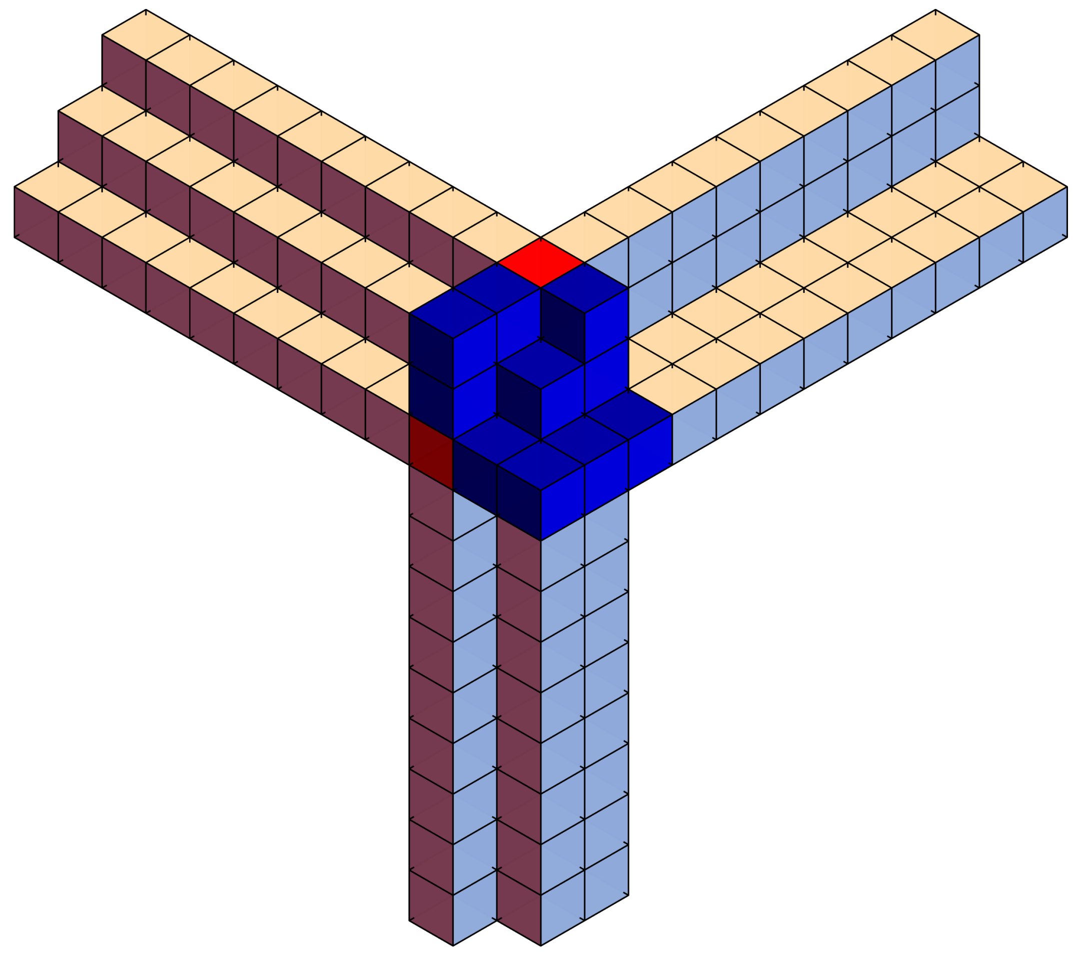

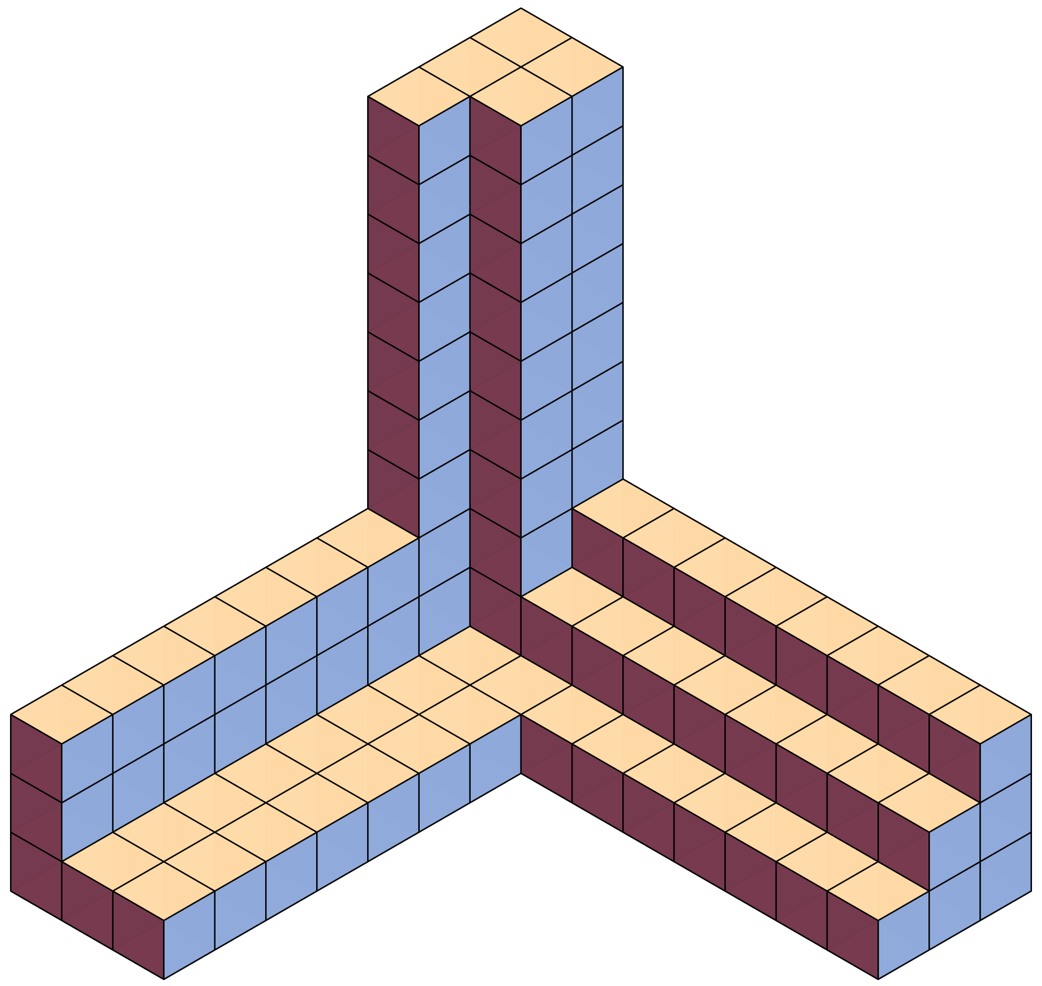

Vortex Origami

Based on work in progress
with Jiaqun Jiang, Taro Kimura, Go Noshita and Jiahao Zheng

Satoshi Nawata, Fudan University

Plan of Talk

1. Counting boxes to Tetrahedron Instanton

2. Donaldson-Thomas and Pandharipande–Thomas

3. DT/PT Correspondence

4. Vortex Origami

Plan of Talk

1. Counting boxes to Tetrahedron Instanton

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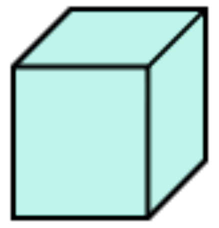
MacMahon Function

The MacMahon function is the generating function of plane partitions:

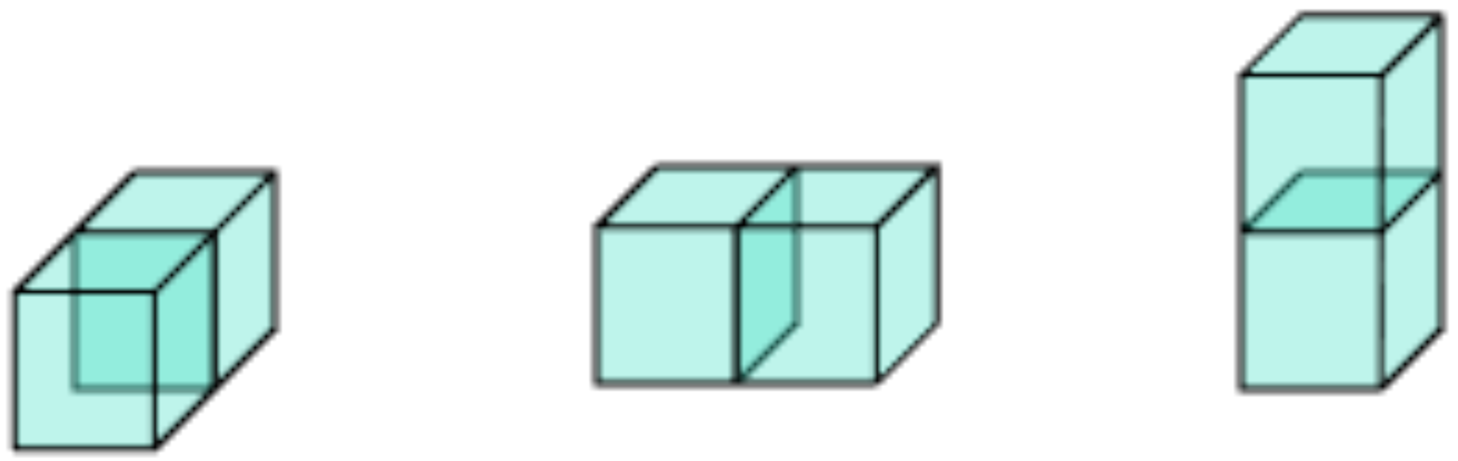
$$M(\mathfrak{q}) = \sum_{\pi \in \mathcal{PP}} \mathfrak{q}^{|\pi|} = \prod_{n=1}^{\infty} \frac{1}{(1-\mathfrak{q}^n)^n} = \mathrm{PE}\left[\frac{\mathfrak{q}}{(1-\mathfrak{q})^2}\right], \qquad |\mathfrak{q}| < 1.$$

$$= 1 + \mathfrak{q} + 3\mathfrak{q}^2 + 6\mathfrak{q}^3 + 13\mathfrak{q}^4 + 24\mathfrak{q}^5 + 48\mathfrak{q}^6 + \cdots$$

