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New Scheme Based On AICTE Flexible Curricula

Mechanical Engineering, IV-Semester

ME404- FLUID MECHANICS

[1] Introduction, fluid and the continuum, fluid properties, surface tension, bulk modulus and thermodynamic properties, Newton's laws of viscosity and it's coefficients, Newtonian and non Newtonian fluids, hydrostatics and buoyancy, meta center and metacentric height, stability of floating bodies.

[2] Fluid kinematics, Langragian and Eularian mrthod, description of fluid flow, stream line, path line and streak line, types of flow and types of motion, local and connective acceleration, continuity equation, potential flow, circulation, velocity potential, stream function, laplace equation, flow nets.

[3] Fluid dynamics, system and control volume, Reynold transport theorem, Euler's equation, Bernoulli's equation, momentum and moment of momentum equation, their applications, forces on immersed bodies, lift and drag, streamlined and bluff bodies, flow around circular cylinder and aerofoils.

[4] Flow through pipes, Reynold number, laminar and turbulent flow, viscous flow through parallel plates and pipes, Navier Stoke's equation, pressure gradient, head loss in turbulent flow (Darcey's equation), friction factor, minor losses, hydraulic and energy gradient, pipe networks

[5] Introduction to boundary layer theory, description of boundary layer, boundary layer parameters, Von Karman momentum equation, laminar and turbulent boundary conditions, boundary layer separation, compressible flow, Mach number, isentropic flow, stagnation properties, normal and oblique shocks, Fanno and Reyleigh lines, flow through nozzles,

BOOKS:

1. Massy B.S., Mechanics of fluid, Routledge Publication
2. Shames, Fluid Mechanics, Tata McGraw Hills

FLUID MECHANICS (ME – 404)
UNIT – 1 (FLUID STATICS)

I. PHYSICAL PROPERTIES OF FLUID

Due to the development of industries there arose a need for the study of fluids other than water. Theories like boundary layer theory were developed which could be applied to all types of real fluids, under various conditions of flow. The combination of experiments, the mathematical analysis of hydrodynamics and the new theories is known as 'Fluid Mechanics'. **Fluid Mechanics encompasses the study ofall types of fluids under static, kinematic and dynamic conditions.**

The study of properties of fluids is basic for the understanding of flow or static condition of fluids. The important properties are **density, viscosity, surface tension, bulk modulusand vapour pressure.** Viscosity causes resistance to flow. Surface tension leads to capillaryeffects. Bulk modulus is involved in the propagation of disturbances like sound waves in fluids. Vapour pressure can cause flow disturbances due to evaporation at locations of low pressure. It plays an important role in cavitation studies in fluid machinery.

A fluid is defined as a material which will continue to deform with the application of shear force however small the force may be.

COMPRESSIBLE & INCOMPRESSIBLE FLUID

If the density of a fluid varies significantly due to moderate changes in pressure or temperature, then the fluid is called compressible fluid. Generally gases and vapoursunder normal conditions can be classified as compressible fluids. In these phases the distance between atoms or molecules is large and cohesive forces are small. So increase in pressure or temperature will change the density by a significant value.

If the change in density of a fluid is small due to changes in temperature and or pressure, then the fluid is called incompressible fluid. All liquids are classified underthis category.

When the change in pressure and temperature is small, gases and vapours are treated as incompressible fluids. For certain applications like propagation of pressure disturbances, liquids should be considered as compressible.

CONTINUUM

As gas molecules are far apart from each other and as there is empty space between molecules doubt arises as to whether a gas volume can be considered as a continuous matter like a solid for situations similar to application of forces.

Under normal pressure and temperature levels, gases are considered as a continuum (*i.e.,* as if no empty spaces exist between atoms). The test for continuum is to measure properties like density by sampling at different locations and also reducing the sampling volume to low levels. If the property is constant irrespective of the location and size of sample volume, then the gas body can be considered as a continuum for purposes of mechanics (application of force, consideration of acceleration, velocity etc.) and for the gas volume to be considered as a single body or entity. This is a very important test for the application of all laws of mechanics to a gas volume as a whole. When the pressure is extremely low, and when there are only few molecules in a cubic meter of volume, then the laws of mechanics should be applied to the molecules as entities and not to the gas body as a whole.

BASIC FLUID TERMINOLOGIES

***Density (mass density)*:**The mass per unit volume is defined as density. The unit used is kg/m^3.The measurement is simple in the case of solids and liquids. In the case of gases and vapours it is rather involved. The symbol used is ρ. The characteristic equation for gases provides a means to estimate the density from the measurement of pressure, temperature and volume.

***Specific Volume*:**The volume occupied by unit mass is called the specific volume of thematerial. The symbol used is *v*, the unit being m^3/kg. Specific volume is the reciprocal of density.

In the case of solids and liquids, the change in density or specific volume with changes in pressure and temperature is rather small, whereas in the case of gases and vapours, density will change significantly due to changes in pressure and/or temperature.

***Weight Density or Specific Weight*:**The force due to gravity on the mass in unitvolume is defined as Weight Density or Specific Weight. The unit used is N/m^3. The symbol used is Œ. At a location where *g* is the local acceleration due to gravity, **Specific weight, γ = g ρ**

In the above equation direct substitution of dimensions will show apparent non-homogeneity as the dimensions on the LHS and RHS will not be the same. On the LHS the dimension will be N/m^3 but on the RHS it is kg/m^2 s^2. The use of g_o will clear this anomaly. As seen in section 1.1, g_o = 1 kg m/N s^2. The RHS of the equation 1.3.1 when divided by g_o will lead to perfect dimensional homogeneity. The equation should preferably be written as, **Specific weight, γ = (g/g_o) ρ**

Since newton (N) is defined as the force required to accelerate 1 kg of mass by 1/s^2, it can also be expressed as kg m/s^2. Density can also be expressed as Ns^2/m^4 (as kg = Ns^2/m). Beam balances compare the mass while spring balances compare the weights. The mass is the same (invariant) irrespective of location but the weight will vary according to the local gravitational constant. Density will be invariant while specific weight will vary with variations in gravitational acceleration.

***Specific Gravity or Relative Density*:**The ratio of the density of the fluid to thedensity of water—usually 1000 kg/m^3 at a standard condition—is defined as Specific Gravity or Relative Density 6 of fluids. This is a ratio and hence no dimension or unit is involved.

VISCOSITY

A fluid is defined as a material which will continue to deform with the application of a shear force. However, different fluids deform at different rates when the same shear stress (force/ area) is applied.

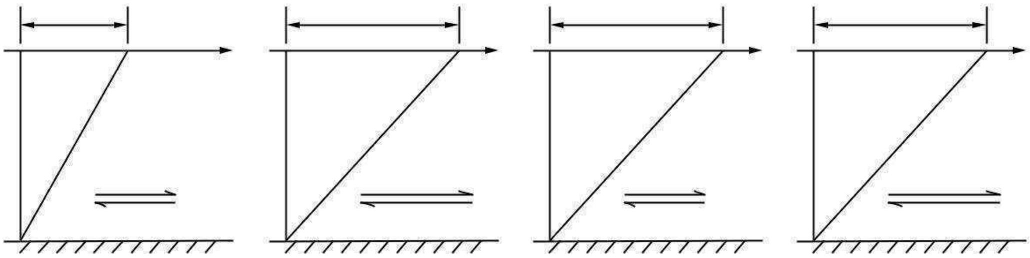
Viscosity is that property of a real fluid by virtue of which it offers resistance to shear force. For a given fluid the force required varies directly as the rate of deformation. As the rate of deformation increases the force required also increases.

The force required to cause the same rate of movement depends on the nature of the fluid. The resistance offered for the same rate of deformation varies directly as the viscosity of the fluid. As viscosity increases the force required to cause the same rate of deformation increases.Newton's law of viscosity states that the shear force to be applied for a deformation rate of (*du/dy*) over an area A is given by,

$$F = \mu A (du/dy)$$

$$(F/A) = \tau = \mu (du/dy) = \mu (u/y)$$

Where, F is the applied force in N, A is area in m^2, du/dy is the velocity gradient (or rate of deformation), 1/s, perpendicular to flow direction, here assumed linear, and μ is the proportionality constant defined as the **dynamic or absolute viscosity** of the fluid.



$u_b > u_a$, $F_b > F_a$ $\mu_a = \mu_b$ $u_a = u_b$, $\mu_a < \mu_b$, $F_b > F_a$

(i) same fluid (*ii*) same velocity

The dimensions for dynamic viscosity μ can be obtained from the definition as Ns/m^2 or kg/ms. The first dimension set is more advantageously used in engineering problems. However, if the dimension of N is substituted, then the second dimension set, more popularly used by scientists can be obtained. The numerical value in both cases will be the same.

$N = kg\ m/s^2$; $\mu = (kg\ m/s^2)\ (s/m^2) = kg/ms$
The popular unit for viscosity is Poise named in honor of Poiseuilli.
Poise = 0.1 Ns/m^2
Centipoise (cP) is also used more frequently as **cP = 0.001 Ns/m^2**

For water the viscosity at 20°C is nearly 1 cP. The ratio of dynamic viscosity to the density is defined as kinematic viscosity, ν, having a dimension of m^2/s. Later it will be seen to relate to momentum transfer. Because of this kinematic viscosity is also called momentum diffusivity. The popular unit used is stokes (in honor of the scientist Stokes). Centistoke is also often used.

1 stoke = 1 $cm^2/s = 10^{-4}\ m^2/s$

Of all the fluid properties, viscosity plays a very important role in fluid flow problems. The velocity distribution in flow, the flow resistance etc. are directly controlled by viscosity. In the study of fluid statics (*i.e.,* when fluid is at rest), viscosity and shear force are not generally involved.

Newtonian and Non Newtonian Fluids

An ideal fluid has zero viscosity. Shear force is not involved in its deformation. An ideal fluid has to be also incompressible. Shear stress is zero irrespective of the value of du/dy. Bernoulli equation can be used to analyses the flow.

Real fluids having viscosity are divided into two group namely Newtonian and non-Newtonian fluids. In Newtonian fluids a linear relationship exists between the magnitude of the applied shear stress and the resulting rate of deformation. It means that the proportionality parameter (in equation, $\tau = \mu\ (du/dy)$), viscosity, μ is constant in the case of Newtonian fluids (other conditions and parameters remaining the same). The viscosity at any given temperature and pressure is constant for a Newtonian fluid and is independent of the rate of deformation.

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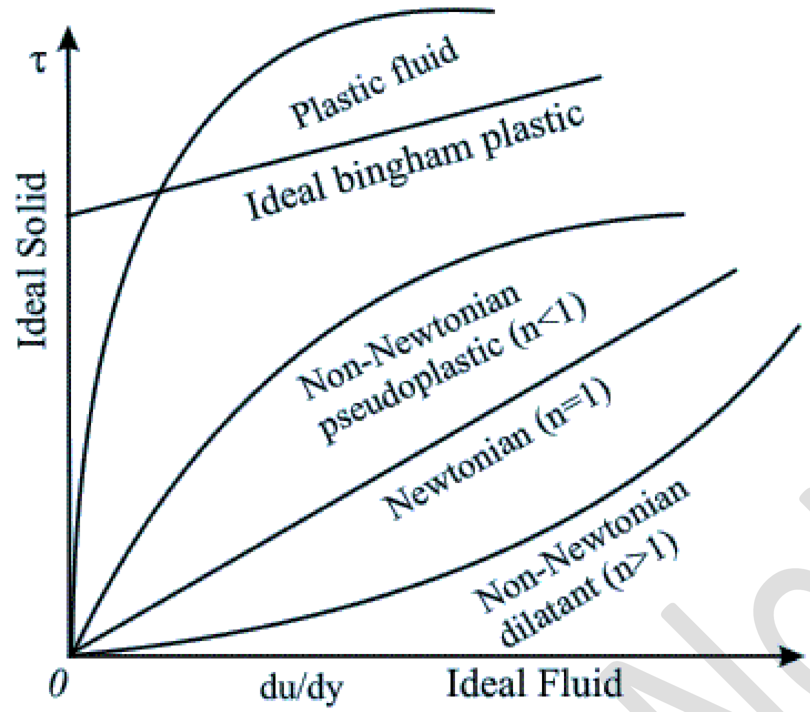


Non Newtonian fluids can be further classified as simple non Newtonian, ideal plastic and shear thinning, shear thickening and real plastic fluids. In non-Newtonian fluids the viscosity will vary with variation in the rate of deformation. Linear relationship between shear stress and rate of deformation (du/dy) does not exist. In plastics, up to a certain value of applied shear stress there is no flow. After this limit it has a constant viscosity at any given temperature. In shear thickening materials, the viscosity will increase with (du/dy) deformation rate. In shear thinning materials viscosity will decrease with du/dy. Paint, tooth paste, printers ink are some examples for different behaviors. Many other behaviors have been observed which are more specialized in nature. The main topic of study in this text will involve only Newtonian fluids.

