学校代码：10246
学　　号：

復旦大學

本 科 毕 业 论 文

链环的 GM 不变量

GM invariants of links

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完 成 日 期：2026 年 1 月 21 日

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摘　要

本文主要通过拓展垂直图封闭流形同调块的定义至有界流形，计算出了部分链环和扭结对应的 Gukov 级数，并在此基础上给出了从链环到链环、扭结到封闭流形的手术公式猜想，使得这种定义不仅限垂直图，离给出 GM 不变量的数学定义更进一步。同时我们对 $T_{4,2}$ 链环的 GM 不变量与 A 多项式进行了对比，将 Gukov 做出的猜想从扭结拓展到了链环上。

关键字：GM 不变量；同调块；Dehn 手术，A 多项式
中图分类号：O413.1

Abstract

In this paper we expand the definition of homological blocks of plumbed graphs into manifold with boundaries, and calculate some F_K of knots and links. Based on such results, we conjectured a surgery formula which can compute a homological block of a closed manifold of a link complement from a F_K of a link. With such surgery formula, we computed the F_k of $T_{4,2}$, and compared it with A polynomial.

Keywords: GM invariant; Homological block; Dehn surgery，A-polynomial

CLC number: O413.1

符号表

$\widehat{Z}_a$	$Spin^C$ 结构表示为 a 的同调块
$\tilde{\Psi}$	赝模函数
F_K	扭结的 GM 不变量
F_L	链环的 GM 不变量

第 1 章　研究背景

Roughly 30 years ago, just after Jones introduced a knot invariant——Jones polynomial, Witten has shown that this topology invariant has deep connection with quantum groups, that's the beginning of quantum topology. In years, Khovanov has developed Khovanov homology, which is an oriented link invariant arising as the homology of a chain complex, the most significant property of such homology is that its Euler characteristic is exactly the Jones polynomial. Despite the definition of such homology is combinatorial，Witten gave a physical description of Khovanov homology, coefficient of Jones polynoamial counts solutions to the Kapustin-Witten equations, and the Khovanova homology is the Morse homology for Haydys equation.

The Witten-Reshetikhin-Turaev invariant is an invariant of 3-manifold with a (clored, oriented, framed) link inside it which is constructed in TQFT. The ways to define it are various: path integral, CFT and affine Lie algebra. This invariant can be seen as an extensiong of Jones polynomial to 3-manifolds. The problem is , it's only defined at roots of unity, and does not have obvious integrality to make it the Euler characteristic of a vector space. Basically, the question we are facing right now can be expressed in following table.

	Quantum invariant	Homological invariant
Knots	Jones polynomials	Khovanov homology
3-manifolds	WRT invariants	???

Recently, based on the physical description of Khovanov homology given by Witten, the problem of categorification of WRT invariants may have a chance to be solved. The theory $6d\mathcal{N} = (2,0)$SCFT of A_1 type on $M \times D^2 \times \mathbb{R}$ with boundary condition a on ∂D^2 is the major motivation of such work. By compactifying M_3, we get a 3d $\mathcal{N} = 2$ theory $T[M]$ on $D^2 \times \mathbb{R}$, and theory T[M] is an invariant of 3-manifold M. That is so-called 3d-3d correspondence. The most interesting invariant rised in this theory is the supersymmetric partition function of $T[M]$ on $D^2 \times_q S^1$:

$$\hat{Z}_a(q) = Z_{T[M]}\left(D^2 \times_q S^1; a\right) \in 2^{-c} q^{\Delta_a} Z[[q]]$$

The conjectured vortex partition $\hat{Z}_b$ in this theory can be regarded as the graded Euler characteristic of the Hilbert space $\mathscr{H}_{BPS}$, and $\mathscr{H}_{BPS}$ can be thought as a 3-manifold analog of Khovanov homology. Therefore, physics predicts a natural categorification of $\hat{Z}_b$

For a 3-manifold Y and a choice of a simply laced root system g, the $\hat{Z}_b$ is a number of q-series with integer coefficients.

$$\hat{Z}_b(Y, \mathfrak{g}; q) = q^{\Delta_b}\left(a_0 + a_1 q + a_2 q^2 + \cdots\right)$$

For each 3-manifold, there will be a series of partition function corresponding to different $Spin^C$ structure, and the WRT invariant can be reconstructed by these $\hat{Z}$. One can say that $\hat{Z}$ is an analytic continuation of the WRT invariant, which is defined on roots of unity, inside the unit disk.

However, the complete definition of $\hat{Z}$ is not found yet, the general method of computing it is using resurgence in Chern-Simons theoryBesides several trivial manifolds such as $S^1 \times S^2$, S^3, $lensspace$, there is a big class of manifolds for which we have exact methods to calculate it, such as

1. negative definite plumbings on trees
2. circle bundle over a Riemann surface
3. some surgery on knots

The first category gives us a big playground to play with and test conjectures, and it includes some important examples like homological spheres and so we are gonna explain it here.

