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New Scheme Based On AICTE Flexible Curricula

Electronics & Communication Engineering IV-Semester

EC404 Control System

Unit-1 Introduction to Control system: Terminology and classification of control system, examples of control system, mathematical modeling of mechanical and electrical systems, differential equations, transfer function, block diagram representation and reduction, signal flow graph techniques.

Feedback characteristics of control systems Open loop and closed loop systems, effect of feedback on control system and on external disturbances, linearization effect of feedback, regenerative feedback

.**Unit-2 Time response analysis** Standard test signals, time response of 1st order system, time response of 2nd order system, steady-state errors and error constants, effects of additions of poles and zeros to open loop and closed loop system.

Time domain stability analysis Concept of stability of linear systems, effects of location of poles on stability, necessary conditions for stability, Routh-Hurwitz stability criteria, relative stability analysis, Root Locus concept, guidelines for sketching Root-Locus.

Unit-3 Frequency response analysis Correlation between time and frequency response, Polar plots, Bode Plots, all-pass and minimum-phase systems, log-magnitude versus Phase-Plots, closed-loop frequency response.

Frequency domain stability analysis : Nyquist stability criterion, assessment of relative stability using Nyquist plot and Bode plot (phase margin, gain margin and stability).

Unit-4 Approaches to system design Design problem, types of compensation techniques, design of phase-lag, phase lead and phase lead-lag compensators in time and frequency domain, proportional, derivative, integral and Composite Controllers.

Unit-5 State space representation of systems, block diagram for state equation, transfer function decomposition, solution of state equation, transfer matrix, relationship between state equation and transfer function, controllability and observability.

Text/Reference Books:

1. Albert D. Helfrick, William David Cooper, "Modern electronic instrumentation and measurement techniques", TMH 2008.
2. Oliver Cage, "Electronic Measurements and Instrumentation", TMH, 2009.
3. Alan S. Morris, "Measurement and Instrumentation Principles", Elsevier (Buterworth Heinmann), 2008.
4. David A. Bell, "Electronic Instrumentation and Measurements", 2nd Ed., PHI, New Delhi 2008.
5. H.S. Kalsi, "Electronics Instrumentation", TMH Ed. 2004
6. A.K.Sawhney, "A Course in Electrical and Electronic Measurements and Instrumentation", Dhanpat Rai.
7. MMS Anand, "Electronic Instruments & Instrumentation Technology", PHI Pvt. Ltd., New Delhi Ed. 2005

CONTROL SYSTEM LAB

Control System performance analysis and applications of MATLAB in Control system performance analysis & design.

Unit 1

Syllabus:

Introduction to Control system

Terminology and classification of control system, examples of control system, Laplace Transform and its application, mathematical modeling of mechanical and electrical systems, differential equations, transfer function, block diagram representation and reduction, signal flow graph techniques.

Feedback characteristics of control systems

Open loop and closed loop systems, effect of feedback on control system and on external disturbances, linearization effect of feedback, regenerative feedback.

1.1 Introduction to Control Systems:

A Control System is a combination of elements, arranged in a planned manner wherein each element causes an effect to produce a desired output. This cause and effect relationship is governed by a mathematical relation. If this relation is linear, control system is called linear control system.

Control systems are systems that are used to maintain a desired result or value. For example, driving a car along a road involves the brain of the driver as a controller comparing the actual position of the car on the road with the desired position and making adjustments to correct any error between the desired and actual position.

A control system consisting of interconnected components is designed to achieve a desired purpose. To understand the purpose of a control system, it is useful to examine examples of control systems through the course of history. These early systems incorporated many of the same ideas of feedback that are in use today. Modern control engineering practice includes the use of control design strategies for improving manufacturing processes, the efficiency of energy use, advanced automobile control, including rapid transit, among others.

System – An interconnection of elements and devices for a desired purpose.

Control System – An interconnection of components forming a system configuration that will provide a desired response.

Process – The device, plant, or system under control. The input and output relationship represents the cause-and-effect relationship of the process.

1.1.1 Classification of Control System:

(a) Open Loop Control System:

These are the systems in which the control action is independent of the output. In an Open Loop Control System the output is neither measured nor fed back for comparison with the input. Faithfulness of an Open Loop Control System depends on the accuracy of input calibration.

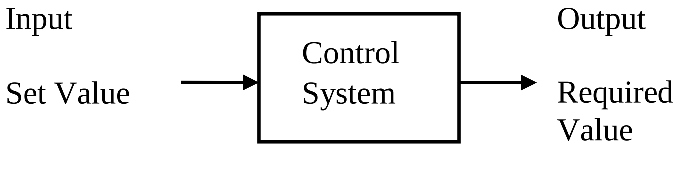


Figure 1.1.1: An open-loop control system

Examples of Open Loop Control System are hand drier, automatic washing machine, electric lift, traffic signal, cold drink bottling coffee server etc.

Advantages of Open Loop Control System are
1) Simple in construction,
2) Economic in operation,
3) No stability problem.

Disadvantages of Open Loop Control System are
1) These are inaccurate,
2) Un reliable
3) Effect of parameter variation and internal disturbances are present in Open Loop Control System.

(b) Close Loop Control System:
Figure below shows the general form of a basic closed-loop system.

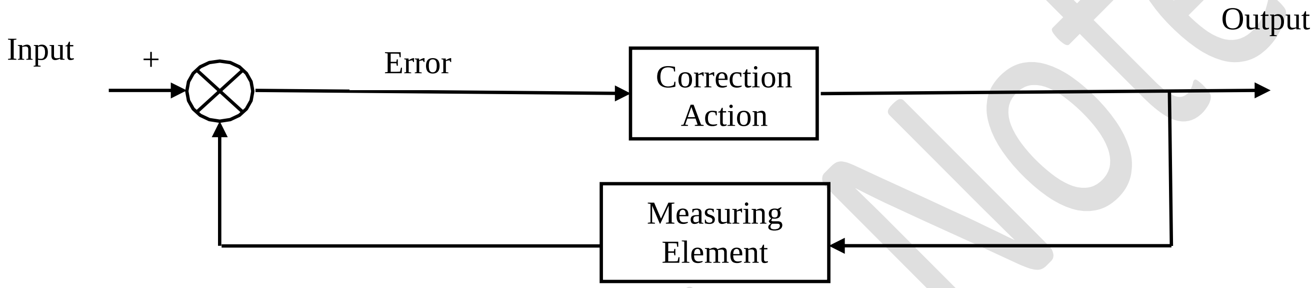


Figure 1.1.2: Basic elements of a closed-loop control

In case of close loop control systems the output of the system depends upon input and previous output itself, or we can say the control action depends upon output of the system. The control action is actuated by an error signal, which is the difference between the input signal and the output signal. This process of comparison between input and the output maintains the output at a desired level through the control action.

The following are the functions of the constituent elements:
(i). Comparison element
This element compares the required value of the variable being controlled with the measured value of what is being achieved and produces an error signal:
Error = reference value signal - measured actual value signal
Thus if the output is the required value then there is no error and so no signal is fed to initiate control. Only when there is a difference between the required value and the actual values of the variable will there be an error signal and so control action initiated.
(ii). Control law implementation element
The control law element determines what action to take when an error signal is received. The control law used by the element may be just to supply a signal which switches on or off when there is an error, as in a room thermostat, or perhaps a signal which is proportional to the size of the error. With a proportional control law implementation, if the error is small a small control signal is produced and if the error is large a large control signal is produced.
(iii). Correction element
The correction element or, as it is often called, the final control element, produces a change in the process which aims to correct or change the controlled condition.
(iv). Process
The process is the system in which there is a variable that is being controlled, e.g. it might be a room in a house with its temperature being controlled.
(v). Measurement element
The measurement element produces a signal related to the variable condition of the process that is being controlled. For example, it might be a temperature sensor with suitable signal processing.

Examples of the close loop control systems are:
1) Automatic Electric Iron,
2) RADAR tracking system,
3) DC speed control,
4) Water level indicator,
5) Auto pilot system.

Advantages of the close loop control systems are:
1) It is accurate and reliable,
2) Effect of parameter variation and internal disturbances is reduced,
3) High bandwidth,
4) Reduces effects of non linearities.

Disadvantages of close loop control systems are:
1) Complex and costly,
2) Instability margin increases i.e. if any open loop control system which is stable and when we feedback the output, then the close loop control system becomes the unstable.

1.2 Comparison between Open Loop and Closed Loop Control Systems:

The differences between the Open loop control system and closed loop control system are as under in the table:

S. No.	Open Loop Control Systems	Closed Loop Control Systems
01	Accuracy depends upon the calibration of the input signal.	As the error signal is continuously measured, these work more accurately.
02	Simple to construct.	Complex
03	Cheap	Costly
04	Operation affected due to non linearity of the elements.	The effects of non linearity present in its elements are adjusted.
05	No change in input with change in output.	Change in output affects the input.
06	Error correction is not possible.	Possible
07	Small bandwidth	Large bandwidth

1.3 Transfer Function of a Control Systems:

The transfer function of a control system is defined as the Laplace transform of the output signal to the Laplace transform of the input signal with assuming all initial conditions to be zero.
The transfer function can also be expressed as the ratio of output quantity to the input quantity.
Suppose that we have a system as shown below:

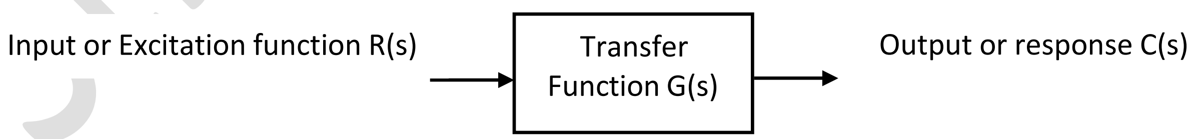


Figure 1.3.1 An open loop control system

The term gain is used to relate the input and output of a system with gain G = output/input. When we are working with inputs and outputs described as functions of s we define the transfer function G(s) as [output C(s)]/input R(s)] when all initial conditions before we apply the input are zero:

Here transfer function $G(s) = \frac{C(S)}{R(S)}$

(a) Transfer function of open loop control system

Suppose that we have an open loop control system as shown in figure 1.3.1
Then the transfer function of this open loop control system is given by:

$$G(s) = \frac{C(s)}{R(s)}$$

A transfer function can be represented as a block diagram (Figure) with X(s) the input, Y(s) the output and the transfer function G(s) as the operator in the box that converts the input to the output. The block represents a multiplication for the input. Thus, by using the Laplace transform of inputs and outputs.

(b) Transfer function of close loop control system

Suppose that we have a closed loop control system as shown in the figure 1.3.2

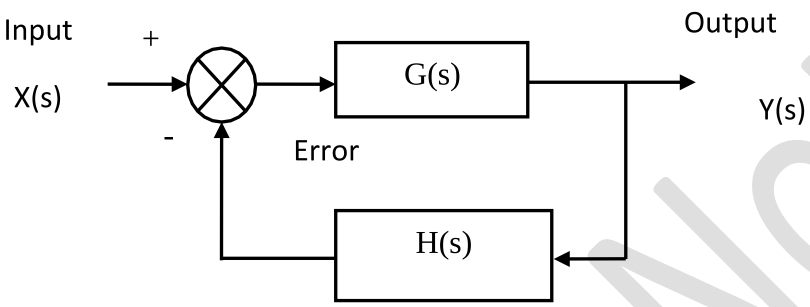


Figure 1.3.2 A closed loop control system

For systems with a negative feedback loop we can have the situation shown in Figure where the output is fed back via a system with a transfer function H(s) to subtract from the input to the system G(s). The feedback system has an input of Y(s) and thus an output of H(s)Y(s). Thus the feedback signal is H(s)Y(s). The error is the difference between the system input signal X(s) and the feedback signal and is thus:

$$Error(s) = X(s) - H(s)Y(s)$$

Overall transfer function of the closed loop system will be:

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

1.4 Laplace Transform:

Definition: The Laplace transform of the function x(t) is denoted by X(s) and given by

$$X(s) = \int_0^{\infty} x(t)e^{-st}\, dt$$

This transform is an operator. It transforms a time domain function $x(t)$ into the frequency domain function $X(s)$.

The Laplace operator id denoted by the script letter $\mathscr{L}$.

$$\mathscr{L}\{x(t)\} = X(s)$$

Inverse Laplace Transform: The Inverse Laplace transform of the function $X(s)$ is denoted by $x(t)$ and given by

$$x(t) = \frac{1}{2\pi j}\int_{\sigma - j\omega}^{\sigma + j\omega} X(s)e^{st}\, ds$$

Ex.1.1 Find the Laplace transform of the following functions

(a) $x(t) = 1$

$$L[x(t)] = \int_0^\infty x(t)e^{-st}dt = \int_0^\infty e^{-st}dt = -\left.\frac{e^{-st}}{-s}\right|_{t=0}^{t=\infty} = \frac{1}{-s}\left[e^{-st}\right]_{t=0}^{t=\infty} = -\frac{1}{s}(0-1) = \frac{1}{s}$$

$$\Rightarrow L(1) = \frac{1}{s} \qquad (\mathrm{Re(s)}) > 0$$

(b) $x(t) = e^{-\alpha t}u(t)$

$$L[e^{-\alpha t}u(t)] = \int_0^\infty e^{-\alpha t}e^{-st}dt = \int_0^\infty e^{-(\alpha+s)t}dt$$

Define a new complex variable $s' = s + \alpha$

$$L[e^{-\alpha t}u(t)] = \int_0^\infty e^{-s't}\,dt$$

We know that $\int_0^\infty e^{-st}dt = \frac{1}{s}$ $\qquad \mathrm{Re}(s) > 0$

$$\Rightarrow \int_0^\infty e^{-s't}dt = \frac{1}{s'} \qquad \mathrm{Re}(s') > 0$$

$$\Rightarrow \int_0^\infty e^{-(\alpha+s)t}dt = \frac{1}{s+\alpha} \qquad \mathrm{Re}(s+\alpha) > 0$$

$$\Rightarrow L[e^{-\alpha t}u(t)] = \frac{1}{s+\alpha} \qquad \mathrm{Re}(s+\alpha) > 0 \quad or \quad \mathrm{Re}(s) > -\mathrm{Re}(\alpha)$$

$$\Rightarrow L[e^{-\alpha t}] = \frac{1}{s+\alpha}$$

(c) $x(t) = \delta(t)$

$$L[\delta(t)] = \int_0^\infty \delta(t)e^{-st}dt$$

$$= e^{-st}\Big|_{t=0} = e^{-\sigma t}e^{-j\omega t}\Big|_{t=0}$$

$$= e^{-\sigma t}(\cos\omega t - j\omega\sin\omega t)\Big|_{t=0}$$

$$= 1$$

Some functions and their Laplace transforms

S. No	Function f(t)	Laplace Transform F(s)
1	1	$\frac{1}{s}$
2	T	$\frac{1}{s^2}$
3	$\frac{t^2}{2}$	$\frac{1}{s^3}$
4	t^n	$\frac{n!}{s^{n+1}}$
5	e^{at}	$\frac{1}{s-a}$
6	$\sin kt$	$\frac{k}{s^2+k^2}$

7	$\cos kt$	$\frac{s}{s^2+k^2}$
8	$e^{at}\sin kt$	$\frac{k}{(s-a)^2+k^2}$
9	$e^{-at}\cos kt$	$\frac{s}{(s+a)^2+k^2}$
10	$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$
11	$\sin hkt$	$\frac{k}{s^2-k^2}$
12	$\cos hkt$	$\frac{k}{s^2-k^2}$

1.5 Properties of Laplace Transform:

The properties of the Laplace Transform are as under…

1. Linearity Property:

Property f is a linear transform;

Assume $x(t) = a_1x_1(t) + a_2x_2(t)$ (a_1 and a_2 are time independent)

And $X_1(s) = L[x_1(t)]$, $X_2(s) = L[x_2(t)]$

then $X(s) = L[x(t)] = a_1X_1(s) + a_2X_2(s)$

2. Laplace Transforms of Derivatives

Assume $X(s) = L[x(t)]$

Then $L\left[\frac{dx(t)}{dt}\right] = sX(s) - x(0^-)$

3. Laplace Transform of an integral

Assume $y(t) = \int_{-\infty}^{t} x(\lambda)d\lambda,$ $X(s) = L[x(t)]$

Then $L\left[\int_{-\infty}^{t} x(\lambda)d\lambda\right] = \frac{X(s)}{s} + \frac{y(0^-)}{s}$

where $y(0^-) = \int_{-\infty}^{0^-} x(\lambda)d\lambda$

4. Complex Frequency shift (s-shift) Theorem

Assume $y(t) = x(t)e^{-\alpha t}$

$X(s) = L[x(t)]$ $Y(s) = L[y(t)]$

Then $Y(s) = X(s+\alpha)$

5. Delay Theorem

Assume $L[x(t)] \equiv L[x(t)u(t)] = X(s)$

Then $L[x(t-t_0)u(t-t_0)] = e^{-st_0}X(s)$ $(t_0 > 0)$

(If $(t_0 < 0)$, it will not be a delay!)

Proof:

$$L[x(t-t_0)u(t-t_0)]$$

$$= \int_0^{\infty} x(t-t_0)u(t-t_0)e^{-st}dt$$

$$= \int_0^{t_0} x(t-t_0)u(t-t_0)e^{-st}dt + \int_{t_0}^{\infty} x(t-t_0)u(t-t_0)e^{-st}dt$$

$$= \int_{t_0}^{\infty} x(t-t_0)e^{-st}dt = \int_{t_0}^{\infty} x(t-t_0)e^{-s(t-t_0)-st_0}dt$$

$$= e^{-st_0}\int_{t_0}^{\infty} x(t-t_0)e^{-s(t-t_0)}d(t-t_0)$$

$$put \qquad \tau = t-t_0$$

$$= e^{-st_0}\int_0^{\infty} x(\tau)e^{-s\tau}d\tau = e^{-st_0}\int_0^{\infty} x(t)e^{-st}dt = e^{-st_0}X(s)$$

6. Initial value theorem

$$If, x(t) = L^{-1}[X(s)]$$

$$then, x(0^+) = \lim_{s\to\infty} sX(s)$$

7. Final value theorem

$$If, x(t) = L^{-1}[X(s)]$$

$$then, x(\infty^+) = \lim_{s\to 0} sX(s)$$

Ex. 1.2 Find $x(t) = L^{-1}[X(s)] = L^{-1}\dfrac{s+8}{s^2+6s+13}$

Solution:

$$X(s) = \frac{s+8}{s^2+6s+13} = \frac{(s+3)+5}{s^2+6s+9+4}$$

$$= \frac{s+3}{(s+3)^2+2^2} + \frac{(5/2)\times 2}{(s+3)^2+2^2}$$

$$x(t) = L^{-1}[X(s)] = e^{-3t}\cos 2t + \frac{5}{2}e^{-3t}\sin 2t \qquad (t>0)$$

1.6 Modeling of Control Systems

The control systems can be represented with a set of mathematical equations known as mathematical model. These models are useful for analysis and design of control systems. Analysis of control system means finding the output when we know the input and mathematical model.

The following mathematical models are mostly used.

- Differential equation model
- Transfer function model

Differential Equation Model

Differential equation model is a time domain mathematical model of control systems. Follow these steps for differential equation model.

- Apply basic laws to the given control system.
- Get the differential equation in terms of input and output by eliminating the intermediate variable(s).

Consider the following electrical system as shown in the following figure 5. This circuit consists of resistor, inductor and capacitor. All these electrical elements are connected in series. The input voltage applied to this circuit is Vi and the voltage across the capacitor is the output voltage Vo.

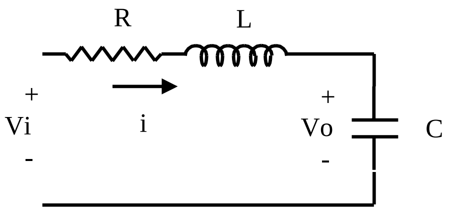


Figure 5: RLC Circuit

Mesh equation for this circuit is

$$V = R + L\frac{d}{dt} + \frac{1}{c}\int dt$$

Substitute, the current passing through capacitor

$$_c = \frac{dV_0}{dt}$$

in the above equation.

$$V = R + L\frac{d}{dt} + V_0$$

$$V = RC\frac{dV_0}{dt} + LC\frac{d^2V_0}{dt^2} + V_0$$

The above equation is a second order differential equation.

1.7 Mechanical Systems:

There are two types of mechanical systems based on the type of motion.

(i) Translational mechanical systems

(ii) Rotational mechanical systems

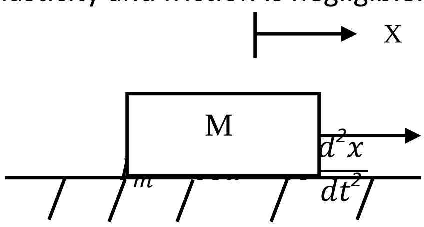
Modeling of Translational Mechanical Systems

Translational mechanical systems move along a straight line. These systems mainly consist of three basic elements. Those are mass, spring and dashpot or damper.

If a force is applied to a translational mechanical system, then it is opposed by opposing forces due to mass, elasticity and friction of the system. Since the applied force and the opposing forces are in opposite directions, the algebraic sum of the forces acting on the system is zero.

Mass

Mass is the property of a body, which stores kinetic energy. If a force is applied on a body having mass **M**, then it is opposed by an opposing force due to mass. This opposing force is proportional to the acceleration of the body. Assume elasticity and friction is negligible.

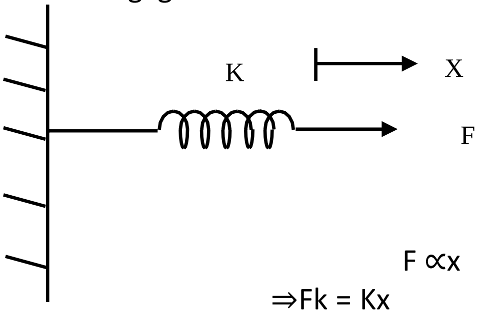


Where,

- **F** is the applied force
- **F_m** is the opposing force due to mass
- **M** is mass
- **a** is acceleration
- **x** is displacement

Spring

Spring is an element, which stores potential energy. If a force is applied on spring **K**, then it is opposed by an opposing force due to elasticity of spring. This opposing force is proportional to the displacement of the spring. Assume mass and friction is negligible.

