

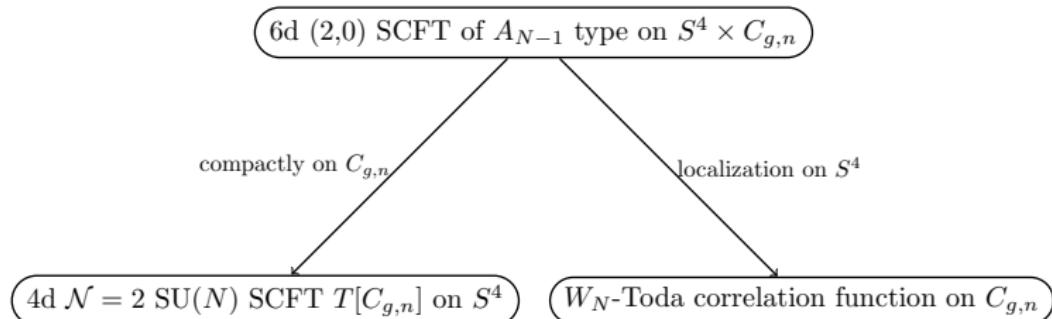
Givental J -functions, Quantum integrable systems, AGT relation with surface operator

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AGT relation



4d partition function is equivalent to Today correlation function [Alday-Gaiotto-Tachikawa, Wyllard]

$$\begin{aligned} Z^{4d}[T[C_{g,n}]] &= \langle V_{\beta_1}(z_1) \cdots V_{\beta_n}(z_n) \rangle_{C_{g,n}} \\ \int da \ Z_{\text{1-loop}}(a, m, \epsilon_1, \epsilon_2) |Z_{\text{inst}}(a, m, \epsilon_1, \epsilon_2; z)|^2 &= \int d\alpha \ C(\alpha, \beta, b) |F(\alpha, \beta, b : z)|^2 \end{aligned}$$

Defects in 6d perspective

- co-dimension four defects: M2-branes attach to 2d submanifold of M5-branes
- co-dimension two defects: intersection of M5-branes supported on 4d submanifold

	4d	2d	4d	2d
codim 4	2	0	surface op. line op.	deg. field Verlinde loop op.
codim 4	1	1		
codim 2	4	0	change theory	?
codim 2	3	1	domain wall	loop op.
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Co-dimension two surface operator

surface operator can be described in two ways [Gukov-Witten]

- imposing singular behavior of gauge field

$$A_\mu dx^\mu \sim \text{diag}(\alpha_1, \dots, \alpha_N) i d\theta ,$$

and inserting the phase factor $\exp(i\eta_i m')$ in the path integral

- 4d $\mathcal{N} = 2$ gauge theory coupled to 2d (2,2) gauge theory on the surface

Co-dimension two surface operator

- If the parameter $\vec{\alpha} = (\alpha_1, \dots, \alpha_N)$ is specified by

$$\vec{\alpha} = \left(\underbrace{\alpha_{(1)}, \dots, \alpha_{(1)}}_{N_1 \text{ times}}, \underbrace{\alpha_{(2)}, \dots, \alpha_{(2)}}_{N_2 \text{ times}}, \dots, \underbrace{\alpha_{(M)}, \dots, \alpha_{(M)}}_{N_M \text{ times}} \right),$$

the gauge group is broken to the commutant of $\vec{\alpha}$ on the surface \mathcal{C} :

$$\mathbb{L} = S[U(N_1) \times U(N_2) \times \dots \times U(N_M)],$$

\mathbb{L} is the Levi part of a parabolic subgroup \mathcal{P} of the complexified Lie group $G_{\mathbb{C}}$

- ▶ $\vec{\alpha} = (\alpha, \dots, \alpha, (N-1)\alpha) \rightarrow \mathbb{L} = SU(N-1) \times U(1)$: simple
- ▶ all α_i are distinct $\rightarrow \mathbb{L} = U(1)^N$: full
- In addition, we insert the phase factor $\exp(i\eta_I m^I)$ where η_I are “electric parameters” or “2d theta angles”, and m^I are magnetic fluxes on \mathcal{C}

$$m^I = \frac{1}{2\pi} \int_{\mathcal{C}} F^I \quad (I = 1, \dots, M),$$

where $\sum_I m^I = 0$.

Instanton moduli space

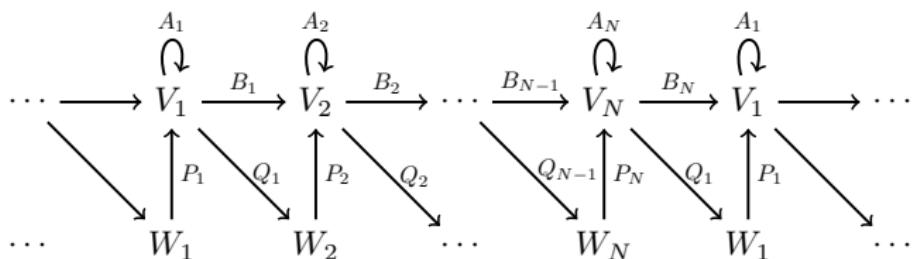
- the moduli space $\mathcal{M}_{\vec{N}, k, \vec{m}}$ of instanton with the boundary condition of the gauge field on the surface is called affine Laumon space [Feigin-Finkelberg-Negut-Rybnikov]

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- it admits quiver representation called chain-saw quiver [Finkelberg-Rybnikov]



where $\dim W_I = N_I$, $\dim V_I = k_I$, with $k_M = k$, $k_{I+1} = k_I + m^{I+1}$.

$$\mathcal{M}_{\vec{N}, \vec{k}} = \{(A_I, B_I, P_I, Q_I) | \mathcal{E}_{\mathbb{C}}^{(I)} = 0, \text{ stability condition}\} / \text{GL}(k_1, \mathbb{C}) \otimes \dots \otimes \text{GL}(k_M, \mathbb{C})$$

Instanton partition function

- the action of the Cartan torus $U(1)^2 \times U(1)^N$ of the spacetime and the gauge symmetry
- Due to the orbifold spaces, the equivariant parameter

$$(z_1, z_2) \rightarrow (e^{i\epsilon_1} z_1, e^{i\epsilon_2/M} z_2) .$$

- non-trivial holonomy shifts the equivariant parameters (a_1, \dots, a_N) of $U(1)^N$ by

$$a_{s,I} \rightarrow a_{s,I} - \frac{I-1}{M}\epsilon_2 , \quad (s = 1, \dots, N_I) .$$

- character of the equivariant action at the fixed points yields the Nekrasov instanton partition function

$$\mathcal{Z}_{\text{inst}}[\vec{N}] = \sum_{\vec{k}} \prod_{I=1}^M z_I^{k_I} \mathcal{Z}_{\vec{N}, \vec{k}}(\epsilon_1, \epsilon_2, a, m) ,$$

