Alka

ENGINEERING MECHANICS

(THIRD EDITION)

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PART—I

1

Introduction to Engineering Mechanics

The state of rest and state of motion of the bodies under the action of different forces has engaged the attention of philosophers, mathematicians and scientists for many centuries. The branch of physical science that deals with the state of rest or the state of motion is termed as **Mechanics.** Starting from the analysis of rigid bodies under gravitational force and simple applied forces the mechanics has grown to the analysis of robotics, aircrafts, spacecrafts under dynamic forces, atmospheric forces, temperature forces etc.

Archimedes (287–212 BC), Galileo (1564–1642), Sir Issac Newton (1642–1727) and Einstein (1878–1955) have contributed a lot to the development of mechanics. Contributions by Varignon, Euler, D. Alembert are also substantial. The mechanics developed by these researchers may be grouped as.

(*i*) Classical mechanics/Newtonian mechanics

(*ii*) Relativistic mechanics

(*iii*) Quantum mechanics/Wave mechanics.

Sir Issac Newton, the principal architect of mechanics, consolidated the philosophy and experimental findings developed around the state of rest and state of motion of the bodies and put forth them in the form of three laws of motion as well as the law of gravitation. The mechanics based on these laws is called **Classical mechanics** or **Newtonian mechanics.**

Albert Einstein proved that Newtonian mechanics fails to explain the behaviour of high speed (speed of light) bodies. He put forth the theory of **Relativistic Mechanics.**

Schrödinger (1887–1961) and Broglie (1892–1965) showed that Newtonian mechanics fails to explain the behaviour of particles when atomic distances are concerned. They put forth the theory of **Quantum Mechanics.**

Engineers are keen to use the laws of mechanics to actual field problems. Application of laws of mechanics to field problem is termed as **Engineering Mechanics.** For all the problems between atomic distances to high speed distances Classical/Newtonian mechanics has stood the test of time and hence that is the mechanics used by engineers. Hence in this text classical mechanics is used for the analysis of engineering problems.

1.1. CLASSIFICATION OF ENGINEERING MECHANICS

Depending upon the body to which the mechanics is applied, the engineering mechanics is classified as

(*a*) Mechanics of Solids, and

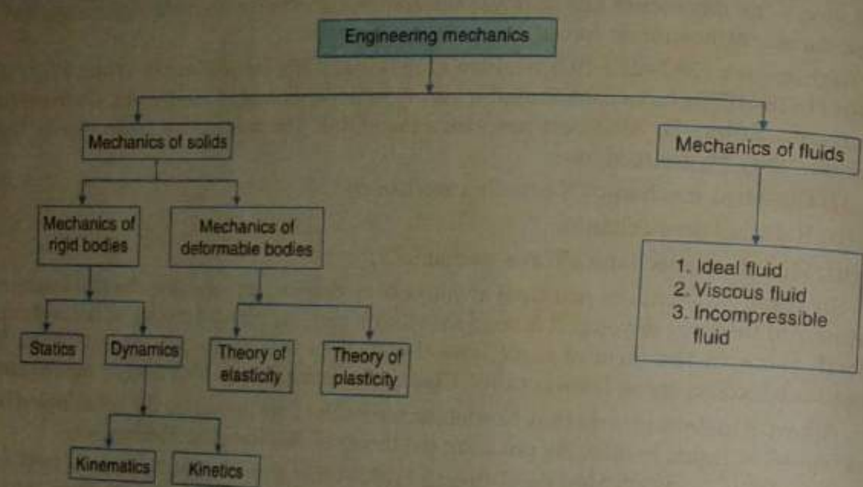
(*b*) Mechanics of Fluids.

The solid mechanics is further classified as mechanics of rigid bodies and mechanics of deformable bodies. The body which will not deform or the body in which deformation can be neglected in the analysis, are called as **Rigid bodies**. The mechanics of the rigid bodies dealing with the bodies at rest is termed as **Statics** and that dealing with bodies in motion is called

Dynamics. The dynamics dealing with the problems without referring to the forces causing the motion of the body is termed as **Kinematics** and if it deals with the forces causing motion also, is called **Kinetics**.

If the internal stresses developed in a body are to be studied, the deformation of the body should be considered. This field of mechanics is called **Mechanics of Deformable Bodies/Strength of Materials/Solid Mechanics**. This field may be further divided into **Theory of Elasticity** and **Theory of Plasticity**.

Liquid and gases deform continuously with application of very small shear forces. Such materials are called **Fluids**. The mechanics dealing with behaviour of such materials is called **Fluid Mechanics**. Mechanics of ideal fluids, mechanics of viscous fluid and mechanics of incompressible fluids are further classification in this area. The classification of mechanics is summarised below in flow chart.



1.2. BASIC TERMINOLOGIES IN MECHANICS

The following are the terms basic to study mechanics, which should be understood clearly:

Mass

The quantity of the matter possessed by a body is called mass. The mass of a body will not change unless the body is damaged and part of it is physically separated. When a body is taken out in a spacecraft, the mass will not change but its weight may change due to change in gravitational force. Even the body may become weightless when gravitational force vanishes but the mass remain the same.

Time

Time is the measure of succession of events. The successive event selected is the rotation of earth about its own axis and this is called a day. To have convenient units for various activities, a day is divided into 24 hours, an hour into 60 minutes and a minute into 60 seconds. Clocks are the instruments developed to measure time. To overcome difficulties due to irregularities in the earth's rotation, the unit of time is taken as second which is defined as the duration of 9192631770 period of radiation of the cesium-133 atom.

Space

The geometric region in which study of body is involved is called **space**. A point in the space may be referred with respect to a predetermined point by a set of linear and angular measurements. The reference point is called the origin and set of measurements as '*coordinates*'. If coordinates involve only in mutually perpendicular directions they are known as *Cartesian coordinates*. If the coordinates involve angle and distances, it is termed as *polar coordinate system*.

Length

It is a concept to measure linear distances. The diameter of a cylinder may be 300 mm, the height of a building may be 15 m. Actually metre is the unit of length. However depending upon the sizes involved micro, milli or kilo metre units are used for measurement. A metre is defined as length of the standard bar of platinum-iridium kept at the International Bureau of Weights and Measures. To overcome difficulties of accessibility and reproduction, now meter is defined as 1690763.73 wavelength of krypton-86 atom.

Displacement

Displacement is defined as the distance moved by a body/particle in the specified direction. Referring to Fig. 1.1, if a body moves from position A to position B in the x-y plane shown, its displacement in x-direction is AB' and its displacement in y-direction is $B'B$.

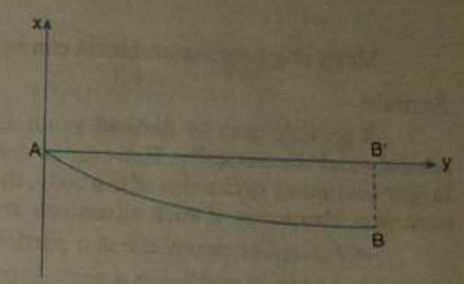


Fig. 1.1

Velocity

The rate of change of displacement with respect to time is defined as velocity.

Acceleration

Acceleration is the rate of change of velocity with respect to time. Thus

$$a = \frac{dv}{dt}, \text{ where } v \text{ is velocity} \qquad ...(1.1)$$

Momentum

The product of mass and velocity is called momentum. Thus

$$\text{Momentum} = \text{Mass} \times \text{Velocity} \qquad ...(1.2)$$

Continuum

A body consists of several matters. It is a well known fact that each particle can be subdivided into molecules, atoms and electrons. It is not possible to solve any engineering problem by treating a body as a conglomeration of such discrete particles. The body is assumed to consist of a continuous distribution of matter. In other words, the body is treated as **continuum.**

Rigid Body

A body is said to be rigid, if the relative positions of any two particles in it do not change under the action of the forces. In Fig. 1.2 (a) points A and B are the original position in a body.

After application of a system of forces F_1, F_2, F_3, the body takes the position as shown in Fig. 1.2 (*b*). A' and B' are the new positions of A and B. If the body is treated as rigid, the relative position of $A'B'$ and AB are the same *i.e.*,

$$A'B' = AB.$$

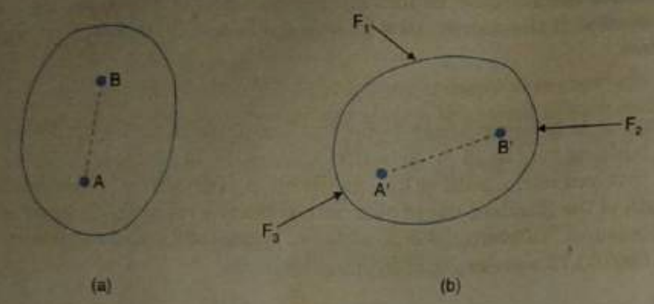


Fig. 1.2

Many engineering problems can be solved satisfactorily by assuming bodies rigid.

Particle

A particle may be defined as an object which has only mass and no size. Such a body cannot exist theoretically. However in dealing with problems involving distances considerably larger compared to the size of the body, the body may be treated as particle, without sacrificing accuracy. Examples of such situations are

— A bomber aeroplane is a particle for a gunner operating from the ground.

— A ship in mid sea is a particle in the study of its relative motion from a control tower.

— In the study of movement of the earth in celestial sphere, earth is treated as a particle.

1.3. LAWS OF MECHANICS

The following are the fundamental laws of mechanics:

Newton's first law

Newton's second law

Newton's third law

Newton's law of gravitation

Law of transmissibility of forces, and

Parallelogram law of forces.

Newton's First Law

It states that every body continues in its state of rest or of uniform motion in a straight line unless it is compelled by an external agency acting on it. This leads to the definition of force as the external agency which changes or tends to change the state of rest or uniform linear motion of the body.

Newton's Second Law

It states that the rate of change of momentum of a body is directly proportional to the impressed force and it takes place in the direction of the force acting on it. Thus according to this law,

Force ∝ rate of change of momentum. But momentum = mass × velocity

As mass do not change,

Force ∝ mass × rate of change of velocity

i.e., Force ∝ mass × acceleration

$$F \propto m \times a \qquad ...(1.3)$$

Newton's Third Law

It states that for every action there is an equal and opposite reaction. Consider the two bodies in contact with each other. Let one body applies a force F on another. According to this law the second body develops a reactive force R which is equal in magnitude to force F and acts in the line same as F but in the opposite direction. Figure. 1.3 shows the action of the ball and the reaction from the floor. In Fig. 1.4 the action of the ladder on the wall and the floor and the reactions from the wall and floor are shown.

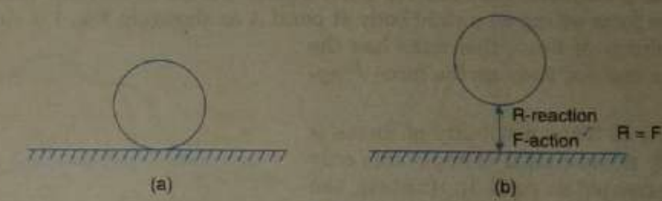


Fig. 1.3

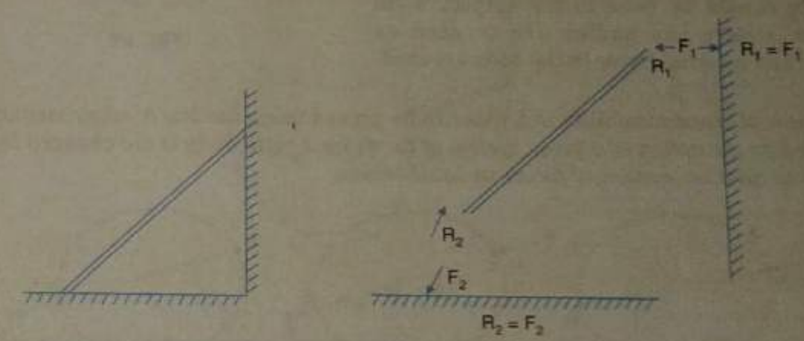


Fig. 1.4

Newton's Law of Gravitation

Everybody attracts the other body. The force of attraction between any two bodies is directly proportional to their masses and inversely proportional to the square of the distance between them. According to this law the force of attraction between the bodies of mass m_1 and mass m_2 at a distance d as shown in Fig. 1.5 is

$$F = G\,\frac{m_1 m_2}{d^2} \qquad ...(1.4)$$

where G is the constant of proportionality and is known as constant of gravitation.

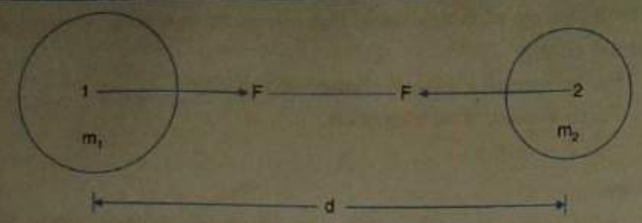


Fig. 1.5

Law of Transmissibility of Force

According to this law *the state of rest or motion of the rigid body is unaltered if a force acting on the body is replaced by another force of the same magnitude and direction but acting anywhere on the body along the line of action of the replaced force.*

Let F be the force acting on a rigid body at point A as shown in Fig. 1.6. According to the law of transmissibility of force, this force has the same effect on the state of body as the force F applied at point B.

In using law of transmissibility of forces it should be carefully noted that it is applicable only if the body can be treated as rigid. In this text, the engineering mechanics is restricted to study of state of rigid bodies and hence this law is frequently used. Same thing cannot be done in the subject 'solid mechanics' where the bodies are treated as deformable and internal forces in the body are studied.

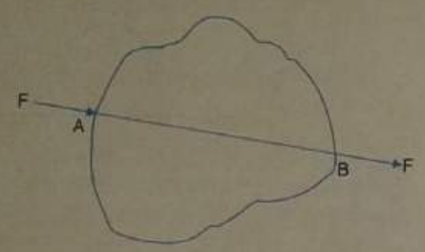


Fig. 1.6

The law of transmissibility of forces can be proved using *the law of superposition*, which can be stated as *the action of a given system of forces on a rigid body is not changed by adding or subtracting another system of forces in equilibrium.*

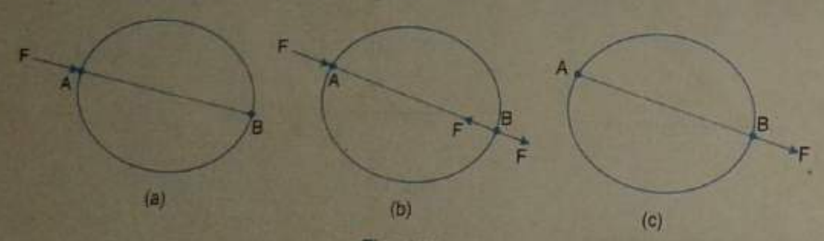


Fig. 1.7

Consider the rigid body shown in Fig. 1.7 (*a*). It is subjected to a force F at A. B is another point on the line of action of the force. From the law of superposition it is obvious that if two equal and opposite forces of magnitude F are applied at B along the line of action of given force F, [Ref. Fig. 1.7 (*b*)] the effect of given force on the body is not altered. Force F at A and opposite force F at B form a system of forces in equilibrium. If these two forces are subtracted from the system, the resulting system is as shown in Fig. 1.7 (*c*). Looking at the system of forces in Figs. 1.7 (*a*) and 1.7 (*c*), we can conclude the law of transmissibility of forces is proved.

Parallelogram Law of Forces

The parallelogram law of forces enables us to determine the single force called resultant which can replace the two forces acting at a point with the same effect as that of the two forces. This law was formulated based on experimental results. Though Stevinces employed it in 1586, the credit of presenting it as a law goes to Varignon and Newton (1687). This law states that *if two forces acting simultaneously on a body at a point are presented in magnitude and direction by the two adjacent sides of a parallelogram, their resultant is represented in magnitude and direction by the diagonal of the parallelogram which passes through the point of intersection of the two sides representing the forces.*

In Fig. 1.8 the force F_1 = 4 units and force F_2 = 3 units are acting on a body at point A. Then to get resultant of these forces parallelogram $ABCD$ is constructed such that AB is equal to 4 units to linear scale and AC is equal to 3 units. Then according to this law, the diagonal AD represents the resultant in the direction and magnitude. Thus the resultant of the forces F_1 and F_2 on the body is equal to units corresponding to AD in the direction α to F_1.

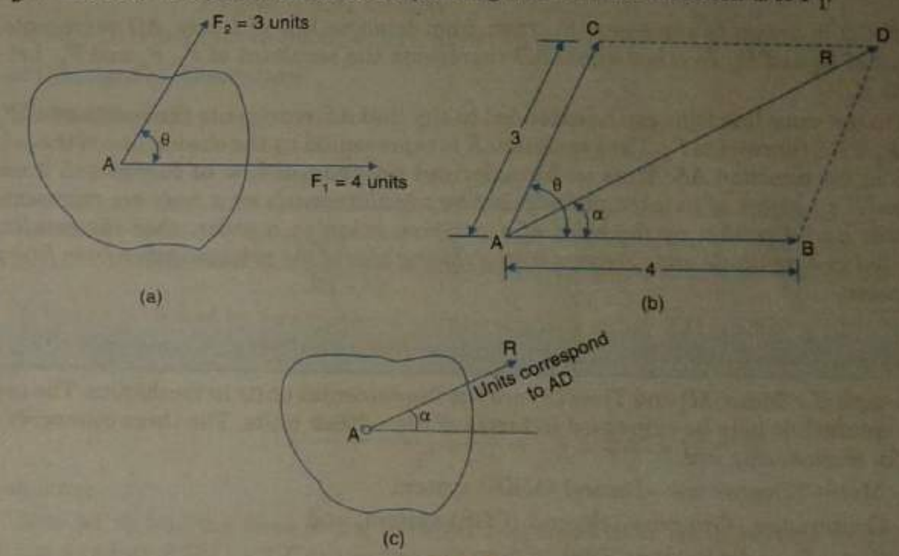


Fig. 1.8

1.4. DERIVED LAWS

Referring to Fig. 1.8 (*b*), we can get the resultant AD by constructing triangle ABD. Line AB is drawn to represent F_1 and BD to represent F_2. Then AD should represent the resultant of F_1 and F_2. Then we have derived triangle law of forces from fundamental law parallelogram law of forces. The **Triangle Law of Forces** may be stated as *If two forces acting on a body are represented one after another by the sides of a triangle, their resultant is represented by the closing side of the triangle taken from first point to the last point.*

If more than two concurrent forces are acting on a body, two forces at a time can be combined by triangle law of forces and finally resultant of all the forces acting on the body may be obtained.

A system of 4 concurrent forces acting on a body are shown in Fig. 1.9. AB represents F_1 and BC represents F_2. Hence according to triangle law of forces AC represents the resultant of F_1 and F_2, say, R_1.

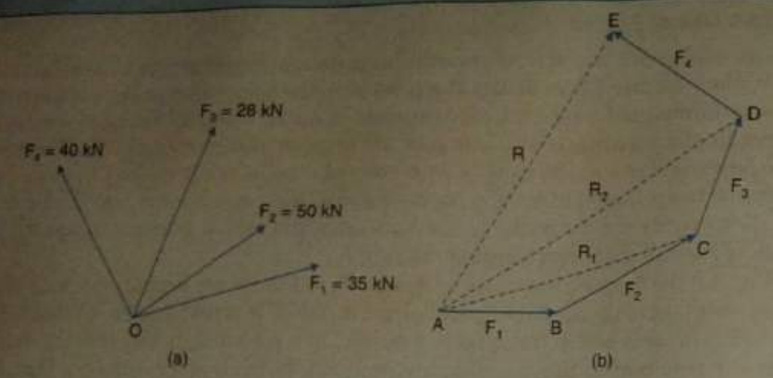


Fig. 1.9

If CD is drawn to represent F_3, then from triangle law of forces AD represents, the resultant of R_1 and F_3. In other words AD represents the resultant of F_1, F_2 and F_3. Let it be called as R_2.

On the same line logic can be extended to say that AE represents the resultant of F_1, F_2, F_3 and F_4 if DE represents F_4. Thus resultant R is represented by the closing line of the polygon $ABCDE$ in the direction AE. Thus we have derived **polygon of law of forces** and it may be stated as *If a number of concurrent forces acting simultaneously on a body are represented in magnitude and direction by the sides of a polygon, taken in a order, then the resultant is represented in magnitude and direction by the closing side of the polygon, taken from first point to last point.*

1.5. UNITS

Length (L), Mass (M) and Time (S) are the fundamental units in mechanics. The units of all other quantities may be expressed in terms of these basic units. The three commonly used systems in engineering are

— Metre-Kilogramme—Second (MKS) system

— Centimetre—Gramme—Second (CGS) system, and

— Foot—Pound—Second (FPS) system.

The units of length, mass and time used in the system are used to name the systems. Using these basic units, the units for other quantities can be found. For example, in MKS the units for the various quantities are as shown below:

Quantity	*Unit*	*Notation*
Area	Square metre	m^2
Volume	Cubic metre	m^3
Velocity	Metre per second	m/sec
Acceleration	Metre per second per second	m/sec^2

Unit of Forces

Presently the whole world is in the process of switching over to SI *system of units*. SI stands for System Internationale d' units or International System of units. As in MKS system, in SI system also the fundamental units are metre for length, kilogramme for mass and second

for time. The difference between MKS and SI system arise mainly in selecting the unit of force. From eqn. (1.3), we have

$$\text{Force} \propto \text{Mass} \times \text{Acceleration} = k \times \text{Mass} \times \text{Acceleration} \quad \text{...(1.5)}$$

In SI system unit of force is defined as that force which causes 1 kg mass to move with an acceleration of $1 m/sec^2$ and is termed as 1 Newton. Hence the constant of proportionality k becomes units. Unit of force can be derived from eqn. (1.5) as

$$\text{Unit of Force} = kg \times m/sec^2 = kg - m/sec^2$$

In MKS, the unit of force is defined as that force which makes a mass of 1 kg to move with gravitational acceleration 'g' m/sec^2. This unit of force is called kilogramme weight or kg/wt. Gravitational acceleration is $9.81\ m/sec^2$ near the earth surface. In all the problems encountered in engineering mechanics the variation in gravitational acceleration is negligible and may be taken as $9.81\ m/sec^2$. Hence the constant of proportionality in eqn. (1.5) is 9.81, which means

$$1 \text{ kg-wt} = 9.81 \text{ newton} \quad \text{...(1.6)}$$

It may be noted that in public usage, kg-wt force is called as kg only.

Unit of Constant of Gravitation

From eqn. (1.4),

$$F = G\frac{m_1 m_2}{d^2} \quad \text{or} \quad G = \frac{Fd^2}{m_1 m_2}$$

$$\therefore \quad \text{Unit of } G = \frac{N \times m^2}{kg \times kg} = Nm^2/kg^2$$

It has been proved by experimental results that the value of $G = 6.673 \times 10^{-11}\ Nm^2/kg^2$. Thus if two bodies one of mass 10 kg and the other of 5 kg are at a distance of 1 m, they exert a force

$$F = \frac{6.673 \times 10^{-11} \times 10 \times 5}{1^2} = 33.365 \times 10^{-10}\ N$$

on each other.

Now let us find the force acting between 1 kg-mass near earth surface and the earth. Earth has a radius of 6371×10^3 m and has a mass 5.96506×10^{24} kg. Hence the force between the two bodies is

$$= \frac{6.673 \times 10^{-11} \times 1 \times 5.96504 \times 10^{24}}{(6371 \times 10^3)^2} = 9.80665\ N.$$

In common usage we call the force exerted by earth on a body as *weight of the body*. Thus weight of 1 kg mass on earth surface is 9.80665 N, which is approximated as 9.81 N for all practical problems. Compared to this force the force exerted by two bodies near earth surface is negligible as may be seen from the example of 10 kg and 5 kg mass bodies.

Denoting the weight of the body by W, from eqn. (1.4), we get

$$W = \frac{GmM_e}{r^2}$$

where m is the mass of body

M_e is the mass of the earth, and

r is the radius of the earth

Denoting $\frac{GM_e}{r^2}$ by g, we get

$$W = mg = 9.81\ m \qquad ...(1.7)$$

Unit of g can be obtained as follows:

$$g = \frac{GM_e}{r^2}$$

$$\text{Unit of } g = \frac{Nm^2}{(kg)^2} \times \frac{kg}{m^2} = \frac{N}{kg}$$

as unit of Newton force is kg-m/sec^2, we get

$$\text{Unit of } g = \frac{kgm/sec^2}{kg} = m/sec^2$$

Hence g may be called as acceleration due to gravity. Any body falling freely near earth surface experiences this acceleration. The value of g is 9.81 m/sec^2 near the earth surface as can be seen from eqn. (1.7).

The prefixes used in SI system when quantities are too big or too small are shown in Table 1.1.

Table 1.1: Prefixes and Symbols of Multiplying Factors in SI

Multiplying Factor	*Prefix*	*Symbol*
10^{12}	tera	T
10^{9}	giga	G
10^{6}	mega	M
10^{3}	kilo	k
10^{0}	–	–
10^{-3}	milli	m
10^{-6}	micro	μ
10^{-9}	nano	n
10^{-12}	pico	p
10^{-15}	femto	f
10^{-18}	atto	a

1.6. CHARACTERISTICS OF A FORCE

From Newton's first law, we defined the force as the agency which tries to change state of stress or state of uniform motion of the body. From Newton's second law of motion we arrived at practical definition of unit force as the force required to produce unit acceleration in a body of unit mass. Thus 1 newton is the force required to produce an acceleration of 1 m/sec^2 in a body of 1 kg mass. It may be noted that a force is completely specified only when the following four characteristics are specified:

— *Magnitude*
— *Point of application*
— *Line of action, and*
— *Direction*

In Fig. 1.10 AB is a ladder kept against a wall. At point C, a person weighing 600 N is standing. The force applied by the person on the ladder has the following characters:

— magnitude is 600 N
— the point of application is at C which is 2 m from A along the ladder.
— the line of action is vertical, and
— the direction is downward.

Note that the magnitude of the force is written near the arrow. The line of the arrow shows the line of application and the arrow head represents the point of application and the direction of the force.

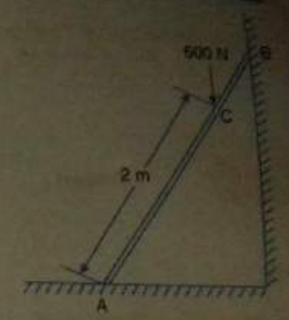


Fig. 1.10

1.7. SYSTEM OF FORCES

When several forces act simultaneously on a body, they constitute a *system of forces*. If all the forces in a system do not lie in a single plane they constitute the *system of forces in space*. If all the forces in a system lie in a single plane, it is called a *coplanar force system*. If the line of action of all the forces in a system pass through a single point, it is called a *concurrent force system*. In a *system of parallel forces* all the forces are parallel to each other. If the line of action of all the forces lie along a single line then it is called a *collinear force system*. Various system of forces, their characteristics and examples are given in Table 1.2 and shown in Fig. 1.11.

Table 1.2: System of Forces

Force System	*Characteristics*	*Examples*
Collinear forces	Line of action of all the forces act along the same line.	Forces on a rope in a tug of war.
Coplanar parallel forces	All forces are parallel to each other and lie in a single plane.	System of forces acting on a beam subjected to vertical loads (including reactions).
Coplanar like parallel forces	All forces are parallel to each other, lie in a single plane and are acting in the same direction.	Weight of a stationary train on a rail when the track is straight.
Coplanar concurrent forces	Line of action of all forces pass through a single point and forces lie in the same plane.	Forces on a rod resting against a wall.
Coplanar non-concurrent forces	All forces do not meet at a point, but lie in a single plane.	Forces on a ladder resting against a wall when a person stands on a rung which is not at its centre of gravity.
Non-coplanar parallel forces	All the forces are parallel to each other, but not in same plane.	The weight of benches in a classroom.
Non-coplanar concurrent forces	All forces do not lie in the same plane, but their lines of action pass through a single point.	A tripod carrying a camera.
Non-coplanar non-concurrent forces	All forces do not lie in the same plane and their lines of action do not pass through a single point.	Forces acting on a moving bus.

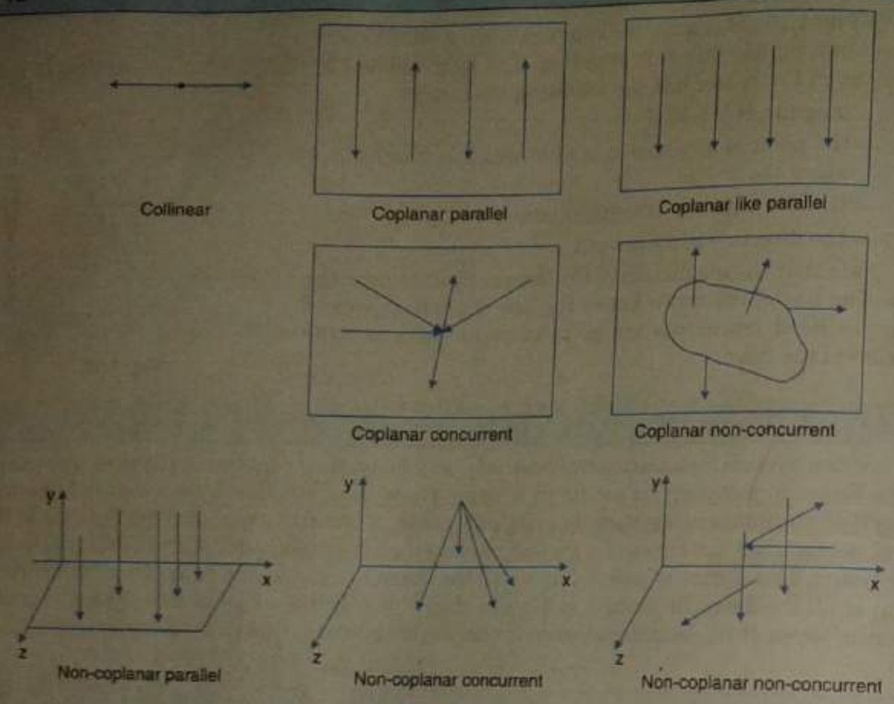


Fig. 1.11

1.8. VECTORS

Various quantities used in engineering mechanics may be grouped into scalars and vectors. A quantity is said to be *scalar* if it is completely defined by its magnitude alone. Examples of scalars are length, area, time and mass.

A quantity is said to be *vector* if it is completely defined only when its magnitude as well as direction are specified. Hence force is a vector. The other examples of vector are velocity, acceleration, momentum etc.

1.9. DIMENSIONAL HOMOGENEITY

The qualitative description of physical variable is known as dimension while the quantitative description is known as unit. We come across several relations among the physical quantities. Some of the terms may be having dimensions and some others may be dimensionless. However in any equation dimensions of the terms on both sides must be the same. This is called dimensional homogenity. The branch of mathematics dealing with dimensions of quantities is called dimensional analysis.

There are two systems of dimensional analysis *viz. absolute system* and *gravitational system*. In absolute system the basic quantities selected are Mass, Length and Time. Hence it is known as *MLT-system*. In gravitational system the basic quantities are Force, Length and Time. Hence it is termed as *FLT*-system.

The dimension of acceleration is $\frac{L}{T^2} = LT^{-2}$ since its unit is m/sec^2. From Newton's law we have physical relation

$$\text{Force} = \text{Mass} \times \text{Acceleration}$$

Hence the dimensional relation is,

$$F = \frac{ML}{T^2} \quad \text{...(1.8a)}$$

or

$$M = \frac{FT^2}{L} \quad \text{...(1.8b)}$$

Eqn. (1.8) helps in converting dimensions from one system to another. The dimensions of some of the physical quantities are listed in Table 1.3.

Table 1.3: Dimensions of Quantities

Sr. No.	*Quantity*	*MLT-system*	*FLT-system*
1.	Velocity	LT^{-1}	LT^{-1}
2.	Acceleration	LT^{-2}	LT^{-2}
3.	Momentum	MLT^{-1}	FT
4.	Area	L^2	L^2
5.	Volume	L^3	L^3
6.	Force	MLT^{-2}	F
7.	Gravitational Constant	$M^{-1}L^3T^{-2}$	$F^{-1}L^4T^{-4}$

Checking Dimensional Homogenity

As stated earlier all the terms in an equation to the left and right side should have the same dimensions. In other words if,

$$X = Y + Z$$

the terms, X, Y and Z should have same dimension. If,

$$X = bY$$

and if X and Y do not have same dimension, b is not dimensionless constant. The value of this constant will be different in different system of units.

Example 1.1. *Verify whether the following equation has dimensional homogenity:*

$$v^2 - u^2 = 2as$$

where v is final velocity, u is initial velocity, a is acceleration and s is the distance moved.

Solution. Dimensions of velocity $= LT^{-1}$

Dimensions of acceleration $= LT^{-2}$

and dimension of distance $= L$

Substituting these dimensions in the given equation, we get

$$LT^{-1} - LT^{-1} = 2LT^{-2}\,L$$

$$LT^{-1} \equiv LT^{-1} \equiv LT^{-1}$$

Hence it is dimensionally homogeneous equation.

Example 1.2. *In the following equation verify, whether 9.81 is dimensionless constant. If it is not so, what should be its dimension?*

$$s = ut + \frac{1}{2}\ 9.81\ t^2$$

where s = *distance*
u = *initial velocity*
t = *time*

Solution. Dimensions of various terms are

$$s = L$$
$$u = LT^{-1}$$
$$t = T$$

Substituting these in the given equation, we get,

$$L = LT^{-1}\,T + \frac{1}{2} \times 9.81\ T^2$$

$$L = L + \frac{1}{2} \times 9.81\ T^2$$

Hence, 9.81 cannot be dimensionless constant. Its dimension is given by

$$L \equiv \frac{1}{2}\ 9.81\ T^2$$

9.81 should have dimensions LT^{-2}, same as that of acceleration. We know this is gravitational acceleration term in SI unit *i.e.*, it is in m/sec^2 term. Hence the given equation cannot be straightway used is FPS system or CGS system.

1.10. IDEALISATIONS IN MECHANICS

A number of ideal conditions are assumed to exist while applying the principles of mechanics to practical problems. In fact without such assumptions it is not possible to arrive at practical solutions. The following idealisations are usually made in engineering mechanics.

1. The body is rigid.
2. The body can be treated as continuum.
3. If the size of the body is small compared to other distances involved in the problem, it may be treated as a particle.
4. If the area over which force is acting on a body is small compared to the size of the body, it may be treated as a point force. For example, in Fig. 1.9, 600 N force is the weight of a man. Actually the man cannot apply his weight through a single point. There is certain area of contact, which is, however, small compared to the other dimensions in the problem. Hence, the weight of the man is treated as a point load.
5. Support conditions are idealised (which will be discussed later) as simple, hinged, fixed etc.

1.11. ACCURACY OF CALCULATIONS

As explained in Art. 1.9, a number of ideal conditions are assumed to exist while applying the principles of mechanics to engineering problems. These assumptions have some effect on the accuracy of final results. Further in engineering analysis, many loads are estimated

loads. There are some imperfection in construction and fabrications of structures. To take care all these uncertainities engineers multiply the load by a number (1.5 to 2.25), which is called factor of safety. In view of all these, there is no need in noting the calculations beyond four digits. It is found that 0.2% accuracy is more than sufficient, in the calculations.

1.12. APPROACHES TO SOLUTION

There are two approaches for the solution of engineering mechanics problems *i.e.* classical approach and vector approach. **Classical approach** gives physical feel of the problem. It gives confidence to engineers in accepting the results presented by others and quickly take decisions on site. Development of such feel of the problem is very much essential for engineers. However it becomes difficult to solve three dimensional problems by classical approach. Vector approach is ideally suited for the analysis of three dimensional problems. But the disadvantage of vector approach is physical feel of the problem is lost and the ability of site engineers in quick decision is not developed. Hence there are two school of academicians, one advocating for classical approach and the other advocating for vector approach. In this book author has used classical approach for the solution of engineering mechanics problems.

Important Definitions and Concepts

1. *Displacement* is defined as the distance moved by a body or particle in the specified direction.
2. The rate of change of displacement with time is called *velocity*. $\vec{v} = \frac{ds}{dt}$
3. *Acceleration* is the rate of change of velocity with respect to time.
4. The product of mass and velocity is called *momentum*.
5. A body is said to be treated as *continuum*, if it is assumed to consist of continuous distribution of matter.
6. A body is said to be *rigid*, if the relative position of any two particles in it do not change under the action of the forces.
7. *Newton's first law* states that everybody continues in its state of rest or of uniform motion in a straight line unless it is compelled by an external agency acting on it.
8. *Newton's second law* states that the rate of change of momentum of a body is directly proportional to the impressed force and it takes place in the direction of the force acting on it. $F = \frac{dp}{dt}$
9. *Newton's third law* states for every action there is an equal and opposite reaction.
10. *Newton's law of gravitation* states everybody attracts the other body, the force of attraction between any two bodies is directly proportional to their mass and inversely proportional to the square of the distance between them.
11. According to the *law of transmissibility of force*, the state of rest or motion of a rigid body is unaltered, if a force acting on a body is replaced by another force of the same magnitude and direction but acting anywhere on the body along the line of action of the replaced force.
12. The *parallelogram law of forces* states that if two forces acting simultaneously on a body at a point are represented by the two adjacent sides of a parallelogram, their resultant is represented in magnitude and direction by the diagonal of the parallelogram which passes through the point of intersection of the two sides representing the forces.
13. The qualitative description of physical variable is known as *dimension* while the quantitative description is known as *unit*.

14. A quantity is said to be *scalar*, if it is completely defined by its magnitude alone.
15. A quantity is said to be *vector* if it is completely defined only when it's magnitude as well as direction are specified.

QUESTIONS

1. Explain the following terms as used in Engineering Mechanics:
   (*i*) Continuum (*ii*) Rigid Body (*iii*) Particle.
2. State and explain Newton's three laws of motion.
3. State and explain Newton's law of gravitation.
4. State and explain Law of transmissibility of forces.
5. State and explain parallelogram law of forces. From this derive triangle and polygonal laws of forces.
6. Explain the term 'Force' and list its characteristics.
7. Explain the terms—concurrent and non-concurrent force system; planar and non-planar system of forces.

2
Resultant and Equilibrium of System of Coplanar Concurrent Forces

If all the forces in a system lie in a single plane and pass through a single point, they constitute a coplanar concurrent force system. It is possible to find a single force which will have the same effect as that of a number of forces acting. Such single force is called Resultant force and the process of finding the resultant force is called composition of forces.

Parallelogram law, triangle law and polygonal law may be used to find the resultant graphically. The graphical method of composition gives a clear picture of the work being carried out. However the main disadvantage is that it needs drawing aids like pencil, scale, drawing sheet, etc. Hence there is need for analytical method. In this chapter analytical method of composition is explained for the system of two forces first and then the general method is explained.

A body may be in equilibrium under the action of a concurrent force system, and the analyst is interested in finding the reactions from the other bodies in contact. Since it is concurrent force system, size of the body will not come into analysis. Hence the body may be referred as a particle and the analysis of equilibrium condition as statics of particle. This process is also explained and illustrated with application to many engineering problems.

2.1. COMPOSITION OF TWO FORCE SYSTEM

Consider the two forces F_1 and F_2 acting on a particle as shown in Fig. 2.1 (*a*). Let the angle between the two forces be θ. If parallelogram *ABCD* is drawn as shown in Fig. 2.1 (*b*), with *AB* representing F_1 and *AD* representing F_2 to same scale, according to parallelogram law of forces *AC* represents the resultant *R*. Drop perpendicular *CE* to *AB*.

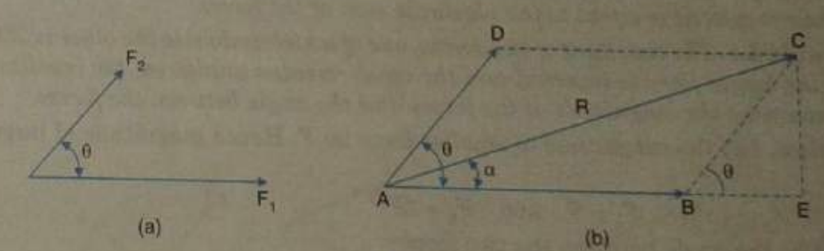


Fig. 2.1

The resultant *R* of F_1 and F_2 is given by:

$$R = AC = \sqrt{AE^2 + CE^2} = \sqrt{(AB + BE)^2 + CE^2}$$

But $AB = F_1$

$$BE = BC \cos\theta = F_2 \cos\theta$$

and $$CE = BC \sin\theta = F_2 \sin\theta$$

$$\therefore \quad R = \sqrt{(F_1 + F_2\cos\theta)^2 + (F_2 \sin\theta)^2}$$

$$= \sqrt{\left(F_1^2 + 2F_1F_2\cos\theta + F_2^2\cos^2\theta + F_2^2\sin^2\theta\right)}$$

$$= \sqrt{F_1^2 + 2F_1F_2\cos\theta + F_2^2} \qquad ...(2.1)$$

Since, $\sin^2\theta + \cos^2\theta = 1$.

The inclination of the resultant to the direction of the force F_1 is given by α, where

$$\tan\alpha = \frac{CE}{AE} = \frac{CE}{AB + BE} = \frac{F_2 \sin\theta}{F_1 + F_2\cos\theta}$$

Hence $$\alpha = \tan^{-1}\frac{F_2\sin\theta}{F_1 + F_2\cos\theta} \qquad ...(2.2)$$

Particular Cases:

1. When $\theta = 90°$ [Ref. Fig. 2.2 *a*], $R = \sqrt{F_1^2 + F_2^2}$
2. When $\theta = 0°$ [Ref. Fig. 2.2 *b*], $R = \sqrt{F_1^2 + 2F_1F_2 + F_2^2} = F_1 + F_2$
3. When $\theta = 180°$ [Ref. Fig. 2.2 *c*], $R = \sqrt{F_1^2 - 2F_1F_2 + F_2^2} = F_1 - F_2$

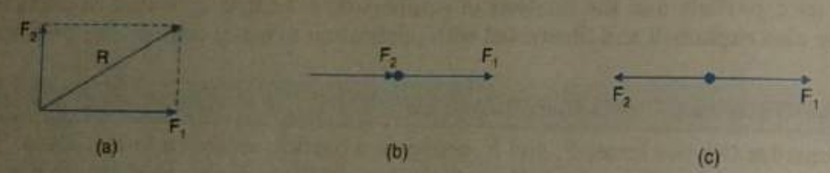


Fig. 2.2

From the particular cases 2 and 3, it is clear that *when the forces acting on a body are collinear, their resultant is equal to the algebraic sum of the forces.*

Example 2.1. *The resultant of two forces, one of which is double the other is 260 N. If the direction of the larger force is reversed and the other remains unaltered, the resultant reduces to 180 N. Determine the magnitude of the forces and the angle between the forces.*

Solution. Let the magnitude of smaller force be F. Hence magnitude of larger force is $2F$.

Thus $F_1 = F$ and $F_2 = 2F$

Let θ be the angle between the two forces

$\therefore$ From the condition 1, we get

$$R = \sqrt{F_1^2 + 2F_1F_2\cos\theta + F_2^2} = 260.$$

$$\therefore \quad F^2 + 2F(2F)\cos\theta + (2F)^2 = 260^2$$

i.e., $$5F^2 + 4F^2\cos\theta = 67600 \qquad ...(i)$$

From condition 2, we have

$$\sqrt{F_1^2 + 2F_1F_2\cos(180° + \theta) + F_2^2} = 180^2$$

i.e., $$F^2 - 2F(2F)\cos\theta + (-2F)^2 = 32400$$

i.e., $$5F^2 - 4F^2\cos\theta = 32400 \qquad ...(ii)$$

Adding equations (*i*) and (*ii*), we get

$$10F^2 = 100000$$

$$\therefore \quad F = 100 \text{ N}.$$

Hence, $\boldsymbol{F_1 = 100}$ **N** and $\boldsymbol{F_2 = 2F =}$ **200 N. Ans.**

Substituting the values of F_1 and F_2 in eqn. (*i*),

we get, $5(100)^2 + 4(100)^2\cos\theta = 67600$

$\therefore \quad \cos\theta = 0.44 \quad \therefore \quad \theta = \mathbf{63.9°}$ **Ans.**

2.2. RESOLUTION OF FORCES

Before taking up the general method of finding resultant of any number of concurrent forces, it is necessary to study resolution of a force into its rectangular components. The **resolution of forces** is exactly the opposite process of composition of forces. It is the process of finding a number of component forces which will have the same effect as the given single force. Exactly the opposite process of composition can be employed to get the resolved component of a given force.

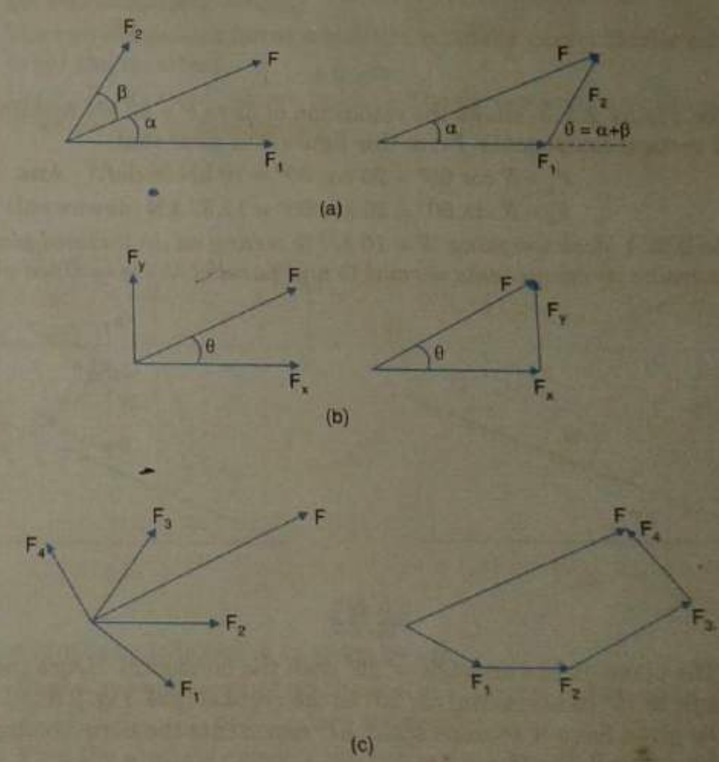


Fig. 2.3

In Fig. 2.3 (*a*), the given force F is resolved into two components making angles α and β with F.

In Fig. 2.3 (*b*) the force F is resolved into its rectangular components F_x and F_y.

In Fig. 2.3 (*c*) the force F is resolved into its four components F_1, F_2, F_3 and F_4.

It should be noted that all component forces act at the same point as the given force. Resolution of forces into its rectangular components is more useful for the analysis. In this case if the force F makes angle θ with x-axis, from Fig. 2.3 (*b*), it is clear that

$$F_x = F \cos \theta$$

and

$$F_y = F \sin \theta \qquad ...(2.3)$$

Example 2.2. *The guy wire of the electric pole shown in Fig. 2.4 (a) makes 60° to the horizontal and is subjected to 20 kN force. Find horizontal and vertical components of the force.*

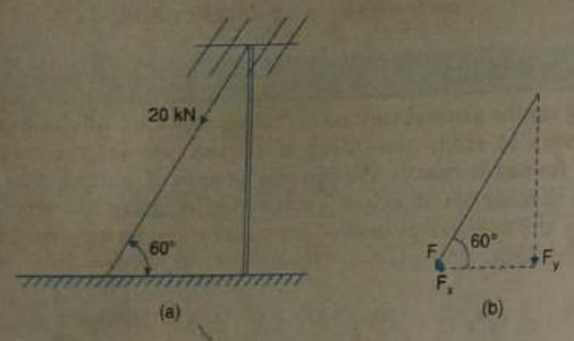


Fig. 2.4

Solution. Figure 2.4 (*b*) shows the resolution of force $F = 20$ kN into its components in horizontal and vertical components. From this figure it is clear that

$$F_x = F \cos 60° = 20 \cos 60° = 10 \text{ kN (to left)} \quad \textbf{Ans.}$$

and

$$F_y = F \sin 60° = 20 \sin 60° = 17.32 \text{ kN (downward)} \quad \textbf{Ans.}$$

Example 2.3. *A block weighing W = 10 kN is resting on an inclined plane as shown in Fig. 2.5 (a). Determine its components normal to and parallel to the inclined plane.*

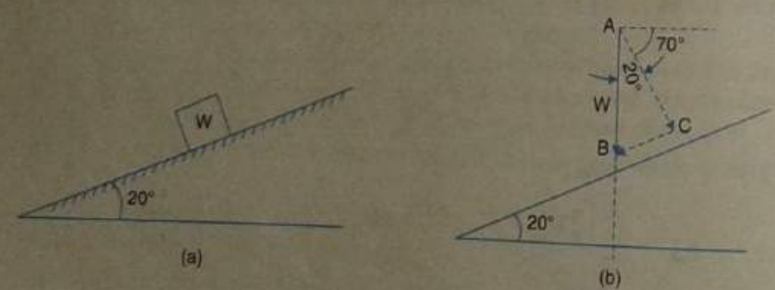


Fig. 2.5

Solution. The plane makes an angle of 20° with the horizontal. Hence the normal to the plane makes an angle of 70° to horizontal *i.e.* 20° to the vertical [Ref. Fig. 2.5 (*b*)]. In Fig. 2.5 (*b*), if AB represents the given force W to some scale, AC represents the normal component and CB represents component parallel to the plane.

From ΔABC,

Normal component $= AC$

$= W \cos 20°$

$= 10 \cos 20°$

$= 9.4$ kN (Thrust on plane) **Ans.**

Parallel component $= W \sin 20° = 10 \sin 20°$

$= 3.42$ kN (down the plane) **Ans.**

From the above two examples, the following points may be noted:

1. Imagine that the arrow drawn represents the given force to some scale.
2. Travel from tail to arrow head of the force in the direction of the coordinates selected.
3. Then the direction of travel gives the direction of component forces.
4. From the triangle of forces, the magnitude of the components can be calculated.

2.3. GENERAL METHOD OF COMPOSITION OF FORCES

The method explained below can be used to determine the resultant if any number of concurrent forces acting at a point in a plane:

Step 1. Determine the component of each force in two mutually perpendicular coordinate directions.

Step 2. Add algebraically components of all the forces in each coordinate direction to get two component forces.

Step 3. The two component forces which are mutually perpendicular can be combined to get the resultant.

Let F_1, F_2, F_3 and F_4 shown in Fig. 2.6 be the system of four forces, the resultant of which is to be determined.

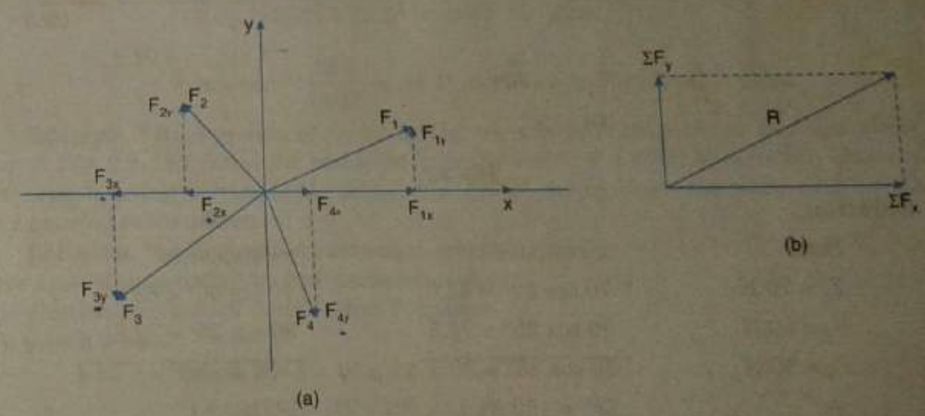


Fig. 2.6

The procedure to be followed is as given below:

Step 1. Find the component of all the forces in x and y directions. Thus F_{1x}, F_{2x}, F_{3x}, F_{4x}, F_{1y}, F_{2y}, F_{3y}, and F_{4y}, are obtained.

Step 2. Find the algebraic sum of the component forces in x and y directions.

$$\Sigma F_x = F_{1x} + F_{2x} + F_{3x} + F_{4x}$$

$$\Sigma F_y = F_{1y} + F_{2y} + F_{3y} + F_{4y}$$

[**Note:** In the above example F_{2x}, F_{3x}, F_{3y} and F_{4y} are having negative values.]

Step 3. Now the system of forces is equal to two mutually perpendicular forces, namely, ΣF_x and ΣF_y as shown in Fig. 2.6 (*b*). Since these two forces are at right angles to each other, the parallelogram of forces becomes a rectangle.

Hence the resultant R is given by:

$$R = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2} \quad ...(2.4)$$

and its inclination to x-axis is given by

$$\alpha = \tan^{-1}\left(\frac{\Sigma F_y}{\Sigma F_x}\right) \quad ...(2.5)$$

Note that $R \cos \alpha = \Sigma F_x$ ...(2.6)

and $R \sin \alpha = \Sigma F_y$ ...(2.7)

i.e., ΣF_x and ΣF_y are the x and y components of the resultant R.

Example 2.4. *Determine the resultant of the three forces acting on a hook as shown in Fig. 2.7 (a).*

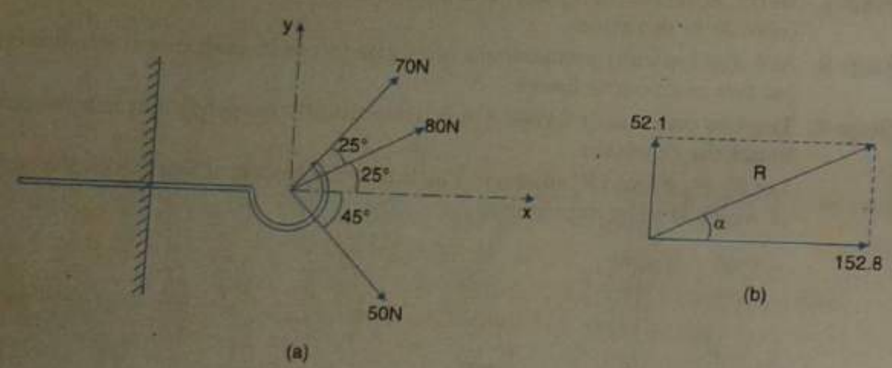


Fig. 2.7

Solution.

Force	*x-component*	*y-component*
$F_1 = 70$ N	$70 \cos 50° = 45.0$	$70 \sin 50° = 53.6$
$F_2 = 80$ N	$80 \cos 25° = 72.5$	$80 \sin 25° = 33.8$
$F_3 = 50$ N	$50 \cos 45° = 35.4$	$-50 \sin 45° = -35.4$
	$\Sigma F_x = 152.8$	$\Sigma F_y = 52.1$

$\therefore$ $R = \sqrt{152.8^2 + 52.1^2} = 161.5$ N **Ans.**

$\alpha = \tan^{-1} \dfrac{52.1}{152.8} = 18.83°$ as shown in Fig. 2.7 (*b*) **Ans.**

Example 2.5. *A system of four forces acting at a point on a body is as shown in Fig. 2.8 (a). Determine the resultant.*

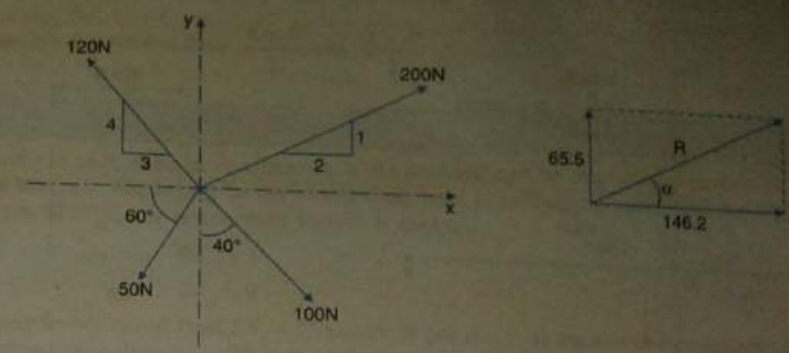


Fig. 2.8

Solution. If θ_1 is the inclination of 200 N force to x-axis.

$\tan \theta = 1/2 \quad \therefore \quad \theta_1 = 26.565°$

Similarly, inclination of 120 N force to x-axis is given by

$\tan \theta_2 = \dfrac{4}{3}$ *i.e.*, $\theta_2 = 53.13°$

$\therefore$ $\Sigma F_x = 200 \cos 26.565° - 120 \cos 53.13° - 50 \cos 60° + 100 \sin 40°$
$= 146.2$ N

$\Sigma F_y = 200 \sin 26.565° + 120 \sin 53.13° - 50 \sin 60° - 100 \cos 40°$
$= 65.5$ N

$\therefore$ $R = \sqrt{146.2^2 + 65.5^2} = 160.2$ N **Ans.**

$\alpha = \tan^{-1} \dfrac{65.5}{146.2} = 24.1°$ as shown in Fig. 2.8 (*b*) **Ans.**

Example 2.6. *A system of forces acting on a body resting on an inclined plane is as shown in Fig. 2.9. Determine the resultant force if $\theta = 60°$, $W = 1000$ N, vertically downward, $N = 500$ newton acting normal to the plane, $F = 100$ N, acting down the plane and $T = 1200$ N, acting parallel to the plane.*

Solution. In this problem, note that coordinate system as parallel to and perpendicular to the plane is convenient. Noting that W makes angle θ with y-axis.

$\Sigma F_x = T - F - W \sin \theta$
$= 1200 - 100 - 100 \sin 60°$
$= 234.0$ N

$\Sigma F_y = N - W \cos 60°$
$= 500 - 1000 \cos 60°$
$= 0$

$\therefore$ Resultant is a force of magnitude 234 N directed up the plane.

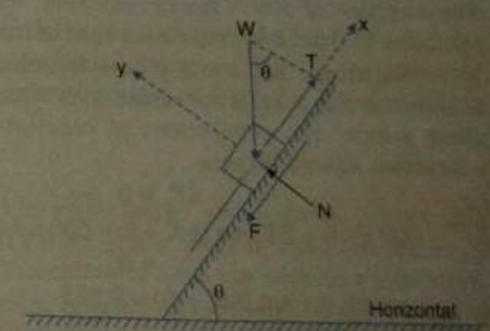


Fig. 2.9

Example 2.7. *Two forces acting on a body are 500 N and 1000 N as shown in Fig. 2.10(a). Determine the third force F such that the resultant of all the three forces is 1000 N directed at 45° to x-axis.*

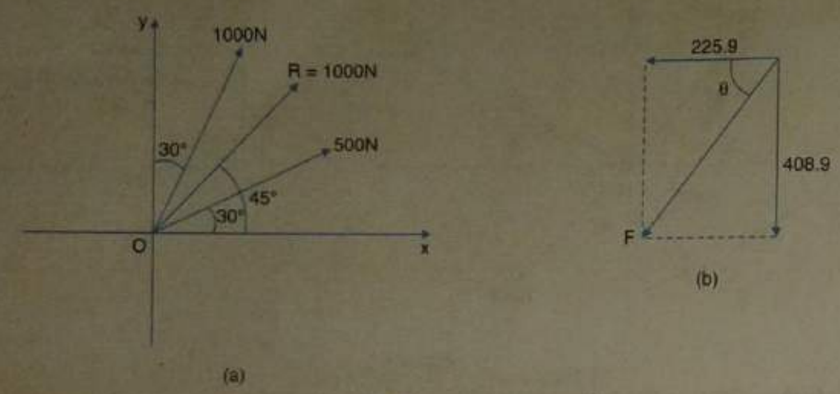


Fig. 2.10

Solution: Let the third force F makes an angle θ with x-axis. Then,

$R \cos \alpha = \Sigma F_x$, gives

$1000 \cos 45° = 500 \cos 30° + 1000 \sin 30° + F \cos \theta$

$\therefore \quad F \cos \theta = -225.9$ N

Similarly, $R \sin 45° = \Sigma F_y$ gives,

$1000 \sin 45° = 500 \sin 30° + 1000 \cos 30° + F \sin \theta$

i.e., $\quad F \sin \theta = -408.9$

$\therefore \quad F = \sqrt{225.9^2 + 408.9^2}$

i.e., $\quad F = 467.2$ N **Ans.**

$$\theta = \tan^{-1}\left(\frac{408.9}{225.9}\right) = 61.08° \text{ as shown in Fig. 2.8 } (b) \quad \textbf{Ans.}$$

Example 2.8. *Three forces acting at centre of gravity of a block are shown in Fig. 2.11. The direction of 300 N forces may vary, but the angle between them is always 40°. Determine the value of θ for which the resultant of the three forces is directed parallel to the plane.*

Solution: Let x and y axes be selected as shown in Fig. 2.11. If the resultant is directed along x-axis, the component of reaction in y-direction should be zero.

i.e. $\quad \Sigma F_y = 0 \rightarrow$

$300 \sin \theta + 300 \sin (40° + \theta) - 500 \sin 30° = 0$

$\sin \theta + \sin (40° + \theta) = 0.833$

i.e., $\quad 2 \sin \dfrac{40° + \theta + \theta}{2} \times \cos \dfrac{40° + \theta - \theta}{2} = 0.833$

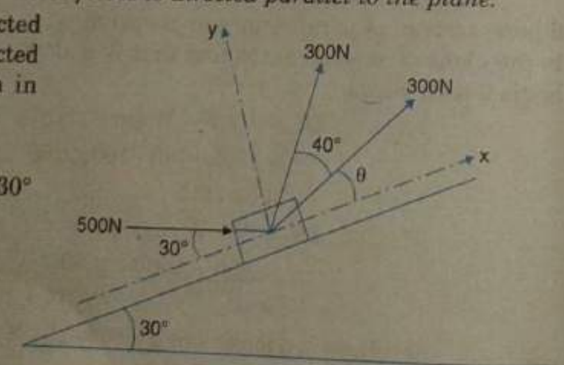


Fig. 2.11

i.e., $\quad 2 \sin (20° + \theta) \cos 20° = 0.833$

$\therefore \quad \theta = 6.31°$ **Ans.**

Note: $\sin A + \sin B = 2 \sin \dfrac{A+B}{2} \cos \dfrac{A-B}{2}$

2.4. EQUILIBRIUM OF BODIES

A body is said to be in equilibrium if its state of rest or of uniform motion in a straight line is not altered. According to Newton's first law, it means no resultant force acts on the body. In case of coplanar concurrent forces, it means,

$$\Sigma F_x = 0$$

and $$\Sigma F_y = 0 \qquad ...(2.8)$$

It may be observed that $\Sigma F_x = 0$ means, $R \cos \alpha = 0$. It means the resultant do not have any component in x-direction. This condition ensures that there is no resultant in any direction, except in y-direction ($\alpha = 90°$). Hence the condition $\Sigma F_y = 0$ also should be satisfied to ensure that the resultant R does not exist in any direction. Hence, if a body is in equilibrium under the action of coplanar concurrent system, eqn. (2.8) should be satisfied.

2.4.1. Types of Forces Acting on a Body

While applying equilibrium equations to a body, it is necessary that all forces acting on a body should be considered. The various forces acting on a body may be grouped as:

— Applied forces

— Non-applied forces.

Applied Forces

Applied forces are the forces applied externally to a body. Each force has got point of contact with the body. If a person stands on a ladder, his weight is an applied force to the ladder. If a temple car is pulled, the force in the rope is an applied force for the car.

Non-applied Forces

There are two types of non-applied forces (*a*) self weight, and (*b*) reactions.

(*a*) **Self weight.** Everybody is subjected to gravitational attraction and hence has got self-weight given by the expression,

$$W = mg \qquad ...(2.9)$$

where m is the mass of the body and g is the gravitational acceleration (9.81 m/sec^2 near the earth surface). Self weight always acts in vertically downward direction. Self weight is treated as a vertically downward force acting through the centre of gravity of the body. If self weight is very small, it may be neglected.

Reactions

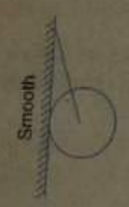
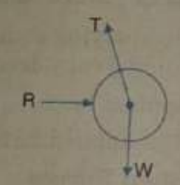
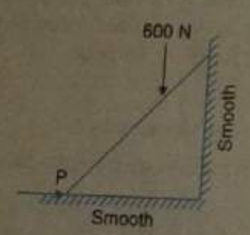
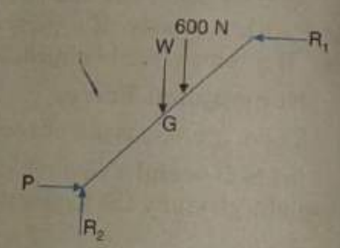
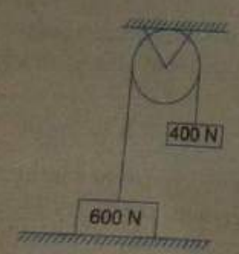
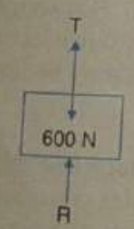
Reactions are self adjusting forces developed by other bodies which are in contact with the body under consideration. According to Newton's third law of motion, reactions are equal and opposite to the actions. The reactions adjust themselves to keep the body in equilibrium.

If the surface of contact is smooth, the direction of reaction is normal to the surface of contact. If the surface of contact is rough, apart from normal reaction, there will be frictional reaction also. Hence the resultant reaction will not be normal to the surface of contact.

2.4.2. Free Body Diagram

For the analysis of equilibrium condition it is necessary to isolate the body under consideration from the other bodies in contact and draw all forces acting on the body. For this, first the body is drawn and then all applied forces, self weight and reactions from the other bodies in contact are drawn. Such *diagram of the body in which the body under consideration is freed from all contact surfaces and is shown with all the forces on it (including self weight, reactions from other contact surfaces) is called the Free Body Diagram* (FBD). Free body diagrams (FBD) are shown for few typical cases in Table 2.1.

Table 2.1: Free Body Diagrams (FBD) for Few Typical Cases

Reacting bodies	*FBD required for*	*FBD*
	Ball	
	Ball	
	Ladder	
	Block weighing 600 N	

2.4.3. Lami's Theorem

If a body is in equilibrium, it may be analysed using equations of equilibrium 2.3. However, if the body is in equilibrium under the action of only three forces, Lami's theorem can be used conveniently.

Lami's theorem states that *if a body is in equilibrium under the action of only three forces, each force is proportional to the sine of the angle between the other two forces.* Thus for the system of forces shown in Fig. 2.12. (*a*),

$$\frac{F_1}{\sin\alpha} = \frac{F_2}{\sin\beta} = \frac{F_3}{\sin\gamma} \qquad ...(2.10)$$

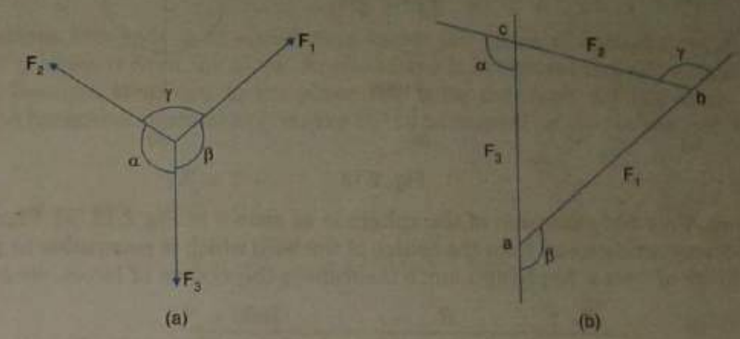


Fig. 2.12

Proof: Draw the three forces F_1, F_2 and F_3 one after the other in direction and magnitude starting from point *a*. Since the body is in equilibrium the resultant should be zero, which means the last point of force diagram should coincide with *a*. Thus, it results in a triangle of forces *abc* as shown in Fig. 2.12 (*b*). Now the external angles at *a*, *b* and *c* are equal to β, γ and α, since *ab*, *bc* and *ca* are parallel to F_1, F_2 and F_3 respectively. In the triangle of forces *abc*,

$$ab = F_1$$
$$bc = F_2 \quad \text{and}$$
$$ca = F_3$$

Applying sine rule for the triangle *abc*,

$$\frac{ab}{\sin(180° - \alpha)} = \frac{bc}{\sin(180° - \beta)} = \frac{ca}{\sin(180° - \gamma)}$$

i.e.,
$$\frac{F_1}{\sin\alpha} = \frac{F_2}{\sin\beta} = \frac{F_3}{\sin\gamma}$$

Note: When a body is in equilibrium under the action of only three forces, these three forces must be concurrent. Proof of this statement is given in the next chapter. However this concept has been used in this article also.

→ The angle is always taken of b/w the two forces

Example 2.9. *A sphere weighing 100 N is tied to a smooth wall by a string as shown in Fig. 2.13 (a). Find the tension T in the string and the reaction R from the wall.*

⇒ forces must be concurrent:

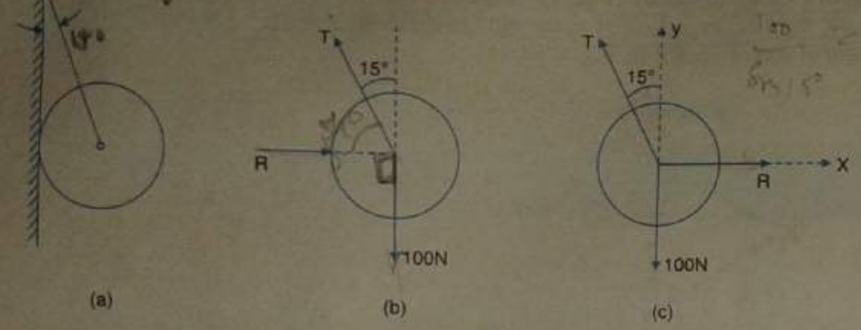


Fig. 2.13

Solution. Free body diagram of the sphere is as shown in Fig 2.13 (*b*). Figure 2.13 (*c*) shows all the forces acting away from the centre of the ball, which is permisible as per the law of transmissibility of forces. Applying Lami's theorem to the system of forces, we get

$$\frac{T}{\sin 90^\circ} = \frac{R}{\sin(180^\circ - 15^\circ)} = \frac{100}{\sin(90^\circ + 15^\circ)}$$

$$T = 103.5 \text{ N} \quad \textbf{Ans.}$$

and

$$R = 26.8 \text{ N} \quad \textbf{Ans.}$$

The above problem may be solved by using equations of equilibrium also. Taking horizontal axis as *x*-axis and vertical direction as *y*-axis as shown in Fig. 2.13 (*c*), we get

$$\Sigma F_y = 0 \rightarrow$$

$$T \cos 15^\circ - 100 = 0$$

$$\therefore \quad T = 103.5 \text{ N} \quad \textbf{Ans.}$$

$$\Sigma F_x = 0 \rightarrow$$

$$R - T \sin 15^\circ = 0$$

$$\therefore \quad R = 103.5 \sin 15^\circ = 26.8 \text{ N} \quad \textbf{Ans.}$$

Note the following points:

1. The string can have only tension in it (it can pull a body), but cannot have compression in it (cannot push a body).
2. The wall reaction is a push, but not a pull on the body.
3. The line of action of reactions should be determined accurately, but the direction can be assumed. If assumed direction is correct the value comes out to be positive. If that is exactly opposite, the value comes out to be negetive. Hence in such cases, the result may be reported with the reversed direction.

Example 2.10. *Determine the horizontal force F to be applied to the block weighing 1500 N to hold it in position on a smooth inclined plane AB which makes an angle of 30° with the horizontal (Ref. Fig. 2.14 (a)).*

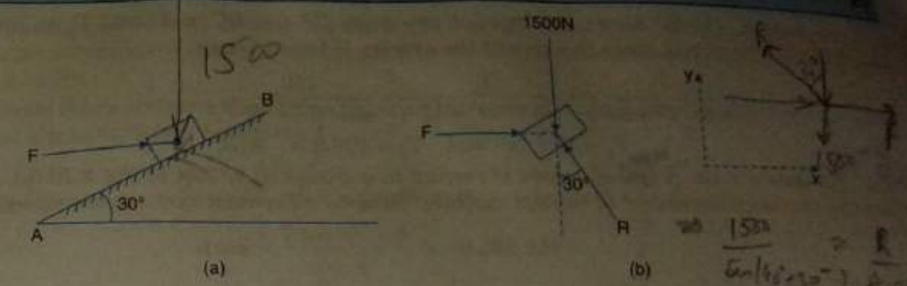


Fig. 2.14

Solution. The body is in equilibrium under the action of applied force *F*, self weight 1500 N and reaction *R* from the plane. Applied force is horizontal and self weight is vertically downward. Reaction is normal to the plane *AB*, since the plane *AB* is smooth. Since plane makes 30° to horizontal, normal to it makes 60° to horizontal *i.e.*, 30° to vertical (Ref. Fig. 2.14 (*b*)).

$$\Sigma F_y = 0 \rightarrow$$

$$R \cos 30^\circ - 1500 = 0$$

$$\therefore \quad R = \frac{1500}{\cos 30^\circ} = 1732.0 \text{ N} \quad \textbf{Ans.}$$

$$\Sigma F_x = 0 \rightarrow$$

$$F - R \sin 30^\circ = 0$$

$$\therefore \quad F = R \sin 30^\circ = 1732 \sin 30^\circ$$

i.e.,

$$F = 866 \text{ N} \quad \textbf{Ans.}$$

Note: Since the body is in equilibrium under the action of only three forces, the above problem can be solved using Lami's theorem also, as shown below:

$$\frac{R}{\sin 90^\circ} = \frac{F}{\sin(180^\circ - 30^\circ)} = \frac{1500}{\sin(90^\circ + 30^\circ)}$$

$$\therefore \quad R = 1732 \text{ N} \quad \text{and} \quad F = 866\text{N} \quad \textbf{Ans.}$$

Example 2.11. *Find the forces developed in the wires, supporting an electric fixture as shown in Fig. 2.15 (a).*

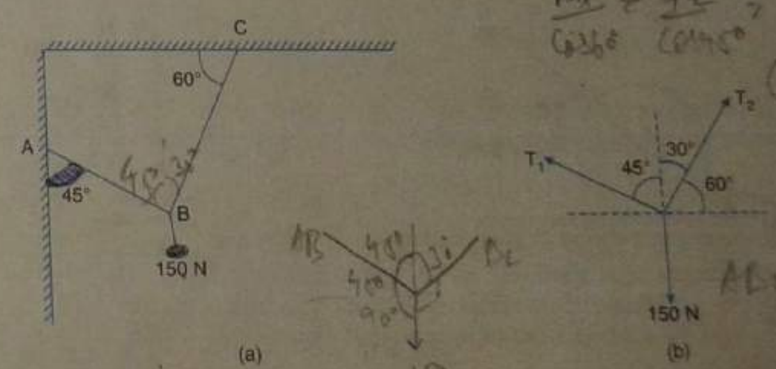


Fig. 2.15

Solution. Let the forces developed in the wires BA and BC be T_1 and T_2 as shown in Fig. 2.15 (*b*). Applying Lami's theorem to the system of forces, we get

$$\frac{T_1}{\sin(90° + 60°)} = \frac{T_2}{\sin(180° - 45°)} = \frac{150}{\sin(45° + 30°)}$$

$$\therefore \quad T_1 = 77.6 \text{ N} \quad \text{and} \quad T_2 = 109.8 \text{ N} \quad \textbf{Ans.}$$

Example 2.12. *A 200 N sphere is resting in a trough as shown in Fig 2.16 (a). Determine the reactions developed at contact surfaces. Assume all contact surfaces are smooth.*

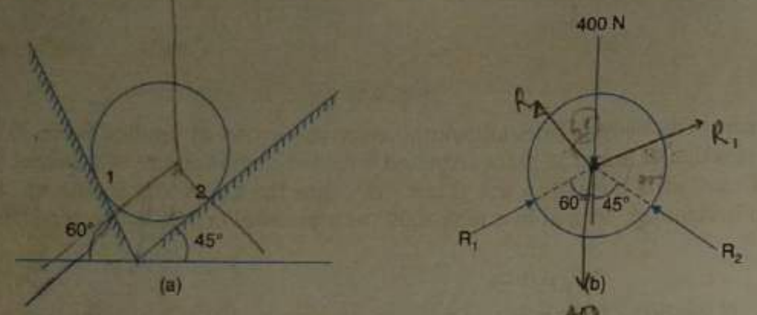


Fig. 2.16

At contact point 1, the surface of contact is making 60° to horizontal. Hence the reaction R_1 which is normal to it, makes 60° with the vertical. Similarly the reaction R_2 at contact point 2, makes 45° to the vertical. FBD is as shown in Fig. 2.16 (*b*). Applying Lami's theorem to the system of forces, we get

$$\frac{R_1}{\sin(180° - 45°)} = \frac{R_2}{\sin(180° - 60°)} = \frac{400}{\sin(60° + 45°)}$$

$$\therefore \quad R_1 = 292.8 \text{ N} \quad \text{and} \quad R_2 = 358.6 \text{ N} \quad \textbf{Ans.}$$

Example 2.13. *A roller weighing 10 kN rests on a smooth horizontal floor and is connected to the floor by the bar AC as shown in Fig. 2.17 (a). Determine the force in the bar AC and reaction from the floor, if the roller is subjected to a horizontal force of 5 kN and an inclined force of 7 kN as shown in the figure.*

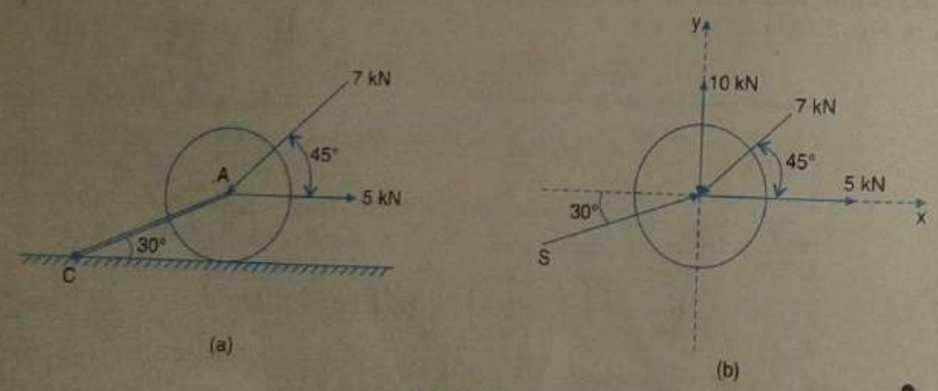


Fig. 2.17

Solution: A bar can develop tensile force or compressive force. Let the force developed be a compressive force S (push on the cylinder). Free body diagram of the cylinder is as shown in Fig. 2.17 (*b*).

Since there are more than three forces in the system, Lami's equations cannot be used. Consider the equilibrium equations.

$$\Sigma F_H = 0 \rightarrow$$

$$S \cos 30° + 5 - 7 \cos 45° = 0$$

$$S = \frac{7 \cos 45° - 5}{\cos 30°} = -0.058 \text{ kN}$$

Since the value is negative the reaction from the bar is not push, but it is a pull (tensile force in the bar) of magnitude 0.058 kN. **Ans.**

Referring to Fig. 2.17 (*b*)

$$\Sigma F_y = 0 \rightarrow$$

$$R - 10 - 7 \sin 45° + S \sin 30° = 0$$

$$R = 10 + 7 \sin 45° - S \sin 30°$$

$$= 10 + 7 \sin 45° - (-0.058) \sin 30°$$

i.e., $$R = 14.980 \text{ kN} \quad \textbf{Ans.}$$

Example 2.14. *A roller of radius r = 300 mm and weighing 2000 N is to be pulled over a curb of height 150 mm, as shown in Fig. 2.18 (a) by applying a horizontal force F applied to the end of a string wound around the circumference of the roller. Find the magnitude of force F required to start the roller move over the curb. What is the least pull F through the centre of the wheel to just turn the roller over the curb?*

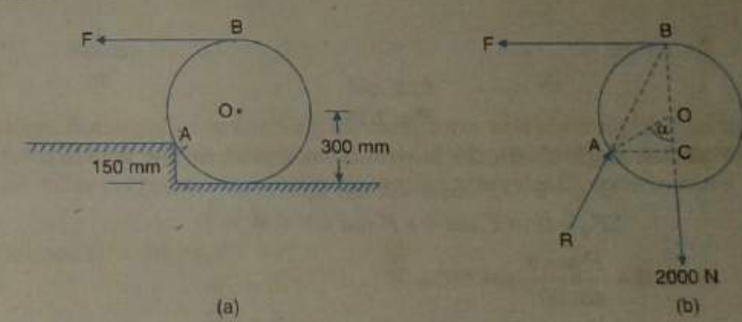


Fig. 2.18

Solution. When the roller is about to turn over the curb, the contact with the floor is lost and hence, there is no reaction from the floor. The body is in equilibrium under the action of three forces, namely,

(*i*) Applied force F, which is horizontal

(*ii*) Self weight, which is vertically downward, acting through the centre of roller, and

(*iii*) Reaction R from the edge of the curb. Since the body is in equilibrium under the action of only three forces, they must be concurrent. It means the reaction at edge A of curb passes through the point B as shown in the Figure 2.18 (*b*).

Referring to Fig. 2.18 (*b*),

→ Always remember the every concept of just in mechanics because it changes the fbd

$$\cos\alpha = \frac{OC}{AO} = \frac{300-150}{300} = \frac{1}{2}$$

$\therefore$ $\alpha = 60°$

Now, in ΔAOB, $\angle OAB = \angle OBA$ Since $OA = OB$ = radius of roller.

but $\angle OAB + \angle OBA = \alpha$

$\therefore$ $2\angle OBA = 60°$ or $\angle OBA = 30°$

i.e., reaction makes 30° with the vertical.

$$\Sigma V = 0 \rightarrow$$

$$R\cos 30° - 2000 = 0$$

$\therefore$ $R = 2309.4$ N.

$$\Sigma H = 0 \rightarrow$$

$F - R\sin 30° = 0$ or $F = 2309.4 \sin 30° = 1154.7$ N **Ans.**

Least force through the centre of Roller.

In this case the reaction from the curb must pass through the centre of the roller since the other two forces pass through that point. Its inclination to vertical is $\theta = 60°$.

