RAJIV GANDHI PROUDYOGIKI VISHWAVIDYALAYA, BHOPAL
New Scheme Based On AICTE Flexible Curricula
Civil Engineering, III-Semester
CE 304 Strength of Materials

UNIT I

Simple Stress and Strains: Concept of Elastic body stress and Strain, Hooke's law, Various types of stress and strains, Elastic constants, Stresses in compound bars, composite and tapering bars, Temperature stresses. Complex Stress and Strains- Two dimensional and three dimensional stress system. Normal and tangential stresses, Principal Planes, Principal Stresses and Strains, Mohr's circle of stresses.

UNIT II

Bending and Shearing Stresses: Theory of simple bending, Concept of pure bending and bending stress, Equation of bending, Neutral axis, Section-Modulus, Differential equation of the elastic curve, Determination of bending stresses in simply supported, Cantilever and Overhanging beams subjected to point load and uniformly distributed loading, Bending stress distribution across a section of beam, Shearing Stress and shear stress distribution across a section in Beams.

UNIT III

Determination of Slope and Deflection of beams by Double Integration Method, Macaulay's Method, Area Moment Method, Conjugate Beam Method, and Strain Energy Method, Castiglione's Method, and Unit Load Method.

UNIT IV

Columns and Struts: Theory of columns, Slenderness ratio**,** Direct and bending stresses in short columns, Kern of a section. Buckling and stability, Euler's buckling/crippling load for columns with different end conditions, Rankin's formula, Eccentric loads and the Secant formula- Imperfections in columns. Thin Pressure Vessels: cylinders and spheres. Stress due to internal pressure, Change in diameter and volume. Theories of failure.

UNIT V

Torsion of Shafts: Concept of pure torsion, Torsion equation, Determination of shear stress and angle of twist of shafts of circular section, Torsion of solid and hollow circular shafts, Analyses of problems based on combined Bending and Torsion. Unsymmetrical Bending: Principal moment of Inertia, Product of Inertia, Bending of a beam in a plane which is not a plane of, symmetry. Shear center; Curved beams: Pure bending of curved beams of rectangular, circular and trapezoidal sections, Stress distribution and position of neutral axis.

Reference books:

1. Punmia B.C., Mechanics of Materials, ,Laxmi Publications (P) Ltd.
2. S.S Bhavikaati, Strength of Materials, Vikas Publisher, new Delhi
3. Rajput R. K., Strength of Materials, S. Chand.
4. S. Ramamrutham, R. Narayanan, Strength of Materials, DhanpatRai Publications.
5. R. Subramaniam, Strength of Materials, Oxford University Press.
6. Sadhu Singh , Strength of Material , Khanna Publishers
7. Mubeen A , Mechanics of solids , Pearsons
8. D.S PrakashRao, Strength of Material , University Press , Hyderabad
9. Debrath Nag, Strength of Material , Wiley
10. Jindal , Strength of Material , Pearsons.
11. Bansal R.K, Strength of Materials, Laxmi Publisher, New Delhi.
12. Nash, W.A., Strength of Materials, Mcgraw hills, New Delhi.
13. Chandramouli, Strength of Materials, PHI learning
14. Dongre A.P., Strength of Materials, Scitech, Chennai
15. Negi L. S ,Strength of Materials, McGraw Hill Professional.
16. Raj Puroshattam, Strength of Material , Pearsons
17. J.M. Gere,.J. G. Barry Mechanics of Material, Cengage Learning

List of Practical

1. Study of Universal testing machine
2. To determine the compressive and tensile strength of materials.
3. To determine the Brinell hardness of materials.
4. To determine the Rockwell hardness of materials
5. To determine the toughness of the materials.
6. To determine the stiffness of the spring.
7. To determine the deflection of beam by the use of deflection-beam apparatus.

Department of Mechanical Engineering

Strength of Material (ME-3002) Class Notes

UNIT I

The mechanics of the member of rigid bodies is primarily concerned with the static and dynamic behavior under external forces/moments of engineering component. They are treated as infinitely strong and undeform able primarily we deal here with the forces and motions accompanied and rigid bodies.

Mechanics under rigid body:
The mechanics of deformable solid rigid body is more related with the internal forces and accompanied changes in the dimension of the components involved. Of particular importance are the properties of the materials used, the strength of which will determine whether the components fails, break and the stiffness of which will determine whether the amount of deformation they suffer is permissible. Here, this subject of strength of material or mechanics of materials is central to the entire activity of engineering design. Generally the objectives in analysis here will be the determination of the stresses, strains, stiffness, bending and deflections produced by loads.

Analysis of stress / strain under load:

Stress: introduce the concept of stress as we know that of the internal resistance of the body, i.e. the nature of forces set up within a body to balance the effect of the externally applied forces.

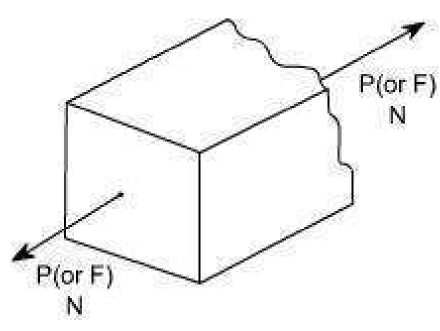
The externally applied forces are noted as load/pressure. These load/pressure may be due to any one of the reason.

(i) Service conditions
(ii) Environment in which the component works
(iii) Contact with other members
(iv) Fluid pressures
(v) Gravity or inertia forces.

So we know that in mechanics of deformable solids, externally forces acts on a body and body suffers a deformation. As for as equilibrium point of view, this action should be opposed or reacted by internal forces which are set up within the particles of material due to cohesion.

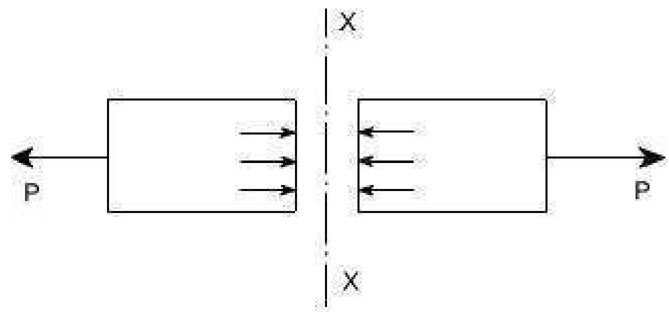
These internal forces provide concept of stress. So so us define a stress Here; so us define a term stress

Stress:



So us consider a rectangular bar ABCD of uniform cross section area and subjected to load P (in Newton)

Now imagine that the same rectangular bar ABCD is assumed to be cut into in between at section XX. The each portion of this rectangular bar is in equilibrium under the action of load P and the internal forces acting at the section XX has been mentioned



So, the stress is defined as the resistive force intensity or force per unit area. Generally e we use a symbol s to represent the stress.

$$\sigma = \frac{P}{A}$$

A is the area of the cross section

There we are using an assumption that the total force or total load carried by the rectangular bar ABCD is equal distributed over cross section.

But the stress distributions may be for from uniform, with local regions of high stress known as stress concentrations.

If the force carried by a component is not equal distributed over its cross – sectional area, A, we must consider a small area, 'dA' which carries a small load dP, of the total force 'P', Then definition of stress is

$$\boxed{\sigma = \frac{\delta F}{\delta A}}$$

As a particular stress generally holds true only at a point, here it is defined mathematically as

$$\boxed{\sigma = \lim_{\delta A \to 0} \frac{\delta F}{\delta A}}$$

Basic Units-

The basic units of stress in S.I units are N / m^2 (or Pa)

Mega Pascal = 10^6 Pa

Gega Paacal = 10^9 Pa

Kilo Pascal = 10^3 Pa

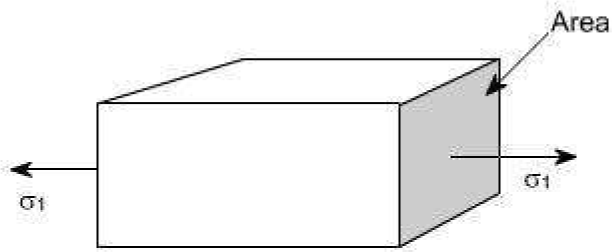
IN S.I unit N / mm^2 units are used, because this is an equivalent to MPa. While US customary unit is FPS.

TYPES OF STRESSES-

Only two basic stresses are exists: (i) normal direct stress and (ii) shear shear stress. Other stresses either are similar to these basic stresses or are a combination of this like bending stress are combination of tensile, compressive and shear stresses and Torsional stress, are encountered in twisting of a shaft is a shearing stress.

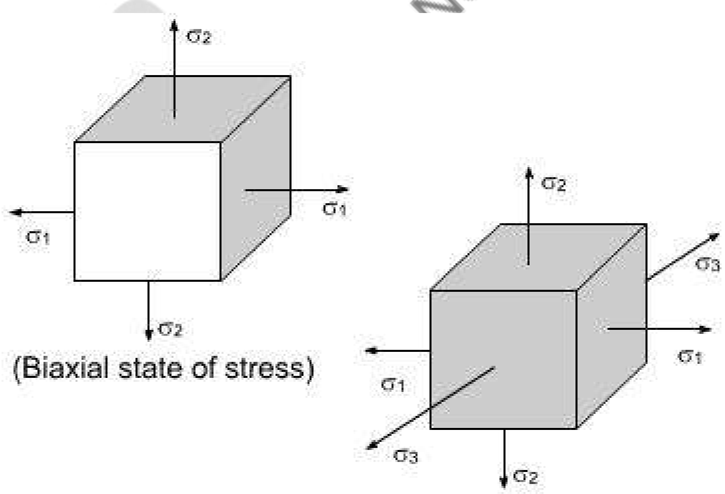
So us define the normal stresses and shear stresses in the following sections.

Normal stresses- We have defined stress as force per unit area. If the stresses are normal to the area in the direction of force, then these are noted as normal stresses. The normal stresses are generally denoted by a



Greek soter(s).

This is uniaxial state of stress, stresses acts only in one direction however, for biaxial and tri axial state of stresses there the two mutually perpendicular normal stresses acts or three mutually perpendicular normal stresses act as mention below:

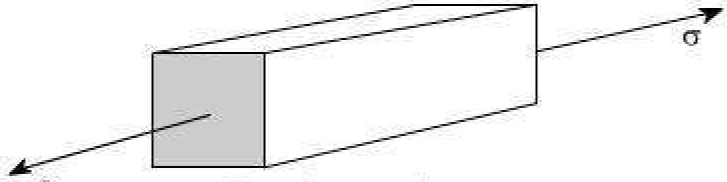


(Biaxial state of stress)

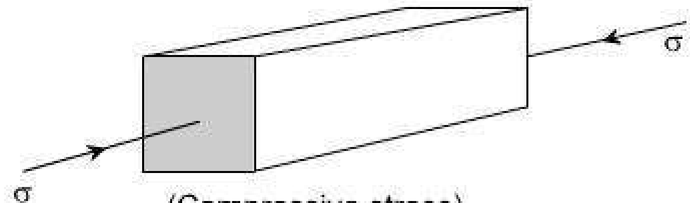
(Triaxial state of stress)

Tensile or compressive stresses:

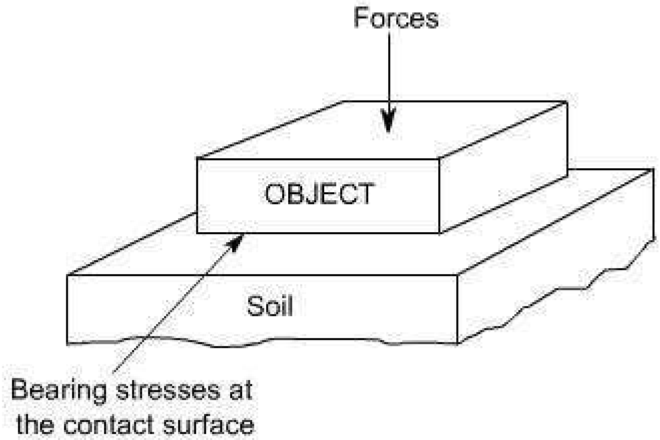
The normal stresses may be either tensile or compressive in nature



(Tensile stress)

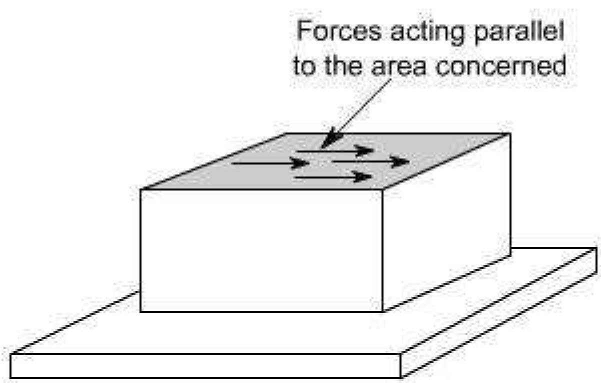


(Compressive stress)



Shear stress-

Now we consider the condition, in which the cross section area of a block of material is subject to a mention forces which are parallel, rather than normal, to the area concerned. This forces are accompanied with a shearing of the material, and are referred to as shear forces. The resulting force interests are known as shear stresses.



The resulting force are known as shear stresses, the mean shear stress being equal to

$$\tau = \frac{P}{A}$$

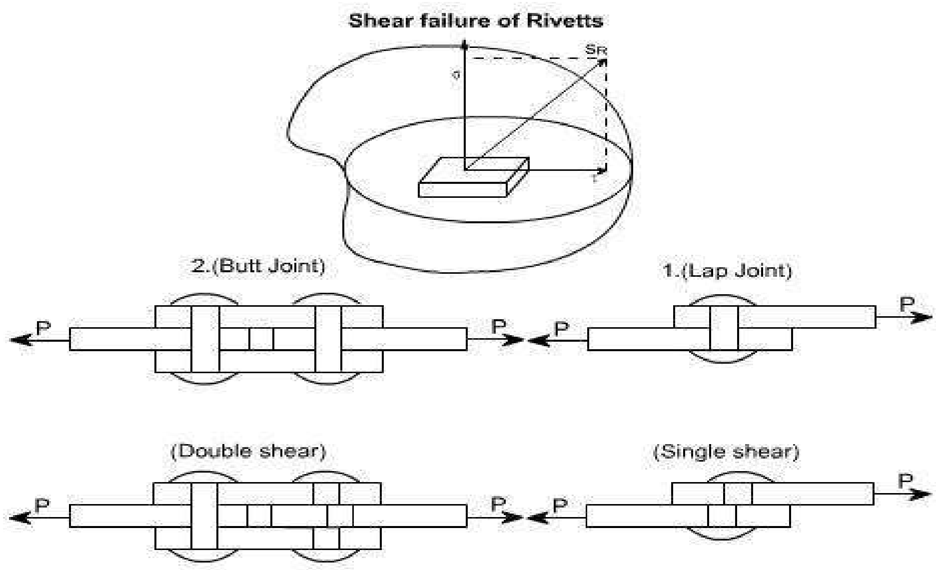
Here P is the total force and A the area over which it acts.

So that the particular stress holds good only at a point here we can define shear stress at a point as

$$\tau = \lim_{\delta A \to 0} \frac{\delta F}{\delta A}$$

The Greek symbol t (tau) (tangential) is used to denote shear stress.

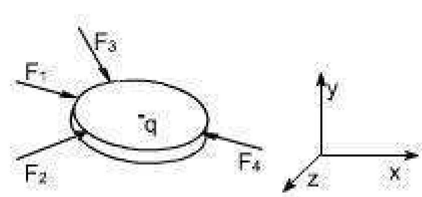
it must be the stress (resultant stress) at any point in a body is basically resolved into two components s and t one acts perpendicular and other parallel to the area, as it is defined in the following figure.



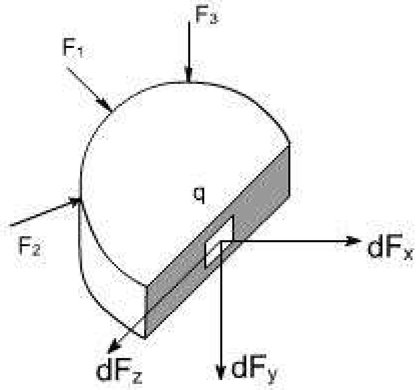
The single shear took place on the single plane and the shear area is the cross - sectional of the rivet, thereas the double shear took place in the case of Butt joints of rivets and the shear area is the twice of the X - sectional area of the rivet.

ANALYSIS OF VARIOUS STERSSES

Against a force Stress at a point in a material body has been defined as a force per unit area. But this definition is somewhat ambiguous since it depends upon what area we consider at that point. So us, consider a point ‘q' in the interior of the body



So pass a cut to plane through a point 'q' perpendicular to the x - axis as mention below



The corresponding force components can be mentioned like this

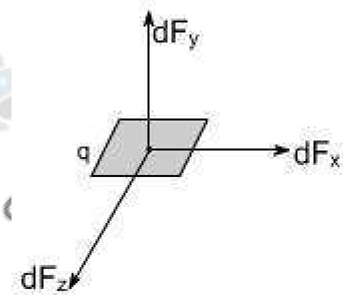
$dF_x = s_{xx}. da_x$

$dF_y = t_{xy}. da_x$

$dF_z = t_{xz}. da_x$

there da_x is the area surrounding the point 'q' when the cut to plane ^ ^{r} is to x - axis.

In a similar way it can be assumed that the cut to plane is passed through the point 'q' perpendicular to the y - axis. The corresponding force components are mentioned below



The corresponding force components may be written as

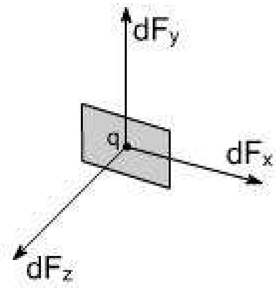
$dF_x = t_{yx}. da_y$

$dF_y = s_{yy}. da_y$

$dF_z = t_{yz}. da_y$

there da_y is the area surrounding the point 'q' when the cut to plane ^ ^{r} is to y - axis.

In the last it can be considered that the cut to plane is passed through the point 'q' perpendicular to the z - axis.



The corresponding force components may be written as

$dF_x = t_{zx}. da_z$

$dF_y = t_{zy}. da_z$

$dF_z = s_{zz}. da_z$

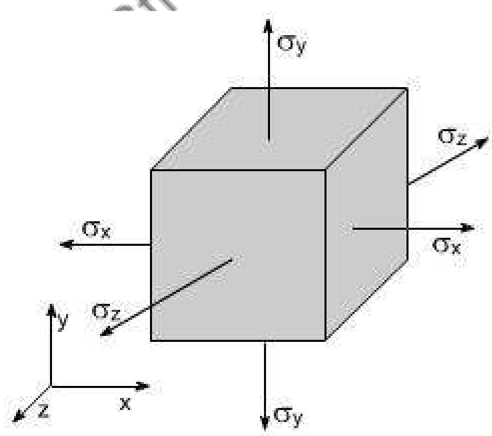
there da_z is the area surrounding the point 'q' when the cut to plane ^ r is to z - axis.

So, from the foregoing discussion it is clear that there is nothing like stress at a point 'q' rather we have a situation there it is a combination of various state of stress at a point. There the stresses on the three mutually perpendicular planes are labelled in the manner as mentioned earlier. The state of stress as depicted earlier is called the general or a tri axial state of stress that can exist at any interior point of a loaded body.

Cartesian co-ordinate system

For the Cartesian co-ordinates system, we make use of the axes, X, Y and Z

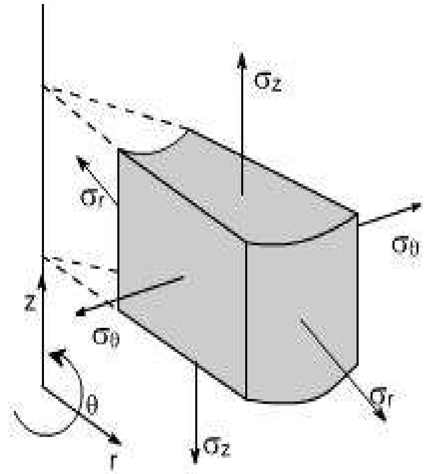
So us consider the small element of the material and show the various normal stresses acting the faces



So, in the Cartesian co-ordinates system the normal stresses have been represented by s_x, s_y and s_z.

Cylindrical co-ordinate system

For the Cylindrical - co-ordinate system we make use of co-ordinates r, q and Z.



S0, in the cylindrical co-ordinates system, the normal stresses i.e components acting over a element is being denoted by s_r, s_q and s_z.