Instanton partition function

- making change of variables

$$\prod_{l=1}^M z_l = q , \quad z_l = e^{t_l - t_{l+1}} ,$$

the instanton partition function can be re-arranged

$$\mathcal{Z}_{\text{inst}}[\vec{N}] = \sum_{k=0}^{\infty} \sum_{\mathbf{m} \in \Lambda_{\mathbb{L}}} q^k e^{t \cdot \mathbf{m}} \mathcal{Z}_{\vec{N}, k, \vec{\mathbf{m}}}(\epsilon_1, \epsilon_2, a, m) ,$$

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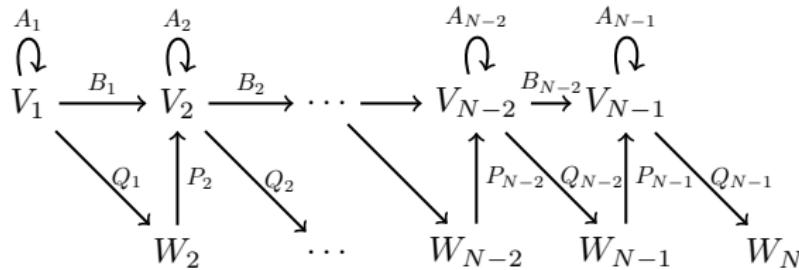
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- when $k = 0$, the partition function encodes only 2d dynamics on the support of the surface operator
- chain-saw quiver demotes to hand-saw quiver when $k = 0$ which describes Laumon space



Dual 2d CFT

- Dual 2d CFT changes since the co-dimension two defect wraps on a Riemann surface $C_{g,n}$ [Alday-Tachikawa, Kozcaz-Pasquetti-Passerini-Wyllard, Wyllard]

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- For a surface operator of type $\vec{N} = [N_1, \dots, N_M]$, the 2d symmetry is the W-algebra $W(\widehat{\mathfrak{sl}}(N), \vec{N})$ obtained by the quantum Drinfeld-Sokolov reduction of $\widehat{\mathfrak{sl}}(N)$ for the embedding $\rho_{\vec{N}} : \mathfrak{sl}(2) \rightarrow \mathfrak{sl}(N)$ corresponding to the partition \vec{N}

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- For example, when $N = 3$,

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- we will show the relation between one-loop determinant and structure constant of $\text{SL}(2, \mathbb{R})$ WZNW model

Quantization

The AGT relation with a surface operator is closely related to quantization of cohomology ring

- cohomology ring of \mathbb{P}^{N-1}

$$H^*(\mathbb{P}^{N-1}) \cong \mathbb{C}[x]/(x^N) .$$

Equivariant cohomology ring

let us define the S^1 -equivariant action on \mathbf{P}^{N-1} by

$$\lambda[z_0 : \cdots : z_{N-1}] = [\lambda^{r_0} z_0 : \cdots : \lambda^{r_{N-1}} z_{N-1}] ,$$

for $\lambda \in S^1$. Then, one can construct bundle

$$(\mathbf{P}^{N-1})_{S^1} = (ES^1 \times \mathbf{P}^{N-1})/S^1 \rightarrow BS^1 ,$$

the S^1 -equivariant cohomology ring of \mathbf{P}^{N-1} is given by

$$H_{S^1}^*(\mathbf{P}^{N-1}) = H^*((\mathbf{P}^{N-1})_{S^1}) \cong \mathbb{C}[x, \hbar] / \left(\prod_{i=0}^{N-1} (x - r_i \hbar) \right) ,$$

where \hbar represents the hyperplane class of the base manifold of the universal S^1 -bundle

$$S^{2\infty+1} = ES^1 \rightarrow BS^1 = \mathbf{P}^\infty ,$$

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Quantum cohomology ring

- the cohomology ring is quantized based on Gromov-Witten theory. The quantum cohomology is ordinary cohomology with a quantum product defined by

$$T_i \circ T_j = \sum_{k,\ell} C_{ijk}(t) \eta^{k\ell} T_\ell ,$$

for a basis T_i of the cohomology group.

- Here the structure constants $C_{ijk}(t) := \frac{\partial^3 F_0}{\partial T_i \partial T_j \partial T_k}$ is the third derivative of the genus-zero prepotential depending on the complexified Kähler parameter t and $\eta_{ij} := \int T_i \cup T_j$ is the metric on the cohomology group.
- the WDVV equation is equivalent to the associativity of the quantum product, and therefore the quantum product can be thought of as quantum deformation of the cup product of cohomology.
- Writing $q = e^t$, the quantum cohomology ring of \mathbf{P}^{N-1} is isomorphic to

$$QH^*(\mathbf{P}^{N-1}) \cong \mathbb{C}[x, q]/(x^N - q) .$$

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S^1 -equivariant Floer homology

- the S^1 -equivariant Floer homology of the universal covering $\widetilde{L\mathbf{P}^{N-1}}$ of the loop space $L\mathbf{P}^{N-1} := \text{Map}(S^1, \mathbf{P}^{N-1})$ of the projective space

$$HF_{S^1}^*(\widetilde{L\mathbf{P}^{N-1}}) = \bigoplus_{m \in \mathbb{Z}} \bigoplus_{k=0}^{N-1} \mathbb{C}[\hbar] \cdot (x - m\hbar)^k \cdot \prod_{j < m} (x - j\hbar)^N .$$

- the S^1 -equivariant Floer homology $HF_{S^1}^*(\widetilde{L\mathbf{P}^{N-1}})$ turns out to be endowed with \mathcal{D} -module structure [Givental]

$$\mathcal{D}/(p^N - q) ,$$

where we define

$$\begin{aligned} p \cdot J(x, \hbar) &= x \cdot J(x, \hbar) , \\ q \cdot J(x, \hbar) &= J(x - \hbar, \hbar) , \end{aligned}$$

for $J(x, \hbar) \in HF_{S^1}^*(\widetilde{L\mathbf{P}^{N-1}})$. From the definition, it is easy to see $[p, q] = \hbar q$ so that p can be regarded as a differential operator $\hbar q \frac{d}{dq}$ on functions of q .

- the \mathcal{D} -module structure can be rephrased as

$$\left[\left(\hbar q \frac{d}{dq} \right)^N - q \right] J(q) = 0 .$$

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- \mathcal{D} -module structure of S^1 -equivariant Floer homology of \mathbf{P}^{N-1}

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Quantization and Givental J -function

- solution of quantum (Dubrovin) connection

$$\left[\left(\hbar q \frac{d}{dq} \right)^N - q \right] J(q) = 0 .$$

is given by the Givental J -function

$$J[\mathbf{P}^{N-1}] = e^{\frac{tx}{\hbar}} \sum_{d=0}^{\infty} \frac{e^{td}}{\prod_{j=1}^d (x + j\hbar)^N} .$$

- The J -function of X is defined by using the *psi class* $\psi = c_1(\mathcal{L}_1)$

$$J(X) = e^{\delta/\hbar} \left(1 + \sum_{\beta \in H_2(X, \mathbb{Z})} \sum_{a=1}^m q^\beta \left\langle \frac{T_a}{\hbar - \psi}, 1 \right\rangle_{0,\beta} T^a \right) ,$$

where $\delta = \sum_{i=1}^r t_i T_i$ and $q^\beta = e^{\int_\beta \delta}$. Thus, it is regarded as a generating function for once-punctured genus zero Gromov-Witten invariants with gravitational descendants.

connection to geometric representation theory

- Braverman and Etingof has constructed invariants equivalent to J -function of complete flag variety Fl_N of degree \vec{k}

$$J[\mathrm{Fl}_N] = \sum_{\vec{k}} z^k \int_{\mathcal{QM}_{\vec{k}}} 1$$

where $\mathcal{QM}_{\vec{k}}$ represents resolution of moduli space of quasi-maps $\mathbf{P}^1 \rightarrow \mathrm{Fl}_N$. This is actually the Laumon space

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- the AGT relation of class \mathcal{S} theories with a surface operator generally provides a rich arena for a vast generalization of Givental theory, and quantum connections therefore appear as differential equations of Knizhnik-Zamolodchikov type.

