Lecture 4

Since we learnt conformal field theories and techniques in the theories, we would like to connect them to string theory, especially by considering vertex operators.

In conformal field theory the conformal symmetry is essential. However, classical conformal symmetry, which is related to WS diff and Weyl sym, breaks at quantum level. We regard this anomaly Weyl anomaly. We will look into important examples of the anomaly.

1 Vertex operators

Let us consider the vertex operators, which has been postponed to do from the second lecture. We learnt the state-operator correspondence, which tells us a state corresponds to a certain local operator (vertex operator). We also learnt that a closed string spectrum includes tachyon, graviton, etc. Why not consider the corresponding local operators of the states ?

Recall a few spectra in a closed string:

$$\begin{aligned}
&\text{Tachyon } \phi && |0;k\rangle\ , \\
&\text{Graviton } G^{\mu\nu} && \alpha^{\mu}_{-1}\widetilde{\alpha}^{\nu}_{-1}|0;k\rangle \quad (\text{symmetric in } \mu \text{ and } \nu, \text{ and traceless})\ , \\
&\text{B-field } B^{\mu\nu} && \alpha^{\mu}_{-1}\widetilde{\alpha}^{\nu}_{-1}|0;k\rangle \quad (\text{anti-symmetric in } \mu \text{ and } \nu)\ , \\
&\text{Dilaton } \Phi && \alpha^{\mu}_{-1}\widetilde{\alpha}_{-1,\mu}|0;k\rangle\ .
\end{aligned} \tag{1}$$

Note that in order to extract physical states we need to contract the polarization tensor $\zeta_{\mu\nu}$, which satisfies $k^{\mu}\zeta_{\mu\nu} = 0$ ($\zeta^{G}_{\mu\nu} = \zeta^{G}_{\nu\mu}$ and $\zeta^{G,\mu}_{\mu} = 0$ etc.). Let us first focus on the tachyon state, which is nothing but a vacuum state with a certain momentum k^{μ}. It was defined by $p|0;k\rangle = k|0;k\rangle$, namely,

$$|0;k\rangle = e^{ik\cdot x}|0;0\rangle \quad \rightarrow \quad e^{ik\cdot X(0,0)}\ . \tag{2}$$

The final replacement is by the state-operator correspondence.

Next we consider the first excited states. Remind ourselves that the closed string mode expansion is given by

$$X^{\mu}(z,\overline{z}) = X^{\mu}_{R}(z) + X^{\mu}_{L}(\overline{z}), \tag{3}$$

$$X^{\mu}_{R}(z) = \frac{1}{2}x^{\mu} - i\frac{\alpha'}{2}p^{\mu}\log z + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\frac{\alpha^{\mu}_{n}}{z^{n}}\ . \tag{4}$$

Therefore, each mode can be expressed as follows.

$$\alpha^{\mu}_{-m} = i\sqrt{\frac{2}{\alpha'}}\oint\frac{dz}{2\pi i}z^{-m}\partial X^{\mu}(z) \quad \rightarrow \quad i\sqrt{\frac{2}{\alpha'}}\frac{1}{(m-1)!}\partial^{m}X(0). \tag{5}$$

At the last replacement we forgot the "classical" mode expansion and regarded $\partial X(z)$ as a local operator. Now we have the correspondence of tachyon, graviton etc. as follows.

$$|0;k\rangle \quad \rightarrow \quad e^{ikX} \tag{6}$$

$$\zeta_{\mu\nu}\alpha^{\mu}_{-1}\widetilde{\alpha}^{\nu}_{-1}|0;k\rangle \quad \rightarrow \quad \zeta_{\mu\nu}\partial X^{\mu}\overline{\partial}X^{\nu}e^{ikX} \tag{7}$$

The string amplitude is

$$A_n = \sum_g \int (\mathcal{D}h_{ab})_{g,n} \int \mathcal{D}X^\mu e^{-S_\sigma[X^\mu, h_{ab}]} \widehat{V}_1 \cdots \widehat{V}_n \ . \tag{8}$$

