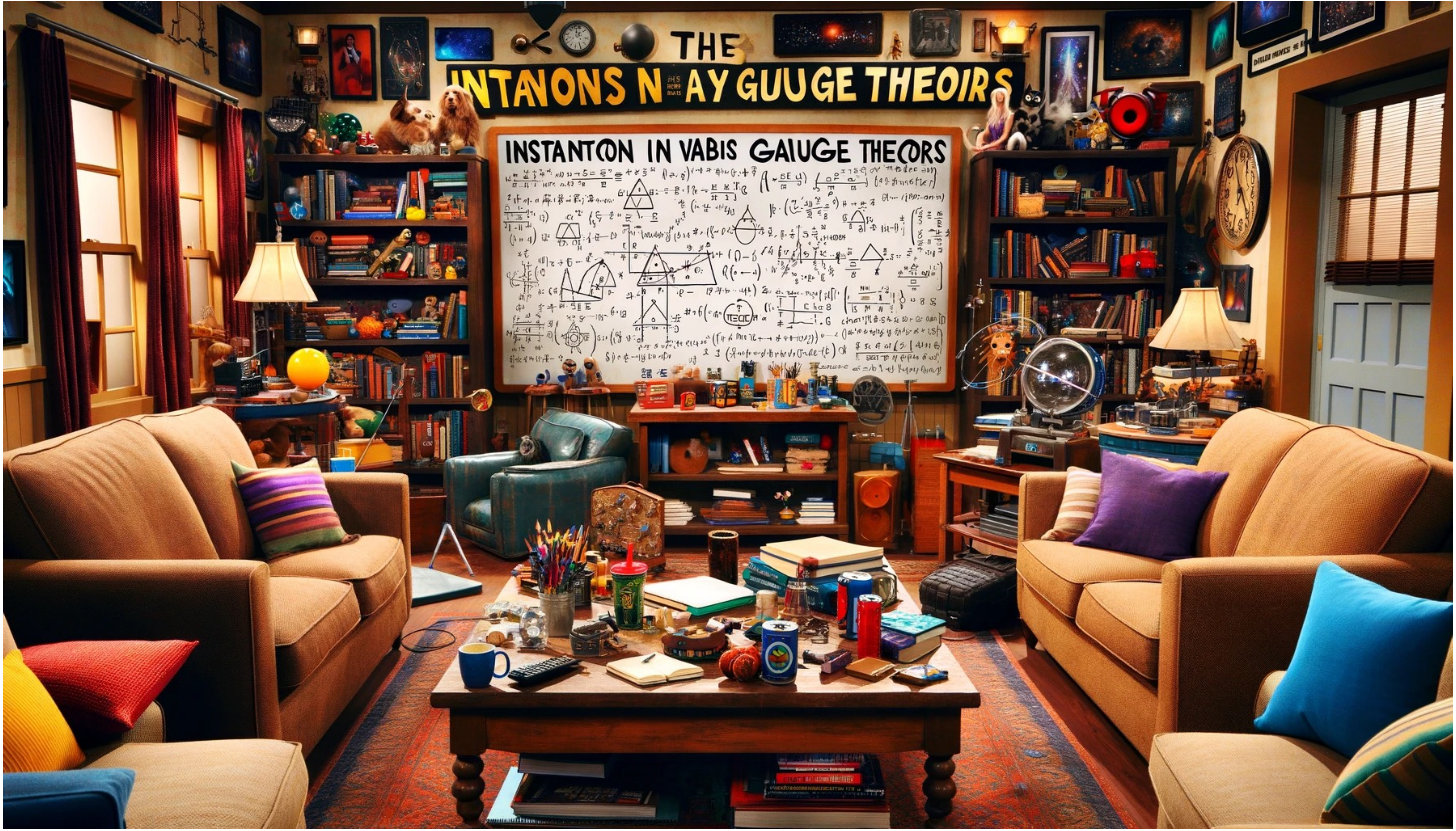
with Rui-Dong Zhu [arXiv:2107.03656]
with Yanyan Chen, Jiaqun Jiang, Yilu Shao [arXiv:2301.02342]
with Kilar Zhang, Rui-Dong Zhu [arXiv:2302.00525]
with Sung-Soo Kim, Xiaobin Li, Futoshi Yagi [arXiv:2403.12525] appeared Today!

Satoshi Nawata, Fudan

Plan of talk

- Instantons in various gauge theories
- qq-characters
- Freezing and BPS jumping



with Rui-Dong Zhu [arXiv:2107.03656]
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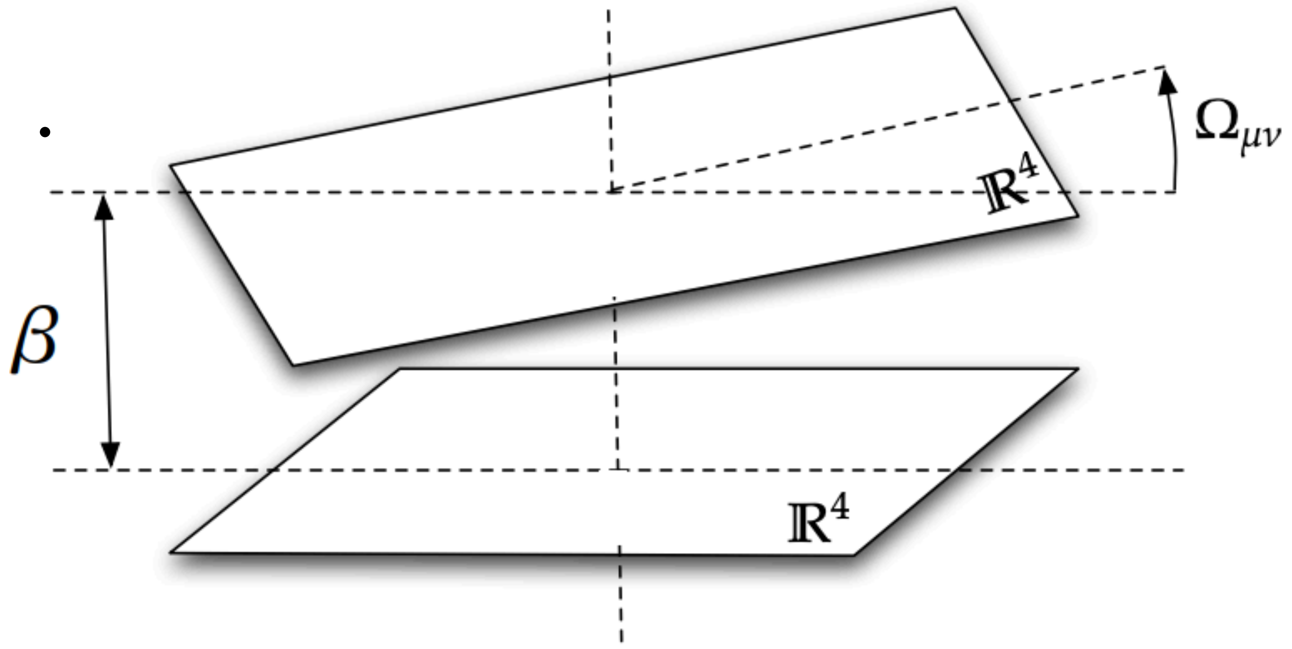
Review of Instanton counting

[Nekrasov]

5d N=1 theory on Ω-background $\longrightarrow$ SQM on instanton moduli space

Instanton partition function can be understood as Witten index

$$Z_{\text{inst}}\left(\epsilon_1,\epsilon_2,a_i,m\right)=\operatorname{Tr}\left[(-1)^{\mathsf{F}}e^{-\beta\{Q,Q^\dagger\}}e^{-\epsilon_1(J_1+J_R)}e^{-\epsilon_2(J_2+J_R)}e^{-a_iC_i}e^{-m\cdot F}\right].$$



$J_{1,2}$ Cartans of SO(4) $\curvearrowright \mathbb{R}^4$

J_R Cartans of $SU(2)_R$

a_i Coulomb branch parameters

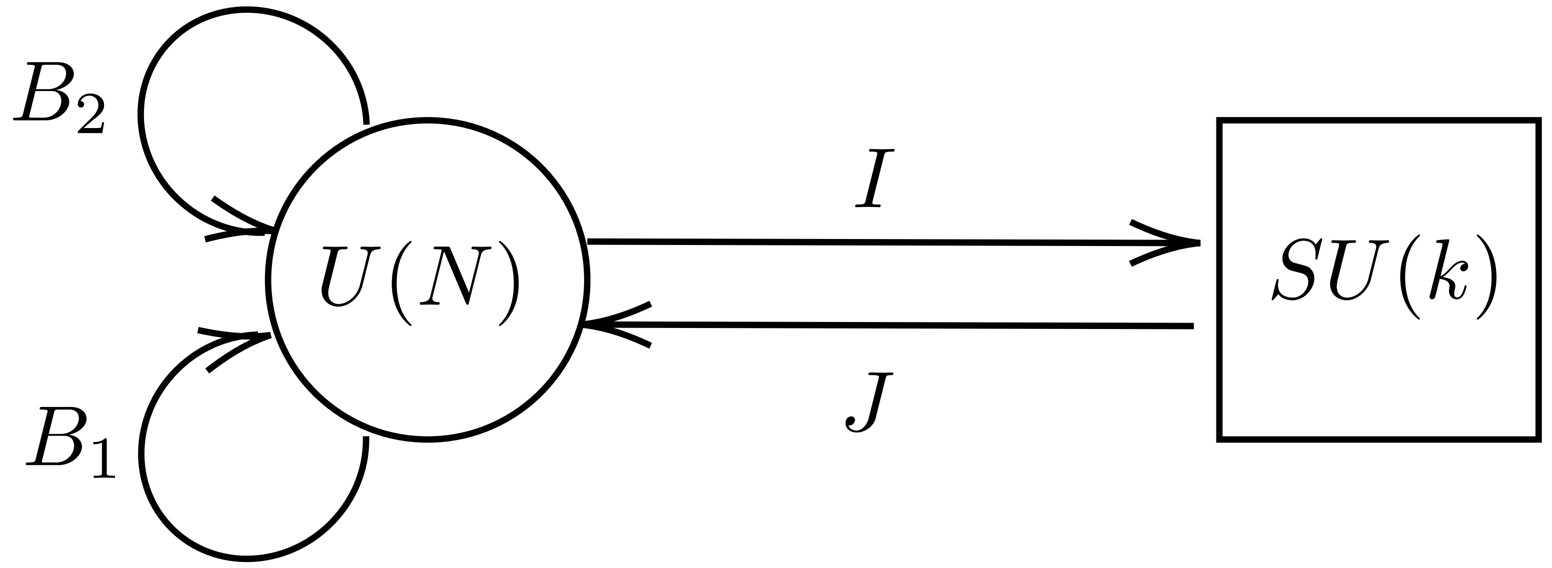
m_i Mass parameters of hypermultiplets

Thermodynamic limit produces prepotential and Seiberg-Witten curve

$$Z_{\text{inst}}\left(\epsilon_1,\epsilon_2,a_i,m\right)\xrightarrow[\epsilon_{1,2}\to 0]{}\exp\left(\frac{\mathcal{F}(a,m)}{\epsilon_1\epsilon_2}+\cdots\right)$$

Review of Instanton counting

[Nekrasov]



Equivariant actions on instanton moduli spaces

$$\mathbb{C}^\times_{\epsilon_1} \times \mathbb{C}^\times_{\epsilon_2} \times \prod_i^{\mathrm{rk}\, G} \mathbb{C}^\times_{a_i} \curvearrowright \mathcal{M}_{\mathrm{inst}}$$

Instanton partition function is the character of the action

$$\begin{aligned} Z_k^{\mathrm{SU}(N)} &= \mathrm{ch}_{\mathbb{C}^\times_{\epsilon_i} \times \mathbb{C}^\times_{a_i}} (\mathcal{M}_{N,k}) \\ &= \frac{1}{N!} \int_{\mathrm{JK}} [\frac{d\phi}{2\pi i}] \frac{\prod_{I \neq J} \mathrm{sh}(\phi_I - \phi_J) \cdot \prod_{I,J} \mathrm{sh}(2\epsilon_+ - \phi_I + \phi_J)}{\prod_{I,J} \mathrm{sh}(\epsilon_{1,2} + \phi_I - \phi_J) \prod_{I=1}^k \prod_{s=1}^N \mathrm{sh}(\epsilon_+ \pm (\phi_I - a_s))} \\ &= \sum_{\sum_s |\lambda^{(s)}| = k} \prod_{s,t=1}^N \prod_{x \in \lambda^{(s)}} \frac{1}{\mathrm{sh}(N_{st})\, \mathrm{sh}(N_{st} - 2\epsilon_+)}, \end{aligned}$$

$$\epsilon_\pm := \frac{\epsilon_1 \pm \epsilon_2}{2}$$

Poles (or fixed points) are classified by N-tuple of Young diagrams

$$N_{st}(x) := a_s - a_t - \epsilon_1 L_{\lambda^{(s)}}(x) + \epsilon_2 (A_{\lambda^{(t)}}(x) + 1) \ .$$

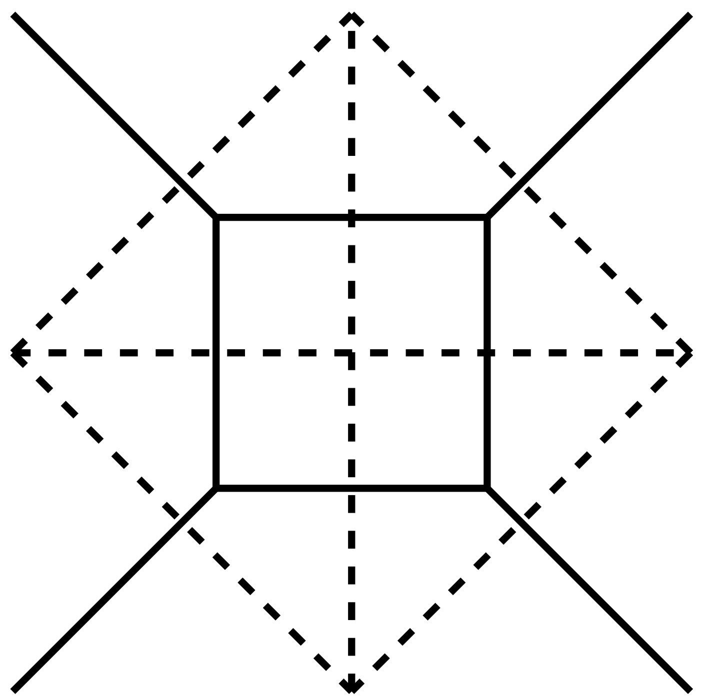
String theory

[Katz-Klemm-Vafa]

Geometric engineering: M-theory on $\underline{S^1 \times \mathbb{C}^2} \times \mathrm{CY3}$

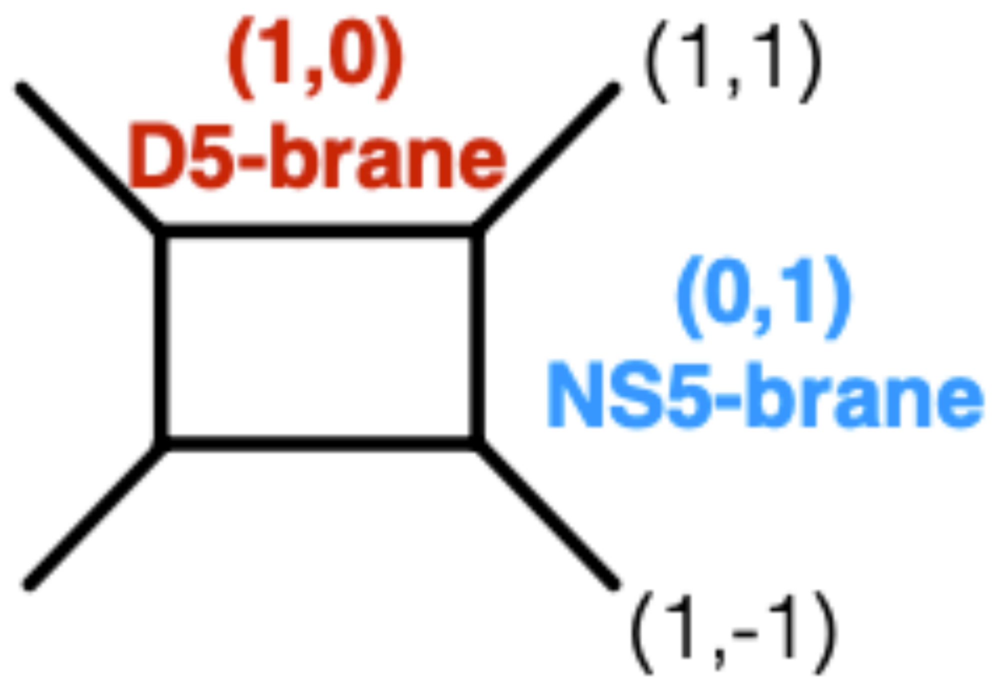
5d N=1 theory

T-dual [Leung-Vafa]

