COUNTING MAGMA EQUATIONS

1. The formula

The symbol $\diamond$ will be used for *the binary operation* of magma equations. The *order of a magma term* is the number of occurrences of $\diamond$ in the term. Thus the order of $(x \diamond y) \diamond (x \diamond z)$ is 3. There is an obvious mapping from terms of order n to plane binary trees of order n. (Additionally the leaves can be naturally labelled by the variables in the term to make the mapping a bijection.) Let

$$T(n) = \frac{1}{n+1} \cdot \binom{2n}{n} \, .$$

This counts the number of plane binary trees of order n.

The *order of a magma equation* is the sum of the orders of the two sides. Thus the order of $(x \diamond y) \diamond (x \diamond z) \approx z \diamond (x \diamond x)$ is 5. The equation $(v \diamond w) \diamond (v \diamond u) \approx u \diamond (v \diamond v)$ obtained by relabelling the variables is considered to be "the same" as the previous equation as far as counting is concerned, as is the equation $z \diamond (x \diamond x) \approx (x \diamond y) \diamond (x \diamond z)$ obtained by switching the two terms.

The *Stirling numbers* $\left\{ {n \atop m} \right\}$ *of the second kind* count the number of ways to partition a set of n elements into m non-empty subsets.

The number of bijections β of a set of n elements to itself is $n!$. The number of involutions β (that is, $\beta = \beta^{-1}$) is given by

$$\mathsf{Invol}(n) = \sum_{0 \leqslant k \leqslant \lfloor n/2 \rfloor} \binom{n}{2k} (2k-1)!!.$$

For non-negative integers a_1, a_2 let $\boxed{E(a_1, a_2)}$ be the number of magma equations $t_1 \approx t_2$ with t_i of order a_i, counting up to relabelling, up to switching terms, and only allowing the equation $x \approx x$ of the form $t \approx t$. Then the following holds:

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- $E(0,0) = 2$.

- If $\boxed{a_1 \neq a_2}$:

$$E(a_1, a_2) = T(a_1) \cdot T(a_2) \cdot \sum_{\substack{1 \leqslant p_i \leqslant a_i + 1 \\ 0 \leqslant s \leqslant \min(p_1, p_2)}} \left\{ {a_1 + 1 \atop p_1} \right\} \left\{ {a_2 + 1 \atop p_2} \right\} \binom{p_1}{s} \binom{p_2}{s} s!\,.$$

- If $\boxed{a_1 = a_2 = a > 0}$:

$$\begin{aligned} E(a,a) = {} & \frac{1}{2} T(a)^2 \cdot \sum_{\substack{1 \leqslant p_1, p_2 \leqslant a+1 \\ 0 \leqslant s \leqslant \min(p_1, p_2)}} \left\{ {a + 1 \atop p_1} \right\} \left\{ {a + 1 \atop p_2} \right\} \binom{p_1}{s} \binom{p_2}{s} s! \\ & + \frac{1}{2} T(a) \cdot \sum_{\substack{1 \leqslant p \leqslant a+1 \\ 0 \leqslant s \leqslant p}} \left\{ {a + 1 \atop p} \right\} \binom{p}{s} \mathsf{Invol}(s) \\ & - T(a) \cdot \sum_{1 \leqslant p \leqslant a+1} \left\{ {a + 1 \atop p} \right\}. \end{aligned}$$

Let $\mathsf{Eqns}(n)$ be the number of magma equations of order n (under the given constraints). Then

$$\mathsf{Eqns}(n) = \sum_{0 \leqslant a \leqslant \lfloor n/2 \rfloor} E(a, n-a).$$

Let $\mathsf{Eqns}^\star(n)$ be the number of magma equations of order $\leqslant n$. Then

$$\mathsf{Eqns}^\star(n) = \sum_{0 \leqslant k \leqslant n} \mathsf{Eqns}(k).$$

Maple calculations for $E(a_1, a_2)$ with $0 \leqslant a_1 \leqslant 2$, $0 \leqslant a_2 \leqslant 5$, and for $\mathsf{Eqns}(n)$, $\mathsf{Eqns}^\star(n)$ with $0 \leqslant n \leqslant 5$:

E	0	1	2	3	4	5
0	2	5	30	260	2842	36834
1		9	104	1015	12278	173880
2			427	8770	115920	1776348

n	$\mathsf{Eqns}(n)$	$\mathsf{Eqns}^\star(n)$
0	2	2
1	5	7
2	39	46
3	364	410
4	4284	4694
5	57882	62576

2. The Proof

A *frame* φ is a 7-tuple $(T_1, \pi_1, S_1, \beta, T_2, \pi_2, S_2)$ where

- the T_i are plane binary trees,
- π_i is a partition of the set $\{1, \ldots, a_i + 1\}$ where a_i is the order of T_i which is $(|T_i| - 1)/2$,
- S_i is a subset of $\{1, \ldots, p_i\}$ where $p_i = |\pi_i|$, the number of blocks in π_i,
- $|S_1| = |S_2|$, and
- $\beta : S_1 \to S_2$ is a bijection.

