Knot homology, Volume conjectures, Physics

§1. Chern-Simons theory & polynomial inv.

§2 Categorifications: knot homology

§3. Volume conjectures & generalizations

§4. Interpretation in string theory and gauge theory

based on work with

Ramadevi, Zodinmawia, Gukov, Stosic, Sulkowski.

Notes for talks at Mcgill University

at Steklov Mathematical Institute

at KdV Institute.

§1. Chern–Simons theory & polynomial inv.

Chern–Simons theory   Witten

M 3-mfd

G semi-simple compact gauge group

A $\mathfrak{g}$-valued gauge connection

The action of Chern-Simons theory

$$S_{CS}(M, A) = \frac{k}{4\pi} \int_M \mathrm{Tr}\left(A dA + \frac{2}{3} A^3 \right)$$

- S_{CS} is metric indep.
- $k \in \mathbb{Z}$ level (inverse of coupling const.)
- eqn of motion : $F_A = 0$ flat connection
- natural metric indep. observable

$$W_R(K) = \mathrm{Tr}_R \, P \exp \oint_K A$$

K a knot

R rep of $\mathfrak{g}$

- Expectation value of $W_R(K)$

$$\langle W_R(K) \rangle = \int_{\mathcal{A}/\mathcal{G}} \mathcal{D}A \; e^{i S_{CS}(M.A)} \, W_R(K)$$

$\mathcal{A}$: space of connections

$\mathcal{G}$: gauge transformations

$\Rightarrow$ Knot invariant.

Suppose M has boundary $\partial M = \Sigma$

$$g^* A = g^{-1} d g + g^{-1} A g$$

$$S_{cs}(g^* A) - S_{cs}(A)$$
$$= \frac{k}{4\pi} \int_{\partial M} \mathrm{Tr}(A \wedge g^{-1} d g) - \frac{k}{12\pi} \int_M \mathrm{Tr}(g^{-1} d g)^3$$

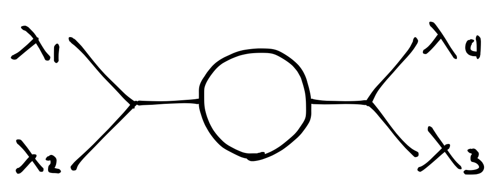
- action of $\hat{\mathfrak{g}}_k$ WZNW model

$$S_\Sigma(g) = \frac{k}{4\pi} \int_\Sigma \mathrm{Tr}(g^{-1} d g \wedge g^{-1} d g) - \frac{k}{12\pi} \int_M \mathrm{Tr}(g^{-1} d g)^3$$

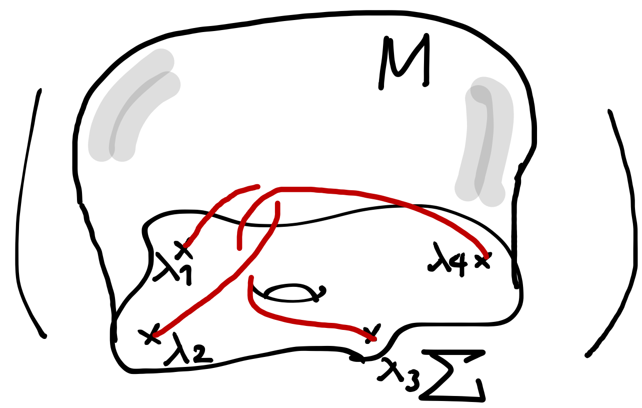
The action is indep of the extension M of Σ

- Chern-Simons partition fn. is an element of Hilbert space living on the bdry Σ

$$\mathcal{H}_\Sigma \ni Z^k_{CS}(M) = \int_{\mathcal{A}/\mathcal{G}} \mathcal{D}A \, e^{i S_{cs}(M, A)}$$
$$\shortparallel$$
$$\{\text{space of conformal blocks}\}.$$



$= Z^k_{CS}\Big($



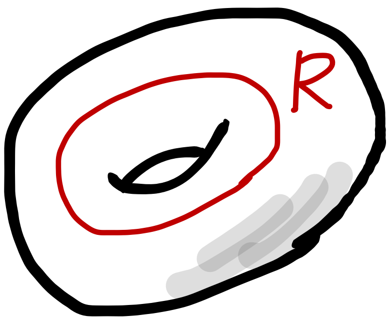
$\Big)$

In particular, $\Sigma = T^2$, $M = S^1 \times D$ (solid Torus)

with Wilson loop colored by R.

$$Z_{CS}(S^1 \times D; R) = \chi_R(\tau) = \mathrm{Tr}_R\, q^{L_0 - \frac{c}{24}}$$

character of integrable rep of $\hat{g}_k$



e.g. $g = sl(2)$ $\quad$ $R = \boxed{\,\square \cdots \square\,}$ less than k

$$\chi_{R_1}(-\tfrac{1}{\tau}) = \sum_{R_2} S_{R_1 R_2}\, \chi_{R_2}(\tau)$$

$$\chi_{R_2}(\tau+1) = \sum_{R_2} T_{R_1 R_2}\, \chi_{R_2}(\tau) = \exp\left[2\pi i\left(\Delta_R - \tfrac{c}{24}\right)\right] \chi_R(\tau)$$

diagonalizable

$sl(2)$: $\quad S_{m,n} = \sqrt{\dfrac{2}{k+2}}\, \sin \dfrac{(m+1)(n+1)\pi}{k+2}$

$$\Delta_n = \frac{n(n+2)}{4(k+2)}$$

$sl(N)$: $\quad S_{\lambda_1 \lambda_2} = s_{\lambda_1}(q^{\rho})\, s_{\lambda_2}(q^{\lambda_1 + \rho})$

$s(x)$: Schur fn

Aganagic - Shakirov : Refined CS thy.

$$S_{\lambda_1 \lambda_2} = M_{\lambda_1}(q^{\rho})\, M_{\lambda_2}(q^{\rho} t^{\lambda_1})$$

Macdonald poly

Partition fn & Expectation value.

1. $S^3 = (S^1 \times D^2) \cup (S^1 \times D^2)$

$$Z_{cs}(S^3) = \langle 0|S|0\rangle = S_{0,0} \left(= \sin\frac{\pi}{k+2}\right)$$

2. unknot

$$\langle W_R(\bigcirc)\rangle = \frac{Z_{cs}(S^3;\bigcirc)}{Z_{cs}(S^3)} = \frac{\langle 0|S|R\rangle}{\langle 0|S|0\rangle} = \frac{S_{0R}}{S_{00}}$$

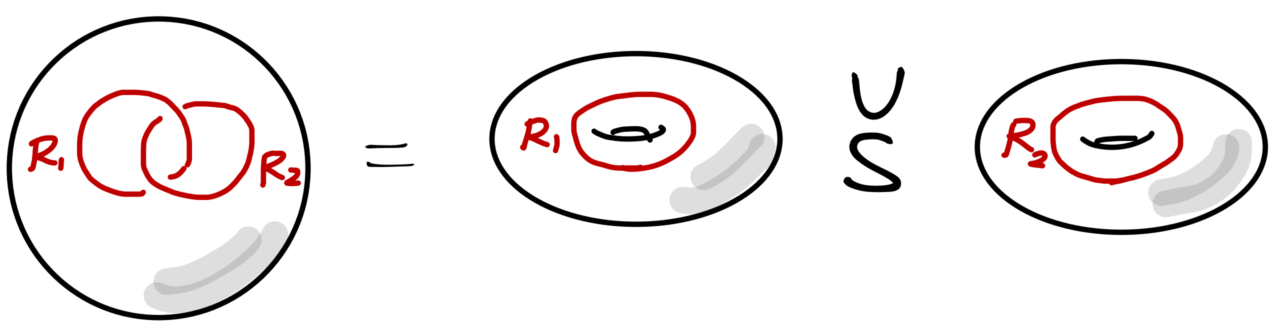
$$= \dim_q R.$$

For $sl(2)$ with $R = \underbrace{\boxed{}\cdots\boxed{}}_{n}$

$$\langle W_n(\bigcirc)\rangle = \frac{\sin\frac{(n+1)\pi}{k+2}}{\sin\frac{\pi}{k+2}} = \frac{q^{n+1} - q^{-(n+1)}}{q - q^{-1}} = [n+1]$$

where $q = \exp\left(\frac{\pi i}{k+2}\right)$

3. Hopf link



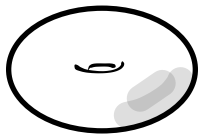
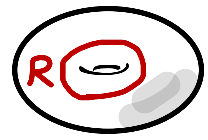
$$\langle W_{R_1R_2}(\text{Hopf})\rangle = \frac{\langle R_1|S|R_2\rangle}{\langle 0|S|0\rangle} = \frac{S_{R_1R_2}}{S_{00}}$$

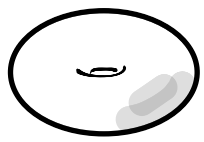
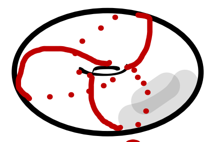
For $sl(2)$. $R_1 = \text{spin}\,\frac{m}{2}$, $R_2 = \text{spin}\,\frac{n}{2}$

$$\langle W_{mn}(\text{Hopf})\rangle = \frac{q^{(m+1)(n+1)} - q^{-(m+1)(n+1)}}{q - q^{-1}} = [(m+1)(n+1)]$$