Note that, naively say,

$$(\mathcal{D}h_{ab})_{g,n} = (\mathcal{D}h_{ab})_{g,0}\, d^2 z_1 \cdots d^2 z_n \tag{9}$$

(naively say, this is because Weyl rescaling can move points on the Riemann surface to anywhere), so we can re-write the amplitude as

$$A_n = \sum_g \int (\mathcal{D}h_{ab})_{g,0} \int \mathcal{D}X^\mu e^{-S_\sigma[X^\mu, h_{ab}]} \prod_{i=1}^{n} \int d^2 z \sqrt{h} V_i \ . \tag{10}$$

Here $\int d^2 z \sqrt{h} V_i$ is an operator of the CFT, and hence, it must be Weyl & WS diff $\sim$ conformal invariant.

1.1 Mass from vertex operator

Now, let us consider a constant scaling $z \to \lambda z$ and $\overline{z} \to \overline{\lambda}\overline{z}$. Under the scaling, a field transforms as $\phi(z, \overline{z}) \to \lambda^{-h} \overline{\lambda}^{-\overline{h}} \phi(z, \overline{z})$, which should compensate the scaling of the measure $dz d\overline{z} \to \lambda\overline{\lambda} dz d\overline{z}$. Namely, $h = \overline{h} = 1$. Conformal dimension of the vertex operators are

Name	$\mathcal{O}$	$(h, \overline{h})$
Tachyon	$e^{ik\cdot X}$	$(\frac{\alpha' k^2}{4}, \frac{\alpha' k^2}{4})$
1st excited states	$\zeta_{\mu\nu} \partial X^\mu \overline{\partial} X^\nu e^{ikX}$	$(1 + \frac{\alpha' k^2}{4}, 1 + \frac{\alpha' k^2}{4})$

(11)

The consistency condition for the tachyon leads that

$$M^2 = -k^2_{\text{Tachyon}} = -\frac{4}{\alpha'} \ . \tag{12}$$

Similarly, that for the first excited states leads

$$M^2 = -k^2_{\text{1st}} = 0 \ . \tag{13}$$

Both results are consistent with the analysis from the first (and the second) lecture.

2 Weyl anomaly

As we have seen the Weyl symmetry ($T^a_a = 0$) is crucial. However, in general matter theories, the symmetry has anomaly. An important example is a string sigma model that is extended to curved space-time(ST):

$$S = \frac{1}{4\pi\alpha'} \int \sqrt{h} d^2 z \ h^{ab} \partial_a X^\mu \partial_b X^\nu g_{\mu\nu}(X) \ . \tag{14}$$

This is called (string) non-linear sigma model(NLSM). The anomaly is characterized by β-functions.

Even if the matter theory has no anomaly on flat world-sheet(WS), $\mathbb{R}^2$, the same theory has anomaly on curved WS. As we will see it is characterized by the central charge $T^a_a = -\frac{c}{12}R^{(2)}$. Therefore, the string sigma model has anomaly. However, the path integral for the metric h_{ab} turns into that of "ghost CFT" due to gauge-fixing, and it leads $c = -26$. This means that the matter theory has to have $c = 26$!

In total we have two types of anomalies:

1. Anomaly from curved WS $T^a_a = -\frac{c}{12}R^{(2)}$,
2. Anomaly from curved ST $\beta[G_{\mu\nu}] \neq 0$.

We will learn these anomalies in this order (ghost CFT will be covered in the next lecture).

2.1 Weyl anomaly from curved WS

As is stated $T^a_a = A \neq 0$. The form of A is highly restricted by symmetries: A should be

- WS diff invariant ,
- zero on $\mathbb{R}^2$,
- WS mass dimension two .

These restriction leads

$$T^a_a = aR^{(2)} \ , \tag{15}$$

where $R^{(2)}$ is a WS Ricci scalar.

