学校代码：10246
学　　号：21307110061

本 科 毕 业 论 文

S 类理论，线算符与库伦从量子化

Class S Theory, Line Operators and Coulomb Branch Quantization

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完 成 日 期：　2025 年 6 月 19 日

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摘　要

在我的学士论文中，我旨在写作一篇综述文章，回顾具有八个超荷（$\mathcal{N} = 2$）的四维超对称规范理论的基础内容以及其最近的进展。

Seiberg–Witten 理论是超对称量子场论发展史上的一块里程碑。这一理论的适用对象为四维 $\mathcal{N} = 2$ 超对称规范理论。从一个的紫外极限为渐进自由或共形的规范理论出发，赛伯格和威腾通过引入 Seiberg–Witten 曲线及其上亚纯的微分一形式，将真空模空间识别为特殊 Kähler 流形，并以几何化方式刻画电—磁对偶与单极子凝聚现象，并以此完整确定理论在库伦丛的低能有效理论[1] [2]。这一理论的提出为量子场论在强耦合区的行为提供了解析的描述，并且揭示了电磁对偶等重要的物理图像。

本文中所涵盖的大多数理论都为 Class $\mathcal{S}$ 理论——这是一类具有八个超荷的四维超对称规范理论，通过将 M5 膜在黎曼曲面上的扭曲紧致化构造而成。由于 M 理论是紫外完备的理论，M 理论的图像包括了理解这一类理论所需要的所有信息。自从 E. Witten 和 D. Gaiotto 的奠基性工作以来，学界在理解这些理论的结构方面取得了重大进展，并由此获得了诸多深刻的物理洞见。[3–4]

从场论角度看，Class $\mathcal{S}$ 理论尤为引人注目，因为它们推广了超对称规范理论中的多个经典的模型，包括超对称版本的量子色动力学（SQCD）。它们源自六维共形场论的构造，不仅为构造新的超对称规范理论提供了一种系统化的方式，还为诸如 AGT 对偶、S 对偶等对偶性以及真空模空间的结构带来了新的见解。[5]此外，这些理论与黎曼曲面及 Hitchin 可积系统的紧密关联，使其具备了丰富的数学结构。[6]特别地，Class $\mathcal{S}$ 理论的库仑支分支模空间可在相差一些离散数据的意义上，与黎曼曲面上的 Hitchin 模空间等同。[7–8]这一关系在超对称量子场论与超 Kähler 几何之间架起了一座桥梁，并且给予了一种系统的方法求解超对称规范理论的 Seiberg-Witten 几何，从此精确求解理论的低能有效作用量。超 Kähler 几何的刚性性质为研究诸如 BPS 希尔伯特空间的穿墙现象等有趣的物理现象提供了有效的解析工具，并可以由此得到了低能理论 BPS 态谱的解析解。其中指的提及的是 Kontsevich–Soibelman 的穿墙公式将弱耦合区与强耦合区的 BPS 态谱联系起来，给出了一种 BPS 谱的计算方法。[9]

四维理论中的另一个关键要素是 BPS 线算符的引入，它们表示在场中传播的无限重 BPS 粒子。在线算符的存在下，理论仅保留原有超对称的一半，这导致 BPS 能量下界被修正。能量达到这一下界的态被称为“框 BPS 态”[10]。有趣的是，类似于局域算符的代数结构，线算符同样具有丰富的代数性质。特别地，当将四维理论的一个维度被紧致化到 S^1 上，并让线算符绕该圆缠绕时，这些算符的期望值可解释为库仑支分支模空间上的全纯函数。此外，通过引入 Ω 背景，线算符的代数将会变为非交换的，这为库仑支的形变量子化提供了自然框架，最初由 E. Witten 和 N. Nekrasov 提出[11]。

多项研究已从不同视角探讨了 BPS 线算符的代数。在紫外极限下，该代数由黎曼曲面上的相对 Skein 代数（relative skein algebra）描述[12–16]，而在红外极限则由量子 Teichmüller 空间和 Chekhov–Fock 代数支配[6]。两者之间的精确对应由框自旋特征标（framed protected spin characters）刻画，这些特征计算了系统中的带框 BPS 态。A. Neitzke 和 F. Yan 提出了一种几何方法来计算这一 UV–IR 映射，为场论的 BPS 态谱提供了全新视角，开启了一个颇具前景的研究领域。[17]

由于线算符代数为库仑支的形变量子化提供了自然框架，它与 Gukov–Witten 的弦束量子化提案相契合。[18–19] 尤其是，线算符的乘积可被理解为拓扑 A 模型中 $(\mathfrak{B}_{cc}, \mathfrak{B}_{cc})$ 弦的融合，而该代数的表示则源自 $(\mathfrak{B}, \mathfrak{B}_{cc})$ 弦。[20] 在这一框架下，还延伸出与细致 Chern–Simons 理论及可积系统的关联。将几何表示论中的 B 型弦与线算符代数的表示联系起来，同样是该领域的核心问题之一。[21]

在若干著名例子中——例如 $\mathcal{N} = 2^*$ SU(2) SQCD 以及具有四个味超多重态的 $\mathcal{N} = 2$ SU(2) SQCD——库仑支的形变量子化与双仿射 Hecke 代数（DAHA）相关[13]，该代数最初由 Cherednik 在研究 Macdonald 猜想时提出[22–24]。这一重要例子深化了我们对超对称理论结构的理解[25]。在笔者和 Satoshi Nawata 老师与黄俊康、庄舒同学长合作的文章中，我们通过对于 DAHA 的表示进行分类，并且通过讨论 DAHA 的表示理论和库仑支中拉格朗日子流形的关系，完成了 $SU(2)$ $\mathcal{N} = 2$ $N_f = 4$ SQCD 库伦丛的膜量子化。这一结果推广了前人关于的结果。进一步汇总这些理论，有望为高秩规范群（尤其是 SU(3) 规范理论）的 BPS 态谱研究提供启示，后者是对现实世界 QCD 的超对称类比。

总之，理解量子场论的强耦合行为是理论物理中最具挑战性的问题之一。在 Class $\mathcal{S}$ 理论中考察 BPS 态谱与动力学，能够为这一复杂课题提供宝贵的物理洞见。

本文旨在为上述内容提供一篇较为系统而细致的综述，其结构安排如下：

第一章为引言，简要介绍本文所探讨的对象及其涵盖的主要内容。第二章系统讲解了超对称性的基本理论，重点介绍了超对称李代数与超对称庞加莱代数的表示理论，特别是中心荷 Z 存在与否对表示构造的影响，并探讨了 CPT 不变性在构造超对称多重态时所带来的限制。由于此部分内容在标准教材与综述中讨论较少，本文在第二章对此进行了补充与深入分析。随后，从超对称多重态出发，本文进一步介绍了超对称量子场理论中所涉及的场算符，并说明了它们在超空间中的构造方法，详尽讨论了超对称变换对场的作用方式，为后续场动力学的展开打下基础。

第三章集中讨论了 $\mathcal{N} = 2$ 超对称量子场论的动力学。首先通过超空间积分形式构造了 $\mathcal{N} = 2$ 超对称规范理论的拉格朗日量，接着分析了该理论的味对称性及其在某些特殊情形下的增强现象。随后探讨了该理论中的 R-对称性与其在微扰阶的重整化行为，并解释了耦合常数仅在一圈图水平上受到修正的原因。同时也指出了该推理在非共形情形下的失效及可能的非微扰修正。之后，本文讨论了超对称场论的经典真空模空间，包括其定义、计算方式及其在量子场论中作为物理观测值背景的基础作用。对于超对称理论，真空模空间可划分为库伦丛、希格斯丛与混合丛，本文主要关注库伦丛的性质，分析了规范群为 $SU(2)$ 时的情形，并推广至更一般的李群。在库伦丛中，由于 Higgs 机制的作用，规范对称性破缺为多个 $U(1)$ 群的乘积，低能下由无质量规范玻色子介导的作用为库伦相互作用，库伦丛亦因此得名。

为了更深入理解理论中的对称性结构，本文进一步讨论了 Witten 效应，并解释了 θ 角对规范电荷的影响。接着分析了超对称性对应的诺特流（即超对称流），通过计算其泊松括号推导出经典中心荷的来源与表达式。随后，以 $SU(2)$ 超对称量子色动力学为例，探讨了满足 BPS 限的 BPS 态，包括微扰激发与非微扰的磁单极、多荷子解，并呈现了 BPS 场构型所满足的方程及其与 Bogomolny 方程的关系。着重分析了磁单极与多荷子在半经典背景下的量子化行为，包括其费米子零模与味对称性表示的联系。

接下来，本文详细讨论了 Seiberg–Witten 理论，首先分析了低能有效理论中的 $SL_2\mathbb{Z}$ 电磁对偶性与守恒荷的变换，继而引入中心荷函数并讨论其与 BPS 谱的关系。本文指出，库伦丛模空间上的几何奇点对应新的无质量自由度，使得 $U(1)$ 有效理论失效；这些奇点在理论绕行后将引发非平凡的对偶变换。这些行

为虽可在微扰理论中观测，但 Seiberg 与 Witten 进一步推测，在强耦合区磁单极可成为新的无质量粒子，并据此确定几何奇点的位置。奇点结构及其绕行行为共同定义了模空间上的电磁荷格点纤维化，此即所谓的 Seiberg–Witten 几何。该几何结构使我们能计算模空间任意一点的低能有效作用量。本文以 $SU(2)$ 纯杨–米尔斯理论与 $SU(2)$ $N_f = 4$ 理论为例，具体求解了不同库伦模空间点上的有效理论，并在章节末尾介绍了更一般的箭图规范理论及其对偶结构，特别是三顶角图在 $SU(2)$ $N_f = 4$ 理论中所扮演的角色。

第四章探讨了这类更广泛的超对称规范理论在 Type IIA 与 M 理论中的构造。Witten 提出的膜构型（$D4$–$NS5$–$D6$）在低能极限下对应四维超对称规范理论，提升至 M 理论后则为 M_5 膜紧致在 Taub-NUT 时空中的黎曼曲面。由于 M 理论的紫外完备性及 M_5 膜的光滑性，其谱与低能行为应完全由 M_5 与 M_2 膜的动力学决定，Seiberg–Witten 曲线可视为 M_5 膜所紧致的黎曼面。Gaiotto 与 Gaiotto–Moore–Neitzke 的工作将该构造推广到一般亏格曲面，构成了 Class $\mathcal{S}$ 理论。

随后，本文进一步探讨了 Class $\mathcal{S}$ 理论中的 Seiberg–Witten 几何与 Hitchin 模空间之间的联系。由于 Hitchin 模空间为超 Kähler 流形，这一联系为我们利用其几何性质理解规范理论的低能有效行为提供了可能。本文讨论了 Hitchin 模空间的构造，其超 Kähler 结构及在不同复结构下的描述，特别分析了 $N_f = 4$ 与 Minahan–Nemeschansky 理论中库伦丛的结构及其对称性增强行为。

第五章聚焦于超对称理论中的 BPS 谱与 BPS 线算符。本文讨论了 BPS 线算符的插入对希尔伯特空间的影响及其所满足的框 BPS 条件，进一步分析了 BPS 线算符的代数结构、形式加法及算符乘积展开。在讨论重 BPS 背景下轻 BPS 态的动力学行为时，本文解释了束缚态的形成机制与束缚态半径的求解。当发生穿墙现象时，束缚态半径趋于无穷，导致希尔伯特空间结构发生突变，此突变赋予 Kontsevich–Soibelman 变换以物理意义。借助其穿墙公式，可以将弱耦合与强耦合区域的 BPS 谱联系起来，进而在部分情形下解析出完整谱。此外，本文还介绍了描述超共形理论中 BPS 态的另一强有力工具——超共形指标。该指标等价于在 $S^1 \times S^3$ 上，带周期性费米边界条件下理论的配分函数，因其拓扑不变量性质，我们可在弱耦合区计算该指标，进而探测强耦合信息。作为例证，本文展示了如何借由超对称指标验证 $N_f = 4$ 理论中的 S 对偶性。

第六章考虑四维规范理论在 $\mathbb{R}^3 \times S^1$ 背景下的低能行为。在能量尺度远小于 S^1 半径倒数时，低能理论可由三维 $\mathcal{N} = 4$ 非线性西格玛模型描述，其靶空间即为 Hitchin 模空间。本文重点讨论了如何通过量子修正计算 Hitchin 模空间度规所满足的自洽方程，并分析了在某些极限下的度规行为。

第七章中，本文进一步将理论约化至二维。通过将四维理论放置在雪茄几何并部分打开 Ω 背景，所得理论成为二维 $\mathcal{N} = (4,4)$ 非线性西格玛模型，并可通过拓扑 A 模型实现其拓扑扭转。本文讨论了 A 模型的一般性质、开弦与保持一半超对称性的边界条件（A 膜），并指出四维中的线算符代数可嵌入为二维理论中的局域算符插入。拓扑 A 模型不仅刻画了线算符代数，还揭示了其表示理论结构，这正是膜量子化的核心思想。最后，本文具体介绍了作者近期的工作，借由膜量子化在 $SU(2)$ $N_f = 4$ 情形下对 Hitchin 模空间的几何结构进行预测，从而建立了几何与表示理论之间的新对应关系。

由于在理解 Hitchin 模空间以及 Taub-Nut 等几何对象时需要大量使用辛几何相关的概念与技术，在本文的附录 A 中对于所需要的技术与概念进行了详细的的解释。

关键词：理论物理; 弦理论; 超对称量子场论; 塞伯格-威滕理论; 膜量子化
中图分类号：O413.1

Abstract

In my bachelor's thesis, I aim to write a review article that surveys the basic topics and recent developments of four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories with eight supercharges.

Seiberg–Witten theory represents a milestone in the history of supersymmetric quantum field theory. It focuses on the four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories whose ultraviolet limits are either asymptotically free or conformal. By introducing the Seiberg–Witten curve and its meromorphic differential, Seiberg and Witten identified the vacuum moduli space as a special Kähler manifold, thus describing the electric–magnetic duality and monopole condensation in a geometric framework. This geometric description allows them fully determined the low-energy effective action on the Coulomb branch[1–2]. Seiberg-Witten theory provides an analytic description of quantum field theories in their strongly-coupled regime and reveals critical physical pictures such as electromagnetic duality.

Most of the theories covered in the content blong to Class $\mathcal{S}$ theory–a family of four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories constructed by compactifying the M5-brane on a Riemann surface with a topological twist. Since M-theory is UV-complete, its brane picture contains all the information needed to understand these theories. Building on the seminal work of E. Witten and D. Gaiotto, the community has made great strides in elucidating their structure, yielding many deep physical insights.[3–4]

From a field-theoretic viewpoint, Class $\mathcal{S}$ theories are particularly interesting because they generalize many classic models in supersymmetric gauge theory, including the supersymmetric analog of QCD. Originating from six-dimensional conformal field theories, they not only furnish a systematic way to engineer new $\mathcal{N} = 2$ theories but also bring fresh insights into dualities, such as the AGT duality and S-duality, as well as the structure of the vacuum moduli space[5]. Their close ties to Riemann surfaces and Hitchin integrable systems endow them with a rich mathematical structure[6]. In particular, the Coulomb branch moduli space of a Class $\mathcal{S}$ theory–up to certain discrete data–coincides with the Hitchin moduli space on the corresponding Riemann surface[7–8]. This correspondence forms a bridge between the supersymmetric quantum field theory and hyperkähler geometry, and further provides a systematic way of determining the Seiberg–Witten geometry of any such gauge theory, thereby exactly solving for its low-energy effective action. The rigidity of hyperkähler geometry also provides an effective analytic tool to study intriguing phenomena like wall-crossing in the BPS Hilbert space, yielding exact solutions for the BPS spectrum via the Kontsevich–Soibelman wall-crossing formula[9].

Another crucial element in four dimensions is the introduction of BPS line operators, which represent infinitely heavy BPS particles propagating in the spacetime. In their presence, half of the original supersymmetry is preserved, which modifies the BPS bound. The states that saturate this bound are called "framed BPS states"[10]. Remarkably, line operators possess a rich algebraic structure akin to that of local operators.

When one dimension of the four-dimensional theory is compactified on an S^1 and line operators wind around this circle, their expectation values become holomorphic functions on the Coulomb-branch moduli space. Moreover, in an Ω-background, algebra of line operators becomes non-commutative, providing a natural framework for the quantization of Coulomb branch—a proposal originally put forth by E. Witten and N. Nekrasov[11].

Many works have explored the algebra of BPS line operators from different angles. In the ultraviolet, it is described by the relative skein algebra on the Riemann surface[12–16], while in the infrared it is governed by the quantum Teichmüller space and the Chekhov–Fock algebra[6]. The precise UV–IR correspondence is captured by framed protected spin characters, which count the framed BPS states in the system. A. Neitzke and F. Yan have introduced a geometric method to compute this UV–IR map, offering a fresh perspective on the field-theoretic BPS spectrum[17].

Because the line-operator algebra provides a natural framework for the quantization of the Coulomb branch, it dovetails with the Gukov–Witten proposal for brane quantization[18–19]. In particular, the product of line operators can be understood as the fusion of $(\mathfrak{B}_{\rm cc}, \mathfrak{B}_{\rm cc})$ strings in the topological A-model, while their representations arise from $(\mathfrak{B}, \mathfrak{B}_{\rm cc})$ strings[20,26]. This framework has also been linked to refined Chern–Simons theory and integrable systems, and relating the B-model strings to the representations of the line-operator algebra remains a key open problem[21].

In several celebrated examples–such as $SU(2)$ $\mathcal{N} = 2^*$ SQCD and $SU(2)$ SQCD with four flavors–the quantization of the Coulomb branch is related to the double affine Hecke algebra (DAHA), originally introduced by Cherednik in his study of the Macdonald conjectures[13,22–24]. These examples deepen our understanding of supersymmetric theory structures[25]. In joint work with Professor Satoshi Nawata , Junkang Huang, and Shutong Zhuang, we classified representations of the DAHA and, by examining their relation to Lagrangian submanifolds on the Coulomb branch, conducted the brane quantization of the Coulomb branch for $SU(2)$ $N_f = 4$ SQCD. This extends earlier results and paves the way for exploring BPS spectra in higher-rank gauge groups–especially SU(3), which serves as a supersymmetric analog of real-world QCD.

In summary, understanding the strong-coupling behavior of quantum field theories is one of the greatest challenges in theoretical physics. Studying BPS spectra and dynamics in Class $\mathcal{S}$ theories offers invaluable physical insights into this intricate problem.

This article offers a systematic and detailed review of the topics outlined above, organized as follows.

Chapter 1 provides an introduction that briefly presents the subject matter and its main themes. Chapter 2 described the preliminaries of supersymmetry in depth: it involves the super-Poincaré algebra and its representation theory, with focus on how the presence or absence of a central charge Z would affect the construction of representations; it also examines the constraints imposed by CPT invariance when building supermultiplets. Because these topics receive relatively little treatment in standard textbooks and reviews, this chapter fills in and expands upon them. Starting from supermultiplets, the chapter then introduces the field contents in the supersymmetric quantum field theory, describes how they are realized in superspace, and discusses in detail how supersymmetry transformations act on these fields, laying the groundwork for later dynamical developments.

Chapter 3 focuses on the dynamics of four-dimensional $\mathcal{N} = 2$ supersymmetric field theories. It begins by constructing the $\mathcal{N} = 2$ supersymmetric gauge Lagrangian via superspace integrals and then analyzes the flavor symmetries of the theory and their enhancements in special cases. Next, it investigates the R-symmetry and its perturbative renormalization behavior, explaining why the coupling constant receives corrections only at one loop level and explaining why that argument breaks down and how non-perturbative effects may enter. The chapter then turns to the classical vacuum moduli space of a supersymmetric theory–its definition, how to compute it, and its role in the field theory. For supersymmetric theories, the vacuum moduli space decomposes into Coulomb, Higgs, and mixed branches; here we concentrate on the Coulomb branch. We provide a description for the Coulomb branch when the gauge group is $SU(2)$, and then generalize to a generic gauge group G. For a generic vacuum on the Coulomb branch, the Higgs mechanism breaks the gauge symmetry to a product of $U(1)$, and thus the low-energy physics is governed by massless abelian gauge bosons interacting via Coulomb interaction, hence the name.

To deepen our understanding of symmetry structures of the theory, the chapter proceeds to discuss the Witten effect and explain how the θ angle shifts electric charges. It then analyzes the supercurrent of the theory, deriving the classical expression for the central charge via its Poisson bracket. Using $SU(2)$ SQCD as an example, it explores BPS states that saturate the BPS bound–both perturbative excitations and non-perturbative magnetic monopoles and dyons–and presents the BPS equations these BPS configurations satisfy, together with their relation to the Bogomolny equation. Special attention is paid to the semiclassical quantization of monopoles and dyons, including fermion zero modes and their flavor symmetry representations.

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The exposition then turns to the relation between Seiberg–Witten geometry in Class

$\mathcal{S}$ theories and the Hitchin moduli space. Since the Hitchin moduli space is a hyperkähler manifold, this correspondence allows one to exploit its rich geometric structure to understand the low-energy effective action of the gauge theory. The chapter details the construction of the Hitchin moduli space, its hyperkähler metric, and its descriptions in different complex structures, and analyzes in particular the Coulomb-branch structure and symmetry enhancements in the $N_f = 4$ and Minahan–Nemeschansky theories.

Chapter 5 turns to BPS spectra and BPS line operators. It discusses how inserting BPS line operators modifies the Hilbert space and imposes framed BPS conditions, then delves into the algebraic structure of these line operators, their additive formalism, and operator-product expansions. When analyzing the dynamics of light BPS states in the background of heavy BPS operators, the chapter explains the formation of bound states and computes their bound-state radii. In wall-crossing phenomena, as the bound-state radius diverges the Hilbert-space structure jumps, giving physical meaning to the Kontsevich–Soibelman transformation. The wall-crossing formula then relates the BPS spectra in weakly and strongly coupled regions, enabling exact spectrum calculations in certain cases. The chapter also introduces the superconformal index—a partition function on $S^1 \times S^3$ with periodic fermion boundary conditions—which, being a topological invariant, can be computed at weak coupling to probe strong-coupling physics; for instance, it provides a check of S-duality in the $N_f = 4$ theory.

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Keywords: theoretical physics; string theory; supersymmetric field theory; Seiberg-Witten theory; brane quantization

CLC code: O413.1

Chapter 1
Introduction

In my bachelor's thesis, I aim to write a review article that surveys the basic topics and recent developments of four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories with eight supercharges.

Seiberg–Witten theory represents a milestone in the history of supersymmetric quantum field theory. It focuses on the four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories whose ultraviolet limits are either asymptotically free or conformal. By introducing the Seiberg–Witten curve and its meromorphic differential, Seiberg and Witten identified the vacuum moduli space as a special Kähler manifold, thus describing the electric–magnetic duality and monopole condensation in a geometric framework. This geometric description allows them fully determined the low-energy effective action on the Coulomb branch[1–2]. Seiberg-Witten theory provides an analytic description of quantum field theories in their strongly-coupled regime and reveals critical physical pictures such as electromagnetic duality.

Most of the theories covered in the content blong to Class $\mathcal{S}$ theory–a family of four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories constructed by compactifying the M5-brane on a Riemann surface with a topological twist. Since M-theory is UV-complete, its brane picture contains all the information needed to understand these theories. Building on the seminal work of E. Witten and D. Gaiotto, the community has made great strides in elucidating their structure, yielding many deep physical insights.[3–4]

From a field-theoretic viewpoint, Class $\mathcal{S}$ theories are particularly interesting because they generalize many classic models in supersymmetric gauge theory, including the supersymmetric analog of QCD. Originating from six-dimensional conformal field theories, they not only furnish a systematic way to engineer new $\mathcal{N} = 2$ theories but also bring fresh insights into dualities, such as the AGT duality and S-duality, as well as the structure of the vacuum moduli space[5]. Their close ties to Riemann surfaces and Hitchin integrable systems endow them with a rich mathematical structure[6]. In particular, the Coulomb branch moduli space of a Class $\mathcal{S}$ theory–up to certain discrete data–coincides with the Hitchin moduli space on the corresponding Riemann surface[7–8]. This correspondence forms a bridge between the supersymmetric quantum field theory and hyperkähler geometry, and further provides a systematic way of determining the Seiberg–Witten geometry of any such gauge theory, thereby exactly solving for its low-energy effective action. The rigidity of hyperkähler geometry also provides an effective analytic tool to study intriguing phenomena like wall-crossing in the BPS Hilbert space, yielding exact solutions for the BPS spectrum via the Kontsevich–Soibelman wall-crossing formula[9].

Another crucial element in four dimensions is the introduction of BPS line operators, which represent infinitely heavy BPS particles propagating in the spacetime. In their presence, half of the original supersymmetry is preserved, which modifies the BPS bound. The states that saturate this bound are called "framed BPS states"[10]. Remarkably, line operators possess a rich algebraic structure akin to that of local operators.

When one dimension of the four-dimensional theory is compactified on an S^1 and line operators wind around this circle, their expectation values become holomorphic functions on the Coulomb-branch moduli space. Moreover, in an Ω-background, algebra of line operators becomes non-commutative, providing a natural framework for the quantization of Coulomb branch—a proposal originally put forth by E. Witten and N. Nekrasov[11].

Many works have explored the algebra of BPS line operators from different angles. In the ultraviolet, it is described by the relative skein algebra on the Riemann surface[12–16], while in the infrared it is governed by the quantum Teichmüller space and the Chekhov–Fock algebra[6]. The precise UV–IR correspondence is captured by framed protected spin characters, which count the framed BPS states in the system. A. Neitzke and F. Yan have introduced a geometric method to compute this UV–IR map, offering a fresh perspective on the field-theoretic BPS spectrum[17].

Because the line-operator algebra provides a natural framework for the quantization of the Coulomb branch, it dovetails with the Gukov–Witten proposal for brane quantization[18–19]. In particular, the product of line operators can be understood as the fusion of $(\mathfrak{B}_{\rm cc}, \mathfrak{B}_{\rm cc})$ strings in the topological A-model, while their representations arise from $(\mathfrak{B}, \mathfrak{B}_{\rm cc})$ strings[20,26]. This framework has also been linked to refined Chern–Simons theory and integrable systems, and relating the B-model strings to the representations of the line-operator algebra remains a key open problem[21].

In several celebrated examples–such as $SU(2)$ $\mathcal{N} = 2^*$ SQCD and $SU(2)$ SQCD with four flavors–the quantization of the Coulomb branch is related to the double affine Hecke algebra (DAHA), originally introduced by Cherednik in his study of the Macdonald conjectures[13,22–24]. These examples deepen our understanding of supersymmetric theory structures[25]. In joint work with Professor Satoshi Nawata , Junkang Huang, and Shutong Zhuang, we classified representations of the DAHA and, by examining their relation to Lagrangian submanifolds on the Coulomb branch, conducted the brane quantization of the Coulomb branch for $SU(2)$ $N_f = 4$ SQCD. This extends earlier results and paves the way for exploring BPS spectra in higher-rank gauge groups–especially SU(3), which serves as a supersymmetric analog of real-world QCD.

In summary, understanding the strong-coupling behavior of quantum field theories is one of the greatest challenges in theoretical physics. Studying BPS spectra and dynamics in Class $\mathcal{S}$ theories offers invaluable physical insights into this intricate problem.

This article offers a systematic and detailed review of the topics outlined above, organized as follows.

Chapter 1 provides an introduction that briefly presents the subject matter and its main themes. Chapter 2 described the preliminary of supersymmetry in depth: it involves the super-Poincaré algebra and it's representation theory, with focus on how the presence or absence of a central charge Z would affects the construction of representations; it also examines the constraints imposed by CPT invariance when building supermultiplets. Because these topics receive relatively little treatment in standard textbooks and reviews, this chapter fills in and expands upon them. Starting from supermultiplets, the chapter then introduces the field contents in the supersymmetric quantum field theory, describes how they are realized in superspace, and discusses in detail how supersymmetry transformations act on these fields, laying the groundwork for later dynamical developments.

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Chapter 2

Preliminary for 4d $\mathcal{N} = 2$ Theory

2.1 4d $\mathcal{N} = 2$ Symmetry Alegbra

In this subsection, we would like to review the basic concepts of 4d $\mathcal{N} = 2$ supersymmetry and to set up the notations. We follow the notation of Wess & Bagger[27] for the superpoincáre algebra, except for the notation of the central charge Z. And we follow[28] for the notation of the superconformal algebra, except for multiplying the SUSY generator and superconformal generator by 2 to agree with Wess & Bagger's notation.

4d $\mathcal{N} = 2$ SUSY Algebra 4d $\mathcal{N} = 2$ SUSY algebra is the fermionic extension of the 4d poincáre algebra, which includes fermionic generators $(Q^I_\alpha, \overline{Q}_{\dot\alpha I})_{I=1,2}$ subjecting to the algebraic relation

$$\{Q^I_\alpha, \overline{Q}_{\dot\alpha J}\} = 2\sigma^\mu_{\alpha\dot\alpha} P_\mu \delta^I_J, \quad \{Q^I_\alpha, Q^J_\beta\} = 2\epsilon_{\alpha\beta}\epsilon^{IJ}\overline{Z}, \quad \{\overline{Q}_{\dot\alpha I}, \overline{Q}_{\dot\beta J}\} = -2\epsilon_{\dot\alpha\dot\beta}\epsilon_{IJ} Z, \tag{2.1}$$

$$(Q^I_\alpha)^\dagger = \overline{Q}_{\dot\alpha I} = \epsilon_{IJ}\overline{Q}^J_{\dot\alpha}. \tag{2.2}$$

where $\{-,-\}$ is the anticommutator and we fix $\epsilon^{12} = \epsilon_{21} = 1$. Schematically, Q_α could be understood as the "square root" of the momentum operator P_μ, while the Lorentz indices μ is split into $\alpha\dot\alpha$.

R Symmetry In the case where $Z = \bar{Z} = 0$, the $\mathcal{N} = 2$ SUSY algebra enjoys an automorphism called R symmetry $U(2)_R = U(1)_r \times SU(2)_R$, acting by rotating the indice I:

$$Q^I_\alpha \to U^I_J Q^J_\alpha, \ \overline{Q}_{\dot\alpha I} \to \overline{Q}_{\dot\alpha J} U^{\dagger J}_I, \tag{2.3}$$

We are going to denote the generator of the $U(2)_R$ symmetry as $\mathscr{R}^I_J$, the generators of it's $SU(2)_R$ subalgebra as $(\mathscr{R}^+, \mathscr{R}^-, \mathscr{R})$, and the generator of it's $U(1)_r$ subalgebra as r. They are related by

$$\mathscr{R}^1{}_2 = \mathscr{R}^+, \quad \mathscr{R}^2{}_1 = \mathscr{R}^-, \quad \mathscr{R}^1{}_1 = \frac{1}{2}r + \mathscr{R}, \quad \mathscr{R}^2{}_2 = \frac{1}{2}r - \mathscr{R}. \tag{2.4}$$

And the $SU(2)_R$ subalgebra is subjected to the commutation relation

$$\left[\mathscr{R}^+, \mathscr{R}^-\right] = 2\mathscr{R}, \quad \left[\mathscr{R}, \mathscr{R}^\pm\right] = \pm\mathscr{R}^\pm. \tag{2.5}$$

For generic central charge $(Z, \bar{Z})$, since the $U(1)_r$ symmetry does not preserve the $\{Q, Q\}$ anticommutator relation, only the $SU(2)_R$ subgroup survives and $U(1)_r$ is not the a part of the symmetry algebra.

It might well be the case that R symmetry is not a part of the symmetry algebra of the $\mathcal{N} = 2$ theory, in the sense that it may not commute with the Hamiltonian. A counterexample is to construct a Hamiltonian that has an indice I. However, in the following discussion, we will only consider the cases where $SU(2)_R$ symmetry exists at least at the classical level.

4d $\mathcal{N} = 2$ Superconformal Algebra In the context of superconformal field theory, the Poincaré symmetry in four-dimensional spacetime is extended to conformal symmetry $SO(2,4)$ by incorporating the additional generators (K_μ, D). The fermionic extension of this symmetry introduces extra fermionic generators called superconformal generators $(S^I_\alpha, \overline{S}^{\dot\alpha}_I)_{I=1,2}$, and the full superconformal algebra is enhanced to $\mathfrak{su}(2,2|2)$. Specifically, the four-dimensional $\mathcal{N} = 2$ superconformal algebra has the following maximal bosonic subalgebra:

$$\mathfrak{su}(2,2|2) \supset \mathfrak{so}(4,2) \oplus \mathfrak{su}(2)_R \oplus \mathfrak{u}(1)_r. \tag{2.6}$$

The non-vanishing fermionic anti-commutation relations are

$$\begin{aligned} \{Q^I_\alpha, \overline{Q}_{\dot\alpha J}\} &= 2\sigma^\mu_{\alpha\dot\alpha} P_\mu \delta^I_J, \quad \{Q^I_\alpha, S^\beta_J\} = 2\left(\delta^I_J \delta^\beta_\alpha D + 2\delta^J_I M_{\mu\nu}\sigma^{\mu\nu\beta}{}_\alpha - 2\delta^\beta_\alpha \mathscr{R}^I_J\right), \\ \{S^\alpha_I, \overline{S}^{\dot\alpha}_J\} &= 2\sigma^\mu_{\alpha\dot\alpha} K_\mu \delta^J_I, \quad \{\overline{Q}^{\dot\alpha}_I, \overline{S}^J_{\dot\beta}\} = 2\left(\delta^J_I \delta^{\dot\alpha}_{\dot\beta} D + 2\delta^I_J M_{\mu\nu}\overline{\sigma}^{\mu\nu\dot\alpha}{}_{\dot\beta} + 2\delta^{\dot\alpha}_{\dot\beta} \mathscr{R}^J_I\right). \end{aligned} \tag{2.7}$$

In particular, the algebraic relation between $\{Q, J\}$ involves $U(1)_r$ generator. Therefore, $U(1)_r$ symmetry has to be a part of the symmetry algebra, and the central charge term in $\{Q, Q\}$ has to vanish.

2.2 Representation of the Superpoincáre algebra

In the standard content of quantum field theory, a one-particle Hilbert space is defined as an irreducible unitary projective representation of the Poincaré group. In order to understand the properties of particles in a relativistic quantum theory, we need to classify these representations. The situation is quite similar in the context of supersymmetric field theory, except that in this case, we are building the Hilbert space for a SUSY multiplet, which includes multiple particles rather than a single particle.

Casemir Element The Casimir elements play a crucial role in classifying the irreducible representations. Since Casimir elements commute with every generator in a representation, by Schur's lemma, they have to take a constant value respectively. Therefore, an irreducible representation could be labeled by the eigenvalues of the Casimir elements.

There are two Casimir elements for the 4d Poincaré algebra:

$$P^2 = P^\mu P_\mu, \quad W^2 = W_\mu W^\mu \tag{2.8}$$

where $W_\mu = \frac{1}{2}\epsilon_{\mu\nu\sigma\rho} M^{\nu\rho} P^\sigma$. W^μ is called the Pauli-Ljubanski polarization vector. P^2 characterizes the mass, and W^2 characterizes the spin/helicity of a particle.

In the $\mathcal{N} = 2$ superpoincaré algebra, however, W^2 does not commute with the fermionic generators, so it's no longer Casimir. For our purpose, we would focus on the eigenvalue of $P^2 = -M^2$ to characterize a SUSY multiplet.

Upon selection of the eigenvalue $P^2 = -M^2$, we would apply the Wigner method to construct the irreducible unitary projective representations of the superpointcaré algebra. The standard Wigner method involves the following four steps:

1. Selecting a typical momentum eigen-state $|p^\mu\rangle$.
2. Identifying the stabilizer(little group) of this state.
3. Construct the (finite-dimensional) projective unitary representation of the stabilizer.
4. Induce the representation to the full Poincaré group.

Moreover, one needs to take into account the CPT invariance, which implies that the full Hilbert space must live in a *real* representation of the Pointcaré group. If the one-particle state is in a complex or pseudo-real representation, then we have to incorporate an antiparticle in the conjugate representation as a remedy.

In the following sections, we will apply this analysis to construct the SUSY multiplet when the central charge $Z = 0$ and $Z \neq 0$. It would also be crucial for the construction of the field contents in the $\mathcal{N} = 2$ theory.

2.2.1 Representation at $Z = 0$

At $Z = 0$, the representation of the superpoincaré group depends on the mass of the particle. It is the standard content of SUSY, but we would repeat the analysis as a warm-up exercise.

Massless Multiplet $M = 0$ For the Hilbert space of a massless SUSY multiplet, we could select the set of momentum eigen-basis $\{|p^\mu\rangle\}$ such that $p^2 = 0$. Following the Wigner method, we could select a typical state such that the momentum of the particle is $p_\mu = [E, 0, 0, E]$. The bosonic little group for the choice of momentum is $ISO(2) \times U(2)_R \supset SO(2) \times U(2)_R$, where the $ISO(2) = SO(2) \rtimes \mathbb{R}^2$ subgroup of the Lorentz group is generated by $SO(2)$ rotation along z-axis $\mathcal{J} = J_z$ and $\mathbb{R}^2$ translations $\mathcal{P}_1 = K_x + J_y, \mathcal{P}_2 = K_y - J_x$.

The representation of the bosonic little group at a specific momentum is interpreted as the polarization state of a particle. Since we believe that a single particle should have finite polarization states, the $\mathbb{R}^2$ part must act trivially in the one-particle Hilbert space. Therefore, we only need to construct one-particle states via $SO(2) \times SU(2)_R$.

On the other hand, since all the fermionic generators commute with the momentum operator, they all live in the stabilizer of the typical momentum state. Under our choice of the representative, the fermionic part of the symmetry reduces to

$$\{Q_1^I, \overline{Q}_{\dot{1}J}\} = 4E\delta_J^I, \quad \{Q_2^I, \overline{Q}_{\dot{2}J}\} = 0. \tag{2.9}$$

By the unitarity of the Hilbert space, the fermionic generator $(Q_2^I, \overline{Q}_{\dot{2}J})$ must act trivially. For the rest of the generators, if we redefine

$$a^I = \frac{1}{\sqrt{4E}} Q_1^I, \quad a_J^\dagger = \frac{1}{\sqrt{4E}} \overline{Q}_{\dot{1}J}. \tag{2.10}$$

Then $(a^I, a_J^\dagger)$ satisfies the Clifford algebra (or fermionic harmonic oscillator)

$$\{a^I, a_J^\dagger\} = \delta_J^I, \quad I = 1, 2. \tag{2.11}$$

The representation theory of Clifford algebra over $\mathbb{C}$ is well-known. It could be constructed by picking a Clifford vacuum $|\Omega\rangle$ such that it is annihilated by the fermionic "annihilation operator"

$$a^I\,|\Omega\rangle = 0, \quad I = 1, 2. \tag{2.12}$$

Then the full representation of the Clifford algebra could be constructed by acting the fermionic "creation operator" on the vacuum

$$\rho_{massless} \,:\, |\Omega\rangle\,, a^{\dagger I}\,|\Omega\rangle\,, \epsilon_{IJ}a^{\dagger I}a^{\dagger J}\,|\Omega\rangle\,. \tag{2.13}$$

By evaluating the commutator$[\mathcal{J}, a^{\dagger I}]$, on can show that each fermionic generator lower the helicity by a half. If we ask $|\Omega\rangle$ to be annihilated by the bosonic subgroup, then as the representation of the bosonic little group $SO(2)\times SU(2)_R$, this massless representation could be presented as

$$\rho_{massless} = (0, \mathbf{0}) \oplus \left(-\frac{1}{2}, \frac{\mathbf{1}}{\mathbf{2}}\right) \oplus (-1, \mathbf{0})\,. \tag{2.14}$$

This massless representation would be the building block for us to construct all the SUSY multiples. A generic representation of a massless multiplet could be constructed by

$$\rho_R = \rho_{\text{massless}} \otimes \mathscr{R}, \tag{2.15}$$

where R is an arbitrary projective representation of $SO(2) \times U(2)_R$ and the internal symmetry group. The projective representation of $SO(2)$ is labelled by the half-integers, which is called the *helicity* of the particle.

Furthermore, the CPT invariance of the Hilbert space requires the representation to be *real*. When ρ_R is a complex representation, we have to incorporate the CPT conjugate of the single particle state $\overline{\rho}_R$(anti-particle) to restore CPT invariance. For instance, if we choose $\mathscr{R}$ to be trivial, then ρ_{massless} is a complex representation. To restore the CPT invariance, the Hilbert space must involve the particle states and it's anti-particle states as

$$\rho_{\text{vect}} = \rho_{\text{massless}} \oplus \overline{\rho_{\text{massless}}} = \underbrace{(-1, \mathbf{0}) \oplus (1, \mathbf{0})}_{\text{vector boson}} \oplus \underbrace{\left(-\frac{1}{2}, \frac{\mathbf{1}}{\mathbf{2}}\right) \oplus \left(\frac{1}{2}, \frac{\mathbf{1}}{\mathbf{2}}\right)}_{\text{two complex fermion}} \oplus \underbrace{2\,(0, \mathbf{0})}_{\text{one complex boson}} \tag{2.16}$$

ρ_{vect} is called the *vector multiplet*, which includes one vector boson state, one complex boson state, and two complex Weyl fermion states rotated by the $SU(2)_R$ symmetry.

As another important example, we choose $R = \left(\frac{1}{2}, \mathbf{0}\right)$, Then the multiplet becomes

$$\rho_R = \left(\frac{1}{2}, \mathbf{0}\right) \oplus \left(-\frac{1}{2}, \mathbf{0}\right) \oplus \left(0, \frac{\mathbf{1}}{\mathbf{2}}\right). \tag{2.17}$$

Generically, if the multiplet lives in the representation R_0 of the internal symmetry group, then we have

$$\rho = \rho_R \otimes R_0 \tag{2.18}$$

An important subtlety is that ρ_R is *pseudo-real* rather than *real*. Therefore, it is not closed under the CPT conjugation for a generic choice of R_0. An obvious remedy for the CPT invariance is to incorporate the anti-particles in $\overline{\rho_R} \otimes \overline{R_0}$. Then the Hilbert space involves the one-particle states and it's anti-particle is as

$$\rho_{\text{hyper}} = \rho \oplus \bar{\rho} = \underbrace{\left\{\left(\frac{1}{2},\mathbf{0}\right) \oplus \left(-\frac{1}{2},\mathbf{0}\right)\right\} \otimes \left(R_0 \oplus \overline{R}_0\right)}_{\text{one complex Weyl fermion in } R_0 \text{ and another in } \overline{R}_0} \oplus \underbrace{\left(0,\frac{\mathbf{1}}{\mathbf{2}}\right) \otimes \left(R_0 \oplus \overline{R}_0\right)}_{\text{two real bosons in } R_0 \text{ and another two in } \overline{R}_0} \tag{2.19}$$

This multiplet is called the *hypermultiplet*, which includes one complex Weyl fermion state in R_0, one complex fermion state in $\overline{R}_0$, two real boson state in R_0 and another two real boson state in $\overline{R}_0$ rotated by $SU(2)_R$ symmetry.

There is another remedy for the CPT invariance. Since the tensor product of two pseudo-real representations is real, the CPT invariance persists if R_0 is pseudo-real. It is called a *half hypermultiplet*, which includes half of the degrees of freedom in a hypermultiplet.

$$\rho_{hh} = \rho_R \otimes R_0, \tag{2.20}$$

with R_0 pseudo-real.

Massive Multiplet $M \neq 0$ The massive representation of the particle could be constructed in a similar fashion. For the Hilbert space of a massive SUSY multiplet, we could select the set of momentum eigen-basis $\{|p^\mu\rangle\}$ such that $p^2 = -M^2$. Afterwards, we select a typical state with momentum $p_\mu = [M, 0, 0, 0]$. Then the bosonic little group for the choice of momentum state is $SO(3) \times SU(2)_R$.

On the other hand, since all the fermionic generators commute with the momentum operator, they all live in the stabilizer of the typical momentum state. Under our choice of the representative, the fermionic part of the symmetry reduces to

$$\{Q_1^I, \overline{Q}_{\dot{1}J}\} = 2M\delta_J^I, \quad \{Q_2^I, \overline{Q}_{\dot{2}J}\} = 2M\delta_J^I. \tag{2.21}$$

If we redefine

$$a_\alpha^I = \frac{1}{\sqrt{2M}} Q_\alpha^I, \quad a_{\dot{\alpha}I}^\dagger = \frac{1}{\sqrt{2M}} \overline{Q}_{\dot{\alpha}I}, \tag{2.22}$$

then they satisfy the Clifford algebra

$$\{a_\alpha^I, a_{\dot{\alpha}J}^\dagger\} = \delta_J^I \delta_{\alpha\dot{\alpha}}, \quad I, J = 1, 2. \tag{2.23}$$

What is different from the massless representation is that now we have two copies of Clifford algebra. As a consequence, the massive multiplet would be sixteen-dimensional rather than four. If we start with a Clifford vacuum annihilated by a_α^I and the bosonic little group, then the multiplet involves the states

$$\rho_{\text{masssive}} : |\Omega\rangle, a_{\dot{\alpha}}^{\dagger I}|\Omega\rangle, a_{[\dot{\alpha}}^{\dagger(I} a_{\dot{\beta}]}^{\dagger J)}|\Omega\rangle, a_{(\dot{\alpha}}^{\dagger[I} a_{\dot{\beta})}^{\dagger J]}|\Omega\rangle, a_{[\dot{\alpha}}^{\dagger(I} a_{(\dot{\beta}]}^{\dagger[J)} a_{\dot{\gamma})}^{\dagger K]}|\Omega\rangle, a_{[\dot{\alpha}}^{\dagger(I} a_{(\dot{\beta}]}^{\dagger[J)} a_{[\dot{\gamma})}^{\dagger(K]} a_{\dot{\delta}]}^{\dagger L)}|\Omega\rangle. \tag{2.24}$$

Complicated as the structure of the indices is, there is a shortcut to repackage the indices using young tableaux and identify the corresponding representations. As a result of this exercise, this massive multiplet could decompose as the bosonic little group of $SO(3) \times SU(2)$ as

$$\rho_{\text{massive}} = (\mathbf{0},\mathbf{0}) \oplus \left(\frac{\mathbf{1}}{\mathbf{2}},\frac{\mathbf{1}}{\mathbf{2}}\right) \oplus (\mathbf{0},\mathbf{1}) \oplus (\mathbf{1},\mathbf{0}) \oplus \left(\frac{\mathbf{1}}{\mathbf{2}},\frac{\mathbf{1}}{\mathbf{2}}\right) \oplus (\mathbf{0},\mathbf{0}). \tag{2.25}$$

A generic multiplet could be constructed via

$$\rho_R = \rho_{\text{massive}} \otimes R \tag{2.26}$$

with R a projective unitary representation of $SO(3) \times SU(2)_R$ and internal symmetry. At the last step, we need to take the CPT invariance of the Hilbert space into account to see if we need to incorporated antiparticles.

As an example, we take R to be the trivial representation. In this case, ρ_{massive} is itself a real representation, so there is no need to incorporate anti-particles. This multiplet is called the *long multiplet*, which involves five real scalars, two complex Dirac fermions and one gauge boson.

2.2.2 Representation at $Z \neq 0$

Now let us move on to the $Z \neq 0$ case, as we would see, the unitary requires the multiplet to be massive $M \neq 0$. Therefore, we only need to consider the irreducible representation where $P^2 = -M^2$. For the Hilbert space of a massive SUSY multiplet, we could select the set of momentum eigen-basis $\{|p^\mu\rangle\}$ such that $p^2 = -M^2$. The full representation could be induced from the typical state with $p_\mu = [M, 0, 0, 0]$. Similar to the analysis in the last section, the bosonic little group of the typical state in $SO(3) \times SU(2)_R$.

Again, since the momentum generators still commute with the fermionic generators, all the fermionic generators persist in the little group. With the choice of the momentum typical state, the fermionic part of the symmetry reduces to

$$\{Q^I_\alpha, \overline{Q}_{\dot\alpha J}\} = 2\delta_{\alpha\dot\alpha} M \delta^I_J, \quad \{Q^I_\alpha, Q^J_\beta\} = 2\epsilon_{\alpha\beta}\epsilon^{IJ}\overline{Z}, \quad \{\overline{Q}_{\dot\alpha I}, \overline{Q}_{\dot\beta J}\} = -2\epsilon_{\dot\alpha\dot\beta}\epsilon_{IJ} Z, \tag{2.27}$$

We could recombine the SUSY generators as follows to

$$Q^{(\zeta,+)I}_\alpha = \zeta^{\frac{1}{2}} Q^I_\alpha + \zeta^{-\frac{1}{2}}\sigma^0_{\alpha\dot\alpha}\overline{Q}^{\dot\alpha I}, \quad Q^{(\zeta,-)I}_\alpha = \zeta^{\frac{1}{2}} Q^I_\alpha - \zeta^{-\frac{1}{2}}\sigma^0_{\alpha\dot\alpha}\overline{Q}^{\dot\alpha I}. \tag{2.28}$$

with $|\zeta| = 1$. Then the algebraic relation is "diagonalized" as

$$\begin{aligned}
\{Q^{(\zeta,+)I}_\alpha, Q^{(\zeta,+)J}_\beta\} &= 4(M + \Re(Z/\zeta))\epsilon_{\alpha\beta}\epsilon^{IJ}, \\
\{Q^{(\zeta,-)I}_\alpha, Q^{(\zeta,-)J}_\beta\} &= 4(-M + \Re(Z/\zeta))\epsilon_{\alpha\beta}\epsilon^{IJ}, \\
\{Q^{(\zeta,+)I}_\alpha, Q^{(\zeta,-)J}_\beta\} &= -4i\epsilon_{\alpha\beta}\epsilon^{IJ}\Im(Z/\zeta),
\end{aligned} \tag{2.29}$$

along with the reality conditions

$$\begin{aligned}
\left(Q^{(\zeta,+)1}_1\right)^\dagger &= -Q^{(\zeta,+)2}_2, \left(Q^{(\zeta,+)1}_2\right)^\dagger = Q^{(\zeta,+)2}_1, \\
\left(Q^{(\zeta,-)1}_1\right)^\dagger &= Q^{(\zeta,-)2}_2, \left(Q^{(\zeta,-)1}_2\right)^\dagger = -Q^{(\zeta,-)2}_1.
\end{aligned} \tag{2.30}$$

The unitarity of the Hilbert space imposes an extra constraint on the Hilbert space

$$\begin{aligned}
\langle p_\mu| \{Q^{(\xi,+)1}_1, -Q^{(\xi,+)2}_2\} |p_\mu\rangle &= 4(M + \Re(Z/\zeta)) \geq 0, \\
\langle p_\mu| \{Q^{(\xi,-)1}_1, Q^{(\xi,-)2}_2\} |p_\mu\rangle &= 4(M - \Re(Z/\zeta)) \geq 0,
\end{aligned} \tag{2.31}$$

which implies that $M \geq -\Re(Z/\zeta)$ and $M \geq \Re(Z/\zeta)$. At this moment, as the phase ζ can be selected arbitrarily, we could thus select the phase ζ such that $\Re(Z/\zeta) = -|Z|$.[①]

① If we insert certain half BPS defects in the theory, then ζ cannot be arbitrary, which modifies the BPS bound.

Therefore, there exists a unitary bound for the mass of the particle in the presence of central charge

$$M \geq |Z|. \tag{2.32}$$

It is called *BPS bound*, and the states saturating the bound are called *BPS states*.① As we will see in the following sections, BPS states play a very important role in the supersymmetric theory. When this special ζ is selected, the SUSY algebra reduces to

$$\begin{aligned}
\{Q_\alpha^{(\zeta,+)I}, Q_\beta^{(\zeta,+)J}\} &= 4(M - |Z|))\epsilon_{\alpha\beta}\epsilon^{IJ}, \\
\{Q_\alpha^{(\zeta,-)I}, Q_\beta^{(\zeta,-)J}\} &= -4(M + |Z|)\epsilon_{\alpha\beta}\epsilon^{IJ}, \\
\{Q_\alpha^{(\zeta,+)I}, Q_\beta^{(\zeta,-)J}\} &= 0.
\end{aligned} \tag{2.33}$$

In the rest of this section, we are going to construct the representation of superpoincaré algebra in the presence of central charge, which could be roughly classified into two types: short representation(BPS representation/multiplet) $M = |Z|$ and long representations $M \geq |Z|$. The technique is almost the same as what we have discussed in the last section.

Short Representation $M = |Z|$ When the BPS bound $M = |Z|$ is saturate, the SUSY algebra reduces to

$$\begin{aligned}
\{Q_\alpha^{(\zeta,+)I}, Q_\beta^{(\zeta,+)J}\} &= 0, \\
\{Q_\alpha^{(\zeta,-)I}, Q_\beta^{(\zeta,-)J}\} &= -8M\epsilon_{\alpha\beta}\epsilon^{IJ}, \\
\{Q_\alpha^{(\zeta,+)I}, Q_\beta^{(\zeta,-)J}\} &= 0.
\end{aligned} \tag{2.34}$$

Again, the unitarity implies that $Q_\alpha^{(\zeta,+)I}$ acts trivially in the representation. $Q_\alpha^{(\zeta,+)I}$ would generate the unbroken supersymmetry in the presence of a BPS multiplet. If we redefine the generators as

$$a_I = \frac{1}{\sqrt{8M}}\epsilon_{IJ}Q_1^{(\zeta,+)J}, a^{\dagger I} = \frac{1}{\sqrt{8M}}Q_2^{(\zeta,+)I}, \tag{2.35}$$

then they satisfy the commutation relation of Clifford algebra

$$\{a_I, a^{\dagger J}\} = \delta_I^J. \tag{2.36}$$

In the following, we would like to construct a minimal representation for the bosonic little group $SO(3) \times SU(2)_R$ and the fermionic generators. To this purpose, notice that we could combine the generators in the following way to construct a Schwinger-like representation for $SO(3)$

$$J_+ = \epsilon_{IJ}a^I a^J, \quad J_z = \frac{1}{4}\epsilon_{IJ}\left(a^I a^{\dagger J} + a^{\dagger I}a^J\right), \quad J_- = \epsilon_{IJ}a^{\dagger I}a^{\dagger J} \tag{2.37}$$

which satisfies the standard $SO(3)$ commutation relation

$$[J_3, J_+] = J_+, \quad [J_3, J_-] = -J_-, \quad [J_+, J_-] = 2J_3. \tag{2.38}$$

① Strictly speaking, the BPS states we constructed are called half-BPS states, as they preserve half of the supersymmetry.

A similar construction could be conducted for $SU(2)_R$ subgroup, which is also the combination of Clifford generators $(a_I, a^{\dagger J})$. Therefore, we can construct the minimal representation by specifying the action of Clifford generators $\left(a_I, a^{\dagger J}\right)$ and imposing the relation (2.37) to realize the bosonic part of the algebra. Upon the selection of a Clifford vacuum $|\Omega\rangle$, (2.37) implies that

$$a^I|\Omega\rangle = 0, \quad J_+|\Omega\rangle = 0. \tag{2.39}$$

Therefore, $|\Omega\rangle$ should also be the highest vector in $SO(3)$ representation. We would take the Clifford vacuum $|\Omega\rangle$ to be the highest weight vector in the $\left(\frac{\mathbf{1}}{\mathbf{2}}, \mathbf{0}\right)$ of $SO(3)\times SU(2)_R$. The full representation could be constructed by acting the fermionic creation operators on the Clifford vacuum

$$\rho_{hh} : |\Omega\rangle, a^{I\dagger}|\Omega\rangle, \epsilon_{IJ}a^{I\dagger}a^{J\dagger}|\Omega\rangle = J_-|\Omega\rangle. \tag{2.40}$$

As the representation of the bosonic little group $SO(3)\times SU(2)_R$, this representation could be represented as

$$\rho_{hh} \cong \left(0, \frac{1}{2}\right) \oplus \left(\frac{1}{2}, 0\right). \tag{2.41}$$

And a generic short multiplet could thus be constructed as

$$\rho_{short} = \rho_{hh} \otimes R, \tag{2.42}$$

where R is arbitrary projective representation of $SO(3)\times SU(2)_R$ and the internal symmetry.

When we take $R = \left(\frac{1}{2}, 0\right)$, the short multiplet decomposes as the representation of the bosonic little group $SO(3)\times SU(2)_R$ as

$$\rho_W = \rho_{hh} \otimes \left(\frac{1}{2}, 0\right) = (0,0) \oplus (1,0) \oplus \left(\frac{1}{2}, \frac{1}{2}\right). \tag{2.43}$$

This is called the *vector multiplet*, which includes one real scalar boson, one massive vector boson(W-boson), and two complex fermions rotated by the $SU(2)_R$ symmetry. Notice that the degrees of freedom in this multiplet are the same as the one we introduced in the absence of central charge, but one scalar boson is "eaten" by the vector boson, while the vector field becomes massive. The standard way of constructing those states using field theory is the Higgs mechanism.

Consider the case where R is taken to be the trivial representation $\rho_{short} = \rho_{hh}$. Since such a representation is pseudo-real, we need to incorporate the anti-particles in the Hilbert space, and the multiplet of particles and antiparticles becomes

$$\rho_{\text{Hyper}} = \rho_{hh} \oplus \overline{\rho_{hh}} = 2\left(\mathbf{0}, \frac{\mathbf{1}}{\mathbf{2}}\right) \oplus 2\left(\frac{\mathbf{1}}{\mathbf{2}}, \mathbf{0}\right). \tag{2.44}$$

This is the *massive hypermultiplet*, which includes two complex scalars and two complex fermions. On the other hand, when R is a pseudo-real representation, ρ_{short} is real and thus respects CPT invariance. This is called the *massive hypermultiplet*. The degrees of freedom in the half-hyper and hyper are the same as those constructed in the last section.

Long Representation $M > |Z|$ The long representation of the theory in the presence of central charge is exactly the same as what we introduced in the last section. The algebra of fermionic operators (2.33) could be rewritten into 2 copies of Clifford Algebra with a proper normalization. Therefore, a generic representation of this algebra could be presented as

$$\rho_{long} = \rho_{hh} \otimes \rho_{hh} \otimes R \tag{2.45}$$

where R is arbitrary projective representation of $SO(3) \times SU(2)_R$ and internal symmetry.

When R is the trivial representation, the long multiplet decomposes as the representation of the bosonic little group $SO(3) \times SU(2)_R$ as

$$\rho_{\text{massive}} = (\mathbf{0}, \mathbf{0}) \oplus \left(\frac{\mathbf{1}}{\mathbf{2}}, \frac{\mathbf{1}}{\mathbf{2}}\right) \oplus (\mathbf{0}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{0}) \oplus \left(\frac{\mathbf{1}}{\mathbf{2}}, \frac{\mathbf{1}}{\mathbf{2}}\right) \oplus (\mathbf{0}, \mathbf{0}) \,. \tag{2.46}$$

It matches with the degrees of freedom in (2.25).

2.3 Character for the Representations of the SUSY algebra

A powerful tool to characterize a representation in group theory is the *character*. In this section, we would like to introduce the character for the representation of SUSY algebra, which will play a very important role in the following discussion.

For a generic representation of the SUSY algebra ρ, the formal character of it is defined as

$$ch(\rho) = \mathrm{Tr}_\rho(x_1^{2J_z} x_2^{2\mathscr{R}}) \tag{2.47}$$

with J_z is the rotation generator along the z-axis and $\mathscr{R}$ is the Cartan generator of $\mathfrak{su}(2)_R$. For instance, for the representations constructed in 2.2.2, character of the multiplets could be evaluated as

$$\begin{aligned} ch(\rho_{hh}) &= x_1 + x_1^{-1} + x_2 + x_2^{-1}, \\ ch(\rho_{long}) &= ch(\rho_{hh})^2 ch(R), \quad ch(\rho_{short}) = ch(\rho_{hh}) ch(R). \end{aligned} \tag{2.48}$$

While the parameters of a theory(such as energy scale) change, a long multiplet may coincidentally saturate the BPS bound $M = |Z|$ and split into the direct sum of the BPS multiples, and vice versa. While this circumstance occurs, a long multiplet splits as

$$\rho_{long} = \rho_{hh} \otimes R' \tag{2.49}$$

such that $R' = R \otimes \rho_{hh}$. It would be helpful to find a quantity Ω that is insensitive under this procedure. Therefore, we expect Ω to vanish on the long multiplets and yield a non-trivial value on the BPS multiples. From 2.48, it is not difficult to find one construction of the Ω as

$$\tilde{\Omega}(\rho, y) = x_1 \partial_{x_1} ch(\rho) \Big|_{x_1 = -x_2 = y} = \mathrm{Tr}_\rho\, 2J_3 (-1)^{2J_3} (-y)^{2\mathscr{J}_3}, \tag{2.50}$$

where $\mathscr{J}_z = J_z + \mathscr{R}$. It is not difficult to check that this quantity vanishes on the long multiplet. On the other hand, it is non-trivial for the short multiplets

$$\tilde{\Omega}(\rho_{short}, y) = (y - y^{-1})\, \mathrm{Tr}_R (-1)^{2J_3} (-y)^{2\mathscr{J}_3}. \tag{2.51}$$

For convenience, we normalize the $\tilde{\Omega}(\rho_{short})$ and introduce the *protected spin character* $\Omega(\rho_{short})$ as

$$\Omega(\rho_{short}, y) = \frac{\tilde{\Omega}(\rho_{short})}{y - y^{-1}} = \mathrm{Tr}_R(-1)^{2J_z}(-y)^{2\mathcal{J}_z}. \tag{2.52}$$

While evaluated on the half-hypermultiplet and vector multiplet, it yields

$$\Omega(\rho_{hh}, y) = 1, \quad \Omega(\rho_{vect}, y) = y + y^{-1} \tag{2.53}$$

An interesting limit occurs when we take the $y = -1$ limit for the protected spin character. By taking this limit and applying l'Hospital's rule in (2.52), we have

$$\Omega(\rho_{short}, -1) = \mathrm{Tr}_R(-1)^{2J_3} = -\frac{1}{2}Tr_\rho(2J_3)^2(-1)^{2J_3}. \tag{2.54}$$

It is called the *second-helicity supertrace*, which coincides with the index of the representation $\boldsymbol{R}$. While evaluated on the half-hypermultiplet and vector multiplet, it yields

$$\Omega(\rho_{hh}, -1) = 1, \quad \Omega(\rho_{vect}, -1) = -2. \tag{2.55}$$

More generally, for a spin-j/2 multiplet $\rho_{j/2} = \rho_{hh} \otimes (\mathbf{j/2}, 0)$[①], the second helicity supertrace yields

$$\Omega(\rho_{j/2}, -1) = (-1)^j(j+1). \tag{2.56}$$

For an arbitrary representation of the SUSY algebra $\mathscr{H}$ (which will be the Hilbert space of a theory), if there exist n_j copies of spin-j multiplets, then the second helicity supertrace over the Hilbert space yields

$$\Omega(\mathscr{H}, -1) = \sum_{j=0}^{\infty} n_j(-1)^j(j+1). \tag{2.57}$$

2.4 Field Representation for $\mathcal{N} = 2$ multiplets

In this subsection, we are going to construct the field content in $\mathcal{N} = 2$ supersymmetric field theory. The philosophy behind constructing a field representation in a quantum field theory is that the modes of the quantum field should generate particle states in an irreducible (projective) representation of Poincaré group. The same logic applies for $\mathcal{N} = 2$ theory. That is, the modes of the quantum fields should generate the particle states corresponding to the irreducible representation of the $\mathcal{N} = 2$ SUSY algebra.

A systematic and efficient method for constructing a 4d $\mathcal{N} = 2$ field representation is the superspace formalism. In this framework, we extend the usual spacetime with bosonic coordinates x^μ to a superspace by introducing additional Grassmannian coordinates $(\theta_I^\alpha, \bar{\theta}_{\dot{\alpha}}^I)$, where $I = 1, \dots, k$. We will refer to this extended space as $\mathcal{N} = k$ superspace.[②] An $\mathcal{N} = k$ superfield is defined as a smooth function on $\mathcal{N} = k$ superspace, which can be expanded as a polynomial in the Grassmannian coordinates:

$$\Phi_a(x, \theta_I^\alpha, \bar{\theta}_{\dot{\alpha}}^I) = \phi_a(x) + \theta_I \psi_a^I + \bar{\theta}^I \bar{\psi}_{aI} + \dots, \quad I = 1, \dots, k. \tag{2.58}$$

① At this moment, we haven't take the CPT invariance into accout

② It is worth mentioning that there is also a coordinate-free definition of superspace. In this formulation, superspace is not a set but rather a locally ringed space, where the structure sheaf contains anticommuting parameters.

Here, a represents all the possible indices, and the coefficients before each Grassmann parameter ($\phi_a(x), \psi_a^I(x), \bar{\psi}_{aI}(x), \dots$) are referred to as the component fields. The smoothness condition on the superfield requires that the component fields must be smooth.

Let us remark on the calculus on 4d $\mathcal{N} = 1$ superspace, which would be applied extensively in the following discussion (for detailed construction, see[27]).

Calculus on $\mathcal{N} = 1$ Superspace A $4d$ $\mathcal{N} = 1$ superspace is characterized by the bosonic and fermionic coordinates $(x^\mu, \theta_\alpha, \overline{\theta}^{\dot{\alpha}})$. The derivative operator would be denoted as

$$\partial_\mu = \frac{\partial}{\partial x^\mu}, \quad \partial_\alpha = \frac{\partial}{\partial \theta^\alpha}, \quad \partial_{\dot{\alpha}} = \frac{\partial}{\partial \overline{\theta}^{\dot{\alpha}}}, \tag{2.59}$$

where the fermionic derivative acts on the *left* and satisfy the graded leibnitz rule. The $\mathcal{N} = 1$ supersymmetry transformations act as the differential operators on the superspace

$$\delta_\xi = \xi Q + \bar{\xi}\bar{Q} = \xi^\alpha \left(\partial_\alpha - i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu\right) + \bar{\xi}_{\dot{\alpha}} \left(\partial^{\dot{\alpha}} - i\theta^\alpha \sigma^\mu_{\alpha\dot{\beta}}\varepsilon^{\dot{\beta}\dot{\alpha}}\partial_\mu\right), \tag{2.60}$$

which transform bosonic component fields into fermionic ones and vice versa. There is another useful operation called (super) covariant derivative

$$D_\alpha = \partial_\alpha + i\sigma_{\alpha\dot{\alpha}}{}^\mu \bar{\theta}^{\dot{\alpha}}\partial_\mu, \quad \overline{D}_{\dot{\alpha}} = -\partial_{\dot{\alpha}} - i\theta^\alpha \sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu. \tag{2.61}$$

This derivative is called covariant in the sense that it anti-commutes with the SUSY transformation

$$\left\{D_\alpha, Q_\beta\right\} = \left\{D_\alpha, \overline{Q}_{\dot{\beta}}\right\} = \left\{\overline{D}_{\dot{\alpha}}, Q_\beta\right\} = \left\{\overline{D}_{\dot{\alpha}}, \overline{Q}_{\dot{\beta}}\right\} = 0. \tag{2.62}$$

In the following, we would also need integration over the $\mathcal{N} = 1$ superspace. The bosonic part satisy the ordinary integration measure on $\mathbb{R}^4$ and the grassmann integral is defined as

$$\int d^2\theta \; \theta\theta = 1, \quad \int d^4\theta \; \theta\theta \; \overline{\theta\theta} = 1, \quad \text{otherwise zero.} \tag{2.63}$$

This integration measure is invariant under the SUSY transformation. Therefore, for any superfield $\boldsymbol{\Phi}$, we have

$$\delta_\xi S = \int d^4x d^4\theta \; \delta_\xi \boldsymbol{\Phi} = 0 \tag{2.64}$$

It provides us a systematic way of constructing the SUSY invariant actions by starting with any superfield $\boldsymbol{\Phi}$ and averaging it over the superspace. We would come back to this point later.

Now we are prepared to introduce various $\mathcal{N} = 1$ superfield. In total, there are two types of $\mathcal{N} = 1$ superfield with spin≤ 1 representing the $\mathcal{N} = 1$ BPS multiplets, called *chiral superfield* and *vector superfield*.

$\mathcal{N} = 1$ **Chiral Superfield** The superfield representing the states in a 4d $\mathcal{N} = 1$ chiral multiplet is known as chiral superfield $\mathcal{Q}$. It is defined as a $\mathcal{N} = 1$ superfield satisfying the constraint

$$\overline{D}_{\dot{\alpha}} Q(x, \theta, \overline{\theta}) = 0 \tag{2.65}$$

This constraint, due to the property of the covariant derivative, is respected by the SUSY transformation. The generic solution to the constraint is

$$\mathcal{Q}^A(y^\mu, \theta) = q^A(y^\mu) + \sqrt{2}\psi^A(y^\mu)\theta + F^A(y^\mu)\theta\theta, \tag{2.66}$$

where A is the flavor index and $y^\mu = x^\mu - i\theta\sigma^\mu\bar{\theta}$. As for the component field, a chiral superfield includes one complex scalar field and one complex Weyl fermion.

The field F^A, known as the auxiliary field, has mass dimension 2. Due to this dimensionality, it does not have a kinetic term in a renormalizable action. Therefore, the auxiliary field would not have the corresponding particle states. Under supersymmetry transformations, F^A transforms as a total derivative.