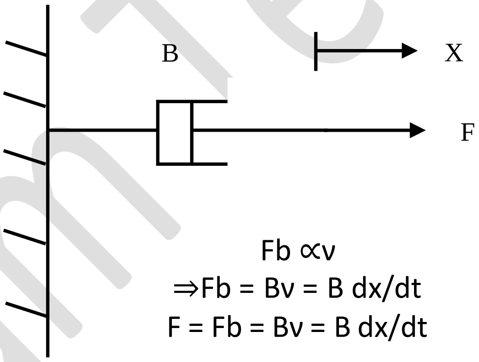


$$F \propto x$$
$$\Rightarrow F_k = Kx$$
$$F = F_k = Kx$$

Where,
- **F** is the applied force
- **F_k** is the opposing force due to elasticity of spring
- **K** is spring constant
- **x** is displacement

Dashpot (Damper)

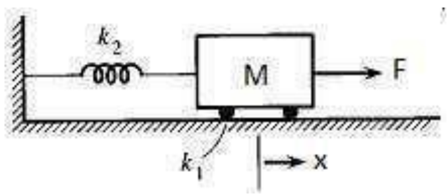
If a force is applied on dashpot **B**, then it is opposed by an opposing force due to **friction** of the dashpot. This opposing force is proportional to the velocity of the body. Assume mass and elasticity negligible.



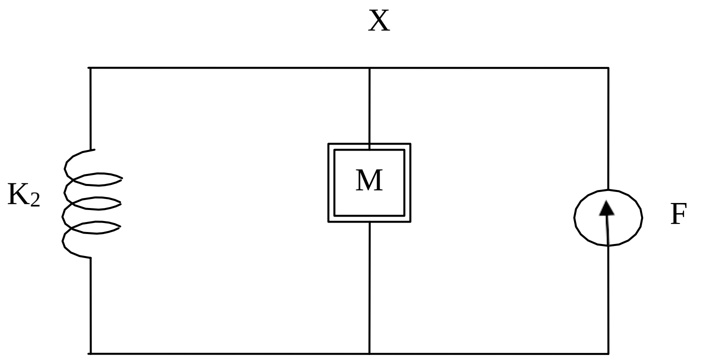
$$F_b \propto v$$
$$\Rightarrow F_b = Bv = B\, dx/dt$$
$$F = F_b = Bv = B\, dx/dt$$

Where,
- **F_b** is the opposing force due to friction of dashpot
- **B** is the frictional coefficient
- **v** is velocity
- **x** is displacement

Ex. 1.3 For the following translational system, find out the mathematical model and write the force equations. Also find out the transfer function.



Solution: As shown in the diagram, a force F is applied on the mass M, due to the force the displacement of the mass is x,
The nodal diagram will be as under



From the nodal diagram, we can write the nodal equations

Therefore in equilibrium condition

$$\boxed{F = M\frac{d^2x}{dt^2} + k_2(x) + k_1(x)}$$

This equation is called the Force equation for the given model. To find out the Transfer function first we need the Laplace Transform of the above equation, with assuming all initial conditions to be zero.

$$F(S) = MS^2X(s) + k_2X(s) + k_1X(S)$$
$$F(S) = [MS^2 + k_2 + k_1]X(s)$$

Therefore the transfer function will be

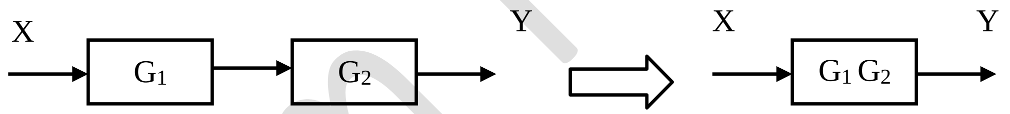
$$\boxed{\frac{X(s)}{F(S)} = \frac{1}{[MS^2 + k_2 + k_1]}}$$

1.8 Block Diagram Reduction Techniques:

The Block diagram Reduction rules are explained as under:

(a) Blocks in series

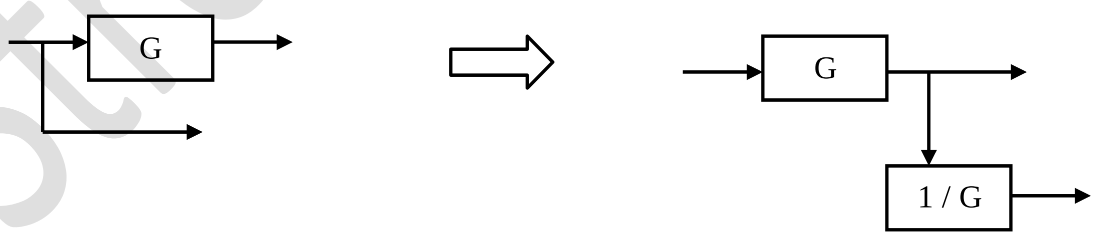
Below drawn Figure shows the basic rule for simplifying blocks in series.



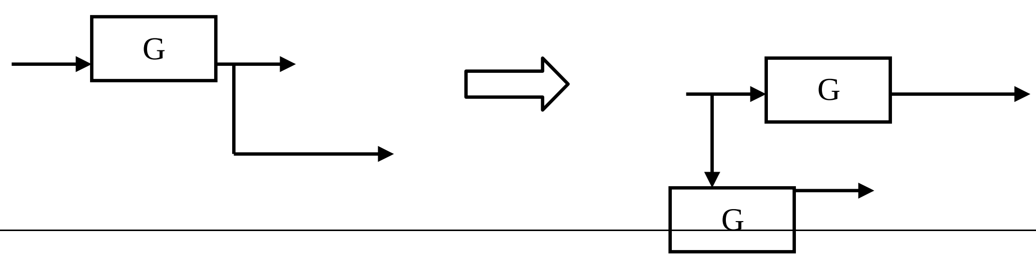
(b) Moving take-off points

As a means of simplifying block diagrams it is often necessary to move takeoff points. The following figures give the basic rules for such movements.

(i) Moving a takeoff point to beyond a block



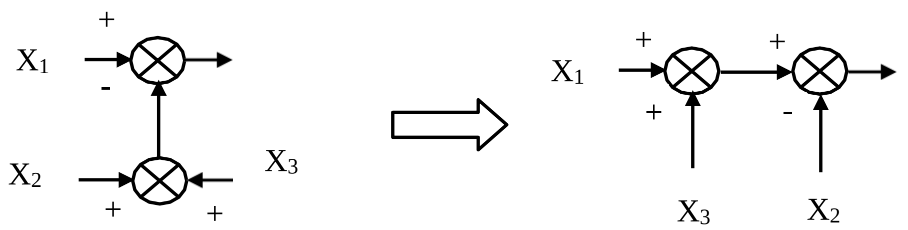
(ii) Moving a takeoff point to ahead of a block



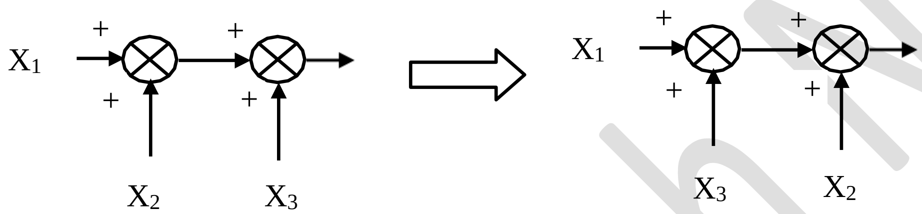
(c) Moving a summing point

As a means of simplifying block diagrams it is often necessary to move summing points. The following figures give the basic rules for such movements.

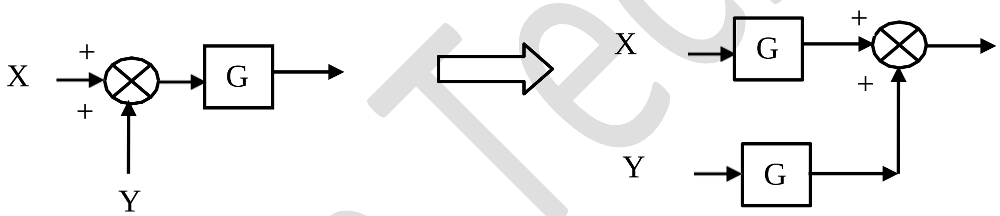
(i) Rearrangement of summing points



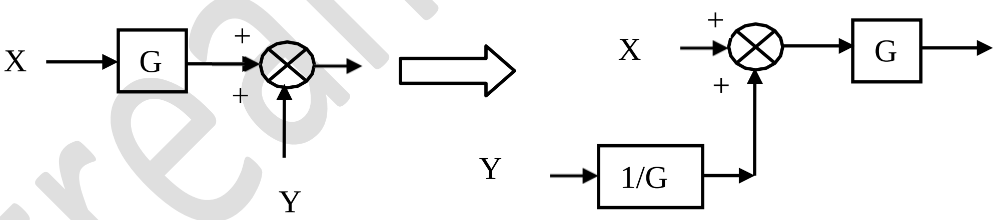
(ii) Interchange of summing points



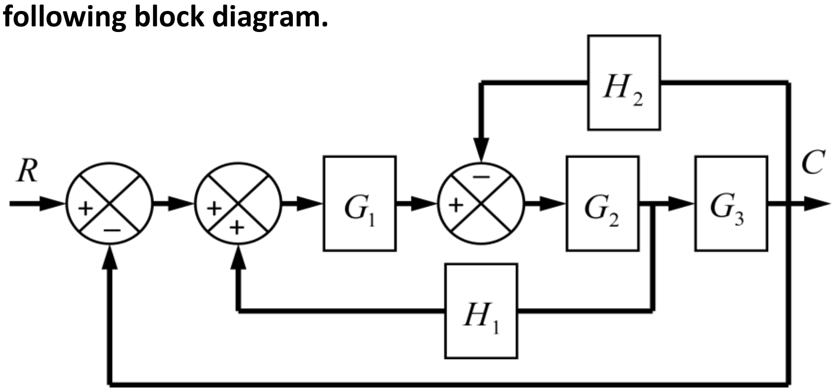
(iii) Moving a summing point beyond of a block



(iv) Moving a summing point ahead of a block

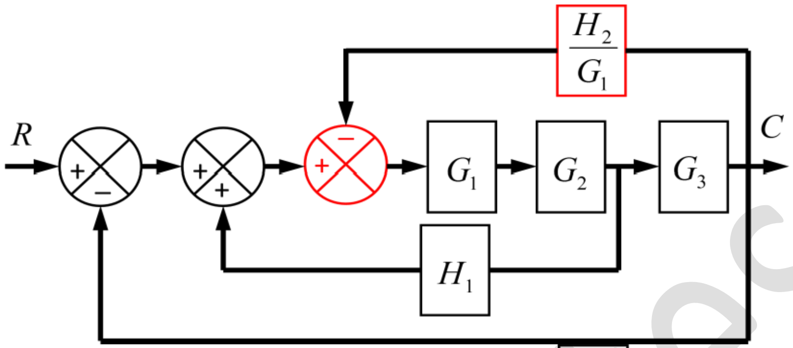


Ex. 1.4 Using the block diagram reduction technique, find the transfer function of the control system represented by the following block diagram.

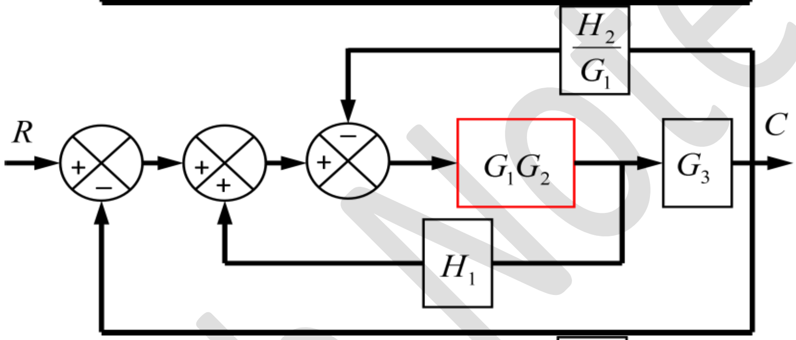


Answer:
We have to find out the Transfer function of the Control System using Block Diagram Reduction Technique.

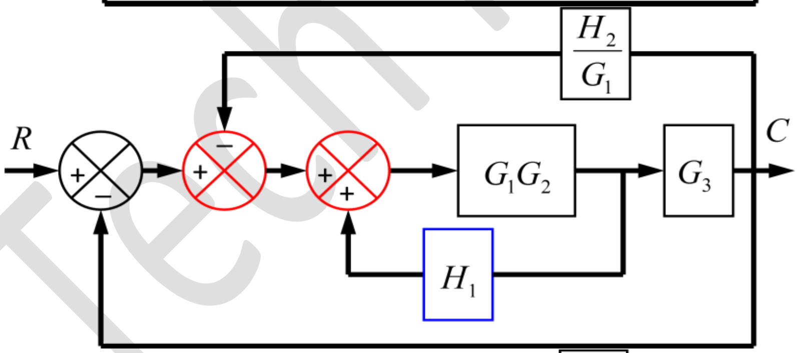
Step:1
Shifting the summing point after block G_1 to before block G_1.



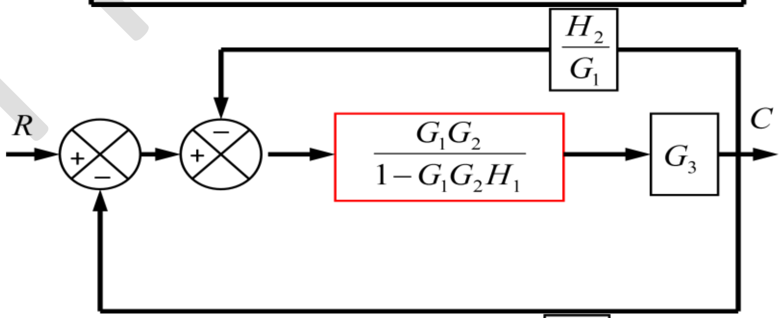
Step:2
Solving G_1 and G_2 in series.



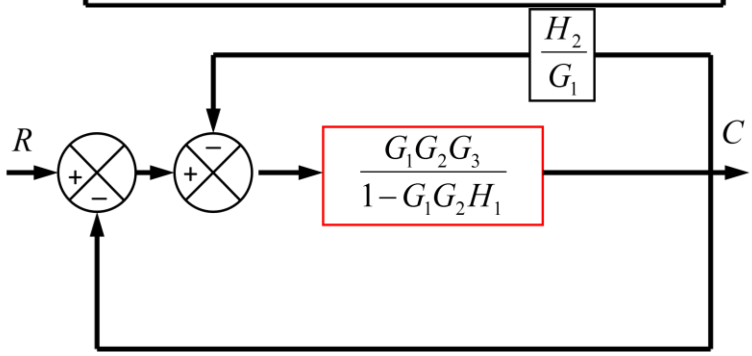
Step:3
Interchanging the two summing points.



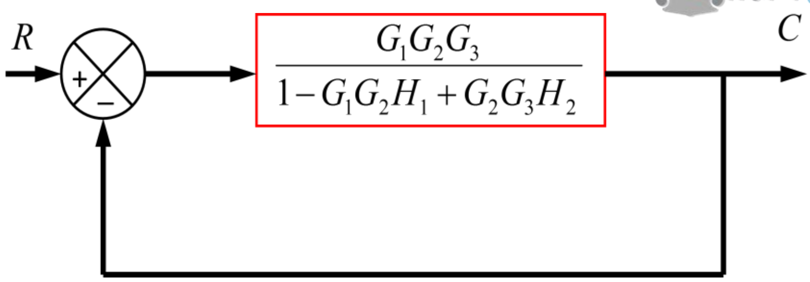
Step:4
Solving the feedback formation.



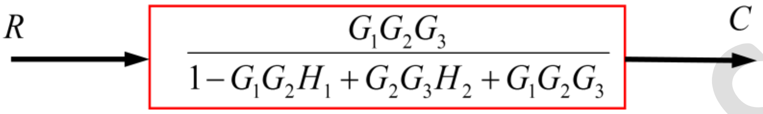
Step:5
Solving the blocks in series.



Step:6
Solving the blocks in feedback.



Step:7
Solving the blocks in feedback again.



Therefore the transfer function of the control system will be as under

$$\frac{C(s)}{R(s)} = \frac{G_1G_2G_3}{1 + G_1G_2H_1 + G_2G_3H_2 + G_1G_2G_3}$$

1.9 Signal Flow Graphs

A signal flow graph is used to represent graphically a set of simultaneous linear algebraic equations. After transforming linear differential equations into algebraic equations in complex variable s the signal flow graph method may be employed for analysis of control systems.

The variables in the algebraic equation are represented by nodes and a graph is formed by connecting the nodes with directed branches in such a way as to satisfy the algebra equations. The signal can flow only in the direction of the arrow of the branch and it is multiplied by a factor indicated along the branch, which happens to be the coefficient of the algebraic equation. The signal flow graph depicts the flow of signals from one point of a system to another in a cause and effect relationships- the variable at the arrow head being the dependent variable.

As an illustration, consider the algebraic equation $x_2 = g_{12}x_1$, where x1 is the independent variable and x2 is the dependent variable. The equation may be represented as a signal flow graph, where signal can flow from x1 to x2 only in the direction of the arrow not the reverse. The magnitude of the signal at x2 is obtained by multiplying the signal x1 with branch gain g12.

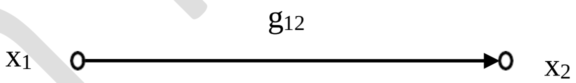


Figure 1.9.1: A simple signal flow graph

A gain formula, known as Mason's gain formula, may be used to obtain the relationships among variables of a control system represented by a signal flow graph. It does not need any reduction of the original graph, as was the case with the block diagram approach.

Node: A node is a point representing a variable or signal, (e.g., x1, x2 in Figure 1.9.1)

Transmittance: The transmittance is a gain, which may be real or complex between two nodes (g12 in Figure 1.9.1).

Branch: A branch is a directed line segment between two nodes. The transmittance is the gain of a branch.

Input node: An input node has only outgoing branches and this represents an independent variable (e.g., x1 in Figure 1.9.1).

Output node: An output node has only incoming branches representing a dependent variable (e.g., x2 in Figure 6). In order to comply with this definition an additional node may be introduced with unity gain to act as an output node (e.g., x3 in Figure 1.9.1).

Mixed node: A mixed node is a node that has both incoming and outgoing branches (e.g., x2, x3)

Path: Any continuous unidirectional succession of branches traversed in the indicated branch direction is called a path.

Loop: A loop is a closed path.

Loop gain: The loop gain is the product of the branch transmittances of a loop.

Non-touching loops: Loops are non-touching if they do not have any common nodes.

Forward path: A forward path is a path from an input node to an output node along which no node is encountered more than once.
Feedback path (loop): A path which originates and terminates on the same node along which no node is encountered more than once is called a feedback path.
Path gain: The product of the branch gains encountered in traversing the path is called the path gain.
Loop gain: The product of the branch gains of the branches forming that loop is called loop gain.

1.9.1 Properties of Signal Flow Graphs

A few important properties of signal flow graphs are noted below:
1. A branch indicates the functional dependence of one variable on another.
2. A node performs summing operation on all the incoming signals and transmits this sum to all outgoing branches.

1.9.2 Signal Flow Graph Algebra

With the help of the fore going terminologies and properties, we can draw the signal flow graph of a linear system. For convenience, the input nodes are placed to the left and the output nodes to the right in the signal flow graph. The independent variables of the equations become the input nodes while the dependent variables become the output nodes in the graph. As already mentioned, the coefficients of the algebraic equations are written as the branch transmittances.
The Mason's gain formula which will be presented shortly is used to find the relationship between any variable considered as input and any other variable taken as output. For applying the Mason's gain formula, we use the following rules:
1. The value of an output node with one incoming branch, as shown in Fig. is found by multiplying the input variable with branch transmittance, x2 = g12x1.
2. The total transmittance of cascaded branches is equal to the gain of the path formed with all the branches. Therefore, the cascaded branches can be replaced by a single branch with transmittance equal to the path gains, as shown in Fig.
3. Parallel branches may be replaced by a single branch with transmittance which is equal to the sum of the transmittances, as shown in Fig.
4. A mixed node may be eliminated, by modifying the graph as shown in Fig.
5. A loop may be eliminated by modifying the graph, as shown in Fig. We find in Fig. that x3 = g23 x2, and x2 = g12 x1 – g32 x3. Hence

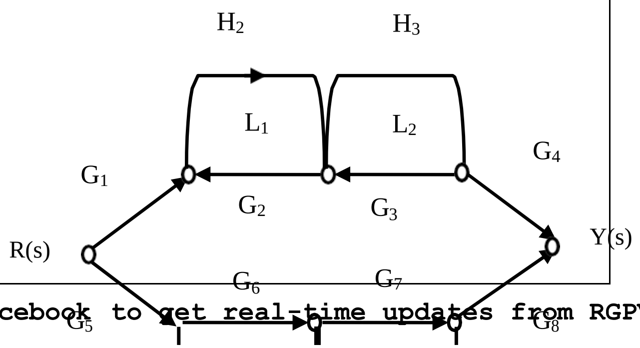
1.10 Mason's Gain Formula

In a control system the transfer functions between any input and any output may be found by Mason's Gain formula. Mason's gain formula is given by

$$T = \frac{C(s)}{R(s)} = \frac{1}{\Delta}\sum_{n} P\Delta$$

Where P_k = path gain of kth forward path
Δ = determinant of the graph
= 1-$\sum(gan\ \mathrm{of}\ the\ ndvdual\ loops)$+ $\sum(gan\ \mathrm{of}\ two\ non\ touc?ng\ loops)$- $\sum(gan\ \mathrm{of}\ three\ non\ touchng\ loops$+….
Δ_K = K^{th} forward path determinant of the graph
= value of Δ for that part of the block diagram that do not touch the K^{th} forward path

Ex. 1.5 Find the transfer function of the Signal flow graph shown in the picture.

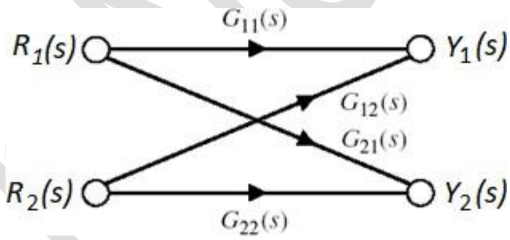


The paths connecting the input R(s) and output Y(s) are
P1 = G1G2G2G4 (path 1) and P2 = G5G6G7G8 (path 2).
There are four self-loops:
L1 = G2H2, L2 = H3G3, L3 = G6f6, and L4 = G7H7.
Loops L, and L2 do not touch L3 and L4. Therefore, the determinant is
A = 1 - (Li + L2 + L3 + L4) + (LtL3 + I4L4 + L2L3 + L2L4).
The cofactor of the determinant along path 1 is evaluated by removing the loops that touch path 1 from A.
Hence, we have
L1 = L2 = 0 and Δ2 = 1 – (L3 + L4)
Similarly, the cofactor for path 2 is
Δ1 = 1 – (L1 + L2)
Therefore, the transfer function of the system is

$$\frac{C(s)}{R(s)} = T(S) = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta}$$

Ex. 1.6 Find out the overall gain of the signal flow graph shown in figure.

