Homework 2: Due at class on March 12

1. We construct vector fields on an n-sphere $S^n \subset \mathbb{R}^{n+1}$ in Example 2.11 of the lecture note.

$$Y = \sum Y^i \frac{\partial}{\partial x^i}$$

where

$$Y^i = \begin{cases} \left(-x^1, x^0, -x^3, x^2, \cdots, -x^{2k+1}, x^{2k}\right) & n = 2k+1 \\ \left(-x^1, x^0, -x^3, x^2, \cdots, -x^{2k+1}, x^{2k}, 0\right) & n = 2k+2 \end{cases} .$$

For $n = 1, 2, 3$, write explicitly the flows generated by this vector field.

2. Write down vector fields that generate the rotation along x-, y-, z-axis in $\mathbb{R}^3$. Find the commutation relations of these vector fields. Compare the theory of angular momenta in quantum mechanics.

3. Let e be the identity element of $SO(3)$. Show that the tangent space $T_eSO(3)$ at e is spanned by tangent vectors of curves in $SO(3)$

$$\exp(tJ_i) = 1 + tJ_i + \frac{1}{2}(tJ_i)^2 + \cdots$$

at $t = 0$ where J_i $(i = x, y, z)$ are defined by

$$J_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \qquad J_x = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \qquad J_x = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} .$$

Let us define the commutator by $[X, Y] = XY - YX$. Then, show that

$$[J_x, J_y] = J_z\ , \qquad [J_y, J_z] = J_x\ , \qquad [J_z, J_x] = J_y\ .$$

4. Show that the tangent space $T_eSU(2)$ is spanned by $i\sigma_x$, $i\sigma_y$ and $i\sigma_z$ (the Pauli matrices by i).

5. Show that $\mathbb{R}P^n$ is non-orientable for even n. In addition, construct an example of unorientable manifolds except the Möbius strip and even-dimensional real projective space.