Homework 4: Due at class on Oct 20

1 bc ghost CFT

1.1 Stress-Energy tensor

Given the bc ghost action (Euclidian signature)

$$S_{\text{gh}} = \frac{1}{2\pi}\int d^2\sigma\sqrt{h}\; b^{ab}\nabla_a c_b \ ,$$

calculate the stress tensor for the bc ghosts by

$$T_{ab} = -\frac{4\pi}{\sqrt{h}}\frac{\delta S}{\delta h^{ab}}$$

Note that the covariant derivative ∇^α contains the Christoffel symbol and b_{ab} is symmetric traceless. Show that it becomes

$$T^{\text{gh}}(z) = -2 : b(z)\partial c(z) : + : c(z)\partial b(z) : \tag{1.1}$$

in the conformally flat metric.

1.2 TT OPE

Using the stress-energy (1.1), derive the TT OPE in the bc ghost CFT

$$T(z)\,T(w) = \frac{-13}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \dots$$

1.3 BRST charge

Using the explicit form of the BRST current

$$j_B = c(z)T^X(z) + : b(z)c(z)\partial c(z) : + \frac{3}{2} : \partial^2 c(z) : \ ,$$

express the BRST charge in terms of the X^μ Virasoro operators and the ghost oscillators as

$$Q_B = \sum_n c_n(L^X_{-n} - \delta_{n,0}) + \sum_{m,n}\frac{m-n}{2} : c_m c_n b_{-m-n} : \ .$$

Show that the OPE between two BRST currents is given by

$$j_B(z)j_B(w) = -\frac{c^X-18}{2(z-w)^3}c\partial c(w) - \frac{c^X-18}{4(z-w)^2}c\partial^2 c(w) - \frac{c^X-26}{12(z-w)}c\partial^3 c(w) + \cdots \ ,$$

where c^X is the central charge of the X^μ bosonic string theory. Use this OPE to determine the anticommutator of the BRST charge with itself. For what value of c^X does this vanish?

1.4 Tj_B OPE

Show that the OPE between the total energy momentum tensor $T = T^X + T^{\text{gh}}$ and j_B is given by

$$T(z)j_B(w) = \frac{c^X - 26}{2(z-w)^4}c(w) + \frac{j_B(w)}{(z-w)^2} + \frac{\partial j_B(w)}{z-w} + \cdots \ .$$

What does the result imply for j_B?

2 $\beta\gamma$ ghost CFT

Now let us consider the same action as the bc ghost system

$$S = \frac{1}{2\pi}\int d^2z \beta\overline{\partial}\gamma + \overline{\beta}\partial\overline{\gamma} \ ,$$

but now β and γ are bosonic fields. Hence, their OPEs are (pay attention to sign)

$$\gamma(z)\beta(w) = -\beta(z)\gamma(w) = \frac{1}{z-w} + \cdots$$

If β and γ are primary fields of weights $(\lambda, 0)$ and $(1-\lambda, 0)$ respectively, the form of the stress energy tensor (holomorphic part) is

$$T(z) =: (\partial\beta)\gamma : -\lambda\,\partial : \beta\gamma :$$

Calculate the the TT OPE to determine the central charge of the CFT in terms of λ.

3 linear fractional transformations

Let us consider the Riemann sphere $S^2 = \mathbb{C}\cup\{\infty\}$. The action of $SL(2,\mathbb{C})$ defined by

$$z \mapsto w = \frac{az+b}{cz+d} \ , \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C}) \ ,$$

maps the Riemann sphere onto itself. These transformations are called linear fractional transformations.

• Given three points z_1, z_2, z_3, find a linear fractional transformation which maps the points to $0, 1, \infty$.

• Given four points z_1, z_2, z_3, z_4, their **cross ratio** is defined by

$$[z_1, z_2, z_3, z_4] = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_2 - z_3)(z_1 - z_4)} \ .$$

Show that the cross ratio is preserved by any linear fractional transformation

$$[z_1, z_2, z_3, z_4] = [w_1, w_2, w_3, w_4] \ .$$