

Lecture 14

1 D-brane dynamics

So far we treated D-branes as a static solid object that is associated to boundary conditions. On the other hand, it actually has dynamics as like a fundamental string. We will learn the dynamics mainly through their action. Therefore, the first half is devoted to derive the action for Dp -branes. Later half is devoted for connection between D-branes, F1-string, and NS5-brane.

1.1 D-brane action

We assume D-branes action is something similar to Nambu-Goto action. Note that when we quantize the string action we used a string sigma action instead of Nambu-Goto action, so that we can evade complexity of the square root. There, the dimension of the world-sheet to be 2 was crucial for quantization. Therefore, we cannot follow the quantization procedure for the fundamental string to quantize Dp -brane, in general. Namely, the action we will learn here is an effective action of D-branes.

What we expect for the action is that

- it contains scalars that is a map from the world-sheet to space-time,
- it also contains vectors living on the D-branes, which arose from an open string massless spectrum,
- it involves B-field because open strings end on D-branes,
- it has supersymmetry (though we only talk about bosonic part in this lecture).

The proposed effective action is called **Dirac-Born-Infeld(DBI) action**. There are several approach to the action. We assume Nambu-Goto action for D-branes and generalize it by utilizing T-duality. Hence, *the action is T-duality manifest*.

We will describe a world-sheet of Dp -branes by $\sigma^a (a = 0, 1, \dots, p)$. $X^\mu(\sigma^a)$ are the scalars. Then, the Nambu-Goto action is

$$S_{Dp} = -T_{Dp} \int d^{p+1}\sigma \sqrt{-\det \left(G_{\mu\nu} \frac{\partial X^\mu}{\partial \sigma^a} \frac{\partial X^\nu}{\partial \sigma^b} \right)}, \quad (1)$$

where T_{Dp} is a Dp -brane tension, which is discussed later. x^μ , which will appear later, are used for space-time coordinate.

Let us consider a simple set up (see Fig. 1). The space-time is $\mathbb{R}_t \times \mathbb{R} \times S^1$ and the metric is $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$ (i.e. $G_{\mu\nu} = \eta_{\mu\nu}$). D1-brane locates at X_2 ($\sim X_2 + 2\pi R$), and D2-brane has Wilson line A_2 ($\sim A_2 + \frac{1}{R}$). Note that here A_2 is not a 2-form but $A_{\mu=2}$. T-duality relates these quantities (see Lecture note 9):

$$X_2 = 2\pi\alpha' A_2. \quad (2)$$

Now consider vibrating D1-brane $X^2 = X^2(X^1)$ (see Fig. 2). It maps to a field strength

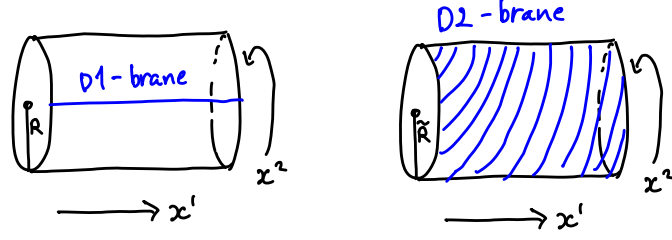


Fig. 1: D1-brane and its T-dual D2-brane on $\mathbb{R} \times S^1$.

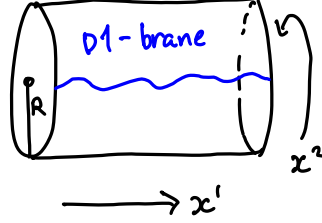


Fig. 2: Vibrating D1-brane.