Sign convention: The tensile forces are noted as positive while the compressive forces are noted as negative.

First sub – script: it mentioned the direction of the normal to the surface.

Second subscript: it mentioned the direction of the stress.

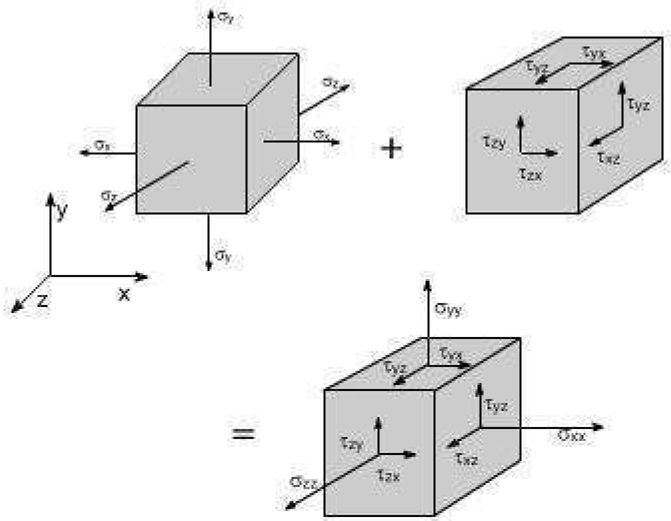
It may be noted that in the case of normal stresses the double script notation may be dispensed with as the direction of the normal stress and the direction of normal to the surface of the element on which it acts is the same. Here, a single subscript notation as used is sufficient to define the normal stresses.

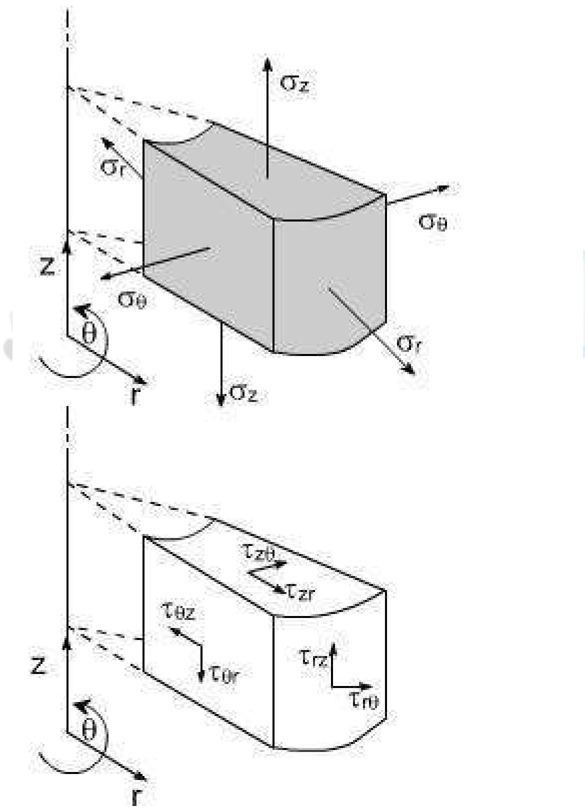
Shear Stresses: With shear stress components, the single subscript notation is not practical, because such stresses are in direction parallel to the surfaces on which they act. We here have two directions to perticolarly, that of normal to the surface and the stress itself. To do this, we stress itself. To do this, we attach two subscripts to the symbol'τ', for shear stresses.

In Cartesian and polar co-ordinates, we have the stress components as mentioned in the figures.

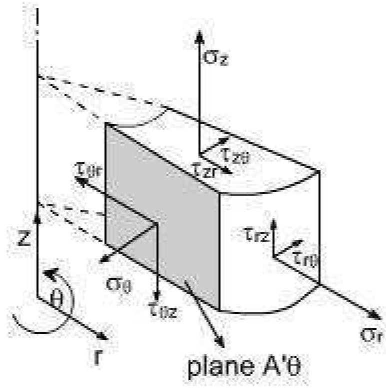
t_{xy} , t_{yx} , t_{yz} , t_{zy} , t_{zx} , t_{xz}

t_{rq} , t_{qr} , t_{qz} , t_{zq} ,t_{zr} , t_{rz}





Now so us combine the normal and shear stress components as mentioned below:



Now so define the state of stress at a point formally.

State of stress at a point: Stress at a point, we mean information which is required at that point such that it remains under equilibrium. Or simply a general state of stress at a point includes all the normal stress components, together with all the shear stress components as mentioned in earlier figures.

Here, we need nine components, to define the state of stress at a point

s_x t_{xy} t_{xz}

s_y t_{yx} t_{yz}

s_z t_{zx} t_{zy}

If we apply the conditions of equilibrium which are as follows:

å $F_x = 0$; å $M_x = 0$

å $F_y = 0$; å $M_y = 0$

å $F_z = 0$; å $M_z = 0$

Then we get

$t_{xy} = t_{yx}$

$t_{yz} = t_{zy}$

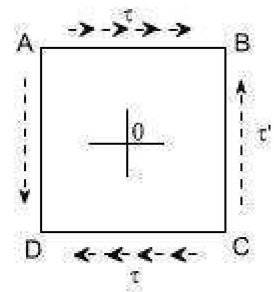
$t_{zx} = t_{xy}$

Only six components to perticolarly the state of stress at a point i.e

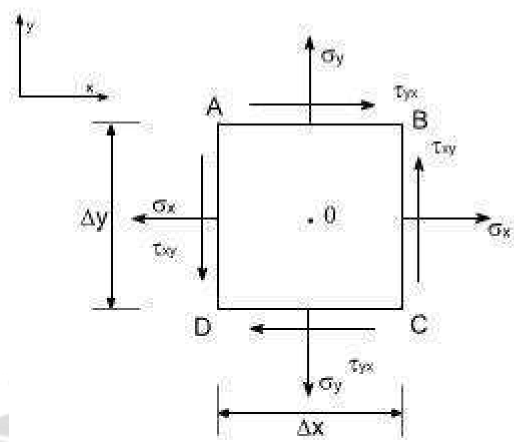
s_x , s_y, s_z , t_{xy} , t_{yz} , t_{zx}

Now so us define the concept of with respect to each other shear stresses.

With respect to each other shear stresses:



on planes AB and CD, the shear stress t acts. To maintain the static equilibrium of this element, on planes AD and BC, t' should act, we shall see that t' which is known as the with respect to each other shear stress would come out to like and alike to the t .



Sign convections for shear stresses:

Direct stresses or normal stresses

Tensile Positive

Compressive Negative

Shear stresses:

- Tending to turn the element C.W Positive.

- Tending to turn the element C.C.W Negative.

We consider the weight of the element is neglected.

Since the element is a static piece of solid body, the moments applied to it must also be in equilibrium. So ‘O' be the centre of the element. So us consider the axis through the point ‘O'. the resultant force accompanied with normal stresses s_x and s_y acting on the sides of the element each pass through this axis, and here, have no moment.

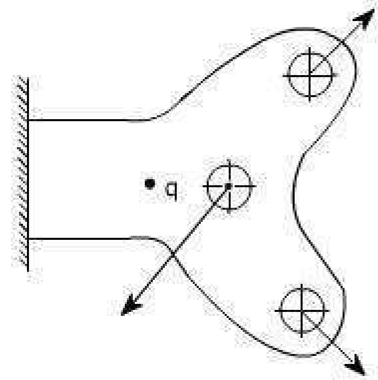
So,t_{yx} . D x . D z . D y = t_{xy} . D x . D z . D y

$$\boxed{\tau_{yx} = \tau_{xy}}$$

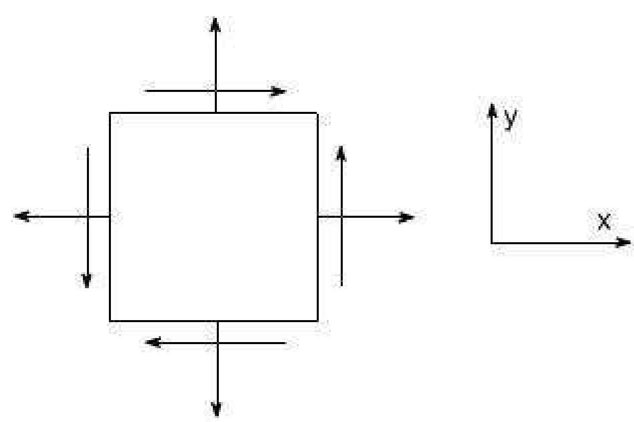
So the with respect to each other shear stresses are equal in its magnitude.

$$\boxed{\begin{aligned}\tau_{zy} &= \tau_{zy}\\ \tau_{zx} &= \tau_{xz}\end{aligned}}$$

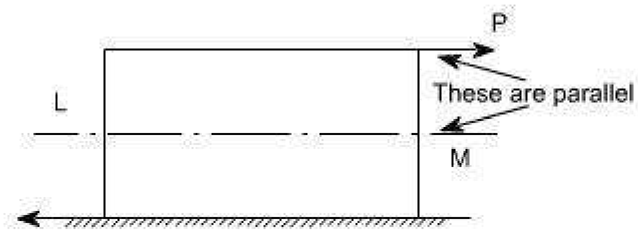
Analysis of Stresses:



Let us take a point 'q' in some sort of structural member like as mentioned in figure below. Assuming that at point exist. 'q' a plane state of stress exist. i.e. the state of state stress is to describe by a parameters s_x, s_y and t_{xy} These stresses could be indicate a on the two dimensional diagram as mentioned below:



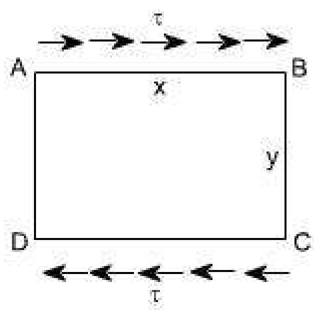
This is a common way of representing the stresses. It must be realize a that the material is unaware of what we have called the x and y axes. i.e. the material has to resist the loads irrespective less of how we wish to name them or whether they are horizontal, vertical or otherwise furthermore, the material will fail when the stresses exceed beyond a permissible value. So, a fundamental problem in engineering design is to determine the highest normal stress or highest shear stress at any particular point in a body. There is no reason to believe that s_x, s_y and t_{xy} are the highest value. Rather the highest stresses may associates themselves with some other planes located at 'q'. So, it becomes imperative to determine the values of s_q and t_q. In order to achieve this so us consider the following.



Shear stress

If the applied load P consists of two like and alike parallel forces not in the same line, than there is a tendency for one part of the body to slide over or shear from the other part across any section LM. If the cross section at LM measured parallel to the load is A, then the average value of shear stress t = P/A. The shear stress is tangential to the area over which it acts.

If the shear stress varies then at a point then t may be defined as $\tau = \lim_{\delta A \to 0} \frac{\delta P}{\delta A}$



With respect to each other shear stress:

So ABCD be a small rectangular element of sides x, y and z perpendicular to the plane of paper so there be shear stress acting on planes AB and CD

It is obvious that these stresses will from a couple (t. xz) y which can only be balanced by tangential forces on planes AD and BC. These are known as with respect to each other shear stresses. i.e. the existence of shear stresses on sides AB and CD of the element implies that there must also be with respect to each other shear stresses on to maintain equilibrium.

So t' be the with respect to each other shear stress induced on planes

AD and BC. Then for the equilibrium (t . xz)y = t' (yz)x

$$\boxed{t = t'}$$

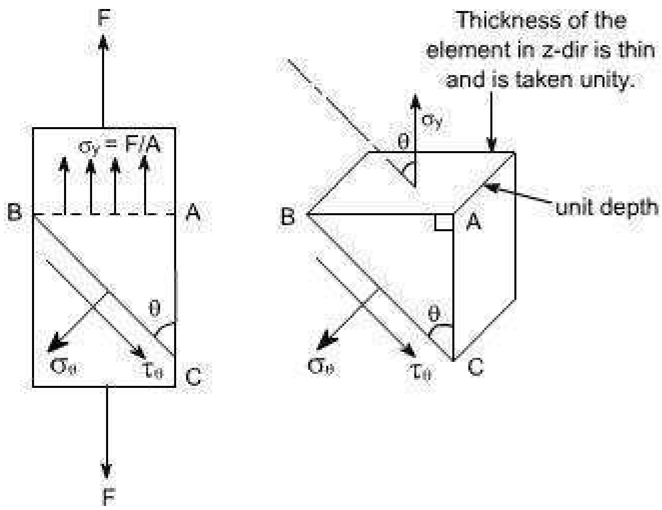
So, every shear stress is accompanied by an equal with respect to each other shear stress.

Stresses on oblique plane: Stresses on oblique plane: until currently we've got controlled either pure traditional direct stress or pure shear stress. In several instances, but each direct and shear stresses acts and also the resultant stress across any section are going to be neither traditional nor tangential to the plane.

$s_z = t_{yz} = t_{zx} = 0$

Example of plane state of stress include plates and shells.

Consider the general case of a bar under direct load F giving rise to a stress s_y vertically



The stresses amendment with the inclination of the planes passing through that time i.e. the strain on the faces of the component vary because the spatial relation of the component changes.

Let the block be of unit depth currently considering the equilibrium of forces on portion rudiment

Resolving forces perpendicular to B.C., gives

$s_q.BC.1 = s_y sinq \,.\, AB.\, 1$

but AB/BC = sinq or AB = BCsinq

Substituting this value in the above equation, we get

$$s_q.BC.1 = s_y sinq \,.\, BCsinq \,.\, 1 \text{ or}$$

$$\boxed{\sigma_\theta = \sigma_y \,.\sin^2 2\theta} \qquad (1)$$

Now resolving the forces parallel to BC

$t_q.BC.1 = s_y \cos q \,.\, ABsinq \,.\, 1$

again AB = BCcosq

$t_q.BC.1 = s_y cosq \,.\, BCsinq \,.\, 1$ or $t_q = s_y sinqcosq$

$$\boxed{\tau_\theta = \frac{1}{2}.\sigma_y.\sin 2\theta} \qquad (2)$$

If $q = 90^0$ the BC will be parallel to AB and $t_q = 0$, i.e. here normal or direct stress.

By examining the equations (1) and (2), the following conclusions may be drawn

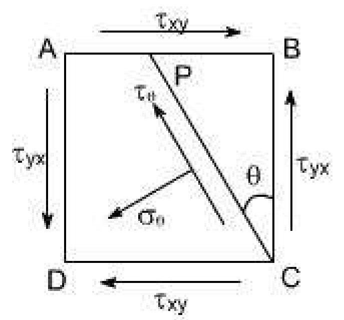
(i) The value of direct stress s_q is highest and is equal to s_y when $q = 90^0$.

(ii) The shear stress t_q has a highest value of 0.5 s_y when $q = 45^0$

(iii) The stresses s_q and s_q are not simply the resolution of s_y

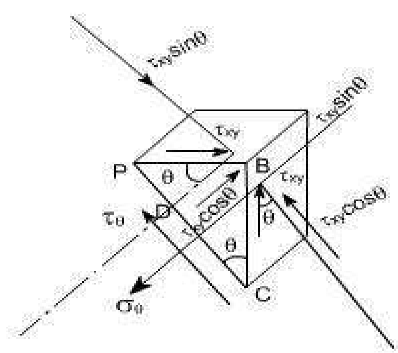
Material subjected to pure shear:

Consider the element mentioned to which shear stresses have been applied to the sides AB and DC



With respect to each other shear stresses of equal price however of alike result ar then found out on the edges AD and B.C. so as to stop the rotation of the component. Since the applied and with respect to each other shear stresses ar of equal price on the x and y planes. Here, they're each pictured by the image txy.

Now think about the equilibrium of portion of PBC



Assuming unit depth and resolving normal to PC or in the direction of s_q

$s_q.PC.1 = t_{xy}.PB.cosq.1+ t_{xy}.BC.sinq.1$

$= t_{xy}.PB.cosq + t_{xy}.BC.sinq$

Now writing PB and BC in terms of PC so that it cancels out from the two sides

PB/PC = sinq BC/PC = cosq

s_q.PC.1 = t_{xy}.cosqsinqPC+ t_{xy}.cosq.sinqPC s_q =

2t_{xy}sinqcosq

s_q = t_{xy}.2.sinqcosq

$$\boxed{\sigma_\theta = \tau_{xy}.\sin 2\theta}$$

Now resolving forces parallel to PC or in the direction t_q.then t_{xy}PC . 1 = t_{xy} . PBsinq - t_{xy} . BCcosq

negative sign has been put because this component is in the same direction as that of t_q.

again converting the various quantities in terms of PC we have

t_{xy}PC . 1 = t_{xy} . PB.sin^{2}q - t_{xy} . PCcos^{2}q

= -[t_{xy} (cos^{2}q - sin^{2}q)]

= -t_{xy}cos2q or

$$\boxed{\tau_\theta = -\tau_{xy}\cos 2\theta}$$

the negative sign means that the sense of t_q is alike to that of assumed one. So us examine the equations (1) and (2) respectively

From equation (1) i.e,

s_q = t_{xy} sin2q

The equation (1) represents that the highest value of s_q is t_{xy} when q = 45^0.

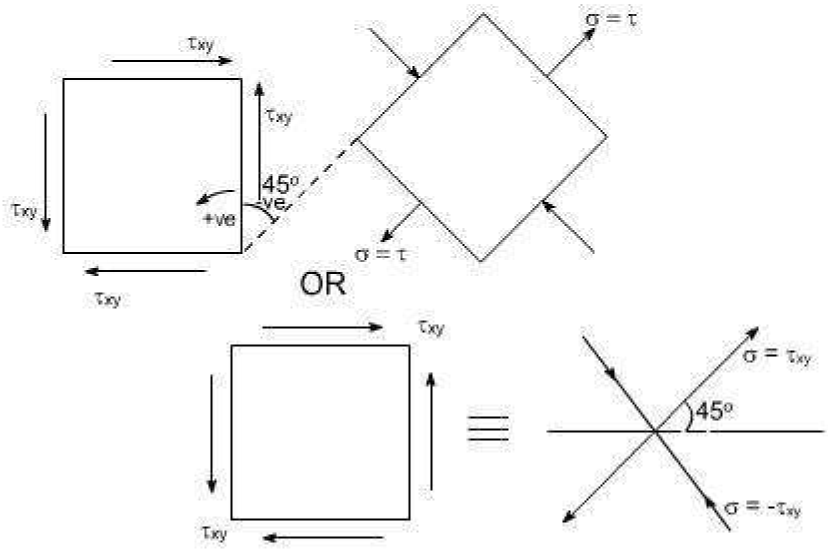
So us take into consideration the equation (2) which states that

t_q = - t_{xy} cos2q

It mentioned that the highest value of t_q is t_{xy} when q = 0^0 or 90^0. it has a value minimum when q = 45^0.

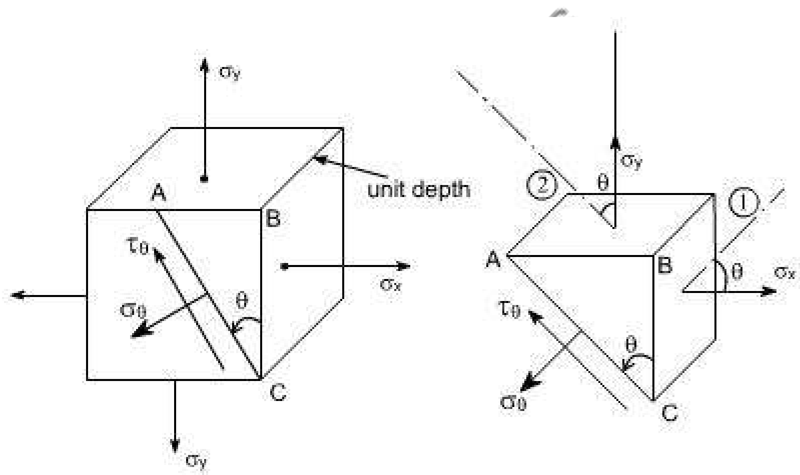
From equation (1) it should be noticed that the traditional part sq has most and minimum values of +txy (tension) and -txy (compression) on plane at ± 450 to the applied shear and on these planes the tangential part tq is minimum.

Hence the system of pure shear stresses produces and equivalent direct stress system, one set compressive and one tensile every settled at 450 to the first shear directions as delineated within the figure below:



<u>**Material under goes to two mutually perpendicular direct stresses:**</u>

Let us consider a rectangular element of unit depth, subjected to a system of two direct stresses both tensile, s_x and s_y acting right perpendicular to each other.



