Homework 5: Due at class on Nov 29

1 Integral

Perform the following integral and show the identity

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{\left[k^2 - \Delta + i\epsilon\right]^n} = (-1)^n \frac{i}{16\pi^2} \frac{1}{(n-1)(n-2)} \frac{1}{\Delta^{n-2}} \,.$$

One can use either way:

- Start with the calculation of the dk^0 integral via Cauchy's theorem and then integrate over d^3k.
- Alternatively the k^0 integration can be done via a Wick rotation and afterwards substituting $k^0 = ik_E^0, (\vec{k} \equiv \vec{k}_E)$. Then the integral can be done in the four-dimensional Euclidean space (with $k^2 = -((k_E^0)^2 + \vec{k}^2)$) For that purpose you will need the integral over the four-dimensional solid angle $\int d\Omega_4 = 2\pi^2$.

2 1-loop correction in scalar QED

The Lagrangian of the scalar QED is given by

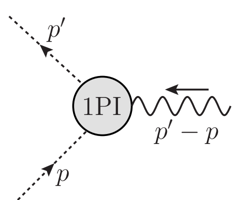
$$\mathcal{L}_{\text{scalar}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_\mu \phi)^* D^\mu \phi - m^2 \phi^* \phi \tag{1}$$

where the covariant derivative is expressed by $D_\mu = \partial_\mu + ieA_\mu$.

2.1

Find the diagrams which contribute to the 1PI diagrams of the scalar vertex in the order $\mathcal{O}(e^3)$ and write down the integral expression for

$$-ieK^\mu(p, p') \equiv$$



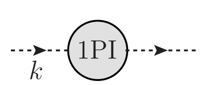
$$= -ie(p + p')^\mu + e^3 K^\mu_{\text{1-loop}}(p, p') + \mathcal{O}(e^5) \,. \tag{2}$$

Perform momentum integration by using the Pauli-Villars regularization.

2.2

Find the diagrams which contribute to the 1PI diagrams of the scalar propagator in the order $\mathcal{O}(e^2)$ and write down the integral expression for $\Pi_\phi^{\text{1-loop}}(k)$

$$i\Pi_\phi(k) \equiv$$



$$= e^2 \Pi_\phi^{\text{1-loop}}(k) + \mathcal{O}(e^4) \,. \tag{3}$$

Perform momentum integration by using the Pauli-Villars regularization.

3 Bremsstrahlung

In the lecture, the IR divergence of the electron vertex is given by

$$\begin{aligned}\Delta F_1(q^2) &= F_1(q^2) - F_1(0) = \frac{\alpha}{2\pi} C_{\mathrm{IR}}(q^2) \times \log\left(\frac{-q^2}{\mu^2}\right) + \text{ regular terms} \\ C_{\mathrm{IR}}(q^2) &= -1 + \int_0^1 d\xi \left(\frac{m^2 - q^2/2}{m^2 - q^2\xi(1-\xi)}\right) .\end{aligned} \tag{4}$$

This cancels with the infrared singularity in the $\mathcal{O}(\alpha)$ cross-section for the single photon emission,

$$d\sigma\left[e^-(p) \to e^-\left(p'\right) + \gamma\right] = d\sigma_0\left[e^-(p) \to e^-\left(p'\right)\right] \times \frac{\alpha}{\pi} I\left(\beta, \beta'\right) \times \frac{dE_\gamma}{E_\gamma}$$

where

$$I\left(\beta, \beta'\right) = \int \frac{d\Omega_{\vec{k}}}{4\pi} |\vec{k}|^2 \left[\frac{2p \cdot p'}{(k \cdot p)\,(k \cdot p')} - \frac{m^2}{(k \cdot p)^2} - \frac{m^2}{(k \cdot p')^2}\right] .$$

Show that $2C_{\mathrm{IR}}(q^2) = I\left(\beta, \beta'\right)$.