Lecture 3

Conformal field theories (CFTs) play distinctive role in quantum field theories, string theory, statistical mechanics, and condensed matter physics. Moreover, 2d CFTs are particularly rich because of infinite dimensional symmetry [BPZ84]. In the previous lecture, we have studied an important property of conformal field theories, the state-operator correspondence. In this lecture, we will learn significant feature of 2d conformal field theories more in detail. However, we glimpse only a tip of iceberg and the subject would actually deserve the entire one semester. If you are interested in this fertile subject, we refer to the standard references [Gin88, FMS12, BP09].

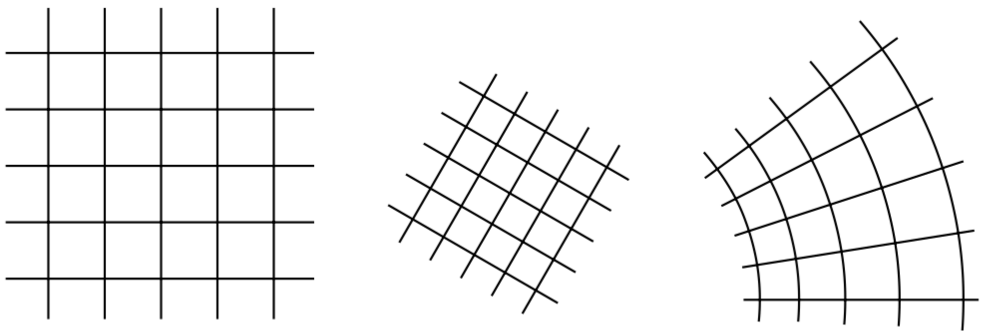
1 Conformal transformations

A CFT in any dimension is a quantum field theory which is invariant under conformal transformations at quantum level. A **conformal transformation** is a change of coordinates $x^a \to \widetilde{x}^a(x)$ such that the metric changes by

$$g_{ab}(x) \to \Omega^2(\sigma) g_{ab}(x) \ . \tag{1.1}$$

This means that the theory behaves the same at all length scales.

In the string sigma model action, the metric is dynamical and the conformal transformations (1.1) consist of a subgroup of diffeomorphisms so that it can be canceled by a Weyl symmetry of the metric. However, in the following, we fix a 2d Euclidian flat metric and the transformation (1.1) should be thought of as physical symmetry, taking the point x^α to point $\widetilde{x}^\alpha$.



2d flat space

In complex coordinate, the 2d flat metric can be written as $ds^2 = dzd\overline{z}$. Under a holomorphic map,

$$z \to f(z) \ ,$$

the metric is transformed as

$$ds^2 = dzd\overline{z} \quad \to \quad ds^2 = \frac{\partial f}{\partial z}\frac{\partial \overline{f}}{\partial \overline{z}} dzd\overline{z} \ .$$

Therefore, all the holomorphic maps are conformal transformations where $\left|\frac{\partial f}{\partial z}\right|^2$ is a conformal factor (corresponding to $\Omega^2(\sigma)$ in (1.1)). Moreover, it is easy to show that all 2d conformal transformations are indeed holomorphic functions (Exercise). This set is infinite-dimensional, corresponding to the the coefficients of the Laurent series of holomorphic functions in some neighborhood. This infinity is what makes conformal symmetry so powerful in two dimensions.

2 Conformal Ward-Takahashi identity

Recall that in a field theory, continuous symmetries correspond to conserved currents. Hence, let us consider the Noether current for conformal symmetry.

For an infinitesimal conformal transformation $x^\mu \to x^\mu + \epsilon^\mu(x)$, the metric is transformed as

$$\eta_{\mu\nu} \to \eta_{\mu\nu} + \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu \ .$$

Since this is conformal transformation, it is proportional to $\eta_{\mu\nu}$ so that we have

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = (\partial_\rho \epsilon^\rho)\eta_{\mu\nu} \ .$$

The current for the conformal transformation can be written as

$$j_\mu = T_{\mu\nu}\epsilon^\nu \ ,$$

where the straightforward calculation provides the stress-energy tensor

$$T_{\mu\nu} = -2\pi\Big[\frac{\partial L}{\partial(\partial^\mu \phi)}\partial_\nu \phi - \delta_{\mu\nu} L\Big] \ . \tag{2.1}$$

