Homework 10: Due at class on Nov 28

1 $\mathrm{SU}(2)$ current algebra with level 1

1.1

Show the OPE in (7.5) of the lecture note.

1.2

Let us define

$$V_{\uparrow} =: e^{\frac{i\varphi(z)}{\sqrt{2}}} : , \qquad V_{\downarrow} =: e^{-\frac{i\varphi(z)}{\sqrt{2}}} : .$$

Show the following OPEs

$$\begin{aligned} J^{+}(z)V_{\downarrow}(0) &\sim \frac{V_{\uparrow}(0)}{z} \\ J^{-}(z)V_{\uparrow}(0) &\sim \frac{V_{\downarrow}(0)}{z} \\ J(z)V_{\uparrow,\downarrow}(0) &\sim \frac{\pm\frac{1}{2}V_{\uparrow,\downarrow}(0)}{z} \end{aligned} \tag{1.1}$$

Can you find the states corresponding to the operators $V_{\uparrow}$ and $V_{\downarrow}$ via the state-operator correspondence?

1.3

We have seen in (4.98) of the lecture note that the partition function

$$Z_R(\tau,\bar{\tau}) = \frac{1}{|\eta(\tau)|^2} \sum_{m,n\in\mathbb{Z}} q^{\frac{1}{2}(\frac{m}{R}+\frac{Rn}{2})^2} \bar{q}^{\frac{1}{2}(\frac{m}{R}-\frac{Rn}{2})^2}. \tag{1.2}$$

of the compactified boson of radius R. Show that at the self-dual radius $R = \sqrt{2}$ it can be written as

$$Z_{R=\sqrt{2}} = |\chi_0(q)|^2 + |\chi_{\frac{1}{2}}(q)|^2 , \qquad \text{where} \quad \chi_0 = \frac{1}{\eta(\tau)}\sum_{n\in\mathbb{Z}} q^{n^2} , \quad \chi_{\frac{1}{2}} = \frac{1}{\eta(\tau)}\sum_{n\in\mathbb{Z}} q^{(n+1/2)^2}$$

In the lecture, we have seen that the highest weight states of the $\mathrm{SU}(2)$ current algebra with level $k = 1$ are $|j = 0, \frac{1}{2}\rangle$. In fact, the χ_j are the spin-j characters of the $\mathrm{SU}(2)$ affine Lie algebra of level $k = 1$.

2 $\mathrm{SU}(2)$ current algebra with level k

The OPE of $\mathrm{SU}(2)$ current with level k is given by

$$J^a(z)J^b(w) \sim \frac{\frac{k}{2}\delta_{ab}}{(z-w)^2} + \frac{i\epsilon^{abc}J^c(w)}{z-w} .$$

Suppose that the energy momentum tensor is defined as

$$T(z) = \frac{1}{k+2} \sum_{a=1}^{3} : J^a J^a(z) :$$

By computing TJ^a OPE (use the generalized Wick theorem in 6.B of the yellow book), find the conformal dimension of the current J^a and derive the commutation relations $[L_n, J^a_m]$. Compute the central charge via the TT OPE.

3 Wakimoto construction (Bonus problem)

3.1

We define the bosonic field $a(z) = i\sqrt{\frac{\kappa}{2}}\partial\varphi(z)$ so that the OPE is

$$a(z)a(w) \sim \frac{\frac{\kappa}{2}}{(z-w)^2}$$

Let us introduce the bosonic fields $\beta(z), \gamma(z)$ where they satisfy the OPE

$$\beta(z)\gamma(w) \sim \frac{1}{z-w}$$

and the rest of OPEs is trivial. Furthermore, let us define operators

$$\begin{aligned} E(z) &= \beta(z) \\ H(z) &= - : \gamma(z)\beta(z) : + a(z) \\ F(z) &= - : \gamma(z)^2\beta(z) : + 2\gamma(z)a(z) + k\gamma'(z) \end{aligned} ,$$

where $\kappa = k+2$. Compute all the possible OPEs of E, F, H and find the relation to current algebra.

3.2

If the energy-momentum tensor is defined as

$$T(z) =: \frac{1}{\kappa}\left\{a(z)^2 - a'(z)\right\} + \beta(z)\gamma'(z) : \ ,$$

compute the central charge. Derive the conformal dimensions of E, F, H. Find the conformal dimension of the vertex operator $V_\alpha(z) =: e^{i\alpha\sqrt{\frac{2}{\kappa}}\varphi(z)} :$.

3.3

Defining the vertex operator

$$V_{\alpha=-1}(z) = e^{-i\sqrt{\frac{2}{\kappa}}\varphi(z)}\beta(z) \ ,$$

compute its OPEs with E, F, H and T. Show that it behaves as a screening current.