Let us derive $T^a_a = -\frac{c}{12}R^{(2)}$. Take the WS to be conformally flat (which is always possible using WS diff)

$$ds^2 = e^{2\Omega(x,y)}(dxdx + dydy) = e^{2\Omega(z,\bar{z})}dzd\bar{z} \ . \tag{16}$$

For the metric we have

$$R^{(2)} = -2\partial^a\partial_a\Omega \ , \tag{17}$$

therefore, the diagonal element of the EM tensor becomes

$$T_{z\bar{z}} = \frac{1}{4}e^{2\Omega}\, T^a_a = -2a\partial\bar{\partial}\Omega \ . \tag{18}$$

The other components can be derived from $\nabla_a T^a_b = 0$.

$$0 = \nabla^z T_{zz} + \nabla^{\bar{z}} T_{\bar{z}z} = \nabla_{\bar{z}} T_{zz} + \nabla_z T_{\bar{z}z} = \bar{\partial}\left(T_{zz} - 2a\left(\partial\partial\Omega - \partial\Omega\partial\Omega\right)\right) \tag{19}$$

$$\therefore \quad T_{zz} = 2a\left(\partial\partial\Omega - \partial\Omega\partial\Omega\right) + T(z) \qquad (T(z):\ \text{holomorphic}) \tag{20}$$

Let us recall that the conformal Ward-Takahashi identity *in flat WS*:

$$\begin{aligned}\delta_{\epsilon,\bar\epsilon}\mathcal{O}(w,\overline{w}) &= \frac{1}{2\pi i}\oint_{\partial M}\left\{dz\ \epsilon(z)T(z) - d\overline{z}\ \overline{\epsilon}(\overline{z})\overline{T}(\overline{z})\right\}\mathcal{O}(w,\overline{w}) \\ &= \int_M \frac{d^2z}{2\pi}\left\{\overline{\partial}\left(\epsilon(z)T(z)\right) + \partial\left(\overline{\epsilon}(\overline{z})\overline{T}(\overline{z})\right)\right\}\mathcal{O}(w,\overline{w})\ . \end{aligned} \tag{21}$$

(Notice that the convention is different from the last lecture.) Note that EM tensor is conserved, $\partial^a T_{ab} = 0$, equivalently, $\overline{\partial}T(z) = 0 = \partial\overline{T}(\overline{z})$ with $T(z) = T_{zz}$, $\overline{T}(\overline{z}) = T_{\overline{z}\overline{z}}$, and $T_{z\overline{z}} = 0$. On the other hand, *in the curved WS*, the current conservation is expressed with a covariant derivative $\nabla^a T_{ab} = 0$. For example,

$$\nabla^a T_{az} = \nabla^z T_{zz} + \nabla^{\overline{z}} T_{\overline{z}z} = \overline{\partial}T(z) = 0 \tag{22}$$

$$\text{with}\quad T(z) = T_{zz} - 2a\left(\partial\partial\Omega - \partial\Omega\partial\Omega\right)\ . \tag{23}$$

Similar for $\overline{T}(\overline{z})$. Hence, we have the same form of conformal Ward-Takahashi identity. Especially,

$$\delta_\epsilon T(z) = \epsilon(w)\partial T(z) + 2\partial\epsilon(z)T(z) + \frac{c}{12}\partial^3\epsilon(z)\ . \tag{24}$$

On the other hand, from the expression (23) we have

$$\delta_\epsilon T(z) = \delta_\epsilon T_{zz}(z) - 2a\left(\partial\partial\delta_\epsilon\Omega - 2\partial\Omega\partial\delta_\epsilon\Omega\right)\ . \tag{25}$$

Using (finite) transformations:

$$z \to \widetilde{z} = z - \epsilon(z)\ , \tag{26}$$

$$T_{zz}(z) \to \widetilde{T}_{\widetilde{z}\widetilde{z}}(\widetilde{z}) = \left(\partial_z\widetilde{z}\right)^{-2} T_{zz}(z)\ , \tag{27}$$

$$\Omega(z) \to \widetilde{\Omega}(\widetilde{z}) = \Omega(z) - \frac{1}{2}\log|\partial_z\widetilde{z}|^2\ , \tag{28}$$

the transformation of (25) becomes

$$\delta_\epsilon T(z) = \epsilon(z)\partial T(z) + 2\partial\epsilon(z)T(z) - a\partial^3\epsilon(z)\ . \tag{29}$$

