MAST90104: Introduction to Statistical Learning

Week 3 Workshop and Lab **Solutions**

1. Suppose that X is a random variable with density function, f, given by

$$f(x) = \sum_{i=0}^{\infty} p(i)g(x;i)$$

where $p(0), p(1), \cdots$ is a discrete probability mass function on $\{0, 1, \cdots\}$ and each $g(x;i)$ is a probability density function. Suppose that $\mu(i), \sigma^2(i), M(t;i)$ are the mean, variance and moment generating function for the density $g(x;i)$. Let $M(t)$ be the moment generating function of X. Suppose also that N is a random variable with probability mass function $p(i), i = 0, 1, \cdots$. Show that

(a) $E(X) = E(\mu(N))$

Solution:

$$\begin{aligned}
E(X) &= \int_{-\infty}^{\infty} xf(x)\,dx \\
&= \int_{-\infty}^{\infty} x\sum_{i=0}^{\infty} p(i)g(x;i)\,dx \\
&= \sum_{i=0}^{\infty}\int_{-\infty}^{\infty} xg(x;i)\,dxp(i) \\
&= \sum_{i=0}^{\infty}\mu(i)p(i) \\
&= E(\mu(N))
\end{aligned}$$

(b) $\text{var}\ (X) = E(\sigma^2(N)) + \text{var}\ (\mu(N))$

Solution:

$$\begin{aligned}
\text{var}\ (X) &= E(X^2) - (E(X))^2 \\
&= \int_{-\infty}^{\infty} x^2\sum_{i=0}^{\infty} p(i)g(x;i)\,dx - (E(X))^2 \\
&= \sum_{i=0}^{\infty}\int_{-\infty}^{\infty} x^2g(x;i)\,dxp(i) - (E(X))^2 \\
&= \sum_{i=0}^{\infty}(\sigma^2(i) + \mu^2(i))p(i) - (E(X))^2 \\
&= E(\sigma^2(N)) + E(\mu^2(N)) - E(\mu(N)))^2 \\
&= E(\sigma^2(N)) + \text{var}\ (\mu(N))
\end{aligned}$$

(c) $M(t) = E(M(t;N))$.

Solution:

$$\begin{aligned}
M(t) &= E(e^{tX}) \\
&= \int_{-\infty}^{\infty} e^{tx} \sum_{i=0}^{\infty} p(i) g(x;i)\, dx \\
&= \sum_{i=0}^{\infty} \int_{-\infty}^{\infty} e^{tx} g(x;i)\, dx p(i) \\
&= \sum_{i=0}^{\infty} M(t;i) p(i) \\
&= E(M(t;N))
\end{aligned}$$

(Hint: You may assume that interchange of infinite sums and integrals is justified.)

2. Section 2.3 states that the density function, f, of a non-central chi-square random variable, X, with degrees of freedom k and non-centrality paramter λ has the form of $f(x)$ in question 1. Further $g(x;i)$ is the central chi-square density with $k+2i$ degrees of freedom and $p(0), p(1), \cdots$ is the pmf of the Poisson distribution with parameter λ. Use the results in question 1 to show that:

   (a) var $(X) = 2k + 8\lambda$

   Solution: From standard facts about the chi-square distribution $\mu(i) = k+2i, \sigma^2(i) = 2k+4i$. Hence, from 1(b),

   $$\begin{aligned}
   \text{var } (X) &= E(2k + 4N) + \text{var } (k + 2N) \\
   &= 2k + 4\lambda + 4\lambda,
   \end{aligned}$$

   since, from standard facts about the Poisson distribution, $E(N) = \text{var } (N) = \lambda$.

   (b) $M(t) = (1 - 2t)^{-k/2} e^{2\lambda t/(1-2t)}$.

   Solution: From standard facts about the chi-square distribution, $M(t;i) = (1 - 2t)^{-k/2-i}$. Further,

   $$\begin{aligned}
   E((1-2t)^{-N}) &= \sum_{i=0}^{\infty} (1-2t)^{-i} \frac{e^{-\lambda}\lambda^i}{i!} \\
   &= e^{-\lambda} \sum_{i=0}^{\infty} \frac{(\lambda/(1-2t))^i}{i!} \\
   &= \exp\{-\lambda + \frac{\lambda}{1-2t}\} \sum_{i=0}^{\infty} e^{-\lambda/(1-2t)} \frac{(\lambda/(1-2t))^i}{i!} \\
   &= \exp\{-\lambda + \frac{\lambda}{1-2t}\} \\
   &= e^{2\lambda t/(1-2t)}
   \end{aligned}$$

   Hence, from 1(c),

   $$\begin{aligned}
   M(t) &= E(M(t;N)) \\
   &= (1-2t)^{-k/2} E((1-2t)^{-N}) \\
   &= (1-2t)^{-k/2} e^{2\lambda t/(1-2t)},
   \end{aligned}$$

   as required.