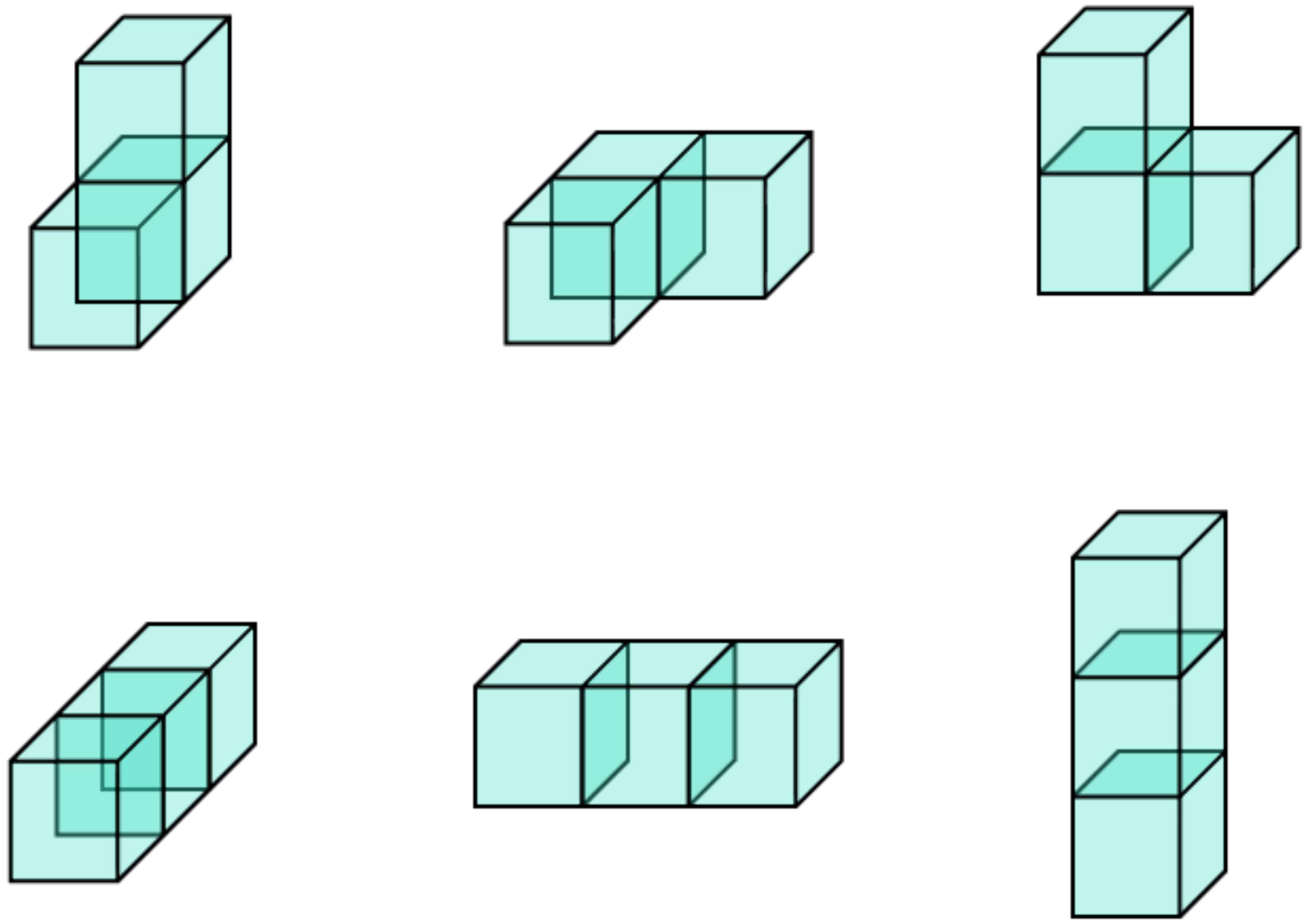
1-box



2-box



3-box

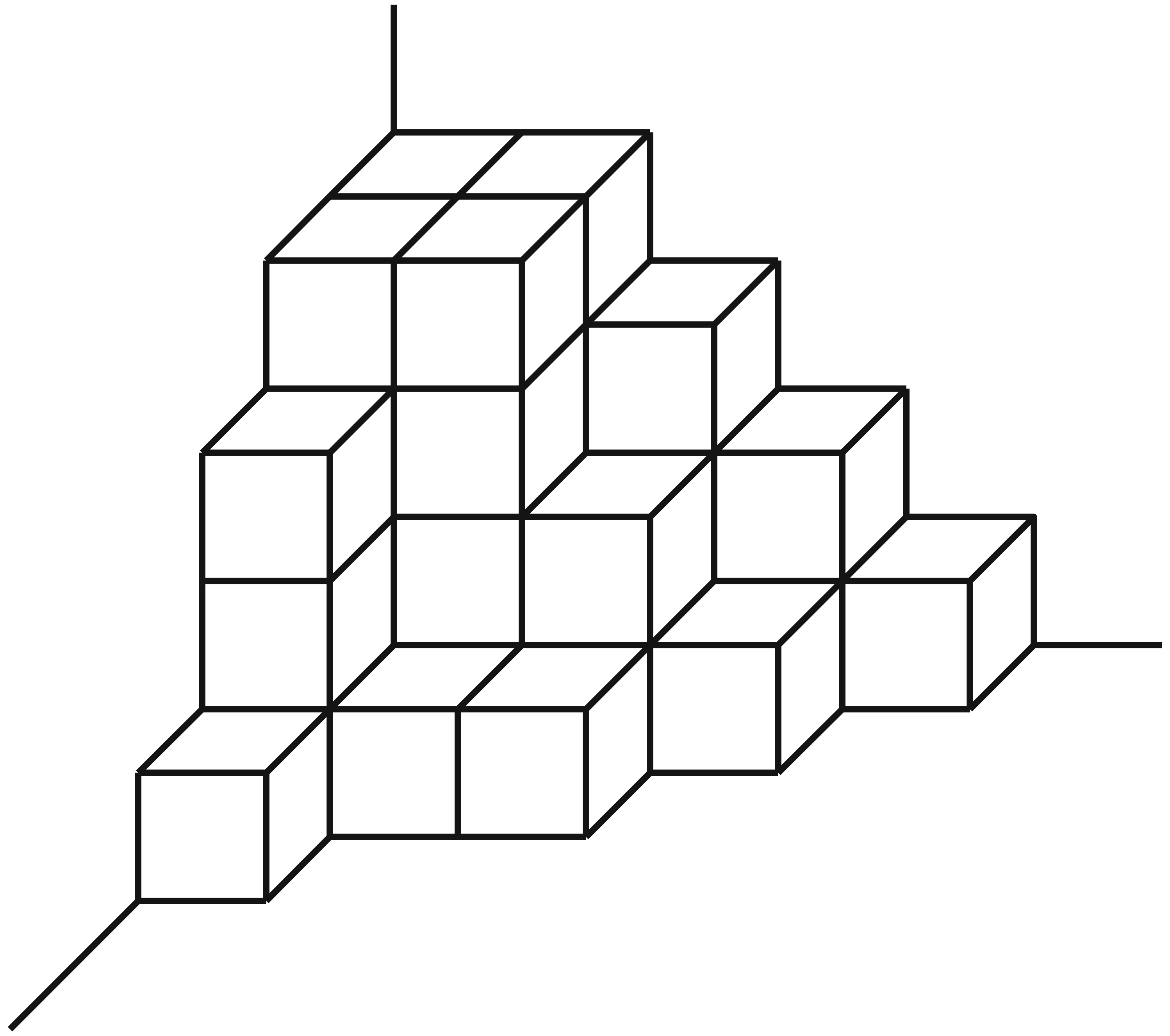


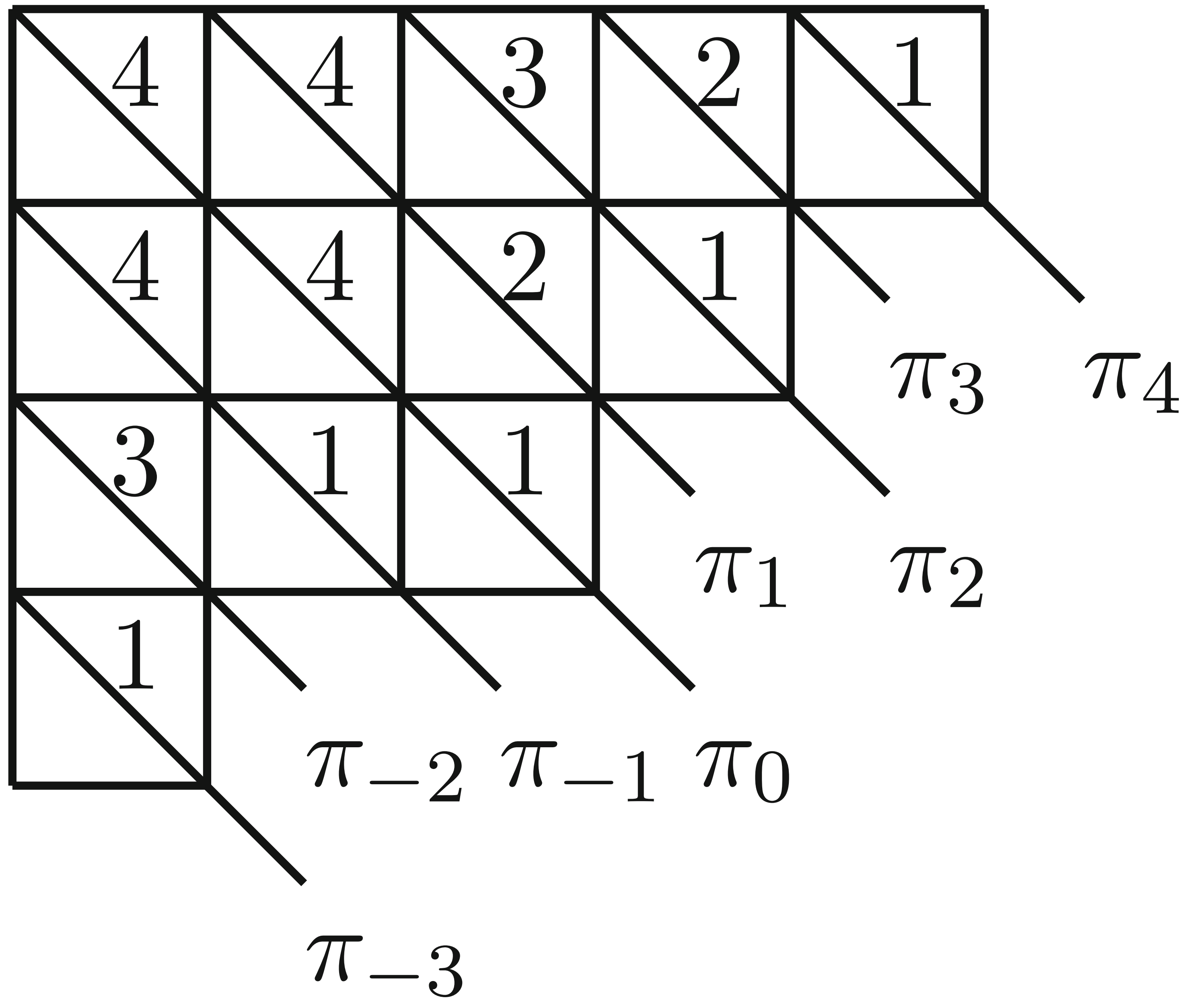
Refined MacMahon Function

We can further introduce a refinement of the MacMahon function,

$$M(q,t) = \sum_{\pi \in \mathcal{PP}} t^{\sum_{i=1}^{\infty} |\pi(-i)|} \, q^{\sum_{j=1}^{\infty} |\pi(j-1)|} = \prod_{i,j=1}^{\infty} \frac{1}{1-q^i t^{j-1}} = \mathrm{PE}\left[\frac{q}{(1-q)(1-t)}\right].$$

Here, $\pi(i)$ denotes the 2d Young diagrams arising from the diagonal slicing of a plane partition.





Further generalization: D6-D0 partition function

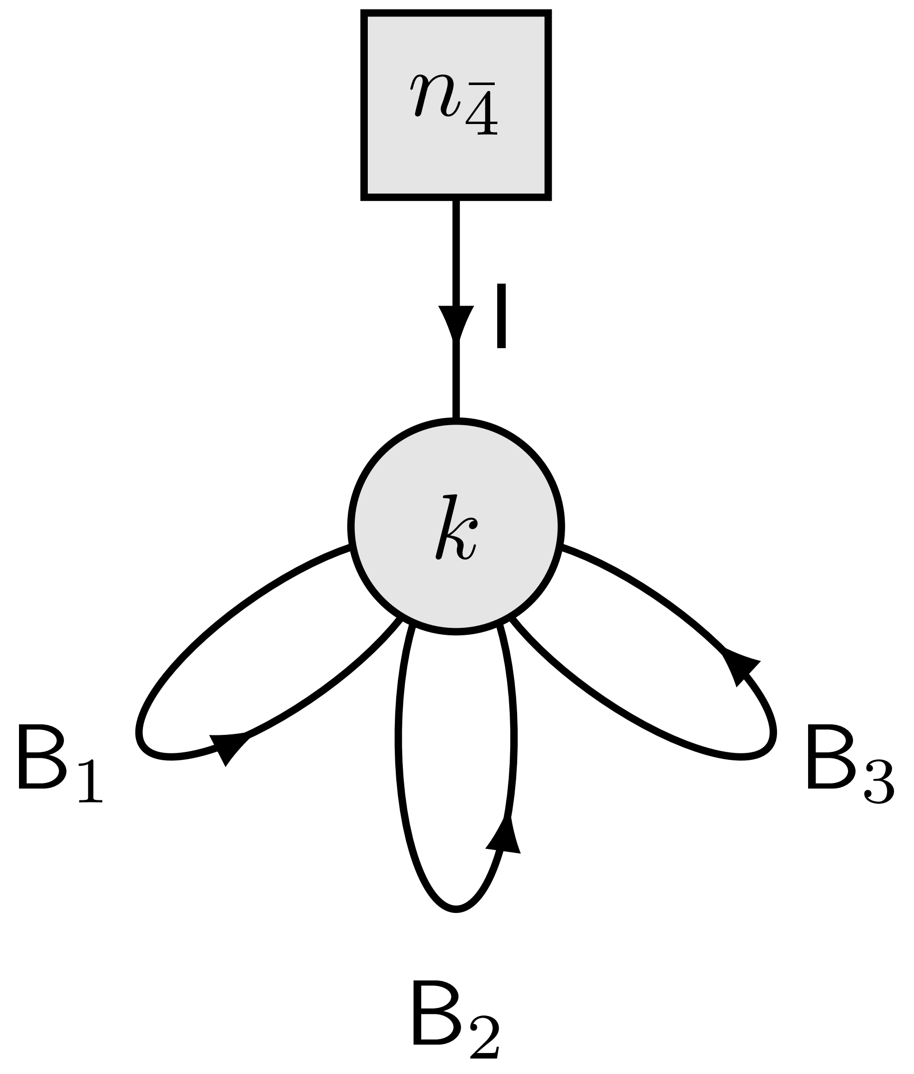
We consider a system of D6–D0 branes in an Ω-background,

$$\mathbb{C}_i : \quad z_i \longrightarrow q_i\, z_i,$$

subject to the CY4 condition $q_1 q_2 q_3 q_4 = 1$.

	$\mathbb{C}_1$		$\mathbb{C}_2$		$\mathbb{C}_3$		$\mathbb{C}_4$		$\mathbb{R}^{1,1}$	
	1	2	3	4	5	6	7	8	9	0
k D0	•	•	•	•	•	•	•	•	•	—
$n_{\bar{4}}$ $\mathrm{D6}_{123}$	—	—	—	—	—	—	•	•	•	—

The low-energy dynamics on the D0-branes is governed by $\mathcal{N}=(2,2)$ supersymmetric quantum mechanics.



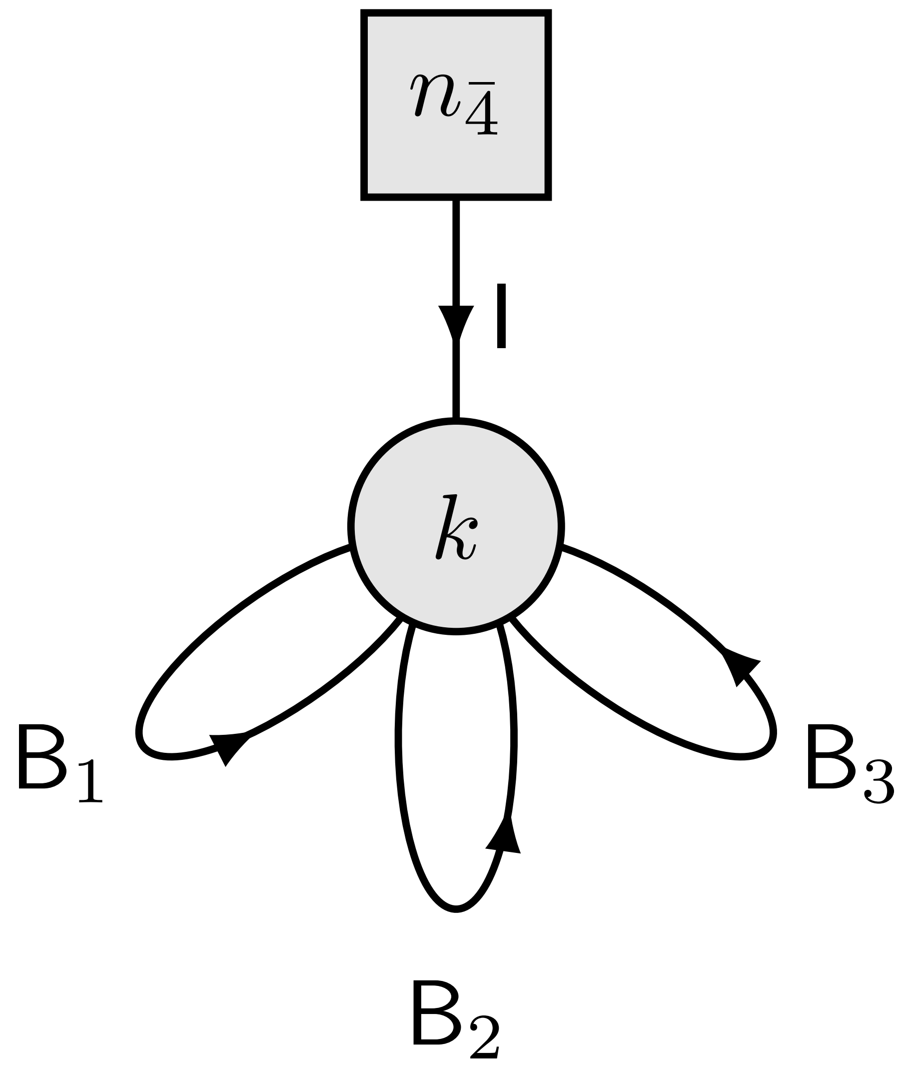
$$W_0 = \mathrm{Tr}\left(B_1\left[B_2, B_3\right]\right)$$

Further generalization: D6-D0 partition function

The equivariant localization on the moduli space of the quiver (Q, W_0) leads to the Jeffrey–Kirwan integral

$$\mathcal{Z}^{\text{D6-D0}} = \sum_{k=0}^{\infty} \mathfrak{q}^k \, \mathcal{Z}^{\text{D6}}[k],$$

$$\mathcal{Z}^{\text{D6}}[k] = \frac{1}{k!} \oint_{\text{JK}} \prod_{I=1}^{k} \frac{d\phi_I}{2\pi i} \, \mathcal{Z}^{\text{D6–D0}}(\{\mathfrak{a}_\alpha\}, \{\phi_I\}) \prod_{I,J} \mathcal{Z}^{\text{D0–D0}}(\phi_I, \phi_J).$$



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Using the notation $\text{sh}(x) := e^{x/2} - e^{-x/2}$, the integrand is expressed as

$$\mathcal{Z}^{\text{D6–D0}}(\{\mathfrak{a}_\alpha\}, \phi_I) = \prod_{\alpha=1}^{n_{\bar{4}}} \prod_{I=1}^{k} \frac{\text{sh}(\phi_I - \mathfrak{a} - \epsilon_4)}{\text{sh}(\phi_I - \mathfrak{a})},$$

$$\mathcal{Z}^{\text{D0–D0}}(\phi_I, \phi_J) = \frac{\text{sh}'(\phi_I - \phi_J) \, \text{sh}(\phi_I - \phi_J - \epsilon_{14,24,34})}{\text{sh}(\phi_I - \phi_J - \epsilon_{1,2,3,4})}$$

Further generalization: D6-D0 partition function

The equivariant localization on the moduli space of the quiver (Q, W_0) leads to the Jeffrey–Kirwan integral

$$\mathcal{Z}^{\text{D6-D0}} = \sum_{k=0}^{\infty} \mathfrak{q}^k \, \mathcal{Z}^{\text{D6}}[k],$$

$$\mathcal{Z}^{\text{D6}}[k] = \frac{1}{k!} \oint_{\text{JK}} \prod_{I=1}^{k} \frac{d\phi_I}{2\pi i} \mathcal{Z}^{\text{D6–D0}}(\{\mathfrak{a}_\alpha\}, \{\phi_I\}) \prod_{I,J} \mathcal{Z}^{\text{D0–D0}}(\phi_I, \phi_J).$$

When $n_{\bar{4}} = 1$, after performing the Jeffrey–Kirwan (JK) residue integral, the result can be conveniently repackaged as

$$\mathcal{Z}^{\text{D6-D0}}[\mathfrak{q}, q_{1,2,3,4}] = \text{PE}\left[\frac{[q_{14}]\,[q_{24}]\,[q_{34}]}{[q_1]\,[q_2]\,[q_3]} \, \frac{1}{[\mathfrak{q} q_4^{-1/2}]\,[\mathfrak{q} q_4^{1/2}]} \right].$$

Nekrasov

Here we use the shorthand notation

$$[x] := x^{1/2} - x^{-1/2}.$$

MacMahon limit

In the unrefined limit $q_4 \to 1$, the single-particle index inside the plethystic exponential simplifies as

$$\frac{[q_{14}][q_{24}][q_{34}]}{[q_1][q_2][q_3]} \frac{\mathfrak{q}}{(1-\mathfrak{q}q_4^{-1/2})(1-\mathfrak{q}q_4^{1/2})} \xrightarrow{q_4 \to 1} \frac{\mathfrak{q}}{(1-\mathfrak{q})^2}.$$

As a result, the partition function reduces to the MacMahon function,

$$\mathcal{Z}^{\text{D6-D0}}[\mathfrak{q}; q_{1,2,3,4}] \xrightarrow{q_4 \to 1} M(\mathfrak{q}).$$

Refined MacMahon limit

To obtain the refined MacMahon function, we first rescale the equivariant parameters as

$$(q_1, q_2, q_3, q_4) \longrightarrow (z^{r_1} q_1,\ z^{r_2} q_2,\ z^{r_3} q_3,\ q_4), \qquad r_1 + r_2 + r_3 = 0.$$

For the region $r_1 \gg r_3 > 0 \gg r_2$, taking the limit $z \to 0$ yields

$$\frac{[q_{14}][q_{24}][q_{34}]}{[q_1][q_2][q_3]} \frac{\mathfrak{q}}{(1 - \mathfrak{q} q_4^{-1/2})(1 - \mathfrak{q} q_4^{1/2})} \xrightarrow{z \to 0} \frac{q_4^{-1/2} \mathfrak{q}}{(1 - \mathfrak{q} q_4^{-1/2})(1 - \mathfrak{q} q_4^{1/2})}.$$

Defining

$$q = \mathfrak{q} q_4^{1/2}, \qquad t = \mathfrak{q} q_4^{-1/2},$$

the partition function reduces to the refined MacMahon function,

$$\mathcal{Z}^{\text{D6-D0}}[\mathfrak{q}; q_{1,2,3,4}] \xrightarrow{z \to 0} M(t, q).$$

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Donaldson-Thomas invariants

- **Donaldson–Thomas (DT) theory** provides an enumerative framework for counting curves on a smooth (quasi-)projective Calabi–Yau threefold X.

