

PROPOSED FORMULA FOR COUNTING MAGMA EQUATIONS

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The symbol $+$ will be used for *the binary operation* of magma equations. The *order of a magma term* is the number of occurrences of $+$ in the term. Thus the order of $(x + y) + (x + z)$ is 3. There is an obvious 1-1 correspondence between magma terms of order n and plane binary trees of order n with the leaves labelled by the variables in the term. Let

$$T(n) := \frac{1}{n+1} \cdot \binom{2n}{n}$$

This counts the number of plane binary trees of order n .

The *order of a magma equation* is the sum of the orders of the two sides. Thus the order of $(x + y) + (x + z) = z + (x + x)$ is 5. The equation $(v + w) + (v + u) = u + (v + v)$ obtained by relabelling the variables is considered to be “the same” as the previous equation as far as counting is concerned, as is the equation $z + (x + x) = (x + y) + (x + z)$ obtained by switching the two terms. The only equation of the form $t = t$ that will be included in the count is $x = x$.

The *Stirling numbers $\left\{ \begin{smallmatrix} n \\ m \end{smallmatrix} \right\}$ of the second kind* count the number of ways to partition a set of n elements into m non-empty subsets.

The number of bijections α of a set of n elements to itself is $n!$. The number of idempotent bijections α (that is, $\alpha^2 = \alpha$) is given by

$$\text{Idem}(n) = \sum_{0 \leq k \leq \lfloor n/2 \rfloor} \binom{n}{2k} (2k-1)!!.$$

For integers $0 \leq a \leq b$ let $\boxed{\text{E}(a,b)}$ be the number of magma equations $t_1 = t_2$ with t_1 of order a , t_2 of order b , counting up to relabelling, up to switching terms, and only allowing the equation $x = x$ of the form $t = t$.

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- If $\boxed{a \neq b}$:

$$E(a, b) = \frac{1}{2}T(a)(T(a) - 1) \cdot \sum_{\substack{1 \leq p \leq a+1 \\ 1 \leq q \leq b+1 \\ 0 \leq s \leq \min(p, q)}} \left\{ \begin{matrix} a+1 \\ p \end{matrix} \right\} \left\{ \begin{matrix} b+1 \\ q \end{matrix} \right\} \binom{p}{s} \binom{q}{s} s! .$$

- If $\boxed{a = b}$:

$$\begin{aligned} E(a, b) = & \frac{1}{2}T(a)^2 \cdot \sum_{\substack{1 \leq p, q \leq a+1 \\ 0 \leq s \leq \min(p, q)}} \left\{ \begin{matrix} a+1 \\ p \end{matrix} \right\} \left\{ \begin{matrix} a+1 \\ q \end{matrix} \right\} \binom{p}{s} \binom{q}{s} s! \\ & + \frac{1}{2}T(a) \cdot \sum_{\substack{1 \leq p \leq a+1 \\ 0 \leq s \leq p}} \left\{ \begin{matrix} a+1 \\ p \end{matrix} \right\} \binom{p}{s} \text{Idem}(s) \\ & - T(a) \cdot \sum_{1 \leq p \leq a+1} \left\{ \begin{matrix} a+1 \\ p \end{matrix} \right\} . \end{aligned}$$

Let $\text{Eqns}(n)$ be the number of magma equations of order n (under the given constraints). Then

$$\text{Eqns}(n) = \sum_{0 \leq a \leq \lfloor n/2 \rfloor} E(a, n-a).$$

Let $\text{Eqns}^*(n)$ be the number of magma equations of order $\leq n$. Then

$$\text{Eqns}^*(n) = \sum_{0 \leq k \leq n} \text{Eqns}(k).$$

Maple calculations for $E(a, b)$ with $0 \leq a \leq 2$, $0 \leq b \leq 5$, and for $\text{Eqns}(n)$, $\text{Eqns}^*(n)$ with $0 \leq n \leq 5$:

E	0	1	2	3	4	5	n	$\text{Eqns}(n)$	$\text{Eqns}^*(n)$
0	2	5	30	260	2842	36834	0	2	2
1		9	104	1015	12278	173880	1	5	7
2			427	8770	115920	1776348	2	39	46
							3	364	410
							4	4284	4694
							5	57882	62576

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