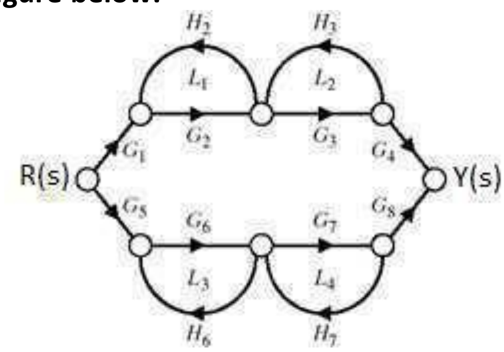
Solution:



This Signal flow graph is a two input two output signal flow graph. Therefore, the gain will be…

$$Y_1(s) = G_{11}(s).R_1(s) + G_{12}(s).R_2(s)$$
$$Y_2(s) = G_{21}(s).R_1(s) + G_{22}(s).R_2(s)$$

Ex. 1.7 Find out the overall gain of the signal flow graph shown in figure below.



We will solve this question by Masons gain formula.
Number of forward paths are two

$P_1 = G_1G_2G_3G_4$ and corresponding $\Delta_1 = 1 - L_3 - L_4$

$P_2 = G_5G_6G_7G_8$and $\Delta_2 = 1 - L_1 - L_2$

And overall

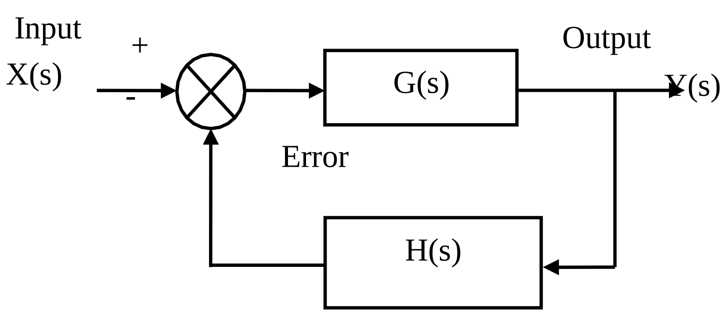
$$\Delta = 1 - L_1 - L_2 - L_3 - L_4 + L_1L_3 + L_1L_4 + L_2L_3 + L_2L_4$$

Therefore according to the Masson's Gain Formula,

$$\frac{Y(S)}{R(s)} = \frac{[G_1G_2G_3G_4(1 - L_3 - L_4)] + [G_5G_6G_7G_8(1 - L_1 - L_2)]}{1 - L_1 - L_2 - L_3 - L_4 + L_1L_3 + L_1L_4 + L_2L_3 + L_2L_4}$$

1.11 Feedback Characteristics of Control Systems

Consider the block diagram of the open-loop and the closed-loop system shown below.



Open loop control System　　　　　　　　　　Closed loop control System

For open-loop system, $C(s) = G(s)R(s)$

For closed-loop system, $C(s) = G(s)E_a(s) = G(s)[R(s) - H(s)C(s)]$

Hence, we have, $C(s) = \dfrac{G(s)}{1+G(s)H(s)}R(s)$ and, $E_a(s) = \dfrac{1}{1+G(s)H(s)}R(s)$

It may be observed from above equations that in order to reduce error, the loop-gain $G(s)H(s)$ should be made large over the range of frequencies of interest, i.e., $|G(s)H(s)| \gg 1$.

1.11.1 Reduction of parameter variations by use of feedback

One important property of negative feedback systems is that the reduction in the sensitivity to the variation in the parameters of the forward path. In control systems, it is important that the transfer function of the closed-loop control system be relatively insensitive to small changes in the values of the parameters of the components in the forward path of the system.

Let μ be a parameter of G(s). Then the sensitivity of G(s) with respect to the parameter μ can be defined as,

$$S_{\mu}^{G} = \frac{\text{Fractional change in } G(s)}{\text{Fractional change in } \mu} = \frac{\Delta G / G}{\Delta\mu / \mu} = \frac{\mu}{G} \cdot \frac{\Delta G}{\Delta \mu}.$$

Now, $$T(s) = \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)};$$

$$S_{\mu}^{T} = \frac{\mu}{T}\cdot\frac{\Delta T}{\Delta\mu} = \frac{\mu}{G}\cdot\frac{G}{T}\cdot\frac{\Delta G}{\Delta\mu}\frac{\Delta T}{\Delta G} = S_{\mu}^{G}\cdot(1+GH)\cdot\frac{1+\cancel{GH}-\cancel{GH}}{(1+GH)^2} = \frac{S_{\mu}^{G}}{1+G(s)H(s)}$$

Thus feedback has reduced sensitivity in the variation in μ by the factor $\dfrac{1}{1+GH}$.

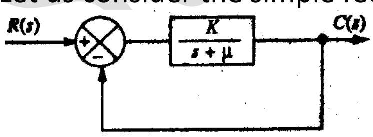
Again, $$S_{\mu}^{T} = \frac{\mu}{T}\cdot\frac{\Delta T}{\Delta\mu} = \frac{\mu}{H}\cdot\frac{H}{T}\cdot\frac{\Delta H}{\Delta\mu}\frac{\Delta T}{\Delta H} = S_{\mu}^{H}\cdot\frac{H(1+GH)}{\cancel{G}}\cdot\frac{-G.\cancel{G}}{(1+GH)^2} = -S_{\mu}^{H}\cdot\frac{GH}{1+GH} \cong -S_{\mu}^{H}.$$

It may be observed that, the magnitude of two sensitivities are nearly equal for the variation of parameter in the feedback path. Thus, feedback does not reduce the sensitivity to variation in the parameter in feedback path.

Therefore, we can conclude that, G(s) in a closed-loop control system may be less rigidly specified. Whereas, on the other hand, we must be careful in accuracy of H(s) in the feedback loop.

1.11.2 Control over system dynamics by use of feedback

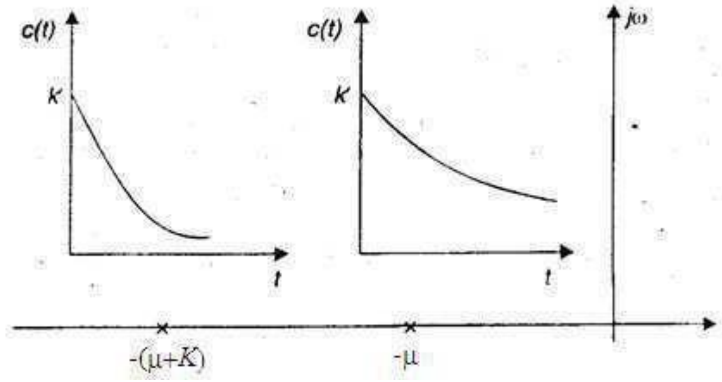
Let us consider the simple feedback system shown below.



The open-loop transfer function is, $G(s) = \dfrac{K}{s+\mu}$.

The impulse response for the non-feedback system would be, $c(t) = Ke^{-\mu t}u(t) = Ke^{-t/\tau_1}u(t)$.

The closed-loop transfer function of the above system is, $T(s) = \dfrac{K}{s+\mu+K}$.



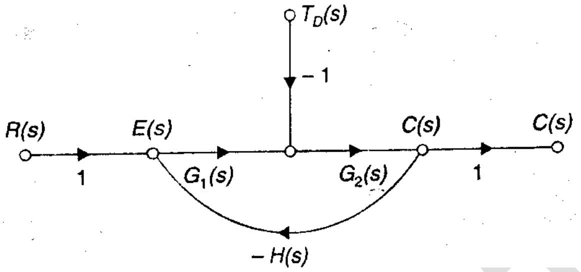
The impulse response of the closed-loop system is, $c(t) = Ke^{-(\mu+K)t}u(t) = Ke^{-t/\tau_2}u(t)$.

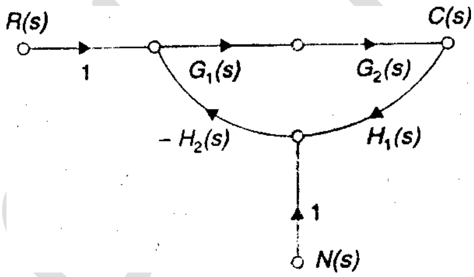
The location of the pole and the dynamic response of the non-feedback and feedback system are shown in Figure below.

It is seen that the time-constant of open-loop system is $\tau_1 = 1/\mu$ and that of closed-loop system is $\tau_2 = 1/(\mu + K)$. As the time-constant of closed-loop system is less, its dynamic response is faster than the same of the open-loop system.

1.11.3 Control of the effect of disturbance signal by use of feedback

A. Disturbance in the forward path





$$\frac{C_d(s)}{T_d(s)} = \frac{-G_2(s)}{1+G_1(s)G_2(s)H(s)} \cong \frac{-1}{G_1(s)H(s)}; \quad \text{or,} \quad C_d(s) = \frac{-T_d(s)}{G_1(s)H(s)}$$

If $G_1(s)$ is made very large, the effect of disturbance on the output will be very small.

B. Disturbance in the feedback path

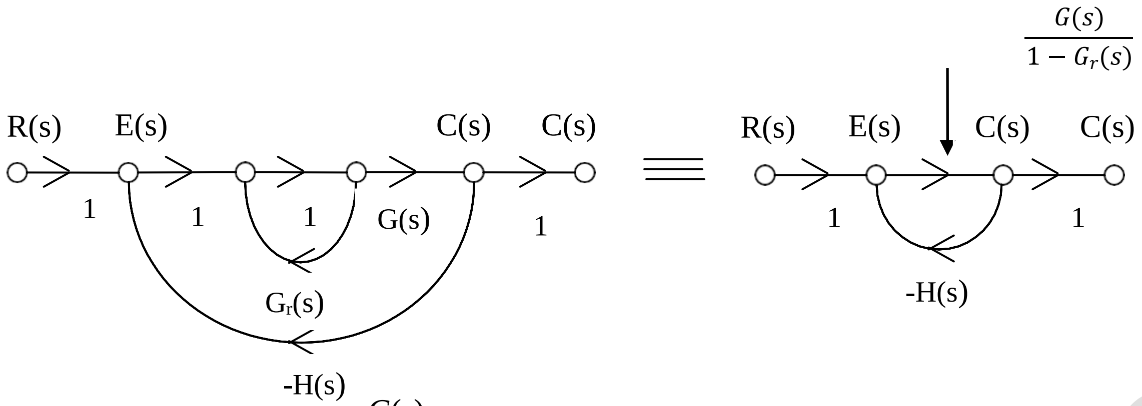
$$\frac{C_n(s)}{N(s)} = \frac{-G_1(s)G_2(s)H_2(s)}{1+G_1(s)G_2(s)H_1(s)H_2(s)} \cong \frac{-1}{H_1(s)}$$

Therefore, the effect of noise on output is, $C_n(s) \cong \dfrac{-1}{H_1(s)} \cdot N(s)$.

Thus, to get the optimum performance of the control system, the measurement sensor should be designed such that feedback gain $H_1(s)$ is maximum. This is equivalent to maximizing the signal to noise ration SNR of the sensor.

1.11.4 Regenerative Feedback

The regenerative feedback is sometimes used for increasing the loop gain of the feedback system. Figure in the following shows a feedback system where regenerative feedback occurs in the inner loop.



The open-loop gain is, $G_o(s) = \dfrac{G(s)}{1-G_a(s)}$.

The system response is obtained as, $C(s) = \dfrac{R(s)\cdot G(s)/1-G_a(s)}{1+G_a(s)G(s)/1-G_a(s)} = \dfrac{R(s)\cdot G(s)}{1-G_a(s)+G(s)H(s)}$

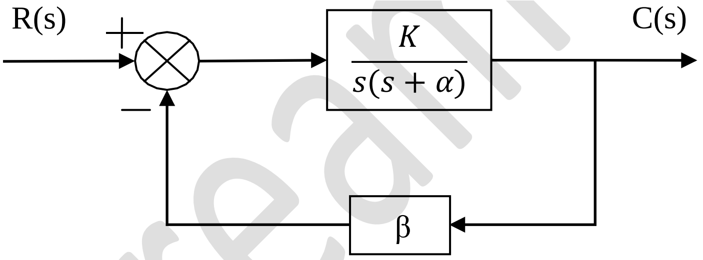
When, $G_a(s) \gg 1$, $C(s) \cong \dfrac{R(s)}{H(s)}$. Due to high loop gain provided by the inner regenerative feedback loop, the closed-loop transfer function becomes insensitive to G(s).

Ex. 1.8 A position control system is shown below. Assume, *K*=10, $\alpha = 2$, $\beta = 1$. Evaluate: $S_K^T, S_\alpha^T, S_\beta^T$. For $r(t) = 2\cos 0.5t$ and a 5% change in K, evaluate the steady-state response and the change in steady-state response.

Here, $G(s) = \dfrac{K}{s(s+\alpha)}$, and $H(s) = \beta$

$$S_K^G = \frac{K}{G}\cdot\frac{dG}{dK} = s(s+\alpha)\cdot\frac{1}{s(s+\alpha)} = 1;$$

$$S_\alpha^G = \frac{\alpha}{G}\cdot\frac{dG}{d\alpha} = \frac{-\alpha}{s+\alpha} = \frac{-2}{s+2};\quad S_\beta^H = \frac{\beta}{H}\cdot\frac{dH}{d\beta} = 1$$



$$S_K^T = \frac{S_K^G}{1+G(s)H(s)} = \frac{s(s+\alpha)}{s(s+\alpha)+K} = \frac{s^2+2s}{s^2+2s+10}$$

Therefore, $$S_\alpha^T = \frac{S_\alpha^G}{1+G(s)H(s)} = \frac{-\alpha}{s+\alpha}\cdot\frac{s(s+\alpha)}{s(s+\alpha)+K} = \frac{-2s}{s^2+2s+10}$$

$$S_\beta^T = \frac{-S_\beta^H\cdot G(s)H(s)}{1+G(s)H(s)} = \frac{-K}{s(s+\alpha)+K} = \frac{-10}{s^2+2s+10}$$

Now, $T(s) = \dfrac{K}{s^2+\alpha s+K\beta} = \dfrac{10}{s^2+2s+10}$; At $s = j0.5$, $T(j0.5) = 1.02e^{-j0.102}$

Thus, $c_{ss}(t) = 2.04\cos(0.5t - 0.102)$

Again, $S_K^T = \frac{K}{T}\cdot\frac{\Delta T}{\Delta K} \Rightarrow \frac{\Delta T}{T} = S_K^T \cdot \frac{\Delta K}{K} = \frac{s^2+2s}{s^2+2s+10}\cdot 0.05$

$\Rightarrow \Delta T(s) = \frac{s^2+2s}{s^2+2s+10}\times 0.05\times\frac{10}{s^2+2s+10} = \frac{0.5s(s+2)}{(s^2+2s+10)^2}$; $\Rightarrow \Delta T(j0.5) = 0.005e^{-j4.672}$

Thus, $\Delta c_{ss}(t) = \Delta T(j0.5)\times 2\cos 0.5t = 0.01\cos(0.5t - 4.672)$ {Answer}

RGPV University
Department of Electronics and Communication Engineering
Sub. Code: EC 404 Sub. Name: Control System

Unit 2

Syllabus: Time response analysis
Standard test signals, time response of 1st order system, time response of 2nd order system, steady-state errors and error constants, effects of additions of poles and zeros to open loop and closed loop system.
Time domain stability analysis
Concept of stability of linear systems, effects of location of poles on stability, necessary conditions for stability, Routh-Hurwitz stability criteria, relative stability analysis, Root Locus concept, guidelines for sketching Root-Locus.

Time Response Analysis

3.1: Standard Test Signals
For time response studies, some specific input test signals are applied to the control systems. These are described below:
(a) Step Function:

It is described as sudden application of input signal, and is given by

$$u(t) = \begin{cases} K \text{ for } t \geq 0 \\ 0 \text{ for } t < 0 \end{cases}$$

If K=1 the function is called Unit Step Function. Its Laplace transform is given by

$$L[\mathrm{u(t)}] = \mathrm{U(S)} = 1 / \mathrm{s}$$

Step function is also called the displacement function.

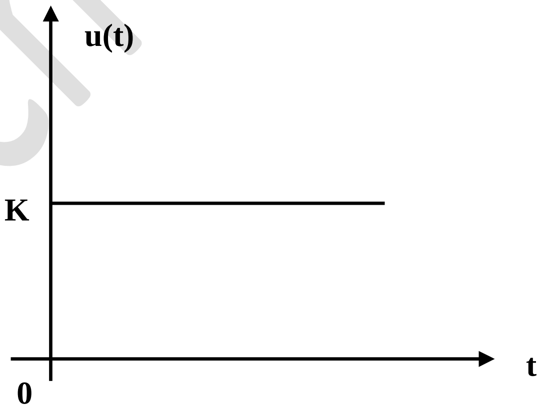


Figure 2.1.1 Step Function

(b) Ramp Function:
Ramp function starts from origin and increases or decreases linearly with time as shown in figure. It is described as gradual application of input signal, and is given by

$$r(t) = \begin{cases} K.t \text{ for } t \geq 0 \\ 0 \text{ for } t < 0 \end{cases}$$

If K=1 the function is called Unit Ramp Function. Its Laplace transform is given by

$$L[\mathrm{r(t)}] = \mathrm{R(s)} = 1 / \mathrm{s}^2$$

Ramp function is also called the Velocity function.

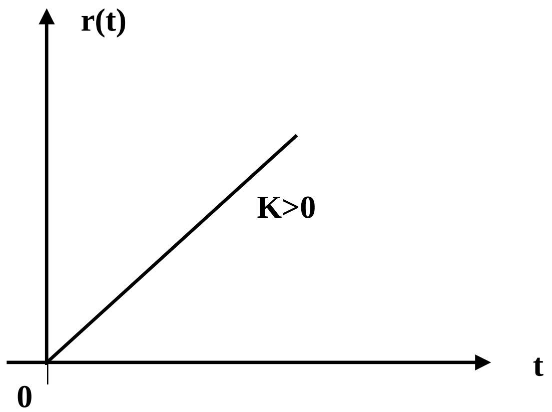


Figure 2.1.2 Ramp Function

(c) Parabolic Function:

Ramp function starts from origin and increases or decreases with time as shown in figure. It is described as more gradual application of input signal, and is given by

$$r(t) = \begin{cases} K.t^2/2 \text{ for } t \geq 0 \\ 0 \text{ for } t < 0 \end{cases}$$

If K=1 the function is called Unit Parabolic Function. Its Laplace transform is given by

$$L[\mathrm{r(t)}] = \mathrm{R(s)} = 1 / s^3$$

Parabolic function is also called the Acceleration function.

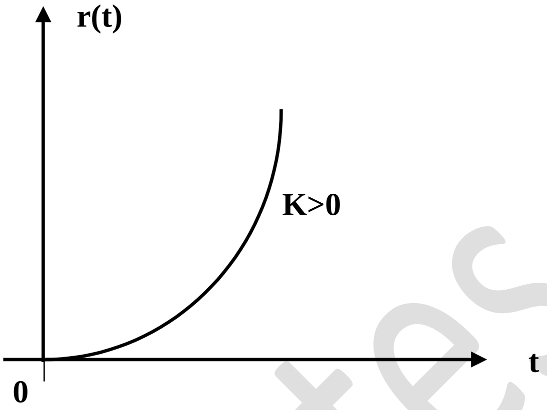


Figure 2.1.3 Parabolic Function

(d) Impulse or delta Function:

In case of Impulse function, the input is applied as a shock for a very short duration. The impulse function is given by:

$$r(t) = \begin{cases} K \text{ for } t = 0 \\ 0 \text{ for } t \neq 0 \end{cases}$$

If K=1 the function is called Unit Impulse Function and it is denoted by δ(t). It's Laplace transform unity.

$$L[\delta\ (\mathrm{t})] = 1$$

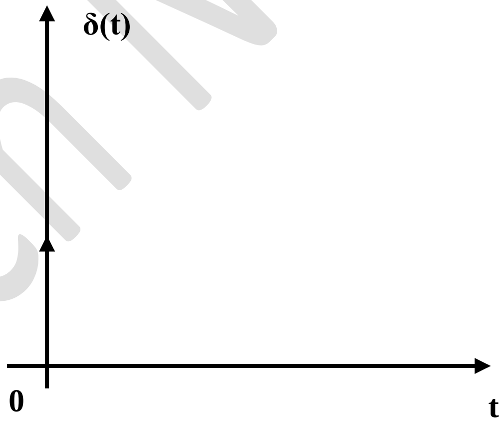


Figure 2.1.4 Impulse Function

3.2: Time response of first order system:

A first order control system is one wherein highest power of S in the denominator equals 1. Suppose we have a First Order control system as shown in figure 2.2.1. Then the transfer function is given by

$$\frac{C(s)}{R(s)} = \frac{1/sT}{1 + 1/sT}$$

Or

$$\frac{C(s)}{R(s)} = \frac{1}{1 + sT} \qquad \text{...2.2.1}$$

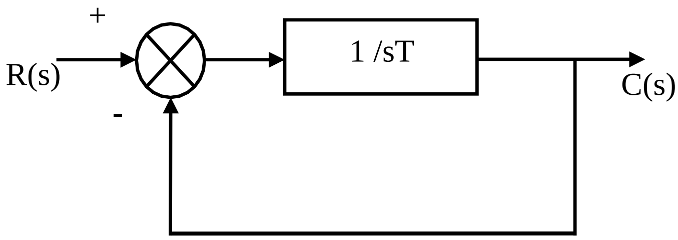


Figure 2.2.1: System with negative feedback

In equation 2.2.1 power of s is one in the denominator term. Hence, the above transfer function is of the first order and the system is said to be the first order system.

Where,

- C(s) is the Laplace transform of the output signal c(t),
- R(s) is the Laplace transform of the input signal r(t), and
- T is the time constant.

(a) Response of the 1st order system when an Impulse function is applied as input:

Consider the unit impulse signal as an input to the first order system. ie. r(t)=δ(t)
Apply Laplace transform on both the sides. R(s)=1
Substitute, R(s)=1 in equation 2.2.1.

$$C(s) = 1.\frac{1}{1+sT}$$

$$C(s) = \frac{1}{T}\frac{1}{s+1/T}$$

Inverse Laplace

$$c(t) = \frac{1}{T}\mathrm{e}^{\frac{-t}{T}}$$

The unit impulse response is shown in the figure 2.2.2.

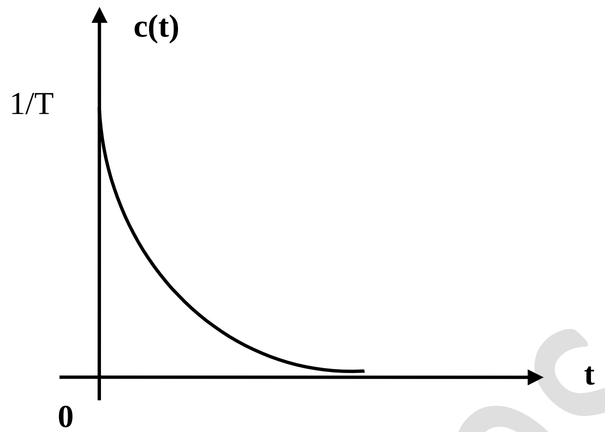


Figure 2.2.2 Impulse Response of First Order System

The unit impulse response, c (t) is an exponential decaying signal for positive values of 't' and it is zero for negative values of 't'.

