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PES1201801585 Linear Algebra Assignment

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Sneha. B. M
Section - J

1. Find the equations of the parabola $y = A + Bx + Cx^2$ that passes through 3 points $(1, 1)$, $(2, -1)$ and $(3, 1)$ using gaussian elimination.

Soln 3 points given are: $(1, 1)$, $(2, -1)$, $(3, 1)$

equation of parabola is $y = A + Bx + Cx^2$

Substitute these three points in equation as it is given that these points passes through equation.

On substituting, we get

$$A + B + C = 1$$

$$A + 2B + 4C = -1$$

$$A + 3B + 9C = 1$$

In Matrix form:

$$Ax = b$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Augmented matrix:

$$[A \quad b] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & -1 \\ 1 & 3 & 9 & 1 \end{array}\right]$$

$$R_2 \to R_2 - R_1 \; ; \; R_3 \to R_3 - R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & -2 \\ 0 & 2 & 8 & 0 \end{array}\right]$$

$$R_3 \to R_3 - 2R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 2 & 4 \end{array}\right]$$

On doing back substitution, we get

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$2C = 4 \Rightarrow \boxed{C = 2}$

$B + 3C = -2 \Rightarrow B + 3(2) = -2$

$B = -2 - 6 \Rightarrow \boxed{B = -8}$

$A + B + C = 1 \Rightarrow A - 8 + 2 = 1$

$A = 1 + 6 \Rightarrow \boxed{A = 7}$

$\boxed{A = 7, B = -8, C = 2}$ $y = 2x^2 - 8x + 7$

$\therefore$ The equation of parabola is $\boxed{y = 2x^2 - 8x + 7}$

2. Find the LU decomposition for the matrix

$$A = \begin{bmatrix} 2 & 5 & 2 & -5 \\ 4 & 12 & 3 & -14 \\ -10 & -29 & -5 & 38 \\ 10 & 21 & 21 & -6 \end{bmatrix}$$

Soln Applying Gaussian Elimination

$$A = \begin{bmatrix} 2 & 5 & 2 & -5 \\ 4 & 12 & 3 & -14 \\ -10 & -29 & -5 & 38 \\ 10 & 21 & 21 & -6 \end{bmatrix}_{4\times 4}$$

$R_2 \to R_2 - 2R_1$; $R_3 \to R_3 + 5R_1$; $R_4 \to R_4 - 5R_1$

$$A \sim \begin{bmatrix} 2 & 5 & 2 & -5 \\ 0 & 2 & -1 & -4 \\ 0 & -4 & 5 & 13 \\ 0 & -4 & 11 & 19 \end{bmatrix}$$

$l_{21} = 2$; $l_{31} = -5$; $l_{41} = 5$

$R_3 \to R_3 + 2R_2$; $R_4 \to R_4 + 2R_2$

$$A \sim \begin{bmatrix} 2 & 5 & 2 & -5 \\ 0 & 2 & -1 & -4 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 9 & 11 \end{bmatrix}$$

$l_{32} = -2$; $l_{42} = -2$

$R_4 \to R_4 - 3R_2$

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$$A \sim \begin{bmatrix} 2 & 5 & 2 & -5 \\ 0 & 2 & -1 & -4 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & -4 \end{bmatrix} = U$$

$l_{43} = 3$

$$\therefore L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -5 & -2 & 1 & 0 \\ 5 & -2 & 3 & 1 \end{bmatrix}$$

$$A = LU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -5 & -2 & 1 & 0 \\ 5 & -2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 5 & 2 & -5 \\ 0 & 2 & -1 & -4 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 5 & 2 & -5 \\ 4 & 12 & 3 & -14 \\ -10 & -29 & -5 & 38 \\ 10 & 21 & 21 & -6 \end{bmatrix}$$

$\therefore A = LU$.

3. Let $T : R^3 \to R^3$ defined by
$T(x, y, z) = (x + 2y - z, y + z, x + y - 2z)$.

i) Find the matrix T relative to the standard basis of R^3.

ii) Find the basis for four fundamental subspaces of T.

iii) Find the eigen values and eigen vectors of T.

iv) Decompose $T = QR$.

