Homework 13

1. Toy models for index theorem

1.1 Index theorem on a point

First let us consider the index theorem in the simplest setting, namely on a point. Since a vector bundle over a point is just a vector space, we consider an index of a linear map. Let $A : \mathbb{R}^r \to \mathbb{R}^s$ be a linear map (an $r \times s$ matrix). Compute

$$\dim \operatorname{Ker} A - \dim \operatorname{Coker} A .$$

1.2 Index theorem on $\mathbb{R}$

Next, we consider the index theorem on $\mathbb{R}$. More explicitly, we consider solutions of ordinary differential equations

$$D = \frac{d}{dx} + A(x) , \qquad D^* = -\frac{d}{dx} + A(x)$$

where A takes the value of symmetric matrices. We require solutions to converge to zero at infinity. For $f, g \in C^\infty(\mathbb{R})$ which decay at infinity, it is easy to see

$$\int_{-\infty}^{\infty} (Df, g) dx = \int_{-\infty}^{\infty} (f, D^*g) dx$$

where we use the fact that A is a symmetric matrix. Therefore, D^* is the adjoint operator of D.

Let $A : \mathbb{R} \to \operatorname{Sym}(r, \mathbb{R})$ be a function taking its value on $r \times r$ symmetric matrices $\operatorname{Sym}(r, \mathbb{R})$, and for sufficiently large R, it is subject to

$$A(x) = \begin{cases} A_- & (x \leq -R) \\ A_+ & (x \geq R) \end{cases}$$

for some matrices A_- and A_+. We consider the following ordinary differential equations for vector valued functions $f(x), g(x)$ i.e., $f, g : \mathbb{R} \to \mathbb{R}^r$:

$$Df(x) = f'(x) + A(x)f(x) = 0 , \tag{0.1}$$
$$D^*g(x) = -g'(x) + A(x)g(x) = 0 . \tag{0.2}$$

Show that these equations are linear. Namely, if $\alpha(x)$ and $\beta(x)$ are solutions of the equation, then $p\alpha(x) + q\beta(x)$ is also a solution for $p, q \in \mathbb{R}$. In other words, the space of solutions is a linear space.

For any initial value $f(x_0), g(x_0)$ for a fixed point $x_0 \in \mathbb{R}$, the theory of ordinary differential equations tells us that the solution exists uniquely. Namely,

$$f(x) = P \exp\left(-\int_{x_0}^{x} A(\tilde{x}) d\tilde{x} \right) \cdot f(x_0) ,$$

$$g(x) = P \exp\left(\int_{x_0}^{x} A(\tilde{x}) d\tilde{x} \right) \cdot g(x_0) ,$$

where P represents the ordered integral. As a result, solutions are determined by an initial value and the space of solutions is therefore an r-dimensional vector space. We denote the spaces of solutions to (0.1) and (0.2) by E and F, respectively. For $f \in E$ and $g \in F$, show that its inner product is independent of a position $x \in \mathbb{R}$:

$$\frac{d}{dx}(f(x), g(x)) = 0.$$

Therefore, one can consider E and F are dual to each other, i.e. $E^* = F$.

We, however, restrict ourselves to solutions which decay at infinity. We write the space of such solutions

$$E \supset E_0 := \{f(x)|f'(x) + A(x)f(x) = 0, \lim_{x \to \pm\infty} f(x) = 0\}.$$
$$F \supset F_0 := \{g(x)|-g'(x) + A(x)g(x) = 0, \lim_{x \to \pm\infty} g(x) = 0\},$$

Changing the function $A(x)$ on a compact interval $-R \leq x \leq R$, the dimensions of E_0 and F_0 may vary. A remarkable fact is that the difference between the dimensions of E_0 and F_0 does not change, even though $A(x)$ varies. Let us see this more precisely by assuming that eigenvalues of $A_\pm$ do not contain zeros.

Suppose that $A_\pm$ has an eigenvalue λ for an eigenvector v. Then, the solution to (0.1) and (0.2) has asymptotic

$$f(x) = e^{-\lambda x}v, \qquad g(x) = e^{\lambda x}v \quad \text{for} \quad x \geq R.$$

Whether or not the solution converges to zero as $x \to \pm\infty$ is dependent of the sign of the eigenvalue λ.

