# Homework 9: Due at class on Nov 21

# 1 Derivation

Derive (6.21), (6.22) and (6.26) of the lecture note.

# 2 $\beta$-functions

## 2.1

Suppose that we perturb the action at a fixed point by an operator $\mathcal{O}$ of scaling dimension $\Delta$

$$S = S^* + \int \frac{d^2x}{2\pi} g\,\mathcal{O}(x)\,.$$

Let us calculate the sub-leading (one-loop) correction to $\beta$-function of the coupling constant $g$. To this end, we introduce the bare coupling $\widehat{g} = a^{2-\Delta} g$ where $a$ as a length-scale and $g$ is now dimensionless. To find the $\beta$-functions, we shall address the question of how to change the coupling constants $g$ under the infinitesimal scale transformation $a \to (1+\delta\lambda)a$ in such a way that the partition function remains invariant.

The perturbative expansion of the partition function is

$$\begin{aligned}\mathcal{Z} &= \int \mathcal{D}\varphi \exp\left[-S^* - \widehat{g}\int \frac{d^2x}{2\pi a^{2-\Delta}}\mathcal{O}(x)\right] \\ &= \mathcal{Z}^*\left[1 - \widehat{g}\int \frac{d^2x}{2\pi a^{2-\Delta}}\langle\mathcal{O}(x)\rangle + \frac{\widehat{g}^2}{2}\int_{|x_1-x_2|>a}\langle\mathcal{O}(x_1)\,\mathcal{O}(x_2)\rangle\frac{d^2x_1}{2\pi a^{2-\Delta}}\frac{d^2x_2}{2\pi a^{2-\Delta}} + \cdots\right]\end{aligned}$$

Here the length-scale $a$ appears both explicitly and implicitly in the integral region.

Under the infinitesimal scale transformation $a \to (1+\delta\lambda)a$, the coupling constant is scaled as

$$\widehat{g} \to (1+\delta\lambda)^{2-\Delta}\widehat{g} \simeq \widehat{g} + (2-\Delta)\widehat{g}\delta\lambda$$

In addition, the integral, after a rescaling of $a$, can be written as

$$\int_{|x_1-x_2|>a(1+\delta\lambda)}[\cdots] = \int_{|x_1-x_2|>a}[\cdots] - \int_{a<|x_1-x_2|<a(1+\delta\lambda)}[\cdots]$$

The first terms produces the original contribution in $\mathcal{Z}$, and the second term can be computed through the operator expansion of the conformal theory. Suppose that the OPE of the operator $\mathcal{O}$ with itself is

$$\mathcal{O}(x_1)\mathcal{O}(x_2) = \frac{\mathbf{C}}{|x_{12}|^{\Delta}}\mathcal{O}(x_2) + \cdots\,.$$

Then, derive that the second term is

$$\begin{aligned}&-\frac{\widehat{g}^2}{2}\mathbf{C}a^{-\Delta}\int_{a<|x_1-x_2|<a(1+\delta\lambda)}\langle\mathcal{O}(x_2)\rangle\frac{d^2x_1}{2\pi a^{2-\Delta}}\frac{d^2x_2}{2\pi a^{2-\Delta}} \\ &= -\frac{1}{2}\delta\lambda\,\widehat{g}^2\mathbf{C}\int\langle\mathcal{O}(x)\rangle\frac{d^2x}{2\pi a^{2-\Delta}}\end{aligned} \tag{2.1}$$

Therefore, the total effect on the coupling constant under the infinitesimal scaling is

$$\widehat{g} \to \widehat{g} + (2-\Delta)\widehat{g}\delta\lambda - \frac{1}{2}\mathbf{C}\widehat{g}^2\delta\lambda + \mathcal{O}(\widehat{g}^3)$$

Consequently, we have the one-loop contribution to the $\beta$ function

$$\frac{d\widehat{g}}{d\lambda} \equiv \beta(\widehat{g}) = (2-\Delta)\widehat{g} - \frac{1}{2}\mathbf{C}\widehat{g}^2 + \mathcal{O}(\widehat{g}^3)$$

Plot the $\beta$-function with respect to $\widehat{g}$ assuming $\Delta < 2$. (A curve $\beta(\widehat{g})$ depends on the sign of $\mathbf{C}$.)

## 2.2

Suppose that the operator $\mathcal{O}$ is a marginal operator $\Delta = 2$. Solve the differential equation for the $\beta$-function

$$\frac{d\widehat{g}}{d\lambda} = -\frac{\mathbf{C}\widehat{g}^2}{2}$$

where we impose the initial condition $\widehat{g}(\lambda = 0) = g_*$. When does the operator $\mathcal{O}$ become marginally relevant or irrelevant?