

Multiplicity-free quantum 6j-symbols for $U_q(sl_N)$

Refs) Kirillov, Reshetikhin : Representation algebra $U_q(sl_2)$,
q-orthogonal polynomials and
invariants of links (1988)

Nanayakkara, Ramadevi, Zodinmawia : Multiplicity-free quantum 6j-symbols
for $U_q(sl_N)$ arXiv:1302.5143

Plan

1. Review of A_1 -case

1.1 q-Clebsch-Gordan for $U_q(sl_2)$

1.2 q-6j symbols for $U_q(sl_2)$

2. Multiplicity-free q-6j for $U_q(sl_N)$

1. A_1 -case.

$$U_q(sl_2) \quad K E K^{-1} = q^{\frac{1}{2}} E$$

For usual convention

$$K F K^{-1} = q^{-\frac{1}{2}} F$$

$$q \rightarrow q^2$$

$$[E, F] = \frac{K - K}{q^{x_2} - q^{-x_2}}$$

Sph-j representation

$$\pi_j(E) e_m^j = ([j-m][j+m+1])^{\frac{1}{2}} e_{m+1}^j$$

$$\pi_j(F) e_m^j = ([j+m][j-m+1])^{\frac{1}{2}} e_{m-1}^j$$

$$\pi_j(K) e_m^j = q^m e_m^j$$

1.1 quantum Clebsch-Gordan Coefficient

decomposition of tensor product into irred. rep's

$$V^{j_1} \otimes V^{j_2} = \bigoplus V^j$$

j satisfy quantum CG condition

$$|j_1 - j_2| \leq j \leq j_1 + j_2$$

$$2j \equiv 2j_1 + 2j_2 \pmod{2}$$

$$e_m^j(j_1, j_2) = \sum_{m_1, m_2} \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix} e_{m_1}^{j_1} \otimes e_{m_2}^{j_2}$$

There are many expressions for q -CG coefficients

- Van der Waerden

$$\begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix}$$

$$= \delta_{m_1+m_2, m} \Delta(j_1, j_2, j)$$

$$\times q^{\frac{1}{2}(j_1+j_2-j)(j_1+j_2+j+1)}$$

$$\boxed{\begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix} = \begin{array}{c} j_1, m_1 \\ \diagdown \\ j_2, m_2 \\ \diagup \\ j, m \end{array}}.$$

graphical rep of q -CG coefficient

$$\times \left\{ [j_1+m]! [j_1-m_1]! [j_2+m_2]! [j_2-m_2]! [j+m]! [j-m]! [2j+1]! \right\}^{\frac{1}{2}}$$

$$\sum_{r \geq 0} (-1)^r q^{-\frac{1}{2}r(j_1+j_2+j+1)}$$

$$\times \frac{1}{[r]! [j_1+j_2+j-r]! [j_1-m_1-r]! [j_2+m_2-r]! [j-j_2+m_1+r]! [j-j_1+r-m_2]!}$$

$$= (-1)^x \frac{\{ \rho(x) \}^{\frac{1}{2}}}{d_n} Q_n(q^{-x}; a, b, N | q)$$

$$Q_n(x; a, b, N | q) = {}_3\psi_2 \left(\begin{matrix} q^{-n}, abq^{n+1}, x \\ aq, q^{-N}, q \end{matrix}; q, q \right)$$

Hahn poly

$$\rho(x) := \rho(a, b, N | q) = \frac{(aq; q)_x (bq; q)_x}{(z; q)_x (q; q)_{N-x}}$$

$$d_n = \frac{(abq^2; q)_n (a q)^n}{(q; q)_n} \frac{(1-abq)(q, b q, abq^{n+2}; q)_n}{(1-abq^{2n+1})(aq, abq, q^{-n}; q)_n} \\ \times (-a q) q^{\binom{n}{2} - N n}$$

$$n = j - m$$

$$a = q^{j_2 - j + m_1}$$

$$x = j_1 - m_1$$

$$b = q^{j_1 - j_2 + m}$$

$$N = j + j_2 - m_1$$

Universal R-matrix.