If we assume ϵ is constant, it is easy to see that the the conservation of the current implies the conservation of the stress-energy tensor:

$$\partial_\mu j^\mu = 0 \quad \to \quad \partial^\mu T_{\mu\nu} = 0 \ . \tag{2.2}$$

For general $\epsilon^\mu(x)$, the conservation of the current gives the traceless condition of $T_{\mu\nu}$:

$$0 = \partial^\mu j_\mu = \frac{1}{2}T_{\mu\nu}(\partial^\mu \epsilon^\nu + \partial^\nu \epsilon^\mu) = \frac{1}{2}T^\mu{}_\mu(\partial_\rho \epsilon^\rho) \quad \to \quad T^\mu{}_\mu = 0 \ . \tag{2.3}$$

In the complex coordinate $z = x^1 + ix^2$, the traceless condition can be written as

$$T_{z\overline{z}} = T_{\overline{z}z} = 0$$

and the conservation of the stress-energy tensor can be written as

$$\partial_{\overline{z}} T_{zz} = 0 \ , \qquad \partial_z T_{\overline{z}\overline{z}} = 0 \ .$$

Thus, the non-vanishing components of the stress-energy tensor factorize to a chiral and anti-chiral field,

$$T(z) := T_{zz} \quad \text{and} \quad \overline{T}(\overline{z}) := T_{\overline{z}\overline{z}} \; .$$

As a result, the Noether currents for conformal transformations $z \to z + \epsilon(z)$ and $\overline{z} \to \overline{z} + \overline{\epsilon}(\overline{z})$ are

$$j(z) = \epsilon(z) T(z) \; , \quad \overline{j}(\overline{z}) = \overline{\epsilon}(\overline{z}) \overline{T}(\overline{z}) \; .$$

The application to the Ward-Takahashi identity leads to **conformal Ward-Takahashi identity**

$$\delta_{\epsilon.\overline{\epsilon}} \mathcal{O}(w, \overline{w}) = \frac{1}{2\pi i} \oint_{C_w} dz \; \epsilon(z) T(z) \mathcal{O}(w, \overline{w}) + \frac{1}{2\pi i} \oint_{C_{\overline{w}}} d\overline{z} \; \overline{\epsilon}(\overline{z}) \overline{T}(\overline{z}) \mathcal{O}(w, \overline{w}) \; , \quad (2.4)$$

where the contour integral is taken as a counter-clockwise circle both in z and in $\overline{z}$ (thereby explaining the sign difference of the second term).

3 Primary fields

Definition 3.1. *Fields depending only on z, i.e. $\phi(z)$, are called **chiral or holomorphic fields** and fields $\overline{\phi}(\overline{z})$ only depending on $\overline{z}$ are called **anti-chiral or anti-holomorphic fields**.*

Definition 3.2. *If a field ϕ that transforms under the scaling $z \to \lambda z$ as*

$$\phi(z, \overline{z}) \to \phi'(\lambda z, \overline{\lambda} \overline{z}) = \lambda^{-h} \overline{\lambda}^{-\overline{h}} \phi(z, \overline{z}) \; , \quad (3.1)$$

*it has **weight** $(h, \overline{h})$. Using these quantities (rather than the scaling dimension), we can define quasi-primary fields.*

Definition 3.3. *Under a conformal transformation $z \to f(z)$, a field transforming according to the rule*

$$\phi(z, \overline{z}) \to \phi'(f(z), \overline{f}(\overline{z})) = \left(\frac{\partial f}{\partial z} \right)^{-h} \left(\frac{\partial \overline{f}}{\partial \overline{z}} \right)^{-\overline{h}} \phi(z, \overline{z}) \quad (3.2)$$

*it is called a **primary field** of weight $(h, \overline{h})$.*

How do primary fields transform infinitesimally? Under the infinitesimal conformal transformation $z \to f(z) = z - \epsilon(z)$, we know that

$$\left(\frac{\partial f}{\partial z} \right)^{-h} = 1 + h \partial_z \epsilon(z) + O(\epsilon^2) \; ,$$
$$\phi(z - \epsilon(z), \overline{z}) = \phi(z) - \epsilon(z) \partial_z \phi(z, \overline{z}) + O(\epsilon^2) \; . \quad (3.3)$$

