Machine Learning -
Theory & Practice

Module 3: Linear Regression

Lecture 1: *Introduction*

Lecture 1 Outline

- Topic 1: Basic concepts
- Topic 2: Model Estimation

Introduction

- **Linear Regression (LR) is a supervised machine learning algorithm for regression tasks.**
- **There are one or more independent variables called *Regressors, Explanatory Variables or Attributes*:**

  $X=\{x_1, x_2 \ldots\}$
- **The response variable *y,* is a continuous-valued random variable**
- **The distribution of *y* is dependent on the Regressors**

Hypothesis

- **The hypothesis, that is the assumed relationship between response *y and* one or more regressors *X =* {x_1,…,x_m}, m≥1, is described by the following:**

Consider $i = 1..n$ instances of data:

$$\boldsymbol{y_i = f(X_i) = w_0 + \sum_{j=1..m} w_j x_{i,j}^k + \epsilon_i}$$

- **Note that in Linear Regression, the relationship is linear in weights, {w_0,w_1,…,w_m}. w_0 is called bias, and the remaining weights are called *Regression Coefficients* - w_j quantifies the strength of the relationship between y and regressor x_j.**

Hypothesis Space & Search Strategy

- **The Hypothesis space is an infinite set of all possible representations of the hypothesis, given training data D={X,y}**

$$y_i = f(X_i) = w_0 + \sum_{j=1..m} w_j x_{i,j}^k + \epsilon_i$$

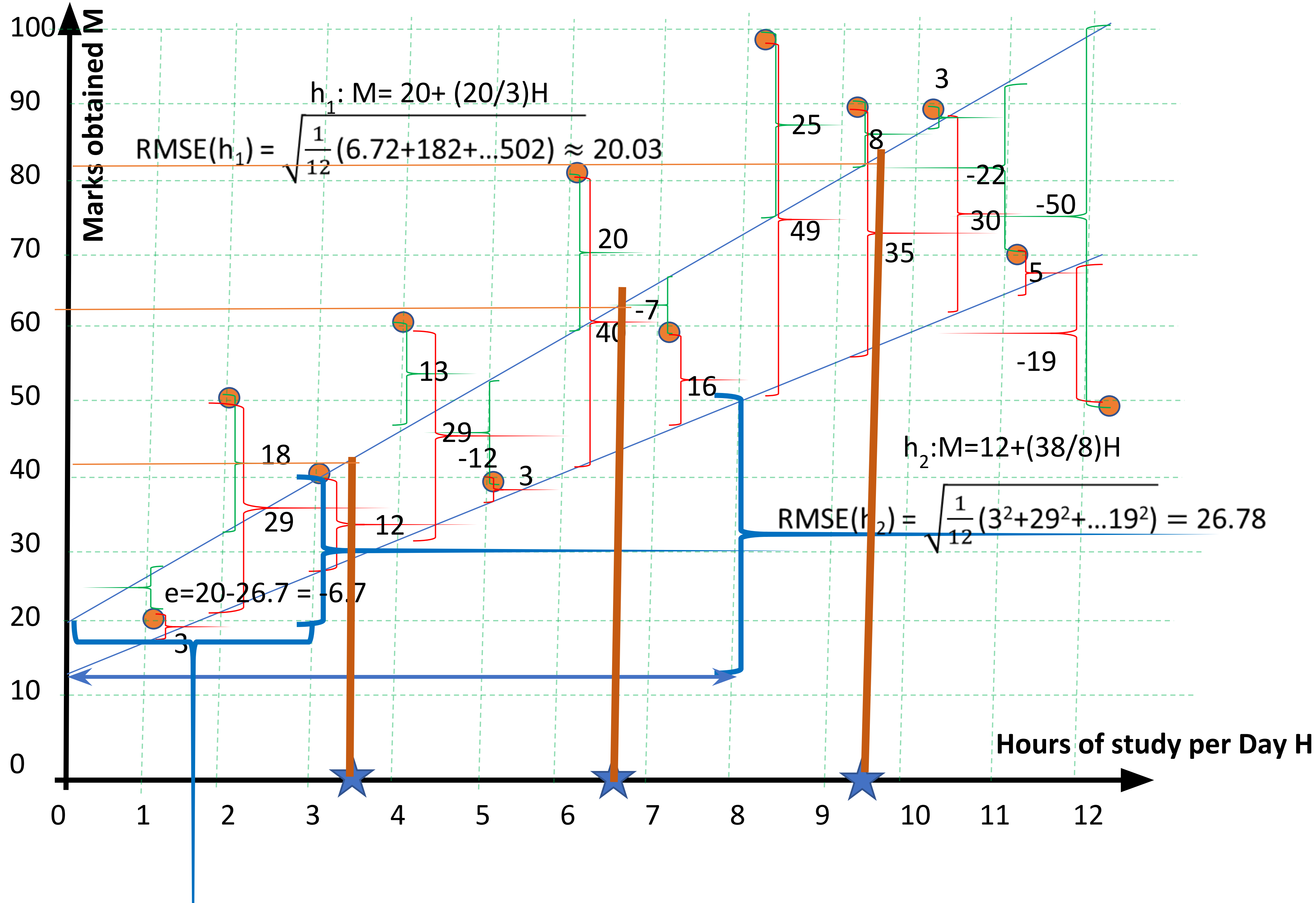
- **To search through the hypothesis space, *LR adjusts w_0 and the weight of each regressor,* to achieve minimum prediction error. This is an optimization process.**

