RAJIV GANDHI PROUDYOGIKI VISHWAVIDYALAYA, BHOPAL
New Scheme Based On AICTE Flexible Curricula
Electronics & Communication Engineering III-Semester
EC304 Network Analysis

Unit-1 Introduction to circuit theory: basic circuit element R,L,C and their characteristics in terms of linearity & time dependant nature, voltage & current sources, controlled & uncontrolled sources KCL and KVL analysis, Steady state sinusoidal analysis using phasors; Concept of phasor & vector, impedance & admittance, Nodal & mesh analysis, analysis of magnetically coupled circuits. Dot convention, coupling coefficient, tuned circuits, Series & parallel resonance
Unit-2 Network Graph theory: Concept of Network graph, Tree, Tree branch & link, Incidence matrix, cut set and tie set matrices, dual networks
Unit-3 Network Theorems: Thevenins & Norton's, Super positions, Reciprocity, Compensation, Substitution, Maximum power transfer, and Millman's theorem, Tellegen's theorem, problems with dependent & independent sources.
Unit-4 Transient analysis: Transients in RL, RC&RLC Circuits, initial& final conditions, time constants. Steady state analysis
Laplace transform: solution of Integro-differential equations, transform of waveform synthesized with step ramp, Gate and sinusoidal functions, Initial & final value theorem, Network Theorems in transform domain.
Unit-5 Two port parameters: Z, Y, ABCD, Hybrid parameters, their inverse & image parameters, relationship between parameters, Interconnection of two ports networks, Reciprocity and Symmetry in all parameter.

Text/Reference Books:
1. M.E. Van Valkenburg, Network Analysis, (Pearson)
2. S P Ghosh A K Chakraborty Network Analysis & Synth. (MGH).
3. http://www.nptelvideos.in/2012/11/networks-and-systems.html

REFERENCE:-
1. Sudhakar-Circuit Network Analysis & Synth(TMH).
2. J. David Irwin Engineering Circuit analysis tenth edition, Wiley india.
2. Kuo- Network Analysis & Synthesis, Wiley India.
4. Robert L Boylestad introductory Circuit analysis, Pearson
5. Smarajit Ghosh, NETWORK THEORY: ANALYSIS AND SYNTHESIS (PHI).
6. Roy Choudhary D; Network and systems; New Age Pub.
7. Bhattacharya and Singh- Network Analysis & Synth (Pearson)

EXPERIMENTS LIST:-
1. To Verify Thevenin Theorem and Superposition Theorem.
2. To Verify Reciprocity Theorem and Millman's Theorem.
3. To Verify Maximum Power Transfer Theorem.
4. To Determine Open Circuit and Short Circuit parameters of a Two Port Network.
5. To Determine A,B, C, D parameters of a Two Port Network.
6. To determine h parameters of a Two Port Network.
7. To Find Frequency Response of RLC Series Circuit RLC parallel Circuit and determine resonance and 3dB frequencies.
8. To determine charging and discharging times of Capacitors.

UNIT 1

Introduction to circuit elements R, L, C and their characteristics in terms of linearity and time dependence, KCL and KVL analysis, dual networks, analysis of magnetically coupled circuits, Dot convention, coupling co-efficient, Tuned circuits, Series and parallel resonance, voltage and current sources, controlled sources

Introduction of Electric Circuit

Objectives

- Familiarity with and understanding of the basic elements encountered in electric networks.
- To learn the fundamental differences between linear and nonlinear circuits.
- To understand the Kirchhoff's voltage and current laws and their applications to circuits.
- Meaning of circuit ground and the voltages referenced to ground.
- Understanding the basic principles of voltage dividers and current dividers.
- Potentiometer and loading effects.
- To understand the fundamental differences between ideal and practical voltage and current sources and their mathematical models to represent these source models in electric circuits.
- Distinguish between independent and dependent sources those encountered in electric circuits.
- Meaning of delivering and absorbing power by the source.

Introduction

The interconnection of various electric elements in a prescribed manner comprises as an electric circuit in order to perform a desired function. The electric elements include controlled and uncontrolled source of energy, resistors, capacitors, inductors, etc. Analysis of electric circuits refers to computations required to determine the unknown quantities such as voltage, current and power associated with one or more elements in the circuit. To contribute to the solution of engineering problems one must acquire the basic knowledge of electric circuit analysis and laws. Many other systems, like mechanical, hydraulic, thermal, magnetic and power system are easy to analyze and model by a circuit. To learn how to analyze the models of these systems, first one needs to learn the techniques of circuit analysis. We shall discuss briefly some of the basic circuit elements and the laws that will help us to develop the background of subject.

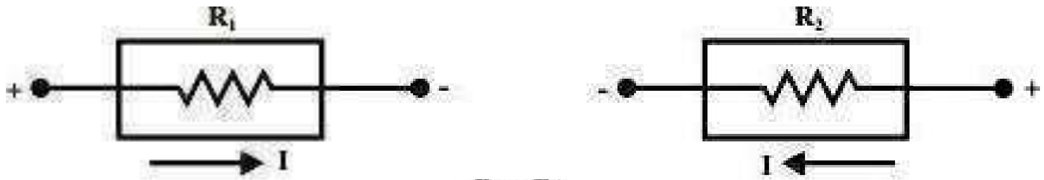
Basic Elements & Introductory Concepts

Electrical Network: A combination of various electric elements (Resistor, Inductor, Capacitor, Voltage source, Current source) connected in any manner what so ever is called an electrical network. We may classify circuit elements in two categories, passive and active elements.

Passive Element: The element which receives energy (or absorbs energy) and then either converts it into heat (R) or stored it in an electric (C) or magnetic (L) field is called passive element.

Active Element: The elements that supply energy to the circuit is called active element. Examples of active elements include voltage and current sources, generators, and electronic devices that require power supplies. A transistor is an active circuit element, meaning that it can amplify power of a signal. On the other hand, transformer is not an active element because it does not amplify the power level and power remains same both in primary and secondary sides. Transformer is an example of passive element.

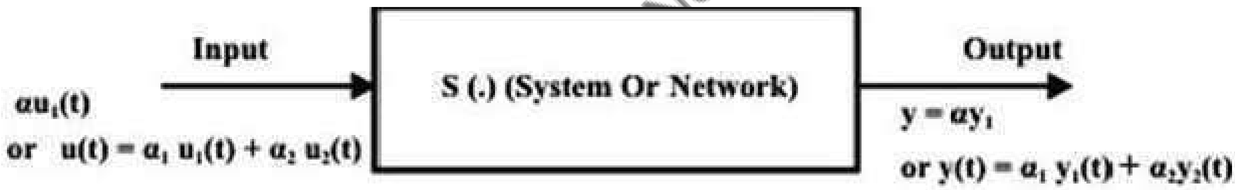
Bilateral Element: Conduction of current in both directions in an element (example: Resistance; Inductance; Capacitance) with same magnitude is termed as bilateral element.



Unilateral Element: Conduction of current in one direction is termed as unilateral (example: Diode, Transistor) element.

Meaning of Response: An application of input signal to the system will produce an output signal, the behavior of output signal with time is known as the response of the system.

Linear and Nonlinear Circuits



Input output behavior of a system

Non-Linear Circuit: Roughly speaking, a non-linear system is that whose parameters change with voltage or current. More specifically, non-linear circuit does not obey the homogeneity and additive properties. Volt-ampere characteristics of linear and non-linear elements are shown in figs. 3.2 - 3.3. In fact, a circuit is linear if and only if its input and output can be related by a straight line passing through the origin as shown in fig.3.2. otherwise, it is a nonlinear system.

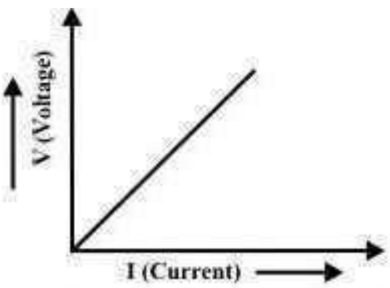
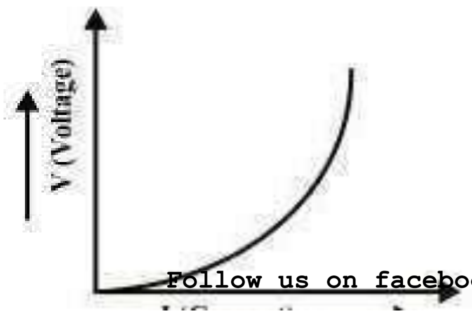


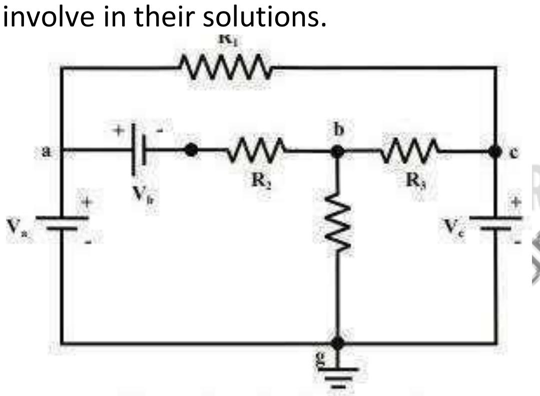
Fig. 3.2: V-I characteristics of linear element.



Potential Energy Difference: The voltage or potential energy difference between two points in an electric circuit is the amount of energy required to move a unit charge between the two points.

Kirchhoff's Laws

Kirchhoff's laws are basic analytical tools in order to obtain the solutions of currents and voltages for any electric circuit; whether it is supplied from a direct-current system or an alternating current system. But with complex circuits the equations connecting the currents and voltages may become so numerous that much tedious algebraic work is involve in their solutions.



Elements that generally encounter in an electric circuit can be interconnected in various possible ways. Before discussing the basic analytical tools that determine the currents and voltages at different parts of the circuit, some basic definition of the following terms are considered.

Node- A node in an electric circuit is a point where two or more components are connected together. This point is usually marked with dark circle or dot. The circuit in fig. 3.4 has nodes a, b, c, and g. Generally, a point, or a node in an circuit specifies a certain voltage level with respect to a reference point or node. Branch- A branch is a conducting path between two nodes in a circuit containing the electric elements. These elements could be sources, resistances, or other elements. Fig.3.4 shows that the circuit has six branches: three resistive branches (a-c, b-c, and b-g) and three branches containing voltage and current sources (a-, a-, and c-g). Loop- A loop is any closed path in an electric circuit i.e., a closed path or loop in a circuit is a contiguous sequence of branches which starting and end points for tracing the path are, in effect, the same node and touches no other node more than once. Fig. 3.4 shows three loops or closed paths namely, a-b-g-a; b-c-g-b; and a-c-b-a. Further, it may be noted that the outside closed paths a-c-g-a and a-b-c- g-a are also form two loops.

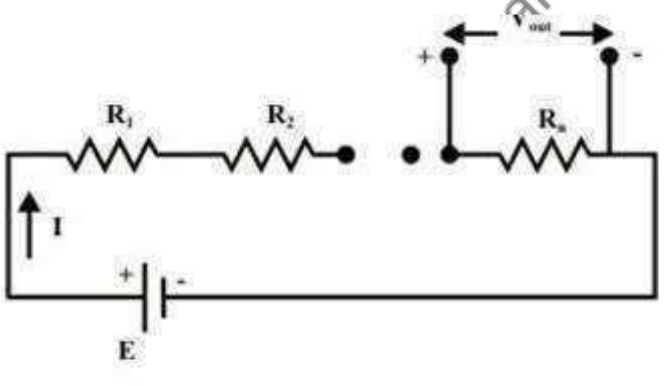
Mesh- a mesh is a special case of loop that does not have any other loops within it or in its interior. Fig. 3.4 indicates that the first three loops (a-b-g-a; b-c-g-b; and a-c-b-a) just identified are also 'meshes' but other two loops (a-c-g-a a-b-c-g-a) are not. With the introduction of the Kirchhoff's laws, a various types of electric circuits can be analyzed.

Kirchhoff's Current Law (KCL): KCL states that at any node (junction) in a circuit the algebraic sum of currents entering and leaving a node at any instant of time must be equal to zero. Here currents entering (+ve sign) and currents leaving (-ve sign) the node must be assigned opposite algebraic signs

Kirchhoff's Voltage Law (KVL): It states that in a closed circuit, the algebraic sum of all source voltages must be equal to the algebraic sum of all the voltage drops. Voltage drop is encountered when current flows in an element (resistance or load) from the higher-potential terminal toward the lower potential terminal. Voltage rise is encountered when current flows in an element (voltage source) from lower potential terminal (or negative terminal of voltage source) toward the higher potential terminal (or positive *A*, *B*, *C*, *and E* with respect to point *D* . Find also the value of voltage source.

In many cases, such as in electronic circuits, the chassis is shorted to the earth itself for safety reasons. Understanding the Basic Principles of Voltage Dividers and Current dividers

Voltage Divider



Current, voltage, power and energy
The most elementary quantity in the analysis of electric circuits is the electric charge. Our interest in electric charge is centered around its motion results in an energy transfer. Charge is the intrinsic property of matter responsible for electrical phenomena. The quantity of charge q can be expressed in terms of the charge on one electron. Which are -1.602 x 10-19 coulombs? Thus, -1 coulomb is the charge on 6.24 x 10^{18}electrons. The current flows through a specified area A and is defined by the electric charge passing through that area per unit time. Thus we define q as the charge expressed in coulombs.

Charge is the quantity of electricity responsible for electric phenomena. The time rate of change constitutes an electric current. Mathematically, this relation is expressed as

$$i(t) = \frac{dq(t)}{dt}$$

$$q(t) = \int_{-\infty}^{t} i(x)\,dx$$

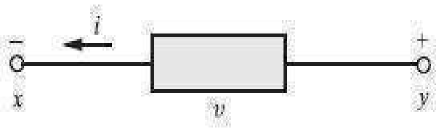
The unit of current is ampere (A); an ampere is 1 coulomb per second. Current is the time rate of flow of electric charge past a given point. The basic variables in electric circuits are current and voltage. If a current flows into terminal of the element shown in Fig., then a voltage or potential difference exists between the two terminals a and b. Normally, we say that a voltage exists across the element.

The voltage across an element is the work done in moving a positive charge of 1 coulomb from first terminal through the element to second terminal. The unit of voltage is volt, V or Joules per coulomb. We have defined voltage in Joules per coulomb as the energy required to move a Positive charge of 1 coulomb through an element. If we assume that we are dealing with a differential amount of charge and energy,

$$v = \frac{dw}{dq}$$

$$vi = \frac{dw}{dq}\left(\frac{dq}{dt}\right) \Rightarrow \frac{dw}{dt} = p$$

Which is the time rate of change of energy or power measured in Joules per second or watts (). P could be either positive or negative. Hence it is imperative to give sign convention for power. If we use the signs as shown in Fig., the current flows out of the terminal indicated by x, which shows the positive sign for the voltage. In this case, the element is said to provide energy to the charge as it moves through. Power is then provided by the element.



Conversely, power absorbed by an element is p = vi, when i is entering through the positive voltage terminal.
Energy is the capacity to perform work. Energy and power are related to each other by the following equation:

$$Energy = w = \int_{-\infty}^{t} p\,dt$$

Linear, active and passive elements
A linear element is one that satisfies the principle of superposition and homogeneity.
Passive Circuit Elements
An element is said to be passive if the total energy delivered to it from the rest of the circuit is either zero or positive. Then for a passive element, with the current flowing into the

positive (+) terminal as shown in Fig. shown above this means that

$$w = \int_{-\infty}^{t} vi\, dt \geq 0$$

Resistors

Resistance is the physical property of an element or device that impedes the flow of current; it is represented by the symbol R. Resistance of a wire element is calculated using the relation:

$$R = \frac{\rho l}{A}$$

Where A is the cross-sectional area, ρ the resistivity, and l the length of the wire. The practical unit of resistance is ohm and represented by the symbol Ω. An element is said to have a resistance of 1 ohm, if it permits 1A of current to flow through it when 1V is impressed across its terminals. Ohm's law, which is related to voltage and current, was published in 1827 as

$$v = Ri$$

$$R = \frac{v}{i}$$

Where v is the potential across the resistive element, i the current through it, and R the resistance of the element. The power absorbed by a resistor is given by

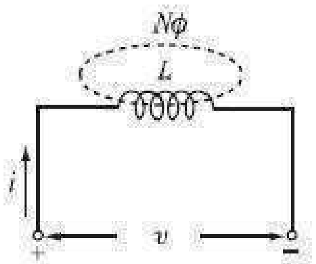
$$p = vi = v\left(\frac{v}{R}\right) = \frac{v^2}{R}$$

$$p = vi = (iR)i = i^2 R$$

Inductors

Whenever a time-changing current is passed through a coil or wire, the voltage across it is proportional to the rate of change of current through the coil. This proportional relationship may be expressed by the equation

$$v = L\frac{di}{dt}$$



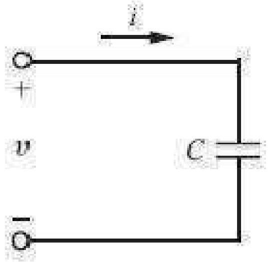
Where L is the constant of proportionality known as inductance and is measured in Henrys (H). Remember v and I am both functions of time. Let us assume that the coil shown in Fig. above has N turns and the core material has a high permeability so that the magnetic flux ϕ is connected within the area A. The changing flux creates an induced voltage in each turn equal to the derivative of the flux ϕ so the total voltage across N turns is

$$v = N\frac{d\phi}{dt}$$

Capacitors

A capacitor is a two-terminal element that is a model of a device consisting of two conducting plates separated by a dielectric material. Capacitance is a measure of the ability of a device to store energy in the form of an electric field.

Capacitance is defined as the ratio of the charge stored to the voltage difference between the two conducting plates or wires,



$$C = \frac{q}{v}$$

The current through the capacitor is given by

$$i = \frac{dq}{dt} = C\frac{dv}{dt}$$

The energy stored in a capacitor is

$$w = \int_{-\infty}^{t} vi\ d\tau$$

Active Circuit Elements (Energy Sources)

An active two-terminal element that supplies energy to a circuit is a source of energy. An ideal voltage source is a circuit element that maintains a prescribed voltage across the terminals regardless of the current flowing in those terminals. Similarly, an ideal current source is a circuit element that maintains a prescribed current through its terminals regardless of the voltage across those terminals.

These circuit elements do not exist as practical devices; they are only idealized models of actual voltage and current sources.

Ideal voltage and current sources can be further described as either independent sources or dependent sources. An independent source establishes a voltage or current in a circuit without relying on voltages or currents elsewhere in the circuit. The value of the voltage or current supplied is specified by the value of the independent source alone.

In contrast, a dependent source establishes a voltage or current whose value depends on the value of the voltage or current elsewhere in the circuit. We cannot specify the value of a dependent source, unless you know the value of the voltage or current on which it depends.

The circuit symbols for ideal independent sources are shown in Fig. 1.8.(a) and (b).

Note that a circle is used to represent an independent source. The circuit symbols for dependent sources are shown in Fig. 1.8.(c), (d), (e) and (f). A diamond symbol is used to represent dependent sources.

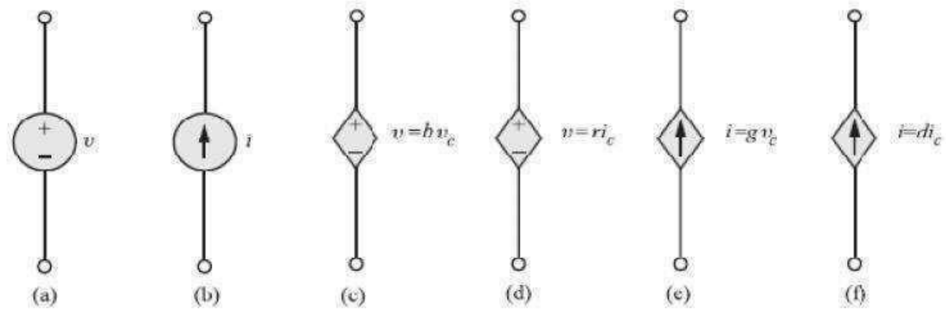


Figure 1.8 (a) An ideal independent voltage source
(b) An ideal independent current source
(c) voltage controlled voltage source
(d) current controlled voltage source
(e) voltage controlled current source
(f) current controlled current source

Unilateral and bilateral networks

A Unilateral network is one whose properties or characteristics change with the direction. An example of unilateral network is the semiconductor diode, which conducts only in one direction.

A bilateral network is one whose properties or characteristics are same in either direction. For example, a transmission line is a bilateral network, because it can be made to perform the function equally well in either direction.

Network simplification techniques

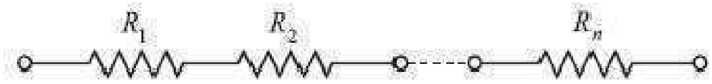
In this section, we shall give the formula for reducing the networks consisting of resistors connected in series or parallel.

Resistors in Series

When a number of resistors are connected in series, the equivalent resistance of the combination is given by

$$R = R_1 + R_2 + \cdots + R_n$$

Thus the total resistance is the algebraic sum of individual resistances.

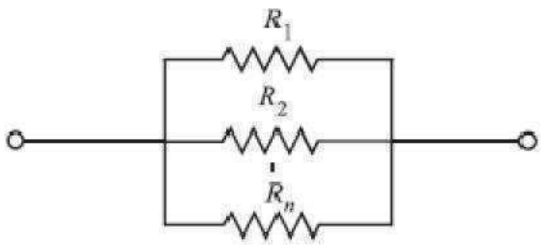


Resistors in Parallel

When a number of resistors are connected in parallel as shown in Fig. below, then the equivalent resistance of the combination is computed as follows:

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \ldots\ldots + \frac{1}{R_n}$$

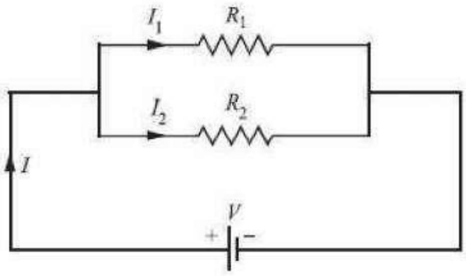
Thus, the reciprocal of a equivalent resistance of a parallel combination is the sum of the reciprocal of the individual resistances. Reciprocal of resistance is conductance and denoted by G. Consequently the equivalent conductance,



Division of Current in a Parallel Circuit

Consider a two branch parallel circuit as shown in Fig. below. The branch currents I1 and I2 can be evaluated in terms of total current I as follows:

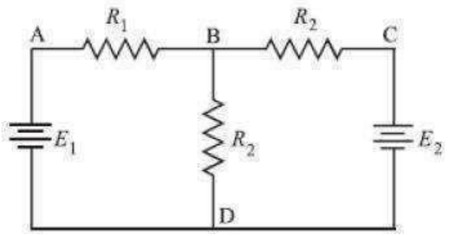
$$I_1 = \frac{IR_2}{R_1 + R_2} = \frac{IG_1}{G_1 + G_2}$$
$$I_2 = \frac{IR_1}{R_1 + R_2} = \frac{IG_2}{G_1 + G_2}$$



That is, current in one branch equals the total current multiplied by the resistance of the other branch and then divided by the sum of the resistances.

Kirchhoff's laws

In the proceeding section, we have seen how simple resistive networks can be solved for current, resistance, potential etc using the concept of Ohm's law. But as the network becomes complex, application of Ohm's law for solving the networks becomes tedious and hence time consuming. For solving such complex networks, we make use of Kirchhoff's laws. Gustav Kirchhoff (1824-1887), an eminent German physicist, did a considerable amount of work on the principles governing the behavior of electric circuits. He gave his findings in a set of two laws: (i) current law and (ii) voltage law, which together are known as Kirchhoff's laws. Before proceeding to the statement of these two laws let us familiarize ourselves with the following definitions encountered very often in the world of electricalcircuits:



Node: A node of a network is an equi-potential surface at which two or more circuit elements are joined. Referring to Fig. above, we find that A,B,C and D qualify as nodes in respect of the above definition.

Junction: A junction is that point in a network, where three or more circuit elements are joined. In Fig. above, we find that B and D are the junctions.

Branch: A branch is that part of a network which lies between two junction points. In Fig. above, BAD,BCD and BD qualify as branches.

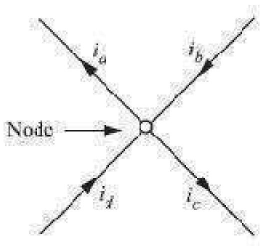
Loop: A loop is any closed path of a network. Thus, in Fig. above, ABDA,BCDB and ABCDA are the loops.

Mesh: A mesh is the most elementary form of a loop and cannot be further divided into other loops. In Fig. above, ABDA and BCDB are the examples of mesh. Once ABDA and BCDB are taken as meshes, the loop ABCDA does not qualify as a mesh, because it contains loops ABDA and BCDB.

Kirchhoff's Current Law

The first law is Kirchhoff's current law (KCL), which states that the algebraic sum of currents entering any node is zero. Let us consider the node shown in Fig. below. The sum of the currents entering the node is

Note that we have – in since the current is leaving the node. If we multiply the foregoing equation by -1, we obtain the expression which simply states that the algebraic sum of currents leaving a node is zero.



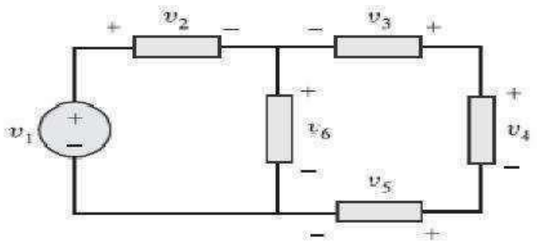
Which states that the sum of currents entering a node is equal to the sum of currents leaving the node? If the sum of the currents entering a node were not equal to zero, then the charge would be accumulating at a node. However, a node is a perfect conductor and cannot accumulate or store charge. Thus, the sum of currents entering a node is equal to zero.

Kirchhoff's Voltage Law

Kirchhoff's voltage law (KVL) states that the algebraic sum of voltages around any closed path in a circuit is zero. In general, the mathematical representation of Kirchhoff's voltage law is

$$\sum_{j=1}^{N} v_j(t) = 0$$

Where Vj (t) is the voltage across the Jth branch (with proper reference direction) in a loop containing N voltages. In Kirchhoff's voltage law, the algebraic sign is used to keep track of the voltage polarity. In other words, as we traverse the circuit, it is necessary to sum the increases and decreases in voltages to zero. Therefore, it is important to keep track of whether the voltage is increasing or decreasing as we go through each element. We will adopt a policy of considering the increase in voltage as negative and a decrease in voltage as positive.



Consider the circuit shown in Fig. above, where the voltage for each element is identified with its sign. The ideal wire used for connecting the components has zero resistance, and thus the voltage across it is equal to zero. The sum of voltages around the loop incorporating v6 , v3 , v4 and v5. The sum of voltages around a loop is equal to zero. A circuit loop is a conservative system, meaning that the work required to move a unit charge around any loop is zero. However, it is important to note that not all electrical systems are conservative. Example of a non conservative system is a radio wave broadcasting system.

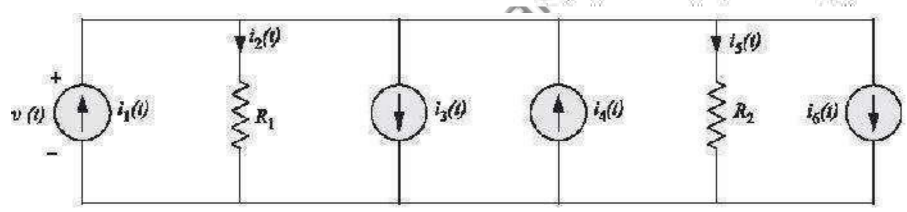
Multiple current source networks

Let us now learn how to reduce a network having multiple current sources and a number of resistors in parallel. Consider the circuit shown in Fig. below. We have assumed that the upper node is v(t) volts positive with respect to the lower node. Applying KCL to upper node yields

$$i_1(t) - i_2(t) - i_3(t) + i_4(t) - i_5(t) - i_6(t) = 0$$

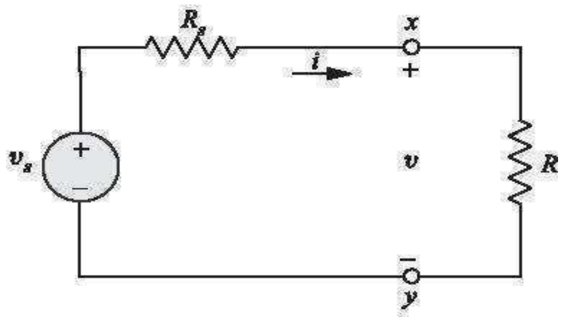
$$i_1(t) - i_3(t) + i_4(t) - i_6(t) = i_2(t) + i_5(t)$$

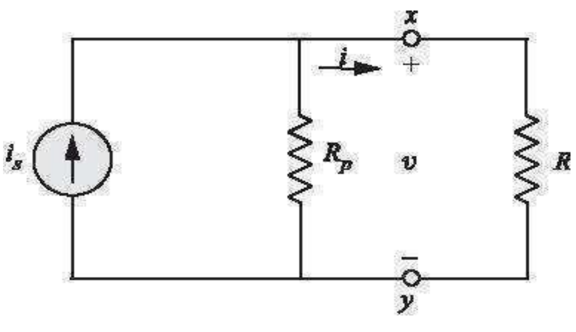
$$i_o(t) = i_2(t) + i_5(t)$$



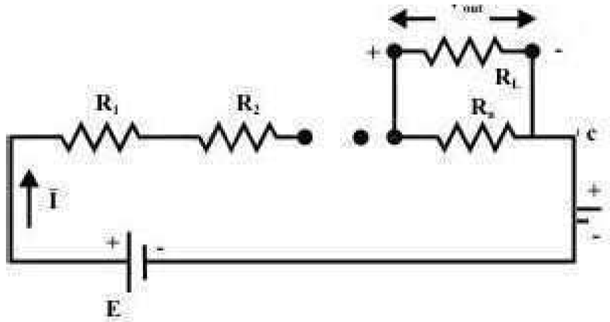
Source transformations

Source transformation is a procedure which transforms one source into another while retaining the terminal characteristics of the original source. Source transformation is based on the concept of equivalence. An equivalent circuit is one whose terminal characteristics remain identical to those of the original circuit. The term equivalence as applied to circuits means an identical e.ect at the terminals, but not within the equivalent circuits themselves.





