i.e., e^{-2st}. So the decay rate of energy depends upon s. The following three characteristics namely logarithmic decrement, relaxation time and quality factor may give the attenuation of a vibrating system.

7.8.1 Logarithmic Decrements

The rate at which the amplitude dies away is measured by logarithmic decrement. The amplitude of damped harmonic oscillator is given by a factor Ae^{-st}. Therefore, at time $t = 0$ the amplitude will be maximum (i.e., $A = A_0$). If $A_1, A_2, A_3, \ldots$ be the amplitude at time $t = T, 2T, 3T, \ldots$ respectively where T is the time period of oscillations, then.

$$A_1 = Ae^{-sT}, A_2 = Ae^{-s(2T)}, A_3 = Ae^{-s(3T)} \text{ and so on.}$$

This yields

$$\frac{A_0}{A_1} = \frac{A_1}{A_2} = \frac{A_2}{A_3} = \ldots = e^{sT} = e^{\lambda} \qquad \text{(where } sT = \lambda\text{)}$$

Here, λ is called *logarithmic decrement.*

Now, by taking the natural logarithmic, we have

$$\ln\frac{A_0}{A_1} = \ln\frac{A_1}{A_2} = \ln\frac{A_2}{A_3} = \lambda$$

Hence, logarithmic decrement is the natural logarithm of ratio between two successive maximum amplitudes, which are separated by one period.

7.8.2 Relaxation Time

It is the time taken by damped harmonic oscillator for decaying total mechanical energy by the factor $\frac{1}{e}$ of its initial value.

The mechanical energy of the oscillator is

$$E = \frac{1}{2}mA^2\omega^2e^{-2st} \qquad \text{(i)}$$

At $\quad t = 0, E = E_0 = \frac{1}{2}mA^2\omega_0^2$

$\therefore \quad$ Total energy, $E = E_0e^{-2st}$ $\qquad$ (ii)

Suppose τ be the relaxation time, then at time $t = \tau$,

$$E = \frac{E_0}{e} \qquad \text{(By definition)}$$

By using Eq. (ii), we get

$$\frac{E_0}{e} = E_0e^{-2st}$$

or $\quad e^l = e^{2s\tau}$

or $\quad 1 = 2s\tau$

or $$\tau = \frac{1}{2s} \qquad \text{(iii)}$$

Therefore, the dissipated energy in terms of relaxation time is written as

$$E = E_0 e^{-t/\tau} \tag{iv}$$

7.8.3 Quality Factor

It is defined as 2π times the ratio of energy stored in the system to the energy lost per cycle. This factor of a damped oscillator shows the quality of oscillator so far as damping is concerned.

$$Q = 2\pi \frac{E}{P_d T} \tag{v}$$

where P_d is the power dissipation and T is the periodic time. Then,

$$Q = 2\pi \frac{E}{(E/\tau)T} = \frac{2\pi\tau}{T} \qquad \left[\because P_d = \frac{E}{\tau}\right]$$

$$Q = \omega\tau \qquad \left[\because \omega = \frac{2\pi}{T}\right] \tag{vi}$$

From the above equation, it is clear that the value of relaxation time τ will be higher (or damping will be lower) for higher value of Q.

For the force constant k and the mass m of the vibrating system

$$\omega = \sqrt{\frac{k}{m}} \text{ and } \tau = \frac{1}{2s} \quad \text{[from Eq. (iii)]}$$

$$\therefore \qquad Q = \frac{1}{2s}\sqrt{\frac{k}{m}}$$

Since lower values of s lead to lower damping, it is clear that for low damping, the quality factor would be higher.

7.9 FORCED VIBRATIONS

LO5

We have discussed earlier the vibrations in which the body vibrates with its own frequency without being placed to any external force. The different situation arises if the body is placed to an external force while it is vibrating. These vibrations are known as *forced vibrations*. For example, if a bob of simple pendulum is held in hand and then given number of swings by the hand. In this case, the pendulum vibrates due to external force and not due to its natural frequency. So forced vibrations can also be defined as the vibrations in which the body vibrates with frequency other than its natural frequency, which is due to some external periodic force. The tuning fork is also one of the examples of forced vibrations.

