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纽结多项式和  $N=(0, 2)$  朗道金兹堡模型  
On knot polynomials and  $N=(0,2)$  Landau-Ginzburg models

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# On knot polynomials and $\mathcal{N} = (0, 2)$ Landau-Ginzburg models

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ABSTRACT: This paper focuses on two topics, knot polynomials and  $\mathcal{N} = (0, 2)$  Landau-Ginzburg models. We first obtain a closed-form expression of HOMFLY-PT polynomial colored by rectangular Young diagrams for double twist knots. We conjecture that there is a cyclotomic expansion of colored HOMFLY-PT polynomial for arbitrary knots. And in a similar fashion, we also conjecture the Poincaré polynomials of HOMFLY-PT homology colored by rectangular Young diagrams for double twist knots. The HOMFLY-PT polynomials are related to the complex Chern-Simons theory with gauge group  $SU(N)$ . When we change the gauge group to  $SO(N)$ , it will lead to the Kauffman polynomials. Using isomorphism of representations of different gauge group, we find a simple rule of grading change which allows us to obtain the  $[r]$ -colored quadruply-graded Kauffman homology from the  $[r^2]$ -colored quadruply-graded HOMFLY-PT homology for thin knots. Along this line of work, we also find the relationship among  $A$ -polynomials of  $SO$  and  $SU$ -type coming from a differential on Kauffman homology. Another way to compute knot polynomials is to use  $6j$ -symbols. We put forward a closed-form expression of  $SO(N)$  ( $N \geq 4$ ) quantum  $6j$ -symbols for symmetric representations, and calculate the corresponding  $SO(N)$  fusion matrices for the cases when representations  $R = \square, \square\square$ . The newly proposed series  $F_K$  is like a analytical continuation of colored HOMFLY-PT polynomial in a sense that its asymptotic behavior agrees with that of the colored HOMFLY-PT polynomials. We derived closed-form expressions of  $(a, t)$ -deformed  $F_K$  for the complements of double twist knots with positive braids from our colored HOMFLY-PT formulae. Using the conjectural expressions, we also derive  $t$ -deformed ADO polynomials. On the other hand, we study  $\mathcal{N} = (0, 2)$  Landau-Ginzburg models. We use techniques involving 't Hooft anomaly matching,  $c$ -extremization and modular invariance of the partition function on the torus to study the low energy physics of the theory. In a certain  $\mathcal{N} = (0, 2)$  Landau-Ginzburg theory, we find an example of modular invariant partition function beyond the ADE classification. This originates from the fact that a part of the left-moving sector is a new conformal field theory which is a variant of the parafermion model.

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## 1 Introduction

Witten's celebrated paper [1] has shown that the Chern-Simons theory provides a natural framework for quantum invariants of 3-manifolds and knots. The Chern-Simons path integral on a 3-manifold  $M_3$  with the action

$$S = \frac{k}{4\pi} \int_{M_3} \text{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right), \quad (1.1)$$

is a quantum invariant of  $M_3$ , known as the Witten-Reshetikhin-Turaev (WRT) invariant [1, 2]. If a Wilson loop is included along a knot in  $S^3$ , the expectation value of the Wilson loop is a quantum invariant of the knot:

$$\bar{J}_R^G(K; q) := \langle W_R(K) \rangle = \frac{\int [\mathcal{D}A] e^{iS} W_R(K)}{\int [\mathcal{D}A] e^{iS}}, \quad (1.2)$$

where  $G$  denotes the gauge group and  $R$  is one of its representation. The parameter  $q$  is expressed in terms of the level  $k$  and the dual Coxeter number  $h^\vee$  of the gauge group  $G$  as

$$q = \exp \left( \frac{2\pi i}{k + h^\vee} \right). \quad (1.3)$$

We can also define the normalized version quantum invariants as

$$J_R^G(K; q) = \frac{\bar{J}_R^G(K; q)}{\bar{J}_R^G(\bigcirc; q)}. \quad (1.4)$$

We will mainly use the knot invariants normalized by the unknot in this paper. Since the Chern-Simons functional integral on a three-manifold with boundary is an element of the Hilbert space on the boundary which is isomorphic to the space of WZNW conformal blocks, quantum knot invariants could be constructed by using braiding and fusion operations on WZNW conformal blocks [3–7]. When  $G = \text{SU}(N)$ , it corresponds to a two-variable colored HOMFLY-PT polynomial  $P_R(K; a, q)$ , where  $R$  is a representation of  $\text{SU}(N)$  and  $a = q^N$ . When  $G = \text{SO}(N)$ , it corresponds to colored Kauffman polynomial  $F_R(K; a, q)$ , where  $R$  is a representation of  $\text{SO}(N)$  and  $a = q^{N-1}$ . The quantum knot invariants have a beautiful property that they are polynomials of  $q$  with integer coefficients. This property led to an important development in knot theory proposed by Khovanov [8], i.e. the categorifications of quantum knot invariants. Khovanov constructed a bi-graded homology which itself is a knot invariant, and its  $q$ -graded Euler characteristic is the Jones polynomial  $J(K; q)$  of a knot  $K$ .

In [9], M. Khovanov and L. Rozansky constructed a triply-graded homology of a knot whose graded Euler characteristic is the HOMFLY-PT polynomial. And in [10], it was conjectured that the HOMFLY-PT homology is endowed with structural properties. Similarly, the existence and structural properties of Kauffman homology were conjectured in [11] although the rigorous definition of a triply-graded homology theory categorifying the Kauffman polynomials has not been given yet. This line of studying knot homology has been pursued in the colored cases. In [12, 13], quadruply-graded colored HOMFLY-PT homology was proposed and various structures and symmetries were uncovered. In [14], it was further conjectured that there exist the quadruply-graded colored Kauffman homologies with rich structural properties.

The first part of this paper investigates knot polynomials. In §2, we mainly study the colored HOMFLY-PT polynomial. Although several methods to compute HOMFLY-PT polynomials for arbitrary color in principle are known, carrying out explicit computation is practically very challenging for general non-torus knots and colors. However, in recent years, studying the structural properties of

colored HOMFLY-PT polynomials, closed form expressions for symmetric representations have been found for a certain class of non-torus knots. Surprisingly, the structural properties become more apparent at the level of HOMFLY-PT homology that categorifies quantum HOMFLY-PT polynomials. Lately, the HOMFLY-PT homology colored by arbitrary representations has been defined in [15]. Although it is difficult to carry out computation of homology from the definition, various structural properties of HOMFLY-PT homology have been uncovered by combining mathematical definitions and physical predictions. In particular, it was proposed in [13, 16] that when colors are specified by rectangular Young diagrams, structural properties become more manifest if  $(a, Q, t_r, t_c)$  quadruple-grading is introduced. In [17], closed form expressions of Poincaré polynomials of HOMFLY-PT homology of the trefoil and the figure-eight colored by rectangular Young diagrams are conjectured. The conjectural formulae in [17] are very simple, being expressed by a summation over Young diagrams inscribed by  $[r^s]$ . We further generalize the formulae to the case of double twist knots. Our conjectural formulae are expressed by a summation over iteratively inscribed Young diagrams with interpolation Macdonald polynomials encoding the twists. We also put forth a conjecture of cyclotomic expansions of HOMFLY-PT polynomials by rectangular Young diagrams for any knot. Like the Rosso-Jones formula [18] for torus knots, we believe that the formulae will find many applications in other areas of mathematics.

In §3, we further investigate the Kauffman polynomial and we find a rule of changing variables that transforms the Poincaré polynomial of  $[r^2]$ -colored HOMFLY-PT homology into that of  $[r]$ -colored Kauffman homology for thin knots. As another important development, the volume conjecture [19, 20] gives a remarkable relationship between quantum invariants of a knot  $K$  and classical geometry of the knot complement  $S^3 \setminus K$ . It states that the large color asymptotic behavior of colored Jones polynomial  $J_{[r]}(K; q)$  of a hyperbolic knot provides the hyperbolic volume of its knot complement. This was further generalized in [21, 22], which gives the relation between the large color limit of colored Jones polynomials and the  $A$ -polynomial for the knot complement. The zero locus of the  $A$ -polynomial  $A(K; x, y)$  of a knot  $K$  determines the character variety of  $\mathrm{SL}(2, \mathbb{C})$ -representation of the knot group [23] so that the generalized volume conjecture paves a way to complex Chern-Simons theory. There are further generalizations of the volume conjecture to incorporate HOMFLY-PT and Kauffman polynomials colored by symmetric representations and their categorifications [14, 24–26], leading to a notion of super- $A$ -polynomials. Since there is a differential that relates colored Kauffman homology to HOMFLY-PT homology, we investigate the influence of the differential on super- $A$ -polynomials. This brings about a conjecture that provides an intriguing relationship among  $A$ -polynomials of  $\mathrm{SU}$  and  $\mathrm{SO}$ -type. As briefly mentioned, quantum invariants can be evaluated with braiding and fusion matrices in the WZNW models. When  $q$  is a root of unity, the fusion matrices of the WZNW models are equivalent to quantum  $6j$ -symbols of the corresponding quantum groups [3, 27–31]. The  $6j$ -symbols were introduced by Wigner [32, 33] and Racah [34] in the study of angular momenta, and they are beautifully expressed by a hypergeometric series  ${}_4F_3$ . In the case of  $\mathfrak{sl}_2$ , the explicit expression for quantum  $6j$ -symbols was obtained in [35], which is a natural quantization with a  $q$ -hypergeometric series  ${}_4\varphi_3$ . Nonetheless, it is notoriously involved to obtain explicit expressions for  $6j$ -symbols from the definition with the Clebsch-Gordan coefficients ( $3j$ -symbols), in particular, for higher ranks [36]. However, the information of quantum  $6j$ -symbols with all symmetric representations was extracted from quantum knot invariants for  $U_q(\mathfrak{sl}_N)$  [37], and it was formulated in terms of the  $q$ -hypergeometric series  ${}_4\varphi_3$  in [38]. We study the  $\mathrm{SO}(N)$  quantum  $6j$ -symbols with symmetric representations along this line. The formulae for the classical  $\mathrm{SO}(N)$   $6j$ -symbols with all representations symmetric were obtained by Ališauskas [39, 40]. We quantize these formulae and compute  $\mathrm{SO}(N)$  fusion matrices of the WZNW models for the case of  $R = \square, \square\square$ . We check that the fusion matrices reproduce the colored Kauffman polynomials of non-torus knots. This gives a piece of evidence that the natural quantization

of the Aliauskas' formulae will be valid for the  $\mathrm{SO}(N)$  quantum  $6j$ -symbols.

In §4, we study a newly proposed two variable series  $F_K$  and its  $(a, t)$ -deformation for double twist knots. The developments of knot invariants and volume conjectures are inseparably bound up with the study of invariants of 3-manifolds. Chern-Simons quantum invariants of 3-manifolds can be computed by a surgery formula of quantum invariants of links in  $S^3$  when  $q$  is a root of unity [2]. In the recent study of the 3d/3d correspondence, an analytic continuation  $\widehat{Z}$  of the WRT invariant has been proposed as a BPS  $q$ -series [41, 42]. This invariant can be thought of as a generating series of BPS states in a 3d  $\mathcal{N} = 2$  theory  $T[M_3]$ . Also it can be regarded as a partition function of complex Chern-Simons theory on  $M_3$  defined on a unit disk  $|q| < 1$  whose radial limit  $q \searrow e^{\pi i/(k+h^\vee)}$  becomes the WRT invariant. Its generalization to a knot complement, denoted as  $F_K(x, q) = \widehat{Z}(S^3 \setminus K)$ , is given in [43] to make surgeries and TQFT method work. It is observed that the asymptotic expansion of  $F_K$  agrees with the Melvin-Morton-Rozansky expansion [44–47] of the colored Jones polynomials. Moreover,  $F_K$  is conjectured to be annihilated by the quantum  $A$ -polynomial [48, 49] of the corresponding knot  $K$ ,

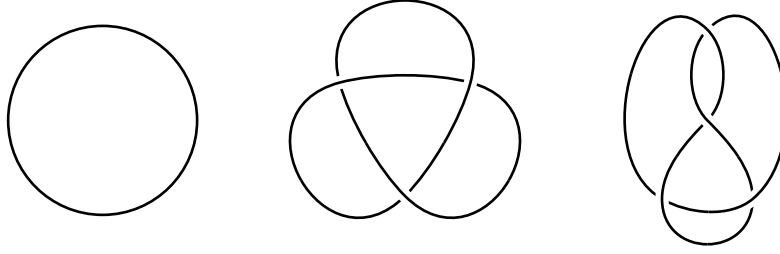
$$\hat{A}(K; \hat{x}, \hat{y}, q) F_K(x, q) = 0. \quad (1.5)$$

Based on [50], it is proposed in [51] that the limit  $q \rightarrow \zeta_p = e^{2\pi i/p}$  of  $F_K$  leads to the  $p$ -th ADO polynomials [52]. The development of  $F_K$  is currently following that of quantum knot invariants although its definition is not given yet. In [53], an extension of  $\widehat{Z}$  and  $F_K$  to arbitrary gauge group  $G$  was studied. Analogous to HOMFLY-PT polynomials and superpolynomials,  $(a, t)$ -deformation of  $F_K$  are put forth in [54]. Following this path, we provide a closed-form expression of the  $(a, t)$ -deformed  $F_K$  for double twist knots with positive braids. We also derive closed-form formulae of  $t$ -deformed ADO polynomials.

Another topic of this paper is  $\mathcal{N} = (0, 2)$  Landau-Ginzburg theories and we elaborate on it in §5.  $\mathcal{N} = (0, 2)$  LG models describe certain phases of  $\mathcal{N} = (0, 2)$  supersymmetric gauge theories. It is shown in [55, 56] that these theories are helpful for understanding the  $\mathcal{N} = (0, 2)$  Calabi-Yau sigma model. Also, the class of  $\mathcal{N} = (0, 2)$  LG models is very rich because firstly they are chiral in general and secondly there is the freedom of interactions due to the  $E$ -terms and  $J$ -terms [57]. Therefore, it is natural to ask how IR CFTs incorporate the richness of  $\mathcal{N} = (0, 2)$  LG models by encoding the information of the  $E$ -terms and  $J$ -terms. Gadde and Putrov paved a way to the understanding of low energy physics of  $\mathcal{N} = (0, 2)$  Landau-Ginzburg models in [58], although they mainly focused on a number of classes of relatively simple LG models. It is shown in [59] that  $\mathcal{N} = (0, 2)$  LG models satisfying certain constraints can be classified. We would like to apply the method of Gadde and Putrov to different classes of  $\mathcal{N} = (0, 2)$  LG models. Our goal is to find the modular invariant partition function of the IR CFT in the NS-NS sector. Since the elliptic genus is invariant under the RG flow, and it is easily computed from the information of the LG model, we can check that whether our modular invariant partition function is consistent with the elliptic genus. For a certain  $\mathcal{N} = (0, 2)$  LG model, we find that, in order to get a modular invariant partition function which agrees with the elliptic genus, we need to go beyond the famous ADE classification, which governs all modular invariant partition functions for conformal field theories associated to the affine  $A_1$  algebra. With careful analysis of the IR Hilbert space, we can show that a part of the left-moving sector is described by a new CFT which is a close cousin of the parafermion model.

## 2 HOMFLY-PT colored by rectangular representations

In mathematical language, a knot is an embedding of  $S^1$  in 3-dimensional Euclidean space  $\mathbb{R}^3$ . Figure 1 shows three simplest knots.



**Figure 1:** Unknot, trefoil, and figure-eight

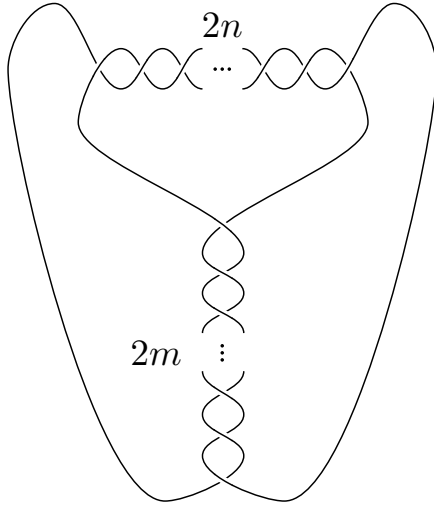
However, when it comes to complicated knots, it is really difficult to distinguish between them, simply looking at the knot diagrams. In practice, knots are often distinguished by knot invariants. And one important kind of knot invariants is called a knot polynomial. Well known examples are Jones polynomials and Alexander polynomials. In this section, we focus on a generalization of these two knot polynomials, called the HOMFLY-PT polynomial. We use the following skein relation for normalized HOMFLY-PT polynomial

$$A H_{\square} \left( \begin{array}{c} \nearrow \searrow \\ \nwarrow \nearrow \end{array} \right) - A^{-1} H_{\square} \left( \begin{array}{c} \nwarrow \nearrow \\ \nearrow \nwarrow \end{array} \right) = (q - q^{-1}) H_{\square} \left( \begin{array}{c} \nearrow \nearrow \\ \searrow \searrow \end{array} \right) ,$$

and the normalized HOMFLY-PT polynomial for unknot is

$$H_{\square}(\bigcirc) = 1 .$$

Here  $\square$  is a Young diagram represents the fundamental representation. We will discuss more about the Young diagrams in the next subsection. In this following part, we will mainly discuss a family of knot, the double twist knot  $K_{m,n}$ . Their corresponding name in Rolfsen table are shown in figure 2.