1.1 homological blocks of negative definite plumbed manifold

Let $Y = Y[\Gamma]$ be a 3-manifold obtained by a negative definite plumbing on a weighted tree Γ, that is to say the this manifold can be seen as surgery on a tree -shaped link. By defining the linking matrix (which must be negative definite) of such plumbed graph

$$M_{v_1,v_2} = \begin{cases} 1, & v_1, v_2 \text{ connected} \\ a_v, & v_1 = v_2 = v, \\ 0, & \text{otherwise} \end{cases} \qquad v_i \in \text{ Vertices of } \Gamma \cong \{1, \dots, L\}$$

Gukov conjectured the homological block of such manifold

$$\widehat{Z}_a(q) = q^{-\frac{3L+\sum_v a_v}{4}} \cdot \mathrm{v} \cdot \mathrm{p} \cdot \int_{|z_v|=1} \prod_{v \in \text{ Vertices}} \frac{dz_v}{2\pi i z_v} \left(z_v - 1/z_v\right)^{2-\deg(v)} \cdot \Theta_b^{-M}(z) \tag{1.1}$$

$$= q^{-\frac{3L+\sum_w a_v}{4}} \cdot \mathrm{v} \cdot \mathrm{p} \cdot \int_{|z_v|=1} \prod_{v \in \text{ Vertices}} \frac{dz_v}{2\pi i z_v} \left(z_v - 1/z_v\right)^2 \times \frac{1}{\left(z_{v_1} - 1/z_{v_1}\right)\left(z_{v_2} - 1/z_{v_2}\right)} \cdot \Theta_b^{-M}(z) \tag{1.2}$$

where $deg(v)$ is the degree of vertex v, and

$$\Theta_b^{-M}(x) = \sum_{\ell \in 2MZ^L+b} q^{-\frac{\left(\vec{l}, M^{-1}\vec{l}\right)}{4}} \prod_{i=1}^{L} x_i^{\ell_i}$$

In the following discussion, the form contributed by vertices and edges is also useful, we first use $\vec{a}$ to represent the $Spin^C$ structure

$$\vec{\ell} = 2M\vec{n} + \vec{a}, \quad \vec{n} = \left(n_v\right)_{v \in \text{ Vert}}, \vec{a} = \left(a_v\right)_{v \in \text{ Vert}} \in \mathbb{Z}^x$$

so

$$\frac{\left(\vec{\ell}, M^{-1}\vec{\ell}\right)}{4} = (\vec{n}, M\vec{n}) + (\vec{a}, \vec{n}) + \frac{\left(\vec{a}, M^{-1}\vec{a}\right)}{4}$$

then we can transform the contribution of vertex degree into the contribution of edge

$$\widehat{Z}_a(q) = (-1)^\pi q^{\frac{3\sigma-\left(\tilde{a},M^{-1}a\right)}{4}} \sum_{n_v} \oint_{|z_v|=1} \frac{dz_v}{2\pi i z_v} \prod_{\text{Vert}}(\dots) \prod_{\text{Edges}}(\dots)$$

the contribution of vertex v is

$$q^{-m_v n_v^2 - \frac{m_v}{4} - a_v n_v} z_v^{2m_v n_v + a_v} \left(z_v - \frac{1}{z_v}\right)^2$$

the contribution of edge(u,v) is

$$q^{-2n_u n_v} \frac{z_u^{2n_v} z_v^{2n_u}}{\left(z_u - \frac{1}{z_u}\right)\left(z_v - \frac{1}{z_v}\right)}$$

In [GM19], Gukov has also defined the homological block $\hat{Z}_a(Y; z, q)$ of knot complement which can be seen as a plumbed manifold with a torus boundary $S^3 \backslash \nu K$. This homological block gives us new sight to the study of knot invariants. The homological block with $Spin^C$ structure $a = 0$ is the most interesting one, some variables of which can be eliminated

$$F_K(x, q) := \widehat{Z}_0\left(Y; x^{1/2}, n, q\right)$$

And it seems has deep connection with known knot invariant.

猜想 1.1 If $\hat{A}(\hat{x}, \hat{y}, q)$ is the quantum A-polynomialG[Gar04] annihilating the unreduced colored Jones polynomials, the

$$\hat{A}(\hat{x}, \hat{y}, q) F_k(x, q) = 0$$

In this paper, we're gonna extend the definition of F_K from knots to links, provide a wider space to study the connection between F_K and known knot invariant.

1.2 Organization of the paper

In Section 2, we will first reproduce the surgery result in [GM19], and perform some consistency checks with the example Lens space L(p,r).

