RAJIV GANDHI PROUDYOGIKI VISHWAVIDYALAYA, BHOPAL

New Scheme Based On AICTE Flexible Curricula

Electronics & Communication Engineering IV-Semester

EC402 Signals & Systems

Unit-1 Introduction of Signals and Systems: Definition of signal, Classification of Signal and representation: Continuous time and discrete time, even/odd, periodic/aperiodic, random/deterministic, energy/power, one/multidimensional, some standard signals, , Basic Operations on Signals for CT/DT signal, transformation of independent & dependent variables,

Definition of system and their classification: CT/DT, linear/non-linear, variant/non-variant, causal and non-causal system state/dynamic system, interconnection of systems. System properties: linearity: additivity and homogeneity, shift-invariance, causality, stability, realizability.

Unit-2 Linear Time- Invariant Systems: Introduction, Impulse Response Representation for LTI Systems, Convolution, Properties of the Impulse Response Representation for LTI Systems, Difference Equation for LTI Systems, Block Diagram Representations(direct form-I, direct form-II, Transpose, cascade and parallel). Impulse response of DT-LTI system and its properties.

Unit-3 z-Transform: Introduction, ROC of finite duration sequence, ROC of infinite duration sequence, Relation between Discrete time Fourier Transform and z-transform, properties of the ROC, Properties of z-transform, Inverse z-Transform, Analysis of discrete time LTI system using zTransform, Unilateral z-Transform.

Unit-4 Fourier analysis of discrete time signals: Introduction, Properties and application of discrete time Fourier series, Representation of Aperiodic signals, Fourier transform and its properties, Convergence of discrete time Fourier transform, Fourier Transform for periodic signals, Applications of DTFT.

Unit-5 State-space analysis and multi-input, multi-output representation. The state-transition matrix and its role. The Sampling Theorem and its implications- Spectra of sampled signals. Reconstruction:

Reference Books:

1. Simon Haykin, “Signals and Systems”, John Wiley.
2. Simon Haykin, “Analog and Digital Communications”, John Willey.
3. Bruce Carlson, “Signals and Systems”, TMH.

4. Oppenheim & Wilsky, “Signals & Systems”, PHI.
5. Taub and Schilling "Principles of communication signals", 2nd ed. New York: Mcgraw-Hill, 1986.

LIST OF EXPERIMENTS

1. Introduction to MATLAB Tool.
2. To implement delta function, unit step function, ramp function and parabolic function for continuous-time.
3. To implement delta function, unit step function, ramp function and parabolic function for discrete-time.
4. To implement rectangular function, triangular function, sinc function and signum function for continuous-time.
5. To implement rectangular function, triangular function, sinc function and signum function for discrete-time.
6. To explore the communication of even and odd symmetries in a signal with algebraic operations.
7. To explore the effect of transformation of signal parameters (amplitude-scaling, time-scaling & shifting).
8. To explore the time variance and time invariance property of a given system.
9. To explore causality and non-causality property of a system.
10. To demonstrate the convolution of two continuous-time signals.
11. To demonstrate the correlation of two continuous-time signals.
12. To demonstrate the convolution of two discrete-time signals.
13. To demonstrate the correlation of two discrete-time signals.
14. To determine Magnitude and Phase response of Fourier Transform of given signals.

Unit-1 Introduction of Signals and Systems:

Definition of signal, Classification of Signal and representation: Continuous time and discrete time, even/odd, periodic/aperiodic, random/deterministic, energy/power, one/multidimensional, some standard signals, , Basic Operations on Signals for CT/DT signal, transformation of independent & dependent variables,
Definition of system and their classification: CT/DT, linear/non-linear, variant/non-variant, causal and non-causal system state/dynamic system, interconnection of systems. System properties: linearity: additivity and homogeneity, shift-invariance, causality, stability, realizability.

Basic Definitions:

Signals : A function of one or more independent variables which contain some information is called signal.

Systems: A system is a set of elements or functional blocks that are connected together and produces an output in response to an input signal.

Classification of Signals :

I. Periodic and Non-Periodic Signals :

A signal that repeats at regular time interval is called as periodic signal. The periodicity of the signal is represented mathematically

$x(t) = x(t+T_0)$; T_0 is the period of the continuous time (CT) signal

$x(n) = x(n+N)$; N is the period of the discrete time (DT) signal

A signal that does not repeats at regular time interval is called non-periodic signal. It does not satisfy the periodicity condition.

$x(t) \neq x(t+T_0)$; for continuous time (CT) signal

$x(n) \neq x(n+N)$; for discrete time (DT) signal

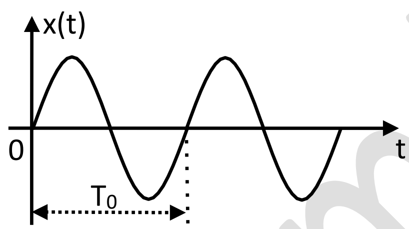


Fig. 1.1 (a) : Periodic Signal

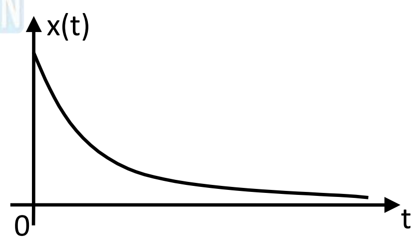


Fig. 1.1 (b) : Non-Periodic Signal

(a) The sum of two continuous-time periodic signals $x_1(t)$ and $x_2(t)$ with period T_1 and T_2 is periodic if the ratio of their respective periods T_1/T_2 is a rational number or ratio of two integers, otherwise not periodic.
(b) The fundamental period is the least common multiple (LCM) of T_1 and T_2.
(c) The sum of two discrete-time periodic sequence is always periodic.

Example: Determine whether the following signals are periodic or not? If periodic find the fundamental period.

(a) $\sin(12\pi t)$ (b) $e^{j4\pi t}$ (c) $\sin(10\pi t) + \cos(20\pi t)$

(a) Given $x(t) = \sin(12\pi t)$, Since x(t) is a sinusoidal signal it is periodic

Comparing x(t) with $\sin(\omega t)$, we get $\omega = 12\pi$ or $T = 2\pi/\omega = 2\pi / 12\pi = 1/6$ sec.

(b) Given $x(t) = e^{j4\pi t}$, Since x(t) is complex exponential signal it is periodic.

Comparing x(t) with $e^{j\omega t}$, we get $\omega = 4\pi$ or $T = 2\pi/\omega = 2\pi / 4\pi = 1/2$ sec.

(c) Given $x(t) = \sin(10\pi t) + \cos(20\pi t)$, Let $x(t) = x_1(t) + x_2(t)$, where $x_1(t) = \sin(10\pi t)$ and $x_2(t) = \cos(20\pi t)$

Comparing $x_1(t)$ and $x_2(t)$ with $\sin(\omega_1 t)$ and $\cos(\omega_2 t)$. we get $\omega_1 = 10\pi$ and $\omega_2 = 20\pi$

$\therefore \frac{T_1}{T_2} = \frac{1/5}{1/10} = 2$, Since T_1/T_2 is a rational number, the given signal is periodic and the fundamental period is $T = T_1 = 2T_2 = 1/5$ Sec.

II. <u>Even and Odd Signals :</u>

A signal is said to be even signal if it is symmetrical about the amplitude axis. The even signal amplitude is not altered when the time axis is inverted.

$$x(t) = x(-t) \quad ; \quad x(n) = x(-n)$$

A signal is said to be odd signal if it is anti-symmetrical about the amplitude axis. The odd signal amplitude is inverted when the time axis is inverted.

$$x(t) = -x(-t) \quad ; \quad x(n) = -x(-n)$$

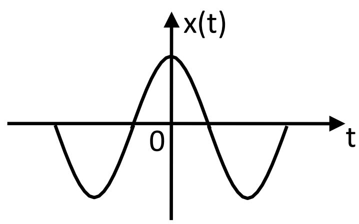


Fig. 1.2 (a) : Symmetric (Even) Signal

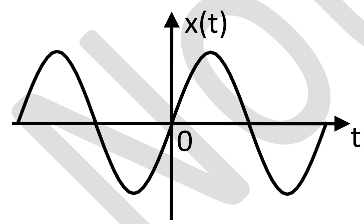


Fig. 1.2 (b) : Anti-Symmetric (Odd) Signal

If the signal does not satisfies either the condition for even signal or the condition for odd signal then it is neither an even signal nor the odd signal. However it contains both even and odd components in the signal. The even and odd components can be calculated as $x_e(t) = \frac{1}{2}[x(t) + x(-t)$; $x_o(t) = \frac{1}{2}[x(t) - x(-t)$

Find even and odd components of following signals

(i) $x(t) = 1 + t + 3t^2 + 5t^3 + 9t^4$ (ii) $x(t) = (1 + t^3)\cos^3(10t)$ (iii) $x(t) = \cos(t) + \sin(t) + \sin(t)\cos(t)$

(i) $x(t) = 1 + t + 3t^2 + 5t^3 + 9t^4$

Now $x(-t) = 1 + (-t) + 3(-t)^2 + 5(-t)^3 + 9(-t)^4$

$x(-t) = 1 - t + 3t^2 - 5t^3 + 9t^4$

$$\therefore x_e(t) = \frac{1}{2}[x(t) + x(-t)]$$

$$= \frac{1}{2}[1 + t + 3t^2 + 5t^3 + 9t^4 + 1 - t + 3t^2 - 5t^3 + 9t^4]$$

$$\boxed{\therefore x_e(t) = 1 + 3t^2 + 9t^4}$$

and $x_o(t) = \frac{1}{2}[x(t) - x(-t)]$

$$= \frac{1}{2}[1 + t + 3t^2 + 5t^3 + 9t^4 - 1 + t - 3t^2 + 5t^3 - 9t^4]$$

$$\boxed{\therefore x_o(t) = t + 5t^3}$$

(ii) $x(t) = (1 + t^3)\cos^3(10t)$

Now $x(t) = (1 + t^3)(\frac{3}{4}\cos(10t) + \frac{1}{4}\cos(30t))$

$$\therefore x(-t) = (1 + (-t)^3)(\frac{3}{4}\cos(-10t) + \frac{1}{4}\cos(-30t))$$

$$x(-t) = (1 - t^3)(\frac{3}{4}\cos(10t) + \frac{1}{4}\cos(30t))$$

To find the even and odd components, consider the equation

$$x_e(t) = \frac{1}{2}[x(t) + x(-t)]$$

$$x_e(t) = \frac{1}{2}\{[(1 + t^3)(\frac{3}{4}\cos(10t) + \frac{1}{4}\cos(30t))] + [(1 - t^3)(\frac{3}{4}\cos(10t) + \frac{1}{4}\cos(30t))]\}$$

$$\boxed{\therefore x_e(t) = (\frac{3}{4}\cos(10t) + \frac{1}{4}\cos(30t))}$$

and

$$x_o(t) = \frac{1}{2}[x(t) - x(-t)]$$

$$x_o(t) = \frac{1}{2}\{[(1 + t^3)(\frac{3}{4}\cos(10t) + \frac{1}{4}\cos(30t))] - [(1 - t^3)(\frac{3}{4}\cos(10t) + \frac{1}{4}\cos(30t))]\}$$

$$\boxed{\therefore x_o(t) = t^3(\frac{3}{4}\cos(10t) + \frac{1}{4}\cos(30t))}$$

(iii) x(t) = cos(t) + sin(t) + sin(t) cos(t)

Now x(-t) = cos(-t) + sin(-t) + sin(-t) cos(-t)

x(-t) = cos(t) - sin(t) - sin(t) cos(t)

$$\therefore x_e(t) = \frac{1}{2}[x(t) + x(-t)]$$

$$= \frac{1}{2}[\cos(t) + \sin(t) + \sin(t)\cos(t) + \cos(t) - \sin(t) - \sin(t)\cos(t)]$$

$$\boxed{\therefore x_e(t) = \cos(t)}$$

and $x_o(t) = \frac{1}{2}[x(t) - x(-t)]$

$$= \frac{1}{2}[\cos(t) + \sin(t) + \sin(t)\cos(t) - \cos(t) + \sin(t) + \sin(t)\cos(t)]$$

$$\boxed{\therefore x_o(t) = \sin(t)\cos(t) + \cos(t)}$$

III. <u>Energy and Power Signals :</u>

The signal which has finite energy and zero average power is called as energy signal.
If x(t) has $0 < E < \infty$ and $P = 0$, then it is a energy signal, where E is the energy and P is the average power of signal x(t).

The signal which has finite average power and infinite energy is called as power signal.
If x(t) has $0 < P < \infty$ and $E = \infty$, then it is a power signal, where E is the energy and P is the average power of signal x(t).

If the signal does not satisfies either the condition for energy signal or the condition for power signal then it is neither an energy signal nor the power signal.

<u>Energy of Signal:</u>

For real CT signal, energy is calculated as

$$E = \int_{-\infty}^{\infty} x^2(t)dt$$

For complex valued CT signal, energy is calculated as

$$E = \int_{-\infty}^{\infty} |x(t)|^2dt$$

For real DT signal, energy is calculated as

$$E = \sum_{n=-\infty}^{\infty} x(n)^2$$

For complex valued DT signal, energy is calculated as

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

<u>Power of Signal:</u>

For real CT signal, average power is calculated as

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} x^2(t)dt$$

For complex valued CT signal, average power is calculated as

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} |x(t)|^2dt$$

For real DT signal, average power is calculated as

$$P = \lim_{N\to\infty} \frac{1}{N} \sum_{n=-\infty}^{\infty} x(n)^2$$

For complex valued DT signal, average power is calculated as

$$P = \lim_{N\to\infty} \frac{1}{N} \sum_{n=-\infty}^{\infty} |x(n)|^2$$

Comparison of Energy Signal & Power Signal

Sl. No.	Energy Signal	Power Signal
1	Total normalized energy is finite and non zero	The normalized average power is finite and non zero
2	Non periodic signals are energy signals	Periodical signals are power signals
3	Power of energy signal is zero	Energy of power signal is infinite

Example : (i) Prove the following (a) The power of the energy signal is zero over infinite time (b) The energy of the power signal is infinite over infinite time

Solution :

(a) Power of the energy signal

Let x(t) be an energy signal

$$\therefore \text{Power } P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2dt = \lim_{T\to\infty} \frac{1}{2T} [\lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2dt]$$

$$P = \lim_{T\to\infty} \frac{1}{2T} \int_{-\infty}^{\infty} |x(t)|^2dt = \lim_{T\to\infty} \frac{1}{2T} [E] \qquad \text{Since } E = \int_{-\infty}^{\infty} |x(t)|^2dt$$

$$\boxed{\therefore \quad P = \frac{1}{2\infty} [E] = 0 \text{ X } E = 0}$$

Thus, the power of the energy signal is zero over infinite time.

(b) Energy of the power signal

Let x(t) be the power signal

$$\therefore \text{Energy } E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

Consider the limits of integration as –T to T and take limit T tends to . this will not change the meaning of above equation

$$\therefore \ E = \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 dt = \lim_{T\to\infty} [2T \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt]$$

$$\therefore \ E = \lim_{T\to\infty} 2T [\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt] = \lim_{T\to\infty} 2TP$$

$$\boxed{\therefore \ E = \infty}$$

Thus, the energy of the power signal is infinite over infinite time.

Example : (ii) Sketch the given signal $x(t) = e^{-a|t|}$ for $a > 0$. Also determine whether the signal is a power signal or energy signal or neither.

Solution : The given signal is $x(t) = e^{-a|t|}$ for $a > 0$

It can be expressed as

$$x(t) = \begin{cases} e^{-at} & \text{for } t > 0 \\ e^{at} & \text{for } t < 0 \end{cases}$$

The sketch of this signal is shown in Fig. (a)

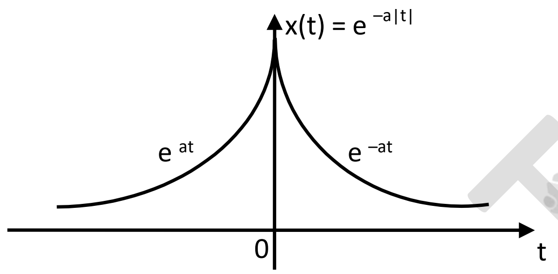


Fig.(a) : Waveforms for Example (ii)

The energy of the signal is expressed as

$$\text{Energy } E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} e^{-2a|t|} dt$$

$$\therefore E = \int_{-\infty}^{0} e^{-2a(-t)} dt + \int_{0}^{\infty} e^{-2a(t)} dt$$

$$\therefore \ E = \int_{0}^{\infty} e^{-2at} dt + \int_{0}^{\infty} e^{-2at} dt = 2 \int_{0}^{\infty} e^{-2at} dt$$

$$\therefore \ E = 2 \left[\frac{e^{-2at}}{-2a}\right]_0^{\infty} = -\frac{1}{a} [e^{-\infty} - e^0] = \frac{1}{a} \text{ Joules}$$

Since energy is finite, the signal is energy signal.