(b) Response of the 1st order system when a unit step function is applied as input:

Consider the unit step signal as an input to first order system.
So $r(t) = u(t)$
Apply Laplace transform on both the sides.

$$\mathrm{R(s)} = 1/\mathrm{s}$$

Consider the equation 2.2.1, and substitute the value of R(s) for unit step input.

$$C(s) = \frac{1}{s}\frac{1}{1+sT} = \frac{1}{s} - \frac{T}{1+sT}$$
$$= \frac{1}{s} - \frac{T}{s+1/T}$$

Therefore $$c(t) = (1-\mathrm{e}^{\frac{-t}{T}})u(t)$$

Error $$\mathrm{e}(t) = r(t) - c(t) = \mathrm{e}^{\frac{-t}{T}}$$

Steady State Error $$\mathrm{e}_{ss} = \lim_{t\to\infty}\mathrm{e}^{\frac{-t}{T}} = 0$$

The response of the first order system will be as shown in figure 2.2.3

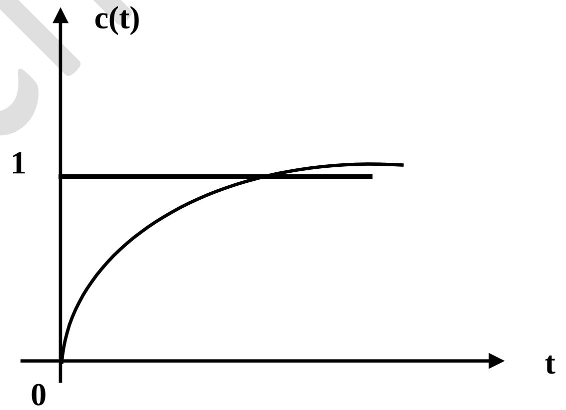


Fig 2.2.3 Unit step Response of First Order System

The value of the **unit step response, c(t)** is zero at t = 0 and for all negative values of t. It is gradually increasing from zero value and finally reaches to one in steady state. So, the steady state value depends on the magnitude of the input.

(c) Response of the 1st order system when a unit ramp function is appliedas input:

Consider the unit ramp signal as an input to first order system.
So $r(t) = t$
Apply Laplace transform on both the sides.

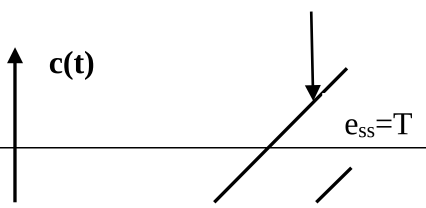
$$\mathrm{R(s)} = 1/\mathrm{s}^2$$

Consider the equation 2.2.1, and substitute the value of R(s) for unit step input.

$$C(s) = \frac{1}{s^2}\frac{1}{1+sT} = \frac{1-sT}{s^2} - \frac{T^2}{1+sT} = \frac{1}{s^2} - \frac{T}{s} + \frac{T}{s+1/T}$$

Then

$$c(t) = t - T + T\mathrm{e}^{\frac{-t}{T}}$$



Therefore Error $\quad e(t) = t - (t - T + Te^{\frac{-t}{T}})$

or $\quad e(t) = T(1 - e^{\frac{-t}{T}})$

Steady State Error $\quad e_{ss} = \lim_{t\to\infty}(T - Te^{\frac{-t}{T}}) = T$

Fig 2.2.4 Unit Ramp Response of First Order System

The response of the first order system will be as shown in figure 2.2.4

The value of the unit ramp response, c(t) is zero at t = 0 and for all negative values of t. It is gradually increasing from zero value.

3.3: Second Order Control System:

Let us consider a Second Order control system as shown in figure 2.3.1. Then the close loop transfer function is given by

$$\frac{C(s)}{R(s)} = \frac{G(S)}{1 + G(s)H(s)}$$

Or

$$\frac{C(s)}{R(s)} = \frac{\frac{\omega_n^2}{s(s+2\omega_n)}}{1 + \frac{\omega^2_n}{s(s+2\omega_n)}}$$

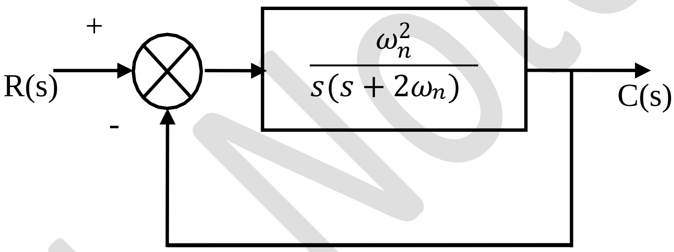


Figure 2.3.1: A Second Order System

Therefore

$$\boxed{\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\omega_n + \omega^2{}_n}} \qquad ...2.3.1$$

Unit Step Response of a Second Order control System:

For s unit step input r(t) = 1 and R(s) = 1/s. Therefore from equation 2.3.1.

$$C(s) = \frac{1}{s}\frac{\omega_n^2}{s^2 + 2\omega_n + \omega^2_n} = \frac{1}{s}\frac{\omega_n^2}{(s + \ _n)^2 + \omega^2_n - {}^2\omega^2_n}$$

Therefore

$$C(s) = \frac{1}{s} - \frac{s + 2\omega_n}{(s + \omega_n)^2 + \omega^2_n(1 - {}^2)} = \frac{1}{s} - \frac{s + 2\omega_n}{(s + \ _n)^2 + \omega_d^2} \qquad \omega_d = \omega_n\sqrt{1 - {}^2}$$

$$C(s) = \frac{1}{s} - \frac{s + \omega_n}{(s + \omega_n)^2 + \omega^2{}_d} - \frac{\omega_n}{(s + \omega_n)^2 + \omega^2{}_d}$$

$$C(s) = \frac{1}{s} - \frac{s + \omega_n}{(s + \omega_n)^2 + \omega^2{}_d} - \frac{\omega_n}{\omega_d}\frac{\omega_d}{(s + \omega_n)^2 + \omega^2_d}$$

Taking the inverse Laplace transform

$$c(t) = 1 - e^{-\omega_n t}\cos{}_d t - \frac{\omega_n}{\omega_d}e^{-\omega_n t}\sin{}_d t \qquad ...2.3.2$$

Since $\omega_d = \omega_n\sqrt{1 - {}^2}$

Therefore $\quad c(t) = 1 - \frac{e^{-\omega_n t}}{\sqrt{1 - {}^2}}[\sqrt{1 - {}^2}\cos\omega_d t + \sin\omega_d t] \qquad ...2.3.3$

Substituting $sin = \sqrt{1 - {}^2}$ and $cos =$

$$c(t) = 1 - \frac{e^{-\omega_n t}}{\sqrt{1 - {}^2}}[sin\ cos\ \omega_d t + cos\ sin\ \omega_d t]$$

$$c(t) = 1 - \frac{e^{-\omega_n t}}{\sqrt{1 - {}^2}} sin(\omega_d t +) \qquad ...2.3.4$$

Now $= tan^{-1}\frac{\sqrt{1-{}^2}}{}$, then

$$c(t) = 1 - \frac{e^{-\omega_n t}}{\sqrt{1 - {}^2}} sin\ (\omega_d t + tan^{-1}\frac{\sqrt{1-{}^2}}{})$$

$$\boxed{c(t) = 1 - \frac{e^{-\omega_n t}}{\sqrt{1 - {}^2}} sin\ (\omega_n \sqrt{1-{}^2}t + tan^{-1}\frac{\sqrt{1-{}^2}}{})} \qquad ...2.3.5$$

Where	ξ	=	Damping ratio of the system
	ω_n	=	Natural Frequency of oscillation
	ω_d	=	Damped Frequency of oscillation

Now, the error is given by $e(t) = r(t) - c(t)$, therefore

$$e(t) = 1 - [1 - \frac{e^{-\omega_n t}}{\sqrt{1 - {}^2}} sin\ (\omega_n \sqrt{1-{}^2}t + tan^{-1}\frac{\sqrt{1-{}^2}}{})] \qquad ...2.3.5$$

$$\boxed{e(t) = \frac{e^{-\omega_n t}}{\sqrt{1 - {}^2}} sin\ (\omega_n \sqrt{1-{}^2}t + tan^{-1}\frac{\sqrt{1-{}^2}}{})}$$

The term $\xi\omega_n = \alpha$ is called the damping factor or the damping coefficient.
The steady state value of output is given by

$$C_{ss} = \lim_{t\to\infty} c(t) = 1$$

Response of a second order system for different values of

We know that the transfer function of a second order system is

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n + \omega_n^2}$$

Therefore the Characteristics equation is

$$s^2 + 2\xi\omega_n + \omega_n^2 = 0$$

Here	ξ	=	Damping ratio of the system		
	ω_n	=	Natural Frequency of oscillation		
	α	=	$\xi\omega_n$	=	damping factor and
	ω_d	=	$\sqrt{1-\xi^2}$	=	Damped Frequency of oscillation

From the characteristics equation

$$s = -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2} = -\alpha \pm j\omega_d$$

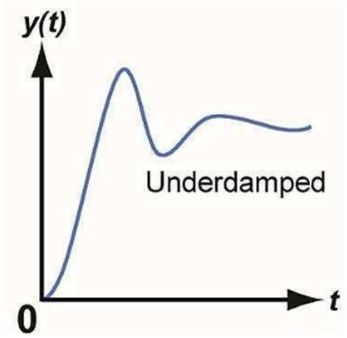
Case I **When** $0 < \ < 1$

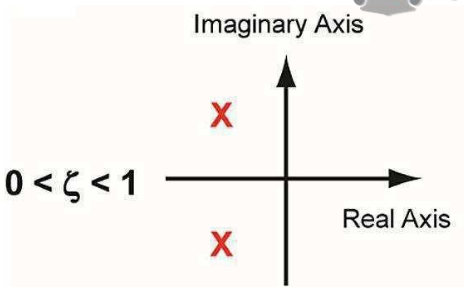
When $0 < \xi < 1$

(i) Roots of the characteristics equation are complex with negative real part.

(ii) The system is called under damped.

The system is stable and for unit step input, its output will be as shown in figure below.





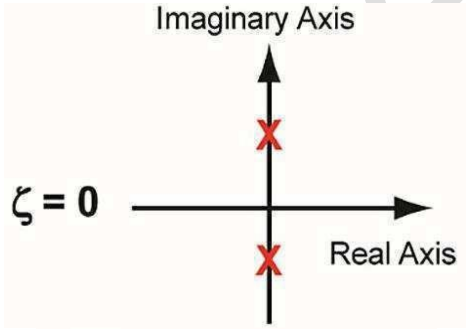
Case II **When** $= 0$

When $\xi = 0$, $s = \pm j\omega_n$

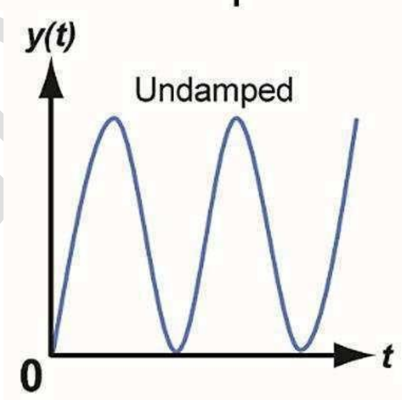
(i) Roots of the characteristics equation will be imaginary and conjugate.

(ii) $\omega_d = \omega_n$

(iii) System is un-damped.



The system is marginally stable and for unit step input, its output will be as shown. Since the damping factor is zero i.e. there will not be any damping, therefore system will be un-damped and we will get the continuous oscillations in the output.



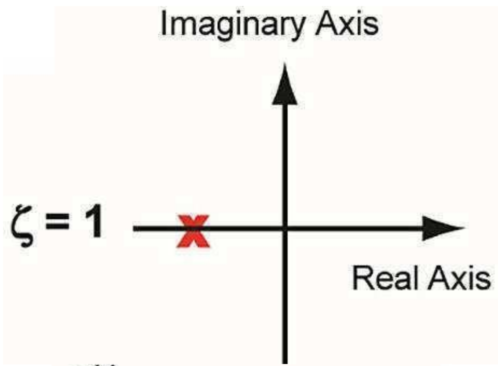
Case III **When** $= 1$

When $\xi = 1$, $s = -\omega_n, -\omega_n$

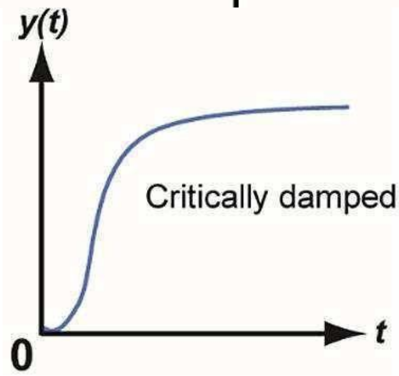
(i) Roots of the characteristics equation are equal real and negative.

(ii) $\omega_d = 0$

(iii) System is critically damped.



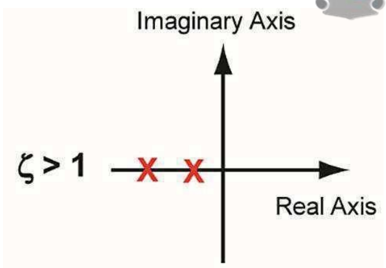
The system is stable and for unit step input, its output will be as shown.



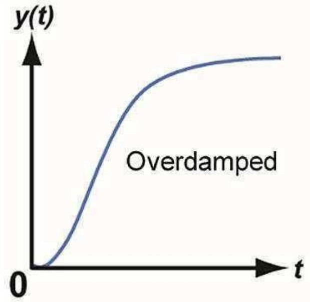
Case IV **When** > 1

When $\xi > 1$,
(i) Roots of the characteristics equation are unequal real and negative.

(ii) System is over damped.

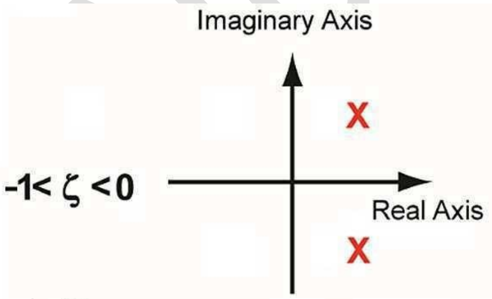


The system is stable but response is sluggish i.e. large time is taken by the output to reach the steady state. For unit step input, its output will be as shown.

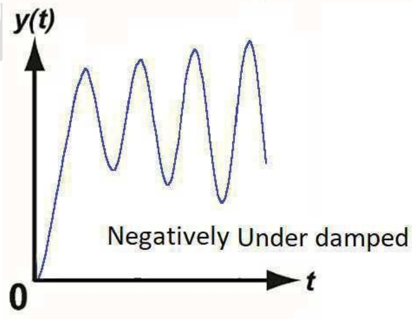


Case V When $-1 < \ < 0$
When$-1 < \xi < 0$,
(i) Roots of the characteristics equation are complex conjugate with positive real part.

(ii) System is called negatively under damped.



The system is unstable. The amplitude of the response increases exponentially. For unit step input, its output will be as shown.



3.4: System Time Response:

After applying the excitation at the input terminals of the system, an output is produced at the output terminals, which varies with time. The response of a second order system for unit step input is shown in figure below:

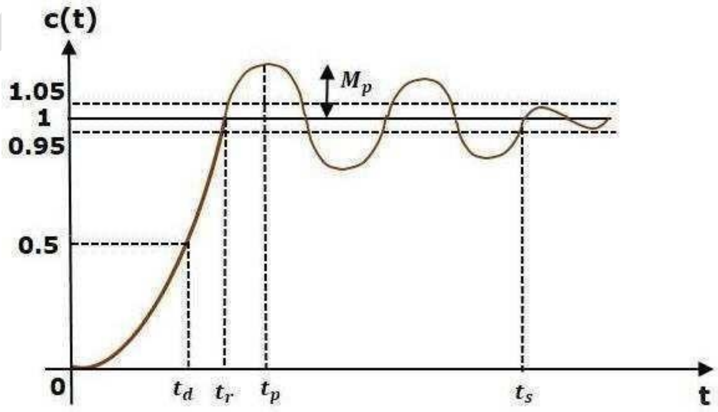


Figure 2.4.1 Output due to Unit step input Vs. Time

Following terms may be defined related to the above figure:

(i) Delay Time (T_d):

The time taken by the system output to reach 50% of its final value is known as delay time.

(ii) Rise Time (T_r):

The time taken by the system output to reach from 10% to 90% of its final value of the output is known as rise time. 0% to 100% & 5% to 95% are also used. For over-damped systems 10% to 90% is used.

(iii) Settling Time (T_s):

The time taken by the system response to settle down and stay within ±2% of its final value is known as settling time.

(iv) Peak Time (T_p):

The time taken by the system response to reach the first maximum value is known as peak time.

(v) Overshoot:

For a step input the ratio of maximum value of the output to the final value of the output is called the overshoot of the system.

The % overshoot is defined as

$$\%\,Overshoot = \frac{Maximum\ output\ overshoot/\ \ Final\ value\ of\ output}{Step\ input} \times 100$$

(vi) Steady State Error:

For a step excitation the difference between the desired output and the final value of the output of any system is termed as steady state error of the system.

3.5 : Analysis of Steady State Error:

A simple closed loop control system with negative feedback is shown in figure 2.5.1.

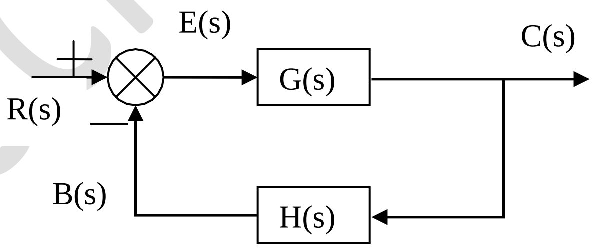


Figure 2.5.1 A closed loop control system

From the figure

$E(s) = R(S) - B(s)$

$\quad\ = R(S) - C(s)H(s)$

$\quad\ = R(S) - E(s)G(s)H(s)$

Therefore

$E(s)[1 + G(s)H(s)] = R(S)$

$$E(s) = \frac{R(S)}{1 + G(s)H(s)}$$

For unity feedback system, i.e. $H(s) = 1$,

$$E(s) = \frac{R(S)}{1 + G(s)}$$

The $E(s)$ is in laplace domain. The error $\mathrm{e}(t)$will be in time domain.

At steady state $t = \infty$

Then steady state error

$$\mathrm{e}_{ss} = \lim_{t \to \infty} \mathrm{e}(t)$$

$$\mathrm{e}_{ss} = \lim_{s \to 0} sE(s) = \lim_{s \to 0} s\,\frac{R(s)}{1 + G(s)H(s)}$$

Hence the steady state error depends upon-

(1) Type and magnitude of R(s),

(2) Open loop transfer function G(s)H(S),

(3) The presence of any non linearity.

3.6: Static Error Coefficients:

The difference between the reference input and output in steady state is called steady state error.

Let we have a unity feedback control system as shown

The Error function is given by

$$E(s) = \frac{R(S)}{1+G(s)} \qquad \text{...2.6.1}$$

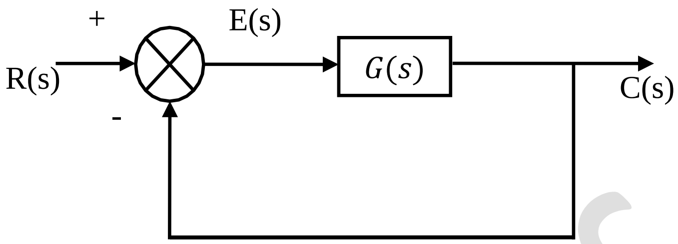


Figure 2.6.1: A Second Order System

And Steady State Error

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s)$$

$$= \lim_{s\to 0} s\frac{R(s)}{1+G(s)} \qquad \text{...2.6.2}$$

Case I When input is unit step i.e. $r(t) = u(t)$

When $r(t) = u(t)$

Then $\quad R(s) = \frac{1}{s}$

From eq. 2.6.2 $\quad e_{ss} = \lim_{s\to 0} s\frac{1/s}{1+G(s)} = \lim_{s\to 0}\frac{1}{1+G(s)} = \frac{1}{1+\lim_{s\to 0} G(s)} = \frac{1}{1+K_P}$

Where $K_P = \lim_{s\to 0} G(s) = G(0)$

K_P is called static position error coefficient.

Case II When input is unit ramp i.e. $r(t) = t$

When $r(t) = t$

Then $\quad R(s) = \frac{1}{s^2}$

From eq. 2.6.2 $\quad e_{ss} = \lim_{s\to 0} s\frac{1/s^2}{1+G(s)} = \lim_{s\to 0}\frac{1}{s[1+G(s)]} = \frac{1}{1+\lim_{s\to 0} sG(s)} = \frac{1}{1+K_v}$

Where $K_v = \lim_{s\to 0} sG(s)$

K_v is called static velocity error coefficient.

Case III When input is unit parabolic i.e. $r(t) = t^2/2$

When $r(t) = t^2/2$

Then $\quad R(s) = \frac{1}{s^3}$

From eq. 2.6.2 $\quad e_{ss} = \lim_{s\to 0} s\frac{1/s^3}{1+G(s)} = \lim_{s\to 0}\frac{1}{s^2[1+G(s)]} = \frac{1}{1+\lim_{s\to 0} s^2G(s)} = \frac{1}{1+K_a}$

Where $K_a = \lim_{s\to 0} s^2G(s)$

K_a is called static acceleration error coefficient.

Time Domain Stability Analysis

3.7: The concept of stability:

A linear time invariant system is said to be stable if the following two conditions are satisfied:

(i) For a bounded input, the output should be bounded, &

(ii) In the absence of the system input, the output should be zero, whatever be the initial conditions.

3.8: Effect of location of poles on stability:

Location of poles has a direct effect on stability. The entire s-plane is taken into account and it is divided as follows:

(a) Left half-plane (LHP), (b) jω-axis, (c) Right half-plane (RHP)
Let us consider the following cases:

(a) LHP Poles

When the poles are located in the left half plane, again there are few possible cases as, the poles are:
(i) On real axis and simple,
(ii) On real axis and multiple,
(iii) Complex conjugates and simple
For all above three cases, the system is stable.

(b) jω-axis

When the poles are located on jω-axis, again there are few possible cases as, the poles are:
(i) On jω-axis and simple,
(ii) On jω-axis and multiple,
(iii) At origin,
(iv) At origin and multiple
In this case, the pole has zero real part. If the poles are not repeated, the system is called marginally stable. If the poles are repeated, the system is unstable.