Soln i) The standard basis of R^3 $(1,0,0), (0,1,0), (0,0,1)$.

$$T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + 2y - z \\ y + z \\ x + y - 2z \end{bmatrix}$$

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$$1\overset{x}{\begin{bmatrix}1\\0\\0\end{bmatrix}} + 2\overset{y}{\begin{bmatrix}0\\1\\0\end{bmatrix}} - 1\overset{z}{\begin{bmatrix}0\\0\\1\end{bmatrix}} \Rightarrow x + 2y - z$$

$$0\overset{x}{\begin{bmatrix}1\\0\\0\end{bmatrix}} + 1\overset{y}{\begin{bmatrix}0\\1\\0\end{bmatrix}} + 1\overset{z}{\begin{bmatrix}0\\0\\1\end{bmatrix}} \Rightarrow y + z$$

$$1\overset{x}{\begin{bmatrix}1\\0\\0\end{bmatrix}} + 1\overset{y}{\begin{bmatrix}0\\1\\0\end{bmatrix}} - 2\overset{z}{\begin{bmatrix}0\\0\\1\end{bmatrix}} \Rightarrow x + y - 2z$$

$$\therefore T = \begin{bmatrix}1 & 2 & -1\\0 & 1 & 1\\1 & 1 & -2\end{bmatrix}$$

ii) $$T = \left[\begin{array}{ccc|c}1 & 2 & -1 & b_1\\0 & 1 & 1 & b_2\\1 & 1 & -2 & b_3\end{array}\right]$$

$R_3 \to R_3 - R_1$

$$[T \quad b] \sim \left[\begin{array}{ccc|c}1 & 2 & -1 & b_1\\0 & 1 & 1 & b_2\\0 & -1 & -1 & b_3 - b_1\end{array}\right]$$

$R_3 \to R_3 + R_2$

$$[T \quad b] \sim \left[\begin{array}{ccc|c}1 & 2 & -1 & b_1\\0 & 1 & 1 & b_2\\0 & 0 & 0 & b_3 - b_1 + b_2\end{array}\right] \quad \cancel{R_1 \to R_1 - 2R_2}$$

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2 pivots variables. Therefore basis for column space is the column containing pivots

$\therefore$ Basis for $C(T) = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \right\}$

Similarly, basis for row space are the rows containing pivots

i.e. basis for $C(T^T) = \left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$

to find Null space, represent in the form $Tx = 0$ and express pivot variables in terms of free variables.

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

u and v are pivot variables. w is a free variable.

$v + w = 0$

$v = -w$

$u + 2v + (-w) = 0$

$u + 2(-w) - w = 0$

$u = 3w$

$$\therefore x = \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 3w \\ -w \\ w \end{bmatrix} = w \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$$

$\therefore$ Basis for Null space $N(T) = \left\{ \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \right\}$

iii) $T = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix}$

The characteristic equation of T is $|T - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 & -1 \\ 0 & 1-\lambda & 1 \\ 1 & 1 & -2-\lambda \end{vmatrix} = 0$$

$= (1-\lambda)((1-\lambda)(-2-\lambda) - 1) - 2(-1) - 1(-(1-\lambda)) = 0$

$= (1-\lambda)(-2-\lambda+2\lambda+\lambda^2-1) + 2 + 1 - \lambda = 0$

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$= (1-\lambda)[\lambda^2+\lambda-3]+3-\lambda=0$

$= \cancel{\lambda^2}+\cancel{\lambda}-\cancel{3}-\lambda^3-\cancel{\lambda^2}+3\lambda+\cancel{3}-\cancel{\lambda}=0$

$= 3\lambda-\lambda^3=0 \Rightarrow \lambda(\lambda^2-3)=0 \Rightarrow \lambda=0$ or $\lambda=\pm\sqrt{3}$

