

# Homework 6 (Due at class on April 23)

## 1 Partition functions

### 1.1 $S^1$ partition function of bosons

Consider two harmonic oscillators on  $S^1$  (time direction is compactified and its period is  $\beta$ ) with the Hamiltonian

$$H = a^\dagger a + b^\dagger b . \quad (1.1)$$

Their commutation relations are

$$[a, a^\dagger] = 1 , \quad [b, b^\dagger] = 2 , \quad (1.2)$$

otherwise zero. Derive the partition function of this theory:

$$Z = \text{tr} \left[ e^{-\beta H} \right] \quad (1.3)$$

### 1.2 $S^1$ partition function of fermions

Consider two fermions on  $S^1$  with Hamiltonian

$$H = d^\dagger d + 2f^\dagger f . \quad (1.4)$$

Non-trivial anti-commutation relations are

$$\{d, d^\dagger\} = 1 , \quad \{f, f^\dagger\} = 1 . \quad (1.5)$$

Derive the partition function of this theory with a periodic boundary condition

$$Z = \text{tr} \left[ (-1)^F e^{-\beta H} \right] , \quad (1.6)$$

as well as an anti-periodic boundary condition

$$Z = \text{tr} \left[ e^{-\beta H} \right] . \quad (1.7)$$

Note that

$$\{(-1)^F, d\} = 0, \quad \{(-1)^F, d^\dagger\} = 0, \quad \{(-1)^F, f\} = 0, \quad \{(-1)^F, f^\dagger\} = 0 , \quad (1.8)$$

and assume

$$(-1)^F |0\rangle = -|0\rangle . \quad (1.9)$$

### 1.3 Torus partition function of $X$

Derive the torus partition function of the matter sector of the bosonic string  $\langle 1 \rangle_X$ .

## 1.4 Torus partition function of ghosts

Derive the torus partition function of holomorphic ghost sector of the bosonic string

$$\text{tr} \left[ (-1)^F b_0 c_0 q^{L_0 - \frac{c_{gh}}{24}} \right] , \quad (1.10)$$

where  $c_{gh} = -26$ , and

$$L_0 = \left( \sum_{n \in \mathbb{Z}} n :b_{-n} c_n: \right) - 1 . \quad (1.11)$$

The non-trivial anti-commutation relations are

$$\{c_m, b_n\} = \delta_{m+n,0} . \quad (1.12)$$

Note that  $c_m$  ( $m \leq 0$ ) and  $b_n$  ( $n < 0$ ) are creation op, and  $c_m$  ( $m > 0$ ) and  $b_n$  ( $n \geq 0$ ) are annihilation op.

## 1.5 Torus partition function of the bosonic string

- Derive the number of the second excited states of the bosonic string from an expansion of  $|\eta(\tau)|^{-48}$ .
- The modular transformations of the  $\tau$ -function are given by

$$T: \quad \eta(\tau + 1) = e^{i\pi/12} \eta(\tau) , \quad (1.13)$$

$$S: \quad \eta(-1/\tau) = \sqrt{-i\tau} \eta(\tau) . \quad (1.14)$$

Show the torus partition function of the bosonic string is modular invariant.

## 2 Wold-sheet supersymmetry

### 2.1 Supersymmetric transformation

Show that the action

$$S^m = \frac{1}{4\pi} \int d^2z \left( \frac{2}{\alpha'} \partial X \bar{\partial} X + \psi^\mu \bar{\partial} \psi_\mu + \bar{\psi}^\mu \partial \bar{\psi}_\mu \right) \quad (2.1)$$

is invariant under the supersymmetric transformation

$$\delta_{\epsilon, \bar{\epsilon}} X^\mu = -\sqrt{\frac{\alpha'}{2}} (\epsilon \psi^\mu + \bar{\epsilon} \bar{\psi}^\mu) , \quad \delta_\epsilon \psi^\mu = \sqrt{\frac{2}{\alpha'}} \epsilon \partial X^\mu , \quad \delta_{\bar{\epsilon}} \bar{\psi}^\mu = \sqrt{\frac{2}{\alpha'}} \bar{\epsilon} \bar{\partial} X^\mu .$$