(c) RHP Poles

There are few possible cases in which poles lie in the right half plane.
(i) On real axis and simple
(ii) On real axis and multiple
(iii) Complex conjugates and simple
In all of the three cases, the system will be unstable.

3.9 : Necessary Conditions for stability:

Let the characteristics equation of a system is given by

$$a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s^1 + a_n s^0 = 0$$

Then the conditions for any system to be stable are:
(i) All the coefficients of the characteristics equation must have same sign.
(ii) All the coefficients of the characteristics equation must be real.
(iii) All the coefficients of the characteristics equation must be non-zero.

3.10 : Routh Hurwitz Stability Criterion:

Routh-Hurwitz stability criterion is having one necessary condition and one sufficient condition for stability. If any control system doesn't satisfy the necessary condition, then we can say that the control system is unstable. But, if the control system satisfies the necessary condition, then it may or may not be stable. So, the sufficient condition is helpful for knowing whether the control system is stable or not.

Necessary Condition for Routh-Hurwitz Stability

The necessary condition for any system to be stable is that the coefficients of the characteristic polynomial should be positive. This implies that all the roots of the characteristic equation should have negative real parts.
Consider the characteristic equation of the order 'n' is -
$a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s^1 + a_n s^0 = 0$
Note that, there should not be any term missing in the nth order characteristic equation. This means that the nth order characteristic equation should not have any coefficient that is of zero value.

Sufficient Condition for Routh-Hurwitz Stability

The sufficient condition is that all the elements of the first column of the Routh array should have the same sign. This means that all the elements of the first column of the Routh array should be either positive or negative.

Routh Array Method

In this method first the Routh array is formed then the number of sign changes are determined in the first column of the Routh Array. The number of sign changes in the first column equal to the number of roots in the right half of the s plane. Therefore if sign changes, in the first column means there are roots in the right half and therefore the system is unstable.

Procedure to form Routh Array

Let the transfer function of any system is given by the following equation:

$$\frac{C(s)}{R(s)} = \frac{a_0s^m + a_1s^{m-1} + a_2s^{m-2} + \cdots + a_{m-1}s^1 + a_ms^0}{b_0s^n + b_1s^{n-1} + b_2s^{n-2} + \cdots + b_{n-1}s^1 + b_ns^0}$$

Ch's equation $\quad b_0s^n + b_1s^{n-1} + b_2s^{n-2} + \cdots + b_{n-1}s^1 + b_ns^0 = 0$

s^n	b_0	b_2	b_4	b_6
s^{n-1}	b_1	b_3	b_5	b_7
s^{n-2}	$c_0 = \frac{b_1b_2 - b_0b_3}{b_1}$	$c_1 = \frac{b_1b_4 - b_0b_5}{b_1}$	$c_2 = \frac{b_1b_6 - b_0b_7}{b_1}$	…
s^{n-3}	$d_0 = \frac{c_0b_3 - c_1b_1}{c_0}$	$d_1 = \frac{c_0b_5 - c_2b_1}{c_0}$	$d_2 = \frac{c_0b_7 - c_3b_1}{c_0}$	…
s^{n-4}	…	…	…	…
s	…	…		
s^0	b_n			

For the system to be stable the sign in the first column should not be change. If the sign changes in first column, then the system is unstable and the number of poles in the right half of the s plane equals the number of the sign changes in the first column.

3.11 : Root Locus:

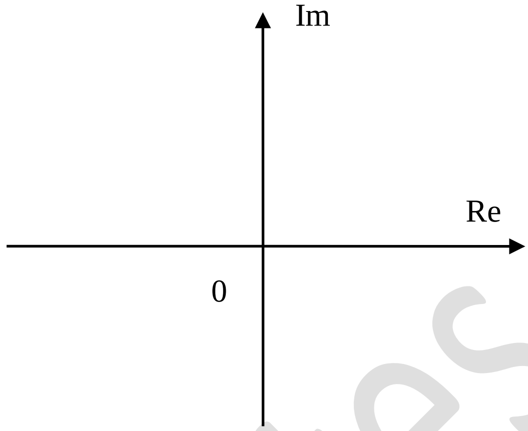
Definition: The root locus is the plot of the roots of system characteristic equation (or the poles of closed-loop transfer function) as a system parameter (K) is varied. The properties of the root locus are as under:

- A root locus starts $(K = 0)$from the open loop poles and ends $(K = \infty)$ on either finite open loop zero or infinity.
- The root locus is symmetrical about the real axis.
- The number of separate branches of the root locus equals either the number of open loop poles or the number of open loop zeros, whichever is greater.

Rules for drawing the root locus:

Following are the rules for constructing a root locus:

Rule 1: Find the number of open loop poles=n and open loop zeros=m.
Rule 2: Find and locate poles and zeros on s plane.
Rule 3: Find the number of root locus ending at infinite = n-m.
Rule 4: Root Locus on real axis:
If the number of (poles + zeros) to the right of a point on real axis are odd then that point lies on the root locus.
Rule 5: Root locus start from O.L. poles with $K = 0$, and ends either on an O.L. zero or on ∞.
Rule 6: No. of asymptotes = n-m.
Asymptotes are the path along with the root locus and it moves towards the∞.



Rule 7: Centroid
It is the point on negative real axis at which the asymptotes converges. It is given by

$$\sigma = \frac{\sum(real\ part\ of\ poles) - \sum(real\ part\ of\ zeros)}{n-m}$$

Rule 8: Angle of Asymptotes
For $K > 0$, $= \frac{(2q+1)\times180^0}{n-m}$ where $q = 0,1,2, \ldots n-m-1$
For $K < 0$, $= \frac{2q\times180^0}{n-m}$ where $q = 0,1,2, \ldots n-m-1$
Rule 9: Break away point or break in point
It is defined as the point at which root locus comes out of the real axis and break in point is defined as the point at which root locus enters the real axis.
To find the breakaway point put $\frac{dK}{ds} = 0$
Break away (in) angle ($'$)

$$' = \frac{180^0}{r}$$

Where r is the number of branches of root locus diverging from the break away (in) point.
Rule 10: Angle of departure (or arrival) for a complex pole (or zero)
Angle of departure (or arrival) is the angle at which the root locus leaves (or arrives) a complex pole (or zero)
Angle of departure

$$\phi_D(s = s_1) = 180^0 + \angle GH' = 180^0 + \angle|G(s)H(S)|'$$

Where $|G(s)H(S)|' = |G(s)H(S)|\ at\ (s = s_1)$excluding point $s = s_1$

Angle of arrival

$$\phi(s = s_1) = 180^0 - \angle GH' = 180^0 - \angle|G(s)H(S)|'$$

Where $\angle|G(s)H(S)|'$ is the angle of $|G(s)H(S)|\ at\ (s = s_1)$excluding point $s = s_1$
Example:1 Draw the root locus of the control system having open loop transfer function

$$G(s)H(s) = \frac{K}{s(s+1)(s+5)}$$

Solution:

Step 1: The number of open loop poles n=3 located at s=0, -1, -5 and open loop zeros m =0.

Step 2: The number of root locus ending at infinite $\mathrm{n} - \mathrm{m} = 3 - 0 = 3$.

Step3: Root Locus on real axis:
If the number of (poles + zeros) to the right of a point on real axis are odd then that point lies on the root locus. Thus root locus will lie on the real axis as shown below

To the right of zero s=0, root locus doesn't exist.
Between s=0 and s=-1, root locus exist.
Between s=-1 and s=-5, root locus doesn't exist.
Between s=-5 and $s = -\infty$, root locus exist.

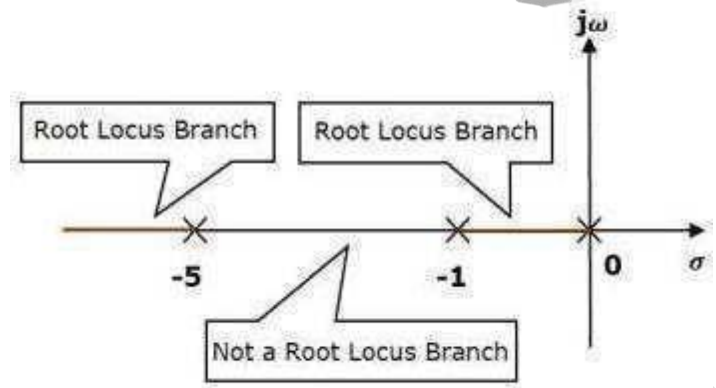


Figure: Root Locus on real axis

Step 4: Number of root locus ending at infinite
$$= n - m = 3 - 0 = 3$$

Step 5: Number of asymptotes
$$= n - m = 3 - 0 = 3$$

Step 6: Centroid
$$\sigma = \frac{(0-1-5)-0}{3-0} = -2$$

Step 7: Angle of Asymptotes
$= \frac{(2q+1)\times 180^0}{3}$ where $q = 0,1,2$
$= 60^0, 180^0, 300^0$

Step 8: Break away point
To find the breakaway point put $\frac{dK}{ds} = 0$
Given $G(s)H(s) = \frac{K}{s(s+1)(s+5)}$
Then we know the characteristics equation is given by $1 + G(s)H(s) = 0$
$$1 + \frac{K}{s(s+1)(s+5)} = 0$$
$$s(s+1)(s+5) + K = 0$$
$$s(s^2 + 6s + 5) + K = 0$$
$$K = -(s^3 + 6s^2 + 5s)$$
$$\frac{dK}{ds} = -(3s^2 + 12s + 5) = 0$$
$$s^2 + 4s + 5/3 = 0$$
after solving s= -3.53, -0.473

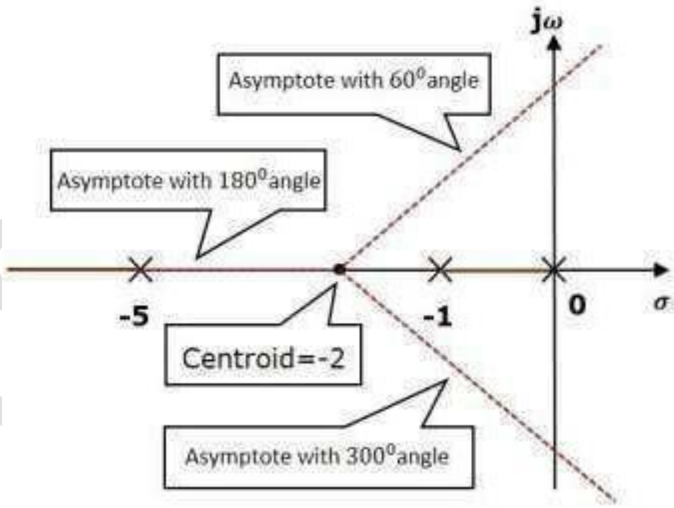


Figure: Root Locus on real Axis and Centroid

Since for s=-3.53 root locus doesn't exist on real axis, we will consider s=-0.473

Step 9: jω Cross over
The characteristics equation of the given function is
$$s^3 + 6s^2 + 5s + K = 0$$
The routh's array will be as under

S^3	1	5
S^2	6	K
S	(30-K)/6	0
S^0	K	0

Therefore the value of K
$$30 - K = 0 \; therefore \; K = 30$$
Then A.E. $6s^2 + K = 0 \; i.e. \; 6s^2 + 30 = 0 \; i.e. \; s^2 + 5 = 0 \; i.e. \; s = \pm j\sqrt{5}$

Therefore the root locus for the given function will be as shown in the figure.

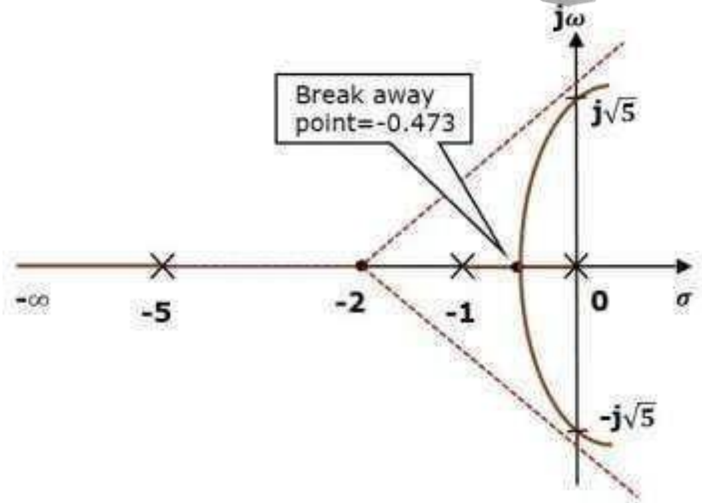


Figure: Root Locus

3.12 : Effects of additions of poles and zeros to open loop and closed loop system

The root locus can be shifted in ‘s’ plane by adding the open loop poles and the open loop zeros. If we add a pole in the open loop transfer function, then some of root locus branches will move towards right half of ‘s’ plane. Because of this, the damping ratio decreases. Which implies, damped frequency ωd increases and the time domain specifications like delay time td, rise time tr and peak time tp decrease.

But, it affects the system stability.

If we add a zero in the open loop transfer function, then some of root locus branches will move towards left half of ‘s’ plane. So, it will increase the control system stability. In this case, the damping ratio increases. Which implies, damped frequency ωd decreases and the time domain specifications like delay time td, rise time tr and peak time tp increase. So, based on the requirement, we can include (add) the open loop poles or zeros to the transfer function.

Unit 3

Syllabus: Frequency response analysis

Correlation between time and frequency response, Polar plots, Bode Plots, all-pass and minimum-phase systems, log-magnitude versus Phase-Plots, closed-loop frequency response.

Frequency domain stability analysis, Nyquist stability criterion, assessment of relative stability using Nyquist plot and Bode plot (phase margin, gain margin and stability).

3.1: Correlation between time and frequency response

For the higher order control system analysis, it is quite difficult to analyze the systems in the time domain, therefore for the higher order systems, frequency domain analysis is used.

The response of a system can be partitioned into both the transient response and the steady state response. We can find the transient response by using Fourier integrals. The frequency response of a system is defined as the steady-state response of the system to a sinusoidal input signal. The sinusoid is a unique input signal, and the resulting output signal for a linear system, as well as signals throughout the system, is sinusoidal in the steady-state; it differs from the input waveform only in amplitude and phase.

3.2: Polar Plot

The Polar plot is a plot, which can be drawn between the magnitude and the phase angle of $G(j\omega)H(j\omega)$ by varying ω from 0 to ∞. The polar graph sheet is shown in the figure 3.2.1 below.

This graph sheet consists of concentric circles and radial lines. The concentric circles and the radial lines represent the magnitudes and phase angles respectively. These angles are represented by positive values in anti-clock wise direction. Similarly, we can represent angles with negative values in clockwise direction. For example, the angle 270^0 in anti-clock wise direction is equal to the angle -90^0 in clockwise direction.

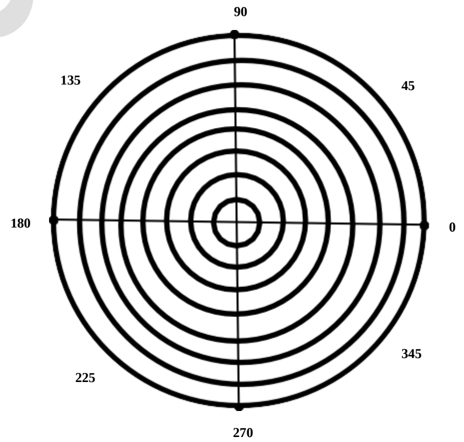


Figure 3.2.1 Polar Graph

Rules for Drawing Polar Plots

Follow these rules for plotting the polar plots.

- Substitute, s=jω in the open loop transfer function.
- Write the expressions for magnitude and the phase of $G(j\omega)H(j\omega)$.
- Find the starting magnitude and the phase of $G(j\omega)H(j\omega)$ by substituting ω=0. So, the polar plot starts with this magnitude and the phase angle.
- Find the ending magnitude and the phase of $G(j\omega)H(j\omega)$ by substituting $\omega = \infty$, so, the polar plot ends with this magnitude and the phase angle.
- Check whether the polar plot intersects the real axis, by making the imaginary term of $G(j\omega)H(j\omega)$ equal to zero and find the value(s) of ω.
- Check whether the polar plot intersects the imaginary axis, by making real term of $G(j\omega)H(j\omega)$ equal to zero and find the value(s) of ω.
- Now find the magnitude and phase of $G(j\omega)H(j\omega)$ by considering the other value(s) of ω.

Example 3.1 Draw the Polar Plot for the closed loop transfer function

$$G(m)H(m) = \frac{20}{s(s+1)(s+2)}$$

Solution: Consider the open loop transfer function of a closed loop control system.

$$G(j\omega)H(j\omega) = \frac{20}{s(s+1)(s+2)}$$

Step 1 − Substitute, $s = j\omega$ in the open loop transfer function.

$$G(j\omega)H(j\omega) = \frac{20}{j\omega(j\omega+1)(j\omega+2)}$$

Magnitude
$$|G(j\omega)H(j\omega)| = \frac{20}{\omega\sqrt{\omega^2+1}\sqrt{\omega^2+4}}$$

Phase angle
$$= -90^0 - \tan^{-1}\omega - \tan^{-1}(\omega/2)$$

Step 2 − The following table shows the magnitude and the phase angle of the open loop transfer function at $\omega = 0$ rad/sec and $\omega = \infty$ rad/sec.

Frequency(rad/sec)	Magnitude	Phase Angle
0	∞	-90^0
∞	0	-270^0
$\sqrt{2}$	10/3	-180^0

So, the polar plot starts at $(\infty, -90^0)$ and ends at $(0,-270^0)$. It intersects the real axis at
$(\sqrt{2}, -180^0)$. The first and the second terms within the brackets indicate the magnitude and phase angle respectively.
The required polar plot is shown in figure ex 3.1.

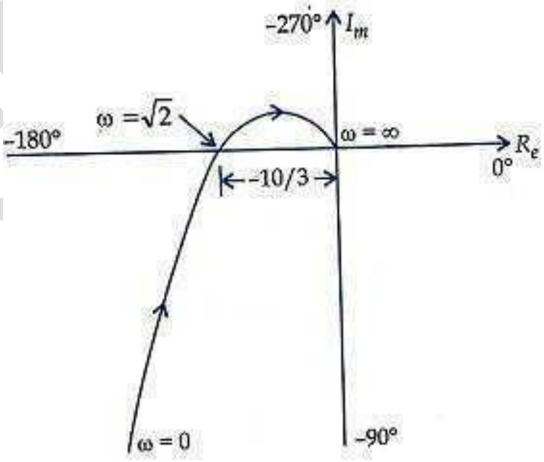


Figure ex 3.1 Polar Plot

3.3: Bode Plot

The Bode plot or the Bode diagram consists of two plots −

- Magnitude Vs. Frequency plot
- Phase Vs. Frequency plot

In both the plots, x-axis represents angular frequency (logarithmic scale). Whereas, y axis represents the magnitude (linear scale) of open loop transfer function in the magnitude plot and the phase angle (linear scale) of the open loop transfer function in the phase plot.

The magnitude of the open loop transfer function in dB is -

$$M = 20log|G(j\omega)H(j\omega)|$$

The phase angle of the open loop transfer function in degrees is -

$$= \angle G(j\omega)H(j\omega)$$

Basic of Bode Plots

The following table shows the slope, magnitude and the phase angle values of the terms present in the open loop transfer function. This data is useful while drawing the Bode plots.

Type of term	$G(m)H(m)$	Slope (dB/dec)	Magnitude (dB)	Phase angle (degrees)
Constant	K	0	20logK	0
Zero at origin	jω	20	20logω	90
'n' zeros at origin	$(j\omega)^n$	20n	20nlogω	90n
Pole at origin	1/jω	−20	−20logω	−90 or 270
'n' poles at origin	$1/(j\omega)^n$	−20n	−20nlogω	−90n or 270n

(i) $G(s)H(s) = K$

Consider the open loop transfer function $G(s)H(s) = K$
Magnitude $M = 20logK\ dB$
Phase angle $= 0$ degrees
If K=1, then magnitude is 0 dB.
If K>1, then magnitude will be positive.
If K<1, then magnitude will be negative.

The following figure shows the corresponding Bode plot.

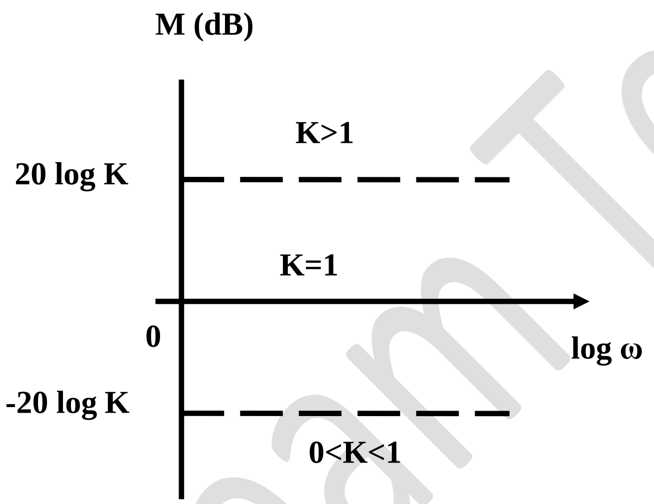


Figure 3.3.2.(a) Magnitude Vs. Frequency

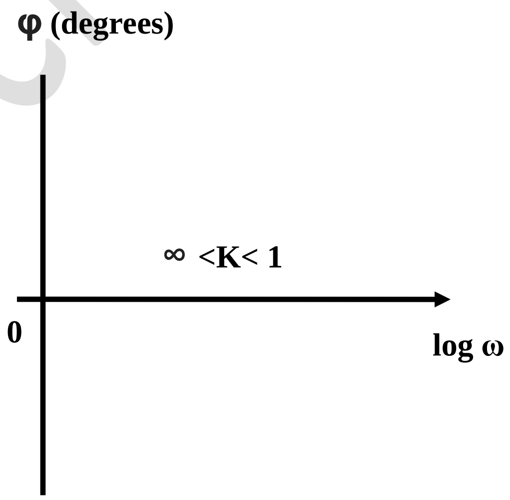


Figure 3.3.2.(b) Phase Vs. Frequency

The magnitude plot is a horizontal line, which is independent of frequency. The 0 dB line itself is the magnitude plot when the value of K is one. For the positive values of K, the horizontal line will shift 20logK dB above the 0 dB line. For the negative values of K, the horizontal line will shift 20logK dB below the 0 dB line. The Zero degrees line itself is the phase plot for all the positive values of K.

(ii) $G(s)H(s) = S$

Consider the open loop transfer function $G(s)H(s) = s$.

Magnitude $M = 20log\omega\ dB$
Phase angle $= 90^0$
At ω=0.1 rad/sec, the magnitude is -20 dB.
At ω=1 rad/sec, the magnitude is 0 dB.
At ω=10rad/sec, the magnitude is 20 dB.

