Homework 11: Due at class on May 21

1. Suppose that we have a connection ∇ on a rank-r vector bundle $\pi : E \to M$. Suppose that we have local trivializations of E over two open sets U_α and U_β in M. On these open sets, the connection can be written in terms of one-form A_α and A_β taking value on $\mathfrak{gl}(r, \mathbb{R})$. Given a transition function $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(r, \mathbb{R})$, show that the connections are related by

$$A_\beta = g_{\alpha\beta}^{-1} A_\alpha g_{\alpha\beta} + g_{\alpha\beta}^{-1} d(g_{\alpha\beta}) ,$$

and the curvature forms are related by

$$F_\beta = g_{\alpha\beta}^{-1} F_\alpha g_{\alpha\beta}$$

on $U_\alpha \cap U_\beta$.

2. Prove that the Bianchi identity

$$dF + A \wedge F - F \wedge A = 0 .$$

3. Show that $\mathrm{U}(n)$ is a direct product $\mathrm{SU}(n) \times \mathrm{U}(1)$ as a manifold, but not as a group.

4. **Hopf fibration and Dirac monopole**

We define $S^{2n+1} = \{(z_0, \cdots, z_n) \in \mathbb{C}^{n+1} | \sum_{i=0}^{n} |z_i|^2 = 1\}$ and let the complex projective space $\mathbb{C}P^n$ be the quotient space $S^{2n+1}/\sim$ where we define an equivalent relation $(z_0, \cdots, z_n) \sim \lambda(z_0, \cdots, z_n)$ with $\lambda \in \mathrm{U}(1)$. We denote a point of $\mathbb{C}P^n$ by $[z_0; \cdots ; z_n]$.

When $n = 1$, we can consider that S^3 is the principal $\mathrm{U}(1)$-bundle over $\mathbb{C}P^1$

$$\begin{array}{ccc} S^1 & \longrightarrow & S^3 \\ & & \downarrow{\scriptstyle \pi} \\ & & \mathbb{C}P^1 \end{array} .$$

First of all, show that $\mathbb{C}P^1$ can be identified with $S^2 \subset \mathbb{R}^3$ by a map

$$[z_0; z_1] \mapsto (z_0\bar{z}_1 + \bar{z}_0 z_1, -i(z_0\bar{z}_1 - \bar{z}_0 z_1), |z_0|^2 - |z_1|^2) .$$

Let us define trivializations of this bundle over $U_N = S^2 \backslash \{\text{north pole}\}$ and $U_S = S^2 \backslash \{\text{south pole}\}$ as

$$\pi^{-1}(U_N) \to U_N \times S^1; (z_0; z_1) \mapsto \left([z_0; z_1], \frac{z_0}{|z_0|} \right)$$
$$\pi^{-1}(U_S) \to U_S \times S^1; (z_0; z_1) \mapsto \left([z_0; z_1], \frac{z_1}{|z_1|} \right) .$$

Find the transition function between these trivializations. Let us consider embedding

$$f_N : U_N \hookrightarrow \pi^{-1}(U_N) : (\theta, \phi) \mapsto \left(\cos\frac{\theta}{2}, \sin\frac{\theta}{2} e^{-i\phi} \right)$$
$$f_S : U_S \hookrightarrow \pi^{-1}(U_S) : (\theta, \phi) \mapsto \left(\cos\frac{\theta}{2} e^{i\phi}, \sin\frac{\theta}{2} \right) ,$$

where (θ, ϕ) is the standard polar coordinate of S^2. Defining a connection

$$A = i\,\mathrm{Im}(\bar{z}_0 dz_0 + \bar{z}_1 dz_1)$$

on $S^3 \subset \mathbb{C}^2$, show that

$$\begin{aligned} f_N^* A &= -\frac{i}{2}(1 - \cos\theta)d\phi \quad \text{on} \quad U_N \\ f_S^* A &= \frac{i}{2}(1 + \cos\theta)d\phi \qquad \text{on} \quad U_S \ . \end{aligned}$$

Show that they are related by gauge transformations and its curvature F is well-defined on S^2. This connection provides **Dirac magnetic monopole**

$$\frac{i}{2\pi}\int_{S^2} F = 1 \ .$$

5. Let us consider the $\mathrm{SO}(2)$ subgroup of $\mathrm{SO}(3)$ as

$$\mathrm{SO}(3) \supset \mathrm{SO}(2) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \ .$$

Show that $S^2 \cong \mathrm{SO}(3)/\mathrm{SO}(2)$ and $\mathrm{SO}(3)$ can be considered as S^1 bundle over S^2. Construct local trivializations of this bundle over $U_N = S^2 \backslash \{\text{north pole}\}$ and $U_S = S^2 \backslash \{\text{south pole}\}$, and find the corresponding transition function at the equator. Construct a $\mathrm{U}(1)$-connection on this bundle and find its curvature.