In Section 3, we extend the surgery formula to links, and demonstrate how to obtain F_K for links from reverse-engineering, and use the quantum $\hat{A}$ polynomial to do the consistency check.

第 2 章　Surgery on knots

The homological block of a manifold with a torus boundary is given as follows

$$\widehat{Z}_a(q) = (-1)^{\pi} q^{\frac{3\sigma-\left(\bar{a},M^{-1}a\right)}{4}} \left[q^{-m_{v_0}n^2-\frac{m_{v_0}}{4}-a_{v_0}n_z 2m_{w_0}n+a_{v_0}} \left(z-\frac{1}{z}\right)\right] \sum_{v\neq v_0} \oint_{|z_v|=1} \frac{dz_v}{2\pi i z_v} \prod_{\text{Vert}}(\dots) \prod_{\text{Edges}}(\dots) \tag{2.1}$$

where v_0 is the vertex corresponding to the torus boundary. And the contribution of the vertex $v \neq v_0$

$$q^{-m_v n_v^2-\frac{m_v}{4}-a_v n_v} z_v^{2m_v n_v+a_v} \left(z_v-\frac{1}{z_v}\right)^2$$

The contribution of edge (u,v) is as follows

$$q^{-2n_u n_v} \frac{z_u^{2n_v} z_v^{2n_u}}{\left(z_u-\frac{1}{z_u}\right)\left(z_v-\frac{1}{z_v}\right)}$$

One significant property of such definition is its additivity. In the manifold with torus boundary, we have so-called standard gluing, which is gluing the longtitude of one torus boundary along the meridian of the other torus boundary. In the situation of plumbed manifold, such process can be expressed in following figure.It's not hard to obtain the additivity of such manifold from the efinition.

$$\widehat{Z}_a(Y;q) = (-1)^{\tau} q^{\xi} \sum_n \oint_{|z|=1} \frac{dz}{2\pi i z} \widehat{Z}_{a^-}\left(Y^-;z,n,q\right) \widehat{Z}_{a^+}\left(Y^+;z,n,q\right) \tag{2.2}$$

To understand the surgery formula provided in [GM19], we also need the homological block of the solid torus with coefficient *p/r*. Write down the integer surgery description of solid torus in the form of continued fraction decomposition

2.1 Surgery formula

With 3.1 and the homological block of the p/r solid torus, we can start reproducing the surgery formula in [GM19]. The homological block of the p/r solid torus is already given as follows

$$\widehat{Z}_a\left(\mathrm{S}_{p/r};z,n,q\right) \cong \begin{cases} \pm\cdot q^{-\frac{(pn+a)^2}{pr}} z^{2j+1} & \text{if } \frac{pn+a}{r} \mp \frac{1}{2r} - \frac{1}{2} = j \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases} \tag{2.3}$$

The *p/r* surgery on a knot complement can be seen as a standard gluing process between a p/r solid torus and a knot complement along the torus boundary, so we write

$$\hat{Z}_a(Y;q) \cong \sum_n \oint_{|z|=1} \frac{dz}{2\pi i z} \hat{Z}_a(\hat{Y};z,n,q)\hat{Z}_0(S_{p/r};z,q)$$

Where we denote the knot complement as $\hat{Y}$, and $\cong$ means it's up to some general factors. For simplicity, we set

$$\hat{Z}_a(\hat{Y};z,n,q) = \sum_m a_m(q,n)z^m$$

So

$$\hat{Z}_a(Y;q) \cong \pm \sum_n \oint \frac{dz}{2\pi i z} \sum_m a_m z^m q^{-\frac{pn^2}{r}-\frac{r}{4p}} z^{\frac{2rn}{p}} = \sum_n \oint \frac{dz}{2\pi i z} \left[\sum_m a_m z^m\right] q^{-\frac{pn^2}{r}-\frac{r}{4p}} z^{\frac{2rn}{p}} z^{\mp\frac{1}{r}}$$

Set

$$\sum_m a_m z^m = \sum_i b_i z^i = Y(z)$$

back to original formula we get

$$\hat{Z}_a(Y;q) \cong \sum_n \oint \frac{dz}{2\pi i z} Y(z) z^{\frac{2rn}{p}\mp\frac{1}{r}} q^{-\frac{pn^2}{r}} = \pm \sum_n \oint \frac{dz}{2\pi i z} \sum_i b_i z^{i+\frac{2rn}{p}\mp\frac{1}{r}} q^{-\frac{pn^2}{r}}$$

Because of the residue theorem, and summation of n is over all the integers, in the non-zero term, we have

$$i + \frac{2rn}{p} \mp \frac{1}{r} = 0 \to \hat{Z}_a(Y;q) \cong \sum_i b_i q^{-\frac{pi^2}{4r}}$$