3. Use question 2 to prove Theorem 3.4.

   Solution: The moment generating function, $M(t)$, of $\sum_{i=1}^{n} X^2_{k_i,\lambda_i}$ is given by

   $$M(t) = \prod_{i=1}^{n} M_i(t)$$

where $M_i(t)$ is the moment generating function of $X^2_{k_i,\lambda_i} \sim \chi^2_{k_i,\lambda_i}$. From question 2,

$$\begin{aligned} M(t) &= \prod_{i=1}^n (1-2t)^{-k_i/2} e^{2\lambda_i t/(1-2t)} \\ &= (1-2t)^{-(\sum_{i=1}^n k_i)/2} e^{2t\sum_{i=1}^n \lambda_i/(1-2t)} \end{aligned}$$

which, as required, is the moment generating function of a noncentral χ^2 distribution with $\sum_{i=1}^n k_i$ degrees of freedom and noncentrality parameter $\sum_{i=1}^n \lambda_i$.

4. Let $\mathbf{y} = \begin{bmatrix} y_1 & y_2 \end{bmatrix}^T$ be a normal random vector with mean and variance

$$\boldsymbol{\mu} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \quad V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Let

$$A = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad B = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

(a) Find the distributions of $\mathbf{y}^T A\mathbf{y}$ and $\mathbf{y}^T B\mathbf{y}$.
Solution: A and B are both idempotent and have rank 1, so $\mathbf{y}^T A\mathbf{y}$ and $\mathbf{y}^T B\mathbf{y}$ have noncentral χ^2 distributions with 1 degree of freedom each and noncentrality parameters

$$\frac{1}{2}\begin{bmatrix} 2 & 4 \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \end{bmatrix} = 9$$

and

$$\frac{1}{2}\begin{bmatrix} 2 & 4 \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \end{bmatrix} = 1$$

respectively.

(b) Are $\mathbf{y}^T A\mathbf{y}$ and $\mathbf{y}^T B\mathbf{y}$ independent?
Solution: $AB = 0$, so they are independent.

(c) What is the distribution of $\mathbf{y}^T A\mathbf{y} + \mathbf{y}^T B\mathbf{y}$?
Solution: $\mathbf{y}^T A\mathbf{y} + \mathbf{y}^T B\mathbf{y}$ has a noncentral χ^2 distribution with 2 degrees of freedom and noncentrality parameter 10.

5. Let $y_1, \ldots, y_n$ be an i.i.d. normal sample. Show that

$$\bar{y} = \tfrac{1}{n}\sum_{i=1}^n y_i \text{ and } s^2 = \tfrac{1}{n-1}\sum_{i=1}^n (y_i - \bar{y})^2$$

are independent. (*Hint*: Express them as a random "vector" and quadratic form respectively.)

Solution: Let $\mathbf{1}$ be the vector made up entirely of 1's, then

$$\begin{aligned} \bar{y} &= \tfrac{1}{n}\mathbf{1}^T\mathbf{y} \\ \mathbf{y} - \bar{y}\mathbf{1} &= (I - \tfrac{1}{n}\mathbf{1}\mathbf{1}^T)\mathbf{y} \\ s^2 &= \tfrac{1}{n-1}[(I - \tfrac{1}{n}\mathbf{1}\mathbf{1}^T)\mathbf{y}]^T(I - \tfrac{1}{n}\mathbf{1}\mathbf{1}^T)\mathbf{y} \\ &= \tfrac{1}{n-1}\mathbf{y}^T(I - \tfrac{1}{n}\mathbf{1}\mathbf{1}^T)^T(I - \tfrac{1}{n}\mathbf{1}\mathbf{1}^T)\mathbf{y} \\ &= \tfrac{1}{n-1}\mathbf{y}^T(I - \tfrac{1}{n}\mathbf{1}\mathbf{1}^T)\mathbf{y} \end{aligned}$$

noting that $I - \frac{1}{n}\mathbf{1}\mathbf{1}^T$ is symmetric and idempotent. It is now easy to check that $B = \frac{1}{n}\mathbf{1}^T$ and $A = \frac{1}{n-1}(I - \frac{1}{n}\mathbf{1}\mathbf{1}^T)$ satisfy $BVA = 0$, where $V = \sigma^2 I = \operatorname{Var}\mathbf{y}$, whence $\bar{y} = B\mathbf{y}$ and $s^2 = \mathbf{y}^T A\mathbf{y}$ are independent.

