2n,\mathbb{R}) | A^{\mathrm{T}} J A = J$ where $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}\}$
- Special unitary group $\mathrm{SU}(n) = \{A \in \mathrm{GL}(n,\mathbb{C}) | A^\dagger A = 1, \ \det A = 1\}$
- Special orthogonal group $\mathrm{SO}(n) = \{A \in \mathrm{GL}(n,\mathbb{R}) | A^T A = 1, \ \det A = 1\}$

2. Write down the definitions of the following Lie algebras: $\mathfrak{sl}(n,\mathbb{R})$, $\mathfrak{sp}(n,\mathbb{R})$, $\mathfrak{su}(n)$, and $\mathfrak{so}(n)$.

3. Show that there are matrices $A, B \in \mathfrak{gl}(n,\mathbb{C})$ such that

$$e^A e^B \neq e^{A+B} .$$

Modify this equation in such a way that the equality holds for those matrices A, B.

4. Let us define

$$\sigma_\mu \equiv (\mathbf{1}, \vec{\sigma})$$

where σ_i are the Pauli matrices. Compute that $\det X$ where

$$X := x^\mu \sigma_\mu = \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix} .$$

Give the definition of the Lorentz group $\mathrm{SO}(1,3)$. $\mathrm{SO}(1,3)$ is indeed not connected. Let us denote a subgroup of Lorentz group by $\mathrm{SO}^+(1,3)$ that satisfies the following two properties:

$$\det \Lambda^\mu{}_\nu = 1 , \qquad \Lambda^t{}_t \geq 1 \qquad \Lambda^\mu{}_\nu \in \mathrm{SO}(1,3) .$$

Show that this subgroup $\mathrm{SO}^+(1,3)$ is isomorphic to $\mathrm{SL}(2,\mathbb{C})/\{\pm \mathrm{Id}\}$. Derive the fundamental group $\pi_1(\mathrm{SO}^+(1,3))$.

5. We have seen that $\mathrm{PSL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})/\{\pm \mathrm{Id}\}$ acts on the upper half plane $(\mathbf{H}, \frac{dz d\bar{z}}{(\mathrm{Im}\ z)^2})$ as an isometry group:

$$\mathrm{PSL}(2,\mathbb{R}) \times \mathbf{H} \to \mathbf{H}; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z \mapsto \frac{az+b}{cz+d} .$$

Show that this action is transitive and find the stabilizer subgroup of the point $i \in \mathbf{H}$. Find the fundamental group of $\mathrm{PSL}(2,\mathbb{R})$.

6. Show that there is a line (rank one vector) bundle L over S^2 such that $TS^2 \oplus L$ is trivial where TS^2 is the tangent bundle of S^2.