Differential Geometry and Topology in Physics: Spring-2017

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Abstract

These are lecture notes of course at Fudan in Spring 2017. Homework sets can be found in the website. These notes are written by referring to various sources without mentioning them. Comments are welcome. If you find typos, please email me.

Contents

1 Prologue

Introduction and overview. Hand-written note.

2 Lecture 2: [1] §5.1, 5.2, 5.3

2.1 Manifolds

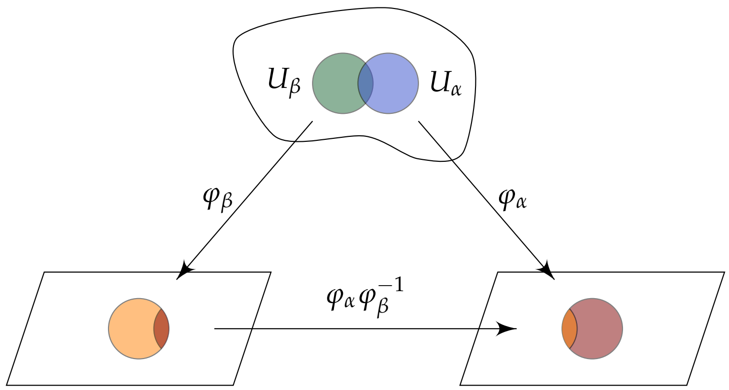
Definition 2.1 (Manifold). Let M be a Hausdorff space. M is called an n-dimensional smooth (differentiable) manifold if it has the following structure:

1. Let $M = \bigcup_\alpha U_\alpha$ be an open covering.
2. There is a continuous and invertible map $\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha) \subseteq \mathbb{R}^n$, where $\varphi(U_\alpha)$ is open in $\mathbb{R}^n$.

3. For all α, β, we have $\varphi_\alpha(U_\alpha \cap U_\beta)$ is open in $\mathbb{R}^n$, and the transition function

$$\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$$

is smooth (C^∞-fucntion).



$(U_\alpha, \varphi_\alpha)$ is called a **coordinate chart** and $\{(U_\alpha, \varphi_\alpha)\}_\alpha$ is called an **atlas**. We can write

$$\varphi_\alpha = (x^1, \cdots, x^n)$$

where each $x_i : U_\alpha \to \mathbb{R}$. We call these the **local coordinates**.

An important point of the definition of a smooth manifold is the following. If $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) are charts in some atlas, and $f : M \to \mathbb{R}$, then $f \circ \varphi_\alpha^{-1}$ is smooth at $\varphi_\alpha(p)$ if and only if $f \circ \varphi_\beta^{-1}$ is smooth at $\varphi_\beta(p)$ for all $p \in U_\alpha \cap U_\beta$.

Example 2.2. Consider the sphere

$$S^n = \{x = (x_0, \cdots, x_n) \in \mathbb{R}^{n+1} : \sum x_i^2 = 1\}.$$

We define an atlas as follows. We define an open covering

$$U^+ = S^n \backslash \{\text{south pole}\}, \quad U^- = S^2 \backslash \{\text{north pole}\}\ .$$

where the south pole is $x_0 = -1$ and the north pole is $x_0 = 1$, and continuous maps

$$\varphi^+ : U^+ \to \mathbb{R}^n; (x_0, \cdots, x_n) \mapsto \frac{1}{1+x_0}(x_1, \cdots, x_n)$$
$$\varphi^- : U^- \to \mathbb{R}^n; (x_0, \cdots, x_n) \mapsto \frac{1}{1-x_0}(x_1, \cdots, x_n)$$

Then, their inverse maps are

$$(\varphi^\pm)^{-1} : \mathbb{R}^n \to U^\pm; (y_1, \dots, y_n) \mapsto \frac{1}{1+|y|^2}(\pm(1-|y|^2), y_1, \dots, y_n)\ .$$

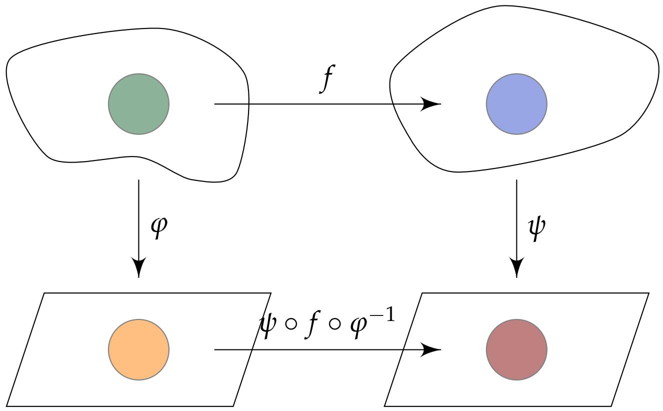
The transformation map

$$\varphi^+ \cdots (\varphi^-)^{-1}(y) = \frac{y}{|y|} \quad \varphi^- \cdots (\varphi^+)^{-1}(y) = \frac{y}{|y|}$$

Let M and N be smooth manifolds, and let $\{(U_\alpha, \varphi_\alpha)\}_\alpha$ and $\{(V_\beta, \psi_\beta)\}_\beta$ be their atlas. We usually consider smooth maps between manifolds.

Definition 2.3 (Smooth map). A map $f : M \to N$ is **smooth** if, for a chart $(U_\alpha, \varphi_\alpha)$ of $p \in M$ and (V_β, ψ_β) of $f(p) \in N$, $\psi_\beta \circ f \circ (\varphi_\alpha)^{-1} : \varphi(U_\alpha \cap f^{-1}(V_\beta)) \to \xi(V_\beta)$ is smooth.

If it has smooth inverse, it is called **diffeomorphism**.



Equivalently, f is smooth at p if $\xi \circ f \circ \varphi^{-1}$ is smooth at $\varphi(p)$ for **any** such charts (U, φ) and (V, ξ).

2.2 Tangent space

Definition 2.4 (Tangent vector). A **tangent vector** X_p at $p \in M$ is a map $X_p : C^\infty(M) \to \mathbb{R}$, which is subject to

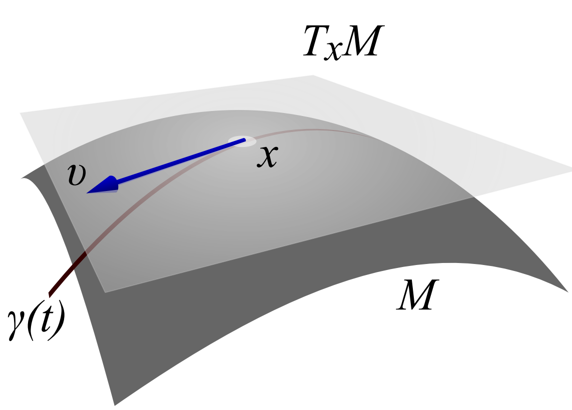
linearity $X_p(\alpha f + \beta g) = \alpha X_p(f) + \beta X_p(g)$, for $\alpha, \beta \in \mathbb{R}$

Leibniz rule $X_p(fg) = f(p)X_p(g) + X_p(f)g(p)$

Namely, a **tangent vector** X_p behaves like a differential operator on $C^\infty(M)$. Then, the set of tangent vectors at p become a vector space

$$(X_p + Y_p)(f) = X_p(f) + Y_p(f) \qquad (\alpha X_p)(f) = \alpha(X_p(f)),$$

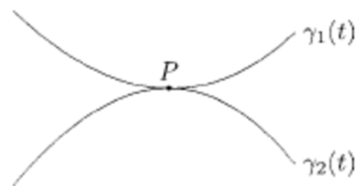
and we call it the **tangent space** T_pM at $p \in M$.



There is another way to think about tangent vectors. Let us consider a **curve**, which is a smooth map $\gamma : I \to M$ with $\gamma(0) = p$ where I is a non-empty open interval. Two curves γ_1, γ_2 are **tangent** at p if

$$\gamma_1(0) = p = \gamma_2(0) \,, \qquad \frac{d}{dt}\varphi(\gamma_1(t))\Big|_{t=0} = \frac{d}{dt}\varphi(\gamma_2(t))\Big|_{t=0}$$

where (U, φ) is a chart around p. We write two tangent curves $\gamma_1 \sim \gamma_2$, which forms an equivalence class. For $^\forall f \in C^\infty(M)$, We can then take the derivative of f along γ



$$X_p(f) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f(\gamma(t)) \,.$$

It is easy to see that X_p satisfies the definition of a tangent vector. So we can provide the definition of the tangent space at p as

$$T_pM = \{\gamma : I \to M | \gamma(0) = p\}/\sim$$

Given a local coordinate $\varphi = (x^1, \cdots, x^n)$, the tangent vector along a curve γ can be written as

$$X_p = \sum_{i=1}^n X_p^i \frac{\partial}{\partial x^i}\Big|_p \quad \text{where} \quad X_p^i = \frac{d}{dt}x^i(\gamma(t))\Big|_{t=0} \,.$$

Therefore, $(\frac{\partial}{\partial x^1}\Big|_p, \cdots, \frac{\partial}{\partial x^n}\Big|_p)$ can be considered as a basis of T_pM. Suppose we also have coordinates $y_1, \cdots, y_n$ near p given by some other chart. Then, we can write

$$\frac{\partial}{\partial y_i}\Big|_p = \sum_{j=1}^n \frac{\partial x_j}{\partial y_i}(p) \, \frac{\partial}{\partial x_j}\Big|_p \,,$$

where $\frac{\partial x_j}{\partial y_i}(p)$ is called **Jacobian** at p.

2.3 Tangent bundles

Let us consider a collection of tangent spaces over every point p on M

$$TM = \bigcup_{p \in M} T_pM = \{(p, X_p) | p \in M \,, X_p \in T_pM\} \,.$$

into a manifold. There is then a natural map $\pi : TM \to M$ sending $X_p \in T_pM$ to p for each $p \in M$, and this is smooth.

We can consider TM as a manifold of dimension $2 \dim M$, which is called the **tangent bundle** of M. Let $x^1, \cdots, x^n$ be coordinates on a chart (U, φ). Then for any $p \in U$ and $X_p \in T_pM$, there are some $\alpha^1, \cdots, \alpha^n \in \mathbb{R}$ such that

$$X^p = \sum_{i=1}^n \alpha^i \, \frac{\partial}{\partial x^i}\Big|^p \,.$$

This gives a bijection

$$
\begin{aligned}
\pi^{-1}(U) &\to \varphi(U) \times \mathbb{R}^n \\
X^p &\mapsto (x^1(p), \cdots, x^n(p), \alpha^1, \cdots, \alpha^n),
\end{aligned}
$$

If (V, ξ) is another chart on M with coordinates $y^1, \cdots, y^n$, then

$$
\left.\frac{\partial}{\partial x^i}\right|^p = \sum_{j=1}^n \frac{\partial y^j}{\partial x^i}(p) \left.\frac{\partial}{\partial y^j}\right|^p .
$$

So we have $\tilde{\xi} \circ \tilde{\varphi}^{-1} : \varphi(U \cap V) \times \mathbb{R}^n \to \xi(U \cap V) \times \mathbb{R}^n$ given by

$$
\tilde{\xi} \circ \tilde{\varphi}^{-1}(x^1, \cdots, x^n, \alpha^1, \cdots, \alpha^n) = \left(y^1, \cdots, y^n, \sum_{i=1}^n \alpha^i \frac{\partial y^1}{\partial x^i}, \cdots, \sum_{i=1}^n \alpha^i \frac{\partial y^n}{\partial x^i} \right),
$$

and is smooth (and in fact fiberwise linear).

2.4 Vector fields

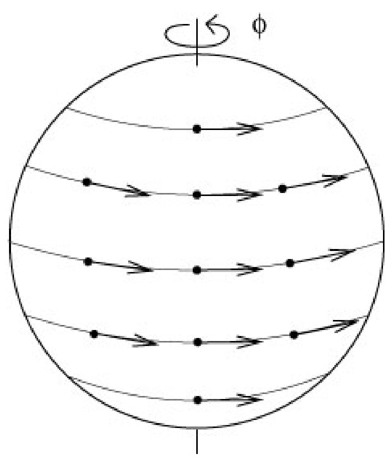
A smooth map $X : M \to TM$ is called a **section** if $X \cdot \pi = id_M$. Since it is actually a smooth assignment $X : p \mapsto X(p)$, it is called a **vector field**.

Example 2.5. Let $S^n \subset \mathbb{R}^{n+1}$ be an n-sphere. We have a vector field

$$
Y = \sum Y^i \frac{\partial}{\partial x^i}
$$

where

$$
Y^i = \begin{cases} (-x^1, x^0, -x^3, x^2, \cdots, -x^{2k}, x^{2k-1}) & n = 2k+1 \\ (-x^1, x^0, -x^3, x^2, \cdots, -x^{2k}, x^{2k-1}, 0) & n = 2k+2 \end{cases} .
$$



We write a set of vector fields by $\mathfrak{X}(M) = \Gamma(TM)$. In fact, a vector field $X \in \mathfrak{X}(M)$ is a map $C^\infty(M) \to C^\infty(M)$, which satisfies

$$
\begin{aligned}
&X(\alpha f + \beta g) = \alpha X(f) + \beta X(g), \quad \text{for} \quad \alpha, \beta \in \mathbb{R} \\
&X(fg) = f X(g) + X(f) g .
\end{aligned}
$$

Let $X, Y \in \mathfrak{X}(M)$, and $f \in C^\infty(M)$. Then we have $X+Y, fX \in \mathfrak{X}(M)$. Therefore, $\mathfrak{X}(M)$ is a $C^\infty(M)$-module.

Note that the product of two vector fields $X, Y \in \mathfrak{X}(M)$ is not a vector field:

$$\begin{aligned} XY(fg) &= X(Y(fg)) \\ &= X(fY(g) + gY(f)) \\ &= X(f)Y(g) + fXY(g) + X(g)Y(f) + gXY(f). \end{aligned}$$

However, that $XY - YX$ is a vector field and we denote it as $[X, Y]$, called the **Lie bracket**. In fact, $\mathfrak{X}(M)$ with Lie bracket satisfies

1. $[\cdot, \cdot]$ is bilinear.
2. $[\cdot, \cdot]$ is antisymmetric, i.e. $[X, Y] = -[Y, X]$.
3. The **Jacobi identity** holds

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

2.5 Flows

An **integral curve** of $X \in \mathfrak{X}(M)$ is a smooth $\gamma : I \to M$ such that I is an open interval in $\mathbb{R}$ and

$$\dot{\gamma}(t) = X_{\gamma(t)}.$$

Example 2.6. A vector field $X = \alpha\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial y}$ in $\mathbb{R}^2$. Then, the integral curve is a translation

$$\gamma_t(x, y) = (x + t\alpha, y + t\beta) .$$

Theorem 2.7 (Existence of integral curves). Let $X \in \mathfrak{X}(M)$ and $p \in M$. Then there exists some open interval $I \subseteq \mathbb{R}$ with $0 \in I$ and an integral curve $\gamma : I \to M$ for X with $\gamma(0) = p$.

Moreover, if $\tilde{\gamma} : \tilde{I} \to M$ is another integral curve for X, and $\tilde{\gamma}(0) = p$, then $\tilde{\gamma} = \gamma$ on $I \cap \tilde{I}$.

Let M be a a compact manifold. A smooth map

$$\gamma : \mathbb{R} \times M \to M; (t, p) \mapsto \gamma_t(p)$$

is called **flow** (or one-parameter group of diffeomorphisms) generated by a vector X if

$$\gamma_{t+s}(p) = \gamma_t(\gamma_s(p)) , \quad \frac{d\gamma_t(p)}{dt} = X_{\gamma_t(p)} .$$

2.6 Orientation

Suppose we have picked two ordered bases $(e_1, \cdots, e_n)$ and $(\tilde{e}_1, \cdots, \tilde{e}_n)$ of T_pM. Then, we define an equivalence class $(e_1, \cdots, e_n) \sim (\tilde{e}_1, \cdots, \tilde{e}_n)$ iff

$$e_i = \sum_j B_{ij}\tilde{e}_j \qquad \det B > 0 .$$

We call the two ordered bases have the **same orientation** if they are the same equivalence class. Therefore, for $p \in M$, we can assign an orientation $\mathcal{O}_p$. If we can give continuous assignment $p \mapsto \mathcal{O}_p$, then M is called **orientable**.

A smooth manifold M is orientable iff there exists an atlas $\{(U_\alpha, \varphi_\alpha)\}_\alpha$ of M such that the determinant of the Jacobian is positive on any $U_\alpha \cap U_\beta$.

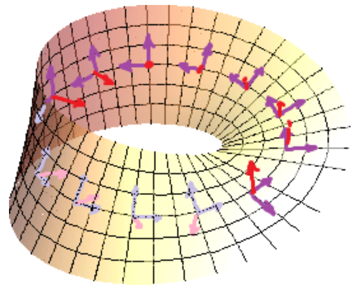


Figure 1: The Möbius strip is not orientable.

3 Lecture 3: [1] §5.4, 5.5

3.1 Cotangent bundles

Given a vector space V, one can take its dual space

$$V^* = \{f : V \to \mathbb{R} | f(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 f(v_1) + \alpha_2 f(v_2)\}$$

The dual space V^* is also a vector space: $\beta_1 f_1 + \beta_2 f_2 \in V^*$ for $f_1, f_2 \in V^*$ and $\beta_1, \beta_2 \in \mathbb{R}$. The dual vector space $T_p^* M$ of the tangent space $T_p M$ is called the **cotangent space**. In fact, given $f \in C^\infty(M)$, we can define its differential df_p at p

$$df_p : T_p M \to \mathbb{R}; X_p \mapsto X_p(f) \,.$$

For a local coordinate $(U, \varphi = (x^1, \cdots, x_n))$, we have seen that $(\frac{\partial}{\partial x^1}\Big|_p, \cdots, \frac{\partial}{\partial x^n}\Big|_p)$ is a basis of $T_p M$. On the other hand, we can take $(dx^1|_p, \cdots, dx^n|_p)$ as a basis of $T_p^* M$ so that

$$dx^i|_p(\frac{\partial}{\partial x^j}\Big|_p) = \delta_i^j \,.$$

Therefore, in this basis, we can write

$$df_p = \sum_{i=1}^n \frac{\partial f}{\partial x^i}(p) dx^i|_p$$

Like the tangent bundle, we can consider a collection of the cotangent spaces

$$T^* M = \cup_{p \in M} T_p^* M$$

which has a manifold structure. We call $T^* M$ the **cotangent bundle** of M. Moreover, the section of the cotangent bundle is called one-form, and we denote the set of one-form by $\Omega^1(M) = \Gamma(T^* M)$. For example, if f is a smooth function on M, then $df \in \Omega^1(M)$, which can take a paring with ${}^\forall X \in \mathfrak{X}(M)$

$$df(X) = X(f) \,.$$

3.1.1 push-forward and pull-back

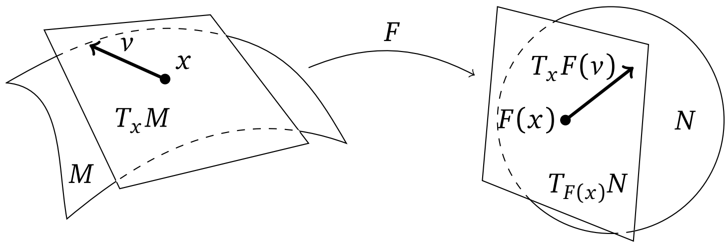
Let $f : M \to N$ be a smooth map between smooth manifolds M and N. It induces a push-forward of tangent vectors

$$f_* : T_p M \to T_{f(p)} N$$

which is defined by

$$f_*(X_p)(g) = X_p(g \circ f)$$

for $g \in C^\infty(N)$.



On the other hand, it induces pull-back of the cotangent space

$$f^* : T^*_{f(p)}N \to T^*_pM$$

which is defined by

$$\langle f^*\omega_{f(p)}, X_p\rangle = \langle \omega_{f(p)}, f_*X_p\rangle$$

for $\omega_{f(p)} \in T_{f(p)}N$ where $\langle\cdot,\cdot\rangle$ is a natural paring between the tangent and cotangent space.

3.2 Differential forms

In general, given a vector space V and its dual vector space V^*, one can consider k-forms, which are alternating k-linear maps $\wedge^k V^*$. Let us start with a product of 1-forms. If α_a are 1-forms (i.e., $\alpha_a \in V^*$), their wedge products are defined by

$$\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_k(v_1, v_2, \cdots, v_k) = \det(\alpha_a(v_b)) \ .$$

More generally, if α is a k-form and β is an ℓ-form,

$$(\alpha \wedge \beta)(v_1, \cdots, v_{k+\ell}) = \frac{1}{(k+\ell)!} \sum_{\sigma \in S_{k+\ell}} \mathrm{sign}\sigma\, \alpha(v_{\sigma(1)}, \cdots, v_{\sigma(k)})\beta(v_{\sigma(k+1)}, \cdots, v_{\sigma(k+l)}) \ .$$

k!l! We can choose a basis of the space of k-forms, $\wedge^k V^*$, as $e_{i_1} \wedge \cdots \wedge e_{i_k}$. Any k-form α can be expanded as

$$\alpha = \frac{1}{k!} \sum_I \alpha_{i_1, \cdots, i_k} e_{i_1} \wedge \cdots \wedge e_{i_k}.$$

Therefore, one can consider the k-th wedge product $\wedge^k T^*_pM$ of the cotangent space T^*_pM and its bundle $\wedge^k T^*M$. We write the set of sections as

$$\Omega^k(M) = \Gamma(\Lambda^k T^*M) = \{k\text{-forms on } M\}.$$

An element of $\Omega^k(M)$ is known as a **differential k-form**. In particular, we have

$$\Omega^0(M) = C^\infty(M).$$

In local coordinates $x^1, \cdots, x^n$ on U we can write $\omega \in \Omega^k(M)$ as

$$\omega = \sum_{i_1 < \ldots < i_p} \omega_{i_1, \ldots, i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p}$$

for some smooth functions $\omega_{i_1,\ldots,i_p}$.

Moreover, there exists a unique linear map called **exterior derivative**

$$d : \Omega^k(M) \to \Omega^{k+1}(M) \,,$$

such that

1. On $\Omega^0(M)$ this is as previously defined, i.e.

$$df(X) = X(f) \text{ for all } X \in \mathfrak{X}(M).$$

2. We have

$$d \circ d = 0 : \Omega^k(M) \to \Omega^{k+2}(M).$$

3. It satisfies the **Leibniz rule**

$$d(\omega \wedge \sigma) = d\omega \wedge \sigma + (-1)^k \omega \wedge d\sigma.$$

In term of local coordinates $x^1, \cdots, x^n$, we can define the exterior derivative as

$$d\left(\sum_{i_1<\ldots<i_p} \omega_{i_1,\ldots,i_p}\, dx^{i_1} \wedge \cdots \wedge dx^{i_p}\right) = \sum d\omega_{i_1,\ldots,i_p} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p} \,.$$

Let $f : M \to N$ be a smooth map between smooth manifolds M and N. The pull-back of differential forms associated to f can be defined as

$$(f^*\omega|_p)(v_1, \cdots, v_k) = \omega|_{f(p)}(df|_p(v_1), \cdots, df|_p(v_k)).$$

for $\omega \in \Omega^k(N)$ and $v_1, \cdots, v_k \in T_pM$. Note that the pull-back f^* has the following property

1. $f^* : \Omega^k(N) \to \Omega^k(M)$ is a linear map.
2. $f^*(\omega \wedge \sigma) = f^*\omega \wedge f^*\omega$.
3. If $g : N \to L$ is a smooth map between two manifolds N and L, then $(g \circ f)^* = f^* \circ g^*$.
4. It commutes with exterior derivative: $df^* = f^*d$.

3.3 Integrals of differential forms

Let M be n-dimensional orientable smooth manifold and $\omega \in \Omega^n(M)$. We would like to define the integral of ω over M. To this end, we define a partition of unity.

Definition 3.1 (Partition of unity). Let $\{U_\alpha\}$ be a locally-finite open cover of a manifold M. A **partition of unity** associated to $\{U_\alpha\}$ is a collection $\chi_\alpha \in C^\infty(M, \mathbb{R})$ such that

1. $0 \le \chi_\alpha \le 1$
2. $\mathrm{supp}(\chi_\alpha) \subseteq U_\alpha$
3. $\sum_\alpha \chi_\alpha = 1$.

If we use local coordinate $(x^1, \cdots, x^n)$ on a chart U_α, we can write

$$\chi_\alpha \omega = f_\alpha(x) x^1 \wedge \cdots \wedge x^n \,.$$

Therefore, we define its integral

$$\int_M \chi_\alpha \omega = \int \cdots \int f_\alpha(x) x^1 \cdots x^n \,.$$

Then, we can define

$$\int_M \omega = \sum_\alpha \int_M \chi_\alpha \omega \,.$$

One can show that this is independent of choice of a locally-finite open covering $\{U_\alpha\}$ and a partition of unity on $\{U_\alpha\}$.

Theorem 3.2 (Stokes' theorem). Let M be an oriented manifold with boundary of dimension n. Then if $\omega \in \Omega^{n-1}(M)$ has compact support, then

$$\int_M d\omega = \int_{\partial M} \omega.$$

In particular, if M has no boundary, then

$$\int_M d\omega = 0 \,.$$

4 Lecture 4: [1] §6.2, 6.3, 6.4, 7.1 and 7.9

4.1 de Rham cohomology

Given a differential operator

$$d : \Omega^k(M) \to \Omega^{k+1}(M) \,,$$

$\omega \in \Omega^k(M)$ is called a **closed form** if $d\omega = 0$, and an **exact form** if there exists $(k-1)$-form such that $\omega = d\eta$. Let us denote the set of all closed k-forms on M by $Z^k(M)$ and the set of all exact k-forms by $B^k(M)$.

$$\begin{aligned} Z^k(M) &= \mathrm{Ker}(d : \Omega^k(M) \to \Omega^{k+1}(M)) \\ B^k(M) &= \mathrm{Im}(d : \Omega^{k-1}(M) \to \Omega^k(M)) \end{aligned}$$

The de Rham cohomology group of M is defined as the quotient space

$$H^k_{dR}(M) = Z^k(M)/B^k(M)$$

In other words, the de Rham cohomology group of M is the cohomology of de Rahm complex

$$0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(M) \to 0$$

Lemma 4.1 (Poincaré Lemma). The de Rham cohomology of $\mathbb{R}^n$ is trivial. That is,

$$H^k(\mathbb{R}^n) = \begin{cases} \mathbb{R} & k = 0 \\ 0 & k \neq 0 \end{cases}$$

This also holds when M is contractible, namely when one can smoothly shrink M to a point. This is not true on a general manifold. However, it is true on each coordinate chart. The issue is how these charts are patched together globally.

