Homework 4: Due at class on March 29

1. Let $\varphi_t : M \to M$ be the flow generated by a vector field $X \in \mathfrak{X}(M)$. Then, show that for another vector field $Y \in \mathfrak{X}(M)$

$$[X, Y] = \lim_{t \to 0} \frac{(\varphi_{-t})_* Y - Y}{t}$$

2. Given a vector field $X \in \mathfrak{X}(M)$, the Lie derivative

$$L_X : \Omega^k(M) \to \Omega^k(M)$$

can be defined by

$$L_X(\omega) = \lim_{t \to 0} \frac{\varphi_t^* \omega - \omega}{t}$$

where $\varphi_t : M \to M$ be the flow as above. Show that it satisfies

$$L_X \omega(X_1, \cdots, X_k) = X\omega(X_1, \cdots, X_k) - \sum_{I=1}^{k} \omega(X_1, \cdots, [X, X_i], \cdots, X_k)$$

for $X_1, \cdots, X_k \in \mathfrak{X}(M)$.

3. **This is a bonus problem with extra 5 points which is NOT mandatory.** Find the preimage of a smooth map $f : SO(3) \to SO(3); g \mapsto g^2$ and the rank of f_* at every point of $SO(3)$. [Hint: An element $g \neq 1 \in SO(3)$ is an rotation along a certain axis in $\mathbb{R}^3$.]

4. Define an n-form ω on the space $\mathbb{R}^{n+1}\backslash\{0\}$, obtained from $\mathbb{R}^{n+1}$ by removing the origin, by

$$\omega = \frac{1}{|x|^{n+1}} \sum_{i=0}^{n} (-1)^i x^i dx^0 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n .$$

Prove that $d\omega = 0$.

5. Let S^n be the unit sphere in $\mathbb{R}^{n+1}$. Find the value of the integral in the case of $n = 2$

$$\int_{S^2} \omega$$

in the case of $n = 2$ where ω is as above.

6. (Example 5.12. of Nakahara)

A symplectic form on a smooth manifold M is non-degenerate closed 2-form ω. "Non-degenerate" means that the mapping $\omega : TM \to T^*M; \ X \mapsto \omega(X, -)$ is an isomorphism. We denote the 1-form $\omega(X, -)$ by $i(X)\omega$.

The couple (M, ω) of a smooth manifold M and a symplectic form ω is called a symplectic manifold. Any symplectic manifold is even dimensional and if $\dim(M) = 2n$, ω^n is a volume-form.

Any smooth function $f \in C^\infty(M)$ gives rise to a vector field X_f defined uniquely by the equation

$$i(X_f)\omega = df \ .$$

This vector field is called the "Hamiltonian vector field" with Hamiltonian f. Given a chart (q_i, p_i) $(i = 1, \cdots, n)$ of M with a symplectic form

$$\omega = \sum_{i=1}^{n} dq_i \wedge dp_i \ , \tag{1}$$

show that the flow of the Hamiltonian vector field X_H associated to a Hamiltonian H can be described by

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i} \ , \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} \ ,$$

which are Hamilton's canonical equations.

For $f, g \in C^\infty(M)$, we define the Poisson product by

$$\{f, g\} = \omega(X_f, X_g) \ .$$

Show that given the symplectic form (1), the Poisson product can be written as the local coordinate

$$\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right) \ .$$

For $f \in C^\infty(M)$, show that the differentiation with respect to the Hamiltonian flow can be expressed by

$$\frac{df}{dt} = -\{H, f\} \ .$$

Let $H = q^2 + p^2$ be the Hamiltonian on the symplectic manifold $(\mathbb{R}^2, \omega = dq \wedge dp)$. Write down the Hamiltonian vector field X_H as well as the Hamilton's canonical equations in terms of the coordinate. Draw the flow generated by the Hamiltonian vector field and show that the Hamiltonian is constant along the flow.