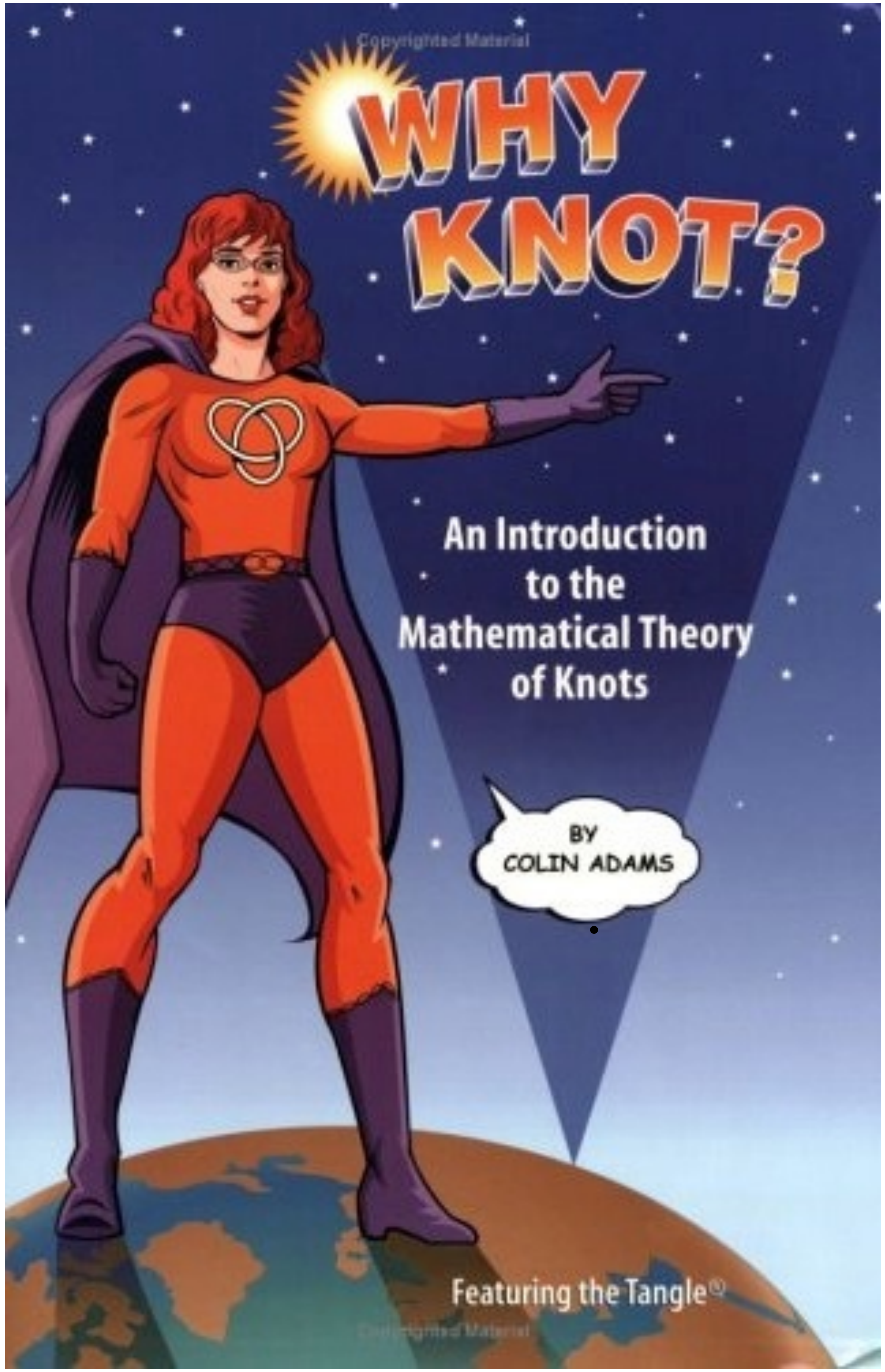
Mirror Symmetries from knots

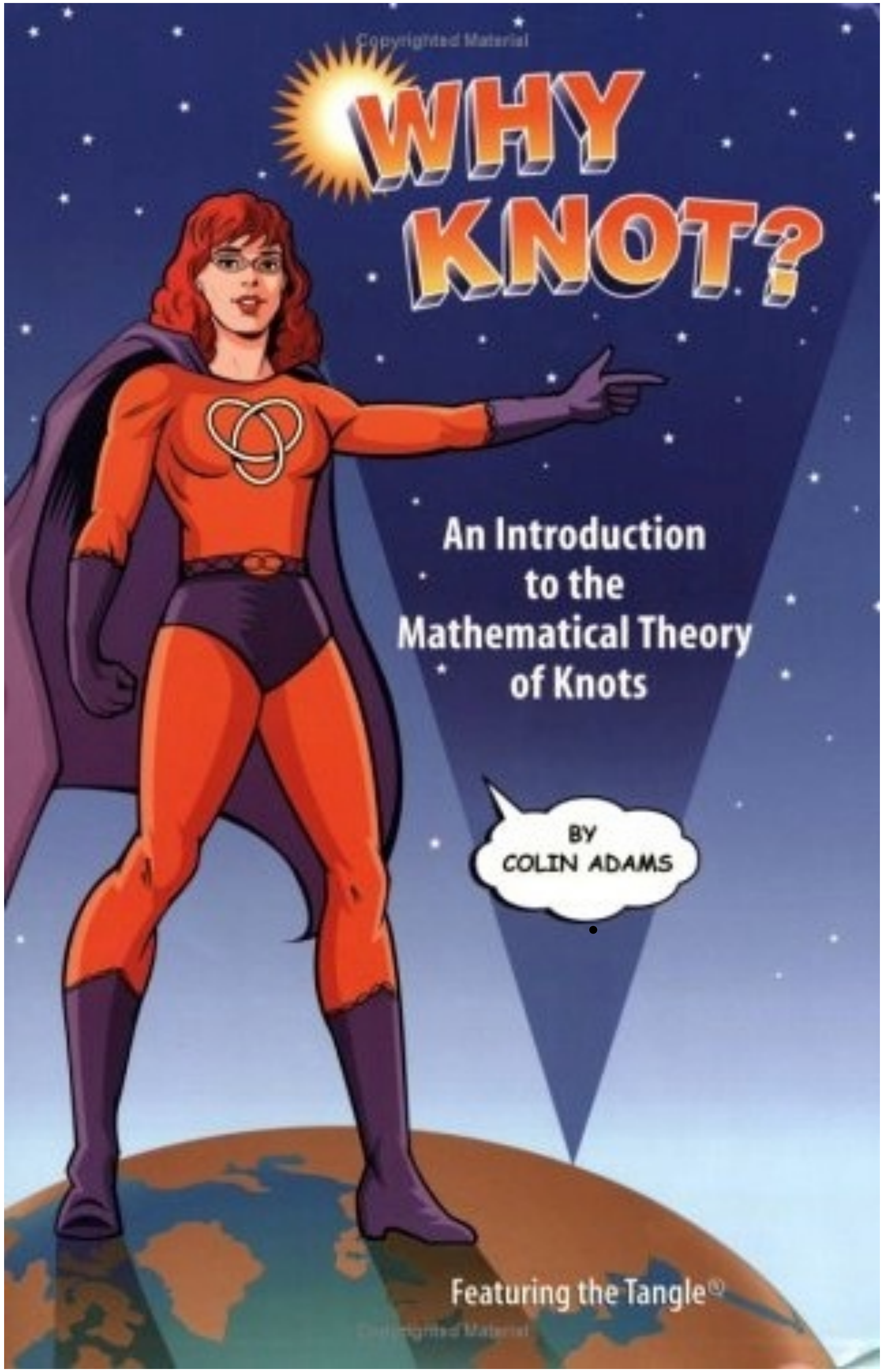
Satoshi Nawata

based on works with

Rama, Zodin, Gukov, Sulkowski

Stosic.

