Introduction to 2d conformal field theories

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Abstract

These are lecture notes on 2d conformal field theories in Fall 2018. A part of Section 8 is written by Daisuke Yokoyama, and the rest of the main notes is written by the other two authors. If you find typos, please email us.

Contents

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1 Introduction

Conformal field theories have been at the centre of much attention during the last few decades since they are relevant for at least three different areas of modern theoretical physics: conformal field theories provide toy models for genuinely interacting quantum field theories, they describe two-dimensional critical phenomena, and they play a central role in string theory. 2d CFTs are moreover placed on mathematical rigorous footing, opening up new areas in mathematics. Surprisingly, we can still see these trends moving forward.

In this course, we will focus on conformal field theories of **two dimensions**. However, the subject is so rich that we can only cover the basics. To begin with, let us first introduce to conformal field theories in physics.

1.1 Renormalization group flow and Wilson-Fisher fixed points

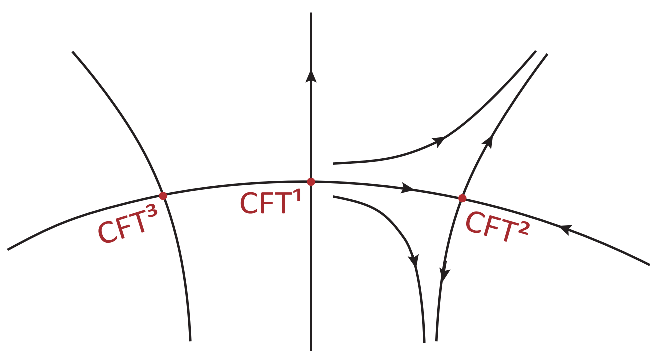
In quantum field theory, a partition function is conceptually defined by using Feynmann path integral. The basic integration variables of the path integral are the Fourier components ϕ_k of a field ϕ. To impose a cutoff Λ, we schematically write

$$\mathcal{Z} = \prod_{|k|<\Lambda} \int d\phi_k \exp\left[-\mathcal{S}_\Lambda[\phi_k]\right].$$

We are interested in relating the coupling constants in a theory having energy cutoff Λ to the coupling constants in a theory having energy cutoff $\Lambda' < \Lambda$. We redefine $\phi \to \phi + \phi'$, where ϕ' has non-zero Fourier modes in $\Lambda' < |k| < \Lambda$ and ϕ has non-zero Fourier modes in $|k| < \Lambda'$. Integrating out the field ϕ' gives us some result written in terms of ϕ.

$$\exp\left(-\mathcal{S}_{\Lambda'}[\phi]\right) \stackrel{\text{def}}{=} \int_{\Lambda' \le |k| \le \Lambda} \mathcal{D}\phi \exp\left[-\mathcal{S}_\Lambda[\phi]\right].$$

Whatever the result is, we include it by changing the Lagrangian to a new, **effective** Lagrangian. The explicit disappearance of the highest energy quantum modes is compensated by some change in the Lagrangian. Integrating out these modes thus has the effect of changing the coefficients of terms in the Lagrangian. Repeatedly integrating out these thin-shells in momentum space corresponds to a smooth motion through this Lagrangian space: this is **renormalization group flow**.



By the naive dimensional analysis, the possible parameter space become finite-dimensional. Although one can consider infinitely many interaction term

$$\mathcal{S}_{int} = \int d^d x \sum g_i \mathcal{O}_i(\phi),$$

the terms $\mathcal{O}_i$ with $d < \Delta^i$ are suppressed at low energy, which are called **irrelevant**. Therefore, we just need to consider finitely many terms $\mathcal{O}_i$ with $d \ge \Delta^i$ at low energy.

To understand the behavior of a theory for these terms, we need to consider the beta function $\beta(g)$ describing the dependence of a coupling parameter on some energy scale μ:

$$\beta(g) = \frac{\partial g}{\partial \log(\Lambda)} = \Lambda \frac{\partial g}{\partial \Lambda}. \tag{1.1}$$

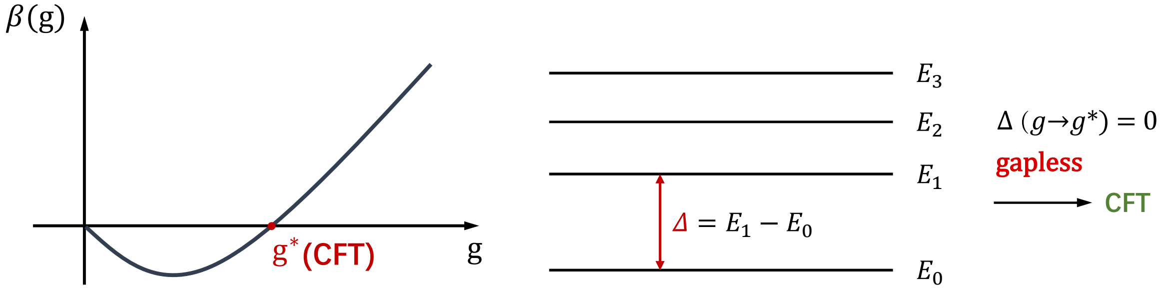
This implies that classical conformal invariance do not maintain conformal invariance quantum mechanically. For example, ϕ^4 theory in $d = 4$ dimensions can be shown to have the one-loop β-function

$$\beta(\lambda) = \frac{3}{16\pi^2}\lambda^2.$$

As we will soon see, the positive sign on this expression means the coupling constant increases with energy. These contrast with the one-loop QCD β-function,

$$\beta(g) = -\frac{9g^3}{16\pi^2}.$$

This β-function says the coupling decreases with energy, which is known as **asymptotic freedom**. Each of these theories, although fine classically, have length scales introduced through quantum effects. However, if $\beta = 0$, the coupling is a constant so that the theory is scale-invariant and does not change with energy scale. The coupling constant g^* with $\beta(g^*) = 0$ is called **Wilson-Fischer fixed point**, and the theory at the fixed point is believed to have conformal symmetry. Therefore, conformal field theories appear at special points in the parameter spaces of quantum field theories, and they play an important role to understand families of quantum field theories.



As we will see below, conformal field theories have more enhanced symmetries than general quantum field theories. Moreover, 2d CFTs are special because conformal symmetry becomes infinite-dimensional. In this lecture, we will study 2d CFTs by making full use of an infinite-dimensional symmetry. However, even in higher dimensions, conformal symmetries are so powerful that they provides great insights into QFTs. Moreover, if a CFT is endowed with enough supersymmetry, it is so much under control that recent study reveals some aspects of QFTs without Lagrangian description.

1.2 Critical phenomena

Let us recall the basics of statistical mechanics. The partition function is defined by

$$\mathcal{Z} = e^{-F/T} = \sum_i \exp\left(-\frac{\mathcal{E}_i}{T}\right)$$

where F is called the free energy and and we set the Boltzmann constant $k_B = 1$. Then, the probability P_i that the system is in a state with energy $\mathcal{E}_i$ is given by the Boltzmann distribution

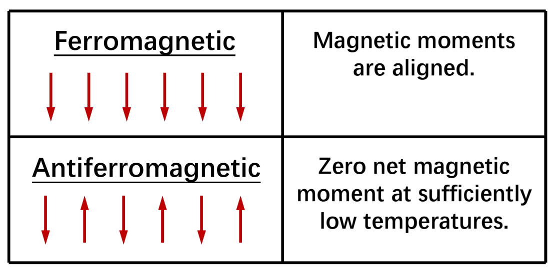
$$P_i = \frac{1}{\mathcal{Z}} \exp\left(-\frac{\mathcal{E}_i}{T}\right) .$$

Therefore, the expectation value of an operator $\mathcal{O}$ is given by

$$\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \sum_i \mathcal{O}_i \, e^{-\mathcal{E}_i/T}$$

For instance, the average energy is

$$E = \langle \mathcal{E} \rangle = -T^2 \frac{\partial}{\partial T} \left(\frac{F}{T}\right) = \frac{1}{\mathcal{Z}} \sum_i \mathcal{E}_i \, e^{-\mathcal{E}_i/T}$$



Now we consider the **Ising model** of a d-dimensional lattice Λ with N lattice sites. For each lattice site $k \in \Lambda$, there is a discrete variable $\sigma_k \in \{+1, -1\}$, representing the site's spin configuration. The Hamiltonian of the Ising model is given by

$$\mathcal{H} = -J \sum_{\langle i,i' \rangle} \sigma_i \sigma_{i'} - B \sum_i \sigma_i \tag{1.2}$$

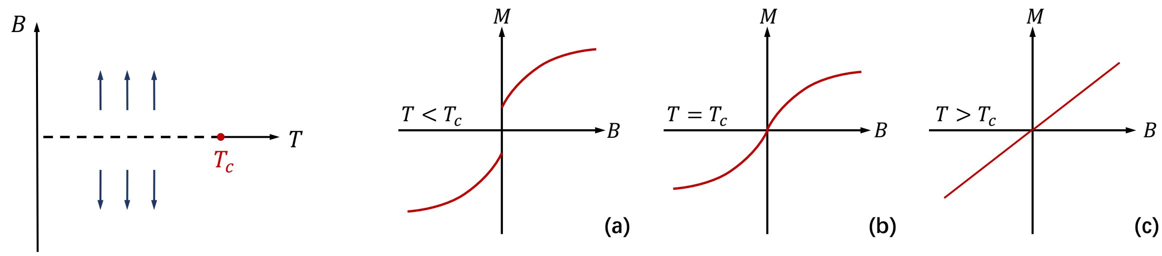
where B is the external magnetic field. If $J > 0$, the alignment of spins is energetically favored and the configuration is called **ferromagnetic**. On the other hand, if $J < 0$, the configuration is called **antiferromagnetic**. In the following, we will focus on the case $J > 0$.

At low temperature T, the system settles in one of two ground states. Therefore, the magnetization M has a jump discontinuity across the line of **first-order** transitions at $B = 0$. A **first-order** phase transitions are the ones in which some first derivative of the free energy F is discontinuous. In this example, the average energy E is discontinuous.

On the other hand, the thermal fluctuations dominate at high T and spins are randomly oriented (**pramagnetic**). Therefore, there is a phase transition at the critical temperature $T = T_c$ called **Curie temperature**. The magnetization is discontinuous across the phase transition, which is called the **order parameter** of the Ising model.

To describe this critical point, we need to introduce a notion of correlation functions. A two-point correlation function is defined as

$$G(r) = \langle \sigma_i \sigma_j \rangle - \langle \sigma_i \rangle \langle \sigma_j \rangle \approx r^{-\tau} \mathrm{e}^{-r/\xi}$$



where $r = |i - j|$ is the distance between the sites i and j. In the 2d Ising model, one has $\tau = 1/2$ for $T > T_c$ and $\tau = 2$ for $T < T_c$. If $r < \xi$, the two spins are correlated, since there is a large probability that they have the same value. On the other hand, for sufficiently large $r \gg \xi$, probability should decrease roughly exponentially with r. Hence, the mean cluster size of correlated spins is ξ, which is called the **correlation length**. As a temperature T approaches T_c, the correlation length ξ diverges, and such a critical point is called **second-order**. At the critical point, fluctuations on all length scales become important, and therefore the spin configuration is a fractal-like structure (Figure 1). Consequently, the corresponding mass scale $1/\xi$ disappears and physics is described by conformal field theory at the critical point [Car84a, ISZ88-No.2].

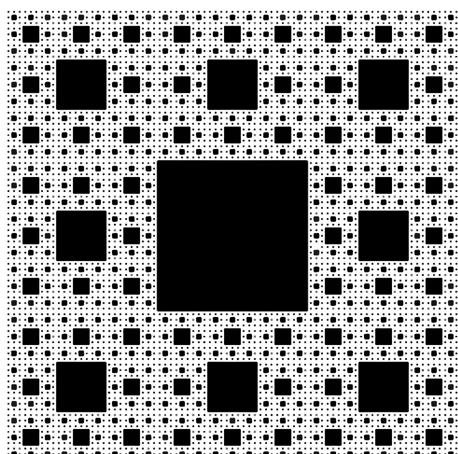


Figure 1: Schematic illustration of spin configuration at the Curie temperature T_c. The figure is taken from Wikipedia:Fractal

Close to the critical point, the correlation lengths and the thermodynamic quantities obey power laws, and the critical behavior near to the critical points can be characterized by the values of the **critical exponents**. Introducing the reduced temperature t and the reduced magnetic field h

$$t := \frac{T - T_c}{T_c} \quad \text{and} \quad h := \frac{B}{T_c} ,$$

the conventional critical exponents $\alpha, \beta, \gamma, \delta, \nu, \eta$ are now defined by

$$\begin{array}{ccc} C \sim t^{-\alpha} & (h=0) & \text{specific heat} \\ M \sim (-t)^{\beta} & (t<0, h=0) & \text{spontaneous magnetization} \\ \chi \sim t^{-\gamma} & (h=0) & \text{zero field susceptibility} \\ M = h^{1/\delta} & (t=0) & \text{magnetization} \\ \xi \sim t^{-\nu} & (h=0) & \text{correlation length} \\ G(r) \sim |r|^{2-d-\eta} & (t=0, h=0) & \text{2pt-function} \end{array}$$

where

$$C := -\frac{T}{N}\frac{\partial^2 F}{\partial T^2}, \quad M := -\frac{1}{N}\frac{\partial F}{\partial B} \quad , \quad \chi := \frac{\partial M}{\partial B} .$$

In fact, there are relations among these critical exponents and there are only two independent exponents ν, η. In the 2d Ising model, one has

$$\alpha = 0 \,, \quad \beta = \frac{1}{8} \,, \quad \gamma = \frac{7}{4} \,, \quad \delta = 15 \,, \quad \nu = 1 \,, \quad \eta = \frac{1}{4}$$

Remarkably, the critical exponents stay the same even if we change the shape of lattice or we include next-nearest-neighbor interactions. The independence of the critical exponents from such microscopic details is referred to as **universality**.

At the critical point, the two-point function can be written as

$$\langle \sigma(x)\sigma(0) \rangle = \frac{C}{|x|^{2\Delta_\sigma}}$$

where Δ_σ is called the **scaling dimension** of σ. One can similarly define a local energy density operator

$$\varepsilon_i = \sum_{i'} J\left(i, i'\right) \sigma_i \sigma_{i'} \,.$$

Then, in the 2d Ising model, their scaling dimensions are

$$\Delta_\sigma = 1/8 \,, \qquad \Delta_\epsilon := 1 \,, \tag{1.3}$$

which we will see in the subsequent lectures.

1.3 String theory

Conformal field theories play a central role in string theory [GSW87, Pol98]. In string theory, string theory propagation on a certain manifold M is described by a non-linear sigma model $\Sigma \to M$ where Σ is a two-dimensional Riemann surface and M is a certain manifold. For instance, in a bosonic string theory, the action is expressed as

$$\mathcal{S} = \frac{1}{4\pi\alpha'} \int d^2x \left(\sqrt{h} h^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X) + i\varepsilon^{ab} \partial_a X^\mu \partial_b X^\nu B_{\mu\nu}(X) + \alpha' \sqrt{h} R^{(2)} \Phi(X) \right) .$$

where h and G are the metrics of Σ and M, respectively, and B is an anti-symmetric field. To describe a consistent string theory, it must be a conformal field theory. To preserve conformal invariance at quantum level, M has to be 26-dimensional in bosonic string theory, and 10-dimensional in a string theory with supersymmetry. Moreover, the vanishing one-loop renormalization group beta-functions of the non-linear sigma model implies that M must be Ricci-flat.

In particular, Calabi-Yau sigma models are described by $\mathcal{N} = 2$ superconformal field theories. It was discovered that $\mathcal{N} = 2$ superconformal field theories associated to a pairs of Calabi-Yau manifolds are equivalent. The existence of such pairs of Calabi-Yau manifolds with specified properties are known to mathematicians as the **mirror symmetry conjecture**, which is now the active subject in mathematics.

Moreover, the equivalence between string theory on an anti-de Sitter space and conformal field theories was proposed, which is called the **AdS/CFT correspondence**. For example, this correspondence states that non-perturbative definition of superstring theory of type IIB on AdS_5 is described by the four-dimensional $\mathcal{N} = 4$ superconformal field theory. The AdS/CFT correspondence has been extensively studied in the last two decades, and now it has huge impact on various areas in physics.

1.4 References

There are too many references on 2d CFTs and each book deals with only a certain aspect of the huge subject. In this lecture, we will first follow [Car88, Gin88a, ZZ89], and the yellow book [FMS97] can be used as a dictionary. In addition, we list some standard references related to statistical mechanics [Car96, Hen13], string theory [BP09, Sch96, Ket95], integrable systems [GRAS05, Mus10]. In mathematics, studying the representation theoretic aspect of 2d CFTs has been led by Kac [Kac94]. Moreover, after the mathematical foundation of 2d CFTs has been given in [Seg88] and [TUY89] (included in [JMT89]), 2d CFTs have shed drastically new insights in various aspects of modern mathematics such as vertex operator algebras, moduli spaces, low-dimensional topology, geometric representation theory (for instance, see [FLM89, Koh02, FBZ04, BD04]). For general conformal field theories of higher dimensions, we refer the reader to [Nak15, Qua15, Ryc16, SD17]

After this course, the reader would be highly recommended to read some of important original papers in [ISZ88] so that the relevant papers in [ISZ88] are indicated in this lecture note. Related but more mathematical papers are collected in [JMT88, JMT89]. Even in this fast-moving community, the importance of reading the classics cannot be overemphasized.

2 Conformal invariance

2.1 Conformal Transformation

A conformal transformation is generally defined as follows: Let us consider differentiable map $\phi : U \to V$, where U and V are open subsets of M and M' respectively. Denoting g and g' as the metric tensors of M and M' respectively, we can pull back g' by using the map ϕ, as we have learnt in differential geometry. A conformal transformation is then defined by such a map that its pull back metric $\phi^* g'$ satisfying the following condition:

$$\phi^* g' = \Lambda g \,. \tag{2.1}$$

Denoting $x' = \phi(x)$ with $x \in U$, we can express the condition in a covariant form:

$$g'_{\rho\sigma}(x') \frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{\nu}} = \Lambda(x) g_{\mu\nu}(x) \,, \tag{2.2}$$

where $\Lambda(x)$ is a positive scalar called the scale factor. We see an appreciable extrapolation from (2.2), that any smooth transformation in one dimension is a conformal transformation. In addition, by using (2.2), the inner product of two arbitrary vectors $V_1'^{\mu}(x')$ and $V_2'^{\mu}(x')$ at the same point of M' has a scaling effect after its pulling back by the conformal map:

$$g_{\mu\nu}(x)V_1^{\mu}(x)V_2^{\nu}(x) = g'_{\mu\nu}(x')V_1'^{\mu}(x')V_2'^{\nu}(x')\Lambda\,, \tag{2.3}$$

where $V_1^{\mu}(x)$ and $V_2^{\mu}(x)$ are the pull-back vectors respectively. From (2.3), it is not difficult to find that the included angle of two vectors of M' remains the same when they are pulled back by a conformal map to M.

In this lecture note, we focus on $M' = M$, which implies g′=g. We also consider M to be flat spaces with a constant metric $\eta_{\mu\nu} = \mathrm{diag}(-1, \ldots, +1, \ldots)$. Thus, the condition for a conformal transformation can be written as

$$\eta_{\rho\sigma}\frac{\partial x'^{\rho}}{\partial x^{\mu}}\frac{\partial x'^{\sigma}}{\partial x^{\nu}} = \Lambda(x)\eta_{\mu\nu}\,. \tag{2.4}$$

The line element transforms under conformal transformation as

$$\mathrm{d}s^2 = \eta_{\mu\nu}\mathrm{d}x^{\mu}\mathrm{d}x^{\nu} \to \eta_{\rho\sigma}\mathrm{d}x'^{\rho}\mathrm{d}x'^{\sigma} = \eta_{\rho\sigma}\frac{\partial x'^{\rho}}{\partial x^{\mu}}\frac{\partial x'^{\sigma}}{\partial x^{\nu}}\mathrm{d}x^{\mu}\mathrm{d}x^{\nu} = \Lambda(x)\mathrm{d}s^2\,. \tag{2.5}$$

Two remarks should be stated here. First of all, using (2.4), we can see the composition of two conformal transformation is still conformal, which may imply that conformal transformations form a group. Secondly, the scale factor $\Lambda(x) = 1$ corresponds to the Poincare group.

2.2 Conditions for Conformal Invariance

In this part, we need to figure out some basic conformal transformations in flat space. To begin with, we study the infinitesimal transformations

$$x'^{\rho} = x^{\rho} + \epsilon^{\rho}(x) + O(\epsilon^2)\,, \tag{2.6}$$

with an infinitesimal variable $\epsilon(x) \ll 1$. Plugging into the conformal condition (2.4), up to first order in ϵ, the infinitesimal form of conformal transformation may be derived as:

$$\partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu} = K(x)\eta_{\mu\nu}\,, \tag{2.7}$$

with $K(x) = \Lambda(x) - 1$, some infinitesimal function. However, since we want to find out the explicit form for $\epsilon(x)$, $K(x)$ need to be cancelled in the constraint formula. By tracing both sides of (2.7), we have

$$K(x) = \frac{2\partial^{\mu}\epsilon_{\mu}}{d}\,. \tag{2.8}$$

Plugging back to (2.7), we find the restriction depending on $\epsilon(x)$ to make the transformation conformal:

$$\partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu} = \frac{2}{d}(\partial\cdot\epsilon)\eta_{\mu\nu}\,. \tag{2.9}$$

2.3 Some Useful Relation

Let us derive two useful equations for later purpose. First of all, by taking ∂^{ν} to both sides of (2.9), we can obtain

$$\partial_{\mu}(\partial\cdot\epsilon) + \Box\epsilon_{\mu} = \frac{2}{d}\partial_{\mu}(\partial\cdot\epsilon)\,. \tag{2.10}$$

We then take ∂_ν to both sides:

$$\partial_\mu\partial_\nu(\partial\cdot\epsilon)+\Box\partial_\nu\epsilon_\mu=\frac{2}{d}\partial_\nu\partial_\mu(\partial\cdot\epsilon)\,. \tag{2.11}$$

Interchanging $\mu\leftrightarrow\nu$, adding back to (2.11) and using (2.9) we get

$$(\eta_{\mu\nu}\Box+(d-2)\partial_\mu\partial_\nu)(\partial\cdot\epsilon)=0\,. \tag{2.12}$$

Finally, tracing spacetime indices gives

$$(d-1)\Box(\partial\cdot\epsilon)=0\,. \tag{2.13}$$

The second useful expression is obtain by taking ∂_ρ of (2.9) and permuting indices:

$$\begin{aligned}
\partial_\rho\partial_\mu\epsilon_\nu+\partial_\rho\partial_\nu\epsilon_\mu&=\frac{2}{d}\eta_{\mu\nu}\partial_\rho(\partial\cdot\epsilon)\,,\\
\partial_\nu\partial_\rho\epsilon_\mu+\partial_\mu\partial_\rho\epsilon_\nu&=\frac{2}{d}\eta_{\rho\mu}\partial_\nu(\partial\cdot\epsilon)\,,\\
\partial_\mu\partial_\nu\epsilon_\rho+\partial_\nu\partial_\mu\epsilon_\rho&=\frac{2}{d}\eta_{\nu\rho}\partial_\mu(\partial\cdot\epsilon)\,.
\end{aligned}$$

Subtracting the first line from the sum of the last two leads to

$$2\partial_\mu\partial_\nu\epsilon_\rho=\frac{2}{d}(-\eta_{\mu\nu}\partial_\rho+\eta_{\rho\mu}\partial_\nu+\eta_{\nu\rho}\partial_\mu)(\partial\cdot\epsilon)\,. \tag{2.14}$$

2.4 Conformal Transformation in $d\geq 3$

For the case $d=1$, the relation (2.13) imposes no constraint to $\epsilon(x)$, which accords with what we have claimed, that any smooth transformation in one dimension is conformal. The case $d=2$ will be studied later in detail. In this section, let's focus on the case $d\geq 3$.

According to (2.13) for $d\geq 3$, we have $\Box(\partial\cdot\epsilon)=0$. Substituting back to condition (2.12) gives $\partial_\mu\partial_\nu(\partial\cdot\epsilon)=0$,implying that $(\partial\cdot\epsilon)$ is at most linear in x^μ, i.e. $(\partial\cdot\epsilon)=A+B_\mu x^\mu$ .When plugging this linear expression back to (2.14), we immediately obtain that $\partial_\mu\partial_\nu\epsilon_\rho$ equals to a constant. This follows that ϵ_μ is at most quadratic in x^ν and so we can make the ansatz:

$$\epsilon_\mu=a_\mu+b_{\mu\nu}x^\nu+c_{\mu\nu\rho}x^\nu x^\rho\,, \tag{2.15}$$

where, $c_{\mu\nu\rho}=c_{\mu\rho\nu}$. For the infinitesimal nature of ϵ_μ, all the constants: $a_\mu,b_{\mu\nu},c_{\mu\rho\nu}\ll 1$.

Noticing that all the terms in (2.9), have only one derivative, and that the conformal constraints for these constants should be independent of x^μ, we can study the various terms in (2.15) separately.

The constant term a_μ in (2.15) is not constrained by (2.9). It describes infinitesimal translation $x'^\mu=x^\mu+a^\mu$, for which the generator is the momentum operator $P_\mu=-i\partial_\mu$ [1].

[1] All the generators of this form can be calculated by the definition :

$$iG_a\Phi=\frac{\delta x^\mu}{\delta\omega_a}\partial_\mu\Phi\,, \tag{2.16}$$

if we suppose that the fields are not affected by the transformation with ω_a the corresponding parameter.

Inserting the linear term into (2.9) , we find

$$b_{\mu\nu} + b_{\nu\mu} = \frac{2}{d}(\eta^{\rho\sigma} b_{\sigma\rho})\eta_{\mu\nu} \,. \tag{2.17}$$

We then split $b_{\mu\nu}$ into symmetric and antisymmetric part, i.e. $b_{\mu\nu} = n_{\mu\nu} + m_{\mu\nu}$, with $n_{\mu\nu} = n_{\nu\mu}$ and $m_{\mu\nu} = -m_{\nu\nu}$. The antisymmetric part $m_{\mu\nu}$ satisfy (2.17) automatically, while the symmetric part gives constraint: $n_{\mu\nu} = \frac{1}{d}(\eta^{\rho\sigma} n_{\sigma\rho})\eta_{\mu\nu}$, telling that the symmetric part should be diagonal. Therefore, we can split $b_{\mu\nu}$ as

$$b_{\mu\nu} = \alpha\eta_{\mu\nu} + m_{\mu\nu} \,. \tag{2.18}$$

The symmetric term $\alpha\eta_{\mu\nu}$ describes infinitesimal scale transformation $x'^{\mu} = (1+\alpha)x^{\mu}$ with generator $D = -ix^{\mu}\partial_{\mu}$. The antisymmetric part $m_{\mu\nu}$ corresponds to infinitesimal rotations $x'^{\mu} = (\delta^{\mu}_{\nu} + m^{\mu}{}_{\nu})x^{\nu}$ with generator being the angular momentum operator $L_{\mu\nu} = i(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu})$.

Now let us consider the quadratic term. By inserting the quadratic term into (2.14), after straight-forward calculation, we find that

$$c_{\mu\nu\rho} = \eta_{\rho\mu} f_{\nu} + \eta_{\mu\nu} f_{\rho} - \eta_{\nu\rho} f_{\mu} \,, \tag{2.19}$$

with $f_{\mu} = \frac{1}{d} c^{\rho}{}_{\rho\mu}$. The resulting transformations are

$$x'^{\mu} = x^{\mu} + 2(x\cdot f)x^{\mu} - (x\cdot x)f^{\mu} \,, \tag{2.20}$$

which are called **special conformal transformations** (SCT) and the corresponding generator is $K_{\mu} = -i(2x_{\mu}x^{\nu}\partial_{\nu} - (x\cdot x)\partial_{\mu})$.

2.5 Finite Conformal Transformation

So far, we have identified all basic infinitesimal conformal transformations and their generators. However, in order to determine the conformal group, we also need the corresponding finite transformations. All the basic finite conformal transformations are shown in Table 1. The finite transformations for translation and rotation are Poincare

Table 1: Finite conformal transformations and corresponding generators

Transformations		Generators
translation	$x'^{\mu} = x^{\mu} + a^{\mu}$	$P_{\mu} = -i\partial_{\mu}$
dilation	$x'^{\mu} = \alpha x^{\mu}$	$D = -ix^{\mu}\partial_{\mu}$
rotation	$x'^{\mu} = M^{\mu}{}_{\nu}x^{\nu}$	$L_{\mu\nu} = i(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu})$
SCT	$x'^{\mu} = \frac{x^{\mu} - (x\cdot x)f^{\mu}}{1 - 2(f\cdot x) + (f\cdot f)(x\cdot x)}$	$K_{\mu} = -i(2x_{\mu}x^{\nu}\partial_{\nu} - (x\cdot x)\partial_{\mu})$

transformations we are familiar with. The finite transformation for dilation can be derived from it infinitesimal form easily. Thus, what we focus on is the finite transformation for SCT. It is hopeless to derive it through its infinitesimal form. But, we can guess this finite form of transformation, if we realize that an inversion $x'^{\mu} = \frac{x^{\mu}}{x\cdot x}$ is also a conformal map. By plugging in to (2.4), we can obtain its scale factor as $\Lambda(x) = (x\cdot x)^{-2}$. Inversion can be one of the basic finite conformal transformation instead of SCT. However, it has neither infinitesimal form nor parameters. Now we want to construct finite SCT. Using the transitive property for conformal map, we can thus first make an inversion of x^μ, then make an translation f^μ, and inverse the result again to construct a new conformal transformation. An illustration in two dimensions is shown in Figure 2. As f^μ goes to zero, x^μ maps to itself. Therefore it should have infinitesimal form.

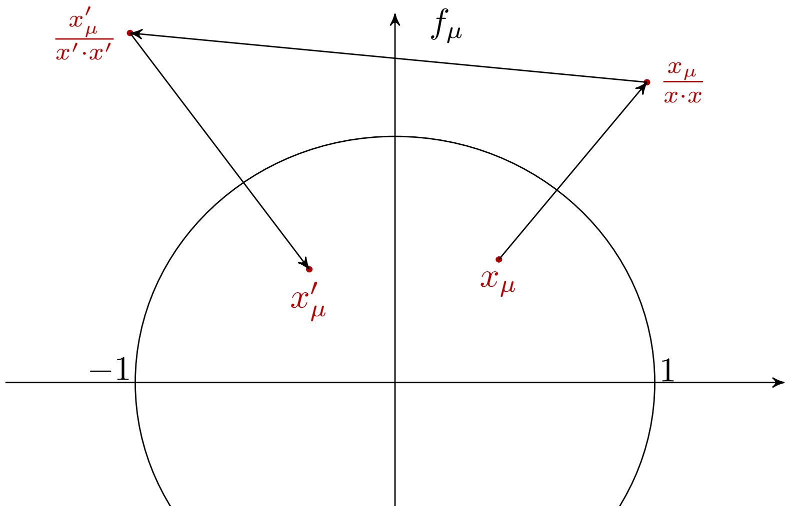


Figure 2: Illustration of a finite Special Conformal Transformation

$$x'^\mu = \frac{\frac{x^\mu}{x\cdot x} - f^\mu}{(\frac{x^\mu}{x\cdot x} - f^\mu)(\frac{x^\mu}{x\cdot x} - f^\mu)} = \frac{x^\mu - (x\cdot x)f^\mu}{1 - 2(f\cdot x) + (f\cdot f)(x\cdot x)}\,. \tag{2.21}$$

The result is what we have shown in Table 1. It is easy to verify that when $f^\mu \to 0$, (2.21) reduces to infinitesimal form of SCT: (2.20). Further more by using (2.4), after a heavy and tedious calculation, the scale factor for SCT can be computed as

$$\Lambda(x) = (1 - 2(f\cdot x) + (f\cdot f)(x\cdot x))^{-2} \tag{2.22}$$

Finally, we observe from (2.21), that the finite SCT are not globally defined. In particular, for a given non-zero vector f^μ, there is a point $x^\mu = \frac{1}{f\cdot f}f^\mu$ such that x^μ is mapped to infinity. (The denominator goes to zero more quickly than the numerator.) Therefore, in order to define the arbitrary conformal transformation, the point at infinity should be included.

2.6 Conformal Groups and Algebras

The conformal transformations form a group, called the conformal group of dimension $d \geq 3$. Actually, the algebra generated by infinitesimal conformal transformations is the Lie algebra of the conformal group, called the conformal algebra.

Let us identify the conformal algebra of dimension $d \geq 3$. The dimension of the algebra is $N = d + 1 + \frac{d(d-1)}{2} + d = \frac{(d+2)(d+1)}{2}$ by counting the generators in Table 1. The commutation rules with no central charge can be obtained by direct calculations.

$$
\begin{aligned}
\left[D, P_\mu\right] &= iP_\mu \,, \\
\left[D, K_\mu\right] &= -iK_\mu \,, \\
\left[K_\mu, P_\nu\right] &= 2i(\eta_{\mu\nu} D - L_{\mu\nu}) \,, \\
\left[K_\rho, L_{\mu\nu}\right] &= i(\eta_{\rho\mu} K_\nu - \eta_{\rho\nu} K_\mu) \,, \\
\left[P_\rho, L_{\mu\nu}\right] &= i(\eta_{\rho\mu} P_\nu - \eta_{\rho\nu} P_\mu) \,, \\
\left[L_{\mu\nu}, L_{\rho\sigma}\right] &= i(\eta_{\rho\nu} L_{\mu\sigma} + \eta_{\mu\sigma} L_{\nu\rho} - \eta_{\rho\mu} L_{\nu\sigma} - \eta_{\nu\sigma} L_{\mu\rho}) \,.
\end{aligned}
$$

To put the above commutation relations into a simpler form, we define the following generators.

$$
\begin{aligned}
J_{\,\mu,\nu} &= L_{\mu,\nu} \,, \\
J_{-1,0} &= D \,, \\
J_{-1,\mu} &= \frac{1}{2}(P_\mu - K_\mu) \,, \\
J_{\,0,\mu} &= \frac{1}{2}(P_\mu + K_\mu) \,.
\end{aligned}
$$

By using (2.23), we can verify that $J_{m,n}$ with $m, n = -1, 0, 1, \ldots d$, satisfy the following commutation relations:

$$
[J_{mn}, J_{rs}] = i(\eta_{ms} J_{nr} + \eta_{nr} J_{ms} - \eta_{ms} J_{ns} - \eta_{ns} J_{mr}) \,. \tag{2.23}
$$

For Euclidean d-dimensional space $\mathbb{R}^d$, we see the metric η_{mn} here is $\eta_{mn} = \mathrm{diag}(-1, 1, \ldots, 1)$ so that we identify (2.23) with the commutation relations of the Lie algebra $\mathfrak{so}(1, d+1)$. We can generalize the result to the space $\mathbb{R}^{p,q}$, with $d = p + q \geq 3$. The corresponding conformal group is $SO(p+1, q+1)$.

2.7 2d Conformal Transformation

In this section, we will focus on the special case of two dimensions with Euclidean metric. From (2.9), the infinitesimal condition for conformal transformation in two dimensions reads as follows:

$$
\partial_1 \epsilon_1 = \partial_2 \epsilon_2 \,, \qquad \partial_1 \epsilon_2 = -\partial_2 \epsilon_1 \,. \tag{2.24}
$$

(2.24) tells us that if we introduce complex variables :

$$
\begin{aligned}
z &= x^1 + ix^2 \,, \qquad & \epsilon &= \epsilon^1 + i\epsilon^2 \,, \\
\bar{z} &= x^1 - ix^2 \,, \qquad & \bar{\epsilon} &= \epsilon^1 - i\epsilon^2 \,,
\end{aligned}
$$

according to Cauchy-Riemann equations in complex analysis, $\epsilon(z)$ is a holomorphic function (in some open set). Equivalently any infinitesimal holomorphic transformation $z' = z + \epsilon(z)$ gives rise to a two-dimensional conformal transformation.

In fact, any holomorphic function on the complex plane is conformal (i.e, preserves angles). To prove this, we should rewrite the definition of a conformal transformation (2.4) with complex variables. Let us first, rewrite the line element:

$$
ds^2 = (\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2 = \mathrm{d}z\mathrm{d}\bar{z} = g_{\alpha\beta} z^\alpha z^\beta \,, \tag{2.25}
$$

with $z^1 = z$, $z^2 = \bar{z}$. The flat space metric in complex coordinates can be obtained:

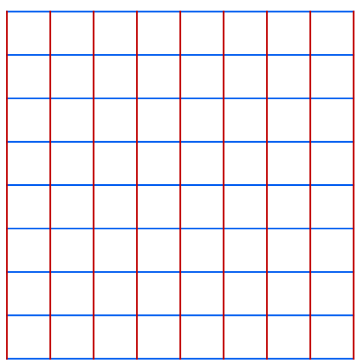
$$g_{\alpha\beta} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \qquad g^{\alpha\beta} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}. \tag{2.26}$$

The condition (2.4) now becomes:

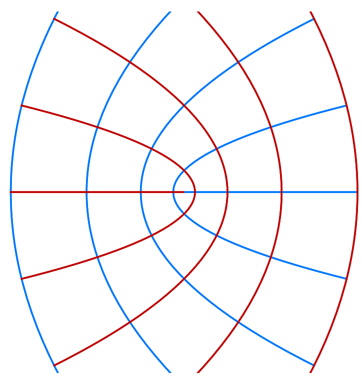
$$g_{\alpha\beta} \frac{\partial z'^{\alpha}}{\partial z^{\gamma}} \frac{\partial z'^{\beta}}{\partial z^{\delta}} = \Lambda(z, \bar{z}) g_{\gamma\delta}\,, \tag{2.27}$$

which is satisfied, if $z' = f(z)$ is a holomorphic function. The conformal factor is then $\Lambda(z, \bar{z}) = \left|\frac{\partial f(z)}{\partial z}\right|^2$. Figure 3 shows some examples on conformal transformations. We can see that an angle at any point is preserved when $z' = f(z)$ is holomorphic.

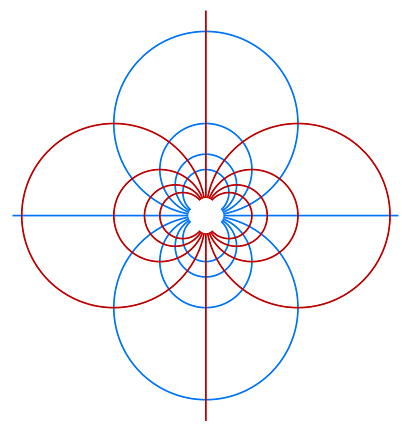
Figure 3: Coordinate transformation: The transformation from square lattice (a), onto the lattice in (b) and (c) is conformal, while the transformation onto the lattice in (d) is not.



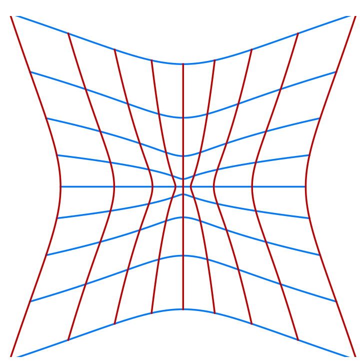
(a) z



(b) $z' = z^2$



(c) $z' = 1/z$



(d) $z' = z|z|$

Witt algebra

As we have seen above, for an infinitesimal conformal transformation in two dimensions, $\epsilon(z)$ has to be holomorphic in some open set. We can locally around say $z = 0$ perform a Laurent expansion of $\epsilon(z)$. A general infinitesimal conformal transformation thus can be written as

$$z' = z + \epsilon(z) = z + \sum_{n\in\mathbb{Z}} \epsilon_n(-z^{n+1}),$$
$$\bar{z}' = z + \bar{\epsilon}(\bar{z}) = \bar{z} + \sum_{n\in\mathbb{Z}} \bar{\epsilon}_n(-\bar{z}^{n+1}),$$

where ϵ_n and $\bar{\epsilon}_n$ are infinitesimal parameters. Supposing a spinless and dimensionless field $\phi(z,\bar{z})$ living on the plane, the effect of such a mapping would be:

$$\phi(z,\bar{z}) \to \phi'(z',\bar{z}') = \phi(z,\bar{z}), \tag{2.28}$$

or

$$\begin{aligned}\delta\phi \equiv \phi'(z,\bar{z}) - \phi(z,\bar{z}) &= \phi'(z' - \epsilon(z), \bar{z}' - \bar{\epsilon}(\bar{z})) - \phi(z,\bar{z}) \\ &= -\epsilon(z)\partial\phi - \bar{\epsilon}(\bar{z})\bar{\partial}\phi + o(\epsilon) = -\sum_{n\in\mathbb{Z}}(\epsilon_n l_n \phi(z,\bar{z}) + \bar{\epsilon}_n \bar{l}_n \phi(z,\bar{z})),\end{aligned}$$

where we have introduced the generators

$$l_n = -z^{n+1}\partial, \qquad \bar{l}_n = -\bar{z}^{n+1}\bar{\partial}. \tag{2.29}$$

We find the number of independent infinitesimal conformal transformations is infinite. The commutators for these local generators can be calculated straightforward then:

$$\begin{aligned}[l_n, l_m] &= (m-n) l_{n+m}, \\ \left[\bar{l}_m, \bar{l}_n\right] &= (m-n)\bar{l}_{m+n}, \\ \left[l_m, \bar{l}_n\right] &= 0.\end{aligned}$$

The first commutation relations define one copy of the so-called Witt algebra. We can observe that the local conformal algebra (2.30) is the direct sum of two Witt algebras.

Möbius transformation

A global conformal transformation in two dimensions is defined to be invertible and well-defined on the Riemann sphere $S^2 \simeq \mathbb{C} \cup \{\infty\}$. Thus, it should be generated by the global-defined infinitesimal generators. At $z = 0$, we find that $l_n = -z^{n+1}\partial_z$, is nonsingular only for $n \geq -1$. At $z = \infty$, let us perform the change of variable $z = -\frac{1}{w}$, and study $w \to 0$. We then observe that $l_n = -(-\frac{1}{w})^{n-1}\partial_w$, is nonsingular at $w = 0$ only for $n \leq 1$. Therefore, globally defined conformal transformations on the Riemann sphere $S^2 = \mathbb{C} \cup \{\infty\}$ are generated by $\{l_{-1}, l_0, l_{+1}\}$. Through the Wit algebra (2.30), we immediately see that these three generators form a close subalgebra of Wit algebra, which implies the corresponding conformal transformations form a globally defined group, called the conformal group in two dimensions. The discussion of antiholomorphic counterpart is similar.

Now, we are going to study the conformal group in two dimensions deeply. From the Laurent expansion (2.28), the operator $l_{-1} = -\partial_z$, generates the infinitesimal transformation $z' = z - \epsilon_{-1}$. Thus, l_{-1} is the generator of translations $z \mapsto z + b$ with b an arbitrary complex constant.

For $n = 0$, the operator $l_0 = -z\partial_z$, generates the linear term of (2.28). It is not difficult to verify that the corresponding finite transformation is $z' = az$, with a an arbitrary complex constant. The modulus of a turns out to lead a dilatation, while the phase of a

gives rise to a rotation. In order to get a geometric intuition of such transformations, we can perform the change of variables $z = re^{i\phi}$ to find

$$l_0 = -\frac{1}{2}r\partial_r + \frac{i}{2}\partial_\phi\,, \qquad \bar{l}_0 = -\frac{1}{2}r\partial_r - \partial_\phi\,. \tag{2.30}$$

We can perform the linear combinations

$$l_0 + \bar{l}_0 = -r\partial_r\,, \qquad i(l_0 - \bar{l}_0) = -\partial_\phi\,. \tag{2.31}$$

Therefore, we see that $l_0 + \bar{l}_0$ is the generator for two-dimensional dilations and that $i(l_0 - \bar{l}_0)$ is the generator of rotations.

Finally, the operator l_{+1} generates the infinitesimal quadratic term. By the variable changing $w = -\frac{1}{z}$, the operator turns out to be $l_{+1} = -\partial_w$, which corresponds to a translation of w: $w \mapsto w - c$. Therefore, the operator l_{+1} generates Special Conformal Transformations: $z \mapsto \frac{z}{cz+1}$.

The complete set of mappings discussed above is of the form

$$z \mapsto \frac{az+b}{cz+d}\,, \tag{2.32}$$

with $a, b, c, d \in \mathbb{C}$. For this transformation to be invertible, we have to require that $ad - bc \neq 0$. If this is the case, we can scale the constants a, b, c, d, such that $ad - bc = 1$. Furthermore, the expression is unaffected by taking all of a, b, c, d to minus themselves. These mappings are called projective transformations. We see each global conformal transformation in 2 dimensions can be associated with a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. For instance:

$$\begin{aligned}
&\text{translation}: && z \mapsto z + b && \Leftrightarrow \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \\
&\text{dilatation}: && z \mapsto az\,, a \in \mathbb{R} && \Leftrightarrow \begin{pmatrix} \sqrt{a} & 0 \\ \sqrt{0} & \sqrt{1/a} \end{pmatrix}, \\
&\text{rotation}: && z \mapsto e^{i\theta}\,, \theta \in \mathbb{R} && \Leftrightarrow \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix}, \\
&SCT: && z \mapsto \frac{z}{cz+1} && \Leftrightarrow \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}.
\end{aligned}$$

We can infer that the conformal group of the Riemann sphere $S^2 = \mathbb{C} \cup \{\infty\}$ is the Möbius group $SL(2, \mathbb{C})/\mathbb{Z}_2$.

3 Basics in Conformal Field Theory

Conformal field theories are quantum field theories invariant under conformal transformation. In this section, we will see this simple statement results in powerful constraints in two-dimensional system.

3.1 Noether theorem and Ward-Takahashi identity

In this subsection, let us review the role of symmetry in quantum field theory.

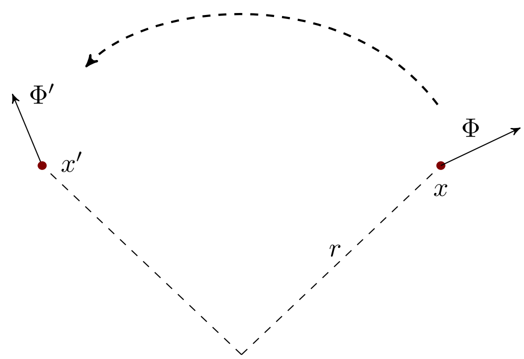
Continuous Symmetry Transformation

In the following sections, we focus on coordinate transformations, while in field theory, we still need to consider the transformation of fields. In the previous sections, we have used transformations of spinless field (2.28), to obtain the corresponding generators. However, sometimes field might have its own non-trivial transformation rules. Figure 4 shows the rotation transformation of a vector field. We see field Φ rotates under the coordinate transformation.

$$x \to x' , \qquad \Phi(x) \to \Phi'(x') = \mathcal{F}(\Phi(x)) . \tag{3.1}$$

Functional change $\mathcal{F}$ can be obtained by studying the representation theory of coordinate transformation group. We will not illustrate more on this, but give some examples for later use.

Figure 4: Rotation transformation of a vector field.



Let us start with a rather trivial one, a translation:

$$x' = x + a , \qquad \Phi'(x + a) = \Phi(x) , \tag{3.2}$$

which implies that functional change $\mathcal{F}$ is trivial here.

Next, we consider a Lorentz transformation (rigid rotation). In general it takes the following form:

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} , \qquad \Phi'(\Lambda x) = U(\Lambda)\Phi(x) , \tag{3.3}$$

where $U(\Lambda)$ is a transformation matrix acting on Φ.[2] The infinitesimal form of $U(\Lambda)$ is

$$U(\Lambda) = 1 - \frac{1}{2} i \omega_{\rho\nu} S^{\rho\nu} , \tag{3.4}$$

where $\omega_{\rho\nu}$ are infinitesimal parameters for Lorentz transformation and $S^{\rho\nu}$ is a representation of the Lorentz algebra.

Finally, under a scale transformations, we have

$$x' = \lambda x , \qquad \Phi'(\lambda x) = \lambda^{-\Delta}\Phi(x) , \tag{3.5}$$

where Δ is the scaling dimension of the field.

[2]In quantum field theory, Φ is more likely to treat as an operator with transformation rule:

$$\Phi'(\Lambda x) = U^{-1}(\Lambda)\Phi(x)U(\Lambda) .$$

Generator

In a field theory, the generator of a symmetry depends on transformation of field as well as of coordinate. The infinitesimal form of (3.1) can be written as:

$$x'^{\mu} = x^{\mu} + \omega_a \frac{\delta x^{\mu}}{\delta \omega_a}\,,$$
$$\Phi'(x') = \Phi(x) + \omega_a \frac{\delta \mathcal{F}}{\delta \omega_a}(x)\,.$$

Here $\{\omega_a\}$ is a set of infinitesimal parameters. Now it is customary to define the generator G_a of a symmetry transformation by the following expression:

$$\delta_{\omega}\Phi(x) \equiv \Phi'(x) - \Phi(x) \equiv -i\omega_a G_a \Phi(x)\,, \tag{3.6}$$

which is just the infinitesimal transformation of field at a same point. Together with (3.6), to first order, we obtain the explicit expression:

$$iG_a\Phi = \frac{\delta x^{\mu}}{\delta \omega_a}\partial_{\mu}\Phi - \frac{\delta \mathcal{F}}{\delta \omega_a}\,. \tag{3.7}$$

For a field satisfying $\Phi'(x') = \Phi(x)$, the expression reduces to (2.16). Using the definition (3.7), and combining with the infinitesimal field of (3.2), (3.3) and (3.5), we can write the corresponding generators in field space

$$\begin{aligned}
&\text{translation}: && P_{\nu} = -i\partial_{\nu}\,,\\
&\text{rotation}: && L^{\rho\nu} = i(x^{\rho}\partial^{\nu} - x^{\nu}\partial^{\rho}) + S^{\rho\nu}\,,\\
&\text{dilatation}: && D = -ix^{\nu}\partial_{\nu} - i\Delta\,.
\end{aligned}$$

We see that when field transformation $\mathcal{F}(\Phi)$ is trivial, three expressions in (3.8) reduce to the first three generators in Table 1.

