FINITE MAGMAS: RIGIDITY AND ASYMPTOTICS

Abstract. Almost all finite magmas are rigid, and the number of finite magmas, up to isomorphism, is asymptotic to $n^{n^2}/n!$.

This note is essentially a rewrite of parts of Ralph Freese's 1990 paper *On the two kinds of probability in algebra* published in Algebra Universalis 27 (1), 70-79. Here the emphasis is on application to magmas.

1. Count estimates and rigidity

First some definitions:

- $\mathsf{I}_n = \{0, 1, \ldots, n-1\}$.
- M_n is the set of n^{n^2} magmas $\mathbf{A} = (\mathsf{I}_n, \diamond)$ on I_n.
- Sym_n is the symmetric group on I_n.
- $\mathsf{Aut}(\mathbf{A})$ is the group of automorphisms of $\mathbf{A} \in \mathsf{M}_n$, a subgroup of Sym_n.
- A magma $\mathbf{A}$ is rigid if the identity map is its only automorphism.
- R_n is the set of rigid magmas $\mathbf{A} \in \mathsf{M}_n$.
- For $\sigma \in \mathsf{M}_n$ let $\mathsf{M}_n(\sigma)$ be the set of magmas $\mathbf{A} \in \mathsf{M}_n$ such that $\sigma \in \mathsf{Aut}(\mathbf{A})$.

We need two simple estimates for $|\mathsf{M}_n(\sigma)|$.

Lemma 1.1.

- *For σ a transposition in Sym_n,*
$$|\mathsf{M}_n(\sigma)| \leqslant n^{n(n-1)}.$$
- *If $\sigma \in \mathsf{Sym}_n$ is not the identity or a transposition then*
$$|\mathsf{M}_n(\sigma)| \leqslant n^{n(n-2)}.$$

Proof. Let σ be the transposition (ij) and suppose $\sigma \in \mathsf{Aut}(\mathbf{A})$. One can then determine the jth row of the table for $\mathbf{A}$ from σ if one is given the ith row because $j \diamond k = \sigma(\sigma^{-1}(j) \diamond \sigma^{-1}(k)) = \sigma(i \diamond \sigma^{-1}(k))$. The total number of ways to fill in the n^2 entries in the rows of the table other than the jth row, with integers from I_n, is clearly $n^{n(n-1)}$. For each such partial filling of the table one then fills in the jth row using σ—the tables that give an $\mathbf{A}$ for which σ is an automorphism will be among these. So $|\mathsf{M}_n(\sigma)| \leqslant n^{n(n-1)}$.

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If $\sigma \in \mathsf{Sym}_n$ is not the identity or a transposition then given an $\mathbf{A} \in \mathsf{M}_n(\sigma)$ one likewise finds that there are at least two rows of the table that are completely determined by the other rows and σ, so $|\mathsf{M}_n(\sigma)| \leqslant n^{n(n-2)}$. □

A key estimate used to prove the main results is the following lemma.

Lemma 1.2.

$$|\mathsf{M}_n| \;\leqslant\; \sum_{\mathbf{A}\in\mathsf{M}_n} |\mathsf{Aut}(\mathbf{A})| \;\leqslant\; |\mathsf{R}_n| + 2\binom{n}{2} n^{(n-1)n} + (n!)^2 \cdot n^{(n-2)n}.$$

Proof. The first inequality is trivial since $1 \leqslant |\mathsf{Aut}(\mathbf{A})|$. For the second inequality break the middle term, the sum over $\mathbf{A} \in \mathsf{M}_n$, into three sums, the first for the $\mathbf{A}$ with $|\mathsf{Aut}(\mathbf{A})| = 1$, the second for $|\mathsf{Aut}(\mathbf{A})| = 2$, and the third for $|\mathsf{Aut}(\mathbf{A})| > 2$.

The first of these sums is just $|\mathsf{R}_n|$ since $|\mathsf{Aut}(\mathbf{A})| = 1$ is precisely the condition for $\mathbf{A}$ to be rigid.

The second sum has $|\mathsf{Aut}(\mathbf{A})| = 2$ which means $\mathsf{Aut}(\mathbf{A})$ consists of a transposition σ and the identity map. The collection of $\mathbf{A} \in \mathsf{M}_n$ with $|\mathsf{Aut}(\mathbf{A})| = 2$ is contained in the union of the $\mathsf{M}_n(\sigma)$ where σ ranges over the transpositions. There are $\binom{n}{2}$ transpositions σ in Sym_n, and for each transposition σ we have $|\mathsf{M}_n(\sigma)| \leqslant n^{n(n-1)}$.