Since $i + \frac{2rn}{p} \mp \frac{1}{r}$ can't be 0 at the same time, in the contour integral, we

$$\pm \sum_n \oint \frac{dz}{2\pi i z} \sum_i b_i z^{i+\frac{2rn}{p}\mp\frac{1}{r}} q^{-\frac{pn^2}{r}} = \sum_n \oint \frac{dz}{2\pi i z} \sum_i b_i z^{i+\frac{2rn}{p}} (z^{\frac{1}{r}} - z^{\frac{-1}{r}}) q^{-\frac{pn^2}{r}}$$

In short, the contour integral can be seen as following "laplace transform",

$$\hat{Z}_a(Y;q) = \mathscr{L}_{p/r}\left[(z^{\frac{1}{r}} - \frac{1}{z^{\frac{1}{r}}})\hat{Z}_a(\hat{Y};q,z)\right]$$

where

$$\mathscr{L}_{p/r} : z^i \to q^{-\frac{pi^2}{4r}} \quad \text{iff } \frac{ri}{p} \in \mathbb{Z}$$

2.2 Consistency check

We use lens space to do the consistency check.

The surgery description of lens space can be represented by the continued fraction decomposition

$$\frac{p}{r} = k_s - \cfrac{1}{k_{s-1} - \cfrac{1}{\ddots - \cfrac{1}{k_1}}}$$

Then we can draw the linking matrix as follows

$$M = \begin{pmatrix} k_1 & 1 & & & & \\ 1 & k_2 & 1 & & & \\ & 1 & k_3 & & & \\ & & & \ddots & & \\ & & & & k_{n-1} & 1 \\ & & & & 1 & k_n \end{pmatrix}$$

Due to the famous affined identification

$$\mathrm{Spin}^c(Y) \cong H^2(Y; \mathbb{Z})$$

The $spin^C$ structure on the solid torus can be expressed in $\vec{l}$ as

$$\vec{l} = (2j+1, 0, \ldots, 0, \pm 1)$$

So in the 1.2, the homological block of lens space can simplified

$$\widehat{Z}_a(q) = q^{-\frac{3L+\sum_v a_v}{4}} \cdot \mathrm{v} \cdot \mathrm{p} \cdot \int_{|z_v|=1} \prod_{v \in \text{ Vertices}} \frac{dz_v}{2\pi i z_v} \left(z_v - 1/z_v\right)^{2-\deg(v)} \cdot \sum_{\pm 1} q^{-\frac{\left(\vec{l}, M^{-1}\vec{l}\right)}{4}} z_1^{2j+1} z_2^{\pm 1}$$

Since the relation between colored Jones polynomial draws the most interest, and in the colored Jones polynomial the general factor is not really important, so in our problem we mainly focus on the following index $(\vec{l}, M^{-1}\vec{l})$, notice $\vec{l}$ has only 2 non-zero vector component $l_1 = 2j+1$ and $l_n = \pm 1$, so in the matrix M^{-1} we only need to compute following component

$$m_{11}^{-1} = \frac{M_{11}}{\det M}, m_{1n}^{-1} = \frac{M_{n1}}{\det M}, m_{nn}^{-1} = \frac{M_{nn}}{\det M}$$

where M_{ab} is the cofactor of M's matrix component m_{ab}

To compute the determinate of M, we first write down following sequence

$$a_s = \begin{vmatrix} k_1 & 1 & & & & \\ 1 & k_2 & 1 & & & \\ & 1 & k_3 & & & \\ & & & \ddots & & \\ & & & & k_{s-1} & 1 \\ & & & & 1 & k_s \end{vmatrix}, a_{s-1} = \begin{vmatrix} k_1 & 1 & & & & \\ 1 & k_2 & 1 & & & \\ & 1 & k_3 & & & \\ & & & \ddots & & \\ & & & & k_{s-1} & 1 \\ & & & & 1 & k_{s-1} \end{vmatrix}$$

One can convince oneself that such sequence has following recursive relation

$$a_n = k_n a_{n-1} - a_{n-2}$$

To the continued fraction decomposition, we set

$$\frac{p_n}{r_n} = k_n - \frac{1}{k_{n-1} - \frac{1}{\ddots - \frac{1}{k_1}}}, \frac{p_{n-1}}{r_{n-1}} = k_{n-1} - \frac{1}{k_{n-2} - \frac{1}{\ddots - \frac{1}{k_1}}}$$

It's not hard to find the recursive relation of sequence $\{p_n\}$

$$p_n = k_n p_{n-1} - p_{n-2}$$

Thus we can conclude the sequences $\{p_n\}$ and $\{a_n\}$ share the same recursive relation, it's also easy to notice that these 2 sequence have the same first term and the second tern(or up to a sign), so we have

$$|detM| = |p|$$

Following the same strategy, we draw the conclusion

$$\left(M^{-1}\right)_{11} = r/p, \quad \left(M^{-1}\right)_{1n} = \pm 1/p, \quad \left(M^{-1}\right)_{nn} = M_{nn}/p$$