Pure Yang-Mills

Instanton partition function of pure Yang-Mills theory in the presence of a full surface operator

$$\mathcal{Z}_{\text{inst}}^{\text{pure}}[1^N] = \sum_{\vec{k}} \left(\prod_{I=1}^N z_I^{k_I} \right) \mathcal{Z}_{[1^N], \vec{k}}^{\text{pure}},$$

where

$$\begin{aligned} \mathcal{Z}_{[1^N], \vec{k}}^{\text{pure}} &= \epsilon_1^{-\sum_{I=1}^N k_I} \oint \prod_{I=1}^N \prod_{s=1}^{k_I} \frac{d\phi_s^{(I)}}{(\phi_s^{(I)} + a_I - \frac{(I-1)\epsilon_2}{N})(\phi_s^{(I)} + a_{I+1} + \epsilon - \frac{I\epsilon_2}{N})} \\ &\quad \prod_{I=1}^N \prod_{s=1}^{k_I} \prod_{t \neq s}^{k_I} \frac{\phi_{st}^{(I)}}{\phi_{st}^{(I)} + \epsilon_1} \prod_{I=1}^N \prod_{s=1}^{k_I} \prod_{t=1}^{k_{I+1}} \frac{\phi_s^{(I)} - \phi_t^{(I+1)} + \epsilon}{\phi_s^{(I)} - \phi_t^{(I+1)} + \frac{\epsilon_2}{N}}. \end{aligned}$$

It satisfies periodic Toda equation

$$\left[\frac{\epsilon_1^2}{2} \sum_{I=1}^N (z_I \partial_I - z_{I+1} \partial_{I+1})^2 + \epsilon_1 \sum_{I=1}^N u_I z_I \partial_I - \sum_{I=1}^N z_I \right] \mathcal{Z}_{\text{inst}}^{\text{pure}}[1^N] = 0$$

Pure Yang-Mills

In fact, making the change of variables

$$\prod_{I=1}^N z_I = \Lambda , \quad z_I = e^{t_I - t_{I+1}} \quad (I = 1, \dots, N-1) ,$$

where Λ can be interpreted as the dynamical scale of the pure Yang-Mills, one can bring the equation into the more familiar form [Braverman-Etingof]

$$\left[2\epsilon_1\epsilon_2\Lambda \frac{\partial}{\partial \Lambda} + \epsilon_1^2 \Delta_{\mathfrak{h}} - 2\left(\Lambda e^{t_N - t_1} + \sum_{\alpha \in \Pi} e^{\langle t, \alpha \rangle}\right) \right] (e^{-\frac{\langle a, t \rangle}{\epsilon_1}} \mathcal{Z}_{\text{inst}}^{\text{pure}}[1^N]) = \langle a, a \rangle (e^{-\frac{\langle a, t \rangle}{\epsilon_1}} \mathcal{Z}_{\text{inst}}^{\text{pure}}[1^N])$$

where Π represents the set of simple roots of $\mathfrak{sl}(N)$ so that $\sum_{\alpha \in \Pi} e^{\langle t, \alpha \rangle} = \sum_{I=1}^N e^{t_I - t_{I+1}}$, and the rest of notations is as follows:

$$\Delta_{\mathfrak{h}} = \sum_{I=0}^{N-1} \frac{\partial^2}{\partial t_I^2} , \quad \langle a, t \rangle = \sum_{I=1}^{N-1} a_I t_I , \quad \langle a, a \rangle = \sum_{I=1}^N a_I^2 .$$

$\mathcal{N} = (2, 2)$ susy gauge theory for full surface operator

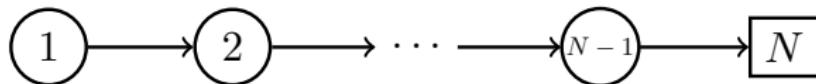
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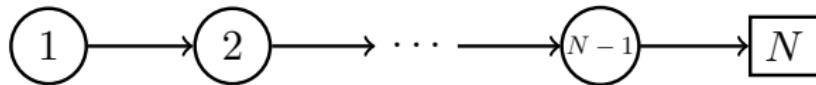
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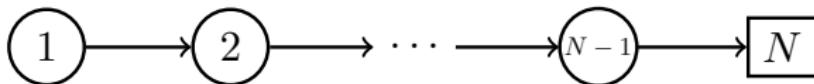
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- The UV description of the 2d theory is given by the quiver diagram where the D-term equation describes complete flag variety

$\mathcal{N} = (2, 2)$ susy gauge theory for full surface operator

- The other way to describe a surface operator is to couple a 4d $\mathcal{N} = 2$ gauge theory to an $\mathcal{N} = (2, 2)$ supersymmetric gauge theory on the surface.
- For the surface operator in the pure Yang-Mills, the 2d theory flows at infrared to the $\mathcal{N} = (2, 2)$ supersymmetric NLSM with complete flag variety $\mathrm{SL}(N, \mathbb{C})/B$ as a target.
- In this description, the combined parameters $\vec{t} = 2\pi i(\vec{\eta} + i\vec{\alpha})$ are identified with the complexified Kähler parameters of the NLSM.



- The UV description of the 2d theory is given by the quiver diagram where the D-term equation describes complete flag variety
- 2d quiver gauge theory is coupled to the 4d pure Yang-Mills by gauging the flavor symmetry $\mathrm{U}(N)$. Hence, the Coulomb branch parameters a_i in the 4d theory become the twisted masses of the fundamentals in the 2d theory.