4.2 Metric

A metric g on M is symmetric and non-degenerate bilinear form

$$g_p : T_pM \times T_pM \to \mathbb{R}$$

in such a way that g_p is smooth with respect to p. Its components are given by $g_{ij} = g(\partial_i, \partial_j)$. If g_{ij} is positive definite, (M, g) is called a Riemannian manifold. In a chart $U, (x^1 \cdots, x^n)$, it can be written locally as

$$ds^2 = g_{ij}(x) dx^i \otimes dx^j \ .$$

On an arbitrary smooth manifold, one can show there exists a Riemannian metric by using a partition of unity. On a Riemannian manifold, one can introduce the notion of the length of a tangent vector $v \in T_pM$

$$||v|| = \sqrt{g(v,v)} \ .$$

For a curve $c : [a,b] \to M$, the length $L(c)$ of curve can be defined by

$$L(c) = \int_a^b ||\dot{c}(t)|| dt = \int_a^b \sqrt{g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt}}$$

For a smooth map $f : M \to N$, the metric g on a smooth manifold N can be pull-back to the metric f^*g on a smooth manifold M in such a way that for $v, w \in T_pM$

$$f^*g(v,w) = g(f_*v, f_*w) \ .$$

Definition 4.2 (Isometry). Let (M, g) and (N, h) be Riemannian manifolds. We say $f : M \to N$ is an *isometry* if it is a diffeomorphism and $f^*h = g$. In other words, for any $p \in M$ and $u, v \in T_pM$, we need

$$h\big(f_*u, f_*v\big) = g(u,v).$$

In fact, given isometries f_1, f_2, its product $f_1 \circ f_2$ is also an isometry so that the set of isometries forms a group, called the **isometry group**.

For instance, any reflection, translation and rotation is a global isometry on Euclidean spaces. The isometry group $E(n)$ of $\mathbb{R}^n$ is the Euclidian group, and therefore it has as subgroups the translational group $T(n)$, and the orthogonal group $O(n)$. Any element of $E(n)$ is a translation followed by an orthogonal transformation (the linear part of the isometry), in a unique way:

$$x \mapsto A(x+b)$$

where $A \in O(n)$ is an orthogonal matrix.

Using a metric g, one have an isomorphism between the vector field and one form

$$\hat{g} : \mathfrak{X}(M) \cong \Omega^1(M)$$

in such a way that for vector fields $v, w \in \mathfrak{X}(M)$ on M,

$$\hat{g}(v)(w) = g(v,w)$$

By using the isomorphism $T_pM \cong T_p^*M$, we can introduce an inner product on T_p^*M.

4.3 Vielbeins, volume form, Hodge $*$ operator

For simplicity, we will assume that the metric g_{ij} is positive definite. For a metric with more general signature, we just have to introduce appropriate sign factors to some of the formulae below. Since the metric g_{ij} is symmetric, we can find a basis $\{e_i^a\}$ $(a = 1, \cdots, n)$ so that

$$g_{ij} = \sum_{a=1}^{n} e_i^a e_j^a \; .$$

This basis is called **the orthonormal frame**. For a given metric, an orthonormal frame is defined modulo $O(n)$.

This can be done at each point p on M. e^a's are called **vielbeins** where viel means many in German, and bein is a leg. (In 4 dimensions, they are also called vierbeins or tetrads. In dimensions other than 4, words like fünfbein, etc. have been used. Vielbein covers all dimensions.) Using the vierbeins, the volume form vol is defined by

$$\mathrm{vol} = e^1 \wedge e^2 \wedge \cdots \wedge e^n \; .$$

Note that it may not be possible to define vol globally on M since it is invariant under $SO(n)$ but not under $O(n)$. It may not be possible to choose a sign factor for vol (associated to $\mathbb{Z}_2 = O(n)/SO(n)$) consistently over M. The volume form is well-defined if and only if M is orientable.

Using coordinates, we can express the volume form as

$$\mathrm{vol} = \sqrt{g} dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n \; ,$$

where $g = \det g$ (we are assuming that the metric is positive definite). There is an isomorphism

$$* : \Omega^k(M) \to \Omega^{n-k}(M)$$

is defined by

$$*(e^1 \wedge \cdots \wedge e^k) = e^{k+1} \wedge \cdots \wedge e^n$$

For a k-form ω, the **Hodge $*$ operator** is defined as

$$(*\omega)_{i_{k+1}\cdots i_n} = \frac{1}{k!} \frac{\epsilon^{j_1\cdots j_k j_{k+1}\cdots j_n}}{\sqrt{g}} \omega_{j_1\cdots j_k} g_{j_{k+1}i_{k+1}} \cdots g_{j_n i_n} \; .$$

Here I used the totally anti-symmetric tensor $\epsilon_{i_1\cdots i_n}$ and $\epsilon^{i_1\cdots i_n}$ normalized as

$$\epsilon_{12\cdots n} = \epsilon^{12\cdots n} = 1$$

The important point is that the Hodge star depends on the metric g. Under coordinate transformations, $\epsilon_{i_1\cdots i_n}$ does not transform as a tensor. However, we can remedy this by multiplying $\sqrt{g}$ to make it into the volume form. The volume form transforms as a tensor if coordinate transformations preserve the orientation. If we change the orientation, we get an extra (-1).

4.4 Adjoint operator

The adjoint operator δ on Ω^k of the exterior differential d is defined by

$$\delta\omega = (-1)^{nk+n+1} * d * \omega \; .$$

We have commutative diagram

$$
\begin{array}{ccc}
\Omega^k(M) & \xrightarrow{\;*\;} & \Omega^{n-k}(M) \\
\big\downarrow{\scriptstyle\delta} & & \big\downarrow{\scriptstyle d} \\
\Omega^{k-1}(M) & \xrightarrow{(-1)^k *} & \Omega^{n-k+1}(M)
\end{array}
$$

We can easily verify the following properties,

$$\delta^2 = 0 \,, \quad *\delta d = d\delta * \,, \quad d * \delta = \delta * d = 0 \,.$$

It is an adjoint operator in the sense that

$$(d\omega, \eta) = (\omega, \delta\eta) \,.$$

where we assume that M is an oriented, compact, closed manifold. where the inner product on $\Omega^k(M)$ is defined by

$$(\rho, \sigma) = \int_M \rho \wedge *\sigma = \int_M \sigma \wedge *\rho$$

On a Riemannian manifold M, an operator defined by

$$\Delta = \delta d + d\delta : \Omega^k(M) \to \Omega^k(M)$$

is called **Laplace-Beltrami operator**. A form $\omega \in \Omega^*(M)$ such that $\Delta\omega = 0$ is called a **harmonic form**. In particular, a function f such that $\Delta f = 0$ is called a **harmonic function**. One can show that a necessary and sufficient condition of harmonic form: $\Delta\omega = 0$ iff $d\omega = 0 = \delta\omega$.

4.5 Hodge theorem and Hodge decomposition

Let us denote the set of all harmonic k-forms on M

$$\mathbb{H}^k(M) = \{\omega \in \Omega^k(M) | \Delta\omega = 0\} \,.$$

Theorem 4.3 (Hodge theorem). On an oriented compact Riemanninan manifold, the natural map $\mathbb{H}^k(M) \to H^k_{dR}(M)$ is isomorphism.

Theorem 4.4 (Hodge decomposition). For an oriented compact Riemanninan manifold, we have the orthogonal decomposition

$$\Omega^k(M) = \mathbb{H}^k(M) \oplus d\Omega^{k-1}(M) \oplus \delta\Omega^{k+1}(M)$$

Applications of the Hodge theorem is as follows.

Theorem 4.5 (Poincare duality). For an connected, oriented, compact n-dimensional Riemanninan manifold, the bilinear map

$$H^k_{dR}(M) \times H^{n-k}_{dR}(M) \to \mathbb{R};\ (\omega, \eta) \mapsto \int_M \omega \wedge \eta$$

is non-degenerate and hence induces an isomorphism

$$H^{n-k}_{dR}(M) \cong H^k_{dR}(M)^*$$

The Hodge theorem tells us that $\omega \in H^*k(M)$ is a harmonic form. Due to $d* = *d$, $\eta = *\omega$ is also a harmonic form. If $\omega \neq 0$, we have

$$\int_M \omega \wedge \eta = ||\omega||^2 \neq 0$$

5 Lecture 5: [1] §7.2, 7.3, and 7.4

5.1 Covariant derivative and parallel transport

Let X and Y be vector fields on M. The symbol $\nabla_X Y$ denotes the derivative of the vector field Y along trajectories of the vector field X. In fact, it is a map

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M); (X, Y) \mapsto \nabla_X Y$$

which satisfies the following conditions

- ∇ is linear in the first variable and additive in the second:

$$\begin{aligned} \nabla_{fX+hY} Z &= f\nabla_X Z + h\nabla_Y Z \\ \nabla_X(Y+Z) &= \nabla_X Y + \nabla_X Z \end{aligned}$$

  where $f, h \in \mathbb{C}^\infty(M)$ are functions and $X, Y \in \mathfrak{X}(M)$ are vector fields.

- ∇ obeys the Leibniz rule in the second variable:

$$\nabla_X(fY) = X(f)Y + f\nabla_X Y .$$

The operator ∇ is called **covariant derivative** or the **connection**.

Let $(U, \{x^i\})$ be a chart on M. Since $\nabla_{\partial/\partial x^i} \frac{\partial}{\partial x^j}$ is a vector field, it can be expressed as a linear combination of the coordinate fields:

$$\nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^k} := \sum_{i=1}^{n} \Gamma^i_{jk} \frac{\partial}{\partial x^i}. \tag{5.1}$$

These are called the **Christoffel symbols**.

Let $\gamma : I \to M$ be a curve on M and we define

$$J(\gamma) = \{\text{vector fields along } \gamma\}.$$

Then, given a connection ∇, we define the **covariant derivative** along γ, which is a map $\frac{D}{dt} : J(\gamma) \to J(\gamma)$ such that

- $\frac{D}{dt}(fX) = \dot{f}X + f\frac{D}{dt}X$ for all $f \in C^\infty(I)$
- If $X \in J(\gamma)$ is induced by a vector field $\tilde{X}$ ($\tilde{X}|_{\gamma(t)} = X_t$ for all $t \in I$), then

$$\frac{D}{dt}(X)\Big|_{t=0} = \nabla_{\dot{\gamma}(0)}\tilde{X}.$$

Given a path $\gamma(t)$ (not necessarily a geodesic), a vector field X is called **parallel**, or constant, along γ if

$$\frac{D}{dt}X = 0. \tag{5.2}$$

In coordinates we can write

$$\frac{d}{dt} = \frac{dx^i}{dt}\frac{\partial}{\partial x^i} \qquad X = X^j \frac{\partial}{\partial x^j}. \tag{5.3}$$

Using the axioms of covariant differentiation, we compute

$$\begin{aligned}
\frac{D}{dt}X &= \nabla_{\frac{dx^i}{dt}\frac{\partial}{\partial x^i}}\left(X^j\frac{\partial}{\partial x^j}\right) \\
&= \frac{dx^i}{dt}\nabla_{\frac{\partial}{\partial x^i}}\left(X^j\frac{\partial}{\partial x^j}\right) \\
&= \frac{dx^i}{dt}\frac{\partial}{\partial x^i}\left(X^j\right)\frac{\partial}{\partial x^j} + \frac{dx^i}{dt}X^j\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j} \\
&= \frac{dX^j}{dt}\frac{\partial}{\partial x^j} + \frac{dx^i}{dt}X^j\Gamma^k_{ij}\frac{\partial}{\partial x^k} \\
&= \left(\frac{dX^k}{dt} + \frac{dx^i}{dt}X^j\Gamma^k_{ij}\right)\frac{\partial}{\partial x^k}.
\end{aligned} \tag{5.4}$$

The Christoffel symbols Γ^k_{ij}, the path γ, and the derivatives $\frac{dx^i}{dt}$ are known. Therefore the parallel transport equation is a system of n first order linear differential equations:

$$\frac{D}{dt}X = 0 \qquad \text{if and only if} \qquad \frac{dX^k}{dt} + X^j\frac{dx^i}{dt}\Gamma^k_{ij} = 0 \quad \text{for all } 1 < k < n. \tag{5.5}$$

This means that given a single vector $X \in T_{\gamma(0)}M$, it can be **transported** along the curve γ by solving this system of equations.

5.2 Levi-Civita Connection

Recall that in order to differentiate vectors, or even tensors on a manifold, we need a connection on the tangent bundle. There is a natural choice for the connection when a Riemannian metric is given.

Definition 5.1 (Levi-Civita connection). Let (M, g) be a Riemannian manifold. The *Levi-Civita connection* is the unique connection ∇ on M satisfying

- Compatibility with metric:
$$Zg(X,Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y),$$
- Symmetry/torsion-free:
$$\nabla_X Y - \nabla_Y X = [X, Y].$$

With a bit more imagination on what the symbols mean, we can write the first property as

$$d(g(X,Y)) = g(\nabla X, Y) + g(X, \nabla Y),$$

while the second property can be expressed in coordinate representation by

$$\Gamma^i_{jk} = \Gamma^i_{kj}\,.$$

On Riemannian manifold (M, g), there exists a unique torsion-free connection ∇ compatible with g. Using the metric compatibility condition

$$\frac{\partial}{\partial x^i}g_{jk} = g\left(\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right) + g\left(\frac{\partial}{\partial x^j}, \nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^k}\right)$$

Changing the index labels, we have the formula

$$\Gamma^k_{ij} = \frac{1}{2}\left(\frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l}\right)g^{lk}. \tag{5.6}$$

Definition 5.2 (Geodesic). A curve $\gamma(t)$ on a Riemannian manifold (M, g) is called a *geodesic curve* if its tangent vectors are parallel transported along the curve itself with respect to the Levi-Civita connection:

$$\frac{D\dot{\gamma}}{dt} = 0.$$

In local coordinates, we write this condition as

$$\ddot{x}_i + \Gamma^i_{jk} \dot{x}^j \dot{x}^k = 0.$$

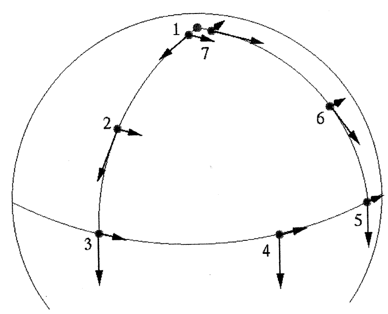
A geodesic is uniquely specified by the initial conditions $p = x(0)$ and $a = \dot{x}(0)$. This equation can be obtained as Euler-Lagrange equation of the action

$$S = \int \sqrt{g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt}}\,,$$

so, on a sufficiently small chart U, the curve is a shortest distant path with respect to the metric g.

5.3 Riemann curvature

We can consider doing some parallel transports on S^n along the loop counterclockwise: We see that after the parallel transport around the loop, we get a *different* vector.



The Riemann curvature is introduced to measure this difference.

The Riemann tensor is a map

$$R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M); (X, Y, Z) \mapsto R(X, Y)\,Z$$

where $R(X, Y)\,Z$ is defined as follows:

$$R(X, Y)\,Z := \nabla_X \nabla_Y Z \;-\; \nabla_Y \nabla_X Z \;-\; \nabla_{[X,Y]} Z. \tag{5.7}$$

Note the obvious fact that R is anti-symmetric in the first two variables:

$$R(X, Y)\,Z \;=\; -R(Y, X)\,Z. \tag{5.8}$$

The Riemann tensor is also called the **curvature tensor**.

On a chart, the Riemann tensor can be written in terms of the coordinate fields:

$$R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) \frac{\partial}{\partial x^k} \;=\; R^l{}_{kij} \frac{\partial}{\partial x^l}. \tag{5.9}$$

In fact, we can show that the Riemann curvature can be expressed in terms of Christoffel symbols

$$R^l{}_{kij} = \Gamma^s_{jk}\Gamma^l_{is} \; - \; \Gamma^s_{ik}\Gamma^l_{js} \; + \; \frac{\partial \Gamma^l_{jk}}{\partial x^i} \; - \; \frac{\partial \Gamma^l_{ik}}{\partial x^j}. \tag{5.10}$$

Essentially the Riemann tensor measures the **failure of commutativity of two derivatives** when applied to vector fields. Let us take infinitesimal parallelogram as in the

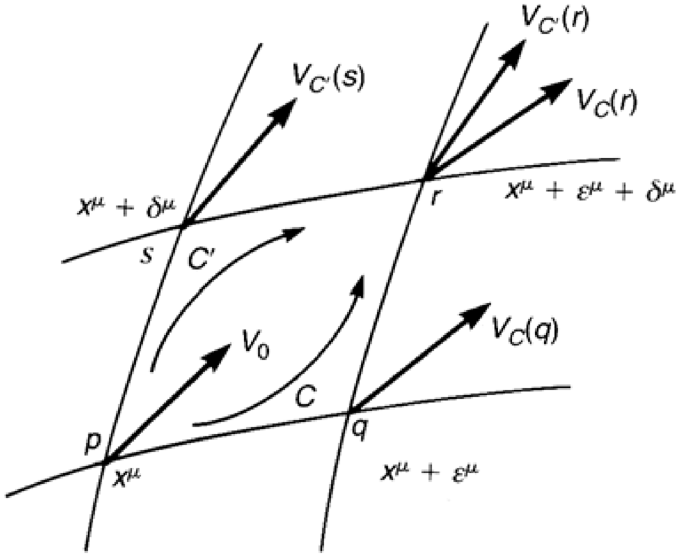


figure. Then, the difference between the two vectors are indeed given by the Riemann curvature

$$V_{C'}(r) - V_C(r) = V_0^j R^i_{jkl} \epsilon^k \delta^l$$

Contracting the first and final indices of the Riemann tensor gives the **Ricci curvature tensor**

$$R_{ij} = \mathrm{Ric}_{ij} := R^s{}_{isj} \; = \; R_{kijl} g^{kj}. \tag{5.11}$$

If there is a constant λ such that $R_{ij} = \lambda g_{ij}$, M, g is called a **Einstein manifold**.

Contracting again, we get the **scalar curvature**

$$R := \mathrm{Ric}_{ij} g^{ij}. \tag{5.12}$$

The scalar curvature is essentially the sum of all sectional curvatures at a point.

5.3.1 Curvature dentities

We can introduce the notation

$$\begin{gathered} R(X, Y, Z, W) = \langle R(X,Y)Z, W \rangle \\ R_{ijk\ell} = R^s{}_{jk\ell} g_{si}. \end{gathered} \tag{5.13}$$

The Riemann tensor satisfy the following identity

$$\begin{aligned}
R(X,Y,Z,W) &= -R(Y,X,Z,W) \\
R(X,Y,Z,W) &= -R(X,Y,W,Z) \\
R(X,Y,Z,W) &= R(Z,W,X,Y) \\
R(X,Y)Z + R(Y,Z)X + R(Z,X)Y &= 0 \qquad \text{(first Bianchi identity)} \\
(\nabla_W R)(X,Y)Z + (\nabla_X R)(Y,W)Z + (\nabla_Y R)(W,X)Z &= 0 \qquad \text{(second Bianchi identity)}
\end{aligned} \tag{5.14}$$

In components, these read

$$\begin{aligned}
R_{ijkl} &= -R_{ijlk} \\
R_{ijkl} &= -R_{jikl} \\
R_{ijkl} &= R_{klij} \\
R_{lijk} + R_{ljki} + R_{lkij} &= 0 \\
R_{lijk,s} + R_{ljsk,i} + R_{lsik,j} &= 0.
\end{aligned} \tag{5.15}$$

Further identities can be derived by 'mixing and matching' these identities. For instance the second Bianchi identity is frequently presented

$$R_{jkli,s} + R_{jlsi,k} + R_{jski,l} = 0, \tag{5.16}$$

which can be derived from our second Bianchi identity and the rule $R_{ijkl} = R_{klij}$.

5.4 Gauss-Bonnet theorem

Let (M,g) be a two-dimensional oriented closed manifold and vol is the volume form. Then Gauss curvature κ is defined by

$$\kappa := -\frac{R_{1212}}{\det g} .$$

Then, the Gauss-Bonnet theorem is

$$\frac{1}{2\pi}\int_M \kappa \,\mathrm{vol} = \chi(M) .$$

It is remarkable in the sense that the integral in LHS is independent of choice of metric g although the Gauss curvature κ depends on the metric g. This formula connects differential geometry to topology.

5.5 Einstein equations

The celebrated Einstein equations can be expressed

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} ,$$

where Λ is called the cosmological constant, G is the Newton constant, c is the speed of light and $T_{\mu\nu}$ is the stress-energy tensor. These are the Euler-Lagrange equations for the Einstein-Hilbert action

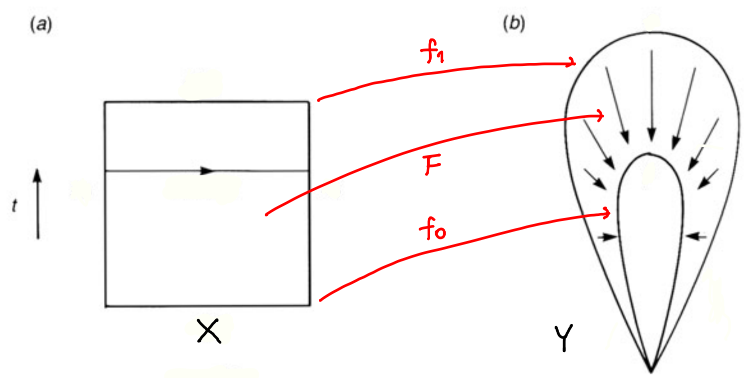
$$S = \int \left[\frac{c^4}{16\pi G}(R - 2\Lambda) + \mathcal{L}_{\mathrm{M}}\right]\sqrt{-g}\,\mathrm{d}^4x .$$

In this equation, a metric becomes dynamical variable and, roughly speaking, the curvature tensor of the spacetime is determined by the stress-energy tensor of matter. The equation is highly non-linear so that it can be solved analytically only in very special situations. The most of the cases requires numerical simulations to solve the Einstein equations. Remarkably, this equation describes nature at terrestrial scale, and predicts the gravitational wave, which was detected by LIGO last year. (The No.1 candidate of Nobel physics prize this year!) It also describes the history of the universe [3].

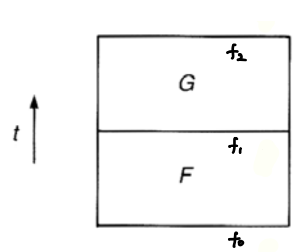
6 Lecture 6: [1] §4

6.1 Homotopy

In topology, we study spaces up to "continuous deformation". Famously, a coffee mug can be continuously deformed into a doughnut, and thus they are considered to be topologically the same. Now we also talk about maps between topological spaces. So a natural question is if it makes sense to talk about the continuous deformations of maps. It turns out we can, and the definition is sort-of the obvious one:



Definition 6.1 (Homotopy). Let X, Y be a topological space. A **homotopy** between $f_0, f_1 : X \to Y$ is a map $F : [0,1] \times X \to Y$ such that $F(0,x) = f_0(x)$ and $F(1,x) = f_1(x)$. If such an F exists, we say f_0 is **homotopic** to f_1, and write $f_0 \simeq f_1$.



As easily can ben seen from the figure above, homotopy $\simeq$ defines an equivalence relation on the set of maps from X to Y. In algebraic topology, we study quantities invariant under homotopies.

Definition 6.2 (Homotopy equivalence). A map $f : X \to Y$ is a **homotopy equivalence** if there is some $g : Y \to X$ such that $f \circ g \simeq \mathrm{id}_X$ and $g \circ f \simeq \mathrm{id}_Y$. We call g the **homotopy inverse** to f.

If $f_0 \simeq f_1 : X \to Y$ and $g_0 \simeq g_1 : Y \to Z$, then $g_0 \circ f_0 \simeq g_1 \circ f_1 : X \to Z$.

$$\begin{tikzcd} X \arrow[r, bend left, "f_0"] \arrow[r, bend right, "f_1"'] & Y \arrow[r, bend left, "g_0"] \arrow[r, bend right, "g_1"'] & Z \end{tikzcd}$$

Therefore, homotopy equivalence is an equivalence relation.

Example 6.3. Let $i : \{0\} \to \mathbb{R}^n$ be the inclusion map. A homotopy equivalence can be constructed by $F : [0,1] \to \mathbb{R}^n \to \mathbb{R}^n$ be

$$F(t, \mathbf{v}) = t\mathbf{v}.$$

We have $F(0, \mathbf{v}) = 0$ and $F(1, \mathbf{v}) = \mathbf{v}$. From the point of view of homotopy, the point $\{0\}$ is equivalent to $\mathbb{R}^n$, so that dimension is irrelevant.

Example 6.4. Let $S^n \subseteq \mathbb{R}^{n+1}$ be the unit sphere, and $i : S^n \hookrightarrow \mathbb{R}^{n+1} \setminus \{0\}$. We show that this is a homotopy equivalence. We define $r : \mathbb{R}^{n+1} \setminus \{0\} \to S^n$ by

$$r(\mathbf{v}) = \frac{\mathbf{v}}{\|\mathbf{v}\|}.$$

It is easy to see $r \circ i = \mathrm{id}_{S^n}$. In the other direction, we need to construct a path from each $\mathbf{v}$ to $\frac{\mathbf{v}}{\|\mathbf{v}\|}$ in a continuous way. We could do so by

$$\begin{aligned} H : [0,1] \times (\mathbb{R}^{n+1} \setminus \{0\}) &\to \mathbb{R}^{n+1} \setminus \{0\} \\ (t, \mathbf{v}) &\mapsto (1-t)\mathbf{v} + t\frac{\mathbf{v}}{\|\mathbf{v}\|}. \end{aligned}$$

We can easily check that this is a homotopy from $\mathrm{id}_{\mathbb{R}^{n+1} \setminus \{0\}}$ to $i \circ r$.

Even though the dimensions are different, homotopy equivalence does not loose information about "hole".

6.2 Fundamental groups

A **path** in a space X is a map $\gamma : I \to X$. If $\gamma(0) = x_0$ and $\gamma(1) = x_1$, we say γ is a path from x_0 to x_1. If $\gamma(0) = \gamma(1)$, then γ is called a **loop** (based at x_0). The idea of fundamental groups is to consider homotopy equivalence of loops at a fixed base point x_0

Definition 6.5 (Fundamental group). Let X be a space and $x_0 \in X$. The **fundamental group** of X (based at x_0), denoted $\pi_1(X, x_0)$, is the set of homotopy classes of loops in X based at x_0 (i.e. $\gamma(0) = \gamma(1) = x_0$). The group operations are defined as follows:

We define an operation by $[\gamma_0][\gamma_1] = [\gamma_0 \cdot \gamma_1]$; inverses by $[\gamma]^{-1} = [\gamma^{-1}]$; and the identity as the constant path $e = [c_{x_0}]$.