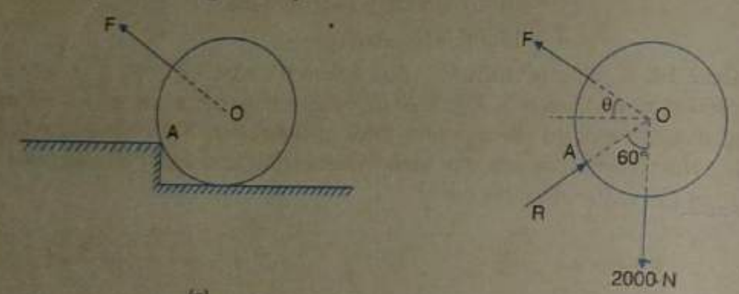


Fig. 2.19

Let force F makes angle θ with the horizontal as shown in Fig. 2.19 (*b*).

$$\Sigma F_H = 0 \rightarrow F\cos\theta = R\sin 60°$$

$$\Sigma F_V = 0 \rightarrow F\sin\theta + R\cos 60° - W = 0$$

i.e., $$F\sin\theta + \frac{F\cos\theta}{\sin 60°}\cos 60° = W$$

i.e., $$F[\sin\theta + \cot 60°\cos\theta] = W$$

$\therefore$ $$\sin\theta + \cot 60°\cos\theta = \frac{W}{P}$$

For $\frac{W}{P}$ to be maximum *i.e.*, P to be least,

$$\frac{d}{d\theta}\left[\frac{W}{P}\right] = 0$$

i.e., $\cos\theta + \cot 60°(-\sin\theta) = 0$

$$\cos\theta = \cot 60°\sin\theta$$

or $$\cot\theta = \cot 60°$$

i.e., $\theta = 60°$

i.e., F is least when it is at right angles to the reaction R.

$\therefore$ $$P_{min} = \frac{W}{\sin 60° + \cot 60°\cos 60°} = \frac{2000\sin 60°}{\sin^2 60° + \cos^2 60°}$$

$= 1732$ N **Ans.**

Example 2.15. *The frictionless pulley A shown in Fig. 2.20 (a), is supported by two bars AB and AC which are hinged at B and C to a vertical wall. The flexible cable hinged at D, goes over the pulley and supports a load of 20kN at G. The angle made by various members of the system are as shown in the figure. Determine the forces in the bars AB and AC. Neglect the size of the pulley.*

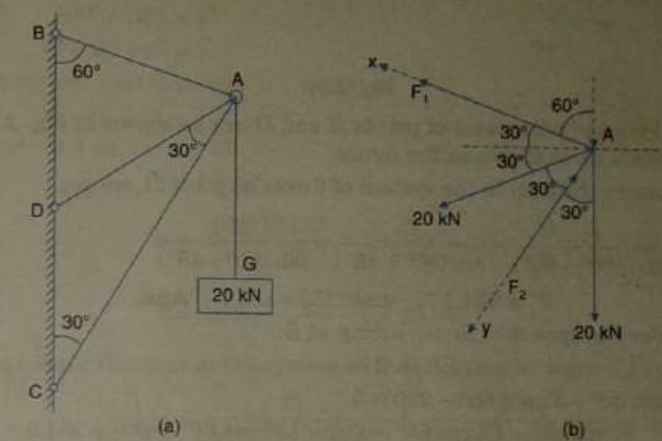


Fig. 2.20

Solution. Since pulley is frictionless, same force exists throughout in the flexible cable. Hence the force in AD is also 20 kN as shown in Fig. 2.20 (*b*). From the figure it may be observed that AC is perpendicular to AB. Selecting AB and AC as cartesian x and y axes,

$$\Sigma F_x = 0 \rightarrow$$

$$F_1 + 20\sin 30° - 20\sin 30° = 0$$

$\therefore$ $F_1 = 0$

$$\Sigma F_y = 0 \rightarrow$$

$$-F_2 + 20\cos 30° + 20\cos 30° = 0$$

$\therefore$ $F_2 = 40\cos 30° = 34.6$ N **Ans.**

2.5. EQUILIBRIUM OF CONNECTED BODIES

When two or more bodies are in contact with one another, the system of forces appear as though it is a non-concurrent force system. However, when each body is considered separately, in many situations it turns out to be a set of concurrent force system. In such cases, first, the body subjected to only two unknown forces is analysed and then it is followed by the analysis of other connected bodies. This type of examples are illustrated below :

Example 2.16. *A system of connected flexible cables shown in Fig. 2.21 (a) is supporting two vertical forces 200 N and 250 N at points B and D. Determine the forces in various segments of the cable.*

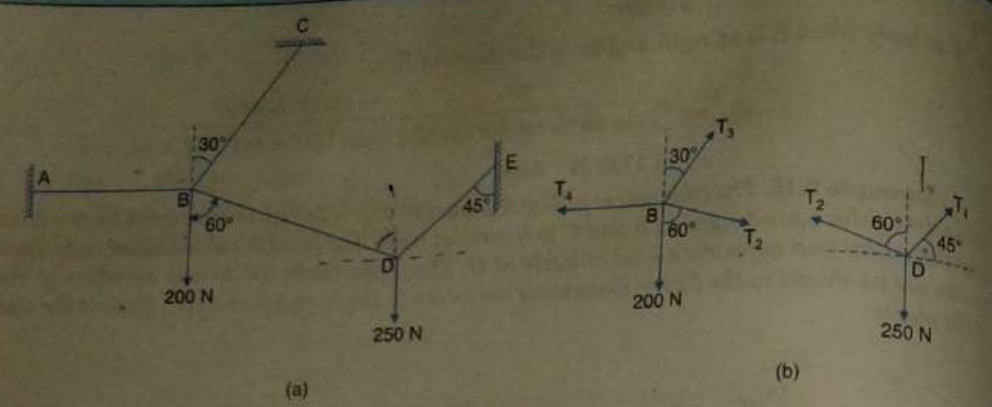


Fig. 2.21

Solution. Free body diagrams of points B and D are as shown in Fig. 2.21 (b). Let the forces in the members be as shown in the figure.

Applying Lami's theorem to the system of forces at point D, we get

$$\frac{T_1}{\sin(180° - 60°)} = \frac{T_2}{\sin(90° + 45°)} = \frac{250}{\sin(60° + 45°)}$$

$\therefore$ $T_1 = 224.1$ N and $T_2 = 183$ N **Ans.**

Now, consider the system of forces acting at B.

$$\Sigma F_V = 0 \rightarrow$$

$$T_3 \cos 30° - T_2 \cos 60° - 200 = 0$$

$$T_3 \cos 30° = T_2 \cos 60° + 200 = 183 \cos 60° + 200 = 291.5$$

$\therefore$ $T_3 = 336.6$ N **Ans.**

$$\Sigma F_H = 0 \rightarrow$$

$$-T_4 + T_3 \sin 30° + T_2 \sin 60° = 0$$

$\therefore$ $T_4 = 336.6 \sin 30° + 183 \sin 60° = 326.8$ N **Ans.**

Example 2.17. *Rope AB shown in Fig. 2.22 (a) is 4.5 m long and is connected at two points A and B at the same level 4 m apart. A load of 1500 N is suspended from a point C on the rope at 1.5 m from A. What load connected at point D on the rope, 1 m from B will be necessary to keep the position CD level?*

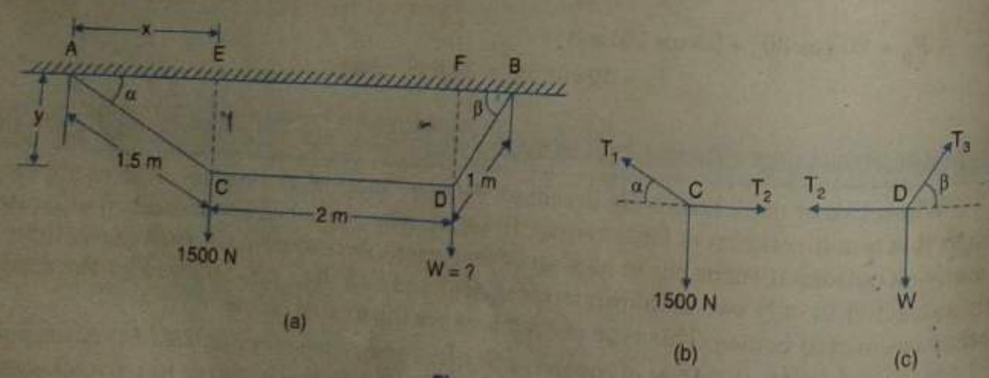


Fig. 2.22

Solution. Drop perpendiculars CE and DF on AB.

Let $CE = y$ and $AE = x$

From ΔAEC, $x^2 + y^2 = 1.5^2 = 2.25$ ...(i)

Now, $AB = 4$ m and $AC + CD + DB = 4.5$ m

$\therefore$ $CD = 4.5 - 1.5 - 1.0 = 2.0$ m

Since $AC = 1.5$ m and $DB = 1$ m

$\therefore$ $EF = 2.0$ m

Now $BF = AB - (AE + EF)$

$= 4 - (x + 2) = 2 - x$ ...(ii)

From ΔBFD,

$$BF^2 + DF^2 = 1^2$$

$$(2 - x)^2 + y^2 = 1 \quad \text{...(iii)}$$

Subtracting eqn. (iii) from eqn. (i), we get

$$x^2 - (2 - x)^2 = 1.25$$

i.e. $x^2 - 4 + 4x - x^2 = 1.25$

$$x = 1.3125 \text{ m.}$$

$\therefore$ $\alpha = \cos^{-1} \dfrac{1.3125}{1.5} = 28.955°$

and $\beta = \cos^{-1} \dfrac{2 - 1.3125}{1} = 46.567°$

Applying Lami's theorem to the system of forces acting at point C [Ref. Fig. 2.22 (b)],

$$\frac{T_1}{\sin 90°} = \frac{T_2}{\sin(90° + 28.955°)} = \frac{1500}{\sin(180° - 28.955°)}$$

$\therefore$ $T_1 = 3098.4$ N

$T_2 = 2711.1$ N

Now, applying Lami's theorem to the system of forces at B [Fig. 2.22 (c)],

$$\frac{T_3}{\sin 90°} = \frac{W}{\sin(180° - 46.567°)} = \frac{T_2}{\sin(90° + 46.567°)}$$

$\therefore$ $T_3 = 3943.4$ N and $W = 2863.6$ N **Ans.**

Example 2.18. *A wire rope is fixed at two points A and D as shown in Fig. 2.23 (a). Weights 20 kN and 30 kN are attached to it at B and C, respectively. The weights rest with portions AB and BC inclined at 30° and 50° respectively, to the vertical as shown in the figure. Find the tension in segments AB, BC and CD of the wire. Determine the inclination of the segment CD to vertical.*

Solution. Figures 2.23 (b) and (c) show free body diagrams of points B and C. Applying Lami's theorem for the system of forces B,

$$\frac{T_1}{\sin 50°} = \frac{T_2}{\sin(180° - 30°)} = \frac{20}{\sin(180° + 30° - 50°)}$$

$\therefore$ $T_1 = 44.80$ kN and $T_2 = 29.2$ kN **Ans.**

Consider the equilibrium of forces at C.

$$\Sigma F_H = 0 \rightarrow$$

$$T_3 \sin\theta = T_2 \sin 50° = 22.4 \quad ...(i)$$

$$\Sigma F_V = 0 \rightarrow$$

$$T_3 \cos\theta + T_2 \cos 50° - 30 = 0$$

i.e.,

$$T_3 \cos\theta = 30 - 29.2 \cos 50° = 11.20 \quad ...(ii)$$

From eqns. (*i*) and (*ii*), we get

$$\tan\theta = 2$$

$$\therefore \quad \theta = 63.43° \quad \textbf{Ans.}$$

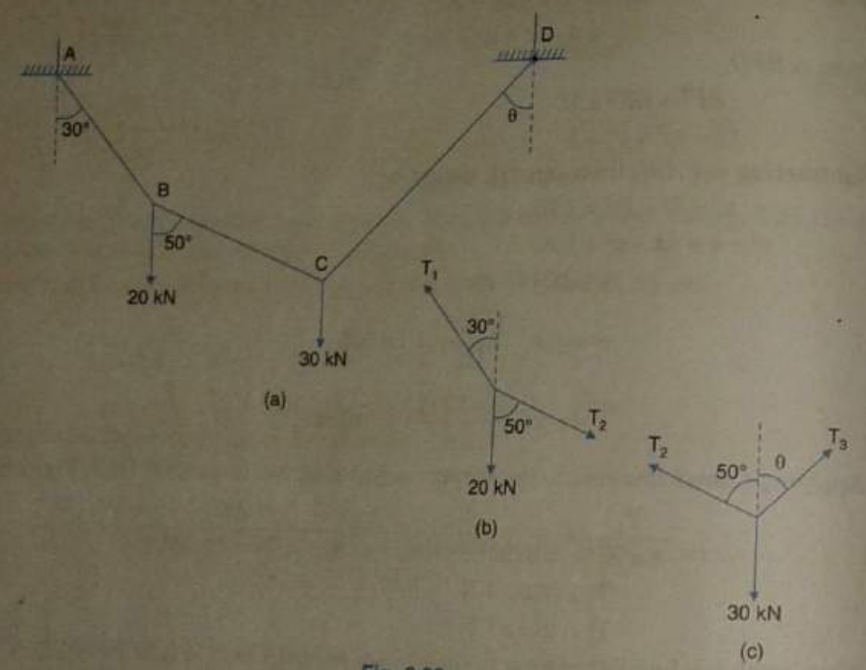


Fig. 2.23

Substituting this value in eqn. (*i*), we get

$$T_3 = 25.04 \text{ kN} \quad \textbf{Ans.}$$

Example 2.19. *A wire is fixed at A and D as shown in Fig. 2.24 (a). Weight 20 kN and 25 kN are supported at B and C respectively. When equilibrium is reached it is found that inclination of AB is 30° and that of CD is 60° to the vertical. Determine the tension in the segments AB, BC and CD of the rope and also the inclination of BC to the vertical.*

Solution. Free body diagrams of points *B* and *C* are shown in Figs. 2.24 (*b*) and (*c*) respectively. Considering equilibrium of point *B*, we get

$$\Sigma F_H = 0 \rightarrow T_2 \sin\theta - T_1 \sin 30° = 0$$

i.e.,

$$T_2 \sin\theta = T_1 \sin 30° \quad ...(i)$$

$$\Sigma F_V = 0 \rightarrow -T_2 \cos\theta + T_1 \sin 30° - 20 = 0$$

i.e.,

$$T_2 \cos\theta = T_1 \sin 30° - 20 \quad ...(ii)$$

Considering the equilibrium of point *C*,

$$\Sigma F_H = 0 \rightarrow T_3 \sin 60° - T_2 \sin\theta = 0$$

i.e.,

$$T_2 \sin\theta = T_3 \sin 60° \quad ...(iii)$$

$$\Sigma F_V = 0 \rightarrow T_3 \cos 60° + T_2 \cos\theta - 25 = 0$$

i.e.,

$$T_2 \cos\theta = -T_3 \cos 60° + 25 \quad ...(iv)$$

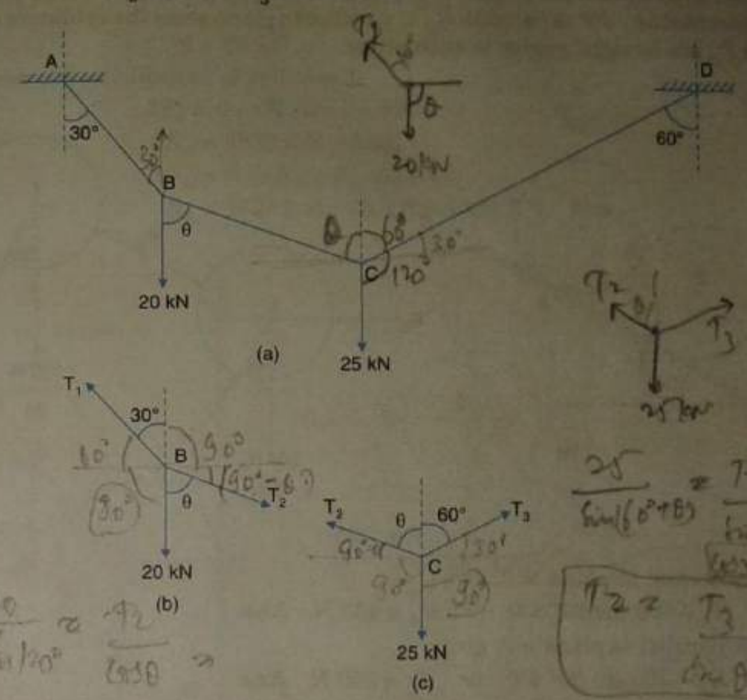


Fig. 2.24

From eqns. (*i*) and (*iii*), we get

$$T_1 \sin 30° = T_3 \sin 60°$$

i.e.,

$$T_1 = \sqrt{3}\, T_3$$

From eqns. (*ii*) and (*iv*), we get,

$$T_1 \cos 30° - 20 = 25 - T_3 \cos 60°$$

$$\sqrt{3} \times T_3 \times \frac{\sqrt{3}}{2} - 20 = 25 - T_3 \times 0.5$$

i.e.,

$$2T_3 = 45 \quad \text{or} \quad T_3 = 22.5 \text{ kN}$$

From eqn. (*v*) $\quad T_1 = 38.97$ kN

From eqn. (*i*) $\quad T_2 \sin\theta = 19.48$

and from eqn. (*ii*) $\quad T_2 \cos\theta = 13.75$

$$\therefore \quad \tan\theta = 1.416$$

$$\therefore \quad \theta = 54.78° \quad \textbf{Ans.}$$

and

$$T_2 = 23.84 \text{ kN} \quad \textbf{Ans.}$$

Example 2.20. *Two identical cylinders, each weighing 500 N are placed in a trough as shown in Fig. 2.25 (a). Determine the reactions developed at contact points A, B, C and D. Assume all points of contact are smooth.*

Solution. Free body diagrams of two cylinders are as shown in Figs. 2.25 (*b*) and (*c*).

Consider the equilibrium of cylinder 1. Since R_A is at right angles to the plane, it makes 60° to the horizontal *i.e.*, 30° to vertical. R_B is parallel to plane since the cylinders are identical. Thus R_A and R_B are at right angles to each other.

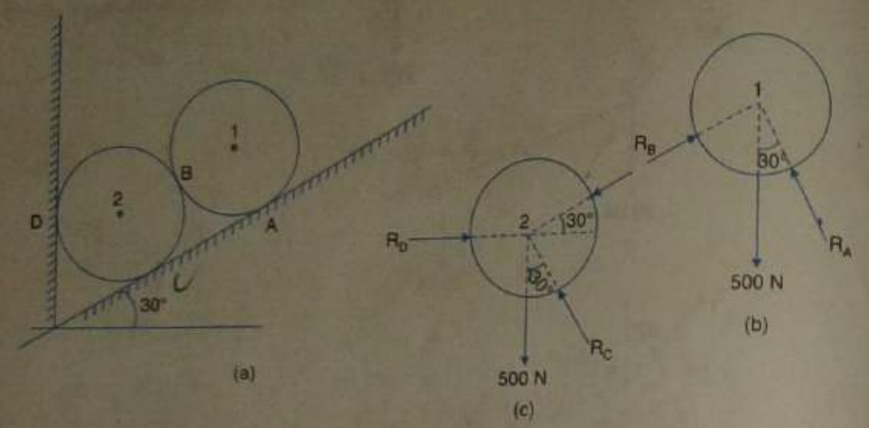


Fig. 2.25

Σ Forces normal to plane = 0, gives

$$R_A - 500 \cos 30° = 0 \quad \text{or} \quad R_A = 433 \text{ N} \quad \textbf{Ans.}$$

Σ Forces parallel to plane = 0, gives

$$R_B - 500 \sin 30° = 0 \quad \text{or} \quad R_B = 250 \text{ N} \quad \textbf{Ans.}$$

Now, consider equilibrium of cylinder 2. R_D is horizontal since the line of contact is vertical. R_C is normal to the plane and R_B is parallel to the plane.

$$\Sigma F_V = 0 \rightarrow$$

$$-500 + R_C \cos 30° - R_B \sin 30° = 0$$

$$\therefore \quad R_C = \frac{500 + 250 \sin 30°}{\cos 30°} = 721.7 \text{ N} \quad \textbf{Ans.}$$

$$\Sigma F_H = 0 \rightarrow$$

$$R_D - R_C \sin 30° - R_B \cos 30° = 0$$

$$R_D = 721.7 \sin 30° + 250 \cos 30°$$

$$\therefore \quad R_D = 577.4 \text{ N} \quad \textbf{Ans.}$$

Example 2.21. *Cylinder 1 of diameter 200 mm and cylinder 2 of diameter 300 mm are placed in a trough as shown in Fig. 2.26 (a). If cylinder 1 weighs 800 N and cylinder B weighs 1200 N, determine the reactions developed at contact surfaces A, B, C and D. Assume all contact surfaces are smooth.*

Solution. Since all contact surfaces reactions are at right angles to the contact surfaces *i.e.*, R_A and R_C are horizontal and R_D makes 45° to horizontal/vertical. R_B should be in the direction of O_1O_2. Let O_1O_2 makes an angle θ with horizontal. Let O_1P and O_2P be vertical and horizontal. Then

$$\cos \theta = \frac{O_2P}{O_1O_2} = \frac{450 - 100 - 150}{100 + 150} = 0.8$$

$$\therefore \quad \theta = 36.87°$$

Consider the equilibrium of cylinder 1,

$$\Sigma F_V = 0 \rightarrow R_B \sin \theta - 800 = 0$$

$$\therefore \quad R_B = 1333.3 \text{ N} \quad \textbf{Ans.}$$

$$\Sigma F_H = 0 \rightarrow R_A - R_B \cos \theta = 0$$

$$\therefore \quad R_A = 1333.3 \cos 36.87° = 1066.7 \text{ N} \quad \textbf{Ans.}$$

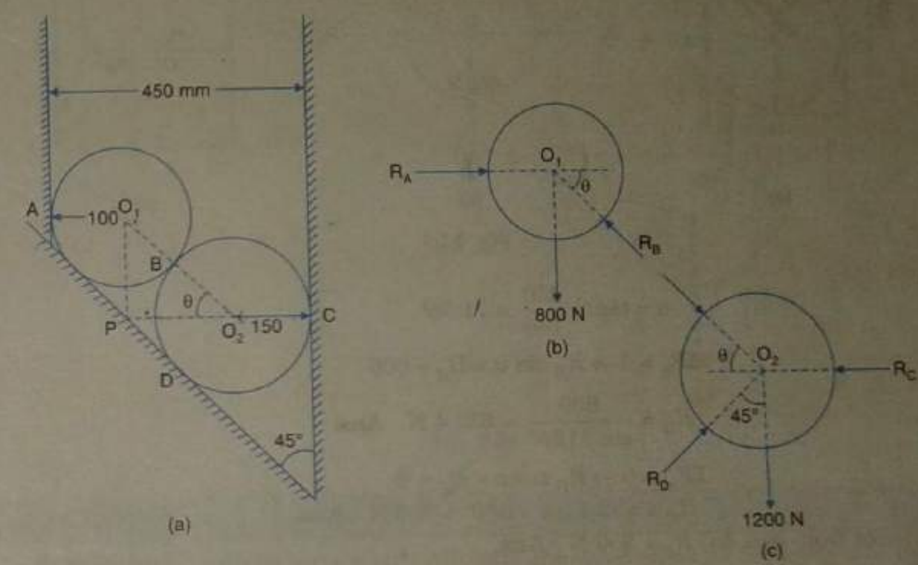


Fig. 2.26

Now consider the equilibrium of cylinder 2,

$$\Sigma F_V = 0 \rightarrow R_D \cos 45° - R_B \sin \theta - 1200 = 0$$

$$\therefore \quad R_D = \frac{1333.3 \sin 36.87° + 1200}{\cos 45°} = 2828.4 \text{ N} \quad \textbf{Ans.}$$

$$\Sigma F_H = 0 \rightarrow R_D \sin 45° + R_B \cos \theta - R_C = 0$$

$$2828.4 \sin 45° + 1333.3 \cos 36.87° - R_C = 0.$$

$$\therefore \quad R_C = 3066.7 \text{ N} \quad \textbf{Ans.}$$

Example 2.22. *A 600 N cylinder is supported by the frame BCD as shown in Fig 2.27 (a). The frame is hinged at D. Determine the reactions developed at contact points A, B, C and D. Neglect the weight of frame and assume all contact surfaces are smooth.*

Solution. Free body diagram of the cylinder and the frame are shown in Figs. 2.27 (b) and (*c*) respectively. Considering the equlibrium conditions of cylinder, we get,

$$\Sigma F_V = 0 \rightarrow R_D = 600 \text{ N} \quad \text{...(i)}$$

$$\Sigma F_H = 0 \rightarrow R_A = R_C \quad \text{...(ii)}$$

Consider the equilibrium of the frame. As it is in equilibrium under the action of three forces only, they must be concurrent forces. In other words, the reaction at D has line of action along OD. Hence its inclination to horizontal is given by

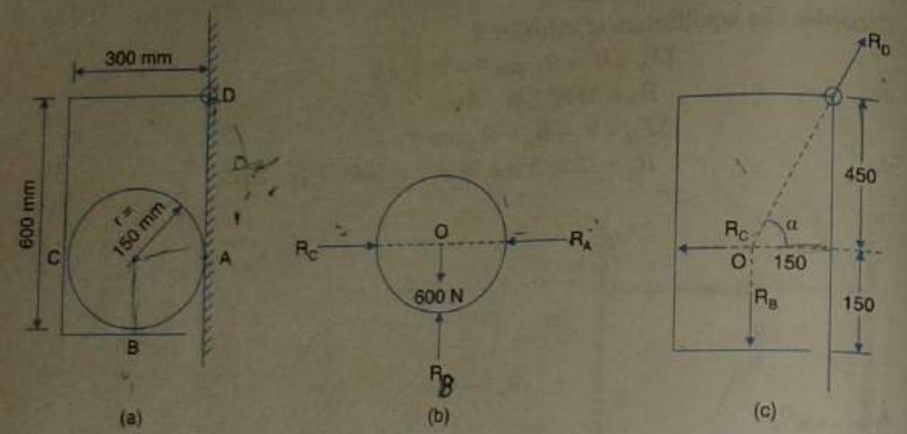


Fig. 2.27

$$\alpha = \tan^{-1}\frac{450}{150} = 71.56°$$

$$\therefore \quad \Sigma F_V = 0 \rightarrow R_D \sin\alpha = R_B = 600$$

$$\therefore \quad R_D = \frac{600}{\sin 71.56°} = 632.4 \text{ N} \quad \textbf{Ans.}$$

$$\Sigma F_H = 0 \rightarrow R_D \cos\alpha - R_C = 0$$

$$\therefore \quad R_C = 632.4 \cos 71.56° = 200 \text{ N} \quad \textbf{Ans.}$$

Hence from eqn. (*ii*) $R_A = 200$ N **Ans.**

Example 2.23. *Cylinder A weighing 4000 N and cylinder B weighing 2000 N rest on smooth inclines as shown in Fig. 2.28 (a). They are connected by a bar of negligible weight hinged to geometric centres of the cylinders by smooth pins. Find the force P to be applied as shown in the figure such that it will hold the system in the given position.*

Solution. Figures 2.28 (*b*) and (*c*) show the free body diagrams of two cylinders. Applying Lami's theorem to the system of forces on cylinder A, we get

$$\frac{C}{\sin(180° - 60°)} = \frac{4000}{\sin(60° + 90° - 15°)}$$

$$\therefore \quad C = 4899 \text{ N}$$

Now consider equilibrium of cylinder B. Summation of forces parallel to the inclined plane = 0, gives

$$P \cos 15° + 2000 \cos 45° - C \cos(45° + 15°) = 0$$

$$\therefore \quad P = \frac{-2000 \cos 45° + 4894 \cos 60°}{\cos 15°} = 1071.8 \text{ N} \quad \textbf{Ans.}$$

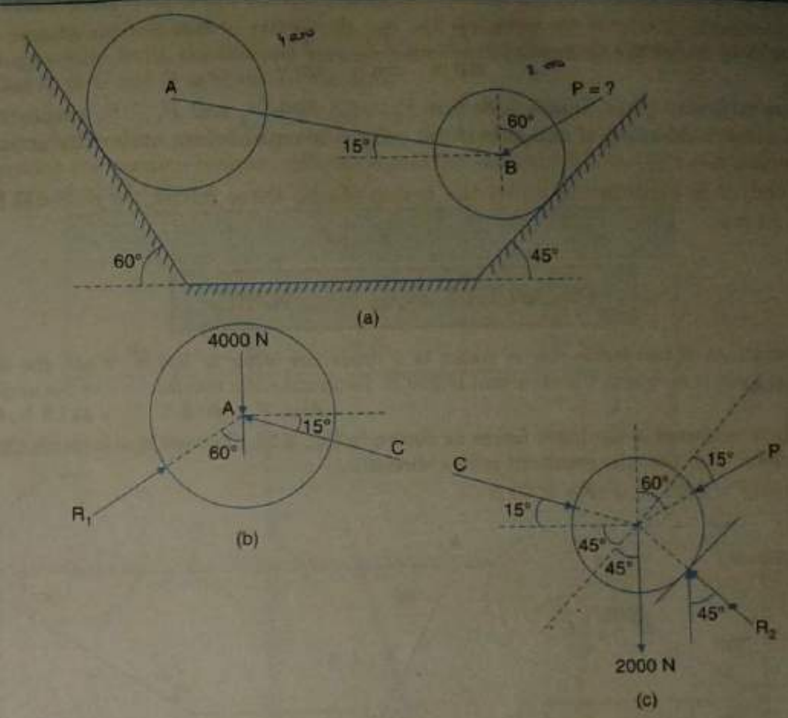


Fig. 2.28

Important Definitions

Resultant: The single force which will have the same effect as the system of forces.

Freebody Diagram: The diagram showing the body freed from all other bodies in contact and showing all the forces acting on it, including self weight and the reactions from the other bodies removed.

Lami's Theorem: If a body is in equilibrium under the action of only three forces, each force is proportional to the sine of the angle between the other two forces.

Important Formulae

1. *Resultant* of any number of concurrent forces may be found using the formulae,

$$R = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2}$$

and $\quad \alpha = \tan^{-1}\dfrac{\Sigma F_y}{\Sigma F_x}$, where α is angle made by resultant with x-axis.

2. *Equilibriant* is same as the resultant in magnitude but its direction is opposite to that of resultant.

3. According to Lami's theorem, $\dfrac{F_1}{\sin\alpha} = \dfrac{F_2}{\sin\beta} = \dfrac{F_3}{\sin\gamma}$

where α, β and γ are angles between $F_2 - F_3$, $F_3 - F_1$ and $F_1 - F_2$ respectively. This formula is to be used, if and only if the body is in equilibrium under the action of three forces only.

4. If a body is in equilibrium under the action of only three forces, they should be concurrent forces.

PROBLEMS FOR EXERCISE

2.1 The resultant of two forces one of which is 3 times the other is 300 N. When the direction of smaller force is reversed, the resultant is 200 N. Determine the two forces and the angle between them. [**Ans.** $F_1 = 80.6$ N, $F_2 = 241.8$ N, $\theta = 50.13°$]

2.2 A body is subjected to the three forces as shown in Fig. 2.29. If possible determine the direction of the force F so that the resultant is in k-direction.

When (*a*) $F = 5000$ N, (*b*) $F = 3000$ N

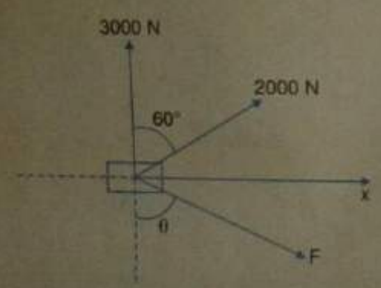


Fig. 2.29

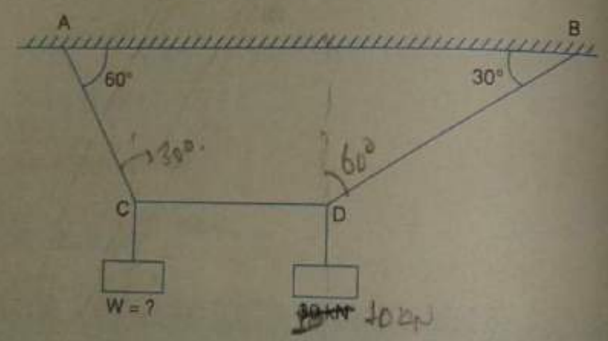


Fig. 2.30

[**Ans.** (*a*) 36.87° ; (*b*) Not possible]

2.3 A chord supported at A and B carries a load of 10 kN at D and a load of W at C as shown in Fig. 2.30. Find the value of W so that CD remains horizontal. [**Ans.** $W = 30$ kN]

2.4 Three bars, hinged at A and D and pinned at B and C as shown in Fig. 2.31 form a four-linked mechanism. Determine the value of P that will prevent movement of bars. [**Ans.** 3047.2 N]

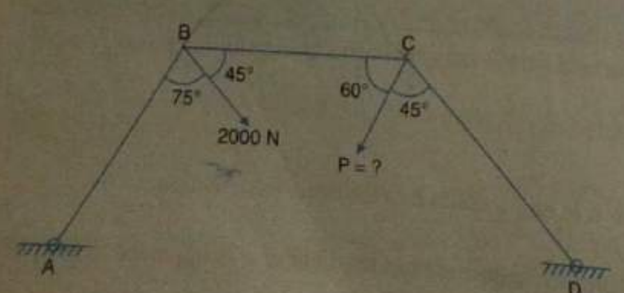


Fig. 2.31

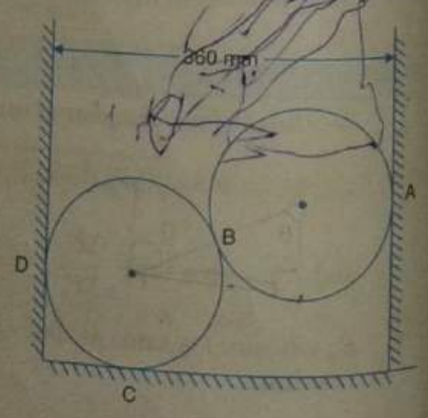


Fig. 2.32

2.5 Two smooth spheres each of radius 100 mm and weighing 100 N, rest in a horizontal channel having vertical walls, the distance between which is 360 mm. Find the reactions at the points of contact A, B, C and D as shown in Fig. 2.32.

[**Ans.** $R_A = 133.33$ N, $R_B = 166.67$ N, $R_C = 200$ N, $R_D = 133.33$ N]

2.6 Three cylinders are placed in a rectangular ditch as shown in Fig. 2.33. Neglecting friction, determine the reaction between cylinder A and the vertical wall. Weights and radii of the cylinders are as given below:

Cylinder	*Weight*	*Radius*
A	75 N	100 mm
B	200 N	150 mm
C	100 N	125 mm

[**Ans.** $R = 400$ N]

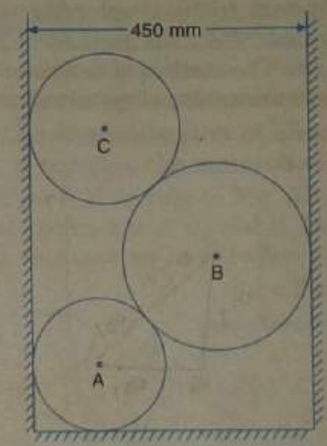


Fig. 2.33

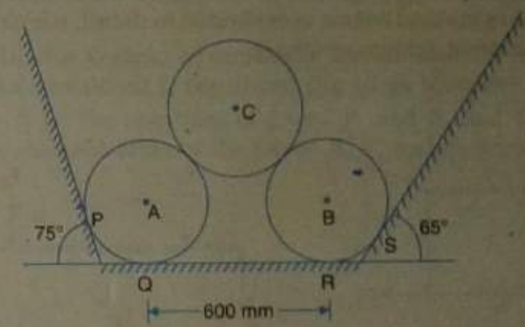


Fig. 2.34

2.7 Three spheres A, B and C having their diameters 500 mm, 500 mm and 800 mm, respectively are placed in a trench with smooth side walls and floor as shown in Fig. 2.34. The centre-to-centre distance of spheres A and B is 600 mm. The cylinders A, B and C weigh 4 kN, 4 kN and 8 kN respectively. Determine the reactions developed at contact points P, Q, R and S.

[**Ans.** $R_P = 2.15$ kN, $R_Q = 7.44$ kN, $R_R = 7.03$ kN and $R_S = 2.29$ kN]

2.8 Three smooth spheres A, B and C weighing 300 N, 600 N and 300 N respectively and having diameters 800 mm, 1200 mm and 800 mm respectively are placed in a trench as shown in Fig. 2.35. Determine the reactions developed at contact points P, Q, R and S.

[**Ans.** $R_P = 61.24$ N, $R_Q = 631.6$ N, $R_R = 1095.2$ N and $R_S = 290.4$ N]

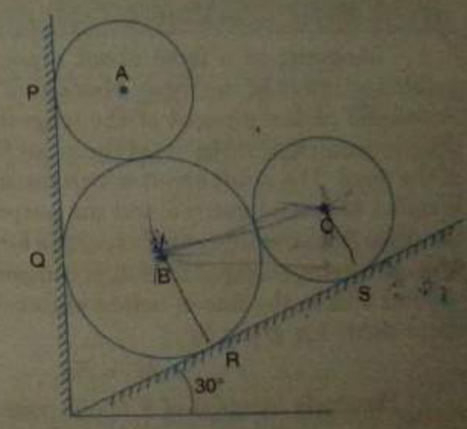


Fig. 2.35

3

Resultant and Equilibrium of System of Coplanar Non-Concurrent Forces

If all the forces in a system lie in a single plane and the lines of action of all the forces do not pass through a single point, the system is said to be **coplanar non-concurrent force system.** Three such systems are shown in Fig. 3.1. The parallel force systems shown in Figs. 3.1(*b*) and (*c*) are special cases of non-concurrent force system. In this chapter few technical terms are explained first, and then preliminaries are treated. Then the method of finding resultant and analysing equilibrium conditions are illustrated. The method of determining the reactions in statical beams is explained in detail, which is a common case of system of coplanar non-concurrent system of forces.

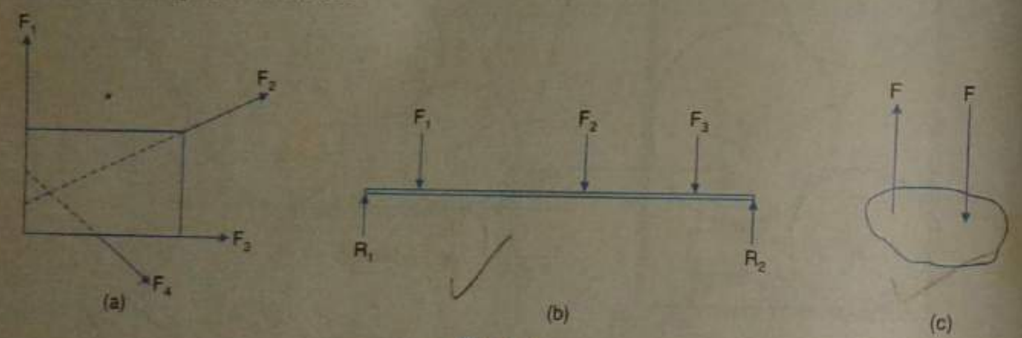


Fig. 3.1

3.1. MOMENT OF A FORCE

Moment of a force about a point is the measure of rotational effect of the force. Moment of a force about a point is defined as the product of the magnitude of the force and the perpendicular distance of the point from the line of action of the force. The point about which the moment is considered is called **moment centre** and the perpendicular distance of the point from the line of action of the force is called **moment arm**. Referring to Fig. 3.2, if d_1 is the perpendicular distance of point 1 from the line of action of force F, the moment of F about point 1 is given by

$$M_1 = Fd_1 \qquad ...(3.1a)$$

Similarly moment of F about point 2 is given by

$$M_2 = Fd_2 \qquad ...(3.1b)$$

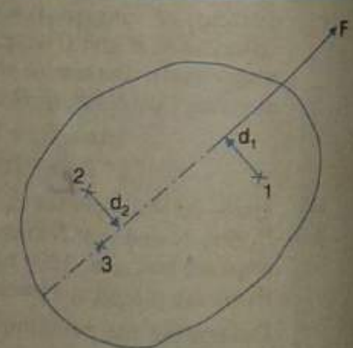


Fig. 3.2

If the moment centre 3 lies on the line of action of the force F, the moment arm is zero and hence

$$M_3 = F \times 0 = 0 \qquad ...(3.1c)$$

Thus it may be noted that, if a point lies on the line of action of a force, the moment of the force about that point is zero.

The moment of a force has got direction also. In Fig. 3.2 it may be noted that M_1 is **clockwise** and M_2 is **anticlockwise**. To find the direction of moment, imagine that the line of action of the force is connected to the point by a rigid rod pinned at the moment centre and is free to move around the point by the force. The direction of the rotation of the rod indicate the direction of the moment by the force.

If the force is in newton unit and the distance is in millimeter, the unit of moment will be N-mm. Commonly used units of moment in engineering are kN-m, N-m, kN-mm and N-mm.

3.2. VARIGNON'S THEOREM

French mathematician Varignon (1654–1722) gave the following theorem which is also known as **principles of moments:**

The algebraic sum of moments of a system of coplanar forces about a moment centre is equal to the moment of their resultant force about the same moment centre.

Proof: Referring to Fig. 3.3, let R be the resultant of forces F_1 and F_2 and O be the moment centre. Let d, d_1 and d_2 be the moment arms of the forces R, F_1 and F_2 respectively. Then in this case we have to prove that

$$Rd = F_1 d_1 + F_2 d_2 \qquad ...(3.2)$$

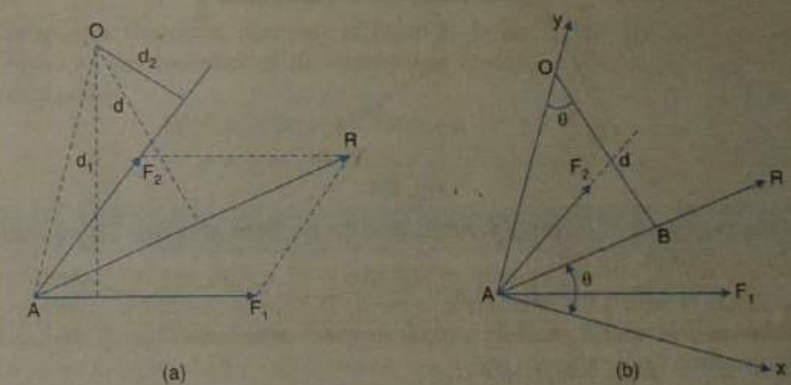


Fig. 3.3

Join OA and consider it as y-axis. Draw x-axis to it with A as origin. [Ref. Fig. 3.3(*b*)]. Let the resultant R makes an angle θ with x-axis. Noting that angle AOB is also θ, we can write

$$\begin{aligned} Rd &= R \times AO \cos\theta \\ &= AO \times (R\cos\theta) \\ &= AO \times R_x \end{aligned} \qquad ...(i)$$

where R_x denotes the component of R in x-direction. Similarly, if F_{1x} and F_{2x} are the components of F_1 and F_2 in x-direction respectively, then

$$F_1 d_1 = AO\, F_{1x} \quad \ldots(ii)$$

and

$$F_2 d_2 = AO\, F_{2x} \quad \ldots(iii)$$

∴ From eqns. (*ii*) and (*iii*) we get,

$$F_1 d_1 + F_2 d_2 = AO\,(F_{1x} + F_{2x})$$

$$= AO\, R_x \quad \ldots(iv)$$

From eqns. (*i*) and (*iv*) we observe

$$F_1 d_1 + F_2 d_2 = Rd$$

Thus we find sum of the moment of forces about a moment centre is same as moment of their resultant about the same moment centre.

If a system of forces consists of more than two forces, the above result can be extended as given below:

Let F_1, F_2, F_3 and F_4 be four concurrent forces and R be their resultant. Referring to Fig. 3.4, d_1, d_2, d_3, d_4 and a be moment arms of F_1, F_2, F_3, F_4 and R about moment centre O.

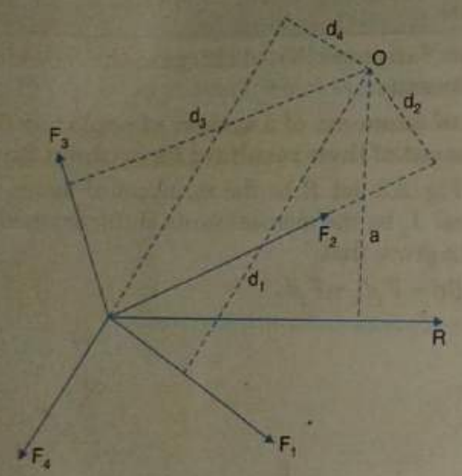


Fig. 3.4

If R_1 is the resultant of forces F_1 and F_2 and its moment arm is a_1, then from the above proof for two force system, we get,

$$R_1 a_1 = F_1 d_1 + F_2 d_2$$

If R_2 is the resultant of R_1 and F_3 and its moment arm from O is a_2, we can say

$$R_2 a_2 = R_1 a_1 + F_3 d_3$$

$$= F_1 d_1 + F_2 d_2 + F_3 d_3$$

Now considering R_2 and F_4, we can write

$$Ra = R_2 a_2 + F_4 d_4$$

$$= F_1 d_1 + F_2 d_2 + F_3 d_3 + F_4 d_4 \quad \ldots(3.3)$$

Thus, the moment of the resultant of a number of forces about a moment centre is equal to the sum of the moments of its component forces about the same moment centre.

Example 3.1. *Determine the moment of 100 N force acting at B about moment centre A as shown in Fig. 3.5.*

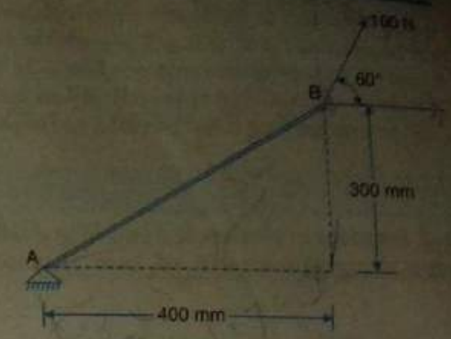


Fig. 3.5

Solution. 100 N force may be resolved into its horizontal components 100 cos 60° and vertical component 100 sin 60°. From Varignon's theorem, moment of 100 N force about moment centre A is equal to sum of moment of its components about A. Taking clockwise moment as positive,

$$M_A = 100 \cos 60° \times 300 - 100 \sin 60° \times 500$$

$$= -28301 \text{ N-mm}$$

i.e., **M_A = 28301 N-mm, anticlockwise.** **Ans.**

Example 3.2. *What will be y-intercept of 500 N force shown in Fig. 3.6, if its moment about A is 8000 N-m ?*

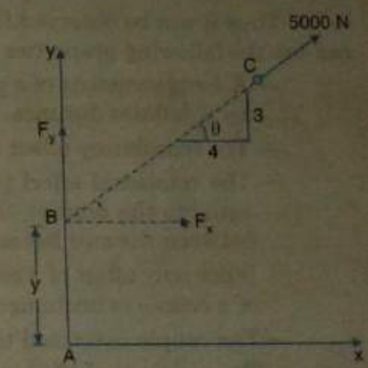


Fig. 3.6

Solution. 5000 N force is shifted to point B along the line of action (law of transmissibility) and then it is resolved into its x and y components F_x and F_y as shown in the figure.

Noting that $\cos\theta = 4/5$ and $\sin\theta = 3/5$,

$$F_x = 5000 \cos\theta = 5000 \times 4/5$$

$$= 4000 \text{ N}$$

and

$$F_y = 5000 \sin\theta = 5000 \times 3/5$$

$$= 3000 \text{ N}$$

By Varignon's theorem, moment of 5000 N force about A is equal to the moment of its component forces about the same point.

∴ $$8000 = 4000 \times y + 3000 \times 0$$

∴ **y = 2 m** **Ans.**

3.3. COUPLE

Two parallel forces equal in magnitude and opposite in direction and separated by a definite distance are said to form a couple. The sum of the forces forming a couple in any direction is zero, which means the translatory effect of the couple is zero. An interesting property of the couple can be observed, if we consider the rotational effect. Let the magnitude of the forces forming the couple be F and the perpendicular distance between the two forces be d. Consider the moment of the two forces constituting a couple about point 1 as shown in Fig. 3.7(*a*).

Let the moment be M_1. Then

$$M_1 = Fd_1 + Fd_2 = F(d_1 + d_2)$$

i.e., $$M_1 = Fd$$

Now consider the moment of the forces about point 2, which is outside the two forces as shown in Fig. 3.7 (*b*). Let M_2 be the moment. Then,

$$M_2 = Fd_3 - Fd_4 = F(d_3 - d_4)$$

i.e., $$M_2 = Fd$$

Couple means to torque

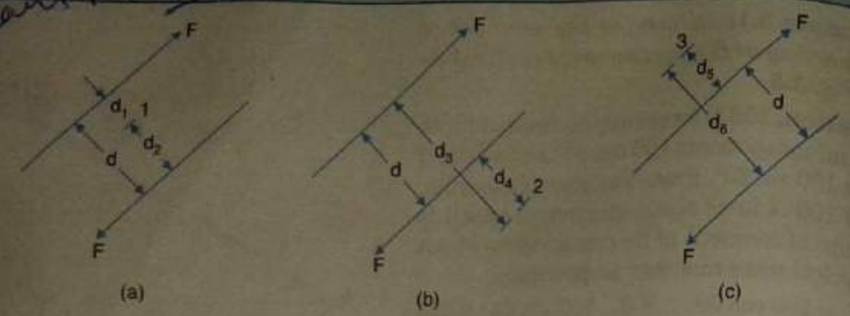


Fig. 3.7

Similarly, referring to Fig. 3.6 (*c*), we get

$$M_3 = Fd_6 - Fd_5 = F(d_6 - d_5)$$

i.e.,

$$M_3 = Fd.$$

Thus it can be observed that moment of a couple about any point is the same. Now we can list the following properties of a couple:

— A couple consists of a pair of equal and opposite parallel forces which are separated by a definite distance.

— The translatory effect of a couple on the body is zero.

— The rotational effect (moment) of a couple about any point is a constant and it is equal to the product of the magnitude of the forces and the perpendicular distance between the two forces.

Since only effect of a couple is the moment, which is the same at all points, the effect of a couple is unchanged if,

— The couple is rotated through any angle.

— The couple is shifted to any other position.

— The couple is replaced by another pair of forces whose rotational effect is the same.

3.4. RESOLUTION OF A FORCE INTO A FORCE AND A COUPLE

It will be advantageous to resolve a force acting at a point on a body into a force acting at some other point on the body and a couple moment. In Fig. 3.8 (*a*), *F* is a force acting on a body at point *A*.

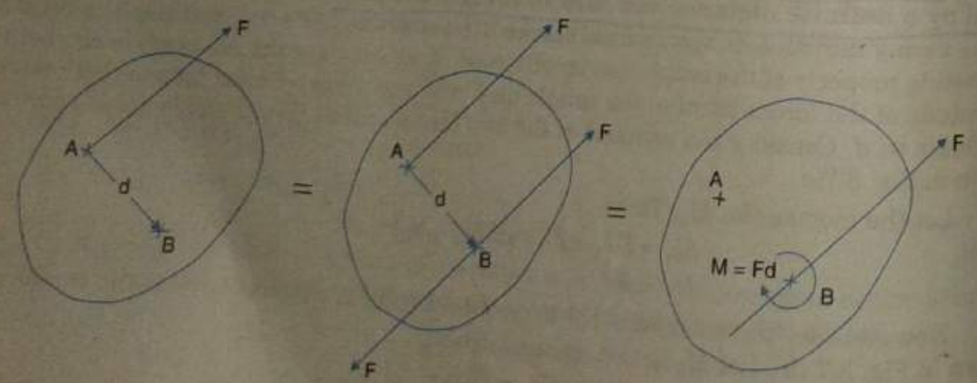


Fig. 3.8

By applying equal and opposite forces of magnitude *F* at *B*, the effect of system of forces is unaltered (Law of superposition). Hence the system of forces in Fig. 3.8 (*b*) is the same as the system given in Fig. 3.8 (*a*). Now the original force *F* at *A* and the opposite force *F* at *B* form a couple moment of magnitude $F \times d$. Hence the system in Fig. 3.8 (*b*) is equal to the system of forces in Fig. 3.8 (*c*). Thus the given force *F* at *A* is equal to a force *F* at *B* and a moment *Fd*.

3.5. RESULTANT OF NON-CONCURRENT FORCE SYSTEM

Resultant of a force system is the one which will have the same rotational and translatory effect as the given system of forces. It may be a single force, a pure moment or a force and a moment.

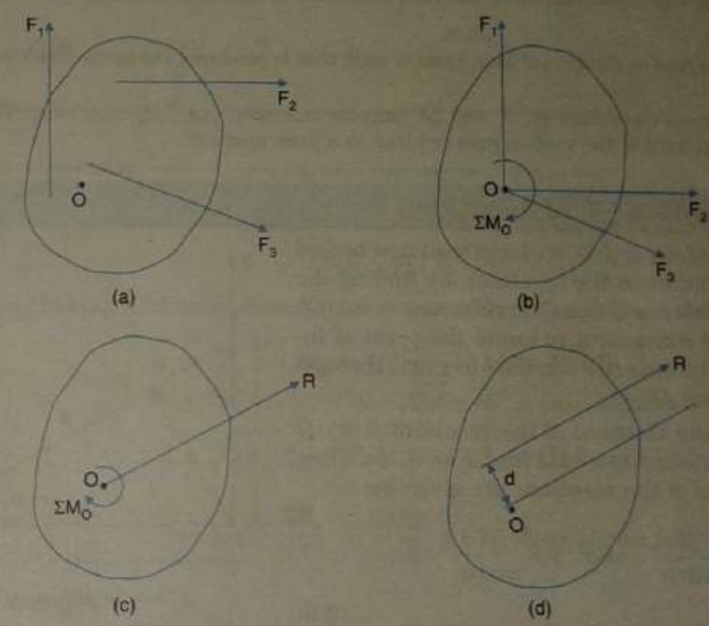


Fig. 3.9

Let F_1, F_2 and F_3, shown in Fig. 3.9 (*a*) constitute a system of forces acting on a body. Each force can be replaced by a force of same magnitude, acting in the same direction at point *O*, and a moment about *O*. Thus the given system in Fig. 3.9 (*a*) is equal to the system shown in Fig. 3.9 (*b*), where ΣM_O is the algebraic sum of the moments of the given forces about *O*.

At *O*, the concurrent forces F_1, F_2 and F_3 can be combined as to get the resultant force *R*. Now the resultant of the given system is equal to a force *R* at *O* and a moment ΣM_O as shown in Fig. 3.9 (*c*).

The force *R* and ΣM_O shown in Fig. 3.9 (*c*), can be replaced by a single force *R* acting at a distance *d* from *O* such that the moment produced by this force *R* about *O* is equal to ΣM_O [Ref. Fig. 3.9 (*d*)].

Thus we get a single force *R* acting at a distance *d* from the point *O* which gives the same effect as the constituent forces of the system. Mathematically,

$$R = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2}$$

$$\alpha = \tan^{-1}\frac{\Sigma F_y}{\Sigma F_x} \quad ...(3.4)$$

and
$$d = \frac{\Sigma M_O}{R}$$

where ΣF_x – algebraic sum of components of all forces in x-direction

ΣF_y – algebraic sum of component of all forces in y-direction

α – inclination of the resultant R to x-direction.

ΣM_O – algebraic sum of all the forces about a point O.

and d – is the distance of the line of action of resultant from point O.

Note:

1. R is marked at distance d from point O such that it produces the same direction of moment as ΣM_O.
2. Sometimes the values of ΣF_x and ΣF_y may become zero, but ΣM_O may exist. This means that the resultant of the system gets reduced to a pure moment.

3.6. x AND y INTERCEPTS OF RESULTANT

In the previous article, we have seen how to find a point of application on the resultant, by finding the perpendicular distance d from the reference point 'O'. Many times it is convenient to locate the point of intersection of the line of action on the x or y axis through the reference point O.

Let d be the distance of the resultant from O and α be its inclination to x-axis (Ref. Fig. 3.10). Then x and y intercepts of the resultant are given by:

$$x = \frac{d}{\sin\alpha} \quad \text{and} \quad y = \frac{d}{\cos\alpha} \quad ...(3.5)$$

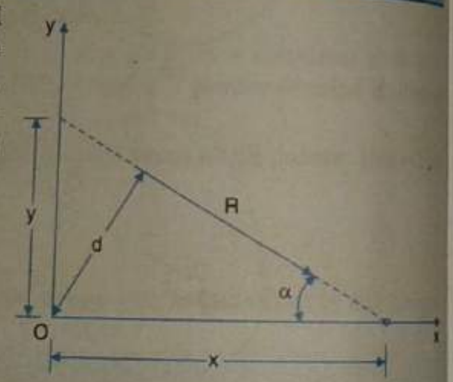


Fig. 3.10

Another method of finding the intercepts is given below:

Let $R_x = \Sigma F_x$ and $R_y = \Sigma F_y$ be the components of the resultant R in x and y directions. Considering the moment of R about O as the sum of its moments of its components at A about O (Varignon's theorem), we get (ref. Fig. 3.11)

$$Rd = \Sigma M_O$$

i.e.,
$$R_x \times O + R_y x = \Sigma M_O$$

$$\therefore \quad x = \frac{\Sigma M_O}{R_y} = \frac{\Sigma M_O}{\Sigma F_y} \quad ...(3.6)$$

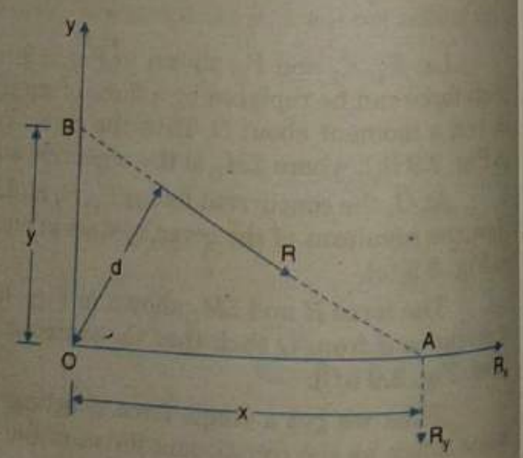


Fig. 3.11

Similarly, resolving the resultant into its components at B (Ref. Fig. 3.11) and then applying Varignon's theorem for moments about O, it can be shown that

$$y = \frac{\Sigma M_O}{R_x} = \frac{\Sigma M_O}{\Sigma F_x} \quad ...(3.7)$$

Note: A resultant in non-concurrent force system is completely determined only when its magnitude (R), direction (finding α) and a point on its line of action (represented by d or x or y intercepts) are determined.

Example 3.3. *Determine the resultant of the system of forces acting on a beam as shown in Fig. 3.12 (a).*

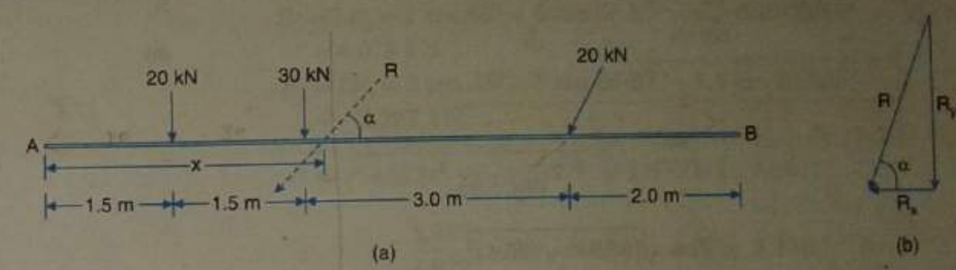


Fig. 3.12

Solution. Taking horizontal direction as x-axis and vertical upward direction as y-axis.

$$R_x = \Sigma F_x = -20\cos 60° = -10 \text{ kN} = \overleftarrow{10 \text{ kN}}$$

$$R_y = \Sigma F_y = -20 - 30 - 20\sin 60° = -67.32 \text{ kN} = 67.32 \text{ kN}\downarrow$$

$$\therefore \quad R = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2} = \sqrt{10^2 + 67.32^2} = 68.06 \text{ kN}$$

$$\alpha = \tan^{-1}\frac{\Sigma F_y}{\Sigma F_x} = \frac{67.32}{10.0} = 81.55°, \text{ as shown in Fig. 3.12(b).}$$

Taking moment about A,

$$\Sigma M_A = 20 \times 1.5 + 30 \times 3.0 + 20\sin 60° \times 6 = 223.92 \text{ kN-m.}$$

x-intercept of the resultant is given by

$$x = \frac{\Sigma M_A}{R_y} = \frac{223.92}{67.32} \cong 3.326 \text{ m, as shown in the figure.}$$

Note: While finding x, signs of ΣM_A and R_y need not be considered, but while showing its location in the figure, see that x is such that the resultant R produces moment of the same nature (clockwise in this case) as ΣM_A.

Example 3.4. *Find the resultant of the force system shown in Fig. 3.13(a) acting on a lamina of equilateral triangular shape.*

Solution.
$$R_x = \Sigma F_x = 60 - 100\cos 60° - 120\cos 30°$$
$$= -93.9 \text{ N} = \overleftarrow{93.9 \text{ N}}$$
$$R_y = \Sigma F_y = -80 + 100\sin 60° - 120\sin 30°$$
$$= -53.40 = 53.40 \text{ N}\downarrow$$

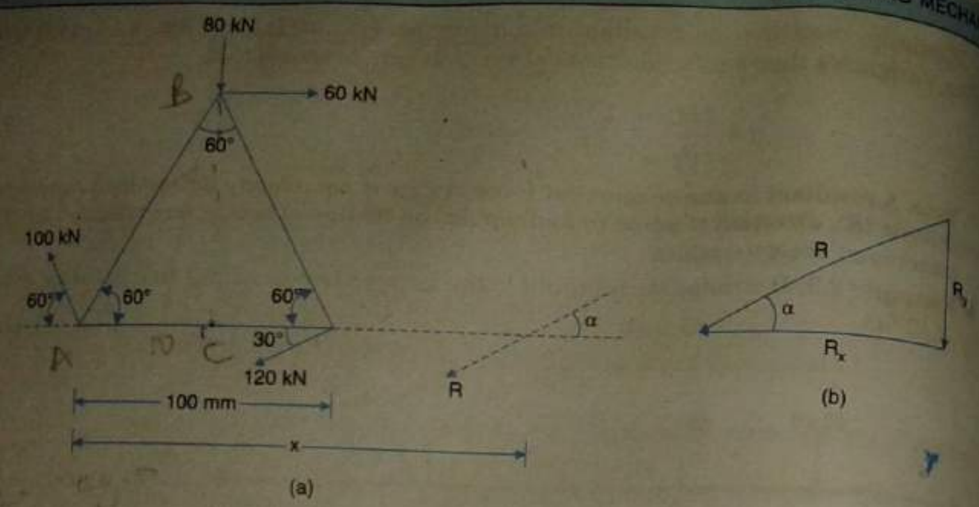


Fig. 3.13

$$\therefore \quad R = \sqrt{(93.9)^2 + (53.4)^2}$$

i.e.,

$$\mathbf{R = 108.0\ N} \quad \textbf{Ans.}$$

$$\alpha = \tan^{-1}\frac{R_y}{R_x} = \tan^{-1}\frac{53.40}{93.9} = 29.62°, \text{ as shown in Fig. 3.13}(b) \quad \textbf{Ans.}$$

Let x be the intercept on x-axis from A. Then taking moment about A

$$x = \frac{\Sigma M_A}{R_y} = \frac{80 \times 100 \cos 60° + 60 \times 100 \sin 60° + 120 \sin 30° \times 100}{53.40}$$

i.e.,

$$x = 284.6 \text{ mm, as shown in Fig. 3.13}(a). \quad \textbf{Ans.}$$

Example 3.5. *Find the resultant of the system of coplanar forces acting on a lamina as shown in Fig. 3.14(a). Each square has a side of 10 mm.*

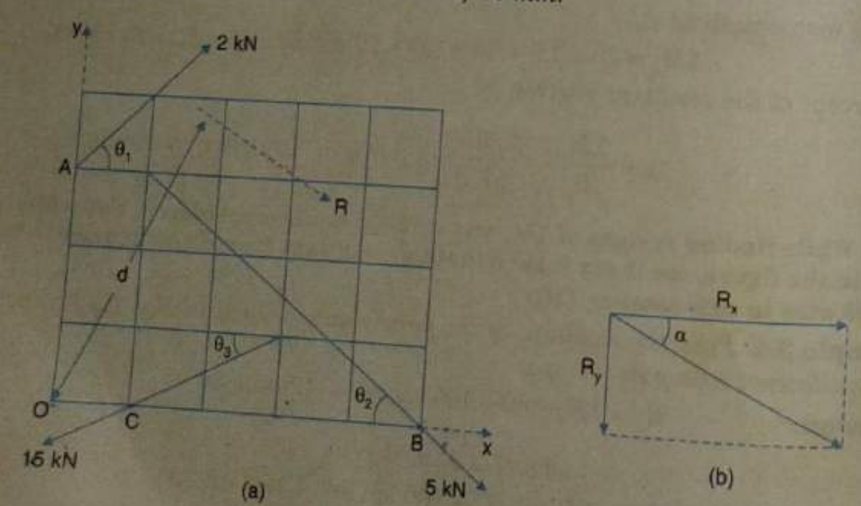


Fig. 3.14

Solution. If θ_1, θ_2 and θ_3 are the inclinations of forces 2 kN, 5 kN and 1.5 kN with respect to x-axis, then

$$\tan\theta_1 = \frac{10}{10} = 1 \quad \therefore \quad \theta_1 = 45°$$

$$\tan\theta_2 = \frac{30}{40} \quad \therefore \quad \theta_2 = 36.87°$$

and

$$\tan\theta_3 = \frac{10}{20} \quad \therefore \quad \theta_3 = 26.565°$$

$$\therefore \quad R_x = \Sigma F_x = 2\cos 45° + 5\cos 36.87° - 1.5\cos 26.565°$$
$$= 4.072 \text{ kN}$$

$$R_y = \Sigma F_y = 2\sin 45° - 5\sin 36.87° - 1.5\sin 26.565°$$
$$= -2.257 \text{ kN}$$

$$\therefore \quad R = \sqrt{(4.072)^2 + (-2.257)^2} = 4.655 \text{ kN} \quad \textbf{Ans.}$$

$$\alpha = \tan^{-1}\frac{2.257}{4.072} = 29°, \text{ as shown in Fig. 3.14}(b), \quad \textbf{Ans.}$$

Resolving the forces into their x, y components at A, B and C as shown in Fig. 3.14a and then finding moment of the forces about O, we get

$$\Sigma M_O = 2\cos 45° \times 30 + 5\sin 36.87° \times 50 + 1.5\sin 26.565° \times 10$$
$$= 199.13 \text{ kN-mm}$$

$\therefore$ Distance d of the resultant from O is given by,

$$d = \frac{199.13}{R} = \frac{199.13}{4.655} = 42.8 \text{ mm as shown in Fig. 3.14}(b) \quad \textbf{Ans.}$$

Example 3.6. *The system of forces acting on a bell crank is shown in Fig. 3.15(a). Determine the magnitude, direction and the point of application of the resultant.*

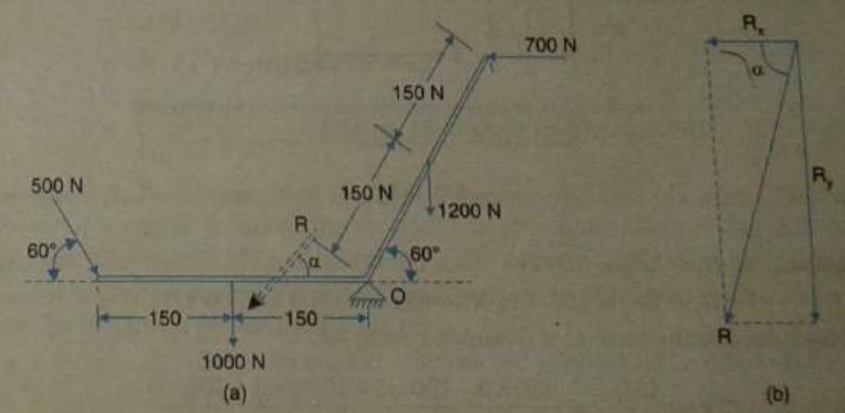


Fig. 3.15

Solution. $R_x = \Sigma F_x = 500 \cos 60° - 700 = -450 \text{ N} = \overleftarrow{450 \text{ N}}$

$R_y = \Sigma F_y = -500 \sin 60° - 1000 - 1200 = -2633 \text{ N} = 2633 \downarrow$

$\therefore \quad R = \sqrt{(450)^2 + (2633)^2} = 2671.2 \text{ N}$ **Ans.**

$$\alpha = \tan^{-1} \frac{2633}{450} = 80.30°, \text{ as shown in Fig. 3.15}(b) \quad \textbf{Ans.}$$

Let the point of application of the resultant be at a distance x from O along the horizontal arm. Then

$$x = \frac{\Sigma M_O}{R_y} = \frac{-500 \sin 60° \times 300 - 1000 \times 150 + 1200 \times 150 \cos 60° - 700 \times 300 \sin 60°}{2633}$$

$\therefore \quad x = 141.2 \text{ mm}$ **Ans.**

Example 3.7. *Various forces to be considered for the stability analysis of a dam are shown in Fig. 3.16. The dam is safe if the resultant force passes through middle third of the base. Verify whether the dam is safe.*

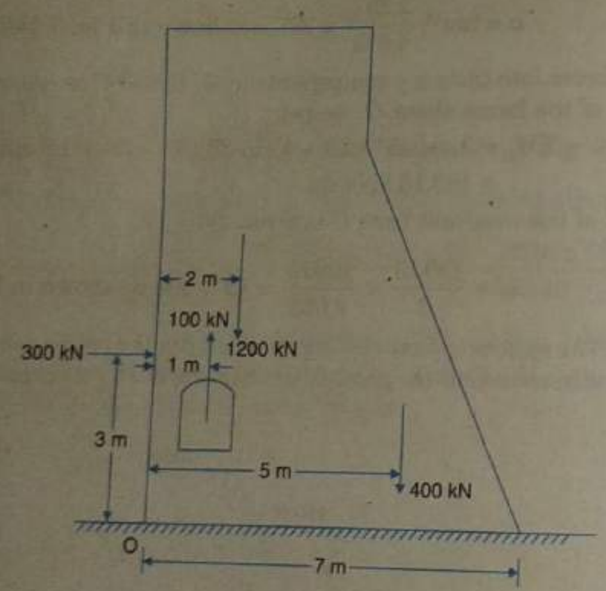


Fig. 3.16

Solution. $R_x = \Sigma F_x = \overrightarrow{300 \text{ kN}}$

$R_y = \Sigma F_y = 100 - 1200 - 400 = -1500 = 1500 \text{ kN} \downarrow$

Let resultant cut the base at a distance x from 'O'.

Then, $$x = \frac{\Sigma M_O}{R_y} = \frac{300 \times 3 - 100 \times 1 + 1200 \times 2 + 400 \times 5}{1500}$$

$= 3.467$ m.

The resultant lies in the middle third of the base *i.e.*, x is between 7/3 and $2 \times 7/3$. Hence the dam is safe. **Ans.**

Example 3.8. *A bracket is subjected to three forces and a couple as shown in Fig 3.17(a). Determine magnitude, direction and the line of action of the resultant.*

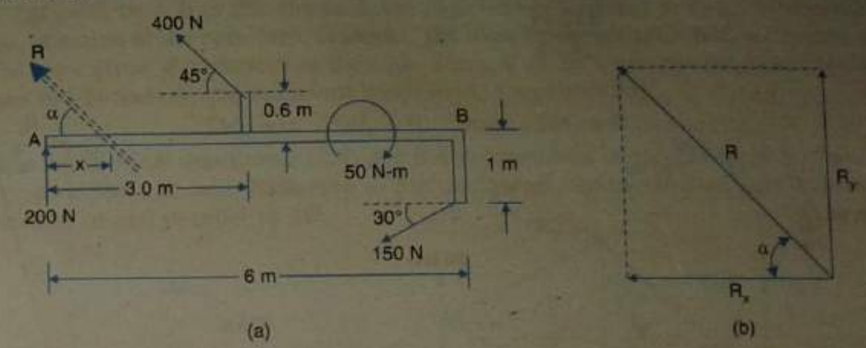


Fig. 3.17

Solution. $R_x = \Sigma F_x = -400 \cos 45° - 150 \cos 30°$

$= -412.7 \text{ N} = \overleftarrow{412.7}$

$R_y = \Sigma F_y = 200 + 400 \sin 45° - 150 \sin 30°$

$= 407.8 \text{ N}$

$\therefore \quad R = \sqrt{(412.7)^2 + (407.8)^2} = \mathbf{580.2 \text{ N}}$

$$\alpha = \tan^{-1}\left(\frac{R_y}{R_x}\right) = \tan^{-1} \frac{407.8}{412.7} = \mathbf{44.76°, \text{ as shown in Fig. 3.17(b)}}$$

Let the resultant intersect arm AB at a distance x from A.

$M_A = -400 \sin 45° \times 3 - 400 \cos 45° \times 0.6 + 50 + 150 \sin 30° \times 6 + 150 \times \cos 30° \times 1$

$= -438.3$ N-m

$= 438.3$ N-m (anticlockwise)

$$\therefore \quad x = \frac{M_A}{R_y} = \frac{438.3}{407.8} = \mathbf{1.074 \text{ m, as shown in Fig. 3.17(a).}}$$

Example 3.9. *Determine the magnitude, direction and line of action of the equilibriant of the given set of coplanar forces acting on a planar structure shown in Fig. 3.18(a).*

Solution. The two 40 kN forces acting on the smooth pulley may be replaced by a pair of 40 kN forces acting at centre of pulley C and parallel to the given forces, since the sum of moments of the two given forces about C is zero.

$\therefore \quad R_x = \Sigma F_x = 20 \cos 45° - 30 \cos 60° - 50 \cos 30° + 40 \cos 20° - 40 \sin 30°$

$= -26.57 \text{ kN}$

$= \overleftarrow{26.57} \text{ kN}$

$R_y = \Sigma F_y = -20 \sin 45° - 20 + 20 - 30 \sin 60° - 50 \sin 30°$

$- 40 \sin 20° - 40 \cos 30°$

$= -113.44$ kN

$= 113.44$ kN ↓

$$\therefore \quad R = \sqrt{(26.57)^2 + (113.44)^2} = 116.51 \quad \textbf{Ans.}$$

$$\alpha = \tan^{-1}\frac{113.44}{26.57} = 76.82° \quad \textbf{Ans.}$$

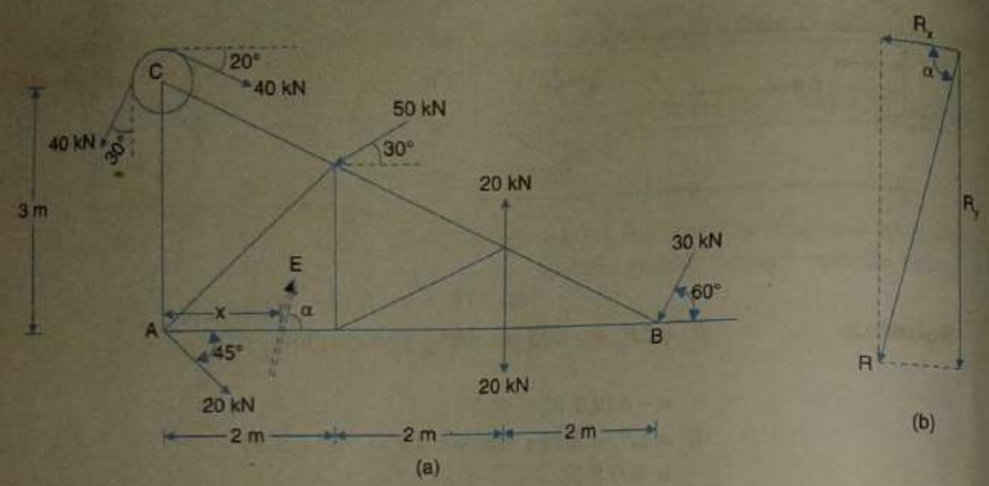


Fig. 3.18

Let the resultant intersect AB at a distance x from A. Then

$$R_y x = \Sigma M_A = 20 \times 4 - 20 \times 4 + 30 \sin 60° \times 6 + 50 \sin 30° \times 2$$
$$- 50 \cos 30° \times 2 + 40 \cos 20° \times 3 - 40 \sin 30° \times 3$$

$$\therefore \quad x = \frac{172.04}{113.44} = 1.516 \text{ m, as shown in Fig 3.18}(a) \quad \textbf{Ans.}$$

The equilibriant is equal and opposite to the resultant. Hence it is as shown in Fig. 3.18(*a*) in which

$$R = 116.51 \text{ kN}, \quad \alpha = 76.82° \quad \text{and} \quad x = 1.516 \text{ m.} \quad \textbf{Ans.}$$

3.7. EQUILIBRIUM OF NON-CONCURRENT SYSTEM OF FORCES

A body is said to be in equilibrium when it does not have any translatory or rotational movement. This means when the body is in equilibrium under the action of coplanar forces, the following conditions are satisfied:

(*i*) The algebraic sum of the components of forces along any two mutually perpendicular directions, is zero.

(*ii*) The algebraic sum of the moments of forces about any point in the plane, is zero.

Mathematically,

$$R_x = \Sigma F_x = 0; \quad R_y = \Sigma F_y = 0 \quad \text{and} \quad \Sigma M_A = 0 \qquad ...(3.8)$$

Referring to Fig. 3.19, A, B, C are three points which are not collinear. Let R be the resultant of the system of forces on the body. Then,

$\Sigma M_A = 0$ means moment of all the forces about point A is zero. In other words, resultant passes through point A.

If $\Sigma M_B = 0$, then B is also the point through which resultant passes. In other words AB is the line of action of the resultant. If point C is not collinear with AB, $\Sigma M_C = 0$ means, R itself should be zero, since R cannot pass through. Thus, if A, B, C are not collinear the following conditions are necessary and sufficient conditions of equilibrium

$$\Sigma M_A = 0, \quad \Sigma M_B = 0 \quad \text{and} \quad \Sigma M_C = 0 \qquad ...(3.9)$$

The equilibrium conditions (3.9) are not independent of conditions (3.8). Two of them are common to the two sets. Referring to Fig. 3.19(*b*), R can be resolved into its components perpendicular to and parallel to AC.

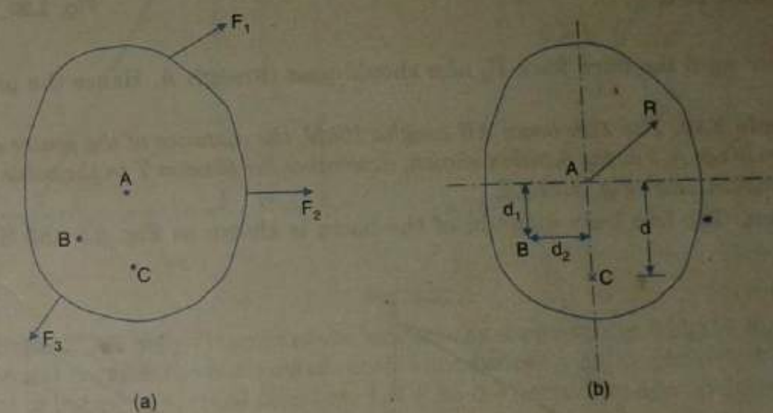


Fig. 3.19

Then, $\quad \Sigma M_C = 0$

$\Rightarrow \quad R_x d = 0$

i.e. $\quad R_x = 0$

Now $\quad \Sigma M_B = 0$

$\Rightarrow \quad R_x d_1 + R_y d_2 = 0$

i.e., $\quad R_y d_2 = 0, \quad \text{since} \quad d_2 \neq 0.$

$\therefore \quad R_y = 0$

Thus $\Sigma M_C = 0$ is equivalent to $R_x = 0$ and

$\Sigma M = 0$ is equivalent to $R_y = 0$. Thus

eqn. (10.9) is same as eqn. (10.8).

In fact any of the following set of equilibrium equations can be used:

1. $R_x = \Sigma F_x = 0$, $R_y = \Sigma F_y = 0$ and $M_A = 0$
2. If line AB is not in y-direction,
$$R_y = \Sigma F_y = 0, \ \Sigma M_A = 0 \text{ and } \Sigma M_B = 0 \qquad ...(3.10)$$
3. If line AB is not in x-direction,
$$R_x = \Sigma F_x = 0, \ \Sigma M_A = 0 \text{ and } \Sigma M_B = 0$$

4. If A, B and C are non-collinear

$$\Sigma M_A = 0, \Sigma M_B = 0 \text{ and } \Sigma M_C = 0$$

Now we are in a position to prove that *if a system is in equilibrium under the action of only three forces, they must be concurrent.*

Let F_1, F_2 and F_3 be the three forces acting on a body and the system is in equilibrium. Then, if A is the point of intersection of forces F_1 and F_2, the equilibrium condition gives

$$\Sigma M_A = 0$$

i.e., $$F_3 d = 0$$

where d is the distance of line of action of F_3 from A. Since F_3 is not zero,

$$d = 0$$

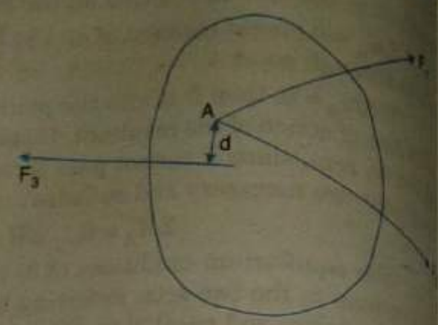


Fig. 3.20

In other word the third force F_3 also should pass through A. Hence the proposition is proved.