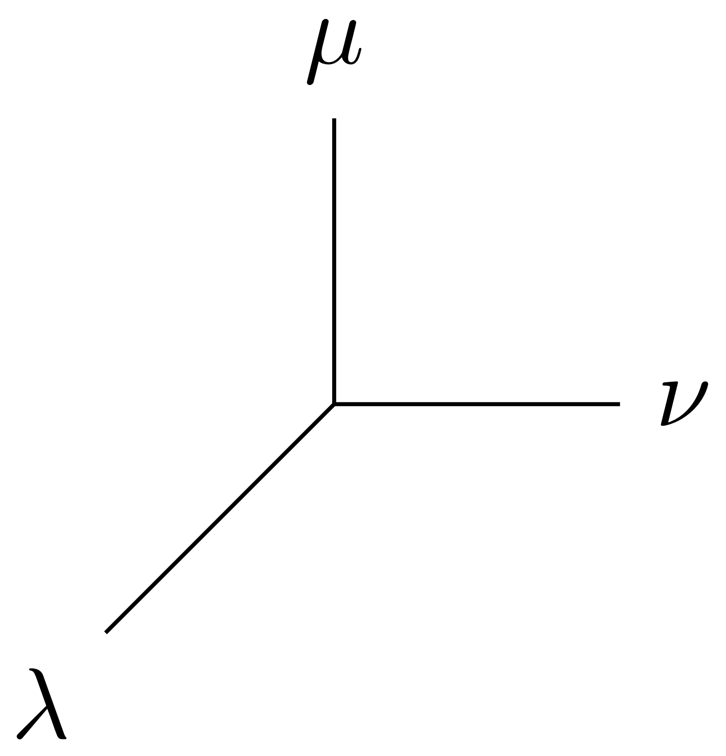


5-brane webs in Type IIB theory

	0	1	2	3	4	5	6	7	8	9
D5/O5	×	×	×	×	×	×				
NS5	×	×	×	×	×		×			
7-brane/O7	×	×	×	×	×			×	×	×



Topological vertex is powerful tool [Aganagic-Klemm-Marino-Vafa]



$$C_{\nu\mu\lambda} \propto s_\lambda(q^{-\rho}) \sum_\sigma s_{\nu^\vee/\sigma}(q^{-\rho-\lambda}) s_{\mu/\sigma}(q^{-\rho-\lambda^\vee})$$

Relation to VOA

[Okouknov-Reshethikin-Vafa]

Topological vertex can be reformulated by free bosons and fermions

$$J_n := \sum_{\alpha \in \mathbb{Z}+1/2} :\psi_{-\alpha}\psi^*_{\alpha+n}:$$

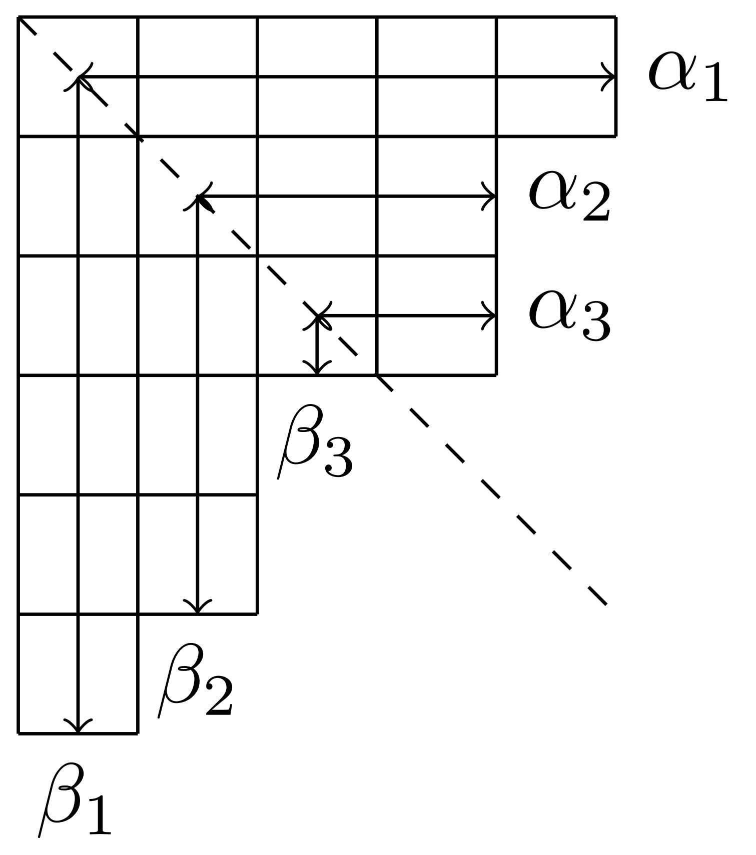
$$\{\psi_\alpha, \psi_\beta\} = \{\psi^*_\alpha, \psi^*_\beta\} = 0, \quad \{\psi_\alpha, \psi^*_\beta\} = \delta_{\alpha+\beta,0},$$

$$[J_n, \psi_\alpha] = \psi_{n+\alpha}, \quad [J_n, \psi^*_\alpha] = -\psi^*_{n+\alpha}, \quad [J_n, J_m] = n\delta_{n+m,0}.$$

We define state corresponding to a Young diagram

$$|\lambda\rangle = \pm\psi^*_{-\beta_1}\psi^*_{-\beta_2}\cdots\psi^*_{-\beta_s}\psi_{-\alpha_s}\psi_{-\alpha_{(s-1)}}\cdots\psi_{-\alpha_1}|0\rangle$$

Frobenius presentation



$$\lambda = (\alpha_1, \alpha_2, \ldots | \beta_1, \beta_2 \ldots)$$

Skew-Schur is expectation value of vertex operator

$$s_{\lambda/\mu}(x) = \langle\mu|V_+(x)|\lambda\rangle = \langle\lambda|V_-(x)|\mu\rangle$$

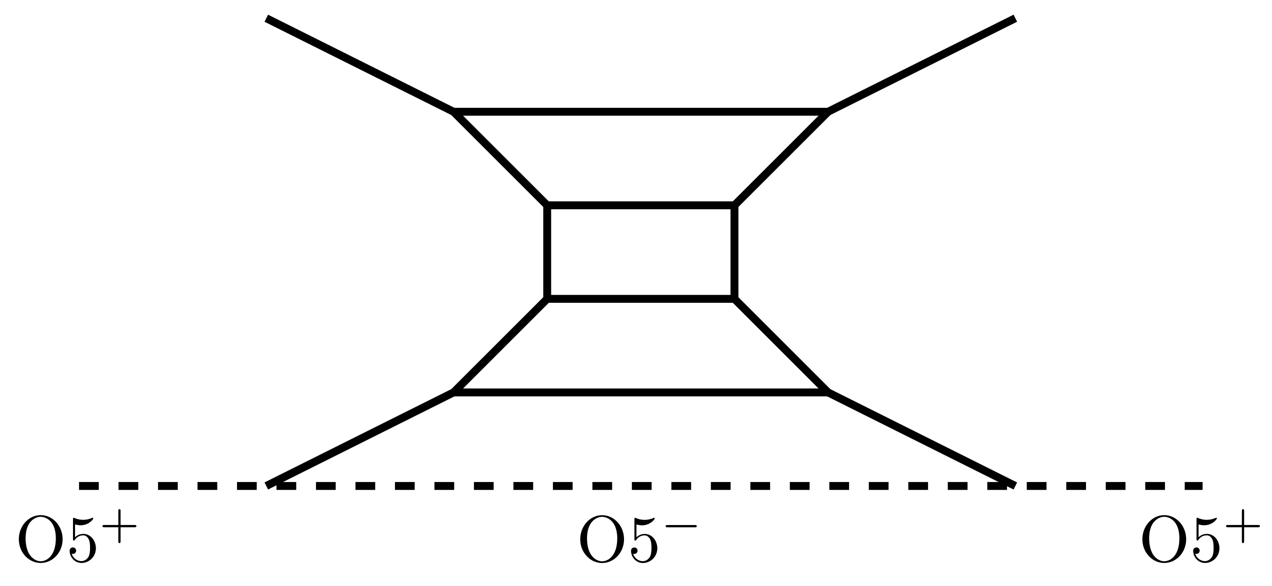
$$V_\pm(x) = \exp\left(\sum_{n=1}^{\infty}\frac{1}{n}x^n J_{\pm n}\right)$$

→ Topological vertex computation is correlation function of $V_\pm$

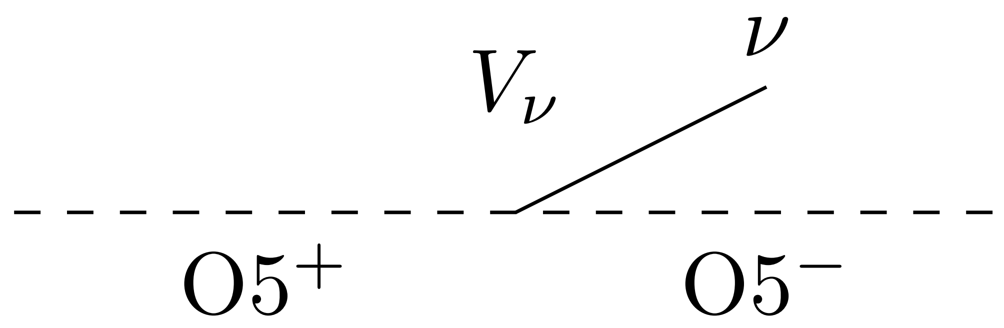
O-vertex

[Hayashi-RuiDong, SN-RuiDong]

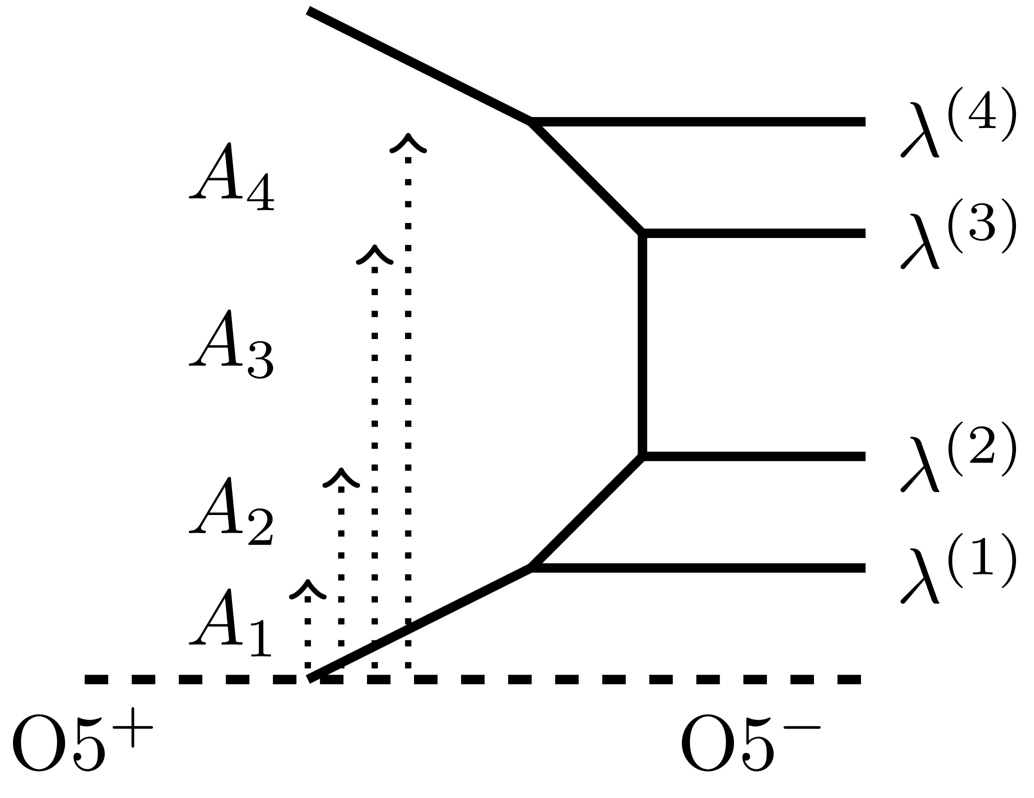
SO(2N) pure Yang-Mills can be realized by N D5 + $\mathrm{O5}^-$



Intersection of O5-plane and 5-branes can be realized by VOA formalism



$$V_\nu(-P)^{|\nu|} = \langle 0| \, \mathbb{O}(P,q) \, |\nu\rangle \,, \qquad \mathbb{O}(P,q) = \exp\left(\sum_{n=1}^{\infty} -\frac{P^{2n}(1+q^n)}{2n(1-q^n)} J_{2n} + \frac{P^{2n}}{2n} J_n J_n \right).$$

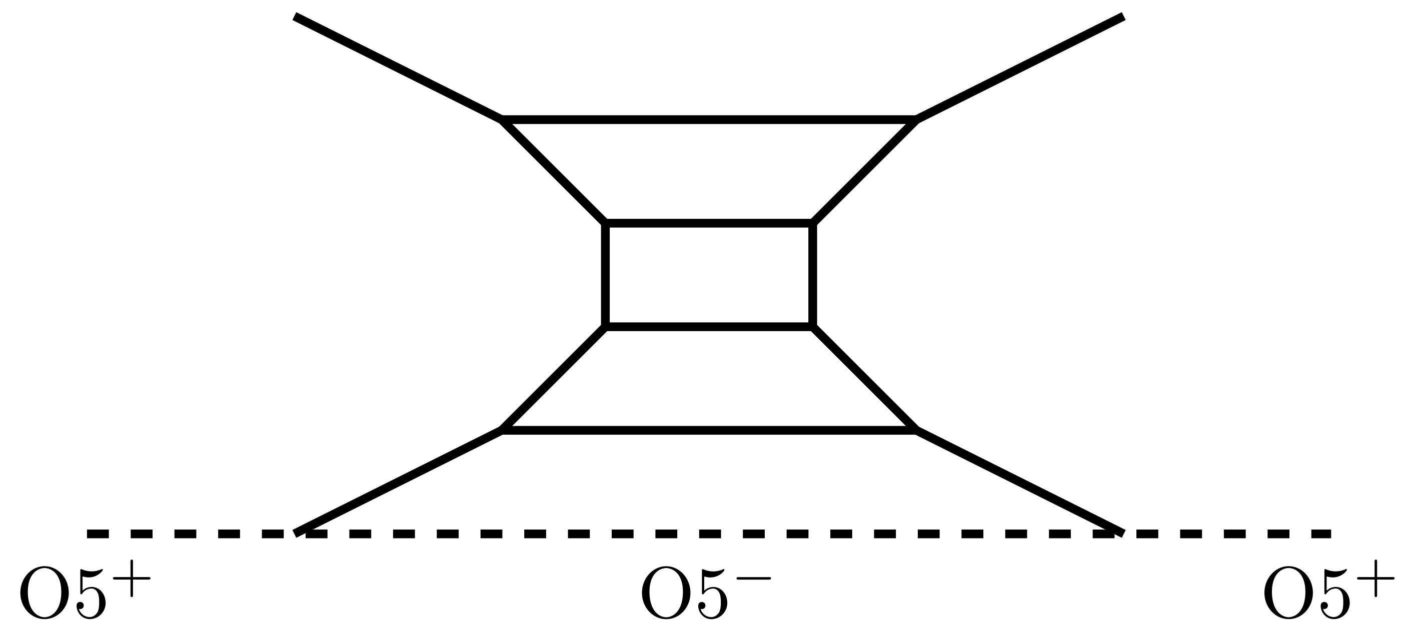


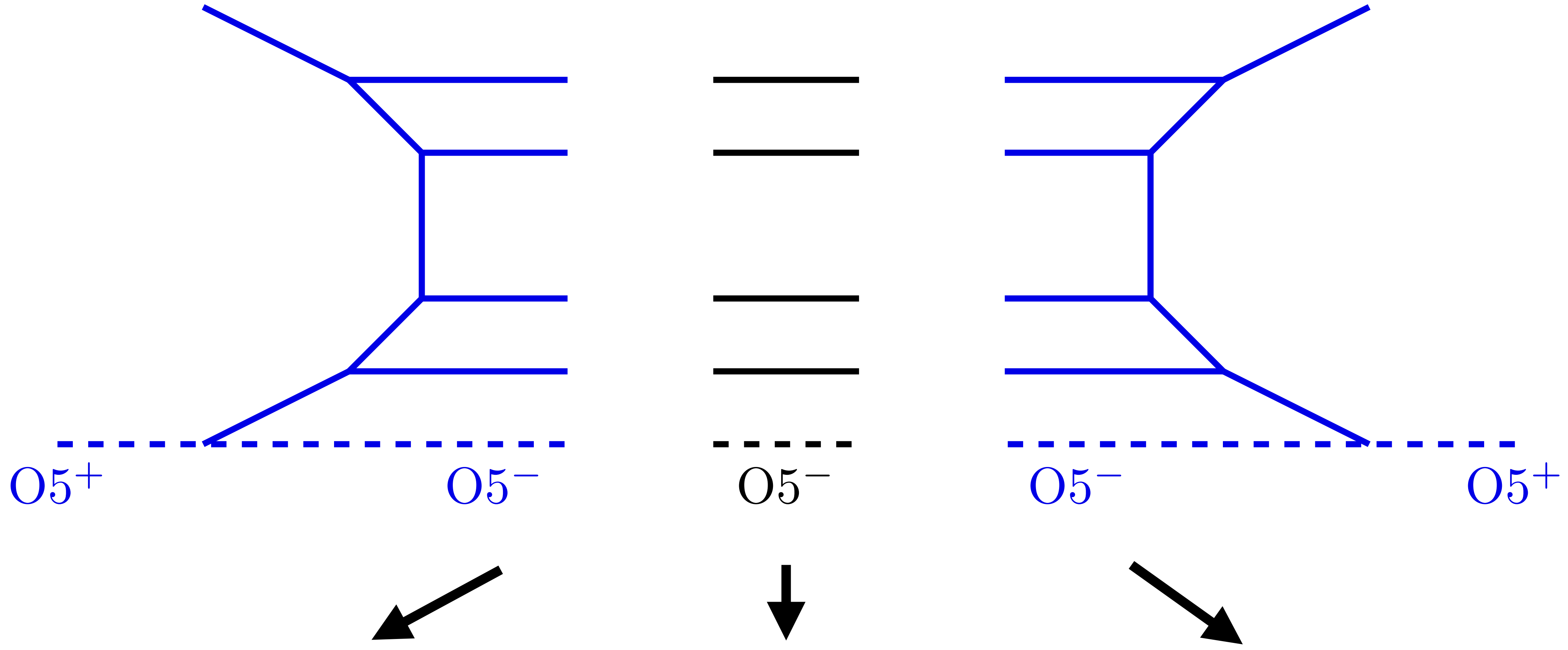
$$= \langle 0| \, \mathbb{O}(Q_0,q) \prod_{i=1}^{N} V_-(A_i A_1^{-1} q^{-\rho-\lambda^{(i)}}) \, |0\rangle$$