Current source connected
to an external resistance R



However, real or practical dc voltage sources do not exhibit such characteristics (see fig. 3.14) in practice. We observed that as the load resistance R connected across the source is decreased, the corresponding load current I_L increases while the terminal voltage across the source decreases (see eq.3.1). We can realize such voltage drop across the terminals with increase in load current provided a resistance element (R_S) present inside the voltage source. Fig. 3.15 shows the model of practical or real voltage source of value .

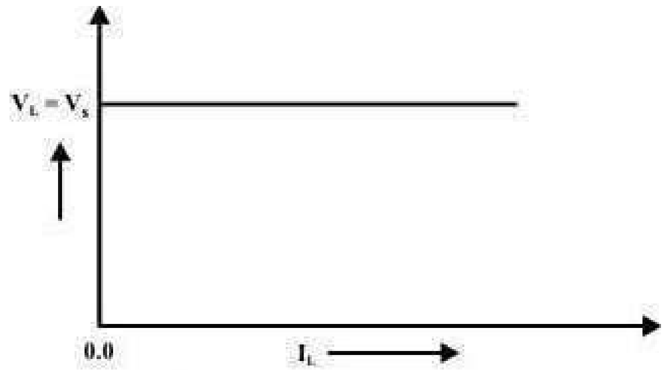


Fig. 3.14: V-I characteristics of ideal voltage source

The terminal $V_s - I$ characteristics of the practical voltage source can be described by an equation $V_L V - I_L R$ (3.1) and this equation is represented graphically as shown in fig.3.16. In practice$_L$, when a load resistance R more than 100 times larger than the source resistance R_S , the source can be considered approximately ideal voltage source. In other words, the internal resistance of the source can be omitted. This statement can be verified using the relation R_L 100R in equation (3.1). The practical voltage source is characterized by two parameters namely known as (i) Open circuit voltage (V) (ii) Internal resistance in the source's circuit model. In many practical situations, it is quite important to determine the source parameters experimentally. We shall discuss briefly a method in order to obtain source parameters.

Method-: Connect a variable load resistance across the source terminals (see fig. 3.15). A voltmeter is connected across the load and an ammeter is connected in series with the load resistance. Voltmeter and Ammeter readings for several choices of load resistances are presented on the graph paper (see fig. 3.16). The slope of the line is $-R_s$, while the curve intercepts with voltage axis (at I_L) is the value of V .

The $V - I$ characteristic of the source is also called the source's **"regulation curve" or "load line"**. The open-circuit voltage is also called the "no-load" voltage, V_{C} . The maximum allowable load current (rated current) is known as full-load current $_{FI}$ and the corresponding source or load terminal voltage is known as "full-load" voltage $_{FL}$. We know that the source terminal voltage varies as the load is varied and this is due to internal voltage drop inside the source. The percentage change in source terminal voltage from no-load to full-load current is termed the "voltage regulation" of the source. It is defined as

For ideal voltage source, there should be no change in voltage from no-load to full-load and this corresponds to "zero voltage regulation". For best possible performance, the voltage source should have the lowest possible regulation and this indicates a smallest possible internal voltage drop and the smallest possible internal resistance.

Ideal and Practical Current Sources

Another two-terminal element of common use in circuit modeling is `current source` as depicted in fig.3.17. An ideal current source, which is represented by a model in fig. 3.17(a), is a device that delivers a constant current to any load resistance connected across it, no matter what the terminal voltage is developed across the load (i.e., independent of the voltage across its terminals across the terminals).

It can be noted from$_{L}$ model of the current source that the current flowing from the source to the load is always constant for any load resistance (see fig. 3.19(a)) i.e. whether $_{L}$ is small (V_L is small is large (V_L is large). The vertical dashed line in fig. 3.18 represents the $V - I$ characteristic of ideal current source. In practice, when a load R is connected across a practical current source, one can observe that the current flowing in load resistance is reduced as the voltage across the current source's terminal is increased, by increasing the load resistance R . Since the distribution of source current in two parallel paths entirely depends on the value of external resistance that connected across the source (current source) terminals. This fact can be realized by introducing a parallel resistances in parallel with the practical current source , as shown in fig. 3.17(b). The dark lines in fig. 3.18 show the $V - I$ characteristic (load-line) of practical current source. The slope of the curve represents the internal resistance of the source. One can apply KCL at the top terminal of the current source in fig. 3.17(b) to obtain the following expression.

respectively. It can be noted from the fig.3.18 that source 1 has a larger internal resistance than source 2 and the slope the curve indicates the internal resistance R_S of the current source. Thus, source 1 is closer to the ideal source. More specifically, if the source internal resistance

then source acts nearly as an ideal current source.

Current source to Voltage Source

Remarks on practical sources: (i) The open circuit voltage that appears at the terminals *A* & *B* for two sources (voltage & current) is same (i.e., V_s).

(ii) When the terminals *A* & *B* are shorted by an ammeter, the shot-circuit results same in both cases (i.e.,).
(iii) If an arbitrary resistor (R_L) is connected across the output terminals *A* & *B* of

either source, the same power will be dissipated in it.
(iv) The sources are equivalent only as concerns on their behavior at the external terminals.
(v) The internal behavior of both sources is quite different (i.e., when open circuit the voltage source does not dissipate any internal power while the current source dissipates. Reverse situation is observed in short-circuit condition).8 Independent and Dependent Sources that encountered in electric circuits Independent Sources
So far the voltage and current sources (whether ideal or practical) that have been

discussed are known as independent sources and these sources play an important role to drive the circuit in order to perform a specific job. The internal values of these sources (either voltage source or current source) – that is, the generated voltage V or

The generated current $_S$ (see figs. 3.15 & 3.17) are not affected by the load connected across the source terminals or across any other element that exists elsewhere in the circuit or external to the source.

Dependent Sources
Another class of electrical sources is characterized by dependent source or controlled source. In fact the source voltage or current depends on a voltage across or a current through some other element elsewhere in the circuit. Sources, which exhibit this dependency, are called dependent sources. Both voltage and current types of sources may be dependent, and either may be controlled by a voltage or a current. In general,

Dependent source is represented by a diamond ()-shaped symbol as not to confuse it with an independent source. One can classify dependent voltage and current sources into four types of sources as shown in fig.3.21. These are listed below:
(i) Voltage-controlled voltage source (VCVS) (ii) Current-controlled voltage source (ICVS)
(iii) Voltage-controlled current source (VCIS) (iv) Current-controlled current source(ICIS)

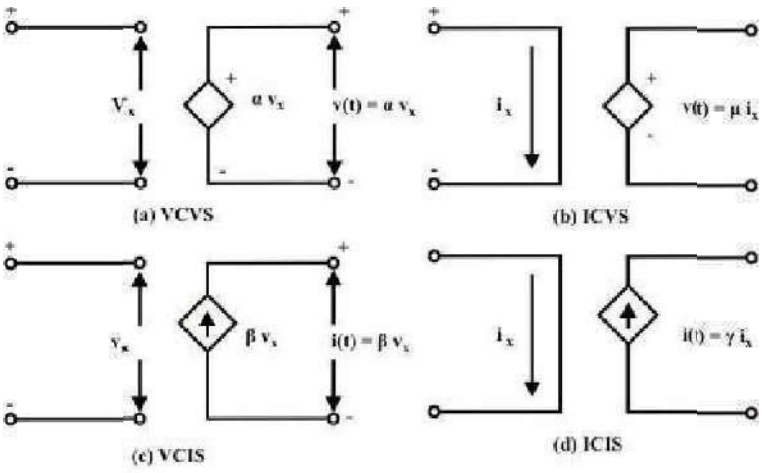


Fig. 3.21: Ideal dependent (controlled) sources

Note: When the value of the source (either voltage or current) is controlled by a voltage (v_X) somewhere else in the circuit, the source is said to be voltage-controlled source. On the other hand, when the value of the source (either voltage or current) is controlled by a current (i_X) somewhere else in the circuit, the source is said to be current-controlled source. KVL and KCL laws can be applied to networks containing such dependent sources. Source conversions, from dependent voltage source models to dependent current source models, or visa-versa, can be employed as needed to simplify the network. One may come across with the dependent sources in many equivalent-circuit models of electronic devices (transistor, BJT (bipolar junction transistor), FET(field-effect transistor) etc.) and transducers.

9 Understanding Delivering and Absorbing Power by the Source. It is essential to differentiate between the absorption of power (or dissipating power) and the generating (or delivering) power. The power absorbed or dissipated by any circuit element when flows in a load element from higher potential point (i.e +ve terminal) toward the lower terminal point (i.e., -ve terminal). This situation is observed when charging a battery or source because the source is absorbing power. On the other hand, when current flows in a source from the lower potential point (i.e., -ve terminal) toward the higher potential point (i.e., +ve terminal), we call that source is generating power or delivering power to the other elements in the electric circuit. In this case, one can note that the battery is acting as a "source" whereas the other element is acting as a "sink".

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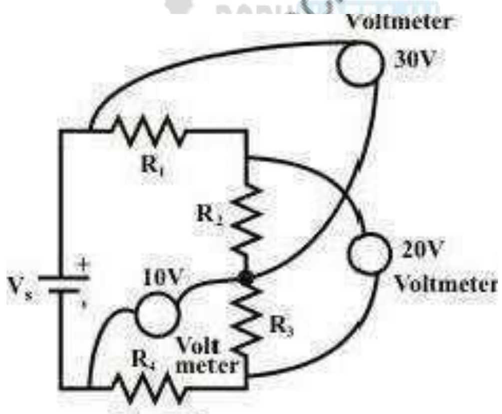


Fig. 3.33

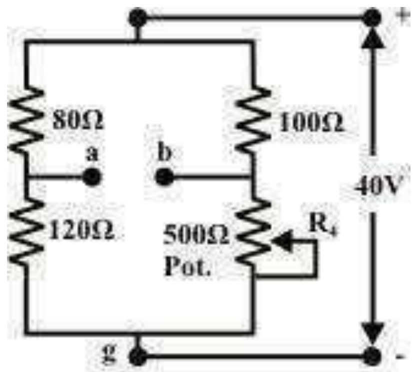


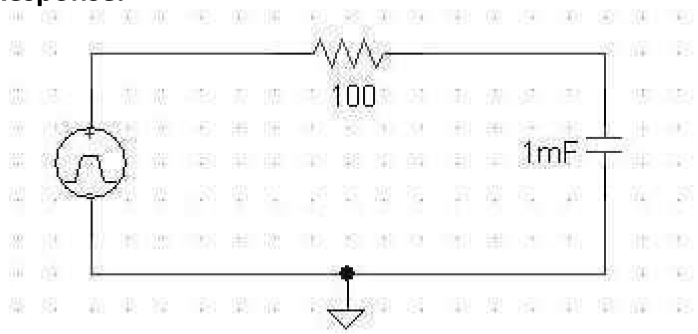
Fig. 3.36

UNIT 3

Transient analysis: Transients in RL, RC and RLC circuits, initial conditions, time constants, networks driven by constant driving sources and their solutions. Steady state analysis: - Concepts of phasors and vectors, impedance and admittance. Node and mesh

The Transient Response of RC Circuits

The Transient Response (also known as the Natural Response) is the way the circuit responds to energies stored in storage elements, such as capacitors and inductors. If a capacitor has energy stored within it, then that energy can be dissipated/absorbed by a resistor. How that energy is dissipated is the Transient Response.



In this circuit, there is a pulse, a resistor, and a capacitor. Assume here that the pulse goes from 10V down to 0V att=0.

Assume also that the circuit is in Steady State at t=0-. This implies that the capacitor is 'open' at t=0-. In order for KVL to be true at t=0- then the capacitor voltage must be 10V at t=0-. This is because there is no current in the circuit, therefore the voltage across the resistor is zero.

$V_c(0-) = V_c(0+) = 10V$

Note that since the Transient Response is the circuit's response toenergies stored in storage elements, we will 'kill' the pulse source. This leaves us with a simple Resitor-Capacitor circuit with an initial 10V on the capacitor at t=0+.

Applying KCL to an RC circuit:

$Cdv/dt + V/R = 0$
$dv/dt + V/(RC) = 0$
$\int dv/V = \int -1/(RC)\, dt$
$\ln V = -t/(RC) + K$
$\ln V(t=0) = K$
$\ln V_o = K$ ← V_o is the voltage on the cap at t=0+.

$\ln V - \ln V_o = -t/(RC)$
$\ln (V/V_o) = -t/(RC)$
$V/V_o = e^{-t/(RC)}$
$V(t) = V_o\, e^{-t/(RC)}$ ← $V_o = 10V$ in this example.

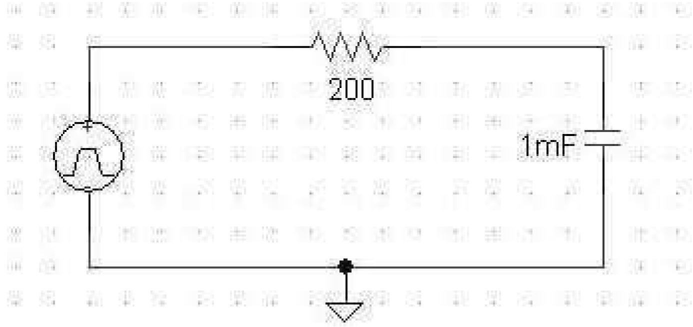
Note that the speed at which the capacitor discharges from 10V to 0V is determined by the product R×C

When t=RC, the voltage on the capacitor is V_o/e or 37% of it's initial value. We call RC the time constant and the symbol is τ

For an RC circuit, τ=RC

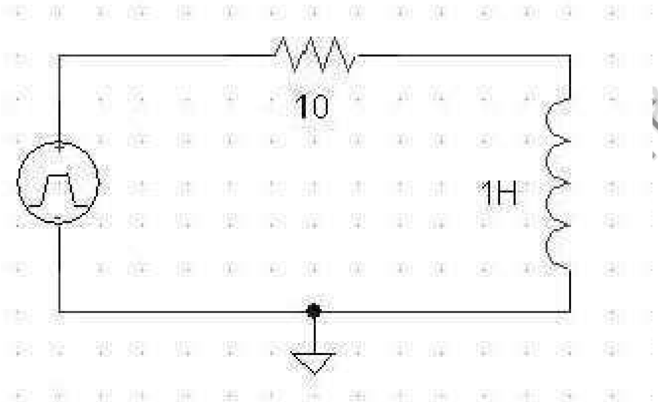
In this particular circuit τ = RC = 100Ω×1mF = 0.1 seconds
This means it takes 0.1 seconds for the capacitor to discharge from 10V down to 3.7V.

Here is the same circuit as that above, except that the resistor value is doubled. This means that τ is also doubled.
τ = RC = 200Ω×1mF = 0.2 seconds
This circuit is twice as slow as the last circuit.



The Transient Response of RL Circuits

The Transient Response (also known as the Natural Response) is the way the circuit responds to energies stored in storage elements, such as capacitors and inductors. If an inductor has energy stored within it, then that energy can be dissipated/absorbed by a resistor. How that energy is dissipated is the Transient Response.



In this circuit, there is a pulse, a resistor, and an inductor. Assume here that the pulse goes from -10V to 0V at t=0.

Assume also that the circuit is in Steady State at t=0-. This implies that the inductor is a 'short' at t=0-. In order for KCL to be true at t=0- the inductor current must be -1A at t=0-.

$I_L(0-) = I_L(0+) = -1A$

Consider the circuit at t=0+, the voltage across the pulse is zero but since $I_L(0+) = -1A$ then $V_R = -10V$.
Therefore for KVL to be true $V_L = +10V$.
Therefore $V_L = +10V$ is the initial voltage across the inductor.
Note that since the Transient Response is the circuit's response to energies stored in storage elements, we will 'kill' the pulse source. This leaves us with a simple Resitor-Inductor circuit with an initial -10A going through the inductor at t=0+.

Applying KVL to an RL circuit:

$iR + Ldi/dt = 0$
$iR/L + di/dt = 0$
$-iR/L = di/dt$
$-R/L\ dt = di/i$
$\int -R/L\ dt = \int di/i$
$-Rt/L + K = \ln i$
$K = \ln i(t=0)$
$K = \ln i_o$
$-Rt/L = \ln i - K$
$-Rt/L = \ln i - \ln i_o$
$-Rt/L = \ln(i/i_o)$
$i/i_o = e^{-Rt/L}$
$i(t) = i_o e^{-Rt/L}$ $\leftarrow i_o$ in this case is -1A

Since the plot on the right is for voltage we will find V_L using $V_L = Ldi/dt$
$V_L = (1H)\ d[i_o e^{-Rt/L}]/dt = (1H)\ (-10)\ i_o e^{-Rt/L}$
$V_L = -10\ e^{-Rt/L}$

When t=L/R, the voltage on the inductor is V_o/e or 37% of it's initial value. We call L/R the time constant and again the symbol is τ

For an RL circuit, $\tau=L/R$
In this particular circuit $\tau = L/R = 1H/10\Omega = 0.1$ seconds
This means it takes 0.1 seconds for the inductor to go from 10V down to 3.7V.

The Complete Response

The Complete Response is the circuit's response to both an independent source as well as energies stored in the circuit.
A circuit driven by an independent source is said to have a forcing function.
$V_{complete\ response} = V_{natural} + V_{forced}$

Here is an RC Circuit with a Forcing Function:

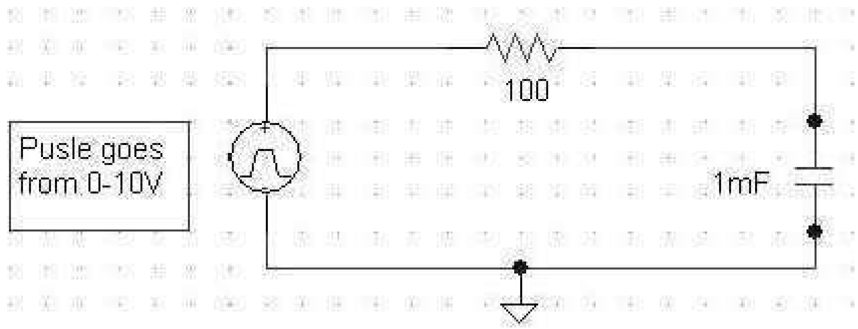
Assume the source is a pulse which goes from0V to 10V at t=0.

If we assume steady state at t=0-, then there is no initial energystored in the circuit.

Intuitively we know that the capacitor is going tocharge up to 10V. When the capacitor gets to 10V then the circuit is again at steady state.

The pulse is forcing the capacitor to 10V, thus the 10V on the capacitor is called the forced response.

The time it takes the capacitor to charge up to 10V is determined by the time constant. The response of getting to 10V is the transient response.



Now we will find the Complete Response for V across the capacitor. This equation will match the curve shown at the right.

From the last section we know that the transient response for an RC circuit is:
$V(t) = V_o\, e^{-t/(RC)} = A\, e^{-t/(RC)}$ Note that A is just some constant.

We also know from inspection that eventually the capacitor will charge up to 10V. Now putting the transient and forced responses together we get:

$V_{complete} = A\, e^{-t/(RC)} + V_{forced}$
$V_{complete} = A\, e^{-t/(RC)} + 10V$

Now we need to find A such that the equation equals V_o at t=0. In other words, the equation must satisfy the initial condition.

$V(t=0) = 0$, therefore:
$0 = A\, e^0 + 10V = A + 10V$
$A = -10V$

$V_{complete} = -10e^{-t/(RC)} + 10V$
$V_{complete} = -10e^{-10t} + 10V$

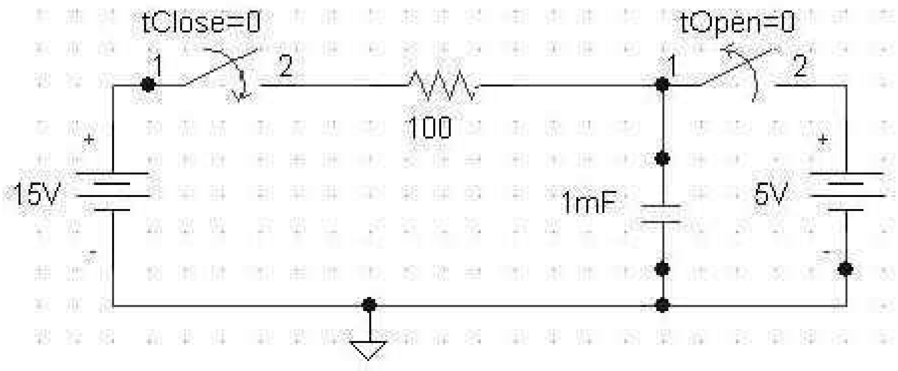
Note that when t>>0, $V_{complete} = 10V$. This intuitively means that when the transient response is gone the forced response still remains.

In this circuit, the capacitor DOES NOT start at 0V. In other words the capacitor has a non-zero initial condition of 5V:

Note that the left switch closes at the same time the right switch opens. Intuitively we can see that the capacitor is going to start at 5V and then charge up to 15V. For t<0 the 5V source is the forcing function and for t>0 the 15V source is the forcing function.

Since this is an RC circuit with a forcing function, the response takes the following form:
$V_{complete}=Ae^{-t/(RC)}+V_{forced}$
By inspection we know that $V_{forced} = 15V$
$V_{complete} = A\, e^{-t/(RC)} + 15V$
Now we need to find A such that the entire equation satisfies the value of V at t=0.
$V(t=0) = 5V = A\, e^0 + 15V$
$A = -10V$
$V_{complete} = -10e^{-t/(RC)} + 15\ V$

$V_{complete} = -10e^{-10t} + 15$ V



Now let's find the voltage across the resistor for the RL circuit to the right.

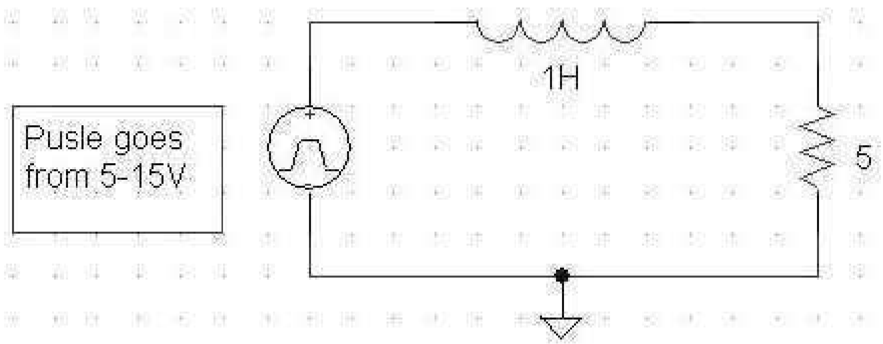
Note that the pulse goes from 5V to 15V at t=0. Assume that the circuit is in steady state at t=0-.
At steady state inductors look like 'shorts' therefore the voltage across the resistor must be equal to the pulse voltage of 5V at t=0-.

At t=0+ the voltage across the resistor is still 5V

Since the current in the inductor is continuous from 0- to 0+, and the current in the resistor is the same as the current in the inductor, and the voltage across the resistor is determined by its current, then we can say that if the resistor's current is continuous then the resistor's voltage must also be continuous.

At t>>0 the voltage across the resistor is 15V.

For t>0 we eventually reach steady state (as the transient response dies away), so we know that at t>>0 the inductor will look like a 'short'. Therefore the voltage across the resistor will equal the voltage of the pulse. Therefore we have both the initial condition and the forced response for the voltage across the resistor:
$V_o = 5$V
$V_{forced} = 15$V



Analysis Steps for finding the Complete Response of RC and RL Circuits

Use these Steps when finding the Complete Response for a 1st-order Circuit:
Step 1: First examine the switch to see if it is opening or closing and at what time.
Step 2: Next draw the circuit right before the switch moves. You will probably assume steady state at this time but not always. The problem needs to tell you to assume steady state.
Step 3: Find all voltages and currents that can not change instantaneously when the switch moves. In other words, Find voltages across all capacitors and currents through all inductors!

Step 4: Now draw the circuit right after the switch moves. Label the circuit with all the capacitor voltages and inductor currents you found in step 3.
Step 5: Now you are ready to find your initial condition(s). Analyze the circuit to find the initial condition(s) of what it is your solving for.
Step 6: Next you will find the transient/natural reponse, or τ. To do this 'kill' all forcing functions. Make all voltage sources 'shorts' and all current sources 'opens'. Remember that the transient response is the circuit's response to energies stored in storage elements, so we need to remove forcing functions to find this. Recall that every voltage and current will have the same τ value. You now have $Ae^{-t/\tau}$ for what your solving for.
Step 7: Now we need to find the forced response. The forced response is the state of the circuit after the switch has moved AND after the transient response has died-off. To find the forced response assume Steady State, i,e, t>>>0. Find the final resting value (forced response - V_F) of whatever it is you are solving for.
Step 8: You should now have an equation which looks like $v(t) = Ae^{-t/\tau} + V_F$ or $i(t) = Ae^{-t/\tau} + I_F$. To find the unknown 'A' you will apply the initial condition to this equation. Usually the initial condition is the value at t=0, so you will plug in t=0 to get the following: I.C. = $A + V_F$ or I.C. = $A + I_F$ You can now solve for A.
Step 9: Plugging the value of A into: $v(t) = Ae^{-t/\tau} + V_F$, you now know the voltage for all time greater than t=0 (assuming that the switch moved at t=0).
Step 10: Using your equation for v(t) or i(t), you can find other things (voltages, currents, power, etc.) using KVL, KCL, and Ohm's Law.

RLC circuit

An **RLC circuit** (or **LCR circuit**) is an electrical circuit consisting of a resistor, an inductor, and a capacitor, connected in series or in parallel. The RLC part of the name is due to those letters being the usual electrical symbols for resistance, inductance and capacitance respectively. The circuit forms a harmonic oscillator for current and will resonate in a similar way as an LC circuit will. The main difference that the presence of the resistor makes is that any oscillation induced in the circuit will die away over time if it is not kept going by a source. This effect of the resistor is called damping. The presence of the resistance also reduces the peak resonant frequency somewhat. Some resistance is unavoidable in real circuits, even if a resistor is not specifically included as a component. A pure LC circuit is an ideal which really only exists in theory.

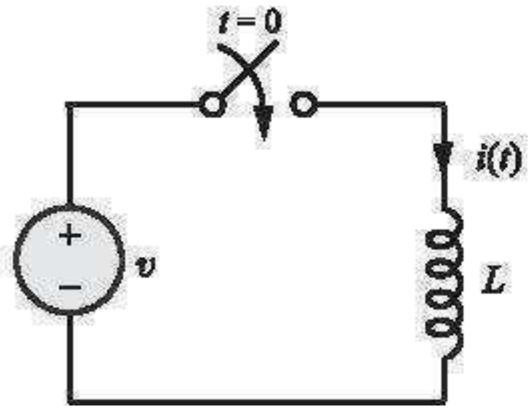
There are many applications for this circuit. They are used in many different types of oscillator circuit. Another important application is for tuning, such as in radio receivers or television sets, where they are used to select a narrow range of frequencies from the ambient radio waves. In this role the circuit is often referred to as a tuned circuit. An RLC circuit can be used as a band-pass filter, band-stop filter, low-pass filter or high-pass filter. The tuning application, for instance, is an example of band-pass filtering. The RLC filter is described as a *second-order* circuit, meaning that any voltage or current in the circuit can be described by a second-order differential equation in circuit analysis.