4. Torus Knots

torus knot operators

$W^{(1,0)}_R$ |  $\rangle$ = |  $\rangle$

$W^{(2,3)}_R$ |  $\rangle$ = |  $\rangle$

R

For general (Q,P)-torus knots.

$$W^{(Q,0)}_R = \sum_V C^V_R(Q)\, W^{(1,0)}_V \qquad \text{adams operation}$$

$$W^{(Q,P)} = T^{P/Q}\, W^{(Q,0)}_R\, T^{-P/Q}$$

$$T^{P/Q} = \begin{pmatrix} 1 & P/Q \\ 0 & 1 \end{pmatrix}$$

Wilson loop along (Q,P)-torus knot

$$\langle W_R(K(Q,P))\rangle = \frac{\langle 0|S W^{(Q,P)}_R|0\rangle}{\langle 0|S|0\rangle}$$

$$= (S_{00})^{-1}\langle 0|S\, T^{P/Q}\sum_V C^V_R(Q)\, W^{(1,0)}_V\, T^{-P/Q}|0\rangle$$

$$= \sum_V C^V_R(Q)\, q^{\frac{P}{Q}\kappa_V}\, \dim_q{}_V R$$

Rosso-Jones formula.

5. colored quantum invariants for Non-Torus knots are very difficult.

$$\langle W_R(K) \rangle = \bar{J}^{\mathfrak{g}}_R(K;q) \qquad q = \exp \frac{\pi i}{k+h^\vee}$$

$h^\vee$ dual coxeter #

$$\bar{J}^{sl(N)}_R(K;q) \xrightarrow{a=q^N} \bar{P}_R(K;a,q) \text{ colored HOMFLY}$$

$$\bar{J}^{so(N)}_R(K;q) \xrightarrow{\lambda=q^{N-1}}$$
$$\bar{F}_R(K;\lambda,q) \text{ colored Kauffman}$$
$$\bar{J}^{sp(N)}_R(K;q) \xrightarrow{\lambda=q^{N+1}}$$

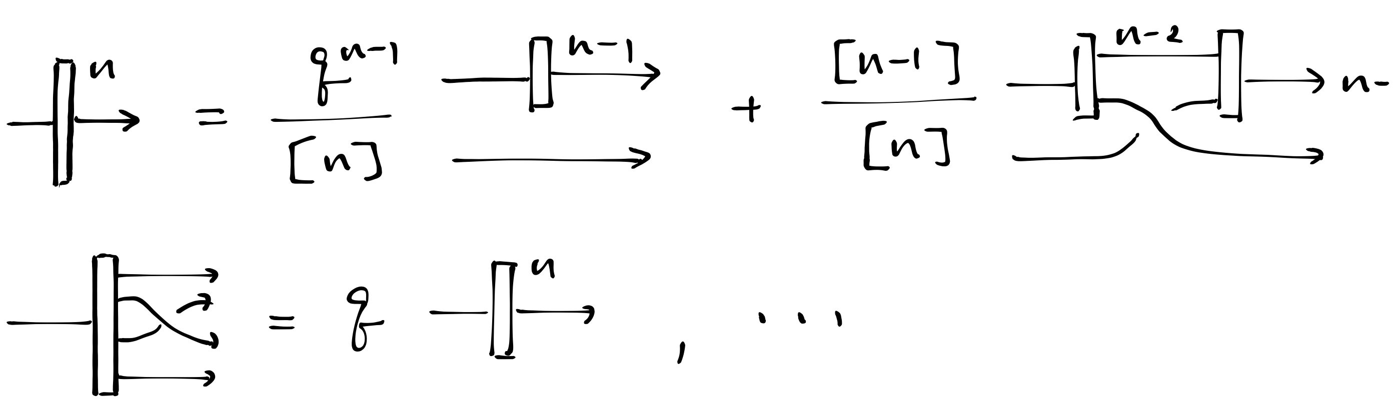
Computational methods

① Reshetikhin-Turaev [ITEP, Itoyama]

$$f : B_n \longrightarrow \mathrm{End}_{U_q(\mathfrak{g})}(V_1 \otimes \cdots \otimes V_m)$$

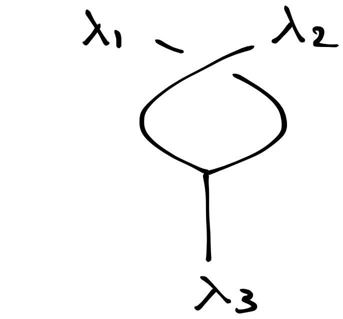
$$\mathrm{Tr}(f) \sim \bar{J}^{\mathfrak{g}}_R(K)$$

② linear Skein relations (Morton, Kawagoe '12)

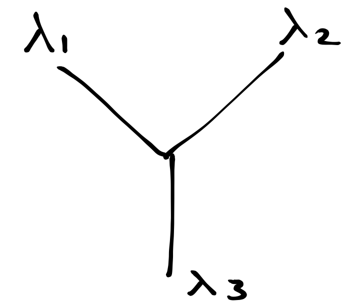


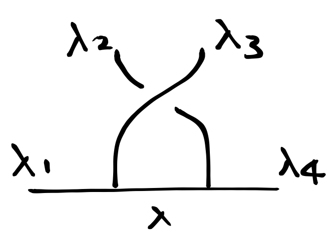
③ Drinfeld - Kohno

Monodromy rep. of Braid group in the space of the KZ eqn is given by Universal $\mathcal{R}$-matrices.

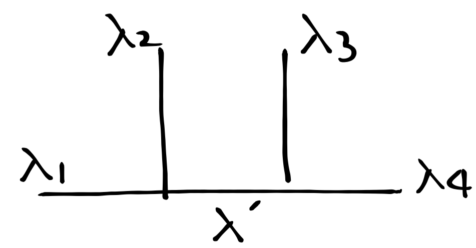


$$= \epsilon\, q^{\kappa_{\lambda_1} - \kappa_{\lambda_2} - \kappa_{\lambda_3}}$$

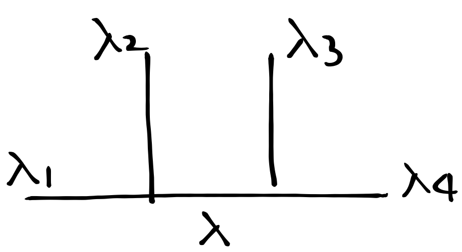




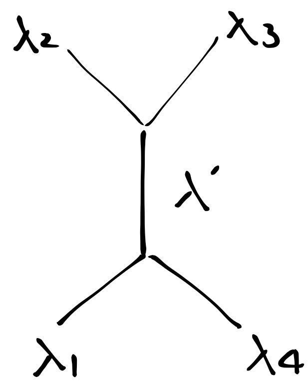
$$= \sum_{\lambda'} B_{\lambda\lambda'}\begin{bmatrix}\lambda_1 & \lambda_2\\ \lambda_3 & \lambda_4\end{bmatrix}$$



Braiding matrix



$$= \sum_{\lambda'} F_{\lambda\lambda'}\begin{bmatrix}\lambda_1 & \lambda_2\\ \lambda_3 & \lambda_4\end{bmatrix}$$



fusion matrix

fusion matrizes are often called quantum 6j-symbols

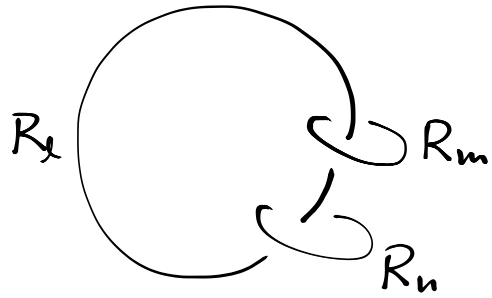
$$F_{\lambda\lambda'}\begin{bmatrix}\lambda_1 & \lambda_2\\ \lambda_3 & \lambda_4\end{bmatrix} \propto \begin{Bmatrix}\lambda_1 & \lambda_2 & \lambda\\ \lambda_3 & \lambda_4 & \lambda'\end{Bmatrix}$$

$U_q(sl_2)$: Kirillov - Reshetikhin '88

$U_q(sl_N)$: the simplest class of multiplicity-free q-6j

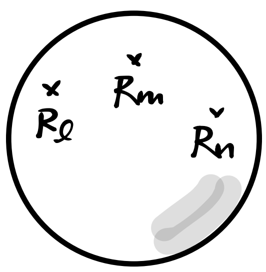
Nawata - Ramadevi - Zodinmawia

Verlinde formula for SU(2).