A **based space** is a pair (X, x_0) of a space X and a **basepoint** $x_0 \in X$. A **map of based spaces**

$$f : (X, x_0) \to (Y, y_0)$$

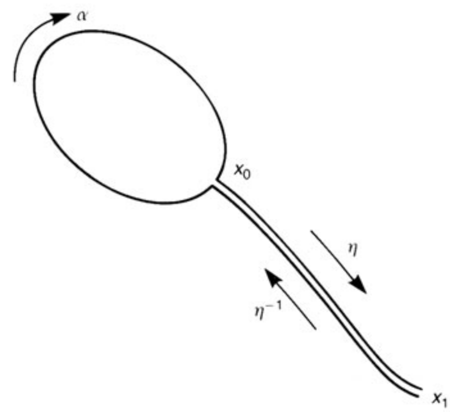
is a continuous map $f : X \to Y$ such that $f(x_0) = y_0$. Given a based map, there is an associated function

$$f_* = \pi_1(f) : \pi_1(X, x_0) \to \pi_1(Y, y_0),$$

defined by $[\gamma] \mapsto [f \circ \gamma]$. Moreover, it satisfies

1. $\pi_1(f)$ is a homomorphism of groups.
2. If $f \simeq f'$, then $\pi_1(f) = \pi_1(f')$.
3. For any maps $(A, a) \xrightarrow{h} (B, b) \xrightarrow{k} (C, c)$, we have $\pi_1(k \circ h) = \pi_1(k) \circ \pi_1(h)$.
4. $\pi_1(\mathrm{id}_X) = \mathrm{id}_{\pi_1(X, x_0)}$

It is easy to see from the figure below that fundamental groups are independent of the choice of a basepoint.

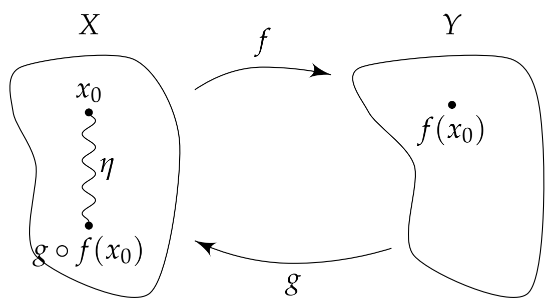


Theorem 6.6. If $f : X \to Y$ is a homotopy equivalence, and $x_0 \in X$, then the induced map

$$f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0)).$$

is an isomorphism.

Proof. Let $g : Y \to X$ be a homotopy inverse. So $f \circ g \simeq_H \mathrm{id}_Y$ and $g \circ f \simeq'_H \mathrm{id}_X$.



We have no guarantee that $g \circ f(x_0) = x_0$, but we know that our homotopy H' gives us $\eta = H'(x_0, \cdot) : x_0 \rightsquigarrow g \circ f(x_0)$.

Applying our previous lemma with id_X for "f" and $g \circ f$ for "g", we get

$$\eta_\# \circ (\mathrm{id}_X)_* = (g \circ f)_*$$

Using the properties of the $_*$ operation, we get that

$$g_* \circ f_* = \eta_\#.$$

However, we know that $\eta_\#$ is an isomorphism. So f_* is injective and g_* is surjective.

Doing it the other way round with $f \circ g$ instead of $g \circ f$, we know that g_* is injective and f_* is surjective. So both of them are isomorphisms. $\square$

Definition 6.7 (Simply connected space). A space X is **simply connected** if it is path connected **and** $\pi_1(X, x_0) \cong 1$ for some (any) choice of $x_0 \in X$.

Example 6.8. Clearly, a point $*$ is simply connected since there is only one path on $*$ (the constant path). Hence, any contractible space is simply connected since it is homotopic to $*$. For example, $\mathbb{R}^n$ is simply connected for all spaces.

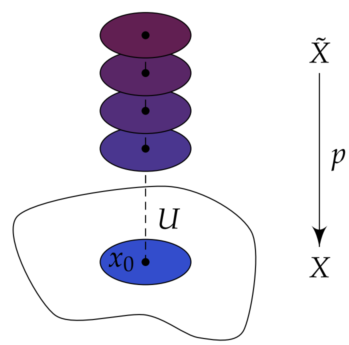
There is an important theorem, called the van Kampen theorem, for computations of fundamental groups. In the following, you will find the simple version of the theorem.

Theorem 6.9 (Simple version of van Kampen theorem). If $X = U_1 \cup U_2$ with U_i open and path-connected, and $U_1 \cap U_2$ path-connected and simply connected, then there is an isomorphism $\pi_1(U_1) * \pi_1(U_2) \cong \pi_1(X)$.

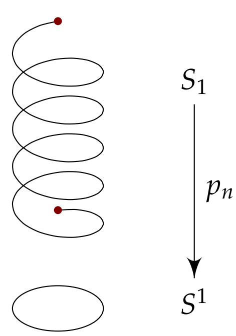
Due to time limit, I have to omit general version of van Kampen theorem. If you are interested in it, you can read the book of Hatcher [5, §1.2].

6.2.1 Covering space

Intuitively, a *covering space* of X is a pair $(\tilde{X}, p : \tilde{X} \to X)$, such that if we take any $x_0 \in X$, there is some neighbourhood U of x such that the pre-image of the neighbourhood is "many copies" of U.



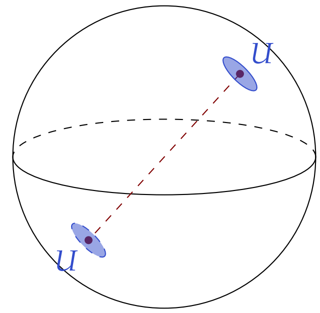
Example 6.10. Consider $p_n : S^1 \to S^1$ (for any $n \in \mathbb{Z} \setminus \{0\}$) defined by $z \mapsto z^n$. We can consider this as "winding" the circle n times, or as the following covering map:



where we join the two red dots together.

This time the pre-image of 1 would be n copies of 1, instead of $\mathbb{Z}$ copies of 1.

Example 6.11. Consider $X = \mathbb{RP}^2$, which is the real projective plane. This is defined by $S^2/\sim$, where we identify every $x \simeq -x$, i.e. every pair of antipodal point. In fact, the quotient map $p : S^2 \to \mathbb{RP}^2$ is indeed a covering map, since the pre-image of a small neighbourhood of any x_0 is just two copies of the neighbourhood.



Definition 6.12 (Universal cover). A covering map $p : \tilde{X} \to X$ is a **universal cover** if $\tilde{X}$ is simply connected.

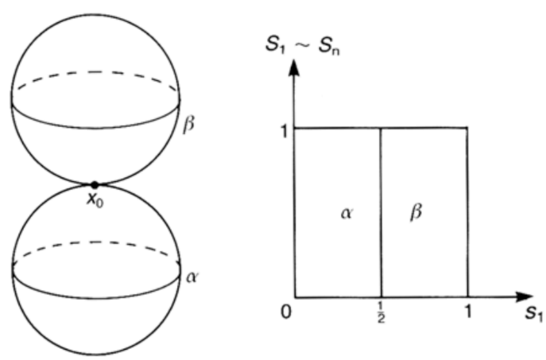
6.3 Homotopy group

Let I^n $(n \geq 1)$ denote the unitn-cube $I \times \cdots \times I$, The boundary ∂I^n is the geometrical boundary of I^n. As in the fundamental group, we assume here that we shall be concerned with continuous maps $\alpha : I^n \to X$, which map the boundary ∂I^n to a point $x_0 \in X$. Since the boundary is mapped to a single point x_0. The map α is called an n-loop at x_0.

Definition 6.13 (Homotopy class). Let X be a topological space. The set of homotopy classes of n-loops $(n \geq 1)$ at $x_0 \in X$ is denoted by $\pi_n(X, x_0)$ and called the n-th homotopy group at x_0. $\pi_n(X, x_0)$ is called the higher homotopy group if $n \geq 2$.

The product $\alpha * \beta$ just defined naturally induces a product of homotopy classes defined by

$$[\alpha] * [\beta] = [\alpha * \beta]$$

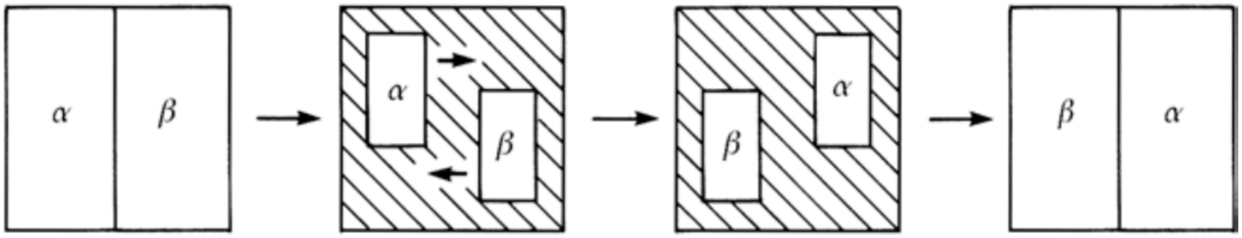


Higher homotopy groups are always Abelian; for any n-loops α and β at $x_0 \in X$, $[\alpha]$ and $[\beta]$ satisfy

$$[\alpha] * [\beta] = [\beta] * [\alpha].$$

6.3.1 Global $SU(2)$ anomaly

There are various applications of homotopy groups to physics. Here I just give you one example.



Let us consider the path integral of a gauge theory with chiral fermion ψ in a representation R of G.

$$Z[A_\mu] = \int [\mathcal{D}\psi][\mathcal{D}\bar{\psi}] e^{-\int \bar{\psi} D_\mu \sigma^\mu \psi} \tag{6.1}$$

To perform a further integration over A_μ consistently, we need

$$Z[A_\mu] = Z[A_\mu^g], \quad A_\mu^g = g^{-1} A_\mu g + g^{-1} \partial_\mu g. \tag{6.2}$$

for any gauge transformation $g : \mathbb{R}^4 \to G$. We will learn it in a subsequent lecture about gauge transformation of non-Abelian gauge theory.

When we consider odd number of chiral fermions in the doublet of SU(2) gauge group, Witten has shown that there is a global anomaly [4]. Since continuous change of g does not change the phase of $[\mathcal{D}\psi][\mathcal{D}\bar{\psi}]$, we need to consider maps $g : \mathbb{R}^4 \to \mathrm{SU}(2)$ up to continuous change. Upon one-point compactifiction of $\mathbb{R}^4$ to S^4, they are characterized by $\pi_4(\mathrm{SU}(2))$, which is known

$$\pi_4(\mathrm{SU}(2)) = \pi_4(S^3) = \mathbb{Z}_2 \,. \tag{6.3}$$

Let $g_0 : \mathbb{R}^4 \to \mathrm{SU}(2)$ be the gauge transformation corresponding to the nontrivial element in this $\mathbb{Z}_2$. It is known that $[\mathcal{D}\psi][\mathcal{D}\bar{\psi}]$ gets a minus sign under this gauge transformation. So, one cannot have an odd number of Weyl fermions in the doublet representation of gauge group SU(2).

7 Lecture 7: [1] §3

7.1 Simplicial complexes

The relevance is that these can be used to define simplexes (which are simple, as opposed to complexes).

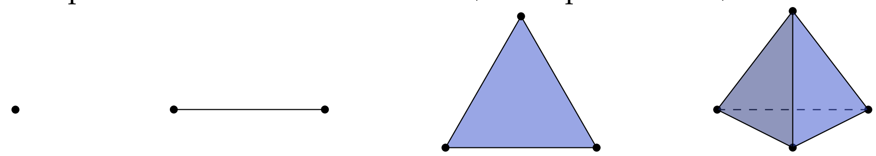
Definition 7.1 (n-simplex). An **n-simplex** is the convex hull of $(n+1)$ affinely independent points $a_0, \cdots, a_n \in \mathbb{R}^m$, i.e. the set

$$\sigma = \langle a_0, \cdots, a_n \rangle = \left\{ \sum_{i=0} t_i a_i : \sum_{i=0}^{n} t_i = 1, t_i \geq 0 \right\}.$$

The points $a_0, \cdots, a_n$ are the **vertices**, and are said to **span** σ. The $(n+1)$ tuples $(t_0, \cdots, t_n)$ are called the **barycentric coordinates** for the point $\sum t_i a_i$.

We often denote an oriented simplex as σ, and then $\bar{\sigma}$ denotes the same simplex with the opposite orientation.

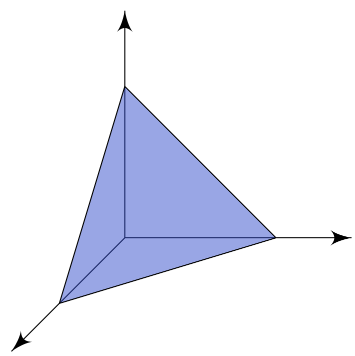
Some examples are as follows. When $n = 0$, it is a point. $n = 1$, it is a line.



$n = 0$ $\qquad$ $n = 1$ $\qquad$ $n = 2$ $\qquad$ $n = 3$

A **face** of a simplex is a subset (or subsimplex) spanned by a subset of the vertices. The **boundary** is the union of the proper faces, and the **interior** is the complement of the boundary. The boundary of σ is usually denoted by $\partial\sigma$, while the interior is denoted by $\mathring{\sigma}$, and we write $\tau \leq \sigma$ when τ is a face of σ. In particular, the interior of a vertex is the vertex itself. Note that this notion of interior and boundary is distinct from the topological notion of boundary.

Example 7.2. The **standard n-simplex** is spanned by the basis vectors $\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$ in $\mathbb{R}^{n+1}$. For example, when $n = 2$, we get the following:

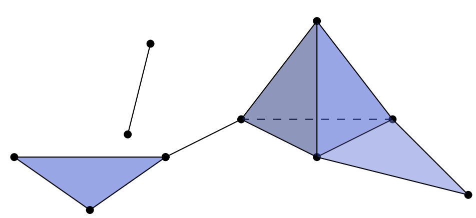


We will now glue simplices together to build **complexes**, or **simplicial complexes**.

Definition 7.3. A **simplicial complex** is a finite set K of simplices in $\mathbb{R}^n$ such that

1. If $\sigma \in K$ and τ is a face of σ, then $\tau \in K$.
2. If $\sigma, \tau \in K$, then $\sigma \cap \tau$ is either empty or a face of both σ and τ.

Example 7.4. This is a simplicial complex:



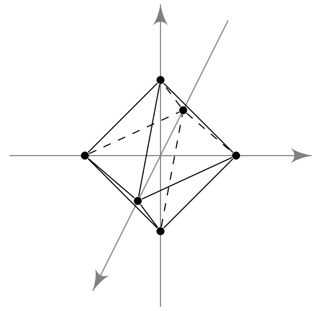
Technically, a simplicial complex is defined to be a set of simplices, which is just collections of points. It is not a subspace of $\mathbb{R}^n$. The **polyhedron** defined by K is the union of the simplices in K, and denoted by $|K|$. The **dimension** of K is the highest dimension of a simplex of K. The **d-skeleton** $K^{(d)}$ of K is the union of the n-simplices in K for $n \leq d$. A **triangulation** of a space X is a homeomorphism $h : |K| \to X$, where K is some simplicial complex.

Example 7.5. Let σ be the standard n-simplex. The boundary $\partial\sigma$ is homeomorphic to S^{n-1} (e.g. the boundary of a (solid) triangle is the boundary of the triangle, which is also a circle) . This is called the **simplicial** $(n-1)$ **sphere**.

We can also triangulate our S^n in a different way:

Example 7.6. In $\mathbb{R}^{n+1}$, consider the simplices $\langle \pm\mathbf{e}_0, \cdots, \pm\mathbf{e}_n \rangle$ for each possible combination of signs. So we have 2^{n+1} simplices in total. Then their union defines a simplicial complex K, and

$$|K| \cong S^n.$$



The nice thing about this triangulation is that the simplicial complex is invariant under the antipodal map. So not only can we think of this as a triangulation of the sphere, but a triangulation of $\mathbb{RP}^n$ as well.

An **oriented** n**-simplex** in a simplicial complex K is an $n+1$-tuple $(a_0, \cdots, a_n)$ of vertices $a_i \in V_k$ such that $\langle a_0, \cdots, a_n \rangle \in K$, where we think of two $n+1$ tuples $(a_0, \cdots, a_n)$ and $(a_{\pi(0)}, \cdots, a_{\pi(n)})$ as the same **oriented** simplex if $\pi \in S^n$ is an **even** permutation.

Definition 7.7 (Chain group $C_n(K)$). Let K be a simplicial complex. Let $\{\sigma_1, \cdots, \sigma_\ell\}$ be the set of n-simplices of K with orientation. Then we define $C_n(K)$ be the free abelian group with basis $\{\sigma_1, \cdots, \sigma_\ell\}$, i.e. $C_n(K) \cong \mathbb{Z}^\ell$.

We define **boundary homomorphisms**

$$\partial_n : C_n(K) \to C_{n-1}(K)$$

by

$$(a_0, \cdots, a_n) \mapsto \sum_{i=0}^{n} (-1)^i (a_0, \cdots, \hat{a}_i, \cdots, a_n),$$

where $(a_0, \cdots, \hat{a}_i, \cdots, a_n) = (a_0, \cdots, a_{i-1}, a_{i+1}, a_n)$ is the simplex with a_i removed. We can show that $\partial_{n+1} \cdot \partial_n = 0$.

We will develop some formalism to help us compute homology groups $H_k(K)$ in lots of examples.

Definition 7.8 (Chain complex and differentials). A **chain complex** $C_\bullet$ is a sequence of chain groups $C_0, C_1, C_2, \cdots$ equipped with boundary maps $\partial_n : C_n \to C_{n-1}$ such that $\partial_{n-1} \circ \partial_n = 0$ for all n.

$$\cdots \xrightarrow[\partial_3]{} C_2 \xrightarrow[\partial_2]{} C_1 \xrightarrow[\partial_1]{} C_0 \xrightarrow[\partial_0]{} 0$$

As in the de Rham cohomology, we define the n**-cycles** is

$$Z_n(C) = \operatorname{Ker} \partial_n.$$

The n**-boundaries** is

$$B_n(C) = \operatorname{Im} \partial_{n+1}.$$

Then, the **n-th homology group** of $C_\bullet$ is defined to be

$$H_n(C) = \frac{\operatorname{Ker} \partial_n}{\operatorname{Im} \partial_{n+1}} = \frac{Z_n(C)}{B_n(C)}.$$

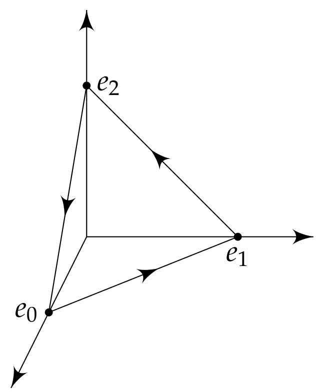
Let $K \to X$ be a triangulation of an n-dimensional compact oriented closed connected manifold X. Then, the n-th homology is $H_n(K, \mathbb{Z}) \cong \mathbb{Z}$ and its generator is denoted by $[X]$ and called the **fundamental class**.

The **Euler characteristic** of a triangulated space $h : |K| \to X$ is the alternative sum of real-valued homology groups of $|K|$

$$\chi(X) = \sum_{i \geq 0} (-1)^i \dim_{\mathbb{R}} H_i(K; \mathbb{R}).$$

As we will see below, this depends only on the homotopy type of X.

Example 7.9. Let K be the standard simplicial 1-sphere, ie, we have the following in $\mathbb{R}^3$.



Our simplices are thus

$$K = \{\langle e_0 \rangle, \langle e_1 \rangle, \langle e_2 \rangle, \langle e_0, e_1 \rangle, \langle e_1, e_2 \rangle, \langle e_2, e_0 \rangle\}.$$

Our chain groups are

$$\begin{aligned} C_0(K) &= \langle (e_0), (e_1), (e_2) \rangle \cong \mathbb{Z}^3 \\ C_1(K) &= \langle (e_0, e_1), (e_1, e_2), (e_2, e_0) \rangle \cong \mathbb{Z}^3. \end{aligned}$$

All other chain groups are zero. Note that our notation is slightly confusing here, since the brackets $\langle \cdot \rangle$ can mean the simplex spanned by the vertices, or the group generated by certain elements. However, you are probably clueful enough to distinguish the two use cases.

Hence, the only non-zero boundary map is

$$\partial_1 : C_1(K) \to C_0(K).$$

We can write down its matrix with respect to the given basis.

$$\begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$$

We have now everything we need to know about the homology groups, and we just need to do some linear algebra to figure out the image and kernel, and thus the homology groups. We have

$$H_0(K) = \frac{\operatorname{Ker}(\partial_0 : C_0(K) \to C_{-1}(K))}{\operatorname{Im}(\partial_1 : C_1(K) \to C_0(K))} \cong \frac{C_0(K)}{\operatorname{Im}\partial_1} \cong \frac{\mathbb{Z}^3}{\operatorname{Im}\partial_1}.$$

After doing some row operations with our matrix, we see that the image of ∂_1 is a two-dimensional subspace generated by the image of two of the edges. Hence we have

$$H_0(K) = \mathbb{Z}.$$

What does this $H_0(\mathbb{Z})$ represent? We initially said that $H_k(K)$ should represent the k-dimensional holes, but when $k = 0$, this is simpler. As for π_0, H_0 just represents the path components of K. We interpret this to mean K has one path component. In general, if K has r path components, then we expect $H_0(K)$ to be $\mathbb{Z}^r$.

Similarly, we have

$$H_1(K) = \frac{\operatorname{Ker}\partial_1}{\operatorname{Im}\partial_2} \cong \operatorname{Ker}\partial_1.$$

It is easy to see that in fact we have

$$\operatorname{Ker}\partial_1 = \langle (e_0, e_1) + (e_1, e_2) + (e_2, e_0) \rangle \cong \mathbb{Z}.$$

So we also have

$$H_1(K) \cong \mathbb{Z}.$$

We see that this $H_1(K)$ is generated by precisely the single loop in the triangle. The fact that $H_1(K)$ is non-trivial means that we do indeed have a hole in the middle of the circle.

We can also consider a map between two simplicial complexes.

Definition 7.10 (Simplicial map). A **simplicial map** $f : K \to L$ is a function $f : V_K \to V_L$ such that if $\langle a_0, \cdots, a_n \rangle$ is a simplex in K, then $\{f(a_0), \cdots, f(a_n)\}$ spans a simplex of L.

Given triangulations $|K| \to X$ and $|L| \to Y$ for manifolds X and Y and a continuous map $f : X \to Y$, we can "approximate" f by a simplicial map $g : K \to L$. (See [5, p.177] for more detail.)

In fact, a simplicial map f induces a map between simplicial complexes.

Definition 7.11 (Chain map). A chain map $f_\bullet : C_\bullet \to D_\bullet$ is a sequence of homomorphisms $f_n : C_n \to D_n$ such that

$$f_{n-1} \circ \partial_n = \partial_n \circ f_n$$

for all n. In other words, the following diagram commutes:

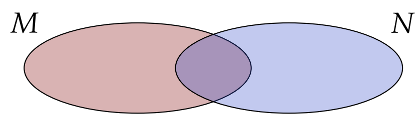
$$\begin{tikzcd}
\cdots \arrow[r, "\partial_3"'] & C_2 \arrow[r, "\partial_2"'] \arrow[d, "f_2"'] & C_1 \arrow[r, "\partial_1"'] \arrow[d, "f_1"'] & C_0 \arrow[r, "\partial_0"'] \arrow[d, "f_0"'] & 0 \\
\cdots \arrow[r, "\partial_3"] & D_2 \arrow[r, "\partial_2"] & D_1 \arrow[r, "\partial_1"] & D_0 \arrow[r, "\partial_0"] & 0
\end{tikzcd}$$

Let $f : K \to L$ be a simplicial map. Then f induces a chain map $f_\bullet : C_\bullet(K) \to C_\bullet(L)$. Hence it also induces $f_* : H_n(K) \to H_n(L); [c] \mapsto [f(c)]$. In fact, given triangulations $|K| \to X$ and $|L| \to Y$ for homotopically equivalent manifolds $f : X \to Y$ with $f \sim \mathrm{id}$, its simplicial approximation $g : K \to L$ induces an isomorphism $g_* : H_n(K) \cong H_n(L)$.

7.2 Mayer-Vietoris sequence

Here I just give you introduction to a powerful theorem called Mayer-Vietoris sequence to compute homology. For more detail, I refer to [5, p.149].

Suppose we have a space $K = M \cup N$, where $M \cap N$ are path-connected.



The theorem tells us how to compute the homology of the union $M \cup N$ in terms of those of M, N and $M \cap N$.

Theorem 7.12 (Mayer-Vietoris theorem). Let K, L, M, N be simplicial complexes with $K = M \cup N$ and $L = M \cap N$. We have the following inclusions and maps:

$$\begin{tikzcd} L \arrow[r, hook, "i"] \arrow[d, hook, "j"] & M \arrow[d, hook, "k"] \\ N \arrow[r, hook, "\ell"] & K. \end{tikzcd}$$

Then there exists some natural homomorphism $p_* : H_n(K) \to H_{n-1}(L)$ that gives the following long exact sequence:

$$\begin{tikzcd} \cdots \arrow[r, "p_*"] & H_n(L) \arrow[r, "i_*+j_*"] & H_n(M) \oplus H_n(N) \arrow[r, "k_*-\ell_*"] & H_n(K) \arrow[dlll, "p_*"] \\ & H_{n-1}(L) \arrow[r, "i_*+j_*"] & H_{n-1}(M) \oplus H_{n-1}(N) \arrow[r, "k_*-\ell_*"] & H_{n-1}(K) \arrow[r] & \cdots \\ & \cdots \arrow[r] & H_0(M) \oplus H_0(N) \arrow[r, "k_*-\ell_*"] & H_0(K) \arrow[r] & 0 \end{tikzcd}$$

Here $A \oplus B$ is the direct sum of the two (abelian) groups, which may also be known as the Cartesian product.

7.3 de Rham theorem and Poincare duality

Given a simplicail complex K and its chain group $C_k(K)$, we can define **cochain group**

$$C^k(K) = \{f : C_k(K) \to \mathbb{Z} \text{ homomorphism}\}$$

In addition, we can define **coboundary** $\delta : C^k(K) \to C^{k+1}(K)$ as $\delta f(c) = f(\partial c)$ were $c \in C_{k+1}(K)$. It is easy to see $\delta \cdot \delta = 0$. We can define **cocycle** $Z^k(K) = \operatorname{Ker} \delta$ and **coboundary** $B^k(K) = \operatorname{Im} \delta$. The cohomology group of K is defined by

$$H^k(K) = Z^k(K)/B^k(K) .$$

There exists bilinear map

$$H_k(K) \otimes H^k(K) \to \mathbb{Z}; ([c], [f]) \mapsto f(c)$$

which is well defined.

Theorem 7.13 (de Rham). Let X be a smooth manifold and $|K| \to X$ be its triangulation. Then, we have an isomorphism

$$H^*_{dR}(X) \cong H^*(K, \mathbb{R}) \ .$$

In fact, there is a relation between homology group and cohomology group. A k-dimensional oriented submanifold $Y \subset X$ without boundary indeed represents a generator $[Y] \in H_k(X)$. The definition of a submanifold is given in section 5.2.7 of Nakahara. There exists $\eta \in H^{n-k}(X)$ such that

$$\int_Y \omega = \int_X \omega \wedge \eta$$

for any $\omega \in H^k(X)$.