- **The search process finds the best hypothesis *h(X) = y,* that matches the mapping *c*(*X*) = *y,* in the training data.**

Tuning the Weight Parameters

Aim is to minimize prediction error. Weights must be tuned by considering:

- **Which regressor(s) have a strong impact on the response, and are therefore significant?**
- **Which regressor(s) barely have any influence and can be eliminated?**
- **Which regressor(s) hold redundant information already captured by another attribute?**


Marks obtained M
Hours of study per Day H
h1: M= 20+ (20/3)H
RMSE(h1) = √(1/12 (6.72+182+...502)) ≈ 20.03
h2:M=12+(38/8)H
RMSE(h2) = √(1/12 (3²+29²+...19²)) = 26.78
e=20-26.7 = -6.7
100
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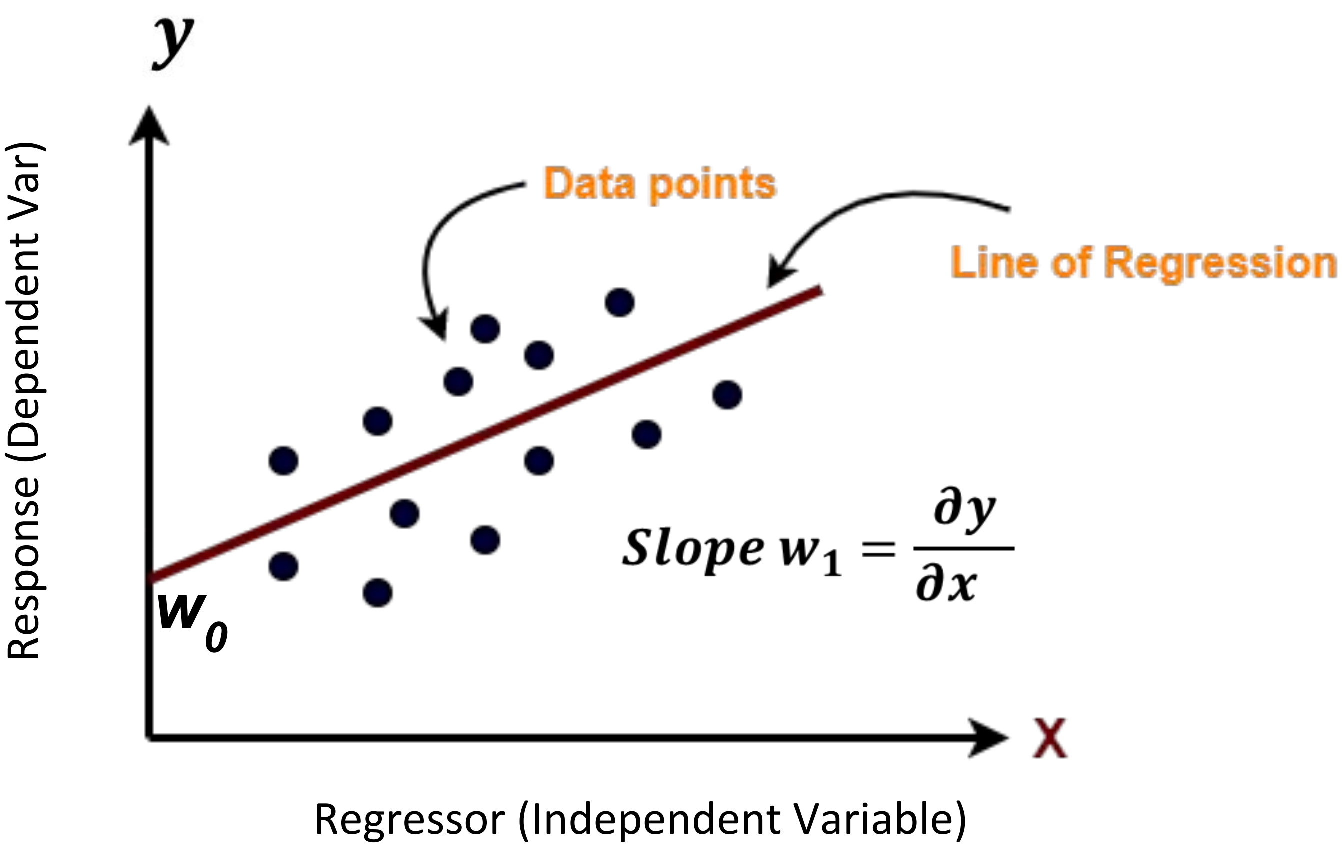
Types of Linear Regression