The following figure shows the corresponding Bode plot.

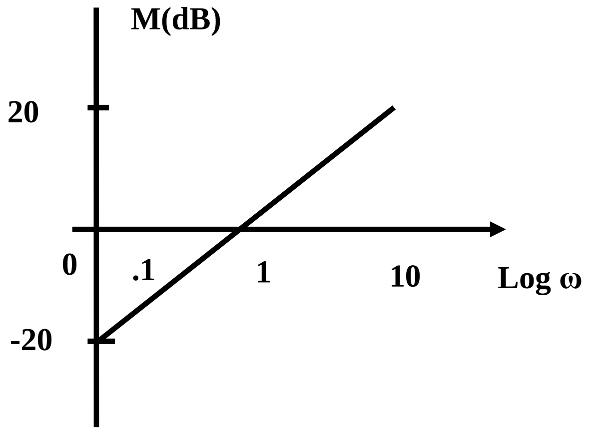


Figure 3.3.3.(a) Magnitude Vs. Frequency

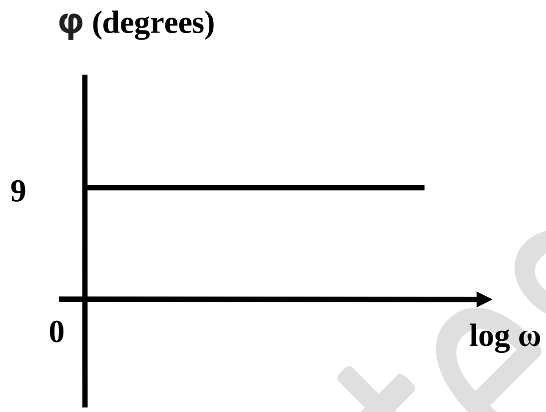


Figure 3.3.3.(b) Phase Vs. Frequency

The magnitude plot is a line, which is having a slope of 20 dB/dec. This line started at ω=0.1 rad/sec having a magnitude of -20 dB and it continues on the same slope. It is touching 0 dB line at ω=1 rad/sec. In this case, the phase plot is 90^0 line.

(iii) $G(s)H(s) = 1 + S$

Consider the open loop transfer function G(s)H(s)=1+sτ.

Magnitude $M = 20\ log\sqrt{(1 + \omega 2r2)}$ dB

Phase angle $= tan^{-1}\omega r$ degrees

For ω<1/τ, the magnitude is 0 dB and phase angle is 0 degrees.

For ω>1/τ, the magnitude is 20logωτ dB and phase angle is 90^0.

The following figure shows the corresponding Bode plot.

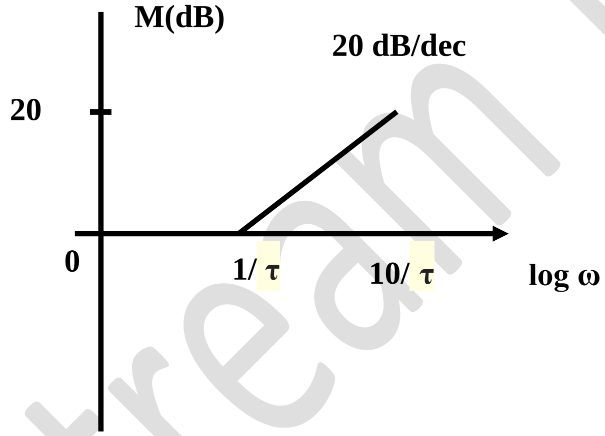


Figure 3.3.4.(a) Magnitude Vs. Frequency

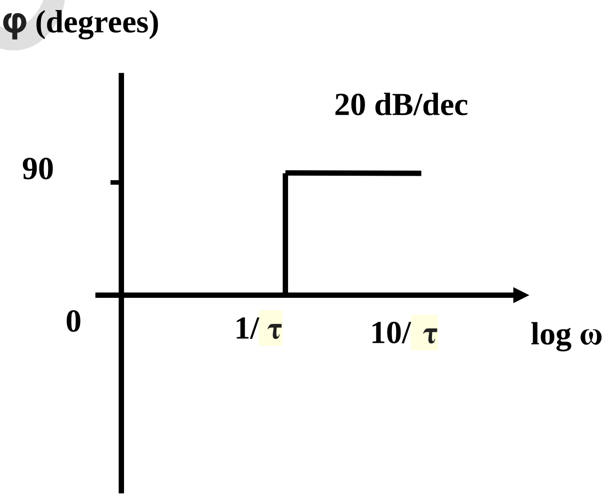


Figure 3.3.4.(b) Phase Vs. Frequency

The magnitude plot is having magnitude of 0 dB up to ω=1/τ rad/sec. From ω=1/τ rad/sec, it is having a slope of 20 dB/dec. In this case, the phase plot is having phase angle of 0 degrees up to ω=1/τ rad/sec and from here, it is having phase angle of 90^0. This Bode plot is called the asymptotic Bode plot.

As the magnitude and the phase plots are represented with straight lines, the Exact Bode plots resemble the asymptotic Bode plots. The only difference is that the Exact Bode plots will have simple curves instead of straight lines.

Example 3.2 Draw the Bode Plot for the system whose open loop transfer function is given as

$$G(s)H(s) = \frac{20}{s(s+1)(s+)}$$

Solution: Consider the open loop transfer function of a closed loop control system.

$$G(s)H(s) = \frac{20}{s(s+1)(s+4)} = \frac{20}{4s(1+s)(1+0.25s)}$$

therefore

$$G(s)H(s) = \frac{5}{s(1+s)(1+0.25s)}$$

This is a type 1 system with K=5. Put $S = j\omega$

therefore

$$G(j\omega)H(j\omega) = \frac{5}{j\omega(1+j\omega)(1+0.25j\omega)}$$

1. The corner frequencies are

$$\omega_1 = \frac{1}{1} = 1\, rad/sec$$

$$\omega_2 = \frac{1}{0.25} = 4\, rad/sec$$

2. Starting frequency

Starting frequency is taken as the lower than the lowest frequency. As the lowest frequency is 1 rad / sec, the starting frequency will be 0.1 rad / secs.

3. Initial Slope

Since the type of the system is one, the initial slope will be $-20bd/decade$.

4. Intersection with 0db axis

For type 1 system,

$$\omega = k = 5\, rad/sec$$

The initial slope will continue from $\omega = 0.1\, rad/sec$ to next corner frequency $\omega = 1\, rad/sec$.

5. Next slope

The next corner frequency $\omega = 1\ rad/sec$ is due to the denominator term $1/(1 + j\omega)$, therefore the slope will change by $-20bd/decade$. The new slope will be now $-40bd/decade$, and it will continue up to the next corner frequency.

6. Next slope

The next corner frequency $\omega = 4\ rad/sec$ is due to the denominator term $1/(4 + j\omega)$, therefore the slope will change by $-20bd/decade$. The new slope will be now $-60bd/decade$, and it will continue for the all frequencies greater than $\omega = 4\, rad/sec$.

7. Phase Angle

The Phase Angle $= \angle G(j\omega)H(j\omega)$ for the frequencies between $\omega = 0.1\ rad/sec$ to $\omega = 4\ rad/sec$ is calculated by

$$= \angle G(j\omega)H(j\omega) = -90^0 - \tan^{-1}\omega - \tan^{-1}0.25\omega$$

Therefore all the details can be summarized in the table below

Factor	Corner Frequency (rad/seconds)	Slope (db/decade)	Angle (degree)
$\frac{5}{j\omega}$	–	$-20bd/decade$	-90^0
$\frac{1}{1+j\omega}$	$1\, rad/sec$	$-20bd/decade$	$-\tan^{-1}\omega$
$\frac{1}{1+0.25j\omega}$	$4\, rad/sec$	$-20bd/decade$	$-\tan^{-1}0.25\omega$

For plotting the phase curve, the total angle is given by

$$= \angle G(j\omega)H(j\omega) = -90^0 - \tan^{-1}\omega - \tan^{-1}0.25\omega$$

Therefore the phase angles for different values of ω are given in the table below:

m	0.1	0.5	1	2	3	5	10
	-97.142^0	-123.69^0	-149.03^0	-180^0	-198.43^0	-220.03^0	-242.48^0

The Magnitude curve and the phase curve are plotted in the diagram given below:

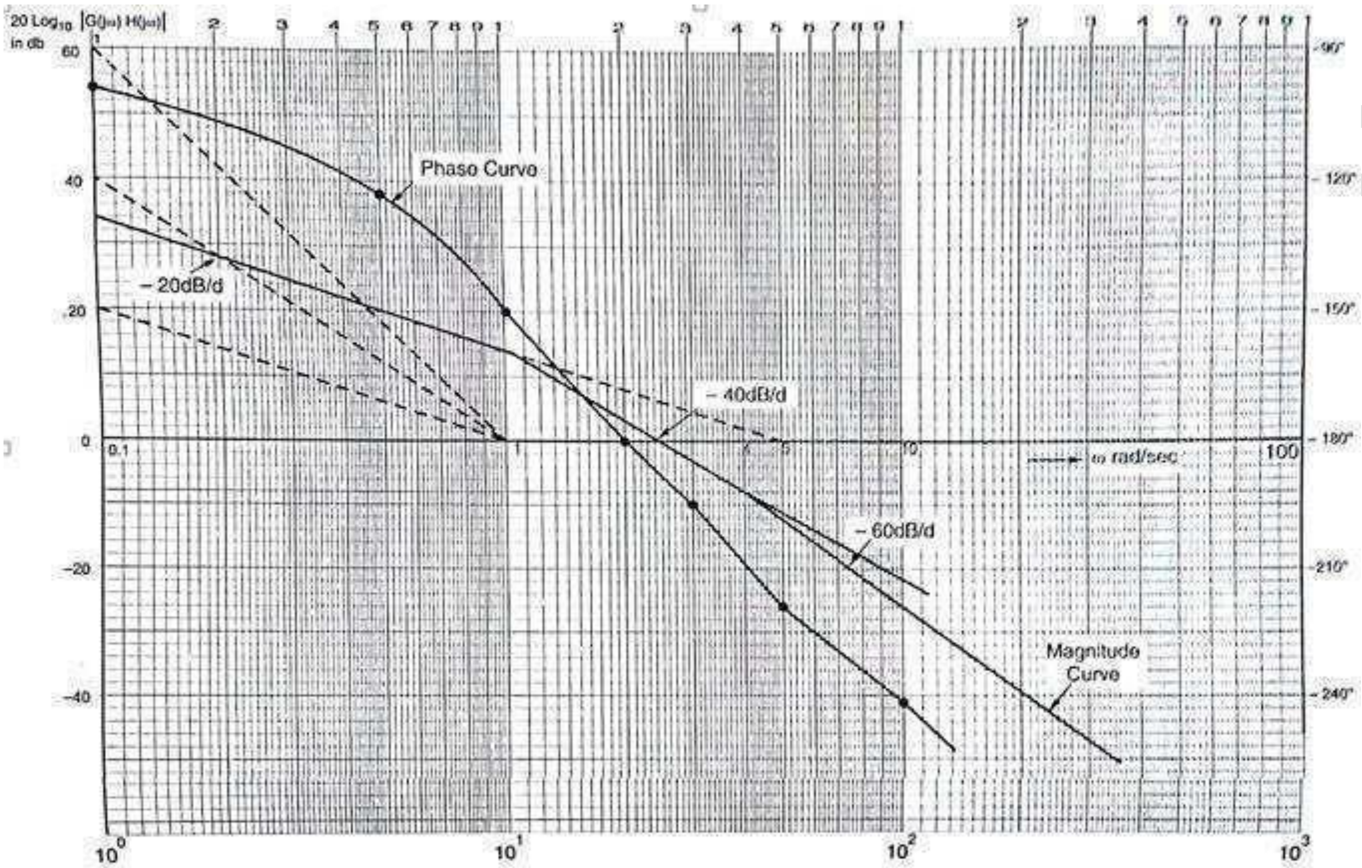


Figure ex 3.2 Bode plot for the Ex. 3.2

From the magnitude curve, the gain cross over frequency is 2.2 rad/sec, i.e. $\omega = 2.2\ rad/sec$.
From the phase curve, the phase cross over frequency is 2 rad/sec, i.e. $\omega_p = 2\ rad/sec$.

$\therefore$	$Gain\ Margin\ = -20\log_{10}\lvert G(J\omega_p)H(j\omega_p)\rvert\ = -2db$	
	$Phase\ Margin = 180^0 + \angle G(j\omega)H(j\omega) = -6^0$	

Since both the Gain Margin and Phase Margin are negative, therefore the system is unstable.

3.4 : Stability Conditions:

1. If both the gain margin and phase margin are positive, then system is stable.
2. It both gain margin and phase margin are zero, then the system is marginally stable.
3. If any of them is negative, system is unstable.

In terms of frequency:

1. The system will be stable if $\omega_p > \omega$.
2. The system will be marginally stable if $\omega_p = \omega$.
3. The system will be unstable if $\omega_p < \omega$.

3.5 : All-pass and minimum-phase systems

(a) All Pass System:

If the zeros of the system are located symmetrically to the poles in right half of S plane, then system is called All Pass System.

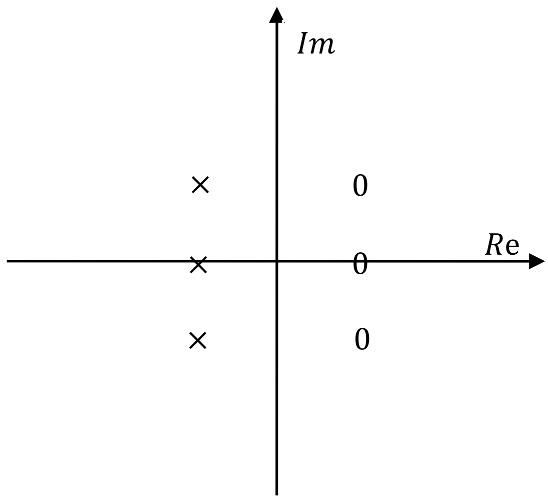


Figure 3.4.1 All Pass System

All pass system is used for phase shifting. Magnitude of transfer function of All Pass System is unity, i.e.

$$\frac{C(s)}{R(s)} = \frac{(S-a)}{(S+a)}$$

All pass system is also known as Non-minimum phase system.

(b) Minimum Phase System

Minimum phase system is a system, which contains all of its close loop poles and zeros in the left half of the S plane.

For example,

$$\frac{C(s)}{R(s)} = \frac{(S+1)(S+4)}{(S+2)(S+5)(S+3)}$$

(c) Non-Minimum Phase System

Minimum phase system is a system, which contains all of its close loop poles in the left half of the S plane and one or more than one zeros in the right hand side of the s plane.

For example,

$$\frac{C(s)}{R(s)} = \frac{(S-1)}{(S+2)(S+3)}$$

3.6: Nyquist Stability Criterion

The Nyquist stability criterion works on the principle of argument. It states that if there are P poles and Z zeros are enclosed by the ‘s’ plane closed path, then the corresponding $G(s)H(s)$ plane must encircle the origin P−) times. So, we can write the number of encirclements N as,

$$N = P - Z$$

- If the enclosed S plane closed path contains only poles, then the direction of the encirclement in the G(s)H(s) plane will be opposite to the direction of the enclosed closed path in the S plane.
- If the enclosed S plane closed path contains only zeros, then the direction of the encirclement in the G(s)H(s) plane will be in the same direction as that of the enclosed closed path in the S plane.

Let us now apply the principle of argument to the entire right half of the S plane by selecting it as a closed path. This selected path is called the Nyquist contour.

We know that the closed loop control system is stable if all the poles of the closed loop transfer function are in the left half of the S plane. So, the poles of the closed loop transfer function are nothing but the roots of the characteristic equation. As the order of the characteristic equation increases, it is difficult to find the roots. So, let us correlate these roots of the characteristic equation as follows.

- The Poles of the characteristic equation are same as that of the poles of the open loop transfer function.

- The zeros of the characteristic equation are same as that of the poles of the closed loop transfer function.

We know that the open loop control system is stable if there is no open loop pole in the right half of the S plane.

I.e. P = 0 ⇒ N = −)

We know that the closed loop control system is stable if there is no closed loop pole in the right half of the S plane.

i.e., Z = 0 ⇒ N = P

Nyquist stability criterion states the number of encirclements about the critical point (1+j0) must be equal to the poles of characteristic equation, which is nothing but the poles of the open loop transfer function in the right half of the S plane. The shift in origin to (1+j0) gives the characteristic equation plane.

3.5.1 Rules for Drawing Nyquist Plots

Follow these rules for plotting the Nyquist plots.

- Locate the poles and zeros of open loop transfer function G(s) H(s) in S plane.
- Draw the polar plot by varying ω from zero to infinity. If pole or zero present at s = 0, then varying ω from 0+ to infinity for drawing polar plot.
- Draw the mirror image of above polar plot for values of ω ranging from −∞ to zero (0^- if any pole or zero present at s=0).
- The number of infinite radius half circles will be equal to the number of poles or zeros at origin. The infinite radius half circle will start at the point where the mirror image of the polar plot ends. And this infinite radius half circle will end at the point where the polar plot starts.

After drawing the Nyquist plot, we can find the stability of the closed loop control system using the Nyquist stability criterion. If the critical point (-1+j0) lies outside the encirclement, then the closed loop control system is absolutely stable.

3.5.2 Stability Analysis using Nyquist Plots

From the Nyquist plots, we can identify whether the control system is stable, marginally stable or unstable based on the values of these parameters.

- Gain cross over frequency and phase cross over frequency
- Gain margin and phase margin

3.5.3 Phase Cross over Frequency

The frequency at which the Nyquist plot intersects the negative real axis (phase angle is 180^0) is known as the phase cross over frequency. It is denoted by ω_{pc}.

3.5.4 Gain Cross over Frequency

The frequency at which the Nyquist plot is having the magnitude of one is known as the gain cross over frequency. It is denoted by ω_c.

The stability of the control system based on the relation between phase cross over frequency and gain cross over frequency is listed below.

- If the phase cross over frequency ω_{pc} is greater than the gain cross over frequency ω_c, then the control system is stable.
- If the phase cross over frequency ω_{pc} is equal to the gain cross over frequency ω_c, then the control system is marginally stable.
- If phase cross over frequency ω_{pc} is less than gain cross over frequency ω_c, then the control system is unstable.

3.5.5 Gain Margin

The gain margin GM is equal to the reciprocal of the magnitude of the Nyquist plot at the phase cross over frequency.

GM=1/Mpc

Where, Mpc is the magnitude in normal scale at the phase cross over frequency.

3.5.6 Phase Margin

The phase margin PM is equal to the sum of 180^0 and the phase angle at the gain cross over frequency.

$$PM = 1800 + {}_c$$

where, ${}_c$ is the phase angle at the gain cross over frequency.

The stability of the control system based on the relation between the gain margin and the phase margin is listed below.

- If the gain margin GM is greater than one and the phase margin PM is positive, then the control system is stable.
- If the gain margin GM is equal to one and the phase margin PM is zero degrees, then the control system is marginally stable.
- If the gain margin GM is less than one and / or the phase margin PM is negative, then the control system is unstable.

3.5.7 Relative Stability:

A transfer function is called minimum phase when all the poles and zeros are LHP and non-minimum-phase when there are RHP poles or zeros.

The gain margin (GM) is the distance on the Bode magnitude plot from the amplitude at the phase crossover frequency up to the 0 dB point. GM=-(dB of GH measured at the phase crossover frequency).

The phase margin (PM) is the distance from -180 up to the phase at the gain crossover frequency. PM=180+phase of GH measured at the gain crossover frequency.

Therefore the system will be stable if in the graph of the Bode Plot, the Gain Margin and the Phase Margin, both are positive. If both are zero, then the system is Marginally Stable, and if any of the GM and PM are negative, the system will be Unstable.

RGPV University

Department of Electronics and Communication Engineering

Sub. Code: EC 404 **Sub. Name: Control System**

Unit 4

Syllabus: Approaches to system design

Design problem, types of compensation techniques, design of phase-lag, phase lead and phase lead-lag compensators in time and frequency domain, proportional, derivative, integral and PID compensation.

4.1 Design Problem

The performance of a control system can be described in terms of the time-domain performance measures or the frequency-domain performance measures. The performance of a system can be specified by requiring a certain peak time Tp, maximum overshoot, and settling-time for a step input. Furthermore, it is usually necessary to specify the maximum allowable steady-state error for several test signal inputs and disturbance inputs. These performance specifications can be defined in terms of the desirable location of the poles and zeros of the closed-loop system transfer function T(s). Thus, the location of the s-plane poles and zeros of T(s) can be specified.

The design of a system is concerned with the alteration of the frequency response or the root locus of the system in order to obtain a suitable system performance. For frequency response methods, we are concerned with altering the system so that the frequency response of the compensated system will satisfy the system specifications. Hence, in the frequency response approach, we use compensation networks to alter and reshape the system characteristics represented on the Bode diagram.

Alternatively, the design of a control system can be accomplished in the s-plane by root locus methods. For the case of the s-plane, the designer wishes to alter and reshape the root locus so that the roots of the system will lie in the desired position in the s-plane.

4.2 Compensation

- A compensator is an additional component or circuit that is inserted into a control system to compensate for a deficient performance.
- In order to obtain the desired performance of the system, we use compensating networks. Compensating networks are applied to the system in the form of feed forward path gain adjustment. Compensate an unstable system to make it stable.
- A compensating network is used to minimize overshoot. These compensating networks increase the steady state accuracy of the system. Also, the increase in the steady state accuracy brings instability to the system. Compensating networks also introduces poles and zeros in the system thereby causes changes in the transfer function of the system

Methods of Compensation

4.2.1 Series Compensation

Connecting compensating circuit between error detector and plants known as series compensation.

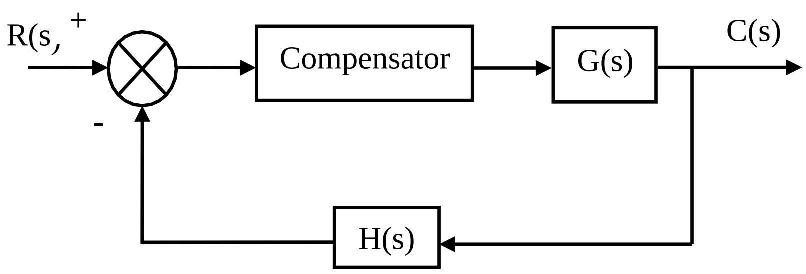


Figure 4.2.1 Series Compensation

When the compensator is placed in series with the forward path gain of the control system, the scheme is called series compensation or the cascade compensation.

In this scheme the signal flows from the lower energy levels towards the higher energy level. Therefore the additional amplifiers are used to increase the gain as well as provide the necessary isolation. A series scheme has more number of components compared to the feedback scheme.