$$R = q^\pm K \otimes K \sum_{n \geq 0} \frac{(1-q^{-1})^n}{[n]!} q^{-\frac{n(n-1)}{4}} E^n \otimes F^n$$

Let us consider $R^{j_1 j_2} = (\pi_{j_1} \otimes \pi_{j_2}) R$ acting on $V_{j_1} \otimes V_{j_2}$

$$(R^{j_1 j_2})_{n_1, n_2}^{n_1, n_2} = \frac{(1-q^{-1})^n}{[n]!} q^{\frac{n_1 - n_2 + 2n}{2}} \cdot \\ \left[\frac{[j_1 - n_1]! [j_1 + n_1 + n]! [j_2 + n_2]! [j_2 - n_2 + n]!}{[j_1 - n_1 - n]! [j_1 + n_1]! [j_2 + n_2 - n]! [j_2 - n_2]!} \right]^{\frac{1}{2}}$$

$$= \sum_{j \in j_1 \otimes j_2} q^{\frac{1}{2} [j_1(j_1 + 1) - j_1(j_1 + 1) - j_2(j_2 + 1)]} \begin{bmatrix} j_1 & j_2 & j \\ n_1 & n_2 & m \end{bmatrix} \begin{bmatrix} j_1 & j_2 & j \\ n_1 + n & n_2 - n & m \end{bmatrix}$$

Orthogonality of q -CG coefficient

$$\sum_{m_1, m_2} \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix} \begin{bmatrix} j_1 & j_2 & j \\ m_1' & m_2' & m \end{bmatrix} = \delta_{j_1 j_1'} \delta_{m_1 m_1'} \delta_{m_2 m_2'}$$

$$\sum_{j, m} \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix} \begin{bmatrix} j_1 & j_2 & j \\ m_1' & m_2' & m \end{bmatrix} = \delta_{m_1 m_1'} \delta_{m_2 m_2'}$$

1.2. quantum 6j-symbols.

$$V_{j_1} \otimes V_{j_2} \otimes V_{j_3} \xrightarrow{\quad} (V_{j_1} \otimes V_{j_2})_{j_{12}} \otimes V_{j_3}$$

$$V_{j_1} \otimes (V_{j_2} \otimes V_{j_3})_{j_{23}}$$

$$= \begin{bmatrix} j_1 & j_2 & j_3 \\ m_{12} & m_3 & m_4 \end{bmatrix} \begin{bmatrix} j_1 & j_2 & j_{12} \\ m_1 & m_2 & m_{12} \end{bmatrix}$$

$$= \begin{bmatrix} j_1 & j_{23} & j_4 \\ m_1 & m_{23} & m_4 \end{bmatrix} \begin{bmatrix} j_2 & j_3 & j_{23} \\ m_2 & m_3 & m_{23} \end{bmatrix}$$

quantum 6j-symbols appear as transformation matrices
(independent of m_i).

$$= \sum_{j_{23}} \epsilon \left([2j_{12}+1] [2j_{23}+1] \right)^{\frac{1}{2}} \left\{ \begin{array}{c} j_1 \ j_2 \ j_{12} \\ j_3 \ j_4 \ j_{23} \end{array} \right\}$$

$$\begin{bmatrix} j_1 & j_2 & j_3 \\ m_{12} & m_3 & m_4 \end{bmatrix} \begin{bmatrix} j_1 & j_2 & j_{12} \\ m_1 & m_2 & m_{12} \end{bmatrix}$$

$$= \sum_{j_{23}} \left\{ \begin{array}{c} j_1 \ j_2 \ j_{12} \\ j_3 \ j_4 \ j_{23} \end{array} \right\} \begin{bmatrix} j_1 & j_{23} & j_4 \\ m_1 & m_{23} & m_4 \end{bmatrix} \begin{bmatrix} j_2 & j_3 & j_{23} \\ m_2 & m_3 & m_{23} \end{bmatrix}$$

