RAJIV GANDHI PROUDYOGIKI VISHWAVIDYALAYA, BHOPAL

New Scheme Based On AICTE Flexible Curricula

B. Tech. First Year (II Semester)

Branch- Common to All Disciplines

BT202	MATHEMATICS-II	3L-1T-0P	4 Credits

OBJECTIVES: The objective of this course is to familiarize the prospective engineers with techniques in Ordinary and partial differential equations, complex variables and vector calculus. It aims to equip the students to deal with advanced level of mathematics and applications that would be essential for their disciplines. More precisely, the objectives are:

- To introduce effective mathematical tools for the solutions of ordinary and partial differential equations that model physical processes.
- To introduce the tools of differentiation and integration of functions of complex variable that are used in various techniques dealing engineering problems.
- To acquaint the student with mathematical tools available in vector calculus needed various field of science and engineering.

Course Contents:

Module 1:Ordinary Differential Equations I :(6 hours) : Differential Equations of First Order and First Degree (Leibnitz linear, Bernoulli's, Exact), Differential Equations of First Order and Higher Degree, Higher order differential equations with constants coefficients, Homogeneous Linear Differential equations, Simultaneous Differential Equations.

Module 2:Ordinary differential Equations II:(8 hours) : Second order linear differential equations with variable coefficients, Method of variation of parameters, Power series solutions; Legendre polynomials, Bessel functions of the first kind and their properties.

Module 3: Partial Differential Equations : (8 hours) : Formulation of Partial Differential equations, Linear and Non-Linear Partial Differential Equations, Homogeneous Linear Partial Differential Equations with Constants Coefficients.

Module 4: Functions of Complex Variable :(8 hours) : Functions of Complex Variables: Analytic Functions, Harmonic Conjugate, Cauchy-Riemann Equations (without proof), Line Integral, Cauchy-Goursat theorem (without proof), Cauchy Integral formula (without proof), Singular Points, Poles & Residues, Residue Theorem, Application of Residues theorem for Evaluation of Real Integral (Unit Circle).

Module 5: Vector Calculus : (10 hours) : Differentiation of Vectors, Scalar and vector point function, Gradient, Geometrical meaning of gradient, Directional Derivative, Divergence and Curl, Line Integral, Surface Integral and Volume Integral, Gauss Divergence, Stokes and Green theorems.

Textbooks/References:

1. G.B. Thomas and R.L. Finney, Calculus and Analytic geometry, 9th Edition, Pearson, Reprint, 2002.
2. Erwin kreyszig, Advanced Engineering Mathematics, 9th Edition, John Wiley & Sons, 2006.
3. W. E. Boyce and R. C. DiPrima, Elementary Differential Equations and Boundary Value Problems, 9th Edn., Wiley India, 2009.
4. S. L. Ross, Differential Equations, 3rd Ed., Wiley India, 1984.
5. E. A. Coddington, An Introduction to Ordinary Differential Equations, Prentice Hall India, 1995.
6. E. L. Ince, Ordinary Differential Equations, Dover Publications, 1958.
7. J. W. Brown and R. V. Churchill, Complex Variables and Applications, 7th Ed., McGraw Hill, 2004.
8. N.P. Bali and Manish Goyal, A text book of Engineering Mathematics, Laxmi Publications, Reprint, 2008.
9. B.S. Grewal, Higher Engineering Mathematics, Khanna Publishers, 36th Edition, 2010.

CLASS NOTES

Module 1: Ordinary Differential Equations-I

Contents
Differential Equations of First Order and First Degree (Leibnitz linear, Bernoulli's, Exact), Differential Equations of First Order and Higher Degree, Higher order differential equations with constants coefficients, Homogeneous Linear Differential equations, Simultaneous Differential Equations.

Introduction: Differential equations arise from many problems in oscillations of mechanical and electrical systems, bending of beams, conduction of heat, velocity of chemical reactions etc.,and as such play a very important role in all modern scientific and engineering studies.

Definitions:
1. **Ordinary Differential Equation**: An ordinary differential equation is that in which all the differential coefficients have reference to a single independent variable.

For example: $\frac{dy}{dx} = x + 1$

2. Order and Degree of Ordinary Differential Equation:
The order of a differential equation is the order of highest derivative appearing in it.
The degree of a differential equation is power of highest derivative, while the highest derivative free from radicals and fractions.

For example: 1. $\frac{dy}{dx} = x^2 + 1$ it is first order and first degree.

2. $\frac{d^2y}{dx^2} - \frac{dy}{dx} = 0$ It is second order and first degree.

In general an ordinary differential equation of nth order is denoted by

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots\ldots \frac{d^ny}{dx^n}\right) = 0$$

Part-I: Solution of First order and First degree Ordinary Differential Equation

Prerequisite Knowledge

[I] Variable Separable Form :-

An differential Equation is of the type $\dfrac{dy}{dx}=\dfrac{f_1(x)}{f_2(y)}$ --------- (1)

can be written as $f_2(y)\,dy = f_1(x)dx$

Integrate both sides

$$\int f_2(y)dy = \int f_1(x)dx + C$$

i.e. $F_2(y) = F_1(x)dx + C$

Where C is the constant of integration .

Which is the required general solution of given equation (1).

Example-1.Solve the differential Equation $x^3\dfrac{dy}{dx} - 3y^2 = xy^2$

Solution**:** Given equation can be written as

$$x^3\frac{dy}{dx} = xy^2 + 3y^2$$

$$x^3\frac{dy}{dx} = y^2(x+3)$$

Or $$\frac{dy}{y^2} = \left(\frac{1}{x^2} + \frac{3}{x^3}\right)dx$$

It is in variable separable form

Therefore integrate both sides

$$\frac{1}{x} - \frac{1}{y} + \frac{3}{2x^2} = C$$

Example-2. Solve the differential Equation $\dfrac{dy}{dx} - x\tan(y-x) = 1$

Solution**:** Put $y - x = t$, therefore differentiate w.r.t. x

$$\frac{dy}{dx}-1=\frac{dt}{dx} \quad \therefore \frac{dy}{dx}=1+\frac{dt}{dx}$$

Hence the given equation becomes

$$1+\frac{dt}{dx}-x\tan t=1$$

$$or\ \cot t\, dt = x dx$$

$$Integrate\ \ both\ \ sides$$

$$\log\sin t=\frac{x^2}{2}+\log c$$

$$or\ \frac{\sin t}{c}=e^{x^2/2}$$

$$or \sin(y-x)=ce^{x^2/2}$$

[II] Homogeneous Form :-

A function, in which each term is of same degree is known as a homogenous function.

A differential Equation is of the type $\frac{dy}{dx}=\frac{f_1(x,y)}{f_2(x,y)}$ --------- (1)

is called a homogeneous differential Equation of order 1 and degree 1, if

$f_1(x,y)$ *and* $f_2(x,y)$ be two different homogeneous functions of x and y .

To find the general solution ,put

$$y=vx\ \ where\ v=f(x)$$

$$\therefore \frac{dy}{dx}=v+x\frac{dv}{dx}$$

Hence the given equation is reduces to variable separable form and we solve them accordingly and we put $v=\frac{y}{x}$ in the last step.

Example . Solve the differential equation $x\frac{dy}{dx}=y+\sqrt{x^2+y^2}$

Solution: Given equation can be written as

$$\frac{dy}{dx}=\frac{y+\sqrt{x^2+y^2}}{x}$$

It is a homogeneous differential Equation of order 1 and degree 1.

To solve put,

$$y=vx \ \ where\ v=f(x)$$

$$\therefore \frac{dy}{dx}=v+x\frac{dv}{dx}$$

$$Hence\ \ the\,, givenequation\ becomes$$

$$v+x\frac{dv}{dx}=\frac{vx+x\sqrt{1+v^2}}{x}$$

$$\frac{dv}{\sqrt{1+v^2}}=\frac{dx}{x}$$

$$It\,is\ \ in\ \ \ \mathrm{var}\,iable\ \ separable\ \ form$$

$$Integrateboth\ sides$$

$$Log\left[v+\sqrt{1+v^2}\right]=\log x+\log c$$

$$or\ \ v+\sqrt{1+v^2}\ =cx$$

$$\frac{y}{x}+\sqrt{1+\frac{y^2}{x^2}}=c\,x$$

$$y+\sqrt{x^2+y^2}=cx^2 .$$

Type-I : Linear differential equation of order 1 and degree 1 :-

A differential equation is said to be Linear if the dependent variable and it's derivatives appear only in the first degree and are not multiplied together**.**

The equation of the type $\frac{dy}{dx}+Py=Q$ ------- (1) is called a linear **differential equation in** y

Where $P\ and\ Q$ be some functions of x .

Solution: Let the solution be $y=uv$ --------- (2) where $u\ and\ v$ be some functions of x.

$$\therefore \frac{dy}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}$$

$\therefore$ Equation (1) becomes

$$u\frac{dv}{dx}+v\frac{du}{dx}+\ P\,uv=Q$$

$$[\frac{du}{dx}+Pu]v+u\frac{dv}{dx}=Q \text{ ----------- (3)}$$

In (2), one of the two functions $u\ and\ v$ is arbitrary,

$\therefore$ Let we choose u in such a way that, $\frac{du}{dx}+Pu=0$

$$\therefore \frac{du}{u}=-Pdx$$

It is in variable separable form

Integrate both sides

$$logu=-\int Pdx$$

$$\therefore\ u=\mathrm{e}^{-\int Pdx}$$

Hence the equation (3) becomes

$$u\frac{dv}{dx}=Q \qquad \therefore\ \frac{dv}{dx}=\frac{Q}{u}$$

$$\frac{dv}{dx}=Q\mathrm{e}^{\int Pdx}$$

$$dv=Q\mathrm{e}^{\int Pdx}\,dx$$

It is in variable separable form

Integrate both sides

$$v=\int Q\mathrm{e}^{\int Pdx}\,dx+c$$

Hence the required solution is

$$y = e^{-\int Pdx}\left[\int Qe^{\int Pdx}\,dx + c\right]$$

Or $\quad ye^{\int Pdx} = \int Qe^{\int Pdx}\,dx + c$

The expression $e^{\int Pdx}$ is called the integrating factor [I.F.].

Example-1: $Solve \quad cos^2x\,\frac{dy}{dx} + y = tanx$

Solution: Given equation can be written as $\frac{dy}{dx} + Sec^2x\;y = Sec^2x\;tanx$

It is of the type $\frac{dy}{dx} + Py = Q$ i.e. it is linear in y

$\therefore P = sec^2x \quad and \quad Q = Sec^2x\;tanx$, I.F. = $e^{\int Pdx} = e^{\int Sec^2x\,dx} = e^{tanx}$

Hence the general solution is

$$ye^{\int Pdx} = \int Qe^{\int Pdx}\,dx + c$$

$$y\;e^{tanx} = \int Sec^2x\;tanx\;e^{tanx}\,dx + c$$

Put $\quad tanx = t$

$$\therefore Sec^2x\;dx = dt$$

Hence the above equation becomes

$$y\;e^{tanx} = \int te^t dt + c$$

$$y\;e^{tanx} = te^t - e^t + c$$

Or $\qquad y\;e^{tanx} = e^{tanx}(tanx - 1) + c$.

Example-2: $Solve\; ydx - xdy + 3x^2y^2e^{x^3}dx = 0$

Solution: Given equation can be written as

$$ydx - xdy + 3x^2y^2e^{x^3}dx = 0$$

$$\Rightarrow (y + 3x^2y^2e^{x^3})dx = xdy$$

$$\Rightarrow x\frac{dy}{dx} = y + 3x^2y^2e^{x^3}$$

$$\Rightarrow \frac{dy}{dx} - \frac{y}{x} = 3xy^2e^{x^3}$$

$$\Rightarrow \frac{1}{y^2}\frac{dy}{dx} - \frac{1}{yx} = 3xe^{x^3}$$

Put $\frac{1}{y} = z \Rightarrow -\frac{1}{y^2}\frac{dy}{dx} = \frac{dz}{dx}$, then

$$-\frac{dz}{dx} - \frac{z}{x} = 3xe^{x^3}$$

$$\Rightarrow \frac{dz}{dx} + \frac{z}{x} = -3xe^{x^3}$$

This is a linear differential equation $in\ z$

I.F. $= e^{\int \frac{1}{x} dx} = e^{\log x} = x$.Hence solution is

$$z \cdot IF = \int (-3xe^{x^3}) \cdot IF\, dx + C$$

$$\Rightarrow zx = -\int 3x^2 e^{x^3} dx + C = -e^{x^3} + C$$

$$\Rightarrow \frac{x}{y} = -e^{x^3} + C$$

Example-3: $Solve\ y(\log y)dx + (x - \log y)dy$=0

Solution: Given equation can be written as

$$y(\log y)\frac{dx}{dy} + x - \log y = 0$$

$$\Rightarrow \qquad \frac{dx}{dy} + \frac{x}{y(\log y)} = \frac{1}{y}$$

This is a linear differential equation in x

I.F. $= e^{\int \frac{1}{y(\log y)} dy}$, Put $\log y = t$, then$e^{\int \frac{1}{t} dt} = e^{\log t} = t = \log y$. Hence solution is

$x \cdot IF = \int \frac{1}{y} \cdot IF dy + c \quad \Rightarrow x \cdot \log y = \int \frac{\log y}{y}\, dy + c$ Put$\log y = t$

$$\Rightarrow x \cdot \log y = \int t dt + c = \frac{t^2}{2} + c = \frac{(\log y)^2}{2} + c$$

Type-II : Non-Linear differential equation of order 1 and degree 1 :-

The equation of the type $\frac{dy}{dx} + Py = Qy^n$--------- (1) is called a Non- linear differential equation **in** y or Bernoulli's Equation or Leibnitz's linear equation. Where $P\ and\ Q$ be some functions of x and $n > 1$.

Solution: To solve, we divide equation (1) by y^n on both sides.

$y^{-n}\frac{dy}{dx} + Py^{1-n} = Q$ -------------- (2)

Put $y^{1-n} = t$

$$\therefore (1-n)y^{-n}\frac{dy}{dx} = \frac{dt}{dx}$$

$$\Rightarrow y^{-n}\frac{dy}{dx} = \frac{1}{1-n}\frac{dt}{dx}$$

Hence, the equation (2) becomes.

$$\frac{1}{1-n}\frac{dt}{dx} + Pt = Q$$

$$\Rightarrow \frac{dt}{dx} + (1-n)Pt = (1-n)Q$$

It is linear in t

∴I.F. = $e^{\int (1-n)P dx}$

Hence the general solution is

$$t.I.F. = \int I.F.(1-n)Q\ dx + C$$

Or

$$y^{1-n}.I.F. = \int I.F.(1-n)Q\ dx + C$$

Example-1: $Solve \quad x\frac{dy}{dx} + y = x^3y^6$ ---- (1)

Solution: To solve, we divide equation (1) by xy^6 on both sides.

∴ $y^{-6}\frac{dy}{dx} + \frac{1}{x}y^{-5} = x^2$ ------- (2)

$$y^{-5} = t$$

$$\therefore -5\,y^{-6}\frac{dy}{dx} = \frac{dt}{dx}$$

Hence, the equation (2) becomes

$$-\frac{1}{5}\frac{dt}{dx} + \frac{t}{x} = x^2$$

$$\text{or} \quad \frac{dt}{dx} - 5\frac{t}{x} = -5x^2$$

It is linear in t

∴I.F. = $e^{\int - -\ d}$ $= ^{-}$

Hence the general solution is

$$t.I.F. = \int I.F.(1-n)Q\ dx + C$$

$$t.^{-} = \int {}^{-}(-5x^2).\ dx + C$$

$$0r \quad t.^{-} = -5\int (x^{-3}).\ dx + C$$

Or $y^{-5-} \quad \frac{=}{2} \ {}^{-2} + C$

Example-2: $Solve \quad xy(1+xy^2)\frac{dy}{dx} = 1$

Solution: Given equation can be written as

$$xy + x^2y^3 = \frac{dx}{dy}$$

$$\Rightarrow \frac{dx}{dy} - xy = x^2y^3$$

$$\Rightarrow \frac{1}{x^2}\frac{dy}{dx} - \frac{y}{x} = y^3$$

Put $-\frac{1}{x} = z \Rightarrow \frac{1}{x^2}\frac{dx}{dy} = \frac{dz}{dy}$

$\Rightarrow \frac{dz}{dy} + yz = y^3$

This is a linear differential equation in z

$IF = e^{\int y dy} = e^{\frac{y^2}{2}}.$

Hence the general solution is

$z \cdot IF = \int y^3 \cdot IF\ dy + c$

$\Rightarrow z \cdot e^{\frac{y^2}{2}} = \int y^3 e^{\frac{y^2}{2}} dy + c$ Put $e^{\frac{y^2}{2}} = t \Rightarrow ye^{\frac{y^2}{2}} dy = dy$

$\Rightarrow \ z \cdot e^{\frac{y^2}{2}} = \int \log t\ dt + c = t(\log t - 1) + c\ .$

Put the values of z and t, then $-\frac{1}{x}e^{\frac{y^2}{2}} = e^{\frac{y^2}{2}}(\frac{y^2}{2} - 1) + c$

Type-III : Exact differential equation of order 1 and degree 1:-

The equation of the type $Mdx + Ndy = 0$--------(1) is called an **exact** differential equation of **order 1 and degree 1 iff** $\frac{M}{y} = \frac{N}{x}$ Where M and N be some functions of x and y.i.e. It is an exact differential **of** some functions of x and y.

Verification: If the equation of the type $Mdx + Ndy = 0$ --------(1) is called an exact differential **of** order 1 and degree 1**.**
Then $Mdx + Ndy = du$, Where u be some functions of x and y.
But $du = \frac{u}{x}dx + \frac{u}{y}dy$
Hence, $Mdx + Ndy = \frac{u}{x}dx + \frac{u}{y}dy$
i.e. $M = \frac{u}{x}$,$N = \frac{u}{y}$
$\therefore \frac{M}{y} = \frac{{}^2u}{yx}$,$\frac{N}{x} = \frac{{}^2u}{xy}$

But, we know that ,$\frac{{}^2u}{yx} = \frac{{}^2u}{xy}$

Hence from above $\frac{M}{y} = \frac{N}{x}$.
Hence proved

The Method of Solution of Exact differential equation of order 1 and degree 1 is

$$\int Mdx_{Keeping\ \ y\ constant} + \int(terms\ \text{of}\ N\ \ not\ containing\ x)dy = C\ .$$

Example-1: $Show\ that\ the\ equation$ $(2x^3 + 3y)dx + (3x + y - 1)dy = 0$is exact. Find it' s general solution.

Solution**:** On comparing the given equation with$Mdx + Ndy = 0$

$\therefore M = 2x^3 + 3y$, $N = 3x + y - 1$

$\frac{M}{y} = 3$, $\frac{N}{x} = 3$

Hence,$\frac{M}{y} = \frac{N}{x}$Therefore ,it is an exact differential equation.
The general solution is

$$\int Mdx_{Keeping\ y\ constant} + \int(terms\ \text{of}\ N\ \ not\ containing\ x)dy = C$$

$\Longrightarrow \int(2x^3 + 3y)dx_{\textit{Keeping y constant}} + \int(y-1)dy = C$

$\Longrightarrow 2\frac{x^4}{4} + 3xy + \frac{y^2}{2} - y = C$.

Example-2**:** *Show that the equation* $(y^2e^{xy^2} + 4x^3)dx + (2xye^{xy^2} - 3y^2)dy = 0$is exact. Find it' s general solution .

Solution**:** On comparing the given equation with$Mdx + Ndy = 0$

$\therefore \quad M = (y^2e^{xy^2} + 4x^3)$ And $N = 2xye^{xy^2} - 3y^2$,

Hence , $\frac{M}{y} = 2ye^{xy^2} + 2xy^3e^{xy^2}$ and $\frac{N}{x} = 2ye^{xy^2} + 2xy^3e^{xy^2}$.

Here, we have $\frac{M}{y} = \frac{N}{x}$, i.e. Equation is exact differential equation. Hence the general solution is

$$\int Mdx + \int(\text{terms of } N \text{ not containing } x)dy = c \Rightarrow \int(y^2e^{xy^2} + 4x^3)dx - \int 3y^2dy = c$$

$\Rightarrow e^{xy^2} + x^4 - y^3 = c.$

Type-VII : Solution of Non-Exact differential equation of order 1 and degree 1 :-

When the equation is of the type $Mdx + Ndy = 0$ is not exact. Then, we multiply this equation by some function of x and y or by a suitable factor so that the equation becomes exact. Then this factor is called I.F. i.e. Integrating factor.

Rule-1 of I.F.:--When the equation is of the type $Mdx + Ndy = 0$ is not exact and if M *and* N be homogeneous functions of x and $y$and $x + Ny \neq 0$, Then $I.F. = \frac{1}{Mx+Ny}$

Example: *Solve the equation* $(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy = 0$

Solution: On comparing the given equation with$Mdx + Ndy = 0$

$\therefore M = x^2y - 2xy^2, N = -(x^3 - 3x^2y)$

$\frac{M}{y} = x^2 - 4xy, \quad \frac{N}{x} = -3x^2 + 6xy$

Hence the given equation is not exact**.**
Since M *and* N be homogeneous functions of x and y.

Now, $Mx + Ny = x^2y^2 \neq 0$ hence $I.F. = \frac{1}{Mx+Ny} = \frac{1}{x^2y^2}$

Therefore, multiply given equation by this I.F

We get, $(\frac{1}{y} - \frac{2}{x})\,dx - (\frac{x}{y^2} - 3/y)\,dy = 0$ is an exact equation.

The general solution is

$$\int_{Keeping\ y\ constant} Mdx + \int (terms\ of\ N\ not\ containing\ x)dy = C$$

$$\Longrightarrow \int_{Keeping\ \ y\ constant} (\frac{1}{y} - \frac{2}{x})\,dx + \int (3/y)dy = C$$

$$\Longrightarrow logy - 2logx + 3logy = C\ .$$

Rule-2 of I.F.:-- When the equation is of the type $Mdx + Ndy = 0$ is not Exact and if it is of the type $f_1(x,y)ydx + f_2(x,y)xdy = 0$ and $Mx - Ny \neq 0$, Then $I.F. = \frac{1}{Mx-Ny}$

Example: $Solve\ the\ equation\ (2xy^2 + y)dx + (x + 2x^2y - x^4y^3)dy = 0$

Solution: On comparing the given equation with $Mdx + Ndy = 0$

$\therefore M = 2xy^2 + y, N = x + 2x^2y - x^4y^3$

$\frac{\partial M}{\partial y} = 4xy + 1, \quad \frac{\partial N}{\partial x} = 1 + 4xy - 4x^3y^3$

Hence the given equation is not exact**.**

Since, It is of the type $f_1(x,y)ydx + f_2(x,y)xdy = 0$.

$\therefore \quad Mx - Ny = x^4y^4 \neq 0$ hence , $I.F. = \frac{1}{Mx-Ny} = \frac{1}{x^4y^4}$

Therefore, multiply given equation by this I.F

We get, $(\frac{2}{x^3y^2} + \frac{1}{x^4y^3})\,dx + (\frac{1}{x^3y^4} + \frac{2}{x^2y^3} - 1/y)\,dy = 0$ is an exact equation.

Hence , the general solution is

$$\int_{Keeping\ y\ constant} Mdx + \int (terms\ of\ N\ not\ containing\ x)dy = C$$

$$\Longrightarrow \int \left(\frac{2}{x^3y^2} + \frac{1}{x^4y^3}\right) dx_{\text{Keeping } y \text{ constant}} \quad + \int -(1/y)dy = C$$

$$\Longrightarrow -\frac{1}{x^2y^2} + \frac{1}{3x^3y^3} - logy = C \ .$$

Rule-3 of I.F.:-- When the equation is of the type $Mdx + Ndy = 0$ is not exact

If $\dfrac{\frac{M}{y} - \frac{N}{x}}{N} = \mathrm{f}(x)$ then, I.F. = $e^{\int \mathrm{f}(x)dx}$.

Rule-4 of I.F.:-- When the equation is of the type $Mdx + Ndy = 0$ is not exact

If $\dfrac{\frac{N}{x} - \frac{M}{y}}{M} = g(y)$ then, I.F. = $e^{\int g(y)dy}$

Example-1 : $Solve\ the\ equation\ (x^2 + y^2 + x)dx + (xy\ )dy = 0$

Solution: On comparing the given equation with $Mdx + Ndy = 0$

$\therefore M = x^2 + y^2 + x \quad , N = xy$

$\dfrac{M}{y} = 2y, \quad \dfrac{N}{x} = y$

Hence the given equation is not exact.

Now, $\dfrac{\frac{M}{y} - \frac{N}{x}}{N} = \dfrac{2y-y}{xy} = \dfrac{1}{x} = \mathrm{f}(x)$ Hence, I.F. = $e^{\int \mathrm{f}(x)dx} = e^{\int \frac{1}{x}dx} = x$

Therefore, multiply given equation by this I.F

We get, $(x^3 + xy^2 + x^2)dx + (x^2y)dy = 0$ is an exact equation.
Hence, the general solution is

$$\int Mdx_{\text{Keeping } y \text{ constant}} \quad + \int (terms \text{ of } N \ \ not\ containing\ x)dy = C$$

$$\Longrightarrow \int (x^3 + xy^2 + x^2)dx_{\text{Keeping } y \text{ constant}} \quad + \int (0)dy = C$$

$\Rightarrow \frac{x^4}{4} + \frac{x^2y^2}{2} + \frac{x^3}{3} = C$.

Example-2 : $Solve\ the\ equation\ (xy^3 + y)dx + 2(x^2y^2 + x + y^4)dy = 0$

Solution: On comparing the given equation with $Mdx + Ndy = 0$

$\therefore M = xy^3 + y$, $N = 2(x^2y^2 + x + y^4)$

$\frac{M}{y} = 3xy^2 + 1$, $\frac{N}{x} = 4xy^2 + 2$

Hence the given equation is not exact.

Now, $\frac{\frac{N}{x} - \frac{M}{y}}{M} = \frac{1}{y} = g(y)$ then, I.F. = $e^{\int g(y)dy} = e^{\int \frac{1}{y}dy} = y$

Therefore, multiply given equation by this I.F

We get, $(xy^4 + y^2)dx + 2(x^2y^3 + xy + y^5)dy = 0$ is an exact equation.

Hence , the general solution is

$$\int_{Keeping\ y\ constant} Mdx + \int (terms\ of\ N\ not\ containing\ x)dy = C$$

$$\Rightarrow \int_{Keeping\ y\ constant} (xy^4 + y^2)dx + \int (2y^5)dy = C$$

$\Rightarrow \frac{x^2y^4}{2} + \frac{xy^2}{1} + \frac{y^6}{3} = C$.

Part-II: Differential Equations of First Order and Higher Degree

The most general form of a differential equation of the first order and of higher degree say of nth degree can be written as

$$\left(\frac{dy}{dx}\right)^n + a_1(x,y)\left(\frac{dy}{dx}\right)^{n-1} + a_2(x,y)\left(\frac{dy}{dx}\right)^{n-2} + \ldots \ \ldots$$

$$\ldots \ \ldots \ \ldots + a_{n-1}(x,y)\frac{dy}{dx} + a_n(x,y) = 0$$

or $\quad p^n + a_1p^{n-1} + a_2p^{n-2} + \ldots\ldots + a_{n-1}p + a_n = 0 \qquad (2.1)$

where $p = \frac{dy}{dx}$ and $a_1, a_2, \ldots, a_n$ are functions of x and y.

(2.1) can be written as

$F(x, y, p) = 0$ (2.2)

[I] First-Order Equations of Higher Degree Solvable for p

Let (2.2) can be solved for p and can be written as

$(p-q_1(x,y))\ (p-q_2(x,y))\ldots\ldots\ldots\ldots(p-q_n(x,y)) = 0$

Equating each factor to zero we get equations of the first order and first degree.

One can find solutions of these equations by the methods discussed in the previous part. Let their solution be given as:

$f_i(x,y,c_i)=0,\ i=1,2,3\ \ldots\ldots\ldots n$ (2.3)

Therefore the general solution of (2.1) can be expressed in the form

$f_1(x,y,c)\ f_2(x,y,c)\ldots\ldots\ldots f_n(x,y,c) = 0$ (2.4)

where c in any arbitrary constant.

It can be checked that the sets of solutions represented by (2.3) and (2.4) are identical because the validity of (2.4) in equivalent to the validity of (2.3) for at least one i with a suitable value of c, namely $c=c_i$

Example 1. Solve $xy\left(\frac{dy}{dx}\right)^2 + (x^2+y^2)\frac{dy}{dx} + xy = 0$ (2.5)

Solution**:** This is first-order differential equation of degree 2. Let $p = \frac{dy}{dx}$

Equation (2.5) can be written as

$xy\ p^2+(x^2+y^2)\ p+xy=0$ (2.6)

$(xp+y)(yp+x)=0$

This implies that

$xp+y=0,\ yp+x=0$ (2.7)

By solving equations in (2.7) we get

$xy=c_1$ and $x^2+y^2=c_2$ respectively

[$x\frac{dy}{dx}+y=0$ or $\frac{dy}{dx}+\frac{1}{x}y=0$,Integrating factor

$I(x)=e^{\int\frac{1}{x}dx} = e^{\log x}$. This gives

$y.x = \int o.x\ dx + c_1$ or $xy=c_1$] ,

[$y\frac{dy}{dx}+x=0$, or $ydy+xdx=0$

By integration we get $\frac{1}{2}y^2+\frac{1}{2}x^2=c$

or $x^2+y^2 = c_2$,]

The general solution can be written in the form

$(x^2+y^2-c_2)\ (xy-c_1)=0$ (2.8)

It can be seen that none of the nontrivial solutions belonging to $xy=c_1$ or $x^2+y^2=c_2$ is valid on the whole real line.

[II] Equations Solvable for y

Let the differential equation given by (2.2) be solvable for y. Then y can be expressed as a function x and p, that is,

$y = f(x, p)$ (2.9)

Differentiating (2.9) with respect to x we get

$$\frac{dy}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial p} \cdot \frac{dp}{dx} \qquad (2.10)$$

(2.10) is a first order differential equation of first degree in x and p. It may be solved. Let solution be expressed in the form

$\varphi(x, p, c) = 0$ (2.11)

The solution of equation (2.9) is obtained by eliminating p between (2.9) and (2.11). If elimination of p is not possible then (2.9) and (2.11) together may be considered parametric equations of the solutions of (2.9) with p as a parameter.

Example 2: Solve y^2-1-p^2=o

Solution: It is clear that the equation is solvable for y, that is

$$y = \sqrt{1 + p^2} \qquad (2.12)$$

By differentiating (2.12) with respect to x we get

$$\frac{dy}{dx} = \frac{1}{2} \frac{1}{\sqrt{1+p^2}} .2p \frac{dp}{dx}$$

or $$p = \frac{p}{\sqrt{1+p^2}} \frac{dp}{dx}$$

or $$p\left[1 - \frac{1}{\sqrt{1+p^2}} \frac{dp}{dx}\right] = 0 \qquad (2.13)$$

(2.13) gives p=o or $1 - \frac{1}{\sqrt{1+p^2}} \frac{dp}{dx} = 0$

By solving p=0 in (2.12) we get

y=1

By $1 - \frac{1}{\sqrt{1+p^2}} \frac{dp}{dx} = 0$

we get a separable equation in variables p and x.

$$\frac{dp}{dx} = \sqrt{1+p^2}$$

By solving this we get

p= sinh (x+c) (2.14)

By eliminating p from (2.12) and (2.14) we obtain

y=cos h (x+c) (2.15)

(2.15) is a general solution.

Solution y=1 of the given equation is a singular solution as it cannot be obtained by giving a particular value to c in (2.15).

[III] Equations Solvable for x

Let equation (2.2) be solvable for x,

that is x=f(y,p) .. (2.16)

Then as argued in the previous section for y we get a function Ψ such that

$\Psi(y, p, c) = 0$ (2.17)

By eliminating p from (2.16) and (2.17) we get a general solution of (2.2). If elimination of p with the help of (2.16) and (2.17) is cumbersome then these equations may be considered parametric equations of the solutions of (2.16) with p as a parameter.

Example 3. Solve $x\left(\frac{dy}{dx}\right)^3 - 12\frac{dy}{dx} - 8 = 0$

Solution: Let $p = \frac{dy}{dx}$, then

$xp^3-12p-8=0$

It is solvable for x, that is,

$$x = \frac{12p+8}{p^3} = \frac{12}{p^2} + \frac{8}{p^3} \quad \ldots \quad (2.18)$$

Differentiating (2.18) with respect to y, we get

$$\frac{dx}{dy} = -2\frac{12}{p^3}\frac{dp}{dy} - 3\frac{8}{p^4}\frac{dp}{dy}$$

$$\text{or } \frac{1}{p} = -\frac{24}{p^3}\frac{dp}{dy} - \frac{24}{p^4}\frac{dp}{dy}$$

$$\text{or } \quad dy = \left(-\frac{24}{p^2} - \frac{24}{p^3}\right)dp$$

$$\text{or } \quad y = \frac{24}{p} + \frac{12}{p^2} + c \quad \ldots \quad (2.19)$$

(2.18) and (2.19) constitute parametric equations of solution of the given differential equation.