Viscosity and Momentum Transfer

In the flow of liquids and gases molecules are free to move from one layer to another. When the velocity in the layers are different as in viscous flow, the molecules moving from the layer at lower speed to the layer at higher speed have to be accelerated. Similarly the molecules moving from the layer at higher velocity to a layer at a lower velocity carry with them a higher value of momentum and these are to be slowed down. Thus the molecules diffusing across layers transport a net momentum introducing a shear stress between the layers. The force will be zero if both layers move at the same speed or if the fluid is at rest.

When cohesive forces exist between atoms or molecules these forces have to be overcome, for relative motion between layers. A shear force is to be exerted to cause fluids to flow.

Viscous forces can be considered as the sum of these two, namely, the force due to momentum transfer and the force for overcoming cohesion. In the case of liquids, the viscous forces are due more to the breaking of cohesive forces than due to momentum transfer (as molecular velocities are low). In the case of gases viscous forces are more due to momentum transfer as distance between molecules is larger and velocities are higher.

Effect of Temperature on Viscosity

When temperature increases the distance between molecules increases and the cohesive force decreases. So, viscosity of liquids decrease when temperature increases.

In the case of gases, the contribution to viscosity is more due to momentum transfer. As temperature increases, more molecules cross over with higher momentum differences. Hence, in the case of gases, viscosity increases with temperature.

Significance of Kinematic Viscosity

Kinematic viscosity, $\nu=\mu/\rho$,
The unit in SI system is m^2/s. $(Ns/m^2)\ (m^3/\ kg) = [(kg\ m/s^2)\ (s/m^2)]\ [m^3/kg] = m^2/s$
Popularly used unit is stoke (cm^2/s) = $10^{-4}\ m^2/s$ named for the scientist Stokes.

Centi stoke is also popular = $10^{-6}\ m^2/s$.
Kinematic viscosity represents momentum diffusivity. It may be explained by modifying equation
$\boldsymbol{\tau = \mu\ (du/dy) = (\mu/\rho) \times \{d\ (\rho u/dy)\} = \nu \times \{d\ (\rho u/dy)\}}$
Where, d $(\rho u/dy)$ represents momentum flux in the *y* direction.

So, $(\mu/\rho) = \nu$ kinematic viscosity gives the rate of momentum flux or momentum diffusivity.
With increase in temperature kinematic viscosity decreases in the case of liquids and increases in the case of gases. For liquids and gases absolute (dynamic) viscosity is not influenced significantly by pressure. But kinematic viscosity of gases is influenced by pressure due to change in density. In gas flow it is better to use absolute viscosity and density, rather than tabulated values of kinematic viscosity, which is usually for 1 atm.

SURFACE TENSION

Many of us would have seen the demonstration of a needle being supported on water surface without it being wetted. This is due to the surface tension of water.

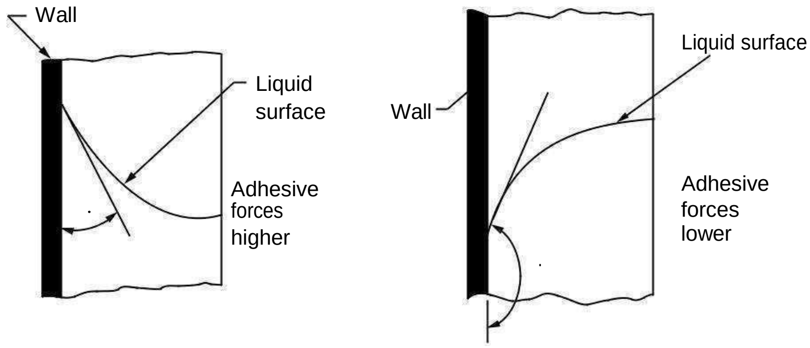
All liquids exhibit a free surface known as meniscus when in contact with vapour or gas. Liquid molecules exhibit cohesive forces binding them with each other. The molecules below the surface are generally free to move within the liquid and they move at random. When they reach the surface they reach a dead end in the sense that no molecules are present in great numbers above the surface to attract or pull them out of the surface. So they stop and return back into the liquid. A thin layer of few atomic thickness at the surface formed by the cohesive bond between atoms slows down and sends back the molecules reaching the surface. This cohesive bond exhibits a tensile strength for the surface layer and this is known as surface tension. Force is found necessary to stretch the surface.

Surface tension may also be defined as the work in Nm/m^2 or N/m required to create unit surface of the liquid. The work is actually required for pulling up the molecules with lower energy from below, to form the surface.

Another definition for surface tension is the force required to keep unit length of the surface film in equilibrium (N/m). The formation of bubbles, droplets and free jets are due to the surface tension of the liquid.

Surface Tension Effect on Solid-Liquid Interface

In liquids cohesive forces between molecules lead to surface tension. The formation of droplets is a direct effect of this phenomenon. So also the formation of a free jet, when liquid flows out of an orifice or opening like a tap. The pressure inside the droplets or jet is higher due to the surface tension.



Liquids also exhibit adhesive forces when they come in contact with other solid or liquid surfaces. At the interface this leads to the liquid surface being moved up or down forming a curved surface. When the adhesive forces are higher the contact surface is lifted up forming a concave surface. Oils, water etc. exhibit such behavior. These are said to be surface wetting. When the adhesive forces are lower, the contact surface is lowered at the interface and a convex surface results as in the case of mercury. Such liquids are called non-wetting.

The angle of contact "β" defines the concavity or convexity of the liquid surface. It can be shown that if the surface tension at the solid liquid interface (due to adhesive forces) is σ_{s1} and if the surface tension in the liquid (due to cohesive forces) is σ_{11} then

$$\mathbf{Cos\beta = [(2\sigma_{s1}/\sigma_{11}) - 1]}$$

At the surface this contact angle will be maintained due to molecular equilibrium. The result of this phenomenon is capillary action at the solid liquid interface. The curved surface creates a pressure differential across the free surface and causes the liquid level to be raised or lowered until static equilibrium is reached.

COMPRESSIBILITY AND BULK MODULUS

Bulk modulus, E_v is defined as the ratio of the change in pressure to the rate of change of volume due to the change in pressure. It can also be expressed in terms of change of density.

$$\mathbf{E_v = -\, dp/(dv/v) = dp/(d\rho/\rho)}$$

Where **dp** is the change in pressure causing a change in volume **dv** when the original volume was **v**. The unit is the same as that of pressure, obviously. Note that $dv/v = -\, d\rho/\rho$.

The negative sign indicates that if dp is positive then dv is negative and vice versa, so that the bulk modulus is always positive (N/m^2). The symbol used in this text for bulk modulus is E_v (K is more popularly used).

This definition can be applied to liquids as such, without any modifications. In the case of gases, the value of compressibility will depend on the process law for the change of volume and will be different for different processes.

The bulk modulus for liquids depends on both pressure and temperature. The value increases with pressure as dv will be lower at higher pressures for the same value of dp. With temperature the bulk modulus of liquids generally increases, reaches a maximum and then decreases. For water the maximum is

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at about 50°C. The value is in the range of 2000 MN/m^2 or 2000 × 10^6 N/m^2 or about 20,000 atm. Bulk modulus influences the velocity of sound in the medium, which equals $(g_o \times E_v/\rho)^{0.5}$.

PRESSURE

Pressure is a measure of force distribution over any surface associated with the force. Pressure is a surface phenomenon and it can be physically visualized or calculated only if the surface over which it acts is specified. Pressure may be defined as the force acting along the normal direction on unit area of the surface. However a more precise definition of pressure, *P* is as below:

$$\text{P} = \lim_{\to a}(\Delta F/\Delta A) = \frac{dF}{d}$$

F is the resultant force acting normal to the surface area A. ‘a’ is the limiting area which will give results independent of the area. This explicitly means that pressure is the ratio of the elemental force to the elemental area normal to it.

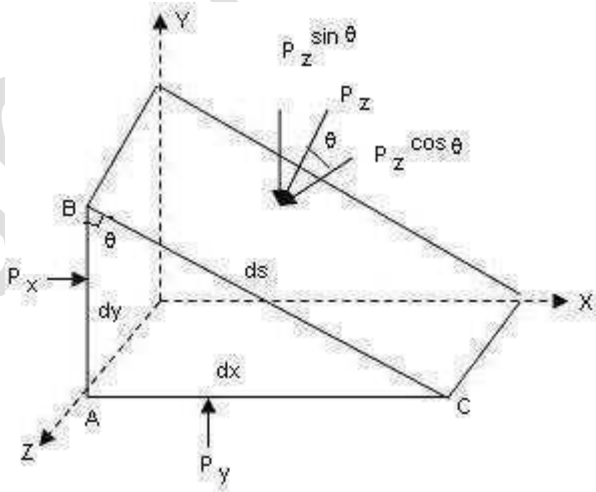
The force *dF* in the normal direction on the elemental area *dA* due to the pressure *P* isdF = P dA

The unit of pressure in the SI system is N/m^2 also called Pascal (Pa). As the magnitude is small kN/m^2 (kPa) and MN/m^2 (MPa) are more popularly used. The atmospheric pressure is approximately 10^5 N/m^2 and is designated as ‘‘bar’’. This is also a popular unit of pressure. In the metric system the popular unit of pressure is kgf/cm^2. This is approximately equal to the atmospheric pressure or 1 bar.

PASCAL’S LAW

In fluids under static conditions pressure is found to be independent of the orientation of the area. This concept is explained by **Pascal’s law** which **states that the pressure at a pointin a fluid at rest is equal in magnitude in all directions**. Tangential stress cannot exist ifa fluid is to be at rest. This is possible only if the pressure at a point in a fluid at rest is the same in all directions so that the resultant force at that point will be zero.

The proof for the statement is given below.



Consider a wedge shaped element in a volume of fluid as shown in Fig. 2.3.1. Let the thickness perpendicular to the paper be **dy**. Let the pressure on the surface inclined at an angle θ**to vertical be**P_θ and its length be **dl**. Let the pressure in the *x*, *y* and *z* directions be P_x, P_y, P_z.

First considering the *x* direction. For the element to be in equilibrium,

$P_\theta \times dl \times dy \times \cos\theta = P_x \times dy \times dz$

But, $dl \times \cos\theta = dz$ So, $P_\theta = P_x$

When considering the vertical components, the force due to specific weight should be considered.

