

Multiplicity-free quantum 6j-symbols for $U_q(\mathfrak{sl}_N)$

Refs) Kirillov, Reshetikhin : Representation algebra $U_q(\mathfrak{sl}_2)$,
q-orthogonal polynomials and
invariants of links (1988)

Nawata, Ramadevi, Zodinmawia : Multiplicity-free quantum 6j-symbols
for $U_q(\mathfrak{sl}_N)$ arXiv:1302.5143

Plan

1. Review of A_1 -case

1.1 q-Clebsch-Gordan for $U_q(\mathfrak{sl}_2)$

1.2 q-6j symbols for $U_q(\mathfrak{sl}_2)$

2. Multiplicity-free q-6j for $U_q(\mathfrak{sl}_N)$

1. A_1 -case.

$$U_q(\mathfrak{sl}_2) \quad KEK^{-1} = q E$$

$$KFK^{-1} = q^{-1} F$$

$$[E, F] = \frac{K - K^{-1}}{q^{1/2} - q^{-1/2}}$$

For usual convention

$$q \rightarrow q^2$$

Sph-j representation

$$\pi_j(E) e_m^j = ([j-m][j+m-1])^{1/2} e_{m+1}^j$$

$$\pi_j(F) e_m^j = ([j+m][j-m+1])^{1/2} e_{m-1}^j$$

$$\pi_j(K) e_m^j = q^m e_m^j$$

1.1 quantum Clebsch-Gordon Coefficient

decomposition of tensor product into irred. rep's

$$V^{j_1} \otimes V^{j_2} = \bigoplus V^j$$

j satisfy quantum CG condition

$$|j_1 - j_2| \leq j \leq j_1 + j_2$$

$$2j \equiv 2j_1 + 2j_2 \pmod{2}$$

$$e_m^j(j_1, j_2) = \sum_{m_1, m_2} \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix} e_{m_1}^{j_1} \otimes e_{m_2}^{j_2}$$

There are many expressions for q -CG coefficients

- Van der Waerden

$$\begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix}$$

$$= \delta_{m_1+m_2, m} \Delta(j_1, j_2, j)$$

$$\times q^{\frac{1}{2}(j_1+j_2-j)(j_1+j_2+j+1)}$$

$$\times \left\{ [j_1+m]! [j_1-m_1]! [j_2+m_2]! [j_2-m_2]! [j+m]! [j-m]! [2j+1]! \right\}^{\frac{1}{2}}$$

$$\sum_{r \geq 0} (-1)^r q^{-\frac{1}{2}r(j_1+j_2+j+1)}$$

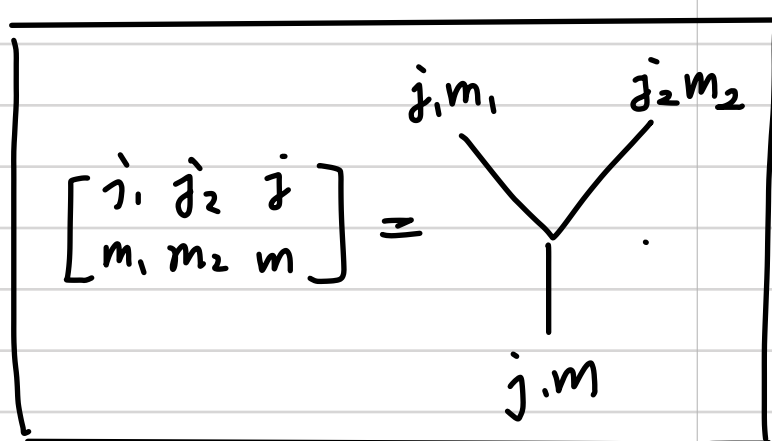
$$\times \frac{1}{[r]! [j_1+j_2+j-r]! [j_1-m_1-r]! [j_2+m_2-r]! [j-j_2+m_1+r]! [j-j_1+r-m_2]!}$$

$$= (-1)^x \frac{\{p(x)\}^{\frac{1}{2}}}{d_n} Q_n(q^{-x}; a, b, N | q)$$

$$Q_n(x; a, b, N | q) = {}_3\psi_2 \left(\begin{matrix} q^{-n}, ab q^{n+1}, x \\ aq, q^{-N} \end{matrix} ; q, q \right)$$