Example 3.10. *The 12m boom AB weighs 10kN, the distance of the centre of gravity G being 6m from hinge A. For the position shown, determine the tension T in the cable BC and the reaction at hinge A. [Ref. Fig. 3.21(a)].*

Solution. The free body diagram of the boom is shown in Fig. 3.21(*b*). Since A is a hinged end,

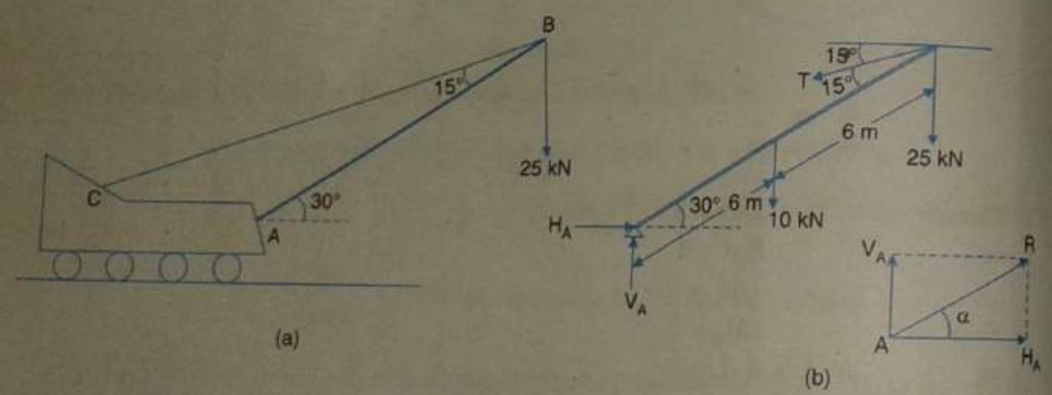


Fig. 3.21

$$\Sigma M_A = 0 \rightarrow$$

$$-T \times 12 \sin 15° + 25 \times 12 \cos 30° + 10 \times 6 \cos 30° = 0$$

$\therefore$ $$T = 100.38 \text{ kN} \quad \textbf{Ans.}$$

$$\Sigma F_H = 0 \rightarrow$$

$$H_A - T \cos 15° = 0$$

$\therefore$ $$H_A = 100.38 \cos 15° = 96.992 \text{ kN.}$$

$$\Sigma F_V = 0 \rightarrow$$

$$V_A - 10 - 25 - T \sin 15° = 0$$

$$V_A = 60.980 \text{ kN}$$

$$R_A = \sqrt{96.992^2 + 60.980^2} = 114.569 \text{ kN}$$

$$\alpha = \tan^{-1} \frac{60.980}{96.992} = 32.16° \text{ as shown in Fig. 3.21 (c).}$$

Example 3.11. *A ladder weighing 100 N is to be kept in the position shown in Fig. 3.22 (a), resting on a smooth floor and leaning on a smooth wall. Determine the horizontal force required at floor level to prevent it from slipping when a man weighing 700 N is at 2m above floor level.*

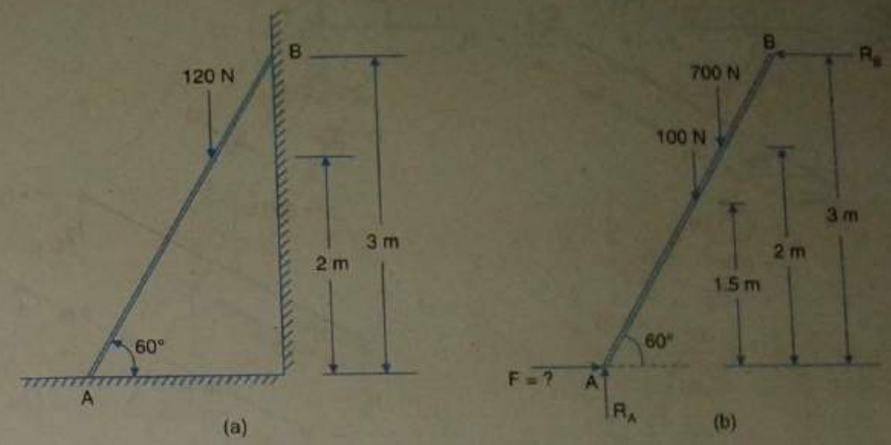


Fig. 3.22

Solution. Free body diagram of the ladder is as shown in Fig. 3.22 (*b*). R_A is vertical and R_B is horizontal because the surfaces of contact are smooth. Self weight of 100 N acts through centre point of ladder in vertical direction. Let F be the horizontal force required to be applied to prevent slipping.

Then, $$\Sigma M_A = 0 \rightarrow -R_B \times 3 + 700 \times 2 \cot 60° + 100 \times 1.5 \cot 60° = 0$$

$\therefore$ $$R_B = 298.3 \text{ N}$$

$$\Sigma F_H = 0 \rightarrow$$

$$F - R_B = 0$$

$\therefore$ $$F = R_B = 298.3 \text{ N} \quad \textbf{Ans.}$$

Example 3.12. In the above ladder problem, if the horizontal force F is to be applied at a height of 1 m above the ground level, how much should F be?

Solution. Figure 3.23 shows the free body diagram of the ladder for this case.

$$\Sigma F_H = 0 \rightarrow F - R_B = 0 \text{ or } F = R_B \quad ...(i)$$

$$\Sigma M_A = 0 \rightarrow -R_B \times 3 + 700 \times 2 \cot 60° + 100 \times 1.5 \cot 60° + F \times 1 = 0 \quad ...(ii)$$

From (*i*) and (*ii*), we get

$$3F - F = (700 \times 2 + 100 \times 1.5) \cot 60°, \text{ since } R_B = F.$$

$\therefore$ $$F = 459 \text{ N} \quad \textbf{Ans.}$$

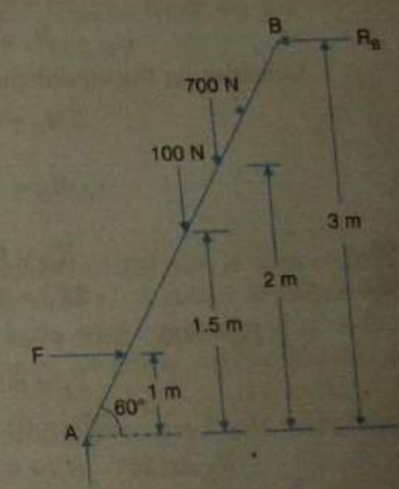


Fig. 3.23

Example 3.13. *A roller weighing 2000 N rests on an inclined bar weighing 800 N as shown in Fig. 3.24(a). Assuming weight of bar AB is negligible, determine the reactions developed at supports C and D.*

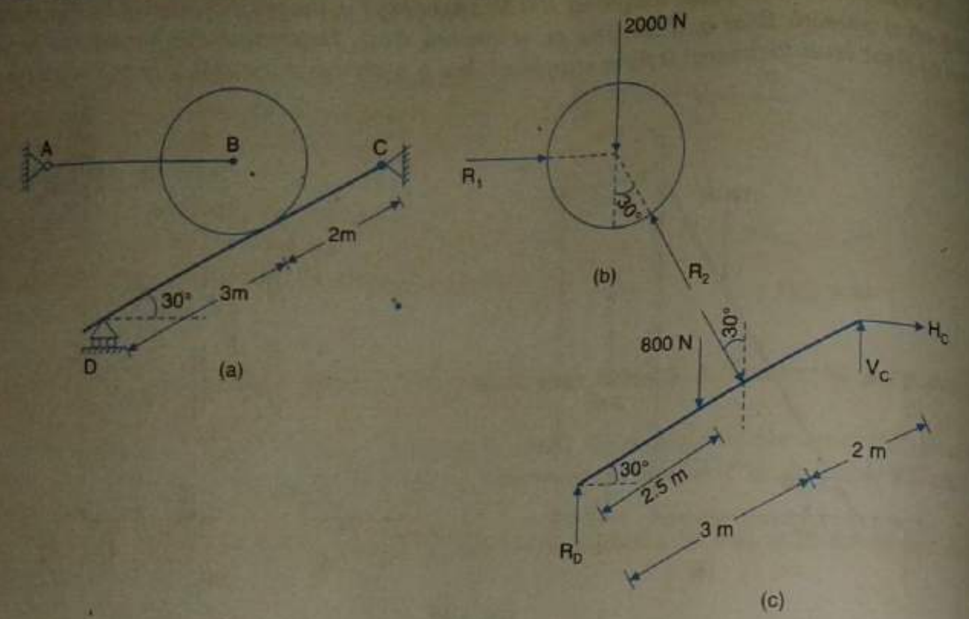


Fig. 3.24

Solution. Free body diagrams of roller and bar *CD* are as shown in Figs. 3.24(*b*) and (*c*).

[**Note:** R_D is vertical since at *D* roller support is horizontal. At *C* reaction can be in any direction since it is hinged support. Hence components of reaction at *C* are taken as H_C and V_C].

For roller,

$$\Sigma F_V = 0 \rightarrow R_2 \cos 30° = 2000$$

$$\therefore \quad R_2 = 2309.4 \text{ N}$$

Considering the equilibrium of the bar *CD*,

$$\Sigma M_C = 0 \rightarrow R_D\, 5 \cos 30° - 800 \times 2.5 \cos 30° - R_2 \times 2 = 0$$

$$R_D = \frac{800 \times 2.5 \cos 30° + 2309.4 \times 2}{5 \cos 30°}$$

i.e., $R_D = 1466.7$ N **Ans.**

$$\Sigma F_V = 0 \rightarrow$$

$$R_D - 800 - 2309.4 \cos 30° + V_C = 0$$

$$\therefore \quad V_C = 800 + 2309.4 \cos 30° - 1466.7 = 1333.3 \text{ N} \quad \textbf{Ans.}$$

$$\Sigma F_H = 0 \rightarrow$$

$$R_2 \sin 30° - H_C = 0.$$

$$\therefore \quad H_C = 2309.4 \sin 30° = 1154.7 \text{ N} \quad \textbf{Ans.}$$

Example 3.14. *A cable car used for carrying materials in a hydroelectric project is at rest on a track formed at an angle of 60° to horizontal. The gross weight of the car and its load is 60 kN and its centroid is at a point 800 mm from the track half way between the axles. The car is held by a cable as shown in Fig. 3.25. The axles of the car are at a distance 1.2 m. Find the tension in the cables and reaction at each of the axles neglecting friction of the track.*

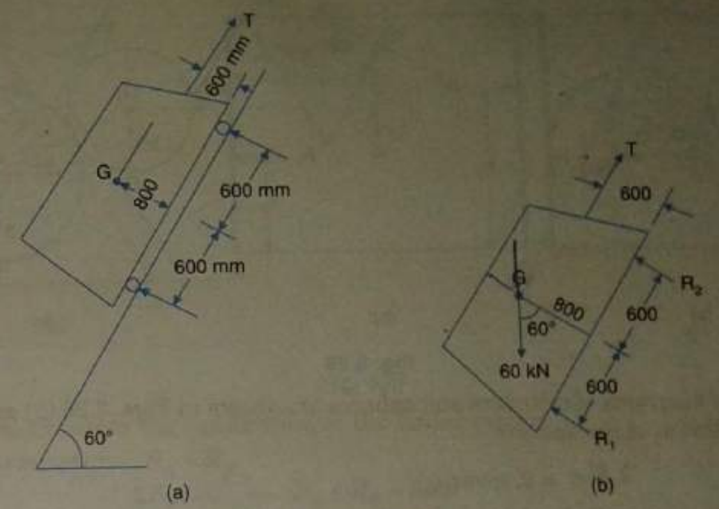


Fig. 3.25

Solution. The free body diagram of the car is shown in Fig. 3.25(*b*). The self weight is vertical and hence makes 60° to the normal to the inclined plane. Let *T* be the tension in the rope which is parallel to track.

Now, ΣForces parallel to track = 0, gives,

$$T - 60 \sin 60° = 0 \quad \text{or} \quad T = 51.961 \text{ kN} \quad \textbf{Ans.}$$

Applying moment equilibrium condition about upper axle point on track, we get

$$R_1 \times 1200 + T \times 600 - 60 \sin 60° \times 800 - 60 \cos 60° \times 600 = 0$$

$$\therefore \quad R_1 = 23.660 \text{ kN.} \quad \textbf{Ans.}$$

ΣForces normal to the plane = 0, gives,

$$R_1 + R_2 - 60 \cos 60° = 0$$

$$\therefore \quad R_2 = 60 \cos 60° - R_1 = 60 \cos 60° - 23.660$$

i.e., $R_2 = 6.34$ kN **Ans.**

Example 3.15. *A hollow right circular cylinder of diameter 1600 mm is open at both ends and rests on a smooth horizontal plane as shown in Fig. 3.26(a). A sphere of radius 600 mm weighing 3 kN is first put in the cylinder and then a sphere weighing 1 kN and of radius 400 mm was placed. Neglecting the friction, find the minimum weight of hollow cylinder so that it will not tip over.*

Solution. Join the centres of spheres O_1 and O_2 and drop O_1D perpendicular to horizontal through O_2.

Now, $O_1O_2 = 400 + 600 = 1000$ mm

$O_2D = 1600 - 400 - 600 = 600$ mm

If α is the inclination of O_2O_1 to horizontal,

$$\alpha = \cos^{-1}\frac{600}{1000} = 53.13°$$

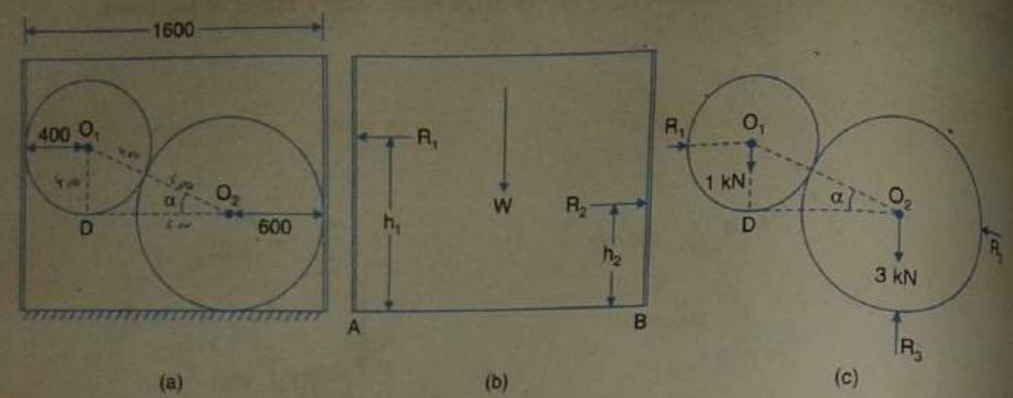


Fig. 3.26

Free body diagrams of cylinders and spheres are shown in Figs. 3.26 (*b*) and (*c*). Considering the equilibrium of the spheres,

$$\Sigma M_{O_2} = 0, \text{ gives}$$

$$R_1 O_1O_2 \sin\alpha - 1 \times O_2D = 0$$

i.e., $$R_1 \times 1000 \sin 53.13° = 1 \times 600$$

$$\therefore \quad R_1 = 0.75 \text{ kN.}$$

$$\Sigma F_H = 0, \text{ gives}$$

$$R_2 = R_1 = 0.75 \text{ kN.}$$

$$\Sigma F_V = 0, \text{ gives}$$

$$R_3 = 1 + 3 = 4 \text{ kN.}$$

Now, consider the equilibrium of the cylinder. Let the minimum weight be *W*. Tipping will take place over point *A*. Hence at this stage there will not be any reaction at point *B*. Hence,

$$\Sigma M_A = 0, \text{ gives,}$$

$$R_1h_1 - R_2h_2 - W \times 800 = 0$$

$$R_1(h_1 - h_2) = W \times 800, \quad \text{since } R_1 = R_2.$$

i.e., $$R_1 \times O_1D = W \times 800$$

$$\therefore \quad W = \frac{0.75 \times 1000 \sin 53.13°}{800} = 0.75 \text{ kN} \quad \textbf{Ans.}$$

Example 3.16. *A 500 N cylinder of 1 m diameter is loaded between the cross pieces which makes an angle of 60° with each other and are pinned at C. Determine the tension in the horizontal rope DE, assuming a smooth floor [Ref. Fig. 3.27(a)].*

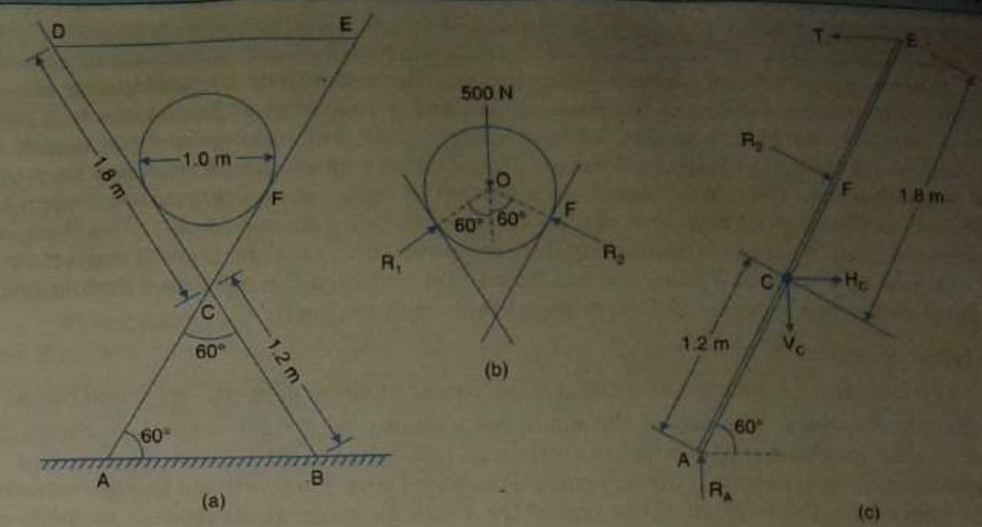


Fig. 3.27

Solution. Consider the equilibrium of the entire system.

Due to symmetry, $R_A = R_B$

$$\Sigma F_V = 0 \rightarrow R_A + R_B = 500$$

$$\therefore \quad R_A = R_B = 250 \text{ N}$$

Now, consider the equilibrium of the cylinder [Ref. Fig. 3.27(*b*)]. The members *AE* and *BD* make 60° with the horizontal. Hence, the reaction R_2 and R_1 make 60° with the vertical.

$$\Sigma F_H = 0, \text{ gives}$$

$$R_1 \sin 60° - R_2 \sin 60° = 0$$

$$\therefore \quad R_1 = R_2$$

$$\Sigma F_V = 0, \text{ gives}$$

$$R_1 \cos 60° + R_2 \cos 60° - 500 = 0.$$

Since, $R_1 = R_2$, we get

$$2R_1 \cos 60° = 500 \quad i.e. \quad R_1 = 500 \text{ N}$$

Hence $$R_2 = 500 \text{ N}$$

Now, consider the equilibrium of the member *AE*. It's free body diagram is shown in Fig. 3.27 (*c*).

$$\Sigma M_C = 0 \rightarrow$$

$$T \sin 60° \times 1.8 - R_2 \times CF - R_A \times 1.2 \cos 60° = 0$$

Now $$CF = OF \tan 60° = 0.5 \tan 60°$$

$$= 0.866 \text{ m.}$$

$$\therefore \quad T = \frac{500 \times 0.866 + 250 \times 1.2 \times 0.5}{1.8 \sin 60°} = 374.0 \text{ N} \quad \textbf{Ans.}$$

Note: If the reactions on the pin at *C* are required, the remaining two equations, namely, $\Sigma F_x = 0$ and ΣF_y, are to be considered.

3.8. APPLICATIONS TO BEAM PROBLEMS

A beam is a structural element which has one dimension (length) considerably l than the other two dimensions in the cross-section and is supported at few points. It is jected to lateral loads. Due to applied loads, reactions develop at supports and the syste forces consisting of applied loads, self weight (many times neglected) and reactions keep beam in equilibrium. The forces constitute a system of coplanar non-concurrent syste equilibrium. If the support reactions can be determined using the equations of equilibr only, then the beam is said to be **statically determinate.** In this article, types of support types of beams and types of loading are explained first and then the method of finding re tions developed at the supports of determinate beams is illustrated.

3.8.1. Types of Supports

Various types of supports and reactions developed at these supports are listed belo

Simple Support. The end of the beam rests simply on a rigid support as shown Fig. 3.28. In the idealised simple support there is no resistance to the force in the direction the support. Hence the reaction is always normal to the support. There will not be any restra from the support for the rotation of the end of the beam. In other words there is no mom resistance at support. [Ref. Fig. 3.28].

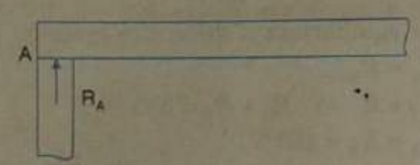


Fig. 3.28. Simple support

Roller Support. In this case, beam end is supported on rollers. In such cases, reactio is always normal to the support, since rollers are free to roll along the supports. In idealise condition rolling friction is neglected and hence there is no resistance in the line of suppo The ends are free to rotate also. Hence there is no resistance to moment. Many mechanic components are provided with roller support which roll between guides. In such cases rea tions can be normal to guides in either directions (Ref. Fig. 3.29).

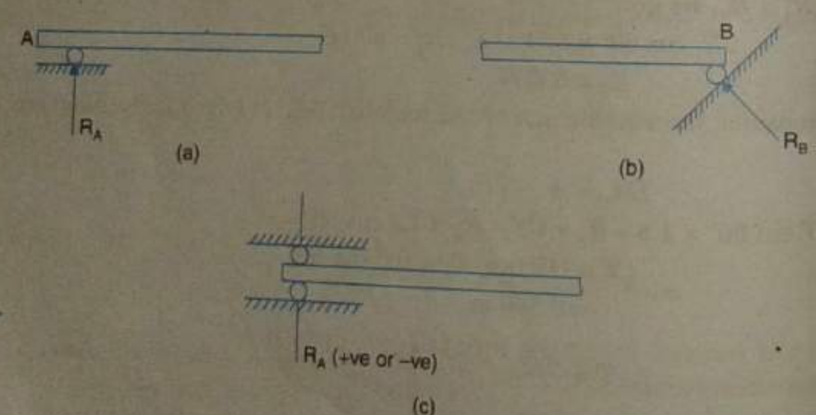


Fig. 3.29. Roller supports

Hinged or Pinned Support. In such cases, the position of the end of the beam is fixed but the end is free to rotate. This idealised conditions can be achieved by using mechanical devices. A typical hinged end condition is shown in Fig. 3.30. At such supports the reaction can be in any direction which is usually represented by its components in two mutually perpendicular directions. This type of support do not provide any resistance to moment, in other words it permits rotation freely at the end.

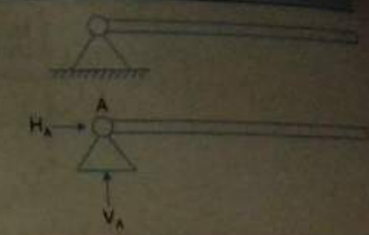


Fig. 3.30. Hinged or Pinned support

Fixed Support. At fixed support the end of the beam is neither permitted to move in any direction nor allowed to rotate. Hence support reactions are a force in any direction and the resisting moment (Ref. Fig. 3.31). Reacting force in any direction is conveniently represented by its components in two mutually perpendicular directions. This end condition may be achieved by taking the end of the beam considerably inside the support or by specially designing the brackets to resist the movement and rotation.

→ Horizontal
→ vertical
→ Moment at the fixed end

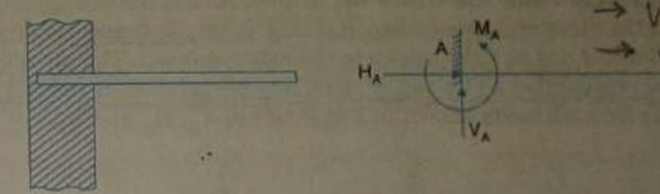


Fig. 3.31. Fixed supports

3.8.2. Types of Beams

Depending upon the types of supports, beams may be classified as following:

(*i*) Cantilever
(*ii*) Simply supported
(*iii*) One end hinged and other on roller
(*iv*) Overhanging
(*v*) Both ends hinged
(*vi*) Propped cantilever, and
(*vii*) Continuous.

The position where the resistance of rotation there exists a Moment

(*i*) **Cantilever.** If a beam is fixed at one end and free at other end, it is called a cantilever beam. In this there are three reaction components at fixed end [V_A, H_A, M_A as shown in Fig. 3.31] and no reaction component at free end.

(*ii*) **Simply Supported Beam.** In this type of beam both ends are simply supported as shown in Fig. 3.32. There is one reaction component at each end (R_A and R_B). They act at right angles to the support. This type of beam can resist forces normal to the beam axis. In other words, the equilibrium condition that summation of forces parallel to axis equal to zero is to be satisfied automatically by the loading condition. Hence two equations of equilibrium are available.

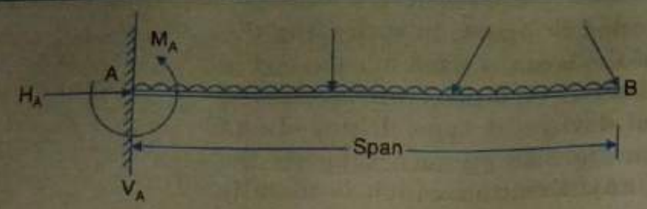


Fig. 3.32. Cantilever beam

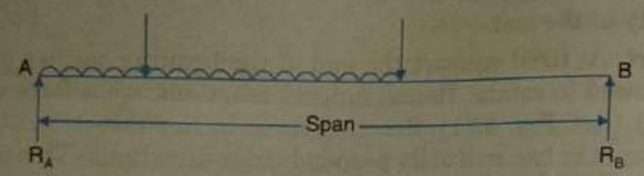


Fig. 3.33. Simple supported beam

(*iii*) **One End Hinged and the Other on Roller.** As the name suggests one end of beam is hinged and the other end is on roller [Ref. Fig. 3.34]. At hinge, reaction can be in direction and at roller end it is at right angles to the roller support. The hinged end reaction any direction can be represented by its two components perpendicular to each other. Thus reaction components for such beam shown in Fig. 3.34 are V_A, H_A and R_B.

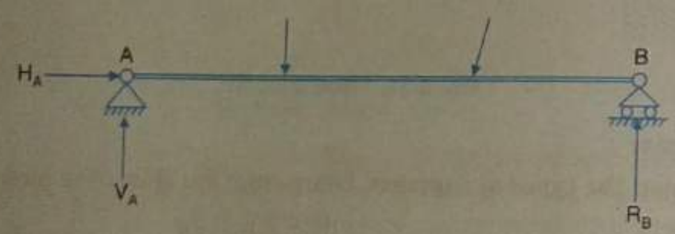


Fig. 3.34. One end hinged other on roller

(*iv*) **Overhanging Beam.** If a beam is projecting beyond the support/supports, it is call an overhanging beam [Fig. 3.35]. The overhang may be on only one side or may be on bo sides.

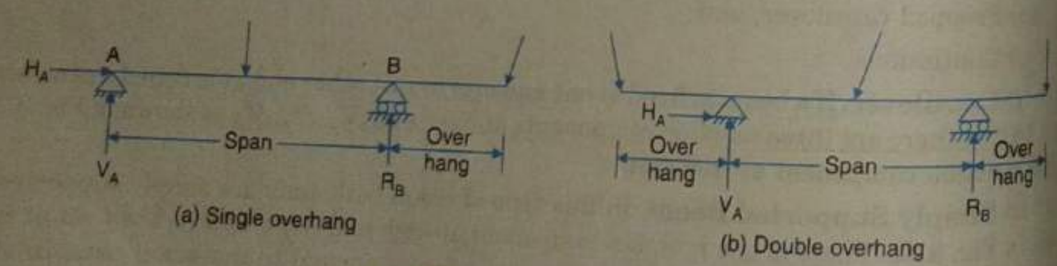


Fig. 3.35. Overhanging beam

(*v*) **Both Ends Hinged.** As the name suggests both ends of the beam are hinged. The re two reaction components at each end, hence total reaction components are four.

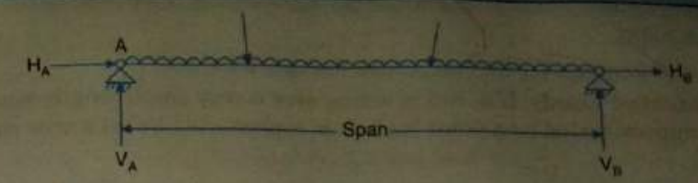


Fig. 3.36. Both ends hinged beam

(*vi*) **Propped Cantilever.** In this type of beam one end of the beam is fixed and the other end is simply supported or is on rollers. It has got four reaction components (V_A, H_A, M_A and R_B) as shown in Fig. 3.37.

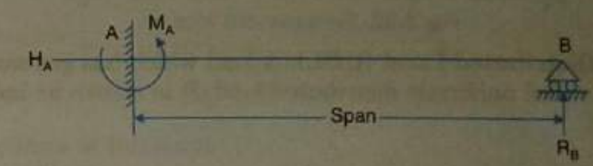


Fig. 3.37. Propped cantilever beam

(*vii*) **Continuous Beam.** A beam having three or more supports is called continuous beam. In such beams three or more reaction components exist. [Ref. Fig. 3.38].

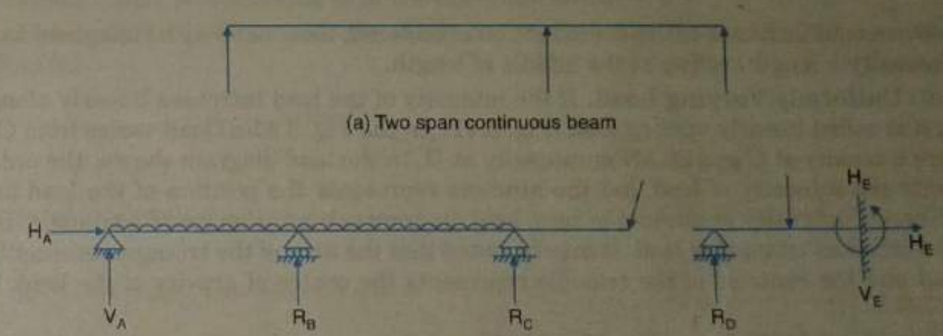


Fig. 3.38. Continuous beams

In cantilever beams, simply supported beams, one end hinged and the other on roller and overhanging beams, number of unknown reactions are equal to the number of independent equations of equilibrium. Hence for any given loading, reactions can be determined by using equations of equilibrium only. Hence such beams are known as statically determinate beams. Thus we can define **statically determinate beams as these beams in which all reaction components can be found using the equations of equilibrium only.**

In case of beams with both ends hinged, propped cantilever and continuous beams, number of unknown reaction components are more than the number of available equations of equilibrium. This type of beams are known as statically indeterminate beams.

3.8.3. Types of Loading

Usual types of loadings on the beams are discussed below:

(*i*) **Concentrated Loads.** If a load is acting over a very small length compared to the of the beam, it is approximated as a point load. It is represented by an arrow as shown in Fig. 3.39.

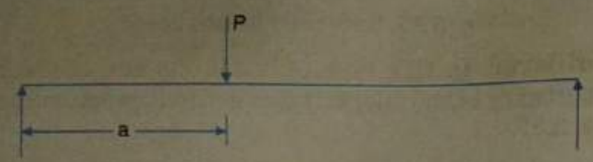


Fig. 3.39. Concentrated load

(*ii*) **Uniformly Distributed Load (UDL).** A load which has got same intensity over considerable length is called uniformly distributed load. It is shown as indicated in Fig. 3.40(*a*) or as in Fig. 3.40(*b*).

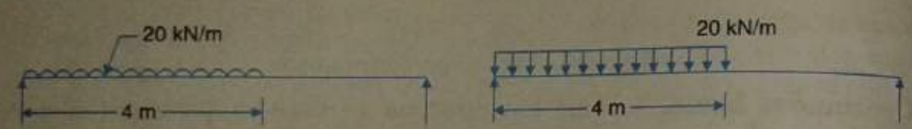


Fig. 3.40. Uniformly distributed load (UDL)

When equilibrium of entire beam is to be considered, this load may be imagined as the load (intensity × length) acting at the middle of length.

(*iii*) **Uniformly Varying Load.** If the intensity of the load increases linearly along the length, it is called linearly varying load (Fig. 3.41). In the Fig. 3.41(*a*) load varies from *C* to *D* with zero intensity at *C* and 20 kN/m intensity at *D*. In the load diagram shown, the ordinate represents the intensity of load and the abscissa represents the position of the load on the beam. Hence if intensity at one end is zero, load diagram takes the shape of a triangle. Hence it may be called as triangular load. It may be noted that the area of the triangle represents the total load and the centroid of the triangle represents the centre of gravity of the load. The total load in Fig. 3.41 (*a*) is $\frac{1}{2} \times 3 \times 20 = 30$ kN and the centre of gravity of this loading is at $\frac{1}{3} \times 3 = 1$ m from *D*. For finding the reaction we can imagine that the given load is equivalent to 30 kN acting at $1 + 3 - 1 = 3$ m from end *A*.

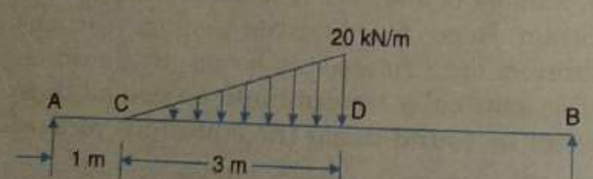


Fig. 3.41. Uniformly varying load

(*iv*) **General Loading.** Figure 3.42 shows the case of general loading. Here also ordinate represents the intensity of loading and abscissa represents the position of the loading. For simplicity, in the analysis such loadings are replaced by a set of equivalent concentrated loads.

Fig. 3.42. General loading

(*v*) **External Moment.** A beam may be subjected to external moments at certain points. In Fig. 3.43 the beam is subjected to a clockwise moment of 30 kN-m at a distance 2 m from the left support.

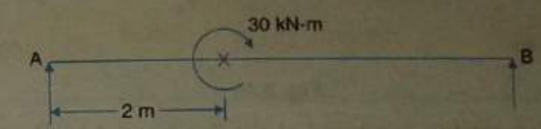


Fig. 3.43. External moment

3.8.4. Finding Reactions at Supports

To resist the applied loads reactions develop at supports of the beam. Reactions are self adjusting forces which will keep the beams in equilibrium. Hence the equations of equilibrium may be written for the system of forces consisting of reactions and the loads. Solutions of these equations give the unknown reactions. The procedure for finding the reactions in statically determinate beams is illustrated with the examples below:

Example 3.17. *Determine the reactions developed in the cantilever beam shown in Fig. 3.44(a).*

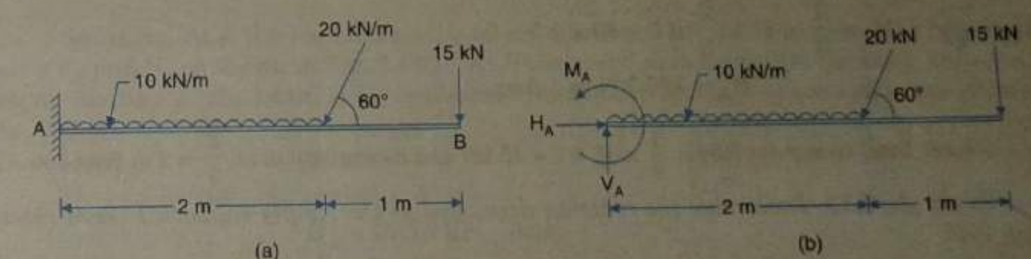


Fig. 3.44

Solution. Let the reactions developed at fixed support *A* be V_A, H_A and M_A as shown in Fig. 3.44 (*b*).

ΣForces in vertical direction = 0 →

$$V_A - 10 \times 2 - 20 \sin 60° - 15 = 0.$$

$$\therefore \quad V_A = \mathbf{52.32\ kN} \quad \textbf{Ans.}$$

ΣForces in horizontal direction = 0 →

$$H_A - 20 \cos 60° = 0 \quad \therefore \quad H_A = \mathbf{10\ kN} \quad \textbf{Ans.}$$

ΣMoments about *A* = 0 →

$$-M_A + 10 \times 2 \times 1 + 20 \sin 60° \times 2 + 15 \times 3 = 0.$$

$$\therefore \quad M_A = \mathbf{99.64\ kN\text{-}m} \quad \textbf{Ans.}$$

[**Note:** *VDL* is treated as a load 10 × 2 kN acting at its centre of gravity which is at 1m from A].

Example 3.18. *Determine the reactions developed in the cantilever beam shown in Fig. 3.45(a).*

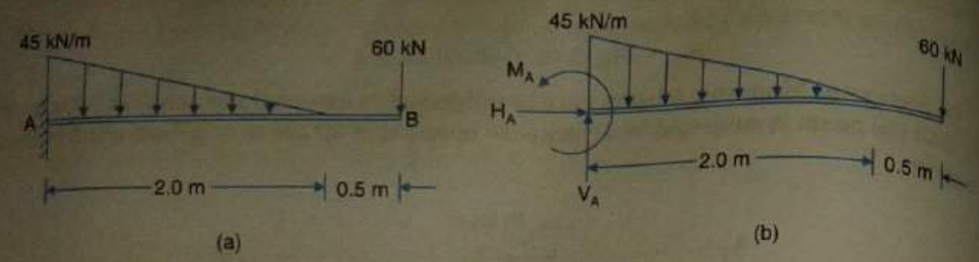


Fig. 3.45

Solution. Let the reactions developed at the fixed end A be V_A, H_A and M_A as shown in Fig. 3.45(*b*).

$$\Sigma V_A = 0 \quad \rightarrow$$

$$V_A - \frac{1}{2} \times 45 \times 2 - 60 = 0$$

$$\mathbf{V_A = 105\ kN \quad Ans.}$$

$$\Sigma H = 0 \quad \rightarrow$$

$$H_A - 0 = 0 \quad \therefore \quad \mathbf{H_A = 0 \quad Ans.}$$

$$\Sigma M_A = 0 \quad \circlearrowright$$

$$-M_A + \frac{1}{2} \times 45 \times \frac{2}{3} \times 2 + 60 \times 2.5 = 0$$

$$\therefore \quad \mathbf{M_A = 180\ kN\text{-}m \quad Ans.}$$

[**Note:** Total triangular load $= \frac{1}{2} \times 45 \times 2 = 45$ kN and its centroid is at $\frac{2}{3} \times 2$ m from end A]

Example 3.19. *Determine the reaction developed in the simply supported beam shown in Fig. 3.46.*

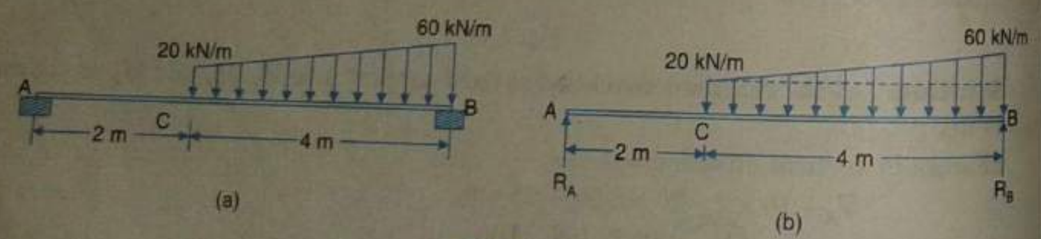


Fig. 3.46

Solution. Let R_A and R_B be the reactions developed at the simply supported ends A and B. The uniformly varying load may be split into a uniformly distributed load of 20kN/m intensity and a triangular load of intensity zero at point C and intensity $60 - 20 = 40$ kN/m at B. Then

$$\Sigma M_B = 0 \quad \rightarrow$$

$$R_A \times 6 - 20 \times 4 \times 2 - \frac{1}{2} \times 4 \times 40 \times \frac{4}{3} = 0$$

$$\therefore \quad \mathbf{R_A = 44.44\ kN \quad Ans.}$$

$$\Sigma V = 0 \quad \rightarrow$$

$$R_A + R_B - 20 \times 4 - \frac{1}{2} \times 4 \times 40 = 0$$

$$\therefore \quad R_B = 80 + 80 - 44.44, \text{ since } R_A = 44.44 \text{ kN}$$

$$\therefore \quad \mathbf{R_B = 115.56\ kN \quad Ans.}$$

Example 3:20. *The beam AB of span 12 m shown in Fig. 3.47 is hinged at A and is on rollers at B. Determine the reactions developed at A and B due to the loading shown in the figure.*

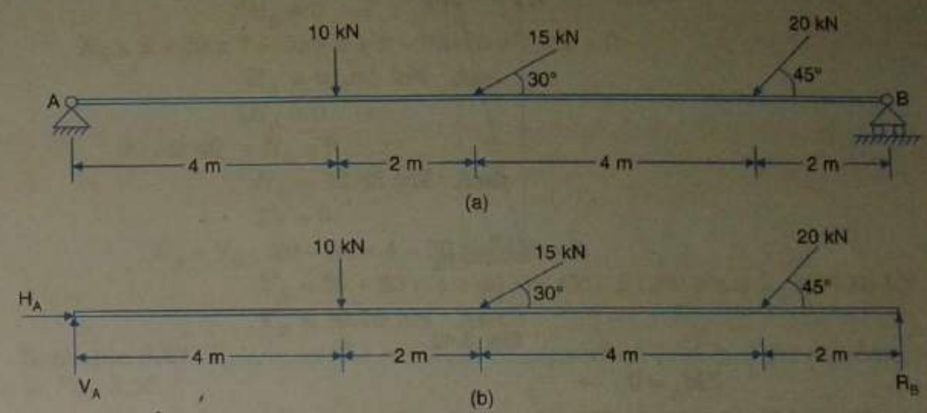


Fig. 3.47

Solution. At A the reaction can be in any direction. Let it be represented by its components V_A and H_A as shown in Fig. 3.47(*b*). At B the reaction is in vertical direction only. Let its magnitude be R_B. The beam is in equilibrium under the action of system of forces shown in Fig. 3.47(*b*).

Now, $\quad \Sigma H = 0 \quad \rightarrow$

$$H_A - 15 \cos 30° - 20 \cos 45° = 0$$

$$\therefore \quad \mathbf{H_A = 27.13\ kN \quad Ans.}$$

$$\Sigma M_A = 0 \quad \rightarrow$$

$$-R_B \times 12 + 10 \times 4 + 15 \sin 30° \times 6 + 20 \sin 45° \times 10 = 0$$

$$\therefore \quad \mathbf{R_B = 18.87\ kN \quad Ans.}$$

$$\Sigma V = 0 \quad \rightarrow$$

$$V_A + R_B - 10 - 15 \sin 30° - 20 \sin 45° = 0$$

$$V_A + 18.87 - 10 - 15 \sin 30° - 20 \sin 45° = 0$$

$$\mathbf{V_A = 12.77\ kN \quad Ans.}$$

Example 3.21. *Find the magnitude and direction of reactions at supports A and B in the beam AB shown in Fig. 3.48.*

Solution. The reaction R_B will be at right angles to the inclined support *i.e.* at $90° - 30° = 60°$ to horizontal as shown in the Fig. 3.48 (*b*). Let the components of reactions at A be V_A and H_A. Then

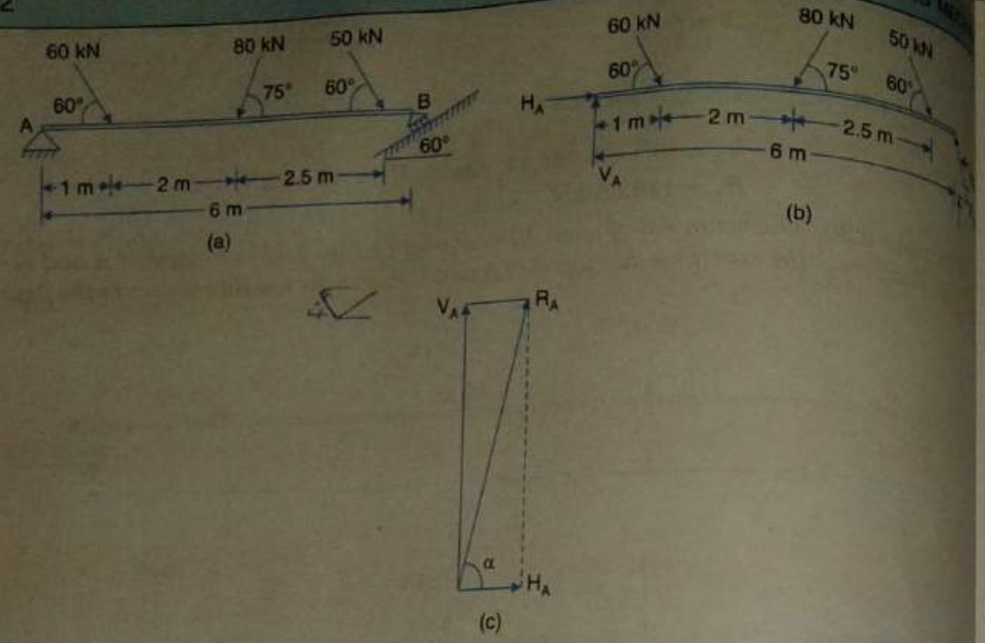


Fig. 3.48

$$\Sigma M_A = 0 \quad \rightarrow$$

$$60 \sin 60° \times 1 + 80 \sin 75° \times 3 + 50 \sin 60° \times 5.5 - R_B \sin 60° \times 6 = 0$$

$\therefore$ $R_B = 100.45$ kN, at 60° to horizontal as shown in figure. Ans.

$$\Sigma H = 0 \quad \rightarrow$$

$$H_A + 60 \cos 60° - 80 \cos 75° + 50 \cos 60° - R_B \cos 60° = 0$$

$$H_A = -60 \cos 60° + 80 \cos 75° - 50 \cos 60° + 100.45 \cos 60°$$

$\therefore$ $H_A = 15.93$ kN

$$\Sigma V = 0 \quad \rightarrow$$

$$V_A + R_B \sin 60° - 60 \sin 60° - 80 \sin 75° - 50 \sin 60° = 0$$

$\therefore$ $V_A = -100.45 \sin 60° + 60 \sin 60° + 80 \sin 75° + 50 \sin 60°$

i.e., $V_A = 85.54$ kN

$\therefore$ $R_A = \sqrt{H_A^2 + V_A^2} = \sqrt{15.93^2 + 85.54^2}$

i.e., $\mathbf{R_A = 87.02\ kN}$

$\alpha_A = \tan^{-1} \dfrac{85.54}{15.93} = 79.45°$, **as shown in Fig. 3.48 c.** Ans.

Example 3.22. *Find the reactions developed at supports A and B of the loaded* [illegible] *shown in Fig. 3.49.*

Solution. Reaction at A is vertical, since roller support is horizontal. Let H_B and [illegible] the components of the reaction at B.

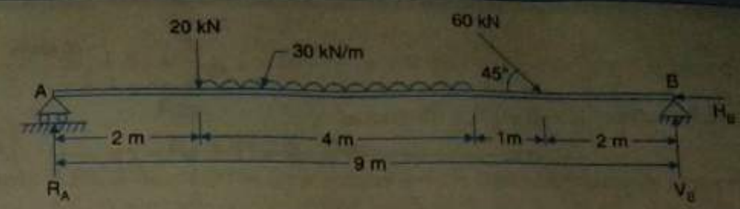


Fig. 3.49.

$$\Sigma M_B = 0 \quad \rightarrow$$

$$R_A \times 9 - 20 \times 7 - 30 \times 4 \times 5 - 60 \sin 45° \times 2 = 0.$$

$\therefore$ $\mathbf{R_A = 91.65\ kN}$ **Ans.**

$$\Sigma H_A = 0 \quad \rightarrow$$

$$60 \cos 45° - H_B = 0.$$

$\therefore$ $\mathbf{H_B = 42.43\ kN}$ **Ans.**

$$\Sigma V = 0$$

$$R_A + V_B - 20 - 30 \times 4 - 60 \sin 45° = 0.$$

$\therefore$ $V_B = 20 + 30 \times 4 + 60 \sin 45° - 91.65$, since $R_B = 91.65$ kN

i.e., $\mathbf{V_B = 90.78\ kN}$ **Ans.**

Example 3.23. *Determine the reactions at supports A and B of the overhanging beam shown in Fig. 3.50.*

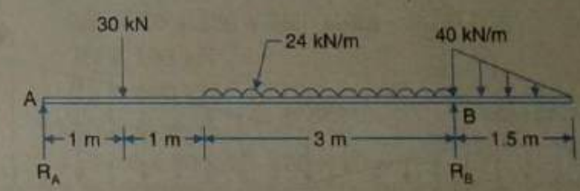


Fig. 3.50

Solution. $\Sigma M_A = 0 \quad \rightarrow$

$$30 \times 1 + 24 \times 3 \times (2 + 1.5) + \frac{1}{2} \times 1.5 \times 40 \times (5 + \frac{1}{3} \times 1.5) - R_B \times 5 = 0.$$

$\therefore$ $\mathbf{R_B = 89.4\ kN}$ **Ans.**

$$\Sigma V = 0 \quad \rightarrow$$

$$R_A + R_B - 30 - 24 \times 3 - \frac{1}{2} \times 1.5 \times 40 = 0.$$

$\therefore$ $R_A = 30 + 72 + 30 - 85.4$, since $R_B = 85.4$ kN

$\mathbf{R_A = 42.6\ kN}$ **Ans.**

Example 3.24. *Determine the reactions developed at supports A and B of overhanging beam shown in Fig. 3.51.*

Solution. Let the reaction components developed at support A be V_A and H_A and that at B be R_B.

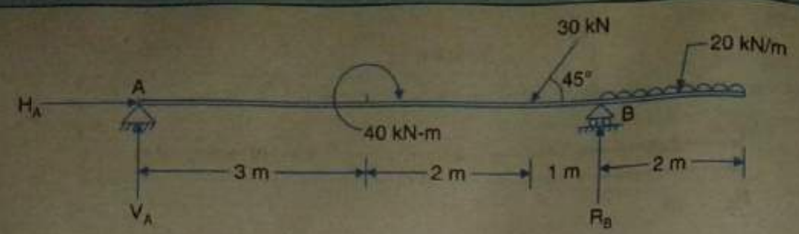


Fig. 3.51

$$\Sigma M_A = 0 \quad \rightarrow$$

$$40 + 30 \sin 45° \times 5 + 20 \times 2 \times 7 - R_B \times 6 = 0$$

$\therefore$ $\mathbf{R_B = 71.01\ kN}$ **Ans.**

$$\Sigma F_H = 0 \quad \rightarrow$$

$H_A - 30 \cos 45° = 0$ $\quad\therefore$ $\mathbf{H_A = 21.21\ kN}$ **Ans.**

$$\Sigma F_V = 0 \quad \rightarrow$$

$$V_A - 30 \sin 45° + R_B - 20 \times 2 = 0.$$

$\therefore$ $V_A = 30 \sin 45° - 71.01 + 40$, since $R_B = 71.01$.

i.e., $V_A = -9.79$ kN

$\mathbf{V_A = 9.79\downarrow\ kN}$ **Ans.**

Example 3.25. *Find the reactions developed at supports A and B of the loaded beam shown in Fig. 3.52.*

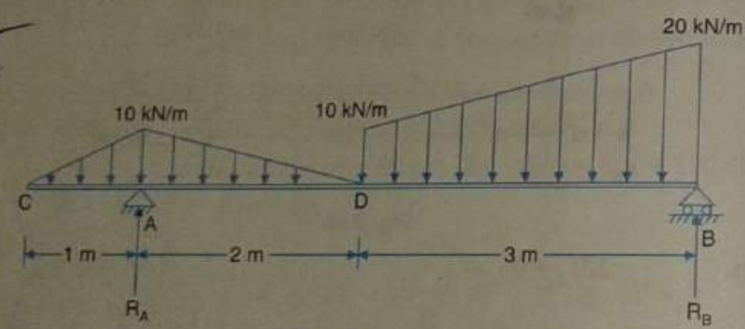


Fig. 3.52

Solution. The load is divided into a triangular load between C to A, another triangular load between A to D, a uniformly distributed load of 10 kN/m over portion DB and a triangular load of intensity zero at D and 10 kN [20 – 10 = 10 kN] at B.

$$\Sigma M_B = 0 \quad \rightarrow$$

$$-\frac{1}{2} \times 1 \times 10 \times \left(\frac{1}{3} + 5\right) - \frac{1}{2} \times 2 \times 10 \times \left(5 - \frac{2}{3}\right) - 10 \times 3 \times 1.5$$

$$-\frac{1}{2} \times 3 \times 10 \times 1 + R_A \times 5 = 0$$

$\therefore$ $\mathbf{R_A = 26\ kN}$ **Ans.**

$$\Sigma F_V = 0 \quad \rightarrow$$

$$R_A + R_B - \frac{1}{2} \times 1 \times 10 - \frac{1}{2} \times 2 \times 10 - 10 \times 3 - \frac{1}{2} \times 3 \times 10 = 0$$

$$26 + R_B - 5 - 10 - 30 - 15 = 0.$$

or $\mathbf{R_B = 34\ kN}$ **Ans.**

Example 3.26. *A beam 20 m long supported on two intermediate supports, 12 m apart, carries a udl of 6 kN/m and two concentrated loads of 30 kN at left end A and 50 kN at the right end B as shown in Fig. 3.53. How far away should the first support C be located from the end A so that the reactions at both the supports are equal.*

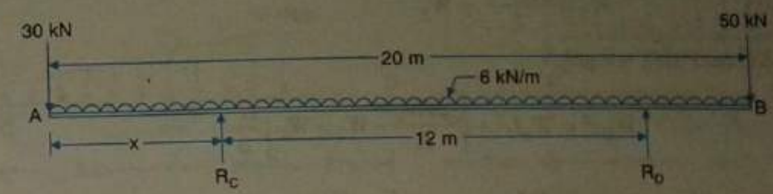


Fig. 3.53

Solution. Let the support C be at a distance x metres from end A.

Now, it is given that $R_C = R_D$

$$\Sigma F_V = 0 \quad \rightarrow$$

$$R_C + R_D - 30 - 6 \times 20 - 50 = 0$$

i.e., $2R_C = 30 + 120 + 50$, since $R_D = R_C$

$\therefore$ $R_C = 100$ kN

Hence $R_D = 100$ kN

$\Sigma M_A = 0$, gives.

$$R_C x + R_D (12 + x) - 6 \times 20 \times 10 - 50 \times 20 = 0$$

$$100x + 100 (12 + x) - 1200 - 1000 = 0, \quad \text{since } R_C = R_D = 100 \text{ kN}$$

$\mathbf{x = 5\ m}$ **Ans.**

Example 3.27. *A beam AB supports uniformly distributed load of intensity W_1 and rests on soil which exerts a uniformly varying upward reaction as shown in Fig. 3.54. Determine W_2 and W_3 corresponding to equilibrium. Note that soil can exert only an upward reaction at any point on beam. Hence, state for what range of values of $\frac{a}{L}$ the results obtained are valid.*

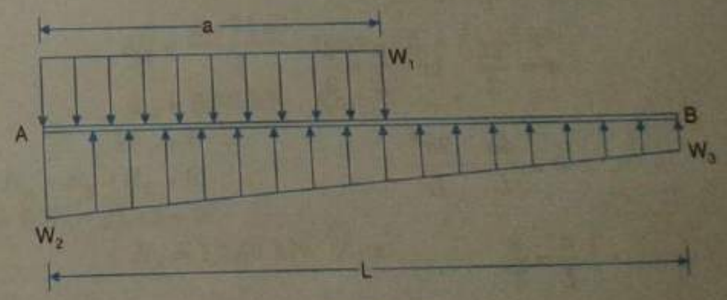


Fig. 3.54

Solution. $\Sigma F_V = 0 \rightarrow aW_1 = \frac{W_2 + W_3}{2} L$

$$\Sigma M_A = 0 \rightarrow aW_1 \frac{a}{2} = W_3 L \times \frac{L}{2} + \frac{1}{2}(W_2 - W_3) L \times \frac{L}{3}$$

i.e., $$W_1 a^2 = W_3 L^2 + (W_2 - W_3)\frac{L^2}{3}$$

From (*i*), $$W_2 = \frac{2aW_1}{L} - W_3$$

From (*ii*) and (*iii*), we get

$$W_1 a^2 = W_3 L^2 + \left[\frac{2aW_1}{L} - W_3 - W_3\right]\frac{L^2}{3}$$

$$= \frac{2aL}{3} W_1 + W_3 \left(1 - \frac{2}{3}\right) L^2$$

$$\therefore \quad W_1 \left(a^2 - \frac{2aL}{3}\right) = W_3 \frac{L^2}{3}$$

$$\therefore \quad \mathbf{W_3 = \frac{3W_1}{L^2}\left(a^2 - \frac{2aL}{3}\right)} \quad \textbf{Ans.}$$

Substituting this value of W_3 in (*iii*), we get

$$W_2 = \frac{2aW_1}{L} - \frac{3W_1}{L^2}\left(a^2 - \frac{2aL}{3}\right)$$

$$\therefore \quad \mathbf{W_2 = \left(\frac{4a}{L} - \frac{3a^2}{L^2}\right) W_1} \quad \textbf{Ans.}$$

For the above expressions to be valid W_2 and W_3 should not be negative. The limiting cases are when they are zero.

$$\therefore \quad 0 = \frac{3W_1}{L^2}\left(a^2 - \frac{2aL}{3}\right)$$

i.e., $$a = \frac{2L}{3} \quad \text{or} \quad \frac{a}{L} = \frac{2}{3}$$

and $$0 = \frac{4a}{L} - \frac{3a^2}{L^2}$$

i.e., $$\frac{a}{L} = \frac{4}{3}$$

Hence the results obtained are valid for the range $\frac{a}{L} = \frac{2}{3}$ to $\frac{4}{3}$ only. However, there is no possibility of loading beyond $\frac{a}{L} = 1$. Hence, **the valid range of $\frac{a}{L}$ is from $\frac{2}{3}$ to 1. Ans.**

Example 3.28. *Determine the reactions at A, B and D of the compound beam shown in Fig. 3.55(a). Neglect the self weight of the members.*

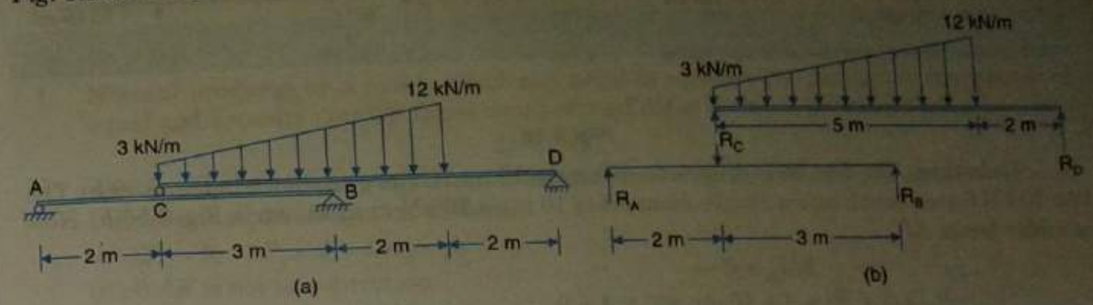


Fig. 3.55

Solution. Free body diagrams of beams *CD* and *AB* are as shown in Fig. 3.55(*b*). The load may be divided into a *udl* of 3 kN/m and a triangular load of intensity zero at *C* and 9 kN/m at 2m from *D*. Consider beam *CD*.

$$\Sigma M_C = 0 \rightarrow$$

$$-R_D \times 7 + 3 \times 5 \times 2.5 + \frac{1}{2} \times 5 \times 9 \times \frac{2}{3} \times 5 = 0$$

$$\therefore \quad \mathbf{R_D = 16.07\ kN} \quad \textbf{Ans.}$$

$$\Sigma F_V = 0 \rightarrow$$

$$R_C + R_D - 3 \times 5 - \frac{1}{2} \times 5 \times 9 = 0$$

$$R_C = 15 + 22.5 - 16.07, \quad \text{since} \quad R_D = 16.07\ \text{kN}$$

i.e., $$R_C = 21.43\ \text{kN}$$

Consider beam *AB*.

$$\Sigma M_A = 0 \rightarrow$$

$$R_C \times 2 - R_B \times 5 = 0.$$

$$R_B = \frac{2 \times 21.43}{5}, \quad \text{since} \quad R_C = 21.43\ \text{kN}$$

$$\therefore \quad \mathbf{R_B = 8.57\ kN} \quad \textbf{Ans.}$$

$$\Sigma F_V = 0 \rightarrow$$

$$R_A + R_B - R_C = 0.$$

$$R_A + 8.57 - 21.43 = 0.$$

$$\therefore \quad \mathbf{R_A = 12.86\ kN} \quad \textbf{Ans.}$$

Example 3.29. *The beam AB and CF are arranged as shown in Fig. 3.56(a). Det the reactions at A, C and D due to the loads acting on the beam as shown in the figure.*

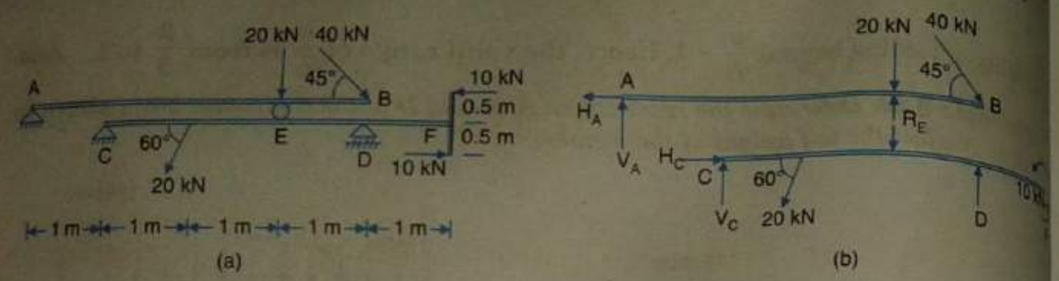


Fig. 3.56

Solution. The free body diagrams of beams *AB* and *CF* are as shown in Fig. 3.56(b). two 10 kN forces constitute a couple moment of 10 × 1 = 10 kN-m as shown in Fig. 3.56(b). consider beam *AB*.

$$\Sigma M_A = 0 \rightarrow$$

$$-R_E \times 3 + 20 \times 3 + 40 \sin 45° \times 4 = 0$$

$$\therefore \quad R_E = 57.71 \text{ kN}$$

$$\Sigma F_H = 0 \rightarrow$$

$$-H_A + 40 \cos 45° = 0.$$

or $\quad$ **H_A = 28.28 kN Ans.**

$$\Sigma F_V = 0 \rightarrow$$

$$V_A - 20 - 40 \sin 45° + R_E = 0$$

$$V_A = 20 + 40 \sin 45° - 57.71, \quad \text{since } R_E = 57.71 \text{ kN}$$

$$\therefore \quad V_A = -9.43 \text{ kN}$$

i.e., $\quad$ **V_A = 9.43 kN downward Ans.**

Consider beam *CF*.

$$\Sigma M_C = 0 \rightarrow$$

$$-R_D \times 3 + 20 \sin 60° \times 1 + R_E \times 2 - 10 = 0$$

$$\therefore \quad R_D = 20 \sin 60° - 10 + 57.71 \times 2, \quad \text{since, } R_E = 57.71 \text{ kN}$$

$\therefore$ $\quad$ **R_D = 40.91 kN Ans.**

$$\Sigma F_H = 0 \rightarrow$$

$$H_C - 20 \cos 60° = 0$$

$\therefore$ $\quad$ **H_C = 10 kN Ans.**

$$\Sigma F_V = 0 \rightarrow$$

$$V_C - 20 \sin 60° - R_E + R_D = 0$$

$$V_C - 20 \sin 60° - 57.71 + 40.91 = 0$$

$\therefore$ $\quad$ **V_C = 34.12 kN Ans.**

Important Definitions

1. **Moment** of a force about a point is the measure of rotational effect of the force and measured as the product of the magnitude of the force and the perpendicular distance the point from the line of action of the force.

2. **Varignon's theorem** states that the sum of moments of a system of coplanar forces about a moment centre is equal to the moment of their resultant about the same moment centre.
3. Two parallel forces equal in magnitude but opposite in direction and separated by a definite distance are said to form a **couple**.
4. **Statically determinate beams** may be defined as beams in which all reaction components can be found using the equation of equilibrium only.

Important Formulae

1. Moment produced by a couple about any point is equal to the product of magnitude of equal and opposite parallel forces and perpendicular distance between them.

$$M = Fd.$$

2. Any one set of following equations of equilibrium may be used for the analysis of non-concurrent coplanar system of forces.

(*i*) $\Sigma F_x = 0, \quad \Sigma F_y = 0, \quad \Sigma M_A = 0.$

(*ii*) If *AB* is not in *x*-direction,

$\Sigma F_x = 0, \quad \Sigma M_A = 0, \quad \Sigma M_B = 0.$

(*iii*) If *AB* is not in *y*-direction,

$\Sigma F_y = 0, \quad \Sigma M_A = 0, \quad \Sigma M_B = 0.$

(*iv*) If *A*, *B* and *C* are not collinear,

$\Sigma M_A = 0, \quad \Sigma M_B = 0, \quad \Sigma M_C = 0.$

PROBLEMS FOR EXERCISE

3.1 Determine the resultant of the parallel coplanar force system shown in Fig. 3.57.

[**Ans.** *R* = 800 N (towards left), *d* = 627.5 mm]

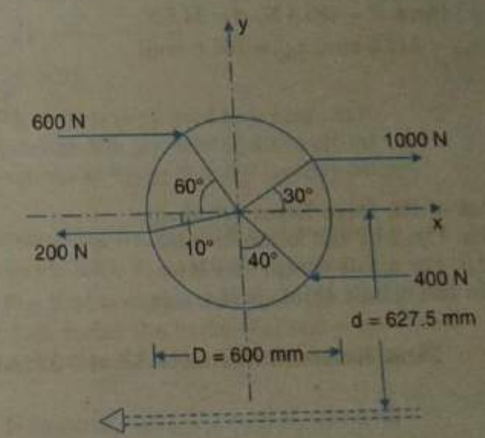


Fig. 3.57

3.2 Determine the magnitude, direction and the line of application of the equilibriant of the set of forces shown in Fig. 3.58, which will keep the plane body *ABCDEFGH* in equilibrium.

[**Ans.** *E* = 23.65 kN, α = 24.37°, *x* from *A* = 1.04 m]

3.3 An equilateral triangular plate of sides 200 mm is acted upon by four forces as shown in Fig. 3.58. Determine the magnitude and direction of the resultant of this system of forces and its point of intersection with the base.

[**Ans.** $R = 57.35$ kN ; $\alpha = 6.70°$, $d = 11.5$ mm]

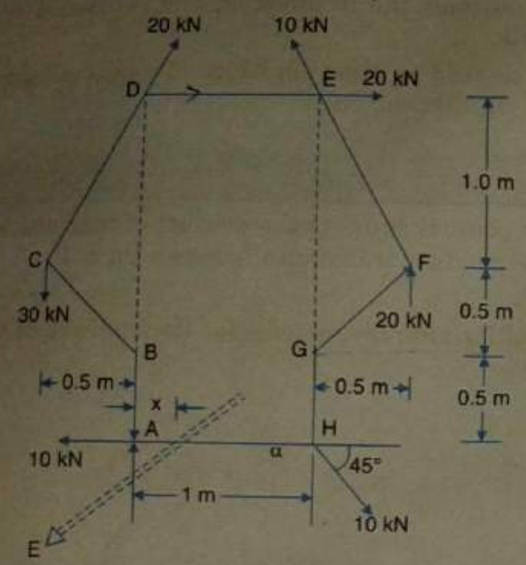


Fig. 3.58

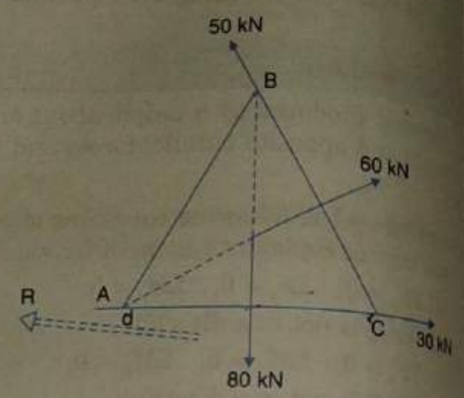


Fig. 3.59

3.4 Determine the resultant of the four forces acting on a body as shown in Fig. 3.60.

[**Ans.** $R = 200$ N, $\alpha = 60°$, at a distance $y = 8.77$ m below O]

3.5 A bracket is subjected to the system of forces and couples as shown in Fig. 3.61. Find the resultant of the system and the point of intersection of its line of action with (*a*) line *AB*, (*b*) line *BC* and line *CD*.

[**Ans.** $R = 485.4$ N, $\alpha = 34.50°$, $y_{CD} = 93.75$ mm, $y_{BA} = 112.5$ mm, $x_{BC} = 163.6$ mm]

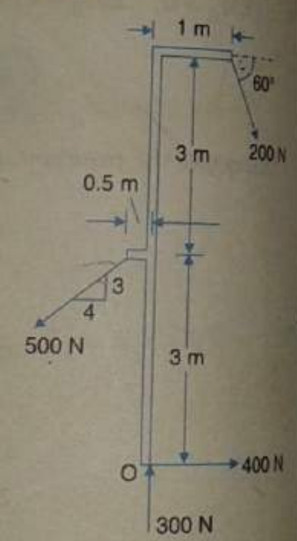


Fig. 3.60

3.6 Determine the resultant of the three forces acting on the dam section shown in Fig. 3.62 and locate its intersection with the base *AB*. For a safe design this intersection should be within the middle third. Is it a safe design ?

[**Ans.** Resultant intersects *AB* at 3.33 m from *A*. It is a good design]

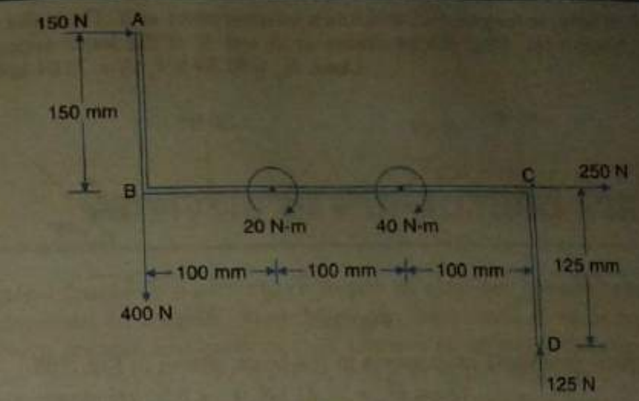


Fig. 3.61

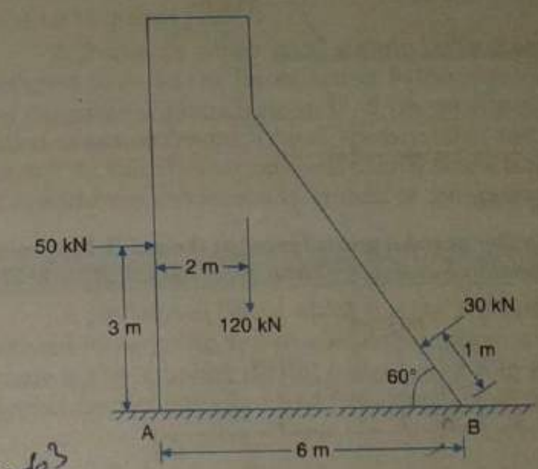


Fig. 3.62

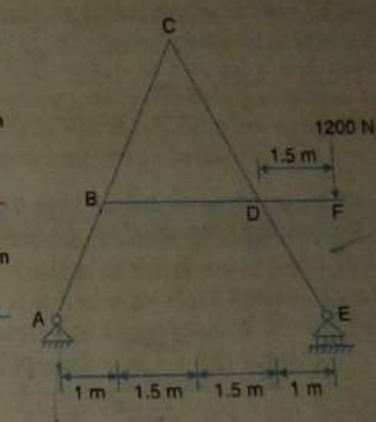


Fig. 3.63

3.7 The frame shown in Fig. 3.63 is supported by a hinge at *A* and a roller at *E*. Compute the horizontal and vertical components of the reactions at hinge *B* and *C* as they act upon member *AC*.

[**Ans.** $H_B = 200$ N (towards right). $V_B = 600$ N downward
$H_C = 200$ N (towards left). $V_C = 480$ N (downward)]

3.8 The frame shown in Fig. 3.64 is supported by a hinge at *E* and by a roller at *D*. Determine the horizontal and vertical components of the reaction at hinge *C* as it acts upon member *BD*.

[**Ans.** $H_C = 140$ N towards right, $V_C = 35$ N, upward]

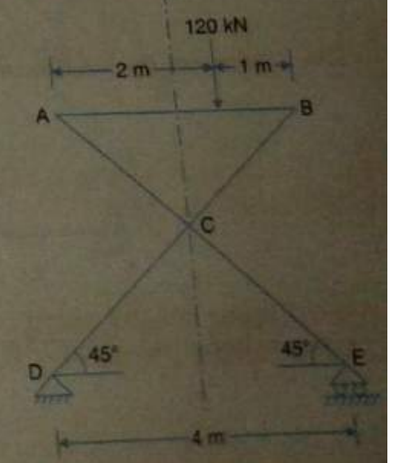


Fig. 3.64

3.9 Beam AB, 8.5 m long, is hinged at A and has a roller support at B. The roller support is at 45° to the horizontal. Find the reactions at A and B, if the loads acting are as shown in Fig. 3.65. [**Ans.** H_A = 51.67 kN, V_A = 73.04 kN, R_B = 101.2

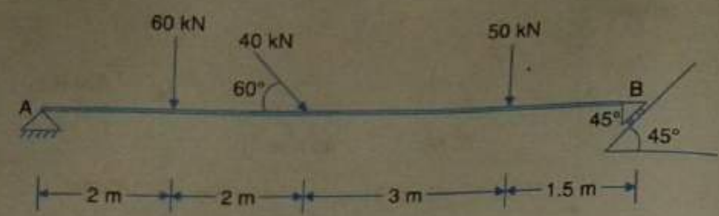


Fig. 3.65

3.10 Determine reactions developed at supports in the beam shown in Fig. 3.66.
[**Ans.** H_A = 21.21 kN, V_A = 9.80 kN, downward, R_B = 71.01

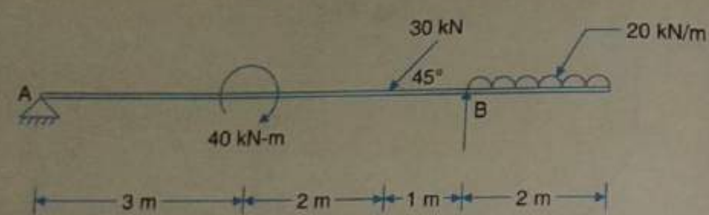


Fig. 3.66

3.11 Overhanging beam shown in Fig. 3.67 is on roller at end A and is hinged at the end B. Determ the reactions developed for the loadings shown in figure. [**Ans.** R_A = 160 kN, R_B = 90

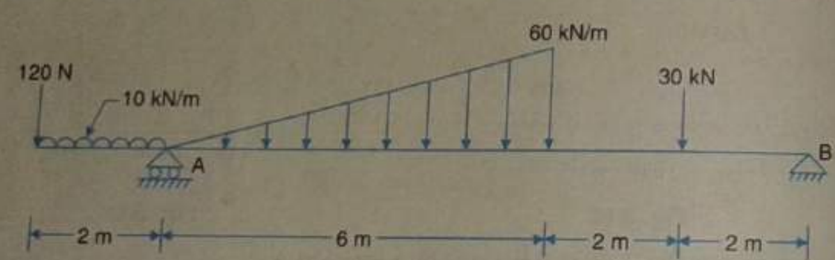


Fig. 3.67

3.12 Determine the reactions at supports A, C and D in the compound beam shown in Fig. 3.68.
[**Ans.** R_A = 20 kN, R_C = 67.5 kN, R_D = 7.5 kN, downw

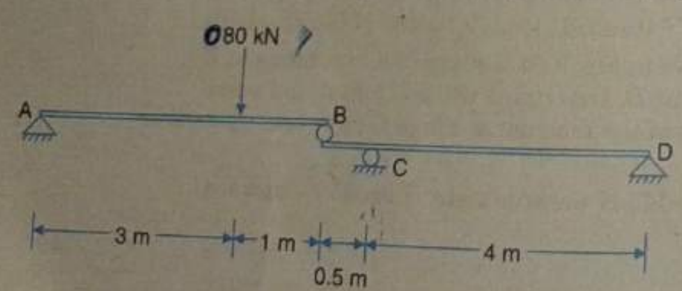


Fig. 3.68

4 Analysis of Pin-Jointed Plane Frames

A pin jointed frame is a structure made of slender (cross-sectional dimensions quite small compared to length) members pin connected at ends and capable of taking loads at joints. Such frames are used as roof trusses to support sloping roofs and as bridge trusses to support deck. Frames are used as arms of many machines. Transmission towers, observation towers are also examples of frames. In case of wooden frames ends of members are connected by nailing or by bolting whereas in steel frames end joints are made by riveting or welding to gusset plates.

A frame, in which all members lie in a single plane, is called **plane frame**. They are designed to resist the forces acting in the plane of frame. Roof trusses and bridge trusses are the examples of plane frames. If all the members of frame do not lie in a single plane, they are called as **space frame**. Tripod, transmission and observation towers are the examples of space frames. In this chapter analysis of only plane frame is dealt. The analysis involves application of equilibrium conditions to system of concurrent and non concurrent forces.

4.1. PERFECT, DEFICIENT AND REDUNDANT FRAMES

A pin jointed frame which has got just sufficient number of members to resist the loads without undergoing appreciable deformation in shape is called a **perfect frame**. Triangular frame is the simplest perfect frame and it has three joints and three members (Ref. Fig. 4.1). Perfect frames with four and five joints are shown in Figs. 4.2 and 4.3 respectively.

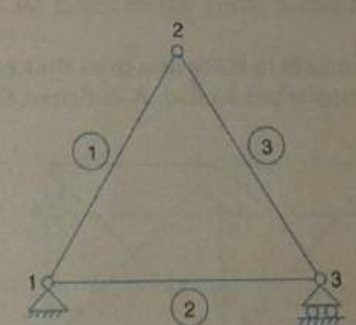


Fig. 4.1

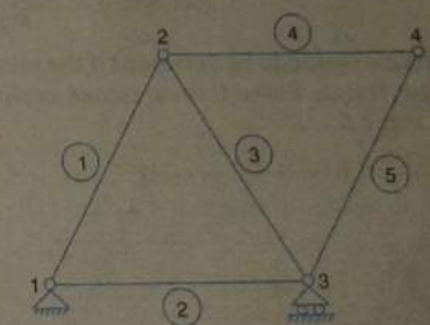


Fig. 4.2

It may be observed that to increase one joint in a perfect frame, two more members are required. Hence the following expression may be written as the relationship between number of joint j, and the number of members m in a perfect frame:

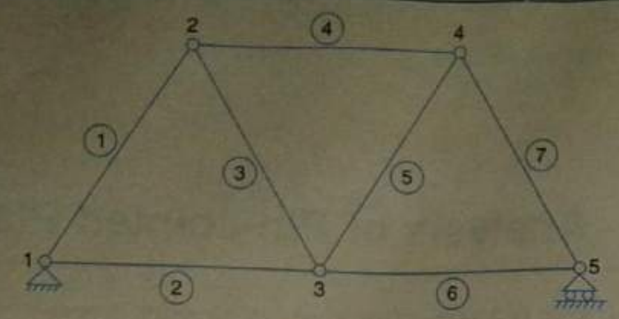


Fig. 4.3

$$m = 2j - 3$$ → m = member & j = joints

However, the above equation gives only a necessary, but not a sufficient condition for a perfect frame. For example, the two frames, shown in Figs. 4.4(*a*) and (*b*) have the same number of members and joints. The frame shown in Fig. 4.4(*a*) is a perfect frame, since it can retain its shape for loading at any joint. The frame shown in Fig. 4.4(*b*), is not capable of retaining its shape, if loaded at joint 6. Hence the necessary and sufficient condition for a perfect frame is that it should retain its shape when load is applied in the plane of frame at any joint.

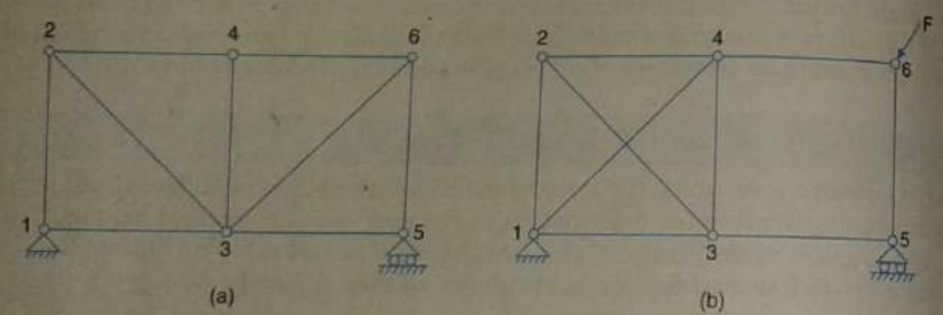


Fig. 4.4

A frame is said to be **deficient** if the number of members in it are less than that required for a perfect frame. These frames cannot retain their shape when loaded. A deficient frame is shown in Fig. 4.5.

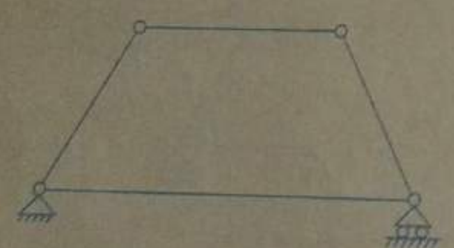


Fig. 4.5

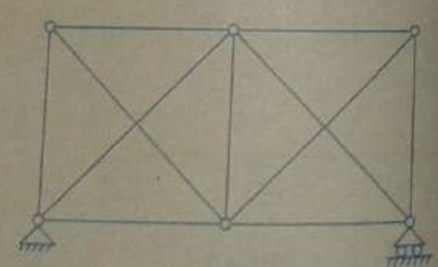


Fig. 4.6

A frame is said to be **redundant**, if the number of members in it are more than the required for a perfect frame. Such frames cannot be analysed by using the equations of equilibrium only. Thus a redundant frame is statically indeterminate frame. Each extra member adds one degree of indeterminancy. For the analysis of such frames consistancy of deformation

is also to be considered. The truss shown in Fig. 4.6 is a typical redundant truss. In this, one diagonal member in each panel is extra. Hence it is two degree redundant frame.