Example : (iii) The signal x(t) is shown in Fig.(b). Determine whether the signal is a power signal or energy signal or neither.

Solution : The given signal x(t) can be expressed as

$$x(t) = \begin{cases} 2 & \text{for } -1 \le t \le 0 \\ 2e^{-t/2} & \text{for } \ t > 0 \end{cases}$$

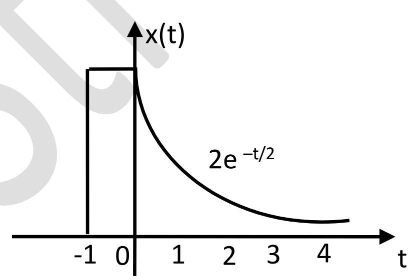


Fig.(b) : Waveforms for Example (iii)

The energy of the signal is expressed as

$$\text{Energy } E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$\therefore \ E = \int_{-1}^{0} 2^2 dt + \int_{0}^{\infty} [2e^{-(t/2)}]^2 dt$$

$$\therefore \ E = \int_{-1}^{0} 4 dt + \int_{0}^{\infty} 4e^{-t} dt$$

$$\therefore \ E = 4[t]_{-1}^{0} + 4 \left[\frac{e^{-t}}{-1}\right]_0^{\infty} = 4 - 4[e^{-\infty} - e^0] = 4 + 4 = 8 \text{ Joules}$$

Since energy is finite, the signal is energy signal.

Example : (iv) The signal x(t) is shown in Fig.(c). Determine whether the signal is a power signal or energy signal or neither.

Solution : The given signal x(t) is a periodic signal with period from -1 to 1 (T=2) , can be expressed as

$$x(t) = t \quad \text{for} -1 \leq t \leq 1$$

Since the signal is periodic, it is a power signal and the average power can be calculated.

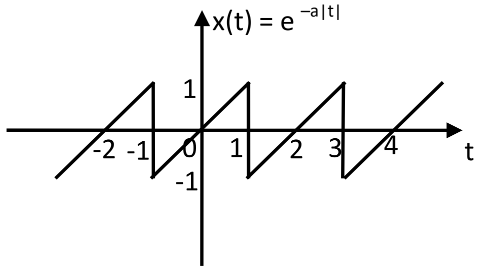


Fig.(c) : Waveforms for Example (iv)

The average power of the signal is expressed as

$$\text{Power P} = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2 dt$$

$$\therefore \quad p = \frac{1}{2}\int_{-T/2}^{T/2} |x(t)|^2 dt = \frac{1}{2}\int_{-1}^{1} t^2 dt$$

$$\therefore \quad P = \frac{1}{2}\left[\frac{t^3}{3}\right]_{-1}^{1} = \frac{1}{6}.2 = \frac{1}{3} \text{ Watts}$$

Since energy is finite, the signal is energy signal.

IV. Deterministic and Random Signals

A signal which has regular pattern and can be completely represented by mathematical equation at any time is call deterministic signal.

i.e., sine wave, exponential signal, square wave and triangular wave etc.

A signal which has uncertainty about its occurrence is called random signal. A random signal cannot be represented by mathematical equation.

i.e., noise is a random signal

Elementary Signals :

(i) *Unit Step Signal :* The unit step signal has a constant amplitude of unity for t ≥ 0 and zero for negative values of t.

The mathematical expression for CT unit step signal : $u(t) = \begin{cases} 1 & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$

The mathematical expression for DT unit step signal : $u(n) = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$

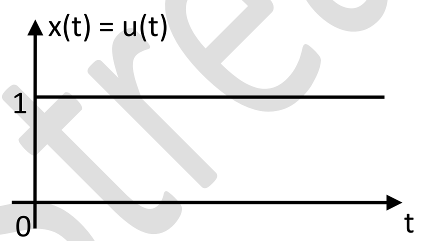


Fig. 1.3 (a) : CT unit Step Signal

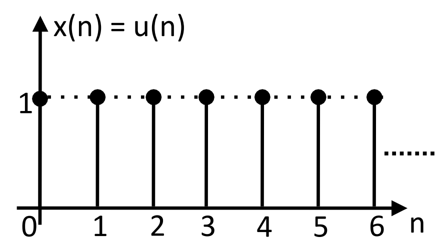
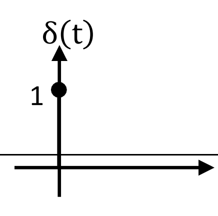
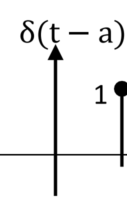


Fig. 1.3 (b) : DT unit Step Signal

(ii) *Unit Impulse Signal:* This signal is most widely used elementary signal in the analysis of systems. It is also called Dirac delta signal. It is defined as

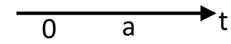
$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$





and $\delta(t) = 0$ for $t \neq 0$ $\delta(t)$

$\delta(t)$ *i.e.,* $\delta(t) = \begin{cases} 1 & \text{for } t = 0 \\ 0 & \text{for } t \neq 0 \end{cases}$



(iii) *Unit Ramp Signal:* The unit ramp signal r(t) is that signal which starts at t=0 and increases linearly with time and is defined as

Continuous time ramp signal $r(t) = \begin{cases} t & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$ or $r(t) = t\,u(t)$

Discrete time ramp sequence $r(n) = \begin{cases} n & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$ or $r(n) = n\,u(n)$

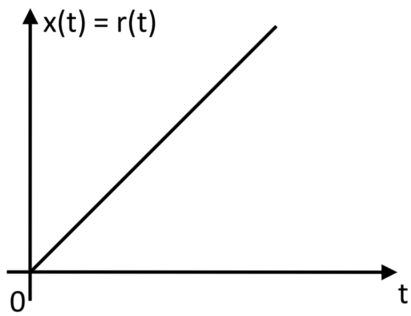


Fig. 1.4 (a) : CT Ramp Signal

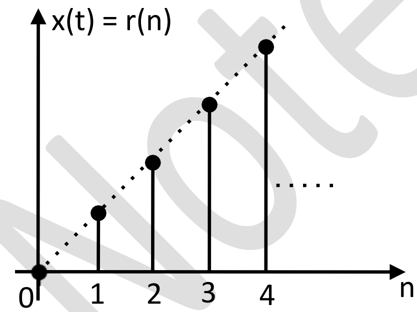
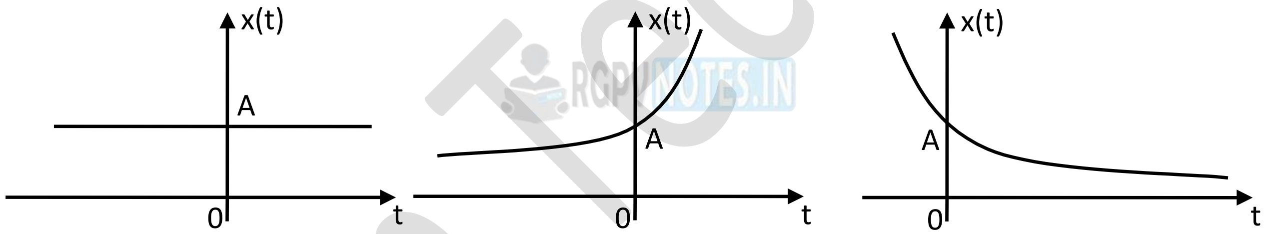


Fig. 1.4 (b) : DT Ramp Signal

(iv) *Exponential Signal :* The continuous-time real exponential signal has general form as $x(t) = A\,e^{\alpha t}$, where both A and α are real number.



$x(t) = A\,e^{\alpha t}$ for $\alpha = 0$ $x(t) = A\,e^{\alpha t}$ for $\alpha > 0$ $x(t) = A\,e^{\alpha t}$ for $\alpha < 0$

Fig. 1.5: CT Exponential Signal

(v) *Sinusoidal Signal:* The CT sinusoidal signal has general form as $x(t) = A \sin(\omega t + \phi)$

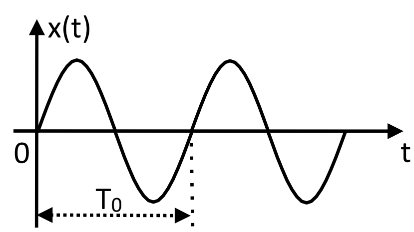


Fig. 1.6: CT Sinusoidal Signal

Relationships between the signals:

Relation between unit step and unit ramp signal:

The unit ramp signal is defined as,

$$r(t) = \begin{cases} t & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

Differentiating r(t) with respect to t,

$$\frac{d\,r(t)}{dt} = \begin{cases} \frac{d\,(t)}{dt} & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases} = \begin{cases} 1 & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

$$\boxed{\therefore\ u(t) = \frac{d\,r(t)}{dt}}$$

The unit step signal is defined as,

$$u(t) = \begin{cases} 1 & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

Integrating u(t) with respect to t,

$$\int u(t)\,dt = \int 1\,dt = t$$

$$\boxed{\therefore\ r(t) = \int u(t)\,dt}$$

Signal Operations and Properties:

Following are the operations performed on the signals;

(i) *Time Shifting (Delay / Advance):* The signal can be delayed or advanced by a constant time factor. The signal x(t) is time delayed if the time factor is having negative value. The signal x(t) is time advanced if the time factor is having positive value.
i.e., x(t) is right shifted if it is represented as x(t-2) and left shifted if it is represented as x(t+2). The Fig. shows the time shifting operations.

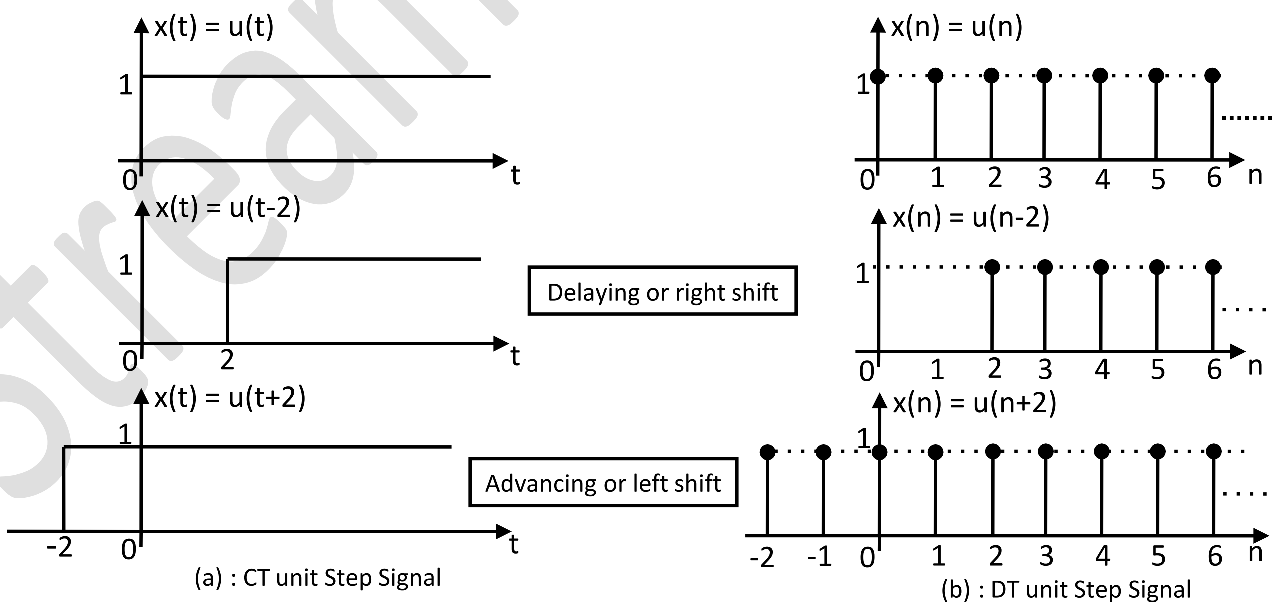
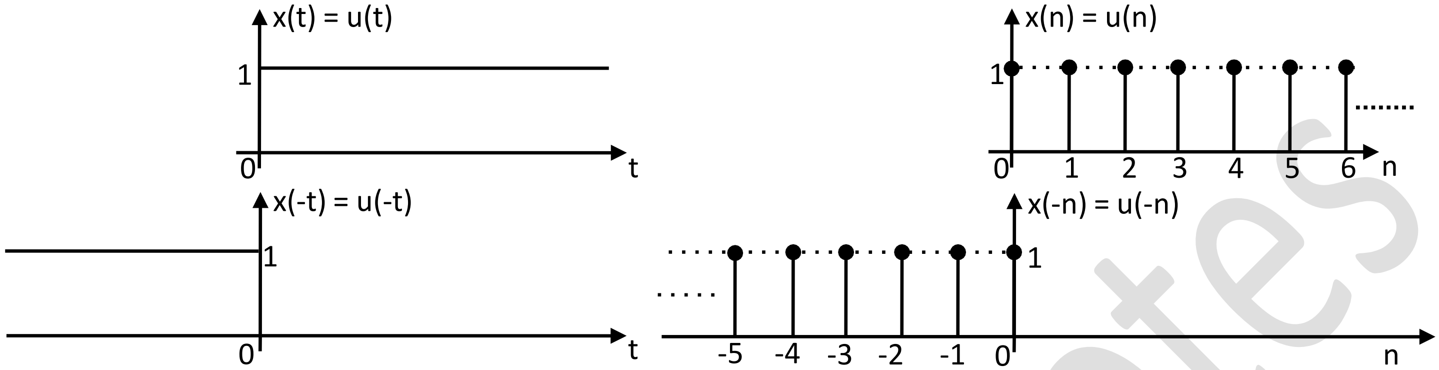


Fig. 1.8: Time Shifting Operation

(ii) *Time Folding:* Time folding is also called as time reversal of signal x(t) and is denoted by x(-t). The signal x(-t) is obtained by replacing t with –t in the given x(t).



(a) : CT unit Step Signal

(b) : DT unit Step Signal

Fig. 1.9: Time Folding Operation

(iii) *Time Scaling (Compression / Expansion):* The time scaling may be compression or expansion of time x(t). It is expressed as y(t) = x(at) where a is the scaling factor.

(iv) *Amplitude Scaling:* The amplitude scaling of the CT signal x(t) is represented as y(t) = A x(t)

(v) *Signal Addition :* The two or more CT signals can be added. The value of new signal is obtained by adding the value of each signal at every instant of time. Subtraction of one signal from other can also be performed in the similar way.

(vi) *Signal Multiplication:* The two signal can be multiplied in its continuous time domain. The value of new signal is obtained by multiplying the two signal values at every instant of time.

Example : Sketch the following signals

(a) $x(t) = 2u(t+2) - 2u(t-3)$ (b) $x(t) = u(t+4)\,u(-t+4)$ (c) $x(t) = r(-t)\,u(t+2)$

(c) $x(t) = r(t) - r(t-1) - r(t-3) + r(t-4)$

Solution: (a) Given $x(t) = 2u(t+2) - 2u(t-3)$

- Consider the elementary signal u(t).
- The signal 2u(t+2) is obtained by shifting u(t) to the left by 2 units and multiplying by 2
- The signal -2u(t-3) is obtained by shifting u(t) to the right by 3 units and multiplying by -2
- The signal x(t) is obtained by adding 2u(t+2) and – 2u(t-3)
- The sketch of all the signals are shown in Fig. (a).