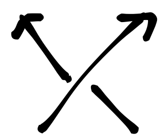
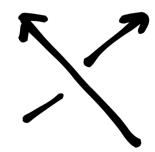

WHY KNOT?
An Introduction to the Mathematical Theory of Knots
BY COLIN ADAMS
Featuring the Tangle©

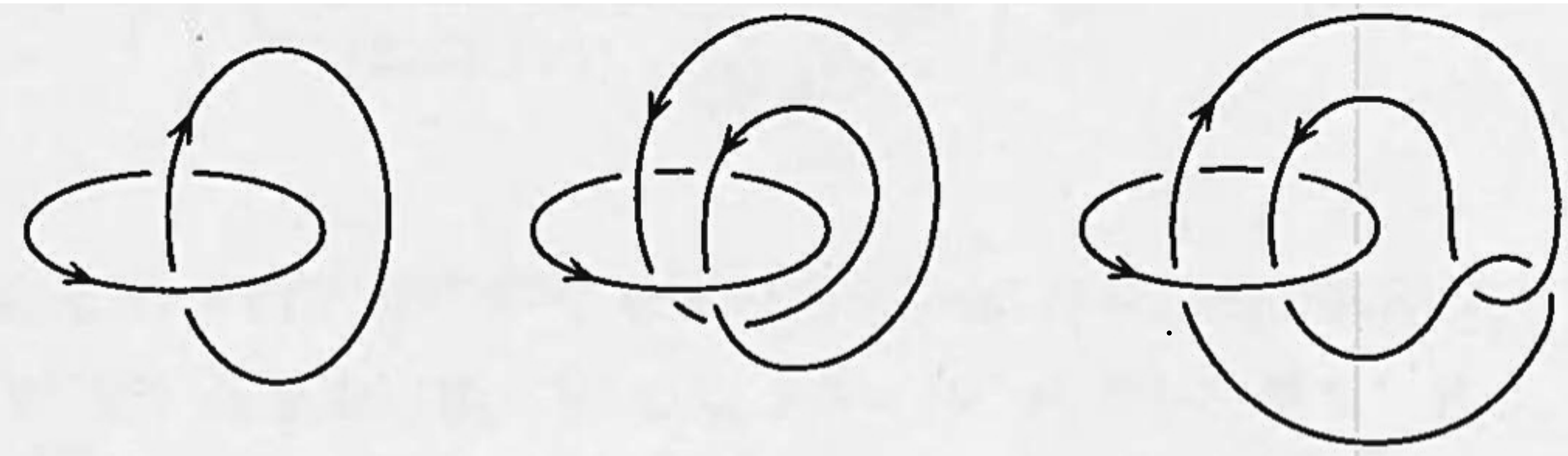


Many branches in Math & Physics interact via Knot theory

Linking number

$$Lk(K_1, K_2) = \frac{1}{2} \sum_{\text{crossing}} \epsilon(\text{crossing})$$

$\epsilon($  $) = +1$ $\qquad$ $\epsilon($  $) = -1$



$Lk=1$ $\qquad$ $Lk=-2$ $\qquad$ $Lk=0$

Gauss Linking Number [1833]

integral form

[4.]

Of the *geometria situs,* which was forseen by LEIBNITZ, and into which only a pair of geometers (EULER and VANDERMONDE) were granted a bare glimpse, we know and have, after a century and a half, little more than nothing.

A principal problem at the *interface* of *geometria situs* and *geometria magnitudinis* will be to count the intertwinings of two closed or endless curves.

Let x, y, z be the coordinates of an undetermined point on the first curve; x', y', z' those of a point on the second and let

$$\iint \frac{(x'-x)(\mathrm{d}y\mathrm{d}z'-\mathrm{d}z\mathrm{d}y')+(y'-y)(\mathrm{d}z\mathrm{d}x'-\mathrm{d}x\mathrm{d}z')+(z'-z)(\mathrm{d}x\mathrm{d}y'-\mathrm{d}y\mathrm{d}x')}{\left[(x'-x)^2+(y'-y)^2+(z'-z)^2\right]^{\frac{3}{2}}} = V$$

then this integral taken along both curves is

$$= 4m\pi$$

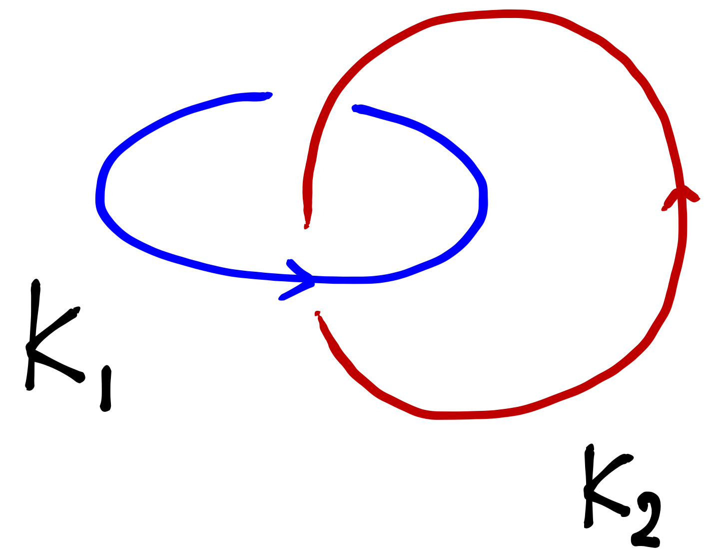
m being the number of intertwinings.

The value is reciprocal, i.e. it remains the same if the curves are interchanged. 1833. Jan. 22.

$$Lk(K_1, K_2) = \frac{1}{4\pi}\int_{K_1} dx_1^i \int_{K_2} dx_2^j \,\epsilon_{ijk} \frac{(x_1 - x_2)^k}{|x_1 - x_2|}$$

$\Longleftrightarrow$ Biot-Savart law

$$\vec{B} = \frac{1}{4\pi}\int_{K_1} \frac{d\vec{x}_1 \times \vec{r}}{r^3}$$



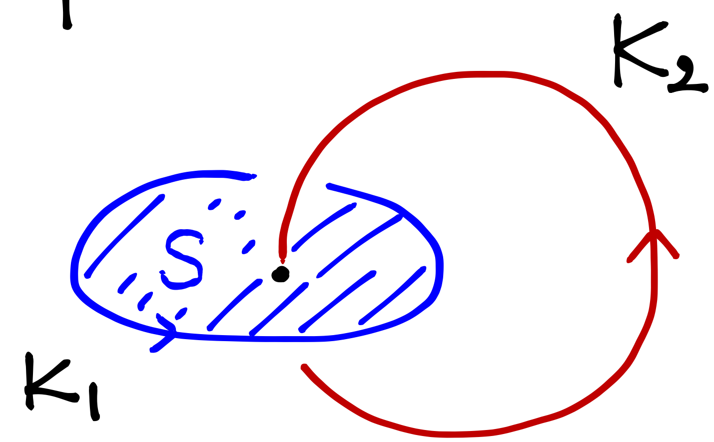
$$f : K_1 \times K_2 \longrightarrow S^2 \ ; \ (x_1, x_2) \longmapsto \frac{x_1 - x_2}{|x_1 - x_2|}$$

$$Lk(K_1, K_2) = \deg(f)$$

Seifert surface

Surface whose boundary is K_1

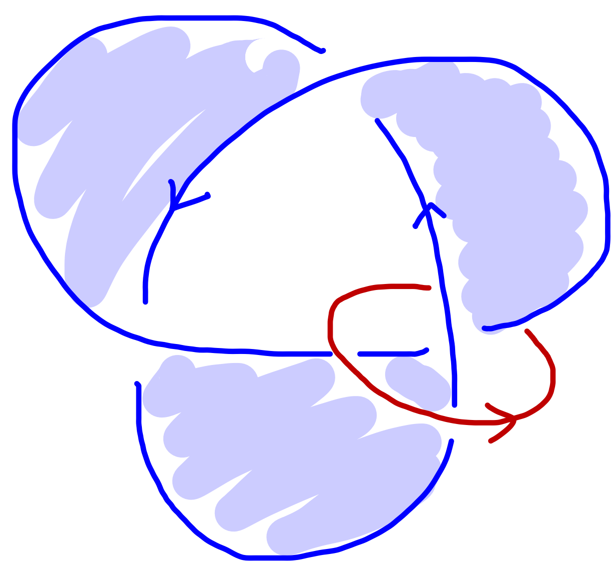
not unique !