4.2. ASSUMPTIONS

The following assumptions are made in the analysis of pin jointed trusses :

(*i*) The ends of the members are pin connected (hinged).

(*ii*) The loads act only at the joints.

(*iii*) Self weight of the members are negligible.

(*iv*) Members are having either uniform cross-sections throughout or if they have varying cross-section the centroid is located along the same longitudinal line.

In reality, the members are connected by bolting, rivetting or by welding. No special care is taken to ensure perfect pin-connections. Assuming pin connected end is quite satisfactory for steel trusses, since the members used are slender. For concrete and steel trusses the results are reasonably satisfactory.

In most of the frames the loads act at the joints. Even if a load is not acting at a joint, it can be replaced by its reaction at the joints and local bending effect on the member. The frame may be analysed for the joint loads and the local bending effect taken care in the design.

In most of the trusses, the self weight is really small compared to the loads they carry. Hence self weight of the members may be neglected.

It is the duty of construction engineer to see that the centroid of all cross-sections of a member lie along a single axis so that the member is held in equilibrium by the two forces at its ends.

4.3. NATURE OF FORCES IN MEMBERS

The members of a pin jointed frames are subjected to either tensile or compressive forces. A typical truss *ABCDE* loaded at joint *E* is shown in Fig. 4.7 (*a*). The member *BC* is subjected to compressive force *C* as shown in Fig. 4.7(*b*). Effect of this force on the joint *B* (or *C*) is equal and opposite to force *C* as shown in Fig. 4.7(*b*). The member *AE* is subjected to tensile force *T*. Its effect on the joints *A* and *E* are as shown in Fig. 4.7(*b*).

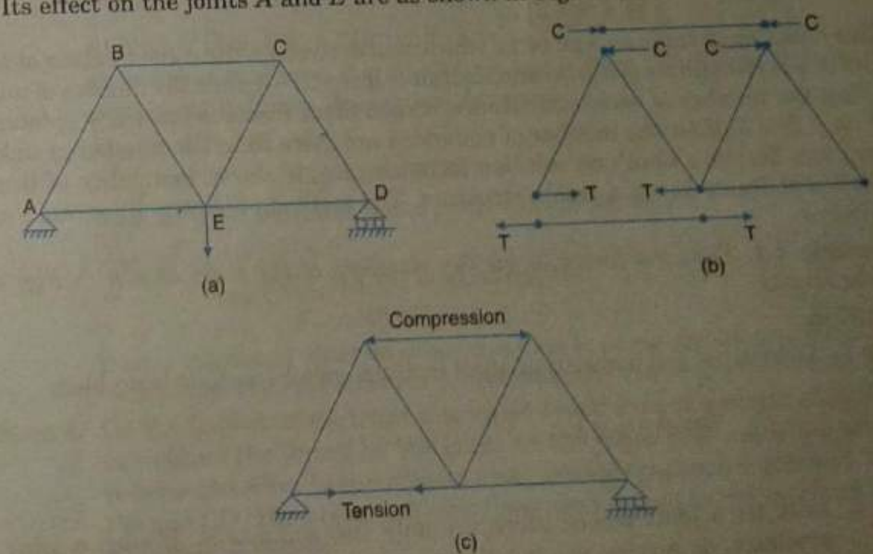


Fig. 4.7

During analysis, we mark the forces on the joints but not forces in the member [Ref. Fig. 4.7 (c)]. Hence, it may be noted that two arrows going away from each other in the figure of frame, show compressive force in the member and two arrows coming towards each other represent the tensile force in the member.

4.4. METHODS OF ANALYSIS

The following three methods are available for the analysis of pin connected plane frames :

(*i*) Method of joint

(*ii*) Method of section

(*iii*) Graphical method.

The first two are analytical methods and are dealt in this chapter. For graphical method any book on graphic static may be referred. However nowadays graphical methods are almost given up since computer aided analysis has become popular.

4.5. METHOD OF JOINT

At any joint, the forces in the members meeting and loads acting, constitute a system of concurrent forces. Hence two equations of equilibrium can be formed at each joint. To start with a joint is selected where there are only two unknown forces. Many times such a joint can be identified only after finding the reactions at supports considering the equilibrium of entire frame. Then using two equations of equilibrium for the system of forces at joint the two unknown forces can be found. Then next joint is selected for the analysis where there are now only two unknown forces. Thus the analysis proceeds from joint to joint to get the forces in all the members.

It may be noted that if there are j number of joints, $2j$ number of equations can be formed. In general, there will be three reactions in a determinate plane frames. The force in each member is unknown. Hence if there are 'm' number of members, the total number of unknowns are $m + 3$. A problem can have unique solution, if there are as many equations as there are unknowns. Hence a pin jointed plane frame analysis is determinate, if

$$\boxed{2j = m + 3} \qquad ...(4.2)$$

This equation is same as eqn. (4.1), which was derived on the consideration of a perfect frame. Hence a perfect frame is a determinate frame. If $m > 2j - 3$, then the number of unknowns is more than the number of equations. Hence, a redundant frame is statically indeterminate frame. If $m \le 2j - 3$, then the number of equations are more than the number of unknowns. Since there can be more than one solution for such case, it shows instability of the frame. Hence a deficient frame is not a stable structure. The method of joint is illustrated with the examples below :

Example 4.1. *Find the forces in all the members of the truss shown in Fig. 4.8 (a). Tabulate the results.*

Solution.

Step 1. Determine the inclination of all inclined members. In this problem

$$\tan\theta = \frac{3}{3} = 1.$$

$$\therefore \qquad \theta = 45°.$$

Step 2. Look for a joint where there are only two unknowns. If such a joint is not available, determine the reactions at supports and then the unknowns at one

of the support may reduce to only two. In this case at joint C, there are only two unknown forces. *i.e.*, forces in members CB and CD, say F_{CB} and F_{CD}.

[**Note:** Usually in cantilever type trusses, we find such joints without need to find support reactions.]

Step 3. Now at joint C there are two equations of equilibrium and two unknown forces. Hence the unknown forces can be found.

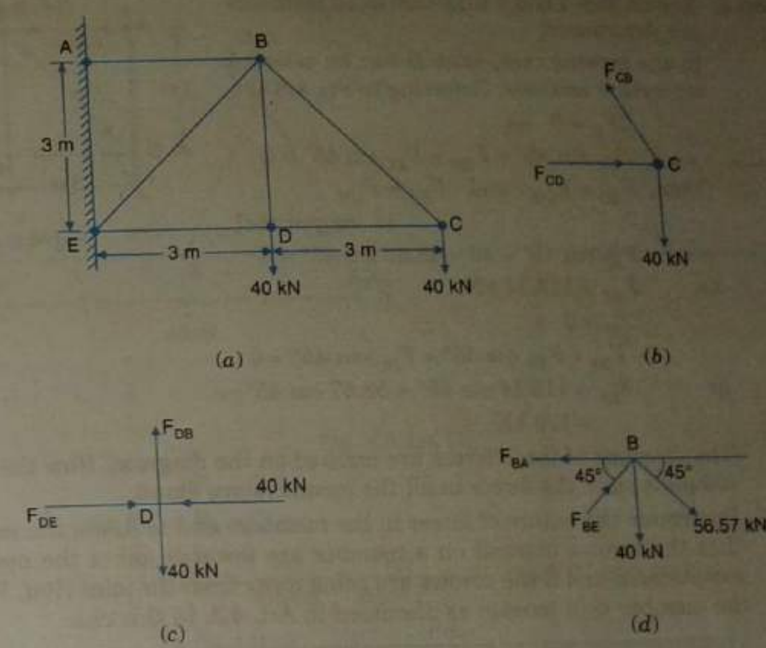


Fig. 4.8

At joint C [Ref. Fig. 4.8 (*b*)], $\Sigma F_V = 0$

shows that the force F_{CB} should act away from joint C so that its vertical component balances the vertical downward load.

$$F_{CB} \sin 45° - 40 = 0$$

$$\therefore \qquad F_{CB} = 56.57 \text{ kN}.$$

Now, $\Sigma F_H = 0$ indicates that F_{CD} should act towards C.

$$F_{CD} - F_{CB} \cos 45° = 0$$

$$\therefore \qquad F_{CD} = F_{CB} \cos 45° = 56.57 \cos 45°$$

$$\therefore \qquad F_{CD} = 40 \text{ kN}.$$

[**Note** : If assumed direction of member force is wrong, the value of member force wil be negative. Then the direction is to be corrected and proceeded further].

Step 4. On the diagram of the truss, mark arrow on the members near the joint analyse to indicate the forces on the joint. At the other end, mark the arrows in th reverse direction. In the present case, arrows are marked at joint C in membe CB and CD. Then reversed directions are marked in the members CB and C near joint B and D.

Step 5. Look for the next joint for analysis where there are only two unknown f In this case, joint D is selected for further analysis. Referring to Fig. 4.

$$\Sigma F_V = 0 \rightarrow$$
$$F_{DB} = 40 \text{ kN}$$
and
$$\Sigma F_H = 0 \rightarrow$$
$$F_{DE} = 40 \text{ kN}.$$

Step 6. Repeat step 4 and 5 till forces in all members are determined.

In the present case, joint B can be selected for further analysis. Referring to Fig 4.8 (d),

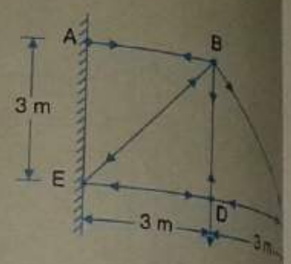


Fig. 4.8 (e)

$$\Sigma F_V = 0 \rightarrow$$
$$F_{BE} \sin 45° - F_{BD} - F_{BC} \sin 45° = 0$$

[Note, $F_{BD} = F_{DB}$ and $F_{BC} = F_{CB}$ in magnitude]

$$\therefore \quad F_{BE} \sin 45° - 40 - 56.57 \sin 45° = 0$$
i.e. $$F_{BE} = 113.14 \text{ kN}.$$
$$\Sigma F_H = 0 \rightarrow$$
$$-F_{BA} + F_{BE} \cos 45° + F_{BC} \cos 45° = 0$$
or $$F_{BA} = 113.14 \cos 45° + 56.57 \cos 45°$$
$$= 120 \text{ kN}.$$

The direction of these forces are marked on the diagram. Now the analysis complete since the forces in all the members are found.

Step 7. Determine the nature of forces in the members and tabulate the results. N that the arrows marked on a member are towards joints the member is compression and if the arrows are going away from the joint [Ref. Fig. 4.8 the member is in tension as discussed in Art. 4.3. In this case,

Member	*Magnitude of Force in kN*	*Nature*
AB	120	Tension
BC	56.57	Tension
CD	40	Compression
DE	40	Compression
BE	113.14	Compression
BD	40	Tension

Example 4.2. *Determine the forces in all the members of the truss shown in Fig. 4.9 and indicate the magnitude and nature of the forces on the diagram of the truss. All incli members are at 60° to horizontal and length of each member is 2 m.*

Solution. In this problem we don't find a joint with only two unknown forces strai way. Hence let us first find the support reactions.

Considering the equilibrium of entire frame.

$$\Sigma M_A = 0 \rightarrow$$
$$-R_D \times 4 + 40 \times 1 + 60 \times 2 + 50 \times 3 = 0$$
$$\therefore \quad R_D = 77.5 \text{ kN}$$
$$\Sigma F_H = 0 \rightarrow H_A = 0$$

$\therefore$ Reaction at A is vertical.

$$\Sigma F_V = 0 \rightarrow$$
$$R_A + R_D - 40 - 60 - 50 = 0$$
$$R_A = 150 - 77.5, \qquad \text{since } R_D = 77.5 \text{ kN}$$
$$= 72.5 \text{ kN}.$$

Now at joint A there are only two unknown forces. Hence consider the equilibrium of joint A [Fig. 4.9 (b)]

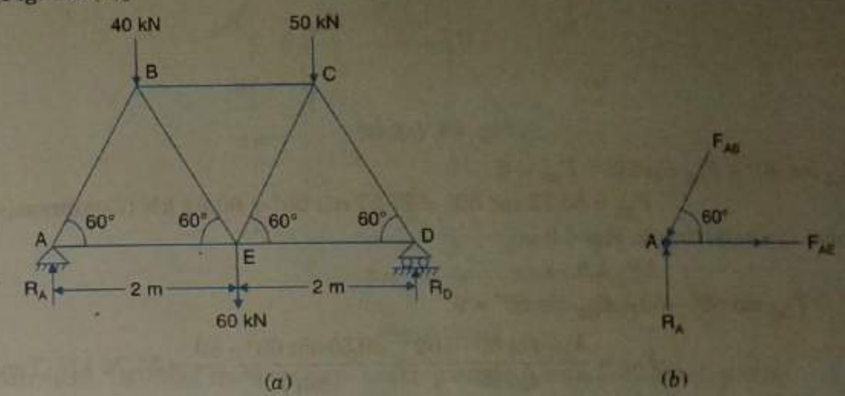


Fig. 4.9. (a), (b)

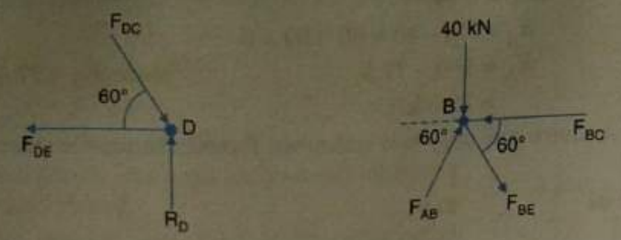
$$\Sigma F_V = 0 \rightarrow$$
$$F_{AB} \sin 60° - R_A = 0$$
$$\therefore \quad F_{AB} = \frac{R_A}{\sin 60°} = \frac{72.5}{\sin 60°} = 83.72 \text{ kN (Compression)}.$$
$$\Sigma F_H = 0 \rightarrow$$
$$F_{AE} - F_{AB} \cos 60° = 0$$
$$\therefore \quad F_{AE} = 83.72 \cos 60° = 41.86 \text{ kN (Tension)}$$

Joint D :
$$\Sigma F_V = 0 \rightarrow$$
$$-F_{DC} \sin 60° + R_D = 0$$
$$\therefore \quad F_{DC} = \frac{R_D}{\sin 60°} = \frac{77.5}{\sin 60°} = 89.5 \text{ kN (Compression)}.$$
$$\Sigma F_H = 0 \rightarrow$$
$$-F_{DE} + F_{DC} \cos 60° = 0.$$
$$\therefore \quad F_{DE} = F_{DC} \cos 60° = 89.5 \cos 60° = 44.75 \text{ kN (Tension)}$$

Joint B : Referring to Fig. 4.9 (d)

$$\Sigma F_V = 0 \rightarrow$$
$$F_{AB} \sin 60° - 40 - F_{BE} \sin 60° = 0$$
$$\therefore \quad F_{BE} = \frac{F_{AB} \sin 60° - 40}{\sin 60°} = \frac{72.5 - 40}{\sin 60°} = 37.53 \text{ kN (Tension)}$$

$$\Sigma F_H = 0 \rightarrow$$



(c) (d)

Fig. 4.9. (c), (d)

$$F_{AB} \cos 60° + F_{BE} \cos 60° - F_{BC} = 0$$

$$\therefore \quad F_{BC} = 83.72 \cos 60° + 37.53 \cos 60° = 60.62 \text{ kN (Compression)}.$$

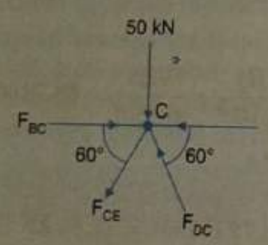
Joint C : Referring to Fig. 4.9 (*e*),

$$\Sigma F_V = 0 \rightarrow$$

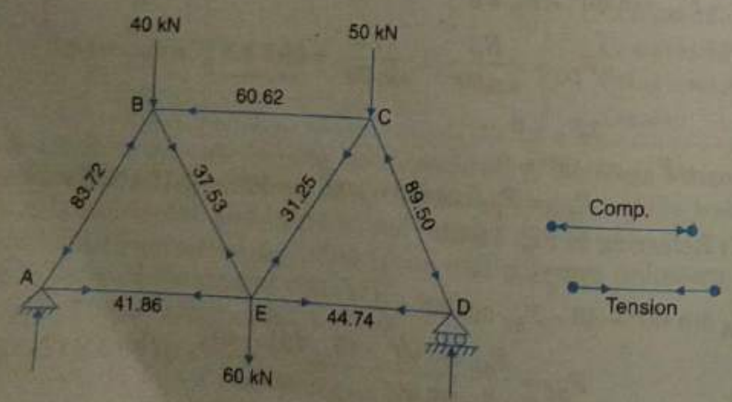
$$F_{DC} \sin 60° - 50 - F_{CE} \sin 60° = 0$$

$$\therefore \quad F_{CE} = \frac{F_{DC} \sin 60° - 50}{\sin 60°} = \frac{89.50 \sin 60° - 50}{\sin 60°} = 31.76 \text{ kN (Tension)}$$

Now the forces in all the members are known. If joint *E* is analysed it will give check for the analysis. The results are shown on diagram of the truss in Fig. 4.9 (*f*).



(e)



(f)

Fig. 4.9. (e), (f)

Example 4.3. *Analyse the truss shown in Fig. 4.10 (a).*

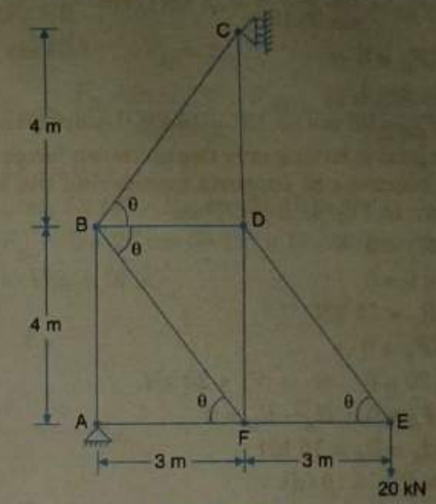


Fig. 4.10 (a)

Solution. Inclined members are making angle θ with horizontal, where

$$\tan \theta = \frac{4}{3}$$

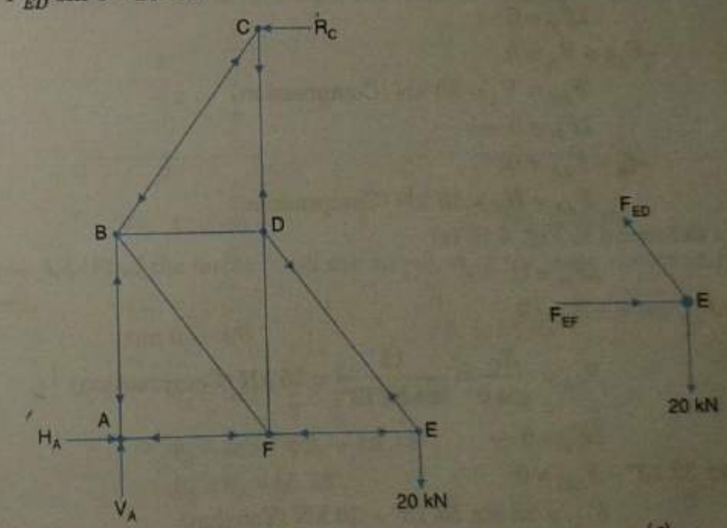
$$\therefore \quad \theta = 53.13°.$$

As soon as a joint is analysed the forces on the joints are marked on members in Fig. 4.10 (*b*).

Joint E : Referring to Fig. 4.10 (*c*)

$$\Sigma F_V = 0 \rightarrow$$

$$F_{ED} \sin \theta - 20 = 0.$$



(b) (c)

Fig. 4.10 (b), (c)

$$F_{ED} = \frac{20}{\sin 53.13°} = 25 \text{ kN (Tension)}$$

$$\Sigma F_H = 0 \rightarrow$$

$$F_{EF} - F_{ED} \cos\theta = 0.$$

$$F_{EF} = 25 \cos 53.13° = 15 \text{ kN (Compression)}$$

At this stage no other joint is having only two unknown forces. Hence no further p is possible. Let us find the reactions at supports considering the equilibrium of entire Let the reactions be as shown in Fig. 4.10 (*b*). Then

$$\Sigma M_A = 0 \rightarrow$$

$$-R_C \times 8 + 20 \times 6 = 0$$

$$\therefore \quad R_C = 15 \text{ kN}$$

$$\Sigma F_V = 0 \rightarrow$$

$$V_A - 20 = 0 \quad \text{or} \quad V_A = 20 \text{ kN}.$$

$$\Sigma F_H = 0, \quad H_A - R_C = 0$$

$$\therefore \quad H_A = R_C = 15 \text{ kN}$$

Joint *A* : Referring to Fig. 4.10 (*d*).

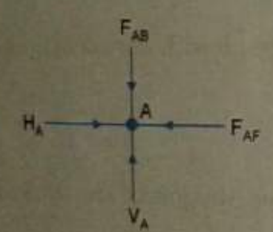


Fig. 4.10 (*d*)

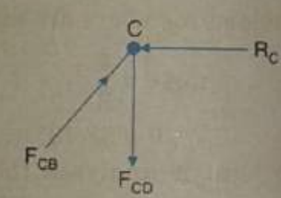


Fig. 4.10 (*e*)

$$\Sigma F_V = 0 \rightarrow$$

$$-F_{AB} + V_A = 0.$$

$$\therefore \quad F_{AB} = V_A = 20 \text{ kN (Compression)}$$

$$\Sigma F_H = 0 \rightarrow$$

$$H_A - F_{AF} = 0.$$

$$\therefore \quad F_{AF} = H_A = 15 \text{ kN (Compression)}$$

Joint *C* : Referring to Fig. 4.10 (*e*)

$$\Sigma F_H = 0 \rightarrow$$

$$F_{CB} \cos\theta - R_C = 0.$$

$$F_{CB} = \frac{R_C}{\cos\theta} = \frac{15}{\cos 53.13°} = 25 \text{ kN (Compression)}$$

$$\Sigma F_V = 0 \rightarrow$$

$$F_{CB} \sin 53.13° - F_{CD} = 0$$

$$\therefore \quad F_{CD} = 25 \sin 53.13° = 20 \text{ kN (Tension)}$$

Joint *B* : Referring to Fig. 4.10 (*f*)

$$\Sigma F_V = 0 \rightarrow$$

$$F_{BF} \sin 53.13° - F_{BC} \sin 53.13° + F_{AB} = 0.$$

$$F_{BF} = \frac{F_{BC} \sin 53.13° - F_{AB}}{\sin 53.13°} = \frac{25 \sin 53.13° - 20}{\sin 53.13°} = 0$$

$$\Sigma F_H = 0 \rightarrow$$

$$F_{BD} - F_{BF} \cos 53.13° - F_{BC} \cos 53.13° = 0$$

$$F_{BD} = 0 + 25 \cos 53.13° = 15 \text{ kN (Tension)}$$

Joint *F* : Referring to Fig. 4.10 (*g*)

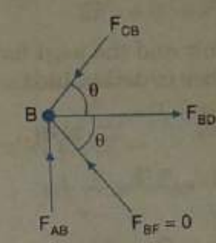


Fig. 4.10 (*f*)

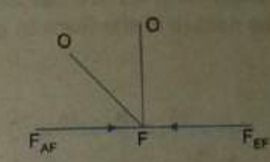


Fig. 4.10 (*g*)

$$\Sigma F_V = 0$$

$$F_{FD} = 0 \quad \text{since } F_{BF} = 0.$$

The magnitude of forces are marked on Fig. 4.10 (*b*).

A member with arrows towards joint is in compression and a member with arrows away from joints is in tension.

Note: When three members are meeting at an unloaded joint and out of them two are collinear, then the force in third member will be zero. Such situations are illustrated in Figs. 4.10 (*h*) and (*i*).

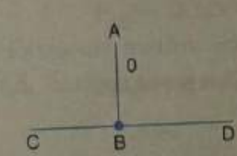


Fig. 4.10 (*h*)

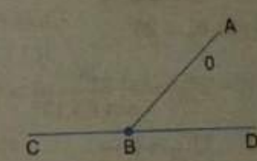


Fig. 4.10 (*i*)

Example 4.4. *Find the forces in all the members of the truss shown in Fig 4.11 (a).*

Solution.

$$\tan\theta_1 = 4/6 \qquad \therefore \quad \theta_1 = 33.69°$$

$$\theta_2 = \tan^{-1} \frac{8}{3} \times \frac{1}{2} = 53.13°$$

$$\theta_3 = \tan^{-1} 4/3 = 53.13°$$

Thus $\quad \theta_2 = \theta_3 = 53.13°$

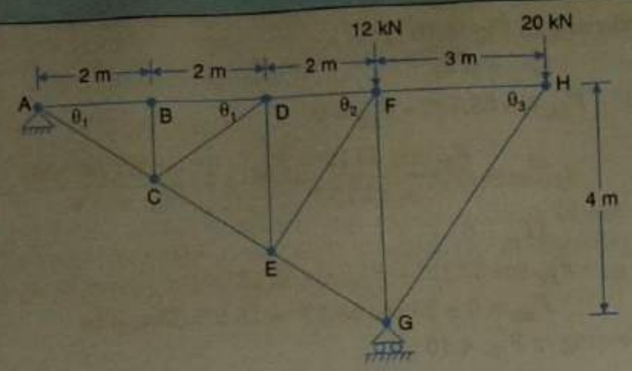


Fig. 4.11 (a)

Joint by joint analysis is carried out as given below and the joint forces are marked in Fig. 4.11 (*b*). Then the nature of the force in each member is determined.

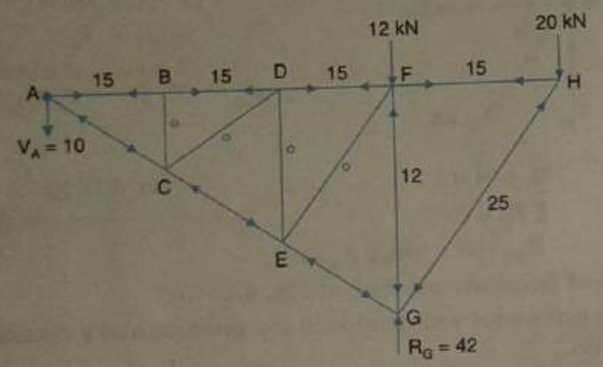


Fig. 4.11 (b)

Joint *H* : $\Sigma F_V = 0 \rightarrow$

$$F_{HG} \sin \theta_3 = 20$$

$$\therefore \quad F_{HG} = \frac{20}{\sin 53.13^\circ} = 25 \text{ kN (Compression)}$$

$$\Sigma F_H = 0 \rightarrow$$

$$F_{HF} - F_{HG} \cos \theta_3 = 0$$

$$\therefore \quad F_{HF} = 25 \cos 53.13^\circ = 15 \text{ kN (Tension)}$$

Now, reactions at supports are to be found.

$$\Sigma M_A = 0 \rightarrow$$

$$-R_G \times 6 + 20 \times 9 + 12 \times 6 = 0$$

$$\therefore \quad R_G = 42 \text{ kN}$$

$$\Sigma F_V = 0 \rightarrow$$

$$V_A + R_G - 12 - 20 = 0$$

$$V_A = 32 - 42, \quad \text{since } R_G = 42 \text{ kN}$$

$$= -10 \text{ kN}$$

$$= 10 \text{ kN} \downarrow$$

$$\Sigma F_H = 0 \rightarrow H_A = 0.$$

Hence reactions are as shown in Fig. 4.11 (*b*)

Joint *A*: $\Sigma F_V = 0 \rightarrow$

$$F_{AC} \sin \theta_1 - 10 = 0$$

$$\therefore \quad F_{AC} = \frac{10}{\sin 33.69^\circ} = 18.03 \text{ kN (Compression)}$$

$$\Sigma F_H = 0 \rightarrow$$

$$F_{AB} - F_{AC} \cos \theta_1 = 0$$

$$\therefore \quad F_{AB} = 18.03 \cos 33.69^\circ = 15 \text{ kN (Tension)}$$

Joint *B*: $\Sigma F_V = 0 \rightarrow F_{BC} = 0$

$$\Sigma F_H = 0 \rightarrow F_{CD} = F_{BA} = 15 \text{ kN (Compression)}$$

Joint *C*:

Σ Forces normal to $AC = 0 \rightarrow$

$$F_{CD} = 0, \quad \text{since } F_{BC} = 0.$$

Σ Forces parallel to $AE \rightarrow$

$$F_{CE} = F_{AC} = 18.03 \text{ kN (Compression)}$$

Joint *D*:

$$\Sigma F_V = 0 \rightarrow F_{DE} = 0$$

$$\Sigma F_H = 0 \rightarrow F_{DF} = F_{BD} = 15 \text{ kN (Tension)}$$

Joint *E*:

ΣForces normal to $CG = 0 \rightarrow$

$$F_{EF} = 0$$

ΣForces parallel to $CG \rightarrow$

$$F_{EG} = F_{CE} = 18.03 \text{ (Compression)}$$

Joint *F*:

$$\Sigma F_V = 0 \rightarrow$$

$$F_{AG} = 12 \text{ kN (Compression)}$$

All member forces are marked on Fig. 4.11 (*b*).

Example 4.5. *Analyse the truss shown in Fig. 4.12 (a). All members are 3 m long.*

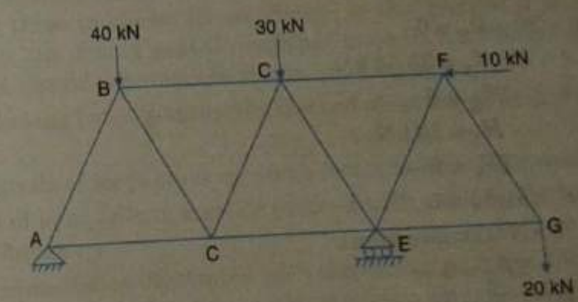


Fig. 4.12 (a)

Solution. Since all members are of same length (3 m), all triangles are equilateral triangles and hence all included angles are 60°. Joint by joint analysis is carried out and the direction of joint forces are marked in Fig. 4.11 (*b*).

Then nature of the force is determined.

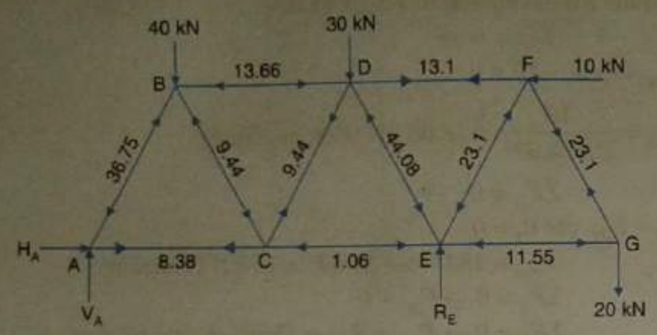


Fig. 4.12 (*b*)

Joint *G*: $\Sigma F_V = 0 \rightarrow$

$F_{GF} = \sin 60° = 20$

$\therefore$ $F_{GF} = 23.1$ kN (Tension)

$\Sigma F_H = 0 \rightarrow$

$F_{GE} - F_{GF} \cos 60° = 0$

$\therefore$ $F_{GE} = 23.1 \cos 60° = 11.55$ kN (Compression)

Joint *F*:

$\Sigma F_V = 0 \rightarrow$

$F_{EF} \sin 60° - F_{GF} \sin 60° = 0.$

$\therefore$ $F_{FE} = F_{GF} = 23.1$ kN (Compression).

$\Sigma F_H = 0 \rightarrow$

$F_{FD} + 10 - F_{GF} \cos 60° - F_{FE} \cos 60° = 0.$

$\therefore$ $F_{FD} = 13.1$ kN (Tension). Since $F_{GF} = F_{FE} = 23.1$ kN.

Now without finding reactions, we can proceed. Hence consider the equilibrium of entire truss.

$-R_E \times 6 - 10 \times 3 \sin 60° + 40 \times 3 \cos 60° + 30 (3 + 3 \cos 60°) + 20 \times 9 = 0$

$\therefore$ $R_E = 58.17$ kN

$\Sigma F_V = 0 \rightarrow$

$V_A - 40 - 30 - 20 + R_E = 0.$

$\therefore$ $V_A = 38.83$ kN

$\Sigma F_H = 0$

$H_A = 10$ kN

Joint *A*: $\Sigma F_V = 0 \rightarrow$

$F_{AB} \sin 60° - 31.83 = 0.$

$\therefore$ $F_{AB} = 36.75$ kN (Compression).

$\Sigma F_H = 0 \rightarrow$

$F_{AC} - F_{AB} \cos 60° + H_A = 0.$

$\therefore$ $F_{AC} = 36.75 \cos 60° - 10$

$= 8.38$ kN (Tension)

Joint *B*: $\Sigma F_V = 0 \rightarrow$

$F_{BC} \sin 60° + F_{AB} \sin 60° - 40 = 0$

$F_{BC} \sin 60° = 40 - F_{AB} \sin 60° = 40 - 36.75 \sin 60°$

$\therefore$ $F_{BC} = 9.44$ kN (Compression)

$\Sigma F_H = 0 \rightarrow$

$F_{AB} \cos 60° - F_{BC} \cos 60° - F_{BD} = 0$

$\therefore$ $F_{BD} = 36.75 \cos 60° - 9.44 \cos 60° = 13.66$ kN (Compression)

Joint *C*: $\Sigma F_V = 0 \rightarrow$

$F_{CD} \sin 60° - F_{BC} \sin 60° = 0$

$F_{CD} = F_{BC} = 9.44$ kN (Tension)

$\Sigma F_H = 0 \rightarrow$

$F_{CD} \cos 60° + F_{BC} \cos 60° - F_{AC} - F_{CE} = 0.$

$\therefore$ $F_{CE} = 9.44 \cos 60° + 9.44 \cos 60° - 8.38$

$= 1.06$ kN (Compression)

Joint *D*: $\Sigma F_V = 0 \rightarrow$

$F_{DE} \sin 60° - F_{CD} \sin 60° - 30 = 0$

$$F_{DE} = \frac{30 + F_{CD} \sin 60°}{\sin 60°} = \frac{30 + 9.44 \sin 60°}{\sin 60°}$$

$= 44.08$ kN (Compression).

Forces are indicated in Fig. 4.12 (*b*).

4.6. METHOD OF SECTION

In the method of section, after determining the support reactions, a section line is drawn passing through not more than three members in which forces are not known, such that the frame is cut into two separate parts. Each part should be in equilibrium under the action of loads, reactions and the forces in the members that are cut by the section line. Equilibrium of any one of these two parts is considered and the unknown forces in the members cut by the section line are determined. The system of forces acting on either part of truss constitutes a non-concurrent force system. Since there are only three independent equations of equilibrium, there should be only three unknown forces. Hence, in this method it is an essential requirement that the section line should pass through not more than three members in which forces are not known and it should separate the frame into two parts.

Under the following two situations the method of section is preferred over the method of joint :

(*i*) In the analysis of large truss in which forces in only few members are required.

(*ii*) If method of joint fails to start or proceed with analysis for not getting a joint with only two unknown forces.

The method of section is illustrated with the examples 4.6 to 4.9. Examples 4.6 and 4.7 are the cases in which method of section is advantageous since forces in only few members are required. Examples 4.8 and 4.9 are the cases in which method of joints fail to start/proceed to get the solution. In practice the frames may be analysed partly by the method of joint and partly by the method of section as illustrated in example 4.9.

Example 4.6. *Determine the forces in the members FH, HG and GI in the truss shown in Fig. 4.13 (a). Each load is 10 kN and all triangles are equilaterals with sides equal to 4 m.*

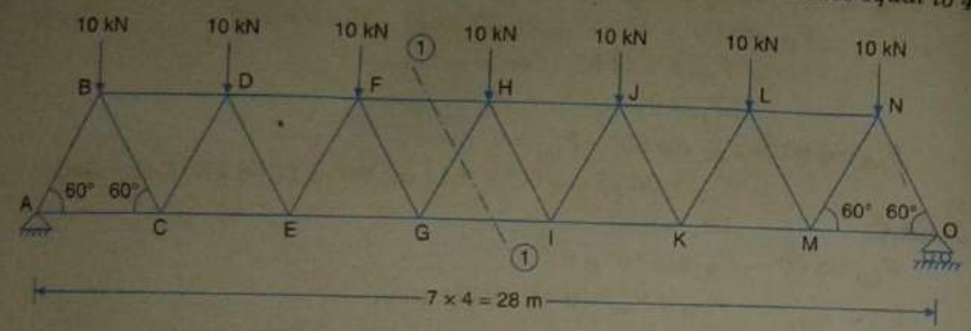


Fig. 4.13 (*a*)

Solution. Due to symmetry,

$$R_A = R_B = \frac{1}{2} \times \text{total load}$$

$$= \frac{1}{2}(10 \times 7) = 35 \text{ kN}$$

There is no horizontal component of reaction at *A*.

Take section (1) – (1) which cuts the members *FH*, *GH* and *GI* and separate the truss into two parts. Consider the equilibrium of left hand side part as shown in Fig. 4.13 (*b*). (Prefer the part in which number of forces are less).

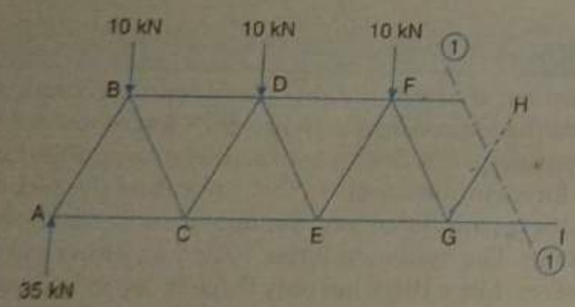


Fig. 4.13 (*b*)

$$\Sigma M_G = 0 \rightarrow$$

$$-F_{FH} \times 4 \sin 60° + 35 \times 12 - 10 \times 10 - 10 \times 6 - 10 \times 2 = 0$$

$$\therefore \quad F_{FH} = 69.28 \text{ kN (Compression)}$$

$$\Sigma F_V = 0 \rightarrow$$

$$35 - 10 - 10 - 10 - F_{GH} \sin 60° = 0$$

$$\therefore \quad F_{GH} = 5.77 \text{ kN (Compression)}$$

$$\Sigma F_H = 0 \rightarrow$$

$$F_{GI} - F_{FH} - F_{GH} \cos 60° = 0$$

$$\therefore \quad F_{GI} = 69.28 + 5.77 \cos 60° = 72.17 \text{ kN (Tension).}$$

Example 4.7. *Find the magnitude and nature of forces in the members U_2U_4, L_3L_4 and U_4L_3 of the loaded truss shown in Fig. 4.14 (a).*

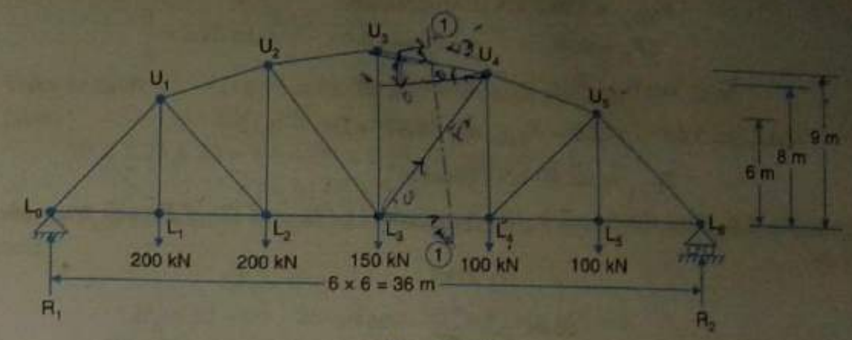


Fig. 4.14 (*a*)

Solution. Considering the equilibrium of entire truss.

$$\Sigma M_{L_0} = 0 \rightarrow$$

$$200 \times 6 + 200 \times 12 + 150 \times 18 + 100 \times 24 + 100 \times 30 - R_2 \times 36 = 0$$

$$\therefore \quad R_2 = 325 \text{ kN}$$

$$\Sigma F_V = 0 \rightarrow$$

$$R_1 + R_2 - 200 - 200 - 150 - 100 - 100 = 0$$

$$R_1 = 200 + 200 + 150 + 100 + 100 - 325, \quad \text{Since } R_2 = 325 \text{ kN}$$

$$= 425 \text{ kN.}$$

Take the section (1) – (1) and consider the equilibrium of right hand side part. Referring to Fig. 4.14 (*b*).

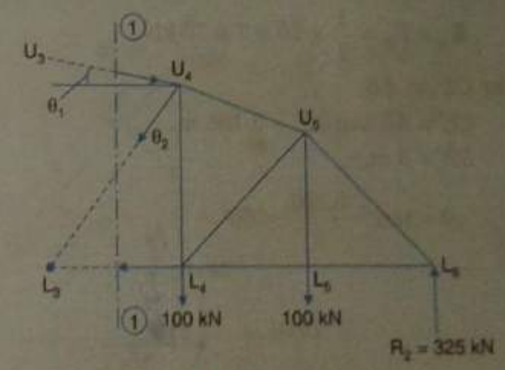


Fig. 4.14 (*b*)

$$\theta_1 = \tan^{-1} 1/6 = 9.46°$$

$$\theta_2 = \tan^{-1} 6/8 = 36.87°.$$

$$\Sigma M_{U_4} = 0 \rightarrow$$

$$F_{L_3L_4} \times 8 + 100 \times 6 - 325 \times 12 = 0$$

$$\therefore \quad F_{L_3L_4} = 412.5 \text{ kN (Tension)}$$

$$\Sigma M_{L_3} = 0 \rightarrow$$

$$F_{U_4U_3} \cos\theta_1 \times 8 + F_{U_3U_4} \sin\theta_1 \times 6 + 100 \times 6 + 100 \times 12 - 325 \times 18 = 0$$

$\therefore$ $F_{U_3U_4} = 456.2$ kN (Compression)

$$\Sigma F_H = 0 \rightarrow$$

$$F_{U_3U_4} \cos\theta_1 - F_{L_3L_4} - F_{U_4L_3} \sin\theta_2 = 0$$

$$456.2 \cos 9.46° - 412.5 - F_{U_4L_3} \sin 36.87° = 0$$

$\therefore$ $F_{U_4L_3} = 62.49$ kN (Tension).

Example 4.8. *Find the forces in the members (1), (2), and (3) of French truss shown in Fig. 4.15 (a).*

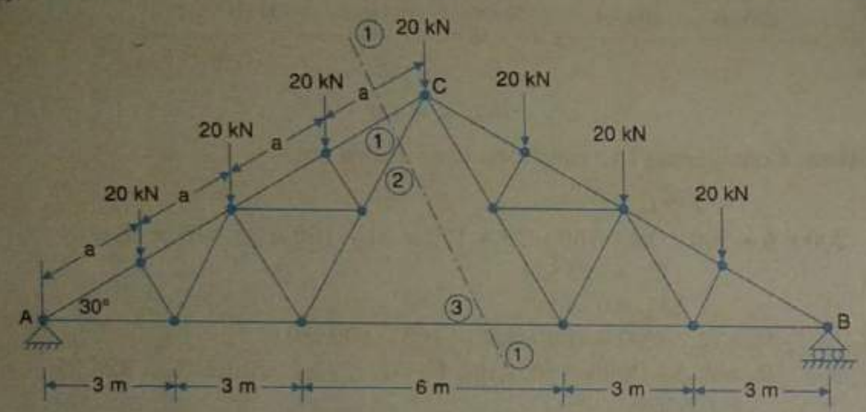


Fig. 4.15 (*a*)

Solution. Due to symmetry,

$$R_A = R_B = \frac{1}{2} \times 20 \times 7 = 70 \text{ kN}.$$

Drop perpendicular CE on AB.

$$CE = AE \tan 30° = 5.196 \text{ m}.$$

$$DE = 3 \text{ m}.$$

$\therefore$ $\theta = \tan^{-1} \dfrac{5.196}{3} = 60°$

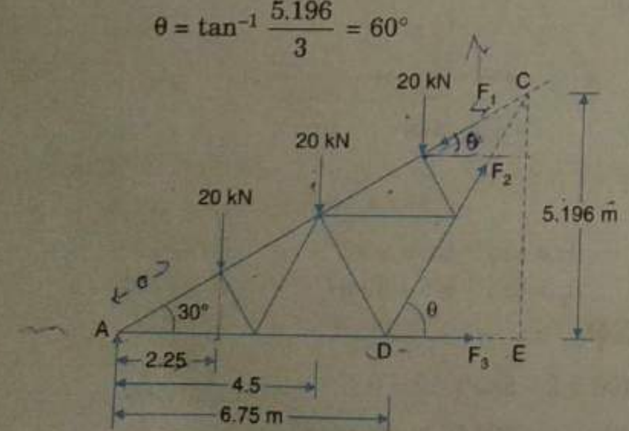


Fig. 4.15 (*b*)

20 kN loads are at equidistances. From $\Delta\, ACE$, we can make out that horizontal distances of loads from A are

$$\frac{9}{4} = 2.25 \text{ m}, \frac{9 \times 2}{4} = 4.5 \text{ m and } \frac{9 \times 3}{4} = 6.75 \text{ m}.$$

Take section (1) – (1) and consider left hand side portion.

Now, $\Sigma M_A = 0 \rightarrow$

$$20 \times 2.25 + 20 \times 4.5 + 20 \times 6.75 - F_2 \sin 60° \times 6 = 0$$

$\therefore$ $F_2 = 51.96$ kN

[**Note:** F_2 is resolved at D in vertical and horizontal components and then moment about A is calculated].

$$\Sigma F_V = 0 \rightarrow$$

$$R_A - 20 - 20 - 20 - F_1 \sin 30° + F_2 \sin 60° = 0$$

$$70 - 20 - 20 - 20 - F_1 \sin 30° + 51.96 \sin 60° = 0$$

$\therefore$ $F_1 = 110$ kN (Compression)

$$\Sigma F_H = 0 \rightarrow$$

$$F_3 + F_2 \cos 60° - F_1 \cos 30° = 0$$

$$F_3 = -51.96 \cos 60° + 110 \times \cos 30°$$

$$= 69.28 \text{ kN (Tension)}$$

[**Note:** In this problem, method of joint alone cannot give complete solution.]

Example 4.9. *Find the forces in all the members of the truss, shown in Fig. 4.16 (a).*

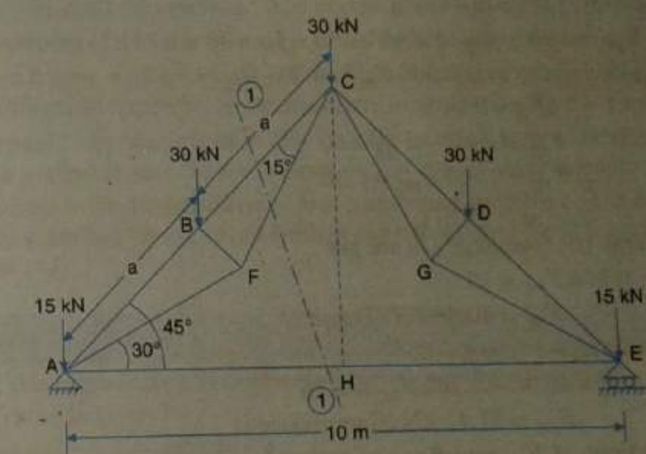


Fig. 4.16 (*a*)

Solution. Due to symmetry the reactions are equal.

$\therefore$ $R_A = R_E = \dfrac{1}{2} \times$ total load

$$= \frac{1}{2}(15 + 30 + 30 + 30 + 15) = 60 \text{ kN}.$$

Drop perpendicular CH to AE.

$$\angle ACH = 45°$$

$\therefore$ $\angle FCH = 45° - 15° = 30°$ *i.e.*, FC makes 30° with vertical.

It is not possible to find a joint with only two unknown forces. Consider section (1) – (1)

For left hand side of the frame, 4.16(*b*)

$$\Sigma M_C = 0 \rightarrow$$

$$-F_{AE} \times 5 + 60 \times 5 - 15 \times 5 - 30 \times 2.5 = 0.$$

$$\therefore \quad F_{AE} = 30 \text{ kN (Tension)}$$

Assuming the directions for F_{FC} and F_{BC} as shown in Fig. 4.16(*b*)

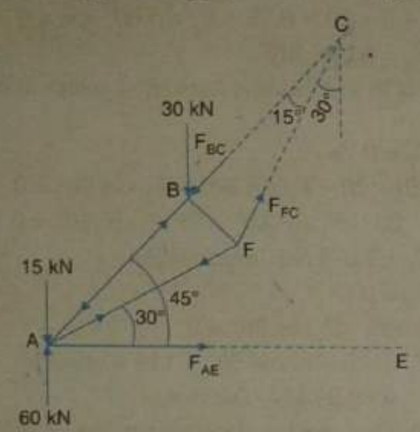


Fig. 4.16 (*b*)

$$\Sigma F_V = 0 \rightarrow$$

$$F_{FC} \cos 30° - F_{BC} \cos 45° + 60 - 15 - 30 = 0$$

$$0.866\, F_{FC} - 0.707\, F_{BC} = -15 \qquad (i)$$

$$\Sigma F_H = 0 \rightarrow$$

$$F_{FC} \sin 30° - F_{BC} \sin 45° + F_{AE} = 0.$$

$$0.5\, F_{FC} - 0.707\, F_{BC} = -30 \qquad ...(ii)$$

Since $\quad F_{AE} = 30$ kN

Subtracting eqn. (*ii*) from eqn. (*i*), we get

$$0.366\, F_{FC} = 15$$

$$\therefore \quad F_{FC} = 40.98 \text{ kN (Tension)}$$

Substituting this value in eqn. (*i*),

$$0.866 \times 40.98 - 0.707\, F_{BC} = -15$$

$$\therefore \quad F_{BC} = 71.41 \text{ kN (Compression)}$$

Assumed directions of F_{FC} and F_{BC} are correct.

Now we can proceed with method of joint to find the forces in other members. Since it is a symmetric truss, analysis of half the truss is sufficient. Other values may be written by making use of symmetry.

Joint *B* :

ΣForces normal to $AC = 0 \rightarrow$

$$F_{BF} - 30 \cos 45° = 0$$

or $\quad F_{BF} = 21.21$ kN (Compression)

ΣForces parallel to $AC = 0 \rightarrow$

$$F_{AB} - F_{BC} - 30 \sin 45° = 0$$

$$F_{AB} = 71.41 + 21.21 = 92.62 \text{ kN (Compression)}$$

Joint *A*: $\quad \Sigma F_V = 0 \rightarrow$

$$F_{AF} \sin 30° - F_{AB} \sin 45° - 15 + 60 = 0.$$

$$\therefore \quad F_{AF} = \frac{92.62 \sin 45° - 45}{\sin 30°} = 40.98 \text{ kN (Tension)}$$

The results are tabulated below :

Member	*Force in kN*
AB and *E*	– 92.62
BC and *DC*	– 71.41
BF and *DG*	– 21.21
AF and *EG*	40.98
FC and *GC*	40.98
AE	30

+ means tension and – means compression.

Important Definitions

1. **A pin jointed frame** is a structure made of slender members pin connected at ends and capable of taking loads at joints.
2. A frame, in which all members lie in a single plane, is called **plane frame.**
3. If all the members of a frame do not lie in a single plane, they are called as space frame.
4. A pin jointed frame, which has got just sufficient number of members to resist the loads without undergoing appreciable deformation in shape is called a **perfect frame.**
5. A frame is said to be **deficient** if the number of members in it are less than that required for a perfect frame. These frames do not retain their shapes when loaded.
6. A frame is said to be **redundant**, if the number of members in it are more than that required for a perfect frame. Such frames cannot be analysed by using the equations of equilibrium only.

Important Formulae

In general the relationship between number of members (m) and number of joints (j) in a perfect frame is given by

$$m = 2j - 3$$

PROBLEMS FOR EXERCISE

4.1 to 4.17: Determine the forces in all the members of the frames shown in Figs. 4.17 to 4.33. Indicate the nature of the forces also. (Tension as + ve and compression as – ve)

4.1 [**Ans.** $F_{AB} = +67.5$ kN; $F_{BC} = +15$ kN ; $F_{CD} = -25$ kN ; $F_{DE} = -30$ kN ; $F_{EF} = -105$ kN; $F_{AE} = +62.5$ kN ; $F_{BE} = -62.5$ kN ; $F_{BD} = +25$ kN]

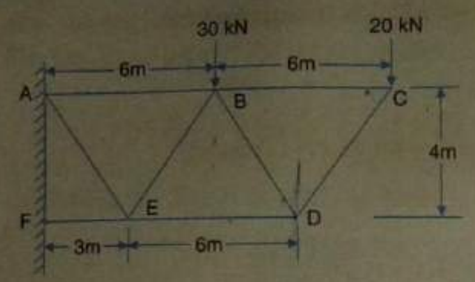


Fig. 4.17 (Prob. 4.1)

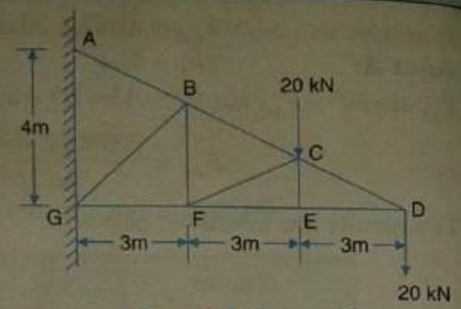


Fig. 4.18 (Prob. 4.2)

4.2 [**Ans.** $F_{AB} = +82.0738$ kN; $F_{BC} = +73.866$ kN; $F_{CD} = 49.2443$ kN; $F_{DE} = -45$ kN; $F_{EF} = -45$ kN; $F_{FG} = -67.5$ kN ; $F_{BG} = -10.0$ kN; $F_{FC} = +24.622$ kN; $F_{CE} = 0$; $F_{BF} = 10$ kN]

4.3 [**Ans.** $F_{AC} = F_{CE} = E_{EG} = +193.1852$ kN; $F_{BD} = F_{DE} = F_{FG} = -193.1852$ kN; All others are zero members]

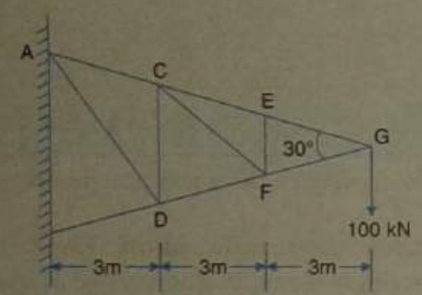


Fig. 4.19 (Prob. 4.3)

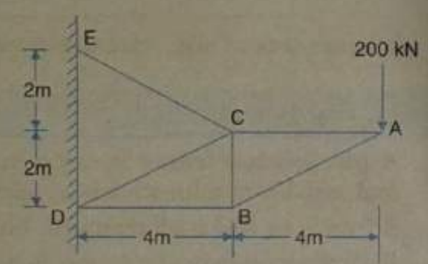


Fig. 4.20 (Prob. 4.4)

4.4 [**Ans.** $F_{BC} = +447.2136$ kN; $F_{CA} = +400$ kN; $F_{AB} = -447.2136$ kN; $F_{BD} = -400$ kN; $F_{CD} = 0$; $F_{CB} = +200$ kN]

4.5 [**Ans.** $F_{DB} = F_{BA} = +5.7735$ kN; $F_{BC} = F_{DE} = -5.7738$ kN; $F_{AC} = -2.8868$ kN; $F_{CE} = -14.4338$ kN; $F_{DC} = +17.3205$ kN; $F_{DF} = +20.0$ kN]

4.6 [**Ans.** $F_{AB} = -30$ kN; $F_{AC} = -160$ kN; $F_{BC} = +50$ kN; $F_{BD} = -200$ kN; $F_{CD} = -50$ kN; $F_{CE} = -120$ kN; $F_{DF} = -266.67$ kN; $F_{DE} = +83.33$ kN]

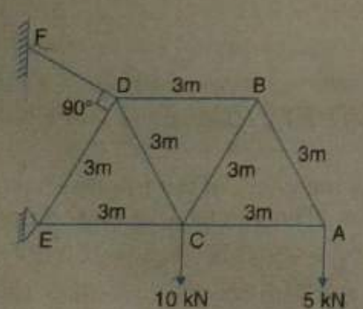


Fig. 4.21 (Prob. 4.5)

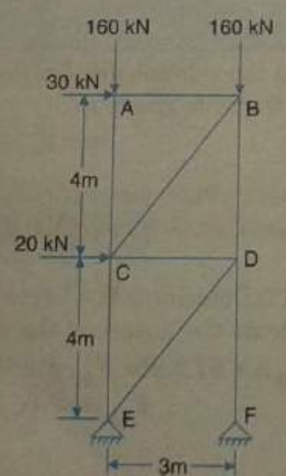


Fig. 4.22 (Prob. 4.6)

4.7 [**Ans.** $F_{AB} = -200$ kN; $F_{AD} = -100$ kN; $F_{BC} = F_{CF} = 0$; $F_{BD} = 100\sqrt{2}$ kN; $F_{BF} = -100\sqrt{2}$ kN; $F_{DE} = -100$ kN; $F_{DG} = 0$; $F_{EF} = +100$ kN; $F_{EH} = -100\sqrt{2}$ kN; $F_{EG} = +100\sqrt{2}$ kN; $F_{GH} = +100$ kN]

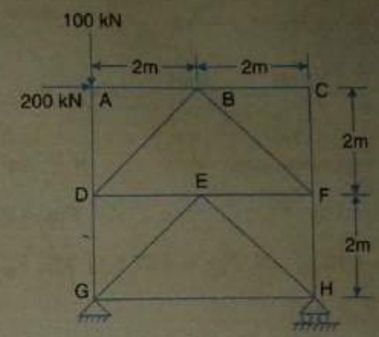


Fig. 4.23 (Prob. 4.7)

4.8 [**Ans.** $F_{BD} = -2\sqrt{2}$ kN; $F_{BA} = +3$ kN; $F_{AC} = +3\sqrt{2}$ kN; $F_{AD} = -3$ kN; $F_{DC} = -2$ kN; $F_{DF} = -5$ kN; $F_{CF} = -\sqrt{2}$ kN; $F_{CE} = +6$ kN; $F_{FE} = +1$ kN; $F_{FH} = -4$ kN; $F_{EH} = -\sqrt{2}$ kN; $F_{EG} = +5$ kN; $F_{GH} = +1$ kN]

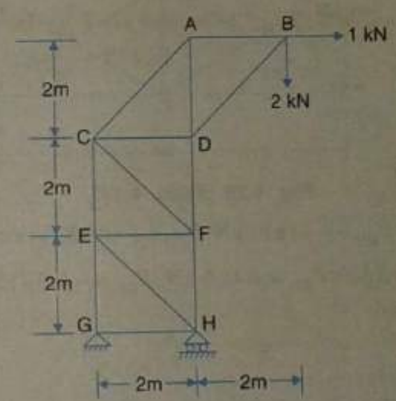


Fig. 4.24 (Prob. 4.8)

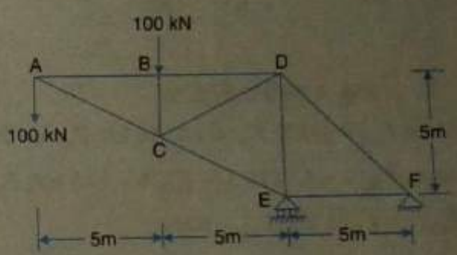


Fig. 4.25 (Prob. 4.9)

4.9 [**Ans.** $F_{AC} = -100\sqrt{5}$ kN; $F_{AB} = +200$ kN; $F_{BD} = +200$ kN; $F_{BC} = -100$ kN; $F_{CD} = +50\sqrt{5}$ kN; $F_{CE} = -150\sqrt{5}$ kN; $F_{DE} = +35.0$ kN; $F_{DF} = 300\sqrt{2}$ kN; $F_{EF} = -300$ kN]

4.10 [**Ans.** $F_{AB} = +5\sqrt{2}$ kN; $F_{AC} = -5$ kN; $F_{BC} = -5$ kN; $F_{BD} = 5$ kN; $F_{CD} = +15\sqrt{2}$ kN; $F_{CE} = -20$ kN; $F_{DE} = -15.0$ kN; $F_{DF} = +20\sqrt{2}$ kN; $F_{EH} = -15$ kN; $F_{EF} = -20$ kN; $F_{FG} = +30\sqrt{2}$ kN; $F_{FH} = +10\sqrt{2}$ kN]

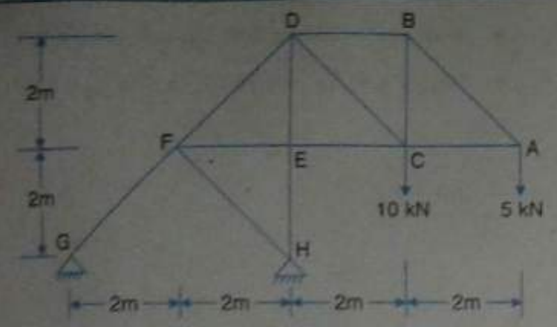


Fig. 4.26 (Prob. 4.10)

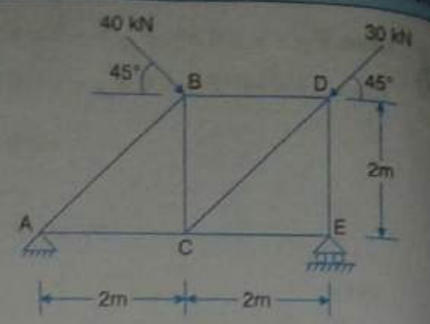


Fig. 4.27 (Prob. 4.11)

4.11. [**Ans.** $F_{AB} = -15$ kN; $F_{AC} = +12\sqrt{2}$ kN; $F_{BD} = -27.5\sqrt{2}$ kN; $F_{BC} = -12.5\sqrt{2}$ kN; $F_{CE} = 0$; $F_{CD} = +25$ kN; $F_{ED} = -27.5\sqrt{2}$ kN]

4.12. [**Ans.** $F_{AB} = -17.32$ kN; $F_{AC} = +5$ kN; $F_{BC} = -20$ kN; $F_{BD} = -17.32$ kN; kN; $F_{CD} = +20$ kN; $F_{CE} = -15$ kN; $F_{DE} = -30$ kN]

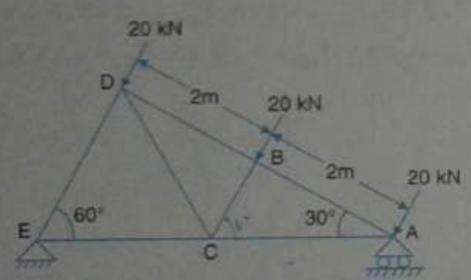


Fig. 4.28 (Prob. 4.12)

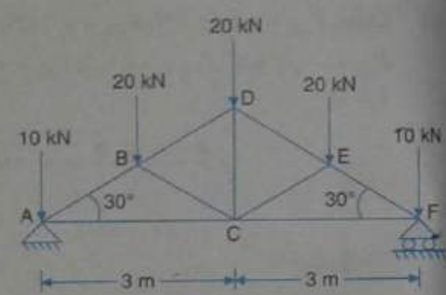


Fig. 4.29 (Prob. 4.13)

4.13. [**Ans.** $F_{AB} = 60$ kN; $F_{AC} = +51.96$ kN; $F_{BC} = -20$ kN; $F_{BD} = -40$ kN; kN; $F_{CD} = +40$ kN; symmetry]

4.14. [**Ans.** $F_{AC} = -4.5\sqrt{13}$ kN; $F_{AB} = +13.5$ kN; $F_{BC} = +6$ kN; $F_{BD} = +13.5$ kN; $F_{CD} = -0.5\sqrt{13}$ kN; $F_{CE} = -4\sqrt{10}$ kN; $F_{DE} = 8$ kN]

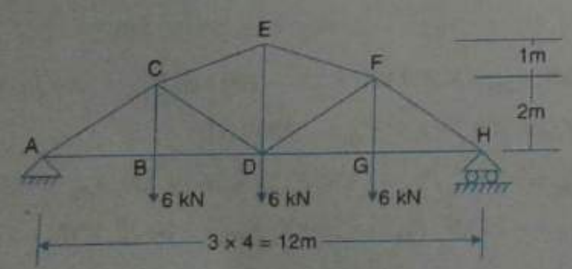


Fig. 4.30 (Prob. 4.14)

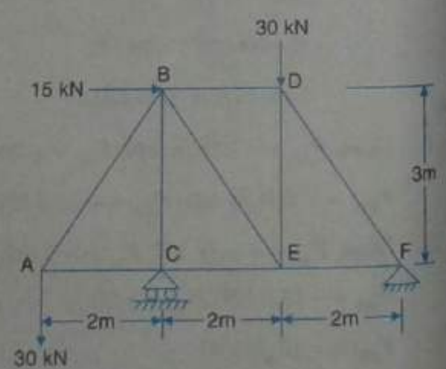


Fig. 4.31 (Prob. 4.15)

4.15. [**Ans.** $F_{AB} = +10\sqrt{13}$ kN; $F_{AC} = -20$ kN; $F_{CB} = -48.75$ kN; $F_{CE} = -20$ kN; $F_{CD} = -7.5$ kN; $F_{BE} = +6.25\sqrt{13}$ kN; $F_{DE} = 18.75$ kN; $F_{DF} = -3.75\sqrt{13}$ kN; $F_{FE} = -7.5$ kN]

4.16. [**Ans.** $F_{AB} = 16.91$ kN; $F_{AF} = +31.55$ kN; $F_{BF} = +23.91$ kN; $F_{BD} = -23.91$ kN; $F_{BC} = +40$ kN; $F_{CD} = -40$ kN; $F_{DE} = -63.1$ kN; $F_{DF} = +23.91$ kN; $F_{EF} = +31.55$ kN]

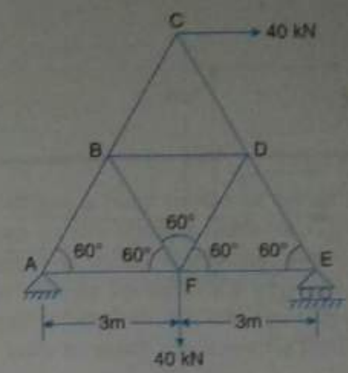


Fig. 4.32 (Prob. 4.16)

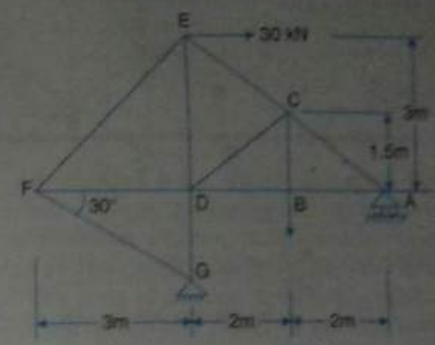


Fig. 4.33 (Prob. 4.17)

4.17. [**Ans.** $F_{AC} = -67.48$ kN; $F_{AB} = +53.99$ kN; $F_{BC} = +10$ kN; $F_{CD} = -8.33$ kN; $F_{CE} = -59.15$ kN; $F_{EF} = -24.5$ kN; $F_{ED} = +52.81$ kN; $F_{FD} = +47.21$ kN; $F_{FG} = -34.64$ kN; $F_{DG} = +47.32$ kN]

4.18. Find the force in the member FG of the triangular Howe truss shown in Fig. 4.34.
(**Hint.** Take section (A′) – (A′) and find force in FD. Then analyse joint F)
[**Ans.** + 28 kN]

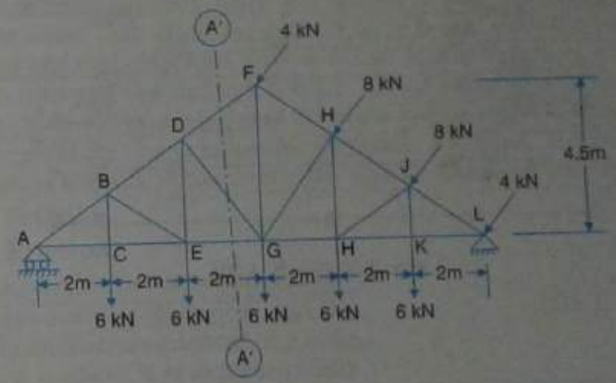


Fig. 4.34 (Prob. 4.18)

4.19. Determine the forces in the members AB, AC, DF and CE of the scissors truss shown in Fig. 4.35.
(**Hint.** Find reaction R_A and analyse joint A. Take section (A′) – (A′) and find forces in DF and CE).
[**Ans.** $F_{AB} = -6.25$ W ; $F_{AC} = 4.51$ W ; $F_{DF} = -3.75$ W ; $F_{CE} + 2.75$ W]

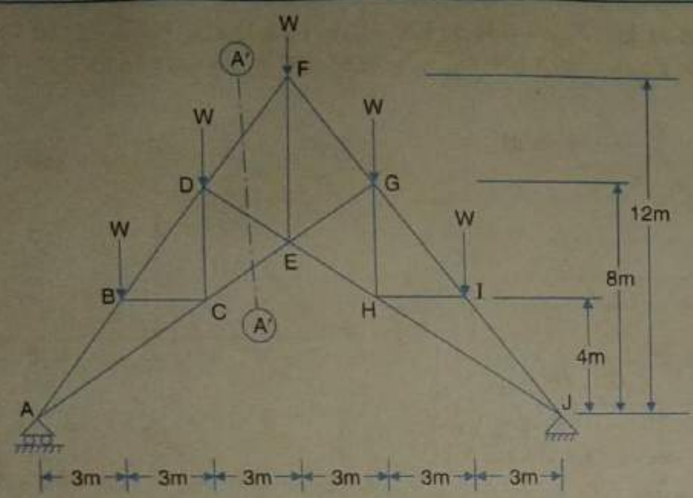


Fig. 4.35 (Prob. 4.19)

4.20. Find the force in member KL of the French truss shown in Fig. 4.36.
(**Hint.** Take section $(A')-(A')$ and find F_{LE} and F_{DE}. From joint D find F_{DL}. Then analyse joint L to get F_{KL}.) [**Ans.** + 41.96 kN]

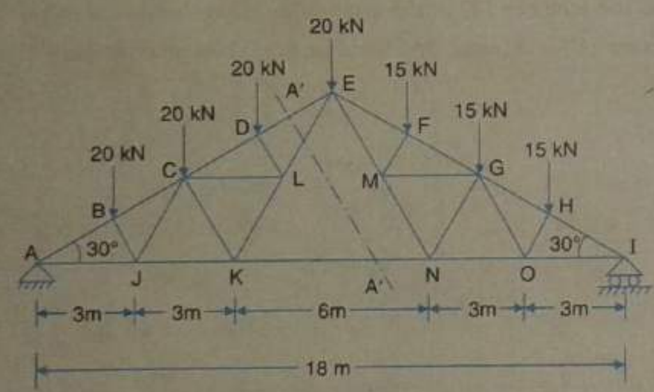


Fig. 4.36 (Prob. 4.20)

5 Friction

When a body moves or tends to move over another body, a force opposing the motion develops at the contact surfaces. This force which opposes the movement or the tendency of movement is called **frictional force** or simply **friction**. Friction is due to the resistance to motion offered by minutely projecting particles at the contact surfaces. In this chapter, the concepts related to friction are explained and the laws of friction presented. Application of these laws to many engineering problems including wedge and rope/belt are illustrated.

5.1. FRICTIONAL FORCE

As defined above frictional force is the resistance offered by minutely projecting particles of a body when it moves over another body. Frictional force has a remarkable property of adjusting itself in magnitude to the force producing or tending to produce the motion so that motion is prevented.

However, there is a limit beyond which the magnitude of this force cannot increase. If the applied force is more than this limit, there will be movement of one body over the other. This limiting value of frictional force when the motion is impending, is known as **Limiting Friction**. It may be noted that when the applied force is less than the limiting friction, the body remains at rest and such frictional force is called **Static Friction**, which will be having any value between zero and the limiting friction. If the value of applied force exceeds the limiting friction, the body starts moving over the other body and the frictional resistance experienced by the body while moving is known as **Dynamic Friction**. Dynamic friction is found to be less than limiting friction. Dynamic friction may be classified into the following two:

(*a*) **Sliding Friction**

(*b*) **Rolling Friction.**

Sliding friction is the friction experienced by a body when it slides over the other body and the rolling friction is the friction experienced by a body when it rolls over a surface. It is experimentally found that the magnitude of limiting friction bears a constant ratio to the normal reaction between the two surfaces and this ratio is called **Coefficient of Friction.** Thus, referring to Fig. 5.1 in which the block has a impending motion,

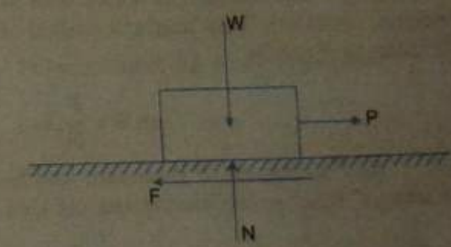


Fig. 5.1

$$\text{Coefficient of friction} = \frac{F}{N}$$

where F is limiting friction and N is normal reaction between the contact surfaces. Coefficient of friction is denoted by μ. Thus

$$\mu = \frac{F}{N}$$

5.2. LAWS OF FRICTION

The principles discussed in Art. 5.1 are mainly due to the experimental studies by Coulomb (1781) and by Morin (1831) on solids in dry conditions. These principles constitute the laws known as Coulomb's laws of friction/laws of dry friction/laws of solid friction. These laws are listed below:

(1) The force of friction always acts in a direction opposite to that in which body tends to move.

(2) Till the limiting value is reached, the magnitude of friction is exactly equal to the force which tends to move the body.

(3) The magnitude of the limiting friction bears a constant ratio to the normal reaction between the two surfaces of contact and this ratio is called coefficient of friction.

(4) The force of friction depends upon the roughness/smoothness of the surfaces.

(5) The force of friction is independent of the area of contact between the two surfaces.

(6) After the body starts moving, the dynamic friction comes into play, the magnitude of which is less than that of limiting friction and it bears a constant ratio with normal force. This ratio is called **coefficient of dynamic friction.**

5.3. ANGLE OF FRICTION, ANGLE OF REPOSE AND CONE OF FRICTION

The above three terms used in this chapter are defined and explained below.

Angle of Friction

Consider the block shown in Fig. 5.2 resting on a horizontal surface and subjected to horizontal pull P. Let F be the frictional force developed and N the normal reaction. Thus at contact surface the reactions are F and N. They can be combined graphically to get the reaction R which acts at angle θ to normal reaction. This angle θ, called the angle of friction is given by

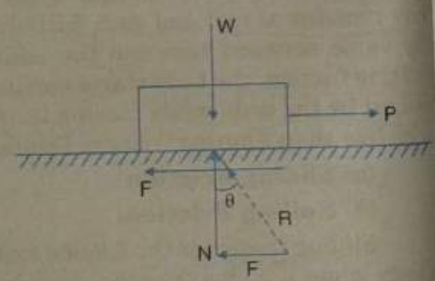


Fig. 5.2

$$\tan\theta = \frac{F}{N}$$

As P increases, F increases and hence θ also increases. θ can reach the maximum value α when F reaches limiting value. At this stage

$$\tan\alpha = \frac{F}{N} = \mu \qquad ...(5.1)$$

and this value of α is called **Angle of Limiting Friction**. Hence the angle of limiting friction may be defined as *the angle between the resultant reaction and the normal to the plane on which the motion of the body is impending.*

Angle of Repose

It is well known that when grains (food grain, soil, sand etc.) are heaped, there exists a limit for the inclination of the surface. Beyond this limiting inclinations the grains start rolling down. This limiting angle upto which the grains repose (slip) is called the angle of repose.

Consider the block of weight W resting on an inclined plane which makes an angle θ with the horizontal as shown in Fig. 5.3. When θ is small the block will rest on the plane. If θ is increased gradually a stage is reached at which the block start sliding down the plane. The angle θ for which motion is impending, is called the angle of repose. Thus *the maximum inclination of the plane on which a body, free from external forces, can repose (sleep) is called* **Angle of Repose.**

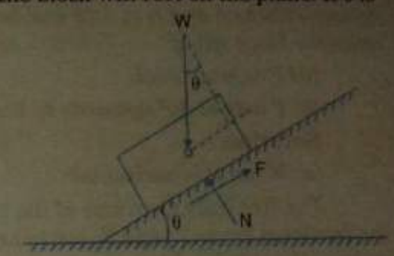


Fig. 5.3

Consider the equilibrium of the block shown in Fig. 5.3. Since the surface of contact is not smooth, not only normal reaction, but frictional force also develops. Since the body tends to slide downward, the frictional force will be up the plane.

ΣForces normal to the plane = 0, gives

$$N = W\cos\theta \qquad ...(5.2)$$

ΣForces parallel to the plane = 0, gives

$$F = W\sin\theta \qquad ...(5.3)$$

Dividing eqn. (5.3) by eqn. (5.2), we get

$$\tan\theta = \frac{F}{N} \qquad ...(5.4)$$

If ϕ is the value of θ when motion is impending, frictional force will be limiting friction and hence

$$\tan\phi = \frac{F}{N} = \mu = \tan\alpha$$

or

$$\phi = \alpha$$

Thus the value of angle of repose is same as the value of limiting angle of repose.

Cone of Friction

When a body is having impending motion in the direction of force P, the frictional force will be limiting friction and the resultant reaction R will make limiting angle α with the normal as shown in Fig. 5.4. If the body is having impending motion in some other direction, the resultant reaction makes limiting frictional angle α with the normal to that direction. Thus when the direction of force P is gradually changed through 360°, the resultant R generates a right circular cone with semi-central angle equal to α.

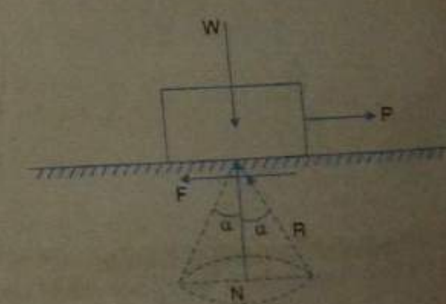


Fig. 5.4

If the resultant R is on the surface of this inverted right circular cone whose semi-central angle is limiting frictional angle (α) the motion of the body is impending. If the resultant is within this cone the body is stationary. This *inverted cone with semi-central angle equal to limiting frictional angle α, is called* **Cone of Friction.**

Example 5.1. *Block A weighing 1000 N rests over block B which weighs 2000 N as shown in Fig. 5.5(a). Block A is tied to wall with a horizontal string. If the coefficient of friction between blocks A and B is 0.25 and between B and floor is 1/3, what should be the value of P to move the block (B), if*

(a) P is horizontal.

(b) P acts at 30° upwards to horizontal ?

Solution.

(*a*) When P is horizontal:

The free body diagrams of the two blocks are shown in Fig. 5.5(*b*). It may be noted that the frictional forces F_1 and F_2 are to be marked in the opposite directions of impending relative motion. Considering block A,

$$\Sigma V = 0 \rightarrow$$

$$N_1 - 1000 = 0 \quad \text{or} \quad N_1 = 1000 \text{ N}$$

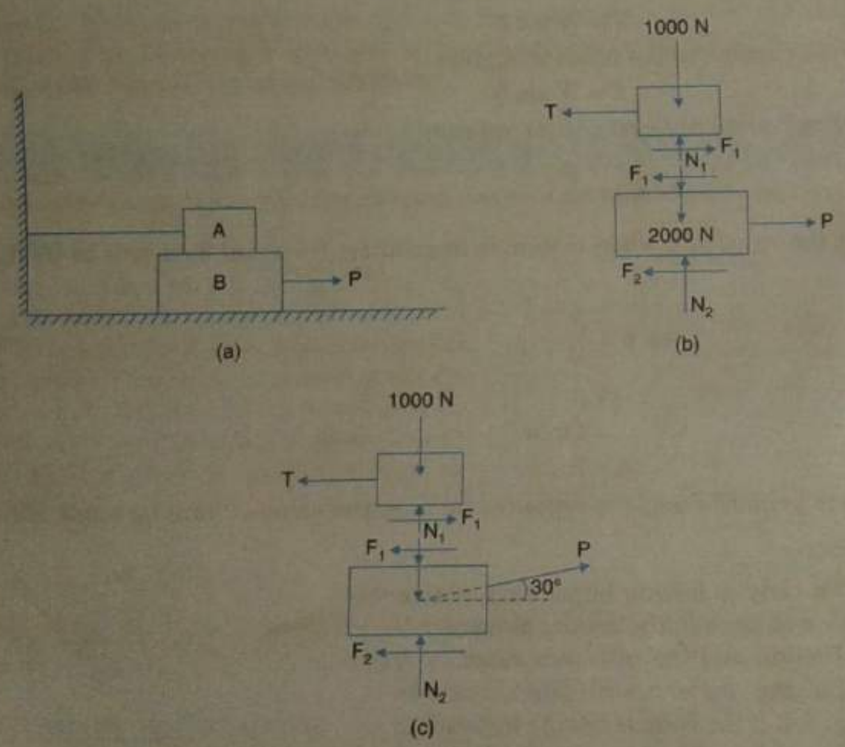


Fig. 5.5

Since F_1 is limiting friction,

$$\frac{F_1}{N_1} = \mu = 0.25$$

$$\therefore \quad F_1 = 0.25\, N_1 = 0.25 \times 1000 = 250 \text{ N}.$$

$$\Sigma H = 0 \rightarrow$$

$$F_1 - T = 0$$

$$\text{or} \quad T = F_1 = 250 \text{ N}$$

Consider equilibrium of block B.

$$\Sigma V = 0 \rightarrow$$

$$N_2 - 2000 - N_1 = 0$$

$$\text{or} \quad N_2 = 2000 + N_1 = 2000 + 1000 = 3000 \text{ N}$$

Since F_2 is limiting friction,

$$F_2 = \mu N_2 = \frac{1}{3} \times 3000 = 1000 \text{ N}$$

$$\Sigma H = 0 \rightarrow$$

$$P - F_1 - F_2 = 0$$

$$\therefore \quad P = F_1 + F_2 = 250 + 1000$$

$$\therefore \quad \mathbf{P = 1250 \text{ N}} \quad \textbf{Ans.}$$

(*b*) When P is inclined:

Free body diagram for this case is shown in Fig. 5.5(*c*).