The three circuit elements can be combined in a number of different topologies. All three elements in series or all three elements in parallel are the simplest in concept and the most straightforward to analyse. There are, however, other arrangements, some with practical importance in real circuits. One issue often encountered is the need to take into account inductor resistance. Inductors are typically constructed from coils of wire, the resistance of which is not usually desirable, but it often has a significant effect on the circuit.

Initial and final conditions in elements

The inductor

The switch is closed at t = 0. Hence t = 0- corresponds to the instant when the switch is just open and t = 0+ corresponds to the instant when the switch is just closed. The expression for current through the inductor is given by



$$i = \frac{1}{L}\int_{-\infty}^{t} v d\tau$$

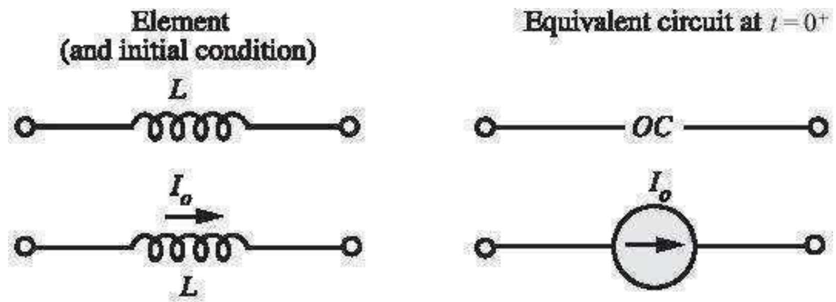
$$\Rightarrow \qquad i = \frac{1}{L}\int_{-\infty}^{0^-} v d\tau + \frac{1}{L}\int_{0^-}^{t} v d\tau$$

$$\Rightarrow \qquad i(t) = i(0^-) + \frac{1}{L}\int_{0^-}^{t} v d\tau$$

Putting $t = 0^+$ on both sides, we get

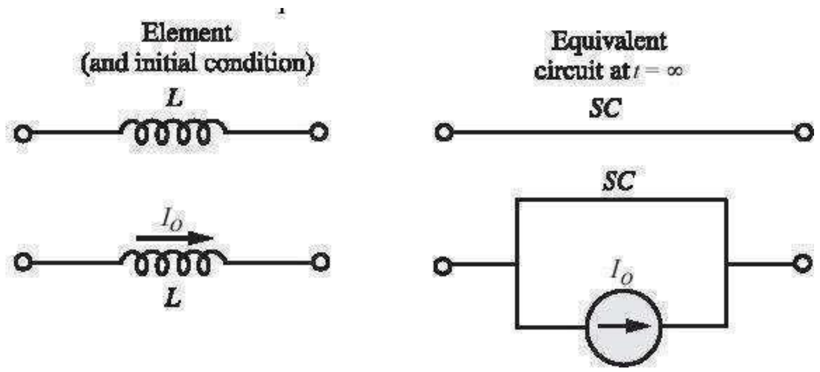
$$i(0^+) = i(0^-) + \frac{1}{L}\int_{0^-}^{0^+} v d\tau$$

$$\Rightarrow \qquad i(0^+) = i(0^-)$$



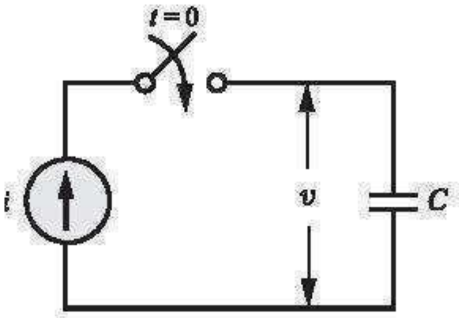
The final-condition equivalent circuit of an inductor is derived from the basic relationship

$$v = L\frac{di}{dt}$$



The capacitor

The switch is closed at t = 0. Hence t = 0- corresponds to the instant when the switch is just open and t = 0+ corresponds to the instant when the switch is just closed. The expression for current through the inductor is given by



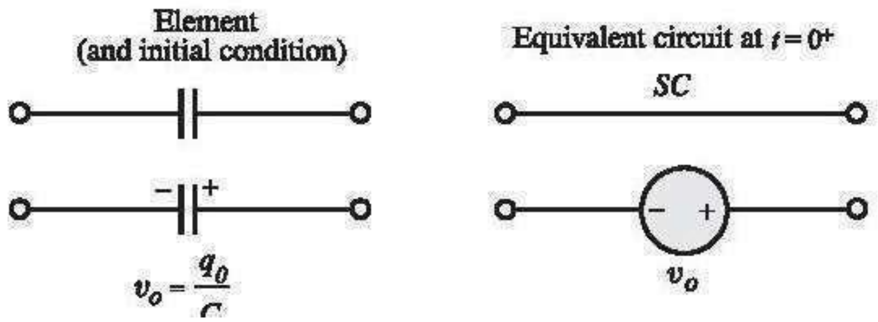
$$v = \frac{1}{C}\int_{-\infty}^{t} i\,d\tau$$

$$\Rightarrow \qquad v(t) = \frac{1}{C}\int_{-\infty}^{0^-} i\,d\tau + \frac{1}{C}\int_{0^-}^{t} i\,d\tau$$

$$\Rightarrow \qquad v(t) = v(0^-) + \frac{1}{C}\int_{0^-}^{t} i\,d\tau$$

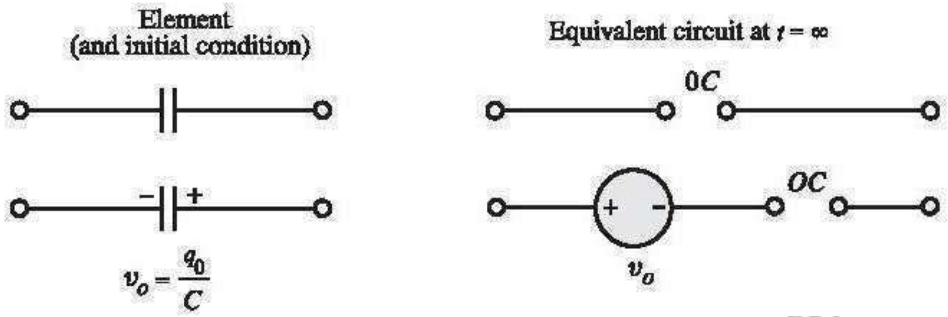
Evaluating the expression at $t = 0^+$, we get

$$v(0^+) = v(0^-) + \frac{1}{C}\int_{0^-}^{0^+} i\,d\tau \quad \Rightarrow \quad v(0^+) = v(0^-)$$



The final–condition equivalent network is derived from the basic relationship

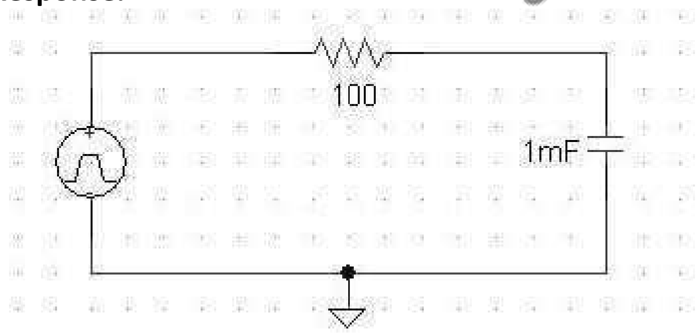
$$i = C\frac{dv}{dt}$$



Unit 4

The Transient Response of RC Circuits

The Transient Response (also known as the Natural Response) is the way the circuit responds to energies stored in storage elements, such as capacitors and inductors. If a capacitor has energy stored within it, then that energy can be dissipated/absorbed by a resistor. How that energy is dissipated is the Transient Response.



n this circuit, there is a pulse, a resistor, and a capacitor. Assume here that the pulse goes from 10V down to 0V at t=0.

Assume also that the circuit is in Steady State at t=0-. This implies that the capacitor is 'open' at t=0-. In order for KVL to be true at t=0- then the capacitor voltage must be 10V at t=0-. This is because there is no current in the circuit, therefore the voltage across the resistor is zero.

$V_c(0-) = V_c(0+) = 10V$

Note that since the Transient Response is the circuit's response toenergies stored in storage elements, we will 'kill' the pulse source. This leaves us with a simple Resitor-Capacitor circuit with an initial 10V on the capacitor at t=0+.

Applying KCL to an RC circuit:

$Cdv/dt + V/R = 0$
$dv/dt + V/(RC) = 0$
$\int dv/V = \int -1/(RC)\, dt$
$\ln V = -t/(RC) + K$
$\ln V(t=0) = K$
$\ln V_o = K$ ← V_o is the voltage on the cap at t=0+.

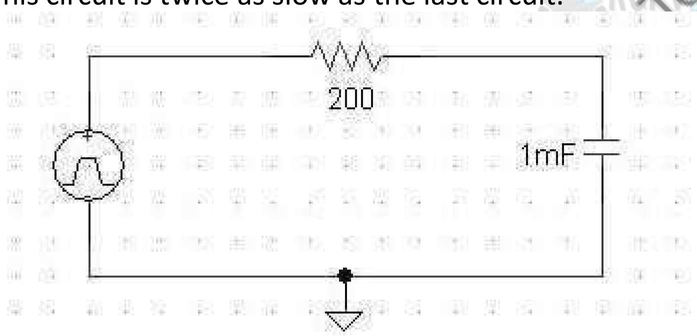
$\ln V - \ln V_o = -t/(RC)$
$\ln (V/V_o) = -t/(RC)$
$V/V_o = e^{-t/(RC)}$
$V(t) = V_o e^{-t/(RC)}$ ← V_o = 10V in this example.

Note that the speed at which the capacitor discharges from 10V to 0V is determined by the product R×C

When t=RC, the voltage on the capacitor is V_o/e or 37% of it's initial value. We call RC the time constant and the symbol is τ

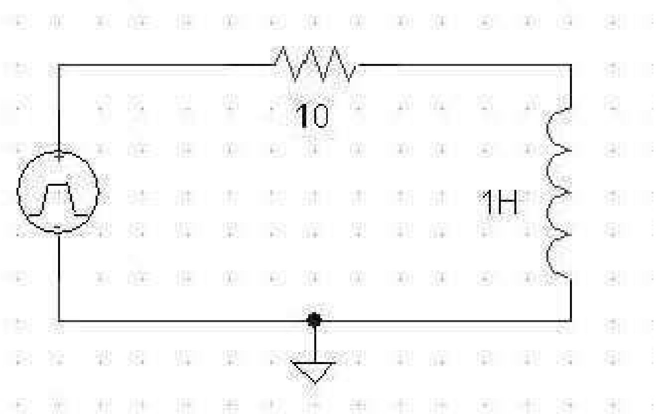
For an RC circuit, τ=RC
In this particular circuit τ = RC = 100Ω×1mF = 0.1 seconds
This means it takes 0.1 seconds for the capacitor to discharge from 10V down to 3.7V.

Here is the same circuit as that above, except that the resistor value is doubled. This means that τ is also doubled.
τ = RC = 200Ω×1mF = 0.2 seconds
This circuit is twice as slow as the last circuit.



The Transient Response of RL Circuits

The Transient Response (also known as the Natural Response) is the way the circuit responds to energies stored in storage elements, such as capacitors and inductors. If an inductor has energy stored within it, then that energy can be dissipated/absorbed by a resistor. How that energy is dissipated is the Transient Response.



In this circuit, there is a pulse, a resistor, and an inductor. Assume here that the pulse goes from -10V to 0V at t=0.

Assume also that the circuit is in Steady State at t=0-. This implies that the inductor is a 'short' at t=0-. In order for KCL to be true at t=0- the inductor current must be -1A at t=0-.
$I_L(0-) = I_L(0+) = -1A$
Consider the circuit at t=0+, the voltage across the pulse is zero but since $I_L(0+) = -1A$ then $V_R = -10V$.
Therefore for KVL to be true $V_L = +10V$.
Therefore $V_L = +10V$ is the initial voltage across the inductor.
Note that since the Transient Response is the circuit's response to energies stored in storage elements, we will 'kill' the pulse source. This leaves us with a simple Resitor-Inductor circuit with an initial -10A going through the inductor at t=0+.

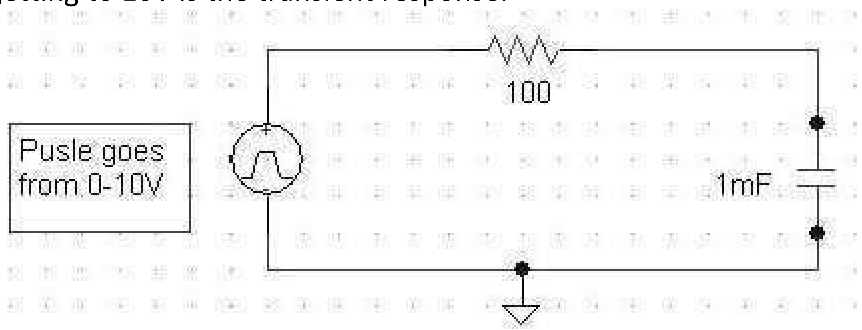
Applying KVL to an RL circuit:
$iR + Ldi/dt = 0$
$iR/L + di/dt = 0$
$-iR/L = di/dt$
$-R/L\ dt = di/i$
$\int -R/L\ dt = \int di/i$
$-Rt/L + K = \ln i$
$K = \ln i(t=0)$
$K = \ln i_o$
$-Rt/L = \ln i - K$
$-Rt/L = \ln i - \ln i_o$
$-Rt/L = \ln(i/i_o)$
$i/i_o = e^{-Rt/L}$
$i(t) = i_o e^{-Rt/L}$ $\leftarrow i_o$ in this case is -1A

Since the plot on the right is for voltage we will find V_L using $V_L = Ldi/dt$
$V_L = (1H)\ d[i_o e^{-Rt/L}]/dt = (1H)\ (-10)\ i_o e^{-Rt/L}$
$V_L = -10\ e^{-Rt/L}$

When t=L/R, the voltage on the inductor is V_o/e or 37% of it's initial value. We call L/R the time constant and again the symbol is τ

For an RL circuit, τ=L/R
In this particular circuit τ = L/R = 1H/10Ω = 0.1 seconds
This means it takes 0.1 seconds for the inductor to go from 10V down to 3.7V.

The Complete Response
The Complete Response is the circuit's response to both an independent source as well as energies stored in the circuit.
A circuit driven by an independent source is said to have a forcing function.
$V_{complete\ response} = V_{natural} + V_{forced}$
Here is an RC Circuit with a Forcing Function:
Assume the source is a pulse which goes from0V to 10V at t=0.
If we assume steady state at t=0-, then there is no initial energystored in the circuit.
Intuitively we know that the capacitor is going tocharge up to 10V. When the capacitor gets to 10V then the circuit is again at steady state.
The pulse is forcing the capacitor to 10V, thus the 10V on the capacitor is called the forced response.
The time it takes the capacitor to charge up to 10V is determined by the time constant. The response of getting to 10V is the transient response.



Now we will find the Complete Response for V across the capacitor. This equation will match the curve shown at the right.

From the last section we know that the transient response for an RC circuit is:
$V(t) = V_o e^{-t/(RC)} = A e^{-t/(RC)}$ Note that A is just some constant.

We also know from inspection that eventually the capacitor will charge up to 10V. Now putting the transient and forced responses together we get:

$V_{complete} = A e^{-t/(RC)} + V_{forced}$
$V_{complete} = A e^{-t/(RC)} + 10V$

Now we need to find A such that the equation equals V_o at t=0. In other words, the equation must satisfy the initial condition.

V(t=0) = 0, therefore:
$0 = A e^0 + 10V = A + 10V$
$A = -10V$
$V_{complete} = -10e^{-t/(RC)} + 10V$
$V_{complete} = -10e^{-10t} + 10V$
Note that when t>>0, $V_{complete} = 10V$. This intuitively means that when the transient response is gone the forced response still remains.

In this circuit, the capacitor DOES NOT start at 0V. In other words the capacitor has a non-zero initial condition of 5V:

Note that the left switch closes at the same time the right switch opens. Intuitively we can see that the capacitor is going to start at 5V and then charge up to 15V. For t<0 the 5V source is the forcing function and for t>0 the 15V source is the forcing function.

Since this is an RC circuit with a forcing function, the response takes the following form:

$V_{complete}=Ae^{-t/(RC)}+V_{forced}$

By inspection we know that V_{forced} = 15V

$V_{complete}$ = A $e^{-t/(RC)}$ + 15V

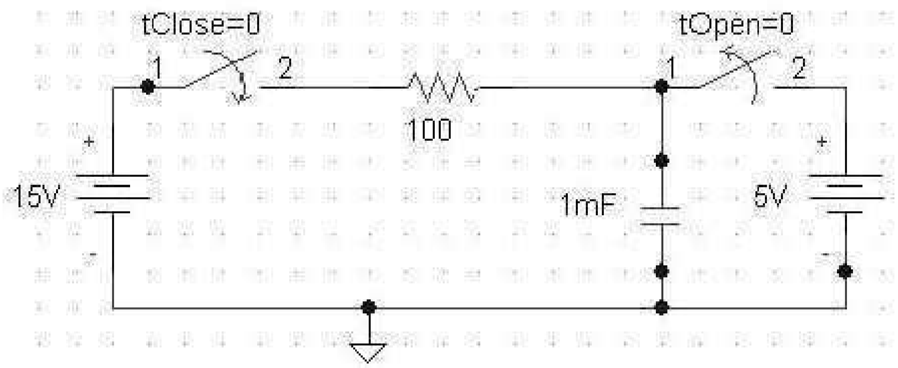
Now we need to find A such that the entire equation satisfies the value of V at t=0.

V(t=0) = 5V = A e^0 + 15V

A = -10V

$V_{complete}$ = $-10e^{-t/(RC)}$ + 15 V

$V_{complete}$ = $-10e^{-10t}$ + 15 V



Now let's find the voltage across the resistor for the RL circuit to the right.

Note that the pulse goes from 5V to 15V at t=0. Assume that the circuit is in steady state at t=0-.

At steady state inductors look like 'shorts' therefore the voltage across the resistor must be equal to the pulse voltage of 5V at t=0-.

At t=0+ the voltage across the resistor is still 5V

Since the current in the inductor is continuous from 0- to 0+, and the current in the resistor is the same as the current in the inductor, and the voltage across the resistor is determined by its current, then we can say that if the resistor's current is continuous then the resistor's voltage must also be continuous.

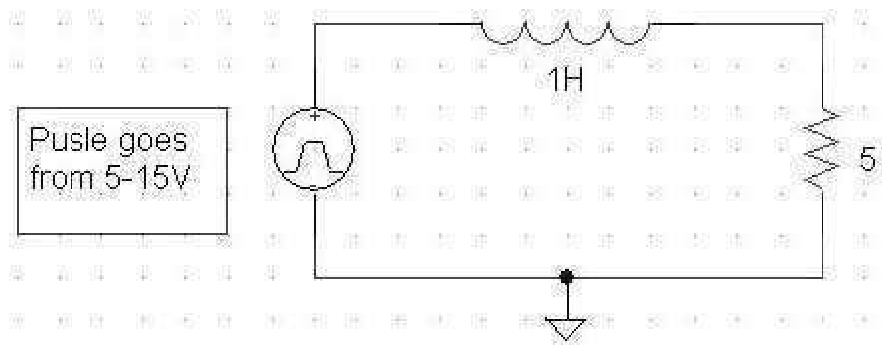
At t>>0 the voltage across the resistor is 15V.

For t>0 we eventually reach steady state (as the transient response dies away), so we know that at t>>0 the inductor will look like a 'short'. Therefore the voltage across the resistor will equal the voltage of the pulse.

Therefore we have both the initial condition and the forced response for the voltage across the resistor:

V_o = 5V

V_{forced} = 15V



Analysis Steps for finding the Complete Response of RC and RL Circuits

Use these Steps when finding the Complete Response for a 1st-order Circuit:

Step 1: First examine the switch to see if it is opening or closing and at what time.
Step 2: Next draw the circuit right before the switch moves. You will probably assume steady state at this time but not always. The problem needs to tell you to assume steady state.
Step 3: Find all voltages and currents that can not change instantaneously when the switch moves. In other words, Find voltages across all capacitors and currents through all inductors!
Step 4: Now draw the circuit right after the switch moves. Label the circuit with all the capacitor voltages and inductor currents you found in step 3.
Step 5: Now you are ready to find your initial condition(s). Analyze the circuit to find the initial condition(s) of what it is your solving for.
Step 6: Next you will find the transient/natural reponse, or τ. To do this 'kill' all forcing functions. Make all voltage sources 'shorts' and all current sources 'opens'. Remember that the transient response is the circuit's response to energies stored in storage elements, so we need to remove forcing functions to find this. Recall that every voltage and current will have the same τ value. You now have $Ae^{-t/\tau}$ for what your solving for.
Step 7: Now we need to find the forced response. The forced response is the state of the circuit after the switch has moved AND after the transient response has died-off. To find the forced response assume Steady State, i,e, t>>>0. Find the final resting value (forced response - V_F) of whatever it is you are solving for.
Step 8: You should now have an equation which looks like $v(t) = Ae^{-t/\tau} + V_F$ or $i(t) = Ae^{-t/\tau} + I_F$. To find the unknown 'A' you will apply the initial condition to this equation. Usually the initial condition is the value at t=0, so you will plug in t=0 to get the following: I.C. = A +V_F or I.C. = A + I_F You can now solve for A.
Step 9: Plugging the value of A into: $v(t) = Ae^{-t/\tau} + V_F$, you now know the voltage for all time greater than t=0 (assuming that the switch moved at t=0).
Step 10: Using your equation for v(t) or i(t), you can find other things (voltages, currents, power, etc.) using KVL, KCL, and Ohm's Law.

RLC circuit

An **RLC circuit** (or **LCR circuit**) is an electrical circuit consisting of a resistor, an inductor, and a capacitor, connected in series or in parallel. The RLC part of the name is due to those letters being the usual electrical symbols for resistance, inductance and capacitance respectively. The circuit forms a harmonic oscillator for current and will resonate in a similar way as an LC circuit will. The main difference that the presence of the resistor makes is that any oscillation induced in the circuit will die away over time if it is not kept going by a source. This effect of the resistor is called damping. The presence of the resistance also reduces the peak resonant frequency somewhat. Some resistance is unavoidable in real circuits, even if a resistor is not specifically included as a component. A pure LC circuit is an ideal which really only exists in theory.

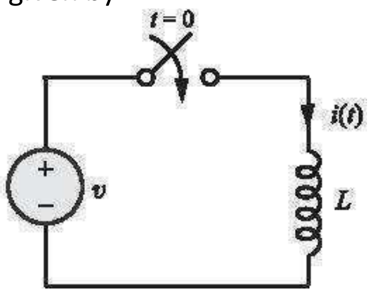
There are many applications for this circuit. They are used in many different types of oscillator circuit. Another important application is for tuning, such as in radio receivers or television sets, where they are used to select a narrow range of frequencies from the ambient radio waves. In this role the circuit is often referred to as a tuned circuit. An RLC circuit can be used as a band-pass filter, band-stop filter, low-pass filter or high-pass filter. The tuning application, for instance, is an example of band-pass filtering. The RLC filter is described as a *second-order* circuit, meaning that any voltage or current in the circuit can be described by a second-order differential equation in circuit analysis.

The three circuit elements can be combined in a number of different topologies. All three elements in series or all three elements in parallel are the simplest in concept and the most straightforward to analyse. There are, however, other arrangements, some with practical importance in real circuits. One issue often encountered is the need to take into account inductor resistance. Inductors are typically constructed from coils of wire, the resistance of which is not usually desirable, but it often has a significant effect on the circuit.

Initial and final conditions in elements

The inductor

The switch is closed at t = 0. Hence t = 0- corresponds to the instant when the switch is just open and t = 0+ corresponds to the instant when the switch is just closed. The expression for current through the inductor is given by



$$i = \frac{1}{L}\int_{-\infty}^{t} v\,d\tau$$

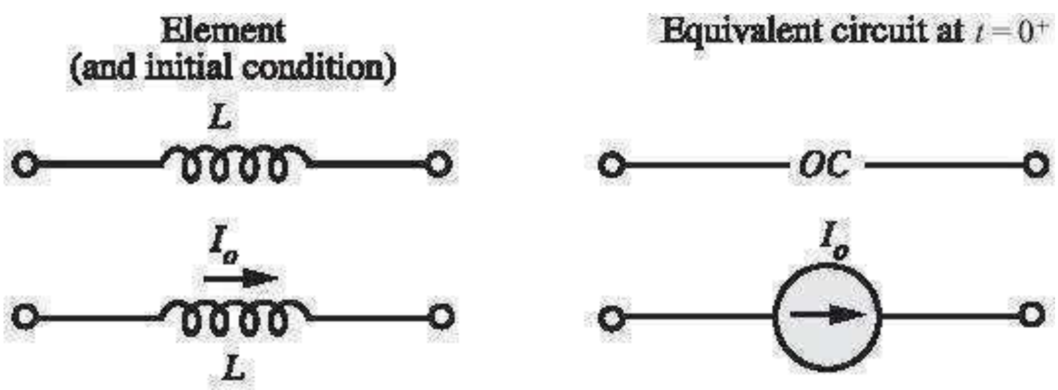
$$\Rightarrow \qquad i = \frac{1}{L}\int_{-\infty}^{0-} v\,d\tau + \frac{1}{L}\int_{0-}^{t} v\,d\tau$$

$$\Rightarrow \qquad i(t) = i(0^-) + \frac{1}{L}\int_{0-}^{t} v\,d\tau$$

Putting $t = 0^+$ on both sides, we get

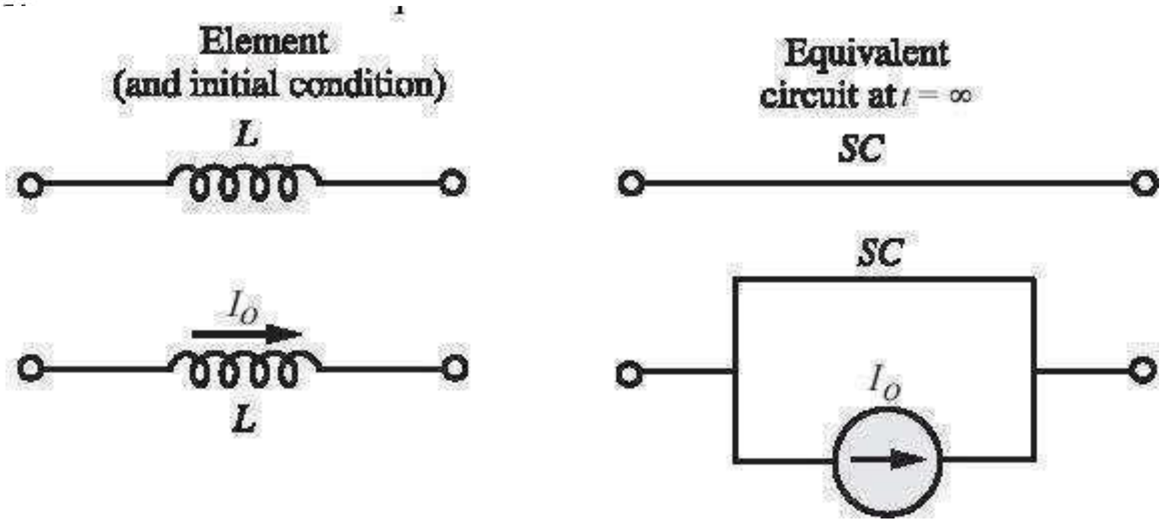
$$i(0^+) = i(0^-) + \frac{1}{L}\int_{0-}^{0+} v\,d\tau$$

$$\Rightarrow \qquad i(0^+) = i(0^-)$$



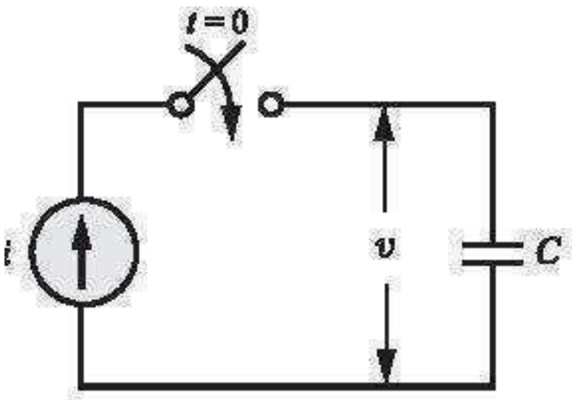
The final-condition equivalent circuit of an inductor is derived from the basic relationship

$$v = L\frac{di}{dt}$$



The capacitor

The switch is closed at t = 0. Hence t = 0- corresponds to the instant when the switch is just open and t = 0+ corresponds to the instant when the switch is just closed. The expression for current through the inductor is given by



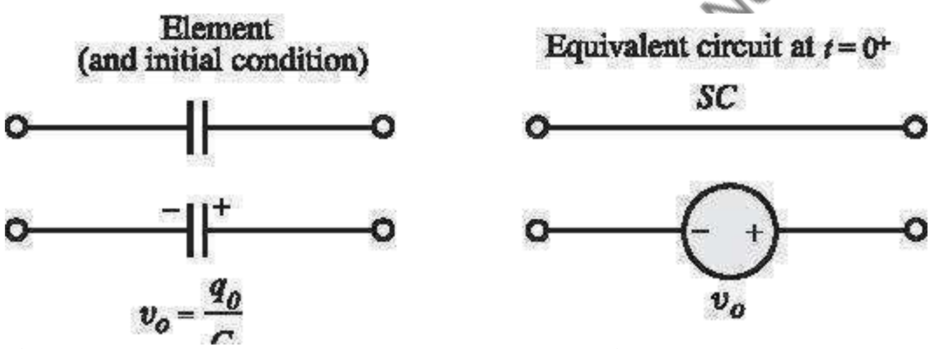
$$v = \frac{1}{C}\int_{-\infty}^{t} i\,d\tau$$

$$\Rightarrow \qquad v(t) = \frac{1}{C}\int_{-\infty}^{0^-} i\,d\tau + \frac{1}{C}\int_{0^-}^{t} i\,d\tau$$

$$\Rightarrow \qquad v(t) = v(0^-) + \frac{1}{C}\int_{0^-}^{t} i\,d\tau$$

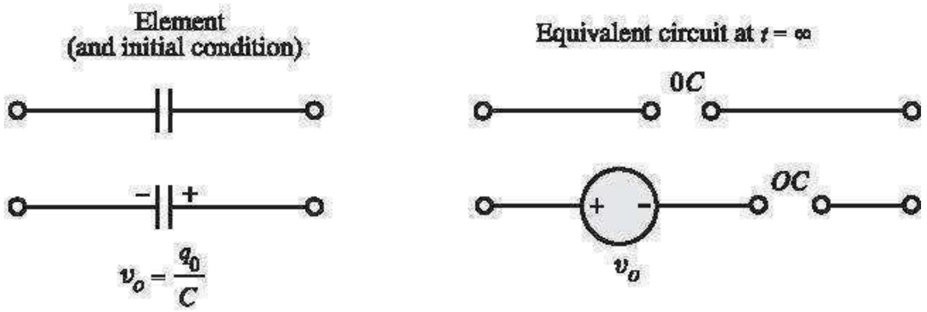
Evaluating the expression at $t = 0^+$, we get

$$v(0^+) = v(0^-) + \frac{1}{C}\int_{0^-}^{0^+} i\,d\tau \quad \Rightarrow \quad v(0^+) = v(0^-)$$



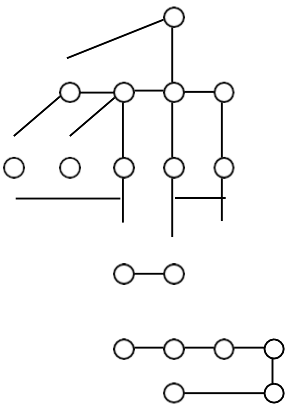
The final–condition equivalent network is derived from the basic relationship

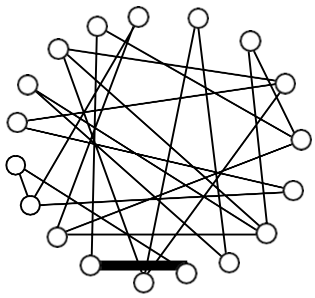
$$i = C\frac{dv}{dt}$$



UNIT 2

Network topology, Concept of Network graph, Tree, tree branches and links, cut set and tie set schedules. Network Theorems – Thevenin, Norton, Superposition, Reciprocity, Compensation, Maximum power transfer and Millmans theorems, problems with controlled sources. This type of simplified picture is called a **graph**. A graph consists of a set of dots, called **vertices**, and a set of **edges** connecting pairs of vertices. While we drew our original graph to correspond with the picture we had, there is nothing particularly important about the layout when we analyze a graph. Both of these graphs are equivalent to the one drawn above.