Theorem 7.14 (Poincare duality). Let X be an n-dimensional compact oriented closed manifold. Then, we have an isomorphism

$$\vartheta : H_k(X) \cong H^{n-k}(K, \mathbb{R}); Y \mapsto \eta$$

Using this isomorphism, one can define an **intersection number** $Y_1 \cdot Y_2$ for $Y_1 \in H^k(X)$ and $Y_2 \in H^{n-k}(X)$.

$$Y_1 \cdot Y_2 := \int_X \eta_1 \wedge \eta_2$$

where $\vartheta(Y_i) = \eta_i$.

7.4 Lefschetz fixed point theorem and Poincare-Hopf theorem

Let X and Y be n-dimensional smooth closed oriented manifolds. A smooth map $f : X \to Y$ induces an homomorphism $f_* : H_*(X) \to H_*(Y)$. In particular, we define **mapping degree** of f by

$$f_*([X]) := (\deg f)[Y] \ .$$

In the language of cohomology group, we can define

$$\deg f := \frac{\int_X f^*\omega}{\int_Y \omega}$$

where ω is the volume form of Y.

Given two knots $K_1, K_2 : S^1 \to \mathbb{R}^3$, we can define $F : K_1 \times K_2 \to S^2$ by

$$F(p,q) = \frac{x_1 - x_2}{|x_1 - x_2|}$$

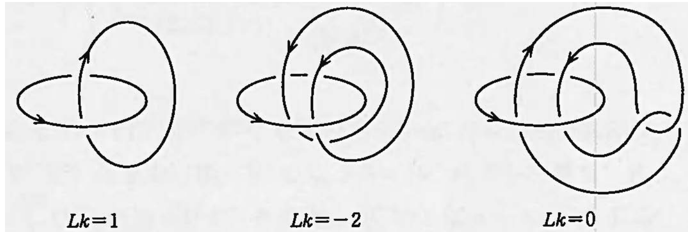
Then, we define the **linking number** of K_1 and K_2 as

$$Lk(K_1, K_2) := \deg F = \frac{1}{4\pi} \int_{K_1 \times K_2} F^*\omega = \frac{1}{4\pi} \int_{K_1} dx_1^i \int_{K_2} dx_2^j \frac{(x_1 - x_2)^k}{|x_1 - x_2|^3} \epsilon_{ijk}$$

where ω is the volume form of S^2.

This number was first introduced by Gauss in the study of electromagnetism. Biot-Savart law tells us that the magnetic field generated by electric current running on K_1 with unit strength is

$$\vec{B}(\vec{x}) = \frac{1}{4\pi} \int_{K_1} \frac{(\vec{x} - \vec{x}_1(t_1))^k \times \frac{d\vec{x}_1}{dt_1}}{|\vec{x} - \vec{x}_1|^3} dt_1$$



Therefore, $Lk(K_1, K_2)$ is the energy which costs to move a magnetic monopole along K_2 under this magnetic field. Gauss has noticed that the integral always provides an integer however K_1 and K_2 are drawn. Moreover, this number stays invariant even though you deform K_1 and K_2!

Let X be an n-dimensional smooth closed oriented manifold. We assume that a map $f : X \to X$ has isolated fixed points $f(p) = p$. We define a map $h : S^{n-1}_\epsilon \to S^{n-1}$ by

$$h(q) = \frac{q - f(q)}{|q - f(q)|}$$

The index of f at the fixed point p is defined by

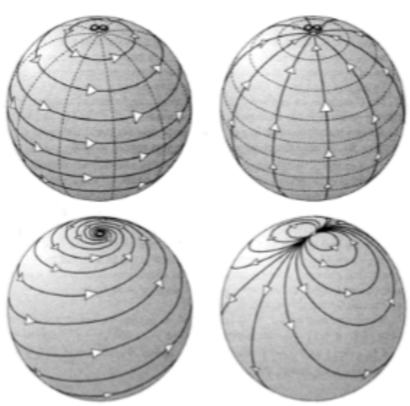
$$\mathrm{ind}_p f := \deg h$$

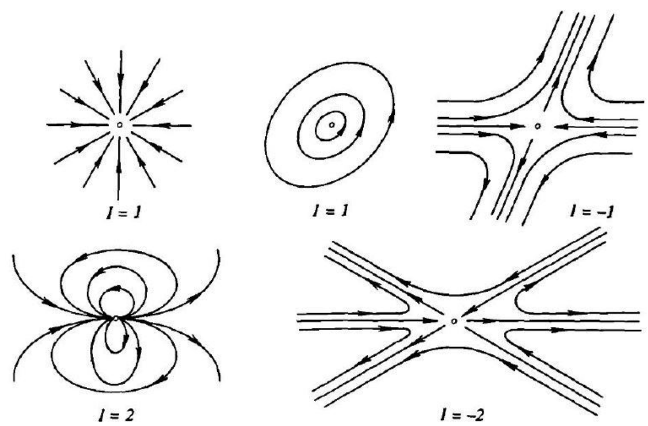
We define the **Lefschetz number** of f as

$$L(f) = \sum_{i \geq 0} (-1)^i \operatorname{tr}(f_* : H_i(X; \mathbb{R}) \to H_i(X; \mathbb{R})).$$

Theorem 7.15 (Lefschetz fixed point theorem). Lefschetz number is the sum of indices over isolated fixed points

$$L(f) = \sum_p \mathrm{ind}_p f$$





Let v is a vector field on X and φ_t is a flow generated by v. Then, the index of a zero point p of v is defined by

$$\mathrm{ind}_p v = \mathrm{ind}_p \varphi_t \ .$$

Since the flow φ_t is homotopic to the identity $\varphi_t \sim id$, $L(\varphi_t)$ is equal to the Euler characteristic.

$$L(\varphi_t) = \sum_{i \geq 0} (-1)^i \operatorname{tr}(id : H_i(X; \mathbb{R}) \to H_i(X; \mathbb{R})). = \sum_{i \geq 0} (-1)^i \dim H_i(X; \mathbb{R}) = \chi(X)$$

Therefore, we obtain:

Theorem 7.16 (Poincare-Hopf theorem). The Euler characteristics is the sum of indices over isolated fixed points

$$\chi(X) = \sum_p \mathrm{ind}_p v$$

This theorem was introduced in the first lecture.

8 Lecture 8: [1] §5.6, 5.7, 9.3

8.1 Lie groups and Lie algebras

Definition 8.1 (Lie group). A **Lie group** is a manifold G with a group structure such that multiplication $m : G \times G \to G$ and inverse $i : G \to G$ are smooth maps. The **dimension** of a Lie group G is the dimension of the underlying manifold.

For each $h \in G$, we define the **left and right translation maps**

$$L_h : G \to G; g \mapsto hg \, ,$$
$$R_h : G \to G; g \mapsto gh \, .$$

These maps are bijections, and in fact diffeomorphisms (i.e. smooth maps with smooth inverses), because they have smooth inverse $L_{h^{-1}}$ and $R_{h^{-1}}$ respectively.

In general, Lie group will be given by subsets of Euclidean space specified by certain algebraic equations, but note that not all subsets given by equations are manifolds, and it is possible that they have some singularities.

Example 8.2. Here are some examples.

- general linear group: $\mathrm{GL}(n, \mathbb{F}) = \{A \in \mathrm{M}_n(\mathbb{F}) | \det A \neq 0\}$
- special linear group: $\mathrm{SL}(n, \mathbb{F}) = \{A \in \mathrm{GL}(n, \mathbb{F}) | \det A = 1\}$
- Unitary group $\mathrm{U}(n) = \{A \in \mathrm{GL}(n, \mathbb{C}) | AA^\dagger = I\}$
- Special unitary group $\mathrm{SU}(n) = \{A \in \mathrm{U}(n) | \det A = 1\}$
- Orthogonal group $\mathrm{O}(n) = \{A \in \mathrm{GL}(n, \mathbb{R}) | AA^T = I\}$
- Special orthogonal group $\mathrm{SO}(n) = \{A \in \mathrm{O}(n) | \det A = 1\}$

Definition 8.3 (Lie algebra). A **Lie algebra** $\mathfrak{g}$ is a vector space (over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$) with a **bracket**

$$[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

satisfying

1. $[\alpha X + \beta Y, Z] = \alpha[X, Z] + \beta[Y, Z]$ for all $X, Y, Z \in \mathfrak{g}$ and $\alpha, \beta \in \mathbb{F}$ (bilinearity)
2. $[X, Y] = -[Y, X]$ for all $X, Y \in \mathfrak{g}$ (antisymmetry)
3. $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ for all $X, Y, Z \in \mathfrak{g}$. (Jacobi identity)

Note that linearity in the second argument follows from linearity in the first argument and antisymmetry.

We now try to get a Lie algebra from a Lie group G, by considering $T_e(G)$. The tangent space of a Lie group G at the identity naturally admits a Lie bracket

$$[\cdot\,,\cdot\,] : T_eG \times T_eG \to T_eG; (X, Y) \mapsto [X, Y] = XY - YX$$

such that

$$\mathfrak{g} = (T_e(G), [\cdot\,,\cdot\,])$$

is a Lie algebra.

Definition 8.4 (Lie algebra of a Lie group). Let G be a Lie group. The **Lie algebra** of G, written $\mathfrak{g}$, is the tangent space T_eG under the natural Lie bracket.

The general convention is that if the name of a Lie group is denoted by capital letters, then the corresponding Lie algebra is the same name with fraktur font. For example, the Lie group of $\mathrm{SO}(n)$ is $\mathfrak{so}(n)$.

Given a finite-dimensional Lie algebra, we can pick a basis B for $\mathfrak{g}$.

$$B = \{T_a : a = 1, \cdots, \dim \mathfrak{g}\}. \tag{8.1}$$

Then any $X \in \mathfrak{g}$ can be written as

$$X = X^a T_a = \sum_{a=1}^{n} X^a T_a,$$

where $X^a \in \mathbb{F}$.

By linearity, the bracket of elements $X, Y \in \mathfrak{g}$ can be computed via

$$[X, Y] = X^a Y^b [T_a, T_b].$$

In other words, the whole structure of the Lie algebra can be given by the bracket of basis vectors. We know that $[T_a, T_b]$ is again an element of $\mathfrak{g}$. So we can write

$$[T_a, T_b] = f_{ab}{}^c T_c,$$

where $f_{ab}{}^c \in \mathbb{F}$ are called the **structure constants**. By the antisymmetry of the bracket, we know

$$f_{ba}{}^c = -f_{ab}{}^c.$$

The Jacobi identity amounts to

$$f_{ab}{}^c f_{cd}{}^e + f_{da}{}^c f_{cb}{}^e + f_{bd}{}^c f_{ca}{}^e = 0.$$

Example 8.5. Take $G = \mathrm{SO}(3)$. Then $\mathfrak{so}(3)$ is the space of 3×3 real anti-symmetric matrices, which one can manually check are generated by

$$T_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

We then have

$$(T_a)_{bc} = -\varepsilon_{abc}.$$

Then the structure constants are $f_{ab}{}^c = \varepsilon_{abc}$.

Given a vector $X \in \mathfrak{g}$ in the tangent space of the identity e, one can generate the vector field by pushing-forward by the left translation L_g. Let us denote the corresponding vector field by X too. Since $(L_g)_* X = X$, it is called a **left-invariant vector field**. The flow generated by the vector field X is called **exponential map**, which can be expressed as a matrix

$$\exp(tX) = \sum_{\ell=0}^{\infty} \frac{1}{\ell!}(tX)^{\ell} \,.$$

Therefore, for any matrix Lie group G, the exponential map defines a map $\exp : \mathfrak{g} \to G$.

Given a Lie algebra $\mathfrak{g}$, it is also natural to think about its dual space $\mathfrak{g}^*$. This can be identified with the set of all left invariant 1-forms ω on G such that $L_g^* \omega = \omega$. Note that $\omega(X)$, $\omega(Y)$ are constant over G for $\omega \in \mathfrak{g}^*$ and $X, Y \in \mathfrak{g}$. Therefore, we have $Y(\omega(X)) = 0 = X(\omega(Y))$ so that

$$d\omega(X,Y) = \frac{1}{2}\omega([X,Y]) \,.$$

Therefore, if we take the basis $\omega_1, \cdots, \omega_{\dim \mathfrak{g}}$ dual to (8.1), we can write it as

$$d\omega_i = -\frac{1}{2}\sum_{j,k} f_{jki}\, \omega_j \wedge \omega_k \,.$$

Moreover, let $\omega \in \Omega^1(G l \mathfrak{g})$ be $\mathfrak{g}$-valued 1-form on G such that $\omega(X) = A$ for $A \in \mathfrak{g}$. Using the above basis, it is describe as

$$\omega = \sum_i \omega_i T^i \,,$$

which is called **Maurer-Cartan form**. Then, the equation above has the following form

$$d\omega = -\frac{1}{2}[\omega, \omega] \,,$$

which is called the **Maurer-Cartan equation**.

8.2 Vector bundles

We have learnt tangent bundles, cotangent bundles and their tensor products. Generalizing these leads to a notion called vector bundles. The notion of vector bundles was introduced by Whiteney. Remarkably, the notion of vector bundles is indispensable for description of non-Abelian gauge theories.

Definition 8.6 (Vector bundle). A **vector bundle** of rank r on M is a smooth manifold E with a smooth **projection** $\pi : E \to M$ such that

1. For each $p \in M$, the fiber $\pi^{-1}(p) = E_p$ is an r-dimensional vector space,
2. For all $p \in M$, there is an open $U \subseteq M$ containing p and a diffeomorphism

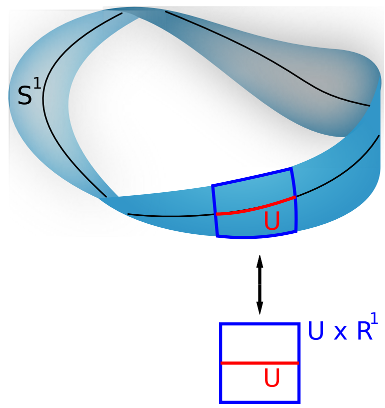
   $$t : E_U = \pi^{-1}(U) \to U \times \mathbb{R}^r$$

   such that

   $$\begin{tikzcd} E_U \arrow[r, "t"] \arrow[d, "\pi"'] & U \times \mathbb{R}^r \arrow[dl, "p_1"] \\ U & \end{tikzcd}$$

commutes, and the induced map $E_q \to \{q\} \times \mathbb{R}^r$ is a linear isomorphism for all $q \in U$.

We call t a **trivialization** of E over U; call E the **total space**; call M the **base space**. Also, for each $q \in M$, the vector space $E_q = \pi^{-1}(\{q\})$ is called the **fiber** over q. If $r = 1$, it is called **line bundle**.



Definition 8.7 (Transition function). Suppose that $t_\alpha : E|_{U_\alpha} \to U_\alpha \times \mathbb{R}^r$ and $t_\beta : E|_{U_\beta} \to U_\beta \times \mathbb{R}^r$ are trivializations of E. Then

$$t_\alpha \circ t_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^r \to (U_\alpha \cap U_\beta) \times \mathbb{R}^r$$

is fiberwise linear, i.e.

$$t_\alpha \circ t_\beta^{-1}(q, v) = (q, \varphi_{\alpha\beta}(q)v),$$

where $\varphi_{\alpha\beta}(q)$ is in $\mathrm{GL}(r, \mathbb{R})$.

In fact, $\varphi_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(r, \mathbb{R})$ is smooth. Then $\varphi_{\alpha\beta}$ is known as the **transition function** from β to α.

We have the following equalities whenever everything is defined:

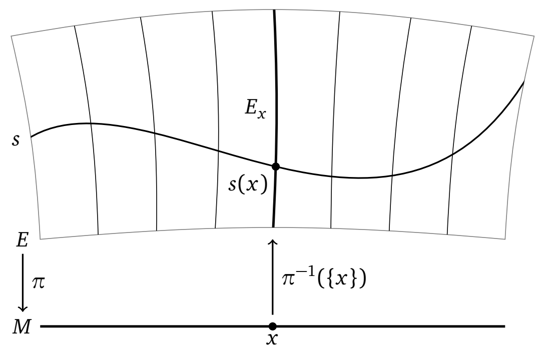
1. $\varphi_{\alpha\alpha} = \mathrm{id}$
2. $\varphi_{\alpha\beta} = \varphi_{\beta\alpha}^{-1}$
3. $\varphi_{\alpha\beta}\varphi_{\beta\gamma} = \varphi_{\alpha\gamma}$, which is called **cocycle condition**.

On the other hand, given an open cover $\{U_\alpha\}$ of open sets of M, suppose we have transition functions $\varphi_{\alpha\beta}$ which satisfy all the above properties. Then, we can glue $U_\alpha \times \mathbb{R}^n$ and $U_\beta \times \mathbb{R}^n$ by the transition functions $\varphi_{\alpha\beta}$ and construct a bundle $E \to M$.

Definition 8.8 (Section). A **section** of a vector bundle $\pi : E \to M$ is a map $s : M \to E$ such that $\pi \circ s = \mathrm{id}$. In other words, $s(p) \in E_p$ for each $p \in M$. We denote a set of sections by $\Gamma(M, E)$.

Definition 8.9 (Bundle map). We can consider about maps between vector bundles. Let $E \to M$ and $E' \to M'$ be vector bundles. A **bundle map** from E to E' is a pair of smooth maps $(F : E \to E', f : M \to M')$ such that the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{F} & E' \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & M' \end{array}.$$



i.e. such that $F_p : E_p \to E'_{f(p)}$ is linear for each p.

Everything we can do on vector spaces can be done on vector bundles, by doing it on each fiber.

Definition 8.10 (Whitney sum of vector bundles). Let $\pi : E \to M$ and $\rho : F \to M$ be vector bundles. The **Whitney sum** is given by

$$E \oplus F = \{(e, f) \in E \times F : \pi(e) = \rho(f)\}.$$

This has a natural map $\pi \oplus \rho : E \oplus F \to M$ given by $(\pi \oplus \rho)(e, f) = \pi(e) = \rho(f)$. This is again a vector bundle, with $(E \oplus F)_x = E_x \oplus F_x$ and again local trivializations of E and F induce one for $E \oplus F$.

Tensor products can be defined similarly.

Definition 8.11 (Tensor product of vector bundles). Given two vector bundles E, F over M, we can construct $E \otimes F$ similarly with fibers $(E \otimes F)|_p = E|_p \otimes F|_p$.

Similarly, we can construct the alternating product of vector bundles $\Lambda^n E$. Finally, we have the **dual** vector bundle.

Definition 8.12 (Dual vector bundle). Given a vector bundle $E \to M$, we define the **dual vector bundle** by

$$E^* = \bigcup_{p \in M} (E_p)^*.$$

Suppose again that $t_\alpha : E|_{U_\alpha} \to U_\alpha \times \mathbb{R}^n$ is a local trivialization. Taking the dual of this map gives

$$t_\alpha^* : U_\alpha \times (\mathbb{R}^n)^* \to E|_{U_\alpha}^*.$$

since taking the dual reverses the direction of the map. We pick an isomorphism $(\mathbb{R}^n)^* \to \mathbb{R}$ once and for all, and then reverse the above isomorphism to get a map

$$E|_{U_\alpha}^* \to U_\alpha \times \mathbb{R}^n.$$

This gives a local trivialization.

One important operation we can do on vector bundles is **pullback**:

Definition 8.13 (Pullback of vector bundles). Let $\pi : E \to M$ be a vector bundle, and $f : N \to M$ a map. We define the **pullback**

$$f^* E = \{(y, e) \in N \times E : f(y) = \pi(e)\}.$$

This has a map $f^*\pi : f^*E \to N$ given by projecting to the first coordinate. The vector space structure on each fiber is given by the identification $(f^*E)_y = E_{f(y)}$. It is a little exercise in topology to show that the local trivializations of $\pi : E \to M$ induce local trivializations of $f^*\pi : f^*E \to N$.

One can introduce a metric g on fibers of a vector bundle E. Namely, we have a non-degenerate symmetric form on a fiber E_x

$$g_x : E_x \times E_x \to \mathbb{R} ,$$

and g_x is differentiable in terms of x. If it is the tangent bundle TM, it is a Riemannian metric. Given a trivialization $\pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^r$, we can take an orthonormal frame $(e_1, \cdots, e_r)$ for $e_i \in \Gamma(U, E)$ with respect to g. Then, the transition function takes the value at $\mathrm{O}(r)$

$$\varphi_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{O}(r) .$$

We can further generalize that the transition function takes the value at an arbitrary Lie group G with a representation $\rho : G \to \mathrm{GL}(V)$.

Definition 8.14 (G-bundle). Let V be a vector space, G a Lie group, and $\rho : G \to \mathrm{GL}(V)$ a representation. Then a G-bundle $\pi : E \to M$ consists of the following data:

1. For each $p \in M$, the fiber is $\pi^{-1}(p) \cong V$.
2. One can take a trivializing cover $\{U_\alpha\}$ with transition functions $t_{\alpha\beta} : (U_\alpha \cap U_\beta) \times V \to (U_\alpha \cap U_\beta) \times V$.
3. The transition functions are constructed by maps $\varphi_{\alpha\beta} : U_\alpha \cap U_\beta :\to G$ satisfying the cocycle conditions with the representation ρ such that $t_{\alpha\beta} = \rho \circ \varphi_{\alpha\beta}$.

9 Lecture 9: [1] §9.4, 10

9.1 Principal G-bundles

Last time, we have studied vector bundles where a fiber is a vector space. We can further generalize it to a **fiber bundle** where a fiber is a general manifold F and a transition function is given by a differomorphism of F. Among fiber bundles, principal G-bundles play an important role in physics.

Definition 9.1 (Principal G-bundle). Let G be a Lie group, and M a manifold. A **principal G-bundle** is a smooth manifold P with a projection $\pi : P \to M$ such that a fiber is $\pi^{-1}(\{x\}) \cong G$ for each $x \in M$. More precisely, we are given an open cover $\{U_\alpha\}$ of M and diffeomorphisms

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "t"] \arrow[d, "\pi"] & U \times G \arrow[dl, "p_1"] \\ U & \end{tikzcd}$$

such that the transition functions

$$t_\alpha \circ t_\beta^{-1} : (U_\alpha \cap U_\beta) \times G \to (U_\alpha \cap U_\beta) \times G$$

is of the form

$$(x, g) \mapsto (x, t_{\alpha\beta}(x) \cdot g)$$

for some $t_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ where G is called the **structure group**.

We cab define an action of G on the total space P to the right

$$P \times G \to P; (u, g) \mapsto ug$$

where each fiber onto itself.

Given a principle G-bundle and a representation $\rho : G \to \mathfrak{gl}(V)$ for a vector space V, we can construct **associated vector bundle**

$$E = P \times_\rho V$$

as follow. Let us consider the direct product $P \times V$ and G action

$$(u, y) \mapsto (us, \rho(s)^{-1} y) \qquad \text{for} \quad s \in G$$

We define the associated bundle as the quotient space $P \times_\rho V := (P \times V)/G$. Conversely, given a G-bundle $E \to M$ with fiber V, there is a canonical way of producing a principal G-bundle by using transition functions.

9.2 Connections and curvatures

In Riemannian geometry, we have learnt Levi-Civita connections and Riemann curvature. Even in vector bundles and principal G-bundles, we can introduce connections and curvatures.

Definition 9.2 (Connection). A **connection** in a vector bundle $\pi : E \to M$ is a bilinear map

$$\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E); (X, s) \mapsto \nabla_X s$$

satisfying

1. $\nabla_{fX}(s) = f\nabla_X(s)$
2. Leibnitz property: $\nabla_X(fs) = (Xf)s + f(\nabla_X s)$

for all $s \in \Gamma(E)$ and $f \in C^\infty(M)$.

If a vector bundle E has a metric g, a connection ∇ is **compatible** with the metric if

$$Xg(s_1, s_2) = g(\nabla_X s_1, s_2) + g(s_1, \nabla_X s_2)$$

for any $X \in \mathfrak{X}(M)$ and $s_1, s_2 \in \Gamma(E)$.

Definition 9.3 (Curvature). A **curvature** in a vector bundle $\pi : E \to M$ is a trilinear map

$$F : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E); (X, Y, s) \mapsto F(X, Y)s$$

defined by

$$F(X, Y)s = \frac{1}{2}\left[\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}\right] s \,.$$

It has the following properties

1. $F(X, Y)s = -F(Y, X)s$
2. $F(fX, gY)(hs) = fghF(X, Y)s$ for $f, g, h \in C^\infty(M)$

for all $s \in \Gamma(E)$.

For a certain open subset of M, we can take a frame $s_1, \cdots, s_r \in \Gamma(\pi^{-1}(U))$. For any vector field X on U, the connection can be locally written as

$$\nabla_X s_j = \sum_{i=1}^{r} A^i_j(X) s_i$$

where $A^i_j \in \Omega^1(U, \mathfrak{gl}(r, \mathbb{R}))$ (1-form on U taking its value on $\mathfrak{gl}(r, \mathbb{R})$) is called **connection form**. We now look at the curvature R from differential forms.

$$F(X,Y)s_j = \sum_{i=1}^{r} F^i_j(X,Y)s_i$$

where $F^i_j \in \Omega^2(U, \mathfrak{gl}(r, \mathbb{R}))$ (2-form on U taking its value on $\mathfrak{gl}(r, \mathbb{R})$) is called **curvature form**. They are related by the following equation:

$$F = dA + A \wedge A \,.$$

The curvature form satisfies the Bianchi identity

$$dF - F \wedge A + A \wedge F = 0 \,.$$

More explicitly, in physics, we write a section $s = \sum_{j=1}^{r} v^j(x) s_j$ on U so that

$$\nabla_{\frac{\partial}{\partial x^\mu}} s = \sum_{i=1}^{r} \left[\frac{\partial}{\partial x^\mu} v^i(x) + (A_\mu)^i{}_j v^j(x) \right] s_i$$

In addition, the curvature can be written in terms of local coordinates

$$F\left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}\right) = F_{\mu\nu} = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu} + [A_\mu, A_\nu] \,.$$

In the case of Maxwell U(1) theory, the last term vanish because it is a commutative group.

It is useful to know how the connection transforms under a change of local trivialization. Given a transition function $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(r, \mathbb{R})$, the gauge fields on U_α and U_β are related by

$$A_\beta = g_{\alpha\beta}^{-1} A_\alpha g_{\alpha\beta} + g_{\alpha\beta}^{-1} d(g_{\alpha\beta}) \,, \tag{9.1}$$

and the curvature forms are related by

$$F_\beta = g_{\alpha\beta}^{-1} F_\alpha g_{\alpha\beta}$$

This construction makes sense for G-bundle, and so we can canonically identify both G and $\mathfrak{g}$ as subsets of $\mathrm{GL}(n, \mathbb{R})$ for some n. Then we have to replace the first term of (9.1) by the adjoint representation of G on $\mathfrak{g}$, and the second of (9.1) by the **Maurer–Cartan form**.

