

Homework 6: Due at class on April 16

1. (Kepler's two-body problem)

Let us consider one of the first examples of integrable systems solved by the Liouville theorem: The Kepler two-body problem of planetary motion. Taking the center-of-mass frame, the potential $V(r)$ of the system depends only on the radius, and the Hamiltonian is given by

$$H = \frac{1}{2} \sum_{i=1}^3 p_i^2 + V(r) .$$

1. Show that the angular momentum

$$\vec{J} = (J_1, J_2, J_3) , \quad J_{ij} = x_i p_j - x_j p_i = \epsilon_{ijk} J_k$$

is conserved.

2. Given the standard symplectic form $\omega = \sum_{i=1}^3 dp_i \wedge dx_i$, compute the Poisson brackets

$$\{J_i, J_j\} = -\epsilon_{ijk} J_k .$$

Show that the following three physical quantities commute under the Poisson bracket

$$H, \quad J_3, \quad J^2 = J_1^2 + J_2^2 + J_3^2$$

3. Rewrite the Liouville 1-form

$$\alpha = \sum_i p_i dx_i = p_r dr + p_\theta d\theta + p_\phi d\phi \tag{0.1}$$

in terms of the polar coordinates

$$x_1 = r \sin \theta \cos \phi, \quad x_2 = r \sin \theta \sin \phi, \quad x_3 = r \cos \theta .$$

Rewrite the conserved quantities H, J_3, J^2 in terms of the polar coordinates and (p_r, p_θ, p_ϕ) .

4. Without loss of generality, we can rotate our coordinate system such that in a new system \vec{J} has only the third component: $\vec{J} = (0, 0, J_3)$. This can be simply done by setting $\theta = \frac{\pi}{2}$. Kepler's 2nd law states that the areal (sectorial) velocity is constant, and in this situation, it is nothing but the conservation of J_3 because the areal velocity is

$$\frac{dA}{dt} = \frac{1}{2} r^2 \dot{\phi} = \frac{1}{2} J_3 .$$

Under this situation, show that an integral of the Liouville 1-form (0.1) becomes

$$S = \int \alpha = \pm \int^r dr \sqrt{2(H - V) - \frac{J^2}{r^2}} + \int^\phi J_3 d\phi \tag{0.2}$$

where the sign \pm is chosen in such a way that it is consistent with p_r . Derive the equations of motion for the angle variables

$$\psi_H = \frac{\partial S}{\partial H}, \quad \psi_J = \frac{\partial S}{\partial J} .$$

Discuss their physical consequence. In particular, under which condition is an orbit of the motion closed?

5. Let us assume that the potential takes the form

$$V(r) = -\frac{k}{r}.$$

Show the Kepler's 1st law: a planet describes an ellipse with the Sun at one focus. Let T be the revolution period of a planet and a be the major semi-axes of ellipse. Show the Kepler's 3rd law:

$$T = \frac{2\pi}{\sqrt{k}} a^{\frac{3}{2}}.$$

Refer to [Wikipedia page](#) for the terminology.

2. (Harmonic oscillator)

Let us consider a linear system of N particles that are coupled by springs, given by the Hamiltonian:

$$H = \frac{1}{2} \sum_{i=1}^N p_i^2 + \frac{\omega^2}{2} \sum_{i=1}^N (q_i - q_{i+1})^2 \quad \text{where } q_{N+1} = q_1$$

The Poisson brackets for the canonical variables associated with this Hamiltonian are given by:

$$\{q_k, q_l\} = 0, \quad \{p_k, p_l\} = 0, \quad \{p_k, q_l\} = -\delta_{k,l}$$

These brackets determine the time evolution of the system.

1. Write down the Hamiltonian equations as

$$\dot{q} = p, \quad \dot{p} = -\omega^2 A q \quad (0.3)$$

Find the explicit form of the matrix A . Find eigenvalues and eigenvectors of A .

2. For the sake of simplicity, let us set $N = 2M + 1$, and let us define

$$Q_k = \frac{1}{\sqrt{N}} \sum_{i=1}^N e^{\frac{2\pi\sqrt{-1}}{N}ki} q_i \quad P_k = \frac{1}{\sqrt{N}} \sum_{i=1}^N e^{\frac{2\pi\sqrt{-1}}{N}ki} p_i \quad (0.4)$$

where $k = -M, \dots, M$. Write down the Hamiltonian in terms of Q_k and P_k . Solve the Hamiltonian equations.

3. Let us define

$$a_k = \frac{1}{\sqrt{2}} \left(P_{-k} + \sqrt{-1}\omega(k)Q_{-k} \right), \quad \bar{a}_k = \frac{1}{\sqrt{2}} \left(P_k + \sqrt{-1}\omega(k)Q_k \right) \quad (0.5)$$

where $\omega(k) = 2\omega \sin \frac{\pi k}{N} = 2\omega \sin \frac{\pi k}{2M+1}$ ($k = \pm 1, \pm 2, \dots, \pm M$). Write down the Hamiltonian in terms of P_0, a_k, \bar{a}_k . Find the Poisson bracket relations

$$\{a_k, a_l\}, \quad \{a_k, \bar{a}_l\}, \quad \{\bar{a}_k, \bar{a}_l\}. \quad (0.6)$$

Show that the followings are conserved:

$$I_0 = \frac{1}{2} P_0^2, \quad I_k = a_{-k} a_k, \quad I_{-k} = \bar{a}_{-k} \bar{a}_k \quad (0.7)$$

where $k = 1, 2, \dots, M$. Show also that

$$\{I_k, I_l\} = 0, \quad k \neq l. \quad (0.8)$$