Therefore,

$$T^a_a = aR^{(2)} = -\frac{1}{12}cR^{(2)}. \tag{30}$$

2.2 Non-linear sigma model (Graviton included)

So far we only consider a flat space-time(ST). However, if we expect that the string theory describe general relativity the action should contain general metric $G_{\mu\nu}(X)$:

$$S[X^\mu, h_{ab}] = \frac{1}{4\pi\alpha'}\int d^2x\left(\sqrt{h}h^{ab}\partial_a X^\mu\partial_b X^\nu G_{\mu\nu}(X)\right)\ . \tag{31}$$

An action of this shape is called non-linear sigma model(NLSM). Now let us consider an almost flat metric:

$$G_{\mu\nu}(X) = \eta_{\mu\nu} + f_{\mu\nu}(X) \ . \tag{32}$$

Then, partition function becomes

$$\begin{aligned} Z &= \int \mathcal{D}h_{ab}\mathcal{D}X^\mu e^{-S} \\ &= \int \mathcal{D}h_{ab}\mathcal{D}X^\mu e^{-S_0}\left(1 + \frac{1}{4\pi\alpha'}\int d^2x \left(\sqrt{h}h^{ab}\partial_a X^\mu \partial_b X^\nu f_{\mu\nu}(X)\right) + \cdots\right) \ . \end{aligned} \tag{33}$$

Notice that the perturbative part is nothing but the graviton operator with wave function $f_{\mu\nu}(X) = \zeta_{\mu\nu}e^{ik\cdot X}$.

Let us again consider general case $G_{\mu\nu}(X)$. As a 2d field theory we can consider the vacuum expectation value(vev) for X, which we set to X_0:

$$\hat{X}(\sigma,\tau) = X_0 + X(\sigma,\tau) \ . \tag{34}$$

On the othat hand, X_0 is a certain point in ST and we will expand the metric around this point. If you choose the coordinate nicely (Riemann normal coordinate) the expansion of the metric can be written as follows.

$$G_{\mu\nu}(X) = G_{\mu\nu} - \frac{1}{3}R_{\mu\lambda\nu\rho}X^\lambda X^\rho + \mathcal{O}(X^3) \ , \tag{35}$$

where $G_{\mu\nu}$ and $R_{\mu\lambda\nu\rho}$ are a metric and a Rimann tensor at X_0, respectively. In a field theory sense those are coupling constants:

$$\begin{aligned} S &= \frac{1}{4\pi\alpha'}\int d^2x \ \partial^a X^\mu \partial_a X^\nu \left(G_{\mu\nu} - \frac{1}{3}R_{\mu\lambda\nu\rho}X^\lambda X^\rho + \cdots\right) \\ &= \frac{1}{2}\int d^2x \ \partial^a X^\mu \partial_a X^\nu \left(G_{\mu\nu} - \frac{2\pi\alpha'}{3}R_{\mu\lambda\nu\rho}X^\lambda X^\rho + \mathcal{O}(\alpha'^2)\right) \ . \end{aligned} \tag{36}$$

We rescaled the field $X \to \sqrt{2\pi\alpha'}X$ so that the expansion looks like "stringy expansion".

2.3 Perturbation theory for NLSM (Weyl anomaly from curved ST)

Discussion here is a simpler version of Hosomichi's note(Lecture 03). Note that discussion on this subject in textbooks(Green-Schwarz-Witten, Polchinski, Hosomichi etc.) are incomplete. If you are interested in the details you should consult [CT88], which is a nice review from TASI lecture 1988.