Explicitly, the SUSY transformation of the component fields in a Chiral multiplet is given by

$$\begin{aligned} \delta_\xi q &= \sqrt{2}\xi\psi \\ \delta_\xi \psi &= i\sqrt{2}\sigma^\mu\bar{\xi}\partial_\mu q + \sqrt{2}\xi F \\ \delta_\xi F &= i\sqrt{2}\bar{\xi}\bar{\sigma}^\mu\partial_\mu\psi. \end{aligned} \tag{2.67}$$

As a concluding remark, the space of chiral superfields forms a ring, as the addition and multiplication of two chiral superfields result in another chiral superfield.

$\mathcal{N} = 1$ **Vector Superfield** $\mathcal{N} = 1$ vector superfield represents the $\mathcal{N} = 1$ vector multiplet, whose field content involves the gauge fields. Given a gauge group G, the vector superfield V is a real superfield in $\mathfrak{g}$ subjected to the gauge transformation

$$V^\dagger = V, \quad e^V \to e^{-i\Lambda^\dagger} e^V e^{i\Lambda}. \tag{2.68}$$

where Λ is a chiral superfield taking value in $\mathfrak{g}$. A chiral superfield coupled to the gauge group in representation ρ transforms as

$$Q \to e^{i\Lambda} Q, \quad Q^\dagger \to Q^\dagger e^{-i\Lambda^\dagger}. \tag{2.69}$$

Under the Wess-Zumino gauge, the $\mathcal{N} = 1$ vector superfield repackages a complex Weyl fermion λ_α and a vector field A_μ in a vector multiplet into a compact form as

$$V = -i\theta\sigma^\mu\bar{\theta}A_\mu + i\theta\theta\bar{\theta}\lambda^\dagger - i\bar{\theta}\bar{\theta}\theta\lambda + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D, \tag{2.70}$$

where D is an auxiliary field.

In order to write down a gauge invariant action for the vector multiplet including the field strength term $F_{\mu\nu}F^{\mu\nu}$, it is convenient to rewrite the vector multiplet V into the gaugino superfield W_α. This is related to V as follows and is covariant under the gauge transformation

$$W_\alpha = -\frac{1}{4}\overline{D}^2 e^{-V} D_\alpha e^V, \quad W_\alpha \to e^{-i\Lambda^\dagger} W_\alpha e^{i\Lambda}. \tag{2.71}$$

Gaugino superfield is a fermionic chiral superfield which has the following expansion:

$$W_\alpha(y^\mu, \theta) = -i\lambda_\alpha + iF_{\mu\nu}(\sigma^{\mu\nu}\theta)_\alpha + D\theta_\alpha + \theta\theta(\sigma^\mu D_\mu \lambda^\dagger)_\alpha, \tag{2.72}$$

whose lowest component λ_α is the Weyl gaugino. Explicitly, the SUSY transformation for the component field is as follows when the gauge group is abelian

$$\begin{aligned} \delta_\xi A_\mu &= i\xi\sigma_\mu\lambda^\dagger + i\bar{\xi}\bar{\sigma}_\mu\lambda \\ \delta_\xi F_{\mu\nu} &= i\left[\left(\xi\sigma_\nu\partial_\mu\lambda^\dagger + \bar{\xi}\bar{\sigma}_\nu\partial_\mu\lambda\right) - \left(\xi\sigma_\mu\partial_\nu\lambda^\dagger + \bar{\xi}\bar{\sigma}_\mu\partial_\nu\lambda\right)\right] \\ \delta_\xi\lambda &= i\xi D + F_{\mu\nu}\sigma^{\mu\nu}\xi \\ \delta_\xi D &= \bar{\xi}\bar{\sigma}^\mu\partial_\mu\lambda - \xi\sigma^\mu\partial_\mu\lambda^\dagger \end{aligned} \tag{2.73}$$

For a vector multiplet for a generic gauge group G and the a chiral multiplet coupled to the gauge field through representation ρ, the SUSY transformation is as follows

$$\begin{aligned} \delta_\xi q &= \sqrt{2}\xi\psi \\ \delta_\xi\psi &= i\sqrt{2}\sigma^\mu\bar{\xi}D_\mu q + \sqrt{2}\xi F \\ \delta_\xi F &= i\sqrt{2}\bar{\xi}\overline{\sigma}^\mu D_\mu\psi + 2iq\bar{\xi}\lambda^\dagger \\ \delta_\xi A_\mu &= -i\lambda^\dagger\bar{\sigma}_\mu\xi + i\bar{\xi}\bar{\sigma}_\mu\lambda \\ \delta_\xi F_{\mu\nu} &= i\left[\left(\xi\sigma_\nu D_\mu\lambda^\dagger + \bar{\xi}\bar{\sigma}_\nu D_\mu\lambda\right) - \left(\xi\sigma_\mu D_\nu\lambda^\dagger + \bar{\xi}\bar{\sigma}_\mu D_\nu\lambda\right)\right] \\ \delta_\xi\lambda &= F_{\mu\nu}\sigma^{\mu\nu}\xi + i\xi D \\ \delta_\xi D &= -\xi\sigma^\mu D_\mu\lambda^\dagger - D_\mu\lambda\sigma^\mu\bar{\xi}. \end{aligned} \tag{2.74}$$

Now we are prepared to construct superfields for $\mathcal{N} = 2$ multiple. It turns out that representing the states in $\mathcal{N} = 2$ multiples via $\mathcal{N} = 2$ superfields is quite involved. However, if we insist on the existence of $SU(2)_R$ symmetry, we can simplify the problem. Notice that imposing $\mathcal{N} = 2$ SUSY is equivalent to imposing $\mathcal{N} = 1$ SUSY along with a suitable $SU(2)_R$ symmetry on the field content. This allows us to reduce the problem to finding an appropriate combination of $\mathcal{N} = 1$ superfields that respects the $SU(2)_R$-symmetry. In the following, we would take all the $\mathcal{N} = 1$ superfields to be defined regarding $(Q^1_\alpha, \overline{Q}^{\dot{\alpha}}_1)$.

$\mathcal{N} = 2$ Vector Multiplet Using $\mathcal{N} = 1$ superfields, we could construct the field representation for $\mathcal{N} = 2$ multiples. A $\mathcal{N} = 2$ vector multiplet can be represented as a combination of an $\mathcal{N} = 1$ vector superfield and an $\mathcal{N} = 1$ chiral superfield:

$$V_{\mathcal{N}=2} \to (V, \Phi) \supset (A_\mu, \lambda_\alpha, \tilde{\lambda}_\alpha, \varphi), \tag{2.75}$$

where the chiral superfield Φ is in the adjoint representation of the gauge group G. As for the component field, a $\mathcal{N} = 2$ vector multiplet contains one complex scalar field, two complex Weyl fermions and one gauge boson. The $SU(2)_R$ symmetry acts trivially on the complex scalar and gauge boson, but rotates the Weyl fermion in the vector multiplet and the Weyl fermion in the chiral multiplet. As a result, such a field representation matches with the construction in (2.16).

For $\mathcal{N} = 2$ vector multiplet, the field contents could be repackaged into an $\mathcal{N} = 2$ superfield,

$$\Phi_{\mathcal{N}=2}(x, \theta, \bar{\theta}, \theta', \bar{\theta}') = \Phi(w, \theta, \bar{\theta}) + \sqrt{2}\theta' W_\alpha(w, \theta, \bar{\theta}) - \theta'\theta' \int d^2\bar{\theta} e^{-2V} \Phi^\dagger e^{2V}. \tag{2.76}$$

where $w^\mu = x^\mu - i\bar{\theta}'\sigma^\mu\theta'$.

$\mathcal{N} = 2$ **Hypermultiplet** On the other hand, an $\mathcal{N} = 2$ hypermultiplet can be represented by two $\mathcal{N} = 1$ chiral superfields in the representation R and $\overline{R}$ of the internal symmetry (both the gauge group and flavor group):

$$H_{\mathcal{N}=2} \to (Q, \tilde{Q}) \supset (q, \tilde{q}, \psi_\alpha, \tilde{\psi}_\alpha). \tag{2.77}$$

It contains one complex scalar q and one complex Weyl fermion ψ_α in R, another complex scalar $\tilde{q}$ and complex Weyl fermion $\tilde{\psi}_\alpha$ in $\overline{R}$. The $SU(2)_R$ symmetry rotates the superfield $(Q, \tilde{Q}^\dagger)$.

When the $\mathcal{N} = 2$ hyper is in a pseudo-real representation of the internal symmetry group, one could define a half-hyper multiplet by imposing the constraint

$$Q^A = \epsilon^{AB}\tilde{Q}_B, \tag{2.78}$$

where A, B are the indices for the internal symmetry. This constraint is compatible with the $\mathcal{N} = 2$ SUSY transformations, and the resulting superfield represents the $\mathcal{N} = 2$ half-hyper multiplet. The analog case does *not* hold when the representation is real.

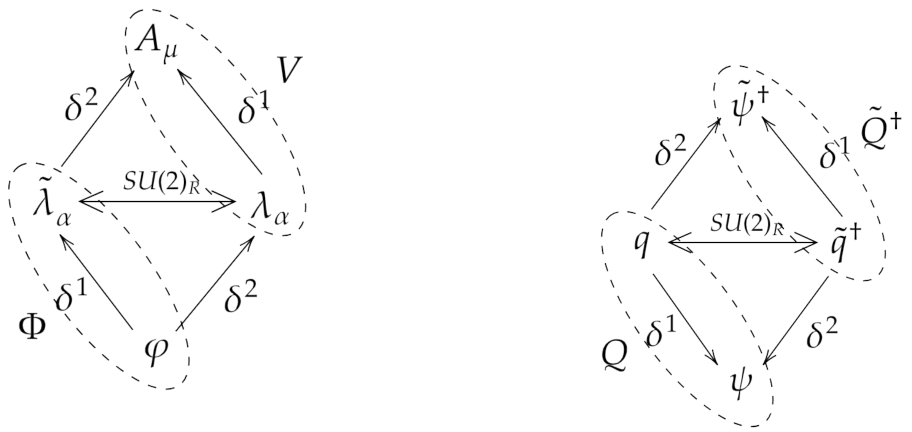

Figure 2-1 (Left) $\mathcal{N} = 2$ Vector Multiplet. (Right) $\mathcal{N} = 2$ Hypermultiplet

Chapter 3

Dynamics of $\mathcal{N} = 2$ Field Theory

3.1 $\mathcal{N} = 2$ Lagrangian

In this section, we will construct the Lagrangian for $\mathcal{N} = 2$ supersymmetric field theory. $\mathcal{N} = 2$ supersymmetry can be enhanced from 4d $\mathcal{N} = 1$ supersymmetric Lagrangian by imposing $SU(2)_R$ symmetry, which is the technique that we would apply in the construction.

The most general 4d $\mathcal{N} = 1$ supersymmetric Lagrangian can be constructed by the Grassmann integration over the $\mathcal{N} = 1$ superspace. A typical $\mathcal{N} = 1$ lagrangian involves a kinetic term for the vector superfields, a Kähler potential that may involve couplings between the chiral superfields and the real vector superfields V, and a superpotential for the chiral superfields.

Schematically, the Lagrangian for a 4d $\mathcal{N} = 2$ gauge theory takes the following form

$$\mathscr{L}_{\mathcal{N}=2} = \mathscr{L}^{\text{SYM}}_{\mathcal{N}=2} + \mathscr{L}^{\text{matter}}_{\mathcal{N}=2}. \tag{3.1}$$

where the first term contains the contribution from 4d $\mathcal{N} = 2$ vector multiplets and the second one contains the contribution from $\mathcal{N} = 2$ hypermultiplets.

Lagrangian for $\mathcal{N} = 2$ Vector Multiplet A $\mathcal{N} = 2$ vector multiplet from (2.75) splits into an $\mathcal{N} = 1$ gaugino superfield W_α and an $\mathcal{N} = 1$ chiral superfield $\boldsymbol{\Phi}$. The corresponding $\mathcal{N} = 2$ super-Yang-Mills Lagrangian takes the form of the following superspace integral

$$\mathscr{L}^{\text{SYM}}_{\mathcal{N}=2} = \left(\frac{1}{8\pi i}\int \mathrm{d}^2\theta\, \mathrm{Tr}\, \tau W_\alpha W^\alpha + \text{ c.c.}\right) + \frac{\mathrm{Im}(\tau)}{4\pi}\int \mathrm{d}^4\theta\, \mathrm{Tr}\left(\boldsymbol{\Phi}^\dagger e^{\mathrm{adj}(V)}\boldsymbol{\Phi}\right). \tag{3.2}$$

The first term is the $\mathcal{N} = 1$ SYM kinetic term, while the second one encodes the $\mathcal{N} = 1$ kinetic term for the chiral superfield in minimal coupling with the vector superfield in the adjoint representation of G. The complexified gauge coupling τ contains both the Yang-Mills coupling and the theta angle,

$$\tau = \frac{\theta_{\text{YM}}}{2\pi} + \frac{4\pi i}{g^2}. \tag{3.3}$$

The relative prefactor in (3.2) is fixed by $SU(2)_R$ symmetry, which ensures that the whole Lagrangian preserves $\mathcal{N} = 2$ supersymmetry. After expanding into the component fields, the action reads[1]

$$\mathscr{L}_{\text{vect}} = \frac{1}{g^2}\operatorname{Tr}\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - i\lambda\sigma^\mu D_\mu\lambda^\dagger + \frac{1}{2}D^2\right] + \frac{\theta_{YM}}{64\pi^2}\operatorname{Tr}\left[\epsilon^{\mu\nu\sigma\rho}F_{\mu\nu}F_{\sigma\rho}\right]$$
$$-\frac{1}{g^2}\operatorname{Tr}\left[\left|D_\mu\varphi\right|^2 - i\tilde{\lambda}^\dagger\sigma^\mu D_\mu\tilde{\lambda} + F^\dagger F - \varphi^\dagger[D,\varphi] + \sqrt{2}i\left(\tilde{\lambda}^\dagger[\lambda^\dagger,\varphi] - \varphi^\dagger\{\lambda,\tilde{\lambda}\}\right)\right]. \tag{3.4}$$

There is another method for constructing the SYM action (which may be non-renormalizable) using the $\mathcal{N}=2$ superfield formalism:

$$\mathscr{L}^{\text{SYM}}_{\mathcal{N}=2} = \mathfrak{I}\int \mathrm{d}^2\theta\mathrm{d}^2\theta' \,\frac{1}{4\pi}\operatorname{Tr}\mathscr{F}(\Phi_{\mathcal{N}=2}), \tag{3.5}$$

where $\mathscr{F}(z)$ is called the *holomorphic prepotential*. This construction describes the most generic $\mathcal{N}=2$ SYM without derivative terms higher than the second order. As we will see, holomorphicity plays a crucial role in controlling the theory' s behavior. If we choose the prepotential to be

$$\mathscr{F}(z) = \frac{1}{2}\tau\Phi^2_{\mathcal{N}=2}, \tag{3.6}$$

we recover the renormalizable SYM Lagrangian (3.2).

Lagrangian for $\mathcal{N}=2$ Hypermultiplet The $\mathcal{N}=2$ Lagrangian for the hypermultiplets contains the kinetic terms for $(Q^A,\tilde{Q}_A)_{A=1\dots N_f}$ of (2.77) in minimal coupling with the real vector superfield V in some representation $(\rho,\bar{\rho})$ of the gauge group G, i.e.,

$$\mathscr{L}^{\text{matter}}_{\mathcal{N}=2} = \int \mathrm{d}^4\theta\left[Q^\dagger_A e^{\rho(V)}Q^A + \tilde{Q}^{\dagger A}e^{\bar{\rho}(V)}\tilde{Q}_A\right] + \int \mathrm{d}^2\theta\, W(Q,\tilde{Q}) + \text{ c.c.}. \tag{3.7}$$

where W is the $\mathcal{N}=1$ superpotential

$$W(Q,\tilde{Q}) = \sqrt{2}\tilde{Q}_A\rho(\Phi)Q^A + \sqrt{2}M^A_B\tilde{Q}^B Q_A \tag{3.8}$$

Superpotential W encodes part of the interaction between an $\mathcal{N}=2$ vector multiplet and a hypermultiplet, and the coupling is fixed to 1 by the $SU(2)_R$ symmetry. Moreover, the $SU(2)_R$ symmetry further imposes $[M,M^\dagger]=0$, which means that the mass matrix could be diagonalized into

$$W(Q,\tilde{Q}) = \sqrt{2}\tilde{Q}_A\rho(\Phi)Q^A + \sqrt{2}\sum_A m_A\tilde{Q}^A Q_A. \tag{3.9}$$

As a remark, the mass parameters m_A could be complex numbers, which would not ruin the unitarity of the theory since we still need to add complex conjugate terms to the action. If we expand the terms in the superpotential faithfully, the genius mass of the multiplet Q_A would be given by $\sqrt{2}|m_A|$.

It is straightforward to see that the only parameters in the general $\mathcal{N}=2$ Lagrangian (at the Lagrangian level) are the complexified gauge coupling τ and the complex mass parameters m_A. As we will see, the renormalization of the theory may introduce an additional parameter, known as the dynamical scale Λ. Nevertheless, the number of parameters in an $\mathcal{N}=2$ gauge theory remains quite restricted.

[1] We normalize the trace such that $\operatorname{Tr}T^aT^b=\delta^{ab}$. Otherwise the kinematic terms are not canonically normalized.

3.2 Flavor Symmetry

Let us now discuss the flavor symmetry of the theory. We will present a detailed analysis of the flavor symmetry in the massless limit $m_I \to 0$ and remark on the case where the mass parameters are generic. Flavor symmetry in the presence of a mass parameter could be analyzed in a similar fashion.

Naively, when there are N_f copies of massless hypermultiplets in the theory, the flavor symmetry of the theory is $U(N_f)$. However, it turns out that the precise flavor symmetry group depends on the properties of the representation ρ of the gauge group. The flavor symmetry is different when ρ is complex, real, or pseudo-real.

ρ is Complex: $G_F = U(N_F)$ When ρ is a complex representation, nothing special happens and the action is invariant under the $U(N_f)$ transformation

$$Q^A \to U^A_B Q^B,\ \tilde{Q}_A \to \tilde{Q}_B U^{\dagger B}_A\ ;\ U^A_B U^{\dagger B}_C = \delta^A_C. \tag{3.10}$$

It is easy to see that the superpotential term and the kinematic term are both invariant under the $U(N_F)$ action.

When ρ is real or pseudo-real, on the other hand, there is an *enhancement* of flavor symmetry. In this case, we have $\bar{\rho} \cong \rho$. Therefore, we could combine the hypermultiplets as

$$\hat{Q}^a_s = (Q^{Aa}, \eta^{ab}\tilde{Q}_{Ab}),\ s = 1, \dots, 2N_f. \tag{3.11}$$

η is the invariant tensor, which is symmetric in a real representation and antisymmetric in a pseudo-real representation. The kinematic terms then become

$$\int d^4\theta\, \hat{Q}^\dagger_s e^{\rho(V)} \hat{Q}^s, \tag{3.12}$$

which is invariant under $U(2N_f)$ rotation on the indices s. $\hat{Q}^a_s$ can be understood as a half-hyper multiplet, in the sense that it represents the half-hyper representation $\rho_{hh} \otimes \Box_{2N_f}$ in the Hilbert space. On the other hand, in order to preserve the superpotential term, the flavor group has to reduce to a subgroup of $U(2N_f)$.

ρ is Real: $G_F = USP(2N_F)$ In a real representation, due to the symmetric invariant tensor, we could always pick an orthonormal basis such that $\rho(\Phi)$ is antisymmetric. Then, the superpotential could be rephrased into

$$W = \rho(\phi)_{[ab]}\tilde{Q}^a_A Q^{bA} = -\frac{1}{2}\rho(\phi)_{[ab]}\hat{Q}^a_s \hat{Q}^b_t \epsilon^{st}. \tag{3.13}$$

Therefore, the superpotential is invariant under the $SP(2N_f)$ rotation over the indices s, t. As a result, the flavor symmetry enhances to $USP(2N_f) := SP(2N_f) \cap U(2N_f)$ for real representation.

ρ is Pseudo-Real: $G_F = O(2N_F)$ In a pseudo-real representation, since there is an invariant antisymmetric invariant tensor, $\rho(\Phi)$ lives inside $\mathfrak{sp}(2N_f, \mathbb{C}) = \{X \mid$

$\epsilon X^T + X\epsilon = 0\}$. It implies that the tensor $\rho(\Phi)_{ab} := \epsilon_{ac}\rho(\Phi)^c_b$ is symmetric. The superpotential could thus be rephrased into

$$W = \rho(\phi)_{(ab)}\tilde{Q}^a_A Q^{bA} = \frac{1}{2}\rho(\phi)_{(ab)}\hat{Q}^a_s\hat{Q}^b_t g^{st}, \tag{3.14}$$

with g^{st} symmetric tensor. Therefore, the superpotential is invariant under the $O(2N_f,\mathbb{C})$ rotation over the indices s,t. As a result, the flavor symmetry enhances to $O(2N_f,\mathbb{R}) := O(2N_f,\mathbb{C})\cap U(2N_f)$ for pesudo-real representation.

Recall that when ρ is pseudo-real, one could define a single half-hyper multiplet. By incorporating half-hyper, one could construct a gauge theory with flavor symmetry $O(2N_f+1)$. In the $SU(2)$ case, the Witten anomaly will come into play and force the number of fermions (and thus the number of half-hypers) to be even.

Generic Mass Parameter In this paragraph, we will remark on the case where a generic mass parameter is turned on. In this case, the superpotential is given by Eq.(3.9). In order to preserve this superpotential, the only flavor symmetry that remains is the diagonal $\prod_A U(1)_{f_A}$ rotation over $(Q_A,\tilde{Q}^A)$. We would normalize such that $(Q_A,\tilde{Q}^A)$ has $U(1)_{f_A}$ charge $(1,-1)$.

3.3 R Symmetry

In this section, we would like to determine the $U(1)^{\mathcal{N}=1}_r$ charges of the $\mathcal{N}=1$ superfields. In the following, we would adopt the notation that Grassmannian variables $(\theta,\bar{\theta})$ have $U(1)^{\mathcal{N}=1}_r$ charge $\left(-\frac{1}{3},\frac{1}{3}\right)$. We would take all the $\mathcal{N}=1$ superfields to be defined regarding $(Q^1_\alpha,\overline{Q}^{\dot{\alpha}}_1)$. Since $U(1)^{\mathcal{N}=1}_r$ only acts on these two supercharges, it should be proportional to $\mathscr{R}^1_1 = \frac{1}{2}r_{\mathcal{N}=2}+\mathscr{R}$. The overall coefficient is fixed by the normalization of it's action on $(\theta,\bar{\theta})$, which yields

$$r_{\mathcal{N}=1} = \frac{1}{3}r_{\mathcal{N}=2} + \frac{2}{3}\mathscr{R}. \tag{3.15}$$

Let's zoom into the vector multiplet. In order to let the field strength term in action (3.2) be invariant under the $U(1)^{\mathcal{N}=1}_r$ symmetry, the field strength W_α must have charge $r_{\mathcal{N}=1}=\frac{1}{3}$, and thus the vector superfield V has charge $r_{\mathcal{N}=1}=0$. By applying $SU(2)_R$ symmetry between the fermions in the vector multiplet and chiral multiplet, we conclude that Φ has $r_{\mathcal{N}=1}=\frac{2}{3}$.

For the hypermultiplet, the possible r charge of the multiple depends on the superpotential. Let us focus on the $m_I=0$ case. When the hypermultiplet is in the trivial representation of the gauge group(or there is no gauge field in the theory), hypermultiplet could be assigned with arbitary $r_{\mathcal{N}=1}$ charges, as it is the case for free $\mathcal{N}=1$ chiral multiplet. When $m_I=0$, and the hypermultiplet is coupled to the gauge field non-trivially, we need to ask the superpotential term 3.9 to respect the symmetry. It imposes chiral superfields $(Q,\tilde{Q})$ to have central charge $r_{\mathcal{N}=1}=0$.

3.4 Renormalization

Now let us focus on the $\mathcal{N}=2$ SYM sector in (3.2), with the complexified gauge coupling defined in (3.3). In a general gauge theory, the gauge coupling g can be corrected perturbatively by the loop diagrams, as well as by non-perturbative contributions of instantons. In particular, the perturbative corrections follow the renormalization group equation

$$E\frac{\mathrm{d}g}{\mathrm{d}E} = -\frac{g^3}{16\pi^2}b + \mathcal{O}(g^5). \tag{3.16}$$

Here, E is the energy scale where g is measured, and the first term on the R.H.S. represents the 1-loop beta function, and the second term comes from higher loop corrections. b is known as the 1-loop beta function coefficient, which can be extracted from the field content in the relevant gauge theory, namely

$$b = \frac{11}{3}T(\mathrm{adj}) - \frac{2}{3}T(\rho_f) - \frac{1}{3}T(\rho_s), \tag{3.17}$$

where ρ_f and ρ_s denote respectively the G representations of the fermions and the scalars in the gauge theory. $T(\rho)$ is the quadratic Casimir invariant in the representation ρ of the gauge group G with the normalization such that $T(\mathrm{adj})$ is equal to the dual Coxeter number of G.[①] In $\mathcal{N} = 2$ supersymmetric field content, b can also be simplify to the following combination

$$b = 2T(\mathrm{adj}) - T(\rho_{\mathrm{matter}}), \tag{3.18}$$

where ρ_{matter} denotes collectively the G representation of the hypermultiplets $H_{\mathcal{N}=2}$.

The non-normalization property from the 4d $\mathcal{N} = 2$ supersymmetry leads to an enormous simplification of the running of the gauge coupling. The statement is that the beta function is 1-loop exact at the perturbative level, and thus we can safely ignore the high-loop perturbative corrections. This is a simple consequence of the holomorphy principle. That is to say, the superpotential in a $\mathcal{N} = 1$ theory, under a certain renormalization scheme, is a holomorphic function of the chiral superfields. In our circumstances, we could apply a trick to regard the complexified coupling τ as a background chiral superfield. Then the term

$$P(\tau, W_\alpha) = \tau W_\alpha W^\alpha \tag{3.19}$$

could be viewed as a superpotential term. The holomorphic principle suggests that the quantum corrections to this superpotential have to be holomorphic in both τ and W_α.

Since perturbative renormalization on the superpotential cannot depend on θ_{YM} as it is associated to a topological term, such contributions at the n-loop order should carry a factor $g^{2-2n} \cong \mathrm{Im}(\tau)^{1-n}$, which does not respect holomorphy unless $n = 1$. Therefore, we conclude that the renormalization of the gauge coupling g is 1-loop exact perturbatively, and τ takes the following form

$$\tau(E) = \tau_{\mathrm{UV}} - \frac{b}{2\pi i}\log\left(\frac{E}{\Lambda_{\mathrm{UV}}}\right) + \cdots, \tag{3.20}$$

where only non-perturbative contributions are unspecified. There is another remarkable thing to be pointed out. As is made explicit in 3.2, the overall coefficient in the Kähler potential is related to superpotential via $SU(2)_R$ symmetry. Therefore, the non-renormalizable property of the superpotential suggests that there is no wave-function

① The quadratic casemir invariant is defined as $\mathrm{Tr}\, T_R^a T_R^b = 2T(R)\delta^{ab}$. For the $G = U(1)$ case, we have $T = q^2/2$ where q is the charge under the gauge $U(1)$ group.

renormalization for the Kähler potential term, which would be present if we consider a generic $\mathcal{N} = 1$ theory.

In the $b \neq 0$ case, the renormalization equation implies that

$$\Lambda^b := E^b e^{2\pi i \tau(E)} \tag{3.21}$$

is invariance under the change of E at the perturbative level. Λ is called the complex dynamical scale.

In the $b = 0$ case, there is a stronger version of the non-renormalizable theorem that ensures the gauge theory Lagrangian is exactly conformal invariant at the non-perturbative level and defines a quantum SCFT. In particular, the gauge coupling τ is exactly marginal and parameterizes a patch of the conformal manifold of the SCFT.

3.5 Vacuum Moduli Space

One interesting aspect of the supersymmetric field theory is its vacuum moduli space. A vacuum state is defined as the state with the lowest energy, and the vacuum moduli space is the collection of the vacuum states (which sometimes can be endowed with the structure of a manifold, algebraic variety, etc). A standard field theoretical argument suggests that one only needs to consider the coherent state of field operators as the vacua.① Namely, a vacuum could be labeled by the expectation value of fields in the theory.

For the vacuum to preserve spacetime symmetry, the expectation values of the spinor and vector fields must vanish.②Consequently, the scalar fields are the only ones that might contribute to the vacuum moduli space. To elucidate this, we can express the Hamiltonian for the scalar field as the sum of two parts: the kinetic energy and the potential energy, $H = T[\varphi, \partial_\mu \varphi] + V[\varphi]$, where φ represents all the scalar field content. Minimizing the Hamiltonian thus involves minimizing each term separately. The kinetic term reaches its minimum when $\partial_\mu \varphi = 0$, meaning that a vacuum configuration corresponds to fields that are independent of position, essentially constant. With the kinetic contribution fixed, the problem reduces to finding the minimum of the potential term by treating the scalar fields as ordinary variables.

The vacuum moduli space plays an extremely important role in the theory. Since the correlators of the field theory would generically depend on the choice of the vacuum, all the correlators could be viewed as functions on the vacuum moduli space. The most straightforward way to see the dependence is from the path integral formalism. The choice of vacuum specifies the boundary condition of the fields at infinity

$$\langle \mathcal{O}(\varphi) \rangle_{\varphi_0} = \int_{\varphi(\infty)=\varphi_0} [D\varphi] \mathcal{O}(\varphi) e^{iS[\varphi]}. \tag{3.22}$$

In some special circumstances, such a function might enjoy certain properties such as holomorphicity, which allows us to understand the correlators by studying the functions on the vacuum moduli space.

For a generic $\mathcal{N} = 2$ theory with both vector multiplet and hypermultiplet, the potential term can be written as the sum of so-called D-term and F-term, coming from

① The matrix element of field operators between one vacuum and another vanishes when the degrees of freedom go to infinity. See Weiberg II.

② Otherwise, the spacetime symmetry is spontaneously broken.

the Gaussian integral over the auxiliary field D and F in the vector multiplet and hypermultiplet, respectively. Assuming the gauge group to be $G = SU(N)$, the scalar potential term is

$$V = \mathrm{tr}\big(D[\varphi]^2\big) + F[\varphi, q, \tilde{q}]\overline{F}[\varphi, q, \tilde{q}], \tag{3.23}$$

where D (F) is a real (complex) functional of fields in the theory. Therefore, the classical vacuum manifold of 4d $\mathcal{N} = 2$ theory is given by

$$\mathscr{M}_{\text{classical}} = \{D[\varphi, q] = 0, F[\varphi, q] = 0\}/\text{Gauge}. \tag{3.24}$$

For $G = SU(N)$, the classical vacuum can be made precise by the careful treatment of the Lagrangian (3.1). The bosonic part of the equations are

$$\begin{aligned} D_V &= \left[\varphi^\dagger, \varphi\right] + g^2 \left(q_A q^{\dagger A} - \tilde{q}_A^\dagger \tilde{q}^A\right)\Big|_{\text{traceless}} = 0 \\ F_\Phi &= q_A \tilde{q}^A\Big|_{\text{traceless}} = 0 \\ F_{Q^A} &= \tilde{q}^A \varphi + m_A \tilde{q}^A = 0 \\ F_{\tilde{Q}^A} &= \varphi q_A + m_A q_A = 0. \end{aligned} \tag{3.25}$$

It can be simplified to the following equations

$$\begin{aligned} &\left[\varphi, \varphi^\dagger\right] = 0, \\ &\begin{cases} \left(q_A \left(q^A\right)^\dagger - \left(\tilde{q}_A\right)^\dagger \tilde{q}^A\right)\Big|_{\text{traceless}} = 0, \\ q_A \tilde{q}^A\big|_{\text{traceless}} = 0, \end{cases} \\ &\begin{cases} \varphi q_A + m_A q_A = \tilde{q}^A \varphi^\dagger + m_A^* \tilde{q}^A = 0, \\ \varphi^\dagger q_A + m_A^* q_A = \tilde{q}^A \varphi + m_A \tilde{q}^A = 0. \end{cases} \end{aligned} \tag{3.26}$$

The vacuum moduli space can be roughly split into three parts depending on the VEV of the scalar in the vector multiplet and hypermultiplet. If $\varphi \neq 0,\ q_A = \tilde{q}_A = 0$ then it's called *Coulomb branch moduli space* $\mathscr{M}_{\text{Coulomb}}$. If $q_A \neq 0,\ \tilde{q}_A \neq 0,\ \varphi = 0$ then it's called *Higgs branch* $\mathscr{M}_{\text{Higgs}}$. For the cases where all the expectation values are nonzero, it's called the *mixed branch* $\mathscr{M}_{\text{Mixed}}$.

The terms Coulomb branch and Higgs branch can be understood as follows. At a generic point on the Coulomb branch, the adjoint scalar field φ develops a vacuum expectation value. Through the Higgs mechanism, some of the vector bosons become massive, while those corresponding to the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ remain massless. Consequently, the gauge group G is higgsed to its maximal torus $T^r = U(1)^{\times r}$. Since $U(1)$ vector bosons mediate Coulomb interactions, as in Maxwell theory, this branch of vacua is called the Coulomb branch. On the other hand, at a generic point on the Higgs branch, the gauge symmetry is completely broken, leaving no massless vector bosons in the low-energy theory. This complete higgsing of the gauge group gives the branch its name, the Higgs branch.

It turns out that, due to the complete Higgsing of the gauge symmetry, the Higgs branch enjoys a rigid property and is thus easier to study. It is an hyper-Kähler manifold with a certain isometry transformation from UV flavor symmetry, which turns out to be rigid enough to prevent it from quantum corrections. On the other hand, the Coulomb branch moduli space does receive quantum corrections. In the following sections, we will focus on the study of the Coulomb branch.

Example 1: $SU(2)$ Pure Yang-Mills Let's consider the pure SYM with gauge group $G = SU(2)$. In this case, there is no hypermultiplet, so the Higgs branch is trivial. The Coulomb branch equation is

$$\left[\varphi, \varphi^\dagger\right] = 0, \quad \varphi \in \mathfrak{su}(2). \tag{3.27}$$

It implies that the matrix φ is normal, so it can be diagonalized as

$$\varphi \sim \begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix}. \tag{3.28}$$

There is a remaining gauge redundancy that flips $a \to -a$. Therefore, to define a gauge-invariant parametrization of the vacua, we could introduce

$$u = \frac{1}{2} \operatorname{Tr} \varphi^2 = a^2. \tag{3.29}$$

The Coulomb branch is thus an affine space $\mathbb{C}$ parametrized by u. Therefore, the 4d $\mathcal{N} = 2$ Coulomb branch on $\mathbb{R}^4$ is often called the u-plane.

Let us identify the unbroken gauge symmetry at each points in the Coulomb Branch moduli space. When the adjoint scalar φ develops the vacuum expectation value, its coupling with the gauge field leads to the higgsing of gauge symmetry. To identify the remaining gauge redundancy, it suffices to work at the level of the vacuum expectation value and ignore the derivative term[①]

$$-\operatorname{Tr}\left(D_\mu \langle\varphi^\dagger\rangle D^\mu \langle\varphi\rangle\right) = -\operatorname{Tr}\left(|[A_\mu, \langle\varphi\rangle]|^2\right). \tag{3.30}$$

It capture the mass of the vector bosons after gauging, and the massless gauge fields correspond to those that commute with $\langle\varphi\rangle$. When $a \neq 0$, the only massless gauge fields are the one proportionate to $\langle\varphi\rangle$, and the gauge group is higgsed to $U(1)$. On the other hand, when $a = 0$, all the elements in $\mathfrak{su}(2)$ commutes with $\langle\varphi\rangle$. Therefore, the gauge group is still $SU(2)$.

Example 2: Pure SYM in a Generic Matrix Gauge Group G For a generic matrix gauge group G, a similar story applies, but one needs more terminology of the Lie algebras to describe it. The Coulomb branch equation is still

$$[\varphi, \varphi^\dagger] = 0, \quad \varphi \in \mathfrak{g} \tag{3.31}$$

It implies that φ is a semisimple element in the group G. The G orbit for the semisimple elements (with respect to the adjoint action) is in one-to-one correspondence with the Weyl group orbit in a Cartan subalgebra

$$\mathfrak{g}//G \cong \mathfrak{h}//W. \tag{3.32}$$

It is a finite-group quotient over an affine plane. On the other hand, the G orbit for the semisimple elements is parametrized by the invariant polynomials, generated by

$$u'_k = \operatorname{Tr} \varphi^k. \tag{3.33}$$

① Just think of the Lagrangian of a free boson. After setting the bosonic field to a constant value, the coefficient of the quadratic term is the mass parameter for the field. The same logic applies here.

where the range of k depends on the property of the group G. The Coulomb branch is still an affine space parametrized by u'_k. Another equivalent way of parametrizing the vacua is to consider the characteristic polynomial

$$\det(\lambda I - \varphi) = \sum u_k \lambda^k. \tag{3.34}$$

The geometric interpretation of u_k will be revisited.

Now let us identify the unbroken gauge redundancy at each points on the Coulomb branch moduli space. The mass term is the same as (3.30), except for replacing the gauge group $SU(2)$ with any matrix Lie group G. The massless vector bosons would be those commuting with $\langle\varphi\rangle$, which forms the centralizer of $\langle\varphi\rangle$, defined as $Z(\langle\varphi\rangle) = \{x \in \mathfrak{g}, [x, \langle\varphi\rangle] = 0\}$. The centralizer subgroup of $\langle\varphi\rangle$ corresponding to the unbroken gauge group.

When $\langle\varphi\rangle$ is a regular semisimple element in $\mathfrak{g}$ whose representative in $\mathfrak{g}//G \cong \mathfrak{h}//W$ does not touch on the walls of fix points for the Weyl group action, the centralizer subgroup is the maximal torus $U(1)^{\times r}$. When the representative of $\mathfrak{g}$ situates on the the walls of fix points for the Weyl group action, there would be unbroken non-abelian gauge redundancy. The Weyl group associated with this non-abelian gauge redundancy would coincide with the stabilizer subgroup for $\langle\varphi\rangle$.

3.6 $U(1)$ Charge and Witten Effect

Before going into the following discussion, let us discuss the $U(1)$ charge in the Maxwell theory in the presence of the θ angle. It would be important in the correct identification of the "electric" charge in the following discussion.

Abelian Gauge Theory The Lagrangian density of the Maxwell theory with θ angle is given by

$$\mathscr{L}_{Maxwell} = -\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + \frac{\theta}{64\pi^2} \epsilon^{\mu\nu\sigma\rho} F_{\mu\nu} F_{\sigma\rho}. \tag{3.35}$$

The variation of the action with respect to A is specified by

$$\delta S = \int d^4x \left(\frac{1}{e^2} \partial_\mu F^{\mu\nu} - \frac{\theta}{16\pi^2} \partial_\mu \epsilon^{\rho\sigma\mu\nu} F_{\rho\sigma} \right) \delta A_\nu. \tag{3.36}$$

The action enjoys the $U(1)$ global symmetry

$$A_\mu \to A_\mu + i\partial_\mu \alpha(x) \tag{3.37}$$

with α any real function, approaching a non-vanishing constant $\alpha(\infty)$ at infinity. To derive the conservation charge associated with this symmetry transformation, we need to add an arbitrary function $\epsilon(x)$ to the variation that vanishes at infinity as

$$\delta A_\mu = i\epsilon(x) \partial_\mu \alpha(x) \tag{3.38}$$

Under this variation, the action changes as

$$\delta S = i \int d^4x \left(\frac{1}{e^2} \partial_\mu F^{\mu\nu} - \frac{\theta}{16\pi^2} \partial_\mu \epsilon^{\rho\sigma\mu\nu} F_{\rho\sigma} \right) \epsilon(x) \partial_\nu \alpha(x) = -i \int d^4x \epsilon(x) \partial_\mu j^\mu[\alpha] \tag{3.39}$$

where the current is specified by

$$j^{\mu}[\alpha] = -\left[\frac{1}{e^2}F^{\mu\nu} - \frac{\theta}{16\pi^2}\epsilon^{\rho\sigma\mu\nu}F_{\rho\sigma}\right]\partial_{\nu}\alpha(x). \tag{3.40}$$

When the field configuration satisfies the classical equation of motion, we have $\delta S = 0$, which leads to the integration equation

$$-i\int d^4x\epsilon(x)\partial_{\nu}j^{\nu}[\alpha] = 0 \tag{3.41}$$

Since the choice of $\epsilon(x)$ is arbitrary, it implies that $j^{\nu}[\alpha]$ is a conservation current

$$\partial_{\nu}j^{\nu}[\alpha] = 0. \tag{3.42}$$

The charge of the conservation current is thus

$$\begin{aligned} Q[\alpha] &= \int_{\mathbb{R}^3} d^3xj^0[\alpha] = \int_{\mathbb{R}^3} d^3x\left(\frac{1}{e^2}E_i - \frac{\theta}{8\pi^2}B_i\right)\partial_i\alpha(x) \\ &= \alpha(\infty)\int_{S^2_{\infty}} d^2S\left(\frac{1}{e^2}E_i - \frac{\theta}{8\pi^2}B_i\right) - \int_{\mathbb{R}^3} d^3x\alpha(x)\left(\frac{1}{e^2}\partial_iE_i - \frac{\theta}{8\pi^2}\partial_iB_i\right) \end{aligned} \tag{3.43}$$

The second term is proportional to the Gaussian constraint, and thus vanishes on a physical configuration. The first term, on the other hand, measures the $U(1)$ charge of the field configuration, which we would call it the electric charge. It is useful to notice that the contribution of the conservation charge only depends on the asymptotic of $\alpha(\infty)$ at infinity.

To simplify the notation, we would denote $Q = n_e$ as the electric charge of a field configuration. Under the $\theta = 0$ limit, the electric charge is measured by the asymptotic of the electric field E_i at infinity. However, after incorporating θ term in the action, the magnetic field contributes to the conservation of charge. This phenomenon is called the *Witten effect*.

On the other hand, the incorporation of the θ term would not change the Bianchi identity of the field strength $\partial_iB_i = 0$. As a consequence, the magnetic charge of a field configuration is still specified by

$$\int d^3x\partial_iB_i = 4\pi n_m. \tag{3.44}$$

Combining (3.44) and (3.43), we conclude that

$$\int d^3x\partial_iE_i = e^2n_e + \frac{\theta e^2}{2\pi}n_m \tag{3.45}$$

It is convenient to combine the divergent term in the following form

$$\frac{1}{e^2}\int d^3x\partial_i(E_i + iB_i) = n_e + \tau n_m \tag{3.46}$$

where τ is the complexified coupling constant.

Application in $\mathcal{N} = 2$ SYM In the last section, we mentioned that for a generic point in the Coulomb branch moduli space, the gauge symmetry is higgsed from G to its Cartan subgroup $U(1)^{\times r}$, where r is the rank of the gauge group. We would like to focus on $G = SU(2)$ case and describe the Noether Charge associated with the global gauge transformation. In particular, we will put an emphasis on the Witten Effect.

The infinitesimal gauge transformation for the vector multiplet is specified by

$$\delta_\alpha A_\mu = -iD_\mu \alpha(x), \quad \delta_\alpha \varphi = i[\alpha(x), \varphi], \quad \alpha(x) \in \mathfrak{g}. \tag{3.47}$$

The global gauge transformation is defined as the ones such that α approaches a fixed element $\alpha(\infty) \in \mathfrak{g}$ at infinity. As we have mentioned in the last section, the vacuum expectation value of φ defines the asymptotic behavior of φ at infinity. Since the unbroken gauge group would be the centralizer of φ, it would preserve the asymptotic behavior of φ, thus preserving the vacuum of the theory. Therefore, the global part of the gauge group would still be a symmetry of the theory.

In $G = SU(2)$ case, there is a convenient choice for the parameter $\alpha(x)$, such that the asymptotics of $\epsilon(x)$ is non-vanishing but $\langle\varphi\rangle$ is left fixed. That is to set $\alpha(x) = \varphi(x)$. Under this choice of parameter, the adjoint scalar φ is invariant under this gauge transformation, and so are the fermions due to Supersymmetry①, which would simplify the calculation. The contribution of the conservation charge only comes from the variation of the gauge field. The variation of the action is given by

$$\delta S = \int d^4x \operatorname{Tr}\left[\left(\frac{1}{g^2} D_\mu F^{\mu\nu} - \frac{\theta_{YM}}{16\pi^2} D_\mu \epsilon^{\rho\sigma\mu\nu} F_{\rho\sigma} - j^\nu_{mat}\right) \delta A_\nu\right]. \tag{3.48}$$

where $j^\mu_{mat}(x) = \frac{\delta S}{\delta A_\mu(x)}$ is the matter current coming from the coupling between the gauge field and matter fields. To derive the Noether current associated with the transformation δ_φ, we add an arbitrary c-number parameter $\epsilon(x)$ to the variation, such that $\delta A_\mu = -i\epsilon(x) D_\mu \varphi$②

$$\begin{aligned}\delta S &= -i \int d^4x \epsilon(x) \operatorname{Tr}\left[\left(\frac{1}{g^2} D_\mu F^{\mu\nu} - \frac{\theta_{YM}}{16\pi^2} D_\mu \epsilon^{\rho\sigma\mu\nu} F_{\rho\sigma} - j^\nu_{mat}\right) D_\nu \varphi\right] \\ &= -i \int d^4x \epsilon(x) \left[\partial_\mu \operatorname{Tr}\left(\frac{1}{g^2} F^{\mu\nu} - \frac{\theta_{YM}}{16\pi^2} \epsilon^{\rho\sigma\mu\nu} F_{\rho\sigma}\right) D_\nu \varphi - \partial_\mu \operatorname{Tr} j^\mu_{mat} \varphi\right]\end{aligned} \tag{3.50}$$

Since the $\epsilon(x)$ is an arbitrary parameter, we conclude that $j^\nu[\varphi]$ is a conservation current

$$\partial_\mu j^\mu[\varphi] = 0, \quad j^\mu[\varphi] = -\operatorname{Tr}\left[\left(\frac{1}{g^2} F^{\mu\nu} - \frac{\theta_{YM}}{16\pi^2} \epsilon^{\rho\sigma\mu\nu} F_{\rho\sigma}\right) D_\nu \varphi\right] + \operatorname{Tr} j^\mu_{mat} \varphi. \tag{3.51}$$

① It follows from the fact that SUSY transformation commutes with gauge transformation.

② To show the second equation, we need to notice that the covariant derivatives in the formula act on the adjoint representation. Since for a covariant derivative in representation ρ, $[D_\mu, D_\nu] = i\rho(F_{\mu\nu})$, the first right hand side of the formula is equivalent to

$$\frac{1}{g^2} D_\mu D_\nu F^{\mu\nu} - \frac{\theta_{YM}}{16\pi^2} \epsilon^{\sigma\rho\mu\nu} D_\mu D_\nu F_{\sigma\rho} = \frac{i}{g^2}[F_{\mu\nu}, F^{\mu\nu}] + \frac{\theta_{YM} i}{16\pi^2} \epsilon^{\sigma\rho\mu\nu}[F_{\mu\nu}, F_{\sigma\rho}] = 0. \tag{3.49}$$

. On the other hand, j^υ_{mat} is expected to satisfy $D_\nu j^\upsilon_{mat}$=0

Therefore, the Noether charge associated with the current is

$$\begin{aligned}Q[\alpha] &= \frac{1}{g^2}\int d^3x\,\mathrm{Tr}\big(E_i D_i\varphi\big) - \frac{\theta_{YM}}{8\pi^2}\int d^3x\,\mathrm{Tr}\big(B_i D_i\varphi\big) - \int d^3x\,\mathrm{Tr}\big(j^0\varphi\big)\\ &= \frac{1}{g^2}\int_{S^2_\infty} d^2Sn^i\,\mathrm{Tr}\big(E_i\varphi\big) - \frac{\theta_{YM}}{8\pi^2}\int_{S^2_\infty} d^2Sn^i\,\mathrm{Tr}\big(B_i\varphi\big) - \int d^3x\,\mathrm{Tr}(G\varphi)\end{aligned} \tag{3.52}$$

with G the Gaussian constraint in the presence of matter.

3.7 Central Charge

In this section, we would like to discuss the central charge Z in the theory, which follows from a faithful evaluation on the supercurrent. Supercurrent could be evaluated by a standard deviation using the Noether theorem, except for the fact that the variation parameter $\xi(x)$ is fermionic rather than bosonic. In the following, we will present the derivation for pure Yang-Mills theory to have a taste on how the central charge Z occurs from the field theory perspective, and how Z is related to the conservation charges of a field configuration.

Pure Yang-Mills The supercurrent of the pure SYM with respect to SUSY transformation δ_1 is given by

$$\begin{aligned}S_1^{\rho\alpha} &= \frac{1}{g^2}\mathrm{Tr}\left(-\frac{i}{2}(\lambda^\dagger\overline{\sigma}^\rho\sigma^{\mu\nu})^\alpha F_{\mu\nu} - \varphi^\dagger(\lambda^\dagger\overline{\sigma}^\rho)^\alpha\varphi + \sqrt{2}(\sigma^\mu\overline{\sigma}^\rho\tilde{\lambda})^\alpha D_\mu\varphi^\dagger\right) - i\frac{\theta}{8\pi^2}\mathrm{Tr}\left(\epsilon^{\rho\mu\nu\sigma}(\lambda^\dagger\overline{\sigma}_\mu)^\alpha F_{\nu\sigma}\right).\\ \overline{S}_1^{\rho\dot\alpha} &= \frac{1}{g^2}\mathrm{Tr}\left(-\frac{i}{2}F_{\mu\nu}(\bar\sigma^{\mu\nu}\bar\sigma^\rho\lambda)^{\dot\alpha} - \varphi^\dagger(\bar\sigma^\rho\lambda)^{\dot\alpha}\varphi + \sqrt{2}(\tilde\lambda^\dagger\overline{\sigma}^\rho\sigma^\mu)^{\dot\alpha} D_\mu\varphi\right) - i\frac{\theta}{8\pi^2}\mathrm{Tr}\left(\epsilon^{\rho\mu\nu\sigma}(\overline{\sigma}_\mu\lambda)^{\dot\alpha} F_{\nu\sigma}\right).\end{aligned} \tag{3.53}$$

Applying $SU(2)_R$ symmetry, we can get another copy of supercurrent by rotating two fermions $(\lambda,\tilde\lambda)\to(\tilde\lambda,-\lambda)$.

$$\begin{aligned}S_2^{\rho\alpha} &= \frac{1}{g^2}\mathrm{Tr}\left(-\frac{i}{2}(\tilde\lambda^\dagger\overline{\sigma}^\rho\sigma^{\mu\nu})^\alpha F_{\mu\nu} + \varphi^\dagger(\tilde\lambda^\dagger\overline{\sigma}^\rho)^\alpha\varphi - \sqrt{2}(\sigma^\mu\overline{\sigma}^\rho\lambda)^\alpha D_\mu\varphi^\dagger\right) - i\frac{\theta}{8\pi^2}\mathrm{Tr}\left(\epsilon^{\rho\mu\nu\sigma}(\tilde\lambda^\dagger\overline{\sigma}_\mu)^\alpha F_{\nu\sigma}\right).\\ \overline{S}_2^{\rho\dot\alpha} &= \frac{1}{g^2}\mathrm{Tr}\left(-\frac{i}{2}F_{\mu\nu}(\bar\sigma^{\mu\nu}\bar\sigma^\rho\tilde\lambda)^{\dot\alpha} - \varphi^\dagger(\bar\sigma^\rho\tilde\lambda)^{\dot\alpha}\varphi - \sqrt{2}(\lambda^\dagger\overline{\sigma}^\rho\sigma^\mu)^{\dot\alpha} D_\mu\varphi\right) - i\frac{\theta}{8\pi^2}\mathrm{Tr}\left(\epsilon^{\rho\mu\nu\sigma}(\overline{\sigma}_\mu\tilde\lambda)^{\dot\alpha} F_{\nu\sigma}\right).\end{aligned} \tag{3.54}$$

The supercharges for the pure Super Yang-Mills are thus defined as the integration of the time component for the supercurrents over the spatial slice

$$Q_I^\alpha = e^{\frac{\pi i}{4}}\int d^3x\, S_I^{0\alpha},\quad \overline{Q}_I^{\dot\alpha} = e^{-\frac{\pi i}{4}}\int d^3x\,\overline{S}_I^{0\dot\alpha}. \tag{3.55}$$

where the overall phase factor is for conventional reasons. To see the contribution of the central charge, we need to faithfully evaluate the (anti)Poisson bracket between Q_1^α and Q_2^α, which yields

$$\left\{Q_{1\alpha},Q_{2\beta}\right\} = \frac{2\sqrt{2}}{g^2}\epsilon_{\alpha\beta}\int d^3x\,\partial_i\,\mathrm{Tr}\left(\left(E_i - iB_i\right)\varphi^\dagger\right) \tag{3.56}$$

The resulting term only depends on the asymptotic of the fields at spatial infinity. While approaching infinity, φ approaches it's vacuum expectation value $\varphi^b(\infty)=a^b$. On the other hand, by Gauss's law and Witten Effect, the electric and magnetic charge of a field configuration can be determined by

$$\int d^3x\partial_i E_i^a = g^2 n_e^a + \frac{\theta_{YM} g^2}{2\pi} n_m^a, \qquad \int d^3x\partial_i B_i^a = 4\pi n_m^a. \tag{3.57}$$

As a consequence, the Poisson bracket between two supercharges yields

$$\left\{Q_{1\alpha}, Q_{2\beta}\right\} = 2\epsilon_{\alpha\beta}\epsilon_{12}\overline{Z}. \tag{3.58}$$

where

$$Z = \sqrt{2}\sum_b (n_e^b a^b + n_m^b \tau a^b). \tag{3.59}$$

This formula could be generalized to the case where the gauge group is a product group $\prod_I G_I$, where each G_I is a simple group or $U(1)$. In this case, the coupling constant is replaced by the coupling "matrix" τ_{IJ} where I, J indices run over different group factors. The central charge is given by

$$Z = \sqrt{2}\left(n_e^{Ib} a_I^b + \tau_{IJ} n_m^{Ib} a^{Jb}\right). \tag{3.60}$$

where b runs over the gauge indices for the I-th gauge group. Especially, when the gauge group is $U(1)^{\times r}$, then the central charge is given by

$$Z = \sqrt{2}\left(n_e^I a_I + \tau_{IJ} n_m^I a^J\right). \tag{3.61}$$

It is worth noticing that in the formula of the central charge, the electric charge and the magnetic charge is put on the same footing. As we would see, it will be a consequence or clue for the electro-magnetic duality of the theory.

Incorporating Hypermultiplets and Mass Deformation Let us remark on the case where there are hypermultiplets. In the presence of hypermultiplets, the supercurrent will be modified, but the only terms that contribute to the central charge are the mass deformation term. The resulting central charge reads

$$Z = \sqrt{2}(n_e a + n_m \tau a + \sum m_i f_i), \tag{3.62}$$

where f_i is the flavor charge for $U(1)_{f_i}$ flavor symmetry.

3.8 Identifying BPS States at Classical Level in $SU(2)$ SQCD

In this section, we are going to identify the states in a field theory that saturate the BPS bound. To be concrete, we will focus on the $G = SU(2)$ case, with the number of flavors $N_f \leq 4$ living in the (anti)fundamental representation of the gauge group. It is a supersymmetry analogy of QCD, and we will call it $SU(2)$ $N_f = N$ SQCD as a terminology. For a generic semisimple gauge group coupled to generic hypermultiplets, the analysis is similar.

3.8.1 Perturbative Excitations and Light BPS States

W-Bosons When the adjoint scalar φ develops the vacuum expectation value, its coupling with the gauge field leads to the higgsing of gauge symmetry. To identify the mass of the W-boson, it suffices to work at the level of the vacuum expectation value and ignore the derivative term[1]

$$-\operatorname{Tr}\left(D_\mu\langle\varphi^\dagger\rangle D^\mu\langle\varphi\rangle\right) = -\operatorname{Tr}\left(|[A_\mu,\varphi]|^2\right). \tag{3.63}$$

If we denote the components of A_μ as follows, where the annoying coefficients come from the canonical normalization of the kinematic term.

$$A_\mu = \begin{bmatrix} A_\mu^{U(1)} & W_\mu^+ \\ W_\mu^- & -A_\mu^{U(1)} \end{bmatrix}, \tag{3.64}$$

then the mass term reads

$$|2\sqrt{2}a|^2 W_\mu^+ W^{-\mu}, \tag{3.65}$$

which indicates that the W-bosons (W^+, W^-) have mass $M = |2\sqrt{2}a|$. On the other hand, the $A_\mu^{U(1)}$ remain massless and thus the gauge symmetry is higgsed from $SU(2)$ to $U(1)$, but the coupling constant is shifted by $\tau^{U(1)} = 2\tau^{SU(2)}$. The central charge formula is thus given by

$$Z = \sqrt{2}(n_e a + n_m \tau a + \sum m_i f_i) \tag{3.66}$$

where (n_e, n_m) is the charge with respect to the unbroken $U(1)$ gauge group. Since the W-boson (W^+, W^-) has electric change $(2, -2)$, the central charge of their Hilbert space is $Z = \pm 2\sqrt{2}a$. Therefore, W-bosons saturate the BPS bound, and so do the SUSY multiplet containing it.

It is not that surprising. By construction, the vector superfield represents the states in a massless multiplet, which has four bosonic degrees of freedom and four fermionic degrees of freedom. By continuously tuning the expectation value of the adjoint scalar φ at the locus where the pertubation theory is still trustable, we expect that the degrees of freedom would not jump, and thus the massive vectors have to live inside a short multiplet(BPS multiplet) containing the same degrees of freedom as the massless multiplet. [2]

Quarks Matter fields $q_A, \tilde{q}^A$ living inside the hypermultiplets $(Q_A, \tilde{Q}^A)$, and we will call them quarks as an analogy to QCD. Since in our setup the hypermultiplet lives in the fundamental representation, while the gauge symmetry is higgsed from $SU(2)$ to $U(1)$, the hyper Q_A^1 has $U(1)$ charge 1 and Q_A^2 has $U(1)$ charge -1.

To identify the mass of the quarks, we follow the same strategy by setting the fields to their vacuum expectation value and identifying the quadratic term. The only terms contributing to the Lagrangian are the superpotential

$$W = \sum_I \sqrt{2}\tilde{Q}_I\langle\rho(\Phi)\rangle Q^I + \sqrt{2}\sum_I m_I \tilde{Q}^I Q_I. \tag{3.67}$$

① Just think of the Lagrangian of a free boson. After setting the bosonic field to a constant value, the coefficient of the quadratic term is the mass parameter for the field. The same logic applies here.

② Convincing as the argument of counting the degrees of freedom might be, the situation is *not* true when we go to the strong-coupling regime, where particles may combine into a bound state.

At the vacua, since fermions and F terms in the chiral multiplet Φ are all set to be zero, we have $\langle\rho(\Phi)\rangle = \langle\rho(\varphi)\rangle$. After Plugging in the vacuum expectation value of the adjoint scalar φ, the superpotential term becomes

$$W = \sqrt{2}(m_I + a)\tilde{Q}^1_I Q^{1I} + \sqrt{2}(m_I - a)\tilde{Q}^2_I Q^{2I} \tag{3.68}$$

Therefore, the masses of the quarks is given by

$$M^1_I = \sqrt{2}|m_I + a|, \quad M^2_I = \sqrt{2}|m_I - a|. \tag{3.69}$$

Compared with the BPS bound, we conclude that the quarks saturate the BPS bound.

3.8.2 Non-Perturbative Solitonic Excitations and Heavy BPS States

In addition to perturbative excitations such as quarks and W bosons, there also exist *non-perturbative solitonic excitations* such as monopoles. At the classical level, these are described by one-soliton solutions of the classical equations of motion with finite energy. Their semi-classical properties can be studied by quantizing the fields around these classical backgrounds at the quadratic order. Upon quantization, the "vacuum" could be interpreted as a one-particle state of the soliton, and the zero modes of the field configuration generate the one-soliton Hilbert space.

BPS Equation for Pure SYM Since we are searching for BPS states that preserve half of the supersymmetry, we need to find the half-BPS solutions for the classical equation of motion. The invariance of the field configuration under half of the SUSY transformations imposes extra differential equations on the fields, which will be called the *BPS Equation*. ① Since the hermitian conjugate of one BPS equation generically implies another equation, it is convenient to apply a parametrization of supercharges that is closed under hermitian conjugate, and the parametrization in (2.28) does the job. Without loss of generality, we look for the half BPS solution invariant under $\xi^\alpha_I Q^{(\zeta,+)I}_\alpha$, for an arbitrary choice of constant spinor ξ.

Following from (2.74), the invariance under the SUSY transformation enforces all the fermionic fields $(\lambda, \tilde{\lambda})$ to vanish. ② On the other hand, the SUSY variation of the fermions gives rise to a non-trivial differential equation between the bosonic fields③

$$e^{\frac{\pi i}{4}}\zeta^{\frac{1}{2}}\left(F_{\mu\nu}\sigma^{\mu\nu\alpha}{}_\beta + i\delta^\alpha_\beta D_V[\varphi, q]\right) + e^{-\frac{\pi i}{4}}\zeta^{-\frac{1}{2}}\left(i\sqrt{2}\sigma^\mu D_\mu\varphi\right) = 0. \tag{3.70}$$

Noticing that $F_{\mu\nu}\sigma^{\mu\nu} = \sigma^i(E_i - iB_i)$ and the four Pauli matrices σ^μ are linear independent, the above equation implies

$$E_i - iB_i = \sqrt{2}D_i\left(\frac{\varphi}{\zeta}\right), \quad D_0\left(\frac{\varphi}{\zeta}\right) = \frac{i}{\sqrt{2}}[\varphi, \varphi^\dagger]. \tag{3.71}$$

① Naively, it seems that imposing the SUSY variation with respect to $Q^I_\alpha, I = 1,2$ leads to a half BPS solution. However, SUSY invariance under $Q^I_\alpha, I = 1,2$ implies the SUSY invariant under another set of generators $Q_{\dot{\alpha}I}, I = 1,2$.

② Remind that this transformation is written with respect to Q^1_α. One needs to correctly identify the field content and restore another copy of SUSY transformation using $SU(2)_R$ symmetry $(\lambda, \tilde{\lambda}) \to (\tilde{\lambda}, -\lambda)$.

③ The phase factor is due to (3.44).

This is the *BPS equation* of the pure Yang-Mills theory, whose solution would preserve half of the supersymmetry.

As another perspective towards the BPS equation, let us consider the Hamiltonian of the theory in the absence of fermions. Under the Coulomb gauge $A^0 = 0$, the Hamiltonian, the Gauss Law constraint, and the Bianchi identity is given by

$$\begin{aligned} H &= \frac{1}{g^2}\int_{\mathbb{R}^3} d^3x \,\mathrm{Tr}\left\{\frac{1}{2}E_i^2 + \frac{1}{2}B_i^2 + \left|D_0\varphi\right|^2 + \left|D_i\varphi\right|^2 - \frac{1}{2}|\left[\varphi,\varphi^\dagger\right]|^2\right\}, \\ D_iE_i &= i\left([\varphi^\dagger, D_0\varphi] - [D_0\varphi^\dagger,\varphi]\right), \quad D_iB_i = 0. \end{aligned} \tag{3.72}$$

Applying Bogomolny's trick to the Hamiltonian, we find that[①]

$$\begin{aligned} H = &\frac{1}{g^2}\int_{\mathbb{R}^3} d^3x \,\mathrm{Tr}\left(\left|\frac{1}{\sqrt{2}}(E_i - iB_i) - D_i\left(\frac{\varphi}{\zeta}\right)\right|^2 + \left|D_0\left(\frac{\varphi}{\zeta}\right) - \frac{i}{\sqrt{2}}[\varphi,\varphi^\dagger]\right|^2\right) \\ &+ \frac{1}{g^2}\int_{\mathbb{R}^3} d^3x \;\mathrm{Tr}\left(\Re\left(\frac{\sqrt{2}}{\zeta}(E_i + iB_i)D_i\varphi\right) + \Re\left(\frac{\sqrt{2}i}{\zeta}[\varphi,\varphi^\dagger]D_0\varphi\right)\right). \end{aligned} \tag{3.73}$$

Due to Gauss's law and Bianchi Identity $D_iB_i = 0$, the term in the second line is a total derivative, which is thus a topological term determined by the asymptotic behavior of the fields at infinity. On the other and, the field configuration minimizing the energy in the bulk leads to the BPS equations. It suggests that the BPS equations automatically imply the classical equation of motion. For a half BPS field configuration, the energy is given by

$$H = \frac{\sqrt{2}}{g^2}\int_{S^2_\infty} d^2S \;\mathrm{Tr}\,\Re\left(\frac{1}{\zeta}(E_i + iB_i)\varphi\right) = -\sqrt{2}\Re\frac{1}{\zeta}\left(n_e a + \tau n_m a\right) = -\Re\frac{Z}{\zeta}, \tag{3.74}$$

which is exactly the BPS bound we have derived from the perspective of the representation theory.

We will focus on the static monopoles and dyons, where the field configuration is time-independent. Under the Coulomb gauge $A^0 = 0$, the BPS equation for the monopole could be simplified into

$$E_i - iB_i = \sqrt{2}D_i\left(\frac{\varphi}{\zeta}\right), \quad [\varphi,\varphi^\dagger] = 0. \tag{3.75}$$

It means that for a static monopole, φ can always be diagonalized under a proper gauge choice.

Semiclassical Feature of the Soliton In this section, we are going to discuss the semiclassical feature of the monopole and dyon in $SU(2)$ SQCD coupled to N_f matters. We will only consider the massless limit, and the case with mass deformation could be analyzed in a similar fashion. As we have explained, to analyze the semiclassical feature of solitons in a field theory, we could expand the fields around the classical solutions at quadratic order. The zero modes of the field configuration generate the one-soliton

① Apply $|a+b|^2 = |a|^2 + |b|^2 + 2\Re(ab^*)$.

Hilbert space upon quantization. Therefore, the central problem is to count the zero modes of the fields under the monopole/dyon background.

For the bosonic fields, are three zero modes coming from the spontaneous breaking of three translation symmetries, and one from a large $U(1)$ gauge transformation.

For the fermionic zero modes, the zero modes can be counted with the help of the index theorem. For the problem at hand, after expanding the field to the quadratic order, we find that the relevant Dirac operator is specified by

$$\begin{aligned} D\lambda &\equiv \left(i\sigma^i D_i + \rho(\varphi)\right)\lambda \\ D^\dagger \lambda^\dagger &\equiv \left(-i\sigma^i D_i + \rho(\varphi^\dagger)\right)\lambda^\dagger . \end{aligned} \tag{3.76}$$

The zero mode for the Dirac operator could be counted with the help of the Callias index theorem. For $G = SU(2)$ case, it tells us there is one zero mode for the fundamental fermion and two zero modes for the adjoint fermion. The mode expansion could be specified by

$$\lambda^I_\alpha = a^I_{0+}\lambda^+_{0\alpha} + a^I_{0-}\lambda^-_{0\alpha}\cdots, \qquad \psi^s_\alpha = b^s_0\psi_{0\alpha} + \dots \tag{3.77}$$

where I is the $SU(2)_R$ index for the adjoint fermion in the vector multiplet and s is the flavor index for the fermions in the half-hypermultiplet $\hat{Q}$. Moreover, the constraint (2.78) implies that $b^{s\dagger}_0 = g_{st}b^t_0 := b_{0t}$.

The canonical quantization of the fermionic field imposes

$$\{a^I_0, a^\dagger_{0J}\} = \delta^I_J, \quad \{b^s_0, b_{0t}\} = \delta^s_t. \tag{3.78}$$

$(a^I_0, a^\dagger_{0J})$ can be understood as the Goldstino associated with the unbroken supersymmetry, since the monopole background breaks half of the supersymmetry by turning on the bosonic fields in the vector multiplet.

Let us look at the zero modes b^s_0 coming from the hypermultiplet. As we can see, b^s_0 generates the action of the Clifford algebra $Cl(2N_f, \mathbb{R})$. There are two irreducible representations for this algebra, corresponding to the spinor/conjugate spinor representation of $Spin(2N_f)$. The spin group surprisingly enters the story!

Naively, by the standard construction of Clifford module as we have repeated for several times, the physical Hilbert space would be 2^{N_f} dimensional representation of the Clifford algebra, which is a direct sum of spinor and conjugate spinor representation. However, as we will see in the next section, the large $U(1)$ gauge transformation would project out half of the states and enforce the physical Hilbert space to be exactly the spinor or conjugate spinor representation of $Spin(2N_f)$. It has an important implication on the duality.

Constraint from $U(1)$ Large Gauge Transformation In $SU(2)$ gauge theory there is a nontrivial central element $z = \mathrm{diag}(-1,-1)$, which acts trivially on any spin-j representation but as multiplication by -1 on any spin-$\left(j+\frac{1}{2}\right)$ representation. We implement this on the Hilbert space by an operator $(-1)^H$. In particular, since the fermions ψ^s_α furnish the $SU(2)$ doublet (i.e. spin-$\frac{1}{2}$) representation, one finds

$$(-1)^H\psi^I_\alpha(-1)^H = -\psi^I_\alpha, \tag{3.79}$$

or equivalently$\{(-1)^H, \psi^I_\alpha\} = 0$. Moreover, $\left((-1)^H\right)^2 = 1$, so this operator plays exactly the role of a "chirality" or γ_5 operator in standard quantum field theory. Consequently, the full 2^{N_f}-dimensional representation of the associated Clifford algebra splits into the ± 1 eigenspaces of $(-1)^H$, which corresponds to the spinor and conjugate-spinor representations of $Spin(2N_f)$.