For the third sum note that it is at most $n!$ times the number of $\mathbf{A} \in \mathsf{M}_n$ which have $|\mathsf{Aut}(\mathbf{A})| > 2$. This collection of $\mathbf{A}$ is contained in the union of the $\mathsf{M}_n(\sigma)$ where the σ in Sym_n range over elements that are not the identity or a transposition. For each such σ we have $|\mathsf{M}_n(\sigma)| \leqslant n^{n(n-2)}$. The number of such σ is clearly at most $n!$, thus the number of $\mathbf{A} \in \mathsf{M}_n$ which have $|\mathsf{Aut}(\mathbf{A})| > 2$ is at most $n! \cdot n^{n(n-2)}$. Combining this with the first line of this paragraph gives the desired estimate for the third sum. □

Theorem 1.1.

$$\lim_{n\to\infty} \frac{|\mathsf{R}_n|}{|\mathsf{M}_n|} \;=\; 1$$

and

$$\lim_{n\to\infty} \frac{1}{|\mathsf{M}_n|} \sum_{\mathbf{A}\in\mathsf{M}_n} |\mathsf{Aut}(\mathbf{A})| \;=\; 1.$$

Proof. Dividing the statement in Lemma 1.2 by $|\mathsf{M}_n|$ which is n^{n^2} gives

$$1 \leqslant \frac{1}{|\mathsf{M}_n|} \sum_{\mathbf{A}\in\mathsf{M}_n} |\mathsf{Aut}(\mathbf{A})| \leqslant \frac{|\mathsf{R}_n|}{|\mathsf{M}_n|} + 2\binom{n}{2} n^{-n} + (n!)^2 \cdot n^{-2n}.$$

The last two terms on the right go to 0 as $n \to \infty$, and $\dfrac{|\mathsf{R}_n|}{|\mathsf{M}_n|} \leqslant 1$. □

The first assertion in the Theorem can be read as *almost all finite magmas are rigid.* (In 1975 Murskii proved a stronger result, that almost all magmas are idemprimal, but that is not needed for this note.)

2. The asymptotics for counting magmas up to isomorphism

Four more definitions:

- $E_n = \dfrac{1}{|\mathsf{M}_n|} \sum_{\mathbf{A} \in \mathsf{M}_n} |\mathsf{Aut}(\mathbf{A})|$. [Note: $E_n \to 1$ by Theorem 1.1]
- Given $\mathbf{A} = (\mathsf{I}_n, \diamond)$ in M_n and $\sigma \in \mathsf{Sym}_n$ let $\mathbf{A}_\sigma$ be the magma $(\mathsf{I}_n, \diamond_\sigma)$ where $\diamond_\sigma$ is defined by

$$i \diamond_\sigma j = \sigma\big(\sigma^{-1}(i) \diamond \sigma^{-1}(j)\big).$$

- Given $\mathbf{A} \in \mathsf{M}_n$ let $\widehat{\mathbf{A}}$ be the set of $\mathbf{B} \in \mathsf{M}_n$ that are isomorphic to $\mathbf{A}$.
- For $\mathsf{P}_n \subseteq \mathsf{M}_n$ let $\widehat{\mathsf{P}}_n$ be the set of $\widehat{\mathbf{A}}$ for $\mathbf{A} \in \mathsf{P}_n$.

The goal of this section is first to prove that for $\mathsf{P}_n \subseteq \mathsf{M}_n$ and P_n closed under isomorphism, $\lim \dfrac{|\mathsf{P}_n|}{|\mathsf{M}_n|}$ exists iff $\lim \dfrac{|\widehat{\mathsf{P}}_n|}{|\widehat{\mathsf{M}}_n|}$ exists, and if so then both limits are equal. Then, with P_n being R_n, Theorem 1.1 quickly proves that $|\widehat{\mathsf{M}}_n|$ is asymptotic to $|\mathsf{M}_n|/n!$, that is to $n^{n^2}/n!$ But first some basic facts.

Lemma 2.1. *For* $\mathbf{A} \in \mathsf{M}_n$

(1) $\widehat{\mathbf{A}} \;=\; \{\mathbf{A}_\sigma : \sigma \in \mathsf{Sym}_n\}$
(2) $|\widehat{\mathbf{A}}| \;=\; \dfrac{n!}{|\mathsf{Aut}(\mathbf{A})|}$
(3) $|\widehat{\mathbf{A}}| \leqslant n!$, *with equality iff* $\mathbf{A} \in \mathsf{R}_n$
(4) $|\mathsf{P}_n| \;\leqslant\; n! \cdot |\widehat{\mathsf{P}}_n|$ *for* $\mathsf{P}_n \subseteq \mathsf{M}_n$
(5) $|\mathsf{R}_n| \;=\; n! \cdot |\widehat{\mathsf{R}}_n|$
(6) $\sum_{\mathbf{A} \in \mathsf{M}_n} |\mathsf{Aut}(\mathbf{A})| \;=\; n! \cdot |\widehat{\mathsf{M}}_n|$

Proof.