Using orthogonality of q -CG

$$\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{23} \end{Bmatrix} = \begin{Bmatrix} j_1 & j_{23} & j_4 \\ m_1 & m_{23} & m_4 \end{Bmatrix}^{-1} \sum_{\substack{m_2, m_3 \\ m_2+m_3 \\ = m_4-m_1}} \begin{Bmatrix} j_2 & j_3 & j_{23} \\ m_2 & m_3 & m_{23} \end{Bmatrix} \begin{Bmatrix} j_{12} & j_3 & j_4 \\ m_{12} & m_3 & m_4 \end{Bmatrix} \begin{Bmatrix} j_1 & j_2 & j_{12} \\ m_1 & m_2 & m_{12} \end{Bmatrix}$$

$m_4 = j_4$ gauge fixing \Rightarrow summation over 4 variables.

First extend to summation over 6 variables.

Then reduces to summation over 1 variables.
by using q -Chu-Vandermonde identity.

$$\begin{Bmatrix} m+n \\ k \end{Bmatrix} = \sum_{i+j=k} \begin{Bmatrix} m \\ i \end{Bmatrix} \begin{Bmatrix} n \\ j \end{Bmatrix} q^{(m-i)j}$$

$$\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{23} \end{Bmatrix} = \Delta(j_1 j_2 j_{12}) \Delta(j_3 j_4 j_{23}) \Delta(j_1 j_4 j_{23}) \Delta(j_2 j_3 j_{23})$$

$$\sum_z (-1)^z \frac{[z+1]!}{[z-j_1-j_2-j_{12}]! [z-j_1-j_4-j_{23}]! [z-j_2-j_3-j_{12}]! [z-j_3-j_4-j_{23}]!}$$

$$= \frac{1}{[j_1+j_2-j_{12}]! [j_1+j_4-j_{23}]! [j_3+j_4-j_{12}]! [j_2+j_3-j_{23}]! [j_{12}+j_{23}-j_2-j_4]! [j_{12}+j_{23}-j_1-j_3]!} \times$$

$$\frac{(-1)^{j_1+j_2+j_3+j_4} [j_1+j_2+j_3+1]!}{\Delta \Delta \Delta \Delta} \Psi_3 \left(\begin{array}{c} q^{-j_1-j_2+j_4}, q^{-j_1-j_4+j_{23}}, q^{-j_4-j_3+j_{12}}, q^{-j_2-j_3-j_{12}} \\ q^{-j_1-j_2-j_3-j_4+1}, q^{j_1+j_2+j_{23}-j_2-j_4+1}, q^{j_1+j_2+j_{23}-j_1-j_3+1} \end{array}; \frac{q}{f}, \frac{q}{f} \right)$$

The properties of $g\text{-}6j$.

$$1. \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{23} \end{Bmatrix} = \begin{Bmatrix} j_2 & j_1 & j_{12} \\ j_4 & j_3 & j_{23} \end{Bmatrix} = \begin{Bmatrix} j_1 & j_4 & j_{23} \\ j_3 & j_2 & j_{12} \end{Bmatrix} = \begin{Bmatrix} j_3 & j_4 & j_{12} \\ j_1 & j_2 & j_{23} \end{Bmatrix}$$

$$2. \begin{Bmatrix} j_1 & j_2 & 0 \\ j_3 & j_4 & j_{23} \end{Bmatrix} = \frac{\epsilon \delta_{j_1 j_2} \delta_{j_3 j_4}}{([2j_2+1][2j_3+1])^{\frac{1}{2}}}$$