On rearranging the various terms we get

$$\boxed{\sigma_\theta = \left(\frac{\sigma_x + \sigma_y}{2}\right) + \left(\frac{\sigma_x - \sigma_y}{2}\right)\cos 2\theta}$$

$$\tau_\theta.AC.1 = [\tau_x \cos\theta \sin\theta - \sigma_y \sin\theta \cos\theta]AC$$

$$\tau_\theta = (\sigma_x - \sigma_y)\sin\theta\cos\theta$$

$$= \frac{(\sigma_x - \sigma_y)}{2}\sin 2\theta$$

$$\text{or} \quad \boxed{\tau_\theta = \frac{(\sigma_x - \sigma_y)}{2}\sin 2\theta}$$

The following result may be drawn from above equation

(i) The highest direct stress will be equal to s_x or s_y whichever is the greater, when $q = 0^0$ or 90^0

(ii) The highest shear stress in the plane of the induced stresses occurs when $q = 45^0$

$$\tau_{max} = \frac{(\sigma_x - \sigma_y)}{2}$$

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Department of Mechanical Engineering

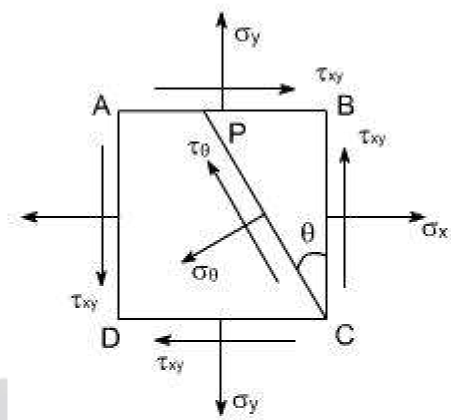
Strength of Material (ME-3002) Class Notes

UNIT II

Material under goes to to combined direct and shear stresses:

Now now a complicated stress system mentioned below, acting on an element of material.

The stresses s_x and s_y may be compressive or tensile and may be the result of direct forces or as a result of bending. The shear stresses may be as mentioned or completely reversed and occur as a result of either shear force or torsion as mentioned in the figure below:



As per the double subscript normal the shear stress on the face BC should be notified as t_{yx} , however, we have already seen that for a pair of shear stresses there is a set of with respect to each other shear stresses generated such that t_{yx} = t_{xy}

By looking at this state of stress, it may be observed that this state of stress is combination of two different cases:

(i) Material under goes to to pure state of stress shear. In this case the various formulas deserved are as follows

s_q = t_{yx} sin2 q

t_q = - t_{yx} cos 2 q

(ii) Material under goes to to two mutually perpendicular direct stresses. In this case the various formulas's derived are as follows.

$$\sigma_\theta = \frac{(\sigma_x + \sigma_y)}{2} + \frac{(\sigma_x - \sigma_y)}{2}\cos 2\theta$$

$$\tau_\theta = \frac{(\sigma_x - \sigma_y)}{2}\sin 2\theta$$

To get the required equations for the case under nowation, let us add the respective equations for the above two cases such that

$$\sigma_\theta = \frac{(\sigma_x + \sigma_y)}{2} + \frac{(\sigma_x - \sigma_y)}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\tau_\theta = \frac{(\sigma_x - \sigma_y)}{2}\sin 2\theta - \tau_{xy}\cos 2\theta$$

These are the equilibrium equations for stresses at a point. They do not depend on material proportions and are equally valid for elastic and inelastic.

This eqn. gives two values of 2q that differ by 180^0 .Hence the planes on which highest and lowest normal stresses occur at 90^0 apart.

For σ_θ to be a maximum or minimum $\frac{d\sigma_\theta}{d\theta} = 0$

Now

$$\sigma_\theta = \frac{(\sigma_x + \sigma_y)}{2} + \frac{(\sigma_x - \sigma_y)}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\frac{d\sigma_\theta}{d\theta} = -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta.2 + \tau_{xy}\cos 2\theta.2$$

$$= 0$$

$$\text{i.e.} -(\sigma_x - \sigma_y)\sin 2\theta + \tau_{xy}\cos 2\theta.2 = 0$$

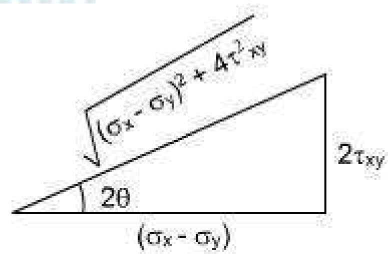
$$\tau_{xy}\cos 2\theta.2 = (\sigma_x - \sigma_y)\sin 2\theta$$

Thus, $\boxed{\tan 2\theta = \frac{2\tau_{xy}}{(\sigma_x - \sigma_y)}}$

From the trimake angle with it may be determined

$$\cos 2\theta = \frac{(\sigma_x - \sigma_y)}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}}$$

$$\sin 2\theta = \frac{2\tau_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}}$$



Substituting the values of cos 2q and sin 2q in equation (5) we get

$$\sigma_\theta = \frac{(\sigma_x + \sigma_y)}{2} + \frac{(\sigma_x - \sigma_y)}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\sigma_\theta = \frac{(\sigma_x + \sigma_y)}{2} + \frac{(\sigma_x - \sigma_y)}{2}\cdot\frac{(\sigma_x - \sigma_y)}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}}$$

$$+ \frac{\tau_{xy}\cdot 2\tau_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}}$$

$$= \frac{(\sigma_x + \sigma_y)}{2} + \frac{1}{2}\cdot\frac{(\sigma_x - \sigma_y)^2}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}}$$

$$+ \frac{1}{2}\frac{4\tau^2_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}}$$

or

$$= \frac{(\sigma_x + \sigma_y)}{2} + \frac{1}{2}\cdot\frac{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}}$$

$$= \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2}\cdot\frac{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}\cdot\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}}$$

$$\sigma_\theta = \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2}\cdot\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}$$

Hence we get the two values of σ_θ, which are designated σ_1 as σ_2 and respectively,therefore

$$\sigma_1 = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}\cdot\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}$$

$$\sigma_2 = \frac{1}{2}(\sigma_x + \sigma_y) - \frac{1}{2}\cdot\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}$$

The σ_1 and σ_2 are termed as the principle stresses of the system.

Substituting the values of $\cos 2\theta$ and $\sin 2\theta$ in equation (6) we see that

$$\tau_\theta = \frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta - \tau_{xy}\cos 2\theta$$

$$= \frac{1}{2}(\sigma_x - \sigma_y)\frac{2\tau_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}} - \frac{\tau_{xy}\cdot(\sigma_x - \sigma_y)}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}}$$

$$\tau_\theta = 0$$

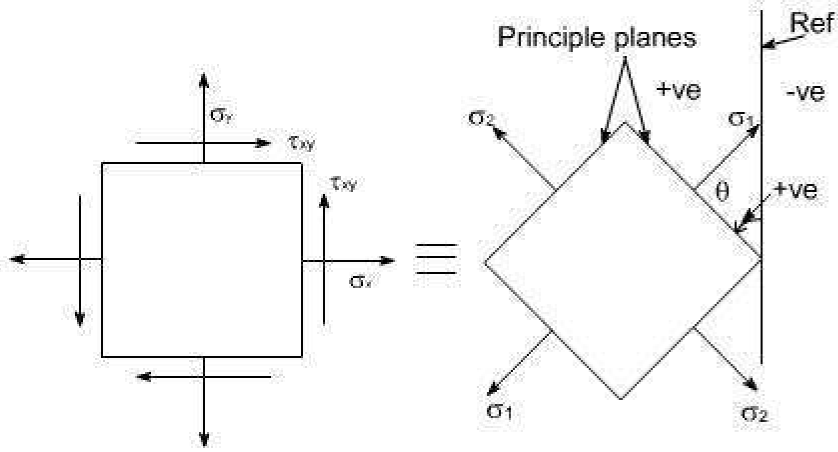
This shows that the values of shear stress are lowest on the principal planes.

Hence the highest and lowest values of normal stresses occur on planes of lowest shearing stress. The highest and lowest normal stresses are called the principal stresses, and the planes on which they act are called principal plane the solution of equation

$$\tan 2\theta_p = \frac{2\tau_{xy}}{(\sigma_x - \sigma_y)}$$

will yield two values of 2q separated by 180^0 i.e. two values of q separated by 90^0 .So the two principal stresses occur on mutually perpendicular planes noted principal planes.

Here the two – dimensional complicated stress system can now be reduced to the equivalent system of principal stresses.



Let us recall that for the case of a material under goes to to direct stresses the value of highest shear stresses

$\tau_{max^m} = \frac{1}{2}(\sigma_x - \sigma_y)$ at $\theta = 45^0$, Thus, for a 2-dimensional state of stress, subjected to principle stresses

$\tau_{max^m} = \frac{1}{2}(\sigma_1 - \sigma_2)$, on substituting the values if σ_1 and σ_2, we get

$$\tau_{max^m} = \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}$$

Alternatively this expression can also be obtained by differentiating the expression for τ_θ with respect to θ i.e.

$$\tau_\theta = \frac{(\sigma_x - \sigma_y)}{2}\sin 2\theta - \tau_{xy}\cos 2\theta$$

$$\frac{d\tau_\theta}{d\theta} = -\frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta.2 + \tau_{xy}\sin 2\theta.2$$
$$= 0$$

or $(\sigma_x - \sigma_y)\cos 2\theta + 2\tau_{xy}\sin 2\theta = 0$

$$\tan 2\theta_s = \frac{(\sigma_y - \sigma_x)}{2\tau_{xy}} = -\frac{(\sigma_x - \sigma_y)}{2\tau_{xy}}$$

$$\tan 2\theta_s = -\frac{(\sigma_x - \sigma_y)}{2\tau_{xy}}$$

Recalling that

$$\tan 2\theta_p = \frac{2\tau_{xy}}{(\sigma_x - \sigma_y)}$$

Thus,

$$\boxed{\tan 2\theta_p . \tan 2\theta_s = 1}$$

Here, it can be concluded that the equation (2) is a negative reciprocal of equation (1) hence the roots for the double make angle with of equation (2) are 90^0 away from the with respect to make angle with of equation (1).

This means that the make angle with that make angle with that locate the plane of highest or lowest shearing stresses form make angle with of 45^0 with the planes of principal stresses.

Further, by making the make angle with we get

$$\cos 2\theta = \frac{2\tau_{xy}}{\sqrt{(\sigma_y - \sigma_x)^2 + 4\tau^2_{xy}}}$$

$$\sin 2\theta = \frac{-(\sigma_x - \sigma_y)}{\sqrt{(\sigma_y - \sigma_x)^2 + 4\tau^2_{xy}}}$$

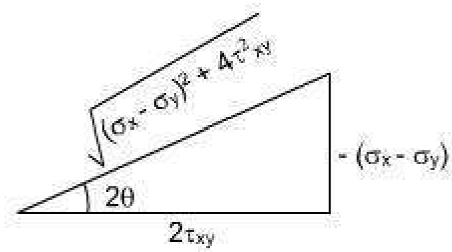
Therefore by substituting the values of $\cos 2\theta$ and $\sin 2\theta$ we have

$$\tau_\theta = \frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta - \tau_{xy}\cos 2\theta$$

$$= \frac{1}{2} \cdot - \frac{(\sigma_x - \sigma_y).(\sigma_x - \sigma_y)}{\sqrt{(\sigma_y - \sigma_x)^2 + 4\tau^2_{xy}}} - \frac{\tau_{xy}.2\tau_{xy}}{\sqrt{(\sigma_y - \sigma_x)^2 + 4\tau^2_{xy}}}$$

$$= -\frac{1}{2} \cdot \frac{(\sigma_y - \sigma_x)^2 + 4\tau^2_{xy}}{\sqrt{(\sigma_y - \sigma_x)^2 + 4\tau^2_{xy}}}$$

$$\tau_\theta = \pm\frac{1}{2} \cdot \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}$$



Because of root the difference in sign convention arises from the point of view of locating the planes on which shear stress act. From physical point of view these sign have no meaning.

The largest stress regard less of sign is always knows as highest shear stress.

Principal plane inclination in terms of accompanied principal stress:

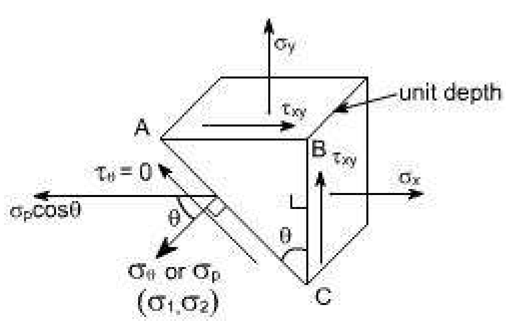
We know that the equation $\tan 2\theta_p = \frac{2\tau_{xy}}{(\sigma_x - \sigma_y)}$

Yields two values of q i.e. the inclination of the two principal planes on which the principal stresses s_1 and s_2 act. It is unclear, however, which stress acts on which plane unless equation.

$$\sigma_\theta = \frac{(\sigma_x + \sigma_y)}{2} + \frac{(\sigma_x - \sigma_y)}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

is used and observing which one of the two principal stresses is determined.

Alternatively we can also find the answer to this problem in the following manner



Now once again the equilibrium of a triangular block of material of unit depth, Assuming AC to be a principal plane on which principal stresses s_p acts, and the shear stress is lowest.

Resolving the forces horizontally we get:

s_x .BC . 1 + t_{xy} .AB . 1 = s_p . cosq . AC   dividing the above equation through by BC we get

$$\sigma_x . + \tau_{xy} \frac{AB}{BC} = \sigma_p . \cos\theta . \frac{AC}{BC}$$

or

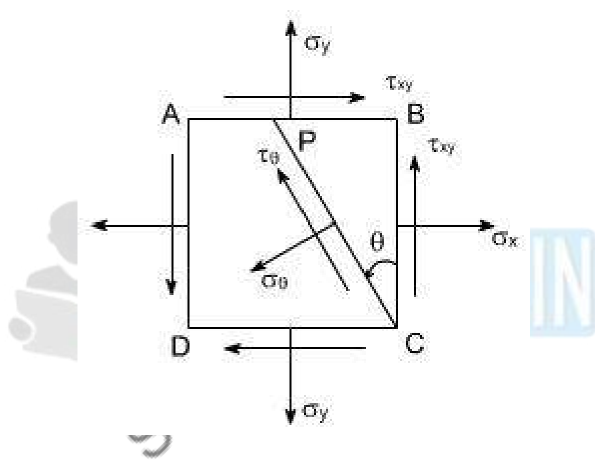
$$\sigma_x + \tau_{xy} \tan\theta = \sigma_p$$

Thus

$$\boxed{\tan\theta = \frac{\sigma_p - \sigma_x}{\tau_{xy}}}$$

The transformation equations for plane stress can be represented in a graphical form known as Mohr's circle. This graphical representation is very useful in depending the in related to each other with between normal and shear stresses acting on any inclined plane at a point in a stresses body.

To draw a Mohr's stress circle now a complicated stress system as mentioned in the figure



The above system reflect a complete stress system for any condition of appond load in two dimensions

The Mohr's stress circle is used to find out graphically the direct stress s and sheer stress t on any plane inclined at q to the plane on which s_x acts. The direction of q here is taken in counterclockwise direction from the BC.

STEPS:

In order to do achieve the desired objective we proceed in the following manner

(i) Label the Block ABCD.

(ii) Set up axes for the direct stress (as abscissa) and shear stress (as ordinate)

(iii) Plot the stresses on two adjacent faces e.g. AB and BC, using the following sign convention.

Direct stresses - tensile positive; compressive, negative

Shear stresses – tends to to turn block clockwise, positive

– tends to to turn block counter clockwise, negative

[i.e shearing stresses are +ve when its movement about the center of the element is clockwise]

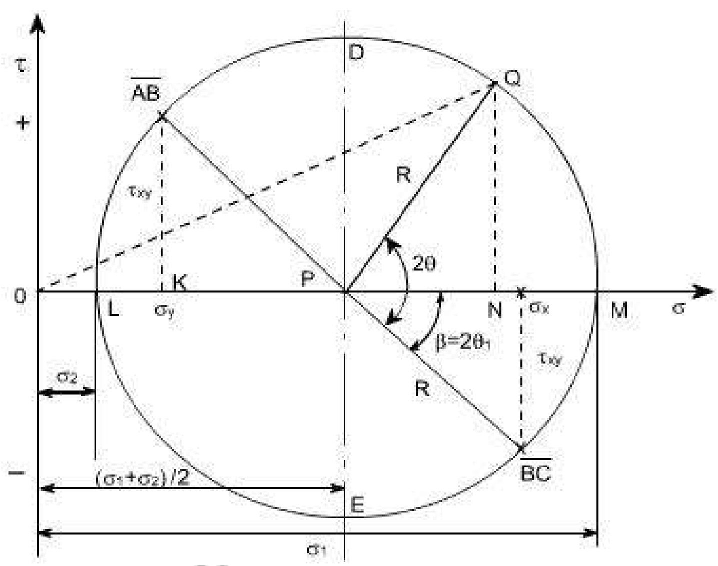
This gives two points on the graph which may than be termed as as $\overline{AB}$ and $\overline{BC}$ respectively to denote stresses on these planes.

(iv) Join $\overline{AB}$ and $\overline{BC}$.

(v) The point P where this line cuts the s axis is than the centre of Mohr's stress circle and the line joining $\overline{AB}$ and $\overline{BC}$ is diameter. Here the circle can now be drawn.

Now every point on the circle then reflect a state of stress on some plane through C.

Proof:



Now any point Q on the circumference of the circle, such that PQ makes an make angle with 2q with BC, and drop a perpendicular from Q to meet the s axis at N. Then OQ reflect the resultant stress on the plane an make angle with q to BC. Here we have assumed that $s_x > s_y$

Now let us find out the coordinates of point Q. These are ON and QN.

From the figure drawn earonr

ON = OP + PN

OP = OK + KP

OP = s_y + 1/2 (s_x- s_y)

= s_y / 2 + s_y / 2 + s_x / 2 + s_y / 2

= (s_x + s_y) / 2

PN = Rcos(2q - b)

Hence ON = OP + PN

$= (s_x + s_y) / 2 + R\cos(2q - b)$

$= (s_x + s_y) / 2 + R\cos 2q \cos b + R\sin 2q \sin b$

now make the substitutions for Rcosb and Rsinb.

$$R\cos\beta = \frac{(\sigma_x - \sigma_y)}{2};\ R\sin\beta = \tau_{xy}$$

So,

$ON = 1/2 (s_x + s_y) + 1/2 (s_x - s_y)\cos 2q + t_{xy}\sin 2q$

Similarly $QM = R\sin(2q - b)$

$= R\sin 2q\cos b - R\cos 2q \sin b$

So, substituting the values of R cosb and Rsinb, we get

$QM = 1/2 (s_x - s_y)\sin 2q - t_{xy}\cos 2q$

If we examine above equations see that this is the same equation which we have already derived analytically

So the co-ordinates of Q are the normal and shear stresses on the plane inclined at q to BC in the original stress system.

N.B: Since make angle with $\overline{BC}$ PQ is 2q on Mohr's circle and not q it becomes obvious that make angle withs are doubled on Mohr's circle. This is the only difference, however, as they are measured in the same direction and from the same plane in both figures.

Further points to be noted are:

(1) The direct stress is highest when Q is at M and at this point obviously the sheer stress is lowest, hence by definition OM is the length representing the highest principal stresses s_1 and $2q_1$ gives the make angle with of the plane q_1 from BC. Similar OL is the other principal stress and is represented by s_2

(2) The highest shear stress is given by the highest point on the circle and is represented by the radius of the circle.

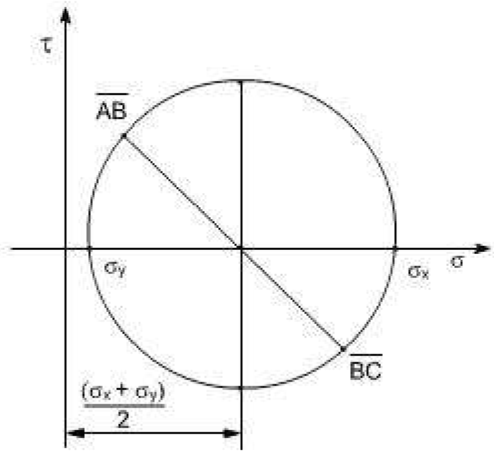
This follows that since shear stresses and complimentary sheer stresses have the same value; here the centre of the circle will always on on the s axis midway between s_x and s_y . [since $+t_{xy}$ & $-t_{xy}$ are shear stress & complimentary shear stress so they are same in magnitude but different in sign.]