- The basic objects are *ideal sheaves of curves* on X, which form the moduli space
$$\mathcal{M}^{\mathrm{DT}}(X) = \{\text{ideal sheaves of curves on } X\}.$$

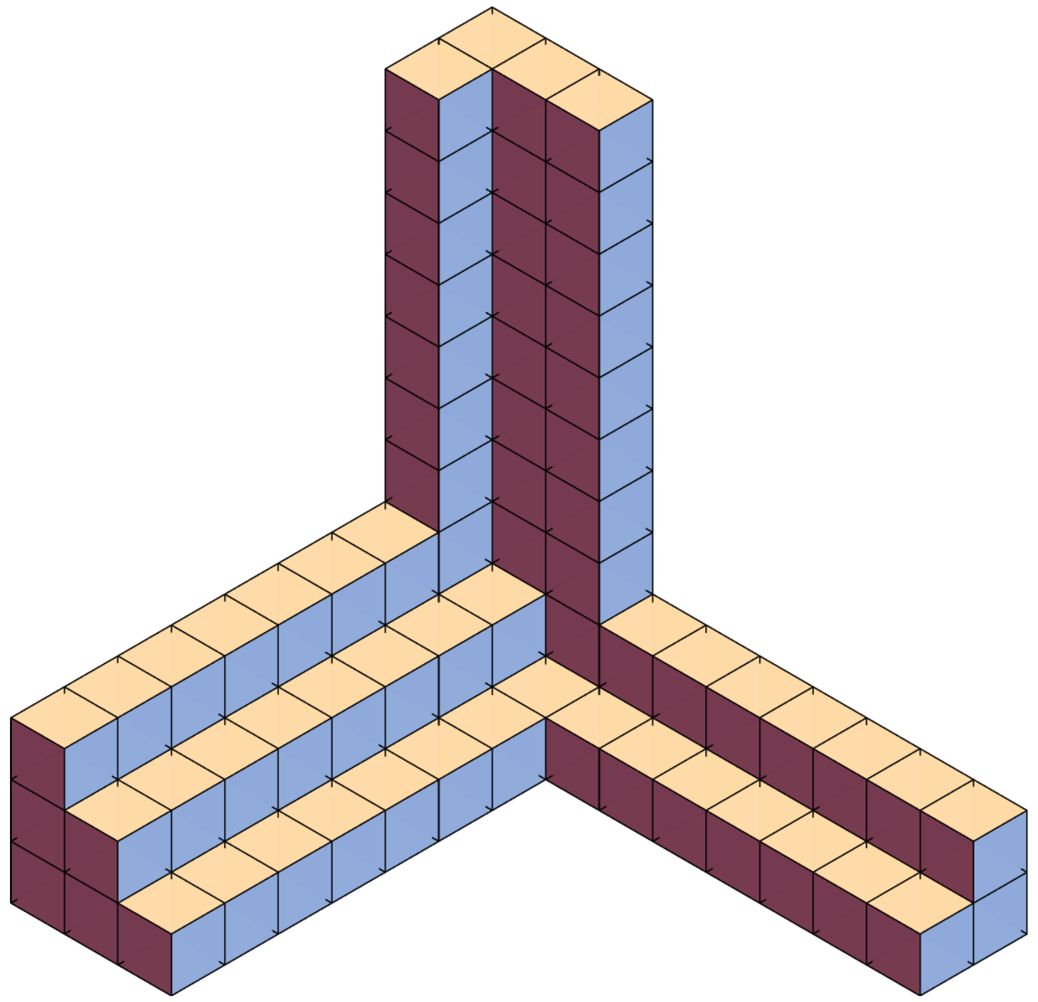
- Any element of $\mathcal{M}^{\mathrm{DT}}(X)$ fits into a short exact sequence
$$0 \longrightarrow \mathcal{I}_Z \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Z \longrightarrow 0,$$
where $\mathcal{I}_Z \subset \mathcal{O}_X$ is the ideal sheaf of a subscheme $Z \subset X$.

Donaldson-Thomas invariants

- When X is **toric**, it is locally modeled by an affine chart $\mathbb{C}^3 \subset X$, and an ideal sheaf takes the form

$$0 \longrightarrow I \longrightarrow \mathbb{C}[q_1, q_2, q_3] \longrightarrow \mathbb{C}[q_1, q_2, q_3]/I \longrightarrow 0.$$

- Torus-fixed points in the DT moduli space are in one-to-one correspondence with plane partitions with prescribed asymptotic boundary conditions (λ, μ, ν).



- We denote the set of plane partitions with fixed asymptotics by

$$\mathcal{DT}_{\lambda\mu\nu} := \{\text{3d partitions with legs } \lambda, \mu, \nu\}.$$

- In the non-equivariant limit with empty boundary conditions, the DT partition function reduces to

$$DT_{\emptyset,\emptyset,\emptyset}(\mathfrak{q}) = \sum_{\pi \in \mathcal{DT}_{\emptyset,\emptyset,\emptyset}} \mathfrak{q}^{|\pi|} = M(\mathfrak{q}),$$

where $M(\mathfrak{q})$ is the MacMahon function.

Donaldson-Thomas invariants

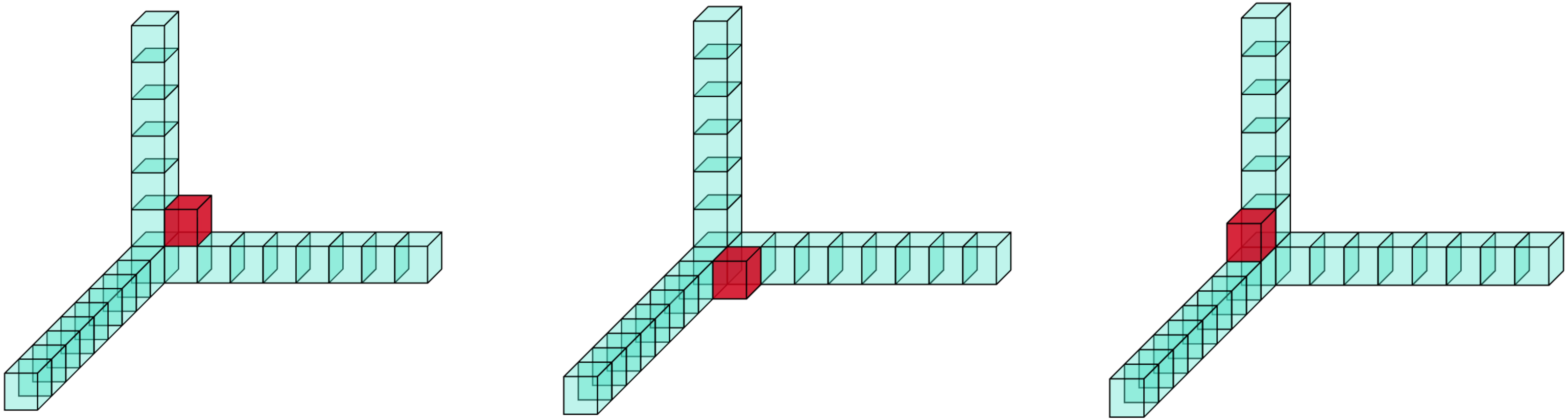
The non-equivariant DT partition function counts three-dimensional partitions with fixed asymptotics (λ, μ, ν):

$$DT_{\lambda\mu\nu}(\mathfrak{q}) := \sum_{\pi \in \mathcal{DT}_{\lambda\mu\nu}} \mathfrak{q}^{|\pi|}.$$

As an example, for single-box boundary conditions we find

1-box

$$DT_{\square,\square,\square}(\mathfrak{q}) = 1 + 3\mathfrak{q} + 9\mathfrak{q}^2 + 22\mathfrak{q}^3 + 52\mathfrak{q}^4 + \cdots .$$



Donaldson-Thomas invariants

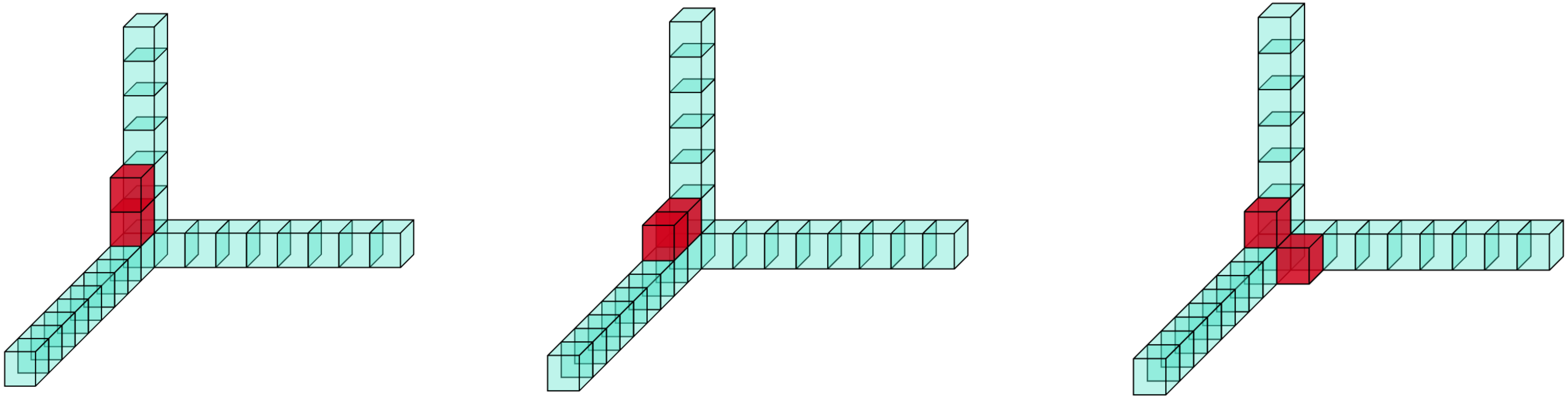
The non-equivariant DT partition function counts three-dimensional partitions with fixed asymptotics (λ, μ, ν):

$$DT_{\lambda\mu\nu}(\mathfrak{q}) := \sum_{\pi \in \mathcal{DT}_{\lambda\mu\nu}} \mathfrak{q}^{|\pi|}.$$

As an example, for single-box boundary conditions we find

$$DT_{\square,\square,\square}(\mathfrak{q}) = 1 + 3\mathfrak{q} + 9\mathfrak{q}^2 + 22\mathfrak{q}^3 + 52\mathfrak{q}^4 + \cdots .$$

2-box



+ cyclic permutations (in total, 9 configurations)

Pandharipande-Thomas invariants

- **Pandharipande–Thomas (PT) theory** provides an alternative framework for curve counting on a Calabi–Yau threefold X.
- The basic objects are *stable pairs*

$$[\,\mathcal{O}_X \xrightarrow{s} \mathcal{F}\,],$$

  where $\mathcal{F}$ is a pure one-dimensional sheaf and the cokernel of s is zero-dimensional.

- The moduli space of stable pairs is

$$\mathcal{M}^{\mathrm{PT}}(X) = \{\text{stable pairs on } X\}.$$

- Each stable pair fits into a short exact sequence

$$0 \longrightarrow \mathcal{I}_C \longrightarrow \mathcal{O}_X \xrightarrow{s} \mathcal{F} \longrightarrow \mathcal{Q} \longrightarrow 0,$$

  where $\mathcal{I}_C$ is the ideal sheaf of a curve C, and $\mathcal{Q}$ is a zero-dimensional quotient.

Pandharipande-Thomas invariants

- On a toric affine chart $\mathbb{C}^3 \subset X$, a stable pair takes the form

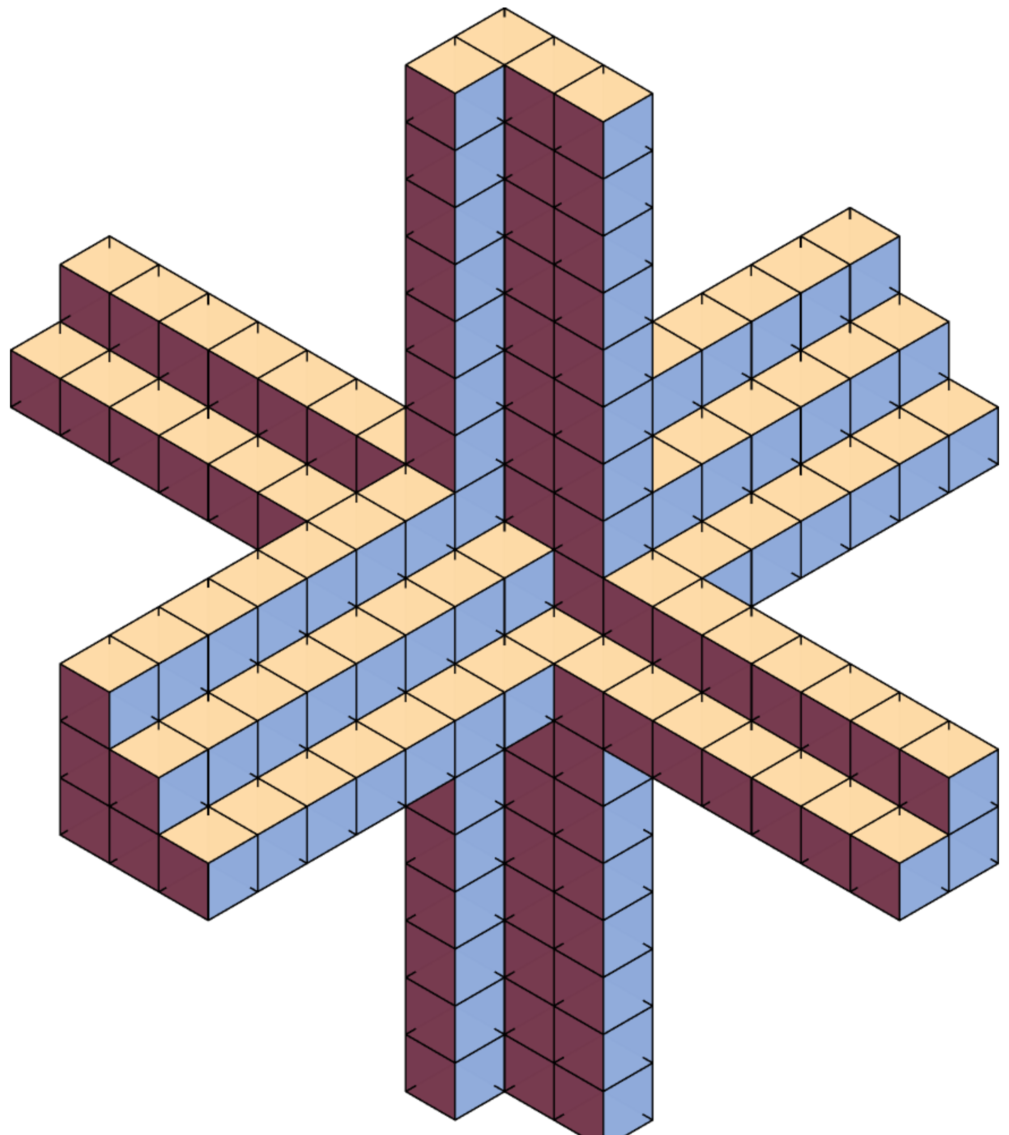
$$0 \longrightarrow I_C \longrightarrow \mathbb{C}[q_1, q_2, q_3] \xrightarrow{s} F \longrightarrow Q \longrightarrow 0,$$

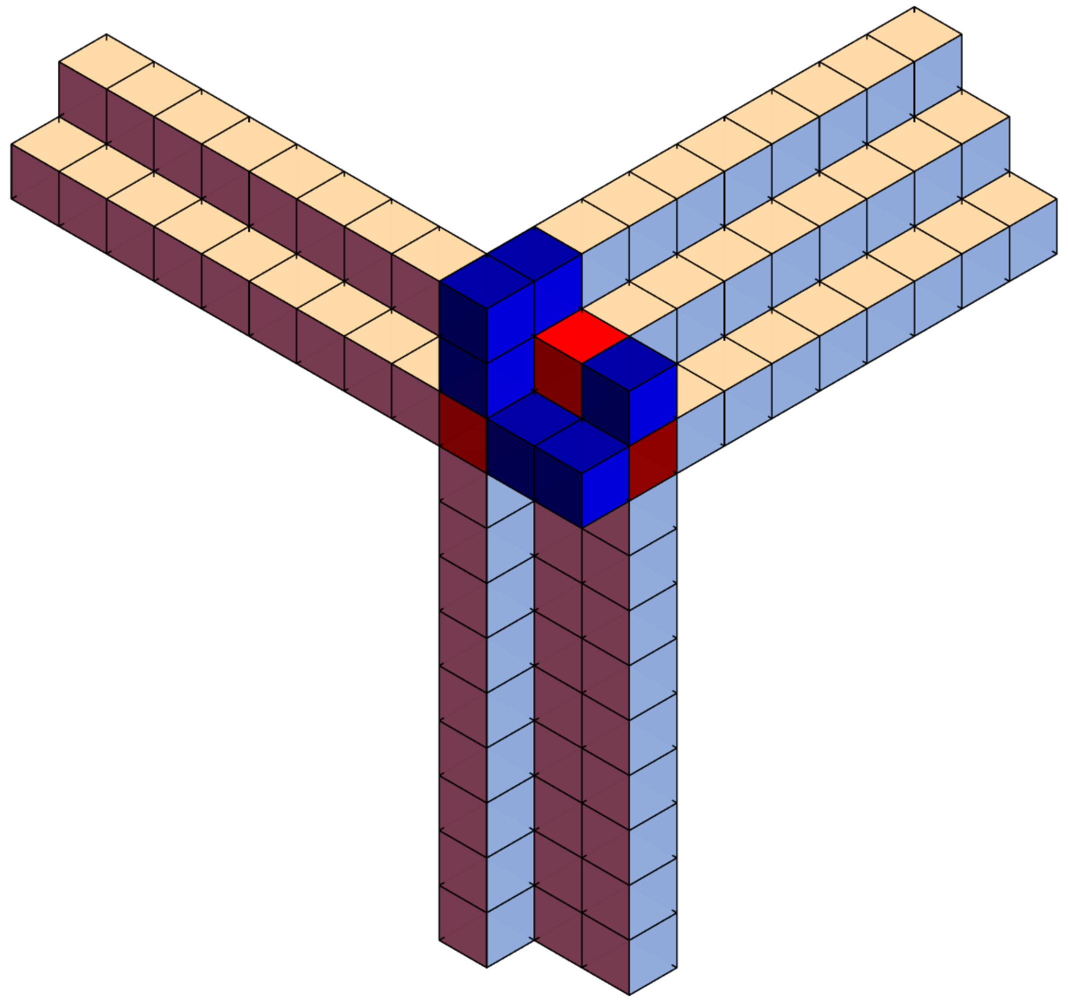
  where I_C is the ideal sheaf of a curve and Q is zero-dimensional.

