2d Superconformal Field Theories

Contents

1 Preliminaries

1.1 Conformal Transformation

Conformal field theory(CFT) is a field theory with conformal symmetry. Roughly speaking, conformal symmetry is invariance of angle under transformation. Mathematically, a conformal transformation is defined as follows. A differential map $\phi : U \to V$ between two manifolds M and M', where U and V are open subsets of M and M' respectively, is called conformal transformation if the metric tensor satisfies $\phi^* g' = \Lambda g$. If we use coordinate x' and x to represent points in V and U, this is expressed in the following way:

$$g'_{\rho\sigma}(x')\frac{\partial x'^{\rho}}{\partial x^{\mu}}\frac{\partial x'^{\sigma}}{\partial x^{\nu}} = \Lambda(x) g_{\mu\nu}(x) \tag{1}$$

Consider $M' = M$ which implies $g' = g$, and consider flat spaces with a constant metric of Lorentz form $g_{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}(-1, +1, +1...)$ or of Euclidean form $g_{\mu\nu} = \delta_{\mu\nu}$. In this case, the condition for a conformal transformation can be written as

$$\eta_{\rho\sigma}\frac{\partial x'^{\rho}}{\partial x^{\mu}}\frac{\partial' x^{\sigma}}{\partial x \nu} = \Lambda(x)\eta_{\mu\nu}$$

Next, consider infinitesimal coordinate transformations which read

$$x'^{\rho} = x^{\rho} + \epsilon^{\rho}(x) + \mathcal{O}(\epsilon^2).$$

Restricting to 2-dimensional space only, the condition (1) for invariance under infinitesimal conformal transformations reads:

$$\partial_0 \epsilon_0 = \partial_1 \epsilon_1, \qquad \partial_0 \epsilon_1 = -\partial_1 \epsilon_0,$$

which we recognise as the Cauchy-Riemann equations. If we introduce complex variables in the following way:

$$z = x^0 + ix^1, \qquad \epsilon = \epsilon^0 + i\epsilon^1, \qquad \partial_z = \frac{1}{2}(\partial_0 - i\partial_1) \tag{2}$$

$$\bar{z} = x^0 - ix^1, \qquad \bar{\epsilon} = \epsilon^0 - i\epsilon^1, \qquad \partial_{\bar{z}} = \frac{1}{2}(\partial_0 + i\partial_1). \tag{3}$$

Then Cauchy-Riemann equations imply $\epsilon(z)$ is holomorphic, so is $f(z) = z + \epsilon(z)$. We conclude that a holomorphic function $f(z) = z + \epsilon(z)$ gives rise to an infinitesimal 2-dimensional conformal transformation. Note that this statement is true both for the Euclidean case and the Lorentz case, as long as we stay in 2-dimensional space.

1.2 Primary Fields

We have discussed the space which we can perform a conformal transformation. Now we move to discuss the fields living in this space. Let us start with a Euclidean 2-dimensional space $\mathbb{R}^2$ and using complex coordinates in (2) to denote the points. Then the fields ϕ of our theory is just

$$\phi(x^0, x^1) = \phi(z, \bar{z}).$$

Fields only depending on z, i.e. $\phi(z)$ are called holomorphic fields or chiral fields, and similarly $\phi(\bar{z})$ are called anti-holomorphic fields or anti-chiral fields. Fields with good transformation properties always play a central role in physics. In conformal field theory, such fields are called the primary field. Mathematically, we give the following definition: If a field transforms under conformal transformations $z \to f(z)$ according to

$$\phi(z,\bar{z}) \to \phi'(z,\bar{z}) = (\frac{\partial f}{\partial z})^h (\frac{\partial \bar{f}}{\partial \bar{z}})^{\bar{h}} \phi(f(z), \bar{f}(\bar{z})), \tag{4}$$

it is called a primary field of the conformal dimension $(h, \bar{h})$. The infinitesimal version of (4) reads

$$\delta_{\epsilon,\bar{\epsilon}}\phi(z,\bar{z}) = (h\partial_z\epsilon + \epsilon\partial_z + \bar{h}\partial_{\bar{z}}\bar{\epsilon} + \bar{\epsilon}\partial_{\bar{z}})\phi(z,\bar{z}).$$

Not all fields in a CFT are primary. Those fields are called secondary fields.

1.3 The Energy-Momentum Tensor

Noether's theorem tells us that for CFT, there is a conserved current which can be written as

$$j_\mu = T_{\mu\nu}\epsilon^\nu.$$

where ϵ^ν represent the infinitesimal coordinate transformation $x^\mu \to x^\mu + \epsilon^\mu(x)$. The associated conserved charges are $Q = \int dx^1 j_0 = \int dx^1 (T_{01}\epsilon^1 + T_{00}\epsilon^0)$ where the integral over x^1 is performed for a fixed x^0 which we identify with time slice. The tensor $T_{\mu\nu}$ can be taken symmetric and is called the energy-momentum tensor. If we let $\epsilon^u = \text{const.}$, then

$$0 = \partial^\mu j_\mu = \partial^\mu (T_\mu \nu \epsilon^\nu) = (\partial^\mu T_{\mu\nu})\epsilon^\nu. \tag{5}$$

So $\partial^\mu T_{\mu\nu} = 0$. For general conformal transformations $\epsilon^\mu(x)$,

$$\begin{aligned} 0 = \partial^\mu j_\mu &= (\partial^\mu T_{\mu\nu})\epsilon^\nu + T_{\mu\nu}(\partial^\mu \epsilon^\nu) \\ &= 0 + \frac{1}{2} T_{\mu\nu}(\partial^\mu\epsilon^\nu + \partial^\nu\epsilon^\mu) \\ &= \frac{1}{d} T^\mu_\mu (\partial \cdot \epsilon). \end{aligned}$$

Since this equation has to be true for arbitrary conformal transformations, we see that in CFT the energy-momentum tensor $T_{\mu\nu}$ is traceless, that is $T^\mu_\mu = 0$.

From now on we will only consider the Euclidean signature case and rewrite everything in complex coordinates of (2). Using $T_{\mu\nu} = \frac{\partial x^\alpha}{\partial x^\mu}\frac{\partial x^\beta}{\partial x^\nu} T_{\alpha\beta}$, together with traceless condition and Euclidean metric, we can get

$$T_{z\bar{z}} = T_{\bar{z}z} = 0, \tag{6}$$

$$T_{zz} = \frac{1}{2}(T_{00} - iT_{10}), \tag{7}$$

$$T_{\overline{zz}} = \frac{1}{2}(T_{00} + iT_{10}), \tag{8}$$

$$\partial_{\bar{z}} T_{zz} = \partial_z T_{\bar{z}\bar{z}} = 0. \tag{9}$$

The last equation allows us to define a chiral field $T(z) := T_{zz}(z, \bar{z})$ and an anti-chiral field $\overline{T}(\bar{z}) := T_{\bar{z}\bar{z}}(z, \bar{z})$. Note that (6) and (9) are invariant under any arbitrary conformal transformation of complex coordinate. In the following, we compactify the $\mathbb{R}^2$ plane by identifying x^1 $x^1 + 2\pi$, making it an infinitely long cylinder. Further, we use another coordinate transformation and a change in notation:

$$w = x^0 + ix^1, \quad z = e^w,$$

thus we map the cylinder onto a complex plane again where the periodic condition for x^1 is retained. We will quantize our theory on this particular space.

1.4 Radial Quantization and the Operator Product Expansion

In quantum theory, the conserved charge is the generator of symmetry transformations. The charge Q discussed above written in the new coordinate reads

$$Q = \frac{1}{2\pi i} \oint_C dz T(z)\epsilon(z) + d\bar{z}\overline{T}(\bar{z})\bar{\epsilon}(\bar{z}). \tag{10}$$

For an operator field $\phi(z, \bar{z})$,

$$\begin{aligned} \delta_{\epsilon,\bar{\epsilon}}\phi(w, \bar{w}) &= [Q, \phi] \\ &= \frac{1}{2\pi i} \oint_C dz[T(z)\epsilon(z), \phi(w, \bar{w})] + \frac{1}{2\pi i} \oint_C d\bar{z}[\overline{T}(\bar{z})\bar{\epsilon}(\bar{z}), \phi(w, \bar{w})] \end{aligned}$$

In classical theory, the integral over z can be performed at an arbitrary circle centered on the origin. However in quantum field theory things in time order is important so we should specify the radius of integration contour according to the order of operators. To this end, we define the radial ordering of two operators as

$$R(A(z)B(w)) := \begin{cases} A(z)B(w) & \text{for} |z| > |w|, \\ B(w)A(z) & \text{for} |w| > |z|. \end{cases}$$

$R(A(z)B(w))$ itself can be treated as operators and often we perform Laurant expansion around w, which results in expressions like the following:

$$R(A(z)B(w)) = \sum_{n>1} \frac{O_n(w)}{(z-w)^n} + \text{regular terms}.$$

Often we are only interested in singular terms and omit regular terms. In this case, we use $\sim$ to denote omission. Such kind of expansion of radial ordered operator is called operator product expansion(OPE).

1.5 The Virasoro Algebra and its representation

Without proof, we state that the general OPE of the chiral energy-momentum tensor with itself is

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} + ... \tag{11}$$

where c is a constant called the central charge and depends on theory. A similar result holds for the anti-chiral part $\overline{T}(\overline{z})$, and the OPE between chiral and anti-chiral part only has non-singular terms. If we perform a Laurent expansion

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2} L_n, \quad L_n = \frac{1}{2\pi i}\oint dz z^{n+1} T(z),$$

then from (11) we can get the Virasoro Algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0} \tag{12}$$

It is the $\mathcal{N} = 0$ case of Superconformal Algebra which we will discuss later.

Given the Virasoro algebra, we want to try and find its representations which we can use to construct the quantum states of the theory. Here we study the highest weight(HW) representation. First, we define the conjugate operators of modes of T

$$(L_n)^\dagger = L_n$$

Define the highest weight states characterised by h as:

$$\begin{aligned} L_0 \left|h\right\rangle &= h \left|h\right\rangle, \\ L_n \left|h\right\rangle &= 0 \quad (n > 0). \end{aligned}$$

Other states in the representation are obtained by acting $L_k (k < 0)$ on $|h\rangle$. Consider a conformal field $\phi(z, \bar{z})$ with weights h and $\bar{h}$, then

$$\left|h, \bar{h}\right\rangle = \phi(0,0)\left|0\right\rangle \tag{13}$$

is a highest weight state with respect to Virasoro Algebra and its anti-holomorphic part. We can verify the condition explicitly.

1.6 The Minimal Model

If the states in the representation all have a non-negative norm, the representation is called unitary. Unitarity can impose a strong constraint on HW representation. Consider the second excitation level (i.e. subspaces contain only states with L_0 eigenvalue $h + 2$), there are linear combinations of states $L_{-2}|h\rangle$ and $(L_{-1})^2|h\rangle$. If all the states in this level have a non-negative norm, then the matrix

$$K_2 = \begin{pmatrix} \langle h| L_{-2}^\dagger L_{-2} |h\rangle & \langle h| L_{-2}^\dagger L_{-1} L_{-1} |h\rangle \\ \langle h| (L_{-1}L_{-1})^\dagger L_{-2} |h\rangle & \langle h| (L_{-1}L_{-1})^\dagger L_{-1} L_{-1} |h\rangle \end{pmatrix}$$

should only have zero or positive eigenvalues. For every excitation level, there is an analogous matrix which cast further constraints through eigenvalues. Detailed argument[BP09] shows that in the region $1 \geqslant c > 0$, $h \geqslant 0$ only discrete sets of h and c can survive these constraints. The allowed central charge and the corresponding conformal dimension is

$$c = 1 - \frac{6}{p(p+1)}, \quad p = 3, 4, 5..., \tag{14}$$

$$h = \frac{[(p+1)r - ps]^2 - 1}{4p(p+1)}, \quad 1 \leqslant s \leqslant r \leqslant p-1. \tag{15}$$

where r and s also take only integer values. Recalling (13), we get that for theory corresponds to these unitary representations, there exist only finitely many possible primary fields. Such theories are called minimal model, denoted by $\mathcal{M}_p$ where p is the integer in (14).