Where the sign of M^{-1}_{1n} is determined by the eigenvalues of M, then the index can be written as

$$-\frac{\left(\vec{\ell}, M^{-1}\vec{\ell}\right)}{4} = -\frac{r}{p}\left(j + \frac{1}{2} \pm \frac{\varepsilon}{2r}\right)^2 + \frac{1}{4pr} - \frac{M_{nn}}{4p}$$

For different $spin^C$ structure (that is to say different j), $\frac{1}{4pr}$ and $-\frac{M_{nn}}{4p}$ won't change, write them in a general factor

$$\alpha(p,r) = \frac{1}{4pr} - \frac{M_{nn}}{4p} + \frac{3\sigma - \sum_v m_v}{4}$$

So

$$\hat{Z}_0(L(p,r)) \cong \oint_{z_n}\oint_{z_1}(z_n - \frac{1}{z_n})z_n^{\pm 1}(z_1 - \frac{1}{z_1})z_1^{2j+1}q^{-\frac{r}{p}(j+\frac{1}{2}\pm\frac{\epsilon}{2r})^2}$$

Since j can be any integral, we set

$$rj \pm \frac{\epsilon}{2} = pn - \frac{r}{2}$$

Then

$$\hat{Z}_0(L(p,r)) \cong \oint_{z_n}\oint_{z_1}(z_n - \frac{1}{z_n})z_n^{\pm 1}(z_1 - \frac{1}{z_1})z_1^{\frac{2pn}{r}}q^{-\frac{pn^2}{r}}$$

then the residue theorem can give the final result

$$\hat{Z}_0(L(p,r);q) \cong \begin{cases} q^{-\frac{p(r+1)^2}{4r}} & \text{if } \frac{r(r+1)}{p} \in \mathbb{Z} \\ -q^{-\frac{p(r-1)^2}{4r}} & \text{if } \frac{r(r+1)}{p} \in \mathbb{Z} \end{cases}$$

In the surgery formula, Lens space L(p,r) can be seen as the result of p/r surgery over an unknot. So

$$\hat{Z}_0(L(p,r);q) \cong \mathscr{L}_{p/r}\left[(z - \frac{1}{z})(z^r - z^{-r})\right] = \begin{cases} q^{-\frac{p(r+1)^2}{4r}} & \text{if } \frac{r(r+1)}{p} \in \mathbb{Z} \\ -q^{-\frac{p(r-1)^2}{4r}} & \text{if } \frac{r(r+1)}{p} \in \mathbb{Z} \end{cases}$$

Consistency check can be made.

第 3 章　GPPV series of links

Now we extend the homological block definition of knot complement into the link complement which is a 3-manifold with numbers of torus boundaries. Imitating the form of the knot complement, we write down

$$\hat{Z}_a(q) = (-1)^{\pi} q^{\frac{3\sigma-\left(\bar{a},M^{-1}a\right)}{4}} \prod_i \left[q^{-m_{v_i}n^2-\frac{m_{v_i}}{4}-a_{v_i}n_z 2m_{v_i}n+a_{v_i}} \left(z-\frac{1}{z}\right)\right] \sum_{v\neq v_i} \oint_{|z_v|=1} \frac{dz_v}{2\pi i z_v} \prod_{\text{Vert}}(\dots) \prod_{\text{Edges}}(\dots) \tag{3.1}$$

where v_i are the vertexes corresponding to the torus boundary. And the contribution of the vertex $v \neq v_0$

$$q^{-m_v n_v^2-\frac{m_v}{4}-a_v n_v} z_v^{2m_v n_v+a_v} \left(z_v-\frac{1}{z_v}\right)^2$$

The contribution of edge (u,v) is as follows

$$q^{-2n_u n_v} \frac{z_u^{2n_v} z_v^{2n_u}}{\left(z_u-\frac{1}{z_u}\right)\left(z_v-\frac{1}{z_v}\right)}$$

3.1　Surgery on links

Based on the surgery result derived on knots, one can have a natural idea that whether we can perform a series of surgery formula on each knot of link to get a homological block of a closed manifold, unfortunately it's not always true, since some Lens space can be obtained by surgery on a Hopf link, one can easily check following result

$$\hat{Z}_0(L(k_1k_2-1,k_2)) \neq q^{\Delta}\mathscr{L}_{k_1}\left[\mathscr{L}_{k_2}\left[(z_1-\frac{1}{z_1})(z_2-\frac{1}{z_2})Z_0(L_{\text{Hopf}},z_1,z_2)\right]\right]$$

A geometry image for such result is that these two knot are "entangled" together, the surgery on one knot will affect the structure on the other torus boundary. One can also check that when one torus boundary is linked with another one, the additivity doesn't hold. In order to obtain a surgery formula that works on links, we focus on one boundary, then can be written as

$$\hat{Z}_a(\hat{L}) = \sum_{l_v\in 2MZ^s+\vec{a}} q^{\frac{3\sigma-\sim_v m_v}{4}} \prod_{\text{Vert}} \frac{dz_v}{2\pi i z_v}(z_v-\frac{1}{z_v})^{1-deg(v)} z_v^{l_v} q^{-\frac{(\vec{l},M^{-1}\vec{l})}{4}} C$$

where C is the contour integral part.