As in the previous case here also,

$$N_1 = 1000 \text{ N}$$

and $F_1 = 250$ N. Consider the equilibrium of block B.

$$\Sigma V = 0 \rightarrow$$

$$N_2 - 2000 - N_1 + P \sin 30° = 0$$

$$N_2 + P \sin 30° = 2000 + N_1$$

$$N_2 + 0.5\, P = 2000 + 1000$$

$$\therefore \quad N_2 = 3000 - 0.5P$$

From law of friction,

$$F_2 = \frac{1}{3} N_2 = \frac{1}{3}(3000 - 0.5P)$$

$$= 1000 - \frac{0.5}{3} P$$

$$\Sigma H = 0 \rightarrow$$

$$P \cos 30° - F_1 - F_2 = 0$$

$$P \cos 30° - 250 - \left(1000 - \frac{0.5}{3} P\right) = 0$$

$$P\left(\cos 30° + \frac{0.5}{3}\right) = 1250$$

$$\text{or} \quad \mathbf{P = 1210.43 \text{ N}} \quad \textbf{Ans.}$$

Example 5.2. *What should be the value of θ in Fig. 5.6(a) that will make the motion of 900 N block down the plane to impend ? The coefficient of friction for all contact surfaces is 1/3.*

Solution. 9000 N block is on the verge of moving downward. Hence frictional forces F_1 and F_2 [Ref. Fig. 5.6 (*b*)] act up the plane on 900 N block. Free body diagram of the blocks is as shown in Fig. 5.6(*b*).

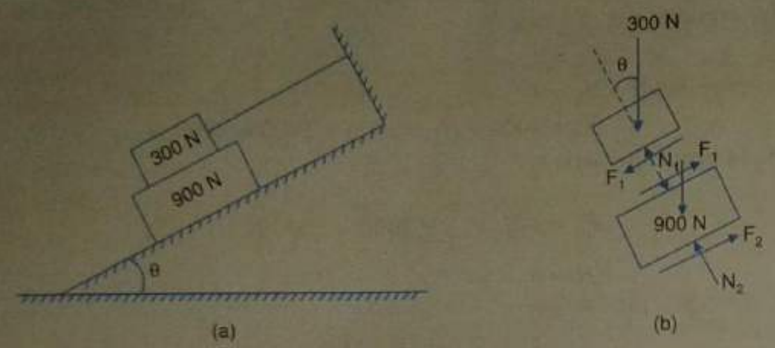


Fig. 5.6

Consider equilibrium of 300 N block.

Σ Forces normal to plane = 0, gives

$$N_1 - 300 \cos\theta = 0$$

or $$N_1 = 300 \cos\theta \quad ...(1)$$

From law of friction, $F_1 = 1/3\, N_1 = 1/3 \times 300 \cos\theta$

$$= 100 \cos\theta \quad ...(2)$$

Consider equilibrium of 900 N block:

ΣForces normal to the plane = 0, gives

$$N_2 - N_1 - 900 \cos\theta = 0$$

$$\therefore \quad N_2 = N_1 + 900 \cos\theta$$

$$= (300 + 900) \cos\theta$$

$$= 1200 \cos\theta \quad ...(3)$$

From law of friction,

$$F_2 = \frac{1}{3} N_2 = \frac{1}{3} \times 1200 \cos\theta = 400 \cos\theta \quad ...(4)$$

ΣForces parallel to plane = 0, gives

$$F_1 + F_2 - 900 \sin\theta = 0$$

$$100 \cos\theta + 400 \cos\theta = 900 \sin\theta$$

$$\therefore \quad \tan\theta = \frac{5}{9}$$

$$\theta = 29.05° \quad \textbf{Ans.}$$

Example 5.3. *A block weighing 500N just starts moving down a rough inclined plane when supported by a force of 200N acting parallel to the plane in upward direction. The same block is on the verge of moving up the plane when pulled by a force of 300N acting parallel to the plane. Find the inclination of the plane and the coefficient of friction between the inclined plane and the block.*

Solution. Free body diagram of the block when it just start moving down is shown in Fig. 5.7(*a*). The direction of frictional force is upward in this case since the direction of impending motion is downward. Since it is limiting case

$$\frac{F}{N} = \mu$$

ΣForce perpendicular to plane = 0, gives

$$N - 500 \cos\theta = 0 \quad \text{or} \quad N = 500 \cos\theta \quad ...(1)$$

From law of friction,

$$F_1 = \mu N = 500\, \mu \cos\theta \quad ...(2)$$

ΣForces parallel to plane = 0, gives

$$200 + F_1 - 500 \sin\theta = 0$$

i.e. $$200 = 500 \sin\theta - F_1$$

$$200 = 500 \sin\theta - 500\, \mu \cos\theta \quad ...(3)$$

When the block starts moving up the plane when 300 N pull is applied, frictional force F_2 is downward. Free body diagram for this case is shown in Fig. 5.7(*b*). In this case,

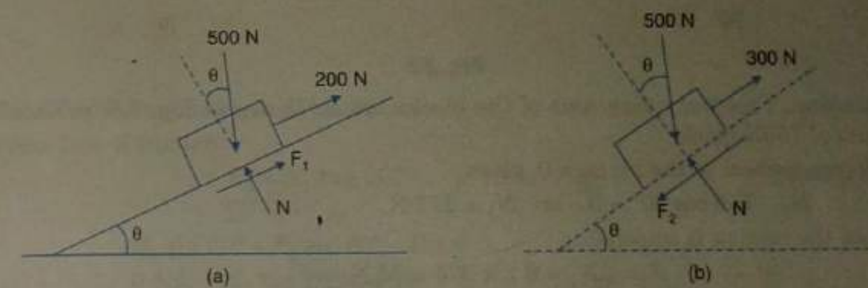


Fig. 5.7

ΣForces normal to plane = 0, gives

$$N - 500 \cos\theta = 0 \quad \text{or} \quad N = 500 \cos\theta \quad ...(4)$$

From the law of friction,

$$F_2 = \mu N = 500\, \mu \cos\theta \quad ...(5)$$

ΣForces parallel to plane = 0, gives

$$300 - F_2 - 500 \sin\theta = 0$$

or $$300 = F_2 + 500 \sin\theta$$

$$= 500\, \mu \cos\theta + 500 \sin\theta \quad ...(6)$$

Adding eqns. (3) and (6), we get

$$500 = 1000 \sin\theta$$

or $$\sin\theta = 0.5$$

$$\therefore \quad \theta = 30° \quad \textbf{Ans.}$$

Substituting it in eqn. (6), we get

$$300 = 500\, \mu \cos 30° + 500 \sin 30°$$

i.e. $$500\, \mu \cos 30° = 300 - 500 \times 0.5 = 50$$

or $$\mu = \frac{50}{500 \cos 30°} = 0.11547 \quad \textbf{Ans.}$$

Example 5.4. *What is the value of P in the system shown in Fig. 5.8(a) to cause the motion of 500N block to the right side ? Assume the pulley is smooth and the coefficient of friction between other contact surfaces is 0.20.*

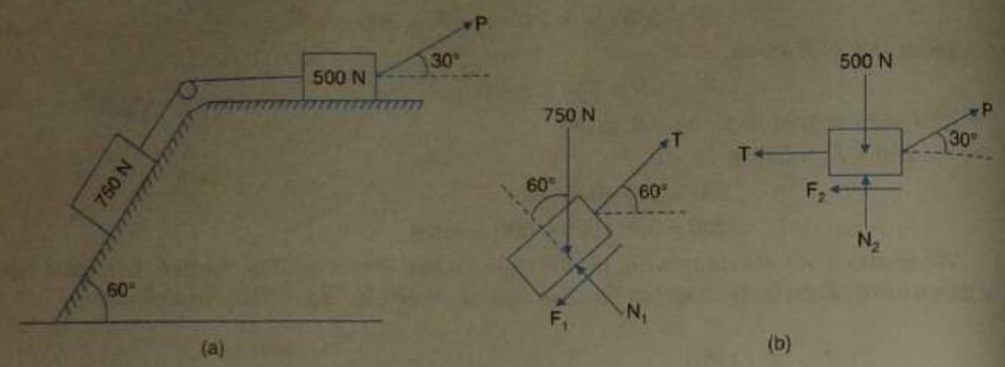


Fig. 5.8

Solution. Free body diagrams of the blocks are as shown in Fig. 5.8(*b*). Consider the equilibrium of 750N block.

ΣForces normal to the plane = 0, gives

$$N_1 - 750 \cos 60° = 0 \quad \text{or} \quad N_1 = 375 \text{ N}.$$

Since the motion is impending

$$F_1 = \mu N_1 = 0.2 \times 375 = 75 \text{ N}.$$

ΣForces parallel to the plane = 0, gives

$$T - F_1 - 750 \sin 60° = 0$$

$$\therefore \quad T = F_1 + 750 \sin 60° = 75 + 750 \sin 60° = 724.52\text{N}.$$

Consider the equilibrium of 500 N block.

$$\Sigma V = 0 \quad \rightarrow$$

$$N_2 - 500 + P \sin 30° = 0$$

or $$N_2 + 0.5P = 500$$

i.e., $$N_2 = 500 - 0.5P$$

From law of friction,

$$F_2 = 0.2\, N_2 = 0.2(500 - 0.5P) = 100 - 0.1P$$

$$\Sigma H = 0 \quad \rightarrow$$

$$P \cos 30° - T - F_2 = 0$$

$$P \cos 30° - 724.52 - (100 - 0.1P) = 0$$

$$\therefore \quad P(\cos 30° + 0.1) = 724.52 + 100 = 824.52$$

$$\therefore \quad \mathbf{P = 853.52 \text{ N} \quad Ans.}$$

Example 5.5. *Two blocks connected by a horizontal link AB are supported on two rough planes as shown in Fig. 5.9(a). The coefficient of friction between the block A and horizontal surface is 0.4. The limiting angle of friction between block B and inclined plane is 20°. What is the smallest weight W of the block A for which equilibrium of the system can exist, if the weight of block B is 5 kN ?*

Solution. Free body diagrams for block *A* and *B* are as shown in Fig. 5.9(*b*).

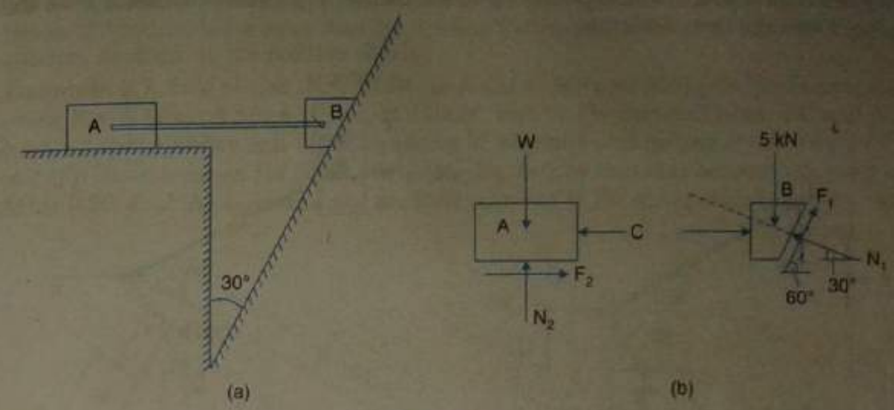


Fig. 5.9

Consider the equilibrium of block *B*.

From law of friction,

$$F_1 = N_1 \tan 20° \qquad [\text{since } \mu = \tan 20°]$$

$$\Sigma V = 0 \quad \rightarrow$$

$$N_1 \sin 30° + F_1 \sin 60° - 5 = 0$$

$$0.5N_1 + N_1 \tan 20° \sin 60° = 5$$

$$\therefore \quad N_1 = 6.133 \text{ kN}$$

$$\therefore \quad F_1 = 6.133 \tan 20° = 2.232 \text{ kN}$$

$$\Sigma H = 0 \quad \rightarrow$$

$$C + F_1 \cos 60° - N_1 \cos 30° = 0$$

$$\therefore \quad C = 6.133 \cos 30° - 2.232 \cos 60°$$

$$= 4.196 \text{ kN}.$$

Now consider the equilibrium of block *A*.

$$\Sigma H = 0 \quad \rightarrow$$

$$F_2 - C = 0$$

or $$F_2 = C = 4.196 \text{ kN}$$

From law of friction

$$F_2 = \mu N_2 = 0.4N_2$$

$$\therefore \quad N_2 = \frac{F_2}{0.4} = \frac{4.196}{0.4} = 10.49 \text{ kN}$$

$$\Sigma V = 0 \quad \rightarrow$$

$$W - N_2 = 0$$

$$\therefore \quad \mathbf{W = N_2 = 10.49 \text{ kN} \quad Ans.}$$

Example 5.6. *Two blocks A and B weighing 2000N each are to be held from slipping by the thrust of two weightless link reds each of which is connected by pin joints at one end to the blocks and interconnected by a pin joint at other end O_2 and subjected to a horizontal force P required to keep the blocks from slipping as shown in Fig. 5.10(a). Coefficient of friction is 0.25 for all contact surfaces.*

Solution. Let C_1 be the force in link AO and C_2 be the force in link OB. The free body diagrams of blocks A and B and the hinge O are shown in Fig. 5.10(*b*). Block A is on the verge of slipping. From the law of friction,

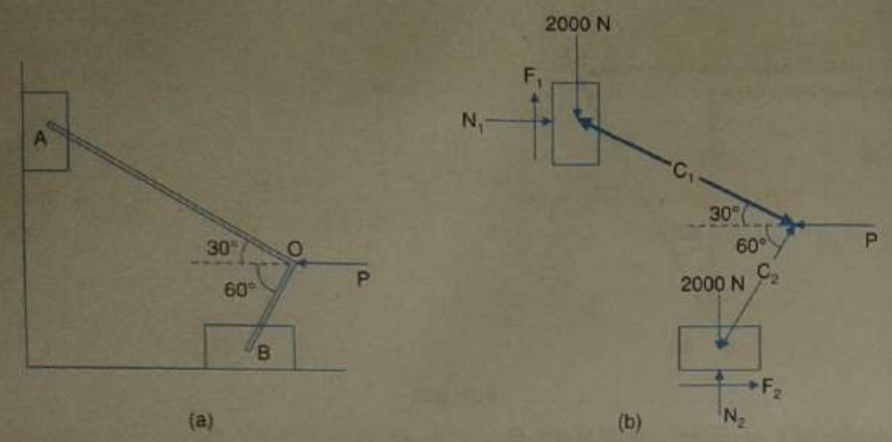


Fig. 5.10

$$F_1 = \mu N_1 = 0.25\, N_1$$

Consider the equilibrium of block A.

$$\Sigma H = 0 \quad \rightarrow$$

$$N_1 - C_1 \cos 30° = 0, \text{ or } \quad N_1 = C_1 \cos 30°$$

$$\Sigma V = 0 \quad \rightarrow$$

$$F_1 + C_1 \sin 30° - 2000 = 0$$

$$0.25\, N_1 + 0.5\, C_1 = 2000$$

$$0.25\, C_1 \cos 30° + 0.5\, C_1 = 2000$$

$$\therefore \qquad C_1 = \frac{2000}{0.25 \cos 30° + 0.5} = 2791.32 \text{ N.}$$

Applying Lami's theorem to the equilibrium of the joint O, we get

$$\frac{P}{\sin 90°} = \frac{C_2}{\sin 150°} = \frac{C_1}{\sin 120°}$$

$\therefore$ **P = 3223.14 N Ans.**

and **C = 1611.57 N Ans.**

The above solution holds good, provided block B is not slipping. To verify this, consider the equilibrium of block B.

$$\Sigma H = 0 \quad \rightarrow$$

$$F_2 - C_2 \cos 60° = 0$$

$$\therefore \qquad F_2 = 1611.57 \cos 60° = 805.79 \text{ N}$$

$$\Sigma V = 0 \quad \rightarrow$$

$$N_2 - 2000 - C_2 \sin 60° = 0$$

$$\therefore \qquad N_2 = 2000 + 1611.57 \sin 60° = 3393.60 \text{ N}$$

Hence limiting friction $= \mu N_2 = 0.25 \times 3393.6 = 842.92$ N.

Since the actual frictional force F_2 developed is less than the limiting frictional force, block B is stationary and hence **P = 3223.14 N is correct answer.**

[**Note:** If F_2 calculated is more than the limiting friction μN_2, there is no possibility of maintaining equilibrium condition in the position shown.]

Example 5.7. *Two planes AC and BC inclined at 60° and 30° to the horizontal meet at C as shown in Fig. 5.11(a). A block of weight 1000N rests on the inclined plane BC and is tied by a rope passing over a pulley to a block weighing W newtons and resting on the plane AC. If the coefficient of friction between the block and plane BC is 0.28 and that between the block and the plane AC is 0.20, find the least and the greatest value of W for the equilibrium of the system.*

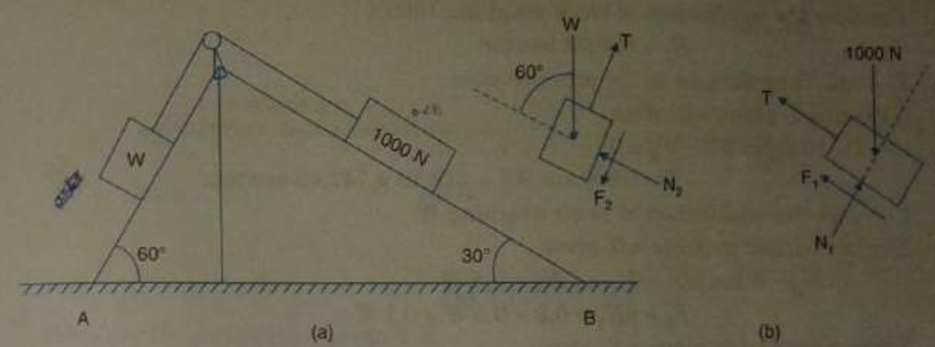


Fig. 5.11

Solution. (*a*) **For the least value of *W*:**

In this case motion of 1000 N block is impending downward. For this case free body diagram of blocks are as shown in Fig. 5.11(*b*).

Consider the equilibrium of 1000 N block:

ΣForces normal to the plane = 0, gives

$$N_1 - 1000 \cos 30° = 0 \quad \therefore \quad N_1 = 866.03 \text{ newton.}$$

From the law of friction,

$$F_1 = \mu N_1 = 0.28 \times 866.03 = 242.49 \text{ newton.}$$

ΣForces parallel to the plane = 0

$$T + F_1 - 1000 \sin 30° = 0$$

$$\therefore \qquad T = -242.49 + 500 = 257.51 \text{ newton.}$$

Now consider the equilibrium of block weighing W:

ΣForces normal to the plane = 0, gives

$$N_2 - W \cos 60° = 0 \quad \therefore \quad N_2 = 0.5\, W$$

$$\therefore \qquad F_2 = \mu N_2 = 0.2 \times 0.5\, W = 0.1\, W$$

ΣForces parallel to the plane = 0, gives

$$T - F_2 - W \sin 60° = 0$$

$$257.51 - 0.1\, W - W \sin 60° = 0$$

$$\mathbf{W} = \frac{\mathbf{257.51}}{\mathbf{(0.1 + \sin 60°)}} = \mathbf{266.57} \quad \textbf{Ans.}$$

(*b*) **For greatest value of *W*:**

In this case 1000N block will be on the verge of moving up the plane. The free body diagram for this case is as shown in Fig. 5.11(*c*).

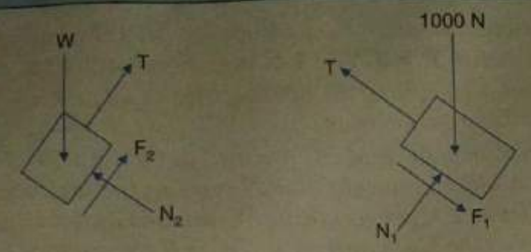


Fig. 5.11(c)

Consider the equilibrium of block weighing 1000N

$$N_1 = 866.03 \text{ newton}$$

and $F_1 = 242.49$ newton, as in the previous case.

ΣParallel to plane = 0, gives

$$T - 1000 \sin 30° - F_1 = 0$$

$$\therefore \quad T = 1000 \sin 30° + 242.49 = 742.49 \text{ newton.}$$

Consider the equilibrium of block weighing W.

ΣForces normal to plane = 0, gives

$$N_2 - W \cos 60° = 0 \quad \therefore \quad N_2 = 0.5\,W$$

$$F_2 = \mu N_2 = 0.2 \times 0.5\,W = 0.1\,W$$

ΣForces parallel to plane = 0, gives

$$T + F_2 - W \sin 60° = 0$$

$$742.49 + 0.1W - W \sin 60° = 0$$

$$\mathbf{W = \frac{742.49}{(0.1 + \sin 60°)} = 969.28 \text{ newton} \quad Ans.}$$

NP

Example 5.8. *Two blocks A and B each weighing 1500N are connected by a unifo horizontal bar weighing 1000N. If the angle of limiting friction for all contact surfaces is 1 find the force P directed parallel to the 60° inclined plane that will cause motion impending the right [Ref. Fig. 5.12 (a)].*

Solution. Free body diagrams of block A, beam AB and block B are as shown Figs. 5.12(*b*), (*c*) and (*d*) respectively.

From vertical equilibrium condition for AB, it may be found that 500N force is tra ferred at A and B, which may be directly added to self weights of the blocks.

Now consider equilibrium of block A.

$$\Sigma V = 0 \quad \rightarrow$$

$$N_A \cos 30° + F_A \sin 30° - (1500 + 500) = 0$$

But from law of friction, $F_A = N_A \tan 15°$

$$\therefore \quad N_A (\cos 30° + \tan 15° \sin 30°) = 2000$$

$$\therefore \quad N_A = 2000 \text{ newton}$$

$$\therefore \quad F_A = 2000 \tan 15° = 535.90 \text{ newton.}$$

$$\Sigma H = 0 \quad \rightarrow$$

$$-C + N_A \sin 30° - F_A \cos 30° = 0$$

$$\therefore \quad C = N_A \sin 30° - F_A \cos 30°$$

$$= 2000 \sin 30° - 535.90 \cos 30° = 535.90 \text{ newton.}$$

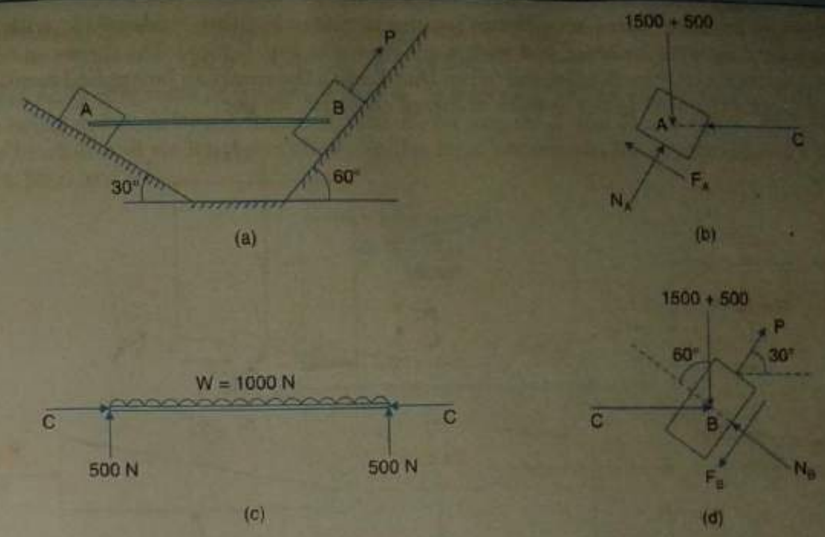


Fig. 5.12

Consider the equilibrium of block B.

ΣForces normal to the inclined plane = 0, gives

$$N_B - 2000 \cos 60° - C \cos 30° = 0$$

$$N_B = 2000 \cos 60° + 535.90 \cos 30° = 1464.10 \text{ newton}$$

$$\therefore \quad F_B = N_B \tan 15° = 1464.90 \tan 15°$$

$$= 392.30 \text{ newton.}$$

ΣForces parallel to the inclined plane = 0, gives

$$P - F_B - 2000 \sin 60° + C \sin 30° = 0$$

$$P = 392.30 + 2000 \sin 60° + 535.90 \sin 30°$$

$$\mathbf{P = 1856.40 \quad Ans.}$$

5.4. WEDGES

Wedges are small pieces of materials with two of their opposite surfaces not parallel. They are used to slightly lift heavy blocks, machinery, precast beams etc. for final alignments or to make place for inserting lifting devices. The weight of the wedge is very small compared to the weight lifted. Hence, in all problems, the self weight of wedge is neglected.

In the analysis, instead of treating normal reaction and frictional force independently, it is advantageous to consider their resultant. If F is limiting friction, then resultant makes limiting frictional angle α with the normal. Its direction should be marked correctly. For this, it should be noted that the tangential component R is the frictional force and it always acts opposite to impending motion. The analysis procedure is illustrated with two examples below:

Example 5.9. *Determine the minimum force required to move the wedge shown in Fig. 5.13(a). The angle of friction for all contact surfaces is 15°.*

Solution. As wedge is driven, it moves towards left and block moves upwards. Force P required to move the system is minimum when the motion is impending and hence at this

stage limiting frictional force acts. Hence resultant makes limiting angle of 15° with normal. The free body diagrams for block and wedge are shown in Fig. 5.13(*b*). The forces on block and wedge are redrawn in Figs. 5.13(*c*) and (*d*) so that Lami's theorem can be applied conveniently. Applying Lami's theorem to the system of forces on block, we get

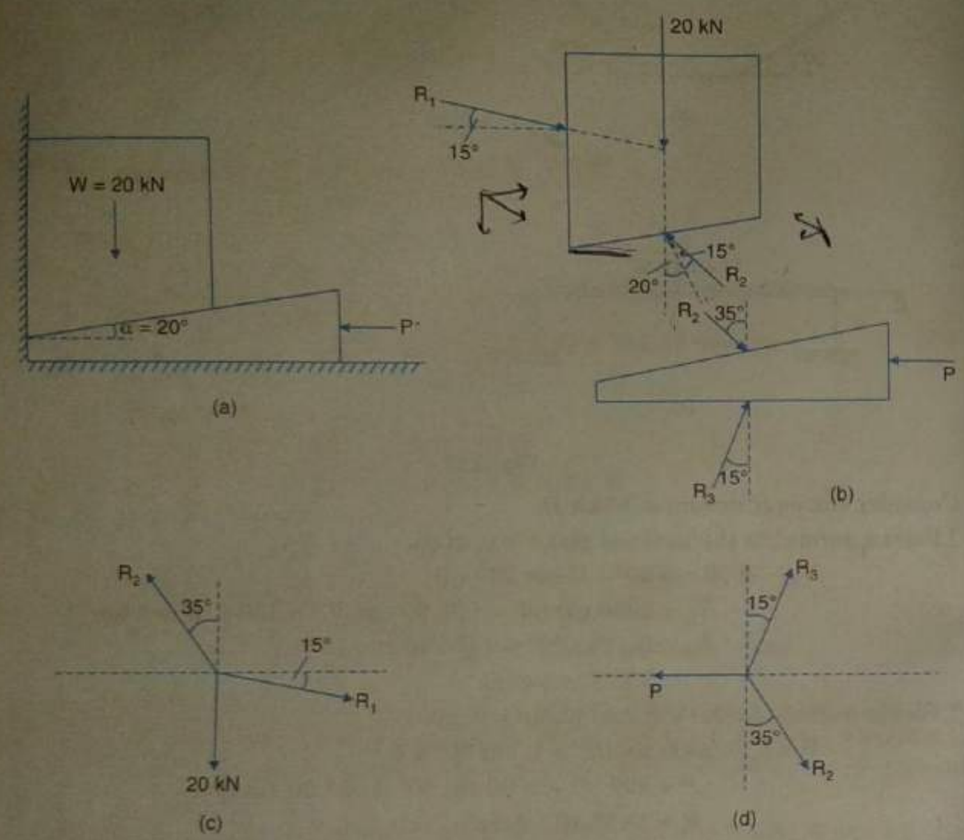


Fig. 5.13

$$\frac{R_1}{\sin 145°} = \frac{R_2}{\sin 75°} = \frac{20}{\sin 140°}$$

$\therefore$ $R_1 = 17.847$ kN

and $R_2 = 30.047$ kN.

Applying Lami's theorem to system of forces on the wedge, we get,

$$\frac{P}{\sin 130°} = \frac{R_2}{\sin 105°}$$

$\therefore$ **P = 23.835 kN Ans.**

Example 5.10. *The block C, weighing 160 kN is to be raised by means of driving wedges A and B as shown in Fig. 5.14(a). Find the value of force P for impending motion of the block upwards, if coefficient of friction is 0.25 for all contact surfaces. Self weight of wedges may be neglected.*

Solution. Let ϕ be the angle of limiting friction. Then

$$\tan \phi = 0.25 \quad \therefore \quad \phi = 14.036°$$

Free body diagrams of wedges *A* and *B*, and block *C* are as shown in Fig. 5.14(*b*). The problem being symmetric, the forces R_1 and R_2 on wedges *A* and *B* are equal. The system of forces on block *C* and wedge *A* are shown in the form convenient for applying Lami's theorem in Figs. 5.14(*c*) and (*d*).

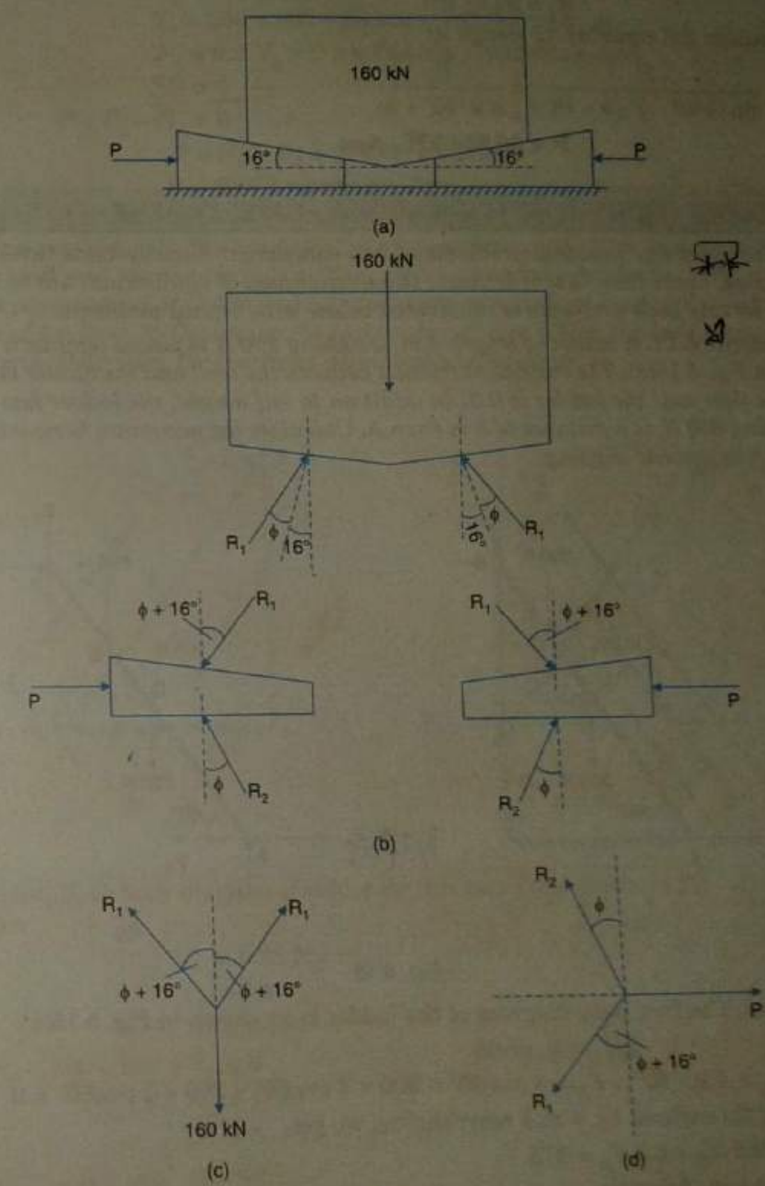


Fig. 5.14

Consider the equilibrium of block C:

$$\frac{R_1}{\sin(180° - 16° - \phi)} = \frac{160}{\sin 2(\phi + 16°)}$$

i.e.
$$\frac{R_1}{\sin 149.96°} = \frac{160}{\sin 60.072°}, \text{ since } \phi = 14.036°$$

$\therefore$ $R_1 = 92.41$ kN.

Consider the equilibrium wedge A:

$$\frac{P}{\sin(180° - \phi - \phi - 16°)} = \frac{R_1}{\sin(90° + \phi)}$$

$\therefore$ **P = 66.256 kN Ans.**

5.5. PROBLEMS INVOLVING NON-CONCURRENT FORCE SYSTEMS

There are many practical problems of non-concurrent force systems involving friction. In these cases, apart from law of friction, three equations of equilibrium are to be used. The method of solving such problems is illustrated below with typical problems.

Example 5.11. *A ladder of length 4 m, weighing 200 N is placed against a vertical wall as shown in Fig. 5.15(a). The coefficient friction between the wall and the ladder is 0.2 and that between the floor and the ladder is 0.3. In addition to self weight, the ladder has to support a man weighing 600 N at a distance of 3 m from A. Calculate the minimum horizontal force to be applied at A to prevent slipping.*

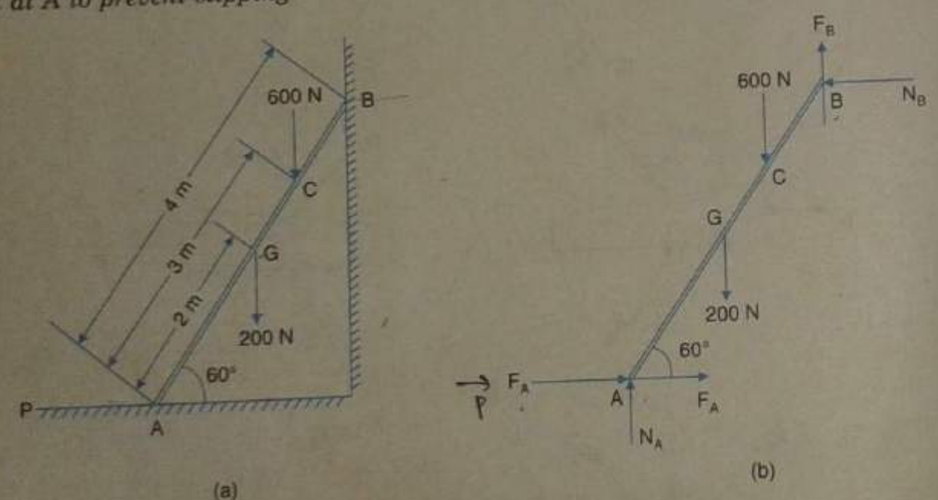


Fig. 5.15

Solution. The free body diagram of the ladder is as shown in Fig. 5.15(*b*).

$\Sigma M_A = 0$, gives

$$N_B \times 4 \sin 60° + F_B \times 4 \cos 60° - 600 \times 3 \cos 60° - 200 \times 2 \cos 60° = 0$$

Dividing throughout by 4 and rearranging, we get,

$$0.866\, N_B + 0.5\, F_B = 275 \quad ...(1)$$

From the law of friction,

$$F_B = 0.2\, N_B \quad ...(2)$$

Substituting this in eqn. (1), we get

$$N_B(0.866 + 0.5 \times 0.2) = 275$$

$\therefore$ $N_B = 284.68$ newton ...(3)

$\therefore$ $F_B = 0.2 \times 284.68 = 56.934$ newton ...(4)

$\Sigma V = 0 \rightarrow$

$$N_A - 200 - 600 + F_B = 0$$

$\therefore$ $N_A = 200 + 600 - 56.934 = 743.066$ newton

$\therefore$ $F_A = 0.3\, N_A = 0.3 \times 743.066 = 222.92$ newton.

$\Sigma H = 0 \rightarrow$

$$P + F_A - N_B = 0$$

$\therefore$ $P = N_B - F_A = 284.68 - 222.92$

i.e. **P = 61.76 newton. Ans.**

Example 5.12. *The ladder shown in Fig. 5.16(a) is 6m long and is supported by a horizontal floor and vertical wall. The coefficient of friction between the floor and the ladder is 0.25 and between wall and the ladder is 0.4. The self weight of the ladder is 200N and may be considered as concentrated at G. The ladder also supports a vertical load of 900N at C which is at a distance of 1m from B. Determine the least value of α at which the ladder may be placed without slipping. Determine the reactions developed at that stage.*

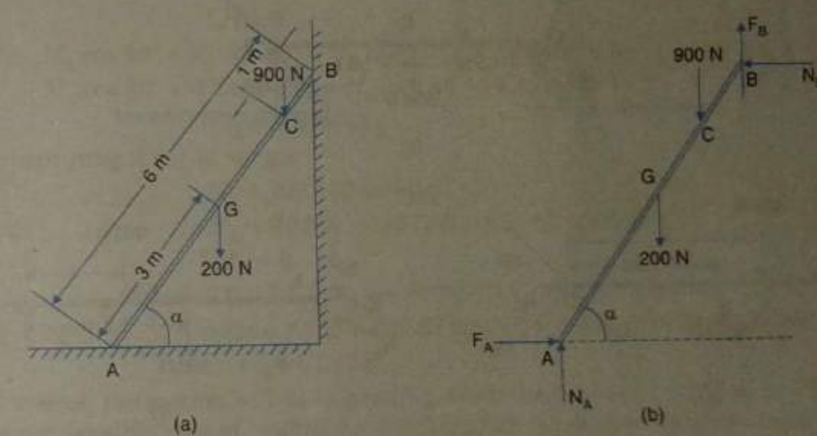


Fig. 5.16

Solution. Free body diagram of ladder for this case is as shown in Fig. 5.16(*b*). From the law of friction,

$$F_A = 0.25\, N_A \quad ...(1)$$

and
$$F_B = 0.4\, N_B \quad ...(2)$$

$\Sigma V = 0 \rightarrow$

$$N_A - 200 - 900 + F_B = 0$$

i.e.
$$N_A + 0.4\, N_B = 1100 \quad ...(3)$$

$\Sigma H = 0 \rightarrow$

$$F_A - N_B = 0$$

i.e.
$$0.25\, N_A = N_B \quad ...(4)$$

From eqns. (3) and (4), we get

$$N_A + 0.4 \times 0.25\, N_A = 1100$$

$\therefore$ $\mathbf{N_A = 1000}$ **newton Ans.**

$\therefore$ $\mathbf{F_A} = 0.25\, N_A = 0.25 \times 1000 = \mathbf{250}$ **newton Ans.**

From eqn. (4), $\mathbf{N_B} = 0.25\, N_A = 0.25 \times 1000 = \mathbf{250}$ **newton Ans.**

Substituting it in eqn. (2), we get

$\mathbf{F_B} = 0.4 \times 250 = \mathbf{100}$ **newton Ans.**

$$\Sigma M_A = 0 \quad \rightarrow$$

$$N_B \times 6 \sin\alpha + F_B \times 6 \cos\alpha - 200 \times 3 \cos\alpha - 900 \times 5 \cos\alpha = 0$$

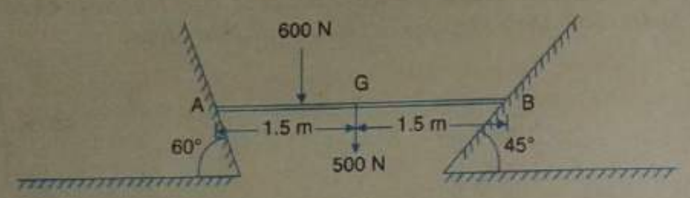
i.e., $$250 \times 6 \sin\alpha + 100 \times 6 \cos\alpha - 600 \cos\alpha - 4500 \cos\alpha = 0.$$

i.e., $$1500 \sin\alpha = (-600 + 600 + 4500) \cos\alpha$$

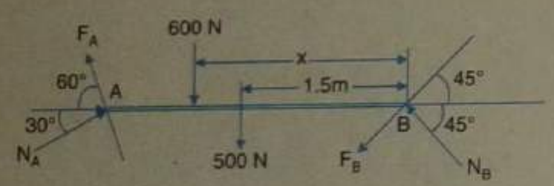
$\therefore$ $$\tan\alpha = 3$$

Hence $\boldsymbol{\alpha = 71.563°}$ **Ans.**

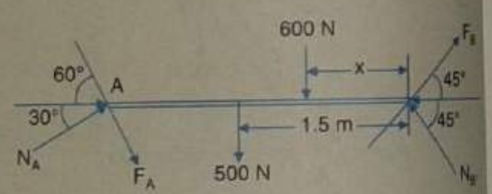
Example 5.13. *A horizontal bar AB of length 3m and weighing 500N is lying in a trou as shown in Fig. 5.17(a). Find how close to end A and B a load of 600N can be placed safely, coefficient of friction between contact surfaces is 0.2.*



(a)



(b)



(c)

Fig. 5.17

Solution. When the load is close to end A, the end A will slip down and end B will sli up. For this impending motion the free body diagram is as shown in Fig. 5.17(b). From law friction

$$F_A = 0.2\, N_A \quad ...(1)$$

and $$F_B = 0.2\, N_B \quad ...(2)$$

$$\Sigma V = 0 \quad \rightarrow$$

$$N_A \sin 30° + F_A \sin 60° + N_B \sin 45° - F_B \sin 45° = 1100$$

$$N_A (\sin 30° + 0.2 \sin 60°) + N_B (\sin 45° - 0.2 \sin 45°) = 1100$$

$$0.6732\, N_A + 0.5657\, N_B = 1100 \quad ...(3)$$

$$\Sigma H = 0 \quad \rightarrow$$

$$N_A \cos 30° - F_A \cos 60° - N_B \cos 45° - F_B \cos 45° = 0$$

$$N_A (\cos 30° - 0.2 \cos 60°) = N_B (\cos 45° + 0.2 \cos 45°)$$

$\therefore$ $$N_A = 1.1077\, N_B \quad ...(4)$$

Substituting this value of N_A in eqn. (3), we get

$$0.6732 \times 1.1077\, N_B + 0.5657\, N_B = 1100$$

$\therefore$ $$N_B = 838.79 \text{ newton}$$

$\therefore$ $$N_A = 1.1077 \times 839.79 = 929.13 \text{ newton}$$

$$\Sigma M_B = 0 \quad \rightarrow$$

$$600x + 500 \times 1.5 - F_A \sin 60° \times 3 - N_A \sin 30° \times 3 = 0$$

$$600x = -750 + 0.2 \sin 60° \times 929.13 \times 3 + 929.13 \sin 30° \times 3$$

$\therefore$ $\mathbf{x = 1.877}$ **m from B. Ans**

When the load is close to end B, the end B may slide down and end A up. Let x be the distance of load from the end B, when motion is impending. For this situation the free body diagram is as shown in Fig. 5.17(a).

$$\Sigma V = 0 \quad \rightarrow$$

$$N_A \sin 30° - F_A \sin 60° + N_B \sin 45° + F_B \sin 45° = 1100$$

$$N_A (\sin 30° - 0.2 \sin 60°) + N_B (\sin 45° + 0.2 \sin 45°) = 1100$$

$$0.3268\, N_A + 0.8485\, N_B = 1100 \quad ...(5)$$

$$\Sigma H = 0 \quad \rightarrow$$

$$N_A \cos 30° + F_A \cos 60° - N_B \cos 45° + F_B \cos 45° = 0$$

$$N_A (\cos 30° + 0.2 \cos 60°) = N_B (\cos 45° - 0.2 \cos 45°)$$

$\therefore$ $$N_A = 0.5856\, N_B \quad ...(6)$$

Substituting it in (5), we get

$$N_B = 1057.82 \text{ newton.}$$

and hence, $$N_A = 0.5856 \times 1057.82 = 619.67 \text{ newton.}$$

$$\Sigma M_B = 0 \quad \rightarrow$$

$$600x + 500 \times 1.5 - N_A \sin 30° \times 3 + F_A \sin 60° \times 3 = 0$$

$$600x = -750 + 619.67 \sin 30° \times 3 - 0.2 \times 6.19.67 \sin 30° \times 3$$

$\therefore$ $$x = -0.237 \text{ m.}$$

It means, the motion will be impending when the load is at 0.237 m to the right of B, which is not possible case of loading for the bar. Hence load can be placed even on point B safely.

Thus 600 N load can be placed anywhere between B and a point 1.85 m from B.

5.6. ROPE FRICTION

The transmission of power by means of belt or rope drives is possible because of friction, which exists between the wheels and the belt. Similarly band brakes stop the rotating discs because of friction between the belts and the discs. All along the contact surface the frictional resistance develops. Hence the tension in the rope is more on the side it is pulled and is less on the other side. Accordingly, the two sides of the rope may be called as tight side and slack side. The relationship between the forces on slack side and tight side can be derived as explained below:

Figure 5.18(*a*) shows a load W being pulled up by a force P over a fixed drum. Let the force on slack side be T_1 and on tight side be T_2 [Ref. Fig. 5.18(*b*)]. T_2 is more than T_1 because frictional force develops between drum and the rope as shown in Fig. 5.18(*c*). Let θ be the angle of contact between rope and the drum. Now consider an elemental length of rope as shown in Fig. 5.18(*d*).

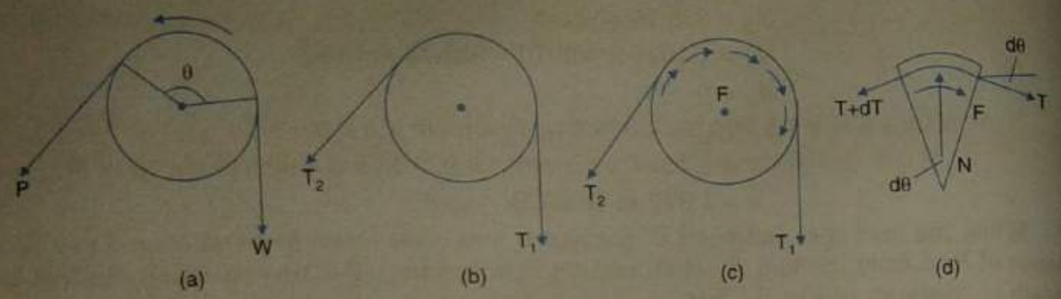


Fig. 5.18

Let T be the force on slack side and $T + dT$ on tight side. There will be normal reaction N on the rope in the radial direction and frictional force $F = \mu N$ in the tangential direction. Then

ΣForces in radial direction = 0, gives

$$N - T\sin\frac{d\theta}{2} - (T + dT)\sin\frac{d\theta}{2} = 0$$

Since $d\theta$ is very small angle, $\sin\frac{d\theta}{2} = \frac{d\theta}{2}$.

$$\therefore \quad N - T\frac{d\theta}{2} - (T + dT)\frac{d\theta}{2} = 0$$

i.e.,

$$N = \left(T + \frac{dT}{2}\right)d\theta \quad \text{...(1)}$$

From the law of friction,

$$F = \mu N = \mu\left(T + \frac{dT}{2}\right)d\theta \quad \text{...(2)}$$

where μ is coefficient of friction.

ΣForces in tangential direction = 0, gives

$$(T + dT)\cos\frac{d\theta}{2} = F + T\cos\frac{d\theta}{2}$$

Since $d\theta$ is very small angle, $\cos\frac{d\theta}{2} = 1$

$$\therefore \quad T + dT = F + T \quad \text{or} \quad dT = F \quad \text{...(3)}$$

From eqns. (2) and (3), we get

$$dT = \mu\left(T + \frac{dT}{2}\right)d\theta$$

Neglecting small quantity of higher order, we get

$$dT = \mu T\, d\theta$$

or

$$\frac{dT}{T} = \mu\, d\theta$$

Integrating both sides over 0 to θ, we get

$$\int_{T_1}^{T_2}\frac{dT}{T} = \int_0^{\theta}\mu\, d\theta$$

$$\Big[\log T\Big]_{T_1}^{T_2} = \mu\Big[\theta\Big]_0^{\theta}$$

$$\therefore \quad \log\frac{T_2}{T_1} = \mu\theta$$

i.e.,

$$\frac{T_2}{T_1} = e^{\mu\theta}$$

or

$$\boxed{T_2 = T_1\, e^{\mu\theta}} \quad \text{...(4)}$$

[**Note:** θ should be in radians.]

Example 5.14. *A rope making $1\frac{1}{4}$ turns around a stationary horizontal drum is used to support a weight W [Fig. 5.19(a)]. If the coefficient of friction is 0.3 what range of weight can be supported by exerting a 600N force at the other end of the rope ?*

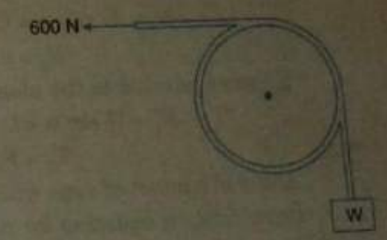


Fig. 5.19

Solution. Angle of contact $\theta = 1.25 \times 2\pi = 2.5\pi$

Case 1. Let the impending motion of the weight be downward. Then

$$T_1 = 600 \text{ newton} \quad \text{and } T_2 = W$$

∴ From law of rope friction,

$$W = 600\, e^{\mu 2.5\pi} = 600\, e^{0.3\times 2.5\pi} = 600\, e^{0.75\pi}$$

$$= 6330.43 \text{ newton.}$$

Case (2). Let the impending motion of weight be upwards. Then

$$T_1 = W \text{ and } T_2 = 600 \text{ newton}$$

$$\therefore \quad T_2 = T_1\, e^{\mu\theta} \text{ gives}$$

$$600 = W\, e^{0.75\pi}$$

$$\therefore \quad W = 56.87 \text{ newton.}$$

Thus a 600 N force can support a range of loads between 56.87 N to 6330.43 N weight on the other side of the drum.

Example 5.15. *In Fig. 5.20(a), the coefficient of friction between the rope and the fixed drum is 0.2 and between other surfaces of contact is 0.3. Determine the minimum weight W to prevent downward motion of the 1000N block.*

Solution. Since 1000 N block is on the verge of sliding down, the rope connecting it is tight side and the rope connecting W is the slack side. Free body diagrams for W and 1000 N block are as shown in Fig. 5.20(*b*).

Now, $\tan \alpha = 3/4 \quad \therefore \quad \alpha = 36.87°$

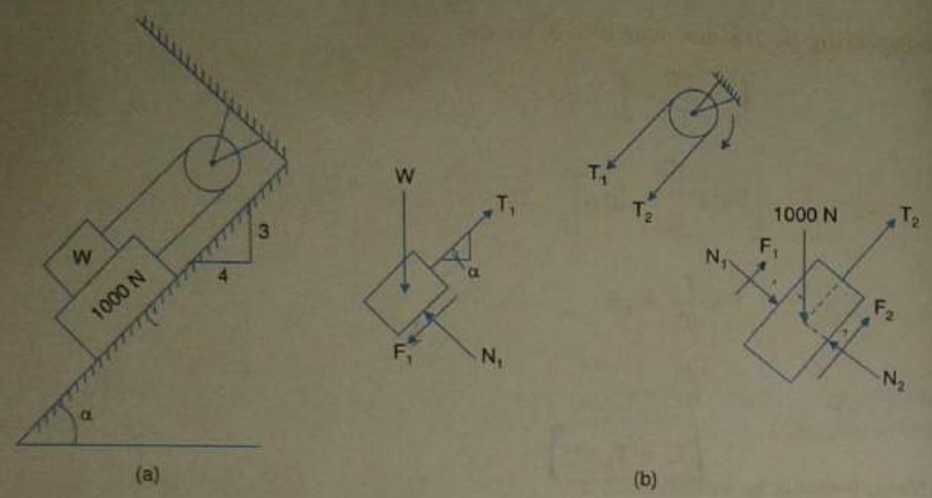


Fig. 5.20

Consider the equilibrium of block W

ΣForces perpendicular to plane = 0, gives

$$N_1 - W\cos\alpha = 0 \quad \text{or} \quad N_1 = W\cos 36.87° = 0.8\,W \qquad ...(1)$$

$$\therefore \qquad F_1 = \mu N_1 = 0.3 \times 0.8\,W = 0.24\,W \qquad ...(2)$$

ΣForces parallel to the plane = 0, gives

$$T_1 - F_1 - W\sin\alpha = 0$$

or

$$T_1 = F_1 + W\sin 36.87° = 0.24\,W + 0.6\,W = 0.84\,W \qquad ...(3)$$

Angle of contact of rope with pulley = 180° = π radians

From friction equation for rope, we get

$$T_2 = T_1 e^{\mu\theta} = T_1 e^{0.3\pi}$$

i.e.,

$$T_2 = 2.566\,T_1$$

Substituting the value of T_1 from eqn. (3),

$$T_2 = 2.566 \times 0.84\,W = 2.156\,W \qquad ...(4)$$

Now consider the equilibrium of 1000 N block.

ΣForces perpendicular to the plane = 0, gives,

$$N_2 - N_1 - 1000\cos\alpha = 0$$

$$N_2 - N_1 = 800, \quad \text{since } \cos\alpha = \cos 36.87° = 0.8$$

Substituting the value of N_1 from eqn. (1), we get

$$N_2 = N_1 + 800 = 0.8\,W + 800$$

$$\therefore \qquad F_2 = 0.3\,N_2 = 0.3 \times 0.8\,W + 0.3 \times 800 = 0.24\,W + 240 \qquad ...(5)$$

ΣForces parallel to the plane = 0, gives

$$F_1 + F_2 - 1000\sin\alpha + T_2 = 0$$

i.e.

$$0.24\,W + 0.24\,W + 240 - 1000\sin 36.87° + 2.156\,W = 0$$

$$\therefore \qquad \mathbf{W = 136.57 \text{ newton} \quad Ans.}$$

Example 5.16. *A torque of 300 N-m acts on a brake drum shown in Fig. 5.21(a). If the brake band is in contact with the brake drum through 250° and the coefficient of friction is 0.3, determine the force P applied at the end of the brake lever for the position shown in the figure.*

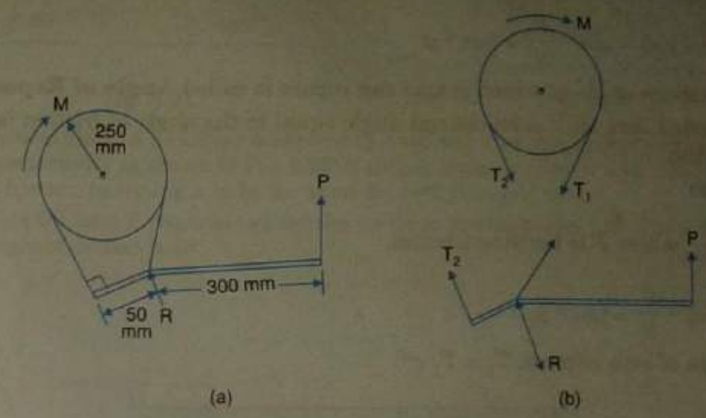


Fig. 5.21

Solution. Figure 5.21(*b*) shows the free body diagram of brake drum and the lever arm.

Now, $\theta = 250 \times \dfrac{\pi}{180}$ radians, $r = 250$ mm.

$$\mu = 0.3$$

$$\therefore \qquad \mu\theta = 0.3 \times 250 \times \frac{\pi}{180} = 1.309$$

From rope friction equation,

$$T_2 = T_1 e^{\mu\theta} = T_1 e^{1.309} = 3.7025\,T_1$$

Now, $(T_2 - T_1)r = M$

i.e., $(3.7025 - 1)\,T_1 \times 250 = 300 \times 10^3$

$$\therefore \qquad T_1 = 444.04 \text{ newton}$$

$$T_2 = 3.7025 \times 444.04 = 1644.06 \text{ newton.}$$

Consider the equilibrium of lever arm

$$T_2 \times 50 = P \times 300$$

$$\therefore \qquad \mathbf{P = \frac{1644.06 \times 50}{300} = 274.0 \text{ newton} \quad Ans.}$$

IMPORTANT DEFINITIONS AND FORMULAE

Definitions

1. The maximum value of the frictional force, which develops between two contacting surfaces when the motion is impending is called **limiting friction**.

2. Between two contacting surfaces, the magnitude of limiting friction bears a constant ratio to normal reaction. This ratio is called **coefficient of friction**. Thus $\mu = \frac{F}{N}$, where F is limiting friction and N normal reaction.
3. The angle between the resultant reaction and the normal reaction is called **Angle of Friction** *i.e.* $\phi = \tan^{-1}\frac{F}{N} = \tan^{-1}\mu$.
4. The maximum angle at which grains can repose is called **Angle of Repose**.
5. The inverted cone with semi-central angle equal to the angle of friction is called **Cone of Friction**.

Formulae:

1. $\mu = \frac{F}{N}$, where F is limiting friction.
2. $\phi = \tan^{-1}\frac{F}{N} = \tan^{-1}\mu$.
3. In case of rope friction, $T_2 = T_1 e^{\mu\theta}$.

PROBLEMS FOR EXERCISE

5.1 A pull of 180 N applied upward at 30° to a rough horizontal plane was required to just move a body resting on the plane while a push of 220 N applied along the same line of action was required to just move the same body downwards. Determine the weight of the body and the coefficient of friction. [**Ans.** $W = 990$ N; $\mu = 0.172$]

5.2 The block A shown in Fig. 5.22 weighs 2000 N. The cord attached to A passes over a frictionless pulley and supports a weight equal to 800 N. The value of coefficient of friction between A and the horizontal plane is 0.35. Solve for horizontal force P: (1) if motion is impending towards the left, and (2) if the motion is impending towards the right. [**Ans.** (1) 1252.82 N; (2) 132.82 N]

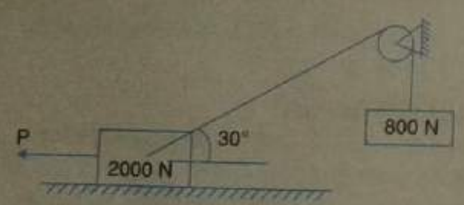


Fig. 5.22

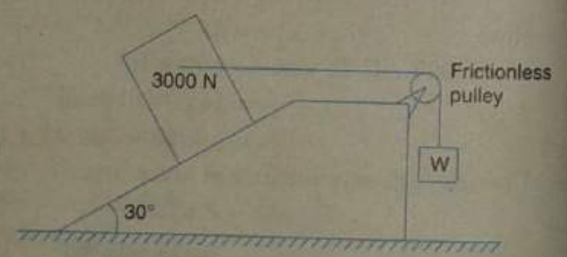


Fig. 5.23

5.3 A 3000 N block is placed on an inclined plane as shown in Fig. 5.23. Find the maximum value of W for equilibrium if tipping does not occur. Assume coefficient of friction as 0.2. [**Ans.** 2636.15]

5.4 Find whether block A is moving up or down the plane in Fig. 5.24 for the data given below. Weight of block A = 300 N. Weight of block B = 600 N. Coefficient of limiting friction between plane AB and block A is 0.2. Coefficient of limiting friction between plane BC and block B is 0.25. Assume pulley as smooth. [**Ans.** The block A is stationary since F developed $< F_{lim}$]

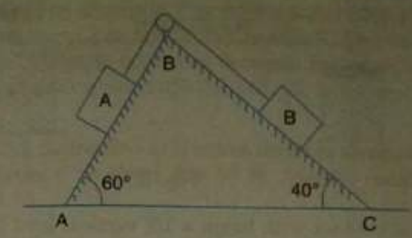


Fig. 5.24

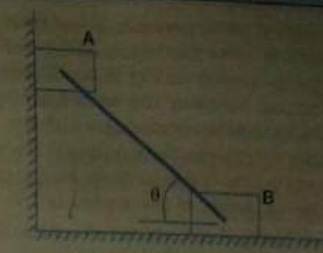


Fig. 5.25

5.5 Two identical blocks A and B are connected by a rod and they rest against vertical and horizontal planes respectively as shown in Fig. 5.25. If sliding impends when $\theta = 45°$, determine the coefficient of friction, assuming it to be the same for both floor and wall. [**Ans.** 0.414]

Determine the force P required to start the wedge as shown in Fig. 5.26. The angle of friction for all surfaces of contact is 15°. [**Ans.** 26.678 kN]

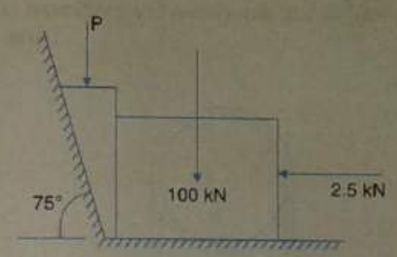


Fig. 5.26

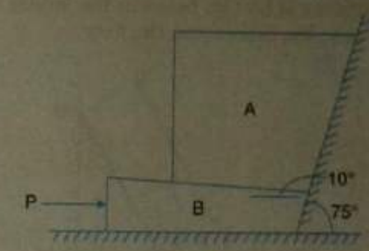


Fig. 5.27

5.7 Two blocks A and B weighing 3 kN, and 15 kN respectively, are held in position against an inclined plane by applying a horizontal force P as shown in Fig. 5.27. Find the least value of P which will induce motion of the block A upwards. Angle of friction for all contact surfaces is 12°. [**Ans.** 14.025 kN]

5.8 In Fig. 5.28, C is a stone block weighing 6 kN. It is being raised slightly by means of two wooden wedges A and B with a force P on wedge B. The angle between the contacting surfaces of the wedge is 5°. If coefficient of friction is 0.3 for all contacting surfaces, compute the value of P required to impend upward motion of the block C. Neglect weight of the wedges. [**Ans.** 2.344 kN]

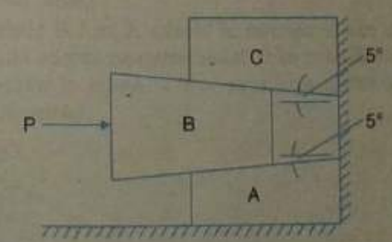


Fig. 5.28

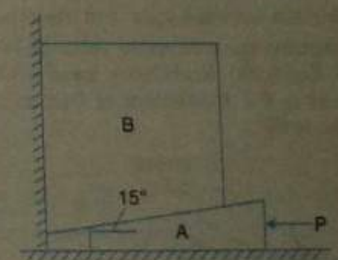


Fig. 5.29

5.9 Find the horizontal force P required to pull the block A of weight 150 N which carries block B of weight 1280 N as shown in Fig. 5.29. Take angle of limiting friction between floor and block A as 14° and that between vertical wall and block B as 13° and coefficient of limiting friction between the blocks as 0.3. [**Ans.** $P = 1294.2$ N]

5.10 The level of a precast beam, weighing 20,000 N is to be adjusted by driving a wedge as shown in Fig. 5.30. If coefficient of friction between the wedge and pier is 0.35 and that between beam and the wedge is 0.25, determine the minimum force P required on the wedge to make adjustment of the beam. Angle of the wedge is 15°.

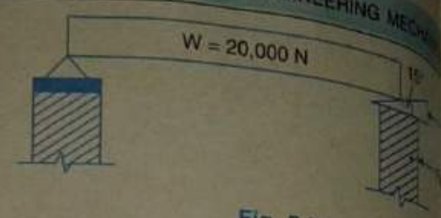


Fig. 5.30

(*Hint:* Vertical component of reaction on wedge at contact with beam = 1/2 vertical load on be = 10,000 kN)

[**Ans.** 9037.4

5.11 A ladder 5 m long rests on a horizontal ground and leans against a smooth vertical wall at angle of 70° with the horizontal. The weight of the ladder is 300 N. The ladder is on the verge sliding when a man weighing 750 N stands on a rung 1.5 m high. Calculate the coefficient friction between the ladder and the floor.

[**Ans.** μ = 0.18

5.12 A 4 m ladder weighing 200 N is placed against a vertical wall as shown in Fig. 5.31. As a m weighing 800 N, reaches a point 2.7 m from A, the ladder is about to slip. Assuming that coefficient of friction between the ladder and the wall is 0.2, determine the coefficient of fric between the ladder and the floor.

[**Ans.** 0.35

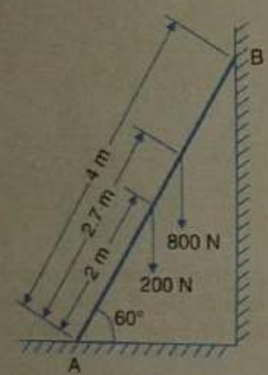


Fig. 5.31

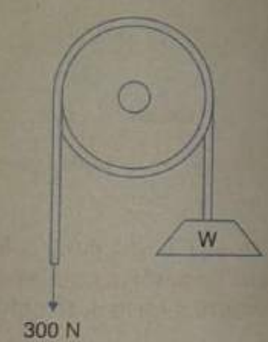


Fig. 5.32

5.13 Determine the maximum weight that can be lowered by a person who can exert a 300 N pull rope if the rope is wrapped $2\frac{1}{2}$ turns round a horizontal spur as shown in Fig. 5.32. Coeffic of friction between spur and the rope is 0.3.

[**Ans.** 33395.35

5.14 Determine the minimum value of W required to cause motion of blocks A and B towards ri (ref. Fig. 5.33). Each block weighs 3000 N and coefficient of friction between blocks and incline planes is 0.2. Coefficient of friction between the drum and rope is 0.1. Angle of wrap over drum is 90°.

[**Ans.** 3065.18

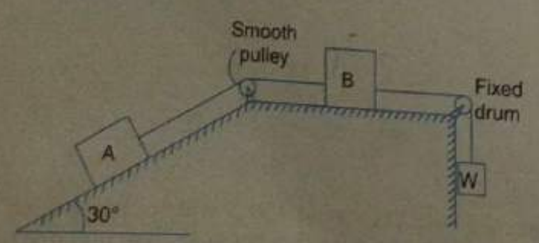


Fig. 5.33

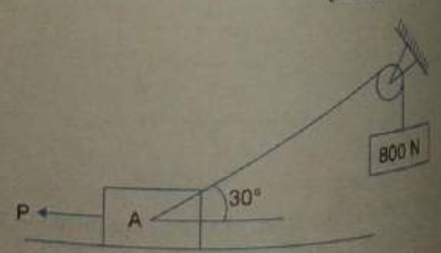


Fig. 5.34

5.15 Block A shown in Fig. 5.34 weighs 2000 N. The cord attached to A passes over a fixed drum and supports a weight equal to 800 N. The value of coefficient of friction between A and the horizontal plane is 0.25 and between the rope and the fixed drum is 0.1. Solve for P: (1) if motion is impending towards the left, (2) if the motion is impending towards the right.

[**Ans.** (1) 1230.94 N; (2) 143.0 N]

5.16 The dimension of a brake drum is as shown in Fig. 5.35. Determine the torque M exerted on the drum if the load P = 50 N. Assume coefficient of kinetic friction between rope and drum to be 0.15.

[**Ans.** 747.685 N-m]

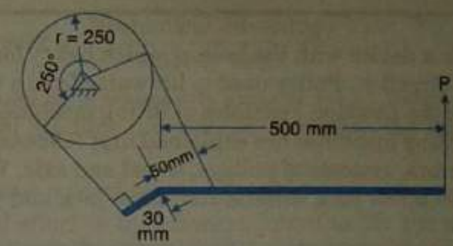


Fig. 5.35

6

Lifting Machines

A lifting machine is a device with the help of which heavy loads are lifted by applying small loads in a convenient direction. Pulley used to lift water from a well and screw jacks used to lift motor car are some of the common examples of lifting machines. In this chapter some of the terms connected with lifting machines are explained first followed by the description of the characteristic features of levers, systems of pulleys, wheel and axle, Weston differential pulley block, inclined plane, simple screw jack, differential screw jack and winch crab.

6.1. DEFINITIONS

The terms commonly used while dealing with lifting machines are defined below:

Load: This is the resistance to be overcome by the machine.

Effort: This is the force required to overcome the resistance to get the work done by the machine.

Mechanical Advantage: This is the ratio of load lifted to effort applied. Thus, if W is the load and P is the corresponding effort, then

$$\text{Mechanical Advantage} = \frac{W}{P} \qquad ...(6.1)$$

Velocity Ratio: This is the ratio of the distance moved by the effort to the distance moved by the load in the same interval of time. Thus,

$$\text{Velocity Ratio} = \frac{D}{d} \qquad ...(6.2)$$

where, D = distance moved by effort

d = distance moved by the load.

Input: The work done by the effort is known as input to the machine. Since work done by a force is defined as the product of the force and the distance moved in the direction of the force,

$$\text{Input} = P \times D \qquad ...(6.3)$$

If force P is in newton and distance D is in metre, the unit of input will be N-m. One N-m work is also known as one Joule (J). (for further details refer Ch. XIV).

Output: It is defined as useful work got out of the machine, *i.e.* the work done by the load. Thus,

$$\text{Output} = W \times d \qquad ...(6.4)$$

Efficiency: This is defined as the ratio of output to the input. Thus, if we use notation η for efficiency,

$$\eta = \frac{\text{Output}}{\text{Input}} = \frac{W \times d}{P \times D} = \frac{W}{P} \times \frac{d}{D}$$

$$= \text{Mechanical Advantage } (MA) \times \frac{1}{\text{Velocity Ratio } (VR)}$$

$$= \frac{MA}{VR} \qquad ...(6.5)$$

i.e.,

$$\text{Efficiency} = \frac{\text{Mechanical Advantage}}{\text{Velocity Ratio}}$$

Ideal Machine: A machine whose efficiency is 1 (*i.e.*, 100%) is called an ideal machine. In other words, in an ideal machine, the output is equal to the input. From eqn. (6.5), in an ideal machine,

Velocity Ratio = Mechanical Advantage

Ideal Effort: Ideal effort is the effort required to lift the given load by the machine assuming the machine to be ideal.

For ideal machine,

$$VR = MA$$

If P_i is the ideal effort, then

$$VR = \frac{W}{P_i}$$

$$\therefore \quad P_i = \frac{W}{VR} \qquad ...(6.6)$$

Ideal Load: Ideal load is the load that can be lifted using the given effort by the machine, assuming it to be ideal.

For the ideal machine,

$$VR = MA$$

If W_i is the ideal load, then,

$$VR = \frac{W_i}{P}$$

$$\therefore \quad W_i = VR \times P \qquad ...(6.7)$$

6.2. PRACTICAL MACHINES

In practice, it is difficult to get an ideal machine. Friction exists between all surfaces of contacts of movable parts. Some of the work done by the effort is utilised to overcome frictional resistance. Hence, the useful work done in lifting the load is reduced, resulting in reduction of efficiency.

Let P = actual effort required

P_i = ideal effort required

W = actual load to be lifted

W_i = ideal load to be lifted

Then, $P - P_i$ is called *effort lost in friction* and $W - W_i$ is called *frictional resistance*.

Now, $$\eta = \frac{MA}{VR} = \frac{W}{P} \times \frac{1}{VR}$$

From eqn. (6.6), $$P_i = \frac{W}{VR}$$

$$\therefore \quad \eta = \frac{P_i}{P}$$

Similarly from eqn. (6.7), $W_i = VR \times P$

$$\therefore \quad \eta = \frac{W}{W_i}$$

Thus, $$\eta = \frac{P_i}{P} = \frac{W}{W_i} \quad \text{...(6.8)}$$

Example 6.1. *In a lifting machine, an effort of 500 N is to be moved by a distance of 20 m to raise a load of 10,000 N by a distance of 0.8 m. Determine the velocity ratio, mechanical advantage and efficiency of the machine. Determine also ideal effort, effort lost in friction, ideal load and frictional resistance.*

Solution. Load, $W = 10{,}000$ N

Effort $P = 500$ N

Distance moved by the effort $D = 20$ m

Distance moved by the load $d = 0.8$ m

Mechanical Advantage,

$$MA = \frac{W}{P} = \frac{10{,}000}{500} = \mathbf{20} \quad \textbf{Ans.}$$

Velocity Ratio, $VR = \frac{D}{d} = \frac{20}{0.8} = \mathbf{25}$ **Ans.**

Efficiency $= \frac{MA}{VR} = \frac{20}{25} = 0.8 = \mathbf{80}$ **percent Ans.**

Ideal effort, $P_i = \frac{W}{VR} = \frac{10{,}000}{25} = \mathbf{400}$ **N Ans.**

Effort lost in friction $= P - P_i$

$= 500 - 400 = \mathbf{100}$ **N Ans.**

Ideal Load, $W_i = P \times VR = 500 \times 25 = \mathbf{12{,}500}$ **N Ans.**

Frictional Resistance $= W_i - W$

$= 12{,}500 - 10{,}000$

$= \mathbf{2500}$ **N Ans.**

6.3. LAW OF MACHINE

The relationship between the load lifted and the effort required in a machine is called the **law of machine.** This is found by conducting experiments in which efforts required for lifting different loads are determined and then load *versus* effort graph as shown in Fig. 6.1 is plotted. This is generally a straight line which does not pass through the origin.

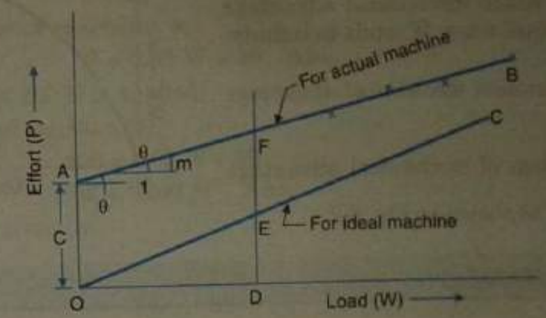


Fig. 6.1

The law of machine can be expressed mathematically in the form:

$$P = mW + C \quad \text{...(6.9)}$$

where, C is the intercept OA and $m = \tan\theta$, the slope of AB. For the ideal machine

$$MA = VR$$

$$\frac{P}{W} = VR$$

This is a straight line relationship passing through the origin and is shown by line OC in Fig. 6.1.

After plotting the law for the actual machine (AB) and the law for ideal machine (OC), it is easy to determine efficiency at any given load. The vertical line DEF corresponding to given load OD is drawn. Then,

$DE = P_i$, effort required in ideal machine

$DF = P$, effort required in actual machine

$\therefore$ Friction loss $= P - P_i$

$= DF - DE = EF$

$$\text{Efficiency, } \eta = \frac{P}{P_i} = \frac{DE}{DF}$$

6.4. VARIATION OF MECHANICAL ADVANTAGE

Mechanical Advantage (MA) is given by:

$$MA = \frac{W}{P}$$

From the law of machine,

$$P = mW + C$$

$$\therefore \quad MA = \frac{W}{mW + C} = \frac{1}{m + \frac{C}{W}} \quad ...(6.1)$$

As the load increases, $\frac{C}{W}$ which is in denominator, decreases and hence mechanical advantage increases. In limiting case when W tends to infinity, $\frac{C}{W} = 0$ and hence *maximum mechanical advantage equals* $\frac{1}{m}$. The variation of mechanical advantage with respect to load is as shown in Fig. 6.2.

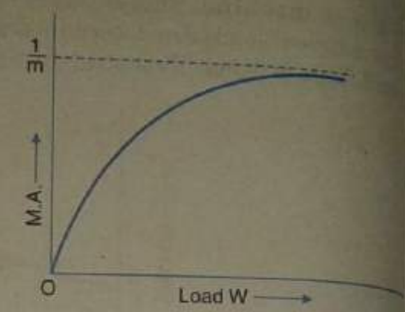


Fig. 6.2

6.5. VARIATION OF EFFICIENCY

From eqn. (6.5), the efficiency of the machine is given by $\eta = \frac{MA}{VR}$. Using the eqn. (6.1)

$$\eta = \frac{1}{VR} \times \frac{1}{m + \frac{C}{W}} \quad ...(6.1)$$

Since the velocity ratio (VR) is constant for a machine, variation of efficiency with load is similar to the variation of mechanical advantage with the load.

The *maximum efficiency* is approached as the load approaches infinity ($W \rightarrow \infty$) and its value is equal to $\frac{1}{VR} \times \frac{1}{m}$. The variation of the efficiency with load is shown in Fig.6.3.

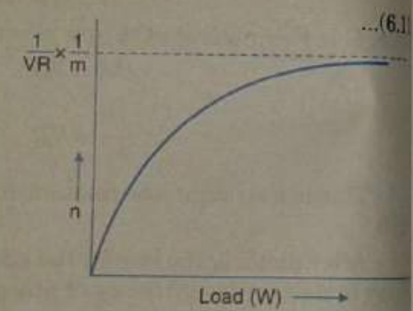


Fig. 6.3

Example 6.2. *In a simple machine, whose velocity ratio is 30, a load of 2400 N is lifted by an effort of 150 N and a load of 3000 N is lifted by an effort of 180 N. Find the law of machine and calculate the load that could be lifted by a force of 200 N. Calculate also:*

(i) The amount of effort wasted in overcoming the friction,

(ii) Mechanical advantage, and

(iii) The efficiency.

Solution. Let the law of machine be

$$P = mW + C$$

In the first case, $P = 150$ N, and $W = 2400$ N

In the second case, $P = 180$ N and $W = 3000$ N

$$\therefore \quad 150 = 2400\, m + C \quad ...(1)$$

$$180 = 3000\, m + C \quad ...(2)$$

Subtracting eqn. (1) from eqn. (2), we get

$$30 = 600\, m$$

$$m = 0.05$$

Substituting this value in eqn. (1), we get

$$150 = 120\, m + C$$

$$\therefore \quad C = 30$$

Hence, **the law of machine is**

$$\mathbf{P = 0.05\, W + 30} \quad \textbf{Ans.} \quad ...(3)$$

When a force of 200 N is applied:

From the law of machine (3),

$$200 = 0.05\, W + 30$$

$$\therefore \quad \mathbf{W = 3400\ N} \quad \textbf{Ans.}$$

Ideal effort is given by:

$$P_i = \frac{W}{VR} = \frac{3400}{30} = 113.33 \text{ N}$$

$\therefore$ **Effort wasted in overcoming the friction**

$$= P - P_i = 200 - 113.33$$

$$\mathbf{= 86.67\ N} \quad \textbf{Ans.}$$

$$\textbf{Mechanical Advantage} = \frac{W}{P} = \frac{3400}{200} = \mathbf{17} \quad \textbf{Ans.}$$

$$\textbf{Efficiency} = \frac{MA}{VR} = \frac{17}{30} = .5667 = \mathbf{56.67\%} \quad \textbf{Ans.}$$

Example 6.3. *In a lifting machine an effort of 150 N raised a load of 7700 N. What is the mechanical advantage? Find the velocity ratio if the efficiency at this load is 60%. If by the same machine, a load of 13,200 N is raised by an effort of 250 N, what is the efficiency? Calculate the maximum mechanical advantage and the maximum efficiency.*

Solution. Effort, $P = 150$ N

Load, $W = 7700$ N

$\therefore$ **Mechanical Advantage:**

$$MA = \frac{W}{P} = \frac{7700}{150} = \mathbf{51.33} \quad \textbf{Ans.}$$

If the efficiency is 60%,

$$\eta = 0.6$$

$$\eta = \frac{MA}{VR}$$

$$\therefore \quad 0.6 = \frac{51.33}{VR}$$

or $$VR = \frac{51.33}{0.6} \quad \textit{i.e.,} \quad \mathbf{VR = 85.55} \quad \textbf{Ans.}$$

When an effort of 250 N raised a load of 13,200 N,

$$MA = \frac{W}{P} = \frac{13,200}{250} = 52.8$$

$$\therefore \quad \eta = \frac{MA}{VR} = \frac{52.8}{85.55} = 0.6172$$

i.e., η = **61.72%** **Ans.**

Let the law of machine be

$$P = mW + C$$

In the first case, $150 = 7700\,m + C$ ...(1)

In the second case, $250 = 13,200\,m + C$ ...(2)

Subtracting eqn. (1) from eqn. (2) we get,

$$100 = 5500\,m$$

$$\therefore \quad m = 0.01818$$

$\therefore$ **Maximum mechanical advantage**

$$= \frac{1}{m} = \frac{1}{0.01818} = \mathbf{55} \quad \textbf{Ans.}$$

Maximum efficiency $= \frac{1}{m} \times \frac{1}{VR} = \frac{1}{0.01818} \times \frac{1}{85.55}$

$= 0.6429 =$ **64.29%** **Ans.**

Example 6.4. *The efforts required for lifting various loads by a lifting machine are tabulated below:*

Load lifted in N	*100*	*200*	*300*	*400*	*500*	*600*
Effort required in N	*16.0*	*22.5*	*28.0*	*34.0*	*40.5*	*46.5*

Determine the law of machine. If the velocity ratio is 25, calculate efficiency at each load and plot efficiency ***versus*** *load curve. From this curve, determine the maximum efficiency.*

Solution. Figure 6.4 shows the graph of effort *versus* load. From this figure, $C = 10$ N

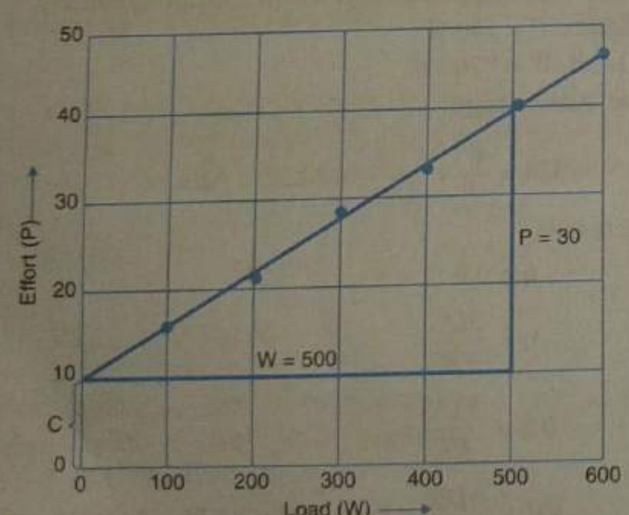


Fig. 6.4

and slope $m = \frac{30}{500} = 0.06$

$\therefore$ **The law of machine is**

$$\mathbf{P = 0.06\,W + 10} \quad \textbf{Ans.}$$

$$\eta = \frac{MA}{VR} = \frac{W}{P} \times \frac{1}{VR} = \frac{W}{P} \times \frac{1}{25} = \frac{W}{25P}$$

Table below shows the calculation of efficiency for various loads:

Load in N	100	200	300	400	500	600
Effort in N	16.0	22.5	28.0	34.0	40.5	46.5
Efficiency in %	25	36.56	42.86	47.06	49.38	51.61

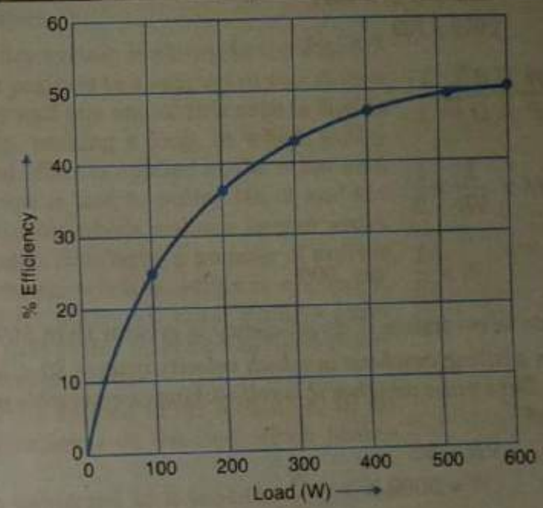


Fig. 6.5

From the graph (Fig. 6.5) maximum efficiency is seen as 50%. Actually if it is plotted for infinitely large load, **maximum efficiency** will be equal to

$$\frac{1}{m} \times \frac{1}{VR} = \frac{1}{0.06} \times \frac{1}{25} = 0.6667 = \mathbf{66.67\%} \quad \textbf{Ans.}$$

6.6. REVERSIBILITY OF A MACHINE

If the removal of effort while lifting results in lowering of the load, the machine is said to be **reversible.** The machine is said to be **self-locking** if the load is not lowered on removal of the effort.

