Homework 5: Due at class on April 14

1 Ghost number anomaly

1.1 Ghost number current

Derive the conserved current for the transformation

$$\delta_g c = \epsilon_g c \,, \quad \delta_g b = -\epsilon_g b \,. \tag{1.1}$$

Holomorphic part of the ghost action is given by

$$S_{gh} = \frac{1}{2\pi} \int d^2 z b \bar{\partial} c \,. \tag{1.2}$$

(You can forget about the anti-holomorphic part for the problem.)

1.2 OPE of EM tensor and the current

Calculate the OPE between ghost EM tensor and the current derived above. The EM tensor is given as follows.

$$T(z) = - :(2b\partial c + \partial b c)(z): \,. \tag{1.3}$$

Furthermore, write down an infinitesimal conformal transformation (with parameter $\epsilon(z)$) of the ghost number current from the OPE result. (Only holomorphic part is enough.)

1.3 Ghost number anomaly from curved WS

Use the assumption $\nabla^a j_a = \kappa R^{(2)}$ derive the current $j_z = -4\kappa \partial \omega + j(z)$. (You can assume that $j_{\bar{z}} = 0$.) Metric is given by

$$ds^2 = e^{2\omega} dz d\bar{z} \,. \tag{1.4}$$

Conformal transformation laws for j_z and ω are given as follows.

$$\widetilde{j_z} = \left(\frac{\partial \widetilde{z}}{\partial z} \right)^{-1} j_z \,, \tag{1.5}$$

$$\widetilde{\omega} = \omega - \frac{1}{2} \log \left| \frac{\partial \widetilde{z}}{\partial z} \right|^2 \,, \tag{1.6}$$

where $\widetilde{z} = z - \epsilon(z)$. Using the transformations, derive the infinitesimal transformation for $j(z) = j_z + 4\kappa \partial \omega$, and confirm $\kappa = -\frac{3}{4}$ by comparing the infinitesimal transformation derived here and the one derived from the OPE calculation.

2 Veneziano amplitude

In this problem, we explore the Veneziano amplitude, which represents the four-point amplitude for open string tachyons scattering on the disk. Unlike its closed-string counterpart, this computation involves summing over the orderings of vertex operator insertions on the disk's boundary. We consider the upper half-plane model, placing vertex operators along the real line.

2.1 3-point amplitude

Before going into the Veneziano amplitude, let us consider the 3-point amplitude of the open string tachyons and show it takes the form

$$S_{D_2}(k_1;k_2;k_3) = g_o^3 e^{-\lambda} \langle : ce^{ik_1\cdot X}(y_1) :: ce^{ik_2\cdot X}(y_2) :: ce^{ik_3\cdot X}(y_3) : \rangle_{D_2} + (k_2 \leftrightarrow k_3) \ .$$

Show that it is independent of the positions of the operators.

2.2 Compute the Veneziano Amplitude

Show the 4-point amplitude of the open string tachyons takes the form

$$S_{D_2}(k_1;k_2;k_3;k_4) = g_o^4 e^{-\lambda} \int_{-\infty}^{\infty} dy_4 \left\langle \prod_{i=1}^{3} : c(y_i)\, e^{ik_i\cdot X(y_i)} :: e^{ik_4\cdot X(y_4)} : \right\rangle + (k_2 \leftrightarrow k_3) \ .$$

Demonstrate that the scattering amplitude $S(k_1,k_2,k_3,k_4)$ for four open string tachyons can be expressed as:

$$\begin{aligned} S(k_1,k_2,k_3,k_4) =& i g_o^4 C_{D_2}^X (2\pi)^{26} \delta^{(26)}(\sum_i k_i) \int_{-\infty}^{\infty} dy_4 \, |y_4|^{-\alpha' u-2} \, |1-y_4|^{-\alpha' t-2} \\ & + (t \leftrightarrow s) \end{aligned}$$

after setting $y_1 = 0, y_2 = 1, y_3 = \infty$ using PSL$(2,\mathbb{R})$ invariance.

2.3 Analyzing the Integral's Ranges

Break down the integral for the Veneziano amplitude into three distinct ranges based on the position of y_4: $-\infty < y_4 < 0, 0 < y_4 < 1, 1 < y_4 < \infty$. Illustrate the corresponding orderings of the vertex operators $1,2,3,4$ for these ranges and for both terms in the amplitude equation.

Discuss how these ranges relate to each other through PSL$(2,\mathbb{R})$ invariance, leading to the final form of the Veneziano amplitude:

$$S(k_1,k_2,k_3,k_4) = 2 i g_o^4 C_{D_2} (2\pi)^{26} \delta^{(26)}(\sum_i k_i)(I(s,t) + I(t,u) + I(u,s))$$

where the integral $I(s,t)$ is defined as:

$$I(s,t) = \int_0^1 dy y^{-\alpha' s-2}(1-y)^{-\alpha' t-2} \equiv B\left(-\alpha' s - 1, -\alpha' t - 1\right) .$$

Note that the Euler Beta-function $B(a, b)$ is related to the Euler Gamma-function Γ as follows:

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

2.4 Residue Analysis of the Veneziano Amplitude

Focus on the term within the Veneziano amplitude that exhibits poles in s and t. Calculate the residue at the pole $t = 0$, and compare this residue with the one obtained in field theory from the exchange of a photon between two charged scalars. Utilize this comparison to determine the gauge coupling in terms of g_o.

2.5 Contribution from the u and t Poles

Analyze the term within the Veneziano amplitude that has poles in u and t. Show that this term contributes an equal and opposite residue to that calculated in the previous problem. Discuss the meaning of this contribution in the context of string theory and gauge interactions.