- The motion of physical bodies is broadly classified into two categories, namely translational motion and vibrational or oscillatory motion. If the position of a body varies linearly with time, then such motions are called translational motions. The examples of translational motion are a ball that rolls on the ground and a train that moves on a straight track. A motion of a body that repeats itself after regular intervals of times and when the body moves back and forth over the same path is called vibrational or oscillatory motion. The example of vibrational motion are the oscillations of the arms of a walking person, the bob of the pendulum clock, beating of heart, etc.
- If the acceleration of a particle in a periodic motion is always directly proportional to its displacement from its equilibrium position and is always directed towards equilibrium position, then the motion of the particle is said to be Simple Harmonic Motion (SHM).
- For f as the linear acceleration of the particle and x as its displacement from the equilibrium position, the essential condition for linear SHM is $f \propto -x$. However, if α be the angular acceleration and θ be the angular displacement from the equilibrium position, then the condition for angular SHM is $\alpha \propto -\theta$.
- For the displacement x and the angular frequency $\omega\,(= \sqrt{k/m}$, where k is the force constant and m is the mass of the particle), $\dfrac{d^2x}{dt^2} + \omega^2 x = 0$ represents the differential equation of the SHM
- The solution $x = A \sin(\omega t + \delta)$ gives the displacement of the particle executing SHM at any instant of time t. Here A represents the maximum displacement of the particle, which is called the amplitude of oscillations. The velocity of the particle is given by $v = \omega\sqrt{(A^2 - x^2)}$ and the acceleration is $f = -\omega^2 x$. The energy of a harmonic oscillator is given by $E = \dfrac{1}{2} m\omega^2 A^2$.

7.7 DAMPED HARMONIC OSCILLATOR

LO3

In the previous section free oscillations were discussed in some oscillating systems. In such systems, no frictional force or resistance is offered. Therefore, the body will keep on vibrating indefinitely and such vibrations are called *free vibrations*. But in real situation, there is always some resistance offered to the oscillating system. In real sense if a body set into vibrations will have its amplitude continuously decreasing due to fractional resistance and so the vibrations will die after some time. The motion is said to be damped by the friction and is called as *damped vibrations*. For example, when a pendulum is displaced from its equilibrium position, it oscillates with a decreasing amplitude and finally comes to the rest.

FIGURE 7.5

Suppose a system has a body of mass m attached to a spring whose force constant is k (Fig. 7.5). Again consider x' to be the displacement of the body from the equilibrium state at any instant of time. Then $\frac{dx'}{dt}$ will be the instantaneous velocity.

Here two types of forces are acting on the body: a restoring force ($F_r = -kx'$) which is proportional to the displacement x' and acts in the opposite direction to the displacement and a damping force $-q'\frac{dx'}{dt}$ which is proportional to the velocity and is oppositely directed to the motion. The total force acting on the body is thus given by

$$F = -kx' - q'\frac{dx'}{dt}$$

or $$m\frac{d^2x'}{dt^2} = -kx' - q'\frac{dx'}{dt}$$

or $$\frac{d^2x'}{dt^2} + \frac{q'}{m}\frac{dx'}{dt} + \frac{k}{m}x' = 0 \quad \text{(i)}$$

If we put $\frac{q'}{m} = 2s$ and $\frac{k}{m} = \omega^2$, then above equation takes the form

$$\frac{d^2x'}{dt^2} + 2s\frac{dx'}{dt} + \omega^2 x' = 0 \quad \text{(ii)}$$

which is the equation of second degree. We assume its solution as

$$x' = Ae^{\alpha t} \quad \text{(iii)}$$

where A and α are arbitrary constants.

Differentiating Eq. (iii) w.r.t. t, we have

$$\frac{dx'}{dt} = A\alpha e^{\alpha t} \text{ and } \frac{d^2x'}{dt^2} = A\alpha^2 e^{\alpha t}$$

By putting these values in Eq. (ii), we have

$$A\alpha^2 e^{\alpha t} + 2sA\alpha e^{\alpha t} + \omega^2 A e^{\alpha t} = 0$$

or $$Ae^{\alpha t}[\alpha^2 + 2s\alpha + \omega^2] = 0$$

$\because$ $$Ae^{\alpha t} \neq 0$$

$\therefore \qquad \alpha^2 + 2s\alpha + \omega^2 = 0$

This gives,

$$\alpha = -s \pm \sqrt{s^2 - \omega^2}$$

Therefore, α has two roots

$$\alpha_1 = -s + \sqrt{s^2 - \omega^2}$$

$$\alpha_2 = -s - \sqrt{s^2 - \omega^2}$$

Therefore, the general solution of Eq. (ii) is given by

$$x' = A_1 e^{(-s+\sqrt{s^2-\omega^2})t} + A_2 e^{(-s-\sqrt{s^2-\omega^2})t} \qquad \text{(iv)}$$

where A_1 and A_2 are arbitrary constants. The actual solution depends upon whether $s^2 > \omega^2$, $s^2 = \omega^2$ and $s^2 < \omega^2$. Now we will discuss all these cases.

Case-A: $s^2 > \omega^2$

The term $\sqrt{s^2 - \omega^2}$ is real and less than s. Therefore both exponents i.e., $[-s + \sqrt{s^2 - \omega^2}]$ and $[-s - \sqrt{s^2 - \omega^2}]$ in Eq. (iv) are negative. Due to this reason the displacement x' continuously decreases exponentially to zero without performing any oscillation [Fig. 7.6(a)]. This motion is known as *overdamped*.