1.7 The Kac-Moody Algebra and the Sugawara construction

Conformal symmetry induces the Virasoro Algebra and its anti-chiral partner, which can be used as a tool for studying CFT. But sometimes CFTs can have other symmetries, forming different symmetry algebra. Here we consider the conserved current form an algebra called Kac-Moody Algebra. Generally, a Kac-Moody algebra $\mathfrak{g}_k$ is an extension of some Lie algebra $\mathfrak{g}$ defined by commutation relations

$$[j^a_m, j^b_n] = i \sum_c f^{abc} j^c_{m+n} + km\delta^{ab}\delta_{m+n,0} \quad ,$$

where f^{abc} are the structure constants of $\mathfrak{g}$ and k is called the level of $\hat{\mathfrak{g}}_k$. We can see that if we let $m = n = 0$, it goes back to the familiar commutation relation of Lie algebra $\mathfrak{g}$. To get the operator product expansions of the fields in CFT, one uses the usual Laurent expansion. Here we define the currents $J^a(z)$ to be chiral fields with conformal dimension $h = 1$:

$$j^a(z) = \sum_{n\in\mathbb{Z}} j^a_n z^{-n-1}, \quad j^a_n = \oint \frac{dz}{2\pi i} z^n j^a(z),$$

Then the commutation relations of $\mathfrak{g}_k$ are equivalent to the OPE

$$j^a(z) j^b(w) \sim \frac{k\delta^{ab}}{(z-w)^2} + \sum_c \frac{i f^{abc}}{z-w} j^c(w).$$

In CFT, this new symmetry algebra realized by currents has to be compatible with the Virasoro algebra. In fact, we can define energy-momentum tensor as following such that the currents have conformal dimension $h = 1$:

$$T(z) = \frac{1}{k + h^\vee_{\mathfrak{g}}} \sum_a : J^a J^a : (z),$$

where $h^\vee_{\mathfrak{g}}$ is the dual Coxeter number of $\mathfrak{g}$. This method of construction is called the Sugawara construction.

1.8 The Unitary Minimal Model in terms of the Coset Model

The coset model is a CFT constructed by the quotient of current algebras. For a general Lie algebra $\mathfrak{g}$ and its subalgebra $\mathfrak{h}$, the energy-momentum tensor is given by Sugawara construction

$$T_{\mathfrak{g}}(z) = \frac{1}{k_{\mathfrak{g}} + h^\vee_{\mathfrak{g}}} \sum_{a=1}^{\dim\mathfrak{g}} : J^a_{\mathfrak{g}} J^a_{\mathfrak{g}} : (z), \tag{16}$$

$$T_{\mathfrak{h}}(z) = \frac{1}{k_{\mathfrak{h}} + h^\vee_{\mathfrak{h}}} \sum_{a=1}^{\dim\mathfrak{h}} : J^a_{\mathfrak{h}} J^a_{\mathfrak{h}} : (z), \tag{17}$$

where $k_{\mathfrak{g}}$ and $k_{\mathfrak{h}}$ are their level, respectively. Computing the TT OPE, we can get the central charge of $\mathfrak{g}$

$$c_{\mathfrak{g}} = \frac{k_{\mathfrak{g}}\mathrm{dim}\mathfrak{g}}{k_{\mathfrak{g}} + h^{\vee}_{\mathfrak{g}}}, \tag{18}$$

and that of the subalgebra $\mathfrak{h}$ in a similar way. The coset model is denoted by $\widehat{\mathfrak{g}}_{k_{\mathfrak{g}}}/\widehat{\mathfrak{h}}_{k_{\mathfrak{h}}}$, of which the energy-momentum tensor is defined by

$$T_{\text{coset}} := T_{\mathfrak{g}} - T_{\mathfrak{h}}.$$

Its TT OPE has a usual form

$$T_{\text{coset}}(z)T_{\text{coset}}(0) \sim \frac{\frac{c}{2}}{z^4} + \frac{T_{\text{coset}}}{z^2} + \frac{\partial T_{\text{coset}}}{z},$$

where the central charge $c = c_{\mathfrak{g}} - c_{\mathfrak{h}}$.

In particular, if we compute the central charge of the coset model $\frac{\widehat{\mathfrak{su}(2)}_{p-2} \oplus \widehat{\mathfrak{su}(2)}_1}{\widehat{\mathfrak{su}(2)}_{p-1}}$, we get

$$c = \frac{(p-2)\times 3}{p} + \frac{3}{3} - \frac{3(p-1)}{p+1} = 1 - \frac{6}{p(p+1)}.$$

Recall that for a unitary minimal model of $0 < c \leqslant 1$, the central charge is

$$c = 1 - \frac{6}{p(p+1)}, \quad p = 3, 4...,$$

which is exactly the same as above. In fact, the unitary minimal model $\mathcal{M}_p$ can be described by $\frac{\widehat{\mathfrak{su}(2)}_{p-2} \oplus \widehat{\mathfrak{su}(2)}_1}{\widehat{\mathfrak{su}(2)}_{p-1}}$. We leave the detail of the proof to the reference[FMS97, 807-811].

1.9 The Representation and the Characters of $\widehat{\mathfrak{su}(2)}_k$

We use $\widehat{\mathfrak{su}(2)}_k$ as an example to study the highest weight representation of current algebra. There are three currents J^3,J^+,J^-, in $\mathfrak{su}(2)$. Their mode expansions are

$$J^a(z) = \sum_{n\in\mathbb{Z}} z^{-n-1} J^a_n, \quad a = 3.\pm$$

The highest weight state $|j\rangle$ is defined by

$$\begin{aligned} &J^3_0 |j\rangle = j\, |j\rangle\,, \\ &J^{\pm}_n |j\rangle = J^3_n |j\rangle = 0 \quad (n > 0), \\ &J^+_0 |j\rangle = (J^-_0)^{2j+1} |j\rangle = 0. \end{aligned}$$

and descendants are states generated from operators

$$J^-_0\,, \quad J^a_{-n} \quad (n > 0, a = 3, \pm).$$

For unitary representation, k, $2j$ must be integers and $0 \leqslant 2j \leqslant k$[BP09]. From the Sugawara construction, we know that the highest weight representation of currents is also the HW representation of the Virasoro algebra and conformal weight h of the states can be determined by j. The highest weight state and its descendants form unitary representation $V(c,j)$, but there are null states $|\chi\rangle \in V(c.j)$ defined as

$$\begin{aligned} J_0^+ |\chi\rangle &= 0, \\ J_n^a |\chi\rangle &= 0. \end{aligned}$$

making the representation reducible. If we let $J(c,j) = \{\text{all null states}\}$, then $L(c,j) = V(c,j)/J(c,J)$ is an irreducible representation of $\widehat{\mathfrak{su}(2)}_k$, which is called the integrable representation of the highest weight $|h,j\rangle$. Define the so-called Weyl-Kac character

$$\chi = \mathrm{Tr}_{L(c,j)}[q^{L_0 - \frac{c}{24}} y^{J_0^3}],$$

then the character of $\widehat{\mathfrak{su}(2)}_k$ can be computed by taking trace[FMS97]

$$\chi = \frac{\Theta_{l+1,k+2} - \Theta_{-l-1,k+2}}{\Theta_{1,2} - \Theta_{-1,2}},$$

where

$$\Theta_{m,k}(\tau,z) = \sum_{n \in \mathbb{Z} + \frac{m}{2k}} q^{kn^2} y^{kn^2}, \tag{19}$$

$$q = e^{2\pi i \tau}, \quad y = e^{2\pi i z}. \tag{20}$$

For coset construction $\frac{\widehat{\mathfrak{su}(2)}_{p-2} \oplus \widehat{\mathfrak{su}(2)}_1}{\widehat{\mathfrak{su}(2)}_{p-1}}$, applying the so-called branching rule of the irreducible representation of algebras, we get

$$L(p-2,l) \bigotimes L(1,t) = \bigoplus_{l'=0}^{p-1} (V^{\text{coset}}_{[l,l';t]} \otimes L(p-1,l')),$$

where l, t and l' denote spins, and $l + t - l' \in 2\mathbb{Z}$. V is some branching function. It follows that their characters satisfy

$$\chi^{\mathfrak{su}(2)_k}_{p-2,l} \chi^{\mathfrak{su}(2)_k}_{1,t} = \sum_{l'=0}^{p-1} \chi^{\text{coset}}_{[l,l';t]} \chi^{\mathfrak{su}(2)_k}_{p-1,l'} \quad ,$$

and we identify $\chi^{\text{coset}}_{[l,l';t]}$ with the character of the coset model. By lengthy calculation and using mathematical identity

$$\Theta_{m,k}(\tau)\Theta_{m',k'}(\tau) = \sum_{r \in \mathbb{Z}_k + k'} \Theta_{mk'-m'k+2rkk',kk'(k+k')} \Theta_{m+m'+2kr,k+k'}, \tag{21}$$

it can be shown that

$$\chi^{\text{coset}}_{[l,l';,t]} = \chi^{\mathcal{M}_p}_{r,s}, \quad \begin{cases} r = l+1 \\ s = l'+1 \end{cases} \tag{22}$$

Here $\chi_{r,s}^{\mathcal{M}_p}$ is the character of minimal model $\mathcal{M}_p$ for a given highest weight $h_{r,s}$

$$\chi_{r,s}(\tau) = \frac{1}{\eta(\tau)}[\Theta_{r(p+1)-sp,p(p+1)}(\tau) - \Theta_{r(p+1)+sp,p(p+1)}(\tau)], \tag{23}$$

where $\Theta(\tau) = \Theta(\tau, z = 0)$.

2 $\mathcal{N} = 1$ Superconformal Field Theory

2.1 $\mathcal{N} = 1$ Superconformal Algebra and Representation

Let us start by giving directly the $\mathcal{N} = 1$ superconformal algebra without the explicit construction of the action. Apart from the energy-momentum tensor, there is a fermionic super current $G(z)$ forming this closed algebra. Their OPE is

$$T(z)T(0) \sim \frac{\frac{1}{2}c}{z^4} + \frac{2T(0)}{z^2} + \frac{\partial T(0)}{z}, \tag{24a}$$

$$G(z)G(0) \sim \frac{\frac{2}{3}c}{z^3} + \frac{2T(0)}{z}, \tag{24b}$$

$$T(z)G(0) \sim \frac{\frac{3}{2}G(0)}{z^2} + \frac{\partial G(0)}{z}. \tag{24c}$$

We use $\sim$ to denote the omission of regular terms. We can perform usual Laurant expansion for T,

$$T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2},$$

however, since G(z) is fermionic, we should pay attention to the boundary condition. Namely in the Neveu-Schwarz sector

$$G(e^{2\pi i}z) = G(z),$$

and in the Ramond sector,

$$G(e^{2\pi i}z) = -G(z),$$

thus we have mode expansion

$$G(z) = \sum_{m\in\mathbb{Z}+\nu} G_m z^{-m-\frac{3}{2}}, \quad \nu = \begin{cases} 1/2 & ,NS \\ 0 & ,R \end{cases}. \tag{25}$$

Using OPE and the contour integral, we get the following $\mathcal{N} = 1$ superconformal algebra(SCA)

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}n(n^2-1)\delta_{m+n,0}, \tag{26a}$$

$$[L_m, G_r] = (\frac{1}{2}m - r)G_{m+r}, \tag{26b}$$

$$\{G_r, G_s\} = 2L_{r+s} + \frac{c}{3}(r^2 - \frac{1}{4})\delta_{r+s,0}. \tag{26c}$$

We define the conjugate operators of modes of T and G as

$$(L_n)^\dagger = L_{-n}, \quad (G_r)^\dagger = G_{-r}.$$

The highest weight states $|h\rangle$ satisfy

$$\begin{aligned} L_0 |h\rangle &= h |h\rangle , \\ L_n |h\rangle &= 0 \quad (n > 0), \\ G_r |h\rangle &= 0 \quad (r > 0), \end{aligned}$$

and descendants are given by acting $L_k(k < 0)$, $G_l(l < 0)$ on $|h\rangle$. There is a unique vacuum in NS sector defined by $|h = 0\rangle$. From(26c), we can see that $G_{+\frac{1}{2}}G_{-\frac{1}{2}} = 2L_0$. Then

$$\left\| G_{-\frac{1}{2}} |0\rangle \right\| = \langle 0| G_{+\frac{1}{2}}G_{-\frac{1}{2}} |0\rangle = 2 \langle 0| L_0 |0\rangle = 0,$$

so for a unitary representation, $G_{-\frac{1}{2}} |0\rangle = 0$.