- A torus-fixed configuration corresponds to a **PT 3d partition**, specified by asymptotic legs (λ, μ, ν) along the coordinate axes.

- We denote the set of such configurations by

$$\mathcal{PT}_{\lambda\mu\nu} := \{\text{PT configurations with legs } \lambda, \mu, \nu\}.$$





Pandharipande-Thomas invariants

- In contrast to DT theory, PT configurations are fewer but geometrically richer, often forming positive-dimensional families rather than isolated fixed points.

- The Pandharipande–Thomas (PT) partition function is given by

$$PT_{\lambda\mu\nu}(\mathfrak{q}) = \widetilde{C}_{\lambda^{\mathrm{T}}\mu^{\mathrm{T}}\nu^{\mathrm{T}}}(\mathfrak{q}),$$

  where $\widetilde{C}_{\lambda\mu\nu}(\mathfrak{q})$ is the unrefined topological vertex, normalized so that its $\mathfrak{q}$-expansion starts from 1. In particular, the empty-leg vertex is trivial:

$$PT_{\emptyset,\emptyset,\emptyset} = 1.$$

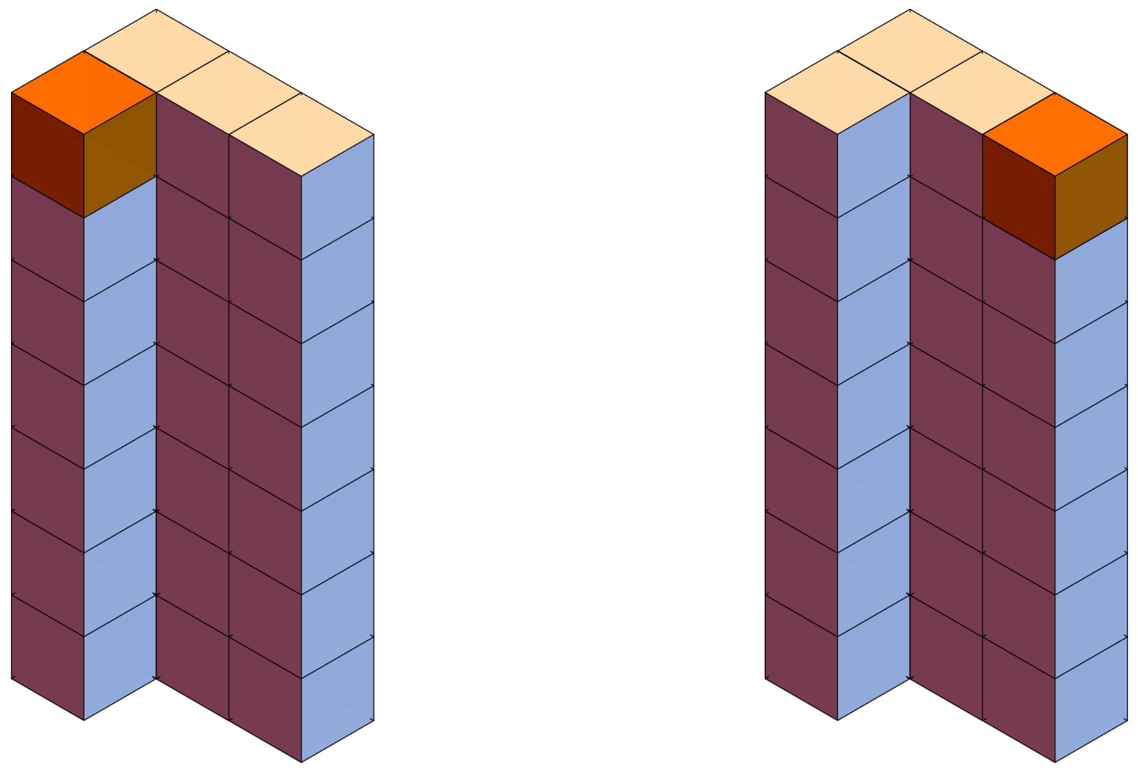
- Recall that the unrefined topological vertex is defined by

$$C_{\lambda\mu\nu}(q) = q^{\kappa(\mu)/2}\, s_{\nu^{\mathrm{T}}}(q^{-\rho}) \sum_{\eta} s_{\lambda^{\mathrm{T}}/\eta}(q^{-\rho-\nu})\, s_{\mu/\eta}(q^{-\rho-\nu^{\mathrm{T}}}).$$

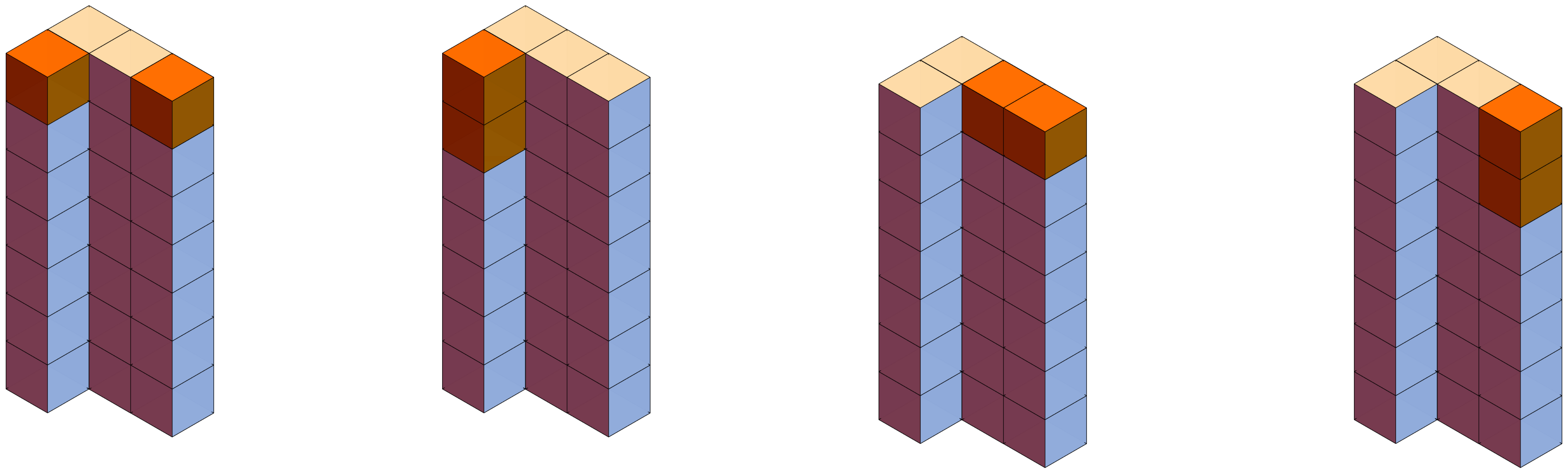
Pandharipande-Thomas invariants

Non-equivariant version: counting rule is simple for 1- and 2-leg cases

1-box: 2 configs



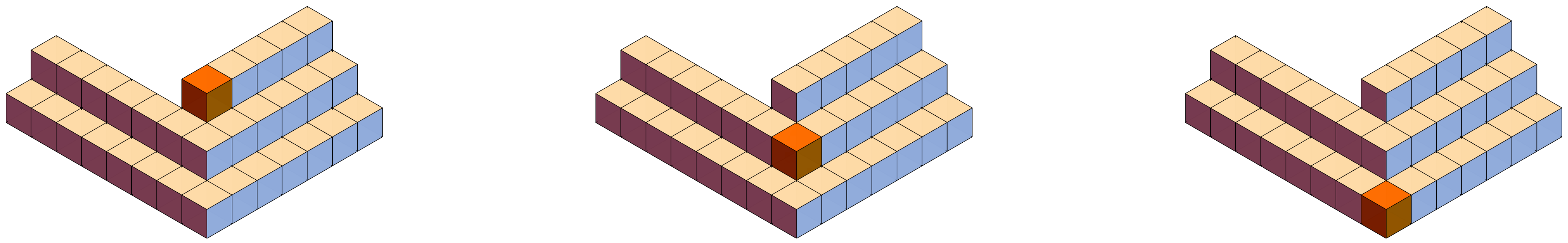
2-box: 4 configs



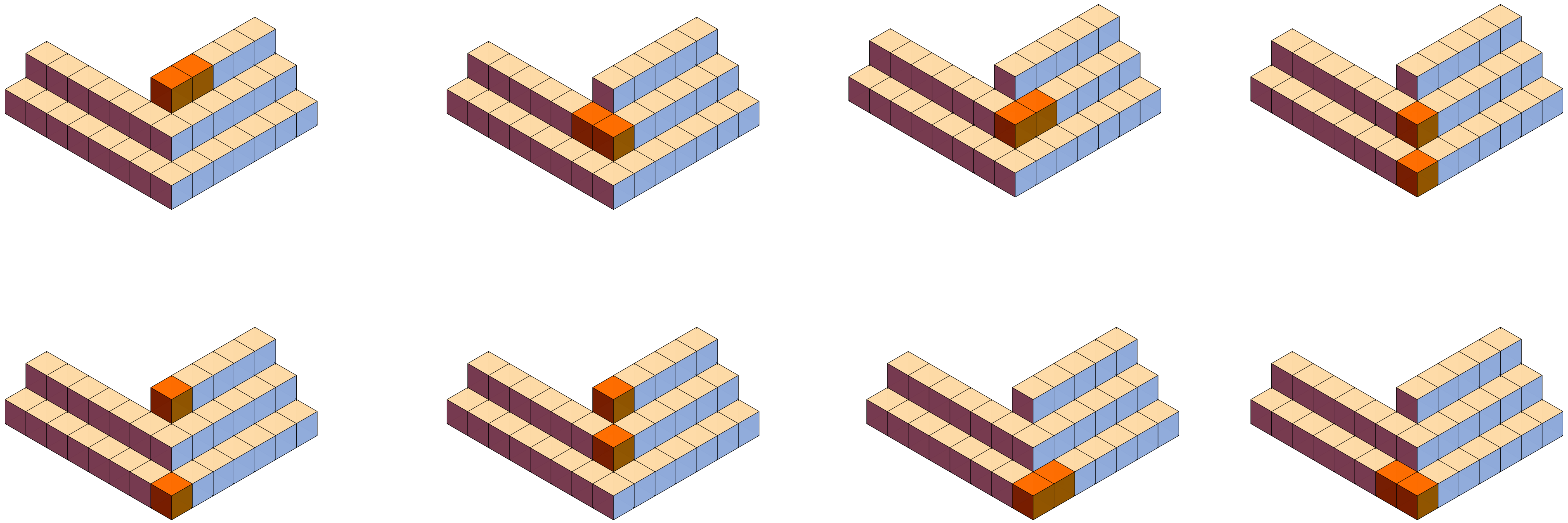
Pandharipande-Thomas invariants

Non-equivariant version: counting rule is simple for 1- and 2-leg cases

1-box: 3 configs

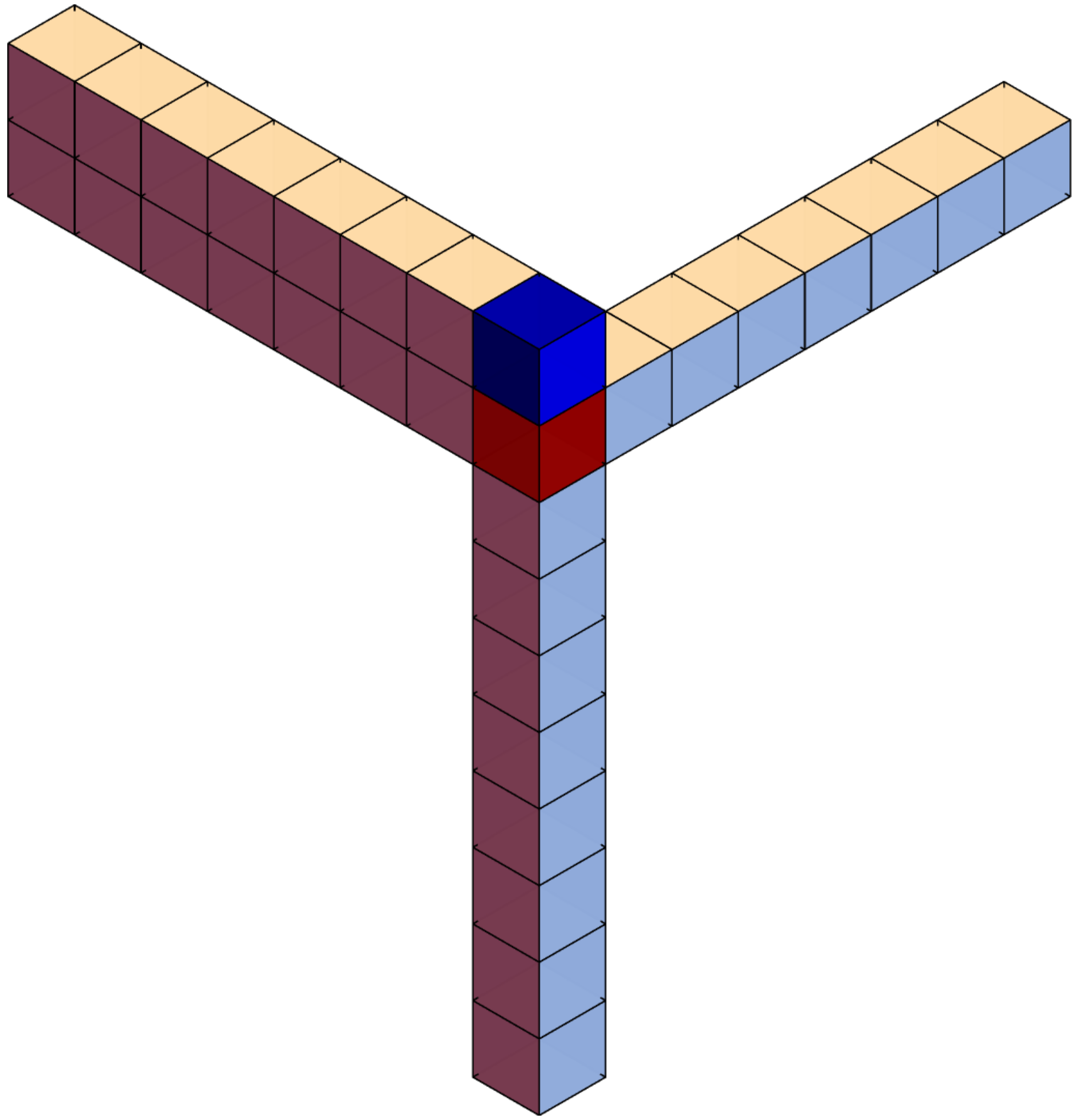


2-box: 8 configs



Pandharipande-Thomas invariants

For 3-leg cases, counting rule is complicated



- **Level 1**:

$$|\epsilon_3\rangle\,, \quad |0^1\rangle$$

- **Level 2**:

$$|0^1, -\epsilon_1\rangle\,, \quad |\epsilon_3, \epsilon_3-\epsilon_2\rangle\,, \quad |0^0, \epsilon_3\rangle$$

- **Level 3**:

$$|\epsilon_3, \epsilon_3-\epsilon_2, \epsilon_3-2\epsilon_2\rangle\,, \quad |\epsilon_3, \epsilon_3-\epsilon_2, 0^0\rangle\,, \quad |\epsilon_3, 0^1, -\epsilon_1\rangle\,,$$
$$|\epsilon_3, 0^3, -\epsilon_3\rangle\,, \quad |0^1, -\epsilon_1, -2\epsilon_1\rangle\,, \quad |0^{-1}, \epsilon_3\rangle$$