Coulomb branch formula

Coulomb branch formula of the partition function is given by [Benini-Cremonesi, Doroud-Gomis-Le Floch-Lee]

$$\begin{aligned} Z[\text{Fl}_N] &= \frac{1}{1! \cdots (N-1)!} \sum_{\vec{B}^{(I)}} \int \prod_{I=1}^{N-1} \prod_{s=1}^I \frac{d\tau_s^{(I)}}{2\pi i} e^{4\pi \xi^{(I)} \tau_s^{(I)} - i\theta^{(I)} B_s^{(I)}} Z_{\text{vector}} Z_{\text{bifund}} Z_{\text{fund}} , \\ Z_{\text{vector}} &= \prod_{I=2}^{N-1} \prod_{s < t}^I \left(\frac{(B_{st}^{(I)})^2}{4} - (\tau_{st}^{(I)})^2 \right) , \\ Z_{\text{bifund}} &= \prod_{I=1}^{N-2} \prod_{s=1}^I \prod_{t=1}^{I+1} \frac{\Gamma\left(\tau_s^{(I)} - \tau_t^{(I+1)} - \frac{B_s^{(I)}}{2} + \frac{B_t^{(I+1)}}{2}\right)}{\Gamma\left(1 - \tau_s^{(I)} + \tau_t^{(I+1)} - \frac{B_s^{(I)}}{2} + \frac{B_t^{(I+1)}}{2}\right)} , \\ Z_{\text{fund}} &= \prod_{s=1}^{N-1} \prod_{t=1}^N \frac{\Gamma\left(\tau_s^{(N-1)} - \frac{B_s^{(N-1)}}{2} - \hbar^{-1} a_t\right)}{\Gamma\left(1 - \tau_s^{(N-1)} - \frac{B_s^{(N-1)}}{2} + \hbar^{-1} a_t\right)} , \end{aligned}$$

J -function of complete flag variety

towers of poles can be dealt by making changes of variables

$$\tau_s^{(I)} = \frac{B_s^{(I)}}{2} - \ell_s^{(I)} - \hbar^{-1} H_s^{(I)},$$

one can manipulate the partition function into

$$Z[\mathrm{Fl}_N] = \frac{1}{1! \cdots (N-1)!} \oint \prod_{l=1}^{N-1} \prod_{s=1}^l \frac{-dH_s^{(l)}}{2\pi\hbar i} (z_l \bar{z}_l)^{\hbar^{-1}|H^{(l)}|} \tilde{Z}_{\text{1-loop}}(H_s^{(l)}) \tilde{Z}_{\text{v}}(H_s^{(l)}) \tilde{Z}_{\text{av}}(H_s^{(l)}),$$

From $\tilde{Z}_{\text{v}}(H_s^{(l)})$, one can obtain the expression of hypergeometric type for J -function of Fl_N
[Bonelli-Sciarappa-Tanzini-Vasko,]

$$J[\mathrm{Fl}_N] = \sum_{\vec{k}^{(I)}} \hbar^{-\sum_{l=1}^{N-2} (|k^{(l)}|(l+1) - |k^{(l+1)}|l) - N|k^{(N-1)}|} \prod_{l=1}^{N-1} z_l^{|k^{(l)}|} \prod_{l=2}^{N-1} \prod_{s \neq t}^l \frac{1}{(\hbar^{-1} H_{st}^{(l)})_{k_s^{(l)} - k_t^{(l)}}} \\ \prod_{l=1}^{N-2} \prod_{s=1}^l \prod_{t=1}^{l+1} \frac{1}{(1 + \hbar^{-1} H_s^{(l)} - \hbar^{-1} H_t^{(l+1)})_{k_s^{(l)} - k_t^{(l+1)}}} \prod_{s=1}^{N-1} \prod_{t=1}^N \frac{1}{(1 + \hbar^{-1} H_s^{(N-1)} - \hbar^{-1} H_t^{(N)})_{k_s^{(N-1)}}}.$$

Here we identify $H_s^{(l)}$ ($s = 1, \dots, l$) with Chern roots to the duals of the universal bundles \mathcal{S}_l :

$$0 \subset \mathcal{S}_1 \subset \mathcal{S}_2 \subset \cdots \subset \mathcal{S}_{N-1} \subset \mathcal{S}_N = \mathbb{C}^N \otimes \mathcal{O}_{\mathrm{Fl}_N}.$$

Higgs branch formula

Higgs branch formula of the partition function is given by

$$\begin{aligned} Z[\text{Fl}_N] &= \frac{1}{1! \cdots (N-1)!} \sum_{\sigma \in S_N} \prod_{I=1}^{N-1} (z_I \bar{z}_I)^{-\hbar^{-1} \sum_{t=1}^I a_{\sigma(t)}} Z_{\text{1-loop}}(a_{\sigma(i)}) Z_v(a_{\sigma(i)}) Z_{av}(a_{\sigma(i)}) , \\ Z_{\text{1-loop}} &= \prod_{s < t}^N \gamma \left(\frac{a_s - a_t}{\hbar} \right) , \\ Z_v &= \sum_{\vec{k}^{(I)}} \hbar^{-\sum_{I=1}^{N-2} (|k^{(I)}|(I+1) - |k^{(I+1)}|I) - N|k^{(N-1)}|} \prod_{I=1}^{N-1} z_I^{|k^{(I)}|} \prod_{I=2}^{N-1} \prod_{s \neq t}^I \frac{1}{(-\hbar^{-1} a_{st})_{k_s^{(I)} - k_t^{(I)}}} \\ &\quad \prod_{I=1}^{N-2} \prod_{s=1}^I \prod_{t=1}^{I+1} \frac{1}{(1-\hbar^{-1} a_{st})_{k_s^{(I)} - k_t^{(I+1)}}} \prod_{s=1}^{N-1} \prod_{t=1}^N \frac{1}{(1-\hbar^{-1} a_{st})_{k_s^{(N-1)}}} \end{aligned}$$

It turns out that the vortex partition function can be obtained from the instanton partition function by setting the instanton number $k = k_N = 0$

$$Z_v[\text{Fl}_N](z_I, a, \hbar) = \mathcal{Z}_{\text{inst}}^{\text{pure}}[1^N](z_I, z_N = 0, a, \epsilon_1 = \hbar)$$

Toda Hamiltonian

We have seen that instanton partition function obeys the periodic Toda equation

$$\left[2\epsilon_1 \epsilon_2 \Lambda \frac{\partial}{\partial \Lambda} + \epsilon_1^2 \Delta_{\mathfrak{h}} - 2 \left(\Lambda e^{t_N - t_1} + \sum_{\alpha \in \Pi} e^{\langle t, \alpha \rangle} \right) \right] (e^{-\frac{\langle a, t \rangle}{\epsilon_1}} \mathcal{Z}_{\text{inst}}^{\text{pure}}[1^N]) = \langle a, a \rangle (e^{-\frac{\langle a, t \rangle}{\epsilon_1}} \mathcal{Z}_{\text{inst}}^{\text{pure}}[1^N])$$

Therefore, the vortex partition function of complete flag variety becomes an eigenfunction of the Toda Hamiltonian [Kim]

$$\left(\hbar^2 \Delta_{\mathfrak{h}} - 2 \sum_{\alpha \in \Pi} e^{\langle t, \alpha \rangle} \right) \left[e^{-\frac{\langle a, t \rangle}{\hbar}} Z_v[\text{Fl}_N] \right] = \langle a, a \rangle \left[e^{-\frac{\langle a, t \rangle}{\hbar}} Z_v[\text{Fl}_N] \right],$$

One-loop determinant

For $Q^2 = \mathcal{R}$ and transversally elliptic operator $D : \Gamma(E_0) \rightarrow \Gamma(E_1)$, we have

$$\mathcal{Z}_{\text{1-loop}} = \frac{\det_{\text{Coker } D} \mathcal{R}}{\det_{\text{Ker } D} \mathcal{R}}$$