On the other hand, upon the canonical quantization, the action of the central element on the Hilbert space could be obtained by exponentiating the Noether charges for the unbroken $U(1)$ gauge group. The global part of the remaining $U(1)$ gauge transformation could be represented by

$$\delta A_\mu = \frac{1}{a} D_\mu \varphi. \tag{3.80}$$

Applying Noether's theorem to this transformation, we could find out the conserved charge associated with the global $U(1)$ transformation, which measures the electric charge n_e of the states. After exponentiating this operator $e^{\pi i n_e}$ to the central element diag$\{-1,-1\}$, the action of $(-1)^H$ on the Hilbert space should agree with this operator. This leads to the following constraint

$$(-1)^H = e^{\pi i n_e}. \tag{3.81}$$

As a consequence, a particle with a certain electric charge n_e must live in an eigen-subspace of the chirality operator $(-1)^H$.

Let us discuss how this constraint would affect the Hilbert space of the soliton. For a monopole with charge $n_e = 0$ or Dyon with charge $n_e = 2n$, the chirality operator $(-1)^H$ must act as $+1$. Therefore, the $2^{N_f - 1}$ dimensional eigen-subspace of $(-1)^H$ with eigenvalue -1 would be projected out, and the physical Hilbert space of a single monopole lives in the spinor representation of the flavor symmetry $Spin(2N_f)$. On the other hand, for a dyon with charge $n_e = 2n + 1$, the chirality operator $(-1)^H$ must act as -1 on the Hilbert space. Thus, those dyons live in the conjugate spinor representation of $Spin(2N_f)$.

An interesting example is to take $N_f = 4$. In this case, the SQCD is superconformal, and the flavor group is $SO(8)$. In this case, the quarks live in the vector(fundamental) representation $\mathbf{8}_V$ of $SO(8)$, while the monopole and charged (2n+1,1) dyon live in the spinor $\mathbf{8}_S$ and conjugate spinor $\mathbf{8}_C$ representation of $Spin(8)$, respectively.

The Lie group $Spin(8)$ possesses an outer automorphism group isomorphic to S_3, known as *triality*. At the level of the Dynkin diagram, this triality symmetry permutes the three outer nodes, thereby rotating among the three eight-dimensional representations: the vector $(\mathbf{8}_V, \mathbf{8}_S, \mathbf{8}_C)$. As we will see, this triality provides an important clue for the electromagnetic duality in $N_f = 4$ SQCD.

3.9 Seiberg Witten Solution for the Low Energy Effective Theory

This chapter gives a short review of the Seiberg-Witten Theory, which provides an exact solution for the low-energy effective field theory with a selected vacuum in the Coulomb branch vacuum moduli space. We mainly focus on $\mathcal{N} = 2$ $SU(2)$ pure gauge theory to get a taste of the great ideas from Seiberg and Witten. One may refer to[1–2] for more details.

In the latter discussion, for simplicity, we will assume $G = SU(2)$ with $N_f = 0$. Although it is not superconformal, we can grasp a lot of main ideas in the analysis of the Seiberg-Witten theory in this example. In this case, the Lie algebra has rank 1 with

Cartan subalgebra given by elements like $\Phi = a\sigma_3$. The first non-trivial G-invariant polynomial is given by $u = 1/2\,\mathrm{tr}\left(\Phi^2\right) = a^2$.

Interesting things occur if one considers the quantum correction, which is done by Seiberg and Witten in[1]. Due to the expectation value of scalar field φ, parts of the gauge field are endowed with mass, leading to the higgsing of gauge invariance $SU(2) \to U(1)$. By the supersymmetry, the entire vector multiplet becomes massive. For each point in the Coulomb branch moduli space, one could consider the low-energy effective theory (LEET) by integrating out all the massive degrees of freedom in the Lagrangian. The resulting theory should generically be described by the $\mathcal{N} = 2$ $U(1)$ pure SQED, where the only massless degrees of freedom are the $U(1)$ vector boson and it's superpartner.

From the $N = 2$ superfield construction, the most generic $\mathcal{N} = 2$ $U(1)$ SQED theory is constructed by holomorphic prepotential $\mathcal{F}(z)$, as mentioned in (3.5). Since the adjoint representation of $U(1)$ is trivial, it is easy to expand the action into the one for $\mathcal{N} = 1$ superfields

$$\mathcal{L}_{eff} = \frac{1}{4\pi}\Im\left[\underbrace{\int \mathrm{d}^4\theta\, \frac{\partial\mathcal{F}(A)}{\partial A}\bar{A}}_{\text{Sigma Model}} + \underbrace{\int \mathrm{d}^2\theta\, \frac{1}{2}\frac{\partial^2\mathcal{F}(A)}{\partial A^2}W_\alpha W^\alpha}_{\text{SYM}}\right]. \tag{3.82}$$

where A is the $\mathcal{N} = 1$ chiral superfield and W_α is the $\mathcal{N} = 1$ gaugino superfield. As a consequence, the gauge coupling of the low-energy theory can be identified with $\tau = \partial^2\mathcal{F}\big/\partial A^2$, where A is the scalar field in the LEET.

3.9.1 Duality in LEET

Duality Group for the Sigma Model The scalar field admits a straightforward physical interpretation. In the IR, the only remaining degrees of freedom are those describing how the three-dimensional "world-sheet" $\mathbb{R}^3$ is embedded into the vacuum manifold. Small fluctuations around a chosen vacuum configuration are then described by a non-linear sigma-model whose target-space metric is precisely that of the vacuum moduli space.

For a 4d $\mathcal{N} = 1$ supersymmetric sigma model, it's action in the superfield formalism could be identified with the Kähler potential of its target manifold. In our case, the Kähler potential can be read off from the $\mathcal{N} = 2$ action $K(A, \bar{A}) = \Im\frac{\partial\mathcal{F}(A)}{\partial A}\bar{A}$. The Kähler metric on the moduli space is thus

$$\mathrm{d}s^2 = \frac{\partial}{\partial A}\frac{\partial}{\partial\bar{A}}K(A,\bar{A})\Big|_{A=a}\, da d\bar{a} = \Im\frac{\partial^2\mathcal{F}(a)}{\partial a^2}\mathrm{d}a\mathrm{d}\bar{a} = \Im\mathrm{d}a_D\mathrm{d}\bar{a} = \frac{1}{2i}(\mathrm{d}a_D\mathrm{d}\bar{a} - \mathrm{d}a\mathrm{d}\bar{a}_D). \tag{3.83}$$

where we use a to denote the coordinate of the target manifold. The holomorphic prepotential $\mathcal{F}$ is identified with a holomorphic function on the target manifold and $a_D := \frac{\partial\mathcal{F}}{\partial a}$. This metric puts an equal footing on a and a_D, which gives us a clue for the duality of the theory. If we combine $a^\alpha = (a_D, a)$, then the metric could be phrased into

$$\mathrm{d}s^2 = \frac{i}{2}\epsilon_{\alpha\beta}da^\alpha d\bar{a}^\beta. \tag{3.84}$$

which admits an isometry transformation

$$a^\alpha \to M^\alpha{}_\beta a^\beta,\ M \in SP(2,\mathbb{R}). \tag{3.85}$$

In other words, there exists a $SP(2,\mathbb{R})$ duality transformation that leaves the metric of the target space invariant. Group $SP(2,\mathbb{R})$ is generated by two elements,

$$T_b = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix},\ S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{3.86}$$

which acts on (a_D, a) as

$$\begin{aligned} T:& \begin{bmatrix} a_D \\ a \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a_D \\ a \end{bmatrix} = \begin{bmatrix} a_D + a \\ a \end{bmatrix}, \quad \tau \to \tau + 1. \\ S:& \begin{bmatrix} a_D \\ a \end{bmatrix} \to \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} a_D \\ a \end{bmatrix} = \begin{bmatrix} a \\ -a_D \end{bmatrix}, \quad \tau \to -\frac{1}{\tau}. \end{aligned} \tag{3.87}$$

We could uplift the $SP(2,\mathbb{R})$ transformation to the action on the $\mathcal{N}=1$ superfield, by replacing (a_D, a) in the (3.86) with $\mathcal{N}=1$ Chiral superfield $(A_D := \frac{\partial\mathcal{F}}{\partial A}, A)$. In the following, we would like to show that the action of $SP(2,\mathbb{R})$ acts naturally as the duality of the sigma model. It suffices to check that the form of the action is invariant under S and T_b duality transformation.

Explicit calculation shows that T_b acts by adding an extra term in the low-energy theory

$$\Delta\mathcal{L} = \frac{b}{4\pi}\Im\left[\int d^4\theta A\overline{A}\right]. \tag{3.88}$$

However, since the extra term $A\overline{A}$ is positive definite, it's imaginary part vanishes. Therefore, T_b would leave the sigma model invariant.

Let us consider the action of S duality. For clarity, we define

$$A_D = \frac{\partial\mathcal{F}}{\partial A} := h(A). \tag{3.89}$$

In the frame we are working on, we recognize A as the dynamical field and A_D as a function of A. After the duality transformation, A_D plays the role of A before duality and we should recognize it as the dynamical field. Similarly, A should be recognized as a function of A_D. The duality transformation suggests us to define a function h_D in the dual frame such that

$$h_D(A_D) = -A. \tag{3.90}$$

h_D is thus minus the inverse function of h. The Sigma model part of the Lagrangian transforms into

$$\mathcal{L}_{\text{Sigma}} = -\frac{1}{4\pi}\Im\left[\int \mathrm{d}^4\theta\ A_D\overline{h_D(A_D)}\right] = \frac{1}{4\pi}\Im\left[\int \mathrm{d}^4\theta\ h_D(A_D)\overline{A_D}\right]. \tag{3.91}$$

Therefore, the theory after the electromagnetic duality is of the same form as the original theory, except that we take A_D as the dynamical field rather than A. We should interpret it as another description for the same theory.

The true duality of the system has to be a subgroup that further leaves the second term, the Super Yang-Mills term, invariant. As we will see, it breaks the $SP(2,\mathbb{R})$ duality group to $SP(2,\mathbb{Z}) \cong SL(2,\mathbb{Z})$.

Duality Group of the Yang-Mills Theory Let us remark on the duality group of the $\mathcal{N} = 1$ SYM. An observation is that T_1 transformation defined in (3.86) acts as a "duality" of the theory. Since the coupling constant in the SYM term is

$$\tau = \frac{\partial^2 \mathcal{F}}{\partial A^2} = \frac{\partial A_D}{\partial A}, \tag{3.92}$$

the T_b transformation would shift coupling constant τ by $T_b : \tau \to \tau + b$. Namely, it shifts the θ angle to $\theta + 2\pi b$. The contribution of the theta angle in the partition function takes the form $\exp\big(i\theta p_1\big)$, where

$$p_1 = \frac{1}{64\pi^2} \int \mathrm{d}^4 x\, \epsilon_{\mu\nu\sigma\rho} \operatorname{Tr}\big(F^{\mu\nu} F^{\sigma\rho}\big) \in \mathbb{Z} \tag{3.93}$$

is the first Pontryagin number of the gauge connection, which is an integer. As a consequence, T_1 transformation contributes to a term $\exp\big(i2\pi p_1\big) = 1$ in the path integral, which leaves the partition function invariant. It requires $b \in \mathbb{Z}$ and breaks the duality group $SP(2, \mathbb{R})$ to $SP(2, \mathbb{Z}) \cong SL(2, \mathbb{Z})$. We would denote the generators of $SL(2, \mathbb{Z})$ as $T = T_1, S$.

The non-trivial duality in the story is the electromagnetic duality. To understand the duality, let us first look at the pure Maxwell theory with $\theta = 0$ as the toy model. The electromagnetic duality states the invariance of the theory under the transformation

$$E_{Di} = B_i, \quad B_{Di} = -E_i, \tag{3.94}$$

It is straightforward to verify that the vacuum Maxwell equations is invariant under the duality transformation. In a Lorentz covariant form, the electric-magnetic dual field strength is given by $F_D = *F$, with the dual gauge field A_D. ①

This duality transformation could be derived via the path-integral formalism, which is convenient to generalize to the cases with a generic θ angle. The action of the Maxwell theory in the presence of θ angle is given by

$$S = \frac{1}{32\pi} \Im \int d^4 x\, \tau (F_{\mu\nu} + i(*F)_{\mu\nu})^2 \tag{3.95}$$

where $\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$ is the complexified coupling constant. In the usual treatment of the path-integral formalism, we take A_μ as the dynamical variable and sum over all the gauge-inequivalent configurations of the A_μ. However, we can take another approach by integrating over the field strength directly. The only subtlety is that the field strength is subjected to the Bianchi identity $\epsilon^{\mu\nu\sigma\rho} \partial_\nu F_{\sigma\rho} = 0$. The path integral writes

$$Z = \int [DF] \delta(\epsilon^{\mu\nu\sigma\rho} \partial_\nu F_{\sigma\rho}) e^{iS_{\text{Maxwell}l}}. \tag{3.96}$$

We could solve the constraint from the Bianchi identity by adding a Lagrangian multiplet $V_{D\mu}$ in the action

$$Z = \int [DF][DV_D] e^{iS_{\text{Maxwell}} + \frac{i}{8\pi} \int d^4 x V_{D\mu} \epsilon^{\mu\nu\sigma\rho} \partial_\nu F_{\sigma\rho}} = \int [DF][DV_D] e^{iS_{\text{Maxwell}} + \frac{i}{8\pi} \int d^4 x F_{D\mu\nu} * F^{\mu\nu}} \tag{3.97}$$

① In the Minkovski signature, $*^2 = -1$.

The action is thus shifted by the term

$$S_{\text{Maxwell}} \to S_{\text{Maxwell}} + \frac{1}{16\pi}\Re \int d^4x(*F_D - iF_D)_{\mu\nu}(F + i * F)^{\mu\nu} \tag{3.98}$$

The resulting action is of Gaussian form in $F + i * F$. Performing the path integral over F is equivalent to imposing the saddle point equation for F

$$F_D = \Im\tau(*F - iF), \tag{3.99}$$

which leads to an identification between the field strength F and the dual field strength F_D. In the component, the electro-magnetic field and its dual is identified by

$$B_i = E_{Di} - \frac{\theta_D}{2\pi}B_{Di}, \quad B_{Di} = -E_i + \frac{\theta}{2\pi}B_i. \tag{3.100}$$

After conducting the Gaussian integral over F, the resulting action would be

$$S_{\text{Maxwell}} = \frac{1}{32\pi}\int d^4x \,\text{Im}\, \frac{-1}{\tau}\left(F_D + i^*F_D\right)^2. \tag{3.101}$$

The dual theory is described by the dual field strength F_D, but the coupling constant is transformed by $\tau \to \tau_D = -\frac{1}{\tau}$. Therefore, the electro-magnetic duality is a strong-weak duality. Starting with a weak coupling constant $\tau \sim i\infty$ or $g \to 0$, the dual theory is strongly coupled $\tau_D \sim i0^+$ or $g \sim +\infty$.

Let us now derive the analogous electro-magnetic duality transformation in the $\mathcal{N} = 1$ SYM part of the action. The Bianchi identity for $\mathcal{N} = 1$ gaugino super field is $\Im DW = \frac{1}{2i}(DW - \overline{DW}) = 0$. We could impose this constraint in the action by adding a real superfield V_D as the Langrangian multiplier and shifting the action by the term

$$\Delta S = \frac{1}{4\pi}\,\text{Im}\int d^4xd^4\theta V_D\mathscr{D}W = \frac{-1}{4\pi}\,\text{Im}\int d^4xd^4\theta\mathscr{D}V_DW. \tag{3.102}$$

The integration could be simplified using the operator equation $\int d^2\overline{\theta} = -\frac{1}{4}\partial^{\dot\alpha}\partial_{\dot\alpha}$. On the other hand, in the $(y, \theta, \overline{\theta})$ coordinate we have $\overline{D}_{\dot\alpha} = \partial_{\dot\alpha}$. Therefore, the action could be simplified into

$$\Delta S = -\frac{1}{4\pi}\,\text{Im}\int d^4xd^2\theta W_DW \tag{3.103}$$

where the dual field strength is $W_D = -\frac{1}{4}\overline{DD}D_\alpha V_D$ by definition. The resulting action is a Gaussian form in W_α, so we can integrate over it in the path integral. The saddle point of the Gaussian integration leads to the identification between the gaugino superfield and its dual. One can check that the electro-magnetic field transforms as 3.100. The resulting action is

$$S_{SYM} = \frac{1}{8\pi}\,\text{Im}\int d^4xd^2\theta\frac{-1}{\tau(A)}W_D^2. \tag{3.104}$$

Since the coupling constant is $\tau = \frac{\partial A_D}{\partial A}$, we notice that the dual coupling constant is

$$\tau_D(A_D) = -\frac{1}{\tau} = -\frac{\partial A}{\partial A_D}. \tag{3.105}$$

It is exactly the result of the S transformation in the Sigma model part. Therefore, S transformation of the Sigma model combined with the electromagnetic duality in the SYM part is a duality of the low-energy effective theory. As a byproduct, we recognize that the physical interpretation of $a_D = \frac{\partial\mathcal{F}}{\partial a}$ defined before is the vev of the scalar field in the magnetic $\mathcal{N} = 2$ vector multiplet.

Summary The analysis above suggests that the low-energy effective theory enjoys an $SL(2,\mathbb{Z})$ duality group, with generator T and S. The action of two generators is specified by

$$\begin{aligned} T: \begin{bmatrix} a_D \\ a \end{bmatrix} &\to \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a_D \\ a \end{bmatrix}, \quad \tau \to \tau + 1. \\ S: \begin{bmatrix} a_D \\ a \end{bmatrix} &\to \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} a_D \\ a \end{bmatrix} = \begin{bmatrix} a \\ -a_D \end{bmatrix}, \quad \tau \to -\frac{1}{\tau}. \end{aligned} \tag{3.106}$$

3.9.2 Transformation of the Electromagnetic charges under $SL_2\mathbb{Z}$ Duality

In this section, we would like to make explicit how the electro-magnetic charges of a particle state would change under the $SL_2\mathbb{Z}$ duality transformation.

T-Duality The electric charge in the presence of the theta angle is measured by the electro-magnetic flux $E_i - \frac{\theta}{2\pi}B_i$ at infinity. Under the T-duality, the θ angle is shifted by 2π, which results in the shift

$$E_i - \frac{\theta}{2\pi}B_i \to E_i - \frac{\theta}{2\pi}B_i - B_i \tag{3.107}$$

As a consequence, the electric charge would be shifted correspondingly such that

$$\left[n_m, n_e\right] \to \left[n_m, n_e - n_m\right] = \left[n_m, n_e\right] \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}. \tag{3.108}$$

S-Duality The duality transformation of the field strength is specified in 3.100. Compared with the definition of the $U(1)$ current, we conclude that under the S transformation, electromagnetic charge swaps as

$$\left[n_m, n_e\right] \to \left[-n_e, n_m\right] = \left[n_m, n_e\right] \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{3.109}$$

An interesting observation is that electro-magnetic charges transform as the inverse of the T, S duality transformation. The physical meaning of this observation will be explained in the next section.

3.9.3 Duality and Central Charge in the LEET

In Section 3.7, we derived the central charge function at the classical level. However, at low energies, quantum corrections modify this classical result. The classical central charge can thus be regarded as the limiting case of a more complete, quantum-corrected expression. To analyze the spectrum of BPS particles in the low-energy regime, it is essential to determine the central charge function within the low-energy effective theory. Remarkably, this quantum-corrected central charge can be constrained—and, in some cases, fully determined—by the $SL_2\mathbb{Z}$ duality of the theory.

Classical Analysis Let us first work at the regime where the perturbative analysis is valid. Therefore, the prepotential in the perturbative regime is described by

$$\mathscr{F}_{\text{cl}}(z) = \frac{1}{2}\tau z^2. \tag{3.110}$$

In the last section, we identified A_D as the chiral superfield in the magnetic $\mathcal{N} = 2$ vector multiplet, which satisfies

$$A_D = \frac{\partial \mathscr{F}(A)}{\partial A} = \tau A. \tag{3.111}$$

Therefore, the vacuum expectation value of A_D (or equivalently, the asymptotic of A_D at infinity) is specified by

$$a_D = \langle A_D \rangle = \tau a. \tag{3.112}$$

The central charge function can be reinterpreted in the following manner

$$Z = \sqrt{2}(n_e a + n_m \tau a) = \sqrt{2}(n_e a + n_m a_D) = \sqrt{2}\begin{bmatrix} n_m & n_e \end{bmatrix}\begin{bmatrix} a_D \\ a \end{bmatrix}. \tag{3.113}$$

As discussed in the previous sections, the $SL_2\mathbb{Z}$ duality transformation would swap the electro-magnetic charge as well as the superfield (A_D, A) and its expectation values. However, the net effect of the electromagnetic duality would leave the central charge function invariant. It is a reasonable result since under the duality transformation, the form of the theory is invariant, and so is the energy spectrum. A BPS multiplet with central charge Z and mass $M = |Z|$ should have the same central charge and mass in the dual description of the same system.

Central Charge in LEET In the low-energy effective theory, it is natural to expect that the central charge function is given by

$$Z = \sqrt{2}(n_e a + n_m a_D). \tag{3.114}$$

This formula could be verified by applying Bogomolny's trick to the low-energy effective action.

On the other hand, since the $SL_2\mathbb{Z}$ is a duality of the theory

3.9.4 Singularity and Monodromy at Coulomb Branch

Pertubative Analysis The analysis in the chapter 3.4 shows that the only perturbative correction is to shift the coupling constant as

$$\tau_{UV}^{SU(2)} \longrightarrow \tau_{\text{1-loop}}^{SU(2)}(E) = -\frac{4}{2\pi i}\log\left(\frac{E}{\Lambda}\right), \tag{3.115}$$

where Λ is the dynamical scale defined in 3.21 and $b = 4$ for the $SU(2)$ pure SYM theory.

At the scale $E < 2\sqrt{2}|a|$, all the massive W bosons are integrated out, and the remaining degrees of freedom do not charge under the $U(1)$ gauge field. As a consequence, there would be no running of coupling below this scale, and the low-energy effective coupling would be[①]

$$\tau^{U(1)} = -\frac{8}{2\pi i}\log\left(\frac{a}{\Lambda}\right) = -\frac{4}{2\pi i}\log\left(\frac{u}{\Lambda^2}\right) \tag{3.116}$$

① Note that we apply the condition $\tau^{U(1)} = 2\tau^{SU(2)}$ from the pertubative analysis before.

The logarithmic function is not single-valued over the complex plane and presents monodromy behavior. When u circle around the infinity $u \to e^{2\pi i}u$, the $\tau^{U(1)}$ would be shifted by an integer

$$\tau^{U(1)} \longrightarrow \tau^{U(1)} - 4. \tag{3.117}$$

Therefore, while circling around the infinity of the moduli space, we arrive at the same theory but *in a dual description* differs by T_{-4} duality. From the low energy coupling constant, a_D could be determined by

$$a_D = \int \frac{\partial a_D}{\partial a} da = \int \tau(a) da = -\frac{8a}{2\pi i}\left(\log\left(\frac{a}{\Lambda}\right) - 1\right) = -\frac{8\sqrt{u}}{2\pi i}\left(\frac{1}{2}\log\left(\frac{u}{\Lambda^2}\right) - 1\right). \tag{3.118}$$

The monodromy at infinity yields

$$\begin{pmatrix} a^D \\ a \end{pmatrix} \to \begin{pmatrix} -1 & 4 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a^D \\ a \end{pmatrix}. \tag{3.119}$$

Metric Singularity on the Moduli Space

$$ds^2 = \Im \tau^{U(1)} da d\bar{a} = -\Im \frac{1}{2\pi i |u|} \log\left(\frac{u}{\Lambda^2}\right) du d\bar{u}. \tag{3.120}$$

At $u \to \infty$, the metric becomes degenerate, and thus the description of the non-linear sigma model is not valid. The interpretation is that there exist extra massless degrees of freedom in the theory, which should be included in the low-energy theory.

If one persists with the classical moduli space, there is an orbifold singularity at $u = 0$. The nice physical interpretation given by Seiberg and Witten is that singularities on the M_{Coulomb} suggest the failure of the low-energy effective theory at that point, which originates from the occurrence of new massless degrees of freedom. Here, $u = 0$ admits a singularity because there is no Higgsing at that point. One may interpret it at some special point where gauge enhancement occurs, therefore, one more gauge field becomes massless.

As the coupling takes the form $\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{g^2}$. To have a well-defined theory in IR, one must ask $\Im\tau > 0$. One of the significant observations is that one cannot define a non-trivial holomorphic function on the whole $\mathscr{M}_{\text{Coulomb}}$ satisfying this condition if there is only one singularity. One of the possible solutions is to assume there are two singularities on the $\mathscr{M}_{\text{Coulomb}}$. The interpretation for those new singularities is that some new degree of freedom becomes massless, as we stated before. One of the exciting candidates for the new degree of freedom is the BPS particles with electric and magnetic charges (n, m), whose mass is protected by the supersymmetry as $M = Z = |na + ma_D|$. The simplest solution is that one singularity comes from the massless monopoles. That merely means that there exists a point where $a_D = \partial\mathscr{F}/\partial a = 0$ from mathematics point of view. However, the physics is somehow revolutionary. Commonly, elementary quarks are light, and monopoles are heavy in UV. However, their role is completely swapped near IR singularities, where IR monopoles become light degrees of freedom! As we will see later, this assumption helps us to solve the IR effective action completely.

The last observation that really helps to solve the low-energy effective theory comes from the knowledge of the Riemann-Hilbert Correspondence. Riemann-Hilbert correspondence tells us that merely knowing the monodromy behavior of (a, a_D) around the singularities is sufficient to solve the torus fibration, or equivalently, the holomorphic prepotential $\mathscr{F}$ uniquely. [1]As we see, singularities occur at the points where the BPS particles become massless. Around the points where electron/monopole becomes massless, one could trust the one-loop perturbative calculation for the coupling constant via the Lagrangian. For example, at $a\to\infty$ where the particle with electric charge $p=1$ becomes massless, the one-loop beta function can be calculated by

$$\tau(a)=\frac{1}{2\pi i}\log\left(\frac{a}{\Lambda_{\mathrm{UV}}}\right). \tag{3.121}$$

The monodromy around $a\to\infty$ clockwise gives $\tau\to\tau-1$. Recall that $\tau=\frac{\partial^2\mathscr{F}}{\partial a^2}=\frac{\partial a_D}{\partial a}$ Therefore, the monodromy around $a\to\infty$ is given by

$$\begin{pmatrix} a^D \\ a \end{pmatrix}\to\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a^D \\ a \end{pmatrix}. \tag{3.122}$$

Applying electro-magnetic duality transformation, we could transform a dyonic state with charge (p,q) to the fundamental quark $(1,0)$ and apply the perturbative analysis. The resulting monodromy matrix is specified by

$$M^{p,q}=\begin{pmatrix} 1+pq & p^2 \\ -q^2 & 1-pq \end{pmatrix} \tag{3.123}$$

Also, the monodromy at the whole moduli space is subject to the constraint.

$$\prod_{\{\gamma\}}\mathscr{M}_\gamma=\mathscr{M}_\infty, \tag{3.124}$$

where $\mathscr{M}_\gamma$ is the monodromy matrix around each singularity in a finite u.

Following from the monodromy condition, the unique family of elliptic curves can be found. It is called the Seiberg-Witten curve. From the argument before, the Seiberg-Witten curve captures (a,a_D) of the theory, thus captures the holomorphic prepotential. Namely, in Seiberg-Witten theory, all the information of LEET is encoded by the geometry of the Seiberg-Witten curve.

3.9.5 LEET from Seiberg-Witten Geometry

One could read off (a,a_D) again from the torus. The classical theory on the elliptic curves tells us that the torus T^2_u at some point in the $\mathscr{M}_{\mathrm{Coulomb}}$ is an algebraic curve written as

$$y^2=x^3+f(u)x+g(u). \tag{3.125}$$

called the Weierstrass form. One could transform the Weierstrass form into the the complex plane coordinate $z\in\mathbb{C}/\Lambda$ by Weierstrass $\mathscr{P}(z)$ function $(y,x)=\left(\mathrm{d}\mathscr{P}(z)\big/\mathrm{d}z\,,\mathscr{P}(z)\right)$.

The complex structure τ of the torus could be read off via

$$d_1=\int_A\mathrm{d}z=\int_A\frac{\mathrm{d}x}{y},\quad d_2=\int_B\mathrm{d}z=\int_B\frac{\mathrm{d}x}{y},\quad \tau=\frac{d_2}{d_1} \tag{3.126}$$

[1] An argument is that the torus fibration is equivalent to the fibration of discrete lattices, so the holonomy along any contractable loop on the regular locus has to vanish. Therefore, the torus fibration is essentially described by a local system where the holonomy along the singular locus determines the fibration uniquely.

On the other hand, from the LEET analysis, the coupling constant in the Yang-Mills theory is given by Then

$$\tau(u) = \frac{\partial^2 \mathscr{F}}{\partial a^2} = \frac{\partial a_D}{\partial a} = \frac{\frac{\partial a_D}{\partial u}}{\frac{\partial a}{\partial u}} \tag{3.127}$$

Therefore, it suggests the identification

$$\frac{\partial a}{\partial u} = f(u) \int_A \frac{\mathrm{d}x}{y}, \quad \frac{\partial a_D}{\partial u} = f(u) \int_B \frac{\mathrm{d}x}{y}, \tag{3.128}$$

where $f(u)$ is the common factor for (a, a_D).

3.10 Prepotential for pure $SU(2)$ SYM

For pure $SU(2)$ theory, the Seiberg-Witten curve is given by

$$y^2 = (x-1)(x+1)(x-u). \tag{3.129}$$

which respect the discrete $\mathbb{Z}_4$ symmetry of the theory. Seiberg-Witten differential could be given by

$$\lambda = -\frac{\sqrt{2}}{2\pi}(x-u)\frac{dx}{y} = \frac{\sqrt{2}}{2\pi}\frac{\sqrt{x-u}}{\sqrt{x^2-1}}dx. \tag{3.130}$$

such that it matches with the standard one-form up to a overall constant

$$\frac{\partial \lambda}{\partial u} = -\frac{\sqrt{2}}{4\pi}\frac{dx}{y}. \tag{3.131}$$

The hypergeometric function admits an integral expression

$$\begin{aligned} {}_2F_1(\alpha, \beta, \gamma; z) &= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 dx x^{\beta-1}(1-x)^{\gamma-\beta-1}(1-zx)^{-\alpha} \\ &= \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \sum_{n\geq 0} \frac{\Gamma(\alpha+n)\Gamma(\beta+n)}{\Gamma(\gamma+n)}\frac{z^n}{n!}. \end{aligned} \tag{3.132}$$

In terms of the hypergeometric function, we can easily see that with a change of variable $x = 2s - 1$

$$a(u) = \int_{-1}^1 \frac{\sqrt{2}}{\pi}\frac{\sqrt{x-u}}{\sqrt{x^2-1}}dx = \sqrt{2(1+u)}F(-1/2, 1/2, 1; 2/(1+u)). \tag{3.133}$$

As for a_D, let us first make the substitution $x = (u-1)t + 1$

$$a_D(u) = \frac{i}{\pi}(u-1)\int_0^1 dt t^{-\frac{1}{2}}(1-t)^{\frac{1}{2}}\left(1 - \frac{1-u}{2}t\right)^{-\frac{1}{2}} \tag{3.134}$$

Comparing this with the expression for the hypergeometric functions, we get

$$a_D(u) = \frac{i}{2}(u-1)F(1/2, 1/2, 2; (1-u)/2). \tag{3.135}$$

3.11 $SU(2)$ $N_f = 4$ SQCD

When the four hypermultiplets are massless, the Seiberg-Witten curve is given by[1]

$$y^2 = \left(x - e_1(\tau)u\right)\left(x - e_2(\tau)u\right)\left(x - e_3(\tau)u\right) , \tag{3.136}$$

with u the parameter of Coulomb branch and $e_i(\tau)$ are roots of the cubic polynomial $4x^3 - g_2(\tau)x - g_3(\tau)$ obeying $e_1+e_2+e_3 = 0$. Note that $g_2(\tau)$ and $g_3(\tau)$ are appropriately normalized Eisenstein series

$$\begin{aligned} g_2(\tau) &= \frac{60}{\pi^4}G_4(\tau) = \frac{60}{\pi^4}\sum_{m,n\in\mathbb{Z}_{\neq 0}}\frac{1}{(m\tau+n)^4} , \\ g_3(\tau) &= \frac{140}{\pi^6}G_6(\tau) = \frac{140}{\pi^6}\sum_{m,n\in\mathbb{Z}_{\neq 0}}\frac{1}{(m\tau+n)^6} . \end{aligned} \tag{3.137}$$

The roots e_i can be expressed by the Jacobi theta functions

$$\begin{aligned} e_1 - e_2 &= \vartheta_3{}^4(0,\tau) , \\ e_3 - e_2 &= \vartheta_2{}^4(0,\tau) , \\ e_1 - e_3 &= \vartheta_4{}^4(0,\tau) , \end{aligned} \tag{3.138}$$

where the Jacobi theta functions are defined by

$$\begin{aligned} \vartheta_2(0,\tau) &= \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}(n+1/2)^2} , \\ \vartheta_3(0,\tau) &= \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}n^2} , \\ \vartheta_4(0,\tau) &= \sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{1}{2}n^2} . \end{aligned} \tag{3.139}$$

At generic mass parameters m_j, the Seiberg-Witten curve is obtained in[1], which is

$$\begin{aligned} y^2 = &\left(x^2 - c_2^2u^2\right)\left(x - c_1u\right) - c_2^2\left(x - c_1u\right)^2\sum_i m_i^2 - c_2^2\left(c_1^2 - c_2^2\right)\left(x - c_1u\right)\sum_{i>j} m_i^2m_j^2 \\ &+ 2c_2\left(c_1^2 - c_2^2\right)\left(c_1x - c_2^2u\right)m_1m_2m_3m_4 - c_2^2\left(c_1^2 - c_2^2\right)^2\sum_{i>j>k} m_i^2m_j^2m_k^2, \end{aligned} \tag{3.140}$$

where $c_1 = \frac{3}{2}e_1$ and $c_2 = \frac{1}{2}(e_3 - e_2)$ and the coefficients are the triality-invariant combination of mass parameters.

3.12 Quiver Gauge Theory

There is a large class of $\mathcal{N}=2$ superconformal field theories with lagrangian description that could be represented by the quiver diagrams. The ingredients of the diagram consist of the flavor node, the gauge node, and the lines between them(with no orientation).

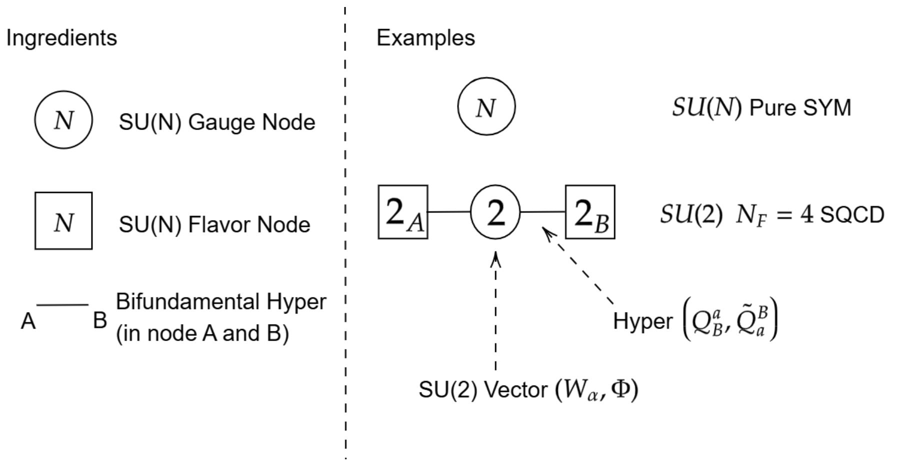

Figure 3-1 Ingredients and examples for the quiver gauge theory

When constructing a superconformal field theory, additional constraints are imposed on the nodes of the quiver diagrams. In particular, the requirement that the one-loop beta function vanishes implies that, for a gauge node with group $SU(N)$, the sum of the numbers in the adjacent nodes must be equal to $2N$. A typical type of superconformal field theory is the linear quiver theory, whose quiver diagram is as Fig.3-2.

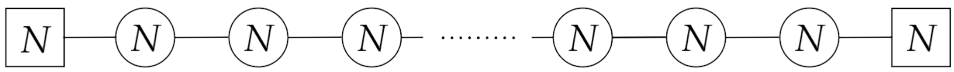

Figure 3-2 Linear Quiver Theory

In the specific case where the gauge group is $G = SU(N_c)$ and there are no flavor nodes, the quiver diagrams can be completely classified. Notably, the condition $b = 0$ in this context coincides with the definition for a harmonic diagram, which is aligned with the condition for an affine ADE Dynkin diagram. As a result, all such gauge theories are classified by the extended Dynkin diagrams of ADE type. One can generalize this construction by incorporating other compact semisimple Lie groups.

3.13 $SU(2)$ SCFT from Trivalent Diagram

In this section, we will focus on the $\mathcal{N} = 2$ SCFT with gauge group several copies of $SU(2)$. $SU(2)$ is quite special as its fundamental representation is pseudo-real, which turns out to give us more perspectives on the SCFT.

Let's consider the bifundamental hypermultiplet in $SU(2) \times SU(2)$. Since the fundamental representation of $SU(2)$ is pseudo-real, the flavor symmetry is enhanced from $SU(2)$ to $SO(4) \cong SU(2) \times SU(2)$. This allows us to combine the pair $(Q, \tilde{Q})$ into a single half-hyper multiple denoted as $\hat{Q}_{i_1 i_2 i_3}$ (see (3.11)). Here, i_1, i_2, i_3 are three $SU(2)$ indices.

This formulation suggests a new perspective on the $SU(2)$ SCFT. We can start with a building block: N copies of trifundamental half-hypermultiplets $\hat{Q}_{i_1 i_2 i_3}$, and then construct a general theory by gauging the $SU(2)$ flavor symmetry. To visualize this process, we can use a trivalent diagram. In this diagram, each vertex represents a trifundamental hypermultiplet. We associate three legs with each vertex, corresponding

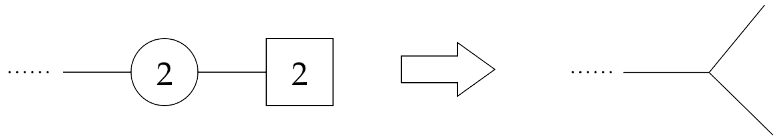

Figure 3-3 From quiver to trivalent diagram

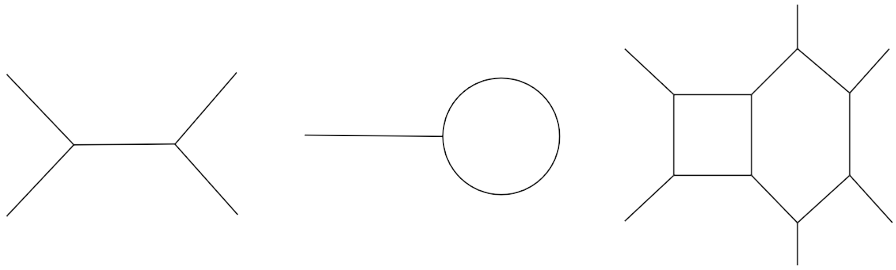

Figure 3-4 Some typical trivalent diagrams. The leftmost corresponds to $N_f = 4$ SQCD. The middle one is $SU(2)$ $N = 4$ theory plus a free halfhyper. The rightmost one is a typical trivalent diagram with loop and external legs.

to the three $SU(2)$ indices of the hypermultiplet. When we connect two vertices, we introduce a $\mathcal{N} = 2$ vector multiplet to gauge the $SU(2)$ flavor symmetry for each half-hypermultiplet. It's relation with quiver gauge theory is illustrated in Fig.3-3.

Therefore, each trivalent diagram would provide a $\mathcal{N} = 2$ SCFT. Let us consider the UV moduli space of the theory. For a trivalent diagram with g loops and n external legs, there are $3g - 3 + n$ gauge couplings in the theory. Those gauge couplings provide a coordinate near a cusp of the UV moduli space. If we assume the UV moduli space is generically smooth, then the analysis here tells us that the dimension of the UV moduli space should be $3g - 3 + n$. As we will see later, this fact could be nicely explained from the five-brane compactification perspective, where the dimension is aligned with that of the Tëchmuller space.

Chapter 4

M-Theory Construction of 4d Theory

4.1 Witten's Type IIA Construction for Linear Quiver Theory

In this subsection, we would like to present Witten's Type IIA construction for the $4d$ $\mathcal{N} = 2$ linear quiver theory. It provides a deep perspective of embedding the field theory into certain brane configurations, providing a top-down approach for probing the problems in the field theory. On the other hand, field theory techniques also help to analyze the BPS sector of the type IIA theory under certain backgrounds.

Starting from the type IIA string theory on $\mathbb{R}^{10}$, we consider the brane construction with D_4 branes and $NS5$ branes. From the Type IIA supergravity description, it is a 1/4 BPS background preserving 8 supercharges.

Under this brane configuration, six dimensions of the NS_5 brane are infinite. D_4 branes, on the other hand, have four infinite directions. A dimension analysis tells us that the NS_5 branes are much heavier than the D_4 branes. Therefore, in the following analysis, we could treat NS_5 brane semi-classically.

The D_4 brane is infinite in four dimensions. While in the x^6 direction, it is sandwiched between two $NS5$ branes. Therefore, the effective description of D_4 brane at long wavelength is a four-dimensional gauge theory with 8 supercharges.

4.1.1 Classical Analysis from NS_5 Brane

Combine $v = x^4 + ix^5$, the worldvolume of NS_5 brane is labeled by

While D_4 branes end on the NS_5 brane, it scratches the brane in the x^6 direction. At large v limit, the scratching could be described by a δ function source to the Laplacian equation of x^6.

$$\partial_v \partial_{\bar{v}} x^6 = \frac{k}{\pi} \sum_{i=1}^{n_L} \delta(v - a_i) - \frac{k}{\pi} \sum_{i=1}^{n_R} \delta(v - b_i). \tag{4.1}$$

NS_5 brane thus behaves as

$$x^6 = k \sum_{i=1}^{n_L} \ln(|v - a_i|) - k \sum_{i=1}^{n_R} \ln(|v - b_i|) + \text{const.} \tag{4.2}$$

A feature is that the well-defined x_6 limit of NS_5 brane exists only if $n_L = n_R$. In this case, there is equal force from the left and the right.

Another piece of information that we could extract from the brane construction is about the IR divergence. Consider a_i, b_i as a function of x^0, x^1, x^2, x^3, representing the

D_4 brane moving slowing on the NS_5 brane. Its contribution to the NS_5 brane kinetic energy yields

$$\int d^4xd^2v\sum_{\mu=0}^{3}\partial_\mu x^6\partial_\mu x^6=k^2\int d^4xd^2v\left|\mathrm{Re}(\sum_{i=1}^{n_L}\partial_\mu a^i\frac{1}{v-a_i}-\sum_{i=1}^{n_R}\partial_\mu b^i\frac{1}{v-b_i})\right|. \tag{4.3}$$

Its convergence thus requires

$$\sum_{i=1}^{n_L}\partial_\mu a^i-\sum_{i=1}^{n_L}\partial_\mu b^i=0. \tag{4.4}$$

Namely,

$$\sum_{i=1}^{n_L}a^i-\sum_{i=1}^{n_L}b^i=q. \tag{4.5}$$

where q should be understood as a characteristic quantity for the NS_5 brane, which is related to the mass parameter of the $4d$ theory.

However, this is only part of the story, as we know the scalar in $4D$ vector multiplet is complex due to the requirement for supersymmetry. Moreover, this complex scalar should vary holomorphically with respect to v. It suggests that we should analytically continue x^6 as a function of v, which is uniquely given by

$$x^6+ix^{10}=R_{10}\sum_{i=1}^{n_L}\ln(v-a_i)-R_{10}\sum_{i=1}^{n_R}\ln(v-b_i)+\text{const} \tag{4.6}$$

RHS of the equation has monodromy, which means that the LHS is well-defined only if $x^{10}\sim x^{10}+2\pi R_{10}$, which suggests that x^{10} should be circle valued. This phenomenon strongly suggests uplifting the type IIA string theory to M theory, with the complex part of the scalar x^{10} understood as the coordinate of the M theory circle.

4.1.2 Classical Analysis from D_4 Brane

Consider the background with $n+1$ NS_5 brane, labeled by $\alpha=0,\dots,n$. We include k_α D_4 brane stacking together and suspending between the $\alpha-1$ and α NS_5 brane. Especially, k_0 D_4 brane are semi-infinite starching on the left-most NS_5 brane and k_{n+1} D_4 brane starching on the right-most NS_5 brane.

The data for $4D$ $\mathcal{N}=2$ theory we would like to extract from the background geometry is as follows.

- The gauge group and vector multiplet W_α,
- The hypermultiplet $(Q,\tilde{Q})$,
- Mass parameter m_α,
- Coupling constant $\tau_\alpha=\frac{\theta_\alpha}{2\pi}+\frac{4\pi i}{g_\alpha^2}$.

Gauge Group We ought to match the gauge group of the world volume theory. Naively, the world volume of D_4 brane is described by the $\prod_{\alpha=1}^{n+1}U(k_\alpha)$ gauge group. However, the condition Eq.4.5 essentially means that there is a constraint equation between the scalar in the vector multiplet of $U(1)_\alpha$ and $U(1)_{\alpha+1}$. This gives $n+1$ constraints. On the other hand, the infinitely scratched branes are heavy, and their position is frozen on a certain background. It implies that all the $U(1)_\alpha$ factor is frozen. By supersymmetry, the whole $U(1)_\alpha$ vector multiplet is frozen. Therefore, the gauge group in the field theory is $\prod_{\alpha=1}^{n+1} SU(k_\alpha)$.

Hypermultiplet The massless hypermutiplet occurs when the D_4 branes scratching on the left and the right of an NS_5 coincide. In this case, the fundamental string scratching between two neighboring D_4 branes is light. Therefore, under the low energy limit, there is a hypermultiplet transforming in the representation

$$R = \oplus_{\alpha=0}^{n} \square_\alpha \otimes \overline{\square}_{\alpha+1} \tag{4.7}$$

Mass Parameter The UV mass parameters of the hypermultiplets are given by

$$m_\alpha = \frac{1}{k_\alpha} \sum a_{i,\alpha} - \frac{1}{k_{\alpha+1}} \sum a_{i,\alpha+1}, \tag{4.8}$$

which is the difference between the average position of four branes on the left and right of the αth NS_5 brane. The argument is that, at the long wavelength limit, the difference between D_4 brane positions becomes negligible While not exactly the same as q_α, it is determined by it.

Coupling Constant The coupling constant of the 4d theory can be obtained by considering the standard DBI action of D_4 brane. Upon integrating over the x^6 direction, we have

$$g_{YM}^{-2} \sim x_\alpha^6 - x_{\alpha-1}^6 \tag{4.9}$$

Upon lifting to the M theory, the holomorphicity of $x^6 + ix^{10}$ suggests that the complex coupling constant should be given by

$$-i\tau_\alpha = s_\alpha - s_{\alpha-1} \tag{4.10}$$

where $s_\alpha = \frac{1}{2\pi R_{11}}(x_\alpha^6 + ix_\alpha^{10})$. The overall coefficient is fixed by matching the periodicity of θ_α and x_α^{10} coordinate.

A key observation is that, according to Eq.(4.6), s is a function of v. By substituting Eq.(4.6) into Eq.(4.10) and taking the large-v limit, we find that

$$-i\tau_\alpha(v) = \frac{1}{2\pi}(2n_\alpha - n_{\alpha+1} - n_{\alpha-1}) \ln(v/l) + constant \tag{4.11}$$

where l is a constant of length dimension. This result is well known in field theory if we interpret the ratio v/l as E/Λ_{UV} and the constant term as $-i\tau_{UV}$. Then, Eq.(4.11) becomes

$$\tau_\alpha = \tau_{\alpha UV} - \frac{b_\alpha}{2\pi i} \ln(\frac{E}{\Lambda_{UV}}) \tag{4.12}$$

with $b_\alpha = -2n_\alpha + n_{\alpha+1} + n_{\alpha-1}$. It matches with the one-loop analysis of the system (3.20) from field theory. This identification suggests that the role of v is specifying a reference point on the $x^4 - x^5$ plane, which introduces a scale in the 4d theory. Also, under this identification, the $v \to \infty$ limit corresponds to the UV limit, which we should apply when we would like to recognize the UV theory form the brane diagram.

While the identification in the coupling constant is clean, it is worth noticing that (4.11) applies only when the separation between NS_5 brane is much larger than R_{10}, where the DBI action on the world volume of D_4 is reliable.

4.1.3 Seiberg-Witten Curve from Type IIA Brane Construction

In the following, we are interested in how to provide a solution for the Coulomb branch effective action from M theory perspective. Therefore, we need to answer which behavior of M_5 brane corresponds to a non-trivial vacuum expectation value of the scalar in the vector multiplet.

To answer this question, it is beneficial to take a look at the type IIA limit. Under this limit, scalars in the vector multiplet parameterize the position of each D_4 brane. When there is a non-trivial expectation value in that scalar, D_4 branes become separated, leading to the Higgsing of the gauge theory $SU(N) \to U(1)^{N-1}$. Fundamental strings scratching between different D_4 branes leads to the massive W bosons in the field theory, with mass $m^2 \sim (a_i - a_j)^2$. Uplifing to the M theory,

One of the key observations from Witten is that when we uplift the configuration to the M theory, both D_4 branes and NS_5 branes become the same object, namely the M_5 brane, with a complicated shape. By the constraint of supersymmetry, the shape of the brane has to be described by a holomorphic curve $S \in \mathbb{C} \times \mathbb{C}^\times$. Such a curve is generically smooth, and thus the singularity of the IIA configuration originated from the contact point between D_4 and NS_5 brane is smoothened. The resolution of the singularities suggests that the brane configuration in the M theory contains enough information for the $4d$ theory.

Indeed, the holomorphicity of the curve S is a strong constraint. At this point, we have collected enough information to determine the low-energy Seiberg-Witten geometry from M theory. Let us denote $t = e^{-s}$ to make it single-valued.

Brane Construction for $SU(2)$ Pure Yang-Mills Theory As the first example, let's consider the configuration with two NS_5 branes, with two D_4 branes suspending between them, and there is no semi-infinity D_4 brane scratching from the left and the right. The fact that there are two NS_5 branes implies that there are generically two solutions for t. Therefore, the curve S should be a quadratic equation in t, which generically writes

$$A(v)t^2 + B(v)t + C(v) = 0. \tag{4.13}$$

Then we will interpret the fact that there are no semi-infinite D_4 branes in the setup. Since there is no D_4 brane on the left, t could not go to infinity when v is finite. Therefore, $A(v)$ could not have zeros on the v-plane and thus is a constant. With out lose of generality, we normalize as $A(v) = 1$. Similarly, since there is no D_4 brane on the rightmost, t could not go to zero when v is finite. Therefore, $C(v)$ could not have zeros either, which implies that $C(v)$ is a constant. Therefore, in this setup, the curve S takes

$$t^2 + B(v)t + 1 = 0. \tag{4.14}$$

Now let us determine the function $B(v)$. Since there are two D_4 branes suspending between NS_5, $B(v)$ should generically have two solutions. Therefore, $B(v)$ is quadratic in v either, which generically takes the following form by a redefinition of v

$$B(v) = v^2 + b_2. \tag{4.15}$$

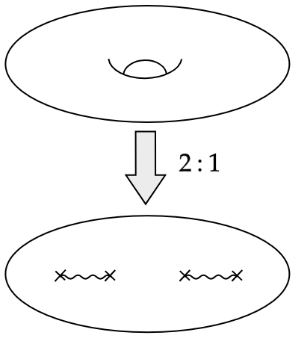

Figure 4-1 Torus geometry as the double cover of a sphere

We could absorb the linear term in t by a change of variable and reorganize the equation of the curve as follows

$$t^2 = \frac{1}{4}B(v)^2 - 1 = \frac{1}{4}\prod_{i=1}^{4}(v - e_i). \tag{4.16}$$

where e_i denotes four roots of the equation $\frac{1}{4}B(v)^2 - 1 = 0$. It is a standard elliptic curve up to a one-point compactification in the v plane, whose topology is a torus (see Fig.4-1).

Under the UV limit where $v \to \infty$, the curve asymptotic to the form

$$t^2 + v^2 t + 1 = 0 \tag{4.17}$$

which factorizes to the two branches, with one NS_5 brane going to infinity on the left and another on the right. Therefore, the complex coupling constant τ in the 4d theory diverges and thus $g \to 0$. It recovers the asymptotic freedom in the pure $SU(2)$ SYM theory. While goes to IR, b_2 occurs as an order parameter of the field theory.

Brane Construction for $SU(2)$ $N_F = 4$ SQCD In order to incorporate four flavors in the theory, we are ought to add branes on the left and right. One possible construction is to put two branes on the left and two branes on the right. The curve for the Riemann surface is thus

$$(v^2 + a_1 v + a_2)t^2 + (b_0 v^2 + b_1 v + v_2)t + (c_0 v^2 + c_1 v + c_2) = 0. \tag{4.18}$$

At UV limit $v \to \infty$, the curve reduces to

$$v^2(t^2 + b_0 t + c_0) = 0. \tag{4.19}$$

The $v = 0$ branch of the solution describes two branes wrapping on the cylinder in $x^6 + ix^{10}$ plane, intersecting with another branch of solution at two points. Compared with the pure Yang-Mills case, there is still one moduli parameter for the UV theory, parameterizing the position of the NS_5 brane. It matches with the feature of the $SU(2)$ $N_F = 4$ SQCD, which is conformal under the UV limit and admit a conformal manifold parametrized by the coupling constant τ.

While flowing to IR, by a redefinition of (t, v), there are five free parameters in total, which should be interpreted as four mass parameters and one

4.1.4 Incorporate D_6 Brane

Let's now remark on the case where D_6 brane is incorporated in Witten's Type IIA construction. We could incorporate D_6 brane on the $x^0, x^1, x^2, x^3, x^7, x^8, x^9$, where 8 supercharges are preserved. From the Type IIA perspective, fundamental strings scratching between D_6 and D_4 brane provides extra massive hypermultiplets in the 4d field theory description. The mass of the extra hyper depends on the position of D_6 brane in the $v = x^4 + ix^5$ plane. Its position on the x^6-plane, on the other hand, is irrelevant to the analysis of the Coulomb branch. [①]

Once we uplift the configuration to M-theory, multiple D_6 branes are resolved into the background multi-center Taub-Nut metric, whose Gibbons-Hawking form is specified by

$$
\begin{aligned}
ds^2 &= \frac{V}{4} d\vec{\mathbf{r}}^2 + \frac{V^{-1}}{4}(d\tau + \vec{\omega} \cdot d\vec{r})^2, \\
V &= 1 + \sum_{a=1}^{d} \frac{1}{|\mathbf{r} - \mathbf{x}_a|}
\end{aligned}
\tag{4.20}
$$

and $*_3 d\omega = dV$. In one complex structure, the Taub-Nut space Q could be described by the affine variety

$$
yz = P(v) = \prod_{a=1}^{d} (v - e_a). \tag{4.21}
$$

One could refer to A.11 for the mathematical details. The $M5$ brane worldvolume is again described by a holomorphic curve $\Sigma \hookrightarrow Q$, which would be given by $F(y, v) = 0$. In our case now, y could be loosely think of as the t parameter before.

For simplicity, we consider the case where there are only two NS_5 branes under the type II A limit, and there is no semi-infinite D_4 brane scratching on the left and right. In this situation, there will be two solution to y for generic choice of v. Thus $F(y, v)$ has to be quodratic in y

$$
F(y, v) = A(v)y^2 + B(v)y + C(v) = 0. \tag{4.22}
$$

where A, B, C are the polynomials in v and we assume that we have divide out all the common factors to make them coprime.

Now we are ought to determine the factors A, B, C from the fact that there is no semi-infinite D_4 brane. As we could see from the brane diagram, this condition means that both y and z could not diverges at any finite value of v. We could solve y as a function of v as

$$
y = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \tag{4.23}
$$

Then the analysis before tells us that there is no zeros of A, and thus $A(v)$ is a constant. With out lose of generality, we normalize as $A(v) = 1$.

Now let's transform the variables to $F(z, v)$. By $z = P(v)/y$, the curve takes

$$
C(v)z^2 + B(v)P(v)z + P(v)^2 = 0. \tag{4.24}
$$

① A similar phenomenon occurs in the Type IIB construction of 3d N=4 theory, which leads to the discovery of Hanany-Witten moves.

The finiteness of z at any finite value of v, similarly, requires that both $B(v)P(v)$ and $P(v)^2$ are divisible by $C(v)$. It means that the most general form of $C(v)$ is to pick out some factors in $P(v)^2 = \prod_{a=1}^{d}(v - e_a)$, which we denote as

$$C(v) = f \prod_{a=1}^{i_0} \left(v - e_a\right)^2 \prod_{b=i_0+1}^{i_1} \left(v - e_b\right). \tag{4.25}$$

with $1 \leq i_0 \leq i_1 \leq d$. On the other hand, since $B(v)P(v)$ is divisible by $C(v)$, $B(v)$ has to take the form

$$B(v) = \tilde{B}(v) \prod_{a=1}^{i_0} \left(v - e_a\right). \tag{4.26}$$

with $\tilde{B}(v)$ any polynomial. After collecting these facts together, the curve takes

$$y^2 + \tilde{B}(v) \prod_{a=1}^{i_0} \left(v - e_a\right) y + f \prod_{a=1}^{i_0} \left(v - e_a\right)^2 \prod_{b=i_0+1}^{i_1} \left(v - e_b\right) = 0. \tag{4.27}$$

By a change of variable $y = \prod_{a=1}^{i_0} \left(v - e_a\right) \tilde{y}$, the resulting curve is

$$\tilde{y}^2 + \tilde{B}(v) y + f \prod_{a=i_0+1}^{i_1} \left(v - e_a\right) = 0. \tag{4.28}$$

If $\tilde{B}(v)$ is a polynomial of degree k (with $2k \geq i_1 - i_0$), then the resulting theory is an $SU(k)$ SQCD with $i_1 - i_0$ flavor. The curve here recovers the previous analysis on the same theory.(Add Ref) In this case, if we identify y and t in a rough sense, then the net effect of incorporating the D_6 brane is to replace $C(v) = 1$ by a polynomial of v constraint by the Taub-nut geometry.

An important observation is that in the final result, only the factors $v - e_a, i_0 + 1 \leq a \leq i_1$ matters, while $a \leq i_0$ and $a \geq i_1 + 1$ are irrelevant. The interpretation is that for the D_6 branes whose center positions are at $e_a, a \leq i_0$ is on the left of all the NS_5 branes. Similarly, D_6 branes whose center positions are at $e_a, a \geq i_1$ are on the right of all the NS_5 branes. Therefore, they do not contribute to the states on the D_4 branes and are thus irrelevant in the analysis of the low energy field theory.

It is not difficult to generalize the argument to a chain of $n + 1$ NS_5 brane, with k_α D_4 brane connecting between $\alpha - 1$th and αth brane and d_α D_6 brane in between $\alpha - 1$th and αth brane. We still restrict ourselves to the case where there is no semi-infinite D_4 brane on the left/right most of the diagram. In this case, the Seiberg Witten curve is given by

$$y^{n+1} + A_1(v)y^n + \cdots + A_{n+1}(v) = 0. \tag{4.29}$$

A similar analysis by asking the finiteness of $z = P(v)/y$ at finite v leads us to the solution for A_i. After the change of variable, the curve takes

$$A_{n+1}(v)z^{n+1} + \cdots + A_\alpha P(v)^{n+1-\alpha} z^\alpha + \cdots + P(v)^{n+1} = 0. \tag{4.30}$$

Again, the finiteness of z asks $A^\alpha P(v)^{n+1-\alpha}$ to be divisible by A_{n+1}.

Now we are ought to provide a generic answer for A_{n+1} by asking $P(v)^{n+1}$ to be divisible by it. Namely, what A_{n+1} looks like is the product of the factors $(v - e_a)^{s_a}$, with multiplicity $0 \leq s_a \leq n+1$. Following from a similar argument as before, it is not hard to see that factors in A_{n+1} with multiplicity 0 and $n+1$ would be irrelevant to our final result, as it represents the D_6 brane locating on the left/right of the NS_5 branes. Therefore, without lose of generality, we assume that factors in A_{n+1} only have multiplicity $1 \leq s_a \leq n$. Introducing integers $i_0, i_1, \dots, i_n$ with $i_0 = 0 \leq i_1 \leq i_2 \leq \dots \leq i_{n-1} \leq i_n = n$ such that if for $1 \leq s \leq n$

$$J_s = \prod_{a=i_{s-1}+1}^{i_s} \left(v - e_a\right), \tag{4.31}$$

then the most general form of A_{n+1} is

$$A_{n+1} = f \prod_{s=1}^{n} J_s^{n+1-s}, \tag{4.32}$$

with f a constant. The generic form of A_α could be determined by asking $A_\alpha P^{n+1-\alpha}$ to be divisible by A_{n+1}, which takes

$$A_\alpha = g_\alpha(v) \prod_{s=1}^{\alpha} J_s^{\alpha-s}, \tag{4.33}$$

where $g_\alpha(v)$ is a polynomial in v. After collecting those facts together, the curves takes

$$y^{n+1} + g_1(v)y^n + g_2(v)J_1(v)y^{n-1} + \cdots + g_\alpha(v)\prod_{s=1}^{\alpha} J_s^{\alpha-s} y^{n+1-\alpha} + \cdots + \prod_{s=1}^{n+1} J_s^{n+1-s} = 0 \tag{4.34}$$

With the degree of g_α denoted as k_α, the gauge group of the four-dimensional field theory is then $G = \prod_{\alpha=1}^{n} SU(k_\alpha)$.

There is another way of looking at this curve, owing to a fantastic analysis in GMN. Since the x_M^6 position of the M-th D_6 brane is irrelevant in the IR theory, we could conduct this limit and move D_6 brane to the $x_M^6 \to \pm\infty$. We call $M \in \mathscr{L}$ if $x_M^6 \to -\infty$ and $M \in \mathscr{R}$ if $x_M^6 \to +\infty$. In the A.11, we considered the geometry of the Taub-Nut space under this limit, which implies

$$y \to t \prod_{M \in \mathscr{L}} (v - v_M). \tag{4.35}$$

where $M \in \mathscr{L}$ and $t \in \mathbb{C}^\times$. Also, $J_\beta(v)$ could be split as $J_\beta(v) = J_{\beta,L}(v) J_{\beta,R}(v)$. If we substitute (4.35) into the Seiberg Witten curve, then the curve could be simplified to

$$\hat{g}_0(v)t^{n+1} + \hat{g}_1(v)t^n + \hat{g}_2(v)t^{n-1} + \cdots + \hat{g}_\alpha t^{n+1-\alpha} + \cdots + \hat{g}_{n+1} = 0, \tag{4.36}$$

where

$$
\begin{aligned}
\hat{g}_0(v) &= \prod_{\beta=1}^{n} J_{\beta,L}^{\beta} \\
\hat{g}_\alpha(v) &= g_\alpha(v)\prod_{\beta=1}^{\alpha} J_{\beta,R}^{\alpha-\beta}\prod_{\beta=\alpha+1}^{n} J_{\beta,L}^{\beta-\alpha} \quad \alpha = 1,\dots,n-1 \\
\hat{g}_n(v) &= g_n(v)\prod_{\beta=1}^{n} J_{\beta,R}^{n-\beta} \\
\hat{g}_{n+1}(v) &= \prod_{\beta=1}^{n} J_{\beta,R}^{n+1-\beta}.
\end{aligned}
\tag{4.37}
$$

The factor $\hat{g}_\alpha(v)$ admits an interpretation as the Hanany-Witten effect. When we move a D_6 brane between $\beta-1$-th and β-th NS_5 brane to the left, it will create $\beta-\alpha$ D_4 brane on the α-th NS_5 brane, contributing $J_{\beta,L}^{\beta-\alpha}$ to the factor $\hat{g}_\alpha(v)$. Compared with free D_4 branes specified by $g_\alpha(v)$, the D_4 brane created in this way is completely specified by the v position of D_6 brane, with no free parameters. Therefore, those D_4 branes are frozen.

At this stage, a residual ambiguity still needs to be resolved. There are many possible ways to distribute the D_6-branes—some moving to the left, others to the right. Gaiotto, Moore, and Neitzke proposed a distinguished prescription known as the *balanced* motion. In this approach, the D_6-branes are moved in such a way that the number of D_4-branes stretched between the $(\alpha-1)$-th and α-th NS_5-branes is constant, denoted by $\hat{K}$.

The feasibility of this motion arises from the invariance of the equation $d_\alpha + K_{\alpha+1} + K_{\alpha-1} - 2K_\alpha = 0$ under Hanany-Witten transitions. After shifting all D_6 branes to the leftmost or rightmost positions, the resulting D_4 brane counts satisfy $\hat{K}_{\alpha+1} + \hat{K}_{\alpha-1} - 2\hat{K}_\alpha = 0$. To construct the balanced motion, it suffices to construct a configuration where $\hat{K}_0 = \hat{K}_1$, which exists and is not hard to construct. This constraint propagates recursively, ensuring that the number of D_4 branes between every pair of adjacent NS_5 branes becomes the same.

4.2 Examples

4.3 4d EFT from 6d Brane Perspective

From the Seiberg-Witten solution for the Coulomb Branch Effective Field theory, the redundant torus fibration over the Coulomb branch magically appears, encoding the coupling constant of the effective theory. As a result, for the generic points in the Coulomb Branch Moduli space, the information of the theory is encoded in the geometry of $\mathbb{R}^4 \times T^2$. From the spirit of string theory, it is natural to think of the underlying brane construction for the 4d theory related to that geometry. As the coulomb branch effective theory now is just $\mathcal{N} = 2$ $U(1)$ gauge theory, the most natural candidate for the brane construction is achieved by wrapping one copy of M5 brane on $\mathbb{R}^4 \times T^2$.

Let's justify the idea by looking at the world volume theory of M5 brane, namely the 6d $\mathcal{N} = (2,0)$ SCFT. 6d $\mathcal{N} = (2,0)$ SCFT plays a very special role in all the SCFTs. It is the maximal superconformal theory that may exist. On one hand, six is the largest dimension where supersymmetry and conformal symmetry are compatible. On the other hand, the classification of superconformal algebra tells us that this theory possesses the largest possible superconformal algebra in 6d, whose real form is given by $\mathfrak{osp}(8^*|4) \supset \mathfrak{so}(2,6) \oplus \mathfrak{so}(5)_R$.

Such free $\mathcal{N} = (2,0)$ theories consist of Abelian tensor multiplets, each containing a real self-dual 2-form gauge field $B_{\alpha\beta}$, spinors λ, and five scalars Φ_I. They live in the singlet, project representation $\mathbf{4}$ and fundamental representation $\mathbf{5}$ respectively for the $\mathfrak{so}(5)_R$ R symmetry. From M theory perspective, five scalars describe the vibration of $M5$ brane along the 5 dimensions where the fivebrane is not occupied, while the $\mathfrak{so}(5)_R$ is simply the rotation along those dimensions.

Let's now consider one single M5 brane wrapped on the $\mathbb{R}^4 \times T^2$. We can conduct dimension reduction along the T^2 direction to get the effective 4d theory. Schematically, one may think of this procedure in the Lagrangian Formalism (although 6d $\mathcal{N} = (2,0)$ theory does not admit Lagrangian description) where we drop the dynamics of fields on the T^2 and do the integration along the Riemann surface in the action. Explicitly, we set all the field content to be

$$\phi(x^i, z, \bar{z}) = \phi(x^i), \tag{4.38}$$

where write the coordinate for $\mathbb{R}^4$ as x^i and for T^2 as $(z, \bar{z})$.

Then the action behaves as

$$S = \int_{\mathbb{R}^4 \times T^2} \mathrm{d}^6 x\, \mathscr{L}(\phi, \partial_I \phi) = \int_{\mathbb{R}^4} \mathrm{d}^4 x\, \tilde{\mathscr{L}}(\phi, \partial_i \phi), \tag{4.39}$$

where

$$\tilde{\mathscr{L}}(\phi, \partial_i \phi) = \int_{T^2} \mathrm{d}^2 z\, \mathscr{L}(\phi, \partial_i \phi). \tag{4.40}$$

The B field could integrate along A-cycle and B-cycle (two generators for $H_1(T^2, \mathbb{Z})$), which gives the $U(1)$ gauge field and its electric-magnetic dual (A, A_D) from 4d perspective. Let's refine this argument a little bit to see that A and A_D are really electric-magnetic dual of each other. For the 6d self-dual 2-form B_{IJ}, we have $G = \mathrm{d}B, *_6 G = G$. In dimension reduction, we may identify $F_{ij} = G_{ij6}$ and $F_{Dij} = G_{ij5}$ as the field strength in 4d. The self-dual constraint thus translates to the relation $F_D = *F$ in 4d, which states that F and F_D are indeed the electric-magnetic dual of each other.

The construction of BPS particles like monopoles and dyons in 4d theory can be clear once we establish the UV completion of the theory. We will come back to this point later.