Hence, under an infinitesimal conformal transformation, the variation of a primary field is given by

$$\delta_\epsilon \phi(z,\overline{z}) = \left(h\partial_z\epsilon + \epsilon\partial_z + \overline{h}\partial_{\overline{z}}\overline{\epsilon} + \overline{\epsilon}\partial_{\overline{z}}\right)\phi(z,\overline{z}). \tag{3.4}$$

Consequently, using simple complex analysis

$$\begin{aligned}(\partial_w\epsilon(w))\phi(w,\overline{w}) &= \frac{1}{2\pi i}\oint_{C_w} dz\ \frac{\epsilon(z)\phi(w,\overline{w})}{(z-w)^2} \\ \epsilon(w)(\partial_w\phi(w,\overline{w})) &= \frac{1}{2\pi i}\oint_{C_w} dz\ \frac{\epsilon(z)\partial_w\phi(w,\overline{w})}{z-w}\ ,\end{aligned} \tag{3.5}$$

one can read off the OPE of a primary operator ϕ of weight $(h,\tilde{h})$ with the stress-energy tensor T (anti-chiral part $\overline{T}$ can be obtained by complex conjugate)

$$T(z)\,\phi(w,\overline{w}) = h\frac{\phi(w,\overline{w})}{(z-w)^2} + \frac{\partial_w\phi(w,\overline{w})}{z-w} + \text{regular terms}\cdots$$

In general, the OPE of an operator $\mathcal{O}$ of weight $(h,\tilde{h})$ with the stress-energy tensor T and $\overline{T}$ takes the form

$$T(z)\,\mathcal{O}(w,\overline{w}) = \quad \cdots + h\frac{\mathcal{O}(w,\overline{w})}{(z-w)^2} + \frac{\partial\mathcal{O}(w,\overline{w})}{z-w} + \cdots$$

One of the main interest in a CFT is to calculate correlation functions of primary fields. Indeed, the conformal Ward-Takahashi identity can be applied to a correlation function of primary fields

$$\langle T(z)\phi_1(w_1,\overline{w}_1)\cdots\phi_n(w_n,\overline{w}_n)\rangle = \sum_{i=1}^n\left(\frac{h_i}{(z-w_i)^2}+\frac{\partial_{w_i}}{z-w_i}\right)\langle\phi_1(w_1,\overline{w}_1)\cdots\phi_n(w_n,\overline{w}_n)\rangle\ .$$

Moreover, conformal symmetry is so powerful that it determines the forms of two-point and three-point functions of primary fields (Exercise).

- **2-point function**

For chiral primary operators ϕ_i with weight h_i $(i=1,2)$, their 2-point function is of form

$$\langle\phi_1(z_1)\phi_2(z_2)\rangle = \delta_{h_1h_2}\frac{d_{12}}{(z_1-z_2)^{2h_1}} \tag{3.6}$$

If d_{12} is non-degenerate, the fields can be normalized such that $d_{12}=\delta_{12}$.

- **3-point function**

A 3-point function is also completely fixed up to the appearance of a **structure constant** C_{ijk},

$$\langle\phi_1(z_1)\phi_2(z_2)\phi_2(z_3)\rangle = \frac{C_{123}}{(z_1-z_2)^{h_1+h_2-h_3}(z_2-z_3)^{h_2+h_3-h_1}(z_3-z_1)^{h_3+h_1-h_2}}$$

The structure constant depends on a CFT and, in general, it is not easy to determine it.

• **Multi-point function**

The computation of multi-point functions involves **conformal blocks** with the 3-point function. The details are explained in [FMS12, BP09].

4 Free scalar field

Now let us study conformal Ward-Takahashi identity in the simplest example, the free scalar field:

$$S = \frac{1}{2\pi\alpha'} \int d^2z \, (\partial X \overline{\partial} X) \ .$$

Let us recall that the stress-energy tensor in 2d free scalar theory is

$$T_{ab} = -\frac{1}{\alpha'} \left(\partial_a X \partial_b X - \frac{1}{2} g_{ab} (\partial X)^2 \right) \ , \tag{4.1}$$