$F_{12} = \partial_1 X_2(X^1) \neq 0$ on D2-brane. Suppose the vibrating D1-brane is described by the Nambu-Goto action

$$\begin{aligned}
 S_{D1} &= -T_{D1} \int d^2\sigma \sqrt{-\det \left(G_{\mu\nu} \frac{\partial X^\mu}{\partial \sigma^a} \frac{\partial X^\nu}{\partial \sigma^b} \right)} \quad \text{with} \quad X^0 = \sigma^0, \quad X^1 = \sigma^1, \\
 &= -T_{D1} \int d\sigma^0 d\sigma^1 \sqrt{1 + \left(\frac{\partial X^2}{\partial \sigma^1} \right)^2}.
 \end{aligned} \tag{3}$$

This expression seems to coincide with

$$\begin{aligned}
 S_{D2} &= -T_{D2} \int d^3\sigma \sqrt{-\det \left(G_{\mu\nu} \frac{\partial X^\mu}{\partial \sigma^a} \frac{\partial X^\nu}{\partial \sigma^b} + 2\pi\alpha' F_{ab} \right)} \\
 &= -T_{D2} \cdot 2\pi\tilde{R} \cdot \int d\sigma^0 d\sigma^1 \sqrt{1 + (2\pi\alpha' F_{12})^2}.
 \end{aligned} \tag{4}$$

D-brane tension

From the dimension analysis D-brane tension should be the following form.

$$T_{Dp} \sim \frac{\text{mass}}{p\text{-dim vol}} \quad \Rightarrow \quad T_{Dp} \sim \frac{1}{l_s^{p+1}}. \tag{5}$$

From the argument above in order for the two action to coincide we need $T_{D1} = 2\pi\tilde{R}T_{D2}$. On the other hand, we do not want T_{Dp} to depend on R because T_{Dp} should be independent of space-time geometry. Note that D-brane effective theory is supposed to reproduce open string amplitude, whose leading contribution is the disk amplitude $\sim e^{-\langle\Phi\rangle}$. Thus, we reach the following form

$$S_{Dp} = -T_{Dp} \int d^{p+1}\sigma e^{-\Phi(X)} \sqrt{-\det (G_{\mu\nu} \partial_a X^\mu \partial_b X^\nu + 2\pi\alpha' F_{ab})}. \tag{6}$$

The ratio of effective tensions of D1 and D2 branes is

$$\frac{T_{D1}^{\text{eff}}}{T_{D2}^{\text{eff}}} = \frac{T_{D1} e^{-\Phi}}{T_{D2} e^{-\tilde{\Phi}}} = \frac{T_{D1}}{T_{D2}} \cdot \frac{\tilde{R}}{l_s} = 2\pi \tilde{R} \quad \Rightarrow \quad T_{D1} = 2\pi l_s \cdot T_{D2} . \quad (7)$$

Note that the dilation field transforms under T-duality as $e^{-\tilde{\Phi}} = e^{-\Phi} \frac{\tilde{R}}{l_s}$ (see Homework 11 Prob. 4). For D-branes in superstring theory the correct normalization is $T_{Dp} = \frac{2\pi}{(2\pi l_s)^{p+1}}$.

RR charge and its normalization

We heavily used the fact that Dp-branes couples to C_{p+1} in previous lectures. Let us consider a concrete coupling. It should be the following form.

$$S_{Dp} = \dots + q_{Dp} \cdot \int d^{p+1} \sigma e^{-\Phi} C_{\mu_1 \dots \mu_{p+1}}(X) \frac{\partial X^{\mu_1}}{\partial \sigma^1} \dots \frac{\partial X^{\mu_{p+1}}}{\partial \sigma^{p+1}} = \dots + q_{Dp} \cdot \int_{Dp} e^{-\Phi} C_{(p+1)} . \quad (8)$$

By considering T-duality we can conclude that $q_{Dp} = \frac{2\pi}{(2\pi l_s)^{p+1}} = T_{Dp}$ (up to a constant). Note that the equality of the charge and the tension is crucial for multiple Dp-branes to co-exist statically. This is because the tension induce gravitational force (graviton & dilaton) between Dp-branes, which is attractive, on the other hand, the RR charge induce repulsive force for positively(negatively) charged objects, which are Dp-branes.