$\therefore$ eigen values $\lambda = 0, \sqrt{3}, -\sqrt{3}$

when $\lambda=0$, $(T-\lambda I)x=0 \Rightarrow Tx=0$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$ consider two independent rows,

$$\frac{x}{3} = \frac{-y}{1} = \frac{z}{1} \quad \left[\because \frac{x}{\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix}}\right]$$

eigen vectors for $\lambda=0$ is $k\begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$

when $\lambda=\sqrt{3}$,

$(T-\lambda I)x=0 \Rightarrow (T-\sqrt{3}I)x=0$

$$\begin{bmatrix} 1-\sqrt{3} & 2 & -1 \\ 0 & 1-\sqrt{3} & 1 \\ 1 & 1 & -2-\sqrt{3} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

consider first two rows,

$$\frac{x}{\begin{vmatrix} 2 & -1 \\ 1-\sqrt{3} & 1 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 1-\sqrt{3} & -1 \\ 0 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 1-\sqrt{3} & 2 \\ 0 & 1-\sqrt{3} \end{vmatrix}}$$

$$\Rightarrow \frac{x}{-\sqrt{3}+3} = \frac{y}{\sqrt{3}-1} = \frac{z}{4-2\sqrt{3}}$$

$\therefore$ eigen vector for $k=\sqrt{3}$ is $k\begin{pmatrix} -\sqrt{3}+3 \\ \sqrt{3}-1 \\ 4-2\sqrt{3} \end{pmatrix}$

when $\lambda=-\sqrt{3}$, $(T-\lambda I)x=0 \Rightarrow (T+\sqrt{3}I)x=0$

$$\begin{bmatrix} 1+\sqrt{3} & 2 & -1 \\ 0 & 1+\sqrt{3} & 1 \\ 1 & 1 & -2+\sqrt{3} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

considering first two independent rows

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$$\frac{x}{\begin{vmatrix} 2 & -1 \\ 1+\sqrt{3} & 1 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 1+\sqrt{3} & -1 \\ 0 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 1+\sqrt{3} & 2 \\ 0 & 1+\sqrt{3} \end{vmatrix}}$$

$$\Rightarrow \frac{x}{3+\sqrt{3}} = \frac{y}{-1-\sqrt{3}} = \frac{z}{4+2\sqrt{3}}$$

$\therefore$ eigen vector for $\lambda = -\sqrt{3}$ is $K\begin{pmatrix} 3+\sqrt{3} \\ -1-\sqrt{3} \\ 4+2\sqrt{3} \end{pmatrix}$

iv) $T = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix}$

let $a = (1, 0, 1)$; $b = (2, 1, 1)$; $c = (-1, 1, -2)$

$$q_1 = \frac{a}{\|a\|}$$

$$\|a\| = \sqrt{1^2 + 0^2 + 1^2} = \sqrt{2}$$

$$q_1 = (1, 0, 1) = \frac{1}{\sqrt{2}} = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$$

$$\boxed{e = b - (q_1^T b)\, q_1}$$

$$= \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} - \left(\begin{bmatrix} 1/\sqrt{2} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}\right)\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} - \frac{3}{\sqrt{2}}\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 3/2 \\ 0 \\ 3/2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1/2 \\ 1 \\ -1/2 \end{bmatrix}$$

$$q_2 = \frac{e}{\|e\|} \qquad \|e\| = \sqrt{\left(\frac{1}{2}\right)^2 + 1^2 + \left(-\frac{1}{2}\right)^2} = \sqrt{\frac{3}{2}}$$

$$= (1/2, 1, -1/2) \cdot \sqrt{2}/\sqrt{3} = \left(\frac{1}{\sqrt{6}}, \frac{\sqrt{2}}{\sqrt{3}}, \frac{-1}{\sqrt{6}}\right)$$

$$\boxed{E = c - (q_1^T c)\, q_1 - (q_2^T c)\, q_2}$$

$$\Rightarrow (q_1^T c)\, q_1$$

$$= \left(\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}\right)\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} = -\frac{3}{\sqrt{2}}\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{pmatrix} -3/2 \\ 0 \\ -3/2 \end{pmatrix}$$