Hahn poly

$$p(x) := p(x, a, b, N | q) = \frac{(aq; q)_x (bq; q)_x}{(q; q)_x (q; q)_{N-x}}$$



graphical rep of q -CG coefficient

$$d_n^2 = \frac{(abq^2; q)_\infty (aq)_\infty^N}{(q; q)_\infty} \frac{(1-abq)(q, bq, abq^{N+2}; q)_\infty}{(1-abq^{2N+1})(aq, abq, q^{-N}; q)_\infty} \\ \times (-aq) q^{\binom{n}{2} - Nn}$$

$$n = j - m$$

$$a = q^{j_2 - j + m_1}$$

$$\alpha = j_1 - m_1$$

$$b = q^{j_1 - j_2 + m}$$

$$N = j + j_2 - m_1$$

Universal R-matrix.

$$R = q^{\frac{1}{2}} K \otimes K \sum_{n \geq 0} \frac{(1 - q^{-1})^n}{[n]!} q^{-\frac{n(n-1)}{4}} E^n \otimes F^n$$

Let us consider $R^{\vec{j}, \vec{j}_2} = (\pi_{j_1} \otimes \pi_{j_2}) R$ acting on $V_{j_1} \otimes V_{j_2}$

$$(R^{\vec{j}, \vec{j}_2})_{n_1+n, n_2-n}^{n_1, n_2} = \frac{(1-q^{-1})^n}{[n]!} q^{\frac{n_1 - n_2 + 2n}{2} n}$$

$$\left[\frac{[j_1 - n_1]! [j_1 + n_1 + n]! [j_2 + n_2]! [j_2 - n_2 + n]!}{[j_1 - n_1 - n]! [j_1 + n]! [j_2 + n_2 - n]! [j_2 - n_2]!} \right]^{\frac{1}{2}}$$

$$= \sum_{j \in j_1 \otimes j_2} q^{\frac{1}{2} [j_1(j_1+1) - j_1(j_1+1) - j_2(j_2+1)]} \begin{bmatrix} j_1 & j_2 & j \\ n_1 & n_2 & m \end{bmatrix} \begin{bmatrix} j_1 & j_2 & j \\ n_1+n & n_2-n & m \end{bmatrix}$$

Orthogonality of q -CG coefficient

$$\sum_{m_1, m_2} \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix} \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix} = \delta_{j, j'} \delta_{m, m'} \delta(j, j_2, j)$$

$$\sum_{j, m} \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix} \begin{bmatrix} j_1 & j_2 & j \\ m_1' & m_2' & m \end{bmatrix} = \delta_{m, m'} \delta_{m_2, m_2'}$$

1.2. quantum 6j-symbols.

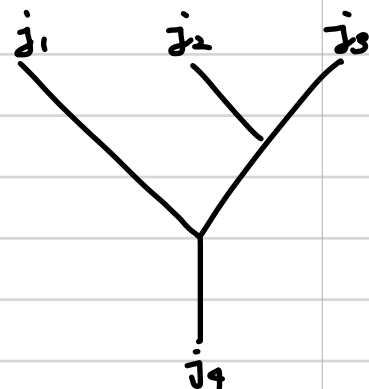
$$V_{j_1} \otimes V_{j_2} \otimes V_{j_3} \begin{cases} \rightarrow (V_{j_1} \otimes V_{j_2})_{j_{12}} \otimes V_{j_3} \\ \rightarrow V_{j_1} \otimes (V_{j_2} \otimes V_{j_3})_{j_{23}} \end{cases}$$

$$\begin{array}{c} j_1 m_1 \quad j_2 m_2 \quad j_3 m_3 \\ \diagdown \quad \diagup \\ \quad \quad \quad \\ \diagup \quad \diagdown \\ j_4 m_4 \end{array} = \begin{bmatrix} j_{12} & j_3 & j_4 \\ m_{12} & m_3 & m_4 \end{bmatrix} \begin{bmatrix} j_1 & j_2 & j_{12} \\ m_1 & m_2 & m_{12} \end{bmatrix}$$

$$\begin{array}{c} j_1 m_1 \quad j_2 m_2 \quad j_3 m_3 \\ \diagdown \quad \diagup \\ \quad \quad \quad \\ \diagup \quad \diagdown \\ j_4 m_4 \end{array} = \begin{bmatrix} j_1 & j_{23} & j_4 \\ m_1 & m_{23} & m_4 \end{bmatrix} \begin{bmatrix} j_2 & j_3 & j_{23} \\ m_2 & m_3 & m_{23} \end{bmatrix}$$

quantum 6j-symbols appear as transformation matrices
(independent of m_i).

