A) CIRCULAR & HYPERBOLIC FUNCTIONS

$$\frac{e^{i\theta} + e^{-i\theta}}{2} = \cos\theta \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$\longrightarrow$ Circular functions

$$\frac{e^{x} + e^{-x}}{2} = \cosh x \qquad \sinh x = \frac{e^{x} - e^{-x}}{2}$$

$\longrightarrow$ Hyperbolic function [By defination]

Relating these 2 : $\theta = ix$

$$\cos ix = \frac{e^{x} + e^{-x}}{2} = \cosh x \quad ; \quad \sin ix = -\frac{1}{i}\left[\frac{e^{x} - e^{-x}}{2}\right] = i\sinh x$$

Result:

$\cos ix = \cosh x$

$\sin ix = i\sinh x$

$\tan ix = i\tanh x$

or $\sinh x = -i\sin ix$

$\boxed{\sinh(ix) = i\sin x}$ *

⟹ TRIGNOMETRIC IDENTIES here:

1. $\cos^2 hx - \sin^2 hx = 1$
2. $\cosh(2x) = \cos^2 hx + \sin^2 hx$
3. $\cosh(2x) = 1 + 2\sin^2 hx$
4. $\cosh(2x) = 2\cos^2 hx - 1$
5. $\sin h(2x) = 2\sin hx \cos hx$
6. $\sin h(3x) = 3\sin hx + 4\sin^3 hx$
7. $\cos h(3x) = 4(\cos hx)^3 - 3\cos h$
8. $\tan h(3x) = \dfrac{3\tan hx + \tan^3 hx}{1 + 3\tan^2 hx}$

⟹ Period of hyperbolic functions

eg: $\sin hx = i\sin ix = \dfrac{e^x - e^{-x}}{2}$

$$\sin h(x + 2n\pi i) = \frac{e^x \cdot \overset{1}{e^{2n\pi i}} - e^{-x} \cdot \overset{1}{e^{-2n\pi i}}}{2} = \frac{e^x - e^{-x}}{2} = \sin hx$$

Period $= 2\pi i$ = [Period of $\sin x$] i

In general: Period of hyperbolic function = [Period of normal] i

{ eg: $\cos hx = 2\pi i$
$\tan hx = \pi i$ }

B) Logarithm of Complex numbers

$$z = x+iy = \sqrt{x^2+y^2} \cdot e^{i\theta}$$

$$\log z = \frac{1}{2}\log[x^2+y^2] + i\theta \Rightarrow \boxed{\log z = \log(x+iy) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\left(\frac{y}{x}\right)}$$

C) Successive Differentiation:

⇒ Standard ones

a) $y = e^{mx} \longrightarrow \boxed{y_n = m^n e^{mx}}$

$$y_1 = me^{mx} = my$$

$$y_2 = my_1 = m^2 y$$

$$y_3 = my_2 = m^3 y$$

$$\vdots$$

$$y_n = m^n y = m^n e^{mx}$$

if base is a then $\boxed{y_n = [m\log a]^n a^{mx}}$

b) $y = \sin(ax+b)$

$$y_1 = a\cos(ax+b) \rightarrow a\sin\left(ax+b+\frac{\pi}{2}\right)$$

$$y_2 = -a^2\sin(ax+b) = -a^2 y \rightarrow a\sin\left(ax+b+2\left(\frac{\pi}{2}\right)\right)$$

$$y_3 = -a^3\cos(ax+b) = -a^2 y'$$

$$y_4 = a^4\sin(ax+b) = -a^2 y_2 = a^4 y$$

$$\vdots$$

$$\boxed{y_n = a^n \sin\left(ax+b+n\frac{\pi}{2}\right)}$$

if $y = \cos(ax+b) \longrightarrow \boxed{y_n = a^n\cos\left(ax+b+\frac{n\pi}{2}\right)}$

c) $y = (ax+b)^m$

$$y_1 = m \cdot a \cdot (ax+b)^{m-1}$$

$$y_2 = [(m)(m-1)]\, a^2 \cdot (ax+b)^{m-2}$$

$$\vdots$$

$$\boxed{y_n = \frac{\lfloor m}{\lfloor m-n} \cdot a^n \, (ax+b)^{m-n}}$$ *

d) $y = \log(ax+b)$

$$y_1 = \frac{a}{ax+b}$$

$$y_2 = \frac{a^2}{(ax+b)^2}(-1)$$

$$y_3 = \frac{a^3}{(ax+b)^3}\, 1\cdot 2$$

$$y_4 = -\frac{a^4}{(ax+b)^4} \cdot 1\cdot 2\cdot 3$$

*
$$\boxed{y_n = (-1)^{n-1} \cdot \frac{a^n}{(ax+b)^n}\{\lfloor n-1\}}$$

$\Rightarrow$ Lebinitz Theorem

$$\cos(ax+b) \longrightarrow a^n \cos\left(ax+b+n\frac{\pi}{2}\right)$$

if $y = (u)(v)$

$$\boxed{y_n = v_0 u_n + {}^nC_1 \cdot u_{n-1} v_1 + {}^nC_2 \cdot u_{n-2} \cdot v_2 \cdots\cdots + {}^nC_{n-1} u_1 v_{n-1} + {}^nC_n \cdot u_0 v_n}$$ *

eg: $\underset{v}{x^3}\ \underset{u}{\cos x}$ — $x^3 \cdot \cos\left(x + \frac{n\pi}{2}\right) + {}^nC_1 \cdot 3x^2 \cdot \cos\left(x + (n-1)\frac{\pi}{2}\right) + {}^nC_2 \cdot 6x \cdot \cos\left(x + (n-2)\frac{\pi}{2}\right)$