Theory of Forced Vibrations

Suppose a particle of mass m is connected to a spring. When it is displaced from its mean position, the oscillations are started and the particle experiences different kinds of forces viz, a restoring force $(-kx)$, a damping force $\left(-q'\frac{dx}{dt}\right)$ and the external periodic force $(F_0 \sin \omega t)$. The total force acting on the particle is, therefore

$$F = F_0 \sin \omega t - q' \frac{dx}{dt} - kx \tag{i}$$

By Newton's second law of motion

$$F = m \frac{d^2x}{dt^2}$$

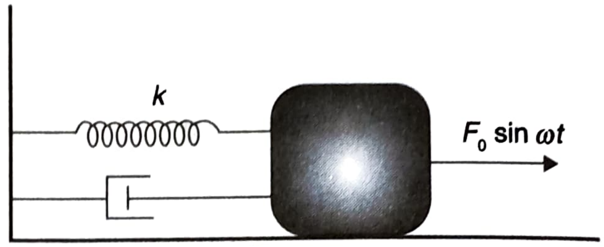

FIGURE 7.7

Hence,

$$m \frac{d^2x}{dt^2} = F_0 \sin \omega t - q' \frac{dx}{dt} - kx$$

or

$$\frac{d^2x}{dt^2} + \frac{q'}{m}\frac{dx}{dt} + \frac{k}{m}x = \frac{F_0 \sin \omega t}{m} \tag{ii}$$

Eq (ii) is the differential equation of motion of the particle.

Substitute $\frac{q'}{m} = 2s$, $\frac{k}{m} = \omega_0^2$ and $\frac{F_0}{m} = f$, then Eq, (ii) becomes

$$\frac{d^2x}{dt^2} + 2s\frac{dx}{dt} + \omega_0^2 x = f \sin \omega t \tag{iii}$$

In the steady state, the solution of the above equation should be

$$x = A \sin (\omega t - \delta) \tag{iv}$$

where A is the amplitude of vibrations in the steady state. By differentiating Eq. (iv) twice w.r.t. t, we have

$$\frac{dx}{dt} = \omega A \cos (\omega t - \delta)$$

and

$$\frac{d^2x}{dt^2} = -\omega^2 A \sin (\omega t - \delta)$$

By substituting the values of x, $\frac{dx}{dt}$ and $\frac{d^2x}{dt^2}$ in Eq. (iii), we have

$$-\omega^2 A \sin (\omega t - \delta) + 2s\,\omega A \cos (\omega t - \delta) + \omega_0^2 A \sin (\omega t - \delta)$$
$$= f \sin \{(\omega t - \delta) + \delta\}$$

or

$$A(\omega_0^2 - \omega^2) \sin (\omega t - \delta) + 2s\omega A \cos (\omega t - \delta)$$
$$= f \sin (\omega t - \delta) \cos \delta + f \cos (\omega t - \delta) \sin \delta \tag{v}$$

If Eq. (v) holds good for all values of t, then the coefficients of $\sin (\omega t - \delta)$ and $\cos (\omega t - \delta)$ must be equal on both the sides, then

$$A(\omega_0^2 - \omega^2) = f \cos \delta \tag{vi}$$

and

$$2s\,\omega A = f \sin \delta \tag{vii}$$

By squaring and adding Eqs. (vi) and (vii), we have

$$A^2(\omega_0^2 - \omega^2)^2 + 4s^2\,\omega^2 A^2 = f^2$$

or $$A = \frac{f}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4s^2\omega^2}} \tag{viii}$$

Dividing Eq. (vii) by Eq. (vi), then we get

$$\tan\delta = \frac{2s\omega}{\omega_0^2 - \omega^2}$$

or phase, $$\delta = \tan^{-1}\left[\frac{2s\omega}{\omega_0^2 - \omega^2}\right] \tag{ix}$$

From Eq. (viii) and (ix), it is clear that the amplitude and phase of the forced oscillations depend upon $(\omega_0^2 - \omega^2)$, i.e., these depend upon the driving frequency (ω) and the natural frequency (ω_0) of the oscillator. The amplitude and the phase are explained as below.

Case - A: Very low driving frequency, i.e., $\omega << \omega_0$. In this case, the amplitude of the vibrations

$$A = \frac{f}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4s^2\omega^2}} \approx \frac{f}{\omega_0^2}$$

$$= \frac{F_0}{m\omega_0^2} \qquad \left[\because f = \frac{F_0}{m}\right]$$

or $$A = \frac{F_0}{k} \qquad \left[\because \omega_0^2 = \frac{k}{m}\right]$$

Hence, the amplitude depends on the force constant of the spring and the magnitude of the applied force.

Phase, $\delta = \tan^{-1}\left[\frac{2s\omega}{\omega_0^2 - \omega^2}\right] = \tan^{-1}\left[\frac{2s\omega}{\omega_0^2}\right]$

Since $\omega_0^2 >> \omega$, $2s\omega/\omega_0^2 \to 0$ and $\delta \to 0$ or ≈ 0. Therefore, under this situation the driving force and the displacement are in phase.

