

# Homework 6: Due at class on Oct 31

## 1 Derivation

### 1.1

Show (5.26) and find the fusion rule  $\phi_{1,2} \times \phi_{r,s} = [\phi_{r,s-1}] + [\phi_{r,s+1}]$ .

### 1.2

Show (5.29).

### 1.3

Following (5.34) of the lecture note, find the explicit fusion rule of the product

$$\phi_{1,3} \times \phi_{2,2}$$

## 2 Correlation functions of vertex operators

A vertex operator  $V_k(z) =: e^{ik\varphi}$  defined by using the normal-ordering can be understood as

$$V_k(z) := V_k^+(z)V_k^-(z)$$

where  $V_k^+(z)$  (resp.  $V_k^-(z)$ ) consists of creation (resp. annihilation) operators

$$V_k^+(z) := \exp \left( k \sum_{n=0}^{\infty} t_n(z) \right), \quad t_0 = i\varphi_0 \quad t_{n>0} = \frac{a_{-n}}{n} z^n,$$
$$V_k^-(z) := \exp \left( k \sum_{n=0}^{\infty} u_n(z) \right), \quad u_0 = a_0 \log z \quad u_{n>0} = -\frac{a_n}{n} z^{-n}.$$

Show that their OPE is

$$V_{k_1}(z)V_{k_2}(w) \sim (z-w)^{k_1 k_2} : V_{k_1}(z)V_{k_2}(w) : ,$$

where

$$V_{k_1}(z)V_{k_2}(w) = V_{k_1}^+(z)V_{k_1}^-(z)V_{k_2}^+(w)V_{k_2}^-(w)$$
$$: V_{k_1}(z)V_{k_2}(w) : = V_{k_1}^+(z)V_{k_2}^+(w)V_{k_1}^-(z)V_{k_2}^-(w) .$$

In addition, show that the two point function of vertex operators is

$$\langle V_{k_1}(z)V_{k_2}(w) \rangle = \frac{\delta_{k_1, -k_2}}{(z-w)^{k_1^2}} .$$

In a similar fashion, show that multi-point correlation function of vertex operators is given by

$$\langle V_{k_1}(z_1)V_{k_2}(z_2) \cdots V_{k_n}(z_n) \rangle = \delta_{\sum k_i, 0} \prod_{1 \leq i < j \leq n} (z_i - z_j)^{k_i k_j} .$$

### 3 Linear dilaton CFT

In the lecture, we have consider the free boson. Now let us consider another important theory whose action is

$$S = \frac{1}{8\pi} \int d^2z \sqrt{g} \nabla_\alpha \varphi \nabla^\alpha \varphi + \frac{Q}{4\pi} \int d^2z \sqrt{g} R \varphi$$

where  $g$  is the metric of the two-dimensional world-sheet and  $R$  is the Ricci scalar of  $g$ . This model is called linear dilaton CFT and  $Q$  is called background charge. Using the definition

$$T_{\alpha\beta} = -\frac{4\pi}{\sqrt{g}} \frac{\delta S}{\delta g^{\alpha\beta}} ,$$

one can show that the energy-momentum tensor in the flat metric  $g_{\alpha\beta} = \delta_{\alpha\beta}$  is

$$T(z) = -\frac{1}{2} : \partial\varphi(z) \partial\varphi(z) : + Q \partial^2 \varphi(z) .$$

(If you derive this, I will give you an extra point.)

#### 3.1

Compute  $TT$  OPE and find the central charge of the linear dilaton CFT.

#### 3.2

Show that  $\partial\varphi$  is not primary unless  $Q = 0$ , but is quasi-primary with conformal dimension  $h = 1$ . Show that the vertex operator  $V_k(z) =: e^{ik\varphi} :$  is primary and compute its conformal dimension. In addition, compute the OPE  $\partial\varphi(z) V_k(w)$ .

#### 3.3

Let us define the state  $|k\rangle = \lim_{z \rightarrow 0} V_k(z) |0\rangle$  that corresponds to the vertex operator  $V_k(z)$ . Show that

$$a_0 |k\rangle = k |k\rangle , \quad L_0 |k\rangle = \frac{k(k+2iQ)}{2} |k\rangle , \quad L_n |k\rangle = 0 \quad \text{for } n > 0 .$$

#### 3.4

When  $\sqrt{2}\alpha_\pm = -iQ \pm \sqrt{2 - Q^2}$ , find that

$$T(z) V_{\sqrt{2}\alpha_\pm}(w) \sim \frac{\partial}{\partial w} \frac{V_{\sqrt{2}\alpha_\pm}(w)}{z - w}$$

so that

$$\left[ T(z), \oint \frac{dw}{2\pi i} V_{\sqrt{2}\alpha_\pm}(w) \right] = 0 .$$

In fact,  $S_\pm = \oint \frac{dw}{2\pi i} V_{\sqrt{2}\alpha_\pm}(w)$  is called the screening charge.

