

Homework 6: Due at class on Dec 21

1 Dimensional regularization

Using dimensional regularization $d = 4 - \epsilon$, and Wick rotation ($p^0 \rightarrow ip_E^0$), show

$$\int \frac{d^d k}{(2\pi)^d} \frac{k^{2a}}{[k^2 - \Delta + i\tilde{\epsilon}]^b} = \frac{i}{16\pi^2} (-\Delta)^{a-b+2} \left(\frac{4\pi}{\Delta}\right)^{\frac{\epsilon}{2}} \frac{\Gamma(2+a-\frac{\epsilon}{2})\Gamma(b-a-2+\frac{\epsilon}{2})}{\Gamma(b)\Gamma(2-\frac{\epsilon}{2})}$$

The integral over the solid angle in d -dimensional euclidean space is given by

$$\int d\Omega_d = \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})}.$$

2 Optical theorem in massless QED

Let us consider the optical theorem in the massless QED. Show that the unpolarized cross-section σ of the scattering $A(k_1)B(k_2) \rightarrow C(p_1)D(p_2)$ in CM-frame namely

$$\sigma = \frac{1}{s} \text{Im}(\mathcal{M})$$

where $i\mathcal{M}$ the unpolarized (average) Feynman amplitude of the process $A(k_1)B(k_2) \rightarrow A(k_1)B(k_2)$.

In the lecture, we have seen that the vacuum polarization takes the form

$$\Pi^{\mu\nu}(q) = \left(q^2 g^{\mu\nu} - q^\mu q^\nu \right) \Pi(q^2)$$

Also, the dimensional regularization $d = 4 - \epsilon$ provides

$$\Pi(q^2) = -\frac{2\alpha}{\pi} \int_0^1 dx x(1-x) \left(\frac{2}{\epsilon} - \gamma + \log(4\pi) - \log \frac{-x(1-x)q^2 - i\epsilon}{\mu^2} \right)$$

where we replace the electric charge e by $e\mu^\epsilon$.

Show that in the case of $e^-e^+ \rightarrow \mu^-\mu^+$ scattering in CM-frame ($m_e = m_\mu = 0$) the unpolarized cross-section (order $\mathcal{O}(\alpha^2)$) is given by

$$\sigma = \frac{4\pi\alpha^2}{3s}$$

using the optical theorem and the expression for $\text{Im}(\Pi(q^2))$.

3 Vacuum polarization in scalar QED

In the scalar QED, find the diagrams which contribute to the 1PI diagrams of the photon propagator in the order $\mathcal{O}(e^2)$ and write down the integral expression for $\Pi_{1\text{-loop}}^{\mu\nu}(k)$

$$i\Pi^{\mu\nu}(k) \equiv \mu \text{ (diagram: a wavy line with a circle labeled 1PI inside) } \nu = e^2 \Pi_{1\text{-loop}}^{\mu\nu}(k) + \mathcal{O}(e^4), \quad (3.1)$$

and perform momentum integration by using the dimensional regularization.