$$\begin{array}{c} j_1 \quad j_2 \quad j_3 \\ \diagdown \quad \diagup \\ \quad \quad \quad \\ \diagup \quad \diagdown \\ j \end{array} = \sum_{j_{23}} \epsilon \left(\frac{[2j_{12}+1][2j_{23}+1]}{[2j+1]} \right)^{\frac{1}{2}} \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{23} \end{Bmatrix}$$



$$\begin{bmatrix} j_{12} & j_3 & j_4 \\ m_{12} & m_3 & m_4 \end{bmatrix} \begin{bmatrix} j_1 & j_2 & j_{12} \\ m_1 & m_2 & m_{12} \end{bmatrix}$$

$$= \sum_{j_{23}} \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{23} \end{Bmatrix} \begin{bmatrix} j_1 & j_{23} & j_4 \\ m_1 & m_{23} & m_4 \end{bmatrix} \begin{bmatrix} j_2 & j_3 & j_{23} \\ m_2 & m_3 & m_{23} \end{bmatrix}$$

Using orthogonality of q -CG

$$\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{23} \end{Bmatrix} = \begin{bmatrix} j_1 & j_{23} & j_4 \\ m_1 & m_{23} & m_4 \end{bmatrix}^{-1} \sum_{\substack{m_2, m_3 \\ m_2+m_3=m_4-m_1}} \begin{bmatrix} j_2 & j_3 & j_{23} \\ m_2 & m_3 & m_{23} \end{bmatrix} \begin{bmatrix} j_{12} & j_3 & j_4 \\ m_{12} & m_3 & m_4 \end{bmatrix} \begin{bmatrix} j_1 & j_2 & j_{12} \\ m_1 & m_2 & m_{12} \end{bmatrix}$$

$m_4 = j_4$ gauge fixing \Rightarrow summation over 4 variables.

First extend to summation over 6 variables.

Then reduces to summation over 1 variables.
by using q -Chu-Vandermonde identity.

$$\begin{bmatrix} m+n \\ k \end{bmatrix} = \sum_{i+j=k} \begin{bmatrix} m \\ i \end{bmatrix} \begin{bmatrix} n \\ j \end{bmatrix} q^{(m-i)j}$$

$$\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{23} \end{Bmatrix} = \Delta(j_1, j_2, j_{12}) \Delta(j_3, j_4, j_{23}) \Delta(j_1, j_4, j_{23}) \Delta(j_2, j_3, j_{23})$$

$$\sum_z (-1)^z \frac{[z+1]!}{[z-j_1-j_2-j_{12}]! [z-j_1-j_4-j_{23}]! [z-j_2-j_3-j_{12}]! [z-j_3-j_4-j_{23}]!}$$

$$\times \frac{1}{[j_1+j_2+j_3+j_4-z]! [j_1+j_3+j_{12}+j_{23}-z]! [j_2+j_4+j_{12}+j_{23}-z]!}$$

$$= \frac{(-1)^{j_1+j_2+j_3+j_4} [j_1+j_2+j_3+1]! \Delta \Delta \Delta \Delta}{[j_1+j_2-j_{12}]! [j_1+j_4-j_{23}]! [j_3+j_4-j_{12}]! [j_2+j_3-j_{23}]! [j_{12}+j_{23}-j_2-j_4]! [j_{12}+j_{23}-j_1-j_3]!}$$

$$\varphi_3 \left(\begin{matrix} q^{-j_1-j_2-j_{12}} & q^{-j_1-j_4-j_{23}} & q^{-j_3-j_4-j_{12}} & q^{-j_{12}-j_3-j_{23}} \\ q^{-j_1-j_2-j_3-j_{12}} & q^{-j_{12}-j_{23}-j_2-j_4+1} & q^{-j_{12}-j_{23}-j_1-j_3+1} & ; q, q \end{matrix} \right)$$

The properties of q -6j.

$$1. \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{23} \end{Bmatrix} = \begin{Bmatrix} j_2 & j_1 & j_{12} \\ j_4 & j_3 & j_{23} \end{Bmatrix} = \begin{Bmatrix} j_1 & j_4 & j_{23} \\ j_3 & j_2 & j_{12} \end{Bmatrix} = \begin{Bmatrix} j_3 & j_4 & j_{12} \\ j_1 & j_2 & j_{23} \end{Bmatrix}$$