We want to check if the theory (NLSM) has Weyl anomaly. As we briefly saw it is an intracting theory, and in general, intracting theories have non-trivial β-functions:

$$\beta[\lambda] \equiv E\frac{\partial}{\partial E}\lambda(E) = \frac{\partial}{\partial(\log E)}\lambda(E) \ , \tag{37}$$

where λ is a coupling constant and E is a characteristic energy scale. When we consider a global scaling of coordinate:$z \to \widetilde{z} = sz = (1-\epsilon)z = e^{-\epsilon}z$, enegy scales oppositely: $E \to \widetilde{E} = \frac{1}{s}E = e^{\epsilon}E$. So the β-function can be written as

$$\beta[\lambda] = \frac{\partial}{\partial \epsilon}\lambda(\epsilon) \ . \tag{38}$$

The variation of the action is expressed in two ways:

$$\delta_\epsilon S = \begin{cases} \int \frac{d^2x}{2\pi}\sqrt{h}\delta_\epsilon h^{ab}T_{ba} = -\epsilon \int \frac{d^2x}{2\pi}T^a_{\ a} \ , \\ \frac{1}{4\pi\alpha'}\int d^2x \ \partial^a X^\mu \partial_a X^\nu \left(\epsilon\frac{\partial}{\partial\epsilon}G_{\mu\nu}(\epsilon) + \cdots\right) \ , \end{cases} \tag{39}$$

where the first variation is a formal transformation of the theory, and in quantum regime, it should be proportional to the trace part of the EM tensor. On the other hand, the second variation is the actual theory with an assumption that ϵ dependence of the theory is only in the coupling constants. Identifying them we have

$$\therefore \qquad T^a_{\ a} = -\frac{1}{2\alpha'}\beta[G_{\mu\nu}]\partial^b X^\mu \partial_b X^\nu + \cdots \ . \tag{40}$$

This shows that the anomaly is parametrized by β-functions.

Let us consider perturbation theory, namely loop corrections to two-point function etc, so that we can see if the theory is anomalous.

$$\begin{aligned} &\langle X^\mu(x_1)X^\nu(x_2)\rangle \\ &\quad = \int \frac{d^2k}{(2\pi)^2}\frac{2\pi\alpha'}{k^2}e^{ik\cdot(x_1-x_2)}\left\{G^{\mu\nu} + \frac{2\pi\alpha'}{3}R^{\mu\nu}\left(\int\frac{d^2p}{(2\pi)^2}\frac{1}{p^2} + \frac{1}{k^2}\int\frac{d^2p}{(2\pi)^2}\right) + \cdots\right\} \ . \end{aligned} \tag{41}$$

We further focus on the logarithmic divergence and introduce regularization parameters:

$$\int_E^\Lambda \frac{d^2p}{(2\pi)^2}\frac{1}{p^2} = \frac{1}{2\pi}\log\left(\frac{\Lambda}{E}\right) \ , \tag{42}$$

where Λ is an ultra-violet(UV) energy scale supposed to be ∞, and E is an infra-red(IR) energy scale supposed to be our life energy scale, which is very law (~ 0).

The divergence can be subtracted by counter terms as follows. Define $\hat{S} = S + S_{\mathrm{ct}}$, which is called bare action:

$$\hat{S} = \frac{1}{4\pi\alpha'}\int d^2x \ \partial^a\hat{X}^\mu\partial_a\hat{X}^\nu\left(\hat{G}_{\mu\nu} - \frac{1}{3}\hat{R}_{\mu\lambda\nu\rho}\hat{X}^\lambda\hat{X}^\rho + \cdots\right) \tag{43}$$

with

$$\hat{X}^\mu = Z^\mu_\nu X^\nu \ , \quad Z^\mu_\nu = \delta^\mu_\nu + \sum_{n=1}^\infty \alpha'^n Z^\mu_{(n),\nu}(\Lambda/E) \ , \tag{44}$$

$$\hat{G}_{\mu\nu} = G_{\mu\nu} + \sum_{n=1}^\infty \alpha'^n G^{(n)}_{\mu\nu}(\Lambda/E) \ , \text{etc.} \tag{45}$$

Physical action is S, which describes IR physics of energy scale E, on the other hand, $\hat{S}$ is called bare action, which describes UV physics of energy scale Λ. Note that the bare action only depends on Λ (not on E), and hence, the bare coupling constants ($\hat{G}_{\mu\nu}$ etc) only depends on high energy Λ. The counter terms leads other contributions

$$\begin{aligned} &\langle X^\mu(x_1) X^\nu(x_2)\rangle \\ &\quad \sim \left\{ G^{\mu\nu} + \frac{\alpha'}{3} R^{\mu\nu} \log\left(\frac{\Lambda}{E}\right) - \alpha'\left(G^{\mu\nu}_{(1)} + Z^{\mu\nu}_{(1)} + Z^{\nu\mu}_{(1)}\right) + \cdots \right\} . \end{aligned} \tag{46}$$