Noether's Theorem

We now derive Noether's theorem, which states that to every continuous symmetry of the action one may associate a conserved current at classical level. Let us consider an action in d dimensions

$$\mathcal{S} = \int \mathrm{d}^d x \mathcal{L}(\Phi, \partial_{\mu}\Phi) \tag{3.8}$$

which is endowed with symmetry generated by $\omega_a G_a$. Under the continuous transformation (3.1), the action becomes

$$\begin{aligned}
\mathcal{S}' &= \int \mathrm{d}^d x' \mathcal{L}(\Phi'(x'), \partial'_{\mu}\Phi'(x'))\\
&= \int \mathrm{d}^d x \left|\frac{\partial x'}{\partial x}\right| \mathcal{L}(\mathcal{F}(\Phi(x)), (\frac{\partial x^{\nu}}{\partial x'^{\mu}})\partial_{\nu}\mathcal{F}(\Phi(x)))\,.
\end{aligned}$$

Using the infinitesimal form for transformation (3.6), we get

$$\mathcal{S}' = \int \mathrm{d}^d x \left(1 + \partial_{\mu}(\omega_a \frac{\delta x^{\mu}}{\delta \omega_a})\right)\mathcal{L}\left(\Phi + \omega_a\frac{\delta \mathcal{F}}{\delta \omega_a}, \left(\delta^{\nu}_{\mu} - \partial_{\mu}(\omega_a \frac{\delta x^{\nu}}{\delta \omega_a})\right)\left(\partial_{\nu}\Phi + \partial_{\nu}(\omega_a \frac{\delta \mathcal{F}}{\delta \omega_a})\right)\right)\,. \tag{3.9}$$

Since $\omega_a G_a$ is symmetry of the action, $\delta\mathcal{S} = \mathcal{S}' - \mathcal{S}$ vanishes if ω_a is a constant. Thus, the contribution from all terms with no derivatives of ω_a in the variation must be zero. Therefore, to first order, the variation involves only the first derivatives of ω_a, obtained by expanding the Lagrangian so that we can write

$$\delta\mathcal{S} = -\int \mathrm{d}^d x\, j_a^\mu \partial_\mu \omega_a\,, \tag{3.10}$$

where j_a^μ is called the current associated with the infinitesimal transformation:

$$j_a^\mu = \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\partial_\nu\Phi - \delta_\nu^\mu\mathcal{L}\right)\frac{\delta x^\nu}{\delta\omega_a} - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\frac{\delta\mathcal{F}}{\delta\omega_a}\,. \tag{3.11}$$

The integral by parts yields

$$\delta\mathcal{S} = \int \mathrm{d}^d x\, \omega_a \partial_\mu j_a^\mu\,. \tag{3.12}$$

If the field configurations satisfy the classical equations of motion, the action is invariant under any variation of the fields. Thus $\delta\mathcal{S}$ should vanish for any parameters $\omega_a(x)$. This implies the conservation law:

$$\partial_\mu j_a^\mu = 0\,. \tag{3.13}$$

The associated charge

$$Q_a = \int \mathrm{d}^{d-1}x\, j_a^0\,, \tag{3.14}$$

is conserved because its time derivative vanishes

$$\frac{d}{dt}Q_a = \int \mathrm{d}^{d-1}x\, \partial_0 j_a^0 = -\int \mathrm{d}^{d-1}x\, \partial_i j^i = 0\,. \tag{3.15}$$

One remark to be mentioned here is that we may freely add to the expression (3.11) the divergence of an antisymmetric tensor without affecting the conservation law:

$$j_a^\mu \to j_a^\mu + \partial_\nu B_a^{\nu\mu}\,, \qquad B_a^{\nu\mu} = -B_a^{\mu\nu}\,. \tag{3.16}$$

Let us now focus on an important example. First of all, if the action is invariant under translation $x'^\mu \to x^\mu + \epsilon^\mu$, we can associate it with a current by using (3.11):

$$T_{\mathrm{c}}^{\mu\nu} = -\eta^{\mu\nu}\mathcal{L} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\partial^\nu\Phi\,, \tag{3.17}$$

which is the **energy-momentum tensor**. However, the canonical energy-momentum tensor $T_{\mathrm{c}}^{\mu\nu}$ is generally not symmetric under μ and ν. However, we have freedom to modify this tensor as illustrated in (3.16), to make it symmetric.

$$T_{\mathrm{B}}^{\mu\nu} = T_{\mathrm{c}}^{\mu\nu} + \partial_\rho B^{\rho\mu\nu}\,, \quad B^{\rho\mu\nu} = -B^{\mu\rho\nu}\,. \tag{3.18}$$

Now consider the conserved current associated with Lorentz transformation. By using the infinitesimal form (3.4), the conserved current becomes

$$j^{\mu\nu\rho} = T_{\mathrm{c}}^{\mu\nu}x^\rho - T_{\mathrm{c}}^{\mu\rho}x^\nu + \frac{i}{2}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}S^{\nu\rho}\Phi\,. \tag{3.19}$$

We can look for $B^{\rho\mu\nu}$ such that this current may be expressed as

$$j^{\mu\nu\rho} = T_{\mathrm{B}}^{\mu\nu}x^\rho - T_{\mathrm{B}}^{\mu\rho}x^\nu\,. \tag{3.20}$$

This relation ensures that $T_{\mathrm{B}}^{\mu\nu} = T_{\mathrm{B}}^{\nu\mu}$ classically, as is easily seen by applying the conservation laws. The energy-momentum tensors with the above identity are called Belinfante tensor. An explicit expression for $B^{\rho\mu\nu}$ can be found:

$$B^{\mu\rho\nu} = \frac{i}{4}\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}S^{\nu\rho}\Phi + \frac{\partial\mathcal{L}}{\partial(\partial_\rho\Phi)}S^{\mu\nu}\Phi + \frac{\partial\mathcal{L}}{\partial(\partial_\nu\Phi)}S^{\mu\rho}\Phi\right]. \tag{3.21}$$

It is worth mentioning that the choice of $B^{\mu\rho\nu}$ is actually not unique.

Finally, if the action of theory is also invariant under dilatation, the current associated with scale invariance can be obtained by using the infinitesimal form of (3.5) and the definition (3.11):

$$j_{\mathrm{D}}^{\mu} = T_{c\ \nu}^{\mu}x^{\nu} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\Delta\Phi\,. \tag{3.22}$$

We can also modify the energy momentum to become

$$j_{\mathrm{D}}^{\mu} = T^{\mu}{}_{\nu}x^{\nu}\,, \tag{3.23}$$

without spoiling the symmetric condition for a Belinfante tensor. Further more, by using conservation laws, we may find such kind of modified energy momentum tensor is traceless classically, i.e.

$$T^{\mu}{}_{\mu} = 0\,. \tag{3.24}$$

In later discussion, we will always assume the energy momentum tensor is symmetric and traceless at classical level.

Correlation Functions

In previous sections, we focus on the effect of a continuous symmetry classically. At quantum level, the main objects of study are correlation functions. A general correlation function is defined as follows

$$\langle\Phi(x_1)\cdots\Phi(x_n)\rangle = \frac{1}{\mathcal{Z}}\int[\mathrm{d}\Phi]\Phi(x_1)\cdots\Phi(x_n)\exp\{-\mathcal{S}[\Phi]\}\,, \tag{3.25}$$

where $\mathcal{Z}$ is the vacuum functional:

$$\mathcal{Z} = \int[\mathrm{d}\Phi]\exp(-\mathcal{S}[\Phi])\,. \tag{3.26}$$

The path integral denotation $[\mathrm{d}\Phi]$ means to integrate out all the field configurations of Φ. The operators in the correlation function $\Phi(x_1)\cdots\Phi(x_n)$ are automatically time-ordered by using path integral method, i.e.

$$\langle\mathcal{T}(\Phi(x_1)\cdots\Phi(x_n))\rangle = \langle\Phi(x_1)\cdots\Phi(x_n)\rangle\,, \tag{3.27}$$

where $\mathcal{T}$ is the time ordering operator which makes the field operator in time order:

$$\mathcal{T}(\Phi(x_1)\cdots\Phi(x_n)) = \Phi(x_1)\cdots\Phi(x_n)\,,\qquad \text{if}\quad t_1 > t_2 > \cdots > t_n\,. \tag{3.28}$$

In Euclidean space, time direction can be arbitrary. We will specify time direction in two-dimensional case later.

A continuous symmetry leads to constraints on the correlation function:

$$\langle\Phi(x_1')\cdots\Phi(x_2')\rangle = \langle\mathcal{F}(\Phi(x_1))\cdots\mathcal{F}(\Phi(x_n))\rangle\,. \tag{3.29}$$

The above identity is the consequence of the invariance of the action and of the path integral measure under the transformation:

$$
\begin{aligned}
\langle \Phi(x_1') \cdots \Phi(x_n') \rangle &= \frac{1}{\mathcal{Z}} \int [\mathrm{d}\Phi] \Phi(x_1') \cdots \Phi(x_n') \exp(-\mathcal{S}[\Phi]) \\
&= \frac{1}{\mathcal{Z}} \int [\mathrm{d}\Phi'] \Phi'(x_1') \cdots \Phi'(x_n') \exp(-S[\Phi']) \\
&= \frac{1}{\mathcal{Z}} \int [\mathrm{d}\Phi] \mathcal{F}(\Phi(x_1)) \cdots \mathcal{F}(\Phi(x_n)) \exp(-\mathcal{S}[\Phi]) \\
&= \langle \mathcal{F}(\Phi(x_1)) \cdots \mathcal{F}(\Phi(x_n)) \rangle \ .
\end{aligned}
$$

On the second line, we just rename the integration variable $\Phi \to \Phi'$. On the third line, the symmetry assumptions for action and functional integral measure were used. (3.29) immediately implies that correlation function is invariant under translation.

Ward-Takahashi Identity

Quantum version of Noether theorem is called the **Ward-Takahashi identity**. We first change the functional field variable in (3.25) by an infinitesimal transformation (3.6)

$$\Phi'(x) = \Phi(x) - i\omega_a G_a \Phi(x) \,, \tag{3.30}$$

The correlation function is invariant under the symmetry so that we can write

$$\langle X \rangle = \frac{1}{\mathcal{Z}} \int [\mathrm{d}\Phi'](X + \delta X) \exp\left\{ -\mathcal{S}[\Phi] - \int \mathrm{d}^d x \omega_a(x) \partial_\mu j_a^\mu \right\} , \tag{3.31}$$

where we denote $\Phi(x_1) \cdots \Phi(x_n)$ of fields in (3.25) by X. When expanded to first order of $\omega_a(x)$, the above yields

$$\langle \delta X \rangle = \int \mathrm{d}^d x \partial_\mu \left\langle j_a^\mu(x) X \right\rangle \omega_a(x) \,. \tag{3.32}$$

The variation δX is explicitly given by

$$
\begin{aligned}
\delta X &= -i \sum_{i=1}^{n} (\Phi(x_1) \cdots G_a \Phi(x_i) \cdots \Phi(x_n)) \omega_a(x_i) \\
&= -i \int \mathrm{d}^d x \, \omega_a(x) \sum_{i=1}^{n} (\Phi(x_1) \cdots G_a \Phi(x_i) \cdots \Phi(x_n)) \delta(x - x_i)
\end{aligned}
$$

Since the above two expression hold for any infinitesimal function $\omega_a(x)$, we may write the following local relation:

$$
\begin{aligned}
&\frac{\partial}{\partial x^\mu} \left\langle j_a^\mu(x) \Phi(x_1) \cdots \Phi(x_n) \right\rangle \\
&\qquad = -i \sum_{i=1}^{n} \delta(x - x_i) \left\langle \Phi(x_1) \cdots G_a \Phi(x_i) \cdots \Phi(x_n) \right\rangle \ .
\end{aligned}
$$

This is the Ward-Takahashi identity for the current j_a^μ.

The Ward-Takahashi identity allows us to identify the conserved charge Q_a defined in (3.14) as the generator of the symmetry transformation in the Hilbert space of quantum states. Let $Y = \Phi(x_2)\cdots\Phi(x_n)$ and suppose that the time $t = x_1^0$ is larger than all the times in Y. We integrate the Ward-Takahashi identity (3.33) in a very thin box bounded by $t_- < t$, by $t_+ > t$, and by spatial infinity, which excludes all the other points $x_2, \cdots, x_n$.

$$\langle Q_a(t_+)\Phi(x_1)Y\rangle - \langle Q_a(t_-)\Phi(x_1)Y\rangle = -i\,\langle G_a\Phi(x_1)Y\rangle\,. \tag{3.33}$$

In the limit $t_- \to t_+$, for arbitrary set of fields Y, we obtain

$$[Q_a, \Phi] = -iG_a\Phi\,. \tag{3.34}$$

We see that, the conserved charge Q_a is the generator in the operator formalism.

3.2 Primary fields

In the previous section, we have reviewed some concepts of field theory in d dimensions, and derived the current expression for three important currents in conformal symmetry. Now, we should fix our concentration in two dimensions. First, we need to define the field transformation under conformal transformation. From (2.30), we find l_0 and $\bar{l}_0$ are the Cartan subalgebra of conformal algebra. Thus we can assume physical states to be eigenstates of l_0 and $\bar{l}_0$ with eigenvalues $\{h, \overline{h}\}$, which is called the **conformal dimension**. Under a scaling transformation $z \to \lambda z, \bar{z} \to \bar{\lambda}\bar{z}$, the field operator transforms as

$$\phi'(\lambda z, \overline{\lambda}\overline{z}) = (\lambda)^{-h}(\overline{\lambda})^{-\overline{h}}\phi(z, \overline{z})\,. \tag{3.35}$$

If a field transforms under an arbitrary global conformal map $SL(2, \mathbb{C})$, $z \to w(z), \overline{z} \to \overline{w}(\overline{z})$,

$$\phi'(w, \overline{w}) = \left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^{-h}\left(\frac{\mathrm{d}\overline{w}}{\mathrm{d}\overline{z}}\right)^{-\overline{h}}\phi(z, \overline{z})\,, \tag{3.36}$$

then we call such field a **quasi-primary field**. Under infinitesimal transformation $z \to z + \epsilon(z)$, we obtain the infinitesimal transformation of quasi-primary field:

$$\delta_{\epsilon,\overline{\epsilon}}\phi \equiv \phi'(z, \overline{z}) - \phi(z, \overline{z}) = -(h\phi\partial_z\epsilon + \epsilon\partial_z\phi) - (\overline{h}\phi\partial_{\overline{z}}\overline{\epsilon} + \overline{\epsilon}\partial_{\overline{z}}\phi)\,. \tag{3.37}$$

A field whose transformation under any local conformal transformation given by (3.36) or equivalently (3.37) is called a **primary field**. Therefore all primary fields are also quasi primary, but the reverse is not true. If an operator is not primary, it is generally called secondary.

The definition (3.36) puts strong constraints to their correlation functions. For the two-point function of chiral quasi-primary fields, the translation invariance implies that the two-point function is of the form

$$\langle\phi_1(z)\phi_2(w)\rangle = g(z - w)\,. \tag{3.38}$$

Then under conformal transformation $z \to \lambda z$, we have

$$\lambda^{h_1+h_2}g(z - w) = g(\lambda(z - w))\,, \tag{3.39}$$

where h_1,h_2 is the conformal dimensions of ϕ_1 and ϕ_2 correspondingly. This leads a constraint for the two point function. since the relation above satisfies for arbitrary λ we obtain that $g(z - w)$ is of the form

$$g(z - w) = \frac{C_{12}}{(z - w)^{h_1+h_2}}\,, \tag{3.40}$$

with C_{12} a constant. Finally, when apply the conformal transformation $z \to 1/z$, for the two-point function to be non-trivial, the conformal dimensions of the two fields must be the same, i.e.

$$\langle \phi_1(z)\phi_2(w)\rangle = \frac{C_{12}}{(z_1-z_2)^{2h}}\,, \quad \text{if} \quad h_1 = h_2 = h\,. \tag{3.41}$$

The same treatment can be used to obtain the form of three-point function for the 2 dimensional primary operators. For chiral part, we have

$$\langle \phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\rangle = \frac{C_{123}}{z_{12}^{h_1+h_2-h_3} z_{23}^{h_2+h_3-h_1} z_{13}^{h_3+h_1-h_2}}\,, \tag{3.42}$$

where $z_{ij} = z_i - z_j$. Although these results are obtained by Polyakov in [Pol70], determining the four-point function and beyond require the use of an infinite-dimensional symmetry [BPZ84, ISZ88-No.1], which will be considered in §5.6.

3.3 Conformal Ward-Takahashi identity

Now, let us apply Ward-Takahashi identities to general conformal invariance, called **conformal Ward-Takahashi identity**. By observing from (3.20) and (3.23), we find for a conformal transformation with infinitesimal variation ϵ^μ, its current can be written as

$$j_a^\mu(x)\omega_a = T^{\mu\nu}\epsilon_\nu\,. \tag{3.43}$$

In fact, the above relation is satisfied for not only translation, dilatation and rotation, but for arbitrary conformal variation. Plugging into (3.32), we obtain the conformal Ward-Takahashi identity

$$\delta_\epsilon \langle X\rangle = \int_D \mathrm{d}^2x \partial_\mu \langle T^{\mu\nu}(x)\epsilon_\nu(x)X\rangle\,. \tag{3.44}$$

Here the integral is taken over the domain D containing the positions of all the fields in the X. In two dimensions, it is convenient to use (anti-)holomorphic coordinate $(z,\bar{z})$ in (2.25). Writing each component explicitly, we have

$$\begin{aligned} T_{zz} =& \frac{1}{4}(T_{00} - 2iT_{10} - T_{11})\,, \\ T_{\bar{z}\bar{z}} =& \frac{1}{4}(T_{00} + 2iT_{10} - T_{11})\,, \\ T_{z\bar{z}} =& T_{\bar{z}z} = \frac{1}{4}(T_{00} + T_{11}) \end{aligned} \tag{3.45}$$

We find the classically traceless of the energy momentum tensor implies that $T_{z\bar{z}} = T_{\bar{z}z} = 0$ classically. From the conservation law $\partial_\mu T^{\mu\nu} = 0$, we see that T_{zz} and $T_{\bar{z}\bar{z}}$ are holomorphic and antiholomorphic respectively. Furthermore we introduce the notation

$$T = -2\pi T_{zz}\,, \quad \overline{T} = -2\pi T_{\bar{z}\bar{z}}\,. \tag{3.46}$$

Using Gauss's theorem and in 2 dimensions, one find

$$\delta_{\epsilon,\bar{\epsilon}} \langle X\rangle = \frac{i}{2}\int_C \{-\mathrm{d}z \langle T^{\bar{z}\bar{z}}\epsilon_{\bar{z}}X\rangle + \mathrm{d}\bar{z}\langle T^{zz}\epsilon_z X\rangle\}\,, \tag{3.47}$$

where we have defined $\epsilon(z) = \epsilon^z$ and $\bar{\epsilon}(\bar{z}) = \epsilon^{\bar{z}}$, respectively. Using (3.46), we get

$$\delta_{\epsilon,\bar{\epsilon}} \langle X\rangle = -\frac{1}{2\pi i}\oint_C \mathrm{d}z\,\epsilon(z)\,\langle T(z)X\rangle + \frac{1}{2\pi i}\oint_C \mathrm{d}\bar{z}\,\bar{\epsilon}(\bar{z})\,\langle \overline{T}(\bar{z})X\rangle\,, \tag{3.48}$$

where the contour $C = \partial D$ should contain all the positions of fields in X. Notice that the validity of (3.48) extends beyond primary fields, thus it may be taken as a definition of the effect of conformal transformations on an arbitrary local field within a correlation function.

Suppose that X is a product of n primary fields $\phi_i(w_i)$,

$$X(w_1, \dots, w_n) := \phi_1(w_1) \cdots \phi_n(w_n) \,.$$

Using simple complex analysis, we write the variation (3.37) of a primary field

$$\begin{aligned}(\partial_{w_i}\epsilon(w_i))\phi_i(w_i,\overline{w}_i) &= \frac{1}{2\pi i}\oint_C dz\, \frac{\epsilon(z)\phi_i(w_i,\overline{w}_i)}{(z-w_i)^2} \\ \epsilon(w_i)(\partial_{w_i}\phi_i(w_i,\overline{w}_i)) &= \frac{1}{2\pi i}\oint_C dz\, \frac{\epsilon(z)\partial_{w_i}\phi_i(w_i,\overline{w}_i)}{z-w_i}\,,\end{aligned} \tag{3.49}$$

Therefore, the conformal Ward identity (3.48) can be written as

$$\langle T(z)X\rangle = \sum_i \left\{ \frac{1}{z-w_i}\partial_{w_i}\langle X\rangle + \frac{h_i}{(z-w_i)^2}\langle X\rangle \right\} + \text{reg.}\,, \tag{3.50}$$

where "reg." stands for a holomorpohic function of z which is regular at $z = w_i$. The antiholomorphic counterpart is similar. This expression gives the singular behavior of the correlator of fields $T(z)$ with a list of primary field X as z approaches the point w_i. This implies the **operator product expansion** (OPE) of the energy-momentum tensor with primary fields by removing the bracket $\langle \cdots \rangle$

$$T(z)X \sim \sum_i \left\{ \frac{1}{z-w_i}\partial_{w_i}X + \frac{h_i}{(z-w_i)^2}X \right\}. \tag{3.51}$$

The symbol $\sim$ means the regular terms are ignored. OPE also contains an infinite numbers of regular terms which, for energy-momentum tensor, can not be obtained from the conformal Ward identity. The **operator product expansion** (OPE) is the representation of a product of operators, which describes the behavior when the two operators approach to each other. In general, we would write the OPE of two operator $A(z)$, $B(w)$ as

$$A(z)B(w) = \sum_{n=-\infty}^{\infty} \frac{\{AB\}_n(w)}{(z-w)^n}\,, \tag{3.52}$$

where the composite operators $\{AB\}_n(w)$ are nonsingular at $w = z$. In particular for a single primary field ϕ with conformal dimensions h and $\overline{h}$, we have:

$$T(z)\phi(w,\overline{w}) \sim \frac{h}{(z-w)^2}\phi(w,\overline{w}) + \frac{1}{z-w}\partial_w\phi(w,\overline{w})\,, \tag{3.53}$$

$$\overline{T}(\overline{z})\phi(w,\overline{w}) \sim \frac{\overline{h}}{(\overline{z}-\overline{w})^2}\phi(w,\overline{w}) + \frac{1}{\overline{z}-\overline{w}}\partial_{\overline{w}}\phi(w,\overline{w})\,. \tag{3.54}$$

The expression above can be treated as an alternative definition for a primary field. For a secondary field, but with conformal dimension h and $\overline{h}$, its OPE with energy-momentum tensor has more singular terms, i.e.

$$T(z)\phi(w,\overline{w}) \sim \cdots + \frac{h}{(z-w)^2}\phi(w,\overline{w}) + \frac{1}{z-w}\partial_w\phi(w,\overline{w})\,, \tag{3.55}$$

$$\overline{T}(\overline{z})\phi(w,\overline{w}) \sim \cdots + \frac{\overline{h}}{(\overline{z}-\overline{w})^2}\phi(w,\overline{w}) + \frac{1}{\overline{z}-\overline{w}}\partial_{\overline{w}}\phi(w,\overline{w})\,. \tag{3.56}$$

3.4 Example: Free boson

Now we will study free boson field as an example. We first review some basic concepts in field theory.

Calculation of Two Point Function

A two point function is a correlation function of two operators. Therefore, from the definition of correlation function (3.25), we can easily write the definition of two point function for bosonic field as

$$\langle \mathcal{T}(\phi(x_1)\phi(x_2))\rangle = \frac{1}{\mathcal{Z}}\int [\mathrm{d}\phi]\phi(x_1)\phi(x_2)\exp\{-\mathcal{S}[\Phi]\}\,. \tag{3.57}$$

The time ordering operation $\mathcal{T}$ is automatically satisfied in a path integral. Then, we can define a generating functional for the theory:

$$\mathcal{Z}[J] = \int [\mathrm{d}\phi]\exp\left\{-(S - \int \mathrm{d}^d x\phi(x)J(x)\right\}, \tag{3.58}$$

where $J(x)$ is an auxiliary "current". Then the two point function can be generated bu functional derivative of J :

$$\langle \phi(x_1)\phi(x_2)\rangle = \frac{1}{\mathcal{Z}[0]}\frac{\delta}{\delta J(x_1)}\frac{\delta}{\delta J(x_2)}\gamma\mathcal{Z}[J]\bigg|_{J=0}\,. \tag{3.59}$$

For calculating the two point function more generally, we first assume the action S take the following form:

$$\mathcal{S} = \frac{1}{2}\int \mathrm{d}^d x\mathrm{d}^d y\,\phi(x)A(x,y)\phi(y)\,. \tag{3.60}$$

Let us introduce a variable transformation $\phi \to \phi + \phi_0$ without changing the path integral measure and result as we have illustrated before. The term in the exponential of generating functional becomes:

$$\begin{aligned}-\mathcal{S} + \int \mathrm{d}^d x\phi(x)J(x) \to &-\frac{1}{2}\int \mathrm{d}^d x\mathrm{d}^d y\phi(x)A(x,y)\phi(y) - \int \mathrm{d}^d x\mathrm{d}^d y\phi(x)A(x,y)\phi_0(y) - \\ &-\frac{1}{2}\int \mathrm{d}^d x\mathrm{d}^d y\phi_0(x)A(x,y)\phi_0(y) + \int \mathrm{d}^d x\phi_0(x)J(x) + \int \mathrm{d}^d x\phi(x)J(x)\,,\end{aligned}$$

Now we choose ϕ_0 to satisfy the equation:

$$\int \mathrm{d}^d yA(x,y)\phi_0(y) = J(x) \;\to \phi_0(x) = \int \mathrm{d}^d yA^{-1}(x,y)J(y)\,. \tag{3.61}$$

Then, the generating functional can be written as

$$\mathcal{Z}[J] = N\exp\left\{\frac{1}{2}\int \mathrm{d}^d x\mathrm{d}^d y\,J(x)A^{-1}(x,y)J(y)\right\}, \tag{3.62}$$

where the factor N is independent of J

$$N = \int [\mathrm{d}\phi]\exp\left\{-\frac{1}{2}\int \mathrm{d}^d x\mathrm{d}^d y\,\phi(x)A(x,y)\phi(y)\right\}. \tag{3.63}$$

Therefore, by using (3.59), we have

$$\langle \mathcal{T}(\phi(x)\phi(y))\rangle = A^{-1}(x,y) \tag{3.64}$$

We will apply the formula above to calculate the two-point function of specific theories.

Two-Point function in Free boson

In two dimensions, the massless free boson has the following Euclidian action:

$$\mathcal{S} = \frac{1}{8\pi}\int d^2x\, \partial_\mu\varphi\partial^\mu\varphi\,, \tag{3.65}$$

which is conformal. Comparing with (3.60), we have

$$A(x,y) = -\frac{1}{4\pi}\delta^{(2)}(x-y)\Box\,, \tag{3.66}$$

From (3.64), we can calculate the two-point function $K(x,y) \equiv \langle\varphi(x_1)\varphi(x_2)\rangle$ by solving the following equation:

$$-\frac{1}{4\pi}\Box K(x,y) = \delta^{(2)}(x-y)\,, \tag{3.67}$$

From (3.29) and the trivial nature of $\mathcal{F}$ for a scalar field, $K(x,y)$ is invariant under translation and rotation. Thus, we can write $K(x,y) \equiv K(\rho)$ with $\rho = |x-y|$, and integrate over x within a Disk of radius r around y. We find

$$\begin{aligned}1 &= \frac{1}{2}\int_0^r d\rho\rho\Big(-\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho K'(\rho))\Big)\\ &= \frac{1}{2}(-rK'(r))\,.\end{aligned}$$

The solution of two-point function for massless free boson can be obtained up to an additive constant,

$$\langle\varphi(x)\varphi(y)\rangle = -\ln|x-y|^2\,. \tag{3.68}$$

Wick's Theorem

In the previous part, we have introduced two-point function for a theory. In this part, we are going to briefly review the context of Wick's Theorem. We will see two-point function is a basic quantity of a theory due to Wick's Theorem.

In quantum field theory, a free(elementary) field operator can be split into two part:

$$\phi(x) = \phi^+(x) + \phi^-(x)\,, \tag{3.69}$$

with $\phi^-(x)$ which contains only annihilation operators and $\phi^+(x)$ contains only creation operators. A **normal-ordering** : : is defined to put all the annihilation operators to the right side of creation operators. For instance,

$$:\phi(x_1)\phi(x_2):= \phi(x_1)^+\phi(x_2)^+ + \phi(x_1)^-\phi(x_2)^- + \phi(x_1)^+\phi(x_2)^- + \phi(x_2)^+\phi(x_1)^-\,.$$

And the time ordering product for $x_1^0 > x_2^0$ gives

$$\mathcal{T}(\phi(x_1)\phi(x_2)) = \phi(x_1)^+\phi(x_2)^+ + \phi(x_1)^-\phi(x_2)^- + \phi(x_1)^+\phi(x_2)^- + \phi(x_1)^-\phi(x_2)^+\,.$$

Thus, we have the relation for time ordering product and normal ordering product for $x_1^0 > x_2^0$ as

$$\mathcal{T}(\phi(x_1)\phi(x_2)) =: \phi(x_1)\phi(x_2) : + \big[\phi(x_1)^-, \phi(x_2)^+\big]\,. \tag{3.70}$$

The commutator in the right side is a constant. In fact, by taking the expectation value of both sides, we find the constant is just a two-point function, i.e.

$$\mathcal{T}(\phi(x_1)\phi(x_2)) =: \phi(x_1)\phi(x_2) : + \langle \phi(x_1)\phi(x_2)\rangle \ . \tag{3.71}$$

Note that, for $x_1^0 < x_2^0$, we get the same expression. It is convenient to introduce **Wick contraction** for computation of two-point functions and OPE

$$\wick{\c\phi_1(x_1)\c\phi_2(x_2)} \equiv \langle \phi_1(x_1)\phi_2(x_2)\rangle \ . \tag{3.72}$$

The Wick's theorem states that the time-ordered product of operators at different points in the space-time is equal to the normal-ordered product, plus all possible ways of contracting pairs of fields within it. For instance,

$$\begin{aligned}\mathcal{T}(\phi_1\phi_2\phi_3\phi_4) = &: \phi_1\phi_2\phi_3\phi_4 : + : \wick{\c\phi_1\c\phi_2}\phi_3\phi_4 : + : \wick{\c\phi_1\phi_2\c\phi_3}\phi_4 : + : \wick{\c\phi_1\phi_2\phi_3\c\phi_4} : \\ &+ : \phi_1\wick{\c\phi_2\c\phi_3}\phi_4 : + : \phi_1\wick{\c\phi_2\phi_3\c\phi_4} : + : \phi_1\phi_2\wick{\c\phi_3\c\phi_4} : + : \wick{\c1\phi_1\c1\phi_2\c2\phi_3\c2\phi_4} : \\ &+ : \wick{\c1\phi_1\c2\phi_2\c1\phi_3\c2\phi_4} : + : \wick{\c1\phi_1\c2\phi_2\c2\phi_3\c1\phi_4} : \ .\end{aligned}$$

The normal-ordering can be applied to the product of two operators at the same point in the space-time. In this situation, we can apply Wick's theorem in the following way

$$\begin{aligned}&\mathcal{T}(: \phi_1(x)\phi_2(x) :: \phi_3(y)\phi_4(y) :) \\ = &: \phi_1(x)\phi_2(x)\phi_3(y)\phi_4(y) : + : \wick{\c\phi_1(x)\phi_2(x)\c\phi_3(y)}\phi_4(y) : + : \wick{\c\phi_1(x)\phi_2(x)\phi_3(y)\c\phi_4(y)} : \\ &+ : \phi_1(x)\wick{\c\phi_2(x)\c\phi_3(y)}\phi_4(y) : + : \phi_1(x)\wick{\c\phi_2(x)\phi_3(y)\c\phi_4(y)} : + : \wick{\c1\phi_1(x)\c2\phi_2(x)\c1\phi_3(y)\c2\phi_4(y)} : \\ &+ : \wick{\c1\phi_1(x)\c2\phi_2(x)\c2\phi_3(y)\c1\phi_4(y)} : \ .\end{aligned}$$

The Wick's theorem for fermion is almost the same except that the exchange of two fermions contributes a minus sign.

OPEs in Free boson

Now we are ready to calculate OPE for the massless free boson. The equation of motion for the action (3.65) is

$$\Box\varphi = 0\,. \tag{3.73}$$

In complex coordinates, we have

$$\partial_{\bar{z}}\partial_z\varphi(z,\bar{z}) = 0\,, \tag{3.74}$$

implying that $\partial_z\varphi$ and $\partial_{\bar{z}}\varphi$ are holomorphic and antiholomorphic respectively. The canonical energy momentum tensor defined by (3.17) is

$$T_{\mu\nu} = \frac{1}{4\pi}\left[\partial_\mu\varphi\partial_\nu\varphi - \frac{1}{2}\eta_{\mu\nu}\partial_\rho\varphi\partial^\rho\varphi\right]. \tag{3.75}$$

It is automatically symmetric and traceless, and its quantum version in complex coordinates is

$$T(z) = -\frac{1}{2} : \partial_z\varphi\partial_z\varphi : \ . \tag{3.76}$$

In order for its vacuum expectation value to vanish, the energy momentum tensor has to be normal ordered. The antiholomorphic part is similar. We can write two point function (3.68) in terms of the complex coordinates, this is

$$\langle \varphi(z,\bar{z})\varphi(w,\overline{w})\rangle = -\{\ln(z-w)+\ln(\bar{z}-\overline{w})\} + \text{const}\,. \tag{3.77}$$

We shall concentrate more on the holomorphic field $\partial_z\varphi$. By taking derivatives ∂_z and ∂_w from above, we have

$$\langle \partial_z\varphi(z,\bar{z})\partial_w\varphi(w,\overline{w})\rangle = -\frac{1}{(z-w)^2}\,. \tag{3.78}$$

Hence, the OPE of $\partial_z\varphi$ with itself is

$$\partial_z\varphi(z)\partial_w\varphi(w) \sim -\frac{1}{(z-w)^2}\,. \tag{3.79}$$

The OPE of $T(z)$ with $\partial_z\varphi$ may be calculated from Wick's theorem:

$$\begin{aligned}
T(z)\partial_w\varphi(w) &= -\frac{1}{2} :\partial_z\varphi(z)\partial_z\varphi(z): \partial_w\varphi(w) \\
&\sim - :\partial_z\varphi(z)\partial_z\varphi(z): \partial_w\varphi(w) \\
&\sim \frac{\partial_z\varphi(z)}{(z-w)^2}\,.
\end{aligned}$$

By expanding $\partial_z\varphi(z)$ around w, we arrive at the OPE

$$T(z)\partial_w\varphi(w) \sim \frac{\partial_w\varphi(w)}{(z-w)^2} + \frac{\partial_w^2\varphi(w)}{z-w}\,. \tag{3.80}$$

Comparing with (3.53), we see $\partial_z\varphi(z)$ is a primary field with conformal dimension h=1. Wick's theorem also allows us to calculate the OPE of the energy momentum tensor with itself.

$$T(z)T(w) \sim \frac{1/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{(z-w)}\,. \tag{3.81}$$

Transformation of the Energy-Momentum Tensor

It is easy to see that the energy momentum tensor is not a primary field, because there is the term

$$\frac{1/2}{(z-w)^4}\,.$$

From (3.55), we see that energy momentum tensor for free massless boson is a secondary field, with conformal dimension $h=2$. For a general 2d conformal field theory, TT OPE takes the form

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{(z-w)}\,, \tag{3.82}$$

with c a constant called the **central charge**. The central charge depends on a 2d CFT, and it is one of the most important information of a 2d CFT. For the free boson, it is equal to one.

By using conformal Ward-Takahashi identity (3.48), we get an infinitesimal variation of T under a local conformal transformation.

$$\begin{aligned}\delta_\epsilon T(w) &= -\frac{1}{2\pi i}\oint_C \mathrm{d}z\,\epsilon(z)T(z)T(w) \\ &= \frac{1}{12}c\partial_w^3\epsilon(w) - 2T(w)\partial_w\epsilon(w) - \epsilon(w)\partial_w T(w)\,.\end{aligned}$$

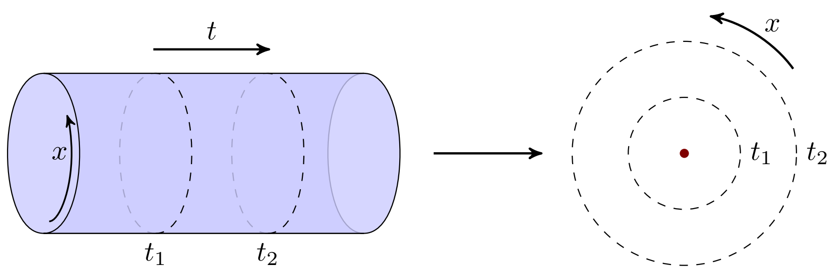
Its finite transformation can be written as $z \to w(z)$ is

$$\widetilde{T}(w) = \left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^{-2}\left[T(z) - \frac{c}{12}\{w;z\}\right], \tag{3.83}$$

where $\{w;z\}$ is the additional term called the **Schwarzian derivative**:

$$\{w;z\} = \frac{(\mathrm{d}^3w/\mathrm{d}z^3)}{\mathrm{d}w/\mathrm{d}z} - \frac{3}{2}\left(\frac{\mathrm{d}^2w/\mathrm{d}z^2}{\mathrm{d}w/\mathrm{d}z}\right)^2. \tag{3.84}$$

Figure 5: Mapping from the cylinder to the complex plane.



3.5 Virasoro Algebra

Radial Quantization

In the previous discussion, we find the operator formalism distinguishes a time direction from a space direction. However, in a 2d CFT, it is most convenient to work in the complex coordinate with Euclidian signature. In fact, there is a natural choice in the context of string theory by using radial quantization. We may first define our theory on an infinite space-time cylinder, with time t going from $-\infty$ to ∞ along the "flat" direction of the cylinder, and space being compactified with a coordinate x going from 0 to ℓ, the points $(0, t)$ and (ℓ, t) being identified. t is still a Euclidean time, and the cylinder can be described by a complex coordinate $\xi = t + ix$. We then map the cylinder onto a complex plane by a conformal transformation

$$z = e^{2\pi\xi/\ell}\,, \tag{3.85}$$

as illustrated in Figure 5 [Car84b, ISZ88-No.18]. The remote past $(t \to -\infty)$ is situated at the origin $z = 0$, whereas the remote future $(t \to +\infty)$ lies at infinity. Now, the theory lives on the complex plane where the radial ray and and the circle around the origin correspond to the t-direction and the circle of the cylinder, respectively. As it is known from quantum mechanics, the generator of time translations is the Hamiltonian which in the present case corresponds to the dilation operator. Similarly, the generator for space translations is the momentum operator corresponding to rotations. Recalling (2.31), we can write

$$H = L_0 + \overline{L}_0\,, \qquad P = i(L_0 - \overline{L}_0)\,, \tag{3.86}$$

where L_0 and $\overline{L}_0$ are the zero mode of energy-momentum tensor as we will see below.

A time ordering product now becomes the radial ordering product:

$$\mathcal{R}(\phi_1(z)\phi_2(w)) = \begin{cases} \phi_1(z)\phi_2(w) & |z| > |w|\,, \\ \phi_2(w)\phi_1(z) & |z| < |w|\,. \end{cases} \tag{3.87}$$

As usual, we will always omit the radial-ordered operator in the correlation function as well as in the OPE expansion. One consequence after specifying time direction is that we can relate OPE to commutation relations. For this, le consider the contour integral around w for two holomorphic field $a(z)$ and $b(w)$

$$\oint_w \mathrm{d}z\, a(z)b(w) = \oint_{C_1} \mathrm{d}z\, a(z)b(w) - \oint_{C_2} \mathrm{d}z\, b(w)a(z) = [A, b(w)]\,, \tag{3.88}$$

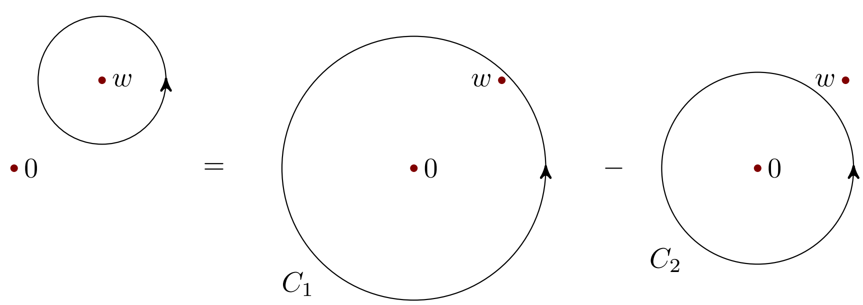
where the operator A is the contour integral of $a(z)$ at a fixed time

$$A = \oint \mathrm{d}z\, a(z)\,. \tag{3.89}$$

Here we take the contours C_1 and C_2 at fixed-radius $|w| + \epsilon$ and $|w| - \epsilon$ with small positive number ϵ as illustrated in Figure 6 . Then, with $B = \oint \mathrm{d}z\, b(z)$, we can generalize the relation (3.88) to

$$[A, B] = \oint_0 \mathrm{d}w [A, b(w)] = \oint_0 \mathrm{d}w \oint_w \mathrm{d}z a(z)b(w)\,. \tag{3.90}$$

Figure 6: Subtraction of contours



State/Operator Correspondence

After the radial quantization, we can generate states living on the circle by inserting operators. The vacuum state $|0\rangle$ corresponds to inserting nothing. The initial state $|\phi_{\text{in}}\rangle$

corresponds to inserting an operator at the origin, since the infinite past on the cylinder is mapped to $z = \overline{z} = 0$,

$$|\phi_{\text{in}}\rangle = \lim_{z,\overline{z}\to 0} \phi(z,\overline{z}) |0\rangle \ . \tag{3.91}$$

As we have learnt, a 2d CFT is endowed with the scaling transformation as a symmetry. The state defined on a circle with fixed radius can be scaled down to the origin by the scaling transformation as in (3.91), which results in an operator. Conversely, an operator at the origin can be scaled up to a state on a circle around the origin. Therefore, there is one-to-one correspondence between a state and an operator in 2d CFT. This is called the **state-operator correspondence**.

On this Hilbert space, we must also define a bilinear product, by defining an asymptotic out state. In Minkowski space, Hermitian conjugation does not affect the space-time coordinates. However, since the the Euclidian time $t_{\text{E}} = it_{\text{M}}$, the Euclidian time must be reversed upon Hermitian conjugation. In radial quantization this corresponds to the mapping $z \to 1/\overline{z}$. Therefore, the definition of Hermitian conjugation is

$$[\phi(z,\overline{z})]^\dagger = \overline{z}^{-2h} z^{-2\overline{h}} \phi(1/\overline{z}, 1/z) \, , \tag{3.92}$$

where h and $\overline{h}$ are conformal dimensions of quasi-primary field ϕ. The prefactors on the r.h.s. can be justified by demanding that the hermitian conjugation exchanges the conformal dimensions for holomorphic and antiholomorphic part. We than define the asymptotic out state:

$$\langle\phi_{\text{out}}| = |\phi_{\text{in}}\rangle^\dagger \ . \tag{3.93}$$

Then, the inner product is well defined:

$$\begin{aligned}\langle\phi_{\text{out}}|\phi_{\text{in}}\rangle &= \lim_{z,\overline{z},w,\overline{w}\to 0} \overline{z}^{-2h} z^{-2\overline{h}} \langle 0| \phi(1/\overline{z}, 1/z)\phi(w,\overline{w}) |0\rangle \\ &= \lim_{\xi,\overline{\xi}\to\infty} \overline{\xi}^{2h} \xi^{2\overline{h}} \langle 0| \phi(\overline{\xi},\xi)\phi(0,0) |0\rangle \ .\end{aligned}$$

According to the expression (3.41), the last expression is independent of ξ, which also justifies the prefactors appearing in (3.92).

The mode expansion of a field $\phi(z,\overline{z})$ with conformal dimension $(h,\overline{h})$ is conventionally given by

$$\phi(z,\overline{z}) = \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} z^{-m-h} \overline{z}^{-n-\overline{h}} \phi_{m,n} \, . \tag{3.94}$$

A straightforward Hermitian conjugation yields

$$\phi(z,\overline{z})^\dagger = \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} \overline{z}^{-m-h} z^{-n-\overline{h}} \phi^\dagger_{m,n} \, , \tag{3.95}$$

while the definition (3.92) gives

$$\phi(z,\overline{z})^\dagger = \overline{z}^{-2h} z^{-2\overline{h}} \phi(1/\overline{z}, 1/z) = \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} \phi_{-m,-n} \overline{z}^{-m-h} z^{-n-\overline{h}} \, . \tag{3.96}$$

Comparing the above two expressions for mode expansions, we get

$$\phi^\dagger_{m,n} = \phi_{-m,-n} \tag{3.97}$$

This is the usual expression for the Hermitian conjugate of modes, not only for primary fields.

Conformal Generators and Virasoro Algebra

In §2.7, we have derived the conformal generators l_n and $\bar{l}_n$ in coordinate space, which form two copies of the Witt algebra (2.30). Now, we shall derive the conformal generators in Hilbert space and its corresponding algebra. We have already seen in (3.34) that the conserved charge defined by (3.14) acts as a generator of the corresponding symmetry in Hilbert space. Since the current for conformal symmetry has the form of $T^{\mu\nu}\epsilon_\nu$, writing in complex coordinate and dropping the antiholomorphic part for convenience, we arrive at the conserved charge by performing the contour integral of equal time as

$$Q_\epsilon = \frac{1}{2\pi i}\oint_C \mathrm{d}z\, \epsilon(z) T(z)\,. \tag{3.98}$$

Then, the conformal Ward-Takahashi identities (3.48) becomes

$$\delta_\epsilon \Phi(z) = -[Q_\epsilon, \Phi(z)]\,, \tag{3.99}$$

as same as (3.34) which verifies that the conserved charge in (3.98) acts like a conformal generator as well. Now we expand the holomorphic part of energy-momentum as following,

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2} L_n\,, \tag{3.100}$$

with L_n equals to

$$L_n = \frac{1}{2\pi i}\oint \mathrm{d}z\, z^{n+1} T(z)\,. \tag{3.101}$$

If we expand $\epsilon(z)$ as

$$\epsilon(z) = \sum_{n\in\mathbb{Z}} z^{n+1}\epsilon_n\,, \tag{3.102}$$

we find the conserved charge

$$Q_\epsilon = \sum_{n\in\mathbb{Z}} \epsilon_n L_n\,. \tag{3.103}$$

Therefore, L_n and its antiholomorphic part $\overline{L}_n$ become a set of conformal generators in Hilbert space. The commutator of L_n can be obtained by using (3.90)

$$[L_n, L_m] = \frac{1}{(2\pi i)^2}\oint_0 \mathrm{d}w\, w^{m+1}\int_w \mathrm{d}z\, z^{n+1} T(z)T(w)\,. \tag{3.104}$$

Plugging TT OPE (3.82), we finally obtain the **Virasoro algebra** [Vir70]

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12} n(n^2-1)\delta_{n+m,0}\,. \tag{3.105}$$

The antiholomorphic part is similar. When central charge c vanishes, the Virasoro algebra reduces to the Witt algebra.

3.6 Verma Module

In this section, we will briefly introduce the Hilbert space of a 2d CFT constructed by primary field operators and generators in Virasoro algebra. We have previously mentioned the vacuum state $|0\rangle$ corresponds to a state with no operator inserted. Now, we want to put more constraints to vacuum state in order to begin the story. A vacuum state should satisfy the following condition:

$$L_n |0\rangle = 0 \qquad \overline{L}_n |0\rangle = 0 . \qquad (n \geq -1) \tag{3.106}$$

This can be recovered from the condition $T(z)|0\rangle$ and $\overline{T}(\overline{z})|0\rangle$ are well-defined as z,$\overline{z}$ goes to zero. We know that asymptotic states can be obtained by acting field operator at the origin on the vacuum state. By inserting primary field, the obtained state becomes the eigenstate of L_0.

$$\left|h,\overline{h}\right\rangle \equiv \phi(0,0)|0\rangle \ . \tag{3.107}$$

A simple demonstration follows from (3.90) and OPE for primary field (3.53):

$$\begin{aligned}[L_n, \phi(w,\overline{w})] &= \frac{1}{2\pi i}\oint_w \mathrm{d}z\, z^{n+1} T(z)\phi(w,\overline{w}) \\ &= h(n+1)w^n \phi(w,\overline{w}) + w^{n+1}\partial\phi(w,\overline{w}) .\end{aligned}$$

With $w \to 0$ above, we immediately reach the conclusion that $\left|h,\overline{h}\right\rangle$ is an eigenstate of L_0.

$$L_0 \left|h,\overline{h}\right\rangle = h\left|h,\overline{h}\right\rangle , \qquad \overline{L}_0 \left|h,\overline{h}\right\rangle = \overline{h}\left|h,\overline{h}\right\rangle . \tag{3.108}$$

Likewise, we have for $n > 0$

$$L_n \left|h,\overline{h}\right\rangle = 0 , \qquad \overline{L}_n \left|h,\overline{h}\right\rangle = 0 . \qquad (n > 0) \tag{3.109}$$

The generators $L_{-m}(m > 0)$ also increase the conformal dimension, by virtue of the Virasoro algebra (3.105):

$$[L_0, L_{-m}] = mL_{-m} . \tag{3.110}$$

This means the excited state with higher conformal dimensions can be obtained by successively applying of L_{-m} on $|h\rangle$

$$L_{-k_1}L_{-k_2}\cdots L_{-k_n}|h\rangle \ , \qquad (1 \leq k_1 \leq k_2 \cdots \leq k_n) \tag{3.111}$$

where L_{-k_i} appear in increasing order by convention. The states defined in (3.111) are called descendants of $|h\rangle$, with eigenvalue of L_0

$$h' = h + k_1 + k_2 + k_3 + \cdots + k_n \equiv h + N , \tag{3.112}$$

where N is called the level of the descendant. A subset of the full Hilbert space generated by the asymptotic state $|h\rangle$ and its descendants is closed under the action of the Virasoro algebra and thus forms a representation of the Virasoro algebra. This subspace is called Verma module.

