Homework 2: Due at class on Mar 19

1 Vertex operator and OPE

Show that $:e^{ikX}:$ is a primary field of weight $h = \overline{h} = \alpha' k^2/4$ in the free scalar theory. In addition, show that $\partial^n X$ ($n \geq 2$) is not a primary field.

2 Virasoro algebra

From the OPE of the stress-energy tensor, derive the Virasoro algebra:

$$T(z)\, T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \ldots$$

$$\longrightarrow \quad [L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m+n,0} \ .$$

3 Witt algebra

A general infinitesimal holomorphic map can be expressed as

$$z' = z - \epsilon(z) = z - \sum_{n\in\mathbb{Z}} \epsilon_n z^{n+1} \ ,$$

with the infinitesimal parameters ϵ_n, and therefore one can define generators of the transformation by $\ell_n = -z^{n+1}\frac{\partial}{\partial z}$. Show that they satisfy the Witt algebra

$$[\ell_m, \ell_n] = (m-n)\ell_{m+n} \ ,$$

so that the Virasoro algebra is the central extension of the Witt algebra.

4 Linear fractional transformations

Let us consider the Riemann sphere $S^2 = \mathbb{C} \cup \{\infty\}$. The action of $SL(2,\mathbb{C})$ defined by

$$z \mapsto w = \frac{az+b}{cz+d} \ , \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C}) \ ,$$

maps the Riemann sphere onto itself. These transformations are called linear fractional transformations.

• Given three points z_1, z_2, z_3, find a linear fractional transformation which maps the points to $0, 1, \infty$.

• Given four points z_1, z_2, z_3, z_4, their **cross ratio** is defined by

$$[z_1, z_2, z_3, z_4] = \frac{(z_1-z_3)(z_2-z_4)}{(z_2-z_3)(z_1-z_4)} \ .$$

Show that the cross ratio is preserved by any linear fractional transformation

$$[z_1, z_2, z_3, z_4] = [w_1, w_2, w_3, w_4] .$$