M a compact 3-mfd, $G = SU(N)$

P principal G-bundle over M.

$$\pi_1(G) = 1 \Rightarrow P = M \times G$$

$\mathcal{A}_M$ space of connections on P. $\quad \mathcal{A}_M \cong \Omega^1(M, \mathfrak{g})$

$\mathcal{G}$ gauge transf. $\quad \mathcal{G} \cong \mathrm{Map}(M, G)$

Chern-Simons action

$$CS(A) = \frac{1}{8\pi^2} \int_M \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right)$$

The bilinear form $B : \Omega^1(M, \mathfrak{g}) \times \Omega^2(M, \mathfrak{g}) \to \mathbb{R}$ defined by

$$B(\alpha, \beta) = \int_M \mathrm{Tr}(\alpha \wedge \beta) \qquad \text{non-deg.}$$

Therefore, $\Omega^2(M, \mathfrak{g}) = (\Omega^1(M, \mathfrak{g}))^*$. The tangent space of $\mathcal{A}_M$ can be identified with $\Omega^1(M, \mathfrak{g})$

$$CS(A + ta) - CS(A) = \frac{t}{4\pi^2} \int \mathrm{Tr}(F_A \wedge a) + O(t^2)$$

Hence,

$$dCS(A) = \frac{1}{4\pi^2} F_A$$

Suppose M has a boundary $\partial M = \Sigma$

$$g^* A = g^{-1} A g + g^{-1} dg \quad \text{gauge transf'n}$$

$$CS(g^* A) - CS(A) = \frac{1}{4\pi} \int_{\partial M} \mathrm{Tr}(A \wedge g^{-1} dg) - \frac{1}{12\pi} \int_M \mathrm{Tr}\left[(g^{-1} dg)^3\right]$$

Especially, $\partial M = \phi$.

$$CS(g^*A) = CS(A) - \int_M g^*\sigma \qquad \sigma : \text{Maurer-Cartan form}$$

Since $\int_M g^*\sigma = \deg(M \to 3\text{-cycle of } G) \in \mathbb{Z}$,

$$CS : \mathcal{A}_M/\mathcal{G} \longrightarrow \mathbb{R}/\mathbb{Z}$$

$\partial M = \Sigma$.

principal G-bundle over Σ is $\Sigma \times G$

$$\Rightarrow \mathcal{A}_\Sigma = \Omega^1(\Sigma . \mathfrak{g}) \qquad \infty\text{-dim'l symp. mfd}$$

$$\omega(\alpha . \beta) = -\frac{1}{8\pi^2}\int_\Sigma \mathrm{Tr}(\alpha \wedge \beta) \qquad \alpha . \beta \in \Omega^1(\Sigma . \beta)$$

symplectic form

For $a \in \mathcal{A}_\Sigma$ and $g \in \mathcal{G}_\Sigma$, we denote A is an extension of a on M and $\tilde{g}$ is an extension of g on M

$$c(a, g) = e^{2\pi i (CS(\tilde{g}^*A) - CS(A))}$$

$$= \exp\left(2\pi i \left\{ \int_\Sigma \frac{1}{8\pi^2} \mathrm{Tr}(g^{-1} a g \wedge g^{-1} dg) - \int_M g^*\sigma \right\}\right)$$

This is indep. of an extension $\tilde{g}$, A, M.

In other words,

$$e^{2\pi i\, CS(g^*A)} = C(a.g|_\Sigma)\, e^{2\pi i\, CS(A)}$$

Therefore, we consider an action of $\mathcal{G}_\Sigma$ on $\mathbb{C}\times\mathcal{A}_\Sigma$

$$\mathcal{G}_\Sigma : \mathbb{C}\times\mathcal{A}_\Sigma \to \mathbb{C}\times\mathcal{A}_\Sigma \;;\; (z,a)\mapsto (c(a.g)\, z,\ g^*a)$$

Chern-Simons functional integral can be regarded as $\mathcal{G}_\Sigma$-invariant holomorphic section of $\mathbb{C}^{\otimes k}\times\mathcal{A}_\Sigma$.