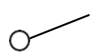
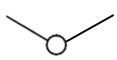
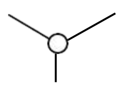
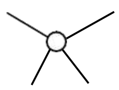
Definitions

Vertex. A vertex is a dot in the graph where edges meet. A vertex could represent an intersection of streets, a land mass, or a general location, like “work” or “school”. Note that vertices only occur when a dot is explicitly placed, not whenever two edges cross. Imagine a freeway overpass – the freeway and side street cross, but it is not possible to change from the side street to the freeway at that point, so there is no intersection and no vertex would be placed.

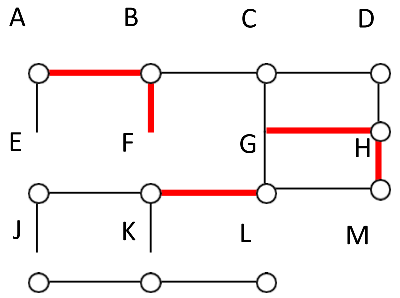
Edges. Edges connect pairs of vertices. An edge can represent a physical connection between locations, like a street, or simply that a route connecting the two locations exists, like an airline flight.

Loop. A loop is a special type of edge that connects a vertex to itself. Loops are not used much in street network graphs.

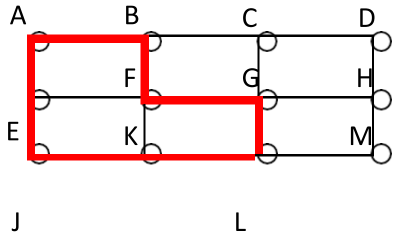
Degree of a vertex. The degree of a vertex is the number of edges meeting at that vertex. It is possible for a vertex to have a degree of zero or larger.

Degree 0	Degree 1	Degree 2	Degree 3	Degree 4
				

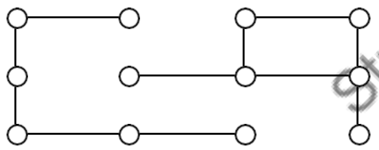
Path. A path is a sequence of vertices using the edges. Usually we are interested in a path between two vertices. For example, a path from vertex A to vertex M is shown below. It is one of many possible paths in this graph.



Circuit. A circuit is a path that begins and ends at the same vertex. A circuit starting and ending at vertex A is shown below.



Connected graph. A graph is connected if there is a path from any vertex to any other vertex. Every graph drawn so far has been connected. The graph below is **disconnected**; there is no way to get from the vertices on the left to the vertices on the right.



Weights. Depending upon the problem being solved, sometimes weights are assigned to the edges. The weights could represent the distance between two locations, the travel time, or the travel cost. It is important to note that the distance between vertices in a graph does not necessarily correspond to the weight of an edge.

Shortest Path

When you visit a website like Google Maps or MapQuest and ask for directions from home to your Aunt's house in Pasadena, you are usually looking for a shortest path between the two locations. These computer applications use representations of the street maps as graphs, with estimated driving times as edge weights.

While often it is possible to find a shortest path on a small graph by guess-and-check, our goal in this chapter is to develop methods to solve complex problems in a systematic way by following **algorithms**. An algorithm is a step-by-step procedure for solving a problem. Dijkstra's (pronounced dike-stra) algorithm will find the shortest path between two vertices.

Terms and definitions

The description of networks in terms of their geometry is referred to as network topology. The adequacy of a set of equations for analyzing a network is more easily determined topologically than algebraically.

Graph (or linear graph): A network graph is a network in which all nodes and loops are retained but its branches are represented by lines. The voltage sources are replaced by short circuits and current sources are replaced by open circuits. (Sources without internal impedances or admittances can also be treated in the same way because they can be shifted to other branches by E-shift and/or I-shift operations.)

Branch: A line segment replacing one or more network elements that are connected in series or parallel.

Node: Interconnection of two or more branches. It is a terminal of a branch. Usually interconnections of three or more branches are nodes.

Path: A set of branches that may be traversed in an order without passing through the same node more than once.

Loop: Any closed contour selected in a graph.

Mesh: A loop which does not contain any other loop within it.

Planar graph: A graph which may be drawn on a plane surface in such a way that no branch passes over any other branch.

Non-planar graph: Any graph which is not planar.

Oriented graph: When a direction to each branch of a graph is assigned, the resulting graph is called an oriented graph or a directed graph.

Connected graph: A graph is connected if and only if there is a path between every pair of nodes.

Sub graph: Any subset of branches of the graph.

Tree: A connected sub-graph containing all nodes of a graph but no closed path. i.e. it is a set of branches of graph which contains no loop but connects every node to every other node not necessarily directly. A number of different trees can be drawn for a given graph.

Link: A branch of the graph which does not belong to the particular tree under consideration. The links form a sub-graph not necessarily connected and is called the co-tree.

Tree compliment: Totality of links i.e. Co-tree.

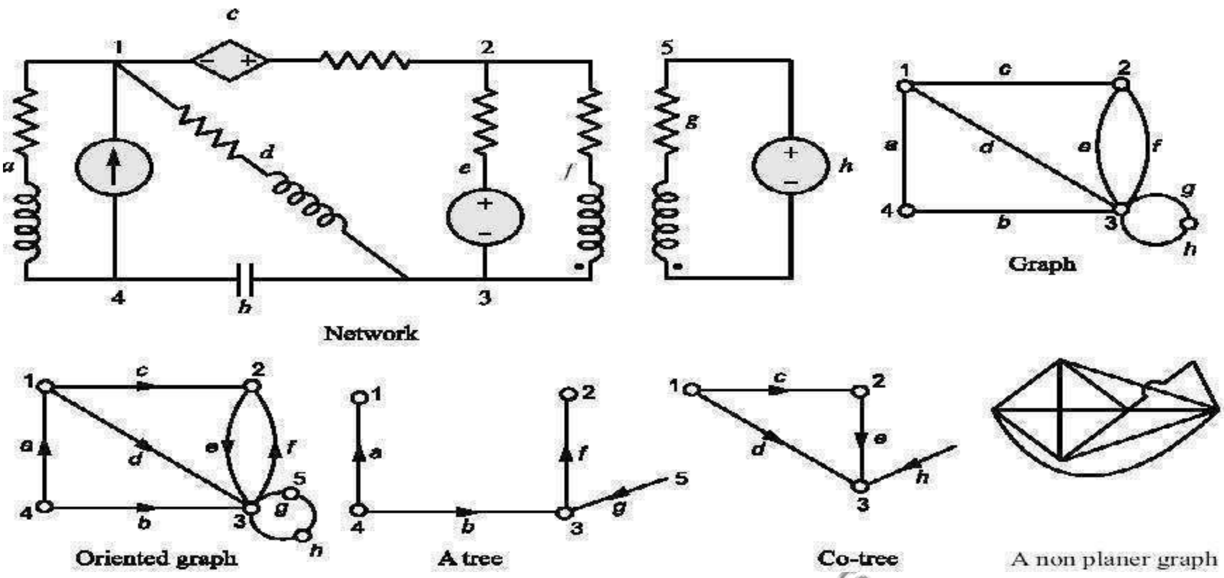
Independent loop: The addition of each link to a tree, one at a time, results one closed path called,an independent loop. Such a loop contains only one link and other tree branches. Obviously, the number of such independent loops equals the number of links.

Tie set: A set of branches contained in a loop such that each loop contains one link and the remainder are tree branches.

Tree branch voltages: The branch voltages may be separated in to tree branch voltages and link voltages. The tree branches connect all the nodes. Therefore if the tree branch voltages are forced to be zero, then all the node potentials become coincident and hence all branch voltages are forced to be zero. As the act of setting only the tree branch voltages to zero forces all voltages in the network to be zero, it must be possible to express all the link voltages uniquely in terms of tree branch voltages. Thus tree branch form an independent set of equations.

Cut set: A set of elements of the graph that dissociates it into two main portions of a network such that replacing any one element will destroy this property. It is a set of branches that if removed divides a connected graph in to two connected sub-graphs. Each cut set contains one tree branch and the remaining being links.

Fig. below shows a typical network with its graph, oriented graph, a tree, co-tree and a non-planar graph.



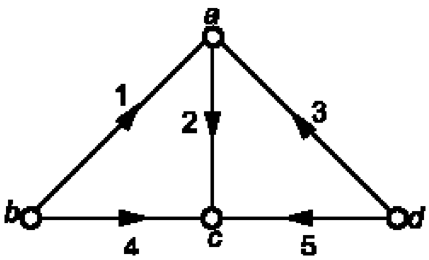
Matrix representation of a graph

For a given oriented graph, there are several representative matrices. They are extremely important in the analytical studies of a graph, particularly in the computer aided analysis and synthesis of large scale networks.

Incidence Matrix An
It is also known as augmented incidence matrix. The element node incidence matrix A indicates in a connected graph, the incidence of elements to nodes. It is an N × B matrix with elements of An = (akj)

akj = 1, when the branch bj is incident to and oriented away from the kth node.
= -1, when the branch bj is incident to and oriented towards the kth node.
= 0, when the branch bj is not incident to the kth node.
As each branch of the graph is incident to exactly two nodes,
∑akj = 0 for j = 1, 2, 3, · · · B.

That is, each column of An has exactly two non zero elements, one being +1 and the other -1. Sum of elements of any column is zero. The columns of An are lineraly dependent. The rank of the matrix is less than N. Significance of the incidence matrix lies in the fact that it translates all the geometrical features in the graph into an algebraic expression. Using the incidence matrix, we can write KCL as An iB = 0, where iB = branch current vector. But these equations are not linearly independent. The rank of the matrix A is N - 1. This property of An is used to define another matrix called reduced incidence matrix or bus incidence matrix. For the oriented graph shown in Fig. below the incidence matrix is as follows:



$$
\begin{array}{cc}
\text{Nodes} \downarrow & \text{branches} \\
& \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\
\mathbf{A}_n = \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} -1 & 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}
\end{array}
$$

Reduced incidence matrix

Any node of a connected graph can be selected as a reference node. Then the voltages of the other nodes (referred to as buses) can be measured with respect to the assigned reference. The matrix obtained from **A**n by deleting the row corresponding to the reference node is the elementbus incident matrix A and is called bus incidence matrix with dimension (N - 1) × B. **A** is rectangular and therefore singular. In **A**n, the sum of all elements in each column is zero. This leads to an important conclusion that if one row is not known in **A**, it can be found so that sum of elements of each column must be zero.

From **A**, we have **A** **i**B = 0, which represents a set of linearly independent equations and there are N - 1 independent node equations.

For the graph shown in Fig 2.3 shown above with d selected as the reference node, the reduced incidence matrix is

$$
\begin{array}{cc}
\text{Nodes} \downarrow & \text{branches} \\
& \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\
\mathbf{A} = \begin{array}{c} a \\ b \\ c \end{array} & \begin{bmatrix} -1 & 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & -1 \end{bmatrix}
\end{array}
$$

Note that the sum of elements of each column in **A** need not be zero. Note that if branch current vector,

$$
\mathbf{i}_B = \begin{bmatrix} j_1 \\ j_2 \\ j_3 \\ j_4 \\ j_5 \end{bmatrix}
$$

Then **A i**B = 0 representing a set of independent node equations.

Another important property of **A** is that determinant AAT gives the number of possible trees of the network. If A = [At : Ai] where At and Ai are sub-matrices of **A** such that At contains only twigs, then det At is either + 1 or -1. To verify the property that det AAT gives the number of all possible trees, consider the reduced incidence matrix **A** of the example considered. That is,

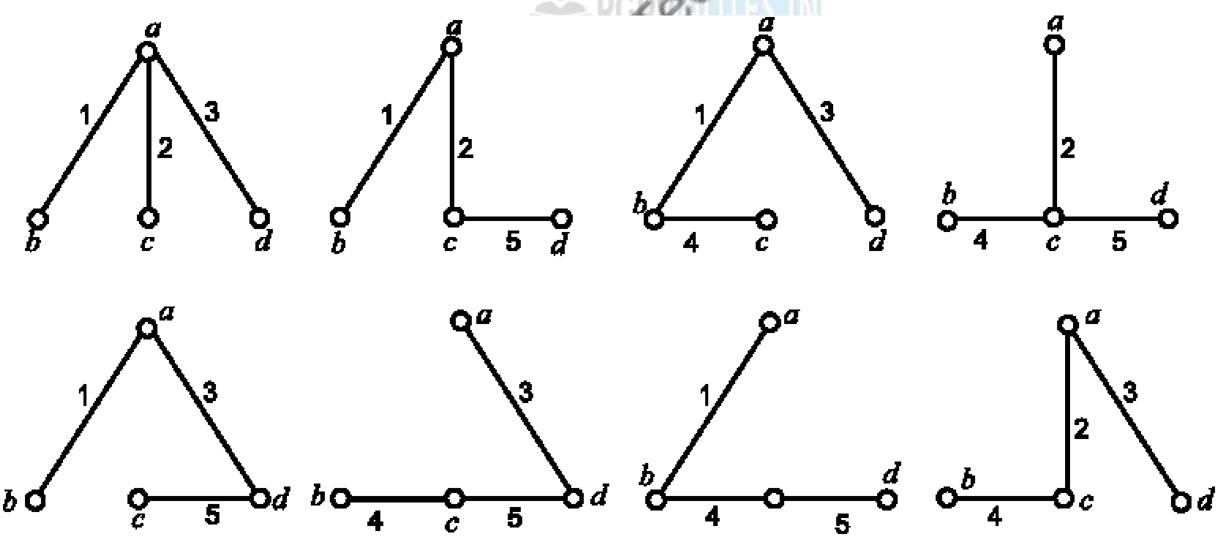
$$\mathbf{A} = \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \left[\begin{array}{ccc:cc} -1 & 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & -1 \end{array}\right] \end{array}$$

$$\underbrace{}_{\mathbf{A}_t} \qquad \underbrace{}_{\mathbf{A}_i}$$

Then,

$$\text{Det } \mathbf{AA}^{\mathrm{T}} = \left| \begin{bmatrix} -1 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}^{\mathrm{T}} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ -1 & -1 \end{bmatrix}^{\mathrm{T}} \right| = 8$$

Fig. shown below shows all possible trees corresponding to the matrix **A**.



To verify the property that the determinant of sub matrix **A**t of **A** = **A**t ; **A**i is +1 or -1. For tree [2, 3, 4]

$$\begin{array}{cc} & \begin{array}{ccccc} 2 & 3 & 4 & 1 & 5 \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \left[\begin{array}{ccccc} +1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ -1 & 0 & -1 & 0 & -1 \end{array}\right] \end{array}$$

Nodes ↓ branches

$$\mathbf{A} = \begin{array}{cc} & \begin{array}{ccccc} 2 & 3 & 4 & 1 & 5 \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \left[\begin{array}{ccccc} +1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ -1 & 0 & -1 & 0 & -1 \end{array}\right] \end{array} = \mathbf{A}_t; \mathbf{A}_i$$

$$\text{Det } \mathbf{A}_i = \begin{vmatrix} 1 & +1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & -1 \end{vmatrix} = 1$$

For another tree [2, 4, 5]

Nodes ↓ branches

$$\mathbf{A} = \begin{array}{cc} & \begin{array}{ccccc} 2 & 4 & 5 & 1 & 3 \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \left[\begin{array}{ccccc} 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 1 & 0 \\ -1 & -1 & -1 & 0 & 0 \end{array}\right] \end{array} = \mathbf{A}_t; \mathbf{A}_i$$

$$\text{Det } \mathbf{A}_i = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & -1 \end{vmatrix} = -1$$

Loop equations and fundamental loop matrix (Tie-set Matrix)

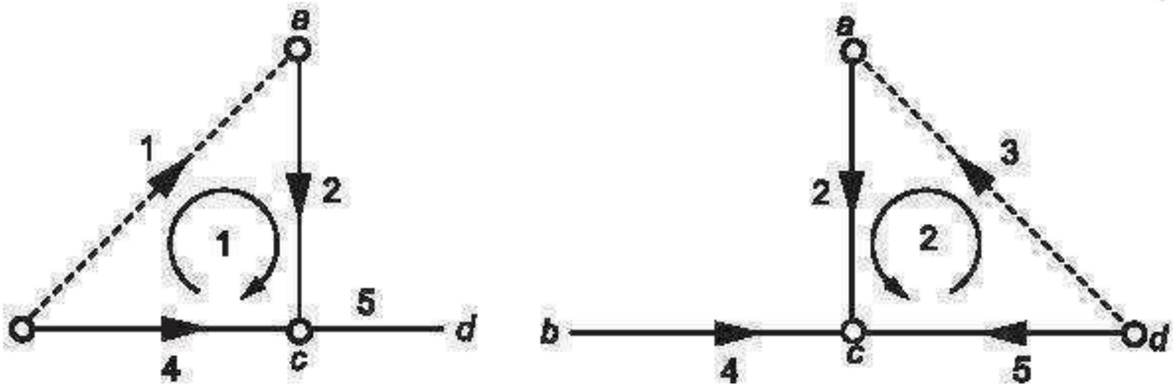
In this matrix, the loop orientation is to be the same as the corresponding link direction. In order to construct this matrix, the following procedure is to be followed.

1. Draw the oriented graph of the network. Choose a tree.
2. Each link forms an independent loop. The direction of this loop is same as that of the corresponding link. Choose each link in turn.
3. Prepare the tie-set matrix with elements biJ, where biJ = 1 bJ when branch i in loop _ and is directed in the same direction as the loop current.

= -1 when branch bJ is in loop i and is directed in the opposite direction as the loop current.

= 0 when branch bJ is not in loop i.

Tie-set matrix is an b x I matrix.



$$\begin{array}{c} \text{Loops} \downarrow \end{array} \qquad \text{branches}$$

$$\mathbf{M} = \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \left[\begin{array}{ccccc} 1 & 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 \end{array}\right] \end{array}$$

$$\mathbf{V}_B = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix}$$

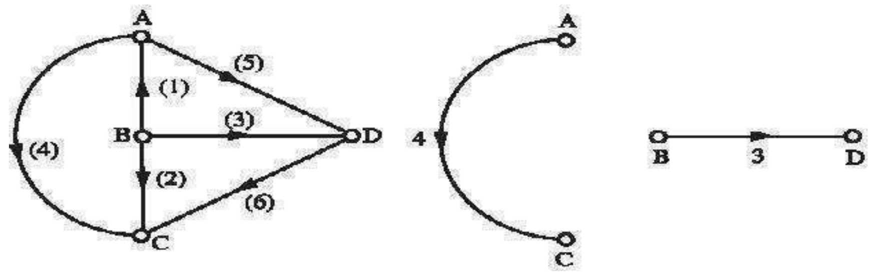
Then MVb gives the following independent loop equations:

$$v_1 + v_2 - v_4 = 0$$
$$v_2 + v_3 - v_5 = 0$$

Looking column wise, we can express branch currents in terms of loop currents. This is done by the following matrix equation.

$$J_B = \mathrm{M}^{\mathrm{T}} \mathbf{I}_L$$

Cut-set matrix and node pair potentials



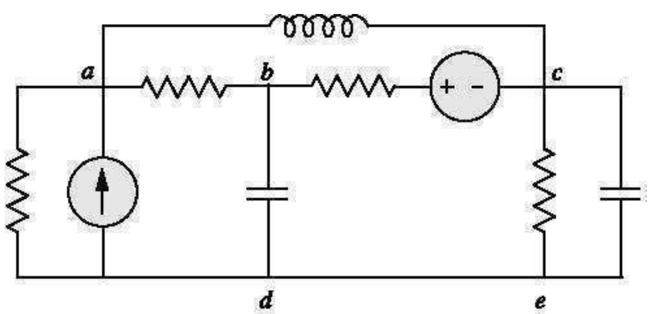
A cut-set of a graph is a set of branches whose removal, cuts the connected graph into two parts such that the replacement of any one branch of the cut-set renders the two parts connected. For example, two separated graphs are obtained for the graph by selecting the cut-set consisting of branches [1, 2, 5, 6]. These seperated graphs are as shown in Just as a systematic method exists for the selection of a set of independent loop current variables, a similar process exists for the selection of a set of independent node pair potential variables.

It is already known that the cut set is a minimal set of branches of the graph, removal of which divides the graph in to two connected sub-graphs. Then it separates the nodes of the graph in to two groups, each being one of the two sub-graphs. Each branch of the tie-set has one of its terminals incident at a node on one sub-graph. Selecting the orientation of the cut set same as that of the tree branch of the cut set, the cut set matrix is constructed row-wise taking one cut set at a time. Without link currents, the network is inactive. In the same way, without node pair voltages the network is active. This is because when one twig voltage is made active with all other twig voltages are zero, there is a set of branches which becomes active. This set is called cut-set. This set is obtained by cutting the graph by a line which cuts one twig and some links. The algebraic sum of these branch currents is zero. Making one twig voltage active in turn, we get entire set of node equations.

This matrix has current values,

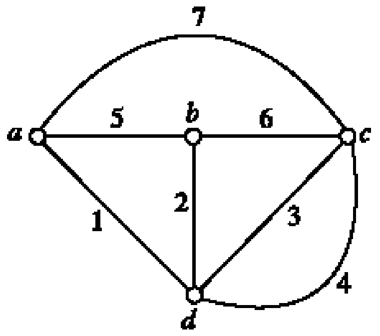
qiJ = 1, if branch J is in the cut-set with orientation same as that of tree branch.
= -1, if branch J is in the cut-set with orientation opposite to that of tree branch.
= 0, if branch J is not in the cut-set.
and dimension is (N – 1) x B

Q. Refer the circuit shown in Fig. below Draw the graph, one tree and its co-tree.

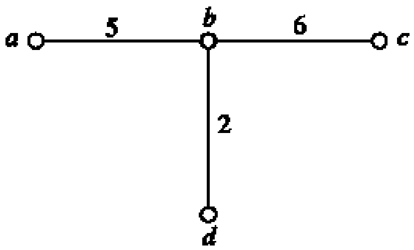


ANS:

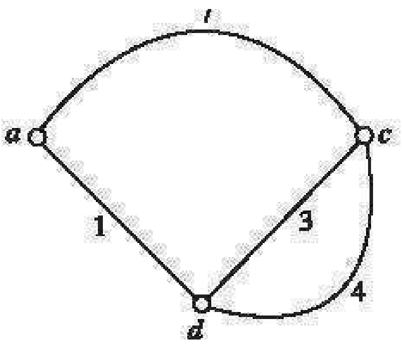
there are four nodes (N = 4) and seven branches (B = 7). The graph is then drawn and appears as shown in Fig. . It may be noted that node d represented in the graph (Fig.) represents both the nodes d and e of Fig. 2. shows one tree of graph shown in Fig. . The tree is made up of branches 2, 5 and 6. The co-tree for the tree of Fig. 2.7 is shown in Fig. . The co-tree has L = B - N +1 = 7 - 4+1 = 4 links.