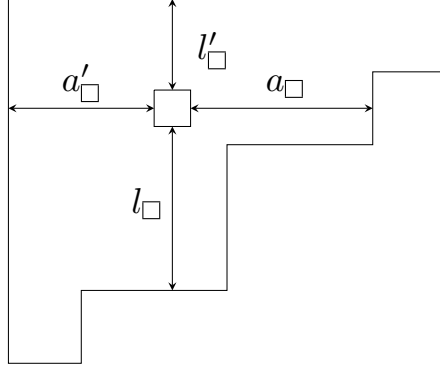


$m$	$n$	Name
1	1	3 <sub>1</sub>
-1	1	4 <sub>1</sub>
2	1	5 <sub>2</sub>
-2	1	6 <sub>1</sub>
3	1	7 <sub>2</sub>
2	2	7 <sub>4</sub>
-3	1	8 <sub>1</sub>
-2	2	8 <sub>3</sub>
4	1	9 <sub>2</sub>
3	2	9 <sub>5</sub>
-4	1	10 <sub>1</sub>
-3	2	10 <sub>3</sub>

**Figure 2:** Double twist knot  $K_{m,n}$  and corresponding knots in Rolfsen Table up to 10 crossings.

## 2.1 Young diagram, Schur polynomial and Macdonald polynomial

There is a one-to-one correspondence between the representation of  $SU(N)$  and the Young diagrams. A Young diagram is a finite collection of boxes, arranged in left-justified row, with the row length in non decreasing order. A Young diagram can be parametrized by a partition, which is a sequence  $\lambda = (\lambda_1, \lambda_2, \dots)$  of non-negative integers in decreasing order,  $\lambda_1 \geq \lambda_2 \geq \dots$ . The highest weight of any irreducible representation can be written in terms of the Dynkin coefficients in the basis of fundamental weights,  $\mu = \sum_k q_k \mu^k$ . Then its corresponding Young diagram is built with  $q^k$  columns of  $k$  boxes. The size of the young diagram  $\lambda$  is denoted by  $|\lambda| = \sum_i \lambda_i$ , which is the number of boxes in the Young diagram. Also we will denote the transpose of the Young diagram by  $\lambda^T$ . In this section, we mainly focus on rectangular Young diagrams and we denote them by  $[r^s]$ , where  $r$  means the number of columns and  $s$  is the number of rows. If  $\lambda, \mu$  are partitions, we shall write  $\mu \subset \lambda$  to mean that the diagram  $\lambda$  contains the diagram  $\mu$ , i.e.  $\lambda_i \geq \mu_i$  for  $i \geq 1$ . We may associate every box in the Young diagram with four functions, arm ( $a$ ), coarm ( $a'$ ), leg ( $l$ ), coleg ( $l'$ ), displayed as in figure 3,



**Figure 3:** Illustration of arm, coarm, leg and coleg

For a Young diagram  $\lambda$ , a Young tableau  $T$  is obtained by filling the boxes of the Young diagram by the numbers in  $\{1, \dots, n\}$ . We denote the entry of a box by  $T_{\square}$ . A Young tableau is called a *reverse tableau* if the entries are weakly decreasing along each row and strongly decreasing along each column. The maximal number of entries in a reverse tableau must be not less than the length of the Young diagram by definition. A reverse tableau  $T$  on  $\lambda$  induces a sequence of Young diagrams

$$\emptyset \equiv \lambda^{(n)} \subset \lambda^{(n-1)} \subset \dots \subset \lambda^{(0)} \equiv \lambda, \quad (2.1)$$

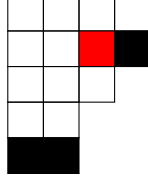
where  $\lambda^{(i)}$  is the sub Young diagram such that all entries satisfy  $T_{\square} > i$ . For example, if we consider the following reverse tableau on  $\lambda = (4, 4, 3, 2, 2)$  with  $n = 6$ ,

6	6	6	2
5	5	2	1
3	3	2	
2	2		
1	1		

then the sequence is

$$\emptyset \subset \begin{array}{|c|c|c|} \hline & & \\ \hline \end{array} \subset \begin{array}{|c|c|} \hline & \\ \hline \end{array} \subset \begin{array}{|c|} \hline \\ \hline \end{array} \subset \begin{array}{|c|c|} \hline & \\ \hline \end{array} \subset \begin{array}{|c|c|c|} \hline & & \\ \hline \end{array} \subset \begin{array}{|c|c|c|c|} \hline & & & \\ \hline \end{array}.$$

For  $\mu \subset \lambda$ , the set-theoretic difference  $\theta = \lambda - \mu$  is called a skew diagram. It is also denoted by  $\lambda/\mu$ . A skew diagram is a horizontal strip if  $\theta_i^T \leq 1$ , which is to say that there is at most one box in each column. For a horizontal strip  $\lambda/\mu$ , we denote sets of rows and columns intersecting with  $\lambda/\mu$  by  $R_{\lambda/\mu}$  and  $C_{\lambda/\mu}$ . Then  $R_{\lambda/\mu} - C_{\lambda/\mu}$  is a set of boxes belonging to  $R_{\lambda/\mu}$  but not to  $C_{\lambda/\mu}$ . For the above sequence  $\lambda^{(k-1)}/\lambda^{(k)}$  are horizontal strips for every  $k = 1, \dots, 6$ . As an example,  $\lambda^{(0)}/\lambda^{(1)}$  and  $R_{\lambda^{(0)}/\lambda^{(1)}} - C_{\lambda^{(0)}/\lambda^{(1)}}$  correspond to the black boxes and the red boxes, respectively.



First, let us introduce the Schur polynomial. Schur polynomial is a symmetric polynomial parametrized by partitions. Given a partition of an integer  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ , where  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0$ , we can define a function

$$a(\lambda_1 + n - 1, \lambda_2 + n - 2, \dots, \lambda_n)(x_1, x_2, \dots, x_n) = \det \begin{bmatrix} x_1^{\lambda_1+n-1} & x_2^{\lambda_1+n-1} & \dots & x_n^{\lambda_1+n-1} \\ x_1^{\lambda_2+n-2} & x_2^{\lambda_2+n-2} & \dots & x_n^{\lambda_2+n-2} \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{\lambda_n} & x_2^{\lambda_n} & \dots & x_n^{\lambda_n} \end{bmatrix}. \quad (2.2)$$

Then, the Schur polynomial is defined as

$$s_\lambda(x_1, x_2, \dots, x_n) = \frac{a(\lambda_1 + n - 1, \lambda_2 + n - 2, \dots, \lambda_n)(x_1, x_2, \dots, x_n)}{a(n - 1, n - 2, \dots, 0)(x_1, x_2, \dots, x_n)}. \quad (2.3)$$

One can also check that the Schur polynomial satisfies the Cauchy formula

$$\prod_{i,j} (1 + x_i y_j) = \sum_{\lambda \subset [r^s]} s_\lambda(x) s_{\lambda^T}(y). \quad (2.4)$$

We further introduce another special polynomial, the Macdonald polynomial. The ordinary Macdonald polynomial  $P_\lambda(x_1, x_2, \dots, x_n; q, t)$  is defined as

$$P_\lambda(x_1, x_2, \dots, x_n; q, t) = \sum_T \psi_T(q, t) \prod_{\square \in \lambda} x_{T_\square}, \quad (2.5)$$

where the sum is taken over all reverse tableau  $T$  on  $\lambda$  with entries in  $\{1, \dots, n\}$  and the coefficients are defined by

$$\psi_T(q, t) = \prod_{i=1}^n \psi_{\lambda^{(i-1)}/\lambda^{(i)}}(q, t), \quad \psi_{\lambda/\mu}(q, t) = \prod_{\square \in R_{\lambda/\mu} - C_{\lambda/\mu}} \frac{b_\mu(\square)}{b_\lambda(\square)}, \quad (2.6)$$

where

$$b_\lambda(\square) = \frac{1 - q^{a_\square} t^{l_\square + 1}}{1 - q^{a_\square + 1} t^{l_\square}}. \quad (2.7)$$

## 2.2 Colored HOMFLY-PT polynomials

It is conjectured in [17] that the  $[r^s]$ -colored HOMFLY-PT polynomial of figure-eight is given by

$$H_{[r^s]}(K_{1,-1}; A, q) = \sum_{\lambda \subset [r^s]} D_{\lambda}^{(s)}(q) D_{\lambda^T}^{(r)}(q^{-1}) \prod_{\square \in \lambda} \{Aq^{r+a'_{\square}-l'_{\square}}\} \{Aq^{a'_{\square}-l'_{\square}-s}\}, \quad (2.8)$$

where  $\{x\} = x - x^{-1}$  and  $D_{\lambda}^{(s)}$  is defined as a specialization of the Schur polynomial,

$$D_{\lambda}^{(s)}(q) = s_{\lambda}(q^{s-1}, q^{s-3}, \dots, q^{-s+3}, q^{-s+1}) = \prod_{\square \in \lambda} \frac{\{q^{s-l'_{\square}+a'_{\square}}\}}{\{q^{l'_{\square}+a_{\square}+1}\}}. \quad (2.9)$$

As a consequence of the Cauchy formula, it is manifest that this HOMFLY-PT polynomial defined this way satisfies the exponential growth property [60],

$$H_{\lambda}(K; A, q = 1) = [H_{\square}(K; A, q = 1)]^{|\lambda|}. \quad (2.10)$$

It is an elegant extension from symmetric representations [61] to rectangular Young diagrams. Hence, as in symmetric representations [16, 62], one can expect that the  $[r^s]$ -colored HOMFLY-PT polynomial of the double twist knot can be expressed by inserting an appropriate twist element into this formula.

For symmetric representations, the twist element has been formulated, using the  $q$ -binomial theorem. For arbitrary representations, the generalized binomial theorem is given by [63]

$$s_{\lambda}(1 + x_1, 1 + x_2, \dots, 1 + x_n) = \sum_{\mu \subset \lambda} d_{\lambda\mu} s_{\mu}(x_1, \dots, x_n), \quad (2.11)$$

where

$$d_{\lambda\mu} = \det M_{ij}, \quad \text{with } M_{ij} = \binom{\lambda_i + n - i}{\mu_j + n - j}. \quad (2.12)$$

Each entry in the matrix is the usual binomial coefficient. We make use of the  $q$ -deformation of  $d_{\lambda\mu}$  and define a factor

$$B_{\lambda,\mu}^{(s)}(A, q) = A^{2|\mu|} q^{(1-s)(|\lambda|-|\mu|)} \left( \prod_{\square \in \mu} q^{4(a'_{\square}-l'_{\square})} \right) \det \widetilde{M}_{ij}, \quad \text{with } \widetilde{M}_{ij} = \left[ \begin{matrix} \lambda_i + s - i \\ \mu_j + s - j \end{matrix} \right]_{q^2}, \quad (2.13)$$

where  $\left[ \begin{matrix} x \\ y \end{matrix} \right]_q$  is the  $q$ -binomial coefficient. When  $\mu$  is trivial representation  $\emptyset$ , we have  $B_{\lambda,\emptyset}^{(s)} = D_{\lambda}^{(s)}$ . We further introduce the factor

$$F_{\lambda}^{(m)}(A, q) = \left( D_{\lambda}^{(s)} \right)^{-1} \sum_{\lambda_m \subset \dots \subset \lambda_1 = \lambda} B_{\lambda_1, \lambda_2}^{(s)} B_{\lambda_2, \lambda_3}^{(s)} \dots B_{\lambda_{m-1}, \lambda_m}^{(s)} D_{\lambda_m}^{(s)}, \quad (2.14)$$

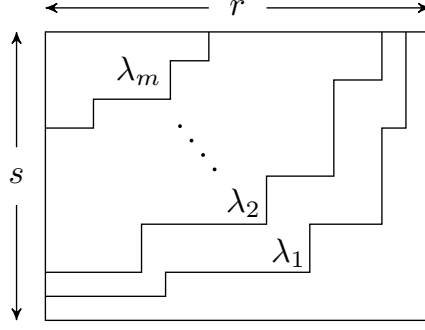
where the summations are taken over iteratively inscribed Young diagrams as illustrated in figure 4. Although the building blocks  $B^{(s)}$  and  $D^{(s)}$  is dependent on  $s$ ,  $F$  is not.

For positive twists, we define the twist factor as

$$\text{Tw}_{\lambda}^{(m)}(A, q) = (-A^2)^{|\lambda|} \left( \prod_{\square \in \lambda} q^{2(a_{\square}-l_{\square})} \right) F_{\lambda}^{(m)}(A, q), \quad m \in \mathbb{Z}_+. \quad (2.15)$$

For negative twists, the twist factor is

$$\text{Tw}_{\lambda}^{(m)}(A, q) = F_{\lambda}^{(-m)}(A^{-1}, q^{-1}), \quad m \in \mathbb{Z}_-. \quad (2.16)$$



**Figure 4:** Illustration of iteratively inscribed Young diagrams

Finally, we conjecture that the HOMFLY-PT polynomial of the double twist knot  $K_{m,n}$  colored by a rectangular Young diagram  $[r^s]$  is of the form,

$$H_{[r^s]}(K_{m,n}; A, q) = \sum_{\lambda \subset [r^s]} \frac{\text{Tw}_{\lambda}^{(m)} \text{Tw}_{\lambda}^{(n)}}{\text{Tw}_{\lambda}^{(1)} \text{Tw}_{\lambda}^{(-1)}} D_{\lambda}^{(s)}(q) D_{\lambda^T}^{(r)}(q^{-1}) \prod_{\square \in \lambda} \{Aq^{r+a'_{\square}-l'_{\square}}\} \{Aq^{a'_{\square}-l'_{\square}-s}\}.$$

This is a natural extensions of the formulae for symmetric representations [16, 62] to rectangular Young diagrams. We have checked this formula with the results in [64–66]

Motivated by this formula, we further generalize the conjecture of cyclotomic expansions of colored HOMFLY-PT polynomials of a knot  $K$  for symmetric representations [67, 68] to rectangular Young diagrams.

**Conjecture 2.1.** *Let  $Y$  be a set of all Young diagrams. For a knot  $K$ , there exists a function,*

$$\begin{aligned} C(K) : Y \times \mathbb{Z}_+ &\longrightarrow \mathbb{Z}[A^{\pm}, q^{\pm}] \\ (\lambda, s) &\longmapsto C_{\lambda,s}(K; A, q) \end{aligned}$$

*which satisfies the following properties*

- $C_{\emptyset,s}(K; A, q) = 1$  for any  $s \in \mathbb{Z}_+$
- $C_{\lambda,s}(K; A, q) = 0$  for  $s$  is less than the length of the partition  $\lambda$

*such that the  $[r^s]$ -colored reduced HOMFLY-PT polynomial of the knot  $K$  is expressed as*

$$H_{[r^s]}(K; A, q) = (A^{\#} q^{\#})^{rs} \sum_{\lambda \subset [r^s]} C_{\lambda,s}(K; A, q) D_{\lambda^T}^{(r)}(q^{-1}) \prod_{\square \in \lambda} \{Aq^{r+a'_{\square}-l'_{\square}}\}. \quad (2.17)$$

*Importantly,  $C_{\lambda,s}(A, q)$  is independent of  $r$ .*

If we can make use of the transposition symmetry [60] of the colored HOMFLY-PT polynomials

$$H_{\lambda}(K; A, q) = H_{\lambda^T}(K; A, q^{-1}), \quad (2.18)$$

then the conjecture can be stated by exchanging  $q \rightarrow q^{-1}$ ,  $s \rightarrow r$  and  $\lambda \rightarrow \lambda^T$ .

### 2.3 Poincaré polynomials of colored HOMFLY-PT homology

Moreover, it is conjectured that Poincaré polynomial of quadruply-graded HOMFLY-PT homology of figure-eight colored by rectangular representation is given by [17]

$$\mathcal{P}_{[rs]}(K_{1,-1}; A, q, t, \sigma) = \sum_{\lambda \subset [rs]} \tilde{D}_\lambda^{(s)}(q, t) \tilde{D}_{\lambda^T}^{(r)}(t^{-1}, q^{-1}) \prod_{\square \in \lambda} \left\{ \frac{A}{\sigma} q^{r+a'_\square} t^{-l'_\square} \right\} \{A\sigma q^{a'_\square} t^{-s-l'_\square}\}, \quad (2.19)$$

where  $\tilde{D}_\lambda^{(s)}$  is a principal specialization of the Macdonald polynomial

$$\tilde{D}_\lambda^{(s)}(q, t) = P_\lambda(t^{s-1}, t^{s-3}, \dots, t^{-s+3}, t^{-s+1}; q, t) = \prod_{\square \in \lambda} \frac{\{t^{s-l'_\square} q^{a'_\square}\}}{\{t^{l_\square+1} q^{a_\square}\}}. \quad (2.20)$$

In order to go to the  $(a, Q, t_r, t_c)$  grading, we need to make change of variables as

$$A \rightarrow a\sqrt{-t_r t_c}, \quad t \rightarrow t_r^{-1}, \quad q \rightarrow -t_c, \quad \sigma \rightarrow t_r^{-s} Q^{-1}. \quad (2.21)$$

The binomial theorem is also generalized for Macdonald polynomials [69], using interpolate Macdonald polynomials defined by

$$P_\lambda^*(x_1, \dots, x_n; q, t) = \sum_T \psi_T(q, t) \prod_{\square \in \mu} t^{1-T_\square} (x_{T_\square} - q^{a'_\square} t^{-l'_\square}). \quad (2.22)$$

The generalized binomial theorem states that

$$\frac{P_\lambda^*(cx_1, \dots, cx_n; q, t)}{P_\lambda^*(c, \dots, c; q, t)} = \sum_\mu \frac{c^{|\mu|}}{t^{(n-1)|\mu|}} \frac{p_\mu^*(q^{-\lambda}; q^{-1}, t^{-1})}{P_\mu^*(q^{-\mu}; q^{-1}, t^{-1})} \frac{P_\mu^*(x_n, \dots, x_1; q^{-1}, t^{-1})}{P_\mu^*(c, \dots, c; q, t)}. \quad (2.23)$$

Inspired by this, we define the refined version of 2.13

$$\begin{aligned} \tilde{B}_{\lambda, \mu}^{(s)}(A, q, t) &= (-t^{(1-s)|\lambda|-|\mu|}) \left( \frac{Aq}{t} \right)^{2|\mu|} \left( \prod_{\square \in \lambda} q^{2a'_\square} t^{-2l'_\square} \right) \left( \prod_{\square \in \mu} q^{2a'_\square} t^{-2l'_\square} \right) \\ &\quad \times \frac{P_\mu^*(q^{2\lambda}; q^2, t^2)}{P_\mu^*(q^{2\mu}; q^2, t^2)} \frac{P_\lambda^*(0; q^{-2}, t^{-2})}{P_\mu^*(0; q^{-2}, t^{-2})}. \end{aligned} \quad (2.24)$$

Notice that  $\tilde{B}_{\lambda, \emptyset} = \tilde{D}_\lambda^{(s)}$ . Then, we define the factor

$$\tilde{F}_\lambda^{(m)}(A, q, t) = (\tilde{D}_\lambda^{(s)})^{-1} \sum_{\lambda_m \subset \dots \subset \lambda_1 = \lambda} \tilde{B}_{\lambda_1, \lambda_2}^{(s)} \tilde{B}_{\lambda_2, \lambda_3}^{(s)} \dots \tilde{B}_{\lambda_{m-1}, \lambda_m}^{(s)} \tilde{D}_{\lambda_m}^{(s)} \quad (2.25)$$

Finally, we define the twist factor for positive twists as

$$\widetilde{\text{Tw}}_\lambda^{(m)}(a, t_r, t_c) = (-a^2 t_r^2 u t_c^2)^{|\lambda|} \left( \prod_{\square \in \lambda} t_c^{2a_\square} t_r^{2l_\square} \right) \tilde{F}_\lambda^{(m)}(a, t_c, t_r), \quad m \in \mathbb{Z}_+, \quad (2.26)$$

while the twist element for negative twists is

$$\widetilde{\text{Tw}}_\lambda^{(m)} = \tilde{F}_\lambda^{(-m)}(a^{-1}, t_c^{-1}, t_r^{-1}), \quad m \in \mathbb{Z}_-. \quad (2.27)$$

We conjecture that the Poincaré polynomial of  $[r^s]$ -colored HOMFLY-PT homology of the double twist knot  $K_{m,n}$  can be expressed as

$$\mathcal{P}_{[r^s]}(K_{m,n}; a, Q, t_r, t_c) = (-t_r t_c)^{-rs(\text{sgn}(m) + \text{sgn}(n))/2} \times \sum_{\lambda \subset [r^s]} \tilde{D}_\lambda^{(s)}(q, t) \tilde{D}_{\lambda^T}^{(r)}(t^{-1}, q^{-1}) \prod_{\square \in \lambda} \left\{ \frac{A}{\sigma} q^{r+a'_\square} t^{-l'_\square} \right\} \{A\sigma q^{a'_\square} t^{-s-l'_\square}\} \Bigg|_{(2.21)} \frac{\widetilde{\text{Tw}}_\lambda^{(m)} \widetilde{\text{Tw}}_\lambda^{(n)}}{\widetilde{\text{Tw}}_\lambda^{(1)} \widetilde{\text{Tw}}_\lambda^{(-1)}}. \quad (2.28)$$

Surprisingly,  $\mathcal{P}_{[r^s]}(K_{m,n})$  is a Laurent polynomial of  $(a, Q, t_r, t_c)$  with *positive coefficients* even though the summands are rational functions of  $(a, Q, t_r, t_c)$  in general. We have checked the structural properties [13] such as self-symmetry, mirror symmetry, refined exponential growth property and colored differentials for a number of examples.