Unfortunately, the equation above cannot fix the ration between $G_{(1)}$ and $Z_{(1)}$. We need further information like 4-pt function etc. to determine the ratio. We simply list the result:

$$G^{(1)}_{\mu\nu} = R_{\mu\nu} \log\left(\frac{\Lambda}{E}\right) , \quad Z^{(1)}_{\mu\nu} = -\frac{1}{3} R_{\mu\nu} \log\left(\frac{\Lambda}{E}\right) , \tag{47}$$

which does cancel the divergent term in (46). From the result we can derive the β-function:

$$G_{\mu\nu}(E,\Lambda) = \hat{G}_{\mu\nu}(\Lambda) - \alpha' R_{\mu\nu} \log\left(\frac{\Lambda}{E}\right) , \tag{48}$$

$$\beta[G_{\mu\nu}] = \frac{\partial}{\partial(\log E)} G_{\mu\nu}(E,\Lambda) = \alpha' R_{\mu\nu} . \tag{49}$$

Therefore, in order for the theory to be anomaly free we need the space-time to be Ricci flat ($R_{\mu\nu} = 0$). This is equivalent for Hilbert-Einstein action to satisfy its E.O.M.

2.4 NLSM (general)

So far only the graviton has been included but not B-field or dilaton. Here, we simply give the action and their β-functions.

$$S = \frac{1}{4\pi\alpha'} \int d^2x \left(\sqrt{h} h^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X) + i\varepsilon^{ab} \partial_a X^\mu \partial_b X^\nu B_{\mu\nu}(X) + \alpha' \sqrt{h} R^{(2)} \Phi(X) \right) . \tag{50}$$

Weyl anomaly:

$$T^a_a = -\frac{1}{2\alpha'} \beta[G_{\mu\nu}] \partial_a X^\mu \partial_b X^\nu - \frac{i}{2\alpha'} \beta[B_{\mu\nu}] \varepsilon^{ab} \partial_a X^\mu \partial_b X^\nu - \frac{1}{2} \beta[\Phi] R^{(2)} \tag{51}$$

β-functions:

$$\beta[G_{\mu\nu}] = \alpha' R_{\mu\nu} + 2\alpha' \nabla_\mu \nabla_\nu \Phi - \frac{\alpha'}{4} H_{\mu\lambda\rho} H_\nu{}^{\lambda\rho} + \mathcal{O}(\alpha'^2) , \tag{52}$$

$$\beta[B_{\mu\nu}] = -\frac{\alpha'}{2} \nabla^\lambda H_{\lambda\mu\nu} + \alpha' \nabla^\lambda \Phi H_{\lambda\mu\nu} + \mathcal{O}(\alpha'^2) , \tag{53}$$

$$\beta[\Phi] = \frac{D-26}{6} - \frac{\alpha'}{2} \nabla^2 \Phi + \alpha' \nabla^\lambda \Phi \nabla_\lambda \Phi - \frac{\alpha'}{24} H_{\mu\lambda\rho} H^{\mu\lambda\rho} + \mathcal{O}(\alpha'^2) . \tag{54}$$

There are a few remarks on them:

- String theory to be consistent requires all the β-functions to be zero.
- Note that 2d gravity has no physical D.O.F, and the Einstein-Hilbert action gives a topological number(Euler characteristic). So if the dilaton is constant we have

$$
\begin{aligned}
S_{\text{dilaton}} &= \frac{1}{4\pi\alpha'}\int d^2x\left(\alpha'\sqrt{h}R^{(2)}\Phi(X)\right) \\
&\to \frac{1}{4\pi}\int d^2x\left(\sqrt{h}R^{(2)}\Phi\right) = \Phi(2-2g)
\end{aligned} \tag{55}
$$

