5. Boolean Algebra :

- Commutative laws : $A + B = B + A$
  $A \cdot B = B \cdot A$

- Associative laws : $A + (B + C) = (A + B) + C = (A + C) + B$
  $A \cdot (B \cdot C) = (A \cdot B) \cdot C = (A \cdot C) \cdot B$

- Distributive law : $A(B + C) = AB + AC$
- OR operations : $A + 0 = A$, $0 + 0 = 0$, $0 + 1 = 1$, $1 + 0 = 1$, $1 + 1 = 1$
- $A + A = A$
  $1 + A = 1$
  $A + \overline{A} = 1$
- AND operations : $A \cdot 1 = A$, $A \cdot A = A$, $A \cdot 0 = 0$, $A\overline{A} = 0$
- Double inversion : $\overline{\overline{A}} = A$
- DeMorgans Theorems :
  $\overline{A + B} = \overline{A} \cdot \overline{B}$
  $\overline{A \cdot B} = \overline{A} + \overline{B}$

- Duality Theorem (without proof) : For a given Boolean relation , another Boolean expression can be derived by -
  (i) Changing OR sign with AND
  (ii) Changing AND sign with OR
  (iii) Complementing any zero and one in the expression.

Example 1 :

$$\underbrace{A \cdot (B + C)} = A.B + A.C$$

⇓

apply duality theorem

$$A + B \cdot C = (A + B) \cdot (A + C)$$

Example 2 : Let $F(A, B) = A \cdot \overline{B} + \overline{A} \cdot B$

$$\therefore \quad F_D(A, B) = (A + \overline{B}) \cdot (\overline{A} + B)$$

$$= \underbrace{A\overline{A}}_{0} + AB + \overline{B} \cdot \overline{A} + \underbrace{B\overline{B}}_{0}$$

$$= A \cdot B + \overline{A}\,\overline{B}$$

- Self-dual : a Boolean function is called self-dual , if its dual is the same function itself .

- Ex-OR and Ex- NOR are duals of each other

A	B	Y
0	0	0
0	1	1
1	0	1
1	1	0

$\therefore$ $Y = \bar{A}B + A\bar{B}$

Let $f(A,B) = A \oplus B = A.\bar{B} + \bar{A}.B$

$$f_D(A,B) = (A+\bar{B}) \cdot (\bar{A}+B) = \underbrace{A\bar{A}}_{0} + AB + \bar{B}\bar{A} + \underbrace{B\bar{B}}_{0} = AB + \bar{A}\bar{B}$$

Now dual of above would be :

$$f_D'(A,B) = (A+B) \cdot (\bar{A}+\bar{B}) = \underbrace{A\bar{A}}_{0} + A\bar{B} + B\bar{A} + \underbrace{B\bar{B}}_{0} = A\bar{B} + \bar{A}B$$

(Hence, proved)

- **Covering and combination:**

Covering means where one term covers the condition of other term so that the other term becomes redundant :

$$A + AB = A$$

$$A(A+B) = AA + AB = A + AB = A$$

Combining Rules : $AB + A\bar{B} = A$; $(A+B)(A+\bar{B}) = \underline{AA} + \underbrace{A\bar{B} + AB} + \underline{B\bar{B}} = A$

- **Consensus Theorem** : This theorem finds a redundant term which is consensus of two other terms. If consensus term is true then any of the remaining term is true and thus, it becomes redundant

Example 1 : $Y = AB + \bar{A}C + \underbrace{BC}$

If $BC = 1$, then $B=1$, $C=1$ $\therefore Y = A + \bar{A} = 1$

Example 2 : $Y = (A+B).(\bar{A}+C).\underbrace{(B+C)}$

$\downarrow$

If $B+C = 0$, the $B=0$, $C=0$

then $Y = A.\bar{A} = 0$

Shanon's expansion Theorem :

$$F(x_1, x_2 \dots x_N) = \bar{x}_1 \; F(0, x_2, \dots x_N) + x_1 \; F(1, x_2, \dots x_N)$$

and its dual

$$F(x_1, x_2, \dots x_N) = [\bar{x}_1 + F(1, x_2 \dots x_N)] . [x_1 + F(0, x_2, \dots x_N)]$$