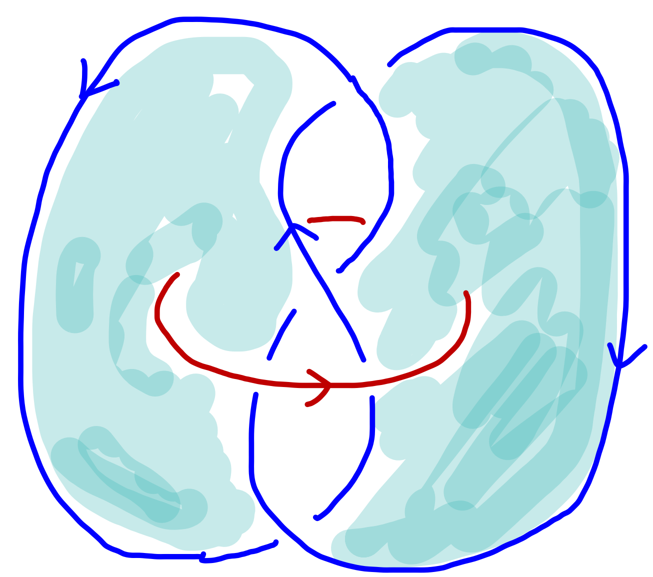


$Lk(K_1, K_2) = \underbrace{S \cdot K_2}$

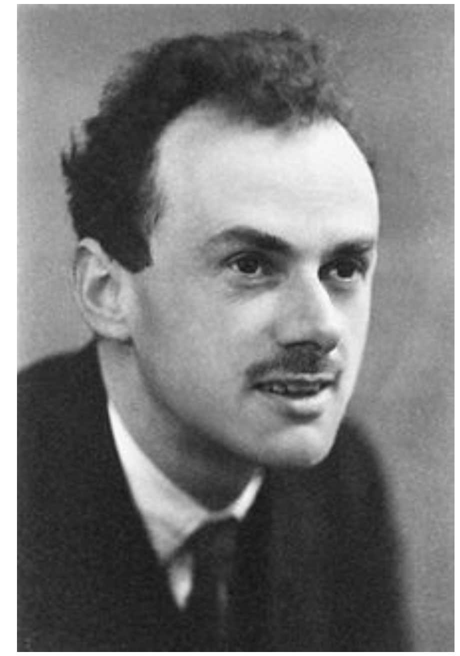
intersection #.