$$= \frac{\prod_{s=1}^{N} \prod_{(i,j)\in\lambda^{(s)}} (1 - A_s^2 q^{i-j+(\lambda^{(s)})^\vee_j - \lambda^{(s)}_i})}{\prod_{1\le s<t\le N} \prod_{(i,j)\in\lambda^{(s)}} (1 - A_s A_t q^{i+j-1-\lambda^{(s)}_i - \lambda^{(t)}_j}) \prod_{(m,n)\in\lambda^{(t)}} (1 - A_s A_t q^{1-m-n+(\lambda^{(t)})^\vee_n + (\lambda^{(s)})^\vee_m})}$$

SO(2N) instanton

[SN-RuiDong]



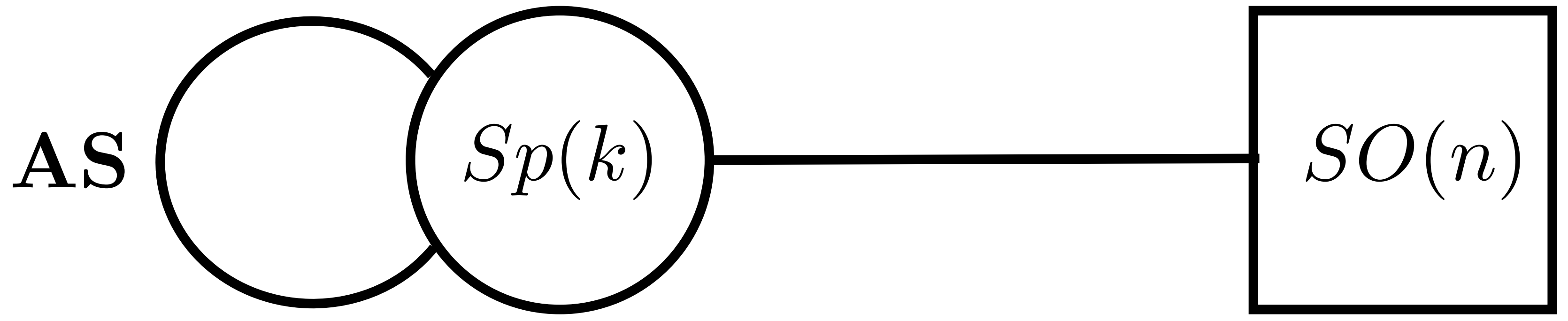


$$\langle 0|\, \mathbb{O}(A_1,q) \prod_{i=1}^{N} V_-(A_i A_1^{-1} q^{-\rho-\lambda^{(i)}})\, |0\rangle \qquad Z_{\vec{\lambda}}^{\mathrm{SU}(N)} \qquad \langle 0|\, \mathbb{O}(A_1,q) \prod_{i=1}^{N} V_-(A_i A_1^{-1} q^{-\rho-\lambda^{(i)}})\, |0\rangle$$

This is compatible with ADHM description

$$Z_k^{\mathrm{SO}(2N)} = \sum_{\vec{\lambda}} \prod_{s,t=1}^{N} \prod_{x\in\lambda^{(s)}} \frac{1}{\mathrm{sh}^2\, N_{st}(x)} \frac{\prod_{s=1}^{N} \prod_{(i,j)\in\lambda^{(s)}} \mathrm{sh}^2(2a_s+\hbar(i-j+(\lambda^{(s)})^\vee_j-\lambda^{(s)}_i))}{\prod_{1\le s<t\le N} \prod_{(i,j)\in\lambda^{(s)}} \mathrm{sh}^2(a_s+a_t+\hbar(i+j-1-\lambda^{(s)}_i-\lambda^{(t)}_j)) \prod_{(m,n)\in\lambda^{(t)}} \mathrm{sh}^2(a_s+a_t+\hbar(1-m-n+(\lambda^{(t)})^\vee_n+(\lambda^{(s)})^\vee_m))}$$

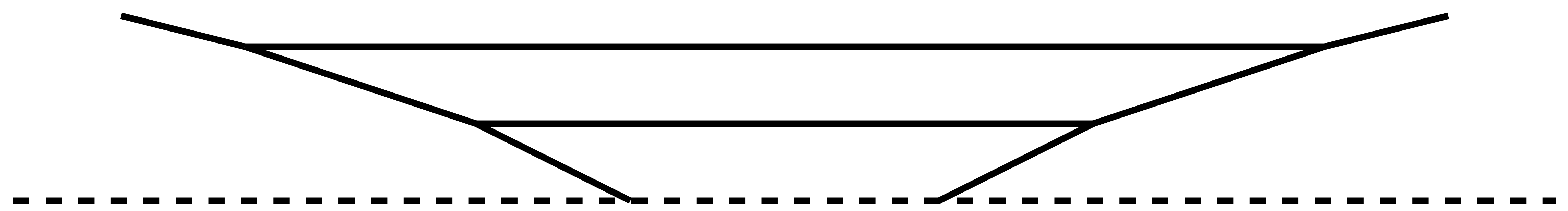
$$=: \sum_{\vec{\lambda}} Z_{\vec{\lambda}}^{\mathrm{SO}(2N)}$$



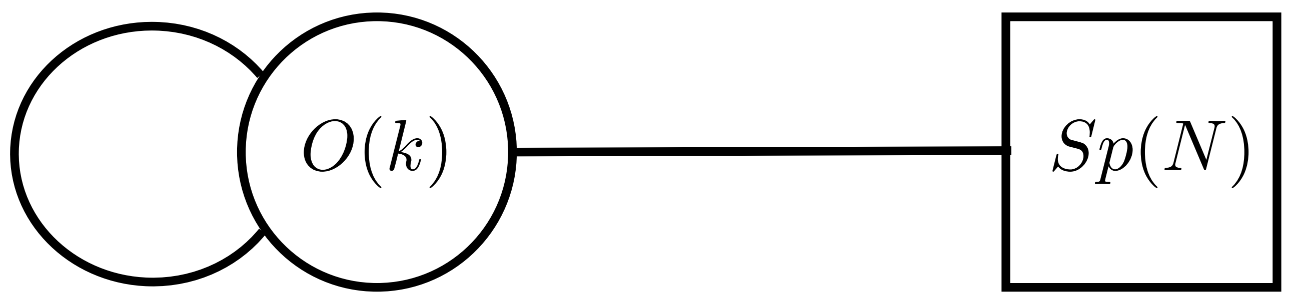
Sp(N) instanton [SN-RuiDong]

Legs talk to each other between O5$^+$

Not straightforward to define O-vertex



$$\pi_4(Sp(N)) = \mathbb{Z}_2 \longrightarrow$$ Discrete θ-angle



We can rely on ADHM description

$$Z^k_{\theta=0} = \frac{Z^k_+ + Z^k_-}{2}, \qquad Z^k_{\theta=\pi} = (-1)^k \frac{Z^k_+ - Z^k_-}{2}$$

Remarkably, each sector is related to SO(2N+8) instanton partition function

$$Z^{\mathrm{Sp}(N)}_{2\ell,+} \propto \sum_{|\vec{\lambda}|=\ell} C^{\mathrm{Sp}}_{\vec{\lambda},\vec{a}} Z^{\mathrm{SO}(2N+8)}_{\vec{\lambda}}\Big|_{a_{N+j}=\frac{\hbar}{2}(+\pi i),\ 0(+\pi i)}$$

$$Z^{\mathrm{Sp}(N)}_{2\ell+1,+} \propto \sum_{|\vec{\lambda}|=\ell} C^{\mathrm{Sp}}_{\vec{\lambda},\vec{a}} Z^{\mathrm{SO}(2N+8)}_{\vec{\lambda}}\Big|_{a_{N+j}=\frac{\hbar}{2}(+\pi i),\ \hbar,\ \pi i}$$

$$Z^{\mathrm{Sp}(N)}_{2\ell+2,-} \propto \sum_{|\vec{\lambda}|=\ell} C^{\mathrm{Sp}}_{\vec{\lambda},\vec{a}} Z^{\mathrm{SO}(2N+8)}_{\vec{\lambda}}\Big|_{a_{N+j}=\frac{\hbar}{2}(+\pi i),\hbar(+\pi i)}$$

$$Z^{\mathrm{Sp}(N)}_{2\ell+1,-} \propto \sum_{|\vec{\lambda}|=\ell} C^{\mathrm{Sp}}_{\vec{\lambda},\vec{a}} Z^{\mathrm{SO}(2N+8)}_{\vec{\lambda}}\Big|_{a_{N+j}=\frac{\hbar}{2}(+\pi i),0,\hbar+\pi i}$$

Sp(N) instanton [SN-RuiDong]

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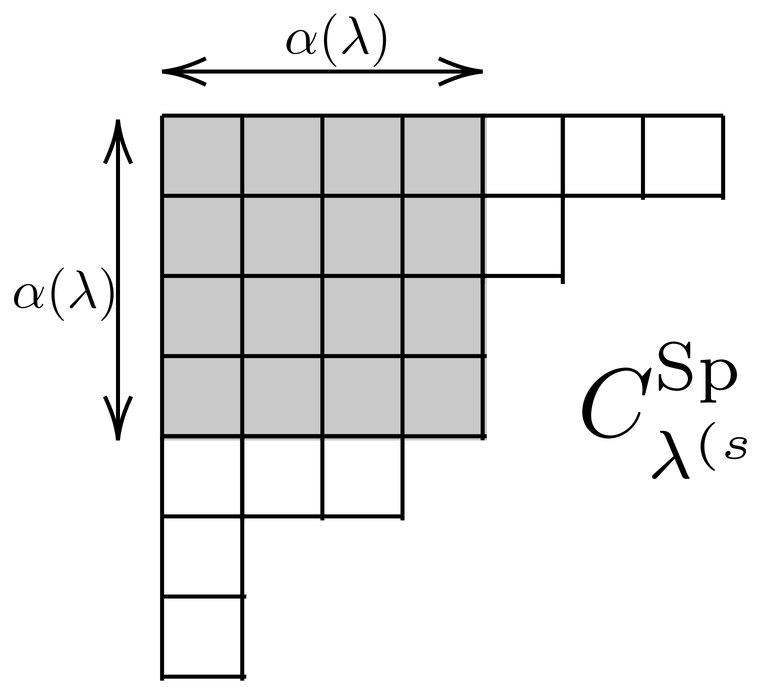
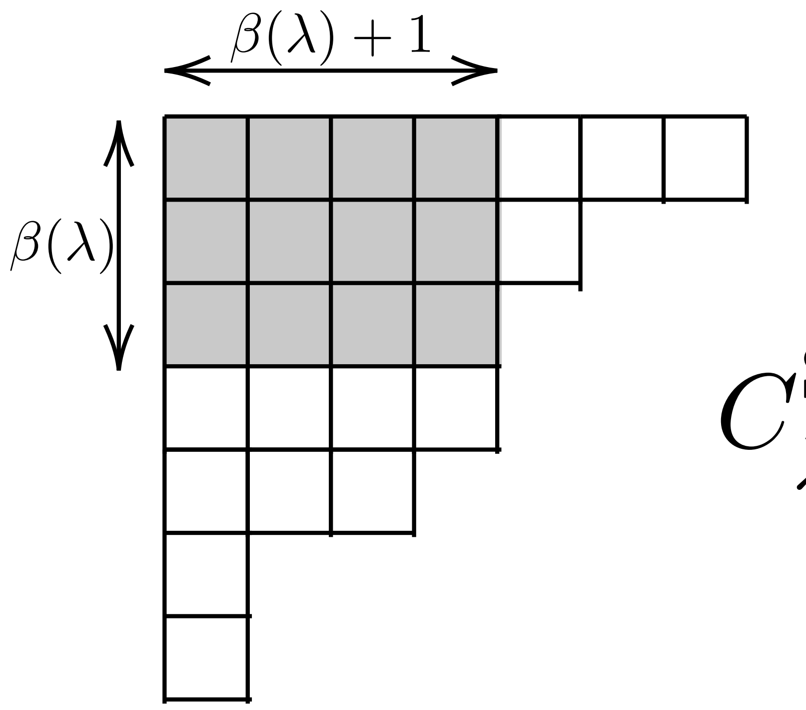
Multiplicity coefficients are given by

$\lambda =$ 

$$C^{\mathrm{Sp}}_{\lambda^{(s)},a_s=0(+\pi i),\frac{\hbar}{2}(+\pi i)} = \frac{2^{2\alpha(\lambda^{(s)})-1}}{\binom{2\alpha(\lambda^{(s)})-1}{\alpha(\lambda^{(s)})-1}}$$

$\lambda =$ 

$$C^{\mathrm{Sp}}_{\lambda^{(s)},a_s=\hbar(+\pi i)} = \frac{2^{2\beta(\lambda^{(s)})}}{\binom{2\beta(\lambda^{(s)})+1}{\beta(\lambda^{(s)})}}$$

Sp(N) instanton

[SN-RuiDong]

Remarkably, each sector is related to SO(2N+8) instanton partition function

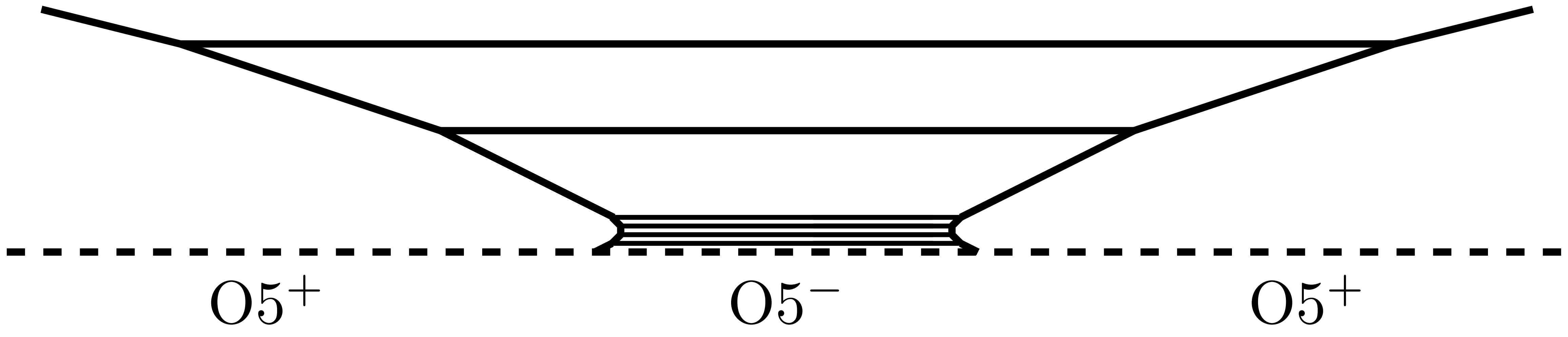
$$Z^{\mathrm{Sp}(N)}_{2\ell,+} \propto \sum_{|\vec{\lambda}|=\ell} C^{\mathrm{Sp}}_{\vec{\lambda},\vec{a}} Z^{\mathrm{SO}(2N+8)}_{\vec{\lambda}}\Big|_{a_{N+j}=\frac{\hbar}{2}(+\pi i),\ 0(+\pi i)}$$