For (1): $\mathbf{A}_\sigma$ is $\mathbf{A}$ when σ is the identity map, and for all $\sigma \in \mathsf{Sym}_n$ we have σ being an isomorphism from $\mathbf{A}$ to $\mathbf{A}_\sigma$.

If σ is an isomorphism from $\mathbf{A}$ to $\mathbf{B} \in \mathsf{M}_n$ then $\mathbf{B} = \mathbf{A}_\sigma$. Thus $\widehat{\mathbf{A}}$ is the set of all $\mathbf{A}_\sigma$, $\sigma \in \mathsf{Sym}_n$.

For (2): The $\mathbf{A}_\sigma$ may not all be distinct magmas, indeed it is routine to show that $\mathbf{A}_\sigma = \mathbf{A}_\tau$ iff $\sigma^{-1}\tau \in \mathsf{Aut}(\mathbf{A})$ for $\sigma, \tau \in \mathsf{Sym}_n$.

Thus $|\widehat{\mathbf{A}}|$, the number of distinct $\mathbf{A}_\sigma$, equals the number of cosets of $\mathsf{Aut}(\mathbf{A})$ in the group Sym_n, which is $n!/|\mathsf{Aut}(\mathbf{A})|$.

For (3): Immediate from (1).

For (4): $\mathsf{P}_n \subseteq \bigcup_{\widehat{\mathbf{A}} \in \widehat{\mathsf{P}}_n} \widehat{\mathbf{A}}$ and $|\widehat{\mathbf{A}}| \leqslant n!$ for $\mathbf{A} \in \mathsf{M}_n$ gives

$$|\mathsf{P}_n| \;\leqslant\; \sum_{\widehat{\mathbf{A}} \in \widehat{\mathsf{P}}_n} |\widehat{\mathbf{A}}| \;\leqslant\; n! \cdot |\widehat{\mathsf{P}}_n|$$

For (5): $\mathsf{R}_n = \bigcup_{\widehat{\mathbf{A}} \in \widehat{\mathsf{R}}_n} \widehat{\mathbf{A}}$, a disjoint union, and $|\widehat{\mathbf{A}}| = n!$ for $\mathbf{A} \in \mathsf{R}_n$ gives

$$|\mathsf{R}_n| \;=\; \sum_{\widehat{\mathbf{A}} \in \widehat{\mathsf{R}}_n} |\widehat{\mathbf{A}}| \;=\; n! \cdot |\widehat{\mathsf{R}}_n|$$

For (6): Note that

$$\begin{aligned}
\sum_{\mathbf{A} \in \mathsf{M}_n} |\mathsf{Aut}(\mathbf{A})| \;&=\; \sum_{\widehat{\mathbf{A}} \in \widehat{\mathsf{M}}_n} \sum_{\mathbf{B} \in \widehat{\mathbf{A}}} |\mathsf{Aut}(\mathbf{B})| \\
&=\; \sum_{\widehat{\mathbf{A}} \in \widehat{\mathsf{M}}_n} |\widehat{\mathbf{A}}| \cdot |\mathsf{Aut}(\mathbf{A})| \\
&=\; \sum_{\widehat{\mathbf{A}} \in \widehat{\mathsf{M}}_n} \frac{n!}{|\mathsf{Aut}(\mathbf{A})|} |\mathsf{Aut}(\mathbf{A})| \\
&=\; n! \cdot |\widehat{\mathsf{M}}_n|
\end{aligned}$$

□

Lemma 2.2. *For $\mathsf{P}_n \subseteq \mathsf{M}_n$ with P_n closed under isomorphism*

$$\frac{|\widehat{\mathsf{P}}_n|}{|\widehat{\mathsf{M}}_n|} \;\geqslant\; \frac{1}{E_n} \frac{|\mathsf{P}_n|}{|\mathsf{M}_n|} \tag{1}$$

$$1 - \frac{|\widehat{\mathsf{P}}_n|}{|\widehat{\mathsf{M}}_n|} \;\geqslant\; \frac{1}{E_n}\Big(1 - \frac{|\mathsf{P}_n|}{|\mathsf{M}_n|}\Big) \tag{2}$$

$$\lim_{n\to\infty} \frac{|\mathsf{P}_n|}{|\mathsf{M}_n|} \text{ exists iff } \lim_{n\to\infty} \frac{|\widehat{\mathsf{P}}_n|}{|\widehat{\mathsf{M}}_n|} \text{ exists. If so, they are equal.} \tag{3}$$

Proof.