independent of choice of S

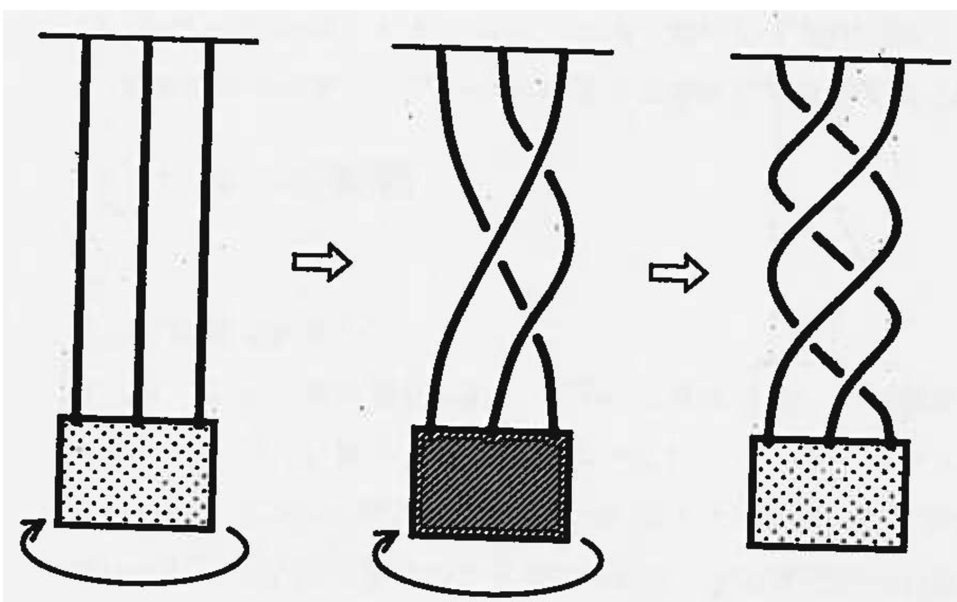




Dirac's String Trick



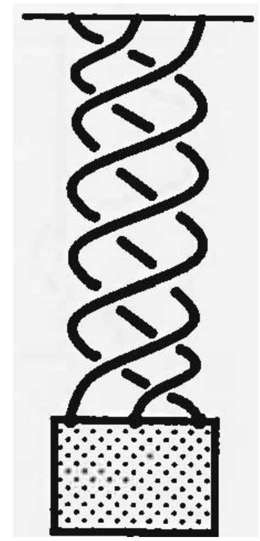
3 strands hang a plate



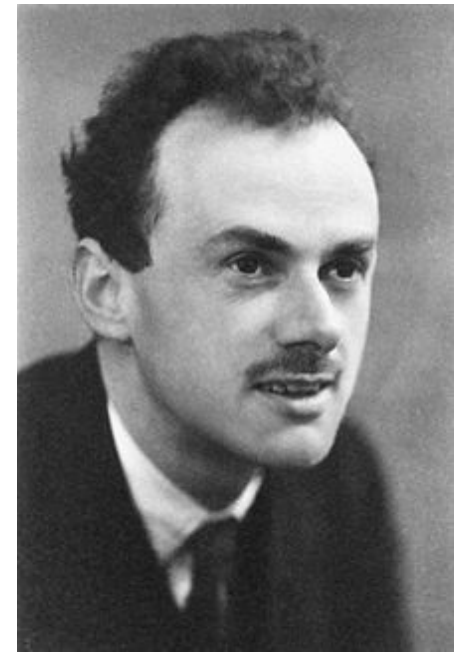
360° holizontal rotation

Dirac's String Trick

360°



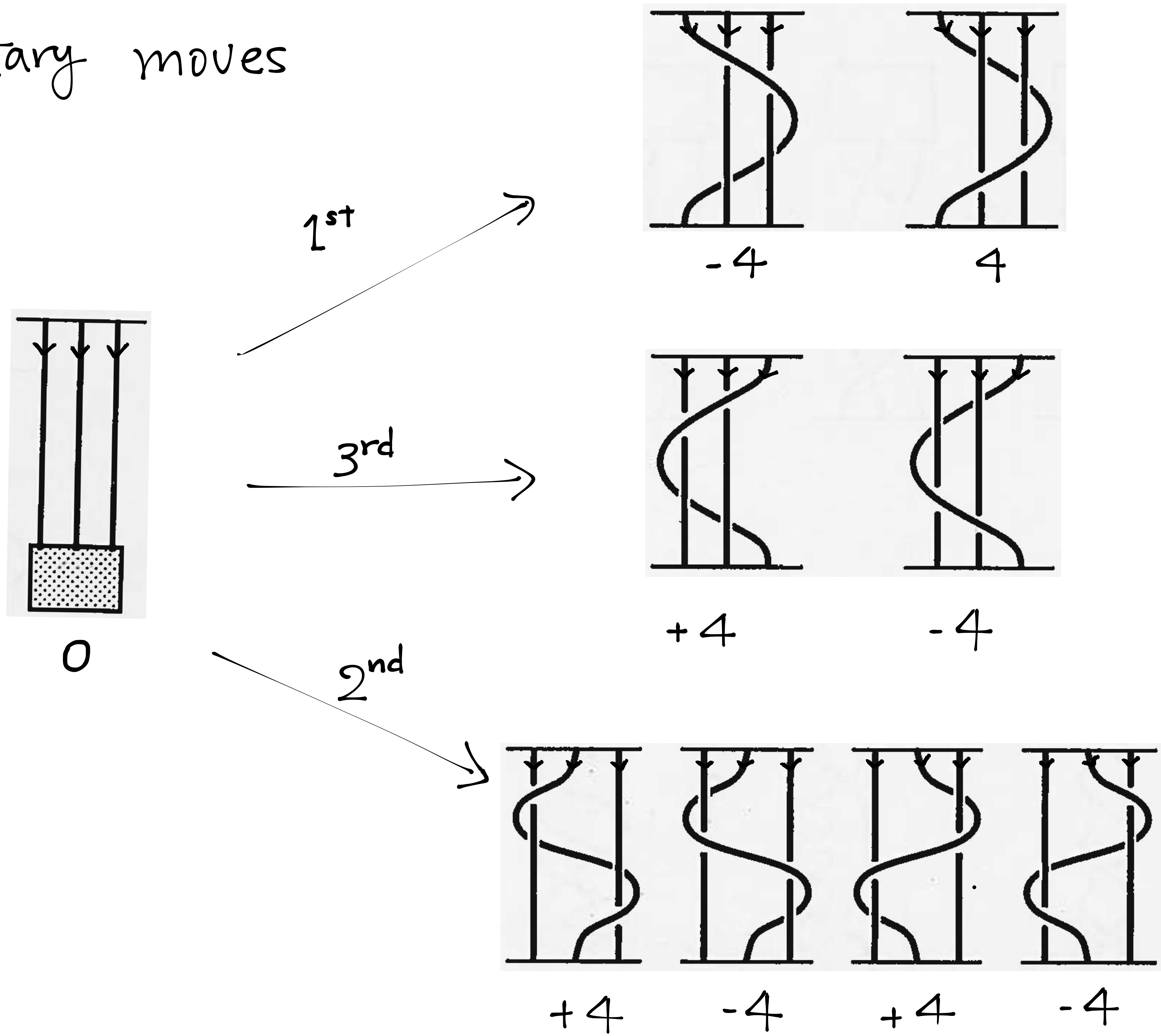
720°



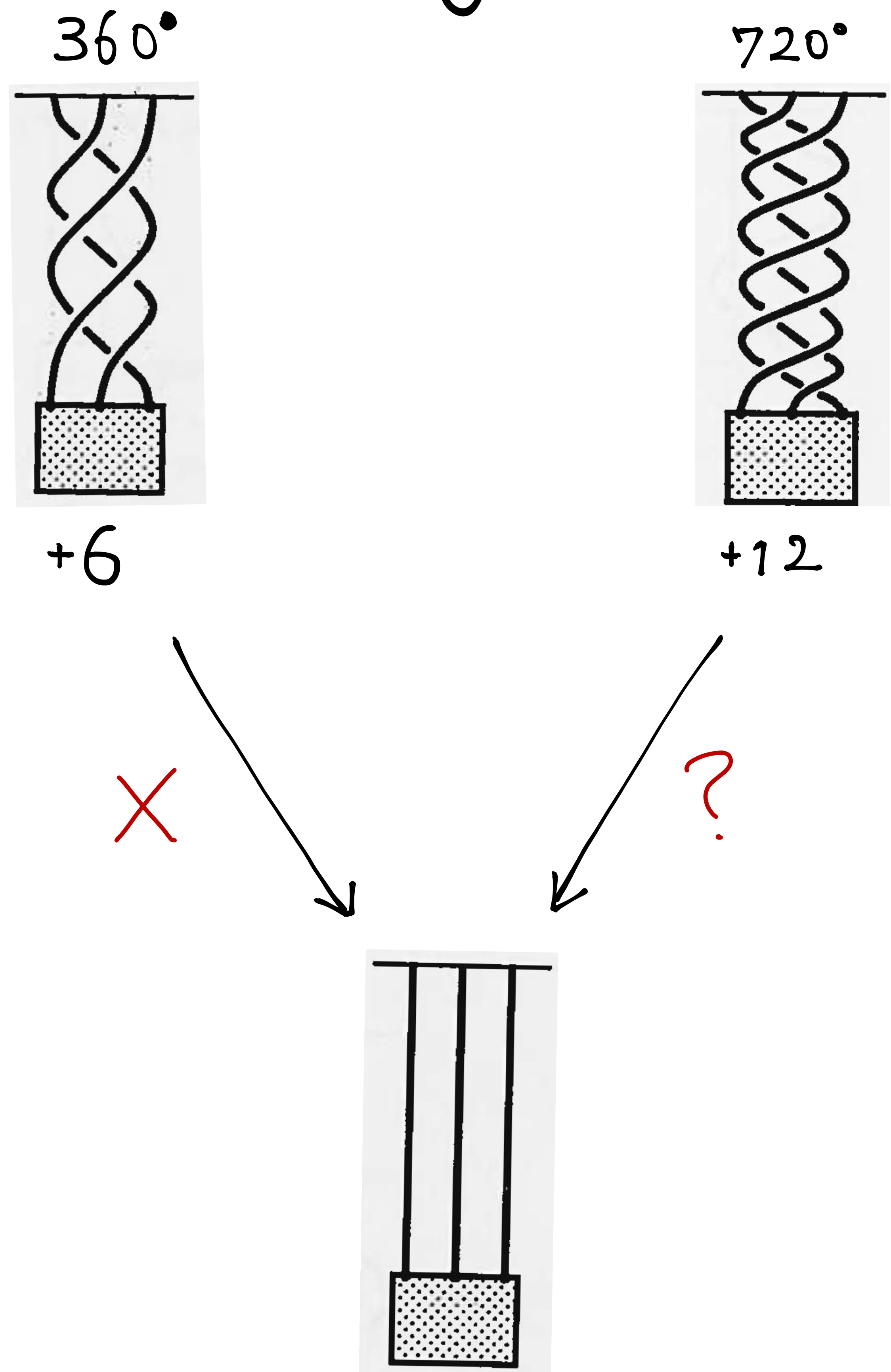
??

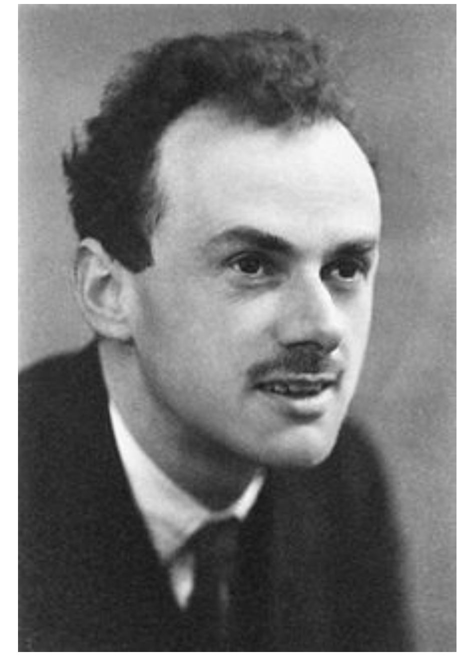
- extend string and pass it under plate
- fix plate do not allow to turn over

elementary moves

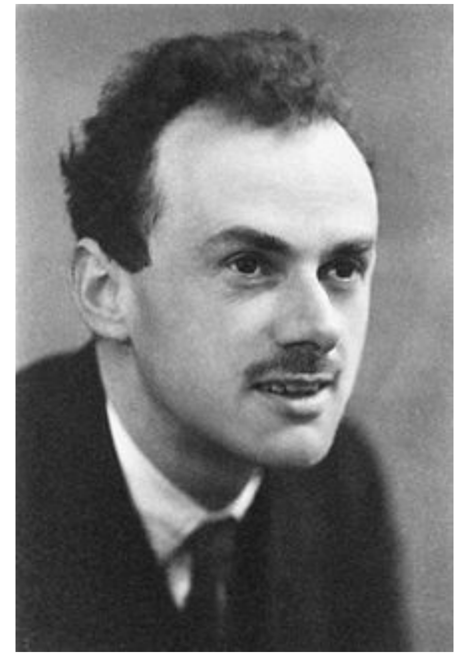


Dirac's String Trick





Dirac's String Trick



Dirac used this trick to explain $SU(2)$ is the double cover of $SO(3)$

$\Rightarrow$ spinor group

Newman proved it in 1940 using Braid group B_n

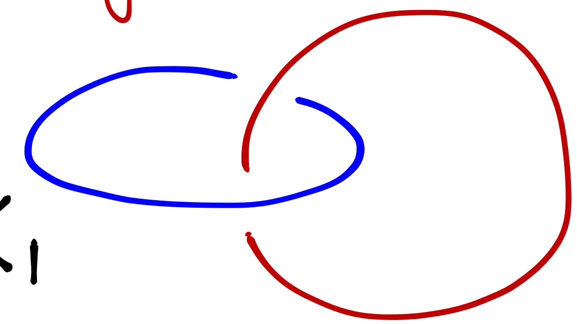
$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \qquad i = 1, \cdots, N-1$$