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DT/PT Correspondence

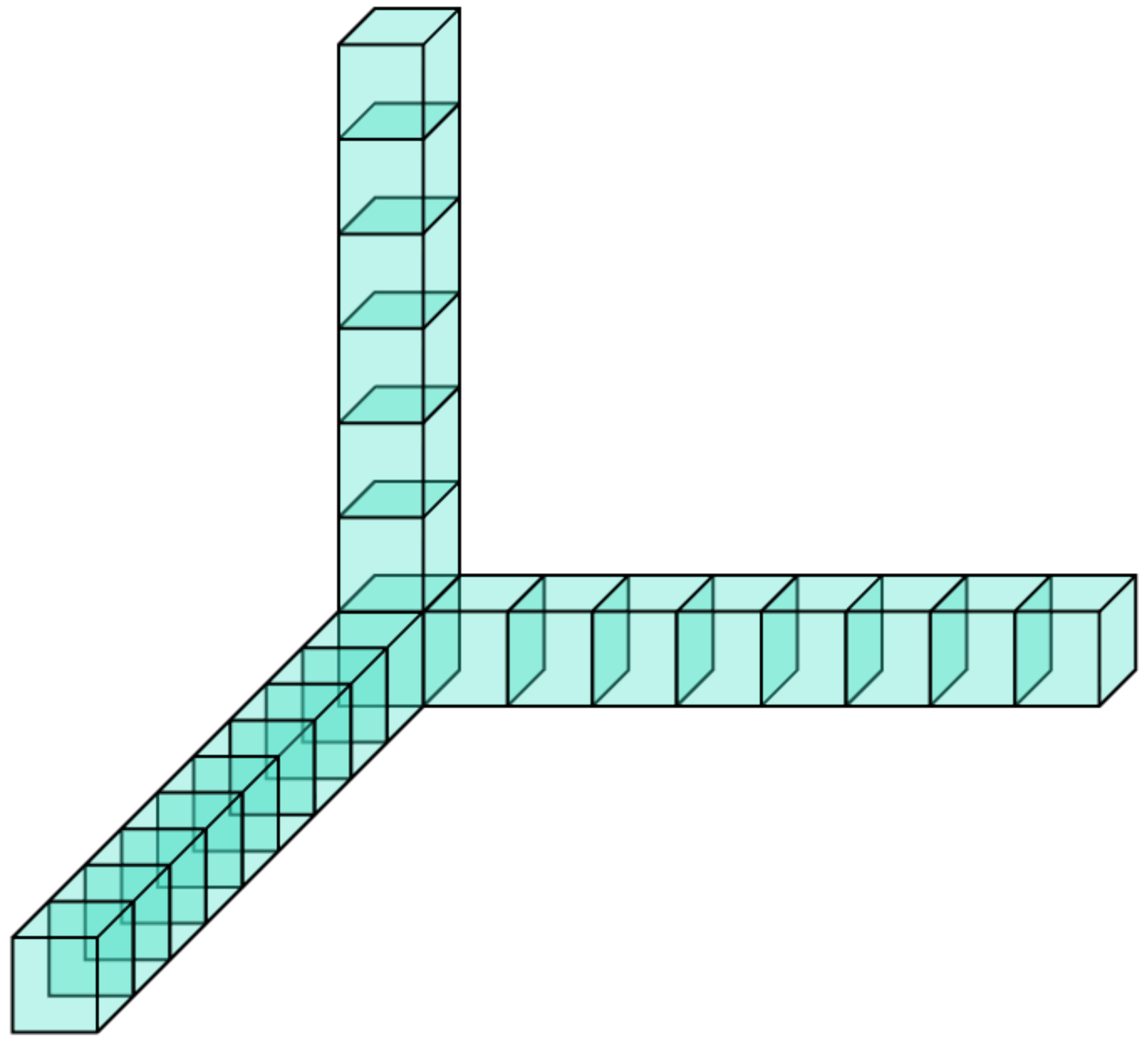
- The DT/PT correspondence states that

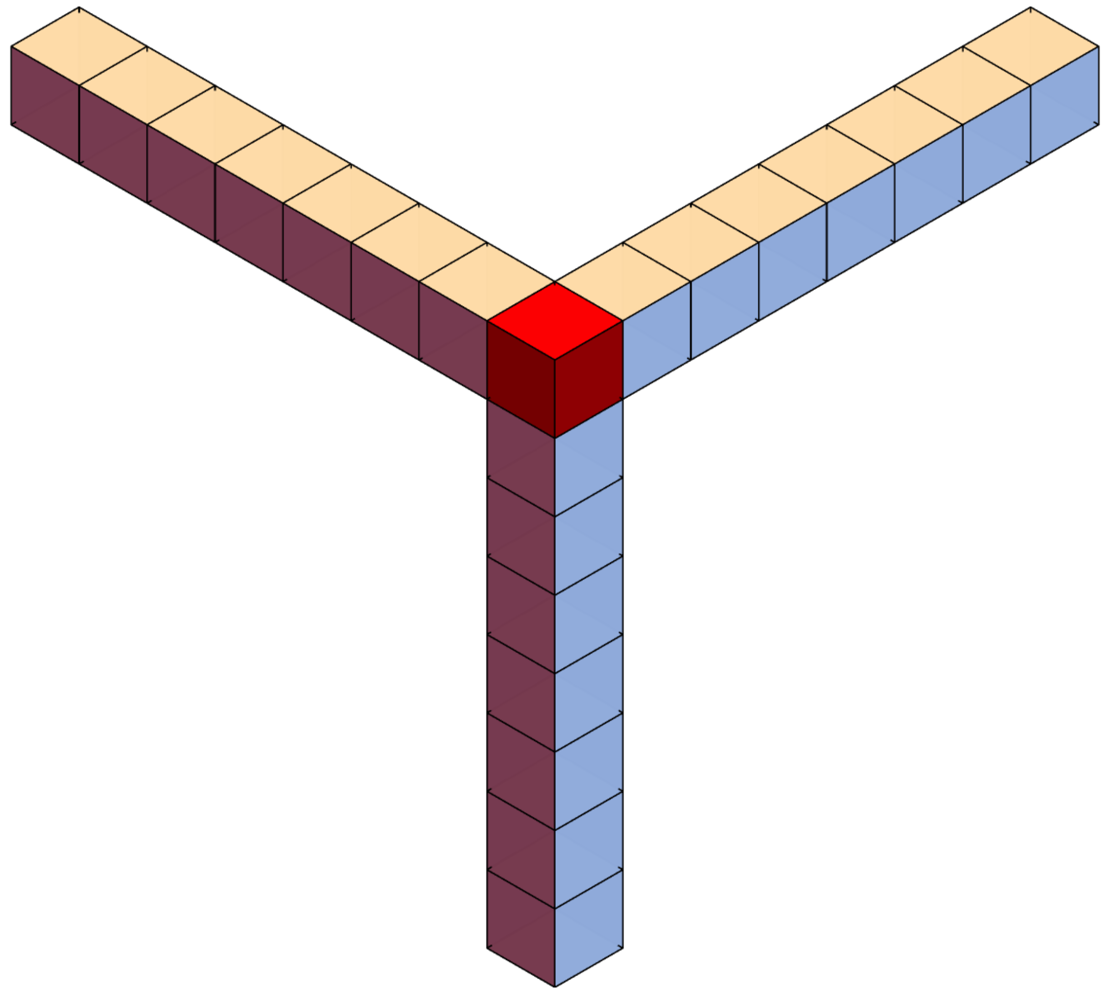
$$DT_{\lambda\mu\nu}(\mathfrak{q}) = DT_{\emptyset,\emptyset,\emptyset}(\mathfrak{q})\, PT_{\lambda\mu\nu}(\mathfrak{q}) = M(\mathfrak{q})\, PT_{\lambda\mu\nu}(\mathfrak{q}).$$

- For the boundary condition $(\square,\square,\square)$, this factorization is explicitly verified:

$$DT_{\square,\square,\square}(\mathfrak{q}) = M(\mathfrak{q})\, PT_{\square,\square,\square}(\mathfrak{q}),$$

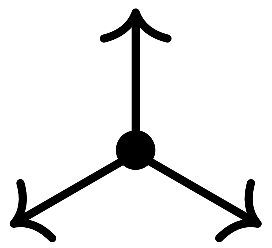
$$1+3\mathfrak{q}+9\mathfrak{q}^2+22\mathfrak{q}^3+52\mathfrak{q}^4+\cdots = \left(1+\mathfrak{q}+3\mathfrak{q}^2+6\mathfrak{q}^3+13\mathfrak{q}^4+\cdots\right) \times \left(1+2\mathfrak{q}+4\mathfrak{q}^2+6\mathfrak{q}^3+9\mathfrak{q}^4+\cdots\right).$$





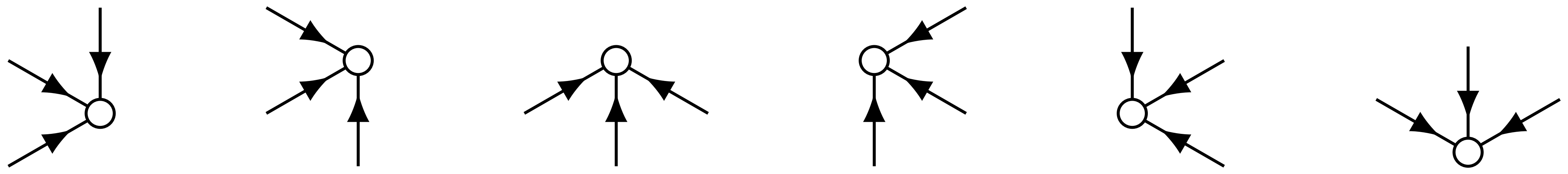
Equivariant Vesion

- **Concave corner** $\Longleftrightarrow$ *outgoing* valence–3 vertex (black):



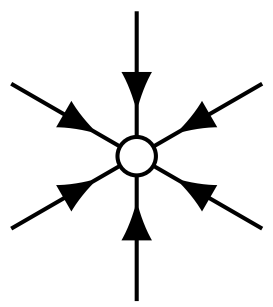
$$\in \; s_1(\vec{Y})$$

- **Half-convex corner** $\Longleftrightarrow$ *incoming* valence–3 vertex (white). There are six possible configurations:

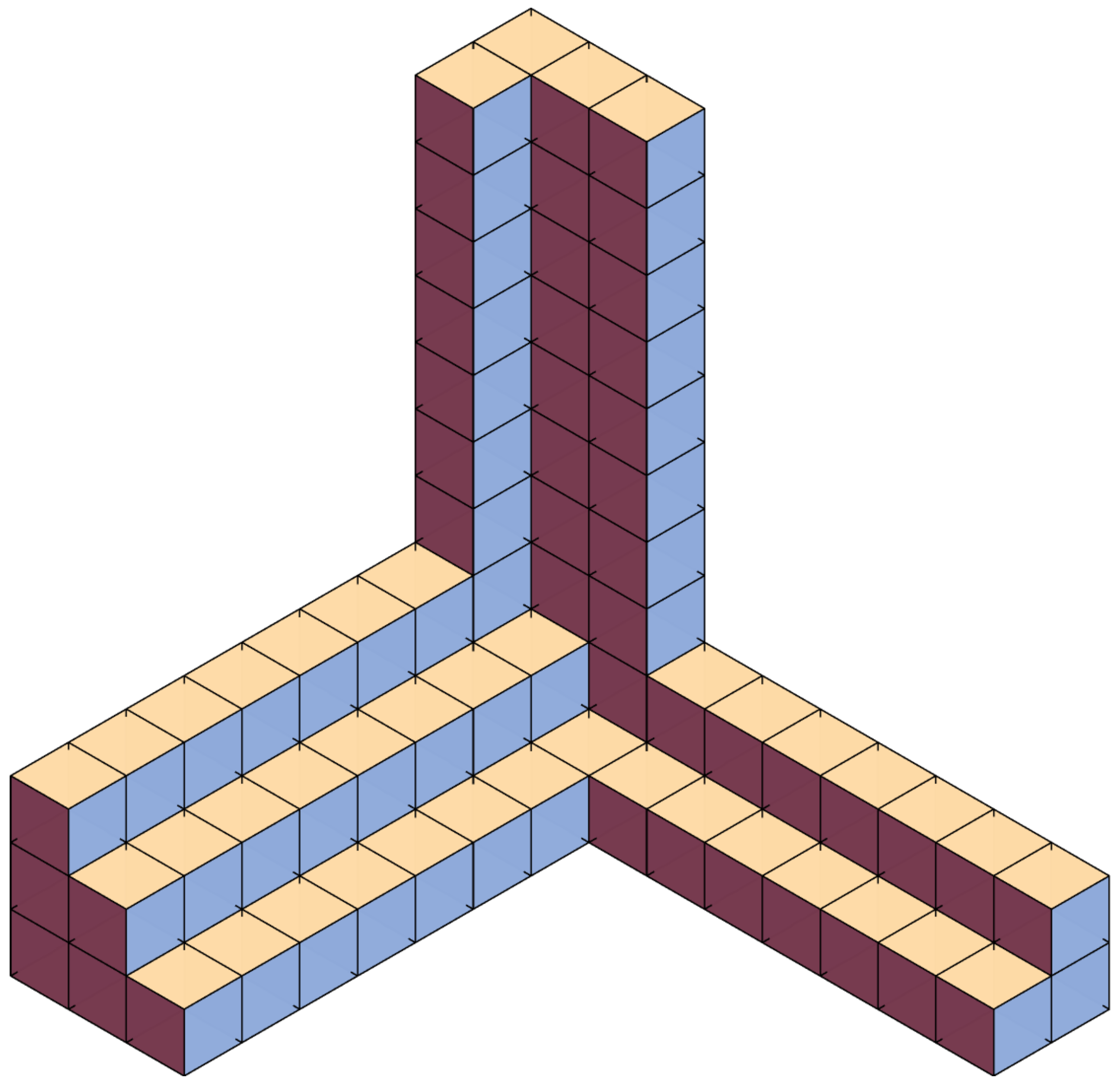


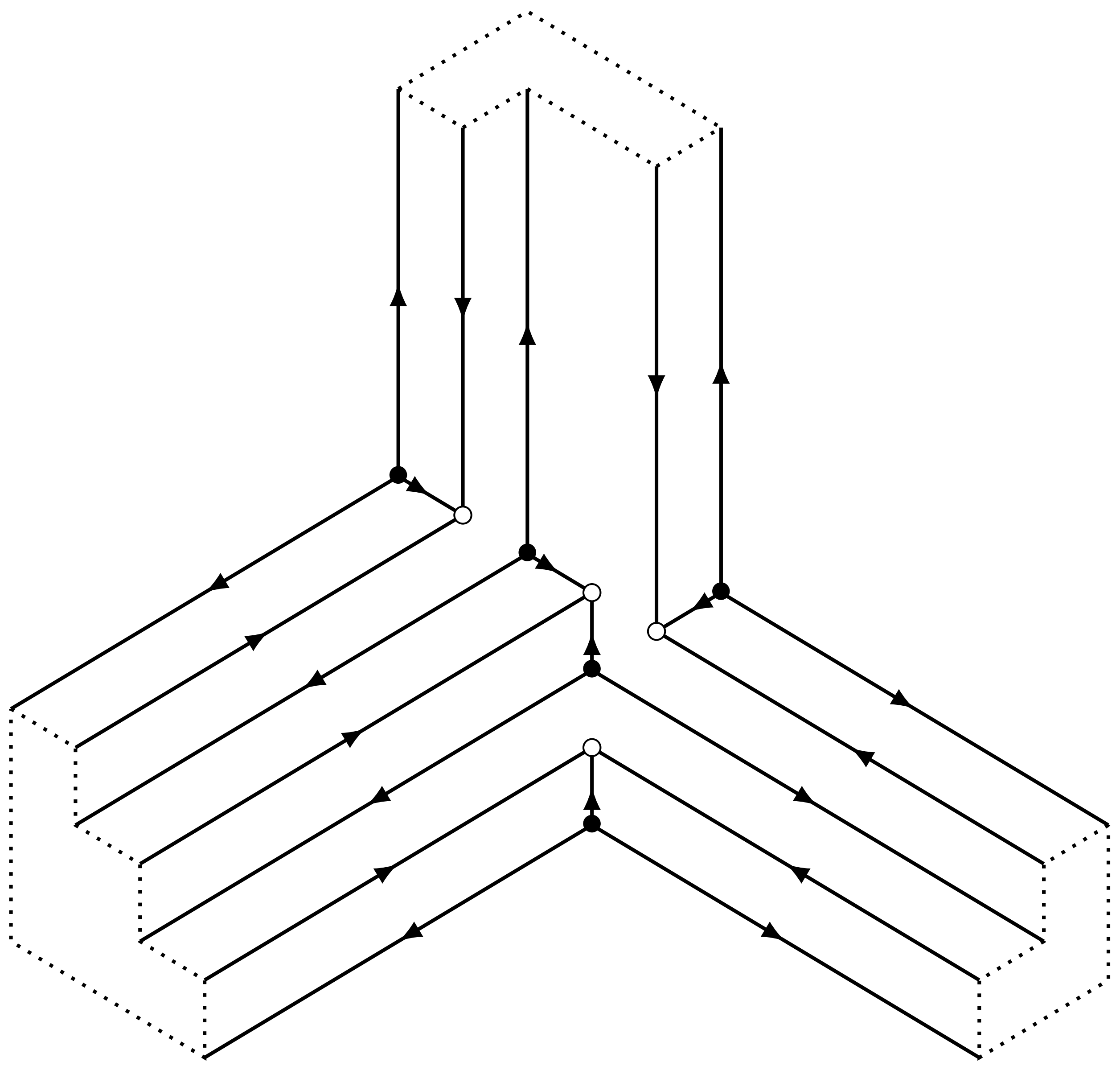
$$\in \; p_1(\vec{Y})$$

- **Fully convex corner** $\Longleftrightarrow$ *incoming* valence–6 vertex (white):