The formulae conjectured in this section cry out for geometric interpretation. In fact, the binomial theorem (2.11) can be interpreted by the first Chern class of the tensor product of line bundles over flag varieties [70]. Recently, a new definition of the uncolored HOMFLY-PT homology is given by  $(q, t)$ -equivariant sheaves on Hilbert schemes of points on  $\mathbb{C}^2$  [71–73]. Although its colored version has yet to be defined, the formula we conjectured strongly suggests that braiding on  $(q, t)$ -equivariant sheaves will be captured by interpolation Macdonald polynomials via the localization of equivariant Grothendieck-Riemann-Roch formula [74] if it is defined. The double twist knots can be obtained by taking surgeries on two components of Borromean rings with framing  $-1/m$  and  $-1/n$ . Actually, the cyclotomic expansions of colored Jones polynomials of the double twist knots have been originally obtained from that of Borromean rings [75]. Therefore the structure in this formula can be extended to Borromean rings and twist links at the level of colored HOMFLY-PT polynomials as in [16, 75].

### 3 From HOMFLY-PT to Kauffman polynomial

The categorifications of quantum knot invariants shed new light on knot theory not only because knot homologies are more powerful than quantum invariants but also because they are functorial [76]. Moreover, they are endowed with rich structural properties [77], and various differentials give remarkable relationships among themselves [10, 12, 13, 78]. In particular, some relations between HOMFLY-PT and Kauffman homology have been uncovered in [11, 14]. The rich structural properties also help us find closed-form expressions of Poincaré polynomials of cyclotomic type [79] for various knots and links.

In this section, we will show a new relationship between colored HOMFLY-PT and Kauffman homology for thin knots. Also, we investigate the effects of differentials on super- $A$ -polynomials of SU and SO-type. And finally, we study the  $\text{SO}(N)$  quantum  $6j$ -symbols.

#### 3.1 Isomorphism between colored HOMFLY-PT and Kauffman homology for thin knots

It has been apparent that the rich structural properties become manifest if we introduce quadruple-gradings to HOMFLY-PT and Kauffman homology. The quadruply-graded HOMFLY-PT homology  $(\mathcal{H}_R^{\text{HOMFLY-PT}}(K))_{i,j,k,l}$  with  $(a, q, t_r, t_c)$ -gradings was introduced in [13]. With this grading, we can associate a  $\delta$ -grading to every generator  $x$  of the  $[r]$ -colored HOMFLY-PT homology defined as,

$$\delta(x) := a(x) + \frac{q(x)}{2} - \frac{t_r(x) + t_c(x)}{2}.$$

A knot  $K$  is called homologically-thin if all generators of  $\mathcal{H}_{[r]}^{\text{HOMFLY-PT}}(K)$  have the same  $\delta$ -grading equal to  $\frac{r}{2}S(K)$ , where  $S(K)$  is the Rasmussen  $s$ -invariant of  $K$  [76]. Otherwise, it is called homologically-thick. In general, thick knots possess more intricate structures [10, 12].

Similarly, the structural properties of quadruply-graded Kauffman homology  $(\mathcal{H}_R^{\text{Kauffman}}(K))_{i,j,k,l}$  were studied in [14]. In the following discussion in this section, we also use the tilde versions,  $\widetilde{\mathcal{H}}_{[r^s]}^{\text{HOMFLY-PT}}(K)$  [13],  $\widetilde{\mathcal{H}}_{[r^s]}^{\text{Kauffman}}(K)$  [14], and their corresponding Poincaré polynomials

$$\widetilde{\mathcal{P}}_{[r^s]}(K; a, Q, t_r, t_c) := \sum_{i,j,k,l} a^i Q^j t_r^k t_c^l \dim \left( \widetilde{\mathcal{H}}_{[r^s]}^{\text{HOMFLY-PT}}(K) \right)_{i,j,k,l}, \quad (3.1)$$

$$\widetilde{\mathcal{F}}_{[r^s]}(K; a, Q, t_r, t_c) := \sum_{i,j,k,l} a^i Q^j t_r^k t_c^l \dim \left( \widetilde{\mathcal{H}}_{[r^s]}^{\text{Kauffman}}(K) \right)_{i,j,k,l}, \quad (3.2)$$

where the gradings of  $\mathcal{H}$  and  $\widetilde{\mathcal{H}}$  are related by

$$\mathcal{F}_{[r^s]}(K; a, q, t_r, t_c) := \widetilde{\mathcal{F}}_{[r^s]}(K; a, q^s, t_r q^{-1}, t_c q), \quad (3.3)$$

$$\mathcal{P}_{[r^s]}(K; a, q, t_r, t_c) := \widetilde{\mathcal{P}}_{[r^s]}(K; a, q^s, t_r q^{-1}, t_c q). \quad (3.4)$$

It turns out that there are differentials, called universal and diagonal in [14], that relate  $[r]$ -colored Kauffman and HOMFLY-PT homology. Furthermore, the well-known isomorphism of the representations [14]

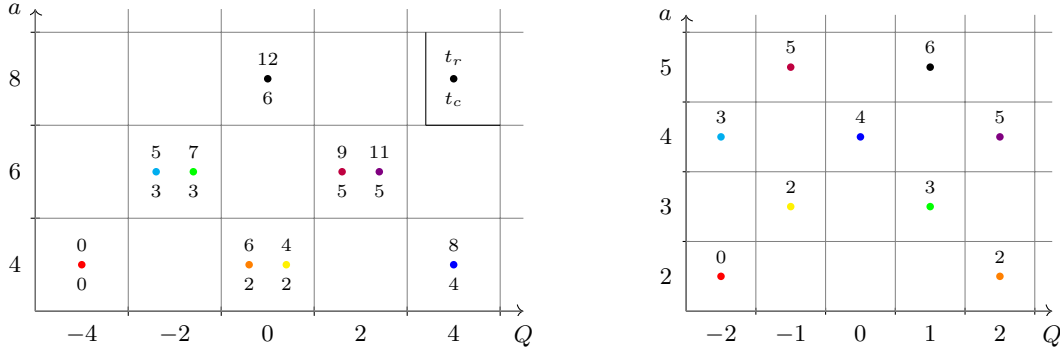
$$(\mathfrak{so}_6, [r]) \simeq (\mathfrak{sl}_4, [r^2]),$$

leads to an isomorphism between bi-graded homologies

$$\mathcal{H}_{\mathfrak{so}_6, [r]}(K) \cong \mathcal{H}_{\mathfrak{sl}_4, [r^2]}(K). \quad (3.5)$$

For a thin knot  $K_{\text{thin}}$ , this can be stated in terms of the Poincaré polynomials [14, (4.57)]

$$\widetilde{\mathcal{F}}_{[r]}(K_{\text{thin}}; a = q^5, Q = q, t_r = q^{-1}, t_c = qt) = \widetilde{\mathcal{P}}_{[r^2]}(K_{\text{thin}}; a = q^4, Q = q^2, t_r = q^{-1}, t_c = qt). \quad (3.6)$$



**Figure 5:** An isomorphism between  $[1^2]$ -colored HOMFLY-PT homology (left) and uncolored Kauffman homology (right) for the trefoil. Generators with the same color are identified under the grading change (3.7).

Moreover, we find an isomorphism between  $[1^2]$ -colored HOMFLY-PT homology and the uncolored Kauffman homology of the trefoil and the figure-eight. For example, there is a one-to-one correspondence between their generators with the same color, shown in Figure 5. This motivates us to uplift (3.5) to an isomorphism between HOMFLY-PT and Kauffman homologies in the case of a thin knot  $K_{\text{thin}}$  with grading change

$$\left( \widetilde{\mathcal{H}}_{[r]}^{\text{Kauffman}}(K_{\text{thin}}) \right)_{i,j,k,l} \cong \left( \widetilde{\mathcal{H}}_{[r^2]}^{\text{HOMFLY-PT}}(K_{\text{thin}}) \right)_{\frac{3}{2}i + \frac{3}{4}j - \frac{1}{2}k, -\frac{5}{2}i - \frac{5}{4}j + \frac{3}{2}k, i + \frac{1}{2}j, l}, \quad (3.7)$$

up to some overall grading shifts proportional to  $\frac{rS(K)}{2}$  in  $(a, Q, t_r)$  degrees. This can be expressed in terms of the Poincaré polynomials

$$\widetilde{\mathcal{F}}_{[r]}(K_{\text{thin}}; a, Q, t_r, t_c) = \left( \frac{Q^3}{at_r^2} \right)^{\frac{rS(K)}{2}} \widetilde{\mathcal{P}}_{[r^2]}(K_{\text{thin}}; a^{\frac{3}{2}} Q^{-\frac{5}{2}} t_r, a^{\frac{3}{4}} Q^{-\frac{5}{4}} t_r^{\frac{1}{2}}, a^{-\frac{1}{2}} Q^{\frac{3}{2}}, t_c) . \quad (3.8)$$

We would like to emphasize that the isomorphism holds only for thin knots, but it is not true for thick knots. For thick knots, the dimensions of  $[r^2]$ -colored quadruply-graded HOMFLY-PT homology and  $[r]$ -colored quadruply-graded Kauffman homology are different [12, Appendix B].

Remark that the relation (3.8) only provides the method to obtain the Poincaré polynomial of  $[r]$ -colored Kauffman homology from that of  $[r^2]$ -colored HOMFLY-PT homology for a thin knot, but it does not allow us to use this relation in the opposite direction since the grading shifts of variables in (3.8) are not linearly independent.

In [17] and in the previous section, the Poincaré polynomial of quadruply-graded HOMFLY-PT homology of the double twist knot  $K_{m,n}$  colored by rectangular Young tableau  $[r^s]$  has been given. Since the double twist knots are known to be homologically-thin, we can obtain the expression of the Poincaré polynomial of  $[r]$ -colored Kauffman homology for the double-twist knots thanks to the relation (3.8). To verify the relations (3.7) and (3.8), we have checked that the colored Kauffman homology obtained by this substitution satisfies all the differentials and properties [14] of the colored Kauffman homology for double twist knots.

### 3.2 Differential on super- $A$ -polynomials of SO-type

The study of the large color behaviors of colored Jones polynomials led us to the volume conjecture [20, 21]. The generalization of the volume conjecture to the large color behavior of the Poincaré polynomial led to the so-called super- $A$ -polynomial [14, 26]. As briefly mentioned above, there are the differentials that relate  $[r]$ -colored Kauffman homology and HOMFLY-PT homology. In this subsection, we will investigate the effect of the differential on super- $A$ -polynomials.

To this end, let us first recall the generalized volume conjecture [21, 22] and the super- $A$ -polynomials [26]. Here we use Poincaré polynomials of triply-graded homology  $\mathcal{Q}_{[r]}(K; a, q, t)$  only with  $t_r$ -grading, also called the superpolynomial, defined as

$$\mathcal{Q}_{[r]}(K; a, q, t) := \mathcal{Q}_{[r]}(K; a, q, t_r = t, t_c = 1) , \quad (3.9)$$

where  $\mathcal{Q} = \mathcal{P}$  or  $\mathcal{F}$  represents Poincaré polynomials of either HOMFLY-PT or Kauffman homology. If a knot  $K$  satisfies the exponential growth property [12, 77] (such as thin knots and torus knots)

$$\mathcal{Q}_{[r]}(K; a, q = 1, t) = \left[ \mathcal{Q}_{[1]}(K; a, q = 1, t) \right]^r ,$$

it is conjectured [80, Conjecture 4.4] that the superpolynomial colored by a symmetric representation can be expressed as

$$\mathcal{Q}_{[r]}(K; a, q, t) = \sum_{l_1 + l_2 + \dots + l_n = r} \left[ \begin{matrix} r \\ l_1, l_2, \dots, l_n \end{matrix} \right]_{q^2} a^{\sum_{i=1}^n a_i l_i} t^{\sum_{i=1}^n t_i l_i} q^{\sum_{i,j=1}^n Q_{i,j} l_i l_j + \sum_{i=1}^n q_i l_i} , \quad (3.10)$$

where  $(a_i, q_i, t_i)$  are constants and  $Q_{i,j}$  is a quadratic form. Here the  $q$ -multinomial is defined by

$$\left[ \begin{matrix} r \\ l_1, l_2, \dots, l_n \end{matrix} \right]_q := \frac{(q; q)_r}{(q; q)_{l_1} (q; q)_{l_2} \dots (q; q)_{l_n}} .$$

In the large color limit

$$q = e^{\hbar} \rightarrow 1, \quad r \rightarrow \infty, \quad a = \text{fixed}, \quad t = \text{fixed}, \quad x = q^{2r} = \text{fixed}, \quad (3.11)$$

a superpolynomial asymptotes to the form

$$\mathcal{Q}_{[r]}(K; a, q, t) \underset{x=q^{2r}}{r \rightarrow \infty, \hbar \rightarrow 0} \exp \left( \frac{1}{2\hbar} \int \log y \frac{dx}{x} + \dots \right), \quad (3.12)$$

where the integral is carried out on the zero locus of classical super-A-polynomial

$$\mathcal{A}(K; x, y, a, t) = 0.$$

The ellipsis in (3.12) represents the subleading terms that possess regular behaviors under the limit  $\hbar \rightarrow 0$ .

If  $\mathcal{Q}_{[r]}(K; a, q, t)$  is written as a summation only over  $n$  variables  $l_1, \dots, l_n$  as in (3.10), then its behavior under the large color limit could be approximated as

$$\mathcal{Q}_{[r]}(K; a, q, t) \sim \int e^{\frac{1}{2\hbar}(\widetilde{\mathcal{W}}(K; z_1, \dots, z_n; x, a, t) + \mathcal{O}(\hbar))} dx \prod_{i=1}^n dz_i, \quad (3.13)$$

where  $z_i = q^{2l_i}$ ,  $x = q^{2r}$ . The leading asymptotic behavior (3.12) with respect to  $\hbar$  comes from the saddle points

$$\exp \left( z_i \frac{\partial \widetilde{\mathcal{W}}(K; z_1, \dots, z_n; x, a, t)}{\partial z_i} \right) \Big|_{z_i = z_i^*} = 1. \quad (3.14)$$

and the zero locus of the classical super-A-polynomial is determined by

$$\exp \left( x \frac{\partial \widetilde{\mathcal{W}}(K; z_1^*, \dots, z_n^*; x, a, t)}{\partial x} \right) = y. \quad (3.15)$$

Now let us consider the relationship between super-A-polynomials of SU-type and SO-type that comes from the universal differential. According to [14, §4.5], the universal differential  $d_{\rightarrow}^{\text{univ}}$  has  $(a, q, t)$ -degree (0,2,1), which relates the  $[r]$ -colored Kauffman homology and the  $[r]$ -colored HOMFLY-PT homology as

$$\mathcal{F}_{[r]}(K; a, q, t) = q^{rS(K)} \mathcal{P}_{[r]}(K; aq^{-1}, q, t) + (1 + q^2 t) f(a, q, t). \quad (3.16)$$

This implies

$$\mathcal{F}_{[r]}(K; a, q, -q^{-2}) = q^{rS(K)} \mathcal{P}_{[r]}(K; aq^{-1}, q, -q^{-2}). \quad (3.17)$$

For a knot satisfying the exponential growth property, the superpolynomial admits an expression (3.10). Therefore, it is straightforward to verify that the leading order of the large color asymptotic behavior of the right hand side of (3.17) is the same as that of the colored HOMFLY-PT polynomial although the change of variables  $a \rightarrow aq^{-1}, t = -q^{-2}$  is non-trivial. As a result, we have the large color asymptotic behavior

$$\mathcal{F}_{[r]}(K; a, q, -q^{-2}) \sim \int e^{\frac{1}{2\hbar}(\widetilde{\mathcal{W}}^{\text{SU}}(K; z_i; x, a) + \mathcal{O}(\hbar))} dx dz_i. \quad (3.18)$$

Thus, it encodes the information about the  $a$ -deformed  $A$ -polynomial  $A^{\text{SU}}(K; x, y, a)$  of SU-type. On the other hand, the decategorification limit  $t = -1$  is just the colored Kauffman polynomial  $\mathcal{F}_{[r]}(K; a, q, -1)$  and its large color limit yields  $a$ -deformed  $A$ -polynomial  $A^{\text{SO}}(K; x, y, a)$  of SO-type. Therefore, the super-A-polynomial  $\mathcal{A}^{\text{SO}}(K; x, y, a, t)$  of SO-type contains both  $A^{\text{SO}}(K; x, y, a)$  and  $A^{\text{SU}}(K; x, y, a)$  at the  $t = -1$  specialization.

**Conjecture 3.1.** *For a knot satisfying the exponential growth property, the zero locus of the super- $A$ -polynomial of SO-type at  $t = -1$*

$$\mathcal{A}^{SO}(K; x, y, a, t = -1) = 0$$

includes two branches  $A^{SO}(K; x, y, a) = 0$  and  $A^{SU}(K; x, y, a) = 0$ .

Let us take the trefoil as an example in which the super- $A$ -polynomial of SO-type can be calculated. The colored superpolynomial of the trefoil [14, (5.8)] and the corresponding twisted superpotential [14, (6.15)] can be written as

$$\begin{aligned} \mathcal{F}_{[r]}(\mathbf{3}_1; a, q, t) &= \sum_{k=0}^r \sum_{j=0}^k \sum_{i=0}^{r-k} a^{i-k+3r} q^{3k-2j(1+r)+r(2r-3)+i(2j+2r-1)} t^{2(i-j+r)} \begin{bmatrix} r \\ k \end{bmatrix}_{q^2} \begin{bmatrix} k \\ j \end{bmatrix}_{q^2} \begin{bmatrix} r-k \\ i \end{bmatrix}_{q^2} \\ &\quad \times (-a^2 t^3 q^{2r-2}; q^2)_j (-q^2 t; q^2)_{r-k} (-a q^{-1} t; q^2)_i . \end{aligned} \quad (3.19)$$

and

$$\begin{aligned} \widetilde{\mathcal{W}}^{SO}(\mathbf{3}_1; x, w, v, z, a, t) &= \log \left( \frac{wx^3}{z} \right) \log a - \log v \log x + (\log x)^2 + \log w \log(vx) \\ &\quad + 2 \log \left( \frac{wx}{v} \right) \log t - \frac{3\pi^2}{6} - \text{Li}_2(x) + \text{Li}_2(v) + \text{Li}_2(zv^{-1}) \\ &\quad + \text{Li}_2(w) + \text{Li}_2(xz^{-1}w^{-1}) + \text{Li}_2(-a^2 t^3 x) - \text{Li}_2(-a^2 t^3 xv) \\ &\quad + \text{Li}_2(-t) - \text{Li}_2(-txz^{-1}) + \text{Li}_2(-at) - \text{Li}_2(-atw) , \end{aligned} \quad (3.20)$$

where we introduce the variables as  $w = q^{2i}$ ,  $v = q^{2j}$  and  $z = q^{2k}$ . Obviously, the  $t = -1$  specialization leads to the large color limit of colored Kauffman polynomial

$$\widetilde{\mathcal{W}}^{SO}(\mathbf{3}_1; x, w, v, z, a, t = -1) = \widetilde{\mathcal{W}}^{SO}(\mathbf{3}_1; x, w, v, z, a) . \quad (3.21)$$

On the other hand, at the  $t = -q^{-2}$  specialization, the expression (3.19) collapses to  $k = r$  and  $i = 0$  due to the term  $(-q^2 t; q^2)_{r-k}$  in the summand so that it is written by the summation only over  $j$ . Correspondingly, the twisted superpotential encodes the large color limit of colored HOMFLY-PT polynomial

$$\widetilde{\mathcal{W}}^{SO}(\mathbf{3}_1; x, w = 1, v, z = x, a, t = -1) = \widetilde{\mathcal{W}}^{SU}(\mathbf{3}_1; x, v, a) . \quad (3.22)$$

The saddle point equations (3.13) and (3.14) yield the classical super- $A$ -polynomial of SO-type of the trefoil [14, (6.19)], and its  $t = 1$  specialization can be consequently written as

$$\mathcal{A}^{SO}(\mathbf{3}_1; x, y, a, t = -1) = (1 - ax) A^{SO}(\mathbf{3}_1; x, y, a) A^{SU}(\mathbf{3}_1; x, y, a) , \quad (3.23)$$

where

$$\begin{aligned} A^{SO}(\mathbf{3}_1; x, y, a) &= a^7 x^7 - a^7 x^6 - a^2 xy + y , \\ A^{SU}(\mathbf{3}_1; x, y, a) &= y^2 - ya^2 + xya^2 - 2x^2 ya^2 - xy^2 a^2 - x^3 a^4 + x^4 a^4 + 2x^2 ya^4 + x^3 ya^4 - x^4 ya^6 . \end{aligned} \quad (3.24)$$

(For  $A^{SU}(\mathbf{3}_1; x, y, a)$ , see [26, (2.25)] with  $a \rightarrow a^2$ .) It is evident that it satisfies Conjecture 3.1.