[IV] Equations of the First Degree in x and y – Lagrange's and Clairaut's Equation.

Let Equation (2.2) be of the first degree in x and y, then

$y = x\varphi_1(p) + \varphi_2(p)$ … (2.20)

Equation (2.20) is known as Lagrange's equation.

If $\varphi_1(p) = p$ then the equation

$$y = xp + \varphi_2(p) \qquad .. \qquad (2.21)$$

is known as Clairaut's equation

By differentiating (2.20) with respect to x, we get

$$\frac{dy}{dx} = \varphi_1(p) + x\varphi_1'(p)\frac{dp}{dx} + \varphi_2'(p)\frac{dp}{dx}$$

$$\text{or } p - \varphi_1(p) = (x\varphi_1'(p) + \varphi_2'(p))\frac{dp}{dx} \qquad ... \qquad (2.22)$$

From (2.22) we get

$$(x + \varphi_2'(p))\frac{dp}{dx} = 0 \quad \text{for } \varphi_1(p) = p$$

This gives

$$\frac{dp}{dx} = 0 \text{ or } x + \varphi_2'(p) = 0$$

$$\frac{dp}{dx} = 0 \text{ gives } p = c \text{ and}$$

by putting this value in (2.21) we get

$$y = cx + \varphi_2(c)$$

This is a general solution of Clairaut's equation.

The elimination of p between $x+\varphi_2'(p) = 0$ and (2.21) gives a singular solution.

If $\varphi_1(p) \neq p$ for any p, then we observe from (2.22) that

$\frac{dp}{dx} \neq 0$ every where. Division by

$[p - \varphi_1(p)]\frac{dp}{dx}$ in (2.22) gives

$$\frac{dx}{dp} - \frac{\varphi_1'}{p - \varphi_1(p)}x = \frac{\varphi_2'(p)}{p - \varphi_1(p)}$$

which is a linear equation of first order in x and thus can be solved for x as a function of p, which together with (2.20) will form a parametric representation of the general solution of (2.20)

Example 4 . Solve $\left(\frac{dy}{dx} - 1\right)\left(y - x\frac{dy}{dx}\right) = \frac{dy}{dx}$

Solution: Let $p = \frac{dy}{dx}$ then,

$$(p-1)(y-xp) = p$$

This equation can be written as

$$y = xp + \frac{p}{p-1}$$

Differentiating both sides with respect to x we get

$$\frac{dp}{dx}\left[x-\frac{1}{(p-1)^2}\right]=0$$

Thus either $\frac{dp}{dx}=0$ or $x-\frac{1}{(p-1)^2}=0$

$\frac{dp}{dx}=0$ gives p=c Since $x-\frac{1}{(p-1)^2}\neq 0$

Putting p=c in the equation we get

$$y=cx+\frac{c}{c-1}$$

(y-cx)(c-1)=c

Which is the required solution.

Part-III: Higher order differential Equations with constant coefficients

A differential equation is said to be Linear if the dependent variable and it's derivatives occur only in the first degree and are not multiplied together.

Thus, the general linear differential equation of the nth order with constant coefficients is of the form

$$\frac{d^n y}{dx^n}+a_1\frac{d^{n-1}y}{dx^{n-1}}+a_2\frac{d^{n-2}y}{dx^{n-2}}+-----a_{n-1}\frac{dy}{dx}+a_n y=\mathrm{X}\text{..............}(1)$$

Or $(D^n+a_1D^{n-1}+a_2D^{n-2}+-----a_{n-1}D+a_n)y=\mathrm{X}$ Since $D\equiv\frac{d}{dx}$

Or f(D)y = X... (1)

Where $\mathrm{f}(D)=D^n+a_1D^{n-1}+a_2D^{n-2}+-----a_{n-1}D+a_n$

and a_1,a_2 ,... .... ,a_{n-1}, a_n be the constants , $\mathrm{X}=\mathrm{f}(x)$

If $\mathrm{X}=0$, then equation (1) is called a homogeneous linear differential equation of the nth order with constant coefficients.

If $\mathrm{X}\neq 0$, then equation (1) is called a Non-homogeneous linear differential equation of the nth order with constant coefficients**.**

Now, the general solution of equation (1) consists of two parts.

i.e. $y=C.F.+P.I$.............................. (2)

Where C.F. = Complementary function and P.I. = Particular Integral.

Method of finding C.F.(Complementary function):-- The Complementary function can written on the basis of nature of roots of auxiliary equation[A.E.] of (1) as

$$f(D = m) = 0$$

i.e. $m^n + a_1 m^{n-1} + a_2 m^{n-2} + - - - - - - a_{n-1} m + a_n = 0$.......................... (3)

On solving the above equation, we get the n roots.

Sr. No.	Roots of A.E.	Form of C.F.
1	$m_1, m_2, m_3, \ldots\ldots\ldots m_n$, (Real and distinct roots)	$c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots \ldots .. c_n e^{m_n x}$
2	$m_1 = m_2, m_3, \ldots\ldots\ldots m_n$, (two equal real roots)	$(c_1 + c_2 x) e^{m_1 x} + c_3 e^{m_3 x} + \cdots \ldots .. c_n e^{m_n x}$
3	$m_1 = m_2 = m_3, \ldots\ldots\ldots m_n$, (three equal real roots)	$(c_1 + c_2 x + c_3 x^2) e^{m_1 x} + c_4 e^{m_4 x} + \cdots \ldots .. c_n e^{m_n x}$
4	$m = \alpha \pm i\beta, m_3, \ldots\ldots\ldots m_n$, (two roots are imaginary)	$e^{\alpha x}[c_1 \cos\beta x + c_2 \sin\beta x] + c_3 e^{m_3 x} + \cdots \ldots .. c_n e^{m_n x}$
5	$\alpha \pm i\beta$, $\alpha \pm i\beta$, $m_5, \ldots\ldots\ldots m_n$, (equal pair of imaginary roots)	$e^{\alpha x}[(c_1 + c_2 x)\cos\beta x + (c_3 + c_4 x)\sin\beta x] + c_5 e^{m_5 x} + \cdots \ldots .. c_n e^{m_n x}$
6	$m = \alpha \pm i\sqrt{\beta}, m_3, \ldots\ldots\ldots m_n$, (two roots a in surds form or irrational)	$e^{\alpha x}[c_1 \cosh\sqrt{\beta}\, x + c_2 \sinh\sqrt{\beta}\, x] + c_3 e^{m_3 x} + \cdots \ldots .. c_n e^{m_n x}$
7	$\alpha \pm i\sqrt{\beta}, \alpha \pm i\beta$ $m5$, ... mn, (equal pair of surds form or irrational)	$e^{\alpha x}[(c_1 + c_2 x)\cos h\sqrt{\beta}\, x + (c_3 + c_4 x) Sinh\sqrt{\beta}\, x] + c_5 e^{m_5 x} + \cdots \ldots .. c_n e^{m_n x}$

Method of finding P.I. [Particular Integral]:--

The Particular Integral of equation $f(D)y = X$ .... (1)

Where $f(D) = D^n + a_1 D^{n-1} + a_2 D^{n-2} + - - - - - - a_{n-1} D + a_n$

and $a_1, a_2, \ldots\ldots, a_{n-1}, a_n$ be the constants , $X = f(x)$

is $\quad$ $\text{P.I.} = \frac{X}{f(D)}$

Particular Integral for different cases of X:-

<u>Case-1</u>: When $X = e^{ax}$

In this case

[i] P.I. $= \frac{e^{ax}}{f(D)} = \frac{e^{ax}}{f(a)}$, $provided f(a) \neq 0$

[ii] If $f(a) = 0$ then P.I. $= \frac{e^{ax}}{f(D)} = x\frac{e^{ax}}{f'(a)}$, $provided f'(a) \neq 0$

Again If $f'(a) = 0$ then P.I. $= \frac{e^{ax}}{f(D)} = x^2\frac{e^{ax}}{f''(a)}$, $provided f''(a) \neq 0$

Case-2 : When $X = cosax$

In this case

[i] P.I. $= \frac{cosax}{f(D^2)} = \frac{cosax}{f(-a^2)}$, $provided f(-a^2) \neq 0$

[ii] If $f(-a^2) = 0$ then P.I. $= \frac{cosax}{f(D^2)} = x\frac{cosax}{f'(-a^2)}$, $provided f'(-a^2) \neq 0$

Again if $f'(-a^2) = 0$ then P.I. $= \frac{cosax}{f(D^2)} = x^2\frac{cosax}{f''(-a^2)}$, $provided f''(-a^2) \neq 0$

Case-3 : When $X = sinax$

In this case

[i] P.I. $= \frac{Sinax}{f(D^2)} = \frac{Sinax}{f(-a^2)}$, $provided$ $f(-a^2) \neq 0$

[ii] If $f(-a^2) = 0$ then P.I. $= \frac{Sinax}{f(D^2)} = x\frac{Sinax}{f'(-a^2)}$, $provided f'(-a^2) \neq 0$

Again if $f'(-a^2) = 0$ then P.I. $= \frac{Sinax}{f(D^2)} = x^2\frac{Sinax}{f''(-a^2)}$, $provided f''(-a^2) \neq 0$

Case-4 : When $X = x^m$, $where\ m$ $is\ a\ positive$ $integer.$

In this case ,

P.I. $= \frac{x^m}{f(D)}$

Take out the lowest degree term common from $f(D)$to make the first term unity.The remaining factor will be of the form $(1 - \emptyset(D))$.Now ,take this factor in the numerator i.e. $(1 - \emptyset(D))^{-1}$. Expand $(1 - \emptyset(D))^{-1}$ in ascending powers of D as far as the term containing D^m and then operate on x^m term by term.

Note:-

1.$(1 - x)^{-1} = 1 + x + x^2 + x^3 + \cdots$ … ….

2. $(1 - x)^{-2} = 1 + 2x + 3\,x^2 + 4x^3 + \cdots$ … ….

Case-5 : When $X = e^{ax}V$ where $V = f(x)$

In this case ,

P.I. $= \frac{e^{ax}V}{f(D)} = e^{ax}\frac{V}{f(D+a)}$

Case-6 : When $X = xV$ where $V = f(x)$

In this case ,

P.I. $= \frac{xV}{f(D)} = x\frac{V}{f(D)} - \frac{f'(D)}{(f(D))^2}V$

Case-7 : $\frac{X}{(D)} = \int X dx$

Case-8 : $\frac{X}{(D-a)} = e^{ax}\int Xe^{-ax}\,dx$

Example-1 :$Solve\ the\ equation\ (D^2 - 4D + 3)y = 0$

Solution: On comparing the given equation with$f(D)y = X$

$$\therefore\ f(D) = (D^2 - 4D + 3)\ ,\qquad X = 0$$

Auxiliary equation is $m^2 - 4m + 3 = 0$

$$m^2 - m - 3m + 3 = 0$$
$$(m-1)(m-3) = 0$$
$$m = 1,3$$

Hence C.F. = $c_1e^x + c_2e^{3x}$

Since $X = 0$ ∴P.I.=0

Hence the complete general solution is

$$y = C.F. + P.I. = c_1e^x + c_2e^{3x} + 0 = c_1e^x + c_2e^{3x}$$

Example-2 : $Solve\ the\ equation(D-2)^2\,y = 8(e^2 + sin2x + x^2)$

Solution: On comparing the given equation with$f(D)y = X$

$$\therefore\ f(D) = (D-2)^2\ ,\qquad X = 8(e^{2x} + sin2x + x^2)$$

Auxiliary equation is $(m-2)^2 = 0$

$$m = 2,2$$

Hence C.F. = $(c_1 + c_2x)\ e^{2x}$

$$\therefore \text{P.I.} = \frac{X}{f(D)} = \frac{8(e^{2x} + \sin 2x + x^2)}{(D-2)^2} = 8\left[\frac{e^{2x}}{(D-2)^2} + \frac{\sin 2x}{(D-2)^2} + \frac{x^2}{(D-2)^2}\right]$$
$$= 8[P.I_1 + P.I_2 + P.I_3].....(1)$$

Now, $P.I_1 = \frac{e^{2x}}{(D-2)^2} = \frac{e^{2x}}{(2-2)^2} = \frac{e^{2x}}{0}$ Hence, this method fails

$$\therefore P.I_1 = \frac{e^{2x}}{(D-2)^2} = x\frac{e^{2x}}{2(D-2)} = x^2\frac{e^{2x}}{2.1} = x^2\frac{e^{2x}}{2}$$

$$P.I_2 = \frac{sin2x}{(D-2)^2} = \frac{sin2x}{D^2 - 4D + 4} = \frac{sin2x}{-2^2 - 4.D + 4} = \frac{sin2x}{-4.D} = \frac{1}{-4}\int sin2x\,dx$$

$$\therefore P.I_2 = \frac{1}{8}Co2x$$

$$P.I_3 = \frac{x^2}{(D-2)^2} = \frac{1}{4}(1-\frac{D}{2})^{-2} x^2 = \frac{1}{4}[1+2\frac{D}{2}+3\frac{D^2}{4} + \cdots \ldots..]x^2$$

$$=\frac{1}{4}[1x^2 + 2\frac{D}{2}x^2 + 3\frac{D}{4}x^{2^2} + \cdots \ldots..]$$

$$=\frac{1}{4}[x^2 + 2\frac{2x}{2} + 3\frac{2}{4} + 0 \ldots \ldots..+0]$$

$$=\frac{1}{4}[x^2 + 2x + \frac{3}{2}]$$

$\therefore$ P.I.= 8 $[P.I_1 + P.I_2 + P.I_3]$

$$=8\,[x^2\frac{e^{2x}}{2} + \frac{1}{8}Co2x + \frac{1}{4}(x^2 + 2x + \frac{3}{2})]$$

Hence the complete general solution is

$$y = C.F. + P.I. = (c_1 + c_2x)\ e^{2x} + 8[x^2\frac{e^{2x}}{2} + \frac{1}{8}Co2x + \frac{1}{4}(x^2 + 2x + \frac{3}{2})]$$

Example-3 : *Solve the equation* $(D^2 - 2D + 1)\,y = x\,e^x Sinx$

Solution: On comparing the given equation with $f(D)y = X$

$$\therefore\ f(D) = D^2 - 2D + 1, \quad X = x\,e^x Sinx$$

Auxiliary equation is $(m-1)^2 = 0$

$$m = 1,1$$

Hence C.F. = $(c_1 + c_2x)\ e^x$

$$\therefore P.I. = \frac{X}{f(D)} = \frac{e^x x\,Sinx}{f(D)} = e^x\frac{x\,Sinx}{f(D+1)} = e^x\frac{x\,Sinx}{(D+1)^2 - 2(D+1)+1} = e^x\frac{x\,Sinx}{(D)^2} =$$

$$= e^x\frac{1}{D}\int x\,Sinx\,dx = e^x\frac{1}{D}[-xcosx + sinx] = e^x[-\int x\,cosx\ dx + \int Sinx\ dx] = e^x[-xsinx - 2cosx]$$

Hence the complete general solution is

$$y = C.F. + P.I = (c_1 + c_2x)\ e^{2x} + e^x[-xsinx - 2cosx]$$

Example-4 : Solve the equation $(D^2 + 4)\,y = x\,Sinx$

Solution: On comparing the given equation with $f(D)y = X$

$$\therefore\ f(D) = (D^2 + 4), \quad X = x\,Sinx$$

Auxiliary equation is $m^2 + 4 = 0$

$$m = \pm 2i = \propto \pm i\beta$$

Hence C.F. = $(c_1 cos2x + c_2 sin2x)$

P.I. $= \frac{X}{f(D)} = \frac{x\,Sinx}{D^2+4} = [x - \frac{1}{D^2+4}2D]\frac{Sinx}{D^2+4} == [x - \frac{1}{D^2+4}2D]\frac{Sinx}{-1+4}$

$$= [x - \frac{1}{D^2+4}2D]\frac{Sinx}{3} = \frac{xSinx}{3} - \frac{1}{D^2+4}2D\frac{Sinx}{3} = \frac{xSinx}{3} - \frac{1}{D^2+4}2\frac{cosx}{3}$$
$$= \frac{xSinx}{3} - \frac{2}{3}\frac{1}{(-1+4)}cosx$$

P.I. $= \frac{xSinx}{3} - \frac{2}{9}cosx$

Hence the complete general solution is

$y = C.F. + P.I = (c_1 cos2x + c_2 sin2x) + \frac{xSinx}{3} - \frac{2}{9}cosx.$

Example-5 : *Solve the equation* $(D^2 - 4)y = 2^x$

Solution: On comparing the given equation with $f(D)y = X$

$$\therefore\ f(D) = D^2 - 4, \quad X = 2^x$$

Auxiliary equation is $m^2 - 4 = 0$

$$m = 2, -2$$

Hence C.F. = $c_1 e^{-2x} + c_2 e^{2x}$

$\therefore$ P.I. $= \frac{X}{f(D)} = \frac{2^x}{D^2-4} = \frac{e^{(log\ 2)x}}{D^2-4} = \frac{e^{(log\ 2)x}}{(lo2)^2-4} = \frac{2^x}{(lo2)^2-4}$ Since $a^x = e^{(loga)x}$

Hence the complete general solution is

$$y = C.F. + P.I = c_1 e^{-2x} + c_2 e^{2x} + \frac{2^x}{(lo2)^2 - 4}$$

Example-6: Solve the equation $(D^2 - 3D + 2)y = x + e^x$

Solution: On comparing the given equation with $f(D)y = X$

$$\therefore\ f(D) = (D^2 - 3D + 2), \quad X = x + e^x$$

Its auxiliary equation is $m^2 - 3m + 2 = 0 \Rightarrow m = 1,2$

$\Rightarrow CF = c_1 e^x + c_2 e^{2x}$ and

$$PI = \frac{1}{D^2 - 3D + 2}(x + e^x) = \frac{1}{2}\left(1 - \frac{3}{2}D + \frac{1}{2}D^2\right)^{-1} \cdot x + \frac{1}{D^2 - 3D + 2}e^x$$

$$= \frac{1}{2}x - \frac{3}{4} + x\frac{1}{2D-3}e^x = \frac{1}{2}x - \frac{3}{4} - xe^x$$

Hence the complete general solution is

$$y = CF + PI = c_1 e^x + c_2 e^{2x} + \frac{1}{2}x - \frac{3}{4} - xe^x.$$

Part-IV: Homogeneous linear differential equations

There are two types of homogeneous linear differential equations, which can be reduced to the linear differential equations with constant coefficient by convenient substitutions**.**

Type-1: Cauchy's Homogeneous linear differential equation:

An equation is of the type

$$x^n\frac{d^n y}{dx^n} + a_1x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + a_2x^{n-2}\frac{d^{n-2}y}{dx^{n-2}} + - - - - + a_n y = X \text{.............} (1)$$

is called a Cauchy's Homogeneous linear differential equation.
Where $a_1, a_2, \ldots\ldots, a_{n-1}, a_n$ be the constants , $X = f(x)$

Steps to solve: Equation (1) can be reduced to a linear differential equation with constant Coefficients by using the following steps:
Step 1: Put $x = e^z$ *so that* $\log x = z \Rightarrow \frac{1}{x} = \frac{dz}{dx}$

Now $\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz}$

$$\therefore\ x\frac{dy}{dx} = \frac{dy}{dz} = Dy \quad \text{where } D \equiv \frac{d}{dz}$$

Similarly $x^2\frac{d^2y}{dx^2} = D(D-1)y,$

$$x^3\frac{d^3y}{dx^3} = D(D-1)(D-2)y \ \text{ and so on } x^n\frac{d^n y}{dx^n} = D(D-1)(D-2)\ldots.(D-n+1)y$$

Step 2 : Putting all the values in equation (1) then it will reduces into linear differential equation with constant Coefficients in terms of z .
Step 3: Find the complete general solution is $y = C.F. + P.I$
Step 4: Finally, we replace z by $\log x$.

Type-2: Legendre's Homogeneous linear differential equation:--

An equation is of the type

$$(ax+b)^n\frac{d^n y}{dx^n}+a_1(ax+b)^{n-1}\frac{d^{n-1}y}{dx^{n-1}}+a_2(ax+b)^{n-2}\frac{d^{n-2}y}{dx^{n-2}}+-----+a_n y = \mathrm{X} \ldots\ldots\ldots\ldots\ldots (1)$$

is called a **Legendre's** Homogeneous linear differential equation.
Where $a_1, a_2, \ldots\ldots\ldots, a_{n-1}, a_n$ be the constants , $\mathrm{X} = \mathrm{f}(x)$

Steps to solve: Equation (1) can be reduced to a linear differential equation with constant Coefficients by using the following steps:
Step 1: Put : $ax+b = e^z$ so that $\log(ax+b) = z \Rightarrow \frac{a}{ax+b} = \frac{dz}{dx}$

Now $\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{a}{ax+b}\frac{dy}{dz}$

$$\therefore (ax+b)\frac{dy}{dx} = a\frac{dy}{dz} = a\,Dy \quad \text{where } D \equiv \frac{d}{dz}$$

Similarly $(ax+b)^2\frac{d^2y}{dx^2} = a^2D(D-1)y,$

$(ax+b)^3\frac{d^3y}{dx^3} = a^3D(D-1)(D-2)y$ *and so on* $(ax+b)^n\frac{d^n y}{dx^n} = a^n D(D-1)(D-2)\ldots.(D-n+1)y$

Step 2 : Putting all the values in equation (1) then it will reduces into linear differential equation with constant Coefficients in terms of z .
Step 3: Find the complete general solution is $y = C.F.+P.I$
Step 4: Finally, we replace z by $\log(ax+b)$.

Example-1 : *Solve the equation* $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 2\log x$

Solution: Given equation is

$$x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 2\log x \ldots\ldots.(1)$$

Now to solve, **Put** $x = e^z$ $so\ that\ log x = z \Rightarrow \therefore\ \frac{1}{x} = \frac{dz}{dx}$

$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz}$

$$\therefore\quad x\ \frac{dy}{dx} = \frac{dy}{dz} = Dy \quad \text{where } D \equiv \frac{d}{dz}$$

Similarly $x^2\frac{d^2y}{dx^2} = D(D-1)y,$

Hence, the equation (1) becomes

$$D(D-1)y - Dy + y = 2z$$

Or $(D^2 - 2D + 1)y = 2z$ -------------(2)

It is reduce into linear differential equation with constant Coefficients in terms of z .

Auxiliary equation is $m^2 - 2m + 1 = 0$

$$(m-1)^2 = 0 \Rightarrow m = 1,1$$

Hence C.F. = $(c_1 + c_2 z)\ e^z$

$\therefore$P.I.$=\frac{Z}{f(D)} = \frac{2z}{(D^2-2D+1)} = 2\,[1-(2D-D^2)]^{-1}z$

$= 2[\ 1 + (2D - D^2) + \cdots.]z$

$= 2[\ z + (2Dz - D^2z) + \cdots.]$

$= 2[\ z + 2 - 0.]$

$= 2z + 4$

Hence the complete general solution is

$$y = C.F. + P.I = (c_1 + c_2 z)\ e^z + 2z + 4 = (c_1 + c_2 log x)x + 2 log x + 4$$

Example-2 : $Solve\ the\ equation$

$$(2x+1)^2\frac{d^2y}{dx^2} - 2(2x+1)\frac{dy}{dx} - 12y = 6x$$

Solution: Given equation is

$$(2x+1)^2\frac{d^2y}{dx^2}-2(2x+1)\frac{dy}{dx}-12y=6x---(1)$$

Put : $2x+1=e^z$ so that $\log(2x+1)=z \Rightarrow \frac{2}{2x+1}=\frac{dz}{dx}$

Now $\frac{dy}{dx}=\frac{dy}{dz}\frac{dz}{dx}=\frac{2}{2x+1}\frac{dy}{dz}$

$$\therefore (2x+1)\frac{dy}{dx}=2\frac{dy}{dz}=2\,Dy \quad \text{where } D\equiv\frac{d}{dz}$$

Similarly $(2x+1)^2\frac{d^2y}{dx^2}=4D(D-1)y=4(D^2-D)y$

Hence, the equation (1) becomes

$$4(D^2-D)y-2\,2Dy-12y=3(e^z-1)$$

$\Rightarrow (D^2-2D-3)y=\frac{3}{4}(e^z-1)$ ------- (2)

It is reduce into linear differential equation with constant Coefficients in terms of z.

Auxiliary equation is $m^2-2m-3=0$

$$(m+1)(m-3)=0 \Rightarrow m=-1,3$$

Hence C.F. = $c_1 e^{-z}+c_2 e^{3z}$

P.I.= $\frac{Z}{f(D)}=\frac{3}{4}\frac{(e^z-1)}{(D^2-2D-3)}=\frac{3}{4}\left[\frac{e^z}{(D^2-2D-3)}-\frac{1}{(-3)}\{1-(\frac{D^2}{-3}-\frac{2}{3}D)\}^{-1}1\right]$

" = $\frac{3}{4}\left[\frac{e^z}{(1^2-2.1-3)}+\frac{1}{(3)}.1\right]$

"= $\frac{3}{4}\left[-\frac{e^z}{4}+\frac{1}{(3)}.\right]$

Hence the complete general solution is

$$y=C.F.+P.I=c_1 e^{-z}+c_2 e^{3z}+\frac{3}{4}\left[-\frac{e^z}{4}+\frac{1}{(3)}.\right]$$

Or $y=c_1(2x+1)^{-1}+c_2(2x+1)^3-\frac{3}{16}(2x+1)+\frac{1}{4}$

Example-3 : *Solve the equation* $(1+x)^2\frac{d^2y}{dx^2}+(1+x)\frac{dy}{dx}+y=2\sin[\log(1+x)]$

Solution: Given equation is $(1+x)^2\frac{d^2y}{dx^2}+(1+x)\frac{dy}{dx}+y=2\sin[\log(1+x)]\ldots\ldots(1)$

Put $1+x=e^t$, i.e. $t=\log(1+x)$, so that $(1+x)\frac{dy}{dx}=\frac{dy}{dt}=Dy$ and $(1+x)^2\frac{d^2y}{dx^2}=D(D-1)y$, where $D\equiv\frac{d}{dt}$.

Hence equation (1) becomes

$D(D-1)y+Dy+y=2\sin(\log t)\Rightarrow(D^2+1)y=2\sin t$...............(2)

It is reduce into linear differential equation with constant Coefficients in terms of z.
It's auxiliary equation is $m^2+1=0\Rightarrow m=\pm$ i. Hence CF is

C.F.=$c_1\cos t+c_2\sin t$

And PI = $\frac{1}{D^2+1}2\sin t=t\frac{2}{2D}\sin t=-t\cos t$

$$\Rightarrow y=CF+PI=c_1\cos t+c_2\sin t-t\cos t$$
$$=c_1\cos[\log(1+x)]+c_2\sin[\log(1+x)]-[\log(1+x)]\cos[\log(1+x)]$$

Part-V: Simultaneous linear differential equations

Important Ordinary Differential Equations arise at times in which there are two or more dependent variables and a single independent variable, usually the time t. We consider here only a system of linear differential equations with constant coefficients, the solutions of which can be obtained by reducing to the solution of one or more separate equations,just analogous to algebraic system of linear equations. The method is well explained by the following examples

Example-1 : $Solve\ the$ simultaneous linear differential equation

$$\frac{dx}{dt}+\frac{dy}{dt}+2x+y=0$$

$$\frac{dy}{dt}+5x+3y=0$$

Also find the particular solution satisfying the conditions $x=1, y=2\ when\ t=0$.

Solution: Given equations can be written as in operator form

$$(D+2)x+(D+1)y=0\ldots\ldots..(1)$$

$$5x+(D+3)y=0\ldots\ldots..\ (2)\text{where}D\equiv\frac{d}{dt}$$

Now operating equation (1) by $(D+3)$ and (2) by $(D+1)$ and then subtracting them, we get

$(D^2+1)x=0$............(3)
Now the
Auxiliary equation is $m^2+1=0 => m=\pm i$

Hence C.F. = $(c_1 \, cost + c_2 \, sint)$ and P.I.= $\frac{0}{f(D)}$ =0

Hence the complete general solution is
$x = C.F.+P.I = c_1 cost + c_2 sint$ ....(4)
Now ,eliminating Dy in $(1) and \ (2) by$ subtracting (2) from (1).

We get , $y=\frac{1}{2}Dx+\frac{3}{2}x$

$$=> y = \frac{1}{2}D(c_1 cost + c_2 sint) + \frac{3}{2}(c_1 cost + c_2 sint)$$
$$=> y = \frac{1}{2}(-c_1 Sint + c_2 cost) + \frac{3}{2}(c_1 cost + c_2 sint)$$
$$=> \ y = -\frac{1}{2}(c_1 + 3c_2)Sint + \frac{1}{2}(c_2 - 3c_1)cost \text{........... } (5)$$

Now it is given that , $x=1, y=2 \text{ when } t=0$,
Hence from equation (4),$c_1 = 1$and by equation (5)$c_2 = 7$
Thus the required particular solution satisfying the given conditions is

$$x = cost + 7sint \quad and \quad y = -11\, sint + 2\, cost$$

Example-2 : $Solve \ the$ simultaneous equation

$\frac{dx}{dt}+5x-2y=t\,;\frac{dy}{dt}+2x+y=0$
Also find the particular solution satisfying the conditions $x=y=0 \text{ when } t=0$.

Solution: Given equations can be written as in operator form

$$(D+5)x-2y=t \dots\dots..(1)$$

$$2x+(D+1)y=0 \dots\dots.. \ (2) \text{ where } D \equiv \frac{d}{dt}$$

Eliminate x as if D were an ordinary algebraic multiplier. Multiplying (1) by 2 and operating on (2) by $D+5$ and then subtracting ,we get

$(D^2+6D+9)y=-2t$..........(3)

Now the Auxiliary equation is $m^2 + 6D + 9 = 0 => m = -3, -3$

Hence C.F. = $(c_1 + c_2 t)e^{-3t}$

$\therefore$ P.I.= $\frac{-2}{f(D)} = \frac{-2}{(D+3)^2} = -\frac{2}{9}(1 + \frac{D}{3})^{-2} t = -\frac{2}{9}(1 - 2\frac{D}{3} + \cdots.)t = -\frac{2}{9}t + \frac{4}{27}$

Hence the complete general solution is
$y = C.F. + P.I = (c_1 + c_2 t)e^{-3t} - \frac{2}{9}t + \frac{4}{27}$..............................(4)

Now to find$x$, either eliminate from (1) and (2) and solve the resulting equation or substitute the value of y in (2).
From (4)

$$Dy = D\left[(c_1 + c_2 t)e^{-3t} - \frac{2}{9}t + \frac{4}{27}\right] = c_2 e^{-3t} + (c_1 + c_2 t)(-3)e^{-3t} - \frac{2}{9}$$

$\therefore$ Substituting for y *and* Dy in (2) ,we get

$x = [(c_1 - \frac{1}{2}c_2) + c_2 t]\, e^{-3t} + \frac{t}{9} + \frac{1}{27}$ ....(5)

Since $x = y = 0$ *when* $t = 0$, *the equations* (4)*and* (5)*give*
$c_1 = -\frac{4}{27}$ and $c_2 = -\frac{2}{9}$

Thus the required particular solution satisfying the given conditions is

$$x = -\frac{1}{27}(1 + 6t)e^{-3t} + \frac{1}{27}(1 + 3t) \quad and \quad y = -\frac{2}{27}(2 + 3t)e^{-3t} + \frac{2}{27}(2 - 3t)$$

Contents

Second order linear differential equations with variable coefficients, Method of variation of parameters, Power series solutions; Legendre polynomials, Bessel functions of the first kind and their properties.

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Part-I : Second-Order Linear Differential Equations with Variable Coefficients

An equation is of the type

$$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R$$

is called a Second-Order Linear Differential Equations with Variable Coefficients.
Where P and Q be some functions of x only.
We shall study the following five methods for solving the Second-Order Linear Differential Equations with Variable Coefficients.

Method-I: [Inspection Method] Complete solution in terms of one known integral belonging to the complementary function :-

Step-1: Put the given equation in the standard form
$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R \ldots\ldots\ldots\ldots\ldots.(1)$ in which coefficient of $\frac{d^2y}{dx^2}$ is unity.

Step-2: Find an integral say u of C.F. by using the following table :

Sr. No.	Condition satisfied	An integral say u of C.Fis
1	$1+P+Q=0$	$u=\mathrm{e}^{x}$
2	$1-P+Q=0$	$u=\mathrm{e}^{-x}$
3	$a^2+aP+Q=0$	$u=\mathrm{e}^{ax}$
4	$P+Qx=0$	$u=x$
5	$2+2Px+Qx^2=0$	$u=x^2$
6	$m(m-1)+Pmx+Qx^2=0$	$u=x^m$ where $m=1,2,3\ldots\ldots$

If an integral say $u$of C.F is given in the question, then this step is omitted .

Step-3: Now, assume that the complete solution of the given equation is

$$y=uv\ldots\ldots\ldots\ldots(2)$$

Where u has been obtained in step (2). Then the given equation reduces to

$$\frac{d^2v}{dx^2}+\left(P+\frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx}=\frac{R}{u}------(3)$$

Step-4: Take $\frac{dv}{dx}=q$ so that $\frac{d^2v}{dx^2}=\frac{dq}{dx}$, ∴Substituting this in (3),it will reduce to first order and first degree and we solve it for q as usual.