For example, while lifting water from the well, the pot falls back if the effort to pull it up is removed whereas the screw jack used to lift the motor car will hold the car at the same position even if the application of the effort is stopped. Hence, the former is a reversible and later is a self-locking type simple lifting machine.

A simple lifting machine will be reversible or self-locking solely based on its efficiency. It can be shown that a lifting machine is reversible if its efficiency is greater than 50 percent and self-locking if its efficiency is less than 50 percent.

Let W – load being lifted
P – effort required
VR – Velocity Ratio
D – distance moved by the effort
d – distance moved by the load

Then,

$$\text{Input} = P \times D$$
$$\text{Output} = W \times d$$
$$\therefore \text{ Work lost in friction} = PD - Wd$$

When effort is removed, the load can start moving down if it can overcome the friction resistance = $PD - Wd$. Hence the condition for the reversibility is:

$$Wd > (PD - Wd)$$
$$\therefore \quad 2Wd > PD$$
$$\left(\frac{W}{P}\right)\left(\frac{d}{D}\right) > \frac{1}{2}$$
$$MA \times \frac{1}{VR} > \frac{1}{2}$$

i.e., $$\eta > \frac{1}{2} \quad \text{or} \quad 50\%.$$

Hence, a machine is reversible if its efficiency is greater than 50%.

Example 6.5. *In a lifting machine in which velocity ratio is 30, a load of 5000 N is lifted with an effort of 360 N. Determine whether it is self-locking or reversible machine. How much the frictional resistance?*

Solution. $VR = 30$
$W = 5000$ N
$P = 360$ N

$$MA = \frac{W}{P} = \frac{5000}{360} = 13.889$$

$$\text{Efficiency, } \eta = \frac{MA}{VR} = \frac{13.889}{30} = 0.4630 = 46.30\%$$

Since the efficiency is less than 50%, **it is self-locking machine. Ans.**

Ideal load, $W_i = P \times VR = 360 \times 30$
$= 10{,}800$ N

$\therefore$ **Frictional resistance** $= W_i - W = 10{,}800 - 5000 =$ **5,800 N Ans.**

6.7. PULLEYS

A systematic arrangement of one or more pulleys may provide a simple and convenient lifting machine. In its simplest form, it consists only one pulley over which a rope or chain passes as shown in the Fig. 6.6. In this case, velocity ratio is equal to one since distance moved by effort is equal to the distance moved by the load. It just changes the direction of the applied force.

Depending on the arrangement, pulleys are classified as:

1. First order pulley system
2. Second order pulley system
3. Third order pulley system.

At times, it may be difficult or may be detour to find velocity ratio directly. In such cases ideal conditions may be assumed (neglecting friction) and mechanical advantage may be found first. Then applying $VR = MA$ for ideal machine, the velocity ratio is found. This method of determining velocity ratio is used for various pulley systems considered here.

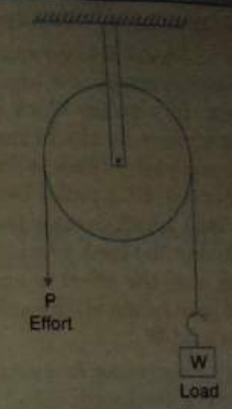


Fig. 6.6

First Order Pulley System

A first order pulley system is shown in the Fig. 6.7. Pulley No. 1 is fixed in position to a support at top. A rope passes over this pulley and one end of this rope is tied to the support at the top, making a loop, in which pulley No. 2 is suspended and effort is applied at the other end. One end of another rope is tied to pulley No. 2 and the other end to the top support which makes a loop in which pulley No. 3 is suspended. Similarly, a number of pulleys can be arranged as shown, when an effort is applied to lift the load except first pulley all other pulleys move vertically. Therefore, first pulley is termed as fixed pulley and the others as movable pulleys.

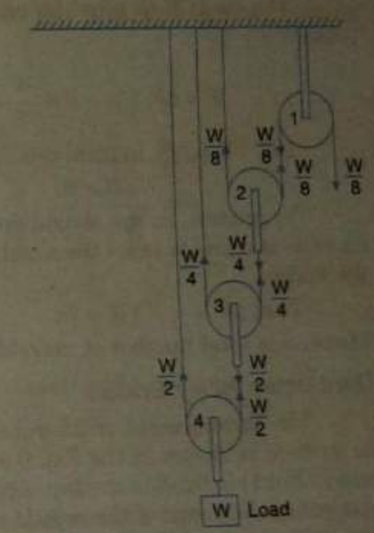


Fig. 6.7

Let an effort P be applied to lift a load W. In an ideal pulley system (friction = 0), the rope which passes over pulley No. 4 is subjected to a tension $\frac{W}{2}$. Then tension in rope which passes over pulley No. 3 is $\frac{W}{4}$ and tension in the rope which passes over pulley No. 2 is $\frac{W}{8}$.

Hence, an effort equal to $\frac{W}{8}$ is required to lift a load W.

$$\therefore \quad P = \frac{W}{8} \text{ and hence } MA = \frac{W}{P} = 8 = 2^3$$

But in an ideal machine, $VR = MA \Rightarrow VR = 2^3$

It is to be noted that in the system considered, there are three movable pulleys and the velocity ratio is 2^3. If there are only two movable pulleys, then velocity ratio would be 4 (i.e., 2^2). In general, in the first order pulley system, velocity ratio (VR) is given by 2^n, where, n is the number of movable pulleys present in the system. Thus, in first order pulley system.

$$VR = 2^n \quad \text{...(6.12)}$$

Second Order Pulley System

Figure 6.8 shows a second order pulley system. This system consists of a top pulley block and a bottom pulley block. In a pulley block pulleys may be arranged side by side or may be one below the other as shown in Fig. 6.8. The top pulley block is fixed in position to the top support whereas bottom pulley block can move vertically along with the load which is attached to it. One end of the rope is attached to the hook provided at the bottom of the top pulley block and the effort is applied at the other end. A single rope goes round all the pulleys. Let an effort P be applied to lift a load W.

Neglecting frictional losses, the tension in the rope all along the length is P. Take the section along (1)–(1) and consider the equilibrium of the bottom pulley block.

The load W is lifted using six ropes having equal tension P

$$\therefore \quad W = 6P, \quad \therefore \quad P = \frac{W}{6}, \quad \therefore \quad MA = \frac{W}{P} = 6.$$

But $VR = MA$ in ideal condition

$$\therefore \quad VR = 6.$$

In general, in the second order pulley system velocity ratio is equal to twice the number of movable pulleys in the system.

That is, $$VR = 2n \qquad ...(6.13)$$

where, n is total number of movable pulley in the system.

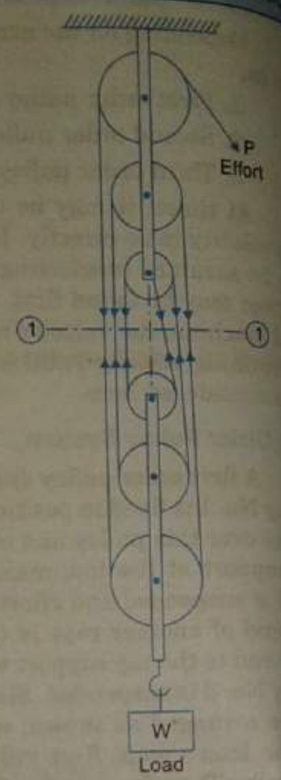


Fig. 6.8

Third Order Pulley System

The arrangement of the pulleys in the third order system is shown in the Fig. 6.9. In this system a pulley (No. 1) is fixed to the top support, over which a rope passes. One end of the rope is attached to a rigid base at the bottom. The other end is attached to a second pulley. Over this pulley another rope passes, whose one end is attached to the same rigid base and the other end to a third pulley as shown. Likewise a series of pulleys can be arranged. The load to be lifted will be attached to the rigid base.

Referring to the Fig. 6.9, let the effort required be P to lift a load W. Then neglecting friction,

Tension in the rope which passes over pulley No. 3 = P

Tension in the rope which passes over pulley No. 2 = $2P$

Tension in the rope which passes over pulley No. 1 = $4P$

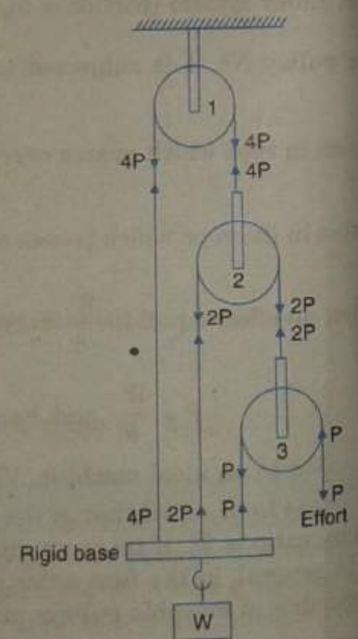


Fig. 6.9

$\therefore$ A total force of $7P$ is acting on the base.

$\therefore$ Lifting force produced on the base = $7P$

Considering the equilibrium of rigid base,

$$7P = W \quad \Rightarrow \quad MA = \frac{W}{P} = 7$$

But in an ideal machine, $VR = MA$, and hence $VR = 7$.

It can be easily seen that, if there are only two pulleys, $VR = 3$ and if there is only one pulley, $VR = 1$. Therefore, in general, for the third order pulley system:

$$VR = 2^n - 1 \qquad ...(6.14)$$

where, n = number of pulleys.

6.8. WHEEL AND AXLE

This machine consists of an axle A having diameter d and a wheel B having diameter D ($D > d$) co-axially fitted as shown in Fig. 6.10. The whole assembly is mounted on ball bearing so that wheel and axle can be rotated.

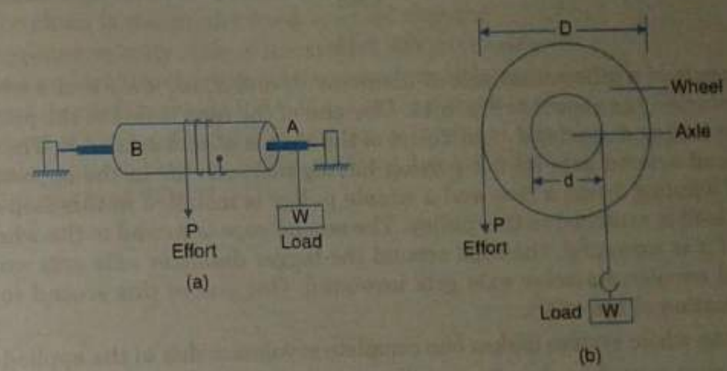


Fig. 6.10

One end of a rope is tied to the pin provided on the wheel and the rope is wound around the wheel. The other end of the rope provides the means for the application of the effort. One end of another rope is tied to the pin provided on the axle and wound around the axle in the opposite direction to that of rope wound to the wheel. To the other end of this rope the load is attached. If the whole assembly is rotated, one rope gets wound up and the other gets unwound.

Suppose the assembly is moved by one complete revolution, then the distance moved by the effort = πD and distance moved by the load = πd

$$\therefore \quad VR = \frac{\text{distance moved by effort}}{\text{distance moved by load}}$$

$$= \frac{\pi D}{\pi d}, \quad \text{or} \quad VR = \frac{D}{d} \qquad ...(6.15)$$

6.9. WHEEL AND DIFFERENTIAL AXLE

An improvement over wheel and axle machine is made by using one more wheel of bigger diameter and it is called wheel and differential axle.

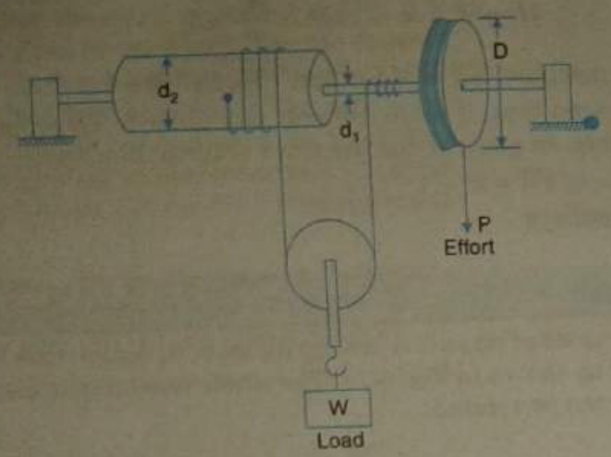


Fig. 6.11

This consists of a differential axle of diameter d_1 and d_2 ($d_1 < d_2$) and a wheel of diameter D, fixed uniaxially as shown in Fig. 6.11. One end of the rope is tied to the pin provided on the axle portion having diameter d_1 and a part of the rope is wound around it. The other end of the rope is wound around axle on the portion having diameter d_2 in the opposite direction. This pattern of winding forms a loop and a simple pulley is installed in this loop as shown in the figure. The load is attached to this pulley. The second rope is wound to the wheel in such a direction that if it is unwound, the rope around the bigger diameter axle gets wound up and the rope around smaller diameter axle gets unwound. One end of this second rope provides means for application of the effort.

Suppose the whole system makes one complete revolution due to the applied effort, then

Total distance moved by the effort at the differential axle $= \pi D$

Length of winding of the rope $= \pi d_2$

Length of unwinding of rope $= \pi d_1$

$\therefore$ Net wound length $= \pi d_2 - \pi d_1$

$= \pi(d_2 - d_1)$

But, the rope is continuous and the load is to be lifted by the pulley block in the loop.

$\therefore$ Total height over which pulley is lifted $= \dfrac{\pi(d_2 - d_1)}{2}$

and hence the distance moved by the load $= \dfrac{\pi(d_2 - d_1)}{2}$

$$\therefore \quad VR = \frac{\pi D}{\dfrac{\pi(d_2 - d_1)}{2}}$$

$$VR = \frac{2D}{d_2 - d_1}$$

Hence, velocity ratio in wheel and differential axle is given by:

$$VR = \frac{2D}{d_2 - d_1} \qquad \text{...(6.16)}$$

6.10. WESTON DIFFERENTIAL PULLEY BLOCK

This is a special type of simple pulley system. It is shown in the Fig. 6.12.

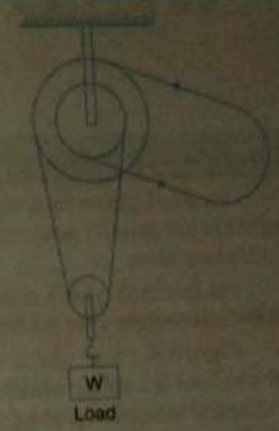


Fig. 6.12

This system consists of two pulley blocks, one at the top attached to the support and the other at the bottom hanging in the chain loop. The top block consists of two wheels of different diameters, but fixed co-axially. The bottom block is a simple pulley to which the load W is attached. An endless chain is wound around the pulley system as shown in the figure. All the wheels are made with teeth so as to accommodate the links of the chain. The chain is essentially used to avoid slipping.

To determine velocity ratio of the system, let us consider pulley block as an ideal machine and determine its mechanical advantage first. In ideal machine $VR = MA$. Let the diameter of the larger wheel of the top block be D and the diameter of the smaller wheel of the top block be d [Fig. 6.13(a)]. Let the effort required to lift the load W be P.

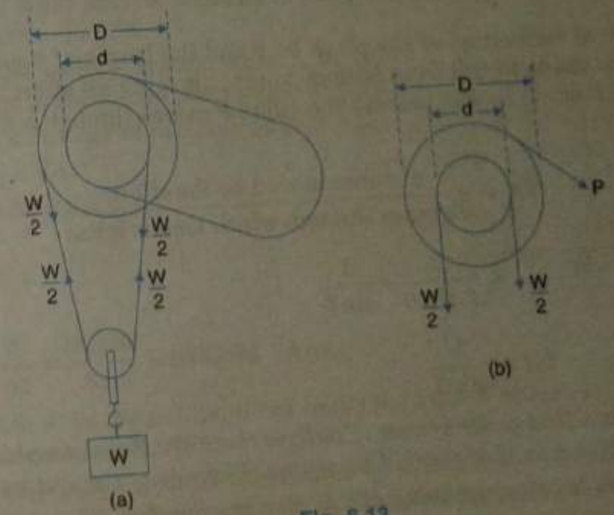


Fig. 6.13

Then the tension in the chain loop in which pulley is hanging is $\dfrac{W}{2}$.

Now, taking moment about the axis of top block [Fig. 6.13(b)],

$$\frac{W}{2} \times \frac{D}{2} = \frac{W}{2} \times \frac{d}{2} + P \times \frac{D}{2}$$

$$\therefore \quad \frac{W}{4}(D-d) = \frac{PD}{2}$$

$$\frac{W}{P} = \frac{2D}{(D-d)}$$

i.e.,

$$MA = \frac{2D}{(D-d)}$$

In an ideal machine,

$$VR = MA = \frac{2D}{(D-d)} \quad \text{...(6.17)}$$

6.11. INCLINED PLANE

Inclined plane is a very simple lifting device. The lift is essentially accomplished with the horizontal displacement.

An inclined plane consists of a plane surface at a definite angle over which the load is to be lifted.

Figure 6.14 shows a typical inclined plane. Here the load is a roller which is to be lifted to a higher elevation. One end of a rope is tied to the roller and the rope is passed over a pulley attached at the top of the inclined plane. At the other end of the rope the effort is applied.

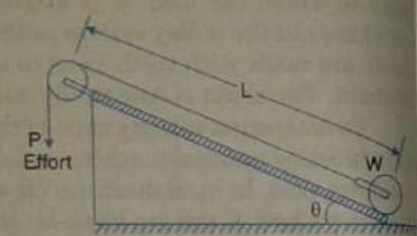


Fig. 6.14

Let the angle of inclination of the plane be θ and the length of the inclined plane be L. Then, if the roller is made to roll from bottom to top, applying an effort P, the load is lifted through a height of $L \sin\theta$. In this process the effort P moves through a distance L vertically downwards.

$$\therefore \quad VR = \frac{\text{distance moved by the effort}}{\text{distance through which load is lifted}}$$

$$= \frac{L}{L\sin\theta} = \frac{1}{\sin\theta}$$

$$\therefore \quad VR = \frac{1}{\sin\theta} \quad \text{...(6.18)}$$

Example 6.6. *In a first order system of pulleys there are three movable pulleys. What is the effort required to raise a load of 6000 N ? Assume efficiency of the system to be 80%.*

If the same load is to be raised using 520 N, find the number of movable pulleys that are necessary.

Assume a reduction of efficiency of 5% for each additional pulley used in the system.

Solution. $VR = 2^n$, where n is the number of movable pulleys.

$$VR = 2^3 = 8$$

Now, $MA = \eta \times VR = 0.8 \times 8 = 6.4$

i.e., $\frac{W}{P} = 6.4$

$$\therefore \quad P = \frac{W}{6.4} = \frac{6000}{6.4}$$

i.e., **P = 937.5 N Ans.**

In the second case,

Effort = 520 N

Efficiency $\eta = 0.80 - n_1 \times 0.05$

where n_1 = number of additional pulleys required and equal to $(n-3)$.

$$MA = \eta \times VR$$

i.e., $\frac{W}{P} = \eta \times VR$

$$\therefore \quad W = P \times \eta \times 2^n$$

$$= P(0.8 - n_1 \times 0.05) \times 2^n$$

$$= P[0.8 - (n-3) \times 0.05]\, 2^n$$

By going for a trial and error solution, starting with one additional pulley *i.e.*, totally with four pulleys,

$$W = 520\,[0.8 - (4-3) \times 0.05]\, 2^4 = 6240 \text{ N}$$

i.e., if four pulleys are used, a load of 6240 N can be raised with the help of 520 N effort.

$\therefore$ **Number of movable pulleys required = 4 Ans.**

Example 6.7. *What force is required to raise the load W shown in Fig. 6.15 ? Assume efficiency of the system to be 85%.*

Solution. The pulley system shown in the Fig. 6.15 is a variation of the second order pulley system.

$$VR = 2 \times \text{number of movable pulleys} = 6$$

$$MA = \eta \times VR$$

$$= 0.85 \times 6 = 5.1$$

i.e., $\frac{W}{P} = 5.1$

$$\therefore \quad \mathbf{P} = \frac{W}{5.1} = \frac{12{,}000}{5.1} = \mathbf{2352.94} \quad \textbf{Ans.}$$

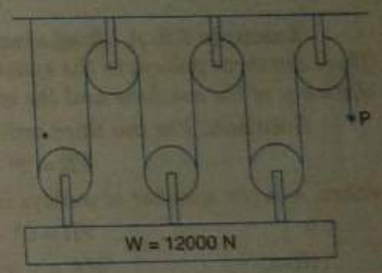


Fig. 6.15

Example 6.8. *Find the pull required to lift the load W shown in Fig. 6.16(a) assuming the efficiency of the system to be 78%.*

Solution. The pulley system shown in Fig. 6.16(*a*) is a combination of a first order system and a second order system as shown in Figs. 6.16(*b*) and (*c*).

Let load W be lifted by a distance x. Consider the first order system portion [Fig. 6.16(*b*)].

Here there are two movable pulleys. Hence

$$VR = 2^2 = 4$$

In this portion P moves by $4x$.

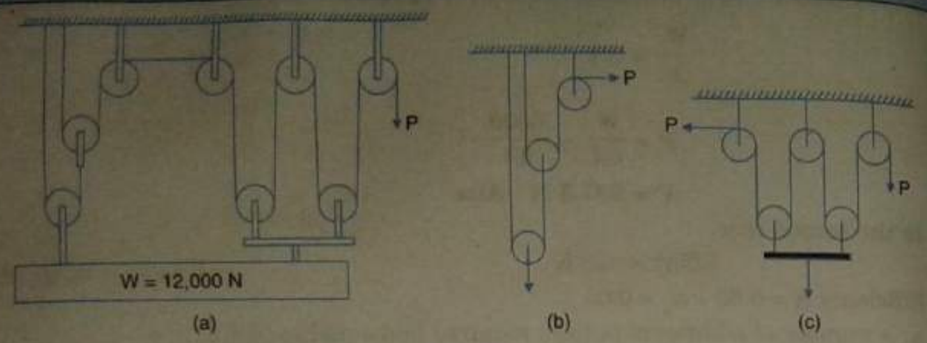


Fig. 6.16

Now, consider the second order pulley system portion [Fig. 6.17(c)]. Here there are two movable pulleys. Hence $VR = 2 \times 2 = 4$.

$\therefore$ Distance moved by the effort in this system $= 4x$.

Hence, the total distance moved by the effort in the given system $= 4x + 4x = 8x$

$$\therefore \quad VR = \frac{8x}{x} = 8$$

Now, $$MA = \eta \times VR = 0.78 \times 8 = 6.24$$

i.e., $$\frac{W}{P} = 6.24$$

$$\therefore \quad \mathbf{P = \frac{12{,}000}{6.24} = 1923.08\ N \quad Ans.}$$

Example 6.9. *A lifting machine consists of pulleys arranged in the third order system. There are three pulleys in the system. A load of 1000 N is lifted by an effort of 180 N. Find the efficiency of the machine and the effort lost in friction.*

Solution. For the third order system of pulleys,

$$VR = 2^n - 1$$

where, n is the number of pulleys in the system.

$$\therefore \quad VR = 2^3 - 1 = 7$$

Now, $$MA = \eta \times VR$$

i.e., $$\frac{W}{P} = \eta \times VR$$

$$\therefore \quad \eta = \frac{W}{P} \times \frac{1}{VR} = \frac{1000}{180} \times \frac{1}{7} = 0.7937$$

i.e., $$\eta = 79.37\%$$

Now,

Ideal effort, $$P_i = \frac{W}{VR} = \frac{1000}{7} = 142.86\ N$$

$\therefore$ **Effort lost in friction** $= P - P_i$

$= 180 - 142.86 = \mathbf{37.14\ N}$ **Ans.**

Example 6.10. *What force P is required to raise a load of 2500 N in the system of pulleys shown in Fig. 6.17(a). Assume efficiency of the system to be equal to 70%.*

Solution. Figure 6.17(*a*) can be split into two simple systems as shown in Figs. 6.17(*b*) and 6.17(*c*).

What is shown in Fig. 6.17(*b*) is a third order pulley system having two pulleys.

$$\therefore \quad VR = 2^n - 1 = 2^2 - 1 = 3$$

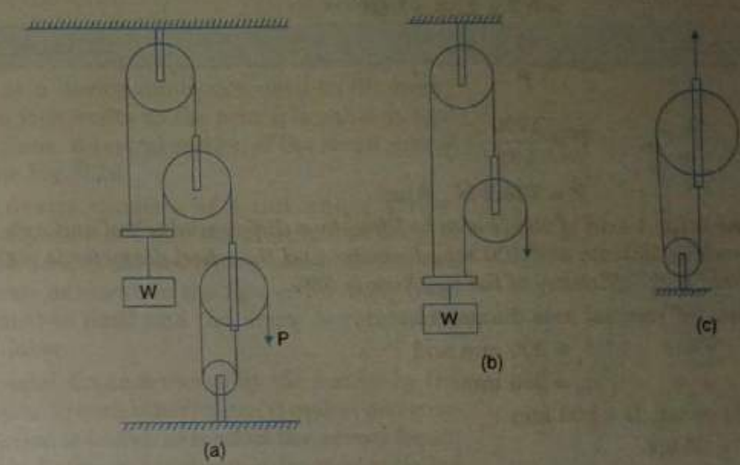


Fig. 6.17

Figure 6.17(c) is also a third order system, having two pulleys.

$$\therefore \quad VR = 2^2 - 1 = 3$$

$\therefore$ VR of the whole system $= 3 + 3 = 6$

Now, $$MA = \eta \times VR$$

i.e., $$\frac{W}{P} = \eta \times VR$$

$$\therefore \quad P = \frac{W}{\eta \times VR} = \frac{2500}{0.7 \times 6}$$

i.e., $$\mathbf{P = 595.24\ N \quad Ans.}$$

Example 6.11. *In a wheel and axle, diameter of the wheel is 500 mm and that of the axle is 200 mm. The thickness of the cord on the wheel is 6 mm and that of the axle is 20 mm. Find the velocity ratio of the machine. If the efficiency when lifting a load of 1200 N with a velocity of 10 metres per minute is 70%, find the effort necessary.*

Solution. Effective wheel diameter

$$= \frac{6}{2} + 500 + \frac{6}{2} = 506\ mm$$

Effective axle diameter $= \frac{20}{2} + 200 + \frac{20}{2} = 220$ mm.

For a wheel and axle, the velocity ratio is given by $\frac{D}{d}$

$$\therefore \quad VR = \frac{512}{220} = 2.33$$

Mechanical advantage = Efficiency × velocity ratio

$= 0.7 \times 2.33 = 1.63$

$$MA = \frac{W}{P}$$

$$\therefore \quad P = \frac{1200}{1.63}$$

i.e., **P = 736.2 N Ans.**

Example 6.12. *A load of 20 kN is to be lifted by a differential wheel and axle. It consists of differential axle of 250 mm and 300 mm diameter and the wheel diameter is 800 mm. Find the effort required if the efficiency of the machine is 55%.*

Solution. Differential axle diameters,

$d_1 = 300$ mm and

$d_2 = 250$ mm

Wheel diameter, $D = 800$ mm

Load, $W = 20$ kN

Efficiency, $\eta = 55\%$

$$\text{Velocity ratio, } VR = \frac{2D}{d_2 - d_1} = \frac{2 \times 800}{300 - 250} = 32$$

Mechanical advantage

MA = Efficiency × velocity ratio

$= 0.55 \times 32 = 17.6$

$$MA = \frac{W}{P}$$

$$\therefore \quad P = \frac{20{,}000}{17.6} \quad \textit{i.e.,} \quad \mathbf{P = 1136.4\ N\ \ Ans.}$$

Example 6.13. *A Weston differential pulley block of diameter 500 mm and 200 mm is used to lift a load of 5000 N. Find the effort required if the efficiency is 60%.*

Solution. Diameter of pulley block $D = 500$ mm, and

$d = 200$ mm

Load, $W = 5000$ N

Efficiency, $\eta = 60\%$

$$\text{Velocity ratio} = \frac{2D}{D-d} = \frac{2 \times 500}{500 - 200} = 3.33$$

Mechanical advantage = Efficiency × Velocity ratio

$= 0.6 \times 3.33 = 2$

$$\text{Effort required, } P = \frac{W}{MA} = \frac{5000}{2}$$

i.e., **P = 2500 N Ans.**

6.12. SCREW JACK

This is a device commonly used to lift heavy loads. Screw jack works on the principle same as that of inclined plane. A typical section of the screw jack is shown in the Fig. 6.18.

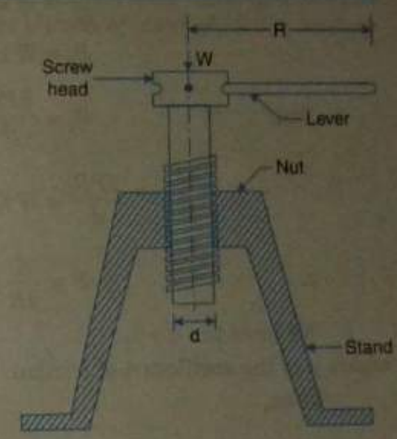


Fig. 6.18

The device consists of a nut and a screw. Monolithically cast nut and stand form the body of the jack. The load is carried by the screw head fitted onto the screw as shown in the figure. The body (consisting of nut) is fixed and the screw is rotated by means of a lever.

The axial distance moved by the nut (or by the screw, relative to each other) when it makes one complete revolution is known as **lead of the screw head.** The distance between consecutive threads is called *pitch* (of a screw thread). If the screw is single threaded, then lead of the screw is equal to the pitch. If the screw is double threaded then lead of the screw is twice the pitch.

Let R be the length of the lever and d be the mean diameter of the screw.

Let a load W be lifted using an effort P.

If an effort P is applied at the lever end, it is equivalent to an effort P_1 at the screw [Fig. 6.19(*a*) and P_1 is given by the condition:

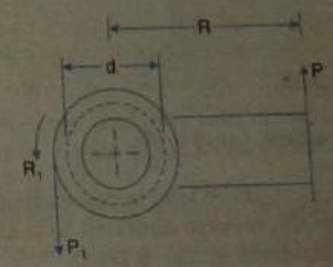


Fig. 6.19(a)

$$P \times R = P_1 \times \frac{d}{2}$$

$$\therefore \quad P_1 = \frac{2PR}{d}$$

Now, consider one complete revolution of the lever. The load W is lifted up by a distance p equal to the lead of the screw.

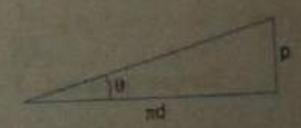


Fig. 6.19(b)

This can be compared with that of inclined plane having

$$\text{inclination} = \tan^{-1}\frac{p}{\pi d}$$

where, p – lead of the screw

d – mean diameter of screw.

Applying an effort P at the end of the lever is as good as applying an effort P_1 (at the screw) on this inclined plane. [Fig. 6.19(c)].

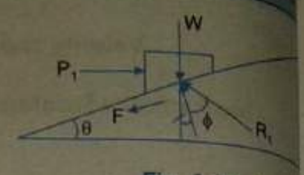


Fig. 6.19(c)

Resolving horizontally *i.e.*, parallel to P_1

$$P_1 = R_1 \sin(\theta + \phi), \qquad ...(6.18)$$

where R_1 is resultant reaction and

ϕ is limiting angle of friction. Resolving vertically

$$W = R_1 \cos(\theta + \phi) \qquad ...(6.18(a))$$

Dividing eqn. (6.18(*a*)) by eqn. (6.18(*b*))

$$\frac{P_1}{W} = W \tan(\theta + \phi)$$

$$\therefore \qquad P_1 = W \tan(\theta + \phi)$$

But,

$$P_1 = \frac{2PR}{d}$$

$$\frac{2PR}{d} = W \tan(\theta + \phi) \qquad ...(6.18(b))$$

$$\therefore \qquad P = \frac{d}{2R} W \tan(\theta + \phi) \qquad ...(6.19)$$

We have $\tan \phi = \mu$

where μ is the coefficient of friction.

Then,

$$P = \frac{d}{2R} W \frac{\tan\theta + \tan\phi}{1 - \tan\theta \tan\phi}$$

$$P = \frac{d}{2R} W \frac{\mu + \tan\theta}{1 - \mu \tan\theta} \qquad ...(6.19(a))$$

where, $\tan\theta = \dfrac{p}{\pi d}$

If the load is descending, then the friction will be acting in the reverse direction so that the resultant reaction R shifts as shown in Fig. 6.19(*d*).

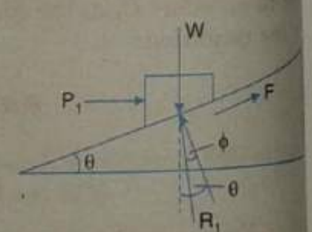


Fig. 6.19(d)

Then eqn. (6.19) changes to

$$P = \frac{d}{2R} W \tan(\theta - \phi) \qquad ...(6.19(b))$$

Torque required, $T = PR$

$$= \frac{d}{2} W \tan(\theta + \phi)$$

Hence torque required while ascending

$$T = \frac{d}{2} W \tan(\theta + \phi) \qquad ...(6.20)$$

and torque required while descending

$$T = \frac{d}{2} W \tan(\theta - \phi) \qquad ...(6.20(a))$$

Now,

$$VR = \frac{\text{Distance moved by the effort}}{\text{Distance moved by the load}} = \frac{2\pi R}{p} \qquad ...(6.21)$$

Example 6.14. *A screw jack raises a load of 40 kN. The screw is square threaded having three threads per 20 mm length and 40 mm in diameter. Calculate the force required at the end of a lever 400 mm long measured from the axis of the screw, if the coefficient of friction between screw and nut is 0.12.*

Solution. Screw diameter, $d = 40$ mm

Lead of the screw, $p = \dfrac{20}{3} = 6.667$ mm

Load, $W = 40$ kN

Lever length, $R = 400$ mm

$\mu = 0.12$

We have $$P = \frac{d}{2R} W \frac{\mu + \tan\theta}{1 - \mu \tan\theta}$$

and $$\tan\theta = \frac{\text{lead}}{\pi D} = \frac{6.667}{\pi \times 40} = 0.05305$$

$$\therefore \qquad P = \frac{40}{2 \times 400} \times 40{,}000 \left[\frac{0.12 + 0.05305}{1 - (0.12 \times 0.05305)}\right]$$

i.e., **P = 348.32 kN** **Ans.**

Example 6.15. *A screw jack has square threads 50 mm mean diameter and 10 mm pitch. The load on the jack revolves with the screw. The coefficient of friction at the screw thread is 0.05.*

(i) Find the tangential force required at the end of 300 mm lever to lift a load of 6000 N.

(ii) State whether the jack is self-locking. If not, find the torque which must be applied to keep the load from descending.

Solution. $$\tan\theta = \frac{p}{\pi d} = \frac{10}{\pi \times 50} = 0.0637$$

$$\therefore \qquad \theta = 3.6426°$$

$$\tan\phi = 0.05$$

$$\therefore \qquad \phi = 2.8624°$$

(*i*) $$P = \frac{d}{2R} \times W \tan(\theta + \phi)$$

$$= \frac{50}{2 \times 300} \times 6000 \tan (3.6426° + 2.8624°)$$

P = 57.01 N Ans.

(*ii*) We have

$$VR = \frac{2\pi R}{p} = \frac{2\pi \times 300}{10} = 188.496$$

$$MA = \frac{W}{P} = \frac{6000}{57.01} = 105.245$$

$$\text{Efficiency} = \frac{MA}{VA} = \frac{105.245}{188.496} = 0.5583$$

i.e., Efficiency = 55.83% > 50.

Hence the screw jack is not self-locking.

∴ The torque required to keep the load from descending

$$= \frac{d}{2} W \tan (\theta - \phi) = \frac{50}{2} \times 600 \times \tan (3.6426 - 2.8624)$$

T = 204.3 N-mm Ans.

6.13. DIFFERENTIAL SCREW JACK

Differential screw jack is an improvement over simple screw jack. A typical differential screw jack is shown in Fig. 6.20. It consists of two threaded elements *A* and *B*. Both *A* and *B* have threads in the same direction (right-handed). The element *A* is a cylinder which has threads on both its outer and inner surfaces. The threads on the outer surface of the element *A* fits into the nut *C* which also functions as the base of the whole mechanism. The threads on the element *B* fit into the threads cut on the inner surface of *A*. Thus, the element *A* acts as a screw for the nut *C* and also as a nut for the element *B*. With the help of a lever inserted in the holes made on the top of the block *D*, which is attached to the element *B*, block *D* can be rotated. When *D* is rotated, *A* rotates with it. Rotation of *B* is prevented by suitable arrangement.

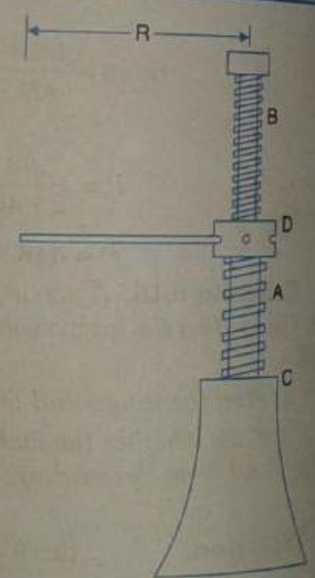


Fig. 6.20

Let *D* and *d* be the mean diameters of the screws *A* and *B*, respectively.

Let p_A and p_B be the pitch of the screws *A* and *B*, respectively and p_A be greater than p_B.

If the lever is rotated through one complete revolution, the height through which the element *A* moves up = p_A

In the mean time, the element *B* moves down with respect to *C*.

The distance through which *B* comes down = p_B

∴ Net height through which load is lifted = $(p_A - p_B)$

Let *R* be the radial distance (from the centre line of *A* and *B*) at which an effort *P* is applied

Now, $$VR = \frac{\text{Distance moved by the effort}}{\text{Distance moved by the load}} = \frac{2\pi R}{p_A - p_B} \quad ...(6.22)$$

It can be seen from eqn. (6.22) that the velocity ratio in the differential screw jack is increased as compared to that of simple screw jack (eqn. 6.21).

Example 6.16. *The following are the specifications for a differential screw jack:*

(i) Pitch of smaller screw, 5.0 mm

(ii) Pitch of larger screw, 10.0 mm

(iii) Lever arm length from centre of screw = 500 mm.

The screw jack raises a load of 15 kN with an effort of 185 N. Determine the efficiency of the differential screw jack at this load.

If the above jack can raise a load of 40 kN with an effort of 585 N, determine the law of machine.

Solution. Now, p_A = 10.0 mm

p_B = 5.0 mm

Lever arm length, *R* = 500 mm

$$VR = \frac{2\pi R}{p_A - p_B} = \frac{2\pi \times 500}{10 - 5.0} = 628.32$$

$$MA = \frac{W}{P} = \frac{15{,}000}{185} = 81.08 \quad \therefore \quad \eta = \frac{MA}{VA} = \frac{81.08}{628.32}$$

= 0.129 = **12.9 percent. Ans.**

To find law of machine:

Let law of machine be $P = mW + C$

From first case: $185 = m \times 15{,}000 + C$ ...(1)

From second case: $585 = m \times 50{,}000 + C$ ...(2)

Subtracting eqn. (1) from eqn. (2), $400 = 35{,}000\, m$.

or $$m = \frac{4}{350}$$

Substituting this value in eqn. (1) we get,

$$185 = 171.43 + C$$

∴ $$C = 13.57 \text{ N}$$

∴ **Law of machine is** $\mathbf{P = \frac{4}{350} W + 13.57}$ **Ans.**

6.14. WINCH CRABS

Winch crabs are lifting machines in which velocity ratio is increased by a gear system. If only one set of gears is used, the winch crab is called a **single purchase winch crab** and if two sets are used it is called **double purchase winch crab.**

Single Purchase Winch Crab

Line diagram of a single purchase winch crab is shown in Fig. 6.21. It consists of a drum of radius r connected to an axle by gears. The toothed wheel on load drum is called wheel and the toothed wheel on axle is called **pinion.** Pinion is always smaller in size and contains less number of teeth as compared to that on the spur wheel. The axle is provided a handle of arm length R. Let the number of teeth on pinion and spur wheel be T_1 and respectively. Let the effort be applied at the end of the handle. When one revolution is the distance moved by the effort is given by:

$$D = 2\pi R$$

When axle makes one revolution, due to gear arrangement load drum moves by T_1 number of teeth, which means that it makes a revolution of $\frac{T_1}{T_2}$.

∴ The distance over which the load moves:

$$d = 2\pi r \times \frac{T_1}{T_2}$$

∴ Velocity ratio,

$$VR = \frac{D}{d} = \frac{2\pi R}{2\pi r \times \frac{T_1}{T_2}}$$

i.e.,

$$VR = \frac{R}{r} \times \frac{T_2}{T_1} \quad ...(6.$$

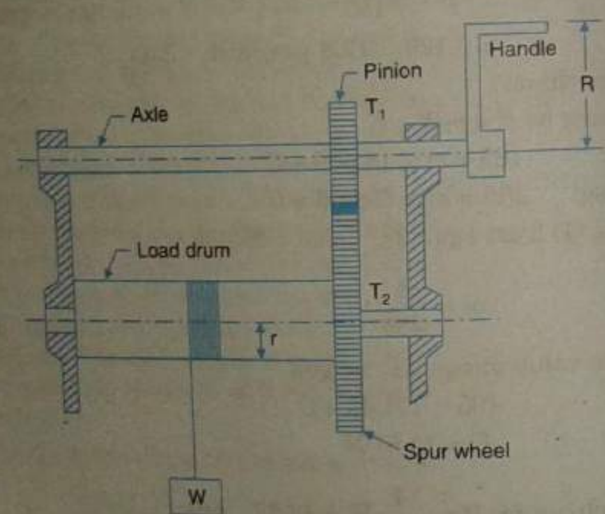


Fig. 6.21

Double Purchase Winch Crab

Velocity ratio of a winch crab can be increased by providing another axle with a pair of pinion and gear as shown in Fig. 6.22. Since two pairs of pinion and gear are used it is called double purchase winch crab. This is used for lifting heavier loads.

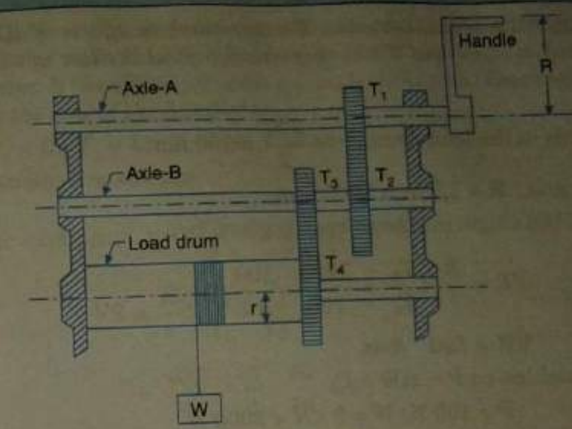


Fig. 6.22

Let the number of teeth on various wheels be T_1, T_2, T_3 and T_4 as shown in Fig. 6.22. Let the handle makes one revolution.

Distance moved by effort P is given by:

$$D = 2\pi R \quad ...(6.24)$$

When axle A makes one revolution, axle B is moved by T_1 teeth, *i.e.*, it makes $\frac{T_1}{T_2}$ revolutions. The number of teeth by which spur wheel is moved is $\frac{T_1}{T_2} \times T_3$ and hence load drum makes $\left(\frac{T_1}{T_2}\right) \times \left(\frac{T_3}{T_4}\right)$ revolutions.

∴ The distance moved by the load

$$d = 2\pi r \times \left(\frac{T_1}{T_2}\right) \times \left(\frac{T_3}{T_4}\right)$$

$$VR = \frac{D}{d} = \frac{2\pi R}{2\pi r \times \left(\frac{T_1}{T_2}\right) \times \left(\frac{T_3}{T_4}\right)}$$

i.e.,

$$VR = \frac{R}{r} \times \left(\frac{T_2}{T_1}\right) \times \left(\frac{T_4}{T_3}\right) \quad ...(6.25)$$

Example 6.17. *Following are the specifications of a single purchase crab:*

Diameter of the load drum = 200 mm

Length of lever arm R = 1.2 m

Number of teeth on pinion, T_1 = 10

Number of teeth on spur wheel, T_2 = 100.

Find the velocity ratio of this machine. On this machine efforts of 100 N and 160 N are required to lift the load of 3 kN and 9 kN, respectively. Find the law of the machine and the efficiencies at the above loads.

Solution. Radius of the load drum, $r = \frac{200}{2} = 100$ mm

Length of lever arm, $R = 1.2$ m $= 1200$ mm

Velocity ratio of the single purchase crab is given by:

$$VR = \frac{R}{r} \times \frac{T_2}{T_1} = \frac{1200}{100} \times \frac{100}{10}$$

VR = 120 Ans.

Let the law of machine be $P = mW + C$

In first case: $P = 100$ N; $W = 3$ kN $= 3000$ N

$\therefore$ $100 = m \times 3000 + C$ ...(1)

In the second case: $P = 160$ N; and $W = 9$ kN $= 9000$ N

$\therefore$ $160 = m \times 9000 + C$ ...(2)

Subtracting eqn. (1) from eqn. (2), we get

$$60 = 6000\, m$$

$$\therefore \quad m = \frac{1}{100} = 0.01$$

Substituting this value of m in eqn. (1), we get

$$100 = \frac{1}{100} \times 3000 + C$$

$$\therefore \quad C = 70$$

Hence, the law of machine is

P = 0.01 W + 70 Ans.

Efficiencies:

In the first case,

$$MA = \frac{W}{P} = \frac{3000}{100} = 30$$

$$\therefore \quad \eta = \frac{MA}{VR} = \frac{30}{120} = 0.25 = \mathbf{25\%} \quad \textbf{Ans.}$$

In the second case,

$$MA = \frac{W}{P} = \frac{9000}{160} = 56.25$$

$$\therefore \quad \eta = \frac{MA}{VR} = \frac{56.25}{120}$$

i.e., $\eta = 0.4688 = \mathbf{46.88\%}$ **Ans.**

Example 6.18. *In a double purchase crab, the pinions have 15 and 20 teeth, while the spur wheels have 45 and 40 teeth. The effort handle is 400 mm long while the effective diameter of the drum is 150 mm. If the efficiency of the winch is 40%, what load will be lifted by an effort of 250 N applied at the end of the handle?*

Solution. $T_1 = 15$; $T_2 = 45$; $T_3 = 20$; $T_4 = 40$

Length of handle, $R = 400$ mm

Radius of the load drum, $r = \frac{150}{2} = 75$ mm

$$\therefore \quad VR = \frac{R}{r} \times \frac{T_2}{T_1} \times \frac{T_4}{T_3}$$

$$= \frac{400}{75} \times \frac{45}{15} \times \frac{40}{20} = 32$$

Now, $\eta = \frac{MA}{VR}$

$$\therefore \quad 0.40 = \frac{MA}{32}$$

or $MA = 12.8$

i.e., $\frac{W}{P} = 12.8$, but $P = 250$ N

$\therefore$ $W = 12.8 \times 250$

i.e., $W = 3200$ N

Applied effort lifts a load of 3200 N Ans.

Important Definitions

1. For load, effort, mechanical advantages, velocity ratio, input, output, efficiency, ideal machine, ideal effort, ideal load—refer page No. 136, 137.
2. **Law of Machine:** The relationship between the load lifted and the effort required in a machine is called law of machine.
3. If the removal of effort while lifting results in lowering of the load, the **machine is** said to be **reversible.** The machine is said to be **self locking** if the load is not lowered on removal of the effort.

Important Formulae

1. Mechanical advantage $= \frac{W}{P}$
2. Velocity ratio $= \frac{D}{d}$
3. Input $= P \times D$
4. Output $= W \times d$.
5. Efficiency $= \frac{MA}{VR} = \frac{P_i}{P} = \frac{W}{W_i}$

6. In ideal machine, $VR = MA$
7. Ideal effort $= \dfrac{W}{VR}$
8. Ideal load $= W_i = VR \times P$
9. Law of machine: $P = mW + C$
10. Maximum mechanical advantage $= \dfrac{1}{m}$
11. Maximum efficiency $= \dfrac{1}{VR} \times \dfrac{1}{m}$
12. A machine is reversible, if $\eta > 50\%$
13. **Machine**

Machine	Velocity ratio
(*a*) First order pulley system	2^n
(*b*) Second order pulley system	2^n
(*c*) Third order pulley system	$2^n - 1$
(*d*) Wheel and axle	$\dfrac{D}{d}$
(*e*) Wheel and differential axle	$\dfrac{2D}{d_2 - d_1}$
(*f*) Weston differential pulley block	$\dfrac{2D}{D - d}$
(*g*) Inclined plane	$\dfrac{1}{\sin\theta}$
(*h*) Screw jack	$\dfrac{2\pi R}{p}$
(*i*) Differential screw jack	$\dfrac{2\pi R}{p_A - p_B}$
(*j*) Winch crabs–single purchase	$\dfrac{R}{r} \times \dfrac{T_2}{T_1}$
–double purchase	$\dfrac{R}{r} \times \dfrac{T_2}{T_1} \times \dfrac{T_4}{T_3}$

PROBLEMS FOR EXERCISE

6.1 In a lifting machine an effort of 1.5 kN is to be moved through a distance of 7.2 m to move a load of 24 kN through a distance of 300 mm. Determine: (1) mechanical advantage (2) velocity ratio, (3) efficiency, (4) ideal effort, (5) effort lost in friction, (6) ideal load, and (7) frictional resistance.

[**Ans.** (1) 16; (2) 24; (3) 66.67%; (4) 1.0 kN; (5) 0.5 kN; (6) 36 kN and (7) 12 kN]

6.2 In a lifting machine an effort of 400 N is required to raise a load of 3000 N and an effort of 640 N, to raise a load of 5000 N. How much load can be lifted with an effort of 760 N? If the velocity ratio is 16, determine the efficiency of the machine when an effort of 760 N is applied. Is it a reversible machine?

[**Ans.** $W = 6000$ N; $\eta = 49.34$; It is not reversible]

6.3 The following observations were made in an experiment on a lifting machine:

Load in N	500	1000	1500	2000	2500	3000
Effort in N	26	47	76	95	105	127

Draw the load *versus* effort graph and determine the law of machine. If the velocity ratio is 30 determine the efficiency while lifting a load of 1800 N.

What is the maximum efficiency of the machine ?

[**Ans.** $P = 0.04W + 6$; $\eta = 51.28\%$; maximum $\eta = 83.33\%$]

6.4 A lifting machine having velocity ratio 28 starts raising a load of 6420 N on applying an effort of 450 N to it. If suddenly the effort is removed find whether the load comes down or not?

[**Ans.** $\eta = 50.95\%$, hence the load comes down]

6.5 In the first order pulley system having three movable pulleys, how much effort is required to raise a load of 5780 N if the same system raises a load of 1200 N with an effort of 200 N? Assume the efficiency to be constant for the pulley system.

[**Ans.** $P = 963.33$ N]

6.6 For the arrangement of pulleys shown in Fig. 6.23, find the effort required to raise the given load 7280 N. Assume efficiency of the system as 75%.

[**Ans.** $P = 2436.7$ N]

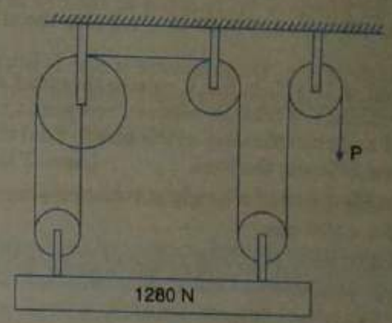


Fig. 6.23

6.7 For the combination of first order and second order pulley system shown in Fig. 6.24 what will be velocity ratio?

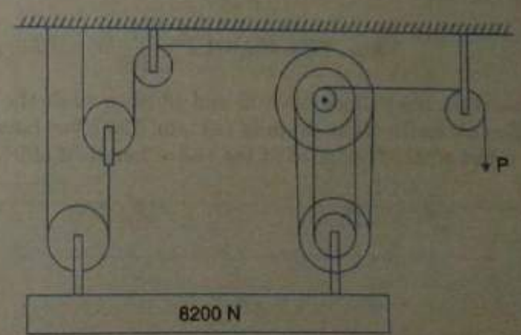


Fig. 6.24

Assuming efficiency to be 80%, calculate what effort is required to raise the load of 8200 N.

[**Ans.** $VR = 8$; $P = 1281.25$ N]

6.8 For a third order pulley system having six movable pulleys, an effort of 720 N is required to raise a load of 30 kN. Calculate the efficiency of the system. [**Ans.** 66.14%]

6.9 For a wheel and axle, the following details are available :

Diameter of the wheel = 540 mm

Diameter of the axle = 270 mm

Thickness of the cord on the wheel = 6 mm

Thickness of the cord on the axle = 20 mm

Calculate the efficiency of the device if an effort of 725 N is required to lift a load of 1000 N.

[**Ans.** η = 73.26%]

6.10 A differential axle and wheel consists of a differential axle of 240 mm and 320 mm diameter, and a wheel of diameter 750 mm. Assuming efficiency to be 57%, find the effort required to raise a load of 24 kN. [**Ans.** P = 2.2456 kN]

6.11 A Weston differential pulley block of diameters 400 mm and 800 mm is used to lift a load of 40,000 N. Find the effort required if efficiency of the system is 60%. [**Ans.** 26,667 N]

6.12 The following particulars refer to a screw jack:

(1) Diameter of the screw rod = 62.5 mm

(2) Length of the handle = 250 mm

(3) Pitch of the square threads = 12.5 mm

(4) Coefficient of friction = 0.05

Find the effort required to lift a load of 5000 N. Find also the efficiency of the jack.

[**Ans.** P = 71.27 N; η = 55.83%]

6.13 A screw jack carries a load of 10 kN. It has a square threaded screw of pitch 25 mm and mean diameter 60 mm. The coefficient of friction between screw and nut is 0.20. Calculate the torque required to raise the load and the efficiency of the screw. Find also the force required at the end of the handle 500 mm long to lower the load. [**Ans.** T = 102.58 kN-mm; P = 0.2052 kN]

6.14 The following are the specifications of a single purchase crab:

Diameter of the load drum = 220 mm

Length of lever arm, R = 1.0 m

Number of teeth on pinion, T_1 = 10

Number of teeth on spur wheel, T_2 = 120

Determine the velocity ratio of the machine. On this machine, it is found in one trial that an effort of 120 N is required to lift a load 4.2 kN and in another trial an effort of 150 N is required to lift a load of 11.4 kN. Establish the law of machine. Also calculate efficiencies corresponding to 120 N and 150 N efforts.

$$\left(\textbf{Ans. } VR = 109.091;\ P = \frac{1}{240}W + 102.5;\ \eta_1 = 32.08\%;\ \eta_2 = 69.67\%\right)$$

6.15 In a double purchase crab, the pinions have 12 and 18 teeth while the spur wheels have 24 and 36 teeth. The effective diametre of the drum is 142 mm. The effort handle is 450 mm. What load can be lifted by an effort of 220 N applied at the end of handle if efficiency of winch is 42%.

[**Ans.** P = 2342.54 N]

IN-COURSE 7

Virtual Work Method

So far in this book we saw application of Newtonian mechanics to many engineering mechanics problems. There were many scientists before and after Newton who found solutions to mechanics problems by completely different approach. Jean Bernoulli (350 BC) and Leonardo da Vinci (1435) are the names associated with this school of thought. They imagined a small displacement of the bodies and found out the work done by various forces in the system and said if the body is in equilibrium then the total work done by the system should be zero. This approach is called 'virtual work method'. The word *virtual* is used since in reality there is no actual displacement. The term *virtual* is defined by Webster as 'being in essence of effect but not in fact.'

The work done is virtual if the displacements are virtual or forces acting are virtual. Hence in virtual work approach we may have the following two approaches:

(*i*) Principles of virtual displacements,

(*ii*) Principles of virtual forces.

In this chapter we consider the principle of virtual displacements.

7.1. WORK DONE BY FORCES AND MOMENTS

Work done by a force = force × distance moved in the direction of force.

In Fig 7.1, various forces acting on a particle are shown. If the particle moves a distance s in x direction, from A to B, then the work done by various forces are as given below:

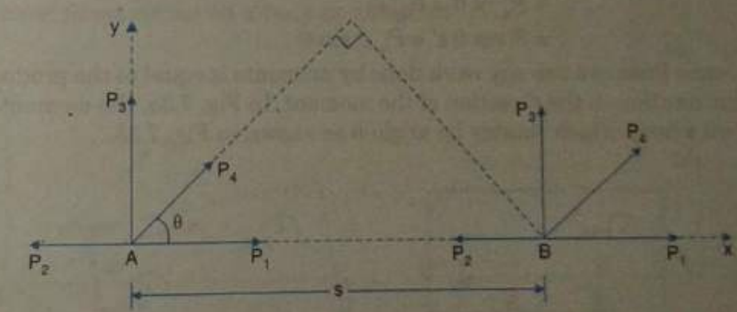


Fig. 7.1

Forces	*Work done*
P_1	$P_1 s$
P_2	$P_2(-s) = -P_2 s$
P_3	$P_3 \times 0 = 0$
P_4	$P_4 s \cos\theta$

Similarly if the particle moves from A to C in vertical direction (Fig. 7.2), the work done by various forces is as given below:

Forces	*Work done*
P_1	$P_1 \times 0 = 0$
P_2	$P_2 \times 0 = 0$
P_3	$P_3 \times s' = P_3 s'$
P_4	$P_4 \times s' \sin\theta = P_4 s' \sin\theta$

If the displacement is at a particular angle, say θ to the direction of the force, as is the case of force P_4 in the above cases, it will be convenient to find the perpendicular and parallel components of the forces to displacement and then find the work done. The component of P_4 in x and y directions are:

$$P_{4x} = P_4 \cos\theta$$
$$P_{4y} = P_4 \sin\theta$$

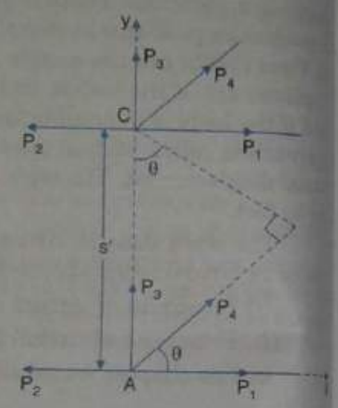


Fig. 7.2

In Fig. 7.1

Displacement in x direction = s

Displacement in y direction = 0

∴ The work done by force P_4, is

$$= P_{4x} s + P_{4y} 0 = P_{4x} s = Ps \cos\theta$$

Similarly in Fig. 7.2

Displacement in x direction = 0

Displacement in y direction = s'

∴ Hence, work done by P_4

$$= P_{4x} \times 0 + P_{4y} s'$$
$$= P \sin\theta\, s' = P_4 s' \sin\theta$$

On the same lines, we can say work done by moments is equal to the product of moment and the angular rotation in the direction of the moment. In Fig. 7.3*a*, the moments M_1, M_2 and M_3 are acting on a body which rotates by angle θ as shown in Fig. 7.3*b*.

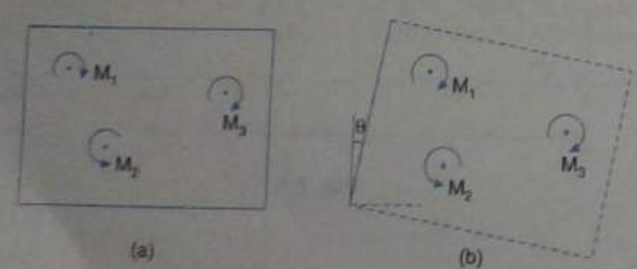


Fig. 7.3

Then the work done by various moments are as given below:

Moments	*Work done*
M_1	$M_1\theta$
M_2	$-M_2\theta$
M_3	$M_3\theta$

1. No work is done by a force if displacement is at right angles to the line of action of force.
2. Work done is – ve if the displacement is in the direction opposite to force.
3. Work done by a moment is – ve if rotation is opposite to the direction of moment.

7.2. THE METHOD

The method consists of the following steps:

1. Give an elemental virtual displacement to the body in equilibrium in suitable direction.
2. Write down the work done by each force in the system.
3. Sum up to get total work done by system of forces and equate it to zero to get required equations. Many times such equations are identical to equations of equilibrium. However, the method of getting these equations is completely different in this method.

A number of problems are solved to explain and to illustrate the virtual work method.

7.3. SIGN CONVENTION

In this chapter the positive senses of moments and displacements are as given below:

1. Forces and displacements are +ve if they are in positive directions of coordinates as shown in Fig. 7.4.
2. Moments and rotations are positive if they are in clockwise directions.

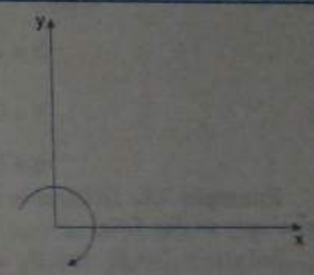


Fig. 7.4

Example 7.1. *Determine the resultant of a system of concurrent forces acting on a body as shown in Fig. 7.5a.*

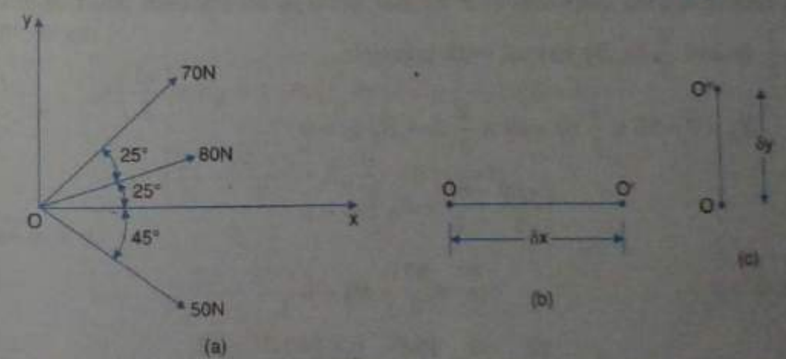


Fig. 7.5

Solution. Let E be the equilibriant of the system which makes anticlockwise angle θ with x-axis. Then the system is in equilibrium with force F. Let us give a virtual displacement of δx in x-direction as shown in Fig. 7.5(b).

Then total virtual work done by the system of given forces is

$$50 \cos 45° \, \delta x + 80 \cos 25° \, \delta x + 70 \cos 50° \, \delta x + E \cos \theta \, \delta x$$

Equating it to zero we get

$$(50 \cos 45° + 80 \cos 25° + 70 \cos 50° + E \cos \theta) \, \delta x = 0$$

Since $\delta x = \neq 0$,

$$50 \cos 45° + 80 \cos 25° + 70 \cos 50° + E \cos \theta = 0$$

or $$-E \cos \theta = 50 \cos 45° + 80 \cos 25° + 70 \cos 50° = 152.86 \text{ N} \quad ...(1)$$

Let us now give virtual displacement δy to point O as shown in Fig. 7.5c. Then virtual work done by the system of forces and the equilibriant are:

$$-50 \sin 45° \, \delta y + 80 \sin 25° \, \delta y + 70 \sin 50° \, \delta y + E \sin \theta \, \delta y$$

Since the system is in equilibrium with E, we get

$$(-50 \sin 45° + 80 \sin 25° + 70 \sin 50° + E \sin \theta) \, \delta y = 0$$

Since $\delta y \neq 0$,

$$-50 \sin 45° + 80 \sin 25° + 70 \sin 50° + E \sin \theta = 0$$

or $$-E \sin \theta = -50 \sin 45° + 80 \sin 25° + 70 \sin 50° = 52.07 \sin 25°$$

Resultant is exactly equal but opposite of equillibriant. Hence, eqns. (1) and (2) can be written as

$$R \cos \theta = 152.86 \text{ N}$$

and $$R \sin \theta = 52.07 \text{ N}$$

$\therefore$ $$R = \sqrt{152.86^2 + 52.07^2} = 161.48 \text{ N}$$

and $$\theta = \tan^{-1} \frac{52.07}{152.86}$$

$\therefore$ $$\theta = 18.81°$$

Example 7.2. *Determine the reactions R_A and R_B developed in the simply supported beam shown in Fig 7.6a.*

Solution. Let R_A and R_B be the reactions at supports A and B. Let δy displacement be given to beam at B without giving any displacement at A. (Ref. Fig. 7.6.)

The corresponding displacements for the beam at 20 kN load point and 40 kN load points are $\frac{1}{3}\delta y$ and $\frac{2}{3}\delta y$. By virtual work principle

$$R_A \times 0 - 20 \times \frac{1}{3} \delta y - 40 \times \frac{2}{3} \delta y + R_B \, \delta y = 0$$

or $$\left(\frac{-20}{3} - \frac{80}{3} + R_B\right) \delta y = 0$$

Since $\delta y \neq 0$ $$\frac{-20}{3} - \frac{80}{3} + R_B = 0$$

or $$R_B = \frac{20}{3} + \frac{80}{3} = \frac{100}{3} = \mathbf{33.33 \text{ kN}} \quad \textbf{Ans.}$$

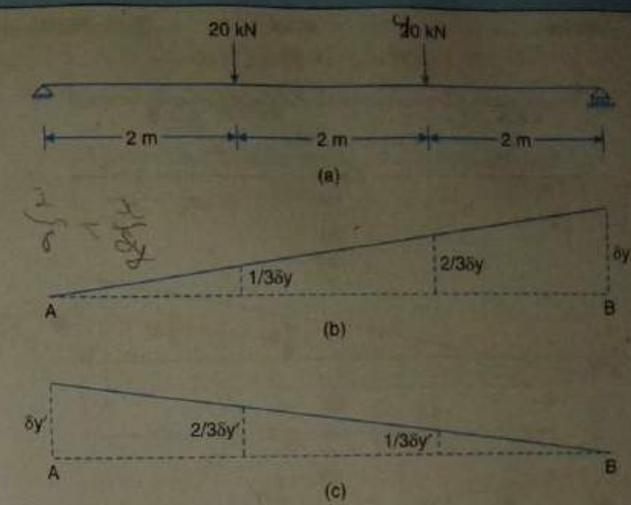


Fig. 7.6

Now let us give virtual displacement to beam as shown in Fig. 7.6c. Since the body is in equilibrium virtual work equation is

$$R_A \, \delta y' - 20 \times \frac{2}{3} \delta y' - 40 \times \frac{1}{3} \delta y' + R_B \times 0 = 0$$

$\therefore$ $$\left(R_A - \frac{40}{3} - \frac{40}{3}\right) \delta y' = 0$$

Since $\delta y \neq 0$, $$R_A - \frac{40}{3} - \frac{40}{3} = 0$$

or $$R_A = \frac{80}{3} = \mathbf{26.667 \text{ kN}} \quad \textbf{Ans.}$$

Example 7.3. *Determine the reactions in the overhanging beam shown in Fig. 7.7a.*

Solution. Let R_A and R_B be the reactions as shown in Fig. 7.7a. Let the virtual displacement δy be given to the beam as shown in Fig. 7.7b. Then the virtual displacement of various loads are as given in the figure. Applying virtual work principle for the beam which is in equilibrium we get

$$20\left(\frac{\delta y}{8}\right) + R_A \times 0 - 60\left(\frac{4}{8} \delta y\right) + R_B \frac{6}{8} \delta y - 30 \times \delta y = 0$$

or $$\left(\frac{20}{8} - \frac{60 \times 4}{8} + R_B \frac{6}{8} - 30\right) \delta y = 0$$

Since $\delta y \neq 0$,

$$\frac{20}{8} - \frac{60 \times 4}{8} + R_B \frac{6}{8} - 30 = 0$$

$\therefore$ $$R_B = \mathbf{76.667 \text{ kN}} \quad \textbf{Ans.}$$

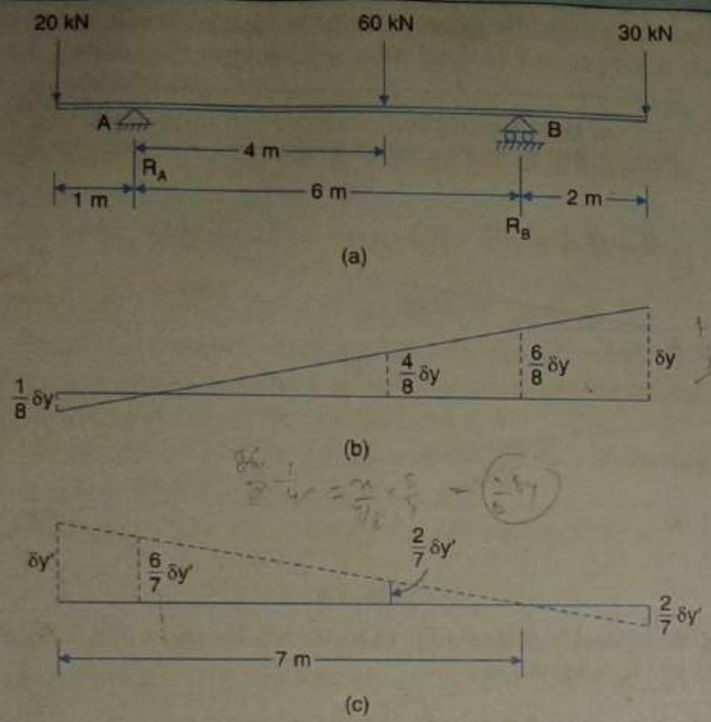


Fig. 7.7

Next, let us give virtual displacement $\delta y'$ to the beam as shown in Fig. 7.7*c*.

The displacements of various load/reaction points are as shown in the figure. From virtual work principle,

$$-20(\delta y') + R_A \frac{6}{7}\delta y' - 60\left(\frac{2}{7}\delta y'\right) + R_B \times 0 - 30\left(-\frac{2}{7}\delta y'\right) = 0$$

$$\left(-20 + R_A \frac{6}{7} - \frac{60 \times 2}{7} + 30 \times \frac{2}{7}\right)\delta y' = 0$$

Since $\delta y' \neq 0$, $$-20 + R_A \frac{6}{7} - \frac{60 \times 2}{7} + 30 \times \frac{2}{7} = 0$$

$$R_A = \mathbf{33.333\ kN} \quad \textbf{Ans.}$$

Example 7.4. *Determine the reaction at A in simply supported beam shown in Fig. 7.8a*

Solution. Let the virtual displacement δy be given at point A as shown in Fig. 7.8*b*.

Total uniformly distributed load = 40 × 2 = 80 kN

Its centre of gravity is at mid point of AC

Displacement of the load at $A = \delta y$

Displacement of the load at $C = \dfrac{\delta y}{2}$

∴ Displacement of 80 kN load at its centroid

$$= \text{Average of } \delta y \text{ and } \frac{\delta y}{2} = \frac{\delta y + \delta y/2}{2} = 0.75\ \delta y$$

Hence, work done by udl

$$= -40 \times 2(0.75\ \delta y) = -60\ \delta y$$

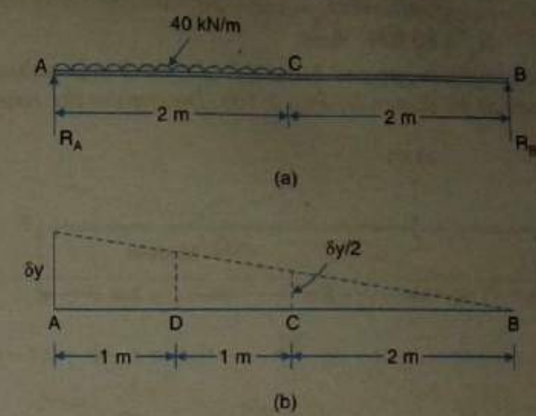


Fig. 7.8

Writing virtual work equation for the system of forces we get

$$R_A\ \delta y - 60\ \delta y + R_B \times 0 = 0$$

$$(R_A - 60)\ \delta y = 0$$

Since, $\delta y \neq 0$ $\quad R_A = \mathbf{60\ kN}$ **Ans.**

Example 7.5. *Determine the reaction at A in the overhanging beam AB as shown in Fig. 7.9.*

Solution. Let the virtual displacement δy be given at point C keeping displacement at B zero as shown in Fig. 7.9*b*. Let R_A and R_B be the reactions at A and B respectively.

The centroid of udl moves down by $\dfrac{0 + 2\delta y}{2} = \delta y$. Total udl = 30 × 6 kN; downward. Hence virtual work equation is

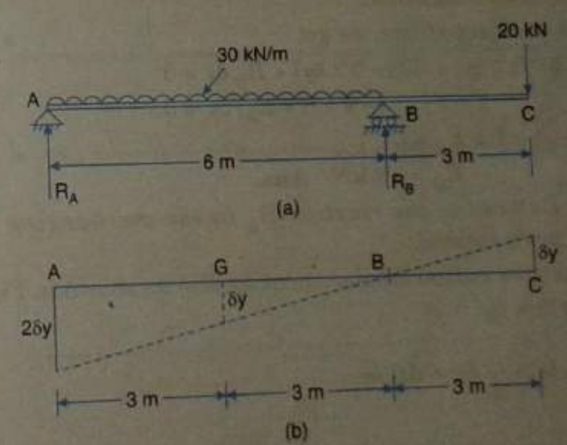


Fig. 7.9

$$R_A(-2\delta y) - 30 \times 6(-\delta y) + R_B \times 0 - 20(\delta y) = 0$$

$$(-2R_A + 180 - 20)\,\delta y = 0$$

Since $\delta y \neq 0$, $\quad -2R_A + 180 - 20 = 0$

$\therefore \quad R_A = \mathbf{80\ kN}$ **Ans.**

Example 7.6. *The simply supported beam AB of span 5 m is subjected to a concentrated load and an external moment as shown in Fig. 7.10a. Determine the reaction at B.*

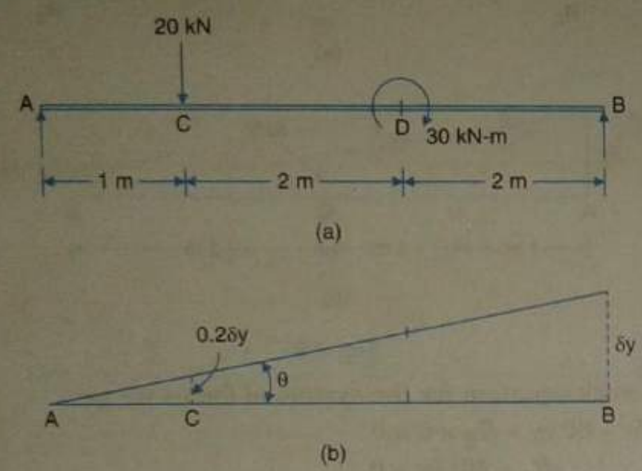


Fig. 7.10

Solution. Let a virtual displacement δy be given at B as shown in Fig. 7.10*b*.

Displacement at $A = 0$

Displacement at $C = 0.2\,\delta y$

Rotation of moment 30 kNm is,

$$\theta = \frac{-\delta y}{5} = -0.2\,\delta y$$

–ve sign since it is anticlockwise.

Writing virtual work equations, we get

$$R_A \times 0 - 20(0.2\,\delta y) + 30(-0.2\,\delta y) + R_B\,\delta y = 0$$

$$(-4 - 6 + R_B)\,\delta y = 0$$

Since $\delta y \neq 0$, $\quad -4 - 6 + R_B = 0$

or $\quad R_B = \mathbf{10\ kN}$ **Ans.**

Example 7.7. *Determine the reaction R_A in the overhanging beam shown in the Fig. 7.11(a) by virtual work method.*

Solution. Let us give virtual displacement δy at A as shown in Fig. 7.11*b*.

Displacement at A is δy

Displacement at D is $\frac{3}{5}\,\delta y = 0.6\,\delta y$.

$\therefore$ Displacement at centroid of udl $= \dfrac{\delta y + 0.6\delta y}{2} = 0.8\,\delta y$

Displacement at $B = 0$

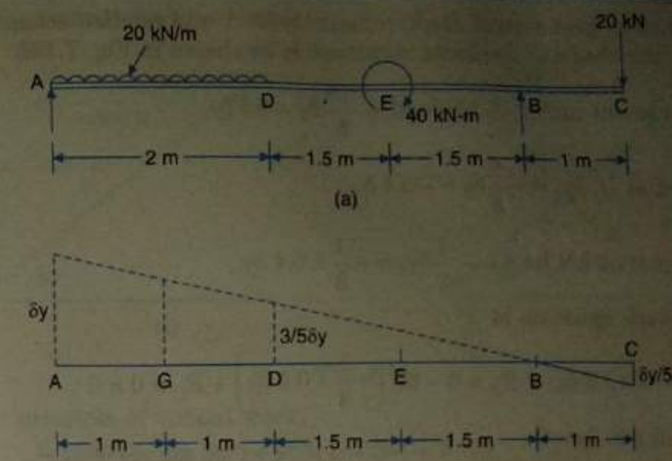


Fig. 7.11

Displacement at C is $-\frac{1}{5}\,\delta y = -0.2\,\delta y$

Rotation of moment $= \frac{\delta y}{5} = 0.2\delta y$

Writing the virtual work equation for the system, we get

$$R_A\,\delta y - 20 \times 2(0.8\,\delta y) + 40(0.2\,\delta y) + R_B \times 0 - 20(-0.2\,\delta y) = 0$$

$$(R_A - 32 + 8 + 0 + 4)\,\delta y = 0$$

Since $\delta y \neq 0$

$$R_A - 32 + 8 + 4 = 0 \quad \text{or} \quad R_A = \mathbf{20\ kN} \quad \textbf{Ans.}$$

Example 7.8. *Determine the reaction R_A in the beam as shown in Fig 7.12a*

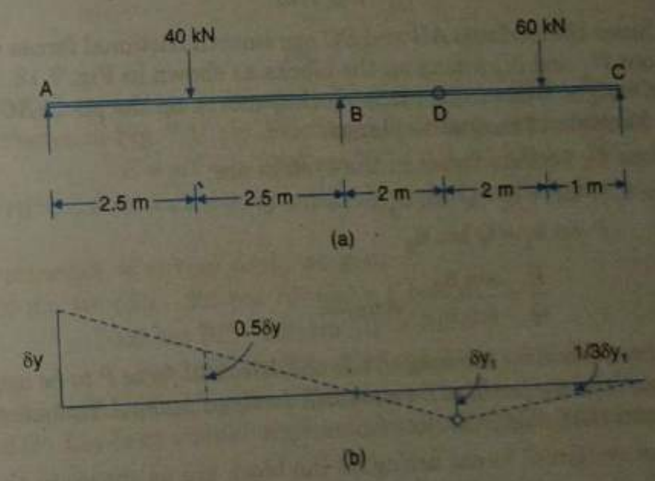


Fig. 7.12

Solution. Let us give virtual displacement δy at A and no displacements at B and C. Since D is hinged, the shape of displaced structure is as shown in Fig. 7.12*b*.

Now, displacement under 40 kN load $= \frac{2.5}{5}\delta y = 0.5\,\delta y$

Displacement at D, $\delta y_1 = -\frac{2}{5}\delta y = -0.4\,\delta y$

Displacement of 60 kN load $= -\frac{1}{3}\delta y_1 = -\frac{1}{3}\times 0.4\,\delta y$

$\therefore$ Virtual work equation is

$$R_A\,\delta y - 40\,(0.5\,\delta y) + R_B \times 0 - 60\left(-\frac{1}{3}\times 0.4\,\delta y\right) + R_C \times 0 = 0$$

$\therefore \quad (R_A - 20 + 8)\,\delta y = 0$

Since $\delta y \neq 0$,

$R_A - 20 + 8 = 0$ or $R_A = \mathbf{12\ kN}$ **Ans.**

Example 7.9. *The surfaces AB and BC shown in Fig 7.13 are smooth. Determine the ratio of loads P and Q for which the system is in equilibrium.*

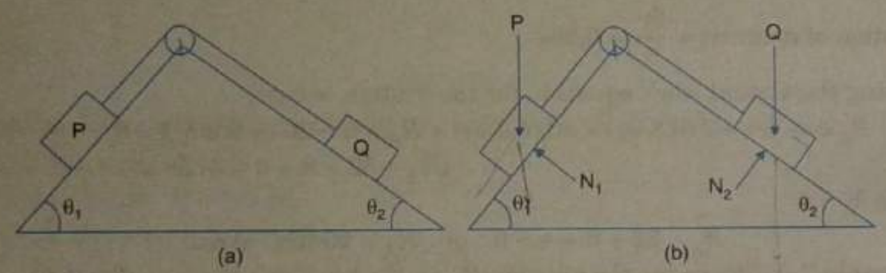


Fig. 7.13

Solution. Since the surfaces AB and BC are smooth frictional forces do not exist. There are normal reactions N_1 and N_2 acting on the blocks as shown in Fig. 7.13. If displacement δs is given to block of weight P down the plane block Q moves up the plane BC by δs. There is no movement in the direction of normal to planes.

The work done by various forces in the system are:

$$-P\sin\theta_1\,(-\delta s) + 0 - Q\sin\theta_2\,(\delta s) + 0 = 0$$

i.e., $\quad P\sin\theta_1 = Q\sin\theta_2$

or $\quad \frac{P}{Q} = \frac{\sin\theta_2}{\sin\theta_1}$ **Ans.**

Example 7.10. *Determine the magnitude of horizontal force P to be applied to a block of weight 1500 N to hold it in the position on a smooth inclined plane AB which makes an angle of 30° with the horizontal (Ref. Fig. 7.14. a).*

Solution. The system of forces acting on the block are as shown in the Fig. 7.14(*b*).