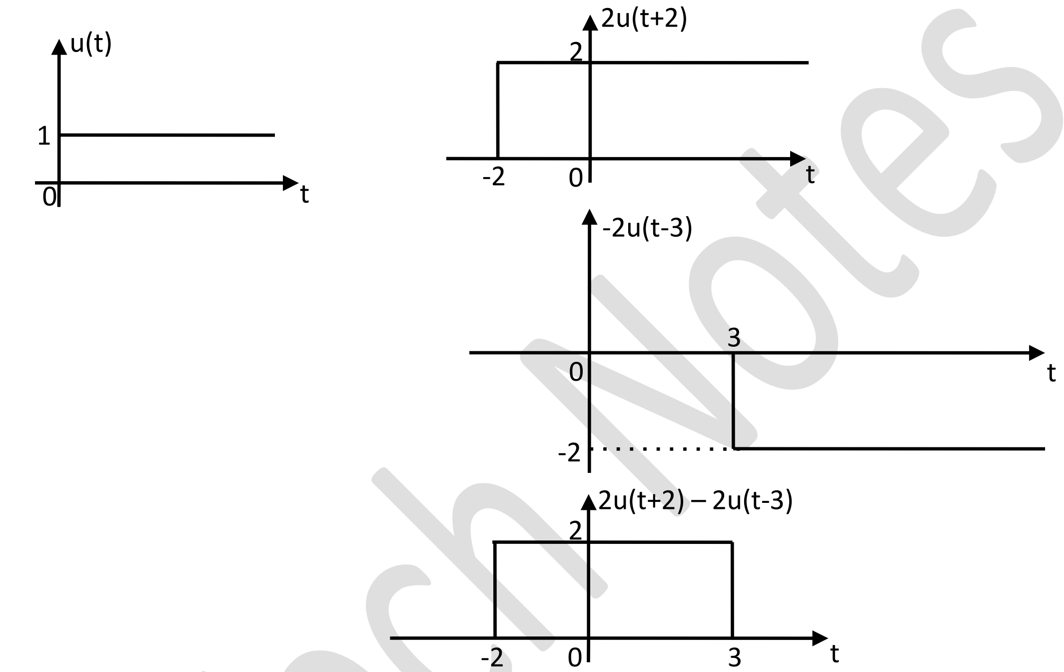


Fig.(a) : Waveforms for Example (a)

Solution: (b) Given $x(t) = u(t+4)\,u(-t+4)$

- Consider the elementary signal u(t).
- The signal u(t+4) is obtained by shifting u(t) to the left by 4 units
- The signal u(-t+4) is obtained by reversing u(t) and then shifting u(-t) to the right by 4 units
- The signal x(t) is obtained by multiplying u(t+4) and u(-t+4)
- The sketch of all the signals are shown in Fig. (b).

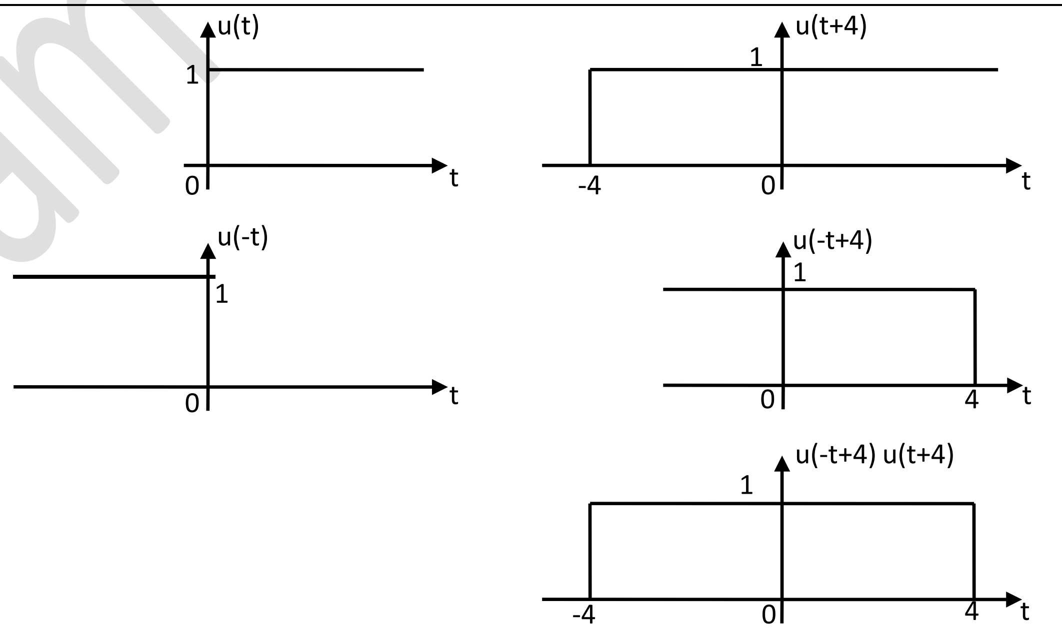


Fig.(b) : Waveforms for Example (b)

Solution: (c) Given x(t) = r(-t) u(t+2)

- Consider the elementary signals u(t) and r(t).
- The signal u(t+2) is obtained by shifting u(t) to the left by 2 units
- The signal r(-t) is obtained by time reversing r(t)
- The signal x(t) is obtained by multiplying r(-t) and u(t+2)
- The sketch of all the signals are shown in Fig. (c).

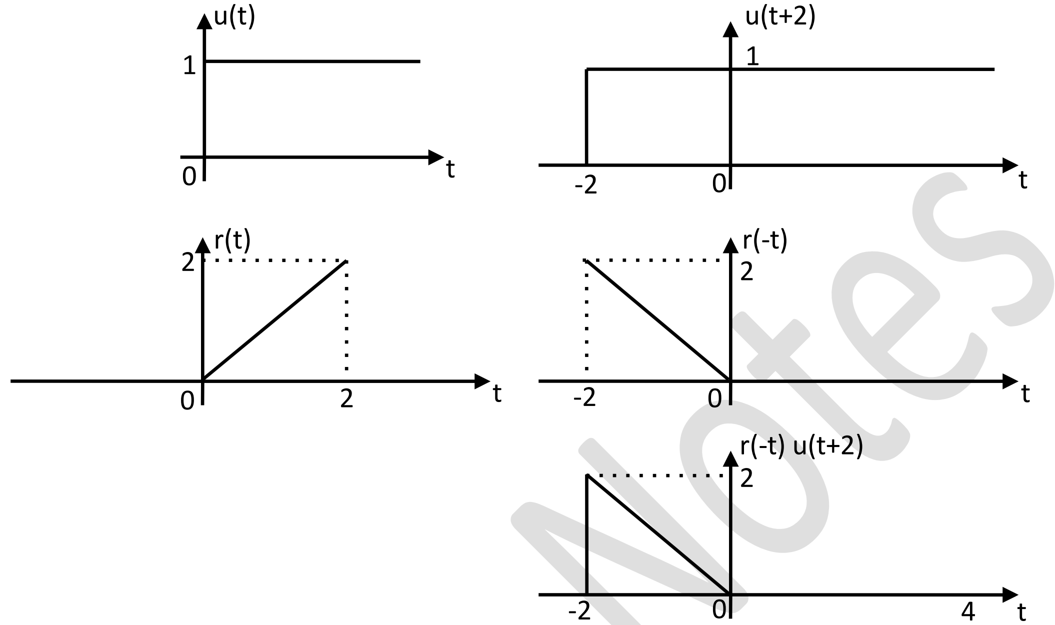


Fig.(c) : Waveforms for Example (c)

(a) Solution: (d) Given x(t) = r(t) – r(t-1) – r(t-3) + r(t-4)

- Consider the elementary signals r(t).
- The signal r(t-1) is obtained by shifting r(t) to the right by 1 unit with slope -1
- Similarly the signal r(t-3) is obtained by shifting r(t) to the right by 3 units with slope -1 and the signal r(t-4) is obtained by shifting r(t) to the right by 4 units with slope 1
- The signal x(t) is obtained by adding r(-t), -r(t-1), -r(t-3) and r(t+4)
- The sketch of all the signals are shown in Fig. (d).

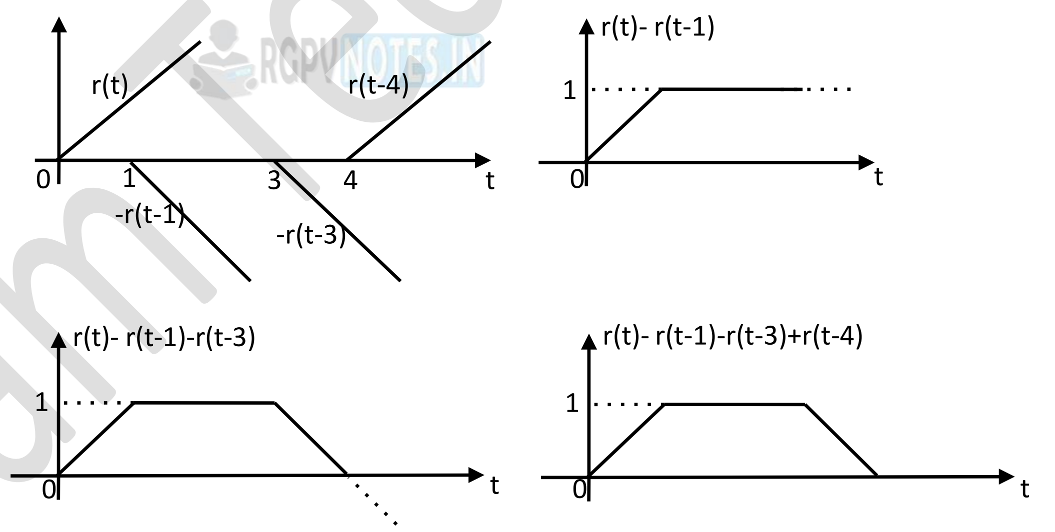


Fig.(d) : Waveforms for Example (d)

Basic System Properties :

(I) Causality (II) Time Invariant & Variant (III) Linearity (IV) Stability (V) Static & Dynamic
(VI) Invertible & Non-Invertible

(I) Causal and Non-causal System : A system is said to be **causal** if its output $y(t)$ at any arbitrary time t_0 depends only on the values of its input $x(t)$ for $t \le t0$. In the causal system the output does not begin before the input signal is applied. If the independent variable represents time, a system must be causal in order to be physically realizable. Noncausal systems can sometimes be useful in practice, however, as the independent variable need not always represent time.

Example : Determine whether the following systems are causal or non-causal

(i) y(t) = 0.2x(t) – x(t-1) (ii) y(t) = 0.8x(t-1) (iii) y(n) = x(n-1)
(iv) y(t) = x(t+1) (v) y(n-2) = x(n) (vi) y(n) = x(n) –x(n+1)

Solution:

(i) Given that y(t) = 0.2x(t) – x(t-1)
In the above equation put t=0 then y(0) = 0.2x(0) – x(-1)
put t=1 then y(1) = 0.2x(1) – x(0)
Since the output y(t) depends on the present and the past input values of x(t), the system is **causal**

(ii) Given that y(t) = 0.8x(t-1)
In the above equation put t=0 then y(0) = 0.8 x(-1)
put t=1 then y(1) = 0.8 x(0)
Since the output y(t) depends on only the past input values of x(t), the system is **causal**

(iii) Given that y(n) = x(n-1)
In the above equation put n=0 then y(0) = x(-1)
put n=1 then y(1) = x(0)
Since the output y(n) depends on only the past input values of x(n), the system is **causal**

(iv) Given that y(t) = x(t+1)
In the above equation put t=0 then y(0) = x(1)
put t=1 then y(1) = x(2)
Since the output y(t) depends on future input values of x(t), the system is **non-causal**

(v) Given that y(n-2) = x(n)
In the above equation put n=0 then y(-2) = x(0)
put n=1 then y(-1) = x(1)
Since the output y(t) depends on future input values of x(t), the system is **non-causal**

(vi) Given that y(n) = x(n) –x(n+1)
In the above equation put n=0 then y(0) = x(0) – x(1)
put n=1 then y(1) = x(1) – x(2)
Since the output y(t) depends on the present and the future input values of x(t), the system is **non-causal**

(II) Time Invariant & Variant System: Let $y(t)$ be the response of a system to the input $x(t)$, and let t_0 be a time-shift constant. If, for any choice of $x(t)$ and t_0, the input $x(t - t_0)$ produces the output $y(t - t_0)$, the system is said to be **time invariant**. A system is time invariant, if a time shift in the input signal results in an identical time shift in the output signal.

(III) **Linear and Non-Linear System:** Let $y_1(t)$ and $y_2(t)$ denote the responses of a system to the inputs $x_1(t)$ and $x_2(t)$, respectively. If, for any choice of $x_1(t)$ and $x_2(t)$, the response to the input $x_1(t)+x_2(t)$ is $y_1(t)+y_2(t)$, the system is said to possess the **additivity** property.

Let $y(t)$ denote the response of a system to the input $x(t)$, and let a denote a complex constant. If, for any choice of $x(t)$ and a, the response to the input $ax(t)$ is $ay(t)$, the system is said to possess the **homogeneity** property.

If a system possesses both the additivity and homogeneity properties, it is said to be **linear**. Otherwise, it is said to be **nonlinear**.

The two linearity conditions (i.e., additivity and homogeneity) can be combined into a single condition known as superposition. Let $y_1(t)$ and $y_2(t)$ denote the responses of a system to the inputs $x_1(t)$ and $x_2(t)$, respectively, and let a and b denote complex constants. If, for any choice of $x_1(t)$, $x_2(t)$, a, and b, the input $ax_1(t)+bx_2(t)$ produces the response $ay_1(t)+by_2(t)$, the system is said to possess the **superposition** property.

To show that a system is linear, we can show that it possesses both the additivity and homogeneity properties, or we can simply show that the superposition property holds.

Example : Determine whether the following systems are linear or non-linear

(i) $y(t) = t.x(t)$ (ii) $y(t) = x^2(t)$ (iii) $y(t) = ax(t) + b$

Solution:

(i) Given that $y(t) = t.x(t)$

Let $y_1(t) = tx_1(t)$ and $y_2(t) = tx_2(t)$

Now, the linear combination of the two outputs will be

$$y_3(t) = a_1\, y_1(t) + a_2y_2(t) = a_1\, tx_1(t) + a_2tx_2(t)$$

Also the response to the linear combination of input will be

$$y_4(t) = f[a_1x_1(t) + a_2x_2(t)] = t[a_1x_1(t) + a_2x_2(t)]$$
$$y_4(t) = a_1t\, x_1(t) + a_2t\, x_2(t)$$

Since the output $y_3(t) = y_4(t)$, the system is **linear system.**

(ii) Given that $y(t) = x^2(t)$

Let $y_1(t) = x_1^2(t)$ and $y_2(t) = x_2^2(t)$

Now, the linear combination of the two outputs will be

$$y_3(t) = a_1\, y_1(t) + a_2y_2(t) = a_1\, x_1^2(t) + a_2\, x_2^2(t)$$

Also the response to the linear combination of input will be

$$y_4(t) = f[a_1x_1(t) + a_2x_2(t)] = [a_1x_1(t) + a_2x_2(t)]^2$$
$$y_4(t) = a_1\, x_1^2(t) + a_2\, x_2^2(t) + 2a_1\, a_2\, x_1(t)\, x_2(t)$$

Since the output $y_3(t) \neq y_4(t)$, the system is **not a linear system.**

(iii) Given that $y(t) = ax(t) + b$

Let $y_1(t) = ax_1(t)+b$ and $y_2(t) = ax_2(t)+b$

Now, the linear combination of the two outputs will be

$$y_3(t) = a_1\, y_1(t) + a_2y_2(t) = a_1(ax1(t)+b) + a_2\, (ax_2(t)+b)$$

Also the response to the linear combination of input will be

$$y_4(t) = f[a_1x_1(t) + a_2x_2(t)] = a[a_1x_1(t) + a_2x_2(t)] + b$$

Since the output $y_3(t) \neq y_4(t)$, the system is **not a linear system.**

(IV) **Memory and Memory-less System:** A system is said to have **memory** if its output $y(t)$ at any arbitrary time $t0$ depends on the value of its input $x(t)$ at any time other than $t = t_0$. If a system does not have memory, it is said to be **memory less**.

(V) **Invertible and Non-invertible System:** A system is said to be **invertible** if its input $x(t)$ can always be uniquely determined from its output $y(t)$. From this definition, it follows that an invertible system will always produce distinct outputs from any two distinct inputs. If a system is invertible, this is most easily demonstrated by finding the inverse system. If a system is not invertible, often the easiest way to prove this is to show that two distinct inputs result in identical outputs.

(VI) **Stability :** The bounded-input bounded-output (BIBO) stability is most commonly defined in system analysis. A system having the input $x(t)$ and output $y(t)$ is **BIBO stable** if, a bounded input produces a bounded output.

Unit-2 Linear Time- Invariant Systems:

Introduction, Impulse Response Representation for LTI Systems, Convolution, Properties of the Impulse Response Representation for LTI Systems, Difference Equation for LTI Systems, Block Diagram Representations (direct form-I, direct form- II, Transpose, cascade and parallel). impulse response of DT-LTI system and its properties.