Step-5: Now ,we replace q by $\frac{dv}{dx}$ and separate the variables v and x. Integrate and determine v. Put this value of v in (2). We get the complete solution of the given equation.

Example-1 :Solve the differential equation

$$x\frac{d^2y}{dx^2} - (2x-1)\frac{dy}{dx} + (x-1)y = 0$$

Solution: Given equations can be written as

$$\frac{d^2y}{dx^2} - (2-\frac{1}{x})\frac{dy}{dx} + (1-\frac{1}{x})\,y = 0 \ldots\ldots\ldots(1)$$

On comparing the above with

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

$\therefore P = -(2-\frac{1}{x})\,,\, Q = (1-\frac{1}{x})\,,\, R = 0$

Here ,$1 + P + Q = 0$, Showing that $y = u = e^x$ ……………………(2) is a part of C.F. of solution of (1).

Let the complete solution of the given equation be

$$y = uv \ldots\ldots\ldots\ldots(3)$$

Now , we know that v is defined by $\frac{d^2v}{dx^2} + (P + \frac{2}{u}\frac{du}{dx})\frac{dv}{dx} = \frac{R}{u}$

$$\therefore \frac{d^2v}{dx^2} + (-2 + \frac{1}{x} + \frac{2}{e^x}\frac{de^x}{dx})\frac{dv}{dx} = 0$$

$$=> \frac{d^2v}{dx^2} + (\frac{1}{x})\frac{dv}{dx} = 0 \ldots\ldots\ldots\ldots\ldots(4)$$

Now to solve the above equation, Put $\frac{dv}{dx} = q$ so that $\frac{d^2v}{dx^2} = \frac{dq}{dx}$

$\frac{dq}{dx} + (\frac{q}{x}) = 0 => \frac{dq}{dx} = -\frac{q}{x}$

$$\text{or } \frac{dq}{q} = -\frac{dx}{x}$$

Integrate both sides

$logq = -logx + logc_1$

Or
$logq = -logx + logc_1$
$log\ qx = logc_1$

Or $qx = c_1$

$\frac{dv}{dx} = \frac{c_1}{x}$
$=> dv = \frac{c1}{x}$dx
Integrate both sides,We get
$v = c_1 logx + c_2$

Thus the complete general solution is

$$y = uv = e^x (c_1 logx + c_2)$$

Example-2 : $Solve\ the$ differential equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0\ , given\ that\ x + \frac{1}{x} \text{ is one } interal$$

Solution: Given equations can be written as

$$\frac{d^2y}{dx^2} + (\frac{1}{x})\frac{dy}{dx} - (\frac{1}{x^2})\,y = 0 \ldots\ldots\ldots.(1)$$

On comparing the above with

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

$\therefore P = (\frac{1}{x})\ , Q = -(\frac{1}{x^2})\ , R = 0$

Here it is given that $y = u = x + \frac{1}{x}$ ....(2)
is a part of C.F. of solution of (1).

Let the complete solution of the given equation be

$$y = uv \ldots\ldots\ldots\ldots(3)$$

Now , we know that v is defined by $\frac{d^2v}{dx^2} + (P + \frac{2}{u}\frac{du}{dx})\frac{dv}{dx} = \frac{R}{u}$

$$\therefore \frac{d^2v}{dx^2} + (\frac{1}{x} + \frac{2}{(x + \frac{1}{x})}(1 - \frac{1}{x^2}))\frac{dv}{dx} = 0$$

$$=> \frac{d^2v}{dx^2} + \left(\frac{3x^2-1}{x(x^2+1)}\right)\frac{dv}{dx} = 0$$

Now to solve the above equation, Put $\frac{dv}{dx} = q$ $so\ that$ $\frac{d^2v}{dx^2} = \frac{dq}{dx}$

$$\frac{dq}{dx} + \left(\frac{3x^2-1}{x(x^2+1)}\right) q = 0$$

Or $\frac{dq}{q} + \left(-\frac{1}{x} + \frac{4x}{(x^2+1)}\right) dx = 0$

Integrate both sides, We get

$log\ q - log x + 2\log(x^2+1) = log c_1$

Or $q = \frac{c_1 x}{(x^2+1)^2}$

$$\therefore \frac{dv}{dx} = \frac{c_1 x}{(x^2+1)^2}$$

$=> dv = \frac{c_1 x}{(x^2+1)^2} dx$

∴Integrate both sides,We get

$v = \frac{-c_1}{2(x^2+1)} + c_2$

Thus the complete general solution is

$$y = uv = \left(x + \frac{1}{x}\right)\left(\frac{-c_1}{2(x^2+1)} + c_2\right)$$

Method-II : [Normal Form] Complete solution by removal of first order derivative :-

If an integral included in the complementary function cannot be determined by Inspection Method,it is sometimes useful to reduce the equation to normal form as follows in which the term containing the first derivative is missing.

Step-1: Put the given equation in the standard form

$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \dots\dots\dots\dots\dots(1)$ in which coefficient of $\frac{d^2y}{dx^2}$ is unity.

Step-2: To remove the first order derivative, we choose $u = e^{-\int(\frac{1}{2})P dx}$

Step-3: Now, we choose complete solution of the given equation be

$$y = uv \dots\dots\dots\dots(2)$$

Then the given equation reduces to normal form

$$\frac{d^2v}{dx^2} + I\,v = S ------(3)$$

Where $I = Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx}$, $S = \frac{R}{u}$

Now , the above equation can be solved by usual methods and we get v.

Step-4 : Thus the complete general solution is

$y = uv.$

Example-1 : *Solve the* differential equation

$$(\frac{d^2y}{dx^2} + y)\,cotx + 2(\frac{dy}{dx} + y\,tanx) = secx$$

Solution: Given equations can be written as

$$\frac{d^2y}{dx^2} + 2tanx\,\frac{dy}{dx} + (1 + 2tan^2x)y = secx\,tanx \text{............}(1)$$

On comparing the above with

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

$\therefore P = 2tanx\,,\, Q = (1 + 2tan^2x)\,,\, R = secx\,tanx$

Now to remove the first order derivative, we choose $u = e^{-\int(\frac{1}{2})Pdx} = e^{-\int(\frac{1}{2})2tanx\;dx} = cosx$

$\therefore$ we choosecomplete solution of the given equation be

$$y = uv \dots\dots\dots\dots (2)$$

Then the given equation reduces to normal form

$$\frac{d^2v}{dx^2} + I\,v = S ------(3)$$

Where $I = Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx} = (1 + 2tan^2x) - \frac{1}{4}\,4tan^2x - \frac{1}{2}\,2\,sec^2x = 0$,

$$S = \frac{R}{u} = sec^2xtanx$$

Hence the equation (3) becomes

$$\frac{d^2v}{dx^2} = sec^2xtanx$$

Integrate both sides, We get

$$\frac{dv}{dx} = \frac{1}{2} sec^2x + c_1$$

Again Integrate both sides

$$v = \frac{1}{2} tanx + c_1x + c_2$$

Thus the complete general solution is

$y = uv = cosx\,(\frac{1}{2} tanx + c_1x + c_2)$.

Example-2 : Solve the differential equation

$$(\frac{d^2y}{dx^2}) - \frac{2}{x}(\frac{dy}{dx}) + (1 + \frac{2}{x^2})\, y = xe^x \text{ ....(1)}$$

Solution: On comparing the above with

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

$\therefore P = -\frac{2}{x}, Q = 1 + \frac{2}{x^2}, R = xe^x$

Now to remove the first order derivative, we choose $u = e^{-\int(\frac{1}{2})Pdx} = e^{-\int(\frac{1}{2})\frac{2}{x}dx} = x$

$\therefore$we choosecomplete solution of the given equation be

$$y = uv \dots\dots\dots (2)$$

Then the given equation reduces to normal form

$$\frac{d^2v}{dx^2} + I\,v = S ------(3)$$

Where $I = Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx} = (1 + \frac{2}{x^2}) - \frac{1}{4}(\frac{4}{x^2}) - \frac{1}{2}(\frac{2}{x^2}) = 1$,

$$S = \frac{R}{u} = e^x$$

Hence the equation (3) becomes

$$\frac{d^2v}{dx^2} + v = e^x \Rightarrow (D^2 + 1)v = e^x \text{ .... (4)}$$

Now the auxiliary equation is $m^2 + 1 = 0 \Rightarrow m = \pm i$

Hence C.F. = $(c_1cosx + c_2sinx)$

$\therefore$ **P.I.**= $\frac{}{f(D)} = \frac{e^x}{D^2+1} = \frac{e^x}{2}$,

The, complete general solution of (4) is

$v = C.F. + P.I. = c_1\cos x + c_2\sin x + \dfrac{e^x}{2}$

Thus the complete general solution of given equation is
$y = uv = x\,(c_1\cos x + c_2\sin x + \frac{e^x}{2})$.

Method-III: Method of change of Independent Variable i.e. Transformation of the equation by changing of Independent Variable:--Some times by the change of Independent Variable, the differential equation may become easily integrable.

Step-1: Put the given equation in the standard form
$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$ ………………(1) in which coefficient of $\frac{d^2y}{dx^2}$ is unity.

Step-2: Now ,we assume a following relation between the new independent variable z and the old independent variable x as

$$\left(\frac{dz}{dx}\right)^2 = +Q$$

Note**:** Carefully that ,we omit -ve sign of Q

Now , we solve the above
$\frac{dz}{dx} = \sqrt{Q}$
Integrate both sides,
$z = \int \sqrt{Q}\, dx$ ……………………………(2)
we omit the constant of Integration ,since ,we are interested in finding just a relation between z and x **.**

Step-3: With the help of above relation (2),We transform the equation (1) as
$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1$ ………………(3)
Where
$P_1 = \dfrac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}$ $Q_1 = \dfrac{Q}{\left(\frac{dz}{dx}\right)^2}$, $R_1 = \dfrac{R}{\left(\frac{dz}{dx}\right)^2}$

Step-4: After solving equation (3) by usual methods,the variable z is *replaced by relation* (2)
.

Example-1 : *Solve the* differential equation

$\left(\frac{d^2y}{dx^2}\right) + \tan x\left(\frac{dy}{dx}\right) - 2\cos^2 x\; y = 2\cos^4 x$…………………(1)
Solution: On comparing the above with

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

$\therefore P = tanx , Q = -2cos^2x, R = 2cos^4x$

Now , we assume a following relation between the new independent variable z and the old independent variable x as

$\left(\frac{dz}{dx}\right)^2 = +Q$ Or $\left(\frac{dz}{dx}\right)^2 = 2cos^2x => \frac{dz}{dx} = \sqrt{2}\,cosx$

$=> dz = \sqrt{2}\,cosx\,dx$

Integrate both sides, We get

$z = \sqrt{2}\,sinx$.................................(2)

Now , with this z ,we know that

$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1$ ... ...(3)

Where

$P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2} = 0, Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} = -1, R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} = 1 - \frac{z^2}{2} =$

Hence the equation (3) becomes,

$$\frac{d^2y}{dz^2} + 0.\frac{dy}{dz} - y = 1 - \frac{z^2}{2}$$

$=> \frac{d^2y}{dz^2} - y = 1 - \frac{z^2}{2}$

$=> (D^2 - 1)y = 1 - \frac{z^2}{2}$..................(4)

Now the auxiliary equation is $m^2 - 1 = 0 => m = \pm 1$

Hence C.F. = $(c_1e^z + c_2e^{-z})$

$\therefore$ P.I. $= \frac{Z}{f(D)} = \frac{1-\frac{z^2}{2}}{D^2-1}$

$= -(1 - D^2)^{-1}\left(1 - \frac{z^2}{2}\right)$

$= -[1 + D^2 + D^4 + \cdots]\left(1 - \frac{z^2}{2}\right)$

$= -\left[1 - \frac{z^2}{2} + D^2\left(1 - \frac{z^2}{2}\right) + 0 + 0 \ldots 0\right]$

$\text{P.I.} = -\left[1 - \frac{z^2}{2} + (0 - 1)\right] = \frac{z^2}{2}$

The complete general solution of (4) is

$$y = C.F. + P.I. = (c_1 e^{z} + c_2 e^{-z}) + \frac{z^2}{2} = (c_1 e^{\sqrt{2}\,sinx} + c_2 e^{-\sqrt{2}\,sinx}) + \frac{(\sqrt{2}\,sinx)^2}{2}$$

Example-2 : *Solve the* differential equation

$$\left(\frac{d^2y}{dx^2}\right) - \frac{1}{x}\left(\frac{dy}{dx}\right) - 4x^2 y = 8x^2 sinx^2 \text{......................}(1)$$

Solution: On comparing the above with

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

$\therefore P = -\frac{1}{x}, Q = -4x^2 , R = 8x^2 sinx^2$

Now , we assume a following relation between the new **independent variable and the old independent variable as**

$\left(\frac{dz}{dx}\right)^2 = +Q$ Or $\left(\frac{dz}{dx}\right)^2 = 4x^2 => \frac{dz}{dx} = 2x$

$$=> dz = 2x\, dx$$

Integrate both sides, We get

$z = x^2$ .... (2)

Now , with this z , we know that

$\frac{d^2y}{dz^2} + P_1 \frac{dy}{dz} + Q_1 y = R_1$ ………….(3)

Where

$P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2} = 0, Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} = -1, R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} = 2sinz$

Hence the equation (3) becomes,

$$\frac{d^2y}{dz^2} + 0.\frac{dy}{dz} - y = 2sinz$$

$$=> \frac{d^2y}{dz^2} - y = 2sinz$$

$=> (D^2 - 1)y = 2sinz$ .... (4)

Now the auxiliary equation is $m^2 - 1 = 0 => m = \pm 1$

Hence C.F. = $(c_1 e^{z} + c_2 e^{-z})$

$\therefore$ P.I.$= \frac{Z}{f(D)} = \frac{2sinz}{D^2-1} = \frac{2sinz}{-1-1} = -sinz$

Thecomplete general solution of (4) is

$$y = C.F. + P.I == (c_1 e^{z} + c_2 e^{-z}) - sinz = (c_1 e^{x^2} + c_2 e^{-x^2}) - sin\, x^2$$

Method-IV: Method of Variation of Parameters:-The Method of variation of parameters is used for finding the complete solution of the equation

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \ldots\ldots\ldots\ldots\ldots(1)$$

,when the complementary function [C.F.] is known.Where , P, Q and R be the some functions of independent variable x

Let the C.F. be C.F.=$(c_1y_1 + c_2y_2)$ .... (2)

Where, y_1 and y_2 be the some functions of independent variable x.

Now according to this P.I. = $A\,y_1 + By_2$ .... (3)

Where, A and B be the some functions of independent variable x are given by

$A = -\int \frac{y_2}{W} R\,dx$ and $B = -\int \frac{y_2}{W} R\,dx$ with W = $\begin{vmatrix} y_1 & y_2 \\ \frac{dy_1}{dx} & \frac{dy_2}{dx} \end{vmatrix} \neq 0$

Thus the complete solution is

$$y = C.F. + P.I. = c_1y_1 + c_2y_2 + A\,y_1 + By_2$$

Example -1: Solve by variation of parameters method $(D^2 - 1)y = 2(1 - e^{-2x})^{-\frac{1}{2}}$.

Sol.: The auxiliary equation $m^2 - 1 = 0 \Rightarrow m = 1, -1$

Hence C.F.= $c_1e^x + c_2e^{-x}$. And

P.I.= $y_1\int \frac{y_2X}{W}\,dx + y_2\int \frac{y_1X}{W}\,dx$, where $y_1 = e^x$, $y_2 = e^{-x}$, $X = 2(1 - e^{-2x})^{-\frac{1}{2}}$ and

$$W = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -2$$

$\Longrightarrow$ P.I. = $e^x\int \frac{e^{-x}\,2(1-e^{-2x})^{-\frac{1}{2}}}{-2}\,dx + e^{-x}\int \frac{e^{x}\,2(1-e^{-2x})^{-\frac{1}{2}}}{-2}\,dx = e^xI_1 + e^{-x}I_2$

Where $I_1 = \int \frac{e^{-x}\,2(1-e^{-2x})^{-\frac{1}{2}}}{-2}\,dx = -\int \frac{e^{-x}}{\sqrt{1-e^{-2x}}}\,dx$ put $e^{-x} = t \Longrightarrow -e^{-x}dx = dt$

$\Longrightarrow I_1 = \int \frac{dt}{\sqrt{1-t^2}} = \sin^{-1} t = \sin^{-1}(e^{-x})$

And $I_2 = \int \frac{e^{x}\,2(1-e^{-2x})^{-\frac{1}{2}}}{-2}\,dx = -\int \frac{e^{2x}}{\sqrt{e^{2x}-1}}\,dx$ put $e^{2x} - 1 = t \Rightarrow 2e^{2x}dx = dt$

$\Rightarrow I_2 = \frac{1}{2}\int \frac{dt}{\sqrt{t}} = \sqrt{t} = \sqrt{e^{2x} - 1}$. Hence P.I. = $e^x \cdot \sin^{-1}(e^{-x}) + e^{-x} \cdot \sqrt{e^{2x} - 1}$

$$y = \text{CF+PI} = c_1 e^x + c_2 e^{-x} + e^x \cdot \sin^{-1}(e^{-x}) + e^{-x} \cdot \sqrt{e^{2x} - 1}$$

Example -2: Solve by variation of parameters method

$$(x-1)\ \frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = (x-1)^2$$

Solution: Given equations can be written as

$$\frac{d^2y}{dx^2} - \frac{x}{(x-1)}\frac{dy}{dx} + \frac{1}{(x-1)}y = (x-1) \dots\dots\dots (1)$$

On comparing the above with

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

$\therefore P = -\frac{x}{(x-1)}, Q = \frac{1}{(x-1)}, R = (x-1)$

$\therefore 1 + P + Q = 0 \ \ P + Qx = 0$,Hence ,$y_1 = e^x$, $y_2 = x$ be the integrals of C.F. of equation (1).

i.e. C.F. $= c_1 y_1 + c_2 y_2 = c_1 e^x + c_2 x$

We have $W = \begin{vmatrix} y_1 & y_2 \\ \frac{dy_1}{dx} & \frac{dy_2}{dx} \end{vmatrix} = \begin{vmatrix} e^x & x \\ e^x & 1 \end{vmatrix} = e^x - xe^x = (1-x)e^x \neq 0$

Now P.I. = $A\,y_1 + By_2$............(2) , where

$$A = -\int \frac{y_2}{W} R\,dx = -\int \frac{x}{(1-x)e^x}(x-1)\,dx = \int x\,e^{-x}dx = [-xe^{-x} - e^{-x}]$$

and

$$B = -\int \frac{e^x}{(1-x)e^x}(x-1)\ dx = -\int 1dx = -x$$

Hence the P.I. = $A\,y_1 + By_2 = [-xe^{-x} - e^{-x}]e^x - x^2 = -(x^2 + x + 1)$

Thus the complete solution is
$y = C.F. + P.I = c_1 e^x + c_2 x - (x^2 + x + 1)$.

Method-V: Method of Undetermined Coefficients:-In the Method of Undetermined Coefficientsfor finding the particular Integral of the equation

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \dots\dots\dots\dots\dots. (1)$$

, a trial solution containing unknown constants is assumed.
Where, P, Q $are\ conatants\ and\ R$ is some function of independent variable$x$

Some of the possible trial solution is given below:-

[1] If $R = ce^{mx}$,then the trial solution is ae^{mx} .
In this case, also consider the following:
[i] If m is a root of A.E. of $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$,then the trial solution is axe^{mx}.
[ii] If both the roots of A.E. of $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$ are m ,then the trial solution is ax^2e^{mx} .
(2)] If $R = c\ sinmx$ or $c\ cosmx$,then the trial solution is $c_1\ cosmx + c_2\ sinmx$.
[3] If $R = c\ x^m$,then the trial solution is $a_1\ x^m + a_2\ x^{m-1} + \cdots \ldots \ldots + a_m\ x + a_{m-1}$.
[4] If R is the sum or product of functions given in [1],[2] ,and [3] above ,then P.I. is chosen as sum or product of functions given in [1],[2] ,and [3] .
Remarks :
[1] The above method is true when no term in the trial solution comes in C.F. In case any term in trial solution comes in C.F. then this trial solution is multiplied by the lowest positive integral power of x ,which is large enough so that no term of trial solution comes in C.F.
[2] The above method fails when $R = tanx$, $cotx, secx, cosecx$.

Example -1: Solve by method of Undetermined Coefficients

$$\frac{d^2y}{dx^2} - 4y = e^x + sin2x$$

Solution: Given equations can be written as

$(D^2 - 4)y = e^x + sin2x$.................. (1)
The auxiliary equation $m^2 - 4 = 0 \Rightarrow m = 2, -2$

Hence C.F .$= c_1e^{2x} + c_2e^{-2x}$.
Now forP.I.Let the trial solution is

$y = P.I. = c_3e^x + c_4cos2x + c_5sin2x$ ....(2) Since $R = e^x + sin2x$

Now to find $c_3, c_4\ and\ c_5$
∴by (2) ,

$$Dy = c_3e^x - 2c_4sin2x + 2c_5cos2x$$

Also $D^2y = c_3e^x - 4c_4cos2x - 4\ c_5sin2x$
Now , putting the values of y , $and\ D^2y\ in\ (1)$, We get

$-3c_3e^x - 8c_4cos2x - 8c_5sin2x = e^x + sin2x$

Now equating the coefficients of like terms on both sides, we get

$-3c_3 = 1$, $-8c_4 = 0, -8c_5 = 1$

$=> c_3 = -\frac{1}{3}$, $c_4 = 0,$ $c_5 = -\frac{1}{8}$

Hence, $P.I. = -\frac{1}{3}e^x + 0.cos2x - \frac{1}{8}sin2x$

Thus the the complete solution is

$$y = C.F. + P.I = c_1e^{2x} + c_2e^{-2x} - \frac{1}{3}e^x - \frac{1}{8}sin2x$$

Example -2: Solve by method of Undetermined Coefficients

$$\frac{d^2y}{dx^2} + 4y = 8x^2$$

Solution: Given equations can be written as

$(D^2 + 4)y = e^x + sin2x$.................. (1)

The auxiliary equation $m^2 + 4 = 0 \Rightarrow m = \pm 2i$

Hence C.F .$= c_1cos2x + c_2sin2x$.

Now for P.I. Let the trial solution is

$y = P.I. = c_3x^2 + c_4x + c_5$ ....(2) Since $R = 8x^2$

Now to find c_3, c_4 and c_5

$\therefore$by (2) ,

$Dy = 2\,c_3x + c_4$

Also $D^2y = 2c_3$

Now , putting the values of y , and D^2y in (1), We get

$$2c_3 + 4\,c_3x^2 + 4c_4x + 4c_5 = 8x^2$$

Now equating the coefficients of like terms on both sides, we get

$$4\,c_3 = 8 => c_3 = 2\,, 4c_4 = 0 => c_4 = 0\ , 2c_3 + 4c_5 = 0 => c_5 = -1$$

Hence, $P.I = 2x^2 - 1$

Thus thethe complete solution is

$y = C.F. + P.I = c_1cos2x + c_2sin2x + 2x^2 - 1$

Example -3: Solve by method of Undetermined Coefficients

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = e^{2x}$$

Solution: Given equations can be written as

$(D^2 - 4D + 4)y = e^{2x}$ ....(1)

The auxiliary equation $m^2 - 4m + 4 = 0 \Rightarrow m = 2,2$

Hence C.F .= $(c_1 + c_2x)e^{2x}$.

Now for P.I. Let the trial solution is

$y = P.I = c_3x^2\, e^{2x}$ ... ... (2) Since $R = e^{2x}$ *and repeated roots*

Now to find c_3,

∴by (2) ,

$Dy = 2\, c_3x\, e^{2x} + 2\, c_3x^2\, e^{2x}$

Also $D^2y = 2c_3\, e^{2x} + 4\, c_3x\, e^{2x} + 4\, c_3x\, e^{2x} + 4\, c_3x^2\, e^{2x}$

Now , putting the values of y , Dy *and* D^2y *in* (1), We *get*

$2c_3\, e^{2x} = e^{2x} => c_3 = \dfrac{1}{2}$

Hence, $P.I = \frac{1}{2}x^2\, e^{2x}$

Thus the complete solution is

$y = C.F. + P.I = (c_1 + c_2\,x)e^{2x} + \frac{1}{2}x^2\, e^{2x}$.

Part-II : Power series solutions

1 Power Series Method

1.1 Power Series

$$\sum_{m=0}^{\infty} a_m\,(x - x_0)^m = a_0 + a_1\,(x - x_0) + a_2\,(x - x_0)^2 + \ldots$$

where $a_0, a_1, \ldots$, are constants (**coefficients**)

x_0 is a constant (**center**)

Taylor's Formula

$$f(x) = \sum_{m=0}^{N} \frac{f^{(m)}(x_0)}{m!} (x - x_0)^m + R_N (x - x_0)$$

If $(x - x_0)$ is sufficiently small, $R_N (x - x_0) \to 0$ as $N \to \infty$, then, we say f(x) is *analytic* at x_0, and

$$f(x) = \sum_{m=0}^{\infty} \frac{f^{(m)}(x_0)}{m!} (x - x_0)^m$$ **Taylor Series**

When $x_0 = 0$ $\Rightarrow$ ***Maclaurin Series***

Examples: $$e^x = \sum_{m=0}^{\infty} \frac{x^m}{m!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

$$\sin x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

1.2 Basic Idea of the Power Series Method

In the previous discussion, the linear differential equations with constant coefficients were solved and shown to have solution for

$$y' + ay' + by = 0$$

They can be anyone of the following 3 forms:

$$y = A_1 e^{m_1 x} + A_2 e^{m_2 x}$$
$$y = (A_1 + A_2 x) e^{mx}$$
$$y = e^{\alpha x} \left(A_1 \cos \beta x + A_2 \sin \beta x \right)$$

But, exponential, sine and cosine functions can be expressed in terms of Maclaurin series or Taylor series expanded around zero.

Example. $y'' + y = 0$

Solution. Assume

$$\boxed{y = \sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + \dots}$$

$$y' = \sum_{m=1}^{\infty} m\, a_m x^{m-1} = a_1 + 2 a_2 x + 3 a_3 x^2 + \dots$$

$$y'' = \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2} = 2a_2 + 6 a_3 x + 12 a_4 x^2 + \dots$$

Since $y'' + y = 0$

$\Rightarrow$ $(2a_2 + 6a_3 x + \dots) + (a_0 + a_1 x + a_2 x^2 + \dots) = 0$

or $(2a_2 + a_0) + (6a_3 + a_1) x + (12a_4 + a_2) x^2 + \dots = 0$

Since **$1, x, x^2, \dots, x^n$ are linearly independent functions**, we have

$2a_2 + a_0 = 0$ coefficients of x^0

$6a_3 + a_1 = 0$ coefficients of x^1

$12a_4 + a_2 = 0$ coefficients of x^2

or $a_2, a_4, a_6, \dots$, can be expressed in terms of a_0

$a_3, a_5, a_7, \dots$, can be expressed in terms of a_1

where a_0 and a_1 are arbitrary constants. After solving the above simultaneous equations, we have

$$a_2 = -\frac{a_0}{2} = -\frac{a_0}{2!}$$

$$a_3 = -\frac{a_1}{6} = -\frac{a_1}{3!}$$

$$a_4 = \dots = \frac{a_0}{4!}; \dots$$

thus $$y = a_0\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right] + a_1\left[x - \frac{x^3}{3!} + \dots\right]$$

$$= a_0 \cos x + a_1 \sin x$$

- Since every linear differential equation with constant coefficients always possesses a valid series solution, it is natural to expect the linear differential equations with variable coefficients to have series solutions too.

- Also, since the majority of series cannot be summed and written in a function form, it is to be expected that some solutions must be left in series form.

$$y'' + p(x)\, y' + q(x)\, y = 0$$

where p(x) and q(x) are expressed in polynomials.

We assume

$$y = \sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + \ldots$$

$$y' = \sum_{m=1}^{\infty} m\, a_m x^{m-1} = a_1 + 2 a_2 x + \ldots$$

$$y'' = \sum_{m=2}^{\infty} m(m-1)\, a_m x^{m-2} = 2 a_2 + 3 \times 2 a_3 x + \ldots$$

(1) Put y, y' and y'' into the differential equation

(2) Collect terms of $x^0, x^1, x^2, \ldots,$

(3) Solve a set of simultaneous equations of $a_0, a_1, a_2, \ldots$

2 Theory of Power Series Method

2.1 Introduction

Power Series:

$$S(x) = \sum_{m=0}^{\infty} a_m (x - x_0)^m = a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + \ldots \quad (1)$$

Partial Sum:

$$S_n(x) = a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + \ldots + a_n (x - x_0)^n \quad (2)$$

Remainder:

$$R_n(x) = a_{n+1}(x - x_0)^{n+1} + a_{n+2}(x - x_0)^{n+2} + \dots \qquad (3)$$

Note that $R_n = S - S_n$ or $|S_n - S| = |R_n|$

Convergence:

Definition 1

If $\lim_{n\to\infty} S_n(x_1) = S(x_1)$, then the series (1) converges at $x = x_1$ and $x_1 \neq x_0$

Definition 2

If the series converges, then for every given positive number (no matter how small, but not zero), we can find a number N such that

$$|S_n - S| < \varepsilon \qquad \text{for every } n > N$$

2.2 Radius of Convergence

We know that for the series $\sum_{m=0}^{\infty} x^m = 1 + x + x^2 + \dots$

$|x| > 1$ divergent

$|x| < 1$ convergent

but for the series $\sum_{m=0}^{\infty} \frac{x^m}{m!} = 1 + x + \frac{x^2}{2!} + \dots \; (= e^x)$

Convergent for all x.

If a series converges for all x in $|x - x_0| < R$

and diverges for $|x - x_0| > R$ $(0 < R < \infty)$

then R = ***radius of convergence***

$R = \infty$ if series converges for all x.

R can be calculated by the following formula:

$$R = \frac{1}{\lim_{m\to\infty} \sqrt[m]{|a_m|}}$$

or $$R = \frac{1}{\lim_{m\to\infty} \left|\frac{a_{m+1}}{a_m}\right|} \quad \text{(Ratio Test)}$$

Ratio Test

$$\rho = \lim_{m\to\infty} \left| \frac{a_{m+1}(x - x_0)^{m+1}}{a_m (x - x_0)^m} \right|$$

$$= \lim_{m\to\infty} \left| \frac{a_{m+1}(x - x_0)}{a_m} \right| = |x - x_0| \lim_{m\to\infty} \left|\frac{a_{m+1}}{a_m}\right|$$

if $\rho > 1$ divergent

$\rho < 1$ convergent

$\rho = 1$ test fails (i.e., inconclusive)

Note that radius of convergence R can be 'calculated' by

since $\rho < 1$: convergence, we need

$$|x - x_0| \lim_{m\to\infty} \left|\frac{a_{m+1}}{a_m}\right| = \lim_{m\to\infty} \left| \frac{a_{m+1}(x - x_0)}{a_m} \right| < 1$$

$$\therefore \quad |x - x_0| < \frac{1}{\lim_{m\to\infty} \left|\frac{a_{m+1}}{a_m}\right|} = R \text{ (radius of convergence)}$$

Example: $e^x = 1 + x + \frac{x^2}{2!} + \ldots = \sum_{m=0}^{\infty} \frac{x^m}{m!}$

$$\rho = \lim_{m\to\infty} \left|\frac{a_{m+1} x}{a_m}\right| = \lim_{m\to\infty} \left| \frac{x/(m+1)!}{1/m!} \right|$$

$$= \lim_{m\to\infty} \frac{x}{m+1} = 0 < 1$$

$\Rightarrow$ The series converges, i.e.,

$$R = \lim_{m\to\infty} \frac{1/m!}{1/(m+1)!} = \lim_{m\to\infty} (m+1) = \infty, \text{ i.e.,}$$

converges for all x.

Example: $\sum_{m=0}^{\infty} x^m = 1 + x + x^2 + x^3 + \ldots$

$$\rho = \lim_{m\to\infty} \left| \frac{x\, a_{m+1}}{a_m} \right| = \lim_{m\to\infty} |x| = |x|$$

thus, converges for $|x| < 1$, diverges for $|x| > 1$

and for x = 1, ratio test fails.

$$|x| < R = \lim_{m\to\infty} \frac{1}{1} = 1$$

i.e., converges for all x in $|x| < 1$.

In fact, this series converges to $\frac{1}{1-x}$ for $-1 < x < 1$.

Example: $\sum_{m=0}^{\infty} m!\, x^m = 1 + x + 2x^2 + 6x^3 + \ldots$

$$\rho = \lim_{m\to\infty} \left| \frac{x\, a_{m+1}}{a_m} \right| = \frac{x\,(m+1)!}{m!} = x\,(m+1) = \infty > 1$$

$$R = \lim_{m\to\infty} \left| \frac{a_m}{a_{m+1}} \right| = \lim_{m\to\infty} \frac{1}{m+1} = 0$$

thus, this series diverges for all $x \neq 0$.