Example : $F(A,B) = A\bar{B} + \bar{A}B$

Using Shanon's theorem : $F(A,B) = \bar{A} \cdot F(0,B) + A \, F(1,B)$

$F(0,B) = 0.\bar{B} + 1.B = B$

$F(1,B) = 1.\bar{B} + 0.B = B'$

$\therefore F(A,B) = \bar{A} \cdot B + A\bar{B}$

Example 1 : Prove that $x + \bar{x} y = x + y$

Let's find $Y = \overline{\overline{(x + \bar{x} y)}}$

$$= \overline{[\bar{x} \cdot \overline{(\bar{x} \cdot y)}\,]}$$

using De Morgan's Theorem

$\overline{A + B} = \bar{A} \cdot \bar{B}$

$$= \overline{\bar{x} \cdot (x + \bar{y})}$$

Using De Morgan's Theorem

$\underbrace{\overline{A \cdot B} = \bar{A} + \bar{B}}$

$$= \overline{\underbrace{\bar{x} x}_{0} + \bar{x} \cdot \bar{Y}}$$

$$= \overline{\bar{x} \cdot \bar{Y}}$$

Use De Morgan's Theorem again

$$= X + Y$$

Hence, proved.

Example 2. Prove that -

$$F = A(\bar{A} + C)(\bar{A}B + C)(\bar{A}BC + \bar{C}) = 0$$

$$F = (\underbrace{A\bar{A}} + AC)(\bar{A}B + C)(\bar{A}BC + \bar{C})$$

$$= (\underbrace{AC\bar{A}B} + \underbrace{ACC})(\bar{A}BC + \bar{C})$$

$$= AC(\bar{A}BC + \bar{C}) = \underbrace{AC\bar{A}BC} + \underbrace{AC\bar{C}} = 0$$

Note : $X + XY$

$= X(1 + Y)$

$= X$

Example 3. Simplify $Y = (A+B)\,\overline{[\bar{A} \cdot (\bar{B} + \bar{C}]} + \bar{A}(B + C)$

$$= (A+B)\underbrace{\{A + BC\}}_{\text{using DeMorgan's theorem}} + \bar{A}(B+C)$$

$$= \underbrace{AA + ABC} + AB + BBC + \bar{A}B + \bar{A}C$$

$$= \underbrace{A + ABC} + AB + BC + \bar{A}B + \bar{A}C$$

$$= \underbrace{A + AB} + BC + \bar{A}B + \bar{A}C$$

$$= A + BC + \bar{A}B + \bar{A}C$$

$$= A + BC + \bar{A}(B + C)$$

$$= \underbrace{A + \bar{A}(B + C)} + BC$$

use $x + \bar{x} y = x + y$

$x = A$

$Y = (B + C)$

$$= A + B + \underbrace{C + BC}$$

$$= A + B + C$$

6. Sum of products equations:

AB, $\bar{A}B$, $A\bar{B}$ and $\bar{A}\bar{B}$ are called fundamental products of two variables

Likewise: fundamental products of three variables are:

A	B	C	
0	0	0	$\bar{A}\bar{B}\bar{C}$
0	0	1	$\bar{A}\bar{B}C$
0	1	0	$\bar{A}B\bar{C}$
0	1	1	$\bar{A}BC$.
1	0	0	$A\bar{B}\bar{C}$
1	0	1	$A\bar{B}C$
1	1	0	$AB\bar{C}$
1	1	1	ABC

These are fundamental products of three variables

Sum of product equation: (example)

$$Y = \bar{A}BC + A\bar{B}C + AB\bar{C} + ABC$$

$$= F(A,B,C)$$

$$= \sum m(3,5,6,7)$$

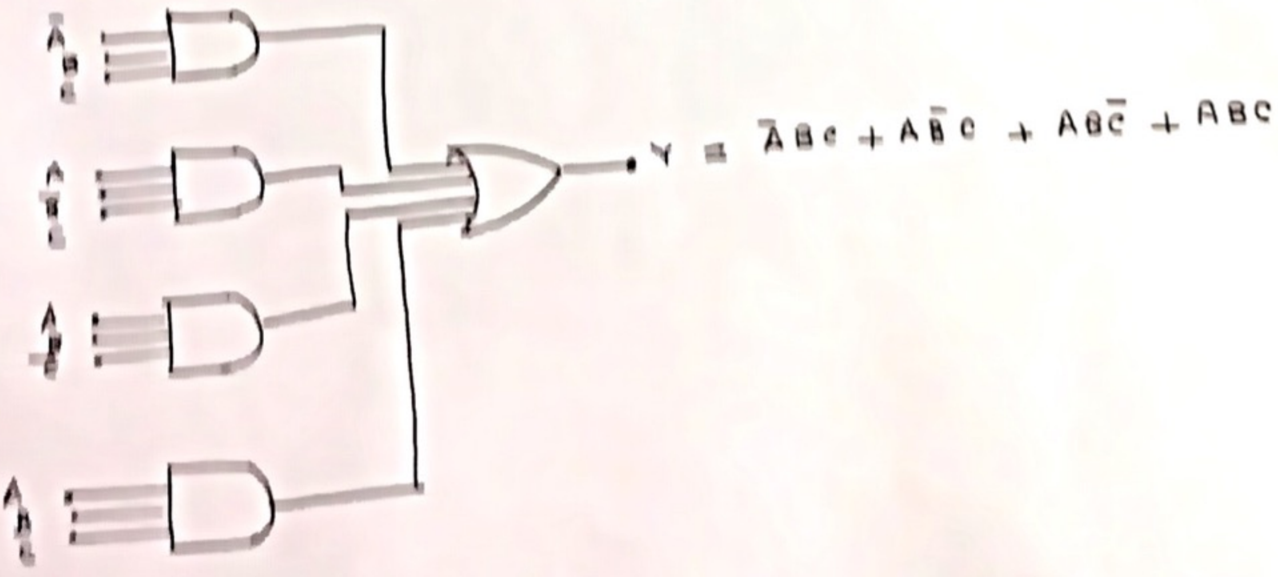
These are known as 'canonical sum forms'.