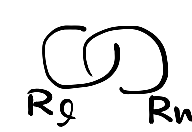
$= L$

$$Z_{CS}(S^3; L) = \sum_k S_\ell^{\,k}\, Z(S^1\times S^2; R_\ell R_m R_n)$$
$$= \sum_k S_\ell^{\,k}\, N_{kmn}$$



$$N_{kmn} := \dim \mathcal{H}_{S^2; R_\ell R_m R_n}.$$

$Z(S^3; R_\ell)\, Z(S^3; L) = Z(S^3;$ 

$)\, Z(S^3;$ 

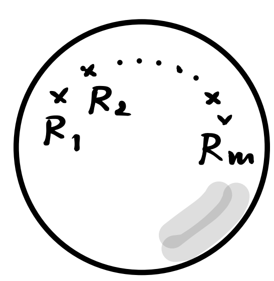
$)$

$$S_{0\ell} \sum_k S_\ell^{\,k} N_{kmn} = S_{\ell m} S_{\ell n}$$

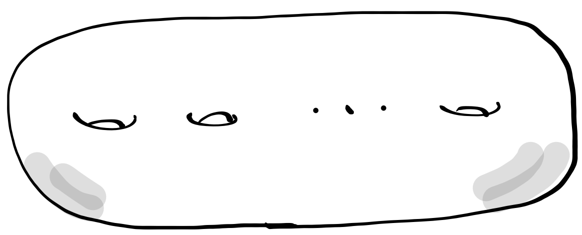
$$\Rightarrow \quad N_{kmn} = \sum_\ell \frac{S_{k\ell} S_{m\ell} S_{n\ell}}{S_{0\ell}}$$

more generally, S^2 with m puncture associated to $(R_1 \cdots R_m)$

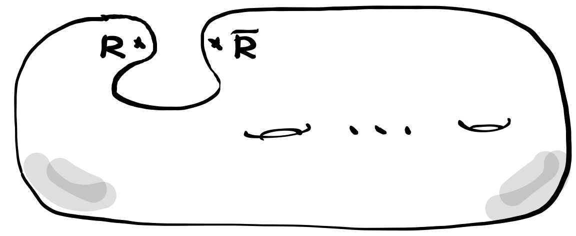
$$\dim \mathcal{H}_{S^2, R_1\cdots R_m} = \sum_\ell \frac{S_{R_1,\ell}\cdots S_{R_m,\ell}}{(S_{0,\ell})^{m-2}}$$



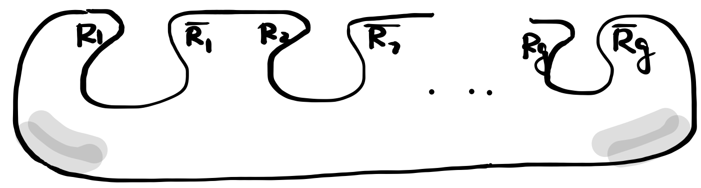
For higher genus Σ_g, one can pinch off a cycle by assigning $(R, \bar{R})$



$$= \sum_R$$



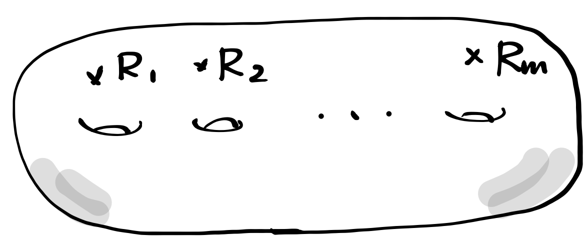
$$= \sum_{R_1 \cdots R_g}$$



$$\dim \mathcal{H}_{\Sigma_g} = \sum_{R_1 \cdots R_g} \sum_{\ell} \frac{S_{R_1 \ell} S_{\bar{R}_1 \ell} \cdots S_{R_g \ell} S_{\bar{R}_g \ell}}{(S_{0\ell})^{2g-2}}$$

$$= \sum_{\ell} \frac{1}{(S_{0\ell})^{2g-2}}$$

general case : Σ_g with $(R_1 \cdots R_m)$



$$\dim \mathcal{H}_{\Sigma_g, R_1 \cdots R_m} = \sum_{\ell} \frac{S_{R_1 \ell} \cdots S_{R_m \ell}}{(S_{0\ell})^{m+2g-2}}$$

Note that this is the case of $SU(2)$. For higher ranks, it's little complicated.

§2 Categorifications : knot homology

$K \rightsquigarrow \mathcal{H}^{\mathfrak{sl}(2)}_{ij}(K)$ bi-graded homology (Khovanov)

this is itself knot invariant.

Jones polynomial is q-graded Euler characteristics.

$$J(K;q) = \sum_{i,j} (-1)^j q^i \dim \mathcal{H}^{\mathfrak{sl}(2)}_{ij}(K)$$

Since homology itself is knot invariant, the Poincare poly is certainly invariant.

$$\mathcal{P}^{\mathfrak{sl}(2)}_{\square}(K;q,t) = \sum_{i,j} q^i t^j \dim \mathcal{H}^{\mathfrak{sl}(2)}_{ij}(K)$$

$\mathfrak{sl}(2)$ homological inv. (Khovanov inv.)

- Status of Categorifications

1. colored $\mathfrak{sl}(2)$ homology $\left(\mathcal{H}^{\mathfrak{sl}(2)}_{R}(K)\right)_{ij}$ — Cooper - Krushkal, Frenkel - Stroppel - Sussan, Webster.

$$J^{\mathfrak{sl}(2)}_{R}(K;q) = \sum_{ij} (-1)^j q^i \dim\left(\mathcal{H}^{\mathfrak{sl}(2)}_{R}(K)\right)_{ij}$$

2. $\mathfrak{sl}(N)$ homology $\left(\mathcal{H}^{\mathfrak{sl}(N)}_{\square}(K)\right)_{ij}$ — Khovanov - Rozansky

$$J^{\mathfrak{sl}(N)}(K;q) = \sum_{ij} (-1)^j q^i \dim\left(\mathcal{H}^{\mathfrak{sl}(N)}(K)\right)_{ij}$$

3. HOMFLY homology

Rasmussen.

Khovanov - Rozansky

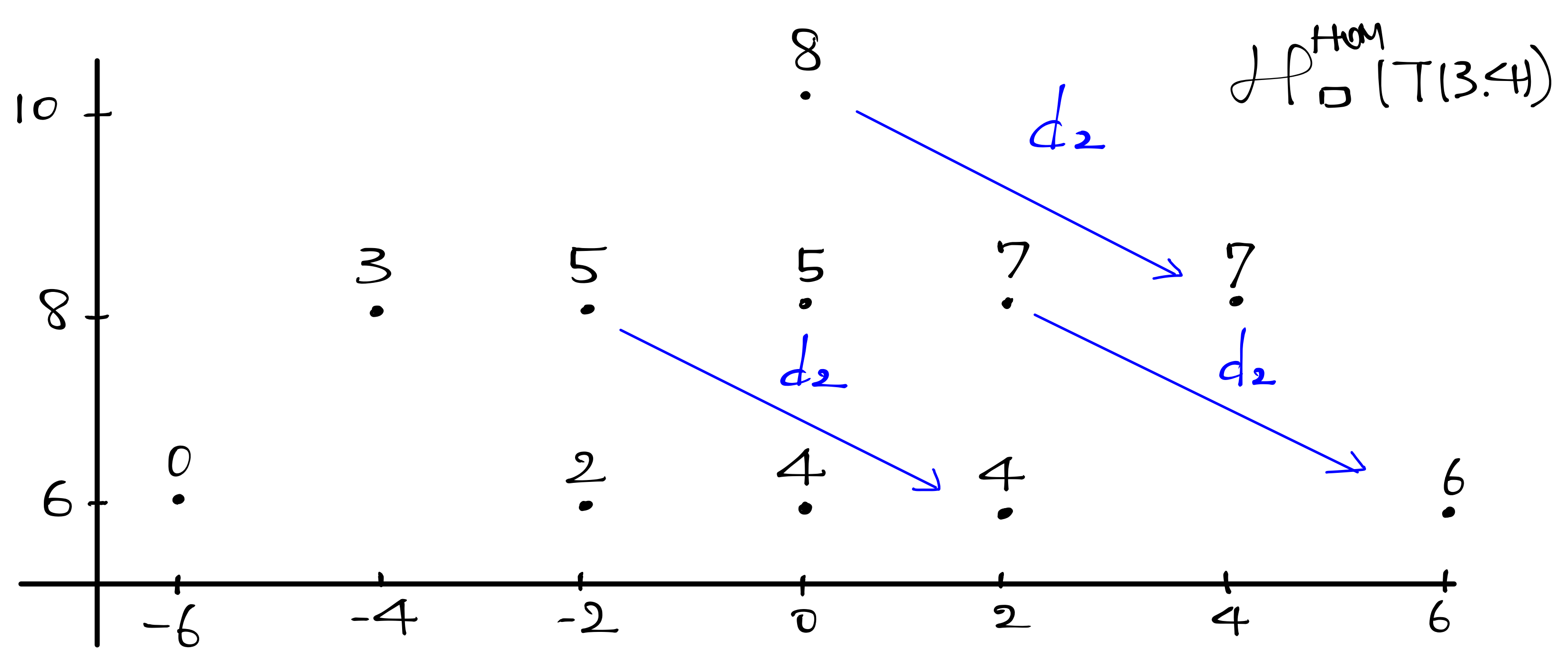
$(\mathcal{H}^{HOM}(K))_{ijk}$ triply-graded homology

Poincare poly (superpolynomial)

$$\mathcal{P}(K;a,q,t) = \sum a^i q^j t^k \dim (\mathcal{H}^{HOM}(K))_{ijk}$$

$$\begin{tikzcd}[column sep=small]
 & \mathcal{P}(K;a,q,t) \arrow[dl, "t=-1"'] \arrow[dr, dashed, "d_N \text{ differential}"] & \\
\begin{array}{c}\text{HOMFLY poly}\\ P(K;a,q)\end{array} \arrow[dr, "a=q^N"'] & & \begin{array}{c}sl(N)\text{ homological inv.}\\ \mathcal{P}^{sl(N)}(K;q,t)\end{array} \arrow[dl, "t=-1"] \\
 & \begin{array}{c}sl(N)\text{ quantum inv.}\\ J^{sl(N)}(K;q)\end{array} &
\end{tikzcd}$$

$$\mathcal{P}^{sl(N)}(K;q,t) = \mathcal{P}(\mathcal{H}^{HOM}(K), d_N)(a=q^N, q, t)$$



Properties of colored HOMFLY homology.