The point of what we have done yet is that we have now lifted the effective 4d Coulomb Branch physics to a 6d free theory compactified on a Riemann surface, translating almost everything into a Brane configuration. It helps us to generalize the argument. We could easily generalize this argument to the Riemann surface with genus r, which describes the $U(1)^r$ Coulomb Branch EFT. Moreover, it helps us to construct the UV completion of the theory, which is called the Class $\mathcal{S}$ construction.

4.4 Class $\mathcal{S}$ Construction

In the last section, we have uplifted the 4d Coulomb Branch EFT to a 6d $\mathcal{N} = (2,0)$ theory with two dimensions compactified on the Riemann Surface Σ_u with $u \in \mathscr{M}_{\text{Coulomb}}$.

Following from the analysis in $\mathcal{N} = 2$ $SU(2)$ pure gauge theory, the UV completion for the Coulomb Branch EFT is just the $SU(2)$ theory we start with. In M theory construction, a theory with gauge group $SU(2)$ is described by two $M5$ brane stacks together with the central mass degree of freedom decoupled. If we tune the energy scale gradually from IR to UV, the single M5 brane in IR must merge together on some surface and behave like "two" M5 brane. Namely, the IR curve[①] of the fivebrane must double cover some UV curve for the fivebrane. The UV curve is the so-called Giaotto curve.

Having awarded of the fact that the UV completion of the IR theory is described by the Gaiotto curve, we can now begin the story with a Gaiotto curve with n M5 brane wrapping on it. This is the input data of the class $\mathcal{S}$ construction $\mathcal{T}(G, \Sigma)$. Quite different from the traditional framework of ordinary quantum field theory, in the Class $\mathcal{S}$ construction, a theory is specified by the geometry. It leads to a gigantic class of theories with rich properties, giving us a way of describing the theory via geometry even it admits no Lagrangian description!

However, placing a 6d $\mathcal{N} = (2, 0)$ theory a generic Gaiotto curve would not preserve 8 supercharges at all! Let's see what goes wrong at this point. Once we put a theory on the curved background, we have to replace all the ordinary derivatives ∂_i with covariant derivative ∇_i. If a theory has supersymmetry in the flat spacetime, then we have $\delta_\epsilon S = 0$ for some **constant** spinor. However, after uplifting the whole story to the curved spacetime, we have to request for the existence of **covariant spinor**, namely the spinor solving the equation $\nabla\epsilon = 0$. Otherwise, there will generically be some extra terms proportion to $\nabla\epsilon$ after the replacement, breaking all the supersymmetry. The existence of covariant spinor turns out to be a picky requirement, which is satisfied only if the background manifold is Calabi-Yau, which is too picky here because even some simple Riemann surfaces such as Riemann Sphere are not Calabi-Yau. Something must be missing from the whole construction.

4.4.1 Reduction of 6d $\mathcal{N} = (0, 2)$ SUSY Algebra and Topological Twist

Topological twist provides a remedy for preserving supersymmetry on the curved background. As we know, the holonomy group of the orientable Riemann surface is $SO(2) = U(1)$. If the fivebrane is flat, $SO(5)_R$ symmetry comes from the rotation in the five dimensions transversal to the fivebrane. In order to match with the $SO(3)_R$ symmetry in 4d, there must be three directions that are still flat and transversal to the $M5$ brane. After compactifing the M5 brane on a Riemann surface, the $SO(5)_R$ symmetry splits into $SO(2)_r \times SO(3)_R$. The $SO(3)_R$ is the legitimated global R-symmetry originated from the rotation in the three transversal directions. On the other hand, the $SO(2)_r$ is the local symmetry on the Riemann surface, described by the principal $SO(2)$ bundle associated the normal bundle of the Riemann surface in the spacetime. [②]

Recall that whether an object is scalar, spinor or vector depends on its behavior under the group action. The supercharge is spinor under the original spacetime symmetry group, but may well be a scalar if we switch to another group action. The topological twist means that we redefine the spacetime symmetry group by mixing the action of $SO(2)_r$ symmetry and $SO(2)$ spacetime symmetry in such a way that they act on

① Curve here refers to the complex curve, which have real dimension 2.

② At this point, we haven't specified the space-time geometry yet. the normal bundle relies on the background geometry the supercharges Q trivially. Then a portion of the supercharge become the scalar on the Riemann surface[①], which will be preserved because we can always find a constant function on the Riemann surface.

Let's now make the argument more explicit. As the starting point, we split 6d gamma matrix as the tensor product of 4d and 2d gamma matrices as $\Gamma^{Ai}_{6\mathrm{d}} = \gamma^A_{4\mathrm{d}} \otimes \sigma^i_{2\mathrm{d}}$. Following from the representation theory of Clifford Algebra, one could define the 6d chirality operator as the tensor product of 4d and 2d chirality operator $\Gamma^7 = \gamma^5 \otimes \sigma^3$. As the 6d theory has supersymmetry $\mathcal{N} = (2,0)$, we shall write $Q = Q^{Aia}_I$ with A the 4d indices, i the $SO(2)$ spacetime symmetry indices (tangent bundle indices) , a the $SO(2)_r$-symmetry indices (normal bundle indices) and $I = 1,2$ the $SO(3)_R$ symmetry indices[②]. All the supercharges live in the positive eigenvalue subspace of Γ^7, which imposes a constraint on the choice of 2d chirality and 4d chirality is not independent. It plays an important role in recovering the supercharges in 4d later.

Let's now zoom into the $SO(2) \times SO(2)_r$ part. The easiest way of acquiring some of the supercharges Q to be in a scalar representation of some subgroup of $SO(2) \times SO(2)_r$ is to ask a to be the cotangent bundle indices, which has the inverse $SO(2)$ charge compared with the tangent bundle. Then the background geometry has to be $\mathbb{R}^4 \times T^*\Sigma \times \mathbb{R}^3$ in order that the normal bundle is exactly the cotangent bundle. The total dimension is eleven, which matches the dimension of the M-theory. The essence of the topological twist is to define the diagonal embedding of the twisted $SO(2)_{\mathrm{twist}}$ into $SO(2) \times SO(2)_r$ and re-identify it as the new spacetime symmetry group. Denote the generator for $SO(2)$ and $SO(2)_r$ as M and r respectively. The generator of $SO(2)_{\mathrm{twist}}$ is then $M + r$. As Q lives in the real spinor representation of both $SO(2)$ and $SO(2)_r$, we could decompose it into the positive eigenvalue part and negative eigenvalue part $i = (+,-)$ and $a = (+,-)$. Let's now count the number of supercharges that survive in the construction. Supercharge with indices Q^{+-} and Q^{-+} now lives in the trivial representation of $SO(2)_{\mathrm{twist}}$, while Q^{++} and Q^{--} possesses $SO(2)_{\mathrm{twist}}$ eigenvalue ± 1. Those two supercharges will not be preserved in 4d generically unless the Riemann Surface has a trivial tangent bundle, which only works for the torus and gives 4d $\mathcal{N} = 4$ theory. In general cases, only Q^{+-} and Q^{-+} survives. Both supercharges have a positive chirality for Γ^7, but they have both positive chirality and negative chirality for 2d gamma matrices. Therefore, they must have both positive and negative chirality for 4d chirality operator γ^5 respectively. It means that the 6d left-handed spinor reduces to the same amount of left-handed and right-handed spinor in 4d. In our notation, Q^{A+-}_I becomes the 4d left-handed supercharges, with $A = 1,2$ the 4d weyl spinor indices, while Q^{A-+}_I becomes the right-handed supercharges. Suppressing the $SO(2) \times SO(2)_r$ indices, the remaining supercharges from 4d perspective is thus Q^α_I and $\bar{Q}^I_{\dot{\alpha}}$ with $I = 1,2$. It is indeed the supercharges in 4d $\mathcal{N} = 2$ theory!

One could easily generalize the story by putting n stalk of M5 brane on the Gaiotto curve, which gives the 4d $\mathcal{N} = 2$ $SU(N)$ gauge theory. 4d theory with other gauge groups like SO or SP can also be constructed if one considers the OM brane wrapping one the Gaiotto curve. Moreover, one could add punctures to the Riemann surface, which can be thought of as a copy of the infinitely long M5 brane, contributing to the Hypermultiplet with certain flavor symmetry in 4d. For details, see[3].

The discussions are summarized as follows. The M-theory setup for class $\mathcal{S}$ theory is given by the background geometry $\mathbb{R}^4 \times T^*\Sigma \times \mathbb{R}^3$, with the fivebrane sitting in

① But still a spinor in $\mathbb{R}^4$ direction

② As Q lives in the **4** of $SO(5)_R$, it should live in the spinor representation of $SO(2)_r \times SO(3)_R$, which asks $I = 1,2$

$\mathbb{R}^4 \times \Sigma$. After the twisted compactification of the $M5$ brane on the Riemann surface, the effective world volume theory is the UV 4d $\mathcal{N} = 2$ theory.

4.4.2 Punctures in Class $\mathcal{S}$ Construction

In general, we can consider point-like defect operators on the Riemann surface called punctures. They can be characterized by the singular behavior of the Higgs field $\boldsymbol{\Phi}_z$ around it. Namely, if we put a puncture at $z = 0$, the Higgs field looks locally like

$$\boldsymbol{\Phi} = \frac{\rho}{2}\frac{dz}{z} + \ldots, \qquad A = \frac{\alpha}{2i}\left(\frac{dz}{z} - \frac{d\bar{z}}{\bar{z}}\right) + \ldots. \tag{4.41}$$

where $\rho \in \mathfrak{sl}(n), \alpha \in \mathfrak{su}(n)$ and $\ldots$ means the regular terms in z.

The residue ρ, or more precisely, the conjugate class of ρ, characterizes the nature of the puncture. Sometimes, people use the terminology that the puncture is decorated by the orbit in the Lie algebra. The eigenvalues of ρ are picked up by the contour integral of the Seiberg-Witten differential around the puncture. They encode the information about the mass of the BPS particles in the IR theory. Therefore, they can be referred to as mass parameters of the puncture. If the eigenvalue is non-zero, this breaks the conformal invariance. The exception is when A is nilpotent, in which case these eigenvalues vanish, and conformal invariance is preserved.

Recall that for $G = SU(N)$,the Seiberg-Witten curve can be written as

$$x^N = \sum_{i=2}^{N} p_i(z) x^{N-i}, \tag{4.42}$$

where $\phi_i(z)$ can be expressed as polynomials of gauge invariant combination of $\boldsymbol{\Phi}_z$ such as $\mathrm{tr}\left(\varphi_z^i\right)$. For each puncture, we can define a pole structure using a set of positive integers $\left\{p_i\right\}$ labeling the order of poles for ϕ_i at the puncture. A Young tableaux can be assigned to each puncture from the pole structure using certain procedure, and vice versa.[4] Then, a flavor symmetry group can be associated to the puncture. Two special types of punctures are called full punctures and simple punctures with their pole structure being $\{1, 2, \cdots, N-1\}$ and $\{1, \cdots, 1\}$ respectively. For the case of simple puncture, the residue of the Higgs field $\boldsymbol{\Phi}_z$ has the structure of

$$A \sim \begin{pmatrix} m & & & \\ & m & & \\ & & \ddots & \\ & & & -(N-1)m \end{pmatrix}. \tag{4.43}$$

The $N-1$ identical mass parameters imply an enhanced flavor symmetry. By tuning the VEV of the field, the mass parameters can be adjusted in such a way that the flavor symmetry gets properly broken. In this way, different punctures can be obtained from the simple puncture.

The punctures discussed above are regular punctures, where $\boldsymbol{\Phi}_z$ only has simple poles. A major generalization of the class $\mathcal{S}$ construction is to include irregular punctures where the Higgs field $\boldsymbol{\Phi}_z$ can have higher and even fractional orders of pole. While the regular punctures can be simply classified, the classification of irregular punctures is much richer and many physical information can be directly read-off in a systematic way.

Adding the irregular punctures to the theory makes it possible to construct a large category of Argyres-Douglas theories using class $\mathcal{S}$ construction. These theories involving irregular punctures usually do not admit any Lagrangian description and show exotic behaviors such as Coulomb branch operators with fractional conformal dimensions. From Seiberg-Witten theory, those theories correspond to the point in the Coulomb branch moduli space where two types of BPS particles becomes massless simultaneously. Those theory are described by the CFT with involves intrinsic interaction and usually admits no free limit. This accounts to the reason why they have no Lagrangian descriptions. Those theories have not been fully understood yet.

There is another way of looking at the puncture. Near the puncture, the theory looks like $6d$ theory on $\mathbb{R}^+ \times S^1$. AS a result, classifying the puncture is the same as classifying the boundary condition of the theory, such that 1/2 of the SUSY is preserved.

4.4.3 From Hitchin System to Class $\mathcal{S}$ Construction

In the last chapter, we find a class of UV theory described by the twisted compactification of the $M5$ brane on the Riemann surface, whose effective world volume theory is 4d UV $\mathcal{N} = 2$ theory. A natural question is how to flow the theory to IR. Equivalently, how should we construct the Seiberg-Witten curve and Seiberg-Witten differential out of the UV configuration?

To that aim, we should first understand how to describe the Coulomb branch in 6d configuration. Recall that for 6d $\mathcal{N} = (2,0)$ theory, there are five scalar φ_I living in the fundamental representation of $SO(5)_R$ describing the position of the $M5$ brane. After compactification, the R-symmetry group splits into $SO(2)_r \times SO(3)_R$, with φ_1 and φ_2 two scalars describing the oscillation of $M5$ brane on the Riemann surface. However, with respect to the $SO(2)_{twist}$, Φ_1 and Φ_2 no longer transform as a scalar but as the component of a one-form. (Intuitively, adding a normal bundle index coming from the $SO(2)_r$, which is indeed the one-form indices. As a result, we may recombine as $\varphi_z = \varphi_1 + i\varphi_2$ such that $\varphi = \varphi_z \, \mathrm{d}z$ is a holomorphic one-form with value in the adjoint vector bundle. In a more mathematical language $\varphi \in H^0(\Sigma, \mathrm{ad}P \otimes T^*\Sigma)$ is called the Higgs Field.

By reversing the renormalization group flow, from the $5d$ SYM perspective, the $\mathbb{R}^3$ supersymmetric translational invariant vacua is described by the following differential equation,called Hitchin equation,

$$\begin{aligned} F + R^2[\varphi, \overline{\varphi}] = 0 \, , \\ \overline{D}_A \varphi = 0 \, . \end{aligned} \tag{4.44}$$

where

$$\overline{D}_A := (\bar{\partial} + [A_{\bar{z}}, -]) \tag{4.45}$$

4.5 Geometry of the Hitchin System

In the last section, we discussed about the Hitchin system on a Riemann surface $\Sigma g, n$. In this section, we will provide a mathematical description of the Hitchin system and Hitchin Moduli space. Hitchin Moduli space is a hyper-Kähler manifold with three complex structures $(I, J, K = IJ)$. Each complex structure coincides with certain interesting moduli spaces with a rich structure.

$$\begin{tikzcd}
& 6d\ \mathcal{N}=(0,2) \text{ on } \mathbb{R}^3\times\Sigma_{g,n}\times S^1 \arrow[dl, "\text{on } S^1"'] \arrow[dr, "\text{on } \Sigma_{g,n}"] & \\
5d\ \mathcal{N}=2 \text{ SYM on } \mathbb{R}^3\times\Sigma_{g,n} \arrow[dr, "\text{on } \Sigma_{g,n}"'] & & 4d\ \mathcal{N}=2\ \mathcal{T}[\Sigma_{g,n}] \text{ on } \mathbb{R}^3\times S^1 \arrow[dl, "\text{on } S^1"] \\
& 3d\ \mathcal{N}=4 \text{ Sigma Model } \mathbb{R}^3 \to \mathcal{M}_H &
\end{tikzcd}$$

Figure 4-2 Reverse Renormalization Group Flow

4.5.1 Definition of the Hitchin Moduli Space

In this section, we will set up the notations and define the Hitchin moduli space. Let $\Sigma_{g,n}$ be a Riemann surface with genus g and n punctures, whose complex structure is denoted as I. We denote the punctures as $p_i, i = 1, \dots n$ and a divisor $D = -\sum n_i p_i$. Let $K = T^*\Sigma_{g,n}$ be the canonical line bundle over Σ_g, and $K[D]$ the twist of K by the divisor D. Also, we use G to denote a compact reductive Lie group (such as $SU(N)$ and $U(N)$) and $G_{\mathbb{C}}$ the complexification of G, whose complex strucutre is denoted as J.

We are going to consider a principal G bundle P over $\Sigma_{g,n}$ with connection A (called gauge field) and a one form field (called higgs field) $\varphi \in \Omega^1(ad(P) \otimes K[D])$, where $ad(P)$ is the associated adjoint vector bundle. Denote the configuration space of the gauge field and Higgs field as $(A, \varphi) \in \mathcal{N}$. Hitchin Moduli space is the space of solutions (A, φ) for the Hitchin Equation

$$\begin{aligned} F + R^2[\varphi, \overline{\varphi}] = 0\ , \\ \overline{D}_A \varphi = 0\ . \end{aligned} \tag{4.46}$$

modulo the action of the gauge group $\mathcal{G} \cong Aut(E)$ via gauge transformation

$$(A(x), \varphi(x)) \mapsto (g(x)Ag^{-1}(x) + g(x)dg^{-1}(x), g(x)\varphi(x)g^{-1}(x)), \tag{4.47}$$

with the boundary condition described by the punctures on the Riemann surface. As a useful fact, the curvature F in the holomorphic coordinate takes

$$F = F_{z\bar{z}} dz \wedge d\bar{z} = (\partial A_{\bar{z}} - \overline{\partial} A_z + [A_z, A_{\bar{z}}]) dz \wedge d\bar{z}. \tag{4.48}$$

4.5.2 Hitchin Moduli Space from hyper-Kähler Quotient

For any $\zeta \in \mathbb{C}^\times$, we could define the complexified connection as

$$\mathscr{A}_\zeta := \frac{R}{\zeta}\varphi + A + R\zeta\bar{\varphi}. \tag{4.49}$$

The configuration space for (A, φ) is an infinite-dimensional affine space, which admits a family of holomorphic symplectic structures

$$\Omega_\zeta = \frac{1}{2\pi}\int_C \mathrm{Tr}\left(\delta\mathscr{A}_\zeta \wedge \delta\mathscr{A}_\zeta\right). \tag{4.50}$$

Explicitly, we could take $\zeta \to \infty$ as a reference point. The symplectic form takes

$$\begin{aligned}\omega_I &= \frac{i}{2\pi}\int_\Sigma |d^2z|\ \mathrm{Tr}\Big(\delta A_{\bar z}\wedge \delta A_z - R^2\delta\overline{\varphi}_{\bar z}\wedge\delta\varphi_z\Big)\,,\\ \Omega_I &= \frac{R}{\pi}\int_\Sigma |d^2z|\,\mathrm{Tr}\left(\delta\varphi_z\wedge\delta A_{\bar z}\right).\end{aligned} \tag{4.51}$$

We recognize that Ω_I is holomorphic in the complex structure I, and ω_I is real in complex structure I. For generic ζ, we could write

$$\Omega_\zeta = \frac{1}{\zeta}\Omega_I + \zeta\overline{\Omega_I} - 2i\omega_I, \tag{4.52}$$

which is holomorphic with respect to the complex structure

$$J^{(\zeta)} = \frac{i(\zeta-\bar\zeta)J + (\zeta+\bar\zeta)K + \left(|\zeta|^2-1\right)I}{1+|\zeta|^2}, \zeta\in\mathbb{C}^\times. \tag{4.53}$$

As we will see, the Hitchin Moduli space could be viewed as the hyper-Kähler quotient of the configuration space $\mathscr{N}$ with respect to the trihamiltonian action of $\mathscr{G}$. This perspective follows from the following straightforward calculation. The infinitesimal $\mathscr{G}$ action generates the vector field on $\mathscr{N}$.

$$V_\epsilon = \int |d^2z|(D_A\epsilon)^a\frac{\delta}{\delta A^a_z} + (\overline{D}_A\epsilon)^a\frac{\delta}{\delta A^a_{\overline{z}}} + [\varphi_z,\epsilon]^a\frac{\delta}{\delta\varphi^a_z} + [\overline{\varphi}_{\overline{z}},\epsilon]^a\frac{\delta}{\delta\overline{\varphi}^a_{\overline{z}}}. \tag{4.54}$$

Then, the internal product yields

$$i_{V_\epsilon}\Omega_I = \frac{R}{\pi}\delta\int_\Sigma |d^2z|\,\mathrm{Tr}\Big(\epsilon\overline{D}_A\varphi_z\Big),\quad i_{V_\epsilon}\omega_I = \frac{i}{2\pi}\delta\int_\Sigma |d^2z|\,\mathrm{Tr}\big(\epsilon(F_{z\overline{z}} + R^2[\varphi_z,\overline{\varphi}_{\overline{z}}])\big), \tag{4.55}$$

from which we read off the Hamiltonian

$$Q^{\mathbb{C}}_\epsilon = \frac{-R}{\pi}\int_\Sigma |d^2z|\,\mathrm{Tr}\Big(\epsilon\overline{D}_A\varphi_z\Big),\quad Q^{\mathbb{R}}_\epsilon = \frac{i}{2\pi}\int_\Sigma |d^2z|\,\mathrm{Tr}\big(\epsilon(F_{z\overline{z}} + R^2[\varphi_z,\overline{\varphi}_{\overline{z}}])\big) \tag{4.56}$$

Notice that there is a non-degenerate pairing

$$\int_\Sigma |d^2z|\,\mathrm{Tr}(-,-) : \mathrm{Lie}\mathscr{G} \mapsto \mathrm{Lie}\mathscr{G}^*, \tag{4.57}$$

which leads to an isomorphism between $\mathrm{Lie}\mathscr{G}$ and $\mathrm{Lie}\mathscr{G}^*$. Using this identification, the moment map could be phrased as

$$\mu^I_{\mathbb{C}}(A,\varphi) = \frac{-R}{\pi}\overline{D}_A\varphi_z,\quad \mu^I_{\mathbb{R}}(A,\varphi) = \frac{i}{2\pi}(F_{z\overline{z}} + R^2[\varphi_z,\overline{\varphi}_{\overline{z}}]). \tag{4.58}$$

Therefore, the Hitchin equation could be viewed as the zeros of the hyper-Kähler moment map. It suggests that Hitchin Moduli space could be viewed as the hyper-Kähler quotient of the configuration space $\mathscr{N}$ with respect to $\mathscr{G}$

$$\mathscr{M}_H \cong \mathscr{N}///\mathscr{G}. \tag{4.59}$$

From this perspective, $\mathscr{M}_H$ is natrually a hyper-Kähler manifold (upto blowup of singularities), which admits a natural hyper-Kähler structure descended from $\mathscr{N}$.

4.5.3 Hitchin Moduli Space in each Complex Structure

Since the Hitchin Moduli space $\mathscr{M}_H$ is hyper-Kähler , it enjoys a $\mathbb{CP}^1$ family of complex structures. In this section, we will provide an explicit description of $\mathscr{M}_H$ in each complex structure J^ζ. The technique we will apply is the Kempf-Ness theorem. It states that instead of taking a quotient on the zeros of three moment maps by the action of the real gauge group $\mathscr{G}$, one could equivalently take a quotient on the zeros of one complexified moment map by the complexified gauge group $\mathscr{G}_\mathbb{C}$.

Complex Structure I: $G_\mathbb{C}$ Higgs Bundle To this purpose, let us first describe the complex moment maps in the limit $\zeta \to \infty$, where we achieve complex structure I. In this complex structure, the zeros of the moment map require that

$$\overline{D}_A\varphi = 0. \tag{4.60}$$

Namely, the zeros of the complexified moment map in $\mathscr{N}$ are described by a holomorphic structure $\overline{D}_A$ on the vector bundle and the holomorphic Higgs field $\varphi \in H^0(ad P \otimes K[D])$. Those data define a $G_\mathbb{C}$-*Higgs bundle* over the Riemann surface $\Sigma_{g,n}$.

The moduli space of $G_\mathbb{C}$ Higgs bundle $\mathscr{M}_{Higgs}$ is defined as the space of holomorphic vector bundles $(E, \overline{D}_A)$[①] along with the holomorphic Higgs field, modulo the $\mathscr{G}_\mathbb{C}$ gauge transformation. Therefore, the Kempf-Ness theorem suggests that $\mathscr{M}_H$ in complex structure I is described by the moduli space of $G_\mathbb{C}$ anti-Higgs bundle.

Complex Structure $-I$: $G_\mathbb{C}$ Anti-higgs Bundle In the complex structure $-I$, the analysis is similar to the $G_\mathbb{C}$ Higgs Bundle. The zeros of the moment map require that

$$D_A\overline{\varphi} = 0, \tag{4.61}$$

which suggest that $\overline{\varphi}$ is an anti-holomorphic section over $ad(P) \otimes \overline{K}$. $(A, \overline{\varphi})$ $\mathscr{G}_\mathbb{C}$ defines an anti-holomorphic version of the Higgs bundle, which is called the anti-Higgs bundle. Therefore, the Kempf-Ness theorem suggests that $\mathscr{M}_H$ in complex structure $-I$ is described by the moduli space of $G_\mathbb{C}$ anti-Higgs bundle.

Complex Structure J^ζ: $G_\mathbb{C}$ flat connection Let us now describe the space in the complex structure J^ζ. In this complex structure, the complex moment map is given by

$$\mu_\mathbb{C}^\zeta(A, \varphi) = F_{\mathscr{A}_\zeta} = d\mathscr{A}_\zeta + \mathscr{A}_\zeta \wedge \mathscr{A}_\zeta, \quad \zeta \in \mathbb{C}^\times, \tag{4.62}$$

where $\mathscr{A}_\zeta$ is the complexified connection defined in 4.49. The boundary condition in the Hitchin equation fixes the conjugacy class of the Holonomy around the punctures. Therefore, the zero locus of the moment map is described by the flat $G_\mathbb{C}$ connection over $\Sigma_{g,n}$, which we would denote as $\mathscr{M}_{flat}(\Sigma_{g,n}, G_\mathbb{C})$. By the Kempf-Ness theorem, Hitchin moduli space is described by the $G_\mathbb{C}$ flat connection over the Riemann surface.

Summary The discussion on the Hitchin moduli space in each complex structure could be summarized as follows (see also Fig.4-3)

- Complex Structure I: $G_\mathbb{C}$ Higgs bundle
- Complex Structure $-I$: $G_\mathbb{C}$ Anti-higgs Bundle
- Complex Structure J^ζ: $G_\mathbb{C}$ flat connection

① Strictly speaking, some stability condition needs to be imposed.

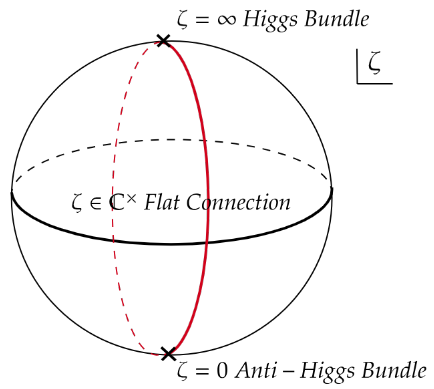

Figure 4-3 Hitchin Moduli space in $\mathbb{CP}^1$ family of complex structure.

4.5.4 More on the Moduli Space of $G_{\mathbb{C}}$ Higgs Bundle

We will explain more about the properties of the Moduli Space of $G_{\mathbb{C}}$ Higgs Bundle and its physical meaning in this part.

Mathematical Motivation To start with, let us present another mathematical motivation towards the construction of the $G_{\mathbb{C}}$ Higgs Bundle. For convenience, we consider the smooth Riemann surface with no punctures. Suppose we are going to study the moduli space $\mathscr{M}_E$ of all the holomorphic vector bundles $(E, \overline{D}_A)$ over a Riemann surface. The discrete data that we can apply to distinguish one bundle from another involves the rank of the vector bundle(dimension of the fibers) and its degree[①]. Therefore, to simplify the problem, we could study the moduli space of holomorphic vector bundles with fixed rank r and degree d separately, which we called $\mathscr{M}_E^{r,d}$. With a certain stability condition imposed, which we will elaborate in the next section, $\mathscr{M}_E^{r,d}$ is a finite-dimensional manifold.

Stability Condition To be precise, in order to have a good moduli space, the vector bundles have to be subjected to a certain *stability condition*. A holomorphic vector bundle E is called stable if any subbundle $F \subset E$ satisfies

$$\frac{\deg F}{\mathrm{rank} F} < \frac{d}{r}, \tag{4.63}$$

which, by the Riemann-Roch theorem, ensures that there exists a holomorphic vector bundle homomorphism between F and E,

$$\deg(F^* \otimes E) = d\,\mathrm{rank}F - r\deg F \geq 0. \tag{4.64}$$

One important property of the stable vector bundle is that the automorphism it is only the fiberwise scalar multiplication. As a consequence, we have

$$\dim H^0(\Sigma, End(E)) = 1, \quad \dim H^0(\Sigma, End(E)) = 0. \tag{4.65}$$

where $End(E)$ is the endormorphism bundle of E, and $End_0(E)$ denote it's traceless part.

① Degree is defined as the first Chern number of the vector bundle. Since a Riemann surface is of complex dimension one, Chern-Weil theory suggests that higher Chern class vanishes for the holomorphic vector bundle over a Riemann surface, and the only non-trivial one is the first Chern class.

Local Description To understand this moduli space, it is helpful to understand the local description of it, which is governed by the (co)tangent bundle. Given a holomorphic vector bundle $V, \overline{D}_A$, the standard deformation theory in complex geometry suggests that the infinitesimal deformation of it is governed by

$$T_V \mathscr{M}_E^{r,d} \cong H^1(\Sigma, \mathrm{End}_0(V)), \tag{4.66}$$

where $End_0(V)$ is the traceless part of the endomorphism bundle of V. Therefore, the cotangent bundle of the moduli space is described by

$$T_V^* \mathscr{M}_E^{r,d} \cong H^1(\Sigma, \mathrm{End}_0(V))^* = H^0(\Sigma, \mathrm{End}_0(V) \otimes K), \tag{4.67}$$

where the second step follows from the Serre duality. From this, we can count the dimension of the moduli space from a standard Riemann-Roch calculation

$$\dim \mathscr{M}_E^{r,d} = \dim T_V^* \mathscr{M}_E^{r,d} = (r^2 - 1)(g - 1). \tag{4.68}$$

Higgs Bundle from the Vector Bundle Perspective Notice that the Higgs bundle is defined as a holomorphic vector bundle over $(E, \overline{D}_A)$ along with a holomorphic Higgs field $\varphi \in H^0(\Sigma, End(E) \otimes K)$ when there is no puncture. The discussion above suggests that a Higgs bundle could be understood as an element in the cotangent bundle of the moduli space of a holomorphic vector bundle

$$(V, \varphi) \in T_V^* \mathscr{M}_E^{r,d}. \tag{4.69}$$

A precise argument shows that there is a birational equivalence[①]

$$T^* \mathscr{M}_E^{r,d} \cong M_{Higgs}. \tag{4.70}$$

Spectrum Data: Spectrum Curve and Holomorphic Line bundle Since the holomorphic Higgs field φ would suffer the $\mathscr{G}_{\mathbb{C}}$ gauge invariance, it is helpful to repackage the data of the holomorphic Higgs field φ to its gauge invariant polynomials. As a rough idea, it might motivate us to define the formal characteristic polynomial for φ_z as

$$\det\big(x - \varphi_z\big) = 0, \tag{4.71}$$

such that the components of this characteristic polynomial are the full set of gauge-invariant polynomials in φ_z. However, this definition has an obvious drawback–it depends on the choice of the coordinate z for the Riemann surface. This problem could be resolved if we regard the formal parameter x also as the component of a one-form $\lambda = x\, dz$. Then the coordinate change would only contribute an overall (non-vanishing) constant to the characteristic polynomial.

Let us make this idea more precise. Given a Higgs field φ, we would like to consider the pull back $\pi^*\varphi$, with π the projection map of the canonical line bundle $K = T^*\Sigma_{g,n}$. On the canonical line bundle, one has the tautological one form λ, which locally satisfies $\lambda = x\, \mathrm{d}z$. With that data, we could define the *spectrum curve* S as a curve inside $T^*\Sigma$

$$S = \{p \in T^*\Sigma \mid \det\big(\lambda - \pi^*\varphi\big)\Big|_p = 0\} \subset T^*\Sigma. \tag{4.72}$$

① Equivalent as a variety after deleting some complex submanifold.

The spectrum curve captures the gauge-invariant data of the Higgs field. When $G = SU(N)$, the spectrum curve generically takes

$$\left(x^N - \sum_{i=2}^{k} p_i(\varphi_z)x^{n-i}\right)dz^{\otimes N} = 0. \tag{4.73}$$

.

Actually, what we can get now is more than the spectrum curve. For each (generic) point $p \in S$, we can associate it with a one-dimensional eigen subspace of Φ_z denoted as L_p. Collecting the eigen subspace over all the points in S defines a holomorphic line bundle $L = \bigcup_{p\in S} L_p$. L is related to the holomorphic vector bundle E by a sheaf theoretical push-forward $\pi_* L = E$. This line bundle should satisfy certain properties related to the choice of group $G_{\mathbb{C}}$. For instance, when $G = SL_n$, the holomorphic vector bundle E must have a trivial determinant $\det E \cong \mathcal{O}(\Sigma_{g,n})$, and thus the line bundle L should satisfy $\det \pi_* L = \det E \cong \mathcal{O}(\Sigma_{g,n})$.

The non-abelian flat gauge field on the holomorphic line bundle can also be mapped to an abelian $U(1)$ gauge field on the line bundle L. Mathematically speaking, this procedure is called the abelianization, where a non-abelian gauge field on Σ splits into an abelian gauge field on S.

A careful sheaf-theoretic argument states that given the spectrum curve S and a line bundle L over S satisfying the condition related to the choice of group $G_{\mathbb{C}}$, one could recover the original Higgs bundle (Σ, P, φ). As a result, there is a one-to-one correspondence between the pair (L, S) and (E, A, φ). (L, S) is called the spectrum data for a Higgs bundle.

Hitchin Fibration In this part, we will introduce a map crucial for the study of the Hitchin moduli space, which is called the Hitchin fibration[29]. It suggests that the Hitchin moduli space could be understood as the total space of an elliptic fibration, which would be useful for understanding the topological property of the Hitchin system.

The fibration structure arises from the one-to-one correspondence between the pair (L, S) and (E, A, φ). To describe the moduli space of Higgs bundles, one can naturally divide the problem into two steps. First, characterize the space of all possible spectral curves S. Second, for each such spectral curve S, describe the space of line bundles L defined over it. This perspective clearly reveals the fibration structure: the moduli space of spectral curves serves as the *base*, and the *fiber* over a point S on the base is the moduli space of line bundles L on the corresponding spectral curve.

Let us make this idea more concrete. The space of all possible Spectrum curves is controlled by the coefficients in the curve, namely $p_k(\varphi) \in H^0(\Sigma, T^*\Sigma^{\otimes k})$. Therefore, the moduli space of spectrum curve is the affine space $\mathscr{A} = \oplus_{i=2}^{k} H^0(\Sigma, T^*\Sigma^{\otimes i})$, called the Hitchin base. The fibration map, called the Hitchin fibration h, is defined as

$$\begin{aligned} h : \mathscr{M}_{g,n} &\longrightarrow \mathscr{A} = \oplus_{i=1}^{k} H^0(\Sigma, T^*\Sigma^{\otimes i}) \\ (E, A,\varphi) &\longmapsto (p_2(\Phi), \cdots, p_k(\Phi)). \end{aligned} \tag{4.74}$$

In the practical calculation, it is convenient to introduce another basis of invariant polynomials given by $\mathrm{Tr}\left(\varphi_z^k\right)$. ①

① Invariant polynomial occurs in another place where physicists might be more familiar with, called the characteristic classes. In the theory of characteristic classes, one knows that some of the characteristic classes can be

For each point $S \in \mathscr{A}$, the fiber $h^{-1}(S)$ describes all the line bundles over S satisfying certain properties. For $G = GL(N)$, there is no extra constraint for L, and the space of holomorphic line bundles is well studied back to the time of Jacobi. The Abel-Jacobi theorem tells us that for a compact complex curve, the space of line bundles with vanishing first Chern class is described by the Jacobian variety $Jac(S)$, which is essentially a torus T^{2g+2} with g the genus of the surface. Therefore, the Hitchin fibration is simply a torus fibration at generic points.

When $G = SL(N)$, the line bundles on the spectrum curve are required to satisfy $\det \pi_* L \cong \mathcal{O}$, which would be a subvariety of the Jacobian variety called the Prym variety $Prym(S) \subset Jac(S)$. When S is a smooth algebraic curve, $Prym(S)$ is also a torus with genus T^{2g}.

Physical Interpretation After introducing the properties of the moduli space of Higgs bundles, let's now consider the interpretation from physics.

As we have discussed before, the Hitchin system naturally arises in the Class S construction, where we compactify the N stack of M_5 branes on a Riemann surface. The adjoint scalar on the $6d$ tensor multiplet, after topological twist, combines into a holomorphic one form $\varphi_z = \varphi_4 + i\varphi_5$, which encodes the position of the M_5 brane in the $T^*\Sigma$ direction.

As mentioned in the construction of linear quiver theory, the shape of the M_5 brane has to be described by a complex algebraic curve inside $T^*\Sigma$ in order to preserve supersymmetry. In the UV theory where N copies of M_5 brane stacks together, the position of the branes is the same and we could normalize such that $\langle \varphi_z \rangle = 0$. The shape of the brane is thus described by the curve

$$x^N = 0. \tag{4.75}$$

On the other hand, when the brane becomes separated in the $T^*\Sigma$ direction, φ_z develops a non-trivial vacuum expectation value which can be locally diagonalized as $\varphi_z = \mathrm{diag}\{x_1(z), x_2(z) \ldots, x_N(z)\}$. The shape of the brane is described by the curve

$$S : \det\bigl(x - \varphi_z\bigr) = (x - x_1(z)) \ldots (x - x_N(z)) = 0. \tag{4.76}$$

As a consequence, the concept of spectrum curve matches neatly with the configuration of the M_5 brane in the $T^*\Sigma$ direction. From the field theory description, the shape of the M_5 brane in the internal direction is identified with the Seiberg-Witten curve of the 4d theory, which suggests that *the spectrum curve of a Hitchin system is the Seiberg-Witten curve for the low-energy effective theory.* The Hitchin base $\mathscr{A}$ parametrizing the spectrum curves is thus identified with the Coulomb branch moduli space(u-plane) in the Seiberg-Witten theory.

However, at this point, the role of the elliptic fibration is still unclear. As we would see, in the section: compactified to 3d, the elliptic fibration arises when we consider the four-dimensional field theory on $\mathbb{R}^3 \times S^1_R$. The expectation value of the Wilson-t'Hooft line contributes $2g$ extra scalars in the $3d$ theory, which enhances the Coulomb branch from the Hitchin base to the entire Hitchin moduli space up to a discrete data from the Dirac quantization condition.

represented by the cohomology class of invariant polynomials of curvatures. One may recall the definition of Chern Class $c(F) = \det\Bigl(\lambda I - \frac{i}{2\pi}F\Bigr)$ and Chern Character $ch(F) = \mathrm{Tr}\Bigl(exp(\frac{i}{2\pi}F)\Bigr)$ via curvature polynomial in order to generate two basis of invariant polynomials. Here, the base change follows from the same reason.

4d Field Theory	Hitchin System
Coulomb Branch on $\mathbb{R}^4$	Hitchin Base $\mathscr{A}$
Coulomb Branch on $\mathbb{R}^3 \times S^1$	Hitchin Moduli Space $\mathscr{M}_{g,n}$/L
Seiberg-Witten Curve	Spectrum Curve S

Table 4-1 Kodiara Singular Fiber

	g	$\mathfrak{g}$	ord(a)	ord(b)	ord(Δ)	sample equation
I_k	$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$	$\mathfrak{su}(k)$	0	0	k	$y^2 = x^3 - 3x + 2 + z^k$
II	$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$	$*$	1	1	2	$y^2 = x^3 + z$
III	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$\mathfrak{su}(2)$	1	2	3	$y^2 = x^3 + zx$
IV	$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$	$\mathfrak{su}(3)$	2	2	4	$y^2 = x^3 + z^2$
I_k^*	$\begin{pmatrix} -1 & -k \\ 0 & -1 \end{pmatrix}$	$\mathfrak{so}(2k+8)$	2	3	$k+6$	$y^2 = x^3 - 3z^2x + 2z^3\left(1 + z^k\right)$
IV^*	$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$	$\mathfrak{e}_6$	3	4	8	$y^2 = x^3 + z^4$
III*	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\mathfrak{E}_7$	3	5	9	$y^2 = x^3 + z^3x$
II*	$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$	$\mathfrak{E}_8$	4	5	10	$y^2 = x^3 + z^5$

4.5.5 More on the Moduli Space of $G_{\mathbb{C}}$ Flat Connection

In this section, we will elaborate more on the properties of the moduli space of $G_{\mathbb{C}}$ flat connections on the Riemann surface $\Sigma_{g,n}$.

Flat Connection = Monodromy Representation To proceed, we will introduce an important fact that the flat $G_{\mathbb{C}}$-connections on a Riemann surface are in one-to-one correspondence with the space of monodromy representations, i.e. the space of homomorphism from the fundamental group of a Riemann surface to $G_{\mathbb{C}}$, which is also referred to as the character variety. Specifically, we have the identification

$$\mathscr{M}_{flat}(\Sigma_{g,n}, G_{\mathbb{C}}) \cong \mathrm{Hom}(\pi_1(\Sigma_{g,n}), G_{\mathbb{C}})//G_{\mathbb{C}}, \tag{4.77}$$

where on the right-hand side, the group $G_{\mathbb{C}}$ acts by conjugation.

The intuitive reasoning behind this fact is as follows: a flat connection on a Riemann surface is determined by its holonomy, $\mathrm{Hol}(\alpha)$, which assigns to each loop α on the surface an element of the group $G_{\mathbb{C}}$. This assignment must respect the fundamental group relations of the loops, meaning that the holonomies must define a homomorphism from the fundamental group $\pi_1(\Sigma_{g,n})$ to $G_{\mathbb{C}}$. Therefore, flat connections correspond to homomorphismsHom$(\pi_1(\Sigma_{g,n}), G_{\mathbb{C}})$. This homomorphism is exactly what we call monodromy representations. Furthermore, gauge transformations act on the holonomy by conjugation. This leads to the quotient over $G_{\mathbb{C}}$, which accounts for the equivalence classes of flat connections under gauge transformations. Thus, the space of flat connections is naturally identified with the quotient $\mathrm{Hom}(\pi_1(\Sigma_{g,n}), G_{\mathbb{C}})//G_{\mathbb{C}}$.

The remaining task is to describe the space $\mathrm{Hom}(\pi_1(\Sigma_{g,n}), G_{\mathbb{C}})//G_{\mathbb{C}}$. Let us take a basis of the cycles $(A_I, B^I, Z_K)_{I=1,\ldots,g;\ K=1,\ldots,n}$ in $H_1(\Sigma_{g,n})$ such that the intersection pairing $\langle A^I, B_J\rangle = \delta^I_J$, and Z_J is the small loop along the punctures with trivial intersection with other loops. The fundamental group of the Riemann surface could be described by

$$\pi_1(\Sigma_{g,n}) \cong \langle A_I, B^I, Z_K \mid \prod_{I=1}^{g}[A_I, B^I] = \prod_{K=1}^{n} Z_K\rangle, \tag{4.78}$$

where $[-,-]$ is the commutator defined as

$$[A, B] := ABA^{-1}B^{-1}. \tag{4.79}$$

The monodromy representation of the Riemann surface is thus described by

$$\mathcal{M}_{flat}(\Sigma_{g,n}, G_{\mathbb{C}}) = \langle M_{A_I}, M_{B^I}, M_{Z_K} \in G_{\mathbb{C}} \mid \prod_{I=1}^{g}[M_{A_I}, M_{B^I}] = \prod_{K=1}^{n} M_{Z_K}\rangle//G_{\mathbb{C}}\ . \tag{4.80}$$

We could take the flat connection as a subspace inside the ambient space $\langle M_{A_I}, M_{B^I}, M_{Z_K} \in G_{\mathbb{C}}\rangle/G_{\mathbb{C}}$. This ambient space is indeed a finite-dimensional affine space. When $G_{\mathbb{C}}$ is a matrix Lie group, the affine space is generated by the trace functions

$$\mathrm{Tr}\Big(M^{\#}_{A_I} M^{\#}_{B^I} M^{\#}_{Z_K} \cdots\Big) \tag{4.81}$$

subject to the trace relations derived from the Cayley-Hamilton theorem

$$M^n - \mathrm{Tr}(M)M^{n-1} + \cdots + (-1)^n \det(M)id = 0, \quad \forall M \in G_{\mathbb{C}}. \tag{4.82}$$

One can prove that only a finite number of trace functions are algebraically independent, and the ring of trace functions is a freely generated polynomial ring. The fundamental group relation, on the other hand, imposes polynomial relations between trace variables. Therefore, $\mathcal{M}_{flat}(\Sigma_{g,n}, G_{\mathbb{C}})$ is an *affine variety*(zeros of polynomial equations), which allows us to use the algebraic geometry techniques to study this space.

Symplectic Structure For $G_{\mathbb{C}} = SL(N)$, there is a holomorphic symplectic structure on the flat connections is given by Goldman

$$\{\mathrm{trHol}(\alpha), \mathrm{trHol}(\beta)\} = \sum_{p\in\alpha\cap\beta} \epsilon(p;\alpha,\beta)\left[\mathrm{trHol}\left(\alpha_p\beta_p\right) - \frac{1}{N}\mathrm{trHol}(\alpha_p)\,\mathrm{trHol}(\beta_p)\right] \tag{4.83}$$

where $\epsilon(p;\alpha,\beta)$ is the intersection number of the loop α, β at p, and α_p, β_p means the loop with a base point at p.

4.5.6 Example: $SU(2)$ Hitchin Moduli Space over Fourth-Punctured Sphere

Complex structure J^ζ: Flat $SL(2,\mathbb{C})$ Connection over $\Sigma_{0,4}$ These monodromy matrices naturally define an $\mathrm{SL}(2,\mathbb{C})$ holonomy representation of the fundamental group of a four-punctured sphere

$$\mathscr{M}_{\text{flat}}(C_{0,4}, \mathrm{SL}(2,\mathbb{C})) = \langle M_1, M_2, M_3, M_4 \in \mathrm{SL}(2,\mathbb{C}) \mid M_1 M_2 M_3 M_4 = \mathrm{Id}\rangle // \mathrm{SL}(2,\mathbb{C}) , \tag{4.84}$$

where the quotient by $\mathrm{SL}(2,\mathbb{C})$ is taken with respect to conjugation.

To describe the character variety geometrically, we introduce holonomy variables as holomorphic functions on $\mathscr{M}_{\text{flat}}(C_{0,4}, \mathrm{SL}(2,\mathbb{C}))$:

$$\begin{aligned} &x = -\operatorname{Tr}\big(M_1 M_2\big) , \quad y = -\operatorname{Tr}\big(M_1 M_3\big) , \quad z = -\operatorname{Tr}\big(M_2 M_3\big) , \\ &\overline{t_j} = \operatorname{Tr}\big(M_j\big) , \qquad (j = 1, 2, 3, 4) . \end{aligned} \tag{4.85}$$

These variables are subject to the trace identity[30]:

$$f(x, y, z) = -xyz + x^2 + y^2 + z^2 + \theta_1 x + \theta_2 y + \theta_3 z + \theta_4 = 0 , \tag{4.86}$$

where

$$\begin{aligned} \theta_1 &= \overline{t_1}\,\overline{t_2} + \overline{t_3}\,\overline{t_4} , \\ \theta_2 &= \overline{t_1}\,\overline{t_3} + \overline{t_2}\,\overline{t_4} , \\ \theta_3 &= \overline{t_1}\,\overline{t_4} + \overline{t_2}\,\overline{t_3} , \\ \theta_4 &= \overline{t_1}^2 + \overline{t_2}^2 + \overline{t_3}^2 + \overline{t_4}^2 + \overline{t_1}\,\overline{t_2}\,\overline{t_3}\,\overline{t_4} - 4 . \end{aligned} \tag{4.87}$$

These θ_i parameters are the characters of the SO(8) representations, which can be attributed to the SO(8) flavor symmetry present in the 4d $\mathcal{N} = 2$ SU(2) theory with $N_f = 4$. Thus, in complex structure J, the character variety $\mathscr{M}_{\text{flat}}(C_{0,4}, \mathrm{SL}(2,\mathbb{C}))$ is an affine variety described by this cubic equation. The four complex structure parameters (t_1, t_2, t_3, t_4) are identified as

$$t_j = \exp\big(-2\pi(\gamma_j + i\alpha_j)\big) , \qquad (j = 1, 2, 3, 4). \tag{4.88}$$

4.5.7 Example: Flat $SL(3)$ connection over $\Sigma_{0,3}$ and E_6 Enhancement in Minahan-Nemeschansky Theory

$$\pi(\Sigma_{0,3}) = \{A, B, C | ABC = id\}. \tag{4.89}$$

Let us define the parameters as

$$\begin{aligned} A_i &= \operatorname{Tr} M_A^i, \quad B_i = \operatorname{Tr} M_B^i, \quad C_i = \operatorname{Tr} M_C^i, \\ x &= \operatorname{Tr} M_A^{-1} M_C - A_1 C_2, \\ y &= \operatorname{Tr} M_A M_C^{-1} - A_2 C_1, \\ z &= \operatorname{Tr}\big(M_C M_B M_A\big)^{-1} - A_1 A_2 - B_1 B_2 - C_1 C_2 + 6. \end{aligned} \tag{4.90}$$

The character is then given by

$$\begin{aligned} 0 = f(x, y, z) = &- xyz + x^3 + \chi_1 x^2 + \big(\chi_3 - \chi_2\big)\, x + y^3 + \chi_2 y^2 + \big(\chi_4 - \chi_1\big)\, y \\ &+ z^2 + \big(\chi_5 - 6\big)\, z + \big(\chi_6 - 3\chi_5 + 9\big) . \end{aligned} \tag{4.91}$$

where $\{\chi_i\}_{i=1,\ldots 6}$ are the characters of E_6 representations associated with each nodes in the Dynkin diagram, which writes

$$
\begin{aligned}
\chi_1 &= A_1B_2 + A_2C_1 + B_1C_2,\\
\chi_2 &= A_2B_1 + A_1C_2 + B_2C_1,\\
\chi_3 &= A_2A_1B_2C_1 + A_1B_1B_2C_2 + A_2B_1C_1C_2 + A_1^2B_1 + B_1^2C_1\\
&\quad + A_2B_2^2 + A_1C_2^2 + A_2^2C_2 + B_2C_1^2 - 2A_2B_1 - 2B_2C_1 - 2C_2A_1,\\
\chi_4 &= A_2A_1B_1C_2 + A_2B_1B_2C_1 + A_1B_2C_1C_2 + A_2^2B_2 + A_1B_1^2\\
&\quad + A_2C_1^2 + A_1^2C_1 + B_1C_2^2 + B_2^2C_2 - 2A_1B_2 - 2B_1C_2 - 2C_1A_2,\\
\chi_5 &= A_1B_1C_1 + A_2B_2C_2 + A_1A_2 + B_1B_2 + C_1C_2 - 3,\\
\chi_6 &= A_1A_2B_1B_2C_1C_2 + A_2^2B_2^2C_1 + A_1^2B_1C_1^2 + A_2B_1^2C_1^2 + A_1^2B_1^2C_2 + A_2^2B_2C_2^2\\
&\quad + A_1B_2^2C_2^2 + A_1^2A_2B_1C_1 + A_1B_1^2B_2C_1 + A_1A_2^2B_2C_2 + A_2B_1B_2^2C_2 + A_1B_1C_1^2C_2 + A_2B_2C_1C_2^2\\
&\quad - 2A_2^2B_1C_1 - 2A_2B_1^2C_2 - 2A_1^2B_2C_2 - 2A_1B_2^2C_1 - 2A_1B_1C_2^2 - 2A_2B_2C_1^2\\
&\quad + A_1A_2B_1B_2 + A_1A_2C_1C_2 + B_1B_2C_1C_2\\
&\quad + A_1^3 + A_2^3 + B_1^3 + B_2^3 + C_1^3 + C_2^3 - 6A_1A_2 - 6B_1B_2 - 6C_1C_2 + 9.
\end{aligned}
\tag{4.92}
$$

Therefore, the $SU(3)^{\times 3}$ flavor symmetry of the theory is enhanced into E_6.

Chapter 5

BPS spectrum and BPS Line Operators

BPS states in 4D N=2 theory could be constructed by M_2 brane wrapping on the cycles on the internal Riemann surface. The central charge of the state is counted via

$$Z_{ij} = T_{M_2} \int dx \wedge dz = \int_{x_j(z)}^{x_i(z)} x dz = \oint_{\partial D} \lambda. \tag{5.1}$$

On the other hand, the mass of the brane is tension times area

$$M_{ij} = T_{M_2} \int |\mathrm{d}x \wedge \mathrm{d}z| = \oint_{\partial D} |\lambda| \tag{5.2}$$

Therefore, the BPS Bound $M = |Z|$ is saturated, if the Seiberg-Witten differential has a constant phase factor such that $e^{-i\theta}\lambda \in \mathbb{R}_+$. Fixing θ, this condition defines a foliation of the Riemann surface, and the BPS states with finite mass correspond to the leaf with finite length.

5.1 BPS Line operators and Framed BPS States

An important ingredient in the gauge theory is the Wilson operator, which represents the infinitely massive charged particle propagating in the field. The Wilson operator is defined as

$$W_l = \rho_R \mathscr{P}\,\mathrm{Exp}\left(i \int_l A\right). \tag{5.3}$$

In a supersymmetric gauge theory, in order to represent the infinitely massive BPS multiplets, we need to consider the line operators preserving half supersymmetry, which is called the BPS Wilson operator. A typical BPS Wilson operator in the 4d $\mathcal{N} = 2$ theory writes

$$L_{R,\zeta,l} = \rho_R \mathscr{P}\,\mathrm{Exp}\left(i \int_l (A + \frac{\varphi}{\sqrt{2}\zeta} + \frac{\zeta}{\sqrt{2}}\overline{\varphi})\right). \tag{5.4}$$

Unlike the Wilson operator, which can be defined along a generic path l, a general choice of l does not preserve half of the supersymmetry. Therefore, we will focus on BPS line operators that extend along the time direction $\mathbb{R}_0$ or its compactification S^1_R, corresponding to infinitely heavy static particles. To streamline the notation, we will omit the path label l from here on. In this case, the BPS Wilson operator (5.4) would be invariant under the SUSY transformation generated by $Q_\alpha^{(\zeta,+)I}$, and thus commute with the supercharge $Q_\alpha^{(\zeta,+)I}$.

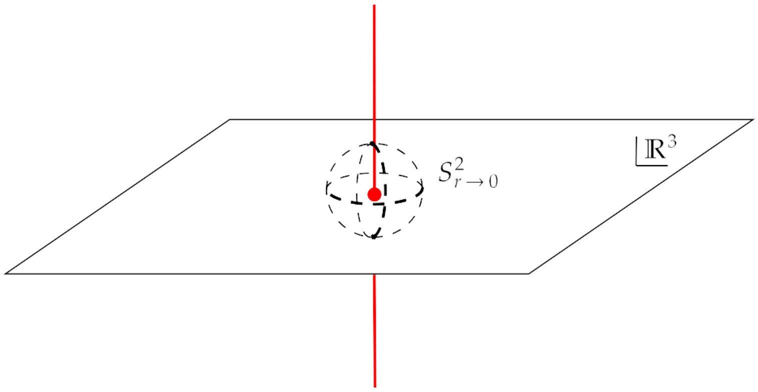

Figure 5-1 Limiting sphere after line operator insertion

We could also define the supersymmetric t'Hooft operator and dyonic operator, which have electromagnetic charge γ. Those operators $L_{\gamma,v}$ could be defined as imposing the half BPS boundary condition for the field configuration on the limiting sphere around the insertion point, such that

$$-\int_{S^2_{r\to 0}} d^2S\, n^i(E_i - \frac{\theta}{8\pi^2}B_i) = g^2 n_e, \quad -\int_{S^2_{r\to 0}} d^2S\, n^i B_i = 4\pi n_m. \tag{5.5}$$

The insertion of the BPS line operator $L_{R,\zeta,l}$ makes a difference to the BPS bound. This is because, in the presence of the BPS line operator, the phase ζ cannot be chosen at will, but rather should be taken as a fixed value. The algebra of supercharge (2.29) and unitary thus implies that the BPS bound is shifted into

$$M \geq |Z| \quad \longrightarrow \quad M \geq -\Re\frac{Z}{\zeta}. \tag{5.6}$$

This is called the *framed BPS bound*. In the presence of a BPS line operator, we could quantize the theory over a spatial slice, and the resulting Hilbert space $\mathscr{H}_{L,\gamma,\zeta}$ would be twisted by the line operator. The states saturating this bound would be called the framed BPS states.

The concept of mutual locality applies to the line operators. If

$$\langle \gamma, \gamma' \rangle \neq 0, \tag{5.7}$$

then two line operators are called mutually non-local.

Consider the line operator wrapping on S^1, its expectation value could be expressed as the trace over the twisted Hilbert space $\mathscr{H}_{u,L_\zeta}$. Therefore, the line operator expectation value could be represented by

$$\left\langle L_{(\mathbf{p},\mathbf{q})} \right\rangle := \mathrm{Tr}_{\mathscr{H}_{(\mathbf{p},\mathbf{q})}}(-1)^{\mathrm{F}} e^{-2\pi RH} \prod_f e^{\mathrm{F}_f m_f} \tag{5.8}$$

One could generalize the partition function by incorporating the fugacity of $U(1)$ rotation symmetry, as reminiscent of the protected spin character.

$$\left\langle L_{(\mathbf{p},\mathbf{q})} \right\rangle := \mathrm{Tr}_{\mathscr{H}_{(\mathbf{p},\mathbf{q})}}(-1)^{\mathrm{F}} e^{-2\pi RH} e^{\epsilon_+ \mathscr{J}_3} \prod_f e^{\mathrm{F}_f m_f} \tag{5.9}$$

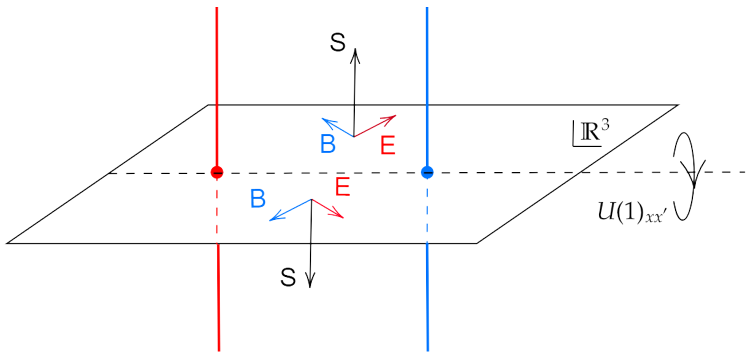

Figure 5-2 Nontrivial angular momentum of the system in the presence of two mutually non-local particle.

The insertion of the rotation generator $e^{\epsilon_+ \mathcal{J}_3}$, in the path integral formalism, imposes a twisted boundary condition such that the $x-y$ plane, after evolution by $2\pi R$ in the time direction, identify with itself by a $\mathcal{J}_3$ rotation. It can be understood as turning on the Ω background on $\mathbb{R}^2$, with Ω background parameter ϵ_+.

5.2 Line Operator Algebra

Just like the operators in the quantum mechanics, which forms the algebra of observables. One could also define the algebraic structure over the line operators.

The summation of two line operators is defined as

$$\mathcal{H}_{L+L'} = \mathcal{H}_L \oplus \mathcal{H}_{L'} \tag{5.10}$$

At the partition function level, it's just the sum over two partition functions.

The multiplication of the line operators could be defined by the operator product expansion, such that the following is satisfied

$$\langle LL'(x)\rangle = \lim_{x_0 \to x, x_1 \to x} \langle L(x_0)L'(x_1)\rangle \tag{5.11}$$

In the present definition, the algebra is certainly commutative. At the Hilbert space level, the twisted Hilbert space after the line operator multiplication is defined as

$$\mathcal{H}^{\mathrm{BPS}}_{L_{\vec{x}} L'_{\vec{x}'}, \gamma_0} = \bigoplus_{\gamma+\gamma'=\gamma_0} \mathcal{H}^{\mathrm{BPS}}_{L,\gamma} \otimes \mathcal{H}^{\mathrm{BPS}}_{L',\gamma'} \otimes N_{\gamma,\gamma'}. \tag{5.12}$$

where $N_{\gamma,\gamma'}$ is a one-dimensional representation of the $U(1)_{xx'}$ labeled by $\langle\gamma,\gamma'\rangle$.(See Fig.5-2)

In a U(1) supersymmetric gauge theory, the line operator algebra is specified by

$$\mathcal{Y}_\gamma \mathcal{Y}_{\gamma'} = (-1)^{\langle\gamma,\gamma'\rangle} \mathcal{Y}_{\gamma+\gamma'}. \tag{5.13}$$

5.3 BPS Bound States and Denef's Formula for Bound State Radius

In this section, we will discuss the microscopic description for the BPS bound states. We would consider a light BPS particle with charge γ_h propagating in the background of a heavy static BPS particle with charge γ_c.

In section 3.8.2, we presented the BPS equation for a static monopole/dyon

$$E_i - iB_i = \sqrt{2}D_i\left(\frac{\varphi}{\zeta}\right), \quad [\varphi, \varphi^\dagger] = 0. \tag{5.14}$$

Consider the gauge where φ is point-wise diagonalized

$$\varphi(x) = \begin{bmatrix} a(x) & 0 \\ 0 & -a(x) \end{bmatrix}. \tag{5.15}$$

Since the $U(1)$ monopole/dyon generates the magnetic field proportionate to σ_3. Then the BPS equation reduces to

$$E_i^{U(1)} - iB_i^{U(1)} = \sqrt{2}\zeta^{-1}\partial_i a(x) \tag{5.16}$$

The typical asymptotic behavior of the field strength in the presence of a static dyon could be represented by

$$\begin{aligned} F &= \frac{1}{2}\left(-\omega_s \otimes \gamma_c + \omega_d \otimes \mathcal{I}(\gamma_c)\right) \\ \omega_s &:= \sin\theta d\theta d\phi \\ \omega_d &:= \frac{drdt}{r^2} \end{aligned} \tag{5.17}$$

Plugging the ansatz into the BPS equation, we have

$$2\sqrt{2}\partial_l\left(\frac{a}{\zeta}\right) = \langle q_e, (i+\mathcal{I})\gamma_c\rangle\partial_l\left(\frac{1}{r}\right) \tag{5.18}$$

Integrating both part along the space slice, we arrive at the equation

$$2\sqrt{2}\zeta^{-1}a(r) = \frac{\langle q_e, (i+\mathcal{I})\gamma_c\rangle}{r} + 2\sqrt{2}\zeta^{-1}a(\infty). \tag{5.19}$$

where q_e is the unit electric charge. We could apply the electromagnetic duality to promote the equation to the duality invariant version. It amounts to replacing

$$\sqrt{2}a(r) = Z_{q_e}(r) \to Z_{\gamma_h}(r), \quad \langle q_e, (i+\mathcal{I})\gamma_c\rangle \to \langle \gamma_h, (i+\mathcal{I})\gamma_c\rangle \tag{5.20}$$

Then we arrive at the following equations

$$\begin{aligned} 2\,\mathrm{Re}\left[\zeta^{-1}Z(\gamma_h; u(r))\right] &= \frac{\langle\gamma_h, \mathcal{I}(\gamma_c)\rangle}{r} + 2\,\mathrm{Re}\left[\zeta^{-1}Z(\gamma_h; u)\right] \\ 2\,\mathrm{Im}\left[\zeta^{-1}Z\left(\gamma_h; u(r)\right)\right] &= \frac{\langle\gamma_h, \gamma_c\rangle}{r} + 2\,\mathrm{Im}\left[\zeta^{-1}Z\left(\gamma_h; u\right)\right] \end{aligned} \tag{5.21}$$

The above analysis focuses on the BPS field equations. In the following, let us investigate the dynamics of the light BPS particle propagating in the field. It could be approximated by the action

$$S = \int ds|Z(\gamma_h, u(r))| + \int \langle\gamma_h, A\rangle. \tag{5.22}$$

where the first term describes the dynamics of the BPS particle with effective mass $M(r) = |Z(\gamma_h, u(r))|$ and the second term describes the coupling to the gauge field. When γ_c is also static, the energy of the light particle is given by

$$\begin{aligned}
E &= |Z(\gamma_h, u(r))| + \frac{\langle \gamma_h, \mathcal{I}(\gamma_c)\rangle}{2r} \\
&= |Z(\gamma_h, u(r))| + \mathrm{Re}\left[\zeta^{-1} Z(\gamma_h; u(r))\right] - \mathrm{Re}\left[\zeta^{-1} Z(\gamma_h; u)\right] \\
&= |Z(\gamma_h, u(r))|(1 - \cos\left(\alpha_c - \alpha_h(r)\right)) - \mathrm{Re}\left[\zeta^{-1} Z(\gamma_h; u)\right] \geq -\mathrm{Re}\left[e^{-i\alpha_c} Z(\gamma_h; u)\right].
\end{aligned} \tag{5.23}$$

where we identify $\zeta = e^{i\alpha_c}$. The energy bound is saturated when the phase α_c and the phase of $Z(\gamma_h; u(r))$ are aligned. When this condition is saturated, the bound state radius can be derived from (5.21)

$$R_{12} = \frac{1}{2}\langle \gamma_h, \gamma_c\rangle \frac{1}{\mathrm{Im}\, Z_{\gamma_h} e^{-i\alpha_c}}. \tag{5.24}$$

The lesson here is that the potential of a heavy dyonic particle *has a minimum at a finite radius* R_{12}, which means that BPS particles *could form a bound state*. On the other hand, when the phase of Z_{γ_h} is aligned with α_c, the bound state radius goes to infinity.

The limitation of this argument is that we have regarded γ_c to be infinitely heavy in order to treat it as the background. When γ_c is of finite mass, the BPS bound state radius is given by Denef

$$R = \frac{1}{2}\langle \gamma_1, \gamma_2\rangle \frac{\left|Z\left(\gamma_1; u\right) + Z\left(\gamma_2; u\right)\right|}{\mathrm{Im}\, Z\left(\gamma_1; u\right)\overline{Z\left(\gamma_2; u\right)}}. \tag{5.25}$$

The technique Denef applied is to regard 4D supersymmetric field theory as the limit of $4d$ $\mathcal{N} = 2$ SUGRA, decoupling the gravitational interactions. Under this limit, the dyonic BPS black holes become dyonic states in the field theory.

When the phase of central charge $Z(\gamma_1; u)$ and $Z(\gamma_2; u)$ is aligned, the bound state radius goes to infinity. It provides a microscopic interpretation for BPS wall crossing

$$|Z_{\gamma_1+\gamma_2}| \longleftrightarrow |Z_{\gamma_1}| + |Z_{\gamma_2}|. \tag{5.26}$$

5.4 KS Wall-Crossing Formula and BPS Spectrum

At a generic point on a wall of marginal stability the symplectomorphisms which enter the WCF are generated by a two-dimensional lattice of charges, $\gamma = (p, q) \in \mathbb{Z}^2$ with canonical symplectic form $\left\langle (p, q), \left(p', q'\right)\right\rangle = pq' - qp'$. We write correspondingly $X_{1,0} = x$, $X_{0,1} = y$. The symplectomorphisms $\mathcal{K}_{p,q}$ are then determined by their action on x and y, which is explicitly

$$\mathcal{K}_{p,q} : (x, y) \to \left(\left(1 - (-1)^{pq} x^p y^q\right)^q x, \left(1 - (-1)^{pq} x^p y^q\right)^{-p} y\right) \tag{5.27}$$

Then there is an identity

$$\mathcal{K}_{-1,0}\mathcal{K}_{0,1} = \mathcal{K}_{0,1}\mathcal{K}_{-1,1}\mathcal{K}_{-1,0}. \tag{5.28}$$

Therefore, after crossing the wall of marginal stability with charge particle $(-1, 0)$ and $(0, 1)$, there is a dyonic bound state with charge $(-1, 1)$ generated. Also, it shows that a single monopole $(0, 1)$ cannot be bound to more than one particle with charge $(-1, 0)$.

5.5 Counting BPS States via Superconformal Index

In Section 2 we have stated that the 4d $\mathcal{N} = 2$ superconformal algebra is $\mathfrak{su}(2,2|2)$. Fermionic generators are supercharges $(Q^I_\alpha, \bar{Q}^{\dot{\alpha}}_I)$ and superconformal charges $(S^I_\alpha, \bar{S}^{\dot{\alpha}}_I)$. The states annihilated by $\bar{Q}^2_{\dot{-}}$ and its conformal superpartner $\bar{S}^{\dot{-}}_2$ is defined as the 1/8 BPS state, as they are killed by 1/8 supercharges.