However, in the Ramond sector things are a little bit complex. From (26c), $G_0G_0 = L_0 - \frac{c}{24}$, for an arbitrary h,

$$\|G_0 |h\rangle\| = \langle h| G_0G_0 |h\rangle = 2 \langle h| L_0 - \frac{c}{24} |h\rangle = (h - \frac{c}{24}) \langle h|h\rangle .$$

So for a unitary representation, h must equal to or bigger than $\frac{c}{24}$. When $h = \frac{c}{24}$, $G_0 |h\rangle = 0$, and $\left|\frac{c}{24}\right\rangle$ is called the Ramond vacuum. When $h > \frac{c}{24}$, noticing that $[L_0, G_0] = 0$, $|h\rangle$ and $G_0 |h\rangle$ are eigenstates of L_0 with the same eigenvalue. In other words, the ground states in this representation are degenerate. The $|h\rangle$ and $G_0 |h\rangle$ are mutually superpartners.

2.2 $\mathcal{N} = 1$ Superconformal Minimal Model

Similar to minimal models of the usual Virasoro algebra, there exists superconformal minimal model $\mathcal{M}_p^{\mathcal{N}=1}$ for SCA which contains only finite number of HW representation. In the regime of $0 < c < \frac{3}{2}$, the allowed central charge and the value of conformal dimension is

$$c = \frac{3}{2}(1 - \frac{8}{p(p+2)}), \quad p = 2, 3, 4..., \tag{27}$$

$$h_{r,s} = \frac{[(p+2)r - ps]^2 - 4}{8p(p+2)} + \frac{\epsilon}{16}. \quad \epsilon = \begin{cases} 0, & NS \\ 1, & R \end{cases} \tag{28}$$

$$\begin{matrix} 1 \leqslant r < p \\ 1 \leqslant s < p+1 \end{matrix} \quad r - s \in \begin{cases} 2\mathbb{Z} & NS \\ 2\mathbb{Z}+1 & R \end{cases}. \tag{29}$$

The integer p is the same in h and c. If $p \in 2\mathbb{Z}$, and we set $(r, s) = (p/2, (p+2)/2)$, then $h_{r,s} = \frac{c}{24}$. So $\mathcal{M}_P^{\mathcal{N}=1}$ has a Ramond vacuum. Furthermore, if we calculate the central charge of the coset model $\frac{\widehat{\mathfrak{su}(2)}_{p-2} \bigoplus \widehat{\mathfrak{su}(2)}_2}{\widehat{\mathfrak{su}(2)}_p}$, we get $c = \frac{3}{2}(1 - \frac{8}{p(p+2)})$, same as (27). It also can be shown that this coset construction describe the $\mathcal{N} = 1$ superconformal minimal model like the case in section (1.2).

We can recover $\mathcal{N} = 1$ SCA from the explicit construction of $\frac{\widehat{\mathfrak{su}(2)}_{p-2} \bigoplus \widehat{\mathfrak{su}(2)}_2}{\widehat{\mathfrak{su}(2)}_p}$. First, we note that $\widehat{\mathfrak{su}(2)}_2$ can be realized by 3 free fermions $\psi^{1,2,3}$, where they satisify the usual OPE

$$\psi^a \psi^b \sim \frac{\delta^{ab}}{z}$$

Define $\widehat{\mathfrak{su}(2)}_2$ currents as $J^a = -\frac{1}{2}\epsilon^{abc}\psi^b\psi^c$, then we get

$$J^a(z)J^b(0) \sim \frac{i\epsilon^{abc}J^c(0)}{z} + \frac{\delta^{ab}}{z^2}.$$

Finally, define

$$G(z) = \frac{2}{\sqrt{p(p+2)}}(\sum_a \psi^a J^a_p + i\frac{p-2}{2}\psi^1\psi^2\psi^3)$$

, where J^a_p is the $\widehat{\mathfrak{su}(2)}_p$ current. All the SCA relation can be retained by above $G(z)$ and the energy-momentum tensor given by Sugawara construction.

Branching rules for $\mathcal{N} = 1$ case read

$$L(p-2, r) \otimes L(2, t) = \bigoplus_{s=0}^{p} V^{\text{coset}}_{[r;s;t]} \otimes L(p, s), \quad r - s + t \in 2\mathbb{Z}$$

where L stands for irreducible representation of $\widehat{\mathfrak{su}(2)}$, and r, s, t are spins. The character relation follows

$$\chi_{p-2,r}\chi_{2,t} = \sum_{s=0}^{p} \chi^{(p)}_{[r,s;t]}\chi_{p,s}, \quad r - s + t \in 2\mathbb{Z}.$$

Here $\chi_{[r,s;t]}$ is not the final result for characters of the coset model. In fact, in NS sector, $t = 0, 2$ and

$$\text{ch}^{(p)NS}_{r,s}(\tau) = \chi_{[r,s;0]} + \chi_{[r,s;2]}, \quad r - s \in 2\mathbb{Z},$$

where ch denotes the true character. For R sector,

$$\text{ch}^{(p)R}_{r,s}(\tau) = \sqrt{2}\chi_{[r,s;1]}, \quad r - s \in 2\mathbb{Z} + 1.$$

2.3 $\mathcal{N} = 1$ Superspace

One explicit realization of $\mathcal{N} = 1$ superconformal field theory is given by the following action

$$S = \frac{1}{4\pi}\int d^2z(\bar{\partial}\phi\partial\phi + \psi\bar{\partial}\psi + \bar{\psi}\partial\bar{\psi}),$$

where ϕ is a boson and $\psi, \bar{\psi}$ are fermions. This action is invariant under supersymmetric transformation, of which the infinitesimal form is

$$\begin{cases} \delta_{\epsilon,\bar{\epsilon}}\phi = -(\epsilon\psi + \bar{\epsilon}\bar{\psi}) \\ \delta_{\epsilon}\psi = \epsilon\partial\phi \\ \delta_{\bar{\epsilon}}\bar{\psi} = \bar{\epsilon}\bar{\partial}\phi \end{cases}. \tag{30}$$

Define

$$Y(z,\bar{z},\theta,\bar{\theta}) = \phi + i\theta\psi + i\bar{\theta}\bar{\psi} + \frac{1}{2}\bar{\theta}\theta F,$$

and two superderivatives

$$D = \partial_\theta + \theta\partial_z, \quad \bar{D} = \partial_{\bar{\theta}} + \bar{\theta}\partial_{\bar{z}},$$

where θ and $\bar{\theta}$ are Grassmann coordinates that satisfy

$$\{\theta,\bar{\theta}\} = 0, \quad \int d\theta d\bar{\theta}\bar{\theta}\theta = 1,$$

and any other function $f(\theta,\bar{\theta})$ integrates to 0. F in Y is an auxiliary field defined on $(z,\bar{z})$. Space spanned by $(z,\bar{z},\theta,\bar{\theta})$ is called the superspace. Then S can be written in a compact form

$$S = \frac{1}{4\pi}\int d^2z d^2\theta \bar{D}YDY,$$

where we set $F = 0$ in the final step of the computation. The super energy-momentum tensor $T(z,\theta)$ can be defined using a variation of the action in terms of its metric. The OPE can be calculated

$$\begin{gathered} T(z_1,\theta_1)T(z_2,\theta_2) \sim \frac{\frac{c}{6}}{z_{12}^3} + (\frac{\frac{3}{2}\theta_{12}}{z_{12}^2} + \frac{\frac{1}{2}D_z + \theta_{12}D_z^2}{z_{12}})T(z_2,\theta_2), \\ \theta_{12} = \theta_1 - \theta_2, \quad z_{12} = z_1 - z_2 - \theta_1\theta_2. \end{gathered} \tag{31}$$

Let $T(z,\theta) = \frac{1}{2}G(z) + \theta T(z)$, then we can recover (24) using (31).

3 $\mathcal{N} = 2$ Superconformal Field Theory

3.1 $\mathcal{N} = 2$ Superconformal Algebra and the Representation

In $\mathcal{N} = 1$ case, the energy-momentum tensor T only have one superpartner $G(z)$. In $\mathcal{N} = 2$ superconformal model, there are two supercurrents G^+ and G^-. Moreover, there is a

$U(1)$ current J forming a closed algebra with them. OPEs are given by

$$T(z)T(0) \sim \frac{\frac{1}{2}c}{z^4} + \frac{2T(0)}{z^2} + \frac{\partial T(0)}{z}, \tag{32a}$$
$$J(z)J(0) \sim \frac{\frac{1}{3}c}{z^2}, \tag{32b}$$
$$T(z)J(0) \sim \frac{J(0)}{z^2} + \frac{\partial J(0)}{z}, \tag{32c}$$
$$G^{\pm}(z)G^{\pm}(0) \sim 0, \tag{32d}$$
$$G^{+}(z)G^{-}(0) \sim \frac{\frac{2}{3}c}{z^3} + \frac{2J(0)}{z^2} + \frac{2T(0) + \partial J(0)}{z}, \tag{32e}$$
$$T(z)G^{\pm}(0) \sim \frac{\frac{3}{2}G^{\pm}(0)}{z^2} + \frac{\partial G^{\pm}(0)}{z}, \tag{32f}$$
$$J(z)G^{\pm}(0) \sim \frac{\pm G^{\pm}(0)}{z}. \tag{32g}$$

Again, by OPEs and the contour integral we can get full $\mathcal{N} = 2$ SCA

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}n(n^2-1)\delta_{m+n,0}, \tag{33a}$$
$$[J_m, J_n] = \frac{c}{3}m\delta_{m+n,0}, \tag{33b}$$
$$[L_m, J_n] = -nJ_{m+n}, \tag{33c}$$
$$\{G_r^{\pm}, G_s^{\pm}\} = 0, \tag{33d}$$
$$\{G_r^{+}, G_s^{-}\} = 2L_{r+s} + (r-s)J_{r=s} + \frac{c}{3}(r^2 - \frac{1}{4})\delta_{r+s,0}, \tag{33e}$$
$$[L_m, G_r^{\pm}] = (\frac{1}{2}m - r)G_{m+r}^{\pm}, \tag{33f}$$
$$[J_m, G_r^{\pm}] = \pm G_{m+r}^{\pm}. \tag{33g}$$

Since $G^{\pm}$ are fermionic, in the Ramond sector, they only have integer modes and in Neveu-Schwarz sector, they have half-integer modes.