The frame $\Phi(t_1 \approx t_2)$ of a magma equation $t_1 \approx t_2$ is defined as follows:

- T_i is the plane binary tree associated with t_i.
- The variables of t_i have locations $1, \ldots, a_i + 1$, counting from the left.
- π_i is the partition of the set $\{1, \ldots, a_i + 1\}$ defined by: j and k are in the same block of π_i iff the variables of t_i at the locations j and k are the same.

  The variable associated to a block of π_i is the variable at all the locations in the block.
- Number the blocks of π_i by $1, \ldots, p_i$ such that for $1 \leqslant j < k \leqslant p_i$ the least member of the jth block is less than the least member of the kth block.
- S_i is the set of block numbers from $\{1, \ldots, p_i\}$ whose associated variables in t_i appear on both sides of the equation.
- For $j \in S_1$, $\beta(j)$ is the number of the block in S_2 whose associated variable is the same as that associated to block number j in S_1.

FACT: Two magma equations $t_1 \approx t_2$ and $t_1' \approx t_2'$ are the same up to a relabelling of the variables iff their frames are equal.

FACT: Every frame φ is the frame of a magma equation.

These facts allow us to compute the number of magma equations up to a relabelling of the variables by counting frames.

Let $\Phi(a_1, a_2)$ be the collection of frames with the order of T_i equal a_i. Then, up to relabelling, the number of magma equations $t_1 \approx t_2$

with $\mathrm{order}(t_i) = a_i$ is $|\Phi(a_1, a_2)|$, which is

$$T(a_1) \cdot T(a_2) \cdot \sum_{\substack{1 \leqslant p_1 \leqslant a_1+1 \\ 1 \leqslant p_2 \leqslant a_2+1 \\ 0 \leqslant s \leqslant \min(p_1,p_2)}} \left\{ {a_1 + 1 \atop p_1} \right\} \left\{ {a_2 + 1 \atop p_2} \right\} \binom{p_1}{s} \binom{p_2}{s} s! \ . \tag{1}$$

Counting equations up to relabelling and symmetry means that one treats $t_1 \approx t_2$ and $t_2 \approx t_1$ as a single equation. This means that when their frames are different only one frame is counted. To accomplish this when counting all equations of order n (that is, when $a_1 + a_2 = n$) the first step is simply to count only one of $\Phi(a_1, a_2)$ and $\Phi(a_2, a_1)$ when $a_1 \neq a_2$, say only count $\Phi(a_1, a_2)$ when $a_1 < a_2$.

The case $a_1 = a_2 = a$, which only occurs when n is even, requires more consideration since the frames of $t_1 \approx t_2$ and $t_2 \approx t_1$ are both in $\Phi(a, a)$, and sometimes they are the same, sometimes they are distinct. Given

$$\Phi(t_1 \approx t_2) = (T_1, \pi_1, S_1, \beta, T_2, \pi_2, S_2)$$

one has

$$\Phi(t_2 \approx t_1) = (T_2, \pi_2, S_2, \beta^{-1}, T_1, \pi_1, S_1).$$

The two equations have the same frame iff

$$T_1 = T_2, \ \pi_1 = \pi_2, \ S_1 = S_2, \ \beta = \beta^{-1}.$$

This means β is an involution. This leads to the count, up to relabelling and symmetry, of equations with frames in $\Phi(a, a)$ being

$$\frac{1}{2} T(a)^2 \cdot \sum_{\substack{1 \leqslant p_1, p_2 \leqslant a+1 \\ 0 \leqslant s \leqslant \min(p_1,p_2)}} \left\{ {a + 1 \atop p_1} \right\} \left\{ {a + 1 \atop p_2} \right\} \binom{p_1}{s} \binom{p_2}{s} s!$$
$$+ \frac{1}{2} T(a) \cdot \sum_{\substack{1 \leqslant p \leqslant a+1 \\ 0 \leqslant s \leqslant p}} \left\{ {a + 1 \atop p} \right\} \binom{p}{s} \mathsf{Invol}(s) \ .$$

To avoid trivial equations $t \approx t$, from this one needs to subtract

$$T(a) \cdot \sum_{1 \leqslant p \leqslant a+1} \left\{ {a + 1 \atop p} \right\} .$$

To then include one trivial equation in the count, namely $x \approx x$, only use this subtraction for $a > 0$.