Homework 3: Due at class on Oct 13

1 Vertex operator and OPE

Show that $:e^{ikX}:$ is a primary field of weight $h = \overline{h} = \alpha' k^2/4$ in the free scalar theory. In addition, show that $\partial^n X$ $(n \geq 2)$ is not a primary field.

2 Virasoro algebra

From the OPE of the stress-energy tensor, derive the Virasoro algebra:

$$T(z)\,T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \ldots$$

$$\longrightarrow \quad [L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m+n,0} \ .$$

3 Witt algebra

A general infinitesimal holomorphic map can be expressed as

$$z' = z - \epsilon(z) = z - \sum_{n\in\mathbb{Z}} \epsilon_n z^{n+1} \ ,$$

with the infinitesimal parameters ϵ_n, and therefore one can define generators of the transformation by $\ell_n = -z^{n+1}\frac{\partial}{\partial z}$. Show that they satisfy the Witt algebra

$$[\ell_m, \ell_n] = (m-n)\ell_{m+n} \ ,$$

so that the Virasoro algebra is the central extension of the Witt algebra.

4 2-point and 3-point function of primary fields

4.1 2-point function

Let us determine the form of the 2-point function of chiral primary operators $\phi_i(z_i)$ with weight h_i $(i = 1, 2)$. The 2-point function is invariant under the translation $z \to z + a$ of the coordinate so that it is a function $g(z_1 - z_2)$ of their relative coordinate $z_1 - z_2$.

Using the property of chiral primary fields under the scaling $z \to \lambda z$, show that the function is of the form

$$g(z_1 - z_2) = \frac{d_{12}}{(z_1 - z_2)^{h_1+h_2}} \ .$$

Furthermore, show that h_1 must be equal to h_2 by using the property under the transformations $z \to -1/z$.

4.2 3-point function

The translation invariance tells us that the 3-point function is also a function $g(z_{12}, z_{23}, z_{31})$ where $z_{ij} = z_i - z_j$. Applying the same argument above, derive the form of the 3-point function

$$\langle \phi_1(z_1)\phi_2(z_2)\phi_2(z_3)\rangle = \frac{C_{123}}{(z_{12})^{h_1+h_2-h_3}(z_{23})^{h_2+h_3-h_1}(z_{31})^{h_3+h_1-h_2}} \ .$$

5 Free fermion

Since we have studied the free scalar theory, now let us study the free fermion theory. The action for a free Majorana fermion reads

$$S = \frac{g}{2}\int d^2x \overline{\Psi}\gamma^a \partial_a \Psi$$

where g is a constant, $\overline{\Psi} = \Psi^\dagger \gamma^0$, and the gamma matrices are given by

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \quad \gamma^1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

being in Euclidean spacetime they satisfy the relation $\{\gamma_i, \gamma_j\} = 2\delta_{ij}$.

- Rewrite the action in terms of $\psi(z,\overline{z}), \overline{\psi}(z,\overline{z})$ where $\Psi = \begin{pmatrix} \psi \\ \overline{\psi} \end{pmatrix}$ where $z = x^0 + ix^1$ and $\overline{z} = x^0 - ix^1$.

- Calculate the equations of motion for $\psi(z,\overline{z}), \overline{\psi}(z,\overline{z})$. Find an explicit expression of the stress-energy tensor. What do they imply?

- The OPE takes the form

  $$\psi(z,\overline{z})\psi(w,\overline{w}) = \frac{1}{2\pi g}\frac{1}{z-w} + :\psi(z,\overline{z})\psi(w,\overline{w}):$$

  Deduce that $\psi(z,\overline{z})$ is a primary field and find its weight.

- Calculate the OPE $T(z)T(w)$.