The alternative approach is to notice that $\bar{y}$ and s^2 are just the usual estimates of $\boldsymbol{\beta}$ and σ^2 for the linear model $\mathbf{y} = \mathbf{1}\boldsymbol{\beta} + \varepsilon$, with $\boldsymbol{\beta} = \mu = Ey_i$ and $\operatorname{Var}\varepsilon = \sigma^2 I$. (This is sometimes called the *null* model.)

6. We model an individual's income at age 30 against the number of years of formal education with a linear model. The following data is collected:

Years of formal education (x)	Income (\$k) (y)
8	8
12	15
14	16
16	20
16	25
20	40

Where possible, solve the following questions in two ways: using matrix calculations as detailed in the lectures, and using the `lm` command in `R`.

(a) Plot the data; is a linear model appropriate?

Solution: See Question 4. From the plot, it is not unreasonable to suppose a linear relationship between income and years of education.

(b) Write down the linear model in matrix form.

Solution: $\mathbf{y} = X\boldsymbol{\beta} + \boldsymbol{\varepsilon}$, where

$$\mathbf{y} = \begin{bmatrix} 8 \\ 15 \\ 16 \\ 20 \\ 25 \\ 40 \end{bmatrix}, X = \begin{bmatrix} 1 & 8 \\ 1 & 12 \\ 1 & 14 \\ 1 & 16 \\ 1 & 16 \\ 1 & 20 \end{bmatrix}, \boldsymbol{\beta} = \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}, \boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix}.$$

(c) Find the normal equations for this model.

Solution:

```
y <- c(8, 15, 16, 20, 25, 40)
 X <- cbind(rep(1, 6), c(8, 12, 14, 16, 16, 20))
t(X)%*%X

##      [,1] [,2]
## [1,]    6   86
## [2,]   86 1316
```

The normal equations are

$$\begin{bmatrix} 6 & 86 \\ 86 & 1316 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} 124 \\ 1988 \end{bmatrix}.$$

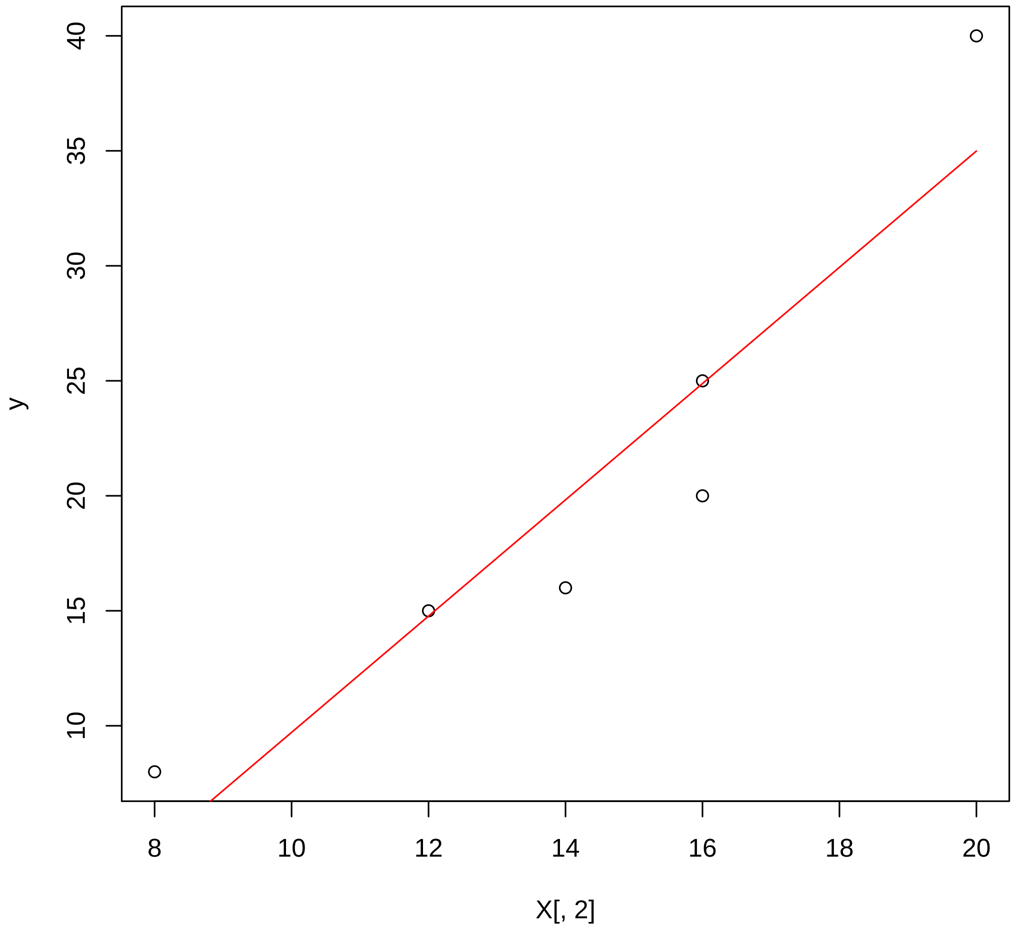
(d) Solve the normal equations to obtain the least squares estimates of the parameters. Add the fitted regression line to your plot (using `curve` for example).