This can be obtained from

$$\text{ind } D = \text{tr}_{\text{Ker } D} e^{\mathcal{R}} - \text{tr}_{\text{Coker } D} e^{\mathcal{R}} ,$$

where we can convert the index into the determinant via

$$\sum_j c_j e^{w_j(\epsilon_1, \epsilon_2, a, m_f)} \rightarrow \prod_j w_j(\epsilon_1, \epsilon_2, a, m_f)^{c_j} ,$$

The index can be computed from Atiyah-Singer index formula

$$\text{ind } D = \sum_{p \in F} \frac{\text{tr}_{E_0(p)} \mathcal{R} - \text{tr}_{E_1(p)} \mathcal{R}}{\det_{T_p M} (1 - \mathcal{R})}$$

One-loop determinant

it is natural to expect that the one-loop computation with a full surface operator can be obtained by the index theorem on $\mathbb{C} \times (\mathbb{C}/\mathbb{Z}_N)$. As in the case of the instanton partition function, the equivariant parameters are shifted by $\epsilon_2 \rightarrow \frac{\epsilon_2}{N}$

$$\Gamma_2(x(a, m_f, \epsilon_1, \epsilon_2) | \epsilon_1, \epsilon_2) \rightarrow \Gamma_2 \left(\tilde{x}(a, m_f, \epsilon_1) + \frac{I\epsilon_2}{N} \middle| \epsilon_1, \frac{\epsilon_2}{N} \right)$$

Then, its \mathbb{Z}_N -invariant part becomes the one-loop determinant on $\mathbb{C} \times (\mathbb{C}/\mathbb{Z}_N)$. Writing $t = e^{i\epsilon_2/N}$, the corresponding index is

$$g(t) = e^{i\tilde{x}} \frac{t^I}{1-t} .$$

The \mathbb{Z}_N -invariant part can be taken by averaging over the \mathbb{Z}_N group

$$\frac{1}{N} \sum_{k=0}^{N-1} g(\omega^k t) = e^{i\tilde{x}} \frac{t^{\lceil \frac{I}{N} \rceil N}}{1-t^N} ,$$

where $\omega = \exp(2\pi i/N)$ is the N -th root of unity and $\lceil x \rceil$ denotes the smallest integer $\geq x$. Subsequently, the one-loop determinant on $\mathbb{C} \times (\mathbb{C}/\mathbb{Z}_N)$ can be written as

$$\mathcal{Z}_{\text{1-loop}}[\mathbb{C} \times (\mathbb{C}/\mathbb{Z}_N)] = \Gamma_2 \left(\tilde{x}(a, m_f, \epsilon_1) + \left\lceil \frac{I\epsilon_2}{N} \right\rceil \middle| \epsilon_1, \epsilon_2 \right) .$$

One-loop determinant

One-loop determinant of vector multiplet [Pestun]

$$\begin{aligned}\mathcal{Z}_{\text{1-loop}}^{\text{pure}} &= \prod_{\alpha \in \Delta} [\Gamma_2(\langle a, \alpha \rangle | \epsilon_1, \epsilon_2) \Gamma_2(\langle a, \alpha \rangle + \epsilon_1 + \epsilon_2 | \epsilon_1, \epsilon_2)]^{-1}, \\ &= \prod_{\alpha \in \Delta} \Upsilon(\langle a, \alpha \rangle | \epsilon_1, \epsilon_2),\end{aligned}$$

To apply the index theorem on $\mathbb{C} \times (\mathbb{C}/\mathbb{Z}_N)$, we shift the parameters

$$\epsilon_2 \rightarrow \frac{\epsilon_2}{N}, \quad a_i \rightarrow a_i - \frac{i-1}{N} \epsilon_2,$$

due to the orbifold operation. This re-parametrization alters the one-loop determinant

$$\begin{aligned}& \prod_{\alpha \in \Delta} \Gamma_2(\langle a, \alpha \rangle | \epsilon_1, \epsilon_2) \Gamma_2(\langle a, \alpha \rangle + \epsilon_1 + \epsilon_2 | \epsilon_1, \epsilon_2) \\ \rightarrow & \prod_{i,j=1, i \neq j}^N \Gamma_2\left(a_i - a_j + \frac{j-i}{N} \epsilon_2 | \epsilon_1, \frac{\epsilon_2}{N}\right) \Gamma_2\left(a_i - a_j + \epsilon_1 + \frac{1+j-i}{N} \epsilon_2 | \epsilon_1, \frac{\epsilon_2}{N}\right).\end{aligned}$$

One-loop determinant

Averaging over the finite group \mathbb{Z}_N , we have the one-loop determinant in the existence of the full surface operator

$$\begin{aligned}\mathcal{Z}_{\text{1-loop}}^{\text{pure}}[1^N] &= \prod_{i,j=1, i \neq j}^N \left[\Gamma_2(a_i - a_j + \left\lceil \frac{j-i}{N} \right\rceil \epsilon_2 | \epsilon_1, \epsilon_2) \Gamma_2(a_i - a_j + \epsilon_1 + \left\lceil \frac{1+j-i}{N} \right\rceil \epsilon_2 | \epsilon_1, \epsilon_2) \right]^{-1} \\ &= \prod_{i,j=1, i \neq j}^N \Upsilon \left(a_i - a_j + \left\lceil \frac{j-i}{N} \right\rceil \epsilon_2 | \epsilon_1, \epsilon_2 \right),\end{aligned}$$

where $\lceil x \rceil$ denotes the smallest integer $\geq x$.

if the 4d contribution $\mathcal{Z}_{\text{1-loop}}^{\text{pure}}$ is subtracted from the one-loop determinant $\mathcal{Z}_{\text{1-loop}}^{\text{pure}}[1^N]$, only the 2d effect $Z_{\text{1-loop}}[\text{Fl}_N]$ should be evident

$$\frac{\mathcal{Z}_{\text{1-loop}}^{\text{pure}}[1^N]}{\mathcal{Z}_{\text{1-loop}}^{\text{pure}}}(\mathbf{a}, \epsilon_1 = \hbar) = \prod_{\alpha \in \Delta^+} \hbar^{\frac{\langle \mathbf{a}, \alpha \rangle}{\hbar} - 1} \gamma \left(\frac{\langle \mathbf{a}, \alpha \rangle}{\hbar} \right) \text{ " = " } Z_{\text{1-loop}}[\text{Fl}_N](\mathbf{a}, \hbar)$$