$$Z_k(M)(a) = \int_{\mathcal{A}_a/\mathrm{Kernel}(\mathcal{G}\to\mathcal{G}_\Sigma)} e^{2\pi i k\, CS(A)}\, \mathcal{D}A.$$

$\mathcal{A}_a$ the space of connexions on M w $A|_\Sigma = a$

$\mathcal{G}_\Sigma$-action on $\mathbb{C}^{\otimes k}\times\mathcal{A}_\Sigma$ is given by

$$Z_k(M)(g^*a) = C(a.g)^k\, Z_k(M)(a)$$

This can be identified with holomorphic section of line bundle over symplectic quotient space.

$$\mathcal{A}_\Sigma /\!/ \mathcal{G}_\Sigma = \mu^{-1}(0)/\mathcal{G}_\Sigma$$

where moment map

$$\mu : \mathcal{A}_\Sigma \to \mathrm{Lie}(\mathcal{G}_\Sigma)^* ;\ A \to F_A$$

$\mu^{-1}(0)/\mathcal{G}_\Sigma = \mathcal{M}_{flat}(\Sigma, G)$ moduli space of G flat connexions over Σ.

> <u>Conclusion</u> : Let M be an oriented 3-manifold with boundary Σ, $k \in \mathbb{Z}$ CS level, then CS functional integral $Z_k(M)$ is considered to be a section of the complex line bundle $\mathcal{L}^{\otimes k}$
>
> $$Z_k(M) \in H^0(\mathcal{M}_{flat}(\Sigma, G), \mathcal{L}^{\otimes k})$$

$H^0(\mathcal{M}_{flat}(\Sigma, G), \mathcal{L}^{\otimes k})$ is called quantum Hilbert space $\mathcal{H}_\Sigma$.

<u>Remark</u> : Strictly speaking, the moduli space $\mathcal{M}_{flat}(\Sigma^J, \mathcal{L}^{\otimes k})$ depends on the complex structure J of Σ. However, if you consider conformal block bundle over Teichmuller space, the bundle has a natural flat connection. In this sense, the quantum Hilbert space is indep. of cpx str. J.

If $M = M_1 \cup M_2$ with $\partial M_1 = \Sigma$ and $\partial M_2 = -\Sigma$

$$\exp(2\pi i k\, CS(A)) = \langle \exp(2\pi i k\, CS(A_1)), \exp(2\pi i k\, CS(A_2)) \rangle$$

where A is a connexion of $M \times G$, A_1 and A_2 are restriction of A to M_1 and M_2 $\Rightarrow$ axiom of TQFT.

Let us consider the case of a link L in a 3-mfd M.

We assign reps R_i to components C_i $(i=1,\cdots,|L|)$ of L.

Wilson loop operator

$$W_{C_j \cdot R_j}(A) = \mathrm{Tr}_{R_j}\left(\mathrm{Hol}_{C_j}(A)\right)$$

The invariant of the link by Witten's formulation.

$$Z_k(M:(C_j \cdot R_j)) = \int \mathcal{D}A \; e^{2\pi i k CS(A)} \prod_{j=1}^{|L|} W_{C_j R_j}(A)$$

The quantum Hilbert space associated with the above integral is given by conformal blocks of $\hat{\mathfrak{g}}_k$ WZW model.

$\mathcal{M}_{p_1 \cdots p_n}$ the vector sp of meromorphic fns on $\mathbb{C}P^1$ with poles of any order at most at $p_1 \cdots p_n$

$$\mathfrak{g}(p_1 \cdots p_n) = \mathfrak{g} \otimes \mathcal{M}_{p_1 \cdots p_n}$$