$(q_2^T c)\, q_2$

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$$= \left(\left[\frac{1}{\sqrt6}, \frac{\sqrt2}{\sqrt3}, \frac{-1}{\sqrt6} \right] \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} \right) \begin{bmatrix} 1/\sqrt6 \\ \sqrt2/\sqrt3 \\ -1/\sqrt6 \end{bmatrix}$$

$$= \frac{3}{\sqrt6} \begin{bmatrix} 1/\sqrt6 \\ \sqrt2/\sqrt3 \\ -1/\sqrt6 \end{bmatrix} = \begin{pmatrix} 1/2 \\ 1 \\ -1/2 \end{pmatrix}$$

$$q_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} - \begin{bmatrix} -3/2 \\ 0 \\ -3/2 \end{bmatrix} - \begin{bmatrix} 1/2 \\ 1 \\ -1/2 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$Q = \begin{bmatrix} | & | & | \\ q_1 & q_2 & q_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1/\sqrt2 & 1/6 & 0 \\ 0 & \sqrt2/\sqrt3 & 0 \\ 1/\sqrt2 & -1/\sqrt6 & 0 \end{bmatrix}$$

$$R = \begin{bmatrix} q_1^Ta & q_1^Tb & q_1^Tc \\ 0 & q_2^Tb & q_2^Tc \\ 0 & 0 & q_3^Tc \end{bmatrix} = \begin{bmatrix} \sqrt2 & 3/\sqrt2 & -3/\sqrt2 \\ 0 & 3/\sqrt6 & 3/\sqrt6 \\ 0 & 0 & 0 \end{bmatrix}$$

$$T = QR = \begin{bmatrix} 1/\sqrt2 & 1/6 & 0 \\ 0 & \sqrt2/\sqrt3 & 0 \\ 1/\sqrt2 & -1/\sqrt6 & 0 \end{bmatrix} \begin{bmatrix} \sqrt2 & 3/\sqrt2 & -3/\sqrt2 \\ 0 & 3/\sqrt6 & 3/\sqrt6 \\ 0 & 0 & 0 \end{bmatrix}$$

4. Fit a best straight line $y = c + dx$ for the following data using least square principles.

x	-4	1	2	3
y	4	6	10	8

Soln In the form of matrix $Ax = b$

$$\begin{bmatrix} 1 & -4 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}_{4\times 2} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 10 \\ 8 \end{bmatrix}$$

$(4>2)$ i.e. $m>n$ system is inconsistent. Hence we use least square method.

Normal equation $A^TA\hat{x} = A^Tb$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ -4 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & -4 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \hat{x} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -4 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 6 \\ 10 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 2 \\ 2 & 30 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 28 \\ 34 \end{bmatrix}$$

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$$\begin{bmatrix} C \\ D \end{bmatrix} = \frac{1}{116}\begin{bmatrix} 30 & -2 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 28 \\ 34 \end{bmatrix}$$

$$\begin{bmatrix} C \\ D \end{bmatrix} = \frac{1}{116}\begin{bmatrix} 772 \\ 80 \end{bmatrix} = \begin{bmatrix} 193/29 \\ 20/29 \end{bmatrix}$$

$\therefore$ Best fit line $y = C + Dx$

$$y = \frac{193}{29} + \frac{20}{9}x$$

5. Find the projection matrices P and Q onto the plane $x_1 + x_2 + 3x_3 + 4x_5 = 0$ and its orthogonal complement respectively.

Soln Equation in the form of matrix $Ax = b$

$$\underbrace{\begin{bmatrix} 1 & 1 & 3 & 0 & 4 \end{bmatrix}}_{1\times5}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}_{5\times1} = \begin{bmatrix} 0 \end{bmatrix}$$

pivot   free

To find a matrix with columns as span of plane we find $N(A) = \{(-1,1,0,0,0), (-3,0,1,0,0), (0,0,0,1,0), (-4,0,0,0,1)\}$.