Example: $\sum_{m=0}^{\infty} \frac{(-1)^m}{8^m} x^{3m}$

This is a series in powers of $t = x^3$ with coefficients $a_m = \frac{(-1)^m}{8^m}$, so that

$$\rho = \lim_{m\to\infty}\left|\frac{t\, a_{m+1}}{a_m}\right| = \frac{|t|}{8}$$

thus, converges for

$$\frac{|t|}{8} < 1 \quad \text{or} \quad |t| < 8 \quad \text{or} \quad R = \lim_{m\to\infty}\left|\frac{a_m}{a_{m+1}}\right| = 8$$

i.e., $|x| < 2$

2.3 Properties of Power Series

(1) A power series may be differentiated term by term (***Term-wise Differentiation***).

i.e., if $$y(x) = \sum_{m=0}^{\infty} a_m (x - x_0)^m, \quad |x - x_0| < R \text{ and } R > 0$$

then $$y'(x) = \sum_{m=0}^{\infty} m\, a_m (x - x_0)^{m-1} = \sum_{m=1} m a_m (x - x_0)^{m-1}$$

(2) Two power series may be added term by term (***Term-wise Addition***).

i.e., if $$f(x) = \sum_{m=0}^{\infty} a_m (x - x_0)^m; \qquad g(x) = \sum_{m=0}^{\infty} b_m (x - x_0)^m$$

then $$f(x) + g(x) = \sum_{m=0}^{\infty} (a_m + b_m)(x - x_0)^m$$

(3) Two power series may be multiplied term by term (***Term-wise Multiplication***).

$$f(x)\, g(x) = \sum_{m=0}^{\infty} (a_0 b_m + a_1 b_{m-1} + \ldots + a_m b_0)(x - x_0)^m$$

(4) Vanishing of all Coefficients (***Linearly Independence***).

if $\qquad f(x) = \sum_{m=0}^{\infty} a_m (x - x_0)^m = 0$

for *all* x in $|x - x_0| < R$, then

$$a_m = 0 \text{ for } \underline{\textit{all}}\ m.$$

Let's ask ourselves a question: Can **all** linear second-order variable coefficient differential equations be solved by power series method? Let us answer this question by the following illustration:

Example: Solve the equations

$$x^2 y'' + a x y' + b y = 0$$

where (i) $a = -2, \quad b = 2$

(ii) $a = -1, \quad b = 1$

(iii) $a = 1, \quad b = 1$

Solution: we assume

$$y = \sum_{m=0}^{\infty} c_m x^m ; \qquad y' = \sum_{m=1}^{\infty} m c_m x^{m-1} ;$$

$$y'' = \sum_{m=2}^{\infty} m(m-1) c_m x^{m-2}$$

$$\therefore \quad x^2 y'' + a x y' + b y =$$

$$\sum_{m=0}^{\infty} [m(m-1) + a m + b] c_m x^m = 0$$

Note that m (m - 1) + a m + b = 0 *is the characteristic equation for Euler equation.*

Case (i) $\quad a = -2, \quad b = 2$

$$\Rightarrow \quad \sum_{m=0}^{\infty} (m^2 - 3m + 2)\, c_m x^m = 0$$

or $\quad (m^2 - 3m + 2)\, c_m = 0$

or $\quad (m-2)(m-1)\, c_m = 0$

$\Rightarrow \quad c_m = 0$ for all $m \neq 1$ or 2

$$(c_0 = c_3 = c_4 = \cdots = 0)$$

$\Rightarrow \quad y = c_1 x + c_2 x^2$

Case (ii) $\quad a = -1, \qquad b = 1$

$$\Rightarrow \quad \sum_{m=0}^{\infty} (m^2 - 2m + 1)\, c_m x^m = 0$$

$\therefore \quad (m-1)^2 c_m = 0$

$\Rightarrow \quad c_m = 0 \qquad$ for all $m \neq 1$

$\Rightarrow \quad y = c_1 x$

In this case, power series method yields only **one solution:** $y = c_1 x$.

We need another linearly independent solution to get the general solution of the differential equation.

$\Rightarrow \quad$ Reduction of order: let $y_2 = x u$

$\Rightarrow \quad x^3 u'' + x^2 u' = 0$

$\Rightarrow \quad u = c \ln|x|$

$\Rightarrow \quad y = A x + B x \ln|x|$

Case (iii) $\quad a = 1, \qquad b = 1$

$\Rightarrow \quad (m^2 + 1)\, c_m = 0$

$\Rightarrow \quad c_m = 0 \qquad$ for all m

i.e., the power series method **fails** completely, but why??

By the way, the general solution of Case (iii) is

$$y = A\cos(\ln|x|) + B\sin(\ln|x|)$$

2.4 Regular Point and Singular Point

Analytic Function: If g is a function defined on an interval I, containing a point x_0, we say that g is analytic at x_0 if g can be expanded in a power series about x_0 which has a positive radius of convergence.

Any polynomial in x is analytic for all x.

Any rational function (ratio of polynomials) is analytic for all values of x which are not zeros of the denominator polynomial.

Question: Are e^x, $\sqrt{x}$, and $\frac{1}{x}$ analytic at $x = 0$?

Theorem *(Existence of Power Series Solutions)*

If the function p, q, r in

$$y'' + p(x)\,y' + q(x)\,y = r(x)$$

are analytic at $x = x_0$*, then every solution y(x) of the above equation is analytic at* $x = x_0$ *and can be represented by a power series of* $x - x_0$ *with radius of convergence* $R > 0$*, i.e.*

$$y = \sum_{m=0}^{\infty} a_m (x - x_0)^m$$

Definition: *Regular Point and Singular Point*

We call $x = 0$ a ***regular point*** (or *ordinary point*) of the differential equation

$$y'' + p(x)\,y' + q(x)\,y = 0$$

when both p(x) and q(x) are analytic at x = 0.

If x = 0 is not a regular point, it is called a ***singular point*** of the differential equation.

Example: $x y'' + 2 y' + x y = 0$

$$y'' + \frac{2}{x} y' + y = 0$$

$\Rightarrow$ $x = 0$ is a singular point!

$\Rightarrow$ may give some trouble in power series method.

If we nonetheless assume **(This is inappropriate!)**

$$y = \sum_{m=0}^{\infty} c_m x^m$$

the differential equation becomes

$$\sum_{m=2}^{\infty} m(m-1)c_m x^{m-1} + \sum_{m=1}^{\infty} 2m c_m x^{m-1} + \sum_{m=0}^{\infty} c_m x^{m+1} = 0$$

Let $m = k+1 \Rightarrow \sum_{m=2}^{\infty} m(m-1)c_m x^{m-1} = \sum_{k=1}^{\infty} (k+1)k\, c_{k+1} x^k$

Let $m = k+1 \Rightarrow \sum_{m=1}^{\infty} 2m c_m x^{m-1} = \sum_{k=0}^{\infty} 2(k+1) c_{k+1} x^k$

$$= 2c_1 + \sum_{k=1}^{\infty} 2(k+1) c_{k+1} x^k$$

Let $m = k-1 \Rightarrow \sum_{m=0}^{\infty} c_m x^{m+1} = \sum_{k=1}^{\infty} c_{k-1} x^k$

Thus we have

$$2c_1 + \sum_{k=1}^{\infty} \{[(k+1)k + 2(k+1)]c_{k+1} + c_{k-1}\} x^k = 0$$

or $$2c_1 + \sum_{k=1}^{\infty} \{(k+1)(k+2)c_{k+1} + c_{k-1}\} x^k = 0$$

$\therefore$ $c_1 = 0$

$$c_{k+1} = \frac{-c_{k-1}}{(k+2)(k+1)} \qquad \text{for } k \geq 1$$

$$\therefore \quad c_3 = c_5 = c_7 = \ldots = 0$$

$$c_2 = -\frac{c_0}{3!} \qquad c_4 = \frac{c_0}{5!}$$

$$\therefore \quad y = c_0 \left\{ 1 - \frac{x^2}{3!} + \frac{x^4}{5!} + \ldots \right\} = c_0 \frac{\sin x}{x}$$

Only one solution is obtained! The other linearly independent solution can be obtained by the method of reduction of order:

$$\Rightarrow \quad y_2 = u \frac{\sin x}{x} \qquad \left(u' = \frac{1}{y_1^2} e^{-\int p(x)dx} \text{ where } y_1 = \frac{\sin x}{x} \text{ and } p = \frac{2}{x} \right)$$

$$\Rightarrow \quad y_2 = \frac{\cos x}{x} \qquad \text{(Exercise!)}$$

$$\therefore \quad y = A\frac{\sin x}{x} + B\frac{\cos x}{x}$$

Note that

$$\frac{\cos x}{x} = x^{-1} \left\{ 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots \right\}$$

This suggests that **we *may* try y = $x^r(c_0 + c_1 + c_2 x^2 + \ldots)$** in the first place to obtain the second linearly independent solution.

3 Frobenius Method

3.1 General Concepts

$$y'' + p(x)\, y' + q(x)\, y = 0$$

If p(x), q(x) are analytic at x = 0

$\Rightarrow$ x = 0 is a **regular point**, two linearly independent exist.

$$\Rightarrow \quad \mathbf{y = \sum_{m=0}^{\infty} a_m x^m}$$

If p(x), q(x) are not analytic at x = 0 $\Rightarrow$ **singular point**

For x = 0 is a singular point, rewrite the differential equation in the following form:

$$y'' + \frac{b(x)}{x} y' + \frac{c(x)}{x^2} y = 0$$

If b(x), c(x) analytic at x =0

$\Rightarrow$ **regular singular point**, at least one solution exist with the following form

$$\Rightarrow \quad y = x^r \sum_{m=0}^{\infty} a_m x^m$$

where r is a parameter which need to be determined. It can be positive or negative.

If b(x), c(x) not analytic at x=0

$\Rightarrow$ **irregular singular point**, a non-trivial solution may or may not exist.

Theorem 1 (Frobenius Method)

Any differential equation of the form

$$y'' + \frac{b(x)}{x} y' + \frac{c(x)}{x^2} y = 0$$

where b(x) and c(x) are analytic at x = 0 (a regular singular point)

has at least one solution of the form

$$y = x^r \sum_{m=0}^{\infty} a_m x^m = x^r (a_0 + a_1 x + a_2 x^2 + \dots),$$

where $a_0 \neq 0$ and r may be any number (real or complex).

x=0 regular singular point!!!

3.2 Indicial Equation

$$y'' + \frac{b(x)}{x} y' + \frac{c(x)}{x^2} y = 0$$

or $$x^2 y'' + x\, b(x)\, y' + c(x)\, y = 0$$

Since b(x) and c(x) are **analytic**, i.e.,

$$b(x) = b_0 + b_1 x + b_2 x^2 + \ldots$$

$$c(x) = c_0 + c_1 x + c_2 x^2 + \ldots$$

We let $y = x^r \sum_{m=0}^{\infty} a_m x^m = x^r\left(a_0 + a_1 x + a_2 x^2 + \cdots\right)$

$$y' = \sum_{m=0}^{\infty} (m+r)\, a_m x^{m+r-1} = x^{r-1}\left[r a_0 + (r+1)\, a_1 x + \ldots \right]$$

$$y'' = \sum_{m=0}^{\infty} (m+r)(m+r-1)\, a_m x^{m+r-2}$$

$$= x^{r-2}\left[r(r-1)\, a_0 + (r+1)\, r\, a_1 x + \ldots \right]$$

Put y, y', y'', b(x), c(x) into the differential equation and collect terms of x^p, we have (for x^r terms)

$$[\, r(r-1) + b_0 r + c_0 \,]\, a_0 = 0$$

Since $a_0 \neq 0$, we have

$$r(r-1) + b_0 r + c_0 = 0$$ **Indicial Equation !!!**

Two roots for r:

one r for $y_1 = x^r \sum_{m=0}^{\infty} a_m x^m$

another r $\Rightarrow$ Theorem 2 for y_2

Theorem 2 Form of the Second Solution

Case 1: r_1 and r_2 differ but not by an integer

$$y_1 = x^{r_1}\left(a_0 + a_1 x + a_2 x^2 + \ldots\ldots \right)$$

$$y_2 = x^{r_2}\,(A_0 + A_1 x + A_2 x^2 + \ldots\ldots)$$

Case 2: $r_1 = r_2 = r, \quad r = \frac{1}{2}(1 - b_0)$

$$y_1 = x^r\,(a_0 + a_1 x + a_2 x^2 + \ldots..)$$

$$y_2 = \mathbf{y_1 \ln x} + x^r\,(A_1 x + A_2 x^2 + \ldots..)$$

Case 3: r_1 and r_2 differ by a nonzero integer, where $r_1 > r_2$

$$y_1 = x^{r_1}\,(a_0 + a_1 x + a_2 x^2 + \ldots\ldots)$$

$$y_2 = \mathbf{k\, y_1 \ln x} + x^{r_2}\,(A_0 + A_1 x + A_2 x^2 + \ldots\ldots)$$

where $r_1 - r_2 > 0$ and **k <u>may</u> or may not be zero!!!**

Note that in Case 2 and Case 3,

the second linearly independent solution y_2 can also be obtained by <u>reduction of order method</u> (i.e., by assuming $y_2 = u\, y_1$).

Case 1: r_1 and r_2 differ but not by an integer

Example: $\quad y'' + \frac{1}{4x} y' + \frac{1}{8x^2} y = 0, \qquad x > 0 \quad$ (Euler Equation)

Solution. $\quad y = x^r\,(a_0 + a_1 x + a_2 x^2 + \ldots) = x^r \sum_{m=0}^{\infty} a_m x^m = \sum_{m=0}^{\infty} a_m x^{m+r}$

$y' = \ \ldots$

$y'' = \ \ldots$

$$\Rightarrow \quad \sum_{m=0}^{\infty} a_m \left[(r+m)(r+m-1) + \frac{1}{4}(r+m) + \frac{1}{8} \right] x^{r+m-2} = 0$$

For m = 0, $a_m \neq 0$ ($a_0 \neq 0$ by Theorem 1)

thus,we have the indicial equation:

$$r(r-1)+\frac{1}{4}r+\frac{1}{8}=0$$

or $r_1 = 1/4$ and $r_2 = 1/2$

Note that in this case, $r_1 \neq r_2$ and $r_1 - r_2$ is not an integer.

$$r_2 - r_1 = \frac{1}{4}$$

For $\mathbf{r = \frac{1}{4}}$, we have

$$y_1 = x^{1/4}(a_0 + a_1 x + a_2 x^2 + \ldots)$$

or $$\sum_{m=0}^{\infty} a_m \left[\left(\frac{1}{4}+m\right)\left(\frac{1}{4}+m-1\right)+\frac{1}{4}\left(\frac{1}{4}+m\right)+\frac{1}{8}\right] x^{\frac{1}{4}+m-2} = 0$$

or $$\sum_{m=0}^{\infty} a_m \left(m^2 - \frac{m}{4}\right) x^{m-\frac{7}{4}} = 0$$

which is valid for all $x > 0$.

Thus, we have

$$a_m m\left[m - \frac{1}{4}\right] = 0 \quad \text{for all } m \ (=0, 1, 2, \ldots)$$

$\Rightarrow$ form $= 0$ a_0 = arbitrary constant

but for $m = 1, 2, \ldots$ $a_m = 0$

$\Rightarrow$ $y_1 = a_0 x^{1/4}$

Similarly, for **r = 1/2**, we have

(by setting $y_2 = x^{1/2}(A_0 + A_1 x + A_2 x^2 + A_3 x^3 + \ldots,)$)

$\Rightarrow$ $y_2 = A_0 x^{1/2}$

Hence, the general solution is

$$y = a_0 x^{1/4} + A_0 x^{1/2}$$

Example: $y'' + \frac{1}{x} y' + \frac{1}{x^2} y = 0, \quad x > 0$

Solution. Letting $y = x^r \sum_{m=0}^{\infty} a_m x^m$, we have

$$\sum_{m=0}^{\infty} a_m [(r + m) (r + m - 1) + (r + m) + 1] x^{r+m-2} = 0$$

The indicial equation is ($m = 0, a_0 \neq 0$)

$$r (r - 1) + r + 1 = r^2 + 1 = 0$$

or $r_1 = i, \quad r_2 = -i, \quad r_1 - r_2 = 2i$ is not an integer.

For $\mathbf{r = i}$

$$\sum_{m=0}^{\infty} a_m [(i + m) (i + m - 1) + (i + m) + 1] x^{i+m-2} = 0$$

or $a_m [(i + m)^2 + 1] = 0$

or $a_m m (m + 2 i) = 0$

$\Rightarrow \quad m = 0 \qquad a_m \neq 0$, i.e., a_0 **is an arbitrary constant**

$m \neq 0 \qquad a_m = 0$

$\Rightarrow \quad y_1 = a_0 x^i = a_0 e^{i \ln x} = a_0 [\cos(\ln x) + i \sin(\ln x)]$

$= \cos(\ln x) + i \sin(\ln x) \qquad$ take $a_0 = 1$

Similarly, for $\mathbf{r = -i}$**,** we have (Exercise!)

$$y_2 = x^{-i} = \cos(\ln x) - i \sin(\ln x)$$

Since the linear combinations of solutions are also solutions of the linear differential equation, thus,

$$y_1^* = \frac{1}{2}(y_1 + y_2) = \cos(\ln x)$$

$$y_2^* = \frac{1}{2i}(y_1 - y_2) = \sin(\ln x)$$

$$\Rightarrow \quad y = c_1 \cos(\ln x) + c_2 \sin(\ln x)$$

Case 2, $r_1 = r_2 = r$**,** Double Roots

Example : $\quad y'' + y' + \frac{1}{4x^2}\, y = 0, \qquad x > 0$

Solution: $\quad$ Letting $y = x^r \sum_{m=0}^{\infty} a_m x^m$, we have

$$\sum_{m=0}^{\infty} a_m (r+m)(r+m-1)\, x^{r+m-2} + \sum_{m=0}^{\infty} a_m (r+m)\, x^{r+m-1}$$

$$+ \sum_{m=0}^{\infty} \frac{1}{4} a_m x^{r+m-2} = 0$$

$$r + m - 1 = r + k - 2 \quad \Rightarrow \quad m = k - 1$$

Since $\displaystyle \sum_{m=0}^{\infty} a_m (r+m)\, x^{r+m-1} = \sum_{k=1}^{\infty} a_{k-1} (r+k-1)\, x^{r+k-2}$

$$= \sum_{m=1}^{\infty} a_{m-1} (r+m-1)\, x^{r+m-2}$$

The differential equation becomes

$$a_0 \left[r(r-1) + \frac{1}{4} \right] x^{r-2} +$$

$$\sum_{m=1}^{\infty} \left\{ a_m \left[(r+m)(r+m-1) + \frac{1}{4} \right] + a_{m-1}(r+m-1) \right\} x^{r+m-2} = 0$$

The indicial equation is

$$r(r-1)+\frac{1}{4} = 0 \quad \text{or} \quad \left[r-\frac{1}{2}\right]^2 = 0$$

or $\quad r_1 = r_2 = r = \frac{1}{2}$

For $\mathbf{r = \frac{1}{2}}$

$$\sum_{m=1}^{\infty}\left\{a_m\left[\left(\frac{1}{2}+m\right)\left(m-\frac{1}{2}\right)+\frac{1}{4}\right]+a_{m-1}\left(m-\frac{1}{2}\right)\right\}x^{m-\frac{3}{2}} = 0$$

or
$$\sum_{m=1}^{\infty}\left\{m^2 a_m+\left[m-\frac{1}{2}\right]a_{m-1}\right\}x^{m-\frac{3}{2}} = 0$$

$$\Rightarrow \quad \mathbf{a_m = -\frac{\left[m-\frac{1}{2}\right]a_{m-1}}{m^2}} \quad \text{for } m \geq 1 \quad \textbf{(recurrence equation)}$$

Hence

$$a_1 = \frac{a_0}{2}, \quad a_2 = -\frac{(3/2)a_1}{2^2} = -\frac{3}{2\cdot 2^2}\left(-\frac{a_0}{2}\right) = \frac{3\,a_0}{2^2\,2^2}, \ldots$$

and
$$y_1 = a_0 x^{\frac{1}{2}}\left\{1-\frac{x}{2}+\frac{3}{2^2}\left[\frac{x}{2}\right]^2+\ldots\right\}$$

$$= x^{1/2}\sum_{m=0}^{\infty}\frac{(2n)!}{(n!)^3}\left[\frac{-x}{4}\right]^n, \qquad x>0$$

Note that we have set $a_0 = 1$ in the above equation.

Approach 1

Since $r = r_1 = r_2$, another solution can be obtained by directly letting

$$y_2 = y_1 \ln x + x^r(A_1 x + A_2 x^2 + \ldots)$$

$$\Rightarrow \quad y_2 = y_1 \ln x + x^{\frac{1}{2}} \left[-\frac{x^2}{16} + \ldots \right]$$

Approach 2

We can also use the method of reduction of order to produce the second linearly independent solution, y_2, by letting

$$y_2 = u\, y_1$$

Put into the differential equation,

$$u''\, y_1 + u' \,(\, 2\, y_1' + y_1 \,) = 0$$

$$\Rightarrow \quad \frac{u''}{u'} = -2\frac{y_1'}{y_1} - 1 = \ldots \text{(long division)} = -\frac{1}{x} - \frac{x}{4} + \ldots$$

i.e., $\ln u' = -\ln x - \frac{x^2}{8} + \ldots$

or $u' = \frac{1}{x} \exp\left\{ -\frac{x^2}{8} + \ldots \right\} = \ldots$

By expanding the exponential function in Taylor series and then integrating

$$u = \ln x - \frac{x^2}{16} + \ldots$$

$$\Rightarrow \quad y_2 = y_1\, u = y_1 \left[\ln x - \frac{x^2}{16} + \ldots \right]$$

$$= y_1 \ln x + \sqrt{x} \left[-\frac{x^2}{16} + \ldots \right]$$

Exercise : $x\, y'' + (\, 1 - x \,)\, y' - y = 0, \quad x > 0$

Case 3: r_1 and r_2 differ by an nonzero integer, $r_1 > r_2$

Example: $x^2\, y'' + x\, y' + \left[x^2 - \frac{1}{4} \right] y = 0$ (Bessel's equation of order 1/2)

Solution . Put $y = x^r \sum_{m=0}^{\infty} a_m x^m = \sum_{m=0}^{\infty} a_m x^{m+r}$

the differential equation becomes

$$\sum_{m=0}^{\infty} a_m (r+m)(r+m-1) x^{r+m-2} + \sum_{m=0}^{\infty} a_m (r+m) x^{r+m-2}$$

$$- \sum_{m=0}^{\infty} \frac{1}{4} a_m x^{r+m-2} + \sum_{m=0}^{\infty} a_m x^{r+m} = 0$$

After substituting $\sum_{m=2}^{\infty} a_{m-2} x^{r+m-2}$ for the last term of the lhs of the above equation, we have

$$a_0 \left[r(r-1) + r - \frac{1}{4} \right] x^{r-2}$$

$$+ a_1 \left[r(r+1) + (r+1) - \frac{1}{4} \right] x^{r-1}$$

$$+ \sum_{m=2}^{\infty} \left\{ a_m \left[(r+m)(r+m-1) + (r+m) - \frac{1}{4} \right] + a_{m-2} \right\} x^{r+m-2}$$

$$= 0$$

Thus, we have the indicial equation:

$$r(r-1) + r - \frac{1}{4} = r^2 - \frac{1}{4} = 0$$

or $\quad r_1 = \frac{1}{2} \qquad r_2 = -\frac{1}{2}$

Note that $r_1 - r_2 = 1$ is ***an integer***!!!

Note also that, when m=1, $r_2(r_2+1) + (r_2+1) - \frac{1}{4} = \left(-\frac{1}{2}\right)\left(\frac{1}{2}\right) + \frac{1}{2} - \frac{1}{4} = 0$!

For $r_1 = \frac{1}{2}$

$$2 a_1 x^{-1/2} + \sum_{m=2}^{\infty} \left[a_m (m^2 + m) + a_{m-2} \right] x^{m - \frac{3}{2}} = 0$$

$$\therefore \quad a_1 = 0 \text{ and } a_m = - \frac{a_{m-2}}{m (m + 1)} \text{ for } m \geq 2$$

$$\Rightarrow \quad y_1 = \frac{\sin x}{\sqrt{x}}$$

For $\mathbf{r_2 = -\frac{1}{2}}$**,** both a_0 and a_1 are arbitrary!

$$\sum_{m=2}^{\infty} \left[a_m (m^2 - m) + a_{m-2} \right] x^{m - \frac{5}{2}} = 0$$

$$\therefore \quad a_m = - \frac{a_{m-2}}{m (m - 1)}$$

or $\quad a_2 = - a_0/2! \qquad a_4 = a_0/4! \qquad a_6 = - a_0/6!$

$\quad a_3 = - a_1/3! \qquad a_5 = a_1/5! \quad \ldots$

$$\Rightarrow \quad y_2 = \frac{1}{\sqrt{x}} (a_0 \cos x + a_1 \sin x)$$

Thus, the linearly independent solution is $\frac{\cos x}{\sqrt{x}}$

(Alternatively, the linearly independent solution y_2 can also be obtained by reduction of order method.)

$$\Rightarrow \quad y = A \frac{\sin x}{\sqrt{x}} + B \frac{\cos x}{\sqrt{x}}$$

Note that k=0 in this case!

Example: $\quad x^2 y'' + x y' + (x^2 - 1) y = 0$ (Bessel's equation of order 1)

Solution. Letting $y = x^r \sum_{m=0}^{\infty} a_m x^m$

we have

$$a_0 [r (r - 1) + r - 1] x^{r-2} + a_1 [r (r + 1) + (r + 1) - 1] x^{r-1}$$

$$+ \sum_{m=2}^{\infty} (a_m ((r + m) (r + m - 1) + (r + m) - 1) - a_{m-2}) x^{r+m-2}$$

$$= 0$$

$\therefore$ The indicial equation is

$$r (r - 1) + r - 1 = r^2 - 1 = 0$$

or $r_1 = 1 \qquad r_2 = -1$

$$r_1 - r_2 = 2$$

$$r_2(r_2+1)+(r_2+1)-1 = -1 \neq 0$$

$$\therefore k \neq 0$$

For **r = 1**, we have

$$3 a_1 + \sum_{m=2}^{\infty} (a_m (m^2 + 2 m) + a_{m-2}) x^{m-1} = 0$$

$\therefore$ $a_1 = 0$ and

$$a_m = - \frac{a_{m-2}}{m (m + 2)} \qquad \text{for } m \geq 2$$

$$\Rightarrow \quad y_1(x) = x \left\{ 1 - \frac{1}{1!\,2!} \left[\frac{x}{2} \right]^2 + \frac{1}{2!\,3!} \left[\frac{x}{2} \right]^4 - \ldots \right\}$$

For **r = − 1**, we have

$$- a_1 x^{-2} + \sum_{m=2}^{\infty} \{ a_m (m^2 - 2 m) + a_{m-2} \} x^{m-3} = 0$$

$\Rightarrow \quad a_1 = 0$ and

$m(m-2)\,a_m = -a_{m-2} \qquad \text{for } m \geq 2$

but **for m = 2, we have 0 = a_0 which is not true.**

Thus we **can not** obtain the second linearly independent solution by setting

$$y = x^r \sum_{m=0}^{\infty} a_m x^m$$

with r = – 1.

Approach 1

From the theorem, we need to *directly* assume that the second solution is of the form:

$$y_2 = \mathbf{k}\, y_1 \ln x + x^{r_2} (A_0 + A_1 x + A_2 x^2 + \ldots)$$

$$= \frac{\mathbf{1}}{\mathbf{4}} y_1 \ln x - \frac{1}{2} x^{-1} + \frac{x}{16} + \ldots \text{ (Exercise!)}$$

Approach 2

Note that the second linearly independent solution can also be obtained by the method of reduction of order (Exercise!):

$$y_2 = u\, y_1$$

$$\frac{u''}{u'} = \frac{-2y_1'}{y_1} - \frac{1}{x} = \frac{-3}{x} + \frac{x}{2} + \ldots$$

or $\quad \ln u' = -3 \ln x + \dfrac{x^2}{4} + \ldots$

$$u' = x^{-3} \exp\left\{\frac{x^2}{4} + \ldots\right\} = x^{-3} + \frac{1}{4} x^{-1} + \ldots$$

or $\quad u = -\dfrac{1}{2} x^{-2} + \dfrac{1}{4} \ln x + \ldots$

Exercise: $x\,y'' + (x - 1)\,y' - 2\,y = 0$

(A) $J_0(0) = 1, J_1(0) = J_2(0) = \ldots = 0, J_0(\infty) = J_1(\infty) = \ldots = 0$

Plots of J_0, J_1, are given in the following figures:

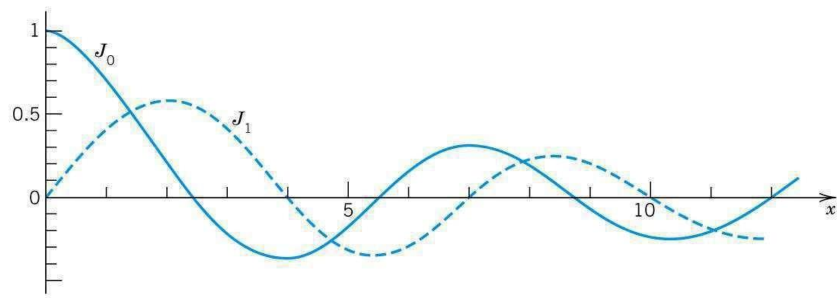

Fig. 103. Bessel functions of the first kind

(B) Bessel Function of Half Integer Order

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\ \sin x \quad , \quad J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\ \cos x$$

4 Legendre's Equation

4.1 Legendre's Differential Equation :

$$(1-x^2)\,y'' - 2xy' + n(n+1)\,y = 0$$

where n is any non-negative real number. Since n(n+1) is unchanged when n is replaced by –(n+1), then (1) solution of n=n′ (where $n' \geq 0$) is the same as n=-(n′+1); (2) solution of n=-n″ (where $n'' \geq 1$) is the same as n=n″-1. The above equation can be written as

$$y'' - \frac{2x}{1-x^2}\,y' + \frac{n(n+1)}{1-x^2}\,y = 0$$

But $$\frac{1}{1-x^2} = 1 + x^2 + x^4 + \ldots$$

Which is analytic at **x = 0** (regular point).

$\therefore$ We can solve the above equation by assuming

$$y = \sum_{m=0}^{\infty} a_m x^m$$

$\Rightarrow$ **Recurrence formula**

$$a_{m+2} = -\frac{(n-m)(n+m+1)}{(m+2)(m+1)}\,a_m, \quad m = 0, 1, \ldots$$

or $$a_2 = -\frac{n(n+1)}{2!}\,a_0 \qquad a_3 = -\frac{(n-1)(n+2)}{3!}\,a_1$$

$$a_4 = \frac{(n-2)n(n+1)(n+3)}{4!}\,a_0$$

$$a_5 = \frac{(n-3)(n-1)(n+2)(n+4)}{5!}\,a_1$$

. . .