Graph

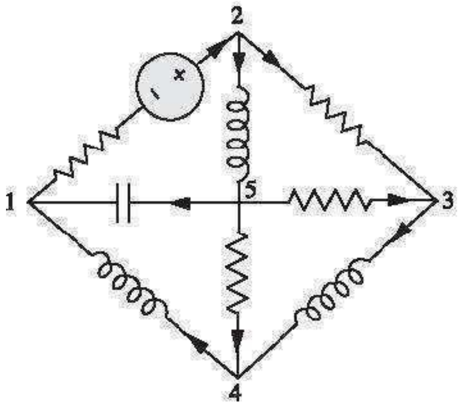


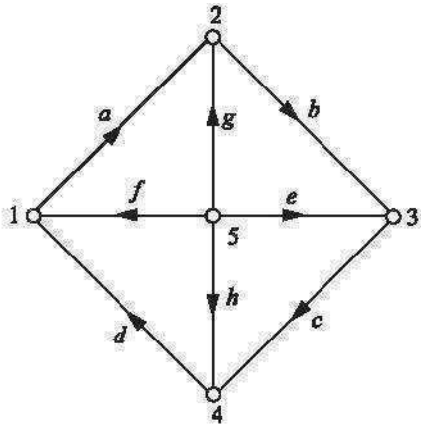
Tree



Cotree

Q. Refer the network shown in Fig. Obtain the corresponding incidence matrix.





ANS:

The network shown in Fig. I has five nodes and eight branches. The corresponding graphappears as shown in Fig. II

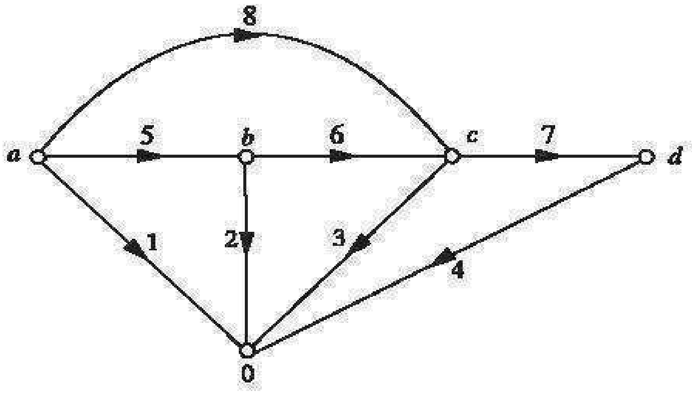
Nodes	Branch numbers							
	a	*b*	*c*	*d*	*e*	*f*	*g*	*h*
1	+1	0	0	−1	0	−1	0	0
2	−1	+1	0	0	0	0	−1	0
3	0	−1	+1	0	−1	0	0	0
4	0	0	−1	+1	0	0	0	−1
5	0	0	0	0	+1	+1	+1	+1

$$\mathbf{A}_5 = \begin{bmatrix} +1 & 0 & 0 & -1 & 0 & -1 & 0 & 0 \\ -1 & +1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & +1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & +1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & +1 & +1 & +1 & +1 \end{bmatrix}$$

Q. For the incidence matrix shown below, draw the graph.

$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \end{bmatrix} \end{array}$$

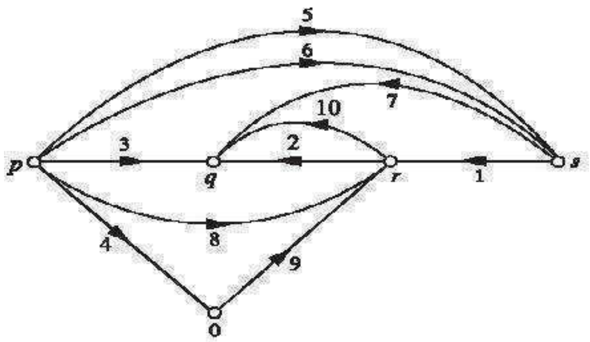
ANS:



Q. Draw the graph of a netwrok of whose the incidence matrix is as shown below.

$$\begin{array}{c} \\ p \\ q \\ r \\ s \end{array}\begin{array}{c} \begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 \\ 1 & 0 & 0 & 0 & -1 & -1 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

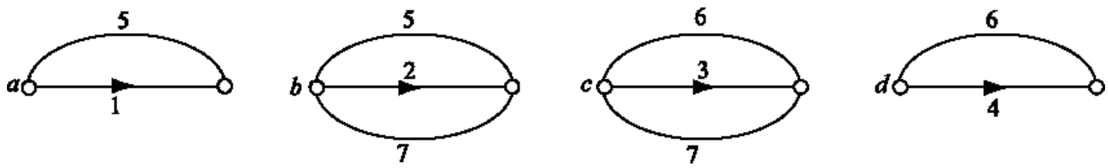
ANS:



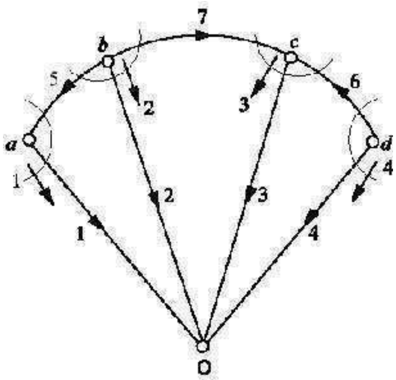
Q. For the given cut-set matrix, draw the oriented graph

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

ANS:

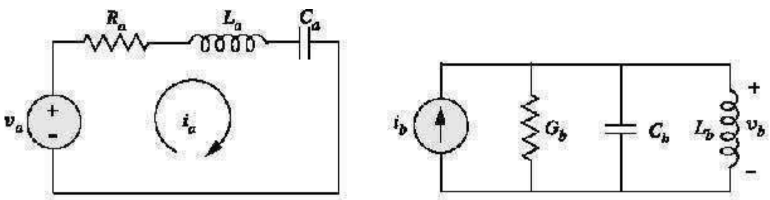


Therefore the graph is



Dual Networks

Two electrical circuits are duals if the mesh equations that characterize one of them have the same mathematical form as the nodal equations that characterize the other. Let us consider the series Ra - La - Ca network excited by a voltage source va as shown in Fig. and the parallel Gb - Cb - Lb network fed by a current source ib as shown in Fig.



$$L_a\frac{di_a}{dt}+R_ai_a+\frac{1}{C_a}\int i_a dt = v_a$$

$$L_a\frac{di_a}{dt}+R_ai_a+\frac{1}{C_a} = v_a$$

$$L_a\frac{di_a}{dt}+R_ai_a+\frac{1}{C_a}\int i_a dt = v_a$$

$$C_b\frac{dv_b}{dt}+G_bv_b+\frac{1}{L_b}\int v_b dt = i_b$$

$$C_b\frac{dv_b}{dt}+G_bv_b+\frac{1}{L_b}\int v_b dt = i_b$$

$$C_b\frac{dv_b}{dt}+G_bv_b+\frac{1}{L_b}\int v_b dt = i_b$$

$$L_a \to C_b,\ R_a \to G_b,\ C_a \to L_b$$

$$v_a \to i_b,\ i_a \to v_b,$$

Loop basis	Node basis
1. A loop made up of several branches	1. A node joining the same number of branches.
2. Voltage sources	2. Current sources
3. Loop currents	3. Node voltages
4. Inductances	4. Capacitances
5. Resistances	5. Conductance
6. Capacitances	6. Inductances

Superposition theorem

The principle of superposition is applicable only for linear systems. The concept of superposition can be explained mathematically by the following response and excitation principle :

$$i_1 \to v_1$$
$$i_2 \to v_2$$
$$i_1 + i_2 \to v_1 + v_2$$

The quantity to the left of the arrow indicates the excitation and to the right, the system response. Thus, we can state that a device, if excited by a current i1 will produce a response v1. Similarly, an excitation i2 will cause a response v2. Then if we use an excitation i1 + i2, we will find a response v1 + v2.

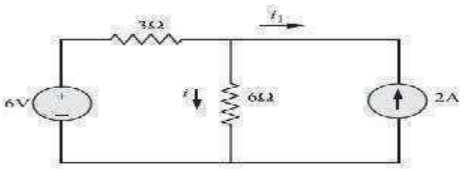
The principle of superposition has the ability to reduce a complicated problem to several easier problems each containing only a single independent source.

Superposition theorem states that,

In any linear circuit containing multiple independent sources, the current or voltage at any point in the network may be calculated as algebraic sum of the individual contributions of each source acting alone.

When determining the contribution due to a particular independent source, we disable all the remaining independent sources. That is, all the remaining voltage sources are made zero by replacing them with short circuits, and all remaining current sources are made zero by replacing them with open circuits. Also, it is important to note that if a dependent source is present, it must remain active (unaltered) during the process of superposition.

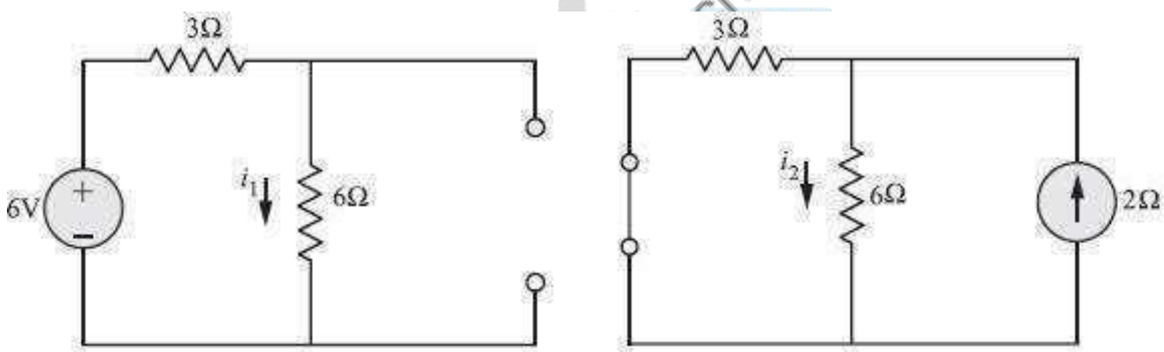
Q. Find the current in the 6 Ω resistor using the principle of superposition for the circuit



ANS:

$$i_1 = \frac{6}{3+6} = \frac{6}{9}\text{A}$$

$$i_2 = \frac{2 \times 3}{3+6} = \frac{6}{9}\text{A}$$

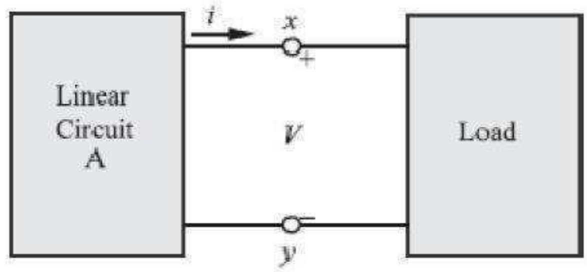


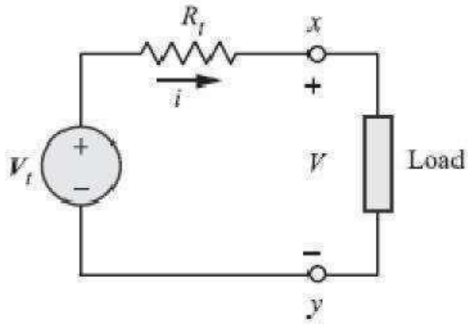
The total current i is then the sum of i_1 and i_2

$$i = i_1 + i_2 = \frac{12}{9}\text{A}$$

Thevenin's theorem

The main objective of Thevenin's theorem is to reduce some portion of a circuit to an equivalent source and a single element. This reduced equivalent circuit connected to the remaining part of the circuit will allow us to find the desired current or voltage. Thevenin's theorem is based on circuit equivalence. A circuit equivalent to another circuit exhibits identical characteristics at identical terminals.



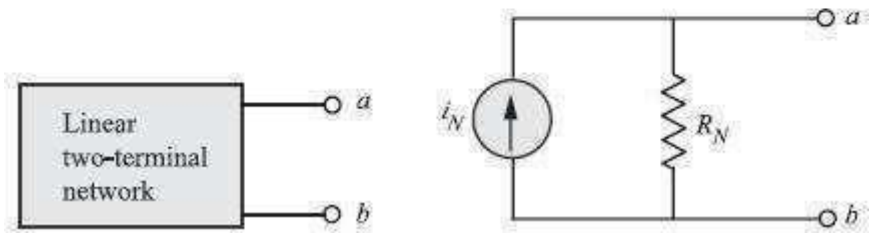


The Thevenin's theorem may be stated as follows:

A linear two–terminal circuit can be replaced by an equivalent circuit consisting of a voltage source Vt ***in series with a resistor*** Rt***, Where*** Vt ***is the open–circuit voltage at the terminals and*** Rt ***is the input or equivalent resistance at the terminals when the independent sources are turned off or*** Rt ***is the ratio of open–circuit voltage to the short–circuit current at the terminal pair.***

Norton's theorem

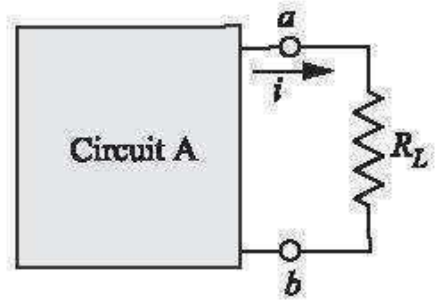
Norton's theorem states that a linear two-terminal network can be replaced by an equivalent circuit consisting of a current source iN ***in parallel with resistor*** RN***, where*** iN ***is the short-circuit current through the terminals and*** RN ***is the input or equivalent resistance at the terminals when the independent sources are turned off. If one does not wish to turn off the independent sources, then*** RN***is the ratio of open circuit voltage to short–circuit current at the terminal pair.***



Maximum Power Transfer Theorem

In circuit analysis, we are some times interestedin determining the maximum power that a circuit can supply to the load. Consider the linear circuit A as shown in Fig.

Circuit A is replaced by its Thevenin equivalent circuit as seen from a and b



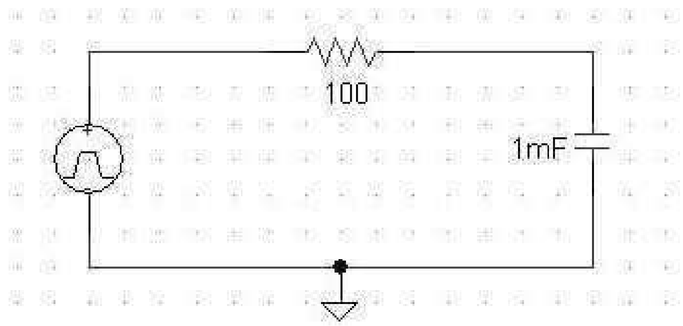
$$p = i^2 R_L = \left[\frac{V_t}{R_t + R_L}\right]^2 R_L$$

The maximum power transfer theorem states that the maximum power delivered by a source represented by its Thevenin equivalent circuit is attained when the load RL***is equal to the Thevenin resistance*** RT,

A simple circuit as shown in is considered to the concept of equivalent circuit and it is always possible to view even a very complicated circuit in terms of much simpler equivalent source and load circuits. Subsequently the reduction of computational complexity that involves in solving the current through a branch for different values of load resistance (RL) is also discussed. In many applications, a network may contain a variable component or element while other elements in the circuit are kept constant. If the solution for current (I) or voltage (V) or power (P) in any component of network is desired, in such cases the whole circuit need to be analyzed each time with the change in component value. In order to avoid such repeated computation, it is desirable to introduce a method that will not have to be repeated for each value of variable component. Such tedious computation burden can be avoided provided the fixed part of such networks could be converted into a very simple equivalent circuit that represents either in the form of practical voltage source known as Thevenin's voltage source (VTh = magnitude of voltage source , RTh = internal resistance of the source) or in the form of practical current source known as Norton's current source (I N = magnitude of current source , RN = int ernal resis tan ce of current source). Intrue sense, this conversion will considerably simplify the analysis while the load resistance changes. Although the conversion technique accomplishes the same goal, it has certain advantages over the techniques that we have learnt in earlier lessons.

The Transient Response of RC Circuits

The Transient Response (also known as the Natural Response) is the way the circuit responds to energies stored in storage elements, such as capacitors and inductors. If a capacitor has energy stored within it, then that energy can be dissipated/absorbed by a resistor. How that energy is dissipated is the Transient Response.



n this circuit, there is a pulse, a resistor, and a capacitor. Assume here that the pulse goes from 10V down to 0V att=0.
Assume also that the circuit is in Steady State at t=0-. This implies that the capacitor is 'open' at t=0-. In order for KVL to be true at t=0- then the capacitor voltage must be 10V at t=0-. This is because there is no current in the circuit, therefore the voltage across the resistor is zero.

$V_c(0-) = V_c(0+) = 10V$

Note that since the Transient Response is the circuit's response toenergies stored in storage elements, we will 'kill' the pulse source. This leaves us with a simple Resitor-Capacitor circuit with an initial 10V on the capacitor at t=0+.

Applying KCL to an RC circuit:

$Cdv/dt + V/R = 0$
$dv/dt + V/(RC) = 0$
$\int dv/V = \int -1/(RC)\, dt$
$\ln V = -t/(RC) + K$
$\ln V(t=0) = K$
$\ln V_o = K$ ←V_o is the voltage on the cap at t=0+.

$\ln V - \ln V_o = -t/(RC)$
$\ln (V/V_o) = -t/(RC)$
$V/V_o = e^{-t/(RC)}$
$V(t) = V_o\, e^{-t/(RC)}$ ←V_o = 10V in this example.

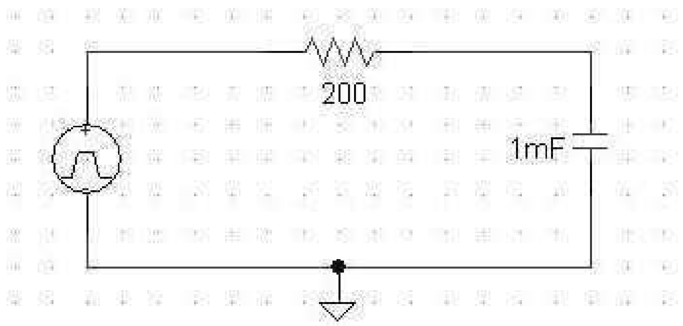
Note that the speed at which the capacitor discharges from 10V to 0V is determined by the product R×C

When t=RC, the voltage on the capacitor is V_o/e or 37% of it's initial value. We call RC the time constant and the symbol is τ

For an RC circuit, τ=RC

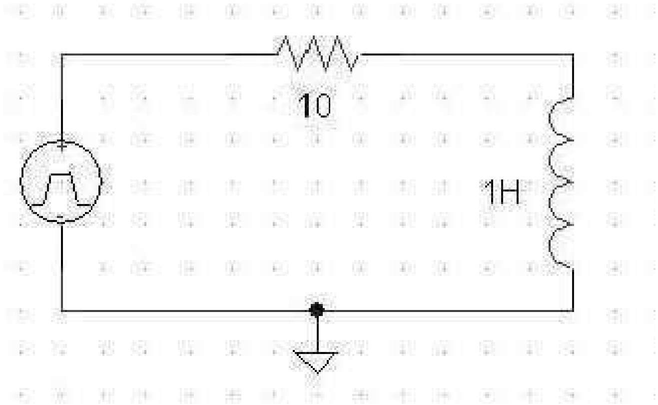
In this particular circuit τ = RC = 100Ω×1mF = 0.1 seconds
This means it takes 0.1 seconds for the capacitor to discharge from 10V down to 3.7V.

Here is the same circuit as that above, except that the resistor value is doubled. This means that τ is also doubled.
τ = RC = 200Ω×1mF = 0.2 seconds
This circuit is twice as slow as the last circuit.



The Transient Response of RL Circuits

The Transient Response (also known as the Natural Response) is the way the circuit responds to energies stored in storage elements, such as capacitors and inductors. If an inductor has energy stored within it, then that energy can be dissipated/absorbed by a resistor. How that energy is dissipated is the Transient Response.



In this circuit, there is a pulse, a resistor, and an inductor. Assume here that the pulse goes from -10V to 0V at t=0.

Assume also that the circuit is in Steady State at t=0-. This implies that the inductor is a 'short' at t=0-. In order for KCL to be true at t=0- the inductor current must be -1A at t=0-.

$I_L(0-) = I_L(0+) = -1A$

Consider the circuit at t=0+, the voltage across the pulse is zero but since $I_L(0+) = -1A$ then $V_R = -10V$. Therefore for KVL to be true $V_L = +10V$.
Therefore $V_L = +10V$ is the initial voltage across the inductor.

Note that since the Transient Response is the circuit's response to energies stored in storage elements, we will 'kill' the pulse source. This leaves us with a simple Resitor-Inductor circuit with an initial -10A going through the inductor at t=0+.

Applying KVL to an RL circuit:

$iR + Ldi/dt = 0$
$iR/L + di/dt = 0$
$-iR/L = di/dt$
$-R/L\ dt = di/i$
$\int -R/L\ dt = \int di/i$
$-Rt/L + K = \ln i$
$K = \ln i(t=0)$
$K = \ln i_o$
$-Rt/L = \ln i - K$
$-Rt/L = \ln i - \ln i_o$
$-Rt/L = \ln(i/i_o)$

$i/i_o = e^{-Rt/L}$

$i(t) = i_o e^{-Rt/L}$ ←i_o in this case is -1A

Since the plot on the right is for voltage we will find V_L using $V_L = Ldi/dt$

$V_L = (1H)\ d[i_o e^{-Rt/L}]/dt = (1H)\ (-10)\ i_o e^{-Rt/L}$

$V_L = -10\ e^{-Rt/L}$

When t=L/R, the voltage on the inductor is V_o/e or 37% of it's initial value. We call L/R the time constant and again the symbol is τ

For an RL circuit, τ=L/R

In this particular circuit τ = L/R = 1H/10Ω = 0.1 seconds
This means it takes 0.1 seconds for the inductor to go from 10V down to 3.7V.

The Complete Response

The Complete Response is the circuit's response to both an independent source as well as energies stored in the circuit.
A circuit driven by an independent source is said to have a forcing function.
Vcomplete response = Vnatural + Vforced

Here is an RC Circuit with a Forcing Function:

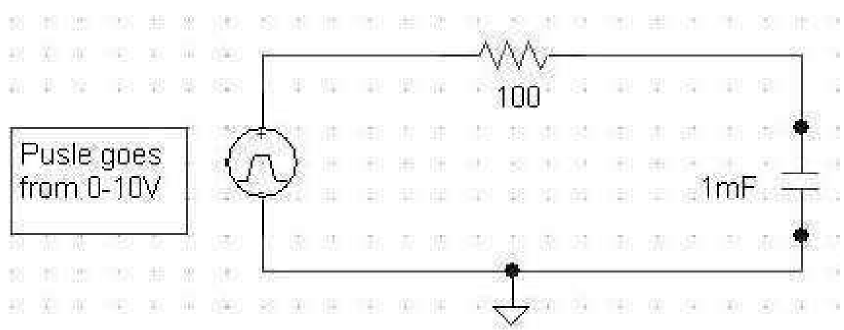
Assume the source is a pulse which goes from0V to 10V at t=0.

If we assume steady state at t=0-, then there is no initial energystored in the circuit.

Intuitively we know that the capacitor is going tocharge up to 10V. When the capacitor gets to 10V then the circuit is again at steady state.

The pulse is forcing the capacitor to 10V, thus the 10V on the capacitor is called the forced response.

The time it takes the capacitor to charge up to 10V is determined by the time constant. The response of getting to 10V is the transient response.



Now we will find the Complete Response for V across the capacitor. This equation will match the curve shown at the right.

From the last section we know that the transient response for an RC circuit is:
$V(t) = V_o e^{-t/(RC)} = A e^{-t/(RC)}$ Note that A is just some constant.

We also know from inspection that eventually the capacitor will charge up to 10V. Now putting the transient and forced responses together we get:

$V_{complete} = A e^{-t/(RC)} + V_{forced}$
$V_{complete} = A e^{-t/(RC)} + 10V$

Now we need to find A such that the equation equals V_o at t=0. In other words, the equation must satisfy the initial condition.

V(t=0) = 0, therefore:
$0 = A e^0 + 10V = A + 10V$
A = -10V

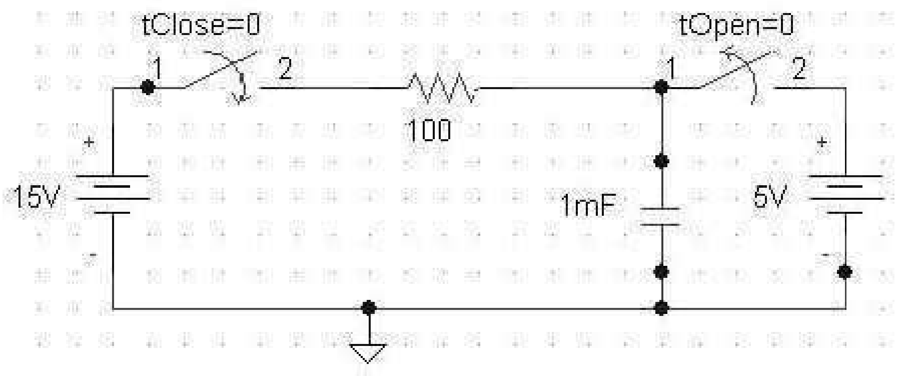
$V_{complete} = -10e^{-t/(RC)} + 10V$
$V_{complete} = -10e^{-10t} + 10V$

Note that when t>>0, $V_{complete} = 10V$. This intuitively means that when the transient response is gone the forced response still remains.

In this circuit, the capacitor DOES NOT start at 0V. In other words the capacitor has a non-zero initial condition of 5V:

Note that the left switch closes at the same time the right switch opens. Intuitively we can see that the capacitor is going to start at 5V and then charge up to 15V. For t<0 the 5V source is the forcing function and for t>0 the 15V source is the forcing function.

Since this is an RC circuit with a forcing function, the response takes the following form:
$V_{complete}=Ae^{-t/(RC)}+V_{forced}$
By inspection we know that $V_{forced} = 15V$
$V_{complete} = A e^{-t/(RC)} + 15V$
Now we need to find A such that the entire equation satisfies the value of V at t=0.
$V(t=0) = 5V = A e^0 + 15V$
A = -10V
$V_{complete} = -10e^{-t/(RC)} + 15 V$
$V_{complete} = -10e^{-10t} + 15 V$



Now let's find the voltage across the resistor for the RL circuit to the right.

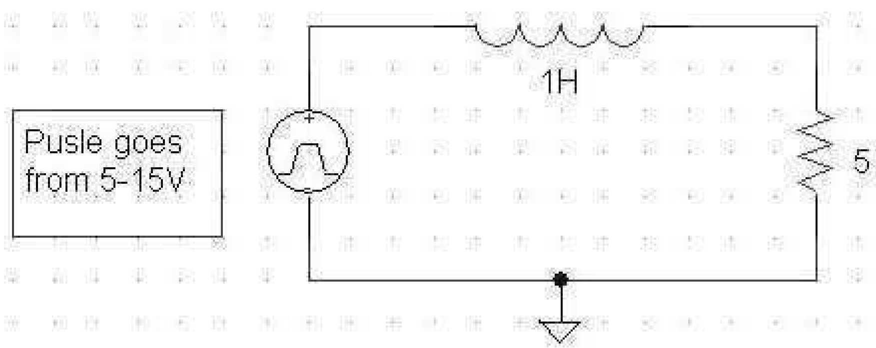
Note that the pulse goes from 5V to 15V at t=0. Assume that the circuit is in steady state at t=0-.
At steady state inductors look like 'shorts' therefore the voltage across the resistor must be equal to the pulse voltage of 5V at t=0-.

At t=0+ the voltage across the resistor is still 5V

Since the current in the inductor is continuous from 0- to 0+, and the current in the resistor is the same as the current in the inductor, and the voltage across the resistor is determined by its current, then we can say that if the resistor's current is continuous then the resistor's voltage must also be continuous.

At t>>0 the voltage across the resistor is 15V.

For t>0 we eventually reach steady state (as the transient response dies away), so we know that at t>>0 the inductor will look like a 'short'. Therefore the voltage across the resistor will equal the voltage of the pulse.
Therefore we have both the initial condition and the forced response for the voltage across the resistor:
$V_o = 5V$
$V_{forced} = 15V$



Analysis Steps for finding the Complete Response of RC and RL Circuits

Use these Steps when finding the Complete Response for a 1st-order Circuit:

Step 1: First examine the switch to see if it is opening or closing and at what time.

Step 2: Next draw the circuit right before the switch moves. You will probably assume steady state at this time but not always. The problem needs to tell you to assume steady state.

Step 3: Find all voltages and currents that can not change instantaneously when the switch moves. In other words, Find voltages across all capacitors and currents through all inductors!

Step 4: Now draw the circuit right after the switch moves. Label the circuit with all the capacitor voltages and inductor currents you found in step 3.

Step 5: Now you are ready to find your initial condition(s). Analyze the circuit to find the initial condition(s) of what it is your solving for.

Step 6: Next you will find the transient/natural reponse, or τ. To do this 'kill' all forcing functions. Make all voltage sources 'shorts' and all current sources 'opens'. Remember that the transient response is the circuit's response to energies stored in storage elements, so we need to remove forcing functions to find this. Recall that every voltage and current will have the same τ value. You now have $Ae^{-t/\tau}$ for what your solving for.

Step 7: Now we need to find the forced response. The forced response is the state of the circuit after the switch has moved AND after the transient response has died-off. To find the forced response assume Steady State, i,e, t>>>0. Find the final resting value (forced response - V_F) of whatever it is you are solving for.

Step 8: You should now have an equation which looks like $v(t) = Ae^{-t/\tau} + V_F$ or $i(t) = Ae^{-t/\tau} + I_F$. To find the unknown 'A' you will apply the initial condition to this equation. Usually the initial condition is the value at t=0, so you will plug in t=0 to get the following: I.C. = A + V_F or I.C. = A + I_F You can now solve for A.

Step 9: Plugging the value of A into: $v(t) = Ae^{-t/\tau} + V_F$, you now know the voltage for all time greater than t=0 (assuming that the switch moved at t=0).

Step 10: Using your equation for v(t) or i(t), you can find other things (voltages, currents, power, etc.) using KVL, KCL, and Ohm's Law.

Unit 4

Laplace transform: solution of Integro-differential equations, transform of waveform synthesized with step ramp, Gate and sinusoidal functions, Initial & final value theorem, Network Theorems in transform domain.

Laplace Transforms and their Applications

Pierre Simon de Laplace was a French mathematician who lived during 1749-1827, and was essentially interested to describe nature using mathematics. The main goal of this chapter is to present those results of Laplace which are used to find solutions of differential and integral equations.