Each term here is called a 'MIN' term

Truth Table

A	B	C	Y
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

Logic circuit for the above function:

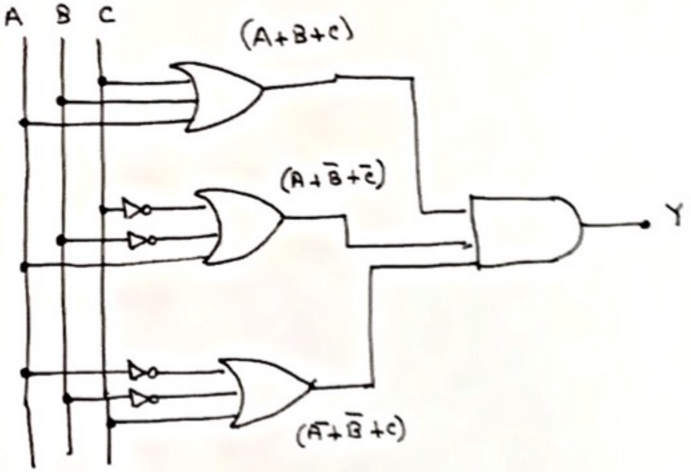


7. Product of Sums (POS) form:

For writing a logical expression in this form, we identify which sums produce a '0' at the output

A	B	C	Y	Max terms	
0	0	0	0	M_0	$(A+B+C)$
0	0	1	1	M_1	
0	1	0	1	M_2	
0	1	1	0	M_3	$(A+\bar{B}+\bar{C})$
1	0	0	1	M_4	
1	0	1	1	M_5	
1	1	0	0	M_6	$(\bar{A}+\bar{B}+C)$
1	1	1	1	M_7	

Logic Circuit:



$$\therefore \; Y = (A+B+C) \cdot (A+\bar{B}+\bar{C}) \cdot (\bar{A}+\bar{B}+C)$$

$$Y = F(A,B,C) = \prod M(0,3,6)$$

Conversion between SOP and POS :

If $Y = F(A,B,C,D) = \prod M(0,3,6,)$ then it is also equal to $\sum m(1,2,4,5,7)$

As another example, if

$Y = F(A,B,C) = \sum m(3,5,6,7)$; it is also equal to $\prod M(0,1,2,4)$.

8. Simplification of Boolean functions using Karnaugh's Maps

Karnaugh's map is a visual display of the fundamental products needed for a 'sum of products' solution.

Consider 2-Variable case

Truth Table

A	B	Y
0	0	0
0	1	0
1	0	1
1	1	1

	$\bar{B}$	B
$\bar{A}$	0	0
A	1	1

K- map representation

3- variable map:

consider three variable Truth table

A	B	C	Y
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	1

	$\bar{C}$	C
$\bar{A}\bar{B}$	0	0
$\bar{A}B$	1	0
AB	1	1
$A\bar{B}$	0	0

K-map

$$Y = \bar{A}B\bar{C} + AB\bar{C} + ABC$$

or

$$Y = F(A,B,C) = \sum m(2,6,7)$$

4-variable map:

$$\text{Let } Y = \bar{A}\bar{B}\bar{C}D + \bar{A}BC\bar{D} + \bar{A}BCD + ABC\bar{D}$$