In particular, for the Maxwell theory, the gauge transformation can be written as $g_{\alpha\beta} = e^{i\lambda_{\alpha\beta}(x)}$ so that

$$A_\beta = A_\alpha + i d\lambda_{\alpha\beta}$$

and the curvature form stays invariant.

9.2.1 Parallel transport and holonomy group

Given a connection ∇ on a vector bundle E, on can define **horizontal** directions in the space $\Gamma(E)$ of sections. A section $s \in \Gamma(E)$ is **parallel** along a path $\gamma : I \to (M)$

$$\nabla_{\dot\gamma(t)} s = 0 \quad \text{for} \quad t \in I \, .$$

In terms of local coordinates, it can be written as

$$\frac{ds_i}{dt} + \sum_{j=1}^{r} (A_\mu)^j{}_i \frac{dx^\mu}{dt} s_j = 0 \, .$$

A theorem of ordinary differential equations tells us that given an initial data $s(t=0) \in E_{\gamma(0)}$, one can do parallel transform along $\gamma(t)$ so that we have a map

$$E_{\gamma(0)} \ni s(t=0) \to s(t=1) \in E_{\gamma(1)} \, .$$

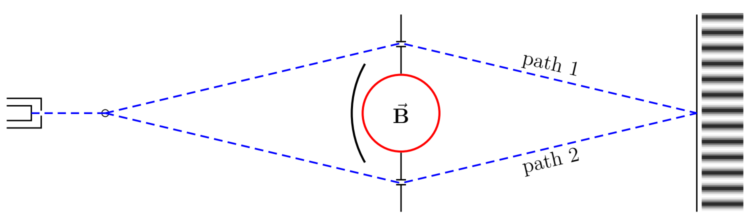
In particular, if we consider a curve $p = \gamma(0) = \gamma(1)$, we obtain amap $\tau_\gamma : E_p \to E_p$. For given two curves γ_1 and γ_2, we can have a multiplication

$$\tau_{\gamma_1 \circ \gamma_2} = \tau_{\gamma_1} \circ \tau_{\gamma_2}$$

and the inverse is defined by

$$\tau_{\gamma^{-1}} = \tau_\gamma^{-1} \, ,$$

so that it forms a group called **holonomy group**. In the case of $\mathrm{U}(1)$, this is the origin of the **Aharonov-Bohm effect**.



9.2.2 Levi-Civita connections

We can consider the tangent bundle TM as a vector bundle and its metric g is indeed a Riemannian metric. Then, we can uniquely determine the natural connection called **Levi-Civita connections**. Let $X_1, \cdots, X_n$ be an orthonormal frame vector field on an open set $U \subset M$ and their dual $e^1, \cdots, e^n \in \Omega^1(U)$. For the Levi-Civita connection ∇, we can write

$$\nabla_{X_j} X_i = \sum_k \mathbf{\Gamma}^k_{ij} X_k \, ,$$
$$R(X_i, X_j) X_k = \sum_l \mathbf{R}^l{}_{ijk} X_l \, .$$

We can define connection one-form and curvature two-form taking their values on $\mathfrak{so}(n, \mathbb{R})$:

$$\omega^k{}_j = \sum_k \mathbf{\Gamma}^k_{ij} e^i \, , \quad \Omega^l_i = \sum_l \mathbf{R}^l{}_{ijk} e^j \wedge e^k \, .$$

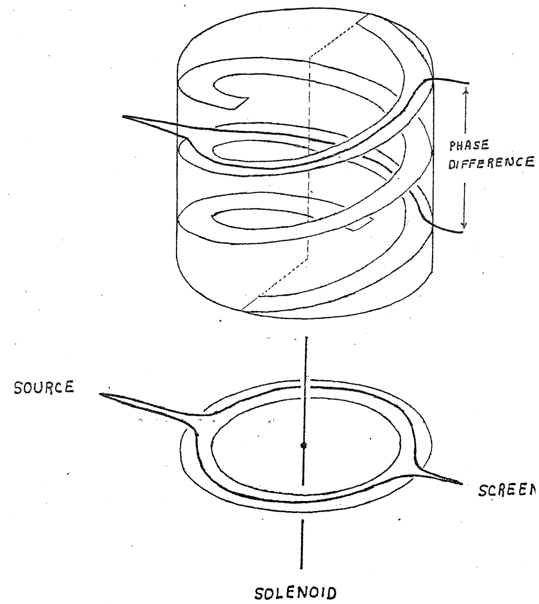


Figure 2: The Aharonov-Bohm solenoid effect takes place when the wave function of a charged particle passing around a long solenoid experiences a phase shift as a result of the enclosed magnetic field, despite the magnetic field being negligible in particle tragectories.

They satisfy the following conditions

$$de^i = -\sum_j \omega^i{}_j \wedge e^j ,$$
$$\Omega^i{}_j = d\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j .$$

9.2.3 Ehresmann connections

Similarly, one can construct theory of connections on principal G-bundles, which are called **Ehresmann connections**. In principle, a Ehresmann connection determine the horizontal direction of a principal G-bundle.

Definition 9.4 (Ehresmann connection). A **Ehresmann connection** A on a principle G-bundle $\pi : P \to M$ is a one-form taking its value on $\mathfrak{g}$, which satisfies the following conditions:

1. Given $X \in \mathfrak{g}$, there is the corresponding vector field $\overline{X}$ on P

$$\overline{X}_u = \frac{d}{dt}(u \cdot \exp tX)\Big|_u \quad \text{for} \quad u \in P .$$

   Then, A is subject to
$$A(\overline{X}) = X \in \mathfrak{g} .$$

2. For $^\forall g \in G$,
$$R_g^*(A) = Ad(g^{-1})A .$$

   In other words,

$$A_{ug}(R_g(Y)) = Ad(g^{-1})(A_u(Y)) , \quad \text{for} \quad Y \in \mathfrak{g} .$$

For an open set $U_\alpha \subset M$, there is a local trivialization $\psi_{U_\alpha} : \pi^{-1}(U_\alpha) \to U_\alpha \times G$. Then, we have a section on $\pi^{-1}(U_\alpha)$

$$\sigma_{U_\alpha} : U_\alpha \to P; x \mapsto \psi_{U_\alpha}^{-1}(x, e) .$$

Then, if we define $A_\alpha = \sigma_{U_\alpha}^*(A)$, we have cocycle condition

$$A_\beta = g_{\alpha\beta}^{-1} A_\alpha g_{\alpha\beta} + g_{\alpha\beta}^{-1} d(g_{\alpha\beta}) , \quad \text{for} \quad U_\alpha \cap U_\beta ,$$

where $g_{\alpha\beta}$ is the transition function on $U_\alpha \cap U_\beta$.

We denote the space of Ehresmann connections on a principal G-bundle P by $\mathscr{A}_P$. The space $\mathscr{G}_P$ of gauge transformations is the section $\Gamma(M, G_P)$ of the bundle $G_P := P \times_{Ad} G$. A gauge transformation $g \in \mathscr{G}_P$ acts on the space of connection $\mathscr{A}_P$ via

$$\mathscr{G}_P \times \mathscr{A}_P \to \mathscr{A}_P; (g, A) \mapsto g^*(A) = \{g^*(A)_U, g^*(A)_V, \cdots\}$$

where

$$g^*(A)_U = g_U^{-1} d(g_U) + g_U^{-1} A g_U .$$

If $A' = g^*(A)$ for $A, A' \in \mathscr{A}_P$, they are physically identical. So we denote the space of physically different connections by

$$\mathscr{B}_P := \mathscr{A}_P / \mathscr{G}_P .$$

9.3 Yang-Mills theory

Now we can describe non-Abelian gauge theory called **Yang-Mills theory**. Let us consider principal G-bundle or its associated G-bundle. In addition, let A be a connection on it and F be its curvature. The classical **Yang-Mills action** can be written as

$$\begin{aligned} S_{YM}[A] &= \frac{1}{2g_{YM}^2} \int_M \mathrm{Tr}\, F \wedge *F \\ &= \frac{1}{2g_{YM}^2} \int_M \mathrm{Tr}\left(F_{\mu\nu} F^{\mu\nu}\right) \sqrt{g} d^d x, \end{aligned} \tag{9.2}$$

where the curvature is 2-form taking its value on the Lie algebra $\mathfrak{g}$, and Tr is taking over Lie algebra $\mathfrak{g}$. The parameter g_{YM}^2 is the Yang-Mills coupling constant. For the flat space, we have $\sqrt{g} = 1$ so that we will drop this term. This is also known as **non-Abelian gauge theory**.

For example, if $G = \mathrm{SU}(N)$, we can choose the basis of the Lie algebra $\mathfrak{su}(N)$

$$\mathrm{Tr}(T_a T_b) = \frac{1}{2} \delta_{ab},$$

and on a local $U \subseteq M$, we have

$$S_{YM}[A] = \frac{1}{4g_{YM}^2} \int F_{\mu\nu}^a F^{b,\mu\nu} \delta_{ab}\, d^d x,$$

with $F_{\mu\nu} = \sum_a F_{\mu\nu}^a T_a$ and

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f_{bc}{}^a A_\mu^b A_\nu^c .$$

Thus, Yang–Mills theory is the natural generalization of Maxwell theory to the non-Abelian case.

At the level of the classical field equations, if we vary our connection by $A_\mu \mapsto A_\mu + \delta a_\mu$, where δa is a matrix-valued 1-form, then we have

$$\delta F_{\mu\nu} = \partial_{[\mu}\delta a_{\nu]} + [A_\mu, \delta a_\nu].$$

The equation of motion can be obtained by taking the variation of the Yang-Mills acion

$$\delta S_{YM}[A] = \frac{1}{2g_{YM}^2}\int \mathrm{Tr}\big(\delta F_{\mu\nu}, F^{\mu\nu}\big)\, d^d x = \frac{1}{2g_{YM}^2}\int \mathrm{Tr}\big(\nabla_\mu \delta a_\nu, F^{\mu\nu}\big)\, d^d x = 0.$$

Therefore, the **Yang–Mills equation** is

$$\nabla^\mu F_{\mu\nu} = \partial^\mu F_{\mu\nu} + [A^\mu, F_{\mu\nu}] = 0\,,$$

or we can write it without coordinates

$$\delta_A F := *d_A * F = *(d+A)*F = 0\,.$$

Recall we also had the Bianchi identity

$$\nabla_\mu F_{\nu\lambda} + \nabla_\nu F_{\lambda\mu} + \nabla_\lambda F_{\mu\nu} = 0\,.$$

Unlike Maxwell's equations, these are non-linear PDE's for A. We no longer have the principle of superposition. This is similar to general relativity.

The Yang-Mills path integral is expressed by

$$Z_{YM} = \int_{\mathscr{A}/\mathscr{G}} DA\ \exp(iS_{YM}[A])\,.$$

10 Lecture 10: [1] §11

10.1 Characteristic Classes

In the last time, we have seen principal $U(1)$-bundles on S^2 are classified by the monopole number n which tells us how the $U(1)$ fibers over the upper hemisphere and the lower hemisphere are glued together. The generalization of this notion for a vector bundle $\pi : E \to M$ leads to a **characteristic class** associated to a cohomology class of M. Characteristic classes was introduced to extract topological information of a base manifold of vector bundles or principal G-bundles from curvature forms. This is called **Chern-Weil theory**.

Characteristic classes are constructed as **invariant polynomials** of the curvature $F = dA + A \wedge A$. Under gauge transformation, F transforms as $F \to g^{-1}Fg$, where $g \in \mathscr{G}_E$. To construct characteristic classes, we need to introduce invariant polynomials $P(X)$ of matrixes, that is invariant under the conjugation, $P(g^{-1}Fg) = P(F)$. Examples of invariant polynomials are $\mathrm{Tr}\, F^k$ ($k = 1, 2, \cdots$) and $\det F$. In fact, we can use these to construct nice bases of invariant polynomials as follows:

(1) $\sigma_k(F)$ defined by

$$\det(1 + tF) = 1 + t\sigma_1(F) + t^2\sigma_2(F) + \cdots + t^r\sigma_r(F)\,.$$

(2) $s_k(F)$ defined by

$$s_k(F) = \operatorname{Tr} F^k \,, \qquad (k = 1, \cdots, r) \,.$$

They are related to each other by Newton's formula,

$$s_1 = \sigma_1 \,, \quad s_2 = \sigma_1^2 - \sigma_2 \,, \quad s_3 = \sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3 \,, \cdots$$

We can also express the invariant polynomials in terms of eigenvalues. If F is a hermitian matrix, we can diagonalize it with eigenvalues $x_1, \cdots, x_k$. Then,

$$\prod_{k=1}^{r}(1 + t x_k) = 1 + t\sigma_1(x) + \cdots + t^r \sigma_r(x).$$

Similarly, $s_k(F) = \sum_{j=1}^{r}(x_j)^k$.

If $P_k(F)$ is an invariant polynomial of degree k, we can use the curvature 2-form F to define a $2k$-form $P_k(F)$ so that it is invariant under the gauge transformation, $F \to g^{-1}Fg$. It is also a closed form because the Bianchi identity

$$dF + [A \wedge F] = 0$$

tells us

$$\begin{aligned} d \ \operatorname{Tr} F^k &= \operatorname{Tr}\Big(dF\, F^{k-1} + F\, dF\, F^{k-1} + \cdots + F^{k-1} dF\Big) \\ &= -\operatorname{Tr}\Big(([A \wedge F])F^{k-1} + \cdots + F^{k-1}([A \wedge F])\Big) \\ &= -\operatorname{Tr}\Big(A \wedge F^k - F^k \wedge A\Big) = 0 \end{aligned}$$

Thus, we find $P_k(F) \in H^{2k}(M)$.

Moreover, $P_k(F)$ is invariant under continuous deformation of the gauge field A as an element of $H^{2k}(M)$. Suppose that we change $A \to A + \eta$ with η being an infinitesimal one-form. Under this deformation, F changes by $\delta F = d\eta + A \wedge \eta + \eta \wedge A$. Therefore,

$$\begin{aligned} \delta \operatorname{Tr} F^k &= \operatorname{Tr}\Big((d\eta + A \wedge \eta + \eta \wedge A)F^{k-1} + \cdots + F^{k-1}(d\eta + A \wedge \eta + \eta \wedge A)\Big) \\ &= k \operatorname{Tr}\Big((d\eta + A \wedge \eta + \eta \wedge A)F^{k-1}\Big) \\ &= k \operatorname{Tr}\Big(d\eta F^{k-1} - \eta dF F^{k-2} - \cdots - \eta F^{k-2} dF\Big) \\ &= k\, d \operatorname{Tr}\Big(\eta F^{k-1}\Big) \,. \end{aligned}$$

Since both η and F transform homogeneously under the gauge transformation, $\operatorname{Tr}\big(\eta F^{k-1}\big)$ is a well-defined $(2k-1)$-form. Thus, under any infinitesimal deformation, $P_k(F)$ changes by an exact form. Thus, $P_k(F)$ depends only on the type of the bundle E and not on a choice of the connection A on E. For a more complete proof, please refer to [2, Proposition 5.28].

10.1.1 Pontryagin Classes

Let us consider a vector bundle $\pi : E \to M$ of rank r. One can always put a metric g on E and consider a connection A which is compatible with the metric g. Therefore, we can consider the curvature form F takes its value on $\mathfrak{so}(r)$. Namely it is a anti-symmetric $(F^i{}_j + F^j{}_i = 0)$ 2-form so that $\operatorname{Tr} F^k = 0$ if k is odd. Therefore, for the invariant polynomial P_k of odd degree, we have

$$P_k(F) = 0 .$$

As a result, the **Pointryagin class** is defined as

$$p_k(E) := \frac{1}{(2\pi)^{2k}} \sigma_{2k}(F) \in H^{4k}(M; \mathbb{R})$$

They may be written as

$$p(E) = \det\left(1 + \frac{1}{2\pi} F\right) = 1 + p_1(E) + p_2(E) + \cdots + p_{[r/2]}(E) .$$

Theorem 10.1 (Hirzebruch signature theorem). Let M be an oriented compact 4-dimensional manifold. Since the Hodge star $* : H^2(M) \to H^2(M)$ satisfies $*^2 = 1$ on $H^2(M)$, we can decompose it into $H^2(M) = H^2_+(M) \oplus H^2_-(M)$ with eigenvalues ± 1 of $*$. Let us define the signature of M by

$$\tau(M) = \dim H^2_+(M) - \dim H^2_-(M) .$$

Then, the signature can be expressed by

$$\tau(M) = \frac{1}{3} \int_M p_1(TM) .$$

10.1.2 Chern Classes

Now let us consider complex vector bundle $\pi : E \to M$ of rank r. Similarly, one can always put a Hermitian metric g on E and consider a connection A which is compatible with the metric g. Then, the curvature form F takes its value $\mathfrak{u}(r)$. Namely it is a skew-Hermitian $(F^i{}_j + \overline{F}^j{}_i = 0)$ 2-form so that $\operatorname{Tr}\left(\frac{F}{2\pi i}\right)^k$ is a real $2k$-form.

Then, **Chern class** is defined as

$$c_k(E) := \left(\frac{-1}{2\pi i}\right)^k \sigma_k(F) \in H^{2k}(M; \mathbb{R}) .$$

This can be written as

$$c(E) = \det\left(1 - \frac{1}{2\pi i} F\right) = c_0(E) + c_1(E) + c_2(E) + \cdots + c_r(E) .$$

For example, we can explicitly write

$$c_0 = 1 , \quad c_1 = \frac{-1}{2\pi i} \operatorname{Tr} F , \quad c_2 = -\frac{1}{8\pi^2} (\operatorname{Tr} F \wedge \operatorname{Tr} F - \operatorname{Tr} F \wedge F), \cdots$$

If the structure group is in $\mathrm{SU}(r) \in \mathrm{U}(r)$, we have a trivial first Chern class $c_1 = 0$ because $\mathfrak{su}(r)$ is traceless.

Instead of σ_k, we can use s_k for invariant polynomials, which defines the **Chern characters**

$$ch_k(E) = \frac{1}{k!} \operatorname{Tr}\left(-\frac{F}{2\pi i}\right)^k \in H^{2k}(M) .$$

We can also write it as

$$ch(E) = ch_0(E) + ch_1(E) + \cdots = \operatorname{Tr} \exp\left(-\frac{F}{2\pi i}\right) .$$

10.1.3 Some properties

The Pontryagin class and Chern class are related by

$$p_k(E) = (-1)^k c_{2k}(E \otimes \mathbb{C}) \in H^{4k}(M; \mathbb{R})$$

where $E \otimes \mathbb{C}$ is the complexification of a real bundle $E \to M$.

One of the important properties of the Pontryagin and Chern classes is that it behaves nicely when we take a direct sum $E_1 \oplus E_2$ of vector bundles E_1, E_2 as,

$$\begin{aligned} p(E_1 \oplus E_2) &= p(E_1) \wedge p(E_2) \ , \\ c(E_1 \oplus E_2) &= c(E_1) \wedge c(E_2) \ . \end{aligned}$$

On the other hand, it does not behave nicely under the direct product $E_1 \otimes E_2$.

The Chern characters behave nicely under both the direct sum and direct product as,

$$\begin{aligned} ch(E_1 \oplus E_2) &= ch(E_1) + ch(E_2) \ , \\ ch(E_1 \otimes E_2) &= ch(E_1) \wedge ch(E_2) \ . \end{aligned}$$

This property plays an important role in **K-theory**.

10.1.4 Euler class

Let us turn to a real vector bundle $\pi : E \to M$ of even rank $2r$. In this case, in addition to Tr and det, we can consider one more way to construct an invariant polynomial, which is called the **Pfaffian**,

$$Pf(F) = \frac{(-1)^r}{2^r r!} \epsilon^{k_1 j_1 i_2 j_2 \cdots i_r j_r} F_{i_1 j_1} F_{i_2 j_2} \cdots F_{i_r j_r} \ .$$

Note that, for antisymmetric matrices, the Pfaffian is a square root of the determinant, $\det F = Pf(F)^2$. If F is real and anti-symmetric, we can block diagonalize it by $\mathrm{SO}(2r)$ as

$$F = \begin{pmatrix} 0 & x_1 & 0 & 0 & \cdots & 0 & 0 \\ -x_1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & x_2 & & & \\ 0 & 0 & -x_2 & 0 & & & \\ \cdot & \cdot & & & \cdots & \cdot & \cdot \\ 0 & 0 & & & \cdots & 0 & x_r \\ 0 & 0 & & & \cdots & -x_r & 0 \end{pmatrix} \tag{10.1}$$

We can then write the Pfaffian as

$$Pf(F) = (-1)^r \prod_{i=1}^{r} x_k.$$

Under the conjugation $F \to g^T F g$, the Pfaffian transforms as $Pf(g^T F g) = \det g \cdot Pf(F)$.

Thus, if $g \in \mathrm{SO}(2r)$, the Pfaffian is invariant. We can now define the **Euler class** by

$$e(E) = Pf(E) \in H^{2r}(M; \mathbb{R}) \ .$$

In fact, the Euler class can be understood as the square root of the highest Pontryagin class $p_r(E)$

$$p_r(E) = e(E)^2 \ .$$

In particular, the tangent bundle TM of an orientable Riemannian manifold M of dimensions $n = 2r$ is an SO$(2r)$ bundle. For example,

$$
\begin{aligned}
n = 2: \quad e(TM) &= \frac{1}{2\pi} R_{12} , \\
n = 4: \quad e(TM) &= \frac{1}{32\pi^2} \epsilon_{\mu\nu\rho\sigma} R^{\mu\nu} \wedge R^{\rho\sigma} ,
\end{aligned}
$$

where the Riemann curvature is regarded as the 2-form as,

$$
R^a{}_b = \frac{1}{2} R_{cd}{}^a{}_b e^a \wedge e^b = \frac{1}{2} R_{\mu\nu}{}^a{}_b dx^\mu \wedge dx^\nu .
$$

The integral of the Euler class $e(TM)$ over M is indeed equal to the Euler characteristic

$$
\chi(M) = \int_M e(TM) ,
$$

which can be considered as the higher dimensional version of the Gauss-Bonnet theorem.

10.1.5 Todd, L- and $\hat{A}$-classes

Sometime we use other characteristic classes, such as **Todd classes, Hirzebruch L-class**, and **$\hat{A}$-class** [6]. These classes are just defined by different basis of invariant polynomials. However, I will give a brief introduction since these classes will show up in the index theorem later.

To describe Todd class, let $x_1, \cdots, x_k$ be eigenvalues of curvature form $\frac{-F}{2\pi i}$ of a complex vector bundle. For example, the total Chern classes can be expressed as

$$
c(F) = \det \left(1 - \frac{F}{2\pi i} \right) = \prod_{i=1}^{r} (1 + x_k).
$$

It is worth mentioning that the right-hand side takes the form $\prod_k c(L_k)$, where L_k is a line bundle with a curvature given by x_k and $c(L_k) = 1 + x_k$. Thus, as far as the Chern classes are concerned, the vector bundle E behaves like a Whitney (direct) sum of the line bundles $L_1 \oplus L_2 \oplus \cdots \oplus L_k$ although they are not isomorphic as bundles. This phenomenon is called the **splitting principle**, and x_i are called **Chern roots**. Using this notation, the Todd class is defined by,

$$
Td(E) = \prod_k \frac{x_k}{1 - e^{-x_k}} = 1 + \frac{1}{2} c_1 + \frac{1}{12} (c_2 + c_1^2) + \cdots ,
$$

Furthermore, for real vector bundle E, the Hirzebruch L-classes are defined by

$$
L(E) = \prod_k \frac{x_k}{\tanh x_k} = 1 + \frac{1}{3} p_1 + \frac{1}{45} (7p_2 - p_1^2) + \cdots ,
$$

where x_i are as in (10.1). Indeed Hirzebruch has introduced this class for the signature theorem of general $4k$-dimensional manifolds.

The $\hat{A}$-classes are defined by,

$$
\hat{A}(E) = \prod_k \frac{x_k/2}{\sinh x_k/2} = 1 + \frac{1}{24} p_1 + \frac{1}{5760} (7p_1^2 - 4p_2) + \cdots .
$$

11 Lecture 11: [1, §11.5, 12] and [2, §6.6]

11.1 Flat connections and holonomy homomorphisms

Let $P \to M$ be a principal G-bundle over a manifold M. An Ehresmann connection A on P is called a **flat connection** If its curvature form F is identically zero $F = 0$. A principal G-bundle equipped with a fiat connection is called a flat G-bundle. As an example of flat bundle, the produce bundle $M \times G$ has a trivial connection, which is flat. This example is too trivial, and there are much richer stories for flat G-bundles.

A connection A determines the horizontal direction of a principal G-bundle. Namely at each point $u \in P$, we define

$$\mathcal{H}_u = \{X \in T_u P | A(X) = 0\} .$$

The collection $\mathcal{H} = \cup_u \mathcal{H}_u$ is called **distribution**. The distribution is called **completely integrable** if

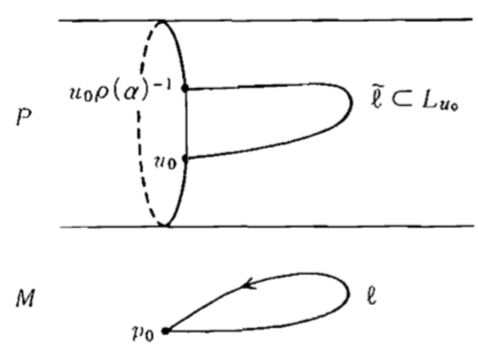
$$\text{for any two vector fields } X, Y \in \mathcal{H} \longrightarrow [X, Y] \in \mathcal{H} .$$

Another way to describe completely integrable distribution $\mathcal{H}$ is that for any point on M there exists an integral manifold containing it. (A submanifold N of M is called an **integral manifold** of $\mathcal{H}$ if $T_u N = \mathcal{H}_u$ for $^\forall u \in N$.)

For horizontal vector field X, Y, we have $A(X) = 0 = A(Y)$.

$$F(X, Y) = dA(X, Y) = \frac{1}{2} \{X(A(Y)) + Y(A(X)) - A([X, Y])\} = -\frac{1}{2} A([X, Y])$$

Therefore, $F = 0 \leftrightarrow [X, Y]$ horizontal vector field. A connection A on a principal G-bundle is fiat if and only if the corresponding distribution $\mathcal{H}$ is completely integrable.