Definition and Fundamental Properties of the Laplace Transform.

The Laplace transform is considered as an extension of the idea of the indefinite integral transform : I $\{f(t)\} = \int_0^x f(t)dt$.

It is defined as follows

Definition: The **Laplace transform** of f(t), provided it exists, denoted by L $\{f(t)\}$ is defined by

$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$, where s is a real number called a **parameter** of the transform.

Remark

(a) Laplace transform takes a function f(t) into a function F(s) of the parameter s.

(b) We represent functions of t by lower case letters f,g, and h, while their respective Laplace transforms by the corresponding capital letter F,G, and H. Thus we write

$L\{f(t)\} = F(s)$ or $F(s) = \int_0^\infty e^{-st} f(t)dt$

(c) The defining equation for the Laplace transform is an improper integral, which is defined as

$$\int_0^\infty e^{-st} f(t)dt = \lim_{T\to\infty} \int_0^T e^{-st} f(t)dt$$

Thus, the existence of the Laplace transform of f depends upon the existence of the limit.

(d) A Laplace transform is rarely computed by referring directly to the definition and integrating.

(e) We shall verify that the Laplace transform is linear, that is, constants factor through the transform, and the transform of a sum of functions, is the sum of the transform of these functions.

$L(f+g) = L(f) + L(g) = F+G$

$L(\lambda f) = \lambda L(f) = \lambda F.$

Example Show that

(i) $L(f(t)) = 1/s$ where, $s>0, f(t)=1$.

(ii) $L(f(t)) = 1/s^2$, where $s>0$, and $f(t)=t$.

(iii) $L(f(t)) = \dfrac{n!}{s^{n+1}}$, $s>0$, whre $f(t)=t^n$

(iv) $L(f(t)) = \dfrac{1}{s^2+1}$, where $f(t) = \sin t$

(v) $L(e^{at}) = \dfrac{1}{s-a}$, $s>a$, where $f(t)=e^{at}$, and a is any real number

Solution: (i) By definition we have

$$L(1) = \int_0^\infty e^{-st}(1)\,dt = \lim_{T\to\infty} \int_0^T e^{-st}\,dt$$

$$= \lim_{T\to\infty} \left[e^{-st}\left(-\frac{1}{s}\right)\right]_0^T$$

$$= \lim_{T\to\infty} \frac{1}{s}\left[-e^{-ST} + e^{-s.0}\right]$$

$$= \frac{1}{s}\left[-\lim e^{-st} + 1\right]$$

$$= \frac{1}{s}[0+1] = \frac{1}{s},\ s>0$$

(i) $L(t) = \int_0^\infty e^{-st}\, t\, dt$

$$= \lim_{T\to\infty} \int_0^T e^{-st}\, t\, dt$$

$$= \lim_{T\to\infty} \left[-\frac{1}{s} e^{-st}\, t\right]_0^T + \frac{1}{s}\int_0^\infty e^{-st}\, dt$$

by applying integration by parts. Since the first term is zero and the second is

$\frac{1}{s^2}$ by part (1) we get

$$L(t) = \frac{1}{s^2}$$

(ii) By Definition 9.1 we have

$$L(t^n) = \int_0^\infty e^{-st}\, t^n\, dt$$

By applying the formula for integration by parts n times we conclude that

$$L(t^n) = \frac{n!}{s^{n+1}}$$

$$\int_0^\infty e^{-st}\, t^n\, dt = \left[-\frac{t^n e^{-st}}{s}\right]_0^\infty + \frac{n}{s}\int_0^\infty e^{-st}\, t^{n-1}\, dt$$

The first term on the right-hand side is equal to zero for n>0 and s>0, so

$$L(t^n) = \int_0^\infty e^{-st}\, t^n\, dt = \frac{n}{s}\int_0^\infty e^{-st}\, t^{n-1}\, dt = \frac{n}{s} L(t^{n-1})$$

Replacing n with n-1 in this equation, we get

$$L\left(t^{n-1}\right) = \left(\frac{(n-1)}{s} L(t^{n-2})\right)$$

Combining values of L (t^n) and L (t^{n-1}) one can write

$$L(t^n) = \frac{n(n-1)}{s^2} L(t^{n-2})$$

Continuing in this way one gets

$$L(t^n) = \frac{n(n-1)(n-2).....3.2.1.}{s^n} L(t^0)$$

Since L (t^o) = L (1) = $\frac{1}{s}$ by hart (i), we obtain

$$L(t^n) = \frac{n!}{s^{n+1}},\ \text{where}$$

n!=n(n-1)(n-2).......3.2.1.

(iv) $L(\sin t) = \int_0^\infty e^{-st} \sin t\, dt$, by Definition 9.1,

T

$$= \lim_{T\to\infty} \int_0 e^{-st} \sin t \, dt.$$

Let $I = \int_0^T e^{-st} \sin t \, dt.$

$$= \left[-\frac{1}{s} e^{-st} \sin t \right]_0^T + \frac{1}{s} \int_0^T e^{-st} \cos t \, dt.$$

$$= \frac{1}{s} e^{-sT} \sin T + \left[-\frac{1}{s^2} e^{-st} \cos t \right]_0^T - \int_0^T \left(-\frac{1}{s^2}\right)(-\sin t) e^{-st} \, dt$$

$$= -\frac{1}{s}e^{-sT}\sin T - \frac{1}{s^2}e^{-sT}\cos T + \frac{1}{s^2} - \frac{1}{s^2}\int_0^T e^{-st}\sin t\,dt$$

$$= -\frac{1}{s}e^{-sT}\sin T - \frac{1}{s^2}e^{-sT}\cos T + \frac{1}{s^2} - \frac{1}{s^2}I$$

Bringing $-\frac{1}{s^2}I$ on the left hand side we get

$$\left(1+\frac{1}{s^2}\right)I = -\frac{1}{s}e^{-sT}\sin T - \frac{1}{s^2}e^{-sT}\cos T + \frac{1}{s^2}$$

By taking the limit as $T \to \infty$ in this equation

we get $\left(1+\frac{1}{s^2}\right)I = \frac{1}{s^2}$

or $I = \frac{s^2}{s^2+1}\frac{1}{s^2} = \frac{1}{s^2+1}$

(v) $L(e^{at}) = \int_0^\infty e^{-st}t^{at}\,dt$, by Definition

$$= \lim_{T\to\infty}\int_0^T e^{(a-s)t}\,dt$$

$$= \lim_{T\to\infty}\left[\frac{1}{a-s}e^{(a-s)t}\right]_0^T dt$$

$$= \lim_{T\to\infty}\left[\frac{1}{a-s} - \frac{1}{a-s}\right]$$

$$= -\frac{1}{a-s} = \frac{1}{s-a} \text{ provided } a-d<0$$

or $s > a$.

Thus the Laplace transform of e^{at} is $F(s) = L(e^{at}) = \frac{1}{s-a}$ if $s>a$.

If may be observed that for $s \le 0$, **does not exist** : Let $s<0$ then the exponent of e is positive for $t>0$. Therefore

$$\lim_{T\to\infty}\int_0^T e^{-st}\,dt = \lim_{T\to\infty}\left[\frac{e^{-st}}{-s}\right]_0^T$$

$$= \lim_{tT\to\infty}\left[-\frac{1}{s}e^{-sT} + \frac{1}{s}\right]$$

$$= \infty$$

which means the integral diverges.

Let $s = 0$, then integral becomes

$$\lim_{T\to\infty}\int_0^T dt = \lim_{T\to\infty}[t]_0^T = -\lim_{T\to\infty} T = \infty$$

Theorem Let $f_1(t)$ and $f_2(t)$ have Laplace transform and let c_1 and c_2 be constants, then

(i) $L(f_1(t) + f_2(t)) = L(f_1(t)) + L(f_2(t))$

(ii) $L(c_1 f_1(t)) = c_1 L(f_1(t))$ and

$L(c_2 f_2(t)) = c_2 L(f_2(t))$.

equivalently

(iii) $L(c_1 f_1(t) + c_2 f_2(t)) = c_1 L(f_1(t)) + c_2 L(f_2(t)$

Proof of (iii): LHS $= L(c_1 f_1(t) + c_2 f_2(t))$

$$= \int_0^\infty e^{-st}\left[c_1 f_1(t) + c^2 f_2(t)\right]dt$$

$$= \int_0^\infty \left[c_1 e^{-st} f_1(t) + c_2\, e^{-st} f_2(t) \right] dt$$

$\int_0^\infty c_1 e^{-st} f_1(t)dt + \int_0^\infty c_1 e^{-st} f_2(t)dt$, by using properties of integrals,

$= c_1 L(f_1(t)) + c_2 L(f_2(t))$.

Laplace Transforms of some Basic Functions

f(t)	**L (f(t))**	**f(t)**	**L (f(t))**
1. I	$\frac{1}{s}$	9 $\cos^2 kt$	$\frac{s^2+2k^2}{s(s^2+4k^2)}$
2. t	$\frac{1}{s^2}$	10 e^{at}	$\frac{1}{s-a}$
3. t^n	$\frac{n!}{s^{n+1}}$	11 sinh k t	$\frac{k}{s^2-k^2}$
4. $t^{-1/2}$	$\sqrt{\frac{\pi}{s}}$	12 cosh kt	$\frac{s}{s^2-k^2}$
5. $t^{1/2}$	$\frac{\sqrt{\pi}}{2s^{3/2}}$	13 $\sinh^2 kt$	$\frac{2k^2}{s(s^2-4k^2)}$
6. sin k t	$\frac{k}{s^2+k^2}$	14 $\cosh^2 kt$	$\frac{s^2-2k^2}{s(s^2-4k^2)}$
7. cos k t	$\frac{s}{s^2+k^2}$	15 $t\,e^{at}$	$\frac{1}{(s-a)^2}$
8. $\sin^2 kt$	$\frac{2k^2}{s(s^2+4k^2)}$	16 $t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$ n a positive integer
17. $e^{at}\sin kt$	$\frac{k}{(s-a)^2+k^2}$	31. $H(t-a)=u_a(t)$	$\frac{e^{-as}}{s}, s>0$
18. $e^{at}\cos kt$	$\frac{s-a}{(s-a)^2+k^2}$	32. $\delta(t)$	I
19. $e^{at}\sinh kt$	$\frac{k}{(s-a)^2-k^2}$	33. $\delta(t-to)$	e^{-st0}
20. $e^{at}\cosh kt$	$\frac{s-a}{(s-a)^2-k^2}$	34. $e^{at} f(t)$	F(s-a)
21. t sin kt	$\frac{2ks}{{}^{y}(s^2+k^2)^2}$	35. f(t-a) H(t-a)	$e^{-as}F(s)$
22. t cos kt	$\frac{s^2-k^2}{(s^2+k^2)^2}$	36. $f^{(n)}(t)$	$s^n F(s)-s^{n-1}f(0)\ldots\ldots-f^{(n-1)}(0)$
23. sin kt + kt cos kt	$\frac{2ks^2}{(s^2+k^2)^2}$	37. $t^n f(t)$	$(-1)^n \frac{d^n}{ds^n}F(s)$
24. sin kt-kt cos kt	$\frac{2k^3}{(s^2+k^2)^2}$	38. $\int_0^t f(u)\,g(t-u)du$	F(s) G(s)
25. t Sinh k t	$\frac{2ks}{(s^2-k^2)^2}$	39. $\frac{\sin at}{t}$	$\arctan\left(\frac{a}{s}\right)$

26. t cosh kt	$\frac{s^2+k^2}{(s^2-k^2)^2}$	40. $\frac{1}{\sqrt{\pi t}} e^{-a^2/4t}$	$\frac{e^{-a\sqrt{s}}}{\sqrt{s}}$
27. $\frac{e^{at}-e^{bt}}{a-b}$	$\frac{1}{(s-a)(s-b)}$		
28. $\frac{ae^{at}-e^{bt}}{a-b}$	$\frac{s}{(s-a)(s-b)}$		
29. 1-cos kt	$\frac{k^2}{s(s^2+k^2)}$		
30. $\frac{e^{at}-e^{bt}}{t}$	$\ln\frac{s-a}{s-b}$		

Definition A function f is said to be **piecewise continuous** on the closed interval [a.b] if the interval can be divided into a finite number of open subintervals (c,d)= {t ε[a. b] / c < t < d} such that

(i) The function is continuous on each subinterval (c,d).

(ii) The function f has a finite limit as t approaches each endpoint from within the interval; that is, $\lim_{t\to d-} f(t)$ and $\lim_{t\to c+} f(t)$ exist.

It is clear that every continuous function is piecewise continuous. f(t) = $\frac{1}{t}$ is not piecewise continuous on any closed interval containing the origin as there is an infinite discontinuity at t = 0.

The function h(t) = $\begin{cases} t, & 0 \le t < 2 \\ 3, & t > 2 \end{cases}$ is piecewise continuous for all $t \ge 0$.

The function g(t)= $\frac{\sin t}{t}$ is discontinuous at t=0 but it is piecewise continuous for all $t \ge 0$.

Remark From calculus we know that a finite number of finite discontinuities of an integrand function do not affect the existence of integral. Therefore the Laplace transform of a piecewise continuous function f(t) can be defined and computed.

Example Find the Laplace transform of the following functions

(a) f(t)= 2t, 0≤t<3

= -1, t>3

(b) h(t) = 1, if 0≤t< $\frac{1}{2}$

=-1, if $\frac{1}{2}$ ≤ t<1

=0, otherwise

Solution (a) L (f(t)) = $\int_0^\infty e^{-st} f(t)dt$

$$= \int_0^3 e^{-st}\, 2t\,dt + \int_3^\infty e^{-st}(-1)dt$$

$$= \left[-\frac{2te^{-st}}{s} - \frac{2e^{-st}}{s^2}\right]_0^3 + \left[\frac{1}{s}e^{-st}\right]_3^\infty$$

where integration by parts has been used to evaluate the first integral on the interval (0,3).

The value of $\frac{1}{s}e^{-st} \to 0$ as $t\to\infty$, if s > 0.

Therefore L (f(t))= $\left[-\frac{6}{s}e^{-3s} - \frac{2}{s^2}e^{-3s}\right]$

$$-\left[0-\frac{2}{s^2}\right]+\left[0-\frac{1}{s}e^{-3s}\right]$$

$$= \frac{2}{s^2} - \frac{2e^{-3s}}{s^2} - \frac{7e^{-3s}}{s}, \; s > 0$$

(b) $L(f(t)) = \int_0^\infty e^{-st} h(t)dt$

$$= \int_0^{1/2} e^{-st}h(t)dt + \int_{1/2}^{1} e^{-st}h(t)dt + \int_1^\infty e^{-st}h(t)dt$$

$$= \int_0^{1/2} e^{-st}(1)dt + \int_{1/2}^{1} e^{-st}(-1)dt + 0$$

$$= \left[-\frac{1}{s}e^{-st}\right]_0^{1/2} - \left[-\frac{1}{s}e^{-st}\right]_{1/2}^{1}$$

$$= -\frac{1}{s}e^{-s/2} + \frac{1}{s} + \frac{1}{s}e^{-s} - \frac{1}{s}e^{-s/2}$$

$$= -\frac{2}{s}e^{-s/2} + \frac{1}{s}e^{-s} + \frac{1}{s}$$

Definition A function f is said to be of **exponential order** if there exist real numbers a, M, and t_0 such that

$|f(t)| \le M e^{at}$ for $t > t_0$.

Example Check whether the following functions are of exponential order.

(a) $f(t) = t^2$

(b) $f(t) = e^t$

(c) $f(t) = \sin t$

(d) $f(t) = e^{t^2}$

Solution (a) Let a be any constant > 0, then $\lim_{t\to\infty} |f(t)|e^{-at} = \lim_{t\to\infty} |t^2|\, e^{-at}$

$$= \lim_{t\to\infty} \frac{2t}{ae^{at}} = \lim_{t\to\infty} \frac{2}{a^2 e^{at}} = 0$$

where the last two limits are obtained by using l' Hospital's rule. Therefore there exists a positive constant M such that $|f(t)|\, e^{-at} \le M$ or $(f(t)) \le M e^{at}$, that is,

$f(t)=t^2$ is of exponential order

(b) $\lim_{t\to\infty} |f(t)|e^{-at}$, $a > 0$,

$= \lim_{t\to\infty} e^t e^{-at}$

$= \lim_{t\to\infty} e^{(1-a)t} \to 0$ for $a > 1$.

Therefore we can find a and M > 0 such that $|f(t)| \le M e^{at}$

(c) $\lim_{t\to\infty} |f(t)|\, e^{-at} \le \lim_{t\to\infty} e^{-at} \to 0$ for $a > 0$ implying there exist a and M > 0 such that $|f(t)| \le M e^{at}$.

(d) $f(t) = e^{t^2}$ is not of exponential order since its graph grows faster than any positive linear power of e for $t > a > 0$.

Now we prove the following basic existence theorem for the Laplace transform of a function f.

Theorem Let f be a piecewise continuous function of exponential order defined on $[0,\infty)$, then its Laplace transform exists for parameter s greater than some constant a.

Proof: Since the function f is of exponential order, we know that there are constants t_o and a and M > 0 such that

$|f(t)| \le M e^{at}$ for $t > t_o$

or $e^{-at} |f(t)| \le M$ for $t > t_o$

or $|e^{-at} f(t)| \le M$ for $t > t_o$

as $|e^{-at} f(t)| = |e^{-at}|\, |f(t)| = e^{-at}\, |f(t)|$

It may be noted that e^{-at} is always positive.

Multiplying by $e^{-st}e^{at}$, we have

$$|e^{-st}f(t)| \le M e^{-st} e^{at}$$

Hence

$$\int_0^\infty |e^{-st} f(t)|\, dt \le \int_0^\infty Me^{-(s-a)t}\, dt = \left[-\frac{Me^{-(s-a)t}}{s-a} \right]_0^\infty$$

$$= -\lim_{t\to\infty} \frac{Me^{-(s-a)t}}{s-a} + \frac{M}{s-a}$$

Since first term is zero for s>a, we have

$$\int_0^\infty |e^{-st} f(t)|dt \le \frac{M}{s-a},\ s > a$$

which implies the existence of the improper integral defining the Laplace transform of f and completes the proof.

The Inverse Laplace Transform

It the previous section we have seen the method for finding the Laplace transform. In this section we discuss the method for reversing the process of the previous section and more precisely we reconstruct a function f(t) whose Laplace transform F(s) is given.

Definition Let f(t) be a function such that L (f(t)) = F(s), then f(t) is called the inverse Laplace transform of F(s). The inverse Laplace transform is designated L^{-1} and we write

$$f(t) = L^{-1}\{F(s)\}.$$

In order to find an inverse transform we must be familiar with the formulas for finding the Laplace transform, see Table 9.1. One should learn to use this table in reverse. However in general the given Laplace transform will not be in the form the allows direct use of the table, so the given F(s) have to be algebraically manipulated in a form that can be found in the table. the most relevant result for this purpose is the linearity property of the inverse Laplace transform which states that

$$L^{-1}\{c_1F_1(s) + c_2F_2(s)\} = c_1 L^{-1}\{F_1(s)\} + c_2 L^{-1}\{F_2(s)\}$$

where c_1 and c_2 are constants.

The proof of this result follows from the definition of the inverse Laplace transform and the corresponding linearity of the Laplace transform.

Example Find

(i) $L^{-1}\left\{\frac{1}{s+2}\right\}$

(ii) $L^{-1}\left\{\frac{1}{s^2+2}\right\}$

(iii) $L^{-1}\left\{\frac{-2s+6}{(s^2+4)}\right\}$

(iv) $L^{-1}\left\{\frac{s+5}{(s^2-2s-3)}\right\}$

(v) $L^{-1}\left\{\frac{s+1}{(s^2-4s)}\right\}$

Solution (i) From Table 9.1 $L(e^{at}) = \frac{1}{s-a}$ Choosing a = -2 we get $L(e^{-2t}) = \frac{1}{s+2}$ and consequently by the definition of the inverse Laplace transform

$$L^{-1}\left\{\frac{1}{s+2}\right\} = e^{-2t}$$

(ii) By Table 9.1 for $k = \sqrt{2}$ and the linearity of the inverse Laplace transform we get

$$L^{-1}\left\{\frac{1}{s^2+2}\right\} = \frac{1}{\sqrt{2}} L^{-1}\left\{\frac{\sqrt{2}}{s^2+2}\right\}$$

$$= \frac{1}{\sqrt{2}} \sin\sqrt{2}\, t$$

(iii) Solution $L^{-1}\left\{\frac{-2s+6}{s^2+4}\right\} = L^{-1}\left\{\frac{-2s}{s^2+4} + \frac{6}{s^2+4}\right\}$

$$= -2L^{-1}\left\{\frac{s}{s^2+4}\right\} + 6L^{-1}\left\{\frac{1}{s^2+4}\right\}$$

By the Linearity of the inverse transform,

$$= -2\cos 2t + \frac{6}{2} L^{-1}\left\{\frac{2}{s^2+4}\right\}$$

$= -2\cos 2t + 3\sin 2t$ by Table 9.1 (6 and 7) .

(iv) Since $s^2 - 2s - 3 = (s-3)(s+1)$ we get

$$\frac{s+5}{s^2-2s-3} = \frac{s+5}{(s-3)(s+1)} = \frac{A}{s-3} + \frac{B}{s+1}$$

where A and B are constants to be determined.

$$\frac{A}{s-3} + \frac{B}{s+1} = \frac{A(s+1)+(s-3)B}{(s-3)(s+1)} = \frac{(A+B)s+(A-3B)}{(s-3)(s+1)}$$

We can write

$$\frac{s+5}{(s-3)(s+1)} = \frac{(A+B)s+(A-3B)}{(s-3)(s+1)}$$

This implies that

$s+5 = (A+B)s + (A-3B)$

This gives $A+B = l$ and $A-3B=5$

Subtracting second from first we get $B = -1$. Putting this value in the first equation we get $A = 2$. Therefore, we have

$$L^{-1}\left\{\frac{s+5}{s^2-2s-3}\right\} = L^{-1}\left\{\frac{2}{s-3} + \frac{-1}{s+1}\right\}$$

$$= 2L^{-1}\left\{\frac{1}{s-3}\right\} - L^{-1}\left\{\frac{1}{s+1}\right\}$$ using linearity of L^{-1}

By Table 9.1 (series no.10, for a = 3 and a = -1) we get

$$L^{-1}\left\{\frac{1}{s-3}\right\} = e^{3t} \text{ and } L^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t}$$

Hence

$$L^{-1}\left\{\frac{s+5}{s^2-2s-3}\right\} = 2e^{3t} - e^{-t}$$

(v) $L^{-1}\left\{\frac{s+1}{s^2-4s}\right\} = L^{-1}\left\{-\frac{1}{4}\frac{1}{s} + \frac{5}{4}\frac{1}{s-4}\right\}$

$$= -\frac{1}{4} L^{-1}\left\{\frac{1}{s}\right\} + \frac{5}{4} L^{-1}\left\{\frac{1}{s-4}\right\}$$

$$= -\frac{1}{4}.1 + \frac{5}{4} e^{4t}$$

Shifting Theorems and Derivative of the Laplace transform

The following theorems are called the shifting theorems.

Theorem (The First Shifting Theorem): Let $L(f(t)) = F(s)$.

Then $L\left\{e^{at} f(t)\right\} = F(s-a)$

Proof : By definition of L $\left\{e^{at} f(t)\right\}$, we write

$$L\left\{e^{at} f(t)\right\} = \int_0^\infty e^{-st}[e^{at} f(t)]\,dt$$

$$= \int_0^\infty e^{-(s-a)t} f(t)\,dt = F(s-a).$$

Theorem (The second shifting theorem). Let $e^{at} f(t)$ (f(t)) = F(s)

Then $L\left\{H(t-a)f(t-a)\right\} = e^{-as}F(s)$

where H is the Heaviside function defined as

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$

Proof $L\left\{H(t-a)f(t-a)\right\} = \int_0^\infty e^{-st} H(t-a)f(t-a)\,dt$

$$= \int_a^\infty e^{-st} f(t-a)\,dt$$

because H(t-a) = 0 for t < a and H(t-a) = 1 for t $\geq$ a. Now let u = t-a in the last integral. We get

$$L\left\{H(t-a)f(t-a)\right\} = \int_0^\infty e^{-s(a+u)} f(u)\,du$$

$$= e^{-as}\int_0^\infty e^{-st} f(u)\,du$$

$$= e^{-as} L(f(u)) = e^{-as} F(s).$$

Example Apply the first shifting theorem to find

(a) $L\left\{e^{3t}\sin t\right\}$

(b) $L\left\{e^{-t} g(t)\right\}$, where

$$g(t) = \begin{cases} 2t, & 0 \leq t < 3 \\ -1, & t \geq 3 \end{cases}$$

(c) $L^{-1}\left\{\dfrac{4}{s^2 - 4s + 20}\right\}$

Solution (a) Since $L\{\sin t\} = \dfrac{1}{s^2+1}$, it follows that

by Theorem $L\left\{e^{3t}\sin t\right\} = \dfrac{1}{(s-3)^2+1}$

(b) By Theorem 9.2 $L\left\{e^{-t} g(t)\right\} = F(s-a)$.

where L (g(t)) = F(s).

$$F(s) = \int_0^\infty e^{-st}g(t)\,dt = \int_0^3 e^{-st}g(t)\,dt + \int_3^\infty e^{-st}g(t)\,dt$$

$$= \int_0^3 e^{-st}\,2t\,dt - \int_3^\infty e^{-st}\,dt$$

$$= 2\left[e^{-st}\left(-\frac{1}{s}\right)t\right]_0^3 - 2\int_0^3 e^{-st}\left(-\frac{1}{s}\right)dt - \left[-\frac{1}{s}e^{-st}\right]_3^\infty$$

$$= \frac{2}{s^2} - \frac{2e^{-3s}}{s^2} - \frac{7e^{-3s}}{s}$$

(c) We have $\dfrac{4}{s^2+4s+20} = \dfrac{4}{(s+2)^2+6}$

$$F(s+2) = \frac{4}{(s+2)^2+16}$$

This means we should choose

$$F(s) = \frac{4}{s^2+16}$$

By the first shifting theorem

$L\{e^{-2t} \sin 4t\} = F(s-(2)) = F(s+2)$

$$= \frac{4}{(s+2)^2+16}$$

and therefore

$$L^{-1}\left\{\frac{4}{(s+2)^2+16}\right\} = e^{-2t} \sin(4t).$$

Example Compute $L^{-1}\left\{\frac{se^{-3s}}{s^2+4}\right\}$.

Solution: By Theorem

$L\{H(t-a)f(t-a)\} = e^{-as}F(s)$

or $H(t-a)f(t-a) = L^{-1}\{e^{-as}F(s)\}$

$$F(s) = \frac{s}{s^2+4}$$

$L^{-1}(F(s))= L^{-1}\left(\frac{s}{s^2+4}\right)$ implies that $f(t)= \cos(2t)$.

Therefore,

$$L^{-1}\left\{\frac{se^{-3s}}{s^2+4}\right\} = H(t-3)\cos(2(t-3)).$$

Derivative of the Laplace Transform

Theorem Let f(t) be piecewise continuous and of exponential order over each finite interval, and let $L(f(t))=F(s)$.

Then F(s) is differentiable and

$F'(s) = L\{-tf(t)\}$.

Proof: Suppose that $|f(t)| \le Me^{at}$, $t>0$ and take any $s_0 > a$.

Then consider

$$\frac{\partial}{\partial s}\left[e^{-st}f(t)\right] = -t\, f(t)e^{-st}.$$

Choose $\varepsilon > 0$ such that $s_0 > a + \varepsilon$. Then we have $|t| < e^{\varepsilon t}$ for all t large enough since in fact

$$\lim_{t\to\infty}\left(\frac{s}{e^{\varepsilon t}}\right) = 0.$$

Thus $|tf(t)| \le M e^{(a+\varepsilon)t}$

for all large t and we find that t f(t) is also of exponential order and

$$\int_0^\infty t\, f(t)e^{-st}\, dt$$

exists by Theorem 9.1, that is, the integral converges uniformly. Hence F(s) is differentiable at s_0 and that

$$F'(s_0) = \int_0^\infty \frac{\partial}{\partial s}\left[e^{-st} f(t)\right] dt \text{ at } s=s_0.$$

Therefore

$$F'(s) = -\int_0^\infty t\, f(t)e^{-st}\, dt$$

$= L(-t\, f(t))$ for all $s > a$

Transforms of Derivatives, Integrals and Convolution Theorem

Transforms of Derivatives and Integrals

The Laplace transform of the derivatives of a differentiable function exist under appropriate conditions. In this section we discuss results that are quite useful in solving differential equations. For solving 2nd order differential equations we need to evaluate the Laplace transforms of $\frac{dy}{dt}$ and $\frac{d^2y}{dt^2}$.