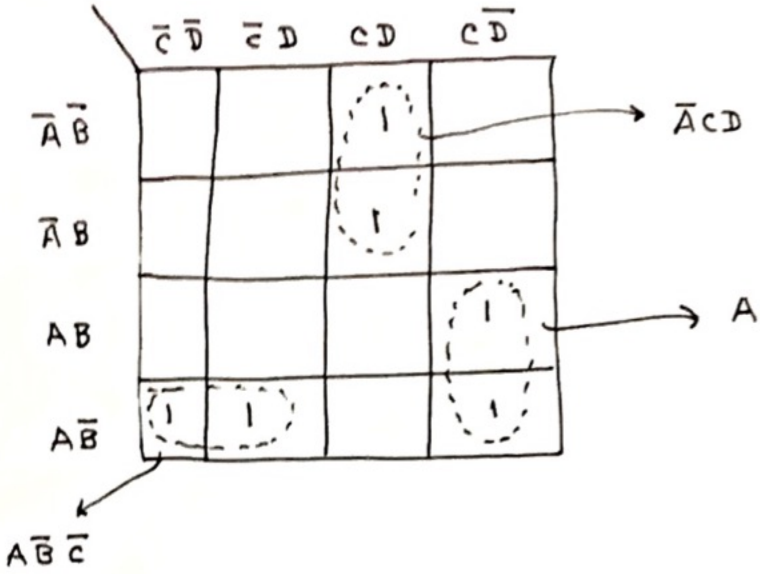
$$= \sum m(1,6,7,14)$$

A	B	C	D	Y
0	0	0	0	0
0	0	0	1	1
0	0	1	0	0
0	0	1	1	0
0	1	0	0	0
0	1	0	1	0
0	1	1	0	1
0	1	1	1	1
1	0	0	0	0
1	0	0	1	0
1	0	1	0	0
1	0	1	1	0
1	1	0	0	0
1	1	0	1	0
1	1	1	0	1
1	1	1	1	0

	$\bar{C}\bar{D}$	$\bar{C}D$	CD	$C\bar{D}$
$\bar{A}\bar{B}$	⓪ 0	① 1	③ 0	② 0
$\bar{A}B$	④ 0	⑤ 0	⑦ 1	⑥ 1
AB	⑫ 0	⑬ 0	⑮ 0	⑭ 1
$A\bar{B}$	⑧ 0	⑨ 0	⑪ 0	⑩ 0

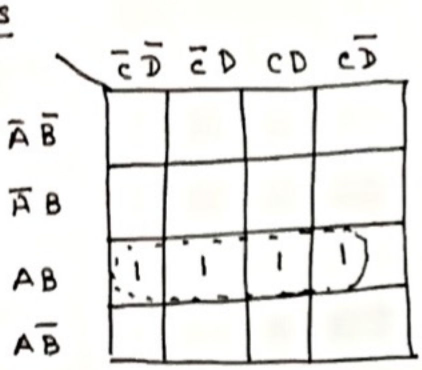
K-pap

- Pairs

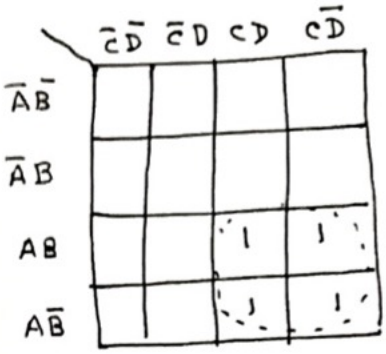


	$\bar{C}\bar{D}$	$\bar{C}D$	CD	$C\bar{D}$
$\bar{A}\bar{B}$			1	
$\bar{A}B$			1	
AB				1
$A\bar{B}$	1	1		1

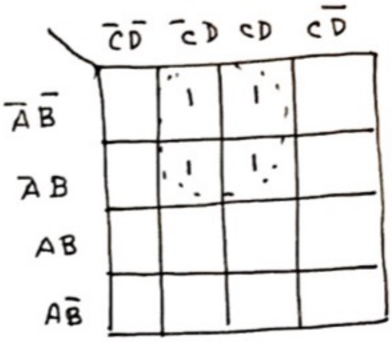
- Quads



	$\bar{C}\bar{D}$	$\bar{C}D$	CD	$C\bar{D}$
$\bar{A}\bar{B}$				
$\bar{A}B$				
AB	1	1	1	1
$A\bar{B}$				



	$\bar{C}\bar{D}$	$\bar{C}D$	CD	$C\bar{D}$
$\bar{A}\bar{B}$				
$\bar{A}B$				
AB			1	1
$A\bar{B}$			1	1



	$\bar{C}\bar{D}$	$\bar{C}D$	CD	$C\bar{D}$
$\bar{A}\bar{B}$		1	1	
$\bar{A}B$		1	1	
AB				
$A\bar{B}$				