- ***Simple LR*: One regressor with degree 1**
- ***Multiple LR:* Multiple regressors all of degree 1**
- ***Polynomial simple/multiple Linear Regression:***
  Regressors can have degree > 1 (x^2, x^3 , x_1x_2…..)
- ***Multivariate simple/multiple Linear Regression:* More than one response variables that are coordinated**

Simple LR $y = w_0 + w_1 x_1 + \epsilon$



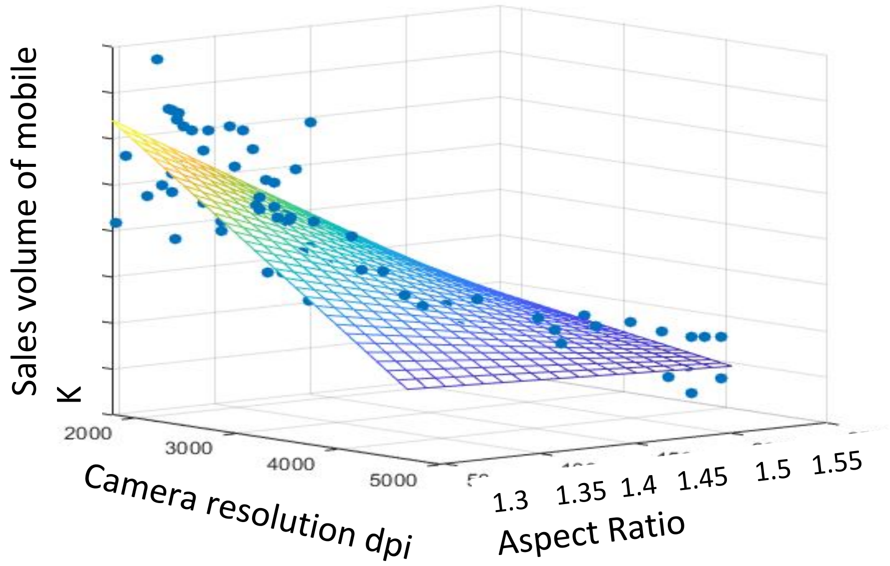
Simple LR example:

Example: Car rental y = base rental w_0 + rate w_1*miles covered x + ϵ

- w_0 may be initial conditions or aggregate effect of all factors
- w_1 is the strength by which X has an impact on y.