Impulse response and convolution integral:

Convolution is a mathematical operation which is used to find the response of LTI system. In the LTI system analysis it relates the impulse response of the system and input signal to the output. Any signal x(t) can be represented as a continuous sum of impulse signals. The response y(t) can then be represented as sum of responses of various impulse components.

The impulse response is denoted as h(t) = T[δ(t)]

Any arbitrary signal can be represented as

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

The system output is given as y(t) = T[x(t)]

$$\therefore y(t) = T\left[\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau\right]$$

For a linear system

$$\therefore y(t) = \int_{-\infty}^{\infty} x(\tau)T[\delta(t-\tau)]d\tau$$

If the system response due to impulse signal is h(t), then the response of the system due to delayed impulse signal is h(t, τ)

$$\therefore y(t) = \int_{-\infty}^{\infty} x(\tau)h(t,\tau)d\tau$$

For a time invariant system, the output due to input delayed by τ sec is equal to the output delayed by τ sec. that is

$$\therefore y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t)$$

Properties of Convolution:

(i) Commutative Property : The commutative property of convolution state that.

$$y(t) = x(t) * h(t) = h(t) * x(t)$$

(ii) Distributive Property : The distributive property of convolution state that.

$$x_1(t) * [x_2(t) + x_3(t)] = x_1(t) * x_2(t) + x_1(t) * x_3(t)$$

(iii) Associative Property : The associative property of convolution state that.

$$x_1(t) * [x_2(t) * x_3(t)] = [x_1(t) * x_2(t)] * x_3(t)$$

Example 1: For an LTI system with unit impulse response $h(t) = e^{-2t}$ for $t \geq 0$, find the system response for the input signal $x(t) = A \quad$ for $\quad 0 \leq t \leq 2$. Sketch the output signal.

Solution :

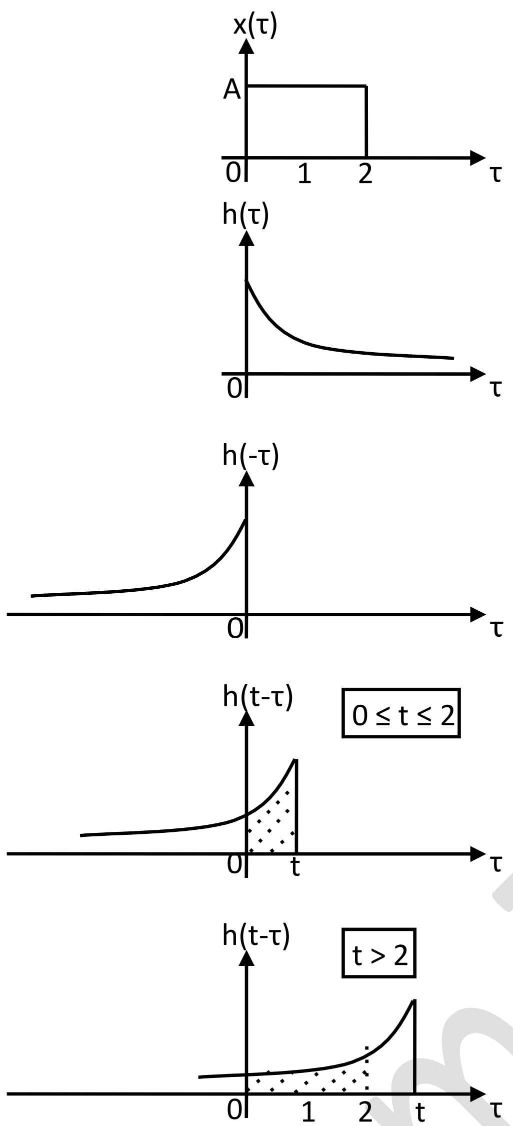


Fig.1a :Evaluation of Convolution Integral

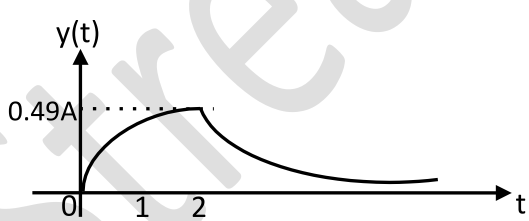


Fig.1b :Sketch of output Signal

Given signals x(t) and h(t) can be written in terms of ' τ ',

$x(\tau) = A \quad \text{for} \quad 0 \leq \tau \leq 2$

and $h(\tau) = e^{-2\tau} \quad \text{for} \ \tau \geq 0$

Now $h(t-\tau) = e^{-2(t-\tau)} \quad \text{for} \ t-\tau \geq 0$

The signals $x(\tau)$ and $h(t-\tau)$ are shown in Fig. .
The output of the system is given by the linear convolution as,

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau) d\tau$$

The integral is non Zero for overlap between

Case – I : For 0 ≤ t ≤ 2

In this case, there will be partial overlap between $x(\tau)$ and $h(t-\tau)$ as shown in the Fig.

$$\therefore y(t) = \int_0^t x(\tau)\, h(t-\tau) d\tau$$

$$\therefore y(t) = \int_0^t A\, e^{-2(t-\tau)} d\tau = Ae^{-2t} \int_0^t e^{2\tau} d\tau$$

$$= Ae^{-2t} \frac{1}{2} [e^{2\tau}]_0^t = Ae^{-2t} \frac{1}{2} (e^{2t} - 1)$$

$$\therefore y(t) = \frac{A}{2} (1 - e^{-2t})$$

Case – I : For t > 2

In this case, there will be complete overlap from 0 to 2 as shown in the Fig.

$$\therefore y(t) = \int_0^2 x(\tau)\, h(t-\tau) d\tau$$

$$= \int_0^2 A\, e^{-2(t-\tau)} d\tau = Ae^{-2t} \int_0^2 e^{2\tau} d\tau$$

$$\therefore y(t) = Ae^{-2t} \frac{1}{2} [e^{2\tau}]_0^2 = 26.8Ae^{-2t}$$

$$\text{Thus } y(t) = \begin{cases} 0 & \text{for} \quad t < 0 \\ \frac{A}{2}(1 - e^{-2t}) & \text{for} \quad 0 \leq t \leq 2 \\ 26.8Ae^{-2t} & \text{for} \quad t < 0 \end{cases}$$

Example 2: Consider $x(t) = 2 \ \text{for} \ 1 \leq t \leq 2$
and $h(t) = 1 \ \text{for} \ 0 \leq t \leq 3$, find x(t) * h(t)

Solution :

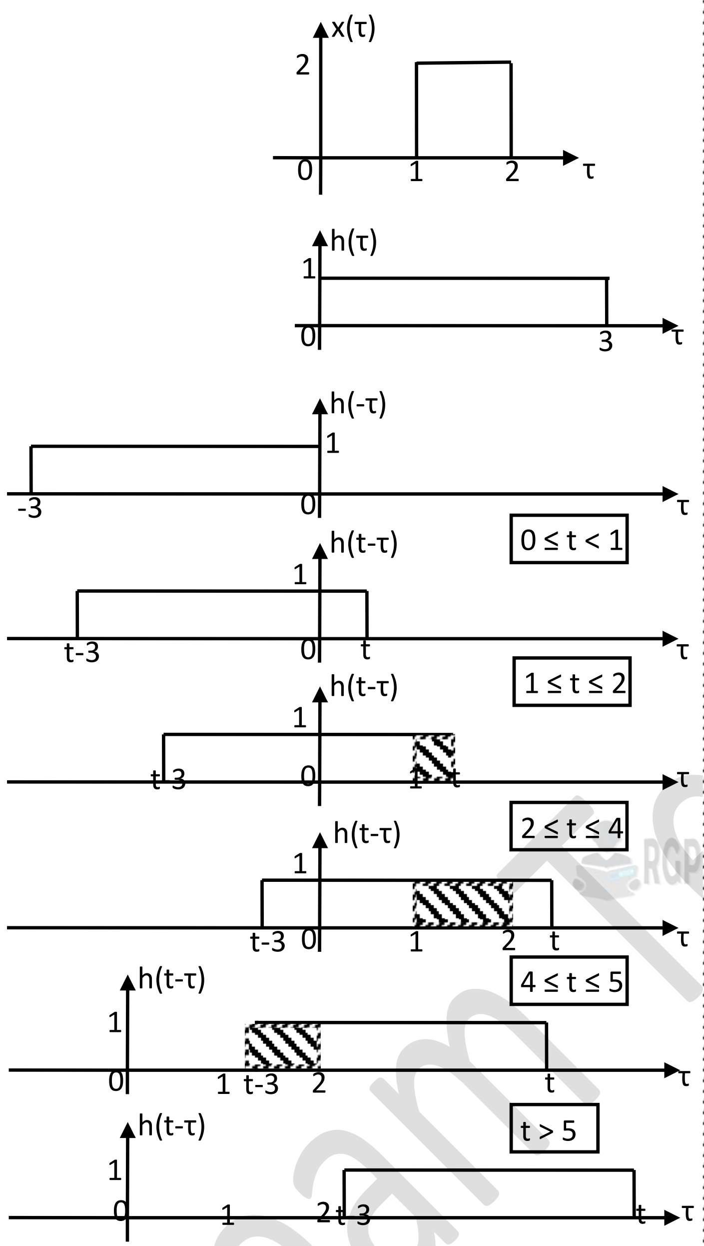


Fig.2a :Evaluation of Convolution Integral

Given signals x(t) and h(t) can be written in terms of 'τ',

$$x(\tau) = 2 \quad \text{for} \quad 1 \le \tau \le 2$$

and $$h(\tau) = 1 \quad \text{for} \quad 0 \le \tau \le 3$$

The signals $x(\tau)$ and $h(t-\tau)$ are shown in Fig.2a.
The output of the system is given by the linear convolution as,

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau) d\tau$$

The integral is non Zero for overlap between $x(\tau)$ and $h(t-\tau)$

Case – I : For 0 ≤ t < 1

In this case, there is no overlap between $x(\tau)$ and $h(t-\tau)$ as shown in the Fig. $\therefore y(t) = 0$

Case – II : For 1 ≤ t ≤ 2

In this case, there is partial overlap between $x(\tau)$ and $h(t-\tau)$ as shown in the Fig. The limits of integration are 1 to t

$$\therefore y(t) = \int_1^t x(\tau)\, h(t-\tau) d\tau$$

$$\therefore y(t) = \int_1^t 2.1.d\tau = 2\int_1^t d\tau = 2\,[\tau]_1^t = 2t$$

$$\boxed{\therefore y(t) = 2t}$$

Case – III : For 2 ≤ t ≤ 4

In this case, there is complete overlap between $x(\tau)$ and $h(t-\tau)$ as shown in the Fig. The limits of integration are 1 to 2

$$\therefore y(t) = \int_1^2 x(\tau)\, h(t-\tau) d\tau$$

$$\therefore y(t) = \int_1^2 2.1.d\tau = 2\int_1^2 d\tau = 2\,[\tau]_1^2 = 2$$

$$\boxed{\therefore y(t) = 2}$$

Case – IV : For 4 ≤ t ≤ 5

In this case, there is partial overlap between $x(\tau)$ and $h(t-\tau)$ as shown in the Fig. The limits of integration are t-3 to 2

$$\therefore y(t) = \int_{t-3}^{2} x(\tau)\, h(t-\tau) d\tau$$

$$\therefore y(t) = \int_{t-3}^{2t} 2.1.d\tau = 2\int_{t-3}^{2t} d\tau = 2\,[\tau]_{t-3}^{2} = 2(1-t)$$

$$\boxed{\therefore y(t) = 2(1-t)}$$

Case – V : For t > 5

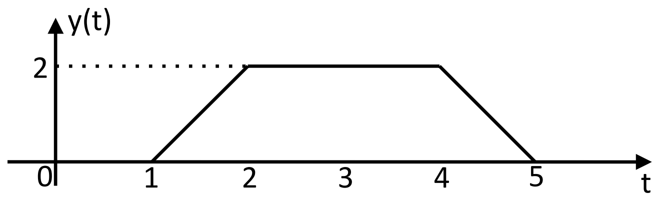


Fig.2b :Sketch of output Signal

In this case, there is no overlap between $x(\tau)$ and $h(t-\tau)$ as shown in the Fig. $\therefore y(t) = 0$

<u>Example 3:</u> Consider $x(t) = u(t+2)$
and $h(t) = u(t-3)$, find x(t) * h(t)

<u>Solution :</u>

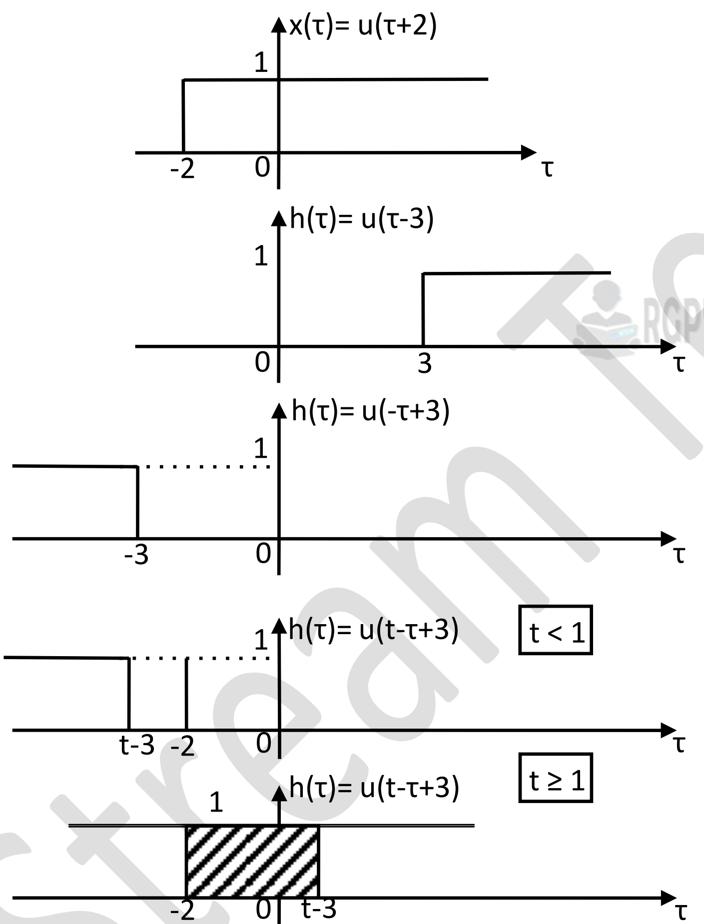


Fig.3a :Evaluation of Convolution Integral

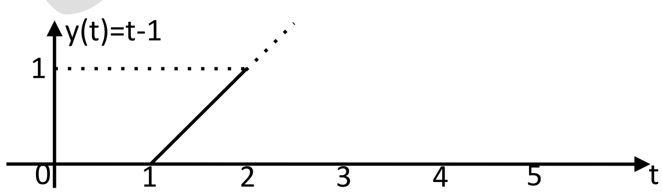


Fig.3b :Sketch of output Signal

Given signals x(t) and h(t) can be written in terms of ' τ ',

$$x(\tau) = u(\tau+2)$$

and $h(\tau) = u(\tau-3)$

The signals $x(\tau)$ and $h(t-\tau)$ are shown in Fig.3a.
The output of the system is given by the linear convolution as,

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau) d\tau$$

The integral is non Zero for overlap between $x(\tau)$ and $h(t-\tau)$

Case – I : For t < 1
In this case, there is no overlap between $x(\tau)$ and $h(t-\tau)$ as shown in the Fig. $\therefore y(t) = 0$

Case – II : For t ≥ 1
In this case, there is overlap between $x(\tau)$ and $h(t-\tau)$ as shown in the Fig. The limits of integration are -2 to t-3

$$\therefore y(t) = \int_{-2}^{t-3} x(\tau)\, h(t-\tau) d\tau$$

$$\therefore y(t) = \int_{-2}^{t-3} 1.1. d\tau = \int_{1}^{t} d\tau = [\tau]_{-2}^{t-3} = (t-3+2)$$

$$\boxed{\therefore y(t) = t-1}$$

Example 4: Obtain the convolution of the following two signals

$$x(t) = \begin{cases} 1 & \text{for } -3 \le t \le 3 \\ 0 & \text{else where} \end{cases} \quad \text{and} \quad h(t) = \begin{cases} 2 & \text{for } 0 \le t \le 3 \\ 0 & \text{else where} \end{cases}$$

Solution :