Then, for each point $u \in P$ if we denote by L_u the maximal integral manifold passing through it, then the projection $\pi : L_u \to M$ becomes a covering map. Now let $p_0 \in M$ be a base point of the base space and choose $u_0 \in \pi^{-1}(p_0)$. Then a homomorphism

$$\rho : \pi_1(M) \to G ,$$

called the **holonomy homomorphism**, is defined as follows. For each element $\alpha \in \pi_1(M)$ of the fundamental group, we choose a closed curve γ with initial point p_0 which represents α. Let $\tilde{\gamma}$ be the lift of γ to L_{u_0} with initial point u_0. Then we can write the end point of $\tilde{\gamma} = u_0 \cdot g$. Since $\pi : L_{u_0} \to M$ is the covering map. we see that the end point of $\tilde{\gamma}$ is determined uniquely by α, and it is independent of choice of $\tilde{\gamma}$. Then, we set

$$\rho(\gamma) = g^{-1} .$$

Theorem 11.1. Via the holonomy homomorphism, the set of flat connections on P is in a one-to-one correspondence with the set of conjugacy classes of homomorphisms $\rho : \pi_1(M) \to G$.

11.2 Chern-Simons theory

Let M be a compact 3-manifold. We will consider a particular physical theory called **Chern-Simons** theory on 3-dimension. Let $P = M \times G$ be a trivial principal G-bundle and we denote a connection on P by A.

The Chern-Simons action for A can be written as

$$\begin{aligned} S_{CS}[M,A] &= \frac{k}{4\pi}\int_M \mathrm{Tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right) \\ &= \frac{k}{8\pi}\int_M \epsilon^{\mu\nu\rho}\,\mathrm{Tr}\left(A_\mu(\partial_\nu A_\rho - \partial_\rho A_\nu) + \frac{2}{3}A_\mu[A_\nu, A_\rho]\right) \end{aligned}$$

The action is independent of metric of M so it gives a topological invariant of M. In fact, the Chern-Simons 3-form is another kind of characteristic class of flat G-bundle. The a parameter k of the theory (inverse of the coupling constant) is called **level**. If G is compact and semi-simple, the level k has to be an integer in order for the action to be gauge invariant. Classically the equations of motion which are the extrema of the action are flat connections:

$$\frac{\delta S}{\delta A} = \frac{k}{2\pi}F = 0\ .$$

11.2.1 Abelian Chern-Simons theory

Let us first consider the case when $G = \mathrm{U}(1)$, namely the Abelian Chern-Simons theory. It has the action,

$$S_{U(1)}[M,A] = \frac{k}{4\pi}\int_M A\wedge dA\ ,$$

Since $\mathrm{U}(1)$ is not a semi-simple group, the level k is not necessarily an integer in this case. The Abelian Chern-Simons theory describes the fractional quantum Hall effect as we will see in the last lecture.

Given an orientable close 3-manifold M, we can consider the path integral of $\mathrm{U}(1)$ Chern-Simons theory

$$Z[M] = \int_{\mathscr{A}/\mathscr{G}} \mathcal{D}A e^{iS_{U(1)}(M,A)}\ .$$

This can be evaluated by so-called **one-loop determinent** and $Z[M]$ turns out to be also topological invariant, called **analytic (Reidemeister) torsion**. This means that Chern-Simons theory provides topological invariants even at quantum level! This was first shown by A. Schwarz in 1978, giving the first construction of what we now call a **topological quantum field theory** [7].

Furthermore, we can consider holonomy group in Chern-Simons theory on $M = S^3$. Given a loop $\gamma : I \to S^3$ with $I(0) = I(1) = p_0$, the parallel transport along K with respect to A provides a holonomy group and it is expressed as

$$u_0 \to u_0 \cdot \exp\left(i\oint_K A\right)\ .$$

We denote it by

$$W_K = \exp\left(i \oint_K A\right)$$

This is called **Wilson loop operator**, which plays an important role in physics. The expectation value of the Wilson loop operator can be expressed by the Feynman path integral

$$\langle W_K \rangle = \int_{\mathscr{A}/\mathscr{G}} \mathcal{D}A\; W_K(A)\; e^{iS_{U(1)}(A)}\;.$$

Let us evaluate the expectation value of two loops K_1 and K_2 in $\mathbb{R}^3$.

$$\langle W_{K_1} W_{K_2} \rangle = \left\langle \exp\left(\oint_{K_1} dx_1^\mu A_\mu \oint_{K_2} dx_2^\nu A_\nu \right) \right\rangle\;. \tag{11.1}$$

Clearly, this expression can be easily evaluated in terms of the two-point correlator (propagator) in S^3

$$\langle A_\mu(x) A_\nu(y) \rangle = \frac{i}{k} \epsilon_{\mu\nu\rho} \frac{(x-y)^\rho}{|x-y|^3}\;.$$

Plugging it into (11.1), the expectation value can be written in terms of linking number

$$\langle W_{K_1}(A) W_{K_2}(A) \rangle = \exp\left(\frac{4\pi i}{k} Lk(K_1, K_2) \right)\;.$$

11.2.2 Non-Abelian Chern-Simons theory

Generalization to non-Abelian Chern-Simons theory has been done by the seminal paper of Witten [8]. Let us consider Wilson loop in non-Abelian Chern-Simons theory where the connections no longer commute. Therefore, the holonomy group should be written

$$u_0 \to u_0 \cdot P \exp\left(i \oint_K A\right)$$

where P is the path-ordered integral due to non-commutativity:

$$P\exp\left(i \oint_K A\right) = \prod_{t=0}^{1} e^{A(\gamma(t'))\,dt'} \equiv \lim_{N\to\infty} \left(e^{A(\gamma(t_N))\Delta t} e^{A(\gamma(t_{N-1}))\Delta t} \cdots e^{A(\gamma(t_1))\Delta t} e^{A(\gamma(t_0))\Delta t} \right)$$

where we subdivide $1 \geq t_N \geq \cdots \geq t_0 \geq 0$ by $\Delta t = \frac{1}{N}$. If the starting point is different $u_0 \to u_0' = u_0 \cdot g$, then holonomy group is

$$P\exp\left(i \oint_K A\right) \to g \cdot P\exp\left(i \oint_K A\right) \cdot g^{-1}\;.$$

Therefore, we can define the operator independent of a starting point by taking trace

$$W_K := \mathrm{Tr}\, P\exp\left(i \oint_K A\right)\;.$$

When $G = \mathrm{SU}(2)$, the expectation value of the Wilson loops

$$\langle W_K \rangle = \int_{\mathscr{A}/\mathscr{G}} \mathcal{D}A\; W_K(A)\; e^{iS_{CS}(A)}\;.$$

gives the Jones polynomial which are invariants of knots and links [8].

Jones polynomials are knot invariants which can be computed by the following skein relation

$$q^2 J\left(\quad \right) - q^{-2} J\left(\quad \right) = (q - q^{-1}) J\left(\quad \right) .$$

where the "quantum" parameter q is expressed as the Chern-Simons level

$$q = \exp\left(\frac{2\pi i}{k+2}\right) .$$

12 Lecture 12: [1]§12

12.1 Index Theorem

Finally, we will study celebrated **Atiyah-Singer index theorem**. The index theorem states the equivalence between the index of an elliptic operator D, which is analytic object, and the characteristic classes, which are topological object. This theorem is one of milestones in mathematics of 20th century. The index theorem plays a crucial role in physics like anomaly, fermion zero modes and supersymmetry. In fact, in the tasteful book [9], you can feel excitement when the index theorem has been formulated. I highly recommend you to read this book.

12.1.1 Symbol, elliptic operator, analytic index

Let E and F be vector bundles of rank m over an n-dimensional M. Let $D\colon \Gamma(E) \to \Gamma(F)$ be a linear differential operator. Namely, on local trivializations $\pi_E^{-1}(U) \cong U \times \mathbb{R}^m$ and $\pi_F^{-1}(U) \cong U \times \mathbb{R}^m$ of E and F, it can be written as

$$D = \sum_{|\alpha| \le k} a_\alpha^{ij}(x) \frac{\partial^\alpha}{\partial x_1^{\alpha^1} \cdots \partial x_n^{\alpha^n}}$$

where $a_\alpha^{ij}(x)$ is an $M_n(\mathbb{R})$-valued function on U. it is said to be of **elliptic type** if for ${}^\forall\xi = (\xi_1, \cdots, \xi_n) \neq 0 \in \mathbb{R}^n$, the matrix

$$\sigma(D)(\xi) = \sum_{|\alpha| \le k} a_\alpha^{ij}(x) \xi_1^{\alpha^1} \cdots \xi_n^{\alpha^n}$$

is non-singular. In other words, $\sigma(D)(\xi) : E_x \to F_x$ is an isomorphism for $\xi \neq 0$. Let us note that $\sigma(D)$ depends only on the highest order portion of D. It is called the **symbol** of D. For example, the Laplacian

$$\sum_i \frac{\partial^2}{\partial x_i^2}$$

is clearly of elliptic type. If M is compact and D is elliptic, it is Fredholm which means that $\ker D$ and $\operatorname{coker} D = \Gamma(F)/D(\Gamma(E))$ are finite-dimensional. We can define the **analytic index**

$$\operatorname{ind}(D) := \dim \operatorname{Ker}(D) - \dim \operatorname{Coker}(D) .$$

12.1.2 de Rham complex

Let M be a compact $2n$-dimensional manifold. Let us define

$$\Omega^{\text{even}}(M) = \Omega^0(M) \oplus \Omega^2(M) \oplus \cdots$$
$$\Omega^{\text{odd}}(M) = \Omega^1(M) \oplus \Omega^3(M) \oplus \cdots$$

Then, the operator

$$d + \delta : \Omega^{\text{even}}(M) \to \Omega^{\text{odd}}(M)$$

is of elliptic type so that we can define the index

$$\text{ind}(d+\delta) = \sum_i (-1)^i H^i_{dR}(M) = \chi(M)$$

so that the index theorem for de Rham complex is

$$\text{ind}(d+\delta) = \int_M e(TM) \ .$$

12.1.3 Dolbeault complex

In this course, I have not introduced to complex manifolds. However, one can complexify the definition of manifolds by taking care of "holomorphicity". In the following, I just explain the index theorem for complex vector bundles, which is called **Hirzebruch-Riemann-Roch** theorem. For more detail, I refer to [1, §8, §12.4].

Let $E \to M$ be holomorphic vector bundle and we consider Cauchy-Riemann operator

$$\overline{\partial} : \Gamma(E) \otimes \Omega^{0,p}(M) \to \Gamma(E) \otimes \Omega^{0,p+1}(M)$$
$$\phi \, d\overline{z}_{i_1} \wedge \cdots \wedge d\overline{z}_{i_p} = \sum_k \frac{\partial \phi}{\partial \overline{z}_k} d\overline{z}_k \wedge d\overline{z}_{i_1} \wedge \cdots \wedge d\overline{z}_{i_p}$$

where $\Omega^{0,p}(M)$ is the set of anti-holomorphic p-forms locally spanned by $d\overline{z}_{i_1} \wedge \cdots \wedge d\overline{z}_{i_p}$. Then, the following long exact sequence is called **Dolbeault complex**

$$0 \to \Gamma(E) \xrightarrow[\overline{\partial}]{} \Gamma(E) \otimes \Omega^{0,1}(M) \xrightarrow[\overline{\partial}]{} \cdots \xrightarrow[\overline{\partial}]{} \Gamma(E) \otimes \Omega^{0,n} \xrightarrow[\overline{\partial}]{} 0 \ .$$

In a similar fashion to de Rham cohomology, one can define Dolbeault cohomology

$$H^{0,*}(M,E) := \text{Ker}(\overline{\partial}) / \text{Im}(\overline{\partial}) \ .$$

If we put Hermitian metric on M and E, we obtain the adjoint operator ϑ of $\overline{\partial}$

$$\vartheta = * \cdot \overline{\partial} \cdot * : \Gamma(E) \otimes \Omega^{0,p}(M) \to \Gamma(E) \otimes \Omega^{0,p-1}(M) \ .$$

Then, the operator

$$\overline{\partial} + \vartheta : \oplus_{p:\text{even}} \Gamma(E) \otimes \Omega^{0,p}(M) \to \oplus_{p:\text{odd}} \Gamma(E) \otimes \Omega^{0,p}(M)$$

turns out to be elliptic, and its analytic index is

$$\text{ind}(\overline{\partial} + \vartheta) = \sum_{p:\text{even}} \dim H^{0,p}(M,E) - \sum_{p:\text{odd}} \dim H^{0,p}(M,E) \ .$$

Then, the Hirzebruch-Riemann-Roch theorem states that the index can be expressed as Chern character and Todd class

$$\text{ind}(\overline{\partial} + \vartheta) = \int_M ch(E) \cdot Td(T^*M \otimes \mathbb{C}) \ .$$

12.1.4 Dirac operator

Another important example is the index of a Dirac operator. Unfortunately, due to time constraint, I cannot explain mathematics of Dirac equations, which involves Clifford algebra, spin group, spin representations, spinor bundle and Dirac operators. Instead, I will use an example in 4-dimension. I refer to [10, 11] for detail. (Nakahara's explanation on Dirac operators is very heuristic and it is not precise.)

In a certain basis, the gamma matrices in $\mathbb{R}^4$ with Euclidean signature can be written as

$$\gamma_\mu = i\begin{pmatrix} 0 & \sigma_\mu \\ -\overline{\sigma}_\mu & 0 \end{pmatrix} , \quad \sigma_\mu = (I, -i\vec{\sigma}) , \quad \overline{\sigma}_\mu = (I, i\vec{\sigma}) .$$

In fact, one can check they satisfy the following algebra

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu} .$$

Generalization of this anti-commutation relation leads to Clifford algebra. The massive Dirac equation on $\mathbb{R}^4$ can be written as

$$(i\gamma^\mu \partial_\mu - m)\psi = 0 ,$$

which is the Euler-Lagrange equation for the action

$$S = \int d^4x\, \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi .$$

The Dirac equation can be interpreted as the square root of the Klein-Gordon equation

$$(\Box + m^2)\phi = 0 .$$

The solutions ψ to the massless Dirac equation

$$i\gamma^\mu \partial_\mu \psi = 0 ,$$

are called **zero modes**. Defining

$$\gamma_5 = -\gamma_1\gamma_2\gamma_3\gamma_4 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

the Dirac spinor ψ can be decomposed into eigenspaces $\mathcal{S}^\pm$ of γ_5 with eigenvalue $\pm$

$$\frac{1}{2}(1+\gamma_5)\psi = \begin{pmatrix} \psi^+ \\ 0 \end{pmatrix} \in \mathcal{S}^+ , \qquad \frac{1}{2}(1-\gamma_5)\psi = \begin{pmatrix} 0 \\ \psi^- \end{pmatrix} \in \mathcal{S}^-$$

which can be understood as left-handed and right-handed fermions, respectively. Now let us consider the Dirac equation on an oriented closed 4-manifold M. Then, the derivative is replaced by the covariant derivative

$$\partial_\mu \to \nabla_\mu = \partial_\mu + \omega_\mu$$

where ω_μ is taking its value on $\mathrm{Spin}(4) = \mathrm{SU}(2) \times \mathrm{SU}(2)$, which is called **spin connection**. The Dirac operator is defined

$$\not{D} = \begin{pmatrix} 0 & \not{\nabla}^\dagger \\ \not{\nabla} & 0 \end{pmatrix}$$

where

$$\bar{\nabla\!\!\!/} = -i\bar{\sigma}^\mu \nabla_\mu \,, \quad \nabla\!\!\!/^\dagger = i\sigma^\mu \nabla_\mu \,.$$

Then, we have the **spin complex**

$$\mathcal{S}^+ \underset{\nabla\!\!\!/}{\overset{\nabla\!\!\!/^\dagger}{\rightleftarrows}} \mathcal{S}^- \,.$$

where $D\!\!\!/$ is of elliptic type. Then, the analytic index of the Dirac operator is

$$\mathrm{ind}(D\!\!\!/) = \dim \mathrm{Ker}\, \nabla\!\!\!/ - \dim \mathrm{Ker}\, \nabla\!\!\!/^\dagger = n_+ - n_-$$

which is the difference between right-handed and left-handed zero modes. Then, the index theorem for the Dirac operator states that the index can be expressed as $\hat{A}$-genus

$$\mathrm{ind}(D\!\!\!/) = \int_M \hat{A}(TM) \,.$$

Usually (like QCD), we consider fermion interacting with non-Abelian gauge fields. To couple fermion to gauge field, we tensor the spinor bundle to a vector bundle E

$$\mathcal{S}^\pm \to \mathcal{S}^\pm \otimes E \,, \qquad \nabla_\mu \to \nabla_\mu + A_\mu \,,$$

Then, one can modify the spin complex accordingly and, in this case, the index theorem is

$$\mathrm{ind}(D\!\!\!/) = \int_M \hat{A}(TM) \cdot ch(E) \,.$$

12.1.5 Anomaly

Although the classical Lagrangian of quantum chromodynamics (QCD) preserves a chiral $U(1)$ symmetry, it is not realized in QCD. The phenomena that the classical symmetry is broken at quantum level is called **anomaly**.

The massless Dirac Lagrangian has a chiral symmetry which can be stated by

$$\partial_\mu j^{\mu 5} = 0 \,, \qquad j^{\mu 5} = \bar{\psi}\gamma^\mu \gamma^5 \psi \,.$$

If we integrate it out

$$\int d^4x \, \partial_\mu j^{\mu 5} = N_R - N_L = 0 \,,$$

the difference in the number of right-handed fermions and left-handed fermions is conserved at classical level. However, if we take quantum effect into account in QCD, this is no longer true.

As we have seen above, the difference between right-handed and left handed fermion zero modes is given by

$$n_+ - n_- = \int_{\mathbb{R}^{1,3}} \hat{A}(\mathbb{R}^{1,3}) \cdot ch(E) = \frac{1}{8\pi^2} \int_{\mathbb{R}^{1,3}} \mathrm{Tr}(F \wedge F) \,.$$

In the presence of instanton effect, the right hand side is no longer zero so that the chiral symmetry is broken. This is called **Adler-Bell-Jackiw**, **chiral** or **triangle anomaly**.

The pion π^0 can be considered as a Goldstone boson for chiral symmetry breaking. The decay rate of the pion into two photon $\pi^0 \to 2\gamma$ can be computed by the index theorem and it is experimentally checked to an accuracy of a few percent.

12.2 Supersymmetric quantum mechanics

Supersymmetry is a symmetry between bosons and fermions. Let us consider the simplest supersymmetric theory where there is only one fermionic field. Hence, we describe

$$\begin{pmatrix}1\\0\end{pmatrix} : \text{ fermionic state} \qquad \begin{pmatrix}0\\1\end{pmatrix} : \text{bosonic state} .$$

The Hilbert space $\mathcal{H}$ of states consists of wave functions of the form

$$\begin{pmatrix}\phi_F(x)\\ \phi_B(x)\end{pmatrix} = \phi_F(x)\begin{pmatrix}1\\0\end{pmatrix} + \phi_B(x)\begin{pmatrix}0\\1\end{pmatrix} ,$$

where it obeys the normalizable (L^2-norm) condition

$$\int_{-\infty}^{\infty} (|\phi_B(x)|^2 + |\phi_F(x)|^2)dx < \infty . \tag{12.1}$$

Let us denote the Pauli matrices as

$$\sigma_3 = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} , \quad \sigma_+ = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} , \quad \sigma_- = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} .$$

Then, $\sigma_\pm$ are the fermion creation $(+)$ and annihilation $(-)$ operators and the fermion number is defined by $F = \frac{1}{2}(1+\sigma_3)$.

Now we define supercharges

$$\begin{aligned} Q &= \frac{1}{\sqrt{2}i}\sigma_+\left(\frac{d}{dx} + W'(x)\right) \\ Q^\dagger &= \frac{1}{\sqrt{2}i}\sigma_-\left(\frac{d}{dx} - W'(x)\right) , \end{aligned} \tag{12.2}$$

where $W(x)$ is called a **superpotential**. It is easy to see that

$$\{Q, Q\} = \{Q^\dagger, Q^\dagger\} = 0 .$$

Since the Hilbert space is decomposed into bosonic states and fermionic states $\mathcal{H}_B \oplus \mathcal{H}_F$, the supercharges are indeed linear maps

$$\begin{tikzcd} \mathcal{H}_B \arrow[r, shift left, "Q"] & \mathcal{H}_F \arrow[l, shift left, "Q^\dagger"] \end{tikzcd} .$$

The Hamiltonian of a superymmstric theory is written as the anti-commutator of supercharges

$$H := \{Q, Q^\dagger\} = \frac{1}{2}\left(-\frac{d^2}{dx^2} + W'(x)^2\right) + \frac{1}{2}\sigma_3 W''(x)$$

In other words, the supercharge is a "square-root" of Hamiltonian. In addition one can check that the supercharges commute with the Hamiltonian

$$[H, Q] = [H, Q^\dagger] = 0 . \tag{12.3}$$

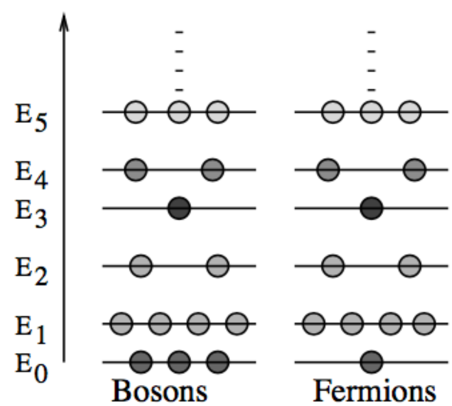
Let us consider a bosonic eigenstate $|b\rangle \in \mathcal{H}_B$ of the Hamiltonian

$$H|b\rangle = \{QQ^\dagger + Q^\dagger Q\}|b\rangle = Q^\dagger Q|b\rangle = E|b\rangle ,$$

for $E > 0$. Now its **superpartner** $|f\rangle = Q|b\rangle$ has the same energy

$$H|f\rangle = \{QQ^\dagger + Q^\dagger Q\}Q|b\rangle = QQ^\dagger Q|b\rangle = E|f\rangle .$$

In a similar fashion, for a fermionic eigenstate $|f\rangle \in \mathcal{H}_F$ with $E > 0$, one can show that its superpartner $|b\rangle = Q^\dagger|f\rangle$ has the same energy. In fact, the commutation relation (12.3) is responsible for the degeneracy.



On the other hands, situation drastically changes for states with $E = 0$. For a bosonic state $|b\rangle$ with zero energy $H|b\rangle = 0$, we have $\langle b|Q^\dagger Q|b\rangle = 0$ so that the supercharge annihilates the state $Q|b\rangle = 0$ and therefore there is no fermonic partner. Similarly, for a fermionic state $|f\rangle$, one can show that $Q^\dagger|f\rangle$ so that there is no bosonic partner. Since these zero energy states $|z\rangle$ are annihilated by both supercharges Q and $Q^\dagger$, they are invariant under supersymmetric transformation

$$\exp(i\epsilon Q)|z\rangle = |z\rangle , \qquad \exp\left(i\epsilon Q^\dagger\right)|z\rangle = |z\rangle .$$

Therefore, they are also called **supersymmetric states**.

Witten considered the difference between bosonic and fermionic zero energy states [12]. To this end, he has introduced the index $\mathrm{Tr}(-1)^F$. Since the excited states do not contribute to the index because there are always boson-fermion pairs. Therefore, the index is equal to

$$\mathrm{Tr}(-1)^F = \dim \mathrm{Ker}\, Q - \dim \mathrm{Ker}\, Q^\dagger$$

Note that the space of bosonic zero energy states can be expressed as $\mathrm{Ker}\, Q$ whereas the space of fermionic zero energy states can be written as $\mathrm{Ker}\, Q^\dagger$.

Since zero energy sates obey

$$\left(\frac{d}{dx} \pm W'(x)\right)\phi(x) = 0 ,$$

we have

$$\phi(x) = \mathrm{const} \times \exp(\mp W(x)) .$$

The normalizability condition (12.1) requires $\phi(x) \to 0$ as $|x| \to \infty$. Suppose that the superpotential $W(x)$ is subject to polynomial growth $W(x) \sim \lambda x^n$ in the region $|x| \gg R$.

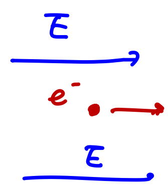
- When n is even, there exists either bosonic or fermionic zero energy states depending on the sign of λ. Therefore, the index is equal to the sign of λ.
- When n is odd, there is no zero energy state because $W(x)$ changes the sign when x moves from $-\infty$ to ∞ so that the index is equal to zero. Although the Hamiltonian is supersymmetric, there is no supersymmetric state for a potential of this type. Therefore, supersymmetry is **spontaneously broken**.

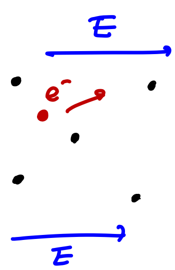
As we have seen, the index is independent of the detail of the superpotential $W(x)$ and it depends only on its asymptotic behavior. This is what we have seen in the first lecture.

13 Epilogue: quantum Hall effects [14]

This section is essentially a write-up of Witten's slides [13].

13.1 What is the Hall effect?





An electron in an electric field is accelerated

$$m\frac{d^2\vec{x}}{dt^2} = e\vec{E} \quad \longrightarrow \quad \vec{v} = \frac{d\vec{x}}{dt} \sim \frac{e\vec{E}}{m}t$$

If the electron is in an obstacle course which randomizes its velocity on an average after a time τ, then the velocity will not go to infinity, but to

$$\langle\vec{v}\rangle \sim \frac{e\vec{E}}{m}\frac{\tau}{2}$$

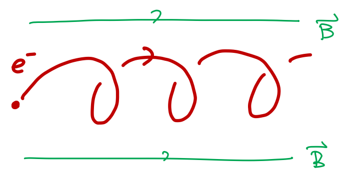
It $\mathcal{N}$ is the number density of electrons that are free to move, The average current will be be roughly

$$\vec{J} = \mathcal{N}e\langle\vec{v}\rangle = \frac{\mathcal{N}e^2\tau}{2m}\vec{E} \quad \longrightarrow \quad \vec{J} = \sigma\vec{E}$$

where the conductivity is $\sigma = \frac{\mathcal{N}e^2\tau}{2m}$ in this approximation. (No topology here - σ depends on all the details.) Now in a magnetic field $\vec{B}$

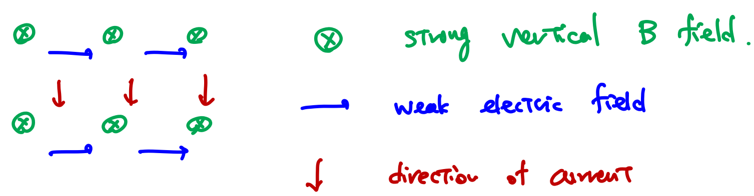
$$m\frac{d^2\vec{x}}{dt^2} = \frac{e}{c}\frac{d\vec{x}}{dt} \times \vec{B}$$

The electron motion is spiral. Uniform motion along $\vec{B}$ and circular oscillations normal



to $\vec{B}$. To get the Hall effect. we want to combine perpendicular $\vec{E}$ and $\vec{B}$ with $\vec{E} \ll \vec{B}$. The electron now drifts out at the page, with velocity

$$\vec{v} = c\frac{\vec{E} \times \vec{B}}{|\vec{B}|^2}$$



With no magnetic field, the current $\vec{J}$ is in the $\vec{E}$ direction, but with a strong magnetic field, the current flow is perpendicular to $\vec{E}$ and $\vec{B}$. With average velocity $\vec{v} = c\frac{\vec{E}\times\vec{B}}{|\vec{B}|^2}$ for each electron, the current is

$$\vec{J} = \mathcal{N}e\vec{v} = \mathcal{N}ec\frac{\vec{E} \times \vec{B}}{|\vec{B}|^2} = \sigma_H \widehat{B} \times \vec{E}$$

where $\widehat{B} = \vec{B}/|\vec{B}|$ the Hall conductivity is $\sigma_H = \mathcal{N}ec/|\vec{B}|$. (no topology here!)