Solution:

```
 n <- 6
p <- 2
(b <- solve(t(X) %*% X, t(X) %*% y))

##         [,1]
## [1,] -15.568
## [2,]   2.528

plot(X[,2], y)
curve(b[1] + b[2]*x, add=TRUE, col="red")
```



Alternatively,

```
income <- data.frame(income=y,education=X[,2])
model <- lm(income ~ education, data=income)
model$coefficients

## (Intercept)   education
##     -15.568       2.528
```

(e) This model is a simple linear regression model. Use the standard linear regression formulae,

$$b_1 = \frac{\overline{xy} - \bar{x}\bar{y}}{\overline{x^2} - \bar{x}^2}, \quad b_0 = \bar{y} - b_1\bar{x},$$

to estimate the parameters again (where the bar indicates the mean). Check that you have the same answers as above.

Solution:

```
(b1 <- (mean(X[,2]*y) - mean(X[,2])*mean(y))/(mean(X[,2]^2) - mean(X[,2])^2))

## [1] 2.528

 (b0 <- mean(y) - b1*mean(X[,2]))

## [1] -15.568
```

(f) Calculate the sample variance s^2.
Solution:

```
e <- y - X %*% b
SSRes <- sum(e^2)
(s2 <- SSRes/(n-p))

## [1] 18.692
```

Alternatively,

```
deviance(model)/model$df.residual

## [1] 18.692
```

(g) Estimate the average income of a person who has had 18 years of formal education.
Solution:

```
xst <- c(1, 18)
t(xst) %*% b

##        [,1]
## [1,] 29.936
```

Alternatively,

```
person <- data.frame(education=18)
predict(model, person)

##      1
## 29.936
```

(h) Calculate the standardised residuals, leverage, and Cook's distance for the first observation. You may need the `R` functions `rstandard`, `influence`, and `cooks.distance`.
Check your numbers against the diagnostic plots produced by R.
Solution:

```
(H <- X %*% solve(t(X) %*% X) %*% t(X))

##        [,1]  [,2]  [,3] [,4] [,5]   [,6]
## [1,]  0.648 0.344 0.192 0.04 0.04 -0.264
## [2,]  0.344 0.232 0.176 0.12 0.12  0.008
## [3,]  0.192 0.176 0.168 0.16 0.16  0.144
## [4,]  0.040 0.120 0.160 0.20 0.20  0.280
## [5,]  0.040 0.120 0.160 0.20 0.20  0.280
## [6,] -0.264 0.008 0.144 0.28 0.28  0.552

# standardised residual
(z1 <- e[1]/sqrt(s2*(1-H[1,1])))

## [1] 1.303668

 # leverage
( H[1,1])

## [1] 0.648

# Cook's distance
(1/p * z1^2 * H[1,1]/(1-H[1,1]))

## [1] 1.564359
```