$$\sigma_i \sigma_j = \sigma_j \sigma_i \qquad |i-j| > 1$$

Abelian Chern-Simons Theory [Polyakov]

$$S_{U(1)} = \frac{k}{4\pi}\int_{S^3} A\,dA$$

K_1



Wilson loop

$$W(K_1, K_2) = \exp\left(i\int_{K_1} A\right)\exp\left(i\int_{K_2} A\right)$$

$$\langle W(K_1, K_2)\rangle = \int_{\mathcal{A}/\mathcal{G}} \mathcal{D}A\; e^{iS_{U(1)}}\, W$$

$$= \exp\left(\frac{i}{2k}\int_{K_1} dx_1^i \int dx_2^j\, \epsilon_{ijk}\, \frac{(x_1 - x_2)^k}{|x_1 - x_2|^3}\right)$$

$$= \exp\left(\frac{2\pi i}{k}\, Lk(K_1, K_2)\right)$$

Chern–Simons Theory [Witten]

M a 3-mfd
G a semi-simple compact gauge grp
A $\mathfrak{g}$-valued gauge connection of P

The action is independent of metric

$$S_{cs}(M,A) = \frac{1}{8\pi^2}\int_M \mathrm{Tr}\left(A\, dA + \frac{2}{3}A^3\right)$$

The partition fn is a 3-mfd inv (WRT inv)

$$Z_k(M) = \int_{\mathcal{A}_M/\mathcal{G}_M} \mathcal{D}A\; e^{2\pi i k S_{cs}(M.A)}$$

- $k \in \mathbb{Z}$ is CS level
- e.o.m $F_A = 0$

Natural observable metric independent

$$W_R(K) = \mathrm{Tr}_R\, P\exp\left(\oint_K A\right) = \mathrm{Tr}_R\, \mathrm{Hol}_K(A)$$

colored $\mathfrak{g}$ quantum invariant of K in S^3

$$J^{\mathfrak{g}}_R(K;q) = \frac{1}{Z_k(S^3)} \int_{\mathcal{A}/\mathcal{G}} \mathcal{D}A\; W_R(K)\, e^{2\pi i k S_{CS}}$$

$$q = \exp\left(\frac{\pi i}{k+h^\vee}\right) \qquad h^\vee : \text{dual coxeter \#.}$$

$\Rightarrow$ Laurent poly'l w.r.t q

$\mathfrak{sl}(2)$ Jones poly $J_R(K;q)$

$\mathfrak{sl}(N)$ HOMFLY inv. $\overline{P}_R(K;a,q)$ replacing q^N by a

$\mathfrak{so}(N)$ Kauffman inv $\overline{F}_R(K;\lambda,q)$ replacing q^{N-1} by λ

Suppose that $\partial M = \Sigma \neq \phi$

$a \in \mathcal{A}_\Sigma \qquad \mathcal{A}_a := \{ A \in \mathcal{A}_M \mid A|_\Sigma = a \}$

$\mathcal{G}_M^* = \{ g \in \mathcal{G}_M \mid g|_\Sigma = id \}$

$$Z_k(M)(a) = \int_{\mathcal{A}_a/\mathcal{G}_M^*} \mathcal{D}A \quad e^{2\pi i k S_{CS}(M.A)}$$

This can be regarded as a section of complex line bdle

$\mathbb{C}^{\otimes k} \times \mathcal{A}_\Sigma \to \mathcal{A}_\Sigma$. The gauge transf. $g \in \mathcal{G}_\Sigma$ acts

$$(Z_k(M)(a), a) \mapsto (e^{2\pi i k C(a.g)} Z_k(M)(a) \quad g^{-1}dg + g^{-1}ag)$$

where $C(a.g) = \frac{1}{8\pi^2} \int_\Sigma Tr(g^{-1}ag \wedge g^{-1}dg) - \int_M (g^{-1}dg)^3 \in \mathbb{R}/\mathbb{Z}$

$$\{\mathcal{G}_M\text{-invariant holom. section of } \mathbb{C}^k \times \mathcal{A}_\Sigma \to \mathcal{A}_\Sigma\}$$

$$= \{\text{holom. section of } \mathcal{L}^{\otimes k} \to \mu^{-1}(0)/\mathcal{G}_\Sigma\}$$ Atiyah - Bott

where the momentum map μ

$$\mu : \mathcal{A}_\Sigma \to \mathrm{Lie}(\mathcal{G}_\Sigma)^* \ ; A \mapsto F_A$$

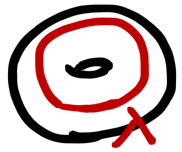
Therefore, $\mathcal{M}_\Sigma = \mu^{-1}(0)/\mathcal{G}_\Sigma = \{A \in \mathcal{A}_\Sigma \mid F_A = 0\}$

moduli sp. of flat connection on Σ

$$Z_k(M) \in H^0(\mathcal{M}_\Sigma, \mathcal{L}^{\otimes k})$$ finite-dim'l

$\shortparallel$

the space of conformal blocks of $\hat{\mathfrak{g}}_k$ WZNW

$Z_{\Lambda k}($  $) = \mathrm{Tr}_{\lambda}\, q^{L_0 - \frac{c}{24}} = \chi_\lambda(\tau)$

character of integrable rep of $\hat{\mathfrak{g}}_k$

Modular transformation

$$\chi_\lambda(-1/\tau) = \sum_\mu S_{\lambda\mu}\, \chi_\mu(\tau)$$

$$\chi_\lambda(\tau+1) = T_{\lambda\lambda}\, \chi_\lambda(\tau)$$

where

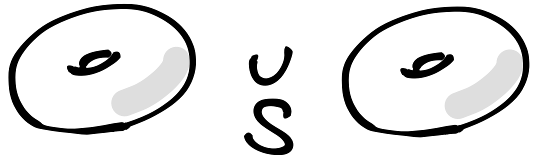
$$S_{\lambda\mu} = s_\lambda(q^\rho)\, s_\mu(q^{\rho-\lambda}) \qquad s_\lambda(x) \text{ Schur fn}$$

$$T_{\lambda\lambda} = q^{\kappa_\lambda} \qquad \kappa_\lambda : \text{ quadratic Casimir}$$

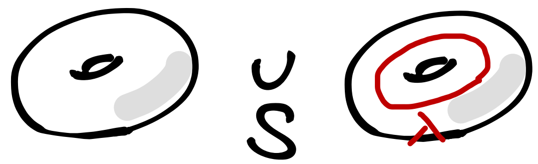
They satisfy $S^2 = (ST)^3 = \mathrm{id}$

For example Verlinde formula

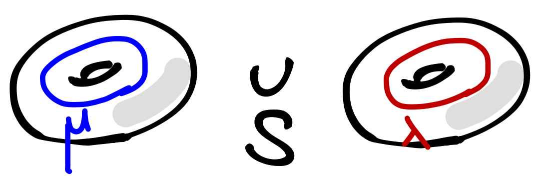
$$\dim H^0(\mathcal{M}_{\Sigma_g}, \mathcal{L}^{\otimes k}) = \sum_j \frac{1}{(S_{0j})^{2g-2}} \quad \text{for } \mathfrak{sl}(2)$$

$Z_k(S^3) =$  $= \langle 0 | S | 0 \rangle$

$$= S_{00} \left(= \sqrt{\frac{2}{k+2}} \sin\left(\frac{1}{k+2}\right) \quad \text{for } \mathfrak{sl}(2) \right)$$

$\langle W_\lambda(\bigcirc) \rangle =$  $= \dfrac{\langle 0 | S | \lambda \rangle}{\langle 0 | S | 0 \rangle}$

$$= \frac{S_{0\lambda}}{S_{00}} \left(= \frac{q^n - q^{-n}}{q - q^{-1}} = [n] \quad \text{for } \mathfrak{sl}(2)\ \lambda = \right.$$

