# Homework 1: Due at class on Mar 17

## 1 Open string spectra

An open string has boundaries so that one needs to impose a boundary condition. There are two boundary conditions one can impose:

- **Neumann boundary condition**: $\partial_\sigma X^\mu = 0 \quad \text{at } \sigma = 0, \pi$
- **Dirichlet boundary condition**: $X^\mu = c^\mu$ (constant) at $\sigma = 0, \pi$

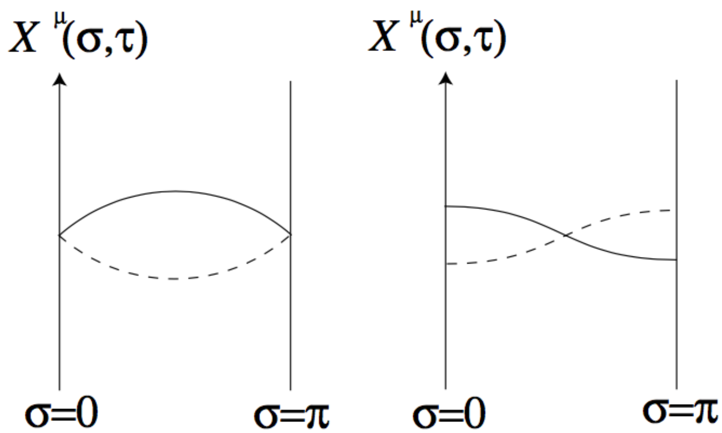


Figure 1: Dirichlet (left) and Neumann (right) boundary conditions

Like the close string, we take the mode expansion for the open string $X^\mu = X_L^\mu(\sigma^+) + X_R^\mu(\sigma^-)$ by

$$\begin{aligned} X_L^\mu(\sigma^+) &= \frac{1}{2}x^\mu + \alpha' p^\mu \sigma^+ + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n} \overline{\alpha}_n^\mu e^{-in\sigma^+} , \\ X_R^\mu(\sigma^-) &= \frac{1}{2}x^\mu + \alpha' p^\mu \sigma^- + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n} \alpha_n^\mu e^{-in\sigma^-} . \end{aligned} \tag{1.1}$$

Note that the second term differs from the closed string by factor of 2. Show that the boundary conditions impose the following requirements

- Neumann boundary condition requires $\alpha_n^a = \overline{\alpha}_n^a$.
- Dirichlet boundary condition requires $x^I = c^I, \quad p^I = 0, \quad \alpha_n^I = -\overline{\alpha}_n^I$.

Actually, this is an essence of the previous probelm.

Now let us study open string mass spectrum in the quantum theory. In the case of open strings, we can define the momentum $\alpha_0^\mu = \sqrt{2\alpha'}\, p^\mu$. Show that the light-cone gauge quantization for (1.1) gives

$$2\alpha_n^- = \sqrt{\frac{1}{2\alpha'}} \frac{1}{p^+} \sum_{m=-\infty}^{\infty} \sum_{i=1}^{D-2} \alpha_{n-m}^i \alpha_m^i .$$

Check that $n = 0$ can be read off

$$M^2 = 2p^+p^- - \sum_{i=1}^{D-2} p^i p^i = \frac{1}{\alpha'} \left( \sum_{n>0} \sum_{i=1}^{D-2} \alpha^i_{-n} \alpha^i_n + \frac{D-2}{2} \sum_{n>0} n \right) .$$

## 2 Vertex operator and OPE

1. Let us define $T(z) = -\frac{1}{\alpha'} :\partial X^\mu \partial X_\mu:$, and $j(z) = a_\mu \partial X^\mu$ in the free scalar theory. Compute the OPE $j(z)X^\nu(w)$, and $T(z)T(w)$.

2. Show that $:e^{ikX}:$ is a primary field of weight $h = \overline{h} = \alpha' k^2/4$ in the free scalar theory. In addition, show that $\partial^n X$ ($n \geq 2$) is not a primary field.