In addition, since the equation of motion for X is $\partial_z \overline{\partial}_{\overline{z}} X = 0$, the general classical solution decomposes as $X(z, \overline{z}) = X(z) + \overline{X}(\overline{z})$. From the equation of motion, we find the conserved chiral and anti- chiral worldsheet currents $j(z) := i\partial X(z)$ and $\overline{j}(\overline{z}) := i\overline{\partial X}(\overline{z})$. Moreover, the stress-energy tensor looks much simpler in complex coordinates. It is simple to check that $T_{z\overline{z}} = 0$ while

$$T(z) = -\frac{1}{\alpha'} \partial_z X(z) \partial_z X(z) \ , \quad \overline{T}(\overline{z}) = -\frac{1}{\alpha'} \overline{\partial}_{\overline{z}} \overline{X}(z) \overline{\partial}_{\overline{z}} \overline{X}(\overline{z}) \ .$$

From the definition (3.2), one can see that $X(z, \overline{z})$ is a primary field of weight (0,0). However, since the weight is of (0,0), the two-point function does not exactly take the form (3.6). Indeed, the OPE XX tells us that the propagator takes the form

$$\langle X(z, \overline{z}) X(w, \overline{w}) \rangle = -\frac{\alpha'}{2} \log |z - w|^2 \ .$$

Also, the currents $\partial X(z)$, $\overline{\partial X}(\overline{z})$ are primary fields of weight (1,0) and (0,1) respectively. An immediate check is their correlation function

$$\langle \partial X(z) \partial X(w) \rangle = -\frac{\alpha'}{2} \frac{1}{(z - w)^2} \ ,$$

which takes the form (3.6). To convince ourselves completely, we need to compute the OPE with the stress-energy tensor by Wick's theorem

$$\begin{aligned} T(z) \, \partial X(w) &= -\frac{1}{\alpha'} : \partial X(z) \partial X(z) : \partial X(w) \\ &= -\frac{1}{\alpha'} \Big[: \partial X(z) \partial X(z) \partial X(w) : + \partial X(z) \langle \partial X(z) \partial X(w) \rangle \Big] \\ &= \frac{\partial X(w)}{(z - w)^2} + \frac{\partial^2 X(w)}{z - w} + \text{regular terms} \cdots \end{aligned} \tag{4.2}$$

This is indeed the OPE for a primary operator of weight $h = 1$.

Finally, lets check to see the OPE of the stress-energy tensor TT. This is again just an exercise in Wick contractions.

$$\begin{aligned} T(z)\,T(w) &= \frac{1}{\alpha'^2} \;:\partial X(z)\,\partial X(z):\,:\partial X(w)\,\partial X(w): \\ &= \frac{2}{\alpha'^2}\left(-\frac{\alpha'}{2}\frac{1}{(z-w)^2}\right)^2 - \frac{4}{\alpha'^2}\frac{\alpha'}{2}\frac{:\partial X(z)\,\partial X(w):}{(z-w)^2} + \dots \end{aligned} \tag{4.3}$$

The factor of 2 in front of the first term comes from the two ways of performing two contractions; the factor of 4 in the second term comes from the number of ways of performing a single contraction. Continuing,

$$\begin{aligned} T(z)\,T(w) &= \frac{1/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} - \frac{2}{\alpha'}\frac{\partial^2 X(w)\,\partial X(w)}{z-w} + \dots \\ &= \frac{1/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \dots \end{aligned} \tag{4.4}$$

We learn that T is **not** a primary operator in the theory of a single free scalar field. It is an operator of weight $(h, \overline{h}) = (2, 0)$, but it fails the primary test on account of the $(z-w)^{-4}$ term. In fact, this property of the stress-energy tensor is general in all CFTs which we now explore in more detail.

5 OPE of stress-energy tensor

For the free scalar field, we have already seen that T has weight $(h, \tilde{h}) = (2, 0)$. This remains true in any CFT. The reason for this is simple: T_{ab} has dimension $\Delta = 2$ because we obtain the energy by integrating over space. It has spin $s = 2$ because it is a symmetric 2-tensor. But these two pieces of information are equivalent to the statement that T is an operator of weight $(2, 0)$. This means that the TT OPE takes the form,