$P_z \times dx \times dy = P_\theta \times dl \times dy \times \sin\theta + 0.5 \times$œ$\times dx \times dy \times dz$

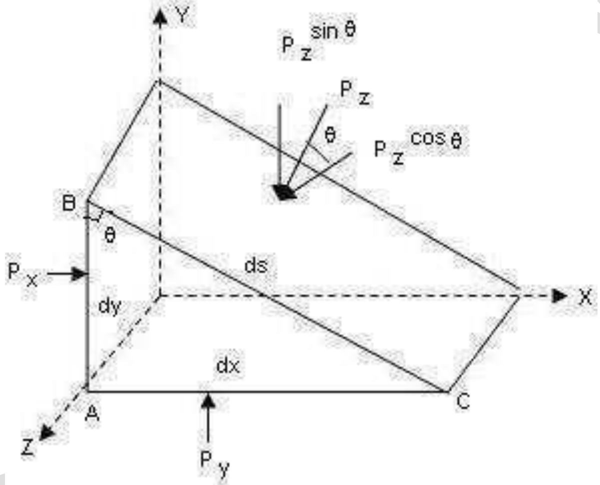
The second term on RHS of the above equation is negligible, its magnitude is one order less compared to the other terms.
Also, $dl \times \sin\theta = dx$
So, $P_z = P_\theta$
Hence, $\mathbf{P_x = P_z = P_\theta}$
Note that the angle has been chosen arbitrarily and so this relationship should hold for all angles. By using an element in the other direction, it can be shown that

$$\mathbf{P_y = P_\theta \quad \text{and} \quad \text{so } P_x = P_y = P_z}$$

Hence, the pressure at any point in a fluid at rest is the same in all directions. The pressure at a point has only one value regardless of the orientation of the area on which it is measured. This can be extended to conditions where fluid as a whole (like a rotating container) is accelerated like in forced vortex or a tank of water getting accelerated without relative motion between layers of fluid. Surfaces generally experience compressive forces due to the action of fluid pressure.

PRESSURE VARIATION IN STATIC FLUID (HYDROSTATIC LAW)

It is necessary to determine the pressure at various locations in a stationary fluid to solve engineering problems involving these situations. **Pressure forces are called surface forces.**
Gravitational force is called body force as it acts on the whole body of the fluid.



Consider an element in the shape of a small cylinder of constant area dA_s along the s direction inclined at angle θ to the horizontal, as shown in Fig. 2.4.1. The surface forces are P at section s and $P + dp$ at section $s + ds$. The surface forces on the curved area are balanced. The body force due to gravity acts vertically and its value is œ × ds × dA_s. A force balance in the s direction (for the element to be in equilibrium) gives
$P \times dA_s - (P + dp) \times dA_s$ – œ × $dA_s \times ds \times \sin\theta = 0$
Simplifying, dp/ds = –œ × sinθ or, dp = –œ × ds × sinθ
This is the fundamental equation in fluid statics.
The variation of specific weight œ with location or pressure can also be taken into account, if these relations are specified as = œ (P, s)
For x axis, θ = 0 and sin θ = 0, $dP/dx = 0$
In a static fluid with no acceleration, the pressure gradient is zero along any horizontal line i.e., planes normal to the gravity direction.
In y direction, θ = 90 and sin θ = 1, dP/dy = –œ = $-\rho g/g_o$
Rearranging and integrating between limits y_1 and y, dP = –œ dy
If œ is constant as in the case of liquids, these being incompressible,
$P - P_1$ = –œ × $(y - y_1) = -\rho\, g\, (y - y_1)/g_o$, (As P_1, y_1 and œ are specified)
For any given situation, P will be constant if y is constant

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This leads to the statement,**The pressure will be the same at the same level in any connected static fluid whose density is constant or a function of pressure only.**
A consequence is that the free surface of a liquid will seek a common level in any container, where the free surface is everywhere exposed to the same pressure.

PRESSURE MEASUREMENT

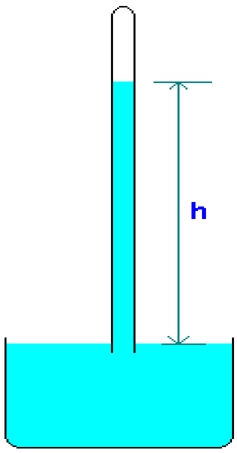
FLUID PRESSURE

In a stationary fluid the pressure is exerted equally in all directions and is referred to as the *static pressure*. In a moving fluid, the static pressure is exerted on any plane parallel to the direction of motion. The fluid pressure exerted on a plane right angles to the direction of flow is greater than the static pressure because the surface has, in addition, to exert sufficient force to bring the fluid to rest. The additional pressure is proportional to the kinetic energy of fluid; it cannot be measured independently of the static pressure.

When the static pressure in a moving fluid is to be determined, the measuring surface must be parallel to the direction of flow so that no kinetic energy is converted into pressure energy at the surface. If the fluid is flowing in a circular pipe the measuring surface must be perpendicular to the radial direction at any point. The pressure connection, which is known as a *piezometer tube*, should flush with the wall of the pipe so that the flow is not disturbed: the pressure is then measured near the walls were the velocity is a minimum and the reading would be subject only to a small error if the surface were not quite parallel to the direction of flow.

The static pressure should always be measured at a distance of not less than 50 diameters from bends or other obstructions, so that the flow lines are almost parallel to the walls of the tube. If there are likely to be large cross-currents or eddies, a *piezometer ring* should be used. This consists of 4 pressure tapings equally spaced at 90^o intervals round the circumference of the tube; they are joined by a circular tube which is connected to the pressure measuring device. By this means, false readings due to irregular flow or avoided. If the pressure on one side of the tube is relatively high, the pressure on the opposite side is generally correspondingly low; with the piezometer ring a mean value is obtained.

BAROMETER



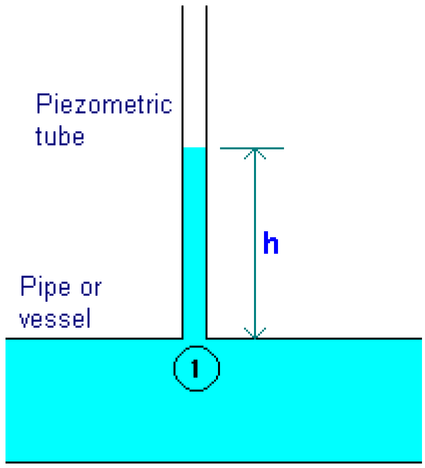
A *barometer* is a device for measuring atmospheric pressure. A simple barometer consists of a tube more than 30 inch (760 mm) long inserted in an open container of mercury with a closed and evacuated end at the top and open tube end at the bottom and with mercury extending from the container up into the tube.

Strictly, the space above the liquid cannot be a true vaccum. It contains mercury vapor at its saturated vapor pressure, but this is extremely small at room temperatures (e.g. 0.173 Pa at 20^{o}C).

The atmospheric pressure is calculated from the relation $P_{atm} = \rho gh$ where ρ is the density of fluid in the barometer.

Piezometer

For measuring pressure inside a vessel or pipe in which liquid is there, a tube may be attached to the walls of the container (or pipe) in which the liquid resides so liquid can rise in the tube. By determining the height to which liquid rises and using the relation $P_1 = \rho gh$, gauge pressure of the liquid can be determined. Such a device is known as *piezometer*. To avoid capillary effects, a piezometer's tube should be about 1/2 inch or greater.It is important that the opening of the device to be tangential to any fluid motion, otherwise an erroneous reading will result.



MANOMETER

A somewhat more complicated device for measuring fluid pressure consists of a bent tube containing one or more liquid of different specific gravities. Such a device is known as *manometer*.

In using a manometer, generally a known pressure (which may be atmospheric) is applied to one end of the manometer tube and the unknown pressure to be determined is applied to the other end.

In some cases, however, the difference between pressures at ends of the manometer tube is desired rather than the actual pressure at the either end. A manometer to determine this differential pressure is known as *differential pressure manometer*.

Manometers – Various Form

(i) Simple U - tube Manometer
(ii) Inverted U - tube Manometer
(iii) U - tube with one leg enlarged
(iv) Two fluid U - tube Manometer
(v) Inclined U - tube Manometer

Simple U - tube Manometer

Equating the pressure at the level XX'(pressure at the same level in a continuous body of fluid is equal),

For the left hand side:

$P_x = P_1 + \rho g(a+h)$

For the right hand side:

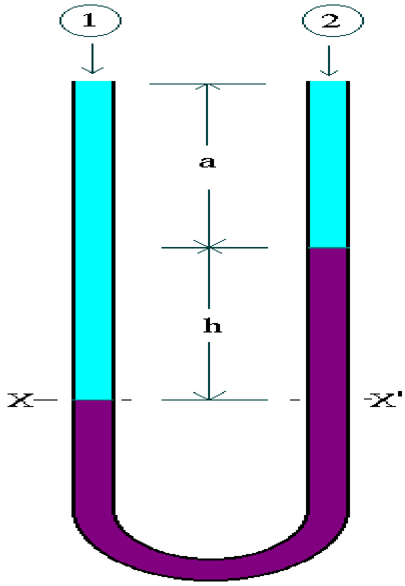
$P_{x'} = P_2 + \rho g a + \rho_m g h$

Since $P_x = P_{x'}$

$P_1 + \rho g(a+h) = P_2 + \rho g a + \rho_m g h$

$P_1 - P_2 = \rho_m g h - \rho g h$

i.e. $P_1 - P_2 = (\rho_m - \rho) g h.$



The maximum value of $P_1 - P_2$ is limited by the height of the manometer. To measure larger pressure differences we can choose a manometer with heigher density, and to measure smaller pressure differences with accuracy we can choose a manometer fluid which is having a density closer to the fluid density.

Inverted U - tube Manometer

Inverted U-tube manometer is used for measuring pressure differences in liquids. The space above the liquid in the manometer is filled with air which can be admitted or expelled through the tap on the top, in order to adjust the level of the liquid in the manometer.

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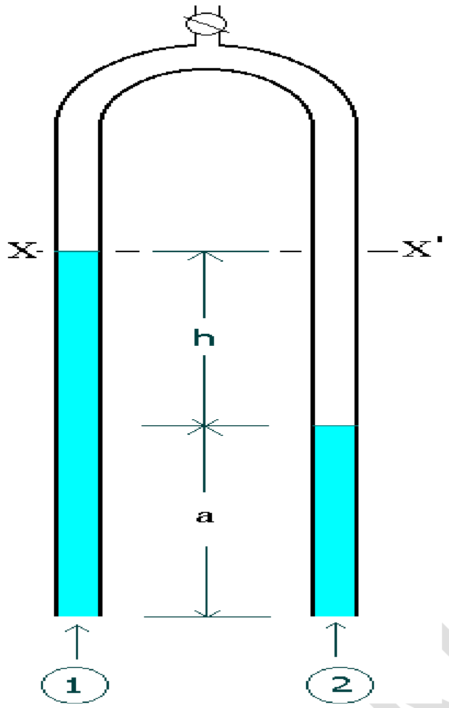

Equating the pressure at the level XX'(pressure at the same level in a continuous body of static fluid is equal),

For the left hand side:

$P_x = P_1 - \cdot g(h+a)$
For the right hand side:
$P_x{}' = P_2 - (\cdot ga + \cdot_m gh)$
Since $P_x = P_{x'}$
$P_1 - \cdot g(h+a) = P_2 - (\cdot ga + \cdot_m gh)$
$P_1 - P_2 = (\cdot - \cdot_m)gh$
If the manometric fluid is choosen in such a way that $\cdot_m << \cdot$ then,
$P_1 - P_2 = \cdot gh.$

For inverted U - tube manometer the manometric fluid is usually air.

U - Tube Manometer with one leg enlarged

Industrially, the simple U - tube manometer has the disadvantage that the movement of the liquid in both the limbs must be read. By making the diameter of one leg large as compared with the other, it is possible to make the movement the large leg very small, so that it is only necessary to read the movement of the liquid in the narrow leg.

