

Homework 1: Due on Mar 11

Caution: When solving homework problems, it is important to show your derivation at each step. Nowadays, many online tools make it easy to find answers, but the primary goal of these assignments is to deepen your understanding through hands-on problem-solving. By working through the calculations yourself, you engage more deeply with the material, making the learning process more meaningful, rather than simply copying answers from external sources.

1 Primary state

Show explicitly that the primary state defined by

$$[K_\mu, O_i(x)] = -i(2x_\mu x^\rho \partial_\rho - x^2 \partial_\mu) O_i(x) - 2i\Delta_i x_\mu O_i(x) + 2ix^\nu (S_{\mu\nu})_i^j O_j(x). \quad (1.1)$$

satisfies the Jacobi identity

$$i[[P_\mu, K_\nu], O_i(x)] = -i[[O_i(x), P_\mu], K_\nu] - i[[K_\nu, O_i(x)], P_\mu]. \quad (1.2)$$

2 CFT on a cylindrical space

1. Construct a conformal map between \mathbb{R}^d and $S^{d-1} \times \mathbb{R}$.
2. Using the conformal map constructed above, show that vector fields for a translation and a special conformal transformation on the cylinder spacetime

$$\begin{aligned} P_\mu &= -ie^{-i\tau} \left[-in_\mu \partial_\tau + (\delta_{\mu\nu} - n_\mu n_\nu) \frac{\partial}{\partial n_\nu} \right], \\ K_\mu &= -ie^{i\tau} \left[+in_\mu \partial_\tau + (\delta_{\mu\nu} - n_\mu n_\nu) \frac{\partial}{\partial n_\nu} \right]. \end{aligned} \quad (2.1)$$

where ∂_τ generate the time translation, and n_μ are orthonormal vectors on S^{d-1} .

3 Correlation functions

Let us consider the correlation functions of a CFT in arbitrary dimensions.

1. Using the invariance under conformal transformations, show a 3-point function takes the form:

$$\langle O_1(x_1) O_2(x_2) O_3(x_3) O_4(x_4) \rangle = \frac{c_{123}}{x_{12}^{\Delta_1 + \Delta_2 - \Delta_3} x_{23}^{\Delta_2 + \Delta_3 - \Delta_2} x_{31}^{\Delta_1 + \Delta_2 - \Delta_2}} \quad (3.1)$$

2. Show cross-ratios

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} \quad (3.2)$$

are dimensionless and invariant not only under translations, rotations, and dilatations as well as under special conformal transformations

3. Using the invariance under conformal transformations, show a 4-point function takes the form:

$$\langle O_1(x_1)O_2(x_2)O_3(x_3)O_4(x_4) \rangle = \frac{g(u, v)}{x_{12}^{\Delta_1 + \Delta_2} x_{34}^{\Delta_3 + \Delta_4}} \left(\frac{x_{14}}{x_{24}} \right)^{-\Delta_1 + \Delta_2} \left(\frac{x_{14}}{x_{13}} \right)^{\Delta_3 - \Delta_4} \quad (3.3)$$

4 Möbius transformation

Consider the Riemann sphere $S^2 = \mathbb{C} \cup \{\infty\}$. The action of $\text{SL}(2, \mathbb{C})$ defined by

$$z \mapsto w = \frac{az + b}{cz + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{C}),$$

maps the Riemann sphere onto itself. These transformations are called fractional linear transformations.

1. Given three points z_1, z_2, z_3 , find a fractional linear transformation that maps the points to $0, 1, \infty$.
2. Given four points z_1, z_2, z_3, z_4 , their **cross ratio** is defined by

$$[z_1, z_2, z_3, z_4] = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_2 - z_3)(z_1 - z_4)}.$$

Show that the cross-ratio is preserved by any fractional linear transformation

$$[z_1, z_2, z_3, z_4] = [w_1, w_2, w_3, w_4].$$