$AB\bar{C} + ABC$

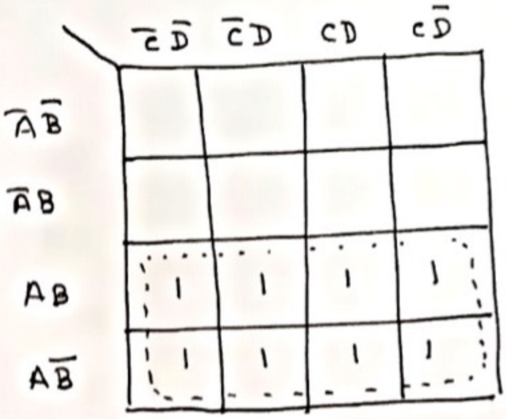
$= AB(\bar{C}+C)$

$= AB$

AC

$\bar{A}D$

- Octets



	$\bar{C}\bar{D}$	$\bar{C}D$	CD	$C\bar{D}$
$\bar{A}\bar{B}$				
$\bar{A}B$				
AB	1	1	1	1
$A\bar{B}$	1	1	1	1

sum of two quods

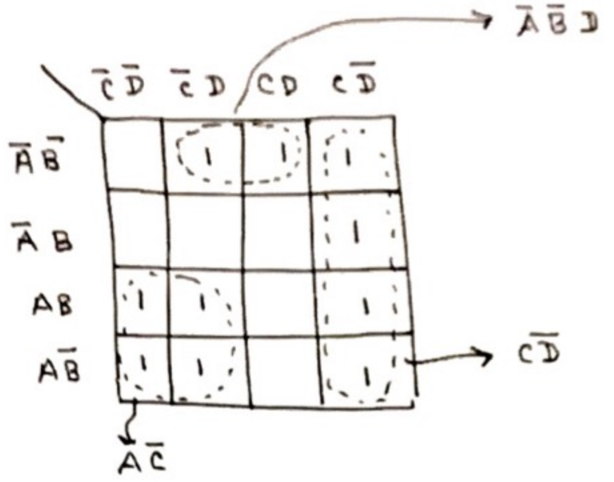
$A\bar{C} + AC$

$= A(\bar{C}+C)$

$= A$

- Example : obtain Simplified expression for Boolean function Y using Karnaugh's map shown :

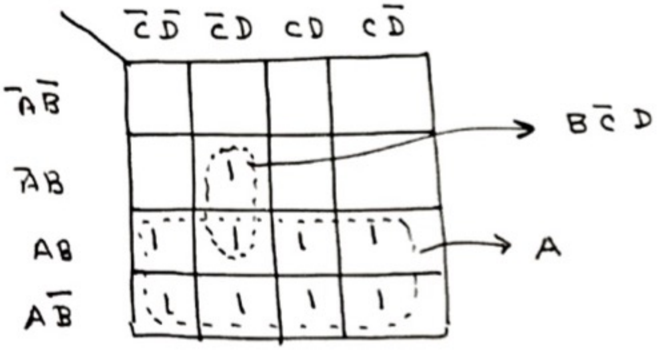
K-map for function Y



	$\bar{C}\bar{D}$	$\bar{C}D$	CD	$C\bar{D}$
$\bar{A}\bar{B}$		1	1	1
$\bar{A}B$				1
AB	1	1		1
$A\bar{B}$	1	1		1

$\therefore Y = \bar{A}\bar{B}D + A\bar{C} + C\bar{D}$

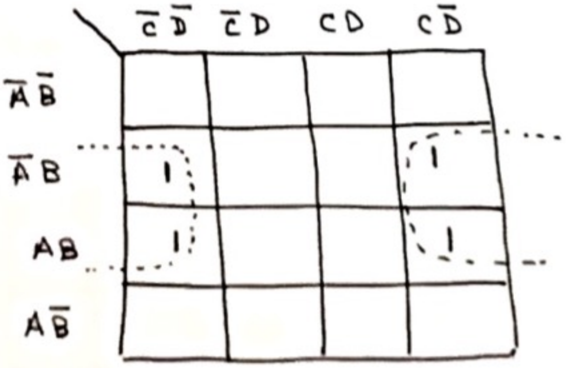
• Overlapping Groups :



	$\bar{C}\bar{D}$	$\bar{C}D$	CD	$C\bar{D}$
$\bar{A}\bar{B}$				
$\bar{A}B$		1		
AB	1	1	1	1
$A\bar{B}$	1	1	1	1

$\therefore \quad Y = A + B\bar{C}D$

• Rolling the map :