Table 2 shows states in a Verma module up to level four, where $p(l)$ is the number of linearly independent states at level l. This is nothing but the partitions of the integer l. In fact, we define the partition function of the free boson

$$\mathrm{Tr}(q^{L_0 - \frac{c}{24}}) = \frac{1}{\eta(q)} \equiv q^{-\frac{1}{24}} \prod_{n=1}^{\infty} \frac{1}{1-q^n} = \sum_{l=0}^{\infty} p(l) q^{l-\frac{1}{24}} , \tag{3.113}$$

which can be also understood as the generating function of the number $p(l)$ of the partition of l. The function

$$\eta(q) := q^{1/24} \prod_{n=1}^{\infty}(1-q^n)$$

Table 2: States of a Verma module

l	$p(l)$	
0	1	$\|h\rangle$
1	1	$L_{-1}\|h\rangle$
2	2	$L_{-1}^2\|h\rangle, L_{-2}\|h\rangle$
3	3	$L_{-1}^3\|h\rangle, L_{-1}L_{-2}\|h\rangle, L_{-3}\|h\rangle$
4	5	$L_{-1}^4\|h\rangle, L_{-1}^2L_{-2}\|h\rangle, L_{-1}L_{-3}\|h\rangle . L_{-2}^2\|h\rangle, L_{-4}\|h\rangle$

is called the **Dedekind η-function**.

We defne the inner product according to our previous definition of the Hermitian conjugate (3.92) : $L_m^\dagger = L_{-m}$. Therefore the dual state $|h\rangle$ satisfies

$$\langle h| L_j = 0 \, . \qquad (j < 0) \tag{3.114}$$

Then the inner product of two states $L_{-k_1} \cdots L_{-k_n} |h\rangle$ and $L_{-l_1} \cdots L_{-l_n} |h\rangle$ is

$$\langle h| L_{k_m} \cdots L_{k_1} L_{-l_1} \cdots L_{-l_n} |h\rangle \, . \tag{3.115}$$

This product can be evaluated using Virasoro algebra. We will see that the inner product of two states vanishes unless they belong to the same level. This inner product depends only on the highest weight h and central charge c. For the representation theory to be unitary, which means it contains no negative-norm states, there must be a strong constraint on (h, c). For instance, consider the norm of the state $L_{-n} |h\rangle$

$$\langle h| L_n L_{-n} |h\rangle = \left[2nh + \frac{1}{12} cn(n^2 - 1) \right] \langle h|h\rangle \, . \tag{3.116}$$

It is easy to see that the norm becomes negative for n sufficiently large if $c < 0$. Moreover, for $n = 1$, we find the theory with negative conformal dimensions is not unitary.

4 Free boson and free fermion

In this section, we are going to demonstrate more simple examples in CFT.

4.1 Free boson

In the previous sections, we have learned calculation of some basic OPEs in the massless free boson theory. We will further study this theory.

Mode Expansion

First let us recall that the action of free boson (3.65)

$$\mathcal{S} = \frac{1}{8\pi} \int \mathrm{d}^2 x \partial_\mu \varphi \partial^\mu \varphi \, . \tag{4.1}$$

First we consider the free boson on the cylinder, in Minkowski space-time. The fact that the fields φ live on the cylinder implies that they may satisfy periodic boundary conditions:

$$\varphi(x + \ell, t) = \varphi(x, t) \, , \tag{4.2}$$

where L is the circumference of the cylinder. The classical equation of motion for φ is

$$[\partial_t^2 - \partial_x^2]\varphi(x,t) = 0\,. \tag{4.3}$$

It is a good exercise to verify that φ of following form satisfies the above two conditions:

$$\varphi_{\text{cyl}} = \varphi_0 + \frac{4\pi}{\ell}\pi_0 t + i\sum_{n\neq 0}\frac{1}{n}\left(a_n e^{-2\pi i n(t-x)/\ell} + \bar{a}_n e^{-2\pi i n(t+x)/\ell}\right), \tag{4.4}$$

where 'cyl' means the field defined on the cylinder. The canonical momentum for the theory is

$$\Pi = \frac{\delta\mathcal{S}}{\delta(\partial_t\varphi)} = \frac{1}{4\pi}\partial_t\varphi\,, \tag{4.5}$$

For quantum fields, we impose the following canonical commutation relations

$$\begin{aligned}
&[\varphi(x,t),\Pi(y,t)] = i\delta(x-y)\,,\\
&[\varphi(x,t),\varphi(y,t)] = 0\,,\\
&[\Pi(x,t),\Pi(y,t)] = 0\,,
\end{aligned}$$

In terms of the modes in (4.4), these are equivalent to

$$\begin{aligned}
&[a_k, a_l] = [\bar{a}_k, \bar{a}_l] = k\delta_{k+l,0}\,,\\
&[a_k, \bar{a}_l] = 0\,,
\end{aligned}$$

for $k, l \neq 0$, and

$$[\varphi_0, \pi_0] = i\,. \tag{4.6}$$

The Hamiltonian for free boson theory is

$$H = \frac{1}{8\pi}\int \mathrm{d}x[:(\partial_t\varphi)^2: + :(\partial_x\varphi)^2:]\,, \tag{4.7}$$

Plugging the mode expansion (4.4), the Hamiltonian is then expressed as

$$H = \frac{2\pi}{\ell}\pi_0^2 + \frac{\pi}{\ell}\sum_{n\neq 0}(:a_{-n}a_n: + :\bar{a}_{-n}\bar{a}_n:)\,, \tag{4.8}$$

The commutation relations (4.6) lead to the relation

$$[H, a_{-m}] = \frac{2\pi}{\ell}m a_{-m}\,, \tag{4.9}$$

which means $a_{-m}(m>0)$ act as a creation operator. When applied to an eigenstate $|E\rangle$ of H of energy $E|E\rangle$, the state $a_{-m}|E\rangle$ is another eigenstate with energy $(E+2m\pi/\ell)a_{-m}|E\rangle$. For the same reason, $a_m(m>0)$ is an annihilation operator.

Transformation to Complex Plane

Now, we move to Euclidian space-time ($t \to -it$) by taking $w = t - ix$ and $\overline{w} = t + ix$. The mode expansion of boson field on the cylinder now is then

$$\varphi_{\text{cyl}}(w,\overline{w}) = \varphi_0 - i\frac{2\pi}{\ell}\pi_0(w+\overline{w}) + i\sum_{n\neq 0}\frac{1}{n}\left(a_n e^{2\pi n w/\ell} + \bar{a}_n e^{2\pi n\overline{w})/\ell}\right), \tag{4.10}$$

with now $w \sim w + i\ell$. We now using a conformal transformation to map all the operators from the cylinder to the complex plane:

$$z = e^{2\pi w/\ell}, \qquad \bar{z} = e^{2\pi \bar{w}/\ell}. \tag{4.11}$$

Since π is invariant under conformal transformations, we finally obtain the expansion by simply replacing w by z.

$$\varphi_{\mathrm{pl}}(z, \bar{z}) = \varphi_0 - i\pi_0 \ln(z\bar{z}) + i \sum_{n \neq 0} \frac{1}{n} (a_n z^{-n} + \bar{a}_n \bar{z}^{-n}). \tag{4.12}$$

Now we concentrate on the holomorphic field $\partial\varphi$

$$i\partial\varphi(z) = \frac{\pi_0}{z} + \sum_{n \neq 0} a_n z^{-n-1}. \tag{4.13}$$

We may write the zero mode by a_0 and $\bar{a}_0$:

$$a_0 \equiv \bar{a}_0 \equiv \pi_0 \tag{4.14}$$

Now the commutation relations of (4.6) can be extended to include the zero mode operator without changing the form of the algebra. The mode expansion of $\partial\varphi$ is consistent with the expansion of a primary field with $h = 1$ (3.94):

$$i\partial\varphi(z) = \sum_n a_n z^{-n-1}. \tag{4.15}$$

We can calculate two-point function by using mode expansion. For ($|z| > |w|$):

$$\langle \varphi(z)\partial\varphi(w) \rangle = \sum_{m,n \neq 0} \frac{1}{n} \langle a_n a_m \rangle z^{-n} w^{-m-1}. \tag{4.16}$$

According to the commutation relation (4.6) and the fact that a_m and a_{-m} are annihilation and creation operators respectively, it follows that

$$\langle \varphi(z)\partial\varphi(w) \rangle = \frac{1}{w} \frac{w/z}{1 - w/z} = \frac{1}{z - w}. \tag{4.17}$$

Its differentiation with respect to z provides the two-point function.

$$\langle \partial\varphi(z)\partial\varphi(w) \rangle = -\frac{1}{(z - w)^2}. \tag{4.18}$$

The holomorphic energy-momentum tensor is given by

$$T(z) = -\frac{1}{2} : \partial\varphi(z)\partial\varphi(z) := \frac{1}{2} \sum_{n,m \in \mathbb{Z}} z^{-n-m-2} : a_n a_m :, \tag{4.19}$$

which implies

$$\begin{aligned} L_n &= \frac{1}{2} \sum_{m \in \mathbb{Z}} : a_{n-m} a_m : \qquad (n \neq 0) \\ L_0 &= \sum_{n > 0} a_{-n} a_n + \frac{1}{2} a_0^2 . \end{aligned}$$

The Hamiltonian (4.8) now can be written as,

$$H = \frac{2\pi}{\ell}(L_0 + \overline{L}_0)\,. \tag{4.20}$$

This confirm the role of L_0 and $\overline{L}_0$ as a Hamiltonian. The mode operators a_m plays a similar role to L_m with respect to L_0, because of the commutation relation $[L_0, a_{-m}] = ma_{-m}$. Therefore its effect on the conformal dimension is the same as that of L_m.

By definition the normal ordering prescription implies that $\langle T(z)\rangle = 0$.[3] We can always define the normal ordering product by subtracting the all the singular terms from the OPE. In fact the general expectation value for energy momentum tensor can be written as

$$\langle T(z)\rangle = -\frac{1}{2}\lim_{\epsilon\to 0}\left(\langle\partial\varphi(z+\epsilon)\partial\varphi(z)\rangle + \frac{1}{\epsilon^2}\right). \tag{4.21}$$

We will see this relation is useful when considering anti-periodic boundary condition later. By plugging in to the two-point function, we see that $\langle T(z)\rangle = 0$, which implies that $(L_0)_{\mathrm{pl}}$ vanishes on the vacuum. We now map the theory back to a cylinder $z \to w = \frac{\ell}{2\pi}\ln z$, by using (3.83):

$$T_{\mathrm{cyl}}(w) = \left(\frac{2\pi}{\ell}\right)^2\left[T_{\mathrm{pl}}(z)z^2 - \frac{1}{24}\right]. \tag{4.22}$$

where we use the central charge $c = 1$ for the free boson. Taking the expectation value on both sides, we gave

$$\langle T_{\mathrm{cyl}}\rangle = -\frac{1}{24}\left(\frac{2\pi}{\ell}\right)^2. \tag{4.23}$$

This implies

$$(L_0)_{\mathrm{cyl}} = \sum_{n>0} a_{-n}a_n - \frac{1}{24}\,. \tag{4.24}$$

Similarly for the anti-holomorphic part. The Hamiltonian is now written as

$$H = \frac{2\pi}{\ell}((L_0)_{\mathrm{cyl}} + (\overline{L}_0)_{\mathrm{cyl}})\,. \tag{4.25}$$

Actually, in general case, Hamiltonian of a theory defined on a cylinder with central charge c can be written as

$$H = \frac{2\pi}{\ell}((L_0)_{\mathrm{pl}} + (\overline{L}_0)_{\mathrm{pl}} - \frac{c}{12})\,. \tag{4.26}$$

Thus, we can infer that the central charge c affects the vacuum energy of a theory [BCN86, ISZ88-No.20].

[3] However, as we will see in the next section, this is not the case for the fermionic theory with anti-periodic boundary condition.

4.2 Free fermion

OPEs in Free fermion

Now we consider another simple model: free fermion. In two dimensions, the action of a free Majorana fermion is

$$\mathcal{S} = \frac{1}{4\pi}\int d^2x \Psi^\dagger \gamma^0 \gamma^\mu \partial_\mu \Psi \,, \tag{4.27}$$

where the gamma matrices γ^μ satisfy the so-called **Clifford algebra**:

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\eta\nu} \,, \tag{4.28}$$

and we impose the **Majorana condition** ($\Psi^* = \Psi$) to the fermionic field to remove a half of the degrees of freedom. In the Euclidean space $\eta^{\mu\nu} = \mathrm{diag}(1,1)$, we take a basis of Dirac matrices as

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^1 = i\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{4.29}$$

Using this basis, we can express the action as

$$\mathcal{S} = \frac{1}{2\pi}\int d^2x (\overline{\psi}\partial\overline{\psi} + \psi\overline{\partial}\psi) \,, \tag{4.30}$$

where we write the two-component spinor Ψ as $(\psi, \overline{\psi})$. Since the equations of motion are $\partial\bar{\psi} = 0$ and $\overline{\partial}\psi = 0$, $\psi(z)$ and $\overline{\psi}(\overline{z})$ holomorphic and antiholomorphic field, respectively.

Now let us calculate the two-point function as in the free boson

$$K_{ij} = \langle \Psi_i(x)\Psi_j(y)\rangle \quad (i,j = 1,2) \,. \tag{4.31}$$

The action can be expressed by

$$\mathcal{S} = \frac{1}{4\pi}\int d^2x d^2y \Psi_i(x) A_{ij}(x,y) \Psi(y) \,, \tag{4.32}$$

where the kernel is

$$A_{ij}(x,y) = \delta(x-y)(\gamma^0\gamma^\mu)_{ij}\partial_\mu \,. \tag{4.33}$$

Recalling (3.64), the propagator K_{ij} is the inverse of A_{ij}. Therefore, we can write the equation for K as

$$\delta(x-y)(\gamma^0\gamma^\mu)_{ik}\frac{\partial}{\partial x^\mu}K_{kj}(x,y) = \delta(x-y)\delta_{ij} \,. \tag{4.34}$$

In terms of complex coordinates, this becomes

$$\frac{1}{\pi}\begin{pmatrix} \partial_{\overline{z}} & 0 \\ 0 & \partial_z \end{pmatrix}\begin{pmatrix} \langle\psi(z,\overline{z})\psi(w,\overline{w})\rangle & \langle\psi(z,\overline{z})\overline{\psi}(w,\overline{w})\rangle \\ \langle\overline{\psi}(z,\overline{z})\psi(w,\overline{w})) & \langle\overline{\psi}(z,\overline{z})\overline{\psi}(w,\overline{w})\rangle \end{pmatrix} = \frac{1}{\pi}\begin{pmatrix} \partial_{\overline{z}}\frac{1}{z-w} & 0 \\ 0 & \partial_z\frac{1}{\overline{z}-\overline{w}} \end{pmatrix}, \tag{4.35}$$

where we have used the complex form of δ function: $\delta(x) = \frac{1}{\pi}\partial_{\overline{z}}\frac{1}{z} = \frac{1}{\pi}\partial_z\frac{1}{\overline{z}}$. Therefore, we obtain the two point functions for the fermionic field,

$$\begin{aligned} \langle\psi(z,\overline{z})\psi(w,\overline{w})\rangle &= \frac{1}{z-w} \,, \\ \langle\overline{\psi}(z,\overline{z})\overline{\psi}(w,\overline{w})\rangle &= \frac{1}{\overline{z}-\overline{w}} \,, \\ \langle\psi(z,\overline{z})\overline{\psi}(w,\overline{w})\rangle &= 0 \,. \end{aligned} \tag{4.36}$$

Thus the OPE of two holomorphic fields can be written as:

$$\psi(z)\psi(w) \sim \frac{1}{z-w} \tag{4.37}$$

In order to see whether fermion field is primary field or not, we can calculate its OPE with energy-momentum tensor. By using (3.17) in complex-coordinate form, we can calculate all the components of energy-momentum tensor.

$$\begin{aligned} T^{\bar{z}\bar{z}} &= 2\frac{\partial \mathcal{L}}{\partial(\bar{\partial}\Phi)}\partial\Phi = \frac{1}{\pi}\psi\partial\psi\,, \\ T^{zz} &= 2\frac{\partial \mathcal{L}}{\partial(\partial\Phi)}\bar{\partial}\Phi = \frac{1}{\pi}\bar{\psi}\bar{\partial}\bar{\psi}\,, \\ T^{z\bar{z}} &= 2\frac{\partial \mathcal{L}}{\partial(\partial\Phi)}\partial\Phi - 2\mathcal{L} = -\frac{1}{\pi}\psi\bar{\partial}\psi\,. \end{aligned} \tag{4.38}$$

The traceless condition $T^{z\bar{z}}$ is preserved when taking into account the equation of motion, as we have discussed. The holomorphic part is defined as:

$$T(z) = -\frac{1}{2} : \psi(z)\partial\psi(z) : \,. \tag{4.39}$$

The normal-ordering product can be written in an equivalent way for free field as follow:

$$: \psi\partial\psi : (z) = \lim_{w\to z}(\psi(z)\partial\psi(w) - \langle\psi(z)\partial\psi(w)\rangle)\,, \tag{4.40}$$

which is the same expression (4.21) as in bosonic field theory. Then we can calculate the OPE between fermion field and energy-momentum tensor directly.

$$T(z)\psi(w) \sim \frac{\frac{1}{2}\psi(w)}{(z-w)^2} + \frac{\partial\psi(w)}{z-w}\,. \tag{4.41}$$

This immediately tells us that in free fermion model, ψ is a primary field with conformal dimension $1/2$. TT OPE can also be obtained directly by calculation.

$$T(z)T(w) \sim \frac{1/4}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)}\,. \tag{4.42}$$

This expression verifies the general form of TT OPE (3.82), with central charge in this model equals to $1/2$.

Mode Expansion

We now study the free fermion field living on a cylinder in analogue with what we have down in the boson field section. We work on a cylinder of circumference L. But now we apply two types of boundary conditions:

$$\begin{aligned} \psi(x+\ell) &\equiv \psi(x) \quad \text{Ramond(R)}\,, \\ \psi(x+\ell) &\equiv -\psi(x) \quad \text{Neveu-Schwarz (NS)}\,. \end{aligned} \tag{4.43}$$

In analogue with free boson, the mode expansion for free fermion takes the form:

$$\psi(x,t) = \sqrt{\frac{2\pi}{\ell}} \sum_{k\in\mathbb{Z}+\nu} b_k e^{-2\pi k w/\ell}\,. \tag{4.44}$$

In the periodic case (R), k takes integer values ($\nu = 0$) whereas in the anti-periodic case (NS) it take all half integer values ($\nu = \frac{1}{2}$). Here we introduce a complex coordinate $w = (t_E - ix)$ as before with t_E now the Euclidian time. For quantum fermionic fields, we introduce the canonical anti-commutation relation

$$\left\{b_k, b_q\right\} = \delta_{k+q,0}\,, \tag{4.45}$$

where b_{-k} for $k > 0$ can be considered as a creation operator and b_k for $k > 0$ as an annihilation one. Note that, in the R sector, there exists a zero mode b_0 which leads to a degeneracy of the vacuum.

Mapping to Complex Plane

By introducing a coordinate transformation $z = e^{2\pi w/\ell}$, the cylinder can be mapped onto the plane. Unlike in the free bosons theory, ψ has conformal dimension $1/2$, which leads to a prefactor given by the definition of conformal transformation for primary field (3.36),

$$\psi_{\text{cyl}}(w) \to \psi_{\text{pl}}(z) = \sqrt{\frac{\ell}{2\pi z}}\psi_{\text{cyl}}(z)\,. \tag{4.46}$$

Therefore, on the plane the mode expansion changes to

$$\psi(z) = \sum_{k\in\mathbb{Z}+\nu} b_k z^{-k-1/2}\,. \tag{4.47}$$

This coincides with the general expansion formula for primary field (3.94). Now the prefactor $z^{-1/2}$ has interchanged the role of two boundary conditions: The NS condition corresponds to a periodic field ($\nu = \frac{1}{2}$) and the R condition to an anti-periodic field ($\nu = 0$)

$$\begin{aligned} \psi\left(e^{2\pi i}z\right) &= -\psi(z) \quad \text{Ramond (R)}\,, \\ \psi\left(e^{2\pi i}z\right) &= \psi(z) \quad \text{Neveu-Schwarz (NS)}\,. \end{aligned} \tag{4.48}$$

We now calculate the two-point function in the NS sector by using the mode expansion:

$$\langle\psi(z)\psi(w)\rangle = \sum_{k,q\in\mathbb{Z}+1/2} z^{-k-1/2}w^{-q-1/2}\left\langle b_k b_q\right\rangle\,. \tag{4.49}$$

Applying the commutation relations (4.45), this can be calculated:

$$\langle\psi(z)\psi(w)\rangle = \sum_{k\in\mathbb{Z}+1/2,k>0} z^{-k-1/2}w^{k-1/2} = \frac{1}{z-w}\,. \tag{4.50}$$

The final result is the same as the two-point function for fermion we have derived before. However, in the Ramond sector, the two point function is different.

$$\langle\psi(z)\psi(w)\rangle = \frac{1}{2\sqrt{zw}} + \sum_{k=1}^{\infty} z^{-k-1/2}w^{k-1/2} = \frac{1}{2}\frac{\sqrt{z/w}+\sqrt{w/z}}{z-w}\,. \tag{4.51}$$

When $w \to z$, this result coincides with the previous one. So, the boundary condition does not affect the two point function locally. However, globally when z or w is taken around the origin, there arise a minus sign , due to the boundary condition. We find different boundary conditions lead to different two-point functions on the plane.

Vacuum Expectation Value

In this section, we will see the boundary condition will affect the vacuum expectation of energy-momentum tensor and finally, affect the vacuum energy. As we have mentioned before, the energy momentum tensor can be calculated by the following normal-ordering prescription:

$$\langle T(z)\rangle = \frac{1}{2}\lim_{\epsilon\to 0}\left(-\langle\psi(z+\varepsilon)\partial\psi(z)\rangle + \frac{1}{\varepsilon^2}\right). \tag{4.52}$$

In the NS sector, the vacuum expectation value of energy momentum tensor vanishes as before.

$$\langle T(z)\rangle = 0\,. \tag{4.53}$$

However, in the R sector, the calculation yields:

$$\langle T(z)\rangle = -\frac{1}{4}\lim_{w\to z}\partial_w\left(\frac{\sqrt{z/w}+\sqrt{w/z}}{z-w}\right) + \frac{1}{2(z-w)^2} = \frac{1}{16z^2}\,. \tag{4.54}$$

This means boundary conditions do affect the expectation value of energy-momentum tensor. Now plugging the mode expansion expression (4.47) into the energy-momentum tensor (4.39):

$$\begin{aligned} T_{\text{pl.}}(z) &= \frac{1}{2}\sum_{k,q\in\mathbb{Z}+\nu}\left(k+\frac{1}{2}\right)z^{-q-1/2}z^{-k-3/2} :b_q b_k: \\ &= \frac{1}{2}\sum_{n\in\mathbb{Z},k\in\mathbb{Z}+\nu}\left(k+\frac{1}{2}\right)z^{-n-2} :b_{n-k}b_k:\ . \end{aligned}$$

We can easily extract the conformal generator from the expression above:

$$L_n = \frac{1}{2}\sum_{k\in\mathbb{Z}+\nu}\left(k+\frac{1}{2}\right) :b_{n-k}b_k:\ . \tag{4.55}$$

We can write L_0 as follows by using the commutation relations (4.45).

$$\begin{aligned} L_0 &= \sum_{k>0} k b_{-k}b_k \quad \left(\text{NS} : k\in\mathbb{Z}+\frac{1}{2}\right), \\ L_0 &= \sum_{k>0} k b_{-k}b_k + \frac{1}{16} \quad (\text{R} : k\in\mathbb{Z})\,, \end{aligned} \tag{4.56}$$

where the constant term is fixed from the vacuum expectation value of energy momentum tensor for NS and R sector, since L_0 is the coefficient of $1/z^2$ in the expansion. As mentioned before, the Hamiltonian of such a theory defined on the cylinder can be determined by the following expression with L_0 calculate on the complex plane (4.26) with c the central charge which is $1/2$, in this theory. From the relation above, we can immediately obtain the vacuum energy in the holomorphic part as follows:

$$\frac{\ell}{2\pi}E_0 = \begin{cases} -\frac{1}{48} & \text{NS sector} \\ +\frac{1}{24} & \text{R sector} \end{cases} \tag{4.57}$$

The same result can be obtained by mapping the energy momentum tensor from complex plane to cylinder, by using (3.83):

$$T_{\text{cyl}}(w) = \left(\frac{2\pi}{\ell}\right)^2\left[T_{\text{pl}}(z)z^2 - \frac{c}{24}\right], \tag{4.58}$$

with $c = 1/2$. Taking expectation value from both sides of above, and then treat NS sector and R sector respectively we find:

$$\langle T_{\text{cyl.}} \rangle = \begin{cases} -\dfrac{1}{48}\left(\dfrac{2\pi}{\ell}\right)^2 \text{NS sector} \\ \dfrac{1}{24}\left(\dfrac{2\pi}{\ell}\right)^2 \text{R sector} \end{cases} \tag{4.59}$$

This implies the same vacuum energy as before.

4.3 Torus partition functions

Moduli of a two-torus

In this chapter, we are going to study CFTs defined on a torus T^2. The easiest way to obtain such a theory while using previous result is to cut out a finite piece of the infinite cylinder described by $w = x^0 + ix^1$, where x^0 is Euclidean time and we identify the circumference of cylinder $l \equiv 2\pi$ in this section. Thus, the conformal mapping $z = e^w$ defines a theory from cylinder to a complex plane. We will now perform the compactification to the torus. A torus can be defined by identifying points w in the complex plane $\mathbb{C}$ as

$$w \sim w + m\alpha_1 + n\alpha_2 , \qquad m, n \in \mathbb{Z} \tag{4.60}$$

where (α_1, α_2) is a pair of complex numbers. This pair spans a lattice whose smallest cell illustrates the shape of a torus as in Figure 7. Thus, we can introduce the **complex structure**

$$\tau = \frac{\alpha_2}{\alpha_1} = \tau_1 + i\tau_2 , \tag{4.61}$$

which parametrizes the shape of a torus. However, there are various choices of pairs (α_1, α_2) giving the same lattice and thus the same torus. If the two pairs (β_1, β_2) and (α_1, α_2) describes the same lattice, then they are related by an invertible 2×2 matrix with integral entries

$$\begin{pmatrix} \beta_2 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha_2 \\ \alpha_1 \end{pmatrix} , \qquad a, b, c, d \in \mathbb{Z} . \tag{4.62}$$

Moreover, because the area is preserved, we have $ad - bc = 1$ so that the matrix is an element of $SL(2, \mathbb{Z})$. Furthermore, the torus illustrated by (α_1, α_2) is the same as the one by $(-\alpha_1, -\alpha_2)$, so that the transformation by the $\mathbb{Z}_2$ subgroup of $SL(2, \mathbb{Z})$ is trivial. All in all, the symmetry of the lattice with fixed shape is $SL(2, \mathbb{Z})/\mathbb{Z}_2$, which is called the **modular group** or the **mapping class group** of the torus. Finally, by choosing $(\alpha_1, \alpha_2) = (1, \tau)$, a modular group acts on the complex structure of the torus as τ as

$$\tau \mapsto \frac{a\tau + b}{c\tau + d} \quad \text{with} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})/\mathbb{Z}_2 . \tag{4.63}$$

Now let us look at some particular cases. T-transformation:

$$T : \tau \to \tau + 1 , \tag{4.64}$$

which makes the lattice $(\alpha_1, \alpha_2) = (1, \tau)$ to $(1, \tau + 1)$, thus define the same lattice. And U-transformation:

$$U : \tau \to \frac{\tau}{\tau + 1} , \tag{4.65}$$

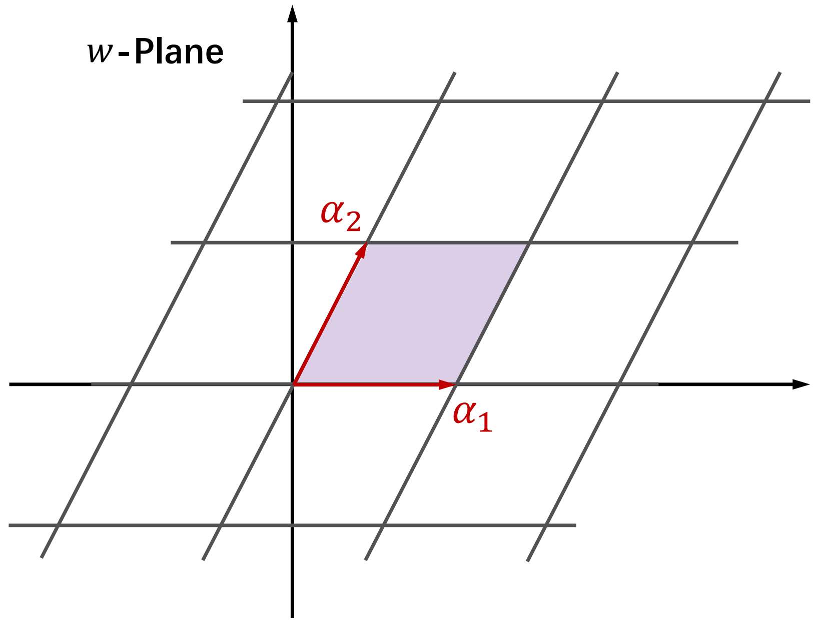


Figure 7: Lattice of a torus

which makes the lattice $(\alpha_1, \alpha_2) = (1, \tau)$ to $(1 + \tau, \tau)$, thus define the same lattice. However, it seems more convenient to work with S- transformation instead of U- transformation.

$$S : \tau \to -\frac{1}{\tau} \tag{4.66}$$

which interchanges the lattice $(\alpha_1, \alpha_2) \leftrightarrow (-\alpha_2, \alpha_1)$. One can easily show that

$$S = UT^{-1}U\,, \qquad S^2 = \mathbb{1} \qquad (ST)^3 = \mathbb{1}\,. \tag{4.67}$$

In fact S and T can prove to be the generators of the modular group $SL(2, \mathbb{Z})/\mathbb{Z}_2$. Therefore, it is sufficient for to check whether a theory is modular invariant by studying its behavior under T and S transformations.

Torus partition function

Now we are in a position to define the partition function $\mathcal{Z}$ in terms of Virasoro generators. This is essentially the same thing for CFTs as in statistical mechanics: a sum over configurations weighted by a Boltzmann factor $\exp(-\beta H)$. It also corresponds to the generating functional in Euclidean QFT due to the fact that the thermodynamic expression can be found by compactifying the time on a circle of radius $R = \beta = 1/T$.

We choose our coodinate system so that the real and imaginary axes to correspond to the spatial and time directions, respectively, and we consider a torus with modular parameter $\tau = \tau_1 + i\tau_2$. For definiteness, we currently choose $\alpha_1 = 1, \alpha_2 = \tau$ (see Figure 7). From this picture, it is clear that a time translation of length τ_2 does not come back to where it started. Instead, it is displaced in space by a factor τ_1. A "closed loop" in time thus also involves a spatial translation. We are therefore motivated to define the CFT partition function as

$$\mathcal{Z} = \mathrm{Tr}_{\mathcal{H}} \left(e^{-2\pi\tau_2 H} e^{2\pi\tau_1 P} \right). \tag{4.68}$$

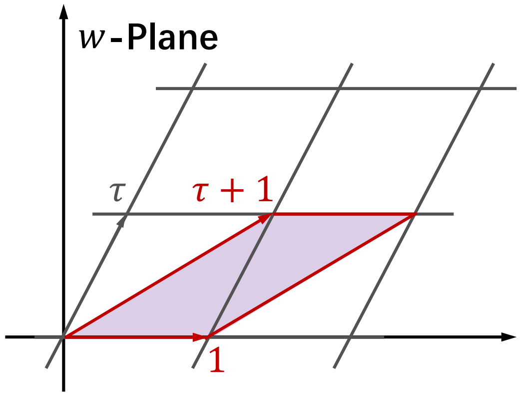


Figure 8: T-transformation

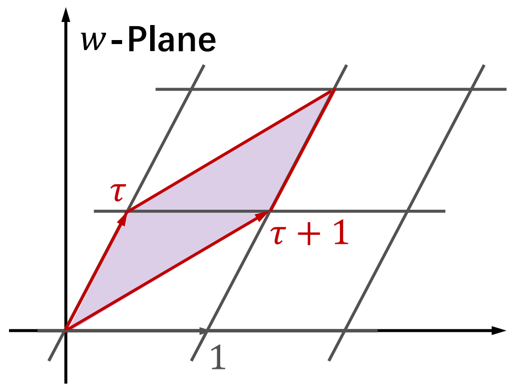


Figure 9: U-transformation

The Hamiltonian H generates time translations, the momentum operator P generates spatial translations, and the trace is taken over all states in the Hilbert space $\mathcal{H}$ of the theory.

Recalling the relations between H, P and Virasoro generators, we know that

$$H_{cyl} = L_{0,cyl} + \overline{L}_{0,cyl}, \qquad P_{cyl} = i(L_{0,cyl} - \overline{L}_{0,cyl}). \tag{4.69}$$

Then we can express the partition function as

$$\mathcal{Z} = \mathrm{Tr}_{\mathcal{H}} \left(e^{2\pi i \tau L_{0,cyl}} e^{-2\pi i \overline{\tau} \overline{L}_{0,cyl}} \right). \tag{4.70}$$

Then by defining $q = \exp(2\pi i \tau)$, we conclude that the partition function for a conformal field theory defined on a torus with modular parameter τ is given by

$$\mathcal{Z} = \mathrm{Tr}_{\mathcal{H}} \left(q^{L_0 - \frac{c}{24}} \overline{q}^{\overline{L}_0 - \frac{\overline{c}}{24}} \right). \tag{4.71}$$

Note that this expression for the partition function involves the characters (3.113). As we will see later, the partition function is indeed expressed in terms of the characters of irreducible representations

$$\mathcal{Z}(\tau) = \sum_{(h,\overline{h})} \mathcal{N}_{h\overline{h}} \chi_h(\tau) \overline{\chi}_{\overline{h}}(\overline{\tau}) \, ,$$

where the multiplicity $\mathcal{N}_{\overline{h}h}$ counts the multiplicity of the representations labelled by $(h, \overline{h})$ in the spectrum $\mathcal{H}$.

Free boson

Having introduced the partition function on the torus and the modular group, we now turn our attention to constraints in specific models. As always, we start with the simplest model: a single free boson. in this theory, it follows from (4.8) that L_0 is expressed as the modes a_m

$$L_0 = \frac{1}{2} a_0 a_0 + \sum_{k=1}^{\infty} a_{-k} a_k. \tag{4.72}$$

In addition, the Hilbert space $\mathcal{H}$ of the free boson can be obtained by acting with creation operators a_{-k} for $k > 0$ on the vacuum $|0\rangle$. From (4.6), we can derive

$$[L_0, a_{-k}] = k a_{-k} \,.$$

Thus, using the geometric progression $(1-x)^{-1} = 1 + x + x^2 + \ldots$, the contribution from states $\bigoplus_\ell a_{-k}^\ell |0\rangle$ to the partition function amounts can be read off as

$$\mathrm{Tr}_{\bigoplus_\ell a_{-k}^\ell |0\rangle}\, q^{L_0} = \sum_{\ell=0}^{\infty} q^{\ell k} = \frac{1}{1-q^k} \,.$$

Since generic states in the Hilbert space $\mathcal{H}$ are obtained by acting with creation operators a_{-k}

$$|n_1, n_2, n_3, \cdots\rangle = a_{-1}^{n_1} a_{-2}^{n_2} \cdots |0\rangle, \quad \text{with} \quad n_i \in \mathbb{Z}_{\geq 0}\,, \tag{4.73}$$

we can calculate the holormorphic part of the torus partition function function

$$\begin{aligned} \mathrm{Tr}_{\mathcal{H}}(q^{L_0 - \frac{c}{24}}) &= q^{-\frac{1}{24}} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \left\langle n_1, n_2, \cdots \middle| q^{L_0} \middle| n_1, n_2, \cdots \right\rangle \\ &= q^{-\frac{1}{24}} \prod_{k=1}^{\infty} \frac{1}{1-q^k} \,. \end{aligned}$$

This is exactly the Dedekind η-function (4.92), and including the anti-holomorphic contribution, we have therefore found the partition function

$$\mathcal{Z}'_{\mathrm{B}}(\tau, \overline{\tau}) = \frac{1}{|\eta(\tau)|^2}. \tag{4.74}$$

However, this is not the end of the story. The torus partition function should be invariant under the modular transformations (4.63) since it does not change the shape of the torus. The modular transformations of the Dedekind η-function is given in (4.93), and it is easy to see that the partition function (4.74) is **not** invariant under the S-transformation. The reason for this is that we miss the contribution the zero mode a_0, and we need to integrate over all the zero modes in the path integral formalism, which yields

$$\int da_0 (q\overline{q})^{\frac{a_0^2}{2}} = \int da_0\, e^{2\pi i \tau_2 a_0^2} = \frac{1}{\sqrt{2\tau_2}} \,. \tag{4.75}$$

Hence, the torus partition function of the free boson is

$$\mathcal{Z}_{\mathrm{B}}(\tau, \overline{\tau}) = \frac{1}{\sqrt{2\tau_2}|\eta(\tau)|^2} \,, \tag{4.76}$$

which is modular-invariant as expected.

Free boson on a circle

The free boson considered above can be understood as a linear sigma model where a bosonic field $\varphi : T^2 \to \mathbb{R}$ is a map from a torus T^2 to a line $\mathbb{R}$. As another example, we consider the free boson φ compactified on a circle S^1 of radius R where a bosonic field

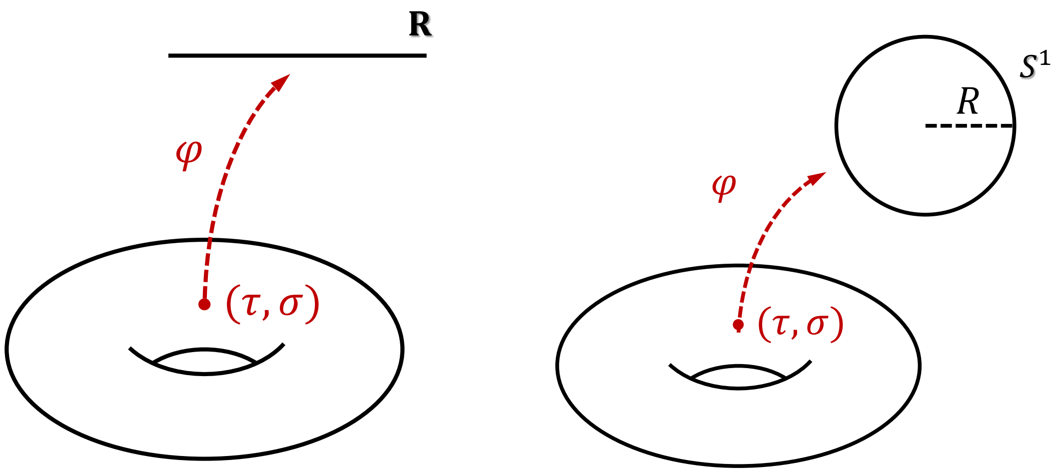


Figure 10: Left: free boson. Right: free boson on a circle.

$\varphi : T^2 \to S^1$ is a map from a torus T^2 to a circle S^1 of radius R. (See Figure 10.) Therefore, we impose the periodicity

$$\varphi(z, \bar{z}) \sim \varphi(z, \bar{z}) + 2\pi R n, \quad n \in \mathbb{Z} . \tag{4.77}$$

To see how this compactification changes the partition function, let us recall the mode expansion (4.10) of the bosonic field φ:

$$\varphi(w, \overline{w}) = \varphi_0 - i\frac{2\pi}{\ell} \left[(a_0 + \bar{a}_0)t - i(a_0 - \bar{a}_0)x\right] + i \sum_{n \neq 0} \frac{1}{n} \left(a_n e^{2\pi n w/\ell} + \bar{a}_n e^{2\pi n \overline{w})/\ell} \right) . \tag{4.78}$$

To find the interesting new constraints, we require that the field φ is invariant up to identifications (4.77) under the rotation $x \to x + \ell$ on a spatial circle of the torus T^2

$$\varphi\left(t, x + \ell\right) = \varphi(t, x) + 2\pi R n , \tag{4.79}$$

which provides

$$a_0 - \bar{a}_0 = R w \quad w \in \mathbb{Z}. \tag{4.80}$$

Since w counts how many times the spatial circle of the torus wraps on the target S^1, it is called the **winding number**.

In addition, it follows from (4.78) that the center of mass momentum of the string is $p = (a_0 + \bar{a}_0)/2$. Since a boson field is compactified on a circle, the momentum is quantized as

$$a_0 + \bar{a}_0 = \frac{2n}{R} \quad n \in \mathbb{Z} ,$$

where n is called the **Kaluza-Klein momentum**.

Thus, a vacuum state is labelled by the winding number and Kaluza-Klein momentum as $|n, w\rangle$ where

$$a_0 |n, w\rangle = \left(\frac{n}{R} + \frac{R w}{2} \right) |n, w\rangle, \qquad \bar{a}_0 |n, w\rangle = \left(\frac{n}{R} - \frac{R w}{2} \right) |n, w\rangle .$$

The contributions from descendants are as before (4.74). However, we now sum over all the zero modes a_0 and $\overline{a}_0$, which is [Yan87, ISZ88-No.40] [DFSZ87, ISZ88-No.43]

$$\mathcal{Z}_R(\tau,\overline{\tau}) = \frac{1}{|\eta(\tau)|^2}\sum_{n,w} q^{\frac{1}{2}(\frac{n}{R}+\frac{Rw}{2})^2}\overline{q}^{\frac{1}{2}(\frac{n}{R}-\frac{Rw}{2})^2}. \tag{4.81}$$

It is straightforward to check that it is invariant under the T-transformation. The invariance under the S-transformation requires the **Poisson resummation formula**

$$\sum_{n\in\mathbb{Z}} \exp\left(-\pi a n^2 + bn\right) = \frac{1}{\sqrt{a}}\sum_{k\in\mathbb{Z}} \exp\left(-\frac{\pi}{a}\left(k+\frac{b}{2\pi i}\right)^2\right). \tag{4.82}$$

The derivation of this expression and its application to deriving invariance of the partition function under the modular S-transformation are left as an exercise.

Remarkably, (4.81) enjoys T-**duality**

$$\mathcal{Z}_{2/R}(\tau,\overline{\tau}) = \mathcal{Z}_R(\tau,\overline{\tau})\,. \tag{4.83}$$

This amounts to the fact that closed strings propagating around a circle cannot distinguish whether the size of the circle is R or $2/R$. The self-dual radius $R=\sqrt{2}$ can be interpreted as a minimal length scale where the symmetry is enhanced from U(1) to SU(2) as we will see in §7.2.

Free fermions

Let us now consider the torus partition function of the free fermion. The mode expansion for a free fermion $\psi(z)$ with Neveu-Schwarz boundary conditions is

$$\psi(z) = \sum_{k\in\mathbb{Z}+\frac{1}{2}} b_k z^{-k-\frac{1}{2}}. \tag{4.84}$$

On the torus with variable w, this expansion corresponds a field with anti-periodic boundary condition. States in the Fock space $\mathcal{F}_{NS}$ of this theory are obtained by acting creation operators b_{-s} on the vacuum $|0\rangle$

$$|n_{\frac{1}{2}},n_{\frac{3}{2}}\cdots\rangle = \left(b_{-\frac{1}{2}}\right)^{n_{\frac{1}{2}}}\left(b_{-\frac{3}{2}}\right)^{n_{\frac{3}{2}}}|0\rangle, \quad n_s = 0,1. \tag{4.85}$$

These occupation numbers reflect the fermionic statistics.

As before, we need to evaluate the eigenvalues of $L_0 = \sum_{s=\frac{1}{2}}^{\infty} s b_{-s} b_s$ in (4.56). Using the anti-commutation relation (4.45) of the fermionic modes, we have

$$\{L_0, b_k\} = k b_k\,.$$

Thus, it is straightforward to compute the character

$$\begin{aligned}\chi_{AA}(\tau) &= \mathrm{Tr}_{\mathcal{F}}\left(q^{L_0-\frac{c}{24}}\right)\\ &= q^{-\frac{1}{48}}\sum_{n_{\frac{1}{2}}=0}^{1}\sum_{n_{\frac{3}{2}}=0}^{1}\cdots\langle n_{\frac{1}{2}},n_{\frac{3}{2}}\cdots|q^{L_0}|n_{\frac{1}{2}},n_{\frac{3}{2}}\cdots\rangle\\ &= q^{-\frac{1}{48}}\prod_{r=0}^{\infty}\left(1+q^{r+\frac{1}{2}}\right) = \sqrt{\frac{\vartheta_3(\tau)}{\eta(\tau)}}\end{aligned} \tag{4.86}$$

For the fermionic character, we impose the anti-periodic boundary condition on the time circle of the torus. The (AA) notation indicates that the anti-periodic boundary condition is imposed on the both time and spatial circle of the torus.

Let us investigate the property of (4.86) under the modular transformation. Using the properties (4.93) of η and ϑ under the modular transformations, we have

$$\begin{aligned} S &: \chi_{AA}(-1/\tau)) = \chi_{AA}(\tau) \\ T &: \chi_{AA}(\tau+1) = e^{-\frac{i\pi}{24}}\sqrt{\frac{\vartheta_4(\tau)}{\eta(\tau)}}. \end{aligned} \tag{4.87}$$

Although the S-transformation exchanges the time and spacial circle of torus, Since we impose the same anti-periodic boundary condition on the both circles, the S-transformation does not change the boundary conditions. On the other hand, the T-transformation maps the time circle to the (1,1)-cycle of the torus as illustrated in Figure 8. The periodic boundary condition is imposed on the (1,1)-cycle because imposing the anti-periodic boundary conditions twice ends up with the periodic boundary condition, *i.e.* $A \times A = P$. Hence, the T-transformation changes the boundary condition, which results in a different ϑ-function.

In order to construct a modular invariant partition function, we need to take into account all the boundary conditions. To do this, we introduce the fermion number operator F such that

$$\{(-1)^F, b_r\} = 0.$$

Then we can define a new partition function $\chi_{PA}(\tau)$, which is often called **an index**

$$\chi_{PA} = \mathrm{Tr}_{\mathcal{F}_{NS}}\left((-1)^F q^{L_0-\frac{c}{24}}\right) = q^{-\frac{1}{48}}\prod_{r=0}^{\infty}\left(1-q^{r+\frac{1}{2}}\right) = \sqrt{\frac{\vartheta_4(\tau)}{\eta(\tau)}}. \tag{4.88}$$

The additional term $(-1)^F$ in the argument of this trace imposes the periodicity condition in the time direction.

Again let us check the modular properties of this new partition function. The modular T-transformation takes the boundary condition back to the (AA) sector because the anti-periodic boundary condition is imposed on the (1,1)-cycle due to $P \times A = A$. Subsequently, we have

$$T: \ \chi_{PA}(\tau+1) = e^{-\frac{i\pi}{24}}\chi_{AA}(\tau)\,.$$

On the other hand, the modular S-transformation has a new effect

$$S: \chi_{PA}(-1/\tau)) = \sqrt{2}q^{\frac{1}{24}}\prod_{r\geq 1}(1+q^r) = \sqrt{\frac{\vartheta_2(\tau)}{\eta(\tau)}}\,, \tag{4.89}$$

because the different boundary conditions are imposed on the time and spatial circle. Since the S-transformation exchange them, we end up with (PA) boundary condition after the S-transformation, *i.e.* the anti-periodic boundary condition on the time circle and the periodic boundary condition on the spacial circle. This is the character of the Ramond sector

$$\chi_{AP} := \mathrm{Tr}_{\mathcal{F}_R}\left(q^{L_0-\frac{c}{24}}\right)$$

where the fermionic modes b_k takes integer values of $k \in \mathbb{Z}$.

In a similar fashion, we can read off the modular properties of this character:

$$T(\chi_{AP}(\tau)) = e^{\frac{i\pi}{12}}\chi_{AP}(\tau), \qquad S(\chi_{AP}(\tau)) = \chi_{PA}(\tau) \tag{4.90}$$

One may wonder what $\chi_{PP}(\tau)$ is. In this case, we need to impose periodic boundary conditions on both time and spatial circle so that there is a zero mode (constant function). Due to the presence of zero mode, the partition function $\chi_{PP}(\tau)$ just vanishes.

We are now in a position to construct the modular invariant partition function. In particular, starting from a free fermion in the NS sector, we have seen that modular invariance requires us to also consider all the boundary conditions, and the final result is

$$\mathcal{Z}_{\mathrm{F}}(\tau,\overline{\tau}) = \frac{1}{2}\left(\left|\frac{\vartheta_3}{\eta}\right| + \left|\frac{\vartheta_4}{\eta}\right| + \left|\frac{\vartheta_2}{\eta}\right|\right). \tag{4.91}$$

The overall factor of $1/2$ is necessary to ensure the NS ground state only appears once so that we avoid overcounting states.