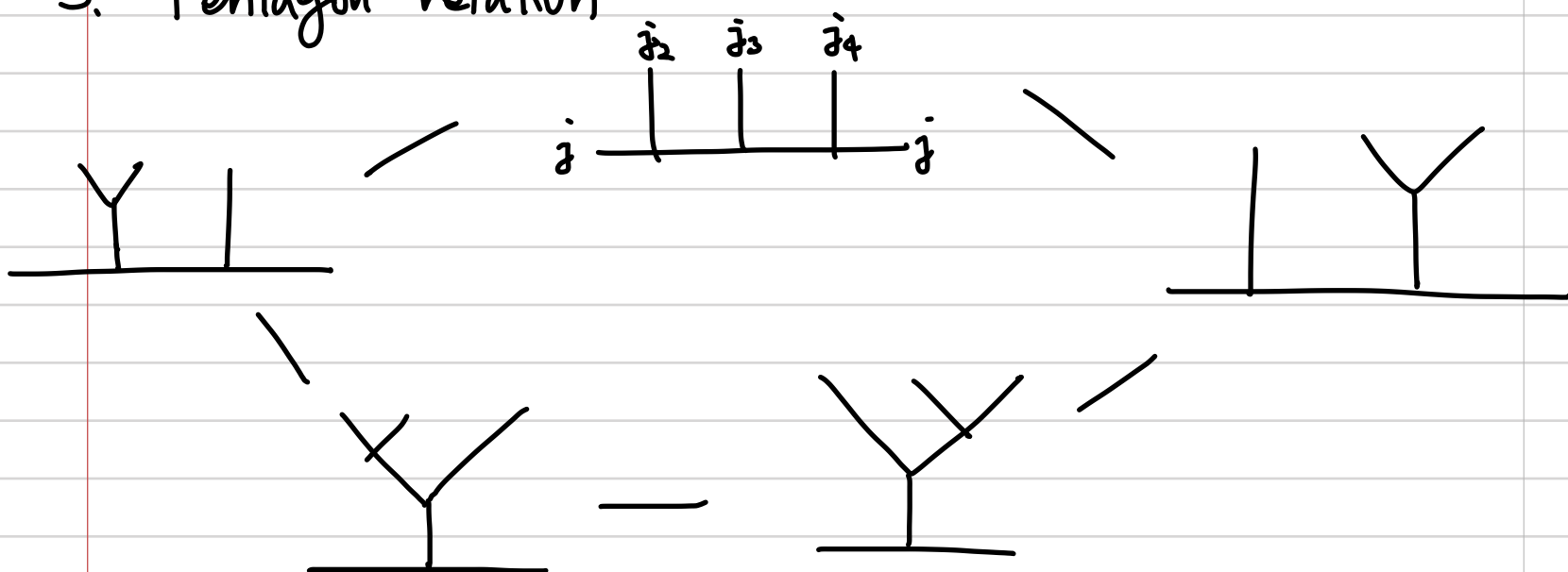
$$2. \begin{Bmatrix} j_1 & j_2 & 0 \\ j_3 & j_4 & j_{23} \end{Bmatrix} = \frac{\in \delta_{j_1 j_2} \delta_{j_3 j_4}}{([2j_2+1][2j_3+1])^{\frac{1}{2}}}$$

$$3. \sum_x [2j_{12}+1][2j_{23}+1] \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{23} \end{Bmatrix} \begin{Bmatrix} j_1 & j_2 & j'_{12} \\ j_3 & j_4 & j'_{23} \end{Bmatrix} = \delta_{j_{23} j'_{23}}$$

orthogonality.

$$4. \sum_{j_{12}} \in [2x+1] q^{C_{12}/2} \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{23} \end{Bmatrix} \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_4 & j_3 & j_{24} \end{Bmatrix} \\ = \begin{Bmatrix} j_3 & j_2 & j_{23} \\ j_4 & j_1 & j_{24} \end{Bmatrix} q^{\frac{C_{12}+C_{24}}{2} - \frac{C_1+C_2+C_3+C_4}{2}}$$

5. Pentagon relation



②. Multiplicity-free q -bj for $U_q(\mathfrak{sl}_N)$.

