

# PROPOSED FORMULA FOR COUNTING MAGMA EQUATIONS

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The symbol  $+$  will be used for *the binary operation* of magma equations. The *order of a magma term* is the number of occurrences of  $+$  in the term. Thus the order of  $(x + y) + (x + z)$  is 3. There is an obvious 1-1 correspondence between magma terms of order  $n$  and plane binary trees of order  $n$  with the leaves labelled by the variables in the term. Let

$$T(n) := \frac{1}{n+1} \cdot \binom{2n}{n}$$

This counts the number of plane binary trees of order  $n$ .

The *order of a magma equation* is the sum of the orders of the two sides. Thus the order of  $(x+y) + (x+z) = z + (x+x)$  is 5. The equation  $(v+w) + (v+u) = u + (v+v)$  obtained by relabelling the variables is considered to be “the same” as the previous equation as far as counting is concerned, as is the equation  $z + (x+x) = (x+y) + (x+z)$  obtained by switching the two terms. The only equation of the form  $t = t$  that will be included in the count is  $x = x$ .

The *Stirling numbers  $\left\{ \begin{smallmatrix} n \\ m \end{smallmatrix} \right\}$  of the second kind* count the number of ways to partition a set of  $n$  elements into  $m$  non-empty subsets.

The number of bijections  $\alpha$  of a set of  $n$  elements to itself is  $n!$ . The number of idempotent bijections  $\alpha$  (that is,  $\alpha^2 = \alpha$ ) is given by

$$\text{Idem}(n) = \sum_{0 \leq k \leq \lfloor n/2 \rfloor} \binom{n}{2k} (2k-1)!!.$$

For integers  $0 \leq a \leq b$  let  $\boxed{E(a,b)}$  be the number of magma equations  $t_1 = t_2$  with  $t_1$  of order  $a$ ,  $t_2$  of order  $b$ , counting up to relabelling, up to switching terms, and only allowing the equation  $x = x$  of the form  $t = t$ .

- $E(0, 0) = 2$ .

- If  $\boxed{a \neq b}$ :

$$E(a, b) = T(a)T(b) \cdot \sum_{\substack{1 \leq p \leq a+1 \\ 1 \leq q \leq b+1 \\ 0 \leq s \leq \min(p, q)}} \left\{ \begin{matrix} a+1 \\ p \end{matrix} \right\} \left\{ \begin{matrix} b+1 \\ q \end{matrix} \right\} \binom{p}{s} \binom{q}{s} s! .$$

- If  $\boxed{a = b > 0}$ :

$$\begin{aligned} E(a, b) = & \frac{1}{2} T(a)^2 \cdot \sum_{\substack{1 \leq p, q \leq a+1 \\ 0 \leq s \leq \min(p, q)}} \left\{ \begin{matrix} a+1 \\ p \end{matrix} \right\} \left\{ \begin{matrix} a+1 \\ q \end{matrix} \right\} \binom{p}{s} \binom{q}{s} s! \\ & + \frac{1}{2} T(a) \cdot \sum_{\substack{1 \leq p \leq a+1 \\ 0 \leq s \leq p}} \left\{ \begin{matrix} a+1 \\ p \end{matrix} \right\} \binom{p}{s} \text{Idem}(s) \\ & - T(a) \cdot \sum_{1 \leq p \leq a+1} \left\{ \begin{matrix} a+1 \\ p \end{matrix} \right\} . \end{aligned}$$

Let  $\text{Eqns}(n)$  be the number of magma equations of order  $n$  (under the given constraints). Then

$$\text{Eqns}(n) = \sum_{0 \leq a \leq \lfloor n/2 \rfloor} E(a, n-a).$$

Let  $\text{Eqns}^*(n)$  be the number of magma equations of order  $\leq n$ . Then

$$\text{Eqns}^*(n) = \sum_{0 \leq k \leq n} \text{Eqns}(k).$$

**Maple calculations** for  $E(a, b)$  with  $0 \leq a \leq 2$ ,  $0 \leq b \leq 5$ , and for  $\text{Eqns}(n)$ ,  $\text{Eqns}^*(n)$  with  $0 \leq n \leq 5$ :

	$n$						$\text{Eqns}(n)$	$\text{Eqns}^*(n)$
$E$	0	1	2	3	4	5		
0	2	5	30	260	2842	36834	0	2
1		9	104	1015	12278	173880	1	5
2			427	8770	115920	1776348	2	39
							3	364
							4	4284
							5	57882
								62576

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