Multiple LR

- $y = w_0 + w_1 x_1 + w_2 x_2 \ldots + \cdots w_n x_n + \epsilon$
- Now, there is a **Plane of Regression**
- w_j is the slope or weight by which x_i has an impact on y



Polynomial LR

$$y = w_0 + w_1 x_1 + w_3 x_1^2 + w_4 x_1 x_2 + \varepsilon$$

Box-Office-Collections = w_0 + w_2*Num-of-viewers + w_3*Actor-popularity_Rating^2

- y Being Linear function of Regression Coefficients, one can solve the polynomial regression problem as a linear problem w.r.t {W}
- Each product term such as $x_1^2, x_1 x_2$ etc can be considered a separate regressor

Practical Applications

I. **Predicting response for new $\vec{X}$**
   - **Predict price of new property**
   - **Predict sales volume of new product**

II. **Explain variation in the response variable y due to variation in the explanatory variables $\vec{X}$ = $\{x_1, x_2 \ldots\ldots\}$**
   - **Explain Stock price variation**

Topic 2: *Model Estimation*

Ordinary Least Squares

- The learning model is found by tuning the regression coefficients: $W=\{w_1,w_2...\}$.
- The Objective is to *Minimize*{*Loss Function J(W)*}
- Error $e_i = y_i - \hat{y}_i$. Loss Function is ½ of MSE over all *n* training samples. Hence OF is:

$$\textit{Minimize}\{J(W)\}=\textit{Minimize}\left\{\left(\frac{1}{2}\times\frac{\sum_{i=1,n} e_i^2}{n}\right)\right\}$$

- This is Ordinary Least Squares (OLS) approach

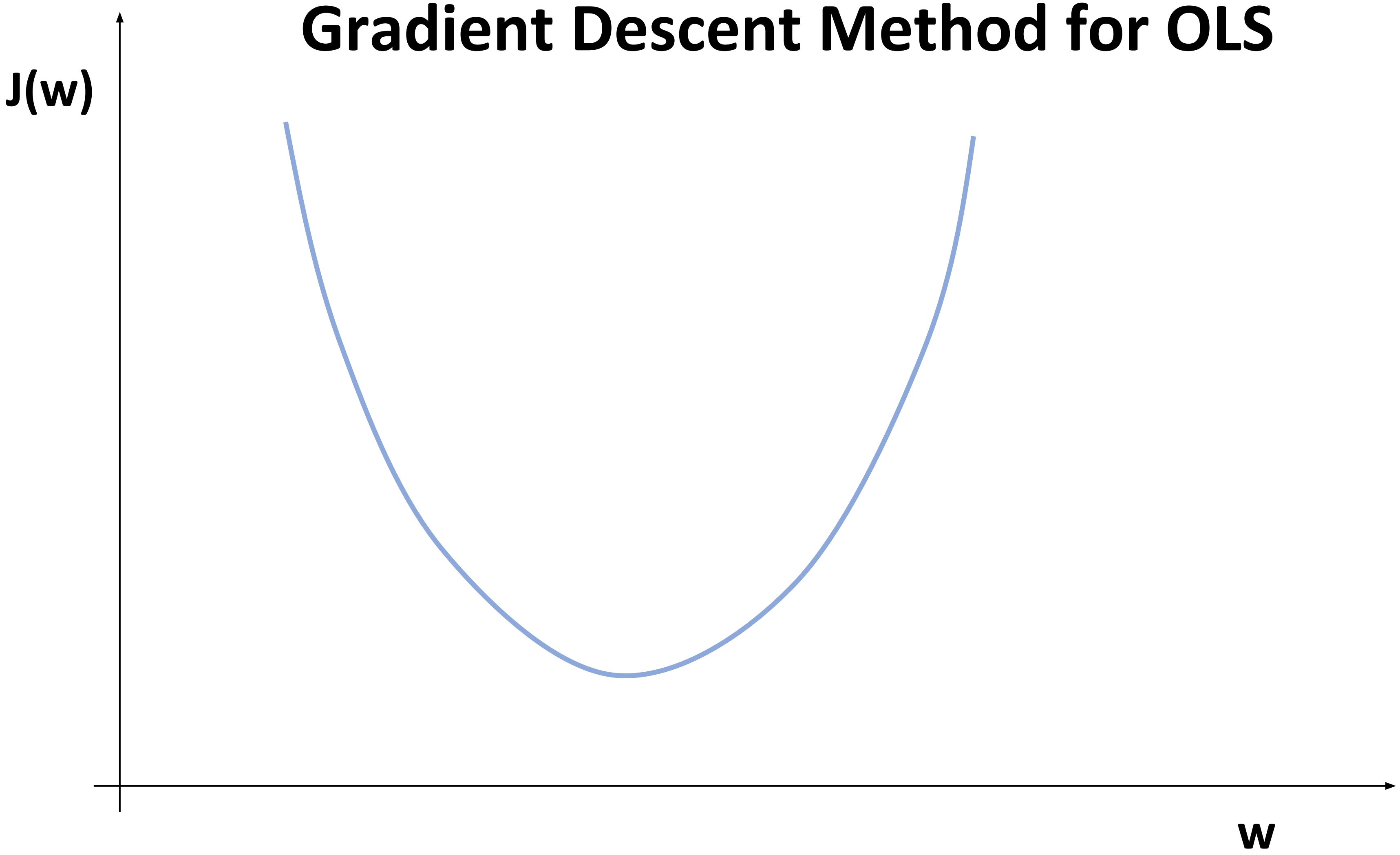
Parabolic relationship between Loss Function and weights