Special functions

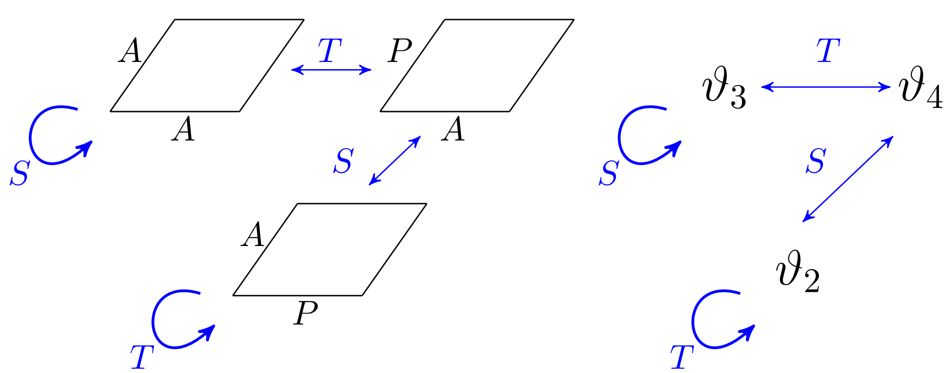


Figure 11: T and S transformation of ϑ function

The special functions we have used in the text are summarized below. You can also consult with Polchinski [Pol98, §7.2] or Blumenhagen & Plauschinn [BP09, §4.2].

The infinite product form of them are

$$\begin{aligned}
\eta(\tau) &= q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n)\,, \\
\vartheta_2(\tau) &= 2q^{\frac{1}{8}}\prod_{n=1}^{\infty}(1-q^n)(1+q^n)^2\,, \\
\vartheta_3(\tau) &= \prod_{n=1}^{\infty}(1-q^n)(1+q^{n-\frac{1}{2}})^2\,, \\
\vartheta_4(\tau) &= \prod_{n=1}^{\infty}(1-q^n)(1-q^{n-\frac{1}{2}})^2\,,
\end{aligned} \tag{4.92}$$

where $q = e^{2\pi i\tau}$. The first one is called Dedekind eta function, the others are called (Jacobi or elliptic or simply without these) theta functions.

Their T and S-transformations are

$$\begin{aligned}
\eta(\tau+1) &= e^{i\pi/12}\eta(\tau)\,, & \eta(-1/\tau) &= \sqrt{-i\tau}\eta(\tau)\,,\\
\vartheta_2(\tau+1) &= e^{i\pi/4}\vartheta_2(\tau)\,, & \vartheta_2(-1/\tau) &= \sqrt{-i\tau}\vartheta_4(\tau)\\
\vartheta_3(\tau+1) &= \vartheta_4(\tau)\,, & \vartheta_3(-1/\tau) &= \sqrt{-i\tau}\vartheta_3(\tau)\\
\vartheta_4(\tau+1) &= \vartheta_3(\tau)\,, & \vartheta_4(-1/\tau) &= \sqrt{-i\tau}\vartheta_2(\tau)\,.
\end{aligned} \tag{4.93}$$

In addition, there are two important identities. One is called Jacobi identity:

$$(\vartheta_3(\tau))^4 = (\vartheta_2(\tau))^4 + (\vartheta_4(\tau))^4\,.$$

The other is called Jacobi triple product identity:

$$\vartheta_2(\tau)\vartheta_3(\tau)\vartheta_4(\tau) = 2\eta^3(\tau)\,,$$

from which you can show the modular transformation of the Dedekind η-function.

5 Minimal Models

This chapter is devoted to particularly simple conformal theory called **minimal models** first studied in the seminal paper [BPZ84, ISZ88-No.1]. These theories are characterized by a Hilbert space made of a finite numbers of representations of the Verma modules.

5.1 Conformal Family

From the previous study, we know that primary fields play a fundamental role in conformal field theory. The asymptotic state is given by $|h\rangle = \phi(0)\,|0\rangle$, which satisfies $L_0\,|h\rangle = h\,|h\rangle$ and $L_n\,|h\rangle = 0$ for $n > 0$. In Verma module section, we also defined descendant states leading by primary $|h\rangle$, by acting a series of L_{-n} on $|h\rangle$. We first study the field operator corresponding to the descendant states. The natural definition of the descendant field associated with the state $L_{-n}\,|h\rangle$ is

$$\phi^{(-n)}(w) \equiv (L_{-n}\phi)\,(w) = \frac{1}{2\pi i}\oint_w dz \frac{1}{(z-w)^{n-1}} T(z)\phi(w)\,. \tag{5.1}$$

By using the OPE (3.53), we can the above expression can be calculated explicitly. In particular, we have

$$\phi^{(0)}(w) = h\phi(w)\,, \quad \text{and} \quad \phi^{(-1)}(w) = \partial\phi(w)\,. \tag{5.2}$$

Now consider the correlation function with one descendant field in it, i.e.

$$\langle (L_{-n}\phi)\,(w)X\rangle\,, \tag{5.3}$$

where $X = \phi_1\,(w_1)\cdots\phi_N\,(w_N)$ is a product of primary fields with conformal dimensions h_i. This correlation function can be calculated by substituting the definition (5.1).

$$
\begin{aligned}
\left\langle \phi^{(-n)}(w)X \right\rangle &= \frac{1}{2\pi i}\oint_w dz(z-w)^{1-n}\langle T(z)\phi(w)X\rangle \\
&= -\frac{1}{2\pi i}\oint_{(w_i)} dz(z-w)^{1-n}\sum_i\{\frac{1}{z-w_i}\partial_{w_i}\langle\phi(w)X\rangle + \frac{h_i}{(z-w_i)^2}\langle\phi(w)X\rangle\} \\
&\equiv \mathcal{L}_{-n}\langle\phi(w)X\rangle \quad (n\geq 1)\,.
\end{aligned}
$$

In the second step, the residue theorem is applied by reversing the contour including w only and summing the contributions from all the poles at w_i. With the help of the relevant OPEs, the definition operator $\mathcal{L}_{-n}$ is given by

$$
\mathcal{L}_{-n} = \sum_i \left\{ \frac{(n-1)h_i}{(w_i-w)^n} - \frac{1}{(w_i-w)^{n-1}}\partial_{w_i} \right\}\,. \tag{5.4}
$$

This result tells us that the evaluation of a correlation function containing one descendant field $\phi^{(-n)}$ is controlled by its primary field. $\mathcal{L}_{-1}$ is equivalent to $\partial/\partial w$, since the operator

$$
\frac{\partial}{\partial w} + \sum_i \frac{\partial}{\partial w_i}\,, \tag{5.5}
$$

annihilates any correlation functions because of the translation invariance.

The descendant states corresponding to $L_{-k}L_{-n}\left|h\right\rangle$ can be defined recursively.

$$
\phi^{(-k,-n)}(w) \equiv (L_{-k}L_{-n}\phi)(w) = \frac{1}{2\pi i}\oint_w dz(z-w)^{1-k}T(z)\,(L_{-n}\phi)(w)\,. \tag{5.6}
$$

In particular, we have

$$
\phi^{(0,-n)}(w) = (h+n)\phi^{(-n)}(w) \quad \text{and} \quad \phi^{(-1,-n)}(w) = \partial_w\phi^{(-n)}(w)\,. \tag{5.7}
$$

And it can be shown directly that

$$
\left\langle \phi^{(-k_1\ldots\ldots-k_n)}(w)X \right\rangle = \mathcal{L}_{-k_1}\cdots\mathcal{L}_{-k_n}\langle\phi(w)X\rangle\,. \tag{5.8}
$$

That is nothing but to apply the differential operators in succession. Consequently, correlation functions of descendant fields attributes to correlation functions of primary field. If we obtain all the correlation functions of primary fields in a theory, then the correlation functions including descendant fields are thus all determined.

Moreover, all the OPEs with $T(z)$ with descendant states can be calculated in a similar fashion. For example, for $\phi^{(-1)} = \partial\phi$

$$
\begin{aligned}
T(z)\partial\phi(w) &\sim \frac{2h\phi(w)}{(z-w)^3} + \frac{(h+1)\partial\phi(w)}{(z-w)^2} + \frac{\partial^2\phi(w)}{z-w} \\
&\sim \frac{2\phi^{(0)}(w)}{(z-w)^3} + \frac{(h+1)\phi^{(-1)}(w)}{(z-w)^2} + \frac{\phi^{(-1,-1)}(w)}{z-w}\,,
\end{aligned}
$$

where we have neglect the regular terms as usual. The set of a primary field ϕ and all of its descendants is called the **conformal family** of ϕ, denoted by $[\phi]$. The OPEs with $T(z)$ are closed in the conformal family.

5.2 Minimal models

In a conformal field theory (or more generally quantum field theories), we want to understand correlation functions in the theory. For free bosons and fermions, all correlation functions can be calculated by directly using partition function. However, it is not so straightforward to determine all the correlation functions for the interacting CFTs. If there are only finitely many highest weight representations of Virasoro algebra (or finitely many primary fields), one can perform exact computations of correlation functions [BPZ84, ISZ88-No.1]. Such CFTs are called **minimal models**. Let us consider a theory with the set of finitely many highest weights $\{h_1, \cdots h_p\}$ so that OPEs take the following form:

$$\phi_1(z,\bar{z})\phi_2(0,0) = \sum_p \sum_{\{k,\bar{k}\}} C_{12}^{p\{k,\bar{k}\}} z^{h_p-h_1-h_2+K} \bar{z}^{\bar{h}_p-\bar{h}_1-\bar{h}_2+\bar{K}} \phi_p^{\{k,\bar{k}\}}(0,0)\,, \tag{5.9}$$

where $K = \sum_i k_i$ and $\overline{K} = \sum_i \bar{k}_i$, and ϕ_i is the primary field with highest weight h_i. As we have seen in the last subsection, all the correlation function can be reduced to the correlation function constructed by primary fields. Hence, we just need to understand OPEs of primary fields in minimal models, and we do not need a concrete realization of the CFT. In this section, we will determine OPEs of primary fields in the minimal models.

Kac Determinant

A representation of the Virasoro algebra is said to be unitary if it contains no negative-norm states. In the section of Verma module, we have mentioned that the negative of central charge and highest weight will make the states non-unitary. As we will see below, the unitarity imposes strong constraints on (h, c).

In addition to unitarity, one has to pay attention to zero-norm states, indeed. If a state $|\chi\rangle$ in a Verma module $V(c, h)$ which is not a highest weight state satisfies the condition

$$L_n|\chi\rangle = 0\,, \quad \text{for } n > 0\,,$$

it is called a **singular vector**. In fact, a singular vector $|\chi\rangle$ is orthogonal to any state $\prod_i L_{-n_i}|h\rangle$ in the Verma module $V(c, h)$

$$\langle\chi|\prod_i L_{-n_i}|h\rangle = 0\,.$$

Therefore, the norm of a singular vector is zero; $\langle\chi|\chi\rangle = 0$ so that it is also called a null vector or a null state.

Suppose that there exists a singular vector $|\chi\rangle$ at level N in the Verma module, namely $L_0|\chi\rangle = (h + N)|\chi\rangle$. Then its descendants

$$L_{-k_1}L_{-k_2}\cdots L_{-k_n}|\chi\rangle\,, \qquad (1 \le k_1 \le k_2 \cdots \le k_n)$$

are all null, and moreover they form a submodule in the Verma module $V(c, h)$. Therefore, in this situation the Verma module is **reducible**. In fact, the Verma module $V(c, h)$ is irreducible if and only if it contains no singular vector. Otherwise, an irreducible representation of the Virasoro algebra is

$$L(c, h) = V(c, h)/J(c, h)$$

where $J(c,h)$ consists of all null states and their descendants

$$J(c,h) = \bigcup_{\chi:\text{null}} \{L_{-k_1} L_{-k_2} \cdots L_{-k_n} |\chi\rangle\} \qquad (1 \le k_1 \le k_2 \cdots \le k_n) \,.$$

To see a Verma module has a null state, let us define a Gram matrix as matrix element $M_{ij} = \langle i|j\rangle$. $|i\rangle, |j\rangle$ are basis. For instance, at level two, we have 2 basis $L_{-2}|h\rangle$, $L_{-1}^2|h\rangle$. Therefore, the Gram matrix is

$$\begin{pmatrix} \langle h|L_2 L_{-2}|h\rangle & \langle h|L_1^2 L_{-2}|h\rangle \\ \langle h|L_2 L_{-1}^2|h\rangle & \langle h|L_1^2 L_{-1}^2|h\rangle \end{pmatrix} = \begin{pmatrix} 4h + c/2 & 6h \\ 6h & 4h(1+2h) \end{pmatrix} . \tag{5.10}$$

We focus on the zero eigenvector of this matrix, which gives a linear combination with zero norm. We write the determinant of this matrix as

$$2\left(16h^3 - 10h^2 + 2h^2 c + hc\right) = 32\,(h - h_{1,1}(c))\,(h - h_{1,2}(c))\,(h - h_{2,1}(c)) \,, \tag{5.11}$$

and

$$\begin{aligned} h_{1,1}(c) &= 0 \\ h_{1,2}(c) &= \frac{1}{16}(5-c) + \frac{1}{16}\sqrt{(1-c)(25-c)} \\ h_{2,1}(c) &= \frac{1}{16}(5-c) - \frac{1}{16}\sqrt{(1-c)(25-c)} \,. \end{aligned} \tag{5.12}$$

The $h = 0$ root is actually due to the null state at level 1, $L_{-1}|0\rangle = 0$, which implies also the vanishing $L_{-1}(L_{-1}|0\rangle) = 0$. This is a general feature: if a null state (a state with zero norm) $|h+n\rangle = 0$ occurs at level n, then at level $N > n$, there are $P(N-n)$ null states $L_{-n_1} \cdots L_{-n_k}|h+n\rangle = 0$ (with $\sum_i n_i = N - n$). Thus a null state for some value of h that first appears at level n implies that the determinant at level N will have a $[P(N-n)]$-th order zero for that value of h.

At level N, the Hilbert space consists of all states of the form

$$\sum_{\{n_t\}} a_{n_1 \cdots n_k} L_{-n_1} \cdots L_{-n_k} |h\rangle \,, \tag{5.13}$$

where $\sum_i n_i = N$. We can pick $P(N)$ basis states to construct a $P(N) \times P(N)$ matrix, with the matrix element as follows.

$$\langle h|L_{m_\ell} \cdots L_{m_1} L_{-n_1} \cdots L_{-n_k}|h\rangle \tag{5.14}$$

where $\sum_{i=1}^{\ell} m_i = \sum_{j=1}^{k} n_j = N$. We denote it by $M_N(c,h)$. If $\det M_N(c,h)$ vanishes, then there exists a linear combination of states with zero norm for that c, h. If negative, then the determinant has an odd number of negative eigenvalues. Then the representation of the Virasoro algebra at those values of c and h includes states of negative norm, and is therefore non-unitary.

Kac has found a closed form expression of the determinant of N level Gram matrix, which is called **Kac determinant** [Kac79, ISZ88-No.7]

$$\det M_N(c,h) = \alpha_N \prod_{r,s \le N} \left(h - h_{r,s}(c)\right)^{P(N-rs)} \,, \tag{5.15}$$

where α_N is a positive constant independent of c and h. The Kac determinant provides the complete information of null states in a Verma module. From this expression, one

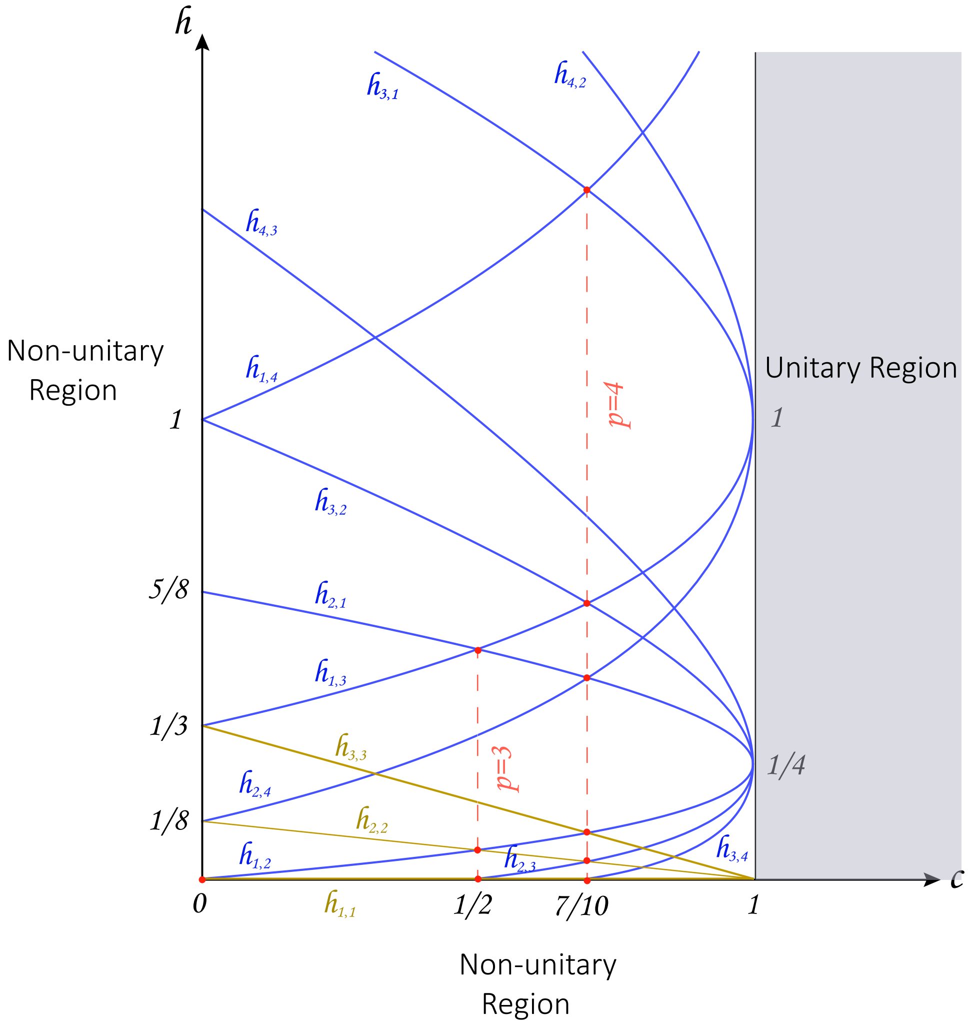


Figure 12: Kac determinant

can see that a null state first appears at level $n = rs$ if $h = h_{r,s}$. Thus at level $N > rs$, the zero order for $h_{r,s}$ is $P(N - rs)$. In fact, there are two useful equivalent way to write $h_{r,s}$; the first expression is

$$h_{r,s}(c) = h_0 + \frac{1}{4}(r\alpha_+ + s\alpha_-)^2 ,$$
$$h_0 = \frac{1}{24}(c-1) ,$$
$$\alpha_\pm = \frac{\sqrt{1-c} \pm \sqrt{25-c}}{\sqrt{24}} ,$$

and the second is

$$h_{r,s}(p) = \frac{[(p+1)r - ps]^2 - 1}{4p(p+1)} ,$$
$$c = 1 - \frac{6}{p(p+1)} ,$$

where m has two branches

$$p = -\frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{25-c}{1-c}} \tag{5.16}$$

From this expression, one can see that when $c = 1$, a Verma module is irreducible if $h \neq n^2/4$ for $n \in \mathbb{Z}$. In addition, when $c = 0$, a Verma module is irreducible if $h \neq (n^2-1)/24$ for $n \in \mathbb{Z}$.

Both of the above expression provide the following result for $h_{r,s}(c)$, which is illustrated in Figure 12

$$h_{r,s}(c) = \frac{1-c}{96}\left[\left((r+s) \pm (r-s)\sqrt{\frac{25-c}{1-c}}\right)^2 - 4\right]. \tag{5.17}$$

One can convince oneself that $h_{1,2}$ and $h_{2,1}$ are as in (5.12). The Kac determinant changes sign when the values of (c, h^2) cross each curve $h = h_{r,s}(c)$.

Minimal models $\mathcal{M}_{p,p'}$

Belavin-Polyakov-Zamolodchikov have noticed that all the correlation functions can be determined by differential equations when the Kac determinant becomes zero. As an example, let us first consider the situation in which there are null states at the level two. In general, a null state $|\chi\rangle$ at level two takes the form

$$|\chi\rangle = L_{-2}|h\rangle + aL_{-1}^2|h\rangle\,. \tag{5.18}$$

By applying the L_1 to the above equation, we have

$$\begin{aligned}[L_1, L_{-2}]\,|h\rangle + a\left[L_1, L_{-1}^2\right]|h\rangle &= 3L_{-1}|h\rangle + a\left(L_{-1}2L_0 + 2L_0L_{-1}\right)|h\rangle \\ &= (3 + 2a(2h+1))L_{-1}|h\rangle = 0\,,\end{aligned}$$

which requires that

$$a = -3/2(2h+1)\,. \tag{5.19}$$

By applying L_2, we find that

$$\begin{aligned}[L_2, L_{-2}]\,|h\rangle + a\left[L_2, L_{-1}^2\right]|h\rangle &= \left(4L_0 + \frac{c}{2}\right)|h\rangle + 3aL_1L_{-1}|h\rangle \\ &= (4h + c/2 + 6ah)|h\rangle = 0\,.\end{aligned}$$

Therefore the central charge must satisfy

$$c = 2(-6ah - 4h) = 2h\frac{5-8h}{2h+1} \tag{5.20}$$

Writing h in terms of c, we have

$$h = \frac{1}{16}\left\{5 - c \pm \sqrt{(c-1)(c-25)}\right\}, \tag{5.21}$$

which are equal to $h_{1,2}$ and $h_{2,1}$ in (5.12).

Since the null state $|\chi\rangle$ is orthogonal to any states in the Verma module, we have

$$\langle\chi(z)X\rangle = 0$$

where X is a product of any primary fields as usual. By using (5.4), the correlation function $\langle \phi(z) X \rangle$ should be governed by the following differential equation

$$\left\{ \mathcal{L}_{-2} - \frac{3}{2(2h+1)} \mathcal{L}_{-1}^2 \right\} \langle \phi(z) X \rangle = 0 \,, \tag{5.22}$$

where $\phi(z)$ has conformal dimension $h_{1,2}$ or $h_{2,1}$. One can write the above expression as a differential equation

$$\left\{ \sum_{i=1}^{N} \left[\frac{1}{z - z_i} \frac{\partial}{\partial z_i} + \frac{h_i}{(z - z_i)^2} \right] - \frac{3}{2(2h+1)} \frac{\partial^2}{\partial z^2} \right\} \langle \phi(z) X \rangle = 0 \,. \tag{5.23}$$

If X is $\phi(w)$ itself, the differential equation for two point function becomes

$$\left\{ \frac{1}{z - w} \partial_w + \frac{h}{(z - w)^2} - \frac{3}{2(2h+1)} \partial_z^2 \right\} \langle \phi(z) \phi(w) \rangle = 0 \,, \tag{5.24}$$

which is satisfied by the general form for two-point function $\langle \phi(z)\phi(w) \rangle = (z - w)^{-2h}$. For three point function, $X = \phi_1(z_1)\phi_2(z_2)$, we have

$$\langle \phi(z) \phi_1(z_1) \phi_2(z_2) \rangle = \frac{C_{h,h_1,h_2}}{(z - z_1)^{h_2 - h - h_1} (z_1 - z_2)^{h - h_1 - h_2} (z - z_2)^{h_1 - h - h_2}} \,. \tag{5.25}$$

The differential equation gives further constraint to the highest weights of field operators in the three-point function.

$$h_2 = \frac{1}{6} + \frac{1}{3} h + h_1 \pm \frac{2}{3} \sqrt{h^2 + 3hh_1 - \frac{1}{2} h + \frac{3}{2} h_1 + \frac{1}{16}} \,. \tag{5.26}$$

For $h = h_{1,2}$, and set $h_1 = h_{r,s}$ in (5.17), we find h_2 equals to $h_{r,s-1}$ or $h_{r,s+1}$. Similarly for $h = h_{2,1}$, we have h_2 equals to $h_{r-1,s}$ or $h_{r+1,s}$. Recall that the two point function has non-vanishing value only when the two field operators within it has the same highest weight. We thus have the following operator algebra

$$\begin{aligned} \phi_{1,2} \times \phi_{r,s} &= [\phi_{r,s-1}] + [\phi_{r,s+1}] \\ \phi_{2,1} \times \phi_{r,s} &= [\phi_{r-1,s}] + [\phi_{r+1,s}] \,, \end{aligned}$$

where $\phi_{r,s}$ means the field operator with highest weight $h_{r,s}$. The expression above means that the fields $\phi_{1,2}$ and $\phi_{2,1}$ act as ladder operators in the operator algebra. In addition, the field operators expanded in the right side not only contain the primary fields but their descendant fields as well. The coefficient on the right side may be zero. For instance,

$$\begin{aligned} \phi_{1,2} \times \phi_{2,1} &= [\phi_{2,0}] + [\phi_{2,2}] \,, \\ \phi_{2,1} \times \phi_{1,2} &= [\phi_{0,2}] + [\phi_{2,2}] \,. \end{aligned}$$

Since the two OPEs are equivalent, this shows the coefficients before $\phi_{2,0}$ and $\phi_{0,2}$ must be zero. Therefore, we have

$$\phi_{1,2} \times \phi_{2,1} = [\phi_{2,2}] \,. \tag{5.27}$$

The above expression can be generalized to the following fusion rules

$$\phi_{r_1,s_1} \times \phi_{r_2,s_2} = \sum_{\substack{k=1+|r_1-r_2| \\ k+r_1+r_2=1 \bmod 2}}^{k=r_1+r_2-1} \sum_{\substack{l=1+|s_1-s_2| \\ l+s_1+s_2=1 \bmod 2}}^{l=s_1+s_2-1} [\phi_{k,l}] \,. \tag{5.28}$$

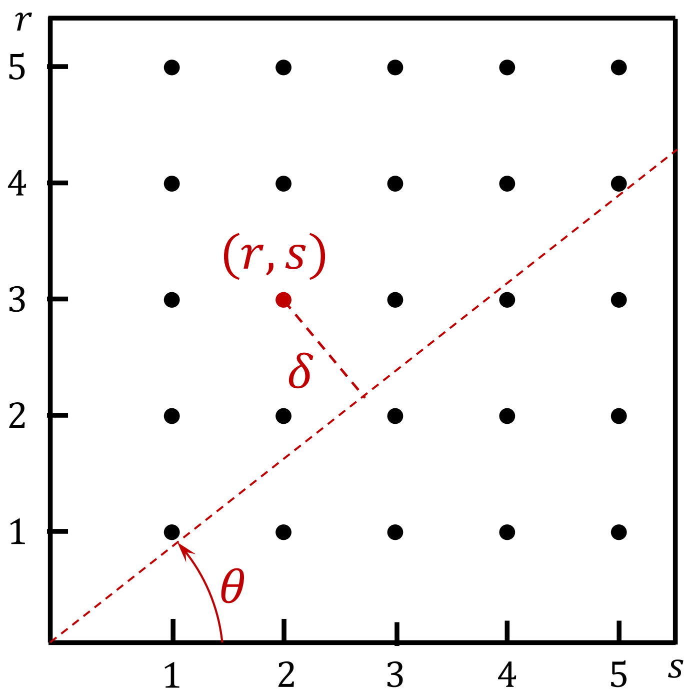


Figure 13: "diagram of dimensions"

The summation variables k and l are incremented by 2. The expression above implies that the conformal family $[\phi_{r,s}]$ forms a closed operator algebra.

The operator algebra (5.28), implies there are infinite number of conformal families present in the theory. However, we want a finite set of highest weight in a minimal model. In order to understand the situation graphically, we consider the "diagram of dimensions" as in Figure 13. The points (r, s) in the first quadrant label the various conformal dimensions appearing in the Kac formula. The dotted line has a slope $\tan\theta = -\alpha_+/\alpha_-$, where $\alpha_\pm$ are defined in (5.16). If δ is the distance between a point (r, s) and the dotted line. It is a good exercise to show the following relation,

$$h_{r,s} = h_0 + \frac{1}{4}\delta^2\left(\alpha_+^2 + \alpha_-^2\right) . \tag{5.29}$$

If the slope $\tan\theta$ is irrational, it will never go through any integer point (r, s), which means δ is always non-zero. There is no periodical property for $h_{r,s}$. However, if the slope $\tan\theta$ is rational. That is, if there exist two coprime integers p and p' such that

$$p'\alpha_- + p\alpha_+ = 0 . \tag{5.30}$$

The dotted line will go through the point (p', p), we have the following periodicity property.

$$h_{r,s} = h_{r+p,s+p'} . \tag{5.31}$$

Therefore, there are finitely many primary fields and we call it the **minimal model** denoted by $\mathcal{M}_{p,p'}$. Using (5.16), the central charge and the Kac formula become

$$c = 1 - 6\frac{(p-p')^2}{pp'}\,,$$
$$h_{r,s} = \frac{(p'r-ps)^2 - (p-p')^2}{4pp'}\,.$$

We can obtain a symmetry property from above:

$$h_{r,s} = h_{p-r,p'-s}\,. \tag{5.32}$$

Now, we get a finite set of conformal families, which closes under fusion rules. The corresponding finite set of conformal dimensions $h_{r,s}$ is given by $1 \leq r < p$ and $1 \leq s < p'$, and thus construct a minimal model. The symmetry above gives

$$\phi_{r,s} = \phi_{p-r,p'-s}\,. \tag{5.33}$$

Hence, there remain $(p-1)(p'-1)/2$ distinct fields in the theory. And the fusion rule is modified to be

$$\phi_{r,s} \times \phi_{m,n} = \sum_{\substack{k=1+|r-m| \\ k+r+m=1 \bmod 2}}^{k_{\max}} \; \sum_{\substack{l=1+|s-n| \\ k+s+n=1 \bmod 2}}^{l_{\max}} [\phi_{k,l}]\,, \tag{5.34}$$

where the upper bound of the summation now is control by the periodical and symmetric condition.

$$\begin{aligned} k_{\max} &= \min\left(r+m-1, 2p-1-r-m\right)\,, \\ l_{\max} &= \min(s+n-1, 2p'-1-s-n)\,. \end{aligned} \tag{5.35}$$

We will see several examples of minimal models in the following subsections.

Unitarity for $c > 1$

We can further use Kac determinant to derive unitarity condition. In fact, the unitary condition gives us great constraint of the choice of h and c, especially in the region $0 \leq c \leq 1$.

The explicit expression for the Kac determinant allow us to prove the representations $(c > 1, h > 0)$ are unitary. The proof is down by three steps:

The first step is to show the curves $C_{r,s} : h = h_{r,s}$ lie below or on the axis $h = 0$ if $c > 1$. if $1 < c < 25$, from (5.17), we see that $h_{r,s}$ is not a real number unless $s = r$, in which $h_{r,s} \leq 0$. On the other hand, if $c \geq 25$ the choice (5.16) implies that $-1 < m < 0$. Then $p(p+1) < 0$ and $[(p+1)r - ps] \geq 1$ which implies $h_{r,s}(p) \leq 0$ according to (5.16). Thus we have shown that all the curves $C_{r,s}$ are located below the $h = 0$ axis if $c > 1$.

The second the step is to prove that the Kac determinant is positive throughout the region. For a given level, we can find a h larger than any $h_{r,s}$, and at such a point Kac determinant is positive. Since none of the curves lies in the region, the determinant sign can not be changed through out the region by crossing any curves. This proves the second point.

The last step is to prove that the Gram matrix at level N, M_N is positive definite in this region, which is equivalent to the theory in this region is unitary. Since the Kac determinant is positive, the number of negative eigenvalues of M_N must be even. Since the number can change only across one of the curves $C_{r,s}$, and consequently must stay the same throughout the region. Therefore, we just need to show that the matrix M_N for each level N is positive definite for at least one point (c, h).

In order to do this, we define the length $n(\alpha)$ of a basis vector $|\alpha\rangle$ as the number of operators L_k used to define it. For instance, L_{-1}^3 has length 3 and $L_{-3}\,|h\rangle$ has length 1. It is possible to show that the dominant behavior in h of inner products is

$$\begin{aligned} \langle\alpha|\alpha\rangle &= c_\alpha h^{n(\alpha)}[1 + O(1/h)] \quad (c_\alpha > 0) \\ \langle\alpha|\beta\rangle &= O\left(h^{(n(\alpha)+n(\beta))/2-1}\right) + \dots . \end{aligned} \tag{5.36}$$

where $|\alpha\rangle$ and $|\beta\rangle$. The above expression immediately shows that M_N is dominant by its diagonal element when h is sufficiently large, and thus M_N is positive definite. Therefore, in this region, a Verma module $V(c, h)$ is irreducible.

Unitarity for $0 \le c \le 1$

In this region, we just roughly gives some statements. First, it is easy to show that, in Figure 12, one may connect any point in the region $0 \le c \le 1$, $h > 0$ to the $c > 1$ region by a path that crosses a single vanishing curve of the Kac determinant at some level. The determinant reverses sign when passing through the vanishing curve. Therefore, there must be a negative norm state at that level. This excludes unitary representations of the Virasoro algebra at all points in this region, except those on the vanishing curves themselves. A more careful analysis shows that there is an additional negative norm state everywhere on the vanishing curves except at certain points where they intersect.

Therefore, the unitary representations of the Virasoro algebra only occur at the values of the central charge [Car84a, ISZ88-No.3]:

$$c = 1 - \frac{6}{p(p+1)}, \quad p = 3, 4, \dots \tag{5.37}$$

with corresponding highest weight

$$h_{r,s}(p) = \frac{[(p+1)r - ps]^2 - 1}{4p(p+1)} . \tag{5.38}$$

where r, s satisfies $1 \le s \le r \le p-1$. The form is exactly the same as (5.16), but now p can only take discrete integers. This result tells us that the unitary condition is satisfied only at the point (h, c) where the Verma module has null states.

Unitary Minimal Models $\mathcal{M}_p$

We now consider the choice of p and p' to make the minimal models unitary. Recalling the admissible conformal dimensions in (5.32), Bezout's lemma states that there exists a couple of integers (r_0, s_0) in the range $1 \le r_0 < p$ and $1 \le s_0 < p'$ such that

$$p' r_0 - p s_0 = 1 . \tag{5.39}$$

Thus, the corresponding conformal dimension

$$h_{r_0,s_0} = \frac{1 - (p - p')^2}{4pp'} , \tag{5.40}$$

which is always negative, expect if $|p - p'| = 1$, in which case it vanishes. In fact, the minimal model is unitary only if $|p - p'| = 1$, and the primary field $h_{r_0,s_0} = 0$ is indeed the identity operator. Hence, we can set $p' = p + 1$ so that the central charge is expressed in (5.37). We denote the unitary minimal model by $\mathcal{M}_p$. We note that the list of unitary representations given in (5.37) and (5.38), coincides with the list of highest weight $h_{r,s}$ of unitary minimal models.

5.3 Characters of minimal models

The character of an irreducible representation of the Virasoro algebra

$$\chi_h(\tau) \equiv \mathrm{Tr}_{L(c,h)}\, q^{L_0 - \frac{c}{24}} = q^{-\frac{c}{24}} \sum_{N=0}^{\infty} d_h(N) q^{h+N} ,$$

where $d_h(N)$ is the number of states at level N, and $d_h(N) \leq P(N)$. If there is no null state, $d_h(N) = P(N)$. Th partition function with periodic boundary conditions on a torus is

$$\begin{aligned} Z_{\mathrm{PP}}(\tau, \overline{\tau}) &= \mathrm{Tr}\left(q^{L_0 - c/24} \overline{q}^{\overline{L}_0 - c/24}\right) \\ &= \sum_{h,\overline{h}} \mathcal{N}_{h\overline{h}}\, \chi_h(\tau) \chi_{\overline{h}}(\overline{\tau}) , \end{aligned} \tag{5.41}$$

where $\mathcal{N}_{h\overline{h}}$ is the number of primary fields of conformal dimension $(h, \overline{h})$ [Car86b, ISZ88-No.25]. Since there is only one vacuum, we set $\mathcal{N}_{00} = 1$. Under the T-transformation, a character behaves as

$$\chi_h(\tau + 1) = e^{2\pi i (h - c/24)} \chi_h(\tau) . \tag{5.42}$$

The invariance of $\mathcal{Z}_{\mathrm{PP}}$ under the T-transformation requires

$$\mathcal{N}_{h\overline{h}} = 0, \quad h - \overline{h} \notin \mathbb{Z} .$$

If we write the S-transformation of characters

$$\chi_h(-1/\tau) = \sum_{h'} S_{hh'}\, \chi_{h'}(\tau) ,$$

then the modular invariance of $\mathcal{Z}_{\mathrm{PP}}$ requires

$$\sum_{h,\overline{h}} \mathcal{N}_{h\overline{h}} S_{hh'} S_{\overline{h}\overline{h}'} = \mathcal{N}_{h'\overline{h}'} . \tag{5.43}$$

Below we will determine the partition function of diagonal type $\mathcal{N}_{hh'} = \delta_{hh'}$ in the unitary minimal models.. We will see one example of non-diagonal type in the tricritical Ising model in §5.5, and the classification of the modular invariant partition functions are discussed in the end of §8.3.

From the Kac determinant, we have learnt that the Verma module $V_{h_{r,s}}$ has a singular vector at level rs. Since $h_{r,s} + rs = h_{-r,s}$, the singular vector generates a submodule $V_{h_{-r,s}}$. Furthermore. since $h_{p-r,p'-s} + (p-r)(p'-s) = h_{2p-r,s}$, we have another submodule $V_{h_{2p-r,s}}$. Moreover, these submodule $V_{h_{-r,s}}$, $V_{h_{2p-r,s}}$ contain other submodules $V_{h_{-2p-r,s}}$, $V_{h_{2p+r,s}}$, and one can see repeatedly this pattern [FF83, ISZ88-No.9].

$$
\begin{tikzcd}
 & (-r,s) \arrow[r] \arrow[dr] & (-2p+r,s) \arrow[r] \arrow[dr] & (-2p-r,s) \arrow[r] \arrow[dr] & (-4p+r,s) \arrow[r] \arrow[dr] & {} \\
(r,s) \arrow[ur] \arrow[dr] & & & & & \\
 & (2p-r,s) \arrow[r] \arrow[uur] & (2p+r,s) \arrow[r] \arrow[uur] & (4p-r,s) \arrow[r] \arrow[uur] & (4p+r,s) \arrow[r] \arrow[uur] & {}
\end{tikzcd}
$$

Therefore, the character of the irreducible representation $L(c,h_{r,s})$ of the Virasoro algebra is

$$
\begin{aligned}
\chi_{r,s}(\tau) &= \mathrm{Tr}_{L(c,h_{r,s})}\, q^{L_0-\frac{c}{24}} \\
&= \frac{1}{q^{-1/24}\eta(\tau)} q^{-\frac{c}{24}} \sum_{k\in\mathbb{Z}} \left[q^{h_{r+2pk,s}} - q^{h_{2pk-r,s}} \right]
\end{aligned}
$$

which is called the **Rocha-Caridi formula** [RC85, ISZ88-No.10].

Defining the theta functions by

$$
\Theta_{m,k}(\tau) = \sum_{n\in\mathbb{Z}} q^{k\left(n+\frac{m}{2k}\right)^2} \tag{5.44}
$$

it can be written by

$$
\chi_{r,s}(\tau) = \frac{1}{\eta(\tau)} \Big(\Theta_{r(p+1)-sp,p(p+1)}(\tau) - \Theta_{r(p+1)+sp,p(p+1)}(\tau) \Big) \tag{5.45}
$$

The behavior of $\chi_{r,s}(\tau)$ under the S-transformation has been obtained in [Car86a, ISZ88-No.29]:

$$
\begin{aligned}
\chi_{r,s}(-1/\tau) &= \sum_{1\leqq s'\leqq r'\leq p-1} S_{(r,s),(r',s')}\, \chi_{r's'}(\tau) \\
S_{(r,s),(r',s')} &= \left(\frac{8}{p(p+1)}\right)^{1/2} (-1)^{(r+s)(r'+s')} \sin\left(\frac{\pi r r'}{p}\right) \sin\left(\frac{\pi s s'}{p+1}\right)
\end{aligned} \tag{5.46}
$$

and the T-transformation is given in (5.42). It is shown that the partition function of diagonal type

$$
\mathcal{Z}^{\mathrm{diag}}_{\mathcal{M}_p} = \sum_{1\leqq r\leqq s\leq p-1} |\chi_{r,s}(\tau)|^2 \tag{5.47}
$$

is invariant under the modular transformations.

5.4 Ising model $(p',p)=(4,3)$

3	$\frac{1}{2}$	0
2	$\frac{1}{16}$	$\frac{1}{16}$
1	0	$\frac{1}{2}$
	1	2

3	ε	$\mathbf{1}$
2	σ	σ
1	$\mathbf{1}$	ε
	1	2

Let us consider the unitary minimal model $\mathcal{M}_{p=3}$ whose the central charge $c=\frac{1}{2}$. The primary fields and their conformal dimensions are described in the table above. By comparing the conformal dimensions with (1.3), ε is the energy density operator and σ is the spin field in the Ising model. Therefore, the unitary minimal model $\mathcal{M}_{p=3}$ describes the Ising model.

Since the Ising model is the most important 2d CFT, let us investigate it more in detail. For the Ising model on a lattice of $(N \times M)$ sites, the statistical partition function is

$$\mathcal{Z} = \sum_{\{\sigma\}} e^{-K\sum_{ij}\sigma_i\sigma_j}$$

where we define $K = J/(k_B T)$. One can use the identity

$$\exp\left[x\sigma_i\sigma_l\right] = \cosh x\left(1 + \sigma_i\sigma_l \tanh x\right)$$

so that the partition function is

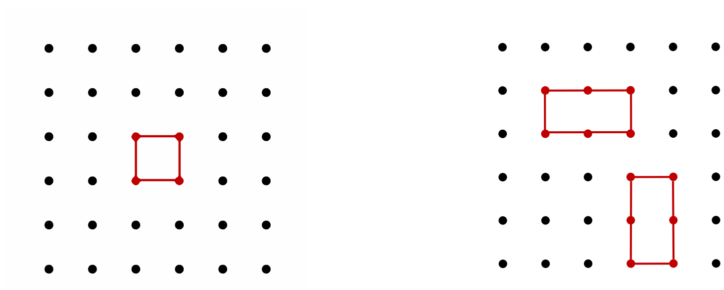
$$\mathcal{Z} = \sum_{\{\sigma\}}\prod_{\{ij\}} \cosh(K)\left(1 + \sigma_i\sigma_j \tanh(K)\right) .$$

At high-temperature, $\tanh K$ is small and one can take expansion. In the expansion, the generic expression of these terms is

$$\tanh(K)^r \sigma_1^{n_1}\sigma_2^{n_2}\sigma_3^{n_3}\ldots$$

where r is the total number of lines connecting the adjacent sites i and j, and n_i is the number of lines where i is the final site. Since each spin σ_i assumes values ± 1, we have a null sum unless all $n_1, n_2, \ldots$ are even numbers, which implies all closed loops P of the lattice as in Figure 5.4. Hence, the partition function is

$$\mathcal{Z}_{\text{high}} = \left[2\cosh(K)\right]^{NM}\sum_P [\tanh(K)]^{\text{lengh}(P)}$$



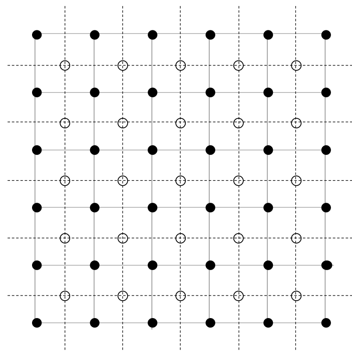
On the other hand, at low temperature, the spins tend to align one with another. Taking the dual lattice, the boader between antiparallel spins form loops P_D, and the partition function is given by

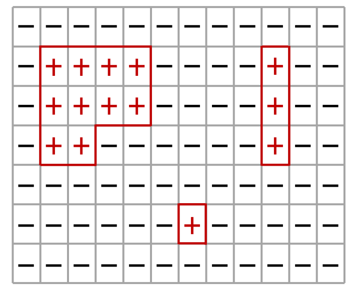
$$\mathcal{Z}_{\text{low}} = 2e^{NMK}\sum_{P_D} e^{-2K(\text{length}(P_D))} .$$

where the contribution of all spins down has been factored out of the sum.

The two phases are indeed dual to each other

$$\mathcal{Z}_{\text{low}}\left(K'\right) = 2(\sinh 2\mathrm{K})^{-NM/2}\mathcal{Z}_{\text{high}}(K)$$





where we identify

$$e^{-2K'} = \tanh K \,.$$

This is called the **Kramers-Wannier duality**.

Let us recall the Hamiltonian (1.2) of the Ising model

$$\mathcal{H} = -J \sum_{\langle i,i' \rangle} \sigma_i^z \sigma_{i'}^z - B \sum_i \sigma_i^x \tag{5.48}$$

Here we assume that the external magnetic field B is parallel to the x-axis. In the low-temperature $J \gg B$, we can ignore the interaction with the external magnetic field B. However, in the high-temperature $J \ll B$, this is not the case. In order to capture the interaction term, it is natural to consider the **disorder operator** μ_i on the dual lattice, which is defined as follows:

$$\mu_a^z \equiv \prod_{i=-N/2}^{i=a} \sigma_i^x \,, \qquad \mu_a^x = \sigma_a^z \sigma_{a+1}^z$$

For instance, if we act μ_a^z on the parallel-spin state

$$|\uparrow\rangle \equiv \prod_{b=-N/2}^{N/2} |\uparrow\rangle_b$$

then we have

$$\mu_a^z |\uparrow\rangle = \prod_{b=-N/2}^{a} |\downarrow\rangle_b \prod_{c=a+1}^{N/2} |\uparrow\rangle_c$$

Thus, the disorder operator μ_a^z changes the ordered spin configurations at the dual site a so that it can be understood as a **soliton** located at the dual site a. Moreover, the Hamiltonian (5.48) can be written as

$$\mathcal{H} = -B \sum_{\langle a,a' \rangle} \mu_a^z \mu_{a'}^z - J \sum_a \mu_a^x$$

so that B and J are exchanged. Hence, in the low-temperature

$$\langle \sigma_a^z \rangle \neq 0, \quad \langle \mu_a^z \rangle = 0, \quad T < T_c$$

whereas in the high-temperature

$$\langle \mu_a^z \rangle \neq 0, \quad \langle \sigma_a^z \rangle = 0, \quad T > T_c \,.$$

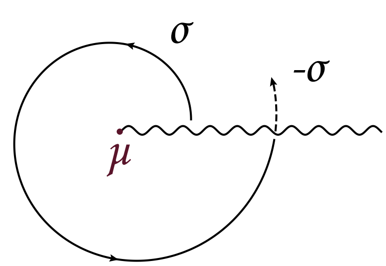
Thus, the Kramers-Wannier duality exchanges the spin and disorder operator $\sigma \leftrightarrow \mu$. Hence, σ and μ have the same conformal dimension. In fact, they are called **mutually non-local**, meaning that one obtains the minus sign

$$\sigma(z,\overline{z})\mu(0,0) \to \sigma\left(e^{2\pi i}z, e^{-2\pi i}\overline{z}\right)\mu(0,0) = -\sigma(z,\overline{z})\mu(0,0)$$

when σ is rotated around μ. From this fact, the fermion shows up in the OPE of σ and μ

$$\sigma(z,\overline{z})\mu(0,0) \sim \frac{1}{|z|^{\frac{1}{4}}}\left(z^{1/2}\psi(0) + \overline{z}^{1/2}\overline{\psi}(0)\right) \tag{5.49}$$

Recalling that the central charge of the minimal model $\mathcal{M}_{p=3}$ is $c = \frac{1}{2}$, one can deduce that the continuum limit of the Ising model is described by a free fermion.



In fact, the various operators are identified as

$$\begin{aligned} \psi(z) &\sim \phi_{2,1}(z) \otimes \phi_{1,1}(\widetilde{z}) \\ \overline{\psi}(\overline{z}) &\sim \phi_{1,1}(z) \otimes \phi_{2,1}(\overline{z}) \\ \psi\overline{\psi} := \varepsilon(z,\overline{z}) &\sim \phi_{2,1}(z) \otimes \phi_{2,1}(\overline{z}) \\ \sigma(z,\overline{z}) \text{ or } \mu(z,\overline{z}) &\sim \phi_{1,2}(z) \otimes \phi_{1,2}(\overline{z}) \end{aligned} \tag{5.50}$$

and the operator algebras are read off

$$\sigma \cdot \sigma = \mathbf{1} + \varepsilon\,, \qquad \varepsilon \cdot \varepsilon = \mathbf{1}\,, \qquad \sigma \cdot \varepsilon = \sigma\,.$$

Moreover, using the associativity of OPEs, one can derive from (5.49)

$$\psi(z)\sigma(w,\overline{w}) \sim \frac{1}{(z-w)^{1/2}}\mu(w,\overline{w})\,.$$

W can compare this with the mode expansion of free fermion on the Ramond sector

$$\psi(z)\sigma(0) = \sum_{r\in\mathbb{Z}} z^{-r-1/2} b_r \sigma(0)\,,$$

we can deduce

$$b_0\sigma(0) = \mu(0)\,.$$

In the presence of the spin field $\sigma(0)$ at the origin, the fermion field obeys the anti-periodic boundary condition

$$\psi(z) \to \psi(e^{2\pi i}z) = -\psi(z)\,.$$

The spin field σ creates a branch cut, which change the boundary condition of the fermion field. Hence, σ can be regarded as the **twist operator**.

Finally, let us consider the partition functions of the Ising model. There are three unitary irreducible representations corresponding to the highest weights $h = 0, \frac{1}{2}, \frac{1}{16}$, and the corresponding characters can be read off from (5.45) as

$$\begin{aligned} \chi_0 &= \frac{1}{2}\left(\sqrt{\frac{\vartheta_3}{\eta}} + \sqrt{\frac{\vartheta_4}{\eta}}\right) = \mathrm{Tr}_{NS}\left(\frac{1+(-1)^F}{2} q^{L_0-\frac{c}{24}}\right), \\ \chi_{\frac{1}{2}} &= \frac{1}{2}\left(\sqrt{\frac{\vartheta_3}{\eta}} - \sqrt{\frac{\vartheta_4}{\eta}}\right) = \mathrm{Tr}_{NS}\left(\frac{1-(-1)^F}{2} q^{L_0-\frac{c}{24}}\right), \\ \chi_{\frac{1}{16}} &= \frac{1}{\sqrt{2}}\sqrt{\frac{\vartheta_2}{\eta}} = \mathrm{Tr}_R\left(q^{L_0-\frac{c}{24}}\right). \end{aligned} \tag{5.51}$$

Using these expressions, the partition function of the Ising model of diagonal type is equal to that of the free fermion (4.91)

$$\mathcal{Z}_{\mathrm{F}}(\tau,\overline{\tau}) = \chi_0\overline{\chi}_0 + \chi_{\frac{1}{2}}\overline{\chi}_{\frac{1}{2}} + \chi_{\frac{1}{16}}\overline{\chi}_{\frac{1}{16}}. \tag{5.52}$$

The structure of this partition function also appears when studying superstrings, and the operator $\frac{1}{2}(1+(-1)^F)$ is known as the **Gliozzi-Scherk-Olive (GSO) projection**.

