Homework 1: Due on Sep 28

Caution: When solving homework problems, it is important to show your derivation at each step. Nowadays, many online tools make it easy to find answers, but the primary goal of these assignments is to deepen your understanding through hands-on problem-solving. By working through the calculations yourself, you engage more deeply with the material, making the learning process more meaningful, rather than simply copying answers from external sources.

Thermodynamics

Suppose that there are n moles of a gas that obey van der Waals' equation, for which the heat capacity C_V is known experimentally to be temperature independent.

1. Write expressions for the infinitesimal increments in entropy dS and internal energy dU based on

$$\begin{aligned} \mathrm{d}S &= \frac{C_V}{T}\,\mathrm{d}T + \left(\frac{\partial P}{\partial T}\right)_V \mathrm{d}V, \\ \mathrm{d}U &= C_V\,\mathrm{d}T + \left[T\left(\frac{\partial P}{\partial T}\right)_V - P\right]\mathrm{d}V. \end{aligned} \tag{0.1}$$

2. Show that C_V does not depend on volume V.

3. Obtain expressions for the entropy S and the internal energy U.

4. Show that the quantity $T\,(V-nb)^{nR/C_V}$ does not change in reversible adiabatic processes.

5. Calculate the temperature change during an adiabatic free expansion from volume V_1 to volume V_2.

Bosonic Bogoliubov diagonalization on a 1D lattice

Consider a one–dimensional bosonic chain with N sites, lattice spacing a, and periodic boundary conditions. Let a_j and $a_j^\dagger$ be bosonic operators at site j satisfying $[a_i, a_j^\dagger] = \delta_{ij}$, and let the jth site be at $R_j = ja$. The Hamiltonian is

$$H = \sum_j \Big\{ J_1\big(a_{j+1}^\dagger a_j + \text{H.c.}\big) + J_2\big(a_{j+1}^\dagger a_j^\dagger + \text{H.c.}\big) \Big\}, \tag{0.2}$$

where "H.c." denotes the Hermitian conjugate and $J_{1,2} \in \mathbb{R}$.

1. **Fourier transform.** Adopt the momentum representation

$$a_j = \frac{1}{\sqrt{N}} \sum_q e^{iqR_j}\, a_q, \qquad q = \frac{2\pi}{Na}\, n,\ n = 0, 1, \ldots, N-1, \tag{0.3}$$

and we consider them in the first Brillouin zone in what follows:

$$q \in \left(-\frac{\pi}{a}, \frac{\pi}{a}\right] \tag{0.4}$$

Rewrite H in the quadratic (BdG) form in the momentum space

$$\begin{aligned} H &= \sum_q A_q\, a_q^\dagger a_q + \sum_{q>0} B_q \big(a_q^\dagger a_{-q}^\dagger + a_{-q} a_q\big) \\ &= \sum_{q>0} A_q \left(a_q^\dagger a_q + a_{-q}^\dagger a_{-q}\right) + B_q \big(a_q^\dagger a_{-q}^\dagger + a_{-q} a_q\big) \end{aligned} \tag{0.5}$$

Determine A_q and B_q explicitly in terms of J_1, J_2, and q.

2. **Bogoliubov transformation.** Diagonalize H via the canonical (bosonic) transformation

$$b_q = u_q\, a_q + v_q\, e^{-i\phi_q}\, a_{-q}^\dagger, \qquad u_q^2 - v_q^2 = 1, \tag{0.6}$$

choosing the phase ϕ_q so that the anomalous terms $b_q b_{-q}$ and $b_q^\dagger b_{-q}^\dagger$ vanish. Show that your choice leads to a real, positive quasiparticle spectrum and express u_q, v_q through a single parameter θ_q (e.g. $u_q = \cosh\theta_q$, $v_q = \sinh\theta_q$) with a condition for θ_q such as $\tanh(2\theta_q) = B_q / A_q$.

3. **Dispersion and stability.** Obtain the diagonal form

$$H = \sum_{q>0} \Big[E_q \left(b_q^\dagger b_q + b_{-q}^\dagger b_{-q}\right) + E_0\Big], \tag{0.7}$$

and derive the dispersion E_q and the vacuum shift E_0 in closed form. State the *stability condition* for the quadratic bosonic Hamiltonian (i.e. the requirement on J_1, J_2, and q ensuring $E_q \in \mathbb{R}_{\geq 0}$ for all q). What happens when $J_1 = J_2$?

Two Fermions (Bogoliubov)

Let a_1, a_2 be two fermionic modes with

$$\{a_i, a_j^\dagger\} = \delta_{ij}, \qquad \{a_i, a_j\} = \{a_i^\dagger, a_j^\dagger\} = 0.$$

Consider the quadratic Hamiltonian

$$H = \epsilon\big(a_1^\dagger a_1 - a_2 a_2^\dagger\big) + \Delta\big(a_1^\dagger a_2^\dagger + \text{H.c.}\big), \tag{0.8}$$

with real parameters ϵ, Δ.

1. **Canonical transformation.** Show that the Bogoliubov transformation

$$c_1 = u\, a_1 + v\, a_2^\dagger, \tag{0.9}$$
$$c_2 = u\, a_2 - v\, a_1^\dagger, \tag{0.10}$$

with real u, v preserves the canonical anticommutation relations $\{c_i, c_j^\dagger\} = \delta_{ij}$ and $\{c_i, c_j\} = 0$ *iff*

$$u^2 + v^2 = 1.$$

2. **Diagonalization.** Using the result of (1), show that for a suitable choice of θ the Hamiltonian (0.8) can be written as

$$H = E\big(c_1^\dagger c_1 + c_2^\dagger c_2 - 1\big), \tag{0.11}$$

and determine E and u, v.

3. **Ground-state energy.** What is the ground-state energy E_0 of (0.11)? (State it in terms of ϵ and Δ.)

4. **Ground state in the *a*-basis.** Let $|0_a\rangle$ be the vacuum annihilated by $a_{1,2}$: $a_{1,2}\,|0_a\rangle = 0$. Let $|\mathrm{GS}\rangle$ be the vacuum annihilated by $c_{1,2}$: $c_{1,2}\,|\mathrm{GS}\rangle = 0$. Find the relation between $|\mathrm{GS}\rangle$ and $|0\rangle$. Verify your state minimizes (0.8).