

# Homework 9: Due at class on Nov 23

## 1 SU(2) current algebra with level 1

### 1.1

Show the OPE in (7.5) of the lecture note.

### 1.2

Let us define

$$V_\uparrow =: e^{\frac{i\varphi(z)}{\sqrt{2}}} : , \quad V_\downarrow =: e^{-\frac{i\varphi(z)}{\sqrt{2}}} : .$$

Show the following OPEs

$$\begin{aligned} J^+(z)V_\downarrow(0) &\sim \frac{V_\uparrow(0)}{z} \\ J^-(z)V_\uparrow(0) &\sim \frac{V_\downarrow(0)}{z} \\ J(z)V_{\uparrow,\downarrow}(0) &\sim \frac{\pm \frac{1}{2}V_{\uparrow,\downarrow}(0)}{z} \end{aligned} \tag{1}$$

Can you find the states corresponding to the operators  $V_\uparrow$  and  $V_\downarrow$  via the state-operator correspondence?

### 1.3

We have seen in (4.98) of the lecture note that the partition function

$$Z_R(\tau, \bar{\tau}) = \frac{1}{|\eta(\tau)|^2} \sum_{m,n \in \mathbb{Z}} q^{\frac{1}{2}(\frac{m}{R} + \frac{Rn}{2})^2} \bar{q}^{\frac{1}{2}(\frac{m}{R} - \frac{Rn}{2})^2}. \tag{2}$$

of the compactified boson of radius  $R$ . Show that at the self-dual radius  $R = \sqrt{2}$  it can be written as

$$Z_{R=\sqrt{2}} = |\chi_0(q)|^2 + |\chi_{\frac{1}{2}}(q)|^2, \quad \text{where } \chi_0 = \frac{1}{\eta(\tau)} \sum_{n \in \mathbb{Z}} q^{n^2}, \quad \chi_{\frac{1}{2}} = \frac{1}{\eta(\tau)} \sum_{n \in \mathbb{Z}} q^{(n+1/2)^2}$$

In the lecture, we have seen that the highest weight states of the SU(2) current algebra with level  $k = 1$  are  $|j = 0, \frac{1}{2}\rangle$ . In fact, the  $\chi_j$  are the spin- $j$  characters of the SU(2) affine Lie algebra of level  $k = 1$ .

## 2 SU(2) current algebra with level $k$

The OPE of SU(2) current with level  $k$  is given by

$$J^a(z)J^b(w) \sim \frac{\frac{k}{2}\delta_{ab}}{(z-w)^2} + \frac{i\epsilon^{abc}J^c(w)}{z-w}.$$

Suppose that the energy momentum tensor is defined as

$$T(z) = \frac{1}{k+2} \sum_{a=1}^3 : J^a J^a(z) :$$

By computing  $TJ^a$  OPE, find the conformal dimension of the current  $J^a$  and derive the commutation relations  $[L_n, J_m^a]$ . Compute the central charge via the  $TT$  OPE.

## 3 Wakimoto construction

### 3.1

We define the bosonic field  $a(z) = i\sqrt{\frac{\kappa}{2}}\partial\varphi(z)$  so that the OPE is

$$a(z)a(w) \sim \frac{\frac{\kappa}{2}}{(z-w)^2}$$

Let us introduce the bosonic fields  $\beta(z), \gamma(z)$  where they satisfy the OPE

$$\beta(z)\gamma(w) \sim \frac{1}{z-w}$$

and the rest of OPEs is trivial. Furthermore, let us define operators

$$\begin{aligned} E(z) &= \beta(z) \\ H(z) &= - : \gamma(z)\beta(z) : + a(z) \\ F(z) &= - : \gamma(z)^2\beta(z) : + 2\gamma(z)a(z) + k\gamma'(z) \end{aligned},$$

where  $\kappa = k+2$ . Compute all the possible OPEs of  $E, F, H$  and find the relation to current algebra.

### 3.2

If the energy-momentum tensor is defined as

$$T(z) =: \frac{1}{\kappa} \{a(z)^2 - a'(z)\} + \beta(z)\gamma'(z) :,$$

compute the central charge. Derive the conformal dimensions of  $E, F, H$ . Find the conformal dimension of the vertex operator  $V_\alpha(z) =: e^{i\alpha\sqrt{\frac{2}{\kappa}}\varphi(z)} :.$

### 3.3

Defining the vertex operator

$$V_{\alpha=-1}(z) = e^{-i\sqrt{\frac{2}{\kappa}}\varphi(z)}\beta(z) ,$$

compute its OPEs with  $E, F, H$  and  $T$ . Show that it behaves as a screening current.