5.5 Other examples

Yang-Lee singularity $(p', p) = (2, 5)$

Among the minimal non-unitary models, a simple but particularly significant example is given by the model $\mathcal{M}_{2,5}$ with central charge is $c = -22/5$. In addition to the identity operator, there is only a field $\phi_{1,2}$ of conformal dimension $h = -1/5$.

The partition function of a statistical model defined on a lattice is an analytic function of its parameters as long as the number N of the fluctuating variables is finite. Let us consider the Ising model at a given value T of the temperature and in the presence of an external magnetic field B

$$\mathcal{H} = -J\sum_{\langle i,i'\rangle}\sigma_i\sigma_{i'} - B\sum_i \sigma_i$$

The Yang-Lee theorem states that the zeroes of the partition function Z are all on the imaginary axis $\mathrm{Re}B = 0$. Fisher showed that the points $B = \pm iB_c$ are new critical points of the theory, which are called **Yang-Lee edge singularities**. Furthermore, he has argued that the effective action is given by the Landau-Ginzburg theory

$$\mathcal{S} = \int d^2z\left[\frac{1}{2}(\partial\Sigma)^2 + i\,(B-B_c)\,\Sigma + ig\Sigma^3\right]$$

where the non-unitarity of the model manifests itself in the imaginary value of the coupling constant. This model is described by the non-unitary minimal model $\mathcal{M}_{2,5}$ [Car85, ISZ88-No.13].

$$
\begin{array}{c|ccc}
4 & \frac{3}{2} & \frac{7}{2} & 0 \\
3 & \frac{3}{5} & \frac{3}{80} & \frac{1}{10} \\
2 & \frac{1}{10} & \frac{3}{80} & \frac{3}{5} \\
1 & 0 & \frac{7}{16} & \frac{3}{2} \\
\hline
0 & 1 & 2 & 3
\end{array}
\qquad\qquad
\begin{array}{c|ccc}
4 & G & \sigma' & \mathbf{1} \\
3 & t & \sigma & \varepsilon \\
2 & \varepsilon & \sigma & t \\
1 & \mathbf{1} & \sigma' & G \\
\hline
0 & 1 & 2 & 3
\end{array}
$$

Tricritical Ising model $(p', p) = (5, 4)$

Let us consider the variant of the Ising model whose Hamiltonian is given by

$$
\mathcal{H} = -J \sum_{\langle i,j \rangle}^{N} \sigma_i \sigma_j t_i t_j - B \sum_{i=1}^{N} \sigma_i t_i - \mu \sum_{i=1}^{N} t_i
$$

where a spin variable σ_k takes values ± 1, and a vacancy variable t_k, with values 0 and 1. This variable specifies whether the site is empty (0) or occupied (1). The chemical potential μ controls the density of the vacancy.

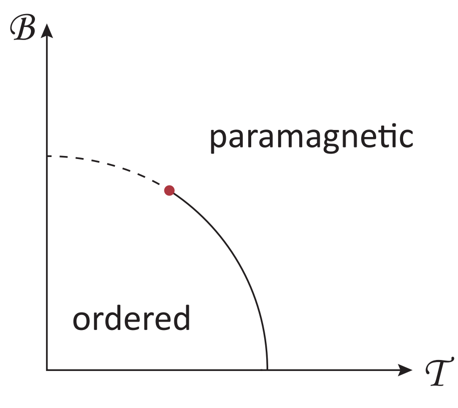


Figure 14: Phase diagram of the tricritical Ising model. The (broken) first-order line and the (full) line of Ising critical points meet at the tricritical point (B_t, T_t), shown as black dot

A typical phase diagram of this model is sketched in Figure 14, where the lines of first-order and second-order transitions meet in the tricritical point, located at $B = B_t$ and $T = T_t$. While at a normal critical point two phases become indistinguishable, at a tricritical point three physically distinct phases merge. For that reason, this model is called the **tricritical Ising model**. The phenomenology of the tricritical Ising model is described in [Car96, §4].

Remarkably, the tricritical model is described by $\mathcal{N} = 1$ superconformal field theory where G is the supercurrent [FQS85, ISZ88-No.5].

3-state Potts model $(p',p)=(6,5)$

$$\begin{array}{c|cccc}
5 & 3 & \frac{7}{5} & \frac{2}{5} & 0 \\
4 & \frac{13}{8} & \frac{21}{40} & \frac{1}{40} & \frac{1}{8} \\
3 & \frac{2}{3} & \frac{1}{15} & \frac{1}{15} & \frac{2}{3} \\
2 & \frac{1}{8} & \frac{1}{40} & \frac{21}{40} & \frac{13}{8} \\
1 & 0 & \frac{2}{5} & \frac{7}{5} & 3 \\
\hline
 & 1 & 2 & 3 & 4
\end{array}$$

On a square lattice, the hamiltonian of the three-state Potts model is given by

$$\mathcal{H}=-\frac{J}{2}\sum_{x,\alpha}(\sigma_x\overline{\sigma}_{x+\alpha}+\overline{\sigma}_x\sigma_{x+\alpha})$$

where σ takes the values ω^i (i=1,2,3) with third root of unity $\omega=e^{2\pi i/3}$. This model undergoes a second-order phase transition at $J_c=\frac{2}{3}\log\left(\sqrt{3}+1\right)$, which is endowed with the $\mathbb{Z}_3$ symmetry [Dot84, ISZ88-No.12]. The lattice theory is exactly solvable and consequently all critical exponents are known. For instance,

$$h_\sigma=h_{\overline{\sigma}}=\frac{1}{15}\,,\qquad h_\varepsilon=\frac{2}{5}\,.$$

It is tempting to identify the primary field of the unitary minimal model $\mathcal{M}_{p=5}$. However, the exact solution of the lattice model does not have operators with conformal dimensions $\frac{1}{8}$, $\frac{1}{40}$, $\frac{21}{40}$, and $\frac{13}{8}$ [vGR86, ISZ88-No.22].

In fact, the partition function of the 3-state Potts model is described by the one of the non-diagonal type [Car86a, ISZ88-No.29]

$$\mathcal{Z}_{\text{3-Potts}}=\sum_{r=1,2}\left\{|\chi_{r,1}+\chi_{r,5}|^2+2\,|\chi_{r,3}|^2\right\}\,, \tag{5.53}$$

so that the primary fields are [Dot84, ISZ88-No.12]

$$(0,0),\left(\frac{2}{5},\frac{2}{5}\right),\left(\frac{7}{5},\frac{7}{5}\right),2\times\left(\frac{1}{15},\frac{1}{15}\right),2\times\left(\frac{2}{3},\frac{2}{3}\right)(0,3),(3,0),\left(\frac{2}{5},\frac{7}{5}\right),\left(\frac{7}{5},\frac{2}{5}\right)$$

It is noteworthy that there are chiral spin-3 current $W(z)$ and $\overline{W}(\overline{z})$ corresponding to (0,3), (3,0) [Zam85, ISZ88-No.6]. In fact, $W(z)$ with the energy momentum tensor $T(z)$ form W_3-algebra [FZ87].

Other models

It was shown in [Hus84, ISZ88-No.32] that a general unitary minimal model $\mathcal{M}_p$ of diagonal type describes an RSOS model constructed in [ABF84, ISZ88-No.31]. Furthermore, the CFT with $\mathbb{Z}_k$ symmetry has been constructed in [FZ85, ISZ88-No.14], which we will see in §7.7, where the central charge is

$$c=\frac{2(k-1)}{k+2}\,. \tag{5.54}$$

When $k = 4$, the central charge is $c = 1$ and the corresponding CFT is called the **Ashkin-Teller model** where the Hamiltonian is written by the two spin fields σ_i and τ_i as

$$\mathcal{H} = -J_2 \sum_{\langle i,j \rangle} \left(\sigma_i \sigma_j + \tau_i \tau_j \right) - J_4 \sum_{\langle i,j \rangle} \sigma_i \sigma_j \tau_i \tau_j \, .$$

The $\mathbb{Z}_4$ symmetry is given by

$$s_j \to e^{\frac{i\pi k}{2}} s_j \, , \qquad \text{where} \qquad s_j \equiv \frac{\sigma_j + i\tau_j}{\sqrt{2}} \, .$$

This Ashkin-Teller model can be described by the free boson on an orbifold space $S^1/\mathbb{Z}_2$, which will be studied in §8.2. All in all, these models are main characters in the papers listed in [ISZ88].

5.6 Conformal blocks and bootstrap

The Operator Algebra

The main object of a field theory is the calculation of correlation functions. We have seen how the coordinate dependence of two- and three-point function. We have mentioned in the Minimal model section that one way to construct all correlation functions is to find the corresponding operator algebra: The complete OPE (including all regular terms) of all primary fields with each other. Minimal model concerns about the primary fields with its fusion rule in a theory, but doesn't tell anything about the coefficients of operator algebra. The goal of this section is to spell out which of its coefficients can be fixed by conformal invariance, which are not.

We know that two-point correlation function vanishes if the conformal dimensions of the two fields are different. If the conformal dimensions are the same for a finite set of primary fields ϕ_α, the correlators are

$$\left\langle \phi_\alpha(w, \overline{w}) \phi_\beta(z, \overline{z}) \right\rangle = \frac{C_{\alpha\beta}}{(w - z)^{2h} (\overline{w} - \overline{z})^{2\overline{h}}} \, , \tag{5.55}$$

where $h_\alpha = h_\beta = h$. Since the coefficients are symmetric, we are free to choose a basis so that $C_{\alpha\beta} = \delta_{\alpha\beta}$. Thus, conformal families associated with different ϕ_α are orthogonal in the sense of the two point function.

The general form of operator algebra is given before (5.9)

$$\phi_1(z, \overline{z}) \phi_2(0, 0) = \sum_p \sum_{\{k, \overline{k}\}} C_{12}^{p\{k, \overline{k}\}} z^{h_p - h_1 - h_2 + K} \overline{z}^{\overline{h}_p - \overline{h}_1 - \overline{h}_2 + \overline{K}} \phi_p^{\{k, \overline{k}\}}(0, 0) \, , \tag{5.56}$$

where h_1, h_2 need not to be the same and $K = \sum_i k_i$ and $\overline{K} = \sum_i \overline{k}_i$; the expression k means a collection of indices k_i. The sum is over all the possible highest operators and its descendants, and the pre-factors ensure the invariance under scaling transformation.

Now consider the three point function. By choosing suitable conformal transformation, we can send two of the coordinate to 0 and ∞. Therefore, we have

$$\begin{aligned} \langle \phi_r \left| \phi_1(z, \overline{z}) \right| \phi_2 \rangle &= \lim_{w, \overline{w} \to \infty} w^{2h_r} \overline{w}^{2\overline{h}_r} \left\langle \phi_r(w, \overline{w}) \phi_1(z, \overline{z}) \phi_2(0, 0) \right\rangle \\ &= \frac{C_{r12}}{z^{h_1 + h_2 - h_r} \overline{z}^{\overline{h}_1 + \overline{h}_2 - \overline{h}_r}} \end{aligned}$$

On the OPE side, the only contributing term is $p\{k,\bar{k}\} = r\{0,0\}$, because of the orthogonality of the Verma modules and the comparison of pre-factors in (5.56). We thus have

$$C_{12}^{p\{0,0\}} = C_{p12} \tag{5.57}$$

which tells us that the most singular term of the operator algebra is the coefficient of the three-point function. Since the correlations of descendants are built on the correlation of the primaries, we expect the coefficient to have the form:

$$C_{12}^{p\{k,\bar{k}\}} = C_{12}^{p}\beta_{12}^{p\{k\}}\overline{\beta}_{12}^{p\{\bar{k}\}}\,. \tag{5.58}$$

Holomorphic and antiholomorphic parts are separated as usual, and by convention we set $\beta_{ij}^{p\{0\}} = 1$. $\beta_{ij}^{p\{k\}}$ can be determined by BPZ equation (5.8) as functions of central charge c and of conformal dimensions.

Therefore, the only coefficients we can not determine are the coefficients C_{pnm} in the three-point functions, which must be obtained from another source for instance through the conformal bootstrap.

Conformal Blocks

The conformal blocks are used to build four-point function or more. Consider a general four-point function.

$$\left\langle \phi_1\left(z_1,\bar{z}_1\right)\phi_2\left(z_2,\bar{z}_2\right)\phi_3\left(z_3,\bar{z}_3\right)\phi_4\left(z_4,\bar{z}_4\right)\right\rangle\,, \tag{5.59}$$

We can perform a suitable transformation to set $z_4 = 0, z_1 = \infty, z_2 = 1$ and $z_3 = x$ so as to reduce 4 variables to only x. The correlation function we want calculate becomes:

$$G_{34}^{21}(x,\bar{x}) = \lim_{z_1,\bar{z}_1\to\infty} z_1^{2h_1}\bar{z}_1^{2\bar{h}_1}\left\langle \phi_1\left(z_1,\bar{z}_1\right)\phi_2(1,1)\phi_3(x,\bar{x})\phi_4(0,0)\right\rangle\,, \tag{5.60}$$

where the order in which the indices of G appear is important. Applying the operator algebra (5.56) to $\phi_3(x,\bar{x})\phi_4(0,0)$, the function G_{34}^{21} becomes

$$G_{34}^{21}(x,\bar{x}) = \sum_p C_{34}^{p}C_{12}^{p}\mathcal{F}_{34}^{21}(p|x)\overline{\mathcal{F}}_{34}^{21}(p|\bar{x})\,, \tag{5.61}$$

where

$$\mathcal{F}_{34}^{21}(p|x) = x^{h_p-h_3-h_4}\sum_{\{k\}}\beta_{34}^{p(k)}x^{K}\frac{\left\langle h_1\left|\phi_2(1)L_{-k_1}\cdots L_{-k_N}\right|h_p\right\rangle}{\left\langle h_1\left|\phi_2(1)\right|h_p\right\rangle}\,, \tag{5.62}$$

which are called conformal blocks. The denominator is simply equal to $(C_{21}^{p})^{1/2}$. They can be simply calculated by commuting the Virasoro generators over the field $\phi_2(1)$, and can be represented in terms of central charge and conformal dimensions. Since we combine ϕ_3 and ϕ_4 to produce intermediate operator $[\phi_p]$, and $[\phi_p]$ then interacts with ϕ_1 and ϕ_2, this process can be displayed in diagrammatic language. More precisely, the diagram shows in Fig. 15 corresponds to $\mathcal{F}_{ij}^{\ell m}(p|x)\overline{\mathcal{F}}_{ij}^{\ell m}(p|\bar{x})$. One may write the conformal block as a power series in x:

$$\mathcal{F}_{34}^{21}(p|x) = x^{h_p-h_3-h_4}\sum_{K=0}^{\infty}\mathcal{F}_K x^K\,. \tag{5.63}$$

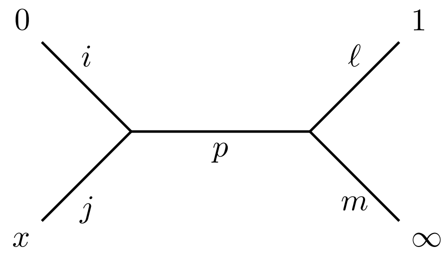


Figure 15: Diagram for the conformal block

and list several leading coefficients in the above series which can be calculated by (5.62).

$$\mathcal{F}_0 = 1\,, \qquad \mathcal{F}_1 = \frac{\left(h_p + h_2 - h_1\right)\left(h_p + h_3 - h_4\right)}{2h_p} \tag{5.64}$$

$$\begin{aligned}\mathcal{F}_2 = &\frac{\left(h_p + h_2 - h_1\right)\left(h_p + h_2 - h_1 + 1\right)\left(h + h_3 - h_4\right)\left(h + h_3 - h_4 + 1\right)}{4h_p\left(2h_p + 1\right)} \\ &+ 2\left(\frac{h_1 + h_2}{2} + \frac{h_p\left(h_p - 1\right)}{2\left(2h_p + 1\right)} - \frac{3\left(h_1 - h_2\right)^2}{2\left(2h_p + 1\right)}\right)^2 \\ &\times \left(\frac{h_3 + h_4}{2} + \frac{h_p\left(h_p - 1\right)}{2\left(2h_p + 1\right)} - \frac{3\left(h_3 - h_4\right)^2}{2\left(2h_p + 1\right)}\right)^2 \left(c + \frac{2h_p\left(8h_p - 5\right)}{2h_p + 1}\right)^{-1}\end{aligned}$$

Crossing Symmetry and the Conformal Bootstrap

In defining the function $G^{21}_{24}(x, \overline{x})$, we have chosen a specific order for the four fields ϕ_{1-4} within the correlation function. By using different conformal transformation as before, we can send z_2 to 0 and z_4 to 1, and then z_3 proves to be $1 - x$. We combine ϕ_2 and ϕ_3 at first in the calculation process, thus get the same answer. Therefore, we obtain the following identity.

$$G^{21}_{34}(x, \overline{x}) = G^{41}_{32}(1 - x, 1 - \overline{x})\,. \tag{5.65}$$

or more precisely

$$\sum_p C^p_{21} C^p_{34} \mathcal{F}^{21}_{34}(p|x)\overline{\mathcal{F}}^{21}_{34}(p|\overline{x}) = \sum_q C^q_{41} C^q_{32} \mathcal{F}^{41}_{32}(q|1 - x)\overline{\mathcal{F}}^{41}_{32}(q|1 - \overline{x}) \tag{5.66}$$

This relation is represented graphically on Fig.16

If the conformal block $\mathcal{F}$ is known, then (5.65) yields a system of equations that can determine C_{ijk}'s and $h, \overline{h}$'s. The program of calculating the correlation functions simply by assuming crossing symmetry is known as the bootstrap approach. For minimal models, the bootstrap equations can be solved completely.

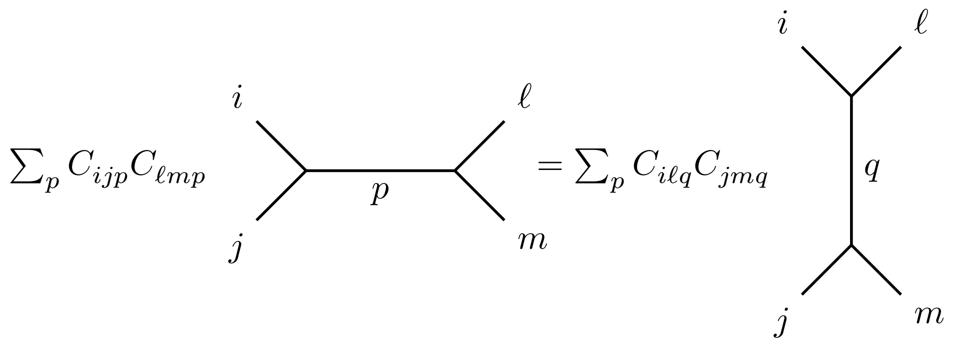


Figure 16: Crossing symmetry

6 Renormalization group flow and c-theorem

6.1 Renormalization group flow

Idea

In the introduction, we have briefly discussed that conformal field theories arise at fixed points of renormalization group flow. The study of renormalization group flows tells us the special role of conformal field theories (See [Zam89] included in [JMT89] and [EY89]). In this section, we shall glimpse this in two dimensional systems.

The name of renormalization group is actually misleading in the following two reasons:

- the transformations of the renormalization "group" are irreversible and therefore they do not form a group.
- moreover, they do not necessarily concern the renormalization of a theory, i.e. the cure of the divergencies of the perturbative series.

Rather, the concept of renormalization group flow describes how families of physical theories behave under change of the length-scale. The effect of averaging over short-distance physics can be summarized in the change of finitely many parameters in the long-distant physics of interest.

There is a wonderful introduction to renormalization group flow in [Car96, §3]. Following Cardy, we shall explain the concept of renormalization group flow by coarse-graining for the Ising model. Let us consider a statistical system defined on a d-dimensional regular lattice of lattice spacing a, with degrees of freedom is placed on its sites. Let $H(\{\sigma_i^{(1)}\}, g_k^{(1)})$ be the Hamiltonian of the system, where $g_k^{(1)}$ are the coupling constants of the various interactions among the spins $\sigma_i^{(1)}$. Now we divide the original lattice into blocks of length ba, denoted by $\mathcal{B}_k$, each of them made of b^d spins, and we assign a new spin variable $\sigma_k^{(2)}$ by averaging over all the spins $\sigma_i^{(1)} \in \mathcal{B}_k$. We are going to do coarse-graning of the original lattice by repeating this procedure. More precisely, we define the new spin variable by

$$\sigma_k^{(n+1)} = A^{(n)} \sum_{i \in \mathcal{B}_k} \sigma_i^{(n)} \tag{6.1}$$

where A is a suitable normalization constant.

To define the effective hamiltonian for the coarse lattice, we introduce an operator

$$T\left(\sigma_k^{(n+1)}, \sigma_i^{(n)}\right) = \begin{cases} 1\,, & \text{if (6.1)} \\ 0\,, & \text{otherwise} \end{cases}$$

so that it satisfies

$$\sum_{\left\{\sigma_k^{(n+1)}\right\}} T\left(\sigma_k^{(n+1)}, \sigma_i^{(n)}\right) = 1$$

Then, the effective hamiltonian $H^{(n+1)}\left(\left\{\sigma_k^{(n+1)}\right\}, g_k^{(n+1)}\right)$ can be defined by averaging the spins $\sigma_i^{(n)}$ with their Boltzmann factor

$$\exp\left[-H^{(n+1)}\left(\left\{\sigma_k^{(n+1)}\right\}, g_k^{(n+1)}\right)\right] = \sum_{\left\{\sigma_i^{(n)}\right\}} \prod_{\text{blocks}} T\left(\sigma_k^{(n+1)}, \sigma_i^{(n)}\right) \exp\left[-H^{(n)}\left(\left\{\sigma_i^{(n)}\right\}, g_i^{(n)}\right)\right]$$

in this way. Then, ignoring an overall constant, we have the equivalence of the statistical partition functions

$$\sum_{\left\{\sigma_k^{(n+1)}\right\}} \exp\left[-H^{(n+1)}\left(\left\{\sigma_k^{(n+1)}\right\}, g_k^{(n+1)}\right)\right] = \sum_{\left\{\sigma_i^{(n)}\right\}} \exp\left[-H^{(n)}\left(\left\{\sigma_i^{(n)}\right\}, g_i^{(n)}\right)\right]$$

This equality also holds for the expectation value of any observable $\mathcal{O}$. In this process, the new coupling constants are expressed by the previous one

$$\left\{g^{(n+1)}\right\} = \mathcal{R}\left(\left\{g^{(n)}\right\}\right)$$

where $\mathcal{R}$ is, in general, a complicated nonlinear transformation.Starting from the original coupling constants $\{g_k^{(1)}\}$, the procedure of the coarse-graining provides the sequence $\{g_k^{(2)}\}$, $\{g_k^{(3)}\}$, ..., which is called a **renormalization group trajectory**. In fact, a fixed point is a point in the space of the coupling constants that remains invariant

$$g^* = \mathcal{R}\left(g^*\right)$$

Since the coarse-graining has no effect on the theory, it is scale-invariant, which implies that the theory is conformal. As briefly explained in the introduction, critical phenomena are described by these fixed points.

One-dimensional Ising model

As an concrete example, let us examine the one-dimensional Ising model whose Hamiltonian is

$$H\left(s_i; J\right) = -J \sum_i s_i s_{i+1}$$

Each pair of spins has the Boltzmann weight

$$W\left(s_i, s_{i+1}; v\right) = e^{J s_i s_{i+1}} = \cosh J\left(1 + v s_i s_{i+1}\right)$$

where $v = \tanh J$.

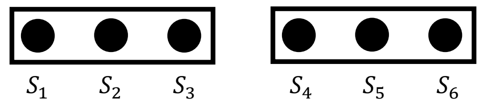


Figure 17: coarse-graining

As illustrated in Figure 17, we divide the lattice of three spins. Denoting $\sigma_1 = s_2$ and $\sigma_2 = s_5$, we sum over s_3 and s_4

$$\begin{aligned}\sum_{s_3,s_4} e^{J\sigma_1 s_3} e^{J s_3 s_4} e^{J s_4 \sigma_2} &= \sum_{s_3,s_4} (\cosh J)^3 \left(1 + v\sigma_1 s_3\right)\left(1 + v s_3 s_4\right)\left(1 + v s_4 \sigma_2\right) \\ &= 2^2 (\cosh J)^3 \left(1 + v^3 \sigma_1 \sigma_2\right)\end{aligned} \tag{6.2}$$

Beside a normalization constant, one can assign the new Boltzmann weight $W(\sigma_1, \sigma_2; \widetilde{v})$ for the new block spins σ_1, σ_2 with the new coupling constant

$$\widetilde{v} = v^3 ,$$

which is called the **renormalization group equation**. It has two fixed points: $v_1^* = 0$ and $v_2^* = 1$ corresponding to the high-temperature phase $T \to \infty$, and the low-temperature phase $T = 0$.

In addition, the new Hamiltonian of the system is given by

$$H\left(\sigma_i; \widetilde{J}\right) = \widetilde{N} p(J) - \widetilde{J} \sum_i \sigma_i \sigma_{i+1}$$

where $\widetilde{N} = N/3$ is the number of new sites, and

$$\widetilde{J} = \tanh^{-1}\left[(\tanh J)^3\right] , \quad p(J) = -\log\left[\frac{(\cosh J)^3}{\cosh \widetilde{J}}\right] - 2\log 2 .$$

β-functions and Callan-Symanzik equation

In general, the idea described above ends up with mathematical equations describing renormalization group flows in the space of parameters of a physical system. Let us derive these equations by using path integrals. Writing the action of a conformal field theory at a fixed point by S^*, the perturbed action can be written as

$$\mathcal{S} = \mathcal{S}^* + \sum_a \int d^2x \, g_a \mathcal{O}_a(x) \tag{6.3}$$

where the coupling constants g_a are sufficiently small. If $\mathcal{O}_a$ is of scaling dimension Δ_a the coupling constant g_a has scaling dimension $2 - \Delta_a$. Under an infinitesimal scaling transformation (dilatation) $x \to (1 + d\lambda)x$, the perturbed action behaves as

$$\begin{aligned}\mathcal{S} \to& \mathcal{S}^* + \sum_a (1 + d\lambda)^2 (1 + d\lambda)^{-\Delta_a} \int d^2x \, g_a \mathcal{O}_a(x) \\ &\simeq \mathcal{S} + d\lambda \sum_a (2 - \Delta_a) \int d^2x \, g_a \mathcal{O}_a(x)\end{aligned} \tag{6.4}$$

where we assume that S^* is scale-invariant. On the other hand, we recall the variation under the scaling transformation

$$\mathcal{S} \to \mathcal{S} + d\lambda \int d^2x \ (T_{11}(x) + T_{22}(x)) = S + 2\pi d\lambda \int d^2x\, \Theta(x)$$

where $\Theta := 2\pi T^{\mu}{}_{\mu}$ is the trace of the energy-momentum tensor. Therefore, we can write

$$\Theta(x) = 2\pi \sum_a \beta_a(g)\mathcal{O}_a(x) \tag{6.5}$$

where β_a are first-order approximations of so-called β-functions

$$\beta_a(g) = (2 - \Delta_a)\, g_a \tag{6.6}$$

The idea of renormalization group flow is that the change of a physical system on the length-scale is absorbed into the change of coupling constants, which is measured by the β-functions

$$\frac{dg_a}{d\lambda} = \beta_a(g)$$

Therefore, at a fixed point, we have

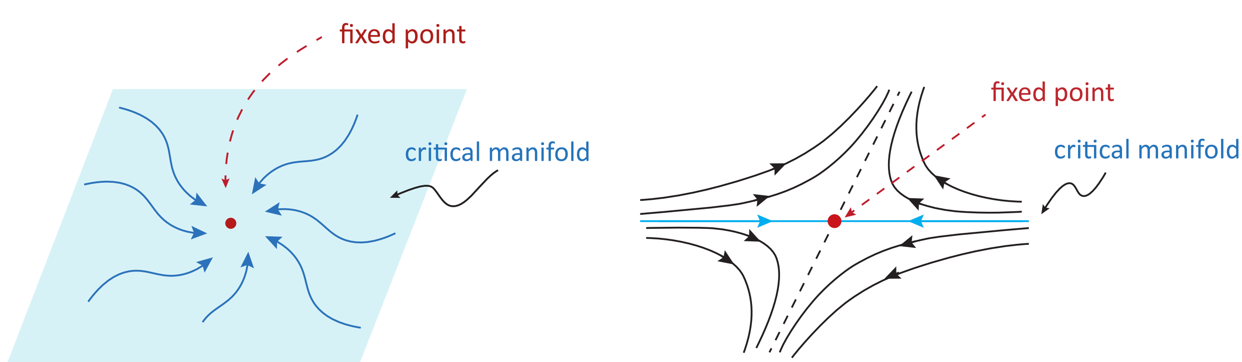
$$\beta_a(g^*) = 0$$

by definition. At the first-order approximation (6.6), the coupling constants behave as

$$g_a \sim e^{(2-\Delta_a)\lambda}$$

around a fixed point $g_a^* = 0$.

At the long distant scale $\lambda \gg 1$ that is called **infra-red** or **IR**, if $\Delta_a < 2$, a coupling constant increases, it is called a **relevant coupling**. If $\Delta_a > 2$, on the contrary, it decreases, is called an **irrelevant coupling**. In the space of coupling constants $\{g_a\}$, the subspace spanned by irrelevant ones is called the **critical manifold**, which is the attractive basin for the fixed point g^*. Under the RG flow, irrelevant operators are scaled out, and it arrives at the fixed point. Hence, all the theories belong to the same universality class.



On the other hand, the instability nature of a critical point is determined by the number of its relevant couplings. Relevant couplings gets amplified as a theory flows to the low-energy region, and it moves away from the fixed point. When $\Delta_a = 2$, an operator $\mathcal{O}_a$ is called **marginal**, and its behavior depends on its β-function, which is determined by taking quantum corrections into account (beyond the first-order approximation). If $\beta_a > 0$ (resp. $\beta_a < 0$), then it is called **marginally relevant** (resp. **marginally irrelevenat**). If $\beta_a = 0$, then it is called **exactly marginal**. Our main interest here is the two-dimensional theories but the results given above can be easily generalized to quantum field theories in any dimensions.

Now let us consider the correlation functions of the local fields $\phi_i(x)$, defined as usual by the functional integral

$$\langle \phi_1(x_1) \dots \phi_n(x_n) \rangle = \int \mathcal{D}\phi \, \phi_1(x_1) \dots \phi_n(x_n) \, e^{-S[\phi]}$$

If the variation of the field ϕ_i under the scaling transformation is

$$\delta\phi_i(x) = \epsilon \left(x^\mu \partial_\mu + \widehat{\Delta}_i \right) \phi_i(x)$$

the corresponding Ward-Takahashi identity (3.33) is

$$\sum_{i=1}^n \left\langle \left(x^\mu \partial_\mu + \widehat{\Delta}_i \right) \phi_1 \dots \phi_n \right\rangle - \int \frac{d^2x}{2\pi} \langle \Theta(x) \phi_1(x_1) \dots \phi_n(x_n) \rangle = 0 \,. \tag{6.7}$$

The fields $\phi_i(x)$ can also depend on the coupling constants and their variation, varying the coupling g_a, is expressed in terms of an operator $\widehat{B}_i^{(a)}$

$$\widehat{B}_i^{(a)} \phi_i(x) = \frac{\partial}{\partial g_a} \phi_i(x)$$

Using (6.5), we can write

$$\sum_a \beta_a \frac{\partial}{\partial g_a} \langle \phi_1 \dots \phi_n \rangle = \sum_{i=1}^n \sum_a \beta_a \left\langle \widehat{B}_i^{(a)} \phi_1 \dots \phi_n \right\rangle - \int \frac{d^2y}{2\pi} \langle \Theta(y) \phi_1 \dots \phi_n \rangle$$

Defining the operator

$$\widehat{\Gamma}_i = \widehat{\Delta}_i + \sum_a \beta_a \widehat{B}_i^{(a)}$$

we can erase Θ from the (6.7), which is called the **Callan-Symanzik equation**

$$\sum_{i=1}^n \left\langle \left(x_i^\mu \frac{\partial}{\partial x_i^\mu} + \widehat{\Gamma}_i(g) \right) \phi_1(x_1) \dots \phi_n(x_n) \right\rangle - \sum_a \beta^a(g) \frac{\partial}{\partial g_a} \langle \phi_1(x_1) \dots \phi_n(x_n) \rangle = 0 \,.$$

Since the energy-momentum tensor is conserved independent of couplings, we have $\widehat{\Gamma}\Theta = 2\Theta$. From here, one can derive

$$\widehat{\Gamma}\mathcal{O}_a(x) = \sum_b \gamma_{ab} \mathcal{O}_b(x) = \sum_b \left(2\delta_{ab} - \frac{\partial \beta_b}{\partial g_a} \right) \mathcal{O}_b(x) \,.$$

In the first-order approximation, it is easy to see

$$\widehat{\Gamma}\mathcal{O}_a = \Delta_a \mathcal{O}_a \,,$$

as expected.

6.2 Zamolodchikov c-theorem

For the deformations of the unitary conformal models there is an important theorem due to A.B. Zamolodchikov, called the c-**theorem** [Zam86b, ISZ88-No.45], associated to the renormalization group flows. The c-Theorem asserts that, under the assumptions of rotation invariance, positivity and energy-momentum conservation in a two-dimensional theory, there exists a quantity C which is non-increasing along a renormalization group trajectory, stationary at a RG fixed point and takes as its values at these fixed points the corresponding central charge c. This theorem tells us the global structure of the space of physical theories, and the role of conformal field theories.

We begin with energy-momentum conservation,

$$\partial_\mu T^{\mu\nu} = 0\,, \tag{6.8}$$

where $T^{\mu\nu}$ is symmetric but not necessary to be traceless. Rewriting this in complex coordinates $z, \bar{z}$ for the 2D case, we have

$$\overline{\partial}_{\bar{z}} T + \frac{1}{4}\partial_z \Theta = 0 \quad , \quad \partial_z \overline{T} + \frac{1}{4}\overline{\partial}_{\bar{z}}\Theta = 0\,, \tag{6.9}$$

where $\Theta := 2\pi T^\mu{}_\mu$. At a fixed point, Θ vanishes and $T = T(z)$. But now we have $T = T(z, \bar{z})$. Let T,Θ and $\overline{T}$ be the components of spin 2, 0 and -2, respectively, of the stress-energy tensor. Rotation invariance spins and dimensions implies for the correlation functions of T and Θ

$$\begin{aligned}
\langle T(z,\bar{z})T(0,0)\rangle &= \frac{F(z\bar{z})}{z^4}\,,\\
\langle \Theta(z,\bar{z})T(0,0)\rangle = \langle T(z,\bar{z})\Theta(0,0)\rangle &= \frac{G(z\bar{z})}{z^3\bar{z}}\,,\\
\langle \Theta(z,\bar{z})\Theta(0,0)\rangle &= \frac{H(z\bar{z})}{z^2\bar{z}^2}\,.
\end{aligned} \tag{6.10}$$

and similarly for $\overline{T}$. Combining the first equation of (6.9) with T, we have

$$\left\langle \overline{\partial}_{\bar{z}} T(z,\bar{z})T(0,0)\right\rangle + \frac{1}{4}\left\langle \partial_z \Theta(z,\bar{z})T(0,0)\right\rangle = 0\,, \tag{6.11}$$

We adopt the notation $\dot{F} := z\bar{z}F'(z\bar{z})$ where the prime stands for the derivative of $F(x)$ with respect to its argument. One finds

$$\dot{F} + \frac{1}{4}\dot{G} - \frac{3}{4}G = 0\,, \tag{6.12}$$

and similarly by combining with Θ

$$\dot{G} - G + \frac{1}{4}\dot{H} - \frac{1}{2}H = 0\,. \tag{6.13}$$

Eliminating G from above two equations, and defining

$$C := 2F - G - \frac{3}{8}H\,. \tag{6.14}$$

One finds

$$\dot{C} = -\frac{3}{4}H \leq 0\,. \tag{6.15}$$

due to positivity. We see C is a non-increasing function of $R := (z\bar{z})^{1/2}$, for fixed values of couplings $\{g\}$. The R-dependence is related to the dependence on the $\{g\}$ via the Callan-Symanzik equation (6.11)

$$\left(R\frac{\partial}{\partial R} - \sum_a \beta_a(g)\frac{\partial}{\partial g_a}\right) C(R,\{g\}) = 0\,. \tag{6.16}$$

Using (6.5) and renormalization group equation, we get

$$\beta^a \frac{\partial}{\partial g^a} C(g) = -\frac{3}{2} G_{ab}(g)\beta^a(g)\beta^b(g)\,, \tag{6.17}$$

where

$$G_{ab}(g) = G_{ab}(1,g), \quad G_{ab}(z\bar{z},g) = (z\bar{z})^2 \langle \mathcal{O}_a(z,\bar{z})\mathcal{O}_b(0,0)\rangle\,, \tag{6.18}$$

is positive-definite, since the theory is unitary. In other words, $G_{ab}(g)$ can be regarded as a metric of the space of the coupling constants. Therefore, it is called the **Zamolodchikov metric**. Moreover, we can write the β-function as

$$\beta_a(g) = -\frac{2}{3}\sum_b G^{-1}_{ab}\frac{\partial C}{\partial g_b}\,, \tag{6.19}$$

so that C can be understood as a Morse function on the space of the coupling constants and β_a can be understood as the gradient vector fields of the Morse function [Mil63].

It follows from (6.17) that the function $C(g)$ decreases under the renormalization group flow. Since β function vanishes at a fix point, C is stationary with respect to R at the fixed point. At the fixed point, the theory is conformal and $F = c/2$, $G = H = 0$. Consequently it is equal to the central charge $C(g^*) = c$ at the fixed point $g = g^*$. This proves the statement of the c-**theorem**. Consequently, the central charge of S is less than or equal to that of S^* in (6.3)

$$c_{\mathrm{UV}} \geq c_{\mathrm{IR}}\,.$$

Hence, one can think that the central charge c encodes a degrees of freedom of the theory, and it decreases under a renormalization group flow since some massive particles are integrated out. There is the corresponding theorem in four dimensions [KS11]. In six dimensions, the corresponding theorem is proven for supersymmetric theories [CDI16].

6.3 Laudau-Ginzburg effective theory and minimal models

Laudau-Ginzburg effective theory

Zamolodchikov has proposed [Zam86a, ISZ88-No.17] that there exists a simple effective Lagrangian description for a special class of minimal models by the following action

$$\mathcal{S} = \int d^2z \left(\frac{1}{2}(\partial\Sigma)^2 + V(\Sigma)\right)$$

where the potential

$$V(\Sigma) = g_1\Sigma + g_2{:}\Sigma^2{:} + \cdots + g_{2(p-2)}{:}\Sigma^{2(p-2)}{:} + g{:}\Sigma^{2(p-1)}{:}$$

Note that the terms ${:}\Sigma^{2p-3}{:}$ can always be removed by a shift of the field $\Sigma \to \Sigma +$ const and absorbed in the linear term $g_1\Sigma$. Here we assume $g > 0$ so that there are $(p-1)$

6					
5					$\phi_{5,5}$
4				$\phi_{4,4}$	$\phi_{5,4}$
3			$\phi_{3,3}$	$\phi_{4,3}$	
2		$\phi_{2,2}$	$\phi_{3,2}$		
1	1	$\phi_{2,1}$			
0	1	2	3	4	5

6					
5					Σ^4
4				Σ^3	Σ^8
3			Σ^2	Σ^7	
2		Σ	Σ^6		
1	1	Σ^5			
0	1	2	3	4	5

Table 3: Relevant operators of conformal dimension $h < 1$ in the minimal model $\mathcal{M}_6$ and the corresponding operators in the Landau-Ginzburg effective theory.

local minima, which corresponds to different phases of the model. This model is called **Laudau-Ginzburg effective theory**.

The theory consists of $2(p-2)$ scalar relevant fields, associated to the operators $:\Sigma^k:$ for $1 \leq k \leq 2(p-1)$. On the other hand, the unitary minimal model $\mathcal{M}_p$ has $2(p-2)$ primary fields with conformal dimension $h < 1$. Hence, we identify the most relevant field $\phi_{2,2}$ of $\mathcal{M}_p$ (with the lowest conformal dimension) with the scalar field Σ in the Landau-Ginzburg theory. The OPE of Σ is

$$\Sigma(z)\Sigma(0) = \frac{1}{|z|^{2h_{2,2}}} + \frac{1}{|z|^{2h_{2,2}-h'}} :\Sigma^2:(0) + \cdots$$

where h' is the conformal dimension of $:\Sigma^2:$. On the other hand, recalling that the fusion rule of primary fields (5.34)

$$\phi_{2,2} \times \phi_{r,s} = [\phi_{r-1,s-1}] + [\phi_{r+1,s+1}] + [\phi_{r+1,s-1}] + [\phi_{r-1,s+1}] \ ,$$

so that

$$\phi_{2,2} \times \phi_{2,2} = [\mathbf{1}] + [\phi_{3,3}] + [\phi_{1,3}] + [\phi_{3,1}] \ .$$

The last two conformal families are irrelevant so that it is natural to identify

$$\phi_{3,3} = :\Sigma^2: .$$

The same procedure leads to

$$\begin{aligned} &\phi_{3,3} = :\Sigma^2: , \quad &&\phi_{4,4} = :\Sigma^3: , \quad &&\cdots \quad &&, \phi_{p-1,p-1} = :\Sigma^{p-2}: \\ &\phi_{2,1} = :\Sigma^{p-1}: , \quad &&\phi_{3,2} = :\Sigma^{q}: , \quad &&\cdots \quad &&, \phi_{p-1,p-2} = :\Sigma^{2p-4}: \end{aligned} \tag{6.20}$$

Moreover, $:\Sigma^{2p-3}:$ can be obtained from the OPE of $\phi_{p-1,p-2}$ and $\phi_{2,2}$. The inversion formula (5.32) identifies $\phi_{1,3} = \phi_{p-1,p-2}$ so that

$$\phi_{2,2} \times \phi_{1,3} = [\phi_{2,2}] + \left[\phi_{p-2,p-3}\right]$$

Therefore, we can identify $:\Sigma^{2p-3}:$ with the descendant $L_{-1}\overline{L}_{-1}\phi_{2,2}$ of $\phi_{2,2}$

$$:\Sigma^{2p-3}: = \partial\bar{\partial}\Sigma$$

This is the equation of motion at the multicritical point ($g_i = 0$) in Landau-Ginzburg theory.

If $g_i = 0$, the theory is invariant under $\mathbb{Z}_2$ symmetry $\Sigma \to -\Sigma$ [Zub86, ISZ88-No.30]. When $p = 3$, it is the Σ^4-theory that describes the Ising model. When $p = 4$, it is the Σ^6-theory that describes the tricritical Ising model. For a generic positive integer p, it describes the RSOS model [ABF84, ISZ88-No.31].

The correspondence between the minimal models and the Landau-Ginzburg models can be also seen even with $\mathcal{N} = 2$ superconformal symmetry. This is related to the ADE classification studied in §8.3, which has deep implication to geometry.

Minimal Models and Renormalization Group Flows

In fact, the minimal models provide a perfect platform to understand renormalization group flows [Zam87]. Let us consider the perturbation of the unitary minimal model $\mathcal{M}_p$ by the primary field $\phi_{1,3}$

$$\mathcal{S} = \mathcal{S}_p + g \int d^2x \phi_{1,3}(x)$$

where we assume $p \gg 1$. Then, the primary field $\phi_{1,3}$ can be considered as a quasi-marginal operator since the scaling dimension is

$$\Delta_{1,3} = 2h_{1,3} = 2 - \frac{4}{p+1} \equiv 2 - \epsilon$$

with $\epsilon \ll 1$. Following the fusion rule (5.34), one can read off

$$\phi_{1,3} \times \phi_{1,3} = 1 + \mathbf{C}_1 \phi_{1,3} + \mathbf{C}_2 \phi_{1,5} \ .$$

Since $\phi_{1,5}$ is an irrelevant operator, the operator expansion above implies the renormalization of the field $\phi_{1,3}$, which does not mix with any other fields. In fact, the structure constant is computed in [DF84, ISZ88-No.15]

$$\begin{aligned} \mathbf{C}_1 &= \frac{4}{\sqrt{3}} \frac{(1-\epsilon)^2}{(1-\epsilon/2)(1-3\epsilon/4)} \left[\frac{\Gamma(1-\epsilon/4)}{\Gamma(1+\epsilon/4)}\right]^{\frac{3}{2}} \left[\frac{\Gamma(1+3\epsilon/4)}{\Gamma(1-3\epsilon/4)}\right]^{\frac{1}{2}} \left[\frac{\Gamma(1+\epsilon/2)}{\Gamma(1-\epsilon/2)}\right]^2 \frac{\Gamma(1-\epsilon)}{\Gamma(1+\epsilon)} \\ &= \frac{4}{\sqrt{3}} \left(1 - \frac{3\epsilon}{4} + O\left(\epsilon^2\right)\right) \end{aligned}$$

In Homework 8, we have seen that the β-function can be written as

$$\frac{dg}{d\lambda} \equiv \beta(g) = \epsilon g - \pi \mathbf{C}_1 g^2 + \mathcal{O}(g^3) \ .$$

Using the gradient flow equation (6.19), one obtains

$$C(g) = c_p + \alpha \left[\frac{\epsilon}{2} g^2 - \frac{\pi}{3} \mathbf{C}_1 g^3\right] + \mathcal{O}(g^4)$$

where c_p is the central charge (5.37) of $\mathcal{M}_p$. Note that α depends on the Zamolodchikov metric G_{ab} in (6.18), and α can be evaluated as follows. The difference of the central charges at UV and IR can be computed by (6.15)

$$\Delta c = -\frac{3}{4} \int_0^\infty d(r^2) r^2 \langle \Theta(r) \Theta(0) \rangle$$

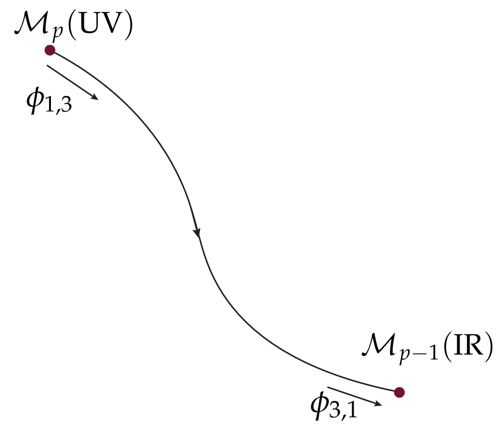


Figure 18: RG-flow between minimal models

Furthermore, (6.5) tells us that

$$\begin{aligned}\Delta c &= -\frac{3}{4}(2\pi\epsilon g)^2\int_0^\infty d(r^2)r^2\langle\phi_{1,3}(r)\phi_{1,3}(0)\rangle \\ &= -\frac{3}{2}(2\pi\epsilon g)^2\int_0^\infty \frac{r^3 dr}{r^{2\Delta_{1,3}}} \\ &= -3\pi^2\epsilon g^2 + \mathcal{O}(\epsilon^0)\,.\end{aligned} \tag{6.21}$$

Therefore, we have $\alpha = -6\pi^2$. At the fixed point $g^* = \epsilon/\pi\mathbf{C}_1$, the Zamolodchikov C-function is

$$\begin{aligned}C(g^*) &= c_p - \frac{\epsilon^3}{\mathbf{C}_1^2} + \mathcal{O}\left(\epsilon^4\right) \\ &= c_p - \frac{3}{4^2}\cdot\left(\frac{4}{p}\right)^3 + \mathcal{O}\left(p^{-4}\right) \\ &= c_p - \frac{12}{p^3} + \mathcal{O}\left(p^{-4}\right) \sim c_{p-1}\end{aligned} \tag{6.22}$$

At this perturbative order, the new value of the central charge coincides with that of the unitary minimal model $\mathcal{M}_{p-1}$. Therefore, the perturbation by $\phi_{1,3}$ leads the renormalization group flow from $\mathcal{M}_p$ to $\mathcal{M}_{p-1}$. Although we have assumed $p \gg 1$ in this analysis, the evidence of RG flow from $\mathcal{M}_p$ to $\mathcal{M}_{p-1}$ for an arbitrary p has been shown by means of factorized S-matrices [Zam91]. In fact, from Callan-Symanzik equation (6.11), one can deduce the conformal dimension, called **anomalous dimension**, of the operator $\phi_{1,3}$ at the fixed point

$$2 - \frac{\partial\beta}{\partial g}\Big|_{g=g^*} = 2 + \epsilon + \mathcal{O}(\epsilon^2)\,,$$

which can be read off as the conformal dimension of $\phi_{3,1}$ in $\mathcal{M}_{p-1}$ at the fixed point. Thus, the relevant operator at UV becomes the irrelevant operator at IR. (See Figure 18.)

In fact, this is consistent with the Landau-Ginzburg effective theory. As we have seen in (6.20), $\phi_{1,3} = \phi_{p-1,p-2} \sim :\Sigma^{2p-4}:$. Hence, the Landau-Ginzburg action for $\mathcal{M}_p$ with the perturbation is

$$\mathcal{S} = \int d^2z \left(\frac{1}{2}(\partial\Sigma)^2 + g_{2p-4}\Sigma^{2p-4} + g_{2p-2}\Sigma^{2p-2} \right) .$$

Then, the operator $:\Sigma^{2p-2}:$ becomes irrelevant and the effective action is

$$\mathcal{S} = \int d^2z \left(\frac{1}{2}(\partial\Sigma)^2 + g_{2p-4}\Sigma^{2p-4} \right) ,$$

which is the effective action for $\mathcal{M}_{p-1}$.