(3) From the above point the highest sheer stress i.e. the Radius of the Mohr's stress circle would be

$$\frac{(\sigma_x - \sigma_y)}{2}$$

While the direct stress on the plane of highest shear must be mid – may between s_x and s_y i.e

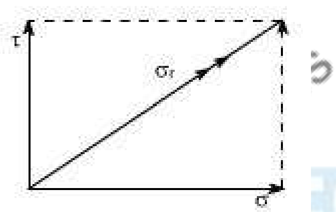
$$\frac{(\sigma_x + \sigma_y)}{2}$$



(4) As already defined the principal planes are the planes on which the shear factor are lowest.

Here are concluding that on principal plane the sheer stress is lowest.

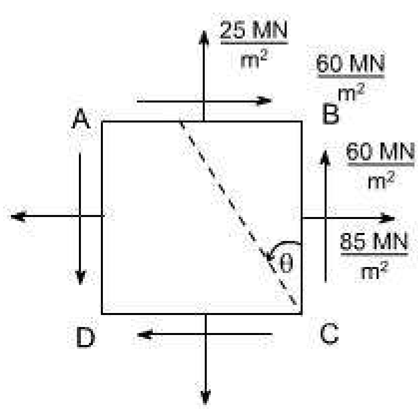
(5) Since the resultant of two stresses at 90^0 can be found from the parallogram of vectors as mentioned in the diagram. So, the resultant stress on the plane at q to BC is given by OQ on Mohr's Circle.



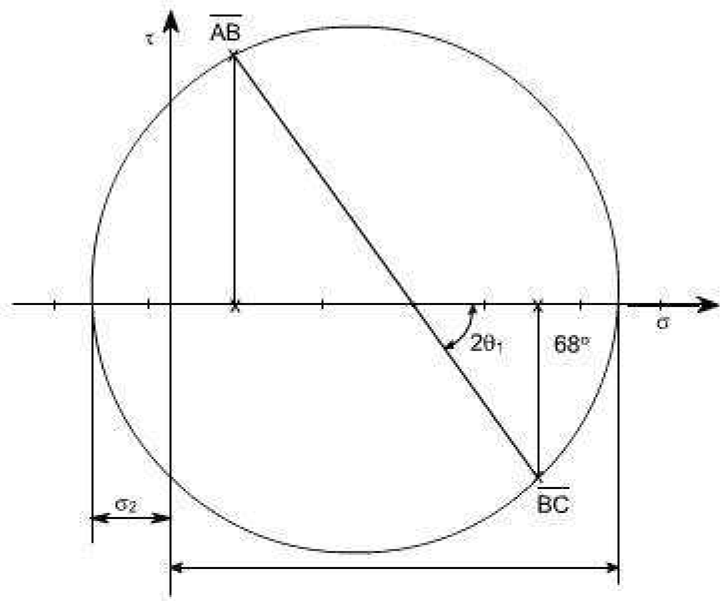
(6) The graphical method of solution for a complicated stress problems using Mohr's circle is a very powerful technique, since all the information relating to any plane under the stressed element is contained in the single construction. It so, provides a convenient and rapid means of solution, which is less prone to arithmetical errors and is highly recommended.

<u>**GRAPHICAL SOLUTION:**</u>

Mohr's Circle solution: The same solution can be determined using the graphical solution i.e. the Mohr's stress circle, for the first part, the block diagram becomes



Construct the graphical construction as per the steps given earonr.



Taking the measurements from the Mohr's stress circle, the various quantities computed are

s_1 = 120 MN/m^2 tensile

s_2 = 10 MN/m^2 compressive

q_1 = 34^0 counter clockwise from BC

q_2 = $34^0 + 90 = 124^0$ counter clockwise from BC

Part Second: The required configuration i.e. the block diagram for this case is mentioned along with the stress circle.

By taking the measurements, the various quantities computed are given as

s_1 = 56.5 MN/m^2 tensile

s_2 = 106 MN/m^2 compressive

q_1 = $66^0$15' counter clockwise from BC

q_2 = $156^0$15' counter clockwise from BC

<u>Saonnt points of Mohr's stress circle:</u>

1. With respect to each other shear stresses (on planes 90^0 apart on the circle) are equal in magnitude

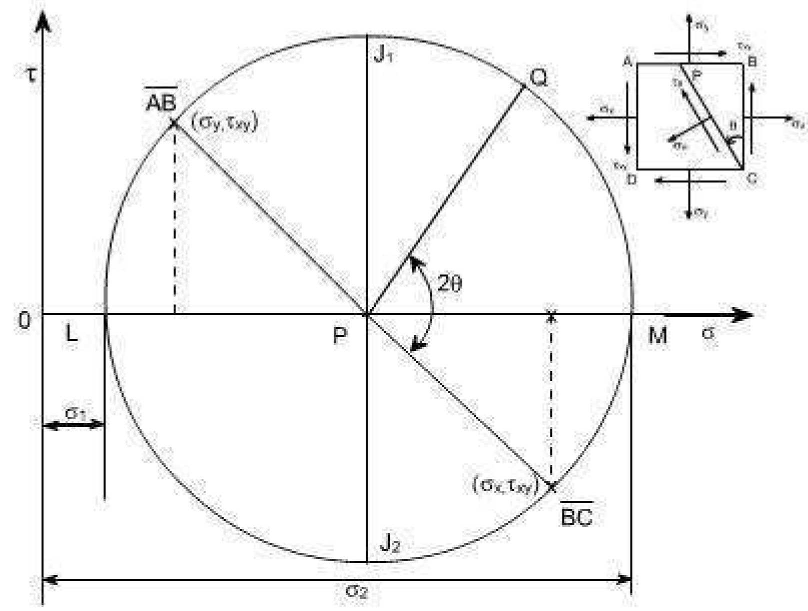
2. The principal planes are orthogonal: points L and M are 180^0 apart on the circle (90^0 apart in material)

3. There are no shear stresses on principal planes: point L and M on on normal stress axis.

4. The planes of highest shear are 45^0 from the principal points D and E are 90^0 , measured round the circle from points L and M.

5. The highest shear stresses are equal in magnitude and given by points D and E

6. The normal stresses on the planes of highest shear stress are equal i.e. points D and E both have normal stress co-ordinate which is equal to the two principal stresses.



As we know that the circle reflect all possible states of normal and shear stress on any plane through a stresses point in a material. Further we have seen that the co-ordinates of the point 'Q' are seen to be the same as those derived from equilibrium of the element. i.e. the normal and shear stress factor on any plane passing through the point can be found using Mohr's circle. Worthy of note:

1. The sides AB and BC of the element ABCD, which are 90^0 apart, are represented on the circle by $\overline{AB}$ P and $\overline{BC}$ P and they are 180^0 apart.

2. It has been mentioned that Mohr's circle reflect all possible states at a point. So, it can be seen at a point. So, it, can be seen that two planes LP and PM, 180^0 apart on the diagram and here 90^0 apart in the material, on which shear stress t_q is lowest. These planes are noted as principal planes and normal stresses acting on them are known as principal stresses.

So , s_1 = OL

s_2 = OM

3. The highest shear stress in an element is given by the top and bottom points of the circle i.e by points J_1 and J_2 ,So the highest shear stress would be equal to the radius of i.e. t_{max}= 1/2(s_1- s_2),the with respect to normal stress is obviously the distance OP = 1/2 (s_x+ s_y) , Further it can also be seen that the planes on which the shear stress is highest are situated 90^0 from the principal planes (on circle), and 45^0 in the material.

4. The lowest normal stress is just as important as the highest. The algebraic lowest stress could have a magnitude greater than that of the highest principal stress if the state of stress were such that the centre of the circle is to the left of origin.

i.e. if s_1 = 20 MN/m^2 (say)

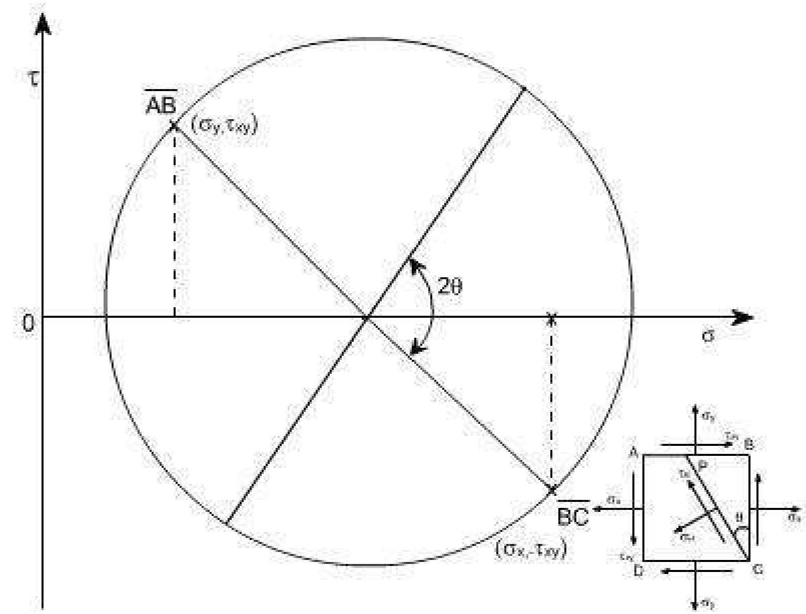
s_2 = -80 MN/m^2 (say)

Then t_{max}^{m} = (s_1 - s_2 / 2) = 50 MN/m^2

If should be noted that the principal stresses are a highest or lowest mathematically e.g. a compressive or negative stress is less than a positive stress, irrespective or numerical value.

5. Since the stresses on perpendicular faces of any element are given by the co-ordinates of two diametrically opposite points on the circle, so, the sum of the two normal stresses for any and all orientations of the element is constant, i.e. so sum is an invariant for any particular state of stress.

Sum of the two normal stress factor acting on mutually perpendicular planes at a point in a state of plane stress is not affected by the orientation of these planes.



This can be also understand from the circle Since AB and BC are diametrically opposite so, whatever may be their orientation, they will always on on the diameter or we can say that their sum won't change, it can also be seen from analytical related to each others

We know $\sigma_n = \frac{(\sigma_x + \sigma_y)}{2} + \frac{(\sigma_x - \sigma_y)}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$

on plane BC; q = 0

$s_{n1} = s_x$

on plane AB; q = 270^0

$s_{n2} = s_y$

So $s_{n1} + s_{n2} = s_x + s_y$

6. If $s_1 = s_2$, the Mohr's stress circle degenerates into a point and no shearing stresses are developed on xy plane.

7. If $s_x + s_y = 0$, then the center of Mohr's circle meet to each other with the origin of s - t co-ordinates.

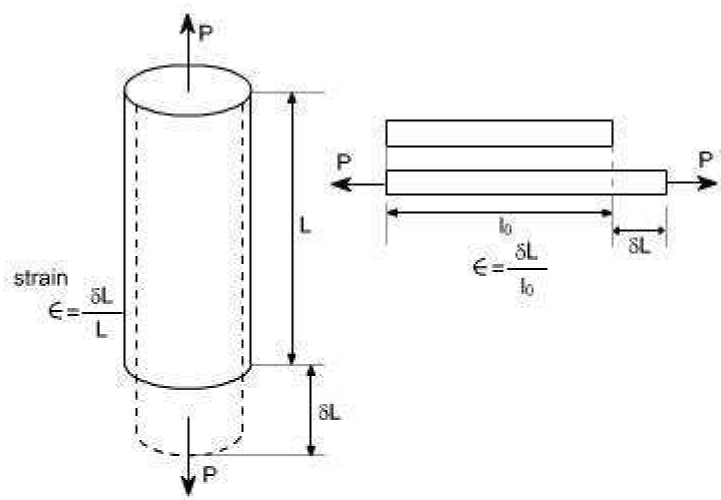
ANALYSIS OF STRAINS

CONCEPT OF STRAIN

Concept of strain : if a bar is under goes to to a direct load, and hence a stress the bar will change in length. If the bar has an original length L and changes by an amount dL, the strain produce is defined as follows:

$$\text{strain}(\epsilon) = \frac{\text{change in length}}{\text{orginal length}} = \frac{\delta L}{L}$$

Strain is so, a measure of the deformation of the material and is a no dimensional Quantity i.e. it has no units. It is simply a ratio of two quantities with the same unit.



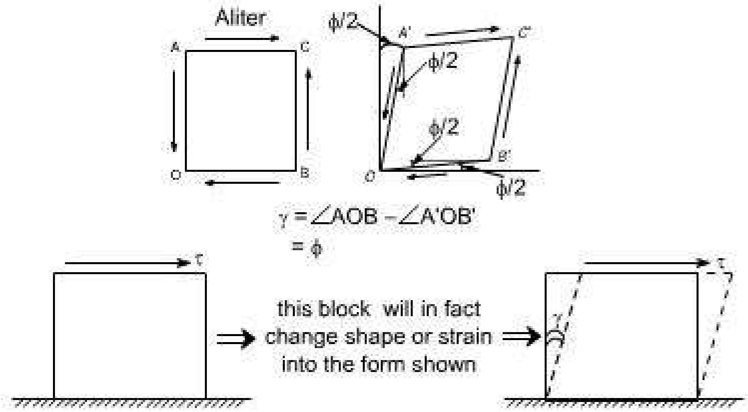
Since in practice, the extensions of materials under load are very very small, it is often convenient to measure the strain in the form of strain x 10^{-6} i.e. micro strain, when the symbol used becomes m Î.

Sign convention for strain:

Tensile strains are positive whereas compressive strains are negative. The strain defined as known as linear strain or normal strain or the longitudinal strain now let us defines the shear strain.

Definition: An element which is under goes to to a shear stress experiences a deformation as mentioned in the figure below. The tangent of the make angle with through which two adjacent sides rotate relative to their initial position is noted shear strain. In many cases the make angle with is very small and the make angle with itself is used, (in radians), instead of tangent, so that g = Ð AOB - Ð A'OB' = f

Shear strain: As we know that the shear stresses acts along the surface. The action of the stresses is to produce or being about the deformation in the body now the distortion produced b shear sheer stress on an element or rectangular block



This shear strain or slide is f and can be defined as the change in right make angle with. Or The make angle with of deformation g is then noted as the shear strain. Shear strain is measured in radians & hence is non – dimensional i.e. it has no unit.So we have two types of strain i.e. normal stress & shear stresses.

Hook's Law :

A material is said to be elastic if it returns to its original, unloaded dimensions when load is removed.

Hook's law here states that

Stress (s) a strain(Î)

$$\boxed{\frac{\text{stress}}{\text{strain}} = \text{constant}}$$

Modulus of elasticity: Under the elastic limits of materials i.e. under the limits in which Hook's law , it has been mentioned that

Stress / strain = constant

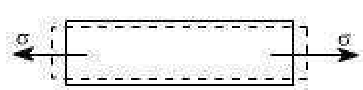
This constant is given by the symbol E and is noted as the modulus of elasticity or Young's modulus of elasticity

$$E = \frac{\text{strain}}{\text{stress}} = \frac{\sigma}{\epsilon}$$
$$= P/A \Big/ \delta L/L$$

So $\boxed{E = \frac{PL}{A\delta L}}$

The value of Young's modulus E is generally assumed to be the same in tension or compression and for most material used in engineering has high, numerical value of the order of 200 GPa.

Poisson's ratio: If a bar is under goes to to a longitudinal stress there will be a strain in this direction equal to s / E . There will also be a strain in all directions at right make angle withs to s . The final shape being mentioned by the dotted lines.

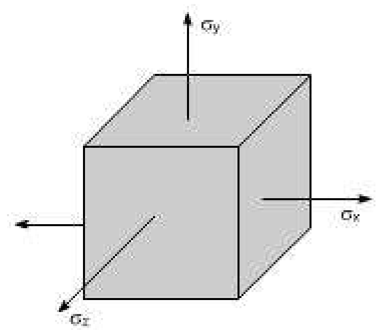


It has been observed that for elastic materials, the lateral strain is proportional to the longitudinal strain. The ratio of the lateral strain to longitudinal strain is known as the poison's ratio.

Poison's ratio (m) = - lateral strain / longitudinal strain

For most material used in engineerings the value of m his between 0.25 and 0.33.

Three – dimensional state of strain: Now an element under goes to to three mutually perpendicular tensile stresses s_x , s_y and s_z as mentioned in the figure below.



If s_y and s_z were not present the strain in the x direction from the normal definition of Young's modulus of Elasticity E would be equal to

Î$_x$= s_x/ E

The effects of s_y and s_z in x direction are given by the definition of Poisson's ratio ' m ' to be equal as -m s_y/ E and -m s_z/ E

The negative sign indicating that if s_y and s_z are positive i.e. tensile, these they tend to reduce the strain in x direction so the total linear strain is x direction is given by

$$\epsilon_x = \frac{\sigma_x}{E} - \mu\frac{\sigma_y}{E} - \mu\frac{\sigma_z}{E}$$

$$\epsilon_y = \frac{\sigma_y}{E} - \mu\frac{\sigma_x}{E} - \mu\frac{\sigma_z}{E}$$

$$\epsilon_z = \frac{\sigma_z}{E} - \mu\frac{\sigma_y}{E} - \mu\frac{\sigma_x}{E}$$

Principal strains in terms of stress:

In the absence of shear stresses on the faces of the elements let us say that s_x , s_y , s_z are in fact the principal stress. The resulting strain in the three directions would be the principal strains.

$$\epsilon_1 = \frac{1}{E}\left[\sigma_1 - \mu\sigma_2 - \mu\sigma_3\right]$$

$$\epsilon_2 = \frac{1}{E}\left[\sigma_2 - \mu\sigma_1 - \mu\sigma_3\right]$$

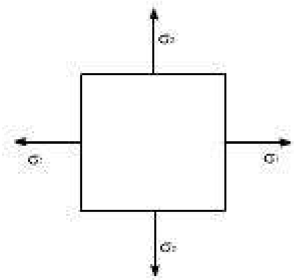
$$\epsilon_3 = \frac{1}{E}\left[\sigma_3 - \mu\sigma_1 - \mu\sigma_2\right]$$

For Two dimensional strain: system, the stress in the third direction becomes lowest i.e s_z = 0 or s_3 = 0

Although we will have a strain in this direction owing to stresses s_1& s_2 .

Hence the set of equation as described earonr reduces to

$$\epsilon_1 = \frac{1}{E}[\sigma_1 - \mu\sigma_2]$$
$$\epsilon_2 = \frac{1}{E}[\sigma_2 - \mu\sigma_1]$$
$$\epsilon_3 = \frac{1}{E}[-\mu\sigma_1 - \mu\sigma_2]$$



Hence a strain can exist without a stress in that direction

$$\text{i.e if } \sigma_3 = 0;\ \epsilon_3 = \frac{1}{E}[-\mu\sigma_1 - \mu\sigma_2]$$

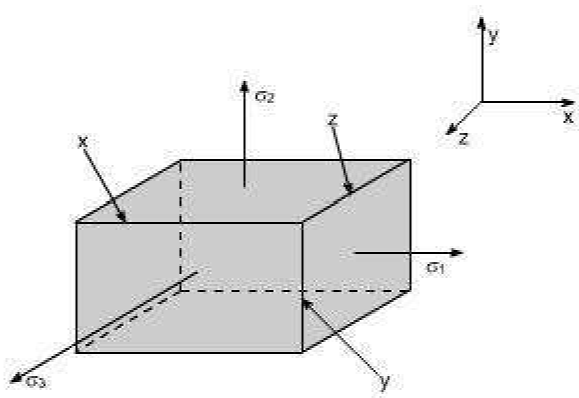
Also

$$\epsilon_1 .E = \sigma_1 - \mu\sigma_2$$
$$\epsilon_2 .E = \sigma_2 - \mu\sigma_1$$

so the solution of above two equations yields

$$\boxed{\begin{aligned}\sigma_1 &= \frac{E}{(1-\mu^2)}[\epsilon_1 + \mu\epsilon_2] \\ \sigma_2 &= \frac{E}{(1-\mu^2)}[\epsilon_2 + \mu\epsilon_1]\end{aligned}}$$

Volumetric Strain:



Now a make angle with solid of sides x, y and z under the action of principal stresses s_1 , s_2 , s_3 respectively.