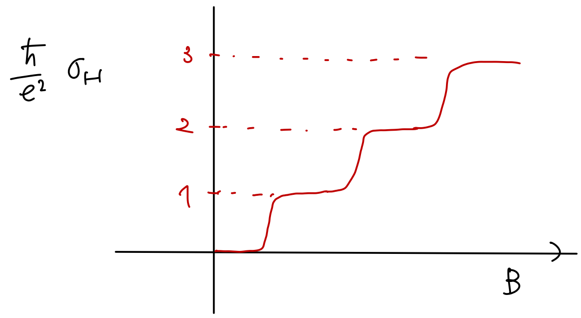
13.2 What is the quantum Hall effect?

The quantum Hall effect arises if we consider a very thin sample: special composition, low temperature, large $\vec{B}$ which effectively is a 2d surface - a molecular monolayer. Here we find, for certain type of material, in certain ranges of $\vec{B}$ and temperature, that the Hall conductivity does have a magic value

$$\sigma_H = \frac{e^2}{\hbar}n \qquad n \in \mathbb{Z}$$

where $\hbar$ is the Planck courant. This value is unchanged as $\vec{B}$, temperature, chemical composition impurity concentration, etc. are varied within certain limits.

The integrality of $\frac{\hbar}{e^2}\sigma_H$, is verified so precisely that it has become the most accurate way to measure $\frac{e^2}{\hbar}$. There is an integer here, so we should seek a topological explanation!



But how can we hope to find one in the messy world of condensed mater physics? In this measurement, we are looking at very large samples observed for very long times, compared to atomic distance and times. One of the main reasons that it is possible in physics is that in any observation made at a certain scale of length and time. Most of microscopic degree of freedom - needed for a much shorter lengths and times - are irrelevant. This notion of **effective theory** is called **universality**.

Familiar example is a rigid body: The only relevant degree of freedom is the time-dependent embedding in space.

Solids vary widely in their relevant degrees of freedom:

- vibrations are always relevant for certain questions, but not for electromagnetic behavior like the Hall effect
- electrons are relevant in conductors where they are free to move, but not in insulators.
- an exotic example of a relevant variable is the trivialization s of the line bundle $\mathcal{L}^2$ in a superconductor
- another exotic example arises in the "fractional quantum Hall effect", when $\sigma_H \frac{\hbar}{e^2}$ is a rational number not an integer.

Whatever the relevant degrees of freedom may be in a particular case, the macroscopic interactions of electromagnetism with the solid are described by Euler-Lagrange equation and quantization from an action

$$L = L_{\text{vacuum}} + L_{\text{solid}} \,.$$

Here L_{vacuum} is the Lagrangian in vacuum of the electromagnetic field. If A is $U(1)$ connection and $F = dA$ is the curvature, then

$$L_{\text{vacuum}} = \frac{\hbar}{4e^2} \int_{\mathbb{R}^{3,1}} F \wedge *F \,.$$

The part of the action due to the solid is

$$L_{\text{solid}} = \int_{\text{solid} \times \text{time}} d\mu \, W(\Phi, A) \;,$$

where W is a local functional of the relevant degrees of freedom Φ as well as the connection A. "Locality" means that the variation of W is a gauge -invariant differential polynomial in Φ, A. The total action is thus

$$\frac{\hbar}{4e^2} \int_{\mathbb{R}^{3,1}} F \wedge *F + \int_{\text{solid}} d\mu \, W(\Phi, A) \;.$$

The Euler-Lagrange equation for A becomes

$$d\,(*F) = -\frac{e^2}{\hbar}\frac{\delta W}{\delta A}\,(\Phi, A)$$

and if we compare to freshman physics

$$*d(*F) = 4\pi J$$

Now what are the relevant degrees of freedom for the quantum Hall effect? (Answer: for the ordinary integer quantum Hall effect, there are none.) Materials with no relevant degrees of freedom in their interaction with electromagnetism are not unusual. (An ordinary pane of class is an example.) It means that $W(\Phi, A)$ reduces to $W(A)$.

13.3 Some condensed matter physics

Before trying to discuss the quantum Hall effect, let us discuss a 3-dimensional piece of class. So we will feel to know what we are doing. The local structure is invariant under rotations and reflections, which will constrain W. A drastic simplification comes from the following. Grade the monomials in W by "dimension"

$$\dim A = 1\,, \qquad \dim\frac{\partial}{\partial x} = \dim\frac{\partial}{\partial t} = 1\,, \qquad \dim\vec{E} = \dim\vec{B} = 2\,.$$

If all fields and frequencies are small on an atomic scale, terms of higher dimension are smaller. This is essentially always true in practice, and anyway otherwise the effective description is not valid. The vacuum part of the action

$$L_{\text{vacuum}} = \frac{\hbar}{4e^2}\int_{\mathbb{R}^{3,1}} F \wedge *F\,.$$

is of dimension four, so we can neglect in $W(A)$ terms of dimension greater than four. Using rotation invariance and reflection symmetry, $W(A)$ cannot have any terms linear in $\vec{E}$ or $\vec{B}$ so the lowest dimension terms are

$$W(A) = \int d^4x(\epsilon\vec{E}^2 - \mu\vec{B}^2)$$

where ϵ and μ are the electric and magnetic susceptibilities. To find how glass interacts with electromagnetism, we just look at the combined Lagrangian

$$\frac{\hbar}{4e^2}\int_{\mathbb{R}^{3,1}} F \wedge *F + \int d^4x(\epsilon\vec{E}^2 - \mu\vec{B}^2) = \int d^4x\Big(\Big(\frac{\hbar}{4e^2}+\epsilon\Big)\vec{E}^2 - \Big(\frac{\hbar}{4e^2}+\mu\Big)\vec{B}^2\Big)\,.$$

The resulting Euler-Langrange equations are Maxwell's equations, but with a reduced speed of light.

If instead of glass, we consider a crystal (still with no relevant degrees of freedom), then W can have general quadratic terms in $\vec{E}$ and $\vec{B}$

$$\int d^4x(\epsilon_{ij}E_iE_j - \mu_{ij}B_iB_j)\,.$$

We get an anisotropic speed of light and "birefringence". If the crystal lacks reflection symmetry, then $W(A)$ can contain terms linear in $\vec{E}$ and $\vec{B}$

$$\Delta W = \int d^4x(\vec{n}\cdot\vec{E} + \vec{m}\cdot\vec{B})$$

where $\vec{n}$ and $\vec{m}$ depend on the crystal axes. Such materials are "ferroelectric" and "ferromagnetic", as one can learn by deriving and solving the Euler-Lagrange equations. By now we know how to study materials with no relevant degrees of freedom: **write the possible terms in $W(A)$ of low dimension. derive and solve the Euler-Lagrange equations.** The hypothesis that there are no relevant degrees of freedom ensures that $W(A)$ is a local functional of A only.

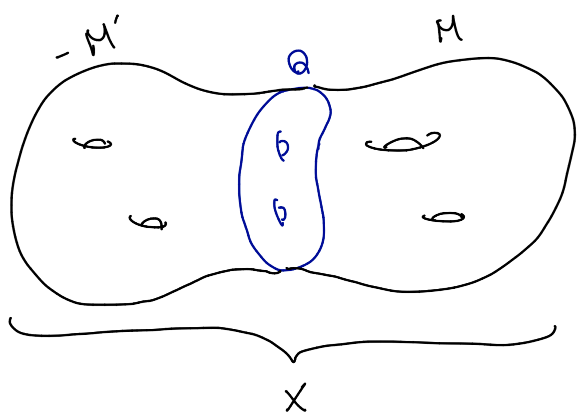
13.4 Some geometry

Now we are ready to study the quantum Hall effect, which involves an atomic monolayer or a 2d surface in space. Hence, the spacetime sweeps out a three-manifold $Q = \Sigma \times \mathbb{R}$ where $\mathbb{R}$ parametrizes time. So we are doing $U(1)$ gauge theory on a three-manifold Q, and the reason that we will get something interesting is that there is an unusual term, of topological interest, that can appear in $Q(A)$. This is the Chern-Simons form. We can assume that Q is the boundary of a 4-manifold M over which the electromagnetic line bundle $\mathcal{L}$ and connection A extend and we write

$$S_M(A) = \frac{1}{4\pi^2} \int_M F \wedge F$$

This depends on M, but only slightly

$$S_M(A) - S_{M'}(A) = \frac{1}{4\pi^2} \int_X F \wedge F \in \mathbb{Z}$$



So as a map to $\mathbb{R}/\mathbb{Z}$, $S_M(A)$ is independent of M and we just call it $S(A)$. Concretely, for a trivial line bundle $\mathcal{L}$, we have

$$S(A) = \frac{1}{4\pi^2} \int_Q A \wedge dA \, .$$

$S(A)$ has a low dimension -three- so if we can include it in $W(A)$, the effective action, then this will be important? Does it make sense to do this, given that $S(A)$ has an additive integer indeterminacy?

In classical mechanics, adding a constant to the action I does not matter as it does not affect the Euler-Lagrange equation. In quantization, one needs to be able to define

$\exp(iI)$. To compute quantum transition amplitude, so I can have an additive ambiguity, but this must be of the form $2\pi k$ for $k \in \mathbb{Z}$. So a conceivable material may have in the electromagnetic effective action $W(A)$ a multiple of $S(A)$, but it must be a very special multiple:

$$W(A) = 2\pi k S(A) = \frac{k}{2\pi}\int A \wedge dA\ ,$$

for some $k \in \mathbb{Z}$ (which depends on the material, the magnetic field, etc.)

13.5 Theory of the quantum Hall effect

Now let us go back to the quantum Hall effect where the gauge group (for the class of materials we are considering) is Abelian. We consider a flat sample in the x-y plane with $k \neq 0$.

$$L = \frac{\hbar}{4e^2}\int_{\mathbb{R}^{1,3}} F \wedge *F + \frac{k}{2\pi}\int_{\text{sample}\times\text{time}} A \wedge dA$$

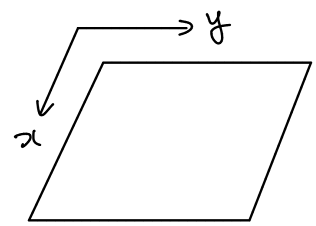
The electromagnetic current is

$$J = \frac{4\pi e^2}{\hbar}\frac{\delta W}{\delta A} = \frac{e^2 k}{\hbar}(*_3 F)$$

where $*_3$ is the Hodge star in the 3-dimensional spacetime for the sample. In a non-relativistic Language, this becomes

$$J_x = \frac{e^2 k}{\hbar}E_y\ , \qquad J_y = -\frac{e^2 k}{\hbar}E_x\ .$$

We have found the quantum Hall effect $\sigma_H = \frac{e^2}{\hbar}k$.



For fractional quantum Hall effects, we add a new $U(1)'$ gauge field B with the old $U(1)$ gauge field A

$$\frac{\hbar}{4e^2}\int F \wedge *F + \frac{k_1}{2\pi}\int A \wedge dA + \frac{k_2}{\pi}\int A \wedge dB + \frac{k_3}{2\pi}\int B \wedge dB\ .$$

where B can be interpreted as an "emergent" gauge field that only propagates in the sample. By integrating out B field, one can check that the conductivity becomes fractional

$$\sigma_H = \frac{e^2}{\hbar}\left(k_1 - \frac{k_2^2}{k_3}\right)\ .$$

14 Extra Lecture 1: [1] §8

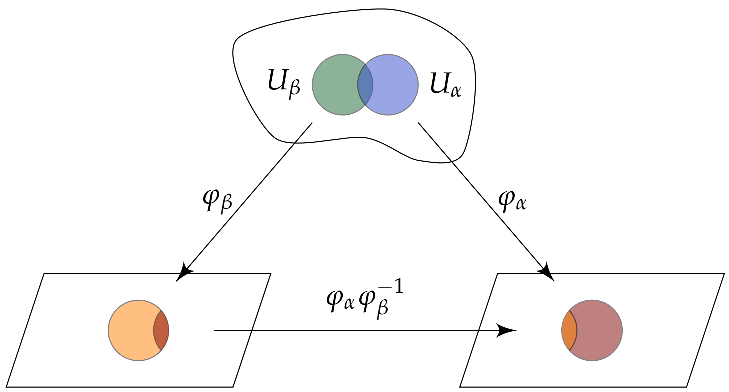
14.1 Complex manifolds

Definition 14.1 (Complex manifold). Let M be an $2n$-dimensional manifold. M is a **complex manifold** of complex dimension n if

1. Let $M = \bigcup_\alpha U_\alpha$ be an open covering.
2. There is a homeomorphism $\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha) \subseteq \mathbb{C}^n$, where $\varphi(U_\alpha)$ is open in $\mathbb{C}^n$.
3. For all α, β, we have $\varphi_\alpha(U_\alpha \cap U_\beta)$ is open in $\mathbb{C}^n$, and the transition function

$$\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$$

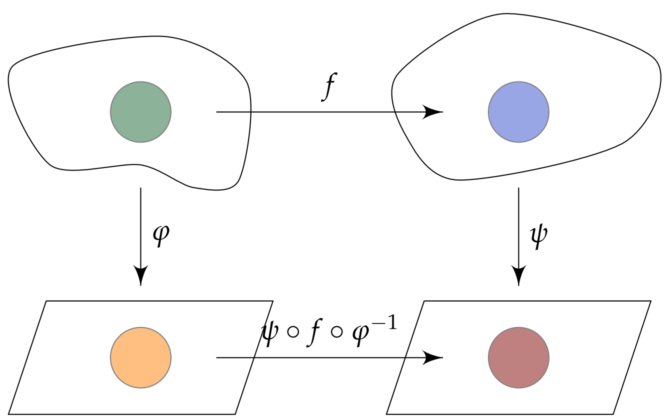
is holomorphic (i.e. they depend only on the z_μ but not on their complex conjugate).



As we consider differentiable (smooth) maps between two smooth manifolds M and N, we can consider holomorphic maps between two complex manifolds M and N. Let M and N be smooth manifolds, and let $\{(U_\alpha, \varphi_\alpha)\}_\alpha$ and $\{(V_\beta, \psi_\beta)\}_\beta$ be their atlas.

Definition 14.2 (holomorphic map). A map $f : M \to N$ is **holomorphic** if, for a chart $(U_\alpha, \varphi_\alpha)$ of $p \in M$ and (V_β, ψ_β) of $f(p) \in N$, $\psi_\beta \circ f \circ (\varphi_\alpha)^{-1} : \varphi(U_\alpha \cap f^{-1}(V_\beta)) \to \xi(V_\beta)$ is holomorphic.

If there is the inverse map that is holomorphic, it is called a **biholomorphic** map, or M and N are **biholomorphic equivalent**.



Example 14.3. Complex projective space $\mathbb{CP}^n$ is a complex manifold. (Check it satisfies the definition.)

Example 14.4. Let us now consider compact complex manifolds. It turns out that submanifolds of $\mathbb{C}^n$ are not interesting, since a connected compact analytic submanifold of $\mathbb{C}^n$ is a point. However, many compact complex manifolds can be constructed as submanifolds of projective spaces $\mathbb{CP}^n$. We saw that $\mathbb{CP}^n$ is compact; all its closed complex submanifolds are also compact. In fact, there is a theorem by Chow stating that any compact submanifold of $\mathbb{CP}^n$ can be realized as the zero loci $F_i(z) = 0$ of a finite number of homogeneous polynomial equations in the homogeneous coordinates z_μ. These compact complex manifolds are called **complex projective variety**. One important example is the **Fermat quintic** in $\mathbb{CP}^4$, given as the zero locus of the equation

$$5\psi z_0 z_1 z_2 z_3 z_4 - \sum_{\mu=0}^{4} (z_\mu)^5 = 0. \tag{14.1}$$

This three-dimensional compact complex manifold turns out to be Calabi–Yau, probably the most studied Calabi–Yau threefold because of mirror symmetry [15].

14.2 Holomorphic vector bundles

Definition 14.5 (Holomorphic vector bundle). Let M be a complex manifold and let $\pi : E \to M$ be a vector bundle of real rank $2r$

1. For each $p \in M$, the fiber $\pi^{-1}(p) = E_p$ is an r-dimensional complex vector space,
2. For all $p \in M$, there is an open $U \subseteq M$ containing p and a diffeomorphism

   $$t : E_U = \pi^{-1}(U) \to U \times \mathbb{C}^r$$

   such that

   $$\begin{tikzcd} E_U \arrow[r, "t"] \arrow[d, "\pi"] & U \times \mathbb{C}^r \arrow[dl, "p_1"] \\ U & \end{tikzcd}$$

   commutes, and the induced map $E_q \to \{q\} \times \mathbb{C}^r$ is a linear isomorphism for all $q \in U$.
3. Suppose that $t_\alpha : E|_{U_\alpha} \to U_\alpha \times \mathbb{C}^r$ and $t_\beta : E|_{U_\beta} \to U_\beta \times \mathbb{C}^r$ are trivializations of E. Then

   $$t_\alpha \circ t_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{C}^r \to (U_\alpha \cap U_\beta) \times \mathbb{C}^r$$

   is fiberwise linear, i.e.

   $$t_\alpha \circ t_\beta^{-1}(q, v) = (q, g_{\alpha\beta}(z)v),$$

   where $g_{\alpha\beta}(z) : U_\alpha \cap U_\beta \to \mathrm{GL}(r, \mathbb{C})$ is a holomorphic function. The transition functions have the following properties

   (a) $g_{\alpha\alpha} = \mathrm{id}$

   (b) $g_{\alpha\beta} = g_{\beta\alpha}^{-1}$

   (c) $g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = 1.$

14.2.1 Holomorphic tangent and cotangent bundle

Let $(U,(z_1,\cdots,z_n))$ be a local coordinate of a complex manifold M and we decompose the complex coordinate $z_j = x_j + iy_j$ into the real and imaginary part. Then, the basis of the tangent bundle TM on an open set $U \subset M$ can be taken by $\frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_j}$ $(j = 1,\cdots,n)$. Now let us complexify the tangent bundle $T_{\mathbb{C}}M := TM \otimes \mathbb{C}$ where we can take the basis on U as

$$\frac{\partial}{\partial z_j} := \frac{1}{2}\Big(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\Big)\,, \qquad \frac{\partial}{\partial \overline{z}_j} := \frac{1}{2}\Big(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\Big)\,.$$

Each fiber of $T_pM \otimes \mathbb{C}$ is a vector space with basis $\frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_j}$ on $\mathbb{C}$. Similarly, one can complexify the cotangent bundle $T^*_{\mathbb{C}}M := T^*M \otimes \mathbb{C}$ where we can take basis dz_j, $d\overline{z}_j$ on U. At each point $p \in M$, we can take the subspace $T_p^{(1,0)}M$ of $T_pM \otimes \mathbb{C}$ spanned by $\{\frac{\partial}{\partial z_j}\}$. A family of the subspace is denoted by $T^{(1,0)}M = \cup_p T_p^{(1,0)}M$ and we can also define $T^{*(1,0)}M = \cup_p T_p^{*(1,0)}M$ in a similar fashion. Given another chart $(V(w_1,\cdots,w_n))$, they transform

$$\frac{\partial}{\partial z_j} = \frac{\partial w_k}{\partial z_j}\frac{\partial}{\partial w_k}\,, \quad dz_j = \frac{\partial z_j}{\partial w_k}dw_k$$

where the transition functions $\frac{\partial w_k}{\partial z_j}$ and $\frac{\partial z_j}{\partial w_k}$ are holomorphic so that they are holomorphic vector bundles. Indeed, we can decompose complxified (co)tangent bundle as

$$T_{\mathbb{C}}M = T^{(1,0)}M \oplus T^{(0,1)}M\,, \qquad T^*_{\mathbb{C}}M = T^{*(1,0)}M \oplus T^{*(0,1)}M\,.$$

We call $T^{(1,0)}M$ (resp. $T^{(0,1)}M$) the **holomorphic (resp. anti-holomorphic) (co)tangent bundle**.

As a real bundle. $T^{(1,0)}M$ is isomorphic to TM. In fact, We can define a map Re : $T_{\mathbb{C}}M \to TM$ by taking real part

$$2\mathrm{Re}\,\frac{\partial}{\partial z_j} = \frac{\partial}{\partial x_j}\,,$$

which is an isomorphism. Since $T^{(1,0)}M$ is a complex vector bundle, we have the multiplication by i, which induces an isomorphism $J : TM \to TM$:

$$2\mathrm{Re}\,\frac{\partial}{\partial z_j} = \frac{\partial}{\partial x_j}\,, \quad 2\mathrm{Re}\,i\frac{\partial}{\partial z_j} = \frac{\partial}{\partial y_j}\,,$$

so that

$$J\Big(\frac{\partial}{\partial x_j}\Big) = \frac{\partial}{\partial y_j}\,, \quad J\Big(\frac{\partial}{\partial y_j}\Big) = -\frac{\partial}{\partial x_j}\,.$$

It is easy to check $J^2 = -\mathrm{id}_{TM}$, and the eigenvalues of J in $T_pM \otimes \mathbb{C}$ are $\pm i$. Hence, we consider $T_p^{(1,0)}M$ (resp. $T_p^{(0,1)}M$) as the eigenspace of J_p with eigenvalue i (resp. $-i$). J is called **almost complex structure**.

An almost complex structure $J_M : TM \to TM$ is a complex structure if for any $p \in m$ there exist an open neighborhood $p \in U \subset M$, $V \subset \mathbb{C}^n$, and a continuous map $\varphi : U \to V$ such that $d\varphi \cdot J_M = J_{\mathbb{C}^n} \cdot d\varphi$. If an almost complex structure J_M is a complex structure, J_M is called to be **integrable**. The Newlander-Nirenberg theorem states that an almost complex structure J is integrable if and only if the Nijenhuis tensor vanishes:

$$N_J(X,Y) = -J^2[X,Y] + J([JX,Y] + [X,JY]) - [JX,JY] = 0$$

for $^\forall X, Y \in \mathfrak{X}(M)$.

The only spheres which admit almost complex structures are S^2 and S^6. In particular, S^4 cannot be given an almost complex structure. In the case of S^2, the almost complex structure comes from the complex structure of the Riemann sphere. The 6-sphere S^6 inherits an almost complex structure from the octonion multiplication;. However, whether S^6 has a complex structure is an open question.

14.2.2 Differential forms

Locally, any differential forms can be expressed as a linear combination of the exterior products of dz_j and $d\bar{z}_k$. In particular, a differential form which can be expanded in terms of

$$dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{i_1} \wedge \cdots \wedge d\bar{z}_{i_q}$$

called (p,q)-form, which is a section of $\bigwedge^p T^{*(1,0)}M \otimes \bigwedge^q T^{*(0,1)}M$. We denote the vector space of (p,q)-forms by $\Omega^{p,q}(M)$. In fact, it is easy to show that the following complexified bundles decompose as

$$\bigwedge^k T^*_{\mathbb{C}}M = \bigoplus_{j=0}^{k} \bigwedge^j T^{*(1,0)}M \otimes \bigwedge^{k-j} T^{*(0,1)}M,$$

On a complex manifold, the exterior derivative also admits a simple decomposition: $d = \partial + \bar{\partial}$, where we defined the operators

$$\partial : \Omega^{p,q}(M) \to \Omega^{p+1,q}(M)\,, \qquad \bar{\partial} : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M)\,.$$

The identity $d^2 = 0$ implies that $\partial^2 = \bar{\partial}^2 = 0$ and $\partial\bar{\partial} + \bar{\partial}\partial = 0$.

A $(p,0)$-form is an element of $\Omega^{p,0}(M)$ and it can be locally written as

$$\varphi = \sum \varphi_{i_1,\cdots,i_p}(z) dz_{i_1} \wedge \cdots \wedge dz_{i_p}$$

where $\varphi_{i_1,\cdots,i_p}(z)$ is holomorphic. Therefore, ϕ is called **a holomorphic p-form**. In particular, when $p = n$,

$$K_M = \bigwedge^n T^{*(1,0)}M$$

is called **the canonical line bundle**. Since TM is isomorphic to $T^{(1,0)}M$, the first Chern class is

$$c_1(M) = c_1(TM) = c_1(T^{(1,0)}M) = c_1(\bigwedge^n T^{(1,0)}M) = -c_1(K_M)$$

Now what is the analog of the de Rham cohomology groups for complex manifolds?

Definition 14.6. Let M be a complex manifold of complex dimension m. As $\bar{\partial}^2 = 0$, we can form the complex

$$0 \xrightarrow{\bar{\partial}} \Omega^{p,0}(M) \xrightarrow{\bar{\partial}} \Omega^{p,1}(M) \xrightarrow{\bar{\partial}} \ldots \xrightarrow{\bar{\partial}} \Omega^{p,n}(M) \xrightarrow{\bar{\partial}} 0.$$

We define the **Dolbeault cohomology groups** $H^{p,q}_{\bar{\partial}}(M)$ of M by

$$H^{p,q}_{\bar{\partial}}(M) = \frac{\mathrm{Ker}(\bar{\partial} : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M))}{\mathrm{Im}(\bar{\partial} : \Omega^{p,q-1}(M) \to \Omega^{p,q}(M))}.$$

Remark that the Dolbeault cohomology groups depend on the complex structure of M. Note also that we could have defined the cohomology groups using ∂ instead of $\bar{\partial}$, this is just a matter of convention since they are complex conjugate.

Theorem 14.7 (Hodge decomposition theorem for Kähler manifolds). For a compact Kähler manifold M, the complex cohomology satisfies

$$H^r(M,\mathbb{C}) \cong \bigoplus_{p+q=r} H^{p,q}_{\bar{\partial}}(M)\,,$$
$$H^{p,q}_{\bar{\partial}}(M) = \overline{H^{q,p}_{\bar{\partial}}(M)} \tag{14.2}$$

where $H^{p,q}_{\bar{\partial}}$ is the Dolbeault cohomology restricted to forms of type (p,q). Further, we could instead choose the de Rham cohomology to obtain the same result, and the cohomology classes contain unique harmonic representatives. Consequently, holomorphic forms are therefore harmonic for any Kähler metric on a compact manifold, and the odd Betti numbers are even.