In principle, we can compute super- $A$ -polynomials of SO-type for the double twist knots using the result in §3.1. However, it is beyond the computational capabilities of current desktops to solve the saddle point equations (3.13) and (3.14) for them. However, since the colored Kauffman homology of

a double twist knot is endowed with the universal differential  $d_{\rightarrow}^{\text{univ}}$ , we can easily check the relations corresponding to (3.21) and (3.22) at the level of the twisted superpotentials for the double twist knots. In this way, we can confirm Conjecture 3.1 for the double twist knots.

Note that the diagonal differentials [14, §4.6] also relate colored Kauffman and HOMFLY-PT homology. However, they do not bring us any interesting results once we take large color limits.

### 3.3 $SO(N)$ quantum $6j$ -symbols

$6j$ -symbols are ubiquitous in physics and mathematics [32, 81–84], and they reveal very rich symmetries. In particular, quantum knot invariants can be evaluated by using quantum  $6j$ -symbols [3, 27, 35]. Despite their importance and versatility, it is difficult to evaluate them in higher ranks. Obtaining a closed-form expression is moreover indispensable to study their symmetries as in [32, 38].

Kauffman polynomials (homology) colored by symmetric representations for non-torus knots obtained in §3.1 in principle encode the information about  $SO(N)$  quantum  $6j$ -symbols. Hence, we attempt to extract the information in this section. First, we quantize  $SO(N)$   $6j$ -symbols with all symmetric representations obtained by Ališauskas [39, 40]. Next, we give some evidence to its validity by using representation theory and computing Kauffman polynomials.

#### 3.3.1 $SO(N)$ quantum $6j$ -symbols for symmetric representations

Multiplicity-free  $6j$ -symbols of a Lie group  $G$  take the form

$$\left\{ \begin{matrix} R_1 & R_2 & R_{12} \\ R_3 & R_4 & R_{23} \end{matrix} \right\}_G,$$

where the arguments are representations of  $G$ , and they obey the fusion rule,  $R_{12} \in (R_1 \otimes R_2) \cap (\bar{R}_3 \otimes \bar{R}_4)$ ,  $R_{23} \in (R_2 \otimes R_3) \cap (\bar{R}_1 \otimes \bar{R}_4)$ .

Aliauskas has obtained the closed-form expression for classical  $SO(N)$   $6j$ -symbols with all representation symmetric. Kirillov and Reshetikhin naturally quantized the classical  $6j$ -symbols of  $SU(2)$ . Therefore, we conjecture that we can naturally quantize Aliauskas' results to a closed-form expression of quantum  $SO(N)$  ( $N \geq 4$ )  $6j$ -symbols for all symmetric representations,

$$\begin{aligned} \left\{ \begin{matrix} a & b & e \\ d & c & f \end{matrix} \right\}_{SO(N)} &= \left( \frac{[2c + N - 2]_q [2d + N - 2]_q [2e + N - 2]_q}{[2]_q^3 \dim_q[c] \dim_q[d] \dim_q[e]} \right)^{1/2} \\ &\times \left( \begin{matrix} c & d & e \\ 0 & 0 & 0 \end{matrix} \right)_N^{-1} \sum_{l'} (-1)^{(c+d-e)/2+f+N+l'} [2l' + N - 3]_q \\ &\times \left\{ \begin{matrix} \frac{1}{2}b & \frac{1}{2}f + \frac{1}{4}N - 1 & \frac{1}{2}d + \frac{1}{4}N - 1 \\ \frac{1}{2}f + \frac{1}{4}N - 1 & \frac{1}{2}(b + N) - 2 & l' + \frac{1}{2}N - 2 \end{matrix} \right\}_{SU(2)} \\ &\times \left\{ \begin{matrix} \frac{1}{2}a & \frac{1}{2}f + \frac{1}{4}N - 1 & \frac{1}{2}c + \frac{1}{4}N - 1 \\ \frac{1}{2}f + \frac{1}{4}N - 1 & \frac{1}{2}(a + N) - 2 & l' + \frac{1}{2}N - 2 \end{matrix} \right\}_{SU(2)} \\ &\times \left\{ \begin{matrix} \frac{1}{2}a & \frac{1}{2}b + \frac{1}{4}N - 1 & \frac{1}{2}e + \frac{1}{4}N - 1 \\ \frac{1}{2}b + \frac{1}{4}N - 1 & \frac{1}{2}(a + N) - 2 & l' + \frac{1}{2}N - 2 \end{matrix} \right\}_{SU(2)} \\ &\times \left( \frac{[l']_q! [N - 3]_q!}{[l' + N - 4]_q!} \right)^{1/2}, \end{aligned} \tag{3.25}$$

where the curly braces on the right hand side are the quantum  $6j$ -symbols of  $SU(2)$ . Since we are only dealing with all symmetric representations,  $a, b, c, d, e, f$  are the number of boxes in the corresponding Young diagram of one row. Here a quantum number is defined as

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}. \quad (3.26)$$

Also, a  $q$ -factorial is defined as the product of quantum numbers

$$[x]_q! = [x]_q [x-1]_q \cdots [x - \lfloor x \rfloor]_q, \quad (3.27)$$

where  $\lfloor x \rfloor$  is the floor of  $x$  so that the definition of  $q$ -factorial works for both integer and half-integer arguments. The quantum dimension of the  $SO(N)$  symmetric representation  $[l]$  is given by

$$\dim_q[l] = \frac{[l + N - 3]_q!}{[l]_q! [N - 2]_q!} ([l + N - 2]_q + [l]_q), \quad (3.28)$$

and the special quantum  $3j$ -symbol are defined as

$$\begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix} = \frac{1}{[N/2 - 1]_q!} \left( \frac{[J + N - 3]_q!}{[N - 3]_q! [J + N/2 - 1]_q!} \right. \quad (3.29)$$

$$\left. \times \prod_{i=1}^3 \frac{[2l_i + N - 2]_q [J - l_i + N/2 - 2]_q!}{[2]_q \dim_q[l_i] [J - l_i]_q!} \right)^{1/2}, \quad (3.30)$$

where  $J = \frac{1}{2}(l_1 + l_2 + l_3)$ .

Note that  $R_{12}, R_{23}$  can be non-symmetric representations even when  $R_1 = R_2 = R_3 = R_4$  are symmetric representations. (Some fusion rules can be found in (3.35).) Furthermore, there are several relations from representation theory, which would lead to identities among the quantum  $6j$ -symbols.

- The isomorphism of representations,  $(\mathfrak{so}_6, [r]) \simeq (\mathfrak{sl}_4, [r, r])$ , would lead to equations,

$$\left\{ \begin{matrix} [r] & [r] & [2l - j, j] \\ [r] & [r] & [2l' - j', j'] \end{matrix} \right\}_{SO(6)} = \left\{ \begin{matrix} [r, r] & [r, r] & [2l, 2l - j, j] \\ [r, r] & [r, r] & [2l', 2l' - j', j'] \end{matrix} \right\}_{SU(4)}, \quad (3.31)$$

where  $0 \leq l, l' \leq r, 0 \leq j \leq l, 0 \leq j' \leq l'$ . We have checked that this property holds using  $SU(4)$  quantum  $6j$ -symbols in [85], and the result in the next subsection. Since now we only have the closed form expression of  $SO(N)$  quantum  $6j$ -symbols for symmetric representations, by setting  $j = j' = 0$ , the above equation reduces to

$$\left\{ \begin{matrix} [r] & [r] & [2l] \\ [r] & [r] & [2l'] \end{matrix} \right\}_{SO(6)} = \left\{ \begin{matrix} [r, r] & [r, r] & [2l, 2l] \\ [r, r] & [r, r] & [2l', 2l'] \end{matrix} \right\}_{SU(4)}, \quad (3.32)$$

where  $0 \leq l \leq r$ . Thus we can easily obtain  $SU(4)$  quantum  $6j$ -symbols of such type, using our closed formula (3.25).

- Also, there is an isomorphism between Lie algebras,  $\mathfrak{so}_4 \simeq \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ . Restricting ourselves to the case where all representations are symmetric, we have

$$\left\{ \begin{matrix} [r] & [r] & [2l] \\ [r] & [r] & [2l'] \end{matrix} \right\}_{SO(4)} = \left( \left\{ \begin{matrix} [r] & [r] & [2l] \\ [r] & [r] & [2l'] \end{matrix} \right\}_{SU(2)} \right)^2, \quad (3.33)$$

where  $0 \leq l, l' \leq r$ .

- Although our formula only works for  $\text{SO}(N)$  ( $N \geq 4$ ), it's still worthy to note that there is another isomorphism of representations,  $(\mathfrak{so}_3, [r]) \simeq (\mathfrak{sl}_2, [2r])$ . This leads to

$$\left\{ \begin{matrix} [r] & [r] & [2l] \\ [r] & [r] & [2l'] \end{matrix} \right\}_{\text{SO}(3)} \Big|_{q \rightarrow q^2} = \left\{ \begin{matrix} [2r] & [2r] & [4l] \\ [2r] & [2r] & [4l'] \end{matrix} \right\}_{\text{SU}(2)}, \quad (3.34)$$

where  $0 \leq l, l' \leq r$ .

### 3.3.2 $\text{SO}(N)$ fusion matrices for symmetric representations

Since the fusion rule of  $\text{SO}(N)$  symmetric representations is given by

$$[r] \otimes [r] = \bigoplus_{l=0}^r \bigoplus_{j=0}^l [2l - j, j], \quad (3.35)$$

we need full  $\text{SO}(N)$  fusion matrices including the case that  $R_{12}$  and  $R_{23}$  are non-symmetric to evaluate Kauffman polynomials colored by symmetric representations. At a root of unity  $q$ , the WZNW fusion matrix and the corresponding quantum  $6j$ -symbol are related by

$$a_{R_{12}R_{23}} \begin{bmatrix} R_1 & R_2 \\ R_3 & R_4 \end{bmatrix} = \epsilon_{\{R_i\}} \sqrt{\dim_q R_{12} \dim_q R_{23}} \begin{bmatrix} R_1 & R_2 & R_{12} \\ R_3 & R_4 & R_{23} \end{bmatrix}, \quad (3.36)$$

where  $\epsilon_{\{R_i\}} = \pm 1$  and  $\dim_q R$  is the quantum dimension of the representation  $R$ . Since  $\text{SO}(N)$  representations are real, for colored knot invariants, we only need to consider fusion matrix of the type  $a_{ts} \begin{bmatrix} R & R \\ R & R \end{bmatrix}$ , where  $t, s \in R \otimes R$ .

To obtain full fusion matrices, we can make use of their properties and symmetries [32]

- The fusion matrix is symmetric,

$$a_{ts} \begin{bmatrix} R & R \\ R & R \end{bmatrix} = a_{st} \begin{bmatrix} R & R \\ R & R \end{bmatrix}. \quad (3.37)$$

- The fusion matrix is orthonormal,

$$\sum_s a_{ts} \begin{bmatrix} R & R \\ R & R \end{bmatrix} a_{sm} \begin{bmatrix} R & R \\ R & R \end{bmatrix} = \delta_{tm}. \quad (3.38)$$

- The entries in the row and column, which corresponds to the trivial representation, are known,

$$a_{\emptyset s} \begin{bmatrix} R & R \\ R & R \end{bmatrix} = \varepsilon_s^R \frac{\sqrt{\dim_q s}}{\dim_q R}, \quad (3.39)$$

where  $\emptyset$  is the trivial representation.

Note that  $6j$ -symbols enjoy other symmetries, such as the Racah identity and the pentagon (Biedenharn-Elliott) identity, which we do not use in this paper. We already know the entries when  $t$  and  $s$  are symmetric representations (3.25), and the entries corresponding to the trivial representation (3.39). Then we can solve the rest entries by using the fact that the fusion matrix is symmetric and orthonormal. In this paper, we will present our results of the  $\text{SO}(N)$  fusion matrices  $a_{ts} \begin{bmatrix} R & R \\ R & R \end{bmatrix}$  for  $R = \square, \square\square$ .

1.  $R = \square$

$$\left\{ \begin{matrix} R & R & t \\ R & R & s \end{matrix} \right\} = \frac{1}{\dim_q R} \left( \begin{array}{c|ccc} & s = \emptyset & \square & \square\square \\ \hline t = \emptyset & 1 & 1 & 1 \\ \square & 1 & \frac{[N]_q}{[N-1]_q([N-2]_q+[2]_q)} & -\frac{1}{[N-1]_q} \\ \square\square & 1 & -\frac{1}{[N-1]_q} & \frac{[N-2]_q}{[N-1]_q([N]_q+[2]_q)} \end{array} \right),$$

where the quantum dimension  $\dim_q R = [N-1]_q + 1$ .

2.  $R = \square\square$

$$\left\{ \begin{matrix} R & R & t \\ R & R & s \end{matrix} \right\} = \frac{1}{\dim_q R} \left( \begin{array}{c|cccccc} & s = \emptyset & \square & \square\square & \square\square & \square\square\square & \square\square\square\square \\ \hline t = \emptyset & 1 & 1 & 1 & 1 & 1 & 1 \\ \square & 1 & b_{11} & b_{12} & b_{13} & b_{14} & b_{15} \\ \square\square & 1 & b_{21} & b_{22} & b_{23} & b_{24} & b_{25} \\ \square\square & 1 & b_{31} & b_{32} & b_{33} & b_{34} & b_{35} \\ \square\square\square & 1 & b_{41} & b_{42} & b_{43} & b_{44} & b_{45} \\ \square\square\square\square & 1 & b_{51} & b_{52} & b_{53} & b_{54} & b_{55} \end{array} \right),$$

where the quantum dimension  $\dim_q R = [N-1]_q + [N-1]_q[N]_q/[2]_q$ , and

$$\begin{aligned} b_{11} &= \frac{-q^4 - q^N - q^{2+N} + q^{4+N} - q^{2+2N} + q^{4+2N} + q^{6+2N} + q^{2+3N}}{(1+q^2)(-1+q^N)(q^4+q^N+q^{2N}+q^{4+N})}, \\ b_{12} &= b_{21} = \frac{q^{N+2} + 1}{q^{N+2} + q^N + q^2 + 1}, \\ b_{13} &= b_{31} = \frac{q^N (q^2 - 1) (q^{N+2} + 1)}{q^{3N} + q^{2N+4} - q^N - q^4}, \\ b_{14} &= b_{41} = \frac{(q^2 - 1)^2 q^N}{q^{2N} + q^{N+4} + q^N + q^4}, \\ b_{15} &= b_{51} = \frac{(q^4 - 1) q^{N-2}}{1 - q^{2N}}, \\ b_{22} &= \frac{-q^{2N} + q^{N+2} + 3q^{N+4} - 2q^{N+6} - q^{N+8} - 2q^{2N+2} + 3q^{2N+4} + q^{2N+6} - 2q^{2N+8} + q^{3N+6} - 2q^N + q^2}{(q^2 + 1)(q^N + 1)(q^N - q^2)(q^{N+4} - 1)}, \\ b_{23} &= b_{32} = -\frac{(q^2 - 1) q^N}{(q^N + 1)(q^N - q^2)}, \\ b_{24} &= b_{42} = \frac{q^N (q^2 - 1)^2 (q^{N+2} + 1)}{(q^2 - q^N)(q^{N+4} + q^{2N+4} - q^N - 1)}, \\ b_{25} &= b_{52} = \frac{(q^4 - 1) q^N}{q^{N+4} + q^{2N+4} - q^N - 1}, \\ b_{33} &= \frac{q^{2N} (q^2 - 1)^2 (q^2 + 1) (q^{2N} - q^{N+2} + q^{N+6} - q^{2N+2} - 2q^{2N+4} - q^{2N+6} + q^{2N+8} + q^{3N+2} - q^{3N+6} + q^{4N+4} + q^4)}{(q^N - 1)(q^N + 1)(q^N - q^2)(q^N + q^4)(q^{2N} - q^6)(q^{2N+2} - 1)}, \\ b_{34} &= b_{43} = -\frac{q^{2N} (q^2 - 1)^2 (q^2 + 1) (q^{N+2} + 1)}{(q^N + 1)(q^{2N} - q^{N+2} + q^{N+4} - q^6)(q^{2N+2} - 1)}, \end{aligned}$$

$$\begin{aligned}
b_{35} = b_{53} &= \frac{(q^2 - 1)^2 (q^2 + 1) q^{2N-2}}{(q^{2N} - 1) (q^{2N+2} - 1)}, \\
b_{44} &= \frac{q^{2N} (q^2 - 1)^2 (q^2 + 1) (2q^{N+2} - 2q^{N+4} + q^{N+6} + q^{2N+4} - q^N - q^2)}{(q^{2N} - q^{N+2} + q^{N+4} - q^6) (q^{2N+2} - 1) (q^{N+4} + q^{2N+4} - q^N - 1)}, \\
b_{45} = b_{54} &= -\frac{(q^2 - 1)^2 (q^2 + 1) q^{2N}}{(q^{2N+2} - 1) (q^{N+4} + q^{2N+4} - q^N - 1)}, \\
b_{55} &= \frac{q^{2N+2} (q^2 - 1)^2 (q^2 + 1) (q^N - q^2)}{(q^{N+6} - 1) (q^{2N+2} - 1) (q^{N+4} + q^{2N+4} - q^N - 1)}.
\end{aligned}$$

Using these two  $\text{SO}(N)$  fusion matrices and braiding operator eigenvalues, we have computed colored Kauffman polynomials for double twist knots when  $R = \square, \square\square$ . The results coincide with  $\mathcal{F}_{[r]}(K; a = q^{N-1}, q, t = -1)$  obtained in §3.1, which comes from the Poincaré polynomial of  $[r^2]$ -colored HOMFLY-PT homology (2.28) by using the relation (3.8).