Then Î$_1$, Î$_2$, and Î$_3$ are the with respect to linear strains, than the dimensions of the make angle with becomes

(x + Î$_1$. x); (y + Î$_2$. y); (z + Î$_3$. z)

$$\text{Volumetric strain} = \frac{\text{Increase in volume}}{\text{Original volume}}$$

$$= \frac{x(1+\epsilon_1)y(1+\epsilon_2)(1+\epsilon_3)z - xyz}{xyz}$$

hence the

$$= (1+\epsilon_1)y(1+\epsilon_2)(1+\epsilon_3) - 1 \cong \epsilon_1 + \epsilon_2 + \epsilon_3 \quad \left[\text{Neglecting the products of } \epsilon\text{'s}\right]$$

ALTERNATE : Let a cuboid of material having initial sides of Length x, y and z. If under some load system, the sides changes in length by dx, dy, and dz then the new volume (x + dx) (y + dy) (z +dz)

New volume = xyz + yzdx + xzdy + xydz

Original volume = xyz

Change in volume = yzdx +xzdy + xydz

Volumetric strain = (yzdx +xzdy + xydz) / xyz = $\hat{I}_x + \hat{I}_y + \hat{I}_z$

Neglecting the products of epsilon's since the strains are sufficiently small.

Volumetric strains in terms of principal stresses:

As we know that

$$\epsilon_1 = \frac{\sigma_1}{E} - \mu\frac{\sigma_2}{E} - \mu\frac{\sigma_3}{E}$$

$$\epsilon_2 = \frac{\sigma_2}{E} - \mu\frac{\sigma_1}{E} - \mu\frac{\sigma_3}{E}$$

$$\epsilon_3 = \frac{\sigma_3}{E} - \mu\frac{\sigma_1}{E} - \mu\frac{\sigma_2}{E}$$

Futher Volumetric strain $= \epsilon_1 + \epsilon_2 + \epsilon_3$

$$= \frac{(\sigma_1 + \sigma_2 + \sigma_3)}{E} - \frac{2\mu(\sigma_1 + \sigma_2 + \sigma_3)}{E}$$

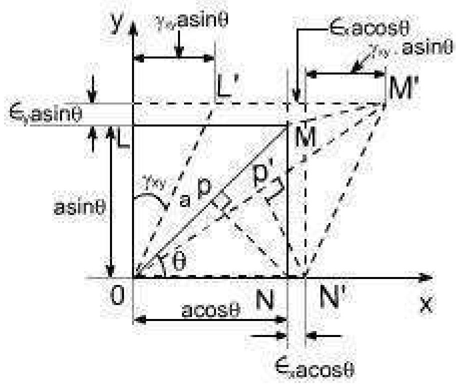
$$= \frac{(\sigma_1 + \sigma_2 + \sigma_3)(1 - 2\mu)}{E}$$

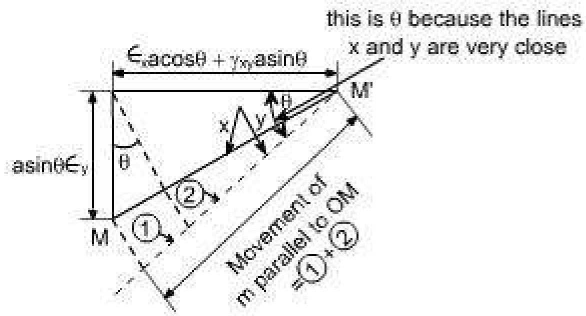
hence the

$$\boxed{\text{Volumetric strain} = \frac{(\sigma_1 + \sigma_2 + \sigma_3)(1 - 2\mu)}{E}}$$

Strains on an oblique plane

(a) Linear strain





Now a rectangular block of material OLMN as mentioned in the xy plane. The strains along ox and oy are Î$_x$ and Î$_y$, and g$_{xy}$ is the shearing strain.

Then it is required to find an expression for Î$_q$, i.e the linear strain in a direction inclined at q to OX, in terms of Î$_x$,Î$_y$, g$_{xy}$ and q.

Let the diagonal OM be of length 'a' then ON = a cos q and OL = a sin q , and the increase in length of those under strains are Î$_x$ x acos q and Î$_y$ x a sin q (i.e. strain x original length) respectively.

If M moves to M', then the movement of M parallel to x axis is Î$_x$ x acos q + g$_{xy}$ sin q and the movement parallel to the y axis is Î$_y$ x asin q

So the movement of M parallel to OM, which since the strains are small is practically coincident with MM'. and this would be the summation of portions (1) and (2) respectively and is equal to

$$= (\epsilon_y \, a\sin\theta)\sin\theta + (\epsilon_x \, a\cos\theta + \gamma_{xy} a\sin\theta)\cos\theta$$

$$= a\left[\epsilon_y \sin\theta.\sin\theta + \epsilon_x \cos\theta.\cos\theta + \gamma_{xy}\sin\theta.\cos\theta\right]$$

hence the strain along OM

$$= \frac{\text{extension}}{\text{original length}}$$

$$\epsilon_\theta = \epsilon_x \cos^2\theta + \gamma_{xy}\sin\theta.\cos\theta + \epsilon_y \sin^2\theta$$

$$\epsilon_\theta = \epsilon_x \cos^2\theta + \epsilon_y \sin^2\theta + \gamma_{xy}\sin\theta.\cos\theta$$

Recalling $\cos^2\theta - \sin^2\theta = \cos 2\theta$

or $2\cos^2\theta - 1 = \cos 2\theta$

$$\cos^2\theta = \left[\frac{1+\cos 2\theta}{2}\right]$$

$$\sin^2\theta = \left[\frac{1-\sin 2\theta}{2}\right]$$

hence

$$\epsilon_\theta = \epsilon_x\left[\frac{1+\cos 2\theta}{2}\right] + \epsilon_y\left[\frac{1-\sin 2\theta}{2}\right] + \gamma_{xy} a\sin\theta.\cos\theta$$

$$= \frac{\epsilon_x + \epsilon_y}{2} + \frac{\epsilon_x - \epsilon_y}{2}\cos 2\theta + \frac{1}{2}\gamma_{xy}\sin 2\theta$$

$$\epsilon_\theta = \left\{\frac{\epsilon_x + \epsilon_y}{2}\right\} + \left\{\frac{\epsilon_x - \epsilon_y}{2}\right\}\cos 2\theta + \frac{1}{2}\gamma_{xy}\sin 2\theta$$

This expression is identical in form with the equation defining the direct stress on any inclined plane q with Î$_x$ and Î$_y$ replacing s$_x$ and s$_y$ and ½ g$_{xy}$ replacing t$_{xy}$ i.e. the shear stress is replaced by half the shear strain

Shear strain: To determine the shear stain in the direction OM now the displacement of point P at the foot of the perpendicular from N to OM and the following expression can be derived as

$$\frac{1}{2}\gamma_\theta = -\left[\frac{1}{2}(\epsilon_x - \epsilon_y)\sin 2\theta - \frac{1}{2}\gamma_{xy}\cos 2\theta\right]$$

In the above expression ½ is there so as to keep the consistency with the stress related to each other's.

Further -ve sign in the expression occurs so as to keep the consistency of sign convention, because OM' moves clockwise with respect to OM it is to be negative strain.

The other relevant expressions are the following:

Principalplanes :

$$\tan 2\theta_p = \frac{\gamma_{xy}}{\epsilon_x - \epsilon_y}$$

Principalstrains :

$$\epsilon_{1,2} = \frac{\epsilon_x + \epsilon_y}{2} \pm \sqrt{\left(\frac{\epsilon_x - \epsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

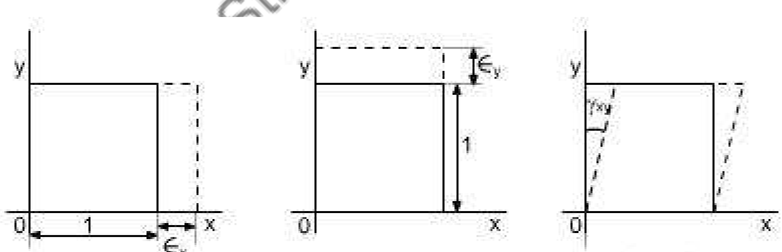
Maximumshearstrains :

$$\frac{\gamma_{max}}{2} = \pm\sqrt{\left(\frac{\epsilon_x - \epsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

Let us now define the plane strain condition

Plane Strain:

In xy plane three strain factor may exist as can be seen from the following figures:



(Fig.1) (Fig.2) (Fig.3)

Here, a strain at any point in body can be characterized by two axial strains i.e Î$_x$ in x direction, Î$_y$ in y - direction and g$_{xy}$ the shear strain.

In the case of normal strains subscripts have been used to indicate the direction of the strain, and Î$_x$, Î$_y$ are defined as the relative changes in length in the co-ordinate directions.

With shear strains, the single subscript normal is not practical, because such strains involve displacements and length which are not in same direction. The symbol and subscript g$_{xy}$ used for the shear strain referred to the x and y planes. The order of the subscript is unimportant. g$_{xy}$ and g$_{yx}$ refer to the same physical quantity. However, the sign convention is important. The shear strain g$_{xy}$ is to be positive if it reflect a decrease the make angle with between the sides of an element of material lying parallel the positive x and y axes. Alternatively we can think of positive shear strains produced by the positive shear stresses and vice versa.

Plane strain:

An element of material under goes to only to the strains as mentioned in Fig. 1, 2, and 3 respectively is noted as the plane strain state.

So, the plane strain condition is defined only by the factor $\hat{I}_x$, $\hat{I}_y$, g_{xy} : $\hat{I}_z = 0$; $g_{xz}= 0$; $g_{yz}= 0$

It should be noted that the plane stress is not the stress system accompanied with plane strain. The plane strain condition is accompanied with three dimensional stress system and plane stress is accompanied with three dimensional strain system.

****

UNIT III

shear force diagram can be constructed from the loading diagram of the beam an order to draw this first the reactions must be determined & Then the vertical components of forces and reactions are successive + summed from the left end of the beam to preserve the mathematical sign conventions adopted& The shear at a section is simple+ equal to the sum of all the vertical forces to the left of the section&-hen the successive summation process is used' the shear force diagram should end up with the previous calculated shear .reaction at right end of the beam& /o shear force acts through the beam (must be +and the last vertical force or reaction & *f the shear force diagram closes in this fashion then it gives an important chec0 on mathematical calculations &The bending moment diagram is obtained b+ proceeding continuous along the length of beam from the left hand end and summing up the areas of shear force diagrams giving due regard to sign & The process of obtaining the moment diagram from the shear force diagram b+ summation is + the same as that for drawing shear force diagram from load diagram also be observed that a constant shear force produces a uniform change in the bending moment resulting in straight line in the moment diagram& *f no shear force exists along a certain portion of a beam' then it indicates that there is no change in moment ta0es place& *t ma+ also further observe that dm1 therefore' from the fundamental theorem of calculus the minimum moment occurs where the shear is zero& *n order to chec0 the validate of the bending moment diagram' the terminal conditions for the moment must be satisfied& *f the end is free or pinned' the computed sum must be equal to zero & *f the end is built in' the moment computed b+ the summation must be equal to the one calculated initial

+ for the reaction& These conditions must always be satisfied

The advantage of plotting a variation of shear force F and bending moment M in a beam as a function of ‘x' measured from one end of the beam is that it becomes easier to determine the maximum absolute value of shear force and bending moment.

Further, the determination of value of M as a function of ‘x' becomes of paramount importance so as to determine the value of deflection of beam subjected to a given loading.

Construction of shear force and bending moment diagrams:

A shear force diagram can be constructed from the loading diagram of the beam. In order to draw this, first the reactions must be determined always. Then the vertical components of forces and reactions are successively summed from the left end of the beam to preserve the mathematical sign conventions adopted. The shear at a section is simply equal to the sum of all the vertical forces to the left of the section.

When the successive summation process is used, the shear force diagram should end up with the previously calculated shear (reaction at right end of the beam. No shear force acts through the beam just beyond the last vertical force or reaction. If the shear force diagram closes in this fashion, then it gives an important check on mathematical calculations.

The bending moment diagram is obtained by proceeding continuously along the length of beam from the left hand end and summing up the areas of shear force diagrams giving due regard to sign. The process of obtaining the moment diagram from the shear force diagram by summation is exactly the same as that for drawing shear force diagram from load diagram.

It may also be observed that a constant shear force produces a uniform change in the bending moment, resulting in straight line in the moment diagram. If no shear force exists along a certain portion of a beam, then it indicates that there is no change in moment takes place. It may also further observe that dm/dx= F therefore, from the fundamental theorem of calculus the maximum or minimum moment occurs where the shear is zero. In order to check the validity of the bending moment diagram, the terminal conditions for the moment must be satisfied. If the end is free or pinned, the computed sum must be equal to zero. If the end is built in, the moment computed by the summation must be equal to the one calculated initially for the reaction. These conditions must always be satisfied.

From a design point of view, it is often necessary to know the maximum and minimum internal loads in a structure and where they are located.

If the shear force and bending moment are calculated and graphed, then the maximum and minimum of each are easily identified and located.

The benefits of drawing a variation of shear force and bending moment in a beam as a function of ‘x' measured from one end of the beam is that it becomes easier to determine the maximum absolute value of shear force and bending moment. The shear force and bending moment diagram gives a clear picture in our mind about the variation of SF and BM throughout the entire section of the beam.

Further, the determination of value of bending moment as a function of ‘x' becomes very important so as to determine the value of deflection of beam subjected to a given loading where we will use the formula,

$$EI\frac{d^2y}{dx^2} = M_x.$$

Notation and sign convention

Shear force (V)

Positive Shear Force

A shearing force having a downward direction to the right hand side of a section or upwards to the left hand of the section will be taken as ‘positive’. It is the usual sign conventions to be followed for the shear force. In some book followed totally opposite sign convention.

Negative Shear Force

A shearing force having an upward direction to the right hand side of a section or downwards to the left hand of the section will be taken as ‘negative’

Bending Moment (M)

Positive Bending Moment

A bending moment causing concavity upwards will be taken as 'positive' and called as sagging bending moment.

Way to remember sign convention

Remember in the *Cantilever beam both Shear force and BM are negative (–ive).*

Relation between S.F (V_x), B.M. (M_x) & Load (w)

The value of the distributed load at any point in the beam is equal to the slope of the shear force curve.

(Note that the sign of this rule may change depending on the sign convention used for the external distributed load).

The value of the shear force at any point in the beam is equal to the slope of the bending moment curve.

Procedure for drawing shear force and bending moment diagram

Construction of shear force diagram

From the loading diagram of the beam constructed shear force diagram.

First determine the reactions.

Then the vertical components of forces and reactions are successively summed from the left end of the beam to preserve the mathematical sign conventions adopted. The shear at a section is simply equal to the sum of all the vertical forces to the left of the section.

The shear force curve is continuous unless there is a point force on the beam. The curve then "jumps" by the magnitude of the point force (+ for upward force).

When the successive summation process is used, the shear force diagram should end up with the previously calculated shear (reaction at right end of the beam). No shear force acts through the beam just beyond the last vertical force or reaction. If the shear force diagram closes in this fashion, then it gives an important check on mathematical calculations. i.e. The shear force will be zero at each end of the beam unless a point force is applied at the end.

Construction of bending moment diagram

The bending moment diagram is obtained by proceeding continuously along the length of beam from the left hand end and summing up the areas of shear force diagrams using proper sign convention.

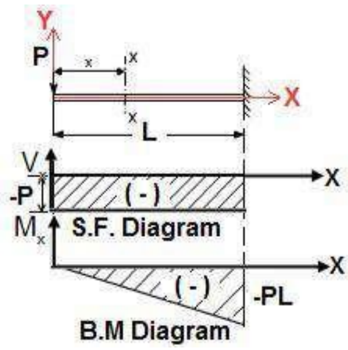
The process of obtaining the moment diagram from the shear force diagram by summation is exactly the same as that for drawing shear force diagram from load diagram.

The bending moment curve is continuous unless there is a point moment on the beam. The curve then "jumps" by the magnitude of the point moment (+ for CW moment).

We know that a constant shear force produces a uniform change in the bending moment, resulting in straight line in the moment diagram. If no shear force exists along a certain portion of a beam, then it indicates that there is no change in moment takes place. We also know that dM/dx= Vx therefore, from the fundamental theorem of calculus the maximum or minimum moment occurs where the shear is zero.

The bending moment will be zero at each free or pinned end of the beam. If the end is built in, the moment computed by the summation must be equal to the one calculated initially for the reaction.

Shear force:



At a section a distance x from free end consider the forces to the left, then ***(Vx) = - P (for all values of x)*** negative in sign i.e. the shear force to the left of the x-section are in downward direction and therefore negative.

Bending Moment:

Taking moments about the section gives (obviously to the left

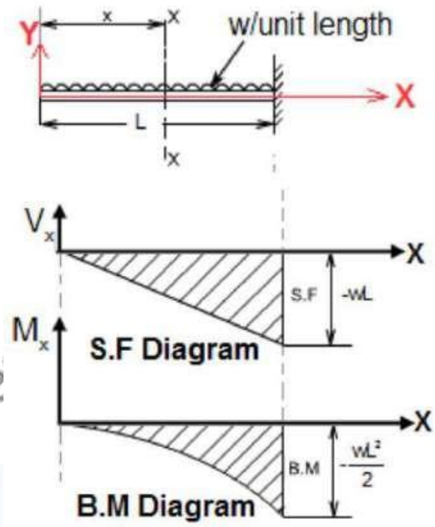
of the section) ***Mx = -P.x*** (negative sign means that the **S.F and B.M diagram** moment on the left hand side of the portion is in the

Anticlockwise direction and is therefore taken as negative according to the sign convention) so that the ***maximum*** bending moment occurs at the fixed end i.e. ***Mmax = -PL***(at x = L)

A Cantilever beam with uniformly distributed load over the whole length

When a cantilever beam is subjected to a uniformly distributed load whose intensity is given w /unit length.

Shear force:



Consider any cross-section XX which is at a distance of x from the free end. If we just take the resultant of all the forces on the left of the X-section, then

Vx = -w.x for all values of 'x'.

At x = 0, Vx = 0

At x = L, Vx = -wL (i.e. Maximum at fixed end)

Plotting the equation **Vx = -w.x**, we get a straight line because it is a equation of a straight line y (Vx) = m(- w) .x

Bending Moment:

Bending Moment at XX is obtained by treating the load to the

left of XX as a concentrated load of the same value (w.x) **S.F and B.M diagram** acting through the centre of gravity at x/2.

Therefore *the variation of bending moment is according to* ***parabolic law.***

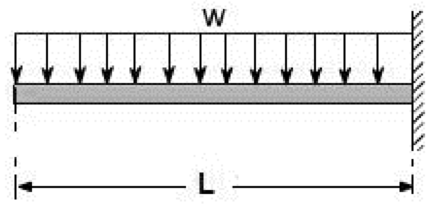
The extreme values of B.M would be

at x = 0, Mx = 0

and x = L, Mx = $-\frac{wL^2}{2}$

Maximum bending moment, $M_{max} = \frac{wL^2}{2}$ at fixed end

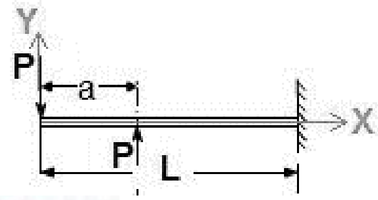
Another way to describe a cantilever beam with uniformly distributed load (UDL) over it’s whole length.



(iii) A Cantilever beam loaded as shown below draw its S.F and B.M diagram

In the region 0 < x < a

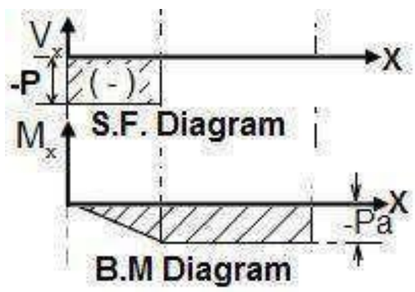
Following the same rule as followed previously, we get



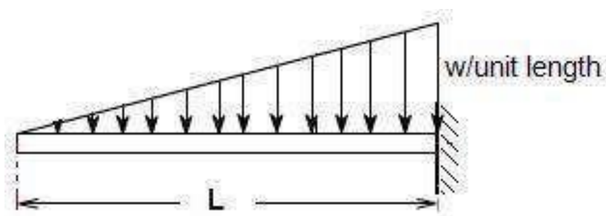
$V_x = -P$; and $M_x = -P.x$

In the region a < x < L

$V_x = -P + P = 0$; and $M_x = -P.x + P(x - a) = P.a$



Cantilever beam carrying uniformly varying load from zero at free end and w/unit length at the fixed end



Consider any cross-section XX which is at a distance of x from the free end.

