Homework 9: Due at class on Nov 24

Prob. 1 Contribution from symplectic representation

Consider

$$P\Lambda^T P = \pm\Lambda \tag{1}$$

for $P = i\begin{pmatrix} 0 & -\mathbf{1}_{k\times k} \\ \mathbf{1}_{k\times k} & 0 \end{pmatrix}$ where $n = 2k$. Derive the dimension(degrees of freedom) of Λ for $+$ and $-$, respectively, and show that $\mathrm{tr}[\Omega_\Lambda] = -n$.

Prob. 2 Orientation flip in superstring

Consider the orientation flip operator Ω for world-sheet fermions of closed string.

- Define the action of the orientation operator on the fermions $\psi^\mu(t,\sigma)$, $\widetilde{\psi}^\mu(t,\sigma)$ in NS- and R-sector properly, and derive the action on their modes. (Hint: the orientation flip is define for c as $\Omega : c(t,\sigma) \to -\bar{c}(t, 2\pi - \sigma)$. The minus sign in from of $\bar{c}$ is coming from relative phase of overall coefficient of the mode expansion in cylinder: $c = i\sum c_n e^{in(it+\sigma)}$ and $\bar{c} = -i\sum \bar{c}_n e^{in(it+\sigma)}$.
- Consider IIB RR-fields as in Prob. 2 of Homework 8, and show that only the 2-form RR-field survives under the Ω projection $\frac{1+\Omega}{2}$. (Hint: you have to consider field strength of n-form RR-fields because RR-fields themselves are not physical degrees of freedom. Note that RR-fields are real valued object so its complex conjugate is itself. Also note that the complex conjugate of fermions gives minus sign $(\psi^\dagger\chi)^\dagger = -\chi^\dagger\psi$ due to their statistics. You can assume that gamma matrices are invariant under Ω. This is because zero modes are identical in L and R so Γ is actually sum of L and R, i.e. $\Gamma = \Gamma^L + \Gamma^R$, which is manifestly invariant under Ω.)

Prob. 3 Op-plane

Let us consider so called orientifold action. It is a combination of the orientation flip Ω as well as a space-time parity ($\mathbb{Z}_2$-orbifold) R_p:

$$R_p : \begin{cases} X^i(t,\sigma) & \leftrightarrow & X^i(t,\sigma) & (i = 0, 1, \cdots, p)\ , \\ X^a(t,\sigma) & \leftrightarrow & -X^a(t,\sigma) & (a = p+1, \cdots, D-1)\ . \end{cases} \tag{2}$$

Note that Ω_{D-1} is Ω itself. The orientifold action $\Omega_p = \Omega \cdot R_p$ is associated to Op-plane (this is why we called the Ω action in superstring theory by O9-plane).

- Write down the orientifold action for $X^i(t,\sigma)$ and $X^a(t,\sigma)$ of closed bosonic string, as well as their modes (do not forget x and p).
- Consider bosonic closed string ($D = 26$) in an existence of O23-plane located at $(X_{24}, X_{25}) = (0,0)$. Illustrate a closed string in (X_{24}, X_{25})-plane, as well as its mirror image. Note that the closed string is either oriented, or unoriented. State which is correct and explain why so.

- Massless states of the oriented closed string are given by

$$|\Phi\rangle = \int \prod_{i,a} dp^i dp^a \Phi^{\pm}_{IJ}(\tau, p^i, p^a) \left(\alpha^I_{-1}\widetilde{\alpha}^J_{-1} \pm \alpha^J_{-1}\widetilde{\alpha}^I_{-1}\right) |p^i, p^a\rangle \ , \tag{3}$$

  where $\Phi^{\pm}_{IJ}(\tau, p^i, p^a)$ are wavefunctions, and I, J runs over the values of both a and i. Find the conditions on $\Phi^{\pm}_{ab}, \Phi^{\pm}_{ia}, \Phi^{\pm}_{ij}$ so that they guarantee Ω_p invariance. The result should be the form of

$$\Phi^{\pm}_{IJ}(\tau, p^i, p^a) = (+ \text{ or } -) \cdot \Phi^{\pm}_{IJ}(\tau, p^i, -p^a) \ . \tag{4}$$

