

# Homework 12 (Due at class on Dec 15)

## Prob. 1 (S-duality)

In the lecture, the low-energy effective action of Type I string is given by

$$\begin{aligned} S_I &= S_{\text{grav}} + S_{\text{YM}} \\ S_{\text{grav}} &= \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-G} \left[ e^{-2\Phi} (R + 4\partial_\mu \Phi \partial^\mu \Phi) - \frac{1}{2} |\tilde{G}_3|^2 \right] \\ S_{\text{YM}} &= -\frac{1}{2g_{10}^2} \int d^{10}x \sqrt{-G} e^{-\Phi} \text{Tr}_V |F_2|^2 . \end{aligned} \quad (1)$$

Also, the low-energy effective action of Heterotic SO(32) is given by

$$\begin{aligned} S_{\text{Het}} &= S_{\text{grav}} + S_{\text{YM}} \\ S_{\text{grav}} &= \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-G} e^{-2\Phi} \left[ R + 4\partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2} |\tilde{H}_3|^2 \right] \\ S_{\text{YM}} &= -\frac{1}{2g_{10}^2} \int d^{10}x \sqrt{-G} e^{-2\Phi} \text{Tr}_V |F_2|^2 \end{aligned} \quad (2)$$

**1.1** Explain why the Yang-Mills action  $S_{\text{YM}}$  in Type I (1) has  $e^{-\Phi}$  whereas  $S_{\text{YM}}$  in Heterotic SO(32) (2) has  $e^{-2\Phi}$ .

**1.2** Show that Type I (1) and Heterotic SO(32) (2) actions are related by the following the field definitions

$$\begin{aligned} G_{\mu\nu}^I &= e^{-\Phi^H} G_{\mu\nu}^H , & \Phi^I &= -\Phi^H \\ \tilde{G}_3^I &= \tilde{H}_3^H , & A^I &= A^H . \end{aligned}$$

## Prob. 2 (Heterotic M-theory)

**2.1** Show that the length of a line interval  $S^1/\mathbb{Z}_2$  in Heterotic M-theory is  $R = g_{\text{Het}}^{\frac{2}{3}} \ell_p$  by using the following duality chain in the lecture note:

$$\text{HE} \xrightarrow{T} \text{HO} \xrightarrow{S} \text{Type I} \xrightarrow{T} \text{Type I'} \xrightarrow{\text{strong coupling}} \text{M-theory}$$

Note that T-duality on a circle  $S^1$  relate radii and string coupling constants as

$$\tilde{R} = \ell_s^2 / R , \quad \tilde{g}_s = \ell_s g_s / R ,$$

whereas S-duality on a circle  $S^1$  relate them as

$$\tilde{R} = R / \sqrt{g_s} , \quad \tilde{g}_s = 1 / g_s ,$$

where  $\ell_s = \sqrt{\alpha'}$  and the definition of  $\ell_p$  is as in Lecture note 12. Note that tilde denotes parameters in the dual theory.

**2.2** Give an explanation how Heterotic strings and fivebranes are related to M2 and M5-branes in Heterotic M-theory up on the compactification on a segment  $S^1/\mathbb{Z}_2$ . Namely, argue how Heterotic strings and fivebranes become M2 and M5-branes in the strong coupling regime and vice versa.

### Prob. 3 (Type I with D1-brane)

Let us consider the massless spectrum of string excitations in the D1-D1 and the D1-D9 sector of a D1-brane along directions  $X^0, X^1$  in Type I string theory and compare this to the massless fields on the worldsheet of the SO(32) heterotic string.

**3.1** Let us first consider the D1-D1 strings. Here  $X^I, \psi_{\pm}^I, I = 2, \dots, 9$  have DD boundary conditions while  $X^\mu, \psi_{\pm}^\mu, \mu = 0, 1$  have NN boundary conditions.

As in the bosonic open string, the NN boundary condition identifies the right and left fermion modes  $\psi_n^\mu = \tilde{\psi}_n^\mu$  so that

$$\psi^\mu(\tau, \sigma) = \sum_{n \in \mathbb{Z} + \nu} \psi_n^\mu (e^{-in(\tau-\sigma)} + e^{-in(\tau+\sigma)}) ,$$

where  $\nu$  takes the values 0 (R) or  $\frac{1}{2}$  (NS). On the other hand, the DD boundary condition identifies the right and left fermion modes by sign  $\psi_n^I = -\tilde{\psi}_n^I$  so that

$$\psi^I(\tau, \sigma) = \sum_{n \in \mathbb{Z} + \nu} \psi_n^I (e^{-in(\tau-\sigma)} - e^{-in(\tau+\sigma)}) .$$

Show that worldsheet parity  $\Omega : \sigma \rightarrow \pi - \sigma$  acts on the modes as

$$\Omega \psi_n \Omega^{-1} = \pm e^{i\pi n} \psi_n , \quad +/- : \text{NN/DD} .$$

The actions of the orientifold  $\Omega$  on the vacua of the D1-string given by

$$\Omega |0\rangle_{NS} = -i|0\rangle_{NS} , \quad \Omega |s_0 = \frac{1}{2}, \mathbf{s}\rangle_R = -e^{i\pi(s_1+s_2+s_3+s_4)} |s_0 = \frac{1}{2}, \mathbf{s}\rangle_R$$

with  $s_0$  the spin in directions  $X^0, X^1$  and  $s_1, \dots, s_4$  the spin in the normal directions. Then, show that the orientifold projection  $(1 + \Omega)/2$  keeps  $\psi_{-\frac{1}{2}}^I |0\rangle$  for the normal directions and removes  $\psi_{-\frac{1}{2}}^\mu |0\rangle$  for the tangent directions in the NS sector. In the R sector, show that the ground states **16** spanned by  $|s_0 = \frac{1}{2}, \mathbf{s}\rangle$  are projected onto **8c** by the orientifold action.

**3.2** Next, let us consider D1-D9 string. Here  $X^I, \psi_{\pm}^I, I = 2, \dots, 9$  have DN boundary conditions while  $X^\mu, \psi_{\pm}^\mu, \mu = 0, 1$  have NN boundary conditions as before. In Homework 7 Problem 1, we have seen that the bosonic open string with DN boundary condition admits the following mode expansion

$$X = c + i\sqrt{\frac{\alpha'}{2}} \sum_{n \in \mathbb{Z} + 1/2} \frac{\alpha_n}{n} (e^{-in\sigma^-} - e^{-in\sigma^+}) .$$

In fact, the supersymmetry requires the periodicity of fermion  $\psi(\tau, \sigma)$  (or mode  $\psi_n$ ) in the R sector to be the same as for  $X(\tau, \sigma)$  (or mode  $\alpha_n$ ). In the NS sector, it is the opposite (modes differ by  $\frac{1}{2}$ ). Using this fact, compute the zero point energy in the NS and R sector. In addition, find the massless spectrum in the D1-D9 string after the GSO projection. Note that since Chan-Patton factors are allowed in the free string end, there are 32 of them.