Homework 5: Due at class on April 7

1. Let $T^2 = S^1 \times S^1$ be the torus. Using the formula for de Rham cohomology of a product space,

$$H^{p+q}_{dR}(M \times N) \cong H^p_{dR}(M) \otimes H^q_{dR}(N) ,$$

find de Rham cohomology $H^*_{dR}(T^2)$. Find the Poincare dual of each generator of $H^*_{dR}(T^2)$.

2. Do parallel transport of a vector along a triangle ΔPQR on a unit sphere (Figure below) and find the angle difference when it comes back. Note that the sphere has the standard metric and we consider the parallel transport with respect to the Levi-Civita connection. Compare it with the area of the triangle. Do the same exercise for a triangle on the upper half plane. Describe the difference between the sphere and the upper half plane.

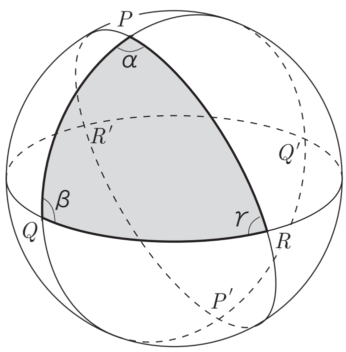

Figure 1: A triangle on a 2-sphere

3. Derive the expression of the Christoffel symbols

$$\Gamma^k_{ij} = \frac{1}{2}\left(\frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l}\right) g^{lk}.$$

from

$$\frac{\partial}{\partial x^i} g_{jk} = g\left(\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right) + g\left(\frac{\partial}{\partial x^j}, \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^k}\right)$$

and its permutations with respect to i, j, k. Show that the Riemann curvature can be written

$$R^l{}_{ijk} = \Gamma^s_{jk}\Gamma^l_{is} - \Gamma^s_{ik}\Gamma^l_{js} + \frac{\partial \Gamma^l_{jk}}{\partial x^i} - \frac{\partial \Gamma^l_{ik}}{\partial x^j}.$$

in terms of local coordinates.

4. Let $S^2 \subset \mathbb{R}^3$ be the 2-sphere with unit radius and the metric on S^2 is induced from the standard metric of $\mathbb{R}^3$ as in Homework 4. Find geodesics on S^2 and compute its Riemann, Ricci and scalar curvature.

5. Let $\mathbf{H} = \{(x, y)|y > 0\}$ be the upper half plane and the metric is given by

$$ds^2 = \frac{dx^2 + dy^2}{y^2} \ . \tag{1}$$

Find geodesics on $\mathbf{H}$ and compute its Riemann, Ricci and scalar curvature.

6. The **Möbius transformation** of the upper half plane $\mathbf{H} = \{z = x + iy \mid y > 0\}$ is a rational function of the form

$$f(z) = \frac{az + b}{cz + d} \ ,$$

where $ad - bc = 1$ with $a, b, c, d \in \mathbb{R}$. If f_1 and f_2 are Möbius transformations, prove that $f_1 \circ f_2$ is also a Möbius transformation. Show that this is an isometry for the metric (1).