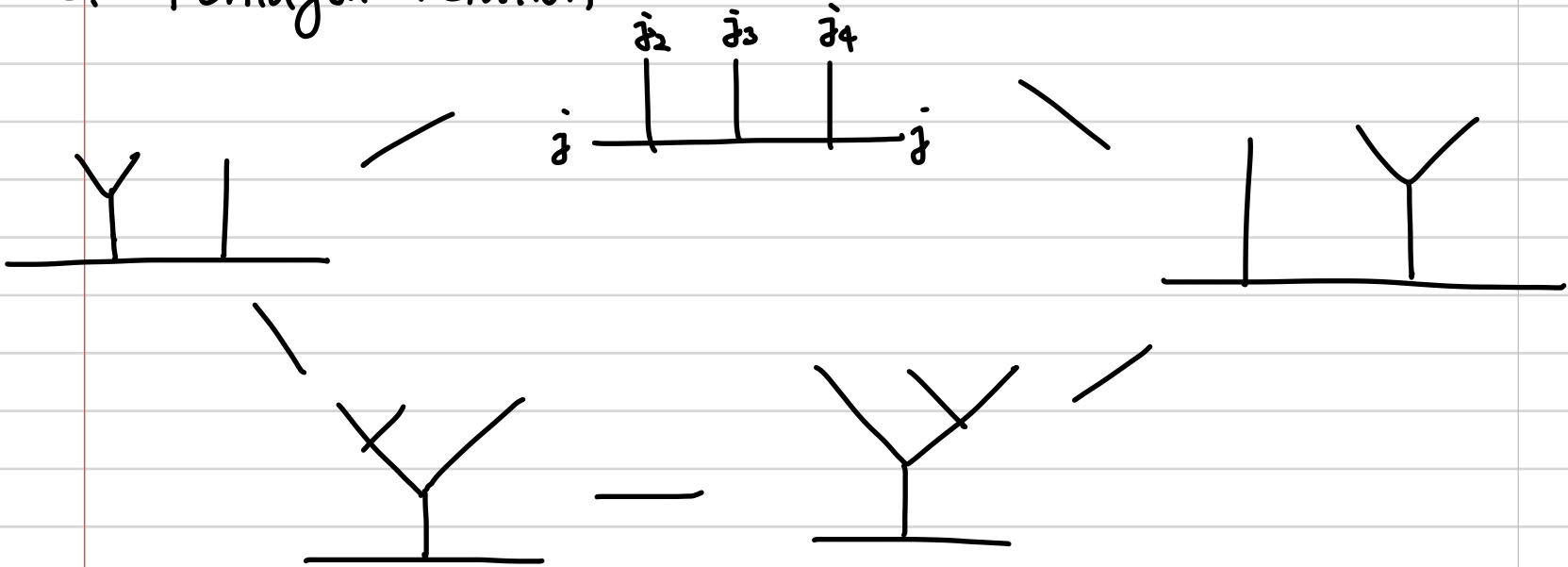
$$3. \sum_x [2j_{12}+1][2j_{23}+1] \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{23} \end{Bmatrix} \begin{Bmatrix} j_1 & j_2 & j'_{12} \\ j_3 & j_4 & j'_{23} \end{Bmatrix} = \delta_{j_{23} j'^{23}}$$

orthogonality.

$$4. \sum_{j_{12}} \in [2x+1] f^{C_{12}/2} \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{23} \end{Bmatrix} \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_4 & j_3 & j_{24} \end{Bmatrix}$$

$$= \begin{Bmatrix} j_3 & j_2 & j_{23} \\ j_4 & j_1 & j_{24} \end{Bmatrix} f^{\frac{C_{24}+C_{23}'}{2} - \frac{C_1+C_2+C_3+C_4}{2}}$$

5. Pentagon relation



② Multiplicity-free f-6j for $V_{\mathbb{Q}}(\mathfrak{sl}_N)$.

For higher ranks, j is replaced by highest weight λ

$$\begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_R \\ \lambda_3 & \lambda_4 & \lambda_{34} \end{Bmatrix}$$

λ_{12} & λ_{34} have to satisfy fusion rules

$$V_{\lambda_{12}} \in (V_{\lambda_1} \otimes V_{\lambda_2}) \cap (V_{\lambda_3^*} \otimes V_{\lambda_4^*})$$

λ^* : conjugate rep of λ

$$V_{\lambda_{34}} \in (V_{\lambda_2} \otimes V_{\lambda_3}) \cap (V_{\lambda_1^*} \otimes V_{\lambda_4^*})$$

$\Rightarrow \begin{Bmatrix} \square & \square & \square \\ \square & \square & \square \end{Bmatrix}$ is not allowed for $N > 2$ for example.

