Homework 5: Due at class on Oct 26

1 ζ-function regularization

To compute Casimir energy of free fermion, the important part is

$$L_0 := \frac{1}{2} \sum_{n \in \mathbb{Z}+\nu} n : b_{-n} b_n :$$

where the cost of normal ordering provides

$$\begin{aligned} -\tfrac{1}{2}\sum_{n=0}^{\infty} n \qquad & \text{R-sector} \\ -\tfrac{1}{2}\sum_{n=0}^{\infty} (n+\tfrac{1}{2}) \qquad & \text{NS-sector} \ . \end{aligned} \tag{1}$$

Though these summations naively provide negative infinity, the proper regularization can avoid infinity. To do that, let us regularize the summations as

$$\sum_{n=0}^{\infty} (n+\theta) \Longrightarrow \sum_{n=0}^{\infty} (n+\theta) - \int_0^{\infty} k dk \ .$$

By introducing the convergence parameter ϵ

$$\begin{aligned} & \sum_{n=0}^{\infty} (n+\theta) e^{-\epsilon(n+\theta)} - \int_0^{\infty} dk k e^{-\epsilon k} \\ = & -\frac{d}{d\epsilon} \sum_{n=0}^{\infty} e^{-e(n+\theta)} - (-1) \frac{d}{d\epsilon} \int_0^{\infty} dk e^{-\epsilon k} \ , \end{aligned}$$

provide the finite numbers to the summations in (1) in the limit of $\epsilon \to +0$, and compare (4.68) in the lecture note.

2 Correlation functions of vertex operators

A vertex operator $V_k(z) =: e^{ik\varphi}$ defined by using the normal-ordering can be understood as

$$V_k(z) := V_k^+(z) V_k^-(z)$$

where $V_k^+(z)$ (resp. $V_k^-(z)$) consists of creation (resp. annihilation) operators

$$\begin{aligned} V_k^+(z) &:= \exp\left(k \sum_{n=0}^{\infty} t_n(z) \right) \ , \quad t_0 = i\varphi_0 \qquad t_{n>0} = \frac{a_{-n}}{n} z^n \ , \\ V_k^-(z) &:= \exp\left(k \sum_{n=0}^{\infty} u_n(z) \right) \ , \quad u_0 = a_0 \log z \qquad u_{n>0} = -\frac{a_n}{n} z^{-n} \ . \end{aligned}$$

Show that their OPE is

$$V_{k_1}(z)V_{k_2}(w) \sim (z-w)^{k_1 k_2} : V_{k_1}(z)V_{k_2}(w) : \ ,$$

where

$$\begin{aligned} V_{k_1}(z)V_{k_2}(w) &= V^+_{k_1}(z)V^-_{k_1}(z)V^+_{k_2}(w)V^-_{k_2}(w) \\ : V_{k_1}(z)V_{k_2}(w) : &= V^+_{k_1}(z)V^+_{k_2}(w)V^-_{k_1}(z)V^-_{k_2}(w) \ . \end{aligned}$$

In addition, show that the two point function of vertex operators is

$$\langle V_{k_1}(z)V_{k_2}(w)\rangle = \frac{\delta_{k_1,-k_2}}{(z-w)^{k_1^2}} \ .$$

In a similar fashion, show that multi-point correlation function of vertex operators is given by

$$\langle V_{k_1}(z_1)V_{k_2}(z_2)\cdots V_{k_n}(z_n)\rangle = \delta_{\sum k_i,0} \prod_{1\le i<j\le n} (z_i - z_j)^{k_i k_j} \ .$$

3 Linear dilaton CFT

In the lecture, we have consider the free boson. Now let us consider another important theory whose action is

$$S = \frac{1}{8\pi}\int d^2z\sqrt{g}\nabla_\alpha\varphi\nabla^\alpha\varphi + \frac{Q}{4\pi}\int d^2z\sqrt{g}R\varphi$$

where g is the metric of the two-dimensional world-sheet and R is the Ricci scalar of g. This model is called linear dilaton CFT and Q is called background charge. Using the definition

$$T_{\alpha\beta} = -\frac{4\pi}{\sqrt{g}}\frac{\delta S}{\delta g^{\alpha\beta}} \ ,$$

one can show that the energy-momentum tensor in the flat metric $g_{\alpha\beta} = \delta_{\alpha\beta}$ is

$$T(z) = -\frac{1}{2} : \partial\varphi(z)\partial\varphi(z) : + Q\partial^2\varphi(z) \ .$$

(If you derive this, I will give you an extra point.)

3.1

Compute TT OPE and find the central charge of the linear dilaton CFT.

3.2

Show that $\partial\varphi$ is not primary unless $Q = 0$, but is quasi-primary with conformal dimension $h = 1$. Show that the vertex operator $V_k(z) =: e^{ik\varphi} :$ is primary and compute its conformal dimension. In addition, compute the OPE $\partial\varphi(z)V_k(w)$.

3.3

Let us define the state $|k\rangle = \lim_{z\to 0} V_k(z)|0\rangle$ that corresponds to the vertex operator $V_k(z)$. Show that

$$a_0|k\rangle = k|k\rangle \ , \qquad L_0|k\rangle = \frac{k(k+2iQ)}{2}|k\rangle \ , \qquad L_n|k\rangle = 0 \quad \text{for } n > 0.$$

3.4

When $\sqrt{2}\alpha_\pm = -iQ \pm \sqrt{2-Q^2}$, find that

$$T(z)V_{\sqrt{2}\alpha_\pm}(w) \sim \frac{\partial}{\partial w}\frac{V_{\sqrt{2}\alpha_\pm}(w)}{z-w}$$

so that

$$\left[T(z), \oint \frac{dw}{2\pi i} V_{\sqrt{2}\alpha_\pm}(w)\right] = 0 \ .$$

In fact, $\mathbf{S}_\pm = \oint \frac{dw}{2\pi i} V_{\sqrt{2}\alpha_\pm}(w)$ is called the screening charge.