Gukov - Stošić.

1) $\mathcal{H}^{\mathrm{HOM}}_{[r]}(K)$ is finite-dim'l

2) $sl(N)$ differential d_N

$$H^*(\mathcal{H}^{\mathrm{HOM}}_{[r]}(K), d_N) \cong \mathcal{H}^{sl(N)}_{[r]}$$

$(a.q.t)$-degree of d_N

$$\deg d_N = (-2, 2N, *)$$

3) Mirror symmetry

$$\left(\mathcal{H}^{\mathrm{HOM}}_{[1^r]}(K)\right)_{i,j,k} \cong \left(\mathcal{H}^{\mathrm{HOM}}_{[r]}(K)\right)_{i,-j,*}$$

$$* = k - (rS(K) + 2(k-i))$$

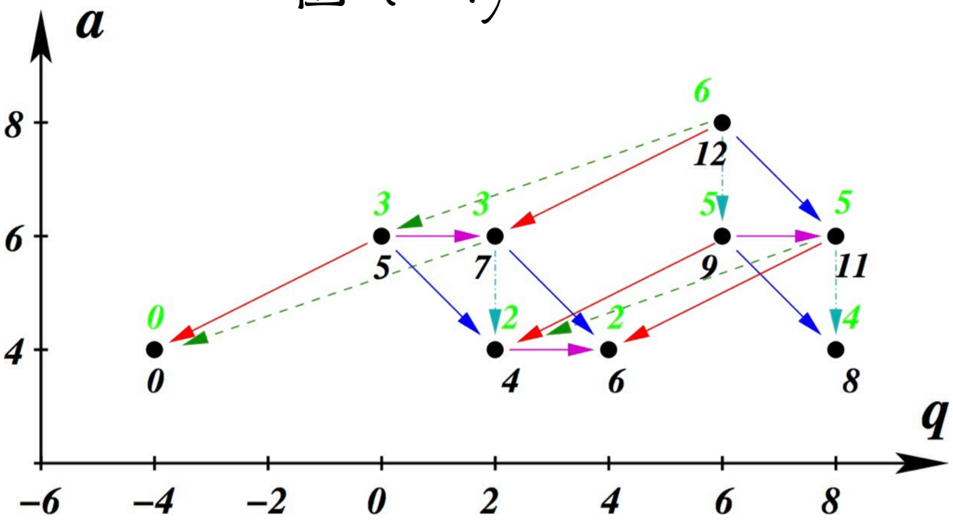
4) Exponential growth properties   K : thin knot or torus knot

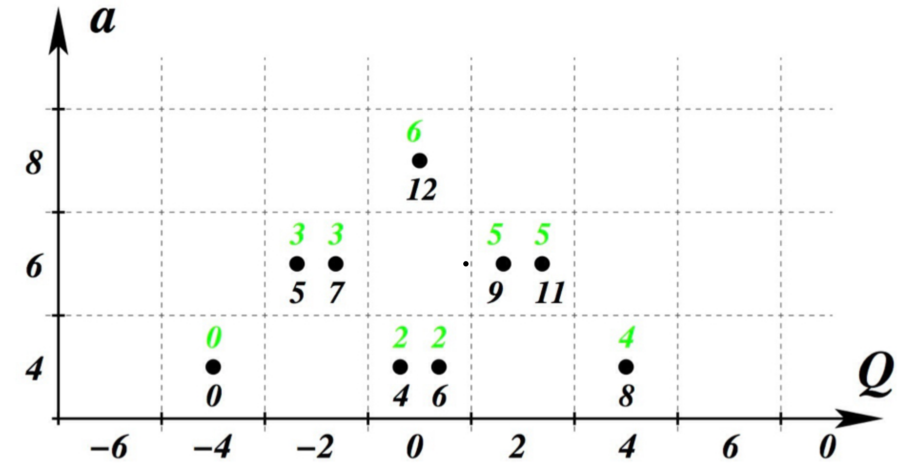
$$\mathcal{P}_{[r]}(K; a.q=1.t) = \left[\mathcal{P}_{\square}(K; a.q=1, t)\right]^r$$

at poly'l level

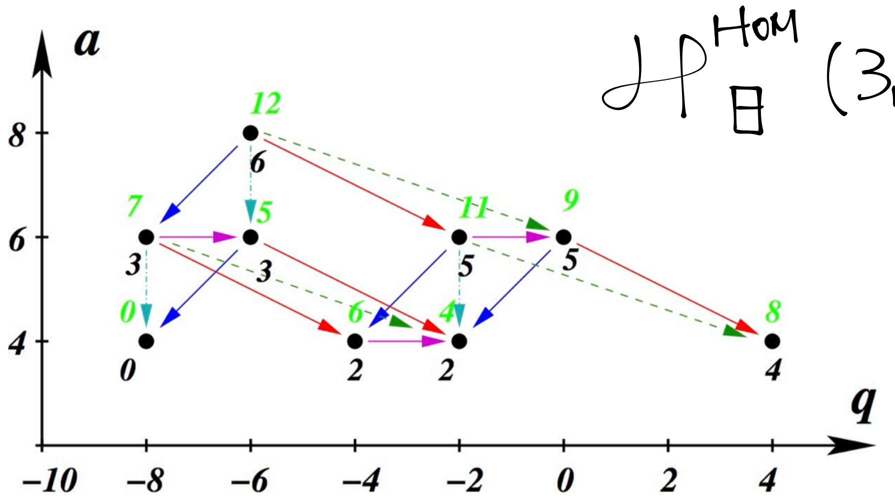
$$P_R(K; a.q=1) = \left[P_{\square}(K; a, q=1)\right]^{|R|}$$

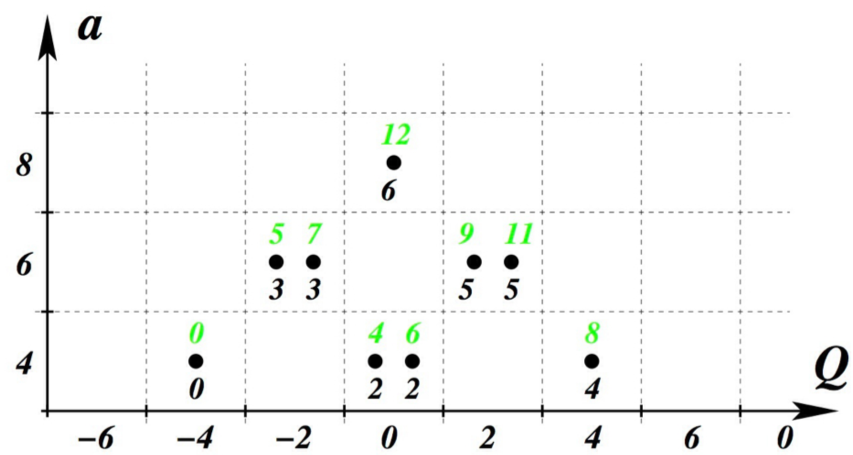
$\mathcal{H}^{\mathrm{HOM}}_{\square}(3_1)$





$\mathcal{H}^{\mathrm{HOM}}_{\boxminus}(3_1)$



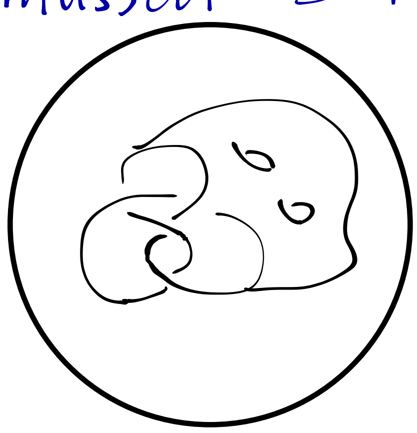


One can introduce 2 homological gradings: T_r- and T_c-gradings

For every element $x \in \mathcal{H}^{\mathrm{HOM}}_{[r]}(K_{\mathrm{thin}})$, one can assign δ-grading

$$\delta(x) = a(x) + \frac{q(x)}{2} - \frac{T_r(x)+T_c(x)}{2} = r\,S(K_{\mathrm{thin}})$$

$S(K)$: Rasmussen S-invariant.