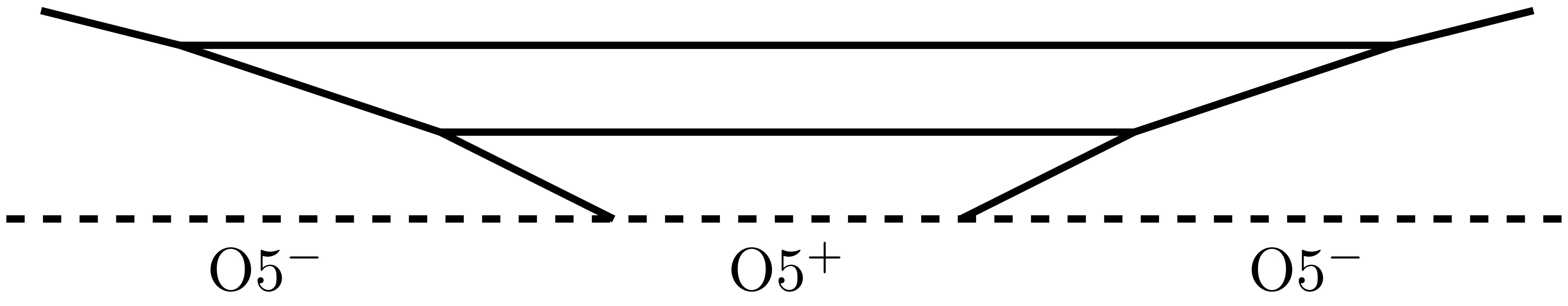
$$Z^{\mathrm{Sp}(N)}_{2\ell+1,+} \propto \sum_{|\vec{\lambda}|=\ell} C^{\mathrm{Sp}}_{\vec{\lambda},\vec{a}} Z^{\mathrm{SO}(2N+8)}_{\vec{\lambda}}\Big|_{a_{N+j}=\frac{\hbar}{2}(+\pi i),\ \hbar,\ \pi i}$$

$$Z^{\mathrm{Sp}(N)}_{2\ell+2,-} \propto \sum_{|\vec{\lambda}|=\ell} C^{\mathrm{Sp}}_{\vec{\lambda},\vec{a}} Z^{\mathrm{SO}(2N+8)}_{\vec{\lambda}}\Big|_{a_{N+j}=\frac{\hbar}{2}(+\pi i),\hbar(+\pi i)}$$

$$Z^{\mathrm{Sp}(N)}_{2\ell+1,-} \propto \sum_{|\vec{\lambda}|=\ell} C^{\mathrm{Sp}}_{\vec{\lambda},\vec{a}} Z^{\mathrm{SO}(2N+8)}_{\vec{\lambda}}\Big|_{a_{N+j}=\frac{\hbar}{2}(+\pi i),0,\hbar+\pi i}$$

Relation between Sp(N) and SO(2N+8) can be seen in 5-brane web



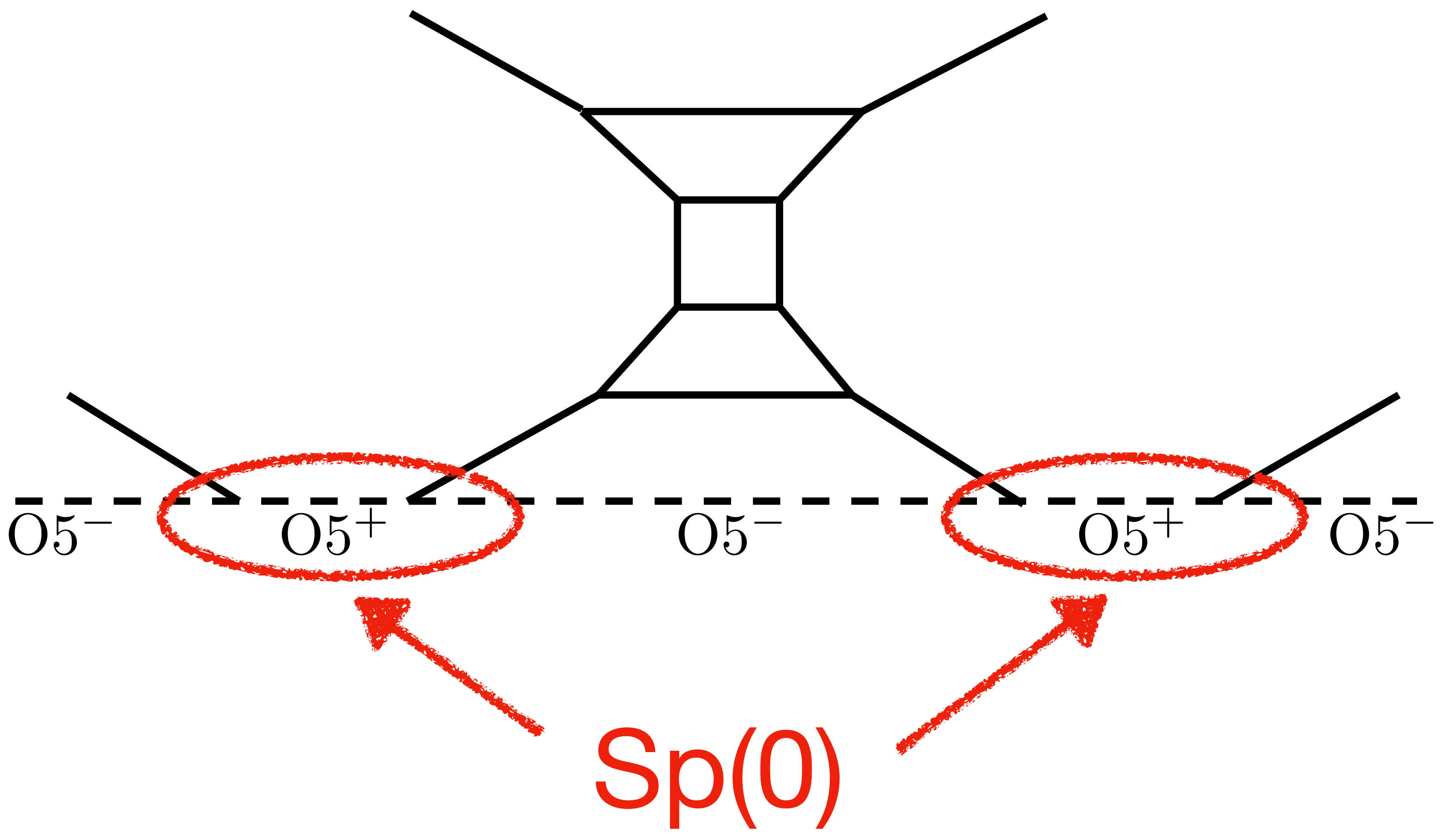


We will come back to this issue later: $\longrightarrow$ I will unfreeze the secret!

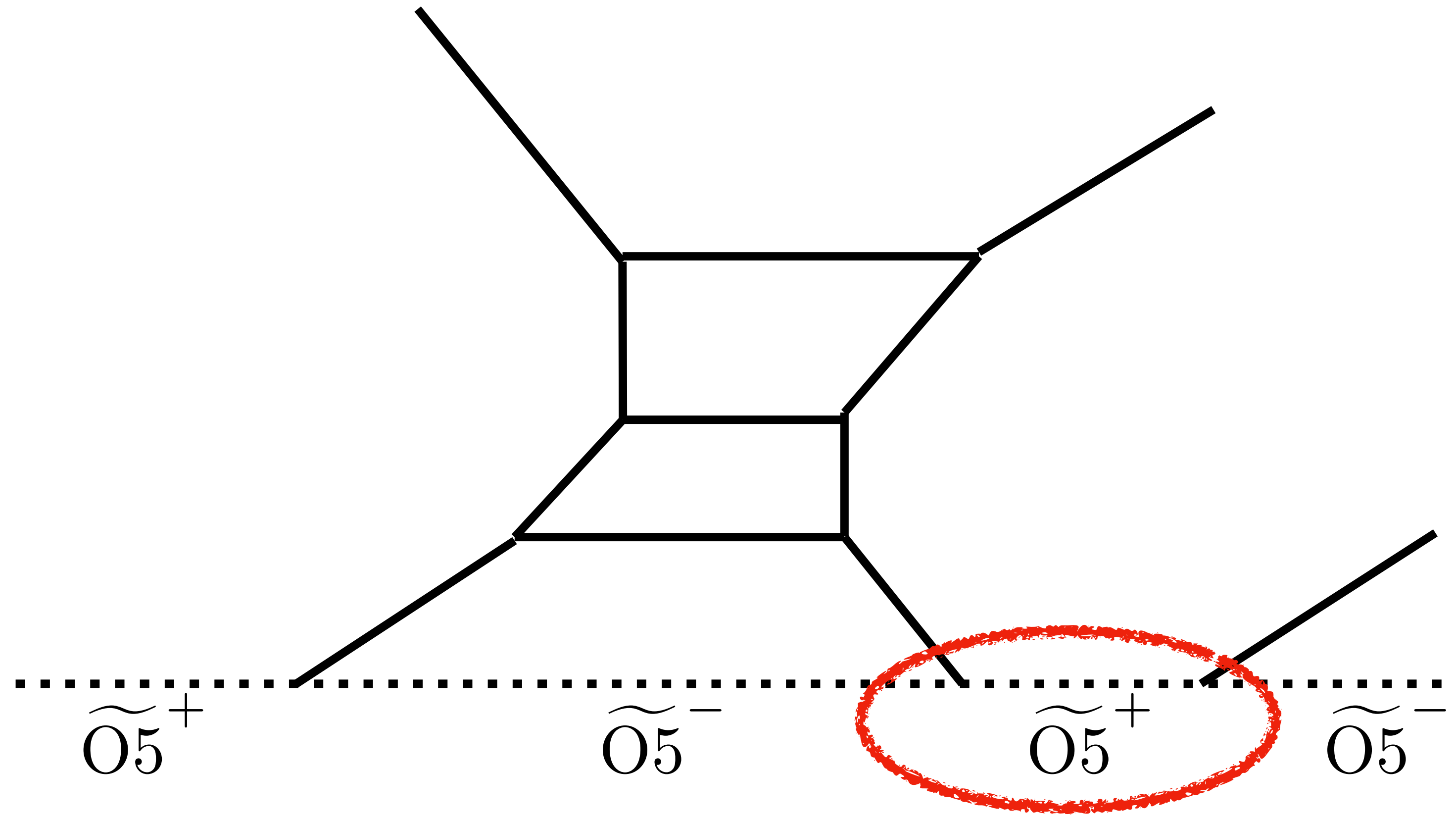
SO(n) instanton with (conjugate) spinor hypermultiplets

[Chen-Jiang-SN-Shao]

5-brane web for SO(2N)+1S+1C



SO(2N+1)+1S



Can be regarded as SO(n)-Sp(0) quiver theory

$\longrightarrow$ Instanton PF can be written as Young-diagram sum

- $(\mathfrak{so}(4), \mathbf{S}) \cong (\mathfrak{su}(2) \oplus \mathfrak{su}(2), \mathbf{F} \oplus \emptyset)$
- $(\mathfrak{so}(5), \mathbf{S}) \cong (\mathfrak{sp}(2), \mathbf{F})$
- $(\mathfrak{so}(6), \mathbf{S}) \cong (\mathfrak{su}(4), \mathbf{F})$

$$\begin{tikzcd}
 & (\mathfrak{so}(8), \mathbf{S}) \arrow[dr, "\text{c.o.v.}"] & \\
(\mathfrak{so}(8), \mathbf{V}) \arrow[ur, "\text{c.o.v.}"] & & (\mathfrak{so}(8), \mathbf{C}) \arrow[ll, "\text{c.o.v.}"]
\end{tikzcd}$$

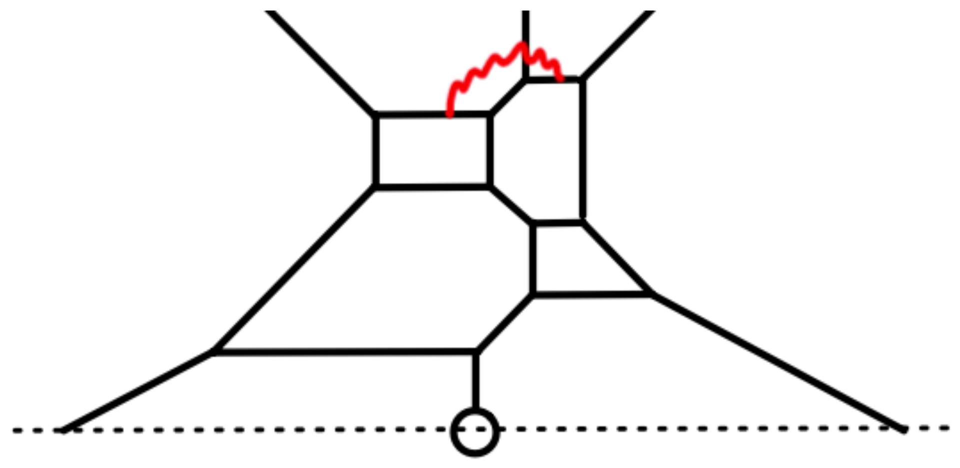
SU(N) instanton with (anti-)symmetric hypermultiplets

[Chen-Jiang-SN-Shao]

SU(N)+1Sym and SU(N)+1AS can be constructed from O7-plane

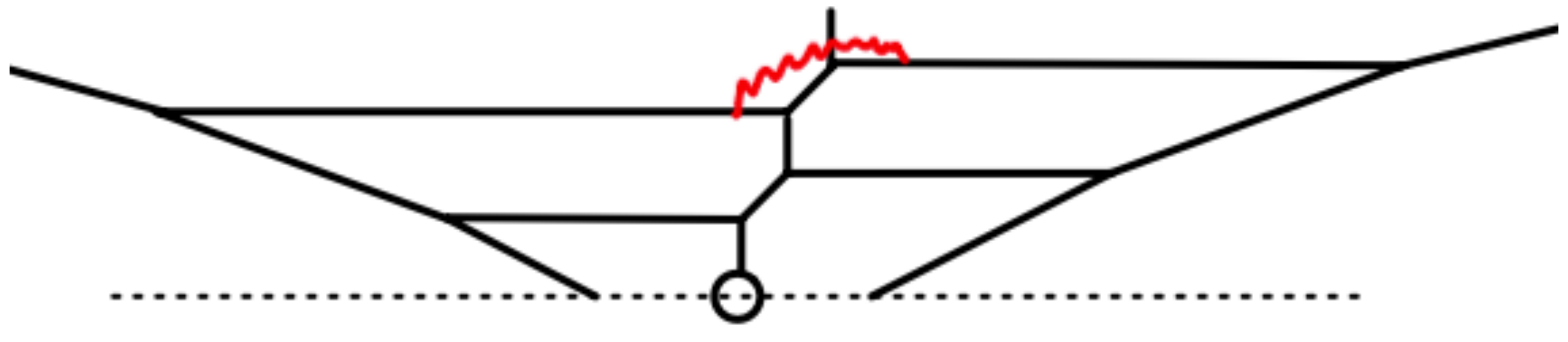
Closed-form expressions for unrefined PFs

SU+1**Sym**



$$
\begin{aligned}
&Z_k^{\mathrm{SU}(N)_\kappa+1\mathbf{Sym}} \\
&= \sum_{|\vec{\lambda}|=k} \prod_{s=1}^{N} \prod_{x\in\lambda^{(s)}} e^{\kappa\phi_s(x)} \frac{\mathrm{sh}\left(2\phi_s(x)+m\pm\hbar\right)\cdot\prod_{t=1}^{N}\mathrm{sh}\left(\phi_s(x)+a_t+m\right)}{\prod_{t=1}^{N}\mathrm{sh}^2\left(N_{s,t}(x)\right)} \\
&\quad\times \prod_{s\le t}^{N} \prod_{\substack{x\in\lambda^{(s)},y\in\lambda^{(t)}\\ x<y}} \frac{\mathrm{sh}\left(\phi_s(x)+\phi_t(y)+m\pm\hbar\right)}{\mathrm{sh}^2\left(\phi_s(x)+\phi_t(y)+m\right)} \\
&=: \sum_{|\vec{\lambda}|=k} Z_{\vec{\lambda}}^{\mathrm{SU}(N)_\kappa+1\mathbf{Sym}},
\end{aligned}
$$