Plane:

$$B = \begin{bmatrix} -1 & -3 & 0 & -4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{5\times4}$$

Projection : $P = B(B^TB)^{-1}B^T$

$$= \begin{bmatrix} -1 & -3 & 0 & -4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\left(\begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ -3 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -4 & 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & -3 & 0 & -4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right)^{-1} \cdot B^T$$

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$$= B\left(\begin{bmatrix} 2 & 3 & 0 & 4 \\ 3 & 10 & 0 & 12 \\ 0 & 0 & 1 & 0 \\ +4 & 12 & 0 & 17 \end{bmatrix}\right) B^T.$$

$$= \begin{bmatrix} -1 & -3 & 0 & -4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{5\times 4} \left(\begin{bmatrix} \frac{26}{27} & \frac{-3}{27} & 0 & \frac{-4}{27} \\ -3/27 & 18/27 & 0 & -12/27 \\ 0 & 0 & 1 & 0 \\ -4/27 & \frac{-12}{27} & 0 & \frac{11}{27} \end{bmatrix}\right)$$

$$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ -3 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -4 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1/27 & -3/27 & 0 & -4/27 \\ 26/27 & -3/27 & 0 & -4/27 \\ -3/27 & 18/27 & 0 & -12/27 \\ 0 & 0 & 1 & 0 \\ -4/27 & -12/27 & 0 & 11/27 \end{bmatrix}_{5\times 4} \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ -3 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -4 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 26/27 & -1/27 & -3/27 & 0 & -4/27 \\ -1/27 & 26/27 & -3/27 & 0 & -4/27 \\ -3/27 & -3/27 & 18/27 & 0 & -12/27 \\ 0 & 0 & 0 & 1 & 0 \\ -4/27 & -4/27 & -12/27 & 0 & 11/27 \end{bmatrix}_{5\times 5}$$

Orthogonal complement of $N(A)$ is $C(A^T)$

$$C(A^T) = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 0 \\ 4 \end{bmatrix} = C$$

$$C = A(A^TA)^{-1}A^T$$

$$= \begin{bmatrix} 1 \\ 1 \\ 3 \\ 0 \\ 4 \end{bmatrix}_{5\times 1} \left(\begin{bmatrix} 1 & 1 & 3 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 3 \\ 0 \\ 4 \end{bmatrix}\right)^{-1} \begin{bmatrix} 1 & 1 & 3 & 0 & 4 \end{bmatrix}_{1\times 5}$$

$$= \begin{bmatrix} 1 \\ 1 \\ 3 \\ 0 \\ 4 \end{bmatrix} (27) \begin{bmatrix} 1 & 1 & 3 & 0 & 4 \end{bmatrix}$$

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$$Q = 27\begin{bmatrix} 1 & 1 & 3 & 0 & 4 \\ 1 & 1 & 3 & 0 & 4 \\ 3 & 3 & 9 & 0 & 12 \\ 0 & 0 & 0 & 0 & 0 \\ 4 & 4 & 12 & 0 & 16 \end{bmatrix}$$

6. For which range of number 'a', the matrix A is positive definite?

$$A = \begin{bmatrix} a & 2 & 2 \\ 2 & a & 2 \\ 2 & 2 & a \end{bmatrix}$$

Which 3X3 matrix (symmetric) B produces this function $f = x^T A x$?
where $f = 2(x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3)$.

Soln consider matrix, $A = \begin{bmatrix} a & 2 & 2 \\ 2 & a & 2 \\ 2 & 2 & a \end{bmatrix}$

By det semi matrices:

$a_1 \Rightarrow a > 0$ - ①

$a_2 \Rightarrow a^2 - 4 > 0$

$a^2 > 2 \Rightarrow \boxed{a > 2}$ - ②

$\llcorner a < 2$ - not a case due to ①

$a_3 \Rightarrow |A| = a(a^2-4) - 2(2a-4) + 2(4-2a) > 0$

$= a^3 - 4a - 4a + 8 + 8 - 4a > 0$

$= a^3 - 12a + 16$

$a > -4$; $a > 2$; $a > 2$

∴ Range of a for A to be positive definite

$\Rightarrow \boxed{a > 2}$ ∴ $a \in (2, \infty)$ $2 < a < \infty$

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$f = x^T \cdot A \cdot x$

given $f(x) = 2(x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3)$

$= 2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_2x_3$ — ①

$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$\therefore f = [x_1 \; x_2 \; x_3] \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ — ②