4.2.2 Feedback Compensation

When a compensator used in a feedback manner called feedback compensation.

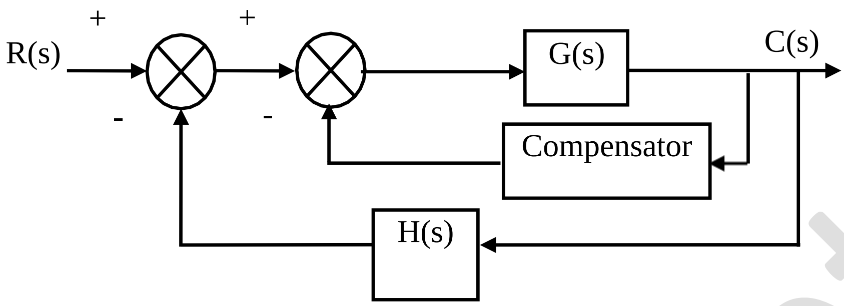


Figure 4.2.2 Feedback compensation

If the feedback is taken from some internal element and a compensator is introduced in such feedback path to provide an additional internal feedback loop, the scheme is known as feedback compensation

4.2.3 Combined Cascade & Feedback Compensation

A combination of series and feedback compensator is called load compensation.

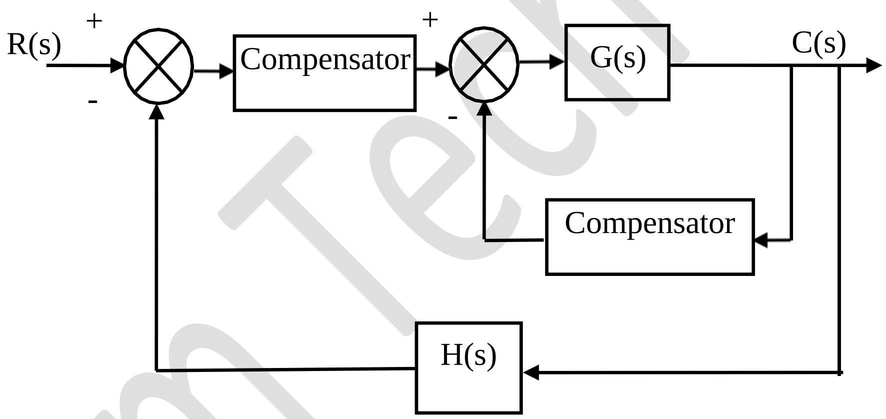


Figure 4.2.3 Combined Cascade & Feedback compensation

Sometimes a situation demands to both type of compensation, series as well feedback. This scheme is called combined series and feedback compensation.

To select a proper compensation scheme, the nature of the signals available in the system, the power levels at various points available components, the economic considerations and the designers experience are important.

4.3 Compensating Network:

- A compensating network is one which makes some adjustments in order to make up for deficiencies in the system. Compensating devices are may be in the form of electrical, mechanical, hydraulic etc. Most electrical compensator is RC filter. The simplest network used for compensator is known as lead, lag network.
- A compensator is a physical device which may be an electrical network, mechanical unit, pneumatic, hydraulic or a combination of various type of devices. The following electrical compensating networks are generally used for series compensation. The following electrical compensating networks are generally used:
  1. Lead Compensation Network
  2. Lag Compensation Network
  3. Lag-Lead Compensation Network

4.3.1 Phase Lead Compensation

The lead compensator is an electrical network which produces a sinusoidal output having phase lead when a sinusoidal input is applied. The lead compensator circuit in the S domain is shown in the following figure.

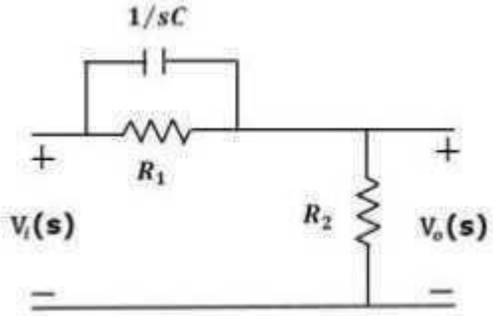


Figure Phase Lead Compensator

Here, the capacitor is parallel to the resistor $R1$ and the output is measured across resistor $R2$.
The transfer function of this lead compensator is –

$$\frac{V_0(s)}{V_i(s)} = \beta\left(\frac{sr+1}{\beta sr+1}\right)$$

Where $r = R_1C$ and $\beta = \frac{R_2}{R_1+R_2}$

From the transfer function, we can conclude that the lead compensator has pole at $s = -1/r$ and a zero at $s = -1/\beta r$.
Substitute, $s = j\omega$ in the transfer function.

$$\frac{V_0(j\omega)}{V_i(j\omega)} = \beta\left(\frac{j\omega r+1}{\beta j\omega r+1}\right)$$

Phase angle $\emptyset = tan^{-1}\,\omega r - tan^{-1}\,\beta\omega r$

We know that, the phase of the output sinusoidal signal is equal to the sum of the phase angles of input sinusoidal signal and the transfer function.

So, in order to produce the phase lead at the output of this compensator, the phase angle of the transfer function should be positive. This will happen when $0 < \beta < 1$. Therefore, zero will be nearer to origin in pole-zero configuration of the lead compensator.

Effects of a Lead Compensator:

1. Since a Lead compensator adds a dominant zero and a pole, the damping of a closed loop system is increased.
2. The less overshoot, less rise time and less settling time are obtained due to increase of damping coefficient and hence there is improvement in the transient response of the closed loop system.
3. It improves the phase margin of the closed loop system.
4. Bandwidth of the closed loop system is increased and hence the response is faster.
5. The steady state error does not get affected.

Limitations:

1. Since an additional increase in the gain is required, it results in larger space, more elements, greater weight and higher cost.
2. From a single lead network, the maximum lead angle available is about 60^0. For lead of more than 70^0 to 90^0, a multistage lead compensator is required.

4.3.2 Phase Lag Compensation

The phase lag compensator is shown in diagram 4.3.2 below

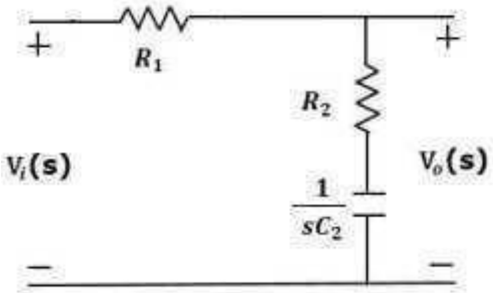


Figure 4.3.2 Phase Lag Compensator

Here, the capacitor C_2 is in series to the resistor R_2 and the output is measured across the series combination of the resistor R_2 and Capacitor C_2 as shown in figure.
The transfer function of this lead compensator is –

$$\frac{V_0(s)}{V_i(s)} = \frac{1}{\beta}\left(\frac{s+\frac{1}{r}}{s+\frac{1}{\beta r}}\right)$$

Where $r = R^2C$ and $\beta = \frac{R_1+R_2}{R_2} > 1$, Generally β is taken to be 10.
From the transfer function, we can conclude that the lead compensator has pole at $s = -1/\beta r$ and a zero at $s = -1/r$.
Substitute, $s = j\omega$ in the transfer function.

$$\frac{V_0(j\omega)}{V_i(j\omega)} = \frac{1}{\beta}\left(\frac{j\omega+\frac{1}{r}}{j\omega+\frac{1}{\beta r}}\right) = \frac{1+j\omega r}{1+j\omega\beta r}$$

Therefore the magnitude $M = \frac{V_0(j\omega)}{V(j\omega)} = \frac{\sqrt{1+\omega^2c^2}}{\sqrt{1+\omega^2\beta^2c^2}}$

And the phase angle is given by, Phase angle $\emptyset = tan^{-1}\,\omega r - tan^{-1}\,\beta\omega r$
The above equations gives the frequencies at which the phase lag is maximum and the $\omega_m$is the geometric mean of two frequencies. Here $\omega_{c1} = \frac{1}{c}$and $\omega_{c2} = \frac{1}{\beta c}$are the two corner frequencies.

Effects and Limitations of the Lag compensator

1. Since Lag compensator allows high gain at low frequencies, it is basically a low pass filter. Therefore it improve the steady state response.
2. In Lag compensation, the attenuation characteristics is used for the compensation, whereas the phase lag characteristic is of no use in compensation.
3. The system is very sensitive to parameters variation.
4. Since a lag compensator approximately acts as a PI controller, it thus tends to make a system less stable.

4.3.3 Lag Lead Compensation Network

Lag-Lead compensator is an electrical network which produces phase lag at one frequency region and phase lead at other frequency region. It is a combination of both the lag and the lead compensators. The lag-lead compensator circuit in the S domain is shown in the following figure.

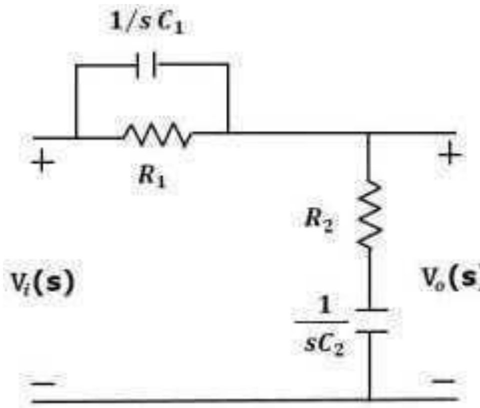


Figure4.3.3 Lag-Lead Compensator

This circuit looks like both the compensators are cascaded. So, the transfer function of this circuit will be the product of transfer functions of the lead and the lag compensators.

$$\frac{V_0(s)}{V_i(s)} = \beta\left(\frac{sr_1+1}{\beta sr_1+1}\right)\frac{1}{\alpha}\left(\frac{s+\frac{1}{r_2}}{s+\frac{1}{\alpha r_2}}\right)$$

$$\frac{V_0(s)}{V_i(s)} = \left(\frac{s+\frac{1}{r_1}}{s+\frac{1}{\beta r_1}}\right)\left(\frac{s+\frac{1}{r_2}}{s+\frac{1}{\alpha r_2}}\right)$$

We know that $\alpha\beta = 1, r_1 = R_1C_1$ and $r_2 = R_2C_2$

Therefore the magnitude $M = \frac{V_0(j\omega)}{V(j\omega)} = \frac{\sqrt{1+\omega^2 c_1^2}\sqrt{1+\omega^2 c_2^2}}{\sqrt{1+\omega^2\beta^2 c_1^2}\sqrt{1+\omega^2\beta^2 c_2^2}}$

And the phase angle is given by, Phase angle $\emptyset = \tan^{-1}\omega r_1 + \tan^{-1}\omega r_2 - \tan^{-1}\beta\omega r_1 - \tan^{-1}\alpha\omega r_2$

The above equations gives the frequencies at which the phase lag is maximum and the ω_m is the geometric mean of four frequencies. Here $\omega_{c1} = \frac{1}{c_1}$, $\omega_{c2} = \frac{1}{c_2}$, $\omega_{c3} = \frac{1}{\beta c_1}$ and $\omega_{c4} = \frac{1}{\beta c_2}$ are the corner frequencies.

Effects of Lag Lead Compensator:

To get fast response and good static accuracy, a lag lead compensator is used. It also increase the low frequency gain, which improves the steady state response. Since it increases the bandwidth of the system, the system response becomes very fast. In general the phase lead portion of this compensator provides large bandwidth & hence shorter rise time and settling time, while the phase lag portion provides the major damping of the system.

4.4 Proportional Controllers

With proportional controllers there are two conditions:

1. Deviation should not be large; it means there should be less deviation between the input and output.
2. Deviation should not be sudden.

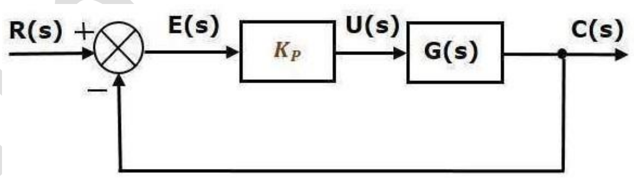


Figure 4.3.1 Proportional Controller

In the proportional controller the output (also called the actuating signal) is directly proportional to the error signal. Now let us analyze proportional controller mathematically. As we know in proportional controller output is directly proportional to error signal, writing this mathematically we have,

$$A(t) \propto e(t)$$

Removing the sign of proportionality we have,

$$A(t) = Kp.e(t)$$

Where, Kp is proportional constant also known as controller gain. It is recommended that Kp should be kept greater than unity. If the value of Kp is greater than unity, then it will amplify the error signal and thus the amplified error signal can be detected easily.

Advantages of Proportional Controller

1. Proportional controller helps in reducing the steady state error, thus makes the system more stable.
2. Slow response of the over damped system can be made faster with the help of these controllers.

Disadvantages of Proportional Controller

1. Due to presence of these controllers we some offsets in the system.
2. Proportional controllers also increase the maximum overshoot of the system.

4.4 Integral Controllers

As the name suggests in integral controllers the output is directly proportional to the integral of the error signal. Now let us analyze integral controller mathematically.

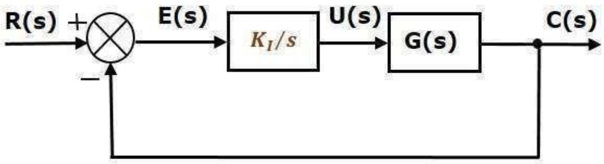


Figure 4.4.1 Integral Controller

As we know in an integral controller output is directly proportional to the integration of the error signal, writing this mathematically we have,

$$A(t) \propto \int e(t)\,dt,$$

Removing the sign of proportionality we have,

$$A(t) = K_i \int e(t)\,dt,$$

Where, Ki is integral constant also known as controller gain. Integral controller is also known as reset controller.

Advantages of Integral Controller

Due to their unique ability they can return the controlled variable back to the exact set point following a disturbance that's why these are known as reset controllers.

Disadvantages of Integral Controller

It tends to make the system unstable because it responds slowly towards the produced error.

4.5 Derivative Controllers

Derivative controller should be used in combinations with other modes of controllers because of its few disadvantages which are written below:

1. It never improves the steady state error.
2. It produces saturation effects and also amplifies the noise signals produced in the system.

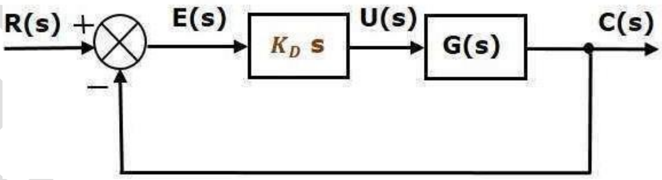


Figure 4.5.1 Derivative Controller

Now, as the name suggests in a derivative controller the output is directly proportional to the derivative of the error signal. Derivative controller output is directly proportional to the derivative of the error signal, writing this mathematically we have,

$$A(t) \propto de(t)/dt$$

Removing the sign of proportionality we have,

$$A(t) = Kd\, de(t)/dt$$

Where, Kd is proportional constant also known as controller gain. Derivative controller is also known as rate controller.

Advantages of Derivative Controller

The major advantage of derivative controller is that it improves the transient response of the system.

4.6 Proportional and Integral Controller

As the name suggests it is a combination of proportional and an integral controller the output is equal to the summation of proportional and integral of the error signal. Now let us analyze proportional and integral controller mathematically.

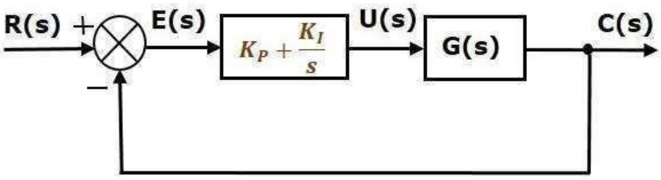


Figure 4.6.1 PI Controller

As we know in a proportional and integral controller output is directly proportional to the summation of proportional of error and integration of the error signal, writing this mathematically we have,

$$A(t) \propto \int e(t)\,dt \;+\; A(t) \;\propto\; e(t)$$

Removing the sign of proportionality we have,

$$A(t) = Ki \int e(t)\,dt \;+\; KP\, e(t),$$

Where, Ki and kp proportional constant and integral constant respectively. Advantages and disadvantages are the combinations of the advantages and disadvantages of proportional and integral controllers.

4.7 Proportional and Derivative Controller

It is a combination of proportional and a derivative controller the output is equals to the summation of proportional and derivative of the error signal. Now let us analyze proportional and derivative controller mathematically.

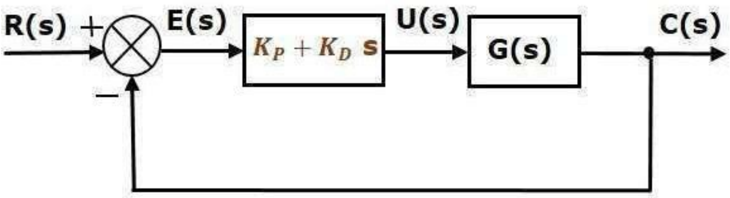


Figure 4.7.1 PD Controller

Proportional and derivative controller output is directly proportional to summation of proportional of error and differentiation of the error signal, writing this mathematically we have,

$$A(t) \propto \left(\frac{de(t)}{dt} + e(t)\right)$$

Removing the sign of proportionality we have,

$$A(t) = K_d\, de(t)/dt \;+ K_p\, e(t)$$

Where, K_d and K_p proportional constant and derivative constant respectively. Advantages and disadvantages are the combinations of advantages and disadvantages of proportional and derivative controllers.

4.8 Proportional Integral and Derivative (PID) Controller

A controller is a device which when introduced in feedback or forward path of system, controls the steady state and transient response as per the requirement. This controller converts the applied input to some other form of error which is proportional to the error due to which steady state and transient response gets improved. The output of the controller is proportional to the amount of error generated by that device.

The performance of this controlling phenomena may be done by means of electrical, mechanical, pneumatic or hydraulic medium.

The proportional integral derivative controller produces an output, which is the combination of the outputs of proportional, integral and derivative controllers.

$$A(t) \propto (\, e(t) + \int e(t)\, dt + \frac{de(t)}{dt})$$

Removing the sign of proportionality we have,

$$A(t) = K_p\, e(t) + K\mathrm{i} \int e(t)\, dt + K_d\, de(t)/dt$$

Where K_p, K_i & K_d and proportional constant, integral constant and derivative constant respectively. Advantages and disadvantages are the combinations of advantages and disadvantages of proportional, integral and derivative controllers.

The block diagram of the unity negative feedback closed loop control system along with the proportional integral derivative controller is shown in the following figure.

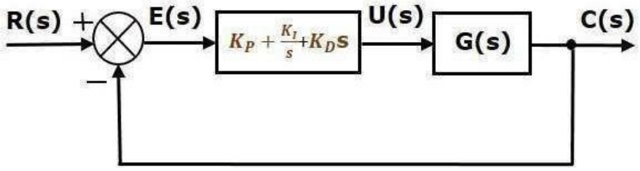


Figure 4.8.1 PID Controller

RGPV University

Department of Electronics and Communication Engineering

Sub. Code: EC 404 **Sub. Name: Control System**

Unit 2

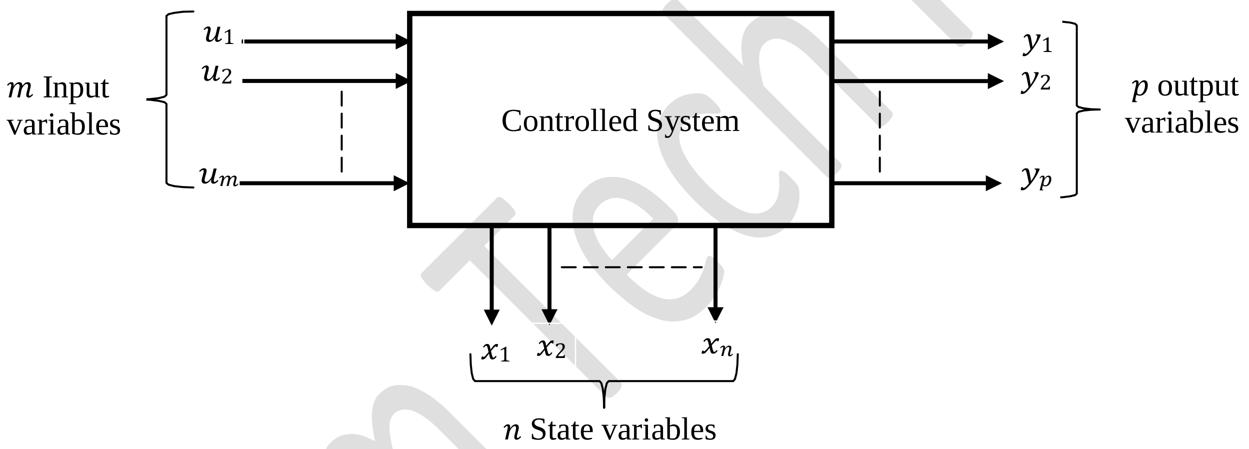
Syllabus: State space analysis

State space representation of systems, block diagram for state equation, transfer function decomposition, solution of state equation, transfer matrix, relationship between state equation and transfer function, controllability and observability.

5.1: State Variables:

The state of a system is defined as the smallest set of variables that must be known at any given instant in order that the future response to any specified input may be calculated from the given dynamic equations. The state may be regarded as a compact representation of the past history of the system, which can be utilized for predicting its future behavior.

Since the complete solution of a differential equation of order n requires precisely n initial conditions, the state of such system will be specified by values of n quantities called state variables.



here

$$u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_m(t) \end{bmatrix}$$ mx1 column matrix called input vector

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$ nx1 column matrix called state vector

and $$y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_p(t) \end{bmatrix}$$ px1 column matrix called output vector

State equation is an arrangement of a set of first order differential equation, with the following form of a LTI system,

$$\dot{x}(t) = A.x(t) + B.u(t)$$

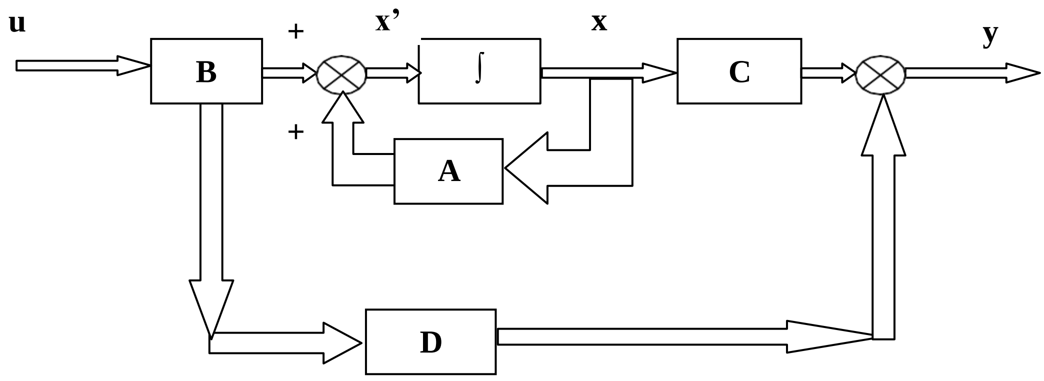
Here A and B are matrices of dimensions nxn and nxm respectively with constant elements.