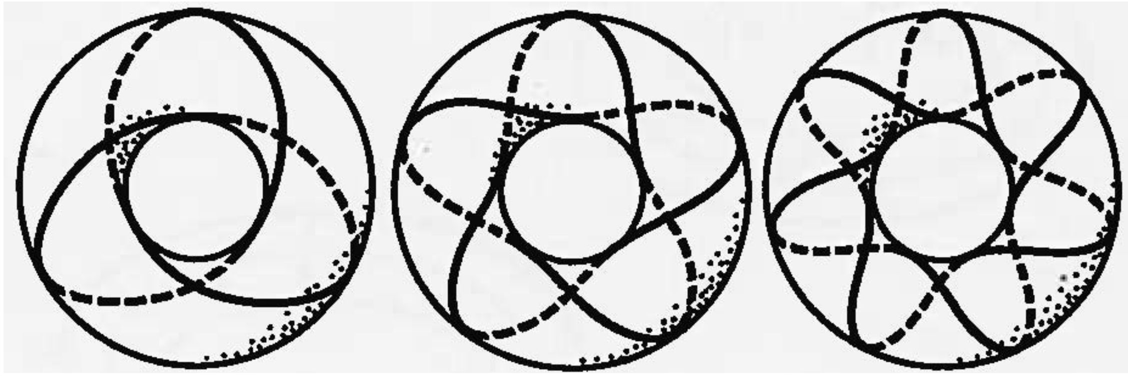
$\langle W($  $) \rangle =$  $= \dfrac{\langle \mu | S | \lambda \rangle}{\langle 0 | S | 0 \rangle}$

$$= \frac{S_{\mu,\lambda}}{S_{00}} \left(= \frac{q^{(m+1)(n+1)} - q^{-(m+1)(n+1)}}{q - q^{-1}} \right.$$

$$\text{for } \mathfrak{sl}(2) \quad \left. \begin{array}{l} \mu = \overbrace{\square\cdots\square}^{m} \\ \lambda = \overbrace{\square\cdots\square}^{n} \end{array} \right)$$

Torus Knots $T(P.Q)$

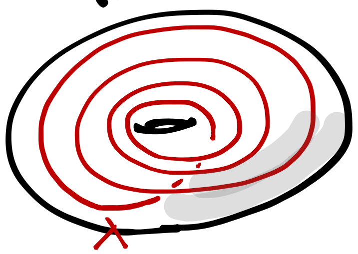
Rosso-Jones, Morton, Lin-Zhang, Gorsky, Brini, Eynard, Marino



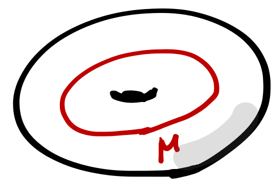
$T(2.3)$ $T(2.5)$ $T(2.7)$

$$(P.Q) = (P.0)\begin{pmatrix} 1 & \frac{Q}{P} \\ 0 & 1 \end{pmatrix}$$

$(P.0)$

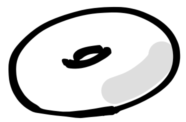


$= \sum C_\lambda^M(P)$

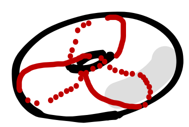


Adams operation

$J_\lambda^{g}(K(P.Q);q) =$



$\cup_S$



$$= \frac{\langle 0|S\, W^{(P.Q)}|0\rangle}{\langle 0|S|0\rangle}$$

$$= (S_{00})^{-1}\langle 0|\, S\, T^{-\frac{Q}{P}} W^{(P.0)}\, T^{\frac{Q}{P}}\, |0\rangle$$

$$= \sum_\mu C_\lambda^M(P)\, q^{-\frac{Q}{P}\kappa_\mu} \dim_q \mu$$

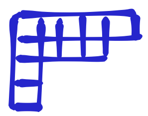
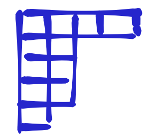
$$P_\square(T(P.Q);a,q) \propto {}_2\varphi_1(q^{1-Q}, q^{1-P}; q^{1-P-Q}; q, a)$$

Conjectures on Dunin-Barkowski, Sleptsov, ITEP, Gukov, Stosic

Properties of HOMFLY (Kauffman) poly'l

- Mirror (Transposition) symmetry.

$$P_\lambda(K; a, q) = P_{\lambda^T}(K; a, -q^{-1})$$

- Exponential growth property

$$P_\lambda(K; a, q=1) = \left[P_\square(K; a, q=1)\right]^{|\lambda|}$$

- Canceling property

$$P_{s\{\boxed{\;\cdots\;}}(K; a=q^s, q) = 1.$$

Volume Conjecture Kashaev, Murakami[2]

"quantum" invariant $\longleftrightarrow$ "classical" 3d geomet

$$2\pi \lim_{n\to\infty} \frac{1}{n} \log \left| J_n(K; e^{2\pi i/n}) \right| = \mathrm{Vol}(S^3 \setminus K)$$

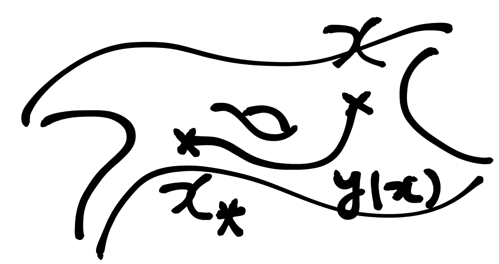
generalization Gukov

$$J_n(K;q) \underset{\substack{q=e^{\hbar}\to 1 \\ n\to\infty}}{\sim} \exp\left(\frac{1}{\hbar} S_{-1}(x) - \frac{3}{2}\log\hbar + \sum \hbar^k S_k(x)\right)$$

$q^n = x$ fixed

$$S_{-1}(x) = \int_{x_*}^{x} \log y \frac{dx}{x}$$

$A(x, y(x)) = 0$



$$\mathcal{M}_{\text{flat}}(T^2, SL(2,\mathbb{C})) \underset{\text{Lag. sub}}{\supset} \mathcal{M}_{\text{flat}}(S^3\setminus K, SL(2,\mathbb{C}))$$

$$\left\{ (x,y) \in \frac{\mathbb{C}^\times \times \mathbb{C}^\times}{\mathbb{Z}_2} \,\middle|\, A(x,y) = 0 \right\}$$

How about $S_k(x)$?

topological recursions!

using perturbative T.R., observation was made

Fuji. Dijgraaf, Manabe

Gap has been filled by non-perturbative T.R.

Eynard Borot

$$J_n(q) \sim B(u) \left[\psi_{\hbar,2}\left(p, 0; \iota(p), \iota(0)\right) \right]^{\frac{1}{2}}$$

$$\psi_{\hbar,2} = \exp\left(\frac{1}{\hbar} \int v\, du + \frac{1}{2} \int_{\cdot\cdot} \omega_2^0 \right) \frac{\vartheta_{\cdot}}{\vartheta}$$

$$\times \sum_{r} \sum_{h_j, k_j, l_j} \frac{\hbar^{\sum 2h_j - 2 + k_j + l_j}}{r!} \bigotimes_{j=1}^{r} \frac{1}{(2\pi i)^{k_j} k_j!\, l_j!} \oint_B \cdots \oint_B \int_{\cdot} \cdots \int_{\cdot} \omega^{h_j}_{k_j + l_j} \; \frac{\vartheta_{\cdot}^{(\sum k_j)}}{\vartheta_{\cdot}}$$

Questions.: Are they related to topological recursion?

LMO invariants

$$Z_k(M) \sim (\text{torsion}) \times \exp \sum \frac{c^l k^{-l}}{(2l)!(3l)!} \sum_{\substack{e(\Gamma)=-l \\ \Gamma:\text{connected}}} \frac{Z_\Gamma}{|\mathrm{Aut}\,\Gamma|}$$

Rozansky-Witten

hyperkähler mfd X^{4k} gives rise to 3d TQFT Z_X. $^\forall$ 3-mfd M, one can associate partition fn $Z_X(M)$. which has perturbative expansion

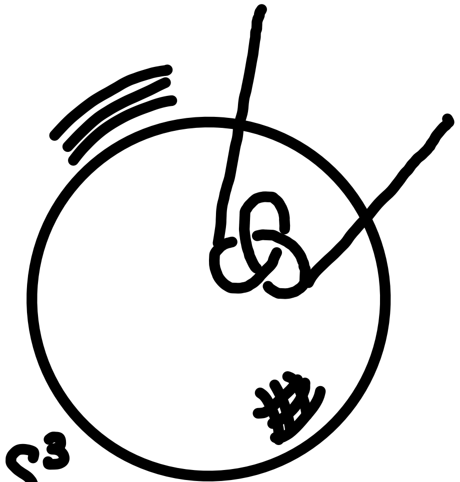
$$Z_X(M) = \sum_\Gamma b_\Gamma(X)\, I_\Gamma(M)$$

Knot in String Theory

$L_K = \{(x,p) \in T^*S^3 \mid x = x(s),\ \dot{x}(s) \perp p\}$

U(N) Chern-Simons theory is realized in topological A-model

A-model can be embedded in M-theory.



