

Homework 3: Due at class on March 22

1. Let (x, y) be the Cartesian coordinate of \mathbb{R}^2 and (r, θ) be the polar coordinate of \mathbb{R}^2 . Write a vector field X in terms of the Cartesian coordinate that generate a flow $\varphi_t : \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} .$$

This is the rotation in \mathbb{R}^2 . In addition, draw the schematic picture of the vector field X .

2. Write down vector fields that generate the rotation along x -, y -, z -axis in \mathbb{R}^3 . Find the commutation relations of these vector fields. Compare the theory of angular momenta in quantum mechanics.

3. Show that the tangent bundle TS^1 of a circle S^1 is diffeomorphic to $S^1 \times \mathbb{R}$.

4. Are there zeros of the vector fields in Example 3.10 of the lecture note?

5. Construct a smooth vector field on S^2 which vanishes only at one point explicitly in terms of local coordinates.

6. Show that the Lie bracket satisfies the Jacobi identity. Show that, for $X_1, X_2 \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$,

$$[fX_1, X_2] = f[X_1, X_2] - X_2(f)X_1 , \quad [X_1, fX_2] = f[X_1, X_2] + X_1(f)X_2 .$$

7. Show that $\mathbb{R}P^n$ is non-orientable for even n . In addition, construct an example of unorientable manifolds except the Möbius strip and even-dimensional real projective space.