Quants Arena
Problem Statement

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Consider the following experimental setting. There are two researchers, Alice and Bob, who are interested in measuring the properties of a given system. Suppose Alice has two measurements a_1 and a_2 at her disposal. She can choose which one to carry out, but she cannot perform them both simultaneously. Similarly, Bob can choose between measurements b_1 and b_2. All the measurements produce an outcome $o_A, o_B \in \{0, 1\}$. At each run of the experiment, Alice and Bob choose a measurement to perform, and record the outcome observed. They then repeat this procedure over and over again, so as to collect probability distributions over the outcomes of the experiment. More formally, the statistics of the experiment can be summarised by a collection of probabilities of the form

$$\mathbb{P}(o_A, o_B \mid a_i, b_j)$$

which express the probability of Alice and Bob obtaining outcomes o_A and o_B when choosing measurement a_i and b_j respectively, with $i, j = 0, 1$. These probabilities can then be then summarized in a probability table:

Table 1: Probability table

A	B	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
a_1	b_1	$\mathbb{P}(0,0 \mid a_1,b_1)$	$\mathbb{P}(1,0 \mid a_1,b_1)$	$\mathbb{P}(0,1 \mid a_1,b_1)$	$\mathbb{P}(1,1 \mid a_1,b_1)$
a_1	b_2	$\mathbb{P}(0,0 \mid a_1,b_2)$	$\mathbb{P}(1,0 \mid a_1,b_2)$	$\mathbb{P}(0,1 \mid a_1,b_2)$	$\mathbb{P}(1,1 \mid a_1,b_2)$
a_2	b_1	$\mathbb{P}(0,0 \mid a_2,b_1)$	$\mathbb{P}(1,0 \mid a_2,b_1)$	$\mathbb{P}(0,1 \mid a_2,b_1)$	$\mathbb{P}(1,1 \mid a_2,b_1)$
a_2	b_2	$\mathbb{P}(0,0 \mid a_2,b_2)$	$\mathbb{P}(1,0 \mid a_2,b_2)$	$\mathbb{P}(0,1 \mid a_2,b_2)$	$\mathbb{P}(1,1 \mid a_2,b_2)$

Notice that each row of the table must sum to 1, effectively defining a probability distribution, which will be referred to as *row-distribution*

Intuitively, when Alice and Bob perform their measurements and observe the corresponding outcomes, they are simply looking at a portion of a predetermined assignment of outcomes to *all* of the measurements, which is completely independent of their choice. This means that, at any given run, we assume that a_1, b_1, a_2, b_2 all have well-defined values, e.g.

$$\{a_1 \mapsto 0, b_1 \mapsto 0, a_2 \mapsto 1, b_2 \mapsto 1\}, \tag{1}$$

even though Alice and Bob can only observe two of them, say

$$\{a_1 \mapsto 0, b_2 \mapsto 1\}.$$

Hence, we expect the probabilities observed over many runs of the experiment to be generated by a larger probability distribution over all measurement assignments like the one in (1). That is, a distribution of the form

Table 2: A distribution over all measurement assignments

$a_1 b_1 a_2 b_2$	$\mathbb{P}$
0000	p_1
0001	p_2
0010	p_3
0011	p_4
0100	p_5
$\vdots$	$\vdots$

such that its marginal for each experimental event $(o_A, o_B \mid a_i, b_j)$ gives back $\mathbb{P}(o_A, o_B \mid a_i, b_j)$.

Exercise 1.
Show that the probability distribution on the left marginalizes to the probability table on the right:

$a_1 b_1 a_2 b_2$	$\mathbb{P}$
0000	$1/8$
0011	$1/4$
0100	$1/4$
1000	$1/4$
1010	$1/8$
Other	0

$\longrightarrow$

A	B	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
a_1	b_1	$3/8$	$3/8$	$1/4$	0
a_1	b_2	$3/8$	$3/8$	$1/4$	0
a_2	b_1	$3/8$	$3/8$	$1/4$	0
a_2	b_2	$5/8$	$1/8$	0	$1/4$

Exercise 2.
Write an algorithm which, given a probability table like the one in Table 1, reconstructs a global probability distribution as in Table 2. The algorithm can be written as pseudo code, however, we encourage you to implement it in your favourite programming language, as this will be useful (although not strictly necessary) in the following problems

Exercise 3.
Apply your algorithm, or any other method, to find the probability distribution giving rise to the following probability tables:

1.

A	B	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
a_1	b_1	$3/4$	0	0	$1/4$
a_1	b_2	$3/4$	0	0	$1/4$
a_2	b_1	$3/4$	0	0	$1/4$
a_2	b_2	$3/4$	0	0	$1/4$

2.

A	B	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
a_1	b_1	$5/8$	$1/4$	$1/8$	0
a_1	b_2	$1/4$	$1/4$	$1/2$	0
a_2	b_1	$3/4$	$1/8$	$1/8$	0
a_2	b_2	$3/8$	$1/8$	$1/2$	0

3.

A	B	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
a_1	b_1	$5/16$	$3/16$	$1/32$	$15/32$
a_1	b_2	$1/8$	$11/32$	$7/32$	$5/16$
a_2	b_1	$3/8$	$1/8$	$9/32$	$7/32$
a_2	b_2	$1/4$	$7/32$	$13/32$	$1/8$

Exercise 4.
Try to apply your algorithm to the following table

A	B	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
a_1	b_1	$1/2$	$1/4$	0	$1/4$
a_1	b_2	$1/2$	0	$1/4$	$1/4$
a_2	b_1	$1/4$	$1/2$	$1/4$	0
a_2	b_2	$1/4$	$1/4$	$1/4$	$1/4$

If you have done things correctly, you will find that this table does not have a global probability distribution which marginalizes to it.

1. Show that Bob's choice of measurement affects Alice's probabilities for measurement a_1
2. In a few paragraphs, explain why this directly implies that no global probability distribution can marginalize to this probability table
3. We say that a probability table is *well-behaved* if the probability of observing any outcome for any of Alice's measurements is independent of Bob's choice of measurement (and viceversa). Give a mathematical definition of this property as an expression about the marginals of each row-distribution.

Exercise 5.
Consider the following table

Table 3

A	B	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
a_1	b_1	$1/2$	0	0	$1/2$
a_1	b_2	$3/8$	$1/8$	$1/8$	$3/8$
a_2	b_1	$3/8$	$1/8$	$1/8$	$3/8$
a_2	b_2	$1/8$	$3/8$	$3/8$	$1/8$

1. Show that this table is well-behaved (see Exercise 4 for the definition)
2. Give a formal mathematical proof that this table does not admit any global distribution.
3. Explain in a few paragraphs how you interpret this lack of a global distribution. Can you think of situations where such a probability table could arise in real life?