compare ① & ②

$a_{11} = 2$ $a_{22} = 2$ $a_{33} = 2$

$a_{12} = -1$ $a_{23} = -1$

$a_{21} = -1$ $a_{32} = -1$

$B = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

7. Find the SVD of A, $U \Sigma V^T$ when

$A = \begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix}$

Soln $A = \begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix}$

$A^T \cdot A = \begin{bmatrix} -3 & 6 & 6 \\ 1 & -2 & -2 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix} = \begin{bmatrix} 81 & -27 \\ -27 & 9 \end{bmatrix}$

eigen values of $A^T \cdot A$ are,

$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 81-\lambda & -27 \\ -27 & 9-\lambda \end{vmatrix} = 0$

$(81-\lambda)(9-\lambda) - 27^2 = 0$

$81 \times 9 - 81\lambda - 9\lambda + \lambda^2 - 27^2 = 0$

$\lambda^2 - 90\lambda = 0 \Rightarrow \lambda = 0, 90$

$\Rightarrow$ eigen values.

to find eigen vectors for $A \cdot A^T$, $\boxed{\begin{matrix} \lambda_1 = 90 \\ \lambda_2 = 0 \end{matrix}}$

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When $\lambda = 90$, $(A - \lambda I)x = 0$.

$$\begin{bmatrix} 81-90 & -27 \\ -27 & 9-90 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -9 & -27 \\ -27 & -81 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-9x - 27y = 0$$

$$x = -3y$$

$$X = \begin{bmatrix} -3y \\ y \end{bmatrix} = y\begin{bmatrix} -3 \\ 1 \end{bmatrix} \qquad X_1 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

When $\lambda = 0$, $Ax = 0$

$$\begin{bmatrix} 81 & -27 \\ -27 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad X = \begin{bmatrix} x \\ 3x \end{bmatrix} = x\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$81x - 27y = 0 \qquad x_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$3x = y$$

$$v_1 = \frac{x_1}{\|x_1\|} \qquad \|x_1\| = \sqrt{(-3)^2 + 1^2} = \sqrt{10}$$

$$= \begin{pmatrix} -3 \\ 1 \end{pmatrix} \frac{1}{\sqrt{10}} = \begin{pmatrix} -3/\sqrt{10} \\ 1/\sqrt{10} \end{pmatrix}$$

$$V_2 = \frac{x_2}{\|x_2\|} \qquad \|x_2\| = \sqrt{1^2 + 3^2} = \sqrt{10}$$

$$= \begin{pmatrix} 1 \\ 3 \end{pmatrix} \frac{1}{\sqrt{10}} = \begin{pmatrix} 1/\sqrt{10} \\ 3/\sqrt{10} \end{pmatrix}, \text{ We know, } V = \begin{bmatrix} | & | \\ V_1 & V_2 \\ | & | \end{bmatrix}$$

$$\therefore V = \begin{bmatrix} -3/\sqrt{10} & 1/\sqrt{10} \\ 1/\sqrt{10} & 3/\sqrt{10} \end{bmatrix} \qquad \sigma_1 = \sqrt{90}, \quad \sigma_2 = 0$$

$$\Sigma = \begin{bmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \sqrt{90} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$u_1 = \frac{Av_1}{\sigma_1} = \frac{\begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix} \begin{bmatrix} -3/\sqrt{10} \\ 1/\sqrt{10} \end{bmatrix}}{\sqrt{90}}$$

$$= \frac{1}{\sqrt{90}} \begin{bmatrix} \sqrt{10} \\ -2\sqrt{10} \\ -2\sqrt{10} \end{bmatrix} = \begin{bmatrix} 1/3 \\ -2/3 \\ -2/3 \end{bmatrix}$$

u_2 and u_3 are orthogonal vectors

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associated with eigen vectors of $A \cdot A^T$.
but since $\sigma_2 = 0$, we cannot find u_2 and u_3.
eigen vectors for $A \cdot A^T$,
when $\lambda = 0$ $(A \cdot A^T - 0 \cdot I)x = 0$.
$A \cdot A^T \cdot x = 0$

$$A \cdot A^T = \begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix} \begin{bmatrix} -3 & 6 & 6 \\ 1 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 10 & -20 & -20 \\ -20 & 40 & 40 \\ -20 & 40 & 40 \end{bmatrix}$$