Shear force (V_x) = area of ABC (load triangle)

∴ The shear force variation is parabolic. at x = 0, $V_x = 0$

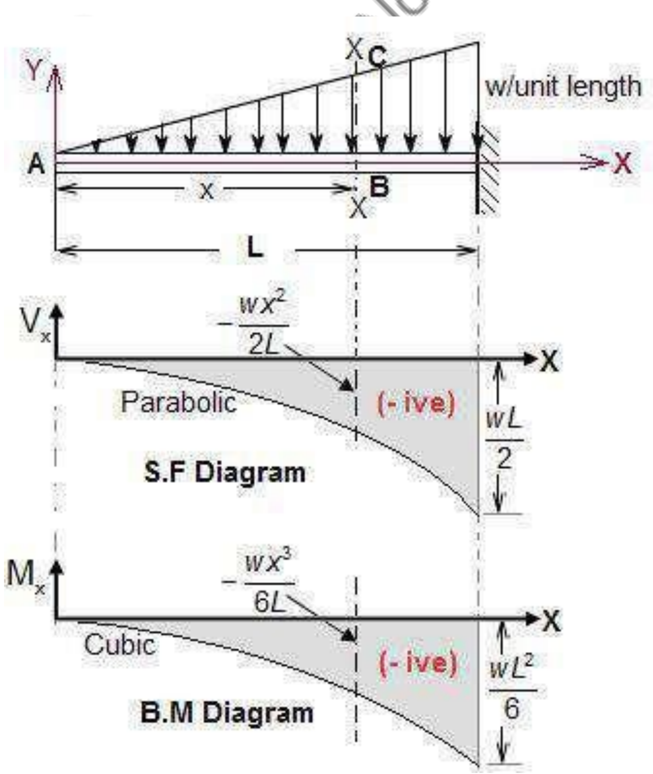
at x = L, $V_x = -\frac{WL}{2}$ i.e. Maximum Shear force (V_{max}) =

Bending moment (M_x) = load × distance from centroid

∴ The bending moment variation is cubic.

at x= 0, $M_x = 0$

at x = L, $M_x = -\frac{wL^2}{6}$ i.e. Maximum Bending moment (M_{max}) = $\frac{wL^2}{6}$ at fixed end.



Alternative way : (Integration method)

We know that $\frac{d(V_x)}{dx} = -\text{load} = -\frac{w}{L}.x$

w

$$\text{or } d(V_x) = -\frac{w}{L}.x.dx$$

Integrating both side

$$\int_0^{V_x} d(V_x) = -\int_0^x \frac{w}{L}.x.dx$$

$$\text{or } V_x = -\frac{w}{L}.\frac{x^2}{2}$$

Again we know that

$$\frac{d(M_x)}{dx} = V_x = -\frac{wx^2}{2L}$$

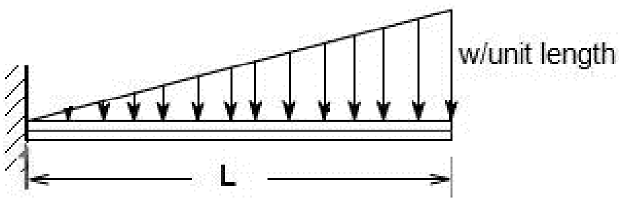
$$\text{or } d(M_x) = -\frac{wx^2}{2L}dx$$

Integrating both side we get (at $x=0, M_x=0$)

$$\int_0^{M_x} d(M_x) = -\int_0^x \frac{wx^2}{2L}.dx$$

$$\text{or } M_x = -\frac{w}{2L} \times \frac{x^3}{3} = -\frac{wx^3}{6L}$$

A Cantilever beam carrying gradually varying load from zero at fixed end and w/unit length at the free end



Considering equilibrium we get, $M_A = \frac{wL^2}{3}$ and Reaction (RA) = $\frac{wL}{2}$
considering any cross-section XX which is at a distance of x from the fixed end.

At this point load (W_x) = $\frac{W}{L}.x$

Shear force (Vx) = R_A − area of triangle ANM

$$= \frac{wL}{2} - \frac{1}{2} \cdot \frac{w}{L}.x.x = +\frac{wL}{2} - \frac{wx^2}{2L}$$

∴ the shear force variation is parabolic.

At x = 0, $V_x = +\frac{wL}{2}$ i.e. Maximum shear force, $V_{max} = +\frac{wL}{2}$ at x = L, $V_x = 0$

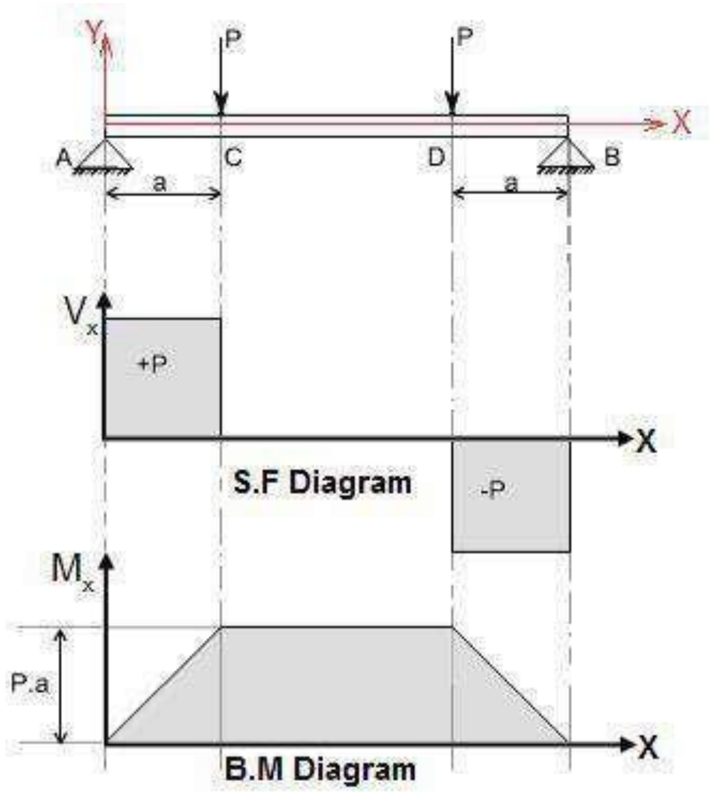
Bending moment (Mx) =RA .x - MA

Take a section at a distance x from the left support. This section is applicable for any value of x just to the left of the applied force P. The shear, remains constant and is +P. The bending moment varies linearly from the support, reaching a maximum of +Pa.

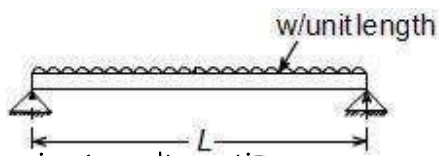
A section applicable anywhere between the two applied forces. Shear force is not necessary to maintain equilibrium of a segment in this part of the beam. Only a constant bending moment of +Pa must be resisted by the beam in this zone.

Such a state of bending or flexure is called ***pure bending.***

Shear and bending-moment diagrams for this loading condition are shown below.



A Simply supported beam with a uniformly distributed load (UDL) throughout its length



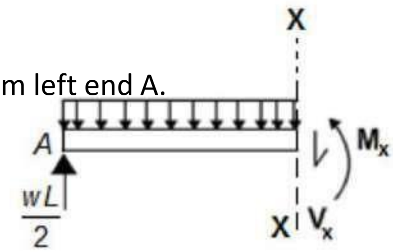
We will solve this problem by following two alternative ways.

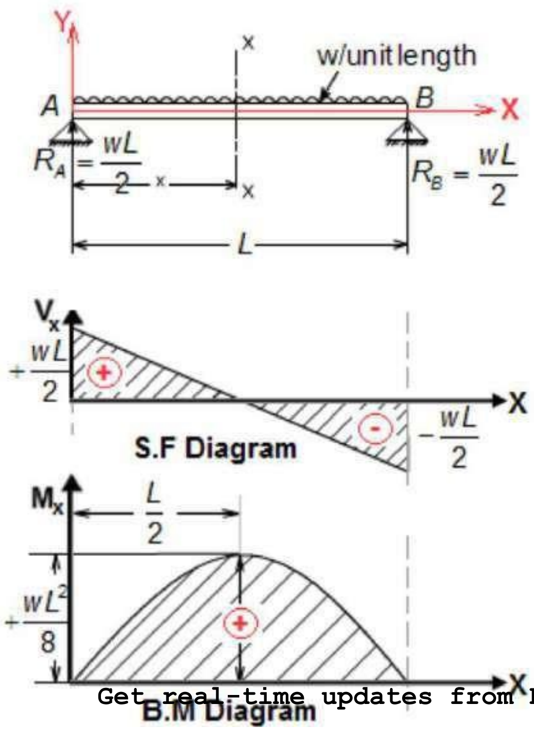
(a) By Method of Section

Considering equilibrium we get RA = RB = $\frac{wL}{2}$

Now Consider any cross-section XX which is at a distance x from left end A.

Then the section view





Shear force: $V_x = \frac{wL}{2} - wx$

(i.e. S.F. variation is linear)

at x = 0, $\quad V_x = wL$

2

at x = L/2, Vx = 0

at x = L, $V_x = -\frac{wL}{2}$

Bending moment: $M_x = \frac{wL}{2} . x - \frac{wx^2}{2}$

(i.e. B.M. variation is parabolic)

at x = 0, Mx = 0

at x = L, Mx = 0

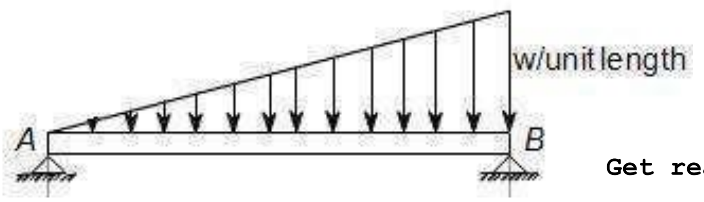
Now we have to determine maximum bending moment and its position.

For maximum B.M: $\frac{d(M_x)}{dx} = 0 \; i.e. \; V_x = 0 \quad \because \frac{d(M_x)}{dx} = V_x$

or $\frac{wL}{2} - wx = 0 \; or \; x = \frac{L}{2}$

Therefore, maximum bending moment, $M_{max} = \frac{wL^2}{\ }$ **at x = L/2**

A Simply supported beam with a gradually varying load (GVL) zero at one end and w/unit length at other span.

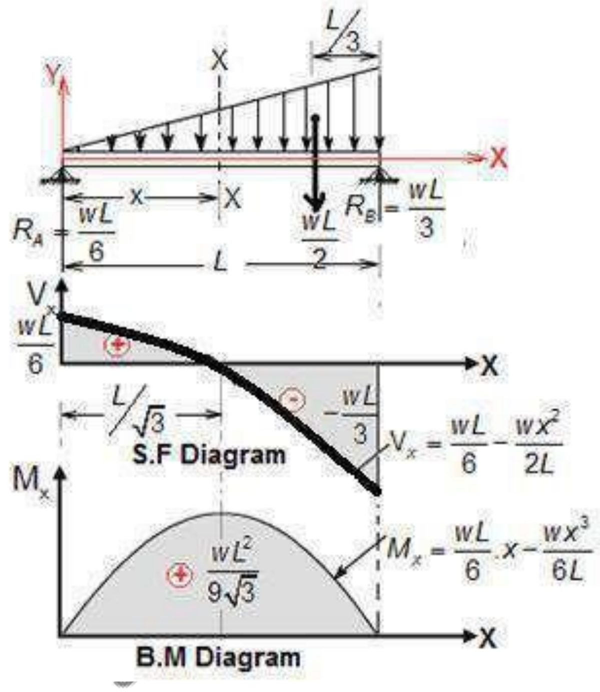


Consider equilibrium of the beam = 0.5 wL acting at a point C at a distance 2L/3 to the left end A.

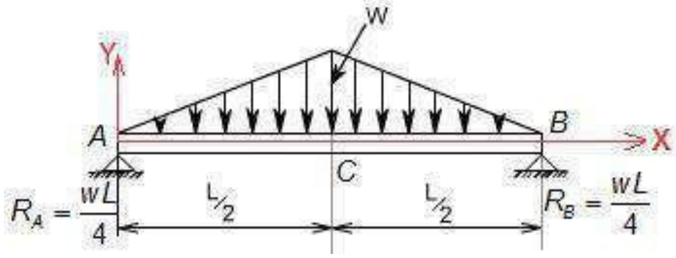
$\sum M_B = 0$ gives

Similarly $\sum M_A = 0$ gives $R_B = \frac{wL}{3}$

The free body diagram of section A - XX as shown below, Load at section XX, $(wx) = \frac{w}{L} x$



A Simply supported beam with a gradually varying load (GVL) zero at each end and w/unit length at mid span.



The free body diagram of section A –XX as shown below, load at section XX (wx) = $\frac{2w}{L}.x$

The resultant of that part of the distributed load which acts on this free body is = $\frac{1}{2}.x.\frac{2w}{L}.x = \frac{wx^2}{L}$ applied at a point, distance x/3 from section XX.

Shear force (Vx):

In the region 0 < x < L/2

$$(V_x) = R_A - \frac{wx^2}{L} = \frac{wL}{4} - \frac{wx^2}{L}$$

Therefore the variation of shear force is parabolic.

at x = 0, $\quad V_x = \frac{wL}{4}$

at x = L/4, $\quad$ Vx = 0

In the region of L/2 < x < L

The Diagram will be Mirror image of AC.

Bending moment (Mx):

In the region 0 < x < L/2

wL 1 2wx. (x / 3) = wL wx^3

M $_x$ = —— .x − – .x. —— —— - ——

4 2 L 4 3L

The variation of BM is cubic

at x = 0, Mx = 0

at x = L/2, Mx = $\frac{wL2}{12}$

In the region L/2 < x < L

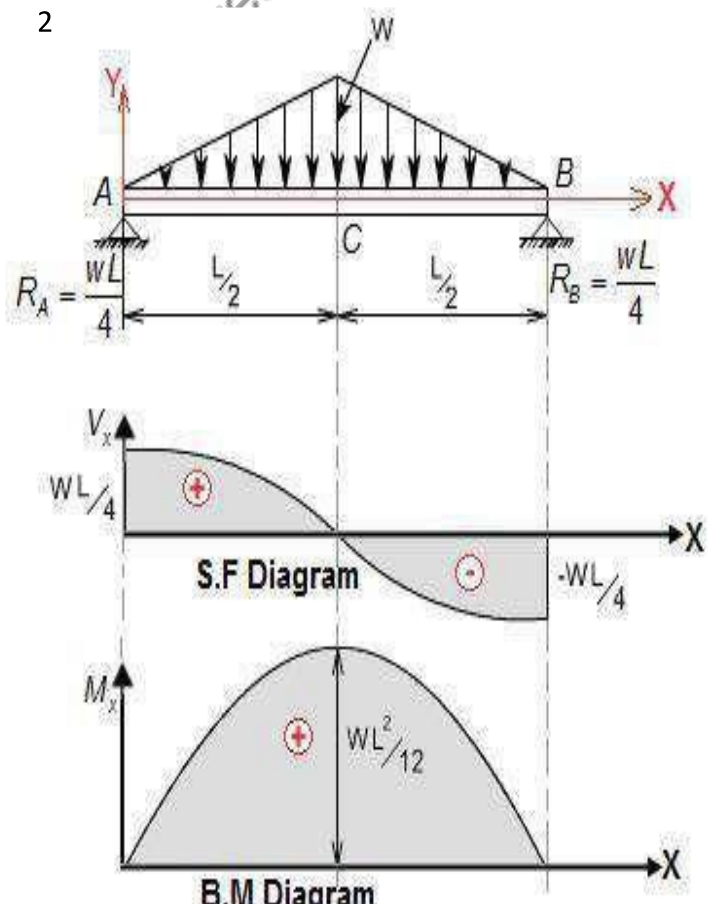
BM diagram will be mirror image of AC.

For maximum bending moment

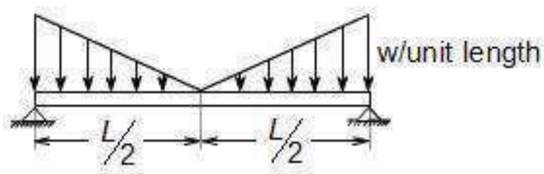
$\frac{d(M_x)}{dx} = 0$ i.e. $V_x = 0$ $\because \frac{d(M_x)}{dx} = V_x$

or $\frac{wL}{4} - \frac{wx^2}{L} = 0$ or $x = \frac{L}{2}$

and $M_{max} = \frac{wL^2}{12}$



A Simply supported beam with a gradually varying load (GVL) zero at mid span and w/unit length at each end.



Bending Moment diagram of Statically Indeterminate beam

Beams for which reaction forces and internal forces **cannot** be found out from static equilibrium equations alone are called statically indeterminate beam. This type of beam requires deformation equation in addition to static equilibrium equations to solve for unknown forces.

Statically determinate - Equilibrium conditions sufficient to compute reactions.

Statically indeterminate - Deflections (Compatibility conditions) along with equilibrium equations should be used to find out reactions.

Load and Shear Force diagram from Bending Moment diagram

If S.F. Diagram for a beam is given, then

If S.F. diagram consists of rectangle then the load will be point load

If S.F diagram consists of inclined line then the load will be UDL on that portion

If S.F diagram consists of parabolic curve then the load will be GVL

If S.F diagram consists of cubic curve then the load distribute is parabolic.

After finding load diagram we can draw B.M diagram easily.

If B.M Diagram for a beam is given, then

If B.M diagram consists of vertical line then a point BM is applied at that point.

If B.M diagram consists of inclined line then the load will be free point load

If B.M diagram consists of parabolic curve then the load will be U.D.L.

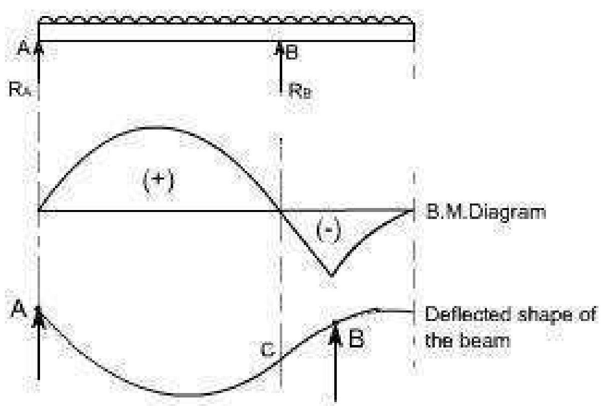
If B.M diagram consists of cubic curve then the load will be G.V.L.

If B.M diagram consists of fourth degree polynomial then the load distribution is parabolic.

Point of Contraflexure

In a beam if the bending moment changes sign at a point, the point itself having zero bending moment, the beam changes curvature at this point of zero bending moment and this point is called the point of contra flexure.

Consider a loaded beam as shown below along with the B.M diagrams and deflection diagram.



In this diagram we noticed that for the beam loaded as in this case, the bending moment diagram is partly positive and partly negative. In the deflected shape of the beam just below the bending moment diagram shows that left hand side of the beam is 'sagging' while the right hand side of the beam is 'hogging'.

The point C on the beam where the curvature changes from sagging to hogging is a point of contraflexure.

There can be more than one point of contraflexure in a beam.

Assumptions in Simple Bending Theory

Beams are initially straight

The material is homogenous and isotropic i.e. it has a uniform composition and its mechanical properties are the same in all directions

The stress-strain relationship is linear and elastic

Young's Modulus is the same in tension as in compression

Sections are symmetrical about the plane of bending

Sections which are plane before bending remain plane after bending

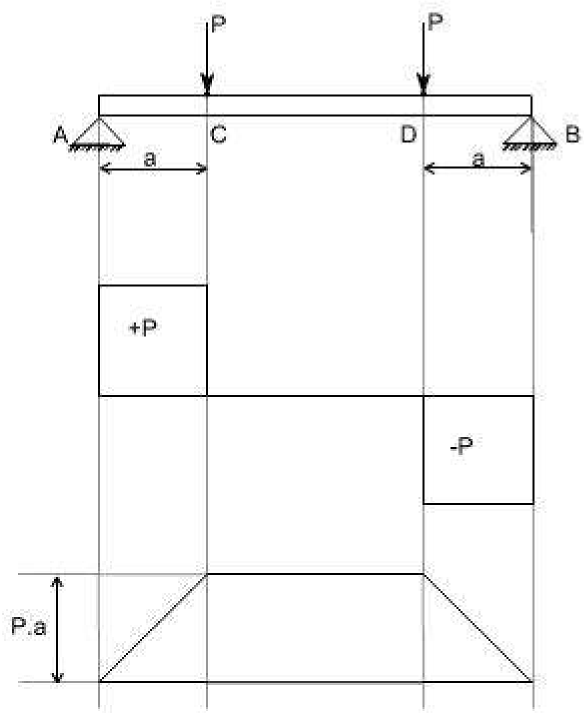
Non-Uniform Bending

In the case of non-uniform bending of a beam, where bending moment varies from section to section, there will be shear force at each cross section which will induce shearing stresses

Also these shearing stresses cause warping (or out-of plane distortion) of the cross section so that plane cross sections do not remain plane even after bending.