Similar to $\mathcal{N} = 0$ and $\mathcal{N} = 1$ case, we define complex conjugate operators as

$$(J_n)^{\dagger} = J_{-n}, \quad (L_n)^{\dagger} = L_{-n}, \quad (G_r^{\pm})^{\dagger} = G_{-r}^{\mp},$$

and highest weight states $|h,q\rangle$ as

$$L_n |h,q\rangle = 0, (n>0) \quad J_m |h,q\rangle = 0, (m>0) \quad G_r^{\pm} |h,q\rangle = 0, (r>0)$$
$$L_0 |h,q\rangle = h |h,q\rangle, \quad J_0 |h,q\rangle = q |h,q\rangle.$$

Using (33e), we can get $[L_0, G_0^{\pm}] = 0$ (R sector), so $|h,q\rangle$ and $G_0^{\pm} |h,q\rangle$ have the same L_0 eigenvalue, but in the Ramond sector, we impose another condition on HW states

$$G_0^{+} |h,q\rangle = 0.$$

And though in general $G_0^- |h,q\rangle \neq 0$, we compute

$$\left\| G_0^- |h,q\rangle \right\| = \langle h,q| \{G_0^+, G_0^-\} |h,q\rangle = 2\langle h,q| (L_0 - c/24) |h,q\rangle = (h - c/24) \langle h,q|h,q\rangle .$$

so when $h = c/24$, $G_0^- |h,q\rangle = 0$, there's no degeneracy in ground states and $|c/24, q\rangle$ is the Ramond vacuum.

In the NS sector, following similar steps, we have

$$\left\| G_{-\frac{1}{2}} |h,q\rangle \right\| = (2h - q) \langle h,q|h,q\rangle , \quad \left\| G_{-\frac{1}{2}} |h,q\rangle \right\| = (2h + q) \langle h,q|h,q\rangle .$$

Unitarity forces $h \geqslant \frac{|q|}{2}$, moreover

$$q > 0, \quad h = \frac{q}{2} \Leftrightarrow G^+_{-\frac{1}{2}} |h,q\rangle = 0, \tag{34}$$

$$q < 0, \quad h = -\frac{q}{2} \Leftrightarrow G^-_{-\frac{1}{2}} |h,q\rangle = 0. \tag{35}$$

The (34) defines chiral primary states $|q/2, q\rangle$ and (35) defines anti-chiral primary states $|-q/2, q\rangle$. We can also get a constraint on q by considering unitarity

$$\left\| G^+_{-\frac{3}{2}} |q/2,q\rangle \right\| = (-2q + \frac{2}{3}c) \langle q/2,q|q/2,q\rangle \geqslant 0$$
$$\to 0 \leqslant q \leqslant \frac{c}{3}$$

3.2 Chiral Ring

Each chiral primary state corresponds to one chiral primary field

$$\lim_{z\to 0} \phi_a(z) |0\rangle = |h_a\rangle$$

As we will see, these chiral fields can form a ring structure. First, recall the general form of the OPE

$$\phi_a(z)\phi_b(w) \sim \sum_c \sum_{n\geqslant 0} \frac{\mathcal{C}^c_{ab} \partial^n \phi_c(w)}{(z-w)^{h_a+h_b-h_c-n}} \tag{36}$$

where $\mathcal{C}^c_{ab}$ are some constants. The conservation of j_0 charges q implies $q_c = q_b + q_a$ leading to

$$h_a + h_b - h_c = \frac{q_a}{2} + \frac{q_b}{2} - h_c \leqslant \frac{q_c}{2} - \frac{|q_c|}{2} \leqslant 0$$

for $h_c \geqslant \frac{|q_c|}{2}$ in unitary representations. So the exponents in the right hand side of (36) are always less than 0, thus in the OPE there are no singular terms. Now

$$\lim_{z\to w} \phi_a(z)\phi_b(w) = \sum_c \mathcal{C}^c_{ab} \phi_c(w)$$

where every ϕ_c appear in the right hand side has conformal weight $h_c = \frac{q_c}{2} = \frac{q_a+q_b}{2}$. Taking limit kills all other terms. In other words, ϕ_c are also chiral fields. Then we can define products among the chiral primaries

$$(\phi_a * \phi_b)(w) = \lim_{z\to w} \phi_a(z)\phi_b(w)$$

so that product $*$ together with usual addition $+$ among fields form a ring (the chiral ring). We note that there's also a ring of anti-chiral fields, and the same structure exists in anti-holomorphic sector of $\mathcal{N}=2$ SCA. So in total there are 4 rings.

3.3 Spectral flow

$\mathcal{N}=2$ SCA admits continuous deformation parameterised by η

$$L_n \to L_n' = L_n + \eta J_n + \frac{c}{6}\eta^2\delta_{n,0}, \tag{37}$$

$$J_n \to J_n' = J_n + \frac{c}{3}\eta\delta_{n,0}, \tag{38}$$

$$G_r^{\pm} \to G_r'^{\pm} = G_{r\pm\eta}^{\pm}. \tag{39}$$

It can be shown that this deformation called the spectral flow doesn't change $\mathcal{N}=2$ SCA. In particular, if $\eta \in \mathbb{Z}+\frac{1}{2}$, the R sector and the NS sector are swapped. Spectral flow can be written in terms of a unitary operator U_η

$$L_m' = U_\eta L_m U_\eta^\dagger, \quad J_m' = U_\eta J_m U_\eta^\dagger, \quad |\phi_\eta\rangle = U_\eta |\phi\rangle .$$

Then

$$\begin{gathered} L_0' |\phi_\eta\rangle = U_\eta L_0 U_\eta^\dagger U_\eta |\phi\rangle = U_\eta L_0 |\phi\rangle = hU_\eta |\phi\rangle = h |\phi_\eta\rangle , \\ J_0' |\phi_\eta\rangle = q |\phi_\eta\rangle . \end{gathered}$$

At the same time,

$$\begin{gathered} L_0' |\phi_\eta\rangle = (L_0 + \eta J_0 + \frac{c}{6}\eta^2) |\phi_\eta\rangle = (h_\eta + \eta q_\eta + \frac{c}{6}\eta^2) |\phi_\eta\rangle , \\ J_0' |\phi_\eta\rangle = (J_0 + \frac{c}{3}\eta) |\phi_\eta\rangle = (q_\eta + \frac{c}{3}\eta) |\phi_\eta\rangle . \end{gathered}$$

Comparing these results, we get

$$h_\eta = h - \eta q + \frac{c}{6}\eta^2, \tag{40}$$

$$q_\eta = q - \frac{c}{3}\eta. \tag{41}$$

Let $\eta = \frac{1}{2}$, then from (40) and (41) we can see

$$\left| h = \frac{q}{2}, q \right\rangle_{\mathrm{NS}} \to \left| h' = \frac{c}{24}, q' = q - \frac{c}{6} \right\rangle_{\mathrm{R}}$$

which means that through $\eta = \frac{1}{2}$ spectral flow, chiral primary states in the NS sector are mapped to the Ramond vacuum in the R sector. Moreover, we can use $\eta = \frac{1}{2}$ spectral flow once again and find

$$\left| h' = \frac{c}{24}, q' = q - \frac{c}{6} \right\rangle \to \left| h'' = -\frac{q''}{2}, q'' = q - \frac{c}{3} \right\rangle$$

which means the Ramond vacuum is brought back to anti-chiral primary states in the NS sector.

3.4 $\mathcal{N}=2$ Superconformal Minimal Model

We have seen that there exist unitary representations for $\mathcal{N}=1$ case which contains finite primary fields. The same situation holds for $\mathcal{N}=2$ SCA representations. Without going into detail , we state that in the regime of $0<c<3$, the allowed central charge is given by

$$c=\frac{3k}{k+2},\quad k=1,2,3....$$

The allowed value of the conformal weights and J_0 charges are given by

$$h^l_{m,s}=\frac{l(l+2)-m^2}{4(k+2)}+\frac{s^2}{8},\quad q_{m,s}=-\frac{m}{k+2}+\frac{s}{2}, \tag{42}$$

where

$$0\geqslant l\geqslant k,\quad 0\geqslant |m-s|\geqslant l,\quad s=\begin{cases}0,2 & NS,\\ \pm 1 & R,\end{cases}.$$

and s is defined modulo 4 while m is defined modulo $2(k+2)$. The coset $\frac{\widehat{\mathfrak{su}(2)}_k\oplus\widehat{\mathfrak{u}(1)}_2}{\widehat{\mathfrak{u}(1)}_{k+2}}$ or equivalently $\frac{\widehat{\mathfrak{su}(2)}_k\oplus\widehat{\mathfrak{so}(2)}_1}{\widehat{\mathfrak{u}(1)}_{k+2}}$ can describe this $\mathcal{M}^{\mathcal{N}=2}_k$. A simple check can be done by computing the central charge. We can recover $\mathcal{N}=2$ SCA by the explicit construction of $\frac{\widehat{\mathfrak{su}(2)}_k\oplus\widehat{\mathfrak{u}(1)}_2}{\widehat{\mathfrak{u}(1)}_{k+2}}$. Firstly, $\widehat{\mathfrak{u}(1)}_2$ or $\widehat{\mathfrak{so}(2)}_1$ can be realised by 2 free fermions ψ^1,ψ^2

$$\psi^{\pm}(z)=\frac{1}{\sqrt{2}}(\psi^1\pm i\psi^2),\quad \psi^+(z)\psi^-(0)\sim\frac{1}{z}.$$

Then $\widehat{\mathfrak{u}(1)}_2$ currents are given by

$$J_{\widehat{\mathfrak{u}(1)}_2}=:\psi^+\psi^-:$$

Moreover,

$$\begin{aligned}T(z)=\frac{1}{k+2}(\sum_a :J^a_{\widehat{\mathfrak{su}(2)}_k}(z)J^a_{\widehat{\mathfrak{su}(2)}_k}(z):-:J_{\widehat{\mathfrak{u}(1)}_{k+2}}(z)J_{\widehat{\mathfrak{u}(1)}_{k+2}}(z):)\\ -\frac{1}{2}(:\psi^+(z)\partial\psi^-(z):+:\psi^-(z)\partial\psi^+(z):),\end{aligned}$$

where

$$J_{\widehat{\mathfrak{u}(1)}_{k+2}}=J^3_{\widehat{\mathfrak{su}(2)}_k}+J_{\widehat{\mathfrak{u}(1)}_2}$$

The two superpartners and the coset current are

$$G^{\pm}=\sqrt{\frac{2}{k+2}}\psi^{\pm}J^{\mp}_{\widehat{\mathfrak{su}(2)}_k},\quad J=J_{\widehat{\mathfrak{u}(1)}_2}-\frac{2}{k+2}J_{\widehat{\mathfrak{u}(1)}_{k+2}}.$$

For future reference, we need to get the explicit form of the character of the $\mathcal{N}=2$ coset model. Let's start from the branching rule as usual

$$L_l^{\widehat{\mathfrak{su}(2)}_k}\otimes L_s^{\widehat{\mathfrak{u}(1)}_2}=\bigoplus_m(L^{\text{coset}}_{k,l,m,s}\otimes L_m^{\widehat{\mathfrak{u}(1)}_{k+2}}),\quad l+m+s\in 2\mathbb{Z},\quad m\in\mathbb{Z}_{2(k+2)},$$

where l, s, m have the same meaning in (42). The character relation is

$$\chi_l^{\widehat{\mathfrak{su}(2)}_k}(\tau)\chi_s^{\widehat{\mathfrak{u}(1)}_2} = \sum_m \chi_{m,s}^l(\tau)\chi_m^{\widehat{\mathfrak{u}(1)}_{k+2}}(\tau).$$

If we use the string function form of characters of $\widehat{\mathfrak{su}(2)}_k$

$$\chi_l^{\widehat{\mathfrak{su}(2)}_k} = \sum_{m=-k+1}^{k} C_{l,m}^{(k)}(\tau)\Theta_{m,k}(\tau), \quad l+m=0 \mod 2$$

and $\chi_s^{\widehat{\mathfrak{u}(1)}_k} = \frac{\Theta_{s,k}(\tau,z)}{\eta(\tau)}$, we get

$$\sum_m^k C_{l,m}^{(k)}(\tau)\Theta_{m,k}(\tau)\frac{\Theta_{s,2}(\tau)}{\eta(\tau)} = \sum_{m=-k-l}^{k+2}\frac{\Theta_{m,k+2}(\tau)}{\eta(\tau)}\chi_{m,s}^l(\tau).$$

Inserting (21), we find

$$\sum_m^k\sum_{j=1}^{k+2} C_{l,m}^{(k)}(\tau)\Theta_{4j+m+s,k+2}\Theta_{4jk-2m+ks,2k(k+2)} = \sum_{m=-k-l}^{k+2}\frac{\Theta_{m,k+2}(\tau)}{\eta(\tau)}\chi_{m,s}^l(\tau).$$

Comparing terms of $\Theta_{m,k+2}$, we finally get the character

$$\chi_{m,s}^l = \sum_{j=1}^{k+2} C_{l,m-4j-s}^{(k)}(\tau)\Theta_{-2m+(4j+s)(k+2),2k(k+2)}$$

4 Mirror symmetry

4.1 Calabi-Yau Manifolds

String theory considers particles not as points but as loops. When these loops propagate through space-time, we get paths that look like two-dimensional surfaces. The surfaces are called worldsheets. We can define some fields which map points on the surface to actual (target) space-time. The most typical example of mirror symmetry, which we will explain in the following section, is discovered in the physical theory whose fields are maps from a two-dimensional Riemann surface Σ to some target manifold M. Such a theory is called the non-linear sigma model and to get a consistent string theory, it must have $\mathcal{N}=2$ supersymmetry.