## 4 Deformed $F_K$ and ADO polynomials

In the study of the 3d/3d correspondence, a new invariant  $\hat{Z}$  of a closed 3-manifold was proposed as a BPS  $q$ -series [41, 42]. To incorporate TQFT methods and surgery formula, an analogous series  $F_K$  for a knot complement  $S^3 \setminus K$  [43] was introduced. Although a mathematical definition of  $\hat{Z}$  or  $F_K$  has not been fleshed out yet, there are many approaches to understand them. A recipe to compute  $\hat{Z}$  for a plumbed 3-manifold with a negative definite linking matrix is given in [42]. We can also use resurgence to calculate  $\hat{Z}$  or  $F_K$  [43, 86]. Also, for a positive braid knot  $K$ ,  $F_K(x, q)$  can be computed using the  $R$ -matrix for Verma modules [87]. In particular, closed-form expressions of  $F_K$  for the double twist knots with positive braids are given in [87], based on the formulae of colored Jones polynomials [88, 89]. The expressions are compatible with Habiro's cyclotomic expansion [79]. In this section, we will generalize the results of [87] to the  $(a, t)$ -deformed case [54], and we will see by explicit computations that the ADO invariants [50–52] are compatible with the cyclotomic expansions.

### 4.1 Deformed $F_K$ of double twist knots

We study the double twist knots,  $K_{s,t}$  ( $s, t \in \frac{1}{2}\mathbb{Z}$ ), whose notation and the corresponding knots in Rolfsen table are shown in figure 6. Among them, two classes possess only positive braids.



**Figure 6:** Double twist knot  $K_{s,t}$  and corresponding knots in Rolfsen table.

**Double twist knot**  $K_{m,n}$  ( $m, n \in \mathbb{Z}_+$ )

The cyclotomic expansions of the  $[r]$ -colored superpolynomials for the double twist knots  $K_{m,n}$  ( $m, n \in \mathbb{Z}_+$ ) are given in [16, 62]. In fact, by substitution  $q^{2r} = x$  and changing the upper limit from  $r$  to  $\infty$ , we can obtain the corresponding  $(a, t)$ -deformed  $F_K$ :

$$F_{K_{m,n}}(x, a, q, t) = (-tx)^{\log_q(-at)-1} \sum_{k=0}^{\infty} g_{K_{m,n}}^{(k)}(x, a, q, t), \quad (4.1)$$

where

$$\begin{aligned} g_{K_{m,n}}^{(k)}(x, a, q, t) &= \sum_{j=0}^k \frac{(x; q^{-1})_k (-at/q; q)_k (-axt^3; q)_j}{(q; q)_k} \\ &\quad \times (-1)^j x^{k-j} a^{-j} q^k t^{-3j+2k} \begin{bmatrix} k \\ j \end{bmatrix}_q \text{Tw}_{K_{m,n}}^{(j)}(a, q, t). \end{aligned} \quad (4.2)$$

The twist factor is defined as

$$\text{Tw}_{K_{m,n}}^{(j)}(a, q, t) = \sum_{l=0}^j (-1)^l q^{\frac{1}{2}l(l+1)-jl} \text{tw}_m^{(l)}(a, q, t) \text{tw}_n^{(l)}(a, q, t) \begin{bmatrix} j \\ l \end{bmatrix}_q, \quad (4.3)$$

with

$$\text{tw}_m^{(l)}(a, q, t) = \sum_{0 \leq b_1 \leq \dots \leq b_{m-1} \leq b_m = l} \prod_{i=1}^{m-1} \left( a^{b_i} q^{b_i(b_i-1)} t^{2b_i} \right) \begin{bmatrix} b_{i+1} \\ b_i \end{bmatrix}_q. \quad (4.4)$$

Note that, to connect the standard notation of  $F_K$  and the ADO invariants, we rescale the variables  $a \rightarrow a^{1/2}$  and  $q \rightarrow q^{1/2}$  from the superpolynomial (3.9)

$$F_{K_{m,n}}(x = q^r, a, q, t) = \mathcal{P}_{[r]}(K_{m,n}; a^{1/2}, q^{1/2}, t).$$

When we write the  $(a, t)$ -deformed  $F_K$  as a four-variable series explicitly, we would usually omit the prefactor  $(-tx)^{\log_q(-at)-1}$ . However, we have to keep it to verify the properties (4.10), (4.11), (4.12) of  $(a, t)$ -deformed  $F_K$ . If we take  $a = -t^{-1}q^N$ , the prefactor simply reduces to  $(-tx)^{N-1}$ .

**Double twist knot**  $K_{m+\frac{1}{2}, -n}$  ( $m, n \in \mathbb{Z}_+$ )

Similarly, we can find the cyclotomic expansions of the  $[r]$ -colored superpolynomial for double twist knot  $K_{m+\frac{1}{2}, -n}$  ( $m, n \in \mathbb{Z}_+$ ) by using the structural properties of HOMFLY-PT homology. Then, we obtain the  $(a, t)$ -deformed  $F_K$  for it:

$$F_{K_{m+\frac{1}{2}, -n}}(x, a, q, t) = (-tx)^{\log_q(-at)-1} \sum_{k=0}^{\infty} g_{K_{m+\frac{1}{2}, -n}}^{(k)}(x, a, q, t), \quad (4.5)$$

where

$$\begin{aligned} g_{K_{m+\frac{1}{2}, -n}}^{(k)}(x, a, q, t) &= \sum_{j=0}^k \frac{(x; q^{-1})_k (-at/q; q)_k (-axt^3; q)_j}{(q; q)_k} \\ &\quad \times x^{k-j} \begin{bmatrix} k \\ j \end{bmatrix}_q q^{\frac{1}{2}j(j-1)+k} t^{-j+2k} \text{tw}_m^{(j)}(a, q, t) \text{tw}_n^{(k)}(x, q, t). \end{aligned} \quad (4.6)$$

A new twist factor is defined as

$$\mathbb{W}_n^{(k)}(x, q, t) = \sum_{0 \leq b_1 \leq \dots \leq b_{n-1} \leq b_n = k} \prod_{i=1}^{n-1} \left( x^{2b_i} q^{-b_i(b_{i+1}-1)} t^{2b_i} \right) \begin{bmatrix} b_{i+1} \\ b_i \end{bmatrix}_q. \quad (4.7)$$

The prefactor becomes simply  $(-tx^n)^{N-1}$  when taking  $a = -t^{-1}q^N$ .

### Properties of $(a, t)$ -deformed $F_K$

The formulae above for  $F_K$  of the double twist knots become the usual  $F_K$  in [87] up on the specialization  $a = q^2, t = -1$ . If we further take  $a = q^N, t = -1$ , it reduces to  $F_K^{\text{SU}(N), \text{sym}}(x, q)$  [53], which is the  $F_K$  with gauge group  $G = \text{SU}(N)$  associated with symmetric representations.

It is conjectured in [54] that the  $(a, t)$ -deformed  $F_K$  has the following property,

$$\hat{A}(K; \hat{x}, \hat{y}, a, q, t) F_K(x, a, q, t) = 0. \quad (4.8)$$

We have studied the classical super- $A$ -polynomial in §3.2, and  $\hat{A}(K; \hat{x}, \hat{y}, a, q, t)$  is its quantum version [62], which annihilates the colored superpolynomial of  $K$ . The operators in the quantum super- $A$ -polynomial act on  $(a, t)$ -deformed  $F_K$  as

$$\hat{x} F_K(x, a, q, t) = x F_K(x, a, q, t), \quad \hat{y} F_K(x, a, q, t) = F_K(xq, a, q, t). \quad (4.9)$$

In the case of  $3_1, 5_2, 7_2$ , we have checked by Mathematica that the conjectural  $(a, t)$ -deformed  $F_K$  can be annihilated by the corresponding super- $A$ -polynomial in [62].

It is proposed in [54] that  $(a, t)$ -deformed  $F_K$  satisfy the following properties:

$$F_K(x, -t^{-1}, q, t) = \Delta_K(x, t), \quad (4.10)$$

$$F_K(x, -t^{-1}q, q, t) = 1, \quad (4.11)$$

$$\lim_{q \rightarrow 1} F_K(x, -t^{-1}q^N, q, t) = \frac{1}{\Delta_K(x, t)^{N-1}}, \quad (4.12)$$

where  $\Delta_K(x, t)$  is the  $t$ -deformed Alexander polynomial introduced in [54], which can be obtained from the superpolynomial,

$$\Delta_K(x, t) = \mathcal{P}_{\square}(K; a^2 = -t^{-1}, q^2 = x, t). \quad (4.13)$$

We have checked that our closed formulae of the  $(a, t)$ -deformed  $F_K$  for the double twist knots satisfy (4.10). Also, we give an analytical proof of (4.11) and (4.12) in Appendix A.

### 4.2 $t$ -deformed ADO polynomials

As studied in [51], at the limit  $q \rightarrow \zeta_p = e^{\frac{2\pi i}{p}}$ ,  $F_K(x, q)$  is equal to the  $p$ -th ADO polynomial [52] up to the Alexander polynomial  $\Delta_K(x^p)$ . For  $(a, t)$ -deformed  $F_K$  at radial limit, it is natural to consider a  $t$ -deformation of the  $p$ -th ADO polynomial. The formal definition is given in [54] as,

$$\text{ADO}_K(p; x, t) = \Delta_K(x^p, -(-t)^p) \lim_{q \rightarrow \zeta_p} F_K(x, -t^{-1}q^2, q, t). \quad (4.14)$$

In this subsection, we derive closed-form expressions of  $t$ -deformed ADO polynomials of the double twist knots. As in Appendix A, it turns out that the closed formulae (4.1) and (4.5) of cyclotomic type are suitable to compute the ADO polynomial up on the limit  $q \rightarrow \zeta_p = e^{\frac{2\pi i}{p}}$  in which an infinite sum truncates to a finite one [90]. (See also [91] for a similar topic.)

$p$	$\text{ADO}_{3_1}(p; x, t)$
1	1
2	$(-tx) + t + (-tx)^{-1}$
3	$\zeta_3^2 \left[ (t\zeta_3 x)^2 + t(t\zeta_3 x) + (t^2 - \zeta_3^{-1}) + t(t\zeta_3 x)^{-1} + (t\zeta_3 x)^{-2} \right]$

**Table 1:**  $t$ -deformed ADO polynomials for the left-handed trefoil

First, we will obtain a closed-form expression of  $t$ -deformed Alexander polynomials. Although  $\Delta_K(x, t)$  can be easily calculated from the superpolynomial by using (4.13), we compute it from (4.1) and (4.5) via the substitution (4.10). Notice that both the formulae share a common term  $(-at/q; q)_k$ , where  $k$  runs from 0 to  $\infty$ . Upon specialization  $a = -t^{-1}$ , it becomes  $(q^{-1}; q)_k$  and vanishes for  $k > 1$ . Therefore, the infinite summation becomes a rather simple finite one. Subsequently, they are given by

$$\begin{aligned}\Delta_{K_{m,n}}(x, t) &= -\frac{1}{t} + \left( t + \frac{1}{t} - tx - \frac{1}{tx} \right) S_m(-t) S_n(-t), \\ \Delta_{K_{m+\frac{1}{2}, -n}}(x, t) &= t^n [n]_{tx} - t^{n-1} [n+1]_{tx} - t^n [n]_{tx} \left( t + \frac{1}{t} - tx - \frac{1}{tx} \right) S_m(-t),\end{aligned}\quad (4.15)$$

where  $S_l(x) = \sum_{i=0}^{l-1} x^i$ , and the definition of quantum number  $[n]_{tx}$  is given in (3.26). We present our formulae this way, so that the Weyl symmetry [54] of the  $t$ -deformed Alexander polynomial becomes manifest,

$$\Delta_K(x^{-1}, t) = \Delta_K(t^{-2}x, t). \quad (4.16)$$

After detailed derivation shown in Appendix A, we obtain closed-form expressions of the  $t$ -deformed ADO polynomials for double twist knots  $K_{m,n}$  and  $K_{m+\frac{1}{2}, -n}$ ,

$$\begin{aligned}\text{ADO}_{K_{m,n}}(p; x, t) &= (-tx)^{1-p} \sum_{k=0}^{p-1} \lim_{q \rightarrow \zeta_p} g_{K_{m,n}}^{(k)}(x, -t^{-1}q^2, q, t), \\ \text{ADO}_{K_{m+\frac{1}{2}, -n}}(p; x, t) &= (-tx^n)^{1-p} \sum_{k=0}^{p-1} \lim_{q \rightarrow \zeta_p} g_{K_{m+\frac{1}{2}, -n}}^{(k)}(x, -t^{-1}q^2, q, t).\end{aligned}\quad (4.17)$$

Note that there is no dependence on  $t$ -deformed Alexander polynomial because the radial limit of  $(a, t)$ -deformed  $F_K$  is proportional to the inverse of the factor  $\Delta_K(x^p, -(-t)^p)$ , which cancels the same factor in (4.14). Especially, we can analytically derive that these formulae give rise to

$$\begin{aligned}\text{ADO}_K(p=1; x, t) &= 1, \\ \text{ADO}_K(p=2; x, t) &= \Delta_K(x, t),\end{aligned}\quad (4.18)$$

which implies that the  $t$ -deformed ADO polynomial is a generalization of  $t$ -deformed Alexander polynomial.

In Table 1, 2, 3, we summarize the results of  $t$ -deformed ADO polynomials for the trefoil,  $5_2$  knot and  $7_2$  knot.

More generally, the higher rank  $t$ -deformed ADO is defined as [54]

$$\text{ADO}_K^{\text{SU}(N)}(p; x, t) = \Delta_K(x^p, -(-t)^p)^{N-1} \lim_{q \rightarrow \zeta_p} F_K(x, -t^{-1}q^N, q, t). \quad (4.19)$$

$p$	$\text{ADO}_{5_2}(p; x, t)$
1	1
2	$(1-t)(-tx) - 1 + t - t^2 + (1-t)(-tx)^{-1}$
3	$\zeta_3^2 \left[ (1+t\zeta_3+t^2)(t\zeta_3x)^2 + (-1+2t+t^2\zeta_3+t^3)(t\zeta_3x) + 2 + \zeta_3 + (-1+\zeta_3)t \right. \\ \left. + 2t^2 + t^3\zeta_3 + t^4 + (-1+2t+t^2\zeta_3+t^3)(t\zeta_3x)^{-1} + (1+t\zeta_3+t^2)(t\zeta_3x)^{-2} \right]$

**Table 2:**  $t$ -deformed ADO polynomials for the  $5_2$  knot

$p$	$\text{ADO}_{7_2}(p; x, t)$
1	1
2	$(1-t+t^2)(-tx) - 1 + 2t - t^2 + t^3 + (1-t+t^2)(-tx)^{-1}$
3	$\zeta_3^2 \left[ (1+t\zeta_3+t^3\zeta_3+t^4)(t\zeta_3x)^2 + (-1+(1-\zeta_3)t+2\zeta_3t^2+t^3+\zeta_3t^4+t^5)(t\zeta_3x) \right. \\ \left. + 2 + \zeta_3 + (-1+2\zeta_3)t + (1-\zeta_3)t^2 + 2\zeta_3t^3 + t^4 + \zeta_3t^5 + t^6 \right. \\ \left. + (-1+(1-\zeta_3)t+2\zeta_3t^2+t^3+\zeta_3t^4+t^5)(t\zeta_3x)^{-1} + (1+t\zeta_3+t^3\zeta_3+t^4)(t\zeta_3x)^{-2} \right]$

**Table 3:**  $t$ -deformed ADO polynomials for the  $7_2$  knot

It is also feasible to derive the closed-form expression for the  $t$ -deformed  $\text{ADO}_K^{\text{SU}(N)}(p; x, t)$  from (4.1) and (4.5)

$$\begin{aligned}
\text{ADO}_{K_{m,n}}^{\text{SU}(N)}(p; x, t) &= \frac{1}{p} \sum_{l=0}^{p-1} (-tx)^{l+1-p} \Delta_{K_{m,n}}(x^p, -(-t)^p)^{\frac{(N-2)(p-1)+l}{p}} S_p(\zeta_p^{N-l-2}) \\
&\quad \times \sum_{k=0}^{p-1} \lim_{q \rightarrow \zeta_p} g_{K_{m,n}}^{(k)}(x, -t^{-1}q^N, q, t), \\
\text{ADO}_{K_{m+\frac{1}{2},-n}}^{\text{SU}(N)}(p; x, t) &= \frac{1}{p} \sum_{l=0}^{p-1} (-tx^n)^{l+1-p} \Delta_{K_{m+\frac{1}{2},-n}}(x^p, -(-t)^p)^{\frac{(N-2)(p-1)+l}{p}} \\
&\quad \times S_p(\zeta_p^{N-l-2}) \sum_{k=0}^{p-1} \lim_{q \rightarrow \zeta_p} g_{K_{m+\frac{1}{2},-n}}^{(k)}(x, -t^{-1}q^N, q, t). \tag{4.20}
\end{aligned}$$

The detailed derivation is also shown in the Appendix A. Especially, we can analytically verify that they become

$$\begin{aligned}
\text{ADO}_K^{\text{SU}(N)}(1; x, t) &= 1, \\
\text{ADO}_K^{\text{SU}(N)}(2; x, t) &= \frac{1}{2} \sum_{l,j=0}^1 (-1)^{j(N-l)} [\Delta_K(x^2, -(-t)^2)]^{\frac{l}{2} + \frac{N}{2} - 1} [\Delta_K(x, t)]^{1-l}, \tag{4.21}
\end{aligned}$$

and they obey the recursion relation [54]

$$\text{ADO}^{\text{SU}(N+p)}(p; x, t) = \Delta_K(x^p, -(-t)^p)^{p-1} \text{ADO}^{\text{SU}(N)}(p; x, t). \tag{4.22}$$

Note that colored Jones polynomials of cyclotomic type [79, 88, 89] at a root of unity  $q = \zeta_p$  provide the corresponding ADO invariants [90] even if a closed-form expression of  $F_K$  is not known. Therefore, one can evaluate the ADO invariants from colored Jones polynomials in a similar fashion for arbitrary double twist knots such as  $K_{m,n}$  with  $m, -n \in \mathbb{Z}_+$  [90].

## 5 New CFT from $\mathcal{N} = (0, 2)$ Landau-Ginzburg models

A 2d CFT is endowed with an infinite-dimensional Lie algebra [92], and modular invariance further constrains its spectrum on torus [93]. Consequently, a number of models have been exactly solved. In an RCFT, a modular invariant partition function consists of finitely many pairs of  $I$  of left-moving and right-moving characters of chiral algebra  $\mathcal{A} \otimes \bar{\mathcal{A}}$ ,

$$Z = \sum_{(i, \bar{i}) \in I} N_{i\bar{i}} \chi_i^{\mathcal{A}} \otimes \bar{\chi}_{\bar{i}}^{\bar{\mathcal{A}}} . \quad (5.1)$$

If we write  $M_{ij}^{\mathcal{A}}$  for actions of  $\text{SL}(2, \mathbb{Z})$  on the spaces of the left-moving characters and  $M_{i\bar{j}}^{\bar{\mathcal{A}}}$  for right-moving characters, then the modular invariance requires that

$$M_{ij}^{\mathcal{A}} N_{j\bar{j}} (M_{\bar{j}\bar{i}}^{\bar{\mathcal{A}}})^* = N_{i\bar{i}} . \quad (5.2)$$

As a result, modular invariant partition functions of  $\text{SU}(2)_k$  WZNW models and unitary Virasoro minimal models admit the celebrated ADE classifications [94–97]. If a CFT is described by a non-chiral coset model [98] involving  $\text{SU}(2)$  and  $\text{U}(1)$ , its modular invariant partition function fits into the ADE classification. As such, one can find modular invariant partition functions for parafermion (PF) models  $\text{SU}(2)_k/\text{U}(1)_k$  [99] and  $\mathcal{N} = 2$  minimal models (MMs)  $(\text{SU}(2)_k \times \text{U}(1)_2)/\text{U}(1)_{k+2}$  [100]. Furthermore, with  $\mathcal{N} = (2, 2)$  supersymmetry, Landau-Ginzburg models with ADE quasi-homogeneous superpotential are described by the MMs of corresponding ADE type in the infra-red (IR) limit [101, 102].

