

## Homework 9: Due at class on May 5

1. Suppose that we have a connection  $\nabla$  on a rank- $r$  vector bundle  $\pi : E \rightarrow M$ . Suppose that we have local trivializations of  $E$  over two open sets  $U_\alpha$  and  $U_\beta$  in  $M$ . On these open sets, the connection can be written in terms of one-form  $A_\alpha$  and  $A_\beta$  taking value on  $\mathfrak{gl}(r, \mathbb{R})$ . Given a transition function  $g_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow \text{GL}(r, \mathbb{R})$ , show that the connections are related by

$$A_\beta = g_{\alpha\beta}^{-1} A_\alpha g_{\alpha\beta} + g_{\alpha\beta}^{-1} d(g_{\alpha\beta}) ,$$

and the curvature forms are related by

$$F_\beta = g_{\alpha\beta}^{-1} F_\alpha g_{\alpha\beta}$$

on  $U_\alpha \cap U_\beta$ .

2. Prove that the Bianchi identity

$$dF + A \wedge F - F \wedge A = 0 .$$

3. Show that there is a line (rank one vector) bundle  $L$  over  $S^2$  such that  $TS^2 \oplus L$  is trivial where  $TS^2$  is the tangent bundle of  $S^2$ . Hint: The tangent bundle of the flat space is trivial  $T\mathbb{R}^n \cong \mathbb{R}^{2n}$ .

### 4. Hopf fibration and Dirac monopole

We define  $S^{2n+1} = \{(z_0, \dots, z_n) \in \mathbb{C}^{n+1} \mid \sum_{i=0}^n |z_i|^2 = 1\}$  and let the complex projective space  $\mathbb{C}P^n$  be the quotient space  $S^{2n+1}/\sim$  where we define an equivalent relation  $(z_0, \dots, z_n) \sim \lambda(z_0, \dots, z_n)$  with  $\lambda \in \text{U}(1)$ . We denote a point of  $\mathbb{C}P^n$  by  $[z_0; \dots; z_n]$ .

When  $n = 1$ , we can consider that  $S^3$  is the principal  $\text{U}(1)$ -bundle over  $\mathbb{C}P^1$

$$\begin{array}{ccc} S^1 & \longrightarrow & S^3 \\ & & \downarrow \pi \\ & & \mathbb{C}P^1 \end{array} .$$

First of all, show that  $\mathbb{C}P^1$  can be identified with  $S^2 \subset \mathbb{R}^3$  by a map

$$[z_0; z_1] \mapsto (z_0 \bar{z}_1 + \bar{z}_0 z_1, -i(z_0 \bar{z}_1 - \bar{z}_0 z_1), |z_0|^2 - |z_1|^2) .$$

Let us define trivializations of this bundle over  $U_N = S^2 \setminus \{\text{north pole}\}$  and  $U_S = S^2 \setminus \{\text{south pole}\}$  as

$$\begin{aligned} \pi^{-1}(U_N) &\rightarrow U_N \times S^1; (z_0; z_1) \mapsto \left( [z_0; z_1], \frac{z_0}{|z_0|} \right) \\ \pi^{-1}(U_S) &\rightarrow U_S \times S^1; (z_0; z_1) \mapsto \left( [z_0; z_1], \frac{z_1}{|z_1|} \right) . \end{aligned}$$

Find the transition function between these trivializations. Let us consider embedding

$$\begin{aligned} f_N : U_N &\hookrightarrow \pi^{-1}(U_N) : (\theta, \phi) \mapsto \left( \cos \frac{\theta}{2}, \sin \frac{\theta}{2} e^{-i\phi} \right) \\ f_S : U_S &\hookrightarrow \pi^{-1}(U_S) : (\theta, \phi) \mapsto \left( \cos \frac{\theta}{2} e^{i\phi}, \sin \frac{\theta}{2} \right) , \end{aligned}$$

where  $(\theta, \phi)$  is the standard polar coordinate of  $S^2$ . Defining a connection

$$A = i \operatorname{Im}(\bar{z}_0 dz_0 + \bar{z}_1 dz_1)$$

on  $S^3 \subset \mathbb{C}^2$ , show that

$$\begin{aligned} f_N^* A &= -\frac{i}{2}(1 - \cos \theta) d\phi \quad \text{on } U_N \\ f_S^* A &= \frac{i}{2}(1 + \cos \theta) d\phi \quad \text{on } U_S . \end{aligned}$$

Show that they are related by gauge transformations and its curvature  $F$  is well-defined on  $S^2$ . This connection provides **Dirac magnetic monopole**

$$\frac{i}{2\pi} \int_{S^2} F = 1 .$$

5. Let us consider the case that a Lie group  $G$  acts on a manifold  $M$

$$G \times M \rightarrow M; (g, p) \mapsto gp .$$

We call the action is **transitive** if any two points of  $M$  can be transformed by the action of  $G$ . Suppose that  $G$  acts on  $M$  transitively, and we pick an arbitrary point  $p \in M$ . We call  $H = \{g \in G | gp = p\}$  a **stabilizer subgroup** of  $G$ . Then, we have diffeomorphism  $M \cong G/H$  and it is called a **homogeneous space**. Moreover,  $G$  can be interpreted as a principle  $H$ -bundle over  $M$ . Let us see this construction in the following examples.

Let us consider the  $\mathrm{SO}(2)$  subgroup of  $\mathrm{SO}(3)$  as

$$\mathrm{SO}(3) \supset \mathrm{SO}(2) = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

Show that  $S^2 \cong \mathrm{SO}(3)/\mathrm{SO}(2)$  and  $\mathrm{SO}(3)$  can be considered as  $S^1$  bundle over  $S^2$ . Construct local trivializations of this bundle over  $U_N = S^2 \setminus \{\text{north pole}\}$  and  $U_S = S^2 \setminus \{\text{south pole}\}$ , and find the corresponding transition function at the equator. Construct a  $\mathrm{U}(1)$ -connection on this bundle and find its curvature.

6. We have seen that  $\mathrm{SL}(2, \mathbb{R})$  acts on the upper half plane  $\mathbf{H}$ :

$$\mathrm{SL}(2, \mathbb{R}) \times \mathbf{H} \rightarrow \mathbf{H}; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z \mapsto \frac{az + b}{cz + d} .$$

Show that this action is transitive and find the stabilizer subgroup of the point  $i \in \mathbf{H}$ . Find the fundamental group of  $\mathrm{SL}(2, \mathbb{R})$ .