SU+1**AS**



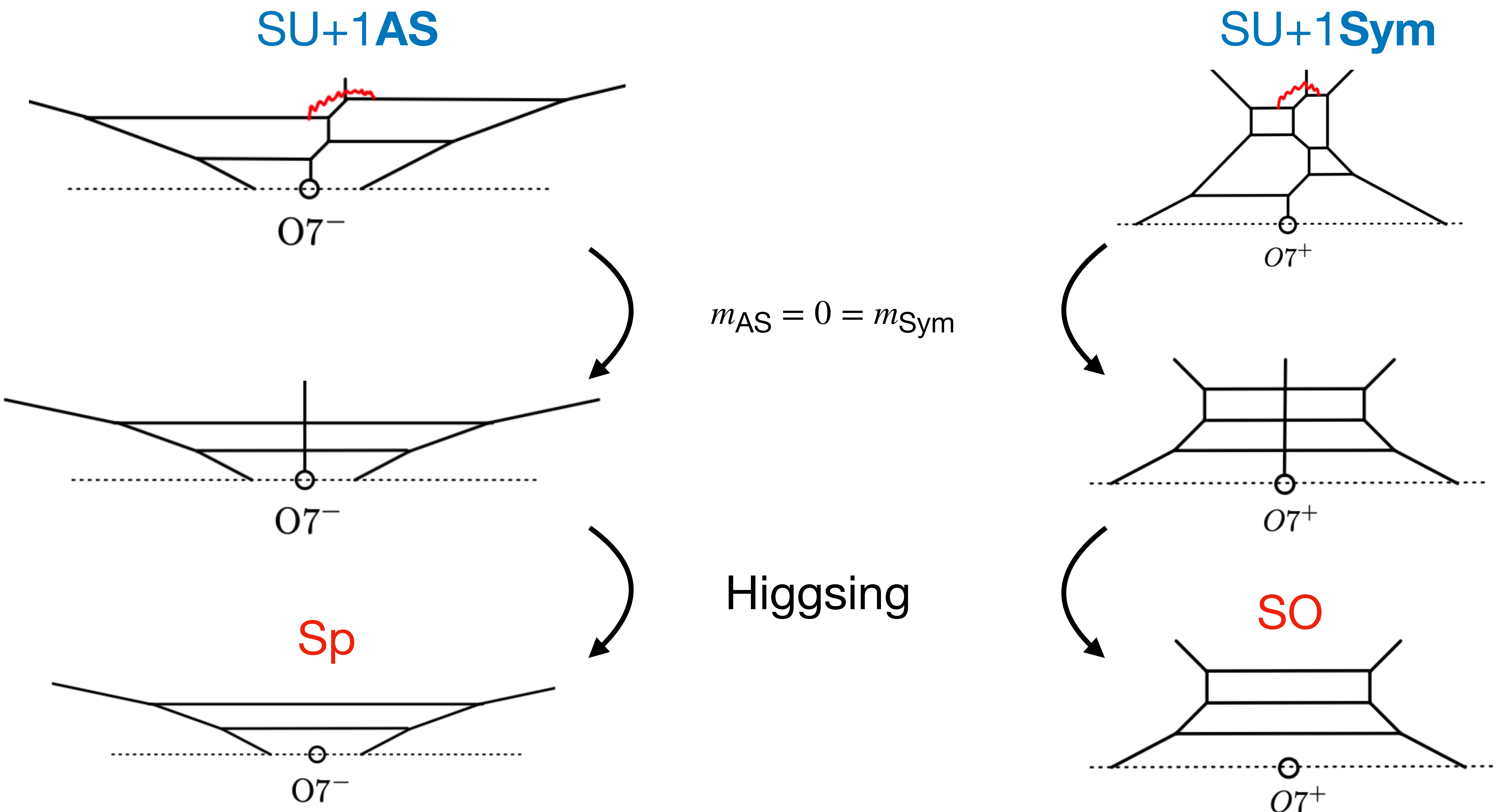
Remarkably, SU(N)+1AS is related to SU(N+8)+1Sym with multiplicity coefficients

$$
Z_k^{\mathrm{SU}(N)_\kappa+1\mathbf{AS}} = \sum_{|\vec{\lambda}|=k} C^{\mathrm{anti}}_{\vec{\lambda},\vec{a}}\, Z_{\vec{\lambda}}^{\mathrm{SU}(N+8)_\kappa+1\mathbf{Sym}}
$$

Higgsing

By Higgsing, we can obtain Sp / SO gauge theories

In this way, we can obtain another expression for Sp / SO gauge theories



with Kilar Zhang, Rui-Dong Zhu [arXiv:2302.00525]

qq-characters

[Nekrasov, Hee-Cheol Kim]

	0	1	2	3	4	5	6	7	8	9
D4/O4	×	×	×	×	×					
D0	×									
D4′	×					×	×	×	×	

D4'-brane creates line defects in 5d N=1* theory.

We take decoupling limit of adj hypermultiplets

$$Z^{\mathfrak{g}}_{\mathrm{defect}}(\zeta) = \sum_{k=0}^{\infty} \mathfrak{q}^k \oint_{\mathrm{JK}} \prod_{i=1}^{k} \frac{d\phi_i}{2\pi i} Z^{(k)}_{\mathrm{vec}} Z^{(k)}_{\mathrm{def}}(\zeta)$$

qq-character is the generating function of VEVs of anti-symmetric Wilson loops

G=SU(N)

$$Z^{(k)}_{\mathrm{def}}(\zeta) = \prod_{\alpha=1}^{N} \mathrm{sh}(\zeta - \mathfrak{a}_\alpha + 2\epsilon_+) \prod_{i=1}^{k} \frac{\mathrm{sh}(\epsilon_- \pm (\zeta - \phi_i))}{\mathrm{sh}(\epsilon_+ \pm (\zeta - \phi_i))}$$

JK poles only from $Z^{(k)}_{\mathrm{vec}}$

One JK pole from $Z^{(k)}_{\mathrm{def}}$

$$Z^{\mathfrak{su}(N)}_{\mathrm{defect}}(z) = z^{-\frac{\dim\square}{2}} \sum_{k=0}^{\dim\square} (-z)^k \langle \mathcal{W}_{\wedge^k} \rangle = \langle Y^A(z) \rangle + \mathfrak{q}(-1)^N \left\langle \frac{1}{Y^A(zq^{-1}t)} \right\rangle,$$

Polynomials in z of degree N

Poles for z cancel each other

qq-characters as quantization of Seiberg-Witten curve

[Nekrasov-Pestun, Nekrasov]

Classical: Seiberg-Witten curve for SU(N) pure YM theory

$$y + \frac{\mathfrak{q}}{y} = t(u), \quad t(u) := \prod_{i=1}^{N} (z - a_i)$$

Quantization: substituting $y = e^{-\hbar \partial_\zeta}$, we obtain Baxter TQ relation

$$\left(e^{-\hbar\partial_\zeta} + \mathfrak{q} e^{\hbar\partial_\zeta} - T(z)\right) Q(z) = Q(zq) + \mathfrak{q} Q(z/q) - T(z)Q(z) = 0$$

$$\longrightarrow \quad Y(z) + \frac{\mathfrak{q}}{Y(zq^{-1})} - T(z) = 0 \qquad\qquad Y(z) := Q(zq)/Q(z)$$

This corresp. to NS limit $\epsilon_1 = \hbar, \epsilon_2 = 0$

Double quantization: qq-character

$$\langle Y^A(z)\rangle + \mathfrak{q}(-1)^N \left\langle \frac{1}{Y^A(zq^{-1}t)} \right\rangle = Z_{\text{defect}}(z)$$

Adding and removing boxes

[Nekrasov, Hee-Cheol Kim]

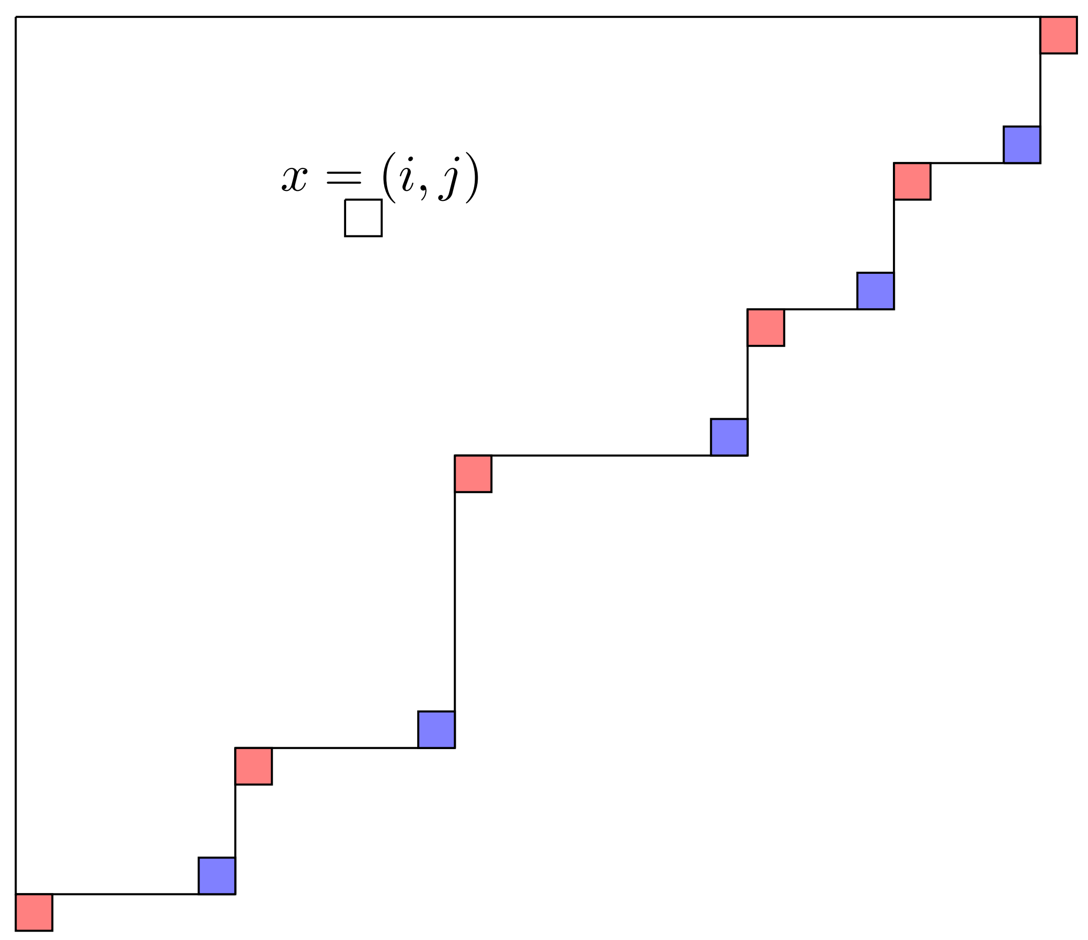
$$Z^{\mathfrak{su}(N)}_{\text{defect}}(z) = z^{-\frac{\dim\square}{2}} \sum_{k=0}^{\dim\square} (-z)^k \langle \mathcal{W}_{\wedge^k} \rangle = \langle Y^A(z) \rangle + \mathfrak{q}(-1)^N \left\langle \frac{1}{Y^A(zq^{-1}t)} \right\rangle ,$$

Instanton PF is expressed as Young diagram sum

$$|\mathfrak{G}\rangle = \sum_{\vec{\lambda}} \left(\mathfrak{q}^{|\vec{\lambda}|} Z^{\text{su(N)}}_{\vec{\lambda}} \right)^{\frac{1}{2}} |\vec{A}, \vec{\lambda}\rangle, \quad \langle \mathfrak{G}|\mathfrak{G}\rangle = Z^{\text{su}(N)}_{\text{inst}}$$

$$\langle \mathcal{O} \rangle := \langle \mathfrak{G}|\mathcal{O}|\mathfrak{G}\rangle.$$

Y-operator receives contribution from adding and removing boxes part



$$\left\langle Y^A(z) \right\rangle := \sum_{\vec{\lambda}} \mathfrak{q}^{|\vec{\lambda}|} Y^A_{\vec{\lambda}}(z) Z^{\mathfrak{su}(N)}_{\vec{\lambda}}$$

Addition of boxes

$$Y^A_{\vec{\lambda}}(z) = \frac{\prod_{x \in A(\vec{\lambda})} \text{sh}(\zeta - \phi_x + \epsilon_+)}{\prod_{x \in R(\vec{\lambda})} \text{sh}(\zeta - \phi_x - \epsilon_+)}, \qquad \phi_x = a_\alpha - \epsilon_+ + (i-1)\epsilon_1 + (j-1)\epsilon_2, \quad x = (i,j) \in \lambda^{(\alpha)}$$

Removal of boxes

Quantum toroidal algebra of $\mathfrak{gl}_1$

[Bourgine-Fukuda-Matsuo-Zhang-Zhu]

generators: $e(z), f(z), \psi^{\pm}(z)$

$$h(z) = \prod_{a=1,2,3} \frac{1 - q_a^{-1} z}{1 - q_a z}$$

$$e(z)e(w) = h(w/z)e(w)e(z), \quad f(z)f(w) = h(z/w)f(w)f(z)$$

$$\psi^{\pm}(z)e(w) = h(w/z)e(w)\psi^{\pm}(z), \quad \psi^{\pm}(z)f(w) = h(z/w)f(w)\psi^{\pm}(z)$$

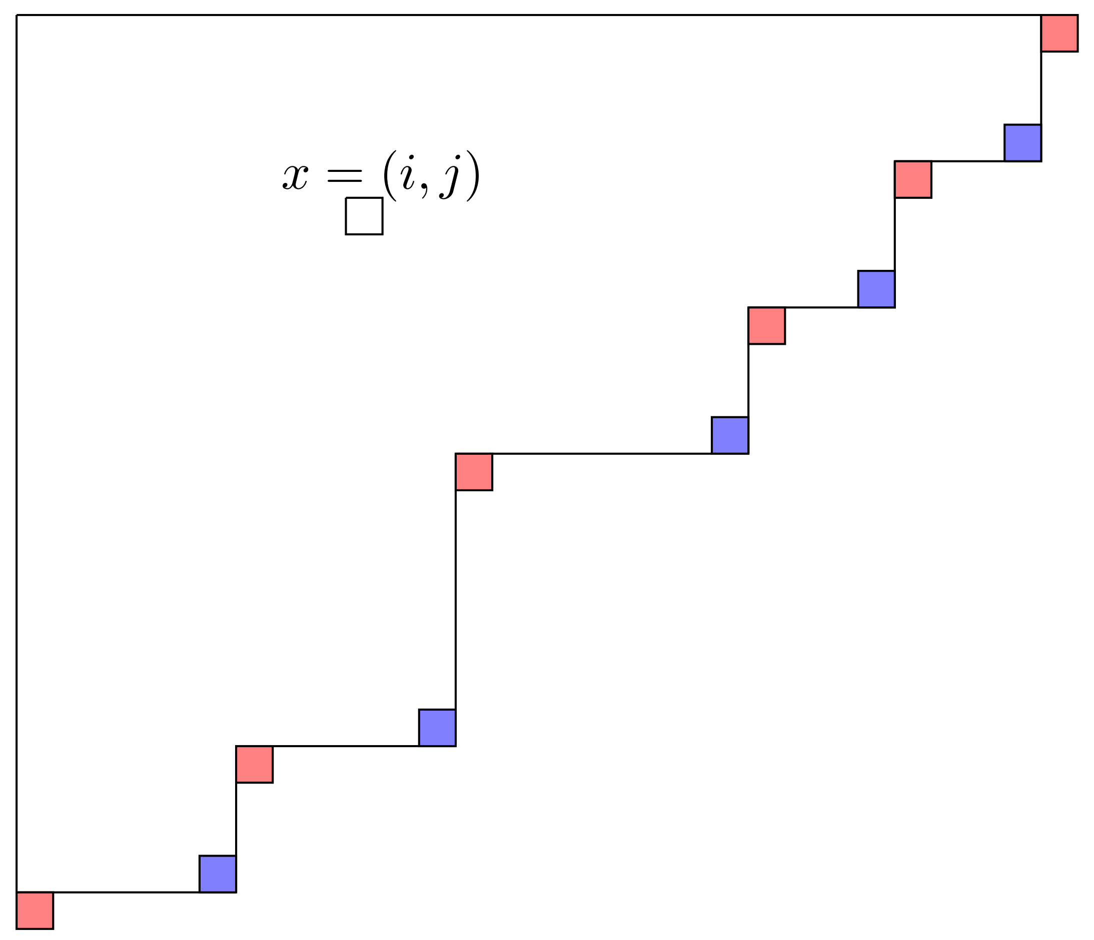
$$[e(z), f(w)] \propto \delta\left(\frac{z}{w}\right)\left(\psi^{+}(z) - \psi^{-}(z)\right)$$