We perform $-1/r$ surgery on the torus boundary v_0 by gluing a 1/r torus, such surgery won't be affected by what's inside the contour integral. We pull out all the factors related to v_0

$$\hat{Z}_0(Y', z_1. \cdots, z_{v-1}, z_{v+1}, \cdots, z_l) = \pm \sum_{n_v} \oint \frac{dz_v}{2\pi i z_v} \prod_i^{deg(v_0)} z_{v_i}^{n_v} F_L'(z_1, \cdots, z_l, q)(z_v - \frac{1}{z_v}) z_v^{2pn_v} q^{-pn_v^2 - \frac{p}{4}} \tag{3.2}$$

$$= \pm \sum_{n_v} \oint \frac{dz_v}{2\pi i z_v} \prod_i^{deg(v_0)} z_{v_i}^{n_v} \sum_m a_m z_v^m (z_v - \frac{1}{z_v}) z_v^{-\frac{1}{r} n_v \mp \frac{1}{r}} q^{\frac{n_v^2}{r} - \frac{p}{4}} \tag{3.3}$$

Where the sign of this expression comes from the $Spin^C$ structure, use the same trick in the knot surgery formula, we can write the sign into $(z_v^r - \frac{1}{z_v^r})$

The residue theorem gives

$$-\frac{2n_v}{r} \pm 1 = -m$$

So the contour integral can be seen as

$$z_v^m \to \prod_i^{deg(v_0)} z_{v_i}^{-\frac{1}{2p} m} q^{\frac{m^2}{4p}}$$

So the link surgery formula can be obtained

$$Z_{L'}\left(z_1, \cdots, z_l, q\right) \cong \mathscr{L}\left[\left(z_v^{\frac{1}{r}} - z_v^{-\frac{1}{r}}\right) Z_L\left(z_0, \cdots, z_l, q\right)\right]$$

where

$$\mathscr{L} : z_v^u \mapsto q^{\frac{ru^2}{4}} \prod_i^{deg(v)} z_i^{rlk(v,i)u} \tag{3.4}$$

And we want to define homological blocks mathemathtically, so we

Now we can move back the Hopf link problem we throw out at the beginning of the chapter to do the consistency check, by the definition of link complement we gave in the beginning of the chapter,

$$Z_0(\mathrm{L_{Hopf}}) = 1$$

Using the Lens space result we get in the last chapter, the consistency check can be made.

$$\mathscr{L}_{-\frac{1}{r_1}}[(z_1^{r_1} - z_1^{-r_2}) \mathscr{L}_{-\frac{1}{r_2}}(z_2^{r_2} - z_2^{-r_2})] = \hat{Z}_0(L(r_2 - r_1, 1))$$

Notice that under this definition the $\hat{Z}_0$s of links are still independent of n, so we set $z_i^2 = x_i$ as GM did on knots, we can obtain

$$F_K(L; x_1, \cdots, x_n, q) = \hat{Z}_0(L; z_1, \cdots, z_n, q)$$

That is the GM invariant or GPPV series of Links.

Another simple example is the torus link $T_{4,2}$ which has 2 knot as its component, perform a $1/r$ surgery on one knot, one can get a $\frac{2r}{1-2r}$ solid torus which has the following GPPV series

$$F_K(S;x_1,q)=\sum_{m\geq 0} q^{r(m+\frac{1}{2})^2+\frac{(m+1)m}{2}}(-1)^m x_1^m$$

and with a not hard calculation, one can obtain

$$\mathscr{L}_{-\frac{1}{r}}[(x_2^{\frac{1}{2r}}-x_2^{-\frac{1}{2r}})x_2^{\frac{1}{2}}x_1^{\frac{1}{2}}\sum_{m\leq 0}(-1)^m q^{\frac{m(m+1)}{2}}x_1^m x_2^m]=\sum_{m\geq 0} q^{r(m+\frac{1}{2})^2+\frac{(m+1)m}{2}}(-1)^m x_1^m$$

Since the $\frac{2r}{1-2r}$ solid torus is also equivalent to a torus knot $T_{2r+1,2}$'s knot complement, it's not hard to use the GPPV series of such torus knot to check the consistency.

