Homework 3: Due at class on March 26

1 Schwarzian derivative

Under a holomorphic transformation $z \to w(z)$, the stress-energy tensor is indeed transformed as

$$\widetilde{T}(w) = \left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^{-2}\left[T(z) - \frac{c}{12}\{w;z\}\right]$$

where $\{w;z\}$ is the additional term called the Schwarzian derivative:

$$\{w;z\} = \frac{(\mathrm{d}^3w/\mathrm{d}z^3)}{\mathrm{d}w/\mathrm{d}z} - \frac{3}{2}\left(\frac{\mathrm{d}^2w/\mathrm{d}z^2}{\mathrm{d}w/\mathrm{d}z}\right)^2$$

1.1

Show that its infinitesimal version provides conformal Ward identity.

1.2 Under $\mathrm{SL}(2,\mathbb{C})$

For an element of $\mathrm{SL}(2,\mathbb{C})$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C})$$

show that

$$\{w;z\} = 0 \quad \text{for } w = \frac{az+b}{cz+d} \,,$$

and

$$\left\{\frac{aw+b}{cw+d};z\right\} = \{w;z\} \,.$$

1.3 Free boson

The energy-momentum tensor of the free boson is

$$T(z) = -\frac{1}{2} :\partial_z X \partial_z X:$$

where the normal ordering can be defined as

$$:\partial_z X \partial_z X: := \lim_{w\to z}\left(\partial_z X(z)\partial_w X(w) + \frac{1}{(z-w)^2}\right) \,.$$

Since $\partial_z X$ is the primary field of conformal dimension one, it transforms as

$$\partial_z X(z)\partial_w X(w) = f'(z)f'(w)\partial_{\widetilde{z}} X(\widetilde{z})\partial_{\widetilde{w}} X(\widetilde{w})$$

under the conformal transformation $z \to \widetilde{z} = f(z)$. Hence we have

$$: \partial_z X(z) \partial_w X(w) : -\frac{1}{(z-w)^2} = f'(z) f'(w) \left[: \partial_{\widetilde{z}} X(\widetilde{z}) \partial_{\widetilde{w}} X(\widetilde{w}) : -\frac{1}{(\widetilde{z}-\widetilde{w})^2} \right]$$

Taking limit $z \to w$, show that

$$\lim_{z \to w} \left[\frac{f'(z) f'(w)}{(f(z) - f(w))^2} - \frac{1}{(z-w)^2} \right] = \frac{1}{6} \{ f(w); w \} .$$

2 2-point and 3-point function of primary fields

2.1 2-point function

Let us determine the form of the 2-point function of chiral primary operators $\phi_i(z_i)$ with weight h_i ($i = 1, 2$). The 2-point function is invariant under the translation $z \to z + a$ of the coordinate so that it is a function $g(z_1 - z_2)$ of their relative coordinate $z_1 - z_2$.

Using the property of chiral primary fields under the scaling $z \to \lambda z$, show that the function is of the form

$$g(z_1 - z_2) = \frac{d_{12}}{(z_1 - z_2)^{h_1 + h_2}} .$$

Furthermore, show that h_1 must be equal to h_2 by using the property under the transformations $z \to -1/z$.

2.2 3-point function

The translation invariance tells us that the 3-point function is also a function $g(z_{12}, z_{23}, z_{31})$ where $z_{ij} = z_i - z_j$. Applying the same argument above, derive the form of the 3-point function

$$\langle \phi_1(z_1) \phi_2(z_2) \phi_2(z_3) \rangle = \frac{C_{123}}{(z_{12})^{h_1 + h_2 - h_3} (z_{23})^{h_2 + h_3 - h_1} (z_{31})^{h_3 + h_1 - h_2}} .$$

3 Free fermion

Since we have studied the free scalar theory, now let us study the free fermion theory. The action for a free Majorana fermion reads

$$S = \frac{1}{4\pi} \int d^2 x \overline{\Psi} \gamma^a \partial_a \Psi ,$$

where $\overline{\Psi} = \Psi^\dagger \gamma^0$, and the gamma matrices are given by

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \gamma^1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} .$$

In Euclidean space-time, they satisfy the relation $\{\gamma_i, \gamma_j\} = 2\delta_{ij}$.

- Rewrite the action in terms of $\psi(z,\bar{z}), \overline{\psi}(z,\bar{z})$ where $\Psi = \begin{pmatrix} \psi \\ \overline{\psi} \end{pmatrix}$ where $z = x^0 + ix^1$ and $\bar{z} = x^0 - ix^1$.

- Calculate the equations of motion for $\psi(z,\bar{z}), \overline{\psi}(z,\bar{z})$. Find an explicit expression of the stress-energy tensor. What do they imply?

- The OPE takes the form

$$\psi(z,\bar{z})\psi(w,\overline{w}) = \frac{1}{z-w} + :\psi(z,\bar{z})\psi(w,\overline{w}):$$

  Deduce that $\psi(z,\bar{z})$ is a primary field and find its weight.

- Calculate the OPE $T(z)T(w)$.