The symmetry of f-6j of \mathfrak{sl}_3

$$\begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_{12} \\ \lambda_3 & \lambda_4 & \lambda_{23} \end{Bmatrix} = \begin{Bmatrix} \lambda_3 & \lambda_1 & \lambda_{23} \\ \lambda_1 & \lambda_4 & \lambda_{12} \end{Bmatrix} = \begin{Bmatrix} \lambda_1^* & \lambda_2^* & \lambda_{12}^* \\ \lambda_3^* & \lambda_4^* & \lambda_{23}^* \end{Bmatrix} \quad (*)$$

$$= \begin{Bmatrix} \lambda_1 & \lambda_4 & \lambda_{23}^* \\ \lambda_3 & \lambda_2 & \lambda_{12}^* \end{Bmatrix} = \begin{Bmatrix} \lambda_2 & \lambda_1 & \lambda_{12} \\ \lambda_4 & \lambda_3 & \lambda_{23}^* \end{Bmatrix}$$

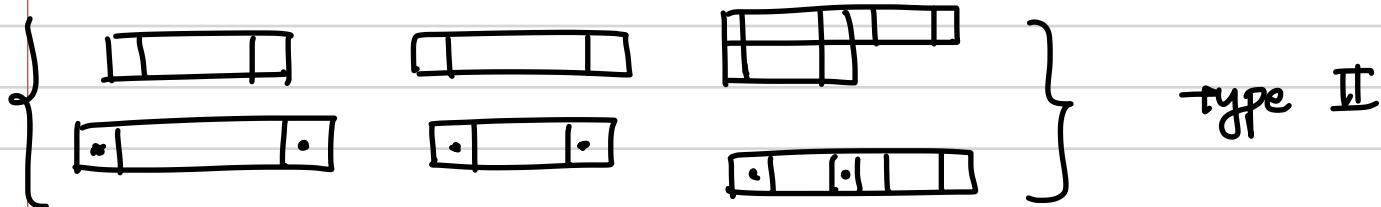
The difficulty of higher ranks stems from multiplicity structure in tensor product.

$$\begin{array}{c} \boxed{\square} \otimes \boxed{\square} = \boxed{\square} \oplus \boxed{\square\square\square} \oplus \boxed{\square\square\square} \oplus \boxed{\square} \\ \oplus \boxed{\square\square\square} \oplus \boxed{\square} \oplus \boxed{\square\square\square} \oplus \boxed{\square} \end{array}$$

Using symmetry (\times), multiplicity-free f-6j amounts to the following two types.

$$\left\{ \begin{array}{c} \boxed{\square}, \quad \boxed{\cdot}, \quad \boxed{\cdot\cdot\cdot}, \\ \boxed{\square}, \quad \boxed{\cdot}, \quad \boxed{\cdot\cdot\cdot\cdot} \end{array} \right\} \text{ type I.}$$

$$\boxed{\square} = \begin{array}{c} \uparrow \\ \vdots \\ N-1 \\ \downarrow \end{array}$$



Using properties of g - $6j$ symbols, we read off the pattern.

$$\left\{ \begin{matrix} \lambda_1 & \lambda_2 & \lambda_{12} \\ \lambda_3 & \lambda_4 & \lambda_{23} \end{matrix} \right\}$$

$$= \Delta(\lambda_1, \lambda_2, \lambda_{12}) \Delta(\lambda_3, \lambda_4, \lambda_{12}) \Delta(\lambda_2, \lambda_3, \lambda_{23}) \Delta(\lambda_1, \lambda_4, \lambda_{23})$$

$$[N-1]! \cdot \sum_z (-1)^z [z+N-1]! C_z(\{\lambda_i\}, \lambda_{12}, \lambda_{23})$$

$$\left\{ [z - \frac{1}{2}\langle \lambda_1 + \lambda_2 + \lambda_{12}, \alpha_1^\vee + \alpha_{N-1}^\vee \rangle]! [z - \frac{1}{2}\langle \lambda_3 + \lambda_4 + \lambda_{12}, \alpha_1^\vee + \alpha_{N-1}^\vee \rangle]! \right.$$