In figure, OO' represents the level of liquid surface when the pressure difference $P_1 - P_2$ is zero. Then when pressure is applied, the level in the right hand limb will rise a distance h vertically.

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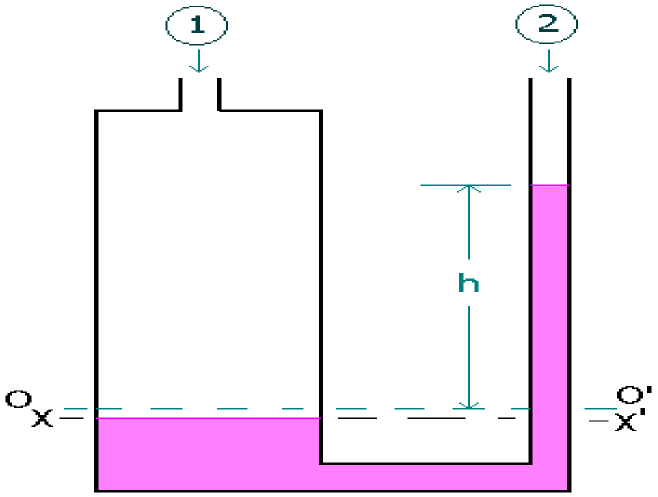


Volume of liquid transferred from left-hand leg to right-hand leg = $h(\pi/4)d^2$

Where d is the diameter of smaller diameter leg. If D is the diameter of larger diameter leg, then, fall in level of left-hand leg= Volume transferred/Area of left-hand leg = $(h(\pi/4)d^2) / ((\pi/4)D^2) = h(d/D)^2$

For the left-hand leg, pressure atX, i.e. $P_x = P_1 + \rho g(h+a) + \rho g\, h(d/D)^2$

For the right-hand leg, pressure at X', i.e. $P_{x'} = P_2 + \rho ga + \rho g(h + h(d/D)^2)$

For the equality of pressure at XX',

$P_1 + \rho g(h+a) + \rho g\, h(d/D)^2 = P_2 + \rho ga + \rho_m g(h + h(d/D)^2)$

$P_1 - P_2 = \rho_m g(h + h(d/D)^2) - \rho gh - \rho g\, h(d/D)^2$

If D>>d then, the term $h(d/D)^2$ will be negligible(i.e approximately about zero)

Then $P_1 - P_2 = (\rho_m - \rho)gh$.

Where h is the manometer liquid rise in the right-hand leg.

If the fluid density is negligible compared with the manometric fluid density (e.g. the case for air as the fluid and water as manometric fluid), then $P_1 - P_2 = \rho_m g h$.

Manometers - Advantages and Limitations

- The manometer in its various forms is an extremely useful type of pressure measuring instrument, but suffers from a number of limitations.
- While it can be adapted to measure very small pressure differences, it cannot be used conveniently for large pressure differences - although it is possible to connect a number of manometers in series and to use mercury as the manometric fluid to improve the range. (limitation)
- A manometer does not have to be calibrated against any standard; the pressure difference can be calculated from first principles. (Advantage)
- Some liquids are unsuitable for use because they do not form well-defined menisci. Surface tension can also cause errors due to capillary rise; this can be avoided if the diameters of the tubes are sufficiently large - preferably not less than 15 mm diameter. (limitation)
- A major disadvantage of the manometer is its slow response, which makes it unsuitable for measuring fluctuating pressures.(limitation)
- It is essential that the pipes connecting the manometer to the pipe or vessel containing the liquid under pressure should be filled with this liquid and there should be no air bubbles in the liquid.(important point to be kept in mind)

STABILITY OF FLOATING BODY

If an object is immersed in or floated on the surface of fluid under static conditions a force acts on it due to the fluid pressure. This force is called **buoyant force**. The calculation of this force is based on Archimedes principle.

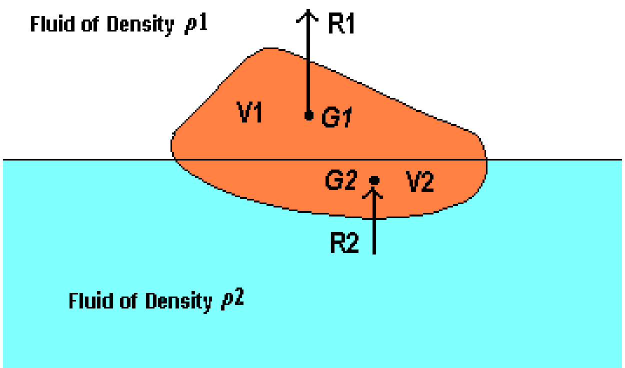
Archimedes principle can be stated as

- *(i)* **A body immersed in a fluid is buoyed upby a force equal to the weight of the fluid displaced**
- *(ii)* **A floating body displaces its own weight of the liquid in which it floats.**

Other possible statements are: The resultant pressure force acting on the surface of a volume partially or completely surrounded by one or more fluids under non flow conditions is defined as **buoyant force and acts vertically on the volume.** The buoyant force is equal to the weight of the displaced fluid and **acts upwards through the centre of gravity of thedisplaced fluid.** This point is **called the centre of buoyancy** for the body.

This principle directly follows from the general hydrostatic equation, *F* = œ*Ah* and is applied in the design of ships, boats, balloons and other such similar systems. The stability of such bodies against tilting over due to small disturbance can be also checked using this principle.

BUYOUNCY:*Up thrust on body = weight of fluid displaced by the body*



If the body is immersed so that part of its volume V1 is immersed in a fluid of density $\cdot_1$ and the rest of its volume V2 in another immiscible fluid of mass density $\cdot_2$,
Up thrust on upper part, R1 = $\cdot_1$gV1 acting through G1, the centroid of V1,
Up thrust on lower part, R2 = $\cdot_2$gV2 acting through G2, the centroid of V2,
Total up thrust = $\cdot_1$gV1 + $\cdot_2$gV2.
The positions of G1 and G2 are not necessarily on the same vertical line, and the Centre of buoyancy of the whole body is, therefore, not bound to pass through the centroid of the whole body.

TOTAL HYDROSTATIC FORCE ON SURFACE

TOTAL HYDROSTATIC FORCE ON PLANE SURFACE

For horizontal plane surface submerged in liquid, or plane surface inside a gas chamber, or any plane surface under the action of uniform hydrostatic pressure, the total hydrostatic force is given by

F=p A

Where *p* is the uniform pressure and *A* is the area.
In general, the total hydrostatic pressure on any plane surface is equal to the product of the area of the surface and the unit pressure at its center of gravity.

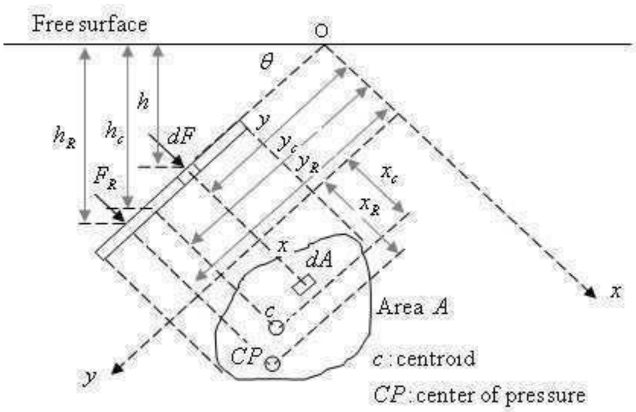
F=Pcg A

Where p_{cg} is the pressure at the center of gravity. For homogeneous free liquid at rest, the equation can be expressed in terms of unit weight *γ* of the liquid.

F=œh¯A

Where h¯ is the depth of liquid above the centroid of the submerged area.

The figure shown below is an inclined plane surface submerged in a liquid. The total area of the plane surface is given by *A*, *cg* is the center of gravity, and *cp* is the center of pressure.



The differential force dF acting on the element dA is
dF=pdA
dF=œhdA

From the figure h=ysinθ
dF=œ(ysinθ)dA

Integrate both sides and note that œ and θ are constants,
F=œsinθ ∫ydA

Recall from Calculus that ∫ydA=Ay¯
F=(œsinθ)Ay¯
F=œ(y¯sinθ)A

From the figure, y¯sinθ=h¯ thus,
F=œh¯A

The product œh¯œh¯ is a unit pressure at the centroid at the plane area, thus, the formula can be expressed in a more general term below.

F=Pcg A

Location of Total Hydrostatic Force (Eccentricity)

From the figure above, *S* is the intersection of the prolongation of the submerged area to the free liquid surface. Taking moment about point *S*.

Fyp=∫ydF

Where
dF=œ(ysinθ)dA
F=œ(y¯sinθ)A

[œ(y¯sinθ)A]y_p=∫y[œ(ysinθ)dA]

(œsinθ)Ay¯y_p=(œsinθ)∫y^2dA

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$A\bar{y}y_p=\int y^2 dA$

Again from Calculus, $\int y^2 dA \int y^2 dA$ is called moment of inertia denoted by I Since our reference point is S,

$A\bar{y}y_p=I_S$

Thus,

$y_p = \frac{Is}{\bar{y}}$

By transfer formula for moment of inertia $Is = Ig + A\tilde{y}^2$, the formula for y_p, will become

$$Is = \frac{I+ \quad \tilde{y}^2}{\tilde{y}}$$

TOTAL HYDROSTATIC FORCE ON CURVED SURFACE

In the case of curved surface submerged in liquid at rest, it is more convenient to deal with the horizontal and vertical components of the total force acting on the surface.

Horizontal Component
F_H=Pcg A

Vertical Component

The vertical component of the total hydrostatic force on any surface is equal to the weight of either real or imaginary liquid above it.

FV= œ V
Total Hydrostatic force = $\sqrt{FH^2 + FV^2}$

Direction of F will be tan θ = $\frac{FH}{FV}$

STABILITY OF FLOATING AND SUBMERGED BODIES

There are three possible situations for a body when immersed in a fluid.

(i) If the **weight of the body is greater** than the weight of the liquid of equal volume then the **body will sink into** the liquid (To keep it floating additional upward force is required).

(ii) If the **weight of the body equals** the weight of equal volume of liquid, then the body will submerge and may **stay at any location** below the surface.

(iii) If the **weight of the body is less** than the weight of equal volume of liquid, then the body will be partly submerged and **will float in** the liquid.

Comparison of densities cannot be used directly to determine whether the body will float or sink unless the body is solid over the full volume like a lump of iron. However the apparent density calculated by the ratio of weight to total volume can be used to check whether a body will float or sink. If apparent density is higher than that of the liquid, the body will sink. If these are equal, the body will stay afloat at any location. If it is less, the body will float with part above the surface.

A submarine or ship though made of denser material floats because, the weight/volume of the ship will be less than the density of water. In the case of submarine its weight should equal the weight of water displaced for it to lay submerged.

Stability of a body: A ship or a boat should not overturn due to small disturbances butshould be stable and return, to its original position. Equilibrium of a body exists when there is no resultant force or moment on the body. A body can stay in three states of equilibrium.

(i) **Stable equilibrium: Small disturbances will create a correcting couple** and the body will go back to its original position prior to the disturbance.
(ii) **Neutral equilibrium Small disturbances do not create any additional force** and so the body remains in the disturbed position. No further change in position occurs in this case.
(iii) **Unstable equilibrium:** A small **disturbance creates a couple which acts toincrease the disturbance** and the body may tilt over completely.

Under equilibrium conditions, two forces of equal magnitude acting along the same line of action, but in the opposite directions exist on a floating/submerged body. These are the gravitational force on the body (weight) acting downward along the centroid of the body and buoyant force acting upward along the centroid of the displaced liquid. Whether floating or submerged, under equilibrium conditions these two forces are equal and opposite and act along the same line.

