

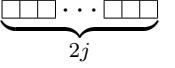
Homework 10: Due at class on Nov 30

1 Derivation

Derive (7.16) for the variation of the action (7.14) under $g \rightarrow g + \delta g$.

2 Solutions to KZ equations

2.1

The spin- j representation V_j of $\mathfrak{su}(2)$ can be labelled by the Young diagrams 

with one row. For instance, the spin- $\frac{1}{2}$ representation is two-dimensional and the basis are spanned by $|+\rangle$ and $|-\rangle$. It corresponds to a single box \square . The tensor product of the two spin- $\frac{1}{2}$ representations is decomposed into

$$(\square)^{\otimes 2} = \square \otimes \square = \square\square \oplus \mathbf{1}$$

where $\mathbf{1}$ is the trivial one-dimensional representation. This can be understood as the fusion rule. Let us write the basis $|j_1\rangle \otimes |j_2\rangle := |j_1, j_2\rangle$ of the tensor product $(\square)^{\otimes 2}$. Write the basis of the representation labelled by $\square\square$ as well as $\mathbf{1}$.

In addition, find the fusion rules of the following tensor products

$$(\square)^{\otimes 3}, \quad (\square)^{\otimes 4}, \quad \text{and} \quad (\square)^{\otimes 5}.$$

2.2

In the lecture, we have learned the correlation functions of N WZW primary fields obey the Knizhnik-Zamolodchikov (KZ) equation

$$\left[\partial_{z_i} + \frac{2}{k+h^\vee} \sum_{j(\neq i)=1}^n \frac{\sum_a t_{\lambda_i}^a \otimes t_{\lambda_j}^a}{z_i - z_j} \right] \langle \phi_{\lambda_1}(z_1) \cdots \phi_{\lambda_n}(z_n) \rangle = 0.$$

Let us consider the situation that $\mathfrak{g} = \mathfrak{su}(2)$ ($h^\vee = 2$) and all the primary fields are labelled by the spin- $\frac{1}{2}$ representation, *i.e.* $\phi_{\lambda_i}(z_i) = \phi_\square(z_i)$. In this situation, we write

$$\Omega_{ij} = \sum_a (t_\square^a)_i \otimes (t_\square^a)_j$$

where $t_\square^a = \frac{1}{2}\sigma^a$ with the Pauli matrices σ^a . By studying the action Ω_{12} on $|\pm, \pm\rangle$ explicitly, show

$$\Omega_{12} = \frac{1}{2} \left(s_{12} - \frac{1}{2} \right)$$

where s_{12} is the exchange of the first and second spin.

2.3

As in Problem 2.1, the fusion rule of n primary fields labelled by \square are

$$\begin{aligned} (\square)^{\otimes n} &= V_{\frac{n}{2}} \oplus (n-1)V_{\frac{n-2}{2}} \oplus \dots \\ &= \underbrace{\square\square\square\dots\square\square}_{n} \oplus (n-1) \underbrace{\square\square\square\dots\square\square}_{n-2} \oplus \dots \end{aligned}$$

Let us find the solutions of the KZ equations corresponding to $V_{\frac{n}{2}}$ and $V_{\frac{n-2}{2}}$. The highest weight state of the representation labelled by $V_{\frac{n}{2}}$ is $|+\dots+\rangle$. Writing down the correlation function corresponding to this state by

$$\Phi_{\frac{n}{2}}(z_1, \dots, z_n) = \psi_0(z_1, \dots, z_n)|+\dots+\rangle ,$$

find the solution of the KZ equation

$$\left[\partial_{z_i} + \frac{2}{k+2} \sum_{j(\neq i)=1}^n \frac{\Omega_{ij}}{z_i - z_j} \right] \Phi_{\frac{n}{2}}(z_1, \dots, z_n) = 0 .$$

The correlation function corresponding to the highest weight state of $V_{\frac{n-2}{2}}$ can be written as

$$\Phi_{\frac{n-2}{2}}(z_1, \dots, z_n) = \psi_0(z_1, \dots, z_n) \sum_{i=1}^n \psi_i(z_1, \dots, z_n) |v_i\rangle$$

where

$$|v_1\rangle = |-+\dots+\rangle , \quad |v_2\rangle = |+-+\dots+\rangle , \quad \dots , \quad |v_n\rangle = |+\dots+-\rangle ,$$

and

$$\sum_{i=1}^n \psi_i(z_1, \dots, z_n) = 0 .$$

Show that the KZ equations

$$\left[\partial_{z_1} + \frac{2}{k+2} \sum_{j=2}^n \frac{\Omega_{1j}}{z_1 - z_j} \right] \Phi_{\frac{n-2}{2}}(z_1, \dots, z_n) = 0$$

reduce to

$$(k+2) \frac{\partial}{\partial z_1} \psi_1(z) + \frac{\psi_2 - \psi_1}{z_1 - z_2} + \frac{\psi_3 - \psi_1}{z_1 - z_3} + \dots + \frac{\psi_N - \psi_1}{z_1 - z_N} = 0 .$$

and

$$(k+2) \frac{\partial}{\partial z_1} \psi_2(z) + \frac{\psi_1 - \psi_2}{z_1 - z_2} = 0 .$$

Show that

$$\psi_i(z) = \int_C dt \prod_{a=1}^n (z_a - t)^{\frac{1}{k+2}} \frac{1}{z_i - t}$$

becomes the solution of the KZ equations. In fact, there are $n-1$ solutions by taking the different contours C .