7 Wess-Zumino-Novikov-Witten Models

This section is devoted to study a very important class of exactly solvable 2d conformal field theories called **Wess-Zumino-Novikov-Witten (WZNW) models** [WZ71, Nov82, Wit84]. Th WZNW models have many applications and they are related to condensed matter physics, AdS/CFT correspondence, knot theory, and mathematical physics.

7.1 U(1) symmetry of free boson

Let us recall the free boson theory in §4.1 whose action is given by

$$\mathcal{S} = \frac{1}{8\pi} \int d^2x\, \partial_\mu\varphi\partial^\mu\varphi = \frac{1}{8\pi} \int d^2z\, \partial_z\varphi\partial_{\bar{z}}\varphi \tag{7.1}$$

This actin has a symmetry

$$\varphi \to \varphi + a \tag{7.2}$$

whose Noether current is $j^\mu = -\partial^\mu\varphi/4\pi$. The current conservation follows from the equation of motion (3.74)

$$\partial_\mu j^\mu = 0 .$$

As in in §4.1, one can decompose the current into holomorphic and anti-holomorphic part

$$J(z) \equiv i\partial_z\varphi , \qquad \bar{J}(\bar{z}) \equiv -i\partial_{\bar{z}}\varphi \tag{7.3}$$

so that the (holomorphic part of) energy-momentum tensor can be written as

$$T(z) = -\frac{1}{2} : \partial\varphi\partial\varphi := \frac{1}{2} : J(z)J(z) : .$$

As we will see below, this is called **Sugawara construction** of the energy-momentum tensor. In addition, the action (7.1) can be written as

$$\mathcal{S}_0 = \frac{1}{8\pi} \int_\Sigma d^2x\, \partial^\mu g^{-1}\partial_\mu g = \frac{1}{8\pi} \int_\Sigma d^2x\, \partial_z g^{-1}\partial_{\bar{z}} g , \tag{7.4}$$

where $g(z, \bar{z})$ is the map from

$$g : \mathbb{C} \to \mathrm{U}(1);\ z \mapsto g(z, \bar{z}) = \exp(i\varphi(z, \bar{z})) .$$

This action can be regarded as the simplest version of WZNW models. In fact, the symmetry can be understood as the U(1) rotation

$$e^{i\varphi(z,\bar{z})} \mapsto e^{i(\varphi(z,\bar{z})+a)} ,$$

so that (7.3) is called the U(1) current. The U(1) group is an abelian group and it is natural to ask if we can generalize this simple construction to non-abelian Lie groups. This section answers to this question.

7.2 SU(2) current algebra

Before moving to the WZNW models, we shall provide the easiest introduction to the non-abelian generalization. In (4.3), we have seen that the compactified boson enjoys the T-duality $R \leftrightarrow 2/R$ (4.83) and $R = \sqrt{2}$ is the self-dual radius. At the self-radius, one can define the vertex operators

$$J(z) = \frac{i\partial\varphi(z)}{\sqrt{2}} , \qquad J^{\pm}(z) =: e^{\pm i\sqrt{2}\varphi(z)} :$$

where they satisfy the OPE

$$\begin{aligned} J(z)J^{\pm}(w) &\sim \frac{\pm J^{\pm}(w)}{z-w} \\ J^{+}(z)J^{-}(w) &\sim \frac{1}{(z-w)^2} + \frac{2J(w)}{z-w}. \\ J(z)J(w) &\sim \frac{1/2}{(z-w)^2} \end{aligned} \tag{7.5}$$

Furthermore, if we write

$$J^{\pm} \equiv J^1 \pm iJ^2, \quad J \equiv J^3 , \tag{7.6}$$

then the OPE amounts to

$$J^a(z)J^b(w) \sim \frac{\frac{1}{2}\delta_{ab}}{(z-w)^2} + \frac{i\epsilon^{abc}J^c(w)}{z-w}$$

where ϵ^{abc} is the anti-symmetric tensor. It is easy to see the connection to the theory of the angular momentum which is associated to the $\mathfrak{su}(2)$ Lie algebra

$$[t^a, t^b] = i\epsilon^{abc}t^c \tag{7.7}$$

where $t^a = \sigma^a/2$ with the Pauli matrices σ^a ($a = 1, 2, 3$). In fact, as we will learn in the homework, the OPE can be generalized to

$$J^a(z)J^b(w) \sim \frac{\frac{k}{2}\delta_{ab}}{(z-w)^2} + \frac{i\epsilon^{abc}J^c(w)}{z-w} .$$

We note that

$$J(z) = \sum_{a=1}^{3} J^a(z)t^a$$

is called the SU(2) current and k is its **level**. As usual, the mode expansion

$$J^a(z) = \sum_{n\in\mathbb{Z}} z^{-n-1}J^a_n$$

leads to the following commutation relations, which is called is called the SU(2) **current algebra**,

$$\left[J^a_m, J^b_n\right] = i\epsilon^{abc} J^c_{m+n} + \frac{k}{2} m\delta^{ab}\delta_{m+n,0}$$

where the zero modes satisfy the $\mathfrak{su}(2)$ Lie algebra (7.7). In mathematics, it is called the $\widehat{\mathfrak{su}(2)}_k$ **affine Lie algebra** with level k. This can be further generalized to so-called **Kac-Moody algebra**.

Let us briefly study the highest weight representation of the SU(2) current algebra. From (7.6), we introduce

$$J^\pm_n \equiv J^1_n \pm i J^2_n$$

where the Hermitian conjugates are defined as

$$\left(J^\pm_n\right)^\dagger = J^\mp_{-n}, \quad \left(J^3_n\right)^\dagger = J^3_{-n} \ .$$

Then, the highest weight state $|h, j\rangle$ is defined as

$$J^3_0|h, j\rangle = j|h, j\rangle \ , \quad J^+_0|h, j\rangle = 0, \qquad J^a_n|h, j\rangle = 0, \quad n > 0, \quad a = 0, \pm \ . \tag{7.8}$$

Note that h is the conformal dimension

$$L_0|h, j\rangle = \frac{j(j+1)}{k+2}|h, j\rangle$$

which is left in homework. Since the zero modes satisfy the $\mathfrak{su}(2)$ Lie algebra, j takes an half-integer value $j = \frac{1}{2}, 1, \frac{3}{2}, 2, \cdots$ associated to the spin-j representation. In addition, it is straightforward to check that the following generators

$$J^+_{(1)} = J^+_{-1}, \quad J^-_{(1)} = J^-_{+1}, \quad J^3_{(1)} = J^3_0 - \frac{k}{2}$$

also form the $\mathfrak{su}(2)$ Lie algebra, and we call it $\mathfrak{su}(2)_{(1)}$. Since the eigenvalue of $2J^3_{(1)} = 2J^3_0 - k$ is an integer, k has to be an integer. Moreover, the unitarity imposes

$$\begin{aligned} 0 \leqq \left| J^+_{(1)}|h, j\rangle \right|^2 &= \langle h, j \left| J^-_{+1} J^+_{-1} \right| h, j\rangle \\ &= \langle h, j \left| [J^-_{+1}, J^+_{-1}] \right| h, j\rangle \\ &= -2\langle j|(J^3_0 - k/2)|h, j\rangle \\ &= -2j + k \end{aligned}$$

Therefore, for a given level $k \in \mathbb{Z}_{>0}$, the spin j is allowed to take the values

$$j = 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots, \frac{k}{2} \ .$$

7.3 A crash course on Lie groups and Lie algebras

It is well-known that the theory of angular momentum in quantum mechanics is described by the representation theory of the $\mathfrak{su}(2)$ Lie algebra (7.7). We want to generalize the current algebra to arbitrary Lie algebras. Thus, we shall briefly review Lie groups and Lie algebras. For more details, we refer the reader to [Kir08].

Definition 7.1 (Lie group). A **Lie group** is a manifold G with a group structure such that multiplication $m : G \times G \to G$ and inverse $i : G \to G$ are smooth maps. The **dimension** of a Lie group G is the dimension of the underlying manifold.

Some of Lie groups will be given by subsets of the space $M_n(\mathbb{F})$ of $n \times n$ matrices where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ specified by certain algebraic equations. For example,

- General linear group: $\mathrm{GL}(n, \mathbb{F}) = \{A \in \mathrm{M}_n(\mathbb{F}) | \det A \neq 0\}$
- Special linear group: $\mathrm{SL}(n, \mathbb{F}) = \{A \in \mathrm{GL}(n, \mathbb{F}) | \det A = 1\}$
- Symplectic group $\mathrm{Sp}(n, \mathbb{F}) = \{A \in \mathrm{GL}(2n, \mathbb{F}) | A^{\mathrm{T}} J A = J \text{ where } J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}\}$
- Unitary group $\mathrm{U}(n) = \{A \in \mathrm{GL}(n, \mathbb{C}) | AA^\dagger = I\}$
- Special unitary group $\mathrm{SU}(n) = \{A \in \mathrm{U}(n) | \det A = 1\}$
- Orthogonal group $\mathrm{O}(n) = \{A \in \mathrm{GL}(n, \mathbb{R}) | AA^T = I\}$
- Special orthogonal group $\mathrm{SO}(n) = \{A \in \mathrm{O}(n) | \det A = 1\}$

In fact, the simple Lie groups are classified by Killing an Cartan, and $\mathrm{SU}(n+1)$, $\mathrm{SO}(2n+1)$, $\mathrm{Sp}(n, \mathbb{R})$ and $\mathrm{SO}(2n)$ are assigned to type A_n, B_n, C_n and D_n, respectively. In addition to these four types, there are five exceptional Lie groups called E_6, E_7, E_8, F_4, and G_2 type. The Dynkin diagrams of type A, D and E are all simply laced, but the other types are not.

Roughly speaking, an element $X \in \mathfrak{g}$ of a Lie algebra $\mathfrak{g}$ describes an element $h \in G$ near the identity element $1 \in G$ via

$$h = 1 + \epsilon X + \mathcal{O}(\epsilon^2) .$$

More precisely, the tangent space $T_1(G)$ of a Lie group G at the identity element 1 naturally admits a Lie bracket

$$[\cdot, \cdot] : T_1 G \times T_1 G \to T_1 G; (X, Y) \mapsto [X, Y] = XY - YX$$

such that the Lie algebra of

$$\mathfrak{g} = (T_1(G), [\cdot, \cdot])$$

is a Lie algebra.

Definition 7.2 (Lie algebra of a Lie group). Let G be a Lie group. The **Lie algebra** of G, written $\mathfrak{g}$, is the tangent space $T_1 G$ under the natural Lie bracket.

Given a vector $X \in \mathfrak{g}$ in the tangent space of the identity element 1, the exponential map defines a map $\exp : \mathfrak{g} \to G$ such that

$$\exp(tX) = \sum_{\ell=0}^{\infty} \frac{1}{\ell!} (tX)^\ell .$$

Without relying on a Lie groups, one can defines a Lie algebra as follows.

Definition 7.3 (Lie algebra). A **Lie algebra** $\mathfrak{g}$ is a vector space (over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$) with a **bracket**

$$[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

satisfying

1. $[\alpha X + \beta Y, Z] = \alpha [X, Z] + \beta [Y, Z]$ for all $X, Y, Z \in \mathfrak{g}$ and $\alpha, \beta \in \mathbb{F}$ (bilinearity)

2. $[X, Y] = -[Y, X]$ for all $X, Y \in \mathfrak{g}$ (antisymmetry)
3. $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ for all $X, Y, Z \in \mathfrak{g}$. (Jacobi identity)

Note that linearity in the second argument follows from linearity in the first argument and antisymmetry.

For instance, we have the corresponding Lie algebras

- $\mathfrak{gl}(n, \mathbb{F}) = \{A \in \mathrm{M}_n(\mathbb{F})\}$
- $\mathfrak{sl}(n, \mathbb{F}) = \{A \in \mathrm{M}_n(\mathbb{F}) | \operatorname{Tr} A = 0\}$
- $\mathfrak{sp}(n, \mathbb{F}) = \{X \in \mathfrak{gl}(2n, \mathbb{F})) | X^T J + JX = 0 \text{ where } J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}\}$
- $\mathfrak{su}(n) = \{A \in \mathrm{M}_n(\mathbb{C}) | \operatorname{Tr} A = 0, A^\dagger = -A\}$
- $\mathfrak{so}(n) = \{A \in \mathrm{M}_n(\mathbb{R}) | A^T = -A\}$

Given a finite-dimensional Lie algebra $\mathfrak{g}$, we can pick a basis for $\mathfrak{g}$:

$$t^a : a = 1, \cdots, \dim \mathfrak{g} \tag{7.9}$$

with

$$\operatorname{Tr}\left(t^a t^b\right) = \frac{1}{2}\delta^{ab} ,$$

which is called the Cartan-Killing form. Then any $X \in \mathfrak{g}$ can be written as

$$X = X^a t^a = i \sum_{a=1}^{n} X^a t^a ,$$

where $X^a \in \mathbb{F}$.

By linearity, the bracket of elements $X, Y \in \mathfrak{g}$ can be computed via

$$[X, Y] = -X^a Y^b [t^a, t^b].$$

In other words, the whole structure of the Lie algebra can be given by the bracket of basis vectors. We know that $[t^a, t^b]$ is again an element of $\mathfrak{g}$. So we can write

$$[t^a, t^b] = i f_{ab}{}^c t^c ,$$

where $f_{ab}{}^c \in \mathbb{F}$ are called the **structure constants**. By the antisymmetry of the bracket, we know

$$f_{ba}{}^c = -f_{ab}{}^c .$$

The Jacobi identity amounts to

$$f_{ab}{}^c f_{cd}{}^e + f_{da}{}^c f_{cb}{}^e + f_{bd}{}^c f_{ca}{}^e = 0.$$

affinization

Hence, one can straightforwardly generalize the current associated to a Lie algebra $\mathfrak{g}$ as follows. First the current taking its value on a Lie algebra $\mathfrak{g}$ can be write

$$J(z) = \sum_a J^a(z) t^a .$$

The OPE of the current takes the form

$$J^a(z)J^b(w) \sim \frac{\frac{k}{2}\delta^{ab}}{(z-w)^2} + \frac{if^{ab}{}_c J^c(w)}{z-w}$$

so that the modes are subject to

$$\left[J^a_n, J^b_m\right] = if^{ab}J^c_{n+m} + \frac{k}{2}\delta^{ab}n\delta_{n+m,0}\,.$$

This is called the G current algebra with level k or the $\widehat{\mathfrak{g}}_k$ affine Lie algebra with level k.

7.4 WZNW models

Nonlinear Sigma Models

In searching for an explicit conformal field theory with the G current algebra, it is natural to first consider the **nonlinear sigma model**

$$\mathcal{S}_0 = \frac{1}{2a^2}\int_\Sigma \mathrm{d}^2x\,\mathrm{Tr}\Big(\partial^\mu g^{-1}\partial_\mu g\Big)\,, \tag{7.10}$$

where a^2 is a positive, dimensionless coupling constant. The bosonic field $g(x)$ takes its value on the Lie group G, namely

$$g : \Sigma \to G$$

where Σ is generally a two-dimensional manifold, called a Riemann surface. Under the variation of the action

$$\delta\mathcal{S}_0 = \frac{1}{a^2}\int_\Sigma \mathrm{d}^2x\,\mathrm{Tr}\Big(g^{-1}\delta g\partial^\mu(g^{-1}\partial_\mu g)\Big)\,, \tag{7.11}$$

which results in the following equation of motion

$$\partial^\mu(g^{-1}\partial_\mu g) = 0\,. \tag{7.12}$$

This implies the conservation of the currents

$$J_\mu = g^{-1}\partial_\mu g\,. \tag{7.13}$$

In the complex coordinate $z = x^0 + ix^1$ and $\bar{z} = x^0 - ix^1$. If we write $J_z = g^{-1}\partial_z g$ and $\bar{J}_{\bar{z}} = g^{-1}\partial_{\bar{z}}g$, we find

$$\partial_z\bar{J}_{\bar{z}} + \partial_{\bar{z}}J_z = 0\,. \tag{7.14}$$

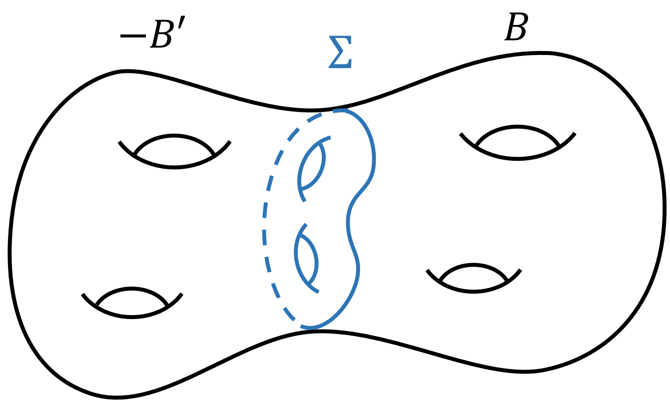
So that the holomorphic and anti-holomorphic currents can not be separately conserved.

WZNW Models

Hence, a more complicated action must be considered to enhance the symmetry. We should add a Wess-Zumino term [WZ71]

$$\Gamma = \frac{-i}{12\pi}\int_B \mathrm{d}^3y\epsilon_{\alpha\beta\gamma}\,\mathrm{Tr}\Big(\widetilde{g}^{-1}\partial^\alpha\widetilde{g}\widetilde{g}^{-1}\partial^\beta\widetilde{g}\widetilde{g}^{-1}\partial^\gamma\widetilde{g}\Big)\,. \tag{7.15}$$

This is defined on a three-dimensional manifold B, whose boundary is the Riemann surface $\partial\Sigma = B$. Here $\widetilde{g} : B \to G$ is an extension of the original field g. Although the extension is not unique, this is actually well-defined. Given another choice $(B', \widetilde{g}')$, we can glue together the three-manifolds B and B' along Σ to get a closed three-manifold $B \cup_\Sigma -B'$. It is easy to check that $\omega \equiv \mathrm{Tr}\big(\widetilde{g}^{-1}\partial^\alpha \widetilde{g}\widetilde{g}^{-1}\partial^\beta \widetilde{g}\widetilde{g}^{-1}\partial^\gamma \widetilde{g}\big)$ is a closed three-form, and moreover it is pull-back of an element $\mathrm{Tr}(\theta \wedge \theta \wedge \theta)$ of the third homology group $H^3(G;\mathbb{Z})$ with integer coefficients [PS86, Prop 4.4.5] via $\widetilde{g} : M \to G$ where θ is the Maurer-Cartan form of G. Hence, the integral of ω over the closed three-manifold $B \cup_\Sigma -B'$ is an integer. Therefore, the Euclidian functional integral, with weight $\exp(-k\Gamma)$ is perfectly well defined if the level k is an integer.



We then consider the action

$$\mathcal{S} = \mathcal{S}_0 + k\Gamma\,, \tag{7.16}$$

where k is an integer. Although the Wess-Zumino term is expressed as a three-dimensional integral, its variation under $g \to g + \delta g$ is a two-dimensional functional, because the variation of its density can be written as a total derivative. The final result is

$$\delta\Gamma = \frac{i}{4\pi}\int_\Sigma \mathrm{d}^2 x \epsilon_{\mu\nu}\,\mathrm{Tr}\Big(g^{-1}\delta g \partial^\mu (g^{-1}\partial^\nu g)\Big)\,. \tag{7.17}$$

The equation of motion of the action (7.16) is then

$$\partial^\mu (g^{-1}\partial_\mu g) + \frac{a^2 ik}{4\pi}\epsilon_{\mu\nu}\partial^\mu (g^{-1}\partial^\nu g) = 0\,. \tag{7.18}$$

In terms of the complex variable z, $\bar{z}$, the equation of motion becomes

$$\left(1 + \frac{a^2 k}{4\pi}\right)\partial_z (g^{-1}\partial_{\bar{z}} g) + \left(1 - \frac{a^2 k}{4\pi}\right)\partial_{\bar{z}} (g^{-1}\partial_z g) = 0\,. \tag{7.19}$$

Thus, for

$$a^2 = 4\pi/k\,, \tag{7.20}$$

we find the desired conservation law

$$\partial_z (g^{-1}\partial_{\bar{z}} g) = 0\,, \qquad \partial_{\bar{z}}((\partial_z g)g^{-1}) = 0\,. \tag{7.21}$$

The action now becomes

$$\mathcal{S} = \frac{k}{8\pi} \int_\Sigma \mathrm{d}^2 x \, \mathrm{Tr}\Big(\partial^\mu g^{-1} \partial_\mu g\Big) + k\Gamma \,, \tag{7.22}$$

which is called the $\widehat{\mathfrak{g}}_k$ **WZNW (WZNW) model**. Since a^2 is positive, k must be a positive integer. The other solution $a^2 = -4\pi/k$, which requires $k < 0$, implies the conservation of the dual currents.

Due to the Noether theorem, the conservation of the currents J_z and $\overline{J}_{\overline{z}}$ (7.21) should implies the invariance of the action. In fact, (7.21) tells us that the field g takes the following factorized form

$$g(z, \overline{z}) = g_L(z) g_R(\overline{z}) \,.$$

We will see, under the transformation

$$g(z, \overline{z}) \to \Omega(z) g(z, \overline{z}) \overline{\Omega}^{-1}(\overline{z}) \,, \tag{7.23}$$

where Ω and $\overline{\Omega}$ are two arbitrary matrices valued in G, the action is invariant. Indeed, under the infinitesimal transformation

$$\Omega(z) = 1 + \omega(z) \,, \qquad \overline{\Omega}(\overline{z}) = 1 + \overline{\omega}(\overline{z}) \,. \tag{7.24}$$

g transforms as follows

$$\delta_\omega g = \omega g \,, \qquad \delta_{\overline{\omega}} g = -g \overline{\omega} \,. \tag{7.25}$$

With $a^2 = 4\pi/k$, the variation of the action for $g \to g + \delta_\omega g + \delta_{\overline{\omega}} g$ is

$$\begin{aligned} \delta \mathcal{S} &= \frac{k}{\pi} \int_\Sigma \mathrm{d}^2 x \, \mathrm{Tr}\Big(g^{-1} \delta g [\partial_z (g^{-1} \partial_{\overline{z}} g)]\Big) \\ &= \frac{k}{\pi} \int_\Sigma \mathrm{d}^2 x \, \mathrm{Tr}\Big[\omega(z) \partial_{\overline{z}} (\partial_z g g^{-1}) - \overline{\omega}(\overline{z}) \partial_z (g^{-1} \partial_{\overline{z}} g)\Big] \,, \end{aligned}$$

which vanishes after an integral by parts. Thus the global $G \times G$ invariance of the sigma model has thus been extended to a local $G(z) \times G(\overline{z})$ invariance.

By rescaling the conserved currents as

$$\begin{aligned} J(z) &\equiv -\frac{k}{2} J_z(z) = -\frac{k}{2} \partial_z g g^{-1} \,, \\ \overline{J}(\overline{z}) &\equiv \frac{k}{2} \overline{J}_{\overline{z}}(\overline{z}) = \frac{k}{2} g^{-1} \partial_{\overline{z}} g \,. \end{aligned} \tag{7.26}$$

We can rewrite (7.26) in the form

$$\delta \mathcal{S} = -\frac{2}{\pi} \int_\Sigma \mathrm{d}^2 x \left\{ \partial_{\overline{z}} (\mathrm{Tr}[\omega(z) J(z)]) + \partial_z (\mathrm{Tr}[\overline{\omega}(\overline{z}) \overline{J}(\overline{z})]) \right\} \,. \tag{7.27}$$

Replace $d^2 x$ by $(-i/2) \mathrm{d}z \mathrm{d}\overline{z}$ and take the holomorphic contour to be counterclockwise and the antiholomorphic contour to be clockwise. This lead to

$$\delta_{\omega, \overline{\omega}} = \frac{i}{\pi} \oint \mathrm{d}z \, \mathrm{Tr}[\omega(z) J(z)] - \frac{i}{\pi} \oint \mathrm{d}\overline{z} \, \mathrm{Tr}[\overline{\omega}(\overline{z}) \overline{J}(\overline{z})] \,. \tag{7.28}$$

Expanding J and ω as well as their dual part in terms of the basis t^α of G

$$J = \sum_a J^a t^a \,, \qquad \omega = \sum_a \omega^a t^a \,, \tag{7.29}$$

The variation of the action now becomes

$$\delta_{\omega,\overline{\omega}} S = -\frac{1}{2\pi i}\oint \mathrm{d}z \sum_a \omega^a J^a + \frac{1}{2\pi i}\oint \mathrm{d}\bar{z} \sum_a \overline{\omega}^a \overline{J}^a \,. \tag{7.30}$$

Let X stand for a list of fields as usual, from (3.12) and (3.32), we have

$$\delta \langle X \rangle = \langle (\delta S) X \rangle \ . \tag{7.31}$$

Hence, we immediately obtain the corresponding Ward-identity

$$\delta_{\omega,\overline{\omega}} \langle X \rangle = -\frac{1}{2\pi i}\oint \mathrm{d}z \sum_a \omega^a \langle J^a X \rangle + \frac{1}{2\pi i}\oint \mathrm{d}\bar{z} \sum_a \overline{\omega}^a \left\langle \overline{J}^a X \right\rangle \ . \tag{7.32}$$

From (7.26) and (7.25), the transformation law for the current is

$$\delta_\omega J = [\omega, J] - \frac{k}{2}\partial_z \omega \,. \tag{7.33}$$

Writing in terms of J^a and ω^a, we have

$$\delta_\omega J^a = \sum_{b,c} i f_{abc} \omega^b J^c - \frac{k}{2}\partial_z \omega^a \,. \tag{7.34}$$

The substitution of this transformation into (7.32), leads to so called current algebra.

$$J^a(z) J^b(w) \sim \frac{\frac{k}{2}\delta_{ab}}{(z-w)^2} + \sum_c i f_{abc} \frac{J^c(w)}{(z-w)} \,. \tag{7.35}$$

Introducing the modes J^a_n from the Laurent expansion

$$J^a(z) = \sum_{n\in\mathbb{Z}} z^{-n-1} J^a_n \,. \tag{7.36}$$

We can easily check (as we have done in the Virasoro case), the commutation relations of J^a_m become

$$\left[J^a_n, J^b_m\right] = \sum_c i f_{abc} J^c_{n+m} + \frac{k}{2} n \delta_{ab} \delta_{n+m,0} \,, \tag{7.37}$$

which is the $\widehat{\mathfrak{g}}$ affine Lie algebra at level k. The dual part $\overline{J}$ is similar.

7.5 Sugawara construction

The energy-momentum tensor of the theory is given by Sugawara construction

$$T(z) = \frac{1}{(k+h^\vee)} \sum_a (J^a J^a)(z) \,. \tag{7.38}$$

The prefactor $\frac{1}{(k+h^\vee)}$ is determined by the usual form of TT OPE, and $h^\vee$ is called the **dual Coxeter number** of $\mathfrak{g}$.

By using the generalized wick theorem which we have encountered in the homework, we can derive the TJ OPE

$$T(z) J^a(w) = \frac{J^a(w)}{(z-w)^2} + \frac{\partial J^a(w)}{(z-w)} \,. \tag{7.39}$$

Therefore, J^a is a Virasoro primary field with conformal dimension 1.

TT OPE can be calculated straightforwardly

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)}, \tag{7.40}$$

with the central charge c

$$c = \frac{k\dim(\mathfrak{g})}{k+h^\vee}. \tag{7.41}$$

The Sugawara energy-momentum tensor can be written in terms of mode expansion the same as before

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2}L_n. \tag{7.42}$$

With the OPEs listed above, we can compute the complete affine Lie and Virasoro algebra

$$\begin{aligned}
[L_n, L_m] &= (n-m)L_{n+m} + \frac{c}{12}(n^3-n)\delta_{n+m,0}, \\
[L_n, J^a_m] &= -mJ^a_{n+m}, \\
\left[J^a_n, J^b_m\right] &= \sum_c if_{abc}J^c_{n+m} + \frac{k}{2}n\delta_{ab}\delta_{n+m,0}.
\end{aligned} \tag{7.43}$$

Finally, for future discussion, we need to write the relation between L_n and J^a_n. From (7.38) and the mode expansion of J^a (7.36), the relation can be obtained:

$$L_n = \frac{1}{(k+h^\vee)}\sum_a\sum_m : J^a_m J^a_{n-m} : . \tag{7.44}$$

7.6 Knizhinik-Zamolodchikov equations

WZNW primary field

A WZNW primary field is defined as a field that transforms covariantly with respect to a $G(z)\times G(\bar{z})$ transformation. From (7.25) and (7.32), we reformulate Jg OPE as following

$$\begin{aligned}
J^a(z)g(w,\overline{w}) &\sim \frac{-t^a g(w,\overline{w})}{z-w}, \\
\overline{J}^a(z)g(w,\overline{w}) &\sim \frac{g(w,\overline{w})t^a}{z-w}.
\end{aligned} \tag{7.45}$$

This can be generalized to define a WZNW primary field. Any WZNW primary field $\phi_{\lambda,\mu}$ with λ and μ specified the representation in the holomorphic and antiholomorphic sector respectively should satisfy the following OPE

$$J^a(z)\phi_{\lambda,\mu}(w,\overline{w}) \sim \frac{-t^a_\lambda \phi_{\lambda,\mu}(w,\overline{w})}{z-w}, \tag{7.46}$$

$$\overline{J}^a(\bar{z})\phi_{\lambda,\mu}(w,\overline{w}) \sim \frac{\phi_{\lambda,\mu}(w,\overline{w})t^a_\mu}{\bar{z}-\overline{w}}, \tag{7.47}$$

where t^a_λ is the matrix t^a in the λ representation. By analogy with (3.51), the above OPE can be extended

$$J^a(z)\phi_1(z_1)\phi_2(z_2)\cdots\phi_n(z_n) \sim -\sum_i \frac{t^a_i}{z-z_i}\phi_1(z_1)\phi_2(z_2)\cdots\phi_n(z_n), \tag{7.48}$$

where we have considered only holomorphic sector, with t_i^a a matrix acting on ϕ_i.

By expanding the currents in terms of the modes evaluated at w,

$$J^a(z) = \sum_n (z-w)^{-n-1} J_n^a(w)\,. \tag{7.49}$$

We can write their OPE with an arbitrary field A as we have done in the Virasoro case

$$J^a(z)A(w) = \sum_n (z-w)^{-n-1} (J_n^a A)(w)\,. \tag{7.50}$$

Thus, from (7.46), for the WZNW holomorphic primary field ϕ_λ, we have

$$\begin{aligned} J_0^a \phi_\lambda &= -t_\lambda^a \phi_\lambda\,, \\ J_n^a \phi_\lambda &= 0 \qquad \text{for} \qquad n>0\,. \end{aligned} \tag{7.51}$$

Associating the state $|\phi_\lambda\rangle$ to the field ϕ_λ as usual

$$\phi_\lambda(0)\,|0\rangle = |\phi_\lambda\rangle\ . \tag{7.52}$$

Then, the condition for a WZNW primary field translate into

$$\begin{aligned} J_0^a\,|\phi_\lambda\rangle &= -t_\lambda^a\,|\phi_\lambda\rangle\ , \\ J_n^a\,|\phi_\lambda\rangle &= 0 \qquad \text{for} \qquad n>0\,. \end{aligned} \tag{7.53}$$

From (7.44), we see that in the expression for L_n, with $n>0$, the rightmost factor J_m^a has $m>0$ which implies that

$$L_n\,|\phi_\lambda\rangle = 0 \qquad \text{for} \qquad n>0\,. \tag{7.54}$$

Therefore, the WZNW primary fields are also Virasora primary fields with the conformal dimension given by

$$L_0\,|\phi_\lambda\rangle = \frac{1}{(k+h^\vee)}\sum_a J_0^a J_0^a\,|\phi_\lambda\rangle = \frac{\sum_a t_\lambda^a t_\lambda^a}{(k+h^\vee)}\,|\phi_\lambda\rangle\ . \tag{7.55}$$

The inversed statement is not true, for example, $J^a(z)$ is a Virasoro primary field but not WZNW primary field which can be seen directly from TJ OPE and JJ OPE we have shown before.

Finally, a WZNW descendant state is defined of the form

$$J_{-n_a}^a J_{-n_2}^b \cdots |\phi_\lambda\rangle\ , \tag{7.56}$$

with $n_1, n_2 \cdots$ all positive integers.

Knizhnik-Zamolodchikov Equations

From the previous learning, we know that there are many constraints to the correlation function, such as the null fields. In this section, we briefly discuss the affine singular vector in level one and the constraint follows form it.

For $n=-1$, from (7.44), we have

$$L_{-1}\,|\phi_i\rangle = \frac{2}{k+h^\vee}\sum_a (J_{-1}^a J_0^a)\,|\phi_i\rangle = \frac{-2}{k+h^\vee}\sum_a (J_{-1}^a t_i^a)\,|\phi_i\rangle\ . \tag{7.57}$$

We consider the insertion of the zero vector

$$|\chi\rangle = \left[L_{-1} + \frac{2}{k+h^\vee}\sum_a (J^a_{-1}t^a_i)\right]|\phi_i\rangle = 0\,. \tag{7.58}$$

First, we need to calculate the contribution of the second term. By using (7.36) , the insertion of the operator J^a_{-1} in the correlator can be expressed as

$$\big\langle\phi_1(z_1)\cdots(J^a_{-1}\phi_i)(z_i)\cdots\phi_n(z_n)\big\rangle = \frac{1}{2\pi i}\oint_{z_i}\frac{\mathrm{d}z}{z-z_i}\left\langle J^a(z)\phi_1(z_1)\cdots\phi_n(z_n)\right\rangle\,. \tag{7.59}$$

Now using (7.48) and perform the residue calculation, we get

$$\big\langle\phi_1(z_1)\cdots(J^a_{-1}\phi_i)(z_i)\cdots\phi_n(z_n)\big\rangle = \sum_{j\neq i}\frac{t^a_j}{z_i-z_j}\left\langle\phi_1(z_1)\cdots\phi_n(z_n)\right\rangle\,. \tag{7.60}$$

Thus

$$\left\langle\phi_1(z_1)\cdots\chi(z_i)\cdots\phi_n(z_n)\right\rangle = \left[\partial_{z_i} + \frac{2}{k+h^\vee}\sum_{j\neq i}\frac{\sum_a t^a_i\otimes t^a_j}{z_i-z_j}\right]\left\langle\phi_1(z_1)\cdots\phi_n(z_n)\right\rangle\,, \tag{7.61}$$

where t^a_j and t^a_i act on ϕ_i and ϕ_j respectively. By construction, it must vanish so that we have

$$\left[\partial_{z_i} + \frac{2}{k+h^\vee}\sum_{j\neq i}\frac{\sum_a t^a_i\otimes t^a_j}{z_i-z_j}\right]\left\langle\phi_1(z_1)\cdots\phi_n(z_n)\right\rangle = 0\,. \tag{7.62}$$

This is called the **Knizhnik-Zamolodchikov equation** [KZ84, ISZ88-No.4]. The solutions to this equation are the correlation functions of WZNW primary fields.

7.7 Coset models

Using current algebras, one can construct an important class of 2d CFTs called **coset construction** [GKO86, ISZ88-No.11]. This construction has many applications in physics like (supersymmetric) unitary minimal models (see the end of §8.3) and Kazama-Suzuki models [KS89].

We start with the current algebra associated to a Lie algebra $\mathfrak{g}$ which contains $\mathfrak{h}$ as subgroups. The Sugawara energy-momentum tensors of affine Lie algebra $\widehat{\mathfrak{g}}_{k_\mathfrak{g}}$ and $\widehat{\mathfrak{h}}_{k_\mathfrak{h}}$ are

$$\begin{aligned}T_\mathfrak{g}(z) &= \frac{1}{k_\mathfrak{g}+h^\vee_\mathfrak{g}}\sum_{a=1}^{\dim\mathfrak{g}} :J^a_\mathfrak{g}J^a_\mathfrak{g}:(z)\\ T_\mathfrak{h}(z) &= \frac{1}{k_\mathfrak{h}+h^\vee_\mathfrak{h}}\sum_{b=1}^{\dim\mathfrak{h}} :J^b_\mathfrak{h}J^b_\mathfrak{h}:(z)\end{aligned} \tag{7.63}$$

We define the energy-momentum tensor of the coset model $\widehat{\mathfrak{g}}_{k_\mathfrak{g}}/\widehat{\mathfrak{h}}_{k_\mathfrak{h}}$ as

$$T_{\mathfrak{g}/\mathfrak{h}} := T_\mathfrak{g} - T_\mathfrak{h}\,.$$

Then, we have

$$\begin{aligned}T_\mathfrak{g}(z)J^b_\mathfrak{h}(w) &= \frac{J^b_\mathfrak{h}(w)}{(z-w)^2} + \frac{\partial_w J^b_\mathfrak{h}(w)}{z-w} + \cdots\\ T_\mathfrak{h}(z)J^b_\mathfrak{h}(w) &= \frac{J^b_\mathfrak{h}(w)}{(z-w)^2} + \frac{\partial_w J^b_\mathfrak{h}(w)}{z-w} + \cdots\end{aligned} \tag{7.64}$$

which yield

$$
\begin{aligned}
(T_{\mathfrak{g}} - T_{\mathfrak{h}})(z) J^b_{\mathfrak{h}}(w) &\sim 0 \\
(T_{\mathfrak{g}} - T_{\mathfrak{h}})(z) T_{\mathfrak{h}}(w) &\sim 0 \,.
\end{aligned} \tag{7.65}
$$

Therefore, one can compute the $T_{\mathfrak{g}/\mathfrak{h}} T_{\mathfrak{g}/\mathfrak{h}}$ OPE so that the central charge of the coset model is

$$
c_{\mathrm{g/h}} = c_{\mathfrak{g}} - c_{\mathrm{h}} = \frac{k_{\mathfrak{g}} \dim \mathfrak{g}}{k_{\mathfrak{g}} + h^\vee_{\mathfrak{g}}} - \frac{k_{\mathfrak{h}} \dim \mathfrak{h}}{k_{\mathfrak{h}} + h^\vee_{\mathfrak{h}}} \,. \tag{7.66}
$$

Parafermion

As the simplest model of the coset model, we consider $\widehat{\mathfrak{su}(2)}_k / \widehat{\mathfrak{u}(1)}_k$. The energy-momentum tensor $T_{\mathfrak{g}/\mathfrak{h}} = T_{\mathfrak{g}} - T_{\mathfrak{h}}$ where

$$
\begin{aligned}
T_{\mathfrak{g}}(z) &= \frac{1}{k+2} \sum_{a=1}^{3} : J^a(z) J^a(z) : \\
T_{\mathfrak{h}}(z) &= \frac{1}{k} : J^3(z) J^3(z) :
\end{aligned} \tag{7.67}
$$

If we define J^3 as

$$
J^3(z) = i\sqrt{\frac{k}{2}} \partial \varphi(z)
$$

then T_H is the energy-momentum tensor of the free boson. It is easy to read off central charge

$$
c_{\mathrm{g/h}} = \frac{3k}{k+2} - 1 = \frac{2(k-1)}{k+2} \,,
$$

which is equal to (5.54). Indeed, this coset model describes the 2d CFT with $\mathbb{Z}_k$ symmetry [FZ85, ISZ88-No.14]. This model contains a current with fractional spin (conformal dimension), which does obey neither bose nor fermi statistics, and is rather subject to para-statistics. Therefore, it is called $\mathbb{Z}_k$ **parafermion**. In terms of the free boson φ and the SU(2) current $J^{\pm}$, the parafermions are defined by

$$
\left(J^+\right)^\ell = \psi_\ell e^{i\ell\sqrt{\frac{2}{k}}X}, \quad \left(J^-\right)^\ell = \psi^\dagger_\ell e^{-i\ell\sqrt{\frac{2}{k}}X}, \quad (\ell = 0, \cdots, k)
$$

where ψ_ℓ and $\psi^\dagger_\ell$ have conformal dimension $h = \ell(k-\ell)/k$.

When $k = 1$, the central charge is $c = 0$ so that the theory is trivial. When $k = 2$, the central charge is $c = \frac{1}{2}$ so that the theory is the free fermion. When $k = 3$, the central charge is $c = \frac{4}{3}$ so that the theory is the 3-state Potts model, which corresponds to the unitary minimal model $\mathcal{M}_5$. In this case, the primary field with conformal dimension $h = 2/3$ corresponds to the parafermion current ψ_1. The partition function of the parafermion is evaluated in [GQ87, ISZ88-No.28].

In fact, the supersymmetric version of the parafermion describes $\mathcal{N} = 2$ unitary minimal models, which plays a crucial role to describe Calabi-Yau sigma models [Gep87, Gep88].

Rational conformal field theories

The minimal models, the WZNW models and the coset models belong to a class of rational conformal field theories (RCFTs). A theory is endowed with a chiral algebra $\mathcal{A} \otimes \overline{\mathcal{A}}$ that contains the Virasoro algebra as a subalgebra. The definition of a RCFT is (roughly) given as follows.

- It is unitary with a unique vacuum.
- There are finitely many primary fields ϕ_i of the chiral algebra $\mathcal{A}$. The same statement holds for anti-holomorphic part.
- The Hilbert space is spanned by finitely many irreducible representations of $\mathcal{A} \otimes \overline{\mathcal{A}}$

$$\mathcal{H} = \bigoplus_{i,j} \mathcal{N}_{i,j} \mathcal{V}_i \otimes \overline{\mathcal{V}}_j$$

  where the irreducible representation $\mathcal{V}_i$ arises from primary fields ϕ_i.
- Their OPEs of primary fields are closed among themselves

$$\phi_i(z)\phi_j(w) \sim \sum_k \frac{C_{ijk}}{(z-w)^{h_i+h_j-h_k}} \phi_j(w) + \cdots$$

It is called a RCFT because conformal dimensions of primary fields of the chiral algebra $\mathcal{A}$ as well as the central charge are rational numbers $\in \mathbb{Q}$ [AM88, Vaf88]. In a RCFT, conformal blocks can be, in principle, determined from structure constants C_{ijk} by bootstrap equations (5.66), and one can evaluate a correlation function on an arbitrary Riemann surface. For more details, we refer to [MS89a, MS89b].

8 Modular Invariant Partition Functions

We have already seen some examples of modular invariant partition functions, such as torus partition functions of the free boson/fermion §4.3, and minimal models §5.3. We first review a free boson on a circle and look into its modular invariance in special cases. Then, we make further investigations on other possibilities.

When we consider torus partition function it should always be modular invariant because it is a consequence of the geometry, torus. Nonetheless, it is non-trivial to see in some cases. Instead, by requiring the modular invariance gives us some useful information, which can be seen in §8.2.

8.1 Free Boson on a Circle

Let us first focus on a single free boson on a circle. The partition function on a circle of radius R is, as we have seen in (4.81):

$$\mathcal{Z}_R(\tau,\overline{\tau}) = \mathrm{Tr}\left(q^{L_0-\frac{c}{24}}\overline{q}^{\overline{L}_0-\frac{c}{24}}\right) = \frac{1}{|\eta(\tau)|^2}\sum_{n,w} q^{\frac{1}{2}(\frac{n}{R}+\frac{Rw}{2})^2}\overline{q}^{\frac{1}{2}(\frac{n}{R}-\frac{Rw}{2})^2}, \tag{8.1}$$

where we changed conventions, and here, n is for KK-momentum and w for winding number. Note that R dependence of the partition function means that the theory depends on the radius R. We will see peculiar features of theories (partition functions) at specific values of the radius. Now, let us study the partition function on a special radius $R = \sqrt{2k}$ with $k \in \mathbb{Z}^+$. Rearranging the summation, the partition function can be written as

$$\begin{aligned}\mathcal{Z}_{\sqrt{2k}}(\tau,\overline{\tau}) &= \frac{1}{|\eta(\tau)|^2}\sum_{n,w\in\mathbb{Z}} q^{\frac{1}{4k}(n+kw)^2}\overline{q}^{\frac{1}{4k}(n-kw)^2} = \frac{1}{|\eta(\tau)|^2}\sum_{\substack{m\in\mathbb{Z}_{2k}\\ n_L,n_R\in\mathbb{Z}}} q^{\frac{1}{4k}(m+2kn_L)^2}\overline{q}^{\frac{1}{4k}(m-2kn_R)^2} \\ &= \frac{1}{|\eta(\tau)|^2}\sum_{m\in\mathbb{Z}_{2k}} \left|\Theta_{m,k}(\tau)\right|^2 \ ,\end{aligned} \tag{8.2}$$

where the range of k is $-k+1 \leq m \leq k$, Θ-function was defined in (5.44). Conformal field theories corresponding to these partition functions are commonly denoted as $\widehat{\mathfrak{u}(1)}_k$, and we use this notation for the subscript from now on. Indeed, for $k=1$, we find

$$\mathcal{Z}_{\widehat{\mathfrak{u}(1)}_1}(\tau,\overline{\tau}) = \frac{1}{|\eta(\tau)|^2}\left(|\Theta_{0,1}|^2 + |\Theta_{1,1}|^2\right) = \chi_0^{(1)}\overline{\chi}_0^{(1)} + \chi_1^{(1)}\overline{\chi}_1^{(1)}\,, \tag{8.3}$$

where $\chi_j^{(1)}$ is the character of $\widehat{\mathfrak{su}(2)}_1$ as we will see below. As we have seen in §7.2, $k=1$ is actually the self-dual point, and the symmetry $\widehat{\mathfrak{u}(1)}_1$ is enhanced to $\widehat{\mathfrak{su}(2)}_1$,

From the definition of Θ-function, it is easy to see the T-transformations for Θ- function, we have

$$\Theta_{m,k}(\tau+1) = e^{\pi i m^2/2k}\Theta_{m,k}(\tau)\,. \tag{8.4}$$

For the modular S-transformation, we have

$$\Theta_{m,k}\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\sum_{m'=-k+1}^{k} S_{m,m'}\Theta_{m',k}(\tau)\,, \tag{8.5}$$

with the modular S-matrix

$$S_{m,m'} = \frac{1}{\sqrt{2k}}\exp\left(-\pi i\frac{mm'}{k}\right). \tag{8.6}$$

For $k=1$ case, the S-transformation for the characters in (8.3) can be inferred from the transformation properties of the Θ- and η-functions

$$\chi_m^{(1)}\left(-\frac{1}{\tau}\right) = \sum_{m'=0}^{1} S_{m,m'}\chi_{m'}^{(1)}(\tau) \tag{8.7}$$

Following what we have done when constructing modular invariant partition function in minimal model, we can write the partition function (8.3) as

$$\mathcal{Z}_{\widehat{\mathfrak{u}(1)}_1}(\tau,\overline{\tau}) = (\chi_0^{(1)},\chi_1^{(1)})\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\overline{\chi}_0^{(1)}\\ \overline{\chi_1}^{(1)}\end{pmatrix} = \vec{\chi}^T M\vec{\overline{\chi}}\,, \tag{8.8}$$

where M is the identity matrix in this case. The S-transformation of partition function now reads

$$\mathcal{Z}_{\widehat{\mathfrak{u}(1)}_1}\left(-\frac{1}{\tau},-\frac{1}{\overline{\tau}}\right) = \vec{\chi}^T S^T M S^*\vec{\overline{\chi}}\,. \tag{8.9}$$

Since $S^T = S$ from the definition of S matrix, the condition for invariance under modular S-transformation for the partition function now reads

$$SMS^\dagger = M\,. \tag{8.10}$$

This is obviously true in this simple case. Because under T-transformations the characters $\chi_m^{(1)}$ only acquire a phase, we have shown that $\mathcal{Z}_{\widehat{\mathfrak{u}(1)}_1}(\tau,\overline{\tau})$ is modular invariant. This process can be generalized to construct more complicated modular invariant partition function.

Let us look into $k=2$ case as well. The modular invariant partition function is given as follows.

$$\mathcal{Z}_{\widehat{u(1)}_2} = \sum_{-1\le m\le 2} \left|\frac{\Theta_{m,2}(\tau)}{\eta(\tau)}\right|^2 = \frac{1}{2}\left\{\left|\frac{\vartheta_2(\tau)}{\eta(\tau)}\right|^2 + \left|\frac{\vartheta_3(\tau)}{\eta(\tau)}\right|^2 + \left|\frac{\vartheta_4(\tau)}{\eta(\tau)}\right|^2\right\} \tag{8.11}$$

Notice that the expression above is exactly the same as the torus partition function of a free complex fermion on a circle (5.52), though the second equality above is non-trivial. This means that a free boson on a circle at $R=2$ is equivalent to a free complex fermion, and it is called "Boson-Fermion correspondence" (or bosonization, fermionization). We have already seen the equivalence in terms of operator correspondence in the homework 4. This is a simple example of "duality", which provides us a pair of theories that describe the same physics (hence their physical quantities like multi-point function and partition function etc. agree each other). It provides not only different physical pictures but also important mathematical formulae, which we will see in homework of this week.

8.2 Orbifold partition function

As we saw in a free boson on a circle case, the theory depends on a "target space"[4] on which the theory lives. Especially, the boundary condition plays an important role. For the circle case it is

$$\varphi(ze^{2\pi i}, \bar{z}e^{-2\pi i}) = \varphi(z,\bar{z}) + 2\pi w R, \qquad w\in\mathbb{Z} \quad : \text{winding number}. \tag{8.12}$$

Now, we want to extend our knowledge to other kinds of CFT, therefore, we consider different target spaces. What we consider here is so called orbifold. For a manifold M and a discrete group Γ, a space defined by a quotient M/Γ with an equivalence relation

$$x\sim gx \qquad g\in\Gamma, \quad x\in M, \tag{8.13}$$

could be singular, and it is, in general, called orbifold.