Note that we have to use proper normalization for RR-fields. The convention used above is called **string normalization**, and the one used in SUGRA is called **canonical normalization**. Let us recall the IIA SUGRA. Appropriate part is, for example,

$$S_{A,R} = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-G} \left[-\frac{1}{2} G_{(2)}^2 \right] + q_{D0} \cdot \int_{D0} C_{(1)} . \quad (9)$$

The same expression in the string normalization is

$$S_{A,R} = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-G} e^{-2\Phi} \left[-\frac{1}{2} G_{(2)}^2 \right] + q_{D0} \cdot \int_{D0} e^{-\Phi} C_{(1)} . \quad (10)$$

The Dirac quantization conditions(for the details see homework 11 Prob. 2.2) for these two expressions are the same:

$$2\kappa_{10}^2 q_e q_m \in 2\pi \mathbb{Z} . \quad (11)$$

($q_e = q_{D0}$ and $q_m = q_{D(6)}$ for the above case.) However, we also define $q^{\text{eff}} = q e^{-\Phi} = q g_s$ in the string normalization, then, the quantization condition is

$$2\kappa_{10}^2 q_e^{\text{eff}} q_m^{\text{eff}} g_s^2 \in 2\pi \mathbb{Z} . \quad (12)$$

In this convention we always assume that Φ is non-dynamical. Confirm that $q_{Dp}^{\text{eff}} = T_{Dp}^{\text{eff}} = \frac{2\pi}{(2\pi l_s)^{p+1} g_s}$ satisfies the quantization condition. Note that some references define $q e^{-\Phi}$ as q (similarly $T_{Dp} e^{-\Phi}$ as T_{Dp}). The reason we used canonical normalization is simply that the expression is much simpler (especially kinetic terms).

For the B-field, the normalization is different and the quantization condition is

$$T_{F1} \cdot T_{NS5} \cdot 2\kappa_{10}^2 g_s^2 \in 2\pi \mathbb{Z} . \quad (13)$$

Since $T_{F1} = \frac{2\pi}{(2\pi l_s)^2}$,

$$T_{NS5} = \frac{2\pi}{T_{F1} \cdot 2\kappa_{10}^2 g_s^2} = \frac{2\pi}{(2\pi l_s)^6 g_s^2} . \quad (14)$$

Generalization of RR coupling and the DBI action

Let us again consider vibrating D1-brane with the RR-coupling. T-duality connects the following two expression.

$$S_{D1} = \cdots + q_{D1} \cdot \int dx^0 dx^1 e^{-\Phi} \left(C_{01} + C_{02} \frac{\partial X^2}{\partial \sigma^1} \right) , \quad (15)$$

$$S_{D2} = \cdots + q_{D2} \cdot \int dx^0 dx^1 dx^2 e^{-\tilde{\Phi}} \left(\tilde{C}_{012} + \tilde{C}_0 \cdot 2\pi\alpha' F_{12} \right) , \quad (16)$$

where $C_{01} \leftrightarrow \tilde{C}_{012}$, $C_{02} \leftrightarrow \tilde{C}_0$, and $X^2 \leftrightarrow 2\pi\alpha' A_2$. This can be understood as follows (see Fig. 3). The vibrating D1-brane consists of straight D1-brane along x^1 and local

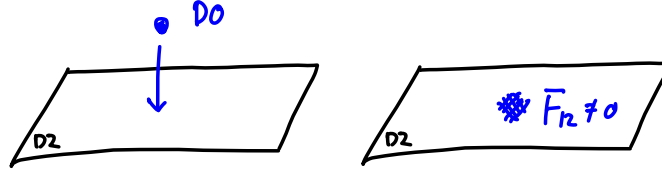


Fig. 3: D0-D2 bound state.

vibration along x^2 . After T-duality along x^2 the vibration part becomes D0-brane and gives $F_{12} \neq 0$. Generalization of the RR-coupling is

$$S_{Dp} = \cdots + q_{Dp} \cdot \int C_{RR} \wedge \exp(2\pi\alpha' F_{(2)}) , \quad (17)$$

where $C_{RR} = \sum_n C_{(n)}$.