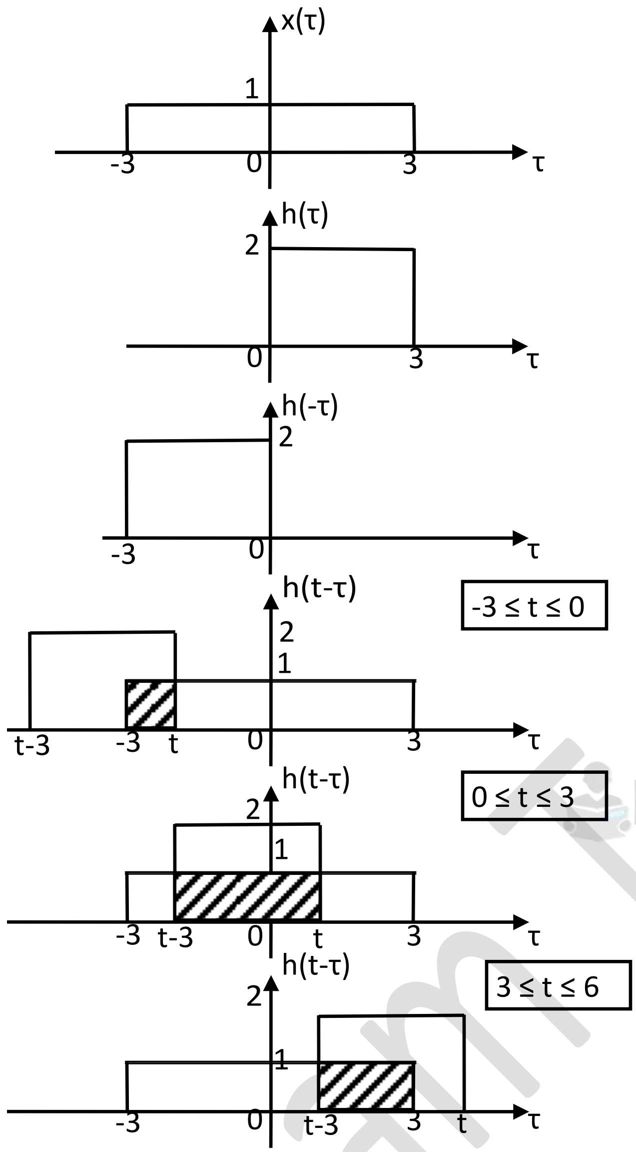


Fig.4a :Evaluation of Convolution Integral

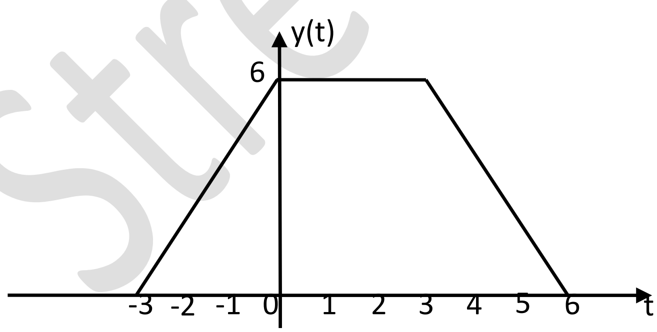


Fig.4b :Sketch of output Signal

Given signals x(t) and h(t) can be written in terms of 'τ',

The signals $x(\tau)$ and $h(t-\tau)$ are shown in Fig.4a.

The output of the system is given by the linear convolution as,

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau) d\tau$$

The integral is non Zero for overlap between $x(\tau)$ and $h(t-\tau)$

Case – I : For t < -3

In this case, there is no overlap between $x(\tau)$ and $h(t-\tau)$ as shown in the Fig. $\therefore y(t) = 0$

Case – II : For -3 ≤ t ≤ 0

In this case, there is overlap between $x(\tau)$ and $h(t-\tau)$ as shown in the Fig. The limits of integration are -3 to t

$$\therefore y(t) = \int_{-3}^{t} x(\tau)\, h(t-\tau) d\tau$$

$$\therefore y(t) = \int_{-3}^{t} 1.2.\, d\tau = 2\int_{-3}^{t} d\tau = 2[\tau]_{-3}^{t} = 2(t+3)$$

$$\boxed{\therefore y(t) = 2(t+3)}$$

Case – III : For 0 ≤ t ≤ 3

In this case, there is overlap between $x(\tau)$ and $h(t-\tau)$ as shown in the Fig. The limits of integration are t-3 to t

$$\therefore y(t) = \int_{t-3}^{t} x(\tau)\, h(t-\tau) d\tau$$

$$\therefore y(t) = \int_{t-3}^{t} 1.2.\, d\tau = 2\int_{t-3}^{t} d\tau = 2[\tau]_{t-3}^{t} = 2(t-t+3)$$

$$\boxed{\therefore y(t) = 6}$$

Case – IV : For 3 ≤ t ≤ 6

In this case, there is overlap between $x(\tau)$ and $h(t-\tau)$ as shown in the Fig. The limits of integration are t-3 to 3

$$\therefore y(t) = \int_{t-3}^{3} x(\tau)\, h(t-\tau) d\tau$$

$$\therefore y(t) = \int_{t-3}^{3} 1.2.\, d\tau = 2\int_{t-3}^{3} d\tau = 2[\tau]_{t-3}^{3} = 2(3-t+3)$$

$$\boxed{\therefore y(t) = 2(-t+6)}$$

Case – V : For t >6

In this case, there is no overlap between $x(\tau)$ and $h(t-\tau)$ as shown in the Fig. $\therefore y(t) = 0$

Example 5: Obtain the convolution of the following two signals shown in Fig.5a.

Solution :

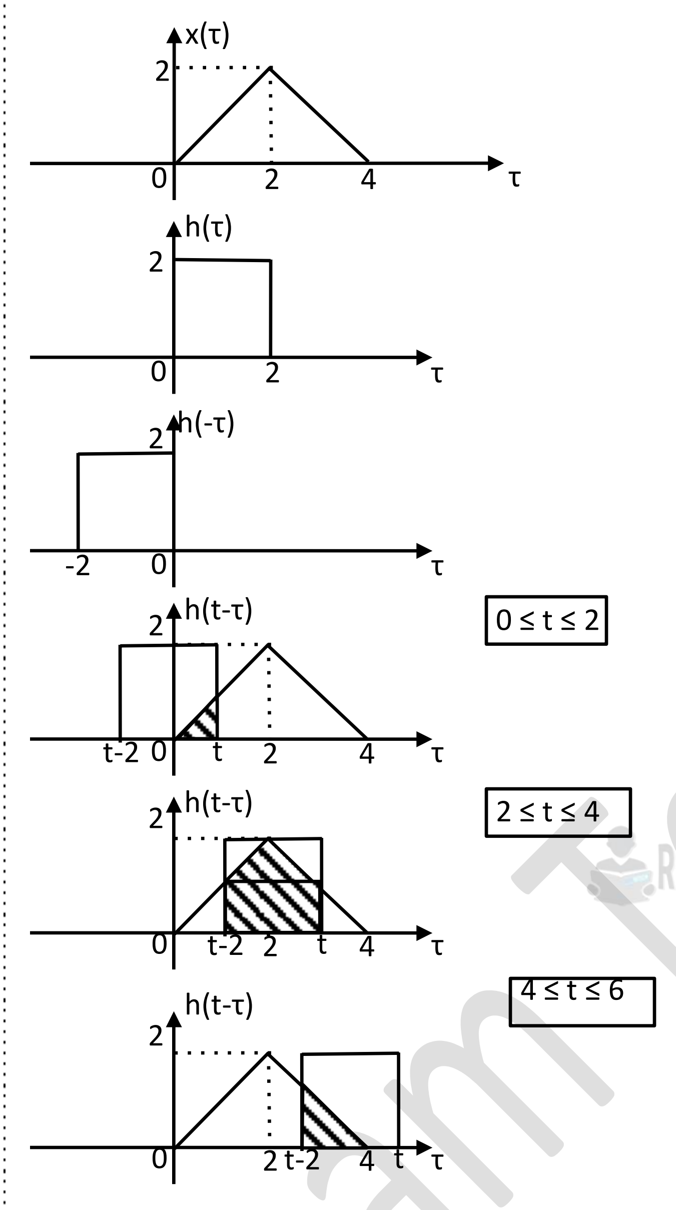


Fig.5a :Evaluation of Convolution Integral

Given signals x(t) and h(t) can be written in terms of 'τ',

The signals $x(\tau)$ and $h(t-\tau)$ are shown in Fig.5a.
The output of the system is given by the linear convolution as,

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau) d\tau$$

The integral is non Zero for overlap between $x(\tau)$ and $h(t-\tau)$

Case – I : For t < 0

In this case, there is no overlap between $x(\tau)$ and $h(t-\tau)$ as shown in the Fig. $\therefore y(t) = 0$

Case – II : For 0 ≤ t ≤ 2

In this case, there is overlap between $x(\tau)$ and $h(t-\tau)$ as shown in the Fig. The limits of integration are 0 to t

$$\therefore y(t) = \int_0^t x(\tau)\, h(t-\tau) d\tau$$

$$\therefore y(t) = \int_0^t \tau . 2 . d\tau = 2\int_0^t \tau d\tau = 2\left[\frac{\tau^2}{2}\right]_0^t = t^2$$

$$\boxed{\therefore y(t) = t^2}$$

Case – III : For 2 ≤ t ≤ 4

In this case, there is overlap between $x(\tau)$ and $h(t-\tau)$ as shown in the Fig. The limits of integration are t-2 to t

$$\therefore y(t) = \int_{t-2}^{t} x(\tau)\, h(t-\tau) d\tau$$

$$\therefore y(t) = \int_{t-2}^{2} \tau . 2 . d\tau + \int_2^t (4-\tau).2.d\tau$$

$$\therefore y(t) = 2\left[\frac{\tau^2}{2}\right]_{t-2}^{2} + 2\left[\frac{(4-\tau)^2}{2}\right]_2^t$$

$$\therefore y(t) = (4-(t-2)^2) + ((4-t)^2 - 4)$$

$$\boxed{\therefore y(t) = 12 - 4t}$$

Case – IV : For 4 ≤ t ≤ 6

In this case, there is overlap between $x(\tau)$ and $h(t-\tau)$ as shown in the Fig. The limits of integration are t-2 to 4

$$\therefore y(t) = \int_{t-2}^{4} x(\tau)\, h(t-\tau) d\tau$$

$$\therefore y(t) = \int_{t-2}^{4} (4-\tau).2.d\tau = 2\left[\frac{(4-\tau)^2}{2}\right]_{t-2}^{4}$$

$$\boxed{\therefore y(t) = -(6-t)^2}$$

Case – V : For t >6

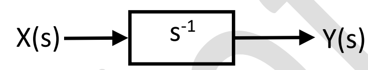
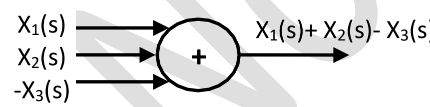
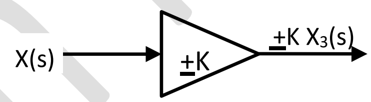
In this case, there is no overlap between $x(\tau)$ and $h(t-\tau)$ as shown in the Fig. $\therefore y(t) = 0$

Block Diagram Representations:

There are four types of system realisation of continuous time LTI systems. They are
(i) Direct form – I (ii) Direct form-II (iii) Cascade form (iv) Parallel form

A transfer function can be realized using either integrators or differentiators. Since the differentiator amplifies the high frequency noise, the integrators are most commonly used in the realization of system as they suppress the high frequency noise. The adder and multipliers are used along with integrator to realize the continuous time systems.

Note: The non-causal systems are not physically realizable. Therefore the system must be causal in this realisation.

Type of Operator	Ideal Transfer Function	Block Representation
Integrator	$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s} = s^{-1}$	
Adder	$X_1(s) + X_2(s) – X_3(s)$	
Multiplier	KX(s)	

(i) Direct Form-I : Direct form-I realization of system is the simplest form of representation of differential equation. In this separate integrators are used for input and output variables.

Example 2.1: Realize the system with the following transfer function in direct form-I:

$$H(s) = \frac{s^2 + 4s + 3}{s^2 + 2s + 5}$$

Solution: To realize the system in direct form-I, the numerator and denominator of given transfer function has to be expressed in power of s^{-1}.

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s^2(1 + 4s^{-1} + 3s^{-2})}{s^2(1 + 2s^{-1} + 5s^{-2})} = \frac{1 + 4s^{-1} + 3s^{-2}}{1 + 2s^{-1} + 5s^{-2}}$$

Cross multiplying the above equation,

$$Y(s)[1 + 2s^{-1} + 5s^{-2}] = X(s)[1 + 4s^{-1} + 3s^{-2}]$$

Let $$\boxed{W(s) = X(s) + 4s^{-1}X(s) + 3s^{-2}X(s)}$$

$$\therefore Y(s)[1 + 2s^{-1} + 5s^{-2}] = W(s)$$

$$\boxed{Y(s) = W(s) - 2s^{-1}Y(s) - 5s^{-2}Y(s)}$$

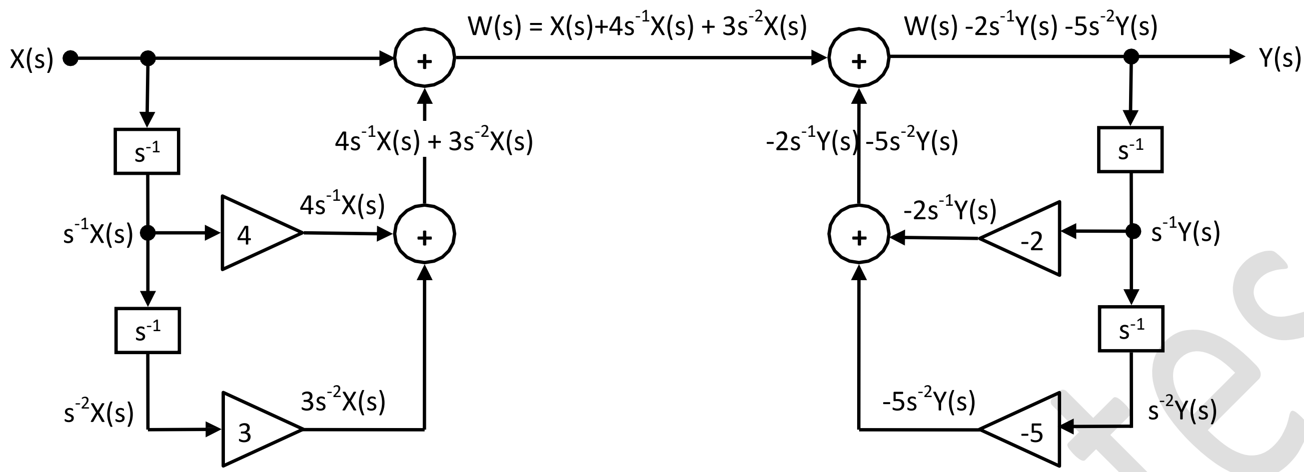


Fig.2.1 : Direct form-I realization of example (2.1)

Example 2.2: Realize the system with the following transfer function in direct form-I:

$$H(s) = \frac{s+5}{s^2 + 2s + 4}$$

Solution: To realize the system in direct form-I, the numerator and denominator of given transfer function has to be expressed in power of s^{-1}.