When the position of the body is disturbed or rocked by external forces (like wind on a ship), the position of the centre of gravity of the body **(with respect to the body)** remains at the same position. But the shape of the displaced volume of liquid changes and so its centre of gravity shifts to a new location. Now these two forces constitute a couple which may correct the original tilt or add to the original tilt. If the couple opposes the movement, then the body will regain or go back to the original position. If the couple acts to increase the tilt then the body becomes unstable.

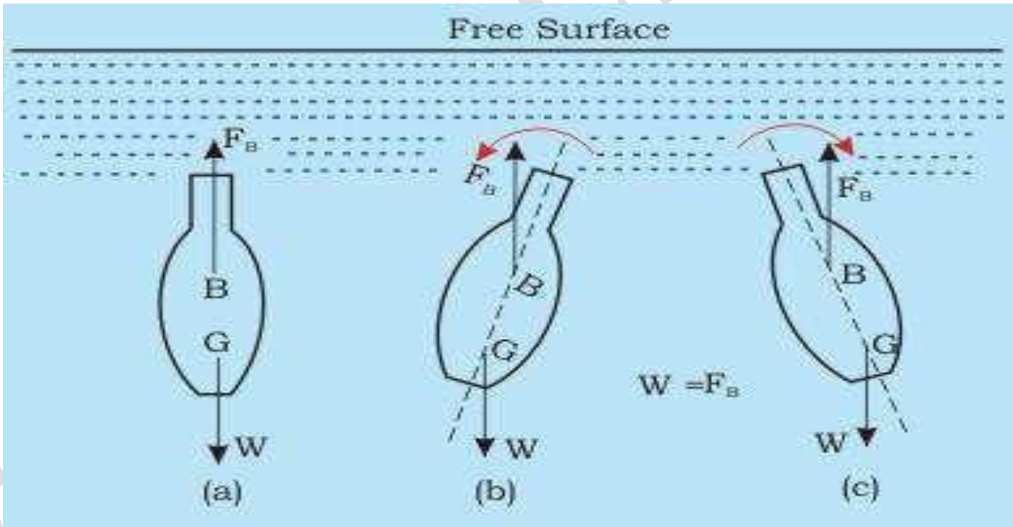


Figure: Stability of Floating and Submerged Bodies

CONDITIONS FOR THE STABILITY OF FLOATING BODIES

(i) **When the centre of buoyancy is above the centre of gravity of the floatingbody, the body is always stable** under all conditions of disturbance. A righting couple isalways created to bring the body back to the stable condition.
(ii) **When the centre of buoyancy coincides with the centre of gravity,** the two forces act at the same point. A disturbance does not create any couple and so **the body justremains in the disturbed position.** There is no tendency to tilt further or to correct the tilt.
(iii) **When the centre of buoyancy is below the centre of gravity** as in the case of ships, **additional analysis is required to establish stable conditions** of floating.

This involves the **concept of metacenter and metacentric height.** When the body is disturbed the centre of gravity still remains on the centroidal line of the body. The shape of the displaced volume changes and the centre of buoyancy moves from its previous position.

The location *M* at which the line of action of buoyant force meets the centroidal axis of the body, when disturbed, is defined as **metacenter.** The distance of this point from the centroid of the body is called metacentric height.

If the metacenter is above the centroid of the body, the floating body will be stable. If it is at the centroid, the floating body will be in neutral equilibrium. If it is below the centroid, the floating body will be unstable.

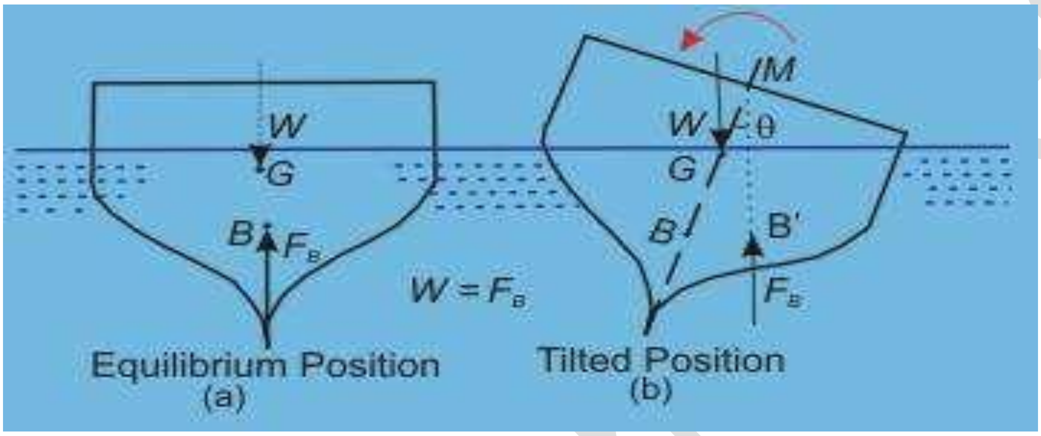


Figure: Metacenter Height and stable condition

When a small disturbance occurs, say clockwise, then the centre of gravity moves to the right of the original centre line. The shape of the liquid displaced also changes and the centre of buoyancy also generally moves to the right. If the distance moved by the centre of buoyancy is larger than the distance moved by the centre of gravity, the resulting couple will act anticlockwise, correcting the disturbance. If the distance moved by the centre of gravity is larger, the couple will be clockwise and it will tend to increase the disturbance or tilting.

The distance between the metacenter and the centre of gravity is known is **metacentricheight**. The magnitude of the righting couple is directly proportional to the metacentric height.Larger the metacentric height, better will be the stability.

The centre of gravity *G* is above the centre of buoyancy *B*. After a small clockwise tilt, the centre of buoyancy has moved to *B'*. The line of action of this force is upward and it meets the body centre line at the metacenter *M* which is above *G*. In this case metacentric height is positive and the body is stable. It may also be noted that the couple is anticlockwise. If *M* falls below *G*, then the couple will be clockwise and the body will be unstable.

METACENTER HEIGHT

In equilibrium $F_B = w$

Due to symmetry of the situation, displaced volume remains unaltered and hence the buoyancy force

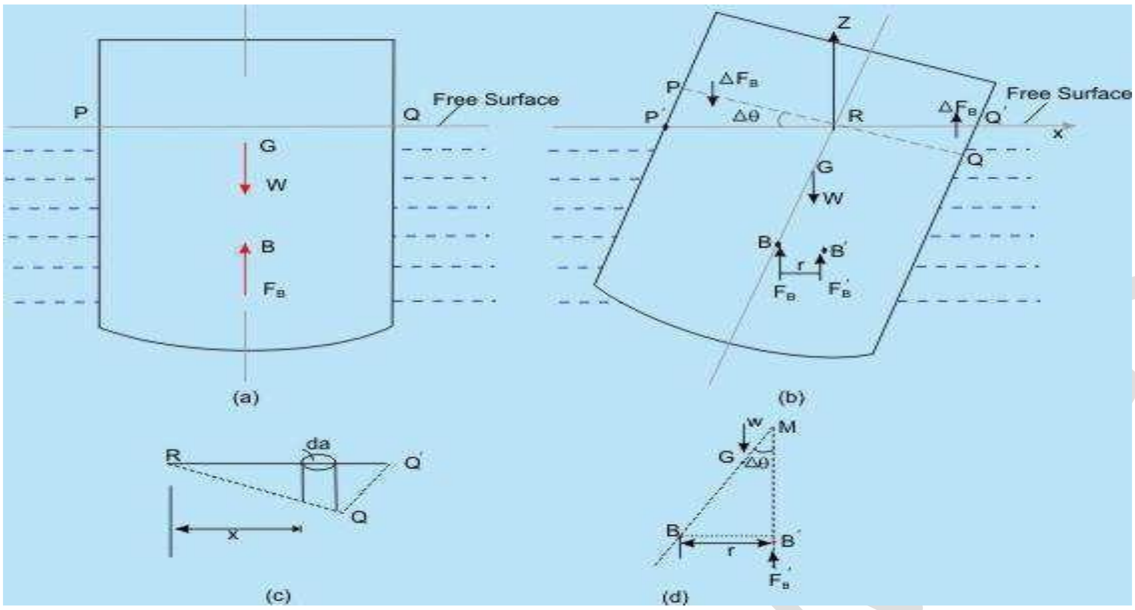
Thus, $F'_B = F_B$

For couple calculation, F_B can be equivalently taken as a sum of

ΔF_B (Upward) due to added volume of fluid RQQ'

ΔF_B (Downward) due to decreased volume RPP'

Let DC be the couple due to these forces. Taking an element of area dA on the surface RQQ' at a distance x from the center line. Corresponding volume element is x DθdA and the buoyant force is ρgxDθ dA. This produces the couple 2ρgx DθdA (due to symmetrically located element on ‘P’).



Therefore, Integrating

$$\Delta C = 2\rho g \Delta\theta \int_{areaRQ'} x^2 dA = \rho g \, \Delta\theta \int_{free\ surface} x^2 \, dA = \rho g \, \Delta\theta \, I_{yy}$$

Now we have $F'_B r = wr = \Delta c$ where r is moment arm of F'_B about B.

Therefore, $wr = \rho g \Delta\theta \, I_{yy} \Rightarrow r = \dfrac{\rho g \, \Delta\theta \, I_{yy}}{w}$

Now from fig (d) $MB = \dfrac{r}{\sin \Delta\theta} = \dfrac{\rho g \, \Delta\theta \, I_{yy}}{w \sin \Delta\theta} = \dfrac{\rho g \, I_{yy}}{w} \quad as\ \Delta\theta \to 0$

Thus $MG + BG = \dfrac{\rho g \, I_{yy}}{w} \Rightarrow MG = \dfrac{\rho g \, I_{yy}}{w} - BG$

This is the metacentric height.

FLUID MECHANICS (ME – 404)
UNIT – 2 (KINEMATICS OF FLOW)

INTRODUCTION

A flow field is a region in which the flow is defined at all points at any instant of time. The means to that is to define the velocities at all the points at different times. It should be noted that the velocity at a point is the velocity of the fluid particle that occupies that point. In order to obtain a complete picture of the flow the fluid motion should be described mathematically. Just like the topography of a region is visualized using the contour map, the flow can be visualized using the velocity at all points at a given time or the velocity of a given particle at different times. It is then possible to also define the potential causing the flow.

Application of a shear force on an element or particle of a fluid will cause continuous deformation of the element. Such continuing deformation will lead to the displacement of the fluid element from its location and this results in fluid flow. The fluid element acted on by the force may move along a steady regular path or randomly changing path depending on the factors controlling the flow. The velocity may also remain constant with time or may vary randomly. In some cases the velocity may vary randomly with time but the variation will be about a mean value. It may also vary completely randomly as in the atmosphere. The study of the velocity of various particles in the flow and the instantaneous flow pattern of the flow field is called flow kinematics or hydrodynamics. Such a study is generally limited to ideal fluids, fluids which are incompressible and inviscid. In real fluid shows, beyond a certain distance from the surfaces, the flow behaves very much like ideal fluid. Hence these studies are applicable in real fluid flow also with some limitations.

LAGRANGIAN AND EULARIAN METHODS OF STUDY OF FLUID FLOW

In the Lagrangian method a single particle is followed over the flow field, the co-ordinate system following the particle. The flow description is particle based and not space based. A moving coordinate system has to be used. This is equivalent to the observer moving with the particle to study the flow of the particle. This method is more involved mathematically and is used mainly in special cases.