Alternatively,

```r
rstandard(model)[1]
```

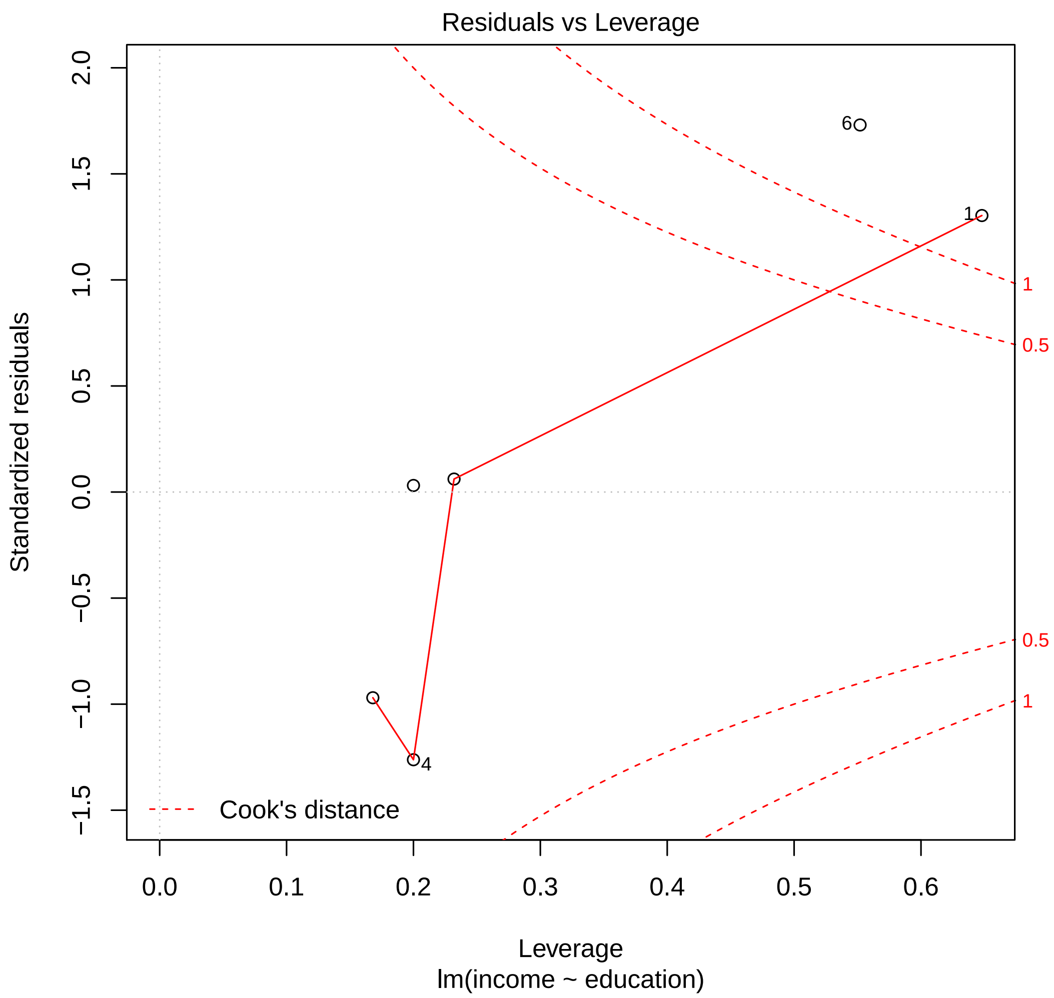
```
##        1
## 1.303668
```

```r
cooks.distance(model)[1]
```

```
##        1
## 1.564359
```

Here is the relevant diagnostic plot:

```r
plot(model, which=5)
```



(i) We know that the least squares estimator $\mathbf{b}$ is an unbiased estimator for $\boldsymbol{\beta}$. Show that $\mathbf{t}^T\mathbf{b}$ is an unbiased estimator for $\mathbf{t}^T\boldsymbol{\beta}$, where $\mathbf{t}$ is a vector of constants.
Solution: We know $E[\mathbf{b}] = \boldsymbol{\beta}$. Therefore $E[\mathbf{t}^T\mathbf{b}] = \mathbf{t}^T E[\mathbf{b}] = \mathbf{t}^T\boldsymbol{\beta}$.

R exercises

Read Sections 5.1–5.3 of spuRs, then attempt the exercises below.

7. The (Euclidean) length of a vector $v = (a_0, \ldots, a_k)$ is the square root of the sum of squares of its coordinates, that is $\sqrt{a_0^2 + \cdots + a_k^2}$. Write a function that returns the length of a vector.

Solution:

```
euclid <- function(x) sqrt(sum(x^2))
```

8. Last week you wrote a program to calculate $h(x, n)$, the sum of a finite geometric series. Turn this program into a *function* that takes two arguments, x and n, and returns $h(x, n)$.