To each pt p_i $(1 \le i \le n)$, we associate the integrable higest weight H_{λ_i}. We define diagonal action Δ of $\mathfrak{g}(p_1 \cdots p_n)$ on the tensor product $H_{\lambda_1} \otimes \cdots \otimes H_{\lambda_n}$

$$\Delta(\varphi)(\xi_1 \otimes \cdots \otimes \xi_n) = \sum_j (\xi_1 \otimes \cdots \otimes i_j(\varphi)\xi_j \otimes \cdots \otimes \xi_n)$$

where $i_j : \mathfrak{g}(p_1 \cdots p_n) \to \hat{\mathfrak{g}}_j$

The space of conformal blocks

$$\mathrm{Hom}_{\mathfrak{g}(p_1 \cdots p_n)}(\mathcal{H}_{\lambda_1} \otimes \cdots \otimes \mathcal{H}_{\lambda_n}, \mathbb{C})$$

is defined to be the space of linear forms

$$\mathcal{H}_{\lambda_1} \otimes \cdots \otimes \mathcal{H}_{\lambda_n} \longrightarrow \mathbb{C}$$

which are invariant under the diagonal action Δ of $\mathfrak{g}(p_1 \cdots p_n)$

Definition of colored quantum invariants

Braid group B_n

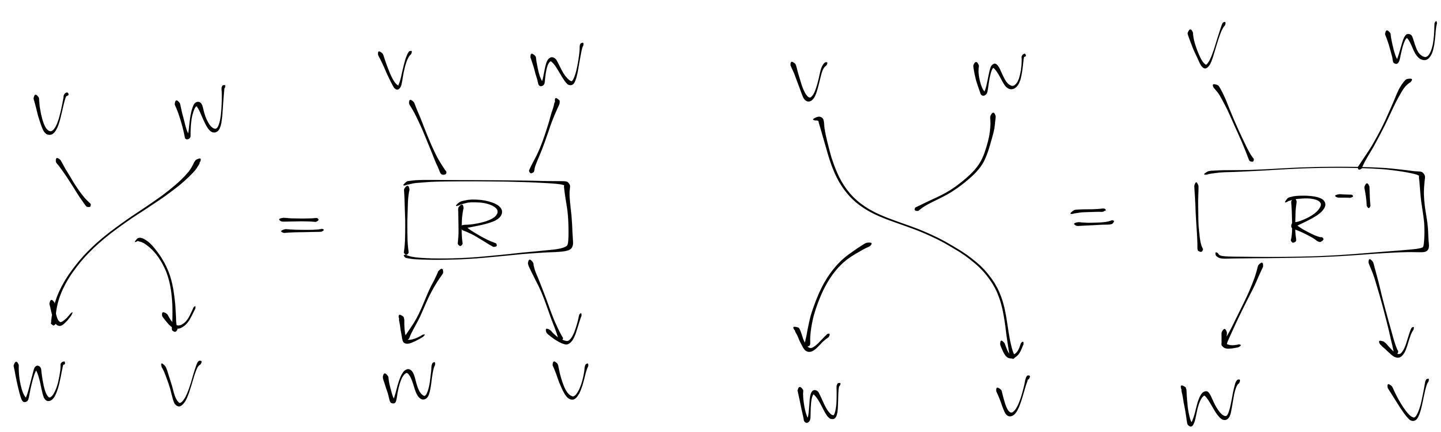
$$B_n = \langle \sigma_1 \cdots \sigma_{n-1} \mid \sigma_i \sigma_j = \sigma_j \sigma_i \ (|i-j|>1),$$
$$\sigma_k \sigma_{k+1} \sigma_k = \sigma_{k+1} \sigma_k \sigma_{k+1} \rangle$$

Th (Alexander) Every link can be realized as a closure of a certain braid.