$(A \cdot A^T)x = 0$

$$\begin{bmatrix} 10 & -20 & -20 \\ -20 & 40 & 40 \\ -20 & 40 & 40 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This is a rank 1 matrix. ∴ Null space for this is calculated as,
$10x - \cancel{20z}\ 20y - 20z = 0$ where x is a pivot variable.
$\therefore x = 2y + 2z$
y and z are free variables.

$$x = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2y + 2z \\ y \\ z \end{bmatrix} = y\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}.$$

$x_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ $x_3 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ are eigen vectors of $A \cdot A^T$ where $\lambda = 0$

We know that for a symmetric matrix, the eigen vectors corresponding to distinct eigen values are orthogonal.
Since $A \cdot A^T$ is a symmetric matrix,
x_2 & x_3 are orthogonal to u_1.
to find u_2, apply Gram-Schmidth process

$u_2 = \dfrac{x_2}{\|x_2\|}$ $\qquad \|x_2\| = \sqrt{2^2 + 1^2 + 0^2} = \sqrt{5}$

$$u_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \frac{1}{\sqrt{5}} = \begin{pmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{pmatrix}$$

to find u_3, find vectors orthogonal to both u_1 & u_2.

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$$c = x_3 - (u_1^T x_3)u_1 - (u_2^T x_3)u_2$$

$$= \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} - \left(\begin{bmatrix} \frac{1}{3} & \frac{-2}{3} & \frac{-2}{3} \end{bmatrix}\right)\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}\begin{bmatrix} 1/3 \\ -2/3 \\ -2/3 \end{bmatrix} - \begin{bmatrix} \frac{2}{\sqrt5} & \frac{1}{\sqrt5} & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}\begin{bmatrix} 2/\sqrt5 \\ 1/\sqrt5 \\ 0 \end{bmatrix}$$

$$c = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} - 0 - \begin{bmatrix} 8/5 \\ 4/5 \\ 0 \end{bmatrix} = \begin{bmatrix} 2/5 \\ -4/5 \\ 1 \end{bmatrix}$$

$$u_3 = \frac{c}{\|c\|} \qquad \|c\| = \sqrt{(2/5)^2 + (-4/5)^2 + 1^2} = \frac{3}{\sqrt5}$$

$$u_3 = \begin{pmatrix} 2/5 \\ -4/5 \\ 1 \end{pmatrix} \frac{\sqrt5}{3} = \begin{pmatrix} 2/3\sqrt5 \\ -4/3\sqrt5 \\ \sqrt5/3 \end{pmatrix}$$

$$U = \begin{bmatrix} 1/3 & 2/\sqrt5 & 2/3\sqrt5 \\ -2/3 & 1/\sqrt5 & -4/3\sqrt5 \\ -2/3 & 0 & \sqrt5/3 \end{bmatrix}$$

$$\therefore A = U \Sigma V^T$$

$$= \begin{bmatrix} 1/3 & 2/\sqrt5 & 2/3\sqrt5 \\ -2/3 & 1/\sqrt5 & -4/3\sqrt5 \\ -2/3 & 0 & \sqrt5/3 \end{bmatrix}\begin{bmatrix} 3\sqrt{10} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} -3/\sqrt{10} & 1/\sqrt{10} \\ 1/\sqrt{10} & 3/\sqrt{10} \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 & 2/\sqrt5 & 2/3\sqrt5 \\ -2/3 & 1/\sqrt5 & -4/3\sqrt5 \\ -2/3 & 0 & \sqrt5/3 \end{bmatrix}\begin{bmatrix} 3\sqrt{10} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} -3/\sqrt{10} & 1/\sqrt{10} \\ 1/\sqrt{10} & 3/\sqrt{10} \end{bmatrix}$$