$\mathcal{N} = 2^*$ theory

Instanton partition function of $\mathcal{N} = 2^*$ theory in the presence of a full surface operator

$$\mathcal{Z}_{\text{inst}}^{\mathcal{N}=2^*}[1^N] = \sum_{\vec{k}} \left(\prod_{l=1}^N z_l^{k_l} \right) \mathcal{Z}_{[1^N], \vec{k}}^{\mathcal{N}=2^*},$$

where

$$\begin{aligned} \mathcal{Z}_{[1^N], \vec{k}}^{\mathcal{N}=2^*} &= \left[\frac{\epsilon_1 - \mu_{\text{adj}}}{\epsilon_1 \mu_{\text{adj}}} \right]^{\sum_{l=1}^N k_l} \\ &\oint \prod_{l=1}^N \prod_{s=1}^{k_l} d\phi_s^{(l)} \frac{(\phi_s^{(l)} + a_l - \frac{(l-1)\epsilon_2}{N} + \mu_{\text{adj}})(\phi_s^{(l)} + a_{l+1} + \epsilon - \frac{l\epsilon_2}{N} - \mu_{\text{adj}})}{(\phi_s^{(l)} + a_l - \frac{(l-1)\epsilon_2}{N})(\phi_s^{(l)} + a_{l+1} + \epsilon - \frac{l\epsilon_2}{N})} \\ &\prod_{l=1}^N \prod_{s=1}^{k_l} \prod_{t \neq s} \frac{\phi_{st}^{(l)} (\phi_{st}^{(l)} + \epsilon_1 - \mu_{\text{adj}})}{(\phi_{st}^{(l)} + \mu_{\text{adj}})(\phi_{st}^{(l)} + \epsilon_1)} \\ &\prod_{l=1}^N \prod_{s=1}^{k_l} \prod_{t=1}^{k_{l+1}} \frac{(\phi_s^{(l)} - \phi_t^{(l+1)} + \epsilon)(\phi_s^{(l)} - \phi_t^{(l+1)} + \frac{\epsilon_2}{N} - \mu_{\text{adj}})}{(\phi_s^{(l)} - \phi_t^{(l+1)} + \frac{\epsilon_2}{N})(\phi_s^{(l)} - \phi_t^{(l+1)} + \epsilon - \mu_{\text{adj}})}. \end{aligned}$$

$\mathcal{N} = 2^*$ theory

- Nekrasov partition function is proportional to $SL(N, \mathbb{R})$ WZNW conformal block on the once-punctured torus.
- If we define

$$\mathcal{Y}(t, q, a, \mu_{\text{adj}}, \epsilon_1, \epsilon_2) := e^{-\frac{\langle a, t \rangle}{\epsilon_1}} f(t, q)^{-\frac{\mu_{\text{adj}}}{\epsilon_1} + 1} \prod_{i=1}^{\infty} (1 - q^i)^{-\frac{\mu_{\text{adj}}(N\epsilon_1 + \epsilon_2 - N\mu_{\text{adj}})}{\epsilon_1 \epsilon_2} + 1} \mathcal{Z}_{\text{inst}}^{\mathcal{N}=2^*}[1^N]$$

it satisfies the KZB equation

$$\left[2\epsilon_1 \epsilon_2 q \frac{\partial}{\partial q} + \epsilon_1^2 \Delta_{\mathfrak{h}} + 2\mu_{\text{adj}}(\mu_{\text{adj}} - \epsilon_1) \sum_{\alpha \in \Delta^+} \left(\frac{1}{4\pi^2} \wp(\langle t, \alpha \rangle; \tau) + \frac{1}{12} \right) \right] \mathcal{Y} = \langle a, a \rangle \mathcal{Y} .$$

$\mathcal{N} = (2, 2)$ susy gauge theory for full surface operator

- Since the $\mathcal{N} = 2^*$ theory is a mass deformation of the $\mathcal{N} = 4$ SCFT, the dynamics on the support of surface operator is also described by a deformation of an $\mathcal{N} = (4, 4)$ supersymmetric gauge theory

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- superpotential is determined by $\mathcal{N} = (4, 4)$ supersymmetry

$$W = \sum_{I=1}^{N-1} \text{Tr } \tilde{Q}^{(I)} \Phi^{(I)} Q^{(I)} + \sum_{I=1}^{N-2} \text{Tr } Q^{(I)} \Phi^{(I+1)} \tilde{Q}^{(I)} ,$$

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- it is deformed by turning the twisted mass m of $\tilde{Q}^{(I)}$ and $\Phi^{(I)}$ ($I = 1, \dots, N-1$)

Coulomb branch formula

Coulomb branch formula of the partition function is given by

$$\begin{aligned}
 Z[T^* \mathrm{Fl}_N] &\propto \sum_{\substack{\vec{B}^{(I)} \\ I=1 \dots N-1}} \int \prod_{l=1}^{N-1} \prod_{s=1}^l \frac{d\tau_s^{(l)}}{2\pi i} e^{4\pi\xi^{(l)}\tau_s^{(l)} - i\theta^{(l)}B_s^{(l)}} Z_{\text{vect}} Z_{\text{adj}} Z_{\text{bifund}} Z_{\text{fund-anti}} , \\
 Z_{\text{vect}} &= \prod_{l=2}^{N-1} \prod_{s < t}^l \left(\frac{(B_{st}^{(l)})^2}{4} - (\tau_{st}^{(l)})^2 \right) , \quad Z_{\text{adj}} = \prod_{l=2}^{N-1} \prod_{s \neq t}^l \frac{\Gamma\left(1 + \tau_{st}^{(l)} - \frac{B_{st}^{(l)}}{2} + \hbar^{-1}m\right)}{\Gamma\left(-\tau_{st}^{(l)} - \frac{B_{st}^{(l)}}{2} - \hbar^{-1}m\right)} . \\
 Z_{\text{bifund}} &= \prod_{l=1}^{N-2} \prod_{s=1}^l \prod_{t=1}^{l+1} \frac{\Gamma\left(\tau_s^{(l)} - \tau_t^{(l+1)} - \frac{B_s^{(l)}}{2} + \frac{B_t^{(l+1)}}{2}\right) \Gamma\left(-\tau_s^{(l)} + \tau_t^{(l+1)} + \frac{B_s^{(l)}}{2} - \frac{B_t^{(l+1)}}{2} - \hbar^{-1}m\right)}{\Gamma\left(1 - \tau_s^{(l)} + \tau_t^{(l+1)} - \frac{B_s^{(l)}}{2} + \frac{B_t^{(l+1)}}{2}\right) \Gamma\left(1 + \tau_s^{(l)} - \tau_t^{(l+1)} + \frac{B_s^{(l)}}{2} - \frac{B_t^{(l+1)}}{2} + \hbar^{-1}m\right)} , \\
 Z_{\text{fund-anti}} &= \prod_{s=1}^{N-1} \left[\frac{\Gamma\left(\tau_s^{(N-1)} - \frac{B_s^{(N-1)}}{2}\right) \Gamma\left(-\tau_s^{(N-1)} + \frac{B_s^{(N-1)}}{2} - \hbar^{-1}m\right)}{\Gamma\left(1 - \tau_s^{(N-1)} - \frac{B_s^{(N-1)}}{2}\right) \Gamma\left(1 + \tau_s^{(N-1)} + \frac{B_s^{(N-1)}}{2} + \hbar^{-1}m\right)} \right]^N
 \end{aligned}$$

