Homework 11: Due at class on Dec 12

1 Derivation

Derive (7.17) for the variation of the action (7.15) under $g \to g + \delta g$.

2 Solutions to KZ equations

2.1

The spin-j representation V_j of $\mathfrak{su}(2)$ can be labelled by the Young diagrams $\underbrace{\Box\Box\cdots\Box\Box}_{2j}$ with one row. For instance, the spin-$\frac{1}{2}$ representation is two-dimensional and the basis are spanned by $|+\rangle$ and $|-\rangle$. It corresponds to a single box $\Box$. The tensor product of the two spin-$\frac{1}{2}$ representations is decomposed into

$$(\Box)^{\otimes 2} = \Box \otimes \Box = \Box\Box \oplus \mathbf{1}$$

where $\mathbf{1}$ is the trivial one-dimensional representation. This can be understood as the fusion rule. Let us write the basis $|j_1\rangle \otimes |j_2\rangle := |j_1, j_2\rangle$ of the tensor product $(\Box)^{\otimes 2}$. Write the basis of the representation labelled by $\Box\Box$ as well as $\mathbf{1}$.

In addition, find the fusion rules of the following tensor products

$$(\Box)^{\otimes 3}\,, \qquad (\Box)^{\otimes 4}\,, \quad \text{and} \quad (\Box)^{\otimes 5}\,.$$

2.2

In the lecture, we have learned the correlation functions of N WZW primary fields obey the Knizhnik-Zamolodchikov (KZ) equation

$$\left[\partial_{z_i} + \frac{2}{k+h^\vee} \sum_{j(\neq i)=1}^{n} \frac{\sum_a t^a_{\lambda_i} \otimes t^a_{\lambda_j}}{z_i - z_j}\right] \langle \phi_{\lambda_1}(z_1) \cdots \phi_{\lambda_n}(z_n)\rangle = 0\,.$$

Let us consider the situation that $\mathfrak{g} = \mathfrak{su}(2)$ ($h^\vee = 2$) and all the primary fields are labelled by the spin-$\frac{1}{2}$ representation, *i.e.* $\phi_{\lambda_i}(z_i) = \phi_\Box(z_i)$. In this situation, we write

$$\Omega_{ij} = \sum_a (t^a_\Box)_i \otimes (t^a_\Box)_j$$

where $t^a_\Box = \frac{1}{2}\sigma^a$ with the Pauli matrices σ^a. By studying the action Ω_{12} on $|\pm, \pm\rangle$ explicitly, show

$$\Omega_{12} = \frac{1}{2}\left(s_{12} - \frac{1}{2}\right)$$

where s_{12} is the exchange of the first and second spin.

2.3

As in Problem 2.1, the fusion rule of n primary fields labelled by $\square$ are

$$
\begin{aligned}
(\square)^{\otimes n} &= V_{\frac{n}{2}} \oplus (n-1) V_{\frac{n-2}{2}} \oplus \cdots \\
&= \underbrace{\square\square\cdots\square\square}_{n} \oplus (n-1) \underbrace{\square\square\cdots\square\square}_{n-2} \oplus \cdots
\end{aligned}
$$

Let us find the solutions of the KZ equations corresponding to $V_{\frac{n}{2}}$ and $V_{\frac{n-2}{2}}$. The highest weight state of the representation labelled by $V_{\frac{n}{2}}$ is $|+\cdots+\rangle$. Writing down the correlation function corresponding to this state by

$$
\Phi_{\frac{n}{2}}(z_1, \ldots, z_n) = \psi_0(z_1, \ldots, z_n) |+\cdots+\rangle \,,
$$

find the solution of the KZ equation

$$
\left[\partial_{z_i} + \frac{2}{k+2} \sum_{j(\neq i)=1}^{n} \frac{\Omega_{ij}}{z_i - z_j} \right] \Phi_{\frac{n}{2}}(z_1, \ldots, z_n) = 0 \,.
$$