For higher ranks, j is replaced by highest weight λ

$$\left\{ \begin{matrix} \lambda_1 & \lambda_2 & \lambda_{12} \\ \lambda_3 & \lambda_4 & \lambda_{34} \end{matrix} \right\}$$

λ_{12} & λ_{34} have to satisfy fusion rules

$$V_{\lambda_{12}} \in (V_{\lambda_1} \otimes V_{\lambda_2}) \cap (V_{\lambda_3}^* \otimes V_{\lambda_4}^*)$$

λ^* : conjugate rep of λ

$$V_{\lambda_{34}} \in (V_{\lambda_2} \otimes V_{\lambda_3}) \cap (V_{\lambda_1}^* \otimes V_{\lambda_4}^*)$$

$\Rightarrow \left\{ \begin{matrix} \square & \square & \square \\ \square & \square & \square \end{matrix} \right\}$ is not allowed for $N > 2$ for example.

The symmetry of q -bj of \mathfrak{sl}

$$\left\{ \begin{matrix} \lambda_1 & \lambda_2 & \lambda_{12} \\ \lambda_3 & \lambda_4 & \lambda_{23} \end{matrix} \right\} = \left\{ \begin{matrix} \lambda_3 & \lambda_2 & \lambda_{23} \\ \lambda_1 & \lambda_4 & \lambda_{12} \end{matrix} \right\} = \left\{ \begin{matrix} \lambda_1^* & \lambda_2^* & \lambda_{12}^* \\ \lambda_3^* & \lambda_4^* & \lambda_{23}^* \end{matrix} \right\} \quad (*)$$

$$= \left\{ \begin{matrix} \lambda_1 & \lambda_4 & \lambda_{23}^* \\ \lambda_3 & \lambda_2 & \lambda_{12}^* \end{matrix} \right\} = \left\{ \begin{matrix} \lambda_2 & \lambda_1 & \lambda_{12} \\ \lambda_4 & \lambda_3 & \lambda_{23}^* \end{matrix} \right\}$$

The difficulty of higher ranks stems from multiplicity structure in tensor product.

$$\begin{array}{c} \begin{array}{ccc} \square & \square & \square \\ \square & \square & \square \end{array} \otimes \begin{array}{ccc} \square & \square & \square \\ \square & \square & \square \end{array} = \begin{array}{ccc} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{array} \oplus \begin{array}{ccc} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{array} \oplus \begin{array}{ccc} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{array} \oplus \begin{array}{ccc} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{array} \\ \oplus \begin{array}{ccc} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{array} \oplus \begin{array}{ccc} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{array} \oplus \begin{array}{ccc} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{array} \oplus \begin{array}{ccc} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{array} \end{array}$$

Using symmetry (*), multiplicity-free q -bj amounts to the following two types.

$$\left\{ \begin{array}{ccc} \square & \square & \square \\ \square & \square & \square \end{array} \quad \begin{array}{ccc} \square & \square & \square \\ \square & \square & \square \end{array} \quad \begin{array}{ccc} \square & \square & \square \\ \square & \square & \square \end{array} \right\} \text{ type I.}$$

$$\square = \begin{array}{c} \uparrow \\ \vdots \\ \vdots \\ \vdots \\ \downarrow \end{array} \begin{array}{c} N-1 \\ \vdots \\ \vdots \\ \vdots \\ 1 \end{array}$$

$$\left\{ \begin{array}{ccc} \begin{array}{|c|c|c|} \hline & & \\ \hline \end{array} & \begin{array}{|c|c|c|} \hline & & \\ \hline \end{array} & \begin{array}{|c|c|c|c|} \hline & & & \\ \hline \end{array} \\ \begin{array}{|c|c|c|c|} \hline \cdot & & & \cdot \\ \hline \end{array} & \begin{array}{|c|c|c|} \hline \cdot & & \cdot \\ \hline \end{array} & \begin{array}{|c|c|c|c|c|} \hline \cdot & & \cdot & & \\ \hline \end{array} \end{array} \right\} \text{type II}$$

Using properties of q-bj symbols, we read off the pattern.