S^3

Spacetime	$S^1 \times \mathbb{R}^4 \times T^*S^3$
N M5	$S^1 \times \mathbb{R}^2 \times S^3$
M5	$S^1 \times \mathbb{R}^2 \times L_K$

Gopalcumar-Vafa

Ooguri-Vafa

At large N, there is geometric transition

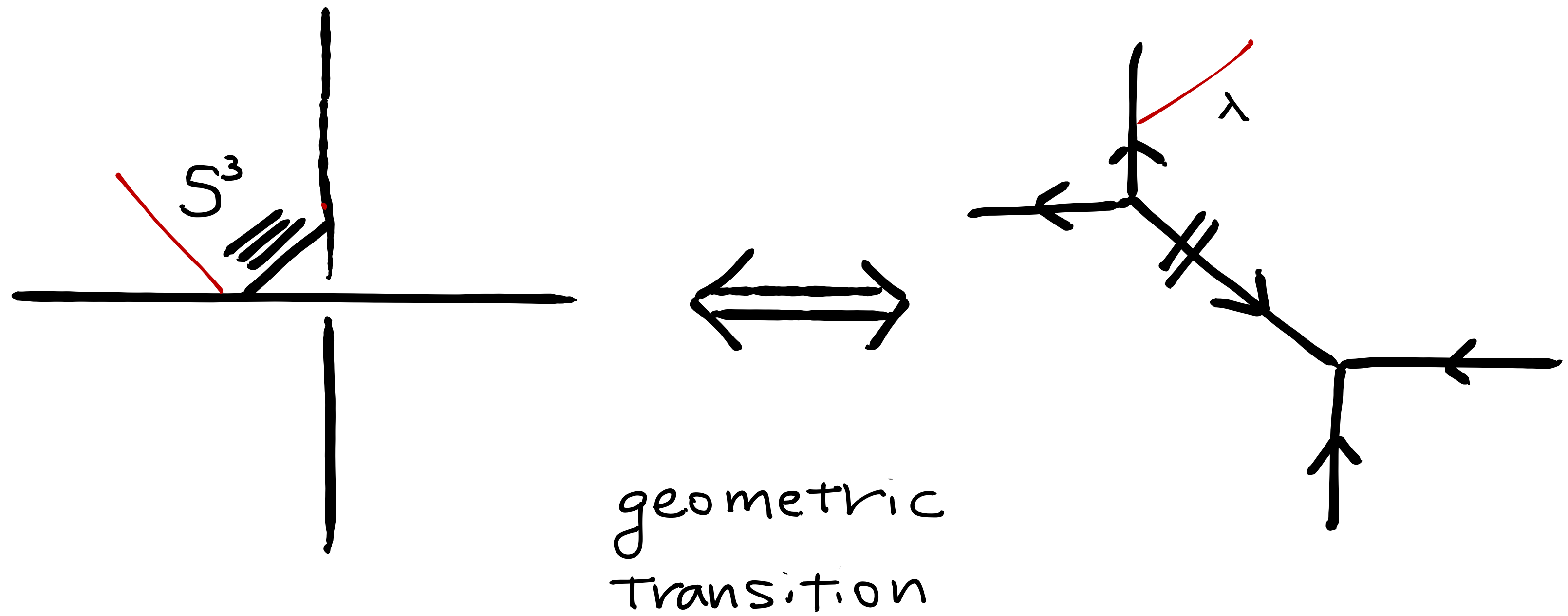
$$T^*S^3 \longrightarrow \begin{array}{c} \mathcal{O}(-1) \oplus \mathcal{O}(-1) \\ \downarrow \\ \mathbb{P}^1 \end{array} = X$$

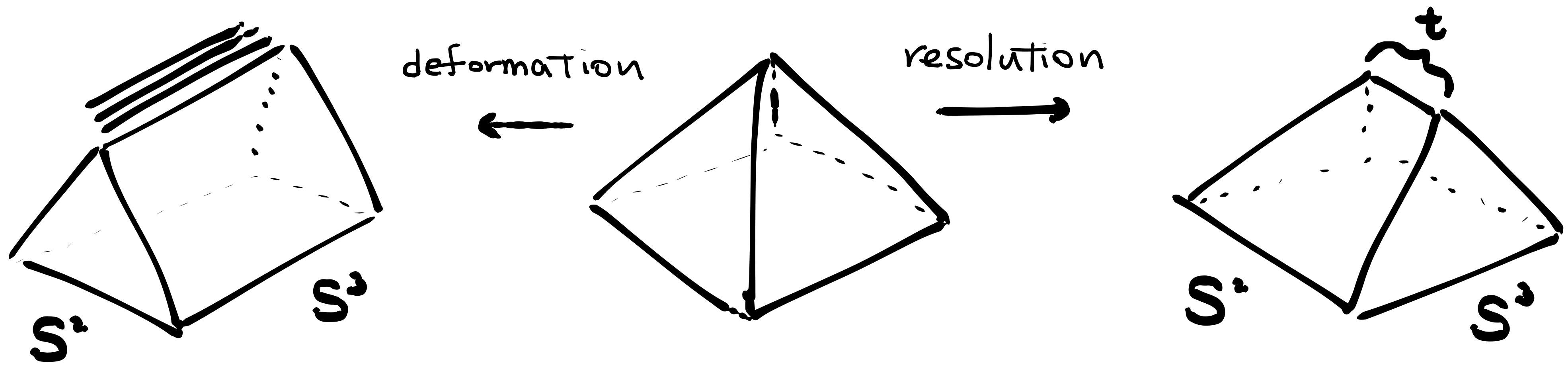
$$\log Z^{CS}(K; a, q) = \mathcal{F}_K(t, g_s)$$

$$q = e^{g_s/2}$$

$$a = e^{t/2}$$

$T^*S^3 \longrightarrow$ resolved conifold $\mathcal{O}(-1)\oplus\mathcal{O}(-1) \to \mathbb{P}^1$





configuration in resolved conifold.

spacetime $\quad S^1 \times \mathbb{R}^4 \times X$

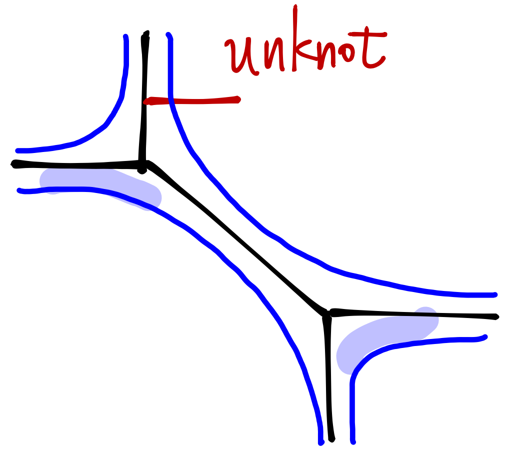
M5 $\quad S^1 \times \mathbb{R}^2 \times L_K$

BPS states associated to M2-M5 bound states can be identified with knot homology Gukov Schwartz Vafa

$$\mathcal{H}_{BPS} \cong \mathcal{H}_{knot}$$

Mirror symmetry is duality b/w A-model & B-model

Aganagic Vafa



$$\xleftrightarrow{\text{mirror}} \quad wZ = 1 - x a^2 - a y + a x y$$

In general, the mirror geometry of non-compact toric CY is described by $wZ = H(x, y; a)$

Given a knot K, there is the corresponding mirror geometry!

$$wZ = H_K(x.y:a)$$

- Brini. Eynard, Mariño conjecture the spectral curve $H_{T(p.q)}(x.y:a)$ for Torus knot can be obtained by $SL(2.\mathbb{Z})$ transformation of the unknot curve.

$$H^{AEM}_{3_1}(x,y;a) = x^3(x-1)^2 - a\,y\,(x-a)^2 = 0 \qquad \text{genus } 0$$

- Aganagic & Vafa conjecture that the spectral curve can be obtained by large color asymptotics of $\square\cdots\square$-colored HOMFLY polynomial.