Def

1. $U_q(\mathfrak{g})$-module V, W

$$\check{R}_{V,W} : V \otimes W \longrightarrow W \otimes V$$



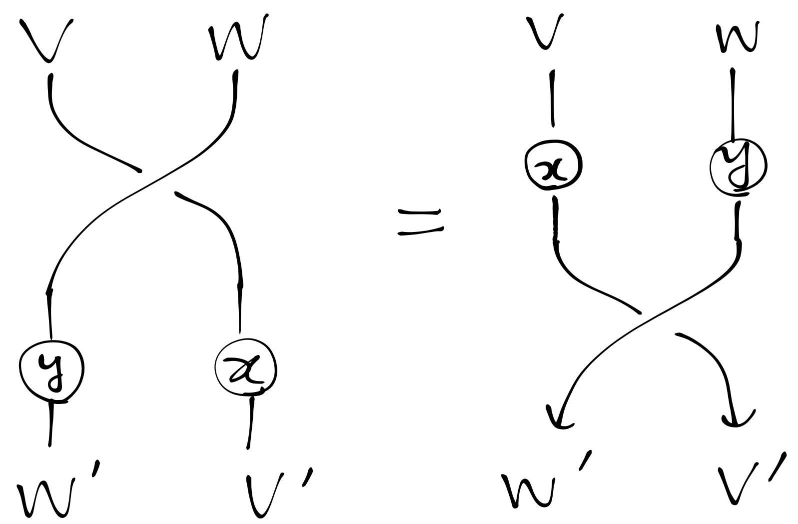
which satisfy

$$\check{R}_{U\otimes V, W} = (\check{R}_{U,W} \otimes id_V)(id_U \otimes \check{R}_{V,W})$$
$$\check{R}_{U, V\otimes W} = (id_V \otimes \check{R}_{U,W})(\check{R}_{U,V} \otimes id_W)$$

It is natural in the sense that

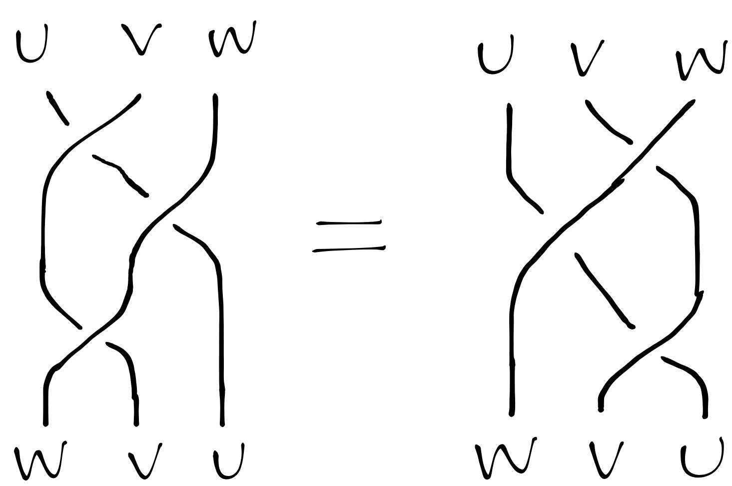
$$(y\otimes x)\,\check{R}_{V,W}=\check{R}_{V',W'}(x\otimes y)$$

for $x\in \mathrm{Hom}_{U_q(\mathfrak{g})}(V,V')$ & $y\in \mathrm{Hom}_{U_q(\mathfrak{g})}(W,W')$



These equs imply Yang–Baxter equs

$$(\check{R}_{V,W}\otimes \mathrm{id}_U)(\mathrm{id}_V\otimes \check{R}_{U,W})(\check{R}_{U,V}\otimes \mathrm{id}_W)$$
$$=(\mathrm{id}_W\otimes \check{R}_{U,V})(\check{R}_{U,W}\otimes \mathrm{id}_V)(\mathrm{id}_U\otimes \check{R}_{V,W})$$



2. There exists an element $K_{2\rho}\in U_q(\mathfrak{g})$

$$K_{2\rho}(v\otimes w)=K_{2\rho}(v)\otimes K_{2\rho}(w)$$

for $v\in V$, $w\in W$

for every $z \in \mathrm{End}_{U_q(\mathfrak{g})}(V \otimes W)$ with $z = \sum x_i \otimes y_i$

$x_i \in \mathrm{End}(V)$ & $y_i \in \mathrm{End}(W)$, the quantum trace is defined

$$\mathrm{Tr}_W(z) = \sum_i \mathrm{Tr}(y_i K_{2\rho})\, x_i \in \mathrm{End}_{U_q(\mathfrak{g})} V$$

Given a link L with the component $L_1, \cdots, L_l$ labeled by $U_q(\mathfrak{g})$-module $V_1, \cdots, V_l$. The braid diagram β of the link L define