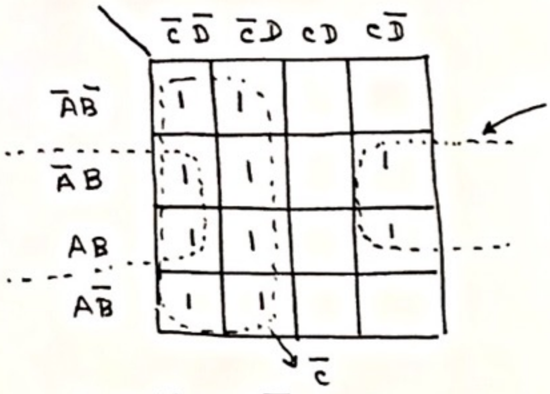


	$\bar{C}\bar{D}$	$\bar{C}D$	CD	$C\bar{D}$
$\bar{A}\bar{B}$				
$\bar{A}B$	1			1
AB	1			1
$A\bar{B}$				

$Y = B\bar{D}$

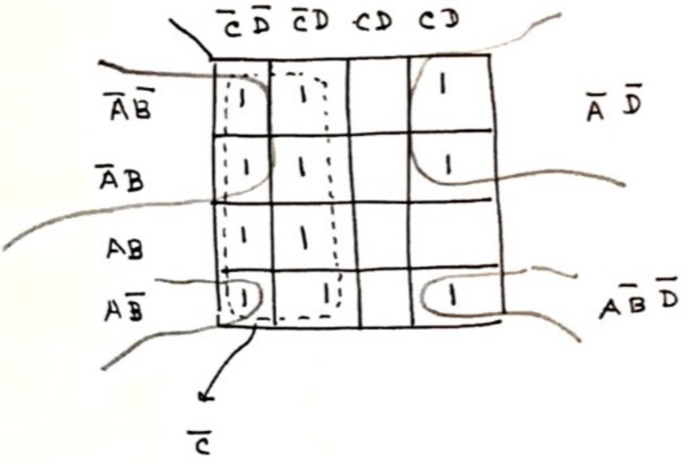
$(Y = B\bar{C}\bar{D} + BC\bar{D}$
$= B\bar{D}(\bar{C}+C)$
$= B\bar{D}$

• More examples: Find Y by simplifying the K-maps shown



	$\bar{C}\bar{D}$	$\bar{C}D$	CD	$C\bar{D}$
$\bar{A}\bar{B}$	1	1		
$\bar{A}B$	1	1		1
AB	1	1		1
$A\bar{B}$	1	1		

$Y = \bar{C} + B\bar{D}$

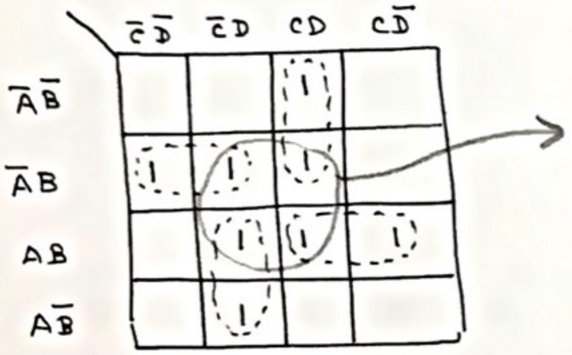


	$\bar{C}\bar{D}$	$\bar{C}D$	CD	$C\bar{D}$
$\bar{A}\bar{B}$	1	1		1
$\bar{A}B$	1	1		1
AB	1	1		
$A\bar{B}$	1	1		1

$\therefore Y = \bar{C} + \bar{A}\bar{D} + A\bar{B}\bar{D}$

or $Y = \bar{C} + \bar{A}\bar{D} + \bar{B}\bar{D}$

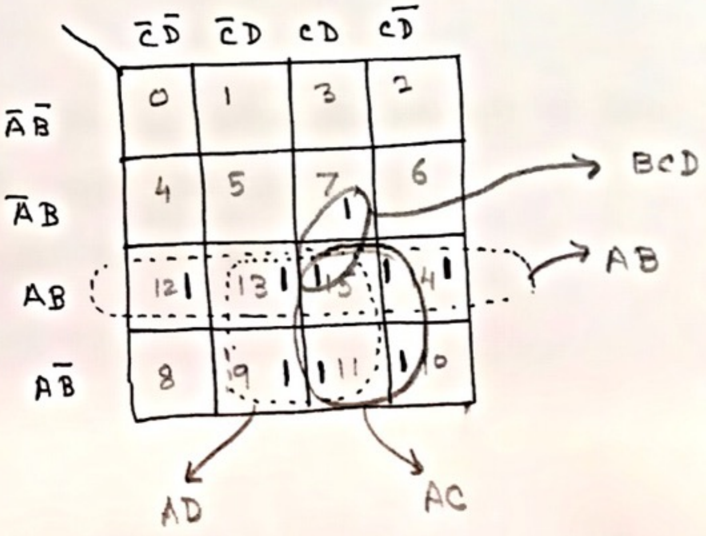
• Eliminating redundant groups:



	$\bar{C}\bar{D}$	$\bar{C}D$	CD	$C\bar{D}$
$\bar{A}\bar{B}$			1	
$\bar{A}B$	1	1	1	
AB		1	1	1
$A\bar{B}$		1		

The quad is redundant as all its 1's have been included in 4 'pairs'.