Finally, full general form of Dp-brane action is given as follows.

$$\begin{aligned} S_{Dp} = & -T_{Dp} \int d^{p+1}\sigma e^{-\Phi(X)} \sqrt{-\det(G_{ab} + 2\pi\alpha' F_{ab} - B_{ab})} \\ & + q_{Dp} \cdot \int C_{RR} \wedge \exp(2\pi\alpha' F_{(2)} - B_{(2)}) , \end{aligned} \quad (18)$$

where $G_{ab} = G_{\mu\nu} \partial_a X^\mu \partial_b X^\nu$ and $B_{ab} = B_{\mu\nu} \partial_a X^\mu \partial_b X^\nu$. The first line is called **DBI action**. Note that we used canonical normalization for RR-fields. B-field should appear with $F_{(2)}$ due to the gauge invariance.

Let us consider a fundamental string action that is coupled to Dp-branes.

$$\begin{aligned} S = & -\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-h} h^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} + \epsilon^{ab} \partial_a X^\mu \partial_b X^\nu B_{\mu\nu} + \cdots \\ & + \int_{\partial\Sigma} d\sigma^0 \partial_0 X^\mu A_\mu \\ = & \cdots - \frac{1}{2\pi\alpha'} \int_\Sigma B_{(2)} + \int_{\partial\Sigma} A_{(1)} . \end{aligned} \quad (19)$$

Important point here is that the gauge transformation of B-field $\delta_B B_{(2)} = d\lambda_{(1)}$ in the action is NOT invariant if there are boundaries, which is exactly the situation we consider now:

$$\delta_B \int_\Sigma B_{(2)} = \int_\Sigma d\lambda_{(1)} = \int_{\partial\Sigma} \lambda_{(1)} \neq 0 . \quad (20)$$

This is compensated if $A_{(1)}$ transform as follows:

$$\delta_B A_{(1)} = \frac{\lambda_{(1)}}{2\pi\alpha'} . \quad (21)$$

Though we focused on the bosonic part so far, there is a fermionic part so that they form space-time supersymmetry. Here we only write down the leading fluctuation:

$$-i \int d^{p+1} \sigma \operatorname{tr} (\bar{\psi} \Gamma^a D_a \psi) . \quad (22)$$

For the full nonlinear supersymmetric form one should consult with, for example [1].

1.2 Branes, Strings ending on Branes

We will look into the RR coupling further from a different view point. Maxwell equation leads charge conservation law as follows.

$$\begin{aligned} d * F_{(2)} &= J_e & \Rightarrow & & dJ_e &= 0 \\ dF_{(2)} &= J_m & & & dJ_m &= 0 \end{aligned} . \quad (23)$$

Charge conservation assure that a world-line of the charged particle does not end (closed path or infinitely long). If we apply this logic to branes we may find the same result for branes. However, charge conservation for SUGRA is quite non-trivial due to the non-linearity of the E.O.Ms. We will see the case of generalized type IIA SUGRA.

Massive IIA SUGRA

When we saw the IIA SUGRA action you may wonder why there is no RR-field corresponding to D8-brane, which is 9-form and its field strength is 10-form $G_{(10)}$. Since it is non-dynamical ($d * G_{(10)} = 0$ leads $*G_{(10)} = G_{(0)} \equiv m$) it has constant contributions to the action, called **massive IIA SUGRA**. m is called **Romans mass** because it is a constant and partly contributes as a mass term in the action.