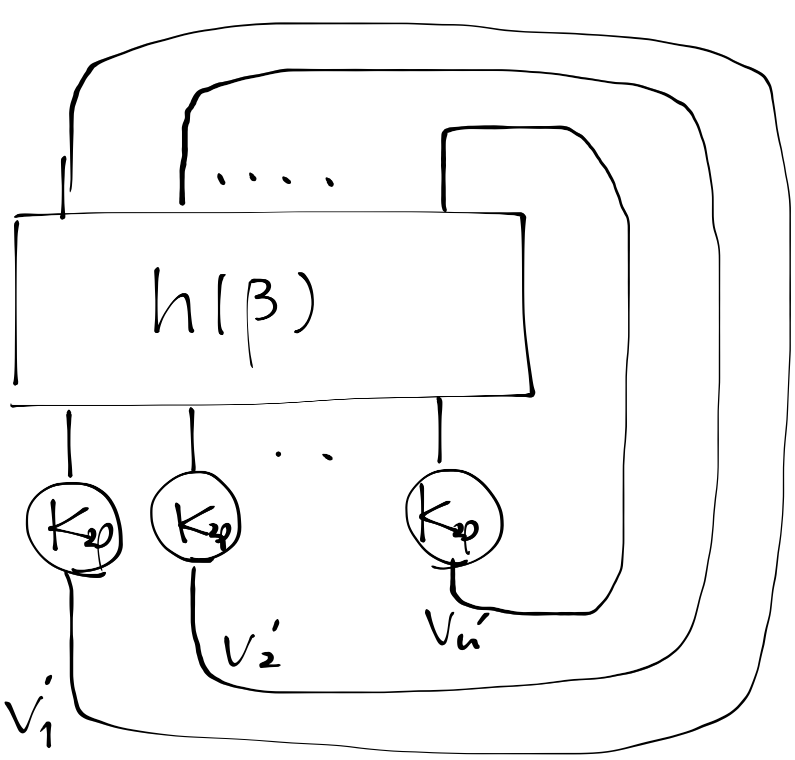
$$\begin{array}{rccc} h: & B_n & \longrightarrow & \mathrm{End}_{U_q(\mathfrak{g})}(V_1' \otimes \cdots \otimes V_n') \\ & \Psi & & \Psi \\ & \beta & \longmapsto & h(\beta) \end{array}$$

Then, the quantum invariant of $(L_1, \cdots, L_l ; V_1, \cdots, V_l)$

$$J_{(\mathfrak{g}; V_1 \cdots V_l)}(L) = \mathrm{Tr}_{V_1' \otimes \cdots \otimes V_n'}(h(\beta))$$

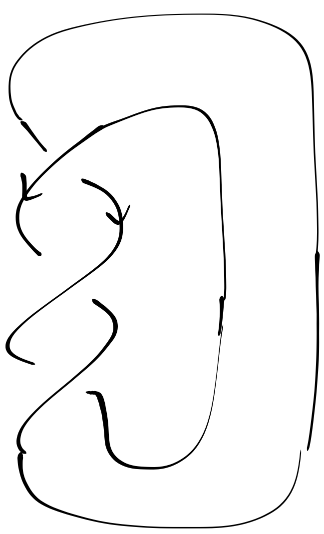
$\uparrow$

framing dependent i.e. invariant under Reidemeister move 2 and 3.



Ex $(\mathfrak{sl}(2), \square)$

$$R = \begin{pmatrix} q^{1/4} & 0 & 0 & 0 \\ 0 & q^{1/4} - q^{-3/4} & q^{-1/4} & 0 \\ 0 & q^{-1/4} & 0 & \\ 0 & 0 & 0 & q^{1/4} \end{pmatrix} \qquad K_{2\rho} = \begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix}$$



$(\mathfrak{sl}(2), \square)$

$$\Rightarrow J_{(\mathfrak{sl}(2), \square)}(3_1) = \mathrm{Tr}_{V_{1/2} \otimes V_{1/2}}\left(R^3 (K_{2\rho} \otimes K_{2\rho}) \right)$$
$$\propto (q^{1/2} + q^{-1/2})(q^{-1} + q^{-3} - q^{-4})$$

Drinfeld constructed universal R-matrix associated to $U_q(\mathfrak{g})$

So, in principle, one can write $\check{R}_{V,W}$ for any rep. V, W.