Vertical representation

$$e(z)\,|\vec{A},\vec{\lambda}\rangle = \sum_{x \in A(\vec{\lambda})} \delta(z/\chi_x)\,\mathrm{Res}_{z\to\chi_x} \frac{1}{Y^A_{\vec{\lambda}}(z)}\,|\vec{A},\vec{\lambda}+x\rangle\,,$$

$$f(z)\,|\vec{A},\vec{\lambda}\rangle = \sum_{x \in R(\vec{\lambda})} \delta(z/\chi_x)\,\mathrm{Res}_{z\to\chi_x} Y^A_{\vec{\lambda}}(zqt^{-1})\,|\vec{A},\vec{\lambda}-x\rangle\,,$$

$$\psi^{\pm}(z)\,|\vec{A},\vec{\lambda}\rangle = \left[\frac{Y_{\vec{\lambda}}(zqt^{-1})}{Y_{\vec{\lambda}}(z)}\right]_{\pm}|\vec{A},\vec{\lambda}\rangle\,,$$



qq-character

$$e_{-}(z)|\mathfrak{G}\rangle \propto Y^A(z)_{-}\,|\mathfrak{G}\rangle$$

$$f_{-}(z)|\mathfrak{G}\rangle \propto \frac{1}{Y^A(zq^{-1}t)}|\mathfrak{G}\rangle,$$

$$\longrightarrow \quad \chi(z) = Y^A(z) + \frac{(-1)^N \mathfrak{q}}{Y^A(zqt^{-1})}$$

qq-character for SO/Sp pure Yang-Mills

[SN-Kilar Zhang-RuiDong]

Young diagram sum for SO/Sp unrefined instanton partition functions

⟶ qq-character makes sense at unrefined level

$$Z^{\mathfrak{g}}_{\text{defect}}(\zeta) = \sum_{k=0}^{\infty} \mathfrak{q}^k \oint_{\text{JK}} \prod_{i=1}^{k} \frac{\mathrm{d}\phi_i}{2\pi i} Z^{(k)}_{\text{vec}} Z^{(k)}_{\text{def}}(\zeta)$$

	0	1	2	3	4	5	6	7	8	9
D4/O4	×	×	×	×	×					
D0	×									
D4′	×					×	×	×	×	

qq-character for SO(n) pure YM

JK poles only from $Z^{(k)}_{\text{vec}}$

One JK pole from $Z^{(k)}_{\text{def}}$

$$Z^{\mathfrak{so}(N)}_{\text{defect}}(z) = z^{-\frac{\dim\square}{2}} \sum_{k=0}^{\dim\square} (-z)^k \langle \mathcal{W}_{\wedge^k} \rangle = \langle Y^{SO}(z) \rangle + \frac{\mathfrak{q}}{\mathrm{sh}(2\zeta \pm \hbar)\,\mathrm{sh}^2(2\zeta)} \left\langle \frac{1}{Y^{SO}(z)} \right\rangle,$$

Polynomials in z of degree N

Poles for z cancel each other

Story is similar to A-type. Y-operator can be constructed by adding/removing boxes

qq-character for SO/Sp pure Yang-Mills

[SN-Kilar Zhang-RuiDong]

qq-character for Sp(N) pure YM is more interesting

Defect partition functions consist of $\pm$ sectors.

The form of defect PFs differ in even and odd instanton numbers

Nonetheless, the defect PFs have universal form:

$$Z^{(k,\pm)}_{\text{defect}}(z) = \langle Y^C \rangle + \frac{\mathfrak{q}^2}{\mathrm{sh}(2\zeta)^2\,\mathrm{sh}(2\zeta \pm \hbar)} \left\langle \frac{1}{Y^C} \right\rangle$$

Not polynomials in z

To get polynomiality, we need to remove extra factor

$$\chi(z) = z^{-\frac{\dim \square}{2}} \sum_{k=0}^{\dim \square} (-z)^k \langle \mathcal{W}_{\wedge^k} \rangle = \langle Y^C \rangle \boxed{- \frac{2\mathfrak{q}}{\mathrm{sh}^2(\zeta)} \langle 1 \rangle} + \frac{\mathfrak{q}^2}{\mathrm{sh}(2\zeta)^2\,\mathrm{sh}(2\zeta \pm \hbar)} \left\langle \frac{1}{Y^C} \right\rangle$$

polynomials in z of degree 2N

Propotional to instanton PF

Quantization of Seiberg-Witten curve!

ABCD of qq-characters

[SN-Kilar Zhang-RuiDong]

qq-character is generating function of VEV of antisymmetric Wilson loops

We can verify isomorphisms of representations

- $\mathfrak{sp}(1) \cong \mathfrak{su}(2)$
- $(\mathfrak{so}(4), \square) \cong (\mathfrak{su}(2) \oplus \mathfrak{su}(2), \square \otimes \square)$
- $(\mathfrak{so}(5), \square \oplus \emptyset) \cong (\mathfrak{sp}(2), \wedge^2 \square)$
- $(\mathfrak{so}(6), \square) \cong (\mathfrak{su}(4), \begin{array}{|c|}\hline \\ \hline \\ \hline\end{array})$

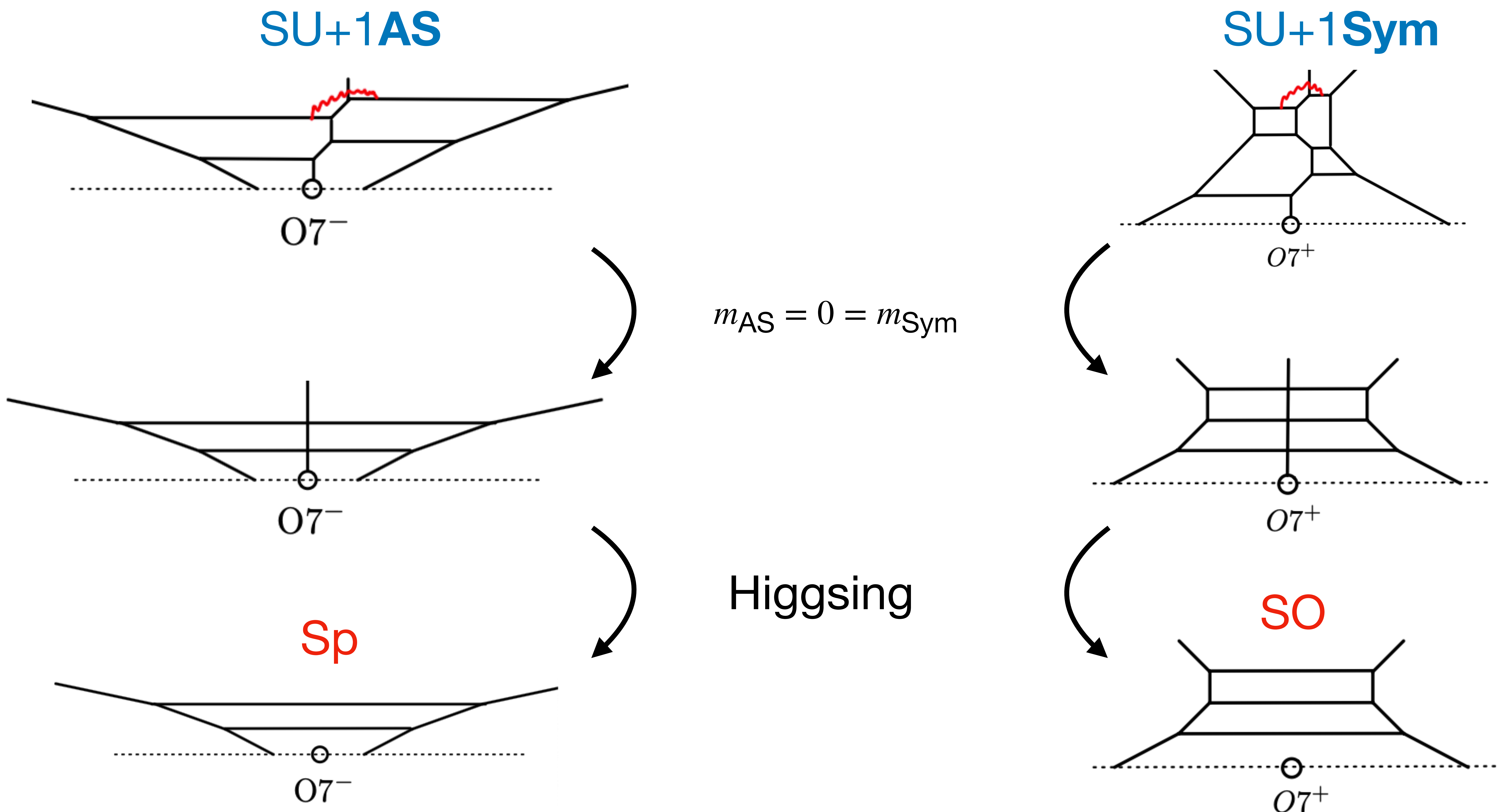
By adding and removing boxes, one can construct

New quantum toroidal algebra for SO(n) instanton partition functions

with Sung-Soo Kim, Xiaobin Li, Futoshi Yagi [arXiv:2403.12525] appeared Today!

O7-plane

With O7-planes, 5-brane webs can describe
SU **AS** or **Sym** and also Sp / SO gauge theories.



$O7^{-}$+8 D7 vs $O7^{+}$ [Kimyeong-SungSoo-Hayashi-Futoshi,Hee-Cheol-Minsung-SungSoo-Zafrir]

It is well known that their brane charges are the same

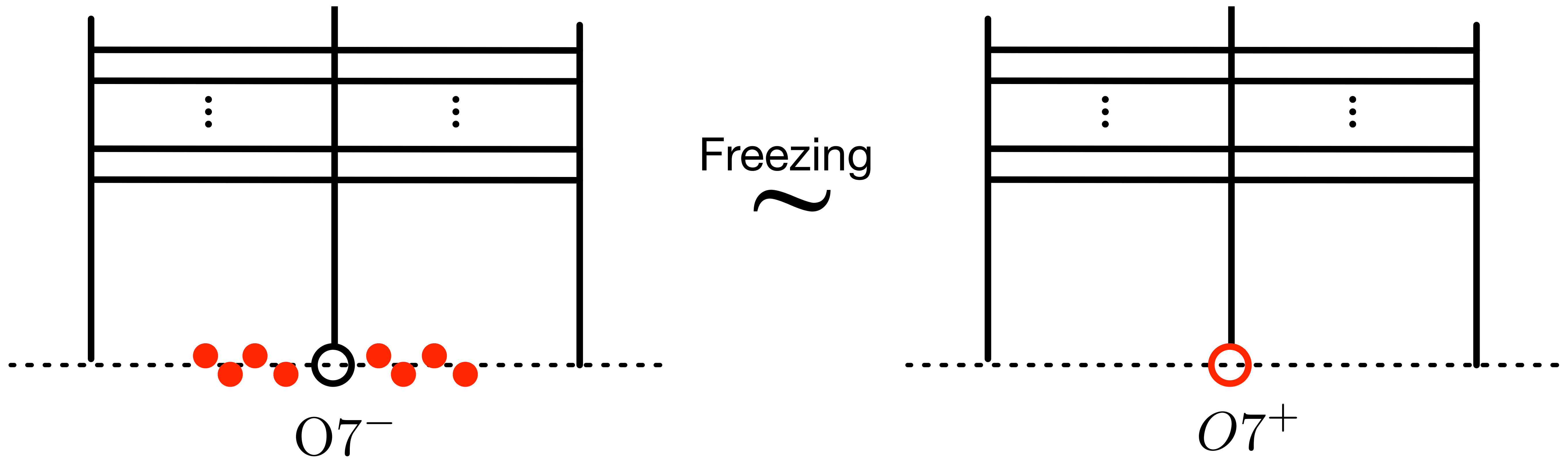
$$O7^{-} + 8\ D7 \longleftrightarrow O7^{+}$$

Also monodromy
$$\begin{pmatrix} -1 & 4 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 8 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -4 \\ 0 & -1 \end{pmatrix}$$

It is tempting to think that there may be some intriguing relations

SU(N)+1**AS**+8**F** $\longleftrightarrow$ SU(N)+1**Sym**

Sp(N)+8**F** $\longleftrightarrow$ SO(2N)



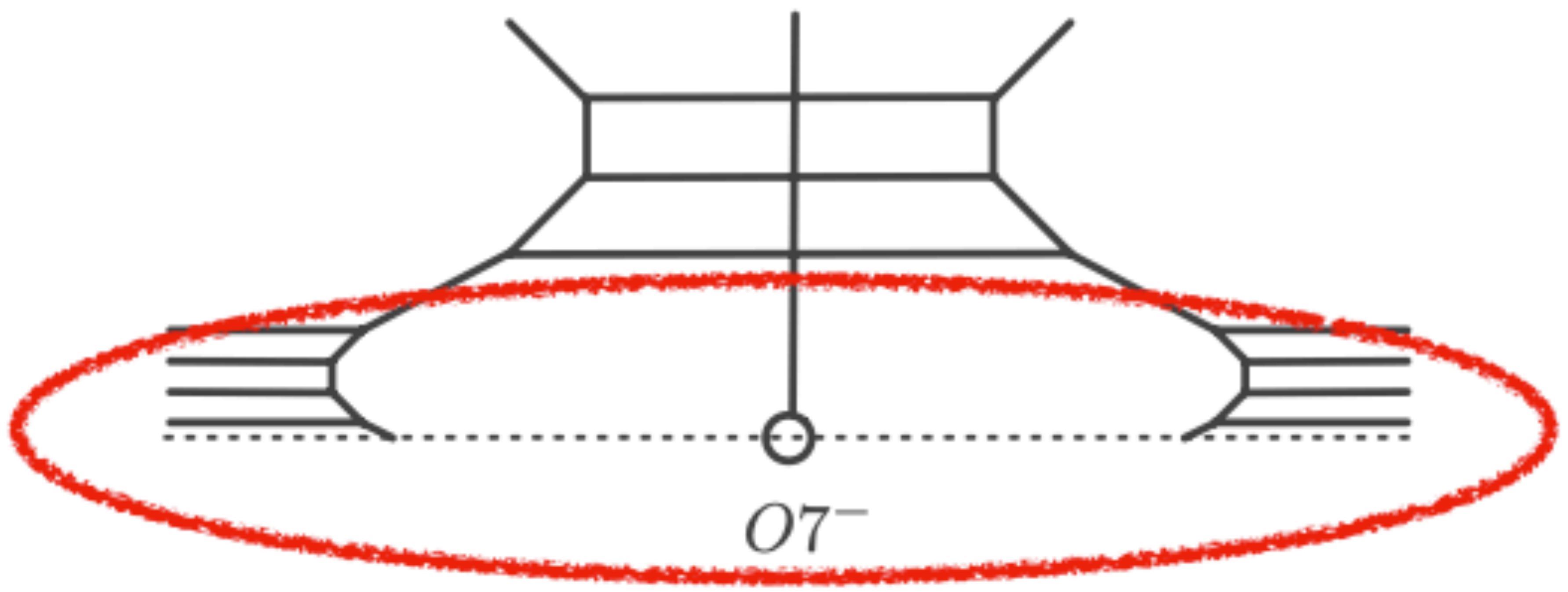
Freezing

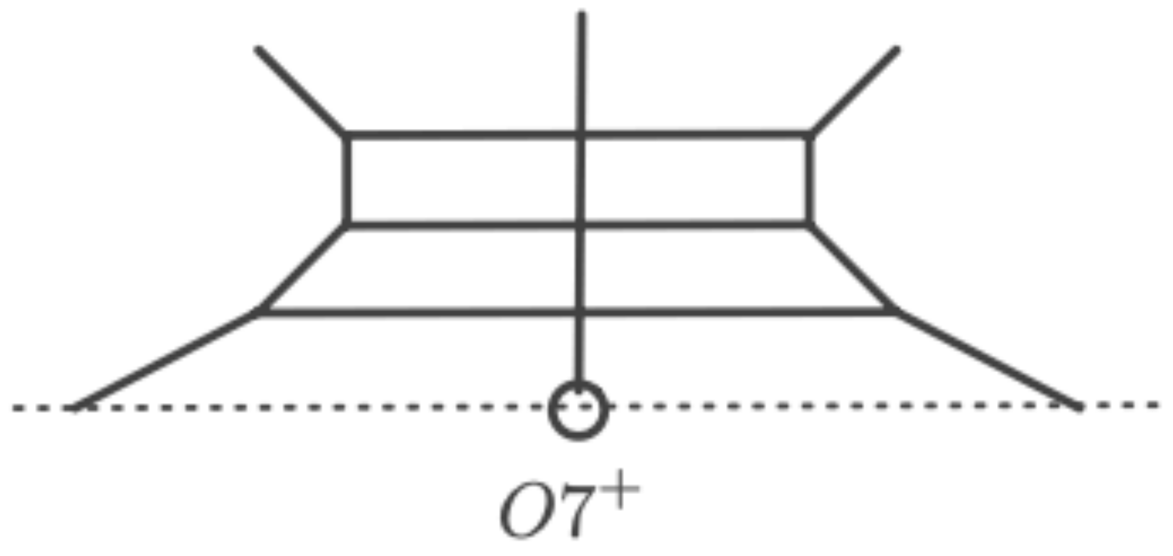
[Kimyeong-SungSoo-Hayashi-Futoshi,Hee-Cheol-Minsung-SungSoo-Zafrir]