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s^2(s^{-1} + 5s^{-2})}{s^2(1 + 2s^{-1} + 4s^{-2})} = \frac{s^{-1} + 5s^{-2}}{1 + 2s^{-1} + 4s^{-2}}$$

Cross multiplying the above equation,

$$Y(s)[1 + 2s^{-1} + 4s^{-2}] = X(s)[s^{-1} + 5s^{-2}]$$

Let $\boxed{W(s) = s^{-1}X(s) + 5s^{-2}X(s)}$

$$\therefore Y(s)[1 + 2s^{-1} + 4s^{-2}] = W(s)$$

$$\boxed{Y(s) = W(s) - 2s^{-1}Y(s) - 4s^{-2}Y(s)}$$

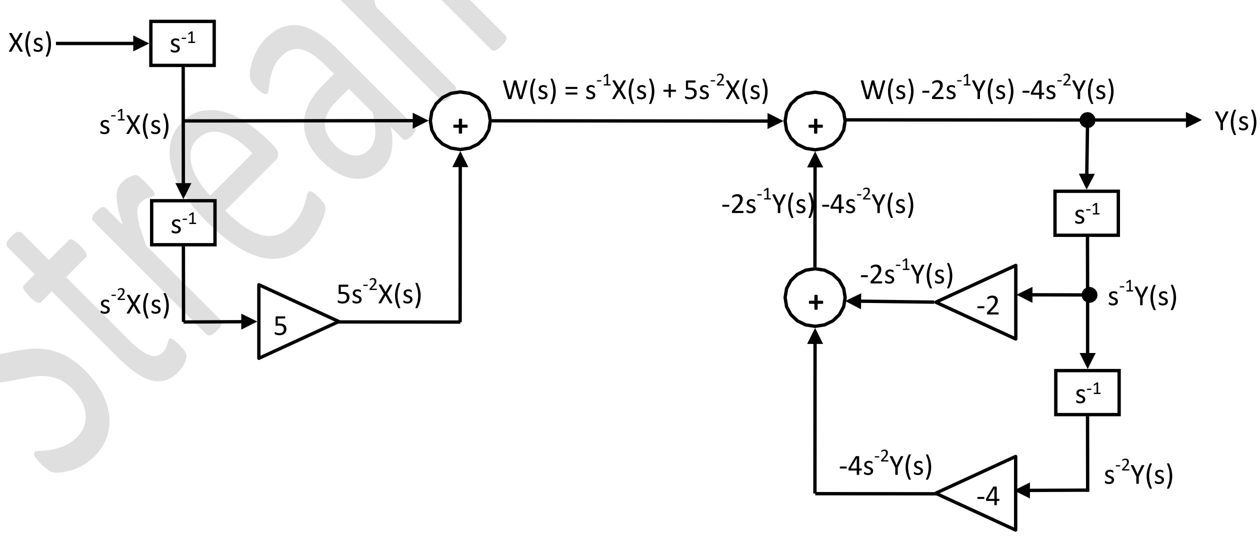


Fig.2.2 : Direct form-I realization of example (2.2)

(ii) *Direct Form-II :* Direct form-II realization of system uses minimum number of integrators. An intermediate variable is used to integrate the input and output variables.

Example 2.3: Realize the system with the following transfer function in direct form-II

$$H(s) = \frac{s^2 + 3s + 4}{s^2 + 5s + 2}$$

Solution: To realize the system in direct form-I, the numerator and denominator of given transfer function has to be expressed in power of s^{-1}.

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s^2(1 + 3s^{-1} + 4s^{-2})}{s^2(1 + 5s^{-1} + 2s^{-2})} = \frac{1 + 3s^{-1} + 4s^{-2}}{1 + 5s^{-1} + 2s^{-2}}$$

Divide the transfer function into two parts,

$$H(s) = \frac{Y(s)}{X(s)} = \frac{Y(s)W(s)}{W(s)X(s)}$$

Where

$$\frac{W(s)}{X(s)} = \frac{1}{1 + 5s^{-1} + 2s^{-2}}$$

and

$$\frac{Y(s)}{W(s)} = 1 + 3s^{-1} + 4s^{-2}$$

Cross multiplying the above equation,

$$\boxed{\begin{aligned} X(s) &= W(s) + 5s^{-1}W(s) + 2s^{-2}W(s) \\ W(s) &= X(s) - 5s^{-1}W(s) - 2s^{-2}W(s) \\ Y(s) &= W(s) + 3s^{-1}W(s) + 4s^{-2}W(s) \end{aligned}}$$

Realize W(s) and Y(s) separately and cascade them to get H(s) as shown in Fig. 5.3(a). Since the inputs and outputs of the integrator are same, they can be commonly used as shown in Fig. 5.3(b).

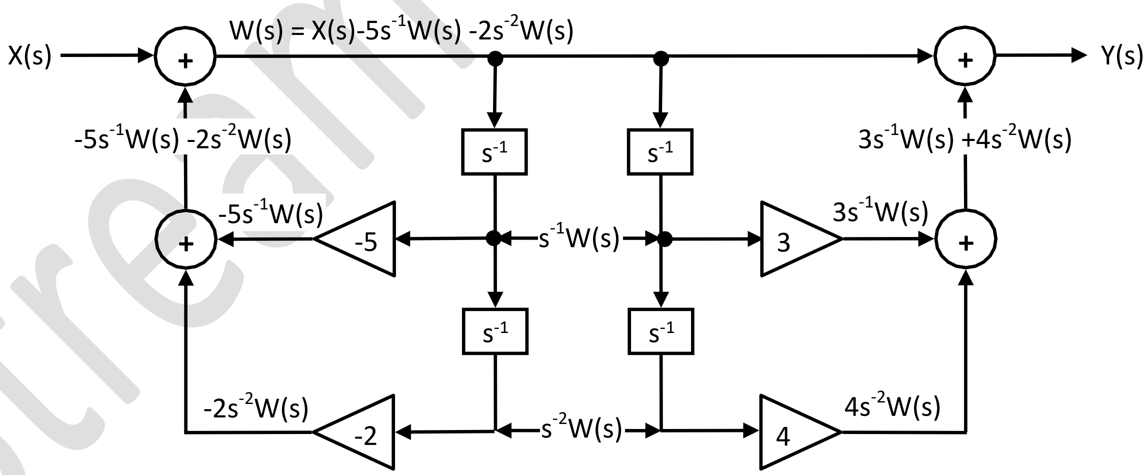


Fig.2.3 (a): W(s) and Y(s) realization of example (2.3)

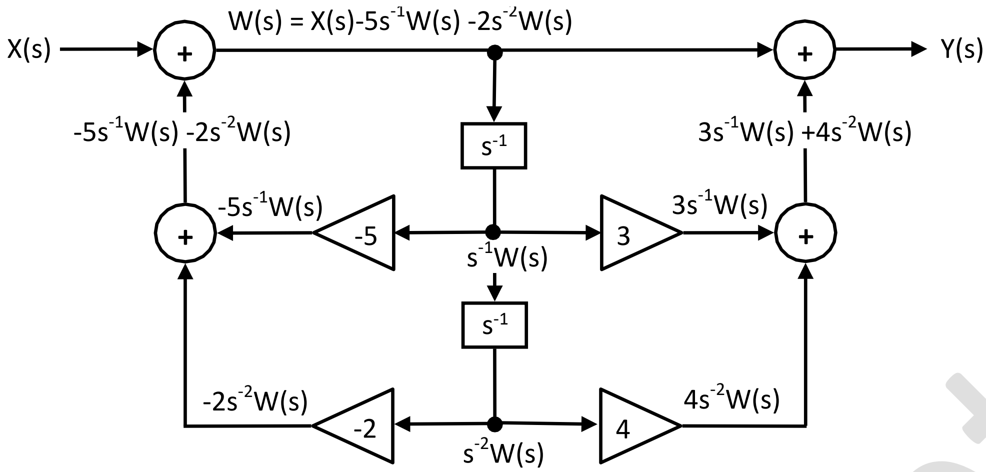


Fig.2.3 (B): Direct Form-II realization of example (2.3)

Example 2.4: Realize the system with the following transfer function in direct form-II

$$H(s) = \frac{s + 4}{s^2 + 3s + 5}$$

Solution: To realize the system in direct form-I, the numerator and denominator of given transfer function has to be expressed in power of s^{-1}.

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s^2(s^{-1} + 4s^{-2})}{s^2(1 + 3s^{-1} + 5s^{-2})} = \frac{s^{-1} + 4s^{-2}}{1 + 3s^{-1} + 5s^{-2}}$$

Divide the transfer function into two parts,

$$H(s) = \frac{Y(s)}{X(s)} = \frac{Y(s)W(s)}{W(s)X(s)}$$

Where

$$\frac{W(s)}{X(s)} = \frac{1}{1 + 3s^{-1} + 5s^{-2}} \quad \text{and} \quad \frac{Y(s)}{W(s)} = s^{-1} + 4s^{-2}$$

Cross multiplying the above equation,

$$\boxed{\begin{aligned} X(s) &= W(s) + 3s^{-1}W(s) + 5s^{-2}W(s) \\ W(s) &= X(s) - 3s^{-1}W(s) - 5s^{-2}W(s) \\ Y(s) &= s^{-1}W(s) + 4s^{-2}W(s) \end{aligned}}$$

The given transfer function is realized in Direct Form-II as shown in Fig. 5.4

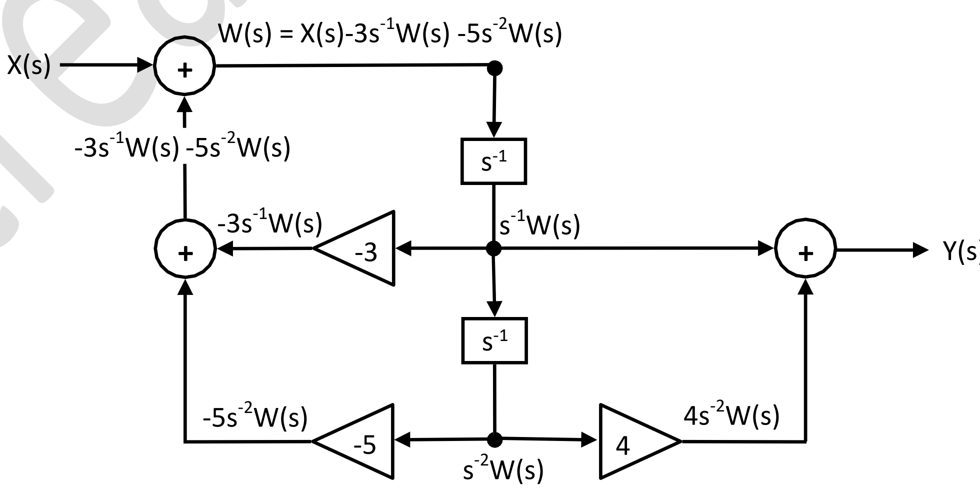


Fig.2.4: Direct Form-II realization of example (2.4)

(iii) *Cascade Form :* The given transfer function is represented as product of several transfer functions. These transfer functions are realized in direct form-II, and then all of them are cascaded to get the complete realization of given transfer function. The input is connected to the first block and output is taken from last transfer function realization.

Example 2.5 : Realize the system with the following transfer function in cascade form

$$H(s) = \frac{4(s^2 + 4s + 3)}{s^3 + 6.5s^2 + 11s + 4}$$

Solution: Express the given equation as product of several transfer functions, we have

$$H(s) = \frac{4(s+1)(s+3)}{(s+0.5)(s+2)(s+4)} = \frac{4}{(s+0.5)} \frac{(s+1)}{(s+2)} \frac{(s+3)}{(s+4)}$$

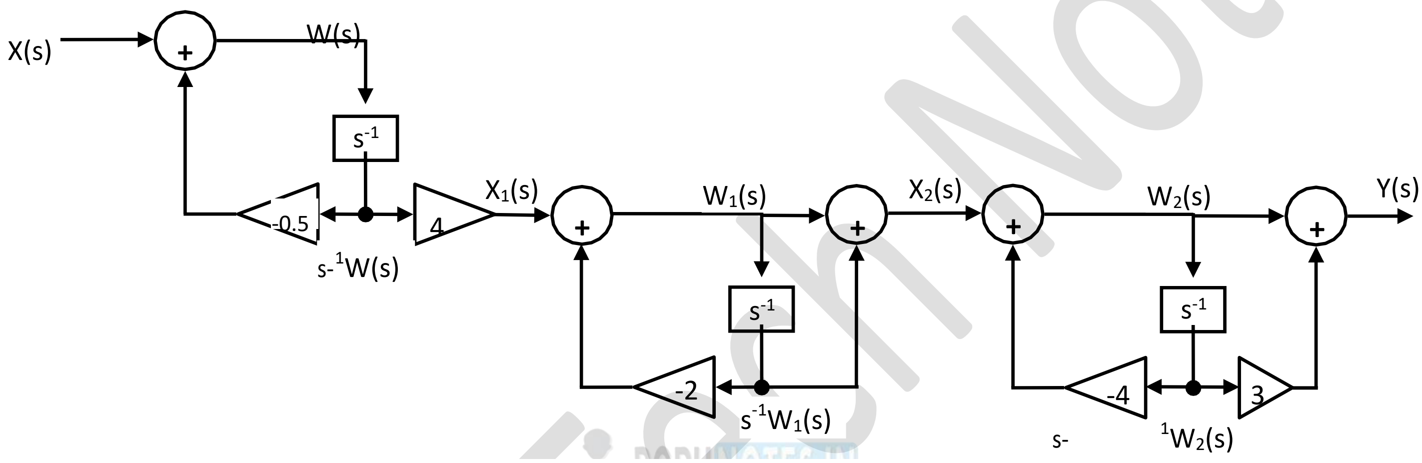


Fig.2.5: Cascade Form realization of example (2.5)

(iv) *Parallel Form :* The given transfer function is expressed into its partial fractions. These factors are realized in direct form-II, and then all of them are connected in parallel to get the complete realization of given transfer function. The input is connected to the each one of these blocks and outputs are added together.

Example 2.6 : Realize the system with the following transfer function in parallel form

$$H(s) = \frac{s(s+1)}{(s+2)(s+3)(s+5)}$$

Solution: Express the given function in its partial fraction form

$$H(s) = \frac{s(s+1)}{(s+2)(s+3)(s+5)} = \frac{A}{(s+2)} + \frac{B}{(s+3)} + \frac{C}{(s+5)}$$

$$s(s+1) = A(s+3)(s+5) + B(s+2)(s+5) + C(s+2)(s+3)$$

where the values of coefficients A ,B and C are calculated as,

$$A|_{s=-2} = \frac{-2(-2+1)}{(-2+3)(-2+5)} = \frac{2}{3}$$

$$B|_{s=-3} = \frac{-3(-3+1)}{(-3+2)(-3+5)} = -3$$

$$C|_{s=-5} = \frac{-5(-5+1)}{(-5+2)(-5+3)} = \frac{10}{3}$$

$$\therefore H(s) = \frac{2/3}{(s+2)} + \frac{-3}{(s+3)} + \frac{10/3}{(s+5)}$$

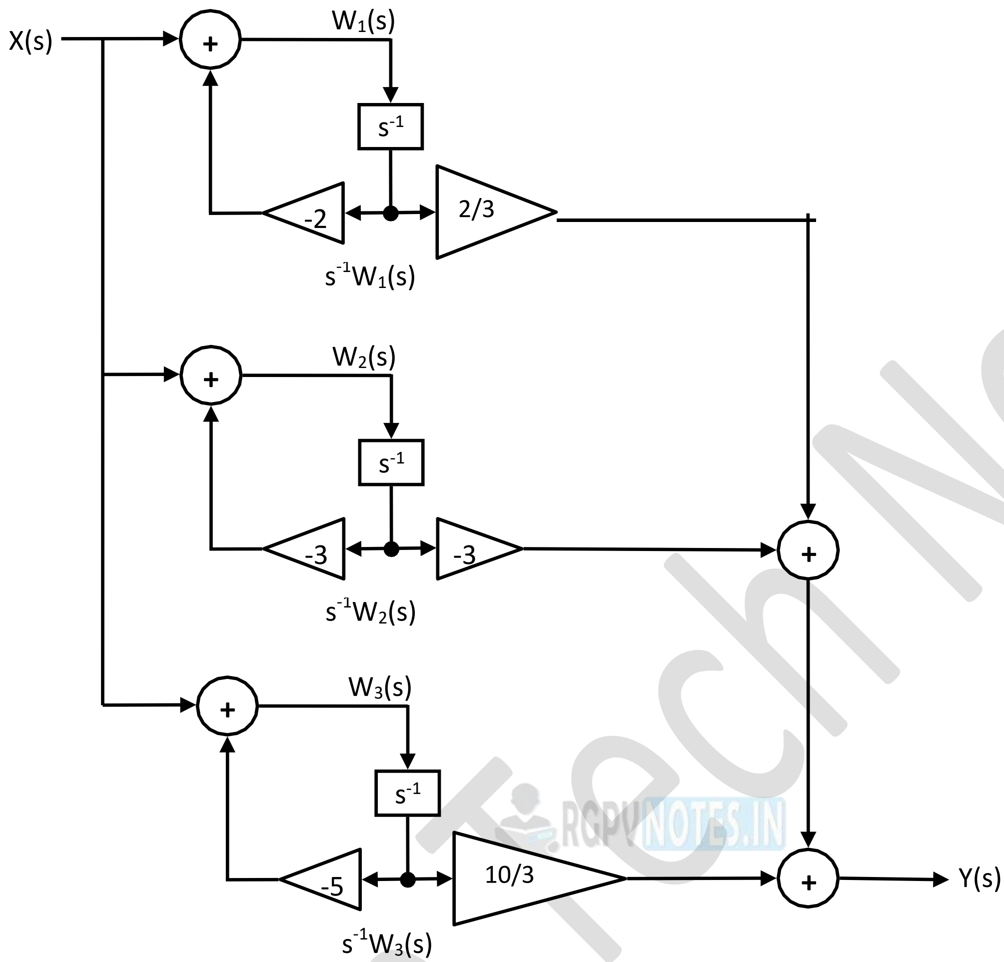


Fig.2.6: Parallel Form realization of example (2.6)