Let us define $E_\pm$ be the solution which converges to zero as $x \to \pm\infty$. Then, show that $E_0 = E_+ \cap E_-$. In addition, using the fact that $F = E^*$ is dual to E, show that $F_0 = E_+^\perp \cap E_-^\perp$. Thus, show that

$$\dim E_0 - \dim F_0 = \dim E_+ + \dim E_- - r.$$

Consequently, show that the index theorem on $\mathbb{R}$ is

$$\dim E_0 - \dim F_0 = \frac{1}{2}(\text{sign}(A_-) - \text{sign}(A_+)),$$

where $\text{sign}(B)$ for a square matrix B denotes the number of positive eigenvalues of B subtracted by the number of negative eigenvalues. This is the index theorem on $\mathbb{R}$, which captures the essential points of supersymmetric quantum mechanics. (Although the supersymmetric quantum mechanics in the lecture note corresponds to the case of $r = 1$, we can generalize it to an arbitrary r.)

2. Superfield formalism

In the lecture, I introduce to supersymmetric quantum mechanics. In this exercise, we will formulate it in a more concise way.

Let us write fermionic creation and annihilation operators as follows:

$$\overline{\psi} = \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \psi = \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad [\overline{\psi}, \psi] = \sigma_3.$$

Then, the Hamiltonian can be written as

$$H = \frac{1}{2}p^2 + \frac{1}{2}W'(x)^2 - \frac{1}{2}[\psi, \overline{\psi}]W''(x),$$

Show that the action can be written as

$$S = \int dt \left[\frac{1}{2}\dot{x}^2 + i\overline{\psi}\partial_t\psi - \frac{1}{2}W'(x)^2 + \frac{1}{2}[\psi, \overline{\psi}]W''(x) \right]. \tag{0.3}$$

Then, show that the action is invariant under the supersymmetric transformation

$$\delta x = \overline{\epsilon}\psi + \overline{\psi}\epsilon,$$
$$\delta\overline{\psi} = \overline{\epsilon}\left(i\partial_t x - W'(x)\right),$$
$$\delta\psi = -\epsilon\left(i\partial_t x + W'(x)\right),$$

where ϵ and $\overline{\epsilon}$ are two infinitesimal anti-commuting parameters.

Let us show that the action (0.3) can be obtained by using superfield formalism. Superfields are defined on the superspace $(x; \theta, \overline{\theta})$ where x is the space coordinate and $\theta, \overline{\theta}$ are Grassmann variables subject to

$$\{\theta, \overline{\theta}\} = \{\theta, \theta\} = \{\overline{\theta}, \overline{\theta}\} = 0.$$

Let us define the following superfield

$$\Phi(x, \theta, \overline{\theta}) = x + i\theta\psi - i\overline{\psi}\overline{\theta} + \theta\overline{\theta}D,$$

where D is introduced as an auxiliary field. Acting the following derivative on it

$$D_\theta = \partial_\theta - i\overline{\theta}\partial_t,$$

show that

$$D_\theta\Phi = i\psi - \overline{\theta}D - i\overline{\theta}\dot{x} + \overline{\theta}\theta\dot{\psi}.$$

Similarly, show that

$$D_{\overline{\theta}}\Phi = -i\overline{\psi} - \theta D + i\theta\dot{x} + \overline{\theta}\theta\dot{\overline{\psi}},$$

where $D_{\overline{\theta}} = \partial_{\overline{\theta}} - i\theta\partial_t$. Then, one can write the action in the superspace

$$S = \int dt d\overline{\theta}d\theta \left(\frac{1}{2}|D_\theta\Phi|^2 - W(\Phi) \right).$$

Using the rule of the Grassmann integral

$$\int \theta d\theta = \int \overline{\theta}d\overline{\theta} = 1, \qquad \int d\theta = \int d\overline{\theta} = 0,$$

show that the action is indeed equal to

$$S = \int dt \left[\frac{1}{2}\dot{x}^2 + \overline{\psi}[\partial_t - W''(x)]\psi + \frac{1}{2}D^2 + DW'(x) \right].$$

By substituting the equation of motion $D = -W'(x)$, we obtain the previous action (0.3)

$$S = \int dt \left[\frac{1}{2}\dot{x}^2 + \overline{\psi}[\partial_t - W''(x)]\psi - \frac{1}{2}W'(x)^2 \right].$$