5-brane web resembles after Hanany-Witten transition

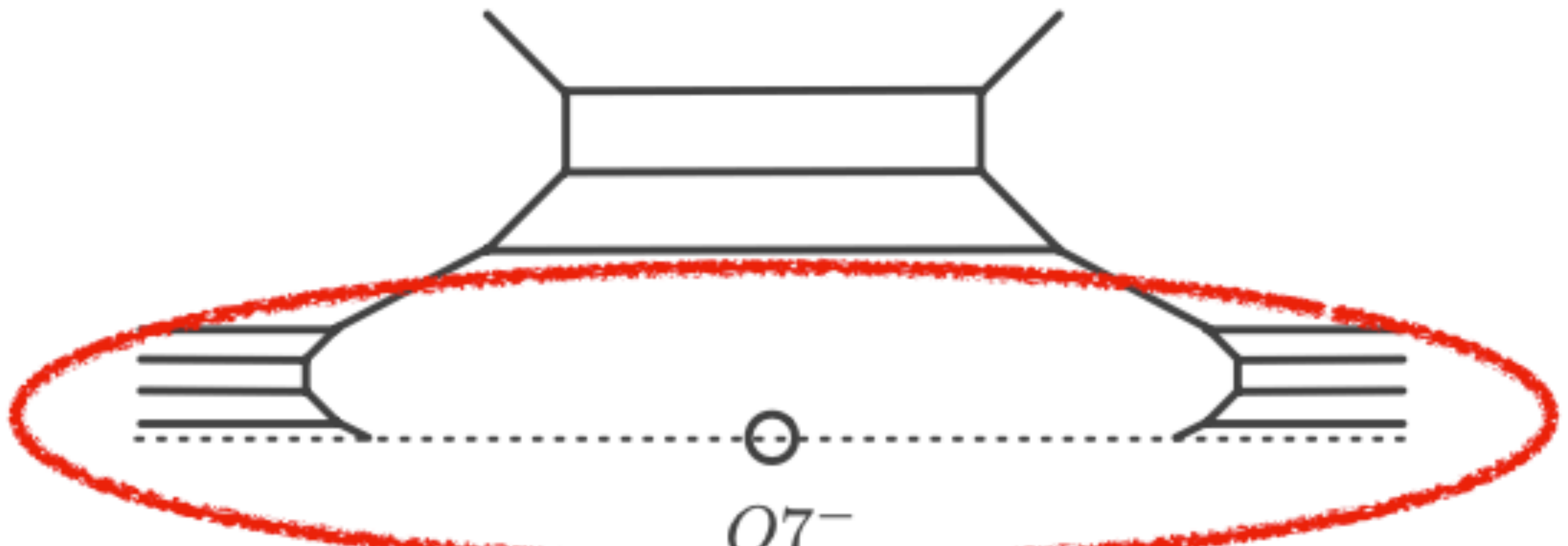
if 8 D7s are near the monodromy cut

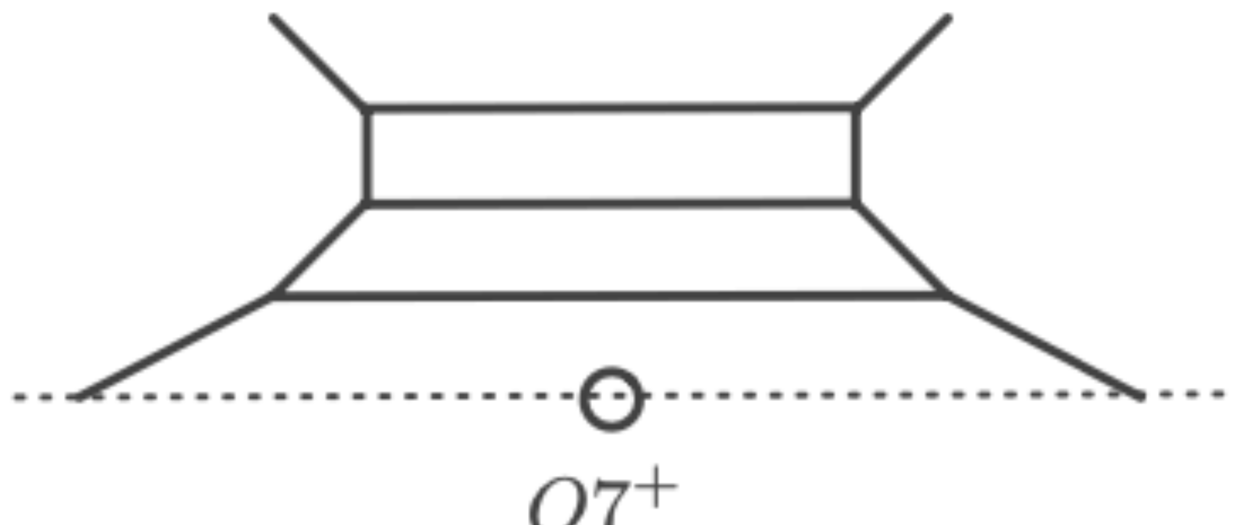
SU(2N) +1**AS** + 8**F** vs. SU(2N) +1**Sym**





Sp(N) + 8**F** vs. SO(2N)





Cubic prepotential

[SungSoo-Xiaobin-SN-Futoshi]

Cubic prepotential: Classical+perturbative contributions

$$\mathcal{F} = \frac{1}{2g_0^2} h_{ij}\phi^i\phi^j + \frac{\kappa}{6} d_{ijk}\phi^i\phi^j\phi^k + \frac{1}{12}\Big(\sum_{\mathrm{Roots}} |R\cdot\phi|^3 - \sum_f \sum_{w\in W_f} |w\cdot\phi + m_f|^3 \Big)$$

$$h_{ij} = \mathrm{Tr}(T_iT_j),\ d_{abc} = \frac{1}{2}\mathrm{Tr}T_a(T_bT_c + T_cT_b), \quad W_f = \text{Weight of } G \text{ in the rep. } r_f$$

First check: cubic prepotentials agree upon mass specializations of 8F

$O7^- + 8\ D7 \longrightarrow O7^+$

$$\mathcal{F}_{\mathrm{SU}(N)_\kappa + 1\mathbf{AS} + 8\mathbf{F}} \xrightarrow[m_{1,\dots,8}=\frac{m}{2}]{\text{Freezing}} \mathcal{F}_{\mathrm{SU}(N)_\kappa + 1\mathbf{Sym}}\ . \qquad m_{\mathbf{AS}} = m = m_{\mathbf{Sym}}$$

$$\mathcal{F}_{\mathrm{Sp}(N) + 1\mathbf{AS} + 8\mathbf{F}} \xrightarrow[m_{1,\dots,8}=0]{\text{Freezing}} \mathcal{F}_{\mathrm{SO}(2N)}$$

Freezing on SU instanton partition functions

[SungSoo-Xiaobin-SN-Futoshi]

AHDM contour integral

$$\mathcal{Z}^{\text{vec}}_{\text{SU}(N)_\kappa,k} = e^{\kappa \sum_{I=1}^k \phi_I} \frac{\prod_{I\neq J} \text{sh}(\phi_I - \phi_J) \cdot \prod_{I,J} \text{sh}(2\epsilon_+ - \phi_I + \phi_J)}{\prod_{I,J} \text{sh}(\epsilon_{1,2} + \phi_I - \phi_J) \prod_{I=1}^k \prod_{s=1}^N \text{sh}(\epsilon_+ \pm (\phi_I - a_s))}$$

$$\mathcal{Z}^{N_f=8}_{\text{SU}(N),k}(m_l) = \prod_{I=1}^k \prod_{l=1}^8 \text{sh}(\phi_I + m_l)$$

$$\mathcal{Z}^{\text{sym}}_{\text{SU}(N),k}(m) = \prod_{I=1}^k \text{sh}\left(2\phi_I + m \pm \epsilon_-\right) \prod_{s=1}^N \text{sh}\left(\phi_I + a_s + m\right) \prod_{I<J}^k \frac{\text{sh}\left(\phi_I + \phi_J + m \pm \epsilon_-\right)}{\text{sh}\left(-\epsilon_+ \pm (\phi_I + \phi_J + m)\right)}$$

$$\mathcal{Z}^{\text{anti}}_{\text{SU}(N),k}(m) = \prod_{I=1}^k \frac{\prod_{s=1}^N \text{sh}\left(\phi_I + a_s + m\right)}{\text{sh}\left(-\epsilon_+ \pm (2\phi_I + m)\right)} \prod_{I<J}^k \frac{\text{sh}\left(\phi_I + \phi_J + m \pm \epsilon_-\right)}{\text{sh}\left(-\epsilon_+ \pm (\phi_I + \phi_J + m)\right)}$$

Freezing at the level of instanton PF

$$\frac{1}{\text{sh}\left(-\epsilon_+ \pm (2\phi_I + m)\right)} \prod_{l=1}^8 \text{sh}\left(\phi_I + m_l\right)\Big|_{m_l = \frac{m\pm\epsilon_\pm}{2}(+\pi i)} = \text{sh}\left(2\phi_I + m \pm \epsilon_-\right)$$

$$Z^{\text{SU}(N)_\kappa + 1\textbf{AS} + 8\textbf{F}}_{\text{inst}} \xrightarrow[m_l = \frac{m\pm\epsilon_\pm}{2}(+\pi i)]{\text{freezing}} Z^{\text{SU}(N)_\kappa + 1\textbf{Sym}}_{\text{inst}} .$$

E-string $\longrightarrow$ M-string

[SungSoo-Xiaobin-SN-Futoshi]

E-string and M-string have 6d origin, but they admit 5d realization

E-string on S^1 = 5d SU(2)+8**F**

M-string on S^1 = 5d SU(2)+1**Sym**

Note that **AS** is trivial in SU(2), and **Sym**=**Adj**

Freezing method gives the connection between PF of E-string and M-string

$$Z^{\text{E-string}} \xrightarrow[m_l=\frac{m\pm\epsilon_\pm}{2}(+\pi i)]{\text{freezing}} Z^{\text{M-string}} .$$

We explicitly check this by 5d instanton partition functions

Freezing on SO/Sp instanton partition functions

[SungSoo-Xiaobin-SN-Futoshi]

AHDM contour integral

$$\mathcal{Z}_{2k,+}^{\mathrm{Sp}(N)+8\mathbf{F}} = \frac{1}{2^k k!} \prod_{I=1}^{k} \frac{\mathrm{sh}\,(2\epsilon_+) \prod_{l=1}^{8} \mathrm{sh}(\pm\phi_I + m_l)}{\mathrm{sh}\,(\epsilon_{1,2})\mathrm{sh}(\epsilon_{1,2} \pm 2\phi_I) \prod_{s=1}^{N} \mathrm{sh}\,(\epsilon_+ \pm \phi_I \pm a_s)} \prod_{I<J}^{k} \frac{\mathrm{sh}(2\epsilon_+ \pm \phi_I \pm \phi_J)\ \mathrm{sh}(\pm\phi_I \pm \phi_J)}{\mathrm{sh}(\epsilon_{1,2} \pm \phi_I \pm \phi_J)}$$

$$\mathcal{Z}_{k}^{\mathrm{SO}(2N)} = \frac{1}{2^k\, k!} \frac{\mathrm{sh}^k(2\epsilon_+)}{\mathrm{sh}^k(\epsilon_{1,2})} \prod_{I=1}^{k} \frac{\mathrm{sh}(2\epsilon_+ \pm 2\phi_I)\ \mathrm{sh}(\pm 2\phi_I)}{\prod_{s=1}^{N} \mathrm{sh}(\epsilon_+ \pm \phi_I \pm a_s)} \prod_{I<J} \frac{\mathrm{sh}(2\epsilon_+ \pm \phi_I \pm \phi_J)\ \mathrm{sh}(\pm\phi_I \pm \phi_J)}{\mathrm{sh}(\epsilon_{1,2} \pm \phi_I \pm \phi_J)}$$

Note that the other sectors drop at this specialization $m_l = 0, \pi i$

$$\left.\frac{\prod_{l=1}^{8} \mathrm{sh}(\pm\phi_I + m_l)}{\mathrm{sh}(\epsilon_{1,2} \pm 2\phi_I)}\right|_{m_l=0(+\pi i),\epsilon_{1,2}(+\pi i),\epsilon_+(+\pi i)} = \mathrm{sh}(2\epsilon_+ \pm 2\phi_I)\ \mathrm{sh}(\pm 2\phi_I)$$

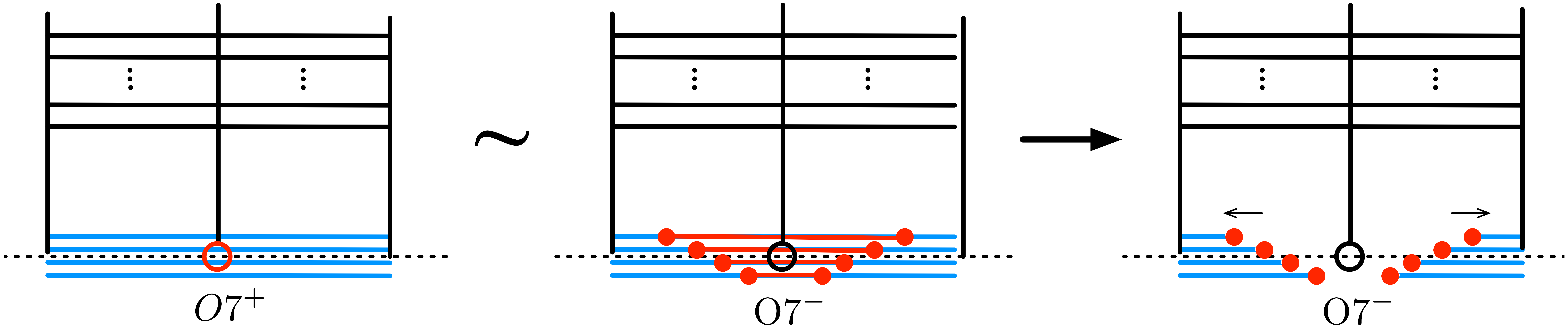
Freezing at the level of instanton PF

$$Z_{\mathrm{inst}}^{\mathrm{Sp}(N)+8\mathbf{F}} \xrightarrow[m_l=0(+\pi i),\epsilon_{1,2}(+\pi i),\epsilon_+(+\pi i)]{\text{freezing}} Z_{\mathrm{inst}}^{\mathrm{SO}(2N)}\ .$$

Unfreezing

[SungSoo-Li-SN-Futoshi]

We can displace 8 D7-branes from $O7^-$



Bring 8 color D5-branes, perform Higgsing, and Hanay-Witten transition

This process gives rise to relation of another kind

$$\mathrm{SU}(N+8)_\kappa + 1\mathbf{Sym} \xrightarrow{\text{unfreezing}} \mathrm{SU}(N)_\kappa + 1\mathbf{AS}$$

$$\mathrm{SO}(2N+8) \xrightarrow{\text{unfreezing}} \mathrm{Sp}(N)$$

Unfreezing on SU instanton partition functions

[SungSoo-Li-SN-Futoshi]

For the sake of simplicity, we focus on unrefined partition functions

$$\mathcal{Z}^{\text{vec}}_{\text{SU}(N)_\kappa,k} = e^{\kappa \sum_{I=1}^{k} \phi_I} \frac{\prod_{I\neq J} \text{sh}(\phi_I - \phi_J) \cdot \prod_{I,J} \text{sh}(-\phi_I + \phi_J)}{\prod_{I,J} \text{sh}(\pm\hbar + \phi_I - \phi_J) \prod_{I=1}^{k} \prod_{s=1}^{N} \text{sh}(\pm(\phi_I - a_s))}$$

$$\mathcal{Z}^{\text{sym}}_{\text{SU}(N),k}(m) = \prod_{I=1}^{k} \text{sh}\left(2\phi_I + m \pm \hbar\right) \prod_{s=1}^{N} \text{sh}\left(\phi_I + a_s + m\right) \prod_{I<J}^{k} \frac{\text{sh}\left(\phi_I + \phi_J + m \pm \hbar\right)}{\text{sh}\left(-\epsilon_+ \pm (\phi_I + \phi_J + m)\right)}$$

$$\mathcal{Z}^{\text{anti}}_{\text{SU}(N),k}(m) = \prod_{I=1}^{k} \frac{\prod_{s=1}^{N} \text{sh}\left(\phi_I + a_s + m\right)}{\text{sh}\left(\pm(2\phi_I + m)\right)} \prod_{I<J}^{k} \frac{\text{sh}\left(\phi_I + \phi_J + m \pm \hbar\right)}{\text{sh}\left(\pm(\phi_I + \phi_J + m)\right)}$$