$\therefore$ The general solution is $y = a_0 y_1 + a_1 y_2$

where

$$y_1(x) = 1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n-2)(n+1)(n+3)}{4!}x^4 - \frac{n(n-2)(n-4)(n+1)(n+3)(n+5)}{6!}x^6 + \cdots$$

$$y_2(x) = x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-1)(n-3)(n+2)(n+4)}{5!}x^5 - \frac{(n-1)(n-3)(n-5)(n+2)(n+4)(n+6)}{7!}x^7 + \cdots$$

If n = 0, 1, 2, . . . (non-negative integer), then

$y = c_1 P_n(x) + c_2 Q_n(x)$

where $P_n(x)$ = Legendre *polynomials* [It is desirable that $P_n(1) = 1$]

$Q_n(x)$ = Legendre functions of the second kind converges in $-1<x<1$, but

$Q_n(\pm 1)$ = unbounded (This is due to the fact that the Legendre equation is not analytic at x=+1 and x=-1)

4.2 Legendre Polynomials $P_n(x)$

Since $a_{m+2} = -\frac{(n-m)(n+m+1)}{(m+2)(m+1)} a_m$ for m = 0, 1, . . .

when n = non-negative **integer**,

$a_{m+2} = 0$ for m = n $\Rightarrow a_{n+2} = 0$

i.e., $a_{n+2} = a_{n+4} = a_{n+6} = \ldots = 0$

when n = even, $y_1 \Rightarrow$ polynomial of degree n

n = odd, $y_2 \Rightarrow$ polynomial of degree n

These polynomials, multiplied by an appropriate constant, are called the **Legendre polynomials $P_n(x)$**, which have the value $P_n(1) = 1$. In other words, let

$$P_n(x) = \begin{cases} \dfrac{y_1(x)}{y_1(1)} & \text{when } n \text{ is even} \\[2ex] \dfrac{y_2(x)}{y_2(1)} & \text{when } n \text{ is odd} \end{cases}$$

Specifically, we have to choose[1]

$$a_n = 1 \qquad \text{if } n = 0$$

$$a_n = \frac{(2n)!}{2^n (n!)^2} = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{n!} \quad ; \quad \text{if } n = 1, 2, \ldots$$

Then, we can obtain the other coefficients in $P_n(x)$

$$a_{n-2} = -\frac{n(n-1)}{2(2n-1)} a_n = \frac{-(2n-2)!}{2^n (n-1)!(n-2)!}$$

. . .

$$a_{n-2m} = (-1)^m \frac{(2n-2m)!}{2^n m!(n-m)!(n-2m)!}$$

Then, the Legendre polynomial of degree n, $P_n(x)$ is given by

$$P_n(x) = \sum_{m=0}^{M} (-1)^m \frac{(2n-2m)!}{2^n m!(n-m)!(n-2m)!} x^{n-2m}$$

where $$M = \begin{cases} \frac{n}{2} & \text{when } n \text{ is even} \\ \frac{n-1}{2} & \text{when } n \text{ is odd} \end{cases}$$

Example:

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$P_4(x) = \frac{1}{8}\left(35x^4 - 30x^2 + 3\right)$$

$$P_5(x) = \frac{1}{8}\left(63x^5 - 70x^3 + 15x\right)$$

In all cases, $P_n(1) = 1$, and $P_n(-1) = (-1)^n$.

$$n = 0 \qquad y_1(1) = 1 \qquad n = 2, 4, 6, \dots \qquad y_1(1) = (-1)^{\frac{n}{2}} \frac{2 \cdot 4 \cdot 6 \cdots n}{1 \cdot 3 \cdot 5 \cdots (n-1)}$$

$$n = 1 \qquad y_2(1) = 1 \qquad n = 1, 3, 5, \dots \qquad y_2(1) = (-1)^{\frac{n-1}{2}} \frac{2 \cdot 4 \cdot 6 \cdots (n-1)}{1 \cdot 3 \cdot 5 \cdots n}$$

4.3 Some Important Properties of $P_n(x)$

(1) Values of $P_n(x)$

$P_n(1) = 1$

$P_n(-1) = (-1)^n$

$P_n(-x) = (-1)^n P_n(x)$

n : even, $P_n(x)$: even function†

n : odd, $P_n(x)$: odd function

$P_n'(-x) = (-1)^{n+1} P_n'(x)$

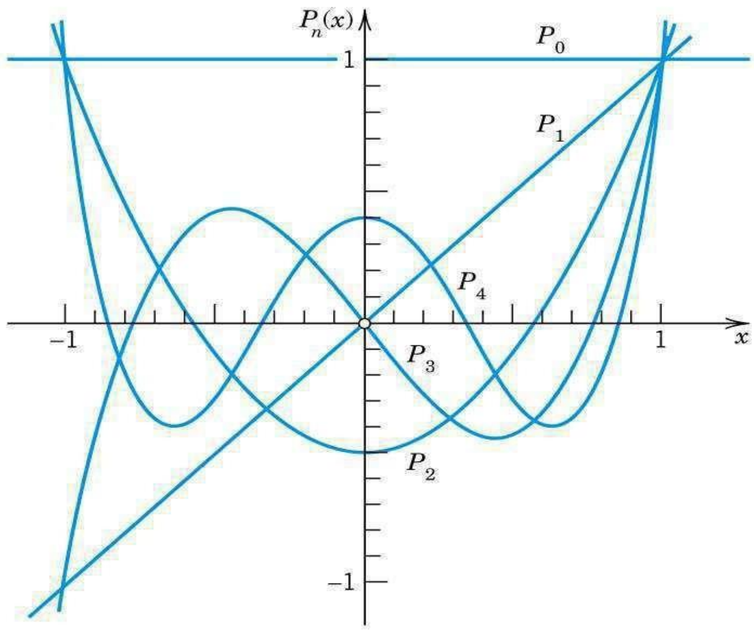

Fig. 101. Legendre polynomials

(2) Rodrigues' Formula

$$P_n(x) = \frac{1}{2^n\, n!}\frac{d^n}{dx^n}\left[(x^2-1)^n\right]$$

Exercise: Show that $P_2(x) = \frac{1}{2}(3x^2 - 1)$

(3) Generating Function for Legendre Polynomials

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)\, t^n$$

(4) Recurrence Formulas

(i) $(n+1)P_{n+1}(x) = (2n+1)\,x\,P_n(x) - n\,P_{n-1}(x)$, $n = 1, 2, \ldots$

(ii) $P_{n+1}'(x) - P_{n-1}'(x) = (2n+1)P_n(x)$

Exercise: starting with $P_0 = 1$, $P_1 = x$, calculate $P_2, P_3, P_4, \ldots$

(5) Integrating Formulas : Orthogonal Property

(i) $$\int_{-1}^{1} P_n^2(x)\,dx = \frac{2}{2n+1} \qquad n = 0, 1, \ldots$$

(ii) $$\int_{-1}^{1} P_m(x)\,P_n(x)\,dx = 0, \qquad m \neq n,\ m, n \in N$$

Contents
Formulation of Partial Differential equations, Linear and Non-Linear Partial Differential Equations, Homogeneous Linear Partial Differential Equations with Constants Coefficients.

Basic concepts and definitions

An equation containing the dependent and independent variables and one or more partial derivatives of the dependent variable is called a **partial differential equation**. In general it may be written in the form

$$F\,(x,y,\ldots\ldots,u,u_x,\ u_y,\ldots\ldots,u_{xx},\ u_{yy},.\quad)=0$$

involving several independent variables x,y,.....,an unknown function u of these variables and the partial derivatives u_x, u_y,......,u_{xx}, u_{xy}, u_{yy} ....of the function. is considered in a suitable subset D of R^n. For the sake of convenience we confine our discussion for n=2. However extension of properties discussed here to higher values of n is possible.

Here as in the case of ordinary differential equations, we define the order of a partial differential equation to be the order of the derivative of highest order occurring in the equation. The power of the highest order derivative in a differential equation is called the degree of the partial differential equation.

Example: (a) $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0$ is a first-order equation in two variables with variable coefficients.

(b) $a\frac{\partial u}{\partial x}+b\frac{\partial u}{\partial y}=c$; where x,y are independent variables, a and b are constants; is partial differential equation of first-order with constant coefficients.

(c) $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}-(x+y)u=0$ is a partial differential equation of first-order.

(d) $a(x)\frac{\partial^2 u}{\partial x^2}+2b(x)\frac{\partial^2 u}{\partial x\partial y}+c(x)\frac{\partial^2 u}{\partial y^2}=x+y+u+\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}$ is a partial differential equation of second-order.

(e) $a(x)\frac{\partial^2 u}{\partial x^2}+2b(x)\frac{\partial^2 u}{\partial x\partial y}+c(x)\frac{\partial^2 u}{\partial y^2}=f(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y})$

where a(x), b(x) and c(x) are functions of x and f(..,..,.,.,.) is a function of x,y,u, $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$, is a partial differential equation of second order.

(f) $u\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial u}{\partial x}=y$ is a partial differential equation of second-order.

(g)(g) $\frac{\partial^2 u}{\partial x^2}+2y\frac{\partial^2 u}{\partial x\partial y}+3x\frac{\partial^2 u}{\partial y^2}=4\sin x$ is a partial differential equation of second-order and degree one.

(h)(h) $\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial y^2}$ is a partial differential of second-order.

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(ii) $\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = 1$ is a partial differential equation of first-order and second degree.

By a solution of a partial differential equation of the type

$$F(x,y,u,u_x,u_y,u_{xx},u_{yy},u_{xy}) = 0$$

we understand functions $u=\varphi(x,y)$ which satisfy identically in D, that is, if we put values of quantities on the left hand side we get right hand side.

The **general solution** of a linear partial differential equation is a linear combination of all linearly independent solutions of the equation with as many arbitrary functions as the order of the equation; a partial differential equation of order 2 has 2 arbitrary functions. A **particular** solution of a differential equation is one that does not contain arbitrary functions or constants. Homogeneous linear partial differential equation has an interesting property that if u is its solution then a scalar multiple of u, that is, cu, where c is a constant, is also its solution. Any equation of the type $F(x,y,u,c_1,c_2)=0$, where c_1 and c_2 are arbitrary constants, which is a solution of a partial differential equation of first-order is called a **complete solution** or a **complete integral** of that equation. An equation $F(\varphi,\psi)=0$ involving arbitrary function. F connecting two known functions φ and ψ of x, y and u, and providing a solution of a first order differential equation is called a **general solution** or **general integral** of that equation.

Formation of Partial Differential Equations:-

(i) By eliminating arbitrary constants :

Let $\varphi(x, y. z. a. b) = 0$…(i)

Where a and b are arbitrary constants, x and y are independent variables and z is dependent variable.

Differentiate $\varphi(x, y. z. a. b) = 0$ w. r. t. x, we get

$$\frac{\varphi}{x} + \frac{\varphi}{z}.\frac{z}{x} = 0$$

Or $\frac{\varphi}{x} + \frac{\varphi}{z}.p = 0$... (ii)

Differentiate $\varphi(x, y. z. a. b) = 0$ w. r. t. y, we get

$$\frac{\varphi}{y} + \frac{\varphi}{z}.\frac{z}{y} = 0$$

Or $\frac{\varphi}{y} + \frac{\varphi}{z}.q = 0$... (iii)

Eliminating a and b from (i), (ii) and (iii), we get

$$Fx, y, z, p, q) = 0$$

Which is required Partial Differential Equation of first order.

(ii) By eliminating arbitrary functions :

Let u and v be two given functions of x,y z connected by the relation

$$f(u, v) = 0$$

Where f is an arbitrary function

The linear PDE is $Pp + Qq = R$

Where,

$$P = \frac{(u, v)}{(y, z)} = \frac{u}{y}.\frac{v}{z} - \frac{u\ v}{z\ y} = \begin{vmatrix} \frac{u}{y} & \frac{u}{z} \\ \frac{v}{y} & \frac{v}{z} \end{vmatrix}$$

$$Q = \frac{(u,v)}{(z,x)} = \frac{u}{z}\cdot\frac{v}{x} - \frac{u}{x}\cdot\frac{v}{z} = \begin{vmatrix} \frac{u}{z} & \frac{u}{x} \\ \frac{v}{z} & \frac{v}{x} \end{vmatrix}$$

$$R = \frac{(u,v)}{(x,y)} = \frac{u}{x}\cdot\frac{v}{y} - \frac{u}{y}\cdot\frac{v}{x} = \begin{vmatrix} \frac{u}{x} & \frac{u}{y} \\ \frac{v}{x} & \frac{v}{y} \end{vmatrix}$$

Example-1: Form partial differential equations from the following equations by eliminating arbitrary constants

(i) $z = ax + by + ab$ (ii) $z = ax + a^2y^2 + b$

Solution:

(i) Given $z = ax + by + ab$

Differentiating partially differentiating w.r.t. x and y, we get

$$\frac{z}{x} = a \text{ and } \frac{z}{y} = b$$

Putting the values of a and b in the given equation, we get

$$\mathrm{z = px + qy + pq}$$

which is required PDE.

(ii) Given $z = ax + a^2y^2 + b$

Differentiating partially differentiating w.r.t. x and y, we get

$$\frac{z}{x} = a \text{ and } \frac{z}{y} = 2a^2y$$

Eliminating a from the above equations, we get

$$\frac{z}{y} = 2\left(\frac{z}{x}\right)^2 y$$

$$q = 2p^2 y$$

Example-2: Form partial differential equations from the following equations by eliminating functions constants (i) $z = f(x^2 - y^2)$ (ii) $z = y^2 + 2f\left(\frac{1}{x} + logy\right)$

Solution:

(i) Given $z = f(x^2 - y^2)$

Differentiating partially differentiating w. r.t. x and y, we get

$$\frac{z}{x} = f(x^2 - y^2).\,2x$$

$$\frac{z}{y} = -f(x^2 - y^2).\,2y$$

From the above equations, we get

$$\frac{p}{q} = -\frac{x}{y}$$

$$py + qx = 0$$

Which is required PDE.

(ii) Given $z = y^2 + 2f\left(\frac{1}{x} + logy\right)$

Differentiating partially differentiating w.r.t. x and y, we get

$$\frac{z}{x} = 2f'\left(\frac{1}{x} + logy\right)\left(-\frac{1}{x^2}\right)$$

$$\frac{z}{y} = 2y + 2f'\left(\frac{1}{x} + logy\right)\left(\frac{1}{y}\right)$$

Eliminating a from the above equations, we get

$$\frac{p}{q-2y} = \frac{-\frac{1}{x}}{\frac{1}{y}}$$

$$\frac{p}{q-2y} = \frac{-y}{x}$$

$$px + y(q-2y) = 0$$

Which is required PDE.

Lagrange's Method :

The general form of first-order linear partial differential equations with variable coefficients is

$P(x,y)u_x+Q(x,y)u_y+f(x,y)u=R(x,y)$

We can eliminate the term in u from by substituting $u=ve^{-\xi(x,y)}$, where $\xi(x,y)$ satisfies the equation

$P(x,y)\ \xi_x(x,y)+ Q\ (x,y)\ \xi_y(x,y)=f(x,y)$

Hence, Eq is reduced to

$P(x,y)u_x+Q\ (x,y)\ u_y =R(x,y)$

where P,Q,R in are not the same as in The following theorem provides a method for solving often called **Lagrange's Method**.

Theorem : The general solution of the linear partial differential equation of first order

$Pp+Qq=R$;

where $p=\dfrac{\partial u}{\partial x}, q=\dfrac{\partial u}{\partial y}$, P, Q and R are functions of x y and u

is $F(\varphi, \psi) = 0$

where F is an arbitrary function and $\varphi(x,y,u)=c_1$ and $\psi(x,y,u)=c_2$ form a solution of the auxiliary system of equations

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{du}{R}$$

Proof: Let $\varphi(x,y,u)=c_1$ and $\psi(x,y,u)=c_2$ satisfy then equations

$$\varphi_x dx+\varphi_y dy+\varphi_u du=0$$

and

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{du}{R}$$

must be compatible, that is, we must have $P\varphi_x+Q\varphi_y+R\varphi_u=0$

Similarly we must have

$P\psi_x+Q\psi_y+R\psi_u=0$

Solving these equations for P,Q, and R, we have

$$\frac{P}{\partial(\varphi,\psi)/\partial(y,u)}=\frac{Q}{\partial(\varphi,\psi)/\partial(u,x)}=\frac{R}{\partial(\varphi,\psi)/\partial(x,y)}$$

where $\partial(\varphi,\psi)/\partial(y,u)= \varphi_y\psi_u - \psi_y\varphi_u \neq 0$ denotes the Jacobian.

Let $F(\varphi,\psi)=0$. By differentiating this equation with respect to x and y, respectively, we obtain the equations

$$\frac{\partial F}{\partial \varphi}\left\{\frac{\partial \varphi}{\partial x}+\frac{\partial \varphi}{\partial u}p\right\}+\frac{\partial F}{\partial \psi}\left\{\frac{\partial \psi}{\partial x}+\frac{\partial \psi}{\partial u}p\right\}=0$$

$$\frac{\partial F}{\partial \varphi}\left\{\frac{\partial \varphi}{\partial y}+\frac{\partial \varphi}{\partial u}q\right\}+\frac{\partial F}{\partial \psi}\left\{\frac{\partial \psi}{\partial y}+\frac{\partial \psi}{\partial u}q\right\}=0$$

and if we now eliminate $\frac{\partial F}{\partial \varphi}$ and $\frac{\partial F}{\partial \psi}$ from these equations, we obtain the equation p $\frac{\partial(\varphi,\psi)}{\partial(y,u)}$ +q $\frac{\partial(\varphi,\psi)}{\partial(u,x)}$ = $\frac{\partial(\varphi,\psi)}{\partial(x,y)}$

Substituting from equations into equation we see that $F(\varphi,\psi)=0$ is a general solution of The solution can also be written as

$\varphi = g(\psi)$ or $\psi = h(\varphi)$,

Example: Find the general solution of the partial differential equation $y^2up + x^2uq = y^2x$

Solution: The auxiliary system of equations is

$$\frac{dx}{y^2u}=\frac{dy}{x^2u}=\frac{du}{xy^2}$$

Taking the first two members we have $x^2dx = y^2dy$ which on integration given $x^3-y^3 = c_1$.

Again taking the first and third members,

we have x dx = u du

which on integration given $x^2-u^2 = c_2$

Hence, the general solution is

$F(x^3-y^3, x^2-u^2) = 0$

Char pit's Method for solving nonlinear Partial Differential Equation of First-Order :

We present here a general method for solving non-linear partial differential equations. This is known as Charpit's method.

Let $F(x,y,u, p.q)=0$

be a general non linear partial differential equation of first-order. Since u depends on x and y, we have

$du=u_x dx+u_y dy = pdx+qdy$

where $p=u_x=\dfrac{\partial u}{\partial x}$, $q = u_y=\dfrac{\partial u}{\partial y}$

If we can find another relation between x,y,u,p,q such that

$f(x,y,u,p,q)=0$

then we can solve for p and q and substitute them in equation This will give the solution provided is integrable.

To determine f, differentiate and w.r.t. x and y so that

$$\frac{\partial F}{\partial x}+\frac{\partial F}{\partial u}p+\frac{\partial F}{\partial p}\frac{\partial p}{\partial x}+\frac{\partial F}{\partial q}\frac{\partial q}{\partial x}=0$$

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial u}p+\frac{\partial f}{\partial p}\frac{\partial p}{\partial x}+\frac{\partial f}{\partial q}\frac{\partial q}{\partial x}=0$$

$$\frac{\partial F}{\partial y}+\frac{\partial F}{\partial u}q+\frac{\partial F}{\partial p}\frac{\partial p}{\partial y}+\frac{\partial F}{\partial q}\frac{\partial q}{\partial y}=0$$

$$\frac{\partial f}{\partial y}+\frac{\partial f}{\partial u}q+\frac{\partial f}{\partial p}\frac{\partial p}{\partial y}+\frac{\partial f}{\partial q}\frac{\partial q}{\partial y}=0$$

Eliminating $\frac{\partial p}{\partial x}$ from, equations and $\frac{\partial q}{\partial y}$ from equations and we obtain

$$\left(\frac{\partial F}{\partial x}\frac{\partial f}{\partial p}-\frac{\partial f}{\partial x}\frac{\partial F}{\partial p}\right)+\left(\frac{\partial F}{\partial u}\frac{\partial f}{\partial p}-\frac{\partial f}{\partial u}\frac{\partial F}{\partial p}\right)p+\left(\frac{\partial F}{\partial q}\frac{\partial f}{\partial p}-\frac{\partial f}{\partial q}\frac{\partial F}{\partial p}\right)\frac{\partial q}{dx}=0$$

$$\left(\frac{\partial F}{\partial y}\frac{\partial f}{\partial q}-\frac{\partial f}{\partial y}\frac{\partial F}{\partial q}\right)+\left(\frac{\partial F}{\partial u}\frac{\partial f}{\partial q}-\frac{\partial f}{\partial u}\frac{\partial F}{\partial q}\right)q+\left(\frac{\partial F}{\partial p}\frac{\partial f}{\partial q}-\frac{\partial f}{\partial p}\frac{\partial F}{\partial q}\right)\frac{\partial p}{dy}=0$$

Adding these two equations and using

$$\frac{\partial q}{\partial x}=\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial p}{\partial y}$$

and rearranging the terms, we get

$$\left(-\frac{\partial F}{\partial p}\right)\frac{\partial f}{\partial x}+\left(-\frac{\partial F}{\partial q}\right)\frac{\partial f}{\partial y}+\left(-p\frac{\partial F}{\partial p}-q\frac{\partial F}{\partial q}\right)\frac{\partial f}{\partial u}+\left(\frac{\partial F}{\partial x}+p\frac{\partial F}{\partial u}\right)\frac{\partial f}{\partial p}$$
$$+\left(\frac{\partial F}{\partial y}+q\frac{\partial f}{\partial u}\right)\frac{\partial f}{\partial q}=0$$

Following arguments in the proof of Theorem we get the auxiliary system of equations

$$\frac{dx}{-\frac{\partial F}{\partial p}} = \frac{dy}{-\frac{\partial F}{\partial q}} = \frac{du}{-p\frac{\partial F}{\partial p} - q\frac{\partial F}{\partial q}} = \frac{dp}{\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial u}} = \frac{dq}{\frac{\partial F}{\partial y} + q\frac{\partial F}{\partial u}} = \frac{df}{0}$$

An Integral of these equations, involving. p or q or both, can be taken as the required equation. p and q determined will make integer able.

Example: Find the general solution of the partial differential equation.

$$\left(\frac{\partial u}{\partial x}\right)^2 x + \left(\frac{\partial u}{\partial y}\right)^2 y - u = 0 \qquad \text{.....................} (1)$$

Solution: Let $p = \frac{\partial u}{\partial x}$, $q = \frac{\partial u}{\partial y}$

The **Char pit's** auxiliary system of equations is

$$\frac{dx}{2px} = \frac{dy}{2qy} = \frac{du}{2(p^2x + q^2y)} = \frac{dp}{p - p^2} = \frac{dq}{q - q^2}$$

Which, we obtain from (1) by putting values of

$$\frac{\partial F}{\partial p} = 2px, \; \frac{\partial F}{\partial q} = 2qy, \frac{\partial F}{\partial x} = p^2, \frac{\partial F}{\partial u} = -1, \frac{\partial F}{\partial y} = q^2$$

and multiplying by -1 throughout the auxiliary system. From first and 4th expression we get

$dx = \frac{p^2dx + 2pxdp}{py}$. From second and 5th expression

$$dy = \frac{q^2dy + 2qydq}{qy}$$

Using these values of dx and dy in 1st and 2nd expression of **Char pit's** auxiliary system , we get

$$\frac{p^2dx + 2pxdp}{p^2x} = \frac{q^2dy + 2qydq}{q^2y}$$

$$\text{or } \frac{dx}{x} + \frac{2}{p}dp = \frac{dy}{y} + \frac{2dq}{q}$$

Taking integral of all terms we get

$\ln|x| + 2\ln|p| = \ln|y| + 2\ln|q| + \ln c$

or $\ln|x| \quad p^2 = \ln|y|q^2c$

or $p^2x = cq^2y$, where c is an arbitrary constant.

for p and q we get $cq^2y + q^2y - u = 0$

$(c+1)q^2y = u$

$$q = \left\{\frac{u}{(c+1)y}\right\}^{1/2} \quad p = \left\{\frac{cu}{(c+1)x}\right\}^{1/2}$$

Therefore

du =p dx +q dy

$$du=\left\{\frac{cu}{(c+1)x}\right\}^{1/2}dx+\left\{\frac{u}{(c+1)y}\right\}^{1/2}dy$$

$$\text{or}\ \left(\frac{1+c}{u}\right)^{1/2}du=\left(\frac{c}{x}\right)^{1/2}dx+\left(\frac{i}{y}\right)^{1/2}dy$$

By integrating this equation, we obtain $((1+c)u)^{1/2}=(cx)^{1/2}+(y)^{1/2}+c_1$

This is a complete solution.

Solutions of special type of nonlinear Partial Differential Equation of First-Order :-

(i) Equations containing p and q only

Let us consider a partial differential equation of the type

$F(p,q)=0$

The auxiliary system of equations of Charpit's method takes the form

$$\frac{dx}{F_p}=\frac{dy}{F_q}=\frac{du}{pF_p+qF_q}=\frac{dp}{0}=\frac{dq}{0}$$

It is clear that p=c is a solution of these equations. Putting value of p in we have

$F(c,q)=0$

So that q=G(c) where c is a constant

Then observing that

du= cdx +G(c) dy

we get the solution u=cx +G(c) y+c_1,

where c_1 is another constant.

Example: Solve $p^2+q^2=1$

Solution: The auxiliary system of equation is

$$-\frac{dx}{-2p}=\frac{dy}{2q}=\frac{du}{-2p^2-2q^2}=\frac{dp}{0}=\frac{dq}{0}$$

or $$\frac{dx}{p}=\frac{dy}{q}=\frac{du}{p^2+q^2}=\frac{dp}{0}=\frac{dq}{0}$$

Using dp =0, we get p=c and q= $\sqrt{1-c^2}$, and these two combined with

du =pdx+qdy yield

u=cx+y $\sqrt{1-c^2}$ + c_1 which is a complete solution.

Using $\frac{dx}{du}$ = p , we get du = $\frac{dx}{c}$ where p= c

Integrating the equation we get u = $\frac{x}{c}+c_1$

Also du = $\frac{dy}{q}$, where q = $\sqrt{1-p^2}=\sqrt{1-c^2}$

or du = $\frac{dy}{\sqrt{1-c^2}}$. Integrating this equation we get u = $\frac{1}{\sqrt{1-c^2}}$ y +c_2

This cu = x+cc_1 and $u\sqrt{1-c^2}$ = y + c_2 $\sqrt{1-c^2}$

Replacing cc_1 and c_2 $\sqrt{1-c^2}$ by - α and -β respectively, and eliminating c, we get

$u^2 = (x-\alpha)^2 + (y-\beta)^2$

This is another complete solution.

This is another complete solution.

(ii) Clairaut's equations

An equation of the form

u=px+qy+f(p,q) or F=px+qy+f(p,q)-u=0

is known as Clairaut's equation.

The auxiliary system of equations for Clairaut's equation takes the form

$$\frac{dx}{x+f_p} = \frac{dy}{y+f_q} = \frac{du}{px+qy+pf_p+qf_q} = \frac{dp}{0} = \frac{dq}{0}$$

From here we find that

dp=0, dq=o implying

$p=c_1$, $q=c_2$

If we put these values of p and q in we get

$u = c_1 x + c_2 y + f(c_1, c_2)$

Therefore, $F(x,y,u,c_1,c_2) = c_1x + c_2y + f(c_1,c_2) - u = 0$ is a complete solution of

(iii) Equations not containing x and y

Consider a partial differential equation of the type

$F(u,p,q) = 0$

The auxiliary system of equations take the form

$$\frac{dx}{F_p} = \frac{dy}{F_q} = \frac{du}{pF_q + qF_q} = \frac{dp}{-pF_u} = \frac{dq}{-qF_u}$$

The last two terms yield $\frac{dp}{p} = \frac{dq}{q}$

i.e. $p = a^2q$ where a^2 is an arbitrary constant

This equation together with 11.43 can be solved for p and q and we proceed as in previous cases.

Example: Solve $u^2 + pq - 4 = 0$

Solution: The auxiliary system of equations is

$$\frac{dx}{q} = \frac{dy}{p} = \frac{du}{2pq} = \frac{dp}{-2up} = \frac{dq}{-2uq}$$

The last two equations yield $p = a^2q$.

Substituting in $u^2+pq - 4 = 0$ gives

$$q = \pm\frac{1}{a}\sqrt{4-u^2} \text{ and } p = \pm a\sqrt{4-u^2}$$

Then du = pdx+qdy yields

$$du = \pm\sqrt{4-u^2}\left(a\,dx + \frac{1}{a}\,dy\right)$$

$$\text{or } \frac{du}{\sqrt{4-u^2}} = \pm\left(a\,dx + \frac{1}{a}\,dy\right)$$

$$\text{Integrating we get } \sin^{-1}\frac{u}{2} = \pm\left(ax + \frac{1}{a}y + c\right)$$

$$\text{or } u = \pm 2\sin\left(ax + \frac{1}{a}y + c\right)$$

which is the required complete solution.

(iv) Equations of the type

f(x,p) = g(y,q)

Then each of these functions must be constant, that is

f(x, p) = g(y, q) = C

Solving for p and q, and using du=pdx+qdy we can obtain the solution

Example: Solve $p^2(1-x^2)-q^2(4-y^2) = 0$

Solution: Let $p^2(1-x^2) = q^2(4-y^2) = a^2$

This gives $p = \dfrac{a}{\sqrt{1-x^2}}$ and $q = \dfrac{a}{\sqrt{4-y^2}}$

(neglecting the negative sign).

Substituting in du = pdx + q dy we have

$$du = \frac{a}{\sqrt{1-x^2}}dx + \frac{a}{\sqrt{4-y^2}}dy$$

Integration gives $u = a\left(\sin^{-1}x + \sin^{-1}\frac{y}{2}\right) + c.$

This is the required complete solution.

Partial Differential Equations Of Higher Order With Constant Coefficients:-

Partial differential equations (PDEs) are equations involving functions of more than one variable and their partial derivatives with respect to those variables. Most (but not all) physical models in engineering that result in partial differential equations are of at most second order and are often linear. (Some problems such as elastic stresses and bending moments of a beam can be of fourth order). In this course we shall have time to look at only a very small subset of second order linear partial differential equations.

Homogeneous Linear Equations with constant Coefficients.

A homogeneous linear partial differential equation of the nth order is of the form

$$c_0 \frac{\partial^n z}{\partial x^n} + c_1 \frac{\partial^n z}{\partial x^{n-1}\partial y} + \ldots\ldots + c_n \frac{\partial^n z}{\partial y^n} = F(x,y) \qquad \text{---------} (1)$$

where c_0, c_1,---------, c_n are constants and F is a function of 'x' and 'y'. It is homogeneous because all its terms contain derivatives of the same order. Equation (1) can be expressed as

$$(c_0 D^n + c_1 D^{n-1} D' + \ldots\ldots + c_n D'^n)\, z = F(x,y)$$

$$\text{or} \quad f(D,D')\, z = F(x,y) \qquad \text{---------}(2),$$

$$\text{where,} \quad \frac{\partial}{\partial x} \equiv D \text{ and } \frac{\partial}{\partial y} \equiv D'$$

As in the case of ordinary linear equations with constant coefficients the complete solution of (1) consists of two parts, namely, the complementary function and the particular integral.

The complementary function is the complete solution of f (D,D') z = 0 ------ (3)

Which must contain n arbitrary functions as the degree of the polynomial f(D,D'). The particular integral is the particular solution of equation (2).

Finding the complementary function:

Let us now consider the equation f(D,D') z = F (x,y). The auxiliary equation of (3) is obtained by replacing D by m and D' by 1.

$$\text{i.e, } c_0 m^n + c_1 m^{n-1} + \ldots\ldots + c_n = 0 \qquad \text{---------}(4)$$

Solving equation (4) for „m", we get „n" roots. Depending upon the nature of the roots, the Complementary function is written as given below:

Roots of the auxiliary equation	Nature of the roots	Complementary function(C.F)
$m_1, m_2, m_3 \ldots\ldots, m_n$	distinct roots	$f_1(y+m_1x)+f_2(y+m_2x) + \ldots\ldots + f_n(y+m_nx)$.
$m_1 = m_2 = m$, $m_3, m_4, \ldots, m_n$	two equal roots	$f_1(y+m_1x)+xf_2(y+m_1x) + f_3(y+m_3x) + \ldots +$ $f_n(y+m_nx)$.
$m_1 = m_2 = \ldots\ldots = m_n = m$	all equal roots	$f_1(y+mx)+xf_2(y+mx) + x^2f_3(y+mx)+\ldots$ $+\ldots+x^{n-1} f_n(y+mx)$

Finding the particular Integral

Consider the equation $f(D,D')\,z = F(x,y)$.

Now, the P.I is given by $\dfrac{1}{f(D,D')} F(x,y)$

Case (i) : When $F(x,y) = e^{ax+by}$

$$P.I = \frac{1}{f(D,D')} e^{ax+by}$$

Replacing D by 'a' and D' by 'b', we have

$$P.I = \frac{1}{f(a,b)} e^{ax+by}, \quad \text{where } f(a,b) \neq 0.$$

Case (ii) : When $F(x,y) = \sin(ax+by)$ (or) $\cos(ax+by)$

$$P.I = \frac{1}{f(D^2, DD', D'^2)} \sin(ax+by) \text{ or } \cos(ax+by)$$

Replacing $D^2 = -a^2$, $DD'^2 = -ab$ and $D' = -b^2$, we get

$$P.I = \frac{1}{f(-a^2, -ab, -b^2)} \sin(ax+by) \text{ or } \cos(ax+by), \text{ where } f(-a^2, -ab, -b^2) \neq 0.$$

Case (iii) : When $F(x,y) = x^m y^n$,

$$P.I = \frac{1}{f(D,D')} x^m y^n = [f(D,D')]^{-1} x^m y^n$$

Expand [f (D,D')]-1 in ascending powers of D or D' and operate on $x^m y^n$ term by term.

Case (iv) : When F(x,y) is any function of x and y.

$$P.I = \frac{1}{f(D,D')} F(x,y)$$

Resolve $\frac{1}{f(D,D')}$ into partial fractions considering f (D,D') as a function of D alone.

Then operate each partial fraction on F(x,y) in such a way that

$$\frac{1}{D-mD'} F(x,y) = \int F(x,c-mx)\,dx$$

where c is replaced by y+mx after integration

Example 1. Solve $(D^3 - 3D^2D' + 4D'^3)\, z = e^{x+2y}$

Solution: The auxillary equation is $m^3 - 3m^2 + 4 = 0$

The roots are m = -1,2,2

Therefore the $C.F = f_1(y-x) + f_2(y+2x) + xf_3(y+2x)$.

$$P.I = \frac{e^{x+2y}}{D^3-3D^2D'+4D'^3} \text{ (Replace D by 1 and D' by 2)}$$

$$= \frac{e^{x+2y}}{1-3(1)(2)+4(2)^3}$$

$$= \frac{e^{x+2y}}{27}$$

Hence, the solution is z = C.F. + P.I

ie, $$z = f_1(y-x) + f_2(y+2x) + x f_3(y+2x) + \frac{e^{x+2y}}{27}$$

Example2. Solve $(D^2 - 4DD' + 4D'^2)\, z = \cos(x - 2y)$

Solution: The auxiliary equation is $m^2 - 4m + 4 = 0$

Solving, we get m = 2, 2.