$$\in \; p_2(\vec{Y})$$





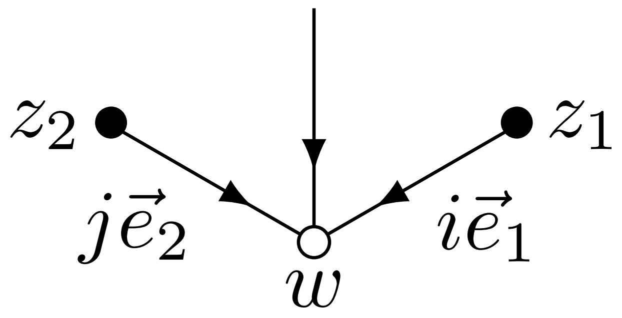
Equivariant Vesion

The superpotential is modified as

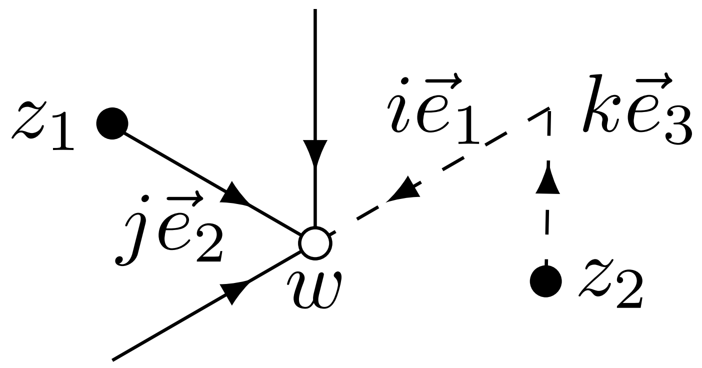
Galakov-Li-Yamazaki

$$\mathsf{W} = \mathsf{W}_0 + \sum_{w \in p_{1,2}(\vec{Y})} \delta \mathsf{W}_w, \qquad \mathsf{W}_0 = \operatorname{Tr} \mathsf{B}_1 [\mathsf{B}_2, \mathsf{B}_3].$$

- For $w \in p_1(\vec{Y})$ (half-convex corners), the local superpotential deformation takes the form



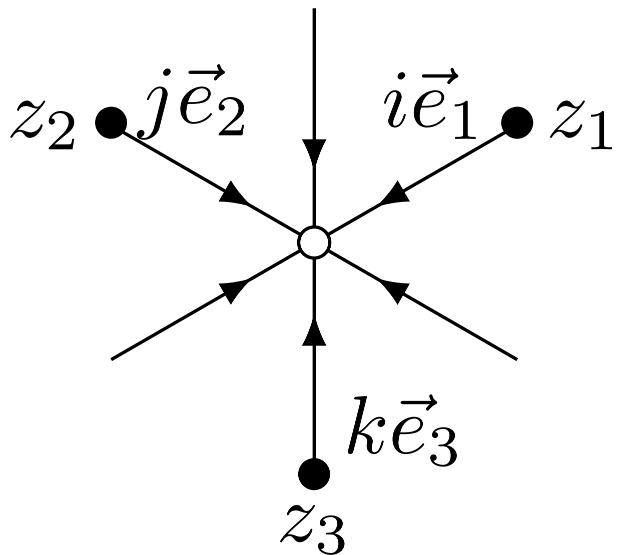
$$\delta \mathsf{W}_w = \mathsf{J}_w \big(\mathsf{B}_1^i \mathsf{I}_{z_1} - \mathsf{B}_2^j \mathsf{I}_{z_2} \big),$$



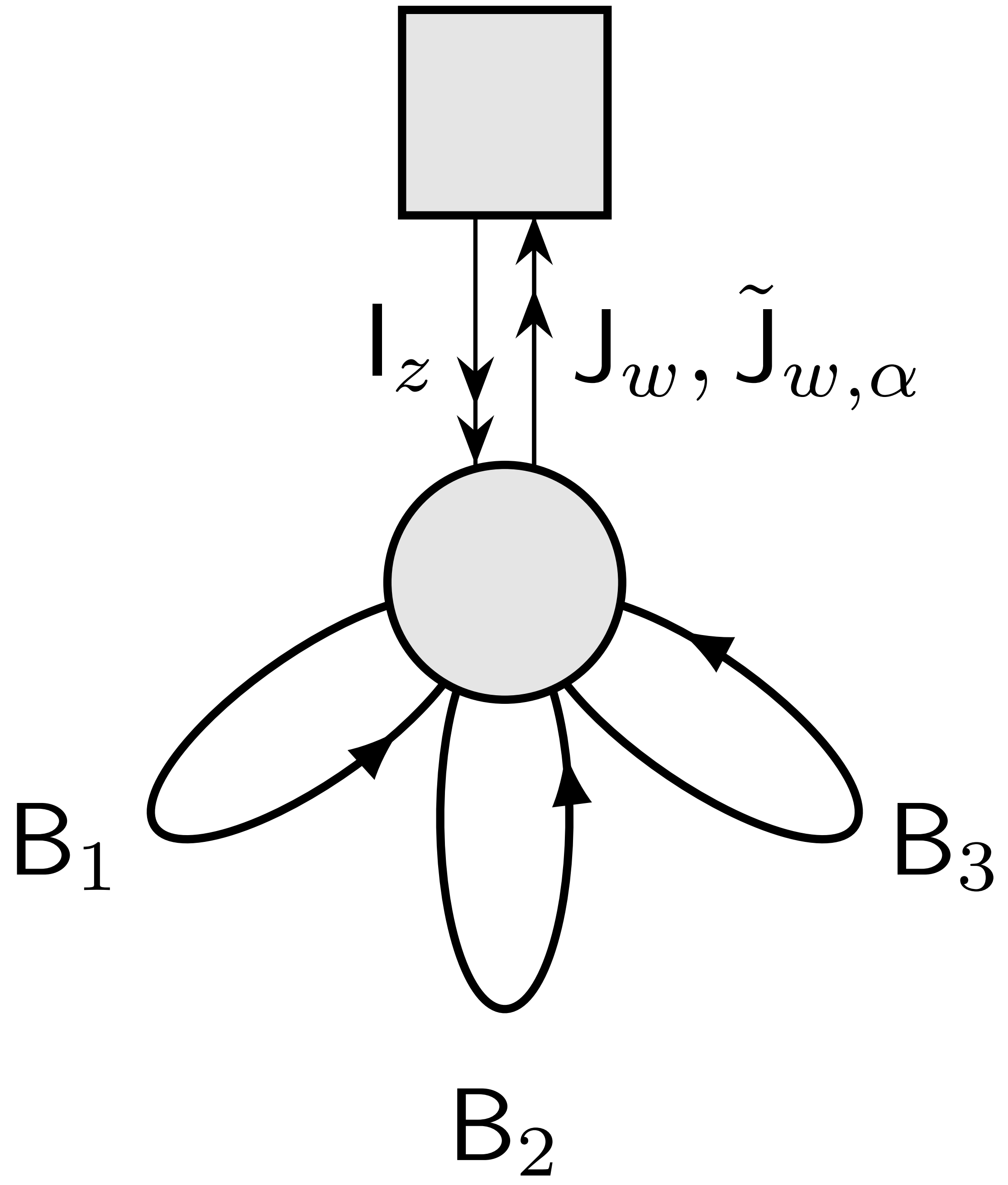
$$\delta \mathsf{W}_w = \mathsf{J}_w \big(\mathsf{B}_2^j \mathsf{I}_{z_1} - \mathsf{B}_1^i \mathsf{B}_3^k \mathsf{I}_{z_2} \big).$$

Other cases follow by triality symmetry.

- For $w \in p_2(\vec{Y})$ (fully convex corners), the superpotential term is



$$\delta \mathsf{W}_w = \tilde{\mathsf{J}}_{w,1} \big(\mathsf{B}_1^i \mathsf{I}_{z_1} - \mathsf{B}_2^j \mathsf{I}_{z_2} \big) + \tilde{\mathsf{J}}_{w,2} \big(\mathsf{B}_1^i \mathsf{I}_{z_1} - \mathsf{B}_3^k \mathsf{I}_{z_3} \big).$$



Equivariant Vesion

The chiral multiplets I_z, J_w, and $\tilde{\mathsf{J}}_{w,\alpha}$ contribute to the one-loop determinant as follows:

$$\mathsf{I}_z \quad (z \in s(\vec{Y})): \qquad \frac{\mathrm{sh}\big(\phi_I - c_{\bar{4},\mathfrak{a}}(\cube_z) - \epsilon_4\big)}{\mathrm{sh}\big(\phi_I - c_{\bar{4},\mathfrak{a}}(\cube_z)\big)} = \mathcal{Z}_{\mathsf{I}_z},$$

$$\mathsf{J}_w \quad (w \in p_1(\vec{Y})): \qquad \frac{\mathrm{sh}\big(c_{\bar{4},\mathfrak{a}}(\cube_w) - \phi_I\big)}{\mathrm{sh}\big(\epsilon_4 + c_{\bar{4},\mathfrak{a}}(\cube_w) - \phi_I\big)} = \mathcal{Z}_{\mathsf{J}_w},$$

$$\tilde{\mathsf{J}}_{w,1,2} \quad (w \in p_2(\vec{Y})): \qquad \left(\frac{\mathrm{sh}\big(c_{\bar{4},\mathfrak{a}}(\cube_w) - \phi_I\big)}{\mathrm{sh}\big(\epsilon_4 + c_{\bar{4},\mathfrak{a}}(\cube_w) - \phi_I\big)}\right)^2 = \mathcal{Z}_{\tilde{\mathsf{J}}_{w,1,2}}.$$

Here

$$c_{\bar{4},\mathfrak{a}}(\cube) = \mathfrak{a} + (i-1)\epsilon_1 + (j-1)\epsilon_2 + (k-1)\epsilon_3, \qquad \cube = (i,j,k).$$

Equivariant DT Vertex

The equivariant DT3 vertex is defined as

$$\mathcal{Z}^{\mathrm{DT}}_{\bar{4};\lambda\mu\nu}[\mathfrak{q}, q_{1,2,3,4}] = \sum_{k=0}^{\infty} \mathfrak{q}^k \, \mathcal{Z}^{\mathrm{DT}}_{\bar{4};\lambda\mu\nu}[k].$$

Here the k–instanton sector of the DT3 partition function (with a single D6–brane) is given by the contour integral

$$\mathcal{Z}^{\mathrm{DT}}_{\bar{4};\lambda\mu\nu}[k] = \frac{1}{k!} \oint_{\eta_0} \prod_{I=1}^{k} \frac{d\phi_I}{2\pi i} \prod_{I=1}^{k} \mathcal{Z}^{\mathrm{D6}_{\bar{4}}\text{–D2–D0}}_{\lambda\mu\nu}(\mathfrak{a}, \phi_I) \prod_{I,J=1}^{k} \mathcal{Z}^{\mathrm{D0–D0}}(\phi_I, \phi_J).$$

The flavor-node contribution factorizes as

$$\mathcal{Z}^{\mathrm{D6}_{\bar{4}}\text{–D2–D0}}_{\lambda\mu\nu}(\mathfrak{a}, \phi_I) = \prod_{z \in s(\vec{Y})} \mathcal{Z}_{\mathsf{I}_z} \prod_{w \in p_1(\vec{Y})} \mathcal{Z}_{\mathsf{J}_w} \prod_{w \in p_2(\vec{Y})} \mathcal{Z}_{\tilde{\mathsf{J}}_{w,1,2}}.$$

The integration contour is specified by the JK residue prescription with reference vector

$$\eta_0 = (1, \ldots, 1).$$

Equivariant PT Vertex

The equivariant PT3 vertex is defined by

$$\mathcal{Z}^{\mathrm{PT}}_{\bar{4};\lambda\mu\nu}[\mathfrak{q}, q_{1,2,3,4}] = \sum_{k=0}^{\infty} \mathfrak{q}^k \, \mathcal{Z}^{\mathrm{PT}}_{\bar{4};\lambda\mu\nu}[k].$$

The k–instanton sector of the PT3 partition function (with a single D6–brane) is computed by the same integrand,

$$\mathcal{Z}^{\mathrm{PT}}_{\bar{4};\lambda\mu\nu}[k] = \frac{1}{k!} \oint_{\tilde{\eta}_0} \prod_{I=1}^{k} \frac{d\phi_I}{2\pi i} \prod_{I=1}^{k} \mathcal{Z}^{\mathrm{D6}_{\bar{4}}\text{–D2–D0}}_{\lambda\mu\nu}(\mathfrak{a}, \phi_I) \prod_{I,J=1}^{k} \mathcal{Z}^{\mathrm{D0–D0}}(\phi_I, \phi_J).$$

Crucially, the integration contour is fixed by the Jeffrey–Kirwan residue prescription with reference vector

$$\tilde{\eta}_0 = (-1, \ldots, -1).$$

In comparison, the DT_3 vertex differs *only* by the choice of the JK reference vector.

Equivariant DT/PT Correspondence

- The equivariant DT/PT correspondence states that

$$\mathcal{Z}^{\mathrm{DT}}_{\bar{4};\lambda\mu\nu}[\mathfrak{q}, q_{1,2,3,4}] = \mathcal{Z}^{\mathrm{DT}}_{\bar{4};\emptyset,\emptyset,\emptyset}[\mathfrak{q}, q_{1,2,3,4}]\, \mathcal{Z}^{\mathrm{PT}}_{\bar{4};\lambda\mu\nu}[\mathfrak{q}, q_{1,2,3,4}].$$

where $\mathcal{Z}^{\mathrm{DT}}_{\bar{4};\emptyset,\emptyset,\emptyset}$ is the D6-D0 partiion function

$$\mathcal{Z}^{\mathrm{DT}}_{\bar{4};\emptyset,\emptyset,\emptyset}[\mathfrak{q}, q_{1,2,3,4}] = \mathrm{PE}\left[\frac{[q_{14}]\,[q_{24}]\,[q_{34}]}{[q_1]\,[q_2]\,[q_3]}\,\frac{1}{[\mathfrak{q}q_4^{-1/2}]\,[\mathfrak{q}q_4^{1/2}]}\right].$$

- A mathematical proof was given by Kühn–Liu–Thimm using wall-crossing techniques.
- However, the existing proof is technically involved, and there is clear room for conceptual simplification and improvement. This motivates a more transparent derivation based on localization and JK residue techniques.

Plan of Talk

1. Counting boxes to Tetrahedron Instanton
2. Donaldson-Thomas and Pandharipande–Thomas
3. DT/PT Correspondence
4. Vortex Origami

D6-D2-D0 brane system

We consider the $\mathrm{D2}_{1,2,3}$–$\mathrm{D6}_{123}$ system:

Brane	$\mathbb{C}_1$		$\mathbb{C}_2$		$\mathbb{C}_3$		$\mathbb{C}_4$		$\mathbb{R}^1 \times \mathbb{S}^1$	
	1	2	3	4	5	6	7	8	9	0
k D0	•	•	•	•	•	•	•	•	•	—
N_1 $\mathrm{D2}_1$	—	—	•	•	•	•	•	•	•	—
N_2 $\mathrm{D2}_2$	•	•	—	—	•	•	•	•	•	—
N_3 $\mathrm{D2}_3$	•	•	•	•	—	—	•	•	•	—
$\mathrm{D6}_{123}$	—	—	—	—	—	—	•	•	•	—

- D6-brane: a rank-1 background object filling $\mathbb{C}^3$, corresponding to the structure sheaf $\mathcal{O}_{\mathbb{C}^3}$.