Beams under shear stress:

Theory which has been discussed earlier, while we discussed the bending stresses in beams was for the case of pure bending i.e. constant bending moment acts along the entire length of the beam.



Take a consideration of the beam AB transversely loaded as shown in the figure above. Together with shear force and bending moment diagrams we note that the middle potion CD of the beam is free from shear force and that it’s bending moment. M = Pxa is uniform between the portion C and D. This condition is called the pure bending condition.

Since shear force and bending moment are related to each other F= dM/dX (eq) therefore if the shear force changes than there will be a change in the bending moment also, and then this won't be the pure bending.

Bending Equation:

Hence one can conclude from the pure bending theory was that the shear force at each X-section is zero and the normal stresses due to bending are the only ones produced.

In the case of non-uniform bending of a beam where the bending moment varies from one X-section to another, there is a shearing force on each X-section and shearing stresses are also induced in the material. The deformation associated with those shearing stresses causes " warping " of the x-section so that the assumption which we assumed while deriving the relation $\frac{\sigma}{y}=\frac{M}{I}=\frac{E}{R}$ that the plane cross-section after bending remains plane is violated. Now due to warping the plane cross=section before bending do not remain plane after bending. This complicates the problem but more elaborate analysis shows that the normal stresses due to bending, as calculated from the equation. $\frac{\sigma}{y}=\frac{M}{I}=\frac{E}{R}$

So the above equation gives the distribution of stresses which are normal to the cross-section that is in x-direction or along the span of the beam are not greatly altered by the presence of these shearing stresses. Thus, it is justifiable to use the theory of pure bending in the case of non-uniform bending and it is accepted practice to do so and similarly non bending equation and cases can be solved.

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Department of Mechanical Engineering
Strength of Material ME-3002
Unit-IV

Introduction

• We know that the axis of a beam deflects from its initial position under action of applied forces.

• In this chapter we will learn how to determine the elastic deflections of a beam.

Selection of co-ordinate axes

We will not introduce any other co-ordinate system. We use general co-ordinate axis as shown in the figure. This system will be followed in deflection of beam and in shear force and bending moment diagram. Here downward direction will be negative i.e. negative Y-axis. Therefore downward deflection of the beam will be treated as negative.

To determine the value of deflection of beam subjected to a given loading where we will use the formula =*EI d y M x*.

We use above Co-ordinate system Some books fix a co-ordinate axis as shown in the following figure. Here downward direction will be positive i.e. positive Y-axis. Therefore downward deflection of the beam will be treated as positive. As
beam is generally deflected in downward directions and this co-ordinate system treats downward deflection is positive deflection.

Why to calculate the deflections?

• To prevent cracking of attached brittle materials

• To make sure the structure not deflect severely and to “appear” safe for its occupants

• To help analyzing statically indeterminate structures

• Information on deformation characteristics of members is essential in the study of vibrations of
Machines

Several methods to compute deflections in beam

• Double integration method (*without* the use of singularity functions)

• Macaulay’s Method (*with* the use of singularity functions)

• Moment area method

• Method of superposition

• Conjugate beam method

• Castigliano’s theorem

• Work/Energy methods

Each of these methods has particular advantages or disadvantages.

Assumptions in Simple Bending Theory

• Beams are initially straight

• The material is homogenous and isotropic i.e. it has a uniform composition and its mechanical properties are the same in all directions

• The stress-strain relationship is linear and elastic

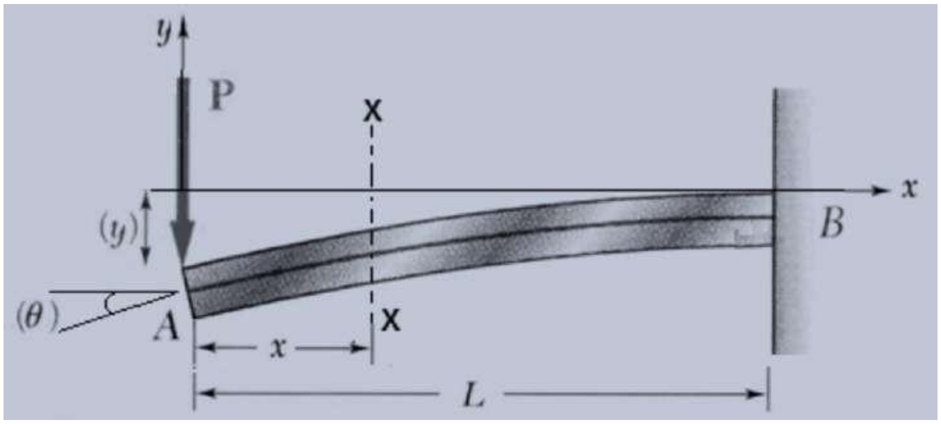
• Young’s Modulus is the same in tension as in compression

• Sections are symmetrical about the plane of bending
• Sections which are plane before bending remain plane after bending

Non-Uniform Bending
• In the case of non-uniform bending of a beam, where bending moment varies from section to section,
there will be shear force at each cross section which will induce shearing stresses
• Also these shearing stresses cause warping (or out-of plane distortion) of the cross section so that
plane cross sections do not remain plane even after bending

step procedure to solve deflection of beam problems by double integration method
Step 1: Write down boundary conditions (Slope boundary conditions and displacement boundary
conditions), analyze the problem to be solved
Step 2: Write governing equations
Step 3: Solve governing equations by integration, results in expression with unknown integration constants
Step 4: Apply boundary conditions (determine integration constants)
Following table gives boundary conditions for different types of support.
Using double integration method we will find the
deflection and slope of the following loaded beams one by one.
(i) A Cantilever beam with point load at the free end.
(ii) A Cantilever beam with UDL (uniformly distributed load)
(iii) A Cantilever beam with an applied moment at free end.
(iv) A simply supported beam with a point load at its midpoint.
(v) A simply supported beam with a point load NOT at its midpoint.
(vi) A simply supported beam with UDL (Uniformly distributed load)
(vii) A simply supported beam with triangular distributed load (GVL) gradually varied load.
(viii) A simply supported beam with a moment at mid span.
(ix) A simply supported beam with a continuously distributed load the intensity of which at any point 'x' along the beam

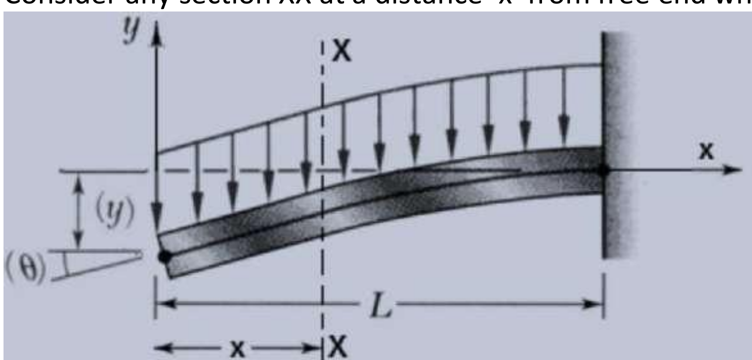
A Cantilever beam with point load at the free end.
We will solve this problem by double integration method. For that at first we have to calculate (Mx).
Consider any section XX at a distance 'x' from free end which is left end as shown in figure.



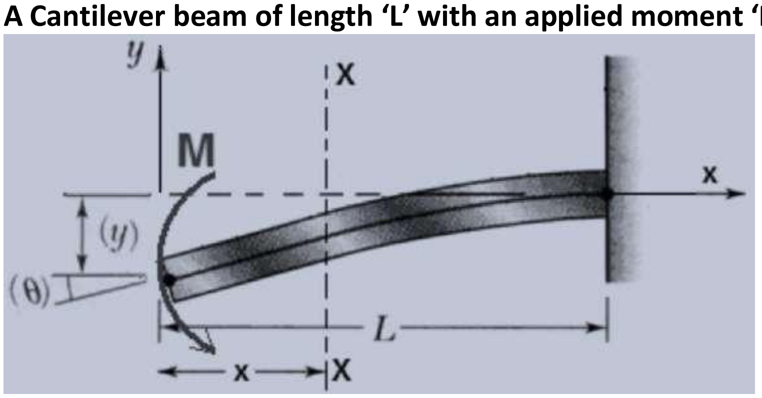
A Cantilever beam with UDL (uniformly distributed load)

We will now solve this problem by double integration method, for that at first we have to calculate (Mx).

Consider any section XX at a distance 'x' from free end which is left end as shown in figure.

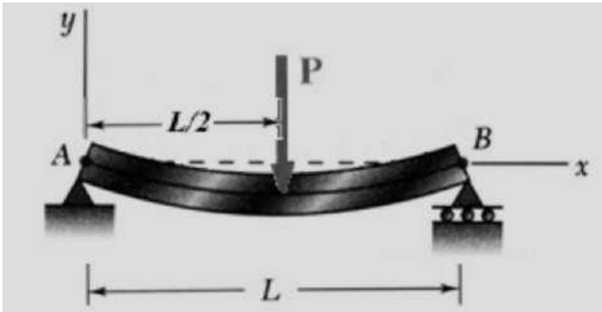


A Cantilever beam of length 'L' with an applied moment 'M' at free end.

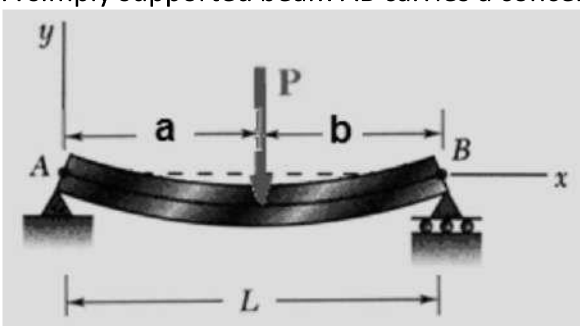


A simply supported beam with a point load P at its midpoint.

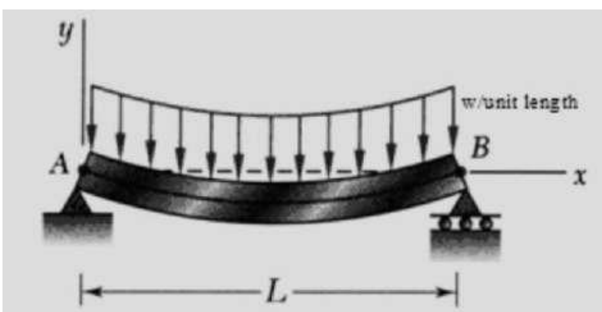
A simply supported beam AB carries a concentrated load P at its midpoint as shown in the figure.



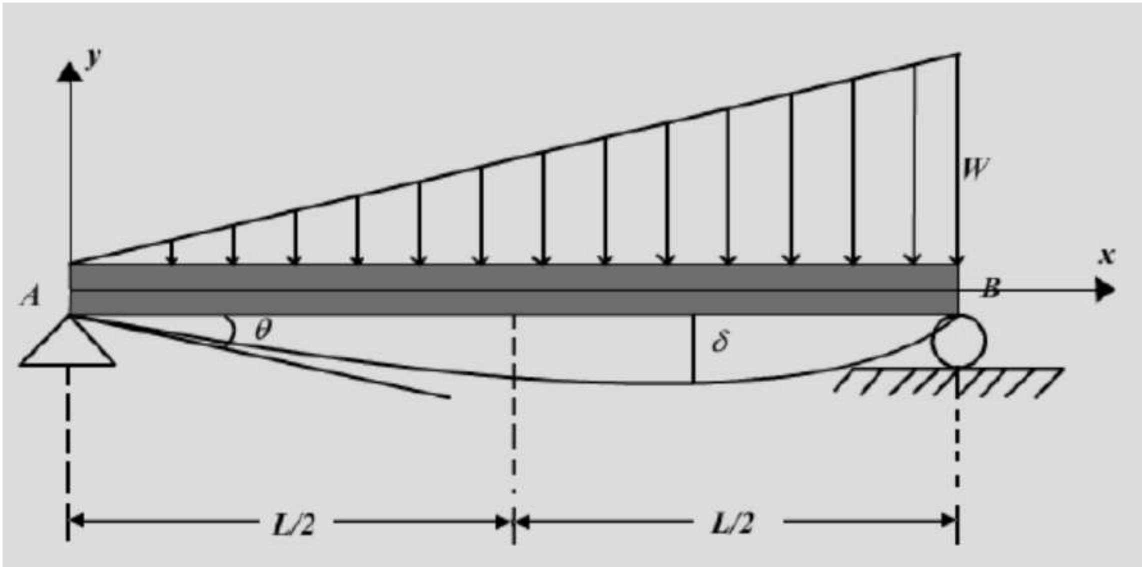
A simply supported beam with a point load ‘P’ NOT at its midpoint.
A simply supported beam AB carries a concentrated load P as shown in the figure.



A simply supported beam with UDL (Uniformly distributed load)
A simply supported beam AB carries a uniformly distributed load (UDL) of intensity w/unit length over its
whole span L as shown in figure. We want to develop the equation of the elastic curve and find the maximum deflection δ at the middle of the span.

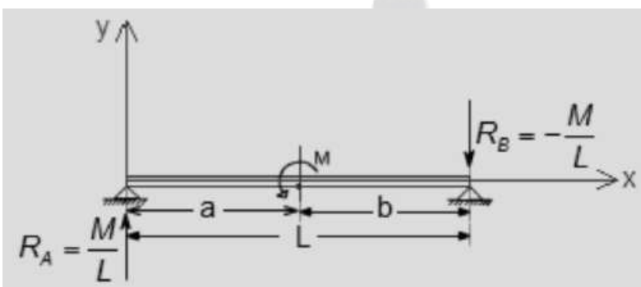


A simply supported beam with triangular distributed load (GVL) gradually varied load.
A simply supported beam carries a triangular distributed load (GVL) as shown in figure below. We have to
find equation of elastic curve and find maximum deflection (δ).

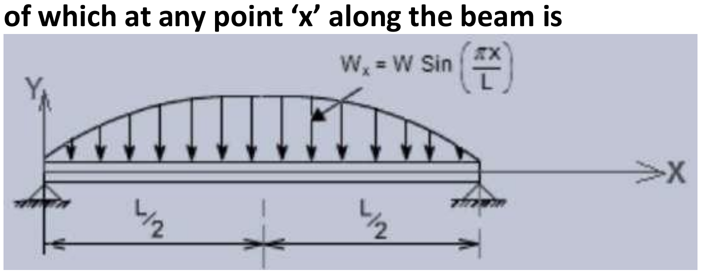


A simply supported beam with a moment at mid-span

A simply supported beam AB is acted upon by a couple M applied at an intermediate point distance ‘a’ from
the equation of elastic curve and deflection at point where the moment acted.



A simply supported beam with a continuously distributed load the intensity of which at any point ‘x’ along the beam is



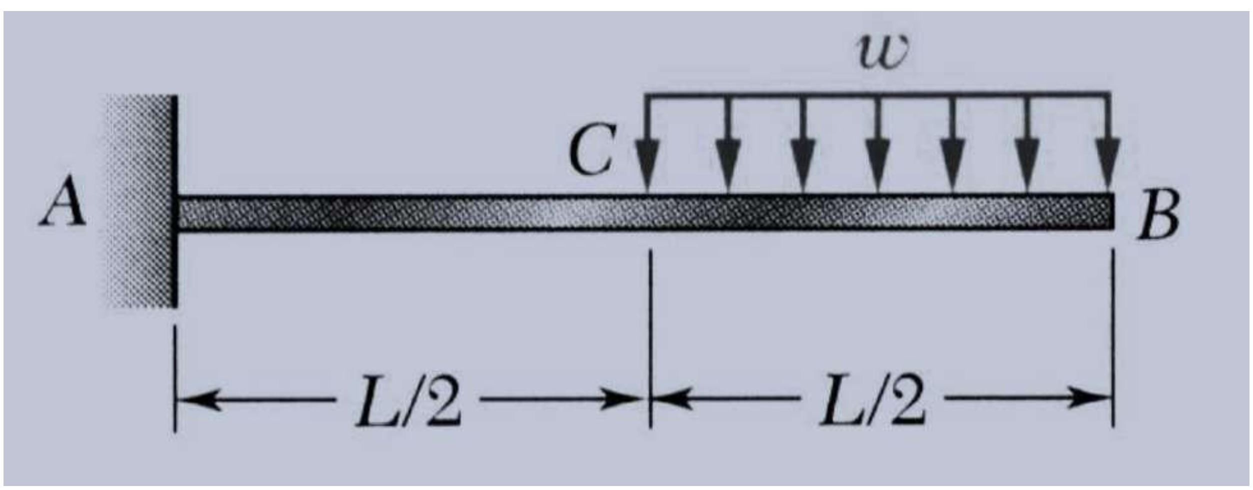
Macaulay's Method (Use of singularity function)

• When the beam is subjected to point loads (but several loads) this is very convenient method for
determining the deflection of the beam.

• In this method we will write single moment equation in such a way that it becomes continuous for

entire length of the beam in spite of the discontinuity of loading.
• After integrating this equation we will find the integration constants which are valid for entire length of the beam. This method is known as ***method of singularity constant.***

Procedure to solve the problem by Macaulay's method
Step – I: Calculate all reactions and moments
Step – II: Write down the moment equation which is valid for all values of x. This must contain brackets.
Step – III: Integrate the moment equation by a typical manner. Integration of (x-a)
Step – IV: After first integration write the first integration constant (A) after first terms and after second
time integration write the second integration constant (B) after A.x . Constant A and B are valid for all
values of x.
Step – V: Using Boundary condition find A and B at a point x = p if any term in Macaulay's method, (x-a) is
negative (-ive) the term will be neglected.
Method of superposition
Assumptions:
• Structure should be linear
• Slope of elastic line should be very small.
• The deflection of the beam should be small such that the effect due to the shaft or rotation of the
line of action of the load is neglected.
Principle of Superposition:
• Deformations of beams subjected to combinations of loadings may be obtained as the linear combination of the deformations from the individual loadings
• Procedure is facilitated by tables of solutions for common types of loadings and supports.



Conjugate beam method

In the ***conjugate beam method*,** the **length** of the conjugate beam is the same as the length of the actual
beam, the **loading diagram** (showing the loads acting) on the conjugate beam is simply the bendingmoment
diagram of the actual beam divided by the flexural rigidity *EI* of the actual beam, and the **corresponding support condition** for the conjugate beam is given by the rules as shown below.
Corresponding support condition for the conjugate beam

Conjugates of Common Types of Real Beams

(a)

By the conjugate beam method, the slope and deflection of the actual beam can be found by using
the following two rules:

• The **slope** of the actual beam at any cross section is equal to the **shearing force** at the corresponding cross section of the conjugate beam.
• The **deflection** of the actual beam at any point is equal to the **bending moment** of the conjugate
beam at the corresponding point.