Unfreezing at the level of instanton PF

$$\text{sh}\left(2\phi_I + m \pm \hbar\right) \prod_{s=N+1}^{N+8} \frac{\text{sh}(\phi_I + a_s + m)}{\text{sh}(\pm(\phi_I - a_s))}\Bigg|_{a_s = -\frac{m}{2}(+\pi i),\, \frac{-m\pm\hbar}{2}(+\pi i)} = \text{sh}\left(\pm(2\phi_I + m)\right)$$

$$Z_{\text{inst}}^{\text{SU}(N+8)_\kappa + 1\textbf{Sym}} \xrightarrow[a_s = -\frac{m}{2}(+\pi i),\, \frac{-m\pm\hbar}{2}(+\pi i)]{\text{unfreeezing}} Z_{\text{inst}}^{\text{SU}(N)_\kappa + 1\textbf{AS}}$$

BPS jumping

[SungSoo-Xiaobin-SN-Futoshi]

Closed-form expressions for both unrefined PFs as Young diagram sum

$$\begin{aligned} Z_k^{\mathrm{SU}(N)_\kappa+1\mathbf{Sym}} &= \sum_{|\vec{\lambda}|=k} \prod_{s=1}^{N} \prod_{x\in\lambda^{(s)}} e^{\kappa\phi_s(x)} \frac{\mathrm{sh}\left(2\phi_s(x)+m\pm\hbar\right)\cdot\prod_{t=1}^{N}\mathrm{sh}\left(\phi_s(x)+a_t+m\right)}{\prod_{t=1}^{N}\mathrm{sh}^2\left(N_{s,t}(x)\right)} \\ &\times \prod_{s\le t}^{N} \prod_{\substack{x\in\lambda^{(s)},y\in\lambda^{(t)} \\ x<y}} \frac{\mathrm{sh}\left(\phi_s(x)+\phi_t(y)+m\pm\hbar\right)}{\mathrm{sh}^2\left(\phi_s(x)+\phi_t(y)+m\right)} \\ &=: \sum_{|\vec{\lambda}|=k} Z_{\vec{\lambda}}^{\mathrm{SU}(N)_\kappa+1\mathbf{Sym}} \end{aligned}$$

They are related by

$$Z_k^{\mathrm{SU}(N)_\kappa+1\mathbf{AS}} = \sum_{|\vec{\lambda}|=k} C^{\mathrm{anti}}_{\vec{\lambda},\vec{a}}\, Z_{\vec{\lambda}}^{\mathrm{SU}(N+8)_\kappa+1\mathbf{Sym}}\Big|_{a_s=-\frac{m}{2}(+\pi i),\frac{-m\pm\hbar}{2}(+\pi i)}$$

Due to degenerate poles in JK residue, non-trivial multiplicity coefficients emerge

This leads to BPS jumping

$$Z_k^{\mathrm{SU}(N+8)_\kappa+1\mathbf{Sym}} = \sum_{|\vec{\lambda}|=k} Z_{\vec{\lambda}}^{\mathrm{SU}(N+8)_\kappa+1\mathbf{Sym}}$$

$$\xrightarrow[\text{BPS jump}]{a_s=-\frac{m}{2}(+\pi i),\frac{-m\pm\hbar}{2}(+\pi i)} \sum_{|\vec{\lambda}|=k} C^{\mathrm{anti}}_{\vec{\lambda},\vec{a}}\, Z_{\vec{\lambda}}^{\mathrm{SU}(N+8)_\kappa+1\mathbf{Sym}}\Big|_{a_s=-\frac{m}{2}(+\pi i),\frac{-m\pm\hbar}{2}(+\pi i)}$$

Recall the previous slide

[SN-RuiDong]

Remarkably, each sector is related to SO(2N+8) instanton partition function

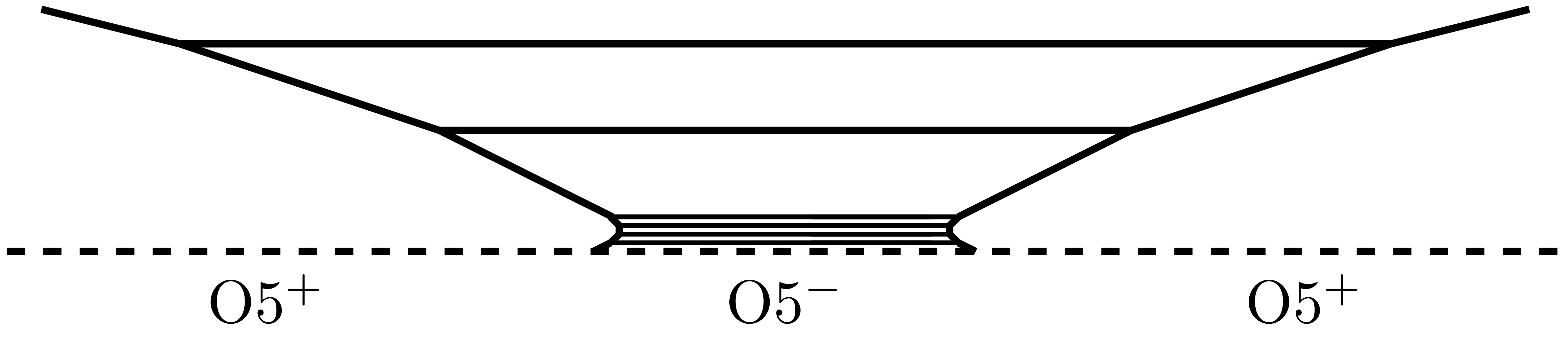
$$Z^{\mathrm{Sp}(N)}_{2\ell,+} \propto \sum_{|\vec{\lambda}|=\ell} C^{\mathrm{Sp}}_{\vec{\lambda},\vec{a}} Z^{\mathrm{SO}(2N+8)}_{\vec{\lambda}}\Big|_{a_{N+j}=\frac{\hbar}{2}(+\pi i),\ 0(+\pi i)}$$

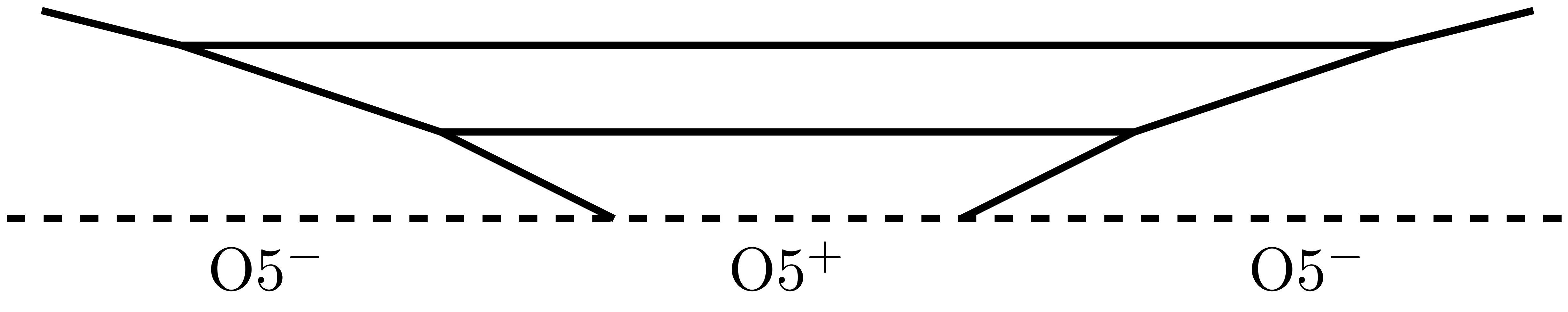
$$Z^{\mathrm{Sp}(N)}_{2\ell+1,+} \propto \sum_{|\vec{\lambda}|=\ell} C^{\mathrm{Sp}}_{\vec{\lambda},\vec{a}} Z^{\mathrm{SO}(2N+8)}_{\vec{\lambda}}\Big|_{a_{N+j}=\frac{\hbar}{2}(+\pi i),\ \hbar,\ \pi i}$$

$$Z^{\mathrm{Sp}(N)}_{2\ell+2,-} \propto \sum_{|\vec{\lambda}|=\ell} C^{\mathrm{Sp}}_{\vec{\lambda},\vec{a}} Z^{\mathrm{SO}(2N+8)}_{\vec{\lambda}}\Big|_{a_{N+j}=\frac{\hbar}{2}(+\pi i),\hbar(+\pi i)}$$

$$Z^{\mathrm{Sp}(N)}_{2\ell+1,-} \propto \sum_{|\vec{\lambda}|=\ell} C^{\mathrm{Sp}}_{\vec{\lambda},\vec{a}} Z^{\mathrm{SO}(2N+8)}_{\vec{\lambda}}\Big|_{a_{N+j}=\frac{\hbar}{2}(+\pi i),0,\hbar+\pi i}$$

Relation between Sp(N) and SO(2N+8) can be seen in 5-brane web



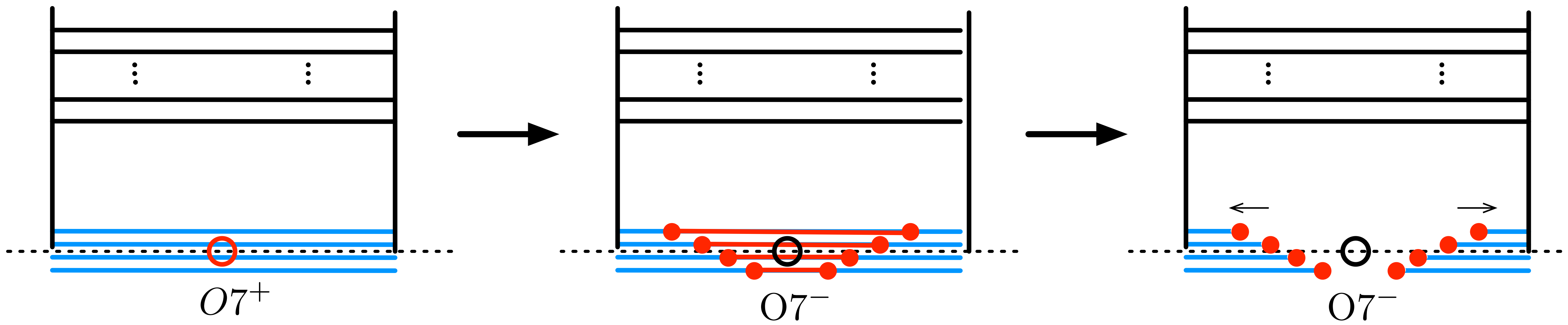


We will come back to this issue later: $\longrightarrow$ I will unfreeze the secret!

Unfreezing on SO/Sp instanton partition functions

[SungSoo-Xiaobin-SN-Futoshi]

This relation can be explained by unfreezing



Adjusting Coulomb branch parameters of ADHM integral $\mathscr{Z}_k^{\mathrm{SO}(2N+8)}$

One can obtain each sector of ADHM integral of Sp(N) instanton

$$Z_k^{\mathrm{SO}(2N+8)} \xrightarrow[a_{N+j}=\frac{\hbar}{2}(+\pi i),\ 0(+\pi i)]{\text{unfreezing}} Z_{2k}^{\mathrm{Sp}(N),+} .$$

Upon unfreezing, degenerate poles appear in JK integral $\longrightarrow$ multiplicity coefficients

$$Z_\ell^{\mathrm{SO}(2N+8)} = \sum_{|\vec{\lambda}|=\ell} Z_{\vec{\lambda}}^{\mathrm{SO}(2N+8)}$$

$$\xrightarrow[\text{BPS jump}]{a_{N+j}=\frac{\hbar}{2}(+\pi i),\ 0(+\pi i)} \quad \sum_{|\vec{\lambda}|=\ell} C^{\mathrm{Sp}}_{\vec{\lambda},\vec{a}} Z_{\vec{\lambda}}^{\mathrm{SO}(2N+8)}\Big|_{a_{N+j}=\frac{\hbar}{2}(+\pi i),\ 0(+\pi i)}$$

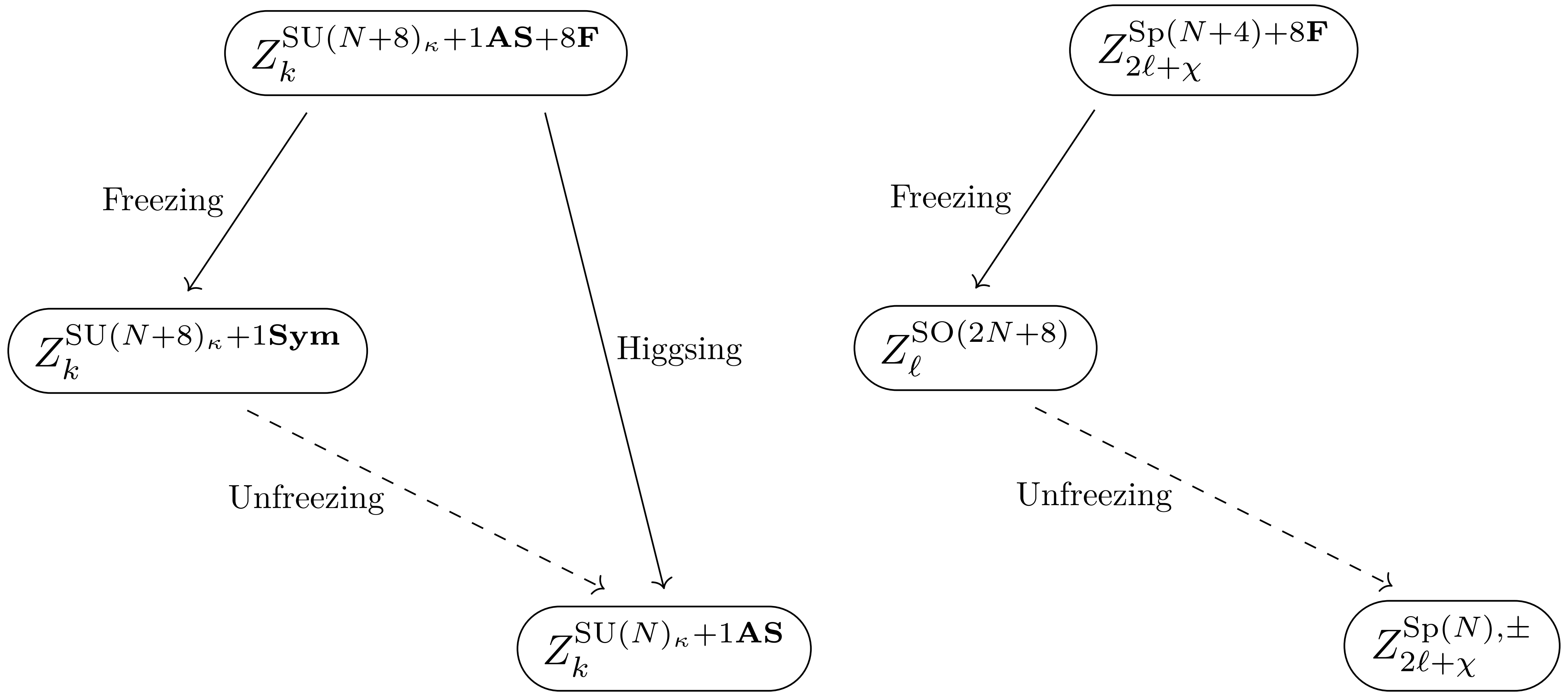
Summary

- Even after Nekrasov, we are slowly making progress on our understanding of instanton counting

- At unrefined level, closed-form expression of instanton partition functions are available as Young diagram sums.

- Young diagram sums are useful for many purposes
  Eg: topological vertex and qq-characters

- 5-brane web with O7-plane provides interesting relations: O7$^-$+8 D7 vs O7$^+$



- We find novel BPS jumping that require further investigation

Summary

- Even after Nekrasov, we are slowly making progress on our understanding of instanton counting

- At unrefined level, closed-form expression of instanton partition functions are available as Young diagram sums.

- Young diagram sums are useful for many purposes
  Eg: topological vertex and qq-characters

- 5-brane web with O7-plane provides interesting relations: $O7^{-}$+8 D7 vs $O7^{+}$

- We find novel BPS jumping that require further investigation

Thanks for listening!

Thanks for wonderful collaborations!