The superconformal index serves the purpose of counting 1/8 BPS states. The strategy is as follows. Observed that

$$\delta = \{\bar{S}^{2\dot{-}}, \bar{Q}_{2\dot{-}}\} = \frac{1}{2}D + M^{\dot{-}}{}_{\dot{-}} + R^1{}_1 = \frac{1}{2}D + M^{\dot{-}}{}_{\dot{-}} + \frac{1}{2}r - R. \tag{5.29}$$

For the conventional reason, we write the eigenvalue of δ as

$$W_\delta = \frac{1}{2}(\Delta - 2j_2 - 2R + r). \tag{5.30}$$

Recall that P_μ and K_μ are BPZ conjugate of each other in the ordinary CFT. By straight calculation in superfield formalism or educative guess, it is easy to see that their superpartners, Q and S are BPZ conjugate of each other. Therefore, δ is semipositive. Once a state saturate the bound $\delta |\psi\rangle = 0$, then $\bar{S}^{1\dot{-}} |\psi\rangle = 0$ and $\bar{Q}_{1\dot{-}} |\psi\rangle = 0$, i.e. the state is 1/8 BPS.

To count the kernel of δ, a clever way is to consider δ as the Hamiltonian of the system and calculate the Witten index

$$I = \mathrm{tr}_{\mathcal{H}} \left((-1)^F e^{-2\beta\delta} \right). \tag{5.31}$$

To avoid some square root in the result, we add an extra 2 to the exponential. It can be understood as a twisted version of partition function. By standard argument, the Witten index enjoy some topological property. It is invariant under the continuous change of β unless wall-crossing happens.

The index tells us the number of 1/8 BPS state in a straightforward way. However, those 1/8 BPS states could possess more good quantum number, which is endowed by the generators in superconformal algebra that commutes with δ. Those generators form a subalgebra $C(\delta)$, the centralizer of δ. The experiences in statistical mechanics motivate us to add all the possible commuting hermitian generators by associating it with a new variable, which we called fugacity before, in the Witten index to real off more information of the system. This complicated partition function is called superconformal index.

By the abuse of notation between eigenvalue and its generator, the 4d superconformal index writes

$$\mathcal{I}^{4\mathrm{d}}_{\mathcal{H}}(p,q,t) = \mathrm{Tr}(-1)^F e^{-2\beta\delta} p^{j_1 - j_2 - r} q^{j_1 + j_2 - r} t^{R+r} \prod_a z_a^{f_a}, \tag{5.32}$$

where p, q, t are gauge fugacities, z_a are flavor fugacities. The superconformal index could be understood as the partition function of the theory on $S^3 \times S^1$, with a periodic boundary condition for the fermionic field.

Recall that 4d $\mathcal{N} = 2$ theory is built up by the building blocks: vector multiplets and hyper multiplets. The superconformal index for those multiplets serves as the building block for us to compute the index for more complicated theories. The contribution of two multiplets to the superconformal index writes:

1. Half-hypermultiplet with representation λ

$$\mathcal{I}^{4\text{d}}_{\frac{1}{2}H}(z;p,q,t) = \prod_{w\in\lambda}\prod_{i,j=0}^{\infty}\frac{1-z^{-w}p^{i+1}q^{j+1}/\sqrt{t}}{1-z^{w}\sqrt{t}p^{i}q^{j}} = \prod_{w\in\lambda}\Gamma\left(z^{w}\sqrt{t}\right), \tag{5.33}$$

where w runs over the weights of the representation λ.

2. Full-hypermultiplet with representation λ

$$\mathcal{I}^{4\text{d}}_{H}(z;p,q,t) = \Gamma(z^{\pm}\sqrt{t}). \tag{5.34}$$

3. Vector multiplet

$$\mathcal{I}^{4\text{d}}_{\text{vec}}(z;p,q,t) = \frac{\kappa^{\text{rk}G}\Gamma\left(\frac{pq}{t}\right)^{\text{rk}G}}{\left|W_G\right|}\prod_{\alpha\in\Delta}\frac{\Gamma\left(z^{\alpha}\frac{pq}{t}\right)}{\Gamma\left(z^{\alpha}\right)}, \quad \kappa=(p;p)(q;q), \tag{5.35}$$

where Δ represents the set of roots associated with the gauge group G, and $\left|W_G\right|$ is the order of the Weyl group of G.

Then, the 4d $\mathcal{N}=2$ superconformal index of a quiver gauge theory can be schematically expressed as the contour integral

$$\mathcal{I}^{4\text{d}}(\mathbf{a};q) = \oint_{|z|=1}\prod_{\text{gauge}}\frac{dz}{2\pi iz}\mathcal{I}^{4\text{d}}_{\text{vec}}(z;p,q,t)\prod_{\text{matter}}\mathcal{I}^{4\text{d}}_{\frac{1}{2}H}(z;p,q,t). \tag{5.36}$$

Here the contour integration over gauge fugacities picks up the terms independent of gauge fugacities. It can be understand as inserting a projection operator in the superconformal index in order to trace over only physical, gauge invariant Hilbert space.

5.6 Schur limit

Superconformal index admit various interesting limits. One of the limits is called the Schur Index. It is obtained by taking $q=t$. Under this limit,

$$\mathcal{I}^{4\text{d}}_{\text{Schur}}(p,q,t) = \text{Tr}(-1)^{F}e^{-2\beta\delta}p^{j_1-j_2-r}q^{R+j_1+j_2}\prod_{a}z_a^{f_a}, \tag{5.37}$$

In the subspace $\mathscr{H}_0$ where $\Delta-2j_2-2R+r=0$, the weight of q writes $\Delta-2j_1-2R-r$, which exactly comes from $\{S^{1+},\bar{Q}_{1\dot{+}}\}$, another pair of BPZ conjugates. As the result, we can go through the same argument over $\mathscr{H}_0$ and justify that only states annihilated by $S^{1+},\bar{Q}_{1\dot{+}},\bar{S}^{2\dot{-}},\bar{Q}_{2\dot{-}}$ contributes in the Schur index. Therefore, Schur index counts $\frac{1}{4}$ BPS states, which is also called Schur states.

Under the Schur limit, the superconformal index reduces as follows:

1. Hypermultiplet

$$\mathcal{I}^{4\text{d}}_{H} = \Gamma\left(z^{\pm}\sqrt{t}\right) := \Gamma(z\sqrt{t})\Gamma\left(z^{-1}\sqrt{t}\right) \quad \xrightarrow{t\to q} \quad \mathcal{I}^{\text{Schur}}_{H} = \frac{\eta(q)}{\vartheta_4(z)}, \tag{5.38}$$

where $\eta(q)$ is the Dedekind eta function.

2. Vector multiplet

$$\mathcal{I}_{\text{vec}}^{4\text{d}} = \frac{\kappa^{\text{rk}G}\Gamma\left(\frac{pq}{t}\right)^{\text{rk}G}}{|W_G|}\prod_{\alpha\in\Delta}\frac{\Gamma\left(z^\alpha\frac{pq}{t}\right)}{\Gamma(z^\alpha)} \xrightarrow{t\to q} \mathcal{I}_{\text{vec}}^{\text{Schur}} = \frac{\eta(q)^{2\text{rk}G}}{|W_G|}\prod_{\alpha\in\Delta}i\frac{\vartheta_1(z^\alpha)}{\eta(q)}. \tag{5.39}$$

By state operator correspondence, a Schur state corresponds to a Schur operator with the same quantum number inserted at the origin. In general CFT, we move the operator around by action translation operator on it. However, as translation operator does not commute with rotation generator M, it will break BPS condition.

As the result, one has to introduce the twisted translation operator, which preserves the 1/4 BPS conditions as follows

$$\mathcal{O}(z,\bar{z}) := e^{-zL_{-1}-\bar{z}\hat{L}_{-1}}\mathcal{O}(0)e^{+zL_{-1}+\bar{z}\hat{L}_{-1}}, \tag{5.40}$$

where

$$L_{-1} = P_{++}, \quad \hat{L}_{-1} = P_{--} + R_1^2.$$

The significance is that, due to the BPS condition, one cannot move Schur operator around in the full $\mathbb{R}^4$ as what we are familiar with in the construction of field operators in generic CFT. Rather, Schur operators can only be defined on a two dimensional plane inside $\mathbb{R}^4$.

The twisted translated Schur operator $\mathcal{O}(z,\bar{z})$ is also annihilated by the two supercharges

$$\mathbb{Q}_1 := Q_-^1 + \tilde{S}_1^-, \quad \mathbb{Q}_2 := \tilde{Q}_-^1 - S_1^-. \tag{5.41}$$

Moreover, the $\bar{z}$-dependence of $\mathcal{O}(z,\bar{z})$ turns out to be $\mathbb{Q}_{1,2}$-exact. Consequently, at the level of $\mathbb{Q}_{1,2}$-cohomology, the cohomology class $\mathcal{O}(z) := [\mathcal{O}(z,\bar{z})]$ depends on the location holomorphically in z. Furthermore, the OPE coefficients of these Schur operators (as cohomology classes) are also holomorphic, forming a 2d vertex operator algebra (VOA)/chiral algebra on the plane $\mathbb{C}_{z,\bar{z}}$. It is clear that $\{L_{-1}, L_0, L_1\}$ can be defined follow from the twisted translation operators and it's BPZ conjugate , which generates the $SL(2,\mathbb{C})$. One of the significant point is that this $SL(2,\mathbb{C})$ action can be enhanced to the full Virasoro symmetry by identifying some twisted $SU(2)_R$ current in 4d with the energy momentum tensor in 2d. The resulting 2d theory is thus a holomorphic 2d Virasoro CFT.

As the Schur index essentially counts the contribution of only Schur operators, it has to equal to the vacuum characters of the corresponding VOA, where each Schur operator contributes to one state by state operator correspondence. It turns out to be a powerful invariant for the 4d theory independent of the coupling constant, helping us to probe the physics of the strong coupling regime.

5.7 Application: Test of S-Duality in SQCD via Superconformal Index

As an example, the superconformal index can be used to check the S-duality of $\mathcal{N}=2$ SQCD.[31] From the class $\mathcal{S}$ perspective, this duality states that theories coming from the different pants decomposition of the Riemann surface are equivalent to each other. The simplest case of the duality is for $SU(2)$ theory with eight half-hypermultiplets.

These half-hypermultiplets transform as a representation of $SO(8)$. We can focus on the decomposition of the vector representation $\mathbf{8}_v = (\mathbf{2}_a \otimes \mathbf{2}_b) \oplus (\mathbf{2}_c \otimes \mathbf{2}_d)$ where the subgroup $SO(8) \supset SU(2)_a \times SU(2)_b \times SU(2)_c \times SU(2)_d$ is considered. The character of this representation is

$$\chi_{\text{hyp}} = \left(a + \frac{1}{a}\right)\left(b + \frac{1}{b}\right) + \left(c + \frac{1}{c}\right)\left(d + \frac{1}{d}\right). \tag{5.42}$$

Then the superconformal index can be written as

$$\mathcal{I}(a; p, q, t) = \kappa \Gamma(\frac{pq}{t}) \oint \frac{\mathrm{d}z}{4\pi i z} \frac{\Gamma(\frac{pq}{t} z^{\pm 2})}{\Gamma(z^{\pm 2})} \Gamma(\sqrt{t} z^{\pm} a^{\pm} b^{\pm}) \Gamma(\sqrt{t} z^{\pm} c^{\pm} d^{\pm}), \tag{5.43}$$

where we used the shorthand notation $\Gamma(x^{\pm}) = \Gamma(x)\Gamma(x^{-1})$.

If we change the vector representation to a spinor or conjugate spinor representation of the half-hypermultiplets

$$\mathbf{8}_s = (\mathbf{2}_a \otimes \mathbf{2}_c) \oplus (\mathbf{2}_b \otimes \mathbf{2}_d), \quad \mathbf{8}_c = (\mathbf{2}_a \otimes \mathbf{2}_d) \oplus (\mathbf{2}_b \otimes \mathbf{2}_c), \tag{5.44}$$

the outcome should be the same theory at different coupling scales. Since the superconformal index is not sensitive to the coupling, it is expected to be invariant under these changes. To be more precise, this triality implies a nontrivial invariance of the index under $b \leftrightarrow c$ and $b \leftrightarrow d$. A quick check can be performed at order by order expansion of the expression with respect to $x \equiv \sqrt{pq}$.

This simple example demonstrates the power of the superconformal index. Before this tool, the dualities in the theory could only be conjectured from physical interpretation. Now, with the help of the index, these dualities can be checked by explicit calculation. This provides solid proof for the duality properties. It is especially useful when testing such strong-weak coupling duality due to the coupling-independent property of the index. The coupling can be continuously tuned while the superconformal index is kept, and thus we could perform the calculation at the weak coupling regime to understand the strong coupling behavior of the theory.

Chapter 6
Compactification to Three Dimensions

In this section, we would consider the S^1 compactification of the Class $\mathcal{S}$ theory to 3d $\mathcal{N}=4$ theory. Upon this compactification, extra scalars in the vector multiplet occur, which enhances the Coulomb branch moduli space into a larger space. As we will see in the following, the Coulomb branch of the 3d theory coincides with the full Hitchin moduli space $\mathcal{M}_H$, which is a hyper-Kähler manifold. Under the low-energy limit, the theory is described by a non-linear sigma model. Under this perspective, the hyper-Kähler metric of the Hitchin moduli space could be captured by the coupling constants of this sigma model, which provides a physical way of constructing the hyper-Kähler metric. Remarkably, the continuity of the metric implies the Kontsevich-Soibelman wall crossing formula.

6.1 Low energy 3d $\mathcal{N}=4$ Sigma Model at Large Radius Limit

It turns out that the Sigma model knows a lot about the geometry of the Hitchin moduli space. The $\zeta \in \mathbb{CP}^1$ family of holomorphic differentials with respect to J^ζ could be captured by the supercharges $Q^{(\zeta)}$ in the SUSY algebra, which is rotated by the $SU(2)_C = SP(1)$ R symmetry. Under this identification, BPS objects with respect to $Q^{(\zeta)}$ define the holomorphic functions on the Hitchin moduli space.

GMN 4D wall crossing from 3D: Compactify 4D Coulomb branch low energy effective theory on S^1, and integrate over the massive quarks/monopoles to correct the effective sigma model. Read off the metric from the effective sigma model. Especially, KS wall crossing formula can be recovered by asking the metric to be smooth.

Field content in 3d: While compactifying the 4d N=2 theory on S^1, the field content reduces as follows. For the 4d vector multiplet $V \supset (A_\mu, \lambda_\alpha, \tilde{\lambda}_\alpha, \phi^I)$. While reduced to 3d, the 3rd component of the gauge field A_3 becomes a 3d scalar. It is convenient to think of it as the Wilson loop θ_e

$$\theta_e = \int_{S^1_R} A_3 = 2\pi R A_3, \tag{6.1}$$

which makes it manifestly gauge invariant. In this expression, θ_e is an element in the $U(1)$ gauge group, with period normalized as $\theta_e \sim \theta_e + 2\pi$.

Meaning of the KS wall-crossing formula: while compactified into three dimensions, the KS wall-crossing formula arises naturally as the physical condition of asking the Hyperkähler metric to be smooth in the Coulomb branch parameter u. In GMN, they start with the large Radius approximation of the effective 3d theory. Under this limit, the effective 3d theory could be obtained by discarding the dependence of the fields on the x^3 coordinate. The bosonic part of the coordinate writes

$$\mathscr{L}^{(3)} = (\operatorname{Im}\tau)\left(-\frac{R}{2}|da|^2 - \frac{R}{2}F^{(3)}\wedge\star F^{(3)} - \frac{1}{8\pi^2 R}d\theta_e^2\right) + (\operatorname{Re}\tau)\left(\frac{1}{2\pi}d\theta_e\wedge F^{(3)}\right). \tag{6.2}$$

We could consider the electromagnetic dual of the action. What is different in three dimensions is that the electromagnetic dual of the gauge field $A^{(3)} = \sum_{i=0}^{2} A_i dx^i$ is a scalar θ_m, satisfying

$$d\theta_m = *_3 dA^{(3)}. \tag{6.3}$$

Another way of viewing θ_m is to consider the electromagnetic dual of the theory in 4d. In this duality frame, θ_m could be viewed as the magnetic Wilson loop

$$\theta_m = \int_{S^1} A_{D3} \tag{6.4}$$

After applying the electromagnetic duality transformation to the Lagrangian (6.2), all the bosonic degrees of freedom become 3d scalars. The resulting action thus is a 3d $\mathcal{N}=4$ non-linear sigma model.

$$\mathscr{L}^{(3)}_{\text{dual}} = -\frac{R}{2}(\operatorname{Im}\tau)|da|^2 - \frac{1}{8\pi^2 R}(\operatorname{Im}\tau)^{-1}\left|d\theta_m - \tau d\theta_e\right|^2 \tag{6.5}$$

From the sigma model action, we could read off the metric of the target space at large Radius limit. This metric is what GMN called the semi-flat metric g^{sf}.

$$g^{\text{sf}} = R(\operatorname{Im}\tau)|da|^2 + \frac{1}{4\pi^2 R}(\operatorname{Im}\tau)^{-1}|dz|^2, \tag{6.6}$$

where

$$dz_I = d\theta_{mI} - \tau_{IJ}d\theta_e^J \tag{6.7}$$

I, J is the charge lattice index and the $|dz|^2 = \langle dz, d\bar{z}\rangle$.

It is beneficial to look at certain limits for this metric, although the following result is not restricted to this specific case. Under the limit where $G = SU(2)$ and $\theta = 0$, the metric reduces to

$$g^{sf} = \frac{4\pi R}{g^2}|da|^2 + \frac{1}{\pi R g^2}d\theta_e^2 + \frac{g^2}{16\pi^3 R}d\theta_m^2 \sim \frac{1}{g^2}|da|^2 + \frac{1}{g^2}d\theta_e^2 + g^2 d\theta_m^2. \tag{6.8}$$

There is another way of looking at the geometry by regarding it as a S^1 fibration over the base space $\mathbb{R}^2\times S^1 \ni (a, \theta_e)$. This perspective is helpful in the following discussion.

6.2 Semi-Flat Coordinate

The semi-flat Kähler structure is equivalently encoded in the semi-flat coordinate or say "Darboux Coordinate", such that the corresponding metric and HK form are obtained by the pullback of the standard symplectic form ω^T on a complex torus to the Coulomb branch. Let us denote the Darboux coordinate as $\mathcal{X}^{\text{sf}}$, such that

$$\mathcal{X}^{\text{sf}*}\omega^T = \omega^{sf}(\zeta). \tag{6.9}$$

$\mathcal{X}_\gamma$ could be understood as the holomorphic Fourier mode on the torus. The semiflat coordinate is given by

$$\mathcal{X}^{\mathrm{sf}}_\gamma = \mathrm{Exp}\left(i\theta_\gamma + i\pi R(\frac{Z_\gamma}{\zeta} + \zeta\overline{Z_\gamma})\right). \tag{6.10}$$

Therefore, the problem of constructing the Hyperkähler metric transforms to the problem of finding the exact Darboux coordinate $\mathcal{X}_\gamma$.

6.3 Correction of Mutually Local Particles to the Sigma Model

Let us computed the correction of the mutually local particles by effective field theory. While moving around the point on the Moduli space where a single hypermultiplet becomes light, we could apply the EFT description to find out the correction to the 3d metric and the holomorphic Fourier modes $\mathcal{X}^{\mathrm{sf}}$. The correction of the additional hyper enters the story as a one-loop correction to the coupling constant $\Im\tau$ in the metric.

While there is a hypermultiplet with an electric charge, the electric $U(1)_e$ symmetry is broken. However, the magnetic $U(1)_m$ symmetry, rotating the θ_m coordinate,e is preserved. As this symmetry action commutes with the SUSY algebra, it preserves the symplectic forms on the target space in all three complex structures. Namely, it provides a tri-Hamiltonian action on the target space. One could thus apply the Gibbons-Hawking ansatz to the Hyperkähler metric.

For comparison with the Gibbons-Hawking ansatz, we introduce a vector $\vec{x}$ by

$$a = x^1 + ix^2, \quad \theta_e = 2\pi R x^3 \tag{6.11}$$

where a is identified with the Coulomb branch parameter in $4d$.

Over an open subset of θ_m, the Gibbons-Hawking ansatz states that the Hyperkähler metric is of the form[①]

$$ds^2 = V(\vec{x})d\vec{x}^2 + V(\vec{x})^{-1}\left(\frac{d\theta_m}{2\pi} + A(\vec{x})\right)^2, \tag{6.12}$$

where $V(x)$ is the positive harmonic function on $\mathbb{R}^3$ identified as the coupling constant $\frac{4\pi R}{g^2}$ and A is a connection of $U(1)$ bundle over $\mathbb{R}^3$ satisfying

$$d_3 A(\vec{x}) = *dV(\vec{x}). \tag{6.13}$$

which results in the broken $U(1)_e$ symmetry after integrating out the hypermultiplet.

Now, we are going to determine the function $V(x)$. In the presence of the hyper, the 4d effective field theory is no longer free. Therefore, there is a one-loop correction after integrating out the hypermultiplet. It could be evaluated by the standard Feynman diagram computation for the propagator of the photon. For the BPS hyper with charge q, its mass is given by $M = |Z| = \sqrt{2}q|a|$.

$$g^2 = \frac{g_0^2}{1 + g_0^2 q^2 I_{1loop}} \tag{6.14}$$

① To be precise, the analysis here is a slight generalization of the standard one in the Gibbons-Hawking Ansatz, as the base space is $\mathbb{R}^2 \times S^1$.

$$\frac{1}{g^2} = \frac{1}{g_0^2} + q^2 I_{1loop} \tag{6.15}$$

where

$$I_{1loop} = \frac{1}{2\pi R}\sum_n \int \frac{d^3p}{(2\pi)^3}\frac{1}{(p^2+2q^2|a|^2+(\frac{q\theta_e}{2\pi R}+\frac{n}{R})^2)^2} = \sum_n \frac{1}{16\pi^2 R}\frac{1}{\sqrt{2q^2|a|^2+(\frac{q\theta_e}{2\pi R}+\frac{n}{R})^2}}. \tag{6.16}$$

Remember that $V = \frac{4\pi R}{g^2}$ in the Gibbons-Hawking ansatz, then the V takes

$$V = \frac{q^2 R}{4\pi}\sum_{n=-\infty}^{\infty}\left(\frac{1}{\sqrt{2q^2R^2|a|^2+\left(q\frac{\theta_e}{2\pi}+n\right)^2}} - \kappa_n\right) \tag{6.17}$$

where κ_n is the regularization constant. We would take $\kappa_n = \frac{1}{\sqrt{\tilde{\Lambda}^2+n^2}}$ as the Pauli-Villars regularization. It is convenient to conduct Poisson resummation for V by viewing it as a periodic function in θ_e, which yields

$$\begin{aligned} V &= V^{\mathrm{sf}} + V^{\mathrm{inst}} \\ V^{\mathrm{sf}} &= -\frac{q^2R}{4\pi}\left(\log\frac{\sqrt{2}a}{\Lambda} + \log\frac{\sqrt{2}\bar{a}}{\bar{\Lambda}}\right), \\ V^{\mathrm{inst}} &= \frac{q^2R}{2\pi}\sum_{n\neq 0} e^{inq\theta_e}K_0(2\sqrt{2}\pi R|nqa|), \end{aligned} \tag{6.18}$$

where K_0 is the modified Bessel function of the second type. To specify the metric fully we must also solve $A(\vec{x})$, which gives

$$A = A^{\mathrm{sf}} + A^{\mathrm{inst}} \tag{6.19}$$

where

$$\begin{aligned} A^{\mathrm{sf}} &= \frac{iq^2}{8\pi^2}\left(\log\frac{\sqrt{2}a}{\Lambda} - \log\frac{\sqrt{2}\bar{a}}{\bar{\Lambda}}\right)d\theta_e, \\ A^{\mathrm{inst}} &= -\frac{q^2R}{4\pi}\left(\frac{da}{a} - \frac{d\bar{a}}{\bar{a}}\right)\sum_{n\neq 0}(\mathrm{sgn}\, n)e^{inq\theta_e}|a|K_1(2\sqrt{2}\pi R|nqa|). \end{aligned} \tag{6.20}$$

With the Gibbons-Hawking metric, the Kähler forms take

$$\omega^\alpha = dx^\alpha \wedge \left(\frac{d\theta_m}{2\pi} + A(\vec{x})\right) + V(x)\frac{1}{2}\epsilon_{\alpha\beta\gamma}dx^\beta \wedge dx^\gamma \tag{6.21}$$

The Kähler form could be rephrased by incorporating the twistor coordinate ζ

$$\omega(\zeta) = \frac{1}{\zeta}\Omega_+ - 2i\omega_3 + \zeta\Omega_-, \tag{6.22}$$

The holomorphic symplectic form $\omega(\zeta)$ is then

$$\omega(\zeta) = \frac{1}{2\pi^2 i R} \xi_m \wedge \xi_e \tag{6.23}$$

where

$$\begin{aligned} \xi_m &= i d\theta_m + 2\pi i A(\vec{x}) - \sqrt{2}\pi i V \left(\frac{1}{\zeta} da - \zeta d\bar{a} \right) \\ \xi_e &= i d\theta_e + \sqrt{2}\pi i R \left(\frac{1}{\zeta} da + \zeta d\bar{a} \right) \end{aligned} \tag{6.24}$$

With the one-loop corrected metric, we would like to find the corresponding holomorphic Darboux coordinate $\mathscr{X}_\gamma$, such that

$$\omega(\zeta) = \mathscr{X}(\zeta)^* \omega^T \tag{6.25}$$

The physical meaning of this Darboux coordinate is simply the expectation value of the BPS line operator with charge in the U(1) theory. The electric part is easy to find: it simply asks

$$d \ln \mathscr{X}_e = \xi_e \tag{6.26}$$

which yields

$$\mathscr{X}_e = \operatorname{Exp}\left(i\theta_e + \sqrt{2}\pi i R (\frac{1}{\zeta} a + \zeta \bar{a}) \right) \tag{6.27}$$

It matches exactly with the semi-flat analysis, which means that $\mathscr{X}_e$ is not corrected. The physcial interpretation for the electric coordinate is clear. Compared with the formula (5.4), we find that $\mathscr{X}_e$ is the expectation value of $U(1)$ BPS wilson line.

6.3.1 The Magnetic Coordinate

Let's now move on to the magnetic part. What we ought to do is simply find a solution to the equation

$$d \ln \mathscr{X}_m(\zeta) = f(\zeta)\xi_e(\zeta) + \xi_m(\zeta). \tag{6.28}$$

with any appropriate value of $f(\zeta)$.

Let's postpone the procedure of solving the equation to the end of this subsection and present the solution for the magnetic coordinate

$$\begin{aligned} \mathscr{X}_m(\zeta) = \mathscr{X}_m^{\text{sf}}(\zeta) \exp\Bigg[& \frac{iq}{4\pi} \int_{\ell_+} \frac{d\zeta'}{\zeta'} \frac{\zeta' + \zeta}{\zeta' - \zeta} \log\left[1 - \mathscr{X}_e\left(\zeta'\right)^q\right] \\ & - \frac{iq}{4\pi} \int_{\ell_-} \frac{d\zeta'}{\zeta'} \frac{\zeta' + \zeta}{\zeta' - \zeta} \log\left[1 - \mathscr{X}_e\left(\zeta'\right)^{-q}\right] \Bigg] \end{aligned} \tag{6.29}$$

where $\mathscr{X}_m^{\text{sf}}$ is the semi-flat approximation of the magnetic coordinate

$$\mathscr{X}_m^{\text{sf}}(\zeta) = \exp\left[-i \frac{Rq^2}{2\zeta} \left(a \log \frac{a}{\Lambda} - a \right) + i\theta_m + i \frac{\zeta R q^2}{2} \left(\bar{a} \log \frac{\bar{a}}{\bar{\Lambda}} - \bar{a} \right) \right] \tag{6.30}$$

and $\ell_\pm$ is the contour from 0 to ∞ such that it is inside the region

$$\mathscr{U}_\pm = \left\{ \zeta \, : \, \pm \operatorname{Re} \frac{a}{\zeta} < 0 \right\}. \tag{6.31}$$

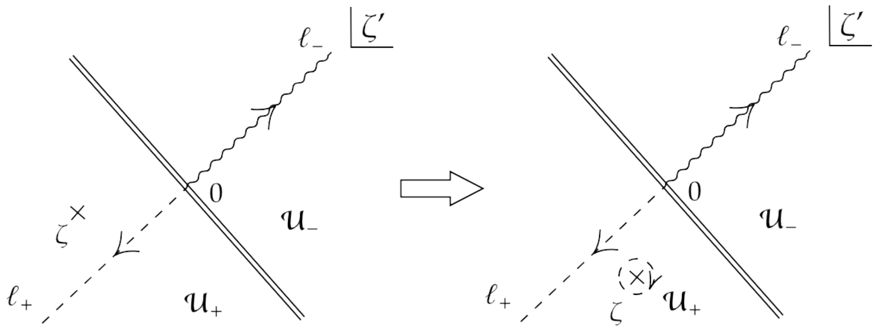

Figure 6-1 Integration domain in the formula of the magnetic coordinate.

A remarkable feature of the magnetic coordinate is the discontinuity in ζ. When we rotate the phase of ζ such that it crosses the wall $\mathscr{W} = \{\zeta \in \mathbb{C}^\times \mid \frac{a}{\zeta} < 0\}$ in the anit-clockwise direction, the integration in ℓ_+ pick up an extra residue as demonstrated in the Fig.6-1. If we denote $\mathcal{X}_m^+$ as the limit of ζ approaching the wall in the clockwise direction and $\mathcal{X}_m^-$ as the anti-clockwise direction, then they are related by

$$\mathcal{X}_m^+ = \mathcal{X}_m^-(1 - \mathcal{X}_e^q)^{-q}. \tag{6.32}$$

Compared with the formula for the KS transformation, we find that two coordinates differ exactly by a KS factor $\mathscr{K}_{q,0}$ on the image complex torus. The calculation here provides a physical interpretation for the KS transformation.

Derivation of the magnetic coordinate. ξ_m is clearly complicated and involves some series of Bessel functions. However, it can be simplified with the following integration transformation

$$\int_0^\infty \frac{dt}{t} e^{-x(t+1/t)} = 2K_0(2x)\,, \text{if } x > 0. \tag{6.33}$$

Apply this formula to V^{inst}, we find

$$V^{\text{inst}} = \frac{q^2 R}{4\pi} \sum_{n>0} \int_0^\infty \frac{d\zeta'}{\zeta'} e^{inq\theta_e - \pi Rnq|a|\zeta' - \pi Rnq|a|/\zeta'} + \frac{q^2 R}{4\pi} \sum_{n<0} \int_0^\infty \frac{d\zeta'}{\zeta'} e^{inq\theta_e + \pi Rnq|a|\zeta' + \pi Rnq|a|/\zeta'} \tag{6.34}$$

Let's deal with the summation over the positive n as an example. We could change the integration path into the one $l = \{\zeta \in \mathbb{C}^\times \mid \Re a/\zeta < 0\}$.

$$\sum_{n>0} \int_l \frac{d\zeta'}{\zeta'} e^{inq\theta_e + \pi Rnqa\zeta' + \pi Rnqa/\zeta'} = \int_l \frac{d\zeta'}{\zeta'} (1 - \mathcal{X}_e^q)^{-1}. \tag{6.35}$$

The summation over negative n could be treated in exactly the same way, which gives

$$V^{\text{inst}} = \frac{q^2 R}{4\pi} \int_l \frac{d\zeta'}{\zeta'} (1 - \mathcal{X}_e^q)^{-1} + \frac{q^2 R}{4\pi} \int_l \frac{d\zeta'}{\zeta'} (1 - \mathcal{X}_e^{-q})^{-1} \tag{6.36}$$

A^{inst} could be simplified similarly, using the integration formula

$$\int_0^\infty dt e^{-x(t+1/t)} = 2K_1(2x)\,, \text{if } x > 0. \tag{6.37}$$

By applying electric-magnetic duality transformation in $4d$, we could transform an arbitrary charge to the electrical charge particle and conduct the same evaluation as above. Therefore, we could generalize the argument to particles with electrical magnetic charge γ and propose that when ζ cross the wall defined by $\mathscr{W}_\gamma = \{\zeta \mid Z_\gamma/\zeta < 0\}$, the coordinate $\mathscr{X}$ picks up the KS factor $\mathscr{K}_\gamma$.

6.3.2 Incorporate Higher Spin Multiplets

In the last section, we analyze the instanton correction to the magnetic coordinates arising from a single hypermultiplet with charge q.

In this subsection, we extend our analysis to include instanton corrections from mutually local particles. This task involves two steps. First, we account for the contributions from all electrically charged hypermultiplets. Second, we consider the corrections from multiplets with higher spin. We will address these two issues in the following section.

To incorporate the contribution from other electrically charged hypermultiplets, we only need to repeat the argument. It is an easy generalization, since each hypermultiplets contributes to the correction of V independently. In the $SU(2)$ case, after taking it into account, the electric coordinate remains uncorrected and the magnetic coordinate becomes

$$\begin{aligned}\mathscr{X}_m(\zeta) = \mathscr{X}_m^{\mathrm{sf}}(\zeta)\exp\Bigg[\sum_{q\in\mathbb{Z}} \frac{iq}{4\pi}\int_{\ell_+}\frac{d\zeta'}{\zeta'}\frac{\zeta'+\zeta}{\zeta'-\zeta}\log\left[1-\mathscr{X}_e\left(\zeta'\right)^q\right] \\ -\frac{iq}{4\pi}\int_{\ell_-}\frac{d\zeta'}{\zeta'}\frac{\zeta'+\zeta}{\zeta'-\zeta}\log\left[1-\mathscr{X}_e\left(\zeta'\right)^{-q}\right]\Bigg]\end{aligned} \tag{6.38}$$

Contributions from mutually local multiplets with different spins can be evaluated using the following trick. Consider the multiplets in a 4d $\mathcal{N}=4$ theory, where a massive vector multiplet consists of an $\mathcal{N}=2$ vector multiplet and two $\mathcal{N}=2$ hypermultiplets. In the $\mathcal{N}=4$ theory, the $SO(6)_R$ symmetry mixes the six scalars, thereby interchanging the Coulomb and Higgs branches. Since the standard argument for non-renormalization applies to the Higgs branch, the metric on the Coulomb branch remains uncorrected. This implies that the contribution of the $\mathcal{N}=2$ vector multiplet cancels against that of the two hypermultiplets. [①]

A corollary of this argument is that the contribution from vector multiplet is the same as that of the hypermultiplet up to a weight factor. A similar argument shows that the instanton corrections from the spin-j multiplet is also proportionate to the contribution of a single hyper, weighted by a factor a_j. The analysis before shows that

$$a_0 = 1,\ a_1 = -2. \tag{6.39}$$

Then the free from corrections in $\mathcal{N}=4$ theory implies a recursion relation between the weight factors

$$a_j + 2a_{j+1} + a_{j+2} = 0, \tag{6.40}$$

from which we solve

$$a_j = (-1)^j(j+1). \tag{6.41}$$

① The vector multiplet here contains the W boson, which is massive and could be charged under the unbroken $U(1)$ gauge group.

Thus, after including the instanton corrections from charged q particles with all possible spins, the net correction introduces a weight factor before the contribution of a single hypermultiplet. This weight factor is equivalent to the second helicity supertrace $\Omega_\gamma = \sum_{j=0}^{\infty} n_j a_j = \mathrm{Tr}_{\mathscr{H}_{BPS,\gamma}}(-1)^{2J_3}$, (see Eq. (2.57)), where n_j counts the number of BPS multiplets with spin j.

For instance, if we consider the contribution of mutually local particles with electric charges in the $SU(2)$ case, then the instanton correction to the magnetic coordinate yields

$$\begin{aligned}\mathcal{X}_m(\zeta) = \mathcal{X}_m^{\mathrm{sf}}(\zeta)\exp\Bigg[&\sum_{q\in\mathbb{Z}} \Omega_q \frac{iq}{4\pi}\int_{\ell_+}\frac{d\zeta'}{\zeta'}\frac{\zeta'+\zeta}{\zeta'-\zeta}\log\left[1-\mathcal{X}_e\left(\zeta'\right)^q\right] \\ &-\Omega_q\frac{iq}{4\pi}\int_{\ell_-}\frac{d\zeta'}{\zeta'}\frac{\zeta'+\zeta}{\zeta'-\zeta}\log\left[1-\mathcal{X}_e\left(\zeta'\right)^{-q}\right]\Bigg].\end{aligned} \tag{6.42}$$

For the mutually local particle with positive charge q, all of the wall $\mathscr{W}_{(q,0)}$ alignes. Therefore, while ζ is approaching the wall $\mathscr{W}_{(q,0)}$ from the clockwise direction and the anti-clockwise direction, the magnetic coordinate would be differ by

$$\mathcal{X}_m^+ = \mathcal{X}_m^- \prod_{q\in\mathbb{Z}}(1-\mathcal{X}_e^q)^{-q\Omega_q}, \tag{6.43}$$

which matches exactly as the KS wall-crossing formula.

$$\mathcal{X}_m^+ = \mathcal{X}_m^- \prod_{q\in\mathbb{Z}} \mathcal{K}_{(q,0)} \tag{6.44}$$

6.4 Incorporate Mutually-Nonlocal Multiplets

For mutually non-local particles, it is hard to find a Lagrangian description for the EFT. However, we could specify the discontinuity of the coordinates along ζ plane using electro-magnetic duality. It formulates a Riemann-Hilbert problem, where the solution to the problem provides the exact coordinate $\mathcal{X}$, and equivalently, the exact metric. Eventually, it is rephrased into an integration equation.

$$\mathcal{X}_\gamma(\zeta) = \mathcal{X}_\gamma^{\mathrm{sf}}(\zeta)\exp\left[-\frac{1}{4\pi i}\sum_{\gamma'}\Omega\left(\gamma';u\right)\langle\gamma,\gamma'\rangle\int_{\ell_{\gamma'}}\frac{d\zeta'}{\zeta'}\frac{\zeta'+\zeta}{\zeta'-\zeta}\log\left(1-\sigma\left(\gamma'\right)\mathcal{X}_{\gamma'}\left(\zeta'\right)\right)\right]. \tag{6.45}$$

The solution to this integration equation yields the exact Darboux coordinate.

Chapter 7
2D Topological string

7.1 Basics of 2D Topological string Thery

7.1.1 $\mathcal{N} = (2,2)$ VOA

Topological string theory is one of the easiest string theories that one may have.

Let's consider the 2D $\mathcal{N} = (2,2)$ sigma model with world sheet Σ and target space M. The worldsheet theory is described by a 2d $\mathcal{N} = (2,2)$ superconformal algebra (SCA). It is the enhancement of the global symmetry, described by the (anti)holomorphic currents $\{T(z), G^{\pm}(z), J(z)\}$, where $T(z)$ is the energy-momentum tensor, $G^{\pm}(z)$ are the superconformal currents, $J(z)$ is the $U(1)$ R symmetry current. We denote its mode expansion as

$$T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}, \quad G^{\pm}(z) = \sum_{n\in\mathbb{Z}} G^{\pm}_{n+a} z^{-(n\pm a)-3/2}, \quad J(z) = \sum_{n\in\mathbb{Z}} J_n z^{-n-1} \tag{7.1}$$

here $a = 0$ corresponds to the Ramond sector, while $a = 1/2$ corresponds to the Neveu-Schwartz sector. $G^{\pm}(z)$ describes the charge of superconformal current under the $U(1)$ R symmetry.

The following set of OPEs describe the algebraic structure of the currents

$$\begin{aligned}
T(z)T(w) &\sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} \\
T(z)G^{\pm}(w) &\sim \frac{3/2}{(z-w)^2} G^{\pm}(w) + \frac{\partial_w G^{\pm}(w)}{z-w} \\
T(z)J(w) &\sim \frac{J(w)}{(z-w)^2} + \frac{\partial_w J(w)}{z-w} \\
G^{+}(z)G^{-}(w) &\sim \frac{2c/3}{(z-w)^3} + \frac{2J(w)}{(z-w)^2} + \frac{2T(w) + \partial_w J(w)}{z-w} \\
J(z)G^{\pm}(w) &\sim \pm\frac{G^{\pm}(w)}{z-w} \\
J(z)J(w) &\sim \frac{c/3}{(z-w)^2}
\end{aligned} \tag{7.2}$$

The first OPE is the standard TT OPE structure. As one can see from the OPE, $G^{\pm}(z)$ has conformal dimension 3/2, and $U(1)_R$ current has the conformal dimension 1 as expected. The JG OPE shows that $G^{\pm}(z)$ has $U(1)_R$ charge ± 1 respectively.

One of the key operations we will do for the topological string is the topological twist. It means that we recombine the energy-momentum tensor $T(z)$ with the R-current$J(z)$ into a twisted one $T_{tw}(z)$, such that part of the superconformal currents lives in the trivial representation of the new Lorentz group. It is useful for either preserving part of the supersymmetry in the dimension reduction process or in constructing topological field theory in various dimensions.

Living in the trivial representation in the spacetime rotation group can be realized by asking the superconformal current to have conformal dimension 1 with respect to $T_{tw}(z)$. As the integration over the space slice has dimension -1, the charge will be dimension zero. There are two ways of doing it.

$$T(z) \to T(z) + \frac{1}{2}\partial J(z), \quad J(z) \to J(z), \quad G^{\pm}(z) \to G^{\pm}(z) \tag{7.3}$$

and

$$T(z) \to T(z) - \frac{1}{2}\partial J(z), \quad J(z) \to -J(z), \quad G^{\pm}(z) \to G^{\pm}(z) \tag{7.4}$$

Under the first twist, G^+ has conformal dimension 1. Under the second twist, G^- has conformal dimension 1. It doesn't matter which twist one take. What matters is whether one take the same or a different twist in the left or right-moving sector. If they are the same, then the resulting theory is called the topological A model, otherwise it is called the topological B model.

We can always stick to the case where the left-moving sector takes the first twist. In this case, we have the supercharges (BRST charge)

$$Q_\gamma = \int_\gamma G^+ dz \tag{7.5}$$

The integration result depends on the homology cycle of $\gamma \in H_1(\Sigma)$. By the construction, Q_γ have zero conformal dimension. As the OPE between G^+ and G^+ is trivial, one has naturally $Q_\gamma^2 = 0$. Therefore, the BRST charge is nilpotent.

Moreover, follow from the OPE between superconformal generators, one finds that

$$\int dz\{G^+(z), G^-(w)\} = \oint dz \frac{2T_{tw}(z)}{z-w} = 2T_{tw}(w) \tag{7.6}$$

Namely,

$$T_{tw}(w) = \frac{1}{2}\{Q, G^-(w)\} \tag{7.7}$$

Namely, after the topological twist, the twisted energy momentum tensor is Q exact. The significant aspect lies in that to reach this conclusion, we do not specify to a certain type of lagrangian. All the decription here is fully algebraic, thus more universal.

This property immediately leads to the result that the correlation function of any Q closed operators is independent of the choice of the worldsheet metric. This is because

$$\begin{aligned}\frac{1}{\sqrt{g}}\frac{\delta}{\delta g}\langle \mathcal{O}_1\mathcal{O}_2\ldots\mathcal{O}_n\rangle &= \int [DX]\frac{1}{\sqrt{g}}\frac{\delta}{\delta g}e^{-S}\mathcal{O}_1\mathcal{O}_2\ldots\mathcal{O}_n = \frac{1}{4\pi}\int [DX]e^{-S}T_{tw}\mathcal{O}_1\mathcal{O}_2\ldots\mathcal{O}_n \\ &= \frac{1}{8\pi}\langle\{Q,G^-\}\mathcal{O}_1\mathcal{O}_2\ldots\mathcal{O}_n\rangle = \frac{1}{8\pi}\sum_k(-1)^{k-1}\langle G^-\mathcal{O}_1\ldots\{Q,\mathcal{O}_k\}\ldots\mathcal{O}_n\rangle = 0\end{aligned} \tag{7.8}$$

Therefore, if one restricts himself to the space of operators annihilated by the supercharges, then the correlation functions are all topological. Those operators are called BPS operators or chiral operators.

We can derive the bound for the BPS operators explicitly. Following from GG OPE, we can derive the commutation relation

$$\{G_0^+, G_0^-\} = \oint \frac{dw}{2\pi i} \oint_w \frac{dz}{2\pi i} z^{1/2} w^{1/2} G^+(z) G^-(w) = 2L_0^{tw} - \frac{c}{12} + J_0 \tag{7.9}$$

For a primary state, we can label it by the conformal dimension and the $U(1)_R$ charge $|h, j\rangle_R$. Therefore, the chiral primary state satisfy

$$|G_0^- \, |h, j\rangle_R|^2 = \langle h, j|_R \{G_0^+, G_0^-\} \, |h, j\rangle_R = 2h + j - \frac{c}{12} \tag{7.10}$$

By unitarity, we have $2h + j - c/12 \geq 0$. For generic (h, j) satisfying this condition, we get a short multiplet (BPS multiplet) generated by $\{|h, j\rangle_R, G_0^- \, |h, j\rangle_R\}$.

Especially, the Ramond vacuum state satuate the bound $2h + j - c/12 = 0$.

Once we restrict ourselves to the BPS operators (Q closed operators), then the correlation function between them won't change if we add any Q exact operators. Without lose of generality, after replacing $\mathcal{O}_1$ with $\mathcal{O}_1 + \{Q, A\}$, the correlation function becomes

$$\begin{aligned} \langle (\mathcal{O}_1 + \{Q, A\}) \mathcal{O}_2 \dots \mathcal{O}_n \rangle &= \langle \mathcal{O}_1 \mathcal{O}_2 \dots \mathcal{O}_n \rangle + \sum_k (-1)^{k-1} \langle A \mathcal{O}_1 \dots \{Q, \mathcal{O}_k\} \dots \mathcal{O}_n \rangle \\ &= \langle \mathcal{O}_1 \mathcal{O}_2 \dots \mathcal{O}_n \rangle \end{aligned} \tag{7.11}$$

Therefore, adding Q exact terms can be viewed as the gauge redundancy in the space of BPS operators. The physical Hilbert space is thus the Q-cohomology $\mathcal{H}_{\text{BPS}} = \ker Q / \text{Im} Q$.

Once the redundancy of Q exact operator is noticed, $\mathcal{H}_{\text{BPS}}$ becomes very restrictive. Follow from Eq. 7.7, we have

$$L_0 = \frac{1}{2} \{Q, G_{-1}^-\} \tag{7.12}$$

Therefore, for BPS operators with non-trivial conformal dimension h, one have

$$\mathcal{O} = \frac{1}{h}[L_0, \mathcal{O}] = \frac{1}{2h}[\{Q, G_{-1}^-\}, \mathcal{O}] = \frac{1}{2h}\{Q, [G_{-1}^-, \mathcal{O}]\} \tag{7.13}$$

Thus it has to be Q-closed. Therefore, the non-trivial representative in $\mathcal{H}_{\text{BPS}}$ has conformal dimension 0.

7.1.2 $\mathcal{N} = (2, 2)$ sigma model

To construct the Lagrangian for $\mathcal{N} = (2, 2)$ sigma model, we consider the $\mathcal{N} = (2, 2)$ superfield in 2d. Superfield is defined as the space with both bosonic coordinates and fermionic coordinates. $\mathcal{N} = (2, 2)$ superconformal algebra admit a differential operator representation over the superspace $\mathbb{R}^{2|4}$, with two bosonic coordinates and four fermionic coordinates. The $\mathcal{N} = (2, 2)$ sigma model admit chiral superfield $\Phi(y, \theta)$ as the field content. Explicitly, one has

$$\Phi^i \left(y^\pm, \theta^\pm\right) = X^i \left(y^\pm\right) + \psi^i \left(y^\pm\right) \theta^+ + \bar{\psi}^i \left(y^\pm\right) \theta^- + F^i \left(y^\pm\right) \theta^+ \theta^- \tag{7.14}$$

where $y^+ = z - i\theta^+\bar{\theta}^+$ and $y^- = z - i\theta^-\bar{\theta}^-$. If one lift the construction to the Riemann surface Σ, then one should replace the functions with the sections of a certain vector bundle. Explicitly, it writes

$$\begin{aligned}
&X \in \Gamma(\Sigma, M) \\
&\psi \in \Gamma\left(\Sigma, X^* TM^{(1,0)} \otimes S\right) \\
&\chi \in \Gamma\left(\Sigma, X^* TM^{(0,1)} \otimes S\right) \\
&F \in \Gamma\left(\Sigma, X^* TM^{(1,0)} \otimes S \otimes \bar{S}\right) \\
&\quad (\psi \leftrightarrow \bar{\psi}, \chi \leftrightarrow \bar{\chi}, S \leftrightarrow \bar{S})
\end{aligned} \tag{7.15}$$

We use i, j, k, l for the indices of the holomorphic tangent bundle, $\bar{i}, \bar{j}, \bar{k}, \bar{l}$ for the indices of the antiholomorphic tangent bundle, α, β, γ for the left-handed spinor indice, $\dot{\alpha}, \dot{\beta}, \dot{\gamma}$ for the right-handed spinor indices.

The $\mathcal{N} = (2, 2)$ sigma model we consider only includes the Kahler potential term. The action writes

$$S_0 = \int_{\Sigma^{(2|2)}} d^4\theta d^2 x K(\Phi^i, \bar{\Phi}^{\bar{i}}) \tag{7.16}$$

Conducting the integration over the Grassmannian coordinates, one has

$$\begin{aligned}
S = \int_\Sigma d^2 z [&\sqrt{\eta} G_{i\bar{j}} \eta^{\mu\upsilon} \partial_\mu X^i \partial_\upsilon X^{\bar{j}} - i G_{i\bar{j}} \bar{\chi}^{\bar{j}} D_z \bar{\psi}^i + i G_{i\bar{j}} \chi^{\bar{j}} D_{\bar{z}} \psi^i \\
&- \frac{1}{2} R_{i\bar{j}k\bar{l}} \psi^i \bar{\psi}^k \bar{\chi}^{\bar{j}} \chi^{\bar{l}} + G_{i\bar{j}} \left(F^i - \Gamma^i_{jk} \psi^j \bar{\psi}^k\right) \left(F^{\bar{j}} - \Gamma^{\bar{j}}_{\bar{k}\bar{l}} \bar{\chi}^{\bar{k}} \chi^{\bar{l}}\right)\Big].
\end{aligned} \tag{7.17}$$

where $g_{i\bar{j}} = \frac{\partial^2 K}{\partial \Phi^i \partial \bar{\Phi}^{\bar{j}}}$, D_z is the covariant derivative given by the spin connection of the tangent bundle and the covariant derivative over the worldsheet $D_z = \nabla_z + i_{\partial_z X} \Gamma$. R is the Riemann curvature over the complexified tangent bundle.

By the knowledge of Kähler geometry, one can define such the metric $g_{i\bar{j}} = \frac{\partial^2 K}{\partial \Phi^i \partial \bar{\Phi}^{\bar{j}}}$ on the target manifold if and only if the target space is a special Kähler manifold.

One could integrate F field out and achieve at the action

$$\begin{aligned}
S_0 = \int_\Sigma d^2 z [&\sqrt{\eta} G_{i\bar{j}} \eta^{\mu\upsilon} \partial_\mu X^i \partial_\upsilon X^{\bar{j}} - i G_{i\bar{j}} \bar{\chi}^{\bar{j}} D_z \bar{\psi}^i + i G_{i\bar{j}} \chi^{\bar{j}} D_{\bar{z}} \psi^i \\
&- \frac{1}{2} R_{i\bar{j}k\bar{l}} \psi^i \bar{\psi}^k \bar{\chi}^{\bar{j}} \chi^{\bar{l}}\Big].
\end{aligned} \tag{7.18}$$

The Noether current of the theory takes the form

$$\begin{aligned}
\mathcal{J} &= G_{i\bar{j}} \left(\psi^i \chi^{\bar{j}} + \bar{\psi}^i \bar{\chi}^{\bar{j}}\right) \\
\mathcal{G}^+ &= G_{i\bar{j}} \left(\psi^i \partial_z X^{\bar{j}} + \bar{\psi}^i \partial_{\bar{z}} X^{\bar{j}}\right) \\
\mathcal{G}^- &= G_{i\bar{j}} \left(\chi^{\bar{j}} \partial_z X^i + \bar{\chi}^{\bar{j}} \partial_{\bar{z}} X^i\right) \\
\mathcal{T} &= G_{i\bar{j}} \left(\partial_z X^i \partial_{\bar{z}} X^{\bar{j}} + i \chi^{\bar{j}} \partial_z \psi^i + i \bar{\chi}^{\bar{j}} \partial_{\bar{z}} \bar{\psi}^i\right)
\end{aligned} \tag{7.19}$$

The holomorphic part of the Noether current looks like

$$\begin{aligned} J(z) &= G_{i\bar{j}}\psi^i\chi^{\bar{j}} \\ G^+(z) &= G_{i\bar{j}}\psi^i\partial_z X^{\bar{j}} \\ G^-(z) &= G_{i\bar{j}}\chi^{\bar{j}}\partial_z X^i \\ T(z) &= \frac{1}{2}G_{i\bar{j}}\partial_z X^i\partial_{\bar{z}}X^{\bar{j}} + i\chi^{\bar{j}}\partial_z\psi^i \end{aligned} \tag{7.20}$$

One may rewrite the noether current in a more covariant form. Let's rewrite the chiral spinor into Dirac spinor $\Psi^a = \begin{bmatrix}\psi^i \\ \chi^{\bar{i}}\end{bmatrix}$, $\bar{\Psi}^a = \begin{bmatrix}\bar{\psi}^i \\ \bar{\chi}^{\bar{i}}\end{bmatrix}$. It takes value in the Clifford bundle $S \oplus \bar{S}$. In this way, the Noether charges take the form

$$\begin{aligned} G^{\pm}(z) &= \frac{1}{2}(G(\Psi, \partial X) \pm i\omega(\Psi, \partial X)) \\ \bar{G}^{\pm}(\bar{z}) &= \frac{1}{2}(G(\bar{\Psi}, \bar{\partial}\bar{X}) \pm i\omega(\bar{\Psi}, \bar{\partial}\bar{X})) \end{aligned} \tag{7.21}$$

where ω is the Kahler form associated with metric G and almost complex structure I of the target manifold, defined as $\omega(\ ,\ ) = G(\ , I\ )$. In physics notation, one may regard everything as matrices and write $I = G^{-1}\omega$. This form is useful when we consider the boundary conditions.

7.2 A-Model

As we mentioned before, $\mathcal{N} = (2, 2)$ sigma model admits topological twist.

$$T(z) \to T(z) + \frac{1}{2}\partial J(z), \quad J(z) \to J(z), \quad G^{\pm}(z) \to G^{\pm}(z) \tag{7.22}$$

It is equivalent to adding an extra term to the action. One can find out the extra term by writing the twisted action as $S_{tw} = S_0 + S^{'}$ and solving the equation

$$\frac{4\pi}{\sqrt{\eta}}\frac{\delta S^{'}}{\delta\eta_{z\bar{z}}} = \frac{1}{2}\partial J(z) \tag{7.23}$$

Notice that the spin connection on the Riemann surface has nice properties. The only non-vanishing components are $\gamma = \gamma^z_{zz}$ and $\bar{\gamma} = \gamma^{\bar{z}}_{\bar{z}\bar{z}}$, which takes the form $\gamma = \partial_z log(\eta_{z\bar{z}})$ and $\bar{\gamma} = \partial_{\bar{z}} log(\eta_{z\bar{z}})$. Therefore, one has

$$\left.\frac{\delta\int_\Sigma d^2z\gamma(z,\bar{z})J(z)}{\delta\eta_{w\bar{w}}(w)}\right|_{\eta=\eta_0} = -\eta^{-1}_{w\bar{w}}\partial J(w) \tag{7.24}$$

Therefore, one could compensate an extra term to the Lagrangian

$$S_{tw} = S_0 - \int d^2z(\gamma(z,\bar{z})J(z) + \bar{\gamma}(z,\bar{z})\bar{J}(\bar{z})) \tag{7.25}$$

After adding this term to the action, one finds that this term cancels out with part of the covariant derivatives in the dynamic term of fermions. Therefore, the previous fermions which are the sections in the world sheet spinor bundle should be identified with the sections of new bundles.

One can verify this interpretation from the OPE perspective. Consider the $T\psi$ OPE, then one has

$$T_{tw}(z)\psi^i(w) \sim \frac{\frac{1}{2}\psi^i(w)}{(z-w)^2} + \frac{\partial\psi^i(w)}{z-w} + \partial\left(\frac{\psi^i(w)}{z-w}\right) = \frac{\partial\psi^i(w)}{z-w} \tag{7.26}$$

Thus ψ^i have conformal dimension 0 under the twisted energy-momentum tensor.

A similar check can be done for other fields. Explicitly, $\psi^i(z)$ have conformal dimension 0, while $\chi^{\bar{i}}(z)$ have conformal dimension 1. Therefore, one may reinterpret $\psi^i(z)$ and $\bar{\chi}^{\bar{i}}(z)$ as follows,

$$\begin{aligned} &\psi \in \Gamma\left(\Sigma, X^*TM^{1,0}\right) \\ &\chi \in \Gamma\left(\Sigma, X^*TM^{0,1} \otimes \Omega^{1,0}(\Sigma)\right) \end{aligned} \tag{7.27}$$

where $\Omega^{1,0}(\Sigma)$ is the cotangent bundle of the Riemann surface.

One can do the same exercise and determine the conformal dimension for the anti-holomorphic field

$$\begin{aligned} &\bar{\psi} \in \Gamma\left(\Sigma, X^*TM^{1,0} \otimes \Omega^{0,1}(\Sigma)\right) \\ &\bar{\chi} \in \Gamma\left(\Sigma, X^*TM^{0,1}\right) \end{aligned} \tag{7.28}$$

The BRST charge can thus be written as $Q_{BRST} = \oint_\gamma G^+ dz + \oint_\gamma \bar{G}^+ d\bar{z}$.

Following from the discussion of $\mathcal{N}=2$ operator algebra, the energy-momentum tensor is Q-closed. Therefore, the action satisfies the differential equation

$$\frac{4\pi}{\sqrt{\eta}}\frac{\delta S_{tw}}{\delta\eta_{z\bar{z}}} = \frac{1}{2}\{Q, G^-(z)\}. \tag{7.29}$$

It means that S_{tw} admits a factorization into one term that depends on the worldsheet metric but is Q exact and another term S_{top} doesn't depend on the worldsheet metric.

$$S_{tw} = \int_\Sigma d^2z\{Q, V[\eta]\} + S_{top} \tag{7.30}$$

. At "large volume limit" where the target tends to have a vanishing curvature, $V[\eta]$ admits a simple solution as

$$V = G_{i\bar{j}}\left(\chi_z^{\bar{j}}\partial_{\bar{z}}X^i + \partial_z X^{\bar{j}}\bar{\psi}_{\bar{z}}^i\right) \tag{7.31}$$

Straightforward calculation shows that

$$S_{top} = S_{tw} - \int_\Sigma d^2z\{Q, V[\eta]\} = i\int_\Sigma X^*(\omega) \tag{7.32}$$

Let's now look at the correlation function of BPS operators in the theory. One may introduce an extra overall t parameter in the theory and write it as

$$\begin{aligned} \langle \mathcal{O}_1 \dots \mathcal{O}_n \rangle_t &= \int [DX] e^{-iS_{top} - t\int_\Sigma d^2z\{Q,V[\eta]\}} \mathcal{O}_1 \dots \mathcal{O}_n \\ &= \sum_{\beta\in H_2(M)} e^{-i\langle\omega,\beta\rangle} \int_{[X(\Sigma)]=\beta} [DX] e^{-t\int_\Sigma d^2z\{Q,V[\eta]\}} \mathcal{O}_1 \dots \mathcal{O}_n \end{aligned} \tag{7.33}$$

The significance of the Q-exact action is that one can apply the localization technique. An interesting observation is that

$$\begin{aligned}\frac{d}{dt}\langle \mathcal{O}_1 \dots \mathcal{O}_n \rangle_t &= -\sum_{\beta \in H_2(M)} e^{-i\langle \omega, \beta \rangle} \int_{[X(\Sigma)]=\beta} [DX] e^{-t\int_\Sigma d^2 z \{Q, V[\eta]\}} \{Q, V[\eta]\} \mathcal{O}_1 \dots \mathcal{O}_n \\ &= -\sum_{\beta \in H_2(M)} e^{-i\langle \omega, \beta \rangle} \int_{[X(\Sigma)]=\beta} [DX] \{Q, e^{-t\int_\Sigma d^2 z \{Q, V[\eta]\}} \mathcal{O}_1 \dots \mathcal{O}_n\} = 0\end{aligned} \tag{7.34}$$

The last step gives zero because the integration measure and integration region are invariant under BRST transformation. One may view the last term as a SUSY version of the total derivative, which vanishes after integration.

Therefore, the correlation function indeed does not depend on t. Thus the integration can be evaluated at $t \to \infty$ limit. At this limit, the contribution in the integration localizes into the region where $\{Q, V[\eta]\} = 0$. A simple check shows that it corresponds to the classical equation of motion including $\partial_{\bar{z}} X = 0$. Therefore, the integration is localized over the space of holomorphic embedding to the target space M.

Once we ignore the instanton corrections and conduct quantization at a large volume limit, the Hilbert space turns out can be identified with the de-Rham cohomology of the target manifold. The argument can be refined as follows. Following from the general argument on the $2d$ $\mathcal{N} = (2,2)$ current algebra, the only possible operators that represent a non-trivial class in $\mathcal{H}_Q$ are those with conformal dimension 0. It is clear that all the local operator with zero conformal dimension takes the form

$$\mathcal{O} = a_{i_1,\dots,i_m,\bar{i}_1,\dots,\bar{i}_n}(X) \psi^{i_1} \dots \psi^{i_m} \bar{\chi}^{\bar{i}_1} \dots \bar{\chi}^{\bar{i}_n} \tag{7.35}$$

It can be interpreted as (m, n) form on the target space. SUSY acts on the local operator as

As a result, one could identify the Hilbert space of the closed topological A model with the de Rham cohomology of the manifold at a large volume limit. Once instanton corrections are involved, the Hilbert space gets modified into the quantum cohomology of the target manifold.

7.3 A-model Open String and A-Brane

Now we consider the open string A model. Several modifications can be made to the theory. Firstly, one may consider the coupling of the gauge field on the ends of the open string. Namely, one can add terms $\int_{\partial\Sigma} X^* A = \int ds \frac{\partial A}{\partial X^a} \frac{dX^a}{ds}$ Secondly, extra conditions have to be added such that world sheet $\mathcal{N} = 2$ SUSY is preserved. The boundary condition on the worldsheet description can be identified with a special type of brane in the target space, which we will call A brane. The key problem now is to derive the condition of A brane.

For an open string, the world sheet is $[0, \pi] \times \mathbb{R}$. By conformal transformation, one can map the strip to the upper half-plane. The boundary is given by $z = \bar{z}$. Taking variation over the action and asking the variation of the boundary term to vanish gives the boundary condition. Denote the boundary manifold as Y, the induced metric on Y as g, and the curvature of connection A as F. The boundary condition takes the form

$$\begin{aligned}\partial X &= R(\bar{\partial} X) \\ \Psi &= R(\bar{\Psi})\end{aligned} \tag{7.36}$$

where $R = -id_{N_Y} \oplus (g - F)^{-1}(g + F)$.

Such a boundary condition has a clear physical interpretation. If we look at the normal direction, the first condition is equivalent to asking $\partial_x X^a = 0$, where a denotes the normal bundle index. It means that the normal position remains unchanged along the boundary of the string. Therefore, the endpoint moves on the Y. The tangent direction can be rewritten into $g(\partial_z - \partial_{\bar{z}})X = F(\partial_z + \partial_{\bar{z}})X$ The transformation of Ψ is due to the $\mathcal{N} = 1$ SUSY on the worldsheet.

Let's show that R preserves the target space metric. Namely

$$R^T G R = G. \tag{7.37}$$

We may test it in the normal direction and tangent direction respectively. On the normal direction $R = -id$, thus the equation holds trivially. On the tangent direction, the metric takes $G\big|_Y = g$. One has

$$[(g-F)^{-1}(g+F)]^T g(g-F)^{-1}(g+F) = (g-F)(g+F)^{-1}g(g-F)^{-1}(g+F) \tag{7.38}$$

As one has

$$(g - F)(g + F)^{-1}(g + F)(g - F)^{-1}(g + F) = g + F \tag{7.39}$$

and

$$(g - F)(g + F)^{-1}(g - F)(g - F)^{-1}(g + F) = g - F \tag{7.40}$$

Adding two equations above and dividing by 2 gives

$$(g - F)(g + F)^{-1}g(g - F)^{-1}(g + F) = g. \tag{7.41}$$

Namely, R preserves the target space metric. By using this property, there are some relations among the SUSY charges. Explicitly, one has

$$G^+ + G^- = \bar{G}^+ + \bar{G}^- \tag{7.42}$$

In order to preserve $\mathcal{N} = 2$ SUSY, we have to ask R-symmetry to be preserved. For A model, we have to ask the current $J(z)+\bar{J}(\bar{z})$ is conserved. This amounts to imposing boundary condition $J = -\bar{J}$ on the boundary. In order to respect the boundary condition of R current, the condition above gets restricted to

$$G^+ = \bar{G}^-; G^- = \bar{G}^+. \tag{7.43}$$

It is equivalent to asking

$$R^T \omega R = -\omega. \tag{7.44}$$

It is called the *A brane condition*. Locally we can assume ω^{-1} takes the form

$$\omega^{-1} = \begin{bmatrix} A & B \\ -B^T & C \end{bmatrix} \tag{7.45}$$

where A, C are both antisymmetric matrices. Then one has the following matrix equations.

$$\begin{aligned} A &= 0 \\ BF &= 0 \\ gCg &= FCF \end{aligned} \tag{7.46}$$

The first condition indicates that $\omega\big|_{N_Y}$ vanishes. Therefore, Y is a coisotropic submanifold. Let's denote $\mathscr{L} = \ker(\omega\big|_Y)$ and $\mathscr{F} = TY/\mathscr{L}$. $\mathscr{L}$ can be understood as the part pairing with the vectors at a normal bundle, whose rank is equal to the codimension of Y. The second condition indicates that if we consider F as a map from TY to TY^*, then pulling back its image to NY gives zero. Therefore, F can be viewed as a section of $\Lambda^2\mathscr{F}$. Denote this section as f. The Kahler form also descends to a section σ over $\Lambda^2\mathscr{F}$. As the kernel of ω has been mod out, by construction it is a non-degenerate section. Therefore, $\mathscr{F}$ is a sympletic vector bundle.

To see the implication of the third condition, we use the metric g to split the tangent bundle as $TY = \mathscr{L} \oplus \mathscr{F}$. By construction $C = 0 \oplus \sigma^{-1}$. By $-id = I^2 = G^{-1}\omega G^{-1}\omega$, restricting the equation to Y gives $g(0 \oplus \sigma^{-1})g = -0 \oplus \sigma$. Compared with the third equation, then one has $f\sigma^{-1}f = -\sigma$. Namely, $(\sigma^{-1}f)^2 = -1$. $J = \sigma^{-1}f$ defines a natural almost complex structure over $\mathscr{F}$.

The coistropic condition indicates that $\dim_{\mathbb{R}} Y - \frac{1}{2}\dim_{\mathbb{R}} X$ is a non-negative integer. The existence of the almost complex structure implies that $\mathscr{F}$ has even rank. Moreover, once we consider the complexification $\Lambda^2\mathscr{F} \otimes \mathbb{C}$, then one can calculate the action of almost complex structure on f and σ. The result is that f and σ has eigenvalue 1 under the almost complex structure of $\Lambda^2\mathscr{F} \otimes \mathbb{C}$. As both f and σ is real, they are of type $(2,0) \oplus (0,2)$. In order to have two non-degenerate sections of this type, $\mathscr{F}$ has to be of complex rank even. Therefore, $\dim_{\mathbb{R}} Y - \frac{1}{2}\dim_{\mathbb{R}} X$ is a even integer.

Examples of A-brane Let's now see some examples. For $4d$ target space, a brane can be an A brane if it is coisotropic and of real dimension 2 or 4. One of the extreme cases is when $\dim_{\mathbb{R}} Y = \frac{1}{2}\dim_{\mathbb{R}} X$. In this case, $\mathscr{L} = TY$. The second condition simply states that $F = 0$, which means that the bundle over the boundary is flat. Therefore, a middle dimension A-brane is a lagrangian submanifold with a unitary flat bundle E.

Another extreme cases is when $\dim_{\mathbb{R}} Y = \dim_{\mathbb{R}} X$. In this case, we have $\mathscr{L} = 0$, while $\mathscr{F} = TY$. The first two conditions are now trivially implied. The last condition implies the existence of an almost complex structure $J = \omega^{-1}F$.

It can be argued that J is indeed integrable. The existence of an integrable complex structure J shows that the target is a complex manifold with respect to J. Note that the target is a complex manifold with respect to I from the construction. The new complex structure J is different from I, as the Kahler form has degree $(1,1)$ with respect to I but has $(2,0) + (0,2)$ with respect to J.