3.2 Quamtum A polynomial of $\mathrm{T}_{4,2}$

As we said in the introduction part, one significant property of GPPV series of knots Gukov has conjectured is that they can be annihilated by quantum A polynomial $\hat{A}$ of corresponding knots, since we've already obtained some GPPV series of links, a nature idea arise that whether GPPV series of links satisfy this property.

The A polynomial can be derived by the recursion relation of an unreduced colored Jones polynomial which can be seen as a degenerate version of refined Chern-Simons invariant. The refined Chern-Simons invariant of Torus link is given in [], which is written as

$$\begin{aligned}
\overline{\mathrm{rCS}}^{(-)}_{[r_1|\cdot|r_2|.}(T_{2,2p};a,q,t) &:= \langle W^{\mathrm{ref}}_{.[1^{r_1}],|1^{r_2}]}(T_{2,2p})\rangle(A=-a^2t^3,q_1=\frac{1}{q^2z^2},q_2=\frac{1}{q^2}) \\
&= (-t^3)^{r_1+r_2}a^{2(r_1+r_2)}q^{2(r_1-1)(r_2-1)-2}\sum_{\ell=0}^{\min(r_1,r_2)}\frac{q^{2\ell(r_1+r_2-\ell+1)+2p\ell-2p(\ell-r_1)(\ell-r_2)}t^{2p\ell}}{(q^{-2};q^{-2})_{r_2-\ell}(q^{-2};q^{-2})_{r_1+r_2-2\ell}} \\
&\times\frac{(q^{-2}t^{-2};q^{-2})_{r_2-\ell}(q^{-2(r_1-\ell+1)};q^{-2})_{r_2-\ell}(-a^{-2}q^2t^{-1};q^{-2})_\ell(-a^{-2}t^{-3};q^{-2})_{r_1+r_2-\ell}}{(q^{-2(r_1-\ell+1)}t^{-2};q^{-2})_{r_2-\ell}(q^{-2(\ell-r_1-r_2-1)}t^2;q^{-2})_\ell(q^{2\ell};q^{-2})_\ell}
\end{aligned}$$

where r_1 and r_2 are the representation, if we set $t=-1,a=q^2,p=2$ this invariant will degenerate into the unreduced colored Jones we want.Here for simplicity, we assume $r_1\geq r_2$, under this assumption, the colored Jones can be written as

$$\mathrm{J}_{\mathrm{r}_1,\mathrm{r}_2}=\frac{1}{1-\mathrm{q}^2}\sum_{1=0}^{\mathrm{r}_2}\mathrm{q}^{-41^2+\mathrm{r}_1+\mathrm{r}_2+1(6+4\mathrm{r}_1+4\mathrm{r}_2)}(1-\mathrm{q}^{2(1-21+\mathrm{r}_1+\mathrm{r}_2)})$$

We notice that the 2 parts of this summation can be considered as the same function with different a independent variable, let $S_{r_1,r_2}=\sum_{l=1}^{r_2}q^{-(-\frac{3}{2}+2l-r_1-r_2)^2}$, the recursive relation of S is not hard to obtain

$$\mathrm{S}_{\mathrm{r}_1,\mathrm{r}_2+2}=\quad\mathrm{S}_{\mathrm{r}_1,\mathrm{r}_2}+\mathrm{q}^{-(\frac{7}{2}+\mathrm{r}_1+\mathrm{r}_2)^2}\quad(1-\mathrm{q}^{2(3+2\mathrm{r}_1)(2+\mathrm{r}_2)})$$

$$S_{r_1,r_2+1}=S_{r_1,r_2}+q^{-(\frac{7}{2}+r_1+r_2)^2}(1-q^{2(5+2r_1)(1+r_2)})$$

So the unreduced colored Jones can be written as

$$\begin{aligned} \mathrm{J}_{\mathrm{r}_1,\mathrm{r}_2} &= \frac{\mathrm{q}^{(\mathrm{r}_1+\mathrm{r}_2+2)^2-\frac{7}{4}}}{1-\mathrm{q}^2}(\mathrm{S}_{\mathrm{r}_1,\mathrm{r}_2} - \mathrm{S}_{\mathrm{r}_1-1,\mathrm{r}_2}) \\ &= \frac{\mathrm{q}^{(\mathrm{r}_1+\mathrm{r}_2+2)^2-\frac{7}{4}}}{1-\mathrm{q}^2}(\mathrm{S}_{\mathrm{r}_1,\mathrm{r}_2} - \sum_{1=0}^{\mathrm{r}_2-1}\mathrm{q}^{-(-\frac{1}{2}+21-\mathrm{r}_1-\mathrm{r}_2)^2} - \mathrm{q}^{-(-\frac{5}{2}+\mathrm{r}_2-\mathrm{r}_1)^2}) \\ &= \frac{\mathrm{q}^{(\mathrm{r}_1+\mathrm{r}_2+2)^2-\frac{7}{4}}}{1-\mathrm{q}^2}(\mathrm{S}_{\mathrm{r}_1,\mathrm{r}_2} - \mathrm{S}_{\mathrm{r}_1,\mathrm{r}_2-1} - \mathrm{q}^{-(-\frac{5}{2}+\mathrm{r}_2-\mathrm{r}_1)^2}) \end{aligned}$$

