PROPOSED FORMULA FOR COUNTING MAGMA EQUATIONS

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The symbol $+$ will be used for *the binary operation* of magma equations. The *order of a magma term* is the number of occurrences of $+$ in the term. Thus the order of $(x+y)+(x+z)$ is 3. There is an obvious 1-1 correspondence between magma terms of order n and plane binary trees of order n with the leaves labelled by the variables in the term. Let

$$T(n) := \frac{1}{n+1} \cdot \binom{2n}{n}$$

This counts the number of plane binary trees of order n.

The *order of a magma equation* is the sum of the orders of the two sides. Thus the order of $(x+y)+(x+z) = z+(x+x)$ is 5. The equation $(v+w)+(v+u) = u+(v+v)$ obtained by relabelling the variables is considered to be "the same" as the previous equation as far as counting is concerned, as is the equation $z+(x+x) = (x+y)+(x+z)$ obtained by switching the two terms. The only equation of the form $t = t$ that will be included in the count is $x = x$.

The *Stirling numbers* $\left\{ {n \atop m} \right\}$ *of the second kind* count the number of ways to partition a set of n elements into m non-empty subsets.

The number of bijections α of a set of n elements to itself is $n!$. The number of idempotent bijections α (that is, $\alpha^2 = \alpha$) is given by

$$\mathsf{Idem}(n) = \sum_{0 \leqslant k \leqslant \lfloor n/2 \rfloor} \binom{n}{2k} (2k-1)!!.$$

For integers $0 \leqslant a \leqslant b$ let $\boxed{\text{E(a,b)}}$ be the number of magma equations $t_1 = t_2$ with t_1 of order a, t_2 of order b, counting up to relabelling, up to switching terms, and only allowing the equation $x = x$ of the form $t = t$.

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- $E(0,0) = 2$.

- If $\boxed{a \neq b}$:

$$E(a,b) = \frac{1}{2}T(a)(T(a)-1) \cdot \sum_{\substack{1 \leqslant p \leqslant a+1 \\ 1 \leqslant q \leqslant b+1 \\ 0 \leqslant s \leqslant \min(p,q)}} \left\{ {a+1 \atop p} \right\} \left\{ {b+1 \atop q} \right\} \binom{p}{s} \binom{q}{s} s!\ .$$

- If $\boxed{a = b > 0}$:

$$\begin{aligned} E(a,b) = & \frac{1}{2}T(a)^2 \cdot \sum_{\substack{1 \leqslant p,q \leqslant a+1 \\ 0 \leqslant s \leqslant \min(p,q)}} \left\{ {a+1 \atop p} \right\} \left\{ {a+1 \atop q} \right\} \binom{p}{s} \binom{q}{s} s! \\ & + \frac{1}{2}T(a) \cdot \sum_{\substack{1 \leqslant p \leqslant a+1 \\ 0 \leqslant s \leqslant p}} \left\{ {a+1 \atop p} \right\} \binom{p}{s} \mathsf{Idem}(s) \\ & - T(a) \cdot \sum_{1 \leqslant p \leqslant a+1} \left\{ {a+1 \atop p} \right\}\ . \end{aligned}$$

Let $\mathsf{Eqns}(n)$ be the number of magma equations of order n (under the given constraints). Then

$$\mathsf{Eqns}(n) = \sum_{0 \leqslant a \leqslant \lfloor n/2 \rfloor} E(a, n-a).$$

Let $\mathsf{Eqns}^\star(n)$ be the number of magma equations of order $\leqslant n$. Then

$$\mathsf{Eqns}^\star(n) = \sum_{0 \leqslant k \leqslant n} \mathsf{Eqns}(k).$$

Maple calculations for $E(a,b)$ with $0 \leqslant a \leqslant 2$, $0 \leqslant b \leqslant 5$, and for $Eqns(n)$, $Eqns^\star(n)$ with $0 \leqslant n \leqslant 5$:

E	0	1	2	3	4	5
0	2	5	30	260	2842	36834
1		9	104	1015	12278	173880
2			427	8770	115920	1776348

n	$\mathsf{Eqns}(n)$	$\mathsf{Eqns}^\star(n)$
0	2	2
1	5	7
2	39	46
3	364	410
4	4284	4694
5	57882	62576

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