$g_s(K) \geq \frac{1}{2}|S(K)|$.

Slice (Murasugi) genus.

We introduce an auxiliary grading Q. Gorsky-Gukov-Stošić.

$$(a.Q.t_r.t_c)\text{-grading} \longleftarrow (a.q.t_r,t_c)\text{-grading}$$

$$Q = \frac{q+t_r-t_c}{\rho}$$

We define tilde-version of HOMFLY homology with $(a.Q.t_r.t_c)$

$$\left(\widetilde{\mathcal{H}}^{\mathrm{HOM}}_{[r^\rho]}(K)\right)_{i.j.k.\ell} := \left(\mathcal{H}^{\mathrm{HOM}}_{[r^\rho]}(K)\right)_{i,\rho j-k+\ell,k,\ell}$$

$$\widetilde{\mathcal{P}}_{[r^\rho]}(K;a.Q.t_r.t_c) = \sum_{i.j.k.\ell} a^i Q^j t_r^k t_c^\ell \dim \widetilde{\mathcal{H}}^{\mathrm{HOM}}_{[r^\rho]}(K).$$

Then the structural properties of HOMFLY homology becomes transparent.

1). Self-symmetry.

$$\left(\mathcal{H}^{\mathrm{HOM}}_{[r^\rho]}(K)\right)_{i.j.k.\ell} \cong \left(\widetilde{\mathcal{H}}^{\mathrm{HOM}}_{[r^\rho]}(K)\right)_{i.-j.k-\rho j.\ell-rj}$$

2). Mirror Symmetry

$$\left(\widetilde{\mathcal{H}}^{\mathrm{HOM}}_{[\rho^r]}(K)\right)_{i.j.k.\ell} \cong \left(\widetilde{\mathcal{H}}^{\mathrm{HOM}}_{[r^\rho]}(K)\right)_{i.j.\ell.k}$$

$$\cong \left(\widetilde{\mathcal{H}}^{\mathrm{HOM}}_{[r^\rho]}(K)\right)_{i.}$$

3) Refined exponential growth property.

$$\widetilde{\mathcal{P}}_{[r^\rho]}(K; a, Q, T_r, T_c = 1) = \left[\widetilde{\mathcal{P}}_{[1^\rho]}(K; a, Q, T_r, T_c = 1)\right]^r$$

$$\widetilde{\mathcal{P}}_{[r^\rho]}(K; a, Q, T_r = 1, T_c) = \left[\widetilde{\mathcal{P}}_{[1^\rho]}(K; a, Q, T_r = 1, T_c)\right]^\rho$$

thin or Torus knot.

$$\dim \widetilde{\mathcal{H}}^{HOM}_{[r^\rho]}(K) = \left[\dim \mathcal{H}^{HOM}_{\square}(K)\right]^{r\rho}$$

4) Colored differentials $d^{\pm}_{[r^\rho]\to[\ell^\rho]}$ & $d^{\pm}_{[r^\rho]\to[r^\sigma]}$

$$H^*\left(\widetilde{\mathcal{H}}^{HOM}_{[r^\rho]}, d^{\pm}_{[r^\rho]\to[\ell^\rho]}\right) \cong \widetilde{\mathcal{H}}^{HOM}_{[\ell^\rho]}(K)$$

$$H^*\left(\widetilde{\mathcal{H}}^{HOM}_{[r^\rho]}, d^{\pm}_{[r^\rho]\to[r^\sigma]}\right) \cong \widetilde{\mathcal{H}}^{HOM}_{[r^\sigma]}(K)$$

5) Universal colored differentials.

– Power of refined exp. growth properties.

$$\mathcal{P}_{[r]}(3_1; a, Q, T_r, T_c = 1) = \left[a^2 Q^{-2}\left(1 + Q^4 T_r^2 (1 + a^2 Q^{-2} T_r\right)\right]^r$$

$$= a^{2r} Q^{-2r} \sum_{k=0}^{r} Q^{4k} T_r^{2k} \binom{r}{k} \left(1 + a^2 Q^{-2} T_r\right)^k$$

One has to restore T_c-grading carefully.

$$\mathcal{P}_{[r]}(3_1; a, Q, T_r, T_c) = a^{2r} Q^{-2r} \sum_{k=0}^{r} Q^{4k} T_r^{2k} T_c^{2rk} \begin{bmatrix} r \\ k \end{bmatrix}_{T_c^2} \left(-a^2 Q^{-2} T_r T_c; T_c^2\right)_k$$

Status of colored HOMFLY homology

- [r]-colored homology for $T(2,2p+1)$, twist knots, $T(3,4)$, $T(3,5)$
- [r,r]-colored homology for 3_1 & 4_1.

Fuji, Gukov, Sulkowski, Stosic,

Nawata, Ramadevi, Zodinmawia.

Remark on Kauffman homology.

Colored Kauffman homology $\widetilde{\mathcal{H}}^{\text{Kauff}}_{[r]}(K)$ holds similar properties.

1) Mirror-Symmetry
2) Refined Exponential growth property for thin knots and torus knots
3) Colored differentials, d_N differentials.

The most interesting fact is that [r]-colored Kauffman homology contains [r]-colored HOMFLY homology.

More precisely, there exist differentials such that

$$H^*(\widetilde{\mathcal{H}}^{\text{Kauff}}_{[r]}, d^{\text{univ}}_{\rightarrow}) \cong \widetilde{\mathcal{H}}^{\text{HOM}}_{[r]}(K)$$

$$H^*(\widetilde{\mathcal{H}}^{\text{Kauff}}_{[r]}, d^{\pm}_{\text{diag}}) \cong \widetilde{\mathcal{H}}^{\text{HOM}}_{[r]}(K)$$

§3. Volume conjectures & generalizations

<u>Volume conjecture</u> Kashaev, Murakami[2]

$$2\pi \lim_{r\to\infty} \frac{1}{r} \log \left| J_{[r]}(K; q = e^{\frac{2\pi i}{r}}) \right| = \mathrm{vol}(S^3 \setminus K)$$

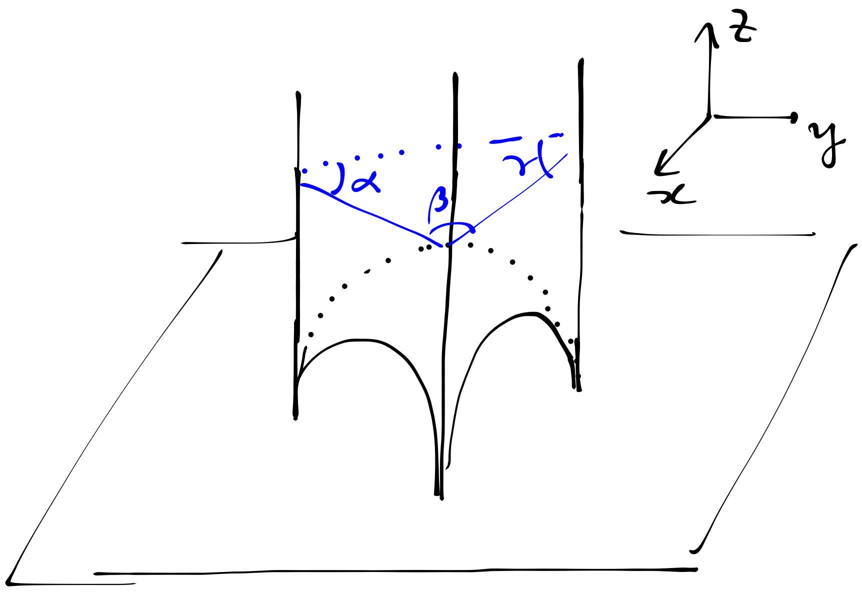
[Thurston]

hyperbolic knot $:=$ $S^3\setminus K$ admits a hyperbolic str $R_{ij} = -2 g_{ij}$

i) geodesically complete

ii) finite volume

iii) Simplicial decomp by ideal tetrahedra $T(\alpha,\beta,\gamma)$



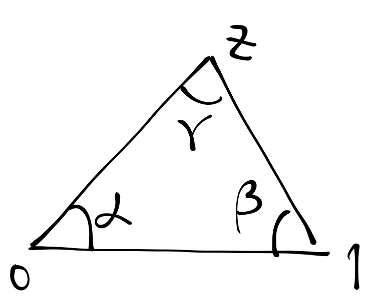
$$ds^2 = \frac{dx^2 + dy^2 + dz^2}{z^2}$$