??

$$P_n(K;a.q) \underset{\substack{n\to\infty \\ q^n = x \text{ fixed}}}{\overset{q=e^{\hbar}\to 1}{\sim}} \exp\left(\frac{1}{\hbar}\int \log y \frac{dx}{x} + \cdots\right)$$

$H_K(x.y:a)$

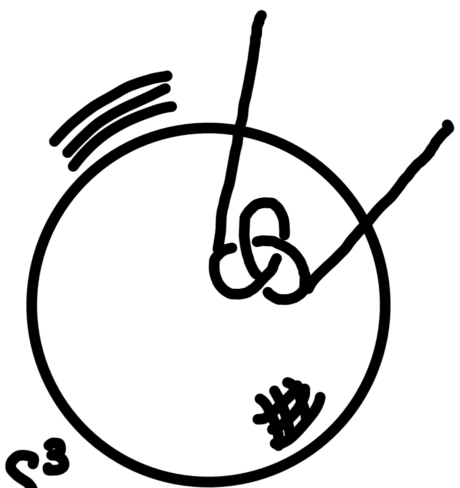
$$H^{AV}_{3_1}(x.y;a) = (1-ax)y^2 + \cdots \qquad \text{genus } 1$$

Knot in String Theory

$$L_K = \{(x,p) \in T^*S^3 \mid x = x(s),\ \dot{x}(s) \perp p\}$$

U(N) Chern-Simons theory is realized in topological A-model

A-model can be embedded in M-theory.



Spacetime	$S^1 \times \mathbb{R}^4 \times T^*S^3$
N M5	$S^1 \times \mathbb{R}^2 \times S^3$
M5	$S^1 \times \mathbb{R}^2 \times L_K$

Gopalkumar-Vafa

Ooguri-Vafa

At large N, there is geometric transition

$$T^*S^3 \longrightarrow \begin{array}{c} \mathcal{O}(-1) \oplus \mathcal{O}(-1) \\ \downarrow \\ \mathbb{P}^1 \end{array} = X$$

$$\log Z^{CS}(K; a, q) = \mathcal{F}_K(t, g_s)$$

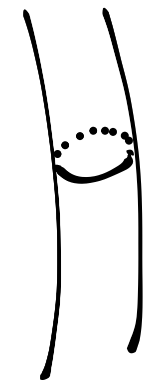
$$q = e^{g_s/2}$$

$$a = e^{t/2}$$

3d / 3d correspondence

Gukov, Dimofte, Hollands, Terashima. Yamazaki

2 M5 $S^1 \times \mathbb{R}^2 \times (S^3 \setminus K)$



K

$\mathcal{N}=2$ 3d gauge theory on $S^1\times\mathbb{R}^2$ $T_2[K]$

$SL(2,\mathbb{C})$ Chern-Simons theory on $S^3\setminus K$.

$A \sim \begin{pmatrix} x & * \\ & x^{-1} \end{pmatrix} d\theta$

- identity of partition fns.

$$\mathrm{Tr}(-1)^F q^{-J+\frac{R}{2}} x^e a^f t^g = Z_q^{CS}(x; a, q, t)$$

- moduli space

$$\mathcal{M}_{vac}(T_2[K]) = \mathcal{M}_{flat}(S^3\setminus K, SL(2,\mathbb{C}))$$

3d Mirror symmetry

$$P_n(5_2;a,q)=\sum_{k=0}^{n}\sum_{j=0}^{k} q^k\frac{(aq^{-1};q)_k}{(q;q)_k}(q^{1-n};q)_k(aq^{n-1};q)_k \times a^j q^{j(j-1)}\begin{bmatrix}k\\ j\end{bmatrix}_q$$

$$=\sum_{k=0}^{n}\sum_{j=0}^{k} q^k\frac{(aq^{-1};q)_k}{(q;q)_k}(q^{1-n};q)_k(aq^{n-1};q)_k \times(-1)^j a^{2j} q^{\frac{3}{2}j(j-1)}\frac{1-aq^{2j-1}}{(aq^{j-1};q)_k}\begin{bmatrix}k\\ j\end{bmatrix}_q$$

	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6	ϕ_7	ϕ_8	ϕ_9	ϕ_{10}
$U(1)_x$	−	+			+	−				
$U(1)_a$			+	−	+	−	−	+		
$U(1)$ gauge		−		−		−		+		+
$U(1)$ gauge							−	+	+	−

	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6	ϕ_7	ϕ_8
$U(1)_x$	−	+			+	−		
$U(1)_a$			+	−	+	−		
$U(1)$ gauge		−		−		−		+
$U(1)$ gauge							+	−

Fenchel-Nielsen v.s. Fock-Goncharov

There are two coords of moduli space of flat connection on Riemann surf.

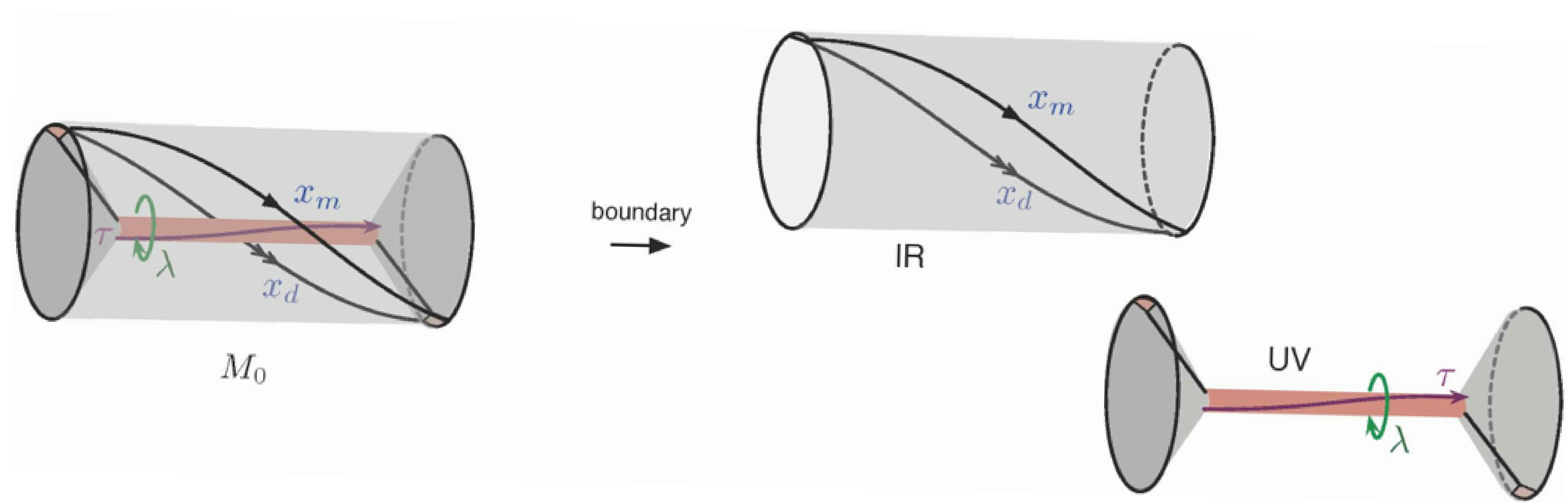
F.N. is suitable for UV description.

F.G is appropriate for IR description

Dimofte
van der Veen
Gaitto

UV | IR

RG wall



hyperbolic geometry

quantum groups

Yang-Baxter equations

cluster algebra.

Chern-Simons theory

Math

Physics

R-matrices

WZW model

Hilbert schemes

topological strings

Khovanov homology

Gromov-Witten invariants

categorifications

WRT invariants

BPS states

modular forms

topological recursions

mirror symmetry