$$\left. [z - \frac{1}{2}\langle \lambda_3 + \lambda_2 + \lambda_{23}, \alpha_1^\vee + \alpha_{N-1}^\vee \rangle]! [z - \frac{1}{2}\langle \lambda_1 + \lambda_4 + \lambda_{23}, \alpha_1^\vee + \alpha_{N-1}^\vee \rangle]! \right.$$

$$[\frac{1}{2}\langle \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4, \alpha_1^\vee + \alpha_{N-1}^\vee \rangle - z]! [\frac{1}{2}\langle \lambda_1 + \lambda_2 + \lambda_{12} + \lambda_{23}, \alpha_1^\vee + \alpha_{N-1}^\vee \rangle - z]!$$

$$[\frac{1}{2}\langle \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4, \alpha_1^\vee + \alpha_{N-1}^\vee \rangle - z]! [\frac{1}{2}\langle \lambda_1 + \lambda_2 + \lambda_{12} + \lambda_{23}, \alpha_1^\vee + \alpha_{N-1}^\vee \rangle - z]! \}$$

$$\Delta(\lambda_1, \lambda_2, \lambda_3) =$$

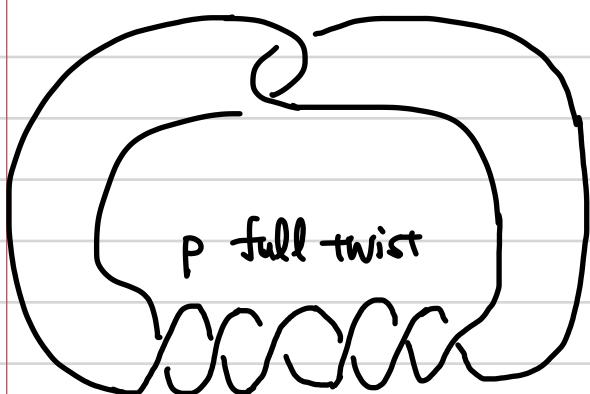
$$\frac{[\frac{1}{2}\langle -\lambda_1 + \lambda_2 + \lambda_3, \alpha_1 + \alpha_{N-1} \rangle]! [\frac{1}{2}\langle \lambda_1 - \lambda_2 + \lambda_3, \alpha_1 + \alpha_{N-1} \rangle]! [\frac{1}{2}\langle \lambda_1 + \lambda_2 - \lambda_3, \alpha_1 + \alpha_{N-1} \rangle]!}{[\frac{1}{2}\langle \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4, \alpha_1^\vee + \alpha_{N-1}^\vee \rangle + N-1]!}$$

$$[\frac{1}{2}\langle \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4, \alpha_1^\vee + \alpha_{N-1}^\vee \rangle + N-1]!$$

C_z depends on type I & II. Details can be found in

arXiv:1302.5143.

Check has been done.



twist knot k_p

Using the expression of multiplicity-free $q\text{-}6j$.

One can obtain colored HOMFLY polynomials.

These are computed by linear Skein theory (mathematically rigorous)

P	-2	-1	0	1	2
k_p	6_1	4_1	0_1	3_1	5_2

Kawagoe arXiv:1210.7574

$$P_{\underbrace{\text{---}}_r} (k_p; a, q) = \sum_{k=0}^{\infty} \sum_{l=0}^k q^k \frac{(aq^{-1}; q)_k}{(q; q)_k} (q^{-r}; q)_k (aq^r; q)_r \\ \times (-1)^l a^{pl} q^{(p+\frac{1}{2})l(l-1)} \frac{1 - aq^{2l-1}}{(aq^{l-1}; q)_{k+1}} \begin{bmatrix} k \\ l \end{bmatrix}_q$$

We check multiplicity-free $q\text{-}6j$ reproduce this formula up to $r=4$.

Minor Symmetry ($\lambda = \boxed{\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}} \leftrightarrow \lambda^T = \boxed{\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}}$) of $g\text{-}6j$.