$$J(w) = \left(\frac{\sum_{i=1,n} e_i^2}{2 * n}\right)$$

$$= \left(\frac{\sum_{i=1,n}(y_i - \hat{y}_i)^2}{2 * n}\right)$$

$$= \left(\frac{\sum_{i=1,n}(y_i - (w_0 + w_1 x_1 + \cdots .))^2}{2 * n}\right)$$

Gradient Descent Method for OLS



Learning the Bias w_0

- **GD or Delta Learning Rule:** $$\Delta w_0 \propto -\frac{\partial J(w)}{\partial w_0}$$
- **Gradient of Loss function J*(W)* w.r.t. w_0 is :**

$$\frac{\partial J(w)}{\partial w_0} = \frac{\partial}{\partial w_0}\left(\frac{\sum_{i=1,n}\left(y_i - (w_0 + w_1 x_{1,i} + \cdots)\right)^2}{2 * n}\right)$$

Therefore,

$$\Delta w_0 = -\eta \frac{1}{n} \sum_i -\left(y_i - \left(w_0 + w_1 x_{1,i} + \cdots\right)\right)$$

$$\Delta w_0 = +\eta \frac{1}{n} \sum_i e_i$$

Delta Learning Rule for w_1

- **GD or Delta Learning Rule:** $\Delta w_1 \propto -\frac{\partial J(w)}{\partial w_1}$
- **Gradient of J*(W)* w.r.t. w_1 is :**

$$\frac{\partial J(w)}{\partial w_1} = \frac{\partial}{\partial w_1}\left(\frac{\sum_{i=1,n}\left(y_i - (w_0 + w_1 x_{1,i} + \cdots)\right)^2}{2 * n}\right)$$