We consider, as the simplest example, the $\mathbb{Z}_2$-orbifold of a circle with radius R [Yan87, ISZ88-No.40]. In this case $M=S^1$ and $\Gamma=\mathbb{Z}_2$ whose elements are $g=\{1,-1\}$. The non-trivial equivalence relation is, therefore, $x\sim -x$. Resultant orbifold is denoted by $S^1/\mathbb{Z}_2$ (see Fig. 19). Our goal is to calculate a partition function of a free boson on the orbifold. In order to achieve this we consider a free boson on a circle $\varphi\sim\varphi+2\pi R$, and define a $\mathbb{Z}_2$ symmetry (orbifold) operator $\mathcal{R}$:

$$\mathcal{R}: \varphi(z,\bar{z}) \mapsto -\varphi(z,\bar{z}), \qquad \mathcal{R}^2 = \mathrm{Id}. \tag{8.14}$$

We calculate a partition function with states invariant under the orbifold action. Namely, we insert an orbifold projection operator $\frac{1}{2}(1+\mathcal{R})$ into the trace in (8.1), and the partition function reads

$$\begin{aligned}\mathcal{Z}_{\rm orb}(\tau,\bar\tau) &= \mathrm{Tr}\left(\frac{1+\mathcal{R}}{2} q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{c}{24}}\right)\\ &= \frac{1}{2}\mathcal{Z}_R(\tau,\bar\tau) + \frac{1}{2}\mathrm{Tr}\left(\mathcal{R} q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{c}{24}}\right).\end{aligned} \tag{8.15}$$

[4]Target space is a space that fields take their values. In the case of a free boson on a circle, φ takes a value on the circle $x\sim x+2\pi R$. Hence, the target space is the circle.

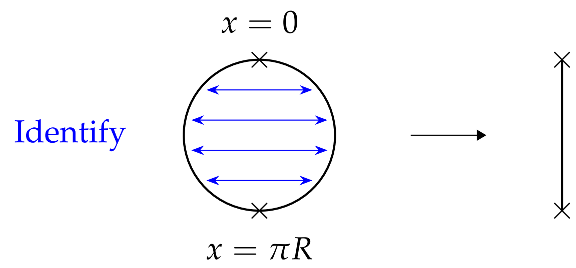


Figure 19: From a circle (S^1) with radius $2\pi R$ to a $\mathbb{Z}_2$-orbifold ($S^1/\mathbb{Z}_2$) of the circle. We identify points x and $-x$ in the circle, which leads to a finite line. There are two singular points, which are illustrated by crosses in the figure.

The first term contains the partition function of a free boson on a circle which we already computed in (8.1). Let us focus on the second term.

The action of $\mathcal{R}$ on the Laurent modes a_n of $i\partial\varphi(z)$ can be found

$$\mathcal{R}a_n\mathcal{R} = -a_n\,. \tag{8.16}$$

Hence

$$\begin{aligned}\mathcal{R}\left|n_1, n_2, n_3, \ldots\right\rangle &= (\mathcal{R}a_{-1}\mathcal{R})^{n_1}(\mathcal{R}a_{-2}\mathcal{R})^{n_2}\ldots\mathcal{R}\left|\Omega\right\rangle \\ &= (-1)^{n_1+n_2+n_3+\cdots}\left|n_1, n_2, n_3, \ldots\right\rangle\,.\end{aligned} \tag{8.17}$$

where $|\Omega\rangle$ is a ground state, and we assumed that it is invariant under $\mathcal{R}$, i.e $\mathcal{R}\,|\Omega\rangle = |\Omega\rangle$. Now, we consider the action $\mathcal{R}$ on the ground states in details. Remember that the ground state is characterized by KK-momentum and winding number, $|\Omega\rangle = |n, w\rangle$. Since the zero mode operator a_0 gives eigenvalues of the states

$$a_0\,|n, w\rangle = \left(\frac{n}{R} + \frac{Rw}{2}\right)|n, w\rangle\,, \tag{8.18}$$

we see that

$$\begin{aligned}&a_0\mathcal{R}\,|n, w\rangle = \mathcal{R}(\mathcal{R}a_0\mathcal{R})\,|n, w\rangle = \mathcal{R}(-a_0)\,|n, w\rangle = -\left(\frac{n}{R} + \frac{Rw}{2}\right)\mathcal{R}\,|n, w\rangle\,, \\ &\therefore\quad \mathcal{R}\,|m, n\rangle \propto |-m, -n\rangle\,.\end{aligned} \tag{8.19}$$

Therefore, only states with $|m = 0, n = 0\rangle$ will contribute in the calculation of the partition function. Then following the same steps in (4.74), we replace the result of the free boson as

$$q^{-\frac{1}{24}}\prod_{n=1}^{\infty}\frac{1}{1-q^n} \quad\to\quad q^{-\frac{1}{24}}\prod_{n=1}^{\infty}\frac{1}{1-(-q^n)} = \sqrt{2}\sqrt{\frac{\eta(\tau)}{\vartheta_2(\tau)}}\,. \tag{8.20}$$

We thus write the partition function as

$$\mathcal{Z}(\tau,\overline{\tau}) = \frac{1}{2}\mathcal{Z}_R(\tau,\overline{\tau}) + \left|\frac{\eta(\tau)}{\vartheta_2(\tau)}\right|. \tag{8.21}$$

Notice that the result (8.21) can not be the correct partition function because the second term is not invariant under modular transformation. Recalling the free fermions partition function result, ϑ-functions enjoy the modular transformations illustrated in Fig.11. In order for a free boson on the orbifold to have modular invariance we are required to add so-called twisted sector, whose contributions should be

$$\mathcal{Z}_{\rm tw}(\tau,\overline{\tau}) = \left|\frac{\eta(\tau)}{\vartheta_4(\tau)}\right| + \left|\frac{\eta(\tau)}{\vartheta_3(\tau)}\right|. \tag{8.22}$$

Then, the modular invariant partition function of a free boson on the Z_2-orbifold reads

$$\mathcal{Z}_{\rm orb}(\tau,\overline{\tau}) = \frac{1}{2}\mathcal{Z}_R(\tau,\overline{\tau}) + \left|\frac{\eta(\tau)}{\vartheta_2(\tau)}\right| + \left|\frac{\eta(\tau)}{\vartheta_4(\tau)}\right| + \left|\frac{\eta(\tau)}{\vartheta_3(\tau)}\right|. \tag{8.23}$$

Although we derived the correct result by requiring modular invariance here, it is instructive to see what is the physical realization of the twisted sector.

Twisted sector

Recall that when we calculate torus partition function of a free fermion on a circle, we considered Neveu-Schwarz (NS) sector as well as Ramond (R) sector, which are realized as follows.

$$\psi\left(e^{2\pi i}z\right) = \psi(z) \qquad \text{NS}, \tag{8.24}$$

$$\psi\left(e^{2\pi i}z\right) = -\psi(z) \qquad \text{R}, \tag{8.25}$$

where only the holomorphic parts are shown. We realize that the NS sector corresponds to the boundary condition of a free boson (8.12), and wonder if there is R sector like boundary condition. There, indeed, exists due to the operation $\mathcal{R}$:

$$\varphi(ze^{2\pi i},\overline{z}e^{-2\pi i}) = \mathcal{R}\varphi(z,\overline{z}) + 2\pi wR = -\varphi(z,\overline{z}) + 2\pi wR. \tag{8.26}$$

In order to realize this boundary condition the mode have to become "half-integers" modes:

$$\begin{aligned} \varphi(z,\overline{z}) &= \varphi(z) + \overline{\varphi}(\overline{z}), \\ \varphi(z) &= \frac{x}{2} + i\sum_{n\in\mathbb{Z}+\frac{1}{2}} \frac{1}{n}\frac{a_n}{z^n}. \end{aligned} \tag{8.27}$$

There are two possible value for the zero mode x; one is $x = 0$ ($w = 0$) and the other is $x = \pi R$ ($w = 1$). Due to the shift of the modes the zero point energy is also modified as follows (recall homework 5: ζ-function regularization).

$$\frac{1}{2}\sum_{n=0}^{\infty} n = -\frac{1}{24} \quad\rightarrow\quad \frac{1}{2}\sum_{n=0}^{\infty}(n+\frac{1}{2}) = -\frac{1}{24} + \frac{1}{16} = \frac{1}{48}. \tag{8.28}$$

Therefore, we have following contribution:

$$\begin{aligned} &\frac{1}{2}\,\mathrm{Tr}_{\rm tw}\left[q^{L_0-\frac{1}{24}}\overline{q}^{\overline{L_0}-\frac{1}{24}}\right] \quad \text{with} \quad L_0 = \sum_{n=1}^{\infty} a_{-n+\frac{1}{2}}a_{n-\frac{1}{2}} + \frac{1}{16} \\ &= \frac{1}{2}\cdot 2\cdot |q|^{2\cdot\frac{1}{48}}\prod_{n=1}^{\infty}\left|\frac{1}{1-q^{n-\frac{1}{2}}}\right|^2 = \left|\frac{\eta(\tau)}{\vartheta_4(\tau)}\right|, \end{aligned} \tag{8.29}$$

where the factor of two in the coefficient is coming from the two contributions of zero modes $x = 0$ and $x = \pi R$. Similarly,

$$\begin{aligned}
&\frac{1}{2}\,\mathrm{Tr}_{\mathrm{tw}}\left[\mathcal{R} q^{L_0-\frac{1}{24}}\bar{q}^{\overline{L_0}-\frac{1}{24}}\right] \quad \text{with} \quad L_0 = \sum_{n=1}^{\infty} a_{-n+\frac{1}{2}} a_{n-\frac{1}{2}} + \frac{1}{16} \\
&= |q|^{\frac{1}{24}} \prod_{n=1}^{\infty} \left|\frac{1}{1+q^{n-\frac{1}{2}}}\right|^2 = \left|\frac{\eta(\tau)}{\vartheta_3(\tau)}\right|. \qquad (8.30)
\end{aligned}$$

As we expected we have correct contributions from the twisted sector. There are several comments:

- $\mathcal{Z}_{\mathrm{orb}}$ depends on radius R only through $\mathcal{Z}_R$, and hence, it enjoys T-duality: $\mathcal{Z}_{\mathrm{orb}}(R) = Z_{\mathrm{orb}}(2/R)$.
- One can see that $\mathcal{Z}_{\mathrm{orb}}(R = \sqrt{2}) = \mathcal{Z}_R(R = 2\sqrt{2})$. Namely, a free boson on a circle with $R = \sqrt{2}$ is equivalent to that on an $S^1/\mathbb{Z}_2$ with $R = 2\sqrt{2}$. As we stressed before, theories depend on their radii, and two branches of a free boson theory intersect at the point of moduli space of the theory (see Fig. 20).
- The $\mathcal{N} = 2$ superconformal symmetry arises at $R_{\mathrm{circ}} = \sqrt{3}$, and $\mathcal{N} = 1$ superconformal symmetry arises at $R_{\mathrm{orb}} = \sqrt{3}$ [FS89, ISZ88-No.39] [YZ87].
- In fact, a CFT with $c = 1$ can be constructed by taking a quotient of the $\mathrm{SU}(2)_1$ current algebra by its finite subgroup. There is one-to-one correspondence between the finite subgroups of $\mathrm{SU}(2)$ and the ADE Dynkin diagrams, which is called the **McKay correspondence**. The CFTs corresponding to T (E_6), O (E_7), and I (E_8) are isolated (does not allow deformation) [Pas87, ISZ88-No.33] [Gin88b, ISZ88-No.41].

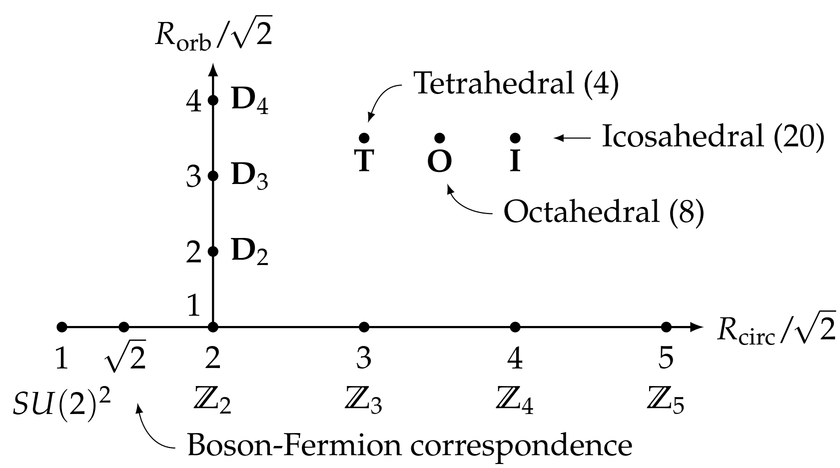


Figure 20: Moduli space of 2d CFTs with $c = 1$. The horizontal allow expresses so called toroidal branch, and the vertical allow does orbifold branch. R_{circ} is radius of a circle and R_{orb} is that of the orbifold. $SU(2)^2$ is the symmetry that the theory has at the self-dual point. $\mathbb{Z}_k$, $\mathbf{D}_k$, T, O, and I are the subgroup of the $SU(2)^2$ at corresponding points illustrated above.. For more details see [Gin88b, ISZ88-No.41].

So far, only a specific orbifold example are illustrated. For general orbifold Γ we can do parallel computations as follows. Let us consider a field in a cylinder coordinates $\Phi(\sigma, \tau)$, and for $g, h \in \Gamma$, we have

$$\begin{aligned}
\Phi(\sigma, \tau + 2\pi) &= g \cdot \Phi(\sigma, \tau), \\
\Phi(\sigma + 2\pi, \tau) &= h \cdot \Phi(\sigma, \tau).
\end{aligned} \tag{8.31}$$

Note that σ and τ are commutative on torus, hence, g and h should also be commutative, i.e $gh = hg$. As we have seen before, time-like orbifold condition (the first line above) is realized by operator insertions, and the space-like orbifold condition (the second line above) is realized by twisted sectors (trace over twisted Hilbert space):

$$\mathcal{Z}_{(g,h)}(\tau, \overline{\tau}) = \mathrm{Tr}_{\mathcal{H}_h} \left[g q^{L_0 - \frac{c}{24}} \overline{q}^{\overline{L_0} - \frac{c}{24}} \right], \tag{8.32}$$

and the full partition function is given by sum over all possible values of g and h:

$$\mathcal{Z}_{M/\Gamma}(\tau, \overline{\tau}) = \frac{1}{|\Gamma|} \sum_{\substack{g,h \in \Gamma \\ gh = hg}} \mathcal{Z}_{(g,h)}(\tau, \overline{\tau}), \tag{8.33}$$

where $|\Gamma|$ is the degree of a discrete group Γ. One should check that this formula reproduces our result of $S^1/\mathbb{Z}_2$.

8.3 $\widehat{\mathfrak{su}(2)}_k$ characters

It has been briefly mentioned in §8.1 that $\widehat{\mathfrak{u}(1)}_1$ theory has actually $\widehat{\mathfrak{su}(2)}_1$ symmetry, therefore, its partition function can be expressed by so called $\widehat{\mathfrak{su}(2)}_1$ character. Here, we derive the $\widehat{\mathfrak{su}(2)}_1$ character by considering a free boson on a circle with $R = \sqrt{2}$.

Recall that the theory can be characterized by three operators (for more details see §7.2):

$$J(z) = \frac{i\partial\varphi(z)}{\sqrt{2}}, \qquad J^{\pm}(z) =: e^{\pm i\sqrt{2}\varphi(z)} : . \tag{8.34}$$

OPE calculation leads to following commutation relations

$$\begin{aligned}
&[J^3_n, J^{\pm}_m] = \pm J^{\pm}_{n+m}, \\
&[J^+_n, J^-_m] = 2J^3_{n+m} + n\delta_{n+m,0}, \\
&[L_0, J^a_n] = -nJ^a_n \qquad (a = 3, \pm).
\end{aligned} \tag{8.35}$$

Let us consider the highest weight representation of the theory. Notice that L_0 and J^3_0 commutes, hence, the highest weight vector(states) should be characterized by their eigenvalues h and j, and the vector is defined as follows.

$$\begin{aligned}
&J^{\pm}_n |h, j\rangle = J^3_n |h, j\rangle = 0 \qquad (n \geq 1), \\
&J^+_0 |h, j\rangle = \left(J^-_0\right)^{2j+1} |h, j\rangle = 0, \\
&J^3_0 |h, j\rangle = j |h, j\rangle .
\end{aligned} \tag{8.36}$$

Note that from the argument of §7.2 j can only be $j = 0, \frac{1}{2}$ for $k = 1$. For $j = 0$, the corresponding conformal dimension is $h = 0$, and for $j = \frac{1}{2}$, that is $h = \frac{1}{4}$. Basically, we can construct an integrable representation by acting the following operators to the highest weight vectors:

$$J_n^- \quad (n \leq 0), \qquad J_m^3,\ J_m^+ \quad (m < 0). \tag{8.37}$$

We organize these creation operators in a following manner.

$$J_{(n)}^+ = J_{-n}^+, \qquad J_{(n)}^- = J_{+n}^-, \qquad J_{(n)}^3 = J_0^3 - \frac{n}{2}. \tag{8.38}$$

Notice that each $J_{(n)}^a$ forms $\mathfrak{su}(2)$ algebra (call them $\mathfrak{su}(2)_{(n)}$).

Let us consider one $|0,0\rangle$ of the highest vectors. Since $J_{(0)}^3 |0,0\rangle = 0$ the vector is singlet under the $\mathfrak{su}(2)_{(0)}$. On the other hand, $J_{(1)}^3 |0,0\rangle = -\frac{1}{2}|0,0\rangle$, $J_{(1)}^- |0,0\rangle = 0$, and hence, it should be a spin-$\frac{1}{2}$ representation of $\mathfrak{su}(2)_{(1)}$. Therefore, there is exactly one other states $J_{(1)}^+ |0,0\rangle$. Repeating similar procedures for $J_{(1)}^+ |0,0\rangle$, and it turns that $J_{(1)}^+ |0,0\rangle$ is singlet under $\mathfrak{su}(2)_{(2)}$ but doublet under $\mathfrak{su}(2)_{(3)}$. In summary, we have a tower of $J_{(n)}^+$ for odd n (let us set $n = 2p-1$):

$$\begin{aligned}
&J_{(1)}^+ J_{(3)}^+ J_{(5)}^+ \cdots J_{(2p-1)}^+ |0,0\rangle := |p;0,0\rangle\,, \\
&L_0 |p;0,0\rangle = \sum_{m=1}^{p} (2m-1) |p;0,0\rangle = p^2 |p;0,0\rangle\,, \\
&J_0^3 |p;0,0\rangle = p\,|p;0,0\rangle\,.
\end{aligned} \tag{8.39}$$

We can also consider negative $n = -2p+1$ ($p \in \mathbb{Z}_{\geq 1}$), and the resultant tower is

$$\begin{aligned}
&J_{(-1)}^- J_{(-3)}^- J_{(-5)}^- \cdots J_{(-2p+1)}^- |0,0\rangle := |-p;0,0\rangle\,, \\
&L_0 |-p;0,0\rangle = \sum_{m=1}^{p} (2m-1) |-p;0,0\rangle = p^2 |-p;0,0\rangle\,, \\
&J_0^3 |-p;0,0\rangle = -p\,|-p;0,0\rangle\,.
\end{aligned} \tag{8.40}$$

Notice that these tower corresponds to vertex operators:

$$\begin{aligned}
|p;0,0\rangle \quad &\leftrightarrow \quad :e^{+ip\sqrt{2}\varphi(z)}:, \\
|-p;0,0\rangle \quad &\leftrightarrow \quad :e^{-ip\sqrt{2}\varphi(z)}:, \\
|0;0,0\rangle := |0,0\rangle \quad &\leftrightarrow \quad 1.
\end{aligned} \tag{8.41}$$

The other highest weight vector $\left|\frac{1}{4},\frac{1}{2}\right\rangle$ is doublet under $\mathfrak{su}(2)_{(0)}$. $\left|\frac{1}{4},\frac{1}{2}\right\rangle$ is singlet under $\mathfrak{su}(2)_{(1)}$ and doublet under $\mathfrak{su}(2)_{(2)}$. On the other hand, $J_{(0)}^- \left|\frac{1}{4},\frac{1}{2}\right\rangle$ is singlet under $\mathfrak{su}(2)_{(-1)}$ and doublet under $\mathfrak{su}(2)_{(-2)}$. Therefore, similar tower for $\left|\frac{1}{4},\frac{1}{2}\right\rangle$ is ($n = 2q \geq 2$)

$$\begin{aligned}
&J_{(2)}^+ J_{(4)}^+ \cdots J_{(2q)}^+ \left|\frac{1}{4},\frac{1}{2}\right\rangle := \left|q;\frac{1}{4},\frac{1}{2}\right\rangle, \\
&L_0 \left|q;\frac{1}{4},\frac{1}{2}\right\rangle = \left(\frac{1}{4} + \sum_{m=1}^{q} 2m\right) \left|q;\frac{1}{4},\frac{1}{2}\right\rangle = \left(q+\frac{1}{2}\right)^2 \left|q;\frac{1}{4},\frac{1}{2}\right\rangle, \\
&J_0^3 \left|q;\frac{1}{4},\frac{1}{2}\right\rangle = \left(q+\frac{1}{2}\right) |p;0,0\rangle\,,
\end{aligned} \tag{8.42}$$

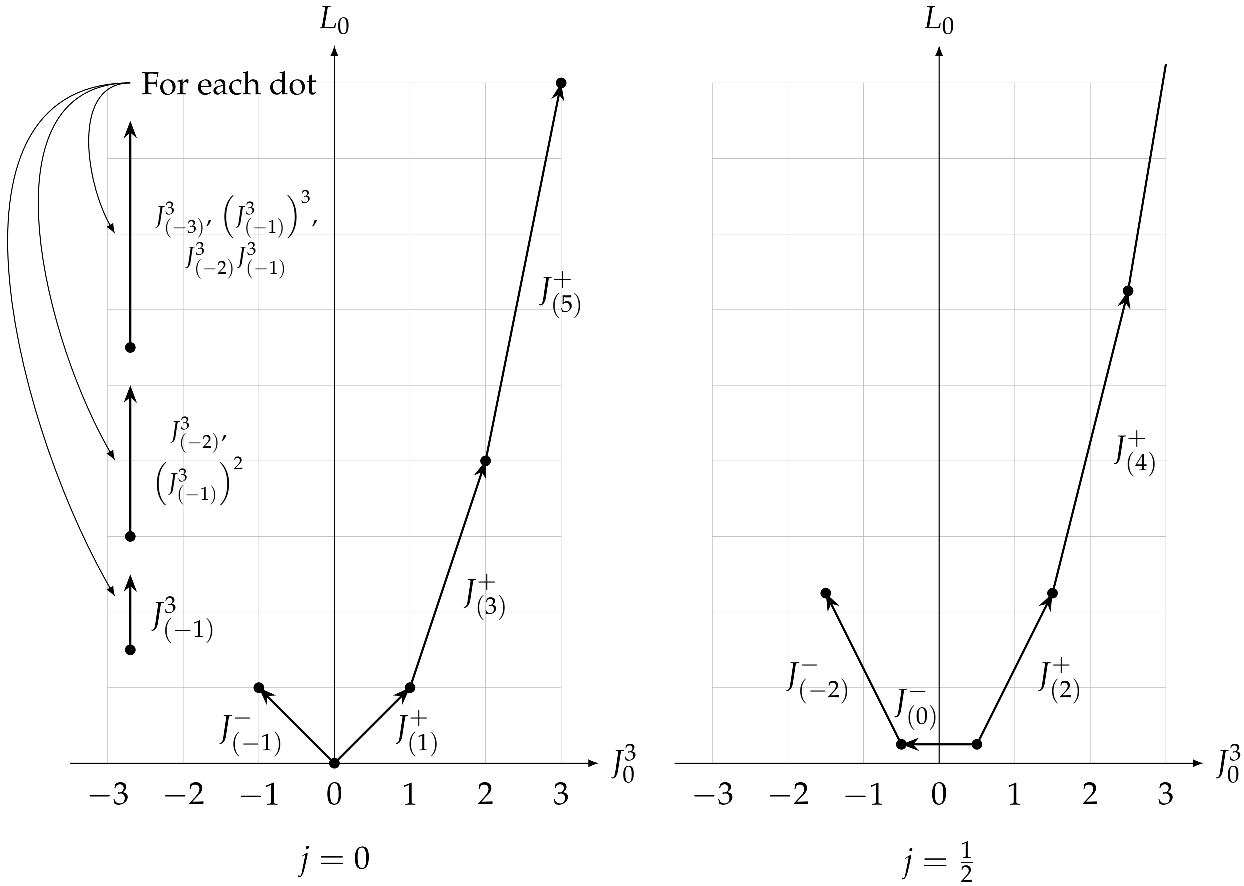


Figure 21: Highest weight representation of $\widehat{\mathfrak{su}(2)}_1$. For each dot there must be degenerate vertical arrows with integer length, which express descendant fields.

and for $J^-_{(0)}\left|\frac{1}{4},\frac{1}{2}\right\rangle$ $(n = -2q \leq 0)$

$$\begin{aligned}
&J^-_{(0)}J^-_{(-2)}J^-_{(-4)}\cdots J^-_{(-2q)}\left|\frac{1}{4},\frac{1}{2}\right\rangle := \left|-q;\frac{1}{4},\frac{1}{2}\right\rangle, \\
&L_0\left|-q;\frac{1}{4},\frac{1}{2}\right\rangle = \left(\frac{1}{4}+\sum_{m=1}^{q}2m\right)\left|q;\frac{1}{4},\frac{1}{2}\right\rangle = \left(q+\frac{1}{2}\right)^2\left|q;\frac{1}{4},\frac{1}{2}\right\rangle, \\
&J^3_0\left|q;\frac{1}{4},\frac{1}{2}\right\rangle = -\left(q+\frac{1}{2}\right)|p;0,0\rangle\,. \qquad (8.43)
\end{aligned}$$

Again, we can see corresponding vertex operators

$$\begin{aligned}
\left|q;\frac{1}{4},\frac{1}{2}\right\rangle &\quad\leftrightarrow\quad :e^{+i\frac{2q-1}{\sqrt{2}}\varphi(z)}:, \\
\left|-q;\frac{1}{4},\frac{1}{2}\right\rangle &\quad\leftrightarrow\quad :e^{-i\frac{2q+1}{\sqrt{2}}\varphi(z)}:\,. \qquad (8.44)
\end{aligned}$$

Recall that we still have J^3_{-n} $(n \geq 1) \sim \varphi^{(n)} = \frac{\partial^n}{\partial z^n}\varphi(z)$, which can freely act on each tower. Consequently, the character of the spin-j highest weight representation of $\widehat{\mathfrak{su}(2)}_1$

$$\chi_{2j}^{(1)}(\tau,z)=\mathrm{Tr}_j\left[q^{L_0-\frac{1}{24}}y^{J_0^3}\right],\quad (y=e^{2\pi iz}) \tag{8.45}$$

can be read off

$$\begin{aligned}\chi_0^{(1)}(\tau,z)&=\frac{\Theta_{0,1}(\tau,z)}{\eta(\tau)},\\ \chi_1^{(1)}(\tau,z)&=\frac{\Theta_{1,1}(\tau,z)}{\eta(\tau)},\end{aligned}$$

where

$$\Theta_{l,k}(\tau,z):=\sum_{n\in\mathbb{Z}+\frac{l}{2k}}q^{kn^2}y^{kn}$$

reduce to the one introduced in (5.44) at $z=0$. It is now obvious that the η-function is coming from the action of $\varphi^{(n)}$ $(n\geq 1)$ and the rest is from $:e^{in\sqrt{2}\varphi(z)}:$ $(n\in\mathbb{Z}+j)$. See Fig. 21 for the summary.

In general, irreducible representations of $\widehat{\mathfrak{su}(2)}_k$ require more careful treatment. Given a highest weight state $|h,j\rangle$ in (7.8), one can obtain the Verma module $V(h,j)$ by acting J_0^- and J_n^a for $n<0$. However, like the Virasoro modules considered in §5.2, the Verma module is generally not irreducible because it contains null states $|\chi\rangle$ which satisfy

$$J_0^+|\chi\rangle=0,\qquad J_n^a|\chi\rangle=0,\quad n>0,\quad a=0,\pm\,.$$

The Verma module $J(h,j)$ generated by the null states $|\chi\rangle$ become a (maximal proper) submodule of $V(h,j)$. Subsequently, the quotient module

$$L(h,j):=V(h,j)/J(h,j)$$

becomes irreducible, which is called the **integrable representation** of highest weight $|h,j\rangle$. For a general affine Lie algebra $\widehat{\mathfrak{g}}_k$, one can construct an irreducible module in a similar fashion and its character is called the **Weyl-Kac character formula**. By refering the reader to [FMS97, §14.4] for the derivation, we just write the Weyl-Kac character formula of the integrable representation of highest weight $|h,\frac{l}{2}\rangle$ for $\widehat{\mathfrak{su}(2)}_k$ for $0\leq l\leq k$

$$\chi_l^{(k)}(\tau,z)=\frac{\Theta_{l+1,k+2}(\tau,z)-\Theta_{-l-1,k+2}(\tau,z)}{\Theta_{1,2}(\tau,z)-\Theta_{-1,2}(\tau,z)}\,. \tag{8.46}$$

We can calculate the modular S-matrix for the character (8.46) as

$$S_{ll'}^{(k)}=\sqrt{\frac{2}{k+2}}\sin\left(\frac{\pi}{k+2}(l+1)(l'+1)\right)\qquad \text{with}\quad l,l'=0,\ldots,k\,. \tag{8.47}$$

We now want to construct modular invariant partition functions out of the characters (8.46). This amounts to determining all matrices $M_{ll'}$ such that

$$\mathcal{Z}_{\widehat{\mathfrak{su}(2)}_k}(\tau,\overline{\tau})=\sum_{l,l'}\chi_l^{(k)}(\tau)M_{ll'}^{(k)}\overline{\chi_{l'}}^{(k)}(\overline{\tau})\,. \tag{8.48}$$

is modular invariant. The invariance under the S-transformation requires

$$S^{(k)T}M^{(k)}S^{(k)*}=M^{(k)}\,. \tag{8.49}$$

$S^{(k)}$ in (8.47) is symmetric and real, thus we find the condition

$$\left[M^{(k)}, S^{(k)}\right] = 0\,. \tag{8.50}$$

We have already encountered this relation in minimal model modular invariant formula (5.43). There are further constraints for entries $M^{(k)}_{ll'}$. Since $M^{(k)}_{ll'}$ have the interpretation as the number of degeneracies of states in the Hilbert space, they must be non-negative integers. In addition, for the vacuum to only appear once, we have to require $M^{(k)}_{00} = 1$. Furthermore, it turns out that in order for (8.48) to be invariant under T-transformations, one has to satisfy the level-matching condition $h_l - \overline{h}_{l'} \in \mathbb{Z}$.

Level	Partition function ($\mathcal{Z}_{\widehat{\mathfrak{su}(2)}_k}$)	Type
$k=n$	$\sum_{l=0}^{n} \lvert\chi_l\rvert^2$	$A_{n+1}\,,\ n \geq 1$
$k=4n$	$\sum_{l=0}^{n-1} \lvert\chi_{2l}+\chi_{k-2l}\rvert^2 + 2\lvert\chi_{\frac{k}{2}}\rvert^2$	$D_{2n+2}\,,\ n \geq 1$
$k=4n-2$	$\sum_{l=0}^{k/2} \lvert\chi_{2l}\rvert^2 + \sum_{l=0}^{2n-2} \chi_{2l+1}\overline{\chi}_{k-2l-1}$	$D_{2n+1}\,,\ n \geq 2$
$k=10$	$\lvert\chi_0+\chi_6\rvert^2 + \lvert\chi_3+\chi_7\rvert^2 + \lvert\chi_4+\chi_{10}\rvert^2$	E_6
$k=16$	$\lvert\chi_0+\chi_{16}\rvert^2 + \lvert\chi_4+\chi_{12}\rvert^2 + \lvert\chi_6+\chi_{10}\rvert^2$ $+(\chi_2+\chi_{14})\overline{\chi}_8 + \chi_8(\overline{\chi}_2+\overline{\chi}_{14}) + \lvert\chi_8\rvert^2$	E_7
$k=28$	$\lvert\chi_0+\chi_{10}+\chi_{18}+\chi_{28}\rvert^2$ $+\lvert\chi_6+\chi_{12}+\chi_{16}+\chi_{22}\rvert^2$	E_8

Table 4: All $\widehat{\mathfrak{su}(2)}_k$ modular invariant partition functions

ADE classification of modular invariant partition functions

Cappelli, Itzykson and Zuber have conjectured [CIZ87a, ISZ88-No.26] that all matrices M with properties above for $\widehat{\mathfrak{su}(2)}_k$ characters are in one-to-one correspondence with the Dynkin diagrams of ADE types, which has been proven by themselves [CIZ87b] and Kato [Kat87]. The corresponding modular invariant partition functions are listed in Table 4, which is known as the ADE classification. In this classification, the Coxeter number h of the Dynkin diagram is related to the level via $h = k+2$, and the Coxeter exponents ℓ_n correspond to the characters that appears in the partition function. (In this lecture, the explanation of the Coxeter numbers and exponents are delegated to [FMS97].) The ADE classification appears in the minimal $\mathcal{N} = 2$ superconformal field theories ($c < 3$) which admits geometric interpretation as Arnold's singularities [LVW89]. This interpretation plays an important role for a discovery of **mirror symmetry**. (See Itzykson's review in [JMT89], and [CZ09] and the references therein.)

In fact, the classification of modular invariant partition functions of the unitary minimal model $\mathcal{M}_p$ attributes to the ADE classification of the modular invariant combinations of the $\widehat{\mathfrak{su}(2)}_k$ characters. This stems from the fact that the unitary minimal model $\mathcal{M}_p$ can be described by the coset model

$$\frac{\widehat{\mathfrak{su}(2)}_{p-2} \oplus \widehat{\mathfrak{su}(2)}_1}{\widehat{\mathfrak{su}(2)}_{p-1}} .$$

As an easy check, the central charge of the coset model can be computed by (7.66), yielding

$$c = 1 - \frac{6}{p(p+1)} ,$$

which is equal to (5.37). We refer the reader to [FMS97, §18.3] for the detail, and we just state the result in the following. As seen in §5.3, the modular invariant partition functions of $\mathcal{M}_p$

$$\mathcal{Z}_{\mathcal{M}_p} = \sum_{\substack{1 \le s \le r \le p-1 \\ 1 \le \widetilde{s} \le \widetilde{r} \le p-1}} \mathcal{N}^{(p)}_{(r,s),(\widetilde{r},\widetilde{s})} \chi_{(r,s)} \overline{\chi_{(\widetilde{r},\widetilde{s})}}$$

are combinations of the Rocha-Caridi character $\chi_{r,s}$ in (5.45). This partition function turns out to be written in terms of two modular invariant partition combinations of the $\widehat{\mathfrak{su}(2)}_k$ characters

$$\begin{aligned}
\mathcal{Z}_{\widehat{\mathfrak{su}(2)}_{p-1}} &= \sum_{r,\widetilde{r}=1}^{p-1} M^{(p-1)}_{r,\widetilde{r}} \chi^{(p-1)}_{r-1} \overline{\chi^{(p-1)}_{\widetilde{r}-1}} \\
\mathcal{Z}_{\widehat{\mathfrak{su}(2)}_{p-2}} &= \sum_{s,\widetilde{s}=1}^{p-2} M^{(p-2)}_{s,\widetilde{s}} \chi^{(p-2)}_{s-1} \overline{\chi^{(p-2)}_{\widetilde{s}-1}} ,
\end{aligned}$$

via

$$\mathcal{N}^{(p)}_{(r,s),(\widetilde{r},\widetilde{s})} = M^{(p-1)}_{r,\widetilde{r}} M^{(p-2)}_{s,\widetilde{s}} .$$

Hence, the modular invariant partition functions of $\mathcal{M}_p$ can be specified by a pair of ADE Dynkin diagrams with adjacent levels as in Table 5. Note that an A-series is always present in a pair. In particular, the partition functions (5.47) of diagonal type is of (A_{p-1}, A_p) type whereas the non-diagonal partition function (5.53) of the 3-state Potts model is of (A_4, D_4) type.

9 Entanglement Entropy

In this section, we will learn how 2d CFTs are related to entanglement entropy [CC04] and holography. There are nice reviews [CC09, NRT09, Tak12, Sol11, Har16, RT17]. At the end of this section, we will introduce the celebrated Ryu-Takayanagi entanglement entropy formula [RT06b, RT06a].

9.1 Introduction to Entanglement Entropy

In this subsection, we briefly introduce some basic concepts and properties in entanglement entropy.

In quantum mechanics, the state of a quantum system is represented by a state vector, denoted $|\psi\rangle$, whose time evolution is governed by the Schrödinger equation

$$i\frac{\partial}{\partial t}|\Psi\rangle = H|\Psi\rangle$$

p	Partition function ($\mathcal{Z}_{\mathcal{M}_p}$)	Type
≥ 3	$\frac{1}{2}\sum_{r=1}^{p-1}\sum_{s=1}^{p}\lvert\chi_{r,s}\rvert^2$	(A_{p-1}, A_p)
$4\ell+1$	$\sum_{r=1}^{p-1}\sum_{a=0}^{\ell}\lvert\chi_{r,2a+1}\rvert^2+\sum_{r=1}^{p-1}\sum_{a=0}^{\ell-1}\chi_{r,2a+1}\overline{\chi}_{p-r,2a+1}$	$(A_{p-1}, D_{2\ell+2})$
$4\ell+2$	$\sum_{a=0}^{\ell}\sum_{s=1}^{p}\lvert\chi_{2a+1,s}\rvert^2+\sum_{a=0}^{\ell-1}\sum_{s=1}^{p}\chi_{2a+1,s}\overline{\chi}_{2a+1,m+1-s}$	$(D_{2\ell+2}, A_p)$
$4\ell+3$	$\sum_{r=1}^{p-1}\sum_{a=0}^{\ell}\lvert\chi_{r,2a+1}\rvert^2+\sum_{r=1}^{2\ell+1}\lvert\chi_{r,2\ell+2}\rvert^2+\sum_{r=1}^{p-1}\sum_{a=1}^{\ell}\chi_{r,2a}\overline{\chi}_{p-r,2a}$	$(A_{p-1}, D_{2\ell+3})$
$4\ell+4$	$\sum_{a=0}^{\ell}\sum_{s=1}^{p}\lvert\chi_{2a+1,s}\rvert^2+\sum_{s=1}^{2\ell+2}\lvert\chi_{2\ell+2,s}\rvert^2+\sum_{a=1}^{\ell}\sum_{s=1}^{p}\chi_{2a,s}\overline{\chi}_{2a,m+1-s}$	$(D_{2\ell+4,A_p})$
11	$\frac{1}{2}\sum_{r=1}^{10}\left[\lvert\chi_{r,1}+\chi_{r,7}\rvert^2+\lvert\chi_{r,4}+\chi_{r,8}\rvert^2+\lvert\chi_{r,5}+\chi_{r,11}\rvert^2\right]$	(A_{10}, E_6)
12	$\frac{1}{2}\sum_{s=1}^{12}\left[\lvert\chi_{1,s}+\chi_{7,s}\rvert^2+\lvert\chi_{4,s}+\chi_{8,s}\rvert^2+\lvert\chi_{5,s}+\chi_{11,s}\rvert^2\right]$	(E_6, A_{12})
17	$\frac{1}{2}\sum_{r=1}^{16}\Big[\lvert\chi_{r,1}+\chi_{r,17}\rvert^2+\lvert\chi_{r,5}+\chi_{r,13}\rvert^2+\lvert\chi_{r,7}+\chi_{r,11}\rvert^2$ $+\lvert\chi_{r,9}\rvert^2+(\chi_{r,3}+\chi_{r,15})^*\chi_{r,9}+\chi_{r,9}^*(\chi_{r,3}+\chi_{r,15})\Big]$	(A_{16}, E_7)
18	$\frac{1}{2}\sum_{s=1}^{18}\Big[\lvert\chi_{1,s}+\chi_{17,s}\rvert^2+\lvert\chi_{5,s}+\chi_{13,s}\rvert^2+\lvert\chi_{7,s}+\chi_{11,s}\rvert^2$ $+\lvert\chi_{9,s}\rvert^2+(\chi_{3,s}+\chi_{15,s})^*\chi_{9,s}+\chi_{9,s}^*(\chi_{3,s}+\chi_{15,s})\Big]$	(E_7, A_{18})
29	$\frac{1}{2}\sum_{r=1}^{28}\left[\lvert\chi_{r,1}+\chi_{r,11}+\chi_{r,19}+\chi_{r,29}\rvert^2+\lvert\chi_{r,7}+\chi_{r,13}+\chi_{r,17}+\chi_{r,23}\rvert^2\right]$	(A_{28}, E_8)
30	$\frac{1}{2}\sum_{s=1}^{30}\left[\lvert\chi_{1,s}+\chi_{11,s}+\chi_{19,s}+\chi_{29,s}\rvert^2+\lvert\chi_{7,s}+\chi_{13,s}+\chi_{17,s}+\chi_{23,s}\rvert^2\right]$	(E_8, A_{30})

Table 5: Modular invariant partition functions of unitary minimal models $\mathcal{M}_p$.

A quantum system with a state vector $|\psi\rangle$ is called a **pure state**.

However, it is also possible for a system to be in a statistical ensemble of different state vectors: For example, there may be a 50% probability that the state vector is $|\psi_1\rangle$ and a 50% chance that the state vector is $|\psi_2\rangle$. This system would be in a **mixed state**. The density matrix $\rho_{\rm tot}$ is especially useful for mixed states, because any state, pure or mixed, can be characterized by a single density matrix.

A density matrix $\rho_{\rm tot}$ is a matrix that describes the statistical state of a system in quantum mechanics. For instance, the density matrix in the canonical ensemble of a system with Hamiltonian H and temperature T is

$$\rho_{\rm tot} = \frac{e^{-\beta H}}{\mathcal{Z}} \, .$$

Note that we always normalize $\mathrm{Tr}_{\mathcal{H}_{\rm tot}} \rho_{\rm tot}$. Then, the entropy can be expressed in terms of the density matrix

$$S = \frac{E-F}{T} = \frac{-1}{\mathcal{Z}} \left(\mathrm{Tr}_{\mathcal{H}_{\rm tot}} \left[\beta H e^{-\beta H} \right] + Z \log Z \right) = -\rho_{\rm tot} \log \rho_{\rm tot} \tag{9.1}$$

which is called **von Neumann entropy**.

If we express a pure state by the density operator

$$\rho = |\phi\rangle \langle \phi| \, . \tag{9.2}$$

The expectation value of operator $\mathcal{O}$ can be expressed by density operator

$$\langle \mathcal{O} \rangle_\rho := \mathrm{Tr}(\rho \mathcal{O}) \, . \tag{9.3}$$

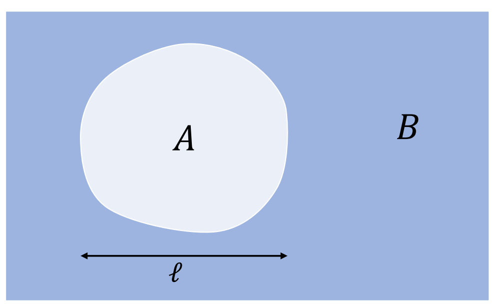


Figure 22:

Let us consider the situation in which a system is divided into two subsystems A and B where the Hilbert space is simply a direct product

$$\mathcal{H}_{\rm tot} = \mathcal{H}_A \otimes \mathcal{H}_B$$

Then, we can consider the density operator ρ_A restricted to the subsystem A by taking the trace over $\mathcal{H}_B$

$$\rho_A := \mathrm{Tr}_{\mathcal{H}_B}(\rho_{\rm tot}) \, . \tag{9.4}$$

Then, the **entanglement entropy** in the subsystem A is defined as

$$S_A := -\operatorname{Tr}_{\mathcal{H}_A}(\rho_A \log \rho_A)\,, \tag{9.5}$$

like the von Neumann entropy (9.1).

For example, let $\mathcal{H}_A$ and $\mathcal{H}_B$ be two Hilbert spaces spanned by $\{|0\rangle_A, |1\rangle_A\}$ and $\{|0\rangle_B, |1\rangle_B\}$, respectively. Considering a state

$$|\phi\rangle = \frac{1}{\sqrt{2}}(|1\rangle_A \otimes |0\rangle_B + |0\rangle_A \otimes |1\rangle_B) \tag{9.6}$$

However long the subsystems are apart, if measurement in A is $|1\rangle_A$, then then the measurement in B is bound to measure $|0\rangle_B$. Hence the subsystem A is is **entangled** with the subsystem B. This is called **quantum entanglement**.

In fact, the density matrix for the state (9.6),

$$\begin{aligned}
\rho_{\text{tot}} =& |\phi\rangle\langle\phi| \\
=& \frac{1}{2}(|0\rangle_A \langle 0|_A \otimes |1\rangle_B \langle 1|_B + |0\rangle_A \langle 1|_A \otimes |1\rangle_B \langle 0|_B \\
& + |1\rangle_A \langle 0|_A \otimes |0\rangle_B \langle 1|_B + |1\rangle_A \langle 1|_A \otimes |0\rangle_B \langle 0|_B)\,,
\end{aligned}$$

and the restricted one is

$$\rho_A := \operatorname{Tr}_{\mathcal{H}_B}(\rho_{\text{tot}}) = \frac{1}{2}(|0\rangle_A \langle 0|_A + |1\rangle_A \langle 1|_A)\ . \tag{9.7}$$

Thus the entanglement entropy can be evaluated as

$$S_A = \log 2\,. \tag{9.8}$$

This is indeed related to the fact that both the Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ are two-dimensional. In fact, the maximal entanglement entropy is given by $\min(\log\dim\mathcal{H}_A, \log\dim\mathcal{H}_B)$. Therefore, the state (9.6) is maximally entangled.

On the other hand, a state $|\psi\rangle$ is called **separate** if it can be written as the direct product of two pure states, i.e

$$|\psi\rangle = |\phi\rangle_A \otimes |\phi\rangle_B\,, \qquad |\phi\rangle_A \in \mathcal{H}_A \,\&\, |\phi\rangle_B \in \mathcal{H}_B\,. \tag{9.9}$$

In this case, it is easy to check that the entanglement entropy vanishes, which means there is no entanglement between both systems.

In conclusion, the information lost by the trace over the Hilbert space $\mathcal{H}_B$ is somehow restored in S_A by means of quantum entanglement. Hence, The entanglement entropy plays a very important role when we cannot make observation of a subsystem B.

There are some equivalent formula that will be useful in later calculation:

$$S_A = \lim_{n\to 1} \frac{\operatorname{Tr}_{\mathcal{H}_A} \rho_A^n - 1}{1-n} = -\frac{\partial}{\partial n} \operatorname{Tr}_{\mathcal{H}_A} \rho_A^n|_{n=1} = -\frac{\partial}{\partial n} \log \operatorname{Tr}_{\mathcal{H}_A} \rho_A^n|_{n=1}\,. \tag{9.10}$$

Properties of Entanglement Entropy

There are several useful properties which the entanglement entropy enjoys generally. We summarize some of them as follows.

- If the density matrix ρ_{tot} is constructed from a pure state such as in the zero temperature system, then we find the following relation assuming B is the complement of A
$$S_A = S_B\,. \tag{9.11}$$
This shows that the entanglement entropy is not an extensive quantity. For a mixed state, this relation does not hold.
- For any three subsystems A, B and C that do not intersect each other, the following inequalities hold:
$$\begin{aligned} S_{A+B+C} + S_B \leq S_{A+B} + S_{B+C}\,, \\ S_A + S_C \leq S_{A+B} + S_{B+C}\,. \end{aligned} \tag{9.12}$$
These inequalities are called the **strong subadditivity**.
- By setting B empty in (9.12), we can find the subadditivity relation
$$S_{A+B} \leq S_A + S_B\,. \tag{9.13}$$
- The subadditivity allows us to define an quantity called **mutual information** $I(A,B)$ by
$$I(A,B) = S_A + S_B - S_{A+B} \geq 0\,. \tag{9.14}$$

9.2 Conformal Field Theory on $\mathbb{Z}_n$ Orbifold

Before studying entanglement entropy in conformal field theory, we shall generalize the twist operator discussed in §5.4 to the $\mathbb{Z}_n$ orbifold. In §5.4, the continuum limit of the Ising model is described by the free fermion ψ, and the presence of a spin field σ introduce the $\mathbb{Z}_2$ twist condition to the free fermion in the complex plane:
$$\psi(e^{2\pi i}z) = -\psi(z)\,, \tag{9.15}$$
which means the field operator produces a minus sign once being taken around the origin. This is a global effect and the field operator is double-valued in the space time. The $\mathbb{Z}_2$ twist operator σ can be generalized to any finite group. In particular, we are interested in $\mathbb{Z}_n$ twists, which is defined as following
$$X(e^{2\pi i}z, e^{-2\pi i}\bar{z}) = e^{2\pi k i/N}X(z,\bar{z})\,, \tag{9.16}$$
where k is the integer called monodromy of the field X. For the definition of (9.16) makes sense, X must be complex. Therefore, We combine two copies of free boson φ_1, φ_2 to introduce the complex free boson:
$$X = \varphi_1(z,\bar{z}) + i\varphi_2(z,\bar{z})\,, \qquad \overline{X} = \varphi_1(z,\bar{z}) - i\varphi_2(z,\bar{z})\,. \tag{9.17}$$
φ_1 φ_2 are real scalar field. We have study these field in Chapter 4. Thus, the properties for complex free boson are easy to derive. Let us list some of them in brief. First the holomorphic fields for complex free boson are as following:
$$\partial X(z) = \partial\varphi_1(z) + i\partial\varphi_2(z)\,. \tag{9.18}$$
The holomorphic energy-momentum tensor is
$$T(z) = \frac{1}{2}\big(:\partial\varphi_1\partial\varphi_1: + :\partial\varphi_2\partial\varphi_2:\big) = -\frac{1}{2}:\partial X\partial\overline{X}:\,, \tag{9.19}$$

which is just the sum of the energy-momentum tensor of two real scalar field. The central charge of this theory becomes 2. The OPE of ∂X and $\partial \overline{X}$ is

$$\partial X(z)\partial X(w) \sim -\frac{2}{(z-w)^2}\,. \tag{9.20}$$

In a similar fashion to (4.21), the vacuum expectation value of energy-momentum is

$$\langle T(z)\rangle := \left[\left\langle -\frac{1}{2}\partial X(z)\partial \overline{X}(w)\right\rangle - \frac{1}{(z-w)^2}\right]_{z=w}. \tag{9.21}$$

To generalize the definition, we need to replace the original vacuum state with the twist ground state. The twist ground state for $\mathbb{Z}_n$ orbifold is defined by acting a twist operator on the vacuum state at the zero point.

$$\sigma_{k/N}(0)\,|0\rangle = |\sigma_{k/N}\rangle\,. \tag{9.22}$$

$\sigma_{k/N}$ is defined to twist the field X by $e^{2\pi i k/N}$ when X turn around it counterclockwise. The conjugate state then is given by inserting anti-twist operator $\sigma_{-k/N}$ at infinity which is denoted by $\langle\sigma_{-k/N}|$. By introducing twist operators, the global ground effect can be illustrated by inserting local operator [DFMS87, ISZ88-No.47].