For (1): Since

$$n! \cdot |\widehat{\mathsf{P}}_n| \geqslant |\mathsf{P}_n| \quad \text{and} \quad n! \cdot |\widehat{\mathsf{M}}_n| \;=\; \sum_{\mathbf{A} \in \mathsf{M}_n} |\mathsf{Aut}(\mathbf{A})|$$

we have

$$\frac{|\widehat{\mathsf{P}}_n|}{|\widehat{\mathsf{M}}_n|} \;\geqslant\; \frac{|\mathsf{P}_n|}{\sum_{\mathbf{A} \in \mathsf{M}_n} |\mathsf{Aut}(\mathbf{A})|} \;=\; \frac{|\mathsf{P}_n|}{|\mathsf{M}_n|} \cdot \frac{|\mathsf{M}_n|}{\sum_{\mathbf{A} \in \mathsf{M}_n} |\mathsf{Aut}(\mathbf{A})|} \;=\; \frac{1}{E_n} \cdot \frac{|\mathsf{P}_n|}{|\mathsf{M}_n|}.$$

For (2): Replace P_n by $\mathsf{M}_n \backslash \mathsf{P}_n$ in the previous steps for proving (1).

Then $\widehat{\mathsf{M}_n \backslash \mathsf{P}_n} = \widehat{\mathsf{M}}_n \backslash \widehat{\mathsf{P}}_n$ gives $|\widehat{\mathsf{M}_n \backslash \mathsf{P}_n}| = |\widehat{\mathsf{M}}_n| - |\widehat{\mathsf{P}}_n|$, and thus

$$1 - \frac{|\widehat{\mathsf{P}}_n|}{|\widehat{\mathsf{M}}_n|} \geqslant \frac{1}{E_n} \cdot \big(1 - \frac{|\mathsf{P}_n|}{|\mathsf{M}_n|}\big).$$

<u>For (3):</u> Combining (1) and (2) gives

$$\frac{1}{E_n} \cdot \frac{|\mathsf{P}_n|}{|\mathsf{M}_n|} \ \leqslant\ \frac{|\widehat{\mathsf{P}}_n|}{|\widehat{\mathsf{M}}_n|} \ \leqslant\ 1 - \frac{1}{E_n} \cdot \big(1 - \frac{|\mathsf{P}_n|}{|\mathsf{M}_n|}\big).$$

Since $E_n \to 1$, if $\dfrac{|\mathsf{P}_n|}{|\mathsf{M}_n|} \to \lambda$ then $\dfrac{|\widehat{\mathsf{P}}_n|}{|\widehat{\mathsf{M}}_n|} \to \lambda$.

Combining (1) and (2) in a different way gives

$$1 - E_n \cdot \big(1 - \frac{|\widehat{\mathsf{P}}_n|}{|\widehat{\mathsf{M}}_n|}\big) \ \leqslant\ \frac{|\mathsf{P}_n|}{|\mathsf{M}_n|} \ \leqslant\ E_n \cdot \frac{|\widehat{\mathsf{P}}_n|}{|\widehat{\mathsf{M}}_n|}.$$

Again since $E_n \to 1$, if $\dfrac{|\widehat{\mathsf{P}}_n|}{|\widehat{\mathsf{M}}_n|} \to \lambda$ then $\dfrac{|\mathsf{P}_n|}{|\mathsf{M}_n|} \to \lambda$. □

Theorem 2.1. $|\widehat{\mathsf{M}}_n|$ *is asymptotic to* $\dfrac{|\mathsf{M}_n|}{n!}$, *that is to* $\dfrac{n^{n^2}}{n!}$.

Proof. Since

$$\frac{|\mathsf{R}_n|}{|\mathsf{M}_n|} \to 1, \quad \text{and therefore} \quad \frac{|\widehat{\mathsf{R}}_n|}{|\widehat{\mathsf{M}}_n|} \to 1,$$

it follows that

$$\frac{|\mathsf{R}_n|}{|\mathsf{M}_n|} \cdot \frac{|\widehat{\mathsf{M}}_n|}{|\widehat{\mathsf{R}}_n|} \to 1.$$

This gives

$$\frac{|\widehat{\mathsf{M}}_n|}{|\mathsf{M}_n|} \cdot \frac{|\mathsf{R}_n|}{|\widehat{\mathsf{R}}_n|} \to 1, \text{ hence } |\widehat{\mathsf{M}}_n| \cdot \frac{n!}{|\mathsf{M}_n|} \to 1.$$

□