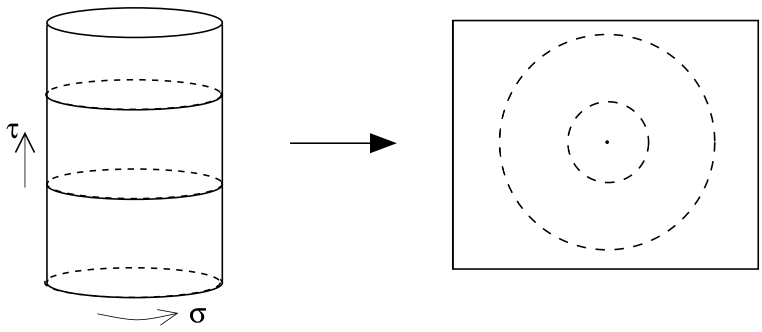
$$T(z)\,T(w) = \dots + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \dots$$

and similar for $\overline{T}\overline{T}$. What other terms could we have in this expansion? Since each term has dimension $\Delta = 4$, the unitarity indeed tells us that the singular part of the OPE takes

$$T(z)\,T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \dots$$

From the OPE of the stress-energy tensor, one can see its variation under an infinitesimal conformal transformation $z \to z - \epsilon(z)$

$$\begin{aligned} \delta_\epsilon T(w) &= \frac{1}{2\pi i}\oint_{C_w} dz\; \epsilon(z) T(z) T(w) \\ &= \epsilon(w)\partial T(w) + 2\epsilon'(w) T(w) + \frac{c}{12}\epsilon'''(w) \end{aligned} \tag{5.1}$$



One can verify by a straightforward computation that this is the infinitesimal version of the following transformation under finite transformation $z \to w(z)$:

$$T'(w) = \left(\frac{\partial w}{\partial z}\right)^{-2} \left[T(z) - \frac{c}{12} S\left(w, z\right)\right], \tag{5.2}$$

where the **Schwarzian derivative** S is defined as

$$S(w, z) := \frac{1}{(\partial_z w)^2} \left((\partial_z w)(\partial_z^3 w) - \frac{3}{2}(\partial_z^2 w)^2\right). \tag{5.3}$$

Using this conformal transformation law, one can see that the mapping from the plane to the cylinder $z = e^{-iw}$ $(w = \sigma + it)$ leads to

$$T_{cyl}(w) = -z^2 T(z) + \frac{c}{24}. \tag{5.4}$$

The Laurent mode expansion of the stress-energy tensor on the cylinder is therefore

$$T_{cyl}(w) = -\sum_{n \in \mathbb{Z}} \left(L_n - \frac{c}{24}\delta_{n,0}\right) e^{inw}. \tag{5.5}$$

We want to look at the Hamiltonian, which is defined by

$$H \equiv \int d\sigma \; T_{\tau\tau} = -\int d\sigma \left(T_{ww} + \overline{T}_{\overline{ww}}\right)$$

The conformal transformation then tells us that the ground state energy on the cylinder is

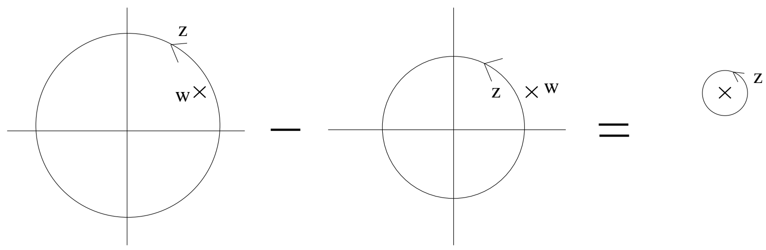
$$E = -\frac{c + \overline{c}}{24}$$

This is indeed the (negative) Casimir energy on a cylinder. For a free scalar field, we have $c = \overline{c} = 1$ and the energy density $E = -1/12$. This is what we have seen in the quantization of bosonic string theory!

Virasoro algebra

The mode expansion of the stress-energy tensor is expressed as

$$T(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$$



It is natural to find the commutation relation of the generators L_m. The commutator can be computed by

$$[L_m, L_n] = \left(\oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} - \oint \frac{dw}{2\pi i} \oint \frac{dz}{2\pi i} \right) z^{m+1} w^{n+1} T(z) T(w)$$

A clever manipulation of the contour makes life easier

$$\begin{aligned} [L_m, L_n] &= \oint \frac{dw}{2\pi i} \oint_w \frac{dz}{2\pi i} z^{m+1} w^{n+1} T(z) T(w) \\ &= \oint \frac{dw}{2\pi i} \operatorname{Res} \left[z^{m+1} w^{n+1} \left(\frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \ldots \right) \right] \end{aligned} \tag{5.6}$$

A simple computation (Exercise) leads to the **Virasoro algebra**

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{c}{12} m(m^2-1) \delta_{m+n,0}$$

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