Let us see the massive IIA SUGRA action (we omit wedge product \wedge in this lecture):

$$S_{A,NS} = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-G} e^{-2\Phi} \left[R + 4\partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2} H_{(3)}^2 \right] , \quad (24)$$

$$S_{A,R} = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-G} \left[-\frac{1}{2} m^2 - \frac{1}{2} G_{(2)}^2 - \frac{1}{2} G_{(4)}^2 \right] , \quad (25)$$

$$\begin{aligned} S_{A,CS} = \frac{1}{2\kappa_{10}^2} \int & \left[-\frac{1}{2} B_{(2)} G_{(4)} G_{(4)} + \frac{1}{2} B_{(2)}^2 G_{(2)} G_{(4)} - \frac{1}{6} B_{(2)}^3 G_{(2)}^2 \right. \\ & \left. - \frac{m}{6} B_{(2)}^3 G_{(4)} + \frac{m}{8} B_{(2)}^4 G_{(2)} - \frac{m^2}{40} B_{(2)}^5 \right] , \end{aligned} \quad (26)$$

where

$$\begin{aligned} H_{(3)} &= dB_{(2)} , \\ G_{(2)} &= dC_{(1)} + mB_{(2)} , \\ G_{(4)} &= dC_{(3)} + dC_{(1)} B_{(2)} + \frac{1}{2} m B_{(2)}^2 . \end{aligned} \quad (27)$$

RR- field

From the massive IIA action we have following equation of motion for $C_{(1)}$ and $C_{(3)}$:
E.O.M

$$-d * G_{(2)} = H_{(3)} * G_{(4)} , \quad (28)$$

$$d * G_{(4)} = H_{(3)} G_{(4)} . \quad (29)$$

Since we know the relation between the field strengths and their gauge fields (27) we have following Bianchi identities:

$$dG_{(2)} = mH_{(3)} , \quad (30)$$

$$dG_{(4)} = H_{(3)} G_{(2)} . \quad (31)$$

Now we relabel m by $G_{(0)}$ and define the dual field strengths:

$$G_{(10)} = *G_{(0)} , \quad G_{(8)} = - *G_{(2)} , \quad G_{(6)} = *G_{(4)} . \quad (32)$$

Then, the E.O.M and Bianchi ids are re-written as

$$dG_{(2n)} = G_{(2n-2)} H_{(3)} . \quad (33)$$

Let us define following formal sum of RR-fields

$$G_{\text{even}} = G_{(0)} + G_{(2)} + G_{(4)} + G_{(6)} + G_{(8)} + G_{(10)} . \quad (34)$$

Using the formal sum we can express (33) by single expression

$$dG_{\text{even}} = H_{(3)} G_{\text{even}} . \quad (35)$$

This equation can be solved as follows.

$$G_{\text{even}} = e^{B_{(2)}} (m + dC_{\text{odd}}) , \quad (36)$$

$$C_{\text{odd}} = C_{(1)} + C_{(3)} + C_{(5)} + C_{(7)} + C_{(9)} . \quad (37)$$

B-field

Field strength of the B-field is defined by

$$H_{(3)} = dB_{(2)} , \quad (38)$$

hence, the Bianchi id is $dH_{(3)} = 0$. E.O.M is given as follows.

$$d(e^{-2\Phi} * H_{(3)}) = m * G_{(2)} + *G_{(4)} G_{(2)} - \frac{1}{2} G_{(4)}^2 . \quad (39)$$

If we define the dual field strength $H_{(7)} = e^{-2\Phi} * H_{(3)}$, then, the E.O.M becomes

$$dH_{(7)} = -\frac{1}{2} [(\mathcal{T} G_{\text{even}}) G_{\text{even}}]_{(8)} \quad (40)$$

where

$$\mathcal{T}(dx^{i_1} \cdots dx^{i_n}) = (dx^{i_n} \cdots dx^{i_1}) , \quad (41)$$

which is a “transpose” of differential forms.