In this section, we study the IR fixed point of a certain  $\mathcal{N} = (0, 2)$  LG model. We will use techniques such as  $c$ -extremization, rational transformation, and modular invariance to obtain a modular invariant partition function, which turns out to be beyond the ADE classification. With careful analysis of the Hilbert space we will show that a part of the left-moving sector is described by a new CFT which is a variation of the parafermion model.

### 5.1 $\mathcal{N} = (0, 2)$ LG model and its classification

Landau-Ginzburg theories with  $\mathcal{N} = (0, 2)$  supersymmetry describe certain phases of  $\mathcal{N} = (0, 2)$  supersymmetric gauge theories. Its Lagrangian is built from chiral superfields  $\phi_i$  and Fermi superfields  $\psi_a$ . The chiral multiplet contains a complex scalar and a complex right-moving fermion and the Fermi multiplet consists of a single complex left-moving fermion. The theories are rich because there are various choice of interaction terms. Due to the constraint of supersymmetry, in these theories, we can consider two types of interactions,  $J$ -terms and  $E$ -terms [57]. The  $J$ -term is like the superpotential term,

$$\int d\theta^+ \sum_a \psi_a J_a(\phi_i) + \text{c.c.} , \quad (5.3)$$

where  $J_a$  are holomorphic functions of  $\phi_i$  and c.c. denotes complex conjugate. On the other hand,  $E$ -term is induced as supersymmetric variation of the Fermi field, by requiring  $\bar{D}_+ \psi_a = E_a(\phi_i)$  instead of the usual  $\bar{D}_+ \psi_a = 0$ . In our settings, we will turn off the  $E$ -terms.

In [59], all  $\mathcal{N} = (0, 2)$  LG models flowing to compact IR fixed points with  $c = \bar{c} < 3$  are classified as follows,

- Class a: the  $J$ -terms are given by  $\phi_1^m + \phi_2^n, \phi_1\phi_2$ ,
- Class b: the  $J$ -terms are given by  $\phi_1^k + \phi_2^2, \phi_1^2\phi_2$ ,
- Class c: the  $J$ -terms are given by  $\phi_1^3 + \phi_2^2, \phi_1^3\phi_2$ ,
- Class d: the  $J$ -terms are given by  $\phi_1^3 + \phi_2^2, \phi_1^2\phi_2^2$ ,
- Class e: the  $J$ -terms are given by  $\phi_1^3 + \phi_2^2, \phi_1\phi_2^2$ .

Since the theories are chiral and flow to IR fixed points, we can determine the central charge, using  $c$ -extremization [103]. Taking the Class a theory as an example, the superpotential is given by

$$W = \psi_1(\phi_1^m + \phi_2^n) + \psi_2\phi_1\phi_2. \quad (5.4)$$

We first assign  $\mathcal{R}$ -charge  $r$  to the Fermi field  $\psi_1$  and the trial right-moving central charge is

$$\bar{c} = 3 \left( \left(1 - \frac{1-r}{n}\right)^2 + \left(1 - \frac{1-r}{m}\right)^2 - r^2 - \left(1 - \frac{1-r}{n} - \frac{1-r}{m}\right) \right). \quad (5.5)$$

Extremizing with respect to  $r$ , we obtain the extrema

$$r = \frac{2}{mn+2}, \quad \bar{c} = \frac{3mn}{mn+2}. \quad (5.6)$$

Then the right-moving central charge can be fixed by gravitational anomaly  $\text{Tr}\gamma_3 = \bar{c} - c = 0$ .

## 5.2 LG/CFT correspondence

Since A.B. Zamolodchikov has identified the LG/CFT correspondence [104], it has given drastically new insights in quantum field theories and mathematical physics. In [58], they explored the low energy physics and calculate the modular invariant partition function for several simple  $\mathcal{N} = (0, 2)$  LG models. We followed their line of work and for Class a theory, their method worked out. However, when it comes to Class b with  $k = 4$ , there is a little twist. In the following part, we will focus on Class b  $\mathcal{N} = (0, 2)$  Landau-Ginzburg model with  $k = 4$ . The theory consists of two chiral multiplets  $\phi_1, \phi_2$  and two Fermi multiplets  $\psi_1, \psi_2$ . The superpotential is given by

$$W = \psi_1(\phi_1^4 + \phi_2^2) + \psi_2\phi_1^2\phi_2. \quad (5.7)$$

Since the numbers of chiral and Fermi multiplets are equal, the vanishing of the gravitational anomaly  $\text{Tr}\gamma_3 = \bar{c} - c = 0$  guarantees the equality of the left-moving and right-moving central charges. Furthermore, the  $c$ -extremization gives

$$c = \bar{c} = \frac{25}{9}. \quad (5.8)$$

The  $\mathcal{R}$ -charges of all the multiplets are listed as below.

	$\phi_1$	$\phi_2$	$\psi_1$	$\psi_2$
$\text{U}(1)_{\mathcal{R}}$	$\frac{5}{27}$	$\frac{10}{27}$	$\frac{7}{27}$	$\frac{7}{27}$
$\text{U}(1)_{\ell}$	1	2	-4	-4

There is also a left-moving  $U(1)_\ell$  global symmetry with 't Hooft anomaly 27. Therefore, these data suggest that, in the IR fixed point, the right-moving sector is the  $\mathcal{N} = 2$  Minimal model  $MM_{25}$  with level  $k = 25$ , and the left-moving sector is the  $U(1)_{\frac{27}{2}}$  WZNW model with level  $k = 27/2$  and a CFT of central charge  $16/9$ . It is tempting to identify the CFT of central charge  $16/9$  with the parafermion model  $PF_{25}$  of level 25 as in [58], and we will indeed write a modular invariant partition function using characters of  $PF_{25}$  in the following discussion. So let us first assume that the low energy theory takes the form,

$$\left( PF_{25} \times U(1)_{\frac{27}{2}} \right) \otimes \overline{\left( \frac{SU(2)_{25} \times U(1)_2}{U(1)_{27}} \right)} \quad (5.9)$$

However, as we will see later, it is not exactly the  $PF_{25}$ , but a certain variant of  $PF_{25}$ .

Let us extract more information about the IR CFT from the UV data. Since an elliptic genus is protected under the RG flow [105], it can be computed from the information of the LG model. We evaluate it in the NS-NS sector

$$\begin{aligned} \text{EG}(\tau, z) &= \text{Tr}_{\text{NSNS}} (-1)^F q^{L_0 - \frac{c}{24}} y^{J_0} e^{-\beta(\bar{L}_0 - \frac{1}{2}\bar{J}_0)} \\ &= q^{-\frac{c}{24}} \frac{\prod_a \theta(y J_0^{\psi_a} q^{\frac{1+r_{\psi_a}}{2}}; q)}{\prod_i \theta(y J_0^{\phi_i} q^{\frac{r_{\phi_i}}{2}}; q)} , \end{aligned} \quad (5.10)$$

where  $J_0$  is the  $U(1)_\ell$  charge,  $r$  is the  $\mathcal{R}$  charge,  $q = e^{2\pi i\tau}$ ,  $y = e^{2\pi iz}$  and  $\theta$  is given by,

$$\theta(x; q) = \prod_{i=0}^{\infty} (1 - xq^i)(1 - q^{i+1}/x) . \quad (5.11)$$

The numerator of (5.10) receives contributions from all the Fermi fields and the denominator receives contributions from all the chiral fields. Note that in our special case,

$$\begin{aligned} \text{EG}(\tau, z) &= q^{-\frac{25}{216}} \frac{\theta(y^{-4} q^{17/27}; q)^2}{\theta(y q^{5/54}; q) \theta(y^2 q^{5/27}; q)} \\ &= q^{-\frac{25}{216}} (1 + y q^{5/54} + 2y^2 q^{5/27} + 2y^3 q^{5/18} + y^4 q^{10/27} + y^5 q^{25/54} + \dots) . \end{aligned} \quad (5.12)$$

Among chiral primary states ( $\bar{L}_0 = \bar{J}_0/2$ ) in the right moving sector that contribute to the elliptic genus, the state subject to  $L_0 = q/2$  in the left-moving sector form the topological heterotic ring  $\mathcal{H}_{\text{top}}$  [106, 107] where  $q$  is equal to the  $U(1)_{\mathcal{R}}$  charge  $r_\phi$  for a chiral field and  $r_\psi - 1$  for a Fermi field. Since the numbers of chiral and Fermi multiplets are equal in the LG theory, it receives contributions only from chiral multiplets with  $L_0 = \bar{J}_0/2$ , which is isomorphic to the Jacobi ring of the  $J$ -term,

$$\mathcal{H}_{\text{top}} = \mathbb{C}[\phi_1, \phi_2] / (\phi_1^4 + \phi_2^2, \phi_1^2 \phi_2) \cong \text{Span}[\phi_1^i]_{i=0}^5 \oplus \text{Span}[\phi_2, \phi_1 \phi_2] . \quad (5.13)$$

In fact, the holomorphic part of the stress-energy tensor [105, 108] is written as

$$T = \sum_{i=1}^2 \left[ \left( 1 - \frac{r_{\phi_i}}{2} \partial \phi_i \partial \bar{\phi}_i - \frac{r_{\phi_i}}{2} \phi_i \partial^2 \bar{\phi}_i \right) \right] + \sum_{a=1}^2 \left[ \frac{i}{2} (1 + r_{\psi_a}) \psi_a \partial \bar{\psi}_a - \frac{i}{2} (1 - r_{\psi_a}) \partial \psi_a \bar{\psi}_a \right] , \quad (5.14)$$

and the OPE of a generator of  $\mathcal{H}_{\text{top}}$  with the stress-energy tensor shows that it is a primary state with  $L_0 = \bar{J}_0/2$ .

### 5.3 Modular invariant partition function

Our goal is to find the modular invariant partition function of the IR CFT in the NS-NS sector as the following form,

$$Z = \text{Tr}_{\text{NS,NS}} q^{L_0 - \frac{c}{24}} y^{J_0} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}} \bar{x}^{\bar{J}_0} \\ = \sum_{\text{wts}} N_{\ell\bar{\ell}}^{\text{SU}(2)} N_{\nu\lambda\bar{\beta}}^{\text{U}(1)} \chi_{\ell,\nu}^{\text{PF}_{25}}(\tau) \chi_{\lambda}^{\text{U}(1)\frac{27}{2}}(\tau, z) \cdot \bar{\chi}_{\bar{\ell},\bar{\beta}}^{\text{MM}_{25}}(\bar{\tau}, \bar{w}), \quad (5.15)$$

which is consistent with the elliptic genus (5.12). Here we are summing over all the weights labeled by  $\ell, \bar{\ell}, \nu, \lambda, \bar{\beta}$ . To obtain an elliptic genus from a partition function, we fix the right-moving sector to be chiral primary states ( $\bar{L}_0 = \bar{J}_0/2$ ) only from which the elliptic genus receives contributions. Then we insert  $(-1)^F$  or equivalently  $(-1)^{2(L_0 - \bar{L}_0)}$  in each term of the left-moving sector [109].

For this purpose, we shall find the modular invariant combination of  $\text{U}_1$  WZNW characters by following [58, 110]. First, let us review the general procedure for low energy theory of the form

$$\left( \frac{\text{SU}(2)_k}{\text{U}(1)_k} \times \text{U}(1)_{k'} \right) \otimes \overline{\left( \frac{\text{SU}(2)_k \times \text{U}(1)_2}{\text{U}(1)_{k+2}} \right)}. \quad (5.16)$$

The level of  $\text{SU}(2)$  piece is the same on the left-moving and right-moving side, so the characters on both sides should be combined in a modular invariant way. For the  $\text{U}(1)$  factors, the existence of modular invariant pairing follows from the rational equivalence,

$$\begin{pmatrix} k' & 0 \\ 0 & k+2 \end{pmatrix} = R^T \begin{pmatrix} k & 0 \\ 0 & 2 \end{pmatrix} R, \quad (5.17)$$

where

$$R = \frac{1}{g} \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a, b, c, d, g \in \mathbb{Z}. \quad (5.18)$$

Now consider a rational transformation,

$$\begin{pmatrix} x \\ y \end{pmatrix} = R \begin{pmatrix} u \\ v \end{pmatrix}. \quad (5.19)$$

Then the  $\text{U}(1)$  characters, which are shown in the appendix, satisfy the identity,

$$\chi_{\mu}^{\text{U}(1)_k}(\tau, gx) \left( \chi_0^{\text{U}(1)_2}(\tau, gy) + \chi_2^{\text{U}(1)_2}(\tau, gy) \right) \\ = \sum_{(r_1, r_2) \in \mathbb{Z}^2 / \Gamma_R} \chi_{a(\mu+2kr_1)+c(\nu+4r_2)}^{\text{U}(1)_{g^2k'}}(\tau, u) \chi_{b(\mu+2kr_1)+d(\nu+4r_2)}^{\text{U}(1)_{g^2(k+2)}}(\tau, v), \quad (5.20)$$

where  $\Gamma_R$  is the sublattice of  $\mathbb{Z}^2$  generated by  $(a, c)$  and  $(b, d)$ . There is also another identity,

$$\chi_{\lambda}^{\text{U}(1)_{k'}}(\tau, gu) \chi_{\rho}^{\text{U}(1)_{k+2}}(\tau, gv) = \sum_{i_1, i_2 \in \mathbb{Z}_g} \chi_{g(\lambda+2k'i_1)}^{\text{U}(1)_{g^2k'}}(\tau, u) \chi_{g(\rho+2(k+2)i_2)}^{\text{U}(1)_{g^2(k+2)}}(\tau, v). \quad (5.21)$$

Now in our special case, the quadratic forms given by  $\text{U}(1)$  levels are rationally equivalent as,

$$\begin{pmatrix} \frac{27}{2} & 0 \\ 0 & 27 \end{pmatrix} = R^T \begin{pmatrix} 25 & 0 \\ 0 & 2 \end{pmatrix} R, \quad (5.22)$$

where

$$R = \frac{1}{10} \begin{pmatrix} 2 & 10 \\ 25 & -10 \end{pmatrix}. \quad (5.23)$$

This rational equivalence gives rise to an identity among theta functions

$$\begin{aligned} & \chi_\mu^{\text{U}(1)_{25}}(\tau, 2u + 10v) \left( \chi_0^{\text{U}(1)_2}(\tau, 25u - 10v) + \chi_2^{\text{U}(1)_2}(\tau, 25u - 10v) \right) \\ &= \sum_{i=0}^{27 \times 10 - 1} \left\{ \chi_{2\mu+50i}^{\text{U}(1)_{\frac{27 \times 10^2}{2}}}(\tau, u) \chi_{10\mu-20i}^{\text{U}(1)_{27 \times 10^2}}(\tau, v) + \chi_{2\mu+50i}^{\text{U}(1)_{\frac{27 \times 10^2}{2}}}(\tau, u) \chi_{10\mu-20i+27 \times 10^2}^{\text{U}(1)_{27 \times 10^2}}(\tau, v) \right\} \\ &\equiv \sum_{\lambda', \rho'} A_{\mu\lambda'\rho'} \chi_{\lambda'}^{\text{U}(1)_{\frac{27 \times 10^2}{2}}}(\tau, u) \chi_{\rho'}^{\text{U}(1)_{27 \times 10^2}}(\tau, v). \end{aligned} \quad (5.24)$$

Furthermore, there is another identity of theta functions,

$$\begin{aligned} & \chi_\lambda^{\text{U}(1)_{\frac{27}{2}}}(\tau, 10u) \chi_\rho^{\text{U}(1)_{27}}(\tau, 10v) \\ &= \sum_{i_1, i_2 \in \mathbb{Z}_{10}} \chi_{10(\lambda+27i_1)}^{\text{U}(1)_{\frac{27 \times 10^2}{2}}}(\tau, u) \chi_{10(\rho+54i_2)}^{\text{U}(1)_{27 \times 10^2}}(\tau, v) \\ &\equiv \sum_{\lambda', \rho'} B_{\lambda\rho\lambda'\rho'} \chi_{\lambda'}^{\text{U}(1)_{\frac{27 \times 10^2}{2}}}(\tau, u) \chi_{\rho'}^{\text{U}(1)_{27 \times 10^2}}(\tau, v). \end{aligned} \quad (5.25)$$

From these identities, one can construct the  $\text{U}(1)$  modular invariant tensor as

$$N_{\nu\lambda\bar{\beta}}^{\text{U}(1)} = \sum_{\lambda', \beta'} A_{\nu\lambda'\beta'} B_{\lambda\bar{\beta}\lambda'\beta'}, \quad (5.26)$$

which satisfies

$$(M_{\nu'\nu}^{\text{U}(1)_{25}})^* M_{\lambda'\lambda}^{\text{U}(1)_{27/2}} N_{\nu\lambda\bar{\beta}}^{\text{U}(1)} M_{\bar{\beta}\beta'}^{\text{U}(1)_{27}} = N_{\nu'\lambda'\bar{\beta}'}^{\text{U}(1)}, \quad (5.27)$$

for all  $M \in \text{SL}(2, \mathbb{Z})$ . More explicitly, one can write

$$Z = \sum_{\text{wts}} N_{\ell\bar{\ell}}^{\text{SU}(2)} \chi_{\ell, 5m}^{\text{PF}_{25}}(\tau) \chi_{\frac{27m-5n}{2}}^{\text{U}(1)_{\frac{27}{2}}}(\tau, z) \cdot \bar{\chi}_{\bar{\ell}, n}^{\text{MM}_{25}}(\bar{\tau}, \bar{w}), \quad (5.28)$$

where the summation over weights runs  $m \in \mathbb{Z}_{10}$ ,  $n \in \mathbb{Z}_{54}$  and  $\ell, \bar{\ell} \in \mathbb{Z}_{26}$ .

Next, we need to determine the  $\text{SU}(2)$  modular invariant tensor  $N_{\ell\bar{\ell}}^{\text{SU}(2)}$ . For the  $\text{SU}(2)$  level  $k = 25$ , only the diagonal (type-A) combination  $N_{\ell\bar{\ell}}^{\text{SU}(2)} = \delta_{\ell\bar{\ell}}$  is listed in the ADE classification [94–97]. However, with the diagonal  $\text{SU}(2)$  combination, one can check that the partition function does not reproduce the elliptic genus (5.12).

To circumvent this situation, we need to relax some of the assumptions in [94–97] for the classification. We notice that the following matrix commutes with all the modular matrices  $M^{\text{SU}(2)}$ ,

$$N_{\ell\bar{\ell}}^{\text{nd}} = (\delta_{2,\ell} - \delta_{14,\ell} + \delta_{20,\ell})(\delta_{2,\bar{\ell}} - \delta_{14,\bar{\ell}} + \delta_{20,\bar{\ell}}) + (\delta_{5,\ell} - \delta_{11,\ell} + \delta_{23,\ell})(\delta_{5,\bar{\ell}} - \delta_{11,\bar{\ell}} + \delta_{23,\bar{\ell}}), \quad (5.29)$$

where the indices range  $\ell, \bar{\ell} \in \mathbb{Z}_{26}$  and nd means non-diagonal. Then, we set

$$N_{\ell\bar{\ell}}^{\text{SU}(2)} = \delta_{\ell\bar{\ell}} - \frac{1}{3} N_{\ell\bar{\ell}}^{\text{nd}}. \quad (5.30)$$

This clearly violates the assumption that  $N_{i\bar{i}}$  in (5.1) are non-negative integer multiplicities, which has been adopted in the literature including [94–97]. However, if we use (5.30) in (5.28), the partition function is a formal series of  $(q, y, \bar{q}, \bar{x})$  with non-negative integer coefficients and it is moreover consistent with the elliptic genus (5.12). We claim that it is the partition function of the IR CFT.