Procedure for Analysis

• Construct the **M / EI** diagram for the given (real) beam subjected to the specified (real) loading. If a
combination of loading exists, you may use M-diagram by parts
• Determine the ***conjugate*** beam corresponding to the given real beam
Apply the M / EI diagram as the **load on the conjugate** beam as per sign convention
• Calculate the **reactions** at the supports of the **conjugate** beam by applying equations of equilibrium and conditions
• Determine the ***shears*** in the ***conjugate*** beam at locations where **slopes** is desired in the **real** beam, V**conj = θreal**
• Determine the ***bending moments*** in the ***conjugate*** beam at locations where **deflections** is desired in the **real** beam, **Mconj = yreal**

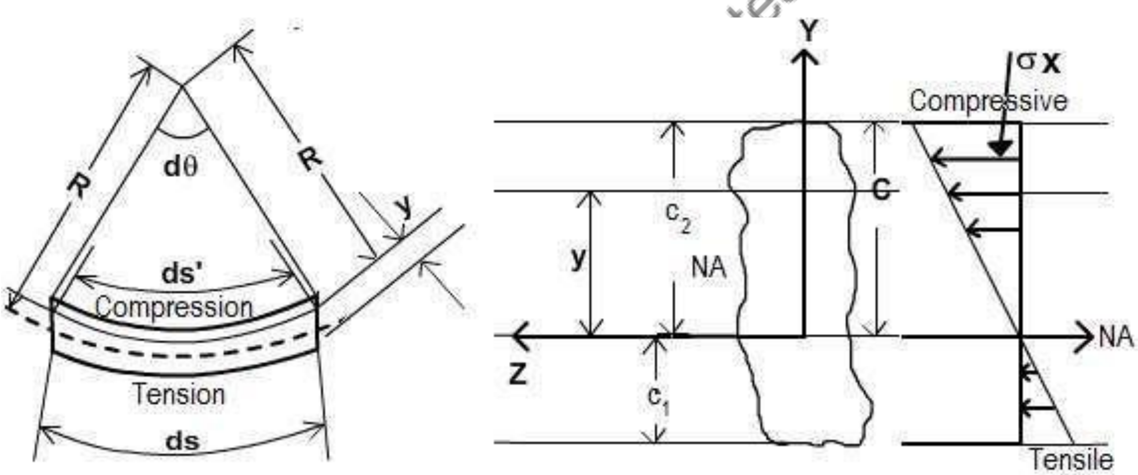
The method of double integration, method of superposition, moment-area theorems, and Castigliano's theorem are all well established methods for finding deflections of beams, but they require that the
boundary conditions of the beams be known or specified. If not, all of them become *helpless*. However,the conjugate beam method is able to proceed and yield a solution for the possible deflections of the beam based on the **support conditions**, rather than the boundary conditions, of the beams.

(i) A Cantilever beam with a point load 'P' at its free end.

For Real Beam: At a section a distance 'x' from free end consider the forces to the left. Taking moments about the

section gives (obviously to the left of the section) ***Mx = -P.x***
(negative sign means that the moment on the left hand side of the portion is in the anticlockwise direction and is therefore taken as negative according to the sign convention)
so that the ***maximum bending moment*** occurs at the fixed end i.e. ***Mmax = - PL*** (at x = L)

Assumptions in Simple Bending Theory
All of the foregoing theory has been developed for the case of pure bending i.e. constant B.M along the
length of the beam. In such case
• The shear force at each c/s is zero.
• Normal stress due to bending is only produced.
• Beams are initially straight
• The material is homogenous and isotropic i.e. it has a uniform composition and its mechanical properties are the same in all directions
• The stress-strain relationship is linear and elastic
• Young's Modulus is the same in tension as in compression
• Sections are symmetrical about the plane of bending
• Sections which are plane before bending remain plane



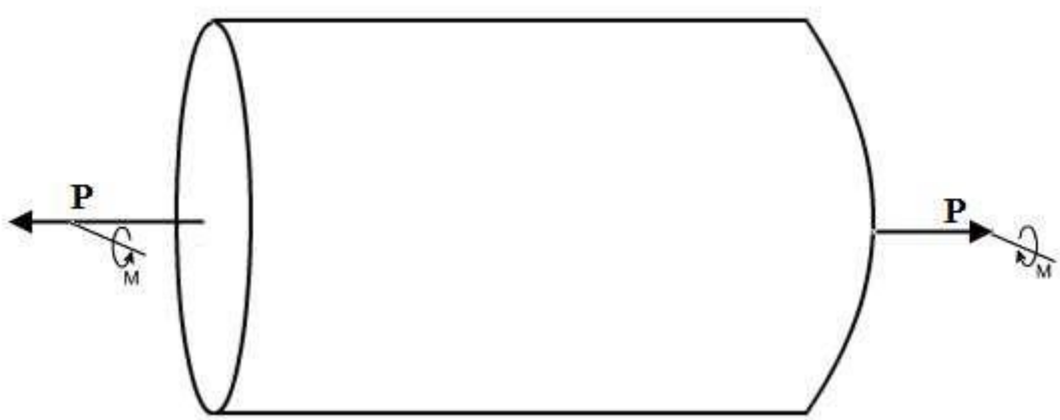
Flexural Rigidity (EI)
Reflects both
• Stiffness of the material (measured by E)
• Proportions of the c/s area (measured by I)
Axial Rigidity = EA

Beam of uniform strength
It is one is which the maximum bending stress is same in every section along the longitudinal axis.
To make Beam of uniform strength the section of the beam may be varied by
• Keeping the width constant throughout the length and varying the depth, *(Most widely used)*

• Keeping the depth constant throughout the length and varying the width
• By varying both width and depth suitably.

Bending stress due to additional Axial thrust (P).

A shaft may be subjected to a combined bending and axial thrust. This type of situation arises in various
machine elements.
If P = Axial thrust



Then direct stress (d σ) = P / A (stress due to axial thrust)
This direct stress (d σ) may be tensile or compressive depending upon the load P is tensile or compressive.
And the bending stress (b σ) = MyI
is varying linearly from zero at centre and extremum (minimum or maximum) at top and bottom fibres.

Statically determinate and indeterminate structures

Beams for which reaction forces and internal forces can be found out from static equilibrium equations
alone are called statically determinate beam.

$\Sigma X_i = \Sigma Y_i = \Sigma$ = to calculate R & RB

Beams for which reaction forces and internal forces **cannot** be found out from static equilibrium
equations alone are called statically indeterminate beam. This type of beam requires deformation
equation in addition to static equilibrium equations to solve for unknown forces.

In machinery, the general term ***"shaft"*** refers to a member, usually of circular crosssection, which supports gears, sprockets, wheels, rotors, etc., and which is subjected to torsion and to transverse or axial loads acting singly or in combination.

• An ***"axle"*** is a rotating/non-rotating member that supports wheels, pulleys,… and carries no torque.
• A ***"spindle"*** is a short shaft. Terms such as *line shaft, head shaft, stub shaft,*

transmission shaft, countershaft, and *flexible shaft* are names associated with special usage.

Torsion of circular shafts

Where J = Polar moment of inertia

τ = Shear stress induced due to torsion T.

G = Modulus of rigidity

θ = Angular deflection of shaft

R, L = Shaft radius & length respectively

Assumptions

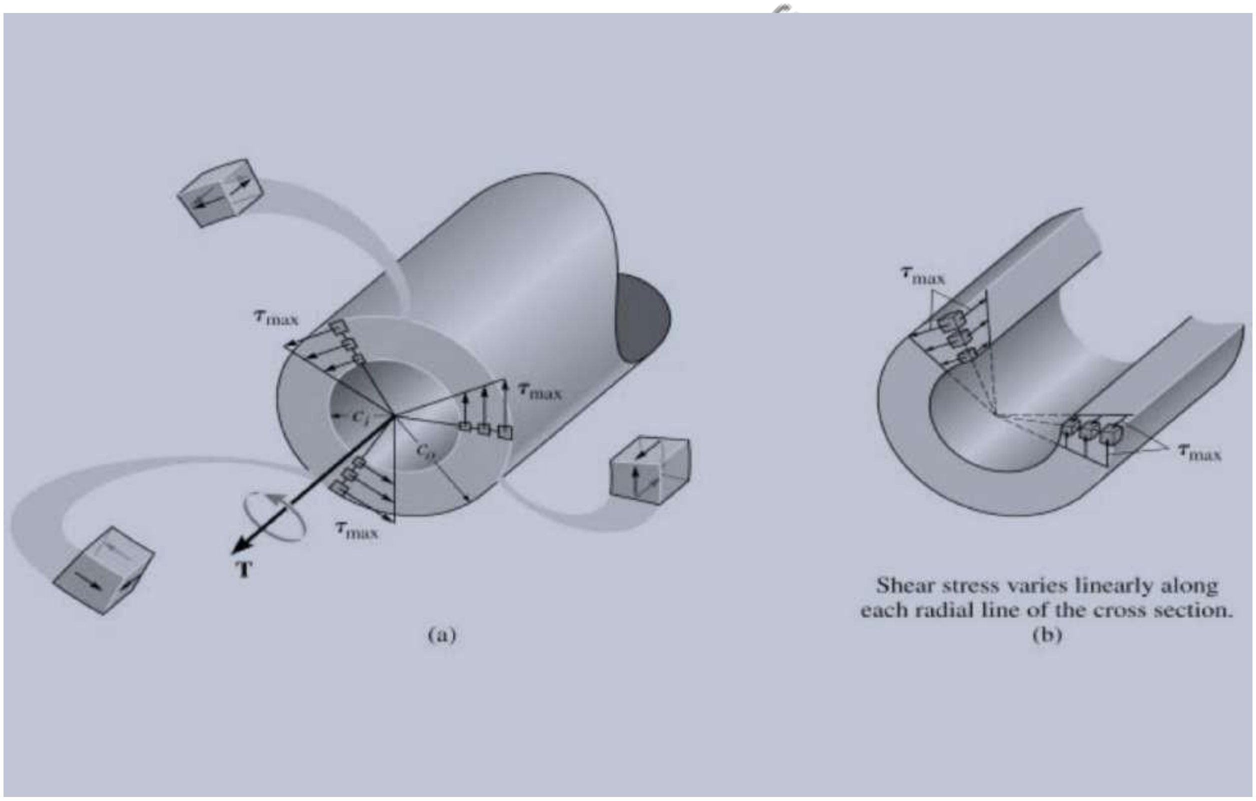
• The bar is acted upon by a pure torque.

• The section under consideration is remote from the point of application of the load and from a change in diameter.

• Adjacent cross sections originally plane and parallel remain plane and parallel after twisting, and any radial line remains straight.

The material obeys Hooke's law

• Cross-sections rotate as if rigid, i.e. every diameter rotates through the same angle



The polar section modulus

Zp= J / c, where c = r = D/2

• For a solid circular cross-section, Zp = π D3 / 16

• For a hollow circular cross-section, Zp = π (Do4 - Di4) / (16Do)

• Then, max τ = T / Zp

• If design shears stress, $d\,\tau$ is known, required polar section modulus can be calculated from:
Zp = T / $d\,\tau$

Torsional Stiffness

The tensional stiffness k is defined as the torque per radius twist

Comparison of solid and hollow shaft

• A Hollow shaft will transmit a greater torque than a solid shaft of the same weight & same material because the average shear stress in the hollow shaft is smaller than the average shear stress in the solid shaft

Shaft in series

1 2 θ =θ +θ

Torque (T) is same in all section

Electrical analogy gives torque(T) = Current (I)

Shaft in parallel

1 2 θ =θ and 1 2 *T* = *T* +*T*

Electrical analogy gives torque(T) = Current (I)

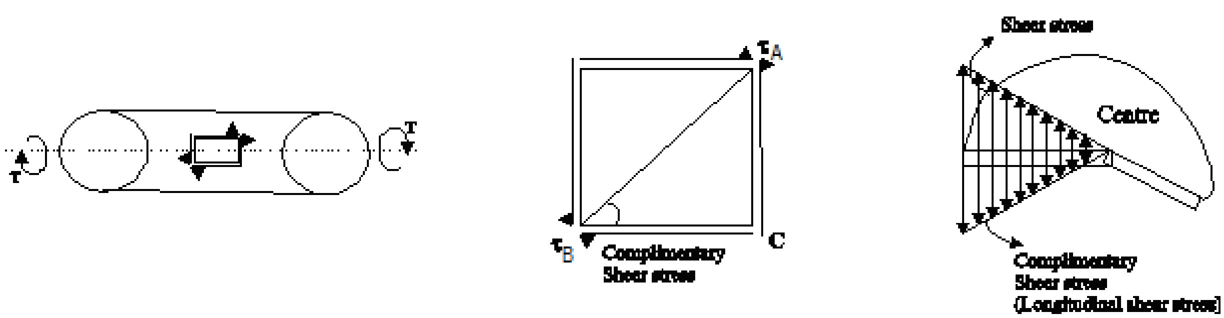
Combined Bending and Torsion

• In most practical transmission situations shafts which carry torque are also subjected **to** bending, if only by virtue of the self-weight of the gears they carry. Many other practical applications occur where bending and torsion arise simultaneously so that this type of loading represents one of the major sources of complex stress situations.

• In the case of shafts, bending gives rise to tensile stress on one surface and compressive stress on the opposite surface while torsion gives rise to pure shear throughout the shaft.

• For shafts subjected to the simultaneous application of a bending moment *M* and torque *T* the *principal stresses* set up in the shaft can be shown to be equal to those produced by an *equivalent bending moment,* of a certain value Me acting alone.

Shaft subjected to twisting moment



Principal stresses at a point on the surface of the shaft = +τ, -τ, 0

i.e $\sigma = \pm\tau \sin 2\theta$

Volumetric strain,

∈ =∈ + ∈ + ∈*v*

• No change in volume for a shaft subjected to pure torque.

Torsional Stresses in Non-Circular Cross-section Members

• There are some applications in machinery for non-circular cross-section members and shafts where a regular polygonal cross-section is useful in transmitting torque to a gear or pulley that can have an axial change in position. Because no key or keyway is needed, the possibility of a lost key is avoided.

• Saint Venant (1855) showed that max τ in a rectangular *b×c* section bar occurs in the middle of the longest side *b* and is of magnitude formula.

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Department of Mechanical Engineering

Strength of Material ME-3002

Unit-V

Theories of Column

Columns and Struts:

- A structural member subjected to an axial compressive force is called strut. As per definition strut may be horizontal, inclined or even vertical.
- The vertical strut is called a column.

Introduction

• **Strut:** A member of structure which carries an axial compressive load.

• **Column:** If the strut is vertical it is known as column.

• A long, slender column becomes unstable when its axial compressive load reaches a value called the critical buckling load.

• If a beam element is under a compressive load and its length is an order of magnitude larger than either of its other dimensions such a beam is called a *columns*.

• Due to its size its axial displacement is going to be very small compared to its lateral deflection called *buckling*.

• *Buckling* does not vary linearly with load it occurs suddenly and is therefore dangerous

• **Slenderness Ratio:** The ratio between the length and least radius of gyration.

• **Elastic Buckling:** Buckling with no permanent deformation.

• Euler buckling is only valid for long, slender objects in the elastic region.

• For short columns, a different set of equations must be used.

Which is the critical load?

At this value the structure is in equilibrium regardless of the magnitude of the angle (provided it stays small)

• Critical load is the only load for which the structure will be in equilibrium in the disturbed position

• At this value, restoring effect of the moment in the spring matches the buckling effect of the axial load represents the boundary between the stable and unstable conditions.

• If the axial load is less than Pcr the effect of the moment in the spring dominates and the structure returns to the vertical position after a small disturbance – stable condition.

• If the axial load is larger than Pcr the effect of the axial force predominates and the structure buckles – unstable condition.

• Because of the large deflection caused by buckling, the least moment of inertia *I* can be expressed as, I = Ak2

• Where: *A* is the cross sectional area and *r* is the *radius of gyration* of the cross sectional area,

Note that the *smallest* radius of gyration of the column, i.e. the *least* moment of inertia *I* should be taken in order to find the critical stress. *l/ k* is called the *slenderness ratio*, it is a measure of the column's flexibility.

Euler's Critical Load for Long Column

Assumptions:

(i) The column is perfectly straight and of uniform cross-section

(ii) The material is homogenous and isotropic

(iii) The material behaves elastically

(iv) The load is perfectly axial and passes through the centroid of the column section.

(v) The weight of the column is neglected.

Limitation of Euler's Formula

- There is always *crookedness* in the column and the load may not be exactly axial.

- This formula does not take into account the axial stress and the buckling load is given by this formula may be much more than the actual buckling load.

Euler’s Buckling (or crippling load)

- The maximum load at which the column tends to have lateral displacement or tends to buckle is known as buckling or crippling load. Load columns can be analysed with the Euler’s column formulas can be given as

$$P_E = \frac{n^2\pi^2 El}{l^2}\,(n = 1, 2, 3, \ldots..)$$

$$or\ P_E = \frac{\pi^2 El}{l_{eff}^2}$$

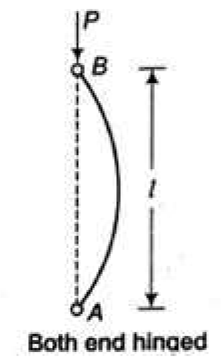
where, E = Modulus of elasticity, *l* = Effective Length of column, and *I* = Moment of inertia of column section.

1. **For both end hinged:**

n=1

$$l_{eff} = l$$

$$p_E = \frac{\pi^2 El}{l^2}$$

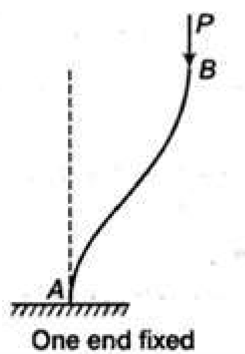


2. **For one end fixed and other free:**

$n = \dfrac{1}{2}$

$l_{eff} = 2l$

$p_E = \dfrac{\pi^2 El}{4l^2}$

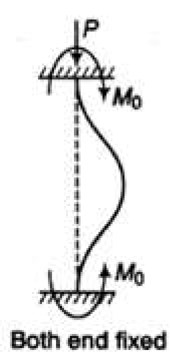


3. **For both end fixed:**

n=2,

$l_{eff} = \dfrac{l}{2}$

$p_E = \dfrac{4\pi^2 El}{El}$



4. **For one end fixed and other hinged:**

$n = \sqrt{2}$

$l_{eff} = \frac{l}{\sqrt{2}}$

$p_E = \frac{2\pi^2 EI}{l^2}$

One end fixed and other hinged

Effective Length for different End conditions

End condition	Both end hinged	One end fixed other free	Both end fixed	One end fixed
Effective length(l_{eff})	L	2L	L/2	L/√2

Modes of failure of Columns

Type of Column	Mode of Failure
Short column	Crushing
Long column	Buckling
Intermediate column	Combined Crushing and Buckling

Slenderness Ratio (λ)

- Slenderness ratio of a compression member is defined as the ratio of its effective length to least radius of gyration.

$$\lambda = \frac{L_\theta}{r_{min}}$$

$L_\theta = Effective\ length$

$r_{min} = \sqrt{(I_{min}/A)}$

r_{min} = Least radius of gyration

Buckling Stress:

$$(\sigma_b) = \frac{P_e}{A} = \frac{\pi^2 E}{\lambda^2}$$

Rankine's Formula for Columns

- It is an empirical formula, takes into both crushing P_{CS} and Euler critical load (P_R).

$$\frac{1}{P_R} = \frac{1}{P_{cs}} + \frac{1}{P_E}$$

- P_R = Crippling load by Rankine's formula
- $P_{cs} = \sigma_{cs}$ A = Ultimate crushing load for column
- $P_E = \frac{\pi^2 El}{l_{eff}^2} =$ Crippling load obtained by Euler's formula

$$P_R = \frac{\sigma_{CS} A}{1 + a\left(\frac{l_e}{k}\right)^2}, l = Ak^2$$

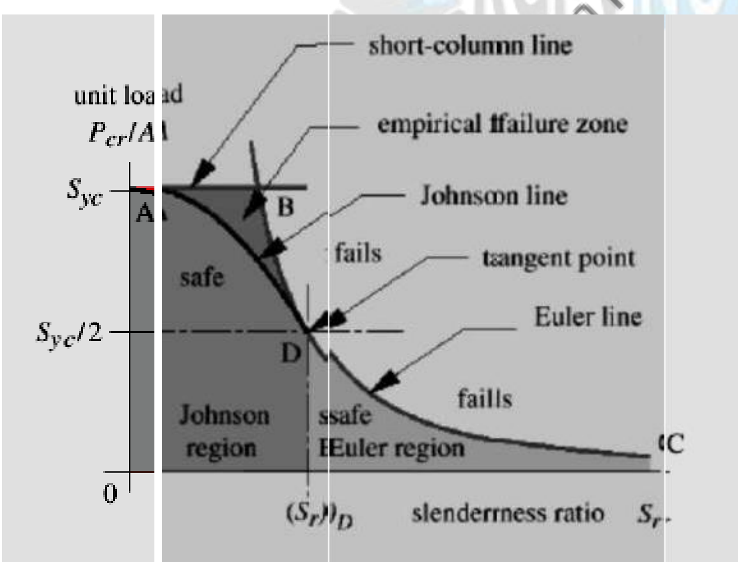
Where, A = Cross-section is of the column, K = Least radius of gyration, and A = Rankine's constant.

The shape of Kern in eccentric loading

- To prevent any kind of stress reversal, the force applied should be within an area near the cross section called as CORE or KERN.
- The shape of Kern for rectangular and I-section is Rhombus and for the square section, the shape is square for circular section shape is circular.

Rankine Crippling Load

Short column up to SR-40, Long column up to SR-120



Construction of column failure lines