We now focus on a particular type of complex manifold called Calabi-Yau(CY) manifold. Instead of a precise mathematical definition, we will first give an explicit construction of CY manifold. $r-1$ dimensional complex projective space is defined by identifying straight lines through the origin

$$\mathbb{C}P^{r-1} = (\mathbb{C}^r - \{0\})/\{z_1, z_2, ..., z_r\} \sim \{\lambda z_1, \lambda z_2, ..., \lambda z_r\} \quad \lambda \neq 0.$$

If we replace the equivalence relation by $\{z_1, z_2, ..., z_r\} \sim \{\lambda^{w_1} z_1, \lambda^{w_2} z_2, ..., \lambda^{w_r} z_r\}$ for some integers $w_1, w_2 ...$, we get a weighted complex projective space $\mathbb{C}P^{r-1}_{w_1,w_2...w_r}$. The solution set to equation $W(z_1, ..., z_r) = 0$ on $\mathbb{C}P^{r-1}_{w_1, w_2 ... w_r}$ is another manifold. If we require the function W to be homogeneous, that is, to satisfy $W(\lambda^{w_1} z_1, \lambda^{w_2} z_2, ..., \lambda^{w_r} z_r) = \lambda^N W(z_1, ..., z_r)$ where $N = \sum_i^r w_i$ then $W = 0$ defines a $r-2$ dimensional CY manifold, written as CY_{r-2}. We note that the conformally invariant non-linear sigma model targeted on a CY manifold has a central charge $c = 3d$, where d is the complex dimension of the manifold[Gre96].

As we have seen, in the regime of $0 < c < 3$, $\mathcal{M}_k^2$ have central charges $c = \frac{3k}{k+2} < 3$. Thus for $d > 1$ it seems that there couldn't be relations between minimal models and CY manifolds. But if we use the tensor product of r conformal theories, we can get the central charge $c = \sum_i^r c_i$. Applying this to the minimal models, we see that $\sum_i^r \frac{3k_i}{k_i+2} = 3d$ can be achieved. Besides, the operation of orbifolding the manifold does not change the central charge. In fact, it has been proven that the minimal model construction yields conformal theories interpretable as non-linear sigma models[Gep88]. We can check an example by calculating the central charge. Consider a case in which complex polynomial function in the definition of CY_{r-2} is $W(\vec{z}) = \sum_i^r z_i^{a_i}$, and let $a_i = \frac{N}{w_i}$, then the manifold is called Fermat Hypersurface of CY_{r-2}. Further we identify $a_i = k_i + 2$, then $\sum_i^{r-2} \frac{3k_i}{k_i+2} = 3(r-2)$. So the central charge of the minimal model construction equals to that of the non-linear sigma model. The corresponding conformal theory $\bigoplus_i M^2_{k_* i} |_{\mathbb{Z}_N \text{orbifold}}$ is called Gepner model.

The above argument shows that the non-linear sigma model establishes a correspondence between conformal field theories and Calabi-Yau manifolds. This correspondence is not one-to-one: two distinct manifolds correspond to the same conformal theory. These 2 Calabi-Yau manifolds M and M^* are said to form a mirror pair. It turns out that they have Hodge diamonds satisfying the relation $h^{p,q}(M) = h^{d-p,q}(M^*)$ where $h^{p,q}$ is the dimension of the (p,q) Dolbeaut cohomology group of M, and d is the complex dimension of the manifold[GP90]. The two Hodge diamonds are thus related by a reflection about a diagonal axis, hence the name mirror manifolds.

In the following section, we will give the explicit construction of mirror pairs in some detail, and discuss its applications in physics briefly.

4.2 Topological Twist of Non-linear Sigma Model

We now give the explicit form of the Lagrangian of the non-linear sigma model(NSLM), which possesses $\mathcal{N} = (2,2)$ supersymmetry.

$$\mathcal{L} = 2t \int_\Sigma d^2z [\frac{1}{2} g_{i\bar{j}} (\partial_z \phi^i \partial_{\bar{z}} \phi^{\bar{j}} + \partial_z \phi^{\bar{j}} \partial_{\bar{z}} \phi^i) + i g_{i\bar{j}} (\psi_-^{\bar{j}} D_z \psi_-^i + \psi_+^{\bar{j}} D_{\bar{z}} \psi_+^i) + R_{i\bar{j}k\bar{l}} \psi_+^i \psi_+^{\bar{j}} \psi_-^k \psi_-^{\bar{l}}],$$

where $z, \bar{z}$ are the holomorphic and anti-holomorphic coordinate respectively on Riemann surface Σ, and ϕ is a bosonic field targeted on manifold M, which typically is CY_3. $\psi_\pm$ are two fermionic fields and g_{ij}, R_{ijkl} are the metric and the Riemann curvature tensor on M, respectively. Covariant derivatives are defined by

$$D_z = \partial_z + \partial_z \phi^j \Gamma^i_{jk},$$
$$D_{\bar{z}} = \partial_{\bar{z}} + \partial_{\bar{z}} \phi^j \Gamma^i_{jk}.$$

We can construct $\mathcal{N} = 2$ SCA from the fields:

$$
\begin{aligned}
T &= -g_{i\bar{j}}\partial_z\phi^i\partial_z\phi^{\bar{j}} + \frac{1}{2}g_{i\bar{j}}\psi^i\partial_z\psi^{\bar{j}} + \frac{1}{2}g_{i\bar{j}}\psi^{\bar{j}}\partial_z\psi^i, \\
G^+ &= \frac{1}{2}g_{i\bar{j}}\psi^i\partial_z\phi^{\bar{j}}, \\
G^- &= \frac{1}{2}g_{i\bar{j}}\psi^{\bar{j}}\partial_z\phi^i, \\
J &= \frac{1}{4}g_{i\bar{j}}\psi^i\psi^{\bar{j}},
\end{aligned}
$$

and similarly for the antiholomorphic part $\overline{T}, \overline{G}^+, \overline{G}^-, \overline{J}$. This particular Lagrangian is invariant under (2,2) supersymmetry transformation:

$$
\begin{aligned}
\delta\phi^i &= i\alpha_-\psi^i_+ + i\alpha_+\psi^i_-, \\
\delta\phi^{\bar{i}} &= i\tilde{\alpha}_-\psi^{\bar{i}}_+ + i\tilde{\alpha}_+\psi^{\bar{i}}_-, \\
\delta\psi^i_\pm &= -\tilde{\alpha}_\mp\partial_z\phi^i - i\alpha_\pm\psi^j_\mp\Gamma^i_{jm}\psi^m_\pm, \\
\delta\psi^{\bar{i}}_\pm &= -\alpha_\mp\partial_z\phi^{\bar{i}} - i\tilde{\alpha}_\pm\psi^{\bar{j}}_\mp\Gamma^{\bar{i}}_{\bar{j}\bar{m}}\psi^{\bar{m}}_\pm,
\end{aligned}
$$

where $\alpha_\pm$ and $\tilde{\alpha}_\pm$ are fermionic generators of (2,2) symmetry.

In order to relate the SCFT operator with the geometric property of the target space in the non-linear sigma model, we need to first construct a topological field theory(TFT). Topological field theory can be constructed from the $\mathcal{N} = 2$ superconformal field theories whose states are those created by operators in the chiral ring or anti-chiral ring. The energy-momentum tensor of the TFT has a form "twisted" from that of SCFT[TSK90]:

$$
T'(z) = T(z) \pm \frac{1}{2}\partial J(z), \tag{43}
$$

$$
\tag{44}
$$

If we take the sign before $\frac{1}{2}$ in (43) to be $+$, it is called A-twist, and the related theory is A-model. The other choice leads to B-twist and B-model similarly.

4.2.1 A-twist side

We now look at A-twist. In A-twist of NLSM, the original ϕ and $\psi_\pm$ are mapped to fields in the topological theory which have now different spins due to the twist:

$$
\begin{aligned}
\phi^i, \phi^{\bar{i}} &\to \phi^i, \phi^{\bar{i}} \\
\psi^i_+ &\to \chi^i \\
\psi^i_- &\to \psi^i_{\bar{z}} \\
\psi^{\bar{i}}_+ &\to \psi^{\bar{i}}_z \\
\psi^{\bar{i}}_- &\to \chi^{\bar{i}},
\end{aligned}
$$

The modified Lagrangian after twist is

$$\mathcal{L}' = 2t\int_{\Sigma} d^2z[\frac{1}{2}g_{i\bar{j}}(\partial_z\phi^i\partial_{\bar{z}}\phi^{\bar{j}} + \partial_z\phi^{\bar{j}}\partial_{\bar{z}}\phi^i) + ig_{i\bar{j}}(\psi_z^{\bar{j}}D_{\bar{z}}\chi^i + \psi_{\bar{z}}^iD_z\chi^{\bar{j}}) + R_{i\bar{j}k\bar{l}}\psi_{\bar{z}}^i\psi_z^{\bar{j}}\chi^k\chi^{\bar{l}}]$$

If we set $\alpha_- = \tilde{\alpha}_+ = 0, \alpha_+ = \tilde{\alpha}_- = \alpha$ in the supersymmetry transformation, then the Lagrangian is still invariant:

$$\begin{aligned}
\delta\phi^i &= i\alpha\chi^i \\
\delta\phi^{\bar{i}} &= i\alpha\chi^{\bar{i}} \\
\delta\chi^i &= 0 \\
\delta\chi^{\bar{i}} &= 0 \\
\delta\psi_z^{\bar{i}} &= -\alpha\partial_z\phi^{\bar{i}} - i\alpha\chi^{\bar{j}}\Gamma_{\bar{j}\bar{m}}^{\bar{i}}\psi_z^{\bar{m}} \\
\delta\psi_{\bar{z}}^i &= -\alpha\partial_{\bar{z}}\phi^i - i\alpha\chi^j\Gamma_{jm}^i\psi_{\bar{z}}^m.
\end{aligned}$$

This symmetry is called BRST symmetry and corresponds to BRST charge Q_A. $Q_A = Q_+ + \bar{Q}_-$ where $Q_\pm, \bar{Q}_\pm$ is the conserved charge of $\mathcal{N} = (2,2)$ supersymmetry. We can verify that $\delta^2 = 0$ which implies $Q_A^2 = 0$, where $\delta X = \{Q_A, X\}$ for some operator X. We can put L' in another form

$$\mathcal{L} = it\int_{\Sigma} dz\,\{Q_A, V\} + t\int_{\Sigma}\phi^*(K),$$

where $V = g_{i\bar{j}}(\psi_z^{\bar{j}}\partial_{\bar{z}}\phi^i + \psi_{\bar{z}}^j\partial_z\phi^{\bar{j}})$ and $\phi^*(K)$ is the pull back of kähler form $K = gi\bar{j}dz^i \wedge d\bar{z}^{\bar{j}}$. Note we can adjust the scale of $g_{i\bar{j}}$ to $\int_\Sigma \phi^*(K) = \int_\Sigma d^2z g_{i\bar{j}}(\partial\phi^i\partial_{\bar{z}}\phi^{\bar{j}} - \partial_{\bar{z}}\phi^i\partial_z\phi^{\bar{j}}) = 2\pi id, d \in \mathbb{Z}^{\geqslant}$.