Let f(t) be differentiable for $t \geq 0$ and let its derivative f'(t) be continuous, then by applying the formula for integration by parts we find that

L {f'(t)}= sF(s)-f(0)

Verification: By definition

$$L\{f'(t)\} = \int_0^\infty e^{-st} f'(t)dt$$

$$= \left[e^{-st} f'(t)dt\right]_0^\infty + s\int_0^\infty e^{-st} f(t)dt$$

= -f(0) + s L (f(t))

=sF(s)-f(0)

Here we have used the fact that

$$\lim_{t\to\infty} e^{-st} f(t) = 0$$

Similarly for a twice differentiable function f(t) such that f''(t) is continuous we can prove that

L {f''(t)} = s^2 F(s)-s f(0) – f'(0)

In fact we can prove the following theorem, repeated by applying integration by parts.

Theorem Let f be piecewise continuous and of exponential order for $t \geq 0$, then

$$L\left\{\int_0^t f(u)du\right\} = \frac{1}{s}L\{f(t)\} = \frac{1}{s}F(s).$$

Proof: Let g(t)= $\{\int_0^t f(u)du\}$. Then g'(t) = f(t) and g(0)=0.

Furthermore, g(t) is of exponential order. By Theorem 9.5 L {g'(t)} = s L {g(t)} -g(0)

$$L\{f(t)\} = sL\left\{\int_0^t f(u)du\right\}$$

$$\text{or } L\left\{\int_0^t f(u)du\right\} = \frac{L\{f(t)\}}{s}$$

Example (a)Using the Laplace transform of f'' find L {sin k t }.

(b) Show that $L^{-1}\left\{\frac{1}{s}F(s)\right\} = \int_0^t f(u)du.$

Solution (a) Let f(t)= sin k t, then f' (t) = k cos k t, f''(t) = $-k^2$ sin k t, f(0)=0 and f'(0)=k.

Therefore $L\{f''(t)\} = s^2 F(s) - sf(0) - f'(0)$

or $L\{f''(t)\} = s^2 F(s) - k$

or $L\{-k^2 \sin kt\} = s^2 F(s) - k$

where F(s)= L {f(t)} = L {sin k t}

L { $-k^2$ sin k t} = s^2 F(s) – k = s^2 L {sin k t} –k. Solving for L {sin k t} we get

$$L\{\sin k t\} = \frac{k}{s^2+k^2}$$

$$\left[\begin{array}{l} -k^2 L\{\sin kt\} = s^2 L\{\sin kt\} - k \text{ or } L\{\sin kt\}\{s^2+k^2\} = k \\ \text{or } L\{\sin kt\} = \dfrac{k}{s^2+k^2} \end{array}\right]$$

(b) By Theorem 9.6

$$L\left\{\int_0^t f(u)ds\right\} = \frac{1}{s}F(s)$$

This implies that $\int_0^t f(u)du = L^{-1}\left\{\frac{1}{s}F(s)\right\}$

Convolution

Definition (Convolution). Let f and g be piecewise continuous functions for $t \geq 0$. Then the **convolution** of f and g denoted by $f*g$, is defined by the integral

$$(f*g)(t) = \int_0^t f(u)g(t-u)du$$

$$= \int_0^t g(u)f(t-u)du$$

$$= (g*f)(t).$$

Theorem (Convolution theorem). Let f and g be piecewise continuous and of exponential order for $t \geq 0$, then the Laplace transform of $f * g$ is given by the product of the Laplace transform of f and the Laplace transform of g. That is

$L\{f * g\} = F(s)\,G(s).$

Proof : Let $F = L(f)$ and $G = L(g)$. Then

$$F(s)G(s) = F(s)\int_0^\infty e^{-st}g(t)dt$$

$$= \int_0^\infty F(s)e^{-su}g(u)du$$

in which we changed variable of integration from t to u and brought F(s) inside the integral

Let us recall that $e^{-su}F(s) = L\{H(t-u)\,f(t-u)\}$

where $F(s) = L\{f(t)\}$ and H (.) is the Heaviside function, see Theorem 9.3.

Substitute this into the integral for F(s)G(s) to get

$$F(s)G(s) = \int_0^\infty L\{H(t-u)\,f(t-u)\}\,g(u)du$$

But, from the definition of the Laplace transform,

$$L\{H(t-u)\,f(t-u)\} = \int_0^\infty e^{-st}\{H(t-u)\,f(t-u)\}\,dt$$

Substituting this into (9.5) we get

$$F(s)G(s) = \int_0^\infty\left[\int_0^\infty e^{-st}H(t-u)f(t-u)dt\right]g(u)du$$

$$= \int_0^\infty\int_0^\infty e^{-st}g(u)H(t-u)f(t-u)dtdu$$

Let us recall that $H(t-u) = 0$ if $0 \leq t < u$ $= 1$ if $t \geq u$

Therefore, $F(s)\,G(s) = \int_0^\infty\int_u^\infty e^{-st}g(u)\,f(t-u)dt\,du.$

The Laplace integral is over shaded region, consisting of points satisfying $0 \leq u \leq t < \infty$. Reversing the order of integration gives us

$$F(s)G(s) = \int_0^\infty\int_0^t e^{-st}g(u)\,f(t-u)du\,dt.$$

$$= \int_0^\infty e^{-st}\left[\int_0^t g(u)f(t-u)du\right]dt$$

$$= \int_0^\infty e^{-st}(f*g)(t)dt$$

$$= L\{f * g\}.$$

It follows immediately from Theorem 9.7 that

Theorem Let $L^{-1}(F)=f$, $L^{-1}(G)=g$. Then

$L^{-1}\{F\,G\} = f * g$

Example Evaluate $L^{-1}\left\{\frac{1}{(s^2+k^2)^2}\right\}$

Solution Let F(s) = G(s) = $\frac{1}{s^2+k^2}$

so that f(t) = g(t) = $\frac{1}{k} L^{-1}\left\{\frac{1}{s^2+k^2}\right\} = \frac{1}{k}\sin kt$

By Convolution Theorem

$$L^{-1}\left\{\frac{1}{(s^2+k^2)^2}\right\} = \frac{1}{k^2}\int_0^t \sin ku \ \sin k(t-u)du$$

$$= \frac{1}{2k^2}\int_0^t [\cos k(2u-t) - \cos kt]du$$

$$= \frac{1}{2k^2}\left[\frac{1}{2k}\sin k(2u-t) - u\cos kt\right]_0^t$$

$$= \frac{\sin kt - kt\cos kt}{2k^3}$$

Unit Impulse and the Dirac Delta Function

Very often one encounters the concept of an impulse, which may be thought as a force of large magnitude applied over an instant of time. Impulse can be defined as follows.

For any positive number ε, the pulse δ_ε is defined by

$$\delta_\varepsilon(t) = \frac{1}{\varepsilon}[H(t) - H(t-\varepsilon)].$$

where H (.) denotes the Heaviside function

This is a pulse of magnitude $\frac{1}{\varepsilon}$ and duration ε.

It may be observed that it is not a function in the conventional sense but it is a more general object called **distribution.** Nevertheless, for historic reason it continues to be called the delta function. It is named for the Nobel laureate physicist P.A.M. Dirac who was also the guide and mentor of another Nobel laureate physicist Abdul Salam–founder Director of the International Centre of Theoretical physics, Trieste, Italy. The shifted delta function $\delta(t-a)$ is zero except for t=a, where it has its infinite spike. It is interesting to note that the Laplace transform of the Dirac delta function $\delta(t)$; that is, L $\{\delta(t)\} = 1$.

Verification:

$$L\{\delta_\varepsilon(t-a)\} = \frac{1}{\varepsilon}\left[\int_0^\infty e^{-st}(H(t-a) - H(t-a-\varepsilon))dt\right]$$

$$= \frac{1}{\varepsilon}\left[\frac{1}{s}e^{-as} - \frac{1}{s}e^{-(a+\varepsilon)s}\right]$$

$$= \frac{e^{-as}(1-e^{-\varepsilon s})}{\varepsilon s}$$

This suggests that we define

$$L(\{\delta(t-a)\} = \lim_{\varepsilon\to 0+}\frac{e^{-as}(1-e^{-\varepsilon s})}{\varepsilon s}$$

$$= e^{-as}$$

In particular choose a=0 we get

L $(\delta(t))=1$.

The following result is known as the filtering property.

The delta function therefore acts as an identity for the product defined by the convolution of two functions. (The convolution defined earlier is treated as a special type of product. The Dirac delta function is its identity).

Theorem Transform of a periodic function

If f(t) is piecewise continuous on $[0,\infty)$, of exponential order and periodic with period T, (f(t+T)=f(t)) then L {f(t)} = $\frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt$

Proof: Writing the Laplace transform of f as:

$L\{f(t)\} = \int_0^T e^{-st} f(t)\,dt + \int_T^\infty e^{-st} f(t)\,dt$

Letting t= u+T in the last integral

$\int_T^\infty e^{-st} f(t)\,dt = \int_0^\infty e^{-(u+T)s} f(u+T)dt$

$= e^{-sT}\int_0^\infty e^{-su} f(u)du = e^{-sT}\ L\{f(t)\}$

Therefore $L\{f(t)\} = \int_0^T e^{-st} f(t)dt + e^{-sT} L\{f(t)\}$

Thus $L\{f(t)\} = \dfrac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)dt$

Example : Find the Laplace Transform of the square wave function E(t) of period T=2 defined as E(t) = $\begin{cases} 1, & 0 \le t < 1 \\ 0, & 1 \le t < 2 \end{cases}$

Solution: $L\{E(t)\} = \dfrac{1}{1-e^{-2s}} \int_0^2 e^{-st} E(t)dt$

$= \dfrac{1}{1-e^{-2s}} \left[\int_0^1 e^{-st}\,1dt + \int_1^2 e^{-st}.0\,dt\right]$

$= \dfrac{1}{1-e^{-2s}}\ \dfrac{1-e^{-s}}{s}$

$= \dfrac{1}{s(1+e^{-s})}$ (using $1-e^{-2s} = (1+e^{-s})(1-e^{-s})$).

Applications to Differential and Integral Equations

In this section we discuss applications of the Laplace transform and related methods in finding solutions of differential equations with initial conditions and integral equations. In view of discussion in Section 9.4, the most important feature of the Laplace method in that the initial value given in the problem is naturally incorporated into the solution process through Theorem 9.5 and particularly equations (9.2) and (9.3). Advantage of this method is that we need not find the general solution first, then solve for the constant to satisfy the initial condition.

General Procedure of the Laplace method for solving initial value problems.

Essentially Laplace transform converts initial value problem to an algebraic problem, incorporating initial conditions into the algebraic manipulations. There are three basic steps:

(i) Take the Laplace transform of both sides of the given differential equation, making use of the linearity property of the transform.

(ii) Solve the transformed equation for the Laplace transform of the solution function.

(iii) Find the inverse transform of the expression F(s) found in step (ii).

Example Apply the Laplace transform to solve the initial-value problem

y'+2y = 0, y(0) = 1.

Solution: Given equation is y'+2y=0. Taking the Laplace transform of both sides of this equation yields

L {y'+2y} = L {0}

or L {y'}+2 L (y) = L {0} by the linearity of the Laplace transform.

Let L {y(t)} = Y(s) and applying equation (9.2), the previous equation takes the form

s Y(s)-1+2 Y(s)=0

Solving for Y(s) we have

$Y(s) = \dfrac{1}{s+2}$

The function y(t) is then found by taking the inverse transform of this equation. Thus

$y(t) = L^{-1}\left\{\dfrac{1}{s+2}\right\} = e^{-2t}$, see Table 9.1

$y(t) = e^{-2t}$ is the solution of the given initial value problem.

Example Apply the Laplace transform to solve the initial-value problem

y'-4y=1, y(0)=1.

Solution: Let L {y(t)}=Y(s)

Taking the Laplace transform of the differential equation, using the linearity of L and equation (9.2) we get

$L\{y'-4y\} = L\{1\}$

or $L\{y'\} - 4L\{y\} = 1/s$

or $sY(s) - y(0) - 4Y(s) = 1/s$

or $sY(s) - 1 - 4Y(s) = 1/s$

or $Y(s)(s-4) = 1 + \frac{1}{s}$

or $Y(s) = \frac{1}{(s-4)} + \frac{1}{s(s-4)}$

Taking the inverse Laplace transform of this equation we have

$$L^{-1}\{Y(s)\} = L^{-1}\left\{\frac{1}{s-4} + \frac{1}{s(s-4)}\right\}$$

By the linearity of L^{-1} we get

$$L^{-1}\{Y(s)\} = L^{-1}\left\{\frac{1}{s-4}\right\} + L^{-1}\{\quad\}$$

By table 9.1 $L^{-1}\left\{\frac{1}{s-4}\right\} = e^{4t}$, $L^{-1}\left\{\frac{1}{s(s-4)}\right\} = \frac{1}{4}L^{-1}\left\{\frac{1}{s-4} - \frac{1}{s}\right\}$

$$= \frac{1}{4}L^{-1}\left\{\frac{1}{s-4}\right\} - \frac{1}{4}L^{-1}\left\{\frac{1}{s}\right\}$$

$$= \frac{1}{4}e^{4t} - \frac{1}{4}$$

Thus

$$y(t) = e^{4t} + \frac{1}{4}e^{4t} - \frac{1}{4}$$

$$= \frac{5}{4}e^{4t} - \frac{1}{4}$$

is the solution of the given initial value problem.

Example Solve $y''+4y=e^{-t}$, $y(0)=2$, $y'(0)=1$

Solution: Let Y(s) = L y((t)). Taking the Laplace transform of both sides of the given differential equation we get

$L\{y''\} + 4L\{y\} = L(e^{-t})$

By equation (9.3) $L\{y''\}=s^2Y(s)-2s-1$ keeping in mind the given initial conditions and so $s^2Y(s)-2s-1+4Y(s) = \frac{1}{s+1}$

Solving for Y(s) we get

$$Y(s) = \frac{1}{s^2+4}\left[1 + \frac{1}{s+1} + 2s\right]$$

$$= \frac{2s^2+3s+2}{(s+1)(s^2+4)}$$

The partial fractions for this are

$$\frac{2s^2+3s+2}{(s+1)(s^2+4)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+4}$$

for which $A = \frac{1}{5}$, $B = \frac{9}{5}$, and $C = \frac{6}{5}$. Thus

$$Y(s)= \frac{1}{5(s+1)} + \frac{9s+6}{5(s^2+4)}$$

$$= \frac{1}{5(s+1)} + \frac{9s}{5(s^2+4)} + \frac{6}{5(s^2+4)}$$

Taking the inverse transform yields

$$y(t) = \frac{1}{5}e^{-t} + \frac{9}{5}\cos 2t + \frac{3}{5}\sin 2t$$

Example Solve

$y''+4y' +3y = e^t$

$y(0) = 0, y'(0) = 2$

Solution: Take the Laplace transform of the given differential equation to get

$L\{y''\} + L\{4y'\} + L\{3y\} = L\{e^t\}$

By equations (9.2), (9.3) and applying initial conditions weget

$L\{y''\} = s^2 Y(s) - sy(0)-y'(0) = s^2 Y(s) -2$ and

$L\{y'\} = sY(s) - y(0)=sY(s)$

Therefore,

$$s^2 Y(s)-2 +4sY(s)+3Y(s) = \frac{1}{s-1}$$

Solving this for Y(s) we get

$$Y(s) = \frac{2s-1}{(s-1)(s^2+4s+3)}.$$

Let $\dfrac{2s-1}{(s-1)(s^2+4s+3)} = \dfrac{A}{s-1} + \dfrac{B}{s+1} + \dfrac{C}{s+3}$

This equation can hold only if, for all s,

$A(s+1)(s+3) + B(s-1)(s+3) + C(s-1)(s+1) = 2s-1.$

Now choose values of s to simplify the task of determining A,B, and C. Let s=1 to get 8A =1, so A= $\frac{1}{8}$. Let s = -1 to get -4B= - 3, so B= $3/4$. Choose s= - 3 to get 8C = - 7, so C= - $\frac{7}{8}$.

Then

$$Y(s)= \frac{1}{8}\frac{1}{s-1} + \frac{3}{4}\frac{1}{s+1} - \frac{7}{8}\frac{1}{s+3}.$$

By Table 9.1 we find that

$$L^{-1}\{Y(s)\} = \frac{1}{8}L^{-1}\left(\frac{1}{s-1}\right) + \frac{3}{4}L^{-1}\left(\frac{1}{s+1}\right) - \frac{7}{8}L^{-1}\left\{\frac{1}{s+3}\right\}$$

$$y(t) = \frac{1}{8}e^t + \frac{3}{4}e^{-t} - \frac{7}{8}e^{-3t}$$

This is the solution of the given initial value problem.

Example Find $L\{f(t) * g(t)\}$, where $f(t) = e^{-t}$ and $g(t) = \sin 2t$.

Solution. By Theorem 9.7 we have

$L\{f(t) * g(t)\}$

$= F(s)G(s)$

where $F(s) = \int_0^\infty e^{-st} e^{-t} dt = \dfrac{1}{s+1}$ (or By Table 9.1)

$G(s) = \left(\dfrac{2}{s^2+4}\right)$ by Table 9.1

Thus $L\{f(t) * g(t)\} = \left(\dfrac{1}{s+1}\right)\left(\dfrac{2}{s^2+4}\right)$

$$= \frac{2}{(s+1)(s^2+4)}$$

Example Evaluate $L\left\{\int_0^t e^u \sin(t-u)\,du\right\}$

Solution: $\int_0^t e^u \sin(t-u)\,du = f(t) * g(t)$ where $f(t) = e^t$ and $g(t) = \sin t$

by definition of the convolution.

By theorem 9.7 we get

$L\{f(t) * g(t)\} = L\{f(t)\}\, L\{g(t)\}$

$= \frac{1}{s+1}\,\frac{1}{s^2+1}$ by Table 9.1

$= \frac{1}{(s-1)(s^2+1)}$

An equation involving an unknown function f(t), known functions g(t) and h(t) and integral of f and g is called a **Volterra integral equation** for f (t) :

$$f(t) = g(t) + \int_0^t f(u)h(t-u)\,du\,.$$

Example Solve the following Volterra integral equation for f(t):

$$f(t) = 3t^2 - e^{-t} - \int_0^t f(u)e^{t-u}du$$

Solution : We identify $h(t-u) = e^{t-u}$ so that $h(t) = e^t$

Take the Laplace transform of both sides, we have

$$L\{f(t)\} = L\{3t^2\} - L\{e^{-t}\} - L\left\{\int_0^t f(u)\,e^{t-u}\,du\right\}$$

$$L\{f(t)\} = F(s),\; L\{3t^2\} = 3\,L\{t^2\} = 3\,\frac{2}{s^3}$$

$$L\{e^{-t}\} = \frac{1}{s+1},\; L\left\{\int_0^t f(u)\,e^{t-u}\,du\right\} = L\{f(t) * h(t)\} = L\{f(t)\}\,L\{h(t)\}$$

by Definition 9.5 and Theorem 9.7

$$L\left\{\int_0^t f(u)\,e^{t-u}\,du\right\} = L\{f(t)\}\,L\{e^t\}$$

$$= F(s).\,\frac{1}{s-1}$$

Therefore,

$$F(s) = \frac{6}{s^3} - \frac{1}{s+1} - \frac{1}{s-1}F(s)$$

$$(s)\left[1+\frac{1}{s-1}\right] = \frac{-s^3+6s+6}{s^3(s+1)}$$

$$F(s)\left[\frac{s}{s-1}\right] = \frac{-s^3+6s+6}{s^3(s+1)}$$

$$F(s) = \frac{6}{s^3} - \frac{6}{s^4} + \frac{1}{s} - \frac{2}{s+1}$$

by carrying out the partial fraction decomposition.

The inverse transform then gives

$$f(t) = 3\,L^{-1}\left\{\frac{2!}{s^3}\right\} - L^{-1}\left\{\frac{3!}{s^4}\right\} + L^{-1}\left\{\frac{1}{s}\right\} - 2\,L^{-1}\left\{\frac{1}{s+1}\right\} = 3t^2 - t^3 + 1 - 2e^{-t}$$

Example Find f(t) such that

$$f(t) = 2t^2 + \int_0^t f(t-u)\, e^{-u}\, du$$

Solution : It is clear that

$$f(t) * g(t) = \int_0^t f(t-u)\, e^{-u}\, du$$

and by Theorem 9.7.

$$L\{f(t) * g(t)\} = L\{f(t)\}\, L\{g(t)\}$$

$$= F(s)\, \frac{1}{s+1}$$

By taking the Laplace transform of both sides of the integral equation we get

$$L\{f(t)\} = L\{2t^2\} + F(s)\, \frac{1}{s+1}$$

or $\quad F(s) = 2.\frac{2}{s^3} + \frac{1}{s+1}\, F(s)$

or $\quad F(s)\left[\frac{s}{s+1}\right] = \frac{4}{s^3}$

or $\quad F(s) = \frac{4(s+1)}{s^4}$

$$= \frac{4}{s^3} + \frac{4}{s^4}$$

Taking inverse Laplace transform we get

$$f(t) = 2\, L^{-1}\left\{\frac{2}{s^3}\right\} + \frac{2}{3}\, L^{-1}\left\{\frac{3!}{s^4}\right\} = 2t^2 + \frac{2}{3} t^3$$

Example Find the function f(t) if

$$f(t) = t + \int_0^t f(u)\sin(t-u)\, du$$

Solution: We can identify the integral as

$f(s)*h(t)$ where $h(t) = \sin t$

Taking the Laplace transform of both sides of the integral equation we get

$L\{f(t)\} = L\{t\} + L\{f(t)*h(t)\}$.

By Theorem 9.7 $L\{f(t)*h(t)\} = L\{f(t)\}\, L\{h(t)\}$

$$= F(s)\, \frac{1}{s^2+1}$$

Thus

$$F(s) = \frac{1}{s^2} + F(s)\, \frac{1}{s^2+1}$$

or $\quad F(s)\, \frac{s^2}{s^2+1} = \frac{1}{s^2}$

or $\quad F(s) = \frac{s^2+1}{s^4} = \frac{1}{s^2} + \frac{1}{s^4}$

Taking the inverse Laplace transform of this equation we get

$$f(t) = t + \frac{1}{6} t^3$$

Example Solve the problem

y''-2y'-8y=f(t); y0)=1,y'(0)=0.

Solution : Apply the Laplace transform inserting the initial values, to obtain

$L(y'' - 2y' - 8y) = (s^2Y(s) - s) - 2(sY(s) - 1) - 8Y(s) = F(s)$.

Then

$$(s^2 - 2s - 8)Y(s) - s + 2 = F(s).$$

So

$$Y(s) = \frac{1}{s^2-2s-8}F(s) + \frac{s-2}{s^2-2s-8}$$

Use a partial fractions decomposition to write

$$Y(s) = \frac{1}{6}\frac{1}{s-4}F(s) - \frac{1}{6}\frac{1}{s+2}F(s) + \frac{1}{3}\frac{1}{s-4} + \frac{2}{3}\frac{1}{s+2}$$

Taking inverse Laplace transform, we get

$$y(t) = \frac{1}{6}e^{4t}*f(t) - \frac{1}{6}e^{-2t}*f(t) + \frac{1}{3}e^{4t} + \frac{2}{3}e^{-2t}$$

This is the solution, for any function f having a convolution with e^{4t} and e^{-2t}

Exercises

Find the Laplace transform of the function

1. $2\sinh t - 4$
2. $t^2 - 3t + 5$
3. $4t\sin 2t$
4. $t - \cos(5t)$
5. $(t+4)^2$
6. $3e^{-t} + \sin 6t$
7. $t^3 - 3t + \cos(4t)$
8. $-3\cos(2t) + 5\sin 4t$
9. te^{4t}
10. t^2e^{-2t}
11. $e^t\cos t$
12. $f(t) = \begin{cases} t, & 0 \le t < 1 \\ 1, & t \ge 1 \end{cases}$
13. $t(t-2)e^{3t}$
14. $t^3 - \sinh 2t$
15. $e^{-2t} + 4e^t$
16. $f(t) = \begin{cases} 1, & 0 \le t < 2 \\ 2, & t \ge 2 \end{cases}$
17. $g(t) = \begin{cases} \sin t & 0 \le t \le \pi \\ 0 & t \ge \pi \end{cases}$
18. $f(t) = \begin{cases} e^t, & 0 \le t < 2 \\ 0, & t \ge 2 \end{cases}$
19. $L(t) = \begin{cases} 1, & 0 \le t < 2 \\ 2, & 2 \le t < 4 \\ 0, & t \ge 4 \end{cases}$
20. $t^{1/2}$
21. $\sinh t$
22. $t\sin 2t$
23. Evaluate $L^{-1}\left\{\frac{s^2}{(s+1)3}\right\}$
24. Evaluate $L^{-1}\left\{ s^2+6s+13 \right\}$
25. Use the first shifting theorem to find the Laplace transform of the following function.

(i) $e^t(\cos 2t - 3\sin 5t)$

(ii) $e^{-2t}\cos 4t$

26. Using the first shifting theorem solve the initial value problem.

$$y''-6y'+9y = t^2e^{3t}$$

$$y(0) = 2,\ y'(0) = 17.$$

27. Solve $f(t) = t - \int_0^t (t-u)f(u)\,du.$

28. Solve $f(t) + 2\int_0^t f(u)\cos(t-u)\,du = 4e^{-t} + \sin t$.

29. Solve $f(t) + \int_0^t f(u)\,du = 1.$

30. Solve $f(t) = 1+t - \frac{8}{3}\int_0^t (u-t)^3 f(u)\,du.$

Unit 5

Two port parameters: Z, Y, ABCD, Hybrid parameters, their inverse & image parameters, relationship between parameters, Interconnection of two ports networks, Reciprocity and Symmetry in all parameter**.**

Introduction:

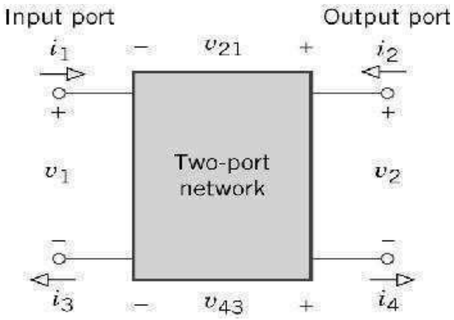
The I/O transfer properties of two-port networks are traditionally defined in terms of one or more of several possible forward transfer characteristics and two driving-point impedance specifications. Among the more commonly used forward transfer specifications is the *voltage gain*, say A_V, which is the ratio of the output voltage developed across the terminating load impedance to the Thévenin equivalent input voltage.

Of these four variables V_1, V_2, i_1 and i_2, two can be selected as independent variables and the remaining two can be expressed in terms of these independent variables. This leads to various two part parameters out of which the following three are more important.

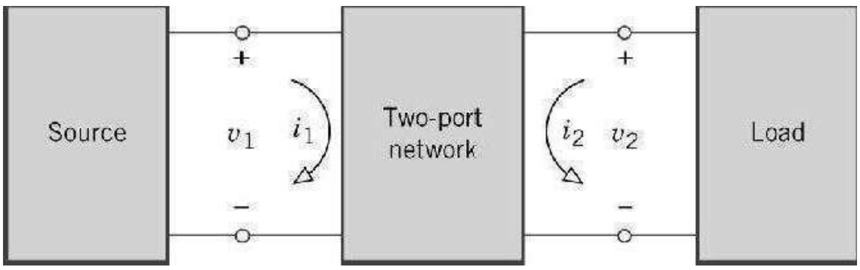
1. Z – Parameters (or) Impedance parameters
2. Y – Parameters (or) Admittance parameters
3. H – Parameters (or) Hybrid parameters.
4. T- Transmission Line Parameters.

Two-ports and impedance parameters

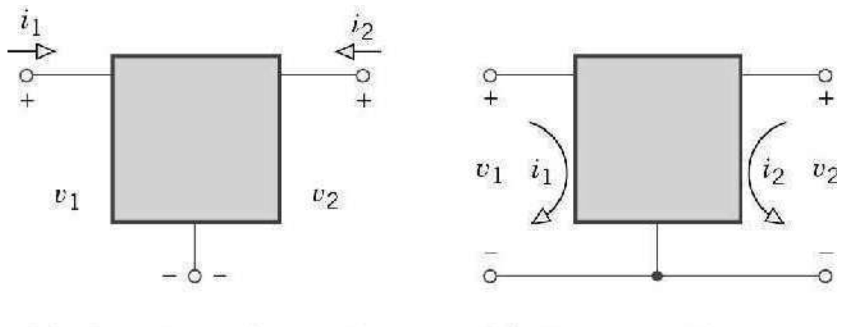
- Two-port networks:



- It is assumed that a two-port network contains no independent sources but may include controlled sources.