Hilbert Space of A-model Let's now discuss the Hilbert space of topological open string. To that aim, we first fix our notations. We fix the target space to be a hyper-Kahler manifold to admit good A-models. Often we consider the target space as an affine variety. (The physics origin comes from the compactification of Class $\mathcal{S}$ theory to 2D, while affine variety is the moduli space of Coulomb branch for 4D Class $\mathcal{S}$ on $\mathbf{R}^3 \times S^1$, see the following section for discussion). As a hyper-Kähler manifold, it has three almost complex structures (I, J, K). They satisfy the quaternion algebra $IJ = K$. For each almost complex structure we have the associated Kahler form $(\omega_I, \omega_J, \omega_K)$. Due to the quoternion algebra, we are gifted with holomorphic sympletic form associated with each complex structure. If we take $\Omega_I = \omega_J + i\omega_K$, $\Omega_J = \omega_K + i\omega_I$ and $\Omega_K = \omega_I + i\omega_J$, then they are holomorphic with respect to the complex structure in the lowering label. The relations between three complex structures are as follows.

$$I^t \omega_J = g(I-, J-) = -g(-, IJ-) = -\omega_K$$
$$I^t \omega_K = g(I-, K-) = -g(-, IK-) = \omega_J \tag{7.47}$$

Similarly,

$$\omega_J I = g(-, JI-) = -\omega_K$$
$$\omega_K I = g(-, KI-) = g(-, J-) = \omega_J \tag{7.48}$$

$$I^t \Omega_I I = I^t(\omega_J + i\omega_K)I = (-\omega_K + i\omega_J) = i\Omega_I I = -\Omega_I \tag{7.49}$$

Therefore, it is a holomorphic sympletic form with respect to I of type $(2, 0)$, because it is holomorphic with respect to two slots. Similar exercise can be conducted for other two complex structures to show that they are holomorphic with respect to the complex structure in their subscript.

The benefit of choosing the target manifold as hyper-Kähler is as follows. We first view the target manifold Y as a sympletic manifold with respect to ω_K.Then the A-brane supported on the whole manifold Y have a nice description. In this case, we consider the third equation $I = \omega_K^{-1}F$. It can be solved by $F = \omega_J$. Therefore, given any hyper-Kähler manifold, we have a natural choice of target space filling A-brane. We call it canonical coisotropic brane $\mathfrak{B}_{cc}$.

Before determining the hilbert space of open string A-model, we can make some general arguments. Given a topological open string, the SUSY transformation of the scalar field takes the form

$$\delta X^k = \{Q, X^k\} = \frac{1}{2}\int dz G_{i\bar{j}}\psi^i \partial_z X^{\bar{j}}, X^k + c.c. = \frac{1}{2}\psi^k + c.c. \tag{7.50}$$

where the third equality follow form $\Pi_{X^i} = G_{i\bar{j}}\partial_z X^{\bar{j}}$. However, notice that we thought of the targer space as a Kahler manifold with respect to complex structure K. To make it explicit, we write

$$\delta X = \frac{1}{4}(1 - iK)\Psi + \frac{1}{4}(1 + iK)\bar{\Psi} \tag{7.51}$$

$\frac{1}{2}(1-iK)$ acts as a projection operator into the holomorphic components while $\frac{1}{2}(1+iK)$ acts as a projection operator into the anti-holomorphic components. To construct the hilbert space, we need to consider the BRST invariant operators. Let's now Consider the operator inserted in the bulk of the string. The most simple bosonic operator with fermion number 0 one can construct takes the form $f(X)$, where f is a holomorphic function. Taking the variation, one get

$$\delta_\epsilon f(X) = \frac{1}{4}\partial_X f(X)[(1 - iK)\epsilon\Psi + (1 + iK)\bar{\epsilon}\bar{\Psi}] \tag{7.52}$$

The latter two terms can not be canceled out in bulk because they are independent degrees of freedom. To get a vanishing result, one can only ask $\partial_X f(X) = 0$, which means that the bulk operator of this type can only be constant functions. Therefore, the interesting operators should be inserted on the boundary. On the boundary, two fermionic degrees of freedom is related with each other via boundary condition. If we consider $\mathfrak{B}_{cc}$ boundary condition, then one have

$$\delta X = \frac{1}{4}(1-iK)\Psi + \frac{1}{4}(1+iK)\bar{\Psi} = \frac{1}{4}((1-iK)(G-F)^{-1}(G+F) + (1+iK))\Psi \tag{7.53}$$

By $G = -K^t \omega_K$ and $F = \omega_J$, one have $(G-F)^{-1}(G+F) = J$. Then the variation is simplified to

$$\delta X = \frac{1}{4}((1-iK)J + (1+iK))\Psi = \frac{1}{4}(J + iI + 1 + iIJ)\Psi = \frac{1}{4}(1+iI)(1+J)\Psi \quad (7.54)$$

This expression means that after the variation, the term survives on the boundary is antiholomorphic with respect to I. Splitting $\delta = \delta^{0,1} + \delta^{1,0}$ with respect to complex structure I, one has

$$\begin{aligned} \delta^{1,0} X &= 0, \\ \delta^{0,1} X &= \rho, \end{aligned} \quad (7.55)$$

where $\rho = \frac{1}{4}(1+iI)(1+J)\Psi$. As for BRST charge, $\delta^2 = 0$, therefore one have $\delta\rho = 0$.

Following from the first variation, the boundary operator of the form $f(X)$ is invariant under BRST transformation if and only if f is a holomorphic function with respect to I. Remember that an operator is BRST exact if it has a non-trivial conformal dimension. Thus we can focus on dimension zero operators. Generically, an dimension zero operator takes the form $f(X, \bar{X})_{\bar{i}_1 \ldots \bar{i}_n} \rho^{\bar{i}_1} \ldots \rho^{\bar{i}_n}$, which has a grading endowed by the fermion number. Similar to the discussion before, Q-closed operators can be interpreted as some geometric quantity. We may identify $\rho^{\bar{i}}$ as $d\bar{X}^{\bar{i}}$ where $\bar{i}$ notation is written with respect to complex structure I. Then the Q action can be identified as the $\bar{\partial}$ operator. The space of Q-exact operator modulo Q-closed operator is thus simply the Dolbeault Cohomology $H^{0,*}_{\bar{\partial}}(Y)$ with respect to the complex structure I.

When the A-brane Y is an affine variety, the higher degree cohomology in $H^{0,*}_{\bar{\partial}}(Y)$ vanishes, and the Hilbert space reduces to the coordinate ring

$$H^{0,*}_{\bar{\partial}}(Y) \cong H^{0,0}_{\bar{\partial}}(Y) \cong \mathscr{O}(Y). \quad (7.56)$$

7.4 Brane quantization

As discussed above, the spherical DAHA $S\ddot{H}$ arises from the deformation quantization of the coordinate ring of

$$\mathfrak{X} = \mathscr{M}_{\text{flat}}(C_{0,4}, \mathrm{SL}(2, \mathbb{C})) \quad (7.57)$$

with respect to the holomorphic symplectic form Ω_J. This space is both an affine variety and a hyper-Kähler manifold. These properties naturally place it into the framework of brane quantization[32] in a 2d sigma-model with target space $\mathfrak{X}$. Brane quantization provides a geometric approach to the representation theory of $S\ddot{H}$, which is the main focus of this paper. We will briefly review the brane quantization method here while referring the reader to sections 2.3 and 2.4 of[25] for a more detailed treatment in the context of DAHA.

We consider the topological A-model on a symplectic manifold $(\mathfrak{X}, \omega_{\mathfrak{X}})$, where quantization is achieved via open strings in the A-model. Brane quantization incorporates both deformation quantization and geometric quantization simultaneously, naturally providing the algebra and its representation, respectively. The boundary conditions of open strings are determined by geometric data in the target space, known as A-branes. Depending on whether one is dealing with deformation quantization or geometric quantization, one considers two types of A-brane: the canonical coisotropic brane $\mathfrak{B}_{\text{cc}}$ and the Lagrangian branes $\mathfrak{B}_{\mathbf{L}}$.

The deformation quantization, which provides the algebra, is achieved by the canonical coisotropic brane, specifically via open $(\mathfrak{B}_{cc}, \mathfrak{B}_{cc})$-strings. The canonical coisotropic brane, which can be figuratively described as the "big A-brane," is a holomorphic line bundle over the target space itself:

$$\mathfrak{B}_{cc}: \quad \begin{array}{c} \mathscr{L} \\ \downarrow \\ \mathfrak{X} \end{array} \qquad c_1(\mathscr{L}) = [F/2\pi] \in H^2(\mathfrak{X}, \mathbb{Z})\,. \tag{7.58}$$

As usual, the curvature F forms a gauge invariant combination with the 2-form B-field in the 2d sigma-model, given by $F + B$, where

$$B \in H^2(\mathfrak{X}, \mathrm{U}(1))\,. \tag{7.59}$$

The parameter of the deformation quantization can be written as

$$q = e^{2\pi i\hbar}, \qquad \hbar \in \mathbb{C}^\times. \tag{7.60}$$

The canonical coisotropic brane $\mathfrak{B}_{cc}$ is parameterized by $\hbar$ on the symplectic manifold $(\mathfrak{X}, \omega_{\mathfrak{X}})$, with the following structure:

$$\Omega := F + B + i\omega_{\mathfrak{X}} = \frac{\Omega_J}{i\hbar}\,. \tag{7.61}$$

At a generic value of $\hbar = |\hbar|e^{i\theta}$, we can express the real and imaginary parts of Ω as

$$\begin{aligned} F + B = \Re\Omega &= \frac{1}{|\hbar|}(\omega_I \cos\theta - \omega_K \sin\theta)\,, \\ \omega_{\mathfrak{X}} = \Im\Omega &= -\frac{1}{|\hbar|}(\omega_I \sin\theta + \omega_K \cos\theta)\,, \end{aligned} \tag{7.62}$$

so the symplectic form $\omega_{\mathfrak{X}}$ depends on the value of $\hbar$. Thanks to the hyper-Kähler structure $\omega_{\mathfrak{X}}^{-1}(F + B) = J$, the brane automatically satisfies the condition $(\omega_{\mathfrak{X}}^{-1}(B + F))^2 = -1$, which is required for a coisotropic A-brane[26]. With this setup, the space of open $(\mathfrak{B}_{cc}, \mathfrak{B}_{cc})$-strings gives rise to the deformation quantization of the coordinate ring on $\mathfrak{X}$, holomorphic in J.

To see the connection between the 4d $\mathcal{N} = 2$ theory and the 2d sigma-model, we compactify the 4d $\mathcal{N} = 2$ theory on $T^2 \cong S^1 \times S^1_q$, as illustrated in Figure 7-1. This compactification yields a 2d sigma-model $\mathbb{R} \times \mathbb{R}_+ \cong \Sigma \to \mathscr{M}_{\text{flat}}(C_{0,4}, \mathrm{SL}(2, \mathbb{C}))$ on the Coulomb branch. Here, $S^1_q \subset \mathbb{R}^2$ is a circle that encircles the axis of the Ω-background, where the loop operators intersect. By the state-operator correspondence, loop operators in the 4d $\mathcal{N} = 2$ theory map to states in $\mathrm{Hom}(\mathfrak{B}_{cc}, \mathfrak{B}_{cc})$. Thus, upon the compactification, the canonical coisotropic brane condition $\mathfrak{B}_{cc}$ naturally emerges from the "axis of the Ω-deformation" $\partial\Sigma$ (or the tip of the cigar, as described in[33]).

The study on the representation theory of $\mathscr{O}^q(\mathfrak{X})$ in the context of the 2d A-model originates from a simple idea: given an A-brane boundary condition $\mathfrak{B}'$, the space of open strings between $\mathfrak{B}_{cc}$ and $\mathfrak{B}'$ forms a vector space $\mathrm{Hom}(\mathfrak{B}', \mathfrak{B}_{cc})$. As illustrated on the right side of Figure 7-2, the joining of $(\mathfrak{B}_{cc}, \mathfrak{B}_{cc})$ and $(\mathfrak{B}_{cc}, \mathfrak{B}')$-string produces another $(\mathfrak{B}_{cc}, \mathfrak{B}')$-string. This implies that the space of $(\mathfrak{B}_{cc}, \mathfrak{B}')$-strings receives an

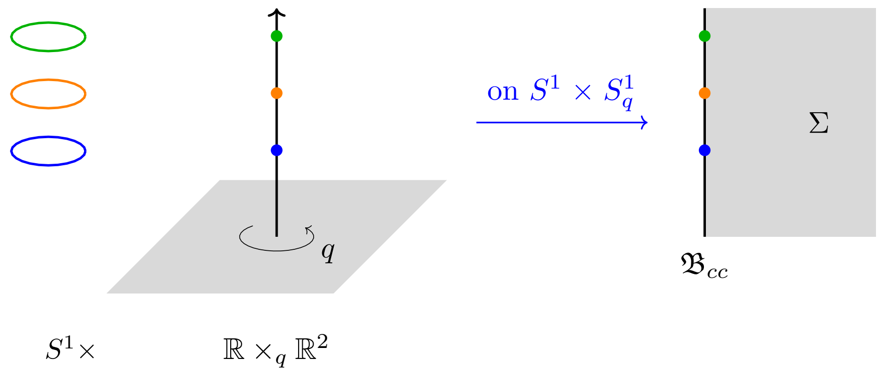

Figure 7-1 In the presence of the Omege background, the line operators localizes to a line, which makes the line operator algebra non-commutative.

action of the algebra of $(\mathfrak{B}_{\rm cc}, \mathfrak{B}_{\rm cc})$ strings[32]. In other words, A-branes $\mathfrak{B}'$ on $\mathfrak{X}$ correspond to representations of $\mathscr{O}^q(\mathfrak{X})$:

$$\begin{array}{ccc} \mathscr{O}^q(\mathfrak{X}) & = & \mathrm{Hom}(\mathfrak{B}_{\rm cc}, \mathfrak{B}_{\rm cc}) \\ \downarrow & & \downarrow \\ \mathscr{B}' & = & \mathrm{Hom}(\mathfrak{B}', \mathfrak{B}_{\rm cc})\,. \end{array} \tag{7.63}$$

In other words, brane quantization naturally proposes a derived functor

$$\mathbf{R}\mathrm{Hom}(-, \mathfrak{B}_{\rm cc}) : D^b A\text{-}\mathsf{Brane}(\mathfrak{X}, \omega_{\mathfrak{X}}) \to D^b\, \mathsf{Rep}(\mathscr{O}^q(\mathfrak{X}))\,, \tag{7.64}$$

which conjecturally provides a derived equivalence between the category of A-branes and the derived category of $\mathscr{O}^q(\mathfrak{X})$-modules.

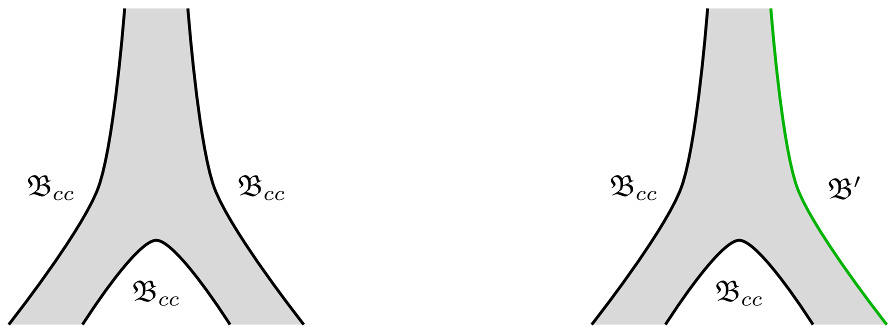

Figure 7-2 (Left) Open strings that start and end on the same brane $\mathfrak{B}_{\rm cc}$ form an algebra. (Right) Joining a $(\mathfrak{B}_{\rm cc}, \mathfrak{B}_{\rm cc})$-string with a $(\mathfrak{B}_{\rm cc}, \mathfrak{B}')$-string leads to another $(\mathfrak{B}_{\rm cc}, \mathfrak{B}')$-string.

As the target symplectic manifold $(\mathfrak{X}, \omega_{\mathfrak{X}})$ is of quaternionic dimension one, A-branes of the other types are all Lagrangian A-branes, namely it has a Lagrangian submanifold $\mathbf{L}$ as its support, endowed with a flat Spin^c-bundle:

$$\mathfrak{B}_{\mathbf{L}} : \quad \begin{array}{c} \mathscr{L}' \otimes K_{\mathbf{L}}^{-1/2} \\ \downarrow \\ \mathbf{L} \end{array} \tag{7.65}$$

where $K_{\mathbf{L}}^{-1/2}$ is a square root of the canonical bundle of $\mathbf{L}$, which gives rise to a Spin^c structure when $\mathscr{L}'$ is not a genuine line bundle. (See[25,32] for more details.) The subtlety of Spin^c structures appears only when we consider bound states of A-branes, and both $\mathscr{L}'$ and $K_{\mathbf{L}}^{-1/2}$ exist as genuine line bundles in most of the examples in this paper since all the Lagrangian submanifolds considered are of real two dimensions. Additionally, a Lagrangian A-brane must satisfy the flatness condition

$$F' + B|_{\mathbf{L}} = 0 \, , \tag{7.66}$$

where F' is the curvature of $\mathscr{L}'$, and $F' + B$ is the gauge-invariant combination as before.

Then, the space of $(\mathfrak{B}_{\text{cc}}, \mathfrak{B}_{\mathbf{L}})$-open string arises from the geometric quantization of $\mathbf{L}$, namely the space of holomorphic sections $\mathfrak{B}_{\text{cc}} \otimes \mathfrak{B}_{\mathbf{L}}^{-1}$ over $\mathbf{L}$. Hence, when the support is a compact Lagrangian submanifold, one can employ the B-model perspective to compute the dimension of the representation space $\text{Hom}(\mathfrak{B}_{\mathbf{L}}, \mathfrak{B}_{\text{cc}})$ using Hirzebruch-Riemann-Roch formula:

$$\begin{aligned} \dim \mathscr{L} &= \dim \text{Hom}(\mathfrak{B}_{\mathbf{L}}, \mathfrak{B}_{\text{cc}}) \\ &= \dim H^0(\mathbf{L}, \mathfrak{B}_{\text{cc}} \otimes \mathfrak{B}_{\mathbf{L}}^{-1}) \\ &= \int_{\mathbf{L}} \text{ch}(\mathfrak{B}_{\text{cc}}) \wedge \text{ch}(\mathfrak{B}_{\mathbf{L}}^{-1}) \wedge \text{Td}(T\mathbf{L}) \, , \end{aligned} \tag{7.67}$$

where $\text{Td}(T\mathbf{L})$ is the Todd class of the tangent bundle of $\mathbf{L}$. Since $\mathbf{L}$ is real two-dimensional in our case, the Todd class $\text{Td}(T\mathbf{L})$ is equal to $\text{ch}(K_{\mathbf{L}}^{-1/2})$. Thus, the dimension formula is simplified to

$$\dim \mathscr{L} = \int_{\mathbf{L}} \text{ch}(\mathfrak{B}_{\text{cc}}) = \int_{\mathbf{L}} \frac{F + B}{2\pi} \, , \tag{7.68}$$

for a real two-dimensional Lagrangian $\mathbf{L}$. In this way, we can explicitly provide the dimension of a finite-dimensional representation corresponding to a Lagrangian A-brane with compact support.

7.5 Brane Quantization of Hitchin Moduli Space

Consider a four-dimensional Class $\mathcal{S}$ theory $\mathscr{T}[\Sigma_{g,n}]$ on $\mathbb{R}^1 \times S^1 \times D_R$. Here D_R is the disk equipped with cigar-like metric

$$ds^2 = d^2r + f(r)d\theta^2, \quad 0 \le r \le R, \quad 0 \le \theta \le 2\pi. \tag{7.69}$$

with $f(r) \to r^2$ in $r \to 0$ limit, and $f(r) \to \rho^2$ asymptotic to a fix value while r is larger than a certain scale $r_0 \le r \le R$. We also expect $R >> \rho, r_0$. Macroscopically, D_R could be approximated by $I \times \tilde{S}^1$. The gauge theory at the macroscopic level is thus described by the theory on the $T^2 = S^1 \times \tilde{S}^1_\rho$ fibration over the strip $\mathbb{R}^1 \times I$. Therefore, the theory at low energy could be described by a sigma model with world volume $\mathbb{R} \times I$.

Similar to the discussion in the three-dimensional situation, the target space of the sigma model is the Hitchin Moduli space over $\Sigma_{g,n}$. A complete description of the low-energy action involves the boundary conditions at the two ends of the strip. The $r = 0$ boundary condition is completely geometrical, which is similar to the D_6 brane in Type IIA construction. The $r = R$ boundary condition comes from the boundary condition in four-dimensional gauge theory.

One can introduce the Ω-background $S^1 \times \mathbb{R} \times_q \mathbb{R}^2$, which effectively introduces a potential around the origin of the Ω-deformation. As illustrated in Figure 7-1, the loop operators are localized along the axis of the Ω-deformation and are forced to cross each other as they exchange positions. Consequently, the algebra of loop operators becomes non-commutative, providing a physical realization of the deformation quantization of the coordinate ring[33–38].

7.6 Example: Brane Quantization of the $SU(2)$ Hitchin Moduli Space over $\Sigma_{0,4}$

In this section, we will investigate the brane quantization of the $SU(2)$ Hitchin Moduli Space over the Fourth Punctured Sphere, which is the Coulomb branch of 4d $\mathcal{N} = 2$ SQCD with four flavors.

The algebra of $(\mathfrak{B}_{\mathrm{cc}}, \mathfrak{B}_{\mathrm{cc}})$ string is described by the deformation quantization of the coordinate ring of the $SL_2\mathbb{C}$ character variety. The deformation quantization in this case is known, which is described by the spherical double affine Hecke algebra of $C^\vee C_1$ type, denoted as $S\ddot{H}$.

7.6.1 Definition of spherical DAHA

The spherical DAHA $S\ddot{H}$ is generated by x, y, z satisfying the defining relations

$$\begin{aligned}
[x, y]_q &= (q^{-1} - q)z + (q^{-\frac{1}{2}} - q^{\frac{1}{2}})\theta_3 \, , \\
[y, z]_q &= (q^{-1} - q)x + (q^{-\frac{1}{2}} - q^{\frac{1}{2}})\theta_1 \, , \\
[z, x]_q &= (q^{-1} - q)y + (q^{-\frac{1}{2}} - q^{\frac{1}{2}})\theta_2 \, ,
\end{aligned} \tag{7.70}$$

where the q-commutator is defined as $[f, g]_q \equiv q^{-\frac{1}{2}} fg - q^{\frac{1}{2}} gf$ and

$$\theta_1 = \overline{t_1}\,\overline{t_2} + \overline{t_3}\,\overline{t_4} \, , \quad \theta_2 = \overline{t_1}\,\overline{t_3} + \overline{t_2}\,\overline{t_4} \, , \quad \theta_3 = \overline{t_1}\,\overline{t_4} + \overline{t_2}\,\overline{t_3} \, , \tag{7.71}$$

using the notation $\overline{f} \equiv f + f^{-1}$. In addition, the generators x, y, z satisfy a Casimir relation

$$-q^{-\frac{1}{2}}xyz + q^{-1}x^2 + qy^2 + q^{-1}z^2 + q^{-\frac{1}{2}}\theta_1 x + q^{\frac{1}{2}}\theta_2 y + q^{-\frac{1}{2}}\theta_3 z + \theta_4(q) = 0 \, , \tag{7.72}$$

where $\theta_4(q) = \overline{t_1}^2 + \overline{t_2}^2 + \overline{t_3}^2 + \overline{t_4}^2 + \overline{t_1}\,\overline{t_2}\,\overline{t_3}\,\overline{t_4} - q - q^{-1} - 2$.

As is evident from (7.70), the spherical DAHA $S\ddot{H}$ becomes commutative in the "classical" limit $q = 1$, and the Casimir relation (7.72) reduces to the equation of an affine cubic surface:

$$-xyz + x^2 + y^2 + z^2 + \theta_1 x + \theta_2 y + \theta_3 z + \theta_4(1) = 0 \, . \tag{7.73}$$

7.6.2 Symmetries of the Spherical DAHA

The spherical DAHA $S\ddot{H}$ exhibits three key types of symmetries, each of which plays an essential role in its structure.

Weyl group $W(D_4)$: The parameters $\theta_1, \theta_2, \theta_3, \theta_4$ can be naturally interpreted as the characters of the vector, spinor, conjugate spinor, and adjoint representations the Lie algebra $\mathfrak{so}(8) = D_4$. The characters are invariant under this Weyl group $W(D_4)$ of $\mathfrak{so}(8)$, and so is the q-deformed version $\theta_4(q)$. Consequently, the spherical DAHA is also invariant under $W(D_4)$.

Braid group B_3: The spherical subalgebra $S\ddot{H}$ is invariant under the B_3 action

$$\begin{aligned} \tau_+ &: (x,y,z,\theta_1,\theta_2,\theta_3,\theta_4) \mapsto \left(x, q^{-\frac{1}{2}}(xy - q^{-\frac{1}{2}}z - \theta_3), y, \theta_1, \theta_3, \theta_2, \theta_4\right) , \\ \tau_- &: (x,y,z,\theta_1,\theta_2,\theta_3,\theta_4) \mapsto \left(q^{-\frac{1}{2}}(xy - q^{-\frac{1}{2}}z - \theta_3), y, x, \theta_3, \theta_2, \theta_1, \theta_4\right) , \\ \sigma &: (x,y,z,\theta_1,\theta_2,\theta_3,\theta_4) \mapsto \left(y, x, q^{-\frac{1}{2}}(xy - q^{-\frac{1}{2}}z - \theta_3), \theta_2, \theta_1, \theta_3, \theta_4\right) . \end{aligned} \tag{7.74}$$

B_3 has a $\mathbb{Z}_3$ cyclic permutation subgroup generated by $(x,y,z,\theta_1,\theta_2,\theta_3,\theta_4) \mapsto (y,z,x,\theta_2,\theta_3,\theta_1,\theta_4)$.

Sign-flip group $\mathbb{Z}_2^{\times 2}$: Lastly, there is a symmetry $\mathbb{Z}_2 \times \mathbb{Z}_2 = \mathbb{Z}_2^{\times 2}$ which flips the signs of x, y, z and $\theta_1, \theta_2, \theta_3$. See[39] for the details.

7.6.3 Polynomial Representations

The spherical DAHA $S\ddot{H}$ acts on $\mathscr{P} \equiv \mathbb{C}_{q,\mathbf{t}}[X, X^{-1}]^{\mathbb{Z}_2}$[40–42], which is the space of symmetric Laurent polynomials with coefficients in the ring $\mathbb{C}_{q,\mathbf{t}}$ of rational functions in q and $\mathbf{t}$. The representation is given by $\mathrm{pol} : S\ddot{H} \to \mathrm{End}(\mathscr{P})$ and the actions of the generators are given by:

$$\begin{aligned} \mathrm{pol}(x) &= X + X^{-1} , \\ \mathrm{pol}(y) &= A(X)(\varpi - 1) + A(X^{-1})(\varpi^{-1} - 1) - q^{\frac{1}{2}} t_1 t_3 - q^{-\frac{1}{2}}(t_1 t_3)^{-1} , \\ \mathrm{pol}(z) &= q^{\frac{1}{2}} \left[X A(X) \varpi + X^{-1} A(X^{-1}) \varpi^{-1}\right] \\ &\quad - \frac{X + X^{-1}}{q^{\frac{1}{2}} + q^{-\frac{1}{2}}} \left[A(X) + A(X^{-1}) + q^{\frac{1}{2}} t_1 t_3 + q^{-\frac{1}{2}} t_1^{-1} t_3^{-1}\right] - \frac{\theta_3}{q^{\frac{1}{2}} + q^{-\frac{1}{2}}} . \end{aligned} \tag{7.75}$$

with $A(X)$ a rational function and ϖ the q-shift operator defined by $\varpi \cdot f(X) = f(qX)$. Under the polynomial representation (7.75), $\mathrm{pol}(y)$ acts diagonally on the Askey-Wilson polynomials. We can define the raising and lowering operators in $S\ddot{H}$ with respect to this basis[43] as

$$R_n = -q^{1-n} x - q t_1 t_3 z + E_n , \quad L_n = -q^{n+2} x - \frac{q}{t_1 t_3} z + D_n , \tag{7.76}$$

where D_n and E_n are two constants. Under the polynomial representation, these operators raise and lower the labels n of the Askey-Wilson polynomials as follows:

$$\mathrm{pol}(R_n) \cdot P_n = q^{1-n} \left(t_1^2 t_3^2 q^{2n+1} - 1\right) P_{n+1} , \tag{7.77}$$

$$\begin{aligned} \mathrm{pol}(L_n) \cdot P_n = &-q^{1-n} \left(q^n - t_1^{-1} t_2 t_3^{-1} t_4\right) \left(q^n - t_1^{-1} t_2 t_3^{-1} t_4^{-1}\right) \left(q^n - t_1^{-1} t_2^{-1} t_3^{-1} t_4\right) \\ &\times \left(q^n - t_1^{-1} t_2^{-1} t_3^{-1} t_4^{-1}\right) \frac{(q^n - 1)(q^n - t_1^{-2})(q^n - t_3^{-2})(q^n - t_1^{-2} t_3^{-2})}{\left(q^{2n} - t_1^{-2} t_3^{-2}\right)^2 (q^{2n-1} - t_1^{-2} t_3^{-2})} P_{n-1} . \end{aligned} \tag{7.78}$$

A common way to construct a finite-dimensional representation is to find null vectors for the raising and lowering operators. The raising operators can never be null, while there are conditions where the lowering operator annihilates some $P_n(x;q,\mathbf{t})$, which becomes the lowest weight state of a sub-representation. In this way, a finite-dimensional $S\ddot{H}$-module appears as the quotient $\mathscr{P}/(P_n)$ of the polynomial representation by the ideal (P_n).

Therefore, we can study finite-dimensional representations by imposing the condition $\mathrm{pol}(L_n)\cdot P_n = 0$. This amounts to the shortening conditions neatly repackaged as

$$q^n = t_1^{-r_1}t_2^{-r_2}t_3^{-r_3}t_4^{-r_4} =: \mathbf{t}^{-r}, \qquad r \in \mathsf{R}(D_4) \cup \{(0,0,0,0)\} \,. \tag{7.79}$$

7.6.4 Global Nilpotent Cone

When the ramification parameters α_j are generic while the others are set to zero, $\beta_j = \gamma_j = 0$ $(j = 1,2,3,4)$, the Hitchin fibration $h : \mathscr{M}_H(C_p, \mathrm{SU}(2)) \to \mathscr{B}_H$ possesses a singular fiber only at the origin of the base, $\mathbf{N} = h^{-1}(0)$, which is referred to as the *global nilpotent cone*. The global nilpotent cone $\mathbf{N}$ is a singular fiber of Kodaira type I_0^* [44], characterized by a configuration of five irreducible components, each of which is topologically $\mathbb{CP}^1$, arranged in the shape of the affine D_4 Dynkin diagram:

$$\mathbf{N} = \mathbf{V} \cup \bigcup_{j=1}^{4} \mathbf{D}_j \,. \tag{7.80}$$

At generic values of α_j, each irreducible component of the global nilpotent cone serves as a generator of the second integral homology group, $H_2(\mathscr{M}_H, \mathbb{Z}) \cong \mathbb{Z}^{\oplus 5}$. Using the basis $\{[\mathbf{D}_1], [\mathbf{D}_2], [\mathbf{D}_3], [\mathbf{D}_4], [\mathbf{V}]\}$, the intersection pairing between these homology classes is represented by the Cartan matrix of the affine D_4 Dynkin diagram and the second homology group $H_2(\mathscr{M}_H, \mathbb{Z})$ can be identified with the root lattice of the affine D_4 Lie algebra. In this identification, the irreducible components correspond to the simple roots of the affine D_4 root system. Furthermore, these components also satisfy a fiber-class relation in $H_2(\mathscr{M}_H, \mathbb{Z})$:

$$[\mathbf{F}] = 2[\mathbf{V}] + \sum_{j=1}^{4}[\mathbf{D}_j] \,, \tag{7.81}$$

where $[\mathbf{F}]$ denotes the homology class of a generic fiber of the Hitchin fibration [44].

As a characteristic of the Hitchin fibration, due to its complete integrability, the period of $\omega_I/2\pi$ over a general fiber $\mathbf{F}$ is one, so $\mathrm{vol}_I(\mathbf{F}) = 1$, and the homology relation(7.81) imposes the constraint

$$2\mathrm{vol}_I(\mathbf{V}) + \sum_{j=1}^{4} \mathrm{vol}_I(\mathbf{D}_j) = 1 \,. \tag{7.82}$$

The relation between the ramification parameters α_j and the volumes of the irreducible components in the global nilpotent cone is subtle and involves the wall-crossing phenomenon. When an irreducible component $\mathbf{V}$ or $\mathbf{D}_j$ shrinks to zero, the Higgs bundle is no longer stable, and the Hitchin moduli space develops a du Val singularity. These singularities correspond to special points in the Kähler moduli space, which are located at codimension-one loci, referred to as walls. Du Val singularities appear precisely at the zeros of the Seiberg-Witten discriminant, which can be neatly summarized as

$$\mathbf{t}^r = 1\,, \quad \forall r \in \mathsf{R}(D_4)\,. \tag{7.83}$$

These walls align with the set of reflection hyperplanes for the affine D_4 weight lattice at level one[45]. In an affine root system, the reflection of an affine weight λ at level one with respect to an affine root $\dot{r} = r - n\delta$

$$s_{\dot{r}}(\lambda) = s_r\left(\lambda - \frac{n}{2}r\right) \tag{7.84}$$

generating the affine Weyl group $\dot{W}(D_4)$. Therefore, a chamber surrounded by the walls corresponds to a Weyl alcove of type D_4.

A du Val singularity arises when a two-cycle shrinks to zero volume, whose condition is linear in the Kähler parameters α_j. This implies that the volume of a compact two-cycle depends linearly on α_j. Then the configurations of the walls in (7.83) uniquely determine the volumes of the cycles. Given a point α_j of the Kähler moduli space, the volumes of the two-cycles $\mathbf{V}$ and $\mathbf{D}_j$ ($j = 1, 2, 3, 4$) are equal to twice the distance from this point to the corresponding walls.

It turns out that, there are 24 chambers in total generically. In the subsequent analysis, we focus on the chamber defined by the following constraints (which we refer to as the center chamber):

$$\begin{aligned} \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 &\leq 1\,, & -\alpha_1 - \alpha_2 + \alpha_3 + \alpha_4 &\geq 0\,, \\ -\alpha_1 + \alpha_2 - \alpha_3 + \alpha_4 &\geq 0\,, & -\alpha_1 + \alpha_2 + \alpha_3 - \alpha_4 &\geq 0\,, \end{aligned} \tag{7.85}$$

together with the restriction $\alpha_1 \in [0, \frac{1}{4}]$. From the identification between the volume functions and the basis of the affine root system, we deduce the explicit volume functions in this chamber:

$$\begin{aligned} \mathrm{vol}_I(\mathbf{D}_j) = \Big(&1 - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4,\ -\alpha_1 - \alpha_2 + \alpha_3 + \alpha_4, \\ &- \alpha_1 + \alpha_2 - \alpha_3 + \alpha_4,\ -\alpha_1 + \alpha_2 + \alpha_3 - \alpha_4\Big), \\ \mathrm{vol}_I(\mathbf{V}) = 2&\alpha_1\,, \end{aligned} \tag{7.86}$$

where the normalization in (7.82) is appropriately applied.

In fact, we can define the simple roots of the D_4 root system with a suitable choice of positive roots as

$$\{e^1, e^2, e^3, e^4\} = \{(-1,-1,1,1), (-1,1,-1,1), (-1,1,1,-1), (2,0,0,0)\}\,, \tag{7.87}$$

Then, the highest root is expressed by $\theta = e^1 + e^2 + e^3 + 2e^4 = (1, 1, 1, 1)$. Using these roots, the volume functions can be concisely written as

$$\begin{aligned} &\Big(\mathrm{vol}_I(\mathbf{D}_1), \mathrm{vol}_I(\mathbf{D}_2), \mathrm{vol}_I(\mathbf{D}_3), \mathrm{vol}_I(\mathbf{D}_4), \mathrm{vol}_I(\mathbf{V})\Big) = \\ &\qquad\qquad \Big(1 - \theta\cdot\square,\ e^1\cdot\square,\ e^2\cdot\square,\ e^3\cdot\square\,, e^4\cdot\square\Big)\,, \end{aligned} \tag{7.88}$$

where $r \cdot \square = \sum_{j=1}^{4} r_j \alpha_j$ is the Euclidean inner product. The homology class of each irreducible component of the I_0^* singular fiber corresponds to an affine D_4 root as

$$[\mathbf{D}_1] \leftrightarrow e^0 = \delta - \theta\,, \quad [\mathbf{D}_2] \leftrightarrow e^1\,, \quad [\mathbf{D}_3] \leftrightarrow e^2\,, \quad [\mathbf{D}_4] \leftrightarrow e^3\,, \quad [\mathbf{V}] \leftrightarrow e^4\,, \tag{7.89}$$

where δ is the imaginary root and the case $\delta \cdot \alpha = 1$ is assumed for the volume function. Using the relation of the second homology classes, the homology class $[\mathbf{F}]$ of a generic Hitchin fiber indeed corresponds to the imaginary root δ.

In the following section, we will show the main result of this project: the explicit correspondence between A-branes and the representations of the $S\ddot{H}$. In particular, we will match the A-brane conditions and the shortening conditions for the generic fibers and global nilpotent cone.

7.6.5 Non-compact Branes and Infinite-dimensional Representations

To identify the A-brane associated with the polynomial representation in (7.75), we consider the classical limit $q \to 1$. In this limit, the lowering operator L_n becomes independent of n, so we denote $L^{(c)} = L_n|_{q\to 1}$. In the classical limit, certain operators act as scalar multiplications:

$$\begin{aligned} \mathrm{pol}(y) &= -t_1t_3 - t_1^{-1}t_3^{-1}, \\ -\mathrm{pol}(L^{(c)}) = 0 &= \mathrm{pol}(x) + \frac{\mathrm{pol}(z)}{t_1t_3} + t_1^{-1}t_2^{-1} + t_1^{-1}t_2 + t_3^{-1}t_4^{-1} + t_3^{-1}t_4. \end{aligned} \tag{7.90}$$

The second equation holds because the lowering operator becomes null in the classical limit $q \to 1$, as demonstrated in (7.78). Geometrically, this describes the support of the brane $\mathfrak{B}_{\mathbf{P}}$, for the polynomial representation:

$$\mathbf{P} = \{y = -t_1t_3 - t_1^{-1}t_3^{-1}, z = -t_1t_3x - t_1t_4^{-1} - t_1t_4 - t_2^{-1}t_3 - t_2t_3\} \tag{7.91}$$

The restriction of Ω_J on $\mathbf{P}$ vanishes. Thus $\mathbf{P}$ is Lagrangian with respect to the symplectic forms ω_I and ω_K. This establishes that $\mathfrak{B}_{\mathbf{P}}$ is an (A, B, A)-brane associated with the polynomial representation.

We can construct new polynomial representations by applying the Weyl group action $W(D_4)$ and the cyclic group $\mathbb{Z}_3$ to $\mathbf{P}$①. Since both group actions act linearly on the coordinates (x, y, z), they map the line $\mathbf{P}$ to other lines. As a result, 24 distinct lines can be generated from $\mathbf{P}$.

A notable feature of these lines is that they all lie in planes where one of the coordinates x, y, or z remains constant. Thus, the slope of each line can be described by a single complex number. To formalize this, we define the slope as follows: if x is constant along the line, the slope is given by $\frac{dy}{dz}$; if y is constant, the slope is $\frac{dz}{dx}$; and if z is constant, the slope is $\frac{dx}{dy}$.

Furthermore, we denote these slopes by $\mathbb{S}_x$, $\mathbb{S}_y$, and $\mathbb{S}_z$, corresponding to lines in planes where x, y, or z is constant, respectively. Interestingly, these sets have a natural interpretation in terms of SO(8) representation theory:

$$\mathbb{S}_x = \{-\mathbf{t}^w \mid w \in \mathsf{P}(\mathbf{8}_V)\}, \quad \mathbb{S}_y = \{-\mathbf{t}^w \mid w \in \mathsf{P}(\mathbf{8}_S)\}, \quad \mathbb{S}_z = \{-\mathbf{t}^w \mid w \in \mathsf{P}(\mathbf{8}_C)\}, \tag{7.92}$$

① The action of braid group will generate more (A, B, A)-branes, see[39] for the details.

where $\mathbf{t}^w \equiv t_1^{w_1} t_2^{w_2} t_3^{w_3} t_4^{w_4}$, and $\mathsf{P}(\mathbf{8}_V)$, $\mathsf{P}(\mathbf{8}_S)$, and $\mathsf{P}(\mathbf{8}_C)$ are the weights of the SO(8) vector, spinor, and cospinor representations, respectively. These weights correspond to the shortest weights in the D_4 weight lattice.

Consequently, the 24 non-zero roots of the D_4 root system are organized into three distinct sets, each in one-to-one correspondence with the weights of the SO(8) vector, spinor, and cospinor representations. Furthermore, a similar argument based on the pullback of Ω_J confirms that the branes corresponding to these 24 lines are (A, B, A)-branes in the cubic surface.

As a result, the lines and their corresponding infinite-dimensional representations can be labeled using the shortest weights $w \in \mathsf{P}(D_4)$. We denote the lines as $\mathbf{P}_w$, the polynomial representations as $\mathscr{P}^w$ and the associated Askey-Wilson polynomials as P_n^w.

The relation between lines and the shortest weights provides a useful criterion to determine whether a finite-dimensional representation, as given in (7.79), can be obtained by truncating an infinite-dimensional representation supported on a line[39]:

Theorem 7.1 *The finite-dimensional representation, labeled by a root $r \in \mathsf{R}(D_4)$, can be obtained by truncating the polynomial representation $\mathscr{P}^w$ if and only if $\langle r, w\rangle > 0$, where $\langle \cdot, \cdot \rangle$ is the standard Euclidean inner product.*

Compact Branes and Finite-dimensional Representations

We make explicit verification of the Claim 7.1 in this section. For simplicity, assume $\hbar$ is real. In this case, a generic fiber $\mathbf{F}$ is Lagrangian with respect to $\mathrm{Im}\Omega = \frac{\omega_K}{\hbar}$, and thus $\mathbf{F}$ can serve as the support of an A-brane $\mathfrak{B}_\mathbf{F}$. Thus, we consider a brane $\mathfrak{B}_\mathbf{F}$ supported on a generic fiber and the corresponding representation $\mathscr{F}$. The dimension formula (7.68) and the unit volume of a generic fiber implies

$$m := \dim \mathrm{Hom}(\mathfrak{B}_\mathbf{F}, \mathfrak{B}_{\mathrm{cc}}) = \int_\mathbf{F} \frac{\omega_I}{2\pi\hbar} = \frac{1}{\hbar}. \tag{7.93}$$

Consequently, the A-brane $\mathfrak{B}_\mathbf{F}$ can exist if and only if $\frac{1}{\hbar} = m \in \mathbb{Z}_{>0}$. By $q = e^{2\pi i\hbar}$, the dimension formula implies $q^m = 1$, which is precisely the shortening condition (7.79) with $r = (0, 0, 0, 0)$. Under this shortening condition, we obtain the finite-dimensional $S\ddot{H}$-module $\mathscr{F}_m = \mathscr{P}/(P_m)$ with

$$\begin{aligned} &P_m(X, q = e^{\frac{2\pi i}{m}}, \mathbf{t}) = X^m + X^{-m} + F_m(\mathbf{t}), \\ &F_m(\mathbf{t}) = \frac{(t_3^m t_4^m + t_3^m/t_4^m)(t_1^{2m} - 1) + (t_1^m t_2^m + t_1^m/t_2^m)(t_3^{2m} - 1)}{(t_1 t_3)^{2m} - 1}. \end{aligned} \tag{7.94}$$

For a non-trivial example, let us consider the singular fiber I_0^*, i.e. the global nilpotent cone that appears when $\beta_j = 0 = \gamma_j$, as considered in §7.6.4. The irreducible components of the I_0^* singular fiber consist of both the moduli space $\mathbf{V}$ of SU(2)-bundles on $C_{0,4}$ and the exceptional divisors $\mathbf{D}_j$ $(j = 1, 2, 3, 4)$.

Object Matching

By combining the dimension formula (7.68) with the volume formula for $\mathfrak{B}_\mathbf{V}$ given in (7.86), we find that the dimension of the corresponding representation is:

$$k := \dim \mathrm{Hom}(\mathfrak{B}_\mathbf{V}, \mathfrak{B}_{\mathrm{cc}}) = \int_\mathbf{V} \frac{F + B}{2\pi} = \int_\mathbf{V} \frac{\omega_I}{2\pi\hbar} = \frac{2\alpha_1}{\hbar}. \tag{7.95}$$

The A-brane $\mathfrak{B}_{\mathbf{V}}$ can exist if and only if k is a positive integer. Using (4.88) and noting that $q = e^{2\pi i\hbar}$, this requirement translates to the condition $q^k = t_1^{-2}$, which corresponds to the shortening condition (7.79), with the D_4 root $r = (2,0,0,0)$. When the shortening condition is satisfied, the corresponding k-dimensional representation, denoted by $\mathscr{V}_k$, can be explicitly constructed as the quotient $\mathscr{V}_k = \mathscr{P}/(P_k)$.

Using the same approach, one can identify an $S\ddot{H}$-module $\mathscr{D}^{(j)}_{\ell_j}$ corresponding to a brane $\mathfrak{B}_{\mathbf{D}_j}$ supported on each exceptional divisor $\mathbf{D}_j$. The dimension of the morphism space is given by:

$$\ell_j := \dim \mathrm{Hom}(\mathfrak{B}_{\mathbf{D}_j}, \mathfrak{B}_{\mathrm{cc}}) = \int_{\mathbf{D}_j} \frac{F+B}{2\pi} = \int_{\mathbf{D}_j} \frac{\omega_I}{2\pi\hbar} = \frac{\mathrm{vol}(\mathbf{D}_j)}{\hbar}. \tag{7.96}$$

The results are summarized in Table 7-1, based on the volume formulas provided in (7.86).

finite-dim rep	shortening condition	A-brane	A-brane condition
$\mathscr{F}^{(x_m)}_m$	$q^m = 1$	$\mathfrak{B}^{(x_m)}_{\mathbf{F}}$	$m = \frac{1}{\hbar}$
$\mathscr{V}_k$	$q^k = t_1^{-2}$	$\mathfrak{B}_{\mathbf{V}}$	$k = \frac{2\alpha_1}{\hbar}$
$\mathscr{D}^{(1)}_{\ell_1}$	$q^{\ell_1} = t_1t_2t_3t_4$	$\mathfrak{B}_{\mathbf{D}_1}$	$\ell_1 = \frac{1}{\hbar} - \frac{\alpha_1+\alpha_2+\alpha_3+\alpha_4}{\hbar}$
$\mathscr{D}^{(2)}_{\ell_2}$	$q^{\ell_2} = t_1t_2t_3^{-1}t_4^{-1}$	$\mathfrak{B}_{\mathbf{D}_2}$	$\ell_2 = \frac{-\alpha_1-\alpha_2+\alpha_3+\alpha_4}{\hbar}$
$\mathscr{D}^{(3)}_{\ell_3}$	$q^{\ell_3} = t_1t_2^{-1}t_3t_4^{-1}$	$\mathfrak{B}_{\mathbf{D}_3}$	$\ell_3 = \frac{-\alpha_1+\alpha_2-\alpha_3+\alpha_4}{\hbar}$
$\mathscr{D}^{(4)}_{\ell_4}$	$q^{\ell_4} = t_1t_2^{-1}t_3^{-1}t_4$	$\mathfrak{B}_{\mathbf{D}_4}$	$\ell_4 = \frac{-\alpha_1+\alpha_2+\alpha_3-\alpha_4}{\hbar}$

Table 7-1 A summary of finite-dimensional $S\ddot{H}$-modules with their shortening conditions and the corresponding A-brane configurations at the I_0^* singular fiber, under the assumption $|q| = 1$.

Beyond a single irreducible component of the I_0^* singular fiber, one can consider a brane supported on a union of several components, denoted by $\mathbf{N}_r$ with r denotes $-a_0\theta + \sum_{i=1}^4 a_i e^i$, which is Lagrangian with respect to $\omega_{\mathfrak{X}} = \omega_K/\hbar$. The homology class of $\mathbf{N}_r$ is represented with the standard basis by

$$[\mathbf{N}_r] = a_0[\mathbf{D}_1] + a_1[\mathbf{D}_2] + a_2[\mathbf{D}_3] + a_3[\mathbf{D}_4] + a_4[\mathbf{V}], \tag{7.97}$$

Here, e^i are the simple roots, and θ is the highest root in the D_4 root system (7.87). Then, it is straightforward to show that $r \in \mathsf{R}(D_4)$ if and only if a cycle $\mathbf{N}_r$ has a self-intersection number minus two: $[\mathbf{N}_r] \cdot [\mathbf{N}_r] = -2$. Using the same approach, we determine the A-brane condition for $\mathbf{N}_r$. Applying the dimension formula (7.68) and the volume formula (7.88), the A-brane condition is given by

$$m := \dim \mathrm{Hom}(\mathfrak{B}_{\mathbf{N}_r}, \mathfrak{B}_{\mathrm{cc}}) = \int_{\mathbf{N}_r} \frac{\omega_I}{2\pi\hbar} = \frac{a_0 + \left(\sum_{i=1}^4 a_i e^i - a_0\theta\right)\cdot\alpha}{\hbar}. \tag{7.98}$$

The corresponding shortening condition is $q^m = \mathbf{t}^{-r}$ with $\mathbf{t}^r = t_1^{r_1}t_2^{r_2}t_3^{r_3}t_4^{r_4}$. Thus, we write the corresponding $S\ddot{H}$-module as $\mathscr{N}^{r,w}_m = \mathscr{P}^w/(P^w_m)$ where $\langle r, w\rangle > 0$ due to Proposition 7.1. As a result, if we consider all the cycles with self-intersection number minus two, all 24 roots in the D_4 root system $\mathsf{R}(D_4)$ are exhausted. In this way, the A-brane conditions in (7.98) recover all the shortening conditions classified in (7.79).

Morphism Matching

In the category of A-branes, a morphism between two A-branes becomes non-trivial when they intersect, forming a bound state[46–47]. Let $\mathfrak{B}_{\mathbf{N}_r}$ and $\mathfrak{B}_{\mathbf{N}_{r'}}$ be A-branes supported on components of the I_0^* singular fiber. The A-brane conditions are given by:

$$m = \dim \mathrm{Hom}(\mathfrak{B}_{\mathbf{N}_r}, \mathfrak{B}_{\mathrm{cc}}), \quad m' = \dim \mathrm{Hom}(\mathfrak{B}_{\mathbf{N}_{r'}}, \mathfrak{B}_{\mathrm{cc}}). \tag{7.99}$$

We denote the corresponding $S\ddot{H}$-modules by $\mathscr{N}_m^{r,w}$ and $\mathscr{N}_{m'}^{r',w'}$ where the shortening conditions are given by $q^m = \mathbf{t}^{-r}$ and $q^{m'} = \mathbf{t}^{-r'}$, respectively. Then we have the following proposition[39]:

Theorem 7.2 *If $\mathfrak{B}_{\mathbf{N}_r}$ and $\mathfrak{B}_{\mathbf{N}_{r'}}$ intersect at a point q, then the morphism space is one-dimensional*

$$\mathrm{Hom}(\mathfrak{B}_{\mathbf{N}_r}, \mathfrak{B}_{\mathbf{N}_{r'}}) = \mathbb{C}\langle q \rangle \, . \tag{7.100}$$

Then, the roots $r, r' \in \mathsf{R}(D_4)$ satisfy $\langle r, r' \rangle = -2$. Moreover, there exists a weight w such that $\langle w, r \rangle > 0$, $\langle s_r(w), r' \rangle > 0$. In this setting, there exists a short exact sequence of finite-dimensional $S\ddot{H}$-modules

$$0 \to \mathscr{N}_{m'}^{r',s_r(w)} \to \mathscr{N}_{m+m'}^{r+r',w} \to \mathscr{N}_m^{r,w} \to 0 \, . \tag{7.101}$$

This exact sequence is uniquely determined up to isomorphism, independent of the choice of w. Thus, $\mathrm{Ext}^1(\mathscr{N}_m^{r,w}, \mathscr{N}_{m'}^{r',s_r(w)})$ is one-dimensional.

As an explicit example, let us consider the morphism space $\mathrm{Hom}^*(\mathfrak{B}_{\mathbf{V}}, \mathfrak{B}_{\mathbf{D}_1})$. Since $\mathbf{D}_1$ and $\mathbf{V}$ intersect at a single point q_1, the geometric perspective predicts that the morphism space is one-dimensional. The A-brane $\mathfrak{B}_{\mathbf{N}_r}$, representing their bound state, is supported on $\mathbf{N}_r = \mathbf{V} \cup \mathbf{D}_1$, with the corresponding root $r = -\theta + e_4 = (1, -1, -1, -1)$. The A-brane condition for $\mathfrak{B}_{\mathbf{N}_r}$ is evaluated as:

$$m = \dim \mathrm{Hom}(\mathfrak{B}_{\mathbf{N}_r}, \mathfrak{B}_{\mathrm{cc}}) = \int_{\mathbf{N}_r} \frac{F + B}{2\pi} = \frac{1}{\hbar} - \frac{-\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4}{\hbar}, \tag{7.102}$$

which translates to the shortening condition $q^m = t_1^{-1} t_2 t_3 t_4$.

Without loss of generality, we can take the weight to be $w = (1, 0, -1, 0)$. Consider the polynomial representation $\mathscr{P}^w$ where the action of lowering operators L_m^w contains a factor $(q^m - t_1^{-2})(q^m - t_1^{-1} t_2 t_3 t_4)$, which can be verified from (7.78). Suppose that we impose the two shortening conditions simultaneously

$$q^k = t_1^{-2} \, , \quad \text{and} \quad q^{k+\ell} = t_1^{-1} t_2 t_3 t_4 \, , \tag{7.103}$$

then we obtain the following short exact sequence from $\mathscr{P}^w$:

$$0 \to \mathscr{D}_\ell^{(1)} \to \mathscr{N}_{k+\ell}^{r,w} \to \mathscr{V}_k \to 0 \, . \tag{7.104}$$

This represents a non-trivial element in $\mathrm{Ext}^1(\mathscr{V}_k, \mathscr{D}_\ell^{(1)})$.

Appendix A
Symplectic Geometry

A.1 From Hamilton Formalism to symplectic Geometry

In classical mechanics, we are familiar with the phase space

$$M = \mathbb{R}^{2n} = \{x^I = (q^1, \dots, q^n, p^1, \dots, p^n), I = 1, \dots 2n\} \tag{A.1}$$

The key concept in the Hamiltonian dynamics is the Poisson bracket $\{-,-\}$. Explicitly, the Poisson bracket between two smooth functions on the phase space (or physical quantities) gives

$$\{f(p,q), g(p,q)\} = \sum_{i=1}^{n} \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p^i} - \frac{\partial f}{\partial p^i}\frac{\partial g}{\partial q^i} \tag{A.2}$$

As is well-known in the context of Hamiltonian mechanics, the Poisson bracket satisfies the following four properties.

$$\begin{aligned}
&1)\text{Anticommutativity}: \{f,g\} = -\{g,f\} \\
&2)\text{Bilinearity}: \{af+bg,h\} = a\{f,h\} + b\{g,h\}, \quad a,b \in \mathbb{R} \\
&3)\text{Leibniz's rule}: \{f,gh\} = \{f,g\}h + g\{f,h\} \\
&4)\text{Jacobi identity}: \{f,\{g,h\}\} = \{g,\{f,h\}\} + \{\{f,g\},h\}.
\end{aligned} \tag{A.3}$$

In order to generalize the concept of phase space, we seek an intrinsic formulation for the Poisson bracket, which is independent of the choice of smooth function f and g. As $\frac{\partial}{\partial q^i}$ admits a natural interpretation as tangent vector, it motivates us to formulate the Poisson bracket as a bivector Π

$$\Pi = \sum_{i=1}^{n} \frac{\partial}{\partial q^i} \otimes \frac{\partial}{\partial p^i} - \frac{\partial}{\partial p^i} \otimes \frac{\partial}{\partial q^i} =: \Pi^{IJ}\frac{\partial}{\partial x^I}\frac{\partial}{\partial x^J}, \tag{A.4}$$

while $\Pi(f,g)$ recovers the formula (A.2). We will call Π a *Poisson bivector* as it encodes the data of the Poisson bracket.

As a natural generalization, we consider a smooth manifold M as the phase space of a physical system. In order to endow the space of smooth functions $C^\infty(M)$ a Poisson bracket, we ask for a bivector Π such that the Poisson bracket between two smooth functions is defined as

$$\{f,g\} = \Pi(f,g). \tag{A.5}$$

To preserve the properties of the Poisson brackets in (A.3), additional constraints must be imposed on the bivector Π. The constraints corresponding to the first three properties are straightforward to describe. Bilinearity and Leibniz's rule are evident from the properties of the tangent vector. Anticommutativity, on the other hand, requires that the Poisson bivector be antisymmetric, i.e., $\Pi \in \Gamma(\Lambda^2(T\mathbb{R}^{2n}))$.

However, a non-trivial constraint arises from the Jacobi identity. If we introduce local coordinates x^I and expand the Poisson bivector as $\Pi = \Pi^{IJ}\partial_I\partial_J$, then the Jacobi identity is equivalent to the condition

$$\Pi^{I[J}\partial_I \Pi^{KL]} = 0. \tag{A.6}$$

We can summarize the discussion above with the following definition.

Definition A.1 *(Possion Bivector) A Poisson bivector over a smooth manifold M is an antisymmetric bivector $\Pi \in \Gamma(\Lambda^2(T\mathbb{R}^{2n}))$ satisfying (A.6). The pair (M,Π) of a smooth manifold M and a Poisson bivector Π is called the Poisson manifold.*

We would proceed by further imposing the non-degeneracy for the Poisson bivector $\det(\Pi^{IJ}(p)) \neq 0, \forall p \in M$. As a consequence, we could define the inverse of Π^{IJ} as

$$\omega_{IJ} = \Pi^{-1}_{IJ} \text{ s.t. } \omega_{IJ}\Pi^{JK} = \Pi^{IJ}\omega_{JK} = \delta^K_I. \tag{A.7}$$

One can show that the condition above implies $\omega_{IJ} = -\omega_{JI}$ and ω_{IJ} is also non-degenerate. By the merit of ω_{IJ}, we could repackage the Jacobi identity (A.6) into

$$0 = \omega_{Ja}\omega_{Kb}\omega_{Lc}\Pi^{I[J}\partial_I \Pi^{KL]} = -\partial_{[a}\omega_{bc]}. \tag{A.8}$$

Therefore, the Jacobi identity is equivalent to asking the two form $\omega = \omega_{IJ}dx^I \wedge dx^J$ to be closed $d\omega = 0$.

As a summary of the discussion above, in order to obtain a Poisson bracket on the phase space M, it suffices to endow M with a closed, non-degenerate two-form ω. The statement here leads us to the definition of the symplectic manifold.

Definition A.2 *(symplectic Manifold) A pair (M, ω) is called a symplectic manifold if M is a smooth manifold and ω is a closed, non-degenerate two form. A Poisson bivector Π associated with a symplectic manifold is an anti-symmetric bivector satisfying (A.7).*

Not all the manifolds admit the symplectic structure, which can be demonstrated as follows. Given a symplectic form ω, there is a canonical choice for the volume form on the manifold M, called the Liouville volume form vol. It is defined as

$$\text{vol} := \frac{\omega^n}{n!}. \tag{A.9}$$

As the symplectic form is non-degenerate, the $2n$-form vol is nowhere vanishing. It demonstrates that the symplectic manifold has to be orientable.

On the other hand, many geometries that we are familiar with are symplectic. Several typical examples are listed as follows

1. $(S^2, \omega = d\cos\theta \wedge d\phi)$ where (θ, ϕ) is the standard angular coordinate on S^2.

2. $(T^2 = S^1 \times S^1, \omega = d\theta_1 \wedge d\theta_2)$ where (θ_1, θ_2) are the angular coordinates of two S^1 respectively.

3. $(T^*M, \omega = d\theta)$ where M is a arbitrary smooth manifold T^*M and θ is the tautological one form on T^*M.

A.2 Hamiltonian Mechanics from symplectic Manifold

At this point, we have only described the structure of the phase space. To specify the evolution of a system, the data we have to specify further is a Hamiltonian $H \in C^\infty(M)$. The geometric language for describing the infinitesimal evolution of a system is the vector field, which motivates us to construct a vector field out of the Hamiltonian H. There is a natural one as follows.

Definition A.3 *(Hamiltonian Vector Field) Let (M, ω) be a symplectic manifold and $H \in C^\infty(M)$. The Hamiltonian vector field $X_H \in \mathfrak{X}(M)$ associated with H is defined by $X_H^I = \Pi^{IJ}(dH)_J$, or equivalently by $i_{X_H}\omega = -dH$. A vector field X is called Hamiltonian, if there exists a smooth function H such that $X = X_H$. The set of Hamiltonian vector fields will be denoted as $\mathfrak{X}_{Ham}(M)$.*

As a remark, the Hamiltonian H here can be an arbitrary smooth function, which may be highly irregular. Consequently, the space of Hamiltonian vector fields is vast.

With the concept of the Hamiltonian vector field, there is an equivalent definition of the Poisson bracket

$$\{f, g\} := \Pi(f, g) = -\omega(X_f, X_g), \forall f, g \in C^\infty(M), \tag{A.10}$$

where X_f and X_g are the Hamiltonian vector field associated with f and g respectively. The second identity in (A.10) can be shown by applying (A.7).

At this point, we are prepared to demonstrate that the Hamiltonian vector field is the correct concept to describe the evolution of the classical mechanics system, in the sense that it recovers the Hamiltonian equation. Let f be a smooth function over M representing the physical quantity, we consider the differential of f over the integration curve of the Hamiltonian vector field

$$\frac{d}{ds} f := X_H(f) = df(X_H) = -i_{X_H} i_{X_f}(\omega) = \{f, H\}. \tag{A.11}$$

It is aligned with the Hamiltonian equation $\frac{d}{ds} f = \{f, H\}$.

As is known in classical mechanics, the evolution of the system preserves the Poisson bracket. As a generalization, it can be shown by evaluating the Lie derivative that the flow of the Hamiltonian vector field preserves the symplectic form,

$$\mathscr{L}_{X_H}\omega = d i_{X_H}\omega + i_{X_H} d\omega = -ddH = 0. \tag{A.12}$$

As a byproduct of (A.12), the Liouville volume form vol $= \frac{\omega^n}{n!}$ is also invariant under the flow of the Hamiltonian vector field.

$$\mathscr{L}_{X_H}\frac{\omega^n}{n!} = \frac{1}{(n-1)!}\omega^{n-1}\mathscr{L}_{X_H}\omega = 0 \tag{A.13}$$

It can be understood as a generalization of Liouville theorem in classical mechanics.

Example A.1 *We consider the symplectic manifold $(S^2, \omega = dh \wedge d\theta)$ and the Hamiltonian $H = h$. By solving the equation $i_{X_H}\omega = -dh$, the Hamiltonian vector field takes $X_H = \frac{\partial}{\partial\theta}$. The flow associated with X_H takes*

$$\begin{aligned} \sigma_t : S^2 &\longrightarrow S^2 \\ (h, \theta) &\mapsto (h, \theta + t) \end{aligned} \tag{A.14}$$

A straightforward evaluation shows that the symplectic form ω is preserved under the flow σ_t

$$\sigma_t^*(\omega) = d\sigma_t^*(h) \wedge d\sigma_t^*(\theta) = dh \wedge d(\theta + t) = \omega. \tag{A.15}$$

A.3 Symplectomorphism and Symplectic Vector Field

As observed in the previous section, the flow of the Hamiltonian vector field provides an example of the diffeomorphisms from M to M that preserves the symplectic form. In this section, we will further investigate maps with this property, which will be referred to as *symplectomorphisms*. It can be viewed as an analog of the isometry in Riemannian geometry.

Definition A.4 *(Symplectomorphism) Let (M, ω) be a symplectic manifold. A* symplectomorphism *is a diffeomorphism $\phi : M \to M$ from (M, ω) to itself such that the symplectic form is preserved $\phi^*\omega = \omega$.*

Let us denote by $\mathrm{Symp}(M) = \{\text{Symplectomorphisms of } M\}$ the set of all symplectomorphisms of M. $\mathrm{Symp}(M)$ admits a natural group structure. The group operation is given by the composition of two maps, and the inverse of each symplectomorphism exists because all such maps are diffeomorphisms.

As illustrated in Sec.A.2, given an arbitrary Hamiltonian $H \in C^\infty(M)$, the flow of the associated Hamiltonian vector fields X_H is a symplectomorphism. It illustrates that $\mathrm{Symp}(M)$ is a huge group, admitting at least all the flows constructed in this way as a subgroup. With an appropriate concept of smoothness structure, $\mathrm{Symp}(M)$ can be understood as an infinite-dimensional Lie group.

It highlights an important difference between Riemannian geometry and symplectic geometry. In Riemannian geometry, the dimension for the isometries groups is always finite, bounded by the maximal symmetric spaces. On the other hand, $\mathrm{Symp}(M)$ is always infinite-dimensional.

To understand the structure of this giant infinite-dimensional Lie group, the first step should always be understanding its Lie algebra $\mathrm{Lie}(\mathrm{Symp}(M)) = T_{id}\mathrm{Symp}(M)$. That is, we need to describe the tangent vector of the group $\mathrm{Symp}(M)$ at the identity element. By definition, we shall consider a curve

$$\begin{aligned} \sigma : (-\epsilon, \epsilon) &\longrightarrow \mathrm{Symp}(M) \\ t &\longmapsto \sigma_t \end{aligned} \tag{A.16}$$

with $\sigma_t^*\omega = \omega$ and $\sigma_0 = id$. The differential over the curve at the identity provides an element in $T_{id}\mathrm{Symp}(M)$, which we denote as $X := \frac{d}{dt}\Big|_0 \sigma_t \in T_{id}\mathrm{Symp}(M)$. Such an element is specified by the following property

$$\mathscr{L}_X\omega := \frac{d}{dt}\bigg|_0 \sigma_t^*\omega = \frac{d}{dt}\bigg|_0 \omega = 0. \tag{A.17}$$

We will refer to the vector fields on M satisfying $\mathscr{L}_X\omega = 0$ as the *symplectic vector fields* and denote the set of symplectic vector fields as $\mathfrak{X}_{Symp}(M)$. Then the discussion above leads us to the following theorem.

Theorem A.1 *$Lie(Symp(M)) = \mathfrak{X}_{Symp}(M) = \{X \in \mathfrak{X}(M) | \mathscr{L}_X \omega = 0\}$.*

In other words, the infinitesimal symplectomorphisms are characterized by the symplectic vector fields. The Lie algebra structure of $\mathfrak{X}_{\text{Symp}}(M)$ is provided by the commutator between vector fields. As a quick check, a straightforward calculation demonstrates that the commutator of two symplectic vector fields is also symplectic.

$$\mathscr{L}_{[X,Y]}\omega = \mathscr{L}_X \mathscr{L}_Y \omega - \mathscr{L}_Y \mathscr{L}_X \omega = 0, \forall X, Y \in \mathfrak{X}_{\text{Symp}}(M). \tag{A.18}$$

As mentioned in Sec.A.2, all the Hamiltonian vector fields preserve the symplectic form and are thus symplectic. Namely, there is an inclusion $\mathfrak{X}_{\text{Ham}}(M) \subset \mathfrak{X}_{\text{Ham}}(M)$. Moreover, we have the following theorem.

Theorem A.2 *Let $X, Y \in \mathfrak{X}_{Symp}(M)$ be two symplectic vector fields, then $[X, Y] \in \mathfrak{X}_{Ham}(M)$ is a Hamiltonian vector field with Hamiltonian $H = \omega(X, Y)$.*

Proof *It suffice to evaluate $i_{[X,Y]}\omega$. By the identity $i_{[X,Y]} = [\mathscr{L}_X, i_Y]$, we have*

$$i_{[X,Y]}\omega = \mathscr{L}_X i_Y \omega - i_Y \mathscr{L}_X \omega = i_X d i_Y \omega + d i_X i_Y \omega. \tag{A.19}$$

As Y is a symplectic vector field,

$$\mathscr{L}_Y \omega = d i_Y \omega + i_Y d\omega = d i_Y \omega = 0 \tag{A.20}$$

Combined with (A.19), we arrive at

$$i_{[X,Y]}\omega = d i_X i_Y \omega = -d\omega(X, Y). \tag{A.21}$$

It demonstrates that $[X, Y]$ is a Hamiltonian vector field with Hamiltonian $\omega(X, Y)$. ∎

A natural question at this point is when the symplectic vector field is Hamiltonian. To answer this question, we have to take a closer look at the symplectic vector field. For X a symplectic vector field, we have

$$0 = \mathscr{L}_X \omega = d i_X \omega + i_X d\omega = d i_X \omega. \tag{A.22}$$

Namely, X is symplectic if and only if $i_X \omega$ is a closed one-form. On the other hand, if X is Hamiltonian, then by definition (A.3), there exists a Hamiltonian $H \in C^\infty(M)$ such that

$$i_X \omega = -dH. \tag{A.23}$$

Namely, X is Hamiltonian if and only if $i_X \omega$ is exact. As a result, the obstruction for symplectic vector fields to be Hamiltonian is the first de-Rham cohomology $H^1_{\text{dR}}(M)$. We could formulate the discussion above as a short exact sequence of vector spaces

$$0 \to \mathfrak{X}_{\text{Ham}}(M) \xrightarrow{i} \mathfrak{X}_{\text{Symp}}(M) \to H^1_{\text{dR}}(M) \to 0. \tag{A.24}$$

To end this section, we provide two examples of the symplectic vector fields.

