A GREEDY CONSTRUCTION IN SKOLEM'S 1920 PAPER

1. Introduction

In the not so well known second section[1] of Skolem's 1920 paper [1] he gives an algorithm to determine the universally quantified first-order sentences that hold in Schröder's Gruppenkalkül. (The Gruppenkalkül was introduced in the first volume of Schröder's famous three-volume *Algebra der Logik*.) From the modern viewpoint the Gruppenkalkül is just a version of what is now called lattice theory. Skolem's version of the Gruppenkalkül is yet another version of modern lattice theory.

2. Skolem's Gruppenkalkül

Skolem axiomatized Schröder's Gruppenkalkül (lattice theory) in a relational language with two ternary relations and one binary relation, but without equality as follows:

I. $x \leqslant x$ for every x

II. $x \leqslant y$ and $y \leqslant z$ imply $x \leqslant z$

$\mathrm{III}_{\times}$.	$Mxyz$ implies $z \leqslant x$ and $z \leqslant y$	III_{+}.	$Jxyz$ implies $x \leqslant z$ and $y \leqslant z$
$\mathrm{IV}_{\times}$.	$Mxyz$ and $u \leqslant x$ and $u \leqslant y$ imply $u \leqslant z$	IV_{+}.	$Jxyz$ and $x \leqslant u$ and $y \leqslant u$ imply $z \leqslant u$
$\mathrm{V}_{\times}$.	$Mxyz$ and $x \leqslant x'$ and $x' \leqslant x$ and $y \leqslant y'$ and $y' \leqslant y$ and $z \leqslant z'$ and $z' \leqslant z$ imply $Mx'y'z'$	V_{+}.	$Jxyz$ and $x \leqslant x'$ and $x' \leqslant x$ and $y \leqslant y'$ and and $y' \leqslant y$ and $z \leqslant z'$ and $z' \leqslant z$ imply $Jx'y'z'$
$\mathrm{VI}_{\times}$.	for all x and y there is a z such that $Mxyz$	VI_{+}.	for all x and y there is a z such that $Jxyz$

Definition 2.1. *A structure* $\mathbf{S} = (S, M, J, \leqslant)$*, where* M *and* J *are ternary relations on* S*, and* $\leqslant$ *is a binary relation on* S*, will be called a* **Skolem structure***.*[2] *A Skolem structure which satisfies Axioms I–VI will be called a* **Skolem lattice***.*

3. Skolem's Algorithm

Skolem views axioms I–V as *production* rules that can only augment the three relations.

Given a finite Skolem structure $\mathbf{S} = (S, M, J, \leqslant)$ he first applies I–V repeatedly until the three relations on S no longer increase. (This a polynomial time procedure.)

Date: May 25, 2025.

[1] *Lösung des Problems zu entscheiden, ob eine gegebene Aussage des Gruppenkalkuls beweisbar ist oder nicht.*

[2] Skolem used the notations (ab), $\overset{\frown}{abc}$ and $\underset{\smile}{abc}$ instead of $a \leqslant b$, $Mabc$ and $Jabc$ used here. In Schröder's *Algebra der Logik* one finds the corresponding notations to be $a ⋹ b$, $ab = c$ and $a + b = c$.

Suppose at this point VI does not hold, say $a, b \in S$ but there is no $c \in S$ such that $Mabc$. He adds a new element, say α, to S, adds $Mab\alpha$, adds $\alpha \leqslant \alpha$, and describes which $s \leqslant \alpha$ and which $\alpha \leqslant s$ to add, where $s \in S$. The bulk of his argument is to show that I–V still hold, and that $\leqslant$ has not increased on the set S.

A similar 1-element extension is used if there are $a, b \in S$ but there is no $c \in S$ such that $Jabc$. By repeated application of these procedures a tower of 1-element extensions can be created whose union satisfies I–VI, with no further augmentation of $\leqslant$ on S since the initial closure under I–V.

The augmentation of $\leqslant$ on S by I–V was Skolem's algorithm to determine for $a, b \in S$ if the information in **S** and the lattice axioms force $a \leqslant b$.

For an example (now using modern notation) he applies this to the two halves

$$a \wedge (b \vee c) \leqslant (a \wedge b) \vee (a \wedge c) \tag{3.1}$$

$$(a \wedge b) \vee (a \wedge c) \leqslant a \wedge (b \vee c) \tag{3.2}$$

of the distributive law. The associated Skolem structure **S** on the set $S = \{a, b, c, d, e, f, g, h\}$ of eight elements is the following:

M	J	$\leqslant$
abd	bcf	
ace	deg	
afh		

Closing **S** under the production rules I–V gives 21 pairs in $\leqslant$ which include gh but not hg. Thus 3.2 holds in lattices, but not 3.1. Consequently the distributive law

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

does not hold in lattices.[3]

Reference

(1) Skolem, Thoralf. *Logisch-kombinatorische Untersuchungen über die Erfüllbarkeit oder Beweisbarkeit mathematischer Sätze nebst einem Theoreme über dichte Mengen.* Skrifter utgitt av Det Norske Videnskaps-Akademi i Oslo, I. Matematisk-naturvidenskabelig Klasse, 1920, no. 4.

Also in "Selected Works in Logic by Th. Skolem", ed. by Jens Erik Fenstad, Scand. Univ. Books, Universitetsforlaget, Oslo, 1970, pp. 103–136. [The first section is translated in: From Frege to Gödel, van Heijenoort, Harvard Univ. Press, 1971, 252–263.]

[3]Originally proved by Schröder in Vol. I of *Algebra der Logik* in a rather tedious manner, using 990 quasi-group equations, to refute a claim of C.S. Peirce a decade earlier that the distributive law held in lattices. Soon after this Dedekind gave a simple proof in his paper on Dualgruppen, a very modern version of lattices, by exhibiting a 5-element lattice that did not satisfy the distributive law. It is not clear if Skolem was aware of Dedekind's proof from the 1890s.