Therefore the $C.F = f_1(y + 2x) + xf_2(y + 2x)$

$$\therefore \text{P.I} = \frac{1}{D^2 - 4DD' + 4D'^2} \cos(x-2y)$$

Replacing D^2 by -1, DD' by 2 and D'^2 by -4, we have

$$\text{P.I} = \frac{1}{(-1) - 4(2) + 4(-4)} \cos(x-2y)$$

$$= -\frac{\cos(x-2y)}{25}$$

The solution is $z = f_1(y + 2x) + xf_2(y + 2x) + \frac{\cos(x-2y)}{25}$

Example 3. Solve $(D^2 - 2DD')\, z = x^3y + e^{5x}$

Solution: The auxiliary equation is $m^2 - 2m = 0$.

Solving, we get m = 0, 2.

Hence the $C.F = f_1(y) + f_2(y + 2x)$.

$$\text{P.I}_1 = \frac{x^3y}{D^2 - 2DD'}$$

$$= \frac{1}{D^2\left[1 - \frac{2D'}{D}\right]}(x^3y)$$

$$= \frac{1}{D^2}\left[1 - \frac{2D'}{D}\right]^{-1}(x^3y)$$

$$= \frac{1}{D^2}\left(1 - \frac{2D'}{D}\right)^{-1} (x^3y)$$

$$= \frac{1}{D^2}\left[1 + \frac{2D'}{D} + \frac{4D'^2}{D^2} + \dots\right](x^3y)$$

$$= \frac{1}{D^2}\left[(x^3y) + \frac{2}{D}D'(x^3y) + \frac{4}{D^2}D'^2(x^3y) + \dots\right]$$

$$= \frac{1}{D^2}\left[(x^3y) + \frac{2}{D}(x^3) + \frac{4}{D^2}(0) + \dots\right]$$

$$\text{P.I}_1 = \frac{1}{D^2}(x^3y) + \frac{2}{D^3}(x^3)$$

$$\text{P.I}_1 = \frac{x^5y}{20} + \frac{x^6}{60}$$

$$\text{P.I}_2 = \frac{e^{5x}}{D^2 - 2DD'} \text{ (Replace D by 5 and } D' \text{ by 0)}$$

$$= \frac{e^{5x}}{25}$$

$$\therefore \text{Solution is } Z = f_1(y) + f_2(y+2x) + \frac{x^5y}{20} + \frac{x^6}{60} + \frac{e^{5x}}{25}$$

Example 4. Solve $(D^2 + DD' - 6D'^2)z = y\cos x$

Solution: The auxiliary equation is $m^2 + m - 6 = 0$

Therefore, m = –3, 2.

Hence the $C.F = f_1(y - 3x) + f_2(y + 2x)$

$$P.I = \frac{y \cos x}{D^2 + DD' - 6D'^2}$$

$$= \frac{y \cos x}{(D + 3D')(D - 2D')}$$

$$= \frac{1}{(D+3D')} \frac{1}{(D - 2D')} y \cos x$$

$$= \frac{1}{(D+3D')} \int (c - 2x) \cos x \, dx, \quad \text{where } y = c - 2x$$

$$= \frac{1}{(D+3D')} \int (c - 2x) \, d(\sin x)$$

$$= \frac{1}{(D+3D')} [(c - 2x)(\sin x) - (-2)(-\cos x)]$$

$$= \frac{1}{(D+3D')} [y \sin x - 2 \cos x)]$$

$$= \int [(c + 3x) \sin x - 2 \cos x] \, dx, \quad \text{where } y = c + 3x$$

$$= (c + 3x)(-\cos x) - (3)(-\sin x) - 2 \sin x$$

$$= -y \cos x + \sin x$$

Hence the complete solution is

$$z = f_1(y - 3x) + f_2(y + 2x) - y \cos x + \sin x$$

Example 5. Solve $r - 4s + 4t = e^{2x+y}$

Given equation is $$\frac{\partial^2 z}{\partial x^2} - 4 \frac{\partial^2 z}{\partial x \partial y} + 4 \frac{\partial^2 z}{\partial y^2} = e^{2x+y}$$

Solution: The auxiliary equation is $m^2 - 4m + 4 = 0$.

Therefore, m = 2, 2

Hence the $C.F$ is $f_1(y+2x) + x f_2(y+2x)$.

$$\text{P.I.} = \frac{e^{2x+y}}{D^2 - 4DD' + 4D'^2}$$

Since $D^2 - 4DD' + 4D'^2 = 0$ for D = 2 and D' = 1, we have to apply the general rule.

$$\therefore \text{P.I.} = \frac{e^{2x+y}}{(D-2D')(D-2D')}$$

$$= \frac{1}{(D-2D')} \frac{1}{(D-2D')} e^{2x+y}$$

$$= \frac{1}{(D-2D')} \int e^{2x+c-2x}\, dx, \quad \text{where } y = c - 2x.$$

$$= \frac{1}{(D-2D')} \int e^c\, dx$$

$$= \frac{1}{(D-2D')} e^c .x$$

$$= \frac{1}{D-2D'} xe^{y+2x}$$

$$= \int xe^{c-2x+2x}\, dx, \quad \text{where } y = c - 2x.$$

$$= \frac{1}{(D-2D')} \int e_c \, dx$$

$$= \frac{1}{(D-2D')} e^c \, .x$$

$$= \frac{1}{D-2D'} xe^{y+2x}$$

$$= \int xe^{c-2x+2x} dx \quad , \quad \text{where } y = c - 2x.$$

$$= \int xe^c \, dx$$

$$= e^c . x^2/2$$

$$= \frac{x^2 e^{y+2x}}{2}$$

Hence the complete solution is

$$z = f_1(y+2x) + f_2(y+2x) + \frac{1}{2} x^2 e^{2x+y}$$

Module 4: Functions of Complex Variable

Contents

Functions of Complex Variables: Analytic Functions, Harmonic Conjugate, Cauchy-Riemann Equations (without proof), Line Integral, Cauchy-Goursat theorem (without proof), Cauchy Integral formula (without proof), Singular Points, Poles & Residues, Residue Theorem, Application of Residues theorem for Evaluation of Real Integral (Unit Circle).

COMPLEX VARIABLES

If $z = x + \mathrm{i}y$, then Z is called a complex variable. Also x and y are respectively called real and imaginary parts of z. sometimes we express z as $z = x + \mathrm{i}y = (x, y)$.

We also write R(z) = x and I(z) = y.

The complex conjugate, or briefly conjugate of $\bar{z} = x - \mathrm{i}y$.

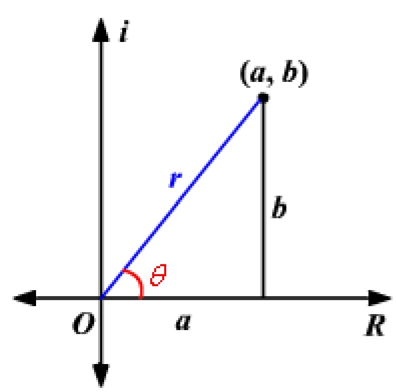
For example, conjugate $z = -3 - 5\mathrm{i}$ is $\bar{z} = -3 + 5\mathrm{i}$

It is easy to verify that: $R(z) = x = \frac{z+\bar{z}}{2}$, and $I(z) = y = \frac{z+\bar{z}}{2\mathrm{i}}$.

POLAR FORMS OF A COMPLEX NUMBER

The polar form of a complex number is another way to represent a complex number. The form $z = a + b\mathrm{i}$ is called the rectangular coordinate form of a complex number.

The horizontal axis is the real axis and the vertical axis is the imaginary axis. We find the real and complex components in terms of r and θ where r is the length of the vector and θ is the angle made with the real axis.



From Pythagorean Theorem:

$$r^2 = a^2 + b^2$$

By using the basic trigonometric ratios:

$cos = a/r$ and $sin = b/r$

Multiplying each side by r:

$rcos = a$ and $rsin = b$

The rectangular form of a complex number is given by $z = a + b\mathrm{i}$

Substitute the values of a and b in $z = a + b\mathrm{i}$, Therefore

$$z = rcos + (rsin)\mathrm{i}$$

i.e. $z = r(cos + \mathrm{i}sin)$ or $z = re^{\mathrm{i}}$

In the case of a complex number, r represents the absolute value or modulus and the angle θ is called the argument of the complex number.

This can be summarized as follows:
The polar form of a complex number $z = a + b\mathrm{i}$ is
$z = r(cos + \mathrm{i}sin)$, where $= |z| = \sqrt{a2 + b2}$, $a = rcos$ and $b = rsin$, and $= tan - 1(b/a)$
for a>0 and

$$= tan - 1(b/a) + \pi$$

Or $= tan - 1(b/a) + 180°$ for $a < 0$.

ANALYTIC FUNCTION

A function f (z) is said to be analytic in a region R of z-plane, if it is uniquely differentiable at every point of R .

Necessary condition or Cauchy-Riemann equations for analytic function :

The necessary condition for the function w = f(z) = u(x,y) + i v(x,y) to be analytic in a region R is that u(x,y) and v(x,y) satisfy the following equations

$$\frac{u}{x} = \frac{v}{y} \text{ and } \frac{u}{y} = -\frac{v}{x}$$

$$u_x = v_y \text{ and } u_y = -v_x$$

Which are known as Cauchy-Riemann equations.

Sufficient condition for analytic function:

If w = f(z) = u(x,y) + i v(x,y) be a function of complex variable defined in the region R such that

I) $\frac{u}{x}, \frac{v}{y}, \frac{u}{y}, \frac{v}{x}$ are continuous functions of x and y in the region R and

II) $\frac{u}{x} = \frac{v}{y}$, $\frac{u}{y} = -\frac{v}{x}$

Then the function w = f(z) = u(x,y) + i v(x,y) is analytic and

$$f'(z) = = \frac{u}{x} + i\frac{v}{x}$$

CAUCHY-RIEMANN EQUATIONS [In Polar Form]:

Let $w = f(z) = u + iv = f(x + iy)$

Put $x = r\cos$ and $y = r\sin$, then

$$u + iv = f(r\cos + ir\sin)$$

$$u + iv = f(re^{i}) \ldots (i)$$

Differentiate equation (i) partially w.r.t. r

$$\frac{u}{r} + i\frac{v}{r} = f'(re^{i})e^{i} \ldots (ii)$$

Differentiate equation (i) partially w.r.t.

$$\frac{u}{} + i\frac{v}{} = f'(re^{i})ire^{i} \ldots (ii)$$

$$\frac{u}{} + i\frac{v}{} = r\left(i\frac{u}{r} - \frac{v}{r}\right)$$

Equating real and imaginary parts, we get

$$\frac{u}{} = \cancel{r}\,\frac{v}{r}$$

$$\frac{v}{} = r\frac{u}{r}$$

Which is the polar form of C-R equations.

Example 1: Test the analytic behavior of $logz$

Solution: let $f(z) = logz$

$$u + iv = \log(x + iy)$$

$$u + iv = \log(r\cos + ir\sin)$$

$$u + iv = \log(re^{i})$$

$$u + iv = \log r + i$$

Equating real and imaginary part

$$u = log r \text{ and } v =$$

Differentiating partially w.r.t r and

$$\frac{u}{r} = \frac{1}{r} \text{ and } \frac{v}{r} = 0$$

$$u = 0 \text{ and } v = 1$$

It is clear that $\frac{u}{r} = \frac{1}{r}\frac{v}{}$ and $\frac{v}{r} = -\frac{1}{r}\frac{u}{}$

The function $logz$ is analytic except at origin.

Example 2 : Show that$f(z) = z^2$ is an analytic function**.**

Solution: Since $z = x + iy$

$$\text{Therefore} \qquad z^2 = (x + iy)^2$$

$$z^2 = x^2 - y^2 + 2ixy$$

$$\text{Hence, } f(z) = u + iv = x^2 - y^2 + 2ixy$$

Equating the real and imaginary parts,

$$\therefore u = x^2 - y^2, \; v = 2xy$$

$$\frac{u}{x} = 2x, \frac{v}{x} = 2y,$$

$$\frac{u}{y} = -2y, \qquad \frac{v}{y} = 2x,$$

Hence, $\frac{u}{x} = \frac{v}{y}, \frac{u}{y} = -\frac{v}{x}$ ie, u & v *satisfy Cauchy – Riemann equations.*
Therefore, f(z) is analytic.

HARMONIC FUNCTION

A real valued function $u = u(x, y)$ is called harmonic function, if

$$\frac{^2u}{x^2} + \frac{^2u}{y^2} = 0$$

i.e. u satisfies Laplace's equation

$$\nabla^2 u = \frac{^2u}{x^2} + \frac{^2u}{y^2} = 0$$

Example : Prove that, if f(z)=u+iv is analytic function in domain D, then u and v are harmonic.

Solution: Since f(z)=u+iv is analytic,it means it is satisfies C-R equations.

$$\text{i.e. } \frac{u}{x} = \frac{v}{y} \text{ and } \frac{u}{y} = -\frac{v}{x}$$

Differentiate w.r.t. x and y, we get

$$\frac{{}^2u}{x^2} = \frac{{}^2v}{xy} \text{ and } \frac{{}^2u}{y^2} = -\frac{{}^2v}{yx}$$

By adding both the equation, we get

$$\frac{{}^2u}{x^2} + \frac{{}^2u}{y^2} = 0$$

Here, u is harmonic

Similarly,

$$\frac{{}^2v}{x^2} + \frac{{}^2v}{y^2} = 0$$

Here, v is harmonic.

METHODS FOR CONSTRUCTING AN ANALYTIC FUNCTION:-

Method-I

If u is given function then to find v:

$$dv = \frac{v}{x}dx + \frac{v}{y}dy$$

$$dv = -\frac{u}{y}dx + \frac{u}{x}dy$$

$$dv = Mdx + Ndy \ldots \text{(i)}$$

Where, $M = -\frac{u}{y}$ and $N = \frac{u}{x}$

Differentiate $\frac{M}{y} = -\frac{{}^2u}{y^2}$ and $\frac{N}{x} = \frac{{}^2u}{x^2}$

Since u is harmonic, therefore

$$\frac{{}^2u}{x^2} + \frac{{}^2u}{y^2} = 0$$

$$\frac{{}^2u}{x^2} = -\frac{{}^2u}{y^2} \Rightarrow \frac{M}{y} = \frac{N}{x}$$

Hence equation (i) is exact differential equation. dv can be solve to get v.

Method-II: Milne Thomson Method

Type-I: To construct analytic function f(z) in terms of z, when real part u is given by the following formula,

$$f(z) = \int[\emptyset_1(z,0) - i\emptyset_2(z,0)]dz + C$$

Where $\emptyset_1(x,y) = \frac{u}{x}$ and $\emptyset_2(x,y) = \frac{u}{y}$

Type-II: To construct analytic function f(z) in terms of z, when imaginary part v is given by the following formula,

$$f(z) = \int[\emptyset_1(z,0) + i\emptyset_2(z,0)]dz + C$$

Where $\emptyset_1(x,y) = \frac{v}{x}$ and $\emptyset_2(x,y) = \frac{v}{y}$

Type-III: To construct analytic function f(z) in terms of z, when U = u – v is given by the following formula,

$$(1+i)f(z) = \int[\emptyset_1(z,0) - i\emptyset_2(z,0)]dz + C$$

Where $\emptyset_1(x,y) = \frac{U}{x}$ and $\emptyset_2(x,y) = \frac{U}{y}$

Type-IV: To construct analytic function f(z) in terms of z, when V = u + v is given by the following formula,

$$(1+i)f(z) = \int[\emptyset_1(z,0) + i\emptyset_2(z,0)]dz + C$$

Where $\emptyset_1(x,y) = \frac{V}{x}$ and $\emptyset_2(x,y) = \frac{V}{y}$

Example: Use C-R equation to find v , where $u = 3x^2y - y^3$.

Solution: Here $u = 3x^2y - y^3$

Differentiating u w.r.t x and y

We have

$$\frac{u}{x} = 6xy = \emptyset_1(x,y)$$

$$\frac{u}{y} = 3x^2 - 3y^2 = \emptyset_2(x,y)$$

Also

$$\frac{{}^2u}{x^2} + \frac{{}^2u}{y^2} = 6y + (-6y) = 0$$

Here u is harmonic

Now putting x = z, y = 0

$$\emptyset_1(x, y) = 0 \text{ and } \emptyset_2(x, y) = 3z^2$$

Hence by Milne Thomson method,

$$f(z) = \int[\emptyset_1(z, 0) - i\emptyset_2(z, 0)]dz + C$$

$$f(z) = \int[0 - 3z^2 i)]dz + C$$

$$f(z) = -z^3 i + C$$

$$f(z) = -(x + iy)^3 i + C$$

$$f(z) = u + iv = (3x^2y - y^3) + i(3xy^2 - x^3 + C)$$

Here $v = (3xy^2 - x^3 + C)$.

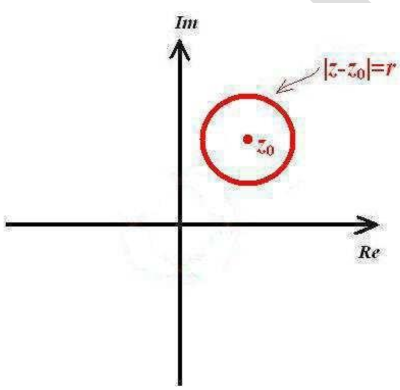
INTEGRATION IN THE COMPLEX PLANE

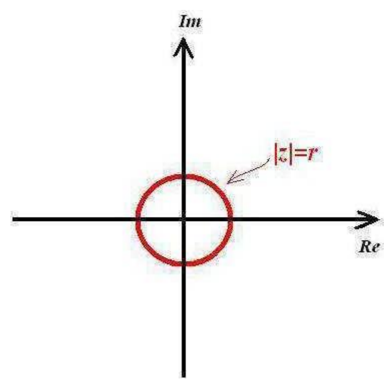
Complex Line Integrals and Some Integral Theorems

For smooth curve C: z=z(t) for $a \le t \le b$, then $\int_c f(z)dz = \int_a^b f(z(t)) \cdot z'(t)dt$

Special case 1C: $|z-z_0|=r \Leftrightarrow z(t)=z_0+re^{it}$, $dz = z'(t)dt=ire^{it}dt$, and $0 \le t \le 2$

Special case 2C: $|z|=r \Leftrightarrow z(t)=re^{it}$, $dz=z'(t)dt=ire^{it}dt$, and $0 \le t \le 2$





Example 1. Find $\oint_c \frac{1}{z}dz$, C: |z|=1.

Solution: $|z| = 1 \Leftrightarrow z = e^{it}, 0 \le t \le 2\pi, z'(t) = ie^{it}$, $\oint_c \frac{dz}{z} = \int_0^{2\pi} \frac{ie^{it}}{e^{it}} dt = 2\pi i$

Example 2: Evaluate: $\oint_C \frac{dz}{z-3i}$, C: |z-3i|= $\frac{1}{3}$.

Solution:

$$z(t) = 3i + \frac{1}{3}e^{it}, z'(t) = \frac{1}{3}ie^{it},$$

$$\oint_C \frac{dz}{z-3i} = \int_0^{2\pi} \frac{1}{\frac{1}{3}e^{it}} \cdot \frac{i}{3} e^{it} dt = 2\pi i$$

Example 3: Evaluate $\oint_c \bar{z}dz$, C: |z|=1.

Solution: $z(t) = e^{it}$, $\bar{z} = e^{-it}$, $z'(t) = ie^{it}$, $\oint_c \bar{z}dz = \int_0^{2\pi} e^{-it} \cdot ie^{it} dt = 2\pi i$

Example 4: Evaluate $\oint_c [z - R_e(z)]dz$, C: |z|=2.

Solution : $|z|=2 \Leftrightarrow z(t)=2e^{it}$, $z'(t)= i2e^{it}$

$$R_e(z) = \frac{1}{2}(z + \bar{z}) = \frac{1}{2}(2e^{it} + 2e^{-it}) \qquad z - R_e(z) = \frac{1}{2}(z - \bar{z}) = \frac{1}{2}(2e^{it} - 2e^{-it})$$

$$\oint_c [z - R_e(z)]dz = \int_0^{2\pi} [z(t) - R_e(z(t))] \cdot z'(t)dt = \int_0^{2\pi} \frac{2}{2}(e^{it} - e^{-it}) \cdot 2ie^{it}dt = 2i \int_0^{2\pi} (e^{2it} - 1)dt = -4\pi i$$

CAUCHY'S INTEGRAL THEOREM

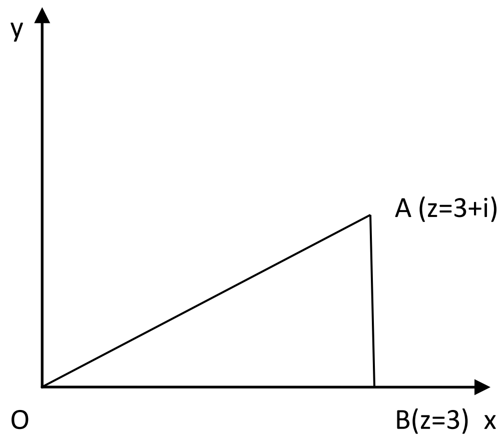
Let f(z) be analytic in a simply-connected domain D, C is a simple closed curve in D, then $\oint_c f(z)dz = 0$.

Example 1: Integrate z^2 along the straight line OA and also along the path OBA consisting of two straight lines OB & BA where O is the origin , B is the point z=3 and A is the point z=3+i. Hence show that the integral of z^2 along the closed path OBAO is zero.

Solution : We have z=x+iy , dz=dx+idy

Along the curve C , We have

$$\int_c z^2 dz = \int_c (x+iy)^2 (dx+idy)$$



$$\int_C z^2 dz = \int_C \left(x^2 - y^2 + 2ixy\right)(dx + idy) \text{ .... (1)}$$

The point A is z=3+I i.e. A(3,1)
The equation of the line OA is

$$y - 0 = \frac{1-0}{3-0}(x - 0)$$

i.e. $x = 3y$

Now, On the line OA, x=3y

dx=3dy y:0 to 1

Therefore from (1) we have,

$$z^2 dz = \int_0^1 (8 + 6i)(3 + i)y^2 dy$$

$$=6+i\frac{26}{3}$$

Again,

$$\int_{OBA} z^2 dz = \int_{OB+BA} z^2 dz$$

$$= \int_0^3 x^2 dx + \int_0^1 (9 - y^2 + 6iy) i\, dy$$

$$=6+i\frac{26}{3}$$

Lastly , The integral of z^2 along the closed path OBAO is given by

$$\int_{OBAO} z^2 dz = \int_{OBA} z^2 dz - \int_{OA} z^2 dz = 0$$

i.e Cauchy Integral theorem is verified.

Application of Cauchy Integral theorem :

Example 1. Evaluate $\oint_c \frac{1}{z} dz$, C: |z-2|=1.

Solution : f(z)= $\frac{1}{z}$ is analytic except z=0. No poles are within C, $\therefore \oint_c \frac{dz}{z} = 0$

Example 2. Evaluate $\oint_{|z|=1} \frac{dz}{z^2-4}$

Solution : f(z)= $\frac{1}{z^2-4}$ is analytic except z=±2. No poles are within C, $\therefore \oint_{|z|=1} \frac{dz}{z^2-4}$ =0

Example 3 . Evaluate $\oint_C \frac{z}{\sin(z)(z-2i)^3} dz$, C: |z-8i|=1.

Solution: $f(z) = \frac{z}{\sin(z)(z-2i)^3}$ is analytic except z=2i, nπ (n=0,±1, ±2, ...)

No poles are within C, $\therefore \oint f(z)dz = 0$

CAUCHY'S INTEGRAL FORMULAE

Let f(z) be analytic in a simply-connected region D, and let C be a simple curve enclosing z_0 in D, then $\oint_c \frac{f(z)}{z-z_0} dz = 2\pi i\, f(z_0)$ and $\oint_c \frac{f(z)dz}{(z-z_0)^n} = \frac{2\pi i}{(n-1)!} \cdot f^{(n-1)}(z_0)$.

Example 1 : Evaluate $\oint_{|z|=3} \frac{e^{2z} dz}{z-2}$.

Solution : $\oint_{|z|=3} \frac{e^{2z}}{z-2}$ =$2\pi i e^{2\cdot 2}$=$2\pi i e^4$

Example 2: Evaluate $\oint_c \frac{e^{iz}}{z^3} dz$, C: |z|=3.

Solution : $\oint_c \frac{e^{iz}}{z^3} dz = i2\pi \cdot \frac{(e^{iz})'}{(3-1)!}\Big|_{z_0=0} = -i\pi$

Example 3: Given $\oint_C \frac{f(z)}{z} dz = 2\pi i$ and $\oint_C \frac{f(z)}{(z-1)^2} dz = 4\pi i$. If let f(z)=a+bz, find a and b

Solution: 2πi=2πi. f(0)=2πai and 4πi=2πi. f'(0)=2πbi, ∴a=1 and b=2.

Example 4: Evaluate $\oint_c \frac{\sin^6(z)}{z-\frac{\pi}{6}}dz$ and $\oint_c \frac{\sin^6(z)}{\left(z-\frac{\pi}{6}\right)^3}dz$ if $C:\left|z-\frac{\pi}{6}\right|=\delta>0$.

Solution: Let f(z)=sin6(z) and n=3 , $\oint_c \frac{\sin^6(z)}{z-\frac{\pi}{6}}dz = 2\pi i\cdot\sin^6\left(\frac{\pi}{6}\right)=\frac{\pi i}{32}$,

$$\oint_c \frac{\sin^6(z)}{\left(z-\frac{\pi}{6}\right)^3}dz=\frac{2\pi i}{2!}\cdot[\sin^6(z)]'\Big|_{z_0=\frac{\pi}{6}}=\frac{21\pi i}{16}$$

Example 5: Let be within C, find $\oint_c \frac{dz}{z-z_0}$ and $\oint_c \frac{dz}{(z-z_0)^n}$, $n \geq 2$.

Solution: Let f(z)=1, f(n-1)(z0)=0$\Rightarrow \oint \frac{dz}{z-z_0}=2\pi i$ and $\oint_c \frac{dz}{(z-z_0)^n}=0$.

Example 6: Evaluate $\oint_c \frac{2\sin(z^2)}{(z-1)^4}dz$, C is a closed curve not passing 1.

Solution: If C does not enclose 1, $\frac{2\sin(z^2)}{(z-1)^4}$ is analytic within C,$\therefore \oint_c \frac{2\sin(z^2)}{(z-1)^4}dz=0$

If C encloses 1, let $f(z)=2\sin(z^2)\Rightarrow \frac{2\sin(z^2)}{(z-1)^4}=\frac{f(z)}{(z-1)^4}$,n=4, n-1=3

$$f^{(3)}(z)=-24z\sin(z^2)-16z^3\cos(z^2)$$

$$\therefore \oint_c \frac{2\sin(z^2)}{(z-1)^4}dz=\frac{2\pi i}{3!}[-24\sin(1)-16\cos(1)]=\frac{\pi i}{3}[-24\sin(1)-16\cos(1)]$$

CAUCHY'S RESIDUE THEOREM

Let f(z) be analytic in D except $z_1, z_2, \ldots, z_n$ and C encloses $z_1, z_2, \ldots, z_n$ within D. Then we have

$$\oint_c f(z)dz = 2\pi i\cdot\sum_{j=1}^{n}\operatorname*{Res}_{z_j}(f)$$

Method of finding Residue:

$\operatorname*{Res}_{z_j}(f)=\frac{1}{(m-1)!}\lim_{z\to z_j}\frac{d^{m-1}}{dz^{m-1}}[(z-z_j)^m\cdot f(z)]$, where m is the order of a pole $z=z_j$.

In case of m=1, $\operatorname*{Res}_{z_j}(f)=\lim_{z\to z_j}[(z-z_j)\cdot f(z)]$.

Example 1: Find the residues of f(z)=$\frac{z}{z-1}$.

Solution: At z=1, m=1, $\operatorname{Res}_{1}(f)=\lim_{z\to 1}[(z-1)\cdot\frac{z}{z-1}]=1$

Example 2: Find the residues of f(z)=$\frac{z}{(z-1)(z+1)^2}$.

Solution : At z=1, m=1, $\operatorname{Res}_{1}(f)=\lim_{z\to 1}[(z-1)\cdot\frac{z}{(z-1)(z+1)^2}]=\frac{1}{4}$

At z=-1, m=2, $\operatorname{Res}_{-1}(f)=\frac{1}{(2-1)!}\lim_{z\to -1}\frac{d}{dz}[(z+1)^2\cdot\frac{z}{(z+1)^2(z-1)}]=\left.\frac{(z-1)-z}{(z-1)^2}\right|_{z=-1}=-\frac{1}{4}$

Example 3: Find the residues of f(z)=$\frac{1}{(z-1)^2(z+1)^2}$.

Solution: At z=1, m=2, $\operatorname{Res}_{1}(f)=\frac{1}{(2-1)!}\lim_{z\to 1}\frac{d}{dz}[(z-1)^2\cdot\frac{1}{(z+1)^2(z-1)^2}]=\left.\frac{-2(z+1)}{(z+1)^4}\right|_{z=1}=-\frac{1}{4}$

At z=-1, m=2, $\operatorname{Res}_{-1}(f)=\frac{1}{(2-1)!}\lim_{z\to -1}\frac{d}{dz}[(z+1)^2\cdot\frac{1}{(z+1)^2(z-1)^2}]=\left.\frac{-2(z-1)}{(z-1)^4}\right|_{z=-1}=\frac{1}{4}$

Example 4 : Find the residues of f(z)=$\frac{z^2-2z}{(z+1)^2(z^2+4)}$.

Solution: f(z)= $\frac{z^2-2z}{(z+1)^2(z^2+4)}=\frac{z^2-2z}{(z+1)^2(z+2i)(z-2i)}$

At z=-1, m=2,

$\operatorname{Res}_{-1}(f)=\frac{1}{(2-1)!}\lim_{z\to -1}\frac{d}{dz}[(z+1)^2\cdot\frac{z^2-2z}{(z+1)^2(z^2+4)}]=\left.\frac{(2z-2)(z^2+4)-(z^2-2z)(2z)}{(z^2+4)^2}\right|_{z=-1}=-\frac{14}{25}$

At z=2i, m=1, $\operatorname{Res}_{2i}(f)=\lim_{z\to 2i}[(z-2i)\cdot\frac{z^2-2z}{(z-2i)(z+2i)(z+1)^2}]=\frac{7+i}{25}$

At z=-2i, m=1, $\operatorname{Res}_{-2i}(f)=\lim_{z\to -2i}[(z+2i)\cdot\frac{z^2-2z}{(z-2i)(z+2i)(z+1)^2}]=\frac{7-i}{25}$

Example 5: Find the residues of f(z)= $\frac{\cot(z)\coth(z)}{z^3}$.

Solution: It is difficult to compute the residue at 0 by $\operatorname{Res}_{z_j}(f)=\frac{1}{(m-1)!}\lim_{z\to z_j}\frac{d^{m-1}}{dz^{m-1}}[(z-z_j)^m\cdot f(z)]$. We utilize $f(z)=\sum_{n=-\infty}^{\infty}a_n(z-z_0)^n=\cdots+\frac{a_{-2}}{(z-z_0)^2}+\frac{a_{-1}}{(z-z_0)}+a_0+a_1(z-z_0)+\cdots$

$$f(z)=\frac{\cot(z)\coth(z)}{z^3}=\frac{\cos(z)\cosh(z)}{z^3\sin(z)\sinh(z)}$$

$$=\frac{\left(1-\frac{z^2}{2!}+\frac{z^4}{4!}-+\cdots\right)\left(1+\frac{z^2}{2!}+\frac{z^4}{4!}+\cdots\right)}{z^3\left(z-\frac{z^3}{3!}+-\cdots\right)\left(z+\frac{z^3}{3!}+\cdots\right)}=\frac{\left(1-\frac{z^4}{6}+\cdots\right)}{\left(z^5-\frac{z^9}{90}+-\cdots\right)}==\frac{1}{z^5}-\frac{7}{45}\cdot\frac{1}{z}+\cdots$$

$$\therefore \operatorname{Res}_0(f)=-\frac{7}{45}$$

Example 6: Evaluate $\oint_c \frac{z^2+1}{z^2-1}dz$, C: |z-1|=1.

Solution: There is only one pole1 within C.

$$\operatorname{Res}_1(f)=\lim_{z\to 1}[(z-1)\cdot\frac{z^2+1}{z^2-1}]=\lim_{z\to 1}\frac{z^2+1}{z+1}=1,\therefore \oint_c \frac{z^2+1}{z^2-1}dz=2\pi i.\quad 1=2\pi i$$

Example 7: Evaluate $\oint_c \frac{\cos(z)}{z^2(z-1)}dz$ for (a) C: |z|= $\frac{1}{3}$, (b) C: |z-1|= $\frac{1}{3}$, (c) C: |z|=2.[

Solution : (a)There is only one pole 0 within C.

$$\operatorname{Res}_0(f)=\frac{1}{(2-1)!}\lim_{z\to 0}\frac{d}{dz}[z^2\cdot\frac{\cos(z)}{z^2(z-1)}]=\left.-\frac{(z-1)\sin(z)-\cos(z)}{(z-1)^2}\right|_{z=0}=-1,\ \oint_c \frac{\cos(z)}{z^2(z-1)}dz=-2\pi i$$

(b) There is only one pole1 within C.

$$\operatorname{Res}_1(f)=\lim_{z\to 1}[(z-1)\cdot\frac{\cos(z)}{z^2(z-1)}]=\cos(1),\quad \oint_c \frac{\cos(z)}{z^2(z-1)}dz=2\pi i.\quad \cos(1)$$

(c) There are two poles 0 and1 within C. $\oint_c \frac{\cos(z)}{z^2(z-1)}dz=2\pi i.\quad [-1+\cos(1)]$

Example 8: Find the residue of $f(z)=\frac{\sin(z)}{(z-i)^3}$ and evaluate $\oint_c \frac{\sin(z)}{(z-i)^3}dz$, C:|z-i|=2.