• Example: Simplify $Y = F(A,B,C,D) = \sum m\,(7, 9, 10, 11, 12, 13, 14, 15)$



	$\bar{C}\bar{D}$	$\bar{C}D$	CD	$C\bar{D}$
$\bar{A}\bar{B}$	0	1	3	2
$\bar{A}B$	4	5	7 1	6
AB	12 1	13 1	15 1	14 1
$A\bar{B}$	8	9 1	11 1	10 1

$Y = AB + AC + AD + BC$

8. Conversion of Decimal number into binary number:

Convert decimal no. 13 to binary :

$$\begin{array}{r} 6 \\ 2\,\overline{)\,13} \\ 12 \\ \hline 1 \end{array} \qquad 1$$

$$\begin{array}{r} 3 \\ 2\,\overline{)\,6} \\ 6 \\ \hline 0 \end{array} \qquad 0$$

$$\begin{array}{r} 1 \\ 2\,\overline{)\,3} \\ 2 \\ \hline 1 \end{array} \qquad 1 \quad \uparrow$$

$$\begin{array}{r} 0 \\ 2\,\overline{)\,1} \\ 0 \\ \hline 1 \end{array} \qquad 1$$

Decimal 13 $= \underbrace{1101}$ (binary)

Check : $1 \times 2^3 + 1 \times 2^2 + 0 \times 10^1 + 1 \times 10^0$

$= 8 + 4 + 0 + 1$

$= 13$

Binary coded decimal (BCD) 8421 code

2^4	2^3	2^2	2^1	2^0	
16	8	4	2	1	
1	0	0	1	1	⇒ 19

Prob. Realise $Y = AB + \overline{C}$ using only one type of Gates..

(i) using NOR gates only

(ii) using NAND gades only

9. Binary Numbers, addition/subtraction :

Decimal	2^3	2^2	2^1	2^0
0	0	0	0	0
1	0	0	0	1
2	0	0	1	0
3	0	0	1	1
4	0	1	0	0
5	0	1	0	1
6	0	1	1	0
7	0	1	1	1
8	1	0	0	0
9	1	0	0	1
10	1	0	1	0
11	1	0	1	1
12	1	1	0	0
13	1	1	0	1
14	1	1	1	0
15	1	1	1	1
	↓ Most Significant bit (MSB)			↓ Least Significant bit (LSB)

Binary addition :

Rules :

0 + 0 = 0

0 + 1 = 1

1 + 0 = 0

1 + 1 = 0 and a carry '1' to the addition of next higher significant bits

Example :

$$\begin{array}{r} 1\;0\;1\;0 \\ +\;0\;1\;1\;0 \\ \hline 1 \\ \hline 1\;0\;0\;0\;0 \end{array} \Rightarrow 16_{10} \text{ (decimal)}$$

$$\underbrace{1\;0\;0\;0\;0}_{2^4\;2^3\;2^2\;2^1\;2^0}$$

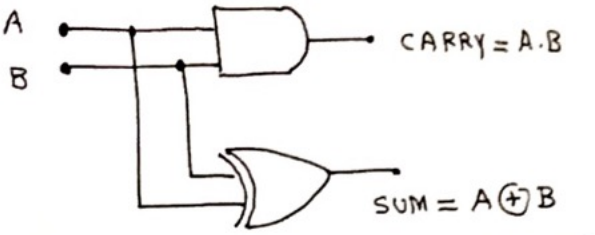
decimal

~~10~~

~~12~~

$$\begin{array}{r} 10 \\ +\;6 \\ \hline 16 \end{array}$$

10. Half adder



$CARRY = A.B$

$SUM = A \oplus B$

$= \bar{A}B + A\bar{B}$ ∴ $SUM = \bar{A}B + A\bar{B} = A \oplus B$

$CARRY = A.B$

Truth Table

A	B	SUM	CARRY
0	0	0	0
0	1	1	0
1	0	1	0
1	1	0	1

11. Full-adder

Truth table

A	B	C	SUM	CARRY
0	0	0	0	0
0	0	1	1	0
0	1	0	1	0
0	1	1	0	1
1	0	0	1	0
1	0	1	0	1
1	1	0	0	1
1	1	1	1	1

$$SUM = \bar{A}\bar{B}C + \bar{A}B\bar{C} + \bar{A}\bar{B}\bar{C} + ABC$$

$$CARRY = \bar{A}BC + A\bar{B}C + AB\bar{C} + ABC$$

$$SUM = C\overbrace{\left[\bar{A}\bar{B} + AB\right]}^{ex-NOR} + \bar{C}\overbrace{\underbrace{\left[\bar{A}B + A\bar{B}\right]}_{\to \text{assume } D}}^{ex-OR}$$