#### 5.4 Hilbert space and a new CFT $\widetilde{\text{PF}}_{25}$

To clarify the multiplicities (5.30) with negative fractional numbers, let us investigate the Hilbert space of the IR CFT. To this end, we denote by  $V_{\ell,m}^{\text{PF}_{25}}$  a highest weight representation of  $\text{PF}_{25}$ . In addition, by taking the direct sum of  $s = 0$  and  $s = 2$  weight of  $\text{U}(1)_2$ , we write by  $V_{\ell,m}^{\text{MM}_{25}}$  a highest weight representation of  $\text{MM}_{25}$  in the NS sector. There are isomorphisms of irreducible modules,

$$\begin{aligned} V_{\ell,m}^{\text{PF}_{25}} &\cong V_{\ell,50-m}^{\text{PF}_{25}} \cong V_{25-\ell,m+25}^{\text{PF}_{25}} , \\ V_{\ell,m}^{\text{MM}_{25}} &\cong V_{\ell,54-m}^{\text{MM}_{25}} \cong V_{25-\ell,m+27}^{\text{MM}_{25}} . \end{aligned} \quad (5.31)$$

First, we note an identity of the parafermion characters,

$$\begin{aligned} \sum_{m=0}^4 \chi_{2,10m}^{\text{PF}_{25}} - \chi_{14,10m}^{\text{PF}_{25}} + \chi_{20,10m}^{\text{PF}_{25}} &= 3, \\ \sum_{m=0}^4 \chi_{5,10m+5}^{\text{PF}_{25}} - \chi_{11,10m+5}^{\text{PF}_{25}} + \chi_{23,10m+5}^{\text{PF}_{25}} &= 3 . \end{aligned} \quad (5.32)$$

Recall that the conformal dimension for parafermion primary states  $|\ell, m\rangle_{\text{PF}_k}$  is given by

$$h_{\ell,m}^{\text{PF}_k} = \frac{\ell(\ell+2)}{4(k+2)} - \frac{m^2}{4k} , \quad \ell = 0, \dots, k, \quad m = -k+1, \dots, k . \quad (5.33)$$

Thus, (5.32) counts the number of primary states  $|\ell, m\rangle_{\text{PF}_{25}}$  with conformal dimension  $h_{\ell,m}^{\text{PF}_{25}} = 2/27$  in  $\text{PF}_{25}$ :

$$\begin{aligned} |\ell, m\rangle_{\text{PF}_{25}} &= |2, 0\rangle, |20, 20\rangle, |20, 30\rangle, \\ &= |23, 25\rangle, |5, 5\rangle, |5, 45\rangle . \end{aligned} \quad (5.34)$$

Hence, roughly speaking, the non-diagonal part of (5.30) adds or eliminates a certain linear combination of these states to or from  $V_{\ell,m}^{\text{PF}_{25}}$ .

To see how the spectrum is organized, we compare the diagonal spectrum

$$\mathcal{H}_{\text{diag}} = \bigoplus_{\ell,m,n} V_{\ell,5m}^{\text{PF}_{25}} \otimes V_{\frac{27m-5n}{2}}^{\text{U}(1)_{\frac{27}{2}}} \otimes \bar{V}_{\ell,n}^{\text{MM}_{25}} , \quad (5.35)$$

where  $N_{\ell\bar{\ell}}^{\text{SU}(2)} = \delta_{\ell\bar{\ell}}$  with the information of the Hilbert space of the IR CFT obtained from the LG model. Note that the diagonal spectrum  $\mathcal{H}_{\text{diag}}$  contains primary states of  $\text{PF}_{25} \times \text{U}(1)_{\frac{27}{2}} \times \text{MM}_{25}$ ,

$$\begin{aligned} &|5s, 5s\rangle_{\text{PF}_{25}} \otimes | -s\rangle_{\text{U}(1)_{\frac{27}{2}}} \otimes |5s, -5s\rangle_{\text{MM}_{25}} , \\ &|5s, 50-5s\rangle_{\text{PF}_{25}} \otimes | -s\rangle_{\text{U}(1)_{\frac{27}{2}}} \otimes |5s, -5s\rangle_{\text{MM}_{25}} , \end{aligned} \quad (5.36)$$

which obey the condition  $L_0 = \bar{J}_0/2 = \bar{L}_0$ . Here we have  $s = 0, 1, \dots, 5$  and these two states are identical to the vacuum state when  $s = 0$ . Thus, there are ten primary states subject to the condition in  $\mathcal{H}_{\text{diag}}$  whereas the topological heterotic string (5.13) of the IR CFT is eight-dimensional as a vector space.

Hence, the diagonal spectrum (5.35) is not the actual Hilbert space. To realize the ring structure of  $\mathcal{H}_{\text{top}}$  in (5.13), let us suppose that  $\phi_1$  and  $\phi_2$  in  $\mathcal{H}_{\text{top}}$  respectively correspond to

$$|5, 5\rangle_{\text{PF}_{25}} + |5, 45\rangle_{\text{PF}_{25}} \quad \text{and} \quad |10, 10\rangle_{\text{PF}_{25}} - |10, 40\rangle_{\text{PF}_{25}} . \quad (5.37)$$

From now on, we will drop the parts of  $U(1)_{\frac{27}{2}}$  and  $MM_{25}$  in (5.36). Then the fusion rule tells us that  $\phi_1^2\phi_2$  corresponds to

$$|20, 20\rangle_{PF_{25}} - |20, 30\rangle_{PF_{25}} , \quad (5.38)$$

which is decoupled from the spectrum due to the equation of motion. In addition, there is no generator in  $\mathcal{H}_{\text{top}}$  corresponding to

$$|5, 5\rangle_{PF_{25}} - |5, 45\rangle_{PF_{25}} . \quad (5.39)$$

Thus, the IR CFT excludes these two states, (5.38) and (5.39), and the identification of (5.37) with  $\phi_1$  and  $\phi_2$  as well as the fusion rule indeed reproduces the topological heterotic ring (5.13).

On the other hand, one can show that  $\phi_1^2\partial\phi_2 \sim -2\phi_1\phi_2\partial\phi_1$  is a primary in the IR CFT from the OPE with the stress energy tensor up to the equations of motion ( $\partial W/\partial\phi_i = 0 = \partial W/\partial\psi_i$  and their complex conjugates). Moreover,  $\phi_1^2\partial\phi_2$  and its descendants contribute to the elliptic genus by

$$(\chi_{|20,20\rangle-|20,30\rangle}^{PF_{25}} - 1)\chi_{-4}^{U(1)_{27/2}} = (\chi_{20,20}^{PF_{25}} - 1)\chi_{-4}^{U(1)_{27/2}} = (q + 3q^2 + 6q^3 + 12q^4 + 21q^5 + \dots)\chi_{-4}^{U(1)_{27/2}} . \quad (5.40)$$

Ignoring the  $U(1)_{27/2}$  part, the subtraction by one means the omission of (5.38), and the primary  $\phi_1^2\partial\phi_2$  contributes to  $q^1$  whereas the subsequent higher order terms count its descendants. This implies that although the IR CFT is not endowed with the parafermion symmetry  $SU(2)_{25}/U(1)_{25}$ , it is still a character of a module of the Virasoro algebra in the left-moving sector. Similarly, it is easy to check from the OPE that  $\phi_1^3\partial\bar{\phi}_2 \sim -2\phi_2\partial\bar{\phi}_1$  is also a primary in the IR CFT, and the contribution from its conformal family to the elliptic genus is  $(\chi_{|5,5\rangle-|5,45\rangle}^{PF_{25}} - 1)\chi_{-1}^{U(1)_{27/2}}$ .

Furthermore, an explicit computation using (5.32) shows that the elliptic genus (5.12) receives all the contributions from the part of  $\ell = 5, n = -5$  in  $\mathcal{H}_{\text{diag}}$  except the states (5.38) and (5.39), and their  $U(1)_{27/2}$  descendants. Indeed, the Hilbert space is organized at the IR fixed point in such a way that the states (5.38) and (5.39) are excluded in the  $PF_{25}$  part but it preserves the Virasoro symmetry and the modular invariance. Denoting the CFT of central charge  $16/9$  by  $\widetilde{PF}_{25}$ , the  $\ell = 5, 20$  parts of the Hilbert space are isomorphic to the quotient spaces

$$\begin{aligned} \mathcal{H}_5^{\widetilde{PF}_{25}} &\cong \bigoplus_{m=0}^4 V_{5,10m+5}^{PF_{25}} / \mathbb{C}(|5, 5\rangle - |5, 45\rangle) , \\ \mathcal{H}_{20}^{\widetilde{PF}_{25}} &\cong \bigoplus_{m=0}^4 V_{20,10m}^{PF_{25}} / \mathbb{C}(|20, 20\rangle - |20, 30\rangle) , \end{aligned} \quad (5.41)$$

as vector spaces graded by  $L_0$ .

In order to keep the modular invariance, one needs to arrange the primary states (5.34) of  $PF_{25}$  according to the non-diagonal part of (5.30),

$$\begin{aligned} \mathcal{H}_2^{\widetilde{PF}_{25}} &\cong \bigoplus_{m=0}^4 V_{2,10m}^{PF_{25}} / \mathbb{C}|2, 0\rangle , \\ \mathcal{H}_{23}^{\widetilde{PF}_{25}} &\cong \bigoplus_{m=0}^4 V_{23,10m+5}^{PF_{25}} / \mathbb{C}|23, 25\rangle , \\ \mathcal{H}_{14}^{\widetilde{PF}_{25}} &\cong \bigoplus_{m=0}^4 V_{14,10m}^{PF_{25}} \oplus \mathbb{C}(|20, 20\rangle + |20, 30\rangle) , \\ \mathcal{H}_{11}^{\widetilde{PF}_{25}} &\cong \bigoplus_{m=0}^4 V_{11,10m+5}^{PF_{25}} \oplus \mathbb{C}(|5, 5\rangle + |5, 45\rangle) . \end{aligned} \quad (5.42)$$

Here  $\cong$  means an isomorphism as vector spaces graded by  $L_0$ . For the order  $\ell \neq 2, 5, 11, 14, 20, 23$ , they are isomorphic to those of  $\text{PF}_{25}$ ,

$$\mathcal{H}_\ell^{\widetilde{\text{PF}}_{25}} \cong \bigoplus_{m=0}^4 V_{\ell, 10m+5(\ell \bmod 2)}^{\text{PF}_{25}}. \quad (5.43)$$

All in all, the Hilbert space of the IR CFT is then expressed as

$$\mathcal{H} = \bigoplus_{\ell, n} \mathcal{H}_\ell^{\widetilde{\text{PF}}_{25}} \otimes V_{\frac{27\ell-5n}{2}}^{\text{U}(1)_{\frac{27}{2}}} \otimes \bar{V}_{\ell, n}^{\text{MM}_{25}}, \quad (5.44)$$

whose generating function is (5.28) with (5.30). This explains the reason why the partition function is a formal power series with non-negative integer coefficients. If we restrict the right-moving sector to be chiral primary states, we have

$$\mathcal{H}|_{\bar{L}_0=\bar{J}_0/2} = \bigoplus_{\ell} \mathcal{H}_\ell^{\widetilde{\text{PF}}_{25}} \otimes V_{16\ell}^{\text{U}(1)_{\frac{27}{2}}}, \quad (5.45)$$

which exactly reproduces the elliptic genus (5.12) by appropriately including signs. In fact, under the equations of motion, the conformal families of two primaries  $\psi_i \partial \bar{\phi}_1$  ( $i = 1, 2$ ) contribute to (5.12)

$$(\chi_{2,0}^{\text{PF}_{25}} - 1) \chi_5^{\text{U}(1)_{27/2}} = (2q + 3q^2 + 6q^3 + 10q^4 + 18q^5 + \cdots) \chi_5^{\text{U}(1)_{\frac{27}{2}}}. \quad (5.46)$$

For  $\ell = 23$ , those of two primaries  $\bar{\psi}_i \phi_1^2 (\partial \phi_2)^2$  ( $i = 1, 2$ ) yield the contribution  $(\chi_{23,25}^{\text{PF}_{25}} - 1) \chi_{17}^{\text{U}(1)_{27/2}}$ . In addition, the conformal family of a primary  $\psi_1 \psi_2$  combines the two irreducible characters of  $\text{PF}_{25}$  into one “irreducible” character of  $\widetilde{\text{PF}}_{25}$ ,

$$(1 + \chi_{14,20}^{\text{PF}_{25}} + \chi_{14,30}^{\text{PF}_{25}}) \chi_8^{\text{U}(1)_{27/2}} = (1 + 2q + 4q^3 + 10q^3 + 20q^4 + 38q^5 + \cdots) \chi_8^{\text{U}(1)_{27/2}}. \quad (5.47)$$

In a similar fashion, that of a primary  $\bar{\psi}_1 \bar{\psi}_2 \phi_1 \phi_2 (\partial \phi_1)^2$  gives the contribution  $(1 + \chi_{11,5}^{\text{PF}_{25}} + \chi_{11,45}^{\text{PF}_{25}}) \chi_{14}^{\text{U}(1)_{27/2}}$ . Hence, this provides a strong evidence that the graded vector spaces (5.42) are decomposed into modules of the Virasoro algebra and  $\widetilde{\text{PF}}_{25}$  preserves the conformal symmetry. In conclusion, the  $\mathcal{N} = (0, 2)$  LG model flows to

$$(\widetilde{\text{PF}}_{25} \times \text{U}(1)_{\frac{27}{2}}) \otimes \overline{\left( \frac{\text{SU}(2)_{25} \times \text{U}(1)_2}{\text{U}(1)_{27}} \right)}, \quad (5.48)$$

and the modular invariant Hilbert space (5.44) on a torus is decomposed into modules of the left-moving Virasoro algebra and the right-moving  $\mathcal{N} = 2$  super-Virasoro algebra.

## 5.5 Discussion

We find the modular invariant partition function beyond the ADE classification because a part of the left-moving sector is the new CFT  $\widetilde{\text{PF}}_{25}$  obtained by breaking the parafermionic symmetry of  $\text{PF}_{25}$ . Certainly, more investigation needs to be carried out to understand  $\widetilde{\text{PF}}_{25}$ . In particular, it is desirable to determine two “irreducible” characters of the primaries,  $\psi_i \partial \bar{\phi}_1$  in  $\widetilde{\text{PF}}_{25}$  whose sum is equal to  $\chi_{2,0}^{\text{PF}_{25}} - 1$  and  $\bar{\psi}_i \phi_1^2 (\partial \phi_2)^2$  in  $\widetilde{\text{PF}}_{25}$  whose sum is equal to  $\chi_{23,25}^{\text{PF}_{25}} - 1$ .

As we mentioned above,  $\mathcal{N} = (0, 2)$  LG models with the same left and right central charges  $\leq 3$  have been classified in [59]. In their classification, IR CFTs of Class a with superpotential

$$\psi_1 (\phi_1^m + \phi_2^n) + \psi_2 \phi_1 \phi_2, \quad m, n \in \mathbb{Z}_+, \quad (5.49)$$

are described by diagonal modular pairing of PFs and  $U(1)$  WZNW models in the left-moving-sector and  $\mathcal{N} = 2$  MMs in the right-moving sector [58]. This is because their topological heterotic rings are simple and it does not contain a mixed generator like  $\phi_1\phi_2$ . Like in our example, the topological heterotic rings of the other classes in [59] are more complicated, and we observe that their elliptic genera cannot be realized by characters of PFs and  $U(1)$  WZNW models except our example. (Another exception is Class b with  $k = 3$ , but it is equivalent to  $\mathcal{N} = (2, 2)$  MM of type  $E_7$ .) It is expected that the left-moving sectors of IR CFTs would be unknown ones so that it requires further study to understand how  $J$ -terms of  $\mathcal{N} = (0, 2)$  LG models are encoded in IR CFTs. It is also worth mentioning that the condition of the same left and right-moving central charges in [59] is rather special in  $\mathcal{N} = (0, 2)$  LG models, and a vast class of general  $\mathcal{N} = (0, 2)$  LG models are waiting to be investigated.

## A Derivation of the ADO polynomials

In this Appendix, we derive the  $t$ -deformed  $p$ -th ADO polynomial (4.14) and its higher rank generalization (4.19) from  $F_K(x, a, q, t)$  (4.1) and (4.5). We use  $F_K$  as a short for,

$$\lim_{q \rightarrow \zeta_p} F_K(x, a = -t^{-1}q^N, q, t) ,$$

and similarly for other functions discussed in this section. We use  $q$  and  $\zeta_p$  interchangeably. When an integer  $k$  is displayed as  $k = k_1p + k_0$ , it is implied that  $k_1, k_0 \in \mathbb{N}$ , and  $0 \leq k_0 \leq p-1$ .

Our closed formulae are built from  $q$ -binomials and  $q$ -Pochhammer symbols and we will first discuss their behavior when  $q$  goes to roots of unity. The  $q$ -Lucas theorem [111] states that,

$$\begin{bmatrix} l \\ k \end{bmatrix}_{q=\zeta_p} = \begin{bmatrix} l_1 \\ k_1 \end{bmatrix} \begin{bmatrix} l_0 \\ k_0 \end{bmatrix}_{q=\zeta_p} , \quad (\text{A.1})$$

where  $l = l_1p + l_0$ ,  $k = k_1p + k_0$ . For  $q$ -Pochhammers, we have

$$(a; q)_k = (1 - a^p)^{k_1} (a; q)_{k_0} , \quad (\text{A.2})$$

where  $q = \zeta_p$ ,  $k = k_1p + k_0$ . In the following discussion, We will break  $F_K(x, a, q, t)$  into parts. As we will see later, these components enjoy similar properties. It turns out that  $F_K$  can be written as a product of a infinite summation over  $k_1$  and a finite one over  $k_0$ , where  $k = k_1p + k_0$  and  $q = \zeta_p$ , and the infinite summation over  $k_1$  can be repackaged by the generalized binomial theorem,

$$\frac{1}{(1-z)^n} = \sum_{i=0}^{\infty} \binom{n+i-1}{i} z^i . \quad (\text{A.3})$$

As a result, an ADO polynomial can be expressed as a finite summation of over  $k_0$ .

### A.1 ADO polynomials of double twist knot $K_{m,n}$ ( $m, n \in \mathbb{Z}_+$ )

For double twist knots  $K_{m,n}$  ( $m, n \in \mathbb{Z}_+$ ), we first analyze  $(a, t)$ -deformed  $F_K$  at radial limit  $q = \zeta_p$ . Each component of (4.1) behaves as follows.