In the Eularian method, the description of flow is on fixed coordinate system based and the description of the velocity etc. are with reference to location and time *i.e.*, $V = V\ (x, y, z, t)$ and not with reference to a particular particle. Such an analysis provides a picture of various parameters at all locations in the flow field at different instants of time. This method provides an easier visualisation of the flow field and is popularly used in fluid flow studies. However the final description of a given flow will be the same by both the methods.

BASIC SCIENTIFIC LAWS USED IN THE ANALYSIS OF FLUID FLOW

Law of conservation of mass: This law when applied to a control volume statesthat the net mass flow through the volume will equal the mass stored or removed from the volume. Under conditions of steady flow this will mean that the mass leaving the control volume should be equal to the mass entering the volume. The determination of flow velocity for a specified mass flow rate and flow area is based on the continuity equation derived on the basis of this law.

Newton's laws of motion: These are basic to any force analysis under variousconditions of flow. The resultant force is calculated using the condition that it equals the rate of change of momentum. The reaction on surfaces are calculated on the basis of these laws. Momentum equation for flow is derived based on these laws.

Law of conservation of energy: Considering a control volume the law can bestated as "the energy flow into the volume will equal the energy flow out of the volume under steady conditions". This also leads to the situation that the total energy of a fluid element in a steady flow field is conserved. This is the basis for the derivation of Euler and Bernoulli equations for fluid flow.

Thermodynamic laws: are applied in the study of flow of compressible fluids.

FLOW OF IDEAL / INVISCID AND REAL FLUIDS

Ideal fluid is no viscous and incompressible. Shear force between the boundary surface and fluid or between the fluid layers is absent and only pressure forces and body forces are controlling.

Real fluids have viscosity and surface shear forces are involved during flow. However the flow after a short distance from the surface is not affected by the viscous effects and approximates to ideal fluid flow. The results of ideal fluid flow analysis are found applicable in the study of flow of real fluids when viscosity values are small.

STEADY AND UNSTEADY FLOW

In order to study the flow pattern it is necessary to classify the various types of flow. The classification will depend upon the constancy or variability of the velocity with time. In the next three sections, these are described. In steady flow the property values at a location in the flow are constant and the values do not vary with time. The velocity or pressure at a point remains constant with time. These can be expressed as $V = V(x, y, z)$, $P = P(x, y, z)$ etc. In steady flow a picture of the flow field recorded at different times will be identical. In the case of unsteady flow, the properties vary with time or $V = V(x, y, z, t)$, $P = P(x, y, z, t)$ where t is time.

In unsteady flow the appearance of the flow field will vary with time and will be constantly changing. In turbulent flow the velocity at any point fluctuates around a mean value, but the mean value at a point over a period of time is constant. For practical purposes turbulent flow is considered as steady flow as long as the mean value of properties do not vary with time.

COMPRESSIBLE AND INCOMPRESSIBLE FLOW

If the density of the flowing fluid is the same all over the flow field at all times, then such flow is called incompressible flow. Flow of liquids can be considered as incompressible even if the density varies a little due to temperature difference between locations. Low velocity flow of gases with small changes in pressure and temperature can also be considered as incompressible flow. Flow through fans and blowers is considered incompressible as long as the density variation is below 5%. If the density varies with location, the flow is called compressible flow. In this chapter the study is mainly on incompressible flow.

LAMINAR AND TURBULENT FLOW

If the flow is smooth and if the layers in the flow do not mix macroscopically then the flow is called laminar flow. For example a dye injected at a point in laminar flow will travel along a continuous smooth line without generally mixing with the main body of the fluid. Momentum, heat and mass transfer between layers will be at molecular level of pure diffusion. In laminar flow layers will glide over each other without mixing.

In turbulent flow fluid layers mix macroscopically and the velocity/temperature/mass concentration at any point is found to vary with reference to a mean value over a time period. For example $u = \cancel{u} + u'$

where u is the velocity at an instant at a location and $\bar{u}$ is the average velocity over a period of time at that location and u' is the fluctuating component. This causes higher rate of momentum/heat/mass transfer. A dye injected into such a flow will not flow along a smooth line but will mix with the main stream within a short distance.

The difference between the flows can be distinguished by observing the smoke coming out of an incense stick. The smoke in still air will be found to rise along a vertical line without mixing. This is the laminar region. At a distance which will depend on flow conditions the smoke will be found to mix with the air as the flow becomes turbulent. Laminar flow will prevail when viscous forces are larger than inertia forces. Turbulence will begin where inertia forces begin to increase and become higher than viscous forces.

CONCEPTS OF UNIFORM FLOW, REVERSIBLE FLOW AND THREE DIMENSIONAL FLOW

If the velocity value at all points in a flow field is the same, then the flow is defined as uniform flow. The velocity in the flow is independent of location. Certain flows may be approximated as uniform flow for the purpose of analysis, though ideally the flow may not be uniform.

If there are no pressure or head losses in the fluid due to frictional forces to be overcome by loss of kinetic energy (being converted to heat), the flow becomes reversible. The fluid can be restored to its original condition without additional work input. For a flow to be reversible, no surface or fluid friction should exist. The flow in a venturi (at low velocities) can be considered as reversible and the pressures upstream and downstream of the venturi will be the same in such a case. The flow becomes irreversible if there are pressure or head losses.

If the components of the velocity in a flow field exist only in one direction it is called one dimensional flow and $V = V(x)$. Denoting the velocity components in x, y and z directions as u, v and w, in one dimensional flow two of the components of velocity will be zero. In twodimensional flow one of the components will be zero or $V = V(x, y)$. In three dimensional flow all the three components will exist and $V = V(x, y, z)$. This describes the general steady flow situation. Depending on the relative values of u, v and w approximations can be made in the analysis. In unsteady flow $V = V(x, y, z, t)$.

VELOCITY AND ACCELERATION COMPONENTS

Two basic and important field variables in the study of fluid mechanics are the velocity and acceleration of the fluid, and they are the focus of the discussion in this section. Both the velocity and acceleration equations are presented in Eulerian viewpoint.

VELOCITY FIELD

Velocity is an important basic parameter governing a flow field. Other field variables such as the pressure and temperature are all influenced by the velocity of the fluid flow. In general, velocity is a function of both the location and time. The velocity vector can be expressed in Cartesian coordinates as

$$V = V(x,y,z,t)$$
$$= u(x,y,z,t)\, i + v(x,y,z,t) j + w(x,y,z,t)\, k$$

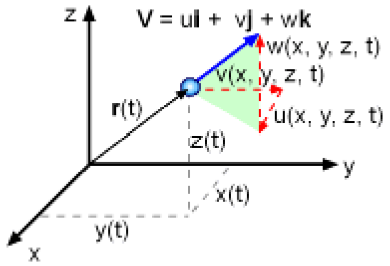


Figure: Velocity Component

Where the velocity components (u, v and w) are functions of both position and time. That is, u = u(x, y, z, t), v = v(x, y, z, t) and w = w(x, y, z, t). This velocity description is called the velocity field since it describes the velocity of all points, Eulerian viewpoint, in a given volume.

For a single particle, Lagrangian viewpoint, the velocity is derived from the changing position vector, or

V = d**r**/dt = = dx/dt **i** + dy/dt **j** + dz/dt **k**

Where **r** is the position vector (**r** = x**i** + y**j** + z**k**). Notice, the velocity is only a function of time since it is only tracking a single particle.

ACCELERATION FIELD

Another important parameter in the study of fluid in motion is the acceleration. Acceleration is related to the velocity, and it can be determined once the velocity field is known. The acceleration is the change in velocity, δ**V**, over the change in time, δt,

a = [d**V**(t + δ) - d**V**(t)] / δt = δ**V**/δt = d**V**/dt

But it is not just a simple derivative of just time since the velocity is a function time, AND space (x,y,z). The change in velocity must be track in both time and space. Using the chain rule of calculus, the change in velocity is,

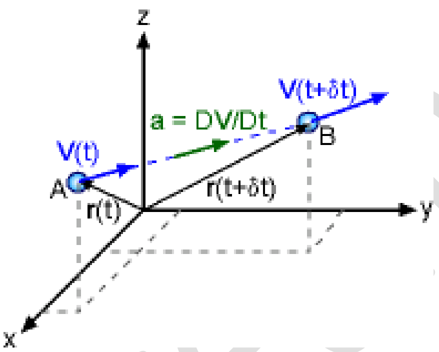


Figure: Acceleration Field

$$d\mathbf{V} = \frac{\partial \mathbf{V}}{\partial x}dx + \frac{\partial \mathbf{V}}{\partial y}dy + \frac{\partial \mathbf{V}}{\partial z}dz + \frac{\partial \mathbf{V}}{\partial t}dt$$

Or

$$\frac{d\mathbf{V}}{dt} = \frac{\partial \mathbf{V}}{\partial x}\frac{dx}{dt} + \frac{\partial \mathbf{V}}{\partial y}\frac{dy}{dt} + \frac{\partial \mathbf{V}}{\partial z}\frac{dz}{dt} + \frac{\partial \mathbf{V}}{\partial t}$$

This can simplified using u, v, and w, the velocity magnitudes in the three coordinate directions. In Cartesian coordinates, the acceleration field is:

$$\mathbf{a} = \frac{\partial \mathbf{V}}{\partial t} + u\frac{\partial \mathbf{V}}{\partial x} + v\frac{\partial \mathbf{V}}{\partial y} + w\frac{\partial \mathbf{V}}{\partial z}$$

This expression can be expanded and rearranged as

$$\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$$

$$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

$$a_y = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

$$a_z = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$

The acceleration equation can also be written in polar coordinates and are given in the Basic Equations appendix.

MATERIAL DERIVATIVE

The time and space derivative used to determine the acceleration field from the velocity is so common in fluid mechanics, it has a special name. It is called the Material or Substantial Derivative and has a special symbol, D ()/Dt. For Cartesian coordinates, it is

$$\frac{D(\,)}{Dt} = \frac{\partial(\,)}{\partial t} + u\frac{\partial(\,)}{\partial x} + v\frac{\partial(\,)}{\partial y} + w\frac{\partial(\,)}{\partial z}$$

Or in vector form,

$$\frac{D(\,)}{Dt} = \frac{\partial(\,)}{\partial t} + \mathbf{V}\cdot\nabla(\,)$$

Where ∇ is the gradient operator from the above equation, it can be seen that the material derivative consists of two terms. The first term ∂t ( )/∂t is referred to as the local rate of change, and it represents the effect of unsteadiness. For steady flow, the local derivative vanishes (i.e., ∂t ( )/∂t = 0).

The second term, $\mathbf{V}\cdot\nabla(\,)$, is referred to as the convective rate of change, and it represents the variation due to the change in the position of the fluid particle, as it moves through a field with a gradient. If there is no gradient (no spatial change) then $\nabla(\,)$ is zero so there is no convective change.As an example, the acceleration field equation can be written as

$$a = \frac{D\mathbf{V}}{Dt} = \frac{\partial\mathbf{V}}{\partial t} + \mathbf{V}\cdot\nabla\mathbf{V}$$

CONTINUITY EQUATION FOR FLOW—CARTESIAN CO-ORDINATES

The continuity equation is an expression of a fundamental conservation principle, namely, that of mass conservation. It is a statement that fluid mass is conserved: all fluid particles that flow into any fluid region must flow out. To obtain this equation, we consider a cubical control volume inside a fluid. Mass conservation requires that the net flow through the control volume is zero. In other words, all fluid that is accumulated inside the control volume (due to compressibility for example) + all fluid that is flowing into the control volume must be equal to the amount of fluid flowing out of the control volume. Accumulation + Flow In = Flow Out.

Consider a fluid element of lengths dx, dy, dz in the direction of x, y, z.

Let u, v, w are the inlet velocity components in x, y, z direction respectively.

