Homework 2: Due on Mar 24

1 Virasoro algebra

From the OPE of the stress-energy tensor, derive the Virasoro algebra:

$$T(z)\,T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \ldots$$

$$\longrightarrow \quad [L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0} \, .$$

2 Witt algebra

A general infinitesimal holomorphic map can be expressed as

$$z' = z - \epsilon(z) = z - \sum_{n\in\mathbb{Z}} \epsilon_n z^{n+1} \, ,$$

with the infinitesimal parameters ϵ_n, and therefore one can define generators of the transformation by $\ell_n = -z^{n+1}\frac{\partial}{\partial z}$. Show that they satisfy the Witt algebra

$$[\ell_m, \ell_n] = (m-n)\ell_{m+n} \, ,$$

so that the Virasoro algebra is the central extension of the Witt algebra.

3 Linear fractional transformations

Let us consider the Riemann sphere $S^2 = \mathbb{C} \cup \{\infty\}$. The action of $SL(2,\mathbb{C})$ defined by

$$z \mapsto w = \frac{az+b}{cz+d} \, , \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C}) \, ,$$

maps the Riemann sphere onto itself. These transformations are called linear fractional transformations.

• Given three points z_1, z_2, z_3, find a linear fractional transformation which maps the points to $0, 1, \infty$.

• Given four points z_1, z_2, z_3, z_4, their **cross ratio** is defined by

$$[z_1, z_2, z_3, z_4] = \frac{(z_1-z_3)(z_2-z_4)}{(z_2-z_3)(z_1-z_4)} \, .$$

Show that the cross ratio is preserved by any linear fractional transformation

$$[z_1, z_2, z_3, z_4] = [w_1, w_2, w_3, w_4] \, .$$

4 2-point and 3-point function of primary fields

4.1 2-point function

Let us determine the form of the 2-point function of chiral primary operators $\phi_i(z_i)$ with weight h_i ($i = 1, 2$). The 2-point function is invariant under the translation $z \to z + a$ of the coordinate so that it is a function $g(z_1 - z_2)$ of their relative coordinate $z_1 - z_2$.

Using the property of chiral primary fields under the scaling $z \to \lambda z$, show that the function is of the form

$$g(z_1 - z_2) = \frac{d_{12}}{(z_1 - z_2)^{h_1 + h_2}} \,.$$

Furthermore, show that h_1 must be equal to h_2 by using the property under the transformations $z \to -1/z$.

4.2 3-point function

The translation invariance tells us that the 3-point function is also a function $g(z_{12}, z_{23}, z_{31})$ where $z_{ij} = z_i - z_j$. Applying the same argument above, derive the form of the 3-point function

$$\langle \phi_1(z_1)\phi_2(z_2)\phi_2(z_3)\rangle = \frac{C_{123}}{(z_{12})^{h_1+h_2-h_3}(z_{23})^{h_2+h_3-h_1}(z_{31})^{h_3+h_1-h_2}} \,.$$

5 Derivation

The derive (4.5) from (4.4) in the latest lecture note. Namely, start with

$$\delta_\epsilon T(z) = \delta_\epsilon T_{zz} - 2k\left(\partial\partial\delta_\epsilon\omega - 2\partial\omega\partial\delta_\epsilon\omega\right).$$

Using (finite) transformations,

$$\begin{aligned}
& z \to \widetilde{z} = z - \epsilon(z), \\
& T_{zz} \to \widetilde{T}_{\tilde{z}\tilde{z}} = \left(\partial_z \widetilde{z}\right)^{-2} T_{zz}, \\
& \omega(z) \to \widetilde{\omega}(\tilde{z}) = \omega(z) - \frac{1}{2}\log\left|\partial_z \tilde{z}\right|^2,
\end{aligned}$$

Show

$$\delta_\epsilon T(z) = \epsilon(z)\partial T(z) + 2\partial\epsilon(z)T(z) - k\partial^3\epsilon(z).$$