$$= C\bar{D} + \bar{C}D$$

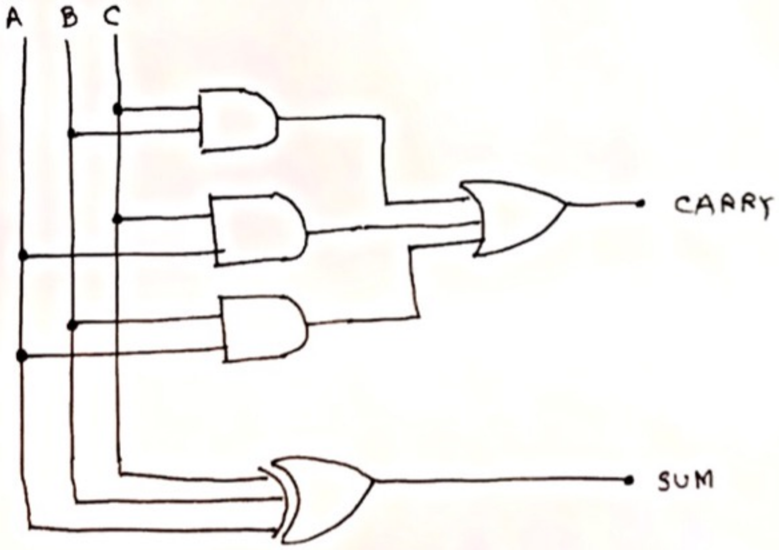
$$= C \oplus D$$

$$= C \oplus (A \oplus B)$$

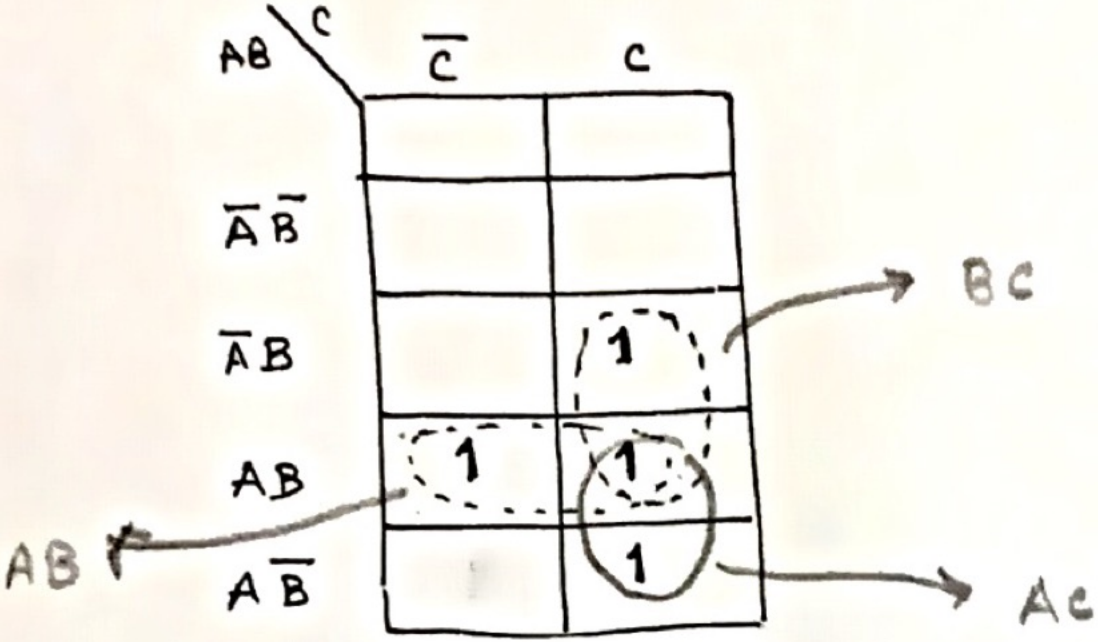
$$CARRY = (\bar{A}B + A\bar{B})C + AB(C + \bar{C})$$

$$= \bar{A}BC + A\bar{B}C + AB$$

$$= AB + BC + AC$$ (use Karnaugh's map)



CARRY



AB \ C	$\overline{C}$	C
$\overline{A}\,\overline{B}$		
$\overline{A}B$		1
AB	1	1
$A\overline{B}$		1

$\therefore$ CARRY $= AB + BC + AC$