Once the Lagrangian is modified, the correlation function will be independent of coordinates $(z, \bar{z})$. To see that, first we examine the one point function $\langle U\rangle = \int \mathcal{D}Xe^{-\mathcal{L}}U$. Here we use X to denote all the fields ϕ, χ, ψ integrated. Q_A is a symmetry of the twisted theory, So $\{Q_A, \mathcal{L}\} = 0$. Then for some infinitesimal transformation ϵ,

$$\begin{aligned}
\langle U\rangle &= \int \mathcal{D}Xe^{-\mathcal{L}}U \\
&= \int \mathcal{D}(\exp(\epsilon Q_A)X)\exp(\epsilon Q_A)e^{-\mathcal{L}}U \\
&= \int \mathcal{D}X\exp(\epsilon Q_A)e^{-\mathcal{L}}U \\
&= \int \mathcal{D}Xe^{-\mathcal{L}}\exp(\epsilon Q_A)U \\
&= \int \mathcal{D}e^{-\mathcal{L}}(U + \epsilon\{Q_A, U\}) \\
&= \langle U\rangle + \epsilon\langle\{Q_A, U\}\rangle \\
&\to \langle\{Q_A, U\}\rangle = 0.
\end{aligned}$$

We proved that if W is Q_A-exact, or $\{Q, U\} = W$ for some operator U, $< W >= 0$.

Next, if there are some Q_A closed observable O_i, or $\{Q_A, O_i\} = 0$, their correlation function is

$$
\begin{aligned}
\langle O_1 O_2 ... O_n \rangle &= \int \mathcal{D}X e^{-\mathcal{L}} O_1 ... O_n \\
&= \int \mathcal{D}X O_1 ... O_n \exp\left(-it \int_\Sigma d^2 z \, \{Q_A, V\} - t \int_\Sigma \phi^*(K) \right) \\
&= \sum_{d=0}^{\infty} e^{-2\pi i d t} (\int_{pd} \mathcal{D}X O_1 ... O_n \exp\left(-it \int_\Sigma d^2 z \, \{Q_A, V\} \right) \\
&= \sum_{d=0}^{\infty} e^{-2\pi i d t} \langle O_1 ... O_n \rangle_d
\end{aligned}
$$

where pd denotes the configuration space in which ϕ satisfy $\int_\Sigma \phi^*(K) = 2\pi d$. $\langle O_1 ... O_n \rangle_d$ is independent of t, which we can verify as follows. If t is varied infinitesimally to $t + \delta t$, then

$$
\begin{aligned}
\langle O_1 ... O_n \rangle_d (t + \delta t) &= \int_{pd} \mathcal{D}X \exp\left[-i(t + \delta t) \int_\Sigma d^2 z \, \{Q_A, V\} \right] O_1 ... O_n \\
&= \int_{pd} \mathcal{D}X e^{-it \int_\Sigma d^2 z \{Q_A, V\}} e^{-i\delta t \int_\Sigma d^2 z \{Q_A, V\}} O_1 ... O_n \\
&= \int_{pd} \mathcal{D}X e^{-it \int_\Sigma d^2 z \{Q_A, V\}} O_1 ... O_n \\
&\quad - i\delta t \int_{pd} \mathcal{D}X e^{-it \int_\Sigma d^2 z \{Q_A, V\}} (\left\{ Q_A, \int_\Sigma d^2 z V \right\} O_1 ... O_n).
\end{aligned}
$$

The last δt term equals 0 because all the O_i are Q_A closed. Hence $\langle O_1 ... O_n \rangle_d (t + \delta t) =< O_1 ... O_n >_d (t)$. So we can take $t \to \infty$ leaving it unchanged.But if we take $t \to \infty$, the original path integral(correlation function) will only have contributions from the classical configuration, since deviation from $\delta \mathcal{L} = 0$ point cause rapid oscillation in phase $e^{-\mathcal{L}}$ that eventually cancels off. The classical configuration of fields ϕ, ψ, χ can be obtained from equations of motion:

$$\partial_z \phi^{\bar{i}} = 0 \tag{45}$$

$$\partial_{\bar{z}} \phi^{i} = 0 \tag{46}$$

$$D_{\bar{z}} \chi^{i} = 0 \tag{47}$$

$$D_z \chi^{\bar{i}} = 0 \tag{48}$$

$$D_z \psi^{i}_{\bar{z}} = 0 \tag{49}$$

$$D_{\bar{z}} \psi^{\bar{i}}_{z} = 0 \tag{50}$$

Equation (46) implies that $\phi : \Sigma \to M$ is a holomorphic map. (47) tells us that $\chi^i \in H^0(\Sigma, \phi^{-1}(TM)$, where TM is holomorphic vector bundle on M. (50) implies that $\psi^{\bar{i}}_z \in H^0(\Sigma, (T\Sigma)^* \otimes \phi^{-1}(\overline{TM})) = H^1(\Sigma, \phi^{-1}(TM))$, with second equality due to Serre duality.

Let $\mathcal{M}_\Sigma(M, d)$ be the moduli space of the holomorphic map $\left\{ \phi : \Sigma \to M, \int_\Sigma \phi^*(K) = 2\pi i d \right\}$.

The virtual dimension of $\mathcal{M}_\Sigma(M,d)$ is defined to be

$$\begin{aligned}
\text{Vir dim}(\mathcal{M}) &= \dim(H^0(\Sigma, \phi^{-1}(TM)) - \dim(H^1(\Sigma, \phi^{-1}(TM))) \\
&= \dim_{\mathbb{C}}(M) * (1 - g_\Sigma) + \int_\Sigma \phi^*(C_1(M)) \\
&= 3(1 - g_\Sigma)
\end{aligned}$$

where g denotes genus and C_1 is the first Chern class which vanishes on CY manifold. Now we can write $\langle O_1 ... O_n \rangle_d$ as integration localized to classical configuration

$$\langle O_1 ... O_n \rangle_d = \int_{\mathcal{M}} \mathcal{D}\phi_0 \mathcal{D}\chi_0 \mathcal{D}\psi_0 e^{-\mathcal{L}_{\text{classical}}} O_1 ... O_n.$$

Subscript 0 represents that configuration satisfies the equation of motion.

Given a (p,q)-form on the target manifold M, we can construct a corresponding operator:

$$\begin{aligned}
W &= W_{i_1...i_p\bar{j}_1...\bar{j}_q}(x) dx^{i_1} \wedge ... dx^{i_p} \wedge d\bar{x}^{\bar{j}_1} \wedge ... d\bar{x}^{\bar{j}_q} \quad (p,q) - \text{form} \\
\Leftrightarrow O_W &= W_{i_1...i_p\bar{j}_1...\bar{j}_q}(\phi(z)) \chi^{i_1}(z) ... \chi^{i_p}(z) \chi^{\bar{j}_1}(z) ... \chi^{\bar{j}_q}(z) \quad \text{operator.}
\end{aligned}$$

x^i denotes holomorphic coordinate of M. This construction gives relation $\{Q_A, O_W\} = O_{dW}$, where d is the differential operator on the form W. If $dW = 0$, then $W \in H^{p,q}(M; \mathbb{R})$, and $\{Q_A, O_W\} = 0$. Now we consider the correlation function $\langle O_{W_1} ... O_{W_n} \rangle_d, W_i \in H^{p_i, q_i}(M, \mathbb{R})$. Following the argument of Q_A-closed operator,

$$\langle O_{W_1} ... O_{W_n} \rangle_d = \int_{\mathcal{M}_\Sigma(M,d)} \mathcal{D}\phi_0 \mathcal{D}\chi_0 \mathcal{D}\psi_0 e^{-\mathcal{L}_{\text{classical}}} O_{W_1} ... O_{W_n}$$

We denote the number of independent solutions of (47) as $a_d = \dim(H^0(\Sigma, \phi^{-1}(Tm)))$ and similarly for $b_d = \dim(H^1(\Sigma, \phi^{-1}(TM)))$. Integration measure $\mathcal{D}\chi_0 \mathcal{D}\psi_0$ can be written as $d\chi_{0,1} ... d\chi_{0,a_d} d\psi_{0,1} d\psi_{0,b_d} d\bar{\chi}_{0,1} ... d\bar{\chi}_{0,a_d} d\bar{\psi}_{0,1} ... d\bar{\psi}_{0,b_d}$. In order for the correlation function not vanishing, the number of χ field in O_i must satisfy the selection rule:

$$\sum_i^n p_i = \sum_i^n q_i = a_d - b_d.$$

4.2.2 B-twist side

On the topological B-twist side, the original ϕ and ψ are mapped by different way:

$$\begin{aligned}
\phi^i, \phi^{\bar{i}} &\to \phi^i, \phi^{\bar{i}}, \\
\psi_+^i &\to \psi_z^i \\
\psi_-^i &\to \psi_{\bar{z}}^i \\
\psi_+^{\bar{i}} &\to \psi_1^{\bar{i}} \\
\psi_-^{\bar{i}} &\to \psi_2^{\bar{i}}.
\end{aligned}$$

Further we define

$$\begin{aligned}
\eta^{\bar{i}} &= \psi_1^{\bar{i}} + \psi_2^{\bar{i}}, \\
\theta_i &= g_{i\bar{j}}(\psi_1^{\bar{j}} + \psi_2^{\bar{j}}), \\
\rho_z^i &= \psi_z^i. \\
\rho_{\bar{z}}^i &= \psi_{\bar{z}}^i.
\end{aligned}$$

Thus we can write the modified Lagrangian in B-model as:

$$\mathcal{L} = t\int_\Sigma d^2z[g_{i\bar{j}}(\partial_z\phi^i\partial_{\bar{z}}\phi^{\bar{j}} + \partial_z\phi^{\bar{j}}\partial_{\bar{z}}\phi^i) + i\eta^{\bar{j}}(D_z\rho_{\bar{z}}^i + D_{\bar{z}}\rho_z^i)g_{i\bar{j}} + i\theta_i(D_{\bar{z}}\rho_z^i - D_z\rho_{\bar{z}}^i) + R_{i\bar{j}k\bar{l}}\rho_z^i\rho_{\bar{z}}^k\eta^{\bar{j}}\theta_m g^{m\bar{l}}]. \quad (51)$$

This Lagrangian is invariant under (2,2) transformation in which we set $\alpha_+ = \alpha_- = 0, \tilde{\alpha}_+ = \tilde{\alpha}_- = \alpha$:

$$\begin{aligned}
\delta\phi^i &= 0 \\
\delta\phi^{\bar{i}} &= i\alpha\eta^{\bar{i}} \\
\delta\eta^{\bar{i}} &= 0 \\
\delta\theta_i &= 0 \\
\delta\rho_z^i &= -\alpha\partial_z\phi^i \\
\delta\rho_{\bar{z}}^i &= \alpha\partial_{\bar{z}}\phi^i
\end{aligned}$$