$+ {}^nC_3 \cdot 6 \cdot \cos\left(x + (n-3)\frac{\pi}{2}\right) + 0 \ldots$

D) Expansion:

⟹ Taylor series

$$f(a+h) = f(a) + \frac{f'(a)}{\lfloor 1} \cdot h + \frac{f''(a)}{\lfloor 2} \cdot h^2 \ldots\ldots + \frac{f_n(a)}{\lfloor n} \cdot h^n \ldots\ldots$$

eg: $\log(1+x) = f(1) + \frac{f'(1)}{\lfloor 1} \cdot x + \frac{f^2(1)}{\lfloor 2}\, x^2 \ldots\ldots$

$\log(1+x)$

$$\left[0 + \frac{1}{1} \cdot x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \ldots\ldots\right]$$

$f(1) = \log(1) = 0$

$f'(x) = \frac{1}{x}$

$f''(x) = -\frac{1}{x^2}$

$f'''(x) = \frac{2}{x^3}$

$f^4(x) = -\frac{6}{x^4}$

$\Rightarrow$ <u>Mclaurien Series:</u> put $a = 0$ $h = x$

$$f(x) = f(0) + \frac{f'(0)}{\lfloor 1} \cdot x + \frac{f''(0)}{\lfloor 2} \cdot x^2 \cdots \frac{f^n(0)}{\lfloor n} \cdot x^n \cdots\cdots$$

eg: $\log(1+x) = f(x)$

$$\frac{1}{1+x} = f'(x)$$

$$\frac{-1}{(1+x)^2} = f''(x) \rightarrow f'''(x) = \frac{+2}{(1+x)^3}$$

$$f^4(x) = \frac{-2 \cdot 3}{(1+x)^4} \cdots\cdots$$

$$\rightarrow \log(1+x) = \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} \cdots\cdots$$

<u>Proof of Taylor</u>

$$f(a+x) = A_0 + A_1 x + A_2 x^2 + A_3 x^3 + A_4 x^4 \cdots\cdots$$

$$f'(a+x) = A_1 + 2A_2 x + 3A_3 x^2 + 4A_4 x^3 \cdots\cdots$$

$$f_2(a+x) = 2A_2 + 6A_3 x + 12A_4 x^2 \cdots\cdots$$

$$f_3(a+x) = 6A_3 + 24A_4 x \cdots\cdots$$

Put $x = 0$

$$f(a) = A_0$$

$$\frac{f'(a)}{\lfloor 1} = A_1$$

$$\frac{f''(a)}{\lfloor 2} = A_2$$

$$\frac{f'''(a)}{\lfloor 3} = A_3 \cdots$$

Put

IMP $\checkmark$ $$\left[f(a+x) = f(a) + \frac{f'(a)}{\lfloor 1} \cdot x + \frac{f''(a)}{\lfloor 2} x^2 \cdots\cdots \right]$$

Questions

Q1: if $\tan(x+iy) = \cos\alpha + i\sin\alpha$ prove that

a) $x = \frac{n\pi}{2} + \frac{\pi}{4}$

b) $y = \frac{1}{2}\log\tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right)$

Q2 find the value of $\tanh(\log x)$ for $x = \sqrt{3}$.

Q3 Show $\log(-\log i) = \log \frac{\pi}{2} - \frac{i\pi}{2}$

Q4 Separate $Re(z)$ & $Im(z)$ if $z = \sqrt{i}^{\sqrt{i}}$

Q5 Show that $\tan\left[i \log\left(\frac{a-ib}{a+ib}\right)\right] = \frac{2ab}{a^2-b^2}$

Q6 If $y = e^{ax}\cos^2 x \sin x$ find y_n.

Q7 If $y = x^3 \cos x$ find y_n.

Q8 if $y = [\sin^{-1}x]^2$; P.T. $(1-x^2)y_{n+2} - (2n+1)x \cdot y_{n+1} - x^2 y_n = 0$

Q9 if $y = \sin(m\sin^{-1}x)$; P.T. $(1-x^2)y_2 - xy_1 + m^2 y = 0$ & deduce $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2-m^2)y_n = 0$

Q10 Expand : $e^x \cos x$

Q11 Expand :

(A)	in powers of	(B)
$\log \sin x$		$x-2$
$\tan^{-1} x$		$x-\pi/4$
$\log(x+h)$		x
$2x^3+7x^2+x-1$		$x-2$
$\sin x$		$x-\pi/2$