Example A.2 *Let us consider the symplectic manifold $(\mathbb{R}^2, \omega = dx \wedge dy)$ along with a vector field over it $X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$. We are going to show that X is Hamiltonian, and thus symplectic. To proceed, let us evaluate $i_X\omega$.*

$$i_X \omega = -xdx - ydy = -d\frac{1}{2}(x^2 + y^2). \tag{A.25}$$

It demonstrates that X is a Hamiltonian Vector field, whose Hamiltonian is $H = \frac{1}{2}(x^2 + y^2)$. This Hamiltonian corresponds precisely to the one-dimensional harmonic oscillator if we interpret x as the position and y as the momentum.

Example A.3 *Let us consider the symplectic manifold* $(T^2 = S^1_{\theta_1} \times S^1_{\theta_2}, \omega = d\theta_1 \wedge d\theta_2)$ *along with a vector field over it* $X = \frac{\partial}{\partial\theta_2}$*. We are going to show that X is symplectic but not Hamiltonian. To proceed, we shall evaluate* $i_X\omega$*.*

$$i_X\omega = i_{\partial_{\theta_2}} d\theta_1 \wedge d\theta_2 = -d\theta_1 \tag{A.26}$$

The RHS of the equation is closed. Therefore, X is a symplectic vector field. It is tempting to identify θ_1 *as the Hamiltonian associated with* X*. However,* θ_1 *is not a well-defined smooth function on* T^2 *as it is not periodic in* $S^1_{\theta_1}$ *direction. Rather,* $d\theta_1$ *represents a non-trivial class in* $H^1_{dR}(M)$*, as its paring with the first homology class is non-vanishing* $\int_{S^1_1} d\theta_1 = 2\pi$*. As a result,* $X = \frac{\partial}{\partial\theta_2}$ *is symplectic but not Hamiltonian.*

A.4 Symplectic G-Action and Moment map

In the discussion in Sec.A.3, we consider the infinitesimal version of symplectomorphism, namely the symplectic vector fields. We could take a step further by considering the flow(or the exponential map) of the vector field. Given a symplectic vector field X, the flow associated with X provides a one-parameter family of symplectomorphism $\{\sigma_t : M \to M, t \in \mathbb{R}\}$ satisfying

$$\sigma_t \circ \sigma_s = \sigma_{t+s}, \quad \sigma_0 = id. \tag{A.27}$$

As a result, the flow of the symplectic vector field X suggests a Lie group homomorphism

$$\begin{aligned} \sigma_. : \mathbb{R} &\to \mathrm{Symp}(M) \\ t &\mapsto \sigma_t \end{aligned} \tag{A.28}$$

As a terminology, we would say that the symplectic manifold admits a symplectic $\mathbb{R}$-action specified by $\sigma_.$. It is natural to generalize the concept of group action to an arbitrary Lie group, which leads to the following definition.

Definition A.5 *(symplectic G-Action) Let G be a connected Lie group. A symplectic G-action is a Lie group homomorphism*

$$\begin{aligned} \psi : G &\longrightarrow Symp(M) \\ g &\mapsto \psi_g. \end{aligned} \tag{A.29}$$

To understand the symplectic G-action, it is often helpful to first consider the infinitesimal G-action. From the geometric language, we shall examine the push-forward ψ_* at the identity:

$$\begin{aligned} \psi_* : T_eG = \mathfrak{g} &\longrightarrow \mathfrak{X}_{\mathrm{Symp}}(M) \\ y &\mapsto \psi_* y =: X_y. \end{aligned} \tag{A.30}$$

By virtue of ψ being a Lie group homomorphism, ψ_* is a Lie algebra homomorphism.

In the following discussion, we first restrict ourselves to the case where G is connected. As a generalization to the $\mathbb{R}$ action case, we would like to establish an appropriate notion of "Hamiltonian", which will eventually represent the conserved charges Q of a physical system once an appropriate Hamiltonian function H is chosen. Intuitively, we would like to ask the following two conditions for the Hamiltonian G action:

1) $\forall y \in \mathfrak{g}$, $\psi_* y$ is Hamiltonian. That is, there exists a charge function Q_y such that $\psi_* y = X_{Q_y}$.

2) Q_y respect the lie algebra structure of $\mathfrak{g}$. That is, $\{Q_x, Q_y\} = Q_{[x,y]}$.

It leads us to the definition of Hamiltonian G-action.

Definition A.6 *(Hamiltonian G-Action) A symplectic G-action is called Hamiltonian if there exists a Lie algebra homomorphism*

$$\begin{aligned} Q \,:\, \mathfrak{g} &\longrightarrow C^\infty(M) \\ x &\mapsto Q_x \end{aligned} \tag{A.31}$$

such that two conditions in (A.4) *is satisfied. Equivalently, the following diagram of Lie algebras commutes*

$$\begin{tikzcd} \mathfrak{g} \arrow[d, "Q"'] \arrow[dr, "\psi_*"] & \\ C^\infty(M) \arrow[r, "X_\cdot"] & \mathfrak{X}_{Symp}(M) \end{tikzcd} \quad . \tag{A.32}$$

Q is often called the comoment map *or charges.*

Apart from the comoment map, mathematicians often consider the moment map, which is defined as follows

Definition A.7 *(Moment Map) Let ψ be a Hamiltonian G-action with the comoment map Q. The map*

$$\begin{aligned} \mu \,:\, M &\longrightarrow \mathfrak{g}^* \\ p &\mapsto \mu(p) \end{aligned} \tag{A.33}$$

is called the moment map, if it satisfies $\mu(p)(x) = Q_x, \forall x \in \mathfrak{g}$.

Before delving into concrete examples, it is important to first examine the properties of the moment map. The following two theorems could be helpful in the latter discussions.

Theorem A.3 *Let G be a connected Lie group and (M, ω) be a symplectic manifold admitting a Hamiltonian G-action ψ. Then the moment map μ is G-equivariant. That is,*

$$\mu(\psi_g(m)) = (Ad_g^* \mu)(m), \ \forall m \in M, \tag{A.34}$$

where Ad_g^* is the coadjoint action of G on $\mathfrak{g}^*$, defined as

$$\langle Ad_g^* x, y \rangle := \langle x, Ad_{g^{-1}} y \rangle, \ \forall x \in \mathfrak{g}^*, y \in \mathfrak{g}. \tag{A.35}$$

and $\langle -, \rangle$ is the natural pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$.

Proof *As G is a connected Lie group, it suffices to prove the infinitesimal version of the statement:*

$$\mu_* \psi_*(y) = ad_y^*(\mu), \ \forall y \in \mathfrak{g}. \tag{A.36}$$

Notice that the statement concerns an equality involving elements of $\mathfrak{g}^*$, the dual space of $\mathfrak{g}$, consisting of linear functional on $\mathfrak{g}$. Therefore, it suffices to demonstrate that *the LHS and RHS yield the same value when evaluated on any element of* $\mathfrak{g}$. *Specifically, for any* $z \in \mathfrak{g}$, *we aim to show that:*

$$\langle \mu_* \psi_*(y), z\rangle = \langle ad_y^*(\mu), z\rangle. \tag{A.37}$$

By identifying z *as an element in* $(\mathfrak{g}^*)^*$, *LHS could be simplified as*

$$LHS = \langle \mu_* \psi_*(y), z\rangle = \langle \psi_*(y), \mu^* z\rangle = \psi_*(y)(\mu^* z) = \{\mu^* z, Q_y\}. \tag{A.38}$$

The last equality holds because $\psi_*(y)$ *is a Hamiltonian vector field with Hamiltonian* Q_y. *The following equation further demonstrates that* $\mu^* z = Q_z$*:*

$$\begin{aligned} \mu^* z \,:\, M &\xrightarrow{\mu} \mathfrak{g}^* \xrightarrow{z} \mathbb{C} \\ m &\mapsto \mu_m \mapsto \mu_m(z) := Q_z(m) \end{aligned} \tag{A.39}$$

and thus we arrive at

$$LHS = \{\mu^* z, Q_y\} = \{Q_z, Q_y\} = -Q_{[y,z]}. \tag{A.40}$$

On the other hand, RHS can be simplified into

$$RHS = \langle ad_y^*(\mu), z\rangle = -\langle \mu, ad_y z\rangle = -\langle \mu, [y, z]\rangle = -Q_{[y,z]}. \tag{A.41}$$

Compared with (A.40), we reach the conclusion that LHS=RHS. It concludes the proof. ∎

As a corollary, we could consider $\mu^{-1}(0)$, a subset of the manifold M. Physically, $\mu^{-1}(0)$ corresponds to the states that are neutral under the group G. theorem A.3 implies that $\mu^{-1}(0)$ is invariant under the G-action as a set. Specifically, for any point $p \in \mu^{-1}(0)$, we have $\psi_g(p) \in \mu^{-1}(0)$. This follows from a straightforward application of theorem A.3:

$$\mu(\psi_g(p)) = \mathrm{Ad}_g^*(\mu)(p) = \mathrm{Ad}_g^*(0) = 0 \in \mathfrak{g}^*. \tag{A.42}$$

In the next theorem, we consider the product manifold $M_1 \times M_2$ of two symplectic manifolds (M_1, ω_1) and (M_2, ω_2), which admit a natural symplectic structure via $\omega = \pi_1^* \omega_1 + \pi_2^* \omega_2$.

$$\begin{tikzcd} & M_1 \times M_2 \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ M_1 & & M_2 \end{tikzcd} \tag{A.43}$$

Physically, the product manifold $(M_1 \times M_2, \omega)$ represents the phase space of a dynamical system formed by combining two independent systems. In this context, we consider the case where both symplectic manifolds M_1 and M_2 admit Hamiltonian G-actions. The following theorem illustrates that under the diagonal G-action on $(M_1 \times M_2, \omega)$, the charge of a state $(p_1, p_2) \in M_1 \times M_2$ is simply the sum of the charges of the individual states p_1 and p_2.

Theorem A.4 *Let* (M_1, ω_1) *and* (M_2, ω_2) *be two symplectic manifolds admitting the Hamiltonian* G *actions* $\psi^{(1)}$ *and* $\psi^{(2)}$ *respectively, with moment map* $\mu^{(1)}$ *and* $\mu^{(2)}$ *respectively. Then the diagonal action* ψ *of* G *on* $(M_1 \times M_2, \omega = \pi_1^* \omega_1 + \pi_2^* \omega_2)$ *is a Hamiltonian* G*-action, whose moment map* μ *satisfies*

$$\mu(p_1, p_2) = \mu^{(1)}(p_1) + \mu^{(2)}(p_2). \tag{A.44}$$

Proof *As ψ is the diagonal action of G on $M_1 \times M_2$, the vector field $\psi_*(x)$ satisfies*

$$\pi_{1*}(\psi_*(x)) = \psi_{1*}(x),\ \pi_{2*}(\psi_*(x)) = \psi_{2*}(x)\ \forall x \in \mathfrak{g}. \tag{A.45}$$

To show that the diagonal G action is Hamiltonian, it suffices to evaluate $i_{\psi_*(x)}\omega$. This fact follows from the following calculations.

$$i_{\psi_*(x)}\omega = i_{\psi_*(x)}\pi_1^*\omega_1 + i_{\psi_*(x)}\pi_2^*\omega_2. \tag{A.46}$$

For the first term in (A.46), we have

$$(i_{\psi_*(x)}\pi_1^*\omega_1)(-) = \pi_1^*\omega_1(\psi_*(x), -) = \omega_1(\pi_{1*}\psi_*(x), \pi_{1*}-) = \omega_1(\psi_{1*}(x), \pi_{1*}-) \tag{A.47}$$

As $\psi_{1*}(x)$ is a Hamiltonian vector field on M_1 with Hamiltonian $Q_x^{(1)}$, we have

$$\omega_1(\psi_{1*}(x), \pi_{1*}-) = -dQ_x^{(1)}(\pi_{1*}-) = -\pi_1^* dQ_x^{(1)}(-) = -d(\pi_1^* Q_x^1)(-), \tag{A.48}$$

from which we concludes $i_{\psi_*(x)}\pi_1^*\omega_1 = -d(\pi_1^* Q_x^1)$. Similar analysis applies for the second term in (A.46), which yields $i_{\psi_*(x)}\pi_2^*\omega_2 = -d(\pi_2^* Q_x^{(2)})$. Therefore, (A.46) implies

$$i_{\psi_*(x)}\omega = -d(\pi_1^* Q_x^{(1)} + \pi_2^* Q_x^{(2)}) \tag{A.49}$$

Therefore, $\psi_*(x)$ is a Hamiltonian vector field on $M_1 \times M_2$, with Hamiltonian $\pi_1^* Q_x^{(1)} + \pi_2^* Q_x^{(2)}$. It suggests that the comoment map associated with ψ is $Q = \pi_1^* Q^{(1)} + \pi_2^* Q^{(2)}$. It remains to show that $\{Q_x, Q_y\} = Q_{[x,y]}$ is satisfied, which follows from the fact that $Q^{(1)}$ and $Q^{(2)}$ are the honest comoment maps. Thus we conclude that the diagonal action ψ is a Hamiltonian G-action. By the definition of the moment map (A.7), the moment map for ψ is given by

$$\mu(p_1, p_2) = \pi_1^*\mu^{(1)}(p_1, p_2) + \pi_2^*\mu^{(2)}(p_1, p_2) = \mu^{(1)}(p_1) + \mu^{(2)}(p_2). \tag{A.50}$$

∎

After discussing the properties of the moment map, let us explore several intuitive examples of Hamiltonian G-actions and moment maps.

Example A.4 *Let us consider $(\mathbb{C}^2, \omega = dp \wedge dq)$, where we denote two complex coordinates for $\mathbb{C}^2$ as (p, q). We take the group G to be $SL(2, \mathbb{C})$. The elements in its Lie algebra $\mathfrak{sl}_2$ are the traceless 2×2 matrices*

$$\mathfrak{sl}_2 = \left\{ \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \mid a, b, c \in \mathbb{C} \right\} \tag{A.51}$$

Three generators of $\mathfrak{sl}_2$ can be selected to be

$$E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{A.52}$$

They satisfy the following commutation relations

$$[E, F] = H, \quad [H, E] = 2E, \quad [H, F] = -2F. \tag{A.53}$$

Next, we specify the action of $SL(2,\mathbb{C})$ *on* $\mathbb{C}^2$ *as the matrix product. For any* $A \in SL(2,\mathbb{C})$*, the action* ψ_A *is defined as*

$$\begin{aligned} \psi_A : \mathbb{C}^2 &\longrightarrow \mathbb{C}^2 \\ \begin{bmatrix} p \\ q \end{bmatrix} &\mapsto A \cdot \begin{bmatrix} p \\ q \end{bmatrix} \end{aligned} \tag{A.54}$$

We are going to show that the matrix product defines a Hamiltonian $SL(2,\mathbb{C})$*-action and evaluate the corresponding moment map. It follows from the three steps: evaluating the vector field associated with the G-action, evaluating the Hamiltonian associated with the vector field, and rephrasing the Hamiltonian into the moment map.*

Step 1: Evaluating the Vector Field *In the first step, we shall first evaluate the vector fields associated with the group action. As* ω *is holomorphic, for our purpose of evaluating Hamiltonian, it suffices to take* $g \in \mathcal{O}(\mathbb{C}^2)$ *an arbitrary holomorphic function. Let's first evaluate the vector field associated with* $H \in \mathfrak{sl}_2$

$$\psi_* H \cdot g = \left(\frac{d}{dt}\bigg|_0 e^{tH} \right) \cdot g := \frac{d}{dt}\bigg|_0 g\left(e^{tH} \cdot \begin{bmatrix} p \\ q \end{bmatrix} \right) = \frac{d}{dt}\bigg|_0 g\left(\begin{bmatrix} (1+t)p \\ (1-t)q \end{bmatrix} \right) = (p\partial_p - q\partial_q) g \tag{A.55}$$

Therefore, $\psi_* h = p\partial_p - q\partial_q$*.*

A similar calculation yields the vector fields associated with $E \in \mathfrak{sl}_2$

$$\psi_* E \cdot g = \left(\frac{d}{dt}\bigg|_0 e^{tE} \right) \cdot g := \frac{d}{dt}\bigg|_0 g\left(e^{tE} \cdot \begin{bmatrix} p \\ q \end{bmatrix} \right) = \frac{d}{dt}\bigg|_0 g\left(\begin{bmatrix} p + tq \\ q \end{bmatrix} \right) = q\partial_p g \tag{A.56}$$

Therefore, $\psi_* E = q\partial_p$*.*

A similar calculation yields the vector fields associated with $F \in \mathfrak{sl}_2$

$$\psi_* F \cdot g = \left(\frac{d}{dt}\bigg|_0 e^{tF} \right) \cdot g := \frac{d}{dt}\bigg|_0 g\left(e^{tF} \cdot \begin{bmatrix} p \\ q \end{bmatrix} \right) = \frac{d}{dt}\bigg|_0 g\left(\begin{bmatrix} p \\ q + tp \end{bmatrix} \right) = p\partial_q g \tag{A.57}$$

Therefore, $\psi_* F = p\partial_q$*.*

Step 2: Evaluating the Hamiltonian *We proceed by proving that* $\{\psi_* H, \psi_* E, \psi_* F\}$ *are Hamiltonian, and thus symplectic. Namely, we need to show that* $i_X \omega$ *is exact, for* X *the vector fields constructed above.*

We first evaluate $i_{\psi_* H}\omega$

$$i_{\psi_* H}\omega = pdq + qdp = -d(-pq). \tag{A.58}$$

Therefore,$\psi_* H$ *is a Hamiltonian vector field with Hamiltonian* $Q_H = -pq$*.*

Next, we evaluate $i_{\psi_* E}\omega$

$$i_{\psi_* E}\omega = qdq = -d(-\frac{1}{2}q^2). \tag{A.59}$$

Therefore,$\psi_* E$ *is a Hamiltonian vector field with Hamiltonian* $Q_E = -\frac{1}{2}q^2$*.*

Next, we evaluate $i_{\psi_* F}\omega$

$$i_{\psi_* F}\omega = -pdp = -d(\frac{1}{2}p^2). \tag{A.60}$$

Therefore, $\psi_* F$ is a Hamiltonian vector field with Hamiltonian $Q_F = \frac{1}{2}p^2$.

In the next step, let us evaluate the Poisson bracket between the charges to show that condition 2) in (A.4) holds. The Poisson bivector Π associate with ω is

$$\Pi = \frac{\partial}{\partial q} \otimes \frac{\partial}{\partial p} - \frac{\partial}{\partial p} \otimes \frac{\partial}{\partial q}. \tag{A.61}$$

As an illustration, we evaluate the Poisson bracket between Q_E and Q_F, which yields

$$\{Q_E, Q_F\} = -\frac{1}{4}\{q^2, p^2\} = -pq = Q_H = Q_{[E,F]}. \tag{A.62}$$

A similar calculation demonstrates that

$$\{Q_H, Q_F\} = 2Q_F, \{Q_H, Q_E\} = -2Q_E. \tag{A.63}$$

Therefore, we reach a conclusion that the action ψ_A is a Hamiltonian action, as it respect the communication relation in $\mathfrak{sl}_2$.

Step 3: Rephrasing Hamiltonian into moment map *In the next step, let's evaluate the moment map μ associated with ψ. As $\mathfrak{sl}_2$ is a semisimple Lie algebra, the Killing form $K(X,Y) = \mathrm{Tr}\, XY$ is non-degenerate. Therefore, we could identify the element in $\mathfrak{sl}_2$ and $\mathfrak{sl}_2^*$ with the Killing form. Explicitly, we introduce the map*

$$\begin{aligned} * : \mathfrak{sl}_2 &\longrightarrow \mathfrak{sl}_2^* \\ X &\longmapsto X^*, s.t. X^*(Y) := K(X,Y). \end{aligned} \tag{A.64}$$

which specifies an isomorphism of vector spaces between $\mathfrak{sl}_2$ and $\mathfrak{sl}_2^*$. It naturally endows $\mathfrak{sl}_2^*$ a basis as a vector space as $\{H^*, E^*, F^*\}$, whose action on the elements $X \in \mathfrak{g}$ is specified by

$$H^*(X) = K(H,X),\ E^*(X) = K(E,X),\ F^*(X) = K(F,X). \tag{A.65}$$

By definition (A.7), the moment map satisfies

$$\mu_{(p,q)}(H) = Q_H = -pq,\ \mu_{(p,q)}(E) = Q_E = -\frac{1}{2}q^2,\ \mu_{(p,q)}(F) = Q_F = \frac{1}{2}p^2 \tag{A.66}$$

Therefore, under this canonical basis $\{H^*, E^*, F^*\}$, the moment map could be represented by

$$\mu_{(p,q)} = -\frac{1}{2}pqH^* - \frac{1}{2}q^2F^* + \frac{1}{2}p^2E^* \tag{A.67}$$

Or explicitly, we have

$$\mu_{(p,q)} = \frac{1}{2}\begin{bmatrix} -pq & p^2 \\ -q^2 & pq \end{bmatrix}^* \tag{A.68}$$

An interesting feature is that the image of the moment map is always a nilpotent matrix. This can be seen from the fact that both the determinant and the trace of the matrix vanish. Indeed, $\mu_{(p,q)}$ defines a double cover of the sl_2 nilpotent cone at generic points, while two branches meets on at the origin.

Such an interesting phenomenon is related to the springer resolution of the nilpotent cone.

Example A.5 *Let us consider the symplectic manifold* $(\mathbb{C}^n, \omega = \sum_{j=1}^n \frac{i}{2} dz_j \wedge d\overline{z}_j)$*. It admits a natural* $T^n = U(1)^{\times n}$ *action, specified by*

$$\Psi_{(e^{i\theta_1},\ldots,e^{i\theta_n})}(z_1,\ldots,z_n) = (e^{ik_1\theta_1}z_1,\ldots,e^{ik_n\theta_n}z_n) \tag{A.69}$$

where $(k_1,\ldots,k_n) \in \mathbb{Z}^n$*. It is easy to see that* T^n *action preserves the symplectic form* ω *and thus is a symplectic action. In the following analysis, we are going to show that* T^n *action is a Hamiltonian action and evaluate the corresponding moment map* μ*.*

Step 1: Evaluating the Vector Field *To evaluate the vector field associated with the* T^n*-action, we select* $f \in C^\infty(\mathbb{C}^n)$ *an arbitrary smooth function which depends on* z_j *and* $\overline{z}_j$*. Notice that Lie*$(T^n) = \mathbb{R}^n$*, the elements in Lie*(T^n) *can be represented by* n *real numbers* $(\theta_1,\ldots,\theta_n)$*. Then the vector field associated with* $(\theta_1,\ldots,\theta_n)$ *could be evaluated by the following standard calculations.*

$$\begin{aligned}\psi_*(\theta_1,\ldots,\theta_n)\Big|_{(z_1,\ldots,z_n)}(f) &= \frac{d}{ds}\Big|_{s=0}\Psi_{(e^{i\theta_1 s},\ldots,e^{i\theta_n s})}(z_1,\ldots,z_n)(f)\\ &= \frac{d}{ds}\Big|_{s=0} f(e^{ik_1\theta_1 s}z_1,\ldots,e^{ik_n\theta_n s}z_n, e^{-ik_1\theta_1 s}\overline{z}_1,\ldots,e^{-ik_n\theta_n s}\overline{z}_n)\\ &= \sum_{j=1}^n i\theta_j k_j (z_j\partial_{z_j} - \overline{z}_j\partial_{\overline{z}_j})f\end{aligned} \tag{A.70}$$

Therefore, we reach the conclusion that the vector field associated with $(\theta_1,\ldots,\theta_n)$ *is given by*

$$\psi_*(\theta_1,\ldots,\theta_n)\Big|_{(z_1,\ldots,z_n)} = \sum_{j=1}^n i\theta_j k_j (z_j\partial_{z_j} - \overline{z}_j\partial_{\overline{z}_j}) \tag{A.71}$$

Step 2: Evaluating the Hamiltonian *Next, let us evaluate the Hamiltonian associated with the vector field. It suffices to evaluate* $i_{\psi_*(\theta_1,\ldots,\theta_n)}\omega$ *as follows*

$$i_{\psi_*(\theta_1,\ldots,\theta_n)}\omega = \sum_{j=1}^n i\theta_j k_j (z_j i_{\partial_{z_j}} - \overline{z}_j i_{\partial_{\overline{z}_j}})\omega. \tag{A.72}$$

An explicit calculation shows that

$$i_{\partial_{z_j}}\omega = \frac{i}{2}d\overline{z}_j,\ i_{\partial_{\overline{z}_j}}\omega = -\frac{i}{2}dz_j. \tag{A.73}$$

As a result, (A.72) can be further simplified into

$$i_{\psi_*(\theta_1,\ldots,\theta_n)}\omega = -d\left(\sum_{j=1}^n \frac{1}{2}\theta_j k_j |z_j|^2\right) \tag{A.74}$$

Thus we conclude the Hamiltonian associated with $(\theta_1,\ldots,\theta_n)$ *is given by*

$$Q_{(\theta_1,\ldots,\theta_n)} = \sum_{j=1}^n \frac{1}{2}\theta_j k_j |z_j|^2 \tag{A.75}$$

In the next step, we would further demonstrate that the T^n action is Hamiltonian. To this aim, we shall prove

$$\{Q_{(\theta_1,\ldots,\theta_n)}, Q_{(\phi_1,\ldots,\phi_n)}\} = Q_{[(\theta_1,\ldots,\theta_n),(\phi_1,\ldots,\phi_n)]} = 0. \tag{A.76}$$

It follows from the straightforward evaluation via the Poisson bivector Π

$$\Pi = 2i\sum_{j=1}^{n} \frac{\partial}{\partial z_j} \otimes \frac{\partial}{\partial \bar{z}_j} - \frac{\partial}{\partial \bar{z}_j} \otimes \frac{\partial}{\partial z_j}. \tag{A.77}$$

Then the explicit calculation demonstrates

$$\begin{aligned}\{Q_{(\theta_1,\ldots,\theta_n)}, Q_{(\phi_1,\ldots,\phi_n)}\} &= 2i\sum_{j=1}^{n} \frac{\partial}{\partial z_j}Q_{(\theta_1,\ldots,\theta_n)}\frac{\partial}{\partial \bar{z}_j}Q_{(\phi_1,\ldots,\phi_n)} - \frac{\partial}{\partial \bar{z}_j}Q_{(\theta_1,\ldots,\theta_n)}\frac{\partial}{\partial z_j}Q_{(\phi_1,\ldots,\phi_n)} \\ &= 2i\sum_{j} \frac{1}{4}k_j^2\theta_j\phi_j|z_j|^2 - \frac{1}{4}k_j^2\phi_j\theta_j|z_j|^2 = 0.\end{aligned} \tag{A.78}$$

Therefore, we conclude that the T^n action is a Hamiltonian action.

Step 3: Rephrase Hamiltonian into Moment Map *Since Lie$(T^n) = \mathbb{R}^n$, there exists a natural pairing between $\mathbb{R}^n$ and its dual space $\mathbb{R}^{n*}$, provided by the Euclidean inner product. Consequently, the moment map can be expressed as*

$$\mu(z) = (\frac{1}{2}k_1|z_1|^2, \ldots, \frac{1}{2}k_n|z_n|^2) \in \mathbb{R}^{n*}, \tag{A.79}$$

As a double check, the pairing of $\mu(z)$ with the element in the Lie algebra $(\theta_1, \ldots, \theta_n)$ yields

$$\mu(z)(\theta_1, \ldots, \theta_n) = \sum_{j=1}^{n} \frac{1}{2}\theta_j k_j |z_j|^2 = Q_{(\theta_1,\ldots,\theta_n)}. \tag{A.80}$$

It confirms that $\mu(z)$ is the correct moment map associated with the T^n action.

Example A.6 *Let us consider the symplectic manifold to be $\mathfrak{gl}_n = \mathbb{C}^{n^2}$. Each element in $\mathfrak{gl}_n$ is a $n \times n$ matrices A, and the symplectic form on $\mathfrak{gl}_n$ is given by*

$$\omega = \sum_{j,k=1}^{n} \frac{i}{2} dA_{jk} \wedge d\bar{A}_{jk} = \sum_{j,k=1}^{n} \frac{i}{2} dA_{jk} \wedge dA^{\dagger}_{kj} = \frac{i}{2}\operatorname{Tr}\left(dA \wedge dA^{\dagger}\right). \tag{A.81}$$

Let us consider the $SU(n)$ action ψ on $\mathfrak{gl}_n$ via conjugation. Namely, the $SU(n)$ action is specified by

$$\psi_g(A) = gAg^{\dagger},\ \psi_g(A) = gA^{\dagger}g^{\dagger}\ \forall g \in SU(n). \tag{A.82}$$

Indeed, ψ_g preserves the symplectic form ω and thus defines a symplectic action. It can be proved from the following calculation

$$\psi_g^*\omega = \frac{i}{2}\operatorname{Tr}\left(dgAg^{\dagger} \wedge dgA^{\dagger}g^{\dagger}\right) = \frac{i}{2}\operatorname{Tr}\left(dA \wedge dA^{\dagger}\right) = \omega. \tag{A.83}$$

In the following analysis, we are going to show that $SU(n)$ action is a Hamiltonian action and evaluate the corresponding moment map μ.

Step 1: Evaluating the Vector Field *To evaluate the vector field associated with the $SU(n)$ action, we select $f \in C^\infty(\mathfrak{gl}_n)$ an arbitrary smooth function which depends on A and $A^\dagger$. The Lie algebra of $SU(n)$ is given by the $n\times n$ traceless Hermitian matrices*

$$\mathfrak{su}(n) = Lie(SU(n)) = \{X \in Mat_n(\mathbb{C}) \mid X^\dagger = X, \operatorname{Tr} X = 0\}. \tag{A.84}$$

Then the vector field associated with the Lie algebra element X could be evaluated with the standard calculation

$$\begin{aligned} \psi_*(X)\Big|_A (f) = \frac{d}{ds}\Big|_{s=0} \psi_{e^{iXs}}(A)(f) = \frac{d}{ds}\Big|_{s=0} f(e^{iXs}Ae^{-iXs}, e^{iXs}A^\dagger e^{iXs}) \\ = i[X,A]_{jk}\frac{\partial f}{\partial A_{jk}} + i[X,A^\dagger]_{jk}\frac{\partial f}{\partial A^\dagger_{jk}}. \end{aligned} \tag{A.85}$$

Therefore, we conclude that the vector field associated with $X \in \mathfrak{su}(n)$ is given by

$$\psi_*(X)\Big|_A = i[X,A]_{jk}\frac{\partial}{\partial A_{jk}} + i[X,A^\dagger]_{jk}\frac{\partial}{\partial A^\dagger_{jk}}. \tag{A.86}$$

Step 2: Evaluating the Hamiltonian *In the next step, let us evaluate the Hamiltonian associated with the vector field $\psi_*(X)$. It suffices to evaluate $i_{\psi_*(X)}\omega$, which yields the following result.*

$$\begin{aligned} i_{\psi_*(X)}\omega &= -\frac{1}{2}[X,A]_{jk} dA^\dagger_{kj} + \frac{1}{2}[X,A^\dagger_{jk}]dA_{kj} \\ &= -\frac{1}{2}\operatorname{tr}\big([X,A]dA^\dagger - [X,A^\dagger]dA\big) \\ &= -\operatorname{Tr}\left(\frac{1}{2}X[A,dA^\dagger] - X[A^\dagger dA]\right) \\ &= -d\operatorname{Tr}\Big(\frac{1}{2}X[A,A^\dagger]\Big). \end{aligned} \tag{A.87}$$

Therefore, we conclude that the Hamiltonian associated with X is given by $Q_X(A) = \operatorname{Tr}\Big(\frac{1}{2}X[A,A^\dagger]\Big)$.

Step 3: Rephrase Hamiltonian into Moment Map *As $\mathfrak{su}(n)$ is a semisimple Lie algebra, the killing form $K(X,Y) = \operatorname{tr} XY$ is non-degenerated. As a result, the Lie algebra $\mathfrak{su}(n)$ and its dual $\mathfrak{su}(n)^*$ can be identified. As in the Example A.4, we introduce the map $*$*

$$\begin{aligned} * : \mathfrak{su}(n) &\longrightarrow \mathfrak{su}(n)^* \\ X &\longmapsto X^*, s.t. X^*(Y) := \operatorname{Tr}(XY). \end{aligned} \tag{A.88}$$

which specifies an isomorphism of vector spaces $\mathfrak{su}(n)$ and $\mathfrak{su}(n)^*$. The moment map associated with the conjugate action can then be expressed in the neat form:

$$\mu(A) = \frac{1}{2}[A,A^\dagger]^* \tag{A.89}$$

As a final remark, the set $\mu^{-1}(0)$ is particularly noteworthy. It precisely corresponds to the set of normal matrices in $\mathfrak{gl}_n$. A similar term occurs in the Coulomb branch equations for 4d $\mathcal{N} = 2$ supersymmetric gauge theory. We will revisit this example in the context of symplectic reduction.

Let (M, ω) be a symplectic manifold admitting a Hamiltonian G-action ψ. To end this section, we would like to consider the Hamiltonian $H \in C^\infty(M)$ compatible with the G-action, which describes the evolution of a physical system. Namely, we want H to be G-invariant, or "have a G-symmetry".

$$\psi_g^* H = H. \tag{A.90}$$

In Hamiltonian dynamics, the renowned Noether's theorem states a one-to-one correspondence between continuous symmetries and conserved charges. In the language of symplectic geometry, there is also a proper generalization of Noether's theorem. The statement is as follows.

Theorem A.5 *(Noether's theorem) Let (M, ω) be a Hamiltonian G-space with the comoment map Q. Let H be a G-invariant smooth function. Then*

$$\frac{d}{ds} Q_x := X_H(Q_x) = 0, \ \forall x \in \mathfrak{g}. \tag{A.91}$$

Proof *The proof follows from the straightforward evaluation*

$$X_H(Q_x) = \{Q_x, H\} = -\{H, Q_x\} = -\psi_*(x)(H) = \mathscr{L}_{\psi_*(x)} H = 0, \tag{A.92}$$

where the last equality follows from the infinitesimal version of (A.90). ∎

As a final remark, Noether's theorem justifies our notation for the comoment map Q_x, as it indeed represents a conserved charge along the time evolution, once the G-invariant Hamiltonian H is appropriately specified.

A.5 System with Constraints and Symplectic Quotient

In this section, we will analyze the system with constraints, which are the crucial components of dynamical systems. There are two main scenarios in which constraints are introduced:

1) We construct a system by imposing constraints on a larger system.
2) We reduce the number of dynamical variables by assigning specific values to the conservation charges.

It motivates us to have a systematic analysis of the constrained system.

General Formalism for System with Constraints Let (M, ω) be a symplectic manifold, and let $\phi_a \in C^\infty(M), a = 1, \ldots, m$ be m independent smooth functions describing the constraints. We aim to restrict the dynamical variables to $\Sigma = \bigcap_{a=1}^m \phi_a^{-1}(0) = \{p \in M \mid \phi_a(p) = 0, a = 1, \ldots m\}$, which, for simplicity, we assume to be an embedded submanifold. For convenience, we will introduce the terminology "weakly zero", which is defined as follows.

Definition A.8 *A smooth function σ is called weakly zero if $\sigma \in I(\phi^a)$, where*

$$I(\phi^a) = \{\sigma(x) \in C^\infty(M) \mid \exists \alpha_a(x) \in C^\infty, \sigma(x) = \sum \alpha_a(x)\phi_a(x)\}. \tag{A.93}$$

In other words, a smooth function $\sigma(x)$ is weakly zero if its restriction on Σ is zero. From the algebra perspective, $I(\phi^a)$ defines an ideal in $C^\infty(M)$. Up to this point, the constraint function ϕ_a has been m arbitrary smooth functions, which can exhibit wild behaviors. To simplify the discussion, we will focus on a specific class of constraints, known as the first-class constraints. This class is general enough to encompass many of the systems we are interested in.

Definition A.9 *(First/Second-Class Constraint) A set of independent constrains $\{\phi_a\}_{a=1,\ldots m}$ is called the first-class constrain, if $\{\phi_a, \phi_b\}$ is weakly zero. Namely, we have $\{\phi_a, \phi_b\} = \sum_{c=1}^m C_{abc}(x)\phi_c$. Otherwise, the set of constraints is called the second-class constraint.*

Unless otherwise specified, we will use the notation ϕ_a to denote a set of first-class constraints.

At this stage, we would like to endow the symplectic manifold with a Hamiltonian $H \in C^\infty(M)$. However, not every Hamiltonian is appropriate in this context. It is crucial that the Hamiltonian respects the constraint $\phi_a(x) = 0$. Specifically, if the system begins at a point on the submanifold Σ, we require that the state's evolution under the Hamiltonian flow remains confined to Σ. To ensure this, it is sufficient to require that

$$\begin{cases} \frac{d}{ds}\phi_a = \{\phi_a, H\} \in I(\phi), \\ \phi(s=0) \in \Sigma. \end{cases} \tag{A.94}$$

Then the uniqueness of the solution to the ODE ensures that $\frac{d}{ds}\phi = 0$ for any s, and thus the Hamiltonian flow remains confined to Σ. This leads us to the following definition.

Definition A.10 *The Hamiltonian $H \in C^\infty(M)$ is called compatible with the first-class constraints $\{\phi_a\}_{a=1,\ldots,m}$, if $\{\phi_a, H\}$ is weakly zero. Equivalently, $\{\phi_a, H\} = \sum_{b=1}^m h_{ab}(x)\phi_b(x)$.*

Let us remark on a special case where the compatibility condition is saturated, that is when the constraints ϕ_a Poisson commute with the Hamiltonian, i.e., $\{\phi_a, H\} = 0$. When this condition holds, the ϕ_a can be interpreted as conserved charges along the Hamiltonian flow. Thus, imposing constraints is equivalent to assigning specific values to these conserved charges.

Before proceeding, it is helpful to consider the following example.

Example A.7 *Consider the symplectic manifold to be $(T^*\left(\mathbb{R}^\times \times \mathbb{R}^\times\right), \omega = dx \wedge dp_x + dy \wedge dp_y)$, where $\mathbb{R}^\times = \mathbb{R} - \{0\}$. We impose the following constraint*

$$\phi(x) = p_\varphi - 1 = xp_y - yp_x - 1. \tag{A.95}$$

It is straightforward to observe that a single constraint is always of the first class, as $\{\phi, \phi\} = 0$. We take the Hamiltonian to be the free Hamiltonian.

$$H = \frac{p_r^2}{2m} + \frac{p_\varphi^2}{2mr^2}. \tag{A.96}$$

An explicit evaluation shows that $\{\phi, H\} = 0$, which verifies that the free Hamiltonian is compatible with the constraint. Physically, the compatibility simply means that the *angular momentum is conserved for a free particle. After imposing the constraint, the Hamiltonian reduces to*

$$H = \frac{p_r^2}{2m} + \frac{1}{2mr^2}, \tag{A.97}$$

which describes a particle moving in the potential $V(r) = \frac{1}{2mr^2}$.

Let us make an important remark. By imposing the constraint $\phi(x) = 0$, *the dynamical variables are now restricted to* $(r, \varphi, p_r, p_\varphi) = (r, \varphi, p_r, 1)$, *which defines a three-dimensional submanifold of* $T^*\mathbb{R}^2$. *This implies that the restriction of the symplectic form to this submanifold is* degenerate, *and the symplectic structure is lost. On the other hand, the reduced Hamiltonian does not depend on the choice of variable* φ, *so* φ *is redundant. Further discarding the redundant variable* φ *yields a two-dimensional submanifold, which admits a standard symplectic form* $\omega_{red} = dr \wedge dp_r$. *As a result, the symplectic structure can be restored by further discarding the redundant variables. As we will see, it is a general feature of the constrained system.*

Given a symplectic manifold (M, ω), a set of first-class constraints $\{\phi_a\}_{a=1,\ldots,m}$ and a Hamiltonian H compatible with the constraint, by the definition of compatibility A.10, we have

$$\frac{d}{ds}\phi_a = \{\phi_a, H\} \in I(\phi^a), \tag{A.98}$$

which implies that

$$\mathscr{L}_{X_{\phi_a}}(H) = X_{\phi_a}(H) = -\{\phi_a, H\} \in I(\phi^a). \tag{A.99}$$

Namely, when restricted to the submanifold Σ, a compatible Hamiltonian remains constant along the flow of the vector field X_{ϕ_a}. As a result, the flow direction becomes *redundant* in describing the system's evolution. This can be viewed as analogous to the gauge redundancy in gauge theory, where two configurations of the gauge field are considered physically identical if they can be connected by a gauge transformation. This observation leads us to introduce an equivalence relation $\sim$ on Σ, where two points are equivalent if they can be connected by the flow of X_{ϕ_a}. Then the physical phase space should be identified as

$$M_{red} := \Sigma/\sim . \tag{A.100}$$

As the constraint $\phi_a = 0$ could be wild, the resulting quotient space could be very singular. For simplicity, we restrict ourselves to the locus away from the singularities such that the quotient space M_{red} is a manifold. In this case, the dimension of the manifold could be counted. Starting from a $2n$-dimensional manifold M, we first impose m independent constraints to reach the $2n - m$ dimensional manifold $\tilde{\Sigma}$. m constrains introduces m independent Hamiltonian vector field, which specifies the equivalence relation $\sim$. Therefore, the quotient manifold M_{red} is of $2n - 2m$-dimensional, which is always an even number.

The following theorem further illustrates that there is always a natural symplectic form ω_{red} on M_{red}.

Theorem A.6 *There exists a unique symplectic form* ω_{red} *on the reduced symplectic manifold* M_{red} *such that it is related to the original symplectic form* ω *via*

$$\begin{tikzcd} \Sigma \arrow[r, "i"] \arrow[d, "\pi"] & M \\ M_{red} & \end{tikzcd} \qquad i^*\omega = \pi^*\omega_{red}. \tag{A.101}$$

Proof *We are going to show that the condition (A.101) admits a solution and defines a symplectic form on M_{red}. To demonstrate the existence of the solution, it suffices to show that $i^*\omega$ is identical on each orbit of the flow. This follows from the fact that the flow is generated by the Hamiltonian vector field, which preserves the symplectic form.*

To show that ω_{red} is symplectic, we need to show that ω_{red} is closed and non-degenerate. Indeed, we have

$$\pi^* d\omega_{red} = di^*\omega = i^* d\omega = 0 \tag{A.102}$$

As π is surjective, the pullback π^* is injective. Therefore, Eq.(A.102) implies that $d\omega_{red} = 0$, namely, ω_{red} is closed.

Next, to show that ω_{red} is non-degenerate, we need to examine the tangent space of Σ. For any point $p \in \Sigma$, the tangent space $T_p\Sigma$ can be decomposed as

$$T_p\Sigma = \ker \pi_*\Big|_p \oplus H_p, \tag{A.103}$$

where H_p is isomorphic to $T_{\pi(p)}M_{red}$ as a vector space, and $\ker \pi_*\big|_p$ is an m-dimensional vector space by dimensional considerations. Therefore, to prove that ω_{red} is non-degenerate, it suffices to show that for every $X \in H_p$, there exists a $Y \in H_p$ such that $\pi^*\omega_{red}(X, Y)$ is non-vanishing. From equation (A.101), we have

$$\pi^*\omega_{red}(X, Y) = i^*\omega(X, Y) = \omega(i_*X, i_*Y). \tag{A.104}$$

Since the tangent space $T_{i(p)}M$ can be decomposed as

$$T_{i(p)}M = N_{i(p)}\Sigma \oplus i\left(\ker \pi_*\Big|_p\right) \oplus i\left(H_p\right), \tag{A.105}$$

it suffices to show that $\omega\big|_{i(H_p)}$ is non-degenerate, i.e., that $i(H_p)$ is a symplectic subspace, which can be shown by proving it's compliment $N\Sigma \oplus i\left(\ker \pi_*\big|_p\right)$ to be a symplectic subspace. It's straightforward to see that $i\left(\ker \pi_*\big|_p\right)$ is spanned by the X_{ϕ_a}. On the other hand, elements $X \in N\Sigma$ satisfy that $\exists a, X(\phi_a) \neq 0$. Therefore, we have

$$\exists a, X(\phi_a) = d\phi_a(X) = \omega(X, X_{\phi_a}) \neq 0. \tag{A.106}$$

which shows that $N_{i(p)}\Sigma \oplus i\left(\ker \pi_*\big|_p\right)$ is a symplectic subspace. We conclude the proof of the non-degeneracy by the fact that the complement of a symplectic subspace is symplectic. ∎

The discussion so far remains somewhat abstract, making explicit calculations challenging. To address this, we will provide a concrete description of ω_{red}, at the expense of going to a local description. Let $U \subset M_{\text{red}}$ be a small enough open subset inside the regular locus of M_{red}. In this case, we could take a local section $s : U \to \Sigma$ such that $\pi \circ s = id$. Intuitively, the local section selects one representative for each equivalent class, which is an analog of the gauge fixing. From (A.101), we have

$$\omega_{\text{red}}\Big|_U = s^*\pi^*\omega_{\text{red}}\Big|_U = (i \circ s)^*(\omega\Big|_{i\circ s[U]}) \tag{A.107}$$

In this case, after selecting a local section s, we embed a small portion of $M_{\rm red}$ back into the original symplectic manifold M, which we will denote as $\tilde{\Sigma} = i \circ s[U] \subset \Sigma \subset M$. It allows us to conduct calculations on $\omega_{\rm red}$ by making use of the original symplectic form ω. Explicitly, let $p \in \tilde{\Sigma}$, we could decompose the tangent space T_pM as

$$T_pM = N_p\tilde{\Sigma} \oplus T_p\tilde{\Sigma},\ p \in \tilde{\Sigma}. \tag{A.108}$$

By identifying U and $\tilde{\Sigma}$, the reduced symplectic form $\omega_{\rm red}$ could thus be identified with the restriction of the symplectic form ω on $T_p\tilde{\Sigma}$.

We could make it more explicit by specifying the locus of $\tilde{\Sigma}$ with the zeros of the smooth functions. When U is small enough, the locus $\tilde{\Sigma} \subset M$ could be represented by the zeros of $2m$ independent functions $\{\phi_a = 0, \chi_a = 0\}$ inside M (with a proper domain). The first m constraints $\{\phi_a = 0\}$ always hold on $\tilde{\Sigma}$ because $\tilde{\Sigma} \subset \Sigma$. What the remaining m constraints $\{\chi_a = 0\}, a = 1, \dots, m$ do is that they select the representative for each flow of X_{ϕ_a}. Therefore, χ_a explicitly introduces gauge fixing. (See LHS of the Fig.A-1). It is indeed the formalism familiar to physicists when dealing with gauge theory. At this point, extra conditions have to be imposed in order that the gauge choice $\{\chi_a(x) = 0\}$ intersects with each orbit only once. Locally, it suffices to ask $\chi_a(x)$ flows monotonically along X_{ϕ_a}. (See the LHS of Fig.A-1). Namely, we have to ask

$$\det\Big(X_{\phi_b}(\chi_a)(x)\Big) = \det\{\chi_a(x), \phi_b(x)\} \neq 0, \forall x \in \Sigma \tag{A.109}$$

By the definition A.9, Eq.(A.109) implies that the set of constraints $\Phi_I = \{\phi_a, \chi_a\}_{a=1,\dots,m}$ is of the second-class.

Moreover, the constraints $\Phi_I = \{\phi_a, \chi_a\}_{a=1,\dots,m}$ introduce a natural frame field $\{X_{\Phi_I}\}$ for $N\tilde{\Sigma}$, which can be seen as follows. We consider the $2m{\times}2m$ matrix $X_{\Phi_J}(\Phi_I), I, J = 1, \dots 2m$. By functional independence, vector fields X_{Φ_J} are nowhere dependent. In order to show that X_{Φ_J} provides a frame for the normal bundle $N\tilde{\Sigma}$, it suffices to show that the matrix $X_{\Phi_J}(\Phi_I)$ is non-degenerate. Indeed, while restricted to $\tilde{\Sigma}$, the matrix takes the form

$$\mathbf{C}_{IJ} = X_{\Phi_J}(\Phi_I) = \{\Phi_I, \Phi_J\} = \left[\begin{array}{c|c} \mathbf{0} & \mathbf{C}_{ab} \\ \hline -\mathbf{C}_{ab} & * \end{array},\right] \tag{A.110}$$

where $\mathbf{C}_{ab} = X_{\phi_a}(\chi_b) = \{\chi_b, \phi_a\}$. The upper left corner of the matrix (A.110) is zero because ϕ_a is the first-class constraint, whose Poisson bracket vanishes when restricted to $\tilde{\Sigma}$. The determinant of the matrix is thus proportional to $\det\big(\mathbf{C}_{ab}\big)^2$, which is non-vanishing by the condition in Eq.(A.109).

At this stage, we are prepared to examine the Poisson bracket on the reduced symplectic manifold $M_{\rm red}$. However, it is common that while smooth functions on M are straightforward to describe, their counterparts on $M_{\rm red}$ are significantly more difficult to handle. To resolve this problem, we introduce the concept of the Dirac bracket.

Definition A.11 *(Dirac Bracket) Given two smooth functions $f, g \in C^\infty(M)$, the Dirac bracket is defined as*

$$\{f, g\}_D := \{f\Big|_{\tilde{\Sigma}}, g\Big|_{\tilde{\Sigma}}\}_{red}, \tag{A.111}$$

where $f\big|_{\tilde{\Sigma}}$ and $g\big|_{\tilde{\Sigma}}$ are the restriction of the smooth function f and g on $\tilde{\Sigma}$, and $\{-,-\}_{red}$ is the Poisson bracket on (M_{red}, ω_{red}).

We will denote $r = i \circ s$ as the embedding of $\tilde{\Sigma}$ in M, such that the restriction of a function f to $\tilde{\Sigma}$ is given by $f\big|_{\tilde{\Sigma}} = r^* f$. In the following discussion, we will provide an explicit description of the Dirac bracket, which follows from the following theorem.

Theorem A.7 *(Working Formula for the Dirac Bracket) Let (M, ω) be a symplectic manifold, ϕ_a be a set of first-class constraints and χ_a the local gauge fixing satisfying (A.109). Denote $\boldsymbol{\Phi}_I = (\phi_a, \chi_a)$. Then the Dirac bracket between two smooth function $f, g \in C^\infty(M)$ could be evaluated as*

$$\{f, g\}_D = \{f, g\} - \{f, \boldsymbol{\Phi}_I\}\mathbf{C}^{IJ}\{\boldsymbol{\Phi}_J, g\}, \tag{A.112}$$

where $\mathbf{C}_{IJ} = \{\boldsymbol{\Phi}_I, \boldsymbol{\Phi}_J\}$ and $\mathbf{C}^{IJ} = (\mathbf{C}_{IJ})^{-1}$.

The benefit of this formula is that all the Poisson bracket involved on the right-hand side of the formula is expressed via the Poisson bracket on M, which can always be evaluated explicitly.

Proof *From the definition of the Dirac bracket (A.10), we have*

$$\{f, g\}_D := \{r^* f, r^* g\}_{red} = -\omega_{red}(X_{r^* f}, X_{r^* g}). \tag{A.113}$$

On the other hand, by the definition of the Hamiltonian vector field, $X_{r^* f}$ satisfy the following relations

$$\omega_{red}(X_{r^* f}, -) = -dr^* f = -r^* df = r^*(\omega(X_f, -)) = \omega(X_f, r_* -). \tag{A.114}$$

Therefore, to prove the theorem, it suffices to evaluate the vector field $r_* X_{r^* g}$. It can be conducted by calculating the symplectic pairing between $r_* X_{r^* g}$ and other vector fields. As we discussed above, $X_{\boldsymbol{\Phi}_I}$ introduces a natural frame for the normal bundle $N\Sigma$. An explicit evaluation shows that the pairing between $r_* X_{r^* g}$ and $X_{\boldsymbol{\Phi}_I}$ vanishes.

$$\omega(r_* X_{r^* g}, X_{\boldsymbol{\Phi}_I}) = d\boldsymbol{\Phi}_a(r_* X_{r^* g}) = d(r^* \boldsymbol{\Phi}_I)(X_{r^* g}) = 0 \tag{A.115}$$

where the last equation follows from the fact that $r^* \boldsymbol{\Phi}_I$ is a zero function. We proceed by evaluating the symplectic pairing between $r_* X_{r^* g}$ and elements in $T\Sigma$. Every element in $T\Sigma$ can be represented by $r_* v$. As a result, the pairing between $r_* X_{r^* g}$ and $X_{\boldsymbol{\Phi}_I}$ yields

$$\omega(r_* X_{r^* g}, r_* v) = (r^* \omega)(X_{r^* g}, v) = \omega_{red}(X_{r^* g}, v) = dr^* g(v) = dg(r_* v) = \omega(X_g, r_* v). \tag{A.116}$$

Namely, the pairing between $r_* X_{r^* g}$ and any element in $T_p\tilde{\Sigma}$ is identical to the pairing between X_g and that element. This demonstrates that $r_* X_{r^* g}$ is precisely the projection of X_g onto $T\tilde{\Sigma}$. Utilizing the frame for the normal bundle $X_{\boldsymbol{\Phi}_I}$, we can explicitly express the projection action as

$$r_* X_{r^* g} = X_g + X_{\boldsymbol{\Phi}_I}\mathbf{C}^{IJ}\omega(X_{\boldsymbol{\Phi}_J}, X_g), \tag{A.117}$$

where $\mathbf{C}^{IJ} = (\mathbf{C}^{-1})_{IJ}$ is the inverse of the matrix in (A.110). Such an inverse matrix exists as we have shown above that $\mathbf{C}_{IJ}$ is non-degenerate. It is easy to see that the terms $X_{\boldsymbol{\Phi}_I}\mathbf{C}^{IJ}\omega(X_{\boldsymbol{\Phi}_J}, X_g)$ subtract the components of X_g on the normal bundle by evaluating the symplectic pairing. Consequentially, by combining (A.117) and (A.113), we have

$$\begin{aligned} \{f, g\}_D &= -\omega(X_f, r_* X_{r^* g}) = -\omega(X_f, X_g) - \omega(X_f, X_{\boldsymbol{\Phi}_I})\mathbf{C}^{IJ}\omega(X_{\boldsymbol{\Phi}_J}, X_g) \\ &= \{f, g\} - \{f, \boldsymbol{\Phi}_I\}\mathbf{C}^{IJ}\{\boldsymbol{\Phi}_J, g\}. \end{aligned} \tag{A.118}$$

It concludes the proof. ∎

To clarify, the gauge choice $\chi_a(x)$ may still exhibit global ambiguity even if it satisfies the condition in Eq.(A.109), as illustrated on the right-hand side of Fig.A-1. In this example, we consider the symplectic manifold to be S^2, equipped with the standard symplectic form. The constraint $\phi = 0$ restricts the variables to the rec circle, which, after applying the equivalence relation, reduces to a single point. The gauge choice $\chi = 0$, represented by the black curve, satisfies (A.109). However, this gauge choice intersects a single orbit twice, demonstrating the presence of global ambiguity.

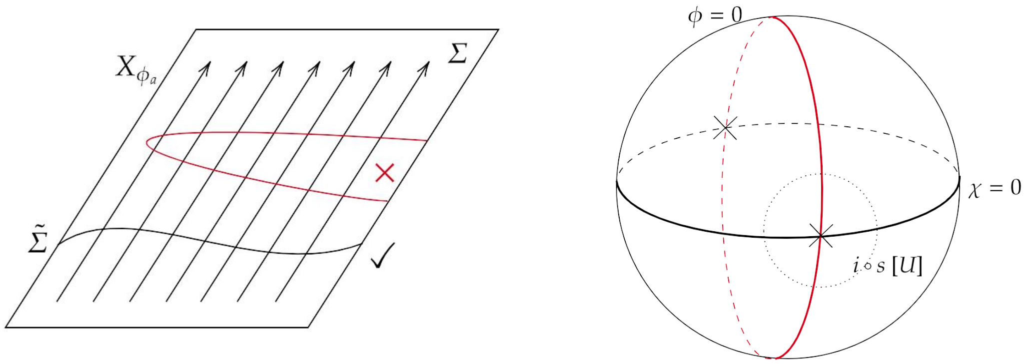

Figure A-1 LHS: There is an equivalence relation on Σ, provided by the flow of the Hamiltonian vector field X_{ϕ_a}. The red line illustrates an improper gauge fixing, which doesn't intersect with each orbit of the flow once. The black line illustrates a proper gauge fixing, which provides a local description $\tilde{\Sigma}$ for the reduced phase space. RHS: A global ambiguity might exist for the gauge choice $\chi = 0$, even if the condition (A.109) is satisfied.

Sympletic Quotient In the following discussion, we will apply the general formalism of the constraint system to a specific class of constraints provided by the symmetries of the phase space.

The setup is as follows: Let (M, ω) be a Hamiltonian G-space with moment map μ and comoment map Q, and let ψ denote the action of G on M. We consider the set of constraints to be $\{Q_x\}_{x\in\mathfrak{g}}$, meaning we focus on the physical states that are charge neutral under the group G. For any point $m \in M$ that is charge neutral under G, we have

$$Q_x(m) = \langle \mu(m), x \rangle = 0, \quad \forall x \in \mathfrak{g}.$$

Thus, the charge neutrality condition is equivalent to $\mu(m) = 0$. At this stage, the constraint surface Σ is given by

$$\Sigma = \bigcap_{x\in\mathfrak{g}} Q_x^{-1}[0] = \mu^{-1}[0].$$

To construct the reduced symplectic manifold, we must further quotient out the equivalence provided by the constraint equations. Under this framework, the Hamiltonian vector fields associated with the constraints Q_x are

$$X_{Q_x} = \psi_*(x),$$

which corresponds to the infinitesimal transformation under the G action. Therefore, the equivalence relation $\sim$ at this point is provided by the G action on $\mu^{-1}[0]$. Indeed, following from Prop.A.3, $\mu^{-1}[0]$ admits a G-action. The reduced symplectic manifold is thus provided by the quotient space $\mu^{-1}[0]\big/G$. The precise statement is formulated by Marsden, Weinstein, and Meyer as follows

Theorem A.8 *(Symplectic Quotient) Let (M, ω) be a Hamiltonian G space with G a compact Lie group with moment map μ. Assume that Lie group G has a free and proper action on M. Then the following statements hold:*

1) *The quotient space $\mu^{-1}[0]/G$ is a manifold.*
2) *$\left(\pi : \mu^{-1}[0] \longrightarrow \mu^{-1}[0]/G\right)$ is a principal G-bundle.*
3) *There exists a sympletic form ω_{red} on $\mu^{-1}[0]/G$, satisfying*

$$\begin{array}{ccc} \mu^{-1}[0] & \overset{i}{\hookrightarrow} & M \\ \downarrow \pi & & \\ \mu^{-1}[0]/G & & \end{array} \qquad i^*\omega = \pi^*\omega_{red}. \tag{A.119}$$

Example A.8 *Let us consider the symplectic manifold $(\mathbb{C}^{n+1}, \omega = \frac{i}{2}\sum_{j=0}^{n} dz_j \wedge d\overline{z}_j)$. The Lie group $U(1)$ acts on $\mathbb{C}^{n+1}$ via*

$$\psi_{e^{i\theta}}(z_0, \ldots, z_n) = (e^{i\theta} z_0, \ldots, e^{i\theta} z_n).$$

We will now evaluate the moment map μ associated with the $U(1)$ action, and proceed to compute the symplectic quotient M_{red} and the reduced symplectic form ω_{red}.

Evaluating the Moment Map *The action of $U(1)$ on $\mathbb{C}^{n+1}$ can be understood as a diagonal action on the $(n+1)$-fold product of the symplectic manifold $\mathbb{C}$. As demonstrated in Example A.5, the $U(1)$ action on the symplectic manifold $(\mathbb{C}_j, \omega = \frac{i}{2} dz_j \wedge d\overline{z}_j)$ is Hamiltonian, with the moment map given by*

$$\mu^{(j)}(z_j) = \frac{1}{2}|z_j|^2. \tag{A.120}$$

By theorem A.4, the diagonal $U(1)$ action on $\mathbb{C}^{n+1}$ is also Hamiltonian, and its moment map is given by

$$\mu(z_0, \ldots, z_n) = \sum_{j=0}^{n} \mu^{(j)}(z_j) = \frac{1}{2}\sum_{j=0}^{n} |z_j|^2. \tag{A.121}$$

It is also straightforward to verify that for the $U(1)$ action, an undetermined constant $\xi \in \mathbb{R}$ can be added to the moment map, resulting in a family of moment maps μ_ξ associated with the same $U(1)$ action.

$$\mu_\xi(z_0, \ldots, z_n) = \frac{1}{2}\sum_{j=0}^{n} |z_j|^2 - \xi. \tag{A.122}$$

Evaluate the Reduced Symplectic Manifold M_{red} *In the following discussion, we will discuss the reduced symplectic manifold M_{red} with different choices of parameter ξ.*

The situation is not interesting when $\xi \leq 0$. In the cases where $\xi < 0$, the set $\mu_\xi^{-1}[0]$ is empty set, and thus the reduced symplectic manifold M_{red} is trivial. In the case where $\xi = 0$, the set $\mu_\xi^{-1}[0]$ is a single point $\{(0, \ldots, 0)\}$. $U(1)$ acts trivially on the set $\mu_\xi^{-1}[0]$, thus the reduced symplectic manifold is also a point.

Let us now focus on the case where $\xi > 0$. Without loss of generality, we set $\xi = \frac{1}{2}$. In this case, the set $\mu^{-1}[0]$ is given by

$$\mu^{-1}[0] = \{z \in \mathbb{C}^{n+1} \mid \sum_{j=0}^{n} |z_j|^2 = 1\} = S^{2n+1} \subset \mathbb{C}^{n+1}. \tag{A.123}$$

To apply the symplectic quotient, we need to verify further that $U(1)$ acts freely on S^{2n+1}. To do this, for any point $z = (z_0, \dots, z_n) \in S^{2n+1}$, we must check that its stabilizer is trivial. Specifically, suppose $e^{i\theta}$ is an element of the stabilizer of z, then

$$(z_0, \dots, z_n) = \psi_{e^{i\theta}}(z_0, \dots, z_n) := (e^{i\theta} z_0, \dots, e^{i\theta} z_n), \tag{A.124}$$

which implies $e^{i\theta} = 1$. Therefore, the stabilizer of z is trivial, confirming that $U(1)$ acts freely on S^{2n+1}. By the symplectic reduction procedure (Prop.A.8), the quotient space S^{2n+1}/S^1 is a manifold, which is exactly the complex projective space $\mathbb{C}P^n$. Moreover, we have

$$\begin{tikzcd} S^{2n+1} \arrow[r, "i"] \arrow[d, "\pi"] & \mathbb{C}^{n+1} \\ \mathbb{C}P^n & \end{tikzcd} \qquad \pi^*\omega_{red} = i^*\omega. \tag{A.125}$$

Evaluate the Reduced Symplectic Form $\omega_{\mathbf{red}}$ *We proceed to discuss the sympletic structure on the reduced symplectic manifold $\mathbb{C}P^n$. As we will see, the reduced symplectic form can be compactly expressed as*

$$\omega_{red} = \frac{i}{2}\partial\bar{\partial}\log\left(|z|^2\right), \tag{A.126}$$

where $|z| = \left(\sum_{m=0}^{n} |z_m|^2\right)^{1/2}$. This symplectic form is known as the Fubini-Study form *on $\mathbb{C}P^n$, which plays a fundamental role in complex geometry.*

The evaluation of ω_{red} is sketched as follows. For our purpose, it is beneficial to construct a section $s : \mathbb{C}P^n \to S^{2n+1}$ such that the following diagram holds

$$\begin{tikzcd} S^{2n+1} \arrow[r, "i"] \arrow[d, shift left, "\pi"] & \mathbb{C}^{n+1} \\ \mathbb{C}P^n \arrow[u, shift left, "s"] \arrow[ur, "i\circ s"'] & \end{tikzcd} \qquad \omega_{red} = (i \circ s)^*\omega. \tag{A.127}$$

Remember that every point on $\mathbb{C}P^n$ could be represented by the homogeneous coordinate $[z_0, \dots, z_n]$. The section can thus be specified by

$$\begin{aligned} s : \mathbb{C}P^n &\longrightarrow S^{2n+1} \\ [z_0, \dots, z_n] &\mapsto \left(\frac{z_0}{|z|}, \dots, \frac{z_n}{|z|}\right), \end{aligned} \tag{A.128}$$

It is easy to show that s satisfies the property $\pi \circ s = id$ and thus defines a section. Then *the reduced sympletic form follows from the following straightforward evaluations*

$$
\begin{aligned}
\omega_{red} &= (i\circ s)^*\omega = (i\circ s)^*\Big(\frac{i}{2}\sum_{j=0}^{n} dz_j\wedge d\bar{z}_j\Big) = \frac{i}{2}\sum_{j=0}^{n} d\left(\frac{z_j}{|z|}\right)\wedge d\left(\frac{\bar{z}_j}{|z|}\right)\\
&= \frac{i}{2}\sum_{j=0}^{n}|z|^{-2}dz_j\wedge d\bar{z}_j + \frac{i}{2}\sum_{j=0}^{n}\bar{z}_j|z|^{-1}dz_j\wedge d|z|^{-1} - \frac{i}{2}\sum_{j=0}^{n} z_j|z|^{-1}d\bar{z}_j\wedge d|z|^{-1}\\
&= \frac{i}{4}\sum_{j=0}^{n}\left(|z|^{-2}dz_j\wedge d\bar{z}_j + \bar{z}_j dz_j\wedge d|z|^{-2}\right) + \frac{i}{4}\sum_{j=0}^{n}\left(|z|^{-2}dz_j\wedge d\bar{z}_j - z_j d\bar{z}_j\wedge d|z|^{-2}\right)\\
&= -\frac{i}{4}d\left(\frac{\bar{z}_j}{|z|^2}dz_j\right) + \frac{i}{4}d\left(\frac{z_j}{|z|^2}d\bar{z}_j\right) = \frac{i}{2}\partial\bar{\partial}log(|z|^2).
\end{aligned}
\tag{A.129}
$$

Example A.9 *(Non-Example for Symplectic Reduction Theorem) Let us consider the $SU(n)$ action on the symplectic manifold $\left(\mathfrak{gl}_n, \omega = \frac{i}{2}\mathrm{Tr}\big(dA\wedge dA^\dagger\big)\right)$ by conjugation. As is discussed in Example A.6, the conjugation action is Hamiltonian, whose moment map is given by*

$$\mu(A) = \frac{1}{2}[A, A^\dagger]^*. \tag{A.130}$$

Now we are going to perform the symplectic quotient for $SU(n)$ action. For this purpose, we shall first evaluate the set $\mu^{-1}[0]$. As we can see, $\mu^{-1}[0]$ is exactly the set of semisimple elements (or normal matrices) in $\mathfrak{gl}_n$

$$\mu^{-1}[0] = \{A\in\mathfrak{gl}_n \mid [A, A^\dagger] = 0\}. \tag{A.131}$$

It is well known that normal matrices can be diagonalized by unitary matrices. Therefore, the representatives for the elements in each orbit of the $SU(n)$ action can be taken as diagonal matrices, with the diagonal entries being the n eigenvalues of the matrices. However, there remains a residual S_n permutation symmetry for the eigenvalues, induced by the $SU(n)$ action. Thus, the quotient space by the conjugation action is given by

$$
\begin{aligned}
\mu^{-1}[0]/G &= \left\{A\in\mathfrak{gl}_n \mid [A, A^\dagger] = 0\right\}/G\\
&= \left\{\begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{pmatrix} \mid \lambda_i\in\mathbb{C}\right\}/S_n\\
&= \mathbb{C}^n/S_n.
\end{aligned}
\tag{A.132}
$$

The resulting geometry is not a manifold but rather an orbifold. *This arises because the $SU(n)$ action on $\mathfrak{gl}_n$ is not free, and thus the symplectic reduction theorem does not apply directly.*

A.6 Kähler Quotient and hyperKähler Quotient

In this section, we will consider two important generalizations for the symplectic quotient, which is tailored to the cases where the symplectic manifold M admits Kähler or hyperKähler structure. Loosely speaking, two theorem states that when the G action on

M is Hamiltonian with respect to Kähler form and preserve the (hyper)Kähler metric, then one could perform the symplectic quotient machinery to produce a quotient (hyper)Kähler manifold M_{red}, whose (hyper)Kähler structure is inherited from the original manifold M.

In the following section, we will only state the content of the theorem and provide several examples for the theorem. We follow $Addcite$ for the notation of the quoternion. For the proof of two theorems, one could consult (AddREF).