Also, show that the combination of supersymmetric transformations provides derivatives

$$[\delta_{\epsilon_1, \bar{\epsilon}_1}, \delta_{\epsilon_2, \bar{\epsilon}_2}] \mathcal{O} = 2\epsilon_1 \epsilon_2 \partial \mathcal{O} + 2\bar{\epsilon}_1 \bar{\epsilon}_2 \bar{\partial} \mathcal{O} .$$

## 2.2 Superspace formalism

This theory can be formulated in terms of superspace  $(z, \bar{z}, \theta, \bar{\theta})$  where  $(\theta, \bar{\theta})$  are anti-commuting Grassmann coordinates. We can introduce the superfield

$$Y^\mu(z, \bar{z}, \theta, \bar{\theta}) = X^\mu(z, \bar{z}) + i\theta\psi^\mu(z, \bar{z}) + i\bar{\theta}\bar{\psi}^\mu(z, \bar{z}) + \frac{1}{2}\bar{\theta}\theta F^\mu(z, \bar{z}),$$

as well as the superderivative

$$D = \frac{\partial}{\partial\theta} + \theta\partial_z, \quad \bar{D} = \frac{\partial}{\partial\bar{\theta}} + \bar{\theta}\partial_{\bar{z}}.$$

The field  $F^\mu$  is called an **auxiliary field**. Then, show that the action (2.1) at  $\alpha' = 2$  is equivalent to

$$S = \frac{1}{4\pi} \int d^2z d^2\theta \bar{D}Y^\mu D Y_\mu,$$

where the superspace integral is defined as  $d^2\theta = d\theta d\bar{\theta}$ , and

$$\int d\theta d\bar{\theta} \bar{\theta}\theta = 1.$$

## 2.3 Supersymmetric ghost

Let us define the ghost superfields as

$$B = \beta + \theta b, \quad C = c + \theta\gamma.$$

Show that the supersymmetric ghost action can be written as

$$S^{\text{gh}} = \frac{1}{2\pi} \int d^2z d^2\theta B \bar{D}C.$$

## 3 $\mathcal{N} = 1$ superconformal algebra

In general, a 2d superconformal field theory has the following OPEs for the stress-energy tensor  $T(z)$  and the supercurrent  $G(z)$

$$\begin{aligned} T(z)T(w) &\sim \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial_w T(w) \\ T(z)G(w) &\sim \frac{3/2}{(z-w)^2}G(w) + \frac{1}{z-w}\partial_w G(w) \\ G(z)G(w) &\sim \frac{2c/3}{(z-w)^3} + \frac{2}{z-w}T(w). \end{aligned} \tag{3.1}$$

As in the lecture, we carry out the mode expansion

$$T_B(z) = \sum_{m \in \mathbb{Z}} \frac{L_m}{z^{m+2}}, \quad G(z) = \sum_{r \in \mathbb{Z}+\nu} \frac{G_r}{z^{r+3/2}}$$

where  $\nu = 0$  and  $\nu = \frac{1}{2}$  correspond to Ramond sector and Neveu-Schwarz sector, respectively. From the OPEs (3.1), derive  $\mathcal{N} = 1$  superconformal algebra

$$\begin{aligned}[L_m, L_n] &= (m - n)L_{m+n} + \frac{c}{12}m(m^2 - 1)\delta_{m+n,0} \\ [L_m, G_r] &= \left(\frac{1}{2}m - r\right)G_{r+m} \\ \{G_r, G_s\} &= 2L_{r+s} + \frac{c}{3}(r^2 - \frac{1}{4})\delta_{r+s,0}.\end{aligned}$$