Case-B: $s^2 = \omega^2$

For this case, Eq. (iv) does not satisfy differential Eq. (ii).Suppose that $\sqrt{s^2 - \omega^2}$ is not exactly zero but it is equal to very small quantity β. Now, Eq. (iv) gives

$$x' = A_1 \exp(-s + \beta)t + A_2 \exp(-s - \beta)t$$

$$x' = e^{-st}[A_1 e^{\beta t} + A_2 e^{-\beta t}]$$

$$= e^{-st}[A_1(1 + \beta t + \ldots) + A_2(1 - \beta t + \ldots)]$$

Here we have neglected the terms containing β^2, β^3, β^4 is very small

$\therefore \qquad x' = e^{-st}[(A_1 + A_2) + \beta t(A_1 - A_2) + \ldots]$

$$x' = e^{-st}[P' + Q't]$$

where $P' = (A_1 + A_2)$ and $Q' = \beta(A_1 - A_2)$ (v)

Equation (v) represents a possible solution of this form. From this equation, it is clear that as t increases the term $(P' + Q't)$ increases but the term e^{-st} gets decreased. Because of this fact the displacement x' first increases due to the term $(P' + Q't)$ but it decreases because of the exponential term e^{-st} and finally it approaches to zero as t increases. When we compare cases A and B it is noticed that the exponent is $-st$ in case B while it as more than $-st$ in case of A. Therefore, in case A the particle acquires its position of equilibrium rapidly than in case B[Fig. 7.6(b)]. This type of motion is called *critical damped motion.* Examples are voltmeter, ammeter, etc. In these instruments, pointer moves to the correct position and comes to the rest without any oscillations.

***Case-C:* $s^2 < \omega^2$**

The term $\sqrt{s^2 - \omega^2}$ is imaginary, which can be written as

$$\sqrt{s^2 - \omega^2} = i\sqrt{\omega^2 - s^2} = i\beta'$$

where $\beta' = \sqrt{\omega^2 - s^2}$ and $i = \sqrt{-1}$

Now, Eq. (iv) becomes

$$x' = A_1 e^{(-s + i\beta')t} + A_2 e^{(-s - i\beta')t}$$

$$x' = e^{-st}[A_1 e^{i\beta' t} + A_2 e^{-i\beta' t}]$$

$$= e^{-st}\,[A_1 \cos \beta' t + i \sin \beta' t) + A_2(\cos \beta' t - i \sin \beta' t)$$

$$= e^{-st}[(A_1 + A_2) \cos \beta' t + i[(A_1 - A_2) \sin \beta' t]$$

$$= e^{-st}[(A \sin \delta \cos \beta' t + A \cos \delta \sin \beta' t]$$

where $A \sin \delta = A_1 + A_2$ and $A \cos \delta = i(A_1 - A_2)$

$$\therefore \quad x' = e^{-st} A \sin (\beta' t + \delta)$$

Putting the value of β' in the above questions, we get

$$x' = Ae^{-st} \sin [\sqrt{(\omega^2 - s^2)}t + \delta] \tag{vi}$$

The above equation shows the oscillatory motion and represents the damped harmonic oscillator. The oscillations are not simple harmonic because the amplitude (Ae^{-st}) is not constant but decreases with time (t). However, the decay of amplitude depends upon the damping factor s. The motion is known as *under damped motion* [Fig. 7.6(c)]. The motion of a pendulum in air, the motion of the coil of ballistic galvanometer, etc. are the example of this case.

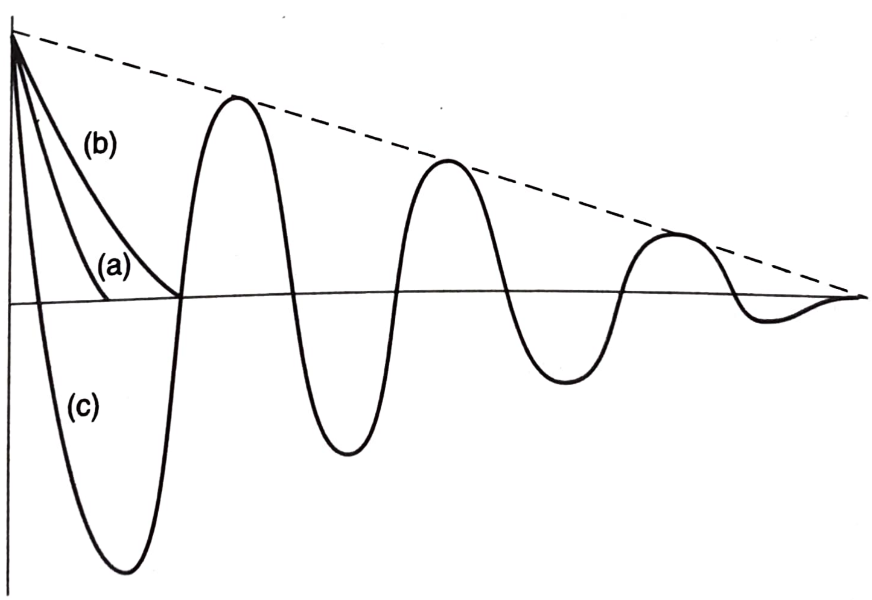

FIGURE 7.6