Again, Q_B is the conserved charge corresponding to α and $Q_B = \overline{Q}_+ + Q_-$. Then the Lagrangian can be written as:

$$\mathcal{L} = it\int_\Sigma \{Q_B, V\} + tW,$$

where

$$\begin{aligned}
V &= g_{i\bar{j}}(\rho_z^i\partial_{\bar{z}}\phi^{\bar{j}} + \rho_{\bar{z}}\partial_z\phi^{\bar{j}}), \\
W &= \int_\Sigma d^2z[i\theta_i(D_{\bar{z}}\rho_z^i - D_z\rho_{\bar{z}}^i) + R_{i\bar{j}k\bar{l}}\rho_z^i\rho_{\bar{z}}^k\eta^{\bar{j}}\theta_m g^{m\bar{l}}] \\
&= \int_\Sigma [-\theta_i D\rho^i - \frac{i}{2}R_{i\bar{j}k\bar{l}}\rho^i \wedge \rho^k\eta^{\bar{j}}\theta_m g^{m\bar{l}}] \\
D &= dz \wedge D_z + d\bar{z} \wedge D_{\bar{z}}, \quad d^2z = -idz \wedge d\bar{z}.
\end{aligned}$$

Then we can redefine $\theta \to \frac{\theta}{t}$ to absorb the dependence on t into W. Similar to A-model case, if there are some Q_B-closed operator O_i, then their correlation function $\langle O_1 ... O_m\rangle$ will independent of t. Again, we take the $t \to \infty$ limit, which confines the field configuration to the classical one. Fixed point of BRST transformationsatisfies

$$\partial_z\phi^i = \partial_{\bar{z}}\phi^i = \partial_z\phi^{\bar{i}} = \partial_{\bar{z}}\phi^{\bar{i}} = 0$$

$$D_z\eta^{\bar{j}} = D_{\bar{z}}\eta^{\bar{j}} = 0 \tag{52}$$

$$D_z\theta_i = D_{\bar{z}}\theta_i = 0 \tag{53}$$

which is a constant map $\phi : \Sigma \to M$. It follows that Moduli space $\mathcal{M}_B(\Sigma) = M$

Given a tensor $V \in \Omega^p(M, \Lambda^q TM)$ on manifold M which takes the form:

$$V = V^{j_1 \dots j_q}_{\bar{i}_1 \dots \bar{i}_p}(x) dx^{\bar{i}_1} \wedge \dots dx^{\bar{i}_p} \otimes \partial_{j_1} \wedge \dots \partial_{j_q},$$

we can construct a corresponding operator

$$O_V = \eta^{\bar{i}_1} \dots \eta \bar{i}_p V^{j_1 \dots j_q}_{\bar{i}_1 \dots \bar{i}_p}(\phi(z)) \theta_{j_1} \dots \theta_{j_q} \leftrightarrow V \tag{54}$$

This construction gives $\{Q_B, O_V\} = O_{\bar{\partial}V}$. If $\bar{\partial}V = 0$, $V \in H^{0,p}(M, \Lambda^q TM)$, then $\{Q_B, O_V\} = 0$. Correlation functions of such Q_B-closed operators confine to fixed point configuration:

$$\begin{aligned} \langle O_1 \dots O_m \rangle &= \int \mathcal{D}\phi \mathcal{D}\eta \mathcal{D}\theta \mathcal{D}\rho e^{-\mathcal{L}} O_{V_1} \dots O_{Vm} \\ &= \int_{\mathcal{M}} d\phi_0 \int d\eta_0^{\bar{1}} \dots d\eta_0^{\bar{n}} d\theta_{0,1} \dots d\theta_{0,n} O_{V_1} \dots O_{V_m}, \end{aligned}$$

where $\eta_0^{\bar{i}}$,$\theta_{0.i}$ are n zero modes of equation (52) and (53). In order for this correlation function not vanishing, the number of fields in operators O_{V_i} should satisfy selection rule:

$$\sum_{i=1}^m p_i = \sum_{i=1}^m q_i = n$$

We can further relate not only the operators O but the correlation functions to integrations of forms on target manifold M through the expression

$$\langle O_{V_1} O_{V_2} \dots O_{V_n} \rangle = \langle 0 | O_{V_1} O_{V_2} \dots O_{V_n} | 0 \rangle .$$

Once we find the appropriate geometric object on M to match the vacuum $|0\rangle$ in field theory, this is essentially done. And for Calabi-Yau manifold, that particular object we want is holomorphic $(n, 0)$-form Ω

$$\Omega = \Omega_{i_1 i_2 \dots i_n} dx^{i_1} \wedge dx^{i_2} \dots \wedge dx^{i_n}$$

It can be shown that Ω is uniquely determined up to a multiplicative constant on Calabi-Yau manifold. And this property is also exactly what we expect of a vacuum state. Recalling the operator-tensor correspondence of (54), we can easily see that $O_{V_1} O_{V_2} \dots O_{V_n} = O_{V_1 \wedge V_2 \dots \wedge V_n}$. So

$$O_{V_1} O_{V_2} \dots O_{V_n} |0\rangle \leftrightarrows V_1 \wedge V_2 \wedge \dots V_n \wedge \Omega$$

where we define the wedge product between $V \in H^{0,p}(M, \wedge^q TM)$ and $W \in H^{0,r}(M, \wedge^s (TM)*)$ as:

$$\begin{aligned} V \wedge W = &(V^{j_1 \dots j_q}_{\bar{i}_1 \dots \bar{i}_p}(x) dx^{\bar{i}_1} \wedge \dots dx^{\bar{i}_p} \otimes \partial_{j_1} \wedge \dots \partial_{j_q}) \\ &\wedge (W_{\bar{k}_1 \dots \bar{k}_r l_1 \dots l_2}(x) dx^{\bar{k}_1} \wedge \dots dx^{\bar{k}_r} \otimes dx^{l_1} \dots \wedge dx^{l_s}) \\ \equiv &V^{j_1 \dots j_q}_{\bar{i}_1 \dots \bar{i}_p}(x) W_{\bar{k}_1 \dots \bar{k}_r l_1 \dots l_2}(x) dx^{\bar{i}_1} \wedge \dots dx^{\bar{i}_p} \\ &\wedge dx^{\bar{k}_1} \wedge \dots dx^{\bar{k}_r} \otimes dx^{l_1} \dots \wedge dx^{l_{s-1}}. \end{aligned}$$

Finally, if

$$\text{inner product with } \langle 0| \leftrightarrow \int_M \Omega \wedge,$$

then we get

$$\langle O_{V_1} O_{V_2} ... O_{V_n} \rangle = \langle 0| O_{V_1} O_{V_2} ... O_{V_n} |0\rangle = \int_M \Omega \wedge (V_1 \wedge V_2 ... \wedge V_n) \wedge \Omega.$$

Thus the integrated differential form is (n, n)-form and the integration is well-defined for topological field theory.

4.3 Correlation functions on Calabi-Yau Threefold mirror pair

From now on, we will settle down in Calabi-Yau threefold constructed in $\mathbb{C}P^4$ by

$$X_1^5 + X_2^5 + X_3^5 + X_4^5 + X_5^5 = 0$$

and name it as M. Next, we need a group G acting on $\mathbb{C}P^4$ to construct its mirror pair. If $g(a_1, a_2, a_3, a_4, a_5) \in G$,

$$gX = (e^{\frac{2\pi i a_1}{5}} X_1 : e^{\frac{2\pi i a_2}{5}} X_2 : e^{\frac{2\pi i a_3}{5}} X_3 : e^{\frac{2\pi i a_4}{5}} X_4 : e^{\frac{2\pi i a_5}{5}} X_5)$$

where $(X_1 : ... : X_5)$ is the homogeneous coordinate of $\mathbb{C}P^4$. We need another manifold $\tilde{M}$ defined by

$$X_1^5 + X_2^5 + X_3^5 + X_4^5 + X_5^5 - 5\psi X_1 X_2 X_3 X_4 X_5 = 0.$$

Finally, the mirror pair of M is given by orbifolding: $M^* = \tilde{M}/G$. We note that there is a parameter ψ in the definition of M^*. This parameter corresponds to the freedom of deformation of the complex structure of M^*. On the other hand, M has freedom of deformation of kähler structure corresponding to one parameter t.

It can be shown from topology that by this construction M and M^* have equal Hodge diamond after a diagonal reflection.[GP90] In particular $h^{1,1}(M) = h^{2,1}(M^*) = 1, h^{2,1}(M) = h^{1,1}(M^*) = 101$. $h^{2,1}$ can be seen as the dimension of the moduli space of the complex structure, while $h^{1,1}$ is that of moduli space of kähler structure. In another word, the dimension of moduli space of complex(kähler) structure of M is equal to that of kähler(complex) structure of M^*.

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 1 & & 101 & & 101 & & 1 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} \tag{55}$$

The Hodge diamond of M

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 101 & & 0 & \\
1 & & 1 & & 1 & & 1 \\
 & 0 & & 101 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array} \tag{56}
$$

The Hodge diamond of M^*

Noticing this fact, Candelas et al[CDLOGP91] conjectured that not only the dimension but the moduli space of the corresponding structure of M and M^* itself is isomorphic to each other. In the last section, we see that operators of A model and B model can be related to tensors on a manifold. We can take the A-model side manifold to be M and B-model side manifold to be M^*. The correlation functions of each side are related to integrals of forms on the manifold, and the forms on M and M^* are further related if we accept the above conjecture.

From now we will focus only on the three point function. On B side, there is $\lambda_{\psi\psi\psi} = \left\langle O_\psi O_\psi O_\psi \right\rangle(\psi)$, which depends on the M^* parameter ψ. Using the correspondence we obtained in last section,

$$
\lambda_{\psi\psi\psi} = \int_{M^*} \Omega_{ijk} u^p_{(\psi),\bar{l}} u^q_{(\psi),\bar{m}} u^r_{(\psi),\bar{n}} \Omega_{pqr} dx^i \wedge dx^j \wedge dx^k \wedge dx^{\bar{l}} \wedge dx^{\bar{m}} \wedge dx^{\bar{n}},
$$

where u are forms corresponding to some operators. On A side, the three point function $\lambda_{ttt} = \langle O_t O_t O_t \rangle$ depends on parameter t, and from the argument in the last section, $\lambda_{ttt} = \sum_{d=0}^{\infty} \alpha_d e^{2\pi i d t}$ for some real constants α_d.

Now we state the main result of the so called classical mirror symmetry conjecture which was first proposed by Candelas et al[CDLOGP91]: There exsits a map(mirror map) between parameters of 2 moduli spaces

$$
\psi = \psi(t), \tag{57}
$$

such that

$$
\lambda_{ttt}(t) = \lambda_{\psi\psi\psi}(\psi(t)). \tag{58}
$$

This conjecture comes from believing that the kähler structure moduli space of M is isomorphic to the complex structure moduli space of M^*. Actually, the above equation needs slight modifications to be true, but the spirit holds anyway. To "prove" why this is true we must give the so-called mirror map(57) explicitly, which is the subject of the next section.