Case - B: Same driving and natural frequencies, i.e., $\omega = \omega_0$. This frequency is called resonant frequency. Under this situation, the amplitude of vibrations

$$A = \frac{f}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4s^2\omega^2}}$$

$$= \frac{f}{2s\omega_0} = \frac{F_0/m}{(q'/m)\omega_0} = \frac{F_0}{q'\omega_0}$$

Hence, the amplitude of vibrations depends upon the damping and applied force. Now

Phase, $\delta = \tan^{-1}\left[\frac{2s\omega}{(\omega_0^2 - \omega^2)}\right] = \tan^{-1}\left[\frac{2s\omega}{0}\right]$

$$= \tan^{-1}[\infty] = \frac{\pi}{2}$$

Thus, the displacement lags behind the force by a phase of $\frac{\pi}{2}$, as $x = A \sin(\omega t - \delta)$ and the applied force if $F_0 \sin \omega t$.

Case - C: Very large driving frequency, i.e., $\omega >> \omega_0$. Here, the amplitude of vibrations

$$A = \frac{f}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4s^2\omega^2}}$$

Since ω is very large $\omega^4 >> 4\omega^2 s^2$

$$A = \frac{f}{\omega^2} = \frac{F_0}{m\omega^2}$$

Phase,
$$\delta = \tan^{-1}\left[\frac{2s\omega}{(\omega_0^2 - \omega^2)}\right] = \tan^{-1}\left[\frac{2s\omega}{-\omega^2}\right]$$

$$\approx \tan^{-1}\left[\frac{2s}{-\omega}\right] = \tan^{-1}[-0] = \pi \qquad \text{[Since } \omega \text{ is very large, } 1/\omega = 0]$$

Therefore, under the situation $\omega >> \omega_0$, the displacement lags behind the force by a phase of π.

7.10 RESONANCE

LO6

So far we have discussed various types of oscillations. It is clear that if a vibrating system consists of a number of oscillators coupled together, the resultant motion would be complicated. However, if we choose the starting conditions correctly, it is possible to cause the system vibrate in such a way that every part has the same frequency. Such simple vibrations are known as normal oscillations or *normal modes* and the associated frequencies are called the normal frequencies or the *natural frequencies*. So this is evident that all mechanical structures, for example, buildings, airplanes, bridges, etc. have one or more natural frequencies. Now if such a structure is subject to a driving frequency, which is equal to one of the natural frequencies, the resulting oscillations will have large amplitude that can have disastrous consequences. Shattering a wine glass with a sound wave that matches one of the natural frequencies of the glass is one demonstration of this phenomenon of resonance. Another outcome of this effect is the collapse of roadways and bridges in earthquakes.

In order to make clear the phenomenon of resonance, we take an example of forced oscillations that occur at the frequency of the external force (driving frequency ω) and not at the natural frequency (ω_0) of the vibrating system. As has been seen, however, the amplitude of oscillations depends on the relationship between the driving frequency of the applied force and the natural frequency. A succession of small impulses, if applied at the proper frequency, can develop an oscillation of large amplitude. You can take an example of pushing a friend on a swing. You would have noticed that by applying pushes precisely at the same time in each cycle, you can cause your friend to move in an increasingly large arc.

In Fig.7.8, we show the variation of amplitude of the forced vibrations with the driving frequency ω for three cases of large damping, medium damping and small damping. Medium damping corresponds to twice the damping force and large damping corresponds to four times the damping force. It is clear from the figure that the amplitude decreases if the frequency ω is far from the condition $\omega = \omega_0$, called condition of

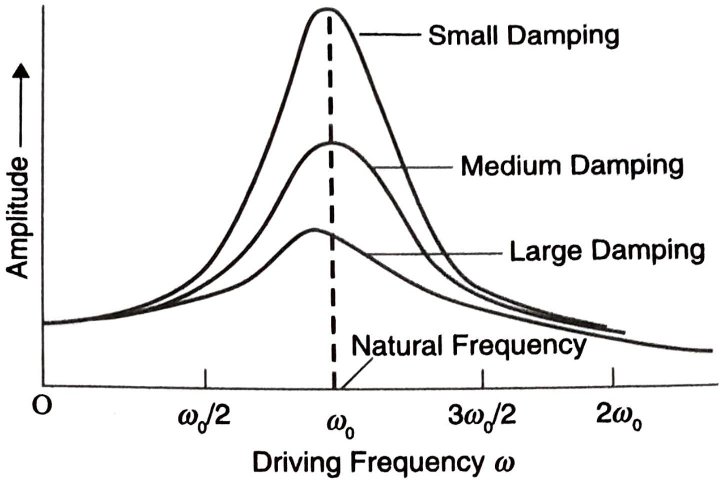

FIGURE 7.8

resonance. Moreover, when the damping is small, the amplitude of the forced oscillations increases rapidly as ω approaches ω_0. The amplitude reaches its maximum when $\omega = \omega_0$. For medium damping also, the amplitude gets increased but it does not increase so rapidly near the resonance ($\omega = \omega_0$). However, for the largest damping the resonant frequency is displaced slightly from the natural frequency.