## 4 Feigin-Fuchs representation

We shall derive the Kac determinant (5.15) of the lecture note by following Feigin-Fuchs.

### 4.1 Kac determinant

Show that the parameters in the Kac determinant (5.15) obey

$$\alpha_+ + \alpha_- = 2\sqrt{-h_0}, \quad \alpha_+\alpha_- = -1.$$

### 4.2 Screening charge

Let us consider the linear dilaton CFT in the previous problem where the background charge is set to

$$Q = \sqrt{2h_0} = i\sqrt{\frac{1-c}{12}}.$$

Recalling that a vertex operator  $V_k(z) =: e^{ik\varphi(z)} :$  in the linear dilaton CFT has conformal dimension  $h = k(k + 2iQ)/2$ , derive that the conformal dimension of the vertex operator  $V_{\sqrt{2}\alpha_{\pm}}(w)$  is equal to one. Hence,  $\mathbf{S}_{\pm} = \oint \frac{dw}{2\pi i} V_{\sqrt{2}\alpha_{\pm}}(w)$  are the screening charge. Show that the state

$$\mathbf{S}_{\pm}^r |k\rangle \quad \text{for } r \in \mathbb{Z}_{>0}$$

is a singular vector where  $|k\rangle = \lim_{z \rightarrow 0} V_k(z)|0\rangle$  if it is not zero.

### 4.3 Integral representation

It admits the integral representation as

$$\begin{aligned} \mathbf{S}_+^r |k\rangle &= \frac{1}{(2\pi i)^r} \int dz_1 \cdots dz_r V_{\sqrt{2}\alpha_+}(z_1) \cdots V_{\sqrt{2}\alpha_+}(z_r) |k\rangle \\ &= \frac{1}{(2\pi i)^r} \int dz_1 \cdots dz_r \prod_{1 \leq i < j \leq r} (z_i - z_j)^{2\alpha_+^2} : V_{\sqrt{2}\alpha_+}(z_1) \cdots V_{\sqrt{2}\alpha_+}(z_r) : |k\rangle. \end{aligned}$$

Let us recall that

$$a_0 |k\rangle = k |k\rangle, \quad e^{ip\varphi_0} |k\rangle = |k+p\rangle, \quad a_n |k\rangle = 0, \quad (n > 0).$$

In addition, the normal-ordering can be understood as

$$: V_{\sqrt{2}\alpha_+}(z_1) \cdots V_{\sqrt{2}\alpha_+}(z_r) := V_{\sqrt{2}\alpha_+}^+(z_1) \cdots V_{\sqrt{2}\alpha_+}^+(z_r) V_{\sqrt{2}\alpha_+}^-(z_1) \cdots V_{\sqrt{2}\alpha_+}^-(z_r).$$

Here we write the creation operator  $V_k^+(z)$

$$V_{\sqrt{2}\alpha_+}^+(z) = \exp \left( \sqrt{2}\alpha_+ \sum_{n=0}^{\infty} t_n(z) \right) = \left( \sum_{n=0}^{\infty} p_n z^n \right) e^{i\sqrt{2}\alpha_+ \varphi_0}.$$

Then, show that the integral expression can be written as

$$\mathbf{S}_+^r |k\rangle = \frac{1}{(2\pi i)^r} \int dz_1 \cdots dz_r \prod_{1 \leq i < j \leq r} (z_i - z_j)^{2\alpha_+^2} \sum_{m_i \geq 0} \prod_{i=1}^r p_{m_i} z_i^{m_i + k\sqrt{2}\alpha_+} |k + r\sqrt{2}\alpha_+\rangle .$$

In order for this state to be non-zero, the integral should be invariant under the scaling  $z_i \rightarrow \lambda z_i$ , which requires

$$r + \frac{r(r-1)}{2} 2\alpha_+^2 + \sum_{i=1}^r m_i + rk\sqrt{2}\alpha_+ = 0 .$$

If we assume that the level  $\sum_{i=1}^r m_i$  of the singular vector  $\mathbf{S}_+^r |k\rangle$  is equal to  $rs$ , show that  $k$  has to be

$$k = \frac{1-r}{2} \sqrt{2}\alpha_+ + \frac{1+s}{2} \sqrt{2}\alpha_- . \quad (4.1)$$

This implies that  $\mathbf{S}_+^r |k\rangle$  is a non-zero singular vector at level  $rs$  if  $k$  is subject to (4.1). Compute the conformal dimension of  $\mathbf{S}_+^r |k\rangle$  and show that it is equal to  $h_{r,s}$  in the Kac determinant (5.15) of the lecture note.