The correlation function corresponding to the highest weight state of $V_{\frac{n-2}{2}}$ can be written as

$$
\Phi_{\frac{n-2}{2}}(z_1, \ldots, z_n) = \psi_0(z_1, \ldots, z_n) \sum_{i=1}^{n} \psi_i(z_1, \ldots, z_n) |v_i\rangle
$$

where

$$
|v_1\rangle = |-+\cdots+\rangle \,, \quad |v_2\rangle = |+-+\cdots+\rangle \,, \quad \cdots \,, \quad |v_n\rangle = |+\cdots+-\rangle \,,
$$

and

$$
\sum_{i=1}^{n} \psi_i(z_1, \ldots, z_n) = 0 \,.
$$

Show that the KZ equations

$$
\left[\partial_{z_1} + \frac{2}{k+2} \sum_{j=2}^{n} \frac{\Omega_{1j}}{z_1 - z_j} \right] \Phi_{\frac{n-2}{2}}(z_1, \ldots, z_n) = 0
$$

reduce to

$$
(k+2) \frac{\partial}{\partial z_1} \psi_1(z) + \frac{\psi_2 - \psi_1}{z_1 - z_2} + \frac{\psi_3 - \psi_1}{z_1 - z_3} + \cdots + \frac{\psi_N - \psi_1}{z_1 - z_N} = 0 \,.
$$

and

$$
(k+2) \frac{\partial}{\partial z_1} \psi_2(z) + \frac{\psi_1 - \psi_2}{z_1 - z_2} = 0 \,.
$$

Show that

$$
\psi_i(z) = \int_C dt \prod_{a=1}^{n} (z_a - t)^{\frac{1}{k+2}} \frac{1}{z_i - t}
$$

becomes the solution of the KZ equations. In fact, there are $n-1$ solutions by taking the different contours C.

3 Verlinde algebra of Ising model

There are finitely many primary fields in a rational conformal field theory like a minimal model and WZW model. The fusion rule of primary fields are closed under themselves

$$[\phi_i] \times [\phi_j] = \sum_k \mathcal{N}_{ij}^k \, [\phi_k] \ ,$$

where $\mathcal{N}_{ij}^k$ are called the fusion coefficients. As we have seen in the minimal models and WZW models, one can construct a highest weight representation associated to a primary field ϕ_i, and we write the corresponding character by $\chi_i(\tau)$. The modular transformations are

$$\chi_i(-1/\tau) = \sum_j S_{ij} \, \chi_j(\tau) \ , \qquad \chi_i(\tau+1) = \sum_j T_{ij} \, \chi_j(\tau) \ .$$

E. Verlinde has found the remarkable relationship between fusion rule and the modular S-matrices [Ver88]

$$\mathcal{N}_{ij}^k = \sum_l \frac{S_{jl} S_{il} \left(S^{-1}\right)_{lk}}{S_{0l}} \ . \tag{3.1}$$

3.1 Ising model revisited

Let us recall that the Ising model is the unitary minimal model $\mathcal{M}_{p=3}$ where the primary fields are associated to

$$\begin{aligned} 1 \quad &\Leftrightarrow \phi_1 := \phi_{1,1} \\ \epsilon \quad &\Leftrightarrow \phi_2 := \phi_{2,1} \\ \sigma \quad &\Leftrightarrow \phi_3 := \phi_{2,2} \end{aligned}$$

where ϵ is the energy density filed and σ is the spin field. The corresponding characters are given in (5.89) of the lecture notes

$$\begin{aligned} \chi_1(\tau) := \chi_0(\tau) &= \frac{1}{2}\left(\sqrt{\frac{\vartheta_3(\tau)}{\eta(\tau)}} + \sqrt{\frac{\vartheta_4(\tau)}{\eta(\tau)}}\right) \\ \chi_2(\tau) := \chi_{\frac{1}{2}}(\tau) &= \frac{1}{2}\left(\sqrt{\frac{\vartheta_3(\tau)}{\eta(\tau)}} - \sqrt{\frac{\vartheta_4(\tau)}{\eta(\tau)}}\right) \\ \chi_3(\tau) := \chi_{\frac{1}{16}}(\tau) &= \frac{1}{\sqrt{2}}\sqrt{\frac{\vartheta_2(\tau)}{\eta(\tau)}} \end{aligned}$$

Using the properties of the ϑ and η-functions, find the 3×3 S-matrix of the Ising model. Then, compute the three 3×3 fusion matrices $\mathcal{N}_{ij}^k$ ($i = 1, 2, 3$) by using the Verlinde formula (3.1). Check they reproduce the fusion rule in (5.87) of §5.4 of the lecture note.

References

[Ver88] E. Verlinde, *Fusion rules and modular transformations in 2d conformal field theory*, Nuclear Physics B **300** (1988) 360–376.