The output equation of the system is given by

$$y(t) = C.x(t) + D.u(t)$$

Here C and D are metrices of dimensions pxn and pxm and y(t) is having dimension px1.

The block diagram representation of a state model of a linear multi input, multi output system is shown below.



5.2: State Transition Matrix:

State transition matrix is denoted by $\emptyset(t)$ and defined as the matrix that satisfy the linear homogeneous state equation

$$\frac{dx(t)}{dt} = A.x(t) \qquad 5.2.1$$

Let $\emptyset(t)$ be the nxn matrix which represent the equation (1), hence

$$\frac{d\emptyset(t)}{dt} = A\emptyset(t) \qquad 5.2.2$$

At t=0, let x(0) denote the initial state then $\emptyset(t)$ can also be defined by matrix equation

$$x(t) = \emptyset(t).x(0) \qquad 5.2.3$$

Which is the solution of the homogeneous equation (1) for $t \geq 0$. Now taking Laplace transform of equation (1)

$$SX(s) - x(0) = A.X(s)$$
$$SX(s) - A.X(s) = x(0)$$
$$[SI - A]X(s) = x(0)$$
$$X(s) = [SI - A]^{-1}x(0) \qquad 5.2.4$$

By eq. (3) $\quad x(0) = x(t)/\emptyset(t)$

Taking inverse Laplace of equation (4)

$$x(t) = L^{-1}[SI - A]^{-1}x(0)$$

Therefore $\quad \boxed{\emptyset(t) = L^{-1}[(SI - A)^{-1}]} \qquad 5.2.5$

Solving the above equation

Therefore $\quad \boxed{\emptyset(t) = e^{t}}$

Properties of State Transition Matrix

1. $\emptyset(0) = e^{\cdot 0} = I$(the identity matrix)2.
$\emptyset^{-1}(t) = \emptyset(-t)$
3. $\emptyset(t_2 - t_1).\emptyset(t_1 - t_0) = \emptyset(t_2 - t_0)$
4. $[\emptyset(t)]$=$\emptyset(kt)$
5. $\emptyset(t_1 + t_2)$=$\emptyset(t_1) + \emptyset(t_2)$

Example 5.1 Obtain the State Transaction Matrix $\emptyset(t)$ for the given system matrix

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Solution:

Given

then

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$[SI - A] = \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} S-1 & 0 \\ -1 & S-1 \end{bmatrix}$$

And

$$|SI - A| = (S-1)^2$$

then

$$\emptyset(S) = [(SI - A)^{-1}] = \frac{1}{(S-1)^2}\begin{bmatrix} S-1 & 0 \\ -1 & S-1 \end{bmatrix} = \begin{bmatrix} \frac{1}{S-1} & 0 \\ \frac{1}{(S-1)^2} & \frac{1}{S-1} \end{bmatrix}$$

Hence

$$\boxed{\emptyset(t) = \begin{bmatrix} e^t & 0 \\ te^t & e^t \end{bmatrix}}$$

5.3: Advantages of state variable approach over transfer function approach:

1. The state variable approach gives the information about internal states of the system.
2. The state variable approach is applicable for linear time varying as well as time invariant systems.
3. The state variable approach is applicable for multiple input multiple output system.
4. The state variable approach takes the initial conditions in to account.
5. It can be used to determine controllability as well as the observability of the system.

Disadvantages

1. It involves the matrix algebra, therefore mathematical calculations are complex.
2. It is difficult to determine poles and zeros of the system.

5.4: State Model:

$$\dot{x}(t) = A.x(t) + B.u(t)$$
$$y(t) = C.x(t) + D.u(t)$$

Where, X = State Matrix
A = System matrix of order $n \times n$
B = Input matrix of order $n \times m$
C = Output matrix of order $p \times n$
D = Transmission matrix of order $p \times m$

Representation of state model:

1. Physical variable representation,
2. Phase variable representation,
3. Canonical representation.

A system can be represented by many state models, therefore state model of a system is not unique. However the transfer function of a system is always unique.

While finding the state model, we will assume that voltage across capacitor and current through inductor is zero for $t < 0$.

5.5: Transfer Function Determination from state model:

Let we have a state model:

$$\dot{x}(t) = A.x(t) + B.u(t) \qquad 5.5.1$$
$$y(t) = C.x(t) + D.u(t) \qquad 5.5.2$$

Taking Laplace transform of equation 5.5.1

$$SX(S) = A.X(S) + B.U(S)$$
$$X(S)[SI - A] = B.U(S)$$
$$X(S) = [SI - A]^{-1}.B.U(S)$$

And from equation 5.5.2

$$Y(S) = C.X(S) + D.U(S)$$
$$Y(S) = C.[SI - A]^{-1}.B.U(S) + D.U(S)$$

$$\boxed{\frac{Y(S)}{U(S)} = C.[SI-A]^{-1}.B + D}$$

$$\boxed{\frac{Y(S)}{U(S)} = \frac{C.\{\mathrm{Adj}[SI-A]^{-1}\}.B + D|SI-A|}{|SI-A|}}$$

Characteristics equation $|SI-A| = 0$
The above equation gives the poles of the system.
Putting $S = \lambda$

$$|\lambda I - A| = 0$$

Here λ represents the Eigen values of A, therefore the Eigen values of A are poles of the System.

Example 5.2 Find out the transfer function of the system given below. Also determine the poles and zeros of the system.

$$\dot{} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 1 \end{bmatrix} + [1]u$$

Solution:
Given

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 \end{bmatrix}, D = [1]$$

Transfer function of the system is given by

$$\frac{Y(S)}{U(S)} = C.[SI-A]^{-1}.B + D$$

Therefore
$$[SI-A] = \begin{bmatrix} S & -1 \\ 2 & S+3 \end{bmatrix}$$

And
$$[SI-A]^{-1} = \frac{1}{S(S+3)+2}\begin{bmatrix} S+3 & 1 \\ -2 & S \end{bmatrix} = \frac{1}{(S+1)(S+2)}\begin{bmatrix} S+3 & 1 \\ -2 & S \end{bmatrix}$$

Now
$$\frac{Y(S)}{U(S)} = C.[SI-A]^{-1}.B + D$$

Therefore
$$\frac{Y(S)}{U(S)} = \begin{bmatrix} 1 & 1 \end{bmatrix} \frac{1}{(S+1)(S+2)} \begin{bmatrix} S+3 & 1 \\ -2 & S \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \end{bmatrix}$$

$$\frac{Y(S)}{U(S)} = \begin{bmatrix} 1 & 1 \end{bmatrix} \frac{1}{(S+1)(S+2)} \begin{bmatrix} 1 \\ S \end{bmatrix} + \begin{bmatrix} 1 \end{bmatrix}$$

$$\frac{Y(S)}{U(S)} = \frac{1}{(S+1)(S+2)} \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ S \end{bmatrix} + \begin{bmatrix} 1 \end{bmatrix}$$

$$\frac{Y(S)}{U(S)} = \frac{1}{(S+1)(S+2)}(S+1) + [1]$$

$$\boxed{\frac{Y(S)}{U(S)} = \frac{(S+1)(S+3)}{(S+1)(S+2)} = \frac{(S+3)}{(S+2)}}$$

Therefore one pole is at $S = -2$
One zero is at $S = -3$
And we have a pole zero cancellation at $S = -1$.

5.6: Solution of the state equation:

Let we have a state model:

$$\dot{x}(t) = A.x(t) + B.u(t) \qquad 5.6.1$$
$$y(t) = C.x(t) + D.u(t) \qquad 5.6.2$$

(i) For homogeneous equation[Unforced equation, u=0]

In this case
$$\dot{x}(t) = A.x(t) \qquad 5.6.3$$
$$y(t) = C.x(t) \qquad 5.6.4$$

Taking Laplace Transform of Equation 5.6.3

$$SX(S) - x(0) = A.X(S)$$

$$X(S)[SI - A] = x(0)$$
$$X(S) = [SI - A]^{-1}.x(0)$$
$$x(t) = L^{-1}[SI - A]^{-1}.x(0)$$
$$\boxed{x(t) = (t).x(0) = e^{at}.x(0)}$$

Where $(t) = e^{at} = L^{-1}[SI - A]^{-1} = STM$

(ii) For Non-homogeneous equation[Unforced equation, u=0]

Taking Laplace transform of equation 5.6.1

$$SX(S) - x(0) = A.X(S) + B.U(S)$$
$$X(S) = [SI - A]^{-1}.x(0) + [SI - A]^{-1}.B.U(S)$$

Taking inverse Laplace

$$x(t) = L^{-1}[SI - A]^{-1}.x(0) + L^{-1}\{[SI - A]^{-1}.B.U(S)\}$$

(iii) Solution of Non-homogeneous state equation in time domain:

Let $$\dot{x}(t) = A.x(t) + B.u(t) \qquad 5.6.1$$

Therefore
$$\frac{dx}{dt} - Ax = Bu$$
$$e^{-t}\frac{dx}{dt} - Axe^{-t} = e^{-t}Bu$$
$$\frac{d}{dt}[xe^{-t}] = e^{-t}Bu$$
$$xe^{-t} = \int_0^t e^{-t}Bu\,dt + k$$

Where, r is the time constant. Not at $t = 0, x(0) = k,$

herefore
$$x(t) = e^{t}x(o) + e^{t}\int_0^c e^{-c}Bu\,dr$$
$$\boxed{x(t) = e^{t}x(o) + \int_0^c e^{(t-c)}Bu\,dr}$$

Example 5.3 Consider a system described by:

$$\dot{} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u$$

Where u is the step input. Compute the solution of this equation, assuming the initial state vector $x(o) = [1 \quad 1]^T$**.**

Solution:

$$[SI - A] = \begin{bmatrix} S & -1 \\ 2 & S+3 \end{bmatrix}$$
$$[SI - A]^{-1} = \frac{1}{(S+1)(S+2)}\begin{bmatrix} S+3 & 1 \\ -2 & S \end{bmatrix}$$
$$[SI - A]^{-1}.x(0) = \frac{1}{(S+1)(S+2)}\begin{bmatrix} S+3 & 1 \\ -2 & S \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$= \frac{1}{(S+1)(S+2)}\begin{bmatrix} S+3+1 \\ -2+S \end{bmatrix}$$
$$= \frac{1}{(S+1)(S+2)}\begin{bmatrix} S+4 \\ S-2 \end{bmatrix}$$

then
$$L^{-1}\{[SI - A]^{-1}.x(0)\} = L^{-1}\begin{bmatrix} \frac{(S+4)}{(S+1)(S+2)} \\ \frac{(S-2)}{[(S+1)(S+2)]} \end{bmatrix} = L^{-1}\begin{bmatrix} \frac{3}{S+1} - \frac{2}{S+2} \\ \frac{-3}{S+1} + \frac{4}{S+2} \end{bmatrix}$$

$$L^{-1}\{[SI-A]^{-1}.x(0)\} = \begin{bmatrix} 3e^{-t} - 2e^{-2t} \\ -3e^{-t} + 4e^{-2t} \end{bmatrix}$$

and
$$[SI-A]^{-1}.B = \frac{1}{(S+1)(S+2)}\begin{bmatrix} S+3 & 1 \\ -2 & S \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{(S+1)(S+2)}\begin{bmatrix} 1 \\ S \end{bmatrix}$$

therefore
$$[SI-A]^{-1}.B.U(S) = \frac{1}{(S+1)(S+2)}\begin{bmatrix} 1 \\ S \end{bmatrix}\begin{bmatrix} \frac{1}{S} \end{bmatrix} = \begin{bmatrix} \frac{1}{S(S+1)(S+2)} \\ \frac{1}{(S+1)(S+2)} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2S} - \frac{1}{S+1} + \frac{\frac{1}{2}}{S+2} \\ \frac{1}{S+1} - \frac{1}{S+2} \end{bmatrix}$$

then
$$L^{-1}\{[SI-A]^{-1}.B.U(S)\} = \begin{bmatrix} \frac{1}{2} - e^{-t} - \frac{1}{2}e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}$$

Then
$$x(t) = L^{-1}\{[SI-A]^{-1}.x(0)\} + L^{-1}\{[SI-A]^{-1}.B.U(S)\}$$

solution

$$x(t) = \begin{bmatrix} 3e^{-t} - 2e^{-2t} \\ -3e^{-t} + 4e^{-2t} \end{bmatrix} + \begin{bmatrix} \frac{1}{2} - e^{-t} - \frac{1}{2}e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}$$

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} + 2e^{-t} - \frac{5}{2}e^{-2t} \\ -2e^{-t} + 3e^{-2t} \end{bmatrix}}$$

5.7: Forming State variable model by phase variables:

(a) Phase variable controllable conical form for numerators term:

Suppose

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^3 + b_1 s^2 + b_2 s + b_3}{s^3 + a_1 s^2 + a_2 s + a_3} \qquad 5.7.1$$

Dividing the equation by the highest power of s, i.e. s^3,

$$\frac{Y(s)}{U(s)} = \frac{b_0 + b_1 s^{-1} + b_2 s^{-2} + b_3 s^{-3}}{1 + a_1 s^{-1} + a_2 s^{-2} + a_3 s^{-3}} \qquad 5.7.2$$

Now introducing a dummy variable X(s) in numerator and denominator,

$$\frac{Y(s)}{U(s)} = \frac{b_0 + b_1 s^{-1} + b_2 s^{-2} + b_3 s^{-3}}{1 + a_1 s^{-1} + a_2 s^{-2} + a_3 s^{-3}}\frac{X(s)}{X(s)}$$

Separating the two equations

$$Y(s) = (b_0 + b_1 s^{-1} + b_2 s^{-2} + b_3 s^{-3})X(s) \qquad 5.7.3$$

And
$$U(s) = (1 + a_1 s^{-1} + a_2 s^{-2} + a_3 s^{-3})X(s)$$

therefore
$$X(s) = U(s) - (a_1 s^{-1} + a_2 s^{-2} + a_3 s^{-3})X(s) \qquad 5.7.4$$

The signal flow graph for equation 5.2.4 is shown below in figure 5.2.1(a)

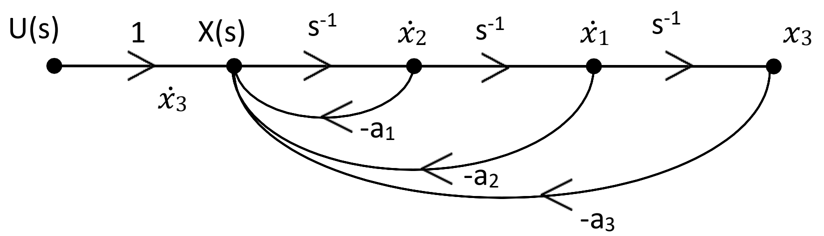


Figure 5.2.1 (a) Signal flow graph for equation 5.2.4

And the signal flow graph for equation 5.2.3 is shown below in figure 5.2.2

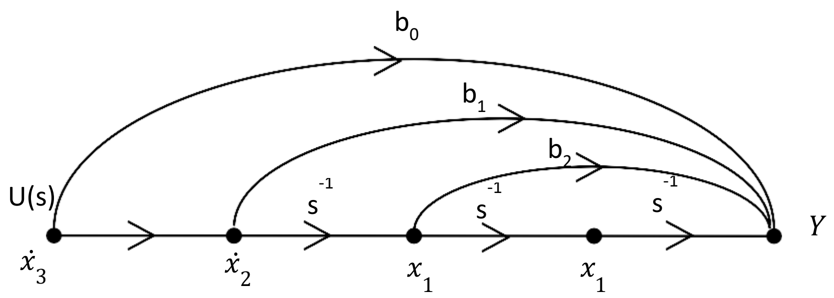


Figure 5.2.1(b) Signal flow graph for equation 5.2.3

Combining the above two signal flow graphs

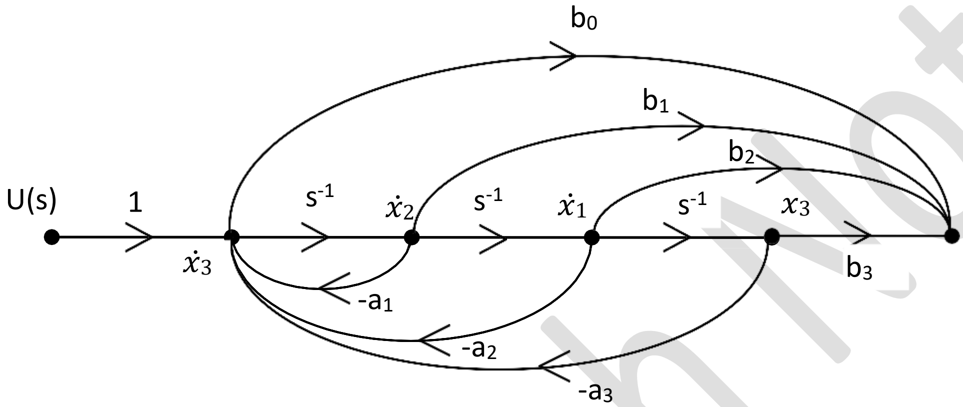


Figure 5.2.1(c) Combination of the SFG of Figure (a) & (b)

From the figure (c) let the various relations are as under

$$\dot{x}_1 = x_2, \dot{x}_2 = x_3, \dot{x}_3 = U(s) - a_1x_3 - a_2x_2 - a_3x_1$$

Therefore

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -a_3 & -a_2 & -a_1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 1\end{bmatrix}[u]$$

1. **State in State Space Analysis**: It refers to smallest set of variables whose knowledge at $t = t_0$ together with the knowledge of input for $t \geq t_0$ gives the complete knowledge of the behavior of the system at any time $t \geq t_0$.
2. **State Variables in State Space analysis**: It refers to the smallest set of variables which help us to determine the state of the dynamic system. State variables are defined by $x1(t), x2(t)$ .... $Xn(t)$.
3. **State Vector**: Suppose there is a requirement of n state variables in order to describe the complete behavior of the given system, then these n state variables are considered to be n components of a vector x(t). Such a vector is known as state vector.
4. **State Space**: It refers to the n dimensional space which has x_1 axis, x_2 axis x_n axis.

5.8: Representation of State Model using Transfer Function

Decomposition: It is defined as the process of obtaining the state model from the given transfer function. Now decompose the transfer function using three different ways:

1. Direct decomposition,
2. Cascade or series decomposition,
3. Parallel decomposition.

In all the above decomposition methods first convert the given transfer function into the differential equations also called the dynamic equations. After converting into differential equations take inverse Laplace transform of the above equation then corresponding to the type of decomposition can create model.

5.9 : Concept of Eigen Values and Eigen Vectors

The roots of characteristic equation are known as Eigen values or Eigen values of matrix A. Now there are some properties related to Eigen values and these properties are written below-

1. Any square matrix A and its transpose at have the same Eigen values.
2. Sum of Eigen values of any matrix A is equal to the trace of the matrix A.
3. Product of the Eigen values of any matrix A is equal to the determinant of the matrix A.
4. If we multiply a scalar quantity to matrix A then the Eigen values are also get multiplied by the same value of scalar.
5. If we inverse the given matrix A then its Eigen values are also get inverses.
6. If all the elements of the matrix are real then the Eigen values corresponding to that matrix are either real or exists in complex conjugate pair.
   Now there exists one Eigen vector corresponding to one Eigen value, if it satisfy the following condition (ek × I - A)Pk = 0. Where k = 1, 2, 3 ...n.

5.10 : State Transition Matrix and Zero State Response

Taking the state equations and taking their Laplace transformation,

$$sX(s) - X(0) = AX(s) - BU(s)$$

Now on rewriting the above equation

$$X(s) = [sI - A]^{-1} x(0) + [sI - A]^{-1} BU(s)$$

Let$[sI - A]^{-1} = (s)$ and taking the inverse Laplace of the above equation

$$X(t) = (t)\, x(0) + L^{-1}\ (t)\, BU(s)$$

The expression (t) is known as state transition matrix.

And $L^{-1}\ (t)\, BU(s)$ = zero state response.

Properties of the state transition matrix.

1. Substitute t = 0 in the above equation ie. $(0) = 1$.
2. Substitute t = -t in the (t) i.e. $(-t) = [\,(t)]^{-1}$.
3. Another important property $[\,(t)]^{n} = (nt)$.

5.11 : Controllability:

A system is said to be controllable if and only if the system states can be changed from a given state to a desired state, over a specific periodic of time, by changing the system input.

Test of Controllability

Let $\quad _c = [B{:}\ AB{:}\ A^2B{:}\ \dots\dots]$

or $\quad _c \cong [B{:}\ AB{:}\ A^2B]$

Here $_c$ is called Controllability Test Matrix.

According to KALMAN, the system will be completely controllable if rank of the matrix $_c$ is equal to order of the system.

If rank of the matrix $_c$ is less than the order of the system, then system is not completely controllable and the number of controllable states are given by $n - r$, where n is the order of the system and r is the rank of the matrix $_c$.

A system is controllable or "Controllable to the origin" when any state x_1 can be driven to the zero state x = 0 in a finite number of steps.

Rank of the system:

Let

$$_c = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

If $ad - bc \neq 0$, then rank of ${}_c$ is 2.
If $ad - bc = 0$, then
- (i) If $a, b, c, d \neq 0$, then rank is 1.
- (ii) If $a, b, c, d =$, then rank is 0.

5.12 : Observability:

A system is said to be observable if and only if any state of the system states can be determined at a given time on the basis of the knowledge of the output of the system.
The observability test matrix is denoted by Q_0 and is given by

$$ {}_o = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \end{bmatrix} $$

According to KALMAN, the system will be completely observable if rank of the matrix ${}_o$ is equal to order of the system.
If rank of the matrix ${}_o$ is less than the order of the system, then system is not completely observable and the number of observable states are given by $n - r$, where n is the order of the system and r is the rank of the matrix ${}_o$.

Example 5.4 Determine the Controllability of the following system

$$\dot{\ } = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u$$

Solution:
Given

$$B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Then

$${}_c = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

As ${}_c = 0$, hence system is not completely controllable.
Rank of the system $r = 1$, and $n = 2$, hence
Number of controllable states $n - r = 2 - 1 = 1$

Example 5.5 Determine the Observability of the following system

$$\dot{\ } = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r$$

$$y = \begin{bmatrix} 1 & 1 \end{bmatrix} + [1] r$$

Solution:
Given

$$C = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

$$CA = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} -2 & -2 \end{bmatrix}$$

Then

$${}_o = \begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix} \; then \; |{}_o| = 0$$

As ${}_o = 0$, hence system is not completely observable.
Rank of the system $r = 1$, and $n = 2$, hence
Number of observable states $n - r = 2 - 1 = 1$