Solution: m=3, $\operatorname{Res}_i(f)=\frac{1}{(3-1)!}\lim_{z\to i}\frac{d^2}{dz^2}[(z-i)^3\cdot\frac{\sin(z)}{(z-i)^3}]=\frac{-1}{2}\sin(i)=-\frac{1}{2}i\sinh(1)$

$$\therefore \oint \frac{\sin(z)}{(z-i)^2}dz=2\pi i\cdot\left(-\frac{1}{2}i\sinh(1)\right)=\pi\sinh(1)$$

Example 9: Evaluate $\oint_c \tan z\,dz$, C: |z|=2.

Solution: There are two poles ±π/2 within C.

$$\operatorname{Res}_{\frac{\pi}{2}}(f) = \lim_{z\to\frac{\pi}{2}}[(z-\frac{\pi}{2})\cdot\tan(z)] = \lim_{z\to\frac{\pi}{2}}[(z-\frac{\pi}{2})\cdot\frac{\sin(z)}{\cos(z)}] = \lim_{z\to\frac{\pi}{2}}\frac{z\sin(z)-\frac{\pi}{2}\sin(z)}{\cos(z)}$$

$$= \lim_{z\to\frac{\pi}{2}}\frac{\sin(z)+z\cos(z)-\frac{\pi}{2}\cos(z)}{-\sin(z)} = -1$$

$$\operatorname{Res}_{-\frac{\pi}{2}}(f) = \lim_{z\to-\frac{\pi}{2}}[(z+\frac{\pi}{2})\cdot\tan(z)] = \lim_{z\to-\frac{\pi}{2}}[(z+\frac{\pi}{2})\cdot\frac{\sin(z)}{\cos(z)}] = \lim_{z\to-\frac{\pi}{2}}\frac{z\sin(z)+\frac{\pi}{2}\sin(z)}{\cos(z)}$$

$$= \lim_{z\to-\frac{\pi}{2}}\frac{\sin(z)+z\cos(z)+\frac{\pi}{2}\cos(z)}{-\sin(z)} = -1$$

$\oint_c \tan z\,dz$ =2πi. [(-1)+(-1)]= -4πi

Example 10: Evaluate $\oint_c \frac{\sin(z)}{z^2(z^2+4)}dz$, C is any piecewise-smooth curve enclosing 0, 2i, and –2i.

Solution : $f(z) = \frac{\sin(z)/z}{z(z+2i)(z-2i)}$, $\therefore \lim_{z\to 0}[\sin(z)/z] = 1$, ∴f(z) has a removable singularity at 0 ⇒ m of the pole z=0 in f(z) is 1. $\operatorname{Res}_0(f) = \lim_{z\to 0} z\cdot\frac{\sin(z)}{z^2(z^2+4)} = \frac{1}{4}$

$$\operatorname{Res}_{2i}(f) = \lim_{z\to 2i}(z-2i)\cdot\frac{\sin(z)}{z^2(z+2i)(z-2i)} = \frac{i}{16}\sin(2i) = -\frac{1}{16}\sinh(2)$$

$$\operatorname{Res}_{-2i}(f) = \lim_{z\to -2i}(z+2i)\cdot\frac{\sin(z)}{z^2(z+2i)(z-2i)} = \frac{i}{16}\sin(2i) = -\frac{1}{16}\sinh(2)$$

$$\oint_c f(z)dz = 2\pi i\left[\frac{1}{4}-\frac{1}{16}\sinh(2)-\frac{1}{16}\sinh(2)\right] = \frac{\pi i}{2}-\frac{\pi i}{4}\sinh(2)$$

Example 11: Evaluate $\oint_c \frac{z^3 e^{\frac{1}{z}}}{z^3+1}dz$, C: |z|=3.

Solution: Let t=1/z, z=1/t, dz=-dt/t2, C: |t|=1/3

$$\oint_c \frac{z^3 e^{\frac{1}{z}}}{z^3+1}dz = \oint_c \frac{(\frac{1}{t})^3 e^t}{(\frac{1}{t})^3+1}\cdot\left(\frac{-dt}{t^2}\right) = \oint_c \frac{e^t}{t^2(t^3+1)}dt.$$

There is only one pole 0 within C.

$$\operatorname{Res}_0(f) = \frac{1}{1!}\lim_{t\to 0}\frac{d}{dt}\left[t^2\cdot\frac{-e^t}{t^2(t^3+1)}\right] = -1, \therefore \oint_c \frac{z^3 e^{\frac{1}{z}}}{z^3+1}dz = 2\pi i\cdot(-1) = -2\pi i$$

Example 12: Evaluate $\oint_c \frac{e^{-z}}{\cos(z)}dz$,C: |z|=2.

Solution: cos(z) has two zeros at z=±π/2 within |z|=2.

$$\operatorname{Res}_{\frac{\pi}{2}}(f) = \lim_{z\to\frac{\pi}{2}}\left[(z-\frac{\pi}{2})\cdot\frac{e_z}{\cos(z)}\right] = \lim_{z\to\frac{\pi}{2}}\frac{ze^z-\frac{\pi}{2}e^z}{\cos(z)} = \lim_{z\to\frac{\pi}{2}}\frac{e^z+ze^z-\frac{\pi}{2}e^z}{-\sin(z)} = -e^{\frac{\pi}{2}}$$

$$\operatorname{Res}_{-\frac{\pi}{2}}(f) = \lim_{z\to-\frac{\pi}{2}}\left[(z+\frac{\pi}{2})\cdot\frac{e^z}{\cos(z)}\right] = \lim_{z\to-\frac{\pi}{2}}\frac{ze^z+\frac{\pi}{2}e^z}{\cos(z)} = \lim_{z\to-\frac{\pi}{2}}\frac{e^z+ze^z+\frac{\pi}{2}e^z}{-\sin(z)} = e^{-\frac{\pi}{2}}$$

$$\therefore \oint_c \frac{e^{-z}}{\cos(z)}dz = 2\pi i\cdot(e^{-\frac{\pi}{2}} - e^{\frac{\pi}{2}})$$

Example 13: Evaluate $\oint_c \frac{e^{\frac{1}{z}}}{z+1}dz$, C: |z|=0.5.

Solution: Let t=1/z, z=1/t, dz= -dt/t2, C: |t|=2

$$\oint_c \frac{e^{\frac{1}{z}}}{z+1}dz = \oint_c \frac{e^t}{(\frac{1}{t})+1}\cdot\left(\frac{-dt}{t^2}\right) = \oint_c \frac{-e^t}{t(t+1)}dt$$. There are twopoles0and -1 within C.

$$\operatorname{Res}_0(f) = \lim_{t\to 0}\left[t\cdot\frac{-e^t}{t(t+1)}\right] = -1,\ \operatorname{Res}_{-1}(f) = \lim_{t\to -1}\left[(t+1)\cdot\frac{-e^t}{t(t+1)}\right] = e^{-1},$$

$$\therefore \oint_c \frac{z^3 e^{\frac{1}{z}}}{z^3+1}dz = 2\pi i\cdot(-1+e^{-1})$$

Evaluation of real Integral by Cauchy Residue Theorem i.e. Integration round the unit circle

Consider the integral $\int_0^{2\pi} f(\cos , \sin)d$ .... (1)

Where $f(\cos , \sin)$ is a rational function of cosθ and sinθ .

Such integrals can be reduced to complex line integrals by the substitution

$$z = e^{i}$$

$$\therefore\ dz = e^{i}\, i d$$

We also know that , $\cos = \frac{e^{i} + e^{-i}}{2} = \frac{1}{2}(z + \frac{1}{z})$

and

$$\sin = \frac{e^{i} - e^{-i}}{2i} = \frac{1}{2i}(z - \frac{1}{z})$$

As θ varies from 0 to 2π ,moves once round the unit circle in the anti-clockwise direction where C is the unit circle |z| = 1.

Hence, after putting the above values in the given real integral (1) ,then it will be reduce in to complex integral and the we, sovle by Cauchy Residue Theorem **.**

Example 1: Evaluate $\int_0^{2\pi} \frac{d\theta}{5 - 3\cos\theta}$.

Solution: $$\int_0^{2\pi} \frac{d\theta}{5 - 3\cos\theta} = \oint_c \frac{dz/iz}{5 - 3\cdot\frac{1}{2}\left(z + \frac{1}{z}\right)} = \oint_c \frac{2idz}{3z^2 - 10z + 3} = \oint_c \frac{2idz}{3(z - \frac{1}{3})(z-3)}$$

There is only one pole $\frac{1}{3}$ within |z|=1.

$$\therefore \int_0^{2\pi} \frac{d\theta}{5 - 3\cos\theta} = 2\pi i \cdot \operatorname{Re}s_{\frac{1}{3}}(f) = 2\pi i \cdot \lim_{z \to \frac{1}{3}} (z - \frac{1}{3}) \cdot \frac{2i}{3(z - \frac{1}{3})(z-3)} = 2\pi i \cdot \frac{2i}{-8} = \frac{\pi}{2}.$$

Example 2: Evaluate $\int_0^{2\pi} \frac{d\theta}{3 - 2\cos\theta + \sin\theta}$.

Solution: $$\int_0^{2\pi} \frac{d\theta}{3 - 2\cos\theta + \sin\theta} = \oint \frac{dz/iz}{3 - 2\cdot\frac{1}{2}\left(z + \frac{1}{z}\right) + \frac{1}{2i}\left(z - \frac{1}{z}\right)} = \oint \frac{2dz}{(1-2i)z^2 + 6iz - 1 - 2i}$$

$$= \oint_c \frac{2dz}{(1-2i)(z-\frac{2-i}{5})[z-(2-i)]}.$$

There is only one pole $\frac{2-i}{5}$ within $|z|=1$.

$$\therefore \int_0^{2\pi} \frac{d\theta}{3-2\cos\theta+\sin\theta}$$

$$= 2\pi i \cdot \operatorname{Res}_{\frac{2-i}{5}}(f) = 2\pi i \cdot \lim_{z\to\frac{2-i}{5}} \{(z-\frac{2-i}{5})\cdot \frac{2}{(1-2i)(z-\frac{2-i}{5})[z-(2-i)]}\} = 2\pi i\cdot(\frac{1}{2i}) = \pi.$$

Example 3: Show that $\int_0^{2\pi} \frac{d\theta}{a+b\sin\theta} = \int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$ if $a>|b|$.

Proof: $\int_0^{2\pi} \frac{d\theta}{a+b\sin\theta} = \oint_c \frac{dz/iz}{a+b\cdot\frac{1}{2i}\left(z-\frac{1}{z}\right)} = \oint_c \frac{2dz}{bz^2+2iaz-b} = \oint_c \frac{2dz}{b(z-z_1)(z-z_2)}$

Poles: $z_1 = \frac{i}{b}\left(-a+\sqrt{a^2-b^2}\right)$ is within C: $|z|=1$, but $z_2 = \frac{i}{b}\left(-a-\sqrt{a^2-b^2}\right)$ is not.

$$\operatorname{Res}_{z_1}(f) = \lim_{z\to z_1}(z-z_1)\cdot\frac{2}{b(z-z_1)(z-z_2)} = \frac{2}{b}\cdot\frac{1}{z_1-z_2} = \frac{2}{b}\cdot\frac{b}{2i\sqrt{a^2-b^2}} = \frac{1}{i\sqrt{a^2-b^2}}$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{a+b\sin\theta} = 2\pi i\cdot\frac{1}{i\sqrt{a^2-b^2}} = \frac{2\pi}{\sqrt{a^2-b^2}}$$

Differentiation of Vectors, Scalar and vector point function, Gradient, Geometrical meaning of gradient, Directional Derivative, Divergence and Curl, Line Integral, Surface Integral and Volume Integral, Gauss Divergence, Stokes and Green theorems.

Vector Differentiation

One-variable vector function: $\vec{R}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$

Multi-variable vector function: $\vec{F}(x,y,z) = F_1(x,y,z)\hat{i} + F_2(x,y,z)\hat{j} + F_3(x,y,z)\hat{k}$

The derivatives of vector functions: $\dfrac{dR(t)}{df(t)} = \dfrac{dR(t)}{dt} \cdot \dfrac{dt}{df(t)} = \dfrac{dR(t)}{dt} / \dfrac{df(t)}{dt}$

$$\frac{\partial F(x,y,z)}{\partial g(x,y,z)} = \frac{\partial F(x,y,z)}{\partial x} \cdot \frac{\partial x}{\partial g(x,y,z)} + \frac{\partial F(x,y,z)}{\partial y} \cdot \frac{\partial y}{\partial g(x,y,z)} + \frac{\partial F(x,y,z)}{\partial z} \cdot \frac{\partial z}{\partial g(x,y,z)}$$

Eg. For $\vec{R}(t) = 2t\hat{i} - \cos(3t)\hat{j} + t^3\hat{k}$, $0 \le t \le 1$, find $R'(t) = dR(t)/dt$.

Let s(t)= $\int_1^0 \sqrt{4 + 9\sin^2(3t) + 9t^4} \cdot dt$, then find $dR(t)/ds(t)$.

Sol: $d\vec{R}(t)/dt = 2\hat{i} + 3\sin(3t)\hat{j} + 3t^2\hat{k}$

$$\therefore \frac{d\vec{R}(t)}{ds(t)} = \frac{d\vec{R}(t)}{dt} \cdot \frac{dt}{ds(t)} = \frac{2\hat{i} + 3\sin(3t)\hat{j} + 3t^2\hat{k}}{\sqrt{4 + 9\sin^2(3t) + 9t^4}}$$

Some theorems of derivatives of vector functions:

1. $(F \cdot G)' = F' \cdot G + F \cdot G'$

2. $(F \times G)' = F' \times G + F \times G'$

3. $(F \times F')' = F \times F''$

Proof: $(F \times F')' = F' \times F' + F \times F'' = F \times F''$

4. For $\vec{R}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$, if $R(t)$ does not change direction, then $R(t) \times R'(t) = 0$, and vice versa.

5. Let $R(t)$ denote the position of a particle at time t. If the particle moves so that equal areas are swept out in equal times, then we have $R(t) \times R''(t) = 0$, and vice versa. (Kepler's law)

Proof:

Area $= \frac{1}{2} R^2 \theta$ and $|R(t+\Delta t) - R(t)| \approx R\theta$

$\therefore$ 2 area $= R^2\theta = R(t) \times [R(t+\Delta t) - R(t)]$

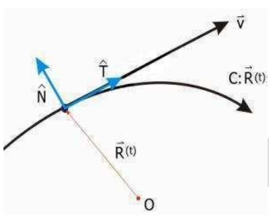
If area $= 0 \Leftrightarrow R(t) \times [R(t+\Delta t) - R(t)] = 0 \Leftrightarrow R(t) \times R'(t) \cdot \Delta t = 0$

$\Leftrightarrow R(t) \times R'(t) = 0$

Equal area in equal time

$\Leftrightarrow R(t) \times R'(t) = \text{Constant} \Leftrightarrow [R(t) \times R'(t)]' = 0 \Leftrightarrow R(t) \times R''(t) = 0$

Differential Geometry



Position vector: $\vec{R}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$

Velocity: $\vec{v}(t) = \vec{R}'(t) = \frac{dx(t)}{dt}\hat{i} + \frac{dy(t)}{dt}\hat{j} + \frac{dz(t)}{dt}\hat{k}$

Arc length: $s(t) = \int_{t_1}^{t_2} |\vec{R}'(t)|\, dt$, $\frac{ds}{dt} = |\vec{R}'(t)| = |\vec{v}(t)|$

Acceleration: $\vec{a}(t) = \vec{v}'(t) = \frac{d^2x(t)}{dt^2}\hat{i} + \frac{d^2y(t)}{dt^2}\hat{j} + \frac{d^2z(t)}{dt^2}\hat{k}$

Curvature: $\kappa = \left|\frac{d\hat{T}}{ds}\right|$, where $\hat{T} = \frac{\vec{R}'(t)}{|\vec{R}'(t)|} = \frac{\vec{v}(t)}{|\vec{v}(t)|} = \frac{dR(t)}{ds(t)}$

Eg. C: $\vec{R}(t) = t\hat{i} + (t-2)\hat{j} + (3t-1)\hat{k}$ is a straight line.

$$\hat{T} = \frac{\vec{v}}{|\vec{v}|} = \frac{\hat{i} + \hat{j} + 3\hat{k}}{\sqrt{11}}, \ \kappa = \left|\frac{d\hat{T}}{ds}\right| = \left|\frac{d\hat{T}}{dt} \cdot \frac{dt}{ds}\right| = 0.$$

Eg. C: $\vec{R}(t) = 2\cos(t)\hat{i} + 2\sin(t)\hat{j} + 4\hat{k}$ is a circle of radius 2 at z=4.

$$\hat{T} = \frac{-2\sin(t)\hat{i} + 2\cos(t)\hat{j}}{2}, \ \kappa = \left|\frac{d\hat{T}}{dt} \cdot \frac{dt}{ds}\right| = \frac{1}{2} = \frac{1}{r}.$$

Theorem : $\vec{a} = \frac{d|\vec{v}|}{dt}\hat{T} + \frac{|\vec{v}|^2}{\rho}\hat{N}$ = tangential acceleration+centripetal acceleration

Proof: $\vec{a}(t) = \dfrac{d\vec{v}(t)}{dt} = \dfrac{d}{dt}[|\vec{v}(t)|\hat{T}] = \dfrac{d|\vec{v}(t)|}{dt}\cdot\hat{T} + |\vec{v}(t)|\cdot\dfrac{d\hat{T}}{dt}$

$= \dfrac{d|\vec{v}(t)|}{dt}\cdot\hat{T} + |\vec{v}(t)|\cdot\left(\dfrac{ds}{dt}\cdot\dfrac{d\hat{T}}{ds}\right) = \dfrac{d|\vec{v}(t)|}{dt}\cdot\hat{T} + |\vec{v}(t)|^2\dfrac{d\hat{T}}{ds}$. Define $\hat{N} = \rho\dfrac{d\hat{T}}{ds}$

$\Rightarrow \vec{a}(t) = \dfrac{d|\vec{v}(t)|}{dt}\hat{T} + \dfrac{|\vec{v}(t)|^2}{\rho}\hat{N} \quad (\hat{N} \perp \hat{T} \Leftarrow \hat{T}\cdot\hat{T} = 1), \ \dfrac{d(\hat{T}\cdot\hat{T})}{ds} = 0$

Eg. : For $\vec{R}(t) = [\cos(t) + t\sin(t)]\hat{i} + [\sin(t) - t\cos(t)]\hat{j} + t^2\hat{k}$**,** t>0, we have

$\vec{v}(t) = t\cos(t)\hat{i} + t\sin(t)\hat{j} + 2t\hat{k}$

$\vec{a}(t) = [\cos(t) - t\sin(t)]\hat{i} + [\sin(t) + t\cos(t)]\hat{j} + 2\hat{k}$

$\hat{T} = \dfrac{\vec{v}(t)}{|\vec{v}(t)|} = \dfrac{1}{\sqrt{5}}\cos(t)\hat{i} + \dfrac{1}{\sqrt{5}}\sin(t)\hat{j} + \dfrac{2}{\sqrt{5}}\hat{k}$

$\rho = \dfrac{1}{\kappa} = \dfrac{1}{\left|\dfrac{d\hat{T}}{dt}\cdot\dfrac{dt}{ds}\right|} = \dfrac{1}{\left|\dfrac{d\hat{T}}{dt}\cdot\dfrac{1}{|\vec{v}(t)|}\right|} = 5t$

$\hat{N} = \rho\dfrac{d\hat{T}}{ds} = \rho\dfrac{dt}{ds}\cdot\dfrac{d\hat{T}}{dt} = \dfrac{\rho}{|\vec{v}(t)|}\cdot\dfrac{d\hat{T}}{dt} = -\sin(t)\hat{i} + \cos(t)\hat{j}$

$\therefore \vec{a}(t) = \sqrt{5}\,\hat{T} + t\,\hat{N}$, $a_t = \sqrt{5}$, $a_n = t$

Binormal vector: $\hat{B} = \hat{T} \times \hat{N}$

Fernet formulae:
$$\begin{cases} \dfrac{d\hat{T}}{ds} = \kappa\hat{N} = \dfrac{\hat{N}}{\rho} & (1) \\ \dfrac{d\hat{N}}{ds} = -\kappa\hat{T} + \tau\hat{B} & (2) \\ \dfrac{d\hat{B}}{ds} = -\tau\hat{N} & (3) \end{cases}$$

Gradient, Divergence and Curl: the Basics

We first consider the position vector, **r**:

r = xi+yj+zK

Where **x**, **y**, and **z** are rectangular unit vectors. Since the unit vectors for rectangular coordinates are constants, we have for dr:

$$\mathbf{dr} = \mathbf{dx\, i} + \mathbf{dy\, j} + \mathbf{dz\, k}.$$

The operator, del: ∇ is defined to be as:

$$\nabla = \partial/\partial x\, \mathbf{i} + \partial/\partial y\, \mathbf{j} + \partial/\partial z\, \mathbf{k},$$

This operator operates as a vector.

1. Gradient

If the del operator, ∇ operates on a scalar function, f(x,y,z), we get the **gradient**:

$$\nabla f = (\partial f/\partial x)\, \mathbf{i} + (\partial f/\partial y)\, \mathbf{j} + (\partial f/\partial z)\, \mathbf{k}.$$

We can interpret this gradient as a vector with the magnitude and direction of the maximum change of the function in space. We can relate the gradient to the differential change in the function:

$$df = (\partial f/\partial x)\, dx + (\partial f/\partial y)\, dy + (\partial f/\partial z)\, dz = \nabla f \bullet \mathbf{dr} = df.$$

Since the del operator should be treated as a vector, there are two ways for a vector to multiply another vector: dot product and cross product. We first consider the dot product:

2. Divergence

The **divergence of a vector** is defined to be:

$$\nabla \bullet \mathbf{A} = [\partial/\partial x\, \mathbf{i} + \partial/\partial y\, \mathbf{j} + \partial/\partial z\, \mathbf{k}] \bullet [A_x\, \mathbf{x} + A_y\, \mathbf{y} + A_z\, \mathbf{z}]$$

$$= (\partial A_x/\partial x) + (\partial A_y/\partial y) + (\partial A_z/\partial z);$$

since the rectangular unit vectors are constant, $\partial\mathbf{x}/\partial x = 0$ (etc.). This will not necessarily be true for other unit vectors in other coordinate systems. We'll see examples of this soon.

3. Curl

The **curl of a vector** is defined to be:

$$\nabla\times\mathbf{A} = [\partial/\partial x\,\mathbf{i} + \partial/\partial y\,\mathbf{j} + \partial/\partial z\,\mathbf{k}] \times [A_x\,\mathbf{i} + A_y\,\mathbf{j} + A_z\,\mathbf{k}]$$

Where we have used the fact that the unit vectors do not change with position ($\partial\mathbf{x}/\partial x = 0$) and the fact that ($\mathbf{x}\times\mathbf{x}=0$ and $\mathbf{x}\times\mathbf{y}=\mathbf{z}$, etc.). For other coordinate systems, unit vectors may change with position.

Remark: i.e. Properties:

Gradient, Divergence and Curl

The Del Operator $\nabla = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k$

- Gradient of a scalar function is a vector quantity. $\nabla f \longrightarrow$ Vector
- Divergence of a vector is a scalar quantity. $\nabla . A$
- Curl of a vector is a vector quantity. $\nabla \times A$
- The Laplacian of a scalar A $\nabla^2 A$

Example 1 : Determine if $\vec{F} = x^2 y\vec{i} + xyz\,\vec{j} - x^2 y^2\,\vec{k}$

$\vec{F} = x^2 y\vec{i} + xyz\,\vec{j} - x^2 y^2\,\vec{k}$ is a conservative vector field.

Solution:

$$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 y & xyz & -x^2 y^2 \end{vmatrix}$$

$$= -2x^2 y\vec{i} + yz\vec{k} - \left(-2xy^2\,\vec{j}\right) - xy\vec{i} - x^2\vec{k}$$

$$= -\left(2x^2 y + xy\right)\vec{i} + 2xy^2\,\vec{j} + \left(yz - x^2\right)\vec{k}$$

$$\neq \vec{0}$$

So, the curl isn't the zero vector and so this vector field is not conservative.

Example 2 : Compute $\text{div}\,\vec{F}$ $\text{div}\,\vec{F}$ for $\vec{F} = x^2 y\vec{i} + xyz\,\vec{j} - x^2 y^2\,\vec{k}$

$\vec{F} = x^2 y\vec{i} + xyz\,\vec{j} - x^2 y^2\,\vec{k}$

Solution: There really isn't much to do here other than compute the divergence.

$$\text{div}\,\vec{F} = \frac{\partial}{\partial x}\left(x^2 y\right) + \frac{\partial}{\partial y}\left(xyz\right) + \frac{\partial}{\partial z}\left(-x^2 y^2\right) = 2xy + xz$$

$$\text{div}\,\vec{F} = \frac{\partial}{\partial x}\left(x^2 y\right) + \frac{\partial}{\partial y}\left(xyz\right) + \frac{\partial}{\partial z}\left(-x^2 y^2\right) = 2xy + xz$$

Example 3: Verify the above fact for the vector field $\vec{F} = yz^2\,\vec{i} + xy\,\vec{j} + yz\,\vec{k}$

$\vec{F} = yz^2\,\vec{i} + xy\,\vec{j} + yz\,\vec{k}$.

Solution: Let's first compute the curl.

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz^2 & xy & yz \end{vmatrix}$$

$$= z\,\vec{i} + 2yz\,\vec{j} + y\,\vec{k} - z^2\vec{k}$$

$$= z\vec{i} + 2yz\,\vec{j} + \left(y - z^2\right)\vec{k}$$

Now compute the divergence of this.

$$\operatorname{div}\left(\operatorname{curl}\vec{F}\right) = \frac{\partial}{\partial x}(z) + \frac{\partial}{\partial y}(2yz) + \frac{\partial}{\partial z}\left(y - z^2\right) = 2z - 2z = 0$$

$$\operatorname{div}\left(\operatorname{curl}\vec{F}\right) = \frac{\partial}{\partial x}(z) + \frac{\partial}{\partial y}(2yz) + \frac{\partial}{\partial z}\left(y - z^2\right) = 2z - 2z = 0$$

Example 4 : Compute the divergence of $F(x, y) = 3x^2i + 2yj$.
Solution: The divergence of $F(x, y)$ is given by $\nabla \bullet F(x, y)$ which is a dot product.

$$\nabla \bullet F(x,y) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$$
$$= \frac{\partial}{\partial x}(3x^2) + \frac{\partial}{\partial y}(2y)$$
$$= 6x + 2$$

Example 5: Find the directional derivative of the function $f(x, y) = 3x^2y$ at a point (-2,1) along the direction $4\hat{i} + 3\hat{j}$.

Solution: The unit vector along $4\hat{i} + 3\hat{j}$ is $\hat{u} = \frac{4\hat{i}+3\hat{j}}{5}$.

Gradient of the given function is $6xy\hat{i} + 3x^2\hat{j}$.

Thus the directional derivative at (x,y) is $\left(\frac{24}{5}\right)xy + \left(\frac{9}{5}\right)x^2$

At (-2,1) its value is -12/5.

Example 6: Find the directional derivative of the function $f(x, y, z) = 3x^3 + 2xy^2 + xyz$ along the direction $2\hat{i} + 4\hat{j} + 4\hat{k}$ at a point (1,1,1)

Solution: The unit vector along $2\hat{i} + 4\hat{j} + 4\hat{k}$ is $\hat{u} = \frac{\hat{i}+2\hat{j}+2\hat{k}}{5}$.

Gradient of the given function is $(9x^2 + 2y^2 + yz)\hat{i} + (4xy + xz)\hat{j} + xy\hat{k}$ At (1,1,1) the gradient is $12\hat{i} + 5\hat{j} + \hat{k}$

Thus the directional derivative at (1,1,1) is 8.

Example 7: Find the equation to the tangent plane to the surface $x^2 + y^2 - z^2 = 7$ at the point (2,2,1).

Solution: The level surface is $x^2 + y^2 - z^2 = 7$.

Normal to the surface is in the direction of gradient which is $4\hat{i} + 4\hat{j} - 2\hat{k}$.

The equation to the tangent plane is given by $f_x(x - x_0) + f_y(y - y_0) + f_z(z - z_0) = 0$, where f_x , f_y and f_z are the partial derivative of the function f(x,y,z)= constant at the point (x_0, y_0, z_0).

In this case $f_x = 2x_0 = 4, f_y = 2y_0 = 4, f_z = -2z_0 = -2$.

Thus the equation is $4(x - 2) + 4(y - 2) - 2(z - 1) = 0$.

Example 8: Find the directional derivative of the function $f(x, y) = e^{x^2 - y^2}$ at the point P = (1, 1) in the direction of the point Q = (0, 0).

Solution: We first need to find a unit vector **u** that travels in the same direction is the vector with an initial point P and terminal point Q. First, we see that

$$\mathbf{v} = \overrightarrow{PQ} = \langle 0-1, 0-1 \rangle = \langle -1, -1 \rangle = -\mathbf{i} - \mathbf{j}$$

Then, since $\|\mathbf{v}\| = \sqrt{(1)^2 + (1)^2} = \sqrt{2}$, we can find the unit vector **u** to be

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\langle -1, -1 \rangle}{\sqrt{2}} = -\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j} = a\,\mathbf{i} + b\,\mathbf{j}$$

Hence, $a = -\frac{1}{\sqrt{2}}$ and $b = -\frac{1}{\sqrt{2}}$. For $f(x, y) = e^{x^2 - y^2}$, we have that

$$f_x(x, y) = 2xe^{x^2-y^2} \text{ and } f_x(x, y) = -2ye^{x^2-y^2}.$$

Hence,

$$\text{Directional derivative } D_u f(x,y) = f_x(x,y)a + f_y(x,y)b$$
$$= 2xe^{x^2-y^2}\left(-\frac{1}{\sqrt{2}}\right) - 2ye^{x^2-y^2}\left(-\frac{1}{\sqrt{2}}\right)$$
$$= -\frac{2}{\sqrt{2}}xe^{x^2-y^2} \quad \frac{2}{\sqrt{2}} \quad x^2-y^2$$

Therefore,

$$\text{Directional derivative at } (1,1) = D_u f(1,1) = -\frac{2}{\sqrt{2}}(1)e^{(1)^2-(-1)^2} + \frac{2}{\sqrt{2}}(1)e^{(1)^2-(-1)^2}$$
$$= -\frac{2}{\sqrt{2}}e^0 + \frac{2}{\sqrt{2}}(1)e^0$$
$$= -\frac{2}{\sqrt{2}}(1) + \frac{2}{\sqrt{2}}(1) = 0$$

Example 9 : Find the constants a, b, c so that

F = $(x + 2y + az)$ **i** + $(bx - 3y - z)$ **j** + $(4x + cy + 2z)$ **k**

is irrotational.

Solution: We have a vector **F** is irrotational, if curl **F=0**. Here

$$\text{curl}\,\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & ax+cy+2z \end{vmatrix} = \mathbf{0},$$

implies

$(c+1)$**i** + $(a - 4)$**j** + $(b - 2)$**k** = 0 **i** + 0 **i** + 0 **i** ,

implies that

$a=4$, $b=2$, $c= -1$

and hence

F = $(x + 2y + 4z)$ **i** + $(2x - 3y - z)$ **j** + $(4x - y + 2z)$ **k**.

LINE INTEGRAL:

Line integral $= \int_c \left(\bar{F}\frac{d\bar{r}}{ds}\right) ds = \int_c \bar{F}\,d\bar{r}$

Note:

1) **Work:** If $\bar{F}$ represents the variable force acting on a particle along arc AB, then the total work done $= \int_A^B \bar{F}\,d\bar{r}$

2) **Circulation:** If $\overline{V}$ represents the velocity of a liquid then $\int_c \overline{V}.\overline{dr}$ is called the circulation of V round the closed curve c.
If the circulation of V round every closed curve is zero then V is said to be irrotational there.

3) When the path of integration is a closed curve then notation of integration is in place of $\int$.

Note: If $\int_A^B \overline{F}.\overline{dr}$ is to be proved to be independent of path, then $\overline{F} = \nabla\emptyset$

here F is called **Conservative** (irrotational) vector field and $\emptyset$ is called the **Scalar potential.**
And $\nabla \times \overline{F} = \nabla \times \nabla\emptyset = 0$

Example 1: Evaluate $\int_c \overline{F}.\overline{dr}$ where $\overline{F} = x^2\hat{i} + xy\hat{j}$ and C is the boundary of the square in the plane $z = 0$ and bounded by the lines $x = 0, y = 0, x = a$ and $y = a$.

