1 Matrices and Linear Algebra

1.1 Partitioning - MM1.1

Matrices can be *partitioned* into smaller (rectangular) *submatrices* and manipulations can be done using the subpieces provided that the dimensions in the partitions permit any matrix multiplications.

1.2 Transposes - MM 1.2

1. $(X^T)^T = X$.
2. $(XY)^T = Y^T X^T \neq X^T Y^T$!
3. A matrix X is *symmetric* if and only if $X^T = X$.

1.3 Inverses - MM 1.3

1. X is singular if and only if it has a 0 determinant, $|X| = 0$.
2. If X is nonsingular, then X^{-1} is nonsingular and $(X^{-1})^{-1} = X$.
3. If X, Y are nonsingular, thenXY is nonsingular and $(XY)^{-1} = Y^{-1}X^{-1} \neq X^{-1}Y^{-1}$.
4. If X is nonsingular, then X^T is nonsingular and $(X^T)^{-1} = (X^{-1})^T$.

1.4 Orthogonality - 1.3

X is an orthogonal matrix if and only if the columns (or rows) of X form an orthonormal set.

1.5 Eigenthings - MM 1.4

1. The eigenvalues of a matrix by solve the *characteristic equation* (this is a polynomial in λ) $|A - \lambda I| = 0$.
2. Find one eigenvector for each eigenvalue as a non-zero solution, $\mathbf{x}$, of the homogeneous system $()A - \lambda I)\mathbf{x} = 0$.
3. If A is (real and) symmetric, then its eigenvalues are all real, and its eigenvectors are orthogonal.
4. If P is an orthogonal matrix of the same size as A, then the eigenvalues of $P^T AP$ are the same as the eigenvalues of A.
5. Let A be a symmetric $k \times k$ matrix. Then an orthogonal matrix P consisting of eigenvectors exists such that

$$P^T AP = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_k \end{bmatrix},$$

where $\lambda_i, i = 1, 2, \dots, k$, are the eigenvalues of A.

1.6 Linear Independence and rank - MM 1.4

1. Vectors $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_k$ are linearly depdendent if there exists some numbers $a_1, a_2, \ldots, a_k$, which are not all zero, such that

$$a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \ldots + a_k\mathbf{x}_k = \mathbf{0}.$$

2. We define the *rank* of a matrix is the the greatest number of linearly independent columns

3. For any matrix X we have $r(X) = r(X^T) = r(X^T X)$.

4. If X is $k \times k$, then X is nonsingular if and only if $r(X) = k$.

5. If X is $n \times k$, P is $n \times n$ and nonsingular, and Q is $k \times k$ and nonsingular, then $r(X) = r(PX) = r(XQ)$.

6. The rank of a diagonal matrix is equal to the number of nonzero diagonal entries in the matrix.

7. $r(XY) \leq r(X), r(Y)$.

1.7 Idempotence - MM 1.5

A square matrix A is *idempotent* if $A^2 = A$

1.8 Trace MM - 1.5

1. The *trace* of a square $k \times k$ matrix X, denoted by $tr(X)$, is the sum of its diagonal entries: $tr(X) = \sum_{i=1}^{k} x_{ii}$.

2. If c is a scalar, $tr(cX) = c\, tr(X)$.

3. $tr(X \pm Y) = tr(X) \pm tr(Y)$.

4. If XY and YX both exist, $tr(XY) = tr(YX)$.

1.9 Theorems

1. The eigenvalues of idempotent matrices are always either 0 or 1.

2. If A is a symmetric and idempotent matrix, $r(A) = tr(A)$.

3. Let $A_1, A_2, \ldots, A_m$ be a collection of symmetric $k \times k$ matrices. Then the following are equivalent:

   - There exists an orthogonal matrix P such that $P^T A_i P$ is diagonal for all $i = 1, 2, \ldots, m$;
   - $A_i A_j = A_j A_i$ for every pair $i, j = 1, 2, \ldots, m$.

4. Let $A_1, A_2, \ldots, A_m$ be a collection of symmetric $k \times k$ matrices. Then any two of the following conditions implies the third:

   - All A_i, $i = 1, 2, \ldots, m$ are idempotent;
   - $\sum_{i=1}^{m} A_i$ is idempotent;

- $A_i A_j = 0$ for $i \neq j$.

5. Let $A_1, A_2, \ldots, A_m$ be a collection of symmetric $k \times k$ matrices. If the conditions in the previous result are true, then $r\left(\sum_{i=1}^m A_i\right) = \sum_{i=1}^m r\left(A_i\right)$.

1.10 Quadratic Forms - MM 2.1

1. Let A be a $k \times k$ matrix and $\mathbf{y}$ a $k \times 1$ vector containing variables, $q = \mathbf{y}^T A \mathbf{y}$ is called a *quadratic form* in $\mathbf{y}$, and A is called the matrix of the quadratic form.

2. $q = \sum_{i=1}^k \sum_{j=1}^k a_{ij} y_i y_j$.

3. If $\mathbf{y}^T A \mathbf{y} > 0$ for all $\mathbf{y} \neq \mathbf{0}$, then we say that the quadratic form $\mathbf{y}^T A \mathbf{y}$ is *positive definite*; we also say that the matrix A is positive definite.

4. If $\mathbf{y}^T A \mathbf{y} \geq 0$ for all $\mathbf{y}$, then we say that the quadratic form $\mathbf{y}^T A \mathbf{y}$ is *positive semi-definite*; we also say that the matrix A is positive semi-definite.

5. A symmetric matrix A is positive (semi-)definite if and only if its eigenvalues are all (non-negative) strictly positive.

1.11 Differentiation by vectors - MM 2.2

1. Suppose we have a vector of variables $\mathbf{y} = (y_1, y_2, \ldots, y_k)^T$, and some scalar function of them: $z = f(\mathbf{y})$. We define the derivative of z with respect to $\mathbf{y}$ as follows: $\frac{\partial z}{\partial \mathbf{y}} = \begin{bmatrix} \partial z / \partial y_1 \\ \partial z / \partial y_2 \\ \vdots \\ \partial z / \partial y_k \end{bmatrix}$.

2. If $z = \mathbf{a}^T \mathbf{y}$ where $\mathbf{a}$ is a vector of constants, then $\frac{\partial z}{\partial \mathbf{y}} = \mathbf{a}$.

3. If $z = \mathbf{y}^T \mathbf{y}$, then $\frac{\partial z}{\partial \mathbf{y}} = 2\mathbf{y}$.

4. If $z = \mathbf{y}^T A \mathbf{y}$, then $\frac{\partial z}{\partial \mathbf{y}} = A\mathbf{y} + A^T \mathbf{y}$. In particular, if A is symmetric, then $\frac{\partial z}{\partial \mathbf{y}} = 2A\mathbf{y}$.