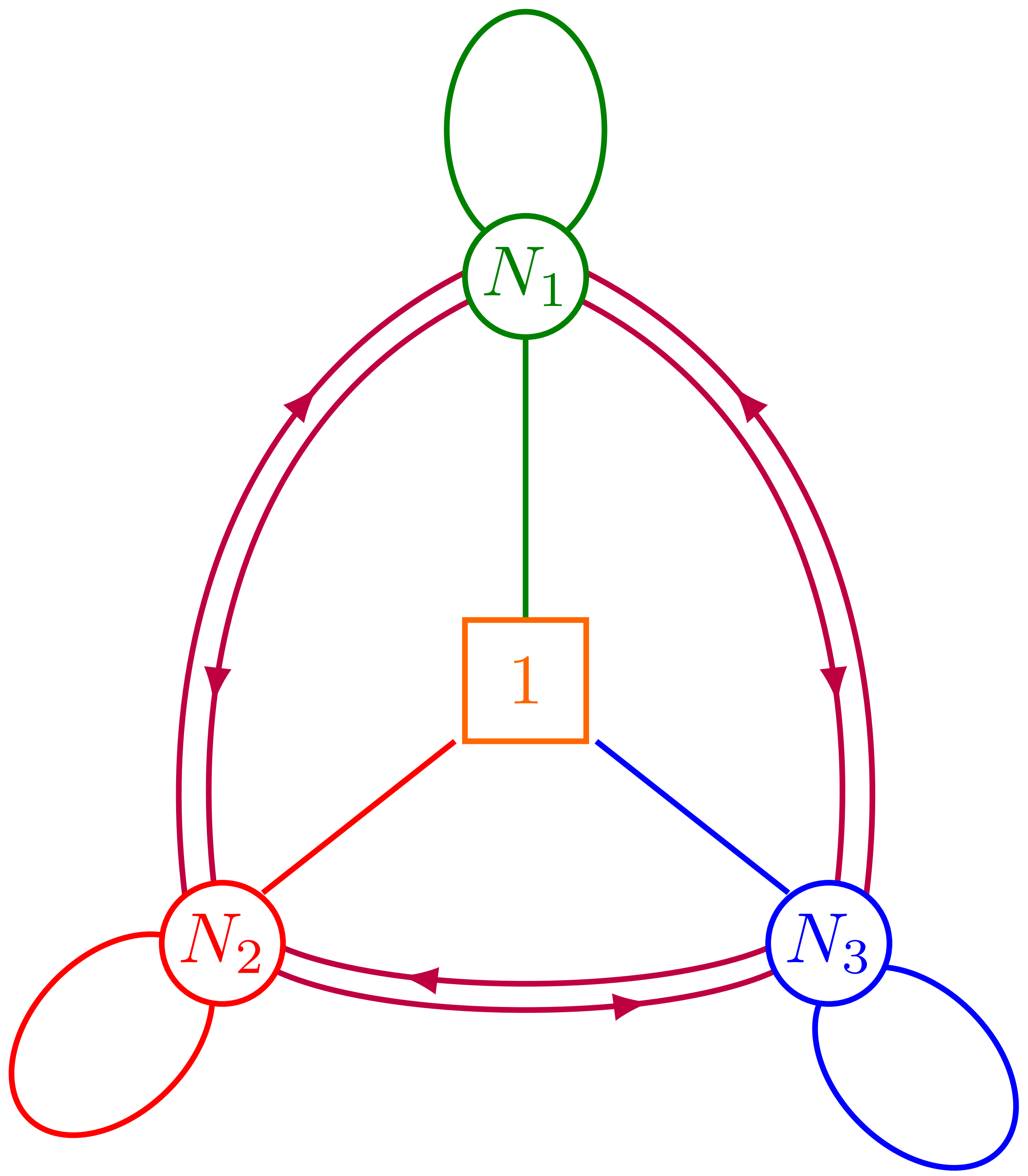
- D2/D0 bound-state data: curve and point degrees of freedom encoded by a stable pair

$$\left(\mathcal{O}_{\mathbb{C}^3} \xrightarrow{s} F\right),$$

where F is a pure one-dimensional sheaf and $\text{coker}(s)$ is zero-dimensional.

Vortex Origami

We now adopt the perspective of the worldvolume theory of the D2-branes. In this description, the relevant enumerative data are encoded in the vortex partition function of the D2-brane theory on $\mathbb{S}^1 \times \mathbb{C}_1 \times \mathbb{C}_2 \times \mathbb{C}_3$. The resulting coupled system is conveniently summarized by the following quiver diagram.



- $\mathbb{S}^1 \times \mathbb{C}_1$, $\mathbb{S}^1 \times \mathbb{C}_2$, $\mathbb{S}^1 \times \mathbb{C}_3$: three decoupled 3d theories.
- Purple links: 1d multiplets on the common $\mathbb{S}^1$ coupling the sectors.

Vortex Origami

The vortex origami partition function is given by

$$Z(N_1,N_2,N_3)=\int_{\mathrm{JK}}\prod_{i=1}^{3}\prod_{I=1}^{N_i}\frac{dy_{i,I}}{2\pi i\,y_{i,I}}\,\mathcal{I}(N_1,N_2,N_3).$$

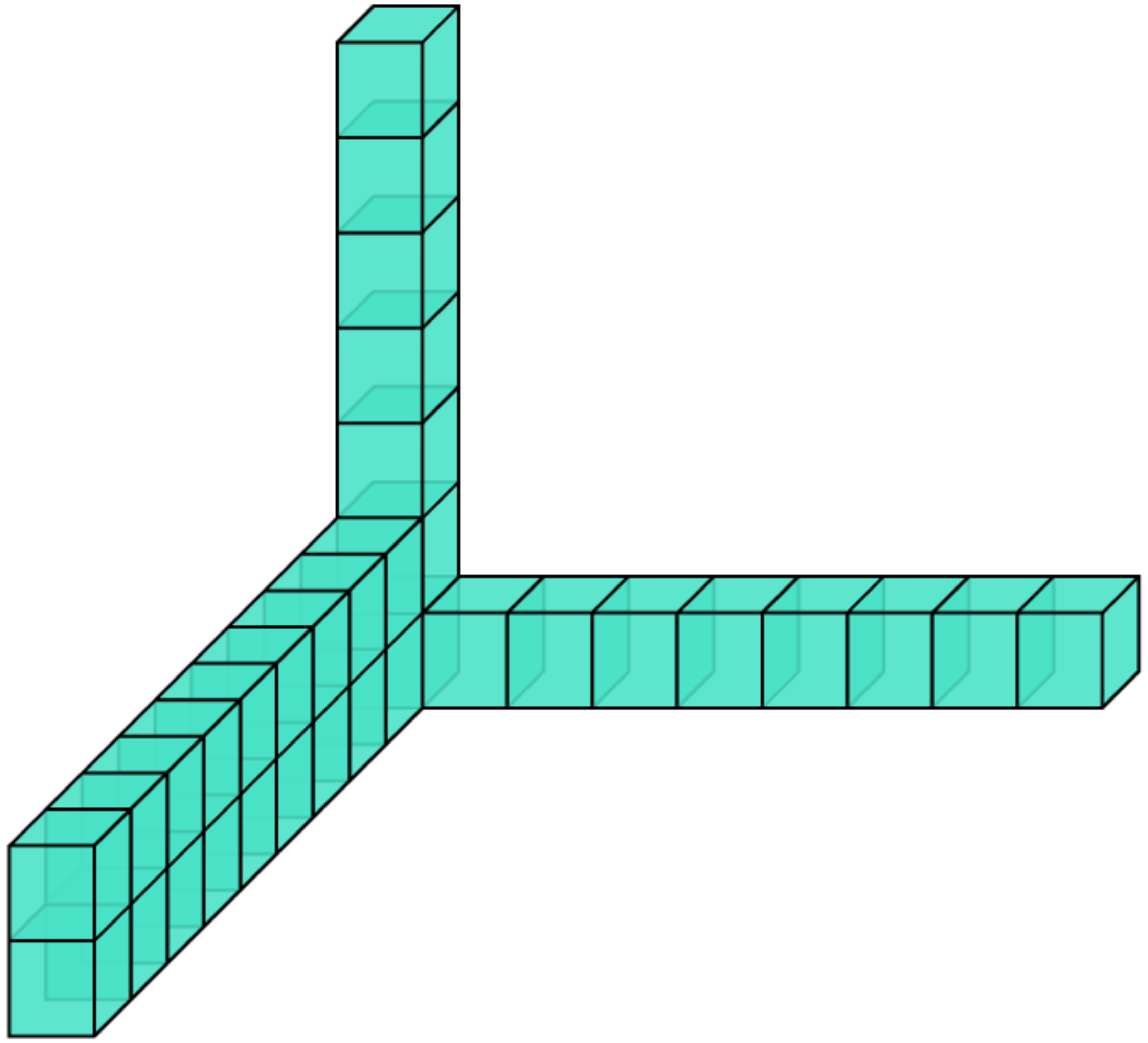
The integrand takes the explicit form

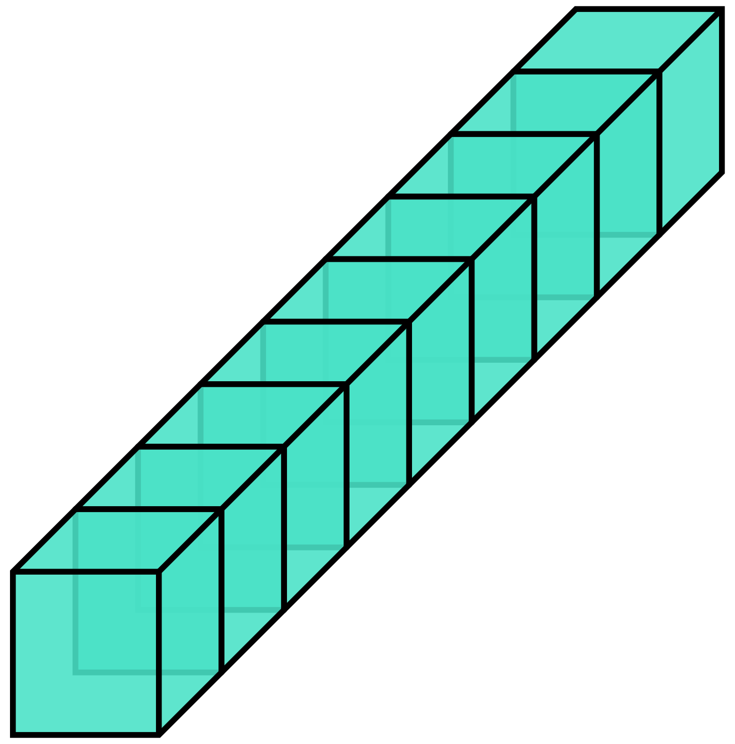
$$\begin{aligned}&\mathcal{I}(N_1,N_2,N_3)\\ &=\prod_{i=1}^{3}\frac{1}{N_i!}\prod_{I,J\neq I}^{N_i}(y_{i,I}y_{i,J}^{-1};q_i)\prod_{I,J}^{N_i}\frac{(q_{i4}y_{i,I}y_{i,J}^{-1};q_i)}{\prod_{s\in\overline{i4}}(q_s^{-1}y_{i,I}y_{i,J}^{-1};q_i)}\prod_{J=1}^{N_i}\frac{(q_iq_4^{-1}x\,y_{i,J}^{-1};q_i)}{(q_ix\,y_{i,J}^{-1};q_i)}\cdot\prod_{i\neq j}\prod_{I=1}^{N_i}\prod_{J=1}^{N_j}\frac{1-q_{i4}^{-1}y_{i,I}y_{j,J}^{-1}}{1-q_i^{-1}y_{i,I}y_{j,J}^{-1}}.\end{aligned}$$

The JK poles are in one-to-one correspondence with boxes of a plane partition with boundary conditions (λ,μ,ν), satisfying

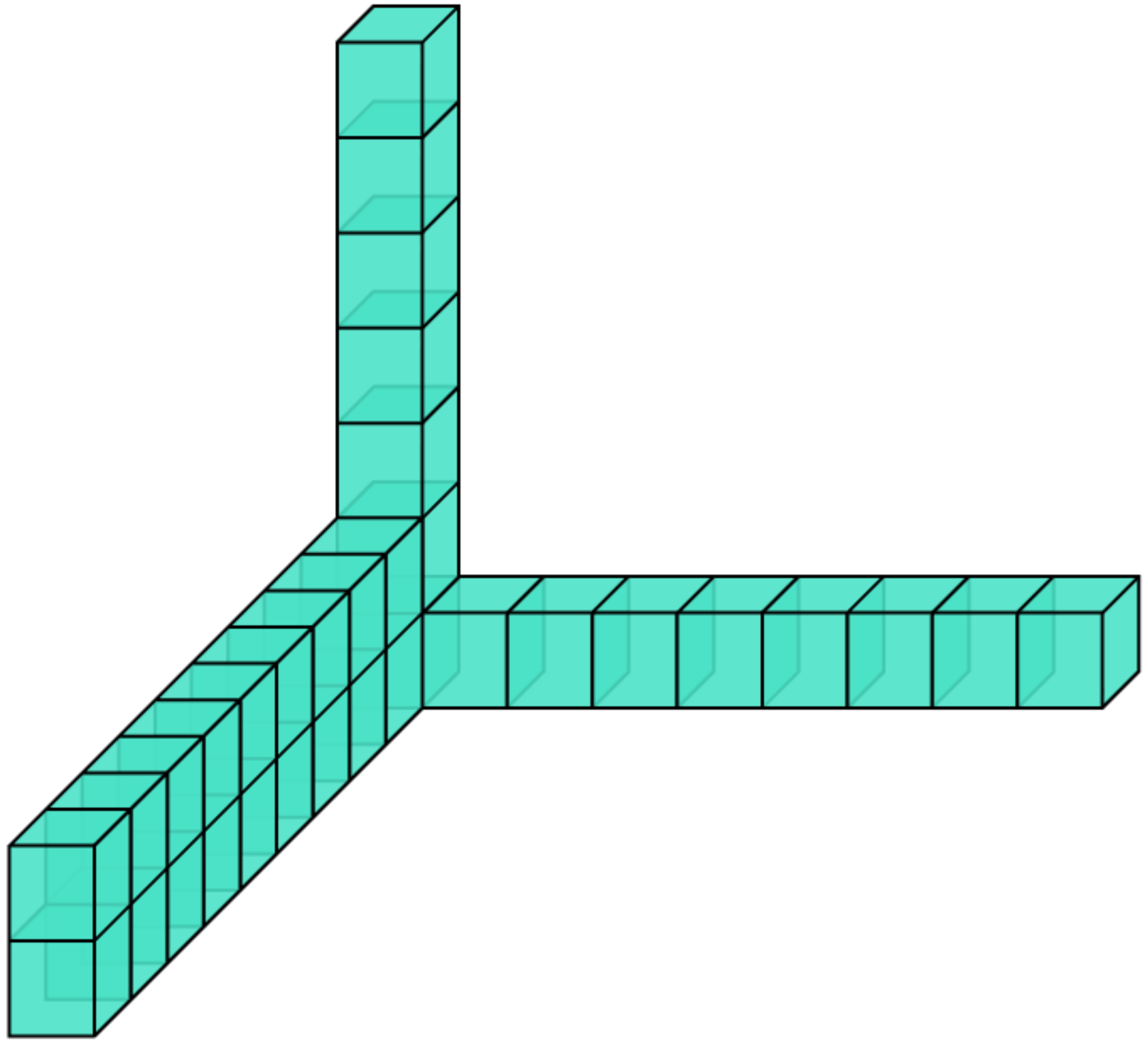
$$|\lambda|=N_1,\qquad|\mu|=N_2,\qquad|\nu|=N_3.$$