Solution: $\int_c \overline{F}.\overline{dr} = \int_{OA} \overline{F}.\overline{dr} + \int_{AB} \overline{F}.\overline{dr} + \int_{BC} \overline{F}.\overline{dr} + \int_{CO} \overline{F}.\overline{dr}$

Here $\quad \bar{r} = x\hat{i} + y\hat{j}, \ \overline{dr} = dx\hat{i} + dy\hat{j}, \ \overline{F} = x^2\hat{i} + xy\mathbf{j}$

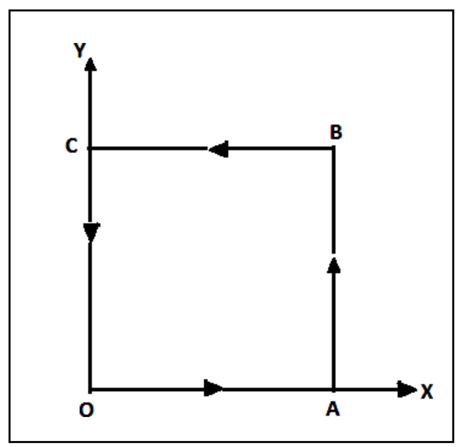
$\overline{F}.\overline{dr} = x^2dx + xydy$ ________(i)

⇨ On $\quad OA, \ y = 0$
$\therefore \overline{F}.\overline{dr} = x^2dx$ (From (i))

$\int_{OA} \overline{F}.\overline{dr} = \int_0^a x^2 dx = [\frac{x^3}{3}]_0^3 = \frac{a^3}{3}$ ________ (ii)

⇨ On $\quad AB, x = a$
$\therefore dx = 0$
$\therefore \overline{F}.\overline{dr} = ay\ dy$ (From (i))

$\int_{AB} \overline{F}.\overline{dr} = \int_0 ay\ dy = a\,[\frac{y^2}{2}]_0^a = \frac{a^3}{2}$ ________ (iii)



⇨ On $\quad BC, y = a$
$\therefore dy = 0$
$\therefore \overline{F}.\overline{dr} = x^2dx$ (From (i))

$\int_{BC} \overline{F}.\overline{dr} = \int_a^0 x^2 dx = [\frac{x^3}{3}]_a^0 = -\frac{a^3}{3}$ ________ (iv)

⇨ On $\quad CO, x = 0$
$\therefore \overline{F}.\overline{dr} = 0$ (From (i))

$\int_{CO} \bar{F}\cdot d\bar{r} = 0$ ________(v)

On adding (ii), (iii), (iv) and (v), we get

$$\int_C \bar{F}\cdot d\bar{r} = \frac{a^3}{3} + \frac{a^3}{2} - \frac{a^3}{3} + 0 = \frac{a^3}{2}$$

Example 2: A vector field is given by

$\bar{F} = (2y+3)\hat{i} + (xz)\hat{j} + (yz-x)\hat{k}$. Evaluate $\int_c \bar{F}\cdot d\bar{r}$ along the path c is $x = 2t,\ y = t,\ z = t^3$ $from\ \ t = 0\ \ to\ \ t = 1.$

Solution:

$$\int_C \bar{F}\cdot d\bar{r} = \int_C (2y+3)dx + (xz)dy + (yz-x)dz$$

$$\left[\begin{matrix} since & x = 2t & y = t & z = t^3 \\ \therefore & \frac{dx}{dt} = 2 & \frac{dy}{dt} = 1 & \frac{dz}{dt} = 3t^2 \end{matrix}\right]$$

$$= \int_0^1 (2t+3)(2\,dt) + (2t)(t^3)dt + (t^4 - 2t)(3t^2dt)$$

$$= \int_0^1 (4t + 6 + 2t^4 + 3t^6 - 6t^3)dt$$

$$= \left[4\frac{t^2}{2} + 6t + \frac{2}{5}t^5 + \frac{3}{7}t^7 - \frac{6}{4}t^4\right]_0^1$$

$$= \left[2t^2 + 6t + \frac{2}{5}t^5 + \frac{3}{7}t^7 - \frac{3}{2}t^4\right]_0^1$$

$$= 2 + 6 + \frac{2}{5} + \frac{3}{7} - \frac{3}{2}$$

$$= 7.32857$$

Example 3: Suppose $\bar{F}(x,y,z) = x^3\hat{i} + y\hat{j} + z\hat{k}$ is the force field. Find the work done by $\bar{F}$ along the line from the (1, 2, 3) to (3, 5, 7).

Solution: Work done $= \int_c \bar{F}\cdot d\bar{r}$

$$= \int_{(1,2,3)}^{(3,5,7)} (x^3\hat{i} + y\hat{j} + z\hat{k}).(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz)$$

$$= \int_{(1,2,3)}^{(3,5,7)} (x^3dx + ydy + zdz)$$

$$= \int_1^3 x^3dx + \int_2^5 y\,dy + \int_3^7 z\,dz$$

$$= \left[\frac{x^4}{4}\right]_1^3 + \left[\frac{y^2}{2}\right]_2^5 + \left[\frac{z^2}{2}\right]_3^7$$

$$= \left[\frac{81}{4} - \frac{1}{4}\right] + \left[\frac{25}{2} - \frac{4}{2}\right] + \left[\frac{49}{2} - \frac{9}{2}\right]$$

$$= \frac{80}{4} + \frac{21}{2} + \frac{40}{2}$$

$$= \frac{202}{4}$$

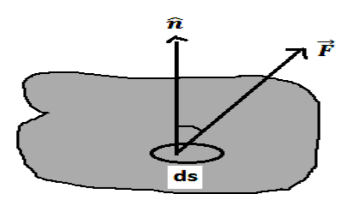
$$= 50.5 \text{ units}$$

Exercise:

1) If a force $\bar{F} = 2x^2y\hat{i} + 3xy\hat{j}$ displaces a particle in the xy-plane from (0, 0) to (1, 4) along a curve $y = 4x^2$. Find the work done.
2) If $\vec{A} = (3x^2 + 6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}$, evaluate the line integral $\vec{A}\cdot d\vec{r}$ from (0, 0, 0) to (1, 1, 1) along the curve C.
3) Show that the integral $\int_{(1,2)}^{(3,4)} (xy^2 + y^3)dx + (x^2y + 3xy^2)dy$ is independent of the path joining the points (1, 2) and (3, 4). Hence, evaluate the integral.

SURFACE INTEGRAL:

Let $\bar{F}$ be a vector function and S be the given surface.
Surface integral of a vector function $\bar{F}$ over the surface S is defined as the integral of the components of $\bar{F}$ along the normal to the surface.



Component of $\bar{F}$ along the normal $= \bar{F}.\hat{n}$
Where n = unit normal vector to an element ds and

$$\hat{n} = \frac{grad\ f}{|grad\ f|} \qquad ds = \frac{dx\ dy}{(\hat{n}.\hat{k})}$$

Surface integral of F over S

$$= \sum \bar{F}.\hat{n} \qquad = \int_S (\bar{F}.\hat{n})ds$$

Note:

1) Flux $= \int_S (\bar{F}.\hat{n})ds$ where, $\bar{F}$ represents the velocity of a liquid.

If $\int_S (\bar{F}.\hat{n})ds = 0$, then $\bar{F}$ is said to be a Solenoidal vector point function.

VOLUME INTEGRAL:

Let $\bar{F}$ be a vector point function and volume V enclosed by a closed surface.

The volume integral $= \iiint_V \bar{F} dv$

Example 1: Evaluate $\iint_S (yz\hat{i} + zx\hat{j} + xy\hat{k}).ds$ where S the surface of the sphere is $x^2 + y^2 + z^2 = a^2$ in the first octant.

Solution: Here, $\emptyset = x^2 + y^2 + z^2 - a^2$

Vector normal to the surface $= \nabla\emptyset$

$$= \hat{i}\frac{\partial\emptyset}{\partial x} + \hat{j}\frac{\partial\emptyset}{\partial y} + \hat{k}\frac{\partial\emptyset}{\partial z}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2 - a^2)$$

$$= 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$\hat{n} = \frac{\nabla\emptyset}{|\nabla\emptyset|} = \frac{2x\hat{i}+2y\hat{j}+2z\hat{k}}{\sqrt{4x^2+4y^2+4z^2}}$$

$$= \frac{x\hat{i}+y\hat{j}+z\hat{k}}{\sqrt{x^2+y^2+z^2}}$$

$$= \frac{x\hat{i}+y\hat{j}+z\hat{k}}{a} \qquad [\because x^2 + y^2 + z^2 = a^2]$$

Here, $\bar{F} = yz\hat{i} + zx\hat{j} + xy\hat{k}$

$$\bar{F}.\hat{n} = (yz\hat{i} + zx\hat{j} + xy\hat{k}).\left(\frac{x\hat{i}+y\hat{j}+z\hat{k}}{a}\right) = \frac{3xyz}{a}$$

Now, $\iint_S \bar{F}.\hat{n}\, ds = \iint_S (\bar{F}.\hat{n})\frac{dx\, dy}{|\hat{k}.\hat{n}|}$

$$= \int_0^a\int_0^{\sqrt{a^2-x^2}} \frac{3xyz\ dx\ dy}{a\left(\frac{z}{a}\right)}$$

$$= 3\int_0^a \int_0^{\sqrt{a^2-x^2}} xy\, dy\, dx$$

$$= 3\int_0^a x\left(\frac{y^2}{2}\right)_0^{\sqrt{a^2-x^2}} dx$$

$$= \frac{3}{2}\int_0^a x\,(a^2 - x^2)dx$$

$$=\frac{3}{2}\left(\frac{a^2x^2}{2}-\frac{x^4}{4}\right)_0^a$$

$$=\frac{3}{2}\left(\frac{a^4}{2}-\frac{a^4}{4}\right)$$

$$=\frac{3a^4}{8}$$

Example 2: If $\bar{F}=2z\hat{i}-x\hat{j}+y\hat{k}$, evaluate $\int_V \bar{F}\,dv$ where, v is the region bounded by the surfaces $x=0,\ y=0,\ x=2,\ y=4,\ z=x^2,\ z=2.$

Solution: $\int_V \bar{F}\,dv = \iiint (2z\hat{i}-x\hat{j}+y\hat{k})\,dx\,dy\,dz$

$$=\int_0^2 dx\int_0^4 dy\int_{x^2}^2 (2z\hat{i}-x\hat{j}+y\hat{k})\,dz$$

$$=\int_0^2 dx\int_0^4 dy\,[z^2\hat{i}-xz\hat{j}+yz\hat{k}]_{x^2}^2$$

$$=\int_0^2 dx\int_0^4 dy\,[4\hat{i}-2x\hat{j}+2y\hat{k}-x^4\hat{i}+x^3\hat{j}-x^2y\hat{k}]$$

$$=\int_0^2 dx\,\left[4y\hat{i}-2xy\hat{j}+y^2\hat{k}-x^4y\hat{i}+x^3y\hat{j}-\frac{x^2y^2}{2}\hat{k}\right]_0^4$$

$$=\int_0^2 (16\hat{i}-8x\hat{j}+16\hat{k}-4x^4\hat{i}+4x^3\hat{j}-8x^2\hat{k})\,dx$$

$$=\left[16x\hat{i}-4x^2\hat{j}+16x\hat{k}-\frac{4x^5}{5}\hat{i}+x^4\hat{j}-\frac{8x^3}{3}\hat{k}\right]_0^2$$

$$=32\hat{i}-16\hat{j}+32\hat{k}-\frac{128}{5}\hat{i}+16\hat{j}-\frac{64}{3}\hat{k}$$

$$=\frac{32\hat{i}}{5}+\frac{32\hat{k}}{3}$$

$$=\frac{32}{15}(3\hat{i}+5\hat{k})$$

Exercise:

1) Evaluate $\iint_S (\bar{F}\cdot\hat{n})\,ds$, where, $\bar{F}=18z\hat{i}-12\hat{j}+3y\hat{k}$ and S is the surface of the plane $2x+3y+6z=12$ in the first octant.
2) If $\bar{F}=(2x^2-3z)\hat{i}-2xy\hat{j}-4x\hat{k}$, then evaluate $\iiint_V \nabla\cdot\bar{F}\,dv$, where V is bounded by the plane $x=0, y=0, z=0$ and $2x+2y+z=4$.

<u>GREEN'S THEOREM</u>: (Without proof)

If $\emptyset(x, y), \Psi(x, y), \frac{\partial \emptyset}{\partial y}$ and $\frac{\partial \Psi}{\partial x}$ be continuous functions over a region R bounded by simple closed curve C in $x - y$ plane, then

$$\oint_C (\emptyset\, dx + \Psi\, dy) = \iint_R \left(\frac{\partial \Psi}{\partial x} - \frac{\partial \emptyset}{\partial y}\right) dx\, dy$$

Note: Green's theorem in vector form

$$\int_c \overline{F}.d\overline{r} = \iint_R (\nabla \times \overline{F}).\hat{k}\, dR$$

Where, $\overline{F} = \emptyset\hat{i} + \Psi\hat{j}$ $\bar{r} = x\hat{i} + y\hat{j}$, $\hat{k}$ is a unit vector along z-axis and $dR = dx\, dy$.

Example 1: Using green's theorem, evaluate $\int_c (x^2 y\, dx + x^2\, dy)$, where c is the boundary described counter clockwise of the triangle with vertices (0,0), (1,0), (1,1).

Solution: By green's theorem, we have

$$\oint_C (\emptyset\, dx + \Psi\, dy) = \iint_R \left(\frac{\partial \Psi}{\partial x} - \frac{\partial \emptyset}{\partial y}\right) dx\, dy$$

$$\int_c (x^2 y\, dx + x^2\, dy) = \iint_R (2x - x^2)\, dx\, dy$$

$$= \int_0^1 (2x - x^2)\, dx \int_0^x dy$$

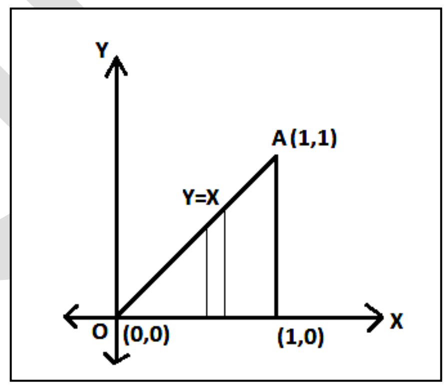
$$= \int_0^1 (2x - x^2)\, dx\, [y]_0^x$$

$$= \int_0^1 (2x^2 - x^3)\, dx$$

$$= \left(\frac{2x^3}{3} - \frac{x^4}{4}\right)_0^1$$

$$= \left(\frac{2}{3} - \frac{1}{4}\right)$$

$$= \frac{5}{12}$$



Example 2: Use green's theorem to evaluate

$\int_c (x^2 + xy)dx + (x^2 + y^2)dy$, where c is the square formed by the lines $y = \pm 1$, $x = \pm 1$.

Solution: By green's theorem, we have

$$\oint_C (\emptyset\, dx + \Psi\, dy) = \iint_R \left(\frac{\partial \Psi}{\partial x} - \frac{\partial \emptyset}{\partial y}\right) dx\, dy$$

$$= \int_{-1}^{1}\int_{-1}^{1} \left[\frac{\partial}{\partial x}(x^2 + y^2) - \frac{\partial}{\partial y}(x^2 + xy)\right] dx\, dy$$

$$= \int_{-1}^{1}\int_{-1}^{1}(2x - x)\,dxdy$$

$$= \int_{-1}^{1}\int_{-1}^{1} x\,dxdy$$

$$= \int_{-1}^{1} x\,dx \int_{-1}^{1} dy$$

$$= \int_{-1}^{1} x\,dx\,(y)_{-1}^{1}$$

$$= \int_{-1}^{1} x\,dx\,(1+1)$$

$$= \int_{-1}^{1} 2x\,dx$$

$$= (x^2)_{-1}^{1}$$

$$= 1 - 1$$

$$= 0$$

Exercise:

1) Apply Green's theorem to evaluate

$\int_C [(2x^2 - y^2)dx + (x^2 + y^2)dy]$, where C is the boundary of the area enclosed by the x-axis and the upper half of circle $x^2 + y^2 = a^2$.

2) A vector field $\bar{F}$ is given by $\bar{F} = \sin y\ \hat{i} + x\,(1 + \cos y)\ \hat{j}$.

Evaluate the line integral $\int_C \bar{F}\cdot d\bar{r}$ where C is the circular path given by $x^2 + y^2 = a^2$.

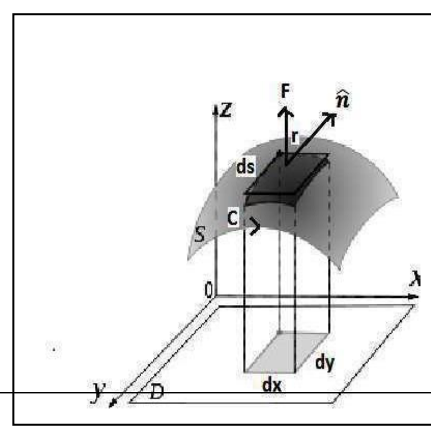
STOKE'S THEOREM: (Relation between Line integral and Surface integral)

<u>(Without Proof)</u>

Surface integral of the component of curl $\bar{F}$ along the normal to the surface S, taken over the surface S bounded by curve C is equal to the line integral of the vector point function $\bar{F}$ taken along the closed curve C.

Mathematically

$$\oint_C \bar{F}\cdot d\bar{r} = \iint_S \text{curl}\,\bar{F}\cdot\hat{n}\,ds$$



Where $\hat n = \cos\alpha\, \hat i + \cos\beta\, \hat j + \cos\gamma\, \hat k$

is a unit external normal to any surface ds.

OR

The circulation of vector F around a closed curve C is equal to the flux of the curve of the vector through the surface S bounded by the curve C.

$$\bar F . d\bar r = \iint_S curl\, \bar F . \hat n\, ds = \iint_S curl\, \bar F . d\bar S$$

Example 1: Apply Stoke's theorem to find the value of

$\int_c (y\, dx + z\, dy + x\, dz)$

Where c is the curve of intersection of $x^2 + y^2 + z^2 = a^2$ and $x + z = a$.

Solution: $\int_c (y\, dx + z\, dy + x\, dz)$

$= \int_c (y\, \hat i + z\, \hat j + x\, \hat k).(\hat i\, dx + \hat j\, dy + \hat k\, dz)$

$= \int_c (y\, \hat i + z\, \hat j + x\, \hat k).d\bar r$

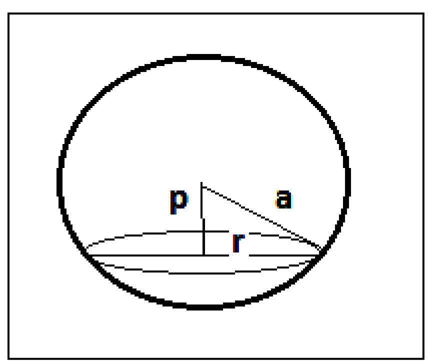
$= \iint_S curl\, (y\, \hat i + z\, \hat j + x\, \hat k).\hat n ds$ (By Stoke's theorem)

$= \iint_S \left(\hat i \frac{\partial}{\partial x} + \hat j \frac{\partial}{\partial y} + \hat k \frac{\partial}{\partial z}\right) \times (y\, \hat i + z\, \hat j + x\, \hat k).\hat n\, ds$

$= \iint_S -(\hat i + \hat j + \hat k).\hat n\, ds$ ________(i)

Where S is the circle formed by the integration of $x^2 + y^2 + z^2 = a^2$ and

$x + z = a.$



$\hat n = \frac{\nabla\phi}{|\nabla\phi|}$

$$= \frac{(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z})(x+z-a)}{|\nabla \emptyset|}$$

$$= \frac{\hat{i}+\hat{k}}{\sqrt{1+1}}$$

$$= \frac{\hat{i}}{\sqrt{2}} + \frac{\hat{k}}{\sqrt{2}}$$

Putting the value of $\hat{n}$ in (i), we have

$$= \iint_S -(\hat{i} + \hat{j} + \hat{k}).\left(\frac{\hat{i}}{\sqrt{2}} + \frac{\hat{k}}{\sqrt{2}}\right) ds$$

$$= \iint_S -\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) ds \quad \left[Use\ r^2 = R^2 - p^2 = a^2 - \frac{a^2}{2} = \frac{a^2}{2}\right]$$

$$= -\frac{2}{\sqrt{2}} \iint_S ds = -\frac{2}{\sqrt{2}} \pi \left(\frac{a}{\sqrt{2}}\right)^2 = -\frac{\pi a^2}{\sqrt{2}}$$

Example 2: Evaluate $\int_C \bar{F}.d\bar{r}$ by stoke's theorem, where

$\bar{F} = y^2\hat{i} + x^2\hat{j} - (x+z)\hat{k}$ and C is the boundary of triangle with vertices at $(0,0,0)$, $(1,0,0)$ and $(1,1,0)$.

Solution: We have, curl $\bar{F} = \nabla \times \bar{F}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x^2 & -(x+z) \end{vmatrix}$$

$$= 0.\hat{i} + \hat{j} + 2(x-y)\hat{k}$$

We observe that z co-ordinate of each vertex of the triangle is zero.

Therefore, the triangle lies in the xy-plane.

$$\therefore \quad \hat{n} = \hat{k}$$

$$\therefore \quad curl\,\bar{F}.\hat{n} = [\hat{j} + 2(x-y)\hat{k}].\hat{k} = 2(x-y).$$

In the figure, only xy-plane is considered.

The equation of the line OB is $y = x$

By Stoke's theorem, we have

$$_C\ \bar{F}.d\bar{r} = {}_S\ (curl\ \bar{F}.\hat{n})ds$$

$$= \int_{x=0}^{1}\int_{y=0}^{x} 2(x-y)dxdy$$

$$= 2\int_{0}^{1} [x^2 - \frac{x^2}{2}]\, dx$$

$$= 2\int_{0}^{1} \frac{x^2}{2} dx$$

$$= \int_{0}^{1} x^2\, dx$$

$$= [\frac{x^3}{3}]_{0}^{1}$$

$$= \frac{1}{3}$$

Exercise:

1) Use the Stoke's theorem to evaluate $\int_C [(x+2y)dx + (x-z)dy + (y-z)dz]$ where C is the boundary of the triangle with vertices (2,0,0), $(0,3,0)$ and (0,0,6) oriented in the anti-clockwise direction.
2) Apply Stoke's theorem to calculate $\int_c 4\,y\,dx + 2z\,dy + 6y\,dz$ Where c is the curve of intersection of $x^2 + y^2 + z^2 = 6z$ and $z = x + 3$
3) Use the Stoke's theorem to evaluate $\int_C y^2\,dx + xy\,dy + xz\,dz$, where C is the bounding curve of the hemisphere $x^2 + y^2 + z^2 = 1$, $z \geq 0$, oriented in the positive direction.

GAUSS'S THEOREM OF DIVERGENCE: (Without Proof)

The surface integral of the normal component of a vector function F taken around a closed surface S is equal to the integral of the divergence of F taken over the volume Venclosed by the surface S.
Mathematically

$$_S\ \bar{F}.\hat{n}\, ds = {}_V\ div\ \bar{F}\, dv$$

Example 1: Evaluate $_S\ \bar{F}.\hat{n}\,ds$ where $\bar{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$ and S is the surface of the cube bounded by $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$.

Solution: By Gauss's divergence theorem,

$$\iint_S \vec{F}.\hat{n}\, ds = \iiint_V (\nabla.\vec{F})\, dv$$

$$= \iiint_v \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).(4xz\hat{i} - y^2\hat{j} + yz\hat{k})\, dv$$

$$= \iiint_v \left[\frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(yz)\right] dx\, dy\, dz$$

$$= \iiint_v (4z - 2y + y)\, dx\, dy\, dz$$

$$= \iiint_v (4z - y)\, dx\, dy\, dz$$

$$= \int_0^1 \int_0^1 \left(\frac{4z^2}{2} - yz\right)_0^1 dx\, dy$$

$$= \int_0^1 \int_0^1 (2z^2 - yz)_0^1\, dx\, dy$$

$$= \int_0^1 \int_0^1 (2 - y)\, dx\, dy$$

$$= \int_0^1 \left(2y - \frac{y^2}{2}\right)_0^1 dx$$

$$= \frac{3}{2}\int_0^1 dx$$

$$= \frac{3}{2}[x]_0^1$$

$$= \frac{3}{2}(1)$$

$$= \frac{3}{2}$$

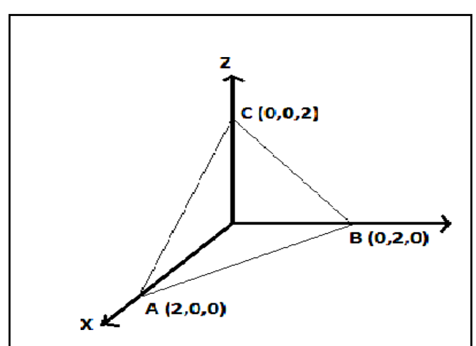
Example 2: Evaluate surface integral $\iint \vec{F}.\hat{n}\, ds$, where $\vec{F} = (x^2 + y^2 + z^2)(\hat{i} + \hat{j} + \hat{k})$, S is the surface of the tetrahedron $x = 0, y = 0, z = 0, x + y + z = 2$ and n is the unit normal in the outward direction to the closed surface S.

Solution: By Gau' dveence teoem,

$$\iint_S \vec{F}.\hat{n}\, ds = \iiint_V div\, \vec{F}.\, dv$$

Where S is the surface of tetrahedron $x = 0, y = 0, z = 0, x + y + z = 2$

$$= \iiint_V \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).(x^2 + y^2 + z^2)(\hat{i} + \hat{j} + \hat{k})dv$$



$$= \iiint_V (2x + 2y + 2z)dv$$

$$= 2 \iiint_V (x + y + z)dx\,dy\,dz$$

$$= 2\int_0^2 dx \int_0^{2-x} dy \int_0^{2-x-y} (x + y + z)\, dz$$

$$= 2\int_0^2 dx \int_0^{2-x} dy\ \left(xz + yz + \frac{z^2}{2}\right)_0^{2-x-y}$$

$$= 2\int_0^2 dx \int_0^{2-x} dy\ \left[2x - x^2 - xy + 2y - xy - y^2 + \frac{(2-x-y)^2}{2}\right]$$

$$= 2\int_0^2 dx\ \left[2xy - x^2y - xy^2 + y^2 - \frac{y^3}{3} - \frac{(2-x-y)^3}{6}\right]_0^{2-x}$$

$$= 2\int_0^2 dx\,\left[2x(2-x) - x^2(2-x) - x(2-x)^2 + (2-x)^2 - \frac{(2-x)^3}{3} + \frac{(2-x)^3}{6}\right]$$

$$= 2\int_0^2 \left[4x - 2x^2 - 2x^2 + x^3 - 4x + 4x^2 - x^3 + (2-x)^2 - \frac{(2-x)^3}{3} + \frac{(2-x)^3}{6}\right]$$

$$= 2\left[2x^2 - \frac{4x^3}{3} + \frac{x^4}{4} - 2x^2 + \frac{4x^3}{3} - \frac{x^4}{4} - \frac{(2-x)^3}{3} + \frac{(2-x)^4}{12} - \frac{(2-x)^4}{24}\right]_0^2$$

$$= 2\left[-\frac{(2-x)^3}{3} + \frac{(2-x)^4}{12} - \frac{(2-x)^4}{24}\right]_0^2$$

$$= 2\left[\frac{8}{3} - \frac{16}{12} + \frac{16}{24}\right]$$

$$= 4$$

Example 3: Verify Stokes' theorem for **F =** $(x^2 - y^2)$ **i +** $2xy$ **j** in the rectangular region in the *xy* plane, the vertices of the rectangle being given by (0, 0), (*a*, 0), (*b*, 0) and (*a*, *b*).

Solution : Here the bounded surface S is the rectangle R given by

$$R : 0 \le x \le a,\ 0 \le y \le b.$$

Given **F =** $(x^2 - y^2)$ **i +** $2xy$ **j** .

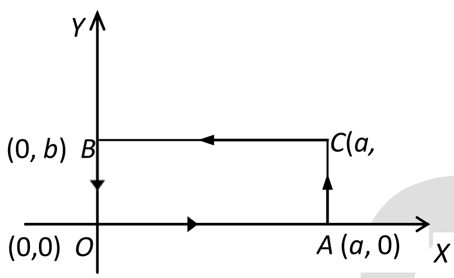
$$\therefore\ \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 - y^2 & 2xy & 0 \end{vmatrix} = \mathbf{i}(0) - \mathbf{j}(0) + \mathbf{k}(4y) = 4y\,\mathbf{k}.$$

The surface is the rectangular region in the *xy* plane. Hence the outward normal to the surface is **k**.

$$\therefore\ \mathbf{n}\, dS = \mathbf{k}\, dx\, dy.$$

$$\therefore \quad \iint_S (\nabla\times \mathbf{F}).\mathbf{S} = \int_0^a\int_0^b (4y\mathbf{k}).(\mathbf{k}\,dydx) \int_0^a\int_0^b 4y\,dydx = 2ab^2. \_\_\_\_\_(1)$$

Now $\displaystyle\int_C \mathbf{F}.d\mathbf{r} = \int_{OA}\mathbf{F}.d\mathbf{r} + \int_{AC}\mathbf{F}.d\mathbf{r} + \int_{CB}\mathbf{F}.d\mathbf{r} + \int_{BO}\mathbf{F}.d\mathbf{r}$ ____(2)



F.d**r =** [$(x^2 - y^2)$ **i** + $2xy$ **j**] . [dx **i** + dy **j** + dz **k**]

= $(x^2 - y^2)dx$ + $2xy\,dy$

(i) In OA, $y = 0$, $dy = 0$ and x varies from 0 to a.

$$\int_{OA}\mathbf{F}.d\mathbf{r} = \int_0^a (x^2 - 0^2)dx + 2x.0.0 = \int_0^a x^2 dx = \frac{1}{3}a^3.$$

Similarly,

$$(ii)\int_{AC}\mathbf{F}.d\mathbf{r} = \int_0^b 2ay\,dy = ab^2. \qquad (\because\ x = a \quad dx = 0.)$$

$$(iii)\int_{CB}\mathbf{F}.d\mathbf{r} = \int_a^0 (x^2 - b^2)dx = -\int_0^a (x^2 - b^2)dx = ab^2 - \frac{1}{3}a^3 \quad (\because\ y = b \quad dy = 0.)$$

$$(iv)\int_{BO}\mathbf{F}.d\mathbf{r} = 0. \qquad (\because\ x = 0 \quad dx = 0.)$$

Using the above values, (2) becomes

$$\int_C \mathbf{F}.d\mathbf{r} = \frac{1}{3}a^3 + ab^2 + ab^2 - \frac{1}{3}a^3 = 2ab^2 \qquad \_\_\_\_(3)$$

From (1) and (3), we get

$$\iint_S (\nabla \times \mathbf{F}).\mathbf{n}\, dS = \oint_C \mathbf{F} . d\mathbf{r},$$

So that Stokes' theorem is verified.

Example 4: Verify Stokes' theorem for **F** = $(2x - y)$ **i** $- yz^2$ **j** $- y^2z$ **k**, where S is the upper half surface of the unit sphere $x^2 + y^2 + z^2 = 1$ and C is its boundary.

Solution

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x - y & -yz^2 & -y^2 z \end{vmatrix} = \mathbf{k}$$

$$\therefore \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \iint_R (\nabla \times \mathbf{F}) \cdot \mathbf{n} \frac{dxdy}{|\mathbf{n} \cdot \mathbf{k}|},$$

where R is the orthogonal projection of S on the xy-plane. i.e., R is the region enclosed by the circle $x^2 + y^2 = 1.$

$$= \iint_R \mathbf{k} \cdot \mathbf{n} \frac{dxdy}{|\mathbf{n} \cdot \mathbf{k}|},$$

$$= \iint_R dxdy$$

Now $\iint_R dxdy$ is the area of the region enclosed by the circle $x^2 + y^2 = 1$ and is $\pi \times 1^2 = \pi.$ hence, we obtain

$$\iint_S (\nabla \times \mathbf{F}).\mathbf{n}\, dS = \pi, \qquad \_\_\_\_(1)$$

The boundary C of S is a circle in the xy plane of radius 1 and center at the origin. Suppose $x = \cos t$, $y = \sin t$, $z = 0$, $0 \le t \le 2\pi$ be the parametric form of C.

Then

$$\int_C F.dr = \int_C (2x - y)dx \qquad (\because \quad z = 0, \quad dz = 0)$$

$$= \int_0^{2\pi} (2\cos t - \sin t)(-\sin t)dt \;\; (\text{since } \; x = \cos t, \; dx = -\sin t dt)$$

$$= \pi \qquad \text{——}(2)$$

From (1) and (2), we get

$$\iint_S (\nabla \times \mathbf{F}).\mathbf{n}\, dS = \oint_C \mathbf{F}.d\mathbf{r}$$

Hence Stokes' theorem is verified.

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