Since the plane is smooth no frictional force acts. Let the block be given a virtual displacement δs up the plane.

Then displacement in vertical direction $= \delta s \sin\theta$

Then displacement in horizontal direction $= \delta s \cos\theta$

Then displacement in normal to plane = 0

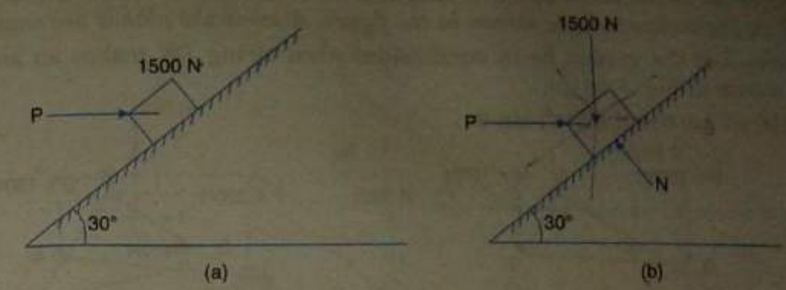


Fig. 7.14

$\therefore$ From principle of virtual work,

$$-1500\,(\delta s \sin 30°) + P\,\delta s \cos 30° + 0 = 0.$$

or $\quad P\cos 30° = 1500 \sin 30°$

$P = 1500 \tan 30° = \mathbf{866.03\ N}$ **Ans.**

Example 7.11. *A block weighing 1000N is to be held on an inclined plane AB which makes angle of 70° with horizontal. For what range of force P it can be kept in equilibrium. Coefficient of friction between the plane and the block is 0.3 (Ref. Fig. 7.15a).*

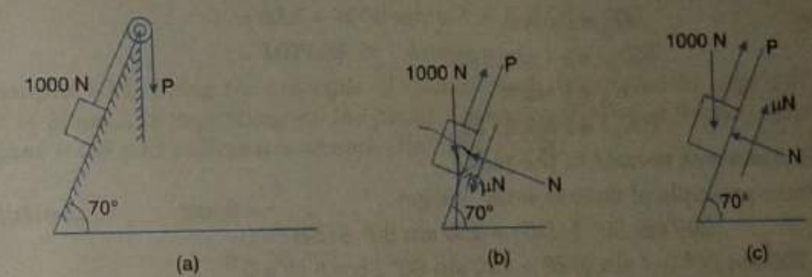


Fig. 7.15

Solution. (*a*) When motion is impending up the plane: The system of forces acting in this situation are shown in Fig. 7.15 (*b*). Frictional force acts down the plane

$$F = \mu N = \mu\,1000\cos 70° = 0.3 \times 1000 \cos 70° = 300 \cos 70°$$

Let us give virtual displacement δs up the plane. Let the direction up the plane be taken +ve.

Then from principle of virtual work, we get

$$-1000 \sin 70°\,(\delta s) - 300 \cos 70°\,(\delta s) + P\,\delta s = 0$$

$\therefore \quad -1000 \sin 70° - 300 \cos 70° + P = 0$

or $\quad P = 1000 \sin 70° + 300 \cos 70° = 1042.3$ N

(*b*) When motion is impending down the plane. The frictional force acts up the plane as shown in Fig. 7.15 (*c*). Let δs be virtual displacement up the plane. Then virtual work equation is,

$$-1000 \sin 70°\,\delta s + F\,\delta s + P\,\delta s = 0$$

$\therefore \quad -1000 \sin 70° + 300 \cos 70° + P = 0$

or $\quad P = 1000 \sin 70° - 300 \cos 70° = 837.1$ N

Hence, the block is in equilibrium for $P = 837.1$ N to 1042.3 N.

Example 7.12. *Ball 1 shown in Fig. 7.16 weighs 200 N and ball 2 weighs 150 N. They are connected by an inextensible string of length l. Determine the position of the balls if the system is left on the inclined plane shown in the figure. Assume the planes are smooth.*

Solution. Let the system be in equilibrium when string DE makes an angle θ with plane AB as shown in Fig. 7.16 (*b*).

Now, $DB = l \cos\theta$ and $EB = l \sin\theta$

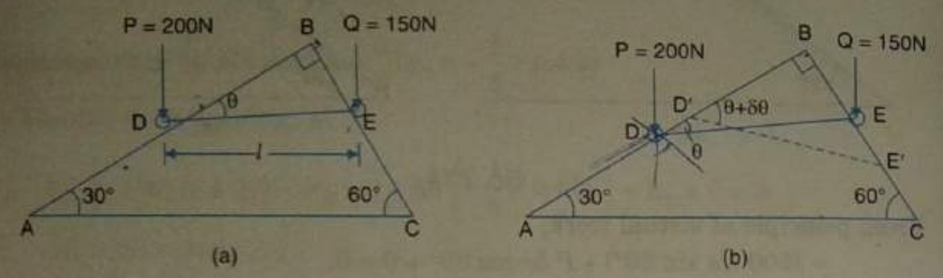


Fig. 7.16

Let a virtual displacement be given to the system of balls so that they move to $D'E'$ i.e. by an angle $\delta\theta$.

Component of P down the plane $= P \sin 30° = 200 \sin 30°$

Component of Q down the plane $= Q \sin 60° = 150 \sin 60°$

Since $\quad D_B = l \cos\theta$

$$\delta(D_B) = -l \sin\theta\, \delta\theta$$

and $\quad E_B = l \sin\theta$

$$\delta(E_B) = l \cos\theta \cdot \delta\theta$$

No displacement normal to the plane.

$\therefore$ From principle of virtual work, we get

$$200 \sin 30° \, \delta(DB) + 150 \sin 60° \, \delta(EB) = 0$$

$$200 \sin 30° (-l \sin\theta)\, \delta\theta + 150 \sin 60° \, l \cos\theta \, \delta\theta = 0$$

or
$$\tan\theta = \frac{150 \sin 60°}{200 \sin 30°} = 1.299$$

$$\theta = \mathbf{52.41°} \quad \textbf{Ans.}$$

Example 7.13. *For the system of connected bodies as shown in Fig. 7.17a determine the force P required to make the motion impending to the left. Coefficient of friction for all contacting surfaces is 0.25. Pulleys are smooth.*

Solution. Since motion is impending to the left, frictional forces are acting in the directions shown in Fig. 7.17(*b*).

$$F_1 = 0.25\, N_1 = 0.25 \times 250 = 62.5 \text{ N}$$
$$F_2 = 0.25\, N_2 = 0.25 \times 1000 \cos 45° = 176.78 \text{ N}$$
$$F_3 = 0.25\, N_3 = 0.25 \times 500 = 12.5 \text{ N}$$

Let us give a virtual displacement towards left. Taking the displacement of the system to left as +ve, virtual work equation is

$$P\delta s - F_1\, \delta s - 1000 \sin 45° \, \delta s - F_2\, \delta s - F_3\, \delta s = 0$$

$$\therefore \quad (P - F_1 - 1000 \sin 45° - F_2 - F_3)\, \delta s = 0$$

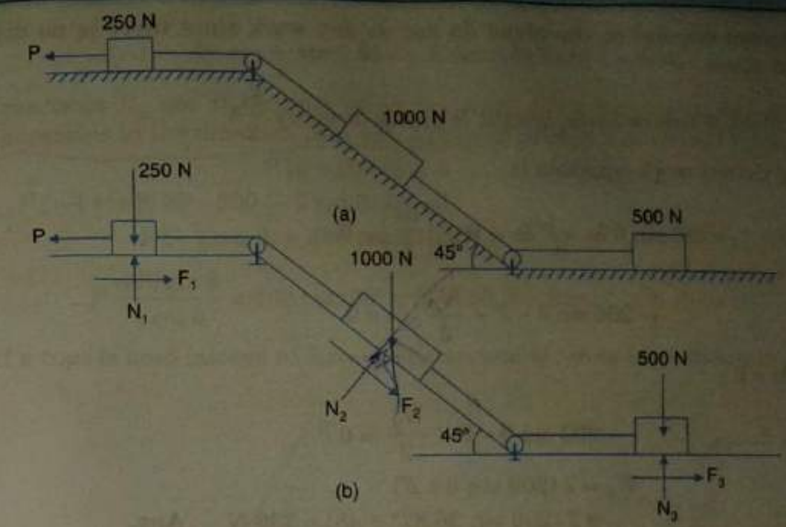


Fig. 7.17

Since $\delta s \neq 0$,

$$P = F_1 + 1000 \sin 45° + F_2 + F_3$$
$$= 62.5 + 1000 \sin 45° + 176.78 + 125$$
$$= \mathbf{1071.38\ N} \quad \textbf{Ans.}$$

Example 7.14. *Using the principle of virtual work determine the weight W_B for which the motion of block A is impending up the plane. Given: coefficient of friction between block A and the plane = 0.3 and pulleys are smooth (Ref. Fig. 7.18a)*

Solution. $\quad \tan\theta = \dfrac{3}{4}$

$$\theta = 36.87°$$
$$F = \mu N = \mu W_A \cos\theta$$
$$= 0.3 \times 200 \cos 36.87° = 48 \text{ N}.$$

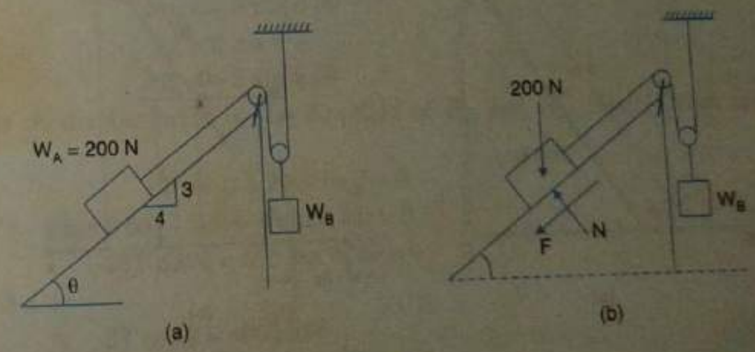


Fig. 7.18

Let us give virtual displacement of δs up the plane to block A.

Component of block A down the plane $= W_A \sin\theta$

Component normal to the plane do not do any work since there is no displacement normal to the plane.

When block A moves by δs, weight B moves down by $\frac{\delta s}{2}$.

Hence, virtual work equation is

$$-200 \sin\theta\, \delta s - F\, \delta s + W_B \left(\frac{\delta s}{2}\right) = 0$$

or
$$\left(-200 \sin\theta - F + \frac{W_B}{2}\right) \delta s = 0$$

Since $\delta s \neq 0$

$$-200 \sin\theta - F + \frac{W_B}{2} = 0$$

or
$$W_B = 2\,(200 \sin\theta + F)$$
$$= 2\,(200 \sin 36.87° + 48) = \mathbf{336\ N} \quad \textbf{Ans.}$$

Example 7.15. *Ladder AB of length 4 m and weight 200 N is held in position by applying force P as shown in Fig. 7.19. Assuming smooth wall and floor, determine the force P if instead of force P for preventing from slipping, a horizontal rope is tied at a distance of 1 m from A, what is the tension in the rope ?*

Solution. (*a*) P at floor level

$$AB = l = 4\text{m}$$

Let ladder rotate by angle $\delta\theta$ as shown in Fig. 7.19(*b*).

Taking C as origin.

Coordinate of $A = x = -4 \sin\theta$

$$\therefore \quad \delta x = -4 \cos\theta\, \delta\theta$$

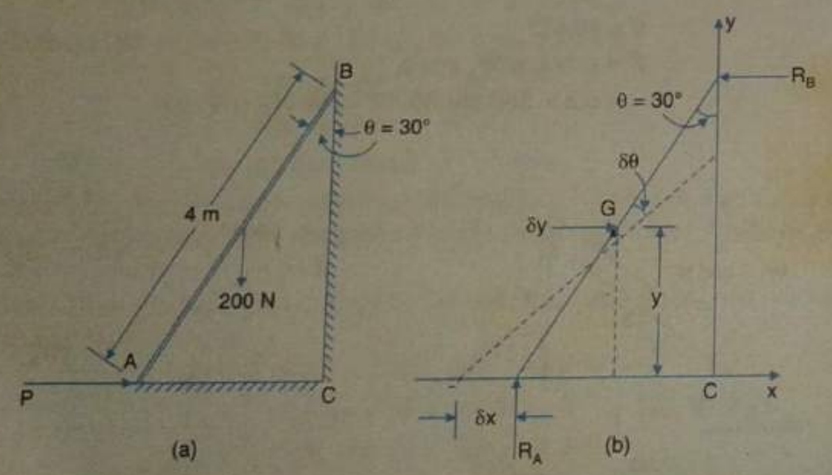


Fig. 7.19

P moves by $-4 \cos\theta\, \delta\theta$

Now,
$$y = \frac{l}{2} \cos\theta$$

$$\delta y = -\frac{l}{2} \sin\theta\, \delta\theta = -2 \sin\theta\, \delta\theta \text{ since } l = 4 \text{ m}.$$

The reactions R_A and R_B are not doing any work since there is no movement of their point of applications in the direction they act. Applying principle of virtual work, we get

$$P\, \delta x - 200\, \delta y = 0$$
$$P\,(-4 \cos\theta)\, \delta\theta - 200\,(-2 \sin\theta)\, \delta\theta = 0$$
$$P\,(-4 \cos\theta) + 400 \sin\theta = 0$$

or
$$P\, \frac{400 \sin\theta}{4 \cos\theta} = 100 \tan 30° = \mathbf{57.735\ N} \quad \textbf{Ans.}$$

(*b*) If a rope is used instead of force P, the system of forces is as shown in Fig. 7.20*a*.

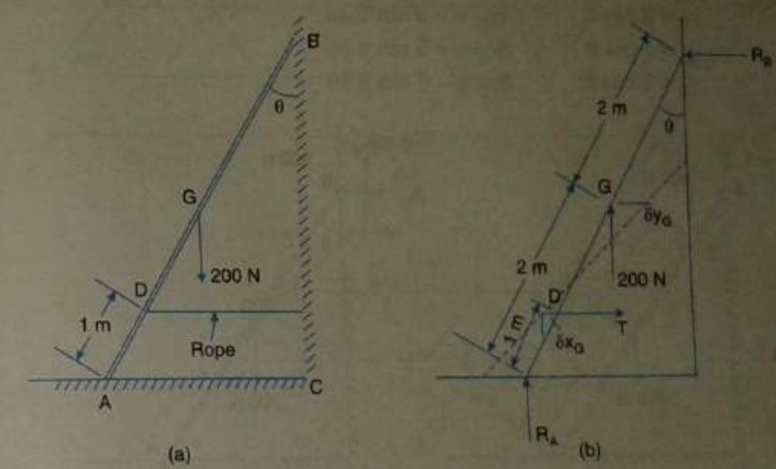


Fig. 7.20

The system of forces in this case are as shown in Fig. 7.20*b*. Taking C as origin,

$$x_D = -3 \sin\theta$$
$$\delta x_D = -3 \cos\theta\, \delta\theta$$
$$y_G = 2 \cos\theta$$
$$\delta y_G = -2 \sin\theta\, \delta\theta$$

There is no displacement in the direction of R_B and R_A. Hence from the principle of virtual work,

$$T\,(\delta x_D) + 200\,(\delta y_G) = 0$$
$$T\,(-3 \cos\theta\, \delta\theta) - 200\,(-2 \sin\theta\, \delta\theta) = 0$$
$$(-3T \cos\theta + 400 \sin\theta)\, \delta\theta = 0$$

Since $\delta\theta \neq 0$

$$3T \cos\theta = 400 \sin\theta$$

$$T = \frac{400}{3} \tan\theta = \frac{400}{3} \tan 30° = \mathbf{76.98\ N} \quad \textbf{Ans.}$$

Example 7.16. *The ladder shown in Fig. 7.21a is 6 m long and is supported by a horizontal floor and vertical wall. The coefficient of friction between the floor and the ladder is 0.4 and between the wall and the ladder is 0.25. The weight of the ladder is 200 N. The ladder also supports a vertical load of 900 N at C which is at a distance of 1 m from B. Determine the least value of θ for which the ladder may be placed without slipping. Determine the reaction at this stage.*

Solution. The various forces acting on the ladder when it is on the verge of slipping are as shown in Fig. 7.21*b*. Note frictional forces always oppose impending motions and they are equal to μ × normal reaction where μ is coefficient of friction. To calculate virtual works we need the displacement in the direction of forces. Normal reactions N_A and N_B are not going to do any work since displacements in direction of those forces is always zero. Taking *O* as origin,

now, $x_A = 6 \sin\theta \qquad \delta x_A = 6 \cos\theta\, \delta\theta$

$y_G = 3 \cos\theta \qquad \delta y_G = -3 \sin\theta\, \delta\theta$

$y_C = 5 \cos\theta \qquad \delta y_C = -5 \sin\theta\, \delta\theta$

and $y_B = 6 \cos\theta \qquad \delta y_B = -6 \sin\theta\, \delta\theta$

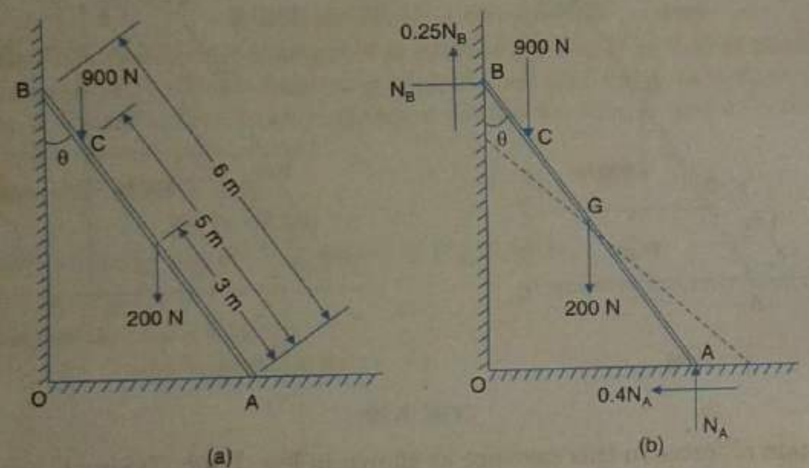


Fig. 7.21

Applying principles of virtual work we get,

$$-0.4\, N_A\, 6 \cos\theta\, \delta\theta - 200\,(-3 \sin\theta\, \delta\theta) - 900\,(-5 \sin\theta\, \delta\theta) + 0.25\, N_B\,(-6 \sin\theta\, \delta\theta) = 0$$

Since $\delta\theta \neq 0$,

$$-2.4\, N_A \cos\theta + 600 \sin\theta + 4500 \sin\theta - 1.5\, N_B \sin\theta = 0 \qquad ...(1)$$

From the equilibrium conditions,

$$\Sigma F_x = 0 \quad \rightarrow 0.4\, N_A = N_B$$

and $$\Sigma F_y = 0 \quad \rightarrow N_A + 0.25\, N_B = 900 + 200 = 1100$$

$$N_A + 0.25 \times 0.4\, N_A = 1100$$

$\therefore$ $$N_A = 1000 \text{ N}$$

Hence $$N_B = 0.4 \times 1000 = 400 \text{ N}$$

Substituting these values in eqn. (1), we get

$$-2.4 \times 1000 \cos\theta + 600 \sin\theta + 4500 \sin\theta - 1.5 \times 400 \sin\theta = 0.$$

i.e., $$4500 \sin\theta = 2400 \cos\theta$$

or $$\tan\theta = \frac{2400}{4500}$$

$$\theta = 28.073 \quad \textbf{Ans.}$$

Example 7.17. *Determine the force in the member FH of the truss shown in Fig 7.22. Each load is 10 kN and all triangles are equilaterals with sides 4 m.*

Solution. Due to symmetry

$$R_A = R_B = \frac{1}{2} \text{ total load} = \frac{1}{2} \times 10 \times 7 = 35 \text{ kN}$$

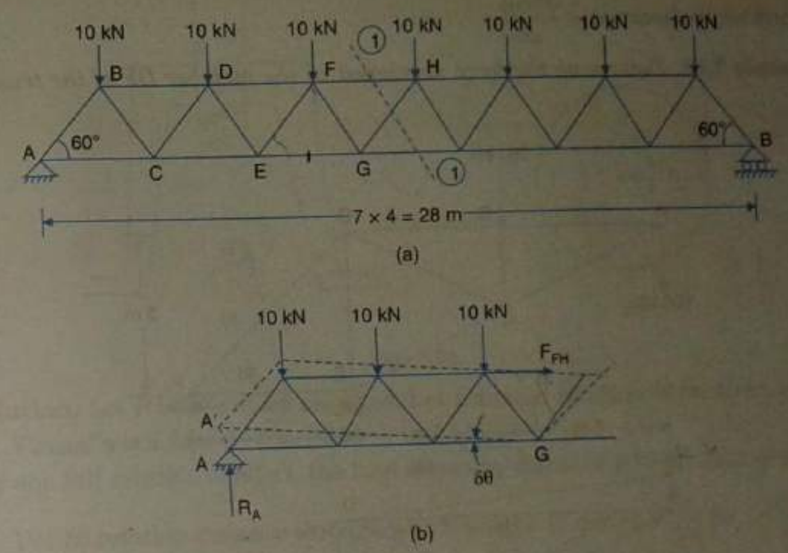


Fig. 7.22

Take section 1-1 as shown in Fig. 7.22 (*a*) and consider left side portion as shown in Fig. 7.22 (*b*). Imagine a rotation of δθ about point *G*.

Vertical displacement of $A = 12\, \delta\theta$

Vertical displacement of $B = 10\, \delta\theta$

Vertical displacement of $D = 6\, \delta\theta$

and Vertical displacement of $F = 2\, \delta\theta$

Horizontal displacement of $F_{FH} = 2 \tan 60°$ towards right

$\therefore$ From the principle of virtual displacement, we have

$$R_A\, 12\, \delta\theta - 10 \times 10\, \delta\theta - 10 \times 6\, \delta\theta - 10 \times 2\, \delta\theta + F_{FH}\, 2 \tan 60° = 0$$

$$2\, F_{FH} \tan 60° = -R_A\, 12 + 100 + 60 + 20$$

$$= -35 \times 12 + 180 = -240$$

$$F_{FH} = -\frac{240}{2 \tan 60°} = -69.28 \text{ kN}$$

i.e., F_{FH} direction is to the reverted

i.e., F_{FH} is compressive force of **69.28 kN** **Ans.**

Note: The difference between the displacement in this case and the ladder problem. Here the rotation is free rotation about a point. Hence

vertical displacement = horizontal coordinate × $\delta\theta$

and horizontal displacement = vertical distance × $\delta\theta$

In ladder problem, the two edges are constrained to move along the specified two edges. In such cases:

Vertical displacement (– y direction) of point D

$$= \frac{d(y_D)}{dy}$$

and horizontal displacement $= \dfrac{d(x_D)}{dx}$

Example 7.18. *Determine the force developed in the member DF of the truss shown in Fig. 7.23.*

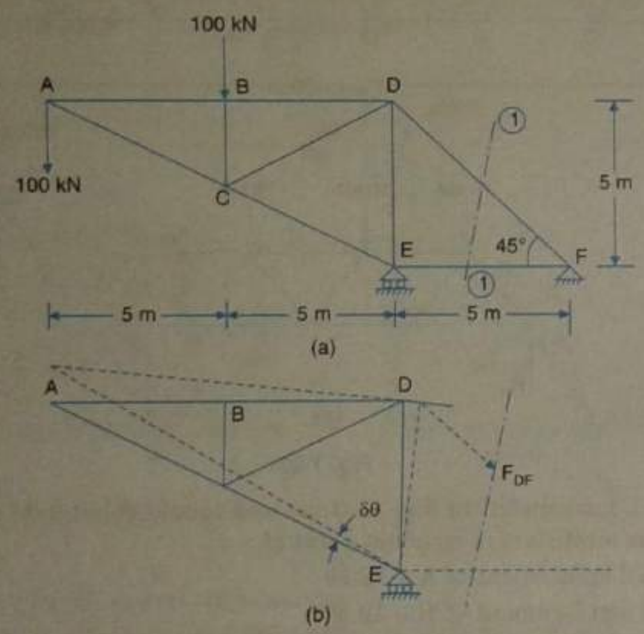


Fig. 7.23

Solution. Take the section 1-1 as shown in Fig. 7.23*a* and consider left side portion. Imagine a virtual rotation $\delta\theta$ at support E. Then

Vertical displacement of point $A = 10\ \delta\theta$

Vertical displacement of point $B = 5\ \delta\theta$

Vertical displacement of point $D = 0$

Horizontal displacement of point $D = 5\ \delta\theta$

∴ From virtual work principle,

$$-100 \times 10\ \delta\theta - 100 \times 5\ \delta\theta + F_{DF} \cos 45° \times 5\ \delta\theta = 0$$

$$\therefore \quad F_{DF}\, 5 \cos 45° = 100 \times 10 + 100 \times 5 = 1500$$

$$\therefore \quad F_{DF} = \frac{1500}{5 \cos 45°} = 300\sqrt{2} = 424.26 \text{ kN (tensile)}.$$

Example 7.19. *Neglecting the friction, determine the effort required to lift a load by the screw jack shown in Fig. 7.24. Take pitch of the jack = p.*

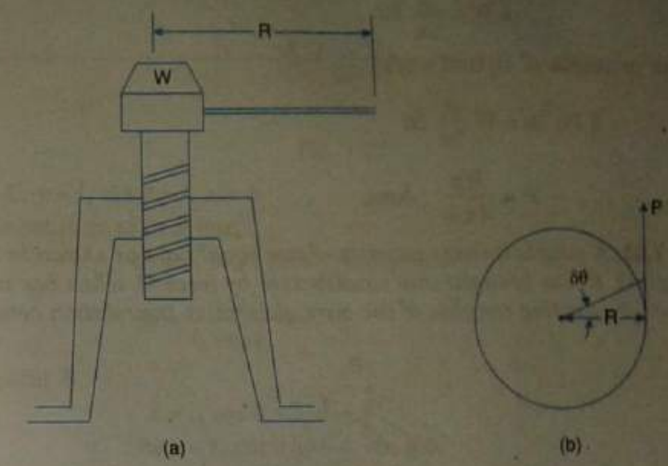


Fig. 7.24

Solution. Let P be the effort required. Let a virtual rotation $\delta\theta$ be given to effort.

∴ Virtual work done by $P = PR\delta\theta$

For one full rotation of effort, the load moves by distance p (definition of pitch).

∴ For $\delta\theta$ rotation distance through which weight W moves $= \dfrac{p}{2\pi}\delta\theta$

From virtual work principle, total work done is zero.

$$\therefore \quad PR\delta\theta = W\frac{p}{2\pi}\delta\theta \quad \text{or} \quad P = \frac{Wp}{2\pi R} \quad \textbf{Ans.}$$

Example 7.20. *Neglecting friction, determine the pressure applied by screw press when effort P is applied (Ref. Fig. 7.25a)*

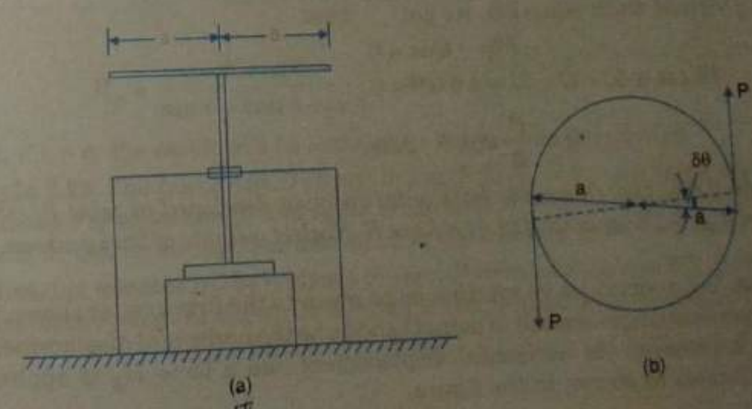


Fig. 7.25

Solution. Let a rotation $\delta\theta$ be given to the screw press by applying efforts P as shown in Fig. 7.25*b*. Work done by effort.

$$= 2Pa\ \delta\theta = 2Pa\ \delta\theta$$

If p is the pitch of the screw, work done by the load W is

$$= W \times \frac{p}{2\pi}\ \delta\theta$$

Hence from principle of virtual work

$$2\ Pa\ \delta\theta = W\ \frac{p}{2\pi}\ \delta\theta$$

or

$$P = \frac{Wp}{4\pi\, a} \quad \textbf{Ans.}$$

Example 7.21. *A simple linkage consists of two equal bars as shown in Fig. 7.26. When a force P was applied, it was brought into equilibrium by force Q, when bar made angle θ as shown in the figure. Neglecting weights of the bars, determine the relation between θ and F.*

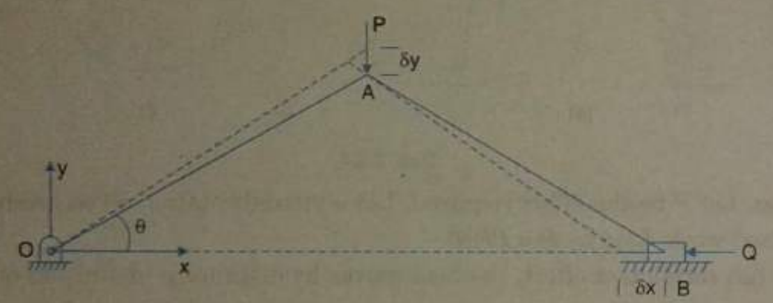


Fig. 7.26

Solution. Let a virtual displacement $\delta\theta$ be given to linkage as shown in the figure. For +ve value of $\delta\theta$ the point P moves up and the force Q moves to the left.

For point A, $\quad y = l \sin\theta$

$\therefore \quad \delta y = l \cos\theta\ \delta\theta$

For point B, $\quad x = l\cos\theta + l\cos\theta = 2l\cos\theta$

$\therefore \quad \delta x = -2l \sin\theta\ \delta\theta$

Writing virtual work equation, we get

$$-P\delta y - Q\delta x = 0$$

$$-Pl\cos\theta\ \delta\theta - Q\,(-2l\sin\theta)\ \delta\theta = 0$$

$$\therefore \quad Q = \frac{P}{2}\cot\theta \quad \textbf{Ans.}$$

Example 7.22. *Determine the horizontal reaction developed at joint B, when the rigid frame shown in Fig. 7.27 (a) is loaded by a force F. Neglect weights of bars and use virtual work method.*

Solution. Let a virtual + ve rotation $\delta\theta$ be given to the first arm as shown in the figure 27 (*b*). 'δy' be vertical displacement of point C and δx be the horizontal displacement of support if restraint is removed for horizontal displacement and a force H_B is applied. Selecting rtesian coordinates as shown in the figure,

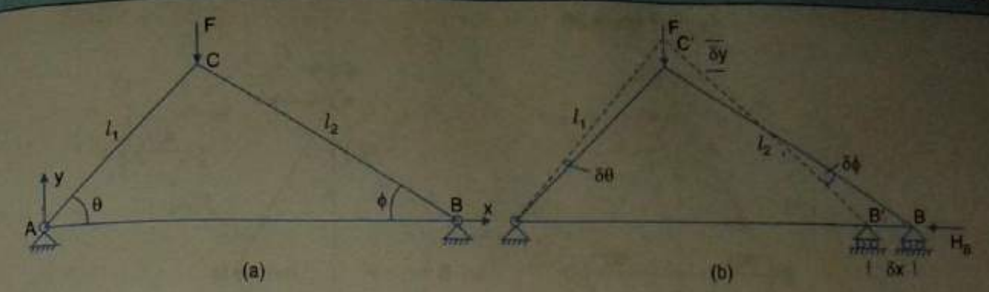


Fig. 7.27

For point C, $y = l_1 \sin\theta = l_2 \sin\phi$ ...(1)

If $\delta\theta$ is the rotation of first bar,

$$\delta y = l_1 \cos\theta\ \delta\theta = l_2 \cos\phi\ \delta\phi \qquad ...(2)$$

$$\therefore \quad \delta\phi = \frac{l_1 \cos\theta}{l_2 \cos\phi}\ \delta\theta \qquad ...(3)$$

Now for point B,

$$x = l_1 \cos\theta + l_2 \cos\phi$$

$$\therefore \quad \delta x = -l_1 \sin\theta\ \delta\theta - l_2 \sin\phi\ \delta\phi$$

$$= -l_1 \sin\theta\ \delta\theta - l_2 \sin\phi\ \frac{l_1}{l_2}\frac{\cos\theta}{\cos\phi}\ \delta\theta$$

$$= \left(-l_1 \sin\theta - l_1 \frac{\sin\phi}{\cos\phi}\cos\theta\right)\delta\theta \qquad ...(4)$$

Writing virtual work equation for the system, we get

$$(-F)\ \delta y - H_B\ \delta x = 0$$

$$(-F)\ l_1 \cos\theta\ \delta\theta - H_B\left(-l_1 \sin\theta - l_1 \frac{\sin\phi}{\cos\phi}\cos\theta\right)\delta\theta = 0$$

$$\therefore \quad H_B = \frac{F\cos\theta}{\sin\theta + \dfrac{\sin\phi}{\cos\phi}\cos\theta} = \frac{F\cos\theta}{\sin\theta + \tan\phi\cos\theta}$$

Thus $\quad \boldsymbol{H_B = \dfrac{F\cos\theta}{\sin\theta + \tan\phi\cos\theta}}$ **Ans.**

[Check, if $\theta = \phi$, the result will be same as in the previous example].

Example 7.23. *The frame ABCD shown in Fig. 7.28 (a) is a pin jointed plane frame with all the five members of equal length and equal cross-section. If joint C is subjected to a vertical downward load W, determine the force developed in the member BD. Use virtual work method.*

Solution. Let member BD be replaced by force P in it, as shown in Fig. 7.28 (*b*). Selecting cartesian coordinate system as shown in the figure and taking length of each member as 'l'

$$x_B = -l\cos\theta$$

$\therefore \quad \delta x_B = l \sin\theta\, \delta\theta$

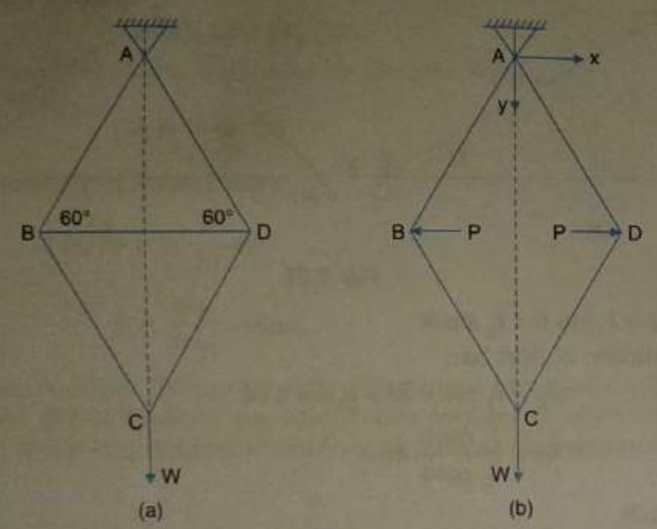


Fig. 7.28

Similarly $x_D = l \cos\theta$

$\therefore \quad \delta x_D = -l \sin\theta\, \delta\theta$

$y_C = 2 \times l \sin\theta$

$\therefore \quad \delta y_C = 2l \cos\theta\, \delta\theta$

Writing virtual work equation for the system, we get

$$-P\,\delta x_B + P\,\delta x_D + W\,\delta y_C = 0.$$

$$-P\,(l \sin\theta\,\delta\theta) + P\,(-l \sin\theta\,\delta\theta) + W\,2l \cos\theta\,\delta\theta = 0$$

i.e., $\quad P \sin\theta = W \cos\theta$

or $\quad P = W \cot\theta$

Since Δ ABD is equilateral triangle, $\theta = 60°$

Hence $\quad \boldsymbol{P = \dfrac{W}{\sqrt{3}}}$ **Ans.**

Example 7.24. *Determine the horizontal and vertical component of reactions developed at end B of the pin jointed plane frame loaded as shown in Fig 7.29 (a).*

Solution. (*i*) Horizontal reaction:

Removing the restraint to support B and allowing it to roll in x direction with a horizontal force H_B, let the angle of rotation of the members of frame be $\delta\theta$.

Now, $\quad y_C = a \sin\theta$

$\therefore \quad \delta y_C = a \cos\theta\, \delta\theta$

$y_D = a \sin\theta$

$\therefore \quad \delta y_D = a \cos\theta\, \delta\theta$

$x = 6\,a \cos\theta$

$\therefore \quad \delta x = -6\,a \sin\theta\, \delta\theta$

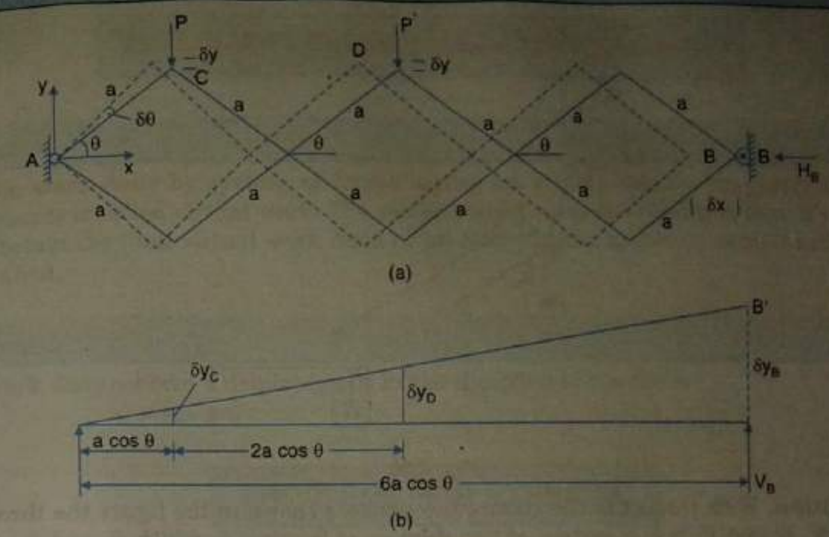


Fig. 7.29

Considering work done by all the active forces, virtual work equation is

$$(-P)\,\delta y_C + (-P)\,\delta y_D + (-H_B)\,\delta x = 0$$

$$-Pa \cos\theta\,\delta\theta - Pa \cos\theta\,\delta\theta - H_B\,(-6a \sin\theta)\,\delta\theta = 0$$

i.e., $\quad \boldsymbol{H_B = \dfrac{2P \cos\theta}{6 \sin\theta} = \dfrac{P}{3} \cot\theta}$ **Ans.**

(*ii*) Vertical reaction at B

Let δy be vertical displacement given to joint-B after removing restraint in vertical direction. The corresponding vertical force at B be V_B. (Ref Fig. 7.29 *b*)

Vertical displacement of C

$$\delta y_C = \frac{a \sin\theta}{6a \sin\theta}\,\delta y_B = \frac{1}{6}\,\delta y_B$$

Vertical displacement of D:

$$\delta y_D = \frac{3a \sin\theta}{6a \sin\theta}\,\delta y_B = \frac{1}{2}\,\delta y_B$$

$\therefore$ Virtual work equation is,

$$(-P)\,\delta y_C + (-P)\,\delta y_D + V_B \delta y_B = 0$$

$$-P\,\frac{1}{6}\,\delta y_B - P\,\frac{1}{2}\,\delta y_B + V_B\,\delta y_B = 0$$

$\therefore \quad \boldsymbol{V_B = \dfrac{4}{6} P = 2/3\,P.}$ **Ans.**

Example 7.25. *An inextensible string is subjected to equal loads W at distance L and 2L from end A. If a horizontal force F is applied at other end, the equilibrium position is as shown in Fig 7.30. Express the angles θ_1 and θ_2 with the vertical in terms of F and W.*

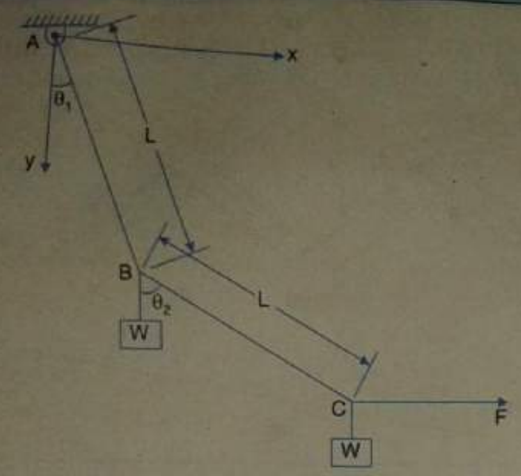


Fig. 7.30

Solution. With respect to the coordinate systems shown in the figure the three active forces are W, W and F. It is a system of two degrees of freedom in which θ_1 and θ_2 can vary independently.

∴ Principle of virtual work is

$$\delta U = \frac{\partial U}{\partial \theta_1}\delta\theta_1 + \frac{\partial U}{\partial \theta_2}\delta\theta_2 = 0$$

Keeping θ_1 constant and varying θ_2, the principle of virtual equation is

$$W\,\delta y_C + F\,\delta x = 0$$

$$W\frac{\partial}{\partial \theta_2}(L\cos\theta_2)\,\delta\theta_2 + F\frac{\partial}{\partial \theta_2}(L\sin\theta_2)\,\delta\theta_2 = 0$$

$$-WL\sin\theta_2 + FL\cos\theta_2 = 0.$$

∴ $\tan\theta_2 = F/W$ **Ans.**

Now, keeping θ_2 constant and varying θ_1,

$$y_B = L\cos\theta_1$$

∴ $$\delta y_B = -L\sin\theta_1\,\delta\theta_1$$

$$y_C = L\cos\theta_1 + L\cos\theta_2$$

Since θ_2 is constant,

$$\delta y_C = -L\sin\theta_1\,\delta\theta_1$$

$$x_C = L\sin\theta_1 + L\sin\theta_2$$

$$\delta x_C = L\cos\theta_1\,\delta\theta_1 \quad \text{(since } \theta_2 \text{ is constant.)}$$

Writing virtual work equation, we get

$$W\,\delta y_B + W\,\delta y_C + F\,\delta x_C = 0$$

i.e., $$W(-L\sin\theta_1)\,\delta\theta_1 + W(-L\sin\theta_1)\,\delta\theta_1 + F(L\cos\theta_1)\,\delta\theta_1 = 0$$

∴ $$\tan\theta_1 = \frac{F}{2W}$$ **Ans.**

IMPORTANT DEFINITIONS AND FORMULAE

Definition

The work done by system of forces acting on a body due to imaginary consistent displacements is called virtual work. The method which uses the condition that, if the body is in equilibrium the total virtual work done by all forces in the system is zero, is called virtual work method.

Equations

Work done = Force × displacement in the direction of the force.

$= Fd$ or $M\theta$

PROBLEMS FOR EXERCISE

7.1 Determine the vertical reaction developed at support B in the beam shown in Fig. 7.31.

[**Ans.** V_B = 35 kN use virtual work method].

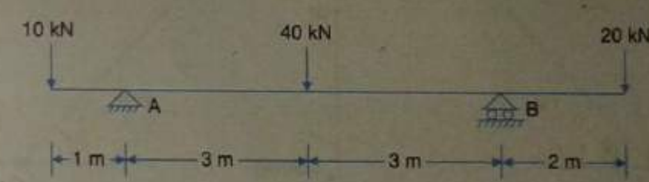


Fig. 7.31

7.2 Determine the reaction V_B in the overhanging beam shown in Fig 7.32 by virtual work method.

[**Ans.** V_B = 38.33 kN]

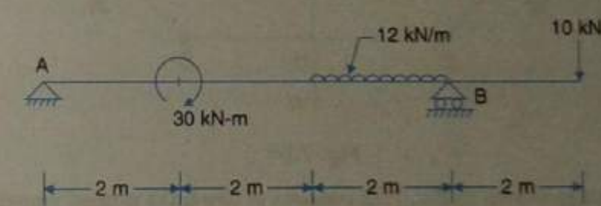


Fig. 7.32

7.3 Using virtual work method determine the force F required to hold the frame in equilibrium when subjected to a force P as shown in Fig 7.33.

$$\left[\textbf{Ans. } F = \frac{P}{2}\tan\theta\right]$$

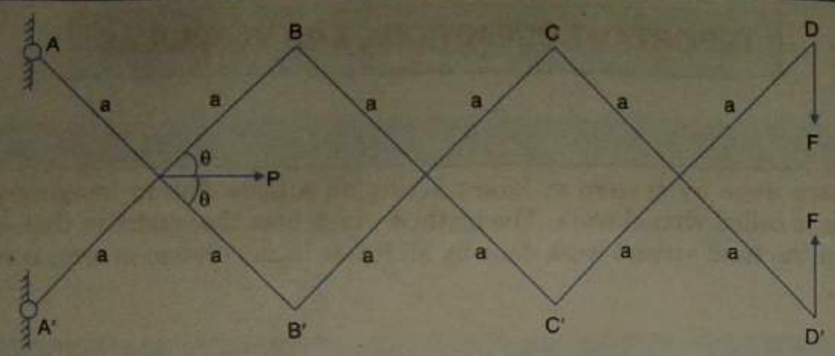


Fig. 7.33

7.4 The frame *ABCDEF* shown in Fig. 7.34 is made up of six identical rods pin connected at ends. The rod *AB* is fixed in horizontal position and the middle points of *AB* and *DE* are connected by an inextensile string so as to keep *DE* also in horizontal position. Determine the tension in string *GH*, if the weight of each rod is *W*. Use virtual work method. [**Ans.** $T = 3W$]

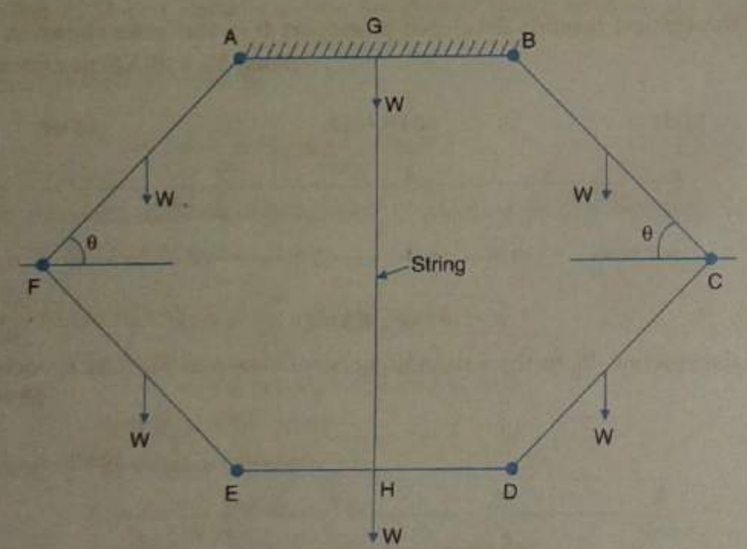


Fig. 7.34

7.5 The double pendulum shown in Fig. 7.35 consists of two links pinned smoothly together and held in position by a horizontal force *F*. If the length of link *AB* is 2 *L* and weighs 2*W* while link *BC* is of length *L* and weighs *W*, determine the equilibrium position as defined by θ_1 and θ_2.

$$\left[\textbf{Ans.}\ \tan\theta_1 = \frac{2W}{P},\ \tan\theta_2 = \frac{W}{2P}\right]$$

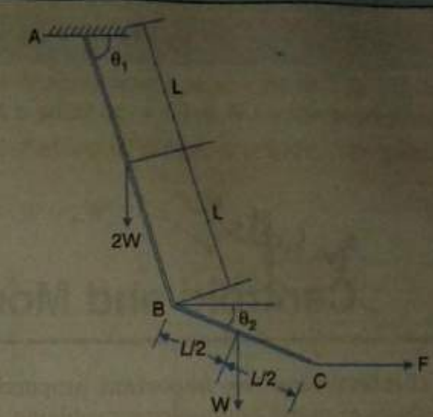


Fig. 7.35

8 Centroid and Moment of Inertia

Centroid and moment of intertia are two important properties of a section, which are required frequently in the analysis of many engineering problems. In this chapter the meaning of centre of gravity and centroid is explained. Procedure for locating centroid of plane figures is illustrated. The term moment of inertia is defined and the method of finding moment of inertia of a given sectional area is explained. Determination of the centroid and the moment of inertia of sections will be of great importance in the study of subjects like strength of materials, structural design and machine design.

8.1. CENTRE OF GRAVITY

Consider a suspended body as shown in Fig. 8.1 The weights of various parts of this body are acting vertically downward. The only upward force is the force in the string. To satisfy the equilibrium condition the resultant weight of the body W must act along the line of the string (1) – (1). Now, if the position is changed and the body is suspended again, it will reach equilibrium in a particular position. Let the line of action of resultant weight be (2) – (2) intersecting line (1) – (1) at G. It is obvious that if the body is suspended in any other position, the line of action of resultant weight W passes through G. This point is called the centre of gravity. Thus, *centre of gravity can be defined as the point through which resultant of force of gravity (Weight) of the body acts.*

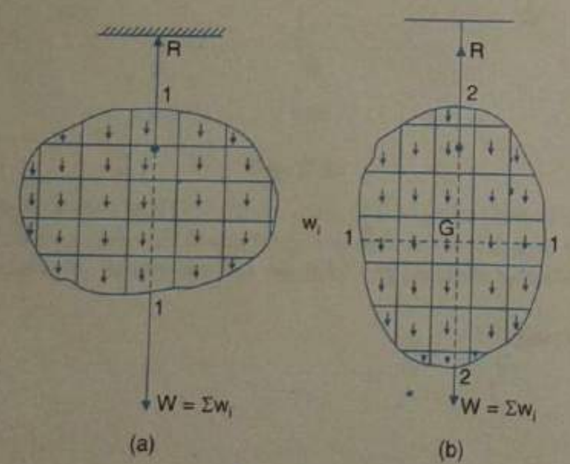


Fig. 8.1

8.2. CENTRE OF GRAVITY OF A FLAT PLATE

Consider a flat plate of thickness t as shown in Fig. 8.2. Let W_i be the weight of any elemental portion acting at a point (x_i, y_i). Let W be the total weight of the plate acting at the point $(\bar{x}, \bar{y})$. According to definition of centre of gravity, the point $(\bar{x}, \bar{y})$ is the centre of gravity. Now,

Total weight $\quad W = \Sigma W_i \qquad$ ...(8.1)

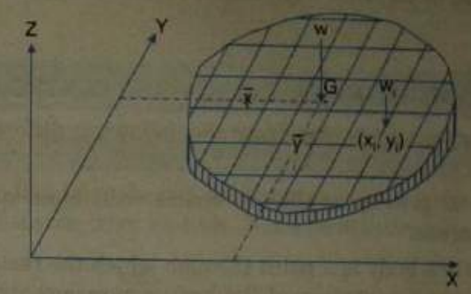


Fig. 8.2

Taking moment about x axis and equating moment of resultant to moment of component forces, we get

$$W_{\bar{y}} = W_1 y_1 + W_2 y_2 + W_3 y_3 + \ldots$$
$$= \Sigma W_i y_i$$

$$\therefore \quad \bar{y} = \frac{\Sigma W_i y_i}{W} \qquad ...(8.2)$$

Similarly, taking moment about y axis we get,

$$W_{\bar{x}} = W_1 x_1 + W_2 x_2 + W_3 x_3 + \ldots$$
$$= \Sigma W_i x_i$$

$$\bar{x} = \frac{\Sigma W_i x_i}{W} \qquad ...(8.3)$$

8.3. CENTROID

Let A_i be the area of the ith element of plate of uniform thickness in the plate. If γ is the unit weight of the material of plate and t its uniform thickness, then

$$W_i = \gamma A_i t$$

$\therefore$ Total weight, $\quad W = \Sigma \gamma A_i t = \gamma t \Sigma A_i$

$$= \gamma t A \qquad ...(8.4)$$

where, $A = \Sigma A_i$ is total area.

From equations 8.2 and 8.3, we get

$$\bar{y} = \frac{\Sigma A_i \gamma t y_i}{\gamma t A} = \frac{\Sigma A_i y_i}{A} \qquad ...(8.5)$$

and

$$\bar{x} = \frac{\Sigma A_i \gamma t x_i}{\gamma t A} = \frac{\Sigma A_i x_i}{A} \qquad ...(8.6)$$

since γ and t are constants.

The terms $\Sigma A_i y_i$ and $\Sigma A_i x_i$ may be considered as moment of area about x axis and y axis. Thus, the distance of the centre of gravity of plate of uniform thickness from an axis can be located by dividing moment of area about that axis by the total area. It will not depend upon the magnitude of thickness of plate and unit weight of material. If the thickness reduces to infinitesimal, then the plate reduces to an area. Still the expressions for finding centre of the area are same as Eqn. 8.5 and 8.6. Centre of gravity is a misnomer for the area. It is to be called as **centroid**. It may be noted that since the moment of area about an axis divided by total area gives the distance of centroid from that axis, the moment of area is zero about any centroidal axis.

8.4. DIFFERENCE BETWEEN CENTRE OF GRAVITY AND CENTROID

From the above discussion we can draw the following differences between centre of gravity and centroid:

(1) The term centre of gravity applies to bodies with mass and weight, and centroid applies to plane areas.

(2) Centre of gravity of a body is a point through which the resultant gravitational force (weight) acts for any orientation of the body whereas centroid is a point in a plane area such that the moment of area about any axis through that point is zero.

8.5. USE OF AXIS OF SYMMETRY

Centroid of an area lies on the axis of symmetry if it exists. This is useful theorem to locate the centroid of an area. This theorem can be proved as follows:

Consider the area shown in Fig. 8.3. In this figure y-axis is the axis of symmetry. From Eqn. 8.6, the distance of centroid from this axis is given by:

$$\frac{\Sigma A_i x_i}{A}$$

Consider the two elemental areas shown in Fig. 8.3, which are equal in size and are equidistant from the axis, but on either side. Now the sum of moments of these areas cancel each other since the areas and distances are the same, but signs of distances are opposite. Similarly, we can go on considering one area on one side of symmetric axis and corresponding image area on the other side, and prove that total moments of area ($\Sigma A_i x_i$) about the symmetric axis is zero. Hence the distance of centroid from the symmetric axis is zero, *i.e.*, centroid always lies on symmetric axis.

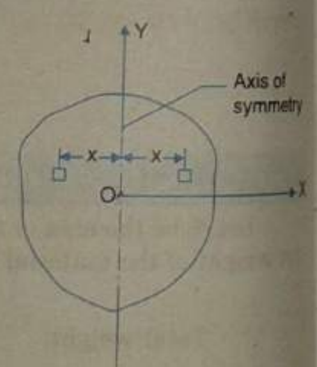


Fig. 8.3

Making use of the symmetry we can conclude that:

(1) Centroid of a circle is its centre (Fig. 8.4);

(2) Centroid of a rectangle of sides b and d is at a distance $\frac{b}{2}$ and $\frac{d}{2}$ from any corner (Fig. 8.5).

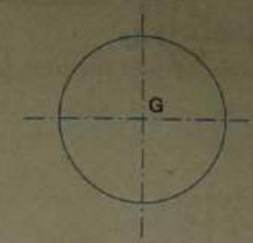


Fig. 8.4

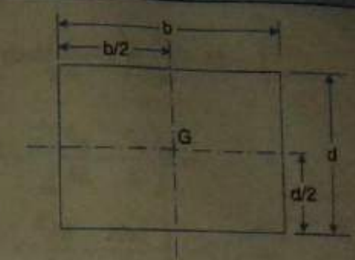


Fig. 8.5

8.6. DETERMINATION OF CENTROID OF SIMPLE FIGURES FROM FIRST PRINCIPLE

For simple figures like triangle and semicircle, we can write general expression for the elemental area and its distance from an axis. Then equations 8.5 and 8.6 reduce to:

$$\bar{y} = \frac{\int y dA}{A} \quad ...(8.7)$$

$$\bar{x} = \frac{\int x dA}{A} \quad ...(8.8)$$

The location of the centroid using the above equations may be considered as finding centroid from first principles. Now, let us find centroid of some standard figures from first principles.

Centroid of a Triangle – Consider the triangle ABC of base width b and height h as shown in Fig. 8.6. Let us locate the distance of centroid from the base. Let b_1 be the width of elemental strip of thickness dy at a distance y from the base. Since ΔAEF and ΔABC are similar triangles, we can write:

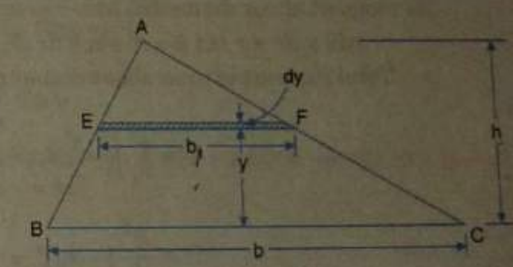


Fig. 8.6

$$\frac{b_1}{b} = \frac{h-y}{h}$$

$$b_1 = \left(\frac{h-y}{h}\right) b = \left(1 - \frac{y}{h}\right) b$$

$\therefore$ Area of the element $= dA = b_1 dy$

$$= \left(1 - \frac{y}{h}\right) b\, dy$$

Area of the triangle $A = \frac{1}{2} bh$

$\therefore$ From Eqn. (8.7)

$$\bar{y} = \frac{\text{Moment of area}}{\text{Total area}} = \frac{\int y dA}{A}$$

Now, $$\int y\,dA = \int_0^h y\left(1-\frac{y}{h}\right) b\,dy = \int_0^h \left(y-\frac{y^2}{h}\right) b\,dy$$

$$= b\left[\frac{y^2}{2}-\frac{y^3}{3h}\right]_0^h = \frac{bh^2}{6}$$

$$\therefore \quad \bar{y} = \int \frac{y\,dA}{A} = \frac{bh^2}{6} \times \frac{1}{\frac{1}{2}bh}$$

$$\therefore \quad \bar{y} = \frac{h}{3}$$

Thus the centroid of a triangle is at a distance $\frac{h}{3}$ from the base (or $\frac{2h}{3}$ from the apex) of the triangle where h is the height of the triangle.

Centroid of a Semicircle – Consider the semicircle of radius R as shown in Fig. 8.7. Due to symmetry centroid must lie on y axis. Let its distance from diametral axis be $\bar{y}$. To find $\bar{y}$, consider an element at a distance r from the centre O of the semicircle, radial width being dr and bound by radii at θ and $\theta + d\theta$.

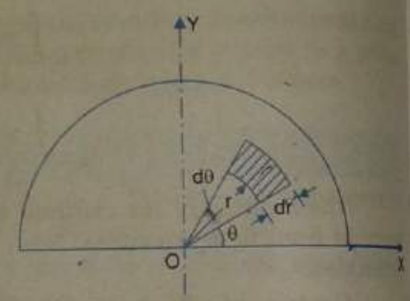


Fig. 8.7

Area of element $= r\,d\theta\,dr$.

Its moment about diametral axis x is given by:

$$rd\theta \times dr \times r\sin\theta = r^2 \sin\theta\,dr\,d\theta$$

$\therefore$ Total moment of area about diametral axis

$$= \int_0^{\pi}\int_0^R r^2 \sin\theta\,dr\,d\theta = \int_0^{\pi}\left[\frac{r^3}{3}\right]_0^R \sin\theta\,d\theta$$

$$= \frac{R^3}{3}[-\cos]_0^{\pi} = \frac{R^3}{3}[1+1] = \frac{2R^3}{3}$$

Area of semicircle $\quad A = \frac{1}{2}\pi R^2$

$$\therefore \quad \bar{y} = \frac{\text{Moment of area}}{\text{Total area}} = \frac{\frac{2R^3}{3}}{\frac{1}{2}\pi R^2} = \frac{4R}{3\pi}$$

Thus, the centroid of the semi-circle is at a distance $\frac{4R}{3\pi}$ from the diametral axis.

Centroid of Sector of a Circle – Consider the sector of a circle of angle 2α as shown in Fig. 8.8. Due to symmetry, centroid lies on x axis. To find its distance from the centre O, consider the elemental area shown.

Area of the element $= r\,d\theta\,dr$

Its moment about y axis

$$= rd\theta \times dr \times r\cos\theta = r^2\cos\theta\,dr\,d\theta$$

$\therefore$ Total moment of area about y axis

$$= \int_{-\alpha}^{\alpha}\int_0^R r^2\cos\theta\,dr\,d\theta$$

$$= \left[\frac{r^3}{3}\right]_0^R [\sin\theta]_{-\alpha}^{\alpha}$$

$$= \frac{R^3}{3}\,2\sin\alpha$$

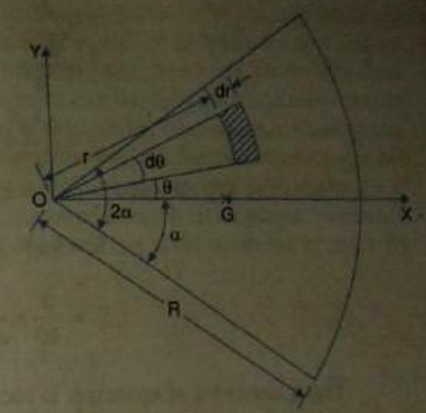


Fig. 8.8

Total area of the sector

$$= \int_{-\alpha}^{\alpha}\int_0^R r\,dr\,d\theta$$

$$= \int_{-\alpha}^{\alpha}\left[\frac{r^2}{2}\right]_0^R d\theta$$

$$= \frac{R^2}{2}[\theta]_{-\alpha}^{\alpha} = R^2\alpha$$

$\therefore$ The distance of centroid from centre O

$$= \frac{\text{Moment of area about } y \text{ axis}}{\text{Area of the figure}}$$

$$\bar{x} = \frac{\frac{2R^3}{3}\sin\alpha}{R^2\alpha} = \frac{2R}{3\alpha}\sin\alpha$$

Centroid of Parabolic Spandrel – Consider the parabolic spandrel shown in Fig. 8.9. Height of the element at a distance x from O is $y = kx^2$

Width of element $= dx$

$\therefore$ Area of the element $= kx^2\,dx$

$\therefore$ Total area of spandrel $= \int_0^a kx^2\,dx$

$$\left[\frac{kx^3}{3}\right]_0^a = \frac{ka^3}{3}$$

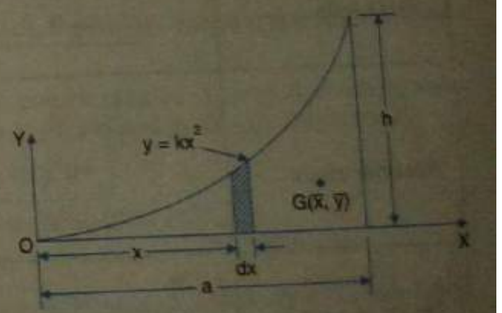


Fig. 8.9

Moment of area about y axis

$$= \int_0^a kx^2\,dx \times x = \int_0^a kx^3\,dx$$

$$= \frac{ka^4}{4}$$

Moment of area about x axis

$$= \int_0^a kx^2\,dx\,\frac{kx^2}{2} = \int_0^a \frac{k^2x^4}{2}\,dx = \frac{k^2a^5}{10}$$

$$\therefore \quad \bar{x} = \frac{ka^4}{4} \div \frac{ka^3}{3} = \frac{3a}{4}$$

$$\bar{y} = \frac{k^2a^5}{10} \div \frac{ka^3}{3} = \frac{3}{10}ka^2$$

From the Fig 8.9, at $x = a$, $y = h$

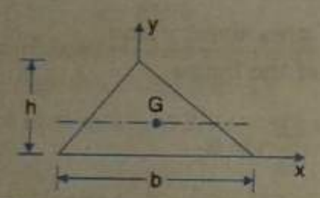
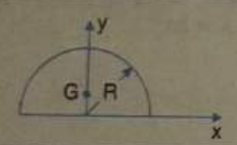
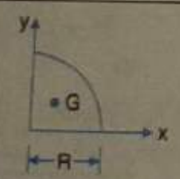
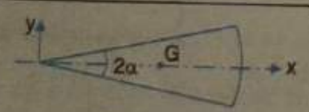
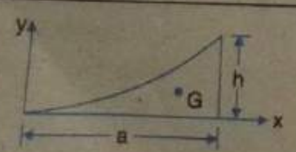
$$\therefore \quad h = ka^2 \quad \text{or} \quad k = \frac{h}{a^2}$$

$$\therefore \quad \bar{y} = \frac{3}{10} \times \frac{h}{a^2}a^2 = \frac{3h}{10}$$

Thus, centroid of spandrel is located at $\left(\frac{3a}{4}, \frac{3h}{10}\right)$

Centroids of some common figures are shown in Table 8.1.

Table 8.1 Centroid of Some Common Figures

Shape	*Figure*	$\bar{x}$	$\bar{y}$	*Area*
Triangle		–	$\frac{h}{3}$	$\frac{bh}{2}$
Semicircle		0	$\frac{4R}{3\pi}$	$\frac{\pi R^2}{2}$
Quarter circle		$\frac{4R}{3\pi}$	$\frac{4R}{3\pi}$	$\frac{\pi R^2}{4}$
Sector of a circle		$\frac{2R}{3\alpha}\sin\alpha$	0	αR^2
Parabolic Spandrel		$\frac{3a}{4}$	$\frac{3h}{10}$	$\frac{ah}{3}$

8.7. CENTROID OF COMPOSITE SECTIONS

So far, the discussion was confined to locating the centroid of simple figures like rectangle, triangle, circle, semicircle, etc. In engineering practice, use of sections whcih are built up with many simple sections is very common. Such sections may be called as built-up sections or composite sections. To locate the centroid of composite sections, one need not go for the first principle (method of integration). The given composite section can be split into suitable simple figures and then the centroid of each simple figure can be found by inspection or using the standard formulae listed in Table 8.1. Assuming the area of the simple figure as concentrated at its centroid, its moment about an axis can be found by multiplying the area with distance of its certroid from the reference axis. After determining moment of each area about reference axis, the distance of centroid from the axis is obtained by dividing total moment of area by total are of the composite section.

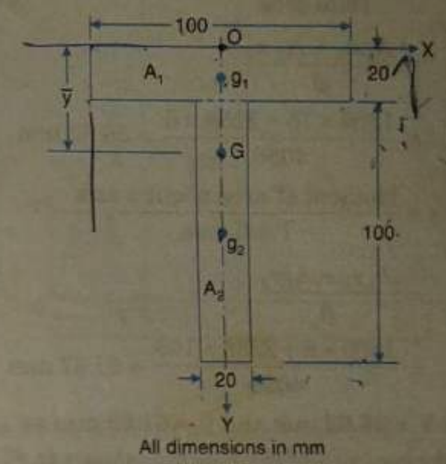


Fig. 8.10

Example 8.1. *Locate the centroid of the T-Section shown in the Fig. 8.10.*

Solution. Selecting the axis as shown in Fig. 8.10, we can say due to symmetry centroid lies on y axis, *i.e.* $\bar{x} = 0$.

Now the given T-section may be divided into two rectangles A_1 and A_2 each of size 100×20 and 20×100. The centroid of A_1 and A_2 are g_1 (0, 10) and g_2 (0, 70), respectively.

$\therefore$ The distance of centroid from top is given by:

$$\bar{y} = \frac{\Sigma a_i y_i}{\Sigma a_i} = \frac{100 \times 20 \times 10 + 20 \times 100 \times 70}{100 \times 20 + 20 \times 100} = 40 \text{ mm}$$

Hence, **centroid of T-section is on the symmetric axis at a distance 40 mm from the top.** **Ans.**

Example 8.2. *Find the centroid of the unequal angle 200 × 150 × 12 mm, shown in Fig. 8.11.*

Solution. The given composite figure can be divided into two rectangles:

$$A_1 = 150 \times 12 = 1800 \text{ mm}^2$$

$$A_2 = (200 - 12) \times 12 = 2256 \text{ mm}^2$$

Total area $\quad A = A_1 + A_2 = 4056 \text{ mm}^2$

Selecting the reference axis x and y as shown in Fig. 8.11. The centroid of A_1 is g_1 (75, 6) and that of A_2 is:

$$g_2\left[6, 12 + \frac{1}{2}(200 - 12)\right]$$

i.e., $\quad g_2\,(6, 106).$

$$\therefore \quad \bar{x} = \frac{\text{Moment of area about } y \text{ axis}}{\text{Total area}}$$

$$= \frac{A_1 x_1 + A_2 x_2}{A}$$

$$= \frac{1800 \times 75 + 2256 \times 6}{4056} = 36.62 \text{ mm}$$

$$\bar{y} = \frac{\text{Moment of area about } x \text{ axis}}{\text{Total area}}$$

$$= \frac{A_1 y_1 + A_2 y_2}{A}$$

$$= \frac{1800 \times 6 + 2256 \times 106}{4056} = 61.62 \text{ mm}$$

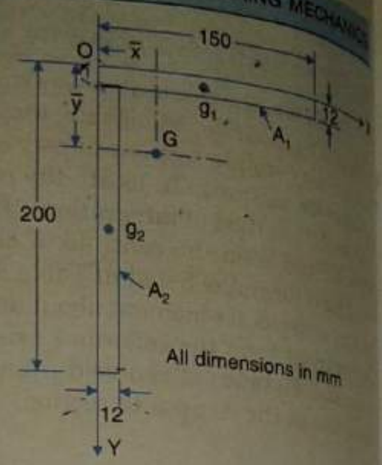


Fig. 8.11

Thus, the centroid is at $\bar{x}$ **= 36.62 mm** and $\bar{y}$ **= 61.62 mm** as shown in the figure. Ans.

Example 8.3. *Locate the centroid of the I-section shown in Fig. 8.12.*

Solution. Selecting the co-ordinate system as shown in Fig. 8.12, due to symmetry centroid must lie on y axis,

i.e., $\quad \bar{x} = 0.$

Now, the composite section may be split into three rectangles.

$$A_1 = 100 \times 20 = 2000 \text{ mm}^2.$$

Centroid of A_1 from the origin is:

$$y_1 = 30 + 100 + \frac{20}{2} = 140 \text{ mm}$$

Similarly $\quad A_2 = 100 \times 20 = 2000 \text{ mm}^2$

$$y_2 = 30 + \frac{100}{2} = 80 \text{ mm}$$

$$A_3 = 150 \times 30 = 4500 \text{ mm}^2, \text{ and}$$

$$y_3 = \frac{30}{2} = 15 \text{ mm}$$

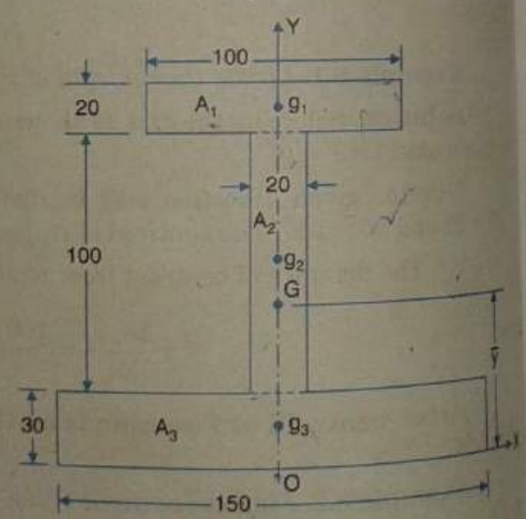


Fig. 8.12

$$\therefore \quad \bar{y} = \frac{A_1 y_1 + A_2 y_2 + A_3 y_3}{A}$$

$$= \frac{2000 \times 140 + 2000 \times 80 + 4500 \times 15}{2000 + 2000 + 4500} = 59.71 \text{ mm}$$

Thus, **the centroid is on the symmetric axis at a distance 59.71 mm from the bottom** as shown in Fig. 8.12. **Ans.**

Example 8.4. *Determine the centroid of the section of the concrete dam shown in Fig. 8.13.*

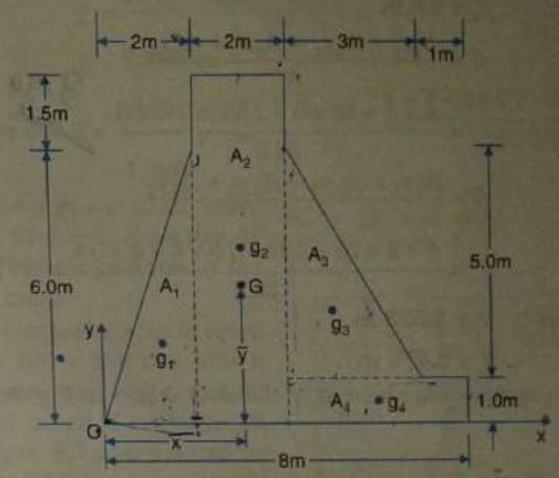


Fig. 8.13

Solution. Let the axis be selected as shown in Fig. 8.13. [Note that it is convenient to take axes in such a way that the centroids of all simple figures are having positive coordinates. If coordinates of any simple figure comes out to be negative, one should be careful in assigning the sign to moment of area of that figure.]

The composite figure can be conveniently divided into two triangles and two rectangles, as shown in Fig. 8.13.

Now,

$$A_1 = \frac{1}{2} \times 2 \times 6 = 6 \text{ m}^2$$

$$A_2 = 2 \times 7.5 = 15 \text{ m}^2$$

$$A_3 = \frac{1}{2} \times 3 \times 5 = 7.5 \text{ m}^2$$

$$A_4 = 1 \times 4 = 4 \text{ m}^2$$

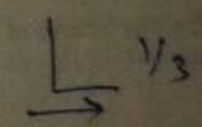
$$A = \text{total area} = 32.5 \text{ m}^2$$

Centroides of simple figures are:

$$x_1 = \frac{2}{3} \times 2 = \frac{4}{3} \text{ m},$$

$$y_1 = \frac{1}{3} \times 6 = 2 \text{ m}$$

$$x_2 = 2 + 1 = 3 \text{ m}$$



$$y_2 = \frac{7.5}{2} = 3.75 \text{ m}$$

$$x_3 = 2 + 2 + \frac{1}{3} \times 3 = 5 \text{ m}$$

$$y_3 = 1 + \frac{1}{3} \times 5 = \frac{8}{3} \text{ m}$$

$$x_4 = 4 + \frac{4}{2} = 6 \text{ m}$$

$$y_4 = 0.5 \text{ m}$$

$$\bar{x} = \frac{A_1x_1 + A_2x_2 + A_3x_3 + A_4x_4}{A}$$

$$= \frac{6 \times \frac{4}{3} + 15 \times 3 + 7.5 \times 5 + 4 \times 6}{32.5} = \cancel{3.523} \text{ m} \quad 3.39$$

$$\bar{y} = \frac{A_1y_1 + A_2y_2 + A_3y_3 + A_4y_4}{A}$$

$$= \frac{6 \times 2 + 15 \times 3.75 + 7.5 \times \frac{8}{3} + 4 \times 0.5}{32.5} = 2.777 \text{ m}$$

The centroid is at $\bar{x} = \mathbf{3.523\ m}$ **Ans.**

and $\bar{y} = \mathbf{2.777\ m}$ **Ans.**

Example 8.5. *Determine the centroid of the area shown in Fig. 8.14 with respect to the axes shown.*

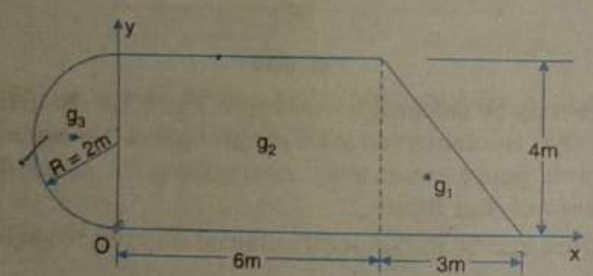


Fig. 8.14

Solution. The composite section is divided into three simple figures, a triangle, a rectangle and a semicircle.

Now, area of triangle $A_1 = \frac{1}{2} \times 3 \times 4 = 6 \text{ m}^2$

Area of rectangle $A_2 = 6 \times 4 = 24 \text{ m}^2$

Area of semicircle $A_3 = \frac{1}{2} \times \pi \times 2^2 = 6.2832 \text{ m}^2$

$\therefore$ Total area $A = 36.2832 \text{ m}^2$

The coordinates of centroids of these three simple figures are:

$$x_1 = 6 + \frac{1}{3} \times 3 = 7 \text{ m}$$

$$y_1 = \frac{4}{3} \text{ m} \quad [\tfrac{1}{3} \times 4]$$

$$x_2 = 3 \text{ m}$$

$$y_2 = 2 \text{ m}$$

$$x_3 = \frac{-4R}{3\pi} = -\frac{4 \times 2}{3\pi} = -0.8488 \text{ m}$$

$$y_3 = 2 \text{ m}$$

(Note carefully the sign of x_3).

$$\bar{x} = \frac{A_1x_1 + A_2x_2 + A_3x_3}{A} = \frac{6 \times 7 + 24 \times 3 + 6.2832 \times (-0.8488)}{36.2832}$$

i.e., $\bar{x} = \mathbf{2.995\ m}$ **Ans.**

$$\bar{y} = \frac{A_1y_1 + A_2y_2 + A_3y_3}{A} = \frac{6 \times \frac{4}{3} + 24 \times 2 + 6.2832 \times 2}{36.2832}$$

i.e. $\bar{y} = \mathbf{1.890}$ **Ans.**

Example 8.6. *In a gusset plate, there are six rivet holes of 21.5 mm diameter as shown in Fig. 8.15. Find the position of the centroid of the gusset plate.*

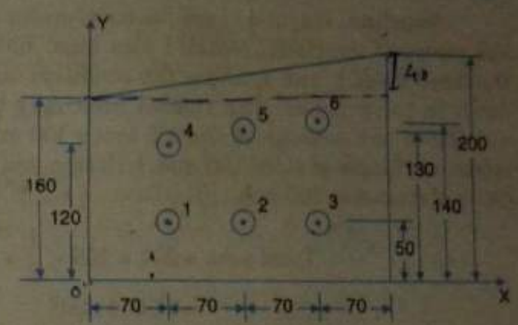


Fig. 8.15

Solution. The composite area is equal to a rectangle of size 160 × 280 mm plus a triangle of size 280 mm base width and 40 mm height and minus areas of six holes. In this case also the Eqns. 8.5 and 8.6 can be used for locating centroid by treating area of holes as negative. The area of simple figures and their centroids are as shown in Table given below:

Figure	*Area in mm²*	*x_i in mm*	*y_i in mm*
Rectangle	$160 \times 280 = 44{,}800$	140	80
Triangle	$\frac{1}{2} \times 280 \times 40 = 5600$	$\frac{2}{3} \times 280$	$160 + \frac{40}{3} = 173.33$
1st hole	$\frac{-\pi \times 21.5^2}{4} = -363.05$	70	50
2nd hole	− 363.05	140	50
3rd hole	− 363.05	210	50
4th hole	− 363.05	70	120
5th hole	− 363.05	140	130
6th hole	− 363.05	210	140

$\therefore$ $A = \Sigma A_i = 48,221.70$

$\therefore$ $\Sigma A_i x_i = 44{,}800 \times 140 + 5600 \times \frac{2}{3} \times 280 - 363.05\,(70 + 140 + 210 + 70 + 140 + 210)$

$= 70,12,371.3 \text{ mm}^3$

$$\bar{x} = \frac{\Sigma A_i x_i}{A} = 145.42 \text{ mm}$$

$$\Sigma A_i y_i = 44,800 \times 80 + 5600 \times 173.33 - 363.05(50 \times 3 + 120 + 130 + 140)$$
$$= 43,58,601 \text{ mm}^3$$

$$\bar{y} = \frac{\Sigma A_i y_i}{A} = \frac{43,58,601}{48221.70} = 90.39 \text{ mm}$$

Thus, the coordinates of centroid of composite figure is given by:

$$\bar{x} = \mathbf{145.42\ mm}$$

$$\bar{y} = \mathbf{90.39} \qquad \textbf{Ans.}$$

Example 8.7. *Determine the coordinates x_c and y_c of the centre of a 100 mm diameter circular hole cut in a thin plate so that this point will be the centroid of the remaining shaded area shown in Fig. 8.16 (All dimensions are in mm).*

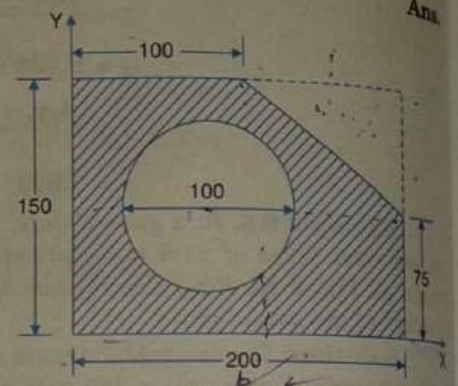


Fig. 8.16

Solution. If x_c and y_c are the coordinates of the centre of the circle, centroid also must have the coordinates x_c and y_c as per the condition laid down in the problem. The shaded area may be considered as a rectangle of size 200 mm × 150 mm *minus* a triangle of sides 100 mm × 75 mm and a circle of diameter 100 mm.

$$\therefore \quad \text{Total area} = 200 \times 150 - \frac{1}{2} \times 100 \times 75 - \left(\frac{\pi}{4}\right) \times 100^2$$
$$= 18,396 \text{ mm}^2$$

$$\bar{x} = x_c = \frac{(200 \times 150 \times 100) - \frac{1}{2} \times 100 \times 75 \times \left[200 - \left(\frac{100}{3}\right)\right] - \frac{\pi}{4} \times 100^2 \times x_c}{18,396}$$

$$\therefore \quad x_c(18,396) = 200 \times 150 \times 100 - \frac{1}{2} \times 100 \times 75 \times 166.67 - \frac{\pi}{4} \times 100^2 x_c$$

$$x_c = \frac{23,75000}{26,250} = \mathbf{90.48\ mm} \qquad \textbf{Ans.}$$

Similarly,

$$18,396\, y_c = 200 \times 150 \times 75 - \frac{1}{2} \times 100 \times 75 \times (150 - 25) - \frac{\pi}{4} \times 100^2\, y_c$$

$$\therefore \quad y_c = \frac{17,81250.0}{26,250} = \mathbf{67.86\ mm} \qquad \textbf{Ans.}$$

Centre of the circle should be located at (90.48, 67.86) so that this point will be the centroid of the remaining shaded area shown in Fig. 8.16.

Note: The centroid of the given figure will coincide with the centroid of the figure without circular hole. Hence, the centroid of the given figure can be obtained by determining the centroid of the figure without the circular hole.