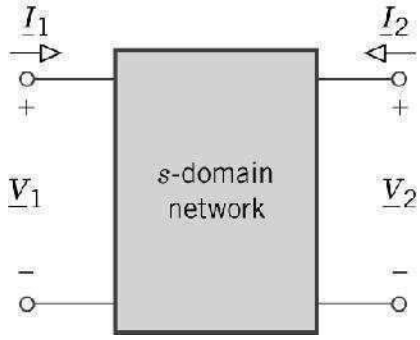


- Three-terminal networks (with a common ground):



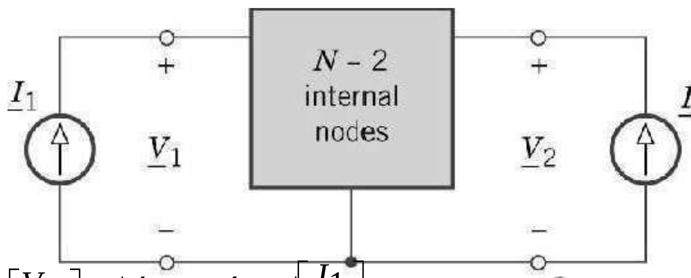
(a) Three-terminal network (b) Two-port with common ground

- Two-port in the *s*-domain:



1. Z – Parameters (or) Impedance parameters

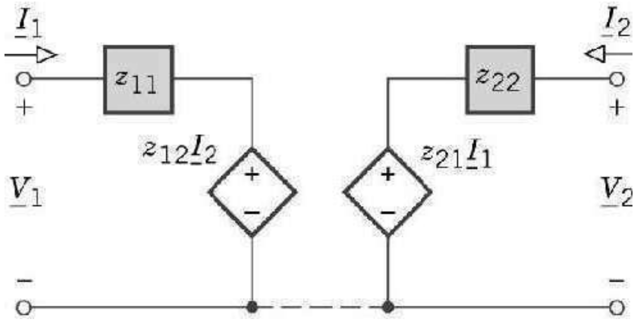
- Impedance parameters: common ground two-port with current sources.



$$\begin{bmatrix} \underline{V}_1 \\ \underline{V}_2 \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \begin{bmatrix} \underline{I}_1 \\ \underline{I}_2 \end{bmatrix}$$

- Open circuit impedance parameters:
- $z_{11} = \underline{V}_1 / \underline{I}_1 |_{\underline{I}_2 = 0}$: input impedance with open output.
- $z_{12} = \underline{V}_1 / \underline{I}_2 |_{\underline{I}_1 = 0}$: reverse transfer impedance with open output.
- $z_{21} = \underline{V}_2 / \underline{I}_1 |_{\underline{I}_2 = 0}$: forward transfer impedance with open output.
- $z_{22} = \underline{V}_2 / \underline{I}_2 |_{\underline{I}_1 = 0}$: output impedance with open input.

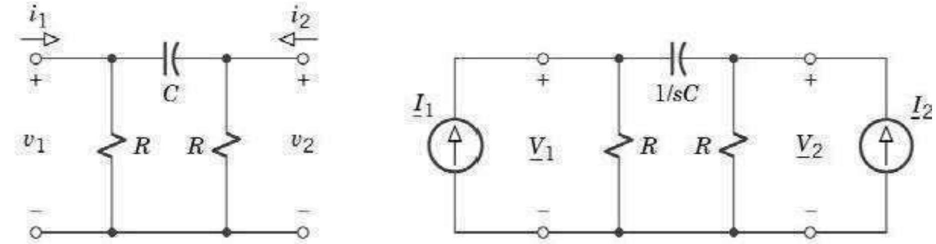
- Equivalent *s*-domain diagram:



6.

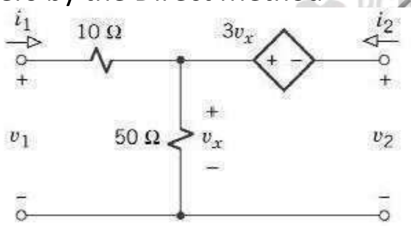
- Indirect method for finding *z* parameters: Treat I_1 and I_2 as source currents and use standard analysis techniques.
- Direct method for finding *z* parameters: open circuit techniques.
- Existence test: independent current sources may be connected to the input and output ports without violating KCL.

Example : z Parameters by the Indirect Method

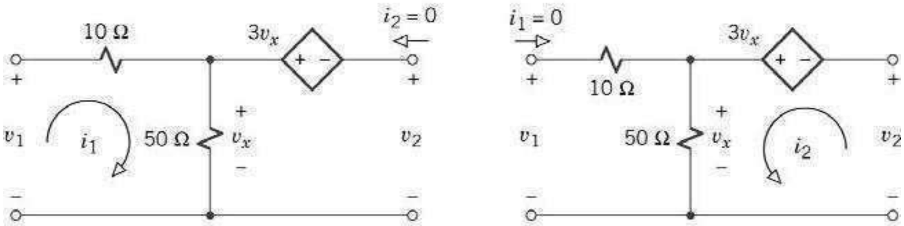


(a) Network for Example 14.1 (b) s-domain diagram with current sources

Example : z Parameters by the Direct Method

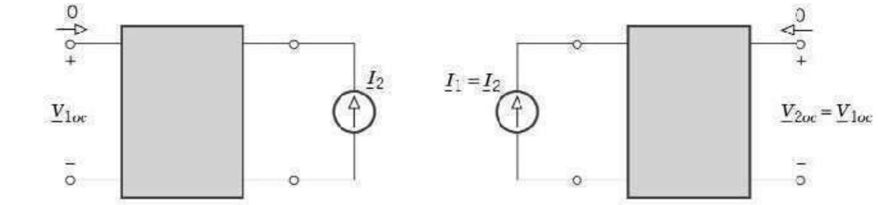


(a) Network for Example 14.2

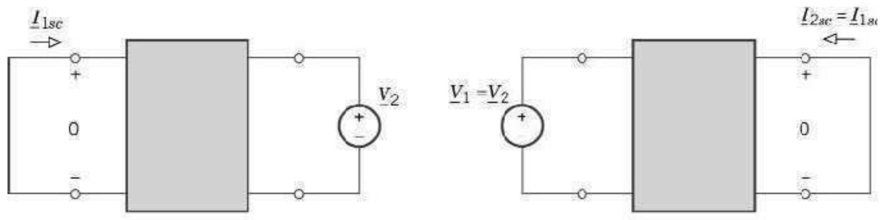


(b) Open-output diagram (c) Open-input diagram

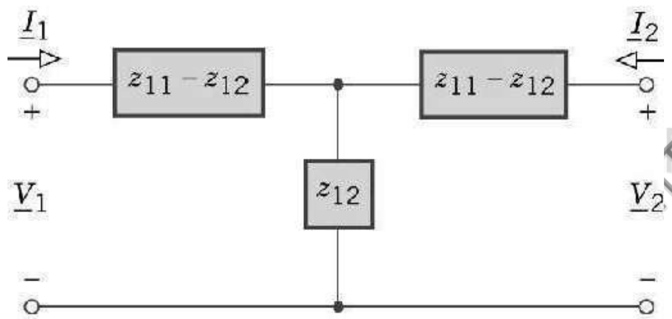
- Reciprocal networks: $z_{12} = z_{21}$.
- Reciprocal theorem:



(a) Interchanging open circuit and current source

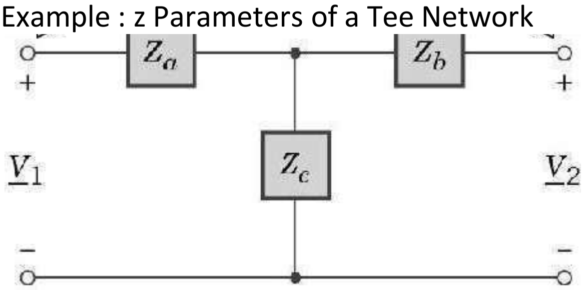


(b) Interchanging short circuit and voltage source



- A linear circuit that contains no controlled sources will be reciprocal.

Example : z Parameters of a Tee Network



12.

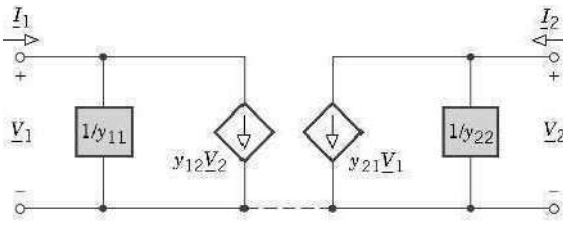
14.2 Admittance, hybrid, and transmission parameters,

- Admittance parameters (*y* parameters): dual of the impedance parameters.

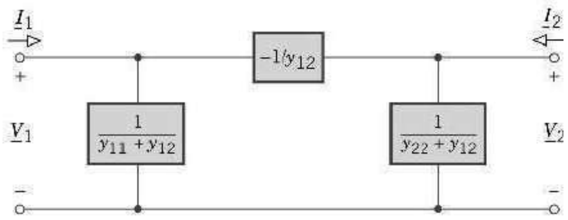
$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

- Short circuit admittance parameters:

- $y_{11} = I_1 / V_1 |_{V_2 = 0}$: input admittance with shorted output.
- $y_{12} = I_1 / V_2 |_{V_1 = 0}$: reverse transfer admittance with shorted output.
- $y_{21} = I_2 / V_1 |_{V_2 = 0}$: forward transfer admittance with shorted output.
- $y_{22} = I_2 / V_2 |_{V_1 = 0}$: output admittance with shorted input.
- Existence test: independent current sources may be connected to the input and output ports without violating KVL.



(a) y-parameter model

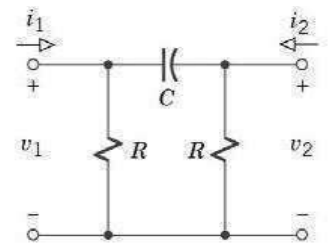


(b) Tee model for reciprocal two-port

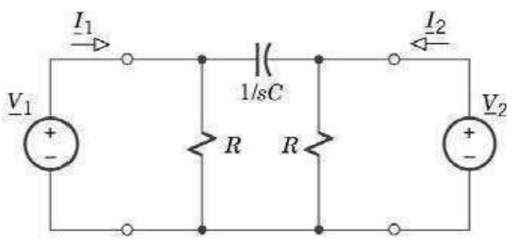
14.

- The y parameters can be found by the indirect and the direct method.

Example 14.4 y Parameters by the Indirect Method



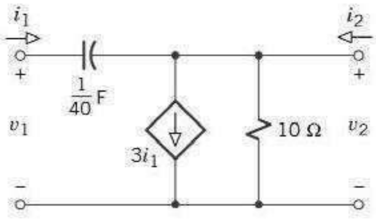
(a) Network for Example 14.4



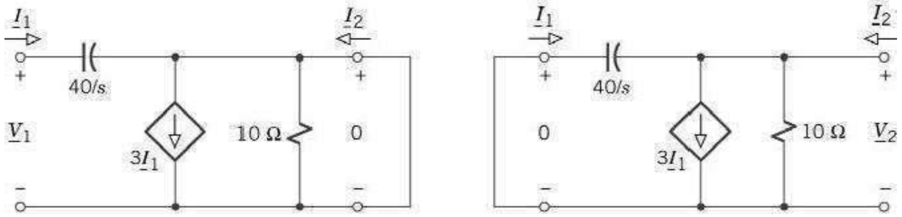
(b) s–domain diagram with voltage sources

15

Example : y Parameters by the Direct Method



(a) Network for Example 14.5



(b) Shorted-output diagram (c) Shorted-input diagram

16

Hybrid parameters (or) h – parameters:-

If the input current i_1 and output Voltage V_2 are takes as independent variables, the input voltage V_1 and output current i_2 can be written as

$V_1 = h_{11}\, i_1 + h_{12}\, V_2$
$i_2 = h_{21}\, i_1 + h_{22}\, V_2$

The four hybrid parameters h_{11}, h_{12}, h_{21} and h_{22} are defined as follows.

$h_{11} = [V_1 / i_1]$ with $V_2 = 0$

= Input Impedance with output part short circuited.

$h_{22} = [i_2 / V_2]$ with $i_1 = 0$

= Output admittance with input part open circuited.

$h_{12} = [V_1 / V_2]$ with $i_1 = 0$

= reverse voltage transfer ratio with input part open circuited.

$h_{21} = [i_2 / i_1]$ with $V_2 = 0$

= Forward current gain with output part short circuited.

The dimensions of h – parameters are as follows:

h_{11} - Ω
h_{22} – mhos

h12, h21 – dimension less.

→ as the dimensions are not alike, (ie) they are hybrid in nature, and these parameters are called as hybrid parameters.

I = 11 = input ; 0 = 22 = output ;
F = 21 = forward transfer ; r = 12 = Reverse transfer.

<u>Notations used in transistor circuits</u>:-

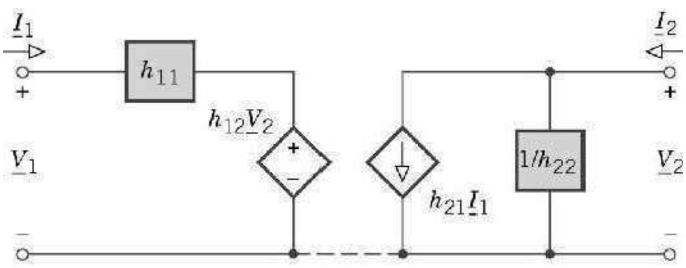
$h_{ie} = h_{11e}$ = Short circuit input impedance
$h_{0e} = h_{22e}$ = Open circuit output admittance
$h_{re} = h_{12e}$ = Open circuit reverse voltage transfer ratio
$h_{fe} = h_{21e}$ = Short circuit forward current Gain.

<u>The Hybrid Model for Two-port Network:-</u>

$V_1 = h_{11} i_1 + h_{12} V_2$
$I_2 = h_1 i_1 + h_{22} V_2$
↓
$V_1 = h_1 i_1 + h_r V_2$
$I_2 = h_f i_1 + h_0 V_2$

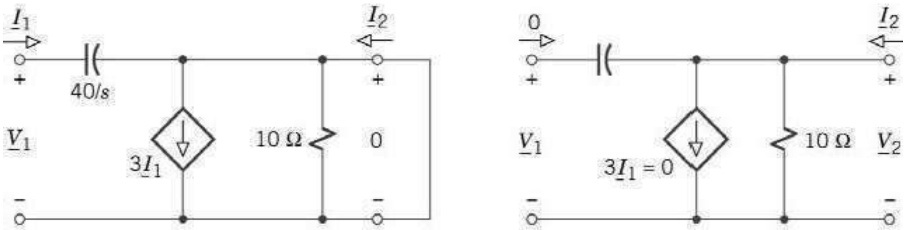
$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$$

- Hybrid parameters:
- $h_{11} = V_1 / I_1 |_{V_2 = 0}$: input impedance with shorted output.
- $h_{12} = V_1 / V_2 |_{I_1 = 0}$: reverse voltage ratio with open input.
- $h_{21} = I_2 / I_1 |_{V_2 = 0}$: forward current ratio with shorted output.
- $h_{22} = I_2 / V_2 |_{I_1 = 0}$: output admittance with open input.
- Existence test: an independent current sources may be connected to the input port and an independent voltage source may be connected to the output port without violating KCL and KVL.



17.

Example 14.6 Calculating *h* paramet rs



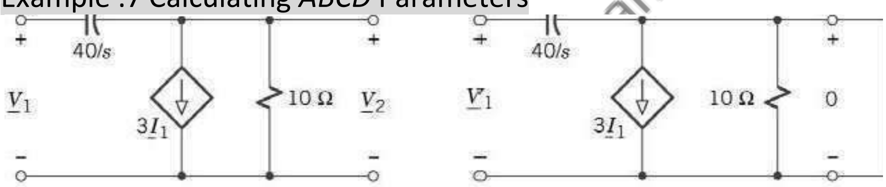
(a) Shorted output *(b)* Open input 18.

- Transmission parameters (*ABCD* parameters):

$$\begin{bmatrix} \underline{V}_1 \\ \underline{I}_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \underline{V}_2 \\ -\underline{I}_2 \end{bmatrix}$$

- Transmission parameters:
- $A = \underline{V}_1 / \underline{V}_2 |_{\underline{I}_2 = 0}$.
- $B = -\underline{V}_1 / \underline{I}_2 |_{\underline{V}_2 = 0}$.
- $C = \underline{I}_1 / \underline{V}_2 |_{\underline{I}_2 = 0}$.
- $D = -\underline{I}_1 / \underline{I}_2 |_{\underline{V}_2 = 0}$.
- *A*: reverse voltage ratio with open output.
- *-B*: reverse transfer impedance with shorted output.
- *C*: reverse transfer admittance with open output.
- *-D*: reverse current ratio with shorted output.
- Existence test: the *ABCD* parameters exist only if $\underline{V}_{2oc} \neq 0$ and $\underline{I}_{2sc} \neq 0$.

Example :7 Calculating *ABCD* Parameters



(a) Open output *(b)* Shorted output 19.

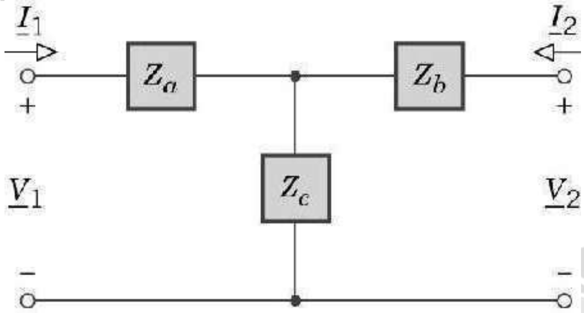
- Parameter conversion: with one set of parameters for a particular two-port, other sets of parameters can be obtained if they exist.

(Table :1 Two-port equations)

- The conversion between z and y can be done by matrix inversion.
- Conversions involving hybrid or transmission parameters need to be done by algebraic manipulation.

(Table :2 Parameter conversions)

Example :8 Parameters of a Tee Net ork



20.

:3 Circuit analysis with two-ports

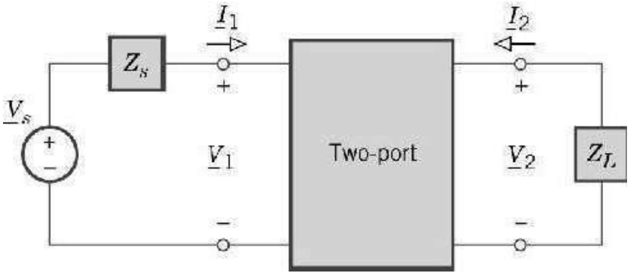
- Terminated two-ports: A two-port is terminated with a source and a load. The source may be replaced by a Thevenin or Norton model and the load may be replaced by an equivalent impedance.
- Typical network functions:

✓ Voltage transfer function: $H_v(s) \equiv \underline{V}_2/\underline{V}_1$.

✓ Current transfer function: $H_i(s) \equiv \underline{I}_2/\underline{I}_1$.

✓ Equivalent input impedance: $Z_i(s) \equiv \underline{V}_1/\underline{I}_1$.

✓ Equivalent output impedance: $Z_o(s) \equiv \underline{V}_2/\underline{I}_2\Big|_{\underline{V}_s=0}$.

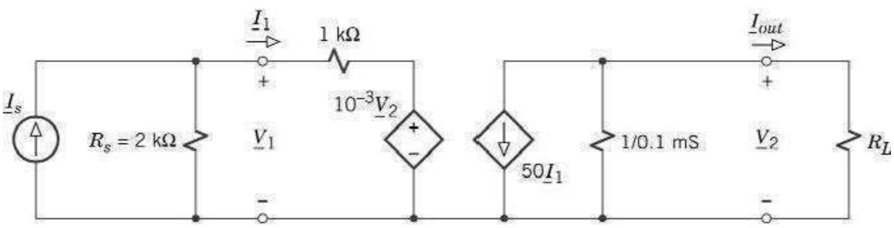


21.

(Table :3 Relations for Terminated Two-Ports)

Example :9 Calculating a Transfer Function

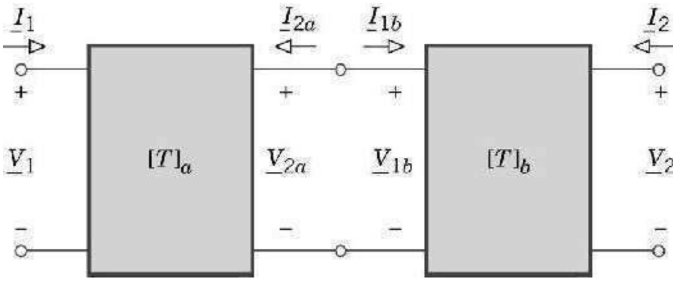
Example :10 A Mid-Frequency Trans stor Amplifier



22.

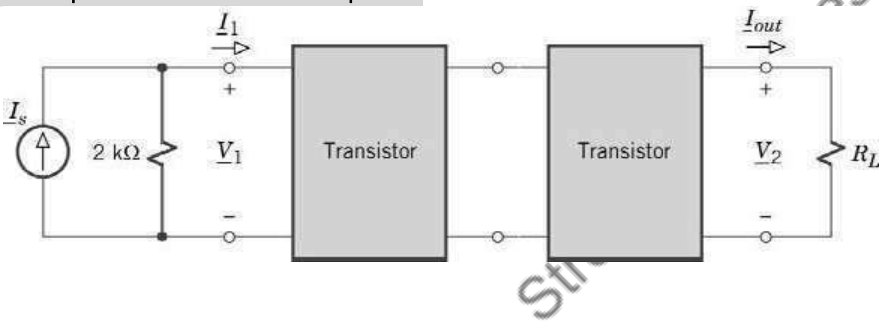
- Interconnected two-ports (assuming the interconnection does not change the properties of individual two-ports).
- Cascade connection:

$$\underline{V}_{1b} = \underline{V}_{2a}, \underline{I}_{1b} = -\underline{I}_{2a}.$$
$$[T]_{cas} = [T]_a [T]_b.$$



24.

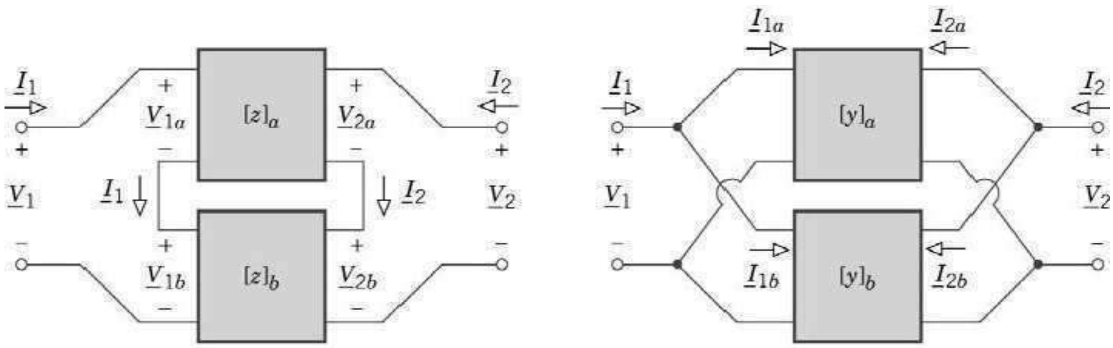
Example :11 A Cascade Amplifier



25.

- Series and parallel connections

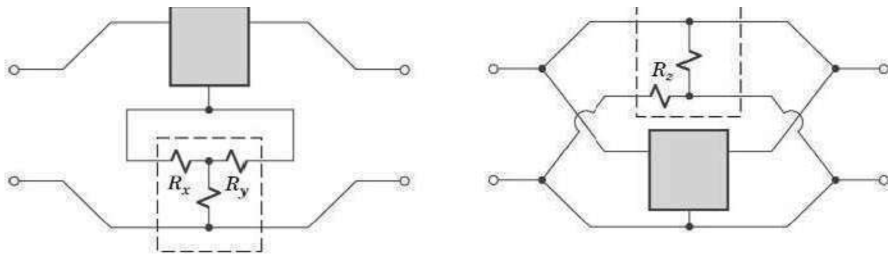
$$[z]_{ser} = [z]_a + [z]_b$$
$$[y]_{par} = [y]_a + [y]_b$$



(a) Series connection of two-ports

(b) Parallel connection of two-ports

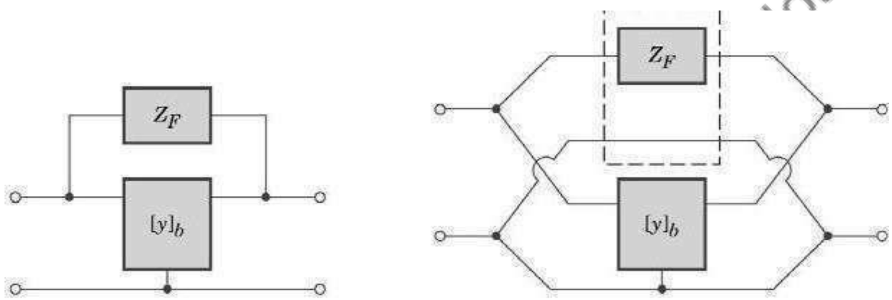
26.



(a) Series

(b) Parallel

27.

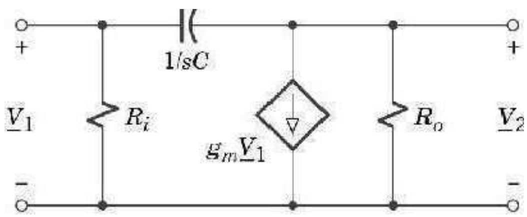


(a) Network with bridging impedance

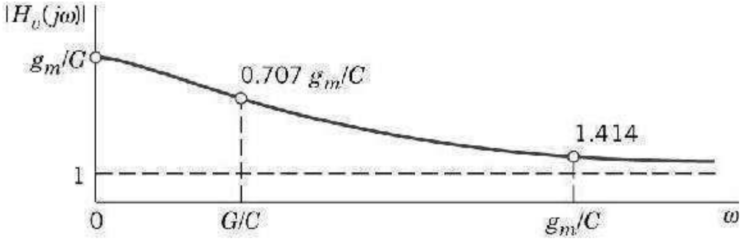
(b) Model as parallel two-ports

28.

Example :12 A High-Frequency Amplifier



29.



30.

Transmission Lines Parameters

The ABCD parameters are also referred to as the transmission parameters, where the following matrix is defined as the transmission matrix:

$$\boldsymbol{T} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{bmatrix}$$

With this notation, becomes:

$$\begin{bmatrix} \boldsymbol{V_1} \\ \boldsymbol{I_1} \end{bmatrix} = \boldsymbol{T} \begin{bmatrix} \boldsymbol{V_2} \\ \boldsymbol{I_2} \end{bmatrix}$$

Two comments about this:

a. It is easy to get the receiving end (V_2, I_2) as a function of the sending end quantities (V_1, I_1):

$$\begin{bmatrix} \boldsymbol{V_2} \\ \boldsymbol{I_2} \end{bmatrix} = \boldsymbol{T}^{-1} \begin{bmatrix} \boldsymbol{V_1} \\ \boldsymbol{I_1} \end{bmatrix}$$ The reason why this is easy is because of the hyperbolic identity:

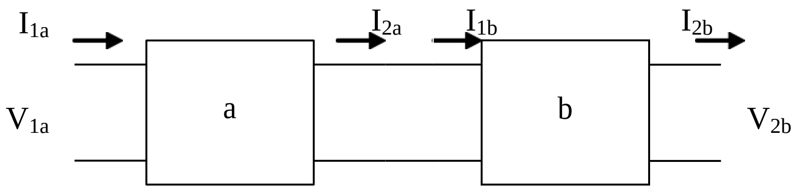
$\cosh^2 x - \sinh^2 x = 1$ and therefore the determinate of T is

$|T| = AD - BC = 1$

and therefore

$$\boldsymbol{T}^{-1} = \begin{bmatrix} \boldsymbol{D} & -\boldsymbol{B} \\ -\boldsymbol{C} & \boldsymbol{A} \end{bmatrix}$$

b. It is easy to express sending end or receiving end quantities as a function of the other when there are multiple two- ports chained together. Figure 1 illustrates the case of two 2-ports chained together.



If we know T_a and T_b, where

$$T_a = \begin{bmatrix} A_a & B_a \\ C_a & D_a \end{bmatrix} \qquad T_b = \begin{bmatrix} A_b & B_b \\ C_b & D_b \end{bmatrix}$$

then it is easy to show that

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = T \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}, \qquad \begin{bmatrix} \boldsymbol{V_2} \\ \boldsymbol{I_2} \end{bmatrix} = \boldsymbol{T}^{-1} \begin{bmatrix} \boldsymbol{V_1} \\ \boldsymbol{I_1} \end{bmatrix}$$

where

$$T = T_a T_b, \qquad T^{-1} = T_a^{-1} T_b^{-1}$$