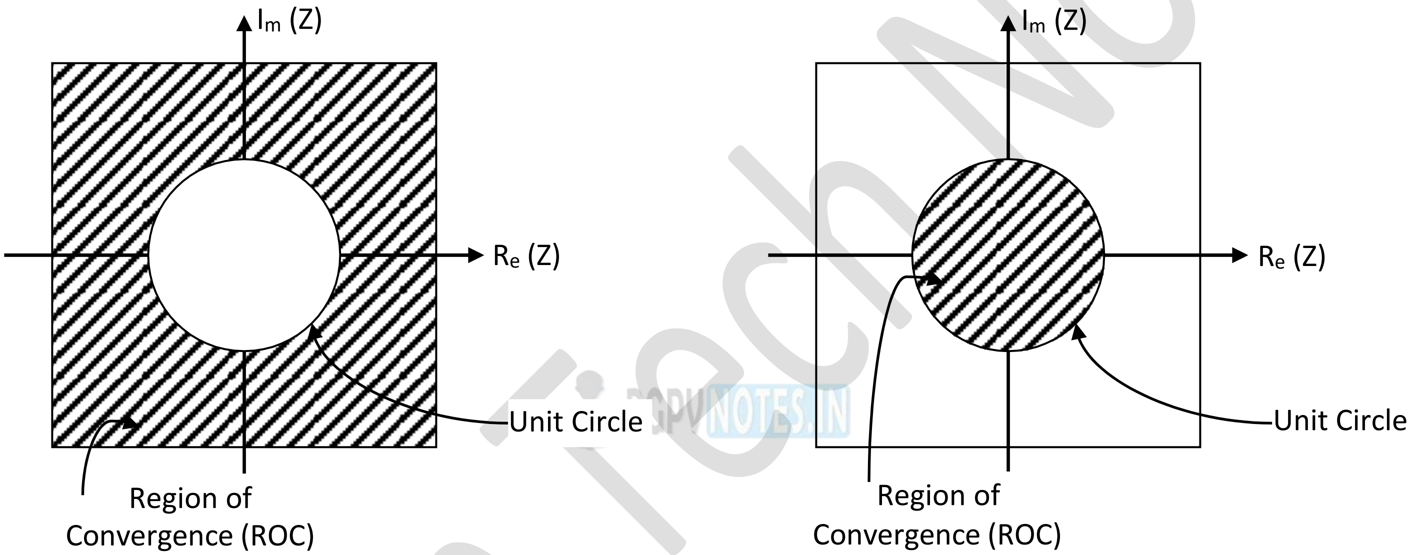
Unit-3 z-Transform: Introduction, ROC of finite duration sequence, ROC of infinite duration sequence, Relation between Discrete time Fourier Transform and z-transform, properties of the ROC, Properties of z-transform, Inverse z-Transform, Analysis of discrete time LTI system using z-Transform, Unilateral z-transform.

Introduction:

The z-transform of x(n) is denoted by X(z). It is defined as,

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

Region of Convergence (ROC): The set of values of z in the z-plane for which the magnitude of X(z) is finite is called the Region of Convergence (ROC). The ROC of X(z) consists of a circle in the z-plane centered about the origin. The Fig.3.1 shows the two possible representation of ROC in z-plane.



(a) : ROC outside Unit Circle

(b): ROC inside Unit Circle

Fig.3.1 : Region of Convergence Plot in Z-Plane

Significance of ROC:

(i) ROC gives an idea about values of z for which z-transform can be calculated

(ii) ROC can be used to test the causality of the system.

(iii) ROC can also be used to test the stability of the system.

Examples: (1) Determine the z-transform of following sequence

(i) $x_1(n) = \{1,2,3,4,5,0,7\}$ (ii) $x_2(n) = \{1,2,3,\underset{\uparrow}{4},5,0,7\}$

Solution: (i) Given that $x_1(n) = \{1,2,3,4,5,0,7\}$

By definition,

$$X_1(z) = \sum_{n=-\infty}^{\infty} x_1[n]z^{-n} \qquad \therefore X_1(z) = \sum_{n=0}^{6} x_1[n]z^{-n}$$

$$\therefore X_1(z) = x(0) + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} + x(4)z^{-4} + x(5)z^{-5} + x(6)z^{-6}$$

$$\boxed{X_1(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 5z^{-4} + 7z^{-6}}$$

$X_1(z)$ is convergent for all values of z, except z=0. Because $X_1(z) = \infty$ for z=0. Therefore the ROC is entire z=plane except z=0.

(ii) Given that $x_2(n) = \{1,2,3,\underset{\uparrow}{4},5,0,7\}$

By definition,

$$X_2(z) = \sum_{n=-\infty}^{\infty} x_2[n]z^{-n} \qquad \therefore X_2(z) = \sum_{n=-3}^{3} x_2[n]z^{-n}$$

$$\therefore X_2(z) = x(-3)z^{3} + x(-2)z^{2} + x(-1)z^{1} + x(0) + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3}$$

$$\boxed{X_1(z) = z^{3} + 2z^{2} + 3z^{1} + 4 + 5z^{-1} + 7z^{-3}}$$

$X_2(z)$ is convergent for all values of z, except z=0 and z=∞. Because $X_2(z) = \infty$ for z=0. Therefore the ROC is entire z=plane except z=0 and z=∞.

Z-Transform of Unit Step, u(n) We know that u(n) = 1 for n≥0 = 0 otherwise By definition $X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$ Now x(n) = u(n) present from n=0 to ∞ $\therefore X(z) = \sum_{n=0}^{\infty} u(n)z^{-n}$ $\therefore X(z) = \sum_{n=0}^{\infty} 1.z^{-n}$ $= 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + \ldots\ldots\ldots$ $= 1 + z^{-1} + (z^{-1})^2 + (z^{-1})^3 + (z^{-1})^4 + \ldots\ldots$ $X(z) = \frac{1}{1 - z^{-1}} = \frac{z}{z - 1}$ The above equation converges if $\|z^{-1}\|<1$ i.e., ROC is $\|z\|>1$. Therefore the ROC is the exterior to the unit circle in the z-plane.	Find Z-transform of given sequence $x(n) = -a^n u(n-1)$ Given that $x(n) = \begin{cases} -a^n & \text{for } n \le 1 \\ 0 & \text{for } n \ge 0 \end{cases}$ By definition $X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$ The given sequence x(n) exists from -∞ to -1 $\therefore X(z) = \sum_{n=-\infty}^{-1} x(n)z^{-n}$ $= \sum_{n=-\infty}^{-1} -a^n z^{-n}$ $= -\sum_{n=-\infty}^{-1} (a^{-1}z)^{-n} = -\sum_{n=1}^{\infty}(a^{-1}z)^n$ $= -\sum_{n=0}^{\infty}(a^{-1}z)^n + 1$ $= -[1 + (a^{-1}z)^1 + (a^{-1}z)^2 + (a^{-1}z)^3 + \ldots\ldots] + 1$ $X(z) = 1 - \frac{1}{1 - a^{-1}z} = \frac{1 - a^{-1}z - 1}{1 - a^{-1}z} = \frac{-a^{-1}z}{1 - a^{-1}z}$ $X(z) = \frac{z}{z - a}$

Relation between Discrete time Fourier Transform and z-transform

The Z-transform of a discrete sequence x(n) is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

The Fourier transform of a discrete sequence x(n) is defined as

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

The X(z) is the unique representation of the sequence x(n) in the complex z-plane. Let $z = re^{j\omega}$

$$\therefore X(z) = \sum_{n=-\infty}^{\infty} x(n)(re^{j\omega})^{-n} = \sum_{n=-\infty}^{\infty} [x(n)r^{-n}]e^{-j\omega n}$$

The RHS of the above equation is the Fourier transform of $x(n)r^{-n}$,
∴ The Z-transform of x(n) is the Fourier transform of $x(n)r^{-n}$.
If r=1, then

$$\therefore X(z) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} = X(\omega)$$

Therefore the Fourier transform of x(n) is same as the Z-transform of x(n) evaluated along the unit circle centered at the origin of the z-plane.

$$\therefore X(\omega) = X(z)|_{z=e^{j\omega}} = \sum_{n=-\infty}^{\infty} x(n)z^{-n}|_{z=e^{j\omega}} = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

For X(ω) to exist, the 'OC must include the unit circle. Since 'OC cannot contain any poles of X(z) all the poles must lie inside the unit circle. Therefore, the Fourier transform can be obtained from Z-transform X(z) for any sequence x(n) if the poles of X(z) are inside the unit circle.

Properties of Z – Transform:

The Z-transform has different properties which can be used to obtain the z-transform of a given sequence. Any complex sequence z-transform can be determined by using the properties, which makes the z-transform a powerful tool for discrete-time system analysis.

(i) Linearity Property

It states that, the Z-transform of a weighted sum of two sequences is equal to the weighted sum of individual Z-transforms.

If $x_1(n) \xleftrightarrow{ZT} X_1(z)$, with ROC $= R_1$

and $x_2(n) \xleftrightarrow{ZT} X_2(z)$, with ROC $= R_2$

then

$$ax_1(n) + bx_2(n) \overset{ZT}{\leftrightarrow} aX_1(z) + bX_2(z), \text{ with ROC} = R_1 \cap R_2$$

Proof: By definition

$$Z[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

$$Z[ax_1(n) + bx_2(n)] = \sum_{n=-\infty}^{\infty} [ax_1(n) + bx_2(n)]z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} ax_1(n)z^{-n} + \sum_{n=-\infty}^{\infty} bx_2(n)z^{-n}$$

$$= a\sum_{n=-\infty}^{\infty} x_1(n)z^{-n} + b\sum_{n=-\infty}^{\infty} x_2(n)z^{-n}$$

$$= aX_1(z) + bX_2(z)\ ; \quad \text{ROC} = R_1 \cap R_2$$

$$\boxed{ax_1(n) + bx_2(n) \overset{ZT}{\leftrightarrow} aX_1(z) + bX_2(z)}$$

(ii) Time Shifting Property

It states that,

If $x(n) \xleftrightarrow{ZT} X(z)$, with ROC $= R$ and with zero initial conditions

then $x(n-m) \xleftrightarrow{ZT} z^{-m} X_2(z)$,

with ROC =R except for the possible addition or deletion of the origin or infinity.

Proof: By definition

$$Z[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

$$Z[x(n-m)] = \sum_{n=-\infty}^{\infty} x(n-m)z^{-n}$$

Put p=n-m in the summation, then n=m+p.

$$Z[x(n-m)] = \sum_{p=-\infty}^{\infty} x(p)z^{-(m+p)}$$

$$Z[x(n-m)] = z^{-m}\sum_{p=-\infty}^{\infty} x(p)z^{-p}$$

$$Z[x(n-m)] = z^{-m}X(z)$$

$$\boxed{\begin{array}{l} Z[x(n-m)] \overset{ZT}{\leftrightarrow} z^{-m}X(z) \\ Z[x(n+m)] \overset{ZT}{\leftrightarrow} z^{m}X(z) \end{array}}$$

(iii) Multiplication by an Exponential Sequence Property (Scaling Property)

It states that,

If $x(n) \overset{ZT}{\leftrightarrow} X(z)$, with ROC $= R$

then $a^n x(n) \overset{ZT}{\leftrightarrow} X\left(\frac{z}{a}\right)$, with ROC $= |a|R$

where 'a' is a complex number

Proof: By definition

$$Z[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

$$Z[a^n x(n)] = \sum_{n=-\infty}^{\infty} a^n x(n)z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\left(\frac{z}{a}\right)^{-n}$$

$$= X\left(\frac{z}{a}\right)$$

$$\boxed{a^n x(n) \overset{ZT}{\leftrightarrow} X\left(\frac{z}{a}\right)}$$

$$e^{j\omega n}x(n) \overset{ZT}{\leftrightarrow} X\left(\frac{z}{e^{j\omega}}\right) = X(e^{-j\omega}z)$$

$$e^{-j\omega n}x(n) \overset{ZT}{\leftrightarrow} X\left(\frac{z}{e^{-j\omega}}\right) = X(e^{j\omega}z)$$

(iv) Time Reversal Property

It states that,

If $x(n) \overset{ZT}{\leftrightarrow} X(z)$, with ROC $= R$

then $x(-n) \overset{ZT}{\leftrightarrow} X\left(\frac{1}{z}\right)$, with ROC $= \frac{1}{R}$

Proof: By definition

$$Z[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

$$Z[x(-n)] = \sum_{n=-\infty}^{\infty} x(-n)z^{n}$$

$$Z[x(-n)] = \sum_{n=-\infty}^{\infty} x(-n)(z^{-1})^{-n}$$

$$\therefore Z[x(-n)] = X(z^{-1})$$

Sl. No.	Time Domain Sequence x(n)	Z-Transform X(z)	ROC
1.	δ(n)	1	Entire Z-plane
2.	u(n)	$\frac{1}{1-z^{-1}}$	|z|>1
3.	$a^n u(n)$	$\frac{1}{1-az^{-1}}$	|z|>|a|
4.	$-a^n u(-n-1)$	$\frac{1}{1-az^{-1}}$	|z|<|a|
5.	$n.a^n u(n)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	|z|>|a|
6.	$-n.a^n u(-n-1)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	|z|<|a|

Inverse z-Transform: Inverse z-transform of X(z) can be obtained by three different methods,

(i) Power Series Expansion (Long Division Method)
(ii) Partial Fraction Expansion
(iii) Contour Integration (Residue Method)

(i) Power Series Expansion (Long Division Method):

Example: Determine the inverse z-transform of the following (i) $X(z) = (1/(1-az^{-1})$, ROC $|z|>|a|$
(ii) $X(z) = (1/(1-az^{-1})$, ROC $|z|<|a|$

Solution: (i)

$1+az^{-1}+a^2z^{-2}+a^3z^{-3}+$ ------

$1-az^{-1}$) 1

$1 - az^{-1}$

$-\quad +$

az^{-1}

$az^{-1} - a^2z^{-2}$

$-\quad +$

a^2z^{-2}

$a^2z^{-2} - a^3z^{-3}$

$-\quad +$

a^3z^{-3}

$a^3z^{-3} - a^4z^{-4}$

$-\quad +$

a^4z^{-4} -----

Thus we have

$$X(z) = \frac{1}{1-az^{-1}} = 1 + az^{-1} + a^2z^{-2} + a^3z^{-3} + - - -$$

Taking Inverse z-transform , $x(n) = \{1,a,a^2,a^3,$-------- $\}$
$x(n) = a^n u(n)$

Solution: (ii)

$-\ a^{-1}z - a^{-2}z^2 - a^{-3}z^3 -$ ------

$-az^{-1} + 1$) 1

$1 - a^{-1}z$

$-\quad +$

$a^{-1}z$

$a^{-1}z - a^{-2}z^2$

$-\quad +$

$a^{-2}z^2$

$a^{-2}z^2 - a^{-3}z^3$

$-\quad +$

$a^{-3}z^3$

$a^{-3}z^3 - a^{-4}z^4$

$-\quad +$

$a^{-4}z^{-4}$ -------

Thus we have

$$X(z) = \frac{1}{1-az^{-1}} = -a^{-1}z - a^{-2}z^2 - a^{-3}z^3 + - - -$$

Taking Inverse z-transform , $x(n) = \{$--------------, $-a^{-3},-a^{-2},-a^{-1},\}$
$x(n) = -a^n u(-n-1)$

(ii) Partial Fraction Method:

Step - 1 : First convert given X(z) into positive powers of z and then write $\frac{X(z)}{z}$

Step - 2 : Using partial fraction method, write the equation in terms of summation of poles. Find the constants in the numerator.

Step - 3 : Rewrite the equation in the form of X(z).

Step - 4 : Based on the condition of ROC, write the inverse z-transform x(n) of X(z).