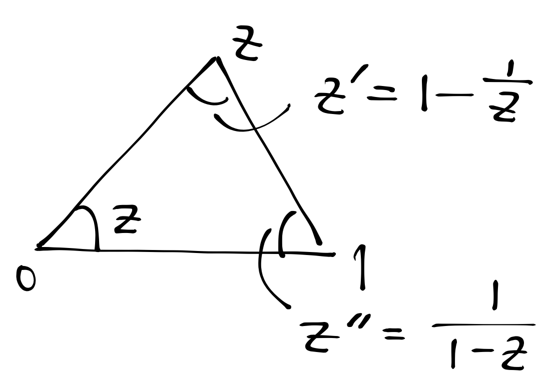
$$\mathrm{Vol}(T(\alpha,\beta,\gamma)) = \int \frac{dx\,dy\,dz}{z}$$

$$= \Lambda(\alpha) + \Lambda(\beta) + \Lambda(\gamma) = D(z)$$

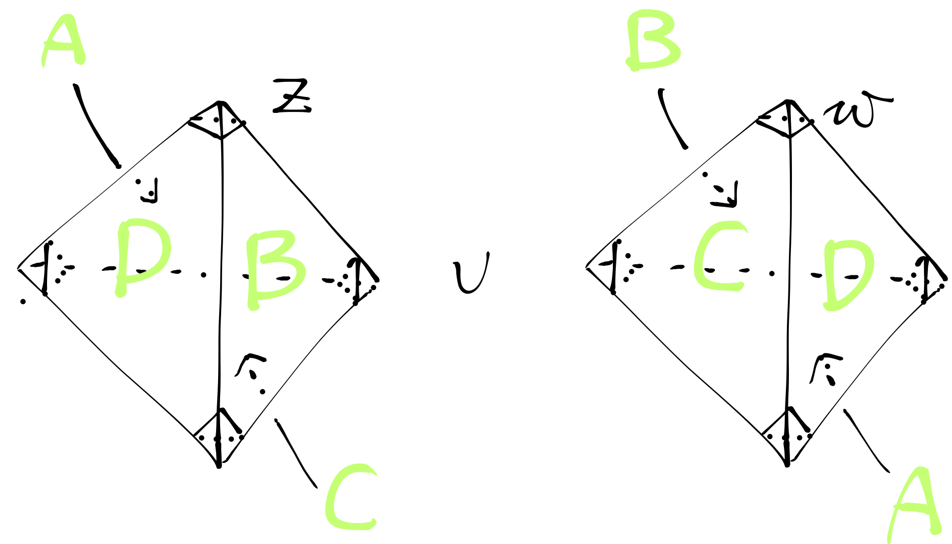
$$\Lambda(\theta) = -\int_0^\theta dt \log|2\sin t| \qquad \text{Lobachevsky}$$

$$D(z) = \mathrm{Im}\, Li_2(z) + \arg(1-z)\log|z| \qquad \text{Bloch-Wigner}$$

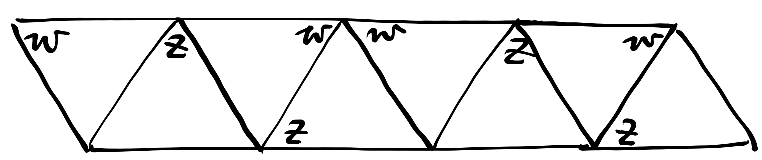




Ex. complement of 4_1

$S^3 \setminus 4_1 =$ 

Developing map



$$\begin{cases} w^2 w'' z' (z'')^2 = 1 & \text{consistency cond.} \\ w = z & \text{completeness} \end{cases}$$

$$\Rightarrow w = z = e^{i\frac{\pi}{3}}$$

$$\therefore \mathrm{Vol}(S^3 \setminus 4_1) = 2D(e^{\frac{\pi i}{3}}) = 6\Lambda\left(\frac{\pi}{3}\right)$$

asymptotics of colored Jones poly'l of 4_1

$$\left|J_{[r]}(4_1; e^{2\pi i/r})\right| \sim e^{\frac{r}{2\pi} \underbrace{\mathrm{Im}\,\mathrm{Li}_2(e^{\frac{\pi i}{3}})}_{= \mathrm{Vol}(S^3 \setminus 4_1)}}$$

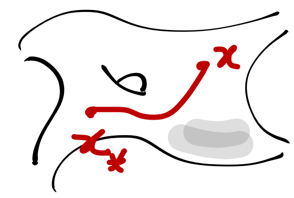
Volume conjecture & A-polynomials

Generalized volume conj. Gukov

$$J_{[r]}(K;q) \xrightarrow[\substack{r\to\infty \\ q^r = x}]{q=e^{\hbar}\to 1} \exp\left(\frac{1}{\hbar}\int_{x_*}^{x}\log y \,\frac{dx}{x} + \cdots\right)$$

$A(K:x.y(x))=0$

x_* complete hypeblic metric

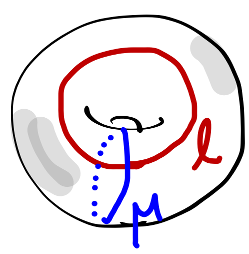


deformation to in complete hyperbolic metric.

$$\mathcal{M}_{flat}(T^2, SL(2.\mathbb{C})) = \frac{\mathbb{C}^\times \times \mathbb{C}^\times}{\mathbb{Z}_2} = \left\{ (x.y) \;\middle|\; \mathbb{Z}_2 : \begin{array}{l} x \to x^{-1} \\ y \to y^{-1} \end{array} \right\}$$

hyper-Kähler manifold.

$$\rho : \pi_1(T^2) = \mathbb{Z}\oplus\mathbb{Z} \longrightarrow SL(2.\mathbb{C})$$



$$\mu \longmapsto \begin{pmatrix} x & * \\ 0 & x^{-1} \end{pmatrix}$$

$$l \longmapsto \begin{pmatrix} y & * \\ 0 & y^{-1} \end{pmatrix}$$

A-poly'l determenes the moduli sp. of $SL(2.\mathbb{C})$ flat conn. $/ S^3\setminus K$.

$$\mathcal{M}_{flat}(S^3\setminus K, SL(2.\mathbb{C})) = \left\{ (x.y) \in \frac{\mathbb{C}^\times \times \mathbb{C}^\times}{\mathbb{Z}_2} \;\middle|\; A(K:x.y)=0 \right\}$$

$\cap$ Lagrangian submfd w.r.t. $\omega = \frac{dx}{x}\wedge\frac{dy}{y}$.

$\mathcal{M}_{flat}(T^2, SL(2.\mathbb{C}))$

quantum volume conjecture a.k.a. AJ conjecture.

The q-diff. eqn of minimal order for colored Jones poly

$$\hat{A}(K:\hat{x}.\hat{y}:q)\,J_{[r]}(K;q)=0$$

where $\hat{x}.\hat{y}$ are defined

$$\hat{x}\,J_{[r]}(K;q) = q^{r}\,J_{[r]}(K;q) \qquad \hat{x}\hat{y} = q\,\hat{y}\hat{x}$$

$$\hat{y}\,J_{[r]}(K;q) = J_{[r+1]}(K;q)$$

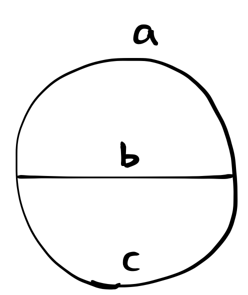
provides quantization of A-polynomial

$$\hat{A}(K:\hat{x}.\hat{y}.q) \xrightarrow[q\to 1]{} A(K:x.y)$$

Gukov. Garoufalidis.

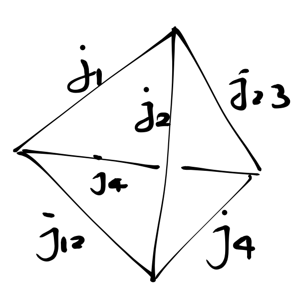
volume conjecture for quantum invariants of trivalent graph.

on-going work with Rama Zodin.



$$\longrightarrow (-1)^{m+n+p}\,\frac{[m+n+p+1]!\,[n]!\,[m]!\,[p]!}{[m+n]!\,[n+p]!\,[p+m]!} \qquad \begin{aligned} a&=m+p\\ b&=m+n\\ c&=n+p\end{aligned}$$

theta graph.



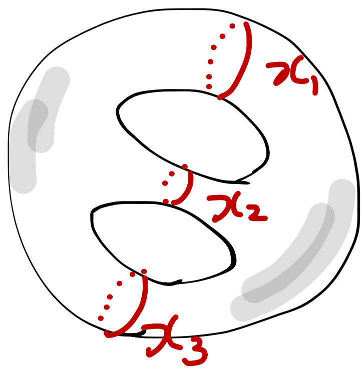
$$\longrightarrow {}_4\varphi_3\left(\begin{matrix} q^{*} & q^{*} & q^{*} & q^{*} \\ q^{*} & q^{*} & q^{*} & \end{matrix}\,;q\right)$$

basic hypergeom. series

Askey-Wilson poly.

quantum 6j-symbol

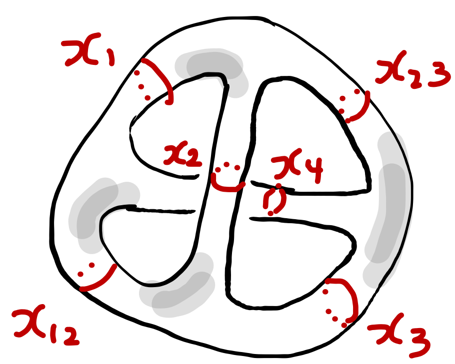
The boundary of tubular neighborhood of Θ-graph or q-$6j$ is a Riemann surface of genus 2 or 3, respectively.



holonomy eigenvalues of these cycles

$$\left.\begin{matrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{matrix}\right\} \text{ conjugate variables.}$$

spans local coordinate of $\mathcal{M}_{\text{flat}}(\Sigma_{g=2}, SL(2,\mathbb{C}))$



$$\left.\begin{matrix} x_1 & x_2 & x_3 & x_4 & x_{12} & x_{23} \\ y_1 & y_2 & y_3 & y_4 & y_{12} & y_{23} \end{matrix}\right\} \text{ conj. variables}$$

local coordinates $\mathcal{M}_{\text{flat}}(\Sigma_{g=3}, SL(2,\mathbb{C}))$

$$\dim \mathcal{M}_{\text{flat}}(\Sigma_g, SL(2,\mathbb{C})) = \dim \mathcal{M}_{\text{Higgs}}(\Sigma_g) = 6g-6.$$

For $g>1$, $\mathcal{M}_{\text{flat}}(\Sigma_g, SL(2,\mathbb{C}))$ has non-trivial topology.

