Homework 2: Due at class on March 17

1. For elements $x = (x_0, \cdots, x_n)$ and $y = (y_0, \cdots, y_n)$ of $\mathbb{R}^{n+1}\backslash 0$, we define an equivalence relation $x \sim y$ by

$$x = \alpha y \qquad \alpha \in \mathbb{R} \ .$$

Let us define $\mathbb{R}P^n$ by $\mathbb{R}^{n+1}\backslash 0/\sim$. Show that $\mathbb{R}P^n$ is a manifold and it is orientable if and only if n is odd. The space is called a real projective space.

2. Let (x, y) be the Cartesian coordinate of $\mathbb{R}^2$ and (r, θ) be the polar coordinate of $\mathbb{R}^2$. Write a vector filed v in terms of the Cartesian coordinate that generate a flow $\varphi_t : \mathbb{R}^2 \to \mathbb{R}^2$

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \ .$$

This is the rotation in $\mathbb{R}^2$. In addition, draw the schematic picture of the vector field v.

3. Let $M_n(\mathbb{R})$ and $M_n(\mathbb{C})$ denote the set of all $n \times n$ matrices over $\mathbb{R}$ and $\mathbb{C}$, respectively. We define

$$\begin{aligned} SU(2) &= \{A \in M_2(\mathbb{C}) |\ A^\dagger A = \mathrm{Id},\ \det A = 1\} \\ SO(3) &= \{A \in M_3(\mathbb{R}) |\ A^T A = \mathrm{Id},\ \det A = 1\} \ . \end{aligned}$$

3.1 Construct a double covering (2-to-1) map $SU(2) \to SO(3)$.
3.2 Show that $SU(2)$ is diffeomorphic to S^3 and $SO(3)$ is diffeomorphic to $\mathbb{R}P^3$.

4. Let e be the identity element of $SO(3)$. Show that the tangent space $T_e SO(3)$ at e is spanned by tangent vectors of curves in $SO(3)$

$$\exp(tJ_i) = 1 + tJ_i + \frac{1}{2}(tJ_i)^2 + \cdots$$

at $t = 0$ where J_i $(i = x, y, z)$ are defined by

$$J_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \qquad J_x = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \qquad J_x = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ .$$

Let us define the commutator by $[X, Y] = XY - YX$. Then, show that

$$[J_x, J_y] = J_z \ , \qquad [J_y, J_z] = J_x \ , \qquad [J_z, J_x] = J_y \ .$$

5. Show that the tangent space $T_e SU(2)$ is spanned by $i\sigma_x$, $i\sigma_y$ and $i\sigma_z$ (the Pauli matrices by i).

6. Write down vector fields that generate the rotation along x-, y-, z-axis in $\mathbb{R}^3$. Find the commutation relations of these vector fields. Compare the theory of angular momenta in quantum mechanics.