Prob. 4 Partition function on $S^1/\mathbb{Z}_2$

Let us consider a partition function on $S^1/\mathbb{Z}_2$ of closed bosonic string. We only consider the direction along the $S^1/\mathbb{Z}_2$. The partition function should naively be given by $\mathbb{Z}_2$-orbifold projection $\frac{R+1}{2}$

$$Z_{\text{orb}} = \text{tr}_{\text{circ}} \left[\frac{1+R}{2} q^{L_0 - \frac{1}{24}} \overline{q}^{\overline{L}_0 - \frac{1}{24}}\right] \ , \tag{5}$$

where R is defined

$$R: \ X(z, \overline{z}) \ \leftrightarrow \ RX(z, \overline{z})R = -X(z, \overline{z}) \ . \tag{6}$$

X on S^1 is given by

$$X(z) = x + i\sqrt{\frac{\alpha'}{2}} \left(-\alpha_0 \log z + \sum_{n\neq 0} \frac{1}{n}\frac{\alpha_n}{z^n}\right) \ , \tag{7}$$

$$\overline{X}(\overline{z}) = \widetilde{x} + i\sqrt{\frac{\alpha'}{2}} \left(-\widetilde{\alpha}_0 \log \overline{z} + \sum_{n\neq 0} \frac{1}{n}\frac{\widetilde{\alpha}_n}{\overline{z}^n}\right) \ , \tag{8}$$

with

$$\alpha_0 = \sqrt{\frac{\alpha'}{2}} \left(\frac{n}{R} + \frac{wR}{\alpha'}\right) \ , \quad \widetilde{\alpha}_0 = \sqrt{\frac{\alpha'}{2}} \left(\frac{n}{R} - \frac{wR}{\alpha'}\right) \ , \tag{9}$$

- Derive S^1 partition function: $Z_{\text{circ}} = \text{tr}_{\text{circ}} \left[q^{L_0 - \frac{1}{24}} \overline{q}^{\overline{L}_0 - \frac{1}{24}}\right]$, where $L_0 = \frac{1}{2}\sum_{n\in\mathbb{Z}} {:}\alpha_{-n}\alpha_n{:}$.
- Write down the action of R on x and α_n. Show that the ground state transforms under R as $R|n, w\rangle = |-n, -w\rangle$.
- Derive $\text{tr}_{\text{circ}} \left[\frac{R}{2} q^{L_0 - \frac{1}{24}} \overline{q}^{\overline{L}_0 - \frac{1}{24}}\right]$, and express it using $\eta(\tau)$ and $\vartheta_2(\tau)$. Confirm that it is modular non-invariant.

In order to get modular-invariant partition function on $S^1/\mathbb{Z}_2$, we need so called twisted sector, which satisfies following boundary condition.

$$X(e^{2\pi i} z) = RX(z)R = -X(z) \ , \tag{10}$$

and similar for $\overline{X}(\overline{z})$.

- Derive mode expansion for X, which is similar to WS fermion in R-sector. However note that X has 0-level ($h = 0$).
- Derive the partition function in twisted sector $Z_{\rm tw} = \mathrm{tr}_{\rm tw}\left[\frac{1+R}{2} q^{L_0 - \frac{1}{24}} \overline{q}^{\overline{L}_0 - \frac{1}{24}}\right]$ using $\eta(\tau)$, $\vartheta_3(\tau)$, and $\vartheta_4(\tau)$, where $L_0 = \frac{1}{2}(\sum_{n\in\mathbb{Z}+\frac{1}{2}} :\alpha_{-n}\alpha_n:) + \frac{1}{16}$.
- Confirm that the sum $Z_{\rm orb} + Z_{\rm tw}$ is modular invariant. (You do not have to show $Z_{\rm circ}$ is modular invariant.)