Higgs branch formula

Higgs branch formula of the partition function is given by

$$\begin{aligned} Z[T^* \mathrm{Fl}_N] &= \frac{1}{1! \cdots (N-1)!} \sum_{\sigma \in S_N} \prod_{I=1}^{N-1} (z_I \bar{z}_I)^{-\hbar^{-1} \sum_{t=1}^I a_{\sigma(t)}} Z_{\text{1-loop}}(a_{\sigma(i)}) Z_v(a_{\sigma(i)}) Z_{\text{av}}(a_{\sigma(i)}) , \\ Z_{\text{1-loop}} &= \prod_{s < t}^N \gamma \left(\frac{a_s - a_t}{\hbar} \right) \gamma \left(\frac{a_t - a_s - m}{\hbar} \right) , \\ Z_v &= \sum_{\vec{k}^{(I)}} \prod_{I=1}^{N-1} z_I^{|k^{(I)}|} \prod_{I=2}^{N-1} \prod_{s \neq t}^I \frac{(1 - \hbar^{-1} a_{st} + \hbar^{-1} m)_{k_s^{(I)} - k_t^{(I)}}}{(-\hbar^{-1} a_{st})_{k_s^{(I)} - k_t^{(I)}}} \\ &\quad \prod_{I=1}^{N-2} \prod_{s=1}^I \prod_{t=1}^{I+1} \frac{(-\hbar^{-1} a_{st} - \hbar^{-1} m)_{k_s^{(I)} - k_t^{(I+1)}}}{(1 - \hbar^{-1} a_{st})_{k_s^{(I)} - k_t^{(I+1)}}} \prod_{s=1}^{N-1} \prod_{t=1}^N \frac{(-\hbar^{-1} a_{st} - \hbar^{-1} m)_{k_s^{(N-1)}}}{(1 - \hbar^{-1} a_{st})_{k_s^{(N-1)}}} \end{aligned}$$

It turns out that the vortex partition function can be obtained from the instanton partition function by setting the instanton number $k = k_N = 0$

$$Z_v[T^* \mathrm{Fl}_N](z_I, a, m, \hbar) = \mathcal{Z}_{\text{inst}}^{\mathcal{N}=2^*}[1^N](z_I, z_N = 0, a, \mu_{\text{adj}} = m + \hbar, \epsilon_1 = \hbar)$$

Calogero-Moser Hamiltonian

- Multiplying the following factor to the vortex partition function

$$Y(t, a, m, \hbar) = e^{-\frac{\langle a, t \rangle}{\hbar}} \prod_{\alpha \in \Delta^+} (1 - e^{\langle t, \alpha \rangle})^{-\frac{m}{\hbar}} Z_v[T^* \mathrm{Fl}_N],$$

it becomes an eigenfunction of the trigonometric Calogero-Moser Hamiltonian
[Negut, Braverman-Maulik-Okounkov]

$$\left[\hbar^2 \Delta_{\mathfrak{h}} - 2m(m + \hbar) \sum_{\alpha \in \Delta^+} \frac{1}{(e^{\langle t, \alpha \rangle / 2} - e^{-\langle t, \alpha \rangle / 2})^2} \right] Y = \langle a, a \rangle Y,$$

where Δ^+ represents the set of positive roots of $\mathfrak{sl}(N)$.

- Furthermore, the monodromy matrices of this differential equation satisfy the affine Hecke algebra. This algebra admits a natural physical interpretation as the action of the loop operators on a full surface operator [Gukov-Witten, Okuda-Honda]

Mirror Landau-Ginzburg model

Let us study the twisted chiral ring in the Landau-Ginzburg (LG) model mirror dual to the NLSM with T^*Fl_N . To bring the partition function into the LG description, let us define

$$\Sigma_s^{(I)} = \sigma_s^{(I)} - i \frac{B_s^{(I)}}{2r} ,$$

which become the twisted chiral multiplet corresponding to the I -th vector multiplet for $U(I)$. In addition, every ratio of Gamma functions can be replaced by [Gomis-Lee]

$$\frac{\Gamma(-ir\Sigma)}{\Gamma(1+ir\bar{\Sigma})} = \int \frac{d^2Y}{2\pi} \exp \left\{ -e^{-Y} + ir\Sigma Y + e^{-\bar{Y}} + ir\bar{\Sigma}\bar{Y} \right\} ,$$

where Y, \bar{Y} represent the twisted chiral fields for the matter sector of the LG model. To study the Coulomb branch of this theory in the infrared, we integrate out the twisted chiral fields Y, \bar{Y} . Performing a semiclassical approximation

$$Y = -\ln(-ir\Sigma) , \quad \bar{Y} = -\ln(ir\bar{\Sigma}) ,$$

we are left with

$$\frac{\Gamma(-ir\Sigma)}{\Gamma(1+ir\bar{\Sigma})} \sim \exp \left\{ \varpi(-ir\Sigma) - \frac{1}{2} \ln(-ir\Sigma) - \varpi(ir\bar{\Sigma}) - \frac{1}{2} \ln(ir\bar{\Sigma}) \right\} ,$$

Twisted chiral ring

Using this prescription, the partition function can be written

$$Z[T^* \mathrm{Fl}_N] \sim \frac{1}{1! \cdots (N-1)!} \int \prod_{l=1}^{N-1} \prod_{s=1}^l \frac{d^2(r\Sigma_s^{(l)})}{2\pi} \left| Q(\Sigma)^{\frac{1}{2}} e^{-\widetilde{\mathcal{W}}(\Sigma)} \right|^2 ,$$

where $\widetilde{\mathcal{W}}(\Sigma)$ is the effective twisted superpotential of the mirror LG model in the Coulomb branch

$$\begin{aligned} \widetilde{\mathcal{W}}(\Sigma) = & \sum_{l=1}^{N-1} \sum_{s=1}^l (-2\pi\xi^{(l)} + i\theta^{(l)})(ir\Sigma_s^{(l)}) + \sum_{l=2}^{N-1} \sum_{s \neq t} \varpi(-ir\Sigma_s^{(l)} + ir\Sigma_t^{(l)} + ir\hat{m}) \\ & + \sum_{l=1}^{N-2} \sum_{s=1}^l \sum_{u=1}^{l+1} [\varpi(-ir\Sigma_s^{(l)} + ir\Sigma_u^{(l+1)}) + \varpi(ir\Sigma_s^{(l)} - ir\Sigma_u^{(l+1)} - ir\hat{m})] \\ & + \sum_{s=1}^{N-1} [\varpi(-ir\Sigma_s^{(N-1)}) + \varpi(ir\Sigma_s^{(N-1)} - ir\hat{m})] . \end{aligned}$$

the twisted chiral ring is given by the equation of supersymmetric vacua [Nekrasvo-Shatashvili]

$$\exp \left(\frac{\partial \widetilde{\mathcal{W}}}{\partial (ir\Sigma_s^{(l)})} \right) = 1 .$$