Therefore,

$$\Delta w_1 = -\eta \frac{1}{n}\sum_i -(y_i - \hat{y}_i)\, x_{1,i}$$

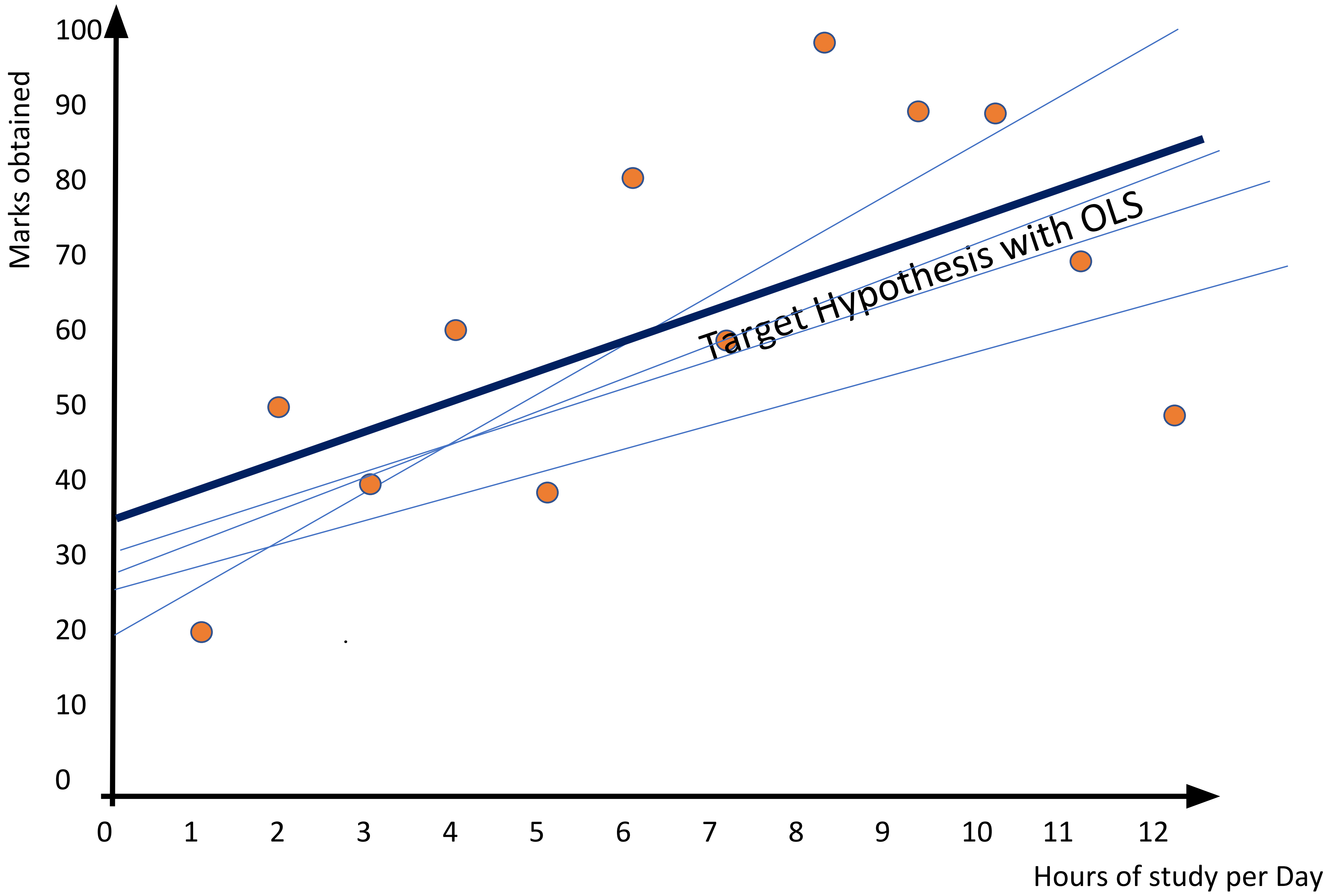
$$\Delta w_1 = +\eta \frac{1}{n}\sum_i e_i\, x_{1,i}$$

Weight Readjustments

$$w_x(t+1) \leftarrow w_x(t)$$

$$w_0(t+1) = w_0(t) + \eta \frac{1}{n} \sum_i (y_i - \hat{y}_i)$$

$$w_i(t+1) = w_i(t) + \eta \frac{1}{n} \sum_i (y_i - \hat{y}_i)\, x_i$$



Marks obtained
100
90
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20
10
0
0
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6
7
8
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10
11
12
Hours of study per Day
Target Hypothesis with OLS

Types of Gradient Descent

- **GD – use all n training samples to readjust weights:**

$$\Delta w_1 = +\eta \frac{1}{n} \sum_{i=1,n} e_i\, x_i \qquad \Delta w_0 = +\eta \frac{1}{n} \sum_{i=1,n} e_i$$