(iii) Contour Integration (Residue Method:

Step-1: Define the function $X_0(z)$ which is rational and its denominator is expanded into product of poles.

$$X_0(z) = X(z)\, z^{n-1}$$

Step-2: (i) For Simple poles, the residue of $X_0(z)$ at pole p_i is given as,

$$\underset{z=p_i}{\mathrm{Res}} = \lim_{z=pi} (z - p_i)\, X_0(z)$$

(ii) For multiple poles of order m_0, the residue of $X_0(z)$ can be calculated as,

$$\underset{z=p_i}{\mathrm{Res}} = \frac{1}{(m-1)!}\left\{\frac{d^{m-1}}{dz^{m-1}}(z - p_i)X_0(z)\right\}_{z=p_i}$$

Step-3: (i) Using residue theorem, calculate x(n), for poles inside the unit circle

$$x(n) = \sum_{i=1}^{N} \underset{z=p_i}{\mathrm{Res}}\ X_0(z) \quad \text{with } n \geq 0$$

(ii) Using residue theorem, calculate x(n), for poles outside the unit circle

$$x(n) = -\sum_{i=1}^{N} \underset{z=p_i}{\mathrm{Res}}\ X_0(z) \quad \text{with } n < 0$$

Solution of Difference Equations using Z-transform:

The difference equations can be easily solved using z-transform.

Examples :

(1) Given that y(-1) =5 and y(-2)=0, solve the difference equation y(n) - 3y(n-1) – 4y(n-2) = 0 , n ≥ 0.

Solution : Consider the given difference equation

$$y(n) - 3y(n-1) - 4y(n-2) = 0$$

Taking unilateral z-transform of the given difference equation

$$Y(z) - 3\,[z^{-1}Y(z) + y(-1)] - 4\,[z^{-2}Y(z) + z^{-1}y(-1) + y(-2)] = 0$$

Put the initial conditions in above equation, we get

$$Y(z) - 3\,[z^{-1}Y(z) + 5] - 4\,[z^{-2}Y(z) + 5z^{-1} + 0] = 0$$

$$Y(z)[1 - 3z^{-1} - 4\,z^{-2}] - 20z^{-1} - 15 = 0$$

$$Y(z) = \frac{15 + 20z^{-1}}{1 - 3z^{-1} - 4z^{-2}} = \frac{z(15z + 20)}{z^2 - 3z - 4}$$

$$\frac{Y(z)}{z} = \frac{(15z + 20)}{z^2 - 3z - 4} = \frac{(15z + 20)}{(z + 1)(z - 4)}$$

Using partial faction method, the above equation can be written as,

$$\frac{(15z + 20)}{(z + 1)(z - 4)} = \frac{A}{(z + 1)} + \frac{B}{(z - 4)}$$

$$(15z + 20) = A\,(z - 4) + B\,(z + 1)$$

$$\therefore A|_{z=-1} = \frac{15z + 20}{(z - 4)} = -1 \qquad \text{and} \qquad B|_{z=4} = \frac{15z + 20}{(z + 1)} = 16$$

$$\therefore \frac{Y(z)}{z} = \frac{-1}{(z + 1)} + \frac{16}{(z - 4)}$$

$$\therefore Y(z) = \frac{-z}{(z + 1)} + \frac{16z}{(z - 4)} = \frac{-1}{(1 + z^{-1})} + \frac{-16}{(1 - 4z^{-1})}$$

Taking Inverse Z-transform of the above equation, we get

$$\boxed{y(n) = - (-1)^n\, u(n) + 16\, (4)^n\, u(n) = [- (-1)^n + 16\, (4)^n\,]\, u(n)}$$

(2) Solve the difference equation using z-transform method x(n-2) – 9x(n-1) + 18x(n) =0. Initial conditions are x(-1)=1 , x(-2) =9.

Solution : Consider the given difference equation

$$x(n-2) - 9x(n-1) + 18x(n) = 0$$

Taking unilateral z-transform of the given difference equation

$$[z^{-2}X(z) + z^{-1}x(-1) + x(-2)] - 9[z^{-1}X(z) + x(-1)] + 18\, X(z) = 0$$

Put the initial conditions in above equation, we get

$$[z^{-2}X(z) + z^{-1} + 9] - 9[z^{-1}X(z) + 1] + 18\, X(z) = 0$$

$$[z^{-2} - 9z^{-1} + 18\,]\, X(z) + z^{-1} = 0$$

$$X(z) = \frac{-z^{-1}}{z^{-2} - 9z^{-1} + 18} = \frac{-z}{1 - 9z + 18z^2}$$

$$\frac{X(z)}{z} = \frac{-1}{18z^2 - 9z + 1} = \frac{-1}{(6z - 1)(3z - 1)}$$

Using partial faction method, the above equation can be written as,

$$\frac{-1}{(6z - 1)(3z - 1)} = \frac{A}{(6z - 1)} + \frac{B}{(3z - 1)}$$

$$-1 = A(3z - 1) + B(6z - 1)$$

$$\therefore A|_{z=\frac{1}{6}} = \frac{-1}{(3z - 1)} = \frac{1}{2} \qquad \text{and} \qquad B|_{z=\frac{1}{3}} = \frac{-1}{(6z - 1)} = -1$$

$$\therefore \frac{X(z)}{z} = \frac{\frac{1}{2}}{(6z - 1)} - \frac{1}{(3z - 1)}$$

$$\text{Now } X(z) = \frac{\frac{1}{2}z}{(6z - 1)} - \frac{z}{(3z - 1)} = \frac{\frac{1}{3}}{(1 - \frac{1}{6}z^{-1})} - \frac{\frac{1}{3}}{(1 - \frac{1}{3}z^{-1})}$$

Taking Inverse Z-transform of the above equation, we get

$$\boxed{x(n) = \frac{1}{3}\left(\frac{1}{6}\right)^n u(n) - \frac{1}{3}\left(\frac{1}{3}\right)^n u(n) = \left[\frac{1}{3}\left(\frac{1}{6}\right)^n - \frac{1}{3}\left(\frac{1}{3}\right)^n\right] u(n)}$$

Unit-4 Fourier analysis of discrete time signals: Introduction, Properties and application of discrete time Fourier series, Representation of Aperiodic signals, Fourier transform and its properties, Convergence of discrete time Fourier transform, Fourier Transform for periodic signals, Applications of DTFT.

The Fourier transform of a discrete sequence x(n) is defined as
The DTFT of x(n) is defined as,

$$F[x(n)] = X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

The inverse DTFT of X(ω) is defined as,

$$F^{-1}[X(\omega)] = x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)\, e^{-j\omega n}\, d\omega$$

Examples: Find the DTFT of the following sequences.

(a) δ(n) (b) u(n) (c) $a^n u(n)$

Solution:

(a) Given, x(n) = δ(n)

$$\delta(n) = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{for } n \neq 0 \end{cases}$$

$$F[\delta(n)] = X(\omega) = \sum_{n=-\infty}^{\infty} \delta(n)e^{-j\omega n}\,|_{n=0} = 1$$

$$\therefore F[\delta(n)] = 1$$

(b) Given, x(n) = u(n)

$$u(n) = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

$$F[u(n)] = X(\omega) = \sum_{n=-\infty}^{\infty} u(n)e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty}(1)e^{-j\omega n} = \frac{1}{1 - e^{-j\omega}}$$

(c) Given, x(n) = $a^n u(n)$

$$x(n) = \begin{cases} a^n & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

$$F[x(n)] = X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} = \sum_{n=-\infty}^{\infty} a^n u(n)e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty}(ae^{-j\omega})^n = \frac{1}{1 - ae^{-j\omega}}$$

Convergence of discrete time Fourier transform:

The Fourier transform of a sequence x(n) exists if and only if the summation is finite value.

$$\text{.., } \quad \sum_{n=-\infty}^{\infty} |x(n)| < \infty$$

The DTFT of the sequence does not exist if the sequence is growing exponentially. The DTFT can be used only for the systems whose system function H(z) has poles inside the unit circle. The Fourier transform represents the frequency components of the sequence x(n). It is unique in the frequency range –π to π .

Properties of DTFT:

(i) Linearity Property

It states that, the DTFT of a weighted sum of two sequences is equal to the weighted sum of individual DTFT.

If $x_1(n) \xleftrightarrow{DTFT} X_1(\omega)$,

and $x_2(n) \xleftrightarrow{DTFT} X_2(\omega)$,

then $ax_1(n) + bx_2(n) \xleftrightarrow{DTFT} aX_1(\omega) + bX_2(\omega)$

Proof: By definition

$$F[x(n)] = X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

$$F[ax_1(n) + bx_2(n)] = \sum_{n=-\infty}^{\infty} [ax_1(n) + bx_2(n)]e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} ax_1(n)e^{-j\omega n} + \sum_{n=-\infty}^{\infty} bx_2(n)e^{-j\omega n}$$

$$= a\sum_{n=-\infty}^{\infty} x_1(n)e^{-j\omega n} + b\sum_{n=-\infty}^{\infty} x_2(n)e^{-j\omega n}$$

$$= aX_1(\omega) + bX_2(\omega)$$

$$\therefore\ ax_1(n) + bx_2(n) \xleftrightarrow{DTFT} aX_1(\omega) + bX_2(\omega)$$

(ii) Time Shifting Property

It states that,

If $x(n) \xleftrightarrow{DTFT} X(\omega)$,

then $x(n-m) \xleftrightarrow{DTFT} e^{-j\omega m} X(\omega)$,

where m is an integer.

Proof: By definition

$$F[x(n)] = X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

$$F[x(n-m)] = \sum_{n=-\infty}^{\infty} x(n-m)e^{-j\omega n}$$

Put p=n-m in the summation, then n=m+p.

$$F[x(p)] = \sum_{p=-\infty}^{\infty} x(p)e^{-j\omega(m+p)}$$

$$F[x(p)] = e^{-j\omega m}\sum_{p=-\infty}^{\infty} x(p)e^{-j\omega p}$$

$$F[x(p)] = e^{-j\omega m}X(\omega)$$

$$\therefore\ x(n-m) \xleftrightarrow{DTFT} e^{-j\omega m} X(\omega),$$

(iii) Frequency Shifting Property

It states that,

If $x(n) \xleftrightarrow{DTFT} X(\omega)$,

then $x(n)e^{j\omega_0 n} \xleftrightarrow{DTFT} X(\omega - \omega_0)$

Proof: By definition

$$F[x(n)] = X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

$$F[x(n)e^{j\omega_0 n}] = \sum_{n=-\infty}^{\infty} \{x(n)e^{j\omega_0 n}\}\, e^{-j\omega n}$$

$$F[x(n)e^{j\omega_0 n}] = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j(\omega-\omega_0)n} = X(\omega - \omega_0)$$

$$\therefore\ x(n)e^{j\omega_0 n} \xleftrightarrow{DTFT} X(\omega - \omega_0)$$

This property is dual of time shifting property.

(iv) Time Reversal Property

It states that,

If $x(n) \xleftrightarrow{DTFT} X(\omega)$,

then $x(-n) \xleftrightarrow{DTFT} X(-\omega)$

Proof: By definition

$$F[x(n)] = X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

$$F[x(-n)] = \sum_{n=-\infty}^{\infty} x(-n)\, e^{-j\omega n}$$

$$F[x(-n)] = \sum_{n=-\infty}^{\infty} x(n)\, e^{j\omega n} = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j(-\omega)n}$$

$$F[x(-n)] = X(-\omega)$$

$$\therefore\ x(-n) \xleftrightarrow{DTFT} X(-\omega)$$

(v) Differentiation in Frequency Domain Property

It states that,

If $x(n) \xleftrightarrow{DTFT} X(\omega)$,

then $n.x(n) \xleftrightarrow{DTFT} j\frac{dX(\omega)}{d\omega}$

Proof: By definition

$$F[x(n)] = X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

Differentiating on both sides w.r.t ω we get,

$$\frac{d}{d\omega}X(\omega) = \frac{d}{d\omega}\sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\frac{d}{d\omega}e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)(-jn)e^{-j\omega n} = -j\sum_{n=-\infty}^{\infty}(n)x(n)e^{-j\omega n}$$

$$j\frac{d}{d\omega}X(\omega) = F(nx(n))$$

$$\therefore n.x(n) \xleftrightarrow{DTFT} j\frac{dX(\omega)}{d\omega}$$

(vi) Time Convolution Property

It states that,

If $x_1(n) \xleftrightarrow{DTFT} X_1(\omega)$,

and $x_2(n) \xleftrightarrow{DTFT} X_2(\omega)$,

then $x_1(n) * x_2(n) \xleftrightarrow{DTFT} X_1(\omega).X_2(\omega)$

Proof: By definition

$$x_1(n) * x_2(n) = \sum_{k=-\infty}^{\infty} x_1(k)x_2(n-k)$$

$$F[x_1(n) * x_2(n)] = \sum_{n=-\infty}^{\infty} x_1(n) * x_2(n)e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty}\{\sum_{k=-\infty}^{\infty} x_1(k)x_2(n-k)\}\}e^{-j\omega n}$$

$$= \sum_{k=-\infty}^{\infty} x_1(k)\sum_{n=-\infty}^{\infty} x_2(n-k)\,e^{-j\omega n}$$

Put n-k=p and n=p+k

$$= \sum_{k=-\infty}^{\infty} x_1(k)\sum_{p=-\infty}^{\infty} x_2(p)\,e^{-j\omega(p+k)}$$

$$= \sum_{k=-\infty}^{\infty} x_1(k)e^{-j\omega k}\sum_{p=-\infty}^{\infty} x_2(p)\,e^{-j\omega p}$$

$$\therefore F[x_1(n) * x_2(n)] = X_1(\omega).X_2(\omega)$$

$$x_1(n) * x_2(n) \xleftrightarrow{DTFT} X_1(\omega).X_2(\omega)$$

(vii) The Modulation Theorem

It states that,

If $x(n) \xleftrightarrow{DTFT} X(\omega)$,

then $x(n)\cos\omega_0 n \xleftrightarrow{DTFT} \frac{1}{2}\{X(\omega+\omega_0) + X(\omega+\omega_0)\}$

Proof: By definition

$$F[x(n)\cos\omega_0 n] = \sum_{n=-\infty}^{\infty} x(n)\cos\omega_0 n e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\frac{(e^{j\omega_0 n} + e^{-j\omega_0 n})}{2}e^{-j\omega n}$$

$$= \frac{1}{2}\sum_{n=-\infty}^{\infty} x(n)e^{-j(\omega-\omega_0)n} + \sum_{n=-\infty}^{\infty} x(n)e^{-j(\omega+\omega_0)n}$$

$$= \frac{1}{2}\{X(\omega+\omega_0) + X(\omega+\omega_0)\}$$

$$\therefore x(n)\cos\omega_0 n \xleftrightarrow{DTFT} \frac{1}{2}\{X(\omega+\omega_0) + X(\omega+\omega_0)\}$$

Examples: Using properties of DTFT, find the DTFT of the following (i) $n(1/2)^n u(n)$ (ii) $u(n+1)-u(n+2)$ (iii) $e^{j3n}\, u(n)$

Solution: (i) Using Differentiation in frequency domain property, we have

$$F\left\{n\left(\frac{1}{2}\right)^n u(n)\right\} = j\frac{d}{d\omega}\left[F\left\{\left(\frac{1}{2}\right)^n u(n)\right\}\right]$$

$$= j\frac{d}{d\omega}\left[\frac{1}{1-\left(\frac{1}{2}\right)e^{-j\omega}}\right]$$

$$= j\left[\frac{-\left[-\left(\frac{1}{2}\right)e^{-j\omega}(-j)\right]}{\left\{1-\left(\frac{1}{2}\right)e^{-j\omega}\right\}^2}\right]$$

$$= \left[\frac{\left(\frac{1}{2}\right)e^{-j\omega}}{\left\{1-\left(\frac{1}{2}\right)e^{-j\omega}\right\}^2}\right]$$

(ii) Using time shifting property

$$F\{u(n+1)-u(n+2)\} = F\{u(n+1)\} - \{u(n+2)\}$$

$$= e^{j\omega}F\{u(n)\} - e^{j2\omega}\{u(n)\}$$

$$= \frac{e^{j\omega}}{1-e^{-j\omega}} - \frac{e^{j2\omega}}{1-e^{-j\omega}}$$

(iii) Using Frequency shifting property

$$F\{e^{j3n}u(n)\} = F\{u(n)\}|_{\omega=\omega-3}$$

$$= \left\{\frac{1}{1-e^{-j\omega}}\right\}_{\omega=\omega-3} = \left\{\frac{1}{1-e^{-j(\omega-3)}}\right\}$$