$$\left\{ \begin{array}{ccc} \lambda_1 & \lambda_2 & \lambda_{12} \\ \lambda_3 & \lambda_4 & \lambda_{23} \end{array} \right\}$$

$$= \Delta(\lambda_1 \lambda_2 \lambda_{12}) \Delta(\lambda_3 \lambda_4 \lambda_{12}) \Delta(\lambda_2 \lambda_3 \lambda_{23}) \Delta(\lambda_1 \lambda_4 \lambda_{23})$$

$$[N-1]! \sum_z (-1)^z [z+N-1]! C_z(\{\lambda_i\}, \lambda_{12} \lambda_{23})$$

$$\left\{ [z - \frac{1}{2} \langle \lambda_1 + \lambda_2 + \lambda_{12}, \alpha_1^\vee + \alpha_{N-1}^\vee \rangle]! [z - \frac{1}{2} \langle \lambda_3 + \lambda_4 + \lambda_{12}, \alpha_1^\vee + \alpha_{N-1}^\vee \rangle]! \right.$$

$$[z - \frac{1}{2} \langle \lambda_3 + \lambda_2 + \lambda_{23}, \alpha_1^\vee + \alpha_{N-1}^\vee \rangle]! [z - \frac{1}{2} \langle \lambda_1 + \lambda_4 + \lambda_{23}, \alpha_1^\vee + \alpha_{N-1}^\vee \rangle]!$$

$$[\frac{1}{2} \langle \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4, \alpha_1^\vee + \alpha_{N-1}^\vee \rangle - z]! [\frac{1}{2} \langle \lambda_1 + \lambda_2 + \lambda_{12} + \lambda_{23}, \alpha_1^\vee + \alpha_{N-1}^\vee \rangle - z]!$$

$$[\frac{1}{2} \langle \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4, \alpha_1^\vee + \alpha_{N-1}^\vee \rangle - z]! [\frac{1}{2} \langle \lambda_1 + \lambda_2 + \lambda_{12} + \lambda_{23}, \alpha_1^\vee + \alpha_{N-1}^\vee \rangle - z]! \Big\}$$

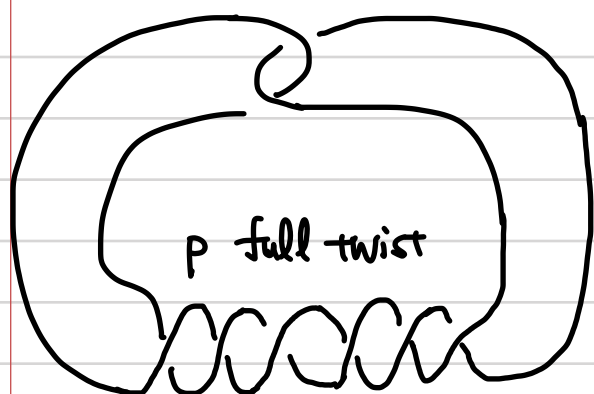
$$\Delta(\lambda_1 \lambda_2 \lambda_3) =$$

$$\frac{[\frac{1}{2} \langle -\lambda_1 + \lambda_2 + \lambda_3, \alpha_1 + \alpha_{N-1} \rangle]! [\frac{1}{2} \langle \lambda_1 - \lambda_2 + \lambda_3, \alpha_1 + \alpha_{N-1} \rangle]! [\frac{1}{2} \langle \lambda_1 + \lambda_2 - \lambda_3, \alpha_1 + \alpha_{N-1} \rangle]!}{[\frac{1}{2} \langle \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4, \alpha_1^\vee + \alpha_{N-1}^\vee \rangle + N-1]!}$$

$$[\frac{1}{2} \langle \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4, \alpha_1^\vee + \alpha_{N-1}^\vee \rangle + N-1]!$$

C_z depends on type I & II. Details can be found in arXiv:1302.5143.

Check has been done.



twist knot K_p

Using the expression of multiplicity-free q -6j.

One can obtain colored HOMFLY polynomials.

These are computed by linear Skein theory (mathematically rigorous)

Kawagoe arXiv:1210.7574

P	-2	-1	0	1	2
K_p	6_1	4_1	0_1	3_1	5_2

$$P_{\underbrace{\square \dots \square}_r}(K_p; a, q) = \sum_{k=0}^{\infty} \sum_{l=0}^k q^k \frac{(aq^{-1}; q)_k}{(q; q)_k} (q^{-r}; q)_k (aq^r; q)_r \\ \times (-1)^l a^{pl} q^{(p+\frac{1}{2})l(l-1)} \frac{1 - aq^{2l-1}}{(aq^{l-1}; q)_{k+1}} \begin{bmatrix} k \\ l \end{bmatrix}_q$$

We check multiplicity-free q -6j reproduce this formula up to $r=4$.