Betti numbers are given by Hitchin

$$\sum_{i=1}^{6g-6} b_i t^i = \frac{(1+t^3)^{2g}}{(1-t^2)(1-t^4)} - \frac{t^{4g-2}}{4(1-t^2)(1-t^4)} \times \left\{(1+t^2)^2(1+t)^{2g} - (1+t)^4(1-t)^{2g}\right\}$$
$$-(g-1)t^{4g-3}\frac{(1+t)^{2g-2}}{1-t} + 2^{2g-1}t^{4g-4}\left\{(1+t)^{2g-2} - (1-t)^{2g-2}\right\}$$

$\mathcal{M}_{flat}(S^3 \setminus \Theta, SL(2.\mathbb{C}))$ is determined by large color asymptotics of quantum invariant of Θ-graph.

$$A_m(y_m, \vec{x}) = (-1+x_m)(-1+x_m x_n x_p) + (-1+x_m x_n)(-1+x_m x_p) y_m$$
$$A_n(y_n, \vec{x}) = (m \to n)$$
$$A_p(y_p, \vec{x}) = (m \to p).$$

Furthermore, recursion relations provide quantization of

$$\mathcal{M}_{flat}(\Sigma_{g=2}, SL(2.\mathbb{C}))$$

The same statement holds for q-$6j$. Since it is expressed by an Askey-Wilson poly'l, it satisfies recursion rel'n

$$A(q)\begin{Bmatrix} j_1+2, & j_2 & j_{12} \\ j_3 & j_4 & j_{23} \end{Bmatrix} + B(q)\begin{Bmatrix} j_1+1, & j_2 & j_{12} \\ j_3 & j_4 & j_{23} \end{Bmatrix} + C(q)\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{23} \end{Bmatrix} = 0$$

Something complicated

classical limit.

$$\Rightarrow$$

$$\mathcal{M}_{flat}(S^3 \setminus H_{g=3}, SL(2.\mathbb{C})) : A_k(\vec{x}, y_k) = 0 \quad k = 1, \cdots, 6$$

$\cap$ Lagrangian submfd

$$\mathcal{M}_{flat}(\Sigma_{g=3}, SL(2.\mathbb{C}))$$

Volume conjecture for homological invariants.

Super-A-polynomials $\mathcal{A}(K;x.y,a.t)$

large color asymptotic of Poincare poly of $[r]$-colored HOMFLY homology

$$\mathcal{P}_{[r]}(K;q.a.t) \xrightarrow[\substack{q^r=x \\ a.t:\text{fixed.}}]{\substack{r\to\infty \\ q=e^{\hbar}\to 1}} \exp\left(\frac{1}{\hbar}\int_{\mathcal{A}(K:x.y:a.t)=0} \log y \frac{dx}{x} + \cdots\right)$$

The quantization of super-A-poly'l is given by q-difference eqn.

$$\hat{\mathcal{A}}(K:\hat{x}.\hat{y}:a.q.t)\,\mathcal{P}_{[r]}(K:a.q.t)=0.$$

where $\hat{\mathcal{A}}(K:\hat{x}.\hat{y}:a.q.t) \xrightarrow{q\to 1} \mathcal{A}(K:x.y:a.t)$

$$\underset{\text{super-A-poly.}}{\mathcal{A}(K:x.y:a.t)} \xrightarrow[t=-1]{} \underset{\substack{\| \\ \text{augumentation polynomial} \\ \text{in knot contact homology}}}{A^{a\text{-def}}(K:x.y:a)} \longrightarrow \underset{\text{ordinary A-poly}}{A(K:x.y)}$$

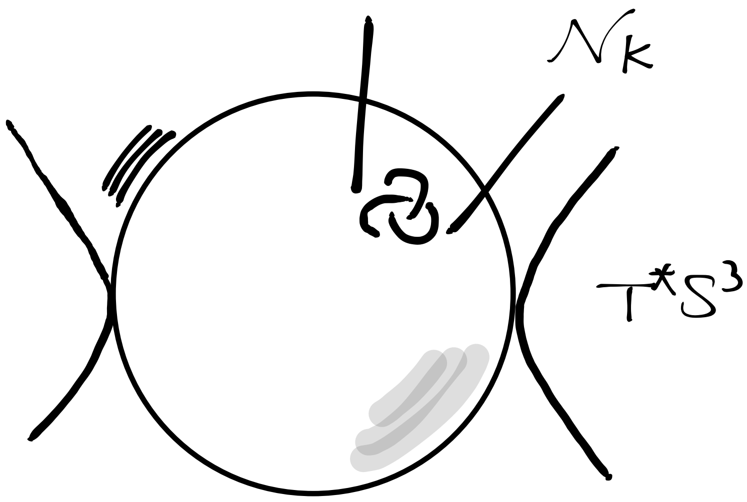
Aganagic-Vafa. Ng.

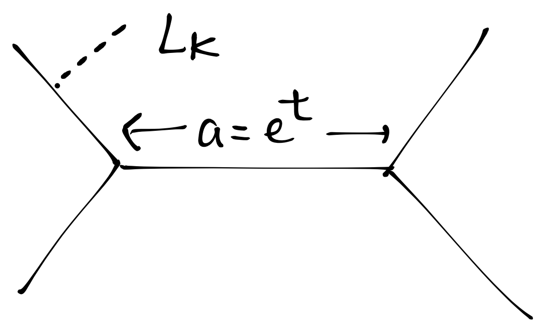
§4. Interpretation in string theory and gauge theory

$U(N)$ Chern–Simons theory on S^3 can be realized.

in A-model topological string theory. on T^*S^3.

deformed conifold $\xleftrightarrow{\text{geometric transition}}$ resolved conifold





$$N_K = \{(x.p) \in T^*S^3 \mid x = x(s),\ \dot{x}(s) \perp p\}$$

$$\mathcal{O}(-1) \oplus \mathcal{O}(-1) = X$$
$$\downarrow$$
$$\mathbb{P}^1$$

$S^1 \times \mathbb{R}^4 \times T^*S^3$

N M5 $\quad S^1 \times \mathbb{R}^2 \times S^3$

M5 $\quad S^1 \times \mathbb{R}^2 \times N_K$

$\xrightarrow{\text{Large } N}$

$$\boxed{\begin{array}{l} S^1 \times \mathbb{R}^4 \times X \\ \\ \text{M5} \quad S^1 \times \mathbb{R}^2 \times L_K \end{array}}$$

$\downarrow$ geom. engineering.

$U(1)$ gauge theory w/ a surface operator S_K associated to knot K

superpoly of $(m.n)$-torus knot.

$$\overline{\mathcal{P}}_{\square}(K(m,n) : a.q.t) = Z^{\text{Nek}}_{U(1), S_{K(m,n)}}(a.q.t)$$

Gorsky–Negut.

resolved conifold $\longrightarrow$ 4d $U(1)$ gauge thy with a surf. op S_K
associated to a knot K

Klemm – Katz – Vafa.

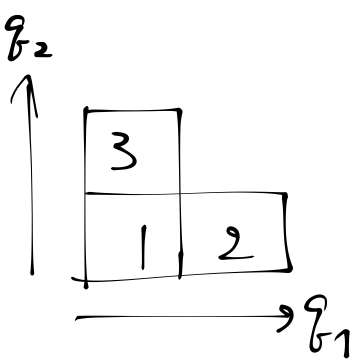
A : Coulomb branch parameter

(q_1, q_2) : equivariant parameter $\begin{array}{ccc} U(1)_{q_1} & \times & U(1)_{q_2} \\ \circlearrowleft & & \circlearrowleft \\ \mathbb{R}^2 & & \mathbb{R}^2 \end{array}$

Gorsky - Negut

$$Z^{Nek}_{U(1),\, S_{K(m,n)}}(A, q_1, q_2) = \sum_{SYT} \frac{\prod x_i^{S(i)} (1-Ax_i)(q_1 x_i - t)}{\left(1-\frac{q_1 x_2}{q_2 x_1}\right) \cdots \left(1-\frac{q_1 x_n}{q_2 x_{n-1}}\right)} \prod_{i<j} \omega\left(\frac{x_i}{x_j}\right)$$

where sum is over SYT of size n



x_i - (q_1, q_2)-content of box labelled i

$x_1 = 1 \quad x_2 = q_1$

$x_3 = q_2$

$$\omega(x) = \frac{(1-q_1 x)(1-q_2 x)}{(1-x)(1-q_1 q_2 x)}$$

$$S(i) = \left\lfloor \frac{im}{n} \right\rfloor - \left\lfloor \frac{(i-1)m}{n} \right\rfloor$$

After suitable change of variables, it is equal to
Poincaré poly'l of HOMFLY homology of $T(m,n)$

$$\overline{P}(T(m,n), a, q, t) = Z^{Nek}_{U(1),\, S_{T(m,n)}}(A = -at,\ q_1 = q,\ q_2 = qt^2)$$

proof of geometric transition!