We now define the **Hodge numbers** to be $h^{p,q} = \dim H^{p,q}_{\bar{\partial}}(M)$. The Hodge numbers of a complex manifold are summarized in what is commonly called the **Hodge diamond**:

$$\begin{array}{ccccccc}
 & & & h^{n,n} & & & \\
 & & h^{n,n-1} & \vdots & h^{n-1,n} & & \\
h^{n,0} & & \cdots & & \cdots & & h^{0,n} \\
 & & h^{1,0} & \vdots & h^{1,0} & & \\
 & & & h^{0,0} & & &
\end{array}$$

Theorem 14.8 (Hard Lefschetz Theorem). Let (M,ω) be a Káhler manifold. Let L be exterior multiplication by the Kähler form ω. Then

$$L^k : H^{n-k}(M) \to H^{n+k}(M)$$

is an isomorphism for $1 \le k \le n$. Further, if we define the primitive cohomology $P^j(M)$ by

$$P^j(M) = \ker L^{n-j+1} : H^j \to H^{2n-j+2}\,,$$

then we have the Lefschetz decomposition

$$H^m(M) = \bigoplus_k L^k P^{m-2k}(M)\,.$$

Thus, the Betti numbers of a compact Kähler manifold have a pyramid structure, called the **Hodge pyramid**.

14.3 Kähler manifolds

The condition for a Riemannian metric g to be Hermitian is that for ${}^\forall X, Y \in \mathfrak{X}(M)$, and ${}^\forall \alpha, \beta \in \mathbb{C}$, it satisfies

$$g(\alpha X, \beta y) = \alpha\overline{\beta} g(X,Y)\,.$$

This condition amounts to the case when $\alpha = \beta = i$. Since the multiplication by i corresponds to J, the Hermitian metric can be defined as follows.

Definition 14.9. Let (M, J) be a complex manifold, and let g be a Riemannian metric on M. We call g a **Hermitian metric** if $g(v, w) = g(Jv, Jw)$ for all vector fields v, w on M

In other words, a Hermitian metric is a positive-definite inner product $T^{(1,0)}M \otimes T^{(0,1)}M \to \mathbb{C}$ at every point on a complex manifold M. In local coordinate, we can express g on $T_{\mathbb{C}}M$ by

$$g_{j\bar{k}} = g\Big(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_k}\Big) ,$$

so that $g_{j\bar{k}}$ is a Hermitian metric.

Using this Hermitian metric g, we can define a two-form ω on M called the **Hermitian form** by $\omega(v, w) = g(Jv, w)$ for all vector fields on v, w on M. In local coordinate, it can be written as

$$\omega = ig_{j\bar{k}} dz_j \wedge d\bar{z}_k$$

Therefore, ω is a $(1, 1)$-form.

Definition 14.10. Let (M, J) be a complex manifold, and g a Hermitian metric on M, with Hermitian form ω. g is a **Kähler metric** if $d\omega = 0$. In this case we call ω a **Kähler form**, and we call a complex manifold (M, J) endowed with a Kähler metric a **Kähler manifold**.

Theorem 14.11. Any complex submanifold $N \subset M$ of a Kähler manifold (M, ω) is Kähler where the Kähler form is its restriction $\omega_N = \omega\Big|_N$.

For a Kähler manifold M, the holonomy group is $U(n) \subset SO(2n)$. The condition that a Riemannian metric g can be Kähler is equivalent to $\nabla J = 0$ with respect to Levi-Civita connection of g. In addition, it can be shown that locally, the Kähler condition $d\omega = 0$ is equivalent to the condition $\partial_\ell g_{j\bar{k}} = \partial_j g_{\ell\bar{k}}$ and its conjugate equation $\bar{\partial}_{\bar{\ell}} g_{j\bar{k}} = \bar{\partial}_{\bar{k}} g_{j\bar{\ell}}$. Locally, we can always integrate these equations as,

$$g_{j\bar{k}} = \partial_j \bar{\partial}_{\bar{k}} K(z, \bar{z}),$$

for some function $K(z, \bar{z})$, which is known as the **Kähler potential**. It is unique up to Kähler transformation, $K(z, \bar{z}) \to K(z, \bar{z}) + f(z) + f(\bar{z})$ for any holomorphic function $f(z)$. We should note that the Kähler potential cannot be a globally defined smooth function on a compact manifold M.

Since ω is closed, it defines a Dolbeault cohomology class $[\omega] \in H^{1,1}_{\bar{\partial}}(M)$, or a de Rham cohomology class $[\omega] \in H^2_{dR}(M, \mathbb{R})$. Further, the wedge product of n copies of ω, denoted by ω^n, is proportional to the volume form of g. Therefore it defines a non-trivial element in both $H^{n,n}_{\bar{\partial}}(M)$ and $H^{2n}_{dR}(M, \mathbb{R})$ so that $[\omega]$ must be non-trivial. However, $\partial_j \bar{\partial}_{\bar{k}} K(z, \bar{z})$ is exact and therefore zero as a cohomology class. It follows that on a compact Kähler manifold it is impossible to define a Kähler potential globally.

Example 14.12. Complex projective space $\mathbb{CP}^n$ is a Kähler manifold. Consider the function $u(z_0, \cdots, z_m) = \sum_{\mu=0}^{n} |z_\mu|^2$ where z_μ, $\mu = 0, \cdots, n$ are homogeneous coordinates on $\mathbb{C}^{n+1} \backslash \{0\}$. Define a $(1, 1)$-form α by $\alpha = \partial\bar{\partial}(\log u)$. α cannot be the Kähler form of any metric on $\mathbb{C}^{n+1} \backslash \{0\}$, since it is not positive. However, if we consider the projection $\pi : \mathbb{C}^{n+1} \backslash \{0\} \to \mathbb{CP}^n$ defined by $\pi : (z_0, \cdots, z_n) \mapsto [z_0; \cdots; z_n]$, one can show that there exists a unique positive $(1, 1)$-form ω on $\mathbb{CP}^n$ such that $\alpha = \pi^*(\omega)$. ω is a Kähler form on $\mathbb{CP}^n$; its associated Kähler metric is called the **Fubini-Study metric**, and is given in components by $g_{\mu\bar{\nu}} = \partial_j \bar{\partial}_{\bar{k}} \log u$.

$$\omega_{\text{FS}} = i\frac{\sum_j dz_j \wedge d\bar{z}_j - \sum_{k,j}(\bar{z}_j dz_j) \wedge (z_k d\bar{z}_k)}{(|z_0|^2 + \cdots + |z_n|^2)^2} \ .$$

Given the metric in the above form, we can compute the Riemann curvature. Straightforward calculations show that the only non-zero components are $\Gamma^i_{jk} = g^{i\bar{\ell}} \partial_j g_{k\bar{\ell}}$, and its complex conjugate. Thus, the only non-zero components of the curvature tensor are $R^{\ell}{}_{i\bar{j}k} = \partial_{\bar{j}} \Gamma^{\ell}_{ik}$. Furthermore, the Ricci tensor turns out to be

$$R_{j\bar{k}} = R^{\ell}{}_{\ell j\bar{k}} = i\partial_j \bar{\partial}_{\bar{k}} \log \det(g) \ .$$

For a Kähler manifold, the complexified de Rham cohomology decomposes into the Dolbeault cohomology (note that the de Rham cohomology groups are complex, as we are now considering complexified k-forms)

$$H^k_{dR}(M, \mathbb{C}) = \bigoplus_{j=0}^{k} H^{j,k-j}_{\bar{\partial}}(M).$$

This can be shown by using Hodge theorem of a Kähler manifold.

14.4 Calabi–Yau manifolds

We are now ready to study Calabi–Yau manifolds, which are a particular kind of Kähler manifolds.

In 1954, Calabi conjectured that a compact Kähler manifold M with $c_1 = 0$ admits Ricci-flat Kähler metric. Yau proved the conjecture in 1976 by solving the complex Monge-Ampëre equation. Therefore, such a manifold is called a **Calabi–Yau manifold**.

Calabi–Yau manifolds have been studied extensively in the recent decades, particularly because of their importance in string theory [17]. While the mathematical study of Calabi–Yau manifolds has helped us understand compactifications of string theory, the study of string theory has led to fascinating insights in the geometry of Calabi–Yau manifolds, for example the study of the Calabi–Yau moduli space and mirror symmetry. Calabi–Yau manifolds are thus a very good example of the fruitful interactions between mathematics and physics that have been taking place in the recent decades.

Let us first list some of the most common definitions of Calabi–Yau manifolds. A Calabi–Yau manifold of real dimension $2n$ is a compact Kähler manifold (M, J, g):

1. with zero Ricci form,
2. with vanishing first Chern class,
3. with $\text{Hol}(g) = SU(n)$ (or $\text{Hol}(g) \subseteq SU(n)$),
4. with trivial canonical bundle,
5. that admits a globally defined and nowhere vanishing holomorphic n-form.

The Hodge numbers of a Calabi–Yau manifold satisfy a few more properties, which drastically decrease the number of undetermined Hodge numbers. We will now focus on Calabi–Yau threefolds, that is Calabi–Yau manifolds with complex dimension 3, for the sake of brevity. These are the most important Calabi–Yau manifolds in string theory applications. But most results extend straighforwardly to higher dimensional Calabi–Yau manifolds.

The Hodge numbers of Kähler manifolds satisfy a Hodge star duality $h^{p,q} = h^{3-q,3-p}$ and a complex conjugation duality $h^{p,q} = h^{q,p}$. For Calabi–Yau manifolds, there is a further duality, sometimes called **holomorphic duality**. The triviality of the canonical bundle of a Calabi–Yau manifold M implies that $h^{3,0} = 1$, i.e. the existence of a unique holomorphic volume form Ω. Given a $(0,q)$ cohomology class $[\alpha]$, there is a unique $(0, 3-q)$ cohomology class $[\beta]$ such that $\int_M \alpha \wedge \beta \wedge \Omega = 1$ (using Stoke's theorem). Thus $h^{0,q} = h^{0,3-q}$. Therefore, for a Calabi–Yau manifold we have that $h^{3,0} = h^{0,3} = h^{0,0} = h^{3,3} = 1$.

Moreover, one can show that $h^{1,0} = 0$. Thus, $h^{1,0} = h^{0,1} = h^{0,2} = h^{2,0} = h^{2,3} = h^{3,2} = h^{3,1} = h^{1,3} = 0$. Therefore, the only remaining independent Hodge numbers are $h^{1,1}$ and $h^{2,1}$, and the Hodge diamond takes the form:

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{1,1} & & 0 & \\
1 & & h^{2,1} & & h^{2,1} & & 1 \\
 & 0 & & h^{1,1} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

The Euler characteristic of a Calabi–Yau manifold accordingly simplifies. Recall that $\chi = \sum_{k=0}^{2m}(-1)^k b^k$, so we now have that $\chi = 2b^0 - 2b^1 + 2b^2 - b^3 = 2 - 0 + 2h^{1,1} - 2 - 2h^{2,1}$, that is

$$\chi = 2(h^{1,1} - h^{2,1}).$$

Therefore, if the Euler characteristic is easily computed, we only have to compute one of the two independent Hodge numbers to get all the topological information. In fact, the Euler characteristic is given by the integral over M of the top Chern class of M, which is $c_3(M)$ for a Calabi–Yau threefold:

$$\chi = \int_M c_3(M).$$

This formula can be used to compute the Euler characteristic of M.

The Hodge number $h^{1,1}$ classifies infinitesimal deformations of the Kähler structure that, roughly speaking, parametrizes the volume of M. For a Calabi–Yau threefold, $h^{2,1}$ classifies infinitesimal deformations of the complex structure that, roughly speaking, parametrizes the shape of M.

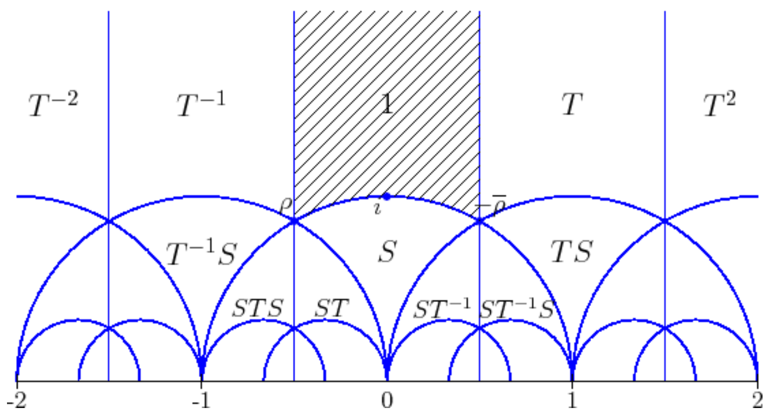
One of the fascinating property of Calabi–Yau threefolds is that they come in mirror pairs, (M, W), such that $H^{2,1}(W) \cong H^{1,1}(M)$ and $H^{1,1}(W) \cong H^{2,1}(M)$. Roughly speaking, the complex structure moduli is exchanged with the Kähler structure moduli. This is the basic idea behind **mirror symmetry**.

14.4.1 Examples

Example 14.13. A torus T^2 is a compact Calabi–Yau manifold one-fold. The Hodge diamond of T^2 is

$$
\begin{array}{ccc}
 & 1 & \\
1 & & 1 \\
 & 1 &
\end{array}
$$

The moduli space of the complex structure is written below.



Example 14.14. T^4 is a compact Calabi–Yau manifold 2-fold. The Hodge diamond of T^4 is

$$\begin{array}{ccccc} & & 1 & & \\ & 2 & & 2 & \\ 1 & & 4 & & 1 \\ & 2 & & 2 & \\ & & 1 & & \end{array}$$

Example 14.15. The other compact Calabi–Yau 2-fold is a K3 surface, which is constructed as follows. Let us consider the quotient space $T^4/\mathbb{Z}_2$. There are 16 singular points and the neighborhood around a singular point is a cone of $\mathbb{RP}^3$. If we consider the set $V = \{p \in TS^2 |\ |v| \le 1\}$ of points with length ≤ 1 in a fiber of TS^2, its boundary is $\partial V \cong \mathbb{RP}^3$. Therefore, we can replace the neighborhood of each singular point by V. Then, the resulting space is smooth and it is a K3 surface. The Hodge diamond of a K3 surface is

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Example 14.16. As mentioned before, a submanifold $\mathbb{CP}^4$ defined by

$$M = \{z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 - 5\psi z_0 z_1 z_2 z_3 z_4 = 0\}$$

is a compact Calabi–Yau 3-fold. Let us define the group

$$G = \{(a_0, \cdots, a_4) \in (\mathbb{Z}_5)^5 | \sum_i a_i = 0\} / (\mathbb{Z}_5 = \{(a, a, a, a, a)\}).$$

Then, the mirror manifold W is constructed by the resolution of singularities of the orbifold M/G where G acts on M by $(z_j) \to (z_j \omega^{a_j})$ with $\omega = e^{2\pi i/5}$. The Hodge diamonds of M and W are given by

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 1 & & 0 & \\
1 & & 101 & & 101 & & 1 \\
 & 0 & & 1 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
\qquad\qquad
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 101 & & 0 & \\
1 & & 1 & & 1 & & 1 \\
 & 0 & & 101 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}$$

Hodge diamonds of M Hodge diamonds of W

where you can see the mirror symmetry.

15 Extra Lecture 2: Symplectic geometry

15.1 Symplectic geometry

A symplectic form on a smooth manifold M is non-degenerate closed 2-form ω. "Non-degenerate" means that the mapping $\omega : TM \to T^*M$; $X \mapsto \omega(X, -)$ is an isomorphism. We denote the 1-form $\omega(X, -)$ by $i(X)\omega$.

The couple (M, ω) of a smooth manifold M and a symplectic form ω is called a **symplectic manifold**. Any symplectic manifold is even dimensional and if $\dim(M) = 2n$, ω^n is a volume-form.

Example 15.1. Let $(q_1, \cdots, q_n, p_1, \cdots, p_n)$ be the standard coordinate of $\mathbb{R}^{2n}$. Then,

$$\omega = \sum_{i=1}^{n} dq_i \wedge dp_i$$

is the symplectic form.

Example 15.2 (cotangent bundle). A symplectic manifold (M, ω) is called **exact** if there exists one form θ such that $\omega = d\theta$ where θ is called **Liouville 1-form**. The vector field Z dual to the Liouville 1-form with respect to ω

$$i_Z\omega = \theta$$

is called **Liouville vector field**. The most important example of exact symplectic manifolds is a cotengent bundle $M = T^*N$ of a smooth manifold N. Given a local coordinate $(q_1, \cdots, q_n)$, they induces the coordinate $(p_1, \cdots, p_n)$ on the fiber T^*_qN. Then, the Liouville 1-form is written in terms of the local coordinate of T^*N

$$\theta = \sum_{i=1}^{n} p_i dq_i$$

so that the symplectic form can be locally written as

$$\omega = \sum_{i=1}^{n} dq_i \wedge dp_i \ .$$

Example 15.3. A Kähler manifold is a symplectic manifold (M, ω) equipped with an integrable almost-complex structure J which is compatible with the symplectic form ω, meaning that the bilinear form

$$g(u, v) = \omega(u, Jv)$$

on the tangent space of M at each point is symmetric and positive definite (and hence a Riemannian metric on M).

Theorem 15.4 (Darboux's theorem). Let (M, ω) be a $2n$-dimensional symplectic manifold, and let p be any point in M. Then there is a coordinate chart $(U, q_1, \cdots, q_n, p_1, \cdots, p_n)$ centered at p such that on U

$$\omega = \sum_{i=1}^{n} dq_i \wedge dp_i \,. \tag{15.1}$$

This theorem states that any symplectic manifold is locally equivalent to an Euclidean space with its standard symplectic structure. As a result, the most important questions in symplectic geometry are the global ones.

15.1.1 Symplectomorphisms

The important maps in symplectic topology are the **symplectomorphisms**. A symplectomorphism between two symplectic manifolds (M, ω_M) and (N, ω_N) is a diffeomorphism $\psi : M \to N$ such that

$$\psi^* \omega_N = \omega_M \,.$$

In fact, this condition is pretty strong. The necessary condition for the existence of symplectomorphisms is $\dim M \leq \dim N$.

Example 15.5 (symplectic group). Let $E = \mathbb{R}^{2n}$ with basis $\{e_1, \cdots, e_n, f_1, \cdots, f_n\}$. Then

$$\omega(e_i, e_j) = 0 \,, \quad \omega(f_i, f_j) = 0 \,, \quad \omega(e_i, f_j) = \delta_{i,j} \,.$$

defines a symplectic structure on E. Then, examples of symplectomorphisms are

$$Sp(E, \omega) = \{g \in GL(2n, \mathbb{R}) \,|g^T J g = J \,, \quad J = \begin{pmatrix} 0 & I_n \\ -I_N & 0 \end{pmatrix}\}$$

which is called the **symplectic group**.

Theorem 15.6 (Moser Stability Theorem). Let (M, ω_t) be a closed manifold with a family of cohomologous symplectic forms. Then there is a family of symplectomorphisms $\psi_t : M \to M$ such that

$$\psi_0 = 1 \,, \psi_t^* \omega_t = \omega_0 \,.$$

Moreover, if $\omega_t(q) = \omega_0(q)$ for all points q on a compact submanifold Q of M, we may assume ψ_t is the identity on Q.

This theorem says that one cannot change the symplectic form in any important way by deforming it, provided that the cohomology class is unchanged.

15.1.2 Lagrangian submanifolds

Definition 15.7 (Lagrangian submanifold). A Lagrangian submanifold of a sympletic manifold (M, ω) is a submanifold where the restriction of the symplectic form ω to $L \subset M$ is vanishing, i.e. $\omega|_L = 0$ and $\dim L = 1/2 \cdot \dim M$.

If we perturb a Lagrangian submanifold, then it is no longer Lagrangian in general. Thus, they are "rigid" objects in symplectic geometry. Since Lagrangian submanifolds play very important role in symplectic geometry, Weinstein advertised slogan that everything is a Lagrangian submanifold. (A. Weinstein's lagrangian creed.)

Example 15.8. The zero section N of the cotangent bundle $M = T^*N$ is a Lagrangian submanifold. Let $f : Z \hookrightarrow N$ is an embedding. Then, the conormal bundle to Z in T^*N defined as

$$L_Z := \{(x,\alpha) \in T^*N | x \in Z\,, \quad \alpha(v) = 0 \text{ for all } v \in T_xZ\} \subset T^*N\,,$$

is a Lagrangian submanifold.

It is known that the neighborhood of a Lagrangian also has the standard symplectic structure as its cotangent bundle.

Theorem 15.9 (Weinstein's tubular neighbourhood theorem). Every lagrangian submanifold L in a symplectic manifold (M, ω) has a neighbourhood U which is symplectomorphic to a neighbourhood V of the zero section of the cotangent bundle T^*L.

15.1.3 Hamiltonian system

Any smooth function $H \in C^\infty(M)$ gives rise to a vector field X_H defined uniquely by the equation

$$i(X_H)\omega = dH\,.$$

This vector field is called the **Hamiltonian vector field** with Hamiltonian H. Given a symplectic form (15.1), the flow of the Hamiltonian vector field X_H associated to a Hamiltonian H can be described by

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}\,, \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}\,,$$

which are Hamilton's canonical equations.

For $f, g \in C^\infty(M)$, we define the Poisson product by

$$\{f, g\} = \omega(X_f, X_g) = X_f(g) = -X_g(f)\,.$$

Given the symplectic form (15.1), the Poisson product can be written as the local coordinate

$$\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} \right)\,.$$

They satisfy the following properties

Skew symmetry: $\{f, g\} = -\{g, f\}$.

Jacobi identity: $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$.

Leibniz's Rule: $\{fg, h\} = f\{g, h\} + g\{f, h\}$.

Using these properties, one can show that

$$[X_f, X_g] = X_{\{f,g\}}\,.$$

For $f \in C^\infty(M)$, the differentiation with respect to the Hamiltonian flow can be expressed by

$$\frac{df}{dt} = -\{H, f\}\,.$$

If $\{f, H\} = 0$, the flow generated by X_f is a symmetry of the Hamiltonian system with H. Namely, if we denote the flow by φ_s, then we have

$$\frac{dH}{ds} = \{f, H\} = 0 \,,$$

or equivalently,

$$\frac{df}{dt} = -\{H, f\} = 0 \,.$$

This is called **Noether theorem.**

Example 15.10. If they system has spherical symmetry, the potential $V(r)$ is independent of θ and ϕ. Then, the momenta p_θ and p_ϕ are conserved.

15.1.4 Arnold-Liouville theorem

Let us consider Poisson commuting functions (Hamiltonians) $H_1, \cdots, H_k$ with $\{H_i, H_j\} = 0$ for all i, j. To insure that we are not discussing a degenerate situation, we assume that $dH_1 \wedge \cdots \wedge dH_k(x) \neq 0$ for $\forall x \in M$, in which H_i are called **analytically independent**. Then, the maximal number of Poisson commuting functions is a half of dimension of M which is n. An $2n$-dimensional symplectic manifold (M, ω) with n Poisson commuting Hamiltonians is called **completely integrable system**.

Theorem 15.11 (Arnold-Liouville theorem). Let $H_1, \cdots, H_n$ be a completely integrable system on a $2n$-dimensional symplectic manifold (M, ω). Namely, $\pi = (H_1, \cdots, H_n) : M \to \mathbb{R}^n$ are analytically independent Poisson commuting functions with $\{H_i, H_j\} = 0$ for all i, j. Then, a compact connected component of the preimage of a non-singular point of π is a Lagrangian submanifold that is diffeomorphic to a torus T^n.

Let us denote the image of π by B. Then, (compact connected component of) the preimage $L = \pi^{-1}(b)$ of $b \in B$ is diffeomorphic to T^n so that we can take the angle coordinate

$$(\phi_1 \cdots \phi_n) : L \to T^n$$

We can further take local coordinate $(I_1, \cdots, I_n)$ of B that the symplectic form can be locally expressed as

$$\omega = \sum_{i=1}^{n} d\phi_i \wedge dI_i \,,$$

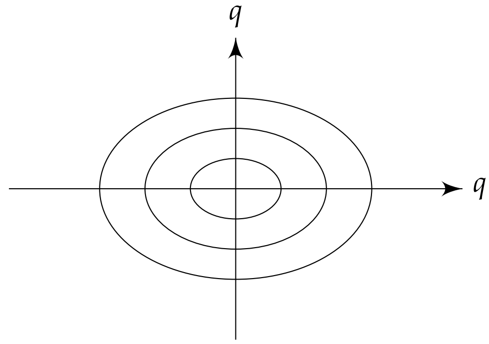
where (ϕ_i, I_i) are called **angle-action** coordinate.

Example 15.12. Consider the harmonic oscillator with Hamiltonian

$$H(q, p) = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2.$$

Since is a 2-dimensional system, so we only need a single first integral. Since H is a first integral for trivial reasons, this is an integrable Hamiltonian system.

We can actually draw the lines on which H is constant — they are just ellipses:



We note that the ellipses are each homeomorphic to S^1. Now we introduce the coordinate transformation $(q, p) \mapsto (\phi, I)$, defined by

$$q = \sqrt{\frac{2I}{\omega}} \sin \phi, \quad p = \sqrt{2I\omega} \cos \phi,$$

Hence, ϕ is the angle coordinate that parametrizes S^1 in the Arnold-Liouville theorem.

We can manually show that this transformation is canonical, but it is merely a computation and we will not waste time doing that. In these new coordinates, the Hamiltonian looks like

$$\tilde{H}(\phi, I) = H(q(\phi, I), p(\phi, I)) = \omega I.$$

The Hamiltonian is independent of ϕ! Therefore, the Hamilton equations become

$$\dot{\phi} = \frac{\partial \tilde{H}}{\partial I} = \omega, \quad \dot{I} = -\frac{\partial \tilde{H}}{\partial \phi} = 0.$$

We can integrate up to obtain

$$\phi(t) = \phi_0 + \omega t, \quad I(t) = I_0.$$

It is interesting to consider the integral along paths of constant H:

$$\begin{aligned}
\frac{1}{2\pi} \oint p \, \mathrm{d}q &= \frac{1}{2\pi} \int_0^{2\pi} p(\phi, I) \left(\frac{\partial q}{\partial \phi} \, \mathrm{d}\phi + \frac{\partial q}{\partial I} \, \mathrm{d}I \right) \\
&= \frac{1}{2\pi} \int_0^{2\pi} p(\phi, I) \left(\frac{\partial q}{\partial \phi} \, \mathrm{d}\phi \right) \\
&= \frac{1}{2\pi} \int_0^{2\pi} \sqrt{\frac{2I}{\omega}} \sqrt{2I\omega} \cos^2 \phi \, \mathrm{d}\phi \\
&= I \,.
\end{aligned}$$

We could always have performed the integral $\frac{1}{2\pi} \oint p \, \mathrm{d}q$ along paths of constant H without knowing anything about I and ϕ, and this would have magically gave us the new coordinate I. That's why it is called **integrable system**.

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