Define $g_{\text{str}} = e^{\Phi}$ and the action leads

$$
e^{-S_{\text{dilaton}}} = g_{\text{str}}^{2g-2} \ . \tag{56}
$$

g_{str} can be understood as a string coupling as follows. See Fig. 1. n-point string tree amplitude can be understood $n-2$ cylinders attaching to a cylinder. So the

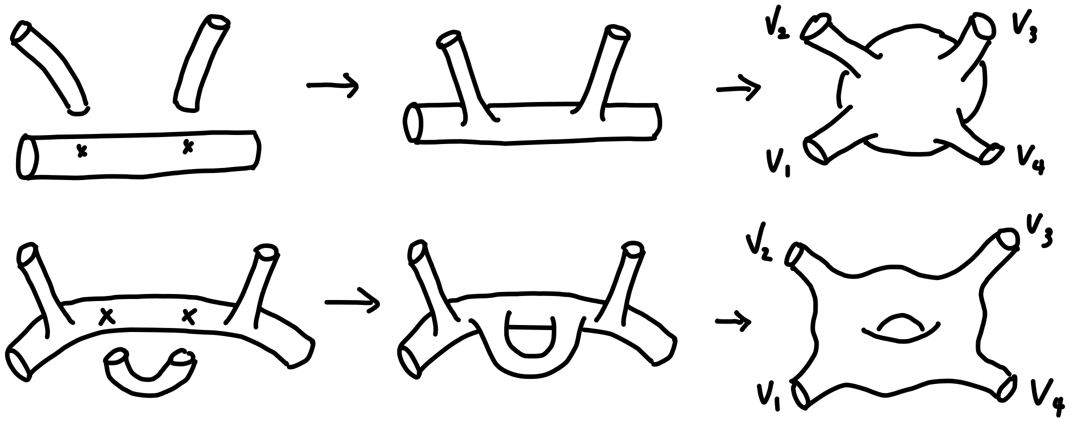

Fig. 1: 4-pt amplitude example. The upper one is a construction of 4pt tree amplitude from cylinders. The lower one is a construction of 4-pt 1-loop amplitude from the tree amplitude.

amplitude should proportional to g_{str}^{n-2}. Higher loop (higher genus) amplitude can be derived by attaching g cylinders to the tree amplitude, and the amplitude should be $\hat{A}_{n,g} \propto g_{\text{str}}^{n-2+2g}$. Usually, vertex operators are re-normalized so that g_{str}^{n} is included in the definition of $V_1\cdots V_n$. Therefore, the re-normalized amplitude should be

$$
A_{n,g} \propto g_{\text{str}}^{2g-2} \ , \tag{57}
$$

which coincide with the dilaton action.

- Trivial β-function ($\beta = 0$) is equivalent to the E.O.M of the following ST action.

$$
S_{\text{eff}} = \frac{1}{2\kappa_D^2}\int d^DX\sqrt{-G}e^{-2\Phi}\left[\frac{2(D-26)}{3\alpha'} + R - \frac{1}{12}H_{\mu\lambda\rho}H^{\mu\lambda\rho} + 4\nabla^{\lambda}\Phi\nabla_{\lambda}\Phi + \mathcal{O}(\alpha'^2)\right] \ . \tag{58}
$$

- B-field is a higher dimensional analogy of gauge fields. It has gauge transformation

$$
\delta B_{\mu\nu} = \partial_{\mu}\Lambda_{\nu} - \partial_{\nu}\Lambda_{\mu} \ , \tag{59}
$$

and field strength $H_{\mu\nu\lambda}$:

$$H_{\mu\nu\lambda} = \partial_\mu B_{\nu\lambda} + \partial_\nu B_{\lambda\mu} + \partial_\lambda B_{\mu\nu} \ . \tag{60}$$

B-field plays an important role with open string.

References

[CT88] Curt Callan and Lrus Thorlacius. *SIGMA MODELS AND STRING THEORY.* TASI Lecture, 1988. (The link below is a direct link to the pdf file of 45MB) http://www.damtp.cam.ac.uk/user/tong/string/sigma.pdf. 5