4.3.1 Mirror map

First, we try to solve $\lambda_{\psi\psi\psi}$. In order to do that, we need the holomorphic $(3,0)$ form $\Omega(\psi)$ on M^*. Take the local coordinates in $\mathbb{C}P^4$ to be $x_i = \frac{X_i}{X_5}$ for $i = 1,2,3,4$. In this coordinate system, the definition equation for M^* becomes

$$
P_\psi = x_1^5 + x_2^5 + x_3^5 + x_4^5 + 1 - 5\psi x_1 x_2 x_3 x_4 = 0
$$

Then we can get the $\Omega(\psi)$

$$\Omega(\psi) = -\frac{5\psi}{\frac{\partial P_\psi}{\partial x_4}} dx^1 \wedge dx^2 \wedge dx^3 = \frac{1}{x_1 x_2 x_3 - \psi^{-1} x_4^4} dx^1 \wedge dx^2 \wedge dx^3$$

Note x_4 should be written in terms of x_1, x_2, x_3 in above expression via $P_\psi = 0$. Next, we need the integral $\int_C \Omega(\psi)$ of Ω over homology 3-cycles C of M^*. From the Hodge diamond of M^*, we know the cohomology group $H^3(M^*) = \oplus_{p+q=3} H^{p,q}(M^*)$ has dimension 4. Poincare duality says the homology group $H_3(M^*)$ also has dimension 4, so we can take 4 basis of 3-cycles of M^*. In particular, we use A^1, A^2, B_1, B_2 which satisfy

$$A^a \cap B_b = \delta^a_b, \quad A^a \cap A^b = 0, \quad B_a \cap B_b = 0$$

as the basis. Define the period integrals over these cycles:

$$\begin{aligned} f^1(\psi) &= \int_{A^1} \Omega(\psi), \quad f^2(\psi) = \int_{A^2} \Omega(\psi), \\ g_1(\psi) &= \int_{B_1} \Omega(\psi), \quad g_2(\psi) = \int_{B_2} \Omega(\psi). \end{aligned}$$

These functions $f(\psi)$ and $g(\psi)$ are very important in constructing the mirror map. By Poincare duality we can set the basis α_a, β^b of the cohomology group $H^3(M^*)$ satisfying

$$\int_{M^*} \alpha_a \wedge \beta^b = \delta^b_a, \quad \int_{M^*} \alpha_a \wedge \alpha_b = 0, \quad \int_{M^*} \beta^a \wedge \beta^b = 0,$$

with α_a and β_b dual to A^a and B_b respectively. Using these basis we expand $\Omega(\psi)$ as follows:

$$\Omega = f^a(\psi)\alpha_a - g_a(\psi)\beta^a \tag{59}$$

thus we absorb the dependence of Ω on ψ into f and g, which is easier to analyse since they are merely functions of ψ rather than differential forms. If we set the B_2 as

$$B_2 = \left\{ |x_1| = |x_2| = |x_3| = \delta, x_4 \text{ determined by } P_\psi = 0 \right\},$$

where δ is a small real number, f and g turn out to satisfy the so called Picard-Fuchs equation for some function $w(\psi)$:

$$\begin{aligned} &\left(\frac{d^4}{d\psi^4} + C_3(\psi)\frac{d^3}{d\psi^3} + C_2(\psi)\frac{d^2}{d\psi^2} + C_1(\psi)\frac{d}{d\psi} + C_0(\psi) \right) w(\psi) = \\ &\left(\frac{d^4}{dz^4} - \frac{2(4z-3)}{z(1-z)}\frac{d^3}{dz^3} - \frac{72z-35}{5z^2(1-z)}\frac{d^2}{dz^2} - \frac{24z-5}{5z^3(1-z)}\frac{d}{dz} - \frac{24}{625z^3(1-z)} \right) w(\psi) = 0, \end{aligned} \tag{60}$$

here $z = \psi^{-5}$. This is a fourth order linear differential equation meaning that it has 4 linearly independent solutions. $f(\psi)$ and $g(\psi)$ are just the solutions up to a multiplicative constant. After all these preparations, we can move to solve $\lambda_{\psi\psi\psi}$. $\Omega(\psi)$ satisfies Kodaira-Spenser equation :

$$\frac{\partial \Omega(\psi)}{\partial \psi} = \tilde{u}^p_{(\psi),\bar{l}} \Omega_{pij}(\psi) dx^i \wedge dx^j \wedge dx^{\bar{l}} + h(\psi)\Omega(\psi)$$

where $h(\psi)$ is some holomorphic function of ψ. Differentiating twice gives us

$$\frac{\partial^3\Omega(\psi)}{\partial\psi^3} = \tilde{u}^p_{(\psi),\bar{l}}\tilde{u}^q_{(\psi),\bar{m}}\tilde{u}^r_{(\psi),\bar{n}}\Omega_{pqr}dx^{\bar{l}} \wedge dx^{\bar{m}} \wedge dx^{\bar{n}}$$
$$+ (\text{differential forms of anti-holomorphic order lower than 2}).$$

Integration over M^* is non-zero only when the integrated form is (3,3)-form. So

$$\lambda_{\psi\psi\psi}(\psi) = \int_{M^*} \Omega \wedge \frac{\partial^3\Omega}{\partial\psi^3}$$

and

$$\int_{M^*} \Omega \wedge \frac{\partial^2\Omega}{\partial\psi^2} = 0.$$

Differentiate twice,

$$\int_{M^*} \frac{\partial^2\Omega}{\partial\psi^2} \wedge \frac{\partial^2\Omega}{\partial\psi^2} + 2\int_{M^*} \frac{\partial\Omega}{\partial\psi} \wedge \frac{\partial^3\Omega}{\partial\psi^3} + \int_{M^*} \Omega \wedge \frac{\partial^4\Omega}{\partial\psi^4} = 0$$
$$\Rightarrow \int_{M^*} \frac{\partial\Omega}{\partial\psi} \wedge \frac{\partial^3\Omega}{\partial\psi^3} = -\frac{1}{2}\int_{M^*} \Omega \wedge \frac{\partial^4\Omega}{\partial\psi^4}.$$

Differentiate $\lambda_{\psi\psi\psi}$:

$$\frac{d\lambda_{\psi\psi\psi}}{d\psi} = \int_{M^*} \frac{\partial\Omega}{\partial\psi} \wedge \frac{\partial^3\Omega}{\partial\psi^3} + \int_{M^*} \Omega \wedge \frac{\partial^4\Omega}{\partial\psi^4}$$
$$= \frac{1}{2}\int_{M^*} \Omega \wedge \frac{\partial^4\Omega}{\partial\psi^4}.$$

Now we use the expansions(59) and Picard-Fuchs equation,

$$\frac{1}{2}\int_{M^*} \Omega \wedge \frac{\partial^4\Omega}{\partial\psi^4} = -\frac{1}{2}f^a(\psi)\frac{d^4g_a(\psi)}{\psi^4} + \frac{1}{2}g_a(\psi)\frac{d^4f_a(\psi)}{d\psi^4}$$
$$= -\frac{C_3}{2}(f^a(\psi)\frac{d^3g_a(\psi)}{\psi^3} - g_a(\psi)\frac{d^3f_a(\psi)}{d\psi^3})$$
$$= -\frac{C_3}{2}\int_{M^*} \Omega \wedge \frac{\partial^3\Omega}{\partial\psi^3}.$$

We finally get the differential equation of $\lambda_{\psi\psi\psi}$

$$\frac{d\lambda_{\psi\psi\psi}(\psi)}{d\psi} = -\frac{1}{2}C_3(\psi)\lambda_{\psi\psi\psi}(\psi)$$

The solution is

$$\lambda_{\psi\psi\psi} = C\frac{\psi^2}{1-\psi^5},$$

where C is a constant.

Now we are only one step away from the mirror map. Recall that the period integrals are the solution of the Picard-Fuchs equation which can be rewritten as follows:

$$\left((z\frac{d}{dz})^4 - z(z\frac{d}{dz}+\frac{4}{5})(z\frac{d}{dz}+\frac{3}{5})(z\frac{d}{dz}+\frac{2}{5})(z\frac{d}{dz}+\frac{1}{5})\right)w(z) = 0$$

If we define

$$w(z,\epsilon) = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{5n}(j+5\epsilon)}{5^{5n}\prod_{j=1}^{n}(j+\epsilon)^5} z^{n+\epsilon},$$

then it can be easily verified that $w_j(z,0) = \frac{\partial^j w(z,\epsilon)}{\partial \epsilon^j}|_{\epsilon=0}, (j=0,1,2,3)$ are just the 4 independent solutions. Besides, if we write $w(z,\epsilon) = z^{\epsilon}u(z,\epsilon)$, then $u(z,\epsilon)$ is series of z and

$$w_j(z) = \sum_{i=0}^{j}\binom{j}{i}(\log z)^i \frac{\partial^{j-i}u(z,\epsilon)}{\partial\epsilon^{j-i}}|_{\epsilon=0} = (\log z)^j w_0(z) + \ldots.$$

$\psi(t)$ can be determined if we find $t(z)$ since $z = \psi^{-5}$. And Candelas proposed that

$$t(z) = \frac{1}{2\pi i}\frac{C' w_0(z) + w_1(z)}{w_0(z)} \tag{61}$$

$$= \frac{1}{2\pi i}(\log z + C' + \frac{\frac{\partial u(z,\epsilon)}{\partial \epsilon}|_{\epsilon=0}}{w_0(z)}), \tag{62}$$

where C' is a constant. This form of $t(z)$ satisfies two points: (1) $z=0$ corresponds to $t \to \infty$ i.e. the limit in which M has infinite volume.(2) When z go around 0 counterclockwise, $t \to t+1$, which is a natural requirement since λ_{ttt} is invariant under such a change. These points originate from examining the global behavior of the period integrals but can be easily verified. C_1 can be determined as $-5\log 5$ also from the global behavior of f and g.

Finally, we should put the neglected factors back into the relation $\lambda_{ttt} \sim \lambda_{\psi\psi\psi}$. Again there are two points : (1) $\lambda_{\psi\psi\psi}$ transforms as rank-3 symmetric tensor under coordinate transformation. (2) $\Omega(\psi)$ should be divided by $w_0(\psi)$ when translate $\lambda_{\psi\psi\psi}$ into λ_{ttt} because it contains period integral f and g which act as homogeneous coordinates of the moduli space. Take these requirements into account, we arrive at the conclusion:

$$\lambda_{ttt} = \frac{1}{(w_0(\psi(t)))^2}\lambda_{\psi\psi\psi}(\psi(t))(\frac{d\psi(t)}{dt})^3 \tag{63}$$

$$= \frac{C}{(w_0(\psi(t)))^2}\frac{(\psi(t))^2}{1-(\psi(t))^5}(\frac{d\psi(t)}{dt})^3 \tag{64}$$

The constant C can be determined by calculating the first constant term of the expansion $\lambda_{ttt} = \sum_{d=0}\alpha_d e^{2\pi i d t}$ in M. And it can be shown that $\alpha_0 = 5$, so $C = 5(\frac{5}{2\pi i})^3$. Then plugging the mirror map worked out from (61) into (63), expanding it as series of $e^{2\pi i t}$, we get

$$\lambda_{ttt} = \alpha_0 + \alpha_1 e^{2\pi i t} + \alpha_2 e^{4\pi i t} + \ldots \tag{65}$$

$$= 5 + 2875 e^{2\pi i t} + 4876875 e^{4\pi i t} + \ldots. \tag{66}$$

In principle we can get α_d to any order we want.

In physics, the λ_{ttt} and $\lambda_{\psi\psi\psi}$ are equal to Yukawa couplings of heterotic string theory compactified on M and M^* respectively. It is known that $\lambda_{\psi\psi\psi}$ can be strictly calculated but λ_{ttt} can not except the first classical term. The left terms involve quantum correction, and are unknown until the mirror symmetry. More interestingly, in algebraic geometry, there's a problem of counting the number n_d of possible rational curve of order d living on Calabi-Yau threefold M, and it turns out that $n_1 = 2875$, $n_2 = 609250$. We can see that

$$\alpha_1 = n_1, \quad \alpha_2 = 2^3 n_2 + n_1.$$

Motivated by this fact, Candelas et al[CDLOGP91] conjectured that

$$\lambda_{ttt} = \sum_{d=0} \alpha_d e^{2\pi i d t} = 5 + \sum_{d=1}^{\infty} \frac{n_d d^3 e^{2\pi i d t}}{1 - e^{2\pi i d t}}.$$

Through this, we can calculate n_d to any order in principle. It was soon verified that n_3 obtained this way is correct for the number of 3rd order rational curves. It appears to all that the compactification of heterotic string theory and the calculation of correlation functions has nothing to do with the number of some curves existing on the manifold. Yet mirror symmetry tells us that they do have some mysterious relation with each other.

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