In that case, we can derive following recursion relation

$$\begin{aligned} J_{r_1,r_2+1} &= \frac{q^{(r_1+r_2+3)^2-\frac{7}{4}}}{1-q^2}(s_{r_1,r_2+1} - s_{r_1,r_2} - q^{-(-\frac{3}{2}+r_2-r_1)^2}) \\ &= \frac{q^{-\frac{7}{4}+(3+r_1+r_2)^2}(S_{r_1,r_2-1} - S_{r_1,r_2} - q^{(\frac{5}{2}+r_1+r_2)^2}(1-q^{2(3+2r_1)(1+r_2)}) - q^{-(-\frac{3}{2}+r_2-r_1)^2})}{1-q^2} \\ &= \frac{1}{1-q^2}q^{-\frac{7}{4}+(3+r_1+r_2)^2}(s_{r_1,r_2-1} - s_{r_1,r_2} + q^{-(\frac{5}{2}+r_1-r_2)^2} - q^{-(\frac{5}{2}+r_1-r_2)^2} + q^{-(\frac{5}{2}+r_1+r_2)^2} \\ &- q^{2(3+2r_1)(1+r_2)-(\frac{5}{2}+r_1+r_2)^2} - q^{-(-\frac{3}{2}+r_2-r_1)^2}) \\ &= -q^{5+2r_1+2r_2}\mathrm{J}_{r_1,r_2} + \frac{q^{1+r_1+r_2}(1-q^{2(3+2r_1)(1+r_2)})}{1-q^2} - \frac{q^{1+11r_2+r_1(1+4r_2)}(1+q^{2(2+r_1-r_2)})}{1-q^2} \\ &= J_{r_1+1,r_2} - \frac{q^{1+11r_2+r_1(1+4r_2)}(1+q^{2(2+r_1-r_2)})}{1-q^2} \end{aligned}$$

Following the same strategy, another recursion relation can be obtained

$$J_{r_1+1,r_2} = J_{r_1+2,r_2} - \frac{q^{1+11r_2-11+(r_1+1)(1+4r_2)}(1+q^{2(2+r_1-r_2)})}{1-q^2}$$

Combine the 2 formulas above, the desired form of recursion relation can be obtained.

$$(J_{r_1+2,r_2} - J_{r_1+1,r_2+1}) = q^{12}q^{4(r_1+r_2)}(J_{r_1+1,r_2-1} - J_{r_1,r_2})$$

Which can be written in the form of $\hat{A}$ polynomial

$$\hat{A}J_{r_1,r_2} = (\hat{y}_1^2 - \hat{y}_1\hat{y}_2 - q^{12}\hat{x}_1^2\hat{x}_2^2\hat{y}_2^{-1} + q^{12}\hat{x}_1\hat{x}_2)J_{r_1,r_2} = 0$$

where

$$\hat{y}_1 J_{r_1,r_2} = J_{r_1+1,r_2}, \hat{y}_2 J_{r_1,r_2} = J_{r_1,r_2+1}, \hat{x}_1 J_{r_1,r_2} = q^{r_1}J_{r_1,r_2}, \hat{x}_2 J_{r_1,r_2} = q^{r_2}J_{r_1,r_2}$$

In the case of GPPV series, we imitate the definition of operators in knot GPPV series, we give

$$\hat{y}_1 F_k(x,y,q) = F_k(xq,y,q), \hat{y}_2 F_k(x,y,q) = F_k(x,yq,q), \hat{y}_2^{-1}F_k(x,yq^{-1},q), \hat{x}_i F_K = x_i F_K$$

It will not be hard to verify the following conclusion

$$\widehat{A}F_K(T_{4,2}) = 0$$

So we can make a conjecture as following

猜想 3.1 If $\hat{A}$ is the quantum A-polynomial[Gar04] annihilating the unreduced colored Jones polynomials of links, the

$$\hat{A}F_k(x_1,\cdots,x_l,q) = 0$$

3.3 总结与展望

So far we have obtained homological blocks of several closed manifold and knot complement, and we followed the path that GM has opened, extended the definition of homological blocks form knots to links, and formulated the surgery formula on links, by using such formula, the GPPV series of Hopf link and Torus link $T_{4,2}$, and fortunately, the relation between quantum $\hat{A}$ polynomial and GPPV series seems still hold in the link situation. We believe the surgery formula can make us one step closer to the mathematical definition of GPPV series and homological block.

参考文献

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