There is a relation b/w symmetric rep & anti-symmetric rep.

$$\left\{ \begin{array}{c} \boxed{\square} \quad \boxed{\square} \quad \boxed{\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}} \\ \downarrow \begin{smallmatrix} N-2 \\ \vdots \\ N-1 \end{smallmatrix} \quad \downarrow \begin{smallmatrix} N-1 \\ \vdots \\ N-2 \end{smallmatrix} \quad \downarrow \begin{smallmatrix} N-2 \\ \vdots \\ N-1 \end{smallmatrix} \end{array} \right\} = \frac{[2]^2}{[N+1][N][N-1][N-2]} \frac{[3][N-3][N-4] - [N][N+1]}{[N] + [3][N-4]}$$

$\uparrow \quad \downarrow$

$$\left\{ \begin{array}{c} \boxed{\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}} \quad \boxed{\begin{smallmatrix} \square & \square & \square \\ \square & \square & \square \end{smallmatrix}} \quad \boxed{\begin{smallmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{smallmatrix}} \\ \downarrow \begin{smallmatrix} \square \\ \square \end{smallmatrix} \quad \downarrow \begin{smallmatrix} \square \cdot \square \cdot \square \\ \square \end{smallmatrix} \quad \downarrow \begin{smallmatrix} \square \cdot \square \cdot \square \\ \square \end{smallmatrix} \end{array} \right\} = \frac{[2]^2}{[N-1][N][N+1][N+2]^2} \frac{[3][N+3][N+4] - [N][N-1]}{[N] + [3][N+4]}.$$

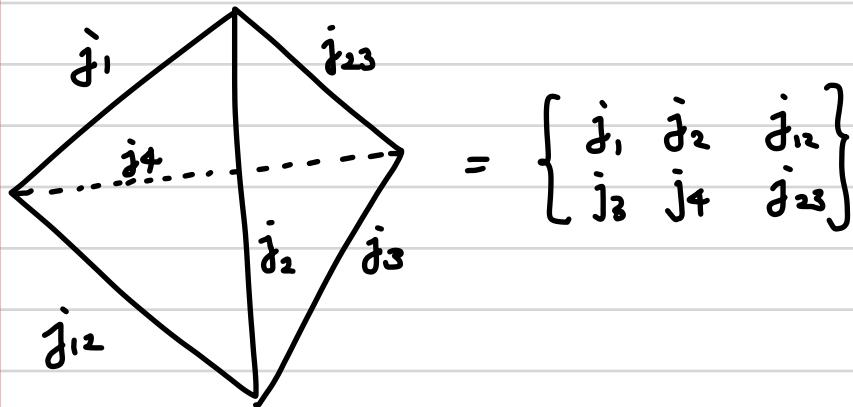
This explains the mirror symmetry of colored HOMFLY polynomial of a link L .

$$P_{\boxed{\dots}}(L; a, g) = P_{\boxed{\vdots}}(L; a, g')$$

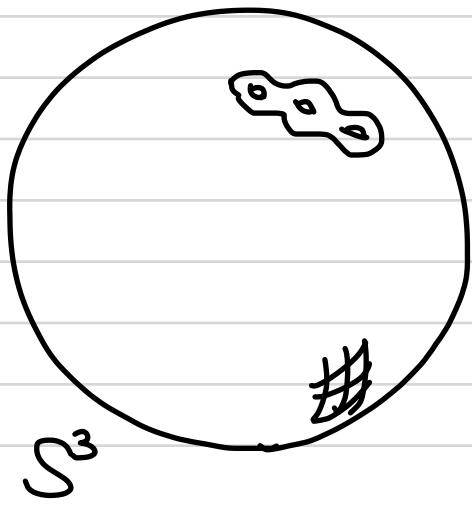
Using the mirror symmetry, one can obtain multiplicity-free $g\text{-}6j$ involving anti-symmetric reps.

Discussion

$g\text{-}6j$ can be associated to a tetrahedron with colored edges.



Volume conjecture tells us the large color behavior of $g\text{-}6j$ will describe moduli space of $SL(2, \mathbb{C})$ flat connection of complement of genus-3 handlebody in S^3 .



\Rightarrow First example of higher-dim character varieties.

\Rightarrow recursion of $g\text{-}6j$ provides quantization of character variety.

Work in progress!