Homework 4: Due at class on March 26

The Einstein summation convention is applied in all the given problems.

1. Let us consider the following action

$$S = \int_a^b dt \left(g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} \right) \tag{0.1}$$

The equation of motion minimizes the action, namely $\delta S = 0$ under the variation $\tilde{x}(t) = x(t) + \delta x(t)$ where we fix the initial and final condition $x(a) = x_i$, $x(b) = x_f$. Show that the equation of motion is equivalent to the geodesic equation.

2. Derive the expression of the Christoffel symbols

$$\Gamma^k_{ij} = \frac{1}{2} \left(\frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l} \right) g^{lk}$$

from

$$\frac{\partial}{\partial x^i} g_{jk} = g\left(\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k} \right) + g\left(\frac{\partial}{\partial x^j}, \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^k} \right)$$

and its permutations with respect to i, j, k. Show that the Riemann curvature can be written

$$R^l{}_{ijk} = \Gamma^s_{jk} \Gamma^l_{is} - \Gamma^s_{ik} \Gamma^l_{js} + \frac{\partial \Gamma^l_{jk}}{\partial x^i} - \frac{\partial \Gamma^l_{ik}}{\partial x^j}$$

in terms of local coordinates.

3. A vector field X is a Killing field if the Lie derivative with respect to X of the metric g vanishes:

$$L_X g = 0 \,.$$

Show that this is equivalent to

$$g\left(\nabla_Y X, Z\right) + g\left(Y, \nabla_Z X\right) = 0$$

for all vectors Y and Z where ∇ is the Levi-Civita connection. In terms of local coordinates where $X = X^j \partial_j$, show that this amounts to

$$\nabla_{\partial_i} X_j + \nabla_{\partial_j} X_i = 0 \,.$$

Also, show that this is equivalent to

$$X^k \partial_k g_{ij} + g_{kj} \partial_i X^k + g_{ik} \partial_j X^k = 0 \tag{0.2}$$

4. Let $\iota : S^2 \hookrightarrow \mathbb{R}^3$ be the inclusion of the 2-sphere with unit radius. Let $g : ds^2 = \sum_{i=0}^2 dx^i \otimes dx^i$ be the standard metric of $\mathbb{R}^3$. Find the induced metric $\iota^* g$ on S^2 in terms of the polar coordinate of $\mathbb{R}^3$.

$$x^0 = r \sin\theta \cos\phi$$

$$x^1 = r \sin\theta \sin\phi$$
$$x^2 = r \cos\theta$$

Given this metric, find geodesics on S^2 and compute its Riemann, Ricci and scalar curvature.

Do parallel transport of a vector along a triangle ΔPQR on a unit sphere (Figure 1) with respect to the Levi-Civita connection of the metric and find the angle difference when it comes back. Compare it with the area of the triangle (see Homework 1).

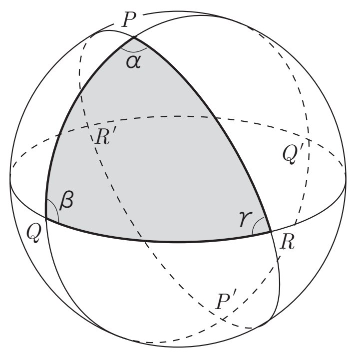

Figure 1: A triangle on a 2-sphere