Example 8.8. *Determine the coordinates of the centroid of the plane area shown in Fig. 8.17 with reference to the axes shown. Take x = 40 mm.*

Solution. The composite figure is divided into the following simple figures:

(1) A rectangle $\quad A_1 = (14x) \times (12x) = 168\, x^2$

$x_1 = 7x;\ y_1 = 6x$

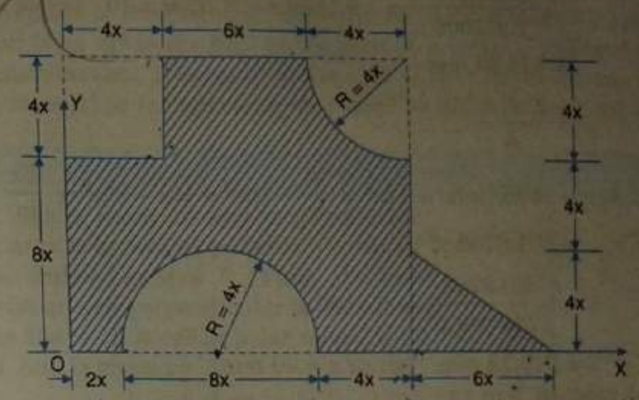


Fig. 8.17

(2) A triangle $\quad A_2 = \frac{1}{2}(6x) \times (4x) = 12x^2$

$x_2 = 14x + 2x = 16x$

$y_2 = \frac{4x}{3}$

(3) A rectangle to be subtracted

$A_3 = (-4x) \times (4x) = -16x^2$

$x_3 = 2x;\ y_3 = 8x + 2x = 10x$

(4) A semicircle to be subtracted

$A_4 = -\frac{1}{2}\pi(4x)^2 = -8\pi x^2$

$x_4 = 6x$

$y_4 = \frac{4R}{3\pi} = 4 \times \frac{(4x)}{3\pi} = \frac{16x}{3\pi}$

(5) A quarter of a circle to be subtracted

$A_5 = -\frac{1}{4} \times \pi(4x)^2 = -4\pi x^2$

$x_5 = 14x - \frac{4R}{3\pi} = 14x - (4)\left(\frac{4x}{3\pi}\right) = 12.3023\, x$

$y_5 = 12x - 4 \times \left(\frac{4x}{3\pi}\right) = 10.3023\, x$

Total area $\quad A = 168x^2 + 12x^2 - 16x^2 - 8\pi x^2 - 4\pi x^2$

$= 126.3009\, x^2$

$$\bar{x} = \frac{\Sigma A_i x_i}{A}$$

$$\Sigma A_i x_i = 168x^2 \times 7x + 12x^2 \times 16x - 16x^2 \times 2x - 8\pi x^2 \times 6x - 4\pi x^2 \times 12.3023x$$

$$= 1030.6083x^3$$

$$\therefore \quad \bar{x} = \frac{1030.6083\,x^3}{126.3009\,x^2} = 8.1599\,x = 8.1599 \times 40 \quad (\text{since } x = 40 \text{ mm})$$

$$= 326.40 \text{ mm}$$

$$\bar{y} = \frac{\Sigma A_i y_i}{A}$$

$$\Sigma A_i y_i = 168\,x^2 \times 6x + 12x^2 \times \frac{4x}{3} - 16x^2 \times 10x - 8\pi x^2 \times \frac{16x}{3\pi} - 4\pi x^2 \times 10.3023x$$

$$= 691.8708\,x^3$$

$$\therefore \quad \bar{y} = \frac{6918708}{126.3009\,x^2} = 5.4780\,x$$

$$= 219.12 \text{ mm} \quad (\text{since } x = 40 \text{ mm})$$

Centroid is at **(326.40, 219.12)** Ans.

8.8. MOMENT OF INERTIA

Consider the area shown in Fig. 8.18(*a*). dA is an elemental area with coordinates as x and y. The term $\Sigma y^2\,dA$ is called **moment of inertia** of the area about x axis and is denoted as I_{xx}. Similarly, the moment of inertia about y axis is $I_{yy} = \Sigma x^2\,dA$

In general, if r is the distance of elemental area dA from the axis AB [Fig. 8.18(*b*)], the sum of the terms $\Sigma r^2\,dA$ to cover the entire area is called moment of inertia of the area about the axis AB. If r and dA can be expressed in general terms, for any element then the sum becomes an integral. Thus,

$$I_{AB} = \Sigma r^2\,dA = \int r^2\,dA \qquad ...(8.9)$$

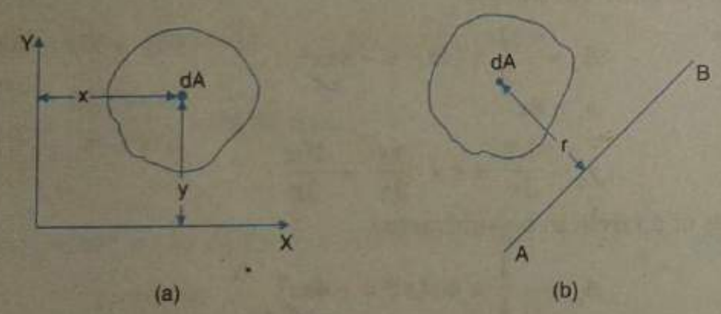


Fig. 8.18

Ther term rdA may be called as moment of area, similar to moment of a force, and hence r^2dA may be called as moment of moment of area or the second moment of area. Thus, the moment of inertia of a plane figure is nothing but second moment of area. In fact, the term **'second moment of area'** appears to correctly signify the meaning of the expression $\Sigma r^2\,dA$. The term 'moment of inertia' is rather a misnomer. However, term moment of inertia has come to stay for long time and hence it will be used in this book also.

Though moment of inertia of plane area is a purely mathematical term, it is one of the important properties of areas. The strength of members subject to bending depends on the moment of inertia of its cross-sectional area. Students will find this property of area very useful when they study subjects like strength of materials, structural design and machine design.

The moment of inertia is a fourth dimensional term since it is a term obtained by multiplying area by the square of the distance. Hence, in SI units, if metre (m) is the unit for linear measurements used then m^4 is the unit of moment of inertia. If millimetre (mm) is the unit used for linear measurements, then mm^4 is the unit of moment of inertia. In MKS system m^4 or cm^4 and FPS system ft^4 or in^4 are commonly used as units for moment of inertia.

8.9. POLAR MOMENT OF INERTIA

Moment of inertia about an axis perpendicular to the plane of an area is known as **polar moment of inertia.** It may be denoted as J or I_{zz}. Thus, the moment of inertia about an axis perpendicular to the plane of the area at O in Fig. 8.19 is called polar moment of inertia at point O, and is given by

$$I_{zz} = \Sigma r^2\,dA \qquad ...(8.10)$$

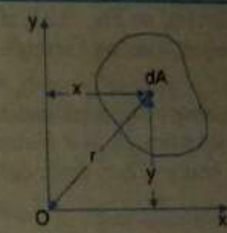


Fig. 8.19

8.10. RADIUS OF GYRATION

Radius of gyration is a mathematical term defined by the relation

$$k = \sqrt{\frac{I}{A}} \qquad ...(8.11)$$

where k = radius of gyration
I = moment of inertia
and A = the cross-sectional area.

Suffixes with moment of inertia I also accompany the term radius of gyration k. Thus, we can have,

$$k_{xx} = \sqrt{\frac{I_{xx}}{A}}$$

$$k_{yy} = \sqrt{\frac{I_{yy}}{A}}$$

$$k_{AB} = \sqrt{\frac{I_{AB}}{A}}$$

and so on.

The relation between radius of gyration and moment of inertia can be put in the form:

$$I = Ak^2 \qquad ...(8.12)$$

From the above relation a geometric meaning can be assigned to the term 'radius of gyration.' We can consider k as the distance at which the complete area is squeezed and kept as a strip of negligible width (Fig. 8.20) such that there is no change in the moment of inertia.

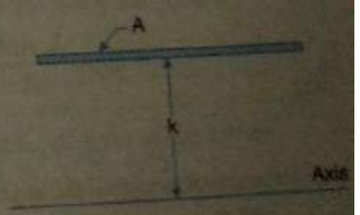


Fig. 8.20

8.11. THEOREMS OF MOMENT OF INERTIA

There are two theorems of moment of inertia:

(1) Perpendicular axis theorem, and

(2) Parallel axis theorem.

These are explained and proved below.

Perpendicular Axis Theorem

The moment of inertia of an area about an axis perpendicular to its plane (polar moment of inertia) at any point O is equal to the sum of moments of inertia about any two mutually perpendicular axis through the same point O and lying in the plane of the area.

Referring to Fig. 8.21, if $z - z$ is the axis normal to the plane of paper passing through point O, as per this theorem,

$$I_{zz} = I_{xx} + I_{yy} \qquad ...(8.13)$$

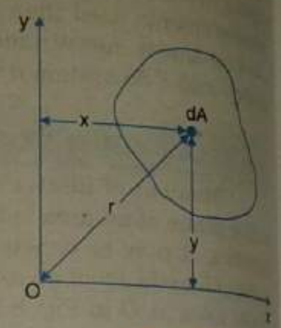


Fig. 8.21

The above theorem can be easily proved. Let us consider an elemental area dA at a distance r from O. Let the coordinates of dA be x and y. Then from definition:

$$I_{zz} = \Sigma r^2 dA = \Sigma(x^2 + y^2)\, dA$$
$$= \Sigma x^2 dA + \Sigma y^2 dA$$
$$I_{zz} = I_{xx} + I_{yy}$$

Parallel Axis Theorem

Moment of inertia about any axis in the plane of an area is equal to the sum of moment of inertia about a parallel centroidal axis and the product of area and square of the distance between the two parallel axes. Referring to Fig. 8.22, the above theorem means:

$$I_{AB} = I_{GG} + A\, y_c^2 \qquad ...(8.14)$$

where I_{AB} – moment of inertia about axis AB

I_{GG} – moment of inertia about centroidal axis GG parallel to AB.

A – The area of the plane figure given and

y_c – the distance between the axis AB and the parallel centroidal axis GG.

Proof: Consider an elemental parallel strip dA at a distance y from the centroidal axis (Fig. 8.22).

Then, $I_{AB} = \Sigma(y + y_c)^2\, dA = \Sigma\,(y^2 + 2yy_c + y_c^2)dA$

$$= \Sigma\, y^2 dA + \Sigma\, 2yy_c\, dA + \Sigma\, y_c^2\, dA$$

Now,

$\Sigma y^2\, dA$ = Moment of inertia about the axis GG

$= I_{GG}$

$$\Sigma\, 2\, yy_c\, dA = 2\, y_c\, \Sigma y\, dA$$

$$= 2y_c A\, \frac{\Sigma\, ydA}{A}$$

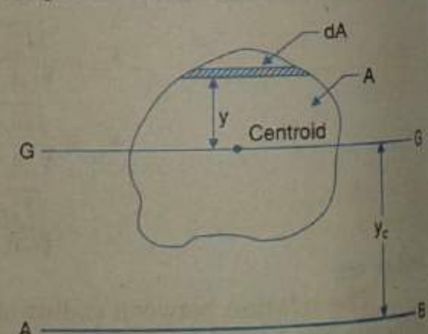


Fig. 8.22

In the above term $2y_c A$ is constant and $\frac{\Sigma\, ydA}{A}$ is the distance of centroid from the reference axis GG. Since GG is passing through the centroid itself $\frac{ydA}{A}$ is zero and hence the term $\Sigma 2yy_c\, dA$ is zero.

Now, the third term,

$$\Sigma y_c^2 dA = y_c^2 \Sigma dA = Ay_c^2$$

$$\therefore \qquad I_{AB} = I_{GG} + Ay_c^2$$

Note: The above equation cannot be applied to any two parallel axes. One of the axes (GG) must be centroidal axis only.

8.12. MOMENT OF INERTIA FROM FIRST PRINCIPLES

For simple figures, moment of inertia can be obtained by writing the general expression for an element and then carrying out integration so as to cover the entire area. This procedure is illustrated with the following three cases:

(1) Moment of inertia of rectangle about the centroidal axis

(2) Moment of inertia of a triangle about the base

(3) Moment of inertia of a circle about a diametral axis.

Moment of Inertia of a Rectangle about the Centroidal Axis—Consider a rectangle of width b and depth d (Fig. 8.23). Moment of inertia about the centroidal axis $x - x$ parallel to the short side is required.

Consider an elemental strip of width dy at a distance y from the axis. Moment of inertia of the elemental strip about the centroidal axis xx is:

$$= y^2\, dA$$
$$= y^2\, b\, dy$$

$$\therefore \qquad I_{xx} = \int_{-\frac{d}{2}}^{\frac{d}{2}} y^2 b\, dy$$

$$= b\left[\frac{y^3}{3}\right]_{-\frac{d}{2}}^{\frac{d}{2}}$$

$$= b\left[\frac{d^3}{24} + \frac{d^3}{24}\right]$$

$$I_{xx} = \frac{bd^3}{12}$$

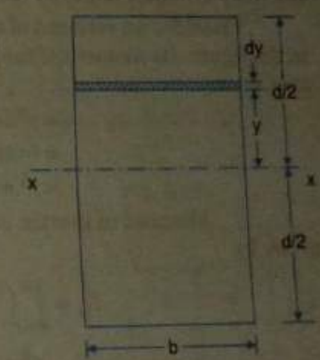


Fig. 8.23

Moment of Inertia of a Triangle About its Base—Moment of inertia of a triangle with base width b and height h is to be determined about the base AB (Fig. 8.24)

Consider an elemental strip at a distance y from the base AB. Let dy be the thickness of the strip and dA its area. Width of this strip is given by:

$$b_1 = \frac{(h - y)}{h} \times b$$

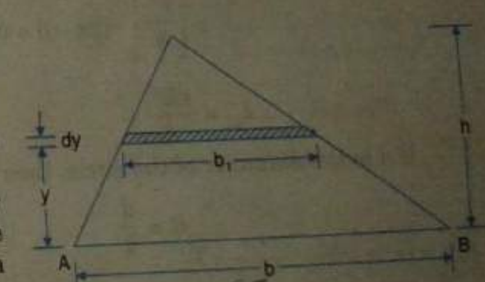


Fig. 8.24

Moment of inertia of this strip about AB

$$= y^2 dA = y^2 b_1 dy$$

$$= y^2 \frac{(h-y)}{h} \times b \times dy$$

$\therefore$ Moment of inertia of the triangle about AB,

$$I_{AB} = \int_0^h \frac{y^2 (h-y)\, b dy}{h} = \int_0^h \left(y^2 - \frac{y^3}{h} \right) b dy$$

$$= \left[\frac{y^3}{3} - \frac{y^4}{4h} \right]_0^h b = b \left[\frac{h^3}{3} - \frac{h^4}{4h} \right]$$

$$I_{AB} = \frac{bh^3}{12}$$

Moment of Inertia of Circle about its Diametral Axis—Moment of inertia of a circle of radius R is required about it's diametral axis as shown in Fig. 8.25.

Consider an element of sides $rd\theta$ and dr as shown in the figure. Its moment of inertia about the diamentral axis $x - x$:

$$= y^2 dA$$

$$= (r \sin \theta)^2\, r\, d\theta\, dr$$

$$= r^3 \sin^2 \theta\, d\theta\, dr$$

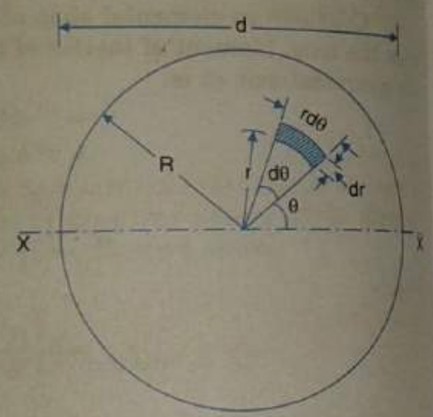


Fig. 8.25

$\therefore$ Moment of inertia of the circle about $x - x$ is given by

$$I_{xx} = \int_0^R \int_0^{2\pi} r^3 \sin^2 \theta\, d\theta\, dr$$

$$= \int_0^R \int_0^{2\pi} r^3 \frac{(1 - \cos 2\theta)}{2} d\theta\, dr$$

$$= \int_0^R \frac{r^3}{2} \left[\theta - \frac{\sin 2\theta}{2} \right]_0^{2\pi} dr$$

$$= \left[\frac{r^4}{8} \right]_0^R [2\pi - 0 + 0 - 0] = \frac{2\pi}{8} R^4$$

$$I_{xx} = \frac{\pi R^4}{4}$$

If d is the diameter of the circle, then

$$R = \frac{d}{2}$$

$$\therefore \quad I_{xx} = \frac{\pi}{4} \left(\frac{d}{2} \right)^4$$

$$I_{xx} = \frac{\pi d^4}{64}$$

8.13. MOMENT OF INERTIA OF STANDARD SECTIONS

Rectangle – Refferring to Fig. 8.26,

(*a*) $I_{xx} = \dfrac{bd^3}{12}$ as derived in Art. 8.12.

(*b*) $I_{yy} = \dfrac{db^3}{12}$ can be derived on the same lines.

(*c*) *About the base AB*, from parallel axis theorem,

$$I_{AB} = I_{xx} + A y_c^2$$

$$= \frac{bd^3}{12} + bd \left(\frac{d}{2} \right)^2, \quad \text{since } y_c = \frac{d}{2}$$

$$= \frac{bd^3}{12} + \frac{bd^3}{4}$$

$$= \frac{bd^3}{3}$$

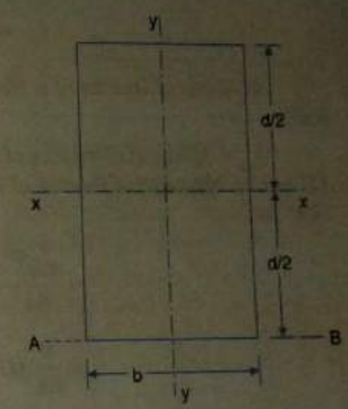


Fig. 8.26

Hollow Rectangular Section – Referring to Fig. 8.27, Moment of inertia I_{xx} = Moment of inertia of larger rectangle – Moment of inertia of hollow portion. That is,

$$= \frac{BD^3}{12} - \frac{bd^3}{12}$$

$$= \frac{1}{12} (BD^3 - bd^3)$$

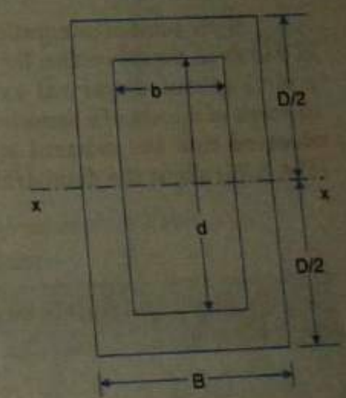


Fig. 8.27

Triangle – Referring to Fig. 8.28,

(*a*) About the base:

As found in Art. 8.12,

$$I_{AB} = \frac{bh^3}{12}$$

(*b*) About centroidal axis, $x - x$ parallel to base:

From parallel axis theorem,

$$I_{AB} = I_{xx} + A y_c^2$$

Now, y_c, the distance between the non-centroidal axis AB and centroidal axix $x - x$, is equal to $\dfrac{h}{3}$

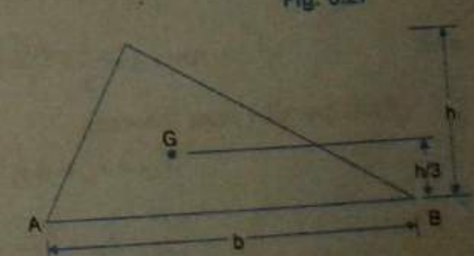


Fig. 8.28

$$\therefore \quad \frac{bh^3}{12} = I_{xx} + \frac{1}{2}bh\left(\frac{h}{3}\right)^2 = I_{xx} + \frac{bh^3}{18}$$

$$\therefore \quad I_{xx} = \frac{bh^3}{12} - \frac{bh^3}{18} = \frac{bh^3}{36}$$

Moment of Inertia of a Circle about any diametral axis

$$= \frac{\pi d^4}{64} \text{ (as found in Art. 8.12)}$$

Moment of Inertia of a Hollow Circle – Referring to Fig. 8.29,

I_{AB} = Moment of inertia of solid circle of diameter D about AB *minus* Moment of inertia of circle of diameter d about AB. That is,

$$= \frac{\pi D^4}{64} - \frac{\pi d^4}{64}$$

$$= \frac{\pi}{64}(D^4 - d^4)$$

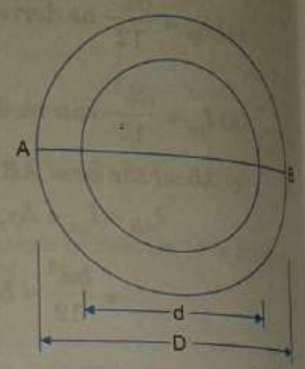


Fig. 8.29

Moment of Inertia of a Semicircle – (a) About Diametral Axis:

If the limit of integration is put as 0 to π instead of 0 to 2π in the derivation for the moment of inertia of a circle about diametral axis (Ref. art. 8.12), the moment of inertia of a semicircle is obtained. It can be observed that the moment of inertia of a semicircle (Fig. 8.30) about the diametral axis AB:

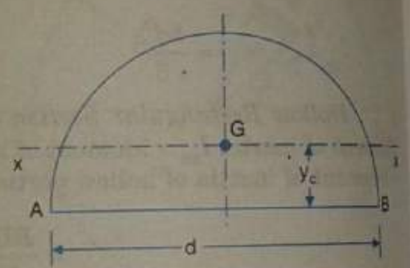


Fig. 8.30

$$= \frac{1}{2} \times \frac{\pi d^4}{64} = \frac{\pi d^4}{128}$$

(*b*) *About centroidal axis* $x - x$: Now, the distance of centroidal axix y_c from the diametral axis is given by:

$$y_c = \frac{4R}{3\pi} = \frac{2d}{3\pi}$$

and, $$\text{Area } A = \frac{1}{2} \times \frac{\pi d^2}{4} = \frac{\pi d^2}{8}$$

From parallel axes theorem,

$$I_{AB} = I_{xx} + A\,y_c^2$$

$$\frac{\pi d^4}{128} = I_{xx} + \frac{\pi d^2}{8} \times \left(\frac{2d}{3\pi}\right)^2$$

$$I_{xx} = \frac{\pi d^4}{128} - \frac{d^4}{18\pi}$$

$$= 0.0068598\, d^4 = 0.11\, R^4$$

Moment of Inertia of a Quarter of a Circle – (a) About the base:

If the limit of integration is put as 0 to $\frac{\pi}{2}$ instead of 0 to 2π in the derivation for moment of inertia of a circle (Ref. art. 8.12), the moment of inertia of a quarter of a circle is obtained. It can be observed that moment of inertia of the quarter of a circle about the base AB:

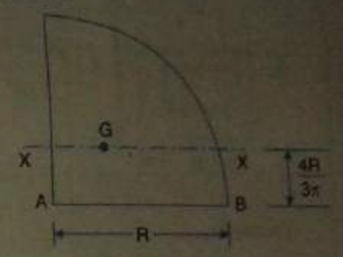


Fig. 8.31

$$= \frac{1}{4} \times \frac{\pi d^4}{64} = \frac{\pi d^4}{256} = \frac{\pi R^4}{16}$$

(*b*) *About centroidal axis* $x - x$

Now, the distance of centroidal axix y_c from the base is given by:

$$y_c = \frac{4R}{3\pi} = \frac{2d}{3\pi}$$

and the area $$A = \frac{1}{4} \times \frac{\pi d^2}{4} = \frac{\pi d^2}{16}$$

From parallel axes theorem,

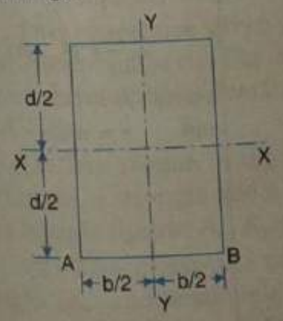
$$I_{AB} = I_{xx} + A\,y_c^2$$

$$\frac{\pi d^4}{256} = I_{xx} + \frac{\pi d^2}{16}\left(\frac{2d}{3\pi}\right)^2$$

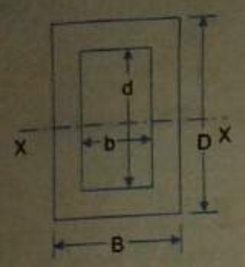
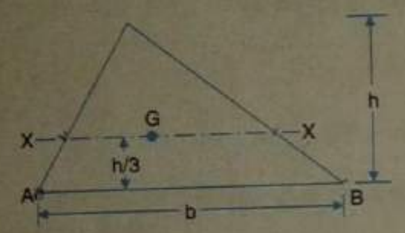
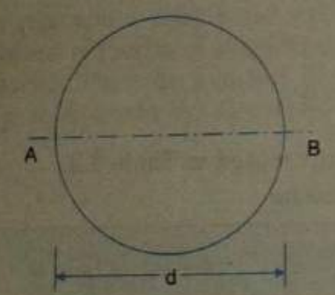
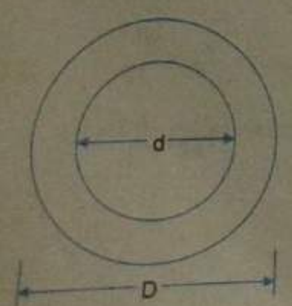
$$I_{xx} = \frac{\pi d^4}{256} - \frac{d^4}{36\pi} = 0.00343 d^4 = 0.055\, R^4$$

The moment of inertia of common standard sections are presented in Table 8.2.

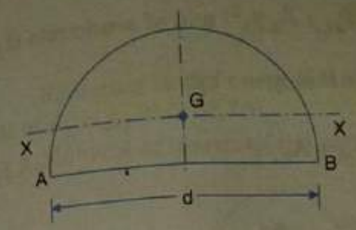
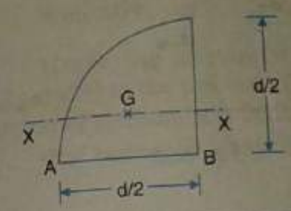
Table 8.2 Moment of Inertia of Standard Sections

Shape	*Axis*	*Moment of Inertia*
Rectangle 	(*a*) Centroidal axis $x - x$	$I_{xx} = \frac{bd^3}{12}$
	(*b*) Centroidal axis $y - y$	$I_{yy} = \frac{db^3}{12}$
	(*c*) $A - B$	$I_{AB} = \frac{bd^3}{3}$

(Contd...)

Hollow Rectangle 	Centroidal axis $x - x$	$I_{xx} = \frac{BD^3 - bd^3}{12}$
Triangle 	(*a*) Centroidal axis $x - x$ (*b*) Base AB	$I_{xx} = \frac{bh^3}{36}$ $I_{AB} = \frac{bh^3}{12}$
Circle 	Diametral axis	$I = \frac{\pi d^4}{64} = \frac{\pi R^4}{4}$
Hollow Circle 	Diametral axis	$I = \frac{\pi}{64}(D^4 - d^4)$ $= \frac{\pi}{4}(R^4 - r^4)$ where $R = D/2$ and $r = d/2$

(Contd...)

Semi Circle 	(*a*) $A - B$ (*b*) Centroidal axis	$I_{AB} = \frac{\pi d^4}{128}$ $I_{xx} = 0.0068598\ d^4$ $= 0.11\ R^4$ where $R = d/2$
Quarter of a Circle 	(*a*) $A - B$ (*b*) Centroidal axis $x - x$	$I_{AB} = \frac{\pi d^4}{256}$ $I_{xx} = 0.00343\ d^4$ $= 0.0055\ R^4$ where $R = d/2$

8.14. MOMENT OF INERTIA OF COMPOSITE SECTIONS

Beams and columns having composite sections are commonly used in structures. Moment of inertia of these sections about an axis can be found by the following steps:

(1) Divide the given figure into a number of simple figures.

(2) Locate the centroid of each simple figure by inspection or using standard expressions.

(3) Find the moment of inertia of each simple figure about its centroidal axis. Add the term Ay^2 where A is the area of the simple figure and y is the distance of the centroid of the simple figure from the reference axis. This gives moment of inertia of the simple figure about the reference axis.

(4) Sum up moments of inertia of all simple figures to get the moment of inertia of the composite section.

The procedure given above is illustrated below. Referring to the Fig. 8.32, it is required to find out the moment of inertia of the section about axis $A - B$.

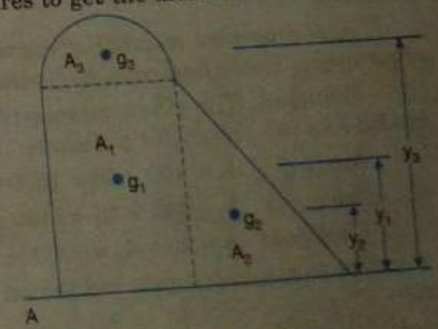


Fig. 8.32

(1) The section in the figure is divided into a rectangle, a triangle and a semicircle. The areas of the simple figures A_1, A_2 and A_3 are calculated.

(2) The centroids of the rectangle (g_1), triangle (g_2) and semicircle (g_3) are located. The distances y_1, y_2 and y_3 are found from the axis AB.

(3) The moment of inertia of the rectangle about its centroid (I_{g1}) is calculated using standard expression. To this, the term $A_1 y_1^2$ is added to get the moment of inertia about the axis AB as:

$$I_1 = I_{g1} + A_1 y_1^2$$

Similarly, the moment of inertia of the triangle ($I_2 = I_{g2} + A_2 y_2^2$) and of semicircle ($I_3 = I_{g3} + A_3 y_3^2$) about axis AB are calculated.

(4) Moment of inertia of the composite section about AB is given by:

$$I_{AB} = I_1 + I_2 + I_3$$
$$= I_{g1} + A_1 y_1^2 + I_{g2} + A_2 y_2^2 + I_{g3} + A_3 y_3^2 \quad ...(8.15)$$

In most engineering problems, moment of inertia about the centroidal axis is required. In such cases, first locate and centroidal axis as discussed in art. 8.7 and then find the moment of inertia about this axis.

Referring to Fig. 8.33, first the moment of area about any reference axis, say AB, is taken and is divided by the total area of the section to locate centroidal axis $x-x$. Then the distances of centroid of individual figures y_{c1}, y_{c2} and y_{c3} from the axis $x-x$ are determined. The moment of inertia of the composite section about the centroidal axis $x-x$ is calculated using the expression:

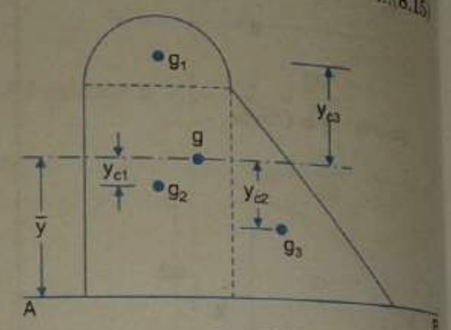


Fig. 8.33

$$I_{xx} = I_{g1} + A_1 y_{c1}^2 + I_{g2} + A_2 y_{c2}^2 + I_{g3} + A_3 y_{c3}^2 \quad ...(8.16)$$

Sometimes the moment of inertia is found about a convenient axis and then using parallel axis theorem, the moment of inertia about centroidal axis is found.

In the above example, the moment of inertia I_{AB} is found and $\bar{y}$, the distance of CG from axis AB is calculated. Then from parallel axis theorem,

$$I_{AB} = I_{xx} + A\,\bar{y}^2$$
$$I_{xx} = I_{AB} - A\,\bar{y}^2$$

where A is the area of composite section.

Example 8.9. *Determine the moment of inertia of the section shown in Fig. 8.34 about an axis passing through the centroid and parallel to the top most fibre of the section.*

Also determine moment of inertia about the axis of symmetry. Hence find radii of gyration.

Solution. The given composite section can be divided into two rectangles as follows:

Area $A_1 = 150 \times 10 = 1500 \text{ mm}^2$

Area $A_2 = 140 \times 10 = 1400 \text{ mm}^2$

Total Area $A = A_1 + A_2 = 2900 \text{ mm}^2$

Due to symmetry, centroid lies on the symmetric axis $y-y$.

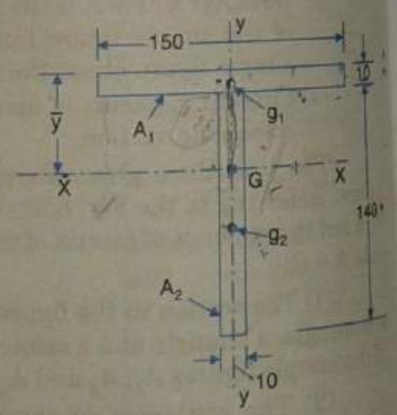


Fig. 8.34

The distance of the centroid from the top most fibre is given by:

$$\bar{y} = \frac{\text{Sum of moment of the areas about the top most fibre}}{\text{Total area}}$$

$$= \frac{1500 \times 5 + 1400\,(10 + 70)}{2900} = 41.21 \text{ mm}$$

Referring to the centroidal axes $x-x$ and $y-y$, the centroid of A_1 is g_1 (0.0, 36.21) and that of A_2 is g_2 (0.0, 38.79).

Moment of inertia of the section about $x-x$ axis I_{xx} = Moment of inertia of A_1 about $x-x$ axis + moment of inertia of A_2 about $x-x$ axis.

$$\therefore \quad I_{xx} = \frac{150 \times 10^3}{12} + 1500\,(36.21)^2 + \frac{10 \times 140^3}{12} + 1400\,(38.79)^2$$

i.e., $\mathbf{I_{xx} = 63{,}72442.5 \text{ mm}^4}$ **Ans.**

Similarly, $$I_{yy} = \frac{10 \times 150^3}{12} + \frac{140 \times 10^3}{12} = \mathbf{28{,}24166.7 \text{ mm}^4}$$ **Ans.**

Hence, the moment of inertia of the section about an axis passing through the centroid and parallel to the top most fibre is 63,72442.5 mm^4 and moment of inertia of the section about the axis of symmetry is 28,24166.66 mm^4.

The radius of gyration is given by:

$$k = \sqrt{\frac{I}{A}}$$

$$\therefore \quad k_{xx} = \sqrt{\frac{I_{xx}}{A}} = \sqrt{\frac{63{,}72442.5}{2900}}$$

$\mathbf{k_{xx} = 46.88 \text{ mm}}$ **Ans.**

Similarly, $$k_{yy} = \sqrt{\frac{28{,}24166.66}{2900}}$$

$\mathbf{k_{yy} = 31.21 \text{ mm}}$ **Ans.**

Example 8.10. *Determine the moment of inertia of the L-section shown in the Fig. 8.33 about its centroidal axes parallel to the legs. Also find the polar moment of inertia.*

Solution. The given section is divided into two rectangles A_1 and A_2.

Area $A_1 = 125 \times 10 = 1250 \text{ mm}^2$

Area $A_2 = 75 \times 10 = 750 \text{ mm}^2$

Total Area $= 2000 \text{ mm}^2$

First, the centroid of the given section is to be located.

Two reference axes (1) – (1) and (2) – (2) are chosen as shown in Fig. 8.35.

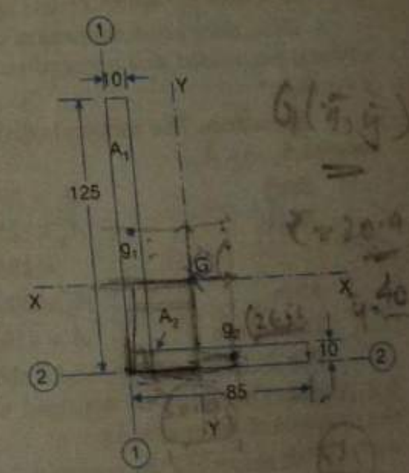


Fig. 8.35

The distance of centroid from the axis (1) – (1)

$$= \frac{\text{sum of moment of areas } A_1 \text{ and } A_2 \text{ about } (1)-(1)}{\text{Total area}}$$

i.e.,

$$\bar{x} = \frac{1250 \times 5 + 750\left(10 + \frac{75}{2}\right)}{2000}$$

$$= 20.94 \text{ mm}$$

Similarly, the distance of the centroid from the axis (2) – (2)

$$= \bar{y} = \frac{1250 \times \frac{125}{2} + 750 \times 5}{2000} = 40.94 \text{ mm}$$

With respect to the centroidal axes $x - x$ and $y - y$, the centroid of A_1 is g_1 (15.94, 21.56) and that of A_2 is g_2 (26.56, 35.94).

$\therefore$ I_{xx} = Moment of inertia of A_1 about $x - x$ axis

+ Moment of inertia of A_2 about

I_{xx} = Moment of inertia of A_1 + Moment of inertia of A_2 + Moment of inertia of A_3 about $x - x$ axis

$$I_{xx} = \frac{200 \times 9^3}{12} + 1800 \times 120.5^2 + \frac{6.7 \times 232^3}{12} + 0 + \frac{200 \times 9^3}{12} + 1800\,(120.5)^2$$

$$I_{xx} = \mathbf{5{,}92{,}69202\ mm^4}\ \textbf{Ans.}$$

Similarly,

$$I_{yy} = \frac{9 \times 200^3}{12} + \frac{232 \times 6.7^3}{12} + \frac{9 \times 200^3}{12}$$

$$I_{yy} = \mathbf{1{,}20{,}05815\ mm^4}\ \textbf{Ans.}$$

Moment of inertia of the section about a centroidal axis perpendicular to $x - x$ and $y - y$ axis is nothing but polar moment of inertia, and is given by:

$$I_{zz} = I_{xx} + I_{yy} = 5{,}92{,}69202 + 1{,}20{,}05815$$

$$I_{zz} = \mathbf{7{,}12{,}75017\ mm^4}\ \textbf{Ans.}$$

Example 8.12. *Compute the*

Then $a_D \sin\theta = Y = 2.4\cos 60° + 15.0\cos 30°$

$= 14.190 \text{ m/sec}^2 \downarrow$

$a_D \cos\theta = 4 + 2.4\sin 60° - 15\cos 30°$

$= -\overrightarrow{1.422} \text{ m/sec}^2$

$= \overleftarrow{1.422} \text{ m/sec}^2$

$a_D = \sqrt{14.190^2 + 1.422^2}$

$= \mathbf{14.261 \text{ m/sec}^2}$ **Ans.**

$\theta = \tan^{-1}\dfrac{14.190}{1.422}$

$= \mathbf{84.277}$ **Ans.**

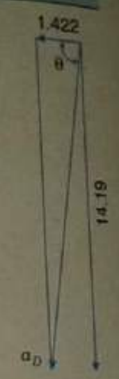


Fig. 20.4(f)

Example 20.3. *A slender beam AB of length 3 m which remains always in the same vertical plane has its ends A and B constrained to remain in contact with a horizontal floor and a vertical wall respectively as shown in Fig. 20.5(a). Determine the velocity and acceleration of the end B at the position shown in the figure, if the point A has a velocity of 2 m/sec and an acceleration of 1.6 m/sec² towards left.*

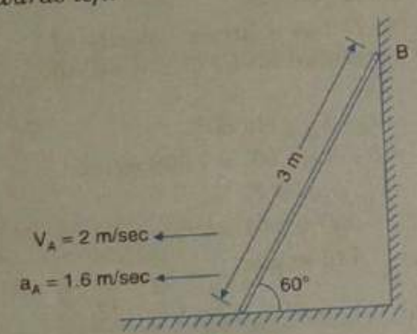


Fig. 20.5(a)

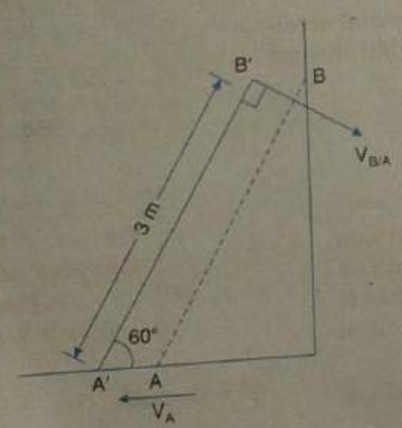


Fig. 20.5(b)

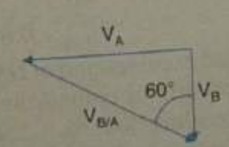


Fig. 20.5(c)

Solution. The plane motion of B can be split into translation of A and rotation about A as shown in Fig. 20.5(b). Due to rotation about A, B has the relative velocity $v_{B/A} = 3\omega$ where ω is angular velocity of B about A. Since the actual velocity of B is vertically downwards the vector diagram for the velocity of B is as shown in Fig. 20.5(c). From vector diagram

$v_B = v_A \cot 60° = 2\cot 60°$

$= \mathbf{1.555 \text{ m/sec}}$ **Ans.**

$v_{B/A} = \dfrac{v_A}{\sin 60°} = \dfrac{2}{\sin 60°} = 2.309 \text{ m/sec}$

Acceleration of point B: There are three components of acceleration as shown in Fig. 20.5(d).

(*i*) Due to translation of $A = a_A = 1.6 \text{ m/sec}^2$

(*ii*) Due to rotation about A,

tangential acceleration $= r\alpha = 3\alpha$ where α is angular acceleration, and

(*iii*) Radial inward acceleration $= \dfrac{v_{B/A}^2}{r} = \dfrac{2.3094^2}{3} = 1.778 \text{ m/sec}^2$

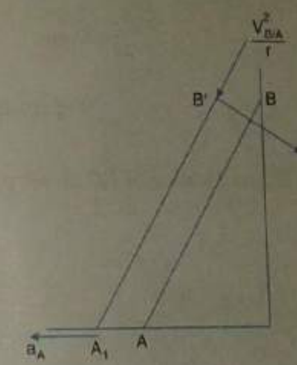


Fig. 20.5(d)

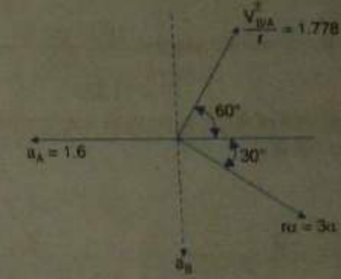


Fig. 20.5(e)

The net resultant acceleration of B, a_b is vertically downwards as shown in vector diagram 20.5(e) Since the resultant acceleration a_B is not having any component in the horizontal direction, considering horizontal forces, we get,

$1.6 + 1.778\cos 60° - 3\alpha\cos 30° = 0$ or $\alpha = 0.958 \text{ rad/sec}^2$.

Considering vertical components, we get,

$a_B = 1.778\sin 60° + 3\alpha\sin 30°$

i.e., $\mathbf{a_B = 2.9766 \text{ m/sec}^2}$ **Ans.**

Example 20.4. *Length of crank AB is 100 mm and that of connecting rod is 250 mm as shown in Fig. 20.6(a). If the crank is rotating in clockwise direction at 1500 rpm, for the position shown in Fig. 20.6(a), determine the angular velocity of the connecting rod and the velocity of the piston.*

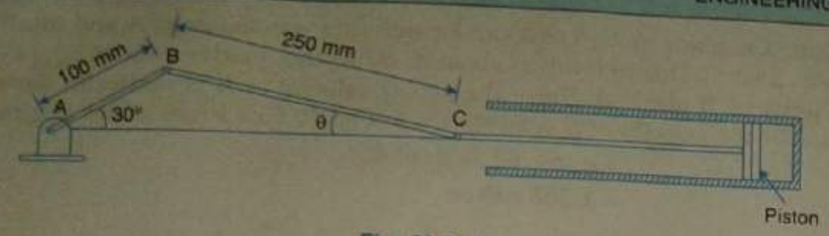


Fig. 20.6(*a*)

Solution. Angular velocity of the crank

$$\omega = 1500 \text{ rpm} = \frac{1500 \times 2\pi}{60} = 157.08 \text{ rad/sec}$$

∴ Tangential velocity of end B (Fig. 20.6(*b*)) is given by

$$v_B = r\omega = 0.100 \times 157.080$$
$$= 15.708 \text{ m/sec}$$

Now consider the motion of connecting rod BC. Inclination of BC to horizontal 'θ' is given by

$$\frac{100}{\sin\theta^\circ} = \frac{250}{\sin 30^\circ}$$

$$\sin\theta = \frac{100 \times \sin 30^\circ}{250}$$

or $$\theta = 11.537^\circ.$$

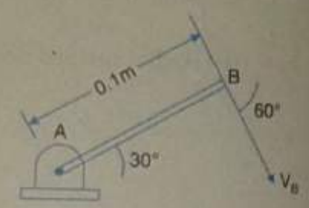


Fig. 20.6(*b*)

The motion of BC can be split into translation of B and rotation of BC about point B as shown in Fig. 20.6(*c*).

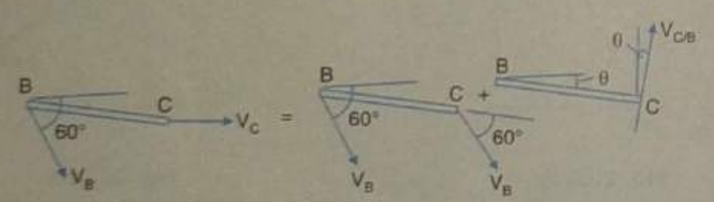


Fig. 20.6(*c*)

Let ω' be the angular velocity of BC. Tangential velocity of point C w.r.t. B, due to rotation, $v_{C/B}$ is given by

$$v_{C/B} = r\omega' = 0.25\,\omega'$$

The resultant velocity of point C is v_C and is horizontal. Considering the vertical component of velocities, we get

$$0 = v_B \sin 60^\circ + v_{C/B} \cos\theta = 15.708 \sin 60^\circ + 0.25\omega' \cos 11.537^\circ$$
$$= 13.6035 + 0.244\omega'$$

i.e., $$\omega' = \mathbf{55.547 \text{ rad/sec}}$$ **Ans.**

Considering the horizontal component of velocities we get,

$$v_C = v_B \cos 60^\circ + v_{C/B} \sin\theta$$
$$= 15.7080 \cos 60^\circ + 0.25 \times 55.547 \times \sin 11.5378^\circ$$

i.e., $$\mathbf{v_C = 10.631 \text{ m/sec}}$$ **Ans.**

20.3. INSTANTANEOUS AXIS OF ROTATION

In the previous article it is shown that plane motion of a point (v_B) can be split into translation of another point (v_A) and rotation of about that point ($r\omega = v_{B/A}$). At any instant it is possible to locate a point in the plane which has zero velocity and hence plane motion of other points may be looked upon as pure rotation about this point. Such point is called **Instantaneous Centre** and the axis passing through this point and at right angles to the plane of motion is called **Instantaneous Axis of Rotation.**

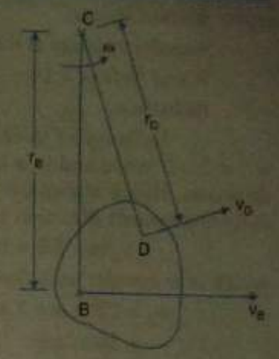


Fig. 20.7

Consider the rigid body shown in Fig. 20.7 which has plane motion. Let B be a point having velocity v_B at the instant considered. Now locate a point C on perpendicular to the direction v_B at B at a distance r_B from the axis of rotation. Now plane motion of the B can be split into translation of C and rotation about C.

∴ $v_B = v_C + r_B\,\omega$ and the direction of the velocity is at right angles to CB. If we write $r_B = v_B/\omega$, then from the above relation we get

$$v_B = v_C + \frac{v_B}{\omega} \cdot \omega \quad i.e., \quad v_C = 0.$$

Thus if point C is selected at a distance v_B/ω along the perpendicular to the direction of velocity at B, the plane motion of B reduces to pure rotation about C.

Hence C is instantaneous centre.

If D is any other point on the rigid body, its velocity will be given by

$$v_D = v_C + r_D\omega$$
$$= r_D\omega \qquad \text{since } v_C = 0,$$

and it will be at right angles to CD.

Thus if the instantaneous centre is located, motion of all other points at that instant can be found by pure rotation about C.

Methods of Locating Instantaneous Centre

Instantaneous centre can be located by any one of the following two methods:

(*i*) if the angular velocity ω and linear velocity v_B are known, instantaneous centre is located at a distance v_B/ω along the perpendicular to the direction of v_B at B.

(*ii*) If the linear velocities of two points of the rigid body are known, say v_B and v_D, drop perpendiculars to them at B and D. The intersection point is the instantaneous centre.

Use of Instantaneous Centre

If instantaneous centre is located, the velocities of all other points on the rigid body at that instant can be determined by treating the plane motion as pure rotation about the instantaneous centre. It may be noted that point C located as instantaneous centre, has zero velocity only at that instant. If it has zero velocity at time t, at time $t + dt$ it has some velocity. It means, the instantaneous centre is not having zero acceleration. Hence for the acceleration calculations we can not treat the plane motion as pure rotation about instantaneous centre. The plane motion can be treated as a case of pure rotation about instantaneous centre only for velocity calculations.

Example 20.5. *Determine the velocities of the points B and D given in example 20.2 by instantaneous centre method. Given*

Angular velocity $\omega = 5$ *rad/sec.*

Wheel radius $= 1$ *m.*

Solution.

$\therefore$ Velocity of geometric Centre A, $v_A = r\omega = 1 \times 5 = 5$ m/sec and it is horizontal in the rightward direction. Hence instantaneous centre is in the vertically downwards direction at a distance

$$= v_A/\omega = 5/5 = 1\text{m}, \quad i.e., \quad \text{at point } C,$$

which is in contact with floor (see Fig. 20.8).

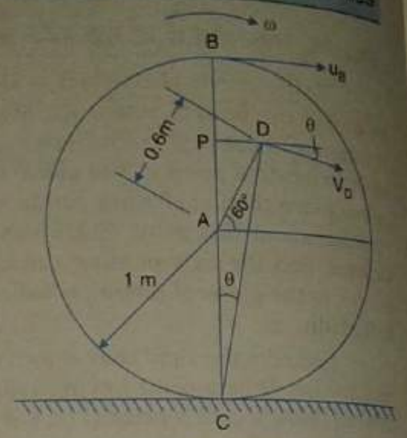


Fig. 20.8

$$\therefore \quad v_B = CB \times \omega = 2 \times 5$$
$$= 10 \text{ m/sec.}$$
$$v_D = CD \times \omega$$

Now $CP = CA + AP$ where P is foot of $\perp r$ of D on CB.

$$\therefore \quad CP = 1 + 0.6 \times \sin 60° = 1.520 \text{ m}$$
$$PD = 0.6 \times \cos 60° = 0.3 \text{ m}$$
$$\therefore \quad CD = \sqrt{CP^2 + PD^2} = \sqrt{1.520^2 + 0.3^2} = 1.549 \text{ m}$$
$$\therefore \quad v_D = CD \times \omega = 1.549 \times 5.0$$

i.e., $\mathbf{v_D = 7.745 \text{ m/sec.}}$ **Ans.**

Its direction is at right angle to CD. Its inclination to horizontal, θ is given by

$$\theta = \angle PCD = \tan^{-1} \frac{0.3}{1.520}$$

i.e., $\theta = \mathbf{11.17°}$ **Ans.**

Example 20.6. *Find the velocity of B in example 20.3 by instantaneous centre method.*

Solution. Since velocity at A (v_A) is horizontal and velocity at B (v_B) is vertical, instantaneous centre is point C, which is obtained by dropping perpendicular to the directions v_A and v_B at points A and B respectively. Now,

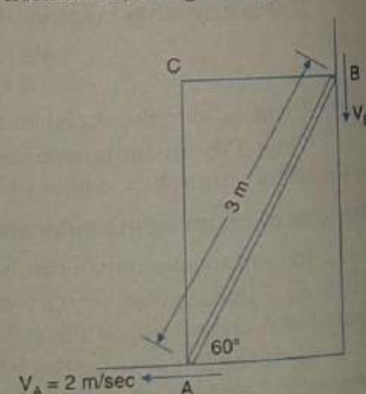


Fig. 20.9

$$v_A = AC \times \omega$$

i.e., $2 = 3 \sin 60° \times \omega$

$$\omega = 0.770 \text{ rad/sec}$$

and $v_B = BC \times \omega = 3 \cos 60° \times 0.770$

i.e., $\mathbf{v_B = 1.555 \text{ m/sec}}$ **Ans.**

20.4. KINETICS OF ROLLING BODIES

It is already known that the plane motion of a rolling body can be split into linear motion and rotation about the geometric axis. Hence the system of forces acting on the body are equal to an effective force $\frac{W}{g}a$ through the geometric centre in the direction of the resultant forces (the direction of motion) and a rotational moment $I\alpha$ as shown in Fig. 20.10(*a*). The reverse

effective forces may be applied on the body along with the system of forces acting on it (see Fig. 20.10(*b*) and then the system may be treated as in equilibrium. This is the *D'Alembert's* principle for rolling bodies.

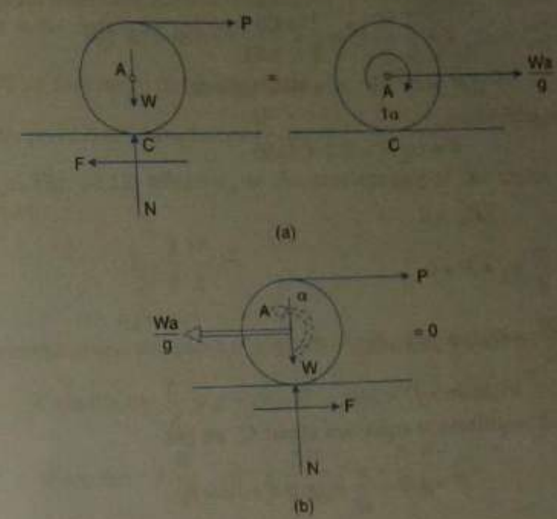


Fig. 20.10

Note: The relation $a_A = r\alpha$, holds good only for the bodies rolling without slipping. If slipping takes place, this relation is not applicable.

Example 20.7. *A solid cylinder weighing 1200 N is acted upon by a force P horizontally as shown in Fig. 20.11(a). Determine the maximum value of P for which there will be rolling without slipping. If P = 1000 N, determine the acceleration of the mass centre and the angular acceleration, given that the coefficient of static friction* $\mu_S = 0.2$ *and the coefficient of kinetic friction* $\mu_K = 0.15$.

Solution. Let the linear acceleration of geometric centre be a_A and the angular acceleration be α. The free body diagram of the cylinder along with reverse effective forces $\frac{w}{g}a$ and $I\alpha$ is shown in Fig. 20.11(*b*).

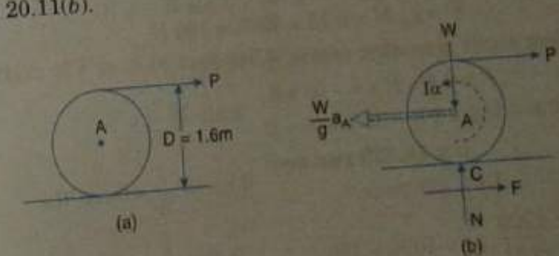


Fig. 20.11

(i) Maximum Value of P for Rolling Without Slipping:

Since there is no slipping.

$$a_A = r\alpha = 0.8\alpha \quad \text{...(i)}$$

$$I = \frac{1}{2}\frac{W}{g}r^2 = \frac{1}{2}\times\frac{1200}{9.81}\times 0.8^2 = 39.144$$

$$\Sigma F_v = 0 \rightarrow N = W = 1200 \text{ N}$$

From the law of friction,

$$F = \mu_S N = 0.2 \times 1200$$
$$= 240 \text{ N}$$

$$\Sigma F_H = 0 \rightarrow$$

$$P - \frac{w}{g}a_A + F = 0$$

$$P - \frac{1200}{9.81}\times 0.8\alpha + 240 = 0$$

$$97.859\alpha - P = 240 \quad \text{...(1)}$$

Writing moment equilibrium equation about C, we get

$$P \times 1.6 - \frac{w}{g}a_A \times 0.8 - I\alpha = 0$$

$$P \times 1.6 - \frac{1200}{9.81}\times 0.8\alpha \times 0.8 - 39.144\alpha = 0$$

i.e. $$1.6 \times P = 117.431\alpha \quad \text{or} \quad P = 73.394\,\alpha \quad \text{...(2)}$$

From (1) and (2), we get

$$97.859\alpha - 73.394\alpha = 240$$

$$\alpha = 9.81 \text{ rad/sec}$$

$$\therefore \quad P = 73.394 \times 9.81$$

$$\mathbf{P = 720\ N} \quad \textbf{Ans.}$$

(*ii*) When $P = 1000$ N;

In this case as slippage occurs the relationship, $a_A = r\alpha$ does not hold good. In this case we have the relationship

$$F = \mu_K N = 0.15 \times 1200 = 180 \text{ N.}$$

Taking moment about geometric centre A, we get

$$P \times 0.8 - F \times r - I\alpha = 0$$

$$1000 \times 0.8 - 180 \times 0.8 - 39.144\,\alpha = 0$$

$$\alpha = \mathbf{16.759\ rad/sec^2} \quad \textbf{Ans.}$$

$$\Sigma F_H = 0 \text{ gives}$$

$$\frac{1200}{9.81}a_A = 1000 + 180$$

$$\boldsymbol{a_A = 9.647\ rad/sec^2} \quad \textbf{Ans.}$$

Example 20.8. *A solid cylinder of weight W and radius r rolls, down an inclined plane which makes θ° with the horizontal axis. Determine the minimum coefficient of friction and the acceleration of the mass centre for rolling, without slipping.*

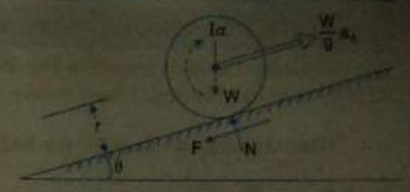


Fig. 20.12

Solution. The free body diagram of the cylinder along with reverse effective forces $\frac{W}{g}a_A$ and $I\alpha$ is shown in Fig. 20.12, where a_A is the acceleration of the mass centre and α is the angular acceleration.

Now, $$I = \frac{1}{2}\frac{W}{g}r^2$$

$$a_A = r\alpha$$

Taking moment about the point of contact of cylinder with the plane, we get

$$W \sin\theta \times r - \frac{W}{g}a_A r - I\alpha = 0$$

$$W \sin\theta \times r = \frac{W}{g}\,.\,r\alpha r + \frac{1}{2}\frac{W}{g}r^2\alpha = \frac{3}{2}\frac{W}{g}r^2\alpha$$

$$\therefore \quad \alpha = \frac{2}{3}\frac{g}{r}\sin\theta$$

$$\therefore \quad a_A = r\alpha = \frac{2}{3}g\sin\theta \quad \textbf{Ans.}$$

Now, Σ Forces normal to the plane = 0, gives

$$N = W\cos\theta$$

$$\therefore \quad F = \mu N = \mu W\cos\theta$$

Σ Forces parallel to the plane = 0, gives

$$F + W\sin\theta - \frac{W}{g}a_A = 0$$

$$F = -W\sin\theta + \frac{W}{g}\times\frac{3}{2}g\sin\theta = -\frac{1}{3}W\sin\theta$$

Hence, direction of F is to be reversed.

$$F = \frac{1}{3}W\sin\theta$$

i.e., $$\mu W\cos\theta = \frac{1}{3}W\sin\theta$$

$$\mu = \frac{1}{3}\tan\theta \quad \textbf{Ans.}$$

20.5. KINETICS OF GENERAL PLANE MOTION

Consider the body shown in Fig. 20.13 which has got plane motion. The plane motion of the body may be split into linear motion and rotation about its mass centre.

Then the effective force on the body is a force $\frac{W}{g}\alpha$ at mass centre and a moment $I\alpha$ as shown in Fig. 20.13(*b*).

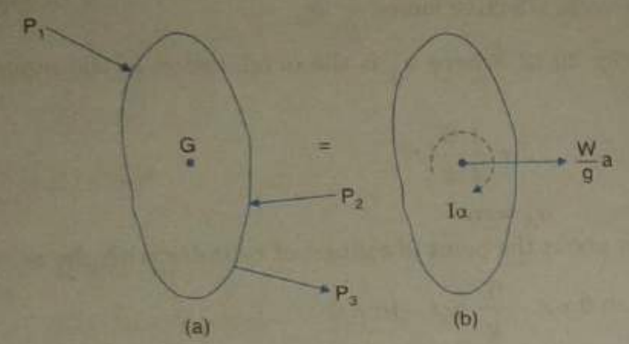


Fig. 20.13

Hence by equating the component of forces acting on the body to the respective components of effective force, $\frac{W}{g}\alpha$, and equating moment of the force about any point to that of effective moment about the same point, kinetic equations may be formed and solved. Another method is one can use D' Alembert's principle, *i.e.* apply reverse effective force to a given force system and solve the equation of equilibrium. The system of force to be considered in this approach are shown in Fig. 20.14.

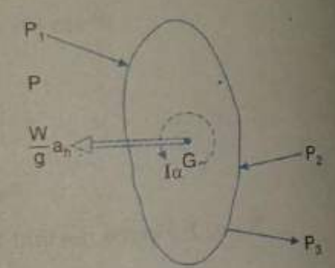


Fig. 20.14

Example 20.9. *A uniform bar 1.6 m long and weighing 300 N is held in position with its ends in contact with smooth surfaces as shown in Fig. 20.15(a). If the bar is released with no velocity, determine the angular acceleration of the rod and the reactions at ends, just after release.*

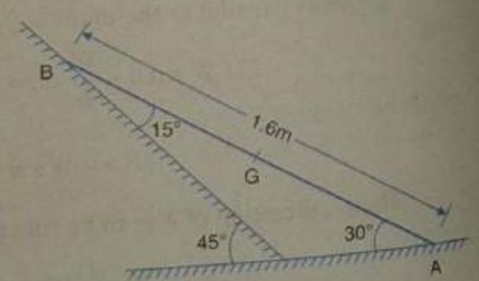


Fig. 20.15(*a*)

Solution. Let α and a be the angular acceleration and linear acceleration of centre of gravity of the rod respectively. The direction of accelerations a_A and a_B of points A and B are known (Fig. 20.15(*b*). The relative velocity of B with respect to A is 1.6 α and is directed at right angles to the position of rod. Hence, from vector diagram of the acceleration shown in Fig. 20.15(*c*), we get

$$\frac{a_A}{\sin 75°} = \frac{a_B}{\sin 60°} = \frac{1.6\alpha}{\sin 45°}$$

$$a_A = 2.1856\alpha$$

and $$a_B = 1.9596\alpha$$

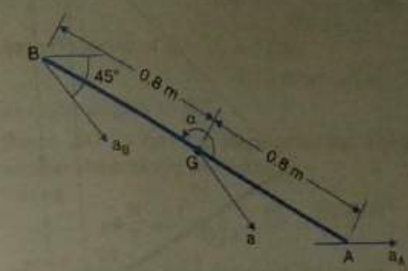


Fig. 20.15(*b*)

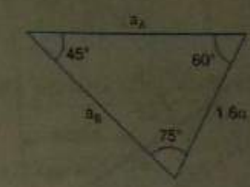


Fig. 20.15(*c*)

Now, the relative acceleration of G with respect to A is directed at right angles to the rod and is equal to

$$a_{G/A} = 0.8\,\alpha.$$

From the vector diagram shown in Fig. 20.15(*d*) for acceleration of G, we get

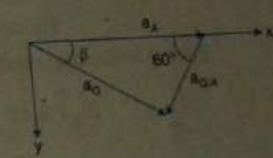


Fig. 20.15(*d*)

$$a_{Gx} = a_A - a_{G/A}\cos 60°$$
$$= 2.1856\alpha - 0.8\alpha\cos 60°$$
$$= 1.7856\alpha$$
$$a_{Gy} = a_{G/A}\sin 60°$$
$$= 0.8\alpha\sin 60° = 0.6928\alpha$$

$$\therefore \quad \tan\beta = \frac{0.6928}{1.7856}$$
$$\beta = 21.21°$$

and $$a_G = \sqrt{1.7856^2 + 0.6928^2} = 1.915\alpha$$

Now consider the kinetics of the rod.

Let E be the point of interaction of the reaction N_A and N_B. Then the required distances for taking moment about E can be easily worked out. Given system of forces (Fig. 20.15(*e*)) are the same as the effective force system shown in Fig. 20.15(*f*).

Now, $$I = \frac{1}{12}\frac{W}{g}l^2 = \frac{1}{12}\times\frac{300}{9.81}\times 1.6^2$$
$$= 6.5240.$$

Equating the moment of the two systems about E, we get,

$$300 \times 0.6928 = \frac{300}{9.81}\times 1.7856\alpha + \frac{300}{9.81}\times 0.6928 + 6.5240\alpha$$
$$= 82.3160\alpha$$

or $$\alpha = \mathbf{2.5249\ rad/sec^2}$$ Ans.

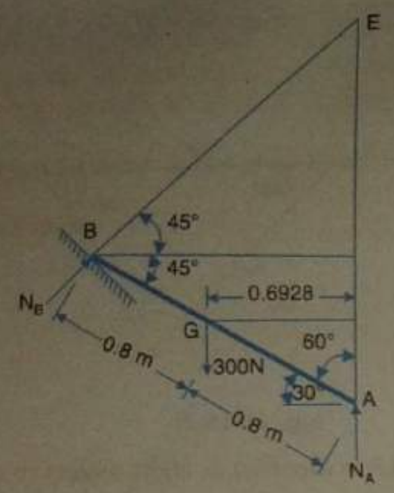


Fig. 20.15(e)

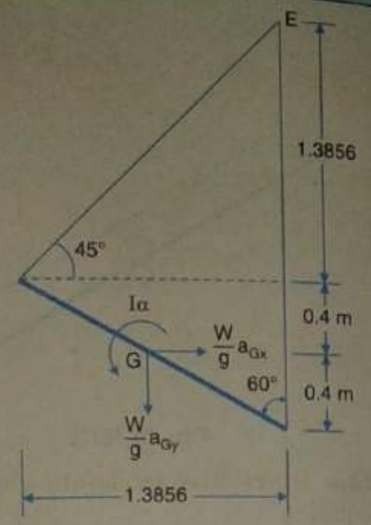


Fig. 20.15(f)

Taking horizontal components, we get,

$$N_B \cos 45° = \frac{w}{g} a_{Gx} = \frac{300}{9.81} \times 1.7856 \times 2.5249$$

$$\mathbf{N_B = 194.98\ N} \qquad \textbf{Ans.}$$

Taking vertical components of the forces, we get

$$N_A + N_B \sin 45° - 300 = \frac{w}{g} a_{Gy}$$

$$N_A + 194.98 \sin 45° - 300 = \frac{300}{9.81} \times 0.6928 \times 2.5249$$

$$\therefore \quad \mathbf{N_B = 215.62\ N} \qquad \textbf{Ans.}$$

Example 20.10. *A 5 N piston in connected to the 80 mm rotating crank by a 200 mm connecting rod. The connecting rod weighs 12 N. In this position shown in Fig. 20.16(a), if a force of 4000 N acts on the piston, and crank rotates at 1800 rpm, determine the reactions at the pins A and B.*

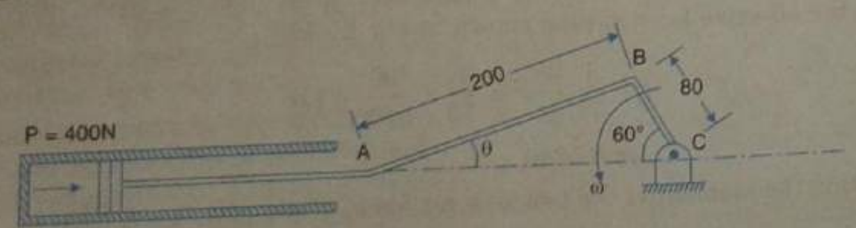


Fig. 20.16(a)

Solution. Let $\angle BAC = \theta°$. Then applying sine rule we get,

$$\frac{80}{\sin\theta} = \frac{200}{\sin 60°} \quad \text{or} \quad \theta = 20.2679°$$

Let ω be the angular velocity of the crank.

$$\omega = \frac{2\pi \times 1800}{60} = 188.4956 \text{ rad/sec}$$

∴ Velocity of B is $v_B = 0.08\omega$

$$= 15.0796 \text{ m/sec}$$

and is at right angles to BC.

Since BC is having pure rotation about C with uniform tangential velocity v_B B has acceleration along BC equal to

$$a_B = \frac{v_B^2}{r} = \frac{15.0796^2}{0.8} = 2842.4292 \text{ m/sec}^2$$

Now consider connecting rod BA shown in Fig. 20.16(b).

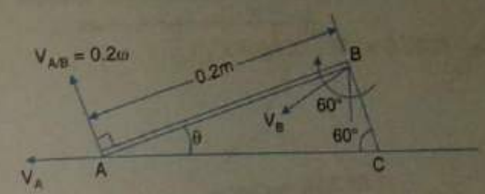


Fig. 20.16(b)

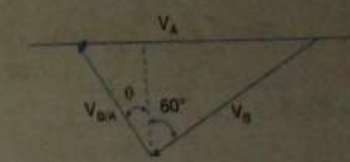


Fig. 20.16(c)

Let ω′ be the angular velocity of BA about B. Then $v_{A/B} = 0.2\omega'$

$$v_A = v_B + v_{A/B}$$

From vector diagram shown in Fig. 20.16(c)

$$v_{A/B} \cos\theta = v_B \cos 60°$$

$$0.2\omega' \cos 20.2679° = 15.0796 \cos 60°$$

$$\omega' = 40.1873 \text{ rad/sec.}$$

$$v_{A/B} = 0.2\omega = 8.0375 \text{ m/sec}$$

and

$$v_A = v_B \sin 60° + v_{A/B} \sin\theta$$

$$= 15.8436 \text{ m/sec.}$$

Consider the acceleration of AB. Let α be angular acceleration (Fig. 20.16(d).

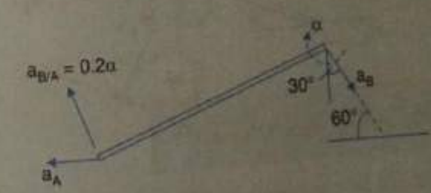


Fig. 20.16(d)

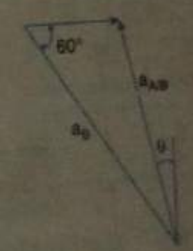


Fig. 20.16(e)

Now, $a_B = 2842.4292$ making 60° with horizontal.

$$a_{A/B} = 0.2\alpha$$

Now,

$$\vec{a}_A = \vec{a}_B + \vec{a}_{A/B}$$

Looking at vector diagram shown in Fig. 19.16(*e*),

$$a_B \sin 60° = a_{A/B} \cos \theta$$

$$2842.4292 \sin 60° = 0.2\alpha \cos 20.2679°$$

$$\alpha = 13120.457 \text{ rad/sec}^2$$

$$\therefore \quad a_{A/B} = 0.2\alpha = 2624.0913$$

$$\therefore \quad a_A = a_B \cos 60° - a_{A/B} \sin 20.2679° = 512.2027 \text{ m/sec}^2.$$

Now the acceleration of the centre of gravity of bar *AB* can be obtained.

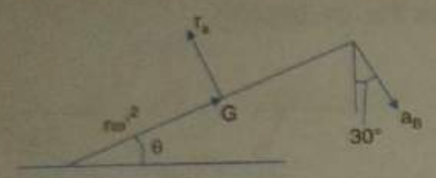


Fig. 20.16(*f*)

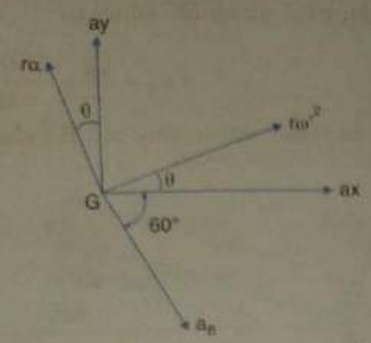


Fig. 20.16(*g*)

Various components of acceleration at *G* are shown in Figures 20.16(*f*) and 20.16(*g*). Let a_X be horizontal component and a_Y be vertical components of acceleration of C.G. of connecting rod *AB*.

Then,

$$a_X = a_B \cos 60° + r\omega^2 \cos \theta - r\alpha \sin \theta$$

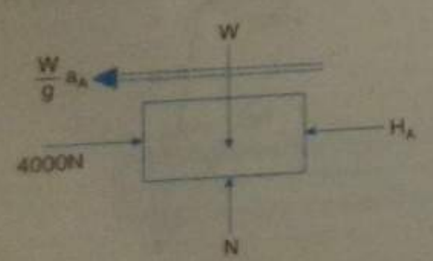
$$= 1118.2109 \text{ m/sec}^2$$

$$a_Y = r\alpha \cos \theta + r\omega^2 \sin \theta - a_B \sin 60°$$

$$= -1174.862 \text{ m/sec}^2$$

i.e., a_Y is downward 1174.862 m/sec^2.

Now consider the dynamic equilibrium of piston *A* (Fig. 20.16(*h*).



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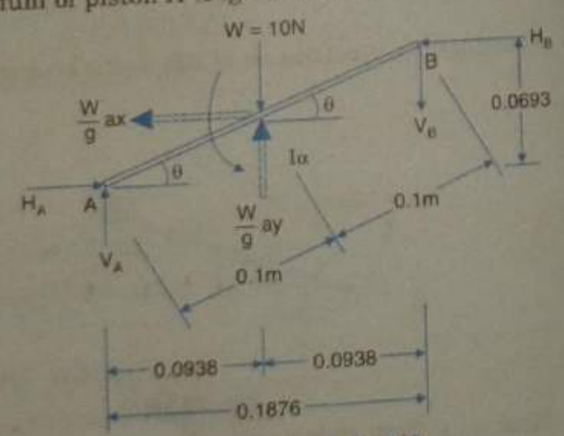


Fig. 20.16(*i*)

$\Sigma F_H = 0$, gives

$$H_A = 4000 - \frac{50}{9.81} \times 512.2027$$

i.e., $\mathbf{H_A = 3738.38 \text{ kN}}$ **Ans.**

Inertia forces are: $\frac{W}{g} a_x = \frac{10}{9.81} \times 1118.2109 = 1139.87\text{N}.$

and $\frac{W}{g} a_Y = \frac{10}{9.81} \times 1174.862 = 1197.62 \text{ N}$

$$I\alpha = \frac{W}{g}\frac{l^2}{12}\alpha = \frac{10}{9.81} \times \frac{0.2^2}{12} \times 13120.457 = 44.8 \text{ N-m}$$

Taking moment about *B*, we get

$$v_A \times 0.1876 - H_A \times 0.0693 - 10 \times 0.0938 + \frac{W}{g} a_Y \times 0.0938$$

$$+ \frac{W}{g} a_x \times 0.03478 - I\alpha = 0$$

or $\mathbf{v_A = 813.95 \text{ N}}$ **Ans.**

$\Sigma F_V = 0$, gives

$$V_B = V_A + \frac{W}{g} a_Y - W$$

i.e., $\mathbf{V_B = 2001.57 \text{ N} \downarrow}$ **Ans.**

$$H_B = H_A - \frac{W}{g} a_x$$

i.e., $\mathbf{H_B = 2598.51 \text{ N}}$ **Ans.**

Important Formulae

When a wheel rotates without slip, the relationship between motion of its geometric centre and its angular motion are

$$s_A = r\theta$$

$$v_A = r\omega$$

$$a_A = r\alpha$$

PROBLEMS FOR EXERCISE

20.1 The wheel of a tractor is 1 m in diameter and the tractor is travelling at 9 kmph. What are the velocities at the top and bottom of the wheel relative to (i) a person seated in tractor; (ii) a person standing on the ground.

[**Ans.** (i) $v_{Top} = 2.5$ m/sec; $v_{bottom} = 2.5$ m/sec; (ii) $v_{Top} = 5.0$ m/sec; $v_{bottom} = 0$]

Looking at vector diagram shown in Fig. 19.16(*e*),

$$a_B \sin 60° = a_{A/B} \cos \theta$$

$$2842.4292 \sin 60° = 0.2\alpha \cos 20.2679°$$

$$\alpha = 13120.457 \text{ rad/sec}^2$$

$$\therefore \quad a_{A/B} = 0.2\alpha = 2624.0913$$

$$\therefore \quad a_A = a_B \cos 60° - a_{A/B} \sin 20.2679° = 512.2027 \text{ m/sec}^2.$$

Now the acceleration of the centre of gravity of bar *AB* can be obtained.

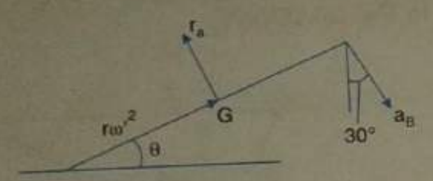


Fig. 20.16(*f*)

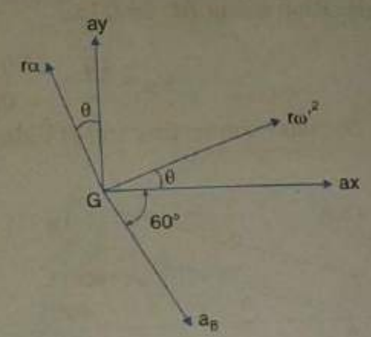


Fig. 20.16(*g*)

Various components of acceleration at *G* are shown in Figures 20.16(*f*) and 20.16(*g*). Let a_X be horizontal component and a_Y be vertical components of acceleration of C.G. of connecting rod *AB*.

Then,

$$a_X = a_B \cos 60° + r\omega'^2 \cos \theta - r\alpha \sin \theta$$

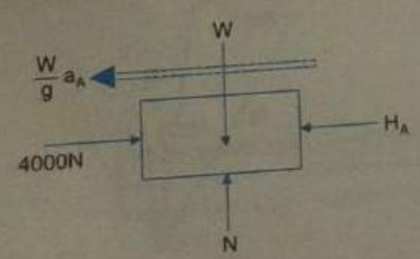
$$= 1118.2109 \text{ m/sec}^2$$

$$a_Y = r\alpha \cos \theta + r\omega'^2 \sin \theta - a_B \sin 60°$$

$$= -1174.862 \text{ m/sec}^2$$

i.e., a_Y is downward 1174.862 m/sec^2.

Now consider the dynamic equilibrium of piston *A* (Fig. 20.16(*h*).



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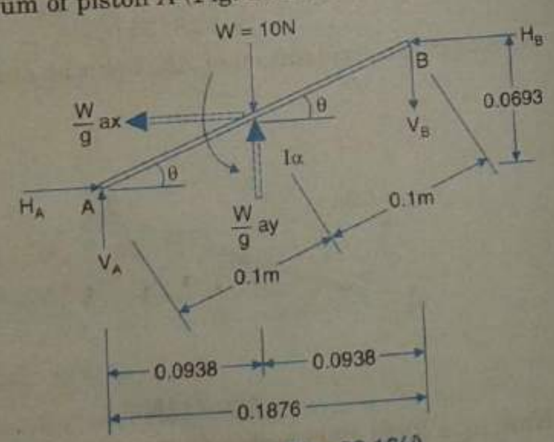

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$$+ \frac{W}{g} a_x \times 0.03478 - I\alpha = 0$$

Ans.

or $\mathbf{v_A = 813.95 \text{ N}}$

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PROBLEMS FOR EXERCISE

20.1 The wheel of a tractor is 1 m in diameter and the tractor is travelling at 9 kmph. What are the velocities at the top and bottom of the wheel relative to (*i*) a person seated in tractor; (*ii*) a person standing on the ground.

[**Ans.** (*i*) $v_{Top} = 2.5$ m/sec; $v_{bottom} = 2.5$ m/sec; (*ii*) $v_{Top} = 5.0$ m/sec; v_{bottom}

20.2 Rod AB shown in Fig. 20.17 is 2 m long and slides with its ends in contact with the floor and the inclined plane. At the instant when AB makes 40° with the horizontal, point A has left ward velocity of 5 m/sec. Determine the angular velocity of the rod and the velocity of end B.

[**Ans.** $\omega = 2.3040$ rad/sec; $v_B = 4.0760$ m/sec.]

20.3 In problem 19.2, if A has acceleration of 3m/sec² to the left at the instant as shown in Fig. 20.17, determine angular acceleration of the rod and resultant acceleration of end B.

[**Ans.** $\alpha = 3.3145$ rad/sec²; $a_B = 13.7438$ m/sec²]

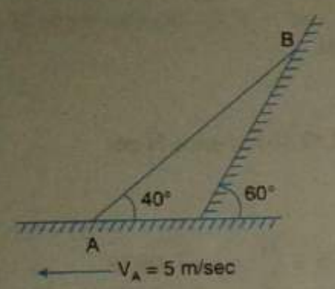

Fig. 20.17

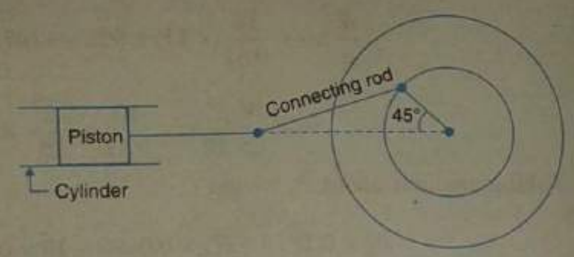

Fig. 20.18

20.4 Figure 20.18 shows a reciprocating pump driven by a driving wheel. If crank is 80 mm long and connecting rod 200 mm, determine the velocity of the piston in the position shown. The driving wheel rotates at 2000 rpm in anticlock direction. [**Ans.** 15.34 m/sec]

20.5 Solve problem 20.2 using instantaneous centre method.

20.6 A solid sphere of weight W and radius r rolls down an inclined plane, which makes angle θ to horizontal. Determine the minimum frictional resistance required to prevent slippage and also the corresponding linear acceleration of mass centre. [**Ans.** $\mu = 1/3 \tan \theta$]

20.7 A solid cylinder of weight 1000 N and radius 1.2 m is to be rolled up an inclined plane which makes 30° with horizontal. Determine the force P to be applied on the periphery in the direction parallel to the plane (see Fig. 20.19), such that it has maximum acceleration but no slippage. Take coefficient of friction of 0.2. [**Ans.** 519.61 N]

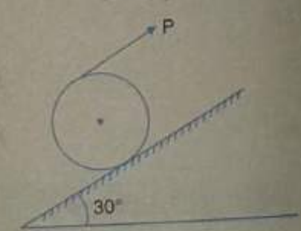

Fig. 20.19

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