Twisted chiral ring

Vacuum equation turns out to be nested Bethe ansatz equations for $\mathfrak{sl}(N)$ spin chain for $I = 1$,

$$\prod_{t=1}^2 \frac{\Sigma_s^{(1)} - \Sigma_t^{(2)}}{\Sigma_s^{(1)} - \Sigma_t^{(2)} - \hat{m}} = e^{-2\pi\xi^{(1)} + i\theta^{(1)}},$$

for $1 < I < N - 1$,

$$\prod_{t \neq s}^I \frac{\Sigma_s^{(I)} - \Sigma_t^{(I)} - \hat{m}}{\Sigma_s^{(I)} - \Sigma_t^{(I)} + \hat{m}} \prod_{t=1}^{I-1} \frac{\Sigma_s^{(I)} - \Sigma_t^{(I-1)} + \hat{m}}{\Sigma_s^{(I)} - \Sigma_t^{(I-1)}} \prod_{t=1}^{I+1} \frac{\Sigma_s^{(I)} - \Sigma_t^{(I+1)}}{\Sigma_s^{(I)} - \Sigma_t^{(I+1)} - \hat{m}} = \pm e^{-2\pi\xi^{(I)} + i\theta^{(I)}},$$

for $I = N - 1$,

$$\prod_{t \neq s}^{N-1} \frac{\Sigma_s^{(N-1)} - \Sigma_t^{(N-1)} - \hat{m}}{\Sigma_s^{(N-1)} - \Sigma_t^{(N-1)} + \hat{m}} \prod_{t=1}^{N-2} \frac{\Sigma_s^{(N-1)} - \Sigma_t^{(N-2)} + \hat{m}}{\Sigma_s^{(N-1)} - \Sigma_t^{(N-2)}} = \pm e^{-2\pi\xi^{(N-1)} + i\theta^{(N-1)}} \left[\frac{\Sigma_s^{(N-1)} - \hat{m}}{\Sigma_s^{(N-1)}} \right]^N.$$

Equivariant quantum cohomology ring

- For the cotangent bundle $T^*\text{Gr}(r, N)$ of a Grassmannian, vacuum equation is BAE for $\mathfrak{sl}(2)$ XXX spin chain [Nekrasov-Shatashvili]

$$\prod_{a=1}^r \frac{\Sigma_i + m_a^f}{\Sigma_i - m_a^{\bar{f}}} = -e^{2\pi i t} \prod_{j=1}^N \frac{\Sigma_i - \Sigma_j - m^{\text{adj}}}{\Sigma_i - \Sigma_j + m^{\text{adj}}}$$

- It is now proven that the Baxter subalgebra of $\mathfrak{sl}(2)$ Yangian is isomorphic to the equivariant quantum cohomology [Maulik-Okounkov]

$$Y(\mathfrak{sl}(2)) \supset B \cong QH_T^*(\coprod_r T^*\text{Gr}(r, N))$$

- Therefore, it is natural to expect

$$Y(\mathfrak{sl}(N)) \supset B \cong QH_T^*(T^*\text{Fl}_N)$$

Correlation function of $SL(2)$ WZNW model

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$$\langle \mathbb{V}_j(x_1; z_1) \mathbb{V}_j(x_2; z_2) \rangle = B(j) \frac{|x_{12}|^{4j}}{|z_{12}|^{4\Delta(j)}}$$

where $\Delta(j) = \frac{j(j+1)}{2(k+2)}$ is the conformal dimension of the primary field and the reflection amplitude $B(j)$ is given by

$$B(j) = -\frac{k+2}{\pi} \frac{\nu_2^{1+2j}}{\gamma\left(\frac{2j+1}{k+2}\right)}, \quad \nu_2 = \pi \frac{\Gamma\left(1 + \frac{1}{k+2}\right)}{\Gamma\left(1 - \frac{1}{k+2}\right)}.$$

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- In addition, the conformal invariance and the affine symmetry determine the three-point function

$$\langle \mathbb{V}_{j_1}(x_1; z_1) \mathbb{V}_{j_2}(x_2; z_2) \mathbb{V}_{j_3}(x_3; z_3) \rangle = D(j_1, j_2, j_3) \frac{|x_{12}|^{2j_1^3} |x_{13}|^{2j_1^2} |x_{23}|^{2j_1^1}}{|z_{12}|^{2\Delta_{12}^3} |z_{13}|^{2\Delta_{13}^2} |z_{23}|^{2\Delta_{23}^1}}$$

where the structure coefficient $D(j_1, j_2, j_3)$ is given by

$$D(j_3, j_2, j_1) = -\frac{\nu_2^{j_1+j_2+j_3+1} \tilde{\Upsilon}_{k+2}(1) \tilde{\Upsilon}_{k+2}(-2j_1-1) \tilde{\Upsilon}_{k+2}(-2j_2-1) \tilde{\Upsilon}_{k+2}(-2j_3-1)}{2\pi^2 \gamma\left(\frac{k+1}{k+2}\right) \tilde{\Upsilon}_{k+2}(-j_1-j_2-j_3-1) \tilde{\Upsilon}_{k+2}(j_3-j_1-j_2) \tilde{\Upsilon}_{k+2}(j_2-j_1-j_3) \tilde{\Upsilon}_{k+2}(j_1-j_2-j_3)}.$$

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- the one-loop determinant of the the $\text{SU}(N)$ SCFT with $N_F = 2N$ in the presence of a full surface operator can be written as

$$\begin{aligned} & \mathcal{Z}_{\text{1-loop}}^{N_F=2N}[1^N] \\ &= \frac{\prod_{\alpha \in \Delta^+} \Upsilon(\langle a, \alpha \rangle + \epsilon_2 | \epsilon_1, \epsilon_2) \Upsilon(-\langle a, \alpha \rangle | \epsilon_1, \epsilon_2)}{\prod_{p,q} \Upsilon\left(\langle a, h_p \rangle + \mu_q + \left\lceil \frac{N-p-q+1}{N} \right\rceil \epsilon_2 | \epsilon_1, \epsilon_2\right) \Upsilon(-\langle a, h_p \rangle - \tilde{\mu}_q + \left\lceil \frac{p-q}{N} \right\rceil \epsilon_2 | \epsilon_1, \epsilon_2)}. \end{aligned}$$

- the parameters between the 4d gauge theory and $\text{SL}(N, \mathbb{R})$ WZNW model are identified with

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- Using this identification, one can easily deduce the form of three-point function

$$\begin{aligned} & D(j_1, \varkappa \omega_{N-1}, j_3) \\ &= A_1 \frac{\left(\tilde{\Upsilon}_{k+N}(1) \right)^{N-1} \tilde{\Upsilon}_{k+N}(-\varkappa - 1) \prod_{\alpha \in \Delta^+} \tilde{\Upsilon}_{k+N}(-\langle j_1 + \rho, \alpha \rangle) \tilde{\Upsilon}_{k+N}(-\langle j_3 + \rho, \alpha \rangle)}{\prod_{p,q=1}^N \tilde{\Upsilon}_{k+N} \left(-\frac{\varkappa}{N} + \langle j_1 + \rho, h_q \rangle + \langle j_3 + \rho, h_p \rangle - \left\lceil \frac{N-p-q+1}{N} \right\rceil (k + N) \right)}. \end{aligned}$$

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- Subsequently, the form of reflection coefficient can be obtained from the three-point function

$$B(j) = D(j, 0, j^*) = \frac{A_2}{\prod_{\alpha \in \Delta^+} \gamma \left(\frac{\langle 2j + \rho, \alpha \rangle}{k+N} \right)}.$$

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Thank you!