The expectation value of energy-momentum in the $\mathbb{Z}_n$ twist ground state now can be evaluated as

$$\frac{\langle\sigma_{-k/N}|\,T(z)\,|\sigma_{k/N}\rangle}{\langle\sigma_{-k/N}|\sigma_{k/N}\rangle} := \left[\frac{\langle\sigma_{-k/N}|-\frac{1}{2}\partial X(z)\partial\overline{X}(w)\,|\sigma_{k/N}\rangle}{\langle\sigma_{-k/N}|\sigma_{k/N}\rangle} - \frac{1}{(z-w)^2}\right]_{z=w}, \tag{9.23}$$

where the normalized factor in the denominator can be adjusted to 1. We have calculated this quantity in the R sector of free fermion. To perform the calculation we need to study mode expansion of X first.

From (9.16), we immediately obtain that

$$\begin{aligned}
\partial X(e^{2\pi i}z, e^{-2\pi i}\bar{z}) &= e^{2\pi i(k/N-1)}\partial X(z,\bar{z})\,,\\
\partial \overline{X}(e^{-2\pi i}\bar{z}, e^{2\pi i}z) &= e^{-2\pi i(k/N+1)}\partial \overline{X}(\bar{z},z)\,.
\end{aligned} \tag{9.24}$$

Therefore, the Laurent expansions of ∂X and $\partial\overline{X}$ must have the form

$$\begin{aligned}
i\partial_z X &= \sum_{m\in\mathbb{Z}} \alpha_{m-k/N} z^{-m-1+k/N}\,,\\
i\partial_z \overline{X} &= \sum_{m\in\mathbb{Z}} \bar{\alpha}_{m+k/N} z^{-m-1-k/N}\,.
\end{aligned}$$

The mode operators have the following canonical commutation relations

$$[\bar{\alpha}_{m+k/N}, \alpha_{n-k/N}] = 2(m+k/N)\delta_{m,-n}\,, \tag{9.25}$$

where 2 comes from the fact that complex boson field consists of two real scalar fields. And the twist ground state $|\sigma_{k/N}\rangle$ is annihilated by all the positive frequency mode operators

$$\begin{aligned}
\alpha_{m-k/N}\,|\sigma_{k/N}\rangle &= 0\,, \qquad m > 0\,,\\
\bar{\alpha}_{m+k/N}\,|\sigma_{k/N}\rangle &= 0\,, \qquad m \geq 0\,.
\end{aligned} \tag{9.26}$$

Using the mode expansion (9.25) and their properties we mentioned above, we can calculate the expectation value in the twist ground state

$$-\frac{1}{2}\langle\sigma_{-k/N}|\,\partial X(z)\bar{\partial}X(w)\,|\sigma_{k/N}\rangle = z^{-(1-k/N)}w^{-k/N}\left[\frac{(1-k/N)z+kw/N}{(z-w)^2}\right]. \tag{9.27}$$

From (9.23), we can compute

$$\langle\sigma_{-k/N}|\,T(z)\,|\sigma_{k/N}\rangle = \frac{1}{z^2}\cdot\frac{1}{2}\frac{k}{N}\left(1-\frac{k}{N}\right). \tag{9.28}$$

From previous study, we know that as a local operator, $T\sigma_{k/N}$ OPE should have the following form

$$T(z)\sigma_{k/N}(0) = \cdots\frac{h\sigma_{k/N}(0)}{z^2}+\cdots, \tag{9.29}$$

where h is the conformal dimension for twist field σ_{k_N}. Plugging into (9.28), we immediately obtain the conformal dimension for $\sigma_{k/N}$ is $\frac{1}{2}\frac{k}{N}(1-\frac{k}{N})$. The same analysis can be applied for for $\sigma_{-k/N}$ and the anti-holomorphic sector.

9.3 Entanglement Entropy in 2d CFTs

Now, we are going to study the Entanglement Entropy in 2d CFTS. We define a subsystem A at fixed time $t=t_0$, and call its complement B. The boundary of A is denoted by ∂A. Then we can define the entanglement entropy S_A by the following formula recall that $\mathrm{Tr}_{\mathcal{H}_A}\rho_A = 1$.

$$\begin{aligned}S_A &= \lim_{n\to 1}\frac{\mathrm{Tr}_{\mathcal{H}_A}\rho_A^n - 1}{1-n}\\ &= -\frac{\partial}{\partial n}\mathrm{Tr}_{\mathcal{H}_A}\rho_A^n|_{n=1} = -\frac{\partial}{\partial n}\log\mathrm{Tr}_{\mathcal{H}_A}\rho_A^n|_{n=1}.\end{aligned}$$

This is called the **replica trick**. Therefore, what we need to do is to evaluate $\mathrm{Tr}_{\mathcal{H}_A}\rho_A^n$ in our 2D system.

This can be done in the path integral formalism as follows. We first assume that A is the single interval $x\in[u,v]$ at $t_E=0$ in the flat Euclidean coordinates $(t_E,x)\in\mathbb{R}^2$. The ground state wave function can be found by path integrating from $t_E=-\infty$ to $t_E=0$

$$\Psi(\phi_0(x)) = \int_{t_E=-\infty}^{\phi(t_E=0,x)=\phi_0(x)} D\phi e^{-\mathcal{S}(\phi)}, \tag{9.30}$$

where $\phi(t_E,x)$ denotes the field which defines the 2D CFT. ϕ_0 is the value of the field which depends on the spacial position x. The total density matrix ρ is given by two copies of the wave function $[\rho]_{\phi_0\phi_0'} = \Psi(\phi_0)\overline{\Psi}(\phi_0')$. The complex conjugate one $\overline{\Psi}$ can be obtained by path-integrating from $t_E=\infty$ to $t_E=0$. To obtain the reduced density matrix ρ_A, we need to integrate ϕ_0 on B assuming $\phi_0(x)=\phi_0'(x)$ when $x\in B$.

$$[\rho_A]_{\phi_+\phi_-} = (\mathcal{Z}_1)^{-1}\int_{t_E=-\infty}^{t_E=\infty} D\phi e^{-\mathcal{S}(\phi)}\prod_{x\in A}\delta(\phi(+0,x)-\phi_+(x))\dot{\delta}(\phi(-0,x)-\phi_-(x)), \tag{9.31}$$

where ϕ_+ and ϕ_- are both boundary function in A, and $\mathcal{Z}_1$ is the vacuum partition function on $\mathbb{R}^2$ in order to make $\mathrm{Tr}\,\rho_A=1$. The computation is sketched in Figure 23.

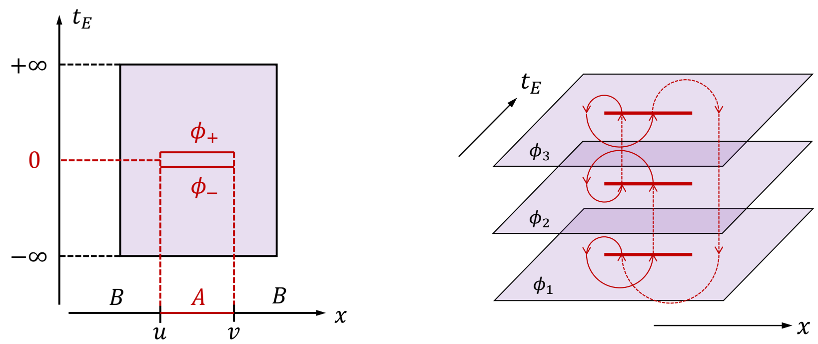


Figure 23: Left: the path integral representation of the reduced matrix $[\rho_A]_{\phi_+\phi_-}$. Right: the n- sheeted Riemann surface $\mathcal{R}_n$

To find $\mathrm{Tr}_{\mathcal{H}_A}\,\rho_A^n$, we can prepare n copies of (9.31)

$$[\rho_A]_{\phi_{1+}\phi_{1-}}[\rho_A]_{\phi_{2+}\phi_{2-}}\cdots[\rho_A]_{\phi_{n+}\phi_{n-}}\,, \tag{9.32}$$

and take the trace successively. In the path-integral formalism this is realized by gluing $\{\phi_{i\pm}(x)\}$ as $\phi_{i-}(x) = \phi_{(i+1)+}(x)(i = 1,2,\cdots,n)$ and integrating $\phi_{i+}(x)$. In this way, $\mathrm{Tr}_{\mathcal{H}_A}\,\rho_A^n$ is given in terms of the path-integral on an n-sheeted Riemann surface $\mathcal{R}_n$ (Figure 23)

$$\mathrm{Tr}_{\mathcal{H}_A}\,\rho_A^n = (\mathcal{Z}_1)^{-n}\int_{(t_E,x)\in\mathcal{R}_n} D\phi e^{-\mathcal{S}(\phi)} \equiv \frac{\mathcal{Z}_n}{(\mathcal{Z}_1)^n}\,. \tag{9.33}$$

To evaluate the path-integral on $\mathcal{R}_n$, it is useful to introduce replica fields. Let us first take n disconnected sheets. The field on each sheet is denoted by $\phi_k(t_E,x)(k=1,2,\cdots n)$. In order to obtain a CFT on the flat complex plane $\mathbb{C}$ which is equivalent to the present one on R_n, we impose the twisted boundary conditions

$$\phi_k(e^{2\pi i}(w-u)) = \phi_{k+1}(w-u)\,,\qquad \phi_k(e^{2\pi i}(w-u)) = \phi_{k-1}(w-v)\,. \tag{9.34}$$

Assuming that ϕ is a complex scalar field with central charge $c=2$, then we can introduce n new fields $\widetilde{\phi}_k = \frac{1}{n}\sum_{l=1}^n e^{2\pi i lk/n}\phi_l$. They obey the boundary condition

$$\widetilde{\phi}_k(e^{2\pi i}(w-u)) = e^{2\pi i k/n}\widetilde{\phi}_k(w-u)\,,\qquad \widetilde{\phi}_k(e^{2\pi i}(w-v)) = e^{-2\pi i k/n}\widetilde{\phi}_k(w-v)\,. \tag{9.35}$$

As we have learned in the last section the system is equivalent to n- disconnected sheets with two twist operators $\sigma_{k/n}$ and $\sigma_{-k/n}$ inserted in the k-th sheet for each values of k. In the end we find

$$\mathrm{Tr}_{\mathcal{H}_A}\,\rho_A^n = \prod_{k=0}^{n-1}\langle\sigma_{k/n}(u)\sigma_{-k/n}(v)\rangle \sim (u-v)^{-4\sum_{k=0}^{n-1}\Delta_{k/n}} = (u-v)^{-\frac{1}{3}(n-1/n)}\,, \tag{9.36}$$

where $\Delta_{k/n} = \frac{1}{2}(\frac{k}{n})^2 + \frac{1}{2}\frac{k}{n}$ is the conformal dimension of $\sigma_{k/n}$. When we have m such complex scalar fields we simply obtain

$$\mathrm{Tr}_{\mathcal{H}_A}\,\rho_A^n \sim (u-v)^{-\frac{c}{6}(n-1/n)}\,, \tag{9.37}$$

setting the central charge $c=2m$. Applying the formula (9.30) to (9.37), we find that

$$S_A \sim \frac{c}{3}\log\ell \tag{9.38}$$

where we set $\ell \equiv u - v$. Refine the UV cut-off ϵ into the expression above we get

$$S_A = \frac{c}{3} \log \frac{\ell}{\epsilon} \,. \tag{9.39}$$

Entanglement Entropy of Finite Size

In previous section, we assume the space direction x is infinite, while in this section we consider the subsystem A is defined in a finite space region.

$$A = \{x | x \in [r, s]\} \,, \tag{9.40}$$

where we assume $-\frac{L}{2} < r < s \leq \frac{L}{2}$. The system is related to the previous system via the conformal map

$$w = \tan\left(\frac{\pi\omega'}{L}\right). \tag{9.41}$$

And $u = \tan\left(\frac{\pi r}{L}\right)$ and $v = \tan\left(\frac{\pi s}{L}\right)$. From (9.36), we can calculate $\mathrm{Tr}_{\mathcal{H}_A} \rho^n_{A\omega'}$ by applying conformal transformation of local twist field.

$$\begin{aligned}
\mathrm{Tr}_{\mathcal{H}_A} \rho^n_{A\omega'} &= \prod_{k=0}^{n-1} \left(\frac{\mathrm{d}w}{\mathrm{d}w'}\right)_r^{2\Delta_{k/n}} \left(\frac{\mathrm{d}w}{\mathrm{d}w'}\right)_s^{2\Delta_{k/n}} \langle \sigma_{k/n}(u) \sigma_{-k/n}(v) \rangle \\
&\sim \left[\frac{L}{\pi} \cos\left(\frac{\pi}{L} r\right) \cos\left(\frac{\pi}{L} s\right) (u - v)\right]^{-\frac{c}{6}(n-1/n)} \\
&\sim \left[\frac{L}{\pi} \sin \frac{\pi}{L} (r - s)\right]^{-\frac{c}{6}(n-1/n)} .
\end{aligned}$$

Therefore, following the same calculation step, we get the entanglement entropy in a finite space region

$$S_A = \frac{c}{3} \cdot \log\left(\frac{L}{\pi\epsilon} \sin\left(\frac{\pi\ell}{L}\right)\right), \tag{9.42}$$

where $\ell = r - s$. It is invariant under the exchange $\ell \to L - \ell$, and thus satisfy the property (9.11).

Entanglement Entropy at Finite Temperature

Considering in the Euclidean 2-dimension theory, the Euclidean time is equivalent to the inverse of temperature. At finite temperature $T = \beta^{-1}$, we can compactify the Euclidean time as $t_E \sim t_E + \beta$. We can map this system to the infinite system via the conformal map

$$w = e^{\frac{2\pi}{\beta} w'} \,. \tag{9.43}$$

And $u = e^{\frac{2\pi r}{\beta}}$, $v = e^{\frac{2\pi s}{\beta}}$. t_E is the imaginary part of w'.This conformal map will lead to the extra factor

$$\left[\frac{\beta}{2\pi} e^{-\frac{\pi}{\beta}}\right]^{-\frac{c}{6}(n-1/n)} \tag{9.44}$$

Follow the similar calculation steps, we obtain

$$S_A = \frac{c}{3} \log\left(\frac{\beta}{\pi\epsilon} \sinh\left(\frac{\pi\ell}{\beta}\right)\right), \tag{9.45}$$

where $\ell = r - s$. In the zero temperature limit $T \to 0$, this reduces to the previous result (9.39). In the high temperature limit $T- > \infty$, it approaches

$$S_A \simeq \frac{c}{3} \log \frac{\beta}{2\pi\epsilon} + \frac{\pi c}{3} \ell T \,. \tag{9.46}$$

This is the same as the thermal entropy for the subsystem A as expected.

9.4 Zamolodchikov c-theorem revisited

In §6.2, we have learned Zamolodchikov c-theorem stating that in a unitary, rotational (Lorentz) invariant 2d theory, there exists a monotonically decreasing function C with respect to the length scale R

$$\frac{dC(R)}{dR} \le 0$$

where $C(R=0)$ is the central charge at ultra-violet and $C(R=\infty)$ is the one at infra-red. This theorem can be shown easily by using the strong subadditivity (9.12) of entanglement entropy.

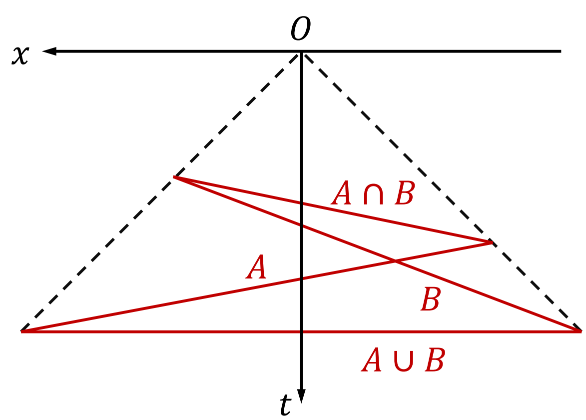


Figure 24: The black lines represent subsystems and the blue lines express the light cone

We consider the subsystems A and B as in Figure 24. Although A and B do not lie at the constant time slice in Figure 24, an appropriate Lorentz boost will bring them at the constant time slice so that we can define the Hilbert space $\mathcal{H}_A$ and $\mathcal{H}_B$. The Hilbert spaces $\mathcal{H}_A \cup \mathcal{H}_B$ and $\mathcal{H}_A \cap \mathcal{H}_B$ are defined on $A \cup B$ and $A \cap B$. Let us write the Lorentz-invariant length of the subsystem A by $\ell(A)$. Then, a simple computation yields

$$\ell(A) \cdot \ell(B) = \ell(A \cup B) \cdot \ell(A \cap B) \,. \tag{9.47}$$

In particular, when

$$\ell(A) = \ell(B) = e^{\frac{a+b}{2}}, \quad \ell(A \cup B) = e^a, \quad \ell(A \cap B) = e^b \,,$$

the strong subadditivity (9.12) implies that

$$2S\left(\frac{a+b}{2}\right) \ge S(a) + S(b) \,.$$

Thus, $S(R)$ is concave and

$$\frac{d^2}{dR^2}S(R) \leq 0$$

where $\ell = e^R$. If we define the function

$$C(R) = 3\frac{dS(R)}{dR} ,$$

then it is monotonically decreasing

$$\frac{dC}{dR} \leq 0 .$$

Comparing with (9.39), it is equal to the central charge at the fixed point. This corresponds to Zamolodchikov C-function (6.14).

9.5 Black hole thermodynamics and Bekenstein-Hawking entropy

The aforementioned Ryu-Takayanagi formula evaluates entangle entropy of a CFT from the holographic viewpoint. First, we shall provide a brief introduction to black hole thermodynamics and AdS/CFT correspondence based on which Ryu-Takayanagi formula is constructed. However, we glimpse only a tip of iceberg and the subject would actually deserve the entire one semester. If you are interested in this fertile subject, we refer to the standard references [Wal10, Tow97, AGM$^+$00].

Let us recall the Schwarzschild black hole of mass M with metric

$$ds^2 = -\left(1 - \frac{r_H}{r}\right)c^2dt^2 + \frac{dr^2}{1-\frac{r_H}{r}} + r^2d\Omega^2$$

where the radius of the horizon is given by

$$r_H = \frac{2GM}{c^2} .$$

To see the thermodynamical property of the Schwarzschild black hole, we use the naive trick. It turns out that much of the interesting physics having to do with the quantum properties of black holes comes from the region near the event horizon. To examine the region *near the horizon* r_H, we analytically continued to the Euclidean metric $t = -it_E$, and we set

$$r - \frac{2GM}{c^2} = \frac{x^2c^2}{8GM} .$$

Then, the metric near the event horizon $x \ll 1$

$$ds_{\rm E}^2 \approx (\kappa c x)^2 dt_E^2 + dx^2 + \frac{1}{4\kappa^2}d\Omega^2 ,$$

where $\kappa = \frac{c^2}{4GM}$ is called the **surface gravity** because it is indeed the acceleration of a static particle near the horizon as measured at spatial infinity. The first part of the metric is just $\mathbb{R}^2$ with polar coordinates if we make the periodic identification

$$t_E \sim t_E + \frac{2\pi}{c\kappa} .$$

Using the relation between Euclidean periodicity and temperature, we can deduce **Hawking temperature** of the Schwarzschild black hole

$$T_H = \frac{\hbar c \kappa}{2\pi} = \frac{\hbar c^3}{8\pi G M} \,. \tag{9.48}$$

This is a very heuristic way to introduce the Hawking temperature which is not found originally in this way.

If the black hole has temperature, then it obeys the thermodynamics law. In general, a stationary black hole can have electric charge Q and angular momentum J. Bardeen, Carter, and Hawking [BCH73] noticed that the laws of black hole mechanics with mass M, angular momentum J, and charge Q bears a striking resemblance with the three laws of thermodynamics. This is quite surprising because *a priori* there is no reason to expect that the spacetime geometry of black holes has anything to do with thermal physics.

(0) Zeroth Law: In thermal physics, the zeroth law states that the temperature T of a body at thermal equilibrium is constant throughout the body. Correspondingly for stationary black holes, one can show that surface gravity κ is constant on the event horizon.

(1) First Law: Energy is conserved, $dE = TdS + \mu dQ + \Omega dJ$, where E is the energy, Q is the charge with chemical potential μ and J is the angular momentum with chemical potential Ω. Correspondingly for black holes, one has $dM = \frac{\kappa}{8\pi G} dA + \mu dQ + \Omega dJ$. Here A is the area of the horizon, and κ is the surface gravity, μ is the chemical potential conjugate to Q, and Ω is the angular velocity conjugate to J.

(2) Second Law: In a physical process the total entropy S never decreases, $\Delta S \geq 0$. Correspondingly for black holes one can prove the area theorem that the net area in any process never decreases, $\Delta A \geq 0$. For example, two Schwarzschild black holes with masses M_1 and M_2 can coalesce to form a bigger black hole of mass M. This is consistent with the area theorem, since the area is proportional to the square of the mass, and $(M_1 + M_2)^2 \geq M_1^2 + M_2^2$. The opposite process where a bigger black hole fragments is however disallowed by this law.

Laws of Thermodynamics	**Laws of Black Hole Mechanics**
Temperature is constant throughout a body at equilibrium. T= constant.	Surface gravity is constant on the event horizon. κ =constant.
Energy is conserved. $dE = TdS + \mu dQ + \Omega dJ$.	Energy is conserved. $dM = \frac{\kappa}{8\pi} dA + \mu dQ + \Omega dJ$.
Entropy never decrease. $\Delta S \geq 0$.	Area never decreases. $\Delta A \geq 0$.

Table 6: Laws of Black Hole Thermodynamics

This result can be understood as one of the highlights of general relativity. Moreover, Hawking has shown this is indeed more than an analogy [Haw75]. There is a deep connection between black hole geometry, thermodynamics and quantum mechanics. Quantum mechanically, a black hole is not quite black. Hawking has applied techniques of quantum field theories on a curved background to the near horizon region of a black hole and showed that a black hole indeed radiates [Haw75]. In a quantum theory, particle-antiparticle are constantly being created and annihilated even in vacuum. Near the horizon, an antiparticle can fall in once in a while and the particle can escapes to infinity. Although this lecture does not deal with Hawking's calculation unfortunately (see [Tow97]), it actually revealed that the spectrum emitted by the black hole is precisely thermal with temperature (9.48).

In fact, because the energy of the black hole is equal to its mass, the first law of thermodynamics can be written as

$$T_H dS = c^2 dM$$

Using the formula for the Hawking temperature (9.48), the Schwarzschild black hole has the entropy

$$S = 4\pi \frac{GM^2}{c} .$$

The entropy is proportional to the area A of the horizon

$$S = \frac{c^3 \,\text{Area}}{4G\hbar} . \tag{9.49}$$

This is a universal result for any black hole, and this remarkable relation between the thermodynamic properties of a black hole and its geometric properties is called the celebrated **Bekenstein-Hawking entropy formula** [Bek72, Bek73, Haw75]. This formula involves all three fundamental constants of nature, and this is the first place where the Newton constant G meets with the Planck constant $\hbar$.

In thermodynamics, the energy and entropy are examples of **extensive properties** *i.e.* they are proportional to the size of the system. However, the Bekenstein-Hawking entropy is proportional to the area. Moreover, in general relativity, the maximal entropy in a certain subsystem R is proportional to the area of ∂R

$$S \le \frac{\text{Area}(\partial R)}{4G} ,$$

which is called **entropy bound**. Subsequently, people come up with the idea that a gravity theory in $(d+1)$-dimension is equivalent to a quantum field theory without gravity in d-dimension, which is called the **holographic principle**.

9.6 AdS/CFT correspondence

The AdS/CFT correspondence [Mal99] is a special case of the holographic principle, which states that a quantum gravity in an AdS$_d$ space is equivalent to a CFT in d-dimensions. Even though the correspondence was originally proposed in the context of string theory, it is now generalized in broader contexts. This subsection briefly introduce to a basics of the AdS/CFT correspondence in order to pave the way to the Ryu-Takayanagi formula.

AdS space

To begin with, let us investigate geometry of AdS spaces. An anti-de Sitter (AdS) space is a maximally symmetric manifold with constant negative scalar curvature. It is a solution of Einstein's equations for an empty universe with negative cosmological constant. The easiest way to understand it is as follows.

A Lorentzian AdS$_{d+1}$ space can be illustrated by the hyperboloid in $(2,d)$ Minkowski space:

$$X_0^2 + X_{d+1}^2 - \sum_{i=1}^{d} X_i^2 = R^2 . \tag{9.50}$$

The metric can be naturally induced from the Minkowski space

$$ds^2 = -dX_0^2 - dX_{d+1}^2 + \sum_{i=1}^{d} dX_i^2 . \tag{9.51}$$

By construction it has SO$(2,d)$ isometry, which is a first connection to the conformal group in d-dimensions studied in §2.4.

Global coordinate

Simple solution to (9.50) is given as follows.

$$\begin{aligned} &X_0^2 + X_{d+1}^2 = R^2 \cosh^2 \rho , \\ &\sum_{i=1}^{d} X_i^2 = R^2 \sinh^2 \rho . \end{aligned}$$

Or, more concretely,

$$\begin{aligned} &X_0 = R \cosh\rho \;\cos\tau , \qquad X_{d+1} = R\cosh\rho \;\sin\tau , \\ &X_i = R\sinh\rho\, \Omega_i \quad (i = 1, \cdots, d, \text{and} \sum_i \Omega_i^2 = 1). \end{aligned}$$

These are S^1 and S^{d-1} with radii $R\cosh\rho$ and $R\sinh\rho$, respectively. The metric is

$$ds^2 = R^2 \left(-\cosh^2\rho\, d\tau^2 + d\rho^2 + \sinh^2\rho\, d\Omega_{(d-1)}^2 \right) .$$

Note that τ is a periodic variable and if we take $0 \le \tau < 2\pi$ the coordinate wrap the hyperboloid precisely once. This is why this coordinate is called **global coordinate**. The manifest sub-isometries are SO(2) and SO(d) of SO$(2,d)$. To obtain a causal spacetime, we simply unwrap the circle S^1, namely, we take the region $-\infty < \tau < \infty$ with no identification, which is called **universal cover** of the hyperboloid.

In literatures another global coordinate is also used, which can be derived by redefinitions $r \equiv R\sinh\rho$ and $dt \equiv Rd\tau$:

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\Omega_{(d-1)}^2 , \qquad f(r) = 1 + \frac{r^2}{R^2} .$$

Poincaré coordinates

There is yet another coordinates, called **Poincaré coordinates**. As opposed to the global coordinate, this coordinate covers only the half of the hyperboloid. It is most easily (but naively) seen in $d = 1$ case:

$$x^2 - y^2 = R^2 ,$$

which is the hyperbolic curve. The curve consists of two isolated parts in regions $x > R$ and $x < -R$. We simply use one of them to construct the coordinate.

Let us get back to general d-dim. We define the coordinate as follows.

$$\begin{aligned}
X_0 &= \frac{1}{2u}\left(1 + u^2\left(R^2 + x_i^2 - t^2\right)\right) , \\
X_i &= Rux_i \qquad (i = 1, \cdots, d-1) , \\
X_d &= \frac{1}{2u}\left(1 - u^2\left(R^2 - x_i^2 + t^2\right)\right) , \\
X_{d+1} &= Rut ,
\end{aligned} \tag{9.52}$$

where $u > 0$. As it is stated the coordinate covers the half of the hyperboloid; in the region, $X_0 > X_d$. The metric is

$$ds^2 = R^2\left(\frac{du^2}{u^2} + u^2(-dt^2 + dx_i^2)\right) = R^2\left(\frac{du^2}{u^2} + u^2 dx_\mu^2\right) . \tag{9.53}$$

The coordinates (u, t, x_i) is called **Poincaré coordinates**. This metric has manifest $ISO(1, d-1)$ and $\mathrm{SO}(1,1)$ sub-isometries of $\mathrm{SO}(2,d)$; the former is the Poincaré transformation and the latter corresponds to the dilatation

$$(u, t, x_i) \to (\lambda^{-1}u, \lambda t, \lambda x_i) . \tag{9.54}$$

If we further define $z = 1/u$ ($z > 0$), then,

$$ds^2 = \frac{R^2}{z^2}\left(dz^2 + dx_\mu^2\right) . \tag{9.55}$$

This is called **the upper (Poincaré) half-plane model**. The hypersurface given by $z = 0$ is called **(asymptotic) boundary** of the AdS space, which corresponds to $u \sim r \sim \rho = \infty$.

AdS/CFT correspondence

The AdS/CFT correspondence [GKP98, Wit98] is the an exact duality between a quantum gravity in an asymptotically AdS_{d+1} spacetime and a CFT_d without gravity. It is **holographic** since the gravitational theory lives in one extra dimension. It is often useful to think that the CFT lives at the **boundary** of the **the bulk** AdS space. Indeed, the CFT lives in a spacetime parameterized by $x = (z, \vec{x})$, whereas gravity fields are functions of $\vec{x}$ and the radial coordinate z. For instance, the scaling transformation of the CFT can be translated into the transformation of the AdS coordinate (9.55)

$$x_\mu \to \lambda x_\mu, \quad z \to \lambda z .$$

Hence, the coordinate z parametrizes the energy scale, and the ultra-violet limit corresponds to the boundary at $z = 0$. As z increases, the energy scales decreases.

The theories are believed to be entirely equivalent: any physical (gauge-invariant) quantity that can be computed in one theory can also be computed in the dual. As typical feature of a duality, computation of physical quantity becomes much easier on one side than on the other side. More precisely, the gravitational partition function on asymptotically AdS space is equal to the generating function of correlation functions of the corresponding CFT:

$$Z_{\mathrm{grav}}[\phi \to \phi_0] = \left\langle \exp\left(\int_{\partial AdS} \overline{\phi}_0 \mathcal{O}\right)\right\rangle_{\mathrm{CFT}} \tag{9.56}$$

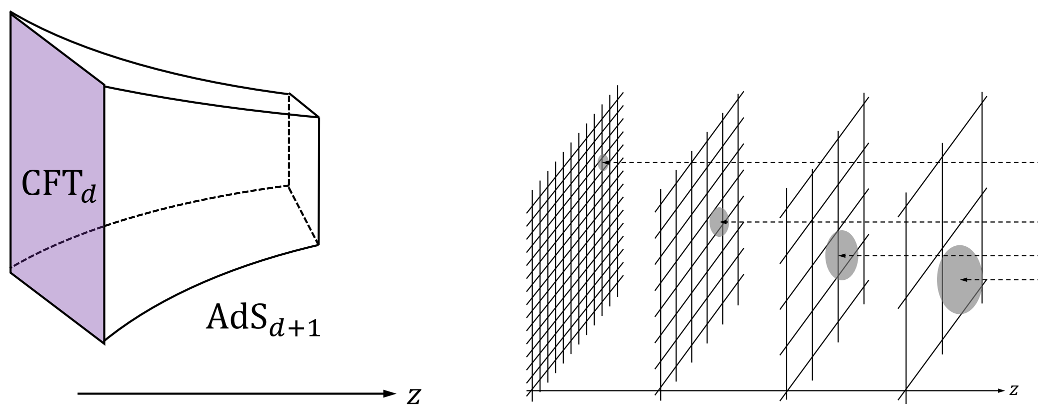


Figure 25:

that is called **GKPW** relation. For any bulk field ϕ in gravity theory on AdS, there exists the corresponding an operator $\mathcal{O}$ in the CFT. The gravitational partition function can be schematically written as

$$Z_{\text{grav}}[\phi \to \phi_0] = \int_{\phi \to \phi_0} \mathcal{D}\phi \, e^{-S_{\text{string}}[\phi]} .$$

Bulk field/boundary operator

Each field propagating on AdS space is in a one to one correspondence with an operator in the field theory. The spin of the bulk field is equal to the spin of the CFT operator; the mass of the bulk field fixes the scaling dimension of the CFT operator. Here are some examples:

- Every theory of gravity has a massless spin-2 particle, the graviton $g_{\mu\nu}$. This is dual the stress tensor $T_{\mu\nu}$ in CFT. This makes sense since every CFT has a stress tensor. The fact that the graviton is massless corresponds to the fact that the CFT stress tensor is conserved.
- If our theory of gravity has a spin-1 vector field A_μ, then the dual CFT has a spin-1 operator J_μ. If A_μ is gauge field (massless), then J_μ is a conserved current so that gauge symmetries in the bulk correspond to global symmetries in the CFT.
- A bulk scalar field is dual to a scalar operator in CFT. The boundary value of acts as a source in CFT.

For more details, we refer the reader to [AGM$^+$00].

9.7 Ryu-Takayanagi formula

The Bekenstein-Hawking entropy encodes the information restored in the interior of the horizon. As explained in §9.1, the entanglement entropy can be used when the observer cannot access to a certain subsystem like the interior of the horizon. Hence, it is natural to seek the relation between the Bekenstein-Hawking entropy and the entanglement entropy. As the first step to understand the relation, one can also ask how the

AdS$_{d+1}$ space time encodes the entanglement entropy in the subsystem A of a CFT$_d$. The answer to this question is the Ryu-Takayanagi formula [RT06b, RT06a]:

$$S_A = \frac{\text{Area}\,(\gamma_A)}{4G^{(d+1)}}. \tag{9.57}$$

The manifold γ_A is the d-dimensional static submanifold with minimal area in AdS$_{d+1}$ whose boundary is given by ∂A as shown in Figure 26. Its area is denoted by Area(γ_A). $G_N^{(d+1)}$ is the $d+1$ dimensional Newton constant. Since $\sqrt{g}$ diverges at the boundary of AdS space $z = 0$, the area γ_A is divergent there. Thus, the result still has divergent of order $\epsilon^{-(d-2)}$

$$\begin{aligned} S_A &\sim \frac{R^{d-1}}{G}\,\text{Area}\,(\partial A)\int_\epsilon \frac{dz}{z^{d-1}} \\ &\sim \frac{R^{d-1}}{G}\frac{\text{Area}\,(\partial A)}{\epsilon^{d-2}} \end{aligned}.$$

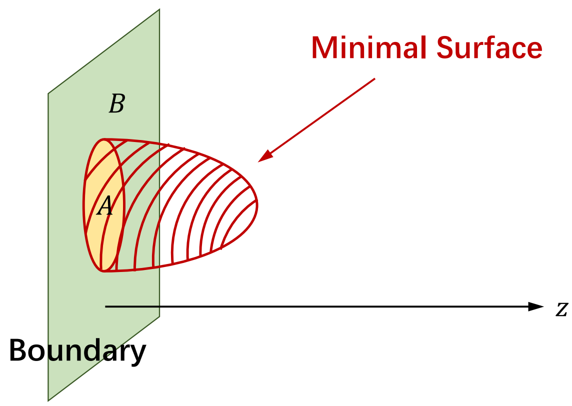


Figure 26: Holographic entanglement entropy

As we will see at the end, when the region A is large enough, the minimal surface (submanifold) wraps the horizon of AdS black hole, and the Ryu-Takayanagi formula can explain the Bekenstein-Hawking entropy as a special case. However, minimal surfaces (submanifolds) are more general so that Ryu-Takayanagi formula can be interpreted as a generalization of the Bekenstein-Hawking entropy formula. Moreover, the entanglement entropy depends on a CFT and it also encodes the quantum entanglement. Therefore, the entanglement entropy in a CFT can be also understood as the quantum correction to the Bekenstein-Hawking entropy in the context of the AdS/CFT correspondence.

Holographic derivation of strong subaddivity

One of the most important properties of the entanglement entropy is the strong subadditivity (9.12). Using the Ryu-Takayanagi formula, one can easily derive strong subadditivity.

Let us start with three regions A, B and C on a spatial slice that do not intersect with each other as shown in Figure 27. We extend this boundary setup toward the bulk AdS. First consider the entanglement entropy S_{A+B} and S_{B+C}. From the Ryu-Takayanagi formula, they are given by the minimal area surfaces γ_{A+B} and γ_{B+C} which satisfy

$\partial\gamma_{A+B} = \partial(A+B)$ and $\partial\gamma_{B+C} = \partial(B+C)$ as before. Then we divide these two minimal surfaces into four pieces and recombine into (i) two surface γ'_B and γ'_{A+B+C} or (ii) two surfaces γ'_A and γ'_C, corresponding to two different ways of the combination. But now γ'_Xs are not the minimal area surface, we have Area$(\gamma'_X) \geq$ Area(γ_X), therefore, we have

$$\begin{aligned}
\text{Area}(\gamma_{A+B}) + \text{Area}(\gamma_{B+C}) &= \text{Area}(\gamma'_B) + \text{Area}(\gamma'_{A+B+C}) \geq \text{Area}(\gamma_B) + \text{Area}(\gamma_{A+B+C}), \\
\text{Area}(\gamma_{A+B}) + \text{Area}(\gamma_{B+C}) &= \text{Area}(\gamma'_A) + \text{Area}(\gamma'_C) \geq \text{Area}(\gamma_A) + \text{Area}(\gamma_C).
\end{aligned}$$

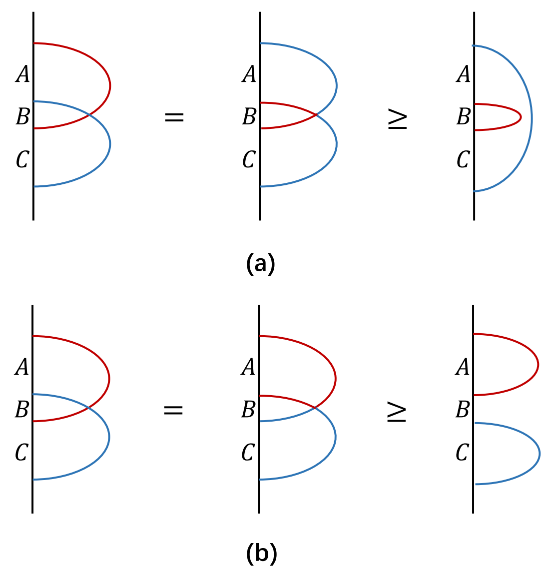


Figure 27: Holographic proof of strong subadditivity

Holographic entanglement entropy

We focus our discussion in AdS_3/CFT_2 case, and the AdS_3 space is defined in the Poincare coordinates.

$$ds^2 = R^2 \left(\frac{dz^2 + dx^2 - dt^2}{z^2} \right).$$

At the fixed time t_0, the whole spatial region of CFT_2 is an infinite line in the x direction. We pick up subsystem A along x direction: $-l/2 \leq x \leq l/2$ with its boundary coordinates given by

$$\begin{aligned}
P : (t, x, z) &= \left(t_0, -\frac{\ell}{2}, \epsilon \right), \\
Q : (t, x, z) &= \left(t_0, \frac{\ell}{2}, \epsilon \right).
\end{aligned}$$

Now Area(γ_A) in (9.57) is the length of geodesic line in the AdS space, with two fixed points P and Q . Therefore, we can apply the Ryu-Takayanagi formula to compute the entanglement entropy for subsystem A

$$S_A = \frac{2R}{4G} \int_{\epsilon}^{\ell/2} \frac{dz}{z\sqrt{1 - 4z^2/\ell^2}} = \frac{R}{2G} \log\left(\frac{\ell}{\epsilon}\right), \tag{9.58}$$

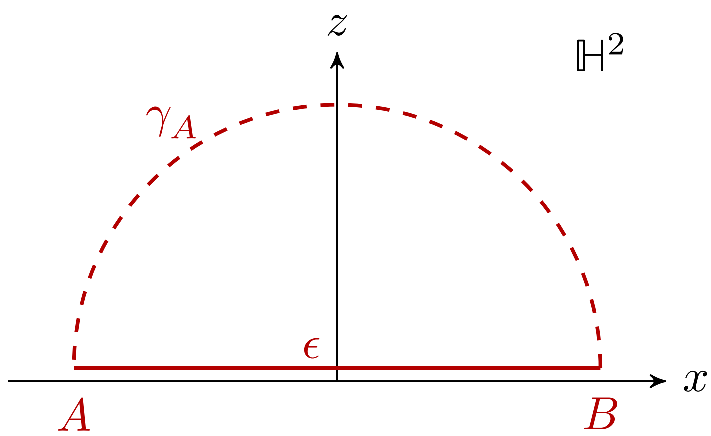


Figure 28: Holographic entanglement entropy in the Poincare coordinates

where we have introduce ϵ as the UV cutoff. Since the AdS_3/CFT_2 correspondence identifies the central charge with the Newton constant

$$c = \frac{3R}{2G},$$

it is equal to (9.39).

Holographic entanglement entropy on a circle

In the global coordinate of AdS_3, the metric of spacetime is written as

$$ds^2 = R^2\left(-\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho d\theta^2\right) \tag{9.59}$$

The CFT_2 is identified with the cylinder (t, θ) at the (regularized) boundary $\rho = \rho_\infty$. The whole spatial region at fixed time is a circle. The subsystem A corresponds to $0 \leq \theta \leq 2\pi\ell/L$. In this case, the minimal surface γ_A is the geodesic line which connects two boundary points at $\theta = 0$ and $\theta = 2\pi l/L$ with t fixed The explicit form of the geodesic AdS_3, expressed in the ambient $\vec{X} \in \mathbb{R}^{2,2}$ space, is

$$\vec{X} = \frac{R}{\sqrt{\alpha^2-1}}\sinh(\lambda/R)\cdot\vec{x} + R\left[\cosh(\lambda/R) - \frac{\alpha}{\sqrt{\alpha^2-1}}\sinh(\lambda/R)\right]\cdot\vec{y},$$

where $\alpha = 1 + 2\sinh^2\rho_0\sin^2(\pi\ell/L)$ and x and y are defined by

$$\begin{aligned}\vec{x} &= (\cosh\rho_\infty\cos t, \cosh\rho_\infty\sin t, \sinh\rho_\infty, 0)\\ \vec{y} &= (\cosh\rho_\infty\cos t, \cosh\rho_\infty\sin t, \sinh\rho_\infty\cos(2\pi\ell/L), \sinh\rho_\infty\sin(2\pi\ell/L)),\end{aligned}$$

The length of geodesic can be found as

$$\cosh\left(\frac{\mathrm{Length}(\gamma_A)}{R}\right) = 1 + 2\sinh^2\rho_\infty\sin^2\frac{\pi\ell}{L}$$

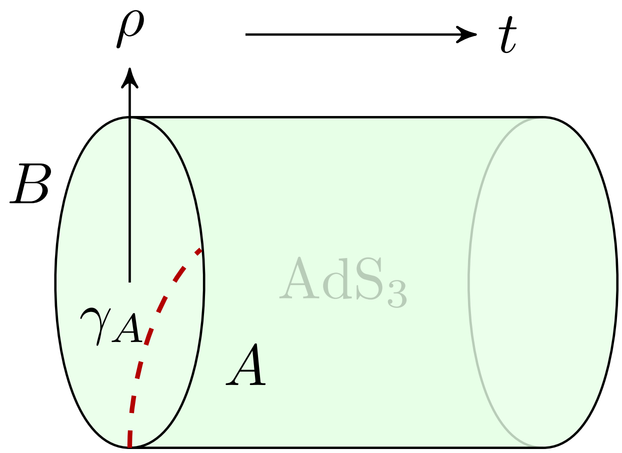


Figure 29: Holographic entanglement entropy on a circle

Assuming that the UV cutoff energy is large $e^{\rho_\infty} \geqq 1$, we can obtain the entropy as follows

$$S_A \simeq \frac{R}{4G} \log\left(e^{2\rho_\infty} \sin^2 \frac{\pi\ell}{L}\right) = \frac{c}{3} \log\left(\frac{L}{\epsilon} \sin \frac{\pi\ell}{L}\right) .$$

This coincides with entanglement entropy in a finite size L (9.42).

BTZ black hole

We have seen that the AdS space has the scaling symmetry (9.54). However, one can also consider an asymptotically AdS space whose dual is a QFT which flows to a CFT at IR $z \to 0$. An typical example of asymptotically AdS spaces is an AdS black hole whose metric can be written as

$$ds^2 = R^2\left(-\frac{f(z)}{z^2}dt^2 + \frac{dz^2}{z^2 f(z)} + \frac{dx^2}{z^2}\right) , \qquad f(z) = 1 - \left(\frac{z}{z_0}\right)^2 ,$$

where the horizon is located at $z = z_0$. It is easy to see that the metric asymptotes to the AdS$_3$ space as $z \to 0$. By the change $r = R/z$ of the coordinates, the metric is written as

$$ds^2 = -(r^2 - r_0^2)dt^2 + \frac{R^2}{r^2 - r_0^2}dr^2 + r^2 dx^2 .$$

We can identify the spatial coordinate $x \sim x + 2\pi$, which is called the **BTZ black hole** and it has temperature

$$T = 1/\beta = \frac{r_0}{2\pi R} .$$

Hence, if we use the Euclidean time $\tau = -it$, the time direction is compactified as $\tau \sim \tau + \beta$. Therefore, the dual CFT$_2$ is living on a torus boundary parametrized by (τ, x) with periodicities $\tau \sim \tau + \beta$ and $x \sim x + 2\pi$.

Let us consider the Ryu-Takayanagi entropy for the subsystem A given by $0 \leq x \leq 2\pi\ell/R$ at the boundary. To evaluate the length of the geodesic from $x = 0$ to $x = 2\pi\ell/R$, one can use the fact that the BTZ metric can be changed to the AdS$_3$ metric (9.59) by

$$r \to r_0 \cosh\rho\,, \quad x \to iRt\,, \qquad \tau \to R\theta\,.$$

Hence, the same analysis above can be applied to compute the length by exchanging $L \leftrightarrow \beta$

$$\cosh\left(\frac{\mathrm{Length}(\gamma_A)}{R}\right) = 1 + 2\cosh^2\rho_\infty \sinh^2\left(\frac{\pi\ell}{\beta}\right)$$

The relation between the cut off ϵ in CFT and the one ρ_∞ of AdS is given by $e^{\rho_\infty} = \frac{\beta}{\epsilon}$ so that it reproduces the CFT result (9.45).

It is also useful to understand these calculations geometrically. The geodesic line in the BTZ black hole takes the form shown in Fig. 30(a). When the size of A is small, it is almost the same as the one in the ordinary AdS$_3$. As the size becomes large, the turning point approaches the horizon and eventually, the geodesic line covers a part of the horizon. This is the reason why we find a thermal extensive behavior of the entropy when $\ell/\beta \gg 1$ in (9.45). The thermal entropy in a conformal field theory is dual to the black hole entropy in its gravity description via the AdS/CFT correspondence. In the presence of a horizon, it is clear that $S_A \neq S_B$ since the corresponding geodesic lines wrap different parts of the horizon (see Fig. 30(b)). As explained below , this is expected for a mixed state like a black hole.

We further expect that when A becomes very large before it coincides with the total system, γ_A becomes separated into the horizon circle and a small half circle localized on the boundary (see Fig. 30(c))

$$S_B - S_A = S_{\mathrm{BH}}\,.$$

Consequently, when A is the entire boundary, the Ryu-Takayanagi formula (9.57) reproduces the Bekenstein-Hawking entropy (9.49).

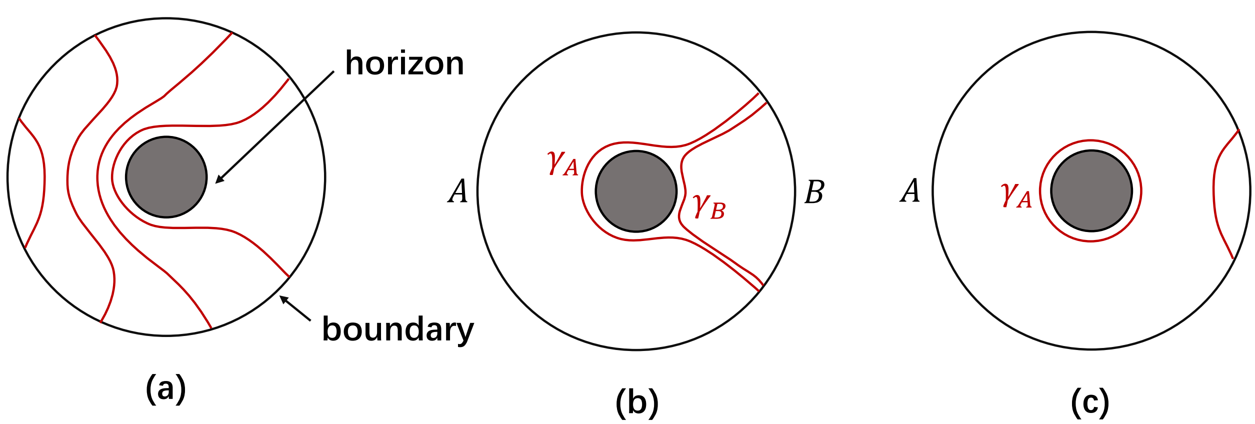


Figure 30: (a) Minimal surfaces γ_A in the BTZ black hole for various sizes of A. (b) γ_A and γ_B wrap the different parts of the horizon. (c) When ∂A gets larger, γ_A is separated into two parts: one is wrapped on the horizon and the other localized near the boundary.

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