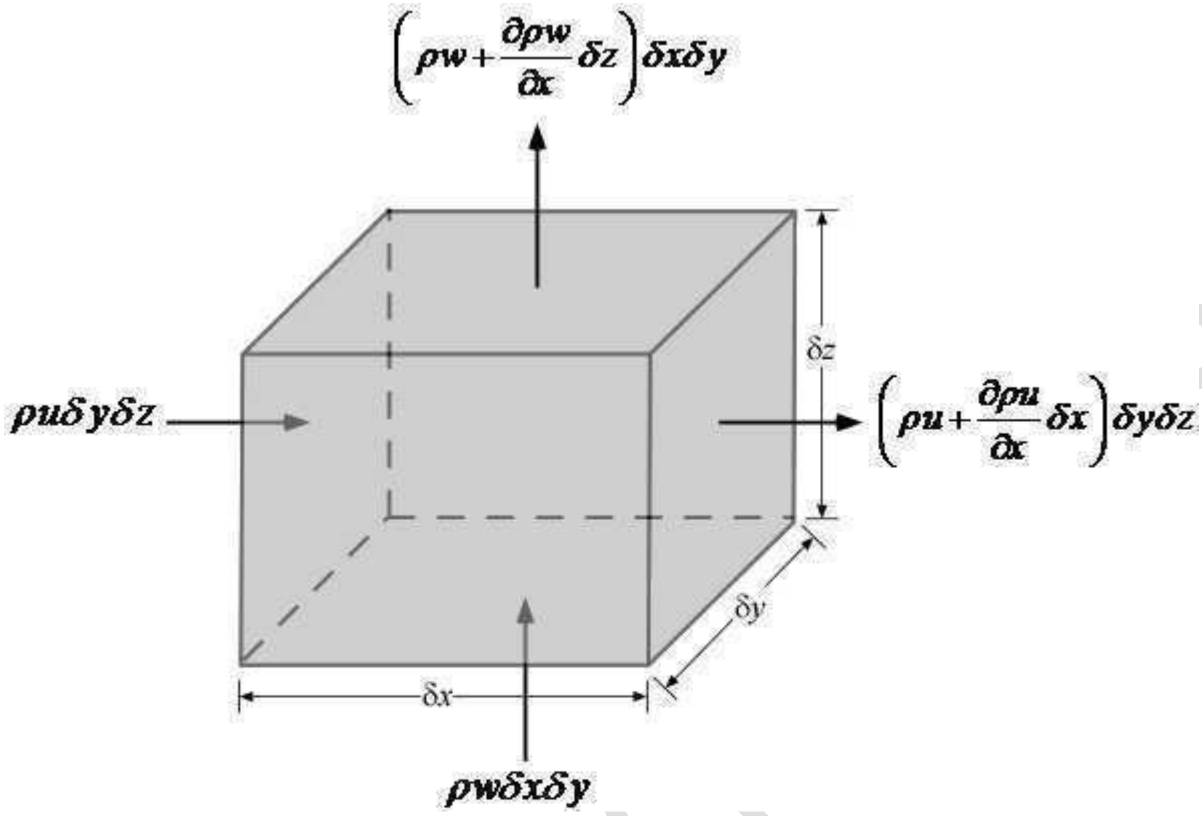
Let, p is mass density of fluid element at particular instate.

Mass of fluid entering the face ABCD (In flow)= Mass density x Velocity x-direction x area of ABCD

= p x u x (∂y x ∂z)

Then mass of fluid leaving the face EFGH (out flow) = (pu ∂y. ∂z) + ∂ / ∂x (pu∂y∂z)

Rate of increases in mass x-direction = Outflow – Inflow

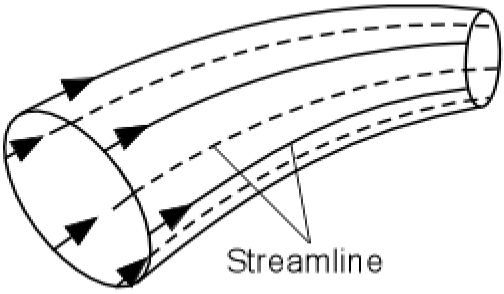


= [(pudydx) + ∂ / ∂_x (pudydz) d_x] - (pudydz)
Rate of increases in mass x direction = ∂ / ∂_x p u d_x d_y d_z
Similarly,
Rate of increase in mass y-direction = ∂ / ∂y p v ∂x ∂y ∂z
Rate of increases in mass z-direction = ∂ / ∂_z p w ∂_x∂y ∂_z
Total rate of increases in mass = ∂_x ∂_y ∂_z [∂p u / ∂_x + ∂p v / ∂_y + ∂p w / ∂_z]
By law of conservation of mass, there is no accumulation of mass, and hence the above quantity must be zero.
∂x. ∂y. ∂z [∂pu / ∂_x + ∂pv / ∂_y + ∂pw / ∂_z] = 0
∂ (pu) / ∂_x +∂ (pv) / ∂_y + ∂ (pw) / ∂_z = 0 (for compressible fluid)
If fluid is incompressible, then is constant
∂_u / ∂_x + ∂_v / ∂_y + ∂_w / ∂_z = 0
This is the continuity equation for three-dimensional flow.

STREAM LINES, STREAM TUBE, PATH LINES, STREAK LINES ANDTIME LINES

The analytical description of flow velocity is geometrically depicted through the concept of stream lines. The velocity vector is a function of both position and time. If at a fixed instant of time a curve is drawn so that it is tangent everywhere to the velocity vectors at these locations then the curve is called a stream line. Thus stream line shows the mean direction of a number of particles in the flow at the same instant of time. **Stream lines are a series of curvesdrawn tangent to the mean velocity vectors of a number of particles in the flow. Since stream lines are tangent to the velocity vector at every point in the flow field, there can be no flow across a stream line.**

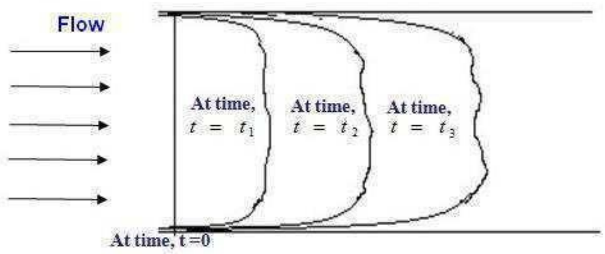
A bundle of neighboring stream lines may be imagined to form a passage through which the fluid flows. Such a passage is called a stream tube. Since the stream tube is bounded on all sides by stream lines, there can be no flow across the surface. Flow can be only through the ends.



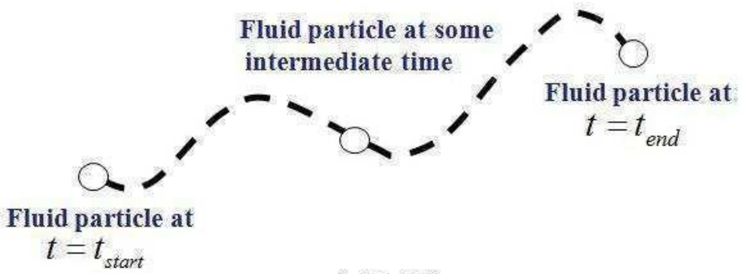
Under steady flow condition, the flow through a stream tube will be constant along the length.

Path line is the trace of the path of a single particle over a period of time. Pathline shows the direction of the velocity of a particle at successive instants of time. In steady flow path lines and stream lines will be identical.

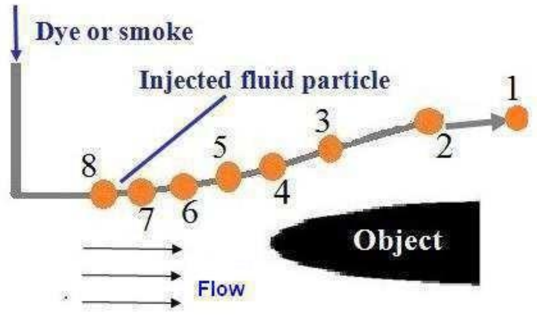
Streak lines provide an instantaneous picture of the particles, which have passed through a given point like the injection point of a dye in a flow. In steady flowthese lines will also coincide with stream lines.



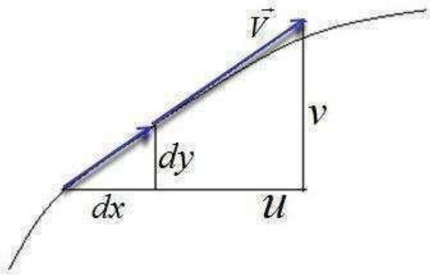
(a) Timelines



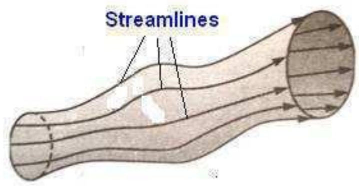
(b) Pathline



(c) Streakline



(d) Streamline



(e) Streamtube

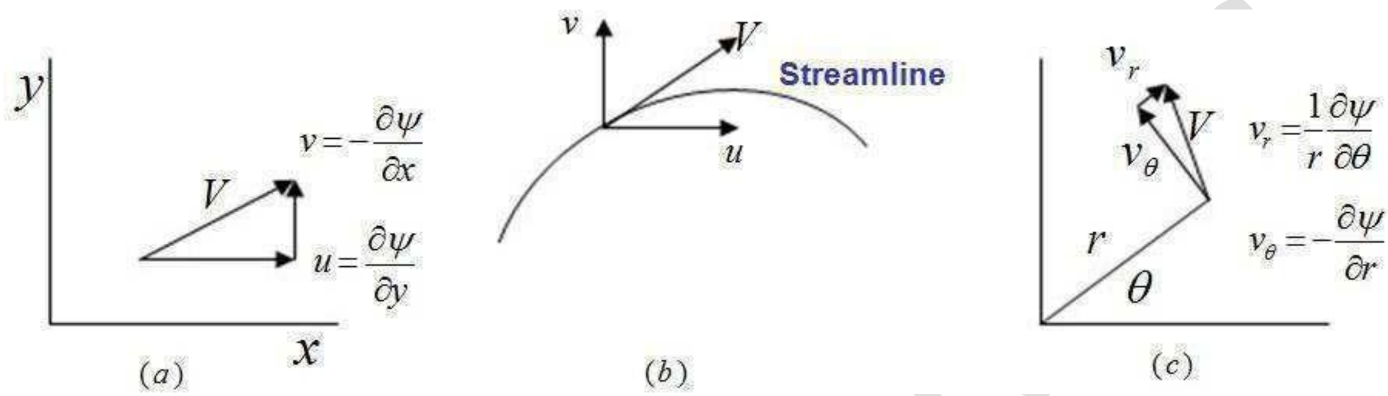
Particles P_1, P_2, P_3, P_4, starting from point P at successive times pass along path lines shown. At the instant of time considered the positions of the particles are at 1, 2, 3 and 4. A line joining these points is the streak line.

If a number of adjacent fluid particles in a flow field are marked at a given instant, they form a line at that instant. This line is called time line. Subsequent observations of the line may provide information about the flow field. For example the deformation of a fluid under shear force can be studied using time lines.

CONCEPT OF STREAM LINE

In a flow field if a continuous line can be drawn such that the tangent at every point on the line gives the direction of the velocity of flow at that point, such a line is defined as a stream line. In steady flow any particle entering the flow on the line will travelonly along this line. This leads to visualisation of a stream line in laminar flow as the path of a dye injected into the flow.

There can be no flow across the stream line, as the velocity perpendicular to thestream line is zero at all points. The flow along the stream line can be considered as one dimensional flow, though the stream line may be curved as there is no component of velocity in the other directions. Stream lines define the flow paths of streams in the flow. The flow entering between two stream lines will always flow between the lines. The lines serve as boundaries for the stream.