Note that those field strengths are invariant under the gauge transformations of B-field as well as RR-fields:

$$\delta_B B_{(2)} = d\lambda_{(1)} , \quad \delta_B C_{\text{odd}} = -\lambda_{(1)} (m + dC_{\text{odd}}) , \quad (42)$$

$$\delta_C B_{(2)} = 0 , \quad \delta_C C_{\text{odd}} = d\lambda_{\text{even}} , \quad (43)$$

where we introduced a formal sum of gauge parameters

$$\lambda_{\text{even}} = \lambda_{(0)} + \lambda_{(2)} + \lambda_{(4)} + \lambda_{(6)} + \lambda_{(8)} . \quad (44)$$

Brane currents

Let us introduce brane currents $J_{(8)}^{\text{F1}}$, $J_{(4)}^{\text{NS5}}$, and

$$J_{\text{odd}} = J_{(1)}^{\text{D8}} + J_{(3)}^{\text{D6}} + J_{(5)}^{\text{D4}} + J_{(7)}^{\text{D2}} + J_{(9)}^{\text{D0}} , \quad (45)$$

and add to the E.O.Ms:

$$dH_{(3)} = J_{(4)}^{\text{NS5}} , \quad (46)$$

$$dH_{(7)} = J_{(8)}^{\text{F1}} - \frac{1}{2}(TG_{\text{even}})G_{\text{even}} , \quad (47)$$

$$dG_{\text{even}} = J_{\text{odd}} + H_{(3)}G_{\text{even}} . \quad (48)$$

From these equations we can derive following “charge conservation” law:

$$dJ_{(4)}^{\text{NS5}} = 0 , \quad (49)$$

$$dJ_{(8)}^{\text{F1}} = [J_{\text{odd}}(\mathcal{T}G_{\text{even}})]_{(9)} , \quad (50)$$

$$dJ_{\text{odd}} = -J_{(4)}^{\text{NS5}}G_{\text{even}} - J_{\text{odd}}H_{(3)} . \quad (51)$$

From the laws we can deduce several facts (see Table 1):

- NS5-brane cannot have boundaries,
- F1 string can end on any D-branes,
- Dp-brane can end on NS5-brane up to $p = 6$ (D8 cannot),
- Dp-brane can end on D($p + 2$)-brane.

The fact that D-brane is coupled to RR-field requires that the brane action should include $S = \int C_{\text{odd}}$. However, it is invariant under the gauge transformations. The invariant form is

$$S = \int \left(e^{2\pi\alpha' F_{(2)} - B_{(2)}} C_{\text{odd}} + m\omega \right) , \quad (52)$$

where

$$\omega = \sum_n \frac{1}{(n+1)!} A_{(1)} F_{(2)}^n . \quad (53)$$

This is consistent with the previous analysis.

Table 1: Branes on which brane ends and branes that end on.

Brane	Branes end on
F1	nothing
NS5-brane	D0, D2, D4, D6
D0-brane	F1
D2-brane	F1, D0
D4-brane	F1, D2
D6-brane	F1, D4
D8-brane	F1, D6

1.3 Bound states of D-branes

As we saw Dp -brane action has following term

$$S \sim \int e^{2\pi\alpha' F_{(2)}} C_{\text{RR}} = \int \delta_{D-p-1}(Dp) e^{2\pi\alpha' F_{(2)}} C_{\text{RR}} , \quad (54)$$

where we set $m = 0 = B_{(2)}$. The fact that the action includes not only $C_{(p+1)}$ but also $C_{(p+1-2n)}$ means that Dp -brane can have $D(p - 2n)$ -brane charges for $n \in \mathbb{Z}_+$.