Theorem A.9 (Kähler Reduction) *Let (ω, g) be a Kähler structure on M^{2n}. Let $\psi : G\times M \to M$ be a Hamiltonian action with respect to ω and an isometry with respect to the metric g. Let $\mu : M \to \mathfrak{g}^*$ be the associated moment map. Suppose that $0 \in \mathfrak{g}^*$ is a clean value of μ and that there is a smooth structure on the orbit space $M_{red} = \mu^{-1}(0)/G$ for which the natural projection $\pi : \mu^{-1}(0) \to \mu^{-1}(0)$ is a smooth submersion. Then there is a unique Kähler structure (ω_{red}, g_{red}) on M_{red} defined by*

$$\begin{tikzcd} \mu^{-1}[0] \arrow[r, hook, "i"] \arrow[d, "\pi"] & M \\ \mu^{-1}[0]/G & \end{tikzcd} \qquad i^*\omega = \pi^*\omega_{red},\ i^*g = \pi^*g_{red}. \tag{A.133}$$

Theorem A.10 *(hyperKähler Quotient) Suppose that $(\omega_I, \omega_J, \omega_K, g)$ is a hyperKähler structure on M and that there is a left action $\psi : G \times M \to M$ that is Hamiltonian with respect to each of the three symplectic forms ω_a (or say trihamiltonian) and an isometry with respect to g. Let $\mu = (\mu_I, \mu_J, \mu_K) : M \to \mathfrak{g}^* \oplus \mathfrak{g}^* \oplus \mathfrak{g}^*$ be the associated moment maps. Suppose that $0 \in \mathfrak{g}^* \oplus \mathfrak{g}^* \oplus \mathfrak{g}^*$ is a clean value for μ and that the quotient $M_{red} = \mu^{-1}(0)/G$ has a smooth structure for which the projection $\pi : \mu^{-1}(0) \to M_{red}$ is a smooth submersion. Then there is a unique hyperKähler structure $(\omega_{Ired}, \omega_{Jred}, \omega_{Kred}, g_{red})$ on M_{red}*

$$\begin{tikzcd} \mu^{-1}[0] \arrow[r, hook, "i"] \arrow[d, "\pi"] & M \\ \mu^{-1}[0]/G & \end{tikzcd} \qquad i^*\omega_a = \pi^*\omega_{ared},\ i^*g = \pi^*g_{red}, a = I, J, K. \tag{A.134}$$

Example A.10 *(Quoternian Space $\mathbb{H}$) Let us consider the simplest hyperKähler manifold $M = \mathbb{H} \ni q = g + ix + jy + kz$. It can be equipped with the standard metric*

$$ds^2 = dq \otimes d\bar{q}. \tag{A.135}$$

and the hyperKähler structure

$$\Omega = \frac{1}{2} dq \wedge d\bar{q} =: \omega_I i + \omega_J j + \omega_K k. \tag{A.136}$$

There are two symmetries of Ω that we are going to discuss.
1. *$\mathbb{R}$ translation action: $\psi_t : q \mapsto q + t, t \in \mathbb{R}$,*
2. *$Sp(1) = SU(2)$ action: $\chi_p : q \mapsto qp, p\bar{p} = 1$.*

It is straightforward to see that two actions preserve the hyperKähler structure. Following from the previous discussion, the associated vector field for two actions is

$$\psi_*(t) = t\partial_g,\ \chi_*(s) = -qs\partial_q + s\bar{q}\partial_{\bar{q}}, \quad t \in \mathbb{R}, s \in \mathfrak{sp}(1) = \mathbb{H}. \tag{A.137}$$

A straightforward evaluation shows that the moment map is given by

$$\mu_\psi(q) = \frac{1}{2}(q - \bar{q}),\ \mu_\chi(q)(s) = \frac{1}{2}qs\bar{q}. \tag{A.138}$$

Especially, we could consider the action of the subalgebra $\mathfrak{u}(1)_i \subset \mathfrak{su}(2)$*. It acts as the right multiplication of* $e^{i\theta}$*. The resulting moment map is thus*

$$\mu_i(q) = \frac{1}{2}qi\bar{q}. \tag{A.139}$$

In another word, μ_i *defines a natural map* $\mathbb{R}^4 \mapsto \mathbb{R}^3 \cong \mathbb{R}^* \oplus \mathbb{R}^* \oplus \mathbb{R}^*$.

For any quaternion q, there exists a polar coordinate $q = ae^{i\phi/2}$*, with* $\phi \in [0, 4\pi)$ *and* $\bar{a} = -a$*. If we denote the coordinate on* $\mathbb{R}^3$ *as*

$$\mathbf{r} = r_1 i + r_2 j + r_3 k = \frac{1}{2}qi\bar{q} = \frac{1}{2}ai\bar{a}, \tag{A.140}$$

then we could see that μ_i *defines an* S^1 *fibration over* $\mathbb{R}^3$*, except for the singular fiber at the origin* $\{0\}$*. The* S^1 *fiber coordinate is given by the angular variable* ϕ*. Under the new coordinate, it can be shown that the flat metric takes the form*

$$ds^2 = \frac{1}{4}\left(\frac{1}{r}\sum_{i=1}^{3} dr_i^2 + r(d\phi + \omega)^2\right), \tag{A.141}$$

where ω *is a one-form satisfying*

$$d\omega = *_3 d\frac{1}{r}. \tag{A.142}$$

In other words, ω *is the connection of the gauge field in the presence of Dirac monopole, which takes*

$$\omega_+ = -(1 - cos\theta)d\phi,\ \omega_- = (1 + cos\theta)d\phi, \tag{A.143}$$

on the north/south part of the $\mathbb{R}^3$ *plane respectively. It is a special case of the hyper-Kähler metric in the Gibbons-Hawking form. This metric would be the starting point for the discussion later.*

Example A.11 *(Muti-centered Taub-Nut Space) Let's consider the flat quaternion manifold* $M = \mathbb{H}^d \times \mathbb{H} \ni (q_a, w)$ *with the quaternion symplectic form*

$$\Omega = \sum_{a=1}^{d} \frac{1}{2} dq_a \wedge d\bar{q}_a + \frac{1}{2} dw \wedge d\bar{w} \tag{A.144}$$

and the standard metric

$$ds^2 = \sum \frac{1}{2} dq_a \otimes d\bar{q}_a + \frac{1}{2} dw \otimes d\bar{w} \tag{A.145}$$

Let's consider the $\mathbb{R}^d$ *action, specified by*

$$\psi_{(\theta_1,\ldots,\theta_d)}(q_a, w) = \left(q_a e^{-i\theta_a}, w + \sum_{a=1}^{d} \theta_a\right). \tag{A.146}$$

Evaluate Vector Field *Let's evaluate the vector field associated with the action. To this purpose, we take an arbitary function $f \in C^{\infty}(M)$. Then*

$$\begin{aligned} X_{(\theta_1,\ldots,\theta_d)}(f) &= \left.\frac{d}{dt}\right|_0 f(q_a e^{-i\theta_a t}, w + \sum_{b=1}^{d}\theta_b t, e^{i\theta_a t}\bar{q}_a, \bar{w} + \sum_{b=1}^{d}\theta_b t) \\ &= \left(\sum_{a=1}^{d} -q_a i\theta_a \partial_{q_a} + i\theta_a \bar{q}_a \partial_{\bar{q}_a} + \theta_a \partial_w + \theta_a \partial_{\bar{w}}\right) f \end{aligned} \tag{A.147}$$

From which we conclude that the symplectic vector field associated with the $\mathbb{R}^d$ action is

$$X_{(\theta_1,\ldots,\theta_d)} = \sum_{a=1}^{d} -q_a i\theta_a \partial_{q_a} + i\theta_a \bar{q}_a \partial_{\bar{q}_a} + \theta_a \partial_w + \theta_a \partial_{\bar{w}}. \tag{A.148}$$

Evaluate the Moment Map *In the next step, let us evaluate the Hamiltonian associated with the vector field. It follows from*

$$\begin{aligned} i_{X_{(\theta_1,\ldots,\theta_d)}}\Omega &= -\sum_{a=1}^{d}\frac{\theta_a}{2}(q_a i d\bar{q}_a + dq_a i\bar{q}_a - \theta_a d\bar{w} + \theta_a dw) \\ &= -\sum_{a=1}^{d}\frac{\theta_a}{2} d(q_a i\bar{q}_a + w - \bar{w}), \end{aligned} \tag{A.149}$$

from which we conclude that the associated Hamiltonian is given by

$$Q_{(\theta_1,\ldots,\theta_d)} = \sum_{a=1}^{d}\frac{\theta_a}{2}(q_a i\bar{q}_a + w - \bar{w}). \tag{A.150}$$

Evaluate Moment Map *From the expression of the Hamiltonian, we conclude that the moment map is given by*

$$\begin{aligned} &\mu : M \longrightarrow \mathbb{R}^d \\ &(q_a, w) \mapsto (q_1 i\bar{q}_1 + w - \bar{w}, \ldots, q_n i\bar{q}_n + w - \bar{w}). \end{aligned} \tag{A.151}$$

In the following discussion, we would like to construct the hyperKähler quotient of M and describe the metric on the quotient space. To this purpose, we introduce the following coordinates

$$q_M = a_M e^{i\phi_M/2},\ \mathbf{r}_M = \frac{1}{2}a_M i\bar{a}_M,\ \mathbf{y} = \frac{1}{2}(w - \bar{w}), y = \frac{1}{2}(w + \bar{w}). \tag{A.152}$$

As a straightforward generalization to the quaternion space, under this coordinate, the flat metric on M takes

$$ds^2 = \sum_{M=1}^{d}\frac{1}{4}\left(\frac{1}{r_M}d\mathbf{r}_M^2 + r_M(d\phi_M + \omega_M)^2\right) + dy^2 + d\mathbf{y}d\bar{\mathbf{y}}. \tag{A.153}$$

where $r_M = |\mathbf{r}_M|$ and $d\omega_M = *_3 d\frac{1}{r_M}$. The moment map takes

$$\mu_M = \mathbf{r}_M + \mathbf{y}. \tag{A.154}$$

Construct HyperKähler Quotient *Now we are going to perform hyperKähler quotient to construct the metric on* $\mu^{-1}(-\mathbf{t})/\mathbb{R}^d$. *As the first step, we shall restrict the flat metric to the locus where* $\mu_M = \mathbf{r}_M + \mathbf{y} = -\mathbf{t}_M$. *Therefore, let us introduce*

$$\mathbf{r} := -\mathbf{y} = \mathbf{r}_M + \mathbf{t}_M. \tag{A.155}$$

Then the restriction of the metric on $\mu^{-1}(0)$ *simplifies to*

$$ds^2 = \sum_{M=1}^{d} \frac{1}{4}\left(\frac{1}{|\mathbf{r}-\mathbf{t}_M|}d\mathbf{r}^2 + |\mathbf{r}-\mathbf{t}_M|(d\phi_M + \omega_M)^2\right) + dy^2 + d\mathbf{r}^2. \tag{A.156}$$

In the next step, we are going to quotient out the $\mathbb{R}^d$ *action. In the present coordinate, the symplectic vector field of the* $\mathbb{R}^d$ *action takes*

$$X_M = 2\partial_{\phi_M} + \partial_y \tag{A.157}$$

It is convenient to introduce the coordinate

$$\tau = \sum_{M=1}^{d} \phi_M - 2y, \tag{A.158}$$

which is invariant under the $\mathbb{R}^d$ *action. If we switch to the new coordinate* (τ, ϕ_M), *then the vector field reduces to*

$$X_M = 2\partial_{\phi_M}. \tag{A.159}$$

And the metric reduces to

$$ds^2 = \left(1 + \sum_{M=1}^{d} \frac{1}{4}\frac{1}{|\mathbf{r}-\mathbf{t}_M|}\right)d\mathbf{r}^2 + \frac{1}{4}\sum_{M=1}^{d} |\mathbf{r}-\mathbf{t}_M|(d\phi_M+\omega_M)^2 + \frac{1}{4}(d\tau - \sum_{M=1}^{d} d\phi_M)^2. \tag{A.160}$$

There is a straightforward way to quotient out X_M *by discarding all the* $d\phi_M$ *term in the metric. However, we would like to do it in another way, by setting* $\phi_M = 0, M = 2, \dots d$ *and select* $d\phi_1$ *appropriately such that the metric reduces to the form from standard Gibbons-Hawking analysis*

$$ds^2 = V(r)d\mathbf{r}^2 + V(r)^{-1}(d\tau + \omega)^2 \tag{A.161}$$

where

$$V(r) = \left(1 + \sum_{M=1}^{d} \frac{1}{4}\frac{1}{|\mathbf{r}-\mathbf{t}_M|}\right),\ \omega = \sum_{M=1}^{d} \omega_M. \tag{A.162}$$

Notice that the following equation is solved automatically

$$*_3 dV(r) = d\omega. \tag{A.163}$$

which confirmed that the metric fits into the Gibbons-Hawking analysis. This metric is known as the multi-center Taub-Nut metric, and the quotient space is recognized as the multi-center Taub-Nut space.

Multi-Center Taub-Nut Space as an Affine Variety *In the following, we are going to describe the multi-center Taub-Nut Space in a selected complex structure I. It is descended from the complex structure i of $M = \mathbb{H}^d \times \mathbb{H} \cong \mathbb{C}^{2d+2}$. In this complex structure, we could write $q_M = u_M + v_M\, j$, with $u_M = g_M + x_M\, i$ and $v_M = y_M + z_M\, i$ the complex variables. Similarly, we have $w = f' + f\, j$, with f, f' the complex variables.*

After selecting a complex structure, the quaternion symplectic form splits into a real one and a holomorphic one

$$\Omega = \omega_I\, i + \omega_J\, j + \omega_K\, k = \omega_I\, i + \Omega_I\, j \tag{A.164}$$

with $\Omega_I = \omega_J + \omega_K\, i$. As a byproduct, the moment map μ_M also splits into a real one $\mu_M^{\mathbb{R}}$ and a complex one $\mu_M^{\mathbb{C}}$, such that

$$\mu_M = \mu_M^{\mathbb{R}}\, i + \mu_M^{\mathbb{C}}\, j. \tag{A.165}$$

A straightforward calculation shows that in our case,

$$\mu_M^{\mathbb{R}} = \frac{1}{2i}(f' - \bar{f}') + \frac{1}{2}(|u_M|^2 - |v_M|^2),\ \mu_M^{\mathbb{C}} = f - u_M v_M. \tag{A.166}$$

To describe the multi-center Taub-NUT space in a single complex structure, it is convenient to apply the Kempf-Ness theorem on the symplectic quotient. This theorem - widely used in analyzing the Higgs branch of 4d $\mathcal{N} = 2$ theories - implies that when conducting a hyperKähler quotient, one can replace the usual procedure of quotienting the set $\mu^{-1}(-\mathbf{t})$ by the real Lie group $\mathbb{R}^d$ with the equivalent approach of quotienting the set $\Sigma = (\mu^{\mathbb{C}})^{-1}(e)$ by its complexified counterpart $\mathbb{C}^d$, where $-\mathbf{t}_M = e'_M + e_M\, j$. The resulting space could be described by the spectrum of the $\mathbb{C}^d$ invariant polynomials on Σ

$$\mu^{\mathbb{C}-1}(e)//\mathbb{C}^d = Spec(\mathcal{O}(\Sigma)^{\mathbb{C}^d}). \tag{A.167}$$

In our case, the $\mathbb{C}^d$ invariant polynomials are generated by the variables

$$u = e^{if'} \prod_{M=1}^{d} u_M,\ v = e^{-if'} \prod_{M=1}^{d} v_M,\ f, \tag{A.168}$$

subjecting to the relation

$$uv = \prod_{M=1}^{d} (f - e_M). \tag{A.169}$$

As a result, the quotient space is a hyperplane inside $\mathbb{C}^3$, specified by the condition (A.169). It provides a description of the multi-center Taub-Nut space in complex structure I.

Remark on the Kempf-Ness Theorem *Let's remark on how this statement relates to the standard form of the Kempf–Ness theorem. To see this equivalence, we split the quotient procedure into two steps. The first step is to restrict the variables to the locus $\Sigma = \cap_{M=1}^{d} \mu_M^{\mathbb{C}-1}[-e_M]$, specified by $v - y_M z_M + e_M = 0$. This locus describes an affine variety inside $\mathbb{C}^{2d+2}$. It is not hard to show that the restriction of ω_I to Σ remains a symplectic form. The $\mathbb{R}^d$ action can be defined on Σ by the $\mathbb{R}^d$-equivariance of the* *complex moment map* $\mu_M^{\mathbb{C}}$. *It is straightforward to demonstrate that the moment map with respect to* (Σ, ω_I) *is* $\tilde{\mu}_M^{\mathbb{R}} = \mu_M^{\mathbb{R}}\big|_{\Sigma}$.

In the standard hyperKähler quotient approach, the next step is to perform the symplectic quotient on $\tilde{\mu}_M^{\mathbb{R}-1}(0)$ *with respect to* $\mathbb{R}^d$. *However, we can apply the Kempf–Ness theorem at this point, which asserts an equivalence between the quotient* $\Sigma^{ps}//\mathbb{C}^d$ *and* $\tilde{\mu}_M^{\mathbb{R}-1}(0)/\mathbb{R}^d$, *where* Σ^{ps} *represents the polystable points on* Σ. *Combining two steps together, we demonstrate the equivalence between* $\mu_{\mathbb{C}}^{-1}(0)//\mathbb{C}$ *and* $\mu^{-1}(0)/\mathbb{R}$.

Relation between two Coordinates *At this point, we have considered two descriptions for the Taub-Nut space: the* S^1 *fibration over* $\mathbb{R}^3$ *with coordinates* $(\mathbf{r}, \tau)$ *and the affine variety inside* $\mathbb{C}^3$ *with ambient coordinates* (u, v, f). *It is important to understand the relation between two coordinates.*

Let us consider a local trivialization of the coordinate on the patch $\mathscr{U}_\epsilon$, *such that*

$$\omega_M = \epsilon_M(1 + \epsilon_M \cos\theta_M)d\phi_M, \tag{A.170}$$

where $\epsilon_a = \pm 1$ *and* (θ_M, ϕ_M) *is the angular coordinate with respect to* $\mathbf{r}_M = \mathbf{t}_M$.

Define

$$z = \mathbf{r}_1 + i\mathbf{r}_2,\ R_{M,\pm} = |\mathbf{r} - \mathbf{t}_M| \pm (\mathbf{r}_3 - \mathbf{t}_{3,M}) \tag{A.171}$$

Then, on this patch, we have

$$u = \prod_M R_{M,+}^{1/2} \prod_{\epsilon_M=1} \frac{z - z_a}{|z - z_a|} e^{\frac{i}{2}\tau + \frac{1}{2}x_3},\ v = \prod_M R_{M,-}^{1/2} \prod_{\epsilon_M=-1} \frac{z - z_M}{|z - z_M|} e^{-\frac{i}{2}\tau - \frac{1}{2}x_3}. \tag{A.172}$$

One can check that (u, v) *is independent of the choice of the local patch, which satisfies the condition*

$$uv = \prod_M (z - z_M). \tag{A.173}$$

Therefore, we identify (u, v, z) *as the ambient coordinate* (u, v, f), *and* z_M *with* e_M. *It explains the relation between two coordinates.*

Asymptotic Behavior of the Coordinate *Let us consider the limit where* $\mathbf{t}_{3,M} \to \pm\infty$. *On an distinguished patch* $\mathscr{U}_\epsilon$, *we consider the limit where* $\mathbf{t}_{3,M} \to \epsilon_M \infty$. *We call the* $M \in \mathscr{L}$ *if* $\epsilon_M = -1$ *and* $M \in \mathscr{R}$ *if* $\epsilon_M = 1$. *In this case, we normalize the coordinate* u, v *as*

$$u = \prod_{M\in\mathscr{R}} (z - z_M) \prod_{M\in\mathscr{R}} \left(\frac{2x_{3,M}}{R_{M,-}}\right)^{1/2} \prod_{M\in\mathscr{L}} \left(\frac{R_{M,+}}{2|x_{3,M}|}\right)^{1/2} e^{\frac{1}{2}i\tau + \frac{1}{2}x_3}. \tag{A.174}$$

Then under $\mathbf{t}_{3,M} \to \epsilon_M\infty$ *limit, we have*

$$u \to t^{-1} \prod_{M\in\mathscr{R}} (z - z_M),\ v \to t \prod_{M\in\mathscr{L}} (z - z_M) \tag{A.175}$$

where we identify $t = e^{\frac{1}{2}i\tau_3 + \frac{1}{2}x_3}$ *as the cylinder coordinate. The geometric consideration is that when the center of the Taub-Nut space is moved to infinity, the background geometry is asymptotic to the cylinder case* $\mathbb{C}^\times \times \mathbb{C}$. *This is relevant to the discussion in incorporating* D_6 *brane in the system.*

A.7 Analysis from Action Principle

In physics, we often begin by formulating a theory through the action. The action principle states that the classical configuration of a system saturates the minimum of the action. Understanding the relationship between the action principle and symplectic geometry is crucial, as symplectic structures inherently emerge in the formulation of classical mechanics. Therefore, we will have a detailed investigation of the action principle in this section.

Hamiltonian Equation from Action Principle Let us briefly review how the Hamiltonian equations could be recovered from the action principle. We will limit ourselves to the local descriptions.

As the setup, let (M, ω) a symplectic manifold, U be a contractable open subset in (M, ω), $\mathbf{I} = [0, 1]$ be the unit interval representing the time direction and $C^\infty(\mathbf{I}, U) = \{f \mid f : \mathbf{I} \xrightarrow{smooth} U\}$ be the set of smooth maps from $\mathbf{I}$ to U. Since U is contractible, the restriction $\omega\big|_U$ is exact by the De Rham theorem. Therefore, we have $\omega\big|_U = d\theta$ for some one-form θ. By the Darboux theorem①, since $\theta \wedge (d\theta)^n = 0$, there exists a local coordinate system $\{q^1 \dots q^n, p_1, \dots p_n\}$ on U called Darboux chart, such that $\theta = \sum_{j=1}^n p_j dq^j$. In the following analysis, we will restrict ourselves to the Darboux chart.

With an appropriate notion②, $C^\infty(\mathbf{I}, U)$ could be viewed as an infinite-dimensional manifold, where various operations such as differential δ admit a natural generalization. Therefore, let us first discuss the calculus on the space $C^\infty(\mathbf{I}, U)$. In particular, we could talk about the tangent vectors and the differential forms on $C^\infty(\mathbf{I}, U)$. Via the local chart on U, any elements in $C^\infty(\mathbf{I}, U)$ could be explicitly represented by $X^I(t) = (q^i(t), p_j(t))$. Schematically, we expect that the tangent space admits a basis as $\left\{\frac{\delta}{\delta X^I(t_0)}\right\}_{t_0\in\mathbf{I}} = \left\{\frac{\delta}{\delta q_i(t_0)}, \frac{\delta}{\delta p_i(t_0)}\right\}_{t_0\in\mathbf{I}}$, whose action on the differentiable functions is specified by the functional derivative

$$
\begin{aligned}
\frac{\delta}{\delta q_i(t_0)}(F[q^j(t), p_k(t)]) &:= \frac{\delta F[q^j(t), p_k(t)]}{\delta q_i(t_0)} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}(F[q^j(t) + \epsilon\delta_{ik}\delta(t - t_0), p_k(t)] - F[q^j(t), p_k(t)]) \\
\frac{\delta}{\delta p_i(t_0)}(F[q^j(t), p_k(t)]) &:= \frac{\delta F[q^j(t), p_k(t)]}{\delta p_i(t)} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}(F[q^j(t), p_k(t) + \epsilon\delta_{ik}\delta(t - t_0)] - F[q^j(t), p_k(t)]).
\end{aligned}
\tag{A.176}
$$

An arbitrary tangent vector Y could thus be spanned under this basis as

$$
Y = \int dt\, Y^I[q^i(t), p_i(t)]\frac{\delta}{\delta X^I(t)}. \tag{A.177}
$$

Similarly, we expect that there is a basis of the one forms on $C^\infty(\mathbf{I}, U)$ as $\{\delta X^I(t_0)\}_{t_0\in\mathbf{I}} =$

① Statement for Darboux theorem is as follows: For a n dimensional manifold M, if a one form θ satisfies $\theta\wedge d\theta^p = 0$, then there exists a local coordinate system $\{x^1, \dots x^{n-p}, y^1, \dots y^p\}$ such that $\theta = \sum_{i=1}^p x^i dy^i$.

② See Fréchet space and Fréchet manifold for related concepts.

$\{\delta q^i(t_0), \delta p_i(t_0)\}_{t_0 \in \mathbf{I}}$, whose pairing with tangent vectors yields

$$\begin{aligned} \delta q^i(t)\left(\frac{\delta}{\delta q^j(t_0)}\right) &= \delta^i_j \delta(t-t_0), \delta q^i(t)\left(\frac{\delta}{\delta p^j(t_0)}\right) = 0, \\ \delta p_i(t)\left(\frac{\delta}{\delta p^j(t_0)}\right) &= \delta^j_i \delta(t-t_0), \delta p_i(t)\left(\frac{\delta}{\delta q^j(t_0)}\right) = 0. \end{aligned} \tag{A.178}$$

An arbitrary one-form $\alpha \in \Omega^1(C^\infty(\mathbf{I}, U))$ can be spanned under the basis as

$$\alpha = \int dt\ \alpha^I[q^j(t), p_k(t)]\delta X^I(t). \tag{A.179}$$

Also, there exists a proper generalization for the exterior differential δ on the space $C^\infty(\mathbf{I}, U)$, which maps a differentiable function to a one-form via

$$\delta F[q^i(t), p_j(t)] = \int_I dt \frac{\delta F}{\delta q^i(t)}\delta q^i(t) + \frac{\delta F}{\delta p^i(t)}\delta p^i(t). \tag{A.180}$$

As U is a symplectic manifold, the symplectic structure on U can also be induced to $C^\infty(\mathbf{I}, U)$, via

$$\omega = \int dt\ \delta p_i(t) \wedge \delta q^i(t). \tag{A.181}$$

It is easy to show that ω is a closed two-form on $C^\infty(\mathbf{I}, U)$, and the non-degeneracy applies if we restrict the domain of ω appropriately. Therefore, $C^\infty(\mathbf{I}, U)$ admits a symplectic structure. As a result, the concept of the symplectic manifold we have discussed before, such as moment map and Hamiltonian G-action, could also be applied to the infinite-dimensional symplectic "manifold" $C^\infty(\mathbf{I}, U)$. For instance, we could evaluate the Poisson bivector Π corresponding to the symplectic form ω, which can be identified as

$$\Pi = \int dt\ \frac{\delta}{\delta q^i(t)} \otimes \frac{\delta}{\delta p_i(t)} - \frac{\delta}{\delta p^i(t)} \otimes \frac{\delta}{\delta q_i(t)}. \tag{A.182}$$

The action S is defined as a real-valued function on $C^\infty(\mathbf{I}, U)$, which we will assume to be differentiable. To recover the Hamiltonian dynamics, we select a specific action as follows to describe the evolution of the system

$$S[q^i(t), p_j(t)] = \int_I dt\ p_i \frac{dq^i}{dt} - \int_I dt\ H(q^i(t), p_j(t)), \tag{A.183}$$

where H is an arbitrary smooth function on U. As expected, the action is independent of the choice of the local coordinates on U. Indeed, there is a coordinate-free formalism for the action as

$$S = \int_I X(t)^*\theta - \int_I dt\ X(t)^* H. \tag{A.184}$$

The action principle states that the physical configuration corresponds to the minimization of the action. In our language, the physical configuration is simply a point in $C^\infty(\mathbf{I}, U)$. Hence, one of the necessary conditions for the action principle is to require

$$\delta S\Big|_{(q^i(t), p_j(t))} = 0. \tag{A.185}$$

A standard evaluation for this condition yields

$$
\begin{aligned}
0 = \delta S &= \int dt\ \delta p_i(t)\frac{dq^i}{dt} + p_i(t)\delta\frac{dq^i}{dt} - \int dt\ \frac{\partial H}{\partial q^i}\delta q^i(t) + \frac{\partial H}{\partial p^i}\delta p^i(t) \\
&= \int dt\ \delta p_i(t)\left(\frac{dq^i}{dt} - \frac{\partial H}{\partial p^i}\right) - \int dt\ \left(\frac{dp_i}{dt} + \frac{\partial H}{\partial q^i}\right)\delta q^i(t) + p_i(t)\delta q^i(t)\bigg|_0^1 .
\end{aligned}
\tag{A.186}
$$

Since we expect that the evolution of a physical system should satisfy locality, the bulk terms (the first two terms) and the boundary term (the last term) should vanish separately. Therefore, locality suggests that an extra boundary condition should be imposed in order to satisfy $p_i(t)\delta q^i(t)\big|_0^1 = 0$. Two common boundary conditions are often considered. The first is $\delta q^i(0) = \delta q^i(1) = 0$, meaning we impose additional constraints on $C^\infty(\mathbf{I}, U)$ such that $q^i(0)$ and $q^i(1)$ are fixed values. Another boundary condition is $p_i(1)\delta q^i(1) = p_i(0)\delta q^i(0)$, implying that we identify the $t = 0$ and $t = 1$ loci on $\mathbf{I}$ to form S^1 and consider the domain of the functional S as $C^\infty(S^1, U)$. For the bulk term, as $\delta p_i(t)$ and $\delta q_i(t)$ are linearly independent, it implies that

$$
\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i},\ \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i} \quad \forall t \in I, \tag{A.187}
$$

which recovers the Hamiltonian equation for the system.

Imposing Constraints in the Action Now, we turn our attention to the constraint Hamiltonian system. In the following discussion, we will use $\phi_a(x)$ to denote the set of first-class constraints of the system, and we assume that H is a compatible Hamiltonian.

There are two main formalisms for imposing constraints in the action analysis. The first formalism involves imposing constraints directly while the second one extends the domain of the action to introduce Lagrangian multipliers.

Specifically, in the first formalism, we impose by restricting the domain of the action to $C^\infty(I, U \cap \Sigma)$, where $\Sigma = \{\phi_a = 0\}$ defines the constraint surface in M. An equivalent but helpful way of doing this is to define a family of constraints $\{\phi_a(q^i(t), p_j(t))\}_{t\in I}$ on the infinite-dimensional symplectic manifold $C^\infty(\mathbf{I}, U)$. We shall put the time t and the parameter a on the same footing, serving as the indices for the constraint. The corresponding constraint surface in $C^\infty(\mathbf{I}, U)$ is $\Sigma = \{\phi_a(q^i(t), p_j(t)) = 0, \forall t \in I\}$. It is a particularly helpful perspective because it reduces the analysis of the action principle to the (infinite-dimensional) symplectic manifold with constraints, which we have already discussed in A.5. Moreover, under the assumption that $\{\phi_a(q^i, p_j)\}$ is a set of first-class constraints on M, $\{\phi_a(q^i(t), p_j(t))\}_{t\in I}$ is also a set of first-class constraints on $C^\infty(\mathbf{I}, U)$. It can be verified by evaluating the Poisson bracket of constraints:

$$
\begin{aligned}
&\{\phi_a(q^i(t_0), p_j(t_0)), \phi_b(q^i(t_1), p_j(t_1))\} \\
&= \int dt\ \frac{\delta\phi_a(q^i(t_0), p_j(t_0))}{\delta q^k(t)}\frac{\delta\phi_b(q^i(t_1), p_j(t_1))}{\delta p_k(t)} - \frac{\delta\phi_b(q^i(t_1), p_j(t_1))}{\delta p_k(t)}\frac{\delta\phi_a(q^i(t_0), p_j(t_0))}{\delta q^k(t)} \\
&= \delta(t_0 - t_1)\left(\frac{\partial\phi_a}{\partial q^i}\frac{\partial\phi_b}{\partial p^i} - \frac{\partial\phi_b}{\partial q^i}\frac{\partial\phi_a}{\partial p^i}\right)\bigg|_{q^i(t_0), p_j(t_0)} \\
&= \delta(t_0 - t_1)C_{abc}(q^i(t_0), p_j(t_0))\ \phi_c(q^i(t_0), p_j(t_0)),
\end{aligned}
\tag{A.188}
$$

which vanishes on the constraint surface Σ and is thus weakly zero.

After imposing the constraints, the action could be schematically written as

$$S[q^i(t),p_j(t)] = \int_{\mathbf{I}} dt \;\; p_i \frac{dq^i}{dt} - H(q^i(t),p_j(t))\Big|_{\phi_a(q^i(t),p_j(t))=0} . \tag{A.189}$$

Following from the discussion in Sec.A.5, whenever a set of first-class constraints $\{\phi_a\}_{a=1,\ldots,m}$ is imposed, there is a gauge redundancy on the constraint surface Σ provided by the flow of the Hamiltonian vector field associated with ϕ_a. A similar logic applies to the discussion here, where the gauge redundancy is provided by the flow of Hamiltonian vector field $X_{\phi_a(q^i(t_0),p_j(t_0))}$. Therefore, we have to check that the action is invariant under the gauge transformation when the Hamiltonian is compatible with the constraints. It can be verified by evaluating the Lie derivative

$$\begin{aligned}\mathscr{L}_{X_{\phi_a(q^i(t_0),p_j(t_0))}}(S) &= \{S,\phi_a(q^i(t_0),p_j(t_0))\}\\ &= \int dt \; \frac{\delta S}{\delta q^k(t)}\frac{\delta\phi_a(q^i(t_0),p_j(t_0))}{\delta p_k(t)} - \frac{\delta S}{\delta p_k(t)}\frac{\delta\phi_a(q^i(t_0),p_j(t_0))}{\delta q^k(t)}\\ &= \frac{d\phi_a}{dt} + \{H,\phi_a\}\Big|_{t_0} .\end{aligned} \tag{A.190}$$

While restricted to the constraint surface Σ, the first term vanishes since ϕ_a vanishes for any t. On the other hand, the second term vanishes precisely when the Hamiltonian is compatible with the constraint. Consequently, the action remains unchanged along the flow of the Hamiltonian vector fields associated with the constraints, namely unchanged under the gauge transformation. As the classical configuration is obtained by minimizing the action, the gauge transformation of a saddle point configuration is also a saddle.

The second formalism for introducing constraints is through the Lagrange multipliers. In this approach, we extend the domain of the action S from $C^\infty(\mathbf{I},U)$ to $C^\infty(\mathbf{I},U\times\mathbb{R}^m)$, where the coordinates in this extended space $C^\infty(\mathbf{I},U\times\mathbb{R}^m)$ can be represented as $(q^i(t),p_j(t),\lambda^a(t))$. Moreover, we extend the action in the following way,

$$S[q^i(t),p_j(t),\lambda^a(t)] = \int_I dt\; p_i\frac{dq^i}{dt} - H(q^i(t),p_j(t)) + \lambda^a(t)\phi_a(q^i(t),p_j(t)), \tag{A.191}$$

whereas $\lambda^a(t)$ is often called the Lagrangian multiplier.

Gauge Fixing and Dirac Bracket Just as in Sec.A.5, we could introduce m functions χ_a to fix the gauge. Again, the condition (A.109) has to be satisfied to ensure that the gauge choice $\chi_a = 0$ intersects with each orbit of the gauge transformation once. In this case, we introduces $2m$ constraints $\Phi_I = (\phi_a,\chi_a)$ simultaneously. The action could be written as

$$\begin{aligned}&\text{Formalism 1: } S[q^i(t),p_j(t)] = \int_{\mathbf{I}} dt \;\; p_i\frac{dq^i}{dt} - H(q^i(t),p_j(t))\Big|_{\Phi_I(q^i(t),p_j(t))=0} ,\\ &\text{Formalism 2: } S[q^i(t),p_j(t),\xi^I(t)] = \int_{\mathbf{I}} dt\; p_i\frac{dq^i}{dt} - H(q^i(t),p_j(t)) + \xi^I(t)\Phi_I(q^i(t),p_j(t)).\end{aligned} \tag{A.192}$$

Additionally, we replace the differential δ on $C^\infty(\mathbf{I}, U)$ with the differential δ' on the constrained space $C^\infty(\mathbf{I}, U \cap \Sigma)$. The resulting action could thus be schematically written as

A.8 Orbit space, Moment map and Springer Theory

In this subsection, we would like to investigate a special type of symplectic geometry related to the complex semisimple Lie group. The discussion here leads to a class of interesting symplectic manifolds and also provides a novel perspective towards the representation theory of complex semisimple lie groups.

Let G be a connected complex semisimple Lie group with Lie algebra $\mathfrak{g}$. $\mathfrak{g}$ admits the Cartan-Weyl decomposition as $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, where $\mathfrak{n}_- = \oplus_{\alpha\in\Delta_-}\mathfrak{g}_\alpha$ is the nilradical spanned by the negative roots $\alpha \in \Delta_-$ and $\mathfrak{n}_+ = \oplus_{\alpha\in\Delta_+}\mathfrak{g}_\alpha$ is spanned by the positive roots. Here, $\mathfrak{g}_\alpha$ is the root space spanned by E_α.

Once we pick a Cartan-Weyl decomposition, there is a subalgebra $\mathfrak{b}$ that we could consider $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_+$. $\mathfrak{b}$ is often called a borel subalgebra. Exponentiating $\mathfrak{b}$[①] leads to a Lie subgroup of G, called the Borel subgroup B.

We will assume that the action of G and B on $\mathfrak{g}$ or $\mathfrak{b}$ is by the adjoint action, and we denote the group action by $g \cdot x$, with g in the Lie group and x in the Lie algebra.

Indeed, the Borel subalgebra is not unique, as there are various ways of choosing the Cartan subalgebra and splitting the roots into positive roots and negative roots. It is, therefore, an interesting question to understand the set of all borel subalgebras in $\mathfrak{g}$, denoted as $\mathscr{B}$. One could prove the following statements:

1. Every two Borel subalgebras are conjugate to each other by the G action.
2. The normalizer of the borel subalgebra B under G action is B itself. Namely $N_G(B) = B$.

The first statement means that G acts transversely on the set $\mathscr{B}$. The second statement means that if we pick a borel subgroup B as a reference point in $\mathscr{B}$, the stabilizer of the G action is B itself. Therefore, we have $\mathscr{B} \cong G/B$ as a set, and the bijection is specified by

$$\begin{aligned} G/B &\longrightarrow \mathscr{B} \\ gB &\mapsto g\mathfrak{b}g^{-1} \end{aligned} \tag{A.193}$$

Such an identification suggests that there is a natural principal B bundle structure over $\mathscr{B}$, given as

$$\begin{tikzcd} B \arrow[r, hook] & G \arrow[d, "\pi"] \\ & G/B \cong \mathscr{B} \end{tikzcd} \tag{A.194}$$

Out of $\mathscr{B}$, there is a canonical symplectic manifold $T^*\mathscr{B}$. The following statement provides a description for $T^*\mathscr{B}$ as an associated vector bundle of the principal B bundle.

Theorem A.11 *There is a G-equivariant vector bundle isomorphism*

$$T^*(G/B) \cong G \times_B \mathfrak{b}^\perp \cong G \times_B \mathfrak{n}_+, \tag{A.195}$$

where $\mathfrak{b}^\perp \subset \mathfrak{g}^*$ is the annihilator of $\mathfrak{b}$, specified by $\mathfrak{b}^\perp = \{u \in \mathfrak{g}^* \mid \langle u, x\rangle = 0, \forall x \in \mathfrak{b}\}$. B acts on $\mathfrak{b}^\perp$ by the coadjoint action.

① Namely, consider the image of $\mathfrak{b}$ under the exponential map.

Proof *To prove the first isomorphism, we need to first check that each fiber of $T^*\mathscr{B}$ and $G\times_B \mathfrak{b}^\perp$ agrees. Take eB, the image of the identity element in G/B, as the base point. Then we have $T_{eB}(G/B) = \mathfrak{g}/\mathfrak{b}$. Therefore $T^*_{eB}(G/B) = (\mathfrak{g}/\mathfrak{b})^* \cong \mathfrak{b}^\perp$.*

Since the G action is transitive, a generic point on G/B could be represented by gB, corresponding to the borel subalgebra $g\mathfrak{b}g^{-1}$. Therefore, we reach at the identification $T^*_{gB}(G/B) \cong g\mathfrak{b}^\perp g^{-1}$. Namely, every element in the cotangent bundle $T^*\mathscr{B}$ could be represented by the pairing $(g, b^\perp) \in G\times\mathfrak{b}^\perp$. Moreover, there is an equivalence relation $(gh, b^\perp) \cong (g, hb^\perp h^{-1})$ with $h \in B$. It holds since h action on $\mathfrak{b}$ would leave it invariant as an algebra, while it maps an element $b^\perp$ in $\mathfrak{b}^\perp$ to $hb^\perp h^{-1}$.

The discussion above suggests a map

$$\begin{aligned} \eta : T^*\mathscr{B} &\longrightarrow G\times_B \mathfrak{b}^\perp \\ (g\mathfrak{b}g^{-1}, gb^\perp g^{-1}) &\longrightarrow (g, b^\perp). \end{aligned} \tag{A.196}$$

It remains to check that the map is an isomorphism of the vector bundle, and we omit the detail.

The second isomorphism follows from the identification between $\mathfrak{g} \cong \mathfrak{g}^*$ using the Killing form. Under this identification, we have $\mathfrak{n}_+ \cong \mathfrak{b}^\perp$. ∎

$T^*\mathscr{B}$ admits a natural symplectic structure coming from the tautological one form. Since $\mathscr{B}$ is a G space, the discussion before shows that $T^*\mathscr{B}$ admits a natural Hamiltonian G action. Let ξ_x be the vector field of G action on $\mathscr{B}$ associated with the Lie algebra element $x \in \mathfrak{g}$. The Hamiltonian Q of the G action on $T^*\mathscr{B}$ is given by

$$Q_x(g\mathfrak{b}g^{-1}, gb^\perp g^{-1}) = gb^\perp g^{-1}(x) \ \ ,(g\mathfrak{b}g^{-1}, gb^\perp g^{-1}) \in T^*\mathscr{B}. \tag{A.197}$$

The first equation holds under the identification between one form and the element in $\mathfrak{b}^\perp$. Therefore, the moment map associated with the Hamiltonian G action is

$$\mu(g\mathfrak{b}g^{-1}, gb^\perp g^{-1}) = gb^\perp g^{-1}. \tag{A.198}$$

In other words, the moment map is aligned with the projection onto the second slot.

Another viewpoint from orbits in the compact semisimple Lie group: Denote G_c as the maximal compact subgroup of G. There is a diffeomorphism $G/B \cong G_c/T$.

RM: Notice that $G_{\mathbb{C}}/B \cong G/T$, where G is the maximal compact subgroup. On the other hand, G/T could be understood as the coadjoint orbit of regular semisimple elements.

For a regular semisimple element $x \in \mathfrak{g}^*$, the G orbit of it only contains one element in $[\lambda] \in \mathfrak{h}^*/W$, or equivalently, one element in the dominant Weyl chamber. Thus, regular semisimple orbits are parametrized by the points in $\mathfrak{h}^*/W$. By construction, the stabilizer of a regular semisimple element is one of the maximum torus T. Therefore, the coadjoint orbit of regular semisimple elements could be identified as $\mathscr{O}_\lambda \cong G_c/T$.

For a coadjoint orbit $\mathscr{O}_\lambda$, there exists a natural Symplectic structure on it, called Kirillov-Kostant-Souriau symplectic form. It is constructed as follows.

Given a coadjoint orbit $x \in \mathscr{O}_\lambda$, it's tangent space is identified as $T_x\mathscr{O}_\lambda \cong T_x(G/G_x) \cong \mathfrak{g}/\mathfrak{g}_x$, where

$$\mathfrak{g}_x = \{u \in \mathfrak{g} \mid ad^*_u(x) = 0\}. \tag{A.199}$$

There is a skew bilinear map upon the choice of x, specified as

$$\begin{aligned} \omega_x : \mathfrak{g}\times\mathfrak{g} &\to \mathbb{C} \\ (u, v) &\mapsto \omega_x(u, v) = x([u, v]). \end{aligned} \tag{A.200}$$

The Kirillov-Kostant-Souriau symplectic form is defined as the restriction of ω_x to $\mathfrak{g}/\mathfrak{g}_x \times \mathfrak{g}/\mathfrak{g}_x$. Such a restriction is well-defined, as we have

$$x([u, v]) = -ad_u^*(x)(v) = 0, \ \forall u \in \mathfrak{g}_x. \tag{A.201}$$

As a result, ω specifies a two form on the coadjoint orbit $\mathscr{O}_\lambda$.

For the coadjoint orbit of two regular semi-simple elements, while they are diffeomorphic as a manifold, they could admit a different symplectic structure. While the parameter of the orbit hits the dominant integral weight, it fits into the quantization condition for the geometric quantization.

The Chevalley restriction theorem states that there is an isomorphism $\mathbb{C}[\mathfrak{g}]^G \cong \mathbb{C}[\mathfrak{h}]^W$.

One could realize it via $G/T \cong \mathscr{O}_\lambda$, with $\lambda \in \mathfrak{t}^*$ a regular element.

A.9 Geometric Quantization

In the discussion above, we argued that there is a natural B-bundle structure $B \hookrightarrow G \to \mathscr{B}$. This section is devoted to understanding the line bundles over $\mathscr{B}$, or say $Pic(\mathscr{B})$. The associated line bundle over $\mathscr{B}$ could be constructed by $G \times_B \mathscr{L}_\lambda$, with $\mathscr{L}_\lambda$ selected as the one-dimensional representation of Borel subgroup B. It is in correspondence with the highest weight space $\mathscr{L}_\lambda$ in each highest weight module of G. Explicitly, state in $\mathscr{L}_\lambda$ is annihilated by all the raising operators, and the Cartan elements act as the scalar multiplication.

Borel-Weil-Bott theorem states that if we take the space of holomorphic sections on the line bundle, it is isomorphic to the representation with highest weight λ. $L_\lambda = H^0(\mathscr{B}, G \times_B \mathscr{L}_\lambda)$.

Another viewpoint from compact semisimple Lie group: Denote G_c as the maximal compact subgroup of G. There is a diffeomorphism $G/B \cong G_c/T$.

RM: Notice that $G_\mathbb{C}/B \cong G/T$, where G is the maximal compact subgroup. On the other hand, G/T could be understood as the orbit of regular semisimple elements. One could realize it via $G/T \cong \mathscr{O}_\lambda$, with $\lambda \in \mathfrak{t}^*$ a regular element.

Appendix B
Wess-Zumino-Witten Model and Affine Lie Algebra

B.1 Introduction to nonlinear sigma model

The simplest example we have encountered in QFT is the free boson. In two-dimension, the action for the free boson is given by

$$S[x] = \int d^2x \; h^{\mu\nu}\partial_\mu X_i \partial_\nu X_j \delta^{ij} \tag{B.1}$$

where i runs from 1 to n. The dynamical variable X in the theory can be understood as the map from the Riemannian manifold $(\mathbb{R}^2, h)$ to the target $(\mathbb{R}^n, \delta)$.

$$X : (\mathbb{R}^2, h) \longrightarrow (\mathbb{R}^n, \delta) \tag{B.2}$$

where h and δ are the flat metric over $\mathbb{R}^2$ and $\mathbb{R}^n$ respectively. A natural generalization to the system is to consider the dynamic variable that describes the map

$$X : (\Sigma, h) \longrightarrow (M, g) \tag{B.3}$$

where (M, g) is an arbitrary Riemannian manifold. The resulting model, the nonlinear sigma model, describes the boson coupled to the curved metric on the targe space. The action is provided by

$$S = \int d^2x \; h^{\mu\nu}\partial_\nu X_i \partial_\mu X_j g^{ij}(X) \tag{B.4}$$

For a generic Riemannian manifold, the non-linear sigma model is extremely interesting but hard to describe. However, it is always beneficial to think of the target space with a dedicated structure to obtain a further understanding of the system. Compact semisimple Lie Group G, as a special case of homogeneous Riemannian manifold, comes as the remedy. For concreteness, we take Lie group G to be $SU(N)$. $SU(N)$ is the group of the $N \times N$ unitary matrices with unit determinant.

$$SU(N) = \{A \in GL(n, \mathbb{C}) | A^\dagger A = I, \det(A) = 1\} \tag{B.5}$$

As one can see, the defining equations of the $SU(N)$ characterize n^2+1 real polynomial equations within $\mathbb{C}^{n^2}$. The metric on $SU(N)$ can thus be constructed as the induced metric of $\mathbb{C}^{n^2}$, provided by

$$\begin{aligned} ds^2 = \Sigma_{ij} dg_{ij} \otimes \overline{dg_{ij}} &= \mathrm{Tr}\big(dg \otimes dg^\dagger\big) \\ &= -\mathrm{Tr}\big(g^{-1}dg \otimes g^{-1}dg\big). \end{aligned} \tag{B.6}$$

Therefore, the action of the nonlinear sigma model with target space $SU(N)$ Lie group can be given by

$$S_0 = \frac{-1}{4a^2} \int d^2x\, h^{\mu\nu}\, \mathrm{Tr}'(g^{-1}\partial_\mu g g^{-1}\partial_\nu g), \tag{B.7}$$

where a^2 is the dimensionless coupling constant, Tr' is the normalized the trace such that $\mathrm{Tr}'(t^a t^b) = 2\delta^{ab}$. The equation of motion for the action can be derived using standard methods of Euler-Lagrangian equation:

$$\partial^\mu(g^{-1}\partial_\mu g) = 0. \tag{B.8}$$

Therefore, there is a one-form conservation current $J = g^{-1}(x)dg(x)$, which is provided by the pull-back of the Maurer-Cartan one-form on the Lie group $J = g^*\omega_{MC}$. By the property of pull-back, one has

$$dJ + J \wedge J = g^*(d\omega_{MC} + \omega_{MC} \wedge \omega_{MC}) = 0. \tag{B.9}$$

B.2 WZW models

For a two-dimensional world-sheet, one could introduce the complex variables

$$z = x^0 + ix^1 \quad \bar{z} = x^0 - ix^1. \tag{B.10}$$

At best, we may expect that the current admits a holomorphic factorization. Namely, we want to have the current in the left(right) moving section which only depends on $z(\bar{z})$. It is equivalent to asking $dJ = d * J = 0$, as the holomorphic function is harmonic. However, the conservation current J does *not* satisfy this equation, due to (B.9). Namely, the conservation current in the non-linear sigma model does not admit holomorphic factorization.

To restore the holomorphic factorization, it turns out that one has to formulate the problem as a 2d-3d coupled system. Explicitly, one could introduce the Wess-Zumino term into the action as $S_{WZW} = S_0 + k\Gamma$. Γ is a topological term called Wess-Zumino term provided by

$$\Gamma = \frac{-i}{24\pi} \int_B \mathrm{Tr}'(g^{-1}dg)^3 \tag{B.11}$$

where B is a three-dimensional manifold with boundary $\partial B = \Sigma$. The ambiguity underlying this step is in the choice of the three-manifold B. To see if the Wess-Zumino term is well-defined, let's consider another three manifold B' with the same boundary Σ.

$$\Gamma' = \frac{-i}{24\pi} \int_{B'} \mathrm{Tr}'(g^{-1}dg)^3 \tag{B.12}$$

The difference between two choices of three-manifolds writes

$$\Delta\Gamma = \frac{-i}{24\pi} \int_M \mathrm{Tr}'(g^{-1}dg)^3 \tag{B.13}$$

where $M = B \sqcup_\Sigma \overline{B}'$ is a closed three-manifold constructed out of the gluing of three-manifolds. The resulting integral is a topological invariance, which under our normalization is in $2\pi\mathbb{Z}$. Therefore, in order to request the partition function of the system independent of the three-manifolds, the coefficient k in the WZW model must be an integer.

Indeed, the Wess-Zumino term

After introducing the Wess-Zumino term, the equation of motion for the system becomes

$$\left(1+\frac{a^2k}{4\pi}\right)\partial_z\left(g^{-1}\partial_{\bar{z}}g\right)+\left(1-\frac{a^2k}{4\pi}\right)\partial_{\bar{z}}\left(g^{-1}\partial_z g\right)=0 \tag{B.14}$$

Therefore, when the coupling constant for the WZW model is selected to be $a^2=4\pi k$, there is an anti-holomorphic conservation current $\bar{J}(\bar{z})=g^{-1}\partial_{\bar{z}}g$ in the theory. Remarkably, the anti-holomorphic current implies another holomorphic current, which can be achieved by

$$\begin{aligned}0=g\partial_z(g^{-1}\partial_{\bar{z}}g)g^{-1}&=-\partial_z gg^{-1}\partial_{\bar{z}}gg^{-1}+(\partial_z\partial_{\bar{z}}g)g^{-1}\\&=\partial_{\bar{z}}(\partial_{\bar{z}}gg^{-1})\end{aligned} \tag{B.15}$$

which defines another conservation current $J(z)=\partial_{\bar{z}}gg^{-1}$. The underlying symmetry transformation corresponding to two conservation currents $J(z)$ and $\overline{J}(\bar{z})$ is specified by

$$g(z,\bar{z})\to\Omega(z)g(z,\bar{z})\overline{\Omega}^{-1}(\bar{z}) \tag{B.16}$$

where Ω and $\overline{\Omega}$ are two arbitrary matrices valued in G. Therefore, the WZW model admits $G(z)\times G(\bar{z})$ global symmetry.

$G(z)$, mathematically, is called the loop group, whose Lie algebra defines the loop algebra. Once we quantize the system, as a usual story, loop algebra admits a projective action on the Hilbert space, which can be formulated as the central extension at the Lie algebra level. Such a Lie algebra is the famous affine lie algebra $\hat{\mathfrak{g}}_k$.

Appendix C
Effective field theory from Integrating out

In the following calculation, we take $\eta = (-1, 1, \ldots, 1)$.

C.1 Schwinger Proper Time

Given $A \in \mathbb{R}$,, it is easy to show the integration formula

$$\frac{i}{A + i\epsilon} = \int_0^\infty d\tau e^{i\tau(A+i\epsilon)}. \tag{C.1}$$

It would provides us another prospective towards the propagator, which turns out to be conceptually significant and practically convenient for constructing the effective action. With this integration formula, a scalar propagator in D-dimensional Minkowski spacetime could be reinterpreted as follows:

$$D_F(x, y) = \int \frac{d^D p}{(2\pi)^D} e^{ip\cdot(x-y)} \frac{i}{-p^2 - M^2 + i\epsilon} = \int_0^\infty d\tau \int \frac{d^D p}{(2\pi)^D} e^{ip\cdot(x-y)} e^{i\tau(-p^2-M^2+i\epsilon)}. \tag{C.2}$$

Performing Gaussian integral over p yields

$$\begin{aligned} D_F(x, y) &= \frac{1}{(2\pi)^D} \int_0^\infty d\tau\, e^{i\tau(-M^2+i\epsilon) - \frac{i}{4\tau}(x-y)^2} \times (-1)^{-1/2} \left(\frac{-i\tau}{\pi}\right)^{-D/2} \\ &= \frac{e^{-\frac{i}{8}\pi(D+4)}}{2^D \pi^{D/2}} \int_0^\infty \frac{d\tau}{\tau^{D/2}} e^{i\tau(-M^2+i\epsilon) - \frac{i}{4\tau}(x-y)^2} \end{aligned} \tag{C.3}$$

This formula provides an equivalent way of expressing the propagator of scalar particle. As a branchmark test, we consider the limit where $M^2 \to 0$. Under this limit, the propagator simplifies to

$$D_F(x, y)\Big|_{M^2=0} = \#|x - y|^{2-d} \tag{C.4}$$

where $\# = \frac{e^{\frac{i}{8}\pi D}}{4\pi^{D/2}} \Gamma(D/2 - 1)$ is some constant factor. It recovers the correlation function of scalar fields in a conformal field theory, whose scaling dimension is known to be $d/2 - 1$.

As an illuminating remark(thanks to witten), there is another way of viewing the propagator from the particle perspective. Consider a one-dimensional gravity theory with cosmological constant $\Lambda = m^2$ couple to bosonic matter fields. In one-dimension, the Ricci curvature is zero, so there is no dynamical term for the metric. The action for the theory writes

$$S = \int dt\sqrt{g}\left(g^{tt}\frac{1}{2}\eta_{IJ}\frac{dX^I}{dt}\frac{dX^J}{dt} - \frac{1}{2}M^2\right). \tag{C.5}$$

By the diffeomorphism invariance, we could fix a coordinate chart such that $g^{tt} = 1$. The only parameter that characterize the one-dimensional manifolds is its length τ.

Let us consider the transition amplitude of a particle from x to y, using this one-dimensional model. It corresponds to evaluating the following matrix element

$$Z(x, y) = \int_0^\infty d\tau \left\langle X^I = y^I\right| e^{i\tau H} \left|X^I = x^I\right\rangle \tag{C.6}$$

where

$$S(\tau) = \int_0^\tau dt\left(-\frac{1}{2}\eta_{IJ}\frac{dX^I}{dt}\frac{dX^J}{dt} - \frac{1}{2}M^2\right). \tag{C.7}$$

In the following, we present two ways of evaluating the matrix element. One is through quantum mechanical method and another is through path integral. For those who are familiar with the calculation could omit this part.

For the problem at hand, after fixing the gauge, the Hamiltonian of the system is specified by

$$H = \frac{1}{2}\left(P_I P^I + M^2\right). \tag{C.8}$$

Therefore, upon the resolution of identity, we could write

$$\begin{aligned} Z(x, y) &= \int \frac{d^D p}{(2\pi)^D}\int_0^\infty d\tau \left\langle X^I = y^I\right| e^{i\tau H}\left|P^I = p^I\right\rangle \left\langle P^I = p^I\middle|X^I = x^I\right\rangle_{S(\tau)} \\ &= \int \frac{d^D p}{(2\pi)^D}\int_0^\infty d\tau e^{-ip(x-y)}e^{\frac{i}{2}(p^2+M^2)} \end{aligned} \tag{C.9}$$

which matches with the propagator in the Schwinger parametrization. Therefore, propagator could be understood precisely as the propagating of a massive particle in the background spacetime.

There is another way of evaluating the matrix element via path integral. It is easier to generalize to the theory in other dimensions. In this procedue, we rephrase the matrix element into the path integral as

$$\int_0^\infty d\tau \left\langle X^I = y^I\right| e^{i\tau H}\left|X^I = x^I\right\rangle_S = \int_0^\infty d\tau \int_{X^I(0)=x^I}^{X^I(\tau)=y^I} [DX]e^{iS(\tau)} \tag{C.10}$$

Let us now perform the path integral honestly. With the boundary condition, we could conduct mode expansion for X^I

$$X^I(t) = x^I + p_0^I t + \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{\tau}t\right)a_n^I. \tag{C.11}$$

where $p_0^I = \frac{y^I - x^I}{\tau}$.

Then we have

$$I(\tau)=\int_{X^I(0)=x^I}^{X^I(\tau)=y^I}[DX]e^{iS(\tau)}=\int\prod_{n=1}^{\infty}da_n\,\mathrm{Exp}\left(\frac{i}{2}(-p_0^2-m^2)\tau-i\sum_{n=1}^{\infty}\frac{1}{2}a_n^I a_{nI}\frac{n^2\pi^2}{\tau^2}\right)\tag{C.12}$$

Integrating over all the modes yields

$$I(\tau)=e^{\frac{i}{2}\tau(-p_0^2-m^2)}\prod_{n=1}^{\infty}\left(\frac{in^2\pi}{2\tau^2}\right)^{-D/2}(-1)^{-1/2}\tag{C.13}$$

The infinite product here could be regularized by the zeta function regularization, or equivalently we could apply

$$\prod_{n=1}^{\infty}\left(\frac{2\pi n}{\beta}\right)^{-2}=\frac{1}{\beta}.\tag{C.14}$$

Then the resulting integration yields

$$I(\tau)\propto\frac{1}{\tau^{D/2}}e^{\frac{i}{2}\tau(-p_0^2-m^2)},\tag{C.15}$$

and the matrix element becomes up to a redefinition of integration varaible $\tau\to2\tau$

$$\left\langle X^I=y^I\middle|X^I=x^I\right\rangle_S\propto\int_0^{\infty}\frac{d\tau}{\tau^{D/2}}e^{-i\tau m^2+\frac{i}{4\tau}(x-y)^2}.\tag{C.16}$$

It matches with the propagator evaluated before, upto an overall normalization.

C.2 Effective Field Theory from path integral

C.2.1 Scalar coupled to U(1) Gauge Field

As an example, let us consider scalar QED in four dimension, whose action is given by

$$S=\int d^4x-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\phi^*(D^2-M^2)\phi.\tag{C.17}$$

where

$$D_\mu=\partial_\mu+ieA_\mu\tag{C.18}$$

We would like to evaluate the effective action $\Gamma[A]$ by integrating out the charged scalar field.

$$Z=\int[DA][D\phi][D\phi^*]e^{iS[A,\phi,\phi^*]}=\int[DA]e^{i\Gamma[A]}\tag{C.19}$$

In the case of QED or QCD, the calculation is easy to conduct, since the integration is an gaussian integral.

$$Z=\int[DA][D\phi][D\phi^*]e^{i\int d^4x-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\phi^*(D^2-M^2)\phi}=\int[DA]e^{-i\int d^4x\frac{1}{4}F^{\mu\nu}F_{\mu\nu}}\,\mathrm{Det}_{\mathscr{H}}\left(\frac{D^2-M^2}{i\pi}\right)^{-1}\tag{C.20}$$

Here $\mathscr{H}$ is the space of scalar field that the operator $D^2 - M^2$ acts on. Notice that in the path-integral, what contributes to the the path integral are the field configuration with finite action. Therefore, $\mathscr{H}$ is specified by

$$\mathscr{H} = \{\phi(x) \in C^\infty(\mathbb{R}^4) \mid \int d^4x\phi^*(-D^2 + M^2)\phi < \infty\} \tag{C.21}$$

up to some completion procedure. Especially, $\mathscr{H}$ is a dense subspace of $L^2(\mathbb{R}^4) \otimes \mathbb{C}$. Therefore, the correction of a charged complex scalar field to the action could be read off as

$$e^{i\Delta_B\Gamma} = \mathrm{Det}_{\mathscr{H}}\left(\frac{D^2 - M^2}{i\pi}\right)^{-1} \tag{C.22}$$

The following task is to evaluate this determinant. Notice that we have an identity

$$i\Delta_B\Gamma = \log \mathrm{Det}_{\mathscr{H}}\left(\frac{D^2 - M^2}{i\pi}\right)^{-1} = -\mathrm{Tr}_{\mathscr{H}}\log\left(\frac{D^2 - M^2}{i\pi}\right) \tag{C.23}$$

The right hand side is nothing but the trace in a quantum mechanical system. We could evaluate it in the position eigenbasis.

$$-\mathrm{Tr}_{\mathscr{H}}\log\left(\frac{D^2 - M^2}{i\pi}\right) = -\int d^4x \langle x|\log\left(\frac{D^2 - M^2}{i\pi}\right)|x\rangle . \tag{C.24}$$

Following from the Schwinger formula, an integration over A yields

$$\int_0^\infty \frac{ds}{s}e^{isA} + C = -\log A. \tag{C.25}$$

where C is a normalization constant, which plays no role in the field theory. Then the integration formula yields

$$i\Delta_B\Gamma = -\int d^4x \langle x|\log\left(\frac{D^2 - M^2}{i\pi}\right)|x\rangle = \int d^4x \int_0^\infty \frac{ds}{s}\langle x| e^{isH}|x\rangle \tag{C.26}$$

where $H_B = -D^2 + M^2$. We shall recognize that the matrix element $\langle x| e^{isH}|x\rangle$ is nothing but the propagator of a scalar particle coupled to the background gauge field $A_\mu(x)$.

Summerizing the calculation above, we conclude that the charges scalar contributes to the effective action by adding a term.

$$\Delta_B\mathscr{L} = -i\int_0^\infty \frac{ds}{s}\langle x| e^{isH}|x\rangle = -i\int_0^\infty \frac{ds}{s}G_A(x, x; s) \tag{C.27}$$

with $G_A(x, x; s)$ the scalar propagator coupled to Gauge field.

C.2.2 Fermion coupled to U(1) Gauge Field

As an example, let us consider the QED in four dimension. The relevant path integral is given by

$$Z = \int [DA][D\psi][D\bar{\psi}]e^{i\int d^4x -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \bar{\psi}(i\not{D}+M)\psi} \tag{C.28}$$

Again, we would like to integrate out fermions to obtain the effective action of gauge field. By the gaussian integration formula for fermion, we arrive at

$$Z = \int [DA] e^{i\int d^4x - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}} \operatorname{Det}_{\mathscr{H}\otimes\mathscr{S}}(-\not{D} + iM) \tag{C.29}$$

where the determinant is performed over the space $\mathscr{H} \otimes \mathscr{S}$, where $\mathscr{H}$ is defined in Eq.C.21 and $\mathscr{S}$ is the space of the Dirac spinor representation. We again transforms the determinant into the trace, and the one-loop determinant transforms into

$$\log \operatorname{Det}_{\mathscr{H}\otimes\mathscr{S}}(-\not{D} + iM) = \operatorname{Tr}_{\mathscr{H}\otimes\mathscr{S}} \log(-\not{D} + iM). \tag{C.30}$$

The trace over $\mathscr{H}$ could be performed in a similar fashion by introducing position eigen basis. A trick that we would apply here is to pick out a factor of $\frac{1}{2}$ and square the operator $(-\not{D} + iM)$, just for calculation convenience.

$$i\Delta_F \Gamma = \operatorname{Tr}_{\mathscr{H}\otimes\mathscr{S}} \log(-\not{D} + iM) = \int d^4x \operatorname{Tr}_{\mathscr{S}} \langle x| \log(-\not{D} + iM) |x\rangle \tag{C.31}$$

To connect to the Schwinger parametrization, we applying the derivative to M

$$i\frac{\partial}{\partial M}\Delta_F \Gamma = i\int d^4x \operatorname{Tr}_{\mathscr{S}} \langle x| \frac{1}{-\not{D} + iM} |x\rangle = i\int d^4x \operatorname{Tr}_{\mathscr{S}} \langle x| \frac{\not{D} - iM}{\not{D}^2 + M^2} |x\rangle \tag{C.32}$$

Since $\not{D}$ term on the numerator always contains odd number of gamma matrices, the result under $\operatorname{Tr}_{\mathscr{S}}$ will always vanishes. Therefore, it leads to a large simplification as

$$i\frac{\partial}{\partial M}\Delta_F \Gamma = M\int d^4x \operatorname{Tr}_{\mathscr{S}} \langle x| \frac{1}{\not{D}^2 + M^2} |x\rangle \tag{C.33}$$

We move M factor to the left and apply schwinger parametrization.

$$i\frac{\partial}{\partial M^2}\Delta_F \Gamma = \frac{1}{2i}\int d^4x \int_0^\infty ds \operatorname{Tr}_{\mathscr{S}} \langle x| e^{is(\not{D}^2+M^2)} |x\rangle \tag{C.34}$$

Integrating both part with M^2, we reaches at

$$i\Delta_F \Gamma = -\frac{1}{2}\int d^4x \int_0^\infty \frac{ds}{s} \operatorname{Tr}_{\mathscr{S}} \langle x| e^{is(\not{D}^2+M^2)} |x\rangle + C. \tag{C.35}$$

with C an arbitary constant which again plays no role in the field theory. As a similar interpretation, $\langle x| e^{is(\not{D}^2+M^2)} |x\rangle$ describes the propagator of a dirac fermion coupled to $U(1)$ gauge field, with Hamiltonian

$$\begin{aligned} H_F &= \not{D}^2 + M^2 = D_\mu D_\nu \gamma^\mu\gamma^\nu + M^2 = \frac{1}{2}[D_\mu, D_\nu]\gamma^\mu\gamma^\nu + \frac{1}{2}D_\mu D_\nu\{\gamma^\mu, \gamma^\nu\} + M^2 \\ &= \frac{ie}{2}F_{\mu\nu}\gamma^\mu\gamma^\nu - D_\mu D^\mu + M^2. \end{aligned} \tag{C.36}$$

And the correction of fermion to the effective action reads

$$\Delta_F \mathscr{L} = \frac{i}{2}\int d^4x \int_0^\infty \frac{ds}{s} \operatorname{Tr}_{\mathscr{S}} \langle x| e^{isH_F} |x\rangle + C \tag{C.37}$$

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Acknowledgements

衷心感谢我的导师 Satoshi Nawata 老师一直以来的悉心指导，让我有机会步入理论物理的世界，领略到其中优美的数学结构。Satoshi 老师撰写的讲义和在学习上的宝贵建议，使我从大二时的懵懂无知，逐步成长为能够独立开展研究课题的学生。在本科学习阶段，与您一次次的深入讨论，是我最宝贵的收获。

感谢裴度老师、Yuji Tachikawa 老师、Junya Yagi 老师等的赏识和帮助，让我有机会能够在学术舞台上展现我自己。感谢 Kentaro Hori 老师、Masahito Yamazaki 老师、包佳康老师、Martin Roček 老师、张昊老师、于兴洋老师，和老师们的交流让我受益匪浅。感谢国家自然科学基金的资助，让我能够有机会看到更加广阔的世界，接触到前沿的研究和极富天赋的研究者们。

感谢臧亦驰和曹子恒一直以来的讨论。感谢 Satoshi Nawata 课题组的学长学姐们，包括黄俊康、郑嘉豪、江加群、庄舒同、邵一路、林易理、杨嘉鑫、张启抟和郭兆婷学长/学姐。感谢李可为和袁榕含、以及贺斌、袁天、袁马轲和黄紫潇学长/学姐。和学长学姐们的读书会、以及平时的交流都让我收获颇丰。

感谢复旦大学物理系和 SIMIS 的老师们，尤其是周洋老师、邵鼎煜老师、周业浩老师、张鹏飞老师、戚扬老师、李晓鹏老师、林志芳老师、万义顿老师。老师们所精心准备的课程很大的拓展了我的学术视野，让我能够了解到这么多有趣的知识和精美的结构。

感谢我的室友臧亦驰、于昊楠和罗熠晨，感谢"卷神课题组"，给我的大学生活增添了很多乐趣和色彩。感谢我的朋友们,包括臧亦驰、李松宇、曹子恒、于昊楠、张成言、范宁玥、俞高航、罗熠晨、夏轩哲、杨远青、施昊哲、张源、童煜智、童士诚、汪明达、黄海洋、莫雨潇、童嘉萱、蒋昕、麻一州、戎天骏、林慧鑫等。感谢张吴承和王雨晨学长在 Princeton 和 Harvard 校园开放日期间提供的帮助。

感谢夏文卓辅导员在大学四年期间所给予的帮助。

感谢我的父母张展和孙澐一直以来对我生活上的帮助，以及一直以来对我选择的支持。

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