CONCEPT OF STREAM FUNCTION

Stream function is a very useful device in the study of fluid dynamics and was arrived at by the French mathematician Joseph Louis Lagrange in 1781. Of course, it is related to the streamlines of flow, a relationship which we will bring out later. We can define stream functions for both two and three dimensional flows. The latter one is quite complicated and not necessary for our purposes. We restrict ourselves to two-dimensional flows.

Consider a two-dimensional incompressible flow for which the continuity equation is given by,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

A stream function is one which satisfies

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial x}\right) \equiv 0$$

VELOCITY POTENTIAL

We have seen that for an irrotational flow

$$curl(\vec{V}) = 0 \quad or \quad \nabla \times \vec{V} = 0$$

It follows from vector algebra that there should be a potential such that:

$$\vec{V} = \nabla \phi$$

Φ is called the Velocity Potential. The velocity components are related to ϕ through the following relations.

$$u = \frac{\partial \phi}{\partial x}, \quad v = \frac{\partial \phi}{\partial y}, \quad w = \frac{\partial \phi}{\partial z}$$

Velocity potential is a powerful tool in analyzingirrotational flows. First of all it meets with the irrotationality condition readily. In fact, it follows from that condition. As a check we substitute the velocity potential in the irrotationality condition, thus,

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial}{\partial x}\frac{\partial \phi}{\partial y} - \frac{\partial}{\partial y}\frac{\partial \phi}{\partial x} \equiv 0$$

The next question we ask is does the velocity potential satisfy the continuity equation? To find out we consider the continuity equation for incompressible flows and substitute the expressions for velocity coordinates in them. Accordingly,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial}{\partial x}\frac{\partial \phi}{\partial x} + \frac{\partial}{\partial y}\frac{\partial \phi}{\partial y} = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}$$

It is clear that to meet with the continuity requirements the velocity potential has to satisfy the equation,

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

In vector notation it is

$$\nabla^2 \phi = 0$$

As with stream functions we can have lines along which potential ϕ is constant. These are called Equipotential Lines of the flow. Thus along a potential line ϕ = C.

The equation is called the Laplace Equation and is encountered in many branches of physics and engineering. A flow governed by this equation is called a Potential Flow. Further the Laplace equation is linear and is easily solved by many available standard techniques, of course, subject to boundary conditions at the boundaries.

RELATIONSHIP BETWEEN Φ AND Ψ

1. We notice that velocity potential **Φ** and stream function **Ψ** are connected with velocity components. It is necessary to bring out the similarities and differences between them.

Stream function is defined in order that it satisfies the continuity equation readily. We do not know yet if it satisfies the irrotationality condition. So we test out below. Recall that the velocity components are given by

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}$$

Substituting these in the irrotationality condition, we have

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial x}\right) - \frac{\partial}{\partial y}\frac{\partial \psi}{\partial y} = -\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2}$$

Which leads to the condition that $\nabla^2\psi = 0$ for irrotationality

Thus we see that the velocity potential ϕ automatically complies with the irrtotatioanlity condition, but to satisfy the continuity equation it has to obey that $\nabla^2\psi = 0$. On the other hand the stream function readily satisfies the continuity condition, but to meet with the irrotationality condition it has to obey $\nabla^2\psi = 0$.

Thus we see that the streamlines too follow the Laplace Equation. So it is possible to solve for a potential flow in terms of stream function.

Table: Properties of stream function and velocity potential

Property	ψ	ϕ
Continuity Equation	Automatically Satisfied	satisfied if $\nabla^2\psi = 0$
Irrotationality Condition	satisfied if $\nabla^2\psi = 0$	Automatically Satisfied

2. Streamlines and equipotential lines are orthogonal to each other. We have seen that the velocity components of the flow are given in terms of velocity potential and stream function by the equations,

$$v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}$$

Those familiar with Complex Variables theory will recognize that these are the Cauchy-Riemann equations and that ϕ = C and ψ = D are orthogonal and that both ϕ and ψ obey Laplace Equation. However, we will prove the orthogonality condition by other means.

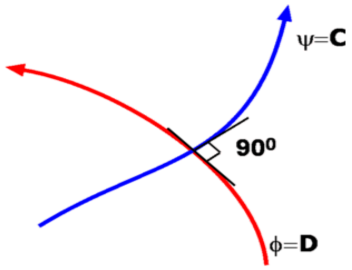


Figure: Orthogonality of Stream lines and equipotential lines

Since ϕ = C, it follows that

$$= \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy$$

= 0

$$= u\ dx + v\ dy$$

The gradient of the equipotential line is hence given by

$$\left(\frac{dy}{dx}\right)_{\phi = C} = -\frac{u}{v}$$

On the other hand the gradient of a stream line is given by

$$\left(\frac{dy}{dx}\right)_{\psi = D} = \frac{v}{u}$$

Thus we find that

$$\left(\frac{dy}{dx}\right)_{\phi = C}\left(\frac{dy}{dx}\right)_{\psi = D} = -1$$

Showing that equipotential lines and streamlines are orthogonal to each other. This enables one to calculate the stream function when the velocity potential is given and vice versa.

Figure shows the flow through a bend where the streamlines and the equipotential lines have been plotted. The two form an orthogonal network.

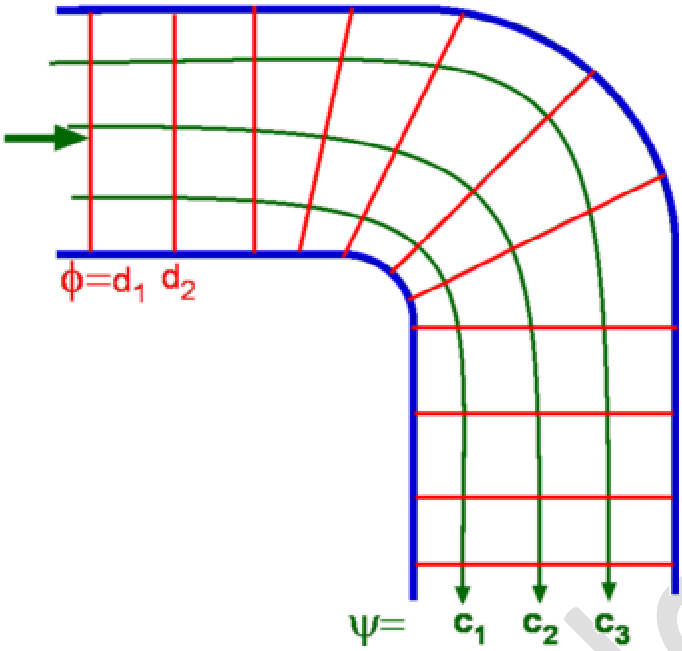


Figure: Stream lines and equipotential lines for flow through a bend

Uniform Flow

Uniform flow is the simplest form of potential flow. For flow in a specific direction, the velocity potential is
$\phi = U (x \cos\alpha + y \sin\alpha)$
While the stream function is $\psi = U (y \cos\alpha - x \sin\alpha)$
Where α represents the angle between the flow direction and the x-axis (as shown in the figure). Recall that the velocity potential and stream functionare related to the component velocity in the 2 dimensional flow field as follows:
Cartesian coordinates:

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}$$

And

$$v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

Cylindrical coordinates:

$$v_r = \frac{\partial \phi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta}$$

And

$$v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r}$$

Note that the lines of the constant velocity potential (equipotential lines) are orthogonal to the lines of the stream function (streamlines).

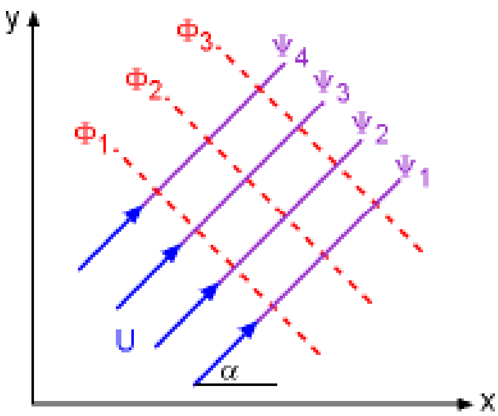


Figure: Uniform Flow

Source and Sink

When a fluid flows radially outward from a point source, the velocities are

$v_r = m / (2ur)$ and $v_\theta = 0$

Where m is the volume flow rate from the line source per unit length. The velocity potential and stream function can then be represented as:

$$\varphi = \frac{m}{2\pi}\ln(r) \qquad \psi = \frac{m}{2\pi}\theta$$

When m is negative, the flow is inward, and it represents a sink. The volume flow rate per unit depth, m, indicates the strength of the source or sink. Note that as r approaches zero, the radial velocity goes to infinity. Hence, the origin represents a singularity. As shown in the figure, the equipotential lines are given by the concentric circles while the streamlines are the radial lines.

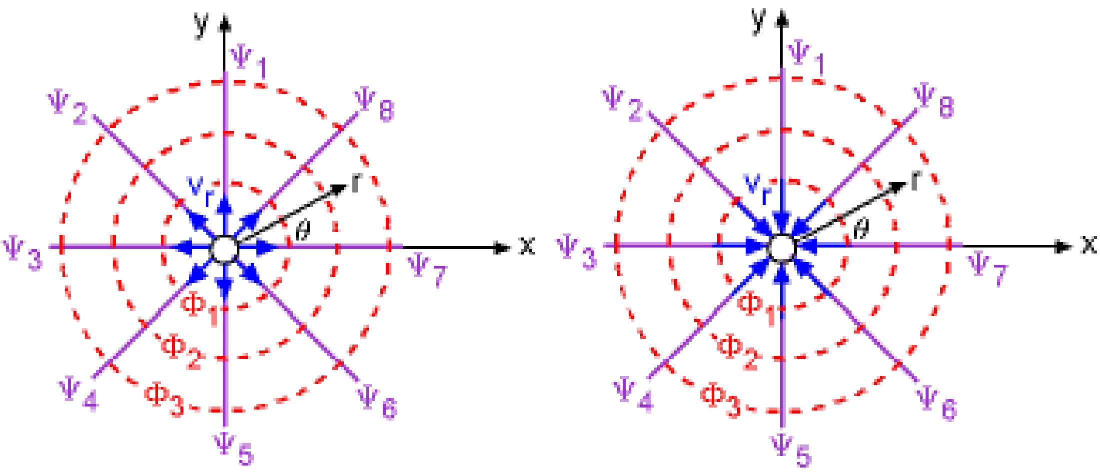


Figure: Source and Sink Flow

Vortex

A vortex can be obtained by reversing the velocity potential and stream functions for a point source such that $\phi = K\theta$ and $\psi = -K \ln(r)$

Where K is a constant indicating the strength of the vortex. Now, the equipotential lines are radial lines while the streamlines are given by the concentric circles. The velocities of a vortex are, $V_r = 0$.

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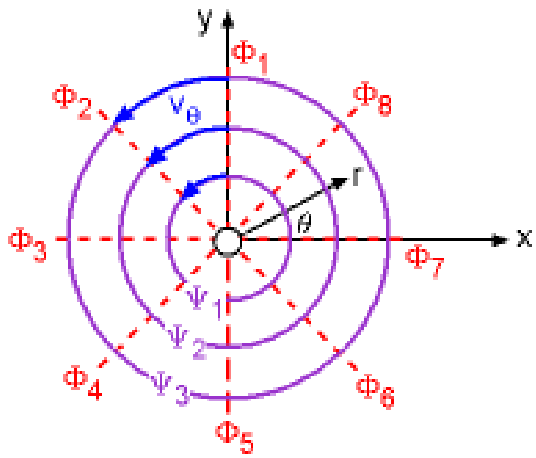

Figure: Vortex Flow