What is $Z^{Nek}_{U(1),Sk(m,n)}$ mathematically?

$\mathrm{Hilb}^n \mathbb{C}^2$ = moduli sp. of codim n ideals in $\mathbb{C}[x,y]$

ideal

$F\mathrm{Hilb}^n(\mathbb{C}^2,0)$ = moduli sp of $\mathcal{O}_{\mathbb{C}^2,0} \supset I_1 \supset I_2 \supset \cdots \supset I_n$

dim $I_k = k$, all supported at 0.

$\mathcal{L}_i \rightsquigarrow I_i / I_{i+1}$ line bundle on $F\mathrm{Hilb}^n(\mathbb{C}^2,0)$

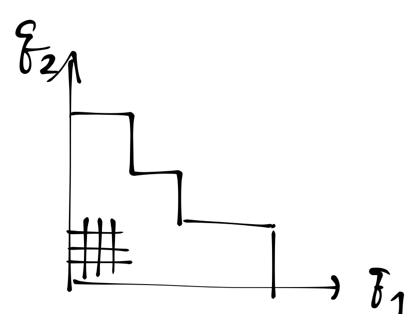
<u>Th</u>

$$P(T(m,n)) = \sum_i \chi_{(\mathbb{C}^\times)^2}\left(F\mathrm{Hilb}^n, \mathcal{L}_1^{S(1)} \cdots \mathcal{L}_n^{S(n)} \otimes \Lambda^i T^*\right) a^i$$

tautological bundle

$$T = \mathbb{C}[x,y] / I_n$$

Fixed pts = flags of monomial ideals.



χ_i (q_1,q_2)-character of $\mathcal{L}_i$

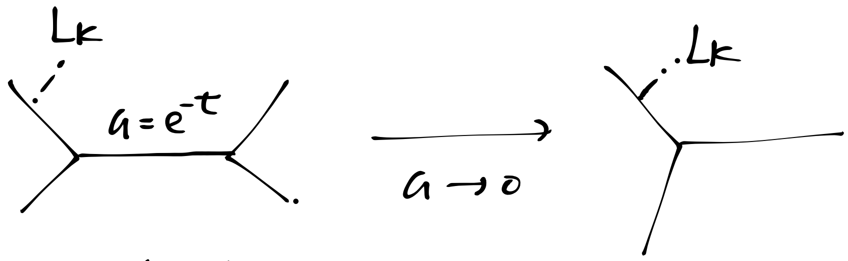
SYT.

<u>issue!</u> Complete intersection resolution

$F\mathrm{Hilb}^2 = \mathbb{CP}^1$

$F\mathrm{Hilb}^3 = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-3) \to \mathbb{P}^1)$ Hirzebruch Surface.

$a \to 0$. limit : turn off 4d gauge dynamics
dynamics on the surface op. S_K remains
$\Rightarrow$ vortex partition fn.



resolved conifold

$\mathbb{C}^3$

Gorsky - Gukov - Stosic.

Vortex Partition function

$$\frac{1}{4\pi}\int_{\mathbb{R}^2} F = r \qquad \text{vortex number.}$$

moduli sp. $\mathcal{M}_r^{U(1)}$ of vortex partition fn with vortex # r

$$\mathcal{M}_r^{U(1)} = \left\{ (A, \phi) \;\middle|\; \begin{array}{c} *F_A = i\phi - it \\ \bar{\partial}_A \phi = 0 \end{array} \right\}$$

$$\Rightarrow \mathcal{M}_r^{U(1)} = \mathrm{Sym}^r \mathbb{C} = \underset{(x_1, \cdots, x_r)}{\underset{\parallel}{\mathbb{C}^r / S_r}} \cong \underset{(a_1, \cdots, a_r)}{\underset{\parallel}{\mathbb{C}^r}}$$

$$f(x) = \prod_{j=1}^{r} (x - x_j) = x^r + a_1 x^{r-1} + \cdots + a_r$$

$U(1)_q$ equivariant character $\mathrm{Ch}_q(\mathcal{M}_r^{U(1)})$ of $\mathcal{M}_r$

(counting monomials w.r.t. $(a_1 \cdots a_r)$)

$$\mathrm{Ch}_q(\mathcal{M}_r^{U(1)}) = \frac{1}{(1-q)\cdots(1-q^r)}$$

$$= \lim_{a \to 0} \overline{P}_{[r]}(\bigcirc; a, q).$$

non-abelian generalization

$U(p+1)$ gauge theory with $(p+1)$ fund. field ϕ_j

$$\mathcal{M}_r^{U(p+1)} = \left\{ (A, \phi) \;\middle|\; \begin{array}{l} *F_A = i \sum_{j=1}^{p+1} \phi_j \phi_j^* - it \\ \bar\partial_A \phi = 0 \end{array} \right\}$$

$$\dim \mathcal{M}_r^{U(p+1)} = 2r(p+1)$$

For general r, explicit construction of $\mathcal{M}_r^{U(p+1)}$ is not known

$r=1$
$$\mathcal{M}_{r=1}^{U(p+1)} = \overset{U(1)_q \atop \curvearrowleft}{\mathbb{C}} \times \mathbb{CP}^p$$

$\mathbb{C}$: center-of-mass motion

$\mathbb{CP}^p$: internal deg. of freedom

$U(1)_t$ is related to homological deg.

Betti number $\displaystyle\sum_{i=0}^{\infty} b_i(\mathbb{CP}^p)\, t^{2i} = \sum_{i=0}^{p} t^{2i}$

$U(1)_q \times U(1)_t$ equiv. charater

$$\mathrm{Ch}_{q,t}\left(\mathcal{M}^{U(p+1)}_{r=1}\right) = \frac{1}{1-q} \underbrace{\sum_{i=1}^{p} t^{2i}}$$

‖ Poincare poly of $\mathbb{C}P^p$
(t: homological degree)

$$= \lim_{a\to 0} \overline{\mathcal{P}}_{\square}\left(T(2,2p+1); a, q, t\right)$$

Conjecture

$$\mathrm{Ch}_{q,t}\left(\mathcal{M}^{U(p+1)}_{r}\right) = \lim_{a\to 0} \overline{\mathcal{P}}_{[r]}\left(T(2,2p+1): a, q, t\right)$$

$$= \frac{q^{-pr}}{(q:q)_r} \sum_{k_1 \cdots k_p} \begin{bmatrix} r \\ k_1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \cdots \begin{bmatrix} k_{p-1} \\ k_p \end{bmatrix}$$

$$\times\, q^{(2r+1)(k_1+\cdots+k_p) - \sum k_i k_{i+1}}\, t^{2(k_1+\cdots+k_p)}$$

Dimofte-Gukov-Hollands

Yamazaki Terashima

Dimofte Gukov Gaiotto.

3d/3d correspondence.

deformed conifold side.

$S^1 \times \mathbb{R}^4 \times T^*S^3$

$\boxed{N \text{ M5} \quad S^1 \times \mathbb{R}^2 \times S^3}$

M5 $\quad S^1 \times \mathbb{R}^2 \times N_K$

on $S^1 \times \mathbb{R}^2$ → $SL(N,\mathbb{C})$ CS on $S^3 \setminus K$

3d/3d duality

on $S^3 \setminus K$ → 3d $\mathcal{N}=2$ gauge theory $T_{sl(N)}[K]$ on $S^1 \times \mathbb{R}^2$

Partition fn of $T_{sl(2)}[K]$ on $S^1 \times \mathbb{R}^2$

$$I_K(a,q,t,\{x\}) = \mathrm{Tr}\,(-1)^F a^{m_1} t^{m_2} q^{\frac{R}{2}-J} x_i^{e_i}$$

$$= \int_\Gamma \frac{ds}{s} \left[\theta(z;q)\cdots\right] B_\Delta(z_1;q)\cdots B_\Delta(z_l;q)$$

CS coupling

FI coupling

chiral field.

$z_i = z_i(s,a,t,\{x\})$

$$B_\Delta(z;q) = \sum_{n=0}^{\infty} \frac{z^{-n}}{(q^{-1};q^{-1})_n}$$ holomorphic block.

Beem-Dimofte-Pasquetti.

$$\mathcal{P}(K;a,q,t) \text{ "=" } I_{T_{sl(2)}[K]}(a,q,t) \;??$$