D0-D2 bound state

Let us consider a concrete example of D2-brane case. The action include following term

$$S \sim \frac{1}{2\pi} \int_{\mathbb{R}_t \times \Sigma} (C_{(3)} + F_{(2)} C_{(1)}) , \quad (55)$$

where Σ is an image of world-sheet space (not time). Note that the flux is quantized;

$$\int_{\Sigma} F_{(2)} = 2\pi n \quad n \in \mathbb{Z} . \quad (56)$$

Now consider a process the Σ shrinks to zero. In this process $C_{(3)}$ part becomes zero as the volume becomes zero. On the other hand, $C_{(1)}$ part remains finite because the flux $F_{(2)}$ is quantized and gives

$$S \sim \frac{1}{2\pi} \int_{\mathbb{R}_t \times \Sigma} (F_{(2)} C_{(1)}) \rightarrow n \int_{\mathbb{R}_t} C_{(1)} . \quad (57)$$

This is nothing but n D0-branes. Namely, when D2-brane with n flux shrinks to a point, n D0-branes remains. The former state can be understood that it is a bound state of D2- and D0-branes (see Fig. 4).

Myers effect

Let us consider the opposite process of the previous argument. When there are n D0-branes they can become D2-brane. This situation can be accelerated by inducing background $C_{(3)}$ flux. If there is $C_{(3)}$ flux, then, being D2-brane is a lower energy state than being D0-branes (see Fig. 5). This is something similar to polarization phenomenon in electro-magnetism, and in this case, it is called **Myers effect**.

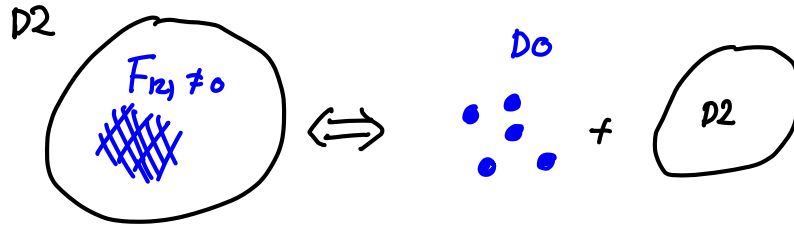


Fig. 4: Transition of D0-D2 bound state and D0 + D2.

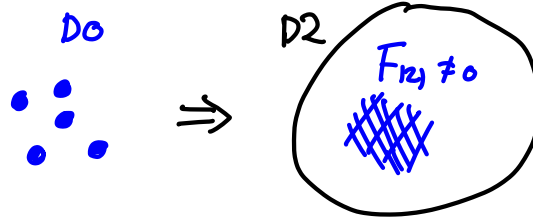


Fig. 5: Myers effect: transition from D0-branes to D0-D2 bound state.

Other bound states

Left it for homework.

Hanany-Witten effect

There is so called Hanany-Witten effect, which is a brane creation/annihilation process when some branes cross each other. Typical example is a cross of NS5 and D5 create/annihilate D3-brane (see Fig. 6). Another example is that Dp and Dp' for $p + p' = 8$

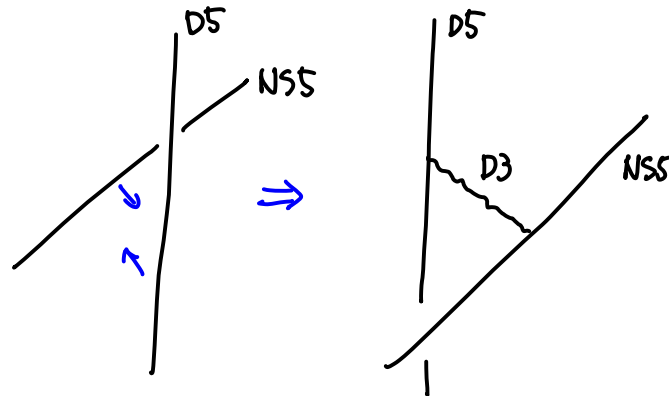


Fig. 6: Hanany-Witten effect: crossing of D5- and NS5-branes.

create/annihilate F1-string. Crossing of two M5-branes create/annihilate M2-brane etc (look it up in the web if you are interested in).

References

- [1] A. A. Tseytlin, “Born-Infeld action, supersymmetry and string theory,” In *Shifman, M.A. (ed.): The many faces of the superworld* 417-452 [hep-th/9908105]. [5](#)