- The twist factor  $\text{tw}_m^{(l)}(a, q, t)$  (4.4) now becomes

$$\text{tw}_m^{(l)} = \sum_{0 \leq b_1 \leq \dots \leq b_{m-1} \leq b_m = l} \prod_{i=1}^{m-1} (-q^N)^{b_i} q^{b_i(b_i-1)} t^{b_i} \begin{bmatrix} b_{i+1} \\ b_i \end{bmatrix}_q . \quad (\text{A.4})$$

We decompose the summation variables as,  $b_i = \alpha_i p + \beta_i$ , and  $l = l_1 p + l_0$ . Using the  $q$ -Lucas theorem, (A.4) can be written as a product of two summations, one over  $\alpha_i$ 's and another one over  $\beta_i$ 's. The summation over  $\beta_i$ 's would give  $\text{tw}_m^{(l_0)}$ . Performing the summation over  $\alpha_i$ 's, we obtain

$$\text{tw}_m^{(l)} = S_m^{l_1}((-t)^p) \text{tw}_m^{(l_0)} , \quad (\text{A.5})$$

where  $S_m^{l_1}(x) := (S_m(x))^{l_1} := \left( \sum_{i=0}^{m-1} x^i \right)^{l_1}$ .

- The twist factor  $\text{Tw}_{K_{m,n}}^{(j)}(a, q, t)$  (4.3) becomes

$$\text{Tw}_{K_{m,n}}^{(j)} = \sum_{l=0}^j (-1)^l q^{\frac{1}{2}l(l+1)-jl} \text{tw}_m^{(l)} \text{tw}_n^{(l)} \begin{bmatrix} j \\ l \end{bmatrix}_q. \quad (\text{A.6})$$

We write the summation variables as  $j = j_1p + j_0$ , and  $l = l_1p + l_0$ . Because of the fact that  $q^p = 1$ , the  $q$ -Lucas theorem and (A.5), we obtain,

$$\text{Tw}_{K_{m,n}}^{(j)} = (1 - S_m((-t)^p) S_n((-t)^p))^{j_1} \text{Tw}_{K_{m,n}}^{(j_0)}. \quad (\text{A.7})$$

- Now  $g_{K_{m,n}}^{(k)}(x, a, q, t)$  in (4.2) becomes

$$g_{K_{m,n}}^{(k)} = \sum_{j=0}^k (x; q^{-1})_k (xt^2 q^N; q)_j (xt^2)^{k-j} q^{k-Nj} \begin{bmatrix} N-2+k \\ k \end{bmatrix}_q \begin{bmatrix} k \\ j \end{bmatrix}_q \text{Tw}_{K_{m,n}}^{(j)}, \quad (\text{A.8})$$

where we have used the fact that

$$\frac{(q^{N-1}; q)_k}{(q; q)_k} = \begin{bmatrix} N-2+k \\ k \end{bmatrix}_q. \quad (\text{A.9})$$

We decompose the variables as  $j = j_1p + j_0$ ,  $k = k_1p + k_0$  and  $N-2 = A_1p + A_0$ . Plugging (A.1), (A.2) and (A.7) into (A.8), we have

$$g_{K_{m,n}}^{(k)} = \begin{pmatrix} A_1 + k_1 \\ k_1 \end{pmatrix} (1 - x^p)^{k_1} [1 - (1 - x^p t^{2p}) S_m((-t)^p) S_n((-t)^p)]^{k_1} g_{K_{m,n}}^{(k_0)}. \quad (\text{A.10})$$

- $F_{K_{m,n}}(x, a, q, t)$  becomes

$$F_{K_{m,n}} = (-tx)^{N-1} \sum_{k=0}^{\infty} g_{K_{m,n}}^{(k)}. \quad (\text{A.11})$$

Given  $k = k_1p + k_0$ ,  $N-2 = A_1p + A_0$ , we have

$$\begin{aligned} F_{K_{m,n}} &= (-tx)^{N-1} \sum_{k_0=0}^{p-1} g_{K_{m,n}}^{(k_0)} \\ &\quad \times \sum_{k_1=0}^{\infty} \begin{pmatrix} A_1 + k_1 \\ k_1 \end{pmatrix} (1 - x^p)^{k_1} [1 - (1 - x^p t^{2p}) S_m((-t)^p) S_n((-t)^p)]^{k_1}. \end{aligned} \quad (\text{A.12})$$

Using the generalized binomial theorem, we obtain

$$F_{K_{m,n}} = \frac{(-tx)^{N-1} \sum_{k=0}^{p-1} g_{K_{m,n}}^{(k)}}{[x^p + (1 - x^p)(1 - x^p t^{2p}) S_m((-t)^p) S_n((-t)^p)]^{A_1+1}}. \quad (\text{A.13})$$

Recall the closed formulae of  $t$ -deformed Alexander polynomials (4.15), we can write

$$F_{K_{m,n}} = \frac{(-tx)^{A_0+1-p}}{\Delta_{K_{m,n}}(x^p, -(-t)^p)^{A_1+1}} \sum_{k=0}^{p-1} g_{K_{m,n}}^{(k)}. \quad (\text{A.14})$$

Before we jump to the ADO polynomials, let us first examine the properties of  $(a, t)$ -deformed  $F_K$ . When  $p = 1$  which leads to  $A_1 = N - 2$ ,  $A_0 = 0$ , we have

$$\lim_{q \rightarrow 1} F_{K_{m,n}}(x, -t^{-1}q^N, q, t) = \frac{g_{K_{m,n}}^{(0)}}{\Delta_{K_{m,n}}(x, t)^{N-1}} = \frac{1}{\Delta_{K_{m,n}}(x, t)^{N-1}}, \quad (\text{A.15})$$

in accordance with (4.12). Finally, for the  $t$ -deformed ADO polynomials, we have

$$\begin{aligned} \text{ADO}_{K_{m,n}}^{\text{SU}(N)}(p; x, t) &= \Delta_{K_{m,n}}(x^p, -(-t)^p)^{N-1} F_{K_{m,n}} \\ &= (-tx)^{A_0+1-p} \Delta_{K_{m,n}}(x^p, -(-t)^p)^{A_1(p-1)+A_0} \sum_{k=0}^{p-1} g_{K_{m,n}}^{(k)}, \end{aligned} \quad (\text{A.16})$$

where  $N - 2 = A_1 p + A_0$ .

Now let us consider some simple cases. If  $p = 1$ , then  $A_0 = 0$ , we have

$$\text{ADO}_{K_{m,n}}^{\text{SU}(N)}(p = 1; x, t) = g_{K_{m,n}}^{(0)} = 1. \quad (\text{A.17})$$

For  $N = 2$  and  $p = 2$ , then  $A_0 = A_1 = 0$ , we have

$$\text{ADO}_{K_{m,n}}^{\text{SU}(2)}(p = 2; x, t) = (-tx)^{-1} \left( g_{K_{m,n}}^{(0)} + g_{K_{m,n}}^{(1)} \right) = \Delta_{K_{m,n}}(x, t). \quad (\text{A.18})$$

## A.2 ADO polynomials of double twist knots $K_{m+\frac{1}{2}, -n}$ ( $m, n \in \mathbb{Z}_+$ )

Following the same procedure in the last subsection, we decompose the closed-form expression of  $F_{K_{m+\frac{1}{2}, -n}}(x, a, q, t)$  (4.5) into parts:

- The twist factor  $\text{tw}_n^{(k)}(x, q, t)$  (4.7) now becomes

$$\text{tw}_n^{(k)}(x, q, t) = \sum_{0=b_0 \leq b_1 \leq \dots \leq b_{n-1} \leq b_n=k} \prod_{i=1}^{n-1} (x^{2b_i} q^{-b_i(b_{i+1}-1)} t^{2b_i}) \begin{bmatrix} b_{i+1} \\ b_i \end{bmatrix}_q. \quad (\text{A.19})$$

We write  $b_i = \alpha_i p + \beta_i$ ,  $k = k_1 p + k_0$ , and we can obtain

$$\text{tw}_n^{(k)} = S_n^{k_1} (x^{2p} t^{2p}) \text{tw}_n^{(k_0)}. \quad (\text{A.20})$$

- Now  $g_{K_{m+\frac{1}{2}, -n}}^{(k)}$  in (4.6) becomes

$$g_{K_{m+\frac{1}{2}, -n}}^{(k)} = \sum_{j=0}^k (x; q^{-1})_k (x t^2 q^N; q)_j x^{k-j} \begin{bmatrix} N-2+k \\ k \end{bmatrix}_q \begin{bmatrix} k \\ j \end{bmatrix}_q q^{\frac{1}{2}j(j-1)+k} t^{-j+2k} \text{tw}_m^{(j)} \text{tw}_n^{(k)}. \quad (\text{A.21})$$

We write  $k = k_1 p + k_0$ ,  $j = j_1 p + j_0$ , and  $N - 2 = A_1 p + A_0$ , and obtain

$$\begin{aligned} g_{K_{m+\frac{1}{2}, -n}}^{(k)} &= \left\{ (1 - x^p) (-t)^p [(-xt)^p + (x^p t^{2p} - 1) S_m((-t)^p)] S_n(x^{2p} t^{2p}) \right\}^{k_1} \\ &\quad \times \begin{pmatrix} A_1 + k_1 \\ k_1 \end{pmatrix} g_{K_{m+\frac{1}{2}, -n}}^{(k_0)}. \end{aligned} \quad (\text{A.22})$$

- $F_{K_{m+\frac{1}{2},-n}}(x, a, q, t)$  now becomes

$$F_{K_{m+\frac{1}{2},-n}} = (-tx^n)^{N-1} \sum_{k=0}^{\infty} g_{K_{m+\frac{1}{2},-n}}^{(k)} . \quad (\text{A.23})$$

We write  $k = k_1 p + k_0$ ,  $N - 2 = A_1 p + A_0$ . Again, with the help of the generalized binomial theorem, we can get rid of the infinite summation over  $k_1$  and obtain

$$F_{K_{m+\frac{1}{2},-n}} = \frac{(-tx^n)^{A_0+1-p}}{\Delta_{K_{m+\frac{1}{2},-n}}(x^p, -(-t)^p)^{A_1+1}} \sum_{k_0=0}^{p-1} g_{K_{m+\frac{1}{2},-n}}^{(k_0)} , \quad (\text{A.24})$$

where the  $t$ -deformed Alexander polynomial is given in (4.15).

When  $p = 1$ , then  $A_1 = N - 2$ ,  $A_0 = 0$ , we have

$$\lim_{q \rightarrow 1} F_{K_{m+\frac{1}{2},-n}}(x, -t^{-1}q^N, q, t) = \frac{g_{K_{m+\frac{1}{2},-n}}^{(0)}}{\Delta_{K_{m+\frac{1}{2},-n}}(x, t)^{N-1}} = \frac{1}{\Delta_{K_{m+\frac{1}{2},-n}}(x, t)^{N-1}} , \quad (\text{A.25})$$

in accordance with (4.12). Finally, for the  $t$ -deformed ADO polynomials, we have

$$\begin{aligned} \text{ADO}_{K_{m+\frac{1}{2},-n}}^{\text{SU}(N)}(p; x, t) &= \Delta_{K_{m+\frac{1}{2},-n}}(x^p, -(-t)^p)^{N-1} F_{K_{m+\frac{1}{2},-n}} \\ &= (-tx^n)^{A_0+1-p} \Delta_{K_{m+\frac{1}{2},-n}}(x^p, -(-t)^p)^{A_1(p-1)+A_0} \sum_{k=0}^{p-1} g_{K_{m+\frac{1}{2},-n}}^{(k)} , \end{aligned} \quad (\text{A.26})$$

where  $N - 2 = A_1 p + A_0$ .

Now we consider some simple cases. When  $p = 1$ ,  $A_0 = 0$ , we have

$$\text{ADO}_{K_{m+\frac{1}{2},-n}}^{\text{SU}(N)}(p = 1; x, t) = g_{K_{m+\frac{1}{2},-n}}^{(0)} = 1 . \quad (\text{A.27})$$

For  $N = 2$ ,  $p = 2$ , we have

$$\text{ADO}_{K_{m+\frac{1}{2},-n}}(p = 2; x, t) = (-tx^n)^{-1} (g_{K_{m+\frac{1}{2},-n}}^{(0)} + g_{K_{m+\frac{1}{2},-n}}^{(1)}) = \Delta_{K_{m+\frac{1}{2},-n}}(x, t) . \quad (\text{A.28})$$

### A.3 Final formulae

In conclusion, for gauge group  $\text{SU}(N)$ , the  $t$ -deformed  $p$ -th ADO polynomials are given by

$$\begin{aligned} \text{ADO}_{K_{m,n}}^{\text{SU}(N)}(p; x, t) &= (-tx)^{A_0+1-p} \Delta_{K_{m,n}}(x^p, -(-t)^p)^{A_1(p-1)+A_0} \sum_{k=0}^{p-1} g_{K_{m,n}}^{(k)} , \\ \text{ADO}_{K_{m+\frac{1}{2},-n}}^{\text{SU}(N)}(p; x, t) &= (-tx^n)^{A_0+1-p} \Delta_{K_{m+\frac{1}{2},-n}}(x^p, -(-t)^p)^{A_1(p-1)+A_0} \sum_{k=0}^{p-1} g_{K_{m+\frac{1}{2},-n}}^{(k)} , \end{aligned}$$

where  $N - 2 = A_1 p + A_0$ ,  $A_1, A_0 \in \mathbb{N}$  and  $0 \leq A_0 \leq p - 1$ . It is easily seen that the recursion relation conjectured in [54] holds,

$$\text{ADO}_K^{\text{SU}(N+p)}(p; x, t) = \Delta_K(x^p, -(-t)^p)^{p-1} \text{ADO}_K^{\text{SU}(N)}(p; x, t) . \quad (\text{A.29})$$

Note that  $\frac{1}{p}S_m(\zeta_p^n) = 1$ , only when  $m|n$ . Otherwise it is zero. Therefore, the  $t$ -deformed  $p$ -th ADO polynomials, can also be written as,

$$\begin{aligned} \text{ADO}_{K_{m,n}}^{\text{SU}(N)}(p; x, t) &= \frac{1}{p} \sum_{l=0}^{p-1} (-tx)^{l+1-p} \Delta_{K_{m,n}}(x^p, -(-t)^p)^{\frac{(N-2)(p-1)+l}{p}} S_p(\zeta_p^{N-l-2}) \sum_{k=0}^{p-1} g_{K_{m,n}}^{(k)} \\ \text{ADO}_{K_{m+\frac{1}{2},-n}}^{\text{SU}(N)}(p; x, t) &= \frac{1}{p} \sum_{l=0}^{p-1} (-tx^n)^{l+1-p} \Delta_{K_{m+\frac{1}{2},-n}}(x^p, -(-t)^p)^{\frac{(N-2)(p-1)+l}{p}} \\ &\quad \times S_p(\zeta_p^{N-l-2}) \sum_{k=0}^{p-1} g_{K_{m+\frac{1}{2},-n}}^{(k)}. \end{aligned} \tag{A.30}$$

Although we write them as summations over  $l$ , there is only one non-vanishing term, which corresponds to that  $l$  is the remainder of  $(N-2)/p$ .

## B Notations for characters

The  $\text{U}(1)_k$  and  $\text{SU}(2)_k$  characters are given by

$$\chi_m^{\text{U}(1)_k}(\tau, z) = \frac{\Theta_{m,k}(\tau, z)}{\eta(\tau)}, \tag{B.1}$$

$$\chi_\ell^{\text{SU}(2)_k}(\tau, z) = \frac{\Theta_{\ell+1,k+2}(\tau, z) - \Theta_{-(\ell+1),k+2}(\tau, z)}{\Theta_{1,2}(\tau, z) - \Theta_{-1,2}(\tau, z)}, \tag{B.2}$$

where the Dedekind  $\eta$ -function is defined as

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n), \tag{B.3}$$

and the theta function is defined as

$$\Theta_{m,k}(\tau, z) = \sum_{n \in \mathbb{Z}} q^{k(n+\frac{m}{2k})^2} y^{k(n+\frac{m}{2k})}. \tag{B.4}$$

The weights of  $\text{U}(1)_k$  and  $\text{SU}(2)_k$  run over  $m = 0, \dots, 2k-1$  and  $\ell = 0, \dots, k$ , respectively. It is well-known that the modular group  $\text{SL}(2, \mathbb{Z})$  is generated by  $T$  and  $S$ , and a  $T$ -transformation on characters of a chiral algebra  $\mathcal{A}$  is always diagonalizable

$$\chi_r^{\mathcal{A}}(\tau + 1) = e^{2\pi i(h_r - c/24)} \chi_r^{\mathcal{A}}(\tau), \tag{B.5}$$

where  $h_r$  is the conformal dimension of the corresponding highest weight state. Under  $S$ -transformation, the characters of  $\mathcal{A}_k = \text{U}(1)_k, \text{SU}(2)_k$  are transformed as

$$\chi_r^{\mathcal{A}_k}\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) = e^{\frac{i\pi k z^2}{2\tau}} \sum_{r'} S_{rr'}^{\mathcal{A}_k} \chi_{r'}^{\mathcal{A}_k}(\tau, z), \tag{B.6}$$

where

$$S_{l,l'}^{\text{SU}(2)_k} \equiv \sqrt{\frac{2}{k+2}} \sin\left(\frac{\pi(l+1)(l'+1)}{k+2}\right),$$

$$S_{m,m'}^{\text{U}(1)_k} \equiv \frac{1}{\sqrt{2k}} e^{-2\pi i \frac{mm'}{2k}} . \quad (\text{B.7})$$

A character of a coset model  $\mathcal{A}/\mathcal{B}$  can be computed via a branching rule

$$V_\ell^{\mathcal{A}} = \bigoplus_m V_m^{\mathcal{B}} \oplus V_{\ell,m}^{\mathcal{A}/\mathcal{B}} \quad (\text{B.8})$$

where  $V_\ell^{\mathcal{A}}$  and  $V_m^{\mathcal{B}}$  are highest weight representations of the chiral algebra  $\mathcal{A}$  and  $\mathcal{B}$ , respectively. By defining the string function  $c_{l,m}^{(k)}$  [99]

$$\chi_\ell^{\text{SU}(2)_k}(\tau, z) = \sum_{m \in \mathbb{Z}_{2k}} c_{\ell,m}^{(k)}(\tau) \Theta_{m,k}(\tau, z) , \quad (\text{B.9})$$

a character of the parafermion is then expressed as

$$\chi_{\ell,m}^{PF_k}(\tau) = \eta(\tau) c_{\ell,m}^{(k)}(\tau) , \quad (\text{B.10})$$

where  $\ell + m \in 2\mathbb{Z}$ , and otherwise  $\chi_{\ell,m}^{PF_k} = 0$ . Note that the characters obey

$$\chi_{\ell,m}^{PF_k} = \chi_{\ell,2k-m}^{PF_k} = \chi_{k-\ell,m+k}^{PF_k} . \quad (\text{B.11})$$

In addition, a character of the  $\mathcal{N} = 2$  minimal model is the NS sector [100] is given by

$$\chi_{\ell,m}^{\text{MM}_k}(\tau, z) = \sum_{r \in \mathbb{Z}_{2k}} c_{\ell,r}^{(k)}(\tau) \Theta_{(k+2)r-km,k(k+2)}\left(\frac{\tau}{2}, \frac{z}{k+2}\right) , \quad (\text{B.12})$$

where the weights  $s = 0, 2$  of  $\text{U}(1)_2$  are summed. Note that the weights are subject to  $\ell + m \in 2\mathbb{Z}$ , and otherwise  $\chi_{\ell,m}^{\text{MM}_k} = 0$ . Note that the characters satisfy  $\chi_{\ell,m}^{\text{MM}_k} = \chi_{\ell,2(k+2)-m}^{\text{MM}_k} = \chi_{k-\ell,m+k+2}^{\text{MM}_k}$ .

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