   Make sure you deal with the case $x = 1$.

   Solution:

```
arithmetic_sum <- function(x, n) {
 # sum of x^k for k = 0, ..., n
if (x == 1) {
return(n + 1)
 } else {
     return((x^(n+1) - 1)/(x - 1))
   }
 }
```

9. In this question we simulate the rolling of a die. To do this we use the function `runif(1)`, which returns a 'random' number in the range (0,1). To get a random integer in the range $\{1, 2, 3, 4, 5, 6\}$, we use `ceiling(6*runif(1))`, or if you prefer, `sample(1:6,size=1)` will do the same job.

   (a) Suppose that you are playing the gambling game of the Chevalier de Méré. That is, you are betting that you get at least one six in four throws of a die. Write a program that simulates one round of this game and prints out whether you win or lose.

   Check that your program can produce a different result each time you run it.

   Solution:

```
win <- FALSE
for (i in 1:4) {
if (sample(1:6, size = 1) == 6) {
     win <- TRUE
   }
 }
if (win) {
   print("win")
 } else {
   print("lose")
 }

## [1] "win"
```

   (b) Turn the program that you wrote in part (a) into a function `sixes`, which returns `TRUE` if you obtain at least one six in n rolls of a fair die, and returns `FALSE` otherwise. That is, the argument is the number of rolls n, and the value returned is `TRUE` if you get at least one six and `FALSE` otherwise.

   How would you give n the default value of 4?

   Solution:

```
sixes <- function(n = 4) {
   # plays the game of the Chevalier de Mere
   # returns TRUE if at least one six in n rolls
   # returns FALSE otherwise
   win <- FALSE
   for (i in 1:n) {
     if (sample(1:6, size = 1) == 6) {
```

```
      return(TRUE)
    }
  }
  return(FALSE)
}
```

Here is a vectorised version.

```
sixes <- function(n = 4) {
  sum(sample(1:6, size = n, replace = TRUE) == 6) > 0
}
```

(c) Now write a program that uses your function `sixes` from part (b), to simulate N plays of the game (each time you bet that you get at least one six in n rolls of a fair die). Your program should then determine the proportion of times you win the bet. This proportion is an estimate of the *probability* of getting at least one six in n rolls of a fair die.

Run the program for $n = 4$ and $N = 100$, 1000, and 10000, conducting several runs for each N value. How does the *variability* of your results depend on N?

The probability of getting no 6's in n rolls of a fair die is $(5/6)^n$, so the probability of getting at least one is $1 - (5/6)^n$. Modify your program so that it calculates the theoretical probability as well as the simulation estimate and prints the difference between them. How does the *accuracy* of your results depend on N?

Solution:

```
p_estimate <- function(N, n = 4) {
  # proportion of wins in N runs of sixes(n)
  total_wins <- 0
  for (i in 1:N) {
    if (sixes(n)) total_wins <- total_wins + 1
  }
  return(total_wins/N)
}
p_accuracy <- function(N, n = 4) {
  # accuracy of p_estimate
  total_wins <- 0
  for (i in 1:N) {
    if (sixes(n)) total_wins <- total_wins + 1
  }
  return(total_wins/N - 1 + (5/6)^n)
}
```

(d) In part (c), instead of processing the simulated runs as we go, suppose we first store the results of every game in a file, then later postprocess the results. You should read spuRs Chapter 4 to see how to read and write text files.

Write a program to write the result of all N runs to a textfile `sixes_sim.txt`, with the result of each run on a separate line. For example, the first few lines of the textfile could look like

```
# TRUE
# FALSE
# FALSE
# TRUE
# FALSE
# .
# .
```

Now write another program to read the textfile `sixes_sim.txt` and again determine the proportion of bets won.

This method of saving simulation results to a file is particularly important when each simulation takes a very long time (hours or days), in which case it is good to have a record of your results in case of a system crash.

Solution:

```
sixes_sim <- function(N, n = 4) {
   # runs sixes(n) N times and saves the results in "sixes_sim.txt"
   cat(file="sixes_sim.txt") # deletes contents of file
   for (i in 1:N) {
     cat(file = "sixes_sim.txt", sixes(n), "\n", append = TRUE)
   }
 }
sixes_sim(100)
results <- scan("sixes_sim.txt", what = TRUE)
mean(results)
```

```
## [1] 0.5
```