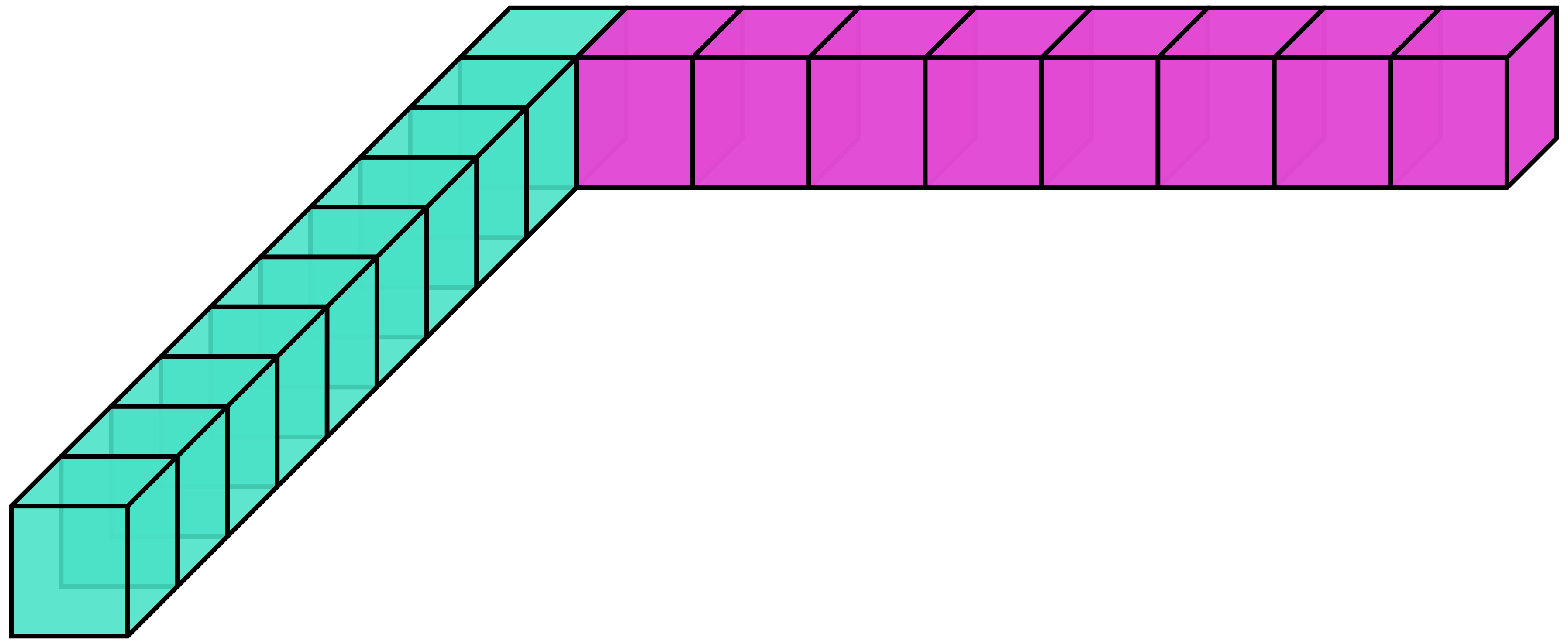
Vortex Origami



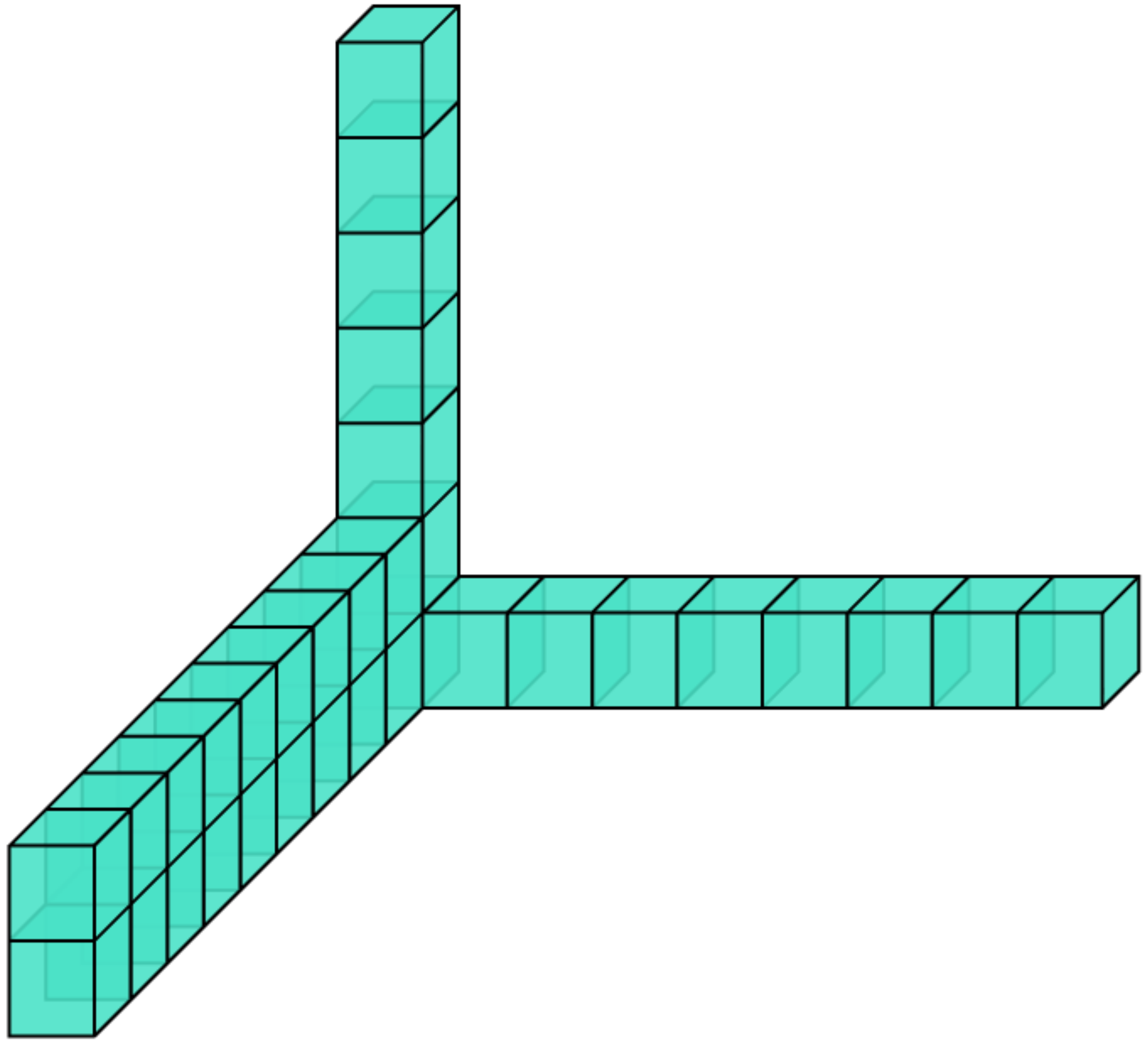


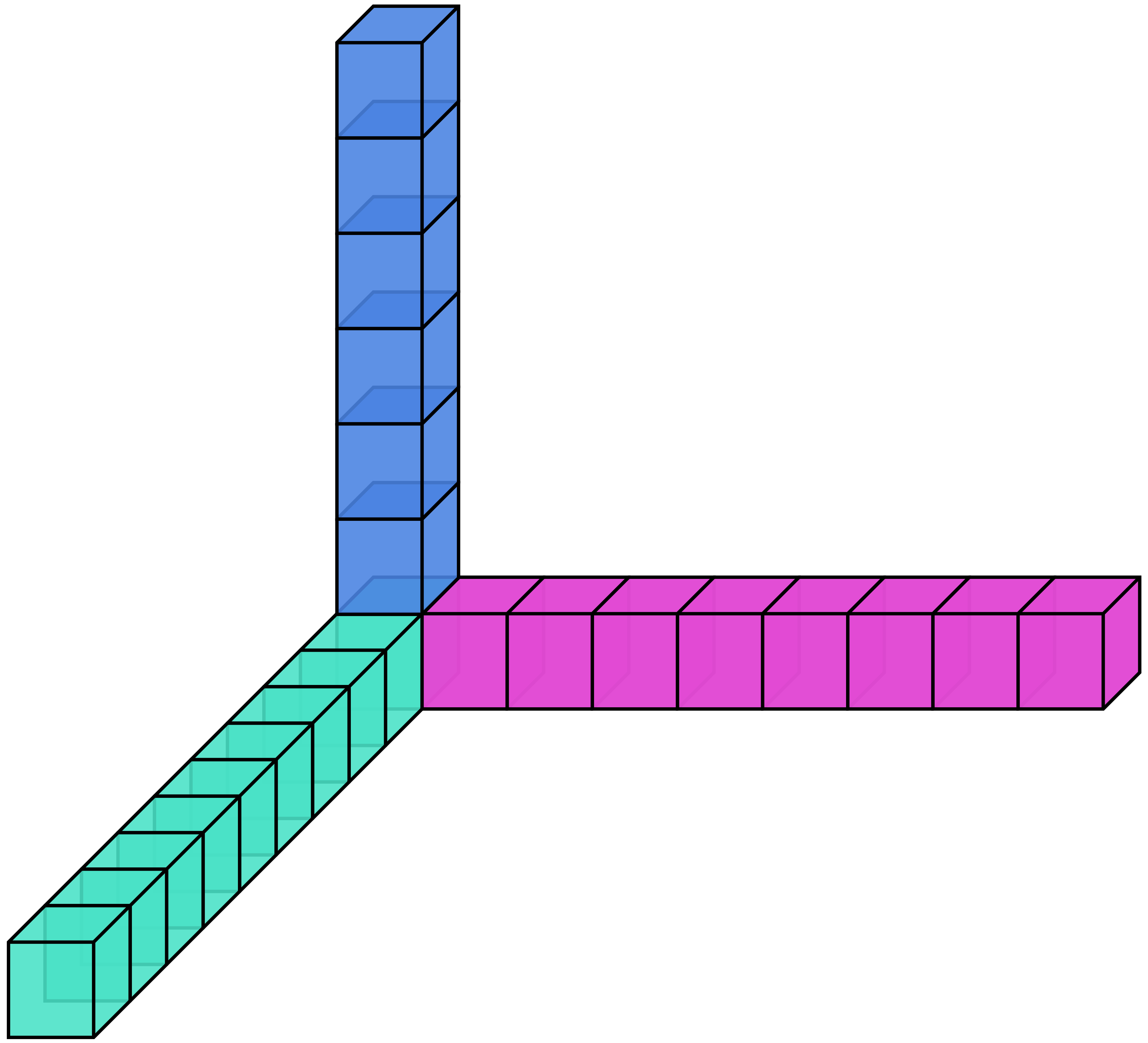
Vortex Origami



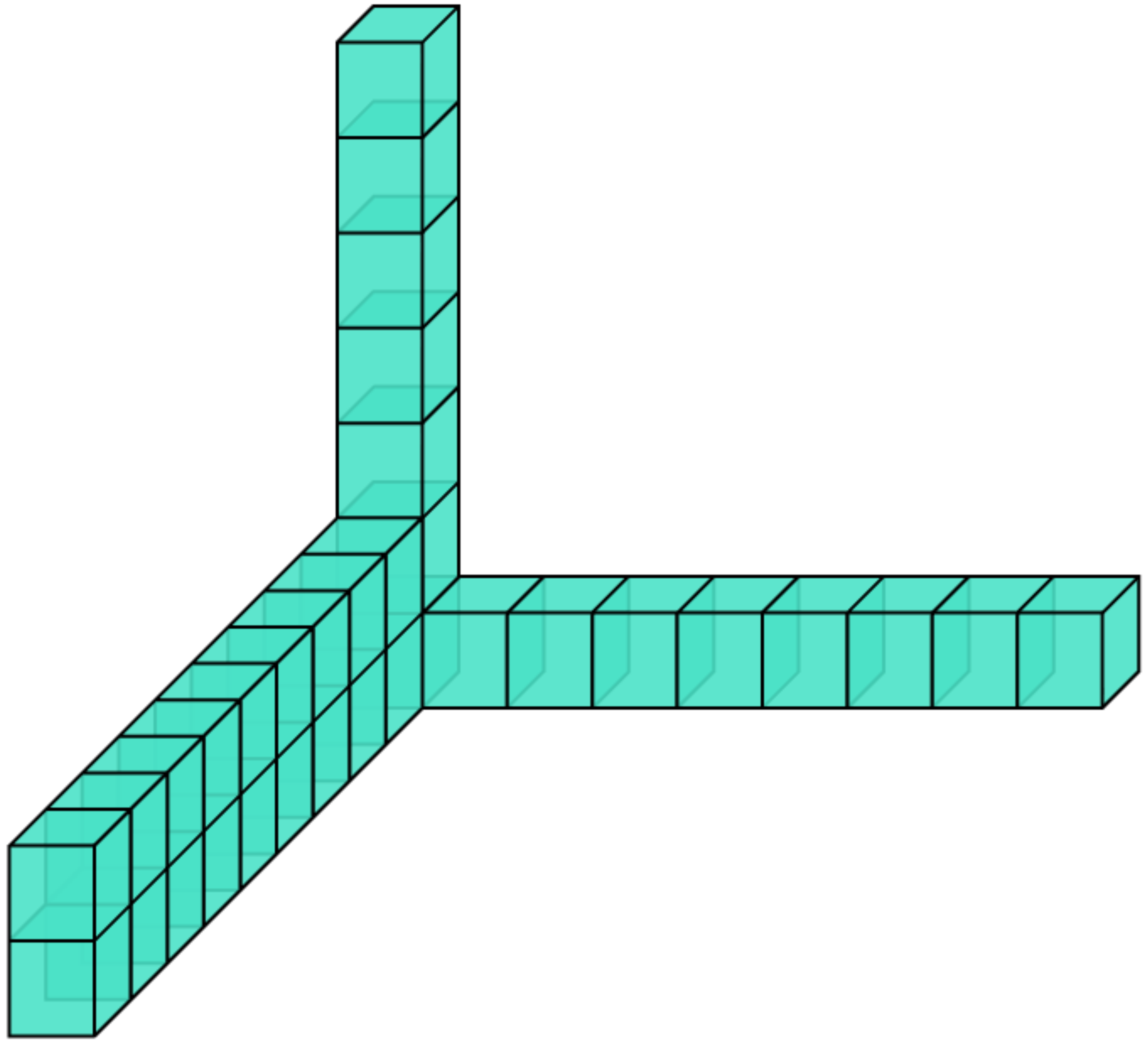


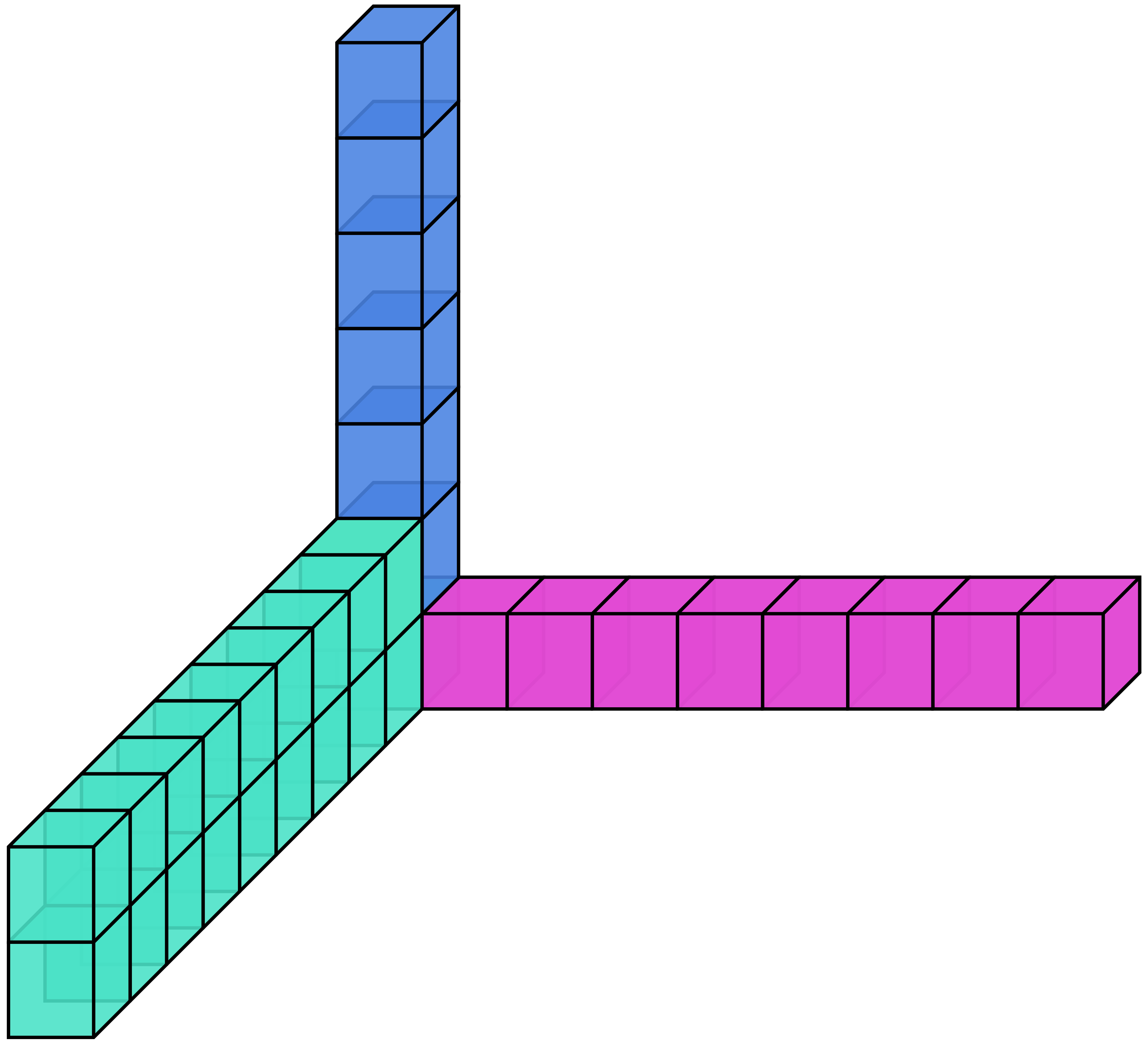
Vortex Origami





Vortex Origami





Vortex Origami

- The vortex origami partition function admits the decomposition

$$Z(N_1, N_2, N_3) = \sum_{\substack{|\lambda|=N_1 \\ |\mu|=N_2 \\ |\nu|=N_3}} Z^{\mathrm{vac}}_{\lambda\mu\nu} \sum_{k=0}^{\infty} Z_{\lambda\mu\nu}[k].$$

- The vortex-k contribution agrees with the equivariant PT vertex carrying D0-brane charge k,

$$Z_{\lambda\mu\nu}[k] \;\simeq\; \mathcal{Z}^{\mathrm{PT}}_{4;\lambda\mu\nu}[k],$$

  up to a normalization constant.

- From this viewpoint, the equivariant PT vertex can be understood as a natural generalization of the topological vertex.

- Vortex origami provides a gauge-theoretic realization of the equivariant PT/topological vertex.

Outlook

- **General brane configurations:** multiple D6-branes and general tetrahedral instanton configurations.
- **CY_4 extension:** vortex origami on Calabi–Yau fourfolds and the *Magnificent Four*.
- **3d mirror symmetry:** mirror descriptions of vortex origami and coupled 1d defect theories.
- **Vertex gluing:** equivariant PT vertex gluing and an origami topological vertex.
- **Algebraic structures:** actions of quantum toroidal algebras, COHA, and shuffle algebras.
- **Integrability:** Bethe/gauge correspondence, R-matrices, and stable envelopes.