$$\mathrm{J}(\mathbf{w}) = \frac{\sum_{i=1,n} (y_i - \hat{y}_i)^2}{2 * n}$$

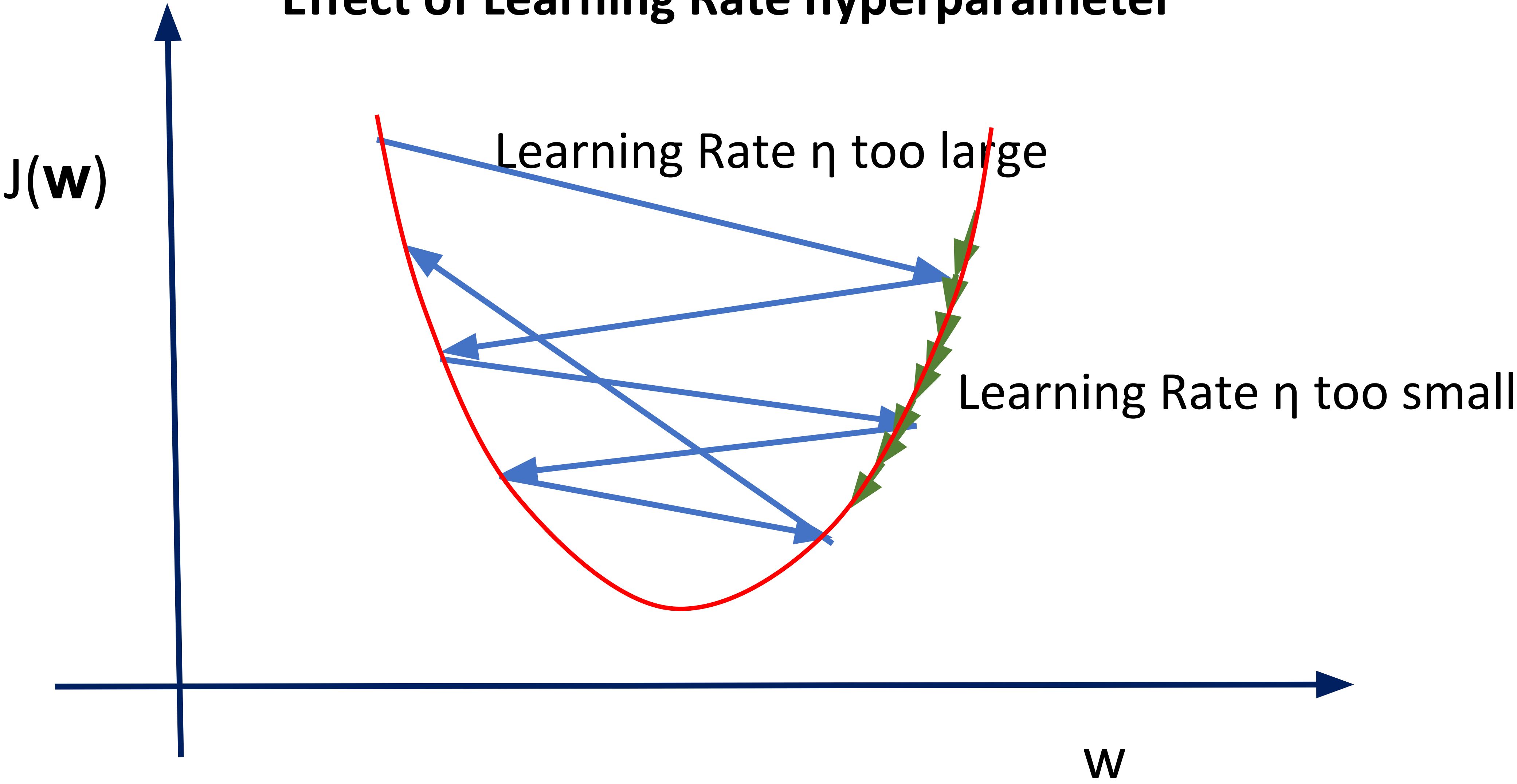
- **Stochastic GD: use any one sample chosen at random time t_i**

$$\Delta w_1 = +\eta e_i x_i \qquad \Delta w_0 = +\eta e_i$$

- **Mini batch GD: use m<<n samples:**

$$\Delta w_1 = +\eta \frac{1}{m} \sum_{i=1,m} e_i\, x_i \qquad \Delta w_0 = +\eta \frac{1}{m} \sum_{i=1,m} e_i$$

Effect of Learning Rate hyperparameter



Recap

- Linear Regression is a supervised regression method, with continuous response and one or more explanatory variables/ regressors/ attributes
- The hypothesis is given by:

$$\boldsymbol{y_i = f(X_i) = w_0 + \sum_{j=1..m} w_j x_{i,j}^k + \epsilon_i}$$

- Hypothesis space search is carried out by adjusting the regression coefficients w_0, w_1……
- LR may be: simple LR with one regressor, Multiple LR with >1 regressors, Polynomial LR with regressors of higher degree or Multivariate LR with multiple coordinated response variables

Recap

- Ordinary Least Squares (OLS) technique, tunes the weight parameters to find the best mapping between attributes and response, with least MSE loss function.
- Gradient Descent is a method for OLS, based on the idea that weights are adjusted in direct proportion to the gradient of the loss function, and in reverse direction
- Stochastic GD method considers one random data point at a time to calculate and perform weight adjustment, Batch GD takes a few of them, and GD uses all of them.