Mirror Symmetry ($\lambda = \begin{array}{|c|} \hline \square \\ \hline \end{array} \leftrightarrow \lambda^T = \begin{array}{|c|} \hline \square \\ \hline \end{array}$) of q -6j.

There is a relation b/w symmetric rep & anti-symmetric rep.

$$\left\{ \begin{array}{ccc} \begin{array}{|c|} \hline \square \\ \hline \end{array} & \begin{array}{|c|} \hline \square \\ \hline \end{array} & \begin{array}{|c|} \hline \square \\ \hline \end{array} \\ \uparrow \downarrow \begin{array}{|c|} \hline \vdots \\ \hline \end{array} & \begin{array}{|c|} \hline \vdots \\ \hline \end{array} \uparrow \downarrow & \begin{array}{|c|} \hline \square \\ \hline \end{array} \end{array} \right\} = \frac{[2]^2}{[N+1][N][N-1][N-2]} \frac{[3][N-3][N-4] - [N][N+1]}{[N] + [3][N-4]}$$

\updownarrow $[N+k] \leftrightarrow [N-k]$

$$\left\{ \begin{array}{ccc} \begin{array}{|c|} \hline \square \\ \hline \end{array} & \begin{array}{|c|} \hline \square \\ \hline \end{array} & \begin{array}{|c|} \hline \square \\ \hline \end{array} \\ \begin{array}{|c|} \hline \square \\ \hline \end{array} & \begin{array}{|c|} \hline \square \\ \hline \end{array} & \begin{array}{|c|} \hline \square \\ \hline \end{array} \end{array} \right\} = \frac{[2]^2}{[N-1][N][N+1][N+2]^2} \frac{[3][N+3][N+4] - [N][N-1]}{[N] + [3][N+4]}.$$

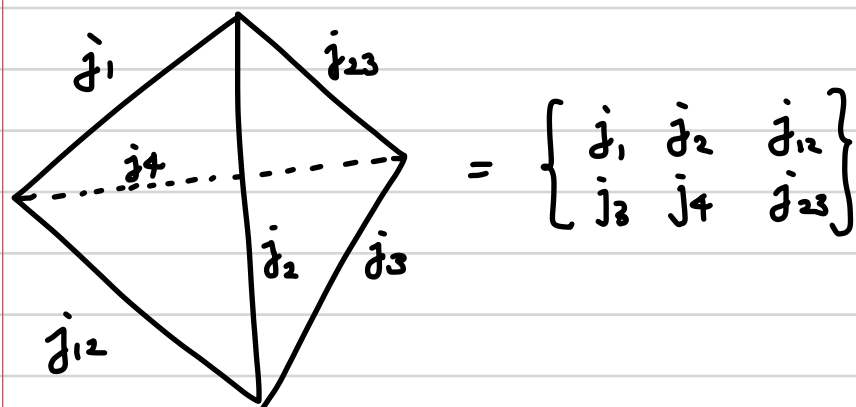
This explains the mirror symmetry of colored HOMFLY polynomial of a link L .

$$P_{\begin{array}{|c|} \hline \square \\ \hline \end{array}}(L; a, q) = P_{\begin{array}{|c|} \hline \square \\ \hline \end{array}}(L; a, q^{-1})$$

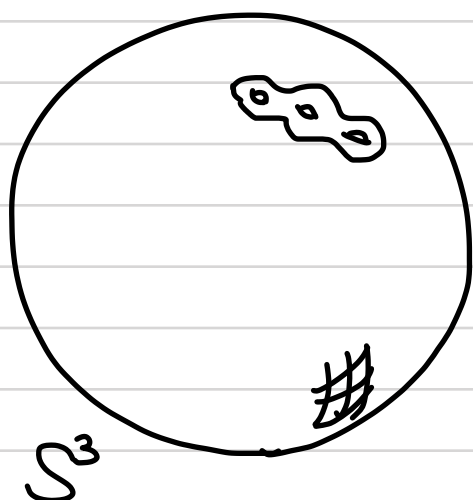
Using the mirror symmetry, one can obtain multiplicity-free q -6j involving anti-symmetric reps.

Discussion

$q-6j$ can be associated to a tetrahedron with colored edges.



Volume conjecture tells us the large color behavior of $q-6j$ will describe moduli space of $SL(2, \mathbb{C})$ flat connection of complement of genus-3 handlebody in S^3 .